The Cauchy problem and wave-breaking phenomenon for a generalized sine-type FORQ/mCH equation

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Abstract
In this paper, we are concerned with the Cauchy problem and wave-breaking phenomenon for a sine-type modified Camassa–Holm (alias sine-FORQ/mCH) equation. Employing the transport equations theory and the Littlewood–Paley theory, we first establish the local well-posedness for the strong solutions of the sine-FORQ/mCH equation in Besov spaces. In light of the Moser-type estimates, we are able to derive the blow-up criterion and the precise blow-up quantity of this equation in Sobolev spaces. We then give a sufficient condition with respect to the initial data to ensure the occurrence of the wave-breaking phenomenon by trace the precise blow-up quantity along the characteristics associated with this equation.

Keywords Sine-mCH equation · Cauchy problem · Wave breaking · Local well-posedness

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1 Introduction

The Camassa–Holm (CH) equation [8]

\[ m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \quad (1.1) \]

which describes the unidirectional propagation of shallow water waves over a flat bottom [8,29,40] or the propagation of axially symmetric waves in hyperelastic rods [25,26], has been extensively investigated in the literature. (1.1) can be derived from the celebrated Korteweg-de Vries (KdV) equation by means of the tri-Hamiltonian duality [45] and has some similar properties to the KdV equation. For instance, it has infinitely many conservation laws and is completely integrable [8,31]. Also, it can be recast as the following bi-Hamilton structure [8]

\[ m_t = J_1 \frac{\delta H_{CH,1}}{\delta m} = K_1 \frac{\delta H_{CH,2}}{\delta m}, \quad J_1 = -\frac{1}{2}(m\partial + \partial m), \quad K_1 = \partial^3 - \partial \]

where

\[ H_{CH,1} = \int_{\mathbb{R}} m u \, dx, \quad H_{CH,2} = \frac{1}{2} \int_{\mathbb{R}} (u^3 + uu_x^2) \, dx. \]

Eq. (1.1) also admits the Lax pair [8], which allows us to solve the Cauchy problem of (1.1) by the inverse scattering transform (IST) [21] as well as to construct the action angle variables of (1.1) [3,5,21,23]. However, there are some significant difference between (1.1) and KdV equation. For example, Eq. (1.1) can model the so-called wave-breaking phenomenon in nature, namely, the wave itself remains bounded, while its slope becomes unbounded in finite time [4,17,50], which can not be derived from the KdV equation. The problem concerning how the solution develops after the wave-breaking has been researched in [6,38] and [7,39], where global conservative solutions or global dissipative solutions are constructed, respectively. The CH equation (1.1) also admits a kind of solutions called peakons [4,8,16,19,20,24,42] whose shapes are stable under small perturbations [24,41]. (1.1) is also related to geometry. Firstly, it represents the geodesic flows. In the aperiodic case and when some asymptotic conditions are satisfied at infinity, Eq. (1.1) can be viewed as the geodesic flow on a manifold of diffeomorphism of the line [15]. In the periodic case, it can represent the geodesic flow on the diffeomorphism group of the circle [22]. Secondly, it can represent the families of pseudo-spherical surfaces [34,36,49]. Thirdly, it appears from a non-stretching invariant planar curve flow in the centro-equiaffine geometry [14,36]. Other mathematical facts about equation (1.1) include the existence of a recursion operator [8,9,30–32], and so on. The local-in-time well-posedness for the initial value problem of (1.1) has been achieved [18,27]. Eq. (1.1) also admits global strong solutions [15,18], and finite time blow-up strong solutions [15,17,18].

The nonlinear terms in Eq. (1.1) are quadratic, while, there do exist other CH-type equations with cubic nonlinearity, for instance, the following modified-CH (mCH)
equation (alias the Fokas-Olver-Rosenau-Qiao (FORQ) equation) \[30,32,45,46\]

\[
m_t + \left[ (u^2 - u_x^2) m \right]_x = 0, \quad m = u - u_{xx},
\]  

(1.2)

which was first derived by operating the tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation [32,45]. Later, it was rederived from the two-dimensional Euler equations [46]. It is also completely integrable. Its bi-Hamiltonian structure reads [36,45,47]

\[
m_t = J_2 \frac{\delta H_{mCH,1}}{\delta m} = K_2 \frac{\delta H_{mCH,2}}{\delta m}, \quad J_2 = -\partial_x m \partial_x^{-1} m \partial_x, \quad K_2 = \partial_x^3 - \partial_x,
\]

where the Hamiltonians are

\[
H_{mCH,1} = \int_{\mathbb{R}} m u dx, \quad H_{mCH,2} = \frac{1}{4} \int_{\mathbb{R}} (u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4) dx.
\]

Also, (1.2) can be solved by using IST since it has the following Lax pair [36,47]. From the viewpoint of geometry, Eq. (1.2) arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [36]. The local-wellposedness, blow-up and wave breaking problems of Eq. (1.2) has been investigated in [12,33,36,44]. The explicit form of the multipeakons of Eq. (1.2) has been discussed in [10]. The peakon solution of equation (1.2) is orbital stability [43,48]. The Hölder continuity of Eq. (1.2) has been addressed in [37].

In this paper, we consider the following Cauchy problem of the sine-type modified Camassa–Holm (alias the sine-FORQ/mCH) equation [1]

\[
\begin{cases}
m_t + \left[ \sin(u^2 - u_x^2) m \right]_x = 0, \quad m = u - u_{xx}, \\
m(0, x) = m_0(x),
\end{cases}
\]

(1.3)

where \(u(t, x)\) denotes the fluid velocity and \(m(t, x) = u - u_{xx}\) stands for the corresponding potential density. Note that Eq. (1.3) admits the conserved quantity \(H_1 = \int_{\mathbb{R}} m u dx\). We know that

- As \(u^2 - u_x^2 \to 0\),

\[
\sin(u^2 - u_x^2) \sim u^2 - u_x^2,
\]

(1.4)

in which Eq. (1.3) just reduces to the FORQ/mCH equation (1.2).

- As \(0 < |u^2 - u_x^2| < 1\),

\[
\sin(u^2 - u_x^2) \sim \sum_{k=1}^{N} \frac{(-1)^{k+1}}{(2k-1)!} (u^2 - u_x^2)^{2k-1} + O((u^2 - u_x^2)^{2N-1}),
\]

as \(0 < |u^2 - u_x^2| < 1\),

(1.5)
in which Eq. (1.3) becomes the higher-order FORQ/mCH equation.

To the best of our knowledge, the Cauchy problem (1.3) has not been researched yet. We aim to explore this problem in this paper. First, following the spirit of [27,28], we will prove the local well-posedness for the strong solutions of equation (1.3) in Besov spaces by the use of the Besov spaces theory and the transport equations theory. Second, the Moser-type estimates in Sobolev spaces enable us to establish a blow-up criterion and the precise blow-up quantity for (1.3). We finally propose a sufficient condition with regard to the initial data to ensure the occurrence of the wave-breaking phenomenon.

Before stating our main results, we first introduce the solution spaces $E^{s}_{p,r}(T)$ defined as follows

$$E^{s}_{p,r}(T) = \begin{cases} C \left([0, T); B^{s}_{p,r}\right) \cap C^{1} \left([0, T); B^{s-1}_{p,r}\right), & \text{if } r < \infty, \\ C_{w} \left([0, T); B^{s}_{p,\infty}\right) \cap C^{0,1} \left([0, T); B^{s-1}_{p,\infty}\right), & \text{if } r = \infty. \end{cases}$$

We next introduce some notations to be used in this paper.

