RECOLLEMENTS IN STABLE ∞-CATEGORIES

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Abstract. We develop the theory of recollements in a stable ∞-categorical setting. In the axiomatization of Beilinson, Bernstein and Deligne, recollement situations provide a generalization of Grothendieck’s “six functors” between derived categories. The adjointness relations between functors in a recollement $D^0 \rightleftarrows D \rightleftarrows D^1$ induce a “recollée” t-structure $t_0 \equiv t_1$ on $D$, given t-structures $t_0, t_1$ on $D^0, D^1$. Such a classical result, well-known in the setting of triangulated categories, is recasted in the setting of stable ∞-categories and the properties of the associated (∞-categorical) factorization systems are investigated. In the geometric case of a stratified space, various recollements arise, which “interact well” with the combinatorics of the intersections of strata to give a well-defined, associative $\equiv$ operation. From this we deduce a generalized associative property for $n$-fold gluing $t_0 \equiv \cdots \equiv t_n$, valid in any stable ∞-category.

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1. Introduction.

Recollements in triangulated categories were introduced by A. Beilinson, J. Bernstein and P. Deligne in [BBD82], searching an axiomatization of the Grothendieck’s “six functors” formalism for derived categories of sheaves on (the strata of a) stratified topological space. [BBD82] will be our main source of inspiration, and reference for classical results and computations; among other recent but standard references, we mention [KS90, Ban07].

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Later, “recollement data” were noticed to appear quite naturally in the context of intersection homology [Pfl01, GM80, GM83] and Representation Theory [PS88, KW01]. In more recent years Beligiannis and Reiten [BR07], adapting to the triangulated setting an old idea of Jans [Jan65], linked recollement data to so-called TTF-triples (i.e. triples \((X, Y, Z)\) such that both \((X, Y)\) and \((Y, Z)\) are t-structures): recollement data, in the form of TTF-triples, appear quite naturally studying derived categories of representations of algebras, see [BR07, Ch. 4].

Here we translate the basic theory of recollements in the stable \(\infty\)-categorical setting and investigate their properties. In particular, inspired by the analysis of geometric recollements data associated with a stratified space, we consider the problem of associativity for iterated recollements, and show how one has associativity as soon as the relevant Beck-Chevalley condition is satisfied. Remarkably, in the geometric situation, this condition is always satisfied so that, as one should maybe expect, geometric iterated recollements do not depend on the order on which recollement data are used to produce the global t-structure on the derived category of the stratified space \(X\). Although probably implicit in the construction, this remark appears not be spelled out explicitly in [BBD82].

2. Classical Recollements.

The aim of this subsection is to present the basic features of “classical” recollements in the setting of stable \(\infty\)-categories ignoring, for the moment, the translation in terms of normal torsion theories which will follow.

**Definition 2.1 :** A (donnée de) recollement consists of the following arrangement of stable \(\infty\)-categories and functors between them:

\[
\begin{array}{ccc}
D^0 & \xrightarrow{i_R} & D & \xrightarrow{q} & D^1 \\
& \xleftarrow{i_L} & & \xleftarrow{q_L} &
\end{array}
\tag{1}
\]

satisfying the following axioms:

1. There are adjunctions \(i_L \dashv i \dashv i_R\) and \(q_L \dashv q \dashv q_R\);
(2) The counit $\epsilon_{(i_L \dashv qL)}: i_L \to 1$ and the unit $\eta_{(i_R \dashv qR)}: 1 \to i_R$ are natural isomorphisms; also, the unit $1 \to qqR$ and counit $qqL \to 1$ are natural isomorphisms;(1)

(3) The (essential) image of $i$ equals the essential kernel of $q$, namely the full subcategory of $D$ such that $qX \cong 0$ in $D^1$;

(4) The natural homotopy commutative diagrams

\[
\begin{array}{ccc}
qLq & \xrightarrow{\epsilon_{(qL \dashv q)}} & \text{id}_D \\
\downarrow \quad \eta_{(iL \dashv i)} & & \downarrow \quad \eta_{(q \dashv qR)} \\
iL & \quad 0 & \quad qRq \\
\end{array}
\]

induced by axioms (1), (2) and (3) are pullouts(2).

**Remark 2.2**: As an immediate consequence of the axioms, a recollement gives rise to various reflections and coreflections of $D$: since by axiom (2) the functors $i, q_L, q_R$ are all fully faithful, $qRq, iiL$ are reflections and $qLq, iiR$ are coreflections. Moreover, axioms (3) and (4) entail that the compositions $i_RqR, qi, i_LqL$ are all “exactly” zero, i.e. not only the kernel of $q$ is the essential image of $i$, but also the kernel of $i_L/i_R$ is the essential image of $qL/R$.

**Remark 2.3**: Axioms (2) and (4) together imply that there exists a canonical natural transformation $i_R \to i_L$, obtained as $i_R(\eta_{(iL \dashv i)})$ (or equivalently, as $i_L(\epsilon_{(i \dashv iR)})$: it’s easy to see that these two arrows coincide). Axiom (4) entails that there is a fiber sequence of natural transformations

\[
\begin{array}{ccc}
i_RqLq & \xrightarrow{i_R} & 0 \\
\downarrow \quad i_L & & \downarrow \\
iL & \quad iLqRq \\
\end{array}
\]

**Notation 2.4**: We will generally use a compact form like

\[(i, q): D^0 \cong D \cong D^1\]

to denote a recollement (1), especially in inline formulas. Variations on this are possible, either to avoid ambiguities or to avoid becoming stodgy.

We will for example say that “$(i, q)$ is a recollement on $D^0$” or that “$D$ is the décollement of $D^0, D^1$” to denote that there exists a diagram like (1) having $D$ as a central object. In other situations we adopt an extremely

(1) With a little abuse of notation we will write $i_Li = \text{id}_{D^0} = i_Ri$, and similarly for $qqL = \text{id}_D = qqR$.

(2) Here and everywhere else the category of functors to a stable ∞-category becomes a stable ∞-category in the obvious way (see [Lur11, Prop. 1.1.3.1]).
compact notation, referring to a (donné de) recollement with the symbol \(\mathcal{R}\) of (the letter \(\text{rae}\) of the Georgian alphabet, in the \(\text{ჟემუღლიანმო}\) script, see [Hew95]).

A geometric example. The most natural example of a recollement comes from the theory of stratified spaces [Wei94, Ban07]:

**Example 2.5**: Let \(X\) be a topological space, \(F \subseteq X\) a closed subspace, and \(U = X \setminus F\) its open complement.

From the two inclusions \(j: F \hookrightarrow X\), and \(i: U \hookrightarrow X\) we obtain the adjunctions \(j^* \dashv j_* \dashv j^!\), \(i^* \dashv i_* \dashv i^!\) between the categories \(\text{Coh}(U), \text{Coh}(X)\) and \(\text{Coh}(F)\) of coherent sheaves on the strata. Passing to their (bounded below-)derived versions we obtain functors\(^{(3)}\)

\[
D(F) \overset{j^*}{\longrightarrow} D(X) \overset{i^*}{\longrightarrow} D(U)
\]

(4)

giving rise to reflections and coreflections

\[
D(F) \overset{j^!}{\longleftarrow} D(X) \overset{i^!}{\longleftarrow} D(U)
\]

D

\[
D(F) \overset{j^*}{\longrightarrow} D(X) \overset{i^*}{\longrightarrow} D(U)
\]

D

(5)

These functors are easily seen to satisfy axioms (1)-(4) above: see [BBD82, 1.4.3.1-5] and [Ban07, 7.2.1] for details.

**Remark 2.6**: The above example, first discussed in [BBD82], is in some sense paradigmatic, and it can be seen as a motivation for the abstract definition of recollement: a generalization of Grothendieck’s “six functors” formalism. Several sources [Han14, BP13, AHKL11, C’14] convey the intuition that a recollement \(\mathcal{R}\) is some sort of “exact sequence” of triangulated categories, thinking \(\text{D}\) as decomposed into two parts, an “open” and a “closed” one. This also motivates the intuition that a donnée de recollement is not symmetric.

An algebraic example. The algebraic counterpart of the above example involves derived categories of algebras: we borrow the following discussion from [Han14].

**Example 2.7**: Let \(A\) be an algebra, and \(e \in A\) be an idempotent element; let \(J = eAe\) be the ideal generated by \(e\), and suppose that

- \(Ae \otimes_J eA \cong J\) under the map \((xe, ey) \mapsto xey);
- \(\text{Tor}^n_J(Ae, eA) \cong 0\) for every \(n > 0\).

\(^{(3)}\)For a topological space \(A\) we denote \(\text{D}(A)\) the derived \(\infty\)-category of coherent sheaves on \(A\) defined in [Lur11, §1.3.2]; we also invariably denote as \(j^* \dashv j_* \dashv j^!\), \(i^* \dashv i_* \dashv i^!\) the functors between stable \(\infty\)-categories induced by the homonym functors between abelian categories.
Then there exists a recollement

$$D(A/J) \xrightarrow{i = - \otimes A/J A/J} D(A) \xrightarrow{q = - \otimes A} D(eAe)$$

(6)

between the derived categories of modules on the rings $A/J, A, eAe$.

Interestingly enough, also this example is paradigmatic in some sense; more precisely, every recollement $\xi : D(A_1) \cong D(A) \cong D(A_2)$ is equivalent, in a suitable sense, to a “standard” recollement where $i_L$ and $q_L$ act by tensoring with distinguished objects $Y \in D(A), Y_2 \in D(A_2)$.

