Preservation of Commutation Relations and Physical Realizability of Open Two-Level Quantum Systems

Luis A. Duffaut Espinosa†, Z. Miao†, I. R. Petersen†, V. Ugrinovskii †, and M. R. James§

Abstract—Coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc. These approaches lack a systematic characterization of quantum realizability. Recently, a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named physical realizability, and it was developed for linear quantum systems satisfying the quantum harmonic oscillator canonical commutation relations. In this context, open two-level quantum systems escape the realm of the current known condition. When compared to linear quantum system, the challenges in obtaining such condition for such systems radicate in that the evolution equation is now a bilinear quantum stochastic differential equation and that the commutation relations for such systems are dependent on the system variables. The goal of this paper is to provide a necessary and sufficient condition for the preservation of the Pauli commutation relations, as well as to make explicit the relationship between this condition and physical realizability.

I. INTRODUCTION

In the last twenty years, the use of quantum feedback control systems has become critical for the development of quantum and nano technologies [1], [4], [5]. However, the majority of approaches consider a classical controller in the feedback loop. In this context, coherent feedback control considers purely quantum controllers in order to overcome disadvantages such as the acquisition of suitable quantum information, quantum error correction, etc [3], [9], [14]. Unfortunately, these approaches lack a systematic characterization of quantum realizability. In [8], a condition characterizing when a system described as a linear stochastic differential equation is quantum was developed. Such condition was named physical realizability, and it was developed specifically for linear systems satisfying the quantum harmonic oscillator canonical commutation relations. The class of systems for which this condition is known to be satisfied is still too limited for applications. In this paper, the focus is on systems describing the dynamics of open two-level quantum systems. Compared to a linear quantum system, the problem is more complicated and requires extra machinery for two basic reasons. The first is that the system being analyzed is bilinear, and the second is that the commutation relations that the system has to obey are now dependent on the system variables, which was not the case for linear quantum systems related to the quantum harmonic oscillator [8], [11]. In [6], a characterization of the physical realizability for open two-level quantum systems was provided. However, it is not clear whether or not such condition imply the preservation in time of the commutation relations for the system variables of the bilinear quantum stochastic differential equation (QSDE) describing the system. Thus, the main contribution of this paper, given in Section V, is to provide a necessary and sufficient condition for the preservation of Pauli commutation relations, as well as to make explicit the relationship between this condition and physical realizability. Furthermore, in Section V the physical realizability condition of open two-level quantum systems is reformulated in terms of the quadrature of the interacting Boson field, which yields a more natural self-adjoint (all component matrices are real) representation of the system and the physical realizability condition.

The paper is organized as follows. Section II presents the basic preliminaries on open quantum systems. In Section III the necessary algebraic machinery to study open two-level quantum systems is given. This is followed by Section IV in which the definition of physical realizability is provided as well as a condition for a bilinear QSDE to be physically realizable. In Section V it is shown that a physically realizable system preserves the commutation relations established for spin operators. Finally, Section VI gives the conclusions.

II. OPEN TWO-LEVEL QUANTUM SYSTEMS

Systems governed by the laws of quantum mechanics that interact with an external environment (e.g., electromagnetic field) are known as open quantum systems. In order to study such systems, one has to give a quantum description of both the system and the interacting environment. The quantum mechanical behavior of the system is based on the notions of observables and states. Observables represent physical quantities that can be measured, as self-adjoint operators on a complex separable Hilbert space $\mathcal{H}$, while states give the current status of the system, as elements of $\mathcal{H}$, allowing the computation of expected values of observables. Here open quantum systems are treated in the context of quantum stochastic processes (see [2], [13] for more information). For this purpose, observables may be thought as quantum random variables that do not in general commute. A measure of the non-commutativity between observables is usually given by the commutator between operators. The commutator of two scalar operators $x$ and $y$ in $\mathcal{H}$ is an antisymmetric bilinear...
operation defined as

\[ [x, y] = xy - yx. \]

Also, for an \( n \)-dimensional vector of operators \( x \) and an \( m \)-dimensional vector of operators \( y \), the commutator of \( x \) and \( y \) is

\[ [x, y] \triangleq xy^T - (yx^T)^T. \]

In particular, the commutator of \( x \) and its adjoint \( x^\dagger \) is the \( n \times n \) matrix of operators

\[ [x, x^\dagger] \triangleq xx^\dagger - (x^\# x^T)^T, \]

where

\[ x^\# \triangleq \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} \]

and * denotes the operator adjoint. In the case of complex vectors (matrices) * denotes the complex conjugate while \( \dagger \) denotes the conjugate transpose. The non-commutativity of observables is a fundamental difference between quantum systems and classical systems in which the former must satisfy certain commutation relations originating from Heisenberg uncertainty principle. The environment consists of a collection of oscillator systems each with annihilation field operator \( w(t) \) and creation field operator \( w^*(t) \) used for the annihilation and creation of quanta at point \( t \), and commonly known as the boson quantum field (with parameter \( t \)). Here it is assumed that \( t \) is a real time parameter. The field operators \( w(t) \) and \( w^*(t) \) satisfy commutation relations as well. That is,

\[ [w(t), w^*(t')] = \delta(t - t'), \]

for all \( t, t' \in \mathbb{R} \), where \( \delta(t) \) denotes the Dirac delta. Its mathematical description is given in terms of a Hilbert space called a Fock space. When the boson quantum field is in the vacuum state, i.e., no physical particles are present, it then represents a natural quantum extension of white noise, and may be described using the quantum Itô calculus [2], [13]. This amounts to have three interacting signals (inputs) in the evolution of the system: the annihilation processes \( W(t) \), the creation process \( W^\dagger(t) \), and the counting process \( \Lambda(t) \). The evolution of an open quantum system (i.e., the system together with the environment) is unitary. That is, if \( \psi \) is a state then \( \psi(t) = U(t)\psi \), where \( U(t) \) is unitary for all \( t \), and is the solution of

\[
\begin{align*}
\frac{dU(t)}{dt} &= \left((S - I) \, d\Lambda(t) + L \, dW^\dagger(t) - L^\dagger S \, dW(t) \right. \\
&\quad \left. - \frac{1}{2}(L^\dagger L + i[H]) \, dt \right) \, U(t),
\end{align*}
\]

with initial condition \( U(0) = I \), \( I \) denoting the identity operator and \( i \) being the imaginary unit. Here, \( H \) is a fixed self-adjoint operator representing the Hamiltonian of the system, and \( L \) and \( S \) are operators determining the coupling of the system to the field, with \( S \) unitary. The evolution of \( \psi \) is equivalent to the evolution of the observable \( X \) given by

\[
X(t) = U^\dagger(t)(X \otimes I) \, U(t),
\]

whose evolution is referred as the Heisenberg picture while the one for \( \psi \) is known as the Schrödinger picture. This paper exclusively takes the point of view of the Heisenberg picture. Quantum stochastic calculus allows then to express the Heisenberg picture evolution of \( X \) as

\[
dX = (S^\dagger XS - X) \, d\Lambda + L(X) \, dt + S^\dagger [X, L] \, dW^\dagger + [L^\dagger, X]S \, dW,
\]

where \( \mathcal{L}(X) \) is the Lindblad operator defined as

\[
\mathcal{L}(X) = -i[X, H] + \frac{1}{2} [L^\dagger[X, L] + [L^\dagger, X]L].
\]

