1. Introduction

Let $X \subset \mathbb{C}^{n+1}$ (resp. $V \subset \mathbb{P}^{n+1}$) be an algebraic hypersurface and set $M_X = \mathbb{C}^{n+1} \setminus X$ (resp. $M_V = \mathbb{P}^{n+1} \setminus V$) where we suppose $n > 0$. The study of the topology of $X$, $V$ and of their complements $M_X$, $M_V$ is a classical subject going back to Zariski. In a sequence of papers Libgober has introduced and studied the Alexander invariants associated to $X$, $V$, see for instance [Li0-3].

In the affine case, let $f = 0$ be a reduced equation for $X$. One can use the results on the topology of polynomial functions, see for instance [B], [ACD], [NZ], [SiT], to study the topology of the complements $M_X$, as in the recent paper by Libgober and Tibăr [LiT].

By taking generic linear sections and using the (affine) Lefschetz theory, see for instance Hamm [H] (and [Li2], [LiT], [D1] for different applications), one can restrict this study to hypersurfaces $X$ having only isolated singularities including at infinity, see [Li2], or in the polynomial framework, to polynomials having only isolated singularities including at infinity with respect to a compactification as in [SiT]. Simple examples show that neither of these two restricted situations is a special case of the other, hence both points of view have their advantages. However, the polynomial point of view embraces larger classes of examples due to the fact that the best compactification of a polynomial function is not usually obtained by passing from the affine space $\mathbb{C}^{n+1}$ to the projective space $\mathbb{P}^{n+1}$. This is amply explained in [LiT].

In the present paper we consider an arbitrary polynomial map $f$ (whose generic fiber is connected) and we study the Alexander invariants of $M_X$ for any fiber $X$ of $f$.

The article has two major messages. First, the most important qualitative properties of the Alexander modules (cf. 4.5, 5.2, 5.4 and 6.9) are completely independent of the behaviour of $f$ at infinity, or about the special fibers. (On the other hand, for particular families of polynomial maps with some additional information about the special fibers or about the behaviour at infinity, one can obtain nice vanishing or connectivity results; see e.g. our case of $h$-good polynomials below).

The second message is that all the Alexander invariants of all the fibers of the polynomial $f$ are closely related to the monodromy representation of $f$. In fact, all the torsion parts of the Alexander modules (associated with all the possible fibers) can be obtained by factorization of a unique universal Alexander module, which is constructed from the monodromy representation. This explains nicely and conceptually all the divisibility properties that have appeared recently in the literature connecting the Alexander polynomials of $M_X$ and the characteristic polynomials of some special monodromy operators, see [Li2] and [LiT]. [Note that the monodromy considered by Libgober in [Li2], section 2, is associated to a Lefschetz pencil and hence quite different from our monodromy associated to an arbitrary polynomial.]

Nevertheless, in order to exemplify our general theory, and also to generalize some connectivity results already present in the literature, we introduce the family of $h$-good polynomials. The family includes e.g. all the “good” polynomials considered by Neumann and...
2. Topological preliminaries, connectivity properties

2.1. Let \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) be a polynomial function with \( n \geq 1 \). It is well known that there is a (minimal) finite bifurcation set \( B_f \) in \( \mathbb{C} \) such that \( f \) is a \( C^\infty \)-locally trivial fibration over \( \mathbb{C} \setminus B_f \). If \( b_0 \in \mathbb{C} \) is not in \( B_f \), then \( F = f^{-1}(b_0) \) is called the generic fiber of \( f \); otherwise \( F_b := f^{-1}(b) \) is called a special fiber.

For any \( b \in \mathbb{C} \) we fix a sufficiently small closed disc \( D_b \) containing \( b \), and a point \( b' \in \partial D_b \). We set \( T_b := f^{-1}(D_b) \), \( T^*_b := T_b \setminus f^{-1}(b) \). Sometimes, it is convenient to identify \( f^{-1}(b') \) with the generic fiber \( F \). Then, we have the obvious inclusions \( F \subset T^*_b \subset T_b \).

By a well-known deformation retract argument (see e.g. (2.3) in [DN1]), the pair \( (\mathbb{C}^{n+1}, F) \) has the homotopy type of the space \( (Y, F) \) obtained by gluing all the pairs \( (T_b, F) \) \((b \in B_f)\). Rudolph [NR], and the polynomials with isolated singularities on the affine space and at infinity in the sense of Siersma-Tibăr [SiT]. This family of h-good polynomials fits perfectly to the study of Alexander invariants, and it is our major source of examples. For different vanishing and connectivity results, see [2.7], [2.10], [4.1] and [4.5(v)].

The content of our paper is the following. In section 2 we establish some properties of the corresponding fundamental groups which basically will guide all the covering properties considered later. Moreover, here we introduce and start to discuss the h-good polynomials. In section 3 we discuss some general facts on the homology groups \( H_\ast(M_X, \mathbb{Z}) \) concentrating on non-vanishing results for \( H_\ast(M_X, \mathbb{Z}) \) and on \( \mathbb{Z} \)-torsion problems. This latter aspect was somewhat neglected recently in spite of the pioneering work by Libgober [Li0] and a famous conjecture on hyperplane arrangement Milnor fibers (see 3.10).

In section 4 we collect some facts on (torsion) Alexander modules and prove one of the main results, Theorem 4.5. In order to emphasize the parallelism of h-good polynomials with the case of hypersurfaces with only isolated singularities including at infinity considered by Libgober, in some of our applications we recall Libgober’s results [Li2] as well.

In the fifth section we explain the relationship between individual monodromy operators and Alexander modules. The two main examples, i.e. the monodromy at infinity and the monodromy around the fiber \( X \) are discussed with special care. These two monodromy operators have been intensively studied recently using various techniques (mixed Hodge structures, D-modules), see the references given in section 5. Via our results, all this information on the monodromy operators yields valuable information on Alexander invariants of \( M_X \). Remark 5.9 relates the homology of the cyclic coverings \( M_{X,d} \) to the \( d \)-suspension of the polynomial \( f \) and in this way the results on the Thom-Sebastiani construction in [DN2] become applicable.

Section 6 introduces into the picture not only individual monodromy operators but also the whole monodromy representation of \( f \). We define two new Alexander modules associated to \( f \), namely the global Alexander module \( M(f) \) which can be regarded as a commutative version of the monodromy representation, and a local Alexander module \( M(f, b) \) associated to any fiber \( X = f^{-1}(b) \). This module \( M(f, b) \) gives a very good approximation of the classical Alexander module \( H_\ast(M_X, \mathbb{Z}) \) of \( X \) (see below for the necessary definitions).

As a convincing example of the power of this new approach, we compute at the end the various Alexander modules for a polynomial \( \mathbb{C}^4 \to \mathbb{C} \) for which a partial information on the monodromy representation is known. This examples shows in particular that the isomorphism \( M(f, b) = H_\ast(M_X, \mathbb{Z}) \) does not always hold.

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along $F$. We denote this fact by
\[(\mathbb{C}^{n+1}, F) \sim \bigvee_{F}(T_{b}, F) \quad (b \in B_{f}). \tag{1}\]

2.2. **Proposition.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map. Then the generic fiber $F$ is connected if and only if $\pi_{1}(T_{b}, F)$ is trivial for any $b \in B_{f}$.

**Proof.** If $\pi_{1}(T_{b}, F) = 1$ for all $b$, then $\hat{H}_{0}(F) = H_{1}(\mathbb{C}^{n+1}, F) = \ominus_{b \in B_{f}} H_{1}(T_{b}, F) = 0$ by (1). Now, assume that $F$ is connected and fix a $b \in B_{f}$. Then we have to show that $j : \pi_{1}(F) \to \pi_{1}(T_{b})$ is onto. Since $T_{b}$ is smooth and $f^{-1}(b) \subset T_{b}$ has real codimension two, one obtains that $i : \pi_{1}(T_{b}^{*}) \to \pi_{1}(T_{b})$ is onto. Since $f$ restricted to $T_{b}^{*}$ is a fiber bundle, the kernel of $f_{*} : \pi_{1}(T_{b}^{*}) \to \mathbb{Z}$ is $\pi_{1}(F)$. Assume that $f^{-1}(b)$ has $r$ irreducible components, and $f - b = \prod_{i=1}^{r} g_{i}^{m_{i}}$. Then one can construct easily elementary loops in $T_{b}^{*}$ around the component $\{g_{i} = 0\}$ representing $x_{i} \in \pi_{1}(T_{b}^{*})$ with properties $f_{*}(x_{i}) = m_{i}$ and $i(x_{i}) = 1$. Set $m := \gcd_{i}(m_{i})$. Then a combination of the $x_{i}$’s provides an $x \in \pi_{1}(T_{b}^{*})$ with $f_{*}(x) = m$ and $i(x) = 1$. The point is that $m = 1$ (otherwise $f - b$ would be an $m$-power of a polynomial whose generic fiber is not connected). The existence of such an $x$ and the surjectivity of $i$ implies the surjectivity of $j$. \qed

In the next paragraphs we fix a $b \in B_{f}$, and we write $X := f^{-1}(b)$ and $M_{X} := \mathbb{C}^{n+1} \setminus X$. For simplicity of the notations, we will assume that $b = 0$.

2.3. **Corollary.** Assume that $F$ is connected. Then

(i) $\pi_{1}(T_{0}^{b}) \to \pi_{1}(M_{X})$ (induced by the inclusion) is onto.

(ii) $\pi_{1}(F) \xrightarrow{i_{X}} \pi_{1}(M_{X}) \xrightarrow{f_{*}} \mathbb{Z} \to (1)$ is an exact sequence (i.e. $\text{im}(i_{X}) = \ker(f_{*})$), where $i_{X}$ is induced by the inclusion $F \subset M_{X}$, and $f_{*}$ by $f : M_{X} \to \mathbb{C}^{*}$.

**Proof.** Similarly as in (1), $M_{X}$ has the homotopy type of a space obtained by gluing $T_{0}^{b}$ and all the “tubes” $T_{b}(b \in B_{f} \setminus \{0\})$ along $F$. Then (i) follows from van Kampen theorem and from the surjectivity of $\pi_{1}(F) \to \pi_{1}(T_{b})$ for each $b$ (cf. 2.2). Part (ii) follows from (i) and the exact sequence $\pi_{1}(F) \to \pi_{1}(T_{b}^{0}) \to \mathbb{Z} \to (1)$. \qed

Let $p : F \to M_{X}$ be the $\mathbb{Z}$-cyclic covering associated to the kernel of the morphism $f_{*} : \pi_{1}(M_{X}) \to \mathbb{Z}$. The notation $F$ is chosen because

(i) $F$ is the homotopy fiber of $f : M_{X} \to \mathbb{C}^{*}$, regarded as a homotopy fibration; and

(ii) in many cases the topology of $F$ is a good approximation for the topology of $F$ (see e.g. the connectivity results below).

Fix a base-point $* \in F$ with $p(*) \in F$. Since $f(F)$ is a point, there is a natural section $s : F \to F$ of $p$ above $F$ with $s(p(*)) = *$. In particular, we can regard $F$ as a subspace of $\mathbb{F}$.

2.4. **Corollary.** Assume that $F$ is connected. Then $s_{*} : \pi_{1}(F) \to \pi_{1}(F)$ is onto, or equivalently, $\pi_{1}(\mathbb{F}, F)$ is trivial.

**Proof.** Compare the exact sequences $(1) \to \pi_{1}(F) \to \pi_{1}(T_{b}^{0}) \to \mathbb{Z} \to (1)$ and $(1) \to \pi_{1}(F) \to \pi_{1}(M_{X}) \to \mathbb{Z} \to (1)$ via 2.3. \qed

Fix an orientation of $S^{1}$, and consider a smooth loop $\gamma : S^{1} \to \mathbb{C} \setminus B_{f}$. Denote by $q : \gamma^{-1}(f) \to S^{1}$ the pull-back of $f$ by $\gamma$, i.e. $\gamma^{-1}(f) = \{(t, x) \in S^{1} \times \mathbb{C}^{n+1} : \gamma(t) = f(x)\}$, and $q(t, x) = t$. 

