SMALL OPERATOR IDEALS ON THE SCHLUMPRECHT AND SCHREIER SPACES

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Abstract. We present a method of building operators on a Banach space $X$ that generate distinct operator ideals in the algebra $\mathcal{B}(X)$ of bounded linear operators on $X$. We show that there are exactly $2^c$ distinct small closed operator ideals on the Schlumprecht space and there is a chain of cardinality $c$ of small closed operator ideals on any Schreier space of finite order.

1. Introduction

The algebras of bounded linear operators on a Banach spaces provide natural examples of non-commutative Banach algebras, thus draw attention to the structure of the lattice of closed operator ideals. We recall here only some of known results, for the thorough survey referring to [BKL, LL, JPS, SZ]. The study of the closed operator ideals dates back to [C], where it was proved that the ideal of compact operators $K(\ell_2)$ is the only non-trivial closed operator ideal on the Hilbert space $\ell_2$. The result was generalized by I. Gohberg, A. Markus and I. Feldman to the case of $c_0$ and $\ell_p$, $1 \leq p < \infty$. The list of Banach spaces with completely described lattice of closed operator ideals includes also non-separable Hilbert spaces ([G, L]), spaces built from classical spaces $\bigoplus_{n=1}^{\infty} \ell_2$, $\ell_1$, $\ell_\infty$, and - of completely different character - the celebrated Argyros-Haydon space $X_{AH}$ with the property $\mathcal{B}(X_{AH})/\mathcal{K}(X_{AH}) \cong \mathbb{R}$, as well as some of spaces built on its bases.

Recently strong results concerning the structure and the cardinality of the lattice of closed operator ideals were obtained, mostly based on the analysis of complemented subspaces of the considered space or of the family of strictly singular operators acting on the space. Recall here that an operator $T : X \to X$ is called strictly singular if none of its restriction to an infinite-dimensional subspace of $X$ is an isomorphism onto its image. The family $\mathcal{S}(X)$ of bounded strictly singular operators on a Banach space $X$ forms a classical example of a closed operator ideal. It turns out to be useful to distinguish operator ideals with respect to the ideal of strictly singular operators; following [JPS] we call an operator ideal in $\mathcal{B}(X)$ small if it is contained in $\mathcal{S}(X)$, otherwise we call it large.

The spaces $\ell_p \oplus \ell_q$, $1 \leq p < q \leq \infty$, and $\ell_p \oplus c_0$, $1 \leq p < \infty$ were shown in [FSZ] to admit continuum many closed operator ideals, with the exception of the case $\ell_1 \oplus c_0$ and $\ell_1 \oplus \ell_\infty$, where only uncountable chains of closed ideals were built ([SW]). Notice that continuum is the maximal length of a chain of closed ideals on a separable Banach space. In [JPS] chains of continuum many small closed ideals on the spaces $L_1(0, 1)$, $L[0, 1]$ and $L_\infty(0, 1)$ were built. Finally in [JS] it was shown that $\mathcal{B}(L_p)$, $1 < p < \infty$, $p \neq 2$, contains $2^c$ many distinct large and $2^c$ small closed operator ideals, answering another longstanding open question of Pietsch. Notice that again it is the maximal cardinality of a family of distinct closed ideals on a separable Banach space.

Spaces less "classical" are also studied with respect to the lattice of closed ideals, as Figiel spaces (cf. [LL, JS]), the Tsirelson space and Schreier spaces of finite order ([BKL]). In the last paper the authors showed that in both cases there are continuum many maximal (large) ideals generated by projections on subspaces spanned by subsequences of the canonical bases of the considered spaces.
In the present paper we consider closed small operator ideals on the Schlumprecht space \( X_S \) ([S]), a fundamental example of Banach space theory, and on Schreier spaces of finite order \( X_{GM} \), \( n \in \mathbb{N} \) ([AS]).

One should notice that by [W], the Schlumprecht space being complementably minimal (i.e. any its closed infinite-dimensional subspace contain a complemented copy of the whole space) admits a unique maximal ideal, which is \( \mathcal{J}(X_S) \), limiting the study on closed ideals on \( X_S \) to the small ones. In the case of the Gowers-Maurey space \( X_{GM} \), the first known hereditarily indecomposable space, defined on the basis of \( X_S \), the ideal \( \mathcal{J}(X_{GM}) \) is also the unique maximal ideal, however, for different reasons. In [AS] strictly singular non-compact operators both on \( X_S \) and \( X_{GM} \) were constructed. Moreover it was noticed that \( \ell_\infty/c_0 \) can be embedded into \( \mathcal{J}(X_{GM})/\mathcal{K}(X_{GM}) \), however, to the best of our knowledge, it was not known how many, if any, there are closed ideals between \( \mathcal{J}(X_S) \) and \( \mathcal{K}(X_S) \) (cf. [LL]). We prove that there is an injective mapping from the family \( 2^\mathbb{N} \) of subsets of \( \mathbb{R} \) into the family of closed small operator ideals that preserves inclusion (Theorem 6.19). In particular there are \( 2^\mathbb{c} \) many distinct closed ideals in \( \mathcal{B}(X_S) \). The proof is based on a refined construction of strictly singular operators of [AS], with use of some techniques of [MP2], and on a general criterion we present in the second section. The cited criterion (Theorem 2.3) is an example of a method of reasoning rather than a set of most optimal conditions, a variant of this method is described also in the case of Schreier spaces of finite order. We choose this way of presenting the method, as more clear for possible adaptation in different settings. In the case of Schreier spaces we show that in every such space there is a chain of cardinality continuum of closed small operator ideals (Theorem 4.8).

