On Classical Solutions to the Cauchy Problem of the Two-Dimensional Barotropic Compressible Navier-Stokes Equations with Vacuum

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Abstract

This paper concerns the Cauchy problem of the barotropic compressible Navier-Stokes equations on the whole two-dimensional space with vacuum as far field density. In particular, the initial density can have compact support. When the shear and the bulk viscosities are a positive constant and a power function of the density respectively, it is proved that the two-dimensional Cauchy problem of the compressible Navier-Stokes equations admits a unique local strong solution provided the initial density decays not too slow at infinity. Moreover, if the initial data satisfy some additional regularity and compatibility conditions, the strong solution becomes a classical one.

Keywords: compressible Navier-Stokes equations; two-dimensional space; vacuum; strong solutions; classical solutions

1 Introduction and main results

We consider the two-dimensional barotropic compressible Navier-Stokes equations which read as follows:

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \mu \Delta u + \nabla ((\mu + \lambda)\text{div}u),
\end{aligned}
\]  

where \( t \geq 0, x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \rho = \rho(x,t) \) and \( u = (u_1(x,t), u_2(x,t)) \) represent, respectively, the density and the velocity, and the pressure \( P \) is given by

\[ P(\rho) = A\rho^\gamma \quad (A > 0, \gamma > 1). \]

The shear viscosity \( \mu \) and the bulk one \( \lambda \) satisfy the following hypothesis:

\[ 0 < \mu = \text{const}, \quad \lambda(\rho) = b\rho^\beta, \]  

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where the constants $b$ and $\beta \geq 0$ satisfy
\[
\begin{cases}
  b > 0, & \text{if } \beta > 0, \\
  \mu + b \geq 0, & \text{if } \beta = 0.
\end{cases}
\]
(1.4)

In the sequel, without loss of generality, we set $\Omega = \mathbb{R}^2$ and we consider the Cauchy problem with $(\rho, u)$ vanishing at infinity (in some weak sense). For given initial data $\rho_0$ and $m_0$, we require that
\[
\rho(x,0) = \rho_0(x), \quad \rho u(x,0) = m_0(x), \quad x \in \Omega = \mathbb{R}^2.
\]
(1.5)

When both the shear and bulk viscosities are positive constants, there are extensive studies concerning the theory of strong and weak solutions for the system of the multi-dimensional compressible Navier-Stokes equations. When the data $\rho_0, m_0$ are sufficiently regular and the initial density $\rho_0$ has a positive lower bound, there exist local strong and classical solutions to the problem (1.1) and the solutions exist globally in time provided that the data are small in some sense. For details, we refer the readers to [5, 9, 16, 17, 19] and the references therein. On the other hand, in the case that the initial density need not be positive and may vanish in open sets, under some additional compatibility conditions, the authors in [2, 4, 18] obtained the local existence and uniqueness of strong and classical solutions for three-dimensional bounded or unbounded domains and for two-dimensional bounded ones. Later, the compatibility conditions on the initial data were further relaxed by Huang-Li-Matsumura [11].

For large initial data, the global existence of weak solutions was first obtained by Lions [13, 15] provided $\gamma$ is suitably large which was improved later by Feireisl-Novotny-Petzeltova [7] (see also [6]). Recently, for the case that the initial density is allowed to vanish, Huang-Li-Xin [12] obtained the global existence of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three spatial dimensions with smooth initial data provided that the initial energy is suitably small.

For large initial data away from vacuum, Vaigant-Kazhikhov [20] first obtained that the two-dimensional system (1.1)-(1.5) admits a unique global strong solution provided that $\beta > 3$ and that $\Omega$ is bounded. Recently, for periodic initial data with initial density allowed to vanish, Huang-Li [10] relaxed the crucial condition $\beta > 3$ of [20] to the one that $\beta > 4/3$. However, for the Cauchy problem (1.1)-(1.5) with $\Omega = \mathbb{R}^2$, it is still open even for the local existence of strong and classical solutions when the far field density is vacuum, in particular, the initial density may have compact support. In fact, this is the aim of this paper.

In this section, for $1 \leq r \leq \infty$, we denote the standard Lebesgue and Sobolev spaces as follows:
\[
L^r = L^r(\mathbb{R}^2), \quad W^{s,r} = W^{s,r}(\mathbb{R}^2), \quad H^s = W^{s,2}.
\]
The first main result of this paper is the following Theorem 1.1 concerning the local existence of strong solutions whose definition is as follows:

**Definition 1.1** If all derivatives involved in (1.1) for $(\rho, u)$ are regular distributions, and equations (1.1) hold almost everywhere in $\mathbb{R}^2 \times (0, T)$, then $(\rho, u)$ is called a strong solution to (1.1).

**Theorem 1.1** Let $\eta_0$ be a positive constant and
\[
\bar{x} \triangleq (e + |x|^2)^{1/2} \log^{1+\eta_0} (e + |x|^2).
\]
(1.6)
For constants $q > 2$ and $a \in (1, 2)$, assume that the initial data $(\rho_0, m_0)$ satisfy that

\[
\rho_0 \geq 0, \quad \bar{x}^a \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \quad \nabla u_0 \in L^2, \quad \rho_0^{1/2} u_0 \in L^2, \tag{1.7}
\]

and that

\[
m_0 = \rho_0 u_0. \tag{1.8}
\]

In addition, if $\beta \in (0, 1)$, suppose that

\[
\lambda(\rho_0) \in L^2, \quad \bar{x}^{\delta_0} \nabla \lambda(\rho_0) \in L^2 \cap L^q, \tag{1.9}
\]

for some $\theta_0 \in (0, \min\{\beta, 1\})$. Then there exists a positive time $T_0 > 0$ such that the problem \((1.1)-(1.5)\) has a unique strong solution $(\rho, u)$ on $\mathbb{R}^2 \times (0, T_0]$ satisfying that

\[
\begin{cases}
\rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), & \bar{x}^a \rho \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{\rho} \nabla u_t \in L^\infty(0, T_0; L^2), \\
\nabla u \in L^2(0, T_0; H^1) \cap L^{(q+1)/q}(0, T_0; W^{1,q}), & \sqrt{\nabla} u \in L^2(0, T_0; W^{1,q}), \\
\sqrt{\rho} u_t, \sqrt{\nabla} u_t, \sqrt{\bar{x}^{-1} u_t} \in L^2(\mathbb{R}^2 \times (0, T_0)), \\
\lambda(\rho) \in C([0, T_0]; L^2), & \bar{x}^{\delta_0} \nabla \lambda(\rho) \in L^\infty(0, T_0; L^2 \cap L^q), \quad \text{for } \beta \in (0, 1),
\end{cases} \tag{1.10}
\]

and that

\[
\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx, \tag{1.11}
\]

for some constant $N > 0$ and $B_N \triangleq \{x \in \mathbb{R}^2 | |x| < N\}$.

Furthermore, if the initial data $(\rho_0, m_0)$ satisfy some additional regularity and compatibility conditions, the local strong solution $(\rho, u)$ obtained by Theorem 1.1 becomes a classical one for positive time, that is, we have

**Theorem 1.2** In addition to \((1.7)-(1.9)\), assume further that

\[
\begin{cases}
\nabla^2 \rho_0, \nabla^2 \lambda(\rho_0), \nabla^2 P(\rho_0) \in L^2 \cap L^q, \\
\bar{x}^{\delta_0} \nabla^2 \rho_0, \bar{x}^{\delta_0} \nabla^2 \lambda(\rho_0), \bar{x}^{\delta_0} \nabla^2 P(\rho_0) \in L^2, \\
\nabla^2 u_0 \in L^\infty(0, T_0; L^2 \cap L^q), \\
t \nabla^3 u \in L^\infty(0, T_0; L^2 \cap L^q), \\
\nabla u_0, \bar{x}^{-1} u_0, t \nabla u_0, t \bar{x}^{-1} u_0 \in L^\infty(0, T_0; L^2), \\
t \nabla^2 (\rho u) \in L^\infty(0, T_0; L^{(q+2)/2}).
\end{cases} \tag{1.12}
\]

A few remarks are in order:
Remark 1.1 First, it follows from (1.10) and (1.14) that
\[ \rho, \bar{x}, \nabla \rho \in C(\mathbb{R}^2 \times [0, T_0]), \] (1.15)
and that
\[ \rho u \in H^1(0, T_0; L^2) \hookrightarrow C([0, T_0]; L^2). \] (1.16)
The Gargilardo-Nirenberg inequality shows that for \( k = 0, 1, \)
\[ \|\nabla^k (\rho u)\|_{C(\mathbb{R}^2)} \leq C\|\rho u\|_{L^2(\mathbb{R}^2)}^{((2-k)q-2k)/(3q+2)} \|\nabla^2 (\rho u)\|_{L^2(\mathbb{R}^2)}^{(k+1)(q+2)/(3q+2)}, \]
which together with (1.16) and (1.14) yields that for any \( \tau \in (0, T_0), \)
\[ \rho u \in C([\tau, T_0]; C^1(\mathbb{R}^2)), \rho_t \in C(\mathbb{R}^2 \times [\tau, T_0]). \] (1.17)
Next, we deduce from (1.10) and (1.14) that
\[ \bar{x}^{-1} u \in L^\infty(0, T_0; H^2) \cap H^1(0, T_0; L^2) \hookrightarrow C(\mathbb{R}^2 \times [0, T_0]), \] (1.18)
and that for any \( \tau \in (0, T_0), \)
\[ \nabla u \in H^1(0, T_0; L^2) \cap L^\infty(0, T_0; H^1) \cap L^\infty(\tau, T_0; W^{2,q}) \hookrightarrow C([\tau, T_0]; C^1(\mathbb{R}^2)). \] (1.19)
Finally, for any \( \tau \in (0, T_0), \) it follows from (1.10) and (1.14) that
\[ \nabla u_t, \bar{x}^{-1} u_t \in H^1(\tau, T_0; L^2) \hookrightarrow C([\tau, T_0]; L^2), \] (1.20)
which combined with \( \nabla^2 u_t \in L^\infty(\tau, T_0; L^2) \) gives
\[ \nabla u_t \in C([\tau, T_0]; L^p), \quad \bar{x}^{-1} u_t \in C(\mathbb{R}^2 \times [\tau, T_0]), \] (1.21)
for any \( p \geq 2. \) This together with (1.15) shows
\[ \rho u_t \in C(\mathbb{R}^2 \times [\tau, T_0]), \]
which, along with (1.15) and (1.17)-(1.21), thus implies that the solution \((\rho, u)\) obtained by Theorem 1.1 is in fact a classical one to the Cauchy problem (1.1)-(1.5) on \( \mathbb{R}^2 \times (0, T_0]. \)

Remark 1.2 To obtain the local existence and uniqueness of strong solutions, in Theorem 1.1, the only compatibility condition we need is (1.8) which is similar to that of (II) and is much weaker than those of [2][4][18] where not only (1.8) but also (1.13) is needed. Moreover, for the local existence of classical solutions, Cho-Kim [4] needs the following additional condition:
\[ \nabla (\rho_0^{-1/2} g) \in L^2(\mathbb{R}^2), \]
besides (1.8) and (1.13). This is in fact stronger than the compatibility conditions listed in our Theorem 1.2.
We now comment on the analysis of this paper. As mentioned by [2–4, 11], the methods in [2–4, 11] cannot be applied directly to our case, since for two-dimensional case their arguments only work for the case that Ω is bounded. In fact, for Ω = \( \mathbb{R}^2 \), it seems difficult to bound the \( L^p(\mathbb{R}^2) \)-norm of \( u \) just in terms of \( \| \rho^{1/2} u \|_{L^2(\mathbb{R}^2)} \) and \( \| \nabla u \|_{L^2(\mathbb{R}^2)} \). The key observations to overcome the difficulties caused by the unbounded domain are as follows: On the one hand, for system (1.1), it is enough to bound the \( L^p(\mathbb{R}^2) \)-norm of the momentum \( \rho u \) instead of just the velocity \( u \), and on the other hand, since \( \rho \) decays for large values of the spatial variable \( x \), the momentum \( \rho u \) decays faster than \( u \) itself. To this end, we first establish a key Hardy-type inequality (see (3.13)) by combining a Hardy-type one due to Lions [14] (see (2.5)) with a spatial weighted mean estimate of the density (see (3.11)). We then construct the approximate solutions to (1.1), that is, for density strictly away from vacuum initially, we consider (1.1) in any bounded ball \( B_R \) with radius \( R > 0 \). To overcome the difficulties caused by the fact that the bulk viscosity \( \lambda \) depends on \( \rho \), we imposed the Navier-slip boundary conditions on (1.1) instead of the usual Dirichlet boundary ones. However, when we extend the approximate solutions by 0 outside the ball, it seems difficult to bound the \( L^2(\mathbb{R}^2) \)-norm of the gradient of the velocity. This will be overcome by putting an additional term \(-R^{-1}u\) on the right-hand side of (1.1). See (2.2) for details. Finally, combining all these ideas stated above with those due to [2–4, 11], we derive some desired bounds on the gradients of the velocity and the spatial weighted ones on both the density and its gradients where all these bounds are independent of both the radius of the balls \( B_R \) and the lower bound of the initial density.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 and 4 are devoted to the a priori estimates which are needed to obtain the local existence and uniqueness of strong and classical solutions. Then finally, the main results, Theorems 1.1 and 1.2, are proved in Section 5.

2 Preliminaries

First, the following local existence theory on bounded balls, where the initial density is strictly away from vacuum, can be shown by similar arguments as in [2-4].

**Lemma 2.1** For \( R > 0 \) and \( B_R = \{ x \in \mathbb{R}^2 | \| x \| < R \} \), assume that \( (\rho_0, u_0) \) satisfies

\[
(\rho_0, u_0) \in H^3(B_R), \quad \inf_{x \in B_R} \rho_0(x) > 0, \quad u_0 \cdot n = 0, \quad \text{rot} u_0 = 0, \quad x \in \partial B_R.
\]

Then there exist a small time \( T_R > 0 \) and a unique classical solution \( (\rho, u) \) to the following initial-boundary-value problem

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P - \mu \Delta u - \nabla ((\mu + \lambda) \text{div} u) &= -R^{-1}u, \\
u \cdot n &= 0, \quad \text{rot} u = 0, \quad x \in \partial B_R, \quad t > 0, \\
(\rho, u)(x, 0) &= (\rho_0, u_0)(x), \quad x \in B_R,
\end{align*}
\]
on \( B_R \times (0, T_R] \) such that

\[
\begin{align*}
\rho &\in C \left( [0, T_R]; H^3 \right), u \in C \left( [0, T_R]; H^3 \right) \cap L^2 \left( 0, T_R; H^1 \right), \\
u_t &\in L^\infty \left( 0, T_R; H^1 \right) \cap L^2 \left( 0, T_R; H^2 \right), \sqrt{pu_{tt}} \in L^2 \left( 0, T_R; L^2 \right), \\
\sqrt{t}u_t &\in L^\infty \left( 0, T_R; H^1 \right), \sqrt{t}u_t \in L^2 \left( 0, T_R; H^2 \right), \sqrt{t}u_{tt} \in L^2 \left( 0, T_R; H^1 \right), \\
tu_{tt} &\in L^\infty \left( 0, T_R; H^1 \right) \cap L^2 \left( 0, T_R; H^2 \right), t\sqrt{pu_{tt}} \in L^2 \left( 0, T_R; L^2 \right), \\
t^{3/2}u_t &\in L^\infty \left( 0, T_R; H^1 \right), t^{3/2}u_{tt} \in L^2 \left( 0, T_R; H^1 \right), \\
t^{3/2}u_{ttt} &\in L^\infty \left( 0, T_R; L^2 \right),
\end{align*}
\]

(2.3)

where we denote \( L^2 = L^2(B_R) \) and \( H^k = H^k(B_R) \) for positive integer \( k \).