Notations. Let $p(x) = \frac{1}{2} e^{-|x|}(x \in \mathbb{R})$ be the fundamental solution of $1 - \partial_{x}^{2}$ on $\mathbb{R}$, and two convolution operators $p_{\pm}$ be [13]

$$p_{\pm} \ast f(x) = \frac{1}{2} e^{\mp x} \int_{-\infty}^{\pm x} e^{y} f(\pm y) dy, \quad (1.6)$$

where the star denotes the spatial convolution. Note that $p$ and $p_{\pm}$ satisfy

$$p = p_{+} + p_{-}, \quad p_{x} = p_{-} - p_{+}, \quad p_{xx} \ast f = p \ast f - f. \quad (1.7)$$

Let $\mathcal{S}$ stand for the Schwartz space and $\mathcal{S}'$ represent the spaces of temperate distributions. Let $L^{p}(\mathbb{R})$ be the Lebesgue space equipped with the norm $\|\cdot\|_{L^{p}}$ for $1 \leq p \leq \infty$ and $H^{s}(\mathbb{R})$ be the Sobolev space equipped with the norm $\|\cdot\|_{H^{s}}$ for $s \in \mathbb{R}$.

Our first result concerning the local well-posedness to (1.3) in Besov spaces reads

**Theorem 1.1** Assume that $1 \leq p, r \leq +\infty, s > \max\{2 + 1/p, 5/2\}$. Let $u_{0} \in B^{s}_{p,r}$. Then there is a time $T > 0$ such that the Cauchy problem (1.3) admits a unique solution $u \in E^{s}_{p,r}(T)$. Furthermore, the data-to-solution map $u_{0} \mapsto u$ is continuous from a neighborhood of $u_{0}$ in $B^{s}_{p,r}$ into

$$C \left([0, T); B^{s'}_{p,r}\right) \cap C^{1} \left([0, T); B^{s'-1}_{p,r}\right)$$

for each $s' < s$ when $r = +\infty$ and $s' = s$ when $r < +\infty$.

Taking $p = r = 2$ in Theorem 1.1, one immediately obtains the following Corollary concerning the local well-posedness of (1.3) in the Sobolev spaces setting, which is more convenience for us to establish our blow-up results.

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Corollary 1.1 Suppose $s > 5/2$ and $u_0 \in H^s$. Then there is a time $T > 0$ such that the Cauchy problem (1.3) admits a unique strong solution $u \in C ([0, T]; H^s) \cap C^1 ([0, T]; H^{s-1})$. Moreover, the data-to-solution map $u_0 \mapsto u$ is continuous from a neighborhood of $u_0$ in $H^s$ into $C ([0, T]; H^s) \cap C^1 ([0, T]; H^{s-1})$.

We next state our result on the finite time blow-up of the strong solution to (1.3).

Theorem 1.2 Let $u_0 \in H^s$ be given as in Corollary 1.1 and $u$ be the corresponding solution to (1.3). Denote by $T^*$ the maximal existence time, then

$$T^* < \infty \Rightarrow \int_0^{T^*} \| m \|^2_{L^\infty} dt = \infty. \quad (1.8)$$

Remark 1.1 Theorems 1.1 and 1.2 also hold true for the sine-mCH equation with the linear dispersive term

$$\begin{cases} m_t + \kappa u_x + [\sin(u^2 - u^2_x)] m_x = 0, & m = u - u_{xx}, \\ m(0, x) = m_0(x), \end{cases} \quad (1.9)$$

where $\kappa$ is the real-valued parameter.

Based on Theorem 1.2, we will establish the following result concerning the wave-breaking phenomenon on the solution of the Cauchy problem (1.3).

Theorem 1.3 Suppose $m_0 \in H^s(\mathbb{R})$ with $s > 1/2$. Let $T^* > 0$ be the maximal existence time of strong solution $m$ to the Cauchy problem (1.3). Let $M(t, x)$ be defined as in (5.7). Set $\bar{M}(t) = M(t, q(t, x_0))$ and $\bar{m}(t) = m(t, q(t, x_0))$, where $q(t, x_0)$ is defined in (5.1). Let $m_0(x) \geq 0$ for all $x \in \mathbb{R}$, and $m_0(x_0) > 0$ for some $x_0 \in \mathbb{R}$. If

$$\bar{M}(0) < 0 \quad \text{and} \quad \frac{\bar{M}(0)}{\bar{m}(0)} \xi + \frac{1}{2} C_1 \xi^2 + \frac{1}{\bar{m}(0)} < 0, \quad (1.10)$$

where $C_1$ is defined below (5.14) and

$$\xi = -\frac{\bar{M}(0)}{C_1 \bar{m}(0)}, \quad (1.11)$$

then the solution $m$ blows up at a time $T^* \in (0, \xi)$.

The rest of this paper is organized as follows. In Sect. 2, we will briefly recall the properties of Besov spaces and some Lemmas on the transport equation theory. Section 3 will deal with the proof of Theorem 1.1. In Sect. 4, the Moser-type estimates in Sobolev spaces will be used to prove Theorem 1.2. Section 5 will provide the proof of the wave-breaking Theorem 1.3 by tracing the precise blow-up quantity along the characteristic associated to (1.3).
2 Preliminaries

Since the local well-posedness for the Cauchy problem (1.3) will be established in Besov-type spaces, we firstly recall some basic properties about the Littlewood–Paley theory and some useful lemmas of the transport equation theory.

2.1 Basics properties of the Littlewood–Paley theory

The definition of Besov spaces and some of their properties will be briefly exhibited in this subsection (see [2,11] for more details).

Let $B(x_0, r)$ be the open ball centered at $x_0$ with radius $r$, $C \equiv \{ \xi \in \mathbb{R}^d \mid |\xi| \leq 8/3 \}$, and $\tilde{C} \equiv B(0, 2/3) + C$. Then there are two radial functions $\chi \in \mathcal{D}(B(0, 4/3))$ and $\varphi \in \mathcal{D}(C)$ satisfying

\[
\left\{ \begin{array}{ll}
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, & 1/3 \leq \chi^2(\xi) + \sum_{q \geq 0} \varphi^2(2^{-q} \xi) \leq 1 \quad (\forall \xi \in \mathbb{R}^d), \\
|q - q'| \geq 2 \Rightarrow \text{Supp} \varphi(2^{-q} \cdot) \cap \text{Supp} \varphi(2^{-q'} \cdot) = \emptyset, & q \geq 1 \Rightarrow \text{Supp} \chi(\cdot) \cap \text{Supp} \varphi(2^{-q} \cdot) = \emptyset, \\
|q - q'| \geq 5 \Rightarrow 2^q \tilde{C} \cap 2^q C = \emptyset.
\end{array} \right.
\]

The dyadic operators $\Delta_q$ and $S_q$ acting on $u(t, x) \in S'(\mathbb{R}^d)$ are defined as

\[
\Delta_q u = \begin{cases} 0, & q \leq -2, \\
\chi(D)u = \int_{\mathbb{R}^d} \tilde{h}(y) u(x-y)dy, & q = -1, \\
\varphi(2^{-q} D) u = 2^{qd} \int_{\mathbb{R}^d} h(2^q y) u(x-y)dy, & q \geq 0,
\end{cases}
\]

\[
S_q u = \sum_{q' \leq q - 1} \Delta_{q'} u,
\]

where $h = \mathcal{F}^{-1} \varphi$ and $\tilde{h} = \mathcal{F}^{-1} \chi$ with $\mathcal{F}^{-1}$ denoting the inverse Fourier transform.

The Besov spaces is $B^s_{p,r}(\mathbb{R}^d) = \{ u \in S' \mid \| u \|_{B^s_{p,r}(\mathbb{R}^d)} = (\sum_{j \geq -1} 2^{rs_j} \| \Delta_j u \|_{L^p(\mathbb{R}^d)})^{1/r} < \infty \}$. With the above-defined Besov spaces, we next recall some of their properties.

**Lemma 2.1** (Embedding property) [2,11] Suppose $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq r_1 \leq r_2 \leq \infty$ and $s$ be real. Then it holds that $B^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow B^{s-d(1/p_1-1/p_2)}_{p_2,r_2}(\mathbb{R}^d)$. If $s > d/p$ or $s = d/p$, $r = 1$, then there holds $B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.

