A New Control Scheme for the Lattice Hydrodynamic Model With the Successive Flux Difference Under Honk Environment

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ABSTRACT Due to the rapid development of the Intelligent Traffic System (for short, ITS), the shared information among successive vehicles can be acquired accurately and reliably. In this paper, a control scheme utilizing the successive flux difference under honk environment (SFDHE) is investigated in the lattice hydrodynamic model. Based on the Hurwitz criteria and the $H_{\infty}$-norm, the sufficient condition for the transfer function is derived with the control theory analysis. The Bode-plot of transfer function indicates that the stability of the traffic flow is increased significantly with the novel controller. Moreover, the numerical simulation results of the new model are compared with that of Peng’s model, which demonstrates that traffic jams can be suppressed efficiently when the SFDHE control signal is taken into account.

INDEX TERMS Control method, successive flux difference, lattice hydrodynamic model, traffic flow.

I. INTRODUCTION

As the acceleration of global urbanization, the traffic congestion problem has become an important issue of modern traffic theory. To address this issue, various traffic flow models have been developed such as car-following models [1]–[6], cellular automaton models [7], [8], the continuum models [9]–[11], lattice hydrodynamic models [12]–[16] and so on. Due to the advantage of convenient implementation, car-following models have been widely applied to describe the individual characteristics of traffic flow with velocity, position and acceleration information. While lattice hydrodynamic models focus on the traffic characteristics through variational relationships among density, flow and velocity variables. Based on the first lattice hydrodynamic model, many lattice models have been developed with various traffic factors such as traffic flux difference [14], [17], [18], traffic density difference [19]–[21], optimal current difference [22]–[24].

Nevertheless, the literature mentioned above only analyze the mechanism of the traffic jam with linear as well as nonlinear stability methods. Konishi et al. [25] firstly studied the traffic jam suppression of the car-following model with a decentralized delayed-feedback control method. After that, many extended car-following models [26]–[28] have been carried out with control theory. Inspired by the development of control methods in car-following models, tremendous attention has been attracted to lattice hydrodynamic models [29]–[33] with the modern control theory [34]–[38].

Hong-Xia et al. [39] utilized the flux difference as a feedback signal to suppress traffic jams. Li et al. [40] proposed a novel control signal with the density change rate difference. Cen et al. [41] proposed the next-nearest-neighbor interactions with the control method. Moreover, to analyze the long-term traffic behaviors, Cen et al. also studied the kink–antikink density waves. In a real traffic system, when a vehicle impedes its rear vehicles to move, the rear vehicles could honk their horns on the congested road. The honk can be taken as a control signal for the preceding driver to adjust the vehicle’s velocity. Therefore, the honk effect has been studied in multi-lane road scenes [42], different driver’s characteristics scenes [43], individual difference scenes [44] and so on. Peng et al. [42]–[45] fully considered the impact of the individual flux difference from maximum flux under honk environment. Simulation results revealed that the control scheme with the difference between the maximum flux and current flux could efficiently increase the stability of traffic flow.

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In recent years, with the rapid development of wireless communication technology and smart vehicles, high-quality shared information among successive vehicles can be acquired reliably and efficiently [46]–[49]. Based on this fact, the novel control scheme is firstly utilized to address the traffic congestion by considering the successive flux difference under honk environment.

The structure of this paper is organized as follows. The extended model with a control scheme will be established in Section II. The control theory analysis of the model will be established in Section III. In Section IV, the numerical simulation results of the new model are compared with that of Peng’s model, which verifies the effectiveness of the new control signal. Finally, the conclusions are drawn in Section V.

II. THE EXTENDED MODEL WITH A CONTROL SCHEME

The basic lattice hydrodynamic model was put forward to describe the evolution of single-lane traffic flow as follows [50]:

$$\frac{\partial \rho_j}{\partial t} + \rho_j \left( \frac{q_j - q_{j-1}}{\rho_j} \right) = 0$$

(1)

$$\frac{\partial q_j}{\partial t} = a \left[ \rho_0 V (\rho_{j+1} - q_j) \right],$$

(2)

where $\rho_0$ and $a$ represent the average density and the driver’s sensitivity, respectively. $q_j = \rho_j v_j$ denotes the flux at lattice $j$ at time $t$. The item $\rho_0 V (\rho_{j+1} - q_j)$ implies the variation of traffic current between the optimal current and the actual current. And the optimal velocity function $V (\rho_{j+1})$ is adopted as

$$V (\rho_{j+1}) = \left( \frac{\nu_{\text{max}}}{2} \right) \left[ \tanh \left( \frac{2 (\rho_{j+1} - \rho_0)}{\rho_0 - 1} \right) + \tanh \left( \frac{1}{\rho_0} \right) \right],$$

(3)

where $\nu_{\text{max}}$ and $\rho_c$ indicate the maximal speed and the critical density, respectively.