The output field is given by

\[
Y(t) = U(t)^\dagger W(t) U(t),
\]

which amount to

\[
dY = L \, dt + S \, dW.
\]

In summary, one can say from the discussion above that the dynamics of an open quantum systems is uniquely determined by the triple of operators \( (S, L, H) \). Hereafter, the operator \( S \) is assumed to be the identity operator \( (S = I) \).

The main focus of this paper is on the dynamics of open two-level quantum systems interacting with one boson quantum field. The Hilbert space for this system is \( \mathcal{H} = \mathbb{C}^2 \), the two dimensional complex vector space. The vector of system variables is

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \triangleq \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix},
\]

where \( \hat{\sigma}_1 \), \( \hat{\sigma}_2 \) and \( \hat{\sigma}_3 \) are spin operators. Given that these operators are self-adjoint, the vector of operators \( x \) satisfies \( x = x^\# \). In particular, a self-adjoint operator \( \hat{\sigma} \) in \( \mathcal{H} \) is spanned by the Pauli matrices [12], i.e.,

\[
\hat{\sigma} = \frac{1}{2} \sum_{i=0}^{3} \alpha_i \sigma_i,
\]

where \( \alpha_0 = \text{Tr}(\hat{\sigma}) \), \( \alpha_i = \text{Tr}(\hat{\sigma} \sigma_i) \), and

\[
\begin{align*}
\sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

denote the Pauli matrices. Thus, \( \alpha_0 \) and \( (\alpha_1, \alpha_2, \alpha_3)^T \in \mathbb{C}^3 \) determine uniquely the operator \( \hat{\sigma} \) with respect to a given basis in \( \mathbb{C}^2 \). The initial value of the system variables can be set to \( x(0) = (\sigma_1, \sigma_2, \sigma_3) \). The product of spin operators satisfy

\[
\sigma_i \sigma_j = \delta_{ij} + \sum_k \epsilon_{ijk} \sigma_k.
\]
for \( i, j, k \in \{1, 2, 3\} \). It is then clear that the commutation relations for Pauli matrices are

\[
[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk} \sigma_k,
\]

where \( \delta_{ij} \) is the Kronecker delta and \( \epsilon_{ijk} \) denotes the Levi-Civita tensor defined as

\[
\epsilon_{ijk} = \begin{cases} +1, & \{i, j, k\} \text{ is an even permutation of } \{1, 2, 3\}, \\ -1, & \{i, j, k\} \text{ is an odd permutation of } \{1, 2, 3\}, \\ 0, & \text{otherwise}. \end{cases}
\]

Due to the fact that the Pauli matrices form a complete orthogonal set, any Hamiltonian and coupling operators of polynomial type are representable as linear functions of \( x \). Therefore, assuming linearity captures a large class of Hamiltonian and coupling operators without much loss of generality, i.e.,

\[
\mathcal{H} = \alpha x \quad \text{and} \quad L = \Lambda x,
\]

where \( \alpha^T \in \mathbb{R}^3 \) and \( \Lambda^T \in \mathbb{C}^3 \). As mentioned before, the coupling operator specifies how the interacting field acts on \( x \). In general, the dimensionality of the coupling matrix \( \Lambda \) depends proportionally on the number of interacting fields.

It is customary to express QSDEs in terms of its interaction with quadrature fields. The quadrature fields are given by the transformation

\[
\begin{pmatrix}
\bar{W}_1 \\
\bar{W}_2
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} W \\ W^\dagger \end{pmatrix},
\]

(4)

where the operators \( \bar{W}_1 \) and \( \bar{W}_2 \) are now self-adjoint. In [7], the Itô table for \( W \) and \( W^\dagger \) is

\[
\begin{pmatrix} dW \\ dW^\dagger \end{pmatrix} (dW \ dW^\dagger) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dt,
\]

which in terms of the quadrature fields is

\[
\begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix} (d\bar{W}_1 \ d\bar{W}_2) = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} dt.
\]

Observe that, in general, the evolution of \( x \) (standard form) falls into a class of bilinear QSDEs expressed as

\[
dx = F_0 \, dt + F \, x \, dt + G_1 x \, d\bar{W}_1 + G_2 x \, d\bar{W}_2,
\]

(5)

where \( F_0 \in \mathbb{R}^3 \) and \( F, G_1, G_2 \in \mathbb{R}^{3 \times 3} \). The fact that all matrices in (5) are real is due to the quadrature transformation (4). The output field is

\[
dY = Hx \, dt + \frac{1}{2} (d\bar{W}_1 + id\bar{W}_2)
\]

with \( H^T \in \mathbb{C}^3 \). Similarly, the quadrature form of the output fields can be obtained from the transformation

\[
\begin{pmatrix}
\bar{Y}_1 \\
\bar{Y}_2
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} Y \\ Y^\dagger \end{pmatrix}.
\]

Thus,

\[
\begin{pmatrix} d\bar{Y}_1 \\ d\bar{Y}_2 \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} x \, dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d\bar{W}_1 \\ d\bar{W}_2 \end{pmatrix},
\]

(6)

where

\[
H_1 = H + H^\# \quad \text{and} \quad H_2 = i(H^\# - H)
\]

are obviously real matrices.

In this context, the goal of the paper can now be stated more specifically. Given a bilinear QSDE as in (5), under what condition there exist \( \mathcal{H} \) and \( L \) such that (5) can be written as (I). Such condition is given in Section IV.