2.5. Corollary. Assume that $\gamma_* : \pi_1(S^1) \to \pi_1(\mathbb{C}^*)$ (i.e. $\gamma_* : \mathbb{Z} \to \mathbb{Z}$) is multiplication by an integer $\ell$. Then one has the following commutative diagram with all the lines and columns exact:

\[
\begin{array}{ccccccccc}
(1) & \to & \pi_1(F) & \to & \pi_1(\gamma^{-1}(f)) & \to & \mathbb{Z} & \to & (1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(1) & \to & \pi_1(F) & \to & \pi_1(M_X) & \to & \mathbb{Z} & \to & (1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(1) & \to & \mathbb{Z}/\ell\mathbb{Z} & \to & \mathbb{Z}/\ell\mathbb{Z} & \to & (1) \\
\end{array}
\]

Proof. The first two lines are the homotopy exact sequences of the corresponding fibrations. Then use 2.4.

Sometimes it is convenient to work with special polynomials with nice behaviour around the special fibers or at infinity. First, we recall the definition of Neumann and Rudolph of good polynomials [NR]. A fiber $f^{-1}(b)$ is called “regular at infinity” if there exist a small disc $D$ containing $b$ and a compact set $K$ such that the restriction of $f$ to $f^{-1}(D) \setminus K$ is a trivial $C^\infty$-fibration. The polynomial $f$ is called good (or “topologically good”) if all its fibers are regular at infinity.

For example, the tame polynomials introduced by Broughton [B], the larger class of Mtame polynomials introduced by Némethi-Zaharia [NZ] are good. We recall that any fiber of a good polynomial is a bouquet of spheres $S^n$, $B_f$ is the set of critical values of $f$, for any $b \in B_f$ the “tube” $T_b$ has the homotopy type of $f^{-1}(b)$, and $f^{-1}(b)$ (homotopically) is obtained from $F$ by attaching some cells of dimension $n + 1$.

For the purpose of the present paper it is enough (and it is more natural) a much weaker assumption.

2.6. Definition. The polynomial $f$ is called “homotopically good” (h-good) if for any $b \in B_f$, the pair $(T_b, F)$ is $n$-connected.

From the above discussion it follows easily that all the good polynomials are h-good. Another example is provided by the polynomials with isolated singularities on the affine space and at infinity in the sense of Siersma-Tibăr [SiT] (see p.776 [loc. cit.]).

In general, for an arbitrary polynomial, it is much easier to handle the properties of the generic fiber and the “tubes” $T_b$ than the properties of the special fibers (see e.g. [DN2]). One of the advantages of the above definition 2.6 is that it requires information only about $F$ and $T_b$’s. (Conversely, this fact also explains that for h-good polynomials one can say very little about the special fibers. E.g. the special fibers of h-good polynomials, in general, are not even reduced, as it happens e.g. for $f(x, y) = x^2y$. For a non-trivial example of a h-good polynomial which has non-isolated singularities, see the polynomial $f_{d,a}$ constructed by tom Dieck and Petrie, cf. [D1], p.175.]

The second advantage of definition 2.6 is that, in fact, it is almost homological. Indeed, for $n = 1$, $f$ is h-good iff $F$ is connected (by 2.4); for $n > 1$ the polynomial $f$ is h-good iff $F$ is simply-connected and $H_q(T_b, F, \mathbb{Z}) = 0$ for all $b$ and $q \leq n$. This second statement follows from 2.2, the next proposition 2.7, and the relative Hurewicz isomorphism theorem (see e.g. [S], p.397).
The above examples and comment show that we cannot expect that the h-good polynomials will share all the properties of the “good” ones. However, the next result will recover one of the most important properties.

2.7. **Proposition.** Assume that $f$ is a h-good polynomial. Then its generic fiber $F$ has the homotopy type of a bouquet of spheres $S^n$.

*Proof.* By (2.1)(1), $H_q(C^n+1, F) = 0$ for $q \leq n$, hence $\tilde{H}_q(F) = 0$ for $q \leq n - 1$. This already proves the statement for $n = 1$. Next, we have to show that $\pi_1(F) = (1)$, provided that $n \geq 2$. The connectivity assumption assures that for each $b \in B_f$, $\pi_1(F) \to \pi_1(T_b)$ is an isomorphism. We denote all these fundamental groups by $G$. If the cardinality $|B_f|$ of $B_f$ is one, then this implies that $\pi_1(F) = G$ is trivial (since in this case, $T_b \sim \mathbb{C}^{n+1}$). If $|B_f| > 1$, then by van Kampen theorem, applied for $\pi$, we get $\pi_1(Y) = G$. But again by (2.1)(1), $\pi_1(Y) = (1)$. Since $F$ has the homotopy type of a finite $n$-dimensional CW complex, the result follows by Whitehead theorem. \hfill $\square$

2.8. **Remark.** (2.7) can be compared with the following classical result of Lê [Lê].

For any projective hypersurface $V$ and a generic hyperplane $H$, the affine hypersurface $X = V \setminus H$ is homotopy equivalent to a bouquet of spheres $S^n$. [For the computation of the number of spheres in this bouquet, in terms of the degree of gradient mappings, see [DPp].]

2.9. Finally, we compare $F$ and $\mathbb{F}$. Since $\mathbb{F}$ is a cyclic covering of $M_X$, and $M_X$ has the homotopy type of a finite CW complex of dimension $\leq (n + 1)$, one has the general fact:

$$H_m(\mathbb{F}) = 0 \text{ for } m > n + 1 \text{ and } H_{n+1}(\mathbb{F}, \mathbb{Z}) \text{ has no } \mathbb{Z}\text{-torsion.} \quad (2)$$

But if $f$ is h-good, we can say more (cf. also with 2.4). With a choice of the base points, we again embed $F$ into $\mathbb{F}$ via the section $s$.

2.10. **Proposition.** If $f$ is h-good then the pair $(\mathbb{F}, F)$ is n-connected. Therefore, $\mathbb{F}$ is $(n - 1)$-connected, and $s_n : H_n(F, \mathbb{Z}) \to H_n(\mathbb{F}, \mathbb{Z})$ is onto. In particular, for $n > 1$, by Hurewicz theorem and the homotopy exact sequence, one has $H_n(\mathbb{F}, \mathbb{Z}) = \pi_n(\mathbb{F}) = \pi_n(M_X)$.

*Proof.* Notice (cf. (2.1)1) and the proof of (2.3) that $M_X$ has the homotopy type of a space obtained by gluing to $T_0^*$ along $F$ all the tubes $T_b$ for $b \in B_f \setminus \{0\}$. Moreover, $p^{-1}(T_0^*)$ has the homotopy type of $F$, and its embedding into $\mathbb{F}$ is homotopically equivalent to the embedding $s : F \to \mathbb{F}$. Therefore, by excision:

$$H_q(\mathbb{F}, F) = H_q(p^{-1}(T_0^* \vee_F (T_b)), p^{-1}(T_0^*)) = \oplus_b H_q(p^{-1}(T_b), p^{-1}(F)),$$

where $\bar{b}$ runs over $B_f \setminus \{0\}$. But $(p^{-1}(T_b), p^{-1}(F)) = \mathbb{Z} \times (T_b, F)$, hence $H_q(\mathbb{F}, F) = 0$ for $q \leq n$. Hence, the connectivity follows from this, (2.4), (2.7), and the relative Hurewicz isomorphism theorem. Finally, the connectivity of $\mathbb{F}$ follows from the connectivity of $F$, cf. (2.7). \hfill $\square$

The same proof (but neglecting $p$), and the Wang exact sequence of $T_0^*$, gives:

2.11. **Corollary.** If $f$ is h-good, then the pair $(M_X, T_0^*)$ is n-connected. In particular, $H_q(M_X, \mathbb{Z}) = 0$ for $1 < q < n$. (In fact, by (2.10), the cyclic covering of $M_X$ is $(n - 1)$-connected, hence $\pi_q(M_X) = 0$ for $1 < q < n$ as well.)

For special cases of this connectivity results, see also [Li2] and [LiT].
3. Preliminaries about $H_*(M_X, \mathbb{Z})$

Let $X$ be a hypersurface in $\mathbb{C}^{n+1}$ and $M_X$ be its complement. The goal of this section is to list some properties of the integral homology of $M_X$, with an extra emphasis on the torsion part and the “interesting part” $H_n(M_X, \mathbb{Z})$. Moreover, we present some constructions which generate examples with non-trivial “interesting part”. Additionally, sometimes we compare the properties of $H_*(M_X)$ with homological properties of hypersurfaces $X$.

We start with the case when $X$ is a (generic or special) fiber of a polynomial $f$.

3.1. Fact. \[\text{[LiT]}\] If $F$ is the generic fiber of an arbitrary polynomial $f$, then $M_F$ has the homotopy type of a join $S^1 \vee S(F)$, where $S(F)$ denotes the suspension of $F$. In particular, 
\[
\tilde{H}_k(M_F) = \tilde{H}_k(S^1) \oplus \tilde{H}_{k-1}(F) \quad \text{for any } k.
\] (1)

In fact, the result \[3.1(1)\] holds for any smooth $X$, as follows from the associated Gysin sequence, see \[\text{[D1]}, p.46\].

The similar result (i.e. the analog of \[3.1(1)\]) for homotopy groups is definitely false; consider for example $\pi_1$, or (for instance) \[S\], p.419, exercise B6, for a reason.

3.2. Example. \[3.1(1)\] is false for special fibers, even for very simple polynomials. Let $f = x_0^2 + \ldots + x_n^2$ and $X = f^{-1}(0)$. The Wang sequence of the global Milnor fibration $F \to M_X \to \mathbb{C}^*$ (see \[\text{[D1]}, p.71-74\]) and the fact that the corresponding monodromy operator $T$ acting on $H_n(F, \mathbb{Z}) = \mathbb{Z}$ is $(-1)^{n+1}Id$, implies the following:

(i) for $n = 2m + 1$ odd, $H^*(M_X) = H^*(S^1 \times S^n)$. In fact, the monodromy is isotopic to the identity and hence we have a diffeomorphism $M_X \cong \mathbb{C}^* \times F$. This implies that $M_X$ has the homotopy type of $S^1 \times S^n$ as claimed in \[\text{[Li2]}, \text{Remark \text{(1.3)}}\].

(ii) for $n = 2m > 0$ even, $H_n(M_X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. In particular, $M_X$ is not homotopy equivalent to the product $S^1 \times S^n$ (contrary to the claim in \[\text{[Li2]}, \text{Remark \text{(1.3)}}\]).

\[\text{[3.1]}\] has the following consequence: when $X$ is the generic fiber of a h-good polynomial then $H_m(M_X, \mathbb{Z}) = 0$ for $1 < m \leq n$, and in fact $H_*(M_X, \mathbb{Z})$ is torsion free (cf. \[\text{2.7}\]). (This can be compared with Lé's result \[\text{2.8}\], which shows that generically an affine hypersurface $X$ has no torsion in homology.)

More generally, it was shown in \[\text{[Li2]}\] that when $X$ has isolated singularities including at infinity, then $H_m(M_X) = 0$ for $1 < m < n$. The same statement holds for the special fiber $X$ of a h-good polynomial by \[\text{2.11}\] (cf. also with \[\text{[LiT]}\]). Hence, in both cases, the first interesting homology group occurs in degree $n$.

We describe now three constructions which provide in a systematic way hypersurfaces $X$ with $H_n(M_X, \mathbb{Z}) \neq 0$.

Below $V$ denotes the projective closure of $X$ and $H$ the hyperplane at infinity.

