Abstract—Tandem-duplication-random-loss (TDRL) is an important genome rearrangement operation studied in evolutionary biology. This paper investigates some of the formal properties of TDRL operations on the symmetric group (the space of permutations over an n-set). In particular, motivated by error correction and reconstruction problems in DNA-based data storage applications, we determine the size of “balls” of radius one in the TDRL metric, as well as the cardinality of the maximum intersection of two balls. The corresponding problems for the so-called mirror (or palindromic) TDRL rearrangement operations are also investigated.

I. INTRODUCTION

The study of genome rearrangements in evolutionary biology is a rich source of mathematical and algorithmic problems that, apart from their relevance for the field they originated in, are also interesting in their own right [11], [17]. In the present paper, we are concerned with the so-called tandem-duplication-random-loss (TDRL) model of genome rearrangement, which is of importance in the study of gene order evolution in mitochondrial genomes [3], [16]. Specifically, we focus on the combinatorial questions of finding the cardinalities of “balls” and intersections of balls in this context, questions that are important primarily from a coding theoretic viewpoint, and in particular for error correction and reconstruction problems. Our main motivation for studying these problems are potential applications in DNA-based data storage [19].

Combinatorial problems inspired by the TDRL rearrangement model have been studied previously in several works; see, e.g., [3], [7], [8], [10], [12].

Notation

For our purposes, genome can be modeled as a permutation on the set \{1, 2, \ldots, n\} [10]. The set of all permutations over \{1, 2, \ldots, n\} is denoted by \Pi(n). Each permutation \pi \in \Pi(n) is regarded simply as a sequence (\pi_1, \pi_2, \ldots, \pi_n), where \{\pi_1, \pi_2, \ldots, \pi_n\} = \{1, 2, \ldots, n\}, and thus the elements of \Pi(n) will sometimes be referred to as sequences. The identity permutation is denoted by \pi^{id}(n) := (1, 2, \ldots, n), or by \pi^{id} if the length n is understood from the context. We say that (\pi_{i_1}, \ldots, \pi_{i_m}), where 1 \leq i_1 < \cdots < i_m \leq n, is a subsequence of length m of the sequence (\pi_1, \pi_2, \ldots, \pi_n).

II. TDRL PERMUTATIONS

A TDRL operation on a sequence \pi \in \Pi(n) is a duplication of the entire sequence \pi, followed by a deletion of one of the two copies of each of the symbols. Thus, each TDRL operation is a permutation of the coordinates of \pi, and the result is another sequence from \Pi(n).

Example 1. An example of a TDRL operation on \pi^{id}(5) is the following:

\[
1 2 3 4 5 \rightarrow 1 2 3 4 5 \
0 1 1 0 1 \quad (1a)
\]

In (1a), the duplicate of the original sequence is overbraced, and the symbols that are not deleted are underlined.

By definition, the symbols that are deleted from the first copy of \pi are not deleted from the second copy, and vice versa. Therefore, a TDRL operation can be specified by a binary pattern indicating the symbols that are not deleted from the first copy of a given sequence, as illustrated in (1a). We will use this binary representation throughout the paper.

Another way to think of a TDRL operation on \pi is as a partition of \pi into two of its subsequences which are then concatenated. For example, in (1a), \pi^{id}(5) is partitioned into (2, 3, 5) and (1, 4), and the final result is (2, 3, 5, 1, 4).

If a sequence \rho is the result of applying a TDRL operation on \pi, we write \pi \rightarrow \rho. We further define \mathcal{S}^{-}(\pi) := \{\rho : \pi \rightarrow \rho\} and \mathcal{S}^{-}(\pi) := \{\rho : \rho \rightarrow \pi\}. The set \mathcal{S}^{-}(\pi^{id}) is illustrated in Table I.

A. Counting TDRL Operations

Define

\[
\mathcal{S}^{-}(n) := |\mathcal{S}^{-}(\pi)|, \quad \mathcal{S}^{-}(n) := |\mathcal{S}^{-}(\pi)|, \\
\mathcal{S}^{-}(n) := |\mathcal{S}^{-}(\pi) \cap \mathcal{S}^{-}(\pi)|. \quad (2)
\]

\mathcal{S}^{-}(n) can be thought of as the number of “reversible” TDRL operations — those TDRL operations that can be inverted by another TDRL operation. We first verify that the quantities \mathcal{S}^{-}(n), \mathcal{S}^{-}(n), \mathcal{S}^{-}(n) are well-defined in that they do not depend on \pi.

