Distribution of complex eigenvalues for symplectic ensembles of non-Hermitian matrices.

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Abstract

Symplectic ensemble of disordered non-Hermitian Hamiltonians is studied. Starting from a model with an imaginary magnetic field, we derive a proper supermatrix $\sigma$-model. The zero-dimensional version of this model corresponds to a symplectic ensemble of weakly non-Hermitian matrices. We derive analytically an explicit expression for the density of complex eigenvalues. This function proves to differ qualitatively from those known for the unitary and orthogonal ensembles. In contrast to these cases, a depletion of eigenvalues near the real axis occurs. The result about the depletion is in agreement with a previous numerical study performed for QCD models.
I. INTRODUCTION

Properties of non-Hermitian random operators or matrices have attracted recently a considerable attention. Non-Hermitian random Hamiltonians can appear as a result of mapping of a model for flux lines in a $(d + 1)$-dimensional superconductor with line defects in a tilted magnetic field on a $d$-dimensional model for bosons in a random potential. Non-Hermitian operators enter Fokker-Planck equations that describe diffusion and advection of classical particles in a spatially random but time-independent velocity field and also determine equations used for study of problems of turbulence.

Ensembles of random complex non-Hermitian and real asymmetric matrices find their application for a description of dissipative quantum maps in neural network dynamics. Recently it was suggested that they could be relevant for QCD where they correspond to a random Dirac operator with a non-zero chemical potential. Starting from the first works, properties of the ensembles of the non-Hermitian matrices were intensively studied in a considerable number of publications.

Unusual properties of the ensembles of the non-Hermitian operators or matrices are related to the fact that eigenvalues of the operators and matrices can be complex. Completely different methods have been used for study of distributions of the eigenvalues on the complex plane. For example, the authors of Refs. applied Green functions methods while in Refs. a method of orthogonal polynomials was used. In Ref., a new regime of a “weak non-Hermiticity” was found and the authors have calculated a joint probability of complex eigenvalues for complex weakly non-Hermitian matrices. For calculations they used the supersymmetry technique and derived a zero-dimensional non-linear $\sigma$-model. An important information about the distribution function of complex eigenvalues of $N \times N$ matrices for the orthogonal, unitary and symplectic chiral random matrix ensembles has been obtained recently numerically.

Although the model with the non-Hermitian Hamiltonian of Ref. differ from those with random matrices, they turn out to be closely related to each other. In Ref., the model with the non-Hermitian Hamiltonian $H$ was studied using the supersymmetry method. This Hamiltonian can be written in the form

$$ H = \frac{\left(\hat{p} + i\hbar\right)^2}{2m} + U(\mathbf{r}) , $$

(1.1)

where $\hat{p} = -i\nabla$, $m$ is the mass of particles, and $U(\mathbf{r})$ is a random potential. The vector $\mathbf{h}$ is proportional to the component of the magnetic field perpendicular to the direction of the line defects in the initial problem of the vortices in superconductors. The Hamiltonian describes a particle moving in an imaginary vector-potential $i\mathbf{h}$ and a real random potential $U(\mathbf{r})$. The distribution of complex eigenvalues on the complex plane can be extracted from the distribution function $P(\epsilon, y)$ defined as follows

$$ P(\epsilon, y) = \frac{1}{V} \left\langle \sum_k \delta(\epsilon - \epsilon'_k)\delta(y - \epsilon''_k) \right\rangle , $$

(1.2)

where $\epsilon'_k$ and $\epsilon''_k$ are the real and imaginary parts of the eigenenergies, respectively and $V$ is the volume of the system. The angle brackets stand for an averaging over the random potential and the sum should be taken over all states.
The problem of calculation of the function $P(\epsilon, y)$ was mapped in Ref. 24 onto a new supermatrix non-linear $\sigma$-model. This model differs from the conventional ones written previously 21 by the presence of new “effective fields”. The symmetry of the matrix $Q$ entering the new $\sigma$-model is the same as that of obtained in Ref. 21 for the orthogonal ensemble. This is not accidental because the Hamiltonian, Eq. (1.1), is real and, hence, time reversal invariant.

To violate the time reversal invariance one can add to the Hamiltonian (1.1) a real magnetic field or/and magnetic impurities. This leads to additional terms in the $\sigma$-model lowering the symmetry of the model. As a result, one gets 24 the $\sigma$-model with the supermatrices $Q$ corresponding to the unitary ensemble. Although the real magnetic interactions in the Hamiltonian with the imaginary vector potential do not correspond to any physical interactions in the initial problem of the vortices, consideration of the $\sigma$-model for the unitary ensemble was interesting from the formal point of view because it allowed to establish important relations with the random matrix models.

