Strong Unique Continuation Property for Stochastic Parabolic Equations*

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Abstract

We establish a strong unique continuation property for stochastic parabolic equations. Our method is based on a new stochastic version of Carleman estimate. As far as we know, this is the first result for strong unique continuation property of stochastic partial differential equations.

2010 Mathematics Subject Classification. 60H15.

Key Words. strong unique continuation property, stochastic parabolic equations, Carleman estimate.

1 Introduction

Let $T > 0$, and $G \in \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given domain. Denote $Q = (0, T) \times G$. Assume that $a^{jk} \in C^{1,2}([0, T] \times G)$ satisfy $a^{jk} = a^{kj}$ ($j, k = 1, 2, \cdots, n$), and for any open subset $G_1$ of $G$, there is a constant $s_0 = s_0(G_1) > 0$ such that

$$\sum_{j,k=1}^{n} a^{jk}(t, x) \xi_j \xi_k \geq s_0 |\xi|^2_{\mathbb{R}^n}, \quad \forall (t, x, \xi^1, \cdots, \xi^n) \in [0, T] \times G_1 \times \mathbb{R}^n. \quad (1.1)$$

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} \triangleq \{\mathcal{F}_t\}_{t \geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined. Assume that $H$ is a Fréchet space. Denote by $L^2_{\mathbb{F}}(0, T; H)$ the Fréchet space consisting all $H$-valued $\mathbb{F}$-adapted process $X(\cdot)$ such that $\mathbb{E}|X(\cdot)|^2_{L^2(0, T; H)} < +\infty$, by $L^\infty_{\mathbb{F}}(0, T; H)$ the Fréchet space of all $H$-valued $\mathbb{F}$-adapted bounded processes and by $L^2_{\mathbb{F}}(\Omega; C([0, T]; H))$ the Fréchet space of all $H$-valued $\mathbb{F}$-adapted continuous processes $X(\cdot)$ with $\mathbb{E}|X(\cdot)|^2_{L^2([0, T]; H)} < +\infty$. All the above spaces are equipped with the canonical quasi-norms.

Consider the following stochastic parabolic equation:

$$dy - \sum_{j,k=1}^{n} (a^{jk} y_{x_j})_{x_k} dt = a \cdot \nabla y dt + by dt + cydW(t) \quad \text{in } Q, \quad (1.2)$$

where $a \in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G; \mathbb{R}^n))$, $b \in L^\infty_{\mathbb{F}}(0, T; L^\infty_{\text{loc}}(G))$ and $c \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}_{\text{loc}}(G))$.

*This work is partially supported by the NSF of China under grants 12025105 and 11931011, and by the Science Development Project of Sichuan University under grant 2020SCUNL201.

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In this paper, for simplicity, we use the notation \( y_{x_j} \equiv y_{x_j}(x) = \partial y(x)/\partial x_j \), where \( x_j \) is the \( j \)-th coordinate of a generic point \( x = (x_1, \ldots, x_n) \) in \( \mathbb{R}^n \). Similarly, we use \( z_{x_j}, v_{x_j}, \) etc. for the partial derivatives of \( z \) and \( v \) with respect to \( x_j \). Also, we use \( C = ((a^{jk})_{1 \leq j,k \leq n}, Q, a, b, c) \) to denote a generic positive constant independent of the solution \( y \), which may change from line to line.

To begin with, we recall the definition of the solution to (1.2).

**Definition 1.1** We call \( y \in L^2_\mathbb{F}(\Omega; C([0, T]; L^2_{\text{loc}}(G))) \cap L^2(0, T; H^1_{\text{loc}}(G)) \) a solution to (1.2) if for any \( t \in [0, T] \), any nonempty open subset \( G' \) of \( G \) and any \( \eta \in H^1_0(G') \), it holds

\[
\int_{G'} y(t, x)\eta(x)dx - \int_{G'} y(0, x)\eta(x)dx
= \int_{0}^{t} \int_{G'} \left\{ -\sum_{j,k=1}^{n} a^{jk}(s, x)y_{x_j}(s, x)\eta_{x_k}(x) + [a(s, x) \cdot \nabla y(s, x) + b(s, x)y(s, x)]\eta(x) \right\} dxds \tag{1.3}
+ \int_{0}^{t} \int_{G'} c(s, x)y(s, x)\eta(x)dxdW(s), \quad \mathbb{P}\text{-a.s.}
\]

In this paper, we study the strong unique continuation property (SCUP for short) for solutions to (1.2). For a given deterministic/stochastic PDE, SCUP means, roughly speaking, if a solution to the equation vanishes to infinite order at a point of a connected open set, then it must vanish identically in that set. SCUP is one of the most fundamental aspects for deterministic PDEs. It is studied extensively in the literature. Classical results are Cauchy-Kovalevskaya theorem and Holmgren’s uniqueness theorem. These results need to assume that the coefficients of the PDE to be analytic. In 1939, T. Carleman introduced in the seminal paper [3] a new method to prove SCUP for two dimensional elliptic equations with \( L^\infty \) coefficients. This landmark work indicates that a non-analytic solution of an elliptic equation can behave in an “analytic” manner in some sense. The technique he used, which is called “Carleman estimate” now, has become a very powerful tool in the study of SCUP for elliptic equations (e.g. [1] [12] [13] [14] [15] [27]) and parabolic equations (e.g. [4] [6] [7] [30] [31]).

It is worth noting that SCUP is an important problem not only in the uniqueness of the solution to a PDE itself, but also in the study of other properties of solutions, such as the nodal sets (e.g. [8] [9]), the Anderson localization (e.g. [2]), etc. Furthermore, it can be applied to solve some application problems, such as controllability problems (e.g. [35]), optimal control problems (e.g. [19]), inverse problems (e.g. [31]) and so on.

Compared with the deterministic PDEs, as far as we know, there is no result concerning SCUP for stochastic PDEs. In our opinion, it would be quite interesting to extend the deterministic SCUP results to the stochastic ones. Nevertheless, there are many things which remain to be done and some of them seem to be challenging.

Before continuing, we give the definition of SCUP for the solution \( y \) to (1.2).

**Definition 1.2** A solution \( y \) to (1.2) is said to satisfy the SCUP if \( y \equiv 0 \) in \( Q \), \( \mathbb{P}\text{-a.s.} \), provided that \( y \) vanishes of infinite order at \( (0, T) \times \{x_0\} \) for some \( x_0 \in G \), i.e., for any \( N \in \mathbb{N} \) and \( r > 0 \), there is a \( C_N > 0 \) such that

\[
\mathbb{E} \int_{(0,T) \times B(x_0,r)} |y(t, x)|^2 dx dt \leq C_N r^{2N}.
\]

The main result of this paper is as follows.

**Theorem 1.1** Solutions \( y \) to (1.2) satisfy SCUP.
In this paper, similar to the deterministic case, we employ a Carleman estimate to establish our SUCP result. In recent years, motivated by the study of unique continuation problems (NOT the strong unique continuation problems), controllability and observability problems, and inverse problems, there are considerable progresses concerning the Carleman estimate for stochastic parabolic equations (see [10, 20, 21, 24, 28, 33]). Despite such developments, the SUCP for stochastic parabolic equations remains an open area. None of the above Carleman estimates can be used to get the SUCP for our equation (1.2) due to the choice of weight functions. Indeed, weight functions in these papers are designed to get some global energy estimate for stochastic parabolic equations with boundary conditions. Moreover, due to the extra difficulties caused by the stochastic setting, such as the requirement of the adaptedness of solutions with respect to the filtration \( \mathcal{F} \), we cannot simply localize the problem as usual because the classical localization technique may change the adaptedness of solutions. To overcome such hindrances, we have taken inspiration from the ideas not only in [28] but also in [2, 5, 30] to prove Theorem 1.1.

There are some other methods to establish the SUCP for parabolic equations (e.g. [4, 18, 26]). However, it seems that these method cannot be applied to get the SUCP for stochastic parabolic equations. For instance, the key step in [4] is to recast equations in terms of parabolic self-similar variables. However, it seems that this cannot be done for stochastic parabolic equations since the related changing of variable with respect to \( t \) will destroy the adaptedness of solutions, which is a key feature in the stochastic setting. The method in [18] is to reduce the SUCP for parabolic equations with time-independent coefficients to the SUCP for elliptic equations. This reduction relies on a representation formula for solutions of parabolic equations in terms of eigenfunctions of the corresponding elliptic operator, and therefore cannot be applied to more general equations with time-dependent coefficients. The difficulty for employing methods in [26] to study the SUCP of (1.2) consists in the fact that one cannot simply localize the problem and do changing of variables as usual because they may also change the adaptedness of solutions.

