SUBCONVEXITY FOR A DOUBLE DIRICHLET SERIES AND NON-VANISHING OF L-FUNCTIONS

ALEXANDER DAHL

Abstract. We study a double Dirichlet series of the form $\sum_d L(s, \chi_d)\chi'(d)d^{-w}$, where $\chi$ and $\chi'$ are quadratic Dirichlet characters with prime conductors $N$ and $M$ respectively. A functional equation group isomorphic to the dihedral group of order 6 continues the function meromorphically to $\mathbb{C}^2$. A convexity bound at the central point is established to be $(MN)^{3/8+\epsilon}$ and a subconvexity bound of $(MN(M+N))^{1/6+\epsilon}$ is proven. The developed theory is used to prove an upper bound for the smallest positive integer $d$ such that $L(1/2, \chi_dN)$ does not vanish, and further applications of subconvexity bounds to this problem are presented.

1. Introduction

The use of multiple Dirichlet series in number theoretic problems, as well as the intrinsic structure they possess, have evolved the subject to be a study in its own right. Indeed, Weyl group multiple Dirichlet series are considered fundamental objects whose structures are intimately linked to their analytic features [BCC'06]. In this paper we are interested in the size of a certain type of double Dirichlet series outside the region of absolute convergence, and some of its arithmetic consequences. For ordinary $L$-functions there is the well-known concept of convexity bound as a generic upper bound in the critical strip, and improvements in the direction of the Lindelöf hypothesis often have deep implications. In the situation of multiple Dirichlet series, the notion of convexity is not so obvious to formalize. Nevertheless there is a fairly natural candidate for a “trivial” upper bound of a multiple Dirichlet series, and it is an important problem to improve this estimate. This has first been carried out in [Blo11] for the Dirichlet series defined by

$$Z(s, w) = \zeta(2)(2s + 2w - 1) \sum_{d \text{ odd}} L^{(2)}(s, \chi_d)d^{-w}$$

at $s = 1/2 + it$, $w = 1/2 + iu$ simultaneously in the $t, u$-aspect, where the $^{(2)}$ superscript denotes that we are removing the Euler factor at 2, and $\chi_d$ is the Jacobi symbol $(d)$. Here we study a non-archimedean analogue. We can expand the $L$-function in the summand as a sum indexed by $n$. Instead of twisting by $n^s, d^u$, we twist by quadratic primitive Dirichlet characters $\chi(n), \chi'(d)$ of conductors $N, M$, respectively. More precisely, we consider

$$Z(s, w; \chi, \chi') := \sum_{d \geq 1, (d, 2MN) = 1} L^{(2MN)}(s, \chi_d\chi)\chi'(d)P^{(\chi)}_{d_0d_1}(s)$$

for sufficiently large $\Re s, \Re w$, where we write $d = d_0d_1^2$ with $d_0$ squarefree. The $P$ factors are a technical complication necessary in order to construct functional equations, which we accomplish thus: if we expand the $L$-function in the numerator and switch the summation, we can relate it to a similar double
Dirichlet series with the arguments and the twisting characters interchanged, with modified correction factors. Next, we have a functional equation obtained by application of the functional equation for Dirichlet $L$-functions, which maps $(s, w)$ to $(1 - s, s + w - \frac{1}{2})$. Applying the switch of summation formula to this, we obtain a functional equation mapping $(s, w)$ to $(s + w - \frac{1}{2}, 1 - w)$. These continue $Z$ to the complex plane except for the polar lines $s = 1$, $w = 1$, and $s + w = 3/2$. They also generate a group isomorphic to $D_6$, the dihedral group of order 6, which is isomorphic to the Weyl group of the root system of type $A_2$. For this reason $Z(s, w; \chi, \chi')$ is considered a type $A_2$ multiple Dirichlet series $\text{[BBFH07]}$. Interestingly, such objects are Whittaker coefficients in the Fourier expansion of a $GL(3)$ Eisenstein series of minimal parabolic type on a metaplectic cover of an algebraic group whose root system is the dual root system of $A_2$ $\text{[BBFH07]}$. In fact, it is conjectured that a similar relationship is true between multiple Dirichlet series of type $A_r$ and $GL(r + 1)$ Eisenstein series. Going down in dimension, type $A_2$ multiple Dirichlet series can be constructed from 1/2-integral weight Eisenstein series $\text{[GH85]}$. As for the correction polynomials $P^{(\chi)}_{d_0, d_1}(s)$, their existence, uniqueness, and construction were studied extensively in $\text{[BFH04]}$ and $\text{[DGH03]}$ in the more general setting of $GL(r)$ multiple Dirichlet series. They are unique in the case of $r$ up to 3.

The notion of convexity is no longer canonical in the case of double Dirichlet series, since our initial bounds for $Z(s, w; \chi, \chi')$ depend on what we know about bounds on their coefficients, and our knowledge here is only partial. Nonetheless, if we use the Lindelöf hypothesis on average (cf. Theorem 3), then through careful choice of initial bounds and functional equation applications (cf. $\text{[33]3}$), we obtain the bound

$$Z(\frac{1}{2}, \frac{1}{2}; \chi, \chi') \ll (MN)^{3/8 + \varepsilon},$$

which we call the convexity bound. In this work, we present the following subconvexity result.

**Theorem 1.** For quadratic Dirichlet characters $\chi$ and $\chi'$ of conductors $N$ and $M$ which are prime or unity, and for $\varepsilon > 0$, we have the bound

$$Z(\frac{1}{2}, \frac{1}{2} + it; \chi, \chi') \ll (1 + |t|)^{2/\varepsilon} (MN(M + N))^{1/6 + \varepsilon}.$$

We first point out that the $t$-aspect bound can be drastically improved, but our focus here is on bounds in the moduli aspect. For purposes of comparison with the convexity bound, we point out that, via the geometric-arithmetic mean inequality, we have $(MN)^{3/8 + \varepsilon} \ll (MN(M + N))^{1/4 + \varepsilon}$. This result is comparable to the subconvexity bound obtained by V. Blomer in the archimedean case, particularly $|sw(s + w)|^{1/6 + \varepsilon}$ for $\Re s = \Re w = \frac{1}{2}$, an improvement upon the convexity bound with 1/4 replaced by 1/6.

A useful arithmetic application of the theory developed for $Z(s, w; \chi, \chi')$ is finding a bound for the least $d$ such that $L(\frac{1}{2}, \chi_d|N)$ does not vanish, where $N$ is a large fixed prime. It is expected that all ordinates of any zeros of $L(s, \chi)$ on the critical line are linearly independent over the rationals, so that in particular it is expected that $L(\frac{1}{2}, \chi)$ is nonzero for any $\chi$. Random matrix theory provides further evidence to this conjecture: The lowest zero in families of $L$-functions (such as $L(s, \chi_d)$) is expected to be distributed like the “smallest” eigenvalue (i.e., closest to unity) of a certain matrix family (depending on the family of $L$-function). Since the corresponding measure vanishes at zero, it suggests that the smallest zero of a Dirichlet $L$-function is “repelled” from the real axis. In the case of quadratic twists of $GL(2)$ automorphic forms, in particular twists of elliptic curves, there is a connection to Waldspurger’s theorem $\text{[Wal81]}$ which states that $L(\frac{1}{2}, f \times \chi)$ is proportional to the squares of certain Fourier coefficients of a half-integral weight modular form, uniformly in $\chi$. We focus here on the simpler case of twists of Dirichlet characters. In particular, we have the following theorem which is proven using the theory of this double Dirichlet series.
Theorem 2. Let \( N \) be an odd prime, and let \( D(N) = d \) denote the smallest positive integer such that \( L\left(\frac{1}{2}, \chi_{dN}\right) \) does not vanish. Then we have \( D(N) \ll N^{1/2+\varepsilon} \).

It is not entirely obvious what should be regarded as the trivial bound in this situation. The most natural approach to non-vanishing would be to prove a lower bound for the first moment \( \sum_{d<X} L\left(\frac{1}{2}, \chi_{dN}\right) \) for \( X \) as small as possible in terms of \( N \). A straightforward argument produces a main term of size \( X \log X \) and an error term of size \( O((NX)^{1/2+\varepsilon}) \) which suggests the trivial bound \( N^{1+\varepsilon} \) for the first non-vanishing twist. This is perhaps unexpectedly weak, since the same bound holds for degree 2 \( L \)-functions \( L\left(\frac{1}{2}, f \times \chi_d\right) \), where \( f \) is an automorphic form of level \( N \). Nevertheless it is not completely obvious how to improve this in the case of Dirichlet characters \( \chi \). Here we follow a modified version of a method presented in [HK10]: Let \( X \) be a large positive number, and \( h(y) \) be a smooth non-negative function with support on \([1, 2]\). By Mellin inversion, we have

\[
\int_{(2)} \tilde{h}(w) Z\left(\frac{1}{2}, w; \chi_N, \psi_1\right) X^w \, dw \approx \sum_{d=1}^{\infty} L\left(\frac{1}{2}, \chi_{dN}\right) h(d/X),
\]

where \( \tilde{h} \) denotes the Mellin transform of \( h \), and \( \psi_1 \) denotes the trivial character. We move the contour to \( \Re w = -\varepsilon \), picking up a double pole at \( w = 1 \). If we apply a symmetric functional equation (3.15) and a bound resulting from use of the Lindelöf principle on average (3.18) to the resulting integral, then we have

\[
(1.1) \quad \sum_{d=1}^{\infty} L\left(\frac{1}{2}, \chi_{dN}\right) h(d/X) \approx a_N X \log X + b_N X + O(N^{1/2+\varepsilon})
\]

for some coefficients \( a_N, b_N \). The idea now is to bound \( a_N \) from below in terms of \( N \), and choose \( X \) so that the main term is greater than the error term. Then it cannot be that \( L\left(\frac{1}{2}, \chi_{dN}\right) \) vanishes for all \( 0 < d < X \) on the left-hand side. We prove the bounds \( a_N, b_N \gg N^{-\varepsilon} \) (cf. Theorem 9), and thus we can choose \( X = N^{1/2+\varepsilon} \). The power of the asymptotic formula (1.1) is that \( a_N, b_N \) can be bounded below by finding an asymptotic for \( \sum_{d=X} L\left(\frac{1}{2}, \chi_{dN}\right) \), but now we can take \( X \) to be very large.

Of course, in the previous discussion there are significant details suppressed for brevity, which include the aforementioned correction factors and the error term arising from truncation of the \( d \)-sum. Most nontrivial, however, are the bounds for \( a_N, b_N \) given by Theorem 9 requiring careful treatment. The techniques used are similar to those in [Jut81], in which an asymptotic formula for \( \sum_{0<d\leq X} L\left(\frac{1}{2}, \chi_d\right) \) is proved, where the sum is over fundamental discriminants.

In order to prove the subconvexity bound Theorem 1, we follow techniques similar to those used in [Blo11], but we must deal with some new complications. As mentioned, we need special correction factors in order to obtain functional equations, whereas in the archimedean case, the correction factors are far simpler. These correction factors appear due to the fact that the characters involved will only be primitive when the summation variable is a fundamental discriminant. Also, the introduction of character twists complicates the functional equations considerably: a pair of twisting characters \( (\psi, \psi') \) will not be static under the variable transformations of the functional equations.