Next, for either \( \Omega = \mathbb{R}^2 \) or \( \Omega = B_R \) with \( R \geq 1 \), the following weighted \( L^p \)-bounds for elements of the Hilbert space \( \tilde{D}^{1,2}(\Omega) \triangleq \{ v \in H^1_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega) \} \) can be found in [14, Theorem B.1].

**Lemma 2.2** For \( m \in [2, \infty) \) and \( \theta \in (1 + m/2, \infty) \), there exists a positive constant \( C \) such that for either \( \Omega = \mathbb{R}^2 \) or \( \Omega = B_R \) with \( R \geq 1 \) and for any \( v \in \tilde{D}^{1,2}(\Omega) \),

\[
\left( \int_{\Omega} \frac{|v|^m}{e + |x|^2}(\log(e + |x|^2))^{-\theta} \, dx \right)^{1/m} \leq C\|v\|_{L^2(B_1)} + C\|\nabla v\|_{L^2(\Omega)}. \tag{2.4}
\]

A useful consequence of Lemma 2.2 is the following weighted bounds for elements of \( \tilde{D}^{1,2}(\Omega) \) which in fact will play a crucial role in our analysis.

**Lemma 2.3** Let \( \bar{x} \) and \( \eta_0 \) be as in (1.6) and \( \Omega \) as in Lemma 2.2. For \( \gamma > 1 \), assume that \( \rho \in L^1(\Omega) \cap L^\gamma(\Omega) \) is a non-negative function such that

\[
\int_{B_{N_1}} \rho \, dx \geq M_1, \quad \int_{\Omega} \rho^\gamma \, dx \leq M_2, \tag{2.5}
\]

for positive constants \( M_1, M_2, \) and \( N_1 \geq 1 \) with \( B_{N_1} \subset \Omega \). Then there is a positive constant \( C \) depending only on \( M_1, M_2, N_1, \gamma, \) and \( \eta_0 \) such that

\[
\|v\bar{x}^{-1}\|_{L^2(\Omega)} \leq C\|\rho^{1/2}v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)}, \tag{2.6}
\]

for \( v \in \tilde{D}^{1,2}(\Omega) \). Moreover, for \( \varepsilon > 0 \) and \( \eta > 0 \), there is a positive constant \( C \) depending only on \( \varepsilon, \eta, M_1, M_2, N_1, \gamma, \) and \( \eta_0 \) such that every \( v \in \tilde{D}^{1,2}(\Omega) \) satisfies

\[
\|v\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\eta}(\Omega)} \leq C\|\rho^{1/2}v\|_{L^2(\Omega)} + C\|\nabla v\|_{L^2(\Omega)}, \tag{2.7}
\]

with \( \bar{\eta} = \min\{1, \eta\} \).

**Proof.** It follows from (2.5) and the Poincaré-type inequality [6, Lemma 3.2] that there exists a positive constant \( C \) depending only on \( M_1, M_2, N_1, \) and \( \gamma, \) such that

\[
\|v\|_{L^2(B_{N_1})} \leq C\int_{B_{N_1}} \rho \, dx + C\|\nabla v\|_{L^2(B_{N_1})},
\]

which together with (2.4) gives (2.6) and (2.7). The proof of Lemma 2.3 is finished.

Finally, the following \( L^p \)-bound for elliptic systems, whose proof is similar to that of [2, Lemma 12], is a direct consequence of the combination of a well-known elliptic theory due to Agmon-Douglis-Nirenberg [1] with a standard scaling procedure.
Lemma 2.4 For $p > 1$ and $k \geq 0$, there exists a positive constant $C$ depending only on $p$ and $k$ such that
\[
\|\nabla^{k+2}v\|_{L^p(B_R)} \leq C\|\Delta v\|_{W^{k,p}(B_R)},
\]
for every $v \in W^{k+2,p}(B_R)$ satisfying either
\[
v \cdot n = 0, \ \text{rot} v = 0, \ \text{on } \partial B_R,
\]
or
\[
v = 0, \ \text{on } \partial B_R.
\]

3 A priori estimates (I)

Throughout this section and the next, for $p \in [1, \infty]$ and $k \geq 0$, we denote
\[
\int f \, dx = \int_{B_R} f \, dx, \quad L^p = L^p(B_R), \quad W^{k,p} = W^{k,p}(B_R), \quad H^k = W^{k,2},
\]
and, without loss of generality, we assume that $\beta > 0$ since all these estimates obtained in this section and the next hold for the case that $\beta = 0$ after some small modifications. Moreover, for $R > 4N_0 \geq 4$, assume that $(\rho_0, u_0)$ satisfies, in addition to (2.1), that
\[
1/2 \leq \int_{B_{N_0}} \rho_0(x) \, dx \leq \int_{B_R} \rho_0(x) \, dx \leq 3/2.
\]
Lemma 2.4 thus yields that there exists some $T_R > 0$ such that the initial-boundary-value problem (2.2) has a unique classical solution $(\rho, u)$ on $B_R \times [0, T_R]$ satisfying (2.3).

In this section, for $\bar{x}$ and $\eta_0 > 0$ as in (1.6) and for $a \in (1, 2)$, $q \in (2, \infty)$, and $\theta_0 > 0$ as in Theorem 1.1, we will use the convention that $C$ denotes a generic positive constant depending only on $\mu, \beta, \gamma, b, q, a, \eta_0, \theta_0, N_0$, and $E_0$, where
\[
E_0 \triangleq \|\rho_0^{1/2} u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + R^{-1/2}\|u_0\|_{L^2} + \|\bar{x}^a \rho_0\|_{L^1 \cap H^1 \cap W^{1,q}} + \|\lambda(\rho_0)\|_{L^2} + \|\bar{x}^{\theta_0} \nabla \lambda(\rho_0)\|_{L^2 \cap L^q},
\]
and we write $C(\kappa)$ to emphasize that $C$ depends on $\kappa$.

Denoting $\nabla^\perp \triangleq (\partial_2, -\partial_1)$, we rewrite the momentum equations (2.2) as
\[
\rho \dot{u} + R^{-1} u = \nabla F + \mu \nabla^\perp \omega, \quad \text{(3.2)}
\]
where
\[
\dot{f} \triangleq f_t + u \cdot \nabla f, \quad F \triangleq (2\mu + \lambda) \text{div} u - P(\rho), \quad \omega \triangleq \nabla^\perp \cdot u,
\]
are the material derivative of $f$, the effective viscous flux, and the vorticity respectively. Thus, (3.2) implies that $\omega$ satisfies
\[
\begin{cases}
\mu \Delta \omega = \nabla^\perp \cdot (\rho \dot{u} + R^{-1} u), & \text{in } B_R, \\
\omega = 0, & \text{on } \partial B_R.
\end{cases}
\]
(3.3)

Applying the standard $L^p$-estimate to (3.3) yields that for $p \in (1, \infty)$,
\[
\begin{align*}
\|\nabla \omega\|_{L^p} & \leq C(p) \left( \|\rho \dot{u}\|_{L^p} + R^{-1} \|u\|_{L^p} \right), \\
\|\nabla^2 \omega\|_{L^p} & \leq C(p) \left( \|\nabla(\rho \dot{u})\|_{L^p} + R^{-1} \|\nabla u\|_{L^p} \right),
\end{align*}
\]
which together with (3.2) gives
\[
\begin{align*}
\|\nabla F\|_{L^p} + \|\rho u\|_{L^p} & \leq C(p) \left( \|\rho u\|_{L^p} + R^{-1}\|u\|_{L^p} \right), \\
\|\nabla^2 F\|_{L^p} + \|\nabla^2 u\|_{L^p} & \leq C(p) \left( \|\nabla (\rho u)\|_{L^p} + R^{-1}\|\nabla u\|_{L^p} \right).
\end{align*}
\tag{3.4}
\]

The main aim of this section is to derive the following key a priori estimate on \(\psi\) defined by
\[
\psi(t) \triangleq 1 + \|\rho^{1/2} u\|_{L^2} + \|\nabla u\|_{L^2} + R^{-1/2}\|u\|_{L^2} + \|\bar{x}^a \rho\|_{L^1 \cap H^1 \cap W^{1,q}} + \|\lambda(\rho)\|_{L^2} + \|\bar{x}^b \nabla \lambda(\rho)\|_{L^2 \cap L^q}.
\tag{3.5}
\]

**Proposition 3.1** Assume that \((\rho_0, u_0)\) satisfies (2.1) and (3.1). Let \((\rho, u)\) be the solution to the initial-boundary-value problem (2.2) on \(B_R \times (0, T_R]\) obtained by Lemma 2.1. Then there exist positive constants \(T_0\) and \(M\) both depending only on \(\mu, \beta, \gamma, b, q, a, \eta_0, \theta_0, N_0,\) and \(E_0\) such that
\[
\sup_{0 \leq t \leq T_0} \psi(t) + \int_0^{T_0} \left( \|\nabla^2 u\|_{L^q}^{(q+1)/q} + t\|\nabla^2 u\|_{L^q}^2 + \|\nabla^2 u\|_{L^2}^2 \right) dt \leq M.
\tag{3.6}
\]

To prove Proposition 3.1 whose proof will be postponed to the end of this section, we begin with the following standard energy estimate for \((\rho, u)\) and preliminary \(L^2\)-bounds for \(\nabla u\).

**Lemma 3.2** Let \((\rho, u)\) be a smooth solution to the initial-boundary-value problem (2.2). Then there exist a \(T_1 = T_1(E_0) > 0\) and a positive constant \(\alpha = \alpha(\gamma, \beta, q) > 1\) such that for all \(t \in (0, T_1]\),
\[
\sup_{0 \leq t \leq \tilde{t}} \left( R^{-1}\|\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) + \int_0^t \rho|u|^2 dx ds \leq C + C \int_0^t \psi \beta ds.
\tag{3.7}
\]

**Proof.** First, applying standard energy estimate to (2.2) gives
\[
\sup_{0 \leq t \leq \tilde{t}} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\rho\|_{L^2}^2 \right) + \int_0^t \left( \|\nabla u\|_{L^2}^2 + R^{-1}\|u\|_{L^2}^2 \right) ds \leq C.
\tag{3.8}
\]

Next, for \(N > 1\) and \(\varphi_N \in C_0^\infty(B_N)\) such that
\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = 1, \text{ if } |x| \leq N/2, \quad |\nabla^k \varphi_N| \leq CN^{-k}(k = 1, 2),
\tag{3.9}
\]
it follows from (3.8) and (3.1) that
\[
\frac{d}{dt} \int \rho \varphi_{2N_0} dx = \int \rho u \cdot \nabla \varphi_{2N_0} dx
\geq -CN_0^{-1} \left( \int \rho dx \right)^{1/2} \left( \int \rho|u|^2 dx \right)^{1/2} \geq -\tilde{C}(E_0),
\tag{3.10}
\]
where in the last inequality we have used
\[
\int \rho dx = \int \rho_0 dx.
\]
due to (2.2) and (2.2)3. Integrating (3.10) gives
\[ \inf_{0 \leq t \leq T_1} \int_{B_{2N_0}} \rho \, dx \geq \inf_{0 \leq t \leq T_1} \int_{B_{2N_0}} \rho \varphi_{2N_0} \, dx \]
\[ \geq \int \rho_0 \varphi_{2N_0} \, dx - \tilde{C}T_1 \geq 1/4, \]
(3.11)
where \( T_1 \triangleq \min\{1,(4\tilde{C})^{-1}\} \). From now on, we will always assume that \( t \leq T_1 \). The combination of (3.11), (3.8), and (2.7) yields that for \( \varepsilon > 0 \) and \( \eta > 0 \), every \( v \in D^{1,2}(B_R) \) satisfies
\[ \|\varphi^{-\eta}\|_{L(2+\varepsilon)/\tilde{\eta}}^2 \leq C(\varepsilon, \eta) \int \rho|v|^2 \, dx + C(\varepsilon, \eta)\|\nabla v\|_{L^2}^2, \]
(3.12)
with \( \tilde{\eta} = \min\{1, \eta\} \). In particular, we have
\[ \|\varphi^\eta u\|_{L(2+\varepsilon)/\tilde{\eta}} + \|\varphi^{\tilde{\eta}} v\|_{L(2+\varepsilon)/\tilde{\eta}} \leq C(\varepsilon, \eta)\psi^{1+\eta}. \]
(3.13)
Next, multiplying equations (2.2) by \( u_t \) and integration by parts yield
\[ \frac{d}{dt} \int ((2\mu + \lambda)(\text{div}u)^2 + \mu \omega^2 + R^{-1}|u|^2) \, dx + \int \rho|u_t|^2 \, dx \]
\[ \leq \int \rho|u|^2 \|
abla u\|^2 \, dx + \int \lambda_t(\text{div}u)^2 \, dx + 2 \int P\text{div}u \, dx. \]
(3.14)
We estimate each term on the right-hand side of (3.14) as follows:
First, the Gagliardo-Nirenberg inequality implies that for all \( p \in (2, +\infty) \),
\[ \|\nabla u\|_{L^p} \leq C(p)\|\nabla u\|_{L^2}^{2/p}\|\nabla u\|_{H^1}^{1-2/p} \]
\[ \leq C(p)\psi + C(p)\psi\|\nabla^2 u\|_{L^2}^{1-2/p}, \]
(3.15)
which together with (3.13) yields that for \( \eta > 0 \) and \( \tilde{\eta} = \min\{1, \eta\} \),
\[ \int \rho^\eta|u|^2 \|
abla u\|^2 \, dx \leq C(\eta)\psi^\eta \|\phi^{\tilde{\eta}} \|_{L^{(2+\varepsilon)}/\tilde{\eta}}^2 \|\nabla u\|_{L^{(2+\varepsilon)}/\tilde{\eta}}^2 \]
\[ \leq C(\varepsilon, \eta)\psi^{4+2\eta} \left( 1 + \|\nabla^2 u\|_{L^2}^{\eta/2} \right) \]
\[ \leq C(\varepsilon, \eta)\psi^{4+2\eta} + \varepsilon\psi^{-2}\|\nabla^2 u\|_{L^2}^2. \]
(3.16)
Then, noticing that \( \lambda = b\rho^{\beta} \) satisfies
\[ \lambda_t + \text{div}(\lambda u) + (\beta - 1)\lambda \text{div} u = 0, \]
(3.17)
we obtain after using (3.16) and (3.15) that
\[ \int \lambda_t(\text{div}u)^2 \, dx \leq C \int \lambda|u|^2 \|
abla u\|^2 \, dx + C \int \lambda|\nabla u|^3 \, dx \]
\[ \leq C(\varepsilon)\psi \int \lambda^2|u|^2 \|
abla u\|^2 \, dx + \varepsilon\psi^{-1}\|\nabla^2 u\|_{L^2}^2 + C\psi^\beta\|\nabla u\|_{L^3}^3 \]
\[ \leq C(\varepsilon)\psi^\alpha + C\varepsilon\psi^{-1}\|\nabla^2 u\|_{L^2}^2, \]
(3.18)
where (and what follows) we use \( \alpha = \alpha(\beta, \gamma, q) > 1 \) to denote a generic constant depending only on \( \beta, \gamma, \) and \( q \), which may be different from line to line.
Finally, since $P$ satisfies
\[ P_t + \text{div}(Pu) + (\gamma - 1)P\text{div}u = 0, \] (3.19)
we deduce from (3.13), (3.15), and the Sobolev inequality that
\[
2 \int P\text{div}u_t dx \\
= 2 \frac{d}{dt} \int Pu dx - 2 \int Pu \cdot \nabla \text{div}u dx + 2(\gamma - 1) \int P(\text{div}u)^2 dx \\
\leq 2 \frac{d}{dt} \int Pu dx + \varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon) \psi^\alpha. 
\] (3.20)
Putting (3.16), (3.18), and (3.20) into (3.14) gives
\[
\frac{d}{dt} \int \left( (2\mu + \lambda)(\text{div}u)^2 + \omega^2 + R^{-1} \|u\|^2 - 2P\text{div}u \right) dx + \int \rho |u_t|^2 dx \\
\leq C\varepsilon \psi^{-1} \|\nabla^2 u\|_{L^2}^2 + C(\varepsilon) \psi^\alpha. 
\] (3.21)
To estimate the first term on the right-hand side of (3.21), it follows from (2.8) and (3.4) that for $p \in [2, q]$, we have
\[
\|\nabla^2 u\|_{L^p} \leq C \|\nabla \omega\|_{L^p} + C \|\nabla \text{div}u\|_{L^p} \\
\leq C \left( \|\nabla \omega\|_{L^p} + \|\nabla ((2\mu + \lambda)\text{div}u)\|_{L^p} + \|\nabla \lambda \text{div}u\|_{L^p} \right) \\
\leq C \left( \|\rho u\|_{L^p} + \|\nabla P\|_{L^p} + R^{-1} \|u\|_{L^p} + \|\nabla \lambda \text{div}u\|_{L^p} \right), 
\] (3.22)
which together with (3.15) and (3.16) leads to
\[
\|\nabla^2 u\|_{L^2} \leq C \psi^{1/2} \|\sqrt{\rho u}\|_{L^2} + C \psi^\alpha + C \|\nabla \lambda\|_{L^q} \|\nabla u\|_{L^{2q/(q-2)}} \\
\leq C \psi^{1/2} \|\sqrt{\rho u}\|_{L^2} + C \psi^\alpha + \frac{1}{2} \|\nabla^2 u\|_{L^2}. 
\] (3.23)
Putting (3.23) into (3.21), integrating the resulting inequality over $(0, t)$, and choosing $\varepsilon$ suitably small yield that
\[
R^{-1} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \int_0^t \int \rho |u_t|^2 dx ds \\
\leq C + C \int_0^t \psi^\alpha ds, 
\]
where in the second inequality we have used
\[
\|P\|_{L^2}^2 \leq \|P(\rho_0)\|_{L^2}^2 + C \int_0^t \|P\|_{L^1}^{1/2} \|P\|_{L^\infty}^{3/2} \|\nabla u\|_{L^2} ds \leq C + C \int_0^t \psi^\alpha ds, 
\]
due to (3.19). The proof of Lemma 3.2 is finished.