**Definition 2.8 [Standard recollement]:** Let $\xi : D(A_1) \cong D(A) \cong D(A_2)$ be a recollement between algebras; it is called a standard recollement generated by a pair $(Y, Y_2)$ if $i_L \cong - \otimes_A Y$, and $q_L \cong - \otimes_{A_2} Y_2$.

**Proposition 2.9:** Let $\xi : D(A_1) \cong D(A) \cong D(A_2)$ be a recollement between algebras; then $\xi$ is equivalent (in the sense of Remark 2.14) to a standard recollement $\xi$ generated by the pair $(Y, Y_2)$.

The proof relies on the following

**Lemma 2.10:** Let $A_1, A, A_2$ be algebras. The derived categories on these algebras are part of a recollement $\xi : D(A_1) \cong D(A) \cong D(A_2)$ if and only if there exist two objects $X_1, X_2 \in D(A)$ such that

- $\text{hom}(X_i, X_i) \cong A_i$ for $i = 1, 2$;
- $X_2$ is an exceptional and compact object, and $X_1$ is exceptional and self-compact;
- $X_1 \in \{X_2\}^\perp$;
- $\{X_1\}^\perp \cap \{X_2\}^\perp = (0)$.

See [Han14, §2] for details.

A homotopical example. Let $Ho(\mathcal{G}Sp)$ be the global stable homotopy category of $[\text{Sch}]$; this is defined as the localization of the category of globally equivariant orthogonal spectra at the homotopical class of global equivalences ([Sch, Def. 1.2]): the homotopical category $\mathcal{G}Sp$ admits a natural forgetful functor $u : \mathcal{G}Sp \to \text{Sp}$ which “forgets the equivariance” (it is the identity on objects, and includes the class of global equivalences in the bigger class of weak equivalences of plain spectra), which has both a left and a right adjoint $u_L, u_R$, and plays the rôle of a $q$-functor in a recollement

$$\text{Sp}_+ \longrightarrow \mathcal{G}Sp \longrightarrow \text{Sp}$$

(7)

where the functor $i : \text{Sp}_+ \to \mathcal{G}Sp$ embeds the subcategory of orthogonal spectra that are stably contractible in the traditional, non-equivariant sense.

**Remark 2.11:** Since in a stable $\infty$-category every pullback is a pushout and vice versa, any functor between stable $\infty$-categories preserving either limits...
or colimits preserves in particular pullout diagrams. Since left adjoints and right adjoints have this property, we find

**Proposition 2.12 [Exactness of recollement functors]:** Each of the functors $i, i_L, i_R, q, q_L, q_R$ in a recollement situation preserves pullout diagrams.

This simple remark will be extremely useful in view of the “standard procedure” for proving results in recollement theory outlined in 2.24.

**Definition 2.13 [The ($\infty$-)category Recol]:** A morphism between two recollements $\mathcal{C}$ and $\mathcal{C}'$ consists of a triple of functors $(F_0, F, F_1)$ such that the following square commutes in every part (choosing from time to time homonymous left or right adjoints):

$$
\begin{array}{ccc}
D^0 & \xrightarrow{F_0} & D \\
F & \downarrow & F_1 \\
D^0' & \xrightarrow{F'} & D'
\end{array}
$$

(8)

This definition turns the collection of all recollement data into a $\infty$-category denoted $\text{Recol}$ and called the ($\infty$-)category of recollements.

**Remark 2.14:** The natural definition of equivalence between two recollement data (all three functors $(F_0, F, F_1)$ are equivalences) has an alternative reformulation (see [PS88, Thm. 2.5]) asking that only two out of three functors are equivalences; nevertheless (loc. cit.) this must not be interpreted as a full 3-for-2 condition.

Equivalently, we can define this notion (see [AHKL11, §1.7]), asking that the essential images of the fully faithful functors $(i, q_L, q_R)$ are pairwise equivalent with those of $(i', q'_L, q'_R)$.

We now concentrate on other equivalent ways to specify a recollement on a stable $\infty$-category, slightly rephrasing Definition 2.1: first of all, [HJ10, Prop. 4.13.1] shows that the localization functor $q_Rq$, which is an exact localization with reflective kernel, uniquely determines the recollement datum up to equivalence; albeit of great significance as a general result, we are not interested in this perspective, and we address the interested readers to [HJ10] for a thorough discussion.

Another equivalent description of a recollement, nearer to our “torsion-centric” approach, is via a pair of $t$-structures on $D$ [Nic08]:

**Definition 2.15 [Stable TTF Triple]:** Let $D$ be a stable $\infty$-category. A stable **$t$-structures** on $D$ is a triple of full subcategories $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of $D$ such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are $t$-structures on $D$. 

Notice in particular that $D$ is reflected on $Y$ via a functor $R^Y$ and coreflected via a functor $S^Y$. The whole arrangement of categories and functors is summarized in the following diagram

$$
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{D} \\
\text{Z}
\end{array}
\begin{array}{c}
S^Y \\
\downarrow i_Y \\
D \\
\downarrow i_Z \\
S_X \\
\downarrow R_Y \\
\downarrow \downarrow R_Z
\end{array}
\begin{array}{c}
\text{X} \\
\text{Y} \\
\text{D} \\
\text{Z}
\end{array}
\
(9)

$$

where $S_Y \dashv i_Y \dashv R_Y$, $i_Z \dashv R_Z$ and $S_X \dashv i_X$.

Stable TTF triples are in bijection with equivalence classes of recollements, as it is recalled in [Nic08, Prop. 4.2.4]; the same bijection holds in the stable setting, 	extit{mutatis mutandis}.

We conclude this introductory section with the following Lemma, which will be of capital importance all along §3: functors in a recollement jointly reflect isomorphisms.

**Lemma 2.16 [Joint conservativity of recollement data]:** Let $D$ be a stable $\infty$-category, and let

$$(i, q) : D^0 \simeq D \simeq D^1$$

be a recollement on $D$. Then the following conditions are equivalent for an arrow $f \in \text{hom}(D)$:

- $f$ is an isomorphism in $D$,
- $q(f)$ is an isomorphism in $D^1$ and $i_R(f)$ is an isomorphism in $D^0$,
- $q(f)$ is an isomorphism in $D^1$ and $i_L(f)$ is an isomorphism in $D^0$.

In other words, the pairs of functors $\{q, i_R\}$ and $\{q, i_L\}$ jointly reflect isomorphisms.

**Proof.** We only prove that if $q(f)$ and $i_L(f)$ are isomorphisms in the respective codomains, then $f$ is an isomorphism in $D$. We need a preparatory sub-lemma, namely that the pair $\{q, i_L\}$ reflects zero objects; the only nontrivial part of this statement is that if $qD \cong 0$ in $D^1$ and $i_LD \cong 0$ in $D^0$, then $D \cong 0$ in $D$, an obvious statement in view of axiom (3) of Def. 2.1, since $qD \cong 0$ entails $D \cong i(D')$, and now $0 \cong i_L(D) = i_LiD' \cong D$.

With this preliminary result, we recall that $f : X \to Y$ is an isomorphism if and only if $\text{fib}(f) \cong 0$, and apply the previous result, together with the fact that recollement functors preserve pullouts.

Replacing $i_L$ with $i_R$, the proof shows a similar statement about the joint reflectivity of $\{q, i_R\}$. □

**Notation 2.17:** We will often use a rather intuitive shorthand, writing $\{q, i_L\}(f)$, or $\{q, i_R\}(f)$ to both functors applied to the same arrow. For example:
• Given (the left classes of) a pair of $t$-structures $D^0_{i=0}, D^1_{i=0}$ we write \( \{q, i_L\}(D) \in D^0_{i=0} \) (see Thm. 2.19) to denote that the object $qD \in D^1_{i=0}$ and $i_L(D) \in D^0_{i=0}$; similarly for $\{q, i_R\}(D) \in D^0_{i=0}$ and other combinations.

• Given (the left classes of) a pair of normal torsion theories $\mathcal{E}_0, \mathcal{E}_1$, we write \( \{q, i_{L/R}\}(f) \in \mathcal{E}'' \) (see Thm. 3.4) to denote that the arrow $f \in \text{hom}(D)$ is such that $qf \in \mathcal{E}_1$ and $i_{L/R}(f) \in \mathcal{E}_0$; similarly for $\{q, i_{L/R}\}(g) \in \mathcal{M}$ and other combinations.

**Remark 2.20:** The joint reflectivity of the recollement functors $\{q, i_L\}$ or $\{q, i_R\}$ can be seen as an analogue, in the setting of an abstract recollement, of the fact that in the geometric case of the recollement induced by a stratification $\emptyset \subset U \subset X$ one has ([PS88, 2.3]) that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{F}'$ on $X$ is uniquely determined by its restrictions $\varphi|_U$ and $\varphi|_{X \setminus U}$.

### 2.1. The classical gluing of $t$-structures

The main result in the classical theory of recollements is the so-called *gluing theorem*, which tells us how to obtain a $t$-structure $t = t_0 \lor t_1$ on $D$ starting from two $t$-structures $t_i$ on the categories $D^i$ of a recollement $\mathcal{H}$.