3.3. The first construction (using duality). Following \[\text{[Li2]}, \text{section } 1\], we consider the isomorphism 
\[
H_m(M_X) = H^{2n-m+1}(V, V \cap H) \quad \text{for all } m.
\] (2)

Assume that $V$ and $V \cap H$ have only isolated singularities (this is exactly the condition on $X$ to have isolated singularities including at infinity in \[\text{[Li2]}\]). The exact sequence 
\[
H^n(V) \to H^n(V \cap H) \to H^{n+1}(V, V \cap H) \to H^{n+1}(V) \to H^{n+1}(V \cap H)
\] (3)
and the isomorphism $H^{n+1}(V \cap H) = H^{n+1}(\mathbb{P}^{n-1})$ (cf. \[\text{[D1]}, p.161\]) imply the following.
Assume that $V$ and $V \cap H$ have only isolated singularities and that $H^{n+1}(V, \mathbb{Z}) \neq H^{n+1}(\mathbb{P}^{n-1}, \mathbb{Z})$. Then $H_n(M_X, \mathbb{Z}) \neq 0$. In particular, the corresponding affine hypersurface is not the generic fiber of a h-good polynomial.

3.4. Example. Let $V$ be a cubic surface in $\mathbb{P}^3$ having two singularities, one of type $A_1$ and the other of type $A_5$. First we take $H$ a generic plane. In this case, using [D1], p.165 we see that the exact sequence $\mathbb{Z} \to \mathbb{Z} \to H_2(M_X, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0$

where the first morphism is multiplication by $\text{deg}(V) = 3$. It follows that $H_2(M_X, \mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$, $X$ is singular and the associated polynomial $f$ is tame.

Secondly, take $H$ to be any plane passing through the 2 singularities on $V$. Then the associated $X$ is smooth, but by 3.3, $X$ is not the generic fiber of a good polynomial.

3.5. The second construction (using finite cyclic coverings and defect). The second approach uses $M_{X,e}$, the cyclic covering of $M_X$ of degree $e$ when $X = f^{-1}(0)$ (cf. also with 4.4(II) and 4.6(4)). It is clear that we can take

$$M_{X,e} = \{(x, u) \in \mathbb{C}^{n+1} \times \mathbb{C}^* | f(x) - u^e = 0\}.$$ 

In some cases we can get a useful approximation of $M_{X,e}$ as follows. Fix a system of positive integer weights $w = (w_0, ..., w_n)$, and let $e$ be the top degree term in $f$ with respect to $w$. Introduce a new variable $t$ of weight 1 and let $\tilde{f}(x, t)$ be the homogenization of $f$ with respect to the weights $(w, 1)$. Consider the affine Milnor fiber $F' : \tilde{f}(x, t) = 1$, which is a smooth hypersurface in $\mathbb{C}^{n+2}$. One has an embedding $j : M_{X,e} \to F'$ given by

$$j(x, u) = (u^{-1} \ast x, u^{-1}),$$

where $\ast$ denotes the multiplication associated to the system of weights $w$. The complement $F' \setminus j(M_{X,e})$ is characterized by $\{t = 0\}$, hence it can be identified with the affine Milnor fiber $F_e : f_e(x) = 1$ (considered in $\mathbb{C}^{n+1}$) defined by the top homogeneous component $f_e$ of $f$. If $f_e$ defines an isolated singularity at the origin, then $f_e$ is a good polynomial, $F_e$ is $(n-1)$-connected and the Gysin sequence of the divisor $F_e$ implies the isomorphisms

$$j_* : H_k(M_{X,e}) \to H_k(F') \text{ for } 1 < k < n + 1. \quad (4)$$

Note that under this isomorphism the action of the natural generator of the covering transformation group on $M_{X,e}$ corresponds to multiplication by $\exp(-2\pi i/e)$ (of all the coordinates) on $F'$. Moreover, notice that $H_k(M_X, \mathbb{Q})$ is isomorphic to the group of invariants of $H_k(M_{X,e}, \mathbb{Q})$ with respect to this action.

Notice that the dimension of $H_n(F')$ is closely related to the defect associated with the singular points of the projective hypersurface $\tilde{f} = 0$ considered in $\mathbb{P}^{n+1}$ (for details, see e.g. [D1]). Hence, this construction emphasizes the connection between $H_n(M_X)$ and superabundance properties.

For more information on the homology of $M_{X,e}$, see also 4.6(4) and 5.3 below.

3.6. Example. Let $f : \mathbb{C}^4 \to \mathbb{C}$ be given by $f(x, y, u, v) = x^3 + y^3 + xy - u^3 - v^3 - uv$. Set $X = \{f = 0\}$. Then $f$ is a tame polynomial and $X$ has 10 nodes. It follows from [D1], p.208-209, (with all the weights $w_i = 1$) that $\dim H_3(F', \mathbb{C}) = 5$ and the multiplication by $\exp(-2\pi i/3)$ on $F'$ induces the trivial action on $H_3(F', \mathbb{C})$. It follows that $\dim H_3(M_X, \mathbb{C}) = \dim H_3(M_{X,3}, \mathbb{C}) = \dim H_3(F', \mathbb{C}) = 5$. 
3.7. The third construction ("counting" the Milnor numbers). For simplicity we will assume that \( f \) is a (topologically) good polynomial. As above, \( F \) is its generic fiber and \( X \) a special fiber. Assume that \( X \) has \( n_X \) singular points with Milnor fibers \( F_i \) and Milnor numbers \( \mu_i \) for \( 1 \leq i \leq n_X \). For each \( i \), let \( \mu_{0,i} \) denote the rank of \( H_n(\partial F_i) \). Set \( \mu_X := \sum \mu_i \) and \( \mu_{0,X} := \sum \mu_{0,i} \). Below, \( \oplus_X \) means \( \oplus_{i=1}^{\mu_X} \).

Finally, let \( \mu \) be the sum of all the Milnor numbers of the singularities of \( f \) (situated on all the singular fibers). With these notations, one has:

The groups \( H_q(M_X) \) \((q = n, n + 1)\) are inserted in the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & \oplus_X H_n(\partial F_i) & \to & \oplus_X H_n(F_i) & \to & \oplus_X H_n(F, \partial F_i) & \to & \oplus_X \tilde{H}_{n-1}(\partial F_i) & \to & 0 \\
0 & \to & H_{n+1}(M_X) & \to & H_n(F) & \to & \oplus_X H_n(F, \partial F_i) & \to & H_n(M_X) & \to & 0 \\
\end{array}
\]

where the first two vertical arrows are monomorphisms, the third is the identity, and the last one is an epimorphism. In particular:

\[
\mu_{0,X} \geq \dim H_n(M_X) \geq \mu_X + \mu_{0,X} - \mu.
\]

Above, the first inequality is not new, and it is not very sharp (cf. e.g. with [GN2] (2.30), or with [Si2] (5.4) and §9; it also follows from 5.9(iii) below).

The second inequality is more interesting, and it can be used in two different ways. First, using the integers \( \mu_X, \mu_{0,X} \) and \( \mu \), if the sum of the first two is strictly larger than the third, one gets a non-vanishing criteria for \( H_n(M_X) \). For example, in the case \( 3.6, \mu = 16, \mu_X = \mu_{0,X} = 10 \), hence the second inequality reads as \( \dim H_n(M_X) \geq 4 \).

Similarly, if one knows \( \mu_X, \mu_{0,X} \) and \( \dim H_3(M_X) \) about the hypersurface \( X \), then these numbers may impose serious conditions about singularities of some other singular fibers of \( f \) (e.g. about their existence).

Proof. Assume that \( X = f^{-1}(b) \), and let \( T_b^\circ \) be the interior of \( T_b \). Clearly, \( M_X \) has the homotopy type of \( \mathbb{C}^{n+1} \setminus T_b^\circ \). Let \( F \) be a fixed generic fiber of \( f \) inside of \( T_b^\circ \). Then one can write the homological long exact sequence of the pair \( (\mathbb{C}^{n+1} \setminus F, \mathbb{C}^{n+1} \setminus T_b^\circ) \). Notice that \( H_q := H_q(\mathbb{C}^{n+1} \setminus F, \mathbb{C}^{n+1} \setminus T_b^\circ) \) equals \( H_q(T_b \setminus F, \partial T_b) \) by excision. Let \( B_i \) be a small Milnor ball of the \( i \)-th singular point of \( X \). Then using the "good"-property of \( f \) and excision one gets \( H_q = \oplus_X H_q(B_i \setminus F_i, \partial B_i \setminus \partial F_i) \). But this is isomorphic to \( \oplus_X H_{q-1}(F_i, \partial F_i) \) by Gysin isomorphism. Use these facts and the Gysin isomorphism \( H_n(F) \to H_{n+1}(M_F) \) to obtain the second line of the above diagram.

Next, assume that the disc \( D_b \) is sufficiently small with respect to the balls \( B_i \), and consider for any ball \( B_i \) the local analog of the above picture; namely the homological long exact sequence of the pair \( (B_i \setminus F, B_i \setminus T_b^\circ) \). This homological sequence admits a natural map to the previous sequence induced by the inclusion. Finally, this "local sequence" is modified by dualities and Gysin isomorphisms.

\[\square\]

3.8. Remarks about the torsion part of \( H_n(M_X, \mathbb{Z}) \). In the final part of this section we discuss the relations between the existence/non-existence of torsion in the homology of \( X \) and \( M_X \) respectively. The following examples show that these relations are not simple even for a homogeneous polynomial \( f \).
3.9. Example. Consider the homogeneous polynomial \( f = x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) \) defined on \( \mathbb{C}^3 \). It is known that its Milnor fiber \( F \) has torsion in homology, more precisely \( H_1(F, \mathbb{Z}) = H_2(M_F, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z} \) and \( H_2(F, \mathbb{Z}) = \mathbb{Z}^3 \) see \([\text{Li0}, \text{DN1}] \) and \([\text{Si}] \). Let \( C \) be the 3-cuspidal quartic in \( \mathbb{P}^2 \) defined by \( f = 0 \). It satisfies \( H_1(\mathbb{P}^2 \setminus C, \mathbb{Z}) = \mathbb{Z}_4 \) and \( H_2(\mathbb{P}^2 \setminus C, \mathbb{Z}) = 0 \) (cf. [loc. cit.]). Set \( X = f^{-1}(0) \). The Gysin sequence of the fibration \( \mathbb{C}^* \to M_X \to \mathbb{P}^2 \setminus C \) yields \( H_1(M_X, \mathbb{Z}) = \mathbb{Z}, H_2(M_X, \mathbb{Z}) = \mathbb{Z}/4\mathbb{Z} \) and \( H_3(M_X) = 0 \). In conclusion, both \( M_F \) and \( M_X \) have torsions in \( H_2 \), but these torsions are different.

If we put together the examples 3.2 and 3.9, we see that for a homogeneous polynomial \( f \) and for \( X = f^{-1}(0) \) the only case not covered is the following.

3.10. Question. Find an example of a homogeneous polynomial \( f \) such that \( M_X \) has no torsion but \( F \) has torsion.

Even in the case when \( f \) is a product of linear forms, the existence of such an example is an open question. (It is known in this latter case that the hyperplane arrangement complement \( M_X \) is torsion free, see [OT], and the corresponding Milnor fiber can be identified to a cyclic covering of \( \mathbb{P}^n \setminus \{ f = 0 \} \), see [CS], [CO]). Notice also, that if we allow \( f \) to be a product of powers of linear forms, then A. Suciu has examples with torsion in the homology of the associated Milnor fiber \( F \).

The following result gives some conditions on the possible torsions that may arise in such a case.

3.11. Proposition. Assume that for a homogeneous polynomial \( f \) of degree \( d \) the complement \( M_X \) has no \( p \)-torsion for some prime \( p \) and that the \( p \)-torsion in a homology group \( H_k(F, \mathbb{Z}) \) has the structure

\[
\mathbb{Z}/p^{k_1}\mathbb{Z} \oplus \mathbb{Z}/p^{k_2}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{k_m}\mathbb{Z}
\]

with \( m \geq 1 \) and \( k_1 \geq k_2 \geq \ldots \geq k_m \geq 1 \). Then

(i) \((p - 1, d) = 1 \) implies \( m \geq 2 \) and \( k_1 = k_2 \);

(ii) \((p - 1)p(p + 1), d) = 1 \) implies \( m \geq 3 \) and \( k_1 = k_2 = k_3 \).

