Numerical evidence for monopoles in 3-dimensional Yang-Mills theory

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Abstract

Recently Anishetty, Majumdar and Sharatchandra have proposed a way of characterizing topologically non-trivial configurations for 2+1-dimensional Yang-Mills theory in a local and manifestly gauge invariant manner. In this paper we develop criteria to locate such objects in lattice gauge theory and find them in numerical simulations.

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Monopoles are expected to play an important role in our understanding of confinement of quarks in non-Abelian gauge theories. To locate these monopole configurations, ’t Hooft advocated the idea of using zeroes of a composite Higgs \[1\]. In lattice simulations of Yang-Mills theories, one usually looks for monopoles by fixing an abelian gauge and looking for U(1) monopoles. Some popular gauge choices are the maximal Abelian gauge, the laplacian Abelian gauge etc. \[2,3\]. Therefore, even though ’t Hooft’s original proposal was gauge invariant, on the lattice one usually looks for a gauge dependent quantity. This is however not very satisfactory because the number of monopoles in different gauges do not agree and it is not clear how to relate the numbers obtained in the different gauges. Therefore, it is important to have a gauge invariant way to detect monopoles.

Recently, progress has been made in this direction by Anishetty, Majumdar and Sharatchandra \[4\]. They have given a criterion for characterizing instanton configurations in (euclidean) 3-dimensional Yang-Mills theory in a local and manifestly gauge invariant way. This was achieved by reformulating the theory in terms of gauge invariant variables closely related to gravity. We expect that such a criterion would be useful for locating the monopoles in the 3+1–dimensional Yang-Mills theory.

The theory in terms of these new variables however does not tell us whether these instanton configurations really occur. Since this is an important dynamical question, we look at it by simulating 3-dimensional SU(2) gauge theory on the lattice. We write down the lattice version of the criterion and check for the instanton configurations envisaged in \[4\].

The rewriting of the SU(2) gauge theory in terms of new variables is also a duality transformation as it neatly separates the “spin wave ” degrees of freedom from the “topological” ones. We see this separation explicitly from our simulation data.
II. METHOD

The partition function for 3-dimensional Yang-Mills theory is

$$Z = \int \mathcal{D}A^a_i(x) \exp\left(-\frac{1}{4g^2} \int d^3xF^a_{ij}(x)F^a_{ij}(x)\right)$$

(1)

where

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + \epsilon^{abc} A^b_i A^c_j$$

(2)

is the usual field strength and $g$ the coupling constant.

To identify the instantons of the theory, one can use the orthogonal set of eigenfunctions of a positive symmetric matrix. Consider the following eigenvalue equation:

$$I_{ik}(x)\chi^A_k(x) = \lambda^A(x)\chi^A_i(x).$$

(3)

Here, $I_{ik}(x)$ is defined at each space-time point $x$ by

$$I_{ik}(x) = F^a_{ij}(x)F^a_{kj}(x).$$

(4)

Isolated points where $I_{ik}$ have triply degenerate eigenvalues are special and have topological significance. At such points, the eigenvector fields are singular. The index of the singular point is the instanton number. Thus the instantons in any Yang-Mills configuration $A^a_i(x)$ can be located in terms of the eigenvectors $\chi^A_i(x)$. One can also construct coordinate systems using integral curves of $\chi^A_i(x)$. The coordinate singularities of this coordinate system then correspond to the instantons.

For a generic instanton, the eigenvectors have the behaviour shown in figure 1(a). To locate them, we need to look at the behaviour of the eigenvectors over many lattice spacings. In this paper, we however consider only “spherical” instantons as in figure 1(b). For the location of those instantons, it is, on the other hand, sufficient to look at the behaviour of

\footnote{Here, $A$ is just a label for the eigenvalues and eigenvectors and is not summed over.}
the eigenvectors around only one lattice point. Therefore, it is easier to identify them even though they are rather rare.

In our simulation, we choose the usual Wilson action for SU(2) gauge theory

$$S_{W} = \frac{\beta}{2} \sum \text{tr}(U_{ij} U_{jk} U_{kl} U_{li})$$

(5)

where $U_{ij}$ are the link variables and the products of the links are taken around plaquettes. Here, the relation between $\beta$ and the gauge coupling $g$ in Eq. (1) is $\beta = 4/g^2$. We measure the elementary plaquette at every site. The plaquette variable can be written as

$$\exp i F_{ij} a^a \sigma^a = \cos(\| F_{ij} \|) + i \hat{F}_{ij} a^a \sin(\| F_{ij} \|).$$

(6)

Therefore

$$F_{ij}^a = \hat{F}_{ij} a^a \cos^{-1} \cos(\| F_{ij} \|).$$

(7)

Once we obtain $F_{ij}^a$ in this way, we can construct $I_{ij}$ using definition (4).

According to the criterion in [4], at the location of the instanton all three eigenvalues of $I_{ij}$ should be degenerate and around that point one of the eigenvectors should have a radial behaviour. On the lattice we do not expect the eigenvalues to become exactly degenerate, but we look for sites where the difference between the eigenvalues are less than some cut-off.

In continuum, slightly away from the centre of an instanton with spherical symmetry, one of the eigenvalues of $I_{ij}$ will become non-degenerate with the other two which would still be degenerate. The eigenvector corresponding to this eigenvalue will show a radial behaviour.

On the lattice we look for spherical instantons by choosing the eigenvector corresponding to the largest eigenvalue and plotting it at the site of the monopole and it’s nearest neighbors to check for this radial behaviour.