The rest of this paper is organized as follows. In Section 2, as a key preliminary, we prove a weighted identity for a stochastic parabolic operator. Section 3 is devoted to establishing a technical generality.

### 2 A weighted identity for a stochastic parabolic operator

First, we introduce the following weighted identity for the stochastic parabolic operator \( "dh - \sum_{j,k=1}^{n}(a^{j}h_{x_k})_{x_k}dt". \)

**Lemma 2.1** Let \( f, \ell \in C^{1,3}(Q) \) and \( \Psi \in C^{1,2}(Q) \). Let \( h \) be an \( H^2(G)\)-valued Itô process. Set \( \theta = e^{\ell} \) and \( v = \theta h \). Then, for any \( t \in [0,T] \) and a.e. \( (x, \omega) \in G \times \Omega \),

\[
2\theta \left[ -\sum_{j,k=1}^{n}(a^{j}v_{x_j})_{x_k} + A\nu \right] \left[ dh - \sum_{j,k=1}^{n}(a^{j}h_{x_k})_{x_k}dt \right] \\
+ 2\sum_{j,k=1}^{n}(fa^{j}v_{x_j}dv)_{x_k} + 2\sum_{j,k=1}^{n} \left\{ \sum_{j',k'=1}^{n} \left[ 2fa^{j}a^{j'}\ell_{x_j}v_{x_j}v_{x_k} - fa^{j}a^{j'}\ell_{x_j}v_{x_j}v_{x_{k'}} + fa^{j}a^{j'}\ell_{x_j}v_{x_{k'}} \right] \right\} dt - d\left( fA\nu^2 + f\sum_{j,k=1}^{n}a^{j}v_{x_j}v_{x_k} \right) \\
\] (2.1)
Theorem 2.1], we introduce an auxiliary function $f$ compared with the widely used weighted identity for stochastic parabolic operator. Remark 2.1 plays a key role in the proof of Theorem 1.1.

From (2.3), we find that

$$\sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k} = 2 \sum_{j,k=1}^{n} a^{jk}v_{x_j}v_{x_k},$$

where

$$A = - \sum_{j,k=1}^{n} a^{jk}x_j x_k - \ell_t,$$

$$B = -2 \sum_{j,k=1}^{n} a^{jk}(fA)_{x_j}x_k - (fA)_t,$$

$$c^{jk} = \sum_{j'=1}^{n} a^{jk'}(fA)_{x_j}x_k - (fA)_{x_k}a^{jk'}x_{j'} - \frac{1}{2}(f^{a^{jk'}})_{x_k}.$$

Remark 2.1 Compared with the widely used weighted identity for stochastic parabolic operator ([28, Theorem 2.1]), we introduce an auxiliary function $f$ in the pointwise identity, which plays a key role in the proof of Theorem 1.1.

Proof of Lemma 2.1 We divide the proof into three steps.

Step 1. Recalling $\theta = e^t$ and $v = \theta h$, one has $dh = \theta^{-1}(dv - \ell_t v)dt$ and $h_{x_j} = \theta^{-1}(v_{x_j} - \ell_{x_j}v), (j = 1, 2, \cdots, n)$. From the symmetry condition of $a^{jk}$, we see that

$$\sum_{j,k=1}^{n} a^{jk}(\ell_{x_j}v_{x_j} + \ell_{x_k}v_{x_k}) = 2 \sum_{j,k=1}^{n} a^{jk}x_{x_k}.$$

From (2.3), we find that

$$\theta \sum_{j,k=1}^{n} a^{jk}h_{x_k} = \theta \sum_{j,k=1}^{n} \left[ \theta^{-1}a^{jk}(v_{x_j} - \ell_{x_j}v) \right]_{x_k}$$

$$= \sum_{j,k=1}^{n} \left[ a^{jk}(v_{x_j} - \ell_{x_j}v) \right]_{x_k} - \sum_{j,k=1}^{n} a^{jk}(v_{x_j} - \ell_{x_j}v)\ell_{x_k}$$

$$= \sum_{j,k=1}^{n} \left\{ (a^{jk}v_{x_j})_{x_k} - 2a^{jk}\ell_{x_j}v_{x_k} + \left[ a^{jk}\ell_{x_j}x_k - (a^{jk}x_k)_{x_k} \right]v \right\}.$$

Recall that $A = - \sum_{j,k=1}^{n} a^{jk}x_{x_k} - \ell_t$ and put

$$I_1 \triangleq - \sum_{j,k=1}^{n} (a^{jk}v_{x_j})_{x_k} + A v,$$

$$I_2 \triangleq dv + \sum_{j,k=1}^{n} \left[ 2a^{jk}\ell_{x_j}v_{x_k} + (a^{jk}x_k)_{x_k} \right]dt.$$
Then, by $dh = \theta^{-1}(dv - \ell_t v)dt$ and (2.4), we get

$$\theta \left[ dh - \sum_{j,k=1}^n (a^{jk} h_{x_k})_{x_k} dt \right] = I_1 dt + I_2.$$  \hspace{1cm} (2.6)

Consequently,

$$2f \theta \left[ - \sum_{j,k=1}^n (a^{jk} v_{x_k})_{x_k} + \mathcal{A}v \right] [du - \sum_{j,k=1}^n (a^{jk} u_{x_k})_{x_k} dt] = 2f I_1^2 dt + 2f I_1 I_2.$$  \hspace{1cm} (2.7)

**Step 2.** In this step, we compute $2f I_1 I_2$. Noting that

$$\sum_{j,k,j',k'=1}^n \left( a^{jk} a^{j'k'} \ell_{x_j' v_{x_j} v_{x_k}} \right) x_{k'} = - \sum_{j,k,j',k'=1}^n \left( a^{j'k'} a^{jk} \ell_{x_{j'} v_{x_j} v_{x_k}} \right) x_{k'},$$

we get

$$4f \sum_{j,k=1}^n a^{jk} \ell_{x_j v_{x_k}} \left[ - \sum_{j,k=1}^n (a^{jk} v_{x_k})_{x_k} + \mathcal{A}v \right]$$

$$= -4 \sum_{j,k=1}^n (f a^{jk} a^{j'k'} \ell_{x_j' v_{x_j} v_{x_k}})_{x_k} + 4 \sum_{j,k=1}^n a^{jk} (f a^{j'k'} \ell_{x_j' v_{x_j} v_{x_k}})_{x_k} v_{x_k}$$

$$+ 4 \sum_{j,k=1}^n f a^{jk} a^{j'k'} \ell_{x_j' v_{x_j} v_{x_k}} + 2f \sum_{j,k=1}^n a^{jk} \ell_{x_j} v^2 \left( a^{jk} v_{x_k} - 2 \sum_{j,k=1}^n (f a^{jk} \ell_{x_j})_{x_k} v^2, \right)$$

and

$$2f \sum_{j,k=1}^n (a^{jk} \ell_{x_j})_{x_k} v \left[ - \sum_{j,k=1}^n (a^{jk} v_{x_k})_{x_k} + \mathcal{A}v \right]$$

$$= -2 \sum_{j,k,j',k'=1}^n \left( f a^{jk} (a^{j'k'} \ell_{x_j' v_{x_j}})_{x_k} v_{x_k} \right)_{x_k} + 2f \sum_{j,k,j',k'=1}^n (a^{j'k'} \ell_{x_j' v_{x_j}} a^{jk} v_{x_k})$$

$$+ 2 \sum_{j,k,j',k'=1}^n \left[ a^{jk} f (a^{j'k'} \ell_{x_j' v_{x_j}})_{x_k} v_{x_k} v + 2f \sum_{j,k=1}^n (a^{jk} \ell_{x_j})_{x_k} \mathcal{A}v^2 \right].$$

Using Itô’s formula, we have

$$2f \left[ - \sum_{j,k=1}^n (a^{jk} v_{x_k})_{x_k} + \mathcal{A}v \right] dv$$

$$= -2 \sum_{j,k=1}^n (f a^{jk} v_{x_k} dv)_{x_k} + 2f \sum_{j,k=1}^n a^{jk} v_{x_k} dv_{x_k} + 2 \sum_{j,k=1}^n a^{jk} f_{x_k} v_{x_k} dv + 2f \mathcal{A}vdv$$  \hspace{1cm} (2.11)
\[= -2 \sum_{j,k=1}^n (fa^{jk}v_{x_j}dv)_{x_k} + d\left( \sum_{j,k=1}^n a^{jk}v_{x_j}v_{x_k} + fA v^2 \right) - \sum_{j,k=1}^n (fa^{jk})_t v_{x_j} v_{x_k} dt \]

\[-(fA)_t v^2 dt - f \sum_{j,k=1}^n a^{jk}dv_{x_j}dv_{x_k} - fA(dv)^2 + 2 \sum_{j,k=1}^n a^{jk}f_{x_j}v_{x_k} dv.\]