**Lemma 3.3** Let $(\rho, u)$ and $T_1$ be as in Lemma 3.2. Then for all $t \in (0, T_1]$,
\[
\sup_{0 \leq s \leq t} s \int \rho |u_t|^2 dx + \int_0^t s \int (|\nabla u|^2 + R^{-1} |u_t|^2) dx ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. 
\] (3.24)
Proof. Differentiating (2.2) with respect to $t$ gives
\begin{equation}
\rho u_t + \rho u \cdot \nabla u_t - \nabla((2\mu + \lambda)\text{div}u) - \mu \nabla u_t^T + R^{-1}u_t
= -\rho_t(u + \mu \cdot \nabla u) - \rho u_t \cdot \nabla u + \nabla(\lambda\text{div}u) - \nabla P_t.
\end{equation}
(3.25)

Multiplying (3.25) by $u_t$ and integrating the resulting equation over $B_R$, we obtain after using (2.2) that
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 \, dx + \int ((2\mu + \lambda)(\text{div}u_t)^2 + \mu |\omega_t|^2 + R^{-1}|u_t|^2) \, dx \\
= -2 \int \rho u_t \cdot \nabla u_t \cdot u_t \, dx - \int \rho u \cdot \nabla (u \cdot \nabla u) \, dx \\
- \int \rho u_t \cdot \nabla u \cdot u_t \, dx - \int \lambda_t \text{div}(\text{div}u_t) \, dx + \int P_t \text{div}u_t \, dx
\leq C \int \rho |u_t| \left( |\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u| \right) \, dx + C \int \rho |u_t|^2 \, dx \\
+ C \int \rho |u_t|^2 |\nabla u| \, dx + C \int |\lambda_t||\text{div}u||\text{div}u_t| \, dx + C \int |P_t||\text{div}u_t| \, dx.
\end{align*}
(3.26)

We estimate each term on the right-hand side of (3.26) as follows:

First, it follows from (3.5), (3.8), (3.12), (3.13), and (3.15) that for $\varepsilon \in (0, 1)$,
\begin{align*}
\int \rho |u_t| \left( |\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u| \right) \, dx \\
\leq C \|\rho^{1/2}u\|_{L^6} \|\rho^{1/2}u_t\|_{L^6} \|\rho^{1/2}u_t\|_{L^6} \left( |\nabla u_t|_{L^2} + |\nabla^2 u|_{L^4} \right) \\
+ C \|\rho^{1/4}u\|_{L^8} \|\rho^{1/4}u_t\|_{L^8} \|\rho^{1/4}u_t\|_{L^8} \|\nabla^2 u|_{L^2}
\leq C \psi^\alpha \|\rho^{1/2}u_t\|_{L^2} \left( \|\rho^{1/2}u_t\|_{L^2} + |\nabla u_t|_{L^2} + |\nabla^2 u|_{L^2} + \psi \right) \\
\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \psi^\alpha \left( \|\nabla^2 u|_{L^2}^2 + \|\rho^{1/2}u_t|_{L^2}^2 + 1 \right).
\end{align*}
(3.27)

Next, Holder’s inequality together with (3.13) and (3.15) yields that
\begin{align*}
\int \rho |u_t|^2 |\nabla u| |\nabla u_t| \, dx &\leq C \|\rho^{1/2}u\|_{L^8} \|\nabla u\|_{L^4} \|\nabla u_t\|_{L^2} \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \left( \psi^\alpha + \|\nabla^2 u|_{L^2}^2 \right).
\end{align*}
(3.28)

Then, Holder’s inequality and (3.12) lead to
\begin{align*}
\int \rho |u_t|^2 |\nabla u| \, dx &\leq \|\nabla u\|_{L^2} \|\rho^{1/2}u_t\|_{L^6} \|\rho^{1/2}u_t\|_{L^6} \|\nabla u_t\|_{L^2} \\
&\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \psi^\alpha \|\rho^{1/2}u_t\|_{L^2}^2.
\end{align*}
(3.29)

Next, we use (3.17) and (3.13) to get
\begin{align*}
\int |\lambda_t||\text{div}u||\text{div}u_t| \, dx \\
\leq C \int \left( \lambda(\text{div}u)^2 |\text{div}u_t| + |\nabla \lambda||u||\text{div}u||\text{div}u_t| \right) \, dx \\
\leq \frac{1}{2} \int \lambda(\text{div}u_t)^2 \, dx + C \psi^\beta \|\nabla u\|_{L^4}^4 \\
+ C \|\rho^{\theta_0} \nabla \lambda\|_{L^q} \|u\|_{L^{4q}((q-2)-\theta_0)} \|\nabla u\|_{L^{4q}((q-2)-\theta_0)} \|\nabla u_t\|_{L^2} \\
\leq \frac{1}{2} \int \lambda(\text{div}u_t)^2 \, dx + \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \psi^\alpha \|\nabla^2 u|_{L^2}^2 + C(\varepsilon) \psi^\alpha.
\end{align*}
(3.30)
Finally, it follows from (3.19) and (3.13) that
\[
\int |P_t||\text{div}u_t|dx \\
\leq C \int (P|\text{div}u| + |\nabla P||u_t|) |\text{div}u_t|dx \\
\leq C \left( ||P||_{L^\infty}||u||_{L^2}^2 + ||\rho\gamma^{-1}||_{L^\infty}||\bar{x}^\delta \nabla \rho||_{L^4}||\bar{x}^{-\alpha}u||_{L^2/\gamma(\gamma-2)} \right) ||\nabla u_t||_{L^2} \\
\leq \varepsilon ||\nabla u_t||_{L^2}^2 + C(\varepsilon)\psi^\alpha,
\]
where in the last inequality we have used (3.13).

Substituting (3.27)-(3.31) into (3.26) and choosing \(\varepsilon\) suitably small lead to
\[
\begin{align*}
\frac{d}{dt} \int \rho|u_t|^2 dx &+ \int \left( (2\mu + \lambda)(\text{div}u_t)^2 + \mu\omega_t^2 + R^{-1}|u_t|^2 \right) dx \\
&\leq C\psi^\alpha \left( 1 + ||\rho^{1/2}u_t||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2 \right) \\
&\leq C\psi^\alpha ||\rho^{1/2}u_t||_{L^2}^2 + C\psi^\alpha,
\end{align*}
\]
where in the last inequality we have used (3.23). Multiplying (3.32) by \(t\), we obtain (3.24) after using Gronwall’s inequality and (3.7). The proof of Lemma 3.3 is completed.

**Lemma 3.4** Let \((\rho, u)\) and \(T_1\) be as in Lemma 3.2. Then for all \(t \in (0, T_1]\),
\[
\sup_{0 \leq s \leq t} \left( ||\bar{x}^\delta u||_{L^1 \cap H^{1,q} \cap W^{1,q}} + ||\lambda(\rho)||_{L^2} + ||\bar{x}^\delta \nabla \lambda(\rho)||_{L^2 \cap L^q} \right) \\
\leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\}.
\]

**Proof.** First, (3.17) gives
\[
||\lambda(\rho)||_{L^2} \leq C||\nabla u||_{L^\infty}||\lambda(\rho)||_{L^2} \leq C||\nabla u||_{W^{1,q}}||\lambda(\rho)||_{L^2},
\]
Next, multiplying (2.24) by \(\bar{x}^\alpha\) and integrating the resulting equality over \(B_R\), we obtain after integration by parts and using (3.8) that
\[
\frac{d}{dt} \int \rho \bar{x}^\alpha dx \leq C \int \rho|u| \bar{x}^{\alpha-1} \log^{1+\gamma_0}(e + |x|^2)dx \\
\leq C \left( \int \rho \bar{x}^{2\alpha-2} \log^{2(1+\gamma_0)}(e + |x|^2)dx \right)^{1/2} \left( \int \rho u^2 dx \right)^{1/2} \\
\leq C \left( \int \rho \bar{x}^\alpha dx \right)^{1/2},
\]
which gives
\[
\sup_{0 \leq t \leq T_1} \int \rho \bar{x}^\alpha dx \leq C.
\]
Next, it follows from the Sobolev inequality and (3.13) that for \(0 < \delta < 1\),
\[
||u \bar{x}^{-\delta}||_{L^\infty} \leq C(\delta) \left( ||u \bar{x}^{-\delta}||_{L^{4/\delta}} + ||\nabla(u \bar{x}^{-\delta})||_{L^3} \right) \\
\leq C(\delta) \left( ||u \bar{x}^{-\delta}||_{L^{4/\delta}} + ||\nabla u||_{L^3} + ||u \bar{x}^{-\delta}||_{L^{4/\delta}} ||\bar{x}^{-1} \nabla \bar{x}||_{L^{12/(4-3\delta)}} \right) \\
\leq C(\delta) \left( \psi^\alpha + ||\nabla^2 u||_{L^2} \right).
\]
One derives from (3.1.1) that $w \triangleq \rho \tilde{x}^\alpha$ satisfies

$$w_t + u \cdot \nabla w - awu \cdot \nabla \log \tilde{x} + \text{div} u = 0,$$

which together with (3.3.6) gives that for $p \in [2, q]$

$$(\|\nabla w\|_{L^p})_t \leq C(1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \tilde{x}\|_{L^\infty})\|\nabla w\|_{L^p} + C\left(\|\nabla u\|_{L^p} + \|u \nabla \log \tilde{x}\|_{L^p} + \|u \nabla u\|_{L^p}\right)\|w\|_{L^\infty}$$

$$\leq C(\psi^\alpha + \|u\|_{L^2 \cap W^{1,q}})\|\nabla w\|_{L^p} + C\left(\|\nabla u\|_{L^p} + \|u \nabla \log \tilde{x}\|_{L^p} + \|u \nabla u\|_{L^p}\right)\|w\|_{L^\infty}$$

(3.37)

$$\leq C(\psi^\alpha + \|\nabla w\|_{L^2 \cap L^q})(1 + \|\nabla w\|_{L^p} + \|\nabla w\|_{L^q}),$$

where in the last inequality we have used (3.35). Similarly, one obtains from (3.17) that

$$(\|\nabla (\tilde{x}^\beta \lambda)\|_{L^2 \cap L^q})_t \leq C(\psi^\alpha + \|\nabla u\|_{L^2 \cap L^q})(1 + \|\nabla (\tilde{x}^\beta \lambda)\|_{L^2 \cap L^q}).$$

(3.38)

Next, we claim that

$$\int_0^t (\|\nabla u\|_{L^2 \cap L^q}^{(q+1)/q} + t\|\nabla u\|_{L^2 \cap L^q}^2) \, dt \leq C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\},$$

(3.39)

which together with (3.37), (3.38), (3.35), (3.34), and the Gronwall inequality yields that

$$\sup_{0 \leq s \leq t} \left( \|\tilde{x}^\alpha \rho\|_{L^1 \cap H^{1 \cap W^{1,q}}} + \|\lambda(\rho)\|_{L^2} + \|\nabla (\tilde{x}^\beta \lambda)\|_{L^2 \cap L^q} \right)$$

$$\leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \right\}. \quad (3.40)$$

One thus directly obtains (3.33) from this and the following simple fact:

$$\|\tilde{x}^\beta \nabla \lambda\|_{L^2 \cap L^q} \leq \|\nabla (\tilde{x}^\beta \lambda)\|_{L^2 \cap L^q} + C\|\tilde{x}^\beta \rho\|_{W^{1,q}},$$

due to $\beta < \alpha\beta$.

