General Helicity Formalism for Two-hadron Production in $e^+e^-$ Collisions and the $\Lambda$ Polarizing Fragmentation Function

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We present the complete structure of the azimuthal dependences and polarization observables for two-hadron production in $e^+e^-$ annihilation processes within a transverse momentum dependent (TMD) approach adopting the helicity formalism. The leading-twist TMD fragmentation functions (TMD-FFs) for spin-1/2 hadrons are fully accounted for. The role of the polarizing FF, together with its extraction from Belle data for the transverse polarization of $\Lambda$’s, are discussed as well. Finally, predictions for SIDIS processes at EIC are presented.

KEYWORDS: TMD fragmentation functions, transverse polarization, polarizing FF

1. Introduction

Hadron production in $e^+e^-$ annihilation processes represents the cleanest way to access the parton-to-hadron fragmentation mechanism. The leading order (LO) expressions for all leading-twist (LT) azimuthal dependences and polarization observables are derived, within a TMD helicity formalism, for $e^+e^- \rightarrow h_1h_2X$ processes, together with the TMD-FFs for spin-1/2 hadrons [1]. We then present a fit of Belle data [2] for the transverse $\Lambda$ polarization, leading to the extraction of the polarizing FF (pFF) [3]: the distribution of a transversely polarized spin-1/2 hadron in the fragmentation of an unpolarized quark. Finally, we show some estimates for SIDIS processes.

2. Formalism

We consider the process $e^+e^- \rightarrow h_1h_2X$, where $h_{1,2}$ are spin-1/2 hadrons produced almost back-to-back. In the hadron-frame configuration, the $\hat{z}_L$-axis is in the opposite direction w.r.t. the hadron $h_2$ and the $\hat{x}_L\hat{z}_L$ lepton plane is determined by the lepton and the $h_2$ directions (with the $e^+e^-$ axis at angle $\theta$). The production plane is fixed by $\hat{z}_L$ and the direction of the hadron $h_1$, with transverse momentum $P_{T1}$, at an angle $\phi_1$ w.r.t. the lepton plane. The master formula at LO is given by

$$\rho^{h_1S_1, h_2S_2} \frac{d\sigma^{e^+e^- \rightarrow h_1h_2X}}{d\cos \theta d\hat{z}_1 d^2p_{T1} d^2p_{T2}} = \sum \frac{1}{32\pi s} \frac{1}{4} \hat{M}_{\lambda h_{1,1,1,1} \lambda h_{2,1,1,1}} \hat{M}^r_{\lambda h_{1,1,1,1} \lambda h_{2,1,1,1}} \hat{D}_{\lambda h_{1,1,1,1}} (\hat{z}_1, p_{T1}) \hat{D}_{\lambda h_{2,1,1,1}} (\hat{z}_2, p_{T2}),$$

where $z_i$ are the hadron light-cone momentum fractions and $p_{T_i}$ are the hadron transverse momenta w.r.t. the parent parton directions of motion; $\rho^{h_i S_i}_{\lambda h_{1,1}}$ is the helicity density matrix of the hadron $h_i$. 


with spin $S_h$; $\hat{M}_{\lambda_q\lambda_q^*\lambda_q^*}$'s are the helicity scattering amplitudes for the process $e^+(\lambda_+) + e^-(\lambda_-) \to q(\lambda_q) + \bar{q}(\lambda_{\bar{q}})$; $\hat{D}_{\lambda_q\lambda_q^*}(z, p_{\perp})$ is the product of the helicity fragmentation amplitudes for the $q \to h + X$ process, defined as

$$\hat{D}_{\lambda_q\lambda_q^*}(z, p_{\perp}) = \int_{X,\lambda_{h}} \hat{D}_{h_1,\lambda_{h_1}:h_2}(z, p_{\perp}) \hat{D}_{\lambda_q\lambda_q^*}(z, p_{\perp}), \quad (2)$$

where $\int_{X,\lambda_{h}}$ stands for a spin sum and phase space integration over all undetected particles. All other details can be found in Ref. [1].