The paper is organized as follows: in the second section we present a general criterion for generating distinct operator ideals by operators of a specific form, the third section is devoted to the case of the Schlumprecht space, the forth section concerns the Schreier spaces of finite order.

2. A GENERAL CRITERION

Recall that for a Banach space \( X \) by \( \mathcal{B}(X) \) we denote the Banach algebra of bounded (linear) operators on \( X \).

We shall consider only operator ideals which are subsets of \( \mathcal{B}(X) \) for some Banach space \( X \), in this setting a non-zero set \( I \subset \mathcal{B}(X) \) is an operator ideal provided it is both a linear subspace and a two-sided ideal in the Banach algebra \( \mathcal{B}(X) \).

For any family \( \mathcal{B} \subset \mathcal{B}(X) \), where \( X \) is a Banach space, by \( I(\mathcal{B}) \) denote the closed operator ideal generated by \( \mathcal{B} \). Recall that \( I(\mathcal{B}) \) is the closure in the operator topology of the set \( \{ \sum_{j=1}^m \sum_{i=1}^{k_j} Q_{j,i} \circ R_j \circ S_{j,i} : Q_{j,i}, S_{j,i} \in \mathcal{B}(X), R_j \in \mathcal{B}, k_1, \ldots, k_l, l \in \mathbb{N} \} \).

Let \( X \) be a Banach space. Let \( (e_n)_n \subset X \) be a normalized basic sequence and let \( Y \) be the closed subspace spanned by \( (e_n)_n \). Denote by \( (e_n)_n \subset Y^* \) the sequence of biorthogonal functionals to \( (e_n)_n \subset Y \).

Notation. Let \( \mathcal{B}_{seq}(X,Y) \subset \mathcal{B}(X) \) denote the family of bounded operators on \( X \) of the form

\[
T : X \ni x \mapsto \sum_{n=1}^\infty f_n(x)e_n \in Y \subset X
\]

(2.1)

for some biorthogonal basic sequences \( (f_n,x_n)_{n \in \mathbb{N}} \subset X^* \times X \) with \( (f_n)_n \subset B_{X^*} \) weakly* null and \( (x_n)_n \) seminormalized (in particular \( T x_n = e_n \) and \( e_n^* \circ T = f_n, n \in \mathbb{N} \)).

Remark 2.1. Obviously the sequence \( (x_n)_n \) in the definition above is not determined by the operator \( T \), however we shall consider an operator \( T \in \mathcal{B}_{seq}(X,Y) \) always with a fixed sequences \( (f_n,x_n)_{n \in \mathbb{N}} \subset X^* \times X \) defining \( T \) as above.
For any operator $T \in \mathcal{B}_{\text{seq}}(X, Y)$ with the associated sequences $(f_n, x_n)_n$ let
\[
c_0(T) := \limsup_{i_1 < \cdots < i_k, i_1, \ldots, k \to \infty} \sup \{ c_1 f_{i_1} + \cdots + c_k f_{i_k} : c_1 = \pm 1, \ldots, c_k = \pm 1 \},
\]
\[
d_k(T) := \limsup_{i_1 < \cdots < i_k, i_1, \ldots, k \to \infty} \| x_{i_1} + \cdots + x_{i_k} \|, \quad k \in \mathbb{N}
\]

**Notation.** Let $\mathcal{F}_\infty(X, Y)$ denote the family of bounded operators $T : X \to Y$ with the following property: for any bounded sequence $(y_m)_m \subset X$ with $\| T y_m \|_{Y, \infty} \to 0$ also $\| T y_m \|_{Y, \infty} \to 0$, where the norm $\| \cdot \|_{Y, \infty}$ on $Y$ is given by $\| \cdot \|_{Y, \infty} = \sup \{ |e_n^*(\cdot)| : n \in \mathbb{N} \}$.

**Remark 2.2.** Notice that for $X$ not containing an isomorphic copy of $c_0$ we have $\mathcal{F}_\infty(X, Y) \subset \mathcal{F}(X)$. On the other hand, if the sequence $(c_n)_n$ is equivalent to the unit vector basis of $c_0$, then any operator of the form $\begin{pmatrix} 1 \\ \vdots \end{pmatrix}$ with $(f_n)_n \subset B_X^*$ belongs to $\mathcal{B}_{\text{seq}}(X, Y) \cap \mathcal{F}_\infty(X, Y)$.

With the abuse of notation we use $Y$ instead of its basis $(c_n)_n$ both in the case of $\mathcal{B}_{\text{seq}}(X, Y)$ and $\mathcal{F}_\infty(X, Y)$, however, it will be clear in the rest of the paper which basis of $Y$ we consider.

Dealing with operators from $\mathcal{F}_\infty(X, Y)$ enables to relate the behaviour of operators of type $\mathcal{B}_{\text{seq}}(X, Y)$ even when composed with other bounded operators, and in consequence yields the following

**Theorem 2.3.** Fix an operator $T \in \mathcal{B}_{\text{seq}}(X, Y)$ defined by $(f_n, x_n)_n$ and a non-empty family $B \subset \mathcal{B}_{\text{seq}}(X, Y) \cap \mathcal{F}_\infty(X, Y)$. Assume
\[
\inf_{k \in \mathbb{N}} \frac{c_0(R) d_k(T)}{k} = 0 \quad \text{for any } R \in B
\]
Then $\| T - Q \| \geq 1/d$ for any $Q$ in the operator ideal $I(B)$ generated by $B$, where $d := \sup_n \| x_n \|$.