The $\sigma$-models corresponding to the Hamiltonian of Eq. (1.1) and its extensions can be written in an arbitrary dimension. Remarkably, the zero-dimensional version of the $\sigma$-model for the unitary ensemble is exactly the same as the zero-dimensional $\sigma$-model derived in Ref. 19 for the weakly non-Hermitian matrices. Complex random non-Hermitian matrices appeared in studies of dissipative quantum maps 10, 15 which justifies an interest to studying the unitary ensemble. By the term “weakly non-Hermitian” the authors of Ref. 19 called matrices $X$ that could be represented in the form

$$X = A + i\alpha N^{-1/2}B,$$

where $A$ and $B$ are Hermitian $N \times N$ matrices and $\alpha$ is a parameter characterizing the non-Hermiticity.

The $\sigma$-model obtained from the ensemble of the matrices, Eq. (1.3), and from the Hamiltonian with the imaginary vector-potential allows us to relate the parameters $h$ and $\alpha$ to each other. A similar correspondence exists for the orthogonal ensemble.

Study of the distributions of the complex eigenvalues revealed a striking difference between the orthogonal and unitary ensembles. The function $P(\epsilon, y)$, Eq. (1.2), is a smooth positive function of $y$ for the unitary (provided the disorder is not very strong, this function does not depend on $\epsilon$). It reaches its maximum at $y = 0$ and monotonously decays with increasing $y$. The corresponding function $P(\epsilon, y)$ for the orthogonal ensemble is a sum of a smooth function and a $\delta$-function of $y$. This means that a finite fraction of the eigenvalues remain real at any degree of the non-Hermiticity.

In all the works done in statistical physics only the orthogonal and unitary were considered. The symplectic ensembles have not been even mentioned, apparently due to the absence of any applications. However, for the random matrix ensembles applied for clarifying properties of QCD models 22, 17, 12, 23, the symplectic ensemble is of the same importance as the orthogonal and unitary ones. Moreover, numerical results for distributions of complex eigenvalues presented in Ref. 23 demonstrate a pronounced difference between the ensembles. The distribution of the complex eigenvalues on the complex plane is homogeneous in the case of the unitary ensemble while it shows an accumulation of the eigenvalues along the real axis. This corresponds to the presence of the $\delta$-function in the function $P(\epsilon, y)$, Eq. (1.2), found.
in Ref. 24. Although the authors of Ref. 23 considered chiral matrices, the dependence of the number of eigenvalues at the real axis on a parameter characterizing the non-Hermiticity was found to be exactly the same as in Ref. 24. This shows that the phenomenon of the accumulation is quite general.

A completely different behavior was found in Ref. 23 for the symplectic ensemble. The distribution function of the complex eigenvalues is in this case smooth but the probability of real eigenvalues turns to zero, which corresponds to a depletion of the eigenvalues along the real axis. This is a new effect that clearly motivates an analytical investigation of non-Hermitian symplectic matrices.

In the present publication the distribution function $P(\epsilon, y)$, Eq. (1.2), is calculated for the ensemble of symplectic non-Hermitian matrices. This is done by writing a proper zero-dimensional $\sigma$-model. We are able to obtain an explicit expression for the function $P(\epsilon, y)$ and demonstrate the depletion of the eigenvalues along the real axis.

The paper is organized as follows: In Sec. II, we introduce the notations and remind to the reader the scheme of the derivation of the $\sigma$-model. In Sec. III, we present the parametrization of the supermatrices $Q$ for the symplectic ensemble. In Sec. IV, the joint probability density of complex eigenvalues is calculated. Sec. V summarizes the results, and in the Appendix the Jacobian of the parametrization is derived.

II. NON-LINEAR $\sigma$-MODEL

The derivation of the $\sigma$ model for the non-Hermitian orthogonal and unitary ensembles has been comprehensively presented in Ref. 24. Addressing to this paper for all details, we repeat some intermediate steps, concentrating on minor changes that have to be done in the symplectic case. The final goal is to derive the joint probability density of complex eigenenergies $P(\epsilon, y)$, Eq. (1.2). Of course, one can derive the zero-dimensional $\sigma$-model from the ensemble of symplectic random matrices but we prefer to start from the Hamiltonian, Eq. (1.1), adding to it spin-orbit impurities.

Due to the non-Hermiticity the Hamiltonian, the notion of advanced and retarded Green functions, $G^A_\epsilon$ and $G^R_\epsilon$ usually used in perturbation theory and in deriving the non-linear $\sigma$-models becomes meaningless since they lose their analytic properties. The difficulty can be overcome by introducing an Hermitian double size operator $\hat{M}$ of the form

$$\hat{M} = \begin{pmatrix} H' - \epsilon & i(H'' - y) \\ -i(H'' - y) & -(H' - \epsilon) \end{pmatrix},$$

where

$$H' = \frac{(H + H^+)}{2}, \quad H'' = -\frac{i(H - H^+)}{2}.$$ (2.2)