**Theorem 2.19 [Gluing Theorem]:** Consider a recollement $\mathcal{H} = (i, q) : D^0 \trianglelefteq D \trianglerightgeq D^1$, and let $t_i$ be $t$-structures on $D^i$ for $i = 0, 1$; then there exists a $t$-structure on $D$, called the *gluing* of the $t_i$ (along the recollement $\mathcal{H}$, but this specification is almost always omitted) and denoted $t_0 \lor t_1$, whose classes \( (D^0 \lor D^1)_{i=0}, (D^0 \lor D^1)_{i=1} \) are given by

\[
(D^0 \lor D^1)_{i=0} = \left\{ X \in D \mid (qX \in D^1_{i=0}) \cap (i_LX \in D^0_{i=0}) \right\},
\]

\[
(D^0 \lor D^1)_{i=1} = \left\{ X \in D \mid (qX \in D^1_{i=1}) \cap (i_RX \in D^0_{i=1}) \right\}. \tag{10}
\]

**Remark 2.20:** Following Notation 2.17 we have that $X \in D_{i=0}$ iff $\{q, i_L\}(X) \in D_{i=0}$ and $Y \in D_{i=1}$ iff $\{q, i_R\}(y) \in D_{i=1}$, which is a rather evocative statement: the left/right class of $t_0 \lor t_1$ is determined by the left/right adjoint to $i$.

**Remark 2.21:** The “wrong way” classes

\[
(D^0 \lor D^1)^\ast_{i=0} = \left\{ X \in D \mid (\{q, i_R\} X \in D_{i=0}) \right\},
\]

\[
(D^0 \lor D^1)^\ast_{i=1} = \left\{ X \in D \mid (\{q, i_L\} X \in D_{i=1}) \right\}. \tag{11}
\]

---

(4) The symbol $\lor$ (pron. glue) reminds the alchemical token describing the process of *amalgamation* between two or more elements (one of which is often mercury): albeit amalgamation is not recognized as a proper stage of the *Magnum Opus*, several sources testify that it belongs to the alchemical tradition (see [RS76, pp. 409-498]).
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do not define a \( t \)-structure in general. However they do in the case the recollement situation \( \mathcal{C} \) is the lower part of a \( 2 \)-recollement, i.e. there exists a diagram of the form

\[
\begin{array}{cccccc}
\mathcal{C}^0 & \xleftarrow{i_1} & \mathcal{C} & \xrightarrow{q_1} & \mathcal{C}^1 \\
\downarrow{i_4} & & \downarrow{q_4} & & \downarrow{i_3} & \quad \text{(12)}
\end{array}
\]

where both

\[
\begin{array}{cccccc}
\mathcal{C}^0 & \xleftarrow{i_1} & \mathcal{C} & \xrightarrow{q_1} & \mathcal{C}^1 \\
\downarrow{i_3} & & \downarrow{q_3} & & \downarrow{i_2} & \quad \text{(13)}
\end{array}
\]

and

\[
\begin{array}{cccccc}
\mathcal{C}^1 & \xleftarrow{q_2} & \mathcal{C} & \xrightarrow{i_2} & \mathcal{C}^0 \\
\downarrow{q_4} & & \downarrow{i_4} & & \downarrow{q_3} & \quad \text{(14)}
\end{array}
\]

are recollements, with \( \mathcal{C} = \mathcal{C}_3 \). Indeed, in this situation one has

\[
(D^0 \vee D^1)_{\leq 0} = \left\{ X \in D \mid \left( \{ q_i \}, q_2 \right) X \in D_{\geq 0} \right\}
\]

\[
= \left\{ X \in C \mid \left( \{ q_3, q_2 \} \right) X \in C_{\geq 0} \right\}
\]

\[
= (C^0 \vee C^1)_{\geq 0}.
\]

More generally, an \( n \)-recollement is defined as the datum of three stable \( \infty \)-categories \( \mathcal{C}^0, \mathcal{C}, \mathcal{C}^1 \) organized in a diagram

\[
\begin{array}{cccccc}
\mathcal{C}^0 & \xleftarrow{i_1} & \mathcal{C} & \xrightarrow{q_1} & \mathcal{C}^1 \\
\downarrow{i_n+2} & & \downarrow{q_{n+2}} & & \downarrow{i_3} & \quad \text{(15)}
\end{array}
\]

with \( n + 2 \) functors on each edge, such that every consecutive three functors form recollements \( \mathcal{C}_{2k} = (i_{2k}, q_{2k}) \), \( \mathcal{C}_{2h+1} = (q_{2h+1}, i_{2h+1}) \), for \( k = 1, \ldots, n - 1 \), \( h = 1, \ldots, n - 2 \), see [HQ14, Def. 2]. Applications of this formalism to derived categories of algebras, investigating the relationships between the recollements of derived categories and the Gorenstein properties of these algebras, can be found in [HQ14, Qin15].

**Notation 2.22**: It is worth to notice that \( D^0 \vee D^1 \) has no real meaning as a category; this is only an intuitive shorthand to denote the pair \( (D, t_0 \vee t_1) \); more explicitly, it is a shorthand to denote the following situation:

The stable \( \infty \)-category \( D \) fits into a recollement \( (i, q); D^0 \rightleftarrows D \rightleftarrows D^1 \), \( t \)-structures on \( D^0 \) and \( D^1 \) have been chosen, and \( D \) is endowed with the glued \( t \)-structure \( t_0 \vee t_1 \).
A proof of the gluing theorem in the classical setting of triangulated categories can be found in [Ban07, Thm. 7.2.2] or in the standard reference [BBD82]. We briefly sketch the argument given in [Ban07] as we will need it in the torsio-centric reformulation of the gluing theorem.

**Proof of Thm. 2.19.** We begin showing the way in which every \( X \in D \) fits into a fiber sequence \( SX \to X \to RX \) where \( SX \in (D^0 \vee D^1)_{\geq 0}, RX \in (D^0 \vee D^1)_{< 0} \). Let \( F_i \) denote the normal torsion theory on \( D^i \), inducing the \( t \)-structure \( t_i \); let \( \eta_i : qX \to R_i qX \) be the arrow in the fiber sequence
\[
S_i qX \xrightarrow{c_i} qX \xrightarrow{\eta_i} R_i qX
\] obtained thanks to \( F_1 \); let \( \hat{\eta} \) be its mate \( X \to qR_1 qX \) in \( D \) under the adjunction \( q \dashv qR \), and let \( W X = \text{fib}(\hat{\eta}) \).

Now, consider \( i_L W X \) in the fiber sequence
\[
S_0 i_L W X \xrightarrow{c_0} i_L W X \xrightarrow{\theta_0} R_0 i_L W X
\] induced by \( F_0 \) on \( D_0 \), and its mate \( \hat{\theta} : WX \to iR_0 i_L W X \); take its fiber \( SX \), and the object \( RX \) defined as the pushout of \( iR_0 i_L W X \xrightarrow{\hat{\theta}} WX \to X \).

To prove that these two objects are the candidate co/truncation we consider the diagram
\[
\begin{array}{ccc}
 SX & \xrightarrow{\theta} & W X \\
 0 & \xrightarrow{iR_0 i_L W X} & RX \\
 0 & \xrightarrow{qR_1 qX} & \end{array}
\]
where all the mentioned objects fit, and where every square is a pullout. We have to prove that \( SX \in (D^0 \vee D^1)_{\geq 0} \) and \( RX \in (D^0 \vee D^1)_{< 0} \). To do this, apply the functors \( q, i_L, i_R \) to (2.1), obtaining the following diagram of pullout squares (recall the exactness properties of the recollement functors, stated in Prop. 2.12):
\[
\begin{array}{cccccccc}
 qSX & \xrightarrow{qW X} & qX & i_L SX & \xrightarrow{i_L WX} & i_L X & i_R SX & i_R WX & i_R X \\
 0 & \xrightarrow{qR X} & 0 & R_0 i_L W X & \xrightarrow{i_L RX} & \end{array}
\]
where we took into account the relations \( qi = 0, i_R q = 0 = i_L qL \). We find that
• \(qSX \simeq qWX \simeq S_1qX \in D_{\geq 0}^1\), since \(0 \to S_1qX\) lies in \(\mathcal{M}_1\), and \(qRX \simeq R_1qX \in D_{<0}^1\);

• \(i_LSX \simeq S_0i_LWX \in D_{\geq 0}^0\), by the pullout square \(\Box\);

• \(i_RRX \simeq R_0i_LWX \in D_{<0}^0\).

It remains to show that the two classes \(D_{\geq 0}, D_{<0}\) are orthogonal; to see this, suppose that \(X \in D_{\geq 0}\) and \(Y \in D_{<0}\). We consider the fiber sequence \(\iota_i RY \to Y \to qRqY\) of axiom (4) in Def. 2.1, to obtain (applying the homological functor \(D(X, -)\))

\[
\begin{array}{c}
D(X, iR Y) \longrightarrow D(X, Y) \longrightarrow D(X, qRqY) \\
\text{H} & \text{H} & \text{H} \\
D(i_LX, iRX) & D(qX, qY) & 0 \quad 0
\end{array}
\]  

(17)

and we conclude, thanks to the exactness of this sequence. \(\square\)

**Remark 2.23**: Strictly speaking, the domain of definition of the gluing operation \(\cup\) is the set of triples \((t_0, t_1, \mathcal{C})\) where \((t_0, t_1) \in \text{ts}(D^0) \times \text{ts}(D^1)\) and \(\mathcal{C} = (i, q)\) is a recollement \(D^0 \rightleftharpoons D \rightleftharpoons D^1\), but unless this (rather stodgy) distinction is strictly necessary we will adopt an obvious abuse of notation.

