The Tripartite-Circle Crossing Number of $K_{2,2,n}$

Charles Camacho
University of Washington, USA

Silvia Fernández-Merchant
California State University, Northridge, USA

Marija Jelić Milutinović
University of Belgrade, Serbia

Rachel Kirsch
George Mason University, USA

Linda Kleist
TU Braunschweig, Germany

Elizabeth Bailey Matson
Alfred University, USA

Jennifer White
Saint Vincent College, USA

Abstract
A tripartite-circle drawing of a tripartite graph is a drawing in the plane, where each part of a vertex partition is placed on one of three disjoint circles, and the edges do not cross the circles. The tripartite-circle crossing number of a tripartite graph is the minimum number of edge crossings among all tripartite-circle drawings. We determine the tripartite-circle crossing number of $K_{2,2,n}$.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics → Graph theory → Graphs and surfaces

Keywords and phrases Complete tripartite graph, crossing number, circle drawing.

1 Introduction

The crossing number of a graph $G$, denoted by $\text{cr}(G)$, is the minimum number of crossings over all drawings of $G$ in the plane. There are multiple variants to the problem of finding the crossing number of a graph. Some problems consider drawings on other surfaces or in other spaces, like drawings on the torus, a cylinder, or a $k$-page book. Others restrict the minimum to specific types of drawings, like rectilinear or geometric drawings, where the edges must be straight line segments. Drawings with few crossings have been studied in connection with readability and VLSI chip design [11]. See [14] for a survey of crossing number variants and some of their applications.

Exact crossing numbers are unknown even for very special graph classes. Famous examples of these are the long-standing Harary-Hill and Zarankiewicz Conjectures on the crossing number of the complete graph $K_n$ [9, 8] and the complete bipartite graph $K_{m,n}$ [10], respectively. Among the best known drawings of complete graphs are 1-circle drawings (or 2-page book drawings) and 2-circle drawings (or cylindrical drawings). This prompted the study of $k$-circle drawings of graphs, that is, drawings in the plane where the vertices are placed on $k$ specified circles and the edges cannot cross these circles. Of particular interest are $k$-partite-circle drawings, where we further require that the vertices on each circle form an independent set. The minimum number of crossings among all $k$-partite-circle drawings of a graph $G$ is known as the $k$-partite-circle crossing number of $G$ and is denoted by $\text{cr}_k(G)$.

This crossing number has been studied for complete bipartite and tripartite graphs. In 1997, Richter and Thomassen [13] settled the balanced case for complete bipartite
The Tripartite-Circle Crossing Number of $K_{2,2,n}$

graphs by showing that $\operatorname{cr}_G(K_{n,n}) = n(n)$. Ábrego, Fernández-Merchant, and Sparks [1] generalized this result to all complete bipartite graphs. In particular, if $m$ divides $n$, then $\operatorname{cr}_G(K_{m,n}) = \frac{1}{12}m(m-1)(2mn-3m-n)$. In previous work [3], we proved lower and upper bounds on $\operatorname{cr}(K_{m,n,p})$: the implied lower and upper bounds for $K_{n,n,n}$ are of order $\sim 5/4 \cdot n^4$ and $\sim 6/4 \cdot n^4$, respectively.

Building on these insights, in this paper, we establish $\operatorname{cr}(K_{2,2,n})$.

Theorem 1. For every integer $n \geq 3$,

$$\operatorname{cr}(K_{2,2,n}) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 3$$

In comparison, the Zarankiewicz Conjecture on $\operatorname{cr}(K_{m,n})$ has only been proved when $\min(m,n) \leq 6$ by Kleitman in 1970 [10], and for $m = 7$ or 8 and $n \leq 10$ by Woodall in 1993 [15]. The current best lower bounds are by de Klerk et al. [11] and Norin and Zwols [12]. The crossing number of the complete tripartite graph is also unknown in general. In 2017, Gethner et al. [6] provided a drawing of $K_{m,n,p}$ with few crossings and conjecture that their drawing is crossing-optimal. A few exact crossing numbers are known for small values of $m$ and $n$, like $\operatorname{cr}(K_{1,3,n}), \operatorname{cr}(K_{1,2,n})$ proved by Asano in 1986 [2]; and $\operatorname{cr}(K_{3,3,n})$ proved by Gim and Miller in 2014 [4].

Although the graphs $K_{2,2,n}$ and $K_{4,n}$ only differ by four edges, their crossing numbers behave in an interesting way. Because $K_{4,n}$ is a subgraph of $K_{2,2,n}$, the known crossing number $\operatorname{cr}(K_{4,n})$ [10] and the upper bound on $\operatorname{cr}(K_{m,n,p})$ in [6] imply that $\operatorname{cr}(K_{4,n}) = \operatorname{cr}(K_{2,2,n}) = 2\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. In contrast, their circle crossing numbers evaluate to $\operatorname{cr}(K_{2,2,n}) = n(3n-6)/2 \approx 3/2 n^2$ and $\operatorname{cr}(K_{4,n}) = n(7n-12)/4 + O(1) \approx 7/4n^2$ and thus differ in a term quadratic in $n$.

Organization. The remainder of our paper is organized as follows: In Section 2, we introduce some of the necessary tools based on previous work. In Section 3, we present our proof. We conclude in Section 4 with a discussion and open problems.

2 Tools for counting the number of crossings

Our tools for counting the number of crossings build on our previous work [3]. For a self-contained presentation, we give a concise summary.

Drawings that minimize the number of crossings are known to be simple. A simple drawing of $G$ is a drawing where no edge crosses itself, two edges that share a vertex do not cross, and two edges with no shared vertices intersect at most once. In a tripartite-circle drawing of $K_{m,n,p}$, we label the three circles $M$, $N$, and $P$, and their numbers of vertices are $m$, $n$, and $p$, respectively. For an illustration consider Figure 1.

