Relativistic mechanics of Casimir apparatuses in a weak gravitational field

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This paper derives a set of general relativistic Cardinal Equations for the equilibrium of an extended body in a uniform gravitational field. These equations are essential for a proper understanding of the mechanics of suspended relativistic systems. As an example, the prototypical case of a suspended vessel filled with radiation is discussed. The mechanics of Casimir apparatuses at rest in the gravitational field of the Earth is then considered. Starting from an expression for the Casimir energy-momentum tensor in a weak gravitational field recently derived by the authors, it is here shown that, in the case of a rigid cavity supported by a stiff mount, the weight of the Casimir energy $E_C$ stored in the cavity corresponds to a gravitational mass $M = E_C/c^2$, in agreement with the covariant conservation law of the regularized energy-momentum tensor. The case of a cavity consisting of two disconnected plates supported by separate mounts, where the two measured forces cannot be obtained by straightforward arguments, is also discussed.

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I. INTRODUCTION

One of the most intriguing predictions of Quantum Electrodynamics is the existence of irreducible fluctuations of the electromagnetic (e.m.) field in the vacuum. It was Casimir’s fundamental discovery [1] to realize that the effects of this purely quantum phenomenon were not confined to the atomic scale, but would rather manifest themselves also at the macroscopic scale, in the form of an attractive force between two parallel discharged metal plates at distance $a$. Under the simplifying assumption of perfectly reflecting mirrors, he obtained a force of magnitude

$$F_{(C)} = \frac{\pi^2 \hbar c}{240 a^3} A,$$  \hspace{1cm} (1.1)

where $A$ is the area of the plates. By modern experimental techniques the Casimir force has now been measured with an accuracy of a few percent (see Refs. [2] and Refs. therein). For detailed reviews of both theoretical and experimental aspects of the Casimir effect, see Refs. [3, 4, 5, 6].

Now, the energy associated with the Casimir force in Eq. (1.1) is

$$E_{(C)} = -\frac{\pi^2 \hbar c}{720 a^3} A,$$  \hspace{1cm} (1.2)

and one may wonder if it is possible to measure this vacuum energy directly, rather than the corresponding force. Experiments of this sort would further enhance one’s confidence in the reality of vacuum fluctuations. Recently, we have proposed an experiment with superconducting cavities, aiming at measuring the variation of Casimir energy that accompanies the superconducting transition [8]. Another line of research concerns the gravitational coupling of the vacuum energy. This problem has been studied by a number of authors, leading to some contradictory conclusions.

In Ref. [9], the authors propose an experiment to measure the Casimir force in the Schwarzschild metric of the galactic center. The experiment is designed to show whether or not virtual quanta follow geodesics. They find gravitational forces that depend on the orientation of the Casimir apparatus with respect to the gravitational field of the earth.

In Ref. [10], the author evaluates scalar Casimir effects in a weak gravitational field, and obtains corrections to the vacuum energy-momentum tensor and attractive force on the plates, resulting from spacetime curvature. He then points out that, if the cosmological constant arises by virtue of zero-point energy, it is susceptible to fluctuations induced by gravitational sources. He uses a curved line element which describes the weak gravitational field in the vicinity of a mass $M$ as a perturbation to Minkowski spacetime rather than the flat metric appropriate for a uniform gravitational field, in the Fermi coordinate system attached to the cavity.

In Ref. [11], the author studies the Casimir vacuum energy density for a massless scalar field confined between two nearby parallel plates in a slightly curved, static spacetime background, employing the weak-field approximation, and obtains the gravity-induced correction to Casimir energy. He then finds that the attractive force between the cavity walls is expected to weaken.

In Ref. [12], a sketchy computation is presented to show that, if one suspends a rigid Casimir cavity, the vacuum fluctuations contribute an extra negative weight...
\[ \tilde{p}^{(C)} = \frac{E^{(C)}}{c^2} \tilde{g}, \]  
\[ \nabla_a (T_{ab}^{(C)}) = 0, \]  
\[ \nabla_a T_{ab} = f_{(\text{vol})}^b, \]  
where \( \tilde{g} \) is the gravity acceleration. In the same paper, the feasibility of such an experiment is also discussed. An important progress was made in Ref. 13, which contains the first detailed quantum-field-theoretic computation of the Casimir energy-momentum tensor \( (T_{ab}^{(C)}) \) (with angle brackets denoting the vacuum expectation value) in a weak gravitational field. This calculation appears rather important since, for quantum field theory in curved spacetime [14], a fundamental task, if not the main problem, is to understand the energy-momentum tensor (see the Introduction in Ref. [15]). To check consistency, covariant conservation of \( (T_{ab}^{(C)}) \), i.e.

\[ \nabla_a (T_{ab}^{(C)}) = 0, \]  
was explicitly verified therein up to first order in \( \epsilon \). Unfortunately, we used incorrectly the expression for \( (T_{ab}^{(C)}) \) derived in Ref. 13, Eqs. 12.13 below, to predict a net push on the Casimir apparatus that is four times the correct value, given in Eq. 13.3, in contradiction with what was stated in Ref. 12. This discrepancy was stressed in a recent paper Ref. 16, where valuable variational methods are used to show that, for the case of parallel conducting plates, the Casimir energy gravitates according to Eq. 13.3. This is in agreement with the early findings of Jaekel and Reynaud, who studied the inertia of Casimir energy in two dimensions 17. No orientation dependence has been found in Ref. 16, accounting clearly for the discrepancy in this respect as compared with the findings in Ref. 9. The work in Ref. 16 and the lack of careful force formulae in our paper 13, led us to undertake a fresh thorough mechanical analysis of the relativistic mechanics of Casimir apparatuses at rest in the gravitational field of the Earth, presented in the next Sections. The problem turns out to be rather subtle, and the most significant result of a careful analysis is that the energy-momentum tensor derived in 13 does indeed lead to the correct result for the gravitational push, as given in Eq. 13.3, in agreement with the findings of Ref. 16.

