

COBORDISM INVARIANCE AND
THE WELL-DEFINEDNESS OF LOCAL INDEX

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ABSTRACT. In the previous papers, Furuta, Yoshida and the author gave a definition of analytic index theory of Dirac-type operator on open manifolds by making use of some geometric structure on an open covering of the end of the open manifold and a perturbation of the Dirac-type operator. In this paper we show the cobordism invariance of the index, and as an application we show the well-definedness of the index with respect to the choice of the open covering.

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1. Introduction

In the series of papers [3,4], Furuta, Yoshida and the author developed an index theory of Dirac-type operators on open Riemannian manifolds when an additional structure, which is called an acyclic compatible system, is given on the end of the manifold. The resulting index, local index (or relative index), has several topological properties. In particular the excision formula leads us to have a localization formula of index of Dirac-type operators. Such a localization formula enables us to have geometric proofs of the localization formula of Riemann-Roch numbers to Bohr-Sommerfeld fibers and singular fibers of Lagrangian fibrations [3], Danilov type formula for 4-dimensional locally toric Lagrangian fibrations [4] and \( [Q,R]=0 \) formula for Hamiltonian torus actions [5]. To describe the index theory one needs to fix an open covering of the end, however there are several choices of open coverings when one would like to apply the theory to individual manifold, for instance, a symplectic manifold with Hamiltonian torus action. It is not clear by definition that the local index does not depend on the choice of the open covering.

In this paper we show that the well-definedness of the local index, that is, if there are two acyclic compatible systems with a specific condition, which we call \( G\)-tangential, on two ends of a single manifold such that the union is also an acyclic compatible system, then two local indices defined by two compatible systems coincide. The well-definedness is shown by the cobordism invariance of the local index. There are several works concerning the cobordism invariance of the analytic index of Fredholm operators [6,7]. The proof of the present paper is a variant of the technique developed by Braverman [1,2]. The technique is a version of Witten-type deformation by using a kind of Morita equivalence and an extra symmetry of Clifford algebras of indefinite metrics. Using the Morita equivalence the formulation of the cobordism invariance can be rephrased as a statement for Clifford module bundles with extra symmetry of Clifford algebras of indefinite metrics. Such a reformulation leads us to have an extension of the cobordism invariance for real Clifford module bundles. The cobordism invariance for the complex Clifford module bundle can be proved by the almost same argument as in [1,2]. We give the proof for the real Clifford module bundle case.

This paper is organized as follows. In Section 2 we recall definitions of Clifford algebras for a vector spaces/bundles with indefinite metrics and the modules over such algebras. In Section 3 we recall definitions of compatible fibrations, compatible systems and the definition of the local index. We introduce a new class of compatible fibrations (resp. systems), \( G\)-tangential compatible fibrations (resp. systems), which is useful to define the local index and describe an application of the well-definedness of the local index. For latter convenience we work with Clifford module bundles with an action of Clifford algebras of Euclidean space with indefinite metric. In Subsection 4.1 and Subsection 4.2 we define the notion of cobordism of compatible fibrations and compatible systems. In Subsection 4.3 we give the statement of the cobordism invariance of the local index (Theorem 4.4). In Section 5 we give a reformulation of the cobordism invariance using extra symmetry. In Section 6 we give the proof of Theorem 5.3 by using a modification of the technique in [1,2] for real Clifford module bundles. In Subsection 6.4 we give comments on how we modify the proofs in [1,2] or that for the real case to prove the complex case. In Section 7 we show the well-definedness of the local index (Theorem 7.1). Finally we give an application of Theorem 7.1 to the product of compatible fibrations, which is convenient to compute the local index.
2. Clifford algebras for indefinite metrics

In this paper we will work with Clifford algebras for indefinite metrics and Clifford modules over such algebras. Let us recall basic definitions.

**Definition 2.1.** Let $\mathbb{R}^{p,q}$ be the real vector space $\mathbb{R}^{p+q}$ with an indefinite metric $\cdot$, defined by

$$u \cdot v = (u_1v_1 + \cdots + u_pv_p) - (u'_1v'_1 + \cdots + u'_qv'_q)$$

for $u = (u_1, \cdots, u_p, u'_1, \cdots, u'_q)$, $v = (v_1, \cdots, v_p, v'_1, \cdots, v'_q) \in \mathbb{R}^{p,q}$. We define the Clifford algebra $Cl(\mathbb{R}^{p,q})$ over $\mathbb{R}^{p,q}$ by the algebra generated by $\mathbb{R}^{p,q}$ over $\mathbb{R}$ with the Clifford relation

$$uv + vu = -2u \cdot v$$

for $u, v \in \mathbb{R}^{p,q}$, which has the universal property among the above relation.

We often denote $Cl_{p,q} = Cl(\mathbb{R}^{p,q})$ for simplicity. By definition we have $Cl_{p,0} = Cl(\mathbb{R}^p)$.

**Definition 2.2.** A (complex) representation of $Cl(\mathbb{R}^{p,q})$ is a pair $(R, c_R)$ consisting of a Hermitian vector space $R$ and an $\mathbb{R}$-algebra homomorphism, the Clifford action (or Clifford multiplication), $c_R : Cl(\mathbb{R}^{p,q}) \to \text{End}_\mathbb{R}(R)$ such that $c_R(u)$ is a skew Hermitian operator for each $u \in \mathbb{R}^{p,0} \subset \mathbb{R}^{p,q}$ and $c_R(v)$ is a Hermitian operator for each $v \in \mathbb{R}^{0,q} \subset \mathbb{R}^{p,q}$. When $R$ has a $\mathbb{Z}/2$-grading and $c_R(u)$ is degree-one map for each $u \in \mathbb{R}^{p,q}$, we call $(R, c_R)$ a $\mathbb{Z}/2$-graded representation.

By using a Euclidean vector space and (skew) symmetric operators, a real representation of $Cl(\mathbb{R}^{p,q})$ can be defined in a similar manner.

**Example 2.3.** Let $W = W^+ \oplus W^-$ be a $\mathbb{Z}/2$-graded representation of $Cl(\mathbb{R}^p)$ and $\epsilon_W : W \to W$ be the map which represents the grading, i.e., $\epsilon|_{W^\pm} = \pm \text{id}_{W^\pm}$. By forgetting the grading $W$ can be regarded as a representation of $Cl(\mathbb{R}^p \otimes \text{Re}_W)$, where $\text{Re}_W$ is the real vector space generated by $\epsilon_W$ with the negative definite metric $\epsilon_W \cdot \epsilon_W = -1$.

It is straightforward to define the Clifford algebra bundle $Cl(E)$ for a real vector bundle $E$ with metric. A ($\mathbb{Z}/2$-graded) complex (or real) $Cl(E)$-module bundle can be also defined. By definition we have $Cl(E_1 \oplus E_2) \cong Cl(E_1) \otimes Cl(E_2)$ for two vector bundles $E_1$ and $E_2$.

**Definition 2.4.** Let $(M, g)$ be a Riemannian manifold and $W$ a $Cl(TM \oplus \mathbb{R}^{p,q})$-module bundle over $M$. A differential operator $D : \Gamma(W) \to \Gamma(W)$ is called Dirac-type if $D$ is a formally self-adjoint operator of order-one whose principal symbol $\sigma(D) : TM \to \text{End}(W)$ satisfies the Clifford relation $\sigma(D)(v)\sigma(D)(v') + \sigma(D)(v')\sigma(D)(v) = -2g(v, v')$ for any $v, v' \in TM$. In other words $\sigma(D)$ is equal to the Clifford action of $Cl(TM) \subset Cl(TM \oplus \mathbb{R}^{p,q})$. If $W$ is $\mathbb{Z}/2$-graded, then we assume that $D$ is of degree-one.

3. Acyclic compatible systems and their local indices

In this section we recall some definitions of compatible fibration, acyclic compatible system and their local indices following [1] [3].