**Proof.** Take operators as in the theorem. Let $(x_n)_n \subset X$ be the seminormalized basic sequence defining $T$.

Assume there is $Q \in I(B)$ with $\| T - Q \| < 1/d$. Then there are $R_j \in B, Q_{j, i}, S_{j, i} \in \mathcal{B}(X), i = 1, \ldots, k_j, j = 1, \ldots, l$ such that $\| T - \sum_{j=1}^l \sum_{i=1}^{k_j} Q_{j, i} \circ R_j \circ S_{j, i} \| < 1/2$. As $T x_n = c_n$ and $(x_n)_n$ is bounded, $n \in \mathbb{N}$, passing to a subsequence if necessary, there are some $j_0 \in \{1, \ldots, l\}, i_0 \in \{1, \ldots, k_{j_0}\}$ such that the sequence $((Q_{j_0, i_0} \circ R_{j_0} \circ S_{j_0, i_0}) x_n)_n$ is seminormalized. It follows that $((R_{j_0} \circ S_{j_0, i_0}) x_n)_n$ is seminormalized as well.

In order to avoid unnecessary indices we write $R := R_{j_0}, S := S_{j_0, i_0}$. Let $(g_n)_n \subset X^*$ be the seminormalized basic sequence defining $R$.

As $R \in \mathcal{F}_\infty(X, Y)$ there is a universal constant $c > 0$ with $\sup_{n \in \mathbb{N}} |e_n^*(R(S x_n))| > c$ for all $n$. For any $n \in \mathbb{N}$ pick $l_n \in \mathbb{N}$ and $c_n = \pm 1$ so that $c_n g_{l_n}(S x_n) = c_n e_n^* \circ R(S x_n) \geq c$ for each $n$.

Notice that for any fixed $n \in \mathbb{N}$ we have $g_{l_n}(S x_m) \to 0$ as $m \to \infty$. Indeed, assume $|g_{l_n}(S x_m)| > \delta$, for some universal $\delta$ and any $m \in M$, with some infinite $M \subset \mathbb{N}$. Without loss of generality we assume $g_{l_n}(S x_m) > \delta$, $m \in M$. Thus for any $k \in \mathbb{N}$ and any $A \subset M$ with $\# A = k$ we have
\[
\| S \| \sum_{m \in A} x_m \| \geq \| S \| \sum_{m \in A} x_m \| \geq g_{l_n}(\sum_{m \in A} S x_m) \geq k \delta
\]
It follows that $\liminf_{k \in \mathbb{N}} \frac{d_k(T)}{k} \geq \delta \| S \|^{-1} > 0$, which contradicts the assumption on $(d_k(T))_k$ (as $\liminf_k c_0(R) > 0$).

It follows that we can pass to subsequence of $(x_n)_n$ on which the mapping $n \mapsto l_n$ is an injection, thus $g_{l_n}(S x_m) \to 0$ as $m \to \infty$. (**) (recall $R \in \mathcal{B}_{\text{seq}}(X, Y)$, thus $(g_{l_n})_n$ is weakly* null).

Using (***) and (**) we easily pass to a further subsequence $(x_n)_n \subset L$ such that $|g_{l_n}(S x_m)| \leq \frac{1}{2^n}$ for any $n \neq m, n, m \in L$. Now for any $k \in \mathbb{N}$ choose $A \subset L$, $\# A = k$, with $\min A > \frac{1}{2}$.
3. Basic definitions and properties. By a tree we shall mean a non-empty partially ordered set \((\mathcal{T}, \preceq)\) for which the set \(\{y \in \mathcal{T} : y \preceq x\}\) is linearly ordered and finite for each \(x \in \mathcal{T}\). The tree \(\mathcal{T}\) is called finite if the set \(\mathcal{T}\) is finite. The root is the smallest element of the tree (if it exists). The terminal nodes are the maximal elements. A branch of \(\mathcal{T}\) is any maximal linearly ordered set in \(\mathcal{T}\). A tree with no infinite branches is called well-founded. The immediate successors of \(x \in \mathcal{T}\), denoted by \(\text{succ}(x)\), are all the nodes \(y \in \mathcal{T}\) such that \(x \prec y\) but there is no \(z \in \mathcal{T}\) with \(x \prec z \prec y\). A tree is finitely branching, if any its non-terminal element has finitely many immediate successors. If \(\mathcal{T}\) has a root, then for any node \(\alpha \in \mathcal{T}\) we define a level of \(\alpha\), denoted by \(|\alpha|\), as the length of the branch linking \(\alpha\) and the root.

We recall the definition of the Schlumprecht space \(X_S\). Let \(K \subset c_{00}\) be the smallest set containing \(\{\pm e_n^* : n \in \mathbb{N}\}\) and such that for any \(n \leq m\) and any block sequence \(f_1 < \cdots < f_n\) of elements of \(K\) also the weighted average \(1/\log_{m+1}(m+1) \sum_{k \in \text{succ}(\gamma)} f_k\) belongs to \(K\). The set \(K\) defines a norm \(|| \cdot ||\) on \(c_{00}\) as its norming set, i.e. \(|| \cdot || = \sup_{f \in K} |f|\)\(,\) where for any \(f = (f_n)_{n \in \mathbb{N}} \in K\) and \(x = (x_n)_{n \in \mathbb{N}} \in c_{00}\) we have \(f(x) = \sum_n f_n x_n\). The Schlumprecht space \(X_S\) is defined as the completion of \((c_{00}, || \cdot ||)\).