Finally, to finish the proof of Lemma 3.4 it only remains to prove (3.39). In fact, on the one hand, it follows from (3.23), (3.7), and (3.24) that

$$\int_0^t \left( \|\nabla u\|_{L^2}^{5/3} + s\|\nabla u\|_{L^2}^2 \right) \, ds$$

$$\leq C \int_0^t (\|\sqrt{\rho} u\|_{L^2}^{5/3} + \psi^{\alpha}) \, ds + C \exp \left\{ C \int_0^t \psi^{\alpha} \, ds \right\} \int_0^t \psi^{\alpha} \, ds$$

$$\leq C \exp \left\{ C \int_0^t \psi^{\alpha} \, ds \right\}.$$  

(3.41)

On the other hand, choosing $p = q$ in (3.24) gives

$$\|\nabla^2 u\|_{L^q} \leq C \left( \|\rho \tilde{u}\|_{L^q} + \|\nabla P\|_{L^q} + R^{-1} \|u\|_{L^q} + \|\nabla \lambda\|_{L^q} \|\nabla u\|_{L^\infty} \right)$$

$$\leq C \left( \|\rho \tilde{u}\|_{L^q} + \psi^\alpha + \psi^{\alpha} \|\nabla u\|_{L^q}^{(2q-2)/q} \right)$$

$$\leq \frac{1}{2} \|\nabla^2 u\|_{L^q} + C \psi^\alpha + C \|\rho \tilde{u}\|_{L^q}.$$  

(3.42)
By (3.12), (3.13), and (3.15), the last term on the right-hand side of (3.42) can be estimated as follows:

\[
\|\rho \dot{u}\|_{L^q} \leq \|\rho u_t\|_{L^q} + \|\rho u \cdot \nabla u\|_{L^q} \\
\leq \|\rho u_t\|_{L^2}^{2(q-1)/(q^2-2)} \|\rho u_t\|_{L^2}^{(q^2-2q)/(q^2-2)} + \|\rho u\|_{L^2} \|\nabla u\|_{L^2} \\
\leq C\psi^\alpha \left( \|\rho^{1/2} u_t\|_{L^2}^{2(q-1)/(q^2-2)} \|\nabla u_t\|_{L^2}^{(q^2-2q)/(q^2-2)} + \|\rho^{1/2} u_t\|_{L^2} \right) \\
+ C\psi^\alpha \left( 1 + \|\nabla^2 u\|_{L^2}^{1-1/q} \right).
\]

This combined with (3.41), (3.24), and (3.7) yields that

\[
\int_0^t \|\rho \dot{u}\|_{L^q}^{q+1/q} dt \leq C \int_0^t \psi^\alpha (t) \|\rho^{1/2} u_t\|_{L^2} \left( t \|\nabla u_t\|_{L^2} \right) \left( \frac{q^2-1}{q(q^2-2)} \right) \left( \frac{q^2-2q}{q^2-2} \right) dt \\
+ C \int_0^t \|\rho^{1/2} u_t\|_{L^2} dt + C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \\
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \int_0^t \left( \psi^\alpha t^{\frac{q^2-1}{q(q^2-2)}} \right) \left( t \|\nabla u_t\|_{L^2} \right) dt \quad (3.43) \\
+ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \\
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\},
\]

and that

\[
\int_0^t \|\rho \dot{u}\|_{L^q}^2 dt \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}. \quad (3.44)
\]

One thus obtains (3.39) from (3.41) - (3.44) and finishes the proof of Lemma 3.4.

Now, Proposition 3.1 is a direct consequence of Lemmas 3.2-3.3.

**Proof of Proposition 3.1.** It follows from (3.8), (3.7), and (3.33) that

\[
\psi(t) \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\}.
\]

Standard arguments thus yield that for \( M \triangleq e^{Ce} \) and \( T_0 \triangleq \min\{T_1, (CM^\alpha)^{-1}\} \),

\[
\sup_{0 \leq t \leq T_0} \psi(t) \leq M,
\]

which together with (3.39), (3.23), and (3.7) gives (3.6). The proof of Proposition 3.1 is thus completed.

### 4 A priori estimates (II)

In this section, in addition to \( \mu, \beta, \gamma, b, q, a, \eta_0, \theta_0, N_0 \), and \( E_0 \), the generic positive constant \( C \) may depend on \( \delta_0, \|\nabla^2 u_0\|_{L^2}, \|\nabla^2 \rho_0\|_{L^2}, \|\nabla^2 \lambda(\rho_0)\|_{L^2}, \|\nabla^2 P(\rho_0)\|_{L^2}, \|\bar{x}^h \nabla^2 \rho_0\|_{L^2}, \|\bar{x}^h \nabla^2 \lambda(\rho_0)\|_{L^2}, \|\bar{x}^h \nabla^2 P(\rho_0)\|_{L^2}, \) and \( \|\bar{g}\|_{L^2} \), where

\[
\bar{g} \triangleq \rho_0^{-1/2} (-\mu \Delta u_0 - \nabla((\mu + \lambda(\rho_0))\text{div} u_0) + \nabla P(\rho_0) + R^{-1} u_0). \quad (4.1)
\]
Lemma 4.1 It holds that

$$\sup_{0 \leq t \leq T_0} \left( \|\bar{x}^\delta \nabla^2 \rho\|_{L^2} + \|\bar{x}^\delta \nabla^2 \lambda\|_{L^2} + \|\bar{x}^\delta \nabla^2 P\|_{L^2} \right) \leq C. \quad (4.2)$$

Proof. First, by virtue of (2.1) and (2.2), defining

$$\rho^{1/2} u_t(x, t = 0) \triangleq -\bar{g} - \rho_0^{1/2} u_0 \cdot \nabla u_0,$$

integrating (3.32) over $(0, T_0)$, and using (3.6) and (3.7), we obtain that

$$\sup_{0 \leq t \leq T_0} \int_0^T \rho |u_t|^2 dx + \int_0^T \left( \|\nabla u_t\|_{L^2}^2 + R^{-1} \|u_t\|_{L^2}^2 \right) dt \leq C. \quad (4.3)$$

This combined with (3.23) and (3.6) gives

$$\sup_{0 \leq t \leq T_0} \|\nabla u\|_{H^1} \leq C, \quad (4.4)$$

which together with (3.36) and (3.6) shows that for $\delta \in (0, 1)$,

$$\|\rho^\delta u\|_{L^\infty} + \|\bar{x}^{-\delta} u\|_{L^\infty} \leq C(\delta). \quad (4.5)$$

Direct calculations yield that for $2 \leq r \leq q$

$$\|(\bar{x}^{(1+a)/2} + |u|) \rho t\|_{L^r} + \|(1 + |u|) \lambda t\|_{L^r} + \|(1 + |u|) P_t\|_{L^r} \leq C, \quad (4.6)$$

due to (3.6), (2.2), (3.17), (3.19), (4.4), and (4.5). It follows from (3.12) and (4.3)–(4.5) that for $\delta \in (0, 1]$ and $s > 2/\delta$,

$$\|\bar{x}^{-\delta} u_t\|_{L^s} + \|\bar{x}^{-\delta} \bar{u}\|_{L^s} \leq C \|\bar{x}^{-\delta} u_t\|_{L^s} + C \|\bar{x}^{-\delta} u\|_{L^\infty} \|\nabla u\|_{L^s} \leq C(\delta, s) + C(\delta, s) \|\nabla u_t\|_{L^2}. \quad (4.7)$$

Next, denoting $v \triangleq \bar{x}^\delta g(\rho)$ with $g(\rho) = \rho^p$ for $p \in [\min\{\beta, 1\}, \max\{\beta, \gamma\}]$, we get from (3.6) that

$$\|v\|_{L^\infty} + \|\nabla v\|_{L^2 \cap L^q} \leq C, \quad (4.8)$$

due to $\delta_0 \leq \theta_0 \leq \min\{1, \beta\}$. It follows from (2.2) that

$$g(\rho)_t + u \cdot \nabla g(\rho) + pg(\rho) \text{div} v = 0, \quad (4.9)$$

which gives

$$v_t + u \cdot \nabla v - \delta_0 v u \cdot \nabla \log \bar{x} + pv \text{div} u = 0.$$

Thus, direct calculations yield that

$$\left( \|\nabla^2 v\|_{L^2} \right)_t \leq C \left( 1 + \|\nabla u\|_{L^\infty} + \|u \cdot \nabla \log \bar{x}\|_{L^\infty} \right) \|\nabla^2 v\|_{L^2} + C \|\nabla^2 u\|_{L^2} \quad (4.10)$$

\[+ C \|\nabla v\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \log \bar{x}\|_{L^2} + C \|\nabla v\|_{L^2} \|\nabla^2 \log \bar{x}\|_{L^2} \]
\[+ C \|\nabla v\|_{L^\infty} \left( \|\nabla^2 (u \cdot \nabla \log \bar{x})\|_{L^2} + \|\nabla^3 u\|_{L^2} \right) \]
\[\leq C \left( 1 + \|\nabla u\|_{L^\infty} \right) \|\nabla^2 v\|_{L^2} + C \|\nabla^2 u\|_{L^{2/(q-2)} \cap L^q} \|\nabla v\|_{L^q} \]
\[+ C \|\nabla v\|_{L^2} \|\nabla u\|_{L^\infty} + C \|\nabla v\|_{L^2} \|\nabla^2 \log \bar{x}\|_{L^\infty} \]
\[+ C \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2} + C \|u \nabla^3 \log \bar{x}\|_{L^2} + C \|\nabla^3 u\|_{L^2} \]
\[\leq C \left( 1 + \|\nabla u\|_{L^\infty} \right) \|\nabla^2 v\|_{L^2} + C \|\nabla^3 u\|_{L^2}.\]
where in the second and third inequalities we have used (4.5) and (4.8). We use (2.8), (4.4), and (3.4) to estimate the last term on the right-hand side of (4.10) as follows:

\[
\|\nabla^3 u\|_{L^2} \leq C\|\nabla (\nabla^2 \omega)\|_{L^2} + C\|\nabla (\nabla \text{div} u)\|_{L^2} + C \\
\leq C \left( \|\nabla^2 \omega\|_{L^2} + \|\nabla^2 P\|_{L^2} + \|\nabla^2 F\|_{L^2} + \|\nabla^2 u\| \right) + C\|\nabla u\|\nabla^2 \lambda\|_{L^2} + C \\
\leq C \left( \|\nabla (\rho \hat{u})\|_{L^2} + \|\bar{x} \nabla^2 P\|_{L^2} + \|\nabla \lambda\|_{L^2} \right) + C\|\nabla u\|\nabla^2 \lambda\|_{L^2} + C \\
\leq C\|\nabla u_t\|_{L^2} + C\|\bar{x} \nabla^2 P\|_{L^2} + \frac{1}{2}\|\nabla^3 u\|_{L^2} + C\|\bar{x} \nabla^2 \lambda\|_{L^2} + C\|\nabla u\|_{L^\infty} + C,
\]

where in the last inequality we have used the following simple fact:

\[
\|\nabla (\rho \hat{u})\|_{L^2} \leq C\|\rho \bar{x} \nabla \hat{u}\|_{L^2} + C\|\bar{x} \nabla \rho\|_{L^2} + C\|\nabla \lambda\|_{L^2} \leq C\|\nabla u_t\|_{L^2} + C,
\]

due to (3.6), (4.7), and (4.4). Noticing that (3.6) leads to

\[
\|\bar{x} \nabla^2 \rho\| + \|\bar{x} \nabla^2 \lambda\| + \|\bar{x} \nabla^2 P\| \leq C\|\nabla^2 (\bar{x} \nabla \rho)\|_{L^2} + C\|\nabla^2 (\bar{x} \nabla \lambda)\|_{L^2} + C\|\nabla^2 (\bar{x} \nabla P)\|_{L^2} + C
\]

we substitute (4.11) into (4.10) to get

\[
\frac{d}{dt} \left( \|\nabla^2 (\bar{x} \nabla \rho)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla \lambda)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla P)\|_{L^2} \right) \\
\leq C(1 + \|\nabla^2 u\|_{L^2}) \left( \|\nabla^2 (\bar{x} \nabla \rho)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla \lambda)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla P)\|_{L^2} \right) \\
+ C\|\nabla u_t\|_{L^2} + C,
\]

which, along with Gronwall’s inequality, (3.6), and (4.3), yields that

\[
\sup_{0 \leq t \leq T_0} \left( \|\nabla^2 (\bar{x} \nabla \rho)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla \lambda)\|_{L^2} + \|\nabla^2 (\bar{x} \nabla P)\|_{L^2} \right) \leq C.
\]

This combined with (4.12) gives (4.2) and finishes the proof of Lemma 4.1.

