Gauge (In)variance, Mass and Parity in D=3 Revisited

S. Deser *

Abstract

We analyze the degree of equivalence between abelian topologically massive, gauge-invariant, vector or tensor parity doublets and their explicitly massive, non-gauge, counterparts. We establish equivalence of field equations by exploiting a generalized Stueckelberg invariance of the gauge systems. Although the respective excitation spectra and induced source-source interactions are essentially identical, there are also differences, most dramatic being those between the Einstein limits of the interactions in the tensor case: the doublets avoid the discontinuity (well-known from D=4) exhibited by Pauli–Fierz theory.

It is a pleasure to dedicate this work to Dieter Brill on the occasion of his 60th birthday. I have learned much from him over the years, not least during our old collaborations on general relativity. I hope he will be entertained by these considerations of related theories in another dimension.

1. Introduction

Perhaps the most paradoxical feature of topologically massive (TM) theories [1, 2] is that their gauge invariance coexists with the finite mass and single helicity, parity violating, character of their excitations. This phenomenon, common to vector (TME) and tensor (TMG) models, is special to 2+1 dimensions because in higher (odd) dimensions the operative Chern–Simons (CS) terms are of at least cubic order in these fields and so do not affect their kinematics; only higher rank tensors could acquire a topological mass there. Equally surprisingly, TM doublets, with mass parameters of opposite sign, are not only invariant under combined parity and field interchanges[1] but they are equivalent in essential respects to non-gauge vector or tensor models [2].

*The Martin Fisher School of Physics, Brandeis University, Waltham, MA 02254, USA. This work was supported by the National Science Foundation under grant #PHY88–04561.

1Doublets of pure (non-propagating) vector CS models have also been studied [3].
with ordinary mass terms: Both their excitation contents and the interactions they induce among their sources are essentially identical, as originally indicated in [2].

To be sure, other mass generation mechanisms exist in gauge theories: in D=2, the Schwinger model — spinor electrodynamics — yields a massive photon, but not at tree level. In D=4, in addition to the Higgs mechanisms, there is one involving a mixed CS-like term bilinear in antisymmetric tensor and vector fields [4]. However, none function, as this one does, purely with a single gauge field. It is also well-known that apparently different free-field theories can have the same excitation spectrum. Indeed, there even exist self-dual formulations of TM theories [5, 6, 7, 8] involving a single field, in which gauge invariance is hidden; however, their minimal couplings to a given source are then inequivalent.

My purpose here is to review and make more explicit how there can be coexistence between those equivalences and the differences in gauge character, as well as to point out that traces of these differences nevertheless remain. On the equivalence side, we will see that the TM field equations can be written in the non-gauge form because their gauge freedom can be hidden through a generalized Stueckelberg field redefinition. We will also discuss three ways in which differences manifest themselves. First, because gauge invariance implies the existence of Gauss constraints, long-range potentials carrying information about the sources survive in the massive doublets, but have no counterpart in the non-gauge models. Second, we will compare the massive excitation spectra with those of the limiting massless theories and explain why normally massive theories always differ from the latter, while TME (but not TMG) is one example in which the number of excitations (if not their helicities) agree with massless theory. Third, the Einstein limits of the tensor models differ dramatically: As is well-known [9] in D=4, massive tensors have components which do not decouple from the sources in this limit and lead to source-source interactions different from those of Einstein theory; TMG, on the other hand, will be shown to avoid this discontinuity. The present analysis is restricted to abelian models; their nonlinear generalizations are not obviously amenable to these considerations. Extension to their supersymmetric generalizations [1, 10], on the other hand, is straightforward.