Based on the above basic lattice hydrodynamic model and control theory, Peng et al. [45] proposed a feedback control signal to the stabilization of traffic flow under honk environment:

$$\frac{\partial \rho_{j+1}}{\partial t} + \rho_{j+1} \left( \frac{q_{j+1} - q_j}{\rho_{j+1}} \right) = 0$$

(4)

$$\frac{\partial q_j}{\partial t} = a \left[ \rho_0 V (\rho_{j+1} - q_j) \right] + u_j$$

(5)

$$u_j = \mu \frac{q_m - q_j}{\tau},$$

(6)

where $\mu$ represents the feedback gain; $\tau$ is the reaction time for the honk effect; $q_m$ means the maximum flux.

As discussed previously, smart vehicles can acquire high-quality shared information among successive vehicles such as traffic density, velocity, flow, and so on.

However, as far as author knowledge is concerned, the control scheme considering the successive flux difference under honk environment has not been explored in the lattice hydrodynamic model. Therefore, a control scheme $u_j$ with more shared information is proposed as follows:

$$u_j = b \left[ (1 + k) q_{j+1} - q_j \right] + \mu \frac{q_m - q_j}{\tau},$$

(7)

where $b$ and $k (k > -1)$ denote a constant parameter and the strength coefficient of flux $q_{j+1}$, respectively. When $k > 0$, the flux effect of the $(j + 1)^{th}$ vehicle is magnified. While $k < 0$, the flux effect of the $(j + 1)^{th}$ vehicle is weakened. In addition, the item $(1 + k) q_{j+1} - q_j$ reflects the real flux difference when $k = 0$. The optimal velocity function $OV$ is chosen as the same as Eq.(3). In Ref [45], Peng’s model concentrated mainly on the difference between the maximum flux $q_m$ and the flux $q_j$. In addition, our control scheme focuses on the successive flux difference $(q_{j+1} - q_j)$.

III. CONTROL THEORY ANALYSIS

In this section, the stability analysis is applied to discuss how the SFDHE control signal suppresses traffic jams by means of control theory. $[\rho_n, q_n]^T = \left[ \rho^*, q^* \right]^T$ is assumed as the steady-state uniform flow solution with the desired density and flux of the traffic flow.

Combining perturbations $[\rho^0, q^0]$ with Eqs.(4), (5) and (7), the controlled system can be linearized as follows:

$$\begin{cases}
\frac{\partial \rho_0}{\partial t} + \rho_0 \left( \frac{q_{0j+1} - q_j}{\rho_0} \right) = 0 \\
\frac{\partial q_j}{\partial t} = a \left[ \rho_0 V (\rho_{j+1} - q_j) \right] + u_j
\end{cases}$$

(8)

where $\rho_0 = \rho_{j+1} - \rho^*; q_j = q - q^*; q_{0j+1} = q_{j+1} - q^*$.

$$\Lambda = \frac{\partial V (\rho_{j+1})}{\partial q_j} \bigg|_{\rho_0=\rho_0} = b \left[ (1 + k) q_{0j+1} - q_j \right] + \mu \frac{q_m - q_j}{\tau}.$$ 

By performing Laplace transformation and ignoring higher-order terms of Eq.(8), one can obtain:

$$s P_{j+1}(s) - q_j(0) + \rho_0 \left[ Q_{j+1}(s) - Q_j(s) \right] = 0$$

(9)

$$s Q_j(s) - q_j(0) = a \left[ \rho_0 \Lambda P_{j+1}(s) - Q_j(s) \right] + b \left[ (1 + k) Q_{j+1}(s) - Q_j(s) \right] + \mu \left( \frac{q_m - Q_j(s)}{\tau} \right),$$

(10)

where $P_{j+1}(s) = L (\rho_{j+1}), Q_{j+1}(s) = L (q_{j+1}), Q_{j+1}(s) = L (q_{j+1}). L (\cdot)$ and $s$ represent the Laplace transform function and the complex number frequency, respectively.