We can iterate the functional equations to obtain one under the map \((s, w) \mapsto (1-s, 1-w)\). Using techniques similar to those in the case of \( L(s, \chi) \) (cf. [IK04] Theorem 5.3), we obtain the approximate functional equation

\[
Z\left(\frac{1}{2}, \frac{1}{2} \chi_{dN}\right) \approx \sum_{d<X} L\left(\frac{1}{2}, \chi_d\right) \chi'(d) + \sum_{d \leq M^2 N/X} L\left(\frac{1}{2}, \chi_d \chi_{dN}\right) \chi \chi_N(d)
\]
We can further apply the approximate functional equation for $L$-functions, 
\[
L\left(\frac{1}{2}, \chi_d\right) \approx \sum_{n \leq (dq)^{1/2}} \frac{\chi(n)}{n^{1/2}}.
\]
We hence see that we can roughly express 
\[
Z\left(\frac{1}{2}, \frac{1}{2}; \chi, \chi'\right)
\]
as a sum of double finite sums of the form 
\[
S(P, Q; \chi, \chi') := \sum_{d \leq P} \sum_{n \leq Q} \frac{\chi_d(n)\chi(n)\chi'(d)}{d^{1/2}n^{1/2}}
\]
for various character pairs $(\chi, \chi')$. Finally, we apply Heath-Brown’s large sieve estimate (cf. Corollary 1) to obtain the subconvexity result.

We note that it is possible to extend these results beyond the case where $M$ and $N$ are prime to arbitrary numbers, but the coefficients of the functional equations in Theorems 7 and 8 would become considerably more complicated due to the Euler factors involved. We thus only treat the case of prime moduli here to simplify the presentation.

**Notation.** The variable $\varepsilon$ will always denote a sufficiently small positive number, not necessarily the same at each occurrence, and the variable $A$ will denote a sufficiently large positive number, not necessarily the same at each occurrence. The numbers $M$ and $N$ will always denote natural numbers that are either odd primes or unity, possibly equal. For a real function $f$, we denote its Mellin transform by $\tilde{f}$. The trivial character modulo unity will be denoted by $\psi_1$, and the primitive character of conductor 4 shall be denoted $\psi_{-1}$. As for the primitive characters modulo 8, we define $\psi_2$ as the character that is unity at exactly 1 and 7, and we set $\psi_{-2} = \psi_2\psi_{-1}$. If $\chi$ is a character, we use the notation $C_{\chi}$ to denote its conductor.

2. Preliminaries

2.1. **Characters.** For a positive integer $d$, we define a character on $(\mathbb{Z}/4d\mathbb{Z})^*$ via the Jacobi symbol by 
\[
\chi_d(n) = \overline{\chi}_n(d) = \left(\frac{d}{n}\right).
\]
For odd positive integers $n$ and $d$, we have the following quadratic reciprocity law for the Jacobi-Kronecker symbol (cf. [Mol98], Theorem 4.2.1, page 157).
\[
\chi_d(n) = \left(\frac{d}{n}\right) = \left(\frac{n}{d}\right)(-1)^{d-1} = \begin{cases} 
\chi_d(n), & d \equiv 1 \pmod{4}; \\
\overline{\chi}_d(-n) = \overline{\chi}_d(n)\psi_{-1}(n), & d \equiv 3 \pmod{4}.
\end{cases}
\]

2.2. **$L$-function results.** Suppose that $\chi$ is a Dirichlet character. We define 
\[
L^{(P)}(s, \chi) = L(s, \chi) \prod_{p \nmid P} \left(1 - \frac{\chi(p)}{p^s}\right).
\]
We define the odd sign indicator function of a Dirichlet character $\chi$ by $\kappa = \kappa(\chi) = \frac{1}{2}(1 - \chi(-1))$. 
2.3. \textbf{\textit{L}-functional bounds and approximate functional equation.} For a primitive character $\chi$ modulo $q$, using an absolute convergence argument for $L(s, \chi)$ in a right half-plane and applying the functional equation, we interpolate via the Phragmén-Lindelöf convexity principle to obtain

\begin{equation}
L(s, \chi) \ll \begin{cases} 
|q(1 + |3s|)|^{1/2 - \Re s}, & \Re s \leq -\varepsilon; \\
|q(1 + |3s|)|^{(1-\Re s)/2 + \varepsilon}, & -\varepsilon < \Re s < 1 + \varepsilon; \\
1, & \Re s \geq 1 + \varepsilon,
\end{cases}
\end{equation}

away from a possible pole at $s = 1$ in the case where $\chi$ is trivial. This bound is known as the convexity bound for Dirichlet $L$-functions.

We shall also need the so-called approximate functional equation for Dirichlet $L$-functions (cf. [IK04], Theorem 5.3). Particularly, if $\chi$ is a quadratic primitive character modulo odd $q$, $\psi$ is a character with conductor dividing 8, and $d_0$ is odd, squarefree and coprime to $q$, then we have the weighted infinite sum

\begin{equation}
L(\frac{1}{2}, \chi d_0 \chi \psi) = 2 \sum_{n=1}^{\infty} \frac{\langle \chi d_0 \chi \psi \rangle(n)}{n^{1/2}} G_{\kappa} \left( \frac{n}{\sqrt{c_0 d_0 q}} \right),
\end{equation}

where

\begin{equation}
\kappa = \kappa(\chi d_0 \chi \psi), \quad c_0 = \begin{cases} 1, & d_0 \equiv 1 \pmod{4}, \psi = \psi_1 \text{ or } d_0 \equiv 3 \pmod{4}, \psi = \psi_{-1}; \\
4, & d_0 \equiv 1 \pmod{4}, \psi = \psi_{-1} \text{ or } d_0 \equiv 3 \pmod{4}, \psi = \psi_{1}; \\
8, & \psi = \psi_2 \text{ or } \psi_{-2},
\end{cases}
\end{equation}

and we have the weight function

\begin{equation}
G_{\kappa}(\xi) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\frac{1/2+s+\kappa}{2})}{\Gamma(\frac{1/2+s}{2})} \xi^{-s} ds,
\end{equation}

satisfying the bound

\begin{equation}
G_{\kappa}(\xi) \ll (1 + \xi)^{-A}
\end{equation}

for arbitrary $A \geq 0$ (cf. [IK04], Proposition 5.4).

We shall also make use of smooth weight functions.

\begin{definition}
We say that $w(x)$ is a smooth weight function if it is a smooth non-negative real function supported on $[1/4, 5/4]$ and unity on $[1/2, 1]$.
\end{definition}

The following bound can be shown via sufficiently many applications of integration by parts. We have

\begin{equation}
\tilde{w}(z) \ll_{A, \Re z} (1 + |z|)^{-A}
\end{equation}

for $A \geq 0$, where $\tilde{w}(z)$ is the Mellin transform of $w(x)$.

2.4. \textbf{\textit{Short double character sums and $L$-function moments.}} We shall need the following adaptation of Theorem 2 from [HB95] which includes a character twist.

\begin{theorem}
Let $\psi$ be a primitive character modulo $j$, and for a positive integer $Q$ define $S(Q)$ to be the set of quadratic primitive Dirichlet characters of conductor at most $Q$. We have

\begin{equation}
\sum_{\chi \in S(Q)} |L(\sigma + it, \chi \psi)|^4 \ll_{\varepsilon} \{Q + (Qj(|t| + 1))^{2-2\sigma}\} \{Qj(|t| + 1)\}^\varepsilon
\end{equation}

for any fixed $\sigma \in [1/2, 1]$ and any $\varepsilon > 0$.
\end{theorem}
Removing even conductors, and using the fact that $2 - 2\sigma \leq 1$, we can restate this as follows.

**Theorem 4.** If $\chi$ is a primitive character with conductor $q$ and $X$ is a positive real number, then

$$\sum_{d_0 \leq X} |L(s, \chi d_0 \chi)|^4 \ll_{\varepsilon} (X q |s|)^{1+\varepsilon}, \quad \sigma \geq \frac{1}{2}$$

for all $\varepsilon > 0$.

An important ingredient for the proof of the subconvexity bound Theorem 1 is a large sieve estimate for quadratic characters due to Heath-Brown. In particular we state here Theorem 1 in [HB95].

**Theorem 5 (Heath-Brown’s large sieve estimate).** Let $P$ and $Q$ be positive integers, and let $(a_n)$ be a sequence of complex numbers. Then

$$\sum_{m \leq P} \sum_{n \leq Q} a_n \left( \frac{n}{m} \right) \ll_{\varepsilon} (PQ)^{\varepsilon} (P + Q)^{1/2+\varepsilon},$$

for any $\varepsilon > 0$, where $\sum^*$ denotes that the sum is over odd squarefree numbers.

Due to the nature of the double Dirichlet series we shall construct, we shall need the following normalization of this result.

**Corollary 1.** If $(a_m), (b_n)$ are sequences of complex numbers satisfying the bound $a_m, b_m \ll m^{-1/2+\varepsilon}$ for some $\varepsilon > 0$, and $P$ and $Q$ are positive real numbers, then

$$\sum_{m \leq P} \sum_{n \leq Q} a_m b_n \left( \frac{n_0}{m} \right) \ll_{\varepsilon} (PQ)^{\varepsilon} (P + Q)^{1/2+\varepsilon},$$

where we have the composition $n = n_0 n_1^2$ with $n_0$ squarefree, uniformly in $P$ and $Q$, for any $\varepsilon > 0$.

### 2.5. Gamma identities

We shall have use for the identity

$$\Gamma\left(\frac{2-z}{2}\right) = \Gamma\left(\frac{1+z}{2}\right) \cot\left(\frac{\pi z}{2}\right), \quad z \in \mathbb{C}.$$  \hfill (2.8)

By Stirling’s formula, in particular (5.113) from [IK04], for $s \in \mathbb{C}$ with fixed real part and nonzero imaginary part, we have

$$\frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \ll_{\Re s} (1 + |s|)^{1/2-\Re s},$$

away from the poles at the odd positive integers. We also have the cotangent bound,

$$\cot(x + iy) = -i \text{sign}(y) + O(e^{-2|y|}), \quad \min_{k \in \mathbb{Z}} |z - \pi k| \geq 1/10.$$  \hfill (2.10)

### 3. Structure and Analytic Properties

#### 3.1. Switch of summation formula

The object we would like to study is

$$Z_0(s, w; \chi, \chi') = \sum_{d \geq 1} \frac{L(s, \chi d \chi') \chi'(d)}{d^w},$$

where $\chi$ and $\chi'$ are quadratic characters with moduli $N$ and $M$ respectively. However, in order to obtain functional equations, we will need to augment this by some correction factors. The exact form of these
and the interchange of summation formula

$$d$$

with the Dirichlet polynomials

$$F$$

Functional equations.

### 3.2. Theorem 6.

Let $$m$$ and $$d$$ be positive integers with $$(md, 2MN) = 1$$, and write $$d = d_0d_1^2$$ and $$m = m_0m_1^2$$, with $$d_0$$, $$m_0$$ squarefree and $$d_1$$, $$m_1$$ positive, and let $$\chi$$ and $$\chi'$$ be characters modulo $$8\text{lcm}(M, N)$$. Then the Dirichlet polynomials

$$P_{d_0, d_1}(s) = \prod_{p^\alpha || d_1} \left[ \sum_{n=0}^{\alpha} \binom{p^{2n}}{p^n} p^{n-2ns} - \sum_{n=0}^{\alpha-1} (\chi_{d_0, \chi})(p^{2n+1}) p^{n-(2n+1)s} \right],$$

$$Q_{m_0, m_1}(w) = \prod_{p^\alpha || m_1} \left[ \sum_{n=0}^{\alpha} \binom{p^{2n}}{p^n} p^{n-2nw} - \sum_{n=0}^{\alpha-1} (\tilde{\chi}_{m_0, \chi'})(p^{2n+1}) p^{n-(2n+1)w} \right]$$

satisfy the functional equations

$$P_{d_0, d_1}(s) = d_1^{1-2s} \chi(d_1^2) P_{d_0, d_1}(1-s)$$ and $$Q_{m_0, m_1}(w) = m_1^{1-2w} \chi'(m_1^2) Q_{m_0, m_1}(1-w),$$

and the interchange of summation formula

$$\sum_{(d, 2MN) = 1} \frac{L((2MN), s, \chi_{d_0, \chi}(d) P_{d_0, d_1}(s))}{d^w} = \sum_{(m, 2MN) = 1} \frac{L((2MN), w, \tilde{\chi}_{m_0, \chi'}(m) Q_{m_0, m_1}(w))}{m^s},$$

for $$\Re s, \Re w > 1$$.