### III. NOTATION AND ALGEBRAIC RELATIONS

In order to continue the description of open two-level quantum systems, some linear algebra identities are needed. Let \( \beta \in \mathbb{C}^3 \) and define the linear mapping \( \Theta : \mathbb{C}^3 \to \mathbb{C}^{3 \times 3} \) as

\[
\Theta(\beta) = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}.
\]

(7)

Note here that this definition allows \( \beta \) to be either a column or a row vector. The fact that \( \beta \) is either a column or a row vector will be clear from the context. It will also be convenient to rewrite \( \Theta(\beta) \) in terms of its columns. That is,

\[
\Theta(\beta) = (\Theta_1(\beta), \Theta_2(\beta), \Theta_3(\beta)).
\]

(8)

The product of Pauli operators can be expressed in a compact matrix form thanks to the mapping \( \Theta \). That is,

\[
x x^T = I + i \Theta(x).
\]

Similarly, the commutation relations for Pauli operators are written as

\[
x x^T = 2i \Theta(x).
\]

Consider now the stacking operator \( \text{vec} : \mathbb{C}^{m \times n} \to \mathbb{C}^{mn} \) whose action on a matrix creates a column vector by stacking its columns below one another. With the help of \( \text{vec} \), the matrix \( \Theta(\beta) \) can be reorganized so that it gives

\[
\text{vec}(\Theta(\beta)) = \begin{pmatrix} \Theta_1(\beta) \\ \Theta_2(\beta) \\ \Theta_3(\beta) \end{pmatrix} = E \beta,
\]

where \( \beta \) is a column vector, \( \Theta_i(\beta) = \bar{e}_i^T \beta \),

\[
E \triangleq \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix}.
\]

and

\[
\bar{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The set \( \{-i\bar{e}_1, -i\bar{e}_2, -i\bar{e}_3\} \) can be identified to be the adjoint representation of \( SU(2) \), which has as generators the Pauli matrices. It is thus that one can rewrite the matrix \( \bar{e}_k \) as

\[
\bar{e}_k = \begin{pmatrix} \epsilon_{k11} & \epsilon_{k12} & \epsilon_{k13} \\ \epsilon_{k21} & \epsilon_{k22} & \epsilon_{k23} \\ \epsilon_{k31} & \epsilon_{k32} & \epsilon_{k33} \end{pmatrix},
\]
where the Levi-Civita tensor is known as the completely antisymmetric structure constant of SU(2). Observe also that
\[
\bar{e}_k = \epsilon_{ijk} (\mathbb{1}_{ji} - \mathbb{1}_{ij})
\]  
(9)
with \(i \neq j \neq k\) and \(\mathbb{1}_{ij} \in \mathbb{R}^{3 \times 3}\) being an elementary matrix (i.e., matrix consisting of 1 in the \((i,j)\) position and 0 everywhere else). In addition, the matrix \(E\) satisfies
\[
E^T E = 2I.
\]  
(10)
If one defines the block matrix \(\mathbb{1}_E = \{\mathbb{1}_{ji}\}_{i,j=1}^{3} \in \mathbb{R}^{9 \times 9}\), then \(E\) also satisfies
\[
\mathbb{1}_E E = -E.
\]  
(12)
The matrix \(\mathbb{1}_E\) can be identified as a tensor permutation matrix, which comes from the fact that the Levi-Civita tensor satisfies the contraction epsilon identity
\[
\sum_{i=1}^3 \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}.
\]  
(13)
The properties of \(\Theta(\beta)\) are summarized in the next lemma.

**Lemma 1:** Let \(\beta, \gamma \in \mathbb{C}^3\) be column vectors. The mapping \(\Theta\) satisfies
1. \(\Theta(\beta) \gamma = -\Theta(\gamma) \beta\),
2. \(\Theta(\beta) \beta = 0\),
3. \(\bar{e}_i \Theta(\beta) = \beta e_i^T - \beta_i I\),
4. \(\Theta(\beta) \Theta(\gamma) = \gamma T - \beta^T \gamma I\),
5. \(\Theta(\Theta(\beta) \gamma) = [\Theta(\beta), \Theta(\gamma)]\),

where \(I\) denotes the identity matrix, and \(e_i\) is an element of the canonical basis of \(\mathbb{R}^3\) with \(i\) indicating the position of the nonzero element.

**Proof:** To show (1), one uses the fact that \(\eta = \eta^T\) when \(\eta \in \mathbb{C}\). If the vector \(\Theta(\beta) \gamma\) is decomposed component-wise, then
\[
\Theta(\beta) \gamma = \begin{pmatrix} \beta^T \bar{e}_1^T \gamma \\ \beta^T \bar{e}_2^T \gamma \\ \beta^T \bar{e}_3^T \gamma \end{pmatrix} = \begin{pmatrix} \gamma^T \bar{e}_1 \beta \\ \gamma^T \bar{e}_2 \beta \\ \gamma^T \bar{e}_3 \beta \end{pmatrix} = -\Theta(\gamma) \beta.
\]
(14)
Property (12) is true since \(\epsilon_{ijj} = 0\) for all \(i\) and \(j\), and
\[
\bar{e}_i \Theta(\beta) \gamma = \beta^T \bar{e}_i \gamma = \beta^T \epsilon_{ijk} (\mathbb{1}_{ji} - \mathbb{1}_{ij}) \gamma = \beta^T \epsilon_{ijk} \epsilon_{imn} (\mathbb{1}_{ji} - \mathbb{1}_{ij}) \gamma = \beta^T \epsilon_{ijk} \epsilon_{imn} (\mathbb{1}_{ji} - \mathbb{1}_{ij}) \gamma = \beta^T (\mathbb{1}_{ji} - \mathbb{1}_{ij}) \gamma = \beta_i \gamma - \beta^T \gamma I.
\]
(11)
which means that \(\bar{e}_i \Theta(\beta) = \beta_i \gamma - \beta^T \gamma I\). Then,
\[
\bar{e}_i \Theta(\beta) \Theta(\gamma) = \gamma T - \beta^T \gamma I.
\]
(12)
Finally, the left-hand-side of (12) can be written using (9) as
\[
\Theta(\Theta(\beta) \gamma) = -\Theta(\beta^T \bar{e}_1 \gamma) = -\Theta(\beta^T \bar{e}_2 \gamma) = -\Theta(\beta^T \bar{e}_3 \gamma).
\]
(13)
The (\(i,j\)) component of \(\Theta(\Theta(\beta) \gamma)\) is then
\[
e^T \Theta(\Theta(\beta) \gamma) e_j = \beta^T (\mathbb{1}_{ji} - \mathbb{1}_{ij}) \gamma = \beta_i \gamma - \beta^T \gamma I \gamma = \gamma_i \beta_j - \beta \gamma_j.
\]  
(14)
Hence, from (14), it follows that
\[
\Theta(\Theta(\beta) \gamma) = \gamma T - \beta^T \gamma I - \beta^T \gamma I = (\gamma T - \beta^T \gamma I) - (\beta^T \gamma I - \beta^T \gamma I) = \Theta(\beta) \Theta(\gamma) - \Theta(\gamma) \Theta(\beta) = [\Theta(\beta), \Theta(\gamma)].
\]
(15)
The explicit computation of the vector fields in (1) is given by the next lemma.
Lemma 2: The component coefficients of equations \((14a)\) and \((14b)\) are