2. Vector Theories

Consider first a topologically massive vector doublet, which will be compared with massive vector (Proca) theory. The respective free Lagrangians are, in \((-+++\)) signature,

\[
L_{VD} = L_{TME}(A^1_{\mu}, m) + L_{TME}(A^2_{\mu}, -m) - \frac{1}{\sqrt{2}} j^\mu(A^1_{\mu} + A^2_{\mu}), \\
L_{TME}(A_{\mu}, m) \equiv -\frac{1}{4} F_{\mu\nu}^2(A) + \frac{1}{2} m \epsilon^{\alpha\mu\nu} A_\alpha F_{\mu\nu}(A) \tag{1}
\]

\(^2\)Their nonlinear parts fail to superpose, and the respective sources cannot simultaneously be covariantly conserved with respect to each gauge field and to their sum.
and

\[ L_p(A_\mu) = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 A_{\mu}^2 - j^\mu A_\mu . \]  

(2)

By gauge invariance, the current must be conserved in TME and the comparison can only be made if, as we assume henceforth, \( \partial_\mu j^\mu = 0 \). Let us review what is known \[\footnote{Invariance of \( L_D \) under combined parity and field interchange conjugations is manifest, since the parity transform is equivalent to a change of sign of \( m \), which is just cancelled by the interchange \( A^1_\mu \leftrightarrow A^2_\mu \). Next, the equivalence of the effective current-current interactions generated by (1) and by (2) follows from the forms of the respective propagators, dropping all (irrelevant) terms proportional to \( p_\mu p_\nu \): That of each TME is the sum of an ordinary (even) Proca term, \( \eta_{\mu\nu}[p^2 + m^2]^{-1} \), and a mass- and parity-odd one \( \sim (m \epsilon_{\mu\nu\alpha\beta} p^\alpha)p^{-2}(p^2 + m^2)^{-1} \), which cancels out in the \( \pm m \) doublet. Hence the interaction is of the usual finite-range form \( \sim j^\mu (p^2 + m^2)^{-1} j^\mu \) in both theories.

It is instructive to establish how equivalence is displayed at the level of the field equations, and especially how the manifest TME gauge invariance can be “hidden” in the process. The field equations are

\[ \partial_\mu F^{\mu\nu}(A_1) + m * F^{\nu}(A_1) = \frac{1}{\sqrt{2}} j^\nu \]  

(3a)

\[ \partial_\mu F^{\mu\nu}(A_2) - m * F^{\nu}(A_2) = \frac{1}{\sqrt{2}} j^\nu . \]  

(3b)

Here \( * F^\nu \equiv \frac{1}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} \) is the dual field strength, with \( F_{\mu\nu} = -\epsilon_{\mu\nu\alpha} F^\alpha \). In terms of the sums and differences, \( A^\pm_\mu = \frac{1}{\sqrt{2}} (A^1_\mu \pm A^2_\mu) \), we have

\[ \partial_\mu F^{\mu\nu}(A_+) + m * F^{\nu}(A_+) = j^\nu \]  

(4a)

\[ \partial_\mu F^{\mu\nu}(A_-) + m * F^{\nu}(A_-) \equiv \epsilon^{\nu\mu\alpha} \partial_\mu \left[ * F_\alpha(A_-) + m A^+_\alpha \right] = 0 . \]  

(4b)

The general solution of (4b) is

\[ * F_\alpha(A_-) + m A^+_\alpha = \partial_\alpha \Lambda + F_\alpha(V) \]  

(5)

where \( V_\mu \) is a solution of the homogeneous Maxwell equation, \( \epsilon^{\mu\alpha\beta} \epsilon^{\nu\lambda\sigma} \partial_\mu \partial^\nu V_\sigma = 0 \). Inserting (5) into (4a) yields the Proca equation in (not quite) Stueckelberg form,

\[ \partial_\mu F^{\mu\nu}(A_+) - m^2 [A^\nu_+ - m^{-1} \partial^\nu \Lambda - m^{-1} * F^{\nu}(V)] = j^\nu . \]  

(6)