Remark 3.1. (1) It follows straightforward from the definition of the Schlumprecht space that the unit vector basis \((e_n)_{n} is 1-unconditional and 1-subsymmetric (i.e. 1-equivalent to any of its subsequences).

(2) By definition of the norming set \(K\) any \(f \in K \setminus \{\pm e_n^* : n \in \mathbb{N}\}\) has a tree-analysis \((f_\gamma)_{\gamma \in \mathcal{R}} \subset K\) where \(\mathcal{R}\) is a well-founded finitely-branching tree with the root \(\emptyset\), \(f_\emptyset = f\), \(f_\gamma = 1/\log_{m_\gamma+1}(m_\gamma+1) \sum_{\zeta \in \text{succ}(\gamma)} f_\zeta\), \(m_\gamma \geq \# \text{succ}(\gamma)\), for any non-terminal \(\gamma \in \mathcal{R}\), \(f_\gamma = \pm e_{n_\gamma}\) for any terminal \(\gamma \in \mathcal{R}\). In such situation a weight \(w(f)\) of \(f\) is given by \(w(f) = m_\gamma\).

(3) \([S]\) We have \(||e_1 + \cdots + e_k|| = \log_2(k+1)\) for any \(k \in \mathbb{N}\). Thus for any \(h \in K\) we have \(#\{i \in \text{supp}\, h : |h(e_i)| \geq \log_2(k+1)\} \leq k\) for any \(k \in \mathbb{N}\).

We recall now a notion from [MP2], implicitly used in [AS].

Definition 3.2. A core tree is any finitely branching tree \(\mathcal{T} \subset \bigcup_n \mathbb{N}^n\) with no terminal nodes and a root \(\emptyset\), considered with the tree order \(\preceq\) and the lexicographic order \(\leq_{\text{lex}}\). For any \(n \in \mathbb{N}\) let \(\mathcal{T}_n = \{\alpha \in \mathcal{T} : |\alpha| \leq n\}\). For any \(\alpha \in \mathcal{T}\) let \(m_\alpha = \# \text{succ}(\alpha)\).
Notation. We enumerate nodes of $T$ according to the lexicographic order as $(\alpha_j)_{j \in \mathbb{N}}$, starting from $\alpha_0 = \emptyset$. With abuse of notation we write $m_j = m_{\alpha_j}$ for each $j \in \mathbb{N}$.

For any $\alpha \in T$, $\alpha \neq \emptyset$, let $c_\alpha = \prod_{\gamma < \alpha} \log_2(m_\gamma + 1)$ and $d_\alpha = \prod_{\gamma < \alpha} \log_2(m_\gamma + 1)m_\gamma^{-1}$. For simplicity we shall write $c_j = c_{\alpha_j}$, $d_j = d_{\alpha_j}$ for any $j \in \mathbb{N}$.

Remark 3.3. Notice that any sequence of parameters $(m_j)_{j \in \mathbb{N}} \subset \mathbb{N} \setminus \{0\}$ defines a unique core tree $T$, for which $\# \text{succ}(\alpha_j) = m_j$ for any $j \in \mathbb{N}$, with the enumeration $(\alpha_j)_{j \in \mathbb{N}}$ of elements of $T$ according to the order $\leq_{\text{lex}}$.

The definition below recalls the notion of "repeated" averages of $AS$.

Definition 3.4. $AS$. We say that a vector $x \in X_S$ has a tree-analysis $(x_\alpha)_{\alpha \in T_n} \subset X_S$ (of height $n \in \mathbb{N}$) with a core tree $T$ with associated parameters $(m_\alpha)_{\alpha \in T} \subset \mathbb{N}$, if the following hold:

1. $x_\emptyset = x$,
2. For any terminal node $\alpha \in T_n$ we have $x_\alpha = e_{t_\alpha}$ for some $t_\alpha \in \mathbb{N}$,
3. For any non-terminal node $\alpha \in T$ the vector $x_\alpha$ is a seminormalized $m_\alpha$-average of $(x_\beta)_{\beta \in \text{succ}(\alpha)}$ of the form $x_\alpha = \frac{1}{\log_2(m_\alpha + 1)} \sum_{\beta \in \text{succ}(\alpha)} x_\beta$,
4. For any nodes $\alpha, \beta \in T$ with $|\alpha| = |\beta|$, $\alpha \leq_{\text{lex}} \beta$ we have $x_\alpha < x_\beta$ (in terms of supports).

Lemma 3.5. $MP2$. Prop. 2.3, Lemma 2.10 Let $T$ be a core tree with the associated sequence of parameters $(m_j)_{j \in \mathbb{N}}$. If for some $(q_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ the sequence $0 < m_0 < q_0 < m_1 < q_1 < \ldots$ increases fast enough, then for any tree-analysis $(x_\alpha)_{\alpha \in T_n}, n \in \mathbb{N}$, of a vector $x \in X_S$ we have

(F1) $1 \leq \|x_\alpha\| \leq 2$ for any $\alpha \in T_n$,
(F2) $|h(x_\alpha)| \leq \frac{2}{\log_2(m_j + 1)}$ for any $h \in K$ with $w(h) \geq q_j$ and any $\alpha_j \in T_n$.