**Lemma 2.2** (Interpolation) [2,11] Let $s_1, s_2$ be real numbers with $s_1 < s_2$ and $\theta \in (0, 1)$. Then there exists a constant $C$ such that

\[
\| u \|_{B^{s_1 + (1-\theta)s_2}_{p,r}} \leq \| u \|_{B^{s_2}_{p_1,r}}^{\theta} \| u \|_{B^{s_2}_{p_2,r}}^{1-\theta},
\]

\[
\| u \|_{B^{s_1 + (1-\theta)s_2}_{p_1,r}} \leq \frac{C}{s_2 - s_1} \frac{1}{\theta(1-\theta)} \| u \|_{B^{s_2}_{p,r}}^{\theta} \| u \|_{B^{s_2}_{p,n}}^{(1-\theta)},
\]

where $(p, r) \in [1, \infty]^2$. 

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Lemma 2.3 (Product law) [2,11] Let \((p, r) \in [1, \infty]^2\) and \(s\) be real. Then
\[
\|uv\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C (\|u\|_{L^\infty(\mathbb{R}^d)} \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)}) + \|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)}),
\]
namely, the space \(L^\infty(\mathbb{R}^d) \cap B^{s}_{p,r}(\mathbb{R}^d)\) is an algebra. Moreover, if \(s > d/p\) or \(s = d/p, r = 1\), then there holds
\[
\|uv\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C \|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \|v\|_{B^{s}_{p,r}(\mathbb{R}^d)}.
\]

Lemma 2.4 (Moser-type estimates) [2,27] Let \(s > \max\{d/p, d/2\}\) and \((p, r) \in [1, \infty]^2\). Then, for any \(a \in B^{s-1}_{p,r}(\mathbb{R}^d)\) and \(b \in B^{s}_{p,r}(\mathbb{R}^d)\), there holds
\[
\|ab\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \leq C \|a\|_{B^{s-1}_{p,r}(\mathbb{R}^d)} \|b\|_{B^{s}_{p,r}(\mathbb{R}^d)}.
\]

The following Lemma is useful for proving the blow-up criterion.

Lemma 2.5 (Moser-type estimates) [35,36] Let \(s \geq 0\). Then one has
\[
\|fg\|_{H^s(\mathbb{R})} \leq C (\|f\|_{H^s(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|g\|_{H^s(\mathbb{R})}),
\]
\[
\|f \partial_x g\|_{H^s(\mathbb{R})} \leq C (\|f\|_{H^{s+1}(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|\partial_x g\|_{H^s(\mathbb{R})}),
\]
where \(C\)'s are constants independent of \(f\) and \(g\).

Lemma 2.6 [2] Let \(s \in \mathbb{R}\) and \(1 \leq p, r \leq \infty\). Then the Besov spaces have the following properties:

- \(B^{s}_{p,r}(\mathbb{R}^d)\) is a Banach space and continuously embedding into \(S'(\mathbb{R}^d)\), where \(S'(\mathbb{R}^d)\) is the dual space of the Schwartz space \(S(\mathbb{R}^d)\);
- If \(p, r < \infty\), then \(S(\mathbb{R}^d)\) is dense in \(B^{s}_{p,r}(\mathbb{R}^d)\);
- If \(u_n\) is a bounded sequence of \(B^{s}_{p,r}(\mathbb{R}^d)\), then an element \(u \in B^{s}_{p,r}(\mathbb{R}^d)\) and a subsequence \(u_{n_k}\) exist such that \(\lim_{k \to \infty} u_{n_k} = u\) in \(S'(\mathbb{R}^d)\) and \(\|u\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq C \lim \inf_{k \to \infty} \|u_{n_k}\|_{B^{s}_{p,r}(\mathbb{R}^d)}\).

2.2 Some lemmas in the theory of the transport equation

We recall some a priori estimates [2,27] for the following transport equation
\[
\begin{cases}
  f_t + v \cdot \nabla f = g, \\
  f|_{t=0} = f_0.
\end{cases}
\]

Lemma 2.7 [2,27] Let \(1 \leq p \leq p_1 < \infty\), \(1 \leq r \leq \infty\) and \(s \geq -d \min(1/p_1, 1 - 1/p)\). Let \(f_0 \in B^{s}_{p,r}(\mathbb{R}^d)\), \(g \in L^1([0, T]; B^{s}_{p,r}(\mathbb{R}^d))\) and \(\nabla v \in L^1([0, T]; B^{s}_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))\), then there exists a unique solution \(f \in L^\infty([0, T]; B^{s}_{p,r}(\mathbb{R}^d))\) to Eq. (2.1) satisfying:
\[
\|f\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq \|f_0\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \int_0^t \|g(t')\|_{B^{s}_{p,r}(\mathbb{R}^d)} + C V_{p_1}(t') \|f(t')\|_{B^{s}_{p,r}(\mathbb{R}^d)} dt',
\]
\[
\|f\|_{B^{s}_{p,r}(\mathbb{R}^d)} \leq \left[\|f_0\|_{B^{s}_{p,r}(\mathbb{R}^d)} + \int_0^t \|g(t')\|_{B^{s}_{p,r}(\mathbb{R}^d)} e^{-C V_{p_1}(t')} dt'\right] e^{C V_{p_1}(t)},
\]
where \( V_{p_1}(t) = \int_0^t \| \nabla v \|_{B^{d/p_1}_p,\infty(\mathbb{R}^d)} dt' \) if \( s < 1 + d/p_1 \), \( V_{p_1}(t) = \int_0^t \| \nabla v \|_{B^{d-1}_p,\infty(\mathbb{R}^d)} dt' \) if \( s > 1 + d/p_1 \) or \( s = 1 + d/p_1, r = 1 \), and \( C \) is a constant depending only on \( s, p, p_1 \), and \( r \).

**Lemma 2.8** [2] Let \( s \geq -d \min(1/p_1, 1 - 1/p) \). Let \( f_0 \in B^{s}_{p,r}(\mathbb{R}^d) \), \( g \in L^1([0, T]; B^{s}_{p,1}(\mathbb{R}^d)) \) and \( v \in L^d([0, T]; B^{M}_{\infty,\infty}(\mathbb{R}^d)) \) for some \( \rho > 1 \) and \( M > 0 \) be a time-dependent vector field satisfying

\[
\nabla v \in \begin{cases} L^1([0, T]; B^{d/p}_{p_1,\infty}(\mathbb{R}^d)), & \text{if } s < 1 + d/p_1, \\
L^1([0, T]; B^{s-1}_{p_1,\infty}(\mathbb{R}^d)), & \text{if } s > 1 + d/p_1 \text{ or } s = 1 + d/p_1 \text{ and } r = 1. 
\end{cases}
\]

Then, Eq. (2.1) has a unique solution \( f \in C([0, T]; B^s_{p,r}(\mathbb{R}^d)) \) for \( r < \infty \), or \( f \in (\bigcap_{s' \leq s} C([0, T]; B^{s'}_{p,\infty}(\mathbb{R}^d))) \cap C_w([0, T]; B^s_{p,\infty}(\mathbb{R}^d)) \) for \( r = \infty \). Furthermore, the inequalities (2.2)-(2.3) hold.

**Lemma 2.9** (A priori estimate in the Sobolev spaces) [2,35] Let \( 0 \leq \sigma < 1 \). Let \( f_0 \in H^\sigma \), \( g \in L^1(0, T; H^\sigma) \) and \( \partial_x v \in L^1(0, T; L^\infty) \). Then the solution \( f \) to Eq. (2.1) belongs to \( C([0, T]; H^{\sigma'}) \). More precisely, there is a constant \( C \) depending only on \( \sigma \) such that

\[
\| f \|_{H^\sigma} \leq \| f_0 \|_{H^\sigma} + \int_0^t \| g(\tau) \|_{H^\sigma} + C V'(\tau) \| f(\tau) \|_{H^\sigma} \, d\tau, \quad V(t) = \int_0^t \| \partial_x v(\tau) \|_{L^\infty} \, d\tau.
\]

### 3 Local well-posedness

In this section, we will give the proof of the local well-posedness in Besov spaces to (1.3), namely, Theorem 1.1.

