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Existence results for a class of nonlocal problems involving $p(x)$-Laplacian

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Abstract We study the existence of weak solutions for a $p(x)$-Kirchhoff problem. The main tool used is the variational method, more precisely, the Mountain Pass Theorem.

Mathematics Subject Classification 35J48 · 35J66

1 Introduction

In this paper, we consider the following problem:

$$-M \left( \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \text{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda (a(x)|u|^{q(x)-2}u + b(x)|u|^{r(x)-2}u) \quad \text{in} \ \Omega,$$

$$u = 0 \quad \text{on} \ \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial \Omega$, $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, $\lambda$ is a positive number, $p$ is Lipschitz continuous on $\Omega$, and $q, r$ are continuous functions on $\Omega$ with $q^- := \inf_{x \in \Omega} q(x) > 1$, $r^- := \inf_{x \in \Omega} r(x) > 1$, $a(x) > 0$ for $x \in \overline{\Omega}$ such that $a \in L^{q^-}(\Omega)$, $a(x) = \frac{p(x)}{p(x)-q(x)}$ and $b \in L^{r^-}(\Omega)$, $r(x) = \frac{p^*(x)}{p^*(x)-r(x)}$. Hereafter,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if} \ p(x) < N; \\ +\infty, & \text{if} \ p(x) \geq N. \end{cases}$$

We will use the notations such as $h^-$ and $h^+$, where

$$h^- := \inf_{x \in \Omega} h(x) \leq h(x) \leq h^+ := \sup_{x \in \Omega} h(x) < +\infty.$$ 

The operator $\nabla_{p(x)} := \text{div}(|\nabla u|^{p(x)-2} \nabla u)$ is called the $p(x)$-Laplacian, and becomes $p$-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated properties than the $p$-Laplacian; for
example, it is inhomogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [21] and image restoration [7]. The problem (1.1) is a generalization of a model introduced by Kirchhoff [18]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which extends the classical D’Alembert’s wave equation, by considering the effects of the changes in the length of the strings during the vibrations. Lions [19] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [19], various equations of Kirchhoff-type have been studied extensively, see [2,5]. The study of Kirchhoff-type equations has already been extended to the case involving the \( p \)-Laplacian (for details, see [3,4,10,11]) and \( p(x) \)-Laplacian (see [9,12]). Motivated by the above papers and the results in [8,20], we consider (1.1) to study the existence of weak solutions.

2 Preliminary

For completeness, we first recall some facts on the variable exponent spaces \( L^{p(x)}(\Omega) \) and \( W^{1,p(x)}(\Omega) \). For more details, see [13–15]. Suppose that \( \Omega \) is a bounded open domain of \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and \( p \in C_+(\overline{\Omega}) \), where

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) \text{ and } \inf_{x \in \overline{\Omega}} p(x) > 1 \right\}.$$

Define the variable exponent Lebesgue space \( L^{p(x)}(\Omega) \) by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} \, dx < +\infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0; \int_\Omega \left| \frac{u}{\tau} \right|^{p(x)} \, dx \leq 1 \right\}.$$

Define the variable exponent Sobolev space \( W^{1,p(x)}(\Omega) \) by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0; \int_\Omega \left( \left| \nabla u \right|^{p(x)} + \left| \frac{u}{\tau} \right|^{p(x)} \right) \, dx \leq 1 \right\}.$$

We denote by \( W^{1,p(x)}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W^{1,p(x)}(\Omega) \). Of course the norm \( \|u\| = |\nabla u|_{L^{p(x)}(\Omega)} \) is an equivalent norm in \( W^{1,p(x)}_0(\Omega) \). In this paper, we denote by \( X = W^{1,p(x)}_0(\Omega) \).

Lemma 2.1 ([15]) Both \( (L^{p(x)}(\Omega), \cdot \cdot_{p(x)}) \) and \( (W^{1,p(x)}(\Omega), \| \cdot \|) \) are separable and uniformly convex Banach spaces.

Lemma 2.2 ([15]) \( \text{Hölder inequality holds, namely} \)

$$\int_\Omega |uv| \, dx \leq \int_\Omega u |v|_{p(x)} \, dx \quad \forall u \in L^{p(x)}(\Omega), v \in L^{p(x)}(\Omega), \text{ where } \frac{1}{p(x)} + \frac{1}{p(x)} = 1.$$

Lemma 2.3 ([6]) Assume that \( h \in L^\infty_c(\Omega), p \in C_+(\overline{\Omega}). \) If \( |u|^{h(x)} \in L^{p(x)}(\Omega) \). Then we have

$$\min\{|u|_{h(x)}^{p(x)}, |u|_{h(x)}^{p(x)}\} \leq |u|^{h(x)}_{p(x)} \leq \max\{|u|_{h(x)}^{p(x)}, |u|_{h(x)}^{p(x)}\}.$$

Lemma 2.4 ([14]) Assume that \( \Omega \) is bounded and smooth.
• Let $p$ be Lipschitz continuous and $p^+ < N$. Then for $h \in L^\infty(\Omega)$ with $p(x) \leq h(x) \leq p^*(x)$ there is a continuous embedding $X \hookrightarrow L^{h(x)}(\Omega)$.