Our paper reinterprets therefore the work in Refs. 12, 13. The content of the paper can be divided into two main parts. The first part, coinciding with Secs. II and III, discusses the mechanics of an extended body at rest in a uniform gravitational field, within Einstein’s Theory of General Relativity. The analysis starts from the assumption that the body satisfies the covariant conservation law expressing the balance of energy and momentum between the body and the fields, Eq. 2.1 below, and we use this to obtain a simple mathematical proof of the general global conditions, the Cardinal Equations, that ensure mechanical equilibrium of the body. Conditions similar to ours were originally obtained in Refs. 18, 19, in the context of any theory of gravitation satisfying the weak Equivalence Principle, by means of an ingenious gedanken experiment. Using these Cardinal Equations, we show that any system, which obeys Eq. 2.1 below, possesses a passive gravitational mass that is equal to its total inertia. In Sec. III we illustrate the conditions obtained in Sec. II to study the mechanical forces in the prototypical case of a rigid suspended vessel, filled with a fluid. The general conditions derived in Sec. II are indispensable for a correct understanding of the forces in a relativistic system, like a vessel filled with radiation, considered in Sec. III, and even more so in the case of Casimir apparatuses, that are studied in Sec. IV.

In the second part of the paper, coinciding with Sec. IV, we study the relativistic mechanics of Casimir apparatuses at rest in the gravitational field of the Earth. Since from our Ref. 12, we now know that vacuum fluctuations satisfy Eq. 1.3, the general theorems derived in the first two sections can be used. In particular, we evaluate the forces exerted by the mounts that hold the apparatus, which represent the actual quantities to be measured in a real experiment. The problem turns out to be rather subtle, because, according to the general Cardinal Equations, the magnitudes of the supporting forces depend on where they are applied, and therefore the answer depends on the setting considered for the mounts. This is a natural phenomenon in general relativity, as it was discovered long ago by Nordtvedt 18, and it turns out to be of fundamental importance in the analysis of an essentially relativistic system like a Casimir cavity. We consider two different settings for the mounts. In the first case, we have just one mount, supporting a rigid cavity; for this case, the general theorems in Sec. II immediately lead to Eq. 1.3 for the “weight” of the Casimir apparatus. In the second setup, the plates of the Casimir cavity are disconnected, and they are supported by separate mounts. Using the expression for \( T_{ab}^{(C)} \) computed in Ref. 13, we obtain the forces exerted by the plates of the respective mounts. Sec. V finally contains our conclusions and a discussion of the results.

II. RELATIVISTIC STATIC CARDINAL EQUATIONS IN A UNIFORM GRAVITATIONAL FIELD

In this Section, we obtain a rigorous proof, within the context of Einstein’s General Theory of Relativity, of the conditions for mechanical equilibrium of an extended body at rest in a uniform gravitational field.

In General Relativity, the equations expressing the balance of energy and momentum between a body and the fields are

\[ \nabla_a T^{ab} = f_{(\text{vol})}^b, \]  
where \( T^{ab} \) is the energy-momentum tensor and \( f_{(\text{vol})}^b \) are the external forces. If the system is also subject to forces \( f_{(\text{surf})}^b \), applied at points of its surface \( \partial \Sigma \), Eqs. 2.1 are
supplemented by the following boundary conditions:

\[ T^{ab} n_b = -f^{a}_{(sur)} \]  
\[ \text{(2.2)} \]

where \( n^a \) is the unit normal at \( \partial \Sigma \) pointing outwards the body.

Now we consider the case of a body at rest in a uniform gravitational field. As we shall see, in this special case it is possible to derive, from the local conditions Eqs. (2.1) and Eq. (2.2), a set of global conditions ensuring mechanical equilibrium of the body. The global conditions that we shall obtain have a form similar to the familiar Cardinal Equations of Statics in Newtonian Theory, and will provide us with a relativistic concept of weight for an extended body. A striking point of departure from classical theory is however the fact, first discovered in Ref. (18), that the weight of a body, intended as the magnitude of the force that must be applied to hold it, depends on where the force is applied.

In what follows, Latin letters from the beginning of the alphabet will denote spacetime and space components of tensors in a definite coordinate system, respectively. Greek letters and Latin letters from the middle of the alphabet will be used as abstract indices for tensors, while \( i \) and \( j \) will denote spacetime and space components of tensors in a definite coordinate system, respectively.

We begin by defining a uniform gravitational field as a field seen in a uniformly accelerating frame. As is well known (20) in Einstein’s General Theory of Relativity, such a field is described by the line element

\[ ds^2 = -c^2 \left( 1 + \frac{A z}{c^2} \right)^2 dt^2 + \delta_{ij} dx^i dx^j, \]  
\[ \text{(2.3)} \]

with \( A > 0 \) the acceleration parameter. This line element describes also the gravitational field of the Earth, in a local Fermi coordinate system, once tidal effects are neglected.

We denote by \( u^a \) the normalized velocity field for observers at rest in the metric (2.3):

\[ u^a = u^0 \left( \frac{\partial}{\partial t} \right)^a = \frac{c}{|g_{00}|^{1/2}} \left( \frac{\partial}{\partial t} \right)^a. \]  
\[ \text{(2.4)} \]

They possess an acceleration \( a^a \) in the upwards \( z \)-direction, i.e.

\[ a^a = \frac{Du^a}{Dt} = \frac{c^2}{|g_{00}|^{1/2}} \partial_z |g_{00}|^{1/2} z^a = \frac{A}{1 + A z/c^2} z^a. \]  
\[ \text{(2.5)} \]

We define the gravity acceleration \( g^a \) as minus the acceleration \( a^a \) of stationary observers, i.e.

\[ g^a (z) = -\frac{Du^a}{Dt} = -\frac{A}{1 + A z/c^2} z^a. \]  
\[ \text{(2.6)} \]

We consider an extended body at rest in the coordinate system of Eq. (2.3). The body being at rest, one has

\[ T^{ab} = \rho u^a u^b + S^{ab}, \]  
\[ \text{(2.7)} \]

where \( \rho \) is the inertial mass density and \( S^{ab} \) is the stress tensor, satisfying the condition

\[ S^a_b u^b = 0. \]  
\[ \text{(2.8)} \]

By virtue of Eq. (2.4), we then have

\[ S^{00} = S^{0i} = S^{00} = 0. \]  
\[ \text{(2.9)} \]

and therefore \( S^{ab} \) is purely spatial.