By eliminating $P_{j+1}(s)$ of Eqs.(9) and (10), the flux equation can be expressed as

$$Q_j(s) = \frac{sb (1 + k) - a \rho_0^2 \Lambda}{s^2 + (a + b) s + \frac{3a + b}{\tau}} Q_{j+1}(s)$$

$$+ \frac{\mu}{s^2 + (a + b) s + \frac{3a + b}{\tau}} Q_j(0)$$

$$+ \frac{ap_0 \Lambda}{s^2 + (a + b) s + \frac{3a + b}{\tau}} Q_{j+1}(0),$$

(11)

where the characteristic polynomial is signified by $D(s) = s^2 + (a + b) s + \frac{3a + b}{\tau} - a \rho_0^2 \Lambda$.

Thus, the transfer relation between $Q_{j+1}(s)$ and $Q_j(s)$ can be written as

$$Q_j(s) = G(s) Q_{j+1}(s).$$

(12)
Let $G(s) = \frac{ab(1+k)-a\rho_0^2\Lambda}{D(s)}$, which is derived as the transfer function.

According to the control theory [51], traffic jams do not occur in the vehicular flow system when the characteristic polynomial $D(s)$ is stable, i.e., the transfer function $\|G(s)\|_\infty \leq 1$. Based on the Hurwitz stability criterion, all the coefficients of $D(s)$ have the same sign to maintain $D(s)$ stable. Assume $\frac{\mu}{\tau}$ is positive. Thus, the traffic flow is stable if $(a+b) > 0$ and $(-a\rho_0^2\Lambda) > 0$. By taking advantage of the properties of the norm, $\|G(s)\|_\infty \leq 1$ is given by

$$\|G(s)\|_\infty = \sup \{ |G(j\omega)| \} \leq 1. \quad (13)$$

The transfer function $G(s)$ can be derived as

$$|G(j\omega)| = \sqrt{G(j\omega)G(-j\omega)}$$

$$= \sqrt{\frac{a^2\rho_0^4\Lambda^2 + b^2\omega^2(1+k)^2}{((a+b)\omega + \frac{\mu}{\tau}\omega)^2 + (\omega^2 + a\rho_0^2\Lambda)^2}} \leq 1. \quad (14)$$

One can obtain

$$g(\omega) = \frac{a^2\rho_0^4\Lambda^2 + b^2\omega^2(1+k)^2}{((a+b)\omega + \frac{\mu}{\tau}\omega)^2 + (\omega^2 + a\rho_0^2\Lambda)^2}, \omega \in [0, \infty]. \quad (15)$$

Obviously, $g(0) = 1$. Then $g(\omega) \leq 1$ should be satisfied, the inequality can be derived as follows:

$$a^2\rho_0^4\Lambda^2 + b^2\omega^2(1+k)^2$$

$$\leq (a+b)^2\omega^2 + \left(\frac{\mu}{\tau}\right)^2\omega^2$$

$$+ 2(a+b)\frac{\mu}{\tau}\omega^2 + \omega^4 + a^2\rho_0^4\Lambda^2 + 2a\rho_0^2\Lambda \omega^2$$

$$b^2(1+k)^2$$

$$\leq \omega^2 + 2a\rho_0^2\Lambda + \left(a+b + \frac{\mu}{\tau}\right)^2 \quad (16)$$

The sufficient condition can be acquired as

$$\omega^2 + 2a\rho_0^2\Lambda + \left(a+b + \frac{\mu}{\tau}\right)^2 - b^2(1+k)^2 \geq 0. \quad (17)$$

The sufficient condition corresponding to Eq.(17) can be rewritten as

$$\left(a+b + \frac{\mu}{\tau}\right)^2 - b^2(1+k)^2 \geq -2a\rho_0^2\Lambda. \quad (18)$$