3.1. Compatible fibration. Let $V$ be a manifold.

**Definition 3.1.** A compatible fibration on $V$ is a collection of the data $\{V_\alpha, \mathcal{F}_\alpha\}_{\alpha \in A}$ consisting of an open covering $\{V_\alpha\}_{\alpha \in A}$ of $V$ and a foliation $\mathcal{F}_\alpha$ on $V_\alpha$ with compact leaves which satisfies the following properties.
(1) The holonomy group of each leaf of \( \mathcal{F}_\alpha \) is finite.

(2) For each \( \alpha \) and \( \beta \), if a leaf \( L \in \mathcal{F}_\alpha \) has non-empty intersection \( L \cap V_\beta \neq \emptyset \), then, \( L \subseteq V_\beta \).

3.2. **Compatible system and its acyclicity.** Let \((V, g)\) be a Riemannian manifold, \(W\) a \( \text{Cl}(TV \oplus \mathbb{R}^{p,q})\)-module bundle over \( V \). In the rest of this paper we impose the following conditions on the Riemannian metric \( g \).

**Assumption 3.2.** Let \( \nu_\alpha = \{ u \in TV_\alpha \mid g(u, v) = 0 \text{ for all } v \in T\mathcal{F}_\alpha \} \) be the normal bundle of \( \mathcal{F}_\alpha \). Then, \( g|_{\nu_\alpha} \) is invariant under holonomy, and gives a transverse invariant metric on \( \nu_\alpha \).

**Definition 3.3.** A compatible system on \((\{V_\alpha, \mathcal{F}_\alpha\}, W)\) is a data \( \{D_\alpha\}_{\alpha \in A} \) satisfying the following properties.

1. \( D_\alpha : \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha}) \) is an order-one formally self-adjoint differential operator.
2. \( D_\alpha \) contains only the derivatives along leaves of \( \mathcal{F}_\alpha \).
3. \( D_\alpha \) is a Dirac-type operator along leaves. Namely the principal symbol of \( D_\alpha \) is given by the composition of the dual of the natural inclusion \( \iota : TV_\alpha \to TV_\alpha \) and the Clifford multiplication \( c : T\mathcal{F}_\alpha \cong TV_\alpha \to TV_\alpha \to \text{End}(W|_{V_\alpha}) \). If \( W \) is \( \mathbb{Z}/2\)-graded, then \( D_\alpha \) is of degree-one.
4. For a leaf \( L \in \mathcal{F}_\alpha \) let \( \tilde{u} \in \Gamma(\nu_\alpha|_L) \) be a section of \( \nu_\alpha|_L \) parallel along \( L \). \( \tilde{u} \) acts on \( W|_L \) by the Clifford multiplication \( c(\tilde{u}) \). Then \( D_\alpha \) and \( c(\tilde{u}) \) anti-commute each other, i.e.

\[
0 = \{D_\alpha, c(\tilde{u})\} := D_\alpha \circ c(\tilde{u}) + c(\tilde{u}) \circ D_\alpha
\]
as an operator on \( W|_L \).
5. \( D_\alpha \) anti-commutes with \( \text{Cl}_{p,q}(\subset \text{Cl}(TM \oplus \mathbb{R}^{p,q}))\)-action.

As in [4] Lemma 3.4] for each leaf \( L \in \mathcal{F}_\alpha \) we have a small open tubular neighborhood \( V_L \) of \( L \) and the finite covering \( q_L : \tilde{V} \to V_L \) such that the induced foliation on \( \tilde{V}_L \) is a bundle foliation with the projection \( \pi_L : \tilde{V}_L \to \tilde{U}_L \).

**Definition 3.4.** A compatible system \( \{D_\alpha\}_{\alpha \in A} \) on \((\{V_\alpha, \mathcal{F}_\alpha\}, W)\) is said to be acyclic if it satisfies the following conditions.

1. The Dirac type operator \( q_L^{-1}\pi_L^{-1}(\tilde{b})D_\alpha|_{\pi_L^{-1}b} \) has zero kernel for each \( \alpha \in A \), leaf \( L \in \mathcal{F}_\alpha \) and \( \tilde{b} \in \tilde{U}_L \).
2. If \( V_\alpha \cap V_\beta \neq \emptyset \), then the anti-commutator \( \{D_\alpha, D_\beta\} := D_\alpha D_\beta + D_\beta D_\alpha \) is a non-negative operator on \( V_\alpha \cap V_\beta \).

3.3. **Definition of the local index.** In [4] Furuta, Yoshida and the author showed that when a manifold whose end is equipped with an acyclic compatible system with some technical conditions, the local index can be defined. In this section we first introduce a class of compatible fibration which gives a sufficient condition to define the local index.

**Definition 3.5.** Suppose that a compact Lie group \( G \) acts on a Riemannian manifold \( V \) in an isometric way. Let \( \{V_\alpha, \mathcal{F}_\alpha\}_{\alpha \in A} \) be a compatible fibration on \( V \). If the following conditions are satisfied, then we call the compatible fibration a \( G \)-tangential compatible fibration (or tangential compatible fibration for short).

- \( \{V_\alpha\}_{\alpha \in A} \) is a \( G \)-invariant open covering of \( V \).
- Each leaf \( L \) of \( \mathcal{F}_\alpha \) has positive dimension for all \( \alpha \in A \).
• For each leaf $L$ of $\mathcal{F}_a$ there exists some $x \in V_a$ such that $L$ is contained in the $G$-orbit $G \cdot x$.

A compatible system on a $G$-tangential compatible fibration is called $G$-tangential compatible system (or tangential compatible system for short).

**Example 3.6.** In [4] Definition 6.7 a class of compatible fibration which is called good compatible fibration is defined. Any non-trivial torus action induces a good compatible fibration, which is a tangential compatible fibration. Moreover the product of two such good compatible fibrations is a tangential compatible fibration which is not good in general.

Though the notion of $G$-tangential compatible fibration can be defined for arbitrary compact Lie group $G$, we do not know examples of tangential compatible fibration for non-Abelian group.

Let $M$ be a Riemannian manifold and $W$ a $\mathbb{Z}/2\mathbb{Z}$-graded $\text{Cl}(TM \oplus \mathbb{R}^{p,q})$-module bundle. As in the same way in the proof Theorem 7.2 and Proposition 7.3 in [4], we have the following.

**Theorem 3.7.** Suppose that $V$ is an open subset of $M$ whose complement is compact. If $V$ is equipped with a $G$-tangential acyclic compatible system $\{V_a, \mathcal{F}_a, D_a\}_{a \in A}$, then we can define the local index $\text{ind}(M, \{V_a, \mathcal{F}_a, D_a\}_{a \in A}, W) = \text{ind}(M, V, W) = \text{ind}(M, V)$, which satisfies the excision formula, sum formula and product formula.

The resulting index is a representation of $\text{Cl}_{p,q}$, i.e., an element of the $K$-group with $\text{Cl}_{p,q}$-action $K(pt, \text{Cl}_{p,q})$, where for any locally compact Hausdorff space $X$, $K$-group $K(X, \text{Cl}_{p,q})$ is defined as the Grothendieck group of the semigroup consisting of equivalence classes of complex vector bundles with fiberwise $\text{Cl}_{p,q}$-action. There is an equivalent description of $K(X, \text{Cl}_{p,q})$. Let $\{e_1, \ldots, e_p, e'_1, \ldots, e'_q\}$ be the standard basis of $\mathbb{R}^{p \times q}$. Let $G_{p,q}$ be the finite group generated by $\{e_1, \ldots, e_p, e'_1, \ldots, e'_q\}$ with relations

\[
\delta^2 = 1, \quad \delta e_i = e_i \delta, \quad \delta e'_j = e'_j \delta, \quad e_i^2 = e'_j^2 = \delta, \quad e_i e_j e_i^{-1} e_j^{-1} = e_i e'_j e'_j^{-1} e_i^{-1} = \delta
\]

for $i, j = 1, \ldots, p$ ($i \neq j$), $i', j' = 1, \ldots, q$ ($i' \neq j'$). Note that a representation of $\text{Cl}_{p,q}$ is equivalent to a representation of $G_{p,q}$ such that $\delta$ acts as $-\text{id}$. In terms of this group, $K(X, \text{Cl}_{p,q})$ can be identified with the subgroup of $G_{p,q}$-equivariant $K$-group $K_{G_{p,q}}(X)$ consisting of vector bundles $E$ on which $\delta$ acts as $-\text{id}$.