**Lemma 4.2** It holds that

\[
\sup_{0 \leq t \leq T_0} \left( \|\nabla u_t\|_{L^2} + R^{-1}\|u_t\|_{L^2} \right)^2 + \int_0^{T_0} t \left( \|\rho^{1/2} u_t\|_{L^2}^2 + \|\nabla^2 u_t\|_{L^2}^2 \right) dt \leq C.
\]

**Proof.** Multiplying (3.23) by \(u_t\) and integrating the resulting equation over \(B_R\) lead to

\[
\frac{1}{2} \frac{d}{dt} \int \left( \langle 2\mu + \lambda \rangle (\text{div} u_t)^2 + \mu \omega_t^2 + R^{-1} |u_t|^2 \right) dx + \int \rho |u_t|^2 dx \\
= - \int (2\mu u_t \cdot \text{grad} u_t + \rho u_t \cdot \nabla u \cdot u_t) dx - \int \rho u \cdot \nabla (u \cdot \nabla u) \cdot u_t dx \\
- \int \rho u_t \cdot \nabla u_t^2 u t dx + \frac{1}{2} \int \lambda_t (\text{div} u_t)^2 dx - \int \lambda_t \text{div} u \text{div} u_t dx \\
+ \int P_t \text{div} u_t dx.
\]
We estimate each term on the right-hand side of (4.15) as follows:

First, it follows from (3.6), (4.3)-(4.5), and (4.7) that

\[
\left| \int (2\rho u \cdot \nabla u_t \cdot u_{tt} + \rho u_t \cdot \nabla u \cdot u_{tt}) \, dx \right| + \left| \int \rho u \cdot \nabla (u \cdot \nabla u) \cdot u_{tt} \, dx \right| \\
\leq \varepsilon \int \rho |u_{tt}|^2 \, dx + \varepsilon \int \rho |u_{tt}|^2 \, dx \\
+ C(\varepsilon) \left( \|\rho^{1/2}u\|^2_{L^\infty} \|\nabla u_t\|^2_{L^2} + \|\rho^{1/2}u_t\|^2_{L^4} \right) \\
+ C(\varepsilon) \int \rho |u|^2 |\nabla u|^4 + \rho |u|^4 |\nabla u|^2 \, dx \\
\leq \varepsilon \int \rho |u_{tt}|^2 \, dx + C(\varepsilon) \left( \|\nabla u_t\|^2_{L^2} + 1 \right).
\]

(4.16)

Next, direct calculations give

\[
\int \rho u \cdot \nabla u_t \cdot u_{tt} \, dx = -\frac{d}{dt} \int \rho u \cdot \nabla u_t \cdot u_{tt} + \int (\rho u_t) \cdot \nabla u \cdot u_{tt} \, dx \\
+ \int \rho u \cdot \nabla u_t \cdot (u_{tt} + u_u \cdot \nabla u + u \cdot \nabla u_t) \, dx.
\]

(4.17)

On the one hand, it follows from (3.6) and (4.5)-(4.7) that

\[
\left| \int (\rho u_t) \cdot \nabla u_t \cdot u_{tt} \right| \leq C \|\rho \tilde{x}^a\|_{L^\infty} \|\tilde{x}^{-a/2}u_t\|_{L^4} \|\tilde{x}^{-a/2}u\|_{L^4} \|\nabla u_t\|_{L^2} \\
+ C \|\tilde{x}^{(a+1)/2}\rho u_t\|_{L^2} \|\tilde{x}^{-1/2}\tilde{u}\|_{L^\infty} \|\tilde{x}^{-a/2}u_t\|_{L^4} \|\nabla u_t\|_{L^4} \\
\leq C(\delta) \|\nabla u_t\|^4_{L^2} + C(\delta) + \delta \|\nabla^2 u_t\|^2_{L^2}.
\]

(4.18)

On the other hand, Cauchy’s inequality and (4.3)-(4.7) lead to

\[
\left| \int \rho u \cdot \nabla u_t \cdot (u_{tt} + u_u \cdot \nabla u + u \cdot \nabla u_t) \, dx \right| \\
\leq \varepsilon \int \rho |u_{tt}|^2 \, dx + C(\varepsilon) \|\rho^{1/2}u\|^2_{L^\infty} \int \|\nabla u_t\|^2 \, dx \\
+ C \|\tilde{x}^{-a/2}u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\tilde{x}^{-a/2}u_t\|_{L^4} \|\nabla u_t\|_{L^4} + C \|\tilde{x}^{-1/2}u\|^2_{L^\infty} \|\nabla u_t\|^2_{L^2} \\
\leq \varepsilon \int \rho |u_{tt}|^2 \, dx + C(\varepsilon) \int \|\nabla u_t\|^2 \, dx + C(\varepsilon).
\]

(4.19)

Putting (4.18) and (4.19) into (4.17) thus shows

\[
\int \rho u \cdot \nabla u_t \cdot u_{tt} \leq -\frac{d}{dt} \int \rho u \cdot \nabla u_t \cdot u_{tt} + \varepsilon \int \rho |u_{tt}|^2 \, dx \\
+ \delta \|\nabla^2 u_t\|^2_{L^2} + C(\varepsilon, \delta) \|\nabla u_t\|^4_{L^2} + C(\varepsilon, \delta).
\]

(4.20)

Next, the Sobolev inequality and (4.6) ensure

\[
\int \lambda_t (\text{div} u_t)^2 \, dx \leq C \|\lambda_t\|_{L^2} \|\nabla u_t\|^2_{L^4} \leq C(\delta) \|\nabla u_t\|^2_{L^2} + \delta \|\nabla^2 u_t\|^2_{L^2}.
\]

(4.21)

Then, (3.17) leads to

\[
\int \lambda_t \text{div}(u_t) \, dx \\
= -\frac{d}{dt} \int \lambda_t \text{div}(u_t) \, dx + \int \lambda_t (\text{div} u_t)^2 \, dx \\
-(\beta - 1) \int (\lambda \text{div} u_t) \text{div}(u_t) \, dx + \int (\lambda u_t) \cdot (\text{div} \text{div} u_t) \, dx.
\]

(4.22)
It follows from (4.6), (4.7), and (4.4) that
\[\left| \int (\lambda \text{div} u_t) \text{div} u_t dx \right| \leq C \left( \| \lambda u \|_{L^2} \| \nabla u \|_{L^2}^2 \| \nabla u_t \|_{L^2} \right) (4.23)\]
and that
\[\left| \int (\lambda u)_t : \nabla (\text{div} u_t) dx \right| \leq C \| \lambda u \|_{L^\infty} (\| \nabla u \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2) (4.24)\]
with \( \tilde{\beta} = \min\{1, \beta\} \). Putting (4.21), (4.23), and (4.24) into (4.22) gives
\[- \int \lambda_t \text{div} u_t dx \leq -d \int \lambda_t \text{div} u_t dx + C \delta \| \nabla^2 u_t \|_{L^2}^2 + C(\delta)(1 + \| \nabla u_t \|_{L^2}^2). (4.25)\]
Finally, it follows from (3.19), (4.6), and (4.4) that
\[\int P_t \text{div} u_t dx \]
\[= d \int P_t \text{div} u_t dx - \int (P u)_t : \nabla u_t dx + (\gamma - 1) \int (P \text{div} u_t) u_t dx \]
\[\leq d \int P_t \text{div} u_t dx + C \sum \| P_t u \|_{L^2}^2 \| \nabla u_t \|_{L^2} \]
\[+ C \| P \|_{L^2} \sum \| \nabla u \|_{L^2} \| \nabla u_t \|_{L^2}^2 \]
\[\leq d \int P_t \text{div} u_t dx + \delta \| \nabla^2 u_t \|_{L^2}^2 + C(\delta)(1 + \| \nabla u_t \|_{L^2}^2). (4.26)\]
Substituting (4.16), (4.20), (4.21), (4.25), and (4.26) into (4.15) and choosing \( \varepsilon \) suitably small lead to
\[\Psi'(t) + \int \rho |u_{tt}|^2 dx \leq C \delta \| \nabla^2 u_t \|_{L^2}^2 + C(\delta)(1 + \| \nabla u_t \|_{L^2}^2. (4.27)\]
where
\[\Psi(t) \triangleq \int \left( (2\mu + \lambda)(\text{div} u_t)^2 + \mu \omega_t^2 + R^{-1} |u_t|^2 \right) dx \]
\[- 2 \int (P_t \text{div} u_t - \lambda_t \text{div} u_t - \rho u \cdot \nabla u_t \cdot \hat{u}) dx \]
satisfies
\[C_0(\mu) \| \nabla u_t \|_{L^2}^2 + R^{-1} |u_t|^2_{L^2} - C \leq \Psi(t) \leq C \| \nabla u_t \|_{L^2}^2 + R^{-1} |u_t|^2_{L^2} + C. (4.28)\]
due to the following simple fact:

\[
\left| \int (P_t \text{div} u_t - \lambda_t \text{div} u_t - \rho u \cdot \nabla u_t \cdot \hat{u}) \, dx \right| \\
\leq C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\lambda_t\|_{L^4} \|\nabla u_t\|_{L^2}^{(q-2)} \|\nabla u_t\|_{L^2} \\
+ C \|\rho^{1/2} u\|_{L^\infty} \|\nabla u_t\|_{L^2} \left( \|\rho^{1/2} u\|_{L^2} + \|\rho^{1/2} \nabla u\|_{L^2} \right) \\
\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon),
\]

which comes from (4.3)-(4.6) and (4.16).

Then, it remains to estimate the first term on the right-hand side of (4.27). In fact, we obtain from (3.17), (4.5), and (4.2) that

\[
\|\nabla \lambda_t\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|\nabla \lambda\|_{L^2} + C \|u \bar{x}^{-\delta_0}\|_{L^\infty} \|\bar{x}^{\delta_0} \nabla^2 \lambda\|_{L^2} + C \|\nabla^3 u\|_{L^2}
\]

(4.29)

where in the last inequality we have used

\[
\|\nabla u\|_{H^2} \leq C + C \|\nabla u_t\|_{L^2},
\]

due to (4.11), (4.13), and (4.4). Similar to (4.29), we have

\[
\|\nabla P_t\|_{L^2} \leq C + C \|\nabla u_t\|_{L^2}.
\]

(4.31)

Using the boundary condition (2.2), we obtain from (3.25) that

\[
\|\nabla ((2\mu + \lambda) \text{div} u_t)\|_{L^2}^2 + \mu^2 \|\nabla^2 \omega_t\|_{L^2}^2 \\
\leq \int \left| \nabla ((2\mu + \lambda) \text{div} u_t) + \mu \nabla^2 \omega_t - R^{-1} u_t \right|^2 \, dx \\
= \int |\rho u_{tt} + \rho_t \hat{u} + \rho u \cdot \nabla u_t + \rho u_t \cdot \nabla u - \nabla(\lambda_t \text{div} u) + \nabla P_t|^2 \, dx \\
\leq C \int \rho |u_{tt}|^2 \, dx + C \|\bar{x}^{-a+1/2} \rho_t\|_{L^4}^2 \|\bar{x}^{-1} \hat{u}\|_{L_{2q/(q-2)}}^2 \\
+ C \|\nabla u_t\|_{L^2}^2 + C \||\bar{x}^{-a} u_t\|_{L^4}^2 \|\nabla u\|_{L^2} + C \|\nabla \lambda_t\|_{L^2} \|\nabla u\|_{L^\infty}^2 \\
+ C \|\lambda_t\|_{L^4} \|\nabla^2 u\|_{L_{2q/(q-2)}}^2 + C \|\nabla P_t\|_{L^2}^2 \\
\leq C \int \rho |u_{tt}|^2 \, dx + C \|\nabla u_t\|_{L^2}^4 + C,
\]

where in the last inequality we have used (4.6), (4.7), and (4.29)-(4.31). This combined with (2.5) and (3.6) yields that

\[
\|\nabla^2 u_t\|_{L^2} \leq C \|\nabla^2 \omega_t\|_{L^2} + C \|\nabla \text{div} u_t\|_{L^2} \\
\leq C \|\nabla^2 \omega_t\|_{L^2} + C \|\nabla ((2\mu + \lambda) \text{div} u_t)\|_{L^2} \\
+ C \|\nabla \text{div} u_t\|_{L_{2q/(q-2)}} \|\nabla \lambda\|_{L^q} \\
\leq C \|\rho^{1/2} u_{tt}\|_{L^2} + C \|\nabla u_t\|_{L^2}^2 + C + \frac{1}{2} \|\nabla^2 u_t\|_{L^2}.
\]

(4.32)

Putting (4.32) into (4.27) and choosing δ suitably small lead to

\[
2\Psi'(t) + \int \rho |u_{tt}|^2 \, dx \leq C \|\nabla u_t\|_{L^2}^4 + C.
\]

(4.33)
Multiplying (4.33) by $t$ and integrating it over $(0, T_0)$, we obtain from Gronwall’s inequality, (4.28), and (4.3) that
\[
\sup_{0 \leq t \leq T_0} t \left( \| \nabla u_t \|_{L^2}^2 + R^{-1} \| u_t \|_{L^2}^2 \right) + \int_0^{T_0} t \| \rho^{1/2} u_t \|_{L^2}^2 \, dt \leq C,
\]
which together with (4.32) and (4.3) gives (4.14) and finishes the proof of Lemma 4.2.

**Lemma 4.3** It holds that
\[
\sup_{0 \leq t \leq T_0} \left( \| \nabla^2 \rho \|_{L^q} + \| \nabla^2 \lambda \|_{L^q} + \| \nabla^2 P \|_{L^q} \right) \leq C. \tag{4.34}
\]

**Proof.** Applying the differential operator $\nabla^2$ to both sides of (4.9), multiplying the resulting equations by $q |\nabla^2 g(\rho)|^{q-2} \nabla^2 g(\rho)$, and integrating it by parts over $B_R$ lead to
\[
\left( \| \nabla^2 g \|_{L^q} \right)_t \leq C \left( \| \nabla u \|_{L^q} \| \nabla^2 g \|_{L^q} + \| \nabla g \|_{L^q} \| \nabla^2 u \|_{L^q} + \| \nabla^2 u \|_{W^{1,q}} \right) \leq C (1 + \| \nabla^2 u \|_{L^q} ) \left( 1 + \| \nabla^2 g \|_{L^q} \right) + C \| \nabla^3 u \|_{L^q}. \tag{4.35}
\]
By (2.38), the last term on the right-hand side of (4.35) can be estimated as follows:
\[
\| \nabla^3 u \|_{L^q} \leq C \left( \| \nabla (\nabla^j \omega) \|_{L^q} + \| \nabla (\nabla \text{div} u) \|_{L^q} \right) + C \| \nabla^2 u \|_{L^q} \\
\leq C \left( \| \nabla^2 \omega \|_{L^q} + \| \nabla^2 F \|_{L^q} + \| \nabla^2 P \|_{L^q} \right) \\
+ C \left( \| \nabla u \| \| \nabla^2 \lambda \|_{L^q} + \| \nabla \lambda \|_{L^q} \right) + C + \frac{1}{4} \| \nabla^3 u \|_{L^q} \tag{4.36}
\]
(4.36) and the following simple fact:
\[
\| |\nabla^2 u| | \| \nabla \lambda \|_{L^q} \leq C \| \nabla^2 u \|_{L^q} \| \nabla \lambda \|_{L^q} \leq \varepsilon \| \nabla^3 u \|_{L^q} + C(\varepsilon)
\]
due to (4.4). For the first term on the right-hand side of (4.36), it follows from the Sobolev inequality, (3.6), (4.7), (4.4), and (4.30) that
\[
\| \nabla (\rho \bar{u}) \|_{L^q} \leq C \| \bar{x}^{-a} \nabla \bar{u} \|_{L^q} + C \| \bar{x}^{-a} \bar{u} \|_{L^q} + C \| \nabla (\bar{x}^{-a} \bar{u}) \|_{L^q} \leq C \| \bar{x}^{-a} \nabla \bar{u} \|_{L^q} + C \| \bar{x}^{-a} \bar{u} \|_{L^q} \leq C \| \nabla \bar{u} \|_{L^q} + C \| \bar{x}^{-a} |u| \| \nabla^2 u \|_{L^q} + C \| \bar{x}^{-a} |u|^2 \|_{L^q} + C \| \nabla \bar{u} \|_{L^2} + C \tag{4.37}
\]
which together with (4.14) and (4.3) yields
\[
\int_0^{T_0} \| \nabla (\rho \bar{u}) \|_{L^q} \frac{1+1/q}{1+1/q} \, dt \leq C \int_0^{T_0} \left( t \| \nabla \bar{u} \|_{L^2}^2 \right)^{1/q} \left( (1+2/q) t^{-1/2} \right)^{1+1/q} \, dt + C \tag{4.38}
\]
\[
\leq C \int_0^{T_0} \left( t \| \nabla^2 \bar{u} \|_{L^2}^2 + t^{-7/4} \right) \, dt + C \leq C.
\]
Putting (4.36) into (4.35), we obtain (4.34) from Gronwall’s inequality, (4.38), and (3.6). The proof of Lemma 4.3 is completed.