For $k$-circle drawings with $k \geq 3$, three or more pairwise nested circles would require that any edges from the outermost to the innermost circle would necessarily cross the middle circle(s). Consequently, tripartite-circle drawings of complete tripartite graphs do not contain three pairwise nested circles. Moreover, note that the drawing in Figure 1 can be transformed by a projective transformation of the plane such that any one circle encloses the other two. Therefore, without loss of generality we consider drawings where the outer circle $P$ contains the inner circles $M$ and $N$. In such a drawing, we label the vertices on circles $M$ and $N$ in clockwise order and the vertices on circle $P$ in counterclockwise order. Likewise, we read arcs of circles in clockwise order for inner circles and in counterclockwise order for outer circles.
2.1 Defining the x- and y-labels

As progressively defined in [1], [13], and [3], we use the following labels. Let $A$, $B$, and $C$ be the three circles and $i$ be a vertex on $A$. The star formed by all edges from $i$ to $B$ together with circle $B$ partitions the plane into disjoint regions, see Figure 2. Exactly one of these regions contains circle $A$. Such a region is enclosed by two edges from $i$ to $B$ and an arc on $B$ between two consecutive vertices. We define the second of these vertices (in clockwise or counterclockwise order depending on whether $B$ is an inner or outer circle, respectively) as $x_i(A,B)$. Similarly, there is exactly one region defined by the star from $i$ that contains the third circle $C$ and the second vertex along circle $B$ (clockwise or counterclockwise as before) on the boundary of this region is $y_i(A,B)$. If the two circles are clear from the context, we may also write $x_i$ or $y_i$.

**Figure 2** The definition of $y_i(A,B)$ illustrated for the cases when (a) $A$ is inside $B$, (b) when $A$ is beside $B$, and (c) $B$ is inside $A$. 
2.2 Counting the crossings

The number of crossings in a simple tripartite-circle drawing of $K_{m,n,p}$ can be found by counting the crossings in the three different, simple bipartite-circle drawings of $K_{m,n}$, $K_{m,p}$ and $K_{n,p}$, along with crossings involving all three circles.

To restate a handy theorem of [3], we briefly introduce some notation. For vertices $k$ and $\ell$ on a circle with $n$ vertices numbered $1, \ldots, n$ clockwise (respectively, counterclockwise), let $d_n(k, \ell) := \ell - k \mod n$ denote the distance from $k$ to $\ell$ in clockwise (respectively, counterclockwise) order on the circle. Let $[n] := \{1, 2, \ldots, n\}$. For any $u, v \in [n]$, define $f_n(u, v) := \left(\frac{d_n(u, v)}{2}\right)^2 + \left(\frac{n - d_n(u, v)}{2}\right)^2$.

For vertices $i$ and $j$ on the inner (respectively, outer) circle $a$, we use $[i, j]$ to denote the arc of $a$ read clockwise (respectively, counterclockwise) from $i$ to $j$. We include $i$ and $j$ in the interval $[i, j]$, whereas $(i, j)$ does not include $i$ and $j$. We similarly define $[i, j)$ and $(i, j]$.

We will later make use of the following fact.

\begin{lemma}[Lemma 3.1, [3]]\end{lemma}

The function $f_n(a, b)$ attains its minimum $M$ if and only if $|a - b| \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$. Among pairs $(a, b)$ such that $|a - b| \notin \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$, the minimum of $f_n$ exceeds $M$ by 1 if $n$ is even and by 2 if $n$ is odd.

A cyclic assignment of $(A, B, C)$ to $(M, N, P)$ is one triple of the following set $t := \{(M, N, P), (N, P, M), (P, M, N)\}$. Let the number of vertices on the circles $A$, $B$, and $C$ be denoted by $a$, $b$, and $c$, respectively. The following theorem counts the total number of crossings.

\begin{theorem}[Theorem 2.4, [3]]\end{theorem}

The number of crossings in a simple tripartite-circle drawing of $K_{m,n,p}$ is given by

$$\sum_{(A, B, C) \in t} \left( \sum_{i < j \in A} f_b(x_i(A, B), x_j(A, B)) + \sum_{i \in A} f_c(y_i(A, C), y_j(B, C)) \right).$$

3 Proof of Theorem 1

In this section, we determine the tripartite-circle crossing number of $K_{2,2,n}$. First, we give a construction to show the upper bound. Then, we consider each case to minimize the number of crossings of $K_{2,2,n}$. Combining the upper bound and lower bound yields the desired result.

3.1 The upper bound

\begin{lemma} For any integer $n \geq 3$, \end{lemma}

$$cr_G(K_{2,2,n}) \leq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3.$$ \begin{proof} We color the edges between $M$ and $N$ red and all others green, see Figure 3. We divide the vertices on $P$ into groups which are called $A, B, C, D$, where $C, D$ are empty in case (a). \end{proof}
All edges are straight line segments except for 1x1, 2x2, and 3x2 in (a) and 1y2, 2y3, 3y4, and 4x1 in (b). If these edges are replaced by straight line segments, the number of crossings is easy to determine. These replacements add 2 crossings to (a) and 4 to (b). For simplicity, we define $c_n := 6 \left( \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \right).

After the replacement, Figure 3(a) has $3\left\lfloor n/2 \right\rfloor + \left\lceil n/2 \right\rceil = 2n - 1$ red-green crossings and no red-red crossing. In order to count the green-green crossings, note that every crossing is determined by any 2 vertices on the inner circles and any 2 vertices in $A$ or any 2 vertices in $B$. Thus there are $c_n$ green-green crossings and the number of crossings in the original drawing is $c_n + 2n - 1 - 2 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 2n - 3$.

After the replacement, Figure 3(b) has $2n$ red-green crossings and 1 red-red crossing. The green-green crossings are determined by either two vertices on the same inner circle and any 2 vertices in $A \cup D$ or any 2 vertices on $B \cup C$; or by 2 vertices on different inner circles and any 2 vertices in $A \cup B$ or any 2 vertices on $C \cup D$. Note that $\{|A \cup D|, |B \cup C|\} = \{|A \cup B|, |C \cup D|\} = \{|n/2|, \lceil n/2 \rceil\}$ and thus there are $c_n$ green-green crossings. The number of crossings in the original drawing is $c_n + 2n + 1 - 4 = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 2n - 3$.