Let us briefly recall the definition of the local index $\text{ind}(M, V, W)$. Let $D : \Gamma(W) \to \Gamma(W)$ be a Dirac-type operator. We may assume that $D$ anti-commutes with $\text{Cl}_{p,q}(\subset \text{Cl}(TM \oplus \mathbb{R}^{p,q}))$-action by taking the average by the finite group $G_{p,q}$. We consider the perturbation $D + t \sum_{a \in A} \rho_a D_a \rho_a$ for $t > 1$, where $\{\rho_a\}_{a \in A}$ is a family of smooth cut-off functions which is constant along leaves of $\mathcal{F}_a$ and satisfies some estimates as in [4] Subsection 4.1. Such a perturbation gives a Fredholm operator on the space of $L^2$-sections of $W$. In [4] Section 7 we explained the details of the definition of $\text{ind}(M, V)$ for good compatible fibrations arising from torus action. The construction can be generalised for tangential compatible fibrations which is not necessarily good. The point is as follows.

• We can take a family of smooth cut-off functions $\{\rho_a\}_{a \in A}$ which satisfies the following.

  - $\rho(x) + \sum_a \rho_a(x) > 0$ for all $x \in M$.
  - For each $a \in A$ we have $\text{supp}(\rho_a) \subset V_a$.
  - Each $\rho_a$ is $G$-invariant.
We can deform the end of $V$ using a proper function together with \( \{ V_{\alpha}, \mathcal{F}_{\alpha}, D_{\alpha} \}_{\alpha \in A} \) so that the end has a cylindrical end and the resulting deformed compatible system has translationally invariance.

**Remark 3.8.** When $V$ has several connected components $V_{(1)}, V_{(2)}, \cdots, V_{(k)}$ we can generalize the definition of $G$-tangential compatible system in such a way that there exists a family of compact Lie groups \( \{ G_{(1)}, G_{(2)}, \cdots, G_{(k)} \} \) and each $G_{(i)}$ acts on the component $V_{(i)}$. Moreover using a suitable technical conditions for $\{ V_{\alpha} \}_{\alpha \in A}$ it may be possible to generalize in such a way that there exists a family of compact Lie groups $\{ G_{\alpha} \}_{\alpha \in A}$ and each $G_{\alpha}$ acts on $V_{\alpha}$.

**Remark 3.9.** We do not assume that $W$, $D$ and $D_{\alpha}$ are $G$-equivariant. We only use the group action to have cut-off functions and to deform the end so that the functions are constant along each leaves and we can extend the restriction of the data to the cylindrical end.

**Remark 3.10.** All constructions in this section can work for real Clifford module bundles. In fact we use real Clifford module bundle case latter.

### 4. Cobordism and cobordism invariance of the local index

In this section we give the definitions of cobordisms of compatible fibrations and compatible systems in a natural way. We also introduce the notion of acyclicity of the cobordism, and by using it we give the statement of our main theorem, **cobordism invariance of the local index**

#### 4.1. Cobordism of compatible fibrations

Let $V_1$ and $V_2$ be smooth manifolds.

**Definition 4.1.** Two compatible fibrations $\{ V_{1,\alpha}, \mathcal{F}_{1,\alpha} \}_{\alpha \in A}$ on $V_1$ and $\{ V_{2,\beta}, \mathcal{F}_{2,\beta} \}_{\beta \in B}$ on $V_2$ are **cobordant** if there exists a data $\{ V, V_{\gamma}, \mathcal{F}_{\gamma} \}_{\gamma \in C}$ satisfying the following conditions.

(i) $V$ is a smooth manifold with $\partial V = V_1 \cup V_2$.

(ii) $\{ V_{\gamma}, \mathcal{F}_{\gamma} \}_{\gamma \in C}$ is a compatible fibration on $V$.

(iii) There exists a tubular neighborhood $V_{0}$ of $\partial V$ which is diffeomorphic to $V_{1} \times (-\varepsilon, 0] \cup V_{2} \times [0, \varepsilon)$ for some small $\varepsilon > 0$ and $V_{0} \cap V_{\gamma}$ consists of leaves of $\mathcal{F}_{\gamma}$ for each $\gamma \in C$.

(iv) Under the diffeomorphism in (iii) the restriction of the foliation $\mathcal{F}_{\gamma}|_{V_{0} \cap V_{\gamma}}$ for each $\gamma \in C$ is isomorphic to the foliation on $V_{1,\alpha} \times (-\varepsilon, 0] \cap V_{\gamma}$ or $V_{2,\beta} \times [0, \varepsilon) \cap V_{\gamma}$ induced from $\mathcal{F}_{1,\alpha}$ or $\mathcal{F}_{2,\beta}$ for some $\alpha \in A$ or $\beta \in B$. In particular each leaf of $\mathcal{F}_{\gamma}|_{V_{0} \cap V_{\gamma}}$ is diffeomorphic to the product of an interval and some leaf of $\mathcal{F}_{1,\alpha}$ or $\mathcal{F}_{2,\beta}$.

We call $\{ V, V_{\gamma}, \mathcal{F}_{\gamma} \}_{\gamma \in C}$ a **cobordism** between $\{ V_{1,\alpha}, \mathcal{F}_{1,\alpha} \}_{\alpha \in A}$ and $\{ V_{2,\beta}, \mathcal{F}_{2,\beta} \}_{\beta \in B}$.

#### 4.2. Cobordism of compatible systems

Let $V_1$ and $V_2$ be smooth Riemannian manifolds. For $i = 1, 2$ let $W_i \rightarrow V_i$ be $\mathbb{Z}/2$-graded $\text{Cl}(TV_i \oplus \mathbb{R}^{p,q})$-module bundles.

**Definition 4.2.** Two compatible systems $\{ V_{1,\alpha}, \mathcal{F}_{1,\alpha}, W_1, D_{1,\alpha} \}_{\alpha \in A}$ on $V_1$ and $\{ V_{2,\beta}, \mathcal{F}_{2,\beta}, W_2, D_{2,\beta} \}_{\beta \in B}$ on $V_2$ are **cobordant** if there is a collection of data $\{ V, V_{\gamma}, \mathcal{F}_{\gamma}, W, D_{\gamma} \}_{\gamma \in C}$ which satisfies the following conditions.

(i) $\{ V, V_{\gamma}, \mathcal{F}_{\gamma} \}_{\gamma \in C}$ is a cobordism between $\{ V_{1,\alpha}, \mathcal{F}_{1,\alpha} \}_{\alpha \in A}$ and $\{ V_{2,\beta}, \mathcal{F}_{2,\beta} \}_{\beta \in B}$.

(ii) The diffeomorphism in Definition (iii) is isometric.

(iii) $W$ is a $\text{Cl}(TV \oplus \mathbb{R}^{p,q})$-module bundle over $V$ (without $\mathbb{Z}/2$-grading).

(iv) $\{ D_{\gamma} \}_{\gamma \in C}$ is a compatible system on $\{ V_{\gamma}, \mathcal{F}_{\gamma}, W \}$. 