• Let $p \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$. Then there is a compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$.

Lemma 2.5 ([16]) Set $\rho(u) = \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx$. Then for $u \in X$, we have

1. $\|u\| < 1$ (respectively $= 1; > 1$) if and only if $\rho(u) < 1$ (respectively $= 1; > 1$);
2. if $\|u\| > 1$, then $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
3. if $\|u\| < 1$, then $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{-}$.

Definition 2.6 A function $u \in X$ is said to be a weak solution of (1.1) if

$$
M \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} a(x)|u|^{q(x)-2}u v \, dx
$$

for all $v \in X$.

The Euler–Lagrange functional associated to (1.1) is

$$
J_{\lambda}(u) = \tilde{M} \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) - \lambda \int_{\Omega} a(x)|u|^{q(x)} \, dx - \lambda \int_{\Omega} b(x)|u|^{r(x)} \, dx,
$$

where $\tilde{M}(t) = \int_0^t M(s) \, ds$. Then

$$
(J_{\lambda}'(u), v) = M \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} a(x)|u|^{q(x)-2}u v \, dx
$$

for all $u, v \in X$, then we know that the weak solution of (1.1) corresponds to the critical point of the functional $J_{\lambda}$. Hereafter, $M(t)$ is supposed to verify the following assumptions:

$(M_0)$ There exists $m_1 \geq m_0 > 0$ and $\mu \geq \nu > 1$ such that

$$
m_0 t^{\nu-1} \leq M(t) \leq m_1 t^{\mu-1}.
$$

$(M_1)$ $\exists 0 < d < 1$ such that

$$
\tilde{M}(t) \geq (1-d)M(t)t \quad \text{for all } t \geq 0.
$$

An example of functions satisfying the assumptions $(M_0)$ and $(M_1)$:

$$
M(t) = t \arctan(t).
$$

Throughout this paper, we assume the condition:

$$
1 < q^- \leq q^+ < v p^-, \quad \max \left\{ \mu p^+, \frac{p^+}{1-d} \right\} < r^- \leq r^+ < p^*(x) \quad \text{and} \quad p^+ < N. \tag{2.1}
$$

For simplicity, we use $C_i, i = 1, 2, \ldots$, to denote the general positive constants whose exact values may change from line to line.
3 Main result

**Theorem 3.1** Assume $p$ is Lipschitz continuous, $q, r \in C_{+}(\overline{\Omega})$ and Condition (2.1) is fulfilled. Then there exists $\lambda^{*} > 0$ such that for any $\lambda \in (0, \lambda^{*})$, Problem (1.1) possesses a nontrivial weak solution.

**Lemma 3.2** There exists $\lambda^{*} > 0$ such that for any $\lambda \in (0, \lambda^{*})$ there exist $\rho, \tau > 0$ such that $J_{\lambda}(u) \geq \tau > 0$ for any $u \in X$ with $\|u\| = \rho$.

**Proof** In view of Lemma 2.4, there exists a positive constant $C_{1}$ such that

$$|u|_{p(x)} \leq C_{1}\|u\|, \quad |u|_{p^*(x)} \leq C_{1}\|u\|, \quad \text{for all } u \in X. \tag{3.1}$$

Fix $\rho \in [0, 1]$ such that $\rho < \frac{1}{C_{1}}$. Then relation (3.1) implies $|u|_{p(x)} < 1$, $|u|_{p^*(x)} < 1$, for all $u \in X$ with $\|u\| = \rho$. By Lemmas 2.2 and 2.3, we obtain

$$\int_{\Omega} a(x)|u|^{|q(x)} dx \leq 2|a_{u}|_{\alpha(x)}|u|_{p(x)}^{q(x)} \leq 2|a_{u}|_{\alpha(x)}|u|_{p^*(x)}^{q^*(x)}, \tag{3.2}$$

and

$$\int_{\Omega} b(x)|u|^{|r(x)} dx \leq 2|b_{u}|_{\gamma(x)}|u|_{r(x)}^{r(x)} \leq 2|b_{u}|_{\gamma(x)}|u|_{r^*(x)}^{r^*(x)}, \tag{3.3}$$