**Proof** We first employ the classical Friedrichs regularization method to construct the approximate solutions to (1.3). Let \( m^{(l+1)} \) solve the following linear transport equation inductively

\[
\begin{align*}
\partial_t m^{(l+1)} + \sin[(u^{(l)}_x)^2 - (u^{(l)}_x)^2] \partial_x m^{(l+1)} &= -2\cos[(u^{(l)}_x)^2 - (u^{(l)}_x)^2] u^{(l)}_x (m^{(l)})^2, \\
m^{(l+1)}_{|t=0} &= m^{(l+1)}_0(x) = S_{l+1} m_0,
\end{align*}
\]

where \( m^{(0)} := 0 \).

Suppose \( m^{(l)} \in L^\infty(0, T; B^{s-2}_{p,r}) \), where \( s - 2 > \max\left\{ \frac{1}{p}, \frac{1}{2} \right\} \) and consequently \( B^{s-2}_{p,r} \) is an algebra. So the right hand side of Eq. (3.1) is in \( L^\infty(0, T; B^{s-2}_{p,r}) \). Hence, by Lemma 2.8 and the high regularity of \( u \), Eq. (3.1) admits a global solution \( m^{(l+1)} \in E^{s-2}_{p,r} \) for all positive \( T \).
Applying (2.3) in Lemma 2.7 to Eq. (3.1) yields

$$\|m^{(l+1)}(t)\|_{B^t_{p,r}} \leq e^{C \int_0^t C \int_0^t \|u(t)\|_{B^t_{p,r}}^2 \, dt} \|m_0\|_{B^t_{p,r}} + C \int_0^t e^{C \int_0^t \|u(t)\|_{B^t_{p,r}}^2 \, dt} \|u_0\|_{B^t_{p,r}} \, dt \cdot$$

for $l = 0, 1, 2, \ldots$

Using Lemma 2.3 concerning the product law in Besov spaces, one obtains

$$\|\sin [(u(t))^2 - (u^l(x))^2]\|_{B^t_{p,r}} \leq \|u(t)^2 - (u^l(x))^2\|_{B^t_{p,r}} \leq \|u(t)^2\|_{B^t_{p,r}}, \quad (3.3)$$

$$\|2\cos [(u(t))^2 - (u^l(x))^2]u_x^l(m(t))^2\|_{B^t_{p,r}} \leq C \|u_x^l(m(t))^2\|_{B^t_{p,r}} \leq C \|u(t)^3\|_{B^t_{p,r}}. \quad (3.4)$$

Substituting (3.3)-(3.4) to (3.1), one finds

$$\|u^{(l+1)}(t)\|_{B^t_{p,r}} \leq e^{C \int_0^t \|u(t)\|_{B^t_{p,r}}^2 \, dt} \|u_0\|_{B^t_{p,r}} + C \int_0^t e^{C \int_0^t \|u(t)\|_{B^t_{p,r}}^2 \, dt} \|u(t)^3\|_{B^t_{p,r}} \, dt \cdot$$

Now, assume $\|u(t)\|_{B^t_{p,r}} \leq a(t)$. Plugging this assumption into (3.5) leads to

$$\|u^{(l+1)}(t)\|_{B^t_{p,r}} \leq e^{C \int_0^t a^2 \, dt} \|u_0\|_{B^t_{p,r}} + C \int_0^t e^{C \int_0^t a^2 \, dt} a^3 \, dt. \quad (3.6)$$

Let the right hand side of inequality (3.6) be equal to $a$, we obtain

$$\begin{cases}
\dot{a} = 2Ca^3, \\
a(0) = \|u_0\|_{B^t_{p,r}}.
\end{cases}$$

Solving this ordinary differential equation, one deduces

$$a(t) = \|u_0\|_{B^t_{p,r}} [1 - 4Ct\|u_0\|_{B^t_{p,r}}^2]^{-1/2}.$$

Therefore, we conclude that the solution sequence $\{u(t)\}_{t=1}^{\infty}$ of Eq. (3.1) is uniformly bounded in $C([0, T]; B^t_{p,r})$ with

$$T < \frac{1}{4C\|u_0\|_{B^t_{p,r}}^2}.$$
Next, we shall prove that \( (m^{(l+1)}_{i=1})^\infty \) is a Cauchy sequence in \( C([0, T]; B_{p,r}^{s-3}) \). In fact, from Eq. (3.1), one derives

\[
\partial_t [m^{(l+1)} - m^{(l+1)}] + \sin [(u^{(l+1)})^2 - (u^{(l+1)})^2] \partial_x [m^{(l+1)} - m^{(l+1)}] \\
= \left\{ \sin [(u^{(l+1)})^2 - (u^{(l+1)})^2] - \sin [(u^{(l+1)})^2 - (u^{(l+1)})^2] \right\} \partial_x m^{(l+1)} \\
- 2 \left\{ \cos [(u^{(l+1)})^2 - (u^{(l+1)})^2] u_x^{(l+1)} (m^{(l+1)})^2 - \cos [(u^{(l+1)})^2 - (u^{(l+1)})^2] u_x^{(l+1)} (m^{(l+1)})^2 \right\} := g.
\]

(3.7)

As a consequence of Lemma 2.7, one deduces

\[
\|m^{(l+1)} - m^{(l+1)}\|_{B_{p,r}^{s-3}} \leq \exp \left\{ C \int_0^t \|\sin [(u^{(l+1)})^2 - (u^{(l+1)})^2]\|_{B_{p,r}^{s-3}} d\tau \right\} \\
\times \left\{ \|m^{(l+1)} - m^{(l+1)}\|_{B_{p,r}^{s-3}} \right\} \\
+ \int_0^t \exp \left\{ - C \int_0^\tau \|\sin [(u^{(l+1)})^2 - (u^{(l+1)})^2]\|_{B_{p,r}^{s-3}} d\tau \right\} \|g\|_{B_{p,r}^{s-3}} d\tau \right\}. \quad (3.8)
\]

Lemma 2.3 enables us to conclude

\[
\|\sin [(u^{(l)})^2 - (u_x^{(l)})^2] - \sin [(u^{(l+1)})^2 - (u_x^{(l+1)})^2]\|_{B_{p,r}^{s-3}} m^{(l+1)} \\
\leq C \|\partial_x m^{(l+1)}\|_{B_{p,r}^{s-3}} \|\sin [(u^{(l)})^2 - (u_x^{(l)})^2] - \sin [(u^{(l+1)})^2 - (u_x^{(l+1)})^2]\|_{B_{p,r}^{s-2}} \\
\leq C \|u^{(l+1)}\|_{B_{p,r}^{s}} \|2\cos \left[ \frac{(u^{(l+1)})^2 - (u_x^{(l+1)})^2 + (u^{(l)})^2 - (u_x^{(l)})^2}{2} \right]\|_{B_{p,r}^{s-2}} \\
\times \sin \left[ \frac{(u^{(l+1)})^2 - (u_x^{(l+1)})^2}{2} \right] \|u^{(l+1)}\|_{B_{p,r}^{s}} \|u^{(l+1)}\|_{B_{p,r}^{s}} \right\} \|u^{(l+1)}\|_{B_{p,r}^{s-1}} \|u^{(l+1)}\|_{B_{p,r}^{s}} \right\}. \quad (3.9)
\]

and

\[
\|\cos [(u^{(l+1)})^2 - (u_x^{(l+1)})^2] u_x^{(l+1)} (m^{(l+1)})^2 - \cos [(u^{(l)})^2 - (u_x^{(l)})^2] u_x^{(l)} (m^{(l)})^2 \|_{B_{p,r}^{s-3}} \\
\leq \|\cos [(u^{(l+1)})^2 - (u_x^{(l+1)})^2] - \cos [(u^{(l)})^2 - (u_x^{(l)})^2]\| u_x^{(l+1)} (m^{(l+1)})^2 \|_{B_{p,r}^{s-3}} \\
+ \|\cos [(u^{(l)})^2 - (u_x^{(l)})^2] u_x^{(l)} (m^{(l+1)})^2 \|_{B_{p,r}^{s-3}} \\
+ \|\cos [(u^{(l)})^2 - (u_x^{(l)})^2] u_x^{(l)} (m^{(l+1)})^2 - (m^{(l+1)})^2 \|_{B_{p,r}^{s-3}}
\]