**Remark 2.24** [A standard technique]: The procedure outlined above is in some sense paradigmatic, and it’s worth to trace it out as an abstract way to deduce properties about objects and arrows fitting in a diagram like (2.1). This algorithm will be our primary technique to prove statements in the “torsio-centric” formulation of recollements:

• We start with a particular diagram, like for example (2.1) or (3.1) below; our aim is to prove that a property (being invertible, being the zero map, lying in a distinguished class of arrows, etc.) is true for an arrow \(h\) in this diagram.

• We apply (possibly only some of) the recollement functors to the diagram, and we deduce that \(h\) has the above property from
  - The recollement relations between the functors (Def. 2.1);
  - The exactness of the recollement functors (Prop. 2.12);
  - The joint reflectivity of the pairs \(\{q, i_L\}\) and \(\{q, i_R\}\) (Lemma 2.16);

3. Stable Recollements.
3.1. The Jacob’s ladder: building co/reflections. The above procedure to build the functors $R, S$ depends on several choices (we forget half of the fiber sequence $S_1qX \to qX \to R_1qX$) and it doesn’t seem independent from these choices, at least at first sight.

The scope of this first subsection is to show that this apparent asymmetry arises only because we are hiding half of the construction, taking into account only half of the fiber sequence (16). Given an object $X \in D$ a dual argument yields another way to construct a fiber sequence

$$S'X \to X \to R'X$$

(18)

out of the recollement data, which is naturally isomorphic to the former $SX \to X \to RX$.

We briefly sketch how this dualization process goes: starting from the coreflection arrow $\epsilon_1: S_1qX \to qX$, taking its mate $q_LS_1qX \to X$ under the adjunction $q_L \dashv q$, and reasoning about its cofiber we can build a diagram which is dual to the former one, and where every square is a pullout:

![Diagram](image)

**Proposition 3.1 [The Jacob’s ladder]:** The two squares of the previous constructions fit into a “ladder” induced by canonical isomorphisms $SX \cong S'X, RX \cong R'X$; the construction is functorial in $X$. The “Jacob’s ladder” is the following diagram:

![Diagram](image)

**Proof.** It suffices to prove that both $SX, S'X$ lie in $D_{\geq 0}$ and both $RX, R'X$ lie in $D_{\leq 0}$; given this, we can appeal (a suitable stable $\infty$-categorical version of) [BBD82, Prop. 1.1.9] which asserts the functoriality of the truncation
functors, i.e. that when the same object $X$ fits into two fiber sequences arising from the same normal torsion theory, then there exist the desired isomorphisms.\(^5\)

The procedure showing this is actually the same remarked in 2.24: we apply $q, i_L, i_R$ to the diagram (3.1) and we exploit exactness of the recollement functors to find pullout diagrams showing that $R'X \in \mathbf{D}_{<0}$ and $S'X \in \mathbf{D}_{\geq 0}$.

Once these isomorphisms have been found, it remains only to glue the two sub-diagrams

\[
\begin{array}{cccccccc}
q_L S_1 q X & \rightarrow & S X & \rightarrow & W X & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i S_0 i_R K X & \rightarrow & C X & \rightarrow & K X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i R_0 q W X & \rightarrow & R X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & q_R R_1 q X
\end{array}
\]

\[
\begin{array}{cccccccc}
q_L S_1 q X & \rightarrow & S' X & \rightarrow & W X & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i S_0 i_R K X & \rightarrow & C X & \rightarrow & K X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i R_0 q W X & \rightarrow & R X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & q_R R_1 q X
\end{array}
\]

to obtain the ladder. Now, this construction is obtained by taking into account the fiber sequence $S_1 q X \rightarrow q X \rightarrow R_1 q X$ as a whole, and since this latter object is uniquely determined up to isomorphism, we obtain a diagram of endofunctors

\[
\begin{array}{cccccccc}
q_L S_1 q & \rightarrow & S & \rightarrow & W & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i S_0 i_R K & \rightarrow & C & \rightarrow & K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & i R_0 q W & \rightarrow & R \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & q R R_1 q
\end{array}
\]

where every square is a pullout (again giving to a category of functors the obvious stable structure [Lur11, Prop. 1.1.3.1]), and where the functorial nature of $W$, $K$ and $C$ is a consequence of their construction. Notice also that this latter diagram of functors uses homogeneously all the recollement functors, and that it is “symmetric” with respect to the antidiagonal (it switches left and right adjoints, as well as reflections and coreflections).  □

\(^5\)In a torsio-centric perspective, this follows from the uniqueness of the factorization of a morphism with respect to the normal torsion theory having reflection $R$ and coreflection $S$. 

The functors $S, R$ are the co/truncations for the recollée $t$-structure, and the normality of the torsion theory is witnessed by the pullout subdiagram

Notation 3.2: From now on, we will always refer to the diagram above as “the Jacob ladder” of an object $X \in D$, and/or to the diagram induced by a morphism $f: X \to Y$ between the ladder of the domain and the codomain, i.e. to three-dimensional diagrams like

3.2. The ntt of a recollement. Throughout this subsection we outline the torsio-centric translation of the classical results recalled above. In particular we give an explicit definition of the $\vee$ operation when it has been “transported” to the set of normal torsion theories, independent from its characterization in terms of the pairs aisle-coaisle of the two $t$-structures. From now on we assume given a recollement

$$D^0 \xleftarrow{i_R} D \xrightarrow{q_R} D^1.$$ 

Given $t$-structures $t_i \in \text{ts}(D^i)$, in view of our “Rosetta stone” theorem [FL15b], there exist normal torsion theories $F_i = (\mathcal{E}_i, \mathcal{M}_i)$ on $D^i$ such that $(D^0, D^1)$ are the classes $(0/\mathcal{E}_1, \mathcal{M}_1/0)$ of torsion and torsion-free objects of $D^i$, for $i = 0, 1$; an object $X$ lies in $(D^0 \vee D^1)_{\geq 0}$ if and only if $qX \in \mathcal{E}_1$ and
\$i_LX \in \mathcal{E}_0$ (6), and similarly an object \$Y \in \mathbf{D}_{\leq 0}$ if and only if \$qY \in \mathcal{M}_1$ and \$i_RY \in \mathcal{M}_0$.

**Remark 3.3**: The \$t$-structure \$t = t_0 \lor t_1$ on \$\mathbf{D}$ must itself come from a normal torsion theory which we denote \$F_0 \lor F_1$ on \$\mathbf{D}$, so that \$(\mathbf{D}^0 \lor \mathbf{D}^1)_{\geq 0}, (\mathbf{D}^0 \lor \mathbf{D}^1)_{< 0}) = (0/(\mathcal{E}_0 \lor \mathcal{E}_1), (\mathcal{M}_0 \lor \mathcal{M}_1)/0)$; in other words the following three conditions are equivalent for an object \$X \in \mathbf{D}$:

- \$X$ lies in \$(\mathbf{D}^0 \lor \mathbf{D}^1)_{\geq 0}$;
- \$X$ lies in \$\mathcal{E}_0 \lor \mathcal{E}_1$, i.e. \$RX \cong 0$ in the notation of (3.1);
- \$\{q, i_L\}(X) \in \mathcal{E}$, following Notation 2.17.

We now aim to a torsio-centric characterization of the classes \$(\mathcal{E}_0 \lor \mathcal{E}_1, \mathcal{M}_0 \lor \mathcal{M}_1)$, relying on the factorization properties of \$(\mathcal{E}_1, \mathcal{M}_1)$ alone; since we proved Thm. 2.19 above, there must be a normal torsion theory \$F_0 \lor F_1 = (\mathcal{E}_0 \lor \mathcal{E}_1, \mathcal{M}_0 \lor \mathcal{M}_1)$ inducing \$t_0 \lor t_1$ as \$(0/(\mathcal{E}_0 \lor \mathcal{E}_1), (\mathcal{M}_0 \lor \mathcal{M}_1)/0)$; in other words,

\[ F_0 \lor F_1 \] is the (unique) normal torsion theory whose torsion/-torsionfree classes are \$(\mathbf{D}^0 \lor \mathbf{D}^1)_{\geq 0}, (\mathbf{D}^0 \lor \mathbf{D}^1)_{< 0})$ of Thm. 2.19.

Clearly this is only an application of our “Rosetta stone” theorem, so in some sense this result is “tautological”. But there are at least two reasons to concentrate in “proving again” Thm. 2.19 from a torsio-centric perspective:

- The construction offered by the Rosetta stone is rather indirect, and only appropriate to show formal statements about the factorization system \$\mathbb{F}(t)$ induced by a \$t$-structure;
- In a stable setting, the torsio-centric point of view, using factorization systems, is more primitive and more natural than the classical one using 1-categorical arguments (i.e., \$t$-structures \$t$ on the homotopy category of a stable \$\mathbf{D}$ are induced by normal torsion theories in \$\mathbf{D}$; in the quotient process one loses important informations about \$t$).

Both these reasons lead us to adopt a “constructive” point of view, giving an explicit characterization of \$F_0 \lor F_1$ which relies on properties of the factorization systems \$F_0, F_1$ alone, independent from triangulated categorical arguments.

In the following section we will discuss the structure and properties of the factorization system \$F_0 \lor F_1$, concentrating on a self-contained and categorically well motivated construction of the classes \$\mathcal{E}_0 \lor \mathcal{E}_1$ and \$\mathcal{M}_0 \lor \mathcal{M}_1$ starting from an obvious *ansatz* which follows Remark 3.3.