Theorem 4.4. Under the above setting we have
\[ \text{ind}(M_1, V_1) = \text{ind}(M_2, V_2). \]

5. COBORDISM INVARIANCE AND EXTRA SYMMETRY OF CLIFFORD ALGEBRAS

In this section we give a reformulation of the cobordism invariance by using an extra symmetry of Clifford algebras for indefinite metrics. In Subsection 5.1 we give the reformulation for the complex case (Theorem 5.1). We also give in Subsection 5.2 a formulation of cobordism invariance of local index for real Clifford module bundles (Theorem 5.3). We will show the real case in the next section. In this section we consider the following set-up.

Set-up. \( M \) is a Riemannian manifold and \( V \) is an open subset of \( M \). Suppose that \( V \) is equipped with a \( G \)-tangential compatible fibration \( \{ V, \mathcal{F}_\gamma, W, D_\gamma \}_{\gamma \in \mathcal{A}} \) for a compact Lie group \( G \). Moreover we assume that there exists a smooth function \( f : M \to \mathbb{R} \) such that \( f|_{M \setminus V} \) is proper and \( f|_V \) is \( G \)-invariant.

5.1. Complex Clifford module bundle case. Let \( \hat{W} \to M \) be a complex \( \text{Cl}(TM \oplus \mathbb{R}e_L \oplus \mathbb{R}e_R \oplus \mathbb{R}e'_L \oplus \mathbb{R}e'_R \oplus \mathbb{R}e''_L \oplus \mathbb{R}e''_R) \)-module bundle, where the metric is defined by \( e_L \cdot e_L = 1, e_R \cdot e_R = e'_R \cdot e'_R = -1 \). Suppose that there exists a \( G \)-tangential acyclic compatible system
\{V_\gamma, \mathcal{F}_\gamma, \hat{W}[V, \hat{D}_\gamma]\}_{\gamma \in \hat{A}} \text{ on } V. \text{ Let } (R_\mathbb{C}, c_{1,2}) \text{ be an irreducible complex representation of } Cl(\mathbb{R}e_\mathbb{L} \oplus \mathbb{R}e_\mathbb{R} \oplus \mathbb{R}e_\mathbb{L}^\mathbb{R}) = Cl_{1,2} \text{ defined by } R_\mathbb{C} := \wedge^* \mathbb{C} = \mathbb{C} \oplus \mathbb{C} \text{ and } \\
c_{1,2}(\varepsilon_\mathbb{L}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_{1,2}(\varepsilon_\mathbb{R}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_{1,2}(\varepsilon_\mathbb{L})^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.

We define \( Cl(TM \oplus \mathbb{R}^{p,q})\)-module bundle

\[ W := \text{Hom}_{Cl_{1,2}}(R_\mathbb{C}, \hat{W}) \to M. \]

Note that we have \( W \otimes R_\mathbb{C} \cong \hat{W} \) as \( Cl(TM \oplus \mathbb{R}^{p,q}) \otimes Cl_{1,2} \cong Cl(TM \oplus \mathbb{R}e_\mathbb{L} \oplus \mathbb{R}^{p,q} \otimes \mathbb{R}e_\mathbb{R} \oplus \mathbb{R}e_\mathbb{R}^\mathbb{R})\)-module bundles. For each regular value \( r \) of \( f \), we put \( M_r := f^{-1}(r) \) and \( V_r := M_r \cap V \). Then \( W_r := W|_{M_r} \) has a structure of \( Cl(TM_r \oplus \mathbb{R}r_r \oplus \mathbb{R}^{p,q})\)-module bundle, where \( v_r \) is the unit tangent vector which is normal to \( TM_r \) with the increasing direction of \( f \). Since we have \( c(v_r)^2 = -1 \), we can define \( \mathbb{Z}/2\)-grading \( W_r = W_r^+ \oplus W_r^- \) by \( c(v_r)|_{W_r^\pm} = \pm \sqrt{-1} \). For each \( \alpha \in \hat{A} \) the Dirac-type operator along leaves \( \hat{D}_\alpha \) induces another Dirac-type operator along leaves \( \hat{D}_\alpha : \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha}) \) defined by \( (\hat{D}_\alpha s)(r) := \hat{D}_\alpha(s(r)) \) for \( s \in \Gamma(W) \) and \( r \in R \). Let \( A_r \) be the subset of \( \hat{A} \) defined by \( A_r = \{ \alpha \in \hat{A} \mid V_\alpha \cap M_r \neq \emptyset \} \).

For \( \alpha \in A_r \) we put \( V_{r,\alpha} := V_\alpha \cap V_r \) and define \( D_{r,\alpha} : \Gamma(W|_{V_{r,\alpha}}) \to \Gamma(W|_{V_{r,\alpha}}) \) by the restriction of \( \hat{D}_\alpha \). Since \( f|_V \) is \( G \)-invariant we have the \( G \)-tangential compatible fibration \( \{ \mathcal{F}_{r,\alpha}\}_{\alpha \in A_r} \), and compatible system \( \{ D_{r,\alpha}\}_{\alpha \in A_r} \) on \( V_r = M_r \cap V \). The following is clear by definition of \( M_r \) and \( D_{r,\alpha} \).

**Lemma 5.1.** The \( G \)-tangential compatible system \( \{ V_{r,\alpha}, \mathcal{F}_{r,\alpha}, W_r, D_{r,\alpha}\}_{\alpha \in A_r} \) over \( V_r \) is acyclic.

Since \( M_r \cap V_r \) is compact the local index \( \text{ind}(M_r, V_r, W_r) \) can be defined as an element of the \( K \)-group with \( Cl_{p,q} \)-action \( K(pt, Cl_{p,q}) \).

**Theorem 5.2.** The local index \( \text{ind}(M_r, V_r, W_r) \) does not depend on the choice of the regular value \( r \).

Given a \( Cl(TM \oplus \mathbb{R}^{p,q})\)-module bundle \( W \) as in Theorem 4.4, we put \( \hat{W} := W \otimes R_\mathbb{C} \), then we have \( \text{Hom}_{Cl_{p,q}}(R_\mathbb{C}, \hat{W}) \cong W \), and hence, Theorem 4.4 is equivalent to the above Theorem 5.2.

### 5.2. Real Clifford module bundle case

Let \( \hat{W} \) be a real \( Cl(TM \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e)\)-module bundle over \( M \), where the metric is defined by \( e_0 \cdot e_0 = e_1 \cdot e_1 = \varepsilon \cdot \varepsilon = -1 \). We regarded \( \hat{W} \) as a \( \mathbb{Z}/2 \)-graded \( Cl(TM \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_0 \oplus \mathbb{R}e_1)\)-module by using the Clifford multiplication \( c(\varepsilon) \). Suppose that there exists a \( G \)-tangential acyclic compatible systems \( \{ V_\gamma, \mathcal{F}_\gamma, \hat{W}[V, \hat{D}_\gamma]\}_{\gamma \in \hat{A}} \) on \( V \). Let \( r \) be a regular value of \( f \). When we put \( M_r := f^{-1}(r) \), we have the \( \mathbb{Z}/2 \)-graded \( Cl(TM_r \oplus \mathbb{R}r_r \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_0 \oplus \mathbb{R}e_1)\)-module bundle \( W_r := \hat{W}|_{M_r} \) as in Subsection 5.1. Let \( (R, c_{1,2}) \) be a realification of the complex representation \( R_\mathbb{C} \) of \( Cl(\mathbb{R}e_\mathbb{L} \oplus \mathbb{R}e_\mathbb{R} \oplus \mathbb{R}e_\mathbb{L}^\mathbb{R}) = Cl_{1,2} \) as in Subsection 5.1 which is an irreducible real representation of \( Cl_{1,2} \). Namely we put \( R := \mathbb{R} \oplus \mathbb{R} \) and define \( c_{1,2} : Cl_{1,2} \to \text{End}(R) \) by

\[ c_{1,2}(v_r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_{1,2}(e_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_{1,2}(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