### 3.2 The lower bound

Now, we turn our attention towards proving the lower bound, i.e., we want to show that this drawing has a minimal number of crossings. To do so, we partition the set of drawings of $K_{2,2,n}$ by the induced subdrawings of $K_{2,2,0}$ depicted in Figure 4(a).

> **Lemma 5.** Up to topological equivalence, there are exactly four simple tripartite-circle drawings of $K_{2,2,0}$.

**Proof.** Up to topological equivalence, there are two different simple bipartite-circle drawings of $K_{2,2}$ on the sphere, namely the ones depicted in Figure 4(a). Placing the third circle in different ‘cells’ of these two drawings yields the four drawings in Figure 4(b) where the third circle is placed in the outer cell.

Any simple tripartite-circle drawing of $K_{2,2,n}$ can be seen as an extension of one of the four drawings of $K_{2,2,0}$ in Figure 4(b). We say that a drawing of $K_{2,2,n}$ is of type $i$ if it is an extension of drawing $i$ in Lemma 4. In our figures, we color the edges of $K_{2,2,0}$ red and the remaining $4n$ edges green. We first count the number of crossings with red edges. For an illustration consider Figure 5, note that some edges are omitted for clarity.
The Tripartite-Circle Crossing Number of $K_{2,2,n}$

**Figure 4** (a) The two simple bipartite-circle drawings of $K_{2,2}$. (b) The four simple tripartite-circle drawings of $K_{2,2,0}$.

**Figure 5** The four simple tripartite-circle drawings of $K_{2,2,n}$. Red edges connect vertices on the two inner circles, all other edges are green (some of them omitted for clarity). The two drawings in the right-hand column have two enclosed vertices. A green edge from these vertices must cross a red edge.
Moreover, the inequality can be strengthened in the following cases: Enclosed vertices in drawings of types 2, 3, and 4 are those separated from the outer circle by red edges; the green edges incident to enclosed vertices must cross at least one red edge. In a drawing of type 1, a green edge from vertex \(i \in \{1, 3\}\) crosses two red edges if the other vertex lies in the interval \([y_i, x_i]\); otherwise it does not cross any edge. Note that the number of these vertices is \(d_n(y_i, x_i)\).

Likewise, a green edge from a vertex \(i \in \{2, 4\}\) crosses two red edges if the other vertex lies in the interval \([x_i, y_i]\). Recall that we consider the vertices on \(P\) in counterclockwise order.

The same holds for green edges incident to vertices 1 or 2 in a drawing of type 4. Hence, for each type of drawing, the number of crossings with red edges is at least

\[
\begin{align*}
2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4)) + 1 & \quad \text{type 1,} \\
2n + 1 & \quad \text{type 2 or 3,} \\
2 \cdot (d_n(y_1, x_1) + d_n(x_2, y_2)) + n & \quad \text{type 4.}
\end{align*}
\]

We partially use Theorem 3 to count the monochromatic green crossings as

\[
f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).
\]

It remains to show that the sum of (1) and (2) is at least \(6\left(\frac{n}{2}\right) \left\lfloor \frac{n}{2} \right\rfloor + 2n - 3\) for any of the four types of drawings. By Lemma 2, each term of (2) is minimized when the corresponding pair of points has a distance of roughly \(\frac{n}{2}\). Hence, for all \(a\) and \(b\) it holds that

\[
f_n(a, b) \geq \left(\frac{n}{2}\right) + \left(\frac{n}{2}\right) = \frac{n}{2} \left\lfloor \frac{n}{2} \right\rfloor + \frac{n}{2} - 1.
\]

We start with two lemmas bounding some of the terms in (1) and (2). In the following lemmas, we write \(Z_i = d_n(y_{i+1}, y_i)\) and \(z_i = \min d_n(y_{i+1}, y_i)\) for \(i = 1, 3\) for notational convenience.

**Lemma 6.** Let \(Z_i = d_n(y_{i+1}, y_i)\). The following inequality holds:

\[
2d_n(y_i, x_i) + 2d_n(x_i+1, y_{i+1}) + f_n(x_i, x_{i+1}) \geq \left(\frac{n}{2}\right) \left\lfloor \frac{n}{2} \right\rfloor + n - 1 - 2Z_i.
\]

Moreover, the inequality can be strengthened in the following cases:

\[
2d_n(y_i, x_i) + 2d_n(x_i+1, y_{i+1}) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 - 2Z_i + \frac{(Z_i - \frac{n-2}{2})^2}{2} \quad \text{if ccw order} \neq x_{i+1}y_{i+1}y_ix_i,
\]

and \(Z_i \geq n/2\)

**Proof.** Note that the (general) statement is equivalent to showing

\[
2\left(\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \right) + n - 1
\]

A short case analysis (by considering the six counterclockwise orders) verifies that for every four points \(a, b, c, d\), the following holds:

\[
d_n(a, b) + d_n(b, c) + d_n(c, d) = \begin{cases} d_n(a, d) & \text{if the ccw order is abcd} \\
d_n(a, d) + 2n & \text{if the ccw order is adcb} \\
d_n(a, d) + n & \text{otherwise.}
\end{cases}
\]
With $a = x_{i+1}, b = y_{i+1}, c = y_i, d = x_i$, Equation (4) implies that
\[
2(d_n(x_{i+1}, y_{i+1}) + d_n(y_{i+1}, y_i) + d_n(y_i, x_i)) + f_n(x_i, x_{i+1}) \\
\geq 2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1})
\tag{5}
\]
The right side of Inequality (5) can be expressed as the quadratic function
\[
2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) = d_n(x_{i+1}, x_i)^2 + (2 - n)d_n(x_{i+1}, x_i) + \left(\frac{n}{2}\right),
\tag{6}
\]
which is minimized for $d_n(x_{i+1}, x_i) = \left\lfloor \frac{n-2}{2} \right\rfloor$. Evaluation at $d_n(x_{i+1}, x_i) = \left\lfloor \frac{n-2}{2} \right\rfloor$ yields
\[
2d_n(x_{i+1}, x_i) + f_n(x_i, x_{i+1}) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1.
\]
This finishes the proof of the general statement.