Remark 3.6. The precise conditions guaranteeing the fast increase of $(m_j, q_j)_j$ are given in $MP2$. The important feature of this notion is that it is "finitely defined" in a sense that if a sequence $m_0 < q_0 < \cdots < m_s < q_s$, can be extended to fast increasing sequence (required in $\text{Lemma 3.5}$), then any sequence $m_0 < q_0 < \cdots < m_s < q_s$, with $q > q_s$, as well, and the same holds true for sequences of the form $m_0 < q_0 < \cdots < q_{s-1} < m_s$.

Remark 3.7. In the situation above it follows that any $x_\alpha$ is an $\ell_1^\alpha$-average with constant 3, with $r = r(m_\alpha) \to \infty$ as $m_\alpha \to \infty$. Thus by standard reasoning for any $\alpha \in T$ there is some $q = q(m_\alpha)$ such that for any $E_1 < \cdots < E_n$, we have $\sum_{s=1}^n \|E_s x_\alpha\| \leq 4$, with $q(m_\alpha) \to \infty$ as $m_\alpha \to \infty$ ( $\text{[S]}$).

Definition 3.8. Given a vector $x$ with a tree-analysis $(x_\alpha)_{\alpha \in T_n}$ as in $\text{Def. 3.4}$, we define in a natural way an associated norming functional $f \in K$ with a tree-analysis $(f_\alpha)_{\alpha \in T_n}$ as follows (with the notation of Def. 3.4):

1. $f_\emptyset = f$,
2. For any terminal node $\alpha \in T$ we set $f_\alpha = e_{t_\alpha}^*$,
3. For any non-terminal node $\alpha \in T$ we set $f_\alpha = \frac{1}{\log_2(m_\alpha + 1)} \sum_{\beta \in \text{succ}(\alpha)} f_\beta$.

In the situation above obviously any $f_\alpha, \alpha \in T_n$, of the above form is in the norming set of $X_S$, thus in particular $f \in K$, and $f(x) = 1$.

3.2. Block sequences of vectors and functionals defined by a core tree. For the rest of this section we fix a core tree $T$ with parameters $(m_j)_j$, a block sequences $(x_n)_n \subset X_S$ and $(f_n)_n \subset X_S^*$ defined by $T$, i.e. such that each $x_n$ has a tree-analysis $(x_\alpha^n)_{\alpha \in T_n}$, and each $f_n$ is an associated functional to $x_n$ with a tree-analysis $(f_\alpha^n)_{\alpha \in T_n}$. We shall estimate in this section finite sums of $(x_n)_n$ and $(f_n)_n$ under certain conditions on parameters associated to the tree $T$.

For a sequence of parameters $(m_j, q_j)_j$, with $0 < m_0 < q_0 < m_1 < q_1 < \ldots$, we define the following conditions of the fast growth: for any tree-analysis $(x_\alpha)_{\alpha \in T_n}, n \in \mathbb{N}$, of a vector $x \in X_S$ we require

(F3) for any $E_1 \cdots < E_{q_{j-1}}$ we have $\sum_{s=1}^{q_{j-1}} \|E_s x_\alpha^n\| \leq 4$, for any $j \in \mathbb{N}$ and $n \geq |\alpha_j|$. 

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(F4) \( \log_2(m_j + 1) \geq 2 \log_2(m_{j-1} + 1) \) for all \( j \in \mathbb{N} \),
(F5) \#\{\alpha : |\alpha| = N\}(= \sum_{|\alpha| = N-1} m_\alpha) \leq \log_2(q_\alpha N + 1) \), where \( \alpha_{N} \) is maximal in \( \{\alpha : |\alpha| = N - 1\} \) with respect to \( \leq_{lex} \).

**Notation.** As in the case of \((m_\alpha)_{\alpha \in \mathcal{T}}\) we write \( q_\alpha \) for \( q_j \) for which \( \alpha = \alpha_j \in \mathcal{T}, j \in \mathbb{N} \).

**Remark 3.9.** Notice that it is possible to construct inductively \((m_j, q_j)\) satisfying (F1)-(F5). Indeed, at each step choose \( q_j, m_j \) big enough to ensure conditions (F1) and (F2) are satisfied (cf. Remark 3.6), and moreover, given \( m_0 < q_0 < \cdots < m_{j-1} < q_{j-1} \) choose \( m_j \) big enough so that (F3) is satisfied (according to Remark 3.7) as well as (F4). Additionally, given \( m_0 < q_0 < \cdots < m_j \), with \( j = N \) for some \( N \in \mathbb{N} \), choose \( q_j \) so that (F5) is satisfied. Moreover, the choice of \((m_j, q_j)\) is "finitely defined" in the sense of Remark 3.6.