Through the transfer functions, one can draw the Bode curves under different parameters, including constant parameters $b$, the ratio of the feedback gain to the reaction time $\frac{\mu}{\tau}$ and the strength coefficient of flux $k$ in Fig. 1. The model stability depends on the peak value of the Bode curve. When the peak value is greater than 1, the model becomes unstable [41]. When $\frac{\mu}{\tau} = 0$, Peng’s model degenerates into Nagatani’s model [50], which is described in a purple curve in Fig. 1(a). The other three solid curves in Fig. 1(a) represent Peng’s model under different values of $\frac{\mu}{\tau}(\frac{\mu}{\tau} = 0.2, 0.4, 0.6)$. Since the parameters of three solid curves $(\frac{\mu}{\tau} = 0, 0.2, 0.4)$ in Fig. 1(a) do not satisfy the stability condition Eq. (18), the peak values of these three Bode curves are greater than 1. This phenomenon will result in congested traffic flow. With the
increasing value of the constant parameter $\mu$, the amplitude of $|G(j\omega)|$ decreases gradually and the stability of the traffic flow increases. Additionally, the traffic congestion doesn’t disappear until $\frac{\mu}{\tau} = 0.6$. To verify the effectiveness of the successive flux difference item in Eq. (7), three solid curves ($\frac{\mu}{\tau} = 0.2, 0.4, 0.6$) of Peng’s model in Fig. 1(a) are preserved in Fig. 1(b)-(d). The dashed Bode curves in Fig. 1(b)-(d) stand for the SFDHE model with the same $b = 0.5$ and different values of the strength coefficient $k$. It can be clearly observed that the amplitude of the $|G(j\omega)|$ decays quickly when the novel feedback control signal Eq. (7) is taken into account on traffic flow. The smaller value of $k$ is, the smaller amplitude of the corresponding $|G(j\omega)|$ is achieved even with different values of $\frac{\mu}{\tau}$. In another word, it can be concluded that the traffic system is more stable with the decreasing value of the strength coefficient $k$. When $k = -0.4$, the peak values of the Bode curves are smaller than 1 in Fig. 1(b)-(d), which demonstrates the designed controller can effectively eliminate the traffic jams.

IV. NUMERICAL SIMULATION

In this section, numerical simulations are carried out under the periodic boundary to demonstrate the effectiveness of the proposed control signal in the lattice hydrodynamic model. According to the Ref [45], the single-lane road is divided into $N$ lattices and the corresponding default parameters are set as: $N = 140, \rho_c = 0.25, a = 1.2, v_{max} = 2$ and $\rho_0 = 0.25$. The initial condition can be given by

$$\rho_j(1) = \begin{cases} 0.5, & 50 \leq j \leq 55 \\ 0.2, & 55 < j \leq 60 \\ 0.25, & j < 50, j > 60. \end{cases} \quad (19)$$

The initial density of lattices $\rho_j(0) = \rho_c = 0.25$.

Figs. 2-3 reveal that the phase space plot $(\rho_j(t) - \rho_j(t - 1), \rho_j(t))$ at lattices $-2, 25, 55$ and 80 between $t = 1 - 800s$ in Peng’s model and our model, respectively. As shown in Fig. 2, it is clear that the scope of the phase space plot all shrinks with the increasing value of $\frac{\mu}{\tau}$ for different lattices. Fig. 3 depicts that the region of the phase space plot gets smaller and smaller with the decreasing value of $k$ for any position of lattice. The region of the dispersed points can reflect the density fluctuation of traffic flow. The smaller distances among the dispersed points are, the more stable the traffic flow system is. Fig. 2 demonstrates that the traffic flow with the pure honk effect (6) does not remain stable until $\frac{\mu}{\tau} = 0.6$. However, the traffic model with the control signal (7) can reach a stable state so long as $\frac{\mu}{\tau} = 0.2$ with $k = -0.4$. Generally speaking, the proposed control scheme could be more conducive to alleviating the traffic congestion even with a lower value of $\frac{\mu}{\tau}$ for the lattice hydrodynamic model.

Fig. 4 shows the oscillation behavior of traffic flow without control at different lattices. It is clear that the kink–antikink density waves oscillate violently at all lattices, which leads to the instability of traffic flow.

In order to highlight the superiority of the proposed control scheme, the fixed value ($\frac{\mu}{\tau} = 0.2$) has been selected to
conduct the comparative traffic behavior analyses between Peng’s model and our novel model.

### A. SHORT-TIME TRAFFIC BEHAVIOR ANALYSIS

The temporal density at different sites \( j = 2, 25, 55 \) and 80 lattices are depicted in detail for short-time \( 1 - 250s \) in Figs. 4 and 5.

**Case 1.** The control signal with pure honk item \( (b = 0) \)

When \( b = 0 \), the control signal in Eq. (7) degenerates into the control signal of Peng’s model [45]. The corresponding density waves at different lattices are shown in Fig. 5(a). Furthermore, it can also be observed that fluctuations of density waves in Fig. 5(a) are all more gentle than the ones at the corresponding lattices in Fig. 4. This means that the pure honk item \( \mu \tau \frac{q_{m} - q_j}{\tau} \) in the control signal Eq. (7) can relieve the traffic congestion to a certain extent. However, the fluctuations of density waves are still vibratory when \( \frac{\mu}{\tau} = 0.2 \), i.e., the traffic flow remains in an unstable state.