For the strengthening, we consider the two cases. If the counterclockwise order is different from $x_{i+1}y_{i+1}y_ix_i$, note that we can add at least a $2n$ term to the right side of the Inequality (5). If the counterclockwise order is $x_{i+1}y_{i+1}y_ix_i$ and $Z_i \geq n/2$, then $d_n(x_{i+1}, x_i) \geq d_n(y_{i+1}, y_i) \geq n/2$ and the expression in Equation (6) is minimized for $d_n(x_{i+1}, x_i) = d_n(y_{i+1}, y_i) = Z_i$.

Now, we show a lower bound on the remaining four $f_n$-terms in (2). To do so, we define
\[
\Delta_k := \begin{cases} 
0 & \text{if } k \text{ even} \\
1 & \text{otherwise}
\end{cases}
\]

\begin{lemma}

Let $z_1 = \min d_n(y_1, y_2), z_3 = \min d_n(y_3, y_4)$ and $S = f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) - 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$.

i) If $y_1, y_2 \in [y_3, y_4]$ or $y_1, y_2 \in [y_4, y_3]$, then it holds that
\[
S \geq z_1^2 + z_3^2 - \Delta_n \Delta_{z_1 + z_3}
\]

ii) If $y_1 \in (y_3, y_4)$ and $y_2 \in (y_3, y_4)$ (or vice versa), then it holds that
\[
S \geq \frac{1}{4} \left( z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2 \right) - \frac{1}{2} \Delta_n
\]

Moreover, in all cases it holds that
\[
S \geq z_1^2 - \Delta_n \Delta_{z_1}.
\]

\end{lemma}

\begin{proof}

First note that exchanging $y_1$ and $y_2$ (or $y_3$ and $y_4$) does not influence $f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4)$. Consequently, by swapping $y_1$ and $y_2$ or $y_3$ and $y_4$, all counterclockwise orders can be transformed to one of the following two: $y_1y_2y_3y_4$ (non-alternating, i.e., Case i) ) and $y_1y_3y_2y_4$ (alternating, i.e., Case ii) )

\begin{case}

Case i): Without loss of generality, we consider the counterclockwise order $y_1, y_2, y_3, y_4$ and define $a := d_n(y_1, y_2), b := d_n(y_2, y_3), c := d_n(y_3, y_4)$, and $d := d_n(y_4, y_1)$, see also Figure 6.

\end{case}

\end{proof}
Then, it holds that

\[
\begin{align*}
&f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) \\
&= 4 \left( \frac{n}{2} \right)^2 - (a + b)(n - (a + b)) - d(n - d) - b(n - b) - (b + c)(n - (b + c)) \\
&= 2n^2 - 2n + a^2 + b^2 + c^2 + d^2 - (a + b + c + d)n + 2b(a + b + c - n) \\
&= 2n^2 - 2n + a^2 + b^2 + c^2 + d^2 - n(z_1 + z_3 + 1) \\
&\geq 2 \left( \frac{n}{2} \right)^2 + z_1^2 + z_3^2 + \Delta_n \Delta_{z_1 + z_3 + 1} + \Delta_n \Delta_{z_1 + z_3}.
\end{align*}
\]

Note that \(a^2 + c^2 + (b - d)^2 \geq z_1^2 + z_3^2 + \Delta_n \Delta_{z_1 + z_3 + 1}\) trivially holds when \(n\) is even. Suppose \(n\) is odd and \(z_1 + z_3\) is even. It holds that either \(z_1 = a, z_3 = c\) or \(z_1 = b + c + d, z_3 = c\) (or symmetrically \(z_1 = a, z_3 = a + b + d\)). If \(z_1 = a, z_3 = c\) and \(z_1 + z_3\) is even, then \(b \neq d\), and hence \(a^2 + c^2 = z_1^2 + z_3^2\) and \((b - d)^2 \geq 1 \geq \Delta_n \Delta_{z_1 + z_3 + 1}\). If \(z_1 = b + c + d < a, z_3 = c\) then it holds that \(a^2 + c^2 \geq (z_1 + 1)^2 + z_3^2 \geq z_1^2 + z_3^2 + 1 \geq z_1^2 + z_3^2 + \Delta_n \Delta_{z_1 + z_3 + 1}\). Therefore, we obtain the term \(+ \Delta_n \Delta_{z_1 + z_3 + 1}\) in the first inequality.

For the second inequality, note that \(+ \Delta_n \Delta_{z_1 + z_3 + 1} = -\Delta_n \Delta_{z_1 + z_3}\). This finishes the proof of part i).

**Case ii):** Without loss of generality, we consider the order \(y_1, y_3, y_2, y_4\) and define \(a := d_n(y_1, y_3), b := d_n(y_3, y_2), c := d_n(y_2, y_4)\), and \(d := d_n(y_4, y_1)\). By symmetry, we may assume that \(z_1 = a + b\) (so \(n - z_1 = c + d\)). Then

\[
2 \left( \frac{z_1}{2} \right)^2 = 2 \left( \frac{a + b}{2} \right)^2 = \frac{1}{2}(a^2 + b^2 + 2ab) = a^2 + b^2 - \frac{1}{2}(a - b)^2 \leq a^2 + b^2 - \frac{1}{2}\Delta_{z_1},
\]

since \((a - b)^2 \geq 1\) when \(z_1\) is odd. Similarly, \(c^2 + d^2 \geq 2(\frac{n-z_1}{2})^2 + \frac{1}{2}\Delta_{n-z_1}\), so we find \(a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_1^2 + (n - z_1)^2 + \Delta_{z_1} + \Delta_{n - z_1})\). By a similar argument, we have \(a^2 + d^2 \geq 2(\frac{n-z_3}{2})^2 + \frac{1}{2}\Delta_{z_3}\) and \(b^2 + c^2 \geq 2(\frac{n-z_3}{2})^2 + \frac{1}{2}\Delta_{n-z_3}\) (or vice versa), so \(a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_3^2 + (n - z_3)^2 + \Delta_{z_3} + \Delta_{n - z_3})\). Therefore \(a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_1^2 + (n - z_1)^2 + z_3^2 + (n -
The Tripartite-Circle Crossing Number of $K_{2,2,n}$