**Proposition 3.10.** Let \((x_n)_n \subset X_\mathcal{S}\) be a block sequence defined by a core tree \( \mathcal{T} \) with \((m_j, q_j)\) satisfying (F1)-(F5). Fix \( j_0 \in \mathbb{N} \) so that \( \alpha_{j_0} \) is minimal in \( \{\alpha \in \mathcal{T} : |\alpha| = |\alpha_{j_0}|\} \) with respect to \( \leq_{lex} \). Then for any \( q_{j_0-1} \leq k \leq m_{j_0} \) and \( j_0 < n_1 < \cdots < n_k \) we have

\[
\frac{1}{2} kd_{j_0} \leq \|x_{n_1} + \cdots + x_{n_k}\| \leq 14 kd_{j_0}
\]

**Proof.** As the bases of \( X_\mathcal{S} \) and \( X_\mathcal{S}^* \) are 1-unconditional, and the vectors \( (x_\alpha^n) \) have non-negative coefficients, we can assume that the functionals analysed below have also non-negative coefficients.

Let \( N := |\alpha_{j_0}| \). Fix \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \) as in the Proposition, in particular \( n_k > \cdots > n_1 > j_0 \geq N \). Let each \( x_{n_i} \) has the tree-analysis \((x_\alpha^n)_{\alpha \in \mathcal{T}_{n_i}}\).

For the left estimate consider the associated functionals \( (f_\alpha)_{\alpha} \) with corresponding tree-analysis \((f_\alpha^n)_{\alpha \in \mathcal{T}_{n_i}}\), \( i = 1, \ldots, k \), and estimate using unconditionality and (F4) as follows

\[
\|x_{n_1} + \cdots + x_{n_k}\| \geq d_{j_0} \|x_{n_1} + \cdots + x_{n_k}\| \geq \frac{1}{2} kd_{j_0} \log_2(km_{j_0} + 1) \sum_{i=1}^{k} \sum_{\alpha \in \mathcal{T}_{n_i}} f_\alpha(x_{n_i})
\]

\[
\geq \frac{1}{2} kd_{j_0} \log_2(m_{j_0} + 1) \sum_{i=1}^{k} \sum_{\alpha \in \mathcal{T}_{n_i}} f_\alpha(x_{n_i})
\]

\[
= \frac{1}{2} kd_{j_0} \sum_{i=1}^{k} f_{n_i}(x_{n_i})
\]

\[
= \frac{1}{2} kd_{j_0}.
\]

For the right estimate we show first that

\[
\|y_1 + \cdots + y_k\| \leq 4kd_{j_0} + 5\|y_1 + \cdots + y_k\| \tag{3.1}
\]

where \((y_1, \ldots, y_k)\) is a block sequence of shifted copies of the vector \( x_N = \sum_{|\alpha| = N} d_\alpha e_\alpha \) and for such \((y_1, \ldots, y_k)\) we prove

\[
\|y_1 + \cdots + y_k\| \leq 2kd_{j_0} \tag{3.2}
\]

For (3.1) take a functional \( \tilde{h} \in K \) with its tree-analysis \( (\tilde{h}_\gamma)_{\gamma \in \mathcal{R}} \). By [MP] we can pick \( h \in K \) with \( \|\tilde{h}\| = \sum_{|\alpha| = N} x_\alpha \) and a tree-analysis \( (h_\gamma)_{\gamma \in \mathcal{R}} \) compatible with the block sequence \((x_\alpha^n)_{|\alpha| = N} \) (meaning that for any \( \gamma \in \mathcal{R}, \alpha \in \mathcal{T} \) with \( |\alpha| = N \) and \( i = 1, \ldots, k \) we have either range \( x_\alpha^i \cap \text{range} \subset \text{range} \hat{h}_\gamma \), range \( x_\alpha^i \cap \text{range} \hat{h}_\gamma \), or range \( x_\alpha^i \cap \text{range} \hat{h}_\gamma = \emptyset \)). We want to estimate

\[
h(x_{n_1} + \cdots + x_{n_k}) = \sum_{\alpha \in \mathcal{T}, |\alpha| = N} d_\alpha h(x_{n_1}^i + \cdots + x_{n_k}^i)
\]
In the sequel we consider only those \((i, j), |\alpha_j| = N\), for which \(h(x_{\alpha_j}^i) \neq 0\). For any \((i, j), |\alpha_j| = N\), let \(\gamma_{i,j} \in \mathcal{R}\) be the node with \(h_{\gamma_{i,j}}\) covering \(x_{\alpha_j}^i\) (i.e. \(\gamma_{i,j}\) is maximal in \(\mathcal{R}\) with range \(h\gamma \supset \mathrm{range}\ h \cap \mathrm{range}\ x_{\alpha_j}^i\)).

Let \(A = \{(i, j) : i = 1, \ldots, k, |\alpha_j| = N, w(h_{\gamma_{i,j}}) \leq q_j-1\}\). Then by (F3) for each \((i, j) \in A\) we have

\[
| \sum_{\beta \in \mathrm{succ}(\gamma_{i,j})} h_\beta(x_{\alpha_j}^i) | \leq 4
\]

thus we can replace in the tree-analysis of \(h\) all successors of \(h_{\gamma_{i,j}}\) with range intersecting the range of \(x_{\alpha_j}^i\) by one functional \(f_{i,j}\), paying the cost of multiplying the action of \(h\) on \(x_{\alpha_1} \cdots x_{\alpha_k}\) by 4. Thus

\[
|h(\sum_{(i,j) \in A} d_j x_{\alpha_j}^i) | \leq 4\|y_1 + \cdots + y_k\| \tag{3.3}
\]

Let \(B = \{(i, j) : i = 1, \ldots, k, |\alpha_j| = N, w(h_{\gamma_{i,j}}) \geq q_j\}\). Then for any \((i, j) \in B\) by (F2) we obtain

\[
|h_{\gamma_{i,j}}(x_{\alpha_j}^i) | \leq \frac{2}{\log_2(m_j + 1)}
\]

Thus

\[
|h(\sum_{(i,j) \in B} d_j x_{\alpha_j}^i) | \leq 2k \sum_{|\alpha_j| = N} \frac{d_j n_j}{\log_2(m_j + 1)} \leq 2kd_{j_0} \tag{3.4}
\]

with the last inequality by (F4).