The expression in Eq. (1) has then to be integrated over the unobserved variables as

$$\frac{d\sigma e^+ e^- \to h(S_1)h(S_2)X}{d\cos \theta dz_1 dz_2 d^2p_{1T}} = \int d^2p_{\perp 1} d^2p_{\perp 2} \delta^2(p_{\perp 1} - P_{1T} + p_{\perp 2}, z_{p_{1T}}/z_{p_{2T}}) \frac{d\sigma e^+ e^- \to h(S_1)h(S_2)X}{d\cos \theta dz_1 dz_2 d^2p_{\perp 1} dz_2 d^2p_{\perp 2}}, \quad (3)$$

where $z_{p_{1T}} = \frac{2E_{h_1}}{\sqrt{s}}$ are the momentum fractions ($z_{h_1} = \frac{2E_{h_1}}{\sqrt{s}}$ are the usual energy fractions).

### 2.1 Quark TMD-FFs for spin-1/2 hadrons

We start defining the TMD-FF for a polarized quark, with spin $s_q$, fragmenting into an unpolarized hadron: $\hat{D}_{h/q,s_q}(z, p_{\perp})$. Then, we combine the parton and hadron helicity density matrices and the generalized FFs, Eq. (2), as [1]

$$\rho^h_{\lambda_q\lambda_q^*} \hat{D}_{h/q,s_q}(z, p_{\perp}) = \sum_{\lambda_q^*} \rho^q_{\lambda_q\lambda_q^*} \hat{D}_{\lambda_q\lambda_q^*}(z, p_{\perp}). \quad (4)$$

Exploiting the left-hand side in Eq. (4), in terms of the polarization components, we can define

$$(P^h_{J} \hat{D}_{h/q,s_T}) = \hat{D}^{h/q}_{S_2/s_T} - \hat{D}^{h/q}_{-S_2/s_T} \equiv \Delta \hat{D}^{h/q}_{S_2/s_T}(z, p_{\perp}) \quad (5)$$

$$(P^h_{J} \hat{D}_{h/q,s_T}) = \hat{D}^{h/q}_{S_2/s_T} - \hat{D}^{h/q}_{-S_2/s_T} \equiv \Delta \hat{D}^{h/q}_{S_2/s_T}(z, p_{\perp}) \quad (6)$$

$$\hat{D}_{h/q,s_T} = D_{h/q}(z, p_{\perp}) + \frac{1}{2} \Delta \hat{D}_{h/q,s_T}(z, p_{\perp}). \quad (7)$$

where $J = X, Y, Z$ ($P^h_j$) are the hadron helicity axes (polarizations), and $s_T$ ($s_z$) stands for the quark transverse (longitudinal) spin component in its helicity frame. These correspond to 8 TMD-FFs for a spin-1/2 hadron, with a clear partonic interpretation.

Taking into account the parity properties of $\hat{D}_{\lambda_q\lambda_q^*}$, only 8 real independent quantities survive, directly related to the above 8 TMD-FFs [1]. These results can be recast as:

$$\hat{D}_{h/q}(z, p_{\perp}) = D_{h/q} = (D_{++}^{++} + D_{++}^{--}) \quad (8)$$

$$\Delta \hat{D}_{h/q,s_T}(z, p_{\perp}) = \Delta \hat{D}_{h/q} \sin (\phi_{s_q} - \phi_h) = 4 \text{Im} D_{++}^{++} \sin (\phi_{s_q} - \phi_h) \quad [\text{Collins FF}] \quad (9)$$

$$\Delta h/q_{S_2/s_T}(z, p_{\perp}) = \Delta D_{S_2/s_T} = (D_{++}^{++} - D_{++}^{--}) \quad (10)$$