**Proof.** This is a straightforward but lengthy computation which we suppress for brevity. We direct the reader to [BFH04] for details. \(\square\)

In light of the above theorem, we now define our double Dirichlet series as follows: Let $$\chi$$ and $$\chi'$$ be characters modulo $$8\text{lcm}(M, N)$$. Then define

$$Z(s, w; \chi, \chi') = \sum_{(d, 2MN) = 1} \frac{L((2MN), s, \chi_{d_0, \chi}(d) P_{d_0, d_1}(s))}{d^w}.$$

We note at this point that it is easily shown that, for $$d = d_0d_1^2$$ and $$m = m_0m_1^2$$ with $$\Re s, \Re w \geq \frac{1}{2}$$, we have the bounds $$|P_{d_0, d_1}(s)| \ll d_1^s$$ and $$|Q_{m_0, m_1}(w)| \ll m_1^{w}$$. Applying the functional equations (3.4), we therefore have

$$|P_{d_0, d_1}(s)| \ll \begin{cases} d_1^{1-2\Re s}, & \Re s < \frac{1}{2}; \\ d_1^s, & \Re s \geq \frac{1}{2}, \end{cases}$$

$$|Q_{m_0, m_1}(w)| \ll \begin{cases} m_1^{1-2\Re w}, & \Re w < \frac{1}{2}; \\ m_1^w, & \Re w \geq \frac{1}{2}. \end{cases}$$

### 3.2. Functional equations.

We recall that $$M$$ and $$N$$ are odd prime numbers or unity, possibly equal. From this point on in the paper, we shall use the following notation: Let $$\chi$$ and $$\chi'$$ be quadratic primitive characters of squarefree conductors $$k$$ and $$j$$ respectively, where $$j, k \mid \text{lcm}(M, N)$$, and let $$\psi, \psi'$$ be primitive characters with conductors dividing 8. We first derive the following expansion of the regions of absolute convergence of the key series involved.

**Lemma 1.** We have the following two series representations of $$Z(s, w; \chi \psi, \chi' \psi')$$. We have

$$Z(s, w; \chi \psi, \chi' \psi') = \sum_{(d, 2MN) = 1} \frac{L((2MN), s, \chi_{d_0, \chi}(d) P_{d_0, d_1}(s))}{d^w}.$$
which is absolutely convergent on the set

\[
R_1^{(1)} := \{ \Re s \leq 0, \Re w + \Re s > 3/2 \} \cup \{ 0 < \Re s \leq 1, \Re s/2 + \Re w > 3/2 \} \cup \{ \Re s, \Re w > 1 \},
\]

except for a possible polar line \{ s = 1 \}, and

\[
Z(s, w; \chi^*, \chi' \psi') = \sum_{(m, 2MN) = 1} \frac{L^{(2MN)}(w, \tilde{\chi}_m \chi' \psi')(\chi \psi)(m)Q^{(\chi' \psi')}(m)}{m^s}
\]

which is absolutely convergent on the set

\[
R_1^{(2)} := \{ \Re w \leq 0, \Re s + \Re w > 3/2 \} \cup \{ 0 < \Re w \leq 1, \Re w/2 + \Re s > 3/2 \} \cup \{ \Re s, \Re w > 1 \},
\]

except for a possible polar line \{ w = 1 \}.

**Proof.** This follows from applying the bounds for the Dirichlet L-function [2.2] and the bounds for the correction polynomials [3.4] to the definition [3.3]. \qed

We now proceed with derivation of functional equations for \( Z \). Due to the summation switch formula [3.2], we have

\[
Z(s, w; \chi^*, \chi' \psi') = \sum_{(m, 2MN) = 1} \frac{L^{(2MN)}(w, \tilde{\chi}_m \chi' \psi')(\chi \psi)(m)Q^{(\chi' \psi')}(m)}{m^s}.
\]

We can further apply the functional equation for the \( Q \) factor [3.1] to obtain

\[
Z(s, w; \chi^*, \chi' \psi') = \sum_{(m, 2MN) = 1} \frac{L^{(2MN)}(w, \tilde{\chi}_m \chi' \psi')(\chi \psi)(m)\chi'(m)^2Q^{(\chi' \psi')}(m_1)}{m^s m_1^{2(w-1/2)}},
\]

which holds for \((s, w) \in R_1^{(2)}\). The next step is to apply the functional equation for Dirichlet L-functions in order to change the \( w \) in the L-function to \( 1 - w \). This would allow us to switch summation again to obtain \( Z \) in its original form, but with a change in variables.

We shall define the following Euler product function: For a character \( \chi^* \) and a positive integer \( P \), we define

\[
K_P(w; \chi^*) = \prod_{p \mid P} \left( 1 - \frac{\chi^*(p)}{p^{1-w}} \right)^{-1} \left( 1 - \frac{\chi^*(p)}{p^w} \right).
\]

Applying the Dirichlet functional equation along with [2.1] and the above, we now have

\[
L^{(2MN)}(w, \tilde{\chi}_m \chi' \psi') = \pi^{w-\frac{1}{2}} \frac{\Gamma \left( \frac{1-w+\kappa'}{2} \right)}{\Gamma \left( \frac{w+\kappa'}{2} \right)} K_{2MN}(w; \tilde{\chi}_m \chi' \psi')(C_{\psi'} jm_0)\frac{1}{2-w} L^{(2MN)}(1-w, \tilde{\chi}_m \chi' \psi'),
\]

where \( \kappa' = \kappa(\tilde{\chi}_m \chi' \psi') \).

We need to break down some of these parts for further manipulation. Recalling that \((m_0, 2MN) = 1\), if \( p \) is prime and does not divide \( m_0 \), we have

\[
K_P(w; \tilde{\chi}_m \chi^*) = \frac{\chi^*(p^2)p - p^2 + \chi_p(m_0)\chi^*(p)(p^2-w - p^{1+w})}{\chi^*(p^2)p^{2w} - p^2}.
\]

If for \( P \in \mathbb{N} \) and a Dirichlet character \( \chi^* \) we set
We note that this holds true even if using (3.10) above, we now have the useful expression
\[ K_p(w; \chi_m \chi^*) = F_p^{(\chi^*)}(w) + \chi_p(m_0)G_p^{(\chi^*)}(w), \]
which holds for prime p not dividing m_0, or p = 1. Noting that \( K_p(w; \chi^*) \) is multiplicative in \( P \), and using (3.10) above, we now have the useful expression
\[ K_{MN}(w; \tilde{\chi}_m \chi \psi^*) = F_M^{(\chi \psi^*)} \cdot F_N^{(\chi \psi^*)}(w) + \chi_M(m_0)F_N^{(\chi \psi^*)}(w) \]
\[ + \chi_N(m_0)F_M^{(\chi \psi^*)} \cdot G_M^{(\chi \psi^*)}(w) + \chi_M(m_0)G_M^{(\chi \psi^*)} \cdot G_N^{(\chi \psi^*)}(w). \]
We note that this holds true even if \( M = N \). Indeed, in this case, by the definition (3.4), we have \( K_{MN}(w; \tilde{\chi}_m \chi \psi^*) = 1 \). This is consistent with (3.11), since according to definition (3.9) we have \( G_N^{(\chi \psi^*)}(w) = 0 \) and \( F_N^{(\chi \psi^*)}(w) = 1 \).

Next, we see from (2.9) that
\[ \Gamma \left( \frac{1-w+\kappa'}{2} \right) = \frac{\Gamma \left( \frac{1-w}{2} \right)}{\Gamma \left( \frac{\kappa'}{2} \right)} \cot \left( \frac{\pi w}{2} \right)^{\kappa'}. \]
We shall find it useful to remove the dependency of \( \kappa' \) on \( m_0 \), or rather, exploit that the dependency is only on its residue modulo 4. Hence, define \( \kappa' = \kappa(\chi \psi^*) \). Now suppose that \( f \) is a function of \( \kappa' \). By sieving out by congruence classes modulo 4, we see that
\[ f(\kappa') = \frac{1}{2}(1 + \psi^{-1}(m_0))f(\kappa') + \frac{1}{2}(1 - \psi^{-1}(m_0))f(1 - \kappa'). \]
Hence we have
\[ \cot \left( \frac{\pi w}{2} \right)^{\kappa'} = \frac{1}{2}(1 + \psi^{-1}(m_0))\cot \left( \frac{\pi w}{2} \right)^{\kappa'} + \frac{1}{2}(1 - \psi^{-1}(m_0))\cot \left( \frac{\pi w}{2} \right)^{(1 - \kappa')}. \]
For brevity, for a character \( \chi^* \), we define
\[ S(s, w; m, \chi^*) := L(2MN, 1 - w; \tilde{\chi}_m \chi \psi^*) \chi^*(m)Q_{\chi^{\prime},m_0}^{(\chi \psi^*)}(1 - w). \]
Applying the functional equation for Dirichlet L-functions (3.7) to the identity (3.5) along with (3.13), and using Lemma 11 we now get
\[ \begin{align*}
Z(s, w; \chi \psi, \chi' \psi^*) &= \frac{1}{2} \pi^{w-1/2} \Gamma \left( \frac{1-w}{2} \right) \Gamma \left( \frac{w}{2} \right) \sum_{(m, 2MN) = 1} K_{2MN}(w; \tilde{\chi}_m \chi' \psi^*) \times S(s, w; m, \chi \psi) \left[ (1 + \psi^{-1}(m_0))\cot \left( \frac{\pi w}{2} \right)^{\kappa'} + (1 - \psi^{-1}(m_0))\cot \left( \frac{\pi w}{2} \right)^{(1 - \kappa')} \right],
\end{align*} \]
for \((s, w) \in R_1^{(2)}\) except for possible polar lines \( \{s = 1\} \) and \( \{w = 1\} \). There are two properties of the function \( S \) which we shall make use of, stated in the following lemma.
Lemma 2. For two characters $\chi^*$ and $\chi^{**}$ and an integer $m$, the following two properties hold for $(s, w) \in R_1^{(2)}$, except for possible polar lines $\{s = 1\}$ and $\{w = 1\}$.

\( (i) \quad S(s, w; m, \chi^*) \chi^{**} = S(s, w; m, \chi^* \chi^{**}), \)

\( (ii) \quad \sum_{(m, 2MN) = 1} S(s, w; m, \chi^*) = Z(s + w - \frac{1}{2}, 1 - w; \chi^*, \chi' \psi'). \)

We can apply the identity (3.11) and Lemma 2 to (3.14), and use the bounds (2.9) and (2.10) to come to the following functional equation.

Theorem 7. Let $\chi$ and $\chi'$ be primitive Dirichlet characters modulo squarefree $k$ and $j$ respectively, where $j, k \mid MN$, and if $M = N$, then $j = k = M = N$, and let $\psi$ and $\psi'$ be Dirichlet characters modulo 8. There exist functions $a_n^{(\chi', \psi')} (w; \psi^*)$ for $n \mid MN$ and $\psi^*$ a Dirichlet character modulo 8 which are holomorphic except for possible poles at the positive integers, and countably many poles on the line $\Re w = 1$, bounded absolutely above by $O((16\pi)^{|\Re s|} (1 + |w|)^{1/2 - \Re s})$ uniformly in $j$ and $k$ away from the poles such that for $(s, w) \in R_1^{(2)}$ away from possible polar lines $\{s = 1\}$ and $\{w = 1\}$ we have

\[ Z(s, w; \chi \psi, \chi' \psi') = j^{1/2 - w} \sum_{n\mid MN} A_n^{(\chi)}(w) \sum_{\psi^* \in (\mathbb{Z}/8\mathbb{Z})^*} a_n^{(\chi', \psi')}(w; \psi^*) Z(s + w - \frac{1}{2}, 1 - w; \chi \chi_n \psi \psi^*, \chi' \psi'), \]

where $R_1^{(2)}$ is defined in Lemma 7, we have

\[ A_1^{(\chi)}(w) = F_1^{(\chi)}(w) |F_N^{(\chi)}(w), \quad A_1^{(\chi')}(w) = F_1^{(\chi')}(w) |G_N^{(\chi')}(w), \]

\[ A_N^{(\chi)}(w) = F_N^{(\chi)}(w) |F_N^{(\chi)}(w), \quad A_N^{(\chi')}(w) = G_N^{(\chi')}(w) |G_N^{(\chi')}(w), \]

and the $F$ and $G$ functions are defined in [5.9].