The gauge-freedom carried by \( \Lambda \) can be field-redefined away by \( A^\nu_+ \rightarrow A^\nu_+ - m^{-1} \partial^\nu \Lambda \) as usual. But so can the additional \( V \) term, by \( A^\nu_+ \rightarrow A^\nu_+ + m^{-1} * F^{\nu}(V) \) since \( \partial_\mu F^{\mu\nu}(**F(V)) \) is just \( \Box * F^{\nu}(V) \), which vanishes for a field strength \( * F^{\nu}(V) \) obeying Maxwell’s equations. So the \( A_+ \) field’s invariance is absorbed by an extended Stueckelberg transformation, leaving the Proca form. Once the \( A_- \) field is determined (up to this gauge) by (6), we can use (4b) to determine \( A_- \) (up to a gauge),
with \( m^*F^\nu(A_+) \) as the effective current determining \( F^\mu\nu(A_-) \) through an ordinary massless Maxwell equation.

The equivalence of the two systems’ degrees of freedom goes as follows. Each TME embodies a single massive particle with helicity \( m/|m| \). Proca theory obviously has \((D-1) = 2\) massive degrees of freedom, and they too have helicities \( \pm 1 \), though this fact requires the same analysis of the full Lorentz algebra, (rather than merely that of the rotation generator \( M^{ij} \)), as was required for TME \([2]\); as we shall see, the naive \( M^{ij} \) is purely orbital for both systems. For completeness, we identify the excitations in terms of the canonical formulation of both models, using first-order form of \((1),(2)\) where \((A_\alpha, F^\mu\nu)\) are initially independent variables. The free Lagrangian

\[
L = \frac{1}{2} F^\mu\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4} F^\mu\nu F^\alpha\beta \eta_\mu_\alpha \eta_\nu_\beta + \frac{2}{3} m^2 A_\mu A_\mu \quad (7)
\]

represents TME or Proca theory, as \( \alpha = 1, \beta = 0 \) or \( \alpha = 0, \beta = 1 \) respectively. The stress tensor associated to \((7)\) is the usual (massive) vector, \( T^\mu_\nu = F^\mu_\alpha F^\nu_\alpha - \frac{1}{4} \eta^\mu_\nu F^2_\alpha_\beta + \beta m^2 (A_\mu A_\nu - \frac{1}{2} \eta^\mu_\nu A^2_\alpha) \) ; \((8)\)

while the transverse-longitudinal decomposition of a spatial vector is expressed by

\[
V^i = V^i_T + V^i_L = \epsilon^{ij} \xi^{-1} \partial_j v_T + \xi^{-1} \partial_i v_L, \quad \xi = (-\nabla^2)^{1/2}. \quad (10)
\]

In both theories, one may eliminate the trivial constraint

\[
F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon^{ij} B = \epsilon^{ij} \xi a_T. \quad (11)
\]

However, in TME, only the gauge-independent pair \((e_T, a_T)\) remains, whereas the longitudinal components also survive in Proca theory. The difference lies in the fact that \( A_0 \) is respectively a Lagrange multiplier or an auxiliary field in the two systems; that is, its variation yields the constraint

\[
- \nabla \cdot E + \frac{\alpha}{2} m B + \beta m^2 A_0 = j^0. \quad (12)
\]

Dropping the source for the moment, we then find for the free actions, which began as

\[
I = - \int d^3x \left( E \cdot (\dot{A} - \nabla A_0) + \frac{1}{2} E^2 + \frac{1}{2} B^2 (A) + \alpha m \left\{ \epsilon^{ij} A_i \dot{A}_j - \frac{1}{2} A_0 B(A) \right\} + \frac{\beta}{2} m^2 (A^2 - A_0^2) \right), \quad (13)
\]