It follows from (2.9), (2.10) and (2.11) that

\[2fI_1I_2 = -2 \sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n \left[ 2fa^{jk}a^{j'k'}\ell_{x_j}v_{x_j}v_{x_k} - fa^{jk}a^{j'k'}\ell_{x_j}v_{x_j}v_{x_k} \right] + fa^{jk}(a^{j'k'}\ell_{x_j})v_{x_j}v_{x_k} \right\} dt \]

\[+ 2 \sum_{j,k,j',k'=1}^n \left[ \frac{1}{2}fa^{jk'}(a^{j'k'}\ell_{x_j})v_{x_j}v_{x_k} \right] \] 

Step 3. Combining (2.7), (2.12), and noting

\[f \sum_{j,k=1}^n a^{jk}dv_{x_j}dv_{x_k} + fA(dv)^2 = f\theta^2 \sum_{j,k=1}^n a^{jk}[(dh_{x_j} + \ell_{x_j}dh)(dh_{x_k} + \ell_{x_k}dh)] + f\theta^2A(dh)^2,\]

we immediately obtain the desired equality. □

3 Carleman estimate for stochastic parabolic equations

Without loss of generality, in what follows, we assume that \(0 \in G\) and \(x_0 = 0\). Let \(t_0 \in (0, T)\). For \(r \in (0, \min_{x \in \overline{G}} |x|_{\mathbb{R}^n})\) and \(\delta_0 \in (0, t_0)\), we set

\[B_r \triangleq \{ x \in G \mid |x|_{\mathbb{R}^n} \leq r \}, \quad Q_{r,\delta_0} \triangleq B_r \times (t_0 - \delta_0, t_0 + \delta_0).\]

To prove Theorem 1.1, we first establish a Carleman estimate by virtue of Lemma 2.1. For simplicity, we denote

\[A(t, x) \triangleq (a^{jk}(t, x))_{1 \leq j, k \leq n}, \quad A_0(t) \triangleq (a^{jk}(t, 0))_{1 \leq j, k \leq n}, \quad (a^{jk})_{1 \leq j, k \leq n}, \quad x \triangleq (x_1, x_2, \cdots, x_n)^\top, \quad A_0(t)^{-1} \triangleq (b^{jk}(t))_{1 \leq j, k \leq n}.\]
For a fixed number $\mu \geq 1$ to be chosen later, define

\[
\begin{cases}
\sigma(x,t) = \sqrt{A_0^{-1} x \cdot x}, \\
\varphi(s) = s \exp \left( \int_0^s \frac{e^{-\mu \tau} - 1}{\tau} d\tau \right), \\
w(x,t) = \varphi(\sigma(x,t)), \quad \phi(s) \triangleq e^{\mu s} = \frac{\varphi(s)}{s \varphi'(s)}.
\end{cases}
\]  
(3.2)

**Remark 3.1** The weight function we use here is the one people used to establish the SUCP for deterministic parabolic equations. However, the proof of the Carleman estimate (Lemma 3.1 below) is not a trivial generalization of the deterministic ones. In the stochastic setting, some extra terms involving the covariation processes of solutions would appear. One needs to handle these terms carefully.

**Lemma 3.1** There exist $r_0 = r_0((a^{jk})_{1 \leq j,k \leq n}) > 0$, $s_1 \in (0,1)$ and $\lambda_0, \mu_0 > 0$ such that $\mu = \mu_0$ and for any $\varepsilon_0 \in (0,s_1 r_0)$, there is a constant $C > 0$ independent on $\varepsilon_0$ so that for all $\lambda \geq \lambda_0$ and

\[
z \in \mathcal{H}_{r_0,\delta_0} \triangleq \left\{ z \in \mathcal{L}_2^a(\Omega; C_0([t_0 - \delta_0, 0]; L^2(\mathcal{B}(r_0)))) \cap \mathcal{L}_2^a(t_0 - \delta_0, t_0 + \delta_0; \mathcal{B}^1(\mathcal{B}(r_0))) \mid z = 0 \text{ in } (t_0 - \delta_0, t_0 + \delta_0) \times [\mathcal{B}(r_0) \cup (\mathcal{B}(r_0) \setminus \mathcal{B}(r_1 r_0))] \right\},
\]

which solves

\[
dz - \sum_{j,k=1}^n (a^{jk} z_{x_j}) x_k dt = g_1 dt + g_2 d\mathbf{W}(t) \quad \text{in} \quad Q_{r_0,\delta_0} \tag{3.3}
\]

for some $g_1 \in \mathcal{L}_2^a(t_0 - \delta_0, t_0 + \delta_0; L^2(\mathcal{B}(r_0)))$ and $g_2 \in \mathcal{L}_2^a(t_0 - \delta_0, t_0 + \delta_0; W^{1,\infty}(\mathcal{B}(r_0)))$, the following inequality holds:

\[
\mathbb{E} \int_{Q_{r_0,\delta_0}} \left( \lambda w^{1-2\lambda} |\nabla z|^2 + \lambda^2 w^{-1-2\lambda} |z|^2 \right) dx dt \\
\leq C \mathbb{E} \int_{Q_{r_0,\delta_0}} w^{2-2\lambda} (g_1^2 + \lambda^2 w^{-2} g_2^2 + |\nabla g_2|^2) dx dt,
\]  
(3.4)

where $C$ depend on $(a^{jk})_{1 \leq j,k \leq n}, r_0, \delta_0$.

**Proof:** The proof is long. We divide it into four steps.

**Step 1.** Let $\ell(t,x) = -\lambda \ln w(t,x)$, $f = \sigma^2 \phi$ and $h = z \in (2,1)$. Integrating (2.1) on $Q_{r_0,\delta_0}$ and taking mathematical expectation, we have that

\[
2\mathbb{E} \int_{Q_{r_0,\delta_0}} f \theta \left[ - \sum_{j,k=1}^n (a^{jk} v_{x_j} x_k + Av) \right] dz - \sum_{j,k=1}^n (a^{jk} z_{x_j} x_k) dt \right] dx + 2\mathbb{E} \int_{Q_{r_0,\delta_0}} \sum_{j,k=1}^n (fa^{jk} v_{x_j} x_k) x_k dx \\
+ 2\mathbb{E} \int_{Q_{r_0,\delta_0}} \sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n \left[ 2fa^{jk} a^{j'k'} \ell_{x_j} v_{x_{k'}}, v_{x_{k'}} - fa^{jk} a^{j'k'} \ell_{x_{j'}}, v_{x_{j'}}, v_{x_{k'}} + fa^{jk} a^{j'k'} \ell_{x_{j'}}, v_{x_{j}} - fa^{jk} a^{j'k'} \ell_{x_{j'}}, v_{x_{j}} \right] \\
- fAa^{jk} \ell_{x_j} v_{x_k} \right] \right\} x_{x_k} dx dt \\
- \mathbb{E} \int_{Q_{r_0,\delta_0}} \left[ fAv^2 + f \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k} \right] dx \tag{3.5}
\]
\[2 \mathbb{E} \int_{Q_{r_0, \delta_0}} \sum_{j,k=1}^n e^{j,k} v_{x_j} v_{x_k} dx dt + \mathbb{E} \int_{Q_{r_0, \delta_0}} B v^2 dx dt \]

\[+ 2 \mathbb{E} \int_{Q_{r_0, \delta_0}} f \left[ - \sum_{j,k=1}^n (a^{j,k} v_{x_j})_{x_k} + A v \right]^2 dx dt \]

\[- \mathbb{E} \int_{Q_{r_0, \delta_0}} \int f \theta^2 \sum_{j,k=1}^n a^{j,k}(d z_j + \ell_{x_j} d z) (d z_{x_k} + \ell_{x_k} d z) dx - \mathbb{E} \int_{Q_{r_0, \delta_0}} f \theta^2 A(d z)^2 dx \]

\[+ 2 \mathbb{E} \int_{Q_{r_0, \delta_0}} \sum_{j,k=1}^n a^{j,k} f_{x_j} v_{x_k} dv dx + 2 \mathbb{E} \int_{Q_{r_0, \delta_0}} \sum_{j,k,j',k'=1}^n a^{j,k} f_{x_j} v_{x_k} (a^{j',k'} \ell_{x_{j'}})_{x_{k'}} v dx dt, \]

where \(a^{j,k}, A \) and \(B \) are given by (2.2).