In equations (2.1) and (2.2), $H$ is the Hamiltonian, Eq. (1.1), and $H^+$ means its Hermitian conjugated. Instead of manipulating the non-Hermitian operator, one can use the Hermitian operator $\hat{M}$ of the “effective Hamiltonian” $\hat{M}$, Eq. (2.1), one can represent the complex eigenvalues distribution function $P(\epsilon, y)$, Eq. (1.2), in a form of a functional
integral over supervectors $\psi(r)$ with the weight $\exp(-L)$ with the Lagrangian $\mathcal{L}$ taking the form

$$\mathcal{L} = -i \int \overline{\psi}(r)[\mathcal{H}_0 + U(r) + V_{so}(r)]\psi(r) \, dr.$$  \hfill (2.3)

Here, $\psi(r)$ and $\overline{\psi}(r)$ are the standard supervector and its charge-conjugated counterpart, respectively, composed from anticommuting and commuting fields. The matrix operator $\mathcal{H}_0$ consists of two terms

$$\mathcal{H}_0 = (H'_0 - \epsilon + i\gamma\Lambda)I + i\Lambda_1 (H''_0 + y\tau_3),$$  \hfill (2.4)

where $H'_0$ and $H''_0$ have the form

$$H'_0 = \frac{\hat{\mathbf{p}}^2}{2m}, \quad H''_0 = -\frac{\hbar\mathbf{p}}{m}. \hfill (2.5)$$

In equation (2.4), $\gamma$ is a small positive number that should be put to zero at the end of calculations. The term $V_{so}$ in Eq. (2.3) stands for spin-orbit impurities. It can be derived from the initial Hamiltonian after a formal inclusion of the interaction $U_{so}(r)$ with the spin-orbit impurities. The simplest form of this interaction can be written as follows

$$U_{so}(r) = \sigma \left[ \nabla u_{so}(r) \times \mathbf{p} \right], \hfill (2.6)$$

where the vector $\sigma$ is formed from the Dirac matrices $\sigma_x$, $\sigma_y$, and $\sigma_z$. The matrices $I$, $\Lambda$, $\Lambda_1$ and $\tau_3$ entering Eq. (2.4) have the form

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \hfill (2.7)$$

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \hfill (2.8)$$

Due to the necessity of considering the spin variables the supervectors $\psi(r)$ have now 16 components. The unit blocks $\mathbf{1}$ in the matrices in Eq. (2.4) have the size $8 \times 8$ and unities 1 entering the matrix $\tau_3$ are $2 \times 2$ matrices.

The distribution of the electric fields $\nabla u_{so}(r)$ is assumed to be Gaussian:

$$\langle \nabla u_{so}(r) \rangle = 0, \quad \langle \partial_i u_{so}(r) \partial_j u_{so}(r) \rangle = \frac{\delta_{ij} \delta(r - r')}{6\pi \nu \tau_{so}}, \hfill (2.9)$$

where the density of states at $\hbar = 0$ at the Fermi surface $\nu = mp_0/2\pi^2$ and $\tau_{so}$ is the spin-orbit scattering time.

Further transformations are performed according to standard rules of the supersymmetry technique. Averaging over the disorder results in an interaction term $\psi^4$ in the Lagrangian $\mathcal{L}$. This term is decoupled by integration over a supermatrix $Q$. Then, one integrates the supervector $\psi$ out, arriving thus at an integral over $Q$ with the weight $\exp(-F[Q])$, where $F[Q]$ is a free energy functional.
The spin-orbit interactions lead to additional “effective fields” of a certain symmetry in the free energy functional $F[Q]$. These fields lower the symmetry of the functional. As a result, a part of fluctuational modes have a gap and their contribution at low energies can be neglected. This is equivalent to putting certain elements of the supermatrix $Q$ to zero. Carrying out this procedure, one comes to a matrix $Q$ with spin blocks proportional to unit matrices. This is equivalent to consideration the model with $8 \times 8$ supermatrices $Q$ having a new symmetry. These are the same supermatrices as those used in Ref. 21 for description of the symplectic case.

Therefore, we should perform calculations similar to those of Ref. 24 but integrating over the supermatrices $Q$ with the symmetry corresponding to the symplectic ensemble. After standard transformations, we reduce the distribution function $P(\epsilon, y)$ to the following integral

$$P(\epsilon, y) = -\frac{\pi \nu}{4\Delta} \int A[Q] \exp (-F[Q]) dQ,$$

(2.10)

$$A[Q] = (Q_{11}^{11} + Q_{12}^{22}) (Q_{24}^{11} + Q_{24}^{22}) - (Q_{12}^{21} + Q_{12}^{12}) (Q_{24}^{12} + Q_{24}^{21})$$

with the zero-dimensional version of the free-energy functional

$$F[Q] = \text{STr} \left( \frac{a^2}{16} [Q, \Lambda_1]^2 - \frac{x}{4} \Lambda_1 \tau_3 Q \right).$$

(2.11)

In equation (2.11), the symbol $[.]$ stands for commutator, $\text{STr}$ for supertrace and we have introduced the following parameters:

$$a^2 = \frac{2\pi D_0 h^2}{\Delta}, \quad x = \frac{2\pi y}{\Delta},$$

(2.12)

where $D_0$ is the classical diffusion coefficient and $\Delta = (2\nu V)^{-1}$ is the mean level spacing (the factor 2 in this expression is due to lifting of the spin degeneracy by the spin-orbit impurities).