$$\Delta h/q_{S_2/s_T}(z, p_{\perp}) = \Delta D_{S_2/s_T} \cos (\phi_{s_q} - \phi_h) = 2 \text{Re} D_{++}^{++} \cos (\phi_{s_q} - \phi_h) \quad (11)$$

$$\Delta h/q_{S_2/s_T}(z, p_{\perp}) = \Delta D_{S_2/s_T} = 2 \text{Re} D_{++}^{--} \quad (12)$$

$$\Delta h/q_{S_2/s_T}(z, p_{\perp}) = \Delta D_{S_2/s_T} \cos (\phi_{s_q} - \phi_h) = (D_{++}^{++} + D_{++}^{--}) \cos (\phi_{s_q} - \phi_h) \quad (13)$$

$$\Delta \hat{D}_{h/q,s_T}(z, p_{\perp}) = \Delta \hat{D}_{h/q} = \Delta \hat{D}_{h/q} = -2 \text{Im} D_{++}^{--} \quad [\text{Polarizing FF}] \quad (14)$$

$$\Delta h/q_{S_2/s_T}(z, p_{\perp}) = \Delta D_{S_2/s_T} \sin (\phi_{s_q} - \phi_h) = (D_{++}^{++} - D_{++}^{--}) \sin (\phi_{s_q} - \phi_h). \quad (15)$$
2.2 Convolutions

By fixing the hadron spins and summing over the helicity indices in Eq. (1), one obtains all azimuthal dependences in terms of pairs of TMD-FFs. A tensorial analysis allows to factor out the measurable azimuthal dependences and to express all quantities in terms of convolutions (see Ref. [1]):

\[ C[w\Delta D^{h_1} \Delta D^{h_2}] = \sum_{q} e_{q}^{2} \int d^{2}p_{\perp 1} d^{2}p_{\perp 2} \delta^{(2)}(p_{\perp 1} - P_{1T} + p_{\perp 2}z_{p_{1}}/z_{p_{2}}) \]
\[ \times w(p_{\perp 2}, P_{1T}) \Delta D_{h_{1}/q}(z_{1}, p_{\perp 1}) \Delta D_{h_{2}/q}(z_{2}, p_{\perp 2}) . \]  

(16)

Here we report only few examples (see also Refs. [4, 5]):

- **Unpolarized cross section**

\[ \frac{d\sigma e^{-h_{1}h_{2}X}}{d\cos \theta dz_{1}dz_{2}d^{2}P_{1T}} = \frac{3\pi \alpha^{2}}{2s} \left( (1 + \cos^{2} \theta)F_{UU} + \sin^{2} \theta \cos(2\phi_{1})F_{UU}^{\cos(2\phi_{1})} \right) , \]  

(17)

with

\[ F_{UU} = \sum_{q} e_{q}^{2} \int d^{2}p_{\perp 2} D_{h_{1}/q}(z_{1}, p_{\perp 1})D_{h_{2}/q}(z_{2}, p_{\perp 2}) = C[D_{h_{1}/q}D_{h_{2}/q}] \]  

(18)

\[ F_{UU}^{\cos(2\phi_{1})} = \sum_{q} e_{q}^{2} \left[ 2(p_{1T} \cdot \hat{P}_{1T})^{2} - 1 \right] \Delta^{N}D_{h_{1}/q} \Delta^{N}D_{h_{2}/q} \]  

(19)

where the latter expression gives access to the Collins FF.