---

(6) Thanks to the Sator lemma we are allowed to use “\$X \in \mathbf{K}$” as a shorthand to denote that either the initial arrow \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) or the terminal arrow \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) lie in a 3-for-2 class \$\mathbf{K} \in \hom(\mathbf{C})$. From now on we will adopt this notation.
The discussion above, and in particular the fact that an initial/terminal arrow $0 \xleftrightarrow{X} X$ lies in $\mathcal{E}_0 \uplus \mathcal{E}_1$ if and only if $\{q, i_L\}(X) \in \mathcal{E}$, suggests that we define $\mathcal{E}_0 \uplus \mathcal{E}_1 = \{ f \in \text{hom}(D) \mid \{q, i_L\}(f) \in \mathcal{E} \}$ and $\mathcal{M}_0 \uplus \mathcal{M}_1 = \{ g \in \text{hom}(D) \mid \{q, i_R\}(g) \in \mathcal{M} \}$. Actually it turns out that this guess is not far to be correct: the correct classes are indeed given by the following:

**Theorem 3.4**: Let $D$ be a stable $\infty$-category, in a recollement

$$(i, q): D^0 \xrightarrow{\cong} D \cong D^1,$$

and let $t_i$ be a $t$-structure on $D_i$. Then the recollé $t$-structure $t_0 \uplus t_1$ is induced by the normal torsion theory $(\mathcal{E}_0 \uplus \mathcal{E}_1, \mathcal{M}_0 \uplus \mathcal{M}_1)$ with classes

$$\mathcal{E}_0 \uplus \mathcal{E}_1 = \{ f \in \text{hom}(D) \mid \{q, i_L\}(f) \in \mathcal{E} \}; \quad (19)$$

$$\mathcal{M}_0 \uplus \mathcal{M}_1 = \{ g \in \text{hom}(D) \mid \{q, i_R\}(g) \in \mathcal{M} \}. \quad (20)$$

**Proof.** We only need to prove the statement for $\mathcal{E}_0 \uplus \mathcal{E}_1$, since the statement for $\mathcal{M}_0 \uplus \mathcal{M}_1$ is completely specular. Thanks to the discussion in section §2, an arrow $f \in \text{hom}(D)$ lies in $\mathcal{E}_0 \uplus \mathcal{E}_1$ if and only if $Rf$ (as constructed in the Jacob ladder (3.2)) is an isomorphism in $D$, so we are left to prove that, given $f \in \text{hom}(D)$:

$$Rf \text{ is an isomorphism in } D \iff \{q, i_L\}(f) \in \mathcal{E}. \quad (21)$$

Equivalently, we have to prove that

$$Rf \text{ is an isomorphism } \iff \{R_1q, R_0i_LW\}(f) \text{ are isomorphisms.} \quad (22)$$

We begin by showing that if $\{R_1q, R_0i_LW\}(f)$ are isomorphisms, then also $Rf$ is an isomorphism. By the joint conservativity of the recollement data (Lemma 2.16) we need to prove that if $\{R_1q, R_0i_LW\}(f)$ are isomorphisms, then both $qRf$ and $i_LRf$ are isomorphisms. Apply the functor $q$ to the Jacob ladder (3.2), to obtain

$$S_{qX} \xrightarrow{qSX} qSY \xrightarrow{qWY} qY \quad (23)$$

$$0 \xrightarrow{R_1qX} R_1qY \xrightarrow{qRY} qKX \xrightarrow{qKY} qX \xrightarrow{qWX} qWY \xrightarrow{qSY} qSY \xrightarrow{S_{qX}} 0$$

$$0 \xrightarrow{qCX} qKY \xrightarrow{qRX} qRX \xrightarrow{R_1qX} R_1qY \xrightarrow{qKY} qX \xrightarrow{qWX} qWY \xrightarrow{qSY} qSY \xrightarrow{S_{qX}} 0$$
Hence $qRf$ is an isomorphism, since it fits into the square

$$
\begin{array}{ccc}
qRX & \longrightarrow & qRY \\
\downarrow & & \downarrow \\
R_1qX & \longrightarrow & R_1qY.
\end{array}
$$

(24)

Now apply the functor $i_L$ to the Jacob ladder, obtaining

$$
\begin{array}{ccc}
0 & \longrightarrow & i_L\text{SY} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & i_L\text{WX} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & i_L\text{RX} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & i_Lq_RR_1qY \\
\end{array}
$$

(25)

As noticed above, $R_1qf$ is an isomorphism, so also $i_Lq_RR_1qf$ is an isomorphism. Then $i_LRf$ is an isomorphism by the five-lemma applied to the morphism of fiber sequences

$$
\begin{array}{ccc}
R_0i_LWY & \longrightarrow & i_LRf \\
\downarrow & & \downarrow \\
R_0i_LRX & \longrightarrow & i_Lq_RR_1qY \\
\end{array}
$$

(26)

Vice versa: assuming $Rf$ is an isomorphism in $\mathbf{D}$, we want to prove that $\{R_1q, R_0i_LW\}(f)$ are isomorphisms. Diagram (23) gives directly that $R_1qf$ is an isomorphism, since the square

$$
\begin{array}{ccc}
qRX & \longrightarrow & qRY \\
\downarrow & & \downarrow \\
R_1qX & \longrightarrow & R_1qY
\end{array}
$$

(27)

is commutative. Then, from diagram (26) we see that, since both $i_Lq_RR_1qf$ and $i_LRf$ are isomorphisms, so is also $R_0i_LWf$. \qed
Remark 3.5: From the sub-diagram

\[
\begin{array}{c}
i_L X \xrightarrow{i_L f} i_L Y \\
i_L K X \xrightarrow{} i_L K Y \\
i_L R X \xrightarrow{} i_L R Y
\end{array}
\]

of diagram (25) one deduces that if \( Rf \) is an isomorphism, then \( i_L f \in \mathcal{E}_0 \), by the 3-for-2 closure property of \( \mathcal{E}_0 \). This means that \( \{ q, i_L W \}(f) \in \mathcal{E} \) implies that \( \{ q, i_L \}(f) \in \mathcal{E} \). The converse implication has no reason to be true in general. However it is true for terminal (or initial) morphisms. Namely, from the Rosetta stone one has that \( X \in \mathcal{E}_0 \vee \mathcal{E}_1 \) if and only if \( X \in (\mathcal{D}_0 \vee \mathcal{D}_1)_{\geq 0} \), and so if and only if \( \{ q, i_L \}(X) \in \mathcal{E} \). On the other hand, \( X \in \mathcal{E}_0 \vee \mathcal{E}_1 \) if and only if \( \{ q, i_L W \}(X) \in \mathcal{E} \). The fact that the condition \( \{ q, i_L \}(X) \in \mathcal{D}_{\geq 0} \) is equivalent to the condition \( \{ q, i_L W \}(X) \in \mathcal{D}_{\geq 0} \) can actually be easily checked directly. Namely, if \( qX \in \mathcal{D}_{\geq 1} \), then \( qR_{\mathcal{R}_1}qX = 0 \) and so \( X = WX \) in this case. Specular considerations apply to the right class \( \mathcal{M}_0 \vee \mathcal{M}_1 \).

4. Properties of recollements.

“Do what thou wilt” shall be the whole of the Law. The study of this Book is forbidden. It is wise to destroy this copy after the first reading. Whosoever disregards this does so at his own risk and peril.

Ankh-ef-en-Khonsu I

In this section we address associativity issues for the \( \vee \) operation: it is a somewhat subtle topic, offering examples of several non-trivial constructions even in the classical geometric case: it is our opinion that in a stable setting the discussion can be clarified by simple, well-known categorical properties.

We start proving a generalization of [Ban07, BBD82] where it is stated that the gluing operation can be iterated in a preferential way determined by a stratification of an ambient space \( X \). This result hides in fact an associativity property for the gluing operation, in a sense which our Thm. 4.2 below makes precise.

Suitably abstracted to a stable setting, a similar result holds true, once we are given a Urizen compass (a certain shape of diagram like in Def. 4.7, implying certain relations and compatibilities between different recollements, which taken together ensure associativity).
4.1. **Geometric associativity of the gluing.** An exhaustive account for the theory of stratified spaces can be found in [Pfl01, Ban07, Wei94]. Here, since we do not aim at a comprehensive treatment, we restrict to a sketchy recap of the basic definitions.

A **stratified space** of length \( n \) consists of a pair \((X, s)\) where

\[
s : \emptyset = U_{-1} \subset U_0 \subset \cdots \subset U_n \subset X = U_{n+1}
\]  

(29)

is a chain of closed subspaces of a space \( X \), subject to various technical assumptions which ensure that the homology theory we want to attach to \((X, s)\) is “well-behaved” in some sense.

All along the following section, we will denote a **pure stratum** of a stratified space \((X, s)\) the set-theoretical difference \( E_i = U_i \setminus U_{i-1} \).

**Remark 4.1:** The definition is intentionally kept somewhat vague in various respects, first of all about the notion of “space”: the definition of stratification can obviously be given in different contexts (topological spaces, topological manifolds, \(PL\)-manifolds, . . .) according to the needs of the specific theory we want to build; when the stratification \( s \) is clear from the context, we indulge to harmless, obvious abuses of notation.