Lemma 1. For all \pi and \pi' \in \Pi(n), \mathcal{S}^{-}(\pi) = |\mathcal{S}^{-}(\pi')|, \mathcal{S}^{-}(\pi') = |\mathcal{S}^{-}(\pi)|, \mathcal{S}^{-}(\pi) \cap \mathcal{S}^{-}(\pi) = |\mathcal{S}^{-}(\pi') \cap \mathcal{S}^{-}(\pi)|.

Proof: A bijection between, e.g., \mathcal{S}^{-}(\pi) and \mathcal{S}^{-}(\pi'), is constructed simply by relabeling the symbols in \{1, 2, \ldots, n\} in such a way that \pi is transformed into \pi'. More precisely, take \sigma \in \Pi(n) such that \sigma \circ \pi = \pi', and notice that \pi \rightarrow \rho if and only if \sigma \circ \pi \rightarrow \sigma \circ \rho.

The same reasoning that was used in the above proof implies that \mathcal{S}^{-}(n) = \mathcal{S}^{-}(n) for every n.
but cannot be partitioned into subsequences (1
\textsuperscript{st}, \ldots, j
\textsuperscript{th}), which is \binom{j}{i}, and among those sequences there is only one, \pi\textsuperscript{id}, which can also be partitioned into \(1, 2, \ldots, j + 1\), \((j + 2), \ldots, n\). Therefore, \(|S^n(\pi\textsuperscript{id})| = 1 + \sum_{j=1}^{n-1} \binom{n}{j} - 1 = 2^n - n\).

Thus, \(2^n - n\) sequences (out of \(n!\)) in \(\Pi(n)\) can be fully sorted, i.e., transformed into \(\pi\textsuperscript{id}\), by a single TDRL operation.

In the following statement we obtain an expression for the number of reversible TDRL operations, or equivalently, for the number of sequences that can both produce \(\pi\textsuperscript{id}\) and be produced by it.

**Theorem 3.** \(S^\text{rev}(n) = 1 + \binom{n}{2} + \binom{n}{3}\).

**Proof:** We first argue that a TDRL operation is reversible if and only if the corresponding binary pattern is of the form \(b = 1^r0^s1^t0^u\), where \(r, s, t, u\) are non-negative integers summing to \(n\). In words, the requirement is that \(b\) has at most two blocks of ones, and if it has exactly two blocks, then the leading block is the same. For the direct part, notice that a TDRL operation \(1^n0^11^0\) is reversible by the TDRL operation \(1^n0^11^0\) itself. Conversely, any binary string can be written as \(1^n0^u1^s\), where \(r, s, t, u\) are strictly positive integers and \(a\) is an arbitrary binary string of length \(n - r - s - t - u\).

Such a TDRL operation produces a sequence that cannot be partitioned into subsequences \((1, 2, \ldots, j), (j + 1, \ldots, n)\), and is therefore not reversible.

Now that we have a characterization of reversible TDRL operations, we can use it to show the desired expression. There is one binary pattern containing no \(1\’\s\), and there are \(\binom{n+1}{2}\) binary patterns containing exactly one block of \(1\’\s\) (a block is determined by its delimiters). Among the latter, there are \(n\) patterns for which this block is the leading block, i.e., patterns of the form \(1^r0^s1^t0^u\), \(r > 0\). As we already know, such patterns correspond to the same TDRL operation as the pattern \(00\cdots0\), while all the other patterns correspond to different TDRL operations. Therefore, there are exactly \(1 + \binom{n+1}{2} - n = 1 + \binom{n}{2}\) different TDRL operations corresponding to binary patterns with at most one block of \(1\’\s\). Finally, there are \(\binom{n}{2}\) binary patterns with exactly two blocks of \(1\’\s\), one of which is the leading block (choose the length of the leading block and then choose the delimiters of the second block), and all of them correspond to different TDRL operations. This completes the proof.

Thus, only an asymptotically vanishing fraction of TDRL operations are reversible, \(\lim_{n\to\infty} S^n(\pi)/S^\text{rev}(n) = 0\).