\[
[x, \mathcal{H}] = -2i\Theta(\alpha)x, \quad (14a) \\
[x, L] = -2i\Theta(\Lambda)x, \quad (14b) \\
[x, L^\dagger] = -2i\Theta(\Lambda^\#)x, \quad (14c) \\
L^\dagger [x, L] = -2i\Theta(\Lambda)\Lambda^\dagger - 2(\Lambda\Lambda^\dagger I - \Lambda^\dagger\Lambda)x, \quad (14d) \\
[x, L^\dagger]L = 2i\Theta(\Lambda)\Lambda + 2(\Lambda\Lambda^\dagger I - \Lambda^T\Lambda^\#)x. \quad (14e)
\]

Proof: For \((14a)\), one has by the definition of the commutator that
\[
x, \mathcal{H} = [x, \alpha x] = x(\alpha x) - ((\alpha x)x^T) = (xx^T)\alpha^T - (xx^T)^T\alpha^T = 2i\Theta(\alpha)x.
\]

Given that the components of \(\alpha\) and \(x\) commute, the commutator \([x, \mathcal{H}]\) is rewritten in standard form by applying property \(i\) of Lemma 1. Thus,
\[
x, \mathcal{H} = -2i\Theta(\alpha)x.
\]

The procedure to compute \((14b)\) and \((14c)\) is identical to the one above. Hence,
\[
x, L = -2i\Theta(\Lambda)x \quad \text{and} \quad [x, L^\dagger] = -2i\Theta(\Lambda^\#)x.
\]

The computation of \((14d)\) is done by directly multiplying the scalar operator \(L^\dagger\) and the vector operator \([x, L]\). Recall that \(x^\dagger = x^T\) since \(x\) is self-adjoint. It then follows that
\[
L^\dagger [x, L] = -2i\Theta(\Lambda)x \quad \text{and} \quad [x, L^\dagger]L = 2i\Theta(\Lambda)\Lambda + 2(\Lambda\Lambda^\dagger I - \Lambda^T\Lambda^\#)x.
\]

Finally, \((14e)\) is computed similarly. That is,
\[
x, L^\dagger]L = -2i\Theta(\Lambda^\#)x, \quad [x, L^\dagger]L = -2i\Theta(\Lambda^\#)x, \quad [x, L] = -2i\Theta(\Lambda^\#)x \quad \text{and} \quad [x, L^\dagger]L = 2i\Theta(\Lambda)\Lambda^\dagger + 2(\Lambda\Lambda^\dagger I - \Lambda^T\Lambda^\#)x.
\]

From \((14a)-(14e)\), one can now write equation \((5)\) as the following bilinear QSDE

\[
dx = -2i\Theta(\Lambda)\Lambda^\dagger dt - 2\Theta(\alpha)x dt + (\Lambda \Lambda^\dagger I + \Lambda^\dagger\Lambda + \Lambda^T\Lambda^\#)x dt + i\Theta(\Lambda^\# - \Lambda)x d\bar{W}_1 - \Theta(\Lambda + \Lambda^\#)x d\bar{W}_2.
\]

Note that \(\Theta(\Lambda)\Lambda^\dagger = -\Theta(\Lambda)\Lambda^\dagger\), which assures that

\[
\text{Re}\{\Theta(\Lambda)\Lambda^\dagger\} = \frac{1}{2}(\Theta(\Lambda)\Lambda^\dagger + (\Theta(\Lambda)\Lambda^\dagger)^*) = 0.
\]

Also, observe that \(\Lambda^\# - \Lambda\) is purely imaginary and \(\Lambda + \Lambda^\#\) is purely real. Therefore, all matrices in \((15)\) are real.

As mentioned in Section III the output fields \(Y_1\) and \(Y_2\) depend linearly on \(L, L^\dagger\) and the fields \(W_1\) and \(W_2\), i.e.,
\[
\begin{bmatrix}
\frac{dY_1}{dt} \\
\frac{dY_2}{dt}
\end{bmatrix} = \begin{bmatrix}
\Lambda + \Lambda^\# \\
i(\Lambda^\# - \Lambda)
\end{bmatrix} x dt + \begin{bmatrix}
\bar{dW}_1 \\
\bar{dW}_2
\end{bmatrix}.
\]

IV. PHYSICAL REALIZABILITY

In an environment where the classical laws of physics apply, standard control techniques such as optimization or a Lyapunov procedures do not worry in general of the nature of the controller they synthesized. In other words, their implementation is always possible since the physics behind them still holds. However, if one desires to implement a controller that obeys the laws imposed by quantum mechanics (quantum coherent control), then such a task is not so easily achieved unless an explicit characterization of those laws is given in terms of the control system vector fields. This is exactly the purpose for introducing the concept of a physically realizable system in the next definition.

Definition 1: System \((5)\) with output equation \((6)\) is said to be physically realizable if there exist \(\mathcal{H} = \alpha x\), with \(\alpha^T \in \mathbb{R}^3\), and \(L = \Lambda x\), with \(\Lambda^T \in \mathbb{C}^3\) such that

\[
F_0 = -2i\Theta(\Lambda)\Lambda^\dagger, \quad \text{(16a)} \\
F = -2\Theta(\alpha) + \Lambda^\dagger\Lambda + \Lambda^T\Lambda^\# - 2\Lambda\Lambda^\dagger I, \quad \text{(16b)} \\
G_1 = \Theta(i(\Lambda^\# - \Lambda)), \quad \text{(16c)} \\
G_2 = -\Theta(\Lambda + \Lambda^\#), \quad \text{(16d)} \\
H_1 = \Lambda + \Lambda^\#, \quad \text{(16e)} \\
H_2 = i(\Lambda^\# - \Lambda). \quad \text{(16f)}
\]

Note by direct inspection that for a physically realizable system \(G_{i}^T = G_{i}\) for \(i = 1, 2\).