We define a \( \mathbb{Z}/2 \)-graded \( Cl(TM_r \oplus \mathbb{R}^{p,q})\)-module bundle \( W_r \) by \( W_r := \text{Hom}_{Cl_{1,2}}(R, \hat{W}_r) \). Note that we have \( W_r \otimes R \cong \hat{W}_r \) as \( Cl(TM_r \oplus \mathbb{R}^{p,q}) \otimes Cl_{1,2} \cong Cl(TM_r \oplus \mathbb{R}r_r \oplus \mathbb{R}^{p,q} \oplus \mathbb{R}e_0 \oplus \mathbb{R}e_1) \).
by attaching the cylinder direction with respect to a Clifford connection on $\pi$ over $M$ is defined as the Grothendieck group of the semi-group consisting of real vector bundles projection to the first factor of $M$. Let $\{\tilde{V}, \tilde{F}_\gamma, \tilde{W}|_V, \tilde{D}_\gamma\}_{\gamma \in A}$ be a real $\mathbb{Z}/2$-graded $\text{Cl}(TM \oplus \mathbb{R}^p \oplus \mathbb{R}e_0 \oplus \mathbb{R}e_1 \oplus \mathbb{R}e)$-module by using the Clifford multiplication of $c(e)$. Suppose that there exists an open subset $V$ of $M$ which has compact complement and is equipped with an acyclic $G$-tangential compatible systems $\{\tilde{V}, \tilde{F}_\gamma, \tilde{W}|_V, \tilde{D}_\gamma\}_{\gamma \in A}$. When we put $V_0 := M_0 \cap V$ we have the induced $G$-tangential acyclic compatible system $\{V_0, \tilde{F}_0, W_0, D_{0,\alpha}\}_{\alpha \in A}$ over $V_0$, where $W_0 = W|M_0 = W_0^+ \oplus W_0^-$ is the $\mathbb{Z}/2$-graded $\text{Cl}(TM_0)$-module.

Though we do not assume that $M$ and $M_0$ are complete, we can deform them so that they are complete. For instance we can deform them into complete manifolds with cylindrical ends. Moreover we can extend the given data, e.g., compatible system, to the data on the completion so that they have translationally invariance on the end. Hereafter we may assume that $M$ and $M_0$ are complete and the given data have uniformity on the end by using such completion. By the excision property, the local indices do not change under the completion.

6.1. The perturbation of Dirac-type operator. Let $\widetilde{M}$ be the manifold obtained from $M$ by attaching the cylinder $M_0 \times [0, \infty)$, i.e., $\widetilde{M} = M \cup M_0 \times [0, \infty)$. Then we can extend $\{V, V_\gamma, F_\gamma, \tilde{W}|_V, \tilde{D}_\gamma\}_{\gamma \in A}$ naturally to $\widetilde{M}$, and we denote them by $\{\tilde{V}, \tilde{V_\gamma}, \tilde{F}_\gamma, \tilde{W}|_{\tilde{V}}, \tilde{D}_\gamma\}_{\gamma \in A}$. We put $W_0 = \text{Hom}_{\text{Cl}_{1,2}}(R, \tilde{W}|_{M_0})$ for an irreducible real representation $R$ of $\text{Cl}_{1,2}$. We take and fix a $\text{Cl}_{p,q}$-anti-invariant Dirac-type operator $D : \Gamma(W_0) \to \Gamma(W_0)$. Let $\tilde{D} : \Gamma(\tilde{W}) \to \Gamma(\tilde{W})$ be a Dirac-type operator such that
\begin{equation}
\tilde{D}|_{M_0 \times (0, \infty)} = \pi^* D \otimes 1 + 1 \otimes c(v_0) \partial_r
\end{equation}
under the isomorphism $\tilde{W}|_{M_0 \times (0, \infty)} \cong \pi^* \tilde{W}|_{M_0} \cong \pi^*(W_0 \otimes R)$, where $\pi$ is the natural projection to the first factor of $M \times (0, \infty)$, $\partial_r$ is the covariant derivative of $(0, \infty)$-direction with respect to a Clifford connection on $\pi^* \tilde{W}|_{M_0}$. We take a family of cut off functions $\{\rho_\gamma^2\}$ for $\widetilde{M} = (\widetilde{M} \setminus \tilde{V}) \cup \bigcup_{\gamma} \tilde{V}_\gamma$ as in the end of Subsection 5.3. Let $D_\gamma$ be
the Dirac-type operator along leaves which is induced by $\tilde{D}_\gamma$ and we put $D'_\gamma := \rho_\gamma D_\gamma \rho_\gamma$. For $t > 0$ we put
\begin{equation}
(6.2) \quad \tilde{D}'_t = \left( \tilde{D} + t \sum_\gamma D'_\gamma \right) c(e_0).
\end{equation}

We take and fix a smooth function $s : \mathbb{R} \to [0, \infty)$ and $p : \tilde{M} \to \mathbb{R}$ such that $s(r) = r$ for $|r| \geq 1$, $s(r) = 0$ for $|r| \leq 1/2$, $p(x) = 0$ for $x \in M \subset \tilde{M}$ and $p(y, r) = s(r)$ for $(y, r) \in M_0 \times (0, \infty)$. For $a \in \mathbb{R}$ we put
\begin{equation}
(6.3) \quad D_{t,a} = \tilde{D}'_t - c((p(x) - a)e_1).
\end{equation}

**Proposition 6.1.** For any $t > 0$ and $a \in \mathbb{R}$ we have
\[ D_{t,a}^2 = (\tilde{D}'_t)^2 + B + |p(x) - a|^2, \]
where $B$ is a uniformly bounded bundle map of $\tilde{W}$ of degree 0.

**Proof.** The proof is almost same as that for [1, Lemma 10.4]. The point is $(\pi^* D + t \sum D'_\gamma) c(e_0)$ anti-commutes with $c((s(r) - a)e_1)$ on $\Gamma(\tilde{W} |_{M_0 \times (0, \infty)})$. \hfill \Box

**Proposition 6.2.** For any $\lambda > 0$ there exists $T > 0$ such that for each $a \in \mathbb{R}$ there exists an open subset $V(a)$ of $\tilde{M}$ whose complement is compact and for each compactly supported section $\phi \in \Gamma(\tilde{W})$ with $\text{supp}(\phi) \subset V(a)$ we have
\[ \|D_{t,a}\phi\|_{L^2(\tilde{W})} \geq \lambda \|\phi\|_{L^2(\tilde{W})}. \]

**Proof.** We take a pre-compact open neighborhood $U$ of $M \setminus V$ in $M$. Let $K$ be the complement $M \setminus U$ and put $K_0 := K \cap M_0$. For each $a \in \mathbb{R}$ we take $b > 0$ large enough so that we have
\[ -\|B\|_{\infty} + |p(x) - a|^2 \geq \lambda \]
on $K(a) := (M_0 \setminus U) \times [b, \infty)$. We define an open subset $V(a)$ of $\tilde{M}$ by the interior part of the union $K_0 \times [0, \infty) \cup K(a)$. By definition the complement $\tilde{M} \setminus V(a) = (M \setminus V) \cup (M_0 \setminus U) \times [0, b]$ is compact. As in the proof of [1, Proposition 4.4], there exists a constant $C > 0$ such that
\[ \|\tilde{D}'_t\phi\|_{L^2(\tilde{W}|_{V})} \geq (t^2 - Ct)\|\phi\|_{L^2(\tilde{W}|_{V})}^2, \]
for any compactly supported section $\phi \in \Gamma(\tilde{W})$ with $\text{supp}(\phi) \subset V$, and hence we have
\[ \|\tilde{D}'_t\phi\|_{L^2(\tilde{W}|_{K})} \geq (t^2 - Ct)\|\phi\|_{L^2(\tilde{W}|_{K})}^2, \]
for any compactly supported section $\phi \in \Gamma(\tilde{W})$ with $\text{supp}(\phi) \subset K$. Let $\phi \in \Gamma(\tilde{W})$ be a compactly supported section with $\text{supp}(\phi) \subset V(a)$. We take $T > 0$ large enough so that
we have \( t^2 - Ct - \|B\|_\infty \geq \lambda \) for any \( t > T \). Together with Proposition 6.3 we have