Clearly, noting that \(w \sim O(\sigma) \) as \(\sigma \to 0\),

\[2 \mathbb{E} \int_{Q_{r_0, \delta_0}} f \theta \left[ - \sum_{j,k=1}^n (a^{j,k} v_{x_j})_{x_k} + A v \right] \left[ d z - \sum_{j,k=1}^n (a^{j,k} z_{x_j})_{x_k} dt \right] dx \]

\[= 2 \mathbb{E} \int_{Q_{r_0, \delta_0}} f \theta \left[ - \sum_{j,k=1}^n (a^{j,k} v_{x_j})_{x_k} + A v \right] g_1 dx dt \]

\[\leq \mathbb{E} \int_{Q_{r_0, \delta_0}} f \left[ - \sum_{j,k=1}^n (a^{j,k} v_{x_j})_{x_k} + A v \right]^2 dx dt + C \mathbb{E} \int_{Q_{r_0, \delta_0}} \theta^2 w^2 g_1^2 dx dt. \]  

(3.6)

Noting that \(z \in \mathcal{H}_{r_0, \delta_0}\), we find that

\[2 \mathbb{E} \int_{Q_{r_0, \delta_0}} \sum_{j,k=1}^n (f a^{j,k} v_{x_j} dv)_{x_k} dx \]

\[+ 2 \mathbb{E} \int_{Q_{r_0, \delta_0}} \sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n \left[ 2 f a^{j,k} a^{j',k'} \ell_{x_{j'}} v_{x_j} v_{x_{k'}} \ell_{x_{j'}} v_{x_j} v_{x_{k'}} + f a^{j,k} (a^{j',k'} \ell_{x_{j'}})_{x_{k'}} v dx dt, \right. \]

\[\left. - f A a^{j,k} \ell_{x_j} v^2 \right\}_{x_k} dx dt = 0. \]  

(3.7)

Recalling that \(\ell(t, x) = -\lambda \ln w(t, x)\) and \(\sigma = \sqrt{A_0^{-1} x \cdot x}\), it is easy to see that

\[\sigma_{x_j} = \frac{1}{\sigma} \sum_{j=1}^n b^{j,k} x_k, \quad \nabla \sigma = A_0^{-1} x \cdot x / \sigma, \]

\[\sigma_{x_j x_k} = \frac{b^{j,k}}{\sigma} - \frac{\sigma_{x_j} \sigma_{x_k}}{\sigma}, \quad \sigma_{x_j} = \frac{\partial_t A_0^{-1} x \cdot x}{2 \sqrt{A_0^{-1} x \cdot x}}, \]

and that

\[\ell_t = -\lambda w^{-1} \ell_t = -\lambda \frac{\sigma_{x_j}}{\sigma \phi}, \quad \ell_{x_j} = -\lambda \frac{\sigma_{x_j}}{\sigma \phi}, \]

\[\ell_{x_j x_k} = -\lambda \frac{b^{j,k}}{\sigma \phi} + \lambda \frac{\sigma_{x_j} \sigma_{x_k}}{\sigma \phi} (2 + \mu \sigma). \]  

(3.8)
It follows from (3.8) and (3.9) that
\[
|f - f_{E}|^{2} = C(\lambda^{2} w^{2} g_{2}^{2} + w^{2} |\nabla g_{2}|^{2}).
\tag{3.10}
\]
Combining (3.5)–(3.7) and (3.10), we get that
\[
2E \int_{Q_{r_{0},\delta_{0}}} \left( \sum_{j,k=1}^{n} a^{jk} v_{x_{j}} v_{x_{k}} \right) dxdt + E \int_{Q_{r_{0},\delta_{0}}} B_{v}^{2} dxdt + 2E \int_{Q_{r_{0},\delta_{0}}} \left( \sum_{j,k=1}^{n} a^{jk} f_{x_{j}} v_{x_{k}} \right) dvdx
\]
\[
+ 2E \int_{Q_{r_{0},\delta_{0}}} \left[ \sum_{j,k,j',k'=1}^{n} \left( a^{jk} (f a^{j'} k_{x_{j'}})_{x_{k'}} \right) v_{x_{j}} v_{x_{k}} \right] dvdx + 2E \int_{Q_{r_{0},\delta_{0}}} \left( - \sum_{j,k=1}^{n} \left( a^{jk} v_{x_{j}} \right)_{x_{k}} + A v \right) dxdt
\]
\[
\leq CE \int_{Q_{r_{0},\delta_{0}}} w^{2} - 2\lambda(g_{1}^{2} + \lambda^{2} w^{2} g_{2}^{2} + |\nabla g_{2}|^{2}) dxdt.
\tag{3.11}
\]
Step 2. Now, we deal with the left hand side of (3.11). Recalling (2.2) for $c^{jk}$, we have
\[
\sum_{j,k=1}^{n} c^{jk} v_{x_{j}} v_{x_{k}} = \sum_{j,k,j',k'=1}^{n} \left[ 2a^{jk'} (f a^{j'k_{x_{j'}}})_{x_{k'}} - (f a^{j} k_{x_{j}} a^{j'k_{x_{j'}}} \sigma_{x_{j'}} - \frac{1}{2} \left( f a^{jk} k_{x_{j}} \right)_{x_{k'}} v_{x_{j}} v_{x_{k}}. \tag{3.12}
\]
For the first and second term in the righthand side of (3.12), we have respectively that
\[
2 \sum_{j,k,j',k'=1}^{n} a^{jk'} (f a^{j'k_{x_{j'}}})_{x_{k'}} v_{x_{j}} v_{x_{k}}
\]
\[
= - 2\lambda \sum_{j,k,j',k'=1}^{n} a^{jk'} \left( \sigma^{2} \phi a^{j'k_{x_{j'}}} \sigma_{x_{j'}} \right) v_{x_{j}} v_{x_{k}}
\]
\[
= - 2\lambda \sum_{j,k,j',k'=1}^{n} a^{jk'} (a^{j'k_{x_{j'}}} \sigma_{x_{j'}}) v_{x_{j}} v_{x_{k}}
\]
\[
= - 2\lambda \sum_{j,k,j',k'=1}^{n} \left[ a^{jk'} a^{j'k_{x_{j'}}} \sigma_{x_{j'}} + \sigma_{x_{j'},x_{j}} v_{x_{j}} v_{x_{k}} + a^{jk'} a^{j'k_{x_{j'}}} \sigma_{x_{j'},x_{j}} v_{x_{j}} v_{x_{k}} \right]
\]
\[
= - 2\lambda \sum_{j,k,j',k'=1}^{n} \left( a^{jk'} a^{j'k_{x_{j'}}} \sigma_{x_{j'}} v_{x_{j}} v_{x_{k}} + a^{jk'} a^{j'k_{x_{j'}}} \sigma_{x_{j'}} v_{x_{j}} v_{x_{k}} \right)
\]
\[
= - 2\lambda AA_{0}^{-1} A \nabla v \cdot \nabla v - 2\lambda \sum_{j,k,j',k'=1}^{n} a^{jk'} a^{j'k_{x_{j'}}} \sigma_{x_{j'}} v_{x_{j}} v_{x_{k}}
\]
and
\[
- \sum_{j,k,j',k'=1}^{n} (f a^{jk})_{x_{k}} a^{j'k_{x_{j'}}} v_{x_{j}} v_{x_{k}}
\]
\[
= - \sum_{j,k,j',k'=1}^{n} (\sigma^{2} \phi a^{jk})_{x_{k}} a^{j'k_{x_{j'}}} \left( - \lambda \frac{\sigma_{x_{j'}}}{\phi} \right) v_{x_{j}} v_{x_{k}}
\tag{3.14}
\]
\[
\lambda \sum_{j,k,j',k'=1}^{n} \left( 2a^{jk}a^{j'k'}\sigma_{x_{j'}}\sigma_{x_{k'}}v_{x_{j'}}v_{x_{k'}} + \lambda \mu a^{jk}a^{j'k'}\sigma_{x_{j'}}\sigma_{x_{k'}}v_{x_{j'}}v_{x_{k'}} + a^{jk}a^{j'k'}\sigma_{x_{j'}}v_{x_{j'}}v_{x_{k'}} \right)
\]

\[
= \lambda (2 + \mu \sigma) \frac{A_{0}^{-1}A_{0}^{-1}x \cdot x}{\sigma^{2}}(A\nabla v \cdot \nabla v) + \lambda \sum_{j,k,j',k'=1}^{n} a^{jk}a^{j'k'}\sigma_{x_{j'}}v_{x_{j'}}v_{x_{k'}}.
\]