The associativity properties of \( \cup \) are deeply linked with the presence of a stratification on a space \( X \), in the sense that a stratification \( s \) is what we need to induce additional recollements “fitting nicely” in the diagram of inclusions determined by \( s \). These recollements define a unique \( t \)-structure \( t_0 \cup \equiv \cdots \cup \equiv t_n \), given \( t_i \) on the derived categories of the pure strata.

To motivate the shape and the strength of the abstract conditions ensuring associativity of \( \cup \), exposed in §4.2, and in particular the definition of a Urizen compass \( 4.7 \), we have to dig into deep in the argument sketched in the geometric case in [BBD82, 2.1.2-3]: we start by recalling

**Theorem 4.2:** [Ban07, p. 158] Let \((X, s)\) be a stratified space, \( \{E_0, \ldots, E_n\} \) the set of its pure strata, and \( t_i \) be a set of \( t \)-structures, one on each \( D(E_i) \), for \( i = 0, \ldots, n \).

Then there exists a uniquely determined \( t \)-structure \( t_0 \cup \cdots \cup t_n \) on \( D(X) \), obtained by an iterated gluing operation as the parenthesization \((\cdots ((t_0 \cup t_1) \cup t_2) \cup \cdots \cup t_{n-1}) \cup t_n \). Following Notation 2.22 we will refer to the pair \((D(X), t_0 \cup \cdots \cup t_n)\) as \( D(E_0) \cup \cdots \cup D(E_n) \).

**Proof.** A stratification of \( X \) as in (29) induces a certain triangular diagram \( G_n \) of the following form, where all maps \( i_k \) are inclusions of the closed subspaces \( U_k \) of \( s \), and all \( j_k \) are inclusions of the pure strata \( E_k \): in the
notation above we obtain

\[
\begin{array}{c}
\xymatrix{ & X \\
& U_n \\
& U_{n-1} \ar[u]_<<<<<<<<<{i_n} \ar[r]_<<<<<<<<<{j_{n-1}} & E_n \\
& U_1 \ar[u]_<<<<<<<<<{i_1} \ar[r]_<<<<<<<<<{j_0} & E_1 \\
E_0 \ar[u]_<<<<<<<<<{i_0} & E_0 \ar[u]_<<<<<<<<<{j_0} & E_0 \ar[u]_<<<<<<<<<{i_0} \\
& E_1 \\
& E_{n+1} \ar[u]_<<<<<<<<<{i_0} \\
& E_{n+1} \\
\end{array}
\]

This diagram can clearly be defined inductively starting from \( n = 1 \) (the diagram of inclusions as in Example 2.5). Given this evident recursive nature, it is sufficient to examine the case \( n = 2 \) of a stratification \( U_0 \subset U_1 \subset X \), depicted as

\[
\begin{array}{c}
\xymatrix{ & D(X) \\
D(U_1) \ar[ru]_<<<<<<<<<{q=j_1^*} & \\
D(E_0) \ar[ru]_<<<<<<<<<{g=j_0^*} & D(E_1) \\
D(E_0) \ar[ru]_<<<<<<<<<{f=i_0^*} & \\
\end{array}
\]

to notice that the \( t \)-structure \( (t_0 \cup t_1) \) obtained by iterated gluing construction is

\[
[(D(E_0) \cup D(E_1)) \cup D(E_2)]_{\geq 0} = \left\{ G \in D(X) \mid \begin{array}{c} qG \in D(E_2)_{\geq 0}, \\
a_L G \in [D(E_0) \cup D(E_1)]_{\geq 0} \\
\end{array} \right\}
\]

\[
= \left\{ G \in D(X) \mid \begin{array}{c} qG \in D(E_2)_{\geq 0}, \\
q(g(a_L G) \in D(E_1)_{\geq 0}, \\
f_L(a_L G) \in D(E_0)_{\geq 0} \end{array} \right\}
\]

\[
(2.17) = \left\{ G \in D(X) \mid \{ q, g a_L, f_L a_L \}(G) \in D_{\geq 0} \right\}
\]

The inductive step simply adds another inclusion (and the obvious maps between derived categories) to these data. \( \square \)

**Remark 4.3:** In the previous proof, in the case \( n = 2 \), we could have noticed that two “hidden” recollement data, given by the inclusions

\[
(E_1 \rightarrow X \setminus U_0, E_2 \rightarrow X \setminus U_0) \text{ and } (E_0 \rightarrow X, X \setminus U_0 \rightarrow X)
\]

\( ^{(7)} \) Here and for the rest of the section, drawing large diagrams of stable categories, we adopt the following shorthand: every edge \( h: E \rightarrow F \) is decorated with an adjoint triple \( h_L \dashv h \dashv h_R: E \rightleftarrows F \).
come into play: the refinement of the inclusions in the diagram above induces an analogous refinement which passes to the derived \( \infty \)-categories,

\[
\begin{array}{c}
\mathcal{D}(X) \\
\mathcal{D}(U_1) \\
\mathcal{D}(E_0) \\
\mathcal{D}(E_1) \\
\mathcal{D}(E_2)
\end{array}
\xymatrix{
\mathcal{D}(X) 
& \mathcal{D}(U_1) \ar[dl]_a \ar[r]^u & \mathcal{D}(X \setminus U_0) \\
\mathcal{D}(E_0) \ar[ur]_f & \mathcal{D}(E_1) \ar[ul]_g & \mathcal{D}(E_2) \ar[ur]_k
}
\]

of functors between derived \( \infty \)-categories on the pure strata. These data induce two additional recollements, \((k, h)\) and \((u, a \circ f)\) which we can use to define a different parenthesization \( t_0 \uplus (t_1 \uplus t_2) \).

**Remark 4.4:** When all the recollements data in (32) are taken into account, we obtain a graph

\[
\begin{array}{c}
\mathcal{D}(X) \\
\mathcal{D}(U_1) \\
\mathcal{D}(E_0) \\
\mathcal{D}(E_1) \\
\mathcal{D}(E_2)
\end{array}
\xymatrix{
\mathcal{D}(X) 
& \mathcal{D}(U_1) \ar[dl]_{a_L} \ar[r]^{u} & \mathcal{D}(X \setminus U_0) \\
\mathcal{D}(E_0) \ar[ur]_{f_L} \ar[ul]_{g} & \mathcal{D}(E_1) \ar[ul]_{h_L} \ar[ur]_{k} & \mathcal{D}(E_2)
}
\]

called the left-winged diagram associated with (32), and defined by taking the left-most adjoint in the string \((-)_L \dashv (-) \dashv (-)_R\), when descending each left “leaf” of the tree represented in diagram (32). In a completely similar fashion we can define the right-winged diagram of (32). We refer to these diagrams as \((l\text{-}32)\) and \((r\text{-}32)\) respectively.

It is now quite natural to speculate about some sort of *comparison* between the two recollements \((t_0 \uplus t_1) \uplus t_2\) and \(t_0 \uplus (t_1 \uplus t_2)\): in fact we can prove with little effort (once the phenomenon in study has been properly clarified) that the two \( t \)-structures are equal, since the square

\[
\begin{array}{c}
E_1 \\
U_1 \\
X
\end{array}
\xymatrix{
E_1 
& X \setminus U_0 \\
U_1 \ar[r] \ar[ur] & X
}
\]

is a fiber product (in a suitable category of spaces) of a proper map with an open embedding, and so there is a “change of base” morphism \( u \circ a \cong h \circ g \) which induces invertible 2-cells \( g \circ a_L \cong h_L \circ u \) and \( g \circ a_R \cong h_R \circ u \) filling the square \( \Box \) in diagram (32): this is a particular instance of the so-called
Beck-Chevalley condition for a commutative square, which we now adapt to the ∞-categorical setting.

**Definition 4.5 [Beck-Chevalley condition]**: Consider the square

\[
\begin{array}{ccc}
A & \xrightarrow{a_L} & B \\
\downarrow{g} & & \downarrow{u} \\
C & \xrightarrow{h_L} & D
\end{array}
\] (35)

in a \((\infty, 2)\)-category, filled by an invertible 2-cell \(\theta: u \circ a \simeq h \circ g\) and such that \(a_L \to a, h_L \to h\); then the square (35) is said to satisfy the left Beck-Chevalley property, or that it is a left Beck-Chevalley square (LBC for short) if the canonical 2-cell

\[
\hat{\theta} : h_L \circ u \xrightarrow{h_L \circ \eta} h_L \circ u \circ a \circ a_L \xrightarrow{h_L \circ \theta \circ a_L} h_L \circ h \circ g \circ a_L \xrightarrow{\epsilon \circ ga_L} g \circ a_L
\] (36)

is invertible as well. Similarly, when \(a \to a_R, h \to h_R\) we define the 2-cell

\[
\hat{\tilde{\theta}} : g \circ a_R \xrightarrow{\eta \circ ga_R} h_R \circ h \circ g \circ a_R \xrightarrow{h_R \circ \theta \circ a_R} h_R \circ u \circ a \circ a_R \xrightarrow{h_R \circ \epsilon \circ a_R} h_R \circ u
\] (37)

and we say that the square (35) is right Beck-Chevalley (RBC for short) when \(\hat{\tilde{\theta}}\) is invertible. We will say that the square (35) is Beck-Chevalley (BC for short) when it is both left and right Beck-Chevalley.