By similar methods, we can obtain a second functional equation under the transformation $(s, w) \mapsto (1 - s, s + w - \frac{1}{2})$ by applying the functional equation (3.11) to the definition (3.3), followed by the functional equation for $L$-functions. The result of this similarly lengthy derivation is the following second functional equation.

Theorem 8. Let $\chi$ and $\chi'$ be Dirichlet characters modulo squarefree $k$ and $j$ respectively, where $j, k \mid MN$, and if $M = N$, then $j = k = M = N$, and let $\psi$ and $\psi'$ be Dirichlet characters modulo 8. There exist functions $b_n^{(\chi, \psi)} (s; \psi^*)$ for $n \mid MN$ and $\psi^*$ a Dirichlet character modulo 8 which are holomorphic except for possible poles at the positive integers, and countably many poles on the line $\Re s = 1$, bounded absolutely above by $O((16\pi)^{|\Re s|} (1 + |s|)^{1/2 - \Re s})$ uniformly in $j$ and $k$ away from the poles such that for $(s, w) \in R_1^{(1)}$ away from possible polar lines $\{s = 1\}$ and $\{w = 1\}$ we have

\[ Z(s, w; \chi \psi, \chi' \psi') = k^{1/2 - s} \sum_{n\mid MN} A_n^{(\chi)}(s) \sum_{\psi^* \in (\mathbb{Z}/8\mathbb{Z})^*} b_n^{(\chi, \psi)}(s; \psi^*) Z(1 - s, s + w - \frac{1}{2}; \chi \psi, \chi' \chi_n \psi \psi^*), \]

where $R_1^{(1)}$ is defined in Lemma 7 and the $A$ functions are as in Theorem 4.

We make note of an analytic subtlety: We note that although the $a_n$, $b_n$, and $A_n$ functions above have poles, they do not contribute poles to $Z(s, w; \chi \psi, \chi' \psi')$; indeed, it will be proven in Proposition 1 that the only possible poles of $Z$ are the polar lines $\{s = 1\}$, $\{w = 1\}$, and $\{s + w = 3/2\}$. Looking at (3.14), though the gamma and cotangent factors together have poles at either the even or odd positive integers, and the $K$ factor has countably many poles along the line $\Re w = 1$ (cf. (3.8)), these poles nonetheless do not produce poles on the right-hand side. The equation (3.14) essentially results from
the application of the functional equation for $L$-functions to the identity, and subsequently sieving out by congruence classes of $m$ modulo 4. The last step introduces coefficients with poles from the gamma and cotangent factors. This is a manifestation of a phenomenon that is observed in the functional equation for $L$-functions: In 3.4, we know that the $L$-function can only have a pole at $w = 1$, yet on the right-hand side, the gamma function produces poles which are mitigated by the trivial zeros of the $L$-function. It is precisely these poles which appear in the coefficients of 3.14. Additionally, although the $K$ function has poles, these are mitigated by corresponding zeros of the $L$ function due to removal of the Euler factors at primes dividing $2MN$.

It shall be useful to note the following properties of the $A$ coefficients, which follow directly from the definitions.

Lemma 3. Let $\chi$ be a Dirichlet character modulo $q$. Then the following properties hold.

1. For a positive integer $P$, if $(q, P) > 1$ then $A_{\chi}^P(w) = 0$.
2. If $q = MN$ then $A_{\chi}^1(w) = 1$.
3. If $q = M$ then $A_{\chi}^1(w) = F_{\chi}^M(w)$ and $A_{\chi}^N(w) = G_{\chi}^N(w)$.
4. If $q = N$ then $A_{\chi}^1(w) = F_{\chi}^N(w)$ and $A_{\chi}^M(w) = G_{\chi}^M(w)$.
5. Moreover, the following asymptotics hold, if $P \neq 1$ and $(P, q) = 1$.

$$|F_{\chi}^P(w)| \approx \begin{cases} 1, & \Re w < 1 - \varepsilon; \\ P - 2\Re w, & \Re w > 1 + \varepsilon, \end{cases}$$

$$|G_{\chi}^P(w)| \approx \begin{cases} P - \Re w, & \Re w < \frac{1}{2}; \\ P^{\Re w - 1}, & \frac{1}{2} \leq \Re w < 1 - \varepsilon; \\ P^{1 - \Re w}, & \Re w > 1 + \varepsilon. \end{cases}$$

It shall also be useful to derive a somewhat symmetric functional equation, obtained by application of Theorem 7 then Theorem 8 followed again by Theorem 7. For quadratic Dirichlet characters $\rho$ and $\rho'$ of conductors $N$ and $M$ respectively which are either prime or unity, possibly equal, we have

\begin{equation}
Z(s, w; \rho \psi, \rho' \psi') = M^{1/2-w} \sum_{n,m,r|MN} C_{\rho \chi}^{1-s-w} C_{\rho' \chi}^{1-s} A_n^{\rho \psi}(w) A_m^{\rho' \psi}(s + w - 1/2) A_r^{(\rho' \chi)}(w) \times \sum_{\psi', \psi'' \in (\mathbb{Z}/2\mathbb{Z})^*} \left(1 - w, 1 - s; \rho \chi, \psi', \psi'' \right) Z(1 - w, 1 - s; \rho \chi, \psi', \psi'') \right)
\end{equation}

where the $A$ functions are as in Theorem 7 the $c$ functions are holomorphic in $\mathbb{C}^2$ except for possible poles for $s$, $w$, or $s + w - 1/2$ equal to positive integers, and countably many poles on the lines $\Re w = 1$, $\Re s = 1$, and $\Re s + \Re w = 3/2$, bounded absolutely above by

\begin{equation}
O((16\pi)^2|\Re s + 1|\Re w)(1 + |s|^{1/2 - \Re s}(1 + |w|^{1/2 - \Re w}(1 + |s + w|)^{1/2 - \Re s - \Re w}).
\end{equation}

3.3. Analytic continuation. We continue $Z(s, w; \chi, \chi')$ to all of $\mathbb{C}^2$ except for the polar lines $s = 1$, $w = 1$, and $s + w = 3/2$. We have

Proposition 1. Let $\chi$ and $\chi'$ be characters modulo $8\text{lcm}(M, N)$. The function

$$Z(s, w; \chi, \chi') = (s - 1)(w - 1)(s + w - 3/2) Z(s, w; \chi, \chi')$$

is holomorphic in $\mathbb{C}^2$ and is polynomially bounded in the sense that, given $C_1 > 0$, there exists $C_2 > 0$ such that $Z(s, w; \chi, \chi') \ll [MN(1 + |\Re s|)(1 + |\Re w|)]C_2$ whenever $|\Re s|, |\Re w| < C_1$.

Proof. We refer to the reader to [BPH04] and [Blo11].
3.4. Convexity bound. The notion of convexity is not canonically defined for double Dirichlet series as it is in the case of (single) Dirichlet series; in the latter case, we have a single functional equation which reflects the region of absolute convergence, and interpolating the bounds produces a convexity bound between the two. In the case of double Dirichlet series, things are more complicated. Firstly, our bounds in the region of absolute convergence depend on our knowledge of the bounds on \( L(s, \chi) \) on average. Secondly, we have 6 functional equations to choose from to apply to this region. If we use the Lindelöf hypothesis on average, namely Theorem \ref{thm:4} then we can carefully choose a functional equation to apply in order to minimize the resulting convexity bound from application of the Phragmén-Lindelöf convexity principle (cf. Theorem 5.53 of [IK04]).

We shall require some initial bounds. We let \( \chi \) and \( \chi' \) be quadratic Dirichlet characters with conductors \( k \) and \( j \) respectively, and \( \psi, \psi' \) are characters modulo 8. We first assume that \( \Re s = 1/2 \) and \( \Re w = 1 + \varepsilon \). We apply the identity \( \sum \), the bounds \( (3.1) \), and the Cauchy-Schwarz inequality with the average bound of Theorem \ref{thm:4} to obtain

\[
Z(s, w; \chi \psi, \chi' \psi') \ll (MN)^{-1/2+\varepsilon} (1 + |s|)^{1/4+\varepsilon}, \quad \Re s = 1/2, \quad \Re w = 1 + \varepsilon,
\]

and by the switch of summation formula \( (3.2) \) we also obtain

\[
Z(s, w; \chi \psi, \chi' \psi') \ll (MN)^{-1/2+\varepsilon} (1 + |w|)^{1/4+\varepsilon}, \quad \Re s = 1 + \varepsilon, \quad \Re w = 1/2.
\]

We use the functional equation Theorem \ref{thm:8} with \( \Re s = -\varepsilon \) and \( \Re w = 1 + \varepsilon \) and apply \( (3.13) \) on the right-hand side in order to obtain a bound for \( Z(s, w; \rho, \rho') \). Looking at the coefficient bounds in Lemma \ref{lem:3} we pick up a factor of \( N^{1/2+\varepsilon} \). Further, we see that the resulting twisting characters on the right-hand side will have conductors \( (k, j) \in \{(N, M), (N, 1)\} \), so that we have

\[
Z(s, w; \rho, \rho') \ll N^{1/2+\varepsilon} M^{1/4+\varepsilon} (1 + |s|)^{1/2+\varepsilon} (1 + |s + w|)^{1/4+\varepsilon}, \quad \Re s = -\varepsilon, \quad \Re w = 1 + \varepsilon.
\]

Likewise with the functional equation Theorem \ref{thm:7} applied to \( (3.17) \), we have the symmetric bound

\[
Z(s, w; \rho, \rho') \ll M^{1/2+\varepsilon} N^{1/4+\varepsilon} (1 + |w|)^{1/2+\varepsilon} (1 + |s + w|)^{1/4+\varepsilon}, \quad \Re s = 1 + \varepsilon, \quad \Re w = -\varepsilon.
\]

We wish to interpolate convexly between these bounds along the two diagonal lines in \( \mathbb{C}^2 \), but we must deal with the potential poles at \( s = 1 \) and \( w = 1 \). To do this we can simply multiply both sides of the bounds by \( (s - 1)(w - 1) \). We obtain the following bound.

**Proposition 2.** For quadratic characters \( \rho \) and \( \rho' \) of prime moduli \( N \) and \( M \) respectively, we have

\[
Z(s, w; \rho, \rho') \ll [(1 + |s|)(1 + |w|)(1 + |s + w|)]^{1/4+\varepsilon} (MN)^{3/8+\varepsilon}, \quad \Re s = \Re w = 1/2,
\]

which we call the convexity bound for the function \( Z(s, w; \rho, \rho') \).