**B. The Reconstruction Problem**

The sequence reconstruction problem, as introduced by Levenshtein [14], is defined as follows: a sequence \(x\) is transmitted through a noisy channel multiple times, and the receiver is required to reconstruct it after it has collected sufficiently many noisy observations. The question is how many different noisy versions of the sequence are necessary in order to guarantee successful and unambiguous reconstruction. In combinatorial terms the problem can be rephrased as follows: what is

| Table I |
| --- |
| Permutations resulting from applying one TDRL operation on the identity permutation \(\pi\text{id}(5)\), and the corresponding binary patterns that define the applied TDRL operations. |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 1\ 1\ 1\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 1\ 1\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 1\ 0\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 0\ 1\ 0\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 1\ 0\ 0\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 1\ 0\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 0\ 0\ 1\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 0\ 0\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((1\ 0\ 1\ 0\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 0\ 1\ 0\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 0\ 1\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 0\ 1\ 0\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 0\ 0\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 0\ 0\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 0\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 1\ 0\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 1\ 0\ 1)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 1\ 1\ 0)\) |
| \(1\ 2\ 3\ 4\ 5\) | \((0\ 1\ 1\ 1\ 1)\) |
the cardinality of the largest possible intersection of sets of channel outputs that two different sequences of length \( n \) can produce? Denoting the cardinality of the mentioned largest intersection by \( N(n) \), one easily concludes that the number of noisy observations that guarantees successful reconstruction in all cases is \( N(n) + 1 \). The problem of determining the largest intersection of two “balls” in a given space is therefore relevant in all situations where one uses a simple repetition scheme to communicate reliably. As argued in [20], this problem naturally arises in DNA-based data storage applications.

In the context of the present paper, the “noise” are the TDRL rearrangement operations and the reconstruction problem reduces to the following: what is the largest possible cardinality of the set \( S^{-}(\pi) \cap S^{-}(\rho) \)? Define:

\[
N(n) := \max_{\pi, \rho \in \Pi(n)} |S^{-}(\pi) \cap S^{-}(\rho)|.
\]

In the following statement we give a solution to the reconstruction problem just defined. For other relevant works on the reconstruction problem for translocation permutation errors, see, e.g., [13], [15], [18].

**Theorem 4.** \( N(n) = 2^{n-1} \).

**Proof:** Consider the sequence \( \pi = (2, 3, \ldots, n, 1) \) obtained from \( \pi^d \) by moving the first symbol to the last symbol (a cyclic shift). Consider some \( \rho \in S^{-}(\pi) \), and suppose that the binary pattern corresponding to the TDRL operation \( \pi \rightarrow \rho \) ends in a 1, i.e., is of the form \( b1 \) for \( b \in \{0, 1\}^{n-1} \). Then it is easy to see that \( \rho \) can also be obtained from \( \pi^d \) via the TDRL operation \( 0b \), and hence \( \rho \in S^{-}(\pi^d) \). Since there are \( 2^{n-1} \) binary strings of the form \( b1 \), and since all of them result in different sequences \( \rho \), we have just shown that \( |S^{-}(\pi^d) \cap S^{-}(\pi)| \geq 2^{n-1} \), and therefore \( N(n) \geq 2^{n-1} \). We now use induction to prove that \( |S^{-}(\pi^d) \cap S^{-}(\pi)| \leq 2^{n-1} \) for every \( n \geq 2 \) and every \( \pi \in \Pi(n) \). Suppose that, for a given \( n \geq 3 \), there is a sequence \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi(n) \) such that \( |S^{-}(\pi^d) \cap S^{-}(\pi)| > 2^{n-1} \). This implies that there are at least \( 2^{n-1} + 1 \) binary patterns describing TDRL operations \( \pi \rightarrow \rho \) such that \( \rho \in S^{-}(\pi^d) \cap S^{-}(\pi) \). If \( \pi_i = j \), denote \( \pi_{i:j} = (\pi_i, \pi_{i+1}, \pi_{i+2} \ldots, \pi_n) \) and \( \sigma_{i:j} = (1, \ldots, j - 1, j + 1, \ldots, n) \), and suppose that \( \pi_{i:j} \neq \sigma_{i:j} \) (if not, choose another index \( i \) for which this holds). (By possibly renaming the symbols, both \( \pi_{i:j} \) and \( \sigma_{i:j} \) can be thought of as sequences/ permutations over \( \{1, 2, \ldots, n-1\} \), in which case \( \sigma_{i:j} \) would be the identity permutation.) By deleting the \( i \)’th bit of each of the mentioned binary patterns, one would get at least \( 2^{n-2} + 1 \) different binary patterns of length \( n - 1 \). Notice that these binary patterns describe TDRL operations on the sequence \( \pi_{i:j} \), and that every sequence \( \rho’ \) that is the result of such an operation can be produced by \( \sigma_{i:j} \), as well, i.e., \( \rho’ \in S^{-}(\sigma_{i:j}) \cap S^{-}(\pi_{i:j}) \). If a binary pattern \( b \) describes a TDRL operation \( \pi \rightarrow \rho \) that produces a sequence \( \rho \) in the intersection \( S^{-}(\pi^d) \cap S^{-}(\pi) \), then it is not difficult to see that the pattern \( b_{i:j} \) describes a TDRL operation \( \pi_{i:j} \rightarrow \rho’ \) that produces a sequence \( \rho’ \) in the intersection \( S^{-}(\sigma_{i:j}) \cap S^{-}(\pi_{i:j}) \). We have thus shown that the assumption \( N(n) > 2^{n-1} \) implies that \( N(n - 1) > 2^{n-2} \). In other words, assuming \( N(n - 1) \leq 2^{n-2} \) implies \( N(n) \leq 2^{n-3} \), and since one can directly verify that \( N(2) = 2 \), the inductive proof that \( N(n) \leq 2^{n-1} \) for every \( n \) is complete.