If we now insert Eq. (2.7) into the l.h.s. of Eq. (2.1), we obtain

\[ \nabla_a (\rho u^a) u^b - \rho g^b = -\nabla_a S^{ab} + f^{(vol)}_{(x, y, z)}. \]  
\[ \text{(2.10)} \]

Let now \( e^a_{(i)} \) be the vector fields for the coordinate spatial axis:

\[ e^a_{(i)} = \left( \frac{\partial}{\partial x^i} \right)^a, i = x, y, z. \]  
\[ \text{(2.11)} \]

Upon multiplying Eq. (2.10) by \( e^b_{(j)} \), we obtain

\[ - \rho g_{ij} = -\nabla_a (S^{ab} e^b_{(j)}) + S^{ab} \nabla_a (e^b_{(j)}) + f^{(vol)}_{(x, y, z)}, \]  
\[ \text{(2.12)} \]

where we define \( g_{ij} = g_{ab} e^a_{(i)} e^b_{(j)} \). A simple computation gives

\[ \nabla_a (e^b_{(j)}) = -\delta_{ij} A (dt^a) (dt_b), \]  
\[ \text{(2.13)} \]

and since \( S^{ab} (dt)_b = 0 \) (see Eq. (2.8)), we see that the second term on the r.h.s. of Eq. (2.12) vanishes. On the other hand, for the first term on the r.h.s. of Eq. (2.12), using well known identities, we find

\[ \nabla_a (S^{ab} e^b_{(j)}) = \frac{1}{|g_{00}|} \partial_j (\sqrt{|g_{00}|} S^j_i) \]  
\[ = \frac{1}{|g_{00}|} \partial_j (\sqrt{|g_{00}|} S^j_i). \]  
\[ \text{(2.14)} \]

Therefore, Eq. (2.12) becomes

\[ - \rho g_{ij} = -\frac{1}{\sqrt{|g_{00}|}} \partial_j (\sqrt{|g_{00}|} S^j_i) + f^{(vol)}_{(x, y, z)}, \]  
\[ \text{(2.15)} \]

Let us now introduce the gravitational red-shift \( r_O (P) \) of the point \( P \) with coordinates \( \{x, y, z\} \) relative to an arbitrary point \( O \) with coordinates \( \{x(O), y(O), z(O)\} \):

\[ r_O (P) = \sqrt{\frac{|g_{00}(P)|}{|g_{00}(O)|}} = \frac{1 + A z/c^2}{1 + A z(O)/c^2}. \]  
\[ \text{(2.16)} \]

Upon multiplying Eq. (2.15) by \( r_O (z) \) we obtain

\[ - \rho r_O g_{ij} (z) = -\partial_j (r_O S^j_i) + r_O f^{(vol)}_{(x, y, z)}. \]  
\[ \text{(2.17)} \]

However, in view of Eq. (2.6), we see that

\[ r_O (z) g_{ij} (z) = g_{ij} (z(O)). \]  
\[ \text{(2.18)} \]

and hence we arrive at the following equation:

\[ - \rho g_{ij} (z(O)) = -\partial_j (r_O S^j_i) + r_O f^{(vol)}_{(x, y, z)}. \]  
\[ \text{(2.19)} \]

Upon integrating the above equation over the body’s volume, and in view of Eqs. (2.2), we obtain our First Cardinal Equation:

\[ \bar{F}_O + \bar{F}^{(tot)}_O = \bar{0}. \]  
\[ \text{(2.20)} \]
In this equation, the total external force is
\[
\vec{F}_{O}^{(\text{tot})} \equiv \int_{\Sigma} d^{3}x \, r_{O} \vec{f}_{(\text{vol})} + \int_{\partial \Sigma} d^{2}\sigma \, r_{O} \vec{f}_{(\text{sur})},
\tag{2.21}
\]
while the weight \(\vec{P}_{O}\) is defined as
\[
\vec{P}_{O} \equiv M \vec{g}_{O},
\tag{2.22}
\]
where \(\vec{g}_{O}\) denotes the gravity acceleration at \(O\) and
\[
M = \int_{\Sigma} d^{3}x \rho.
\tag{2.23}
\]
Now the quantity \(M\) in Eq. (2.22) is, by definition, the passive gravitational mass of the body, and therefore Eq. (2.22) tells us that \(M\) is equal to the total inertia of the body. We stress that the above derivation shows that this identity is a necessary consequence of the covariant equations (2.11). Therefore, for all physical systems which satisfy Eq. (2.17), the passive gravitational mass is equal to the total inertia.