We consider one more set which can have non-empty intersection with the previous ones. Let \(C = \{(i, j) : i = 1, \ldots, k, |\alpha_j| = N\} \setminus (A \cup B \cup C)\). Then for any \((i, j) \in E\) we have \(q_{j-1} < w(h_{\gamma_{i,j}}) < q_j\). Notice that for \((i, j) \in E\) and \((i', j')\) with \(h_{\gamma_{i,j}}(x_{\alpha_j}^i) \neq 0\), we have \(\gamma_{i,j} \preceq \gamma_{i', j'}\) (recall the tree-analysis of \(h\) is compatible with \((x_{\alpha_j}^i)_{i=1,\ldots,k,|\alpha_j|=N}\)), thus either \(h_{\gamma_{i,j}} = h_{\gamma_{i', j'}}\) (and thus \(j = j'\)) or \((i', j') \in C\).

We split \(E\) into two pieces: \(E' = \{(i, j) \in E : h_{\gamma_{i,j}}(x_{\alpha_j}^i) = 0\ \forall (i', j') \in E, (i', j') \neq (i, j)\}\) and \(E'' = E \setminus E'\). By the remark above we have \(#E'' \leq k\) and thus

\[
|h(\sum_{(i,j) \in E''} d_{\alpha_j} x_{\alpha_j}^i) | \leq \|E'' d_{\alpha_j} \leq k d_{j_0} \tag{3.5}
\]

By the definition of \(E''\) we can replace in the tree-analysis of \(h\) each \(h_{\gamma_{i,j}}, (i, j) \in E''\), by the functional \(f_{\gamma_{i,j}}\), not decreasing the action of the resulting functional on \(\sum_{(i,j) \in E'} d_{\alpha_j} x_{\alpha_j}^i\). Thus

\[
|h(\sum_{(i,j) \in E'} d_{\alpha_j} x_{\alpha_j}^i) | \leq \|y_1 + \cdots + y_k\| \tag{3.7}
\]

which ends the proof of (3.1).

For (3.2) take any \(h \in K\) and let \(F = \{i \in \mathbb{N} : |h(e_i)| > \log_2(q_{j_0}^{-1} + 1)^{-1}\}\). Then by Remark (3.1) 3 and the choice of \(k \geq q_{j_0}^{-1}\) we have

\[
|h_F(y_1 + \cdots + y_k) | \leq \#F \cdot d_{j_0} \leq q_{j_0}^{-1} d_{j_0} \leq k d_{j_0} \tag{3.8}
\]
On the other hand estimate, using (F5)
\[
|h|_{N,F}(y_1 + \cdots + y_k) \leq \# \text{supp}(y_1 + \cdots + y_k)\|y_1 + \cdots + y_k\| \leq k\# \{\alpha \in T : |\alpha| = N\}d_j\frac{1}{\log_2(q_{j_0-1} + 1)}
\]
which ends the proof of (3.2).

For any \(j_0 \in \mathbb{N}\) let
\[
A_{j_0} = \{\alpha \in T : |\alpha| = |\alpha_{j_0}|, \alpha \geq_{\text{lex}} \alpha_{j_0}\} \cup \bigcup_{|\alpha| = |\alpha_{j_0}|, \alpha <_{\text{lex}} \alpha_{j_0}} \text{succ} \alpha, \quad A'_{j_0} = A_{j_0} \setminus \{\alpha_{j_0}\}
\]

Remark 3.11. Notice that \(#A_{j_0} \leq \sum_{j < j_0} m_j\) and \(m_{\alpha} \geq m_{j_0} + 1\) for each \(\alpha \in A'_{j_0}\).

Proposition 3.12. Let \((f_n)_n \subset X^*_S\) be a block sequence of functionals given by a core tree \(T\) with parameters \((m_j)_j\). Fix \(\alpha_{j_0} \in T\). Then for any \(k \leq m_{j_0}\) and \(j_0 + 1 < n_1 < \cdots < n_k\) we have
\[
\|f_{n_1} + \cdots + f_{n_k}\| \leq 2 \sum_{j < j_0} m_j
\]
and for any \(k \leq m_{j_0} + 1\) and \(|\alpha_{j_0}| + 1 < n_1 < \cdots < n_k\) we have
\[
\|\sum_{i=1}^{k} f_{n_i} - c_{j_0} \sum_{i=1}^{k} f_{\alpha_{j_0}}^i\| \leq 2 \sum_{j < j_0} m_j
\]