Recalling \(\sigma = \sqrt{A_{0}^{-1}x \cdot x}\), we find that

\[
\frac{A_{0}^{-1}A_{0}^{-1}x \cdot x}{\sigma^{2}}(A\nabla v \cdot \nabla v) I - A_{0}^{-1} = \frac{A_{0}^{-1}(A - A_{0})A_{0}^{-1}x \cdot x}{A_{0}^{-1}x \cdot x} I - (A - A_{0})A_{0}^{-1}.
\]

This, together with \(|A - A_{0}| \leq C\sigma\), implies that

\[
2\lambda \left[ \frac{A_{0}^{-1}A_{0}^{-1}x \cdot x}{\sigma^{2}}(A\nabla v \cdot \nabla v) - A_{0}^{-1}A\nabla \cdot \nabla v \right] \leq C\lambda \sigma A\nabla v \cdot \nabla v.
\]

Combining (3.12) and (3.13), we find that there exists a constant \(\mu_{1} > 0\) such that for any \(\mu \geq \mu_{1}\),

\[
2\mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} c^{jk}v_{x_{j}}v_{x_{k}}dxdt \geq C\lambda \mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} |\nabla v|^{2}dxdt.
\]

Further, by the first equality in (2.2), we have that

\[
A = - \sum_{j,k=1}^{n} a^{jk}x_{j}\sigma_{x_{j}} - \ell_{t}
\]

\[
= - \lambda^{2}\frac{1}{\sigma^{2}\phi^{2}} \sum_{j,k=1}^{n} a^{jk}x_{j}\sigma_{x_{j}} - \lambda \frac{\sigma_{t}}{\sigma_{\phi}}.
\]

This, together with the second equality in (2.2), implies that

\[
B = - 2 \sum_{j,k=1}^{n} a^{jk}(fA)_{x_{j}}x_{k} - (fA)_{t}
\]

\[
= 2\lambda^{3}\frac{1}{\sigma_{\phi}} A\nabla \left( \frac{1}{\phi} A\nabla \sigma \cdot \nabla \sigma \right) \cdot \nabla \sigma - O(\lambda^{2})
\]

\[
\geq C\lambda^{3}\mu \frac{1}{\sigma_{\phi}^{2}}(A\nabla \sigma \cdot \nabla \sigma)^{2} - O(\lambda^{3}) \frac{1}{\sigma_{\phi}}.
\]

Thus, there exists a constant \(\mu_{2} > 0\) such that for any \(\mu \geq \mu_{2}\), we have

\[
\mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} Bv^{2}dxdt \geq C\lambda^{3} \mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} w^{-1}v^{2}dxdt.
\]

Moreover, it is clear that

\[
2\mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} \sum_{j,k,j',k'=1}^{n} \left[ a^{jk}f(a^{j'k'}x_{j'})x_{k'}v_{x_{j'}}v_{x_{k'}} \right] x_{j} v_{x_{k}}dxdt
\]

\[
= 2\lambda \mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} \sum_{j,k,j',k'=1}^{n} \left[ a^{jk}a^{j'k'}\sigma_{x_{j'}} - a^{jk}b^{j'k'}a^{j'k'} + (2 + \mu \sigma)a^{jk}a^{j'k'}a^{j'k'} \right] x_{j} v_{x_{k}}dxdt
\]

\[
\geq - C\lambda^{2} \mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} w|\nabla v|^{2}dxdt - C\lambda^{2} \mathbb{E} \int_{Q_{\tau_{0},\delta_{0}}}^{n} w^{-1}v^{2}dxdt.
\]
From (3.11) and (3.16)–(3.18), by setting \( \mu = \mu_0 = \max\{\mu_1, \mu_2\} \), we find that there \( \lambda_0 > 0 \) such that for any \( \lambda \geq \lambda_0 \), it holds that

\[
2E \int_{Q_{r_0, \delta_0}} \sum_{j,k,j',k'=1} a^{jk} f_{x_j} v_{x_k} dvdx + \lambda \int_{Q_{r_0, \delta_0}} \left( \lambda^2 w^{-1} v^2 + w|\nabla v|^2 \right) dx dt \\
+ \mathcal{E} \int_{Q_{r_0, \delta_0}} f \left[ - \sum_{j,k=1} (a^{jk} v_{x_j})_x + \mathcal{A} v \right]^2 dx dt \\
\leq CE \int_{Q_{r_0, \delta_0}} w^{2-\lambda} (g_1^2 + \lambda^2 w^{-2} g_2^2 + |\nabla g_2|^2) dx dt.
\]

**Step 3.** Now we deal with the first term on the left hand side of (3.19). Clearly,

\[
2E \int_{Q_{r_0, \delta_0}} \sum_{j,k,j',k'=1} a^{jk} f_{x_j} v_{x_k} dvdx = 2E \int_{Q_{r_0, \delta_0}} \sum_{j,k,j',k'=1} a^{jk} (\sigma \phi + \mu \sigma^2 \phi) \sigma_{x_j} v_{x_k} dvdx. \tag{3.20}
\]

From (2.5), (2.6), (3.3), we get that

\[
\theta \left[ du - \sum_{j,k=1} (a^{jk} u_{x_j})_{x_k} \right] dt \\
= \left[ - \sum_{j,k=1} (a^{jk} v_{x_j})_x + \mathcal{A} v \right] dt + dv + \sum_{j,k=1} \left[ 2a^{jk} \ell_{x_j} v_{x_k} + (a^{jk} \ell_{x_j})_{x_k} v \right] dt \tag{3.21}
\]

\[
= \theta \left( g_1 dt + g_2 dW(t) \right).
\]

Noting that \( v = \theta u \), we have that

\[
-\theta \sum_{j,k=1} (a^{jk} u_{x_j})_{x_k} = - \sum_{j,k=1} (a^{jk} v_{x_j})_x + \mathcal{A} v + \ell_t v + \sum_{j,k=1} \left[ 2a^{jk} \ell_{x_j} v_{x_k} + (a^{jk} \ell_{x_j})_{x_k} v \right].
\]

Consequently, recalling (3.9),

\[
2\lambda \frac{1}{\sigma \phi} \sum_{j,k=1} a^{jk} \sigma_{x_j} v_{x_k} \\
= \theta \sum_{j,k=1} (a^{jk} u_{x_j})_{x_k} - \sum_{j,k=1} (a^{jk} v_{x_j})_x + \mathcal{A} v + \ell_t v + \sum_{j,k=1} \left[ 2a^{jk} \ell_{x_j} v_{x_k} + (a^{jk} \ell_{x_j})_{x_k} v \right]. \tag{3.22}
\]

Recalling \( v = \theta u \), we have

\[
dv = \theta du + \theta \ell_t u dt = \theta \left[ g_1 dt + g_2 dW(t) + \sum_{j,k=1} (a^{jk} u_{x_j})_{x_k} dt + \ell_t u dt \right]. \tag{3.23}
\]

Combining with (3.20), (3.22) and (3.23), we obtain

\[
2E \int_{Q_{r_0, \delta_0}} \sum_{j,k,j',k'=1} a^{jk} f_{x_j} v_{x_k} dvdx
\]
where $\hat{\delta}$

From (3.25), we find that

$$z_0, t \geq E \geq E \lambda_1 \delta \left| E a \in C \lambda_0 \int L a \hat{\delta} \geq -Q \lambda_1 \right.;$$

where

$$J_1 = \theta \sum_{j,k=1}^{n} (a^{jk} u_{xj})_{xk} - \sum_{j,k=1}^{n} (a^{jk} v_{xj})_{xk} + \mathcal{A} v + \ell_t v,$$

$$J_2 = \theta \left[ g_1 + \sum_{j,k=1}^{n} (a^{jk} u_{xj})_{xk} + \ell_t u \right].$$

From (3.25), we find that

$$\frac{1}{\lambda} \mathbb{E} \int_{Q_{t_0}, \delta_0} \sigma^2 \phi^2 (1 + \mu \sigma) J_1 J_2 dx dt$$

$$\geq \mathbb{E} \int_{Q_{t_0}, \delta_0} \frac{1}{\lambda} \sigma^2 \phi^2 (1 + \mu \sigma) \left\{ \frac{1}{2} \theta^2 \left[ \sum_{j,k=1}^{n} (a^{jk} u_{xj})_{xk} \right]^2 - \left[ - \sum_{j,k=1}^{n} (a^{jk} v_{xj})_{xk} + \mathcal{A} v \right]^2 \right\}$$

$$- C w^{-2\lambda} g_1^2 - C \lambda^2 w^{-3-2\lambda} u^2$$

$$\geq \mathbb{E} C \lambda^{-1} \int_{Q_{t_0}, \delta_0} \left\{ w^2 \left[ - \sum_{j,k=1}^{n} (a^{jk} v_{xj})_{xk} + \mathcal{A} v \right]^2 - C \left( w^{-2\lambda} g_1^2 + \lambda^2 w^{-1-2\lambda} u^2 \right) \right\} dx dt$$