- **Single-transverse polarized cross section**

\[ P_{T}^{h_{1}} \frac{d\sigma e^{-h_{1}h_{2}X}}{d\cos \theta dz_{1}dz_{2}d^{2}P_{1T}} = \frac{3\pi \alpha^{2}}{2s} \left( (1 + \cos^{2} \theta)\sin(\phi_{1} - \phi_{S_{1}}^{L})F_{TU}^{\sin(\phi_{1} - \phi_{S_{1}}^{L})} \right) \]
\[ + \sin^{2} \theta \left( \sin(\phi_{1} + \phi_{S_{1}}^{L})F_{TU}^{\sin(\phi_{1} + \phi_{S_{1}}^{L})} + \sin(3\phi_{1} - \phi_{S_{1}}^{L})F_{TU}^{\sin(3\phi_{1} - \phi_{S_{1}}^{L})} \right) \]  

(20)

where \( \phi_{S_{1}}^{L} \) is the azimuthal angle of the spin of the hadron \( h_{1} \) and

\[ F_{TU}^{\sin(\phi_{1} - \phi_{S_{1}}^{L})} = C \left[ \left( \frac{z_{p_{1}}P_{1T}}{z_{p_{2}}P_{\perp 1}} \hat{P}_{\perp 2} \cdot \hat{P}_{1T} - \frac{P_{1T}}{P_{\perp 1}} \right) \Delta^{N}D_{h_{1}/q}D_{h_{2}/q} \right] \]  

(21)

\[ 2F_{TU}^{\sin(\phi_{1} + \phi_{S_{1}}^{L})} = C \left[ \left( \frac{z_{p_{1}}P_{1T}}{z_{p_{2}}P_{\perp 1}} \right)^{2} \left( \hat{P}_{\perp 2} \cdot \hat{P}_{1T} \right)^{2} - 3 \left( \hat{P}_{\perp 2} \cdot \hat{P}_{1T} \right) \right] \Delta^{N}D_{h_{2}/q} \]  

(22)

\[ 2F_{TU}^{\sin(3\phi_{1} - \phi_{S_{1}}^{L})} = C \left[ \left( \frac{z_{p_{1}}P_{1T}}{z_{p_{2}}P_{\perp 1}} \right)^{2} \left( \hat{P}_{\perp 2} \cdot \hat{P}_{1T} \right)^{2} - 3 \left( \hat{P}_{\perp 2} \cdot \hat{P}_{1T} \right) \right] \Delta^{N}D_{h_{2}/q} \]  

(23)

The polarization orthogonal to the hadron plane, that is along

\[ \hat{n} = (\cos \phi_{n}, \sin \phi_{n}, 0) = \frac{-P_{2} \times P_{1}}{|P_{2} \times P_{1}|} = -\sin \phi_{1} \hat{x}_{L} + \cos \phi_{1} \hat{y}_{L} , \]  

(24)

can be obtained by identifying \( \phi_{S_{1}}^{L} \equiv \phi_{n} = \frac{\pi}{2} \) in Eq. (20). By integrating over \( P_{1T} \) we get

\[ P_{n}^{h_{1}}(z_{1}, z_{2}) = -\frac{F_{TU}^{\sin(\phi_{1} - \phi_{S_{1}}^{L})}}{F_{UU}} . \]  

(25)
follows the full-data set [2], including in a second phase also the inclusive data set [3]. The best-fit estimates, based on the Belle full-data set [2], of the transverse polarization for $\Lambda/\bar{\Lambda}$ production in $e^+e^- \rightarrow \Lambda \pi X$ vs. $z_\pi$ (left panel) and $e^+e^- \rightarrow \Lambda(\bar{\Lambda}) X$ vs. $p_\perp$ (right panel), for different $z_\Lambda$ bins.

3. Phenomenology

By adopting a Gaussian Ansatz for the TMD-FFs in Eqs. (18), (21), Eq. (25) becomes

$$p_{h1}^{\pi} = \sqrt{\frac{e\pi}{2}} \frac{1}{M_{\text{pol}}} \frac{\langle p_{\perp}^2 \rangle_{\text{pol}}}{\langle p_{\perp} \rangle_{\text{pol}}} \zeta z_2 \sum_q e_q^2 \Delta D_{h^1/q}(z_1) D_{h^2/q}(z_2),$$

where we have used the following parametrizations:

$$D_{h^1/q}(z, p_\perp) = D_{h^1/q}(z) e^{-p_{\perp}^2/\langle p_{\perp}^2 \rangle_{\text{pol}}} \Delta^N D_{h^1/q}(z, p_\perp) = \Delta^N D_{h^1/q}(z) \frac{\sqrt{2}\ z_m}{M_{\text{pol}}} \frac{\langle p_{\perp} \rangle_{\text{pol}}}{\langle p_{\perp} \rangle_{\text{pol}}} \Delta D_{h^2/q}(z).$$

with $\langle p_{\perp}^2 \rangle_{\text{pol}} = \frac{M_{\text{pol}}}{M_{\text{pol}} + \langle p_{\perp}^2 \rangle_{\text{pol}}}$. Another important quantity is the first $p_\perp$-moment of the pFF:

$$\Delta D_{h^1/q}(z) = \int d^2 p_\perp \frac{p_\perp}{2 zm_n} \Delta^N D_{h^1/q}(z, p_\perp) = \sum_q e_q^2 \Delta^N D_{h^1/q}(z, p_\perp).$$

We also consider the inclusive $\Lambda$ production case (within a jet), adopting a simplified approach in terms of TMD-FFs. The transverse polarization (w.r.t. the jet-$\Lambda$ plane) is given as

$$P_T(z, p_\perp) = \frac{\sum_q e_q^2 \Delta^N D_{\Lambda^1/q}(z, p_\perp)}{\sum_q e_q^2 D_{\Lambda^1/q}(z, p_\perp)}.$$

3.1 Results

We first perform a fit of the associated production data alone (see also Ref. [6]), including in a second phase also the inclusive data set [3]. The z-dependent part of the pFF is parameterized as follows ($q = u, d, s, \text{sea}$)

$$\Delta D_{\Lambda^1/q}(z) = N_q e_q^2 (1 - z) \frac{a_q + b_q}{a_q} \frac{a_q + b_q}{b_q} D_{\Lambda^1/q}(z).$$

where $|N_q| \leq 1$. The best parameter choice turns out to be: $N_u$, $N_d$, $N_s$, $N_{\text{sea}}$, $a$, $b$, $b_{\text{sea}}$, with all other $a$, $b$ parameters set to zero. Together with $\langle p_{\perp}^2 \rangle_{\text{pol}}$ (Eq. (27)) we have 8 free parameters.

Some estimates (with their statistical uncertainty bands at 2σ-level), compared against Belle data [2], are shown in Fig. 1. The associated production (full-) data fit leads to a $\chi^2_{\text{ dof}} = 1.26$ (1.94), while the corresponding first moments, Eq. (28), are quite stable (Fig. 2).
We now use the so extracted pFFs to give predictions for the same observable in SIDIS. In such a case the polarization is measured transversely w.r.t. the plane containing the target and the \( \Lambda \) particle. The final result, as a function of \( x_B \) and \( z_h \) and adopting a Gaussian parametrization also for the unpolarized TMD parton distribution, reads:

\[
P_T(x_B, z_h) = \frac{\sqrt{2}e\pi}{2m_p} \frac{\langle p_\perp^2 \rangle_{\text{pol}}}{\langle p_\perp^2 \rangle + z_h^2 \langle k_\perp^2 \rangle} \sum_q \frac{e_q^2 f_{q/p}(x_B)\Delta D_{\Lambda/q}(z_h)}{\sum_q e_q^2 f_{q/p}(x_B)D_{\Lambda/q}(z_h)}.
\]

The corresponding estimates for EIC kinematics are shown in Fig. 3. This would definitely allow for a test of the universality of the pFF as well as of its flavor dependence.

4. Conclusions

The complete azimuthal structure in terms of TMD-FFs for the process \( e^+e^- \rightarrow h_1h_2X \) has been derived within the helicity formalism in full detail and the first extraction of the polarizing FF from Belle data has been presented. Estimates for the transverse \( \Lambda \) polarization in SIDIS are given, emphasising their role in the study of the (universality) properties of the pFF.

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