Proof. Consider the Wang sequence in homology associated to the fibration \( F \to M_X \to \mathbb{C}^* \). The fact that \( M_X \) has no \( p \)-torsion implies that \( T - Id \) is an isomorphism when restricted to the \( p \)-torsion part.

To prove (i) note that since such an isomorphism preserves the orders of the elements, unless the claim (i) above holds, we get an induced automorphism of \( \mathbb{Z}/p\mathbb{Z} \), where this latter group is regarded as the quotient of the \( p \)-torsion part by the subgroup of elements killed by multiplication by \( p^{k_1-1} \). The same is true for the monodromy transformation \( T \).

Denote by \( T_p \) the induced automorphism of \( \mathbb{Z}/p\mathbb{Z} \). It follows that \( T_p \) is not the identity, \( T_p^d = Id \) (since \( f \) is homogeneous of degree \( d \)) and \( T_p^{p-1} = 1 \) (since \( |Aut(\mathbb{Z}/p\mathbb{Z})| = p - 1 \), a contradiction.

For (ii) use the same argument, plus the equality \( |Aut((\mathbb{Z}/p\mathbb{Z})^2)| = (p - 1)^2p(p + 1) \).

3.12. Remark. Other examples involving torsion in the homology of the special fibers of a polynomial can be obtained by suspension, see for details [DN2].

4. Alexander Modules

4.1. Definitions. Let \( Y \) be a connected CW-complex and \( e_Y : \pi_1(Y) \to \mathbb{Z} \) be an epimorphism. We denote by \( Y^c \) the \( \mathbb{Z} \)-cyclic covering associated to the kernel of the morphism \( e_Y \). It follows that a generator of \( \mathbb{Z} \) acts on \( Y^c \) by a certain covering homeomorphism \( h \).
and all the groups \( H_*(Y^c, A), H^*(Y^c, A) \) and \( \pi_j(Y^c) \otimes A \) for \( j > 1 \) become in the usual way \( \Lambda_A \)-modules, where \( \Lambda_A = A[t, t^{-1}] \), for any ring \( A \). These are called the Alexander modules of the pair \((Y, e_Y)\) or simply of \( Y \) when the choice of \( e_Y \) is clear.

If \( Z \) is a second connected CW-complex, \( e_Z : \pi_1(Z) \to \mathbb{Z} \) an epimorphism and \( \phi : Y \to Z \) is a continuous map such that the induced map at the level of \( \pi_1 \) is an epimorphism compatible to the two given epimorphisms \( e_Y \) and \( e_Z \), then \( Y^c \to Y \) can be regarded as a pull-back covering obtained via \( \phi \) from the covering \( Z^c \to Z \). In particular, this gives a lift \( \phi^c : Y^c \to Z^c \) which is compatible with the covering transformations, and hence, an induced morphism of \( \Lambda_A \)-modules, say \( \phi^c_* : H_*(Y^c, A) \to H_*(Z^c, A) \).

If \( A \) is a field then the ring \( \Lambda_A \) is a PID. Hence any finite type \( \Lambda_A \)-module \( M \) has a decomposition \( M = \Lambda_A^k \oplus \bigoplus_p M_p \), where \( k \) is the rank of \( M \) and the second sum is over all the prime elements in \( \Lambda_A \), and \( M_p \) denotes the \( p \)-torsion part of \( M \).

More precisely, for each prime \( p \) with \( M_p \neq 0 \), we have a unique decomposition

\[
M_p = \bigoplus_{i=1, \ell_i} \Lambda_A/p^{k_i}
\]

for \( \ell_p > 0 \) and \( k_1 \geq k_2 \geq \ldots \geq k_{\ell_p} \geq 1 \). The sequence \((k_1, k_2, \ldots, k_{\ell_p})\) obtained in this way will be denoted by \( K(M, p) \). One can define an order relation on such sequences by saying that

\[
(k_1, \ldots, k_a) \geq (m_1, \ldots, m_b)
\]

iff \( a \geq b \) and \( k_i \geq m_i \) for all \( 1 \leq i \leq b \).

Let \( \Delta_p(M) = \prod_{i=1, \ell_i} p^{k_i} \) (resp. \( \Delta(M) = \prod_p \Delta_p(M) \)) be the \( p \)-Alexander polynomial (resp. the Alexander polynomial) of the module \( M \). This latter invariant \( \Delta(M) \) is called the order of \( M \) in \([\text{Li2}]. \) See 4.4(1) for a motivation of this terminology.

With this notation one has the following easy result whose proof is left to the reader.

**4.2. Lemma.** Let \( u : M \to N \) be an epimorphism of \( R \)-modules, where \( R \) is a PID and \( M \) is of finite type. Then \( N \) is of finite type and for any prime \( p \in R \) one has \( K(M, p) \geq K(N, p) \). In particular \( \Delta(N) \) divides \( \Delta(M) \).

**4.3. Example.** For \( A = \mathbb{C} \), we will simply write \( \Lambda \) instead of \( \Lambda_\mathbb{C} \). The prime elements in this case are just the linear forms \( t - a \), for \( a \in \mathbb{C}^* \). Moreover, a \( \Lambda \)-module of the form \( H_q(Y^c, \mathbb{C}) \) is of finite type and torsion iff the corresponding Betti number \( b_q(Y^c) \) is finite. If this is the case, the \((t - a)\)-torsion part of \( H_q(Y^c, \mathbb{C}) \) is determined (and determines) the Jordan block structure of the corresponding automorphism \( h_q \); i.e., a Jordan block of size \( m \) corresponding to the eigenvalue \( a \) produces a summand \( \Lambda/(t - a)^m \). The corresponding Alexander polynomial \( \Delta(H_q(Y^c, \mathbb{C}), h_q) \) is just the characteristic polynomial \( \Delta(h_q)^m \).

When \( \Lambda = \mathbb{Q} \), the corresponding prime elements are the irreducible polynomials in \( \mathbb{Q}[t] \) different from \( t \), hence they are a lot more difficult to describe. However, the knowledge of the \( \Lambda \)-module structure implies easily the \( \Lambda_\mathbb{Q} \)-module structure just by grouping together the polynomials \( t - a \) for those \( a \)'s having the same minimal polynomial over \( \mathbb{Q} \).

**4.4. The Alexander modules of \( M_X \) and local systems.** Coming back to the situation (and notation) of the previous sections, for any hypersurface \( X \) we define \( e_X : \pi_1(M_X) \to \mathbb{Z} \) as follows. In fact, we will distinguish two cases.

(1) First, assume that \( X \) is an arbitrary hypersurface in \( \mathbb{C}^{n+1} \) (and even if \( X = f^{-1}(0) \), we disregard \( f \)). Assume that \( X \) has \( r \) irreducible components \( X_1, \ldots, X_r \). Then \( H_1(M_X, \mathbb{Z}) = \mathbb{Z}^r \), where the generator \( (0, \ldots, 1, \ldots, 0) \) \((1 \text{ on the place } i, 1 \leq i \leq r)\) corresponds to an
elementary oriented circle “around $X_i$”. For each set of integers $m := (m_1, \ldots, m_r) \in \mathbb{Z}^r$ we define $\phi_m : \mathbb{Z}^r \to \mathbb{Z}$ by $(x_1, \ldots, x_r) \mapsto \sum_i m_i x_i$. If $\gcd(i, m_i) = \pm 1$ then

$$e_{X,m} : \pi_1(M_X) \to \mathbb{H}_1(M_X, \mathbb{Z}) \to \mathbb{Z}$$

is onto.

(II) Now, assume that $X = f^{-1}(0)$ for some polynomial $f$. Then define $e_{X,f}$ by $e_{X,f} := f_* : \pi_1(M_X) \to \pi_1(\mathbb{C}^*)$. In fact, this is a particular case of (I): if $f = \prod_{i=1}^r g_i^{m_i}$ (where $g_i$ are irreducible with distinct zero sets) then $e_{X,f} = e_{X,m}$ for $m = (m_1, \ldots, m_r)$.

By a similar argument as in the proof of §2.3, if the generic fiber $F$ of $f$ is connected then $e_{X,f}$ is onto. Therefore, in the sequel, in all our Alexander-module discussions associated with $f$, we will assume that $F$ is connected.

Sometimes, we will use the notations (I) resp. (II) to remind the reader about the corresponding cases. In both cases (I) and (II), let $\mathbb{F} := M_X^e$ be the $\mathbb{Z}$-cyclic covering associated with the kernel of $e_X$ (cf. also with §2). Since $M_X$ has the homotopy type of a finite CW-complex, it follows that all the associated Alexander modules are of finite type over $\Lambda_A$ (but in general not over $A$).

Moreover, for a complex number $a \in \mathbb{C}^*$, we consider the rank one local system $L_a$ on $M_X$ defined by the composed map $\pi_1(M_X) \xrightarrow{e_X} \mathbb{Z} \to \mathbb{C}^*$, where the last map is defined by $1_\mathbb{Z} \mapsto a$. Obviously, if $a = 1$, then $L_a = \mathbb{C}$.

Then, exactly as in [Li3], we have the following long exact sequence

$$\cdots \to H_k(\mathbb{F}, \mathbb{C}) \to H_k(M_X, \mathbb{C}) \to H_k(M_X, L_a) \to H_{k-1}(\mathbb{F}, \mathbb{C}) \to \cdots$$

(2)

where the first morphism is multiplication by $t - a$.

In the next paragraph we summarize the properties of $H_*(M_X, L_a)$ and the Alexander modules $H_*(\mathbb{F}, \mathbb{C})$.

4.5. Theorem. (i) $H_m(M_X, L_a) = 0$ for any $a \in \mathbb{C}^*$ and $m > n + 1$. The Alexander modules $H_k(\mathbb{F})$ are trivial for $k > n + 1$ and $H_{n+1}(\mathbb{F}, \mathbb{C})$ is free (over $\Lambda$).

(ii) The $\Lambda$-rank of $H_{n+1}(\mathbb{F}, \mathbb{C})$ equals $\dim H_{n+1}(M_X, L_a)$ for any generic $a$.

(iii) For any $q \leq n$, the $(t-a)$-torsion in $H_q(\mathbb{F}, \mathbb{C})$ can be non zero only when $a$ is a root of unity.

(iv) Assume that all the Alexander modules $H_k(\mathbb{F}, \mathbb{C})$ are torsion for $k < n + 1$. Denote by $N(a,k)$ the number of direct summands in the $(t-a)$-torsion part of $H_k(\mathbb{F}, \mathbb{C})$. Then

$$\dim H_k(M_X, L_a) = N(a,k) + N(a,k-1) \quad \text{for} \quad k < n + 1 \quad \text{and} \quad \dim H_{n+1}(M_X, L_a) = N(a,n) + |\chi(M_X)|.
$$

Moreover, if either

(I) $X$ has only isolated singularities including at infinity, or

(II) $X$ is the fiber of a $h$-good polynomial,

then

(v) $H_k(\mathbb{F}) = 0$ for $k < n$, and $H_n(\mathbb{F}, \mathbb{C})$ is a $\Lambda$-torsion module. Moreover, one has the isomorphisms of the Alexander $\Lambda_{\mathbb{Z}}$-modules

$$\pi_n(M_X) = \pi_n(\mathbb{F}) = H_n(\mathbb{F}, \mathbb{Z}).$$

Proof. For (i) and the first part of (ii) see §2.9(2); for (ii) use the exact sequence §1.4(2) and the fact that multiplication by $t - a$ is injective, provided that $a$ is generic. The vanishing of $(t-a)$-torsion for a non root of unity follows from §1.4(iv) below. (iv) is straightforward, if we notice that $\chi(M_X, L_a) = \chi(M_X)$ for any local system $L_a$. Claim (v) in case (I) for $m = (1, \ldots, 1)$ is due to Libgober, see [Li2], Theorem 4.3; the case of arbitrary $m$ follows similarly. The case (II) is a consequence of §2.10.
4.6. **Remarks.** (1) Under the assumption (I) and (II) in 4.3, if $H_n(F,\mathbb{Z})$ is $\mathbb{Z}$-torsion free, then it has a finite rank over $\mathbb{Z}$ and the associated Alexander polynomial $\Delta(H_n(F,\mathbb{Z}))$ is the characteristic polynomial $\Delta(h_n)$ as in 4.3. It follows that $H_n(M_X,\mathbb{Z})$ is finite iff $\Delta(h_n)(1) \neq 0$. Moreover, if $H_n(M_X,\mathbb{Z})$ is finite then its order is exactly $|\Delta(h_n)(1)|$.