Recalling $z \in H_{t_0, \delta_0}$ and using Itô formula, we get that

$$\frac{1}{\lambda} \mathbb{E} \int_{Q_{t_0}, \delta_0} \sigma^2 \phi^2 (1 + \mu \sigma) \sum_{j,k=1}^{n} (a^{jk} \ell_{xj})_{xk} v dv dx$$

$$= - \frac{1}{\lambda} \mathbb{E} \int_{Q_{t_0}, \delta_0} \sigma^2 \phi^2 (1 + \mu \sigma) \left[ \sum_{j,k=1}^{n} (a^{jk} \ell_{xj})_{xk} v^2 - \sum_{j,k=1}^{n} (a^{jk} \ell_{xj})_{xk} \theta^2 g_2^2 \right] dx dt$$

$$\geq - C \int_{Q_{t_0}, \delta_0} w^{-2\lambda} (g_2^2 + u^2) dx dt.$$  

**Step 4.** Combining (3.19), (3.24), (3.26) and (3.27), recalling that $v = w^{-\lambda} u$, we get (3.4). 

**4 Proof of Theorem 1.1**

Assume that $z \in L^2_{\mathbb{P}}(\Omega; C([t_0 - \delta_0, t_0 \in 0]; L^2(B_{\delta_0})) \cap L^2_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; H^1(B_{\delta_0}))$ satisfies that

$$dz - \sum_{j,k=1}^{n} (a^{jk} z_{xj})_{xk} dt = \hat{a} \cdot \nabla z dt + \hat{b} \cdot z dt + \hat{c} dz W(t) \quad \text{in} \; Q_{t_0, \delta_0},$$

where $\hat{a} \in L^\infty_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; L^\infty(B_{\delta_0}; \mathbb{R}^n))$, $\hat{b} \in L^\infty_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; L^\infty(B_{\delta_0}))$ and $\hat{c} \in L^\infty_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; W^{1,\infty}(B_{\delta_0})).$ Set

$$M = |\hat{a}|_{L^p_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; L^\infty(B_{\delta_0}; \mathbb{R}^n))} + |\hat{b}|_{L^p_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; L^\infty(B_{\delta_0}))} + |\hat{c}|_{L^p_{\mathbb{P}}(t_0 - \delta, t_0 + \delta; W^{1,\infty}(B_{\delta_0}))}.$$
For any fixed constant $0 < \delta_1 < \delta_0$, in the rest of this section, we denote by $C \triangleq C(\delta_1, \delta_0, M, (a^{jk})_{1 \leq j, k \leq n})$ a generic constant which may change from line to line.

**Lemma 4.1** There exists a constant $C > 1$ such that for all $0 < r_2 < r_1 < 2r_1 < r_0$, it holds that

$$
|z|L^2_\varphi(t_0 - \delta_1, t_0 + \delta_1; L^2(B_{r_1})) \leq C|z|L^2_\varphi(t_0 - \delta_0, t_0 + \delta_0; L^2(B_{r_0}))
$$

\begin{align}
&+ C|z|L^2_\varphi(t_0 - \delta_0, t_0 + \delta_0; L^2(B_{r_0}))^\frac{1 - \varepsilon_0}{\varepsilon_0}
&+ C|z|L^2_\varphi(t_0 - \delta_0, t_0 + \delta_0; L^2(B_{r_0}))^{\frac{1 - \varepsilon_0}{\varepsilon_0}}
&+ C|z|L^2_\varphi(t_0 - \delta_0, t_0 + \delta_0; L^2(B_{r_0}))^{\frac{1 - \varepsilon_0}{\varepsilon_0}}
\end{align}

where $\varphi$ given in \((3.2)\) and

\begin{align}
\varepsilon_0 &= \frac{\ln \varphi(2r_0/3) - \ln \varphi(r_1)}{\ln \varphi(2r_0/3) - \ln \varphi(r_2/2)}.
\end{align}

**Remark 4.1** Lemma 4.1 is a generalization of the classical three cylinder inequality of parabolic equations (e.g., \cite{30}). Besides being an important tool to prove the strong unique continuation, as the deterministic case, it can be used to solve some inverse problems for stochastic parabolic equations. However, this is beyond the scope of this paper and will be presented in our future works.

**Proof:** Let $t_1 \in (0, (\delta_0 - \delta_1)/4)$. Set $T_1 = \delta_0 - t_1$ and $T_2 = \delta_0 - 2t_1$. Let $\psi \in C^2_0(t_0 - \delta_0, t_0 + \delta_0)$ such that

\begin{align}
\psi(t) = \begin{cases}
0, & \text{if } t \in [t_0 - \delta_0, t_0 - T_1] \cup [t_0 + T_1, t_0 + \delta_0], \\
1, & \text{if } t \in [t_0 - T_2, t_0 + T_2], \\
\exp\left(-\frac{\delta_0^3}{(T_1 - t + t_0)^3}t\right), & \text{if } t \in (t_0 + T_2, t_0 + T_1), \\
\exp\left(-\frac{\delta_0^3}{(T_2 + t - t_0)^3}t\right), & \text{if } t \in (t_0 - T_1, t_0 - T_2).
\end{cases}
\end{align}

Let $\alpha_0 = r_2/2$ and let $f \in C^2_0(0, r_0)$ such that

\begin{align}
f(t) = \begin{cases}
0, & \text{if } t \in [0, \alpha_0] \cup [3r_0/4, r_0], \\
1, & \text{if } t \in [3\alpha_0/2, 2r_0/3],
\end{cases}
\end{align}

and that

\begin{align}
|f'| \leq C_f/\alpha_0, & \quad |f''| \leq C_f/\alpha_0^2 & \text{in } [\alpha_0, 3\alpha_0/2], \\
|f'| \leq C_f/r_0, & \quad |f''| \leq C_f/r_0^2 & \text{in } [r_0/2, 3r_0/4],
\end{align}

where $C_f$ is an absolute constant.

Let us choose $\zeta$ as

$$
\zeta(x, t) = f(|x|)\psi(t), \quad \text{if } (x, t) \in Q_{r_0, \delta_0}.
$$

Then $\hat{z} \triangleq \zeta z$ solves

\begin{align}
d\hat{z} - \sum_{j,k=1}^n (a^{jk}\hat{z}_x)_x dt = (\hat{a} \cdot \nabla \hat{z} + \hat{b} \hat{z} - \hat{f}) dt + \hat{c} \hat{z} dW(t) \quad \text{in } Q_{r_0, \delta_0},
\end{align}

where

$$
\hat{f} = \zeta \hat{f} + 2 \sum_{j,k=1}^n a^{jk} \zeta x_j \zeta x_k + \sum_{j,k=1}^n (a^{jk} \zeta x_k) x_j z.
$$
By applying the inequality (4.11) to $\tilde{z}$, we obtain that

$$
\mathbb{E} \int_{Q_{r_0,\delta_0}} (\lambda w^{-2\lambda} |\nabla \tilde{z}|^2 + \lambda^3 w^{-2-2\lambda} \tilde{z}^2) \, dx \, dt
$$

$$
\leq C \mathbb{E} \int_{Q_{r_0,\delta_0}} w^{2-2\lambda} (|\hat{a} \cdot \nabla \tilde{z} + \hat{b} \tilde{z} - \tilde{f}|^2 + |\nabla (\hat{c} \tilde{z})|^2) \, dx \, dt + C \lambda^2 \mathbb{E} \int_{Q_{r_0,\delta_0}} w^{-2\lambda} |\hat{c} \tilde{z}|^2 \, dx \, dt.
$$

(4.10)

Denote by

- $K'_1 = \{ (x, t) \in \mathbb{R}^{n+1} \mid \frac{3}{2} \alpha_0 \leq |x| \leq \frac{2r_0}{3}, \ t \in [t_0 - T_1, t_0 - T_2] \}$,
- $K''_1 = \{ (x, t) \in \mathbb{R}^{n+1} \mid \frac{3}{2} \alpha_0 \leq |x| \leq \frac{2r_0}{3}, \ t \in [t_0 - T_2, t_0 + T_1] \}$,
- $K_2 = \{ (x, t) \in \mathbb{R}^{n+1} \mid \alpha_0 \leq |x| \leq \frac{3\alpha_0}{2}, \ t \in [t_0 - T_1, t_0 + T_1] \}$,
- $K_3 = \{ (x, t) \in \mathbb{R}^{n+1} \mid \frac{2r_0}{3} \leq |x| \leq \frac{3r_0}{4}, \ t \in [t_0 - T_1, t_0 + T_1] \}$,
- $K_4 = \{ (x, t) \in \mathbb{R}^{n+1} \mid \frac{3}{2} \alpha_0 \leq |x| \leq \frac{2r_0}{3}, \ t \in [t_0 - T_2, t_0 + T_2] \}$,
- $K_1 = K'_1 \cup K''_1$, $\ K_5 = Q_{r_0,\delta_0} \setminus \bigcup_{i=1}^5 K_i$.