As exemplified in the previous proof, the intersection \( S^{-}(\pi) \cap S^{-}(\rho) \) is of maximum possible cardinality when \( \pi, \rho \) are cyclic shifts (by one position) of one another. It is not difficult to show that this is also the case for any two sequences \( \pi, \rho \) that differ only by one adjacent transposition, e.g., \( \pi = \pi^d = (1, 2, 3, \ldots, n), \rho = (2, 1, 3, \ldots, n) \).

**Corollary 5.** Let \( n \geq 3 \). Every sequence \( \pi \in \Pi(n) \) is uniquely determined by any \( 2^{n-1} + 1 \) elements of \( S^{-}(\pi) \).

**Proof:** We just have to verify that \( |S^{-}(\pi)| = 2^n - n \geq 2^{n-1} + 1 = N(n) + 1 \) for \( n \geq 3 \).

**C. Bounded TDRL Permutations**

In this subsection we analyze a more general model where a TDRL rearrangement operation is confined to segments of width \( k \) within the original sequence \( \pi^d \). In other words, a TDRL operation is in this case applied on a segment of \( k \) consecutive symbols of a given sequence \( \pi \), while the remaining symbols of \( \pi \) are left intact.

**Example 2.** One possible TDRL operation on \( \pi^d(5) \), applied on the segment \( (2, 3, 4) \) of length \( k = 3 \), is the following:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
\end{array}
\rightarrow
\begin{array}{cccc}
1 & 2 & 4 & 3 \\
\end{array}
\]

where the duplicate segment is overbraced, and the symbols that are not deleted (from the original segment \( (2, 3, 4) \) and its duplicate) are underlined.

In the special case \( k = 2 \), the only non-trivial TDRL operations are adjacent transpositions, i.e., swaps of two adjacent symbols.

Let \( S^{-}(n; k) \) be the number of sequences that can be obtained from \( \pi \in \Pi(n) \) by applying a TDRL operation on an arbitrary segment of \( \pi \) consisting of \( k \) consecutive symbols, and define \( S^{-}(n; k) \) and \( S^{-}(n; k) \) accordingly (see (2)). The same argument that was used in the proof of Lemma \( \Pi \) can be used in this context as well, implying that \( S^{-}(n; k) = S^{-}(n; k) \).

**Theorem 6.** \( S^{-}(n; k) = \frac{n!}{(n-k)!} \left( \frac{2k-1}{2} \right) + 1 \).
The number of sequences that have been double-counted—those that can be produced by a TDRL operation on the segment $2, 3, \ldots, k$—is by Theorem 2 $2^{k-1} - (k - 1)$. We then proceed to find $S^r(n; k)$ as follows: count the sequences that can be produced by a TDRL operation on $1, 2, \ldots, k$ but cannot be produced by a TDRL operation on $2, \ldots, k$ (the latter will be counted in the second window); then add the number of sequences that can be produced by a TDRL operation on $2, 3, \ldots, k + 1$ but cannot be produced by a TDRL operation on $3, \ldots, k + 1$; etc. This is done for the first $n - k$ windows. For the last, $(n - k + 1)$th window there is no need exclude any sequences because the procedure stops and there is no double-counting. We thus get $S^r(n; k) = (n - k)(2^k - k - (2^{k-1} - (k - 1))) + 2^k - k$, which is what we needed to show.