In light of this property enjoyed by diagram \(\Box\) in (32) it's rather easy to show that the two left classes

\[
\begin{align*}
[D(E_0) \vee D(E_1)]_{\geq 0} &= \{G \in D(X) \mid \{ku, ga_L, f_1 a_L\}(G) \in D_{\geq 0}\} \\
[D(E_0) \vee (D(E_1) \vee D(E_2))]_{\geq 0} &= \{G \in D(X) \mid \{ku, h_L u, f_1 a_L\}(G) \in D_{\geq 0}\}
\end{align*}
\]

coincide up to a canonical isomorphism determined by the Beck-Chevalley 2-cell in \(\Box\) of diagram (32).

As a result, both \([D(E_0) \vee D(E_1)]_{\geq 0}\) and \([D(E_0) \vee (D(E_1) \vee D(E_2))]_{\geq 0}\) define the torsion class of the same \(t\)-structure \(D_{\geq 0}^{012}, D_{<0}^{012}\) on \(D(X)\). Since 2-cell in \(\Box\) of diagram (32) is both left and right Beck-Chevalley, the analogous statement holds for the right classes, too. We can state this fact as follows.

**Scolium 4.6**: An object \(G \in D(X)\) lies in \(D_{\geq 0}^{012}\) if and only if \(l_0 G \in D(E_0)_{\geq 0}, l_1 G \in D(E_1)_{\geq 0}, l_2 G \in D(E_2)_{\geq 0}\) where \(l_i\) is any choice of a functor \(D(X) \to D(E_i)\) in the left-winged diagram of (32). An object \(G \in D(X)\) lies in \(D_{<0}^{012}\) if and only if \(r_0 G \in D(E_0)_{<0}, r_1 G \in D(E_1)_{<0}, r_2 G \in D(E_2)_{<0}\) where \(r_i\) is any choice of a functor \(D(X) \to D(E_i)\) in the right-winged diagram of (32).
It is now rather easy to repeat the same reasoning with arbitrarily long chains of strata: given a stratified space \((X,s)\) we can induce the diagram

\[
\begin{array}{cccc}
& X & \\
& & \downarrow \\
& U_n & \leftarrow X \setminus U_0 & \rightarrow X \setminus U_1 & \rightarrow & \cdots & \rightarrow & X \setminus U_n \\
& U_{n-1} & \leftarrow U_n \setminus U_0 & \rightarrow U_n \setminus U_1 & \rightarrow & \cdots & \rightarrow U_n \setminus U_{n-1} & \rightarrow U_0
\end{array}
\]

where leaves correspond to pure strata of the stratification of \(X\), and every square is a pullback of a proper map along an open embedding, so that the Beck-Chevalley condition is automatically satisfied by each square in the corresponding diagram \(D(38)\) of \(\infty\)-categories of sheaves of the various nodes. The diagram \(D(38)\) is equipped with recollement data between its adjacent nodes; we can again define the left-winged and right-winged version of \(D(38)\), which we will refer as \(l-D(38)\) and \(r-D(38)\).

Grouping all these considerations we obtain that

1. There exist “compatible” recollements to give associativity of all the parenthesizations

   \[
   (t_0 \vee \cdots \vee t_n)_\mathcal{P} = (t_0 \vee \cdots \vee t_n)_\mathcal{Q}
   \]

   for each \(\mathcal{P}, \mathcal{Q}\) in the set of all possible parenthesizations of \(n\) symbols. This is precisely the sense in which, as hinted above, geometric stratifications and recollement data “interact nicely” to give canonical isomorphisms between \((t_0 \vee \cdots \vee t_n)_\mathcal{P}\) and \((t_0 \vee \cdots \vee t_n)_\mathcal{Q}\), i.e. a canonical choice for associativity constraints on the \(\vee\) operation.

2. The following characterization for the class \((D(E_0) \vee \cdots \vee D(E_n))_{\geq 0}\) holds:

   \[
   (D(E_0) \vee \cdots \vee D(E_n))_{\geq 0} = \left\{ G \mid l_i(G) \in D(E_i)_{\geq 0}, \forall i = 0, \ldots, n \right\}
   \]

   where \(l_i\) is any choice of a functor \(D(X) \to D(E_i)\) in the left-winged diagram \(l-D(38)\).

   Similarly, the right class \((D(E_0) \vee \cdots \vee D(E_n))_{< 0}\) can be characterized as the class of objects \(G\) such that \(r_i(G) \in D(E_i)_{< 0}\), where \(r_i\) is any choice of a functor \(D(X) \to D(E_i)\) in the right-winged diagram \(r-D(38)\).
4.2. Abstract associativity of the gluing. The geometric case studied above gives us enough information to make an ansatz for a general definition, telling us what we have to generalize, and in which way.

In an abstract, stable setting we have the following definition, which also generalizes, in some sense, 2.1.

Let \( n \geq 2 \) be an integer, and let us denote as \([i, j]\) the interval between \( i, j \in [n] \), i.e., set \( \{ k \mid i \leq k \leq j \} \subset [n] = \{0, 1, \ldots, n\} \) (we implicitly assume \( i \leq j \) and we denote \([i, i] = \{i\}\) simply as \(i\)).

**Definition 4.7 [Urizen compass]**: A Urizen compass of length \( n \) is an arrangement of stable \( \infty \)-categories, labeled by intervals \( I \subseteq [n] \), and functors in a diagram \( G_n \) of the form

\[
\begin{array}{c}
D^{[0,n]} \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\( G_n \)

\[
\begin{array}{c}
D^{[0,1]} \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\( D^{[1,2]} \)

\[
\begin{array}{c}
D^{[1,2]} \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\( D^{[n-1,n]} \)

such that the following conditions hold:

- All the triples \( \{D^I, D^{I\cup J}, D^J\} \), where \( I, J \) are contiguous intervals, form different recollements \( D^I \cong D^{I\cup J} \cong D^J \).
- Every square

\[
\begin{array}{c}
D^{[i,j]} \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array}
\]

\( D^{[i,j+1]} \)

\[
\begin{array}{c}
D^{[i+1,j]} \\
\downarrow \\
\end{array}
\]

\( D^{[i+1,j+1]} \)

is BC in the sense of Definition 4.5.

Note that each row, starting from the base of the diagram, displays all possible intervals of length \( k \). We can think of a Urizen compass as a special

---

(8) In the complicated cosmogony of W. Blake, Urizen represents conventional reason and law; it is often represented bearing the same compass of the Great Architect of the Universe postulated by speculative Freemasonry; see for example the painting *The Ancient of Days*, appearing on the frontispiece of the prophetic book “Europe a Prophecy”.

(9) Two intervals \( I, J \subseteq [n] \) are called contiguous if they are disjoint and their union \( I \cup J \) is again an interval denoted \( I \cup J \).
kind of directed graph (more precisely, a special kind of rooted oriented tree—a multtree if we stipulate that each edge shortens a triple of adjunctions); the root of the tree is the category $D^{[0,...,n]}$; the leaves are the categories $\{D^0,...,D^n\}$ (the “generalized pure strata”).

**Theorem 4.8 [The northern emisphere theorem]**: A Urizen compass of length $n$ induce canonical isomorphisms between the various parenthesizations of $t_0 \cdots \cdot t_n$, giving associativity of the glue operation between $t$-structures.

Rephrasing the above result in a more operative perspective, whenever we have a $n$-tuple $\{(D^i,t_i)\}_{i=0,...,n}$ of stable $\infty$-categories with $t$-structure, such that $\{D^0,...,D^n\}$ are the leaves of a Urizen compass of length $n$, then the gluing operation between $t$-structures gives a unique (up to canonical isomorphism) “glued” $t$-structure on the root $D^{[0,n]}$ of the scheme, resulting as

$$
(D^0 \cdots \cdot D^n)_{\geq 0} = \left\{ X \in D^{[0,n]} \mid l_i(X) \in D^i_{\geq 0}, \forall i = 0,...,n \right\}
$$

$$
(D^0 \cdots \cdot D^n)_{< 0} = \left\{ X \in D^{[0,n]} \mid r_i(X) \in D^i_{< 0}, \forall i = 0,...,n \right\}
$$

where $l_i$ is any choice of a path from the root $D^{[0,n]}$ to the $i$th leaf in the left-winged diagram of $G_n$, and $r_i$ is any choice of a path from the root $D^{[0,n]}$ to the $i$th leaf in the right-winged diagram of $G_n$.

### 4.3. Gluing $J$-families.

Our theory of slicings [FL15a] shows that the set $\text{ts}(D)$ of $t$-structures on a stable $\infty$-category $D$ carries a natural action of the ordered group of integers. This entails that the most natural notion of a “family” of $t$-structures is a equivariant $J$-family of $t$-structures, namely an equivariant map $J \to \text{ts}(D)$ from another $\mathbb{Z}$-poset $J$.

The formalism of equivariant families allows to unify several constructions in the classical theory of $t$-structures: in particular

The semiorthogonal decompositions of [BO95, Kuz11] are described as precisely those $J$-families $t: J \to \text{ts}(D)$ taking

---

(10) In the languages spoken in the northern hemisphere of Tlön, “la célula primordial no es el verbo, sino el adjetivo monosilábico. El sustantivo se forma por acumulación de adjetivos. No se dice luna: se dice aéreo-claro sobre oscuro-redondo o anaranjado-tenue-del cielo o cualquier otra agregación. […] Hay objetos compuestos de dos términos, uno de carácter visual y otro auditivo: el color del naciente y el remoto grito de un pájaro. Los hay de muchos: el sol y el agua contra el pecho del nadador, el vago rosa trémulo que se ve con los ojos cerrados, la sensación de quien se deja llevar por un río y también por el sueño. Esos objetos de segundo grado pueden combinarse con otros; el proceso, mediante ciertas abreviaturas, es prácticamente infinito. Hay poemas famosos compuestos de una sola enorme palabra.” ([Bor44])
values on fixed points of the $\mathbb{Z}$-action; these are equivalently characterized as

- the stable $t$-structures, where the torsion and torsion-free classes are themselves stable $\infty$-categories;
- the equivariant $J$-families where $J$ has the trivial action.