III. RESULTS

In our simulation, we looked at lattice sizes of length between 16 and 64 lattice sites and $\beta$ between 1.5 and 6. In three dimensions, $\beta$ has dimension of length to leading order.
Therefore we kept the ratio between $\beta$ and the lattice size fixed. To check our configurations, we looked at the string tension and matched it with the values quoted in [5]. We also looked at the minimum of the eigenvalues for the various lattice sizes. The results are plotted in fig.2.

From weak coupling expansion, the average plaquette goes as

$$P = \frac{n_g}{4\beta} + O\left(\frac{1}{\beta^2}\right)$$

where $n_g$ is the number of group generators. For SU(2), the eigenvalues which are proportional to $P^2$ should therefore behave like

$$P^2 = \frac{9}{16\beta^2} + O\left(\frac{1}{\beta^4}\right).$$

This indicates that our data matches reasonably well with first order weak coupling expansion. The deviation that we see is due to terms of higher order in $1/\beta$ which we neglect. As expected, the fit is much better for higher values of $\beta$.

Equilibrium time increases with lattice size. The largest lattice we considered is $64^3$. For this lattice size the equilibration time is of the order of 240 updates. Therefore we ignored the first 300 updates and after that took measurements in every successive update for the next 300 updates. We checked that for this particular measurement autocorrelation effects are negligible.

To choose our cutoff for the difference between the eigenvalues, we looked at the distribution of the smallest eigenvalues for the various lattice sizes. Then we chose the cut-off to be half of the mean minimum eigenvalue for each lattice size. We found that the eigenvalue criterion is extremely sensitive to the choice of cut-off. The various values for which we took our data are shown in table 1. The first and the second columns show the various values of $\beta$ and the lattice size respectively. We recorded the lowest eigenvalue of $I_{ij}$ on the lattice for every measurement averaged it over the 300 measurements that we took. That is tabulated in the third column. The fourth column is the cut-off we chose for each lattice size. It's values are half of that of the third.
With these parameters, we typically found one lattice site in every two or three measurements which had the difference of eigenvalues less than the cut-off. For a few configurations (roughly 3 or 4 out of the 300 configurations probed), for every lattice size, we found two or three sites in a single configuration which met the eigenvalue criterion. After this we looked at the eigenvectors corresponding to the largest eigenvalue around the lattice site which satisfied the criterion for the degenerate eigenvalues. For a few of them we found that there are at least three non-coplanar eigenvectors which converge to or diverge from a point. Fig. 3 shows an eigenvector configuration where three eigenvectors converge to the point $C$. To find out whether this is a true congruence in three dimensions, we rotated the configuration and checked if the eigenvectors remained convergent at various different angles. Our data is presented in table 2. Here again the first two columns show the values of $\beta$ and lattice size. The total number of cases when the difference between the eigenvalues were less than the cut-off, over the 300 measurements, is tabulated in the third column. Among these configurations, we looked for radial behaviour of the eigenvector corresponding to the largest eigenvalue. That number is tabulated in column four.

In figure 4 we look at the distribution of the eigenvectors in the x-y plane. A close look shows clearly that most of the spins suffer small deviations and are like spin wave excitations. However, there are a few spin configurations which form vortices. In the figure the two closed boxes show vortex configurations over one plaquette and two plaquettes respectively. We see that the number of vortices extended over more than one plaquette are more common than those over a single plaquette. We expect similar conclusion to hold true for the monopoles as well. This suggests that it is important to try and characterize big monopoles as well. That will be investigated later.

**IV. CONCLUSIONS**

In this note we have looked at simulation data for SU(2) lattice gauge theory in three dimensions. We found that there indeed exist configurations that correspond to the criterion
constructed in [4]. The data also shows that large size objects extending over many cells or plaquettes are favoured compared to the ones over single cell or plaquette. Also only about $\frac{1}{3}$ of the instantons have a spherical symmetry.

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### Table-1

| $\beta$ | lattice size | mean lowest eigenvalue | cut-off eigenvalue |
|---------|--------------|------------------------|-------------------|
| 1.5     | 16           | 0.253                  | 0.1265            |
| 2.25    | 24           | 0.129                  | 0.0646            |
| 3       | 32           | 0.079                  | 0.0397            |
| 3.75    | 40           | 0.0521                 | 0.026             |
| 4.5     | 48           | 0.0368                 | 0.0184            |
| 5.25    | 56           | 0.0264                 | 0.0132            |
| 6       | 64           | 0.0217                 | 0.0109            |

### Table-2

| $\beta$ | lattice size | satisfies eigenvalue | satisfies eigenvector |
|---------|--------------|----------------------|-----------------------|
| 1.5     | 16           | 59                   | 22                    |
| 2.25    | 24           | 58                   | 12                    |
| 3       | 32           | 67                   | 6                     |
| 3.75    | 40           | 63                   | 14                    |
| 4.5     | 48           | 78                   | 16                    |
| 5.25    | 56           | 52                   | 7                     |
| 6       | 64           | 69                   | 18                    |
fig. 1. (a) Eigenvectors around an instanton extended over many lattice spacings.
(b). Behavior of the eigenvectors around a “spherical” instanton centered on one lattice point.
fig. 2. In this figure we plot the lowest eigenvalues of $I_{ij}$ (points with errorbars) along the y-axis for various values of $\beta$ (along the x-axis). The dashed line is first order weak coupling expansion for the square of the plaquette $P^2 = \frac{9}{16\beta^2}$. 
fig. 3. Eigenvector configuration we are looking for. The eigenvectors shown here are scaled to twice their size to show their crossing explicitly. Lattice size is 56.
fig. 4. Snapshot of a x-y plane with eigenvectors projected to the plane. Closed boxes show projection of spherical instanton formed on one plaquette and “non-spherical” instanton extended over two plaquettes on the plane. Lattice size is 16.