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\[ \leq \|m^{(l+i)}\|_{B^{s-3}_{p,r}} \|m^{(l+i)}\|_{B^{s-3}_{p,r}} \|u^{(l+i)}\|_{B^{s-2}_{p,r}} \times 2\sin \left[ \frac{(u^{(l+i)})^2 - (u_x^{(l+i)})^2 + (u^{(l)})^2 - (u_x^{(l)})^2}{2} \right] \]
\[ \sin \left[ \frac{(u^{(l+i)})^2 - (u_x^{(l+i)})^2 - (u^{(l)})^2 + (u_x^{(l)})^2}{2} \right] \|B^{s-2}_{p,r} \]
\[ + \|m^{(l+i)}\|_{B^{s-3}_{p,r}} \|m^{(l+i)}\|_{B^{s-2}_{p,r}} \|u^{(l+i)} - u_x^{(l+i)}\|_{B^{s-2}_{p,r}} \]
\[ + \|m^{(l+i)} - m^{(l)}\|_{B^{s-3}_{p,r}} \|m^{(l+i)} + m^{(l)}\|_{B^{s-2}_{p,r}} \|u^{(l+i)}\|_{B^{s-2}_{p,r}} \]
\[ \leq C\|u^{(l+i)}\|_{B^{s}_{p,r}}^3 \left( \|u^{(l+i)}\|_{B^{s}_{p,r}}^2 - (u^{(l)})^2 \right) + \|u^{(l+i)} - u^{(l)}\|_{B^{s}_{p,r}}^2 - (u_x^{(l)})^2 \|B^{s-2}_{p,r} \]
\[ + C\|u^{(l+i)}\|_{B^{s}_{p,r}}^2 \|u^{(l+i)} - u^{(l)}\|_{B^{s-1}_{p,r}} \]
\[ + C\|u^{(l+i)}\|_{B^{s}_{p,r}}^2 \|u^{(l+i)} - u^{(l)}\|_{B^{s}_{p,r}} \]
\[ + C\|u^{(l+i)}\|_{B^{s}_{p,r}}^2 \|u^{(l+i)} - u^{(l)}\|_{B^{s-1}_{p,r}} \]
\[ + C\|u^{(l+i)}\|_{B^{s}_{p,r}}^2 \|u^{(l+i)} - u^{(l)}\|_{B^{s}_{p,r}} \]
\[ (3.10) \]

Plugging (3.9)-(3.10) into (3.8) leads to
\[ \|u^{(l+i+1)} - u^{(l+1)}\|_{B^{s-1}_{p,r}} \leq CT \left( 2^{-l} + \int_{0}^{t} \|u^{(l+i)} - u^{(l)}\|_{B^{s-1}_{p,r}} \, dt \right). \]  (3.11)

Next, notice that
\[ \|m^{(l+i+1)} - m^{(l+1)}\|_{B^{s-3}_{p,r}} = \|S_{l+i+1}m_0 - S_{l+1}m_0\|_{B^{s-3}_{p,r}} \]
\[ = \|\sum_{q=0}^{l+i+1} \Delta_q m_0\|_{B^{s-3}_{p,r}} \leq C2^{-l}\|m_0\|_{B^{s-3}_{p,r}} \] (3.12)

and \( \{m^{(l)}\} \) is bounded in \( C([0, T]; B^{s-2}_{p,r}) \), we thus deduce
\[ \|m^{(l+i+1)} - m^{(l+1)}\|_{B^{s-3}_{p,r}} \leq CT \left( 2^{-l} + \int_{0}^{t} \|m^{(l+i)} - m^{(l)}\|_{B^{s-1}_{p,r}} \, dt \right). \] (3.13)

So we conclude that
\[ \|m^{(l+i+1)} - m^{(l+1)}\|_{C(0, T; B^{s-3}_{p,r})} \leq C_T \left( \frac{2^{-l}}{2} \sum_{k=0}^{l} \frac{(2TC_T)^k}{k!} \right) \]
\[ + \frac{(TC_T)^{l+1}}{(l+1)!} \|m^{(l)} - m^{(0)}\|_{C(0, T; B^{s-3}_{p,r})}. \] (3.14)
Since \( \{m^{(i)}\} \) is uniformly bounded in \( C(0, T; B^s_{p,r}) \), one can find a new constant \( C'_T \) so that

\[
\|m^{(i+1)} - m^{(i)}\|_{C(0,T;B^s_{p,r})} \leq \frac{C'_T}{2^n}.
\]

Accordingly, \( \{m^{(m)}\} \) is Cauchy in \( C(0, T; B^s_{p,r}) \) and converges to some limit function 
\( m \in C(0, T; B^s_{p,r}) \).

We next show the existence of the solution to equation (1.3) by proving that the obtained limit function \( m \) satisfies Eq. (1.3) in the sense of distribution and belongs to \( E^s_{p,r} \).

Firstly, using Lemma 2.6(iii) and the uniform boundedness of \( \{m^{(i)}\} \) in \( L^\infty(0, T; B^{s-2}_{p,r}) \), one derives that \( m \in L^\infty(0, T; B^{s-2}_{p,r}) \).

Secondly, we claim that \( \{m^{(i)}\} \) converges to \( m \) in \( C(0, T; B^{s'}_{p,r}) \) for all \( s' < s - 2 \). In fact, this is the consequence of the following statement: \( \|m_l - m\|_{B^{s'}_{p,r}} \leq C \|m_l - m\|_{B^{s-3}_{p,r}} \) when \( s' \leq s - 3 \) and \( \|m_l - m\|_{B^{s-2}_{p,r}} \leq C \|m_l - m\|_{B^{s-2}_{p,r}}^\theta \) with \( \theta = s - 2 \) or \( s - 3 < s' \leq s - 2 \) recalling Lemma 2.2. This claim enables us to take limit in equation (3.1) to conclude that the limit function \( m \) does satisfy Eq. (1.3).

Note that Eq. (1.3) can be recast as a transport equation

\[
\partial_t m + \sin(u^2 - u_x^2)\partial_x m = -2\cos(u^2 - u_x^2)u_x m. \tag{3.15}
\]

Since \( m \in L^\infty(0, T; B^{s-2}_{p,r}) \), it is easy to deduce that the right hand side of the above equation also belongs to \( L^\infty(0, T; B^{s-2}_{p,r}) \) in view of the product law in Besov spaces and the Sobolev embedding. Consequently, Lemma 2.8 implies \( m \in C([0, T); B^{s-2}_{p,r}) \) when \( r < \infty \) or \( m \in C_w([0, T); B^{s-2}_{p,r}) \) when \( r = \infty \). On the other hand, from the Moser-type estimates in Lemma 2.4, we infer that \( [\sin(u^2 - u_x^2)]\partial_x m \) is bounded in \( L^\infty(0, T; B^{s-3}_{p,r}) \). Therefore, one knows \( \partial_t m \in C([0, T); B^{s-3}_{p,r}) \) when \( r < \infty \) according to the high regularity of \( u \) and Eq. (1.3) and thus \( m \in E^{s-2}_{p,r} \).

Furthermore, a standard use of a sequence of viscosity approximate solutions \( \{u_\epsilon\}_{\epsilon > 0} \) for (1.3) which converges uniformly in \( C([0, T]; B^{s-2}_{p,r}) \cap C^1([0, T]; B^{s-3}_{p,r}) \) implies the continuity of the solution \( m \) in \( E^{s-2}_{p,r}(T) \).

To complete the proof of Theorem 1.1, we next show the uniqueness.