However, we do not know whether (I) or (II) in 4.5 imply that $H_n(F,\mathbb{Z})$ is $\mathbb{Z}$-torsion free.

(2) The proof of 4.5 is topological in both cases (I) and (II). It follows that the same proof implies that this inequality is not true for a general homogeneous polynomial.

(3) Notice that 4.5(iv) is not true without the assumption about $F$. In other words, it is not true that for a generic $a \in \mathbb{C}^*$ one has $H_m(M_X, L_a) = 0$ for $m \neq n+1$. Indeed, consider a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ as a function $\mathbb{C}^{n+1+k} \to \mathbb{C}$ independent of the last $k$-variables and denote by $M'_X$ the corresponding complement. Then $M'_X = M_X \times \mathbb{C}^k$. In particular, if $f$ is chosen such that $H_{n+1}(M_X, L_a) \neq 0$ for any generic $a$ (i.e. $\chi(M_X) \neq 0$), we get counter-examples to the above claim.

(4) Additionally to the exact sequence (4.3(2)) one has two more (both valid over $\mathbb{Z}$). First notice that the $\mathbb{Z}$-covering $F \to M_X$ homotopically can be identified with the inclusion of the fiber into the total space of a fibration $M_X \to K(\mathbb{Z}, 1) = S^1$ (with fiber $F$). Hence, the (homotopy) Wang exact sequence of the covering $F \to M_X$ gives:

$$
\ldots \to H_k(F, \mathbb{Z}) \to H_k(F, \mathbb{Z}) \to H_k(M_X, \mathbb{Z}) \to H_{k-1}(F, \mathbb{Z}) \to \ldots
$$

where the first morphism is multiplication by $t-1$.

Moreover, let $M_{X,e}$ ($e > 0$) be the cyclic covering of $M_X$ of degree $e$ (i.e. the covering classified by the subgroup $e\mathbb{Z}$ of $\mathbb{C}$). If $\mathbb{Z}$ denotes the transformation group of $F$ above $M_X$, then $F$ is a $\mathbb{Z}$-covering of $M_{X,e}$ with transformation group $e\mathbb{Z}$. Hence, we have an exact sequence

$$
\ldots \to H_k(F, \mathbb{Z}) \to H_k(F, \mathbb{Z}) \to H_k(M_{X,e}, \mathbb{Z}) \to H_{k-1}(F, \mathbb{Z}) \to \ldots
$$

where the first morphism is multiplication by $e^{t-1}$. It follows that there is a close relation between the homology of $M_{X,e}$ and the structure of the Alexander invariants $H_k(F, \mathbb{Z})$, see Example 5.8 below.

4.7. **Example.** Assume that $f$ is a weighted homogeneous polynomial with respect to an integer set of weights (non necessarily strictly positive) such that $d = \text{deg}(f) \neq 0$. Denote by $F$ the associated affine Milnor fiber. Then it is easy to see that $F$ and $F$ are homotopically equivalent, and the covering transformation $h$ corresponds to the Milnor monodromy. In particular $h_*$ is $Id$. It follows that $N(a,k) = \dim \ker(h_* - aId|H_0(F, \mathbb{C}))$ is trivial for $a^d \neq 1$. Moreover, in this case, due to the presence of $S^1$-actions, we have $\chi(M_X) = 0$.

In the case of a central hyperplane arrangement D. Cohen [C] has shown that for any rank one local system $L$ on $M_X$ one has $\dim H_m(M_X, L) \leq \dim H_m(M_X, \mathbb{C})$ for any $m$.

The example of the $A_1$-singularity $f : \mathbb{C}^3 \to \mathbb{C}$, $f = x^2 + y^2 + z^2$, $m = 2$ and $a = -1$, shows that this inequality is not true for a general homogeneous polynomial.

5. RELATIONS TO INDIVIDUAL MONODROMY OPERATORS

5.1. For the convenience of the reader we recall the present set up. Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial whose generic fiber $F$ is connected. Let $X = f^{-1}(0)$ be a fixed special fiber, $M_X$ its complement and $e_X = f_* : \pi_1(M_X) \to \mathbb{Z}$ the episnorphism whose kernel determines the $\mathbb{Z}$-cyclic covering $p : \mathbb{F} \to M_X$. Fix a generic fiber $F \subset M_X$, and a base point $* \in \mathbb{F}$ with $p(*) = b_0 \in F$. Then there is a unique lift (embedding) $s : F \to \mathbb{F}$ of $p$ over $F$ with
$s(b_0) = \ast$. Let $h$ be the covering transformation of $F$ corresponding to the generator $1_Z$. Sometimes we prefer to denote this Alexander $\Lambda_Z$-module by $(H_\ast(F), h_\ast)$.

Now, fix an orientation of $S^1$, and consider an arbitrary smooth map $\gamma : S^1 \to \mathbb{C} \setminus B_f$ (we can even take $\gamma(1) = b_0$). Let $q : \gamma^{-1}(f) \to S^1$ be the pull back of $f$ by $\gamma$. Obviously, $q$ is a fiber bundle over $S^1$ with fiber $F$. The map which covers $\gamma$ is denoted by $\tilde{\gamma} : \gamma^{-1}(f) \to MX$, hence $f \circ \tilde{\gamma} = \gamma \circ q$. We can consider the epimorphism $e_\gamma : \pi_1(\gamma^{-1}(f)) \to \pi_1(S^1) = \mathbb{Z}$. The associated $\mathbb{Z}$-covering has a total space isomorphic to $F \times \mathbb{R}$ and the covering transformation (corresponding to $1_Z$) can be identified with the geometric monodromy of $f$ associated with the oriented loop $\gamma$. This Alexander $\Lambda_Z$-module will be denoted by $(H_\ast(F), T_{\gamma, \ast})$.

Next, we connect these two Alexander modules. First, assume that $\gamma_\ast : \mathbb{Z} \to \mathbb{Z}$ (i.e. $\gamma_\ast : \pi_1(S^1) \to \pi_1(\mathbb{C}^*)$) satisfies $\gamma_\ast(1) = 1$. Then by $\mathbb{Z}$, $\pi_1(\gamma^{-1}(f)) \to \pi_1(MX)$ is onto, and $e_X \circ \pi_1(\tilde{\gamma}) = e_\gamma$. Therefore, by $\mathbb{Z}$ there exists a morphism of $\Lambda_Z$-modules

\[ \gamma_{\Lambda, \ast} : (H_\ast(F), T_{\gamma_\ast}) \to (H_\ast(F), h_\ast) \]

(in the sense that $h_\ast \circ \gamma_{\Lambda, \ast} = \gamma_{\Lambda, \ast} \circ T_{\gamma_\ast}$).

More generally, assume that $\gamma_\ast : \mathbb{Z} \to \mathbb{Z}$ is multiplication with an integer $\ell$ (in other words, $\ell$ is the winding number of $\gamma$ with respect to $0$). Then we can replace $MX$ by the $\mathbb{Z}/\ell\mathbb{Z}$-cyclic covering $M_{X,|\ell}$ (with projection $pr : M_{X,|\ell} \to MX$). By $\mathbb{Z}$, $\bar{\gamma} : \gamma^{-1}(f) \to MX$ can be lifted to $\gamma' : \gamma^{-1}(f) \to M_{X,|\ell}$ (with $pr \circ \gamma' = \bar{\gamma}$). Obviously, $F$ is the cyclic covering of $M_{X,|\ell}$ with transformation group $\ell\mathbb{Z}$, and $\pi_1(\gamma')$ is onto. Hence again we obtain a morphism of $\Lambda_Z$-modules

\[ \gamma_{\Lambda, \ast}^\ell : (H_\ast(F), T_{\gamma_\ast}) \to (H_\ast(F), h_\ast) \]

(i.e. $h_\ast \circ \gamma_{\Lambda, \ast}^\ell = \gamma_{\Lambda, \ast}^\ell \circ T_{\gamma_\ast}$). Above, if $\ell = 0$ then $M_{X,|\ell}$ is obviously $F$ itself.

The above constructions can be made compatible with some base points. (Since all the spaces are connected, this is not really relevant, the details are left to the interested reader).

Notice that the target module $(H_\ast(F), h_\ast)$ is completely independent of $\gamma$. The main point is that even the map $H_\ast(F) \to H_\ast(F)$, as a $\Lambda$-module, is independent of $\gamma$. Indeed, $\gamma^{-1}(f)$ has the homotopy type of $F$ and $\tilde{\gamma}$ is up to homotopy the same as our fixed embedding $s : F \to F$. The above discussion is summarized in the following theorem.

**5.2. Theorem.** Assume that the generic fiber $F$ of $f$ is connected, and $0 \in B_f$. Fix an embedding $s : F \to F$ as above. Let $\gamma : S^1 \to \mathbb{C} \setminus B_f$ be a smooth loop. Assume that $\gamma_\ast : \pi_1(S^1) \to \pi_1(\mathbb{C}^*)$ is multiplication by $\ell = \ell(\gamma)$. Then $s_\ast : H_\ast(F) \to H_\ast(F)$ is compatible with the Alexander $\Lambda_Z$-module structures in the sense that $s_\ast \circ T_{\gamma_\ast} = h_\ast \circ s_\ast$.

One has the following immediate consequence.

**5.3. Corollary.** Assume that two loops $\gamma, \gamma' : S^1 \to \mathbb{C} \setminus B_f$ satisfy $\ell(\gamma) = \ell(\gamma')$ (i.e. they have the same winding number with respect to $0$). Then $T_{\gamma,q} = T_{\gamma',q}$ modulo $\ker(s_q)$, for any $q \geq 0$.

In other words, if the monodromy operators are “very different”, then the image of $s_\ast : H_\ast(F) \to H_\ast(F)$ is forced to be “small”. Conversely, the non-vanishing of $\text{im}(s_\ast)$ imposes some compatibility restrictions on the monodromy operators. For a precise reinterpretation of 5.2 and 5.3 see 5.4.

The above theorem is optimal exactly when $s_q$ is onto in the non-trivial cases $q \leq n$. As we will see, this is the case e.g. for $h$-good polynomials (cf. 5.6 below). On the other hand, we cannot hope that for an arbitrary polynomial $f$ the map $s_q : H_q(F) \to H_q(F)$ ($q \leq n$) is onto, since $H_q(F, \mathbb{C})$ is a $\Lambda$-torsion module, but $H_q(F, \mathbb{C})$ may have a non zero free part.
Nevertheless, the next theorem shows that \( im(s_n) \) is as “large as possible”. Below \( \Delta_{\gamma,\ast}(t) \) denotes the characteristic polynomial of the monodromy operator \( T_{\gamma,\ast} \).

5.4. **Theorem.** Assume that \( F \) is connected, and \( 0 \in B_f \), and \( s : F \to \overline{F} \) is fixed as in 5.2. Let \( (T_s, h_s) \) be the torsion part of the \( \Lambda \)-module \( (H_s(\overline{F}, \mathbb{C}), h_s) \). Then:

(i) \( im(s_n) = T_s \).