From a control perspective, it is necessary to characterize when a bilinear QSDE posses underlying Hamiltonian and coupling operators which allows to express the matrices comprising \((5)\) and \((6)\) as in Definition 1. Thus, the main result of the paper is given in the next theorem, which establishes necessary and sufficient conditions for the physical realizability of a bilinear QSDE.

Theorem 1: System \((5)\) with output equation \((6)\) is physically realizable if and only if

\begin{itemize}
  \item[i.] \(F_0 = \frac{1}{2}(G_1 - iG_2)(H_1 + iH_2)^\dagger\),
  \item[ii.] \(G_1 = \Theta(H_2)\),
  \item[iii.] \(G_2 = -\Theta(H_1)\),
  \item[iv.] \(F + F^T + G_1G_1^T + G_2G_2^T = 0\).
\end{itemize}

In which case, one can identify the matrix \(\alpha\) defining the system Hamiltonian as

\[
\alpha = \frac{1}{8}\text{vec}(F - F^T)TE.
\]
and the coupling matrix can be identified to be
\[
\Lambda = \frac{1}{2}(H_1 + iH_2).
\]

**Proof:** Assuming that (5) and (6) are physically realizable implies that (16) and (17) hold. By property (iv) of Lemma 1, it follows that
\[
G_1G_1^T = \Theta(\Lambda^\# - \Lambda)^2
\]
\[
= \Lambda'\Lambda^\# - \Lambda'\Lambda - \Lambda'\Lambda^\# + \Lambda T\Lambda
\]
\[
- (\Lambda'\Lambda^\# - \Lambda'\Lambda^\# - \Lambda'\Lambda + \Lambda'\Lambda^\#)I.
\]
(17)

Similarly,
\[
G_2G_2^T = -\Theta(\Lambda + \Lambda^\#)^2
\]
\[
= (\Lambda\Lambda^T + \Lambda\Lambda^\# + \Lambda^\#\Lambda^T + \Lambda^\#\Lambda^\#)I
\]
\[
- \Lambda T\Lambda - \Lambda T\Lambda^\# - \Lambda^\#\Lambda - \Lambda^\#\Lambda^\#. \hspace{0.5cm} (18)
\]

Thus, \(G_1G_1^T + G_2G_2^T = 2 \left(2\Lambda\Lambda^I - \Lambda'\Lambda - \Lambda'\Lambda^\# \right)\). One can now rewrite \(F\) in terms of \(G_1\) and \(G_2\) as
\[
F = -2\Theta(\alpha) - \frac{1}{2} \left(G_1G_1^T + G_2G_2^T \right).
\]
Similarly, \(F^T = 2\Theta(\alpha) - \frac{1}{2} \left(G_1G_1^T + G_2G_2^T \right)\) since \(G_1^T = -G_1\). Hence,
\[
F + F^T + G_1G_1^T + G_2G_2^T = 0.
\]

Conversely, one needs to show that if conditions (11)-(12) of Theorem 1 are satisfied, then there exist matrices \(\alpha\) and \(\Lambda\) such that system (3) is physically realizable. Let
\[
\Theta(\alpha) = \frac{1}{4} \left(F - F^T \right). \hspace{0.5cm} (19)
\]

It is trivial to check that the right-hand-side of (19) is antisymmetric with zero diagonal and hence this equation uniquely defines \(\alpha\) via (7). Also, let \(\Lambda = \frac{1}{2}(H_1 + iH_2)\). It follows that
\[
\Lambda^\# \Lambda = \frac{1}{4} \left( \begin{bmatrix} H_1^1H_1 + iH_1^2H_2 - iH_2^2H_1 + H_2^1H_2 \end{bmatrix} \right),
\]
\[
\Lambda \Lambda^\# = \frac{1}{4} \left( \begin{bmatrix} H_1^1H_1^* - iH_1^2H_2^* + iH_2^2H_1^* + H_2^1H_2^* \end{bmatrix} \right)
\]
and
\[
\Lambda\Lambda^\# = \frac{1}{4} \left( \begin{bmatrix} H_1^1H_1 - iH_1^2H_2 - iH_2^2H_1 + H_2^1H_2 \end{bmatrix} \right).
\]

Recall from Section 3 that \(H_i = H_i^1\) and \(G_i = G_i^1\) for \(i = 1, 2\). It then follows that
\[
\Lambda^\# \Lambda + \Lambda\Lambda^\# - 2\Lambda\Lambda^I
\]
\[
= \frac{1}{2} \left( \begin{bmatrix} H_1^1H_1 + H_1^2H_2 - \left( H_1^1H_1^1 + H_2^1H_2 \right) \end{bmatrix} \right).
\]

From conditions (11) and (13), one obtains
\[
G_1G_1^T + G_2G_2^T = -G_1G_1^T - G_2G_2^T
\]
\[
= \left( H_1^1H_1^1 + H_2^1H_2 \right) I - H_1^1H_1 - H_2^1H_2.
\]

Therefore,
\[
G_1G_1^T + G_2G_2^T = 2 \left( 2\Lambda\Lambda^I - \Lambda'\Lambda - \Lambda'\Lambda^\# \right).
\]

From (11), one obtains \(F = -F^T - G_1G_1^T - G_2G_2^T\). Then,
\[
\Theta(\alpha) = \frac{1}{4} \left( -2F^T + G_1G_1^T + G_2G_2^T \right),
\]
which agrees with (16). Moreover, from (8), (10), (19) and by applying the stacking operator to \(\Theta(\alpha)\), \(\alpha\) is explicitly obtained as \(vec(\Theta(\alpha)) = \frac{1}{4} vec(F - F^T)\). Multiplying both sides by \(E^T\) leaves
\[
\alpha = \frac{1}{8} vec(F - F^T)^T E,
\]
which completes the proof.