\[
\|D_{t,a}\phi\|_{L^2(W_t,\phi)}^2 = \int_{V_a} ((|D_t|^2 + B + |p(x)|^2)\phi, \phi) \\
= \left( \int_{K_0 \times [0,\infty)} + \int_{K(a)} \right) ((|D_t|^2 + B + |p(x)|^2)\phi, \phi) \\
\geq \int_{K_0 \times [0,\infty)} ((|D_t|^2 + B)\phi, \phi) + \int_{K(a)} (B + |p(x)|^2)\phi, \phi) \\
\geq (t^2 - Ct - \|B\|_\infty)\|\phi\|_{L^2(W_{K_0 \times [0,\infty)},|\phi|)}^2 + \lambda\|\phi\|_{L^2(W_{K(a)},|\phi|)}^2 \\
= \lambda\|\phi\|_{L^2(W_{K_0 \times [0,\infty]},|\phi|)}^2 + \lambda\|\phi\|_{L^2(W_{K(a)},|\phi|)}^2.
\]

\( \square \)

**Proposition 6.3.** For any \( a \in \mathbb{R} \) and \( T > 0 \) as in Proposition 6.2, the space of \( L^2 \)-solutions \( \ker_{L^2}(D_{t,a}) \) is a \( \mathbb{Z}/2 \)-graded finite-dimensional vector space and its super-dimension \( \text{ind}(D_{t,a}) \) does not depend on \( t > T \) and \( a \).

**Proof.** By [1, Theorem 3.2], \( \ker_{L^2}(D_{t,a}) \) is a \( \mathbb{Z}/2 \)-graded finite-dimensional vector space for each \( t > T \) and \( a \in \mathbb{R} \). Moreover its super-dimension \( \text{ind}(D_{t,a}) \) does not depend on \( t > T \) and \( a \in \mathbb{R} \) by the homotopy invariance and the excision property. \( \square \)

**Proposition 6.4.** Let \( T > 0 \) be the constant as in Proposition 6.5. For any \( t > T \) and \( a \in \mathbb{R} \) we have

\[
\text{ind}(D_{t,a}) = \dim \ker_{L^2}(D_{t,a})^+ - \dim \ker_{L^2}(D_{t,a})^- = 0.
\]

**Proof.** The proof is same as that for [1, Proposition 10.7]. \( \square \)

### 6.2. The model operator on the cylinder.

Consider the natural extension of the \( \mathbb{Z}/2 \)-graded Clifford module bundle \( \tilde{W}|_{M_0 \times (0,\infty)} \) to \( M_0 \times \mathbb{R} \). We denote it by the same notations \( \tilde{W} \). We put \( W_0 := \text{Hom}_{Cl_{1,2}}(\tilde{R}, \tilde{W}) \) for the irreducible representation \( R \) of \( Cl_{1,2} \) as in (5.1).

Let \( D : \Gamma(W_0^+) \to \Gamma(W_0^-) \) be a Dirac-type operator of odd degree, where the \( \mathbb{Z}/2 \)-grading \( W_0 = (W_0^+) \oplus (W_0^-) \) is given by \( c(\epsilon) = \pm 1 \).

**Definition 6.5.** For \( t > 0 \) we define the model operator \( D_t^{\text{mod}} \) which acts on the space of smooth sections of \( \tilde{W} \to M_0 \times \mathbb{R} \) by

\[
D_t^{\text{mod}} = \left( r^*D + c(v_r)\partial_r + t \sum_{\alpha} D_0,\alpha c(e_0) - c(r(x)e_1) \right)
\]

under the isomorphism \( W_0^+ \otimes R \cong \tilde{W} \), where \( r : M_0 \times \mathbb{R} \to \mathbb{R} \) is the projection and we put \( D_0,\alpha = \rho_\alpha D_{0,\alpha} \rho_\alpha \).

As in the same way in Proposition 6.3 there exists \( T^{\text{mod}} > 0 \) such that \( \ker_{L^2}(D_t^{\text{mod}}) \) is a finite-dimensional vector space and its index \( \text{ind}(D_t^{\text{mod}}) \) can be defined for any \( t > T^{\text{mod}} \) as its super-dimension.
Proposition 6.6. As a $\mathbb{Z}/2$-graded vector space $\ker L^2(D_t^{\text{mod}})^2$ is isomorphic to $\ker L^2(D + t \sum_\alpha D'_{0,\alpha})^2 \otimes \mathbb{R}^2$ for any $t > T^{\text{mod}}$. In particular we have

$$\text{ind}(D_t^{\text{mod}}) = 2\text{ind}(D + t \sum_\alpha D'_{0,\alpha}) = 2\text{ind}(M_0, V_0).$$

Proof. Since $(r^*D + t \sum_\alpha D'_{0,\alpha}) c(e_0)$ and $c(v_r)c(e_0)\partial_r - c(r(x)e_1)$ anti-commutes and $c(v_r)c(e_0) = -c(e_1)$ we have

$$(D_t^{\text{mod}})^2 = \left( r^*D + t \sum_\alpha D'_{0,\alpha} \right)^2 + (c(v_r)\partial_r c(e_0) - c(r(x)e_1))^2$$

$$= \left( r^*D + t \sum_\alpha D'_{0,\alpha} \right)^2 + (\partial_r + r(x))^2,$$

and hence,

$$\ker L^2(D_t^{\text{mod}})^2 \cong \ker L^2 \left( D + t \sum_\alpha D'_{0,\alpha} \right)^2 \otimes \ker L^2(\partial_r + r)^2$$

as $\mathbb{Z}/2$-graded vector spaces. Since the space $\ker L^2(\partial_r + r)^2$ is a two-dimensional real vector space generated by $f_1(r) = e^{-\frac{r^2}{2}}$ and $f_2(r) = re^{-\frac{r^2}{2}}$ without $\mathbb{Z}/2$-grading, we complete the proof. $\square$

6.3. Definition of $D_{t,a}^{\text{mod}}$. For $a \in \mathbb{R}$ and $t > 0$ we define $D_{t,a}^{\text{mod}} : \Gamma(W) \to \Gamma(W)$ by

$$(6.5) \quad D_{t,a}^{\text{mod}} = \left( r^*D + c(v_r)\partial_r + t \sum_\alpha D'_{0,\alpha} \right) c(e_0) - (r(x) - a)c(e_1).$$

Let $\tau_a : M_0 \times \mathbb{R} \to M_0 \times \mathbb{R}$, $\tau_a(x, r) = (x, r + a)$ be the translation. Then we have $D_{t,a}^{\text{mod}} = \tau_a^*D_{t,a}^{\text{mod}} \circ \tau_a^*$, and hence, $\text{ind}(D_{t,a}^{\text{mod}}) = \text{ind}(D_t^{\text{mod}})$ for any $t > T^{\text{mod}}$ and $a \in \mathbb{R}$. Let $D_{t,a}^{\pm}$ (resp. $D_{t,a}^{\text{mod},\pm}$ and $D_{t,a}^{\text{mod},\pm}$) be the restriction of $D_{t,a}$ (resp. $D_t^{\text{mod}}$ and $D_t^{\text{mod}}$) to $\Gamma(W^{\pm})$.

For a self-adjoint operator $P$ with discrete spectrum and a real number $\lambda \in \mathbb{R}$, let $N(\lambda, P)$ be the number of eigenvalues of $P$ less than or equal to $\lambda$.