(ii) Therefore, for any loop \( \gamma : S^1 \to \mathbb{C} \setminus B_f \) with \( \ell := \ell(\gamma) \) one has the following epimorphism of \( \Lambda \)-modules:

\[
S_n : (H_s(F, \mathbb{C}), T_{\gamma,\ast}) \to (T_s, h_s^{\ell}).
\]

(iii) In particular, \( \Delta((H_\ast(M_{\gamma,\ast}^C, \mathbb{C}), h_{\ell}^q)(t) \) divides \( \Delta_{\gamma,\ast}(t) \) for any \( q \geq 0 \) (cf. with 5.5).

(iv) Part (iii) applied for \( T_{\gamma,\ast} \) implies that the \( (t-a) \)-torsion of \( H_\ast(M_{\gamma,\ast}^C, \mathbb{C}) \) is zero if \( a \) is not a root of unity.

**Proof.** Notice that part (i) and 5.1(\( \gamma \)) imply (ii), hence all the others as well. Notice that the statement of (i) is independent of the choice of any loop. In order to prove (i), we take a special loop, namely the oriented boundary of \( D_0 \). Then using the notations and the construction of 2.10, we deduce that \( H_s(\overline{F}, F) = \oplus b H_s(\mathbb{Z} \times (T_b, F)) \); in particular this is a free \( \Lambda \)-module. Then the result follows from the homological exact sequence of the pair \( (F, F) \) (considered as a sequence of \( \Lambda \)-modules). \( \square \)

5.5. **Remark.** If \( M = (H, h) \) is a torsion \( \Lambda \)-module with Alexander polynomial \( \Delta(H, h)(t) \) (i.e., if \( h \) acts on \( H \) with characteristic polynomial \( \Delta(H, h)(t) \)), then for any \( \ell \geq 0 \) the polynomial \( \Delta(H, h)(t) \) determines \( \Delta(H, h^{\ell})(t) \) as follows. For any polynomial \( P(t) = \prod (t-a)^{n_a} \) \( (n_a \in \mathbb{N}) \) write \( P(t)^{\ell} : = \prod (t-a^{\ell})^{n_a} \). Then \( \Delta(H, h^{\ell})(t) = \Delta(H, h)(t)^{\ell} \).

Now, we will apply our general results for \( h \)-good polynomials. The next corollary follows directly from 2.10 and 1.2.

5.6. **Corollary.** Assume that \( f \) is a \( h \)-good polynomial, \( 0 \in B_f \), and take a smooth loop \( \gamma : S^1 \to \mathbb{C} \setminus B_f \) with \( \ell = \ell(\gamma) \). Then there is an epimorphism of \( \Lambda_{\mathbb{Z}} \)-modules

\[
S_n : (H_n(F), T_{\gamma,n}) \to (\pi_n(M_X), h_n^{\ell}).
\]

In particular, \( \Delta(\pi_n(M_X))(t)^{\ell} \) divides \( \Delta_{\gamma,n}(t) \).

5.7. **The main examples.** The monodromy around the origin and at infinity. To any polynomial \( f \) (and to a distinguished atypical value \( 0 \in B_f \)) one can associate two distinguished monodromy operators:

(i) the local monodromy of \( f \) at 0, namely \( T_{0,\ast} : H_\ast(F) \to H_\ast(F) \) provided by a loop \( \gamma \) going around the bifurcation point 0 on the boundary of a small disc \( D_0 \) containing no other bifurcation points inside (with positive orientation);

(ii) the monodromy at infinity of \( f \), namely \( T_{\infty,\ast} : H_\ast(F) \to H_\ast(F) \) provided by a loop \( \gamma \) going around the boundary of a large disc \( D_\infty \) containing all the bifurcation points \( B_f \) inside.

They provide two (Alexander) \( \Lambda \)-module structures on \( H_\ast(F, \mathbb{C}) \) denoted by \( A_\ast(f, 0) \), respectively \( A_\ast(f, \infty) \). The corresponding Alexander polynomials, or equivalently, the characteristic polynomials of the monodromy operators \( T_{0,\ast} \) and \( T_{\infty,\ast} \), are denoted by \( \Delta(T_{0,\ast}) \) resp. \( \Delta(T_{\infty,\ast}) \). Note that the Alexander module \( A_\ast(f, 0) \) (resp. \( A_\ast(f, \infty) \)) encodes exactly the Jordan structure of \( T_{0,\ast} \) (resp. \( T_{\infty,\ast} \)) and that this Jordan structure was studied in several papers, e.g. [ACD], [D2], [DN1-2], [DS], [Do], [GN1-3], [Sa].
Obviously, in both cases $\ell = 1$. Therefore, \[3.4\] guarantees that for any $f$ with $F$ connected, and for any $q \geq 0$, the Alexander polynomial

$$\Delta(H_q(M_X^{\infty}, \mathbb{C}))$$

divides the characteristic polynomials $\Delta(T_{0,q})$ and $\Delta(T_{\infty,q})$. \(1\)

This is a generalization of [Li2] (4.3) to arbitrary polynomials and $q$. Let us explain more precisely the relation between this divisibility result (1) and the divisibility results in [Li2].

First consider the local monodromy $T_{0,*}$. In [Li2], $X$ has only isolated singularities including at infinity. In that case only the case $q = n$ is relevant (cf. [15]). It is known that by the localization of the monodromy (see e.g. [NN]) one has

$$\Delta(T_{0,n})(t) = \Delta(\mathcal{M}(0)),$$

where $\Delta_i$ are the characteristic polynomials of the local monodromies associated to the isolated singularities on $V$, and $k = b_n(F) - \sum \deg(\Delta_i)$. In fact, the singularities at infinity (i.e. on $V \cap H$) should be treated slightly different, as explained in [Li2], [LiT]. Moreover, in general $k \geq 0$, and $k = 0$ iff $G_f = \{0\}$. Note that (1) (for $q = n$) and (2) give a similar result to Theorem (4.3) in [Li2], yielding in addition a precise value for $k$.

The discussion for the monodromy at infinity $T_{\infty,*}$ is more involved. It was shown by Neumann and Norbury [NN] that the total space of the fibration $(*) f : f^{-1}(S^1) \to S^1$ for $r \gg 0$ (which provides $T_{\infty,*}$) can be embedded in a natural way as an open subset of $S^2n+1 \setminus f^{-1}(0)$, where $S^2n+1$ is a large sphere in $\mathbb{C}^{n+1}$. Moreover, it was shown in [NZ] that for an $M$-tame polynomial this fibration $(*)$ is equivalent to the Milnor fibration at infinity $\phi : S^{2n+1} \setminus f^{-1}(0) \to S^1$, $\phi(x) = f(x)/|f(x)|$.

Note that in general $S^{2n+1} \setminus f^{-1}(0)$ is not the total space of a fibration over the circle, or, even when it is, it may happen that the corresponding fiber is not $F$, see the case of semi-tame polynomials in Păunescu-Zaharia [PZ].

In [Li2], Libgober considers an infinite cyclic covering $U_{\infty}$ of the knot complement $S^{2n+1} \setminus f^{-1}(0)$ and takes the associated Alexander module $H_n(U_{\infty}, \mathbb{C})$ as the Alexander module at infinity for $f$. From our discussion above it seems that, in general, one cannot hope to identify easily the structure of this module $H_n(U_{\infty}, \mathbb{C})$. However, in the case of $M$-tame polynomials, the module $H_n(U_{\infty}, \mathbb{C})$ is exactly our $A_n(f, \infty)$.

5.8. Example. Recall the situation described in [3.3] Namely, let $f$ be a polynomial such that there exists a system of weights $w$ with the top degree form $f_e$ defining an isolated singularity at the origin. The monodromy at infinity of such a polynomial coincides to the monodromy of the singularity $f_e = 0$, in particular $T_{\infty,n}$ is semisimple and all the eigenvalues are $e$-roots of unity. It follows that in the second exact sequence in \[4.6\](4), for $k = n$, the first morphism is trivial. Hence (for $n > 1$)

$$\pi_n(M_X) \otimes \mathbb{C} = H_n(F, \mathbb{C}) = H_n(M_{X,e}, \mathbb{C}) = H_n(F', \mathbb{C})$$

are isomorphic $\Lambda$-modules. This results should be compared to Corollary (4.9) in [Li2].

As a concrete example, let $f : \mathbb{C}^4 \to \mathbb{C}$ be the tame polynomial considered in Example 3.6. The above discussion and \[5.7\](1) gives:

$$\pi_3(M_X) \otimes \mathbb{C} = H_3(F, \mathbb{C}) = H_3(M_{X,3}, \mathbb{C}) = H_3(F', \mathbb{C}) = (\Lambda/(t-1))^5.$$ 

5.9. Remarks. If one wants to determine the homology groups of $M_X$, either one needs some information about the “tubes” $T_b$ for $b \in B_f \setminus \{0\}$, or (using 3.3) one needs to know the behaviour at infinity of $X$: both rather subtle problems. Therefore, it is rather surprising that, in some cases, all these information is carried by only one monodromy operator $T_{0,n}$. 

Here we present the case of h-good polynomials: we describe completely \( H_*(M_X, \mathbb{Z}) \) in terms of \( T_{0,n} \).

Let \( \mathcal{V}_0 \subset H_n(F, \mathbb{Z}) \) be the subgroup of vanishing cycles at 0 corresponding to a choice of a star in \( \mathbb{C} \) as in [DN1]. It follows as in [loc. cit.] that the morphism \( T_{0,n} - Id \) induces a “variation” morphism \( V : H_n(F, \mathbb{Z}) \rightarrow \mathcal{V}_0 \) and, by restriction to \( \mathcal{V}_0 \), a morphism \( V_0 : \mathcal{V}_0 \rightarrow \mathcal{V}_0 \). Using the definition of h-good polynomials, the connectivity \( \mathbb{Z}_2 \) of \( F \), and the Wang sequence associated to \( T_{0}^n \) (cf. also with \( \mathbb{Z}_1 \)), one can prove the following.

The homology groups of \( M_X \) are trivial except possibly for:

**Case** \( n > 1 \).

(i) \( H_0(M_X, \mathbb{Z}) = H_1(M_X, \mathbb{Z}) = \mathbb{Z} \),

(ii) \( H_{n+1}(M_X, \mathbb{Z}) \) is \( \mathbb{Z} \)-torsion free of rank \( b_n(F) - \text{rank}(V) \) and

(iii) \( H_n(M_X, \mathbb{Z}) = \text{coker}(V) \).

**Case** \( n = 1 \).

(i') \( H_0(M_X, \mathbb{Z}) = \mathbb{Z} \),

(ii') \( H_2(M_X, \mathbb{Z}) \) is \( \mathbb{Z} \)-torsion free of rank \( b_n(F) - \text{rank}(V) \),

(iii') \( H_1(M_X, \mathbb{Z}) = \text{coker}(V) + \mathbb{Z} \), a \( \mathbb{Z} \)-torsion free group of rank equal to the number of irreducible components of \( X \).

Note that for \( n > 1 \), \( H_n(M_X, \mathbb{Z}) \) is a finite group if \( V_0 \) is injective. This happens exactly when, in the notation from \( \mathbb{Z}_7(2) \), one has \( \prod \Delta_i(1) \neq 0 \). Moreover, the epimorphism \( \text{coker}(V_0) \rightarrow \text{coker}(V) \) implies that the order \( |H_n(M_X, \mathbb{Z})| \) divides \( |\prod \Delta_i(1)| = |\text{coker}(V_0)| \).

Let \( g : \mathbb{C}^{n+1} \times \mathbb{C} \rightarrow \mathbb{C} \) be the \( d \)-suspension of the polynomial \( f \), namely \( g(x, y) = f(x) - y^d \). Let \( Y = g^{-1}(0) \). Then writing the Gysin sequences in homology associated to a smooth divisor \( D \) in a complex manifold \( Z \) for the pairs \( (Z, D) = (M_Y, M_X) \), \( (\mathbb{C}^{n+1} \times \mathbb{C}, M_X, d) \) and resp. \( (M_X \times \mathbb{C}^*, \text{graph}(f)) \), and comparing the associated morphisms, we get for all \( q > 0 \) the exact sequence

\[
0 \rightarrow H_q(M_X, \mathbb{Z}) \rightarrow H_q(M_X, d, \mathbb{Z}) \rightarrow H_{q+1}(M_Y, \mathbb{Z}) \rightarrow 0.
\]

The exact sequence (3.\( n+1 \)) is split since the last group in it is free according to (ii) above. The exact sequence (3.\( n \)) is not split, as can be seen in the case \( n = 1, f = x_1 x_2, d = 3 \) when we get \( 0 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow 0 \). This example shows the difficulty of the question 3.10.