V. PRESERVATION OF CANONICAL COMMUTATION RELATIONS

The goal of this section is to show that the conditions presented in Theorem 1 are necessary and sufficient for preserving the Pauli commutation relations (3) by the system (5). To achieve this task, a property of the stacking operator and a lemma are needed. Let \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times l}\) and \(C \in \mathbb{R}^{k \times r}\) for \(n, m, l, r \in \mathbb{N}\). It is well-known that the stacking operator used at the end of Section 3 satisfies
\[
vec(ABC) = (C^T \otimes A)vec(B).
\]

**Lemma 3:** Let \(E = (\bar{e}_1, \bar{e}_2, \bar{e}_3)^T\), and \(A, B \in \mathbb{R}^{n \times n}\) for \(n \in \mathbb{N}\). Then

i. \(E^T(A \otimes B)E = E^T(B \otimes A)E\).

ii. \(E^T(I \otimes A)E = Tr(A)I - A^T\).

iii. \(E^T(A \otimes B)E = A^T B^T + B^T A^T + Tr(A)Tr(B)I - Tr(B)A^T - Tr(A)B^T - Tr(AB)I\).

iv. \(E^T(A \otimes B)E = (A \otimes B)E + (B \otimes A)E\).

**Proof:** To prove (i), a key observation is that the matrix \(I_E\) is a symmetric permutation matrix such that \(I_E(A \otimes B)\) is \((B \otimes A)\). By (12),
\[
E^T(A \otimes B)E = (-I_E)E^T(A \otimes B)(-I_E) = E^T(I_E(A \otimes B)I_E)E = E^T(B \otimes A)E.
\]

Expanding \(E^T(I \otimes A)E\), identity (6) is computed as
\[
E^T(I \otimes A)E = -\sum_{k=1}^{3} \bar{e}_k A \bar{e}_k.
\]

From (13), it then follows that
\[
\left( \begin{array}{ccc}
\sum_{k=1}^{3} \bar{e}_k A \bar{e}_k \\
\end{array} \right)_{ij}
\]
\[
= \sum_{k=1}^{3} \epsilon^T \bar{e}_k A \bar{e}_k \epsilon_j
\]
\[
= \sum_{k=1}^{3} \left( \epsilon_{kj1}, \epsilon_{kj2}, \epsilon_{kj3} \right) A \left( \epsilon_{k1j}, \epsilon_{k2j}, \epsilon_{k3j} \right)
\]
\[ = \sum_{k=1}^{n} \sum_{r,s=1}^{n} \epsilon_{krs} A_{rs} \epsilon_{ksj} \]
\[ = \sum_{r,s=1}^{n} A_{rs} \sum_{k=1}^{n} \epsilon_{krs} \epsilon_{ksj} \]
\[ = \sum_{r,s=1}^{n} A_{rs} (\delta_{rs} \delta_{sj} - \delta_{ij} \delta_{rs}) \]
\[ = A_{ij} \sum_{r=1}^{n} A_{rr} \delta_{ij} \]
\[ = (A^T - \text{Tr}(A) I)_{ij} \]

Therefore, it is clear that
\[ E^T (I \otimes A) E = \text{Tr}(A) I - A^T. \]

From (3) and equation (11), identity (12) is obtained as
\[ E^T (A \otimes B) E = E^T (A \otimes I) (I \otimes B) E + E^T (A \otimes I) (I \otimes B) E - E^T (A \otimes I) (I \otimes B) E + E^T (A \otimes I) (I \otimes B) E = (E^T (A \otimes I) (I \otimes B) E). \]

Finally, identity (12) is obtained using (11) and (12). That is,
\[ EE^T (A \otimes B) E = (I - \mathbb{1}_E) (A \otimes B) E \]
\[ = (A \otimes B) E + \mathbb{1}_E (A \otimes B) \mathbb{1}_E, \]
\[ (B \otimes A) \]
which completes the proof.

In order to be considered a quantum system, the system variables of (3) must preserve (5) for all times. The condition that (5) has to satisfy is
\[ d[x, x^T] - 2i \Theta(dx) = 0. \]

Note by the linearity of the map \( \Theta \) that
\[ \Theta(dx) = \Theta(F_0) dt + \Theta(F x) dt \]
\[ + \Theta(G_1 x) dW_1 + \Theta(G_2 x) dW_2. \]

A condition for system (5) to satisfy (20) is given in the next theorem.

**Theorem 2:** Let \( [x_i(0), x_j(0)] = 2i \sum_k \epsilon_{ijk} x_k(0) \). System (5) implies
\[ [x_i(t), x_j(t)] = 2i \sum_k \epsilon_{ijk} x_k(t) \]
for all \( t \geq 0 \) if and only if
\[ G_1 + G_1^T = G_2 + G_2^T = 0. \]

Proof: From the fact that \( d[x, x^T] = d(xx^T) - (d(xx^T))^T \) and in light of the quantum Itô formula (see [7, Theorem 4.5]), the term \( d(xx^T) \) is expanded as
\[ d(xx^T) = (dx) x^T + x(dx)^T + (dx)(dx)^T \]
\[ + x (F_0 dt + F x dt + G_1 x dW_1 + G_2 x dW_2) x^T \]
\[ + x (F_0^T dt + x F^T dt + x^T G_1^T dW_1 + x^T G_2^T dW_2) \]
\[ + (F_0 dt + F x dt + G_1 x dW_1 + G_2 x dW_2) \]
\[ = (F_0 x^T + x F_0^T) dt + (F x x^T + x^T F^T) dt \]
\[ + (G_1 x x^T + x x^T G_1^T) dW_1 + (G_2 x x^T + x x^T G_2^T) dW_2 \]
\[ + G_1 x x^T G_1^T dW_1 + G_1 x x^T G_2^T dW_2 + G_2 x x^T G_1^T dW_1 + G_2 x x^T G_2^T dW_2 \]
\[ + G_1 x x^T G_1^T dW_1 + G_2 x x^T G_2^T dW_2. \]

The \((i, j)\) component of \( d(xx^T) \) is computed as
\[ e_i^T d(xx^T) e_j \]
\[ = (F_0 x_j + x_j F_0^T) dt + (F x x_j + x_j F^T) dt \]
\[ + (G_1 x x_j + x_j G_1^T) dW_1 + (G_1 x x_j + x_j G_2^T) dW_2 \]
\[ + G_1 x x_j G_1^T dW_1 + G_1 x x_j G_2^T dW_2 + G_2 x x_j G_1^T dW_1 + G_2 x x_j G_2^T dW_2 \]
\[ + G_1 x x_j G_1^T dW_1 + G_2 x x_j G_2^T dW_2. \]