In order to derive the Second Cardinal Equation, we now define the center of mass \(\vec{x}_{CM}\) via the equation
\[
\vec{x}_{CM} = \frac{1}{M} \int_{\Sigma} d^{3}x \rho \vec{x}.
\tag{2.24}
\]
Now we multiply Eq. (2.19), for \(O = CM\), by \(\epsilon_{klj}(x - x_{CM})^{j}\) and integrate the resulting Equation over the body’s volume. On using the identity
\[
\epsilon_{klj}(x - x_{CM})^{j} \partial_{j}(c_{CM}S^{ij}) = \partial_{j}[\epsilon_{klj}(x - x_{CM})^{j} r_{CM}S^{ij}],
\tag{2.25}
\]
implied by the symmetry of \(S^{ij}\), one obtains the Second Cardinal Equation
\[
\vec{r}_{(\text{CM})}^{(\text{tot})} = \vec{0},
\tag{2.26}
\]
where \(\vec{r}_{(\text{CM})}^{(\text{tot})}\) is the total torque of the external forces, relative to to the center of mass:
\[
\vec{r}_{(\text{CM})}^{(\text{tot})} = \int_{\Sigma} d^{3}x (\vec{x} - \vec{x}_{CM}) \times r_{CM} \vec{f}_{(\text{vol})} + \int_{\partial \Sigma} d^{2}\sigma (\vec{x} - \vec{x}_{CM}) \times r_{CM} \vec{f}_{(\text{sur})}.
\tag{2.27}
\]
Equations similar to Eqs. (2.20,2.21) were first obtained in Ref. [18], by exploiting the phenomenon of gravitational red-shift for photons, via an ingenious gedanken experiment involving an ideal electro-mechanical device converting into photons the mechanical work done by a heavy body, as it lowers or rises into the gravitational field. By a similar procedure, the authors of Ref. [19] obtained equations similar to our Eqs. (2.26,2.27), ensuring rotational equilibrium of the body.

As we see, Eqs. (2.20,2.22) look remarkably similar to the analogous Equations of Newtonian theory. The striking difference with respect to classical theory is that when forces and torques are added, one has to multiply each of them by the red-shift of the point where they act, relative to the point where they are added. A remarkable consequence of this is that the force that must be applied to a body to hold it still in a gravitational field, depends on where the force is applied [18]. To see this, define the proper weight \(\vec{P}\) of a body as
\[
\vec{P} \equiv M \vec{g}_{CM},
\tag{2.28}
\]
and suppose that the supporting force \(\vec{f}_{Q}\) is applied at the point \(Q\). Then, from Eq. (2.20) we obtain
\[
\vec{f}_{Q} = -M \vec{g}_{Q} = -r_{Q}(CM) \vec{P},
\tag{2.29}
\]
where in the final passage we used Eq. (2.18), for \(O = Q\) and \(P = CM\). Therefore, the magnitude of the applied force is equal to the proper weight, multiplied by the red-shift of the center of mass \(CM\) relative to the point \(Q\).

We would like to comment now on an alternative possible form of the Cardinal Equations, based on a different rearrangement of Eq. (2.15). As we shall see, the alternative form implies a definition of inertia for a stressed body, which explicitly involves the stresses. For this purpose, we have to consider again the last line of Eq. (2.14). If we perform the partial derivatives, and use Eqs. (2.5) and (2.6), we obtain
\[
\frac{1}{\sqrt{|g_{00}|}} \partial_{j} (\sqrt{|g_{00}|} S_{ij}) = \partial_{j} S_{ij} - \frac{1}{c^{2}} S_{ij} \delta_{ij}(z).
\tag{2.30}
\]
If we substitute this expression into Eq. (2.15), and rearrange the terms, we obtain
\[
- \left( \rho \delta^{i}_{j} + \frac{1}{c^{2}} S_{ij} \right) g_{j} = -\partial_{j} S_{ij} + f_{(\text{vol})j}.
\tag{2.31}
\]
This form of the equation suggests that the inertia of a stressed body is not just \(\rho\), but rather is a tensor \(m_{ij}^{\text{vol}}\) that depends on the stresses, i.e.
\[
m_{ij} = \rho \delta^{i}_{j} + \frac{1}{c^{2}} S_{ij}.
\tag{2.32}
\]
If we integrate this equation over the body’s volume, instead of Eq. (2.20), we obtain
\[
\vec{P} + \vec{F}_{(\text{tot})} = \vec{0},
\tag{2.33}
\]
where the “weight” \(\vec{P}\) is now defined as
\[
\vec{P} = \int_{\Sigma} d^{3}x \, m_{ij}^{\text{vol}} g_{j}(z) \vec{e}_{i},
\tag{2.34}
\]
and \(\vec{F}_{(\text{tot})}\) coincides with the classical expression for the total external force:
\[
\vec{F}_{(\text{tot})} = \int_{\Sigma} d^{3}x \vec{f}_{(\text{vol})} + \int_{\partial \Sigma} d^{2}\sigma \vec{f}_{(\text{sur})}.
\tag{2.35}
\]
It is of course possible to derive from Eq. (2.31), by similar steps as those followed earlier, the analogue of the
Second Cardinal Equation (2.20), but we shall not write it here. We would like to comment, instead, on the conceptual differences between the two approaches. Indeed, the key difference arises from the fact that the two approaches use different expressions for the sum of forces acting at different points and, in particular, for the sum of the contact forces acting on the faces of an infinitesimal cube inside the body. Consider a small cube with vertex at \( \{x, y, z\} \) and sides \( \{dx, dy, dz\} \), and consider a pair of opposite faces, say the \( yz \) faces at \( x \) and \( x + dx \). Then, by definition of stress tensor, the forces in the direction \( i \) acting on these two faces are, respectively, \( dy \, dz \, S^{xi}(x) \) and \( -dy \, dz \, S^{xi}(x + dx) \). The question now is: what do we take for the sum of these two forces? If we thrust the picture outlined in Ref. \[18\] according to which the phenomenon of red-shift for forces represents a general physical effect, we believe that in all situations one should say that the sum at \( O \) of the elementary forces on the \( yz \) faces is

\[
- dy \, dz \, (r_O(x + dx)S^{xi}(x + dx) + r_O(x)S^{xi}(x)) = -dx \, dy \, dz \, \partial_i (r_O(x)S^{xi}(x)).
\]

Then, upon summing over the three pairs of faces, we obtain for the total force \( dF^i_O(x) \) the expression

\[
dF^i_O = -dx \, dy \, dz \, \partial_j (r_O S^{ij}),
\]

which is the one used in Eq. (2.17), which eventually leads to Eqs. (2.20) and (2.26). Note that, with this choice, \( dF^i_O \) depends on the reference point \( O \), which is why it has a suffix \( O \). On the contrary, in the second approach, forces are not red-shifted as they are translated, and therefore one now writes

\[
dF^i = -dx \, dy \, dz \, \partial_j S^{ij},
\]

which is the expression used in classical theory. The extra term that one gets, i.e. the last term on the r.h.s. of Eq. (2.30), is now interpreted as a contribution to the inertia of the matter inside the cube, and is therefore shifted to the l.h.s. of Eq. (2.13), leading to Eq. (2.31), and eventually to the concept of weight in Eq. (2.34).