**Lemma 4.4** It holds that

\[
\sup_{0 \leq t \leq T_0} t \left( \| \nabla^3 u \|_{L^2 \cap L^q} + \| \nabla u_t \|_{H^1} + \| \nabla^2 (\rho u) \|_{L^{(q+2)/2}} \right) \\
+ \int_0^{T_0} t^2 \left( \| \nabla u_{tt} \|_{L^2}^2 + \| u_{tt} \bar{x}^{-1} \|_{L^2}^2 + R^{-1} \| u_{tt} \|_{L^2}^2 \right) dt \leq C. 
\]  

(4.39)

**Proof.** We claim that

\[
\sup_{0 \leq t \leq T_0} t^2 \| \nabla u_t \|_{H^1}^2 + \int_0^{T_0} t^2 \left( \| \nabla u_{tt} \|_{L^2}^2 + R^{-1} \| u_{tt} \|_{L^2}^2 \right) dt \leq C,
\]  

(4.40)

which together with (4.32), (4.14), and (2.6) yields that

\[
\sup_{0 \leq t \leq T_0} t \| \nabla u_t \|_{H^1} + \int_0^{T_0} t^2 \left( \| u_{tt} \bar{x}^{-1} \|_{L^2}^2 \right) dt \leq C. 
\]  

(4.41)

This combined with (4.30), (4.36), (4.37), and (4.34) leads to

\[
\sup_{0 \leq t \leq T_0} t \| \nabla^3 u \|_{L^2 \cap L^q} \leq C,
\]  

(4.42)

which, along with (3.6), (4.34), and (4.2), gives

\[
t \| \nabla^2 (\rho u) \|_{L^{(q+2)/2}}^2 \\
\leq C t \| \nabla \rho \|_{L^{(q+2)/2}} + C t \| \nabla \|_{L^{(q+2)/2}} + C t \| \rho \nabla^2 u \|_{L^{(q+2)/2}} \\
\leq C t \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^{(q+2)/2}} + C t \| \rho \nabla^2 u \|_{L^{(q+2)/2}} \\
+ C t \| \rho \|_{L^2} \| \nabla u \|_{L^{(q+2)/2}} + 2 C t \| \nabla^2 u \|_{L^{(q+2)/2}} \leq C. 
\]  

(4.43)

We thus directly obtain (4.39) from (4.40)-(4.43).

It remains to prove (4.40). In fact, differentiating (3.25) with respect to \( t \) leads to

\[
\rho u_{ttt} + \rho u \cdot \nabla u_t - \nabla ((2 \mu + \lambda) \text{div} u_t) - \mu \nabla \omega_t + R^{-1} u_{tt} \\
= 2 \nabla (\lambda \text{div} u_t) + \nabla (\mu \text{div} u) + \text{div} (\rho u) u_{tt} + \text{div} (\mu u) u_{tt} \\
- 2 (\rho u)_t \cdot \nabla u_t - \rho u_t u \cdot \nabla u - 2 \rho u_t u \cdot \nabla u - \rho u_{tt} \cdot \nabla u - \nabla P_{tt},
\]

which, multiplied by \( u_{tt} \) and integrated by parts over \( B_R \), yields

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_{tt}|^2 dx + \int ((2 \mu + \lambda) (\text{div} u_t)^2 + \mu \omega_t^2 + R^{-1} |u_{tt}|^2) dx \\
= - 2 \int \lambda \text{div} u_t \text{div} u_{tt} dx - \int \lambda u_t \text{div} u \text{div} u_{tt} dx \\
- 4 \int \rho u \cdot \nabla u_t \cdot u_{tt} dx - \int (\rho u)_t \cdot (\nabla (u_t \cdot u_t) + 2 \nabla u_t \cdot u_t) dx \\
- \int (\rho u_t)_t \cdot \nabla (u \cdot u_{tt}) dx - 2 \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \\
- \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} dx + \int P_{tt} \text{div} u_{tt} dx \triangleq \sum_{i=1}^{8} J_i.
\]
We estimate each $J_i (i = 1, \cdots , 8)$ as follows:

First, we deduce from (4.6) that

$$|J_1| \leq C ||\lambda_t||_{L^0} ||\nabla u_t||_{L^{2q/(q-2)}} ||\nabla u_{tt}||_{L^2} \leq \varepsilon ||\nabla u_{tt}||_{L^2}^2 + C(\varepsilon) ||\nabla u_t||_{H^1}^2. \quad (4.45)$$

Next, the Cauchy inequality gives

$$|J_2| \leq \varepsilon ||\nabla u_{tt}||_{L^2}^2 + C(\varepsilon) ||\lambda_t||_{L^2}^2 ||\nabla u||_{L^\infty} \leq \varepsilon ||\nabla u_{tt}||_{L^2}^2 + C(\varepsilon) ||\lambda_t||_{L^2}^2 ||\nabla u||_{L^4}, \quad (4.46)$$

where in the second inequality we have used (4.30). Using (3.17), we estimate the last term on the right-hand side of (4.46) as follows:

$$||\lambda_t||_{L^2} \leq C||u_t||||\nabla u||_{L^2} + C||u||||\nabla \lambda_t||_{L^2} + C||\lambda_t \nabla u||_{L^2} + C||\lambda_t \div u||_{L^2} \leq C\|\|\nabla u\||\|_{L^2} + C\|\nabla u\||_{L^2} + C\|\lambda_t \div u||_{L^2} \leq C + ||\nabla u||_{L^2},$$

where in the last inequality we have used (4.1)-(4.7), (3.6), and the following simple fact that

$$||\nabla^{\beta_0/\gamma} \lambda_t||_{L^2} \leq C ||\nabla^{\beta_0/\gamma} u_t||_{L^2} + C ||\nabla^{\beta_0/\gamma} \lambda_t||_{L^2} + C ||\nabla^{\beta_0/\gamma} \lambda_t ||_{L^2} \leq C \|\|\lambda_t \div u||\|_{L^2}$$

due to (4.5), (4.2), (4.1), and (3.6). Putting (4.47) into (4.46) gives

$$|J_2| \leq \varepsilon ||\nabla u_{tt}||_{L^2}^2 + C(\varepsilon)(1 + ||\nabla u_t||_{L^2}^4). \quad (4.48)$$

Next, the combination of the Cauchy inequality with (4.5) yields that

$$|J_3| \leq C||\rho^{1/2} u||_{L^\infty} ||\rho^{1/2} u||_{L^2}^2 ||\nabla u||_{L^2} \leq \varepsilon ||\nabla u_{tt}||_{L^2}^2 + C(\varepsilon)||\rho^{1/2} u_{tt}||_{L^2}^2. \quad (4.49)$$

Next, noticing that (4.5)-(4.7) lead to

$$||\bar{x}(\rho u)_t||_{L^9} \leq C ||\bar{x}|\rho||_{L^\infty} + C ||\bar{x}|\rho|u||_{L^9} \leq C ||\rho_t \bar{x}^{(a+1)/2}||_{L^a} ||\bar{x}^{(1-a)/2}||_{L^\infty} + C ||\rho \bar{x}^{a/2}||_{L^a} ||u||_{L^\infty} \leq C + ||\nabla u_t||_{L^2}, \quad (4.50)$$

we obtain from Holder’s inequality, (4.6), (2.7), and (3.12) that

$$|J_4| \leq C ||\bar{x}(\rho u)_t||_{L^9} \left(||\bar{x}^{-1} u_{tt}||_{L^{2q/(q-2)}} ||\nabla u||_{L^2} + ||\bar{x}^{-1} u_{tt}||_{L^{2q/(q-2)}} ||\nabla u_t||_{L^2} \right) \leq C \left(1 + ||\nabla u_t||_{L^2}^2 \right) \left(||\rho^{1/2} u_{tt}||_{L^2} + ||\nabla u_{tt}||_{L^2} \right) \leq C(\varepsilon) \left(1 + ||\nabla u_t||_{L^2}^2 \right) + \varepsilon \left(||\rho^{1/2} u_{tt}||_{L^2} + ||\nabla u_{tt}||_{L^2}^2 \right). \quad (4.51)$$
Then, it follows from (4.50), (4.5), (4.4), and (3.12) that

\[
|J_5| \leq C \int |(\rho u_t)||(u||\nabla^2 u||u_t| + |u||\nabla u||\nabla u_t| + |\nabla u|^2|u_t|)|dx \\
\leq C\|\bar{x}(\rho u_t)\|_{L^2}\|\bar{x}^{-1/4}u\|_{L^\infty}\|\nabla^2 u\|_{L^2}\|\bar{x}^{-1/4}u_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
+ C\|\bar{x}(\rho u_t)\|_{L^3}\|\bar{x}^{-1}u\|_{L^\infty}\|\nabla u\|_{L^{2q/(q-2)}}\|\nabla u\|_{L^2} \\
+ C\|\bar{x}(\rho u_t)\|_{L^3}\|\nabla u\|_{L^2}\|\bar{x}^{-1}u_t\|_{L^2}\|\nabla u\|_{L^2} \\
\leq C (1 + \|\nabla u_t\|_{L^2}) \left(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right) \\
\leq C(\varepsilon) (1 + \|\nabla u_t\|_{L^2}^2) + \varepsilon \left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\right).
\]

(4.52)

Next, Cauchy’s inequality together with (4.6), (4.7), and (3.12) gives

\[
|J_6| \leq C \int |\rho_t||u_t||\nabla u||u_t||dx \\
\leq C\|\bar{x}\rho_t\|_{L^2}\|\bar{x}^{-1/4}u_t\|_{L^{4q/(q-2)}}\|\nabla u\|_{L^2}\|\bar{x}^{-1/4}u_t\|_{L^{4q/(q-2)}} \\
\leq C (1 + \|\nabla u_t\|_{L^2}) \left(\|\rho^{1/2}u_t\|_{L^2} + \|\nabla u_t\|_{L^2}\right) \\
\leq C(\varepsilon) (1 + \|\nabla u_t\|_{L^2}^2) + \varepsilon \left(\|\rho^{1/2}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\right).
\]

(4.53)

Finally, similar to (4.47), we have

\[
\|P_t\|_{L^2} \leq C(1 + \|\nabla u_t\|_{L^2}),
\]

which together with direct calculations gives

\[
|J_7| + |J_8| \leq C\|\nabla u\|_{L^\infty} \int \rho|u_t|^2dx + \varepsilon \int |\nabla u_t|^2dx + C(\varepsilon)\|P_t\|_{L^2}^2 \\
\leq C\|\nabla u\|_{L^\infty}\|\rho^{1/2}u_t\|_{L^2}^2 + \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon)(1 + \|\nabla u_t\|_{L^2}^2).
\]

(4.54)

Substituting (4.45), (4.48), (4.49), and (4.51)-(4.54) into (4.41), choosing \(\varepsilon\) suitably small, and multiplying the resulting inequality by \(\beta^2\), we obtain (4.40) after using Gronwall’s inequality and (4.14). The proof of Lemma 4.4 is finished.

5 Proofs of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we will only deal with the case that \(\beta > 0\), since the same procedure can be applied to the case that \(\beta = 0\) after some small modifications.

Proof of Theorem 1.1 Let \((\rho_0, u_0)\) be as in Theorem 1.1. Without loss of generality, assume that

\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]

which implies that there exists a positive constant \(N_0\) such that

\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}.
\]

(5.1)
We construct $\rho_0^R = \hat{\rho}_0^R + R^{-1}e^{-|x|^2}$ where $0 \leq \hat{\rho}_0^R \in C_0^\infty(\mathbb{R}^2)$ satisfies that
\[
\int_{B_{N_0}} \hat{\rho}_0^R \, dx \geq 1/2, \quad (5.2)
\]
and that
\[
\begin{aligned}
\bar{\omega}^0 \rho_0^R &\to \bar{\omega}^0 \rho_0 \quad \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \\
\lambda(\hat{\rho}_0^R) &\to \lambda(\rho_0), \quad \bar{\omega}^0 \nabla \lambda(\hat{\rho}_0^R) \to \bar{\omega}^0 \nabla \lambda(\rho_0) \quad \text{in } L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2),
\end{aligned} \quad (5.3)
\]
as $R \to \infty$.

Since $\nabla u_0 \in L^2(\mathbb{R}^2)$, choosing $v_i^R \in C_0^\infty(B_R)(i = 1, 2)$ such that
\[
\lim_{R \to \infty} \|v_i^R - \partial_i u_0\|_{L^2(\mathbb{R}^2)} = 0, \quad i = 1, 2, \quad (5.4)
\]
we consider the unique smooth solution $u_0^R$ of the following elliptic problem:
\[
\begin{aligned}
-\Delta u_0^R + R^{-1} u_0^R &= -\rho_0^R u_0^R + \sqrt{\rho_0^R} h_R - \partial_i v_i^R, \quad \text{in } B_R, \\
\rho_0^R \cdot n &= 0, \quad \text{on } \partial B_R,
\end{aligned} \quad (5.5)
\]
where $h_R = (\sqrt{\rho_0^R} u_0) \ast j_{1/R}$ with $j_\delta$ being the standard mollifying kernel of width $\delta$.

Extending $u_0^R$ to $\mathbb{R}^2$ by defining 0 outside $B_R$ and denoting $u_0^R \triangleq u_0^R \varphi_R$ with $\varphi_R$ as in (3.9), we claim that
\[
\lim_{R \to \infty} \left( \|\nabla (u_0^R - u_0)\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0^R u_0^R} - \sqrt{\rho_0 u_0}\|_{L^2(\mathbb{R}^2)} \right) = 0. \quad (5.6)
\]

In fact, multiplying (5.5) by $u_0^R$ and integrating the resulting equation over $B_R$ lead to
\[
\int_{B_R} \rho_0^R \nabla u_0^R \cdot \nabla u_0^R \, dx + \int_{B_R} (\nabla u_0^R \cdot \nabla u_0^R) \, dx + \int_{B_R} (\nabla \rho_0^R \cdot \nabla u_0^R) \, dx
\]
\[
\leq \|\nabla u_0^R\|_{L^2(B_R)} \|h_R\|_{L^2(B_R)} + C \|v_i^R\|_{L^2} \|\partial_i u_0^R\|_{L^2}
\]
\[
\leq \varepsilon \|\nabla u_0^R\|_{L^2(B_r)}^2 + \varepsilon \int_{B_R} \rho_0^R |u_0^R|^2 \, dx + C(\varepsilon),
\]
which implies
\[
R^{-1} \int_{B_R} |u_0^R|^2 \, dx + \int_{B_R} \rho_0^R |u_0^R|^2 \, dx + \int_{B_R} |\nabla u_0^R|^2 \, dx \leq C, \quad (5.7)
\]
for some $C$ independent of $R$.