**Case 2.** The control signal of the SFDHE model \( (b \neq 0) \)

Fig. 5(b)-(d) depict density waves of four different lattices for different values of \( k \) when \( b = 0.5 \) in our model. Compared with Fig. 5(a), the fluctuations of the corresponding density waves for different lattices in Fig. 5(b)-(d) are more gentle. Furthermore, the amplitude of the corresponding density waves decreases with the decreasing value of \( k \). That is to say, there is a negative correlation between the strength coefficient of flux \( k \) and the stability of traffic flow in a short-time stage. These phenomena also demonstrate that the successive flux difference item in Eq. (7) can significantly enhance the stability of the traffic system even with the slight honk effect.

Though the fluctuation of density waves becomes more gentle with the decreasing value of \( k \), the fluctuation decays slowly in a short period of time. It may be a long time before the traffic flow reaches its desire density and desire flux. Therefore, the long-time traffic behavior of density waves will be carried out in the following section.

### B. LONG-TIME TRAFFIC BEHAVIOR ANALYSIS

The long-time effect of a control signal on the traffic flow model is explored at different sites \( j = 2, 25, 55, \) and 80 lattices for long-time \( 10000 - 10200s \) in Fig. 6.
For comparative analysis with Peng’s model, the selected value of the constant parameter $\mu \tau$ ($\mu \tau = 0.2$) in the long-time analysis is the same as the one in the short-time analysis.

**Case 1.** The control signal with pure honk item ($b = 0$)

Fig. 6(a) displays that the amplitude of density waves oscillates acutely with the slight value of the constant...
parameter \( \frac{k}{c} = 0.2 \), which is in agreement with the short-time result in Fig. 5(a).

**Case 2.** The control signal of the SFDHE model \((b \neq 0)\). Fig. 6(b)-(d) show the amplitude of density waves with different values of \( k \) under the SFDHE control signal \((7)\) when \( b = 0.5 \) in our model. And the corresponding amplitude of density waves decreased rapidly with the decreasing value of \( k \) at four lattices. The kink-antikink density waves have appeared at the four lattices in Fig. 6(b)-(c), but the amplitudes of density waves are all weaker than that of Fig. 6(a). Until the sufficient condition Eq.(18) is satisfied, the density waves turn into steady curves, as shown in Fig. 6(d). In another word, the traffic flow is more stable as the strength coefficient \( k \) decreases.

Different from the violent fluctuation of density waves with the short-time analysis in Fig. 5(d), the density waves of Fig. 6(d) show a vibration-free feature with the long-time analysis. It is obvious that long-time analysis is more beneficial to reveal the stability characteristic of the traffic behavior.

These comparative analyses under honk environment shown in Fig. 5 and Fig. 6 verify the effectiveness of the SFDHE control scheme \((7)\) and further validate the comparative analysis results demonstrated in Fig. 1. On the whole, these phenomena indicate that the proposed SFDHE control scheme can effectively relieve traffic jams even when the honk effect is slight.

**V. CONCLUSION**

With the rapid development of ITS, high-quality shared information among successive vehicles under honk environment can be acquired reliably. A feedback control scheme that takes the SFDHE control signal into account is proposed to suppress the traffic jam. Based on the control theory, the transfer function \( G(s) \) and the sufficient condition are derived through the Laplace transformation of the control method. Compared with the Bode-plot for the transfer function \( G(s) \) in Peng’s model, the peak values of the Bode curves decline significantly when the proposed SFDHE control signal is taken into account. Meanwhile, the theoretical results indicate that the stability of the traffic flow enhances with the increasing value of the constant parameter \( \frac{k}{c} \). While the increasing value of the strength coefficient \( k \) has the opposite effect. Numerical experiments are further performed from three perspectives, including phase space plot analysis, short-time traffic behavior and long-time traffic behavior, which demonstrate that the novel controller can push forward an immense influence on the stability of the traffic system. Future work will focus on solving the traffic congestion on the design of robust controllers, the application of the \( H_\infty \) or dissipative control. Additionally, the SFDHE control scheme will also be applied in different scenes, such as multi-lane road, driver’s characteristic analyses and so on.

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