As an application of Theorem 6 we next state a sphere-packing bound for codes in $\Pi(n)$ correcting one “TDRL error” of length $k$. Namely, let $C \subseteq \Pi(n)$ be a set of sequences with the property that every sequence from $C$ can be uniquely recovered even after a TDRL operation of length $k$ has been applied on it. Then, by Theorem 6 and a simple sphere-packing argument, we conclude that the cardinality of any such code is upper-bounded as:

$$|C| \leq \frac{|\Pi(n)| - 2^k - k}{2}$$

For $k = 2$ we have $S^r(n; k) = n$, and the above sphere-packing bound reduces to $|C| \leq (n - 1)!$. We note that error-correcting codes in $\Pi(n)$ with respect to various error/rearrangement models have been extensively studied in the literature; see, e.g., [1], [9] and the references therein.

**Theorem 7.** $S^r(n; k) = (n - k + 1)\binom{k}{2} + \binom{k}{1} + 1$.

**Proof:** We can apply the same inclusion-exclusion method of counting as in the proof of Theorem 6. The statement then follows from Theorem 3 after some simple manipulation of the resulting expression.

Note that Theorems 2 and 3 are recovered from Theorems 6 and 7 for $n = k$.

### III. MIRROR-TDRL PERMUTATIONS

A mirror (or palindromic) TDRL operation—MTDRL operation for short—on a sequence $\pi \in \Pi(n)$ is a duplication of the sequence $\pi$, followed by a reversal of the second copy, and by a deletion of one of the two copies of each of the individual symbols [2].

**Example 3.** An example of a MTDRL operation on $\pi^{id}(5)$ is the following:

$$1 \ 2 \ 3 \ 4 \ 5 \ \underline{5} \ 4 \ 3 \ 2 \ 1 \ \rightarrow \ 2 \ 3 \ 5 \ 4 \ 1 \ \quad (6a)$$

$$0 \ 1 \ 1 \ 0 \ 1 \ \quad (6b)$$

where the reversed copy of the original sequence is overbraced, and the symbols that are not deleted are underlined.

The set of sequences resulting from applying a MTDRL operation on $\pi^{id}$ is illustrated in Table II.

### TABLE II

**PERMUTATIONS RESULTING FROM APPLYING ONE MTDRL OPERATION ON THE IDENTITY PERMUTATION $\pi^{id}(4)$, AND THE CORRESPONDING BINARY PATTERNS THAT DEFINE THE APPLIED MTDRL OPERATIONS.**

| $\pi^{id}$ | Binary Patterns |
|------------|-----------------|
| 1 2 3 4    | 1 1 1 1         |
| 1 2 3 4    | 1 1 1 0         |
| 1 2 4 3    | 1 1 0 1         |
| 1 2 4 3    | 1 0 1 1         |
| 1 3 4 2    | 1 0 1 1         |
| 1 3 4 2    | 0 1 0 1         |
| 1 4 3 2    | 0 1 0 1         |
| 1 2 3 4    | 1 0 0 0         |
| 2 3 4 1    | 0 1 1 1         |
| 2 3 4 1    | 0 1 1 0         |
| 2 4 3 1    | 0 1 0 1         |
| 2 4 3 1    | 0 1 0 0         |
| 3 4 2 1    | 0 0 1 1         |
| 3 4 2 1    | 0 0 1 0         |
| 4 3 2 1    | 0 0 0 1         |
| 4 3 2 1    | 0 0 0 0         |

#### A. Counting MTDRL Operations

The quantities $S_M(n), S_M^r(n), S_M^{**}(n)$ in this setting are defined similarly to (2). The fact that $S_M(n) = S_M^r(n)$ is established by the same reasoning as in Lemma 1.

**Theorem 8.** $S_M(n) = S_M^r(n) = 2^{n-1}$.

**Proof:** The binary patterns $b_1$ and $b_0$, for $b \in \{0, 1\}^{n-1}$, always produce the same sequence. This follows from the definition of MTDRL operations (6) (see also Table II). Furthermore, all patterns $b_1, b \in \{0, 1\}^{n-1}$, produce different sequences. Hence, $S_M(n) = 2^{n-1}$.

In the following statement we count the reversible MTDRL operations.