Let \( m = u - u_{xx} \) and \( n = v - v_{xx} \) both be solution to (1.3). Then we have

\[
\partial_t (m - n) + \sin(u^2 - u_x^2)\partial_x (m - n)
= -2[\cos[u^2 - u_x^2]u_x m - \cos(v^2 - v_x^2)v_x n] - [\sin(v^2 - v_x^2) - \sin(u^2 - u_x^2)]n_x := f, \tag{3.16}
\]

where \((m - n)(0) = m_0 - n_0 = 0\).
Using (2.2) in Lemma 2.7, one finds
\[
\|m - n\|_{B^{s-3}_{p,r}} \leq \|m_0 - n_0\|_{B^{s-3}_{p,r}} + C \int_0^t \|m - n\|_{B^{s-3}_{p,r}} \|\partial_x \sin[u^2 - u_x^2]\|_{B^{s-2}_{p,r}} \, d\tau \\
+ C \int_0^t \|f\|_{B^{s-3}_{p,r}} \, d\tau.
\] (3.17)

Again, thanks to the product law in the Besov spaces stated in Lemma 2.3 and the embedding relation in Lemma 2.1, one deduces
\[
\|\partial_x \sin[u^2 - u_x^2]\|_{B^{s-2}_{p,r}} \leq \|\sin(u^2 - u_x^2)\|_{B^{s-1}_{p,r}} \leq \|u^2 - u_x^2\|_{B^{s-1}_{p,r}} \leq C \|u\|_{B^{s}_{p,r}}^2.
\] (3.18)

The Moser-type estimates in Lemma 2.4 leads to
\[
\|\{\cos(u^2 - u_x^2)u_x m^2 - \cos(v^2 - v_x^2)v_x n^2\}\|_{B^{s-3}_{p,r}} \\
\leq C \|\{\cos(u^2 - u_x^2) - \cos(v^2 - v_x^2)\}u_x m^2\|_{B^{s-3}_{p,r}} + C \|\cos(v^2 - v_x^2)u_x - v_x \|_{B^{s-3}_{p,r}} \\
+ C \|\sin(2u^2 - 2u_x^2)v_x(m^2 - n^2)\|_{B^{s-3}_{p,r}} \\
\leq C \|\frac{u^2 - u_x^2 + v^2 - v_x^2}{2} \sin(2u^2 - 2u_x^2 + v^2 - v_x^2)\|_{B^{s-3}_{p,r}} \\
+ \|u\|_{B^{s-1}_{p,r}} \|m\|_{B^{s-2}_{p,r}} \|u_x - v_x\|_{B^{s-2}_{p,r}} + \|v\|_{B^{s-1}_{p,r}} \|m - n\|_{B^{s-3}_{p,r}} \|m + n\|_{B^{s-2}_{p,r}} \\
\leq C \|m\|_{B^{s-1}_{p,r}} \|u\|_{B^{s-2}_{p,r}} + C \|v\|_{B^{s-1}_{p,r}} \|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}} \|u - v\|_{B^{s-1}_{p,r}}
\] (3.19)

and
\[
\|\{\sin(u^2 - u_x^2) - \sin(v^2 - v_x^2)\}n_x\|_{B^{s-3}_{p,r}} \\
\leq C \|n_x\|_{B^{s-3}_{p,r}} \|\cos((u^2 - u_x^2 + v^2 - v_x^2)/2)\sin((u^2 - u_x^2 - v^2 + v_x^2)/2)\|_{B^{s-2}_{p,r}} \\
\leq C \|n_x\|_{B^{s-3}_{p,r}} \|u - v\|_{B^{s-1}_{p,r}} \|u\|_{B^{s}_{p,r}}^2 + \|v\|_{B^{s}_{p,r}}^2.
\] (3.20)

Substituting (3.18)–(3.20) into (3.17) yields
\[
\|u - v\|_{B^{s-1}_{p,r}} \leq \|u_0 - v_0\|_{B^{s-1}_{p,r}} + C \int_0^t \|u - v\|_{B^{s-1}_{p,r}} \|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}}^4 \, d\tau.
\] (3.21)

The Gronwall inequality then implies \(u = v\) or \(m = n\). We thus complete the proof of Theorem 1.1.
4 Blow-up criterion

The proof of the blow-up criterion for the solution to the Cauchy problem (1.3) will be provided in this section.

**Proof** The proof of Theorem 1.2 will be divided into three steps. The method is mainly induction with respect to $s$. Let us recall that Eq. (1.3) can be recast as

$$m_t + \sin(u^2 - u_x^2)\partial_x m = -2\cos(u^2 - u_x^2)u_x m^2. \quad (4.1)$$

**Step 1:** When $s \in (1/2, 1)$, applying Lemma 2.9 to (4.1) yields

$$\|m\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|m\|_{H^s}\|\partial_x \sin(u^2 - u_x^2)\|_{L^\infty} d\tau$$

$$+ C \int_0^t \|2\cos(u^2 - u_x^2)u_x m^2\|_{H^s} d\tau. \quad (4.2)$$

Using $u = (1 - \partial_x^2)^{-1}m = p * m$ and $\|p\|_{L^1} = \|\partial_x p\|_{L^1} = 1$, one finds after employing the Young inequality that for $s \in \mathbb{R}$

$$\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} \leq C\|m\|_{L^\infty},$$

$$\|u\|_{H^s} + \|u_x\|_{H^s} + \|u_{xx}\|_{H^s} \leq C\|m\|_{H^s}. \quad (4.3)$$

Using (4.3), one derives

$$\|\partial_x \sin[u^2 - u_x^2]\|_{L^\infty} = \|2\cos[u^2 - u_x^2]u_x m\|_{L^\infty} \leq C\|m\|_{L^\infty}^2 \quad (4.4)$$

and

$$\|2\cos[u^2 - u_x^2]u_x m^2\|_{H^s} \leq C\|u_x m^2\|_{H^s}$$

$$\leq C\|m\|_{L^\infty}\|u_x m\|_{H^s} + C\|m\|_{H^s}\|u_{xx} m\|_{L^\infty}$$

$$\leq C\|m\|_{L^\infty}\|m\|_{H^s}. \quad (4.5)$$

Combining (4.2), (4.4) and (4.5), one finds

$$\|m\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|m\|_{H^s}\|m\|_{L^\infty}^2 d\tau \quad (4.6)$$

or

$$\|m\|_{H^s} \leq \|m_0\|_{H^s} e^{C \int_0^t \|m\|_{L^\infty}^2 d\tau}. \quad (4.7)$$
Consequently, if \( \int_0^{T^*} \|m(\tau)\|^2_{L^\infty} d\tau < \infty \) for the maximal existence time \( T^* < \infty \), then the inequality (4.7) implies that \( \lim \sup_{t \to T^*} \|m(t)\|_{H^s} < \infty \), which contradicts our assumption on \( T^* \). Thus we complete the proof of Theorem 1.2 for the case \( s \in (1/2, 1) \).

**Step 2:** For \( s \in [1, 2) \), after differentiating (4.1) once with respect to \( x \), we obtain

\[
m_{xt} + \sin(u^2 - u_x^2)\partial_x m = -2\cos(u^2 - u_x^2)u_x mm_x - 2\partial_x[\cos(u^2 - u_x^2)u_x m^2].
\]

(4.8)

Lemma 2.9 again yields

\[
\|m_x\|_{H^{s-1}} \leq \|m_{0x}\|_{H^{s-1}} + C \int_0^t \|m_x\|_{H^{s-1}} \|\sin(u^2 - u_x^2)\|_{L^\infty} d\tau + C \int_0^t \| -2\cos(u^2 - u_x^2)u_x mm_x - 2\partial_x[\cos(u^2 - u_x^2)u_x m^2]\|_{H^{s-1}} d\tau.
\]

(4.9)

Simple computation leads to

\[
\| -2\cos(u^2 - u_x^2)u_x mm_x - 2\partial_x[\cos(u^2 - u_x^2)u_x m^2]\|_{H^{s-1}} \leq C \|u_x mm_x\|_{H^{s-1}} \|u_x m^2\|_{H^s}
\leq C \|\partial_x(u_x m^2)\|_{H^{s-1}} + C \|u_x m^2\|_{H^{s-1}} + C \|m\|_{H^s} \|m\|_{L^\infty}^2
\leq C \|m\|_{H^s} \|m\|_{H^s}^2_{L^\infty}.
\]

(4.10)

Combining (4.4) and (4.9)–(4.10), one deduces

\[
\|m_x\|_{H^{s-1}} \leq \|m_{0x}\|_{H^{s-1}} + C \int_0^t \|m\|_{H^s} \|m\|_{H^s}^2_{L^\infty} d\tau.
\]

(4.11)

Adapting the same argument as in **Step 1**, we conclude that this theorem holds for \( s \in [1, 2) \).