Finally, assume that \( n > 1 \) and \( \prod \Delta_i(\alpha^k) \neq 0 \) for \( \alpha = \text{exp}(2\pi i/d) \) and for any \( k \in \mathbb{Z} \) (cf. \( \mathbb{Z}_7(2) \)). Then one has:

(a) all the groups \( H_n(M_X, \mathbb{Z}), H_n(M_X, d, \mathbb{Z}) \) and \( H_{n+1}(M_Y, \mathbb{Z}) \) are finite; and

(b) the order \( |H_n(M_X, d, \mathbb{Z})| \) divides the product \( |\prod_{1 \leq k \leq d} \prod \Delta_i(\alpha^k)| \).

The proof of these claims follows from the exact sequence (3.q) and the property (iii) above once we know how to compute the variation associated to the special fiber \( Y \). This, in turn, is explained in [DN2]. Note that the claim (b) is similar to Theorem 3 in [Li0].

### 6. Relations to Monodromy Representation

6.1. The results of the previous section already suggest (see e.g. 5.3) that one can obtain finer results about the Alexander modules if one takes the whole monodromy representation instead of individual monodromy operators. The main message of this section is that from the monodromy representation of \( f \) one can construct a universal Alexander module which, in some sense, dominates all the Alexander modules associated with (all) the fibers of \( f \).
Since the case of h-good polynomials with all the involved numerical invariants (cf. 6.2 and 6.3) represents a special interest, we start our detailed discussion with this case. But, thanks to the general results of the previous sections, the next constructions and factorization phenomenon described in the h-good polynomial case, can be repeated word by word in the general case. The general result will be formulated at the end of the section in 6.9.

We start with a h-good polynomial. With the notation of §2, let $\mathcal{S} = \mathbb{C} \setminus B_f$, $E = f^{-1}(S)$ and $g = |B_f|$. Then the locally trivial fibration $f : E \to S$ induces a monodromy representation $\rho : G \to \text{Aut}(\mathcal{H})$, where $G = \pi_1(S, b_0)$ is a free group on $g$ generators, $b_0 \in S$ is a base point, and $\mathcal{H} = H_n(F, \mathbb{Z})$ with $F = f^{-1}(b_0)$. For each $b \in B_f$ write $F_b = f^{-1}(b)$. Let $\gamma_i$ denote an elementary loop around $b_i \in B_f$ and $m_i = \rho(\gamma_i)$ be the corresponding monodromy operators. With a natural choice for $\{\gamma_i\}$ one has $m_1 \cdot m_2 \cdots m_g = T_{\infty,n}$, see [DN1].

For any $H$-module $\mathcal{M}$ of a group $H$, we denote by $\mathcal{M}_H$ the group of coinvariants, namely the quotient of $\mathcal{M}$ by the subgroup spanned by all elements $h \cdot m - m$ for $h \in H$ and $m \in \mathcal{M}$, see Brown [Br]. We denote by $b^k(Y)$ the C-dimension of the $k^{\text{th}}$-cohomology space $H^k_c(Y, \mathbb{C})$ of $Y$ with compact supports.

We start by recalling how the $G$-module $\mathcal{H}$ determines the homology of the space $E$.

### 6.2. Proposition

The reduced homology groups $\tilde{H}_k(E, \mathbb{Z})$ are trivial except at most for $k = 1$, $k = n$ and $k = n + 1$. For these values of $k$ one has the following.

(i) For $n = 1$ one has $H_2(E, \mathbb{Z}) = H_1(G, \mathcal{H})$ and an exact sequence of groups

$$0 \to \mathcal{H}_G \to H_1(E, \mathbb{Z}) \to \mathbb{Z}g \to 0.$$  

In particular, $\mathcal{H}_G$ is a free $\mathbb{Z}$-module with rank($\mathcal{H}_G$) = $\sum_{b \in B_f} (n(F_b) - 1)$, where $n(Y)$ denotes the number of irreducible components of a curve $Y$.

(ii) For $n > 1$ one has $H_1(E, \mathbb{Z}) = \mathbb{Z}g$, $H_n(E, \mathbb{Z}) = H_0(G, \mathcal{H}) = \mathcal{H}_G$ and $H_{n+1}(E, \mathbb{Z}) = H_1(G, \mathcal{H})$. In particular, rank(\mathcal{H}_G) = $\sum_{b \in B_f} b^k_{n+1}(F_b) = \sum_{b \in B_f} b_{n+1}(T_b, \partial T_b)$.

**Proof.** The result follows from the Leray spectral sequence in homology of the fibration $F \to E \to S$ and basic facts on group homology, see Brown [Br]. The claim about the rank of $\mathcal{H}_G$ follows from the long exact sequence

$$\cdots \to H^k_c(E) \to H^k_c(\mathbb{C}^{n+1}) \to H^k_c(\cup F_b) \to H^{k+1}_c(E) \to \cdots$$

Let $H = [G, G] = G'$ be the commutator of $G$ and $S' \to S$ be the corresponding covering space. Let $f' : E' \to S'$ be the fibration (with fiber $F$) obtained from the fibration $f : E \to S$ pull-back. Then the monodromy of the fibration $f'$ corresponds exactly to the $H$-module $\mathcal{H}$ obtained by restriction of $\rho$ to $H$. On the other hand, we can regard $E' \to E$ as being the covering space corresponding to the kernel of the composition $\pi_1(E) \to \pi_1(S) = G \to G/H = \mathbb{Z}g$. It follows that the deck transformation group of $E' \to E$ is $\mathbb{Z}g$ and hence we can regard $H_n(E', R)$ as a $\Lambda_{R,g}$-module, where $\Lambda_{R,g} = R[\mathbb{Z}g]$ is a Laurent polynomial ring in $g$ indeterminates $t_1, \ldots, t_g$. As before, when $R = \mathbb{C}$ we simply write $\Lambda_g$.

To state the result similar to 6.1 for the fibration $f' : E' \to S'$, note that $S' = \mathbb{R}$ for $g = 1$ and $S'$ is homotopy equivalent to a bouquet of infinitely many $S^1$'s for $g > 1$.

### 6.3. Proposition

The reduced homology groups $\tilde{H}_k(E', \mathbb{Z})$ are trivial except at most for $k = 1$, $k = n$ and $k = n + 1$. For these values of $k$ one has the following.

(i) For $n = 1$ one has $H_2(E', \mathbb{Z}) = H_1(H, \mathcal{H})$ and an exact sequence of groups

$$0 \to \mathcal{H}_H \to H_1(E', \mathbb{Z}) \to H_1(H, \mathbb{Z}) \to 0.$$
(ii) For \( n > 1 \) one has \( H_1(E', \mathbb{Z}) = 0 \) for \( g = 1 \) and \( H_1(E', \mathbb{Z}) = H_1(H, \mathbb{Z}) \) for \( g > 1 \), \( H_n(E', \mathbb{Z}) = H_0(\mathcal{H}_n) = H_{n-1} \) and \( H_{n+1}(E', \mathbb{Z}) = H_1(\mathcal{H}_n) \).

Using the description of the \( K(H, 1) \) and of the associated chain complex given in \([Li4]\), (1.2.2.1), it follows that \( H_1(H, \mathbb{Z}) = G'/G'' \) is a submodule of \( \Lambda_{Z,g} \) and hence \( H_1(H, \mathbb{Z}) \) is \( \Lambda_{Z,g} \)-torsion free. This implies that in both cases (i) and (ii) in 6.3, we have \( \mathcal{H}_H = \text{Tors}(H_n(E', \mathbb{Z})) \) (as a \( \Lambda_{Z,g} \)-module).

In the sequel we denote the \( \Lambda_{Z,g} \)-module \( \mathcal{H}_H \) by \( M(f) \), and we call it the \textit{global Alexander module of the polynomial} \( f \).

6.4. Remark. The global Alexander module of the polynomial \( f \) can be regarded as a commutative version of the monodromy representation \( \rho \). Notice also that using Brown \([Br]\), Exercise 3, p.35, it follows that \( M(f)_{\mathbb{Z}^g} = \mathcal{H}_G \).

Assume now that \( 0 \in B_f \). We will construct a new \( \Lambda_{\mathbb{Z}} \)-module \( M(f, 0) \) out of the monodromy representation. The inclusion \( S \to \mathbb{C}^* \) at \( \pi_1 \)-level gives rise to a projection \( p_0 : G \to \mathbb{Z} \). Let \( K_0 \) denote the kernel of this projection. Then \( H \subset K_0 \) and hence we have a tower of covering spaces \( S' \to S^0 \to S \). Let \( E^0 \to S^0 \) be the fibration induced from \( E \to S \) by pull-back. In this way we get a second tower of covering spaces, namely \( E' \to E^0 \to E \).

We have, exactly as in the proof of 6.3, the following isomorphisms of \( \Lambda \)-modules:
\[
\text{Tors}(H_n(E^0, \mathbb{Z})) = \mathcal{H}_{K_0} = (\mathcal{H}_H)_{K_0/H}.
\]

Here \( K_0/H = \mathbb{Z}^{g-1} \) with generators corresponding to elementary loops in \( \mathbb{C} \) around points in \( B_f \) different from \( 0 \). Moreover the \( \mathbb{Z} = G/K_0 \)-action on \( \text{Tors}(H_n(E^0, \mathbb{Z})) \) is induced by the monodromy operator \( T_{b,n} \).

We denote this \( \Lambda_{\mathbb{Z}} \)-module \( \mathcal{H}_{K_0} \) by \( M(f, 0) \), and we call it the \textit{local Alexander module of} \( f \) \textit{at} \( 0 \). (Clearly, similar local Alexander module can be defined for any \( b \in B_f \).)

The above isomorphisms show that the local Alexander module \( M(f, 0) \) of \( f \) at \( 0 \) can be computed from the global Alexander module \( M(f) \) of \( f \). In this sense, the module \( M(f) \) is universal, i.e. contains all the information about the local Alexander modules associated to all the special fibers of \( f \).

The usefulness of this new Alexander module comes from the fact that it can be calculated using the monodromy representation and gives another approximation for the Alexander module \( \pi_n(M_X) = H_n(\mathbb{F}, \mathbb{Z}) \). Before we formulate this statement, let us reinterpret 5.2 and 7.3.

6.5. Theorem 5.2 revisited. Let us explain the meaning of 5.2 in the language of the present section. Clearly, \( \mathcal{H} = H_n(F, \mathbb{Z}) \) is a \( G \)-module, and \( H_n(\mathbb{F}, \mathbb{Z}) \) has a cyclic action generated by \( h_n \). The map \( p_0 : G \to \mathbb{Z} \) can be identified with \( [\gamma] \mapsto \ell(\gamma) \) considered in 5.2. Therefore, if we consider both \( \mathcal{H} \) and \( H_n(\mathbb{F}, \mathbb{Z}) \) as \( G \)-modules (the last one via \( p_0 \)), then 5.2 says that \( s_n \) is a morphism of \( G \)-modules.

In other words, the complicated monodromy representation (i.e. the \( G \)-module \( \mathcal{H} \)), when it is mapped via \( s_n \) into \( H_n(\mathbb{F}, \mathbb{Z}) \), it is collapsed into a modest cyclic action. Since the action in the target is abelian, this already shows that \( s_n : \mathcal{H} \to H_n(\mathbb{F}, \mathbb{Z}) \) has a factorization through \( \mathcal{H}_H = M(f) \).