Of course, both approaches are mathematically correct. However, as we think with the author of Ref. \[18\] that the phenomenon of red-shift for forces represents a genuine physical effect, we believe that in all situations one should accordingly modify the sum of forces acting at different points. Therefore we regard Eq. (2.37) and Eq. (2.21) as the physically correct ones. A further important advantage of this approach is that it leads to a very simple concept of weight that only involves the density of mass \( \rho \) of the body, as in Eq. (2.20). This should be contrasted with the very complicated concept of weight in the second approach, Eq. (2.24), which explicitly involves an average of the stresses (the latter quantity being hard to evaluate in general).

Applications of our formalism are now described in the following sections.

## III. THE CASE OF A VESSEL FILLED WITH A FLUID

It is instructive to use the previous general formulae to examine a system composed by two subsystems, i.e. a rigid vessel, filled with a fluid. We consider, for simplicity, the case of a rectangular box, hanging by a thread. The box is described by an energy-momentum tensor of the same form as in Eq. (2.7): 

\[
T^{ab}_{(box)} = \rho_{(box)} u^a u^b + S^{ab}_{(box)},
\]

while for the fluid one has

\[
T^{ab}_{(fl)} = \rho_{(fl)} u^a u^b + p_{(fl)} \delta^{ab},
\]

where \( p_{(fl)} \) is the pressure. If we consider the total system formed by the box together with the fluid, the only external force is the force \( \tilde{f}^{(thr)}(Q) \) applied by the thread, in the suspension point \( Q \). We can determine \( \tilde{f}^{(thr)}(Q) \) by using the First Cardinal Equation, Eq. (2.20), with \( O = Q \), and we obtain

\[
\tilde{f}^{(thr)}(Q) = -(M_{(box)} + M_{(fl)}) \tilde{g}(Q),
\]

where

\[
M_{(box)} = \int_{(box walls)} d^3 x \, \rho_{(box)},
\]

and

\[
M_{(fl)} = \int_{(fluid)} d^3 x \, \rho_{(fl)}.
\]

A comment on the above equation is now in order. Even though the expression for \( \tilde{f}^{(thr)}(Q) \) has mathematically the form of the sum of two distinct contributions, one from the box and the other from the fluid, it would be wrong to think of the former as the weight of the empty box, i.e. without the fluid in its interior. This is so, because when the box is filled with the fluid, the pressure exerted by the fluid on its internal walls causes a small deformation of the walls, and therefore these pressure forces make some work \( W \) on the box. This work causes a small change \( \delta M_{(box)} \) in the mass of the box, of magnitude \( W/c^2 \), and therefore \( M_{(box)} \neq M_{(empty \ box)} \). However, for a very stiff box, \( \delta M_{(box)} \) is extremely small and therefore, for all practical purposes, one can identify \( M_{(box)} \) with \( M_{(empty \ box)} \). After this identification is made, Eq. (3.3) can be given the standard classical interpretation, according to which the total weight of the system is the sum of the separate weights of the box and of the fluid.

It is interesting now to repeat the analysis, by considering just the box as the whole system. Now, in addition to \( \tilde{f}^{(thr)}(Q) \), the external forces acting on the box include the pressure forces exerted by the fluid filling it. The First Cardinal Equation, again taken for \( O = Q \), now gives

\[
\tilde{f}^{(thr)}(Q) = -M_{(box)} \tilde{g}(Q) - \tilde{f}^{(fl)}_Q,
\]
where

\[ f_Q^{(b)} = \int_{\Sigma_{\text{int}}} d^2 \sigma \, r_Q \, p(b) \, \hat{n}. \]  
(3.7)

Here, \( \Sigma_{\text{int}} \) is the internal surface of the box, and \( \hat{n} \) is the unit normal to \( \Sigma_{\text{int}} \), pointing inside the box, i.e. outwards with respect to the cavity filled with fluid. By symmetry, the lateral walls of the cavity give a vanishing net force, and therefore we have

\[ f_Q^{(b)} = A \left[ r_Q(z_2) \, p(b)(z_2) - r_Q(z_1) \, p(b)(z_1) \right] \hat{z}, \]  
(3.8)

where \( A \) is the area of the base of the box, while \( z_2 \) and \( z_1 \) are the heights of the upper and lower sides of the cavity, respectively. Note, again, that the relativistic sum of the pressures is not equal to their algebraic sum, as it happens in classical theory, because it involves the respective red-shifts. Of course, Eq. (3.6) should eventually reproduce Eq. (3.3). To see it explicitly, we note that by the same steps leading to Eq. (2.19), one finds that Euler’s Equations for the fluid \( \nabla_x T_{ab}^{(fi)} = 0 \) imply

\[ -\rho(b) g(z) = -\frac{d}{dz} (r_Q p(b)). \]  
(3.9)

Upon integrating the above Equation from \( z_1 \) to \( z_2 \), we find

\[ r_Q(z_2) \, p(b)(z_2) - r_Q(z_1) \, p(b)(z_1) = g(z) \int_{z_1}^{z_2} dz \, \rho(b)(z). \]  
(3.10)