**Step 3:** Assume \( 2 \leq l \in \mathbb{N} \). Suppose (1.8) holds when \( l - 1 \leq s < l \). Using induction, we should prove the validity of (1.8) for \( l \leq s < l + 1 \). Applying \( \partial_x^l \) to (4.1) leads to

\[
\partial_t \partial_x^l m + \sin(u^2 - u_x^2)\partial_x^{l+1} m = -\sum_{i=0}^{l-1} C_i^l \partial_x^{l-i} \sin(u^2 - u_x^2)\partial_x^{i+1} m
-2\partial_x^{l}[\cos(u^2 - u_x^2)u_x m^2] := f_2.
\]
Lemma 2.9 once again leads to

\[
\|\partial_t^l m\|_{H^{s-l}} \leq \|m_0\|_{H^s} + C \int_0^t \|\partial_t^l m\|_{H^{s-l}} \|\partial_x \sin(u^2 - u_x^2)\|_{L^\infty} \, d\tau + C \int_0^t \|f_2\|_{H^{s-l}} \, d\tau.
\]

(4.12)

Using Lemma 2.5 and the Sobolev embedding inequality produces

\[
\|\sum_{i=0}^{l-1} C_i^{l-i} \partial_x^{l-i} \sin(u^2 - u_x^2) \partial_x^{i+1} m\|_{H^{s-l}} \\
\leq \sum_{i=0}^{l-1} C_i^{l-i} \|\partial_x^{l-i} \sin(u^2 - u_x^2) \partial_x^{i+1} m\|_{H^{s-l}} \\
\leq \left( \|\partial_x^{l-i} \sin(u^2 - u_x^2)\|_{H^{s-l+1}} \|\partial_x^i m\|_{L^\infty} + \|\partial_x^{l-i} \sin(u^2 - u_x^2)\|_{L^\infty} \|\partial_x^{i+1} m\|_{H^{s-l}} \right) \\
\leq C \|m\|_{H^{l+1/2+\epsilon}} \|\sin(u^2 - u_x^2)\|_{H^{s-l+1}} + C \|m\|_{H^{l-i+1/2+\epsilon}} \|\sin(u^2 - u_x^2)\|_{H^{s-l+1/2+\epsilon}} \\
\leq C \|m\|_{H^{l+1/2+\epsilon}} \|u^2 - u_x^2\|_{H^{s-l+1}} + C \|m\|_{H^{l-i+1/2+\epsilon}} \|u^2 - u_x^2\|_{H^{s-l+1/2+\epsilon}} \\
\leq C \|m\|_{H^{l+1/2+\epsilon}} \|m\|_{H^s},
\]

(4.13)

where \(\epsilon \in (0, 1/8)\) so that \(H^{1/2+\epsilon}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\).

Direct computation gives

\[
\|2\partial_x^l [\cos(u^2 - u_x^2) u_x m^2]\|_{H^{s-l}} \leq C \|u_x m^2\|_{H^s} \leq C \|m\|_{H^s} \|m\|_{H^{l+1/2+\epsilon}}^2.
\]

(4.14)

(4.12) together with (4.13)–(4.14) yields

\[
\|\partial_x^l m\|_{H^{s-l}} \leq \|m_0\|_{H^s} + C \int_0^t \|m\|_{H^s} \|m\|_{H^{l+1/2+\epsilon}}^2 \, d\tau
\]

(4.15)

or

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} \exp \left\{ C \int_0^t \left( \|m(\tau)\|_{H^{l+1/2+\epsilon}}^2 + 1 \right) \, d\tau \right\}.
\]

(4.16)

Therefore, if the maximal existence time \(T^* < \infty\) satisfies \(\int_0^{T^*} \|m(\tau)\|_{L^\infty}^2 \, d\tau < \infty\), then the uniqueness of the solution provided by Theorem 1.1 ensures the uniform boundedness of \(\|m(t)\|_{H^{l+1/2+\epsilon}}\) in \(t \in (0, T^*)\) recalling our induction assumption.
which together with (4.16) implies \( \lim \sup_{t \to T^*} \| m(t) \|_{H^s} < \infty \), a contradiction. We thus complete the proof of Theorem 1.2. \( \square \)

5 Wave-breaking

In this section, we will prove Theorem 1.3. Before doing this, we shall first deduce the precise blow-up quantity for strong solutions of (1.3).

Let \( q(t, x) \) solve the following ordinary differential equation:

\[
\begin{aligned}
\frac{d}{dt} q(t, x) &= \sin(u^2 - u_x^2)(t, q(t, x)), \\
q(x, 0) &= x.
\end{aligned}
\] (5.1)

After differentiating (5.1) with respect to \( x \), we obtain

Lemma 5.1 Suppose \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \). Let \( T^* > 0 \) be the maximal existence time of the solution \( u \) to Eq. (1.3). Then there exists a unique solution \( q \in C^1([0, T^*) \times \mathbb{R}; \mathbb{R}) \) to Eq. (5.1) satisfying

\[
q_x(t, x) = \exp \left\{ 2 \int_0^t \cos(u^2 - u_x^2) u_x ds \right\} > 0.
\] (5.2)

Moreover, we have the expression of \( m(t, q(t, x)) \) as

\[
m(t, q(t, x)) = m_0(x) \exp \left\{ -2 \int_0^t \cos(u^2 - u_x^2) u_x(s, q(s, x)) ds \right\},
\] (5.3)

which implies that the sign and zeros of \( m(x, t) \) are the same as those of \( m_0(x) \).

Proof Applying \( \partial_x \) to Eq. (5.1) leads to

\[
\begin{aligned}
\frac{d}{dt} q_x(t, x) &= 2 \cos(u^2 - u_x^2) u_x m(t, q(t, x)) q_x(t, x), \\
q_x(0, x) &= 1.
\end{aligned}
\] (5.4)

Solving (5.4) produces the solution given by (5.2). The Sobolev embedding inequality gives for \( T' < T^* \)

\[
\sup_{(s,x) \in [0,T') \times \mathbb{R}} |2 \cos(u^2 - u_x^2) u_x m(s, x)| < \infty,
\]

which along with (5.2) yields \( q_x(t, x) \geq \exp(-Ct), (t, x) \in [0, T^*) \times \mathbb{R} \) for some \( C > 0 \), and consequently \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) before the blow-up time, that is, (5.2) holds.
It follows from Eqs. (1.3) and (5.1) that
\[
\frac{d}{dt} m(t, q(t, x)) = m_t(t, q(t, x)) + m_x(t, q(t, x))q_t(t, x)
\]
\[
= m_t(t, q(t, x)) + m_x(t, q(t, x))[\sin(u^2 - u_x^2)](t, q(t, x)) \\
= [m_t + m_x(\sin(u^2 - u_x^2))](t, q(t, x)) \\
= [-2\cos(u^2 - u_x^2)u_x m](t, q(t, x))m(t, q(t, x)).
\] (5.5)

Solving Eq. (5.5) yields Eq. (5.3). We thus complete the proof of Lemma 5.1.

We next deduce the precise blow-up quantity of the Cauchy problem (1.3).