**Theorem 9.** $S_M^{**}(n) = n$.

**Proof:** We need to count all sequences that can both produce $\pi^{id}$ and be produced by it in a single MTDRL operation. First notice that all sequences in $S_M^{**}(\pi^{id})$ are unimodular (first increasing, then decreasing). This follows from the definition of MTDRL operations—each such operation can be seen as selecting a (necessarily increasing) subsequence of $\pi^{id}$ in the first step, and then reading off the remaining subsequence in reverse order. Now, if a sequence $\rho \in S_M^{**}(\pi^{id})$ ends with 1, it is possible to produce $\pi^{id}$ from it only via the pattern 0 1 0 0 1 (because $\pi^{id}$ starts with 1), which implies that $\rho = (n, n - 1, \ldots, 2, 1)$. If a sequence $\rho \in S_M^{**}(\pi^{id})$ ends with 2, then it has to start with 1 because it is unimodular, as we have noted above. It is possible to produce $\pi^{id}$ from such a sequence only via the pattern 1 0 0 1 0 (because $\pi^{id}$ starts with 1, 2), which implies that $\rho = (1, n, n - 1, \ldots, 3, 2)$. Continuing in this way, one concludes that there is exactly one sequence $\rho \in S_M(\pi^{id})$ that ends with $i, i \in \{1, 2, \ldots, n\}$, and that can produce $\pi^{id}$. Therefore, the number of reversible MTDRL operations is $n$. \hfill \blacksquare
B. The Reconstruction Problem

We next determine the maximum cardinality of the intersections $S_M^*(\pi) \cap S_M^*(\rho)$, pertaining to the reconstruction problem as defined in Section II-B. Let

$$N_M(n) := \max_{\pi, \rho \in \Pi(n)} |S_M^*(\pi) \cap S_M^*(\rho)|. \quad (7)$$

As it turns out, $N_M(n) = S_M^*(n)$ for all $n$, and therefore unambiguous reconstruction is in general impossible for MTDRL operations.

**Theorem 10.** $N_M(n) = 2^{n-1}$.

**Proof:** Consider the sequence $\pi = (1, 2, \ldots, n, n-1)$ obtained from $n^{id}$ by swapping its last two elements. Recall that every sequence in $S_M^*(\pi^{id})$ can be produced from $\pi^{id}$ via a MTDRL operation whose binary pattern is of the form $b_1 b_2 \in \{0, 1\}^{n-1}$. Furthermore, it can be easily checked that $\pi^{id} \to \rho$ via $b_0 1$ if and only if $\pi \to \rho$ via $b_1 1$, where $b \in \{0,1\}^{n-2}$. This shows that every $\rho$ that belongs to $S_M^*(\pi^{id})$ also belongs to $S_M^*(\pi)$, and thus $\left|S_M^*(\pi^{id}) \cap S_M^*(\pi)\right| = \left|S_M^*(\pi^{id})\right| = 2^{n-1}$. $\blacksquare$

C. Bounded MTDRL Permutations

Consider now a more general model where MTDRL re-arrangement operations are confined to segments of width $k$ within the original sequence (see Section II-C), and let $S_M^*(n; k), S_M(n; k), S_M^*(n; k)$ be defined accordingly.

**Theorem 11.** $S_M^*(n; k) = S_M^*(n; k) = (n - k + 1)(2^{k-1} - 1) + 1$.

**Proof:** We use the inclusion-exclusion counting method for “sliding window” of width $k$, as for the TDRL model (see the proof of Theorem 6). The main question is how many sequences need to be excluded for a given window in order to avoid double-counting? It turns out that the situation for MTDRL is simpler than for TDRL, and only one sequence needs to excluded — the identity permutation. Namely, any non-trivial MTDRL operation on the window $(2, 3, \ldots, k + 1)$ results in the last symbol $(k + 1)$ being moved to one the preceding positions (see (6)), and applying a MTDRL operation to the window $(1, 2, \ldots, k)$ clearly leaves the symbol $k + 1$ intact. Therefore, only one sequence — $n^{id}(n)$ itself — can be produced by both a MTDRL operation on the segment $(1, 2, \ldots, k)$ and a MTDRL operation on the segment $(2, 3, \ldots, k + 1)$ of $\pi^{id}(n)$. By using this fact and Theorem 8 we then get $S_M^*(n; k) = (n - k)(2^{k-1} - 1) + 2^{k-1}$.

If $C_M \subseteq \Pi(n)$ is a code that is able to recover from one MTDRL operation of length $k$, then, by Theorem 11 and a simple sphere-packing argument, we have the following bound on its cardinality:

$$|C_M| \leq \frac{|\Pi(n)|}{S_M^*(n; k)} = \frac{n!}{(n - k + 1)(2^{k-1} - 1) + 1}. \quad (8)$$

**Theorem 12.** $S_M^*(n; k) = (n - k + 1)(k - 1) + 1$.

**Proof:** The statement follows from Theorem 9 after applying the same method of counting as in the proof of Theorem 11.

\[\blacksquare\]

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