$z_3^2 + \Delta_1 + \Delta_{n-z_1} + \Delta_{z_1} + \Delta_{n-z_3}$). We thus have

$$f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4)$$

$$= 4 \left(\frac{n}{2}\right) - a(n - a) - b(n - b) - c(n - c) - d(n - d)$$

$$= n^2 - 2n + a^2 + b^2 + c^2 + d^2$$

$$\geq n^2 - 2n + \frac{1}{4}(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2)$$

$$+ \frac{1}{4}(\Delta_1 + \Delta_{n-z_1} + \Delta_{z_1} + \Delta_{n-z_3})$$

$$= 4 \left(\frac{n}{2}\right) \left(\frac{n - 1}{2}\right) + \frac{1}{4}(z_1^2 + (n - z_1)^2 + z_3^2 + (n - z_3)^2)$$

$$+ \frac{1}{4}(\Delta_1 + \Delta_{n-z_1} + \Delta_{z_1} + \Delta_{n-z_3} - 4\Delta_n)$$

When $n$ is odd, then for $i = 1, 3$ one of $z_i$ and $n-z_i$ is even and the other odd. Therefore, it holds that $\frac{1}{4}(\Delta_1 + \Delta_{n-z_1} + \Delta_{z_1} + \Delta_{n-z_3} - 4\Delta_n) = -\frac{1}{2}\Delta_n$. Hence, part ii) is proved.

Note that in case ii) it holds that $a^2 + b^2 + c^2 + d^2 \geq \frac{1}{2}(z_1^2 + (n - z_1)^2) \geq z_1^2$. By a similar analysis as in the proofs of parts i) and ii), it can be checked that in both cases i) and ii):

$$S \geq z_1^2 - \Delta_n \Delta_1,$$

which finishes the proof.

Now, we are ready to prove the lower bound.

**Lemma 8.** For any integer $n \geq 3$,

$$cr_{\Phi}(K_{2,2,n}) \geq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3.$$

**Proof.** We consider the drawings depending on their type. We start with the types for which there exist simple arguments and end with the more complicated ones, namely we consider the drawing types in the order 2 or 3, 4, and 1.

For a **drawing of type 2 or 3**, by (1) and (2), the number of crossings is

$$1 + 2n + f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).$$

Using Equation (3), this is bounded from below by $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n + 1 > 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3$, as desired. Note that drawings of these types do not attain the minimum number of crossings.

Next, we consider **drawings of type 4**. The number of crossings is

$$n + 2(d_n(y_1, x_1) + d_n(x_2, y_2)) + f_n(x_1, x_2) + f_n(x_3, x_4) + f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4).$$

We show the lower bound by considering two cases for $Z_1$. In each of the cases, we use Lemma (2) to bound $f_n(x_3, x_4)$.

In the first case, it holds that $Z_1 \leq (n - 1)/2$. By Lemmas (6) and (7) the number of crossings is at least

$$n + 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta Z_1.$$
and \( 2n - 1 - 2Z_1 + Z_1^2 - \Delta_n \Delta Z_1 = 2n - 2 + (Z_1 - 1)^2 - \Delta_n \Delta Z_1 \geq 2n - 3 \). This shows the claim.

In the second case, it holds that \( Z_1 \geq n/2 \). Here we distinguish the subcases whether or not \([x_1, x_2] \subseteq [y_1, y_2]\). If \([x_1, x_2] \subseteq [y_1, y_2]\), using the third inequality of Lemma 6, the number of crossings is at least \(6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] +\)

\[
\begin{align*}
n + n - 1 - 2Z_1 + \left( Z_1 - \frac{n-2}{2} \right)^2 + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} & \\
\geq 2n - 3 + \left\lfloor \frac{1}{n} \right\rfloor (4Z_1 - 3n^2 + (n - 4)^2) & \\
\geq 2n - 3.
\end{align*}
\]

If \( Z_1 \geq n/2 \) and \([x_1, x_2] \not\subseteq [y_1, y_2]\), then the second inequality of Lemma 6 shows that the number of crossings is at least \(6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] +\)

\[
\begin{align*}
n + n - 1 - 2Z_1 + 2n + (n - Z_1)^2 - \Delta_n \Delta_{n-Z_1} & \\
\geq 2n - 1 + 2(n - Z_1) + (n - Z_1)^2 - 1 & \\
= 2n - 3 + (n - Z_1 + 1)^2 & \\
\geq 2n - 3.
\end{align*}
\]

This finishes the proof for drawings of type 4.

It remains to consider drawings of type 1, which have the following number of crossings:

\[
\begin{align*}
2(d_n(y_1, x_1) + d_n(x_2, y_2) + d_n(y_3, x_3) + d_n(x_4, y_4)) + f_n(x_1, x_2) + f_n(x_3, x_4) & \\
+ f_n(y_1, y_3) + f_n(y_1, y_4) + f_n(y_2, y_3) + f_n(y_2, y_4) + 1.
\end{align*}
\]

Without loss of generality, we assume that \( Z_1 \leq Z_3 \).

**Case A**: First, we consider the case that Lemma 7.i applies and distinguish three cases depending on whether \( Z_1 \) and \( Z_3 \) are small \((\leq (n - 1)/2)\) or large \((\geq n/2)\).

**Case A1**: if \( Z_1, Z_3 \leq (n - 1)/2 \). By Lemmas 6 and 7.i, the number of crossings is at least

\[
6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] + (n - 1 - 2Z_1) + (n - 1 - 2Z_3) + Z_1^2 + Z_3^2 + - \Delta_n \Delta_{Z_1+Z_3} + 1
\]

Then, it holds that:

\[
\begin{align*}
2n - 1 - 2Z_1 - 2Z_3 + Z_1^2 + Z_3^2 - \Delta_n \Delta_{Z_1+Z_3} & \\
\geq 2n - 3 + (Z_1 - 1)^2 + (Z_3 - 1)^2 - \Delta_n \Delta_{Z_1+Z_3} & \\
\geq 2n - 3
\end{align*}
\]

Note that in the case \( Z_1 = Z_3 = 1 \), it holds that \( Z_1 + Z_3 \) is even and thus \(- \Delta_n \Delta_{Z_1+Z_3} = 0\).