for all $u \in X$. Combining (3.1), (3.2) and (3.3), we obtain

$$\int_{\Omega} a(x)|u|^{|q(x)} dx \leq 2|a_{u}|_{\alpha(x)}C_{1}^{q^*(x)}\|u\|^{q^*(x)} \quad \text{and} \quad \int_{\Omega} b(x)|u|^{|r(x)} dx \leq 2|b_{u}|_{\gamma(x)}C_{1}^{r^*(x)}\|u\|^{r^*(x)}, \tag{3.4}$$

for all $u \in X$. Hence, from (3.4) and (M0) we deduce that for any $u \in X$ with $\|u\| = \rho$, we have

$$J_{\lambda}(u) \geq \frac{m_{0}}{\nu} \left( \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^{v} - \frac{\lambda}{q^*(x)} \int_{\Omega} a(x)|u|^{q^*(x)} dx - \frac{\lambda}{r^*(x)} \int_{\Omega} b(x)|u|^{r^*(x)} dx \geq \frac{m_{0}}{\nu(p^+(x))^{v}}\|u\|^{p^+(x)} - \frac{\lambda}{q^*(x)}2|a_{u}|_{\alpha(x)}C_{1}^{q^*(x)}\|u\|^{q^*(x)} - \frac{\lambda}{r^*(x)}2|b_{u}|_{\gamma(x)}C_{1}^{r^*(x)}\|u\|^{r^*(x)}.$$ 

Putting

$$\lambda^{*} = \min \left\{ \frac{m_{0}q^{v}\rho^{p^+(x)q^{-}}}{4C_{1}^{q^*(x)}}\|a_{u}\|_{\alpha(x)}^{q^{v}(x)}, \frac{m_{0}r^{v}\rho^{p^+(x)r^{-}}}{4C_{1}^{r^*(x)}}\|b_{u}\|_{\gamma(x)}^{r^*(x)} \right\}, \tag{3.5}$$

for any $u \in X$ with $\|u\| = \rho$, there exists $\tau = \frac{m_{0}q^{v}\rho^{p^+(x)q^{-}}}{4C_{1}^{q^*(x)}}$ such that $J_{\lambda}(u) \geq \tau > 0$ for any $\lambda \in (0, \lambda^{*})$. This completes the proof. □

**Lemma 3.3** There exists $e \in X$ with $\|e\| > \rho$ (where $\rho$ is given in Lemma 3.2) such that $J_{\lambda}(e) < 0$.

**Proof** Let $\varphi \in C_{0}^{\infty}(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$ and $t > 1$. By (M0) we have

$$J_{\lambda}(t\varphi) = \tilde{M} \left( \int_{\Omega} \frac{1}{p(x)} |\nabla t\varphi|^{p(x)} dx \right) - \lambda \int_{\Omega} \frac{a(x)}{q(x)}|t\varphi|^{q(x)} dx - \lambda \int_{\Omega} \frac{b(x)}{r(x)}|t\varphi|^{r(x)} dx \leq \frac{m_{1}}{\mu} \left( \int_{\Omega} \frac{1}{p(x)} |\nabla \varphi|^{p(x)} dx \right)^{\mu} - \lambda \frac{t^{q^{v}}}{q^{v}} \int_{\Omega} a(x)|\varphi|^{q^*(x)} dx - \lambda \frac{t^{r^{v}}}{r^{v}} \int_{\Omega} b(x)|\varphi|^{r^*(x)} dx \leq \frac{m_{1}}{\mu(p^+(x))^{v}}t^{q^{v}} \int_{\Omega} |\nabla \varphi|^{p(x)} dx - \lambda \frac{t^{q^{v}}}{q^{v}} \int_{\Omega} a(x)|\varphi|^{q^*(x)} dx - \lambda \frac{t^{r^{v}}}{r^{v}} \int_{\Omega} b(x)|\varphi|^{r^*(x)} dx.$$ 

Since $\mu p^+ < r^-$, we obtain $\lim_{t \to \infty} J_{\lambda}(t\varphi) = -\infty$. Then for $t > 1$ large enough, we can take $e = t\varphi$ such that $\|e\| > \rho$ and $J_{\lambda}(e) < 0$. □

**Lemma 3.4** The functional $J_{\lambda}$ satisfies the Palais–Smale condition (PS).
Proof Suppose that \((u_n) \subset X\) is a (PS) sequence; that is,
\[
\sup |J_\lambda(u_n)| \leq C_2, \quad J'_\lambda(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\] (3.6)