We deduce from (5.1) and (5.3) that there exists a subsequence $R_j \to \infty$ and a function $w_0 \in \{ w_0 \in H_{\text{loc}}^1(\mathbb{R}^2) \mid \sqrt{\rho_0} w_0 \in L^1(\mathbb{R}^2), \nabla w_0 \in L^2(\mathbb{R}^2) \}$ such that
\[
\begin{aligned}
\sqrt{\rho_0^R} w_0^R \to \sqrt{\rho_0} w_0 \text{ weakly in } L^1(\mathbb{R}^2), \\
\nabla w_0^R \to \nabla w_0 \text{ weakly in } L^2(\mathbb{R}^2).
\end{aligned} \quad (5.8)
\]
It follows from (5.5) and (5.7) that $u_0^R$ satisfies
\[
- \Delta u_0^R + R^{-1} u_0^R = -\rho_0^R u_0^R + \sqrt{\rho_0^R} \varphi_R - \partial_i v_i^R \varphi_R + R^{-1} F^R, \quad (5.9)
\]
which combined with (5.8) implies

$$\int_{\mathbb{R}^2} \partial_t (w_0 - u_0) \cdot \partial_t \psi \, dx + \int_{\mathbb{R}^2} \rho_0 (w_0 - u_0) \cdot \psi \, dx = 0,$$

which yields that

$$w_0 = u_0. \quad (5.10)$$

Furthermore, we get from (5.9) that

$$\limsup_{R_j \to \infty} \int_{\mathbb{R}^2} \left( |\nabla w_0| \right)^2 + \rho_0 |w_0'|^2 \, dx \leq \int_{\mathbb{R}^2} \left( |\nabla u_0| \right)^2 + \rho_0 |u_0|^2 \, dx,$$

which combined with (5.8) implies

$$\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla w_0| \, dx = \lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla u_0|^2 \, dx, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0 |w_0'|^2 \, dx = \int_{\mathbb{R}^2} \rho_0 |u_0|^2 \, dx.$$

This, along with (5.10) and (5.8), gives (5.9).

Then, in terms of Lemma 2.1 the initial-boundary-value problem (2.2) with the initial data \((\rho^R_0, u^R_0)\) has a classical solution \((\rho^R, u^R)\) on \(B_R \times [0, T_R]\). Moreover, Proposition 3.1 shows that there exists a \(T_0\) independent of \(R\) such that (3.10) holds for \((\rho^R, u^R)\).

Extending \((\rho^R, u^R)\) by zero on \(\mathbb{R}^2 \setminus B_R\) and denoting

$$\tilde{\rho}^R \triangleq (\varphi_R)^{4/\beta} \rho^R, \quad \tilde{\varphi}^R \triangleq \varphi_R u^R,$$

with \(\varphi_R\) as in (3.9) and \(\tilde{\beta} = \min\{\beta, 1\}\), we first deduce from (3.10) that

$$\sup_{0 \leq t \leq T_0} \left( \left\| \sqrt{\tilde{\rho}^R} \tilde{w}^R \right\|_{L^2(\mathbb{R}^2)} + \left\| \nabla w^R \right\|_{L^2(\mathbb{R}^2)} \right) \leq C + C \sup_{0 \leq t \leq T_0} \left( \left\| \nabla u^R \right\|_{L^2(B_R)} + C R^{-1} \left\| u^R \right\|_{L^2(B_R)} \right) \quad (5.11)$$

$$\leq C,$$

and that

$$\sup_{0 \leq t \leq T_0} \left( \left\| \tilde{\rho}^R \tilde{x}^a \right\|_{L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)} + \left\| \lambda(\tilde{\rho}^R) \right\|_{L^2(\mathbb{R}^2)} \right) \leq \sup_{0 \leq t \leq T_0} \left( \left\| \rho^R \tilde{x}^a \right\|_{L^1(B_R) \cap L^\infty(B_R)} + \left\| \lambda(\rho^R) \right\|_{L^2(B_R)} \right) \leq C. \quad (5.12)$$

Next, for \(p \in [2, q]\), it follows from (3.40) and (3.6) that

$$\sup_{0 \leq t \leq T_0} \left( \left\| \nabla (\tilde{x}^a \lambda(\tilde{\rho}^R)) \right\|_{L^p(\mathbb{R}^2)} + \left\| \tilde{x}^a \nabla \lambda(\tilde{\rho}^R) \right\|_{L^p(\mathbb{R}^2)} \right)$$

$$\leq C \sup_{0 \leq t \leq T_0} \left( \left\| \nabla (\tilde{x}^a \lambda(\rho^R)) \right\|_{L^p(B_R)} + \left\| \tilde{x}^a \nabla \lambda(\rho^R) \right\|_{L^p(B_R)} \right)$$

$$+ C \sup_{0 \leq t \leq T_0} \left| \tilde{x}^a \nabla \lambda(\rho^R) \right|_{L^p(B_R)}$$

$$\leq C + C \sup_{0 \leq t \leq T_0} \left| \tilde{x}^a \rho^R \right|_{L^{2p/(\beta + 1)}(B_R)} \left\| \nabla \varphi_R \right\|_{L^{2p/(2 - \tilde{\beta})}(B_R)} \leq C,$$
and that
\[
\sup_{0 \leq t \leq T_0} \left( \| \nabla (\bar{x}^a \bar{\rho}^R) \|_{L^p(\mathbb{R}^2)} + \| \bar{x}^a \nabla \bar{\rho}^R \|_{L^p(\mathbb{R}^2)} \right)
\leq C \sup_{0 \leq t \leq T_0} \left( \| \bar{x}^a \nabla \bar{\rho}^R \|_{L^p(B_R)} + \| \bar{x}^a \rho^R \nabla R \|_{L^p(B_R)} + \| \rho^R \nabla \bar{x}^a \|_{L^p(B_R)} \right)
\leq C + C \sup_{0 \leq t \leq T_0} \| \bar{x}^a \rho^R \|_{L^p(B_R)} \leq C.
\] (5.14)

Then, it follows from (3.6) and (5.33) that
\[
\int_0^{T_0} \left( \| \nabla^2 w^R \|_{L^q(\mathbb{R}^2)}^{(q+1)/q} + t \| \nabla^2 w^R \|_{L^q(\mathbb{R}^2)}^2 + \| \nabla^2 w^R \|_{L^2(\mathbb{R}^2)}^2 \right) dt \leq C,
\] (5.15)
and that for \( p \in [2, q] \),
\[
\int_0^{T_0} \| \bar{x}^a \rho^R \|_{L^p(\mathbb{R}^2)}^2 dt \leq C \int_0^{T_0} \left( \| \bar{x} \|_{L^p(\mathbb{R}^2)} \| \nabla \rho^R \|_{L^p(\mathbb{R}^2)}^2 + \| \bar{x} \rho^R \div \nabla R \|_{L^p(\mathbb{R}^2)}^2 \right) dt
\leq C \int_0^{T_0} \| \bar{x}^{1-a} u^R \|_{L^\infty(B_R)}^2 \| \bar{x}^a \rho^R \|_{L^p(B_R)}^2 dt + C
\leq C.
\] (5.16)

Next, one derives from (3.21) and (3.6) that
\[
\sup_{0 \leq t \leq T_0} t \int_{\mathbb{R}^2} \bar{\rho}^R |u_t|^2 dx + \int_0^{T_0} t \| \nabla w^R \|_{L^2(\mathbb{R}^2)}^2 dt
\leq C + C \int_0^{T_0} t \left( \| \nabla u^R \|_{L^2(B_R)}^2 + R^{-2} \| u^R \|_{L^2(B_R)}^2 \right) dt
\leq C.
\] (5.17)

With all these estimates (5.11)-(5.17) at hand, we find that the sequence \((\bar{\rho}^R, w^R)\) converges, up to the extraction of subsequences, to some limit \((\rho, u)\) in the obvious weak sense, that is, as \( R \to \infty \), we have
\[
R^{-1} w^R \to 0, \text{ in } L^2(\mathbb{R}^2 \times (0, T_0)),
\] (5.18)
\[
\bar{x} \bar{\rho}^R \to \bar{x} \rho, \text{ in } C(\overline{B_N} \times [0, T_0]), \text{ for any } N > 0,
\] (5.19)
\[
\bar{x}^a \bar{\rho}^R \to \bar{x}^a \rho, \text{ weakly * in } L^\infty(0, T_0; H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2)),
\] (5.20)
\[
\nabla (\bar{x}^{0a} \lambda(\bar{\rho})) \to \nabla (\bar{x}^{0a} \lambda(\rho)), \text{ weakly * in } L^\infty(0, T_0; L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)),
\] (5.21)
\[
\sqrt{\bar{\rho}^R} w^R \to \sqrt{\rho} u, \nabla w^R \to \nabla u, \text{ weakly * in } L^\infty(0, T_0; L^2(\mathbb{R}^2)),
\] (5.22)
\[
\nabla^2 w^R \to \nabla^2 u, \text{ weakly in } L^{(q+1)/q}(0, T_0; L^q(\mathbb{R}^2)) \cap L^2(\mathbb{R}^2 \times (0, T_0)),
\] (5.23)
\[
t^{1/2} \nabla w^R \to t^{1/2} \nabla^2 u, \text{ weakly in } L^2(0, T_0; L^q(\mathbb{R}^2)),
\] (5.24)
\[
\sqrt{t} \sqrt{\bar{\rho}^R} w^R_t \to \sqrt{t} \sqrt{\rho} u_t, \nabla w^R \to \nabla u, \text{ weakly * in } L^\infty(0, T_0; L^2(\mathbb{R}^2)),
\] (5.25)
\[
\sqrt{t} \nabla w^R_t \to \sqrt{t} \nabla u_t, \text{ weakly in } L^2(\mathbb{R}^2 \times (0, T_0)),
\] (5.26)

with
\[
\bar{x}^a \rho \in L^\infty(0, T_0; L^1(\mathbb{R}^2)), \inf_{0 \leq t \leq T_0} \int_{B_{2N_0}} \rho(x, t) dx \geq \frac{1}{4},
\] (5.27)
Next, for any function \( \phi \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times [0,T_0]) \), we take \( \phi(R) \) as test function in the initial-boundary-value problem (2.2) with the initial data \((\rho_0, u_0)\). Then letting \( R \to \infty \), it follows from (5.13)-(5.27) that \((\rho, u)\) is a strong solution of (1.1)-(1.5) on \( \mathbb{R}^2 \times (0,T] \) satisfying (1.10) and (1.11). The proof of the existence part of Theorem 1.1 is finished.

It only remains to prove the uniqueness of the strong solutions satisfying (1.10) and (1.11). We only treat the case \( \beta > 0 \), since the procedure can be adapted to the case \( \beta = 0 \) after some small modifications. Let \((\rho, u)\) and \((\bar{\rho}, \bar{u})\) be two strong solutions satisfying (1.10) and (1.11) with the same initial data. Subtracting the momentum equations satisfied by \((\rho, u)\) and \((\bar{\rho}, \bar{u})\) yields

\[
\rho U_t + \rho u \cdot \nabla U - \mu \Delta U - \nabla ((\mu + \lambda(\rho))\text{div}U)
= -\rho U \cdot \nabla \bar{u} - H(\bar{u}_t + \bar{u} \cdot \nabla \bar{u}) - \nabla (P(\rho) - P(\bar{\rho})) + \nabla ((\lambda(\rho) - \lambda(\bar{\rho}))\text{div}\bar{u}),
\tag{5.28}
\]

with

\[
H \triangleq \rho - \bar{\rho}, \quad U \triangleq u - \bar{u}.
\]

Since \( \mu + \lambda \geq 0 \), multiplying (5.28) by \( U \) and integrating by parts lead to

\[
\frac{d}{dt} \int \rho|U|^2 dx + 2\mu \int |\nabla U|^2 dx
\leq C \|
\]

\[
|\nabla \bar{u}| L^2 \int \rho|U|^2 dx + C \int |H||U| (|\bar{u}_t| + |\bar{u}||\nabla \bar{u}|) dx
+ C (\|P(\rho) - P(\bar{\rho})\|_{L^2} + \|\nabla \bar{u}\|_{L^\infty} \lambda(\rho) - \lambda(\bar{\rho})) \|\text{div}U\|_{L^2}
\triangleq C \|
\]

\[
\int \rho|U|^2 dx + K_1 + K_2.
\tag{5.29}
\]

We first estimate \(K_1\). Holder’s inequality shows that for \( r \in (1,a) \),

\[
K_1 \leq C \|\nabla \bar{u}\|_{L^2} \|U\|_{L^r} \left( \|\nabla \bar{u}\|_{L^\infty}^r \right)^{1/2} \left( \|\bar{u}_t\|_{L^r}^r \|\nabla \bar{u}\|_{L^\infty} \right)^{1/2}
\leq C(\varepsilon) \left( \|\sqrt{\rho} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^\infty}^2 \right) \|H\bar{x}\|^2_L

\tag{5.30}
\]

where in the second inequality we have used (2.7) and (1.11). Then, subtracting the mass equation for \((\rho, u)\) and \((\bar{\rho}, \bar{u})\) gives

\[
H_t + \bar{u} \cdot \nabla H + H \text{div}\bar{u} + \rho \text{div}U + U \cdot \nabla \rho = 0.
\tag{5.31}
\]

Multiplying (5.31) by \( 2H\bar{x}^2 \) and integrating by parts lead to

\[
\|H\bar{x}\|^2_{L^2} \leq C \left( \|\nabla \bar{u}\|_{L^\infty} + \|\nabla \bar{u}\|_{L^\infty} \right) \|H\bar{x}\|^2_{L^2} + C \|\nabla \bar{u}\|_{L^\infty} \|\text{div}U\|_{L^2} + \|H\bar{x}\|^2_{L^2}

\tag{5.32}
\]

where in the second inequality we have used (1.11), (3.12), and (3.36). This combined with Gronwall’s inequality yields that for all \( 0 \leq t \leq T_0 \)

\[
\|H\bar{x}\|^2_{L^2} \leq C \int_0^t (\|\nabla U\|_{L^2} + \|\sqrt{\rho} U\|_{L^2}^2) ds.
\tag{5.32}
\]
As observed by Germain [8], putting (5.32) into (5.30) leads to
\[
K_1 \leq C(\varepsilon) \left(1 + t\|\nabla \bar{u}_t\|_{L^2}^2 + t\|\nabla^2 \bar{u}\|_{L^2}^2\right) \int_0^t \left(\|\nabla U\|_{L^2}^2 + \|\sqrt{\rho} U\|_{L^2}^2\right) ds \tag{5.33}
\]

Next, we will estimate $K_2$. In fact, one deduces from (3.17) that
\[
(\lambda(\rho) - \lambda(\bar{\rho}))_t + \bar{u} \cdot \nabla (\lambda(\rho) - \lambda(\bar{\rho})) + U \cdot \nabla \lambda(\rho) + \beta(\lambda(\rho) - \lambda(\bar{\rho})) \text{div} \bar{u} + \beta \lambda(\rho) \text{div} U = 0,
\]
which gives
\[
(\|\lambda(\rho) - \lambda(\bar{\rho})\|_{L^2})_t \leq C(1 + \|\nabla \bar{u}\|_{L^\infty}) \|\lambda(\rho) - \lambda(\bar{\rho})\|_{L^2} + C\|U \cdot \nabla \lambda(\rho)\|_{L^2} + C\|\nabla U\|_{L^2}. \tag{5.34}
\]