**Case A2**: if \( Z_1 \leq (n - 1)/2, Z_3 \geq n/2 \). Suppose we do not have the counterclockwise ordering \( x_1y_1y_3x_3 \). By Lemmas 6 and 7.i, the number of crossings is at least

\[
6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] + (n - 1 - 2Z_1) + (n - 1 - 2Z_3) + Z_1^2 + (n - Z_3)^2 + 2n - \Delta_n \Delta_{Z_1+n-Z_3} + 1.
\]
Then, since \( Z_3 \leq n - 1 \) it holds that \( (n - Z_3 + 1)^2 \geq 2^2 \), which implies that
\[
2n - 1 - 2Z_1 + Z_2^2 - 2Z_3 + 2n + (n - Z_3)^2 - \Delta_n \Delta_{Z_1+n-Z_3} \\
= 2n - 3 + (Z_1 - 1)^2 + (n - Z_3 + 1)^2 - \Delta_n \Delta_{Z_1+n-Z_3} \\
> 2n - 3.
\]

Now, suppose we have the counterclockwise ordering \( x_4 y_1 y_3 x_3 \). By Lemmas 6 and 7, the number of crossings is at least
\[
6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] + (n - 1 - 2Z_1) + (n - 1 - 2Z_3) + Z_1^2 + (n - Z_3)^2 \\
+ \left( Z_3 - \frac{n-2}{2} \right)^2 - \Delta_n \Delta_{Z_1+n-Z_3} + 1
\]
Then,
\[
2n - 1 - 2Z_1 - 2Z_3 + Z_1^2 + (n - Z_3)^2 + \left( Z_3 - \frac{n-2}{2} \right)^2 - \Delta_n \Delta_{Z_1+n-Z_3} \\
\geq 2n - 3 + (Z_1 - 1)^2 + \left[ \frac{1}{8}(4Z_3 - 3n)^2 + (n - 4)^2 \right] - \Delta_n \Delta_{Z_1+n-Z_3}.
\]
This is at least \( 2n - 3 \) if \( (Z_1 - 1)^2 \geq 1 \), so we may assume that \( Z_1 = 1 \). This is also at least \( 2n - 3 \) if \( n \) is even or if \( Z_1 + n - Z_3 \) is even, and so we may assume that \( n \) is odd, and further that \( Z_3 \) is odd, since \( Z_1 = 1 \). In this case, we have that
\[
(Z_1 - 1)^2 + \left[ \frac{1}{8}(4Z_3 - 3n)^2 + (n - 4)^2 \right] - \Delta_n \Delta_{Z_1+n-Z_3} = \frac{1}{8}(4Z_3 - 3n)^2 + (n - 4)^2 - 5/4.
\]
Toward contradiction, assume that \( \frac{1}{8}(4Z_3 - 3n)^2 + (n - 4)^2 - 5/4 \leq -1 \). By rearranging, we obtain \( (4Z_3 - 3n)^2 + (n - 4)^2 \leq 2 \), which implies that \( n \in \{3, 5\} \) since \( n \) is odd and \( (n - 4)^2 \leq 2 \).
If \( n = 3 \), since \( \frac{n}{2} \leq Z_3 < n \), this means that \( Z_3 = 2 \), which contradicts the assumption that \( Z_3 \) is odd.
If \( n = 5 \), since \( \frac{n}{2} \leq Z_3 < n \) and \( Z_3 \) is odd, \( Z_3 = 3 \). With these values,
\[
(4Z_3 - 3n)^2 + (n - 4)^2 = (15 - 12)^2 + (5 - 1)^2 \leq 2,
\]
a contradiction. Therefore, \( \frac{1}{8}(4Z_3 - 3n)^2 + (n - 4)^2 - 5/4 \geq 0 \), completing this subcase.

**Case A3:** if \( Z_1, Z_3 \geq n/2 \). By Lemmas 6 and 7, the number of crossings is at least
\[
1 + 6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] + n - 1 - 2Z_1 + B(1) + n - 1 - 2Z_3 + B(3) \\
+ (n - Z_1)^2 + (n - Z_3)^2 - \Delta_n \Delta_{Z_1+Z_3},
\]
where, for \( i = 1, 3 \), \( B(i) = \begin{cases} 2n & \text{if ccw order } \neq x_{i+1} y_{i+1} y_i x_i \\
\left( Z_i - \frac{n-2}{2} \right)^2 & \text{if ccw order is } x_{i+1} y_{i+1} y_i x_i
\end{cases} \)
Let \( A(i) = 1 - 2Z_i + (n - Z_i)^2 + B(i) \). Then the number of crossings in this case is at least
\[
6 \left[ \frac{n}{2} \right] \left[ \frac{n-1}{2} \right] + 2n - 3 + A(1) + A(3) - \Delta_n \Delta_{Z_1+Z_3}.
\]
We conclude that (8) is at least 
\[ A(i) = (n - Z_i)^2 + 2(n - Z_i) + 1 \geq 4 \]
because \( n - 1 \geq Z_i \). On the other hand, if the counterclockwise order is \( x_{i+1} y_{i+1} y_i x_i \) for either \( i = 1 \) or \( i = 3 \), then after simplifying,
\[ A(i) = \frac{1}{16} \left( (4Z_i - 3n)^2 + (n - 4)^2 \right) - \frac{1}{4} \Delta_n \geq -\frac{1}{4}. \]
In particular, if at least one of \( i = 1 \) or \( i = 3 \) does not have the counterclockwise order \( x_{i+1} y_{i+1} y_i x_i \), then (7) is at least
\[ f(x) = \frac{1}{16} \left( (4x - 3n)^2 + (n - 4)^2 \right) - \Delta_n - \frac{1}{4} \Delta_n - \Delta_n \Delta_{Z_i + Z_3}. \]
(8)
If either \( n \) or \( Z_1 + Z_3 \) is even, then (5) is at least
\[ 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3 - 1/2. \]
Since (5) is an integer, it follows that (5) is in fact bounded below by \( 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3 \), as desired. Hence, assume that both \( n \) and \( Z_1 + Z_3 \) are odd. As (5) is an integer, it suffices to show that
\[ f(Z_1) + f(Z_3) - 3/2 > -1. \]
Suppose to the contrary that \( f(Z_1) + f(Z_3) \leq 1/2 \). Then one of \( f(Z_1) \) or \( f(Z_3) \) is at most \( 1/4 \). Assume without loss of generality that \( f(Z_1) \leq 1/4 \). In particular, this means \( (4Z_1 - 3n)^2 + (n - 4)^2 \leq 2 \). Since \( n \) is odd, we must have \( n \in \{3, 5\} \). If \( n = 3 \), then \( Z_1 = 2 \) (as we saw at the end of Case A2), while if \( n = 5 \), then \( Z_1 = 3 \) or \( Z_1 = 4 \). Because \( (4Z_1 - 3n)^2 + (n - 4)^2 \leq 2 \), we must have \( Z_1 = 4 \) when \( n = 5 \). Thus, in both cases of \( n \), we have \( f(Z_1) = 1/4 \). This implies that \( f(Z_3) \leq 1/4 \). The same argument gives us that \( Z_3 = 2 \) if \( n = 3 \) or \( Z_3 = 4 \) if \( n = 5 \). However, in both cases of \( n \), the sum \( Z_1 + Z_3 \) is even, contradicting the assumption that \( Z_1 + Z_3 \) is odd. Therefore, \( f(Z_1) + f(Z_3) - 3/2 > -1 \).
We conclude that (5) is at least \( 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n - 1}{2} \right\rfloor + 2n - 3 \). This completes Case A.