One can compute the \((j, i)\) component similarly. Thus, the \((j, i)\) component of \( d([x, x^T]) \) is
\[ e_j^T d(xx^T) e_i \]
\[ = (F_0 x_j - x_j F_0^T) dt + (F x x_j - F x x_j) dt \]
\[ + (G_1 x x_j - x_j G_1^T) dW_1 + (x_j G_1 x - G_1 x x_j) dW_2 \]
\[ + (G_2 x x_j - x_j G_2^T) dW_2 + (x_j G_2 x - G_2 x x_j) dW_2 \]
\[ + G_1 (x x_j)_1 G_1 (x x_j)_2 dW_1 + G_1 (x x_j)_1 G_1 (x x_j)_2 dW_2 \]
\[ + G_1 (x x_j)_1 G_2 (x x_j)_2 dW_1 + G_1 (x x_j)_1 G_2 (x x_j)_2 dW_2 \]
\[ + G_2 (x x_j)_1 G_1 (x x_j)_2 dW_1 + G_2 (x x_j)_1 G_2 (x x_j)_2 dW_2 \]
\[ = [F x, x_j] dt + [x_i, F x] dt + [G_{1, r}, x_j] dW_1 \]
\[ + [x_i, G_{1, r}] dW_1 + [G_{2, r}, x_j] dW_2 + [x_i, G_{2, r}] dW_2 \]
\[ + [G_{1, r} (x x_j)_1, G_{1, x_j}] dW_1 + [G_{1, r} (x x_j)_1, G_{2, x_j}] dW_2 \]
\[ + [G_{2, r} (x x_j)_1, G_{1, x_j}] dW_1 + [G_{2, r} (x x_j)_1, G_{2, x_j}] dW_2 \]
\[ = \sum_{k=1}^{n} (F_{ik} [x_k, x_j] + F_{kj} [x_i, x_k]) dW_1 \]
\[ + \sum_{k=1}^{n} (G_{1, ik} [x_k, x_j] + G_{1, kj} [x_i, x_k]) dW_1 \].
\[ + \sum_{k=1}^{n} (G_{2k}[x_k, x_j] + G_{2j}[x_k, x_i]) \, dW_2 \\
+ \sum_{k,l=1}^{n} G_{1k}G_{1lj} [x_k dW_1, x_l dW_1] \\
+ 2i \sum_{m} \ell_{klm} x_m \, dt \\
+ \sum_{k,l=1}^{n} G_{2k}G_{2lj} [x_k dW_2, x_l dW_2] \\
+ \sum_{k,l=1}^{n} G_{2k}G_{1lj} [x_k dW_2, x_l dW_1] \\
+ 2i \sum_{m} \ell_{klm} x_m \, dt \]

The variation in time of the commutator \([x, x^T]\) amounts to
\[ d([x, x^T]) = 2i (G_1G_1^T - G_2G_2^T + F(\Theta(x) + \Theta(x)F^T) \\
+ G_1\Theta(x)G_1^T + G_2\Theta(x)G_2^T) \, dt \\
+ 2i (G_1\Theta(x) + \Theta(x)G_1^T) \, dW_1 \\
+ 2i (G_2\Theta(x) + \Theta(x)G_2^T) \, dW_2. \] (23)

Replacing (23) into (20) amounts to
\[ 2i (G_2G_1^T - G_1G_2^T - \Theta(F_0)) + F(\Theta(x) + \Theta(x)F^T) \\
+ G_1\Theta(x)G_1^T + G_2\Theta(x)G_2^T - \Theta(Fx) \, dt \\
+ 2i (G_1\Theta(x) + \Theta(x)G_1^T - \Theta(G_1x)) \, dW_1 \\
+ 2i (G_2\Theta(x) + \Theta(x)G_2^T - \Theta(G_2x)) \, dW_2 = 0. \] (24)

From [13, Proposition 23.7], one can also equate the integrands in (24) to zero. Thus, the equations to be satisfied for preservation of commutation relations are
\[ G_1\Theta(x) + \Theta(x)G_1^T - \Theta(G_1x) = 0 \] (25a)
\[ G_2\Theta(x) + \Theta(x)G_2^T - \Theta(G_2x) = 0 \] (25b)
\[ G_2G_1^T - G_1G_2^T - \Theta(F_0) + F(\Theta(x) + \Theta(x)F^T) \\
+ G_1\Theta(x)G_1^T + G_2\Theta(x)G_2^T - \Theta(Fx) = 0. \] (25c)

For (25a) and (25b), apply the operator vec and multiply by \(E^T\) to the left
\[ E^T (I \otimes G_1)Ex + E^T (G_1 \otimes I)Ex - E^T EG_i x = 0. \] (26)

From identities 0 and 0 of Lemma 3 and 0, (Tr\((G_i)I - G_i^T - G_i\) = 0. \] (27)

Similarly for (25a), one has that
\[ \text{vec}(G_2G_1^T - G_1G_2^T - \Theta(F_0)) \\
+ (I \otimes F)Ex + (F \otimes I)Ex - FX \\
+ (G_1 \otimes G_1)Ex + (G_2 \otimes G_2)Ex = 0. \]

From identities 0 and 0 of Lemma 3 and 0, \(E^T \text{vec}(G_2G_1^T - G_1G_2^T - \Theta(F_0)) + 2 \left( \text{Tr}(F) - F^T - F \right) \\
+ (G_1^T)^2 + (\frac{\text{Tr}(G_1)}{2})^2 I - \text{Tr}(G_1)G_1^T - \frac{\text{Tr}(G_1^2)}{2} I \\
+ (G_2^T)^2 + (\frac{\text{Tr}(G_2)}{2})^2 I - \text{Tr}(G_2)G_2^T - \frac{\text{Tr}(G_2^2)}{2} I \right) x = 0. \]

A key observation is that \(x(0)\) is represented by the linearly independent Pauli matrices, and that any linear combination \(a_1x_1(0) + a_2x_2(0) + a_3x_3(0) \neq 0\) unless \(a_1 = a_2 = a_3 = 0\) for \(a_1, a_2, a_3 \in \mathbb{C}\). In addition, no linear combination of Pauli matrices generates the identity. So, given that \(x(0) \neq 0\), any equation involving the system variables of the form \(Ax = b\) \((A \in \mathbb{C}^{3 \times 3})\) implies \(A\) and \(b\) must be identically 0. Thus,
\[ \text{Tr}(G_1)I - G_1^T - G_1 = 0 \] (28a)
\[ \text{Tr}(G_2)I - G_2^T - G_2 = 0 \] (28b)
\[ G_1G_1^T - G_2G_2^T - \Theta(F_0) = 0 \] (28c)
\[ \text{Tr}(F)I - F^T - F + \sum_{i=1}^{2} \left( (G_i^T)^2 + (\frac{\text{Tr}(G_i)}{2})^2 \right) \\
- \text{Tr}(G_i)G_i^T - \frac{\text{Tr}(G_i^2)}{2} I \right) = 0. \] (28d)