And again

The datum of a single $t$-structure $t: \{\ast\} \to \text{TS}(D)$ is equivalent to the datum of a whole $\mathbb{Z}$-orbit of $t$-structures, namely an equivariant map $\mathbb{Z} \to \text{TS}(D)$.

In light of these remarks, given a recollement $(i, q): D^0 \leftrightarrows D \leftrightarrows D^1$ it is natural to define the gluing of two $J$-families

$$
\begin{array}{ccc}
\text{TS}(D^0) & \overset{t_0}{\longleftarrow} & J \overset{t_1}{\longrightarrow} \text{TS}(D^1)
\end{array}
$$

(44)

to be the $J$-family $t_0 \triangleright t_1: J \to \text{TS}(D); j \mapsto t_0(j) \triangleright t_1(j)$.

It is now quite natural to ask how does the gluing operation interact with the two situations above: is the gluing of two $J$-families again a $J$-family? As we are going to show, the answer to this question is: yes. Indeed, it’s easy to see that the gluing operation is an equivariant map, by recalling that $(\mathcal{E}_0 \triangleright \mathcal{E}_1)[1] = \{ f \in \text{hom}(D) \mid f([-1]) \in \mathcal{E}_1 \}$, and that all of the functors $q, i_L, i_R$ preserves the pullouts (and so commute with the shift). We have

$$(\mathcal{E}_0 \triangleright \mathcal{E}_1)[1] = \{ f \in \text{hom}(D) \mid q, i_L \} (f[-1]) \in \mathcal{E}_0$$

$$(\mathcal{E}_0 \triangleright \mathcal{E}_1)[1] = \{ f \in \text{hom}(D) \mid q f([-1]) \in \mathcal{E}_1, i_L f([-1]) \in \mathcal{E}_0 \}$$

$$(\mathcal{E}_0 \triangleright \mathcal{E}_1)[1] = \{ f \in \text{hom}(D) \mid q f \in \mathcal{E}_1[1], i_L f \in \mathcal{E}_0[1] \}$$

$$(\mathcal{E}_0 \triangleright \mathcal{E}_1)[1] = \mathcal{E}_0[1] \triangleright \mathcal{E}_1[1].$$

Given this, it is obvious that given two semiorthogonal decompositions $t_0, t_1: J \to \text{TS}(D_0)$ on $D^0, D^1$, the $J$-family $t_0 \triangleright t_1$ is again a semiorthogonal decomposition on $D$ (the trivial action on $J$ remains the same; it is also possible to prove directly that if $\mathcal{E}_0, \mathcal{E}_1$ are left parts of two exact normal torsion theories $F_0, F_1$ on $D^0, D^1$, then the gluing $\mathcal{E}_0 \triangleright \mathcal{E}_1$ is the left part of the exact normal torsion theory $F_0 \triangleright F_1$ on $D$). In some sense at the other side is the gluing of two $\mathbb{Z}$-orbits $t_0, t_1: \mathbb{Z} \to \text{TS}(C)$ on $D^0$ and $D^1$. Namely, the glued $t$-structure $t_0 \triangleright t_1$ on $D$ is the $\mathbb{Z}$-orbit $(t_0 \triangleright t_1)[k] = t_0[k] \triangleright t_1[k]$.

The important point here is that this construction can be framed in the more general context of perversity data associated to a recollement, which we now discuss in the attempt to generalize at least part of the classical theory of “perverse sheaves” to the abstract, $\infty$-categorical and torsio-centric setting.
Definition 4.9 [Perversity datum]: Let \( p: \{0, 1\} \to \mathbb{Z} \) be any function, called a perversity datum; suppose that a recollement
\[
(i, q): \mathcal{D}^0 \leftrightarrows \mathcal{D} \leftrightarrows \mathcal{D}^1
\]
is given, and that \( t_0, t_1 \) are \( t \)-structures on \( \mathcal{D}^0, \mathcal{D}^1 \) respectively. We define the \( (p-) \) perverted \( t \)-structures on \( \mathcal{D}^0, \mathcal{D}^1 \) as
\[
P_{t_0} = t_0[p(0)] = (\mathcal{D}^0_{\leq p(0)}, \mathcal{D}^0_{< p(0)})
\]
\[
P_{t_1} = t_1[p(1)] = (\mathcal{D}^1_{\leq p(1)}, \mathcal{D}^1_{< p(1)})
\]

Definition 4.10 [Perverse objects]: Let \( p \) be a perversity datum, in the notation above; the \( (p-) \) glued \( t \)-structure is the \( t \)-structure \( P(t_0 \triangleright t_1) = P_{t_0} \triangleright P_{t_1} \).

The heart of the \( p \)-perverted \( t \)-structure on \( \mathcal{D} \) is called the \( (\infty-) \) category of \( (p-) \) perverse objects of \( \mathcal{D} \).

Notice that saying “the category of \( p \)-perverse objects of \( \mathcal{D} \)” is an abuse of notation: this category indeed does not depend only on \( \mathcal{D} \) and \( p \), but on all of the recollement data and on the \( t \)-structures \( t_0 \) and \( t_1 \). Also notice how for a constant perversity datum \( p(0) = p(1) = k \), the \( p \)-perverted \( t \)-structure is nothing but the \( t \)-structure \( t_0 \triangleright t_1 \) shifted by \( k \).

We can extend the former discussion to the gluing of a whole \( n \)-tuple of \( t \)-structures, using a Urizen compass:

Remark 4.11: In the case of a Urizen compass of dimension \( n \) (diagram 41), whose leaves are the categories \( \{\mathcal{D}^0, \ldots, \mathcal{D}^n\} \), each endowed with a \( t \)-structure \( t_i \); a perversity function \( p: \{0, \ldots, n\} \to \mathbb{Z} \) defines a perverted \( t \)-structure
\[
P(t_0 \triangleright \cdots \triangleright t_n) = t_0[p(0)] \triangleright t_1[p(1)] \triangleright \cdots \triangleright t_n[p(n)]
\]
which is well-defined in any parenthesization thanks to the structure defining the Urizen compass. This result immediately generalizes to the case of a Urizen compass of \( J \)-families of \( t \)-structures, \( t_i: J \to \text{ts}(\mathcal{D}_i) \), with \( i = 0, \ldots, n \). Indeed perversity data act on \( J \)-equivariant families of \( t \)-structures by
\[
P_{t_i}(j) = t_i(j)[p(i)] = (\mathcal{D}^i_{\leq j + p(i)}, \mathcal{D}^i_{< j + p(i)}).
\]
This way, a \( J \)-perversity datum \( p: \{0, \ldots, n\} \to \mathbb{Z} \) induces a \( p \)-perverted \( t \)-structure
\[
P(t_0 \triangleright \cdots \triangleright t_n) = t_0 \triangleright \cdots \triangleright t_n: J \to \text{ts}(\mathcal{D}^{0,n})
\]
on \( \mathcal{D}^{0,n} \).

Remark 4.12 [Gluing of slicings]: Recall that a slicing on a stable \( \infty \)-category \( \mathcal{D} \) consists on a \( \mathbb{R} \)-family of \( t \)-structures \( t: \mathbb{R} \to \text{ts}(\mathcal{D}) \), where \( \mathbb{R} \) is endowed with the usual total order. This means that we are given \( t \)-structures \( t_\lambda = (\mathcal{D}_{> \lambda}, \mathcal{D}_{\leq \lambda}) \), one for each \( \lambda \in \mathbb{R} \), such that \( t_{\lambda+1} = t_\lambda[1] \). Slicings on
are part of the abstract definition of a \textit{t-stability} on a triangulated (or stable) category $D$, see [Bri07, GKR04].

Grouping together all the above remarks, we obtain that the gluing of two slicings $t_i: \mathbb{R} \to T_S(D^i)$ gives a slicing on $D$ every time $D^0 \rightleftarrows D \rightleftarrows D^1$ is a recollement on $D$. Moreover, if $p: \{0, 1\} \to \mathbb{Z}$ is a perversity datum, we have a corresponding notion of \textit{$p$-perverted slicing} on $D$. More generally one has a notion of $p$-perverted slicing on $D^{[0, n]}$ induced by a perversity datum $p$ and by and by a Urizen compass of slicings $G_n$.

Acknowledgements. Version 1 of the present paper is sensibly different from the present one; the unexpected (and actually undue) symmetric behavior of stable recollements (Lemma 4.3 of version 1, therein called the \textit{Rorschach lemma}\(^{(11)}\)) turned out to be the far reaching consequence of a typo in one of the commutative diagrams on page 9. This has now been corrected (i.e., Lemma 4.3, together with all its corollaries, has been removed).

Luckily, this was only minimally affecting the remaining part of the article, which has now been revised accordingly. In particular the section on the associative properties of recollements has been expanded, some additional examples have been added, and several other minor typos have been corrected.

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