We prove that \((u_n)\) is bounded in \(X\). Arguing by contradiction we assume that, passing eventually to a subsequence, still denote by \((u_n)\), \(\|u_n\| \to \infty\) and \(\|u_n\| > 1\) for all \(n\). By (3.6) and (M₀), (M₁), for \(n\) large enough, we have
\[
1 + C_2 \geq J_\lambda(u_n) - \frac{1}{r^-} (J'_\lambda(u_n), u_n) + \frac{1}{r^-} (J'_\lambda(u_n), u_n)
\]
\[
\geq (1 - d) M \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \right) \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx - \lambda \int_\Omega \frac{a(x)}{q(x)} |u_n|^{q(x)} \, dx
\]
\[
- \lambda \int_\Omega \frac{b(x)}{r(x)} |u_n|^{r(x)} \, dx - \frac{1}{r^-} M \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \right) \int_\Omega |\nabla u_n|^{p(x)} \, dx
\]
\[
+ \frac{\lambda}{r^-} \int_\Omega a(x)|u_n|^{q(x)} \, dx + \frac{\lambda}{r^-} \int_\Omega b(x)|u_n|^{r(x)} \, dx + \frac{1}{r^-} (J'_\lambda(u_n), u_n)
\]
\[
\geq \frac{m_0}{v(p^+)} \left( \frac{1 - d}{p^+} - \frac{1}{r^-} \right) \|u_n\|^{p^+} - \lambda \left( \frac{1}{q^-} - \frac{1}{r^-} \right) C_3 |a|_{\alpha(x)} \|u_n\|^{q^-}
\]
\[
- \frac{1}{r^-} \|J_\lambda(u_n)\| \|u_n\|
\]
\[
\geq \frac{m_0}{v(p^+)} \left( \frac{1 - d}{p^+} - \frac{1}{r^-} \right) \|u_n\|^{p^+} - \lambda \left( \frac{1}{q^-} - \frac{1}{r^-} \right) C_3 |a|_{\alpha(x)} \|u_n\|^{q^-} - C_4 \|u_n\|.
\]

Dividing the above inequality by \(\|u_n\|^{p^-}\), taking into account (2.1) holds true and passing to the limit as \(n \to \infty\), we obtain a contradiction. It follows that \((u_n)\) is bounded in \(X\). By the reflexivity of \(X\), for a subsequence still denoted \((u_n)\), we have \(u_n \rightharpoonup u\) in \(X\) and \(u_n \to u\) in \(L^{\hat{h}(x)}(\Omega)\), where \(1 \leq \hat{h}(x) < p^+(x)\). Therefore,
\[
<J'_\lambda(u_n), u_n - u> \to 0,
\] (3.7)
\[
\int_\Omega a(x)|u_n|^{q(x)-2} u_n (u_n - u) \, dx \to 0,
\] (3.8)
and
\[
\int_\Omega b(x)|u_n|^{r(x)-2} u_n (u_n - u) \, dx \to 0.
\] (3.9)

Since \((u_n)\) is bounded in \(X\), passing to a subsequence, if necessary, we may assume that
\[
\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \to h_0 \geq 0 \quad \text{as} \quad n \to \infty.
\]
If \(h_0 = 0\) then \((u_n)\) converges strongly to \(u = 0\) in \(X\) and the proof is complete. If \(h_0 > 0\) then since the function \(M\) is continuous, we obtain
\[
M \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \right) \to M(h_0) \geq 0 \quad \text{as} \quad n \to \infty.
\]

Thus, by (M₀), for sufficiently large \(n\), we have
\[
0 < C_5 \leq M \left( \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx \right) \leq C_6.
\] (3.10)

From (3.7), (3.8), (3.9) and (3.10), we deduce that \(A(u) := \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \, dx \to 0\). According to the fact that \(A\) satisfies Condition \((S^+)^*\) (see [17]), we have \(u_n \to u\) in \(X\). This completes the proof. □
Proof of Theorem 3.1  From Lemmas 3.2 and 3.3, we deduce

$$\max(J_\lambda(0), J_\lambda(e)) = J_\lambda(0) < \inf_{\|u\|=\rho} J_\lambda(u) =: \beta.$$  

By Lemma 3.4 and the Mountain Pass Theorem (see [1]), we deduce the existence of critical points $u$ of $J_\lambda$ associated of the critical value given by

$$c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} J_\lambda(g(t)) \geq \beta,$$

(3.11)

where $\Gamma = \{ g \in C([0,1], X) : g(0) = 0 \text{ and } g(1) = e \}$. This completes the proof. \qed

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