### III. Parametrization for the Supermatrices $Q$

To integrate over all symplectic matrices, proper variables parametrizing the $Q$ supermatrix should be introduced. The parameters have to be chosen so that to cover all the set of the symplectic matrices: Any symplectic matrix has to be reached only once.

Although the parametrization for the unitary and the orthogonal ensembles cannot be used for the symplectic ensemble only minor changes have to be done to adjust the non-Hermitian parametrization [24] to the case under consideration. As in Ref. 23, we represent the $Q$ matrix in the form of the product

$$Q = TYQ_0 Y^T.$$

(3.1)

To fulfill the constrain $Q^2 = 1$, the following equalities must be hold
\[ Q_0^2 = 1, \quad T \overline{T} = 1, \quad Y \overline{Y} = 1. \quad (3.2) \]

As in Ref. 24, the supermatrices \( T \) and \( Y \) are chosen to commute with \( \Lambda_1 \)

\[ [T, \Lambda_1] = [Y, \Lambda_1] = 0. \quad (3.3) \]

The next simplification facilitates the parametrization and enables one to calculate the Jacobians quickly. Namely, we decompose the supermatrix \( Y \) into the product of the matrix \( Y_0 \) containing commuting variables and the matrices \( R \) and \( S \), consisting of the Grassmann ones

\[ Y = Y_0 RS. \quad (3.4) \]

The 2 \( \times \) 2 blocks in the matrices \( R, S \), and \( Q_0 \) are chosen to be diagonal; the necessary symmetry of the \( 2 \times 2 \) blocks \( a, b \) and \( \sigma \) is achieved by a proper choice of \( 2 \times 2 \) blocks of the matrix \( Y_0 \). Thus, the matrices \( Q_0, R, \) and \( S \) can be written in a form similar to the one in 24

\[ Q_0 = \begin{pmatrix} \cos \hat{\phi} & -\tau_3 \sin \hat{\phi} \\ -\tau_3 \sin \hat{\phi} & \cos \hat{\phi} \end{pmatrix}, \quad \hat{\phi} = \begin{pmatrix} \varphi & 0 \\ 0 & i\chi \end{pmatrix}, \quad (3.5) \]

where \( \varphi \) and \( \chi \) are proportional to the unity 2 \( \times \) 2 matrix,

\[ R = \begin{pmatrix} \hat{R} & 0 \\ 0 & \hat{R} \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 1 - 2\rho \overline{\rho} & 2\rho \\ -2\overline{\rho} & 1 + 2\rho \overline{\rho} \end{pmatrix}, \quad (3.6) \]

and

\[ S = \begin{pmatrix} 1 - 2\hat{\sigma}^2 & 2i\hat{\sigma} \\ 2i\hat{\sigma} & 1 - 2\hat{\sigma}^2 \end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix} 0 & \sigma \\ \overline{\sigma} & 0 \end{pmatrix}. \quad (3.7) \]

The matrices \( \rho \) and \( \sigma \) in Eqs. (3.6, 3.7) have the form

\[ \rho = \begin{pmatrix} \rho & 0 \\ 0 & \rho^* \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^* \end{pmatrix}. \quad (3.8) \]

The next step is to represent the supermatrix \( Y_0 \) in Eq. (3.4) as the product

\[ Y_0 = Y_3Y_2Y_1, \quad (3.9) \]

where \( Y_3 \) is the diagonal matrix

\[ Y_3 = \begin{pmatrix} \exp(i\hat{\beta}/2) & 0 \\ 0 & \exp(i\hat{\beta}/2) \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} \beta \tau_3 & 0 \\ 0 & \beta_1 \tau_3 \end{pmatrix}. \quad (3.10) \]

In order to recover the symplectic symmetry we have to choose the matrices \( Y_1 \) and \( Y_2 \) as follows

\[ Y_1 = \begin{pmatrix} \hat{\omega} & 0 \\ 0 & \hat{\omega} \end{pmatrix}, \quad \hat{\omega} = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}, \quad (3.11) \]

\[ w = \begin{pmatrix} \cosh(\mu/2) & -i\sinh(\mu/2) \\ i\sinh(\mu/2) & \cosh(\mu/2) \end{pmatrix}, \]
\[ Y_2 = \begin{pmatrix} \cos(\hat{\theta}_2/2) & -i \sin(\hat{\theta}_2/2) \\ -i \sin(\hat{\theta}_2/2) & \cos(\hat{\theta}_2/2) \end{pmatrix}, \quad (3.12) \]

\[
\hat{\theta}_2 = \begin{pmatrix} \theta_2 \tau_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Finally, the supermatrix \( T \) can be taken as
\[
T = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos(\hat{\theta}/2) & -i \sin(\hat{\theta}/2) \\ -i \sin(\hat{\theta}/2) & \cos(\hat{\theta}/2) \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \quad (3.13)
\]

where
\[
\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i \theta_1 \end{pmatrix},
\]
\[
u = \begin{pmatrix} 1 - 2\eta\eta^* - 2\eta & 2\eta \\ -2\eta & 1 - 2\eta\eta^* \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 - 2\kappa\kappa^* - 2\kappa & 2\kappa \\ -2\kappa & 1 - 2\kappa\kappa^* \end{pmatrix}.
\]