4. Approximate Functional Equations

4.1. A symmetric functional equation. We introduce a succession of applications of the functional equations in the special case of \( (s, w) = (1/2, 1/2 - z) \) for some \( z \in \mathbb{C} \) with \( \Re z > 0 \). We recall that \( \rho \) and \( \rho' \) are primitive quadratic Dirichlet characters of conductors \( N \) and \( M \) respectively. We first apply Theorem \ref{thm:7} which after observing the coefficient properties of Lemma \ref{lem:8} gives

\[
Z(\frac{1}{2}, \frac{1}{2} - z; \rho, \rho') = \sum_{\psi \in (\mathbb{Z}/N\mathbb{Z})^*} \left[ \left( M^2 F_N^{(\rho')} \phi_1^{(\rho', \psi_1)} \left( \frac{1}{2} - z; \psi^* \right) Z\left( \frac{1}{2} - z, \frac{1}{2} + z; \rho^* \rho' \right) \right. \right.
\]

\[
\left. \left. + M^2 G_N^{(\rho')} \left( \frac{1}{2} - z \right) \phi_1^{(\rho', \psi_1)} \left( \frac{1}{2} - z; \psi^* \right) Z\left( \frac{1}{2} - z, \frac{1}{2} + z; \psi^* \rho' \right) \right] \right).
\]
We then apply the functional equation Theorem 8 (and again use Lemma 3) which further gives

\[ Z\left(\frac{1}{2}, \frac{1}{2} - z; \rho, \rho'\right) = \sum_{\psi^*, \psi^{**} \in \mathbb{Z}/(\mathbb{Z}/2\mathbb{Z})^*} \left[ (MN)^{\frac{1}{2} - z} F_N^{(\rho)}(\frac{1}{2} - z) F_M^{(\rho)}(\frac{1}{2} - z) e_{1,1}^{(\rho, \rho)}(z; \psi^*, \psi^{**}) Z(\frac{1}{2} + z, \frac{1}{2}; \rho \psi^*, \rho' \psi^{**}) + (MN)^{\frac{1}{2} - z} F_N^{(\rho)}(\frac{1}{2} - z) G_M^{(\rho)}(\frac{1}{2} - z) e_{1,1}^{(\rho, \rho)}(z; \psi^*, \psi^{**}) Z(\frac{1}{2} + z, \frac{1}{2}; \rho \psi^*, \rho' \psi^{**}) + M^2 G_N^{(\rho)}(\frac{1}{2} - z) F_M^{(\psi_1)}(\frac{1}{2} - z) G_M^{(\psi_1)}(\frac{1}{2} - z) e_{1,1}^{(\rho, \rho)}(z; \psi^*, \psi^{**}) Z(\frac{1}{2} + z, \frac{1}{2}; \rho \psi^*, \rho' \psi^{**}) + (\frac{1}{2} - z) F_M^{(\psi_1)}(\frac{1}{2} - z) G_M^{(\psi_1)}(\frac{1}{2} - z) c_{\nu, \nu, \psi^*}^{(\rho, \psi^*)} \right], \]

where

\[ c_{\nu, \nu, \psi^*}^{(\rho, \psi^*)} = a_{\nu}(\psi^*) \frac{1}{z - \frac{1}{2} z^*} b_{\nu}(\psi^*) \frac{1}{z^*}. \]

We note that, in the case where \( M = N \neq 1 \), we may not apply the functional equation Theorem 8 to \( Z\left(\frac{1}{2} - z, \frac{1}{2} + z; \psi^*, \rho'\right) \) in (4.1) because Theorem 8 is only valid if \( j = k = M = N \), but here we have \( k = 1 \neq N \). Nonetheless, equation (4.2) still holds: Indeed, in the case where \( M = N, \) the term with \( Z\left(\frac{1}{2} - z, \frac{1}{2} + z; \psi^*, \rho'\right) \) of (4.1) vanishes because \( G_N^{(\rho)}(\frac{1}{2} - z) = 0 \), and so only the first term of (4.2) will remain (note also that \( G_M^{(\rho)}(\frac{1}{2} - z) = 0 \) in this case).

Looking at (3.9) for a quadratic character \( \chi^* \) whose modulus is coprime to \( P \), we see that

\[ F_p^{(\chi^*)}(\frac{1}{2} - z) = \frac{P_{p-1}}{P_{p-2z-1} - 1} \quad \text{and} \quad G_p^{(\chi^*)}(\frac{1}{2} - z) = P_{p-2z+1/2} \left( \frac{\chi^*(P)(1 - P_{p-2z})}{P_{p-2z-1} - 1} \right). \]

Therefore, setting

\[ \Phi := \{(\rho, \rho'), (\rho, \psi_1), (\psi_1, \rho'), (\psi_1, \psi_1), (\psi_1, \rho' \rho), (\psi_1, \rho)\} \]

we have

\[ Z\left(\frac{1}{2}, \frac{1}{2} - z; \rho, \rho'\right) = \sum_{(\chi^*, \psi') \in \Phi} \beta_{\chi^*, \psi'}^{(\chi^*, \psi')} \omega_{\psi, \psi'}^{(\chi^*, \psi')} (z) \gamma_{\psi, \psi'}^{(\chi^*, \psi)} Z(\frac{1}{2} + z, \frac{1}{2}; \chi^*, \chi') \]

where we absorb the \( F_p^{(\chi^*)}(\frac{1}{2} - z) \) factors and parenthetical expression of (4.3) for the \( G_p^{(\chi^*)}(\frac{1}{2} - z) \) factors, as well as the \( c_{\nu, \nu, \psi^*}^{(\rho, \psi^*)} \) factors into the \( \omega_{\psi, \psi'}^{(\chi^*, \psi')} (z) \) functions, and collect the remaining factors into the \( \beta_{\chi^*, \psi'}^{(\chi^*, \psi')} \) and \( \gamma_{\psi, \psi'}^{(\chi^*, \psi)} \) coefficients. Hence we see that for \( \Re z > 0 \), the \( \omega_{\psi, \psi'}^{(\chi^*, \psi')} (z) \) functions are holomorphic satisfying the bound

\[ \omega_{\psi, \psi'}^{(\chi^*, \psi')} (z) \ll (1 + |3z|)^{\Re z} \]

uniformly in \( M \) and \( N \). Thus, we obtain the upper bounds in Table 1.
Proof.

Let \( \beta > 0 \) to uniformly in \( \xi \).

The following lemma essentially takes the preceding functional equation a step further by opening the first sum of \( Z \).

Lemma 4. There exist smooth, rapidly decaying functions \( V(\xi; t) \) and \( V^{(x', \chi')}(\xi; t) \) such that for any constant \( X > 0 \) one has

\[
Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') = X^{-it} \sum_{(d,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\rho')}{d^{1/2}} P_{d_0,d_1}(\frac{1}{2}) V \left( \frac{d}{X}; t \right) \\
+ X^{it} \sum_{(\chi', \chi') \in \Phi} \beta^{(x', \chi')}(\psi, \psi') \sum_{(d,2MN)=1} \frac{L^{(2MN)}(\frac{1}{2}, \chi_{d_0}\chi')}{d^{1/2}} Q_{d_0,d_1}(\frac{1}{2}) V^{(x', \chi')}(\psi, \psi') \left( \frac{dX}{\gamma^{(x', \chi')}}, t \right),
\]

where \( \beta^{(x', \chi')} \) and \( \gamma^{(x', \chi')} \) satisfy the bounds listed in Table 1 and we have the bounds

\[
V^{(x', \chi')}(\xi; t) \ll |\xi|^{-B} (1 + |t|)^B \quad \text{and} \quad V(\xi; t) \ll |\xi|^{-B}
\]

uniformly in \( \xi \) and \( t \) for any number \( B > 0 \).

Proof. Let \( B > 0 \), \( H \) be an even, holomorphic function with \( H(0) = 1 \) satisfying the growth estimate \( H(z) \ll_{\Re z, A} (1 + |z|)^{-A} \) for any \( A > 0 \). We consider the integral

\[
I(c, X, t) = \frac{1}{2\pi i} \int_{(1)} X^{cz} \left( \frac{4 \frac{1}{2} + it + cz - 4}{4 \frac{1}{2} + it - 4} \right)^2 Z(\frac{1}{2}, \frac{1}{2} + it + cz; \rho, \rho') H(z) \frac{dz}{z}
\]

for a real number \( c \), a positive real number \( X > 0 \), and a fixed real number \( t \). Examining the expression when \( c = 1 \), the fraction cancels the pole of the \( Z \) factor at \( z = 1/2 - it \). We apply a shift of the contour to \( \Re z = -1 \), picking up the pole at \( z = 0 \), whence we obtain

\[
I(1, X, t) = Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') + \frac{1}{2\pi i} \int_{(-1)} X^z \left( \frac{4 \frac{1}{2} + it + z - 4}{4 \frac{1}{2} + it - 4} \right)^2 Z(\frac{1}{2}, \frac{1}{2} + it + z; \rho, \rho') H(z) \frac{dz}{z}.
\]

We now apply a change of variables \( z \mapsto -z \), arriving at

\[
Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') = I(1, X, t) + I(-1, X, t).
\]

### Table 1. Coefficient upper bounds

| \( M \neq N \) | \( (\chi', \chi') \) bound | \( (\rho, \rho') \) bound | \( (\psi_1, \rho') \) bound | \( (\psi_1, \psi_1) \) bound | \( (\psi_1, \rho \rho') \) bound | \( (\psi_1, \rho) \) bound |
|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \beta^{(x', \chi')} \) | \( M^{-1/2} \) | \( N^{-1/2} \) | \( M^{-1/2} N^{-1/2} \) | \( N^{-1} \) | \( M^{-1/2} N^{-1} \) |
| \( \gamma^{(x', \chi')} \) | \( M^2 N \) | \( M^2 N \) | \( M^2 N^2 \) | \( M^2 N^2 \) |
| \( M = N \) | \( \beta^{(x', \chi')} \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( \gamma^{(x', \chi')} \) | \( N^2 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |
Applying the functional equation (4.4) and the switch of summation formula (3.2) and expanding the $Z$ functions, definition (3.3) gives that $I(-1, X, t)$ equals

$$(4.8) \quad X^{it} \sum_{(\chi, \chi') \in \Phi} \beta(\chi, \chi') \sum_{(m, 2MN) = 1} L(2MN) \left( \frac{1}{2}, \overline{\chi m_0} \chi' \psi \right)(m) \overline{Q(\chi' \psi)}(\frac{1}{2}) \frac{m^s}{\Gamma(\frac{s}{2}, \psi \psi') \left( mX \right)} V(\chi, \chi')(t),$$

where

$$V(\chi, \chi')(\xi; t) = \frac{1}{2\pi i} \int_{(1)} \left( \frac{4^{1/2+it} - 4}{4^{1/2+it} - 4} \right)^2 \xi^{-z} z \left( \zeta - it \right) H(z) \frac{dz}{z}.$$

We wish to obtain an upper bound for $V(\xi, \chi')(\xi; t)$. Moving the contour to $B$ (recalling that there are no poles of $\omega(\xi, \chi')$ in this region), and bounding by taking the absolute value of the integrand and using the bound (4.0), we have

$$V(\xi, \chi')(\xi; t) \ll |\xi|^{-B} (1 + |t|)^B$$

uniformly in $\xi$ and $t$. Changing the summation variable from $m$ to $d$ in (4.8), we obtain the first term in the statement of the lemma.

Looking at $I(1, X, t)$, we have

$$I(1, X, t) = X^{-it} \sum_{(d, 2MN) = 1} \frac{L(2MN)(\frac{1}{2}, \chi_{d_0} \rho)(d) D_{p_0, \chi'}(\frac{1}{2})}{d^{1/2 - \varepsilon}} V \left( \frac{d}{X}, t \right),$$

where

$$V(\xi; t) = \frac{1}{2\pi i} \int_{(1)} \left( \frac{4^{1/2+it} - 4}{4^{1/2+it} - 4} \right)^2 \xi^{-z} H(z) \frac{dz}{z}.$$

Also, it is immediate that we have the bound $V(\xi; t) \ll |\xi|^{-B}$ uniformly in $\xi$ and $t$. □

We now wish to truncate the sums above, accruing an error. This is the object of the following lemma.