Keeping both terms results in a parity-violating theory with two different helicities, as does taking the CS term to be of the form \( \epsilon^{\mu\nu\alpha} \alpha_\lambda A_\alpha F_{\lambda\sigma} \eta_\mu_\lambda \eta_\nu_\sigma \), which is no longer gauge-invariant, nor metric-independent.
the final forms

\begin{align}
I_{\text{TME}} &= \int d^3x [(-e_T)\dot{a}_T - \frac{1}{2} e_T^2 - \frac{1}{2} a_T (m^2 - \nabla^2) a_T], \\
I_P &= I_{\text{TME}} + \int d^3x [\bar{a}_L \dot{\bar{e}}_L - \frac{1}{2} \bar{a}_L^2 - \frac{1}{2} \bar{e}_L (m^2 - \nabla^2) \bar{e}_L].
\end{align}

The doublet action is just the sum of two independent terms (14). To reach (14), we eliminated the longitudinal electric field using the Gauss constraint (12). To obtain (15), we used (12) for $\alpha = 0$, $\beta = 1$, to eliminate $A_0$, and rescaled $\bar{a}_L \equiv m^{-1} a_L$, $\bar{e}_L \equiv m e_L$. The free field content then, is that both theories have two massive degrees of freedom. To determine their helicities requires, in both cases, study of the full Lorentz algebra, for the rotation generators are ostensibly purely orbital because of the peculiarities of 2–space. That is, from its definition,

\begin{equation}
M = \int d^2r \epsilon_{ij} x^i T^j_0,
\end{equation}

it follows from (8) and the canonical reductions that

\begin{equation}
M = \int d^2r \sum_{i=1}^{2} (-e_T^i) \partial_\theta a^i_T
\end{equation}

for the TME doublet and

\begin{equation}
M = \int d^2r [(-e_T) \partial_\theta a_T + \bar{a}_L \partial_\theta \bar{e}_L]
\end{equation}

for the massive vector, as is to be expected since the canonical variables are spatial scalars.

We now reinstate the $j^\mu A_\mu$ term to recover the effective current-current coupling in the present context. The canonical TME doublet action (14) contains a term $\sim \frac{1}{\sqrt{2}} j_T (a^L_T + a^L_T)$ together with the longitudinal interactions, which for our conserved currents would look like

\begin{equation}
\rho \frac{\Box}{(m^2 - \Box)} \nabla^{-2} \rho \sim \rho (-\nabla^{-2}) \rho + \rho \frac{(m^2/\nabla^2)}{m^2 - \Box} \rho.
\end{equation}

This is the usual sum of a Coulomb term and a (retarded) Yukawa interaction, whereas the $j_T a_T$ coupling above results in the purely retarded form $j_T (m^2 - \Box)^{-1} j_T$. The Proca action would have, in addition to the $j^T a_T$ and $j^L a_L$ terms, the remnant of $\rho A_0$, which together of course yield the same $\rho - \rho$ interactions as above for TME doublets. Hence the total current-current coupling is of course the same $j^\mu (m^2 - \Box)^{-1} j_\mu$ in both cases, and limits smoothly to the Maxwell one as $m \to 0$. 

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3. Tensor Theories

The gravitational case is similar to the vector in most respects. A doublet of opposite mass TMG has the same excitation content as a single massive, Pauli–Fierz, tensor theory and leads to essentially equivalent induced source-source interactions \[2\].

The doublet’s action is \[2\]

\[
I_{TD} = I_{TMG}(h_{\mu\nu}^{1}, \mu) + I_{TMG}(h_{\mu\nu}^{2}, -\mu) + \frac{1}{\sqrt{2}} \kappa T_{\mu\nu}(h_{\mu\nu}^{1} + h_{\mu\nu}^{2})
\]

\[
I_{TMG}(h_{\mu\nu}, \mu) = -\kappa^{-2} \int d^{3}x R^{Q}(h) + 1/2\mu \int d^{3}x \epsilon^{\mu\alpha\beta} G_{\alpha} \partial_{\mu}(\kappa h_{\beta\nu}) ,
\]