It follows from (1.10), (1.11), and (2.7) that
\[
\|U \cdot \nabla \lambda(\rho)\|_{L^2} \leq \|U \bar{x}^{-\theta_0\|L^2_{n/(q-2)\theta_0}}\|\bar{x}^{-\theta_0} \nabla \lambda(\rho)\|_{L^2_{n/(q-2)\theta_0}} \leq C\|\nabla U\|_{L^2} + C\|\sqrt{\rho} U\|_{L^2},
\]
which together with (5.34) and Gronwall’s inequality gives
\[
\|\lambda(\rho) - \lambda(\bar{\rho})\|_{L^2} \leq C \int_0^t \left(\|\nabla U\|_{L^2} + \|\sqrt{\rho} U\|_{L^2}\right) ds. \tag{5.35}
\]

Similarly, we have
\[
\|P(\rho) - P(\bar{\rho})\|_{L^2} \leq C \int_0^t \left(\|\nabla U\|_{L^2} + \|\sqrt{\rho} U\|_{L^2}\right) ds,
\]
which combined with (5.35) shows
\[
K_2 \leq \varepsilon\|\nabla U\|_{L^2}^2 + C(\varepsilon) \left(1 + t\|\nabla^2 \bar{u}\|_{L^2}^2\right) \int_0^t \left(\|\nabla U\|_{L^2}^2 + \|\sqrt{\rho} U\|_{L^2}^2\right) ds. \tag{5.36}
\]

Denoting
\[
G(t) \triangleq \|\sqrt{\rho} U\|_{L^2}^2 + \int_0^t \left(\|\sqrt{\rho} U\|_{L^2}^2 + \mu \|\nabla U\|_{L^2}^2\right) ds,
\]
putting (5.33) and (5.36) into (5.29) and choosing $\varepsilon$ suitably small lead to
\[
G'(t) \leq C \left(1 + \|\nabla \bar{u}\|_{L^\infty} + t\|\nabla^2 \bar{u}\|_{L^2}^2 + t\|\nabla \bar{u}_t\|_{L^2}^2\right) G,
\]
which together with Gronwall’s inequality and (1.10) yields $G(t) = 0$. Hence, $U(x, t) = 0$ for almost everywhere $(x, t) \in \mathbb{R}^2 \times (0, T_0)$. Then, (5.32) implies that $H(x, t) = 0$ for almost everywhere $(x, t) \in \mathbb{R}^2 \times (0, T_0)$. The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 Let $(\rho_0, u_0)$ be as in Theorem 1.2. Without loss of generality, assume that
\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]
which implies that there exists a positive constant $N_0$ such that (5.1) holds. We construct \( \rho^R_0 = \rho_0^R + R^{-1}e^{-|x|^2} \) where \( 0 \leq \rho^R_0 \in C_0^\infty(\mathbb{R}^2) \) satisfies (5.2), (5.3), and

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\nabla^2 \rho^R_0 \rightarrow \nabla^2 \rho_0, \quad \nabla^2 \lambda(\rho^R_0) \rightarrow \nabla^2 \lambda(\rho_0), \quad \nabla^2 P(\rho^R_0) \rightarrow \nabla^2 P(\rho_0), \\
\tilde{x}^{\delta R_0} \nabla^2 \rho^R_0 \rightarrow \tilde{x}^{\delta R_0} \nabla^2 \rho_0, \quad \tilde{x}^{\delta R_0} \nabla^2 \lambda(\rho^R_0) \rightarrow \tilde{x}^{\delta R_0} \nabla^2 \lambda(\rho_0), \\
\tilde{x}^{\delta R_0} \nabla^2 P(\rho^R_0) \rightarrow \tilde{x}^{\delta R_0} \nabla^2 P(\rho_0),
\end{array} \right. \\
\text{in } L^q(\mathbb{R}^2),
\end{aligned}
\]

as \( R \rightarrow \infty. \)

Then, we consider the unique smooth solution \( u^R_0 \) of the following elliptic problem:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
-\mu \Delta u^R_0 - \nabla \left( (\mu + \lambda(\rho^R_0)) \text{div} u^R_0 \right) + \nabla P(\rho^R_0) + R^{-1}u^R_0
\\
= -\rho^R_0 u_0 + \sqrt{\rho^R_0} h^R,
\\
u^R_0 \cdot n = 0, \quad \text{rot} u^R_0 = 0,
\end{array} \right. \\
&\text{in } B_R, \quad \text{on } \partial B_R,
\end{aligned}
\]

where \( h^R = (\sqrt{\rho_0 u_0 + g}) * j_{1/R} \) with \( j_\delta \) being the standard mollifying kernel of width \( \delta \). Multiplying (5.38) by \( u^R_0 \) and integrating the resulting equation over \( B_R \) lead to

\[
\begin{aligned}
&\int_{B_R} (\rho^R_0 + R^{-1}) |u^R_0|^2 dx + \mu \int_{B_R} |\text{rot} u^R_0|^2 dx + \int_{B_R} (2\mu + \lambda(\rho^R_0)) (\text{div} u^R_0)^2 dx \\
&\leq \int_{B_R} P(\rho^R_0) |\text{div} u^R_0|^2 dx + \| \rho^R_0 u_0 \|_{L^2(B_R)} \| h^R \|_{L^2(B_R)} \\
&\leq \epsilon \| \nabla u^R_0 \|_{L^2(B_R)}^2 + \epsilon \int_{B_R} \rho^R_0 |u^R_0|^2 dx + C(\epsilon),
\end{aligned}
\]

which implies

\[
R^{-1} \int_{B_R} |u^R_0|^2 dx + \int_{B_R} \rho^R_0 |u^R_0|^2 dx + \int_{B_R} |\nabla u^R_0|^2 dx \leq C,
\]

for some \( C \) independent of \( R \). By (2.8), we have

\[
\begin{aligned}
\| \nabla^2 u^R_0 \|_{L^2(B_R)} &\leq C \| \nabla \text{rot} u^R_0 \|_{L^2(B_R)} + C \| \nabla \left( (2\mu + \lambda(\rho^R_0)) \text{div} u^R_0 \right) \|_{L^2(B_R)} \\
&+ C \| \nabla \lambda(\rho^R_0) \|_{L^2(B_R)} + C \| \nabla (\mu \Delta u^R_0) + \nabla \left( (\mu + \lambda(\rho^R_0)) \text{div} u^R_0 \right) - R^{-1}u^R_0 \|_{L^2} \\
&\leq C \| \mu \Delta u^R_0 + \nabla \left( (\mu + \lambda(\rho^R_0)) \text{div} u^R_0 \right) - R^{-1}u^R_0 \|_{L^2} \\
&+ C \| \nabla \lambda(\rho^R_0) \|_{L^2(B_R)} (1 + \| \nabla^2 u^R_0 \|_{L^2(B_R)}^{1/2}) \\
&\leq C \| \rho^R_0 u_0 \|_{L^2(B_R)} + C \| P(\rho^R_0) \|_{L^2(B_R)} \\
&+ C \| \sqrt{\rho^R_0} h^R \|_{L^2(B_R)} + C + \frac{1}{2} \| \nabla^2 u^R_0 \|_{L^2(B_R)} \\
&\leq C + \frac{1}{2} \| \nabla^2 u^R_0 \|_{L^2(B_R)}^2,
\end{aligned}
\]

which gives

\[
\| \nabla^2 u^R_0 \|_{L^2(B_R)} \leq C.
\]

Next, extending \( u^R_0 \) to \( \mathbb{R}^2 \) by defining 0 outside \( B_R \) and denoting \( u^R_0 \triangleq u^R_0 \varphi_R \) with \( \varphi_R \) as in (3.9), we deduce from (5.39) and (5.40) that

\[
\| \nabla u^R_0 \|_{H^1(\mathbb{R}^2)} \leq C,
\]
which together with (5.39) and (5.37) yields that there exists a subsequence \( R_j \to \infty \)
and a function \( w_0 \in \{ w_0 \in H^2_{\text{loc}}(\mathbb{R}^2) | \sqrt{\rho_0}w_0 \in L^2(\mathbb{R}^2), \nabla w_0 \in H^1(\mathbb{R}^2) \} \) such that
\[
\begin{cases}
\sqrt{\rho_0}w_0 \to \sqrt{\rho_0}w_0 \text{ weakly in } L^2(\mathbb{R}^2), \\
\nabla w_0^{R_j} \to \nabla w_0 \text{ weakly in } H^1(\mathbb{R}^2).
\end{cases}
\] (5.41)

It follows from (5.38) that \( w_0^R \) satisfies
\[
-\mu \Delta w_0^R - \nabla \left( (\mu + \lambda(\rho_0^R)) \text{div} w_0^R \right) + \nabla (P(\rho_0^R) \varphi_R) + R^{-1} w_0^R
= -\rho_0^R w_0^R + \sqrt{\rho_0^R} R \varphi_R + R^{-1} F^R,
\] with \( \| F^R \|_{L^2(\mathbb{R}^2)} \leq C \) due to (5.39) and (5.40). Thus, one can deduce from (5.42), (5.37), and (5.41) that \( w_0 \) satisfies
\[
\mu \Delta w_0 - \nabla \left( (\mu + \lambda(\rho_0)) \text{div} w_0 \right) + \nabla P(\rho_0) + \rho_0 w_0 = \rho_0 u_0 + \sqrt{\rho_0}g,
\]
which combined with (1.13) yields that
\[
w_0 = u_0.
\] (5.43)

Next, we get from (5.42) and (1.13) that
\[
\limsup_{R_j \to \infty} \int_{\mathbb{R}^2} \left( \mu |\text{rot} w_0^{R_j}|^2 + (2\mu + \lambda(\rho_0^{R_j})) (\text{div} w_0^{R_j})^2 + \rho_0^{R_j} |w_0^{R_j}|^2 \right) dx
\leq \int_{\mathbb{R}^2} \left( \mu |\text{rot} u_0|^2 + (2\mu + \lambda(\rho_0)) (\text{div} u_0)^2 + \rho_0 |u_0|^2 \right) dx,
\]
which together with (5.41) implies
\[
\lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla w_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} |\nabla u_0|^2 dx, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0^{R_j} |w_0^{R_j}|^2 dx = \int_{\mathbb{R}^2} \rho_0 |u_0|^2 dx.
\]
This, along with (5.43) and (5.41), yields that
\[
\lim_{R \to \infty} \left( \| \nabla(w_0^R - u_0) \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\rho_0^0} w_0^R - \sqrt{\rho_0} u_0 \|_{L^2(\mathbb{R}^2)} \right) = 0.
\] (5.44)

Similar to (5.44), we can obtain that
\[
\lim_{R \to \infty} \| \nabla^2(w_0^R - u_0) \|_{L^2(\mathbb{R}^2)} = 0.
\]

Finally, in terms of Lemma 2.1, the initial-boundary-value problem (2.2) with the initial data \((\rho_0^R, u_0^R)\) has a classical solution \((\rho^R, u^R)\) on \( B_R \times [0, T_R] \). For \( \tilde{g} \) defined by (1.11) with \((\rho_0, u_0)\) being replaced by \((\rho_0^R, u_0^R)\), it follows from (5.38), (5.39), and (1.13) that
\[
\| \tilde{g} \|_{L^2(B_R)} \leq C,
\]
for some \( C \) independent of \( R \). Hence, there exists a generic positive constant \( C \) independent of \( R \) such that all those estimates stated in Proposition 3.1 and Lemmas 4.1, 4.4 hold for \((\rho^R, u^R)\). Extending \((\rho^R, u^R)\) by zero on \( \mathbb{R}^2 \setminus B_R \) and denoting
\[
\tilde{\rho}^R \triangleq (\varphi_R)^{4/3} \rho^R, \quad w^R \triangleq \varphi_R u^R,
\]
with \( \varphi_R \) as in (3.9) and \( \tilde{\beta} = \min\{\beta, 1\} \), we deduce from (3.6) and Lemmas 4.1, 4.4 that the sequence \((\tilde{\rho}^R, w^R)\) converges weakly, up to the extraction of subsequences, to some limit \((\rho, u)\) satisfying (1.10), (1.11), and (1.14). Moreover, standard arguments yield that \((\rho, u)\) in fact is a strong solution to the problem (1.1)-(1.5). The proof of Theorem 1.2 is completed.
References

[1] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math. 12, 623–727 (1959); II, Comm. Pure Appl. Math. 17, 35–92 (1964)

[2] Cho, Y.; Choe, H. J.; Kim, H. Unique solvability of the initial boundary value problems for compressible viscous fluids. J. Math. Pures Appl. (9) 83 (2004), 243-275.

[3] Choe, H. J.; Kim, H. Strong solutions of the Navier-Stokes equations for isentropic compressible fluids. J. Differ. Eqs. 190 (2003), 504-523.

[4] Cho Y.; Kim H. On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities. Manuscript Math.,120 (2006), 91-129.

[5] Danchin, R. Global existence in critical spaces for compressible Navier-Stokes equations. Invent. Math., 141 (2000), 579-614.

[6] Feireisl, E. Dynamics of viscous compressible fluids. Oxford University Press, 2004.

[7] Feireisl, E.; Novotný, A.; Petzeltová, H. On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. 3 (2001), no. 4, 358-392.

[8] Germain, P. Weak-strong uniqueness for the isentropic compressible Navier-Stokes system. J. Math. Fluid Mech. 13 (2011), no. 1, 137-146.

[9] Hoff, D. Global existence of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Diff. Eqs., 120 (1995), 215–254.

[10] Huang, X.; Li, J. Existence and blowup behavior of global strong solutions to the two-dimensional barotropic compressible Navier-Stokes system with vacuum and large initial data. http://arxiv.org/abs/1205.5342

[11] Huang, X.; Li, J.; Matsumura, A. On the strong and classical solutions to the three-dimensional barotropic compressible Navier-Stokes equations with vacuum. Preprint.

[12] Huang, X.; Li, J.; Xin, Z. P. Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. Comm. Pure Appl. Math. 65 (2012), 549-585.

[13] Lions, P. L. Existence globale de solutions pour les equations de Navier-Stokes compressibles isentropiques. C. R. Acad. Sci. Paris, Sr I Math. 316, 1335C1340 (1993)

[14] Lions, P. L. Mathematical topics in fluid mechanics. Vol. 1. Incompressible models. Oxford University Press, New York, 1996.

[15] Lions, P. L. Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford University Press, New York, 1998.
[16] Matsumura, A.; Nishida, T. The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20(1980), no. 1, 67-104.

[17] Nash, J. Le problème de Cauchy pour les équations différentielles d’un fluide général. Bull. Soc. Math. France. 90 (1962), 487-497.

[18] Salvi, R.; Straškraba, I. Global existence for viscous compressible fluids and their behavior as $t \to \infty$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40 (1993), no. 1, 17-51.

[19] Serrin, J. On the uniqueness of compressible fluid motion. Arch. Rational. Mech. Anal. 3 (1959), 271-288.

[20] Vaigant, V. A.; Kazhikhov. A. V. On existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible viscous fluid. Sib. Math. J. 36 (1995), no.6, 1283-1316.