**Lemma 5.** Let $A$ be a large positive constant, $t$ be a real number, and $V(\xi; t)$ be a rapidly decaying function in $\xi$ satisfying

$$V(\xi; t) \ll |\xi|^{-B} (1 + |t|)^B,$$

uniformly in $\xi$ and $t$ for any number $B > 0$, let $\chi$ be a character modulo $k$, and let $a$ be an arithmetic function satisfying $a(d) \ll d^\varepsilon$ uniformly in $d$. Then we can truncate the double sum

$$\sum_{(d, 2MN) = 1} \frac{L(2MN)(\frac{1}{2}, \chi_{d_0} \chi')(d) a(d)}{d^{1/2}} V \left( \frac{d}{Y}, t \right)$$

at $d < Y^{1+\varepsilon}$, accruing an error that is bounded above by $O((1 + |t|)^{2A/\varepsilon} k^{1/4+\varepsilon} Y^{-A})$.

**Proof.** The $L$-function is bounded asymptotically by $(d_0 k)^{1/4+\varepsilon}$ due to the Phragmén-Lindelöf convexity bound (2.2). The $V$ factor is bounded by its argument to an arbitrarily large power $-B$. Applying this gives an error that is bounded above by

$$(1 + |t|)^{2B} \sum_{d > P^{1+\varepsilon}} \frac{(d_0 k)^{1/4+\varepsilon}}{d^{1/2-\varepsilon}} \left( \frac{d}{Y} \right)^{-B},$$

and the result follows. □
In order to bound \( Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') \), by applying a smooth partition of unity as in [Blo11], it now suffices to bound
\[
D_{W,a}(Y; t, \chi', \chi) := \sum_{d|2MN} \frac{L(2MN)(\frac{1}{2}, \chi d a \chi') \lambda(d)a(d)}{d^{1/2}} W \left( \frac{d}{Y}; t \right)
\]
for \( t \) a real number, \( \psi, \psi' \in (\mathbb{Z}/8\mathbb{Z})^* \), a smooth function \( W \) with support on \([1, 2]\) satisfying
\[
W(x; t) \ll_B x^{-B} (1 + |t|)^B
\]
uniformly in \( x \) and \( t \) for any \( B > 0 \), an arithmetic function \( a \) satisfying the bound \( a(d) \ll d^\varepsilon \), and the following conditions, according to each of the two sums in Lemma 6, either \((\chi, \chi') \in \Phi \) (cf. (4.4)) with conductors \( k \) and \( j \) respectively, and
\[
1 \leq Y \leq \left( \lambda(\chi, \chi') X^{-1} \right)^{1+\varepsilon},
\]
or \((\chi, \chi') = (\rho', \rho) \) and
\[
1 \leq Y \leq X^{1+\varepsilon}.
\]
Expanding according to the Dirichlet functional equation, and further truncating that sum expresses \( D_{W,a}(Y; t, \chi', \chi) \) as a double finite character sum, allowing us to apply Heath-Brown’s large sieve estimate Corollary 1. The result of this is the following lemma.

**Lemma 6.** We have the bound
\[
D_{W,a}(Y; t, \chi', \chi) \ll (1 + |t|)^{2/\varepsilon} (MN)^{\varepsilon} \left( Y^{1+\varepsilon} + (Yj)^{1/2+\varepsilon} \right)^{1/2+\varepsilon}
\]
uniformly in \( t \), \( Y \), \( j \), and \( k \).

**Proof.** Applying Lemma 5 above, we have
\[
D_{W,a}(Y; t, \chi', \chi) = \sum_{(d, 2MN) = 1 \atop d < Y^{1+\varepsilon}} \frac{L(2MN)(\frac{1}{2}, \chi d a \chi') \lambda(d)a(d)}{d^{1/2}} W \left( \frac{d}{Y}; t \right) + O((1 + |t|)^{2/\varepsilon} j^{1/4+\varepsilon} Y^{-1}).
\]
Applying (2.4) and (2.3), we have that \( D_{W,a}(Y; t, \chi', \chi) \) equals, up to an error of \( O((1 + |t|)^{2/\varepsilon} j^{1/4+\varepsilon} Y^{-1}) \),
\[
(4.9) \quad 2 \sum_{(d, 2MN) = 1 \atop d < Y^{1+\varepsilon}} \frac{\lambda(d)}{d^{1/2}} \prod_{p|2MN} \left( 1 - \frac{\chi d a \chi'}{p^{1/2}} \right) a(d) W \left( \frac{d}{Y}; t \right) \sum_{n=1}^\infty \frac{(\chi d a \chi')(n)}{n^{1/2}} G_{\kappa, \chi} \left( \frac{n}{\sqrt{c_0 d a \chi'}} \right),
\]
where \( c_0 \) is given in (2.1) and \( G_{\kappa, \chi} \) is given in (2.9). Because of the rapid decay of \( G_{\kappa, \chi} \) and \( W \), we can truncate the \( n \)-sum at \( n < (Yj)^{1/2+\varepsilon} \).

Applying the bounds (2.6) and (2.7), and \( a(d) \ll d^\varepsilon \), the error obtained by this is bounded by
\[
(MN)^{\varepsilon} (1 + |t|)^B \sum_{d < Y^{1+\varepsilon}} \frac{1}{d^{1/2-\varepsilon}} \left( 1 + \frac{d}{Y} \right)^{-B} \sum_{n > (Yj)^{1/2+\varepsilon}} \frac{1}{n^{1/2}} \left( 1 + \frac{n}{\sqrt{c_0 d a \chi'}} \right)^{-B}
\]
for any large positive number \( B \). Indeed, this is bounded above by
\[
(MN)^{\varepsilon} (1 + |t|)^B Y^{Bj^{1/2}} \sum_{d < Y^{1+\varepsilon}} d^{-B-1/2+\varepsilon} \sum_{n > (Yj)^{1/2+\varepsilon}} n^{-B-1/2} d^{B/2}
\]
\[
\leq (MN)^{\varepsilon} (1 + |t|)^B Y^{Bj^{1/2}} (Y^{1+\varepsilon})^{-B-1/2+\varepsilon}. Y^{j(1/2+\varepsilon)-(B+1/2)}.
\]
We first apply Lemma 4 which gives
\[ (\frac{c_0d_0}{n})^{s/2}a(d)\frac{(\chi\nu(d))(n)\chi'(d)}{d^{1/2+w1/2+s}}\tilde{W}(w;t)Y^w\] 
Looking at the summand, for \( \Re s = \varepsilon \) and \( \Re w = \varepsilon \), we have
\[ (c_0d_0)^{s/2}a(d)\frac{\chi'(d)}{d^{1/2+w}} \ll d^{-1/2+\varepsilon} \] 
and
\[ \frac{\chi(n)}{n^{1/2+s}} \ll n^{-1/2+\varepsilon}, \]
so that we can now apply Corollary 1 and the result follows.

5. Proof of Theorem 1

We first apply Lemma 6 which gives
\[ Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') = X^{it} \sum_{(\chi, \chi') \in \Phi} \beta_{\psi, \psi'}^{(\chi, \chi')} D_V,Q \left( \frac{\gamma_{\psi, \psi'}}{X}; t, \chi', \chi \right) + X^{-it} D_V,P(X; t, \rho, \rho'), \]
where the subscripts for \( D \) in the first term are \( V = V^{(\chi, \chi')}_{\psi, \psi'}(\xi, t) \) and \( Q = Q^{(\psi, \psi')}_{\psi, \psi'}(\sqrt{2}), \) and the subscripts for \( D \) in the second term are \( V = V(\xi, t) \) and \( P = P^{(\psi, \psi')}_{\psi, \psi'}(\frac{1}{2}). \) Applying Lemma 8 further gives
\[ Z(\frac{1}{2}, \frac{1}{2} + it; \rho, \rho') \ll (1 + |t|)^{2/3}(MN)^{\varepsilon} \left[ (X + (NX)^{1/2})^{1/2+\varepsilon} \right] \]
\[ + (1 + |t|)^{2/3}(MN)^{\varepsilon} \max_{(\chi, \chi') \in \Phi} \beta_{\psi, \psi'}^{(\chi, \chi')} \left( \frac{\gamma_{\psi, \psi'}}{X} X^{-1} + \left( \frac{\gamma_{\psi, \psi'}}{X} X^{-1} \right)^{1/2} \right)^{1/2+\varepsilon}. \]

Here, \( \Phi \) is given by \([1, 3] \), and \( j \) is the conductor of \( \chi'. \)

We initially set \( X = M^{a}N^{b} \) and eventually choose optimized values for \( a \) and \( b \), depending on the bounds for the \( \beta \)'s and \( \gamma \)'s from Table 10. We come to an optimal choice of \( X = M^{2/3}N^{1/3} \) and obtain the subconvexity bound of
\[ M^{1/3+\varepsilon}N^{1/6+\varepsilon} + M^{1/6+\varepsilon}N^{1/3+\varepsilon} \approx (MN(M + N))^{1/6+\varepsilon} \]
(even in the case of \( M = N \), compared to the convexity bound of \((MN)^{3/8+\varepsilon}\).
6. Proof of Theorem 2

Here we present an application of the theory of double Dirichlet series as developed. Given a fixed positive prime \( N \), we seek an upper bound on \( d \) such that \( L(1/2, \chi_d\chi_N) \) does not vanish. We follow a modified version of a method outlined in [HK10], and we rigorously prove an important lower bound on coefficients which arise from a residue of \( Z \). Let \( h(y) \) be a smooth weight function as in Definition 1. Expanding as per (3.3) and by Mellin inversion we have

\[
\int_{\gamma} \hat{h}(w)Z(\frac{1}{2}, w; \chi_N, \psi_1)X^w dw = \sum_{(d,2N) = 1} L^{(2N)}(\frac{1}{2}, \chi_{dN})P_{d_0,d_1}(\frac{1}{2})h(d). 
\]

We move the contour of the integral on the left-hand side to \( \Re w = -\varepsilon \), picking up a residue at \( w = 1 \) due to the double pole of \( Z(\frac{1}{2}, w; \chi_N, \psi_1) \) there. If we write its Laurent expansion as

\[
Z(1/2, w; \chi_N, \psi_1) = \frac{\mu_N}{(w - 1)^2} + \frac{\nu_N}{(w - 1)} + \cdots
\]

then the left-hand side of (6.1) equals

\[
[\mu_N \hat{h}(1) + \mu_N \hat{h}'(1)]X + \mu_N \hat{h}(1)X \log X + \int_{(-\varepsilon)} \hat{h}(w)Z(\frac{1}{2}, w; \chi_N, \psi_1)X^w dw.
\]

We now apply the symmetric functional equation (3.15) with \( \rho\psi = \chi_N, \rho' = \psi' = \psi_1, \) and \( (s, w) = (1/2, w) \) with \( \Re w = -\varepsilon \) to \( Z \) in the resulting integral. Using Lemma 3 to bound the coefficients, we see that \( C_{\rho\chi_N} = N \) when \( n = 1 \) and is unity otherwise, and \( C_{\rho'\chi_m} = 1 \) in every case because \( s = 1/2 \). In every case the \( A \) factors will always be bounded above by \( O(N^\varepsilon) \), since \( \Re w = -\varepsilon \). Also, \( A_m^{\rho\chi_N}(s + w - 1/2) \) vanishes when \( m = N \) and \( n = 1 \). Hence, we have

\[
Z(1/2, w; \chi_N, \psi_1) = \sum_{\psi*, \psi**, \psi*** \in (\mathbb{Z}/N\mathbb{Z})^2, m,r|N} \sum_{m,r|N} B(w; N, m, r, \psi*, \psi**, \psi***) \\
\times \left(N^{1/2-w}Z(1-w, 1/2; \chi_N \bar{\chi}_r \psi* \psi**, \psi***) + Z(1-w, 1/2; \bar{\chi}_r \psi* \psi**, \bar{\chi}_m \psi***)\right),
\]

where, due also to the bound (3.16), we have

\[
|B(w; N, m, r, \psi*, \psi**, \psi***)| \ll N^\varepsilon(1 + |w|)^{1+\varepsilon}.
\]

We now apply the bound (3.18), which finally gives

\[
|Z(\frac{1}{2}, -\varepsilon + it; \chi_N, \psi_1)| \ll (1 + |t|)^{1+\varepsilon}N^{1/2+\varepsilon},
\]

which we shall apply to the integral. We thus have

\[
S(X; \chi_N) := \sum_{(d,2N) = 1} L^{(2N)}(\frac{1}{2}, \chi_{dN})P_{d_0,d_1}(\frac{1}{2})h(d) = a_N X \log X + b_N X + O(N^{1/2+\varepsilon})
\]

for certain coefficients \( a_N \) and \( b_N \). More elementary analysis of \( S(X; \chi_N) \) via Theorem 9 (cf. §7) below gives us the lower bounds \( a_N, b_N \gg N^{-\varepsilon} \). If we now assume that

\[
L^{(2N)}(\frac{1}{2}, \chi_{dN}) = L(\frac{1}{2}, \chi_{dN}) \prod_{p|d_1|N} (1 - p^{-1/2})
\]

vanishes for \( d \ll X \), then we get a contradiction as long as \( a_N X \log X \) is greater than the error term. We see we can choose \( X = N^{1/2+\varepsilon} \), as required.
We essentially combine two asymptotic formulas for $S(X; \chi_N)$: Looking at it elementarily via Theorem 9 from §7 gives us a bad error term, but lower bounds on the coefficients. Looking at it analytically as above allows us to take advantage of the bound (3.18) for $Z$ in order to obtain a smaller error term.