Clearly, we have that $Q_{r_0,\delta_0} = \bigcup_{i=1}^5 K_i$ and $K_i \cap K_j = \emptyset$ for $i \neq j, \ i, j = 1, 2, 3, 4, 5$.

It follows from (4.11) that for every $\lambda \geq \lambda_0$,

$$
\mathbb{E} \int_{K_4} (\lambda w^{-2\lambda} |\nabla \tilde{z}|^2 + \lambda^3 w^{-2-2\lambda} \tilde{z}^2) \, dx \, dt
$$

$$
\leq J_1 + J_2 + C \mathbb{E} \int_{K_4} w^{2-2\lambda} (\tilde{z}^2 + |\nabla \tilde{z}|^2) \, dx \, dt,
$$

(4.11)

where

$$
J_1 \triangleq - \mathbb{E} \int_{K_1} (\lambda w^{-2\lambda} |\nabla \tilde{z}|^2 + \lambda^3 w^{-2-2\lambda} \tilde{z}^2) \, dx \, dt
$$

$$
+ C \mathbb{E} \int_{K_1} w^{2-2\lambda} (|\hat{a} \cdot \nabla \tilde{z} + \hat{b} \tilde{z} - \tilde{f}|^2 + |\nabla (\hat{c} \tilde{z})|^2) \, dx \, dt + C \lambda^2 \mathbb{E} \int_{K_1} w^{-2\lambda} |\hat{c} \tilde{z}|^2 \, dx \, dt,
$$

(4.12)

and

$$
J_2 \triangleq C \mathbb{E} \int_{K_2 \cup K_3} w^{2-2\lambda} (|\hat{a} \cdot \nabla \tilde{z} + \hat{b} \tilde{z} - \tilde{f}|^2 + |\nabla (\hat{c} \tilde{z})|^2) \, dx \, dt
$$

$$
+ C \lambda^2 \mathbb{E} \int_{K_2 \cup K_3} w^{-2\lambda} |\hat{c} \tilde{z}|^2 \, dx \, dt.
$$

(4.13)

By (4.11), we obtain that there is a $\lambda_1 \geq 0$ such that for all $\lambda \geq \max\{\lambda_0, \lambda_1\}$,

$$
\mathbb{E} \int_{K_4} [\lambda w^{-2\lambda} |\nabla (z \zeta)|^2 + \lambda^3 w^{-2-2\lambda} z^2 \zeta^2] \, dx \, dt \leq J_1 + J_2.
$$

(4.14)

It follows from (4.13) that

$$
J_2 \leq C \varphi(\alpha_0)^{2-2\lambda} \mathbb{E} \int_{K_2} (|\nabla \tilde{z}|^2 + z^2) \, dx \, dt + C \varphi\left(\frac{3r_0}{4}\right)^{2-2\lambda} \mathbb{E} \int_{K_3} (|\nabla \tilde{z}|^2 + z^2) \, dx \, dt.
$$

(4.15)
Now we estimate $J_1$. Let
\[
\mathcal{E}(t, x; \lambda) = \psi(t)^2 \left[ \mathcal{C}w^{2-2\lambda} + C \left( \frac{\psi'(t)}{\psi(t)} \right)^2 - \lambda^3 w(x, t)^{-2-2\lambda} \right].
\] (4.16)

From (4.12), we get that
\[
J_1 \leq \mathbb{E} \int_{K_1} \mathcal{E}(t, x; \lambda) z^2 \, dx \, dt + \mathbb{E} \int_{K_1} (\mathcal{C}w^{2-2\lambda} - \lambda w^{-2\lambda}) |\nabla z|^2.
\]

It follows from the choice of $w$ that there is a $\lambda_2 > 0$ such that for all $\lambda \geq \lambda_2$,
\[
\mathcal{C}w^{2-2\lambda} - \lambda w^{-2\lambda} \leq 0
\]
and
\[
\lambda^3 w^{2-2\lambda} \geq 2 \mathcal{C}w^{2-2\lambda}.
\]

Thus, for all $\lambda \geq \lambda_2$,
\[
J_1 \leq \mathbb{E} \int_{K_1} \lambda^3 \psi^2 w^{-1-2\lambda} \left[ w^3 \left( \frac{\psi'(t)}{\psi(t)} \right)^2 \frac{C}{\lambda^3} - 1 \right] z^2 \, dx \, dt.
\] (4.17)

We first handle the case that $(t, x) \in K'_1$. The estimate on $K''_1$ would be similar. Recalling $T_1 - T_2 = t_1$, we get that
\[
\psi'(t) = -\psi(t) \left( \frac{\delta^3}{t_1^4} \right) 4(T_2 + t - t_0)^3(T_1 + t - t_0) - 3(T_2 + t - t_0)^4.
\] (4.18)

Set
\[
\varepsilon(t, x; \lambda) \triangleq w^3 \left( \frac{\psi'(t)}{\psi(t)} \right)^2 \frac{C}{\lambda^3} - 1.
\] (4.19)

Then we know that there exists a constant $C_1 > 0$ such that
\[
\varepsilon(t, x; \lambda) \leq -\frac{1}{2} + \frac{C_1 w^3}{\lambda^3(T_1 + t - t_0)^8} \quad \text{on } K'_1.
\] (4.20)

Set
\[
K_{1,\lambda} \triangleq \left\{ (t, x) \in K'_1 \mid -\frac{1}{2} + \frac{C_1 w^3}{\lambda^3(T_1 + t - t_0)^8} \geq 0 \right\}.
\]

Obviously, we have
\[
\mathbb{E} \int_{K'_1} \lambda^3 \psi^2 w^{-1-2\lambda} \left[ w^3 \left( \frac{\psi'(t)}{\psi(t)} \right)^2 \frac{C}{\lambda^3} - 1 \right] z^2 \, dx \, dt \leq \mathbb{E} \int_{K_{1,\lambda}} \lambda^3 \psi^2 w^{2-2\lambda} z^2 \, dx \, dt.
\] (4.21)

From (4.18), we see that $\sigma^2/\psi^{3/2}$ is bounded in $K_{1,\lambda}$. This, together with (4.21), implies that
\[
\mathbb{E} \int_{K'_1} \lambda^3 \psi^2 w^{-1-2\lambda} \left[ w^3 \left( \frac{\psi'(t)}{\psi(t)} \right)^2 \frac{C}{\lambda^3} - 1 \right] z^2 \, dx \, dt \leq \mathbb{E} \int_{K_{1,\lambda}} \lambda^3 w^{2-2\lambda} \psi^{3/2} z^2 \, dx \, dt.
\] (4.22)

On the other hand, when $(t, x) \in K_{1,\lambda}$, we have
\[
\frac{T_1 + t - t_0}{\delta_0} \leq \left( \frac{2C_1 w^3}{\lambda^3 \delta_0^8} \right)^{1/8}.
\] (4.23)
After choosing \( \lambda \geq \lambda_1 = \left[ \frac{9Cw^3}{t_1} \right]^{1/3} \), we get that
\[
|T_2 + t - t_0| = |T_2 - T_1 + T_1 + t - t_0| \geq t_1 - \left( \frac{2Cw^3}{\lambda^3} \right)^{1/8} \geq \frac{t_1}{2}. \tag{4.24}
\]

From (4.24), we know there exists a constant \( \lambda_3 \), independent of \( r_2 \), such that for any \( \lambda \geq \lambda_3 \),
\[
\varphi \left( \frac{3r_0}{4} \right)^{2\lambda-2} \psi^{3/2}w^2 - 2\lambda \leq \exp \left\{ - \left( \frac{\lambda^3 \delta_0^3}{2Cw^3} \right)^{3/8} \frac{C}{t_1^4} \lambda + (2-2\lambda) \left[ \ln w - \ln \varphi \left( \frac{3r_0}{4} \right) \right] \right\} \leq 1 \tag{4.25}
\]