**Case B:** Now, we suppose that we are in the case that Lemma 7 applies; note that this implies \( n \geq 4 \).

Before we consider different cases, note that for \( n \geq 10 \), the desired bound follows easily from Lemma 7ii:
\[
2n - 1 - 2Z_1 - 2Z_3 + 1/4(Z_1^2 + Z_3^2 + (n - Z_1)^2 + (n - Z_3)^2) - \Delta_n/2
\geq -5 + 1/4(Z_1 - 2)^2 + (Z_3 - 2)^2 + (n - Z_1 + 2)^2 + (n - Z_3 + 2)^2) - \Delta_n/2
\geq -5 + n^2/4 - \Delta_n/2
\geq 2n - 3
\]
The Tripartite-Circle Crossing Number of $K_{2,2,n}$

if $n \geq 10$. It remains to consider the cases that $4 \leq n \leq 9$. We consider the same three cases as in Case A.

**Case B1:** if $Z_1, Z_3 \leq (n-1)/2$. Then, by Lemmas 6 and 7ii, the number of crossings is at least

$$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 1 - 2Z_1 - 2Z_3 + 1/4(Z_1^2 + Z_3^2 + (n - Z_1)^2 + (n - Z_3)^2) - \Delta_n/2$$

Note that if $n \geq 4$, then $n(n - Z_1 - Z_3) - \Delta_n \geq n - \Delta_n \geq 4$ since $Z_1, Z_3 \leq (n-1)/2$ implies $n - Z_1 - Z_3 \geq 1$. Here we use the fact that $\Delta_4 = 0$. Therefore, it follows that

$$-2Z_1 - 2Z_3 + 1/4(Z_1^2 + Z_3^2 + (n - Z_1)^2 + (n - Z_3)^2) - \Delta_n/2 \geq -2$$

$$\iff 2(Z_1 - 2)^2 + 2(Z_3 - 2)^2 + 2n^2 - 2nZ_1 - 2nZ_3 - 2\Delta_n \geq 8$$

$$\iff (Z_1 - 2)^2 + (Z_3 - 2)^2 + n(n - Z_1 - Z_3) - \Delta_n \geq 4$$

$$\iff n(n - Z_1 - Z_3) - \Delta_n \geq 4$$

$$\iff n \geq 4$$

**Case B2:** if $Z_1 \leq (n-1)/2$ and $Z_3 \geq n/2$. Firstly, we consider the case that we have the counterclockwise order $x_{i+1}y_{i+1}y_ix_i$ for $i = 3$. By Lemmas 6 and 7ii, the number of crossings is at least

$$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 1 - 2Z_1 - 2Z_3 + \left(\frac{n-n - 2}{2}\right)^2$$

$$+ 1/4(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - 1/2\Delta_n.$$}

We show that

$$-2Z_1 - 2Z_3 + \left(\frac{n-n - 2}{2}\right)^2 + 1/4(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - 1/2\Delta_n \geq -2.$$}

By simplifying and rearranging, it is equivalent to show that

$$(Z_1 - 2)^2 + (Z_3 - 2)^2 + n(n - Z_1 - Z_3) + 2 \left(\frac{n-n - 2}{2}\right)^2 \geq 4 + \Delta_n.$$}

The left side of the claimed inequality is minimized when $Z_1 = n/2 + 2$ and $Z_3 = n/2$. Moreover, as $Z_1 \leq (n-1)/2$ and $Z_3 \geq n/2$, we have that the left side of the inequality decreases as $Z_1$ increases and $Z_3$ decreases. This is seen from the partial derivatives. Since $Z_1 \leq (n-1)/2$, we take $Z_1 = \lfloor (n-1)/2 \rfloor$ and $Z_3 = \lceil n/2 \rceil$ for the location of the minimum of the left side in this case. When $n$ is even, it follows that $Z_1 + Z_3 \leq n - 1$ and thus

$$n(n - Z_1 - Z_3) \geq n \geq 4 + \Delta_n$$

if $n \geq 4$.

When $n = 2k + 1$, the minimum is attained for $Z_1 = k$ and $Z_3 = k + 1$ and the left side of the claimed inequality simplifies to $2k^2 - 6k + 9$, which is at least $5 = 4 + \Delta_{2k+1}$ for $k \geq 1$.