7. An Asymptotic Formula for an $L$-function sum

7.1. Result. Of particular importance for the smallest nonvanishing quadratic central value of twisted $L$-functions result is an asymptotic formula for the weighted and twisted $L$-function sum given by $S(X; \chi)$ in (6.2). Indeed, we already have an asymptotic formula, but it is important that we have a lower bound on the main term coefficients.

Let $N$ be a natural number, $X$ a large positive real number, and let $\chi$ be a quadratic primitive character modulo $N$. Let $h$ be a smooth weight function as defined in Definition 1, and define

$$S(X; \chi) = \sum_{(d, 2N) = 1} L(2N, \frac{1}{2}, \chi_{d_0}) P_{d_0, d_1}(\frac{1}{2}) h(d/X).$$

We seek to obtain information on the main term coefficients of the asymptotic formula (6.2). We shall prove the following Theorem.

Theorem 9. There exists $\delta > 0$ such that we have the asymptotic formula

$$S(X; \chi) = a_N X \log X + b_N X + O(N^{3/8 + \epsilon} X^{1-\delta}),$$

where $N^{-\epsilon} \ll a_N, b_N \ll N^\epsilon$, uniformly in $N$.

7.2. Proof of Theorem 9. According to the approximate functional equation for Dirichlet $L$-functions (2.3), we have

$$L(\frac{1}{2}, \chi_{d_0}) = 2 \sum_{n=1}^\infty \frac{(\chi_{d_0})(n)}{n^{1/2}} G_{\kappa} \left( \frac{n}{\sqrt{cd_0 N}} \right),$$

where $G_{\kappa}$ is given in (2.5), and $c_0$ is given in (2.4). From the definition of $P_{d_0, d_1}^{(\chi)}$ in Theorem 6 we have

$$P_{d_0, d_1}(\frac{1}{2}) = \sum_{f | d_1^2} \mu(f_0)(\chi_{d_0})(f_0) f_0^{1/2},$$

where we write $f = f_0 f_1^2$ with $f_0$ squarefree. We shall also use the expansion

$$1 - \frac{(\chi_{d_0})(2)}{2^{1/2}} = \sum_{g | 2} \mu(g)(\chi_{d_0})(g) g^{1/2}.$$

Applying these expressions to $S(X; \chi)$, we have

(7.1) $$S(X; \chi) = 2 \sum_{(d_1, 2N) = 1} \sum_{(d_0, 2N) = 1} \sum_{f_1 | d_1} \sum_{g | d_0} \frac{\mu(g)(\chi_{d_0})(g)}{h^{1/2}} \sum_{n=1}^\infty \frac{(\chi_{d_0})(n)}{n^{1/2}} G_{\kappa} \left( \frac{n}{\sqrt{cd_0 N}} \right) h \left( \frac{d_0 d_1^2}{X} \right).$$

For a subset $H \subset \mathbb{N}$, we define $S_H(X; \chi)$ to be the same as the expression (7.1) with the added condition in the $n$-sum of $ng \in H$. We use Mellin inversion for $G_{\kappa}$ to separate the variables, and by
moving the $d_0$-sum to the inside, we get that $S_H(X;\chi)$ equals

\begin{equation}
2 \int_{(1/2)^+} N^{\frac{\varepsilon}{2}} \sum_{(d_1,2N)=1} \sum_{f_1|d_1} \sum_{g|2f_1^2} \frac{\mu(g)\chi(g)}{g^{1/2}} \sum_{n|d_0,2N=1} \frac{\chi(n)}{n^{s+1/2}} \sum_{(d_0,2N)=1} \tilde{G}_\kappa(s)\chi_{d_0}(ng)c_0^{s/2}d_0^{-s/2}h\left(\frac{d_0d_1^2}{X}\right)ds.
\end{equation}

The variables $\kappa$ and $c_0$ depend on the residues of $d_0$ and $N$ modulo 4. Thus, given $\ell \in \{\pm 1\}$, we define $\kappa(\ell)$ and $c_0(\ell)$ to be the corresponding values for $d_0 \equiv \ell$ (mod 4). For convenience, for $\ell \in \{\pm 1\}$, we define $S_H(X;\chi,\ell,\ell)$ to be the same as (7.2) except that $\kappa$ and $c_0$ are replaced with $\kappa(\ell)$ and $c_0(\ell)$, and $\psi_i(d_0)$ is multiplied to the summand in the $d_0$-sum.

Observing that $\frac{1}{2}(1 + \psi_{-1}(d_0))$ is the characteristic function of $d_0 \equiv \pm 1$ (mod 4), the $d_0$-sum in the integrand of (7.2) is

\[
\frac{1}{2} \sum_{(d_0,2N)=1} (1 + \psi_{-1}(d_0))\tilde{G}_\kappa(\pm 1)(s)c_0(\pm 1)^{s/2}\tilde{\chi}_{ng}(d_0)d_0^{-s/2}h\left(\frac{d_0d_1^2}{X}\right),
\]

whence we see that

\begin{equation}
S_H(X;\chi) = \frac{1}{2} \sum_{\ell = \pm 1} \sum_{\ell = \pm 1} \text{sgn}(1 + \ell + \ell)S_H(X;\chi,\ell,\ell).
\end{equation}

For treatment of the $d_0$-sum, for a positive real number $Y$ and a Dirichlet character $\psi$ we further define

\[
T(s;Y,\psi) = \sum_{d_0=1}^{\infty} \psi(d_0)d_0^{s/2}h\left(\frac{\bar{d}_0}{Y}\right),
\]

where the sum is over squarefree $d_0$. With these simplifications, we obtain that $S_H(X,\chi,\ell,\ell)$ equals

\begin{equation}
2 \int_{(1/2)^+} (N\tilde{c}_0(\ell))^{s/2}\tilde{G}_\kappa(\ell)(s) \sum_{(d_1,2N)=1} \sum_{f_1|d_1} \sum_{g|2f_1^2} \frac{\mu(g)\chi(g)}{g^{1/2}} \sum_{n|d_0,2N=1} \frac{\chi(n)}{n^{s+1/2}} \sum_{(d_0,2N)=1} T(s;X/d_1^2,\chi_0^{(2N)}\tilde{\chi}_{ng}\psi_i)ds.
\end{equation}

We shall choose $H = \square$ which we use to denote the set of positive squares, and denoting the complement of $H$ in $\mathbb{N}$ by $\bar{H}$, it is clear that

\[
S(X;\chi) = S_{\square}(X;\chi) + S_{\bar{\square}}(X;\chi).
\]

In order to estimate the size of $T(s;Y,\psi)$, we observe that there are order $Y$ squarefree numbers up to $Y$. If $\psi$ is a principal character and $\Re s = \varepsilon$, we therefore expect this sum to be roughly of size $Y^{1+\varepsilon}$. If $\psi$ is non-principal, the oscillations will give us a Pólya-Vinogradov type estimate.

Indeed, with this last point in mind, we step back to explain the main idea of the proof: The sum $S_H(X,\chi,\psi_i,\ell)$ will hence only be large when $\tilde{\chi}_{ng}\psi_i$ is principal, that is, precisely when $H = \square$ and $\ell = 1$, and will be small otherwise. To this end, we have the following asymptotic formula.

**Lemma 7.** For $\Re s > 0$, we have

\[
T(s;Y,\chi^{(m)}) = \frac{\tilde{h}(1 + s/2)}{\zeta(2)} \prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1} Y^{1+s/2} + U(s;Y,m)
\]

with $U(s;Y,m)$ holomorphic and

\[
U(s;Y,m) \ll |s|^{1/2 - \Re s/2 + \varepsilon}m^{Y^{1/2 + \Re s/2 + \varepsilon}}.
\]
uniformly in $Y$ and $\Im s$. Further, if $\psi = \chi_0^{(m)}\tilde{\psi}$, where $\tilde{\psi}$ is a nontrivial quadratic primitive character with conductor $c$, then
\[
T(s; Y, \psi) \ll |c|s^{1/2-\Re s/2+\varepsilon}m^sY^{1/2+\Re s/2+\varepsilon}
\]
uniformly in $\Im s$, $m$, and $Y$.

Proof. Via Mellin inversion, we have
\[
T(s; Y, \chi_0^{(m)}\tilde{\psi}) = \int_{(1+\Re s/2+\varepsilon)} T_s(z)\tilde{h}(z)Y^zdz,
\]
where we have the generating function
\[
T_s(z) = \sum_{d=1}^{\infty} \frac{\mu^2(d)(\chi_0^{(m)}\tilde{\psi})(d)}{d^{z+s/2}} = \frac{L(z-s/2, \tilde{\psi})}{L(2z-s, \psi)} \prod_{p|m} \left(1 + \frac{1}{p^{s/2}}\right)^{-1}.
\]
If $\tilde{\psi}$ is the trivial character then the $L$-function in the numerator is just the zeta function, and therefore has a pole at $z = 1 + s/2$. Due to the $1/L(2z-s, \psi)$ factor, all other poles lie in $\Re z < \Re s/2$. We move the contour to $(1/2 + \Re s/2 + \varepsilon)$, and in the case where $\psi$ is trivial, we pick up the residue
\[
\tilde{h}(1+s/2)\prod_{p|m} \left(1 + \frac{1}{p}\right)^{-1} Y^{1+s/2}.
\]
Due to (2.7) along with the convexity bound
\[
L(z-s/2, \tilde{\psi}) \ll |c(1 + |\Im z - \Im s/2|)|^{1/2-\Re s/2+\varepsilon}
\]
obtained from (2.2), we bound the resulting integral by
\[
|c|s^{1/2-\Re s/2+\varepsilon}m^sY^{1/2+\Re s/2+\varepsilon}.
\]

7.2.1. Main term. As explained above, the main contribution will come from $S_\square(X; \chi, 1, \ell)$, which we will bound from below. Looking at the expansion (7.3) with $\ell = \ell = 1$ and $H = \square$, since $ng$ is square, and recalling that $g$ is squarefree, we have $n = gm^2$ for $m \in \mathbb{N}$, so the inner sum is
\[
\sum_{n\in\square \atop (n,N)=1} \frac{1}{n^{s+1/2}}T(s; X/d_1^2, \chi_0^{(2ngN)}) = \frac{1}{g^{s+1/2}} \sum_{(m,N)=1} \frac{1}{m^{2s+1}}T(s; X/d_1^2, \chi_0^{(2gmN)}).
\]

By Lemma 7, the above expression is
\[
\frac{1}{g^{s+1/2}} \sum_{(m,N)=1} \frac{1}{m^{2s+1}} \prod_{p|2gmN} \left(1 + \frac{1}{p}\right)^{-1} \frac{\tilde{h}(1+s/2)}{\zeta(2)} \left(\frac{X}{d_1^2}\right)^{1+s/2} + U(s; X/d_1^2, 2gmN)
\]
where $U(s; X/d_1^2, 2gmN)$ is holomorphic for $\Re s > 0$ and
\[
U(s; X/d_1^2, 2gmN) \ll |s|^{1/2-\Re s/2+\varepsilon}(gmN)^\varepsilon \left(\frac{X}{d_1^2}\right)^{1/2+\Re s/2}.
\]
Referring to (7.3), and moving the contour to \((\epsilon)\), the error term is bounded above by

\[
\sum_{(d_1,2N)=1} \int_{(\epsilon)} |Nc_0(\ell)|^{\Re s/2} |\tilde{G}_{\kappa(\ell)}(s)||s|^{1/2-\Re s/2+\epsilon} \sum_{f_1|d_1} \sum_{g|2f_1^2} \sum_{(g,N)=1} \frac{1}{g^{s+1}} \sum_{(m,N)=1} (gmN)^\epsilon \left( \frac{X}{d_1^2} \right)^{1/2+\Re s/2} ds.
\]

Bounding the sums absolutely and using the fact that \(\tilde{G}_{\kappa(\ell)}(s)\) decays rapidly in fixed vertical strips, we see that this is bounded above by \(N^\epsilon X^{1/2+\epsilon}\). As for the main term, through a calculation we have the following result.