By using this result into Eq. (3.8), we obtain

\[ f_Q^{(b)} = g(z) A \int_{z_1}^{z_2} dz \, \rho(b)(z) = g(z) M(b). \]  
(3.11)

Upon inserting this formula into Eq. (3.10), we recover Eq. (3.3). It is now interesting to consider the same problem from the point of view of the alternative Eqs. (2.33-2.35). The final result, Eq. (3.3) will of course be the same, but it is instructive to see how this comes about if one considers again the problem from the point of view of the box only. Instead of Eq. (3.10), Eqs. (2.33-2.35) now give

\[ f^{(thr)}(Q) = -\vec{P}^{(b)}_{\text{(box)}} - \vec{F}^{(i)}. \]  
(3.12)

Here, according to Eqs. (2.31) and (2.32)

\[ \vec{P}^{(b)}_{\text{(box)}} = \int_{\text{(box walls)}} d^3 x \, \rho^{(b)} \delta^{(i)} \hat{i} + \frac{1}{c^2} S^{(j)}_{\text{(box)}}, \]  
(3.13)

while, according to Eq. (2.35), for the total force exerted on the box by the fluid we have the classical formula

\[ \vec{F}^{(i)} = \int_{\Sigma_{\text{int}}} d^2 \sigma \, p(b) \, \hat{n} = A \left[ p(b)(z_2) - p(b)(z_1) \right] \hat{z}. \]  
(3.14)

Now, from Eq. (3.9), we find

\[ \rho(b)(z_2) - p(b)(z_1) = g(z) \int_{z_1}^{z_2} dz \, \rho(b), \]  
(3.15)

where in the last passage we have used Eq. (2.10), Eq. (2.25) and Eq. (2.20) to write \( dr_p/dz = g(z)/c^2 \). If the fluid satisfies an equation of state of the form

\[ p(b) = \gamma \, \rho(b) \, c^2, \]  
(3.16)

we then find for \( \vec{F}^{(i)} \)

\[ \vec{F}^{(i)} = \bar{g}(Q) \left( 1 + \gamma \right) M(b), \]  
(3.17)

a result which, at first sight, seems to contradict the implications of Eq. (2.1). (It is interesting to note that, if the box is filled with thermal radiation, \( 1+\gamma = 4/3 \). This is the same “anomalous” factor of 4/3 that occurred in the classical models for the electromagnetic mass of the electron, considered by H.A. Lorentz at the end of the nineteenth century). Of course, there is really no contradiction, because what one measures here is not \( \vec{F}^{(i)} \) by itself, but rather \( f^{(thr)}(Q) \), which includes also the “weight” of the box \( \vec{P}^{(b)}_{\text{(box)}} \). Now, according to Eq. (3.13), \( \vec{P}^{(b)}_{\text{(box)}} \) can be separated in two parts, i.e.

\[ \vec{P}^{(b)}_{\text{(box)}} = \vec{P}^{(1)}_{\text{(box)}} + \vec{P}^{(2)}_{\text{(box)}}, \]  
(3.18)

where

\[ \vec{P}^{(1)}_{\text{(box)}} = \int_{\text{(box walls)}} d^3 x \, \rho^{(b)} \, \bar{g}(z), \]  
(3.19)

and

\[ \vec{P}^{(2)}_{\text{(box)}} = \frac{1}{c^2} \int_{\text{(box walls)}} d^3 x \, S^{(j)}_{\text{(box)}} \bar{g}(z) \hat{x}^j. \]  
(3.20)

Recalling the considerations following Eq. (3.3), for a stiff box \( \vec{P}^{(1)}_{\text{(box)}} \) is independent, to a high degree of precision, of the box being filled or empty, and therefore can be interpreted as a feature of the box, by itself. On the contrary, the second contribution \( \vec{P}^{(2)}_{\text{(box)}} \) depends on the stresses in the box walls, and therefore this term is strongly affected by the pressure of the fluid, whose pressure on the inner surfaces of the walls leads to additional stresses in the walls \( \vec{P}^{(2)}_{\text{(box)}} \). We can see this clearly by explicitly evaluating \( \vec{P}^{(2)}_{\text{(box)}} \). We need not perform any extra calculations, because we can exploit the mathematical equivalence of the two formulations of the first Cardinal Equation to obtain \( \vec{P}^{(2)}_{\text{(box)}} \). Upon comparing the expression for \( f^{(thr)}(Q) \) in Eq. (3.6) with Eq. (3.12), we obtain

\[ \vec{P}^{(2)}_{\text{(box)}} = (f_Q^{(b)} - \vec{F}^{(i)}) + (M_{\text{(box)}} \bar{g}(Q) - \vec{P}^{(1)}_{\text{(box)}}). \]  
(3.21)
Upon using Eqs. (3.11), (3.17), (3.4) and (3.19), we then obtain
\[
\mathbf{\mathcal{T}}^{(2)}(\text{box}) = -\gamma \bar{g}(Q) M_{(0)} + \int_{(\text{box walls})} d^3x \rho_{(\text{box})}(\bar{g}(Q) - \bar{g}(z)).
\] (3.22)
As we see, the first term on the r.h.s cancels the undesirable \(\gamma\)-dependent contribution in Eq. (3.17). However, the fact that \(\mathbf{\mathcal{T}}(\text{box})\) depends, via this term, on the fluid, shows clearly another possible deficiency (or peculiar property) of Eqs. (2.33-2.34). for a system formed by several bodies in contact, Eq. (2.33) leads to a concept of weight for the individual bodies constituting the system that depends strongly on the other bodies with which it interacts. This should be contrasted with the first formulation based on Eq. (2.20), which on the contrary permits, to a high degree of precision (see comments following Eq. (3.3)), to consider the weights of the individual bodies as independent of each other.