Notice that \( K_0 = \ker(p_0) \) constitutes of loops (with base points) \( \gamma \) with \( \ell(\gamma) = 0 \). Corollary 5.3 applied for such a loop \( \gamma \) and for the trivial loop guarantees that \( \rho(\gamma)m = m \in \ker(s_n) \) for any \( m \). In particular, \( s_n : \mathcal{H} \to H_n(\mathbb{F}, \mathbb{Z}) \) has the following factorization of \( G \)-modules:
\[
\mathcal{H} \to \mathcal{H}_H \to \mathcal{H}_{K_0} \to H_n(\mathbb{F}, \mathbb{Z}).
\]
6.6. Corollary. Assume that \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is a h-good polynomial. Then \( s_n : H_n(F,\mathbb{Z}) \to H_n(\mathbb{F},\mathbb{Z}) \) induces an epimorphism \( M(f,0) \to \pi_n(M_X) \) of \( \Lambda\mathbb{Z} \)-modules.

Proof. Since \( s_n \) is epimorphism (cf. 5.4), the result follows from (1) above.

6.7. Remark. Notice that any \( \gamma \) with \( \ell(\gamma) = +1 \) induces the same operator \( \overline{p}(\gamma) \) acting on \( H_{K_0} \); and this operator is the positive generator of the cyclic action on \( H_{K_0} \). E.g., one can take \( T_{0,n} \) or \( T_{\infty,n} \) as well, depending which one is easier to compute. We write \( M(f,0) = (H_{K_0},T_{0,n}) \). Then, for an arbitrary \( [\gamma] \in G \) with \( \ell := \ell(\gamma) = p_0([\gamma]) \) one has the \( \Lambda\mathbb{Z} \)-module epimorphisms:

\[
(H_n(F,\mathbb{Z}),\rho(\gamma)) \to (H_{K_0},T_{0}^\ell) \to (\pi_n(M_X),h_n^\ell).
\]

Evidently, this provides the divisibilities of the corresponding Alexander (or characteristic) polynomials.

6.8. Example. Assume we are in the situation of Example 6.3 with \( n > 0 \) even. Then it is easy to see that \( H_G = \mathbb{Z}/2\mathbb{Z} \) and \( H = H_H = M(f) = M(f,0) = \pi_n(M_X) = \Lambda\mathbb{Z}/(t-1) \).

Using the general result 5.2 and 5.4, one can verify easily that the above factorization (1) is valid for arbitrary polynomials as well.

6.9. Theorem. Let \( f \) be an arbitrary polynomial with \( F \) connected. For any \( q \geq 0 \), consider \( H_q := H_q(F,\mathbb{Z}) \) as a \( G = \pi_1(S,b_0) \)-module provided by the monodromy representation. Define the global Alexander \( \Lambda\mathbb{Z}^g \)-module by \( (H_q)_H \), and the local Alexander module associated with the bifurcation point \( 0 \in B_{f} \) by \( (H_q)_{K_0} \). If we consider \( H_q(\mathbb{F},\mathbb{Z}) \) as a \( G \)-module via \( p_0 \), then \( s_q : H_q \to H_q(\mathbb{F},\mathbb{Z}) \) has the following factorization of \( G \)-modules:

\[
s_q : (H_q)_H \to (H_q)_{K_0} \to H_q(\mathbb{F},\mathbb{Z}).
\]

If one tensor this tower by \( \mathbb{C} \), then the last term \( H_q(\mathbb{F},\mathbb{C}) \) can be replaces by \( T_q \), being the image of \( s_q \).

7. An Example

Let \( f : \mathbb{C}^2 \to \mathbb{C} \) be the polynomial \( f = x + x^2 y^2 + x^2 y^3 \). Then \( B_f = \{ b_1, b_2 \} \) with \( b_1 = -27/16 \) and \( b_2 = 0 \). The fiber \( F_{b_1} \) is irreducible, has a node as a singularity and is regular at infinity. On the other hand, the fiber \( F_0 = F_{b_2} \) is smooth, has two irreducible components, one a copy of \( \mathbb{C} \) the other \( \mathbb{C} \setminus \{0,-1\} \), and has a singularity at infinity with a Milnor number equal to 3. It follows that \( b_1(F) = 4 \) and the Jordan normal form for the monodromy operators \( m_1, m_2 \) and \( T_\infty \) was obtained in [BM]:

\[
m_1 \approx \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad m_2 \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j^2 \end{pmatrix}, \quad T_\infty \approx \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

with \( j = \exp(2\pi i/3) \).

Let \( g : \mathbb{C}^2 \to \mathbb{C} \) be given by \( g = u + u^2 v \). Then \( B_g = \{ 0 \} \), the generic fiber is homotopy equivalent to \( S^1 \) and the corresponding monodromy \( m \) is the identity. Consider the polynomial \( h : \mathbb{C}^4 \to \mathbb{C} \) given by \( h(x,y,u,v) = f(x,y) + g(u,v) \).

It follows from [DN2] that \( B_h \subset B_f \). Notice that \( f \) and \( g \) are not good, but both are h-good. The fact that \( h \) is also a h-good polynomial follows from [DN2], Corollary (4.4), which basically says that the “Thom-Sebastiani sum” of two h-good polynomials is h-good.
Our next aim is to compute the global Alexander module associated with $h$ and to its special fibers. Since the information we have on the monodromy representation of $f$ is over $\mathbb{C}$, we choose this coefficient ring.

The generic fiber of $h$ is the join $(\vee_4 S^1) \ast S^1$, hence it is $\vee_4 S^3$. Since $m = Id$, [DN2] guarantees that the monodromy representation of $h$ can be identified to that of $f$. Using the above Jordan forms:

$$A_3(h, b_1) = \Lambda/(t - 1) \oplus \Lambda/(t - 1) \oplus \Lambda/(t - 1)^2.$$  \hfill (1)

$$A_3(h, 0) = \Lambda/(t - 1) \oplus \Lambda/(t - 1) \oplus \Lambda/(t - j)^2;$$  \hfill (2)

$$A_3(h, \infty) = \Lambda/(t - 1) \oplus \Lambda/(t - 1) \oplus \Lambda/(t + 1)^2;$$  \hfill (3)

Now, we consider the space $M_X$ corresponding to the two special fibers. First, let $X = h^{-1}(0)$. Then $X$ is smooth and applying Theorem (4.7), (ii) in [DN2], we get that $X$ has the homotopy type of $S^2 \vee S^3 \vee S^3$. It follows that $b_3(M_X) = b_2(X) = 1$. Using (5.7) and (4.5) one has $N(1, 3) = 1$, hence:

$$\pi_3(M_X) \otimes \mathbb{C} = H_3(\mathbb{F}, \mathbb{C}) = H_3(M_X, \mathbb{C}) = \Lambda/(t - 1).$$  \hfill (4)

Next, let $X = h^{-1}(b_1)$. Note that $X$ is again smooth but we can no longer apply Theorem (4.7) in [DN2] since $F_{b_1}$ is not smooth. Using the equality $\chi(X) = \chi_c(X)$ we get $\beta_0(X) - \beta_1(X) = 3$. Moreover, it is known that $\beta(X) = \dim \ker(m_1 - 1) = 3$, see [ACD] or [DN1]. It follows that $b(X) = 0$. The exact sequence

$$\cdots \to H^k_c(M_X) \to H^k_c(\mathbb{C}^n + 1) \to H^k_c(X) \to H^k_c(X) \to \cdots$$

and duality for $M_X$ imply that $b_3(M_X) = b_1(X) = 0$. Moreover, (1) follows that only $(t - 1)$-torsion is possible. Hence, via (4.5) and (4.5) one has $N(1, 3) = 1$, hence:

$$\pi_3(M_X) \otimes \mathbb{C} = H_3(\mathbb{F}, \mathbb{C}) = H_3(M_X, \mathbb{C}) = 0.$$  \hfill (5)

Our next aim is to compute the global Alexander module $M(h)$. This is done by using the partial information we have on the monodromy representation $\rho : G \to Aut(\mathcal{H})$, where $G$ is a free group on two generators, and $\mathcal{H}$ is the third homology of the generic fiber of $h$ with $\mathbb{C}$ coefficients (i.e., to simplify notation, we denote by $\mathcal{H}$ the complexification $\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{C})$. In terms of a special basis $e_1, e_2, e_3, e_4$ for $\mathcal{H}$ as in [DN1], (2.5), we can write the monodromy operators $m_1$ and $m_2$ of $h$ in the form

$$m_1 = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & j \\ \gamma & 0 & j^2 \end{pmatrix}.$$  

Checking that $m_1 m_2$ is conjugate to the Jordan normal form for $m_\infty$ given above implies $a \alpha = 0$ and $b c \neq 0$. By an obvious change of base we may take $b = c = 1$ and then $\beta$ and $\gamma$ are determined by the equations $\beta + \gamma = -1, \beta j + \gamma j^2 = 2$.

Let $C = [m_1, m_2] = m_1 m_2 m_1^{-1} m_2^{-1}$. Then $C \in H$ and a direct computation shows that $v_1 = (C - Id)(e_1) = - (\alpha e_2 + \beta e_3 + \gamma e_4)$. Let $\mathcal{H}_0$ be the vector subspace in $\mathcal{H}$ spanned by all the vectors $h(v) - v$ for $h \in H$ and $v \in \mathcal{H}$. It follows that

(i) $v_1 \in \mathcal{H}_0$

(ii) $\mathcal{H}_0$ is a $G$-invariant subspace of $\mathcal{H}$ (this property being always true).

It follows that we have to discuss two cases.
Case 1. \((\alpha \neq 0)\) Then the vectors \(v_1, m_2 v_1\) and \(m_2^2 v_1\) span the same subspace in \(\mathcal{H}\) as the vectors \(e_2, e_3, e_4\). Moreover \(m_1 e_3 = e_1 + e_3 \in \mathcal{H}_0\). Therefore \(\mathcal{H} = \mathcal{H}_0\) and hence \(\mathcal{H}_H = \mathcal{H}/\mathcal{H}_0 = 0\). But this is a contradiction since we have epimorphisms \(M(h) = \mathcal{H}_H \rightarrow M(h, 0) = H_3(M^c_Y, \mathbb{C}) = \Lambda/(t-1)\) by \((3.6)\) and (4) above.

Case 2. \((\alpha = 0)\) As above one shows that \(\mathcal{H}_0\) is spanned by \(e_1, e_3, e_4\) and hence \(\mathcal{H}_H = \mathbb{C}\) with a trivial \(\mathbb{Z}^2\)-action. This implies the following.

7.1. Proposition. For the polynomial \(h : \mathbb{C}^4 \rightarrow \mathbb{C}\) described above one has the following Alexander modules.

(i) For the fiber \(X = h^{-1}(0)\) one has \(\pi_3(M_X) \otimes \mathbb{C} = H_3(M^c_X, \mathbb{C}) = M(f, 0) = \Lambda/(t-1)\).

(ii) For the fiber \(Y = h^{-1}(b_1)\) one has \(\pi_3(M_Y) \otimes \mathbb{C} = H_3(M^c_Y, \mathbb{C}) = 0\). Moreover in this case \(M(f, b_1) = \Lambda/(t-1) \neq H_3(M^c_Y, \mathbb{C}) = 0\).

(iii) The global Alexander module \(M(h)\) is isomorphic to \(\Lambda_2/(t_1-1, t_2-1)\).

Note that:

(i) we succeeded to determine the above data in spite of the fact that the monodromy representation of \(h\) (over \(\mathbb{C}\)) is not completely determined (the value of \(\alpha\) is unknown);

(ii) in the case of \(Y\), we have \(M(h, b_1) \neq H_3(M^c_Y, \mathbb{C})\), in particular we cannot expect isomorphism in \((3.6)\). Nevertheless, the approximation of \(H_3(M^c_Y, \mathbb{C}) = 0\) given by \(M(h, b_1)\) is better than that given by \(A_3(h, b_1)\) since \(\dim_{\mathbb{C}} M(h, b_1) = 1\) while \(\dim_{\mathbb{C}} A_3(h, b_1) = 4\).

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