where \( Q \) indicates that only terms quadratic in \( \kappa h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu}) \) are to be kept in the expansion of the Einstein actions and \( G_{\alpha} \) is linearized. Again, parity conservation is restored by including the interchange \( h_{\mu\nu}^{1} \leftrightarrow h_{\mu\nu}^{2} \). The sign of the Einstein term is “ghost-like,” i.e., opposite of the conventional one, in order that the TMG excitations be non-ghost. The Einstein constant \( \kappa^{2} \) has dimensions of length, while \( \mu \) is dimensionless since the CS term is of third derivative order; the mass \( m \) is \( \mu \kappa^{-2} \), but the “massless” Einstein limit is clearly \( \mu \to \infty \). The Pauli–Fierz action is as in \( D=4 \),

\[
I_{PF} = + \int d^{3}x R^{Q}(k) - \frac{m^{2}}{2} \int d^{3}x (k_{\mu\nu}^{2} - k^{2}) + \kappa \int d^{3}x k_{\mu\nu} T_{\mu\nu} , \quad k \equiv k_{\alpha}^{\alpha}.
\]

The couplings in both cases are the usual minimal ones, and \( T^{\mu\nu} \) is necessarily conserved for consistency with the (linearized) gauge invariance of TMG; henceforth we assume\[\footnote{Of course, just as in normal general relativity, a dynamical \( T_{\mu\nu} \) will no longer be conserved as a result of its coupling and the full nonlinear theory will be required.}\] \( \partial_{\mu} T_{\mu\nu} = 0 \). The propagator of a single TMG field (neglecting terms proportional to \( p^{\mu} p^{\nu} \), which vanish for conserved sources) differs from that of the massive one in two respects: it contains, in addition, a term with numerator \( \sim \mu (\epsilon^{\mu\alpha\beta} p^{\gamma} + \text{symm}) \) and also a term independent of \( \mu \), corresponding to a free (ghost) Einstein field propagator. In the doublet then, the odd terms cancel, leaving two interaction terms, the usual Pauli–Fierz finite range one,

\[
\kappa^{2} \int d^{3}x [T^{\mu\nu}(-\Box + m^{2})^{-1}T_{\mu\nu} - \frac{1}{2} T_{\mu}^{\mu}(-\Box + m^{2})^{-1}T_{\nu}^{\nu}]
\]

and the extra ghost Einstein part,

\[
-\kappa^{2} \int d^{3}x [T^{\mu\nu}(-\Box)^{-1}T_{\mu\nu} - T_{\mu}^{\mu}(-\Box)^{-1}T_{\nu}^{\nu}] .
\]
terms when conservation of $T^{\mu\nu}$ is taken into account \[11\]. The finite-range interaction (20a) is then effectively the entire residue in both systems, and equivalence of interactions is established modulo the (trivial) Einstein part (20b) in the doublet.

Now let us analyze the field equations of the TMG doublet, as we did for TME. They are

$$-G_{\mu\nu}(h_1) + m^{-1}C_{\mu\nu}(h_1) = \frac{1}{\sqrt{2}} \kappa T_{\mu\nu}$$

(21a)

$$-G_{\mu\nu}(h_2) - m^{-1}C_{\mu\nu}(h_2) = \frac{1}{\sqrt{2}} \kappa T_{\mu\nu},$$

(21b)

where $G_{\mu\nu}$ is the linearized Einstein tensor and $C_{\mu\nu}$ is the third-derivative order linearized Cotton–Weyl tensor $C_{\mu\nu} \equiv \epsilon_{\mu\alpha\beta} \partial_\alpha (R_\beta \nu - \frac{1}{4} \delta_\beta^\nu R)$; it is symmetric, traceless and (identically) conserved. We again take the sum and difference of (21a,b) in terms of $h_{\pm\mu\nu} = \frac{1}{\sqrt{2}} (h_1^{\mu\nu} \pm h_2^{\mu\nu})$,

$$-G_{\mu\nu}(h^+) + m^{-1}C_{\mu\nu}(h^-) = \kappa T_{\mu\nu}$$

(22a)

$$-G_{\mu\nu}(h^-) + m^{-1}C_{\mu\nu}(h^+) = 0.$$  

(22b)