\[
\langle T^{00}_{(C)}(z) \rangle = -\frac{\pi^2\hbar c}{a^2} \left[ \frac{1}{720} + \frac{2g_a}{c^2} \left( \frac{1}{1200} - \frac{1}{3600a} - \frac{z}{240\pi} \right) \right],
\]
(4.1)
\[
\langle T^{11}_{(C)}(z) \rangle = \langle T^{22}_{(C)}(z) \rangle = \pi^2\hbar c \left[ \frac{1}{720} + \frac{2g_a}{c^2} \left( \frac{1}{3600} - \frac{z}{1800a} - \frac{z^2}{120\pi} \right) \right],
\]
(4.2)
\[
\langle T^{33}_{(C)}(z) \rangle = -\frac{\pi^2\hbar c}{a^2} \left[ \frac{1}{240} + \frac{2g_a}{c^2} \left( \frac{1}{720} - \frac{z^2}{a} \right) \right].
\]
(4.3)

These equations, obtained after performing a very difficult calculation, based upon the covariant geodesic point separation [21, 22], are on firm ground because the following consistency checks are satisfied:

(i) The photon and ghost Green functions used to build the Casimir energy-momentum tensor obey the perfect-conductor boundary conditions.

(ii) The Ward identity \(G_{\mu\nu}^{(\gamma)} + G_{\nu\mu} = 0\), relating covariant derivatives of photon and ghost Green functions, has been checked to first order in \(ga/c^2\).

(iii) The Casimir energy-momentum tensor satisfies, to first order in \(ga/c^2\), the covariant conservation law
\[
\nabla_a \langle T^{ab}_{(C)}(z) \rangle = 0.
\] (4.4)

Had we considered instead the case of a strong gravitational field, it would have been impossible to perform a Fourier analysis of Green functions as in Ref. [13]. The resulting evaluation of the energy-momentum tensor remains an open problem.

We note that, since \(\langle T^{00}_{(C)}(z) \rangle = \langle T^{00}_{(C)}(z) \rangle = 0\), the Casimir energy-momentum tensor has the form corresponding to a “body” at rest, as in Eq. (2.7). Therefore, all theorems derived in Sec. II automatically apply to a Casimir apparatus. In particular, Eq. (2.4) holds, and therefore the Casimir apparatus has a passive gravitational mass \(M_{(C)}\) which is equal to its total inertia:
\[
M_{(C)} = \int_{cavity} d^3x \rho_{(C)}.
\] (4.5)

Recalling that, according to Eq. (2.7),
\[
\rho_{(C)} = \frac{1}{c^4} \langle T^{ab}_{(C)} \rangle u_a u_b,
\]
(4.6)
we obtain from Eq. (4.1)
\[
M_{(C)} = -\frac{\pi^2\hbar}{720c^3a^3} A + O(ga/c^2) = \frac{E_{(C)}}{c^2} + O(ga/c^2),
\] (4.7)
in agreement with the findings for \(M_{(C)}\) in Ref. [13].

### A. A suspended rigid Casimir cavity

In the first setup we consider, the plates are rigidly connected to each other, forming a unique rigid system, supported by a thread. By steps similar to those used in Sec. II, we obtain an Equation analogous to Eq. (3.3):

IV. FORCES ON CASIMIR APPARATUSES
for the force $\vec{f}^{(\text{thr})}(Q)$ required to support the cavity:

$$\vec{f}^{(\text{thr})}(Q) = -M_{\text{(box)}} \vec{g}(Q) - M_C \vec{g}(Q),$$  \hspace{1cm} (4.8)

where $M_{\text{(box)}}$ is defined as in Eq. (3.4). After bearing in mind the observations made after Eq. (3.3) on the interpretation of $M_{\text{(box)}}$, one can think of the second term on the r.h.s. of Eq. (4.8) as the weight $\vec{F}^{(C)}$ of the “Casimir mass”. Using Eq. (4.7), we obtain to leading order in $g^2/c^2$:

$$\vec{F}^{(C)} \approx -\frac{\pi^2 A h}{720 c^2 a^4} \vec{g} = \frac{E_{\text{(box)}}}{c^2} \vec{g},$$  \hspace{1cm} (4.9)

in agreement with the weak Equivalence Principle. It is interesting to see how the same result is obtained, if we consider the forces acting only on the rigid walls bounding the cavity, analogously to what was done in Sec. II, for the case of a rigid box filled with a fluid. Again, following the same steps that led to Eqs. (4.6) and (4.7), we obtain

$$\vec{f}^{(\text{thr})}(Q) = -M_{\text{(box)}} \vec{g}(Q) - \int_{\Sigma_{\text{(int)}}} d^2 \sigma r_Q \left( \mathbf{T}^{ij}_{\text{(C)}} \right) \hat{n}_j \hat{x}_i,$$  \hspace{1cm} (4.10)

Note again the presence of the red-shift $r_Q$ multiplying the Casimir stresses in the integral on the r.h.s., which is crucial to obtain the right answer, as we shall now see. By symmetry, the lateral walls of the cavity give a net vanishing contribution to the integral on the r.h.s. of the above Equation, and therefore we have

$$\int_{\Sigma_{\text{(int)}}} d^2 \sigma r_Q \left( \mathbf{T}^{ij}_{\text{(C)}} \right) \hat{n}_j \hat{x}_i = \mathcal{A} \left[ r_Q(z_2) \langle T^{33}_{\text{(C)}}(z_2) \rangle - r_Q(z_1) \langle T^{33}_{\text{(C)}}(z_1) \rangle \right] \hat{z}.$$  \hspace{1cm} (4.11)

Now we see from Eq. (2.16) that, to leading order in $g^2/c^2$, the red-shift $r_Q$ is

$$r_Q(z) \approx 1 + \frac{g}{c^2} (z - z_Q).$$  \hspace{1cm} (4.12)

By using this formula in Eq. (4.11), together with the expression for $\langle T^{33}_{\text{(C)}} \rangle$ in Eq. (4.3), we find that the quantity between square brackets in Eq. (4.11) is equal to

$$-\frac{\pi^2 \hbar c}{a^2} \left[ \frac{g}{240 c^2} (z_2 - z_1) - \frac{4 g}{720 c^2} (z_2 - z_1) \right] = \frac{\pi^2 \hbar c}{720 a^2} \vec{g}.$$  \hspace{1cm} (4.13)

Upon using this expression into Eq. (4.10), we recover the same result as Eq. (4.9).