The 2 \times 2 matrices \( \theta \) and \( \theta_1 \) in Eq. (3.13) are proportional to the unit matrix and the matrices \( \eta \) and \( \kappa \) are
\[
\eta = \begin{pmatrix} \eta & 0 \\ 0 & \eta^* \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa^* \end{pmatrix}.
\]

The explicit form of the supermatrix \( Q \) within the parametrization suggested, Eqs. (3.5-3.14), is very similar to that for the orthogonal ensemble and differs from the latter by minor changes in the matrices \( \hat{w}, \hat{\theta}_2, \sigma, \rho, \eta, \) and \( \kappa \).

To ensure the unambiguity of the parametrization, we should specify the variation range of the variables. This is done by comparing the compact and the noncompact sector with those in the standard parametrization. As the result, the variables vary in the following intervals:
\[
-\pi/2 < \varphi < \pi/2, \quad 0 < \chi < \infty, \quad -\pi < \theta < \pi, \quad -\infty < \theta_1 < \infty, \quad (3.15)
\]
\[
0 < \mu < \pi, \quad 0 < \theta_2 < \infty, \quad 0 < \beta < \pi, \quad 0 < \beta_1 < 2\pi.
\]

The only thing that remains to be done to perform explicit calculations for physical quantities is calculation of the Jacobian of the transformation to the variables described by Eqs. (3.5-3.14).

Its derivation presented in the Appendix leads to the following final result for the elementary volume
\[
[dQ] = J_\varphi J_\theta J_\mu J_\chi dR_B dR_F, \quad (3.16)
\]

where
\[ J_\varphi = \frac{1}{8\pi} \frac{\cos \varphi \cosh \chi}{(\sinh \chi + i \sin \varphi)^2}, \quad J_\theta = \frac{1}{32\pi} \frac{1}{\sinh^2 \frac{1}{2}(\theta_1 + i\theta)}, \quad J_\mu = \frac{1}{2^8\pi^2} \frac{\sin \theta_2 \sinh \mu}{(\cos \theta_2 - \cosh \mu)^2}, \quad J_c = \frac{4\sinh^2 \chi}{(\sinh \chi - i \sin \varphi)^2} . \] (3.17)

and

\[ dR_B = d\theta d\theta_1 d\varphi d\chi d\mu d\theta_2 d\beta d\beta_1, \quad dR_F = d\eta d\eta^* d\kappa d\kappa^* d\sigma d\sigma^* d\rho d\rho^*. \] (3.18)

Equations (3.1-3.17) are sufficient for evaluation of any integral over the supermatrices \( Q \) and, with the help of Eqs. (2.10, 2.11), provides a straightforward way of calculating the distribution function of complex eigenvalues \( P(\epsilon, y) \), Eq. (1.2).

### IV. DENSITY OF COMPLEX EIGENVALUES

Before staring the calculations let us introduce more compact notations. As it will be seen in what follows, only the following combinations of the variables describing the parametrization of the supermatrix \( Q \) enter all functions of interest

\[ t = \sin \varphi, \quad z = \sinh \chi, \quad \omega = \cosh \mu, \quad \lambda = \cos \theta_2 . \] (4.1)

Since the matrices \( T \) and \( Y \) commute with \( \Lambda_1 \), the first term in the free energy, Eq. (2.11) does not depend on them. The second term in Eq. (2.11) does not depend on \( T \). As a result, the free energy \( F[Q] \) takes a rather simple form

\[ F[Q] = a^2(t^2 + z^2) + x[(\lambda t - i\omega z) + 4(\sigma \sigma^* + \rho \rho^*)(\omega - \lambda)(t - iz)] . \] (4.2)

The fact that \( F[Q] \) does not depend on \( T \) simplifies the integration over \( Q \) in Eq. (2.10). Using the parametrization, Eqs. (3.13.13), we can also represent the supermatrix \( Q \) as

\[ Q = u\tilde{Q}\pi , \] (4.3)

with \( u \) from Eq. (3.14) and some supermatrix \( \tilde{Q} \). Substituting Eq. (4.3) into Eq. (2.10) for the density of complex eigenvalues \( P(\epsilon, y) \) and integrating over \( \eta \) and \( \eta^* \), one represents this function in the form

\[ P(\epsilon, y) = \frac{\pi \nu}{4\Delta} \int [\text{Str}(\pi_3\Lambda_1\tilde{Q})]^2 \exp(-F[Q]) d\tilde{Q} \]
\[ = \frac{4\pi \nu}{\Delta} \frac{d^2}{dx^2} \int \exp(-F[Q]) dQ . \] (4.4)