One may solve (22b) for either $h^-_{\mu\nu}$ or $h^+_{\mu\nu}$ and insert into (22a) to get an equation in terms of the other; this will yield third or fourth derivative inhomogeneous equations. I will sketch the results, but to save space will omit all the “Stueckelberg” gauge parts as well as required symmetrizations that arise in the process; I will also drop scalar curvatures since it is the Ricci tensor that counts (actually the trace of (22b) shows that $R(h^-)$ vanishes). All “equalities” below that are subject to these caveats will carry $\approx$ signs. Using the fact that $G^{\mu\nu}(h) = -\frac{1}{2} \epsilon^{\mu\alpha\beta} \epsilon_{\nu\lambda\sigma} \partial_\alpha h_{\beta\sigma}$, we learn from (22b) that, in the obvious notation $R^+_{\nu\beta} \equiv R_{\nu\beta}(h^+),$

$$\epsilon^{\mu\alpha\beta} \partial_\alpha (m^{-1} R^+_{\nu\beta} \frac{1}{4} \epsilon_{\nu\lambda\sigma} \partial_\lambda h^-_{\beta\sigma}) \approx 0.$$  

(23)

Apart from homogeneous (gauge) terms, then, we may eliminate $R^+_{\nu\beta}$ in terms of first derivatives of $h^-_{\beta\sigma}$ in (22a) to find the third order equation

$$\epsilon^{\mu\lambda\sigma} \partial_\lambda (2R^-_{\sigma\nu} + m^2 h^-_{\sigma\nu}) \approx 2m\kappa T_{\mu\nu}.$$  

(24a)

The left side is the gauge-invariant curl of a Klein–Gordon form; in harmonic gauge where $2R^-_{\sigma\nu} = -\Box h^-_{\sigma\nu}$, we have

$$\epsilon^{\mu\lambda\sigma} \partial_\lambda (-\Box + m^2) h^-_{\sigma\nu} \approx 2m\kappa T_{\mu\nu}.$$  

(24b)

Alternatively, we may take the curl of (22b) to learn that $mC^-_{\mu\nu} = \Box R^+_{\mu\nu}$, and hence we may write (22a) as

$$(m^2 - \Box) R^+_{\mu\nu} \approx -\kappa m^2 T_{\mu\nu}.$$  

(25a)

This is a (fourth order) Klein–Gordon equation which states that, in harmonic gauge,

$$\Box (m^2 - \Box) h^+_{\mu\nu} \approx 2\kappa m^2 T_{\mu\nu}.$$  

(25b)
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[Of course (24b) and (25b) just reflect the propagator structure of TMG and its doublets that was given earlier.]

Let us now compare the above TMG equations with the Pauli–Fierz ones,

$$G_{\mu\nu}(k) + \frac{m^2}{2}(k_{\mu\nu} - \eta_{\mu\nu}k) = \kappa T_{\mu\nu}.$$  \hspace{1cm} (26)

The analysis goes as in D=4, namely one first notes that (for conserved $T_{\mu\nu}$) the divergence of (26) implies $\partial^\mu(\eta_{\mu\nu}k - k_{\mu\nu}) = 0$; the double divergence is just the scalar curvature $R(k)$, which then also vanishes. Using these constraints, we may write $G_{\mu\nu}(k)$ as follows:

$$G_{\mu\nu}(k) = R_{\mu\nu}(k) = -\frac{1}{2}\left(\Box k_{\mu\nu} - \partial^{\alpha}_{\mu\nu}k^\alpha - \partial^{\alpha}_{\mu\nu}k^\alpha + \partial^{\alpha}_{\mu\nu}k\right)$$

$$= -\frac{1}{2}\left(\Box k_{\mu\nu} - \partial^{\alpha}_{\mu\nu}k\right).$$  \hspace{1cm} (27)