**Lemma 8.** We have the identity

\[
\sum_{(m,N)=1} \frac{1}{m^{2s+1}} \prod_{p|2gmN} \left(1 + \frac{1}{p}\right)^{-1} = \zeta(2s+1)E_0(s)E_1(s;g),
\]

where

\[
E_0(s) = \frac{4}{9} (1 - 2^{-2s-1}) \prod_p (1 + p^{-1}) (1 + p^{-1}(1 - p^{-2s-1})^{-1})^{-1} \prod_p (1 + p^{-1})^{-2(1 + p^{-1} - p^{-2s-1})},
\]

\[
E_1(s;g) = \prod_{p|g} (1 + p^{-1})^{-2(1 - p^{-2s-1})^{-1}(1 + p^{-1} - p^{-2s-1})}.
\]

**Proof.** This is a straightforward but monotonous calculation which we omit. \(\square\)

Applying this, we now have that \(S_{\square}(X,\chi,1,\ell)\) equals, up to an error \(O(N^\epsilon X^{1/2+\epsilon})\),

\[
\frac{2}{\zeta(2)} \int_{(\epsilon)} \tilde{h}(1+\frac{s}{2})(c_0(\ell)N)^{s/2} X^{1+s/2} \tilde{G}_{\kappa(\ell)}(s) \zeta(2s+1) E_0(s) \sum_{(d_1,2N)=1} d_1^{-2-s} \sum_{f_1|d_1} \sum_{g|2f_1^2} \sum_{(g,N)=1} \frac{\mu(g)}{g^{s+1}} E_1(s;g) ds.
\]

Since \(E(s;g)\) is multiplicative in \(g\), we can further collapse the \(g\)-sum above into an Euler product. Hence we have

(7.7)

\[
S_{\square}(X,\chi,1,\ell) = \frac{2}{\zeta(2)} \int_{(\epsilon)} \tilde{h}(1+s/2)(c_0(\ell)N)^{s/2} X^{1+s/2} \tilde{G}_{\kappa(\ell)}(s) \zeta(2s+1) E_0(s) H(s) ds + O(N^\epsilon X^{1/2+\epsilon}),
\]

where

\[
H(s) = \sum_{(d_1,2N)=1} d_1^{-2-s} \sum_{f_1|d_1} \prod_{p|2f_1} \left(1 - p^{-s-1} E_1(s;p)\right).
\]

We have the following estimates for \(H\).

**Lemma 9.** There exists \(K > 0\) such that \(1/3 \leq H(0) \leq K\) and \(|H'(0)| \leq K\).

**Proof.** Let \(\Re s \geq 0\), and for convenience define

\[
H(s;d_1) = \sum_{f_1|d_1} \prod_{p|2f_1} \left(1 - p^{-s-1} E_1(s;p)\right).
\]
Because $0 < E_1(0; p) \leq 4/3$, we have $1/3 \leq 1 - p^{-1}E_1(0; p) < 1$, and so taking $d_1 = 1$, we have $H(0) \geq 1/3$. Hence by the same reasoning,

$$|H(0; d_1)| \leq \sum_{n|d_1} 1 \ll d_1^\varepsilon,$$

and so we see that $H(0)$ is absolutely bounded above. In order to show that $H'(0)$ is absolutely bounded, because we have

$$H'(0) = \sum_{(d_1, 2N) = 1} d_1^{-2} (H'(0; d_1) - H(0; d_1) \log d_1),$$

it suffices to show that $H'(0; d_1) \ll d_1^{-\varepsilon}$, and for this, it suffices to show that $E'_1(0; p)$ is absolutely bounded, which is easily seen via taking the logarithmic derivative. □

We shall also need the following bounds for the $E_0$ function.

**Lemma 10.** We have the bounds

$$N^{-\varepsilon} \ll E_0(0), \quad E'_0(0) \ll N^\varepsilon.$$

**Proof.** We have

$$N^{-\varepsilon} \ll d(N) \ll 2 \frac{\zeta(2)}{\zeta(2)^2} \prod_{p|N} (1 + p^{-1})^{-2} = E_0(0) \ll 1,$$

and the result follows.

To treat the derivative, it is easily shown that the logarithmic derivative at $s = 0$ is bounded below by a constant and above by $N^\varepsilon$, as required. □

We wish to move the contour of integration of (7.7) from $(\varepsilon)$ to $(-\varepsilon)$. In doing so, we pick up a double pole, since $\tilde{G}_{\kappa(\ell)}(s)$ and $\zeta(2s + 1)$ have simple poles at $s = 0$. The residue of this pole shall be our main term. In order to calculate it, we shall need further analysis of the integrand. We define

$$A(s) = \tilde{h}(1 + s/2) N^{s/2} X^{1+s/2} E_0(s) H(s)$$

which is the part of the integrand which is holomorphic at $s = 0$. If the Laurent coefficients of $\zeta(2s + 1)$ and $\tilde{G}_{\kappa(\ell)}$ (centred at 0) are given by $e_n$ and $g_n$ respectively, then the residue is

(7.8) \quad \boxed{R := (g_{-1}e_0 + g_0e_{-1})A(0) + g_{-1}e_{-1}A'(0)},

and

(7.9) \quad A(0) = \tilde{h}(1) E_0(0) H(0) X,$n

(7.10) \quad A'(0) = \frac{1}{2} \tilde{h}'(1) E_0(0) H(0) X + \frac{1}{2} \tilde{h}(1) E_0(0) H(0)(\log N) X

$$\quad + \frac{1}{2} \tilde{h}(1) E_0(0) H(0) X \log X + \tilde{h}(1) E'_0(0) H(0) X + \tilde{h}(1) E_0(0) H'(0) X.$$n

We now prove the following expression of the residue $R$ from the above results.

**Corollary 2.** For sufficiently large $N$, we have

$$R = a_N X \log X + b_N X,$n

where

$$N^{-\varepsilon} \ll a_N, b_N \ll N^\varepsilon.$$
Proof. The first term comes from the $X \log X$ term in (7.10). We see that $g_{-1} = 1$ due to the definition of $G_{\kappa}(s)$ given in (2.5), and we also easily see that $e_{-1} = 1/2$, so that the coefficient for $A'(0)$ in (7.8) is positive. Now the bounds for $a_N$ follow from those for $E_0(0)$ and $H(0)$ from Lemmas 9 and 10 and the fact that $\tilde{h}(1)$ is simply a positive constant.

Next, we prove the bounds for the $X$ term coefficient. The upper bound follows from those for $E_0(0)$ and $H(0)$ in Lemmas 9 and 10 and again from the fact that $\tilde{h}(1)$ and $\tilde{h}'(1)$ are constants. As for the lower bound, there are two difficulties: First, we do not know if the coefficient for $A(0)$ in (7.8) is negative, which would result in a negative $X$ term contribution from (7.9). Secondly, we do not know whether the $\tilde{h}'(1)$ factor in the first term of (7.10) is positive. Nonetheless, we can simply compare the (positive) $X$ coefficient in the second term of (7.10) to that of the two terms just mentioned, and choose $N$ large enough so that the log $N$ factor dominates. The lower bound then follows again from Lemmas 9 and 10.

Now we just need to bound the remaining integral (7.7) with contour $(-\varepsilon)$. Using the triangle inequality and observing that $\tilde{G}_{\kappa}(s)$ decays rapidly in fixed vertical strips, we therefore see that the integral with contour $(-\varepsilon)$ is bounded above by $N^\varepsilon X^{1-\varepsilon/2}$.

7.2.2. Error term. Using the same method for bounding the error term in the previous section, we can bound $S(1; X; \bar{\chi}, -1, \ell)$ from above, since its expansion according to (7.4) will have the factor $T(s; X/d_1^2, \chi_0^{2N}\psi_1)$. By Lemma 4 this becomes (7.10). It now remains to bound $S(X; \bar{\chi}, \ell)$ from above.

According to (7.2) with the integral contour moved to $(3/4 + \varepsilon)$, we see that $S(X; \bar{\chi}, \ell)$ equals

$$2 \int_{(3/4 + \varepsilon)} (c_0(\ell)N)^s/2 \tilde{G}_{\kappa}(s) \sum_{(d_1, 2N) = 1} \sum_{(g, N) = 1} \frac{\mu(g)\chi(g)}{g^{1/2}} \sum_{ng \in \mathbb{N}} \frac{\chi(n)}{n^{s+1/2}} T(s; X/d_1^2, \chi_0^{2N} \psi_1) ds.$$  

Applying Lemma 7, since $ng$ is not square, the character $\chi_0^{2N} \psi_1$ is never principal, so we have

$$T(s; X/d_1^2, \chi_0^{2N} \psi_1) \ll (|s|ng)^{1/4 + \varepsilon} N^\varepsilon (X/d_1^2)^{7/8 + \varepsilon}.$$  

Now we can absolutely bound the sums and ignore the condition that $ng$ is not square, as we did for (7.6). We note that the $n$-sum will absolutely converge since $Re(s) = 3/4 + \varepsilon$, as long as we select a small enough $\varepsilon$ in the bound for the $T$-function above. We then arrive at a bound of

$$S(X; \bar{\chi}, \ell) \ll N^{3/8 + \varepsilon} X^{7/8 + \varepsilon},$$  

whence by (7.4) we have sufficiently bounded $S(X; \bar{\chi}).$

8. Appendix: Subconvexity bounds applied to non-vanishing results

Though the non-vanishing result Theorem 2 does not require the subconvexity bound in Theorem 1, one might ask whether a subconvexity bound could be applied to such a problem. We positively answer this here, outlining another method presented in [HK10].

In [6] we start with (6.11) and move the contour of the integral from (2) to $(-\varepsilon)$, picking up a residue at $w = 1$. Instead, we can move the contour to (1/2). Now it is clear that a good subconvexity bound for $Z(1/2, w; \chi_N, \psi_1)$ with $Re(w) = 1/2$ will produce a non-vanishing result. In our case, the subconvexity bound Theorem 1 will only yield an upper bound of $N^{2/3+\varepsilon}$, which is worse than that already presented. However, this formulation gives motivation for developing subconvexity bounds for...
double Dirichlet series. In particular, an improvement to Theorem 2 would result from an improvement in the $N$ exponent lower than $1/4 + \varepsilon$ in Theorem 1.

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Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3

E-mail address: adahl@yorku.ca