### B. Disconnected plates with separate mounts

In the second setup that we wish to consider, the two plates are disconnected, and are supported by two separate mounts. If the mounts are connected to the outer faces of plates, the forces $\vec{f}_1$ and $\vec{f}_2$ that support the plates are applied at points $Q_1$ and $Q_2$, with heights $w_2 = a + D$ and $w_1 = -D$, respectively. It should be remarked that the forces $\vec{f}_1$ and $\vec{f}_2$ can only be determined by using the explicit expression of $\langle T^{33}_{\text{(C)}} \rangle$ in Eq. (4.3). Upon applying the first Cardinal Equation to each plate separately, we find for the force $\vec{f}_I$ ($I = 1, 2$)

$$\vec{f}_I = -\vec{P}^{(I)}_{Q_I} - \vec{F}^{(C)}_{Q_I},$$  \hspace{1cm} (4.14)

where $\vec{P}^{(I)}_{Q_I}$ is the weight of the $I$-th plate, i.e.

$$\vec{P}^{(I)}_{Q_I} = g(w_I) \int_{A_I} d^2 r \rho_1$$  \hspace{1cm} (4.15)

while $\vec{F}^{(C)}_{Q_I}$ is the contribution from the Casimir pressure

$$\vec{F}^{(C)}_{Q_I} = \int dx dy r_{Q_I}(z_1) \langle T^{33}_{\text{(C)}}(z_1) \rangle \hat{n}_j \hat{x}_i.$$  \hspace{1cm} (4.16)

On using Eq. (4.12) and the expression for $\langle T^{33}_{\text{(C)}}(z) \rangle$ in Eq. (4.3), we obtain for the upper plate, to order $g/c^2$:

$$\vec{f}^{(C)}_{Q_2} \approx -\frac{\pi^2 A h c}{240 a^4} \left[ 1 - \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \hat{z},$$  \hspace{1cm} (4.17)

while for the lower plate we get

$$\vec{f}^{(C)}_{Q_2} \approx \frac{\pi^2 A h c}{240 a^4} \left[ 1 + \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \hat{z}.$$  \hspace{1cm} (4.18)

We note that both forces depend on the thickness of the plates. It is useful to remark also that the previous result for the total force measured by a rigid cavity, Eq. (4.9), is recovered if the forces $\vec{f}^{(C)}_{Q_1}$ and $\vec{f}^{(C)}_{Q_2}$ are added, say at $Q_2$, using the relativistic law in Sec. II, because then

$$\vec{F}^{(C)}_{Q_2} + r_{Q_1}(Q_1) \vec{P}^{(C)}_{Q_1} \approx F_C \left\{ -\left[ 1 - \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] + \left[ 1 - \frac{g}{c^2} \left( 2 D + a \right) \right] \left[ 1 + \frac{g}{c^2} \left( D + \frac{2}{3} a \right) \right] \right\} \hat{z} \approx \frac{1}{3} \frac{g}{c^2} F_C \hat{z} = \frac{E_C}{c^2} \vec{g},$$  \hspace{1cm} (4.19)

which is the same result as Eq. (4.9).

### V. CONCLUDING REMARKS

Einstein’s Equivalence Principle, according to which the Laws of Physics in a uniform gravitational field are the same as in a uniformly accelerated frame, is one of the most powerful and general principles of Physics. Initially formulated within the context of Classical Physics, it is currently regarded as a universal principle, which retains its validity also in the realm of Quantum Physics. Among the quantum phenomena, one of the most fundamental is that of vacuum fluctuations, with the associated unavoidable content of energy. It would clearly
be of great importance to test by experiments whether this quantum vacuum energy conforms to the Equivalence Principle. A convincing way to do that would be to verify whether the energy of vacuum fluctuations existing in a cavity with reflecting walls, i.e. the Casimir energy \( E_{(C)} \), gravitates as other conventional forms of matter-energy. While the feasibility of such an experiment by current weak-force measurement devices was discussed in Ref. [12], the present paper has analyzed in detail, from the point of view of General Relativity, the mechanical forces in a Casimir apparatus suspended in the Earth’s gravitational field. This is an essential step, because these forces are the quantities to be confronted with real experiments. For that purpose, we have derived a set of Cardinal Equations giving the conditions for mechanical equilibrium for any extended body, satisfying the covariant balance of energy and momentum between body and external fields, at rest in a uniform gravitational field. The key feature of these equations is that, in a gravitational field, forces are subject to redshifts, a phenomenon originally discovered by Nordtvedt [18], using heuristic arguments based on the Equivalence Principle. Consideration of this phenomenon is essential in order to obtain the correct values for the forces occurring in an intrinsically relativistic system, such as a Casimir apparatus. On the basis of these Cardinal Equations, we proved rigorously that, for the case of a rigid cavity, the weight associated with the Casimir energy \( E_{(C)} \) is equal to \( \frac{1}{2} E_{(C)}/c^2 \), as expected. Moreover, we considered the case of a Casimir cavity consisting of two disconnected plates, supported by separate mounts. Also for this case the general Cardinal Equations provide the relativistically correct expressions for the forces exerted by the mounts on the plates.

Encouraging agreement is also found between our force formulae, relying upon energy-momentum methods, and the force formulae in Ref. [16], which rely instead upon variational methods pioneered by Schwinger [23, 24].

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