For the symplectic ensemble, one has in Eq. (4.4) an uncertainty of the type \( 0 \times \infty \), since the integrand does not contain the variables \( \kappa \) and \( \kappa^* \) and, on the other hand, the Jacobians \( J_\theta \) and \( J_\mu \) are singular as \( \theta, \theta_1, \theta_2, \) and \( \mu \rightarrow 0 \). To resolve this singularity, we can use the regularization procedure developed for the orthogonal ensemble. All the manipulations are identical to those of Ref. 2, because the free energy, Eq. (4.2) has the form similar to that of the orthogonal ensemble. Moreover, the singularities of the Jacobians, Eqs. (3.17), are the
same as the ones for the orthogonal ensemble. We do not specify the procedure once more and present here only the final result of the regularization with proper changes of notations.

The function \( P(\epsilon, y) \) can be written in the form of a sum of two terms

\[
P(\epsilon, y) = P^{(1)}(\epsilon, y) + P^{(2)}(\epsilon, y),
\]

where

\[
P^{(1)}(\epsilon, y) = \frac{\nu}{4\Delta} \frac{d^2}{dx^2} \int \exp[-a^2(t^2 + z^2) - x(t - iz)] \frac{4z^2dt dz}{(t^2 + z^2)^2}
\]

and

\[
P^{(2)}(\epsilon, y) = \frac{\nu}{4\Delta} \frac{d^2}{dx^2} \int \exp[-a^2(t^2 + z^2) - x(t\omega - i\lambda z)]
\times \frac{(t - iz)^2 z^2 x^2}{(t^2 + z^2)^2} dt \, dz \, d\omega \, d\lambda.
\]

The integration in Eqs. (4.6) and (4.7) is performed in the intervals \(-1 < t < 1, -\infty < z < \infty, 1 < \omega < \infty\), and \(-1 < \lambda < 1\).

To perform the integration in Eq. (4.7) over \( \lambda \), one should introduce an infinitesimal positive \( \delta \), defining \( z_- \) according to \( z_- = z + i\delta \text{sgn}(x) \), so that the integral becomes convergent. After this, the integration over \( \lambda \) and \( \omega \) in Eq. (4.7) is easily carried out. Adding the result of the integration to Eq. (4.6) we obtain

\[
P(\epsilon, y) = \frac{\nu}{4\Delta} \frac{d^2}{dx^2} \int_{-1}^{+1} dt \int_{-\infty}^{+\infty} dz \exp(-a^2(t^2 + z_-^2))[(t + iz_-)^2 \exp(ixz_- - tx) - (iz_- - t)^2 \exp(ixz_- + tx)] \frac{z_-}{it(t^2 + z_-^2)}.
\]

Comparing Eq. (4.8) with its analog for the orthogonal ensemble, we notice an important difference between them: The variable \( z_- \) is present in the numerator in Eq. (4.8), whereas it stands in the denominator of the equation for the orthogonal ensemble. In the latter case, the distribution function \( P(\epsilon, y) \) contains an additional contribution of a \( \delta \)-function after differentiation of \( z_- \) over \( x \). For the symplectic ensemble, the differentiation of \( z_- \) in the numerator leads to no singularity on the real axis and one take the limit \( \delta \to 0 \) before calculating the integral in Eq. (4.8). Thus, only the exponents should be differentiated over \( x \). These differentiations simplifies considerably the integrand and the integration over \( z \) can be easily carried out. After that one obtains

\[
P(\epsilon, y) = \frac{\nu}{\Delta} \frac{x}{4a^2} \sqrt{\pi} \exp \left(-\frac{x^2}{4a^2} \right) \int_0^1 dt \exp(-a^2t^2) \frac{\sinh(tx)}{t}.
\]

Equation (4.9) solves completely the problem involved and is the main result of the present paper.

The following properties of the density function \( P(\epsilon, y) \) are easily checked: It is symmetric with respect to \( y \) and is properly normalized
\[ \int dy P(\epsilon, y) = 1. \tag{4.10} \]

In the limit \( a \gg 1 \)(the limit of a strong non-Hermiticity), one obtains the universal asymptotics valid for all three ensembles
\[ P(\epsilon, y) \simeq \frac{\pi \nu}{2a^2 \Delta} \begin{cases} 1, & 2a \ll |x| < 2a^2 \\ 0, & |x| > 2a^2 \end{cases} \tag{4.11} \]

The form of the density of complex eigenstates, Eq. (4.11), corresponds to the “elliptic law” of Refs. 13, 14.