Combining (4.22) with (4.25), and doing a similar argument on region \( K_1' \), we conclude that
\[
J_1 \leq C \varphi \left( \frac{3r_0}{4} \right)^{2-2\lambda} \int_{K_1} z^2 dxdt. \tag{4.26}
\]

Notice that \( r_1 \in (3\alpha_0/2, 2r_0/3) \) and denote by \( K_4^{(r_1)} \) the region \( \{(x, t) \in K_4 ||x| \leq r_1\} \). By (4.11), (4.15) and (4.26), we obtain that for all \( \lambda \geq \max\{\lambda_3, \lambda_5\} \),
\[
E \int_{K_4^{(r_1)}} z^2 dxdt
\leq \varphi(r_1)^{2\lambda + 2} E \int_{K_4} z^2w^{-2\lambda} dxdt
\leq C \varphi(r_1)^{2\lambda + 2} \left[ \varphi(\alpha_0)^{2-2\lambda} E \int_{K_2} (|\nabla z|^2 + z^2) dxdt + \varphi \left( \frac{3r_0}{4} \right)^{2-2\lambda} E \int_{K_1} z^2 dxdt \right.
\]  
\[+ \varphi \left( \frac{3r_0}{4} \right)^{2-2\lambda} E \int_{K_3} (|\nabla z|^2 + z^2) dxdt \]  
\]

Let \( \zeta_1 \in C_0^\infty(Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}) \) such that \( \zeta_1 = 1 \) in \( K_2 \). By Itô’s formula, we get that
\[
d(\zeta_1^2 z^2) = 2\zeta_1 \zeta_1' tz^2 + 2\zeta_1^2 zdz + \zeta_1^2 (dz)^2. \tag{4.28}
\]

Integrating (4.28) on \( Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0} \) and taking mathematical expectation, we find that
\[
0 = 2E \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1 \zeta_1' tz^2 dxdt + 2E \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1^2 zdzdx
\]  
\[+ E \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1^2 (dz)^2 dx. \tag{4.29}
\]

It follows from (4.1) that
\[
E \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1^2 zdzdx
\]
\[= \sum_{j,k=1}^n \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1^2 \left[ \alpha_{jk} \cdot \nabla z + b_j z \right] dxdt \tag{4.30}
\]  
\[= \sum_{j,k=1}^n \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1^2 \alpha_{jk} z_xz_x dxdt + 2E \int_{Q_{2\alpha_0, \delta_0} \setminus Q_{\alpha_0/2, \delta_0}} \zeta_1 \sum_{j,k=1}^n \alpha_{jk} z_{x_j} z_{x_k} dxdt \]  
\]
Combing (4.29) and (4.30), we find that for any $\varepsilon > 0,$

$$+\mathbb{E}\int_{Q_{2\alpha_0,t_0}\setminus Q_{\alpha_0/2,t_0}} \zeta_1^2 z(a \cdot \nabla z + \hat{b}z) \, dx\,dt.$$  

This, together with (1.1), implies that

$$\epsilon \mathbb{E}\int_{Q_{2\alpha_0,t_0}\setminus Q_{\alpha_0/2,t_0}} \zeta_1^2 |\nabla z|^2 \, dx\,dt + \frac{C}{\epsilon} \mathbb{E}\int_{Q_{2\alpha_0,t_0}\setminus Q_{\alpha_0/2,t_0}} z^2 \, dx\,dt.$$  

By choosing $\varepsilon = s_0/2$, from (4.31) and the definition of $\zeta_1$, we obtain that

$$\mathbb{E}\int_{K_2} |\nabla z|^2 \, dx\,dt \leq C \mathbb{E}\int_{Q_{2\alpha_0,t_0}\setminus Q_{\alpha_0/2,t_0}} z^2 \, dx\,dt.$$  

Similarly, we can get that

$$\mathbb{E}\int_{K_3} |\nabla z|^2 \, dx\,dt \leq C \mathbb{E}\int_{Q_{r_0,t_0}\setminus Q_{r_0/3,t_0}} z^2 \, dx\,dt.$$  

Put

$$\eta = |z|_{L^2_t(L^3_x;\mathbb{R}^n; L^2(B_{r_1})))}, \quad \tilde{\eta} = |z|_{L^2_t(L^3_x;\mathbb{R}^n; L^2(B_{r_0}))).$$

Adding $\mathbb{E}\int_{Q_{r_1,t_1}} z^2 \, dx\,dt$ to both sides of (4.27), from (4.32) and (4.33), we obtain that

$$|z|^2_{L^2_t(L^3_x;\mathbb{R}^n; L^2(B_{r_1})))} \leq C \left[ \left( \frac{\varphi(3\alpha_0/2)}{\varphi(r_1)} \right)^{2-2\lambda} \eta_1^2 + \left( \frac{\varphi(2r_0/3)}{\varphi(r_1)} \right)^{2-2\lambda} \tilde{\eta}^2 \right].$$  

Set

$$\lambda_4 = \frac{\ln \tilde{\eta} - \ln \eta}{\ln \varphi(2r_0/3) - \ln \varphi(r_2/2)}.$$  

If $\lambda_4 \geq \max_{i=1,2,3}\{\lambda_i\}$, then, by choosing in (4.34) $\lambda = \lambda_4$, we get that

$$|z|^2_{L^2_t(L^3_x;\mathbb{R}^n; L^2(B_{r_1})))} \leq C\eta_1^{\varepsilon_0} \tilde{\eta}^{1-\varepsilon_0},$$  

where $\varepsilon_0$ given in (4.4).

If $\lambda_4 < \max_{i=1,2,3}\{\lambda_i\}$, then, by (4.35), we have that

$$|z|^2_{L^2_t(L^3_x;\mathbb{R}^n; L^2(B_{r_1})))} \leq \eta_1 \leq e^{C\ln \varphi(2r_0/3) - \ln \varphi(r_2/3)} \eta.$$  

This, together with (4.36), yields (4.3).

Now we are in a position to prove Theorem 1.1.
Proof of Theorem 1.1: Recalling that for any $N \in \mathbb{N}$ and $r > 0$, it holds that
\[ \mathbb{E} \int_{Q_{r,\delta_0}} |y(t,x)|^2 \, dx \, dt = O(r^{2N}). \] (4.38)

Applying Lemma 4.1 to the equation (1.2), by (4.38), and passing to the limit as $r_2$ tends to 0, we obtain that
\[ \mathbb{E} |y|^2_{L^2(Q_{r_1,\delta_1})} \leq C e^{-CN}, \quad \text{for every } N \in \mathbb{N}, \] (4.39)
where $C$ is independent of $N$. Passing to the limit as $N \to +\infty$, (4.39) yields that $y = 0$ in $Q_{r_1,\delta_1}$, $\mathbb{P}$-a.s. By iteration, it follows that $y = 0$ in $Q$, $\mathbb{P}$-a.s. \qed

5 Further comments

As far as we know, Theorem 1.1 is the first result concerning the SUCP for stochastic PDEs. Compared with the fruitful study of the SUCP for deterministic PDEs, lots of things should be done and some of them seem to be very interesting and challenging.

- The SUCP for stochastic parabolic equations with nonsmooth coefficients. In [5, 16], the authors show that the SUCP for deterministic parabolic equations holds when the coefficients $a$ and $b$ are integrable in some weighted spaces. We believe these results can be generalized in the stochastic setting. However, to this end, one has to develop $L^p$ Carleman estimate for stochastic parabolic equations. It seems to us that this is a fascinating but difficult problem.

- SUCP for other type of stochastic PDEs. SUCP is also studied for some other type of PDEs, such as wave equations (e.g. [17, 32]) and plate equations (e.g. [29]). It is diverting to see whether these results hold for corresponding stochastic PDEs. However, although some Carleman estimates have been obtained for some other stochastic PDEs (e.g. [22, 23, 25, 34]), as far as we know, they cannot be used to establish SUCP for these equations.

- Applications with the SUCP. As we said in the introduction, there are lots of applications of SUCP for deterministic PDEs. It is quite interesting to investigate applications of SUCP for stochastic PDEs. Some of them can be done easily. For example, our result implies approximate controllability of backward stochastic parabolic equations. Another example is that following the idea in [31], one can get some results for some inverse problems of stochastic parabolic equations. Details of these two applications are beyond the scope of this paper and will be investigated in our forthcoming papers.

Acknowledgement

We appreciate Professor Luis Escauriaza for pointing out some references for SUCP for deterministic parabolic equations.

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