Secondly, if we do not have the counterclockwise order $x_{i+1}y_{i+1}y_ix_i$ for $i = 3$, then the number of crossings is at least

$$6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4n - 1 - 2Z_1 - 2Z_3 + 1/4(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - 1/2\Delta_n.$$
And
\[ 2n - 2Z_1 - 2Z_3 - 4 + 4 + \Delta_n. \]

The left side of the claimed inequality is minimized when \( Z_1 = Z_3 = n/2 + 2 \). Therefore, for \( n \geq 4 \), it holds that
\[ 4n + (Z_1 - 2)^2 + (Z_3 - 2)^2 + n(n - Z_1 - Z_3) \geq 4 + \Delta_n. \]

This finishes the proof of B3.

Case B3: By the last argument of case B2, we may assume that we do have the counterclockwise order \( x_{i+1} y_i + y_i x_i \) for \( i = 1, 3 \). Thus, the number of crossings is
\[
6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2n - 1 - 2Z_1 - 2Z_3 + \left( Z_1 - \frac{n-2}{2} \right)^2 + \left( Z_3 - \frac{n-2}{2} \right)^2 \\
+ \frac{1}{4}(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - \frac{1}{2} \Delta_n,
\]
which is minimized if \( Z_1 = Z_3 = n/2 \). In this case, \( Z_1 \geq n/2 \) and \( Z_3 \geq n/2 \), so the minimum occurs when \( Z_1 = Z_3 = \lfloor n/2 \rfloor \). Consequently, we obtain
\[
2(Z_1 - Z_3) + \left( Z_1 - \frac{n-2}{2} \right)^2 + \left( Z_3 - \frac{n-2}{2} \right)^2 \\
+ \frac{1}{4}(Z_1^2 + (n - Z_1)^2 + Z_3^2 + (n - Z_3)^2) - \frac{1}{2} \Delta_n \\
\geq -2(n + \Delta_n) + (1 + \Delta_n) + (1 + \Delta_n) + \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 + \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor^2 - \frac{1}{2} \Delta_n \\
\geq -2n + 2 + 4(n^2 + \Delta_n) - \frac{1}{2} \Delta_n \\
\geq \frac{1}{4} n^2 - 2n + 2 - \frac{1}{4} \Delta_n = \frac{1}{4}(n - 4)^2 - 2 - \frac{1}{4} \Delta_n \geq -2
\]
if \( n \geq 4 \). This finishes the proof of B3 as well as of the lemma.

4 Conclusion

In this paper, we determine the tripartite-circle crossing number of \( K_{2,2,n} \).

Comparison with \( K_{4,n} \), which misses exactly four edges of \( K_{2,2,n} \), shows an interesting phenomenon. While their crossings numbers coincide, i.e., \( \text{cr}(K_{2,2,n}) = \text{cr}(K_{4,n}) \), their circle crossing numbers fall apart by a term quadratic in \( n \), i.e., \( \text{cr}_0(K_{2,2,n}) - \text{cr}_0(K_{4,n}) = \frac{1}{4} n^2 + O(n) \). It would be interesting to study this comparison for larger graphs.

Moreover, it is interesting to extending our work to \( k > 3 \).

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Number DMS 1641020. Silvia Fernández-Merchant was supported by the NSF grant DMS 1400653. Marija Jelić Milutinović received support from Grant 174034 of the Ministry of Education, Science and Technological Development of Serbia.
The Tripartite-Circle Crossing Number of $K_{2,2,n}$

References

1. B. M. Ábrego, S. Fernández-Merchant, and A. Sparks, The cylindrical crossing number of the complete bipartite graph, Graphs and Combinatorics 36 (2019), 205–220. https://doi.org/10.1007/s00373-019-02076-5
2. K. Asano, The crossing number of $K_{1,3,n}$ and $K_{2,3,n}$, Journal of Graph Theory 10 no. 1 (1986), 1–8.
3. C. Camacho, S. Fernández-Merchant, R. Kirsch, E. Matson, M. J. Milutinović, and J. White, Bounding the tripartite-circle crossing number of complete tripartite graphs, Journal of Graph Theory (JGT) (2021+), accepted, to appear.
4. E. De Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, and G. Salazar, Improved bounds for the crossing numbers of $K_{m,n}$ and $K_n$, SIAM Journal on Discrete Mathematics 20 no. 1 (2006), 189–202.
5. F. Duque, H. González-Aguilar, C. Hernández-Vélez, J. Leaños, and C. Medina, The complexity of computing the cylindrical and the t-circle crossing number of a graph, The Electronic Journal of Combinatorics 25 no. 2 (2018), P2–43.
6. E. Gethner, L. Hogben, B. Lidický, F. Pfender, A. Ruiz, and M. Young, On crossing numbers of complete tripartite and balanced complete multipartite graphs, Journal of Graph Theory 84 no. 4 (2017), 552–565.
7. M. Ginn and F. Miller, The crossing number of $K_{3,3,n}$, Congr. Numer. 221 (2014), 49–54.
8. R. K. Guy, A combinatorial problem, Nabla (Bull. Malayan Math. Soc.) 7 (1960), 68–72.
9. F. Harary and A. Hill, On the number of crossings in a complete graph, Proceedings of the Edinburgh Mathematical Society 13 no. 4 (1963), 333–338.
10. D. J. Kleitman, The crossing number of $K_{5,n}$, J. Combin. Theory 9 (1970), 315–323.
11. F. T. Leighton, Complexity Issues in VLSI, MIT press, 1983.
12. S. Norin, Turán brickyard problem and flag algebras, 2013, Video from 13w5091: Geometric and Topological Graph Theory Workshop. Available at http://www.birs.ca/events/2013/5-day-workshops/13w5091/videos/watch/201310011538-Norin.html.
13. R. B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs, The American Mathematical Monthly 104 no. 2 (1997), 131–137.
14. M. Schaefer, The graph crossing number and its variants: A survey, The Electronic Journal of Combinatorics (2021), DS21.
15. D. R. Woodall, Cyclic-order graphs and Zarankiewicz’s crossing-number conjecture, Journal of Graph Theory 17 no. 6 (1993), 657–671.
16. K. Zarankiewicz, On a problem of P. Turán concerning graphs, Fundamenta Mathematicae 1 no. 41 (1955), 137–145.