Proposition 6.7. Let $\lambda_\pm$ be the smallest nonzero eigenvalue of $(D_t^{\text{mod},\pm})^2$. For any $\varepsilon > 0$ there exists $A = A(\varepsilon, M, W, t) > 0$ such that

$$N(\lambda_\pm - \varepsilon, (D_t^{\pm})^2) = \dim \ker L^2(D_t^{\text{mod},\pm})^2$$

for any $t > \max\{T, T^{\text{mod}}\}$ and $a > A$. In particular we have

$$\text{ind}(D_{t,a}^{\pm}) = \text{ind}(D_t^{\text{mod}}).$$

Together with Proposition 6.6 the above Proposition 6.7 implies the following theorem, and hence, the proof of Theorem 5.3 is finished by Proposition 6.4.

Theorem 6.8. For any $t > \max\{T, T^{\text{mod}}\}$ and $a > A$ we have

$$\text{ind}(D_{t,a}) = 2\text{ind}(M_0, V_0).$$
Namely let \( j, \tilde{j} \) be smooth functions as in Subsection 5.1. Namely let \( \hat{J} \) whose complement is compact and \( \hat{\phi} \), \( \hat{\psi} \) functions on \( J \) and \( \hat{r} \). Let \( R \) complex Clifford module bundle as in the set-up in Subsection 5.1. Namely let \( \hat{J} \) be smooth functions such that \( j^2 + \tilde{j}^2 = 1 \), \( j(r) = 1 \) for \( r \geq 3 \) and \( j(r) = 0 \) for \( r \leq 2 \). We also define smooth functions \( J_a, \tilde{J}_a : M_0 \times \mathbb{R} \rightarrow [0, 1] \) by

\[
J_a(x) := j(a^{-1/2}r(x)), \quad \tilde{J}_a(x) = \tilde{j}(a^{-1/2}r(x)),
\]

where \( r : M_0 \times \mathbb{R} \rightarrow M_0 \) is the projection. We will denote by the same letters for smooth functions on \( \hat{M} \) defined by the formula

\[
J_a(x) := j(a^{-1/2}p(x)), \quad \tilde{J}_a(x) = j(a^{-1/2}p(x)),
\]

where \( p : \hat{M} \rightarrow \mathbb{R} \) is the smooth function defined in Subsection 6.1.

The proof of Proposition 6.7 is almost same as that for [1, Proposition 11.6]. In [1] Braverman used IMS localization formula using \( J_a \) and \( \tilde{J}_a \) due to [8]. We only note that we have to change [1, Lemma 11.14] to the following lemma.

**Lemma 6.9.** There exists a positive constant \( K = K(M,W,t) > 0 \) such that

\[
\| [J_a, [J_a, \mathcal{D}_{t,a}^2]] \|, \| [\tilde{J}_a, [\tilde{J}_a, \mathcal{D}_{t,a}^2]] \| \leq \frac{K}{a}
\]

for any \( a > 0 \).

**Proof.** The estimate follows from the fact that the principal symbol of \( \mathcal{D}_{t,a} \) has the form

\[
c \circ (\text{id} + t \sum_{\gamma} a_{\gamma}^2 \varepsilon_{\gamma}^*) c(e_0),
\]

where \( c \) is the Clifford multiplication on \( \hat{W} \) and \( \varepsilon_{\gamma}^* \) is the dual of the natural inclusion from \( T \hat{T}_{\gamma} \) to \( T\hat{\gamma} \).

\[ \square \]

**6.4. Comments on the proof of complex case.** In this subsection we give comments on the proof of Theorem 5.2 which can be proved by the almost same argument as that for Theorem 5.3 or in [1, 2]. We explain how we modify the perturbation etc.

Let \( M \) be a Riemannian manifold with boundary \( \partial M = M_0 \). Let \( W \rightarrow M \) be the complex Clifford module bundle as in the set-up in Subsection 5.1. Namely let \( W \) is a complex \( \text{Cl}(TM \oplus \mathbb{R} e_L \oplus \mathbb{R}^p \oplus \mathbb{R} e_R \oplus \mathbb{R} e_R') \)-module bundle, where the metric is defined by \( e_L \cdot e_L = 1 \) and \( e_R \cdot e_R = e_R' \cdot e_R' = -1 \). Let \( W = \text{Hom}_{C_{1,2}}(R_C, \hat{W}) \rightarrow M \) be the associated \( \text{Cl}(TM \oplus \mathbb{R}^p \oplus e_R') \)-module bundle for the irreducible complex representation \( R_C \) of \( C_{1,2} = C(\mathbb{R} e_L \oplus \mathbb{R} e_R \oplus \mathbb{R} e_R') \). Suppose that there exists an open subset \( V \) of \( M \) whose complement is compact and \( V \) is equipped with an acyclic compatible system \( \{ \tilde{V}_\gamma, \tilde{Q}_\gamma, \hat{W}, \hat{D}_\gamma \}_{\gamma \in \hat{A}} \). By attaching the cylinder \( M_0 \times [0, \infty) \) we have the associated non-compact manifold \( \hat{M} \), its open subset \( \hat{V} \) and the induced \( G \)-tangential acyclic compatible system \( \{ \tilde{V}_\gamma, \tilde{Q}_\gamma, \hat{W}, \hat{D}_\gamma \}_{\gamma \in \hat{A}} \) on \( \hat{V} \). When we put \( V_0 := M_0 \cap V \) we have the induced \( G \)-tangential acyclic compatible system \( \{ \tilde{V}_{0,\alpha}, \tilde{Q}_{0,\alpha}, W_0, D_{0,\alpha} \}_{\alpha \in \hat{A}} \) over \( V_0 \), where \( W_0 = W|_{M_0} = W_0^+ \oplus W_0^- \) is the \( \mathbb{Z}/2 \)-graded \( \text{Cl}(TM_0) \)-module whose grading is given by \( c(v_\gamma) = \pm \sqrt{-1} \). We first take and fix a Dirac-type operator \( D \) on \( \Gamma(W_0) \) and \( \hat{D} \) on \( \Gamma(\hat{W}) \) such that

\[
\hat{D}|_{M_0 \times (0, \infty)} = \sqrt{-1} (\pi^* D + c(v_\gamma) \partial_\gamma) \otimes c_{1,2}(e_L)
\]
under the isomorphism $\widetilde{W}|_{M_0 \times (0, \infty)} \cong \pi^* W_0 \otimes R_C$. For $t > 0$ and $a \in \mathbb{R}$ we consider the perturbations

$$D_t = \widetilde{D} + t \sum_{\gamma \in \hat{A}} \widetilde{D}'_{\gamma} \quad \text{and} \quad D_{t,a} = D_t - c((p(x) - a)e_R)$$

instead of the perturbation (6.2) and (6.3), which acts on $\Gamma(\widetilde{W})$. As in the same way we consider the model operator on $\Gamma(\widetilde{W}|_{M_0 \times \mathbb{R}})$ and its perturbation

$$D_{t}^{\text{mod}} = \sqrt{-1} \left( r^* D + c(v_\tau) \partial_\tau + t \sum_{\alpha \in A_0} D'_{0,\alpha} \right) \otimes c_{1,2}(e_L) - 1 \otimes c_{1,2}((p(x) - a)e_R),$$

$$D_{t,a}^{\text{mod}} = \sqrt{-1} \left( r^* D + c(v_\tau) \partial_\tau + t \sum_{\alpha \in A_0} D'_{0,\alpha} \right) \otimes c_{1,2}(e_L) - 1 \otimes c_{1,2}((p(x) - a)e_R).$$

Using these operators we can give a proof of Theorem 5.2 as for the same argument for the proof of Theorem 5.3.

**7. WELL-DEFINEDNESS OF THE LOCAL INDEX**

Though the local index is invariant under the continuous deformation of the data, it is not clear that it does not depend on the choice of the open covering. We show that the local index does not depend on it for a suitable class of compatible systems.