In the opposite limit \( a \ll 1 \), the function \( P(\epsilon, y) \) can be written as
\[ P(\epsilon, y) = \frac{\nu}{\Delta} \frac{x^2}{4a^3} \sqrt{\pi} \exp \left( -\frac{x^2}{4a^2} \right). \tag{4.12} \]

The behavior of the function \( P(\epsilon, y) \), Eq. (4.13), at small \( y \) (related to \( x \) by Eq. (2.12)) is drastically different from the behavior of the corresponding functions for the orthogonal and unitary ensembles. This function is small at small \( y \), being proportional to \( y^2 \) and turns to zero in the limit \( y \to 0 \). This means that the probability that eigenvalues remain real at finite degree of the non-Hermiticity is zero. In other words, the distribution function of complex eigenvalues exhibits a depletion along the real axis. The depletion region broadens with increasing the non-Hermiticity. The function \( P(\epsilon, y) \) is represented in Fig. 1 for several values of \( a \approx 1 \).

V. CONCLUSIONS

In the present paper, we studied analytically disordered non-Hermitian models with the symplectic symmetry. This is the last of three universality classes that has not been considered yet. Using the supersymmetry technique we derived a proper non-linear \( \sigma \)-model starting from the a model of disorder with a direction. The zero-dimensional version of the non-linear \( \sigma \)-model corresponds the ensemble of random non-Hermitian symplectic matrices. Within the zero-dimensional \( \sigma \)-model, we calculated the joint probability density function of complex eigenvalues. We introduced a convenient parametrization and calculated the Jacobian corresponding to this parametrization.

All this allowed us to derive an explicit expression for the density of complex eigenvalues. Asymptotic behavior of this function demonstrates clearly that the basic properties of the system depend strongly on the ensemble. Introducing the non-Hermiticity in the Hamiltonian affects very differently the spectrum of three ensembles. Only when the non-Hermiticity is very large, the difference is no longer important.

It is known from previous works that the eigenvalues of a system belonging to the unitary ensemble are smoothly distributed around the real axis. The density function for the orthogonal ensemble contains a \( \delta \)-function contribution on the real axis describing an accumulation of the eigenvalues. In contrast to the previous cases, we obtained for the symplectic ensemble a depletion of eigenvalues along the real axis, which is in a good agreement with the results of a numerical study23. These features correspond to a tendency of a system from the orthogonal ensemble to preserve localized behavior. However, after introducing spin-orbit impurities, the system acquires delocalized features.
VI. APPENDIX

The Jacobian of the parametrization specified by Eqs. (3.1-3.14) can be derived from the elementary length \( \text{Str}(dQ)^2 \). The most economical way to proceed is to compare the parametrization involved with that for the orthogonal ensemble \(^{24}\). Two essential differences are easily noticed: the \( 2 \times 2 \) blocks in matrices \( Y_1 \) and \( Y_2 \) are interchanged and all the conjugated Grassmann variables have the opposite sign. The last difference, however, does not lead to any change in the calculation, so long as the contribution to the length \( \text{Str}(dQ)^2 \) from the Grassmann variables is due to terms of the kind \( \bar{\eta} d\kappa, \bar{\eta} d\eta \) etc. Taking this into account, we can immediately reduce the elementary length to the following expression

\[
\text{Str}(dQ)^2 = \text{Str}(\left( dQ_0 \right)^2 + [\delta Z, Q_0]^2),
\]

where all the terms apart from the last one, \( \text{Str}(Y_0 R S) \), are identical to those in Ref. \(^{24}\).

Using Eq. (3.9), we write \( \delta Y_0 \) as

\[
\delta Y_0 = \delta Y_1 + \delta Y_2 + Y_1 Y_2 \delta Y_3 Y_2 Y_1,
\]

which can be rewritten in the form

\[
\delta Y_0 = \frac{1}{2} \left[ \begin{pmatrix} d\beta_1 \tau_3 \cos \theta_2 & 0 \\ 0 & d\kappa \end{pmatrix} - d\mu \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 \end{pmatrix} \right] + \Lambda_1 \left[ - \begin{pmatrix} \tau_2 d\beta_2 \sin \theta_2 & 0 \\ 0 & 0 \end{pmatrix} + d\theta_2 \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} \right],
\]

where

\[
\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

Multiplying three matrices with each other, one obtains

\[
Y_0 \delta TY_0 = 1 \times 2 \left[ \cos \frac{\theta_2}{2} \begin{pmatrix} 0 & d\kappa' \\ 0 & 0 \end{pmatrix} + i \sin \frac{\theta_2}{2} \begin{pmatrix} 0 & \tau_1 d\eta' \\ 0 & 0 \end{pmatrix} \right] + 2i\Lambda_1 \left[ \cos \frac{\theta_2}{2} \begin{pmatrix} 0 & \eta' \\ \tau_1 & 0 \end{pmatrix} - i \sin \frac{\theta_2}{2} \begin{pmatrix} 0 & \tau_1 d\kappa' \\ 0 & 0 \end{pmatrix} \right],
\]

where \( d\kappa' = d\kappa w \exp[i(\beta - \beta_1)/2] \) and \( d\tau' = w d\kappa \exp[i(\beta - \beta_1)/2] \) and analogous for \( d\eta' \) and \( d\eta' \). One should keep in mind that the differentials \( d\eta \) and \( d\kappa \) in Eq. (3.14) are not the initial variables entering Eq. (3.14) but new variables obtained from the initial ones by several replacements and shifts common for all three ensembles. The Jacobian of those transformations \( J_{\theta} \) is given by Eqs. (3.17).