**Lemma 5.2** Suppose \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \). Let \( T^* > 0 \) be the maximal existence time of the solution \( u \) to the Cauchy problem (1.3). Then the solution \( u \) blows up in finite time if and only if
\[
\liminf_{t \to T^*} \left( \inf_{x \in \mathbb{R}} \left( 2\cos(u^2 - u_x^2)u_x m(t, x) \right) \right) = -\infty.
\] (5.6)

**Proof** From the expression of \( m(t, q(t, x)) \) in Eq. (5.3), we conclude that if there exists a positive constant \( K_1 \) such that
\[
\inf_{x \in \mathbb{R}} (-2\cos(u^2 - u_x^2)u_x m(t, x)) \geq -K_1, \quad 0 \leq t \leq T^*,
\]
then
\[
\|m(t)\|_{L^\infty} = \|m(t, q(t, x))\|_{L^\infty} \\
= m_0(x) \exp \left( -2 \int_0^t \left( \cos(u^2 - u_x^2)u_x m(\tau, q(\tau, x))d\tau \right) \right) \\
\leq e^{K_1T^*}\|m_0\|_{L^\infty},
\]
which combined with Theorem 1.2 implies that \( m(t, x) \) will not blow up in a finite time.

However, if (5.6) holds true, then the Sobolev embedding ensure that \( m(t, x) \) will blow up in a finite time. We thus complete the proof of Lemma 5.2.

Now we are in a position to prove Theorem 1.3.

**Proof of Theorem 1.3** Set the precise blow-up quantity as
\[
M = \cos(u^2 - u_x^2)mu_x.
\] (5.7)

Next, we need to estimate the dynamics of \( M \) along the characteristic. To do this, we first compute
\[
(1 - a_x^2)[u_t + \sin(u^2 - u_x^2)u_x]
\]
\[
\begin{align*}
&= m_t + (1 - \partial_x^2)[\sin(u^2 - u_x^2)u_x] \\
&= -\sin(u^2 - u_x^2)(u_x - u_{xxx}) - 2\cos(u^2 - u_x^2)u_x m^2 + \sin(u^2 - u_x^2)u_x \\
&\quad - \partial_x^2[\sin(u^2 - u_x^2)u_x] \\
&= \sin(u^2 - u_x^2)u_{xxx} - 2\cos(u^2 - u_x^2)u_x m^2 - \partial_x^2[\sin(u^2 - u_x^2)u_x] \\
&= -2uM - 2\partial_x(u_x M),
\end{align*}
\]

from which we derive

\[
u_t + \sin(u^2 - u_x^2)u_x = -2p^* (uM) - 2p_x^* (u_x M).
\]

Differentiating (5.9) with respect to \(x\) then produces

\[
u_{xt} + \sin(u^2 - u_x^2)u_{xx} = -2p_x^* (uM) - 2p^* (u_x M).
\]

Recasting (1.3) leads to

\[
m_t + \sin(u^2 - u_x^2)m_x = -2mM.
\]

Combining (5.9)–(5.11), one obtains

\[
\begin{align*}
M_t + \sin(u^2 - u_x^2)M_x &= -\sin(u^2 - u_x^2)mu_x [2u[-2p^* (uM) - 2p_x^* (u_x M)] \\
&\quad - 2u_x[-2p_x^* (uM) - 2p^* (u_x M)]) \\
&\quad + \cos(u^2 - u_x^2)u_x (-2mM) + \cos(u^2 - u_x^2)m[-2p_x^* (uM) - 2p^* (u_x M)] \\
&= 4\sin(u^2 - u_x^2)muu_x [p^* (uM) + p_x^* (u_x M)] \\
&\quad - 4\sin(u^2 - u_x^2)mu_x^2 [p_x^* (uM) + p^* (u_x M)] \\
&\quad - 2M^2 - 2\cos(u^2 - u_x^2)m[p_x^* (uM) + p^* (u_x M)].
\end{align*}
\]

The positivity of the initial data \(m_0\) and the equality \(u = p^* m\) yields \(|u_x| \leq u\), accordingly, one obtains

\[
(5.12) \leq -2M^2 + C(\|u\|_{L^\infty}^5 + \|u\|_{L^\infty}^3)m, \tag{5.13}
\]

which combined with the definition of \(\bar{M}(t)\) gives

\[
\begin{align*}
\frac{d}{dt} \bar{M}(t) &= (M_t + \sin(u^2 - u_x^2)M_x)(t, q(t, x_0)) \\
&\leq -2\bar{M}^2(t) + C(\|u_0\|_{H^1}^5 + \|u_0\|_{H^1}^3)\bar{m} \\
&\leq -2\bar{M}^2(t) + C_1\bar{m},
\end{align*}
\]

where \(C_1 = C(\|u_0\|_{H^1}^5 + \|u_0\|_{H^1}^3)\).
On the other hand, one easily deduces
\[
\frac{d}{dt} \tilde{m}(t) = -2\tilde{m} \tilde{M}. \tag{5.15}
\]
From (5.14)–(5.15), we have
\[
\frac{d}{dt} \begin{bmatrix} \tilde{M}(t) \\ \tilde{m}(t) \end{bmatrix} \leq \begin{bmatrix} \tilde{M}'(t) \tilde{m}(t) - \tilde{M}(t) \tilde{m}'(t) \\ \tilde{m}(t)(-2\tilde{M}^2(t) + C_1\tilde{m}) - \tilde{M}(-2\tilde{m} \tilde{M}) \end{bmatrix} = C_1. \tag{5.16}
\]
Integrating (5.16) from 0 to \(t\) yields
\[
\tilde{M}(t) \leq \tilde{M}(0) + C_1t \tag{5.17}
\]
or
\[
\tilde{M}(t) \leq \left( \frac{\tilde{M}(0)}{\tilde{m}(0)} + C_1 \right) \tilde{m}(t). \tag{5.18}
\]
Consequently, one derives
\[
\frac{d}{dt} \left( \frac{1}{\tilde{m}(t)} \right) = -\frac{1}{\tilde{m}^2(t)} (-2\tilde{m} \tilde{M}) = 2 \frac{\tilde{M}(t)}{\tilde{m}(t)} \leq 2 \left( \frac{\tilde{M}(0)}{\tilde{m}(0)} + C_1t \right). \tag{5.19}
\]
Integrating once again, we obtain
\[
0 < \frac{1}{\tilde{m}(t)} \leq 2 \left( \frac{\tilde{M}(0)}{\tilde{m}(0)} t + \frac{1}{2} C_1 t^2 \right) + \frac{1}{\tilde{m}(0)} : = h(t). \tag{5.20}
\]
Since \(\tilde{M}(0) < 0\), then \(h'(0) < 0\). According to the fact that \(\lim_{t \to \infty} h'(t) = +\infty\) and the continuity of the function \(h'(t)\), there exists a \(\xi > 0\) such that \(h'(\xi) = 0\). Under the assumption (1.10), we have \(h(\xi) < 0\). Note that \(h(0) = \frac{1}{\tilde{m}(0)} > 0\) and \(h(t) \in C[0, +\infty)\), one can find some \(T^* \in (0, \xi)\) such that
\[
0 < \frac{1}{\tilde{m}(t)} \leq h(t) \to 0, \quad \text{as } t \to T^*, \tag{5.21}
\]
which indicates that \(\lim_{t \to T^*} \tilde{m}(t) = +\infty\). Since \(\frac{\tilde{M}(0)}{\tilde{m}(0)} + C_1 T^* < \frac{\tilde{M}(0)}{\tilde{m}(0)} + C_1 \xi = 0\), then by (5.18) and (5.21), we derive
\[
\inf_{x \in \mathbb{R}} [\cos(u^2 - u^2_x)mu_x](t, x) \to -\infty, \quad \text{as } t \to T^*.
\]
Using Lemma 5.2, we conclude that the solution \(m\) blows up at the time \(T^* \in (0, \xi]\). We thus complete the proof of Theorem 1.3. \(\square\)
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Declarations

Conflict of interest The authors declare that there is no conflicts of interest.

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