**Theorem 7.1.** Let $M$ be a Riemannian manifold. Let $W \to M$ be a $\mathbb{Z}/2$-graded $\text{Cl}(TM \oplus \mathbb{R}^{p,q})$-module bundle. Let $V_1$ and $V_2$ be two open subsets whose complements in $M$ are compact. Suppose that for $i = 1, 2$ there exist $G$-tangential acyclic compatible systems $\{V_{1,\alpha}, \mathcal{F}_{i,\alpha}, W|_{V_i}, D_{i,\alpha}\}_{\alpha \in A_i}$. If the union $\{V_{1,\alpha}, V_{2,\beta}, \mathcal{F}_{1,\alpha}, \mathcal{F}_{2,\beta}, W|_{V_1 \cup V_2}, D_{1,\alpha}, D_{2,\beta}\}_{\alpha, \beta \in A_1 \cup A_2}$ is also a $G$-tangential acyclic compatible system on $V_1 \cup V_2$, then we have $\text{ind}(M, V_1) = \text{ind}(M, V_2)$.

**Proof.** We put $\widetilde{M} := M \times [0, 1]$ and $\widetilde{V} := V_i \times [0, 2/3] \cup V_2 \times [1/3, 1]$. Then we have a cobordism $\{V_{1,\alpha} \times [0, 2/3], V_{2,\beta} \times [1/3, 1], \mathcal{F}_{i,\alpha} \times [0, 2/3], \mathcal{F}_{2,\beta} \times [1/3, 1], W|_{V_1 \cup V_2} \times [0, 1], D_{1,\alpha}, D_{2,\beta}\}_{\alpha, \beta \in A_1 \cup A_2}$ between $\{V_{1,\alpha}, \mathcal{F}_{1,\alpha}, W|_{V_1}, D_{1,\alpha}\}_{\alpha \in A_1}$ and $\{V_{2,\alpha}, \mathcal{F}_{2,\alpha}, W|_{V_2}, D_{2,\alpha}\}_{\alpha \in A_2}$, and hence, we have $\text{ind}(M, V_1) = \text{ind}(M, V_2)$ by Theorem 4.4.

**7.1. Application to the product of compatible fibrations.** For $i = 1, 2$ let $M_i$ be Riemannian manifolds and $W_i \to M_i$ $\mathbb{Z}/2$-graded $\text{Cl}(TM_i \oplus \mathbb{R}^{p,q})$-module bundles. Suppose that for each $i = 1, 2$ there exists a compact Lie group $G_i$ which acts on $M_i$ so that $W_i \to M_i$ is a $G_i$-equivariant $\text{Cl}(TM_i \oplus \mathbb{R}^{p,q})$-module bundle. Let $M_i'$ be the complement of the fixed point set, $M_i' = M_i \setminus M_i^{G_i}$. Suppose that there exists a $G_i$-tangential compatible fibration $\{M_{i,\alpha}, \mathcal{F}_{i,\alpha}\}_{\alpha \in A_i}$ and a $G_i$-tangential compatible system $\{D_{i,\alpha}\}_{\alpha \in A_i}$ on it. Let $V_i$ be a $G_i$-invariant open subset of $M_i'$ such that the compatible system $\{D_{i,\alpha}\}_{\alpha \in A_i}$ is acyclic on $V_i$. We assume $M_i \setminus V_i$ is compact. Let $M$ be the product of $M_1$ and $M_2$ and $G$ the product of $G_1$ and $G_2$. As in [4] Section 8 when we take small open neighborhoods $V_i, \infty$ of $M_i \setminus V_i$ for $i = 1, 2$ and put $V := V_1, \infty \times V_2 \cup V_1 \times V_2, \infty \cup V_1 \times V_2$, there exists a structure of $G$-tangential acyclic compatible system on $V$. Under the above assumptions we can define three local indices $\text{ind}(M, V_i), \text{ind}(M, V_2), \text{ind}(M, V)$. 
On the other hand we can define a refinement of compatible fibration on $V$ using the decomposition

$$V = M'_{1,∞} × V_2 ∪ M'_1 × V_2 ∪ V_1 × M'_{2,∞} ∪ V_1 × M'_1,$$

where $M'_{1,∞}$ is a small open neighborhood of $M_i - M'_i = M'_i \oplus$. Here we consider the structure of compatible fibration on $V$ as follows.

- $M'_{1,∞} × V_2$ : product of the trivial foliation on $M'_{1,∞}$ and $\{ F_{2,β}|V_2 \}_{β ∈ A_2}$.
- $M'_1 × V_2$ : product of $\{ F_{1,α}|V_1 \}_{α ∈ A_1}$ and $\{ F_{2,β}|V_2 \}_{β ∈ A_2}$.
- $V_1 × M'_{2,∞}$ : product of $\{ F_{1,α}|V_1 \}_{α ∈ A_1}$ and the trivial foliation on $M'_{2,∞}$.
- $V_1 × M'_1$ : product of $\{ F_{1,α}|V_1 \}_{α ∈ A_1}$ and $\{ F_{2,β}|V_2 \}_{β ∈ A_2}$.

Note that even if the compatible system $\{ D_{1,α}|M'_i \}_{α ∈ A_i}$ is not acyclic, the product of it and $\{ D_{2,β}|V_2 \}_{β ∈ A_2}$ is acyclic.

In this way we have another acyclic compatible system on $V$, which is also $G$-tangential and different from the product of the given compatible systems. We denote the open subset $V$ with this refined structure by $V^{ref}$. We can define the local index $\text{ind}(M, V^{ref})$ for this refined structure.

**Theorem 7.2.** Under the above setting we have

$$\text{ind}(M, V^{ref}) = \text{ind}(M, V) = \text{ind}(M_1, V_1) \text{ind}(M_2, V_2).$$

**Proof.** The first equality follows from Theorem 7.1. The second equality follows from the product formula of local indices ([4, Theorem 8.8]). □

**Remark 7.3.** Since we assume the global group action, when all the data are $G_i$-equivariant the equality in Theorem 7.2 can be regarded as an equality between $G$-equivariant indices.

**Remark 7.4.** It is possible to give a formulation of the similar equality as in Theorem 7.2 for a twisted product of $M_1$ and $M_2$ using a principal bundle over $M_1$.

**Example 7.5.** Let $M_1$ be the product of $S^1$ and the open interval $(-1, 1)$. Consider the standard $S^1$-action on the first factor. It induces a structure of $S^1$-tangential compatible fibration on $M_1$. In this case we have $M'_1 = M_1$. Let $W$ be a $Z/2$-graded Clifford module bundle over $M_1$. Suppose that we have a family of Dirac-type operator $\{ D_{S^1,r} : Γ(W|_{S^1×\{r\}}) \rightarrow Γ(W|_{S^1×\{r\}}) \}_{r ∈ (-1, 1)}$. We assume that $\text{ker}(D_{S^1,r}) = \{0\}$ for all $r ≠ 0$.

Then $\{ D_{S^1,r} \}_{r≠0}$ induce an acyclic compatible system on $V_1 := M_1 \setminus S^1 × \{0\}$. Consider the product $M := M_1 × M_1$. When we put

$$V_{01} := (S^1 × (-1/4, 1/4)) × V_1, \quad V_{10} := V_1 × (S^1 × (-1/4, 1/4)),$$

$$V_{11} := V_1 × V_1, \text{ and } V := V_{01} ∪ V_{10} ∪ V_{11},$$

it induces a structure of compatible system using the following torus actions.

- $V_{01}$ : $\{e\} × S^1$-action.
- $V_{10}$ : $S^1 × \{e\}$-action.
- $V_{11}$ : $S^1 × S^1$-action.

On the other hand when we put

$$V'_{01} = M_1 × V_1, \quad V'_{10} = V_1 × M_1, \text{ and } V^{ref} := V'_{01} ∪ V'_{10} (= V),$$

it induces a structure of compatible system using the $S^1 × S^1$-actions on $V'_{01}$ and $V'_{10}$. See Figure [□].
Theorem 7.2 guarantees that two local indices defined by using these two different structures coincide: \( \text{ind}(M, V) = \text{ind}(M, V_{\text{ref}}) \).

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