Now perform the “Stueckelberg” field redefinition, $k_{\mu\nu} \rightarrow k_{\mu\nu} - m^{-2}\partial^{\alpha}_{\mu\nu}k$ in (26), under which $G_{\mu\nu}$ is of course invariant. This change in form of the mass term just cancels the $\partial^{\alpha}_{\mu\nu}k$ in (27) and yields the standard Klein–Gordon equation

$$(m^2 - \Box)(k_{\mu\nu} - \eta_{\mu\nu}k) = 2\kappa T_{\mu\nu}.$$  \hspace{1cm} (28)

Comparing with the corresponding (schematic!) equations for the TMG doublet, (24b) or (25b), we see there the same basic Klein–Gordon propagation, modified by the higher order character of TMG. Had we kept the $R^+$ terms, the gauge parts, performed Stueckelberg shifts, etc., we would have again deduced from these field equations the equivalence of source-source interactions modulo the extra Einstein “coupling” in TMG.

We will not reproduce the canonical analysis of TMG, which may be found in [2], nor that of the massive theory, well-known from D=4. The pattern is clear from the vector story: in both cases there are two massive excitations, whose helicity $\pm 2$ character cannot be determined from the rotation generators alone.

4. Differences

The correspondence between the TM gauge doublets and their normally massive counterparts is not complete. We discuss three types of differences here. The first deals with existence of long-range Coulomb-like (but locally pure gauge) potentials in the gauge doublets — despite the finite range character of the excitations and interactions — this is the most manifest aspect of their gauge invariance, with no massive counterpart. Secondly, we analyse the zero mass limits of the free excitation spectra, and thirdly, that of the source-source interactions. The latter aspect displays particularly striking differences in the tensor case: TMG leads to a smooth (and hence trivial) Einstein limit, whereas the Pauli–Fierz discontinuity of D=4 [3] not
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only persists in D=3 but now gives a non-trivial, and therefore especially “different,” interaction from that of Einstein gravity.

The Proca and Pauli–Fierz equations consist entirely of a set of Klein–Gordon equations of the form \((m^2 - \Box)\phi_i = \rho_i\), with no differential constraints and hence no long-range components. The TM theories, however, do contain Gauss constraints, but with additional terms. In the vector case, (12) shows (for \(\beta = 0\)) that it is now \(mB\), rather than the short-range \(\nabla \cdot E\) which carries the asymptotic information \[2\], because \(mB\) is a curl whose spatial integral is proportional to the total charge \(Q = \int d^2r j^0\). We therefore expect that in our doublet, it is the difference, \(B(A^-)\), which is relevant. This is indeed the case: while the \(A_\mu^+\)-field is short-range, obeying as it does the Proca equation (6), it follows from (5) that \(A_\mu^+\) is determined then by \(A^-\). Specifically, \(^*F^0(A_-) = B(A_-)\) is proportional to the sum of \(\nabla \cdot E(A_+)\) and the charge density \(j^0\). Consequently, since \(E(A_+)\) is short-range, it is \(A_-\) at spatial infinity that is proportional to \(Q = \int d^2r j^0\). This can also be read off from the time component of (5): the right hand side’s spatial integral vanishes, while the integral of \(A_\mu^+\) is proportional to \(Q\), by the Proca equation.

A very similar picture emerges in the tensor case: the lower derivative part, \(G^0_{00}\), of the TMG field equations (21) carries the asymptotic information about the energy of the source \[2\], just as it does in Einstein gravity. But this time it is the term even in \(m\), and hence \(G^0_{00}(h^+)\) that is relevant. Now \(G^0_{00}(h^+)\) is (like \(B(A)\)) a total spatial divergence, and the source energy is the spatial integral of \(T^0_{0}\), so we need only integrate the (00) components of (22a) over space, noting that the \(C^-\) term falls off too rapidly to contribute at infinity.