After that we pick up the differentials of the Grassmann variables (proportional to the unit matrix), make a shift of the differentials analogous to the one in Ref.\(^{24}\) and introduce the matrix differentials.
\[ d\sigma = \begin{pmatrix} d\sigma_1 & d\sigma_2 \\ -d\sigma_2^* & d\sigma_1^* \end{pmatrix}, \quad d\rho = \begin{pmatrix} d\rho_1 & d\rho_2 \\ -d\rho_2^* & d\rho_1^* \end{pmatrix}, \quad (6.6) \]

where

\[ d\sigma_2 = -i \cos \frac{\theta_2}{2} \sinh \frac{\mu}{2} d\eta - \sin \frac{\theta_2}{2} \cosh \frac{\mu}{2} d\kappa^* \]
\[ d\sigma_2^* = -i \cos \frac{\theta_2}{2} \sinh \frac{\mu}{2} d\eta^* + \sin \frac{\theta_2}{2} \cosh \frac{\mu}{2} d\kappa \]
\[ d\rho_2 = -i \cos \frac{\theta_2}{2} \sinh \frac{\mu}{2} d\kappa + \sin \frac{\theta_2}{2} \cosh \frac{\mu}{2} d\eta^* \]
\[ d\rho_2^* = -i \cos \frac{\theta_2}{2} \sinh \frac{\mu}{2} d\kappa^* - \sin \frac{\theta_2}{2} \cosh \frac{\mu}{2} d\eta \]
\[ (6.7) \]

The Jacobian of the transformation, Eq. (6.7), from \( \eta, \kappa \), to \( \sigma \) and \( \rho \), equals

\[ \tilde{J}_\mu = 4 \cdot \frac{1}{(\cos \theta_2 - \cosh \mu)^2}. \]
\[ (6.8) \]

The supermatrix \( \delta Z \) from Eq. (6.1) can be represented as

\[ \delta Z = \delta Y'_0 + i\Lambda_1 (2d\hat{\sigma} - d\hat{\theta}/2) + 1 \times 2k d\hat{\rho}, \]
\[ (6.9) \]

where

\[ k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

and

\[ \delta Y'_0 = -\frac{1}{2} i \left[ d\beta \sinh \mu \begin{pmatrix} 0 & 0 \\ 0 & \tau_1 \end{pmatrix} \right] + d\mu \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 \end{pmatrix} \]
\[ + \Lambda_1 \frac{1}{2} \left[ -d\beta_1 \sin \theta_2 \begin{pmatrix} \tau_2 & 0 \\ 0 & 0 \end{pmatrix} + d\theta_2 \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} \right]. \]
\[ (6.10) \]

Calculating the length \( \text{Str}(dQ)^2 \) one notices that the anticommuting part of \( \delta Z \), proportional to \( \Lambda_1 \), decouples from the commuting one, proportional to the unit matrix 1. The contribution to the length from the first part is the same as that of Ref. 24 leading to the Jacobian

\[ J_{\varphi \chi} = \frac{1}{2^{24}} \frac{1}{(\sin^2 \varphi + \sinh^2 \chi)^2}, \]
\[ (6.11) \]

whereas the second part of the elementary length equals

\[ \text{Str}[\delta Z \parallel Q_0]^2 = 4 \{ [(d\mu)^2 + (d\beta)^2 \sinh \mu] \sinh^2 \chi + (d\theta)^2 \cos^2 \varphi \]
\[ + (d\theta_1)^2 \cosh^2 \chi + (d\theta_2)^2 + (d\beta_1)^2 \sin^2 \theta_2 \} \]
\[ (6.12) \]

Since in our parametrization the blocks from the commuting variables in the matrices \( Q_0 \) and \( T \) are the same as in Ref. 24, the Jacobian \( J_\theta \) does not change. Combining the contributions from Eqs. (6.8, 6.11, 6.12) with \( J_\theta \), we arrive at the elementary volume, Eq. (3.16–3.18).
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FIGURE CAPTIONS

Figure 1. The dependence of complex eigenvalues of energy $P(\epsilon, y)$ on the imaginary part of energy $y = x\Delta/2\pi$, for several values of non-Hermiticity $a = 2\pi D_0 h^2/\Delta$. 