The trace of \(G_1\) and \(G_2\) can be calculated from (28a) and (28b). That is,
\[ \text{Tr}(G_1 + G_1^T - \text{Tr}(G_1)I) \\
= \text{Tr}(G_1) + \text{Tr}(G_1^T) - \text{Tr}(G_1)\text{Tr}(I) \\
= 2\text{Tr}(G_1) - 3\text{Tr}(G_1) \\
= -\text{Tr}(G_1) = 0. \]

which leaves \(\text{Tr}(G_1) = 0\). Thus, \(G_1 = -G_1^T\). Applying this result to (28d) gives
\[ F^T + F - \text{Tr}(F)I = \sum_{i=1}^{2} \left( (G_i^T)^2 - \frac{\text{Tr}(G_i^2)}{2} \right) \]. (29)

Similarly, applying the trace operator to the previous equation gives
\[ 2\text{Tr}(F) - 3\text{Tr}(F) = \sum_{i=1}^{2} \left( (G_i^T)^2 - \frac{3\text{Tr}(G_i^2)}{2} \right) \],

which amounts to
\[ \text{Tr}(F) = \frac{1}{2} (\text{Tr}(G_1^T) + \text{Tr}(G_2^T)). \] (30)

Replacing (30) into (29) and since \(G_i = -G_i^T\), one has that
\[ F^T + F + G_1G_1^T + G_2G_2^T = 0. \]

Conversely, it is going to be shown that if (22a), (22c) hold then (24) holds as well. One can see from (22a) and (11) of Lemma 3 that
\[ (\text{Tr}(G_i) - G_i^T - G_i) = (E^T (I \otimes G_i)E - G_i) = 0. \]

Applying (11) of Lemma 3 and multiplying on the left by \(E\) and on the right by \(x\) gives
\[ EE^T (I \otimes G_i)Ex - EG_i x = (I \otimes G_i)Ex + (G_i \otimes I)Ex - EG_i x. \]
Let $\text{vec}^{-1}$ denote the inverse of the stacking operator. That is, $\text{vec}^{-1}(\text{vec}(A)) = A$ for an arbitrary square matrix $A$. From the definition of $\Theta$, it follows that

$$
\text{vec}^{-1}
\left((I \otimes G_i)Ex + (G_i \otimes I)Ex - EG_ix\right)
=
\text{vec}^{-1}
\left((I \otimes G_i)\text{vec}(\Theta(x)) + (G_i \otimes I)\text{vec}(\Theta(x)) - \text{vec}(\Theta(G_ix))\right)
=
G_i\Theta(x) + \Theta(x)G_i^T - \Theta(G_ix).
$$

Thus,

$$
G_i\Theta(x) + \Theta(x)^T G_i^T - \Theta(G_ix) = 0, \quad i = 1, 2.
$$  \hfill (31)

Note that (22b) appears explicitly in the first line of (24). Next, computing the trace on both sides of (22c) gives

$$
2\text{Tr}(F) = \text{Tr}\left(G_i^2\right) + \text{Tr}\left(G_2^2\right)
$$

Equations (32) and (42a) allow one to write (22c) as

$$
F + F^T - \text{Tr}(F)I - \sum_{i=1}^{2}(G_i^2)^T + \frac{2}{2}\left(\text{Tr}(G_1)^2\right)I
- \text{Tr}(G_1)G_1^T - \frac{2}{2}\left(\text{Tr}(G_2)^2\right)I = 0.
$$

From (4) and (14) of Lemma 3 it follows that

$$
F - E^T(I \otimes F)E - \frac{1}{2}\sum_{i=1}^{2}E^T(G_i \otimes I)E = 0.
$$

By identity (10) of Lemma 3 and multiplying on the left by $E$ and on the right by $x$, (22c) is equivalent to

$$
EFx - (I \otimes F)Ex - (F \otimes I)Ex
- (G_1 \otimes G_1)Ex - (G_2 \otimes G_2)Ex = 0.
$$

Applying $\text{vec}^{-1}$ gives

$$
F\Theta(x) + \Theta(x)F^T - \Theta(Fx)
- G_1\Theta(x)G_1^T + G_2\Theta(x)G_2^T = 0.
$$  \hfill (33)

Obviously if (31) and (33) hold then (24) is zero, which completes the proof. \hfill \blacksquare

Theorem 3: A physically realizable system satisfies the conditions of Theorem 2.

Proof: It is enough to show that if conditions (11), (12), and (14) of Theorem 1 hold then (22a)–(22c) are satisfied as well. Define $H = \frac{1}{i}(H_1 + iH_2)$. By conditions (11) and (12) of Theorem 1 one has that

$$
F_0 = \frac{1}{2}(G_1 - ig_2)(H_1 + iH_2)^\dagger = -2i\Theta(H)H^\dagger.
$$

Now, from Lemma 1 property (26), $\Theta(F_0)$ is

$$
\Theta(F_0) = -\Theta(2i\Theta(H)H^\dagger)
= -2i\left(\Theta(H)\Theta(H^\dagger) - \Theta(H^\dagger)\Theta(H)\right).
$$

Since $G_i = -G_i^T$, it follows that

$$
G_2G_1^T - G_1G_2^T = -2i(\Theta(H)\Theta(H^\dagger) - \Theta(H^\dagger)\Theta(H)).
$$

Therefore, (22b) holds. Again, from (17) and (18) of Theorem 1 a physically realizable system satisfy $\text{Tr}(G_i) = 0$ and $G_i^T = -G_i$, which imply condition (22a). Finally, it is clear that condition (22a) is equivalent to

$$
F + F + G_1G_1^T + G_2G_2^T = 0,
$$

which concludes the proof. \hfill \blacksquare

VI. CONCLUSIONS

A condition for physical realizability was given for open two-level quantum systems. Under this condition it was shown that there exist operators $H$ and $L$ such that the bilinear QSDE (5) with output equation (6) can be written as in (1). Also, it was shown that physical realizability implies preservation of the Pauli commutation relations for all times. Future work includes extending the formalism for the case of multi-particle spin systems.

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