Next we compare the massive and massless excitation spectra, using the well-known fact that for each gauge symmetry broken by a lower derivative term, one new degree of freedom arises (only one, because the former Lagrange multiplier, unlike the gauge variable, is merely promoted to be an auxiliary field). Thus massive vector theory acquires one degree of freedom beyond the (D–2) of Maxwell theory because \(m^2 A_\mu^2\) breaks its gauge invariance. The massive tensor model has (D–1) additional excitations beyond the \(\frac{1}{2} D(D–3)\) of Einstein theory because \(m^2 (h_{\mu \nu}^2 - h^2)\) breaks all but the longitudinal one of the D gauges of linearized gravity; so there is always a discontinuity (except at D=2, where the Einstein kinetic term is absent altogether). On the other hand, in (singlet) TME, the CS term shares the Maxwell invariance and there is no discontinuity in the number, but only in the helicity, of the single excitation (massless particles always have vanishing helicity \[12\]). [This method of counting also applies in vector theory when a “mass” term is added to a pure CS term \[4\], breaking the latter’s gauge invariance to give rise to a single massive degree of freedom; it is in fact equivalent to TME \[4\], as is also the case if the mass term arises through a Higgs mechanism \[7\].] In TMG, while the Einstein term also preserves gauge invariance of the CS term, it breaks the latter’s conformal invariance, thereby generating the massive graviton discontinuously from its two separately nondynamical parts.
More dramatic are the differences between source-source interactions in our models and in their $m = 0$ or $m = \infty$ counterparts. Whereas the degree of freedom count is a more formal difference – for example, it has long been known from $D = 4$ that the extra degree of freedom in Proca theory decouples from the current as $m \to 0$, so that the limit is (all but gravitationally) indistinguishable from electrodynamics, there is a discontinuity in the induced source couplings already in $D = 4$ between Einstein and Pauli–Fierz models, precisely because not all the extra modes decouple from the stress tensor in the massless limit. Here TMG provides a clear difference from the massive model: As we have seen by considering the propagator, the infinite mass limit of the TMG doublet (where its free action reduces to the Einstein part) leads to just the same $T_{\mu \nu} - T_{\mu \nu}$ coupling (or rather lack of coupling!) as in Einstein theory. On the other hand, the massless (“Einstein”) limit of Pauli–Fierz coupling differs from the Einstein form by the famous $(D-1)^{-1}$ versus $(D-2)^{-1}$ coefficient of the $-T_{\mu}^{\mu} \Box^{-1}T_{\nu}^{\nu}$ term of (20). This is especially dramatic in $D=3$ where it means in particular the difference between, respectively, existence or absence of a Newtonian limit!

5. Summary

We have reviewed the resemblances and differences between the two ways of giving mass to gauge fields in $D=3$, through gauge-invariant Chern–Simons terms or through explicit mass terms. The equivalences between these ways, despite their different gauge properties, were followed also at the level of field equations where the “fading” of gauge invariance into ordinary Klein–Gordon-like expressions was understood as a manifestation of Stueckelberg mechanisms. Most dramatic among the differences was the fact that TMG provides the only example of a smooth limit for the interactions from massive to Einstein form, in contrast to the discontinuity in the explicitly massive tensor theory.

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5For completeness, we note that the $m \to 0$ limit of TMG is to be contrasted with pure CS, rather than Einstein, gravity. Here our doublet is now “discontinuous,” since it leads to a net pure trace, $\sim T_{\mu}^{\mu} \Box^{-1}T_{\nu}^{\nu}$ coupling, whereas pure CS gravity of course cannot even couple to sources whose traces fail to vanish; continuity is of course restored if the source is traceless. The Pauli–Fierz theory (like Proca theory) of course leaves no interactions at all in the infinite mass limit.
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