Proof. Let any \(f_{n_i}\) have the tree-analysis \((f_i^\alpha)_{\alpha \in T_{n_i}}\). Then \(f_{n_i} = \sum_{\alpha \in A'_{\alpha_{j_0}}} c_{\alpha} f_{\alpha_{j_0}}^\alpha + c_{j_0} f_{\alpha_{j_0}}^\alpha\) for each \(i = 1, \ldots, k\), where each \(\alpha \in A'_{\alpha_{j_0}}\) is not terminal in \(T_{n_i}\) as \(n_i > j_0 + 1 \geq |\alpha_{j_0}| + 1\) for any \(i = 1, \ldots, k\). We show the second estimate, as the proof of the first estimate is a simplified version of the proof of the second one. Estimate, using second part of Remark 3.11 as follows
\[
\|\sum_{i=1}^{k} f_{n_i} - c_{j_0} \sum_{i=1}^{k} f_{\alpha_{j_0}}^i\| \leq \|\sum_{\alpha \in A'_{\alpha_{j_0}}} \frac{1}{\log_2(|m_{\alpha}| + 1)} \sum_{\beta \in \text{succ} \alpha} f_{\beta}^j\|
\]
\[
\leq \sum_{\alpha \in A'_{\alpha_{j_0}}} \|\frac{1}{\log_2(|m_{\alpha}| + 1)} \sum_{\beta \in \text{succ} \alpha} f_{\beta}^j\|
\]
\[
\leq \sum_{\alpha \in A'_{\alpha_{j_0}}} \|\frac{2}{\log_2(km_{\alpha} + 1)} \sum_{\beta \in \text{succ} \alpha} f_{\beta}^j\|
\]
\[
\leq 2\#A_{\alpha_{j_0}}
\]
as each functional \(\frac{1}{\log_2(|m_{\alpha}| + 1)} \sum_{\beta \in \text{succ} \alpha} f_{\beta}^j\) is in the norming set \(K\) of \(X_S\).

3.3. Operators defined by a core tree. We fix a core tree \(T\) with parameters \((m_j)_j\). The aim of this section is to study properties of an operator \(T \in \mathcal{B}_{\text{seq}}(X_S) := \mathcal{B}_{\text{seq}}(X_S, X_S)\) where \(X_S\) is considered with the canonical basis \((e_n)_n\), defined by a sequence \((f_n, x_n)_n \subset X_S^* \times X_S\) associated to \(T\).

Remark 3.13. In [MP2] the index set of summation was required to be more lacunary, however, in the situation of the Schlumprecht space we profit from the specific form of involved coefficients \((\log_2(n+1))^{-1}\) and dealing with the family \(A_n\).
For an increasing sequence \((m_j)_j \subset \mathbb{N}\) we define the following properties:

(F6) \(2^j \sum_{t < j} m_t \leq \log_2(m_j + 1), \ j \in \mathbb{N}\),

(F7) \(c_j \leq 1/2^{j+1}, \ j \in \mathbb{N}\).

We recall now estimate used already in [MP2]. Any sequence \((f_n)_{n \in \mathbb{N}}\) defines a family of norms \(\| \cdot \|_j, j \in \mathbb{N}\), on \(X_S\) in the following way:

\[
\| \cdot \|_j = \sup\{ |f_n(\cdot)| + \cdots + |f_n(\cdot)| : n_1 < \cdots < n_j \}
\]

**Proposition 3.14.** Take a block sequence of functionals \((f_n) \subset X_S^*\) associated to a core tree \(T\) with \((m_j)_j\) satisfying (F6)-(F7). Then the operator \(T : X_S \ni x \mapsto \sum_n f_n(x)e_n \in X_S\) is well defined and satisfies the following for any \(j_0 \in \mathbb{N}\):

\[
\|Ty\| \leq \sum_{j=1}^{j_0} \|y\|_{m_j} + \frac{7}{2^{j_0}} \|y\|, \ y \in X_S,
\]

**Proof.** The calculation below proves both statements \((T\) well-defined and the estimate). The calculation for \(j_0 = 1\) and \(y \in X_S\) with finite support show that for any \(y\) finite support we have \(\|Ty\| \leq 8\|y\|\), proving that \(T\) is well-defined on the whole space \(X_S\).

Put \(c_j = \log_2(m_j + 1)^{-1}, j \in \mathbb{N}\), with \(c_0 = 1\). Let each \(f_n\), has the tree-analysis \((f_n^\alpha)_{\alpha \in T_n}\).

Let \(h \in K\). For any \(j \in \mathbb{N}\) let

\[
B_j = \{ n \in \mathbb{N} : c_{j+1} < |h(e_n)| \leq c_j \}, \quad D_j = B_j \cap \{1, \ldots, j+2\},
\]

Notice that \(#B_j \leq m_{j+1}\) for any \(j \in \mathbb{N}\) by Remark 3.13. Therefore by Prop. 3.12 for any \(j \in \mathbb{N}\) we have

\[
\| \sum_{n \in B_j \setminus D_j} \pm(f_n - c_j f_{\alpha_j}^n) \| \leq 2 \sum_{t < j} m_t \tag{3.9}
\]

Now for a fixed \(j_0 \in \mathbb{N}\) and \(y \in X_S\) we have

\[
|h(Ty)| = |h(\sum_{n \in \mathbb{N}} f_n(y)e_n)|
\]

\[
\leq \sum_{j=0}^{j_0-1} \|y\| |\sum_{n \in B_j} f_n(y)h(e_n)| + |h(\sum_{j=j_0}^{\infty} \sum_{n \in B_j} f_n(y)e_n)|
\]

\[
\leq \|y\| \sum_{j=0}^{j_0-1} |\sum_{n \in B_j} (f_n - c_j f_{\alpha_j}^n)(y)h(e_n)| + \|\sum_{j=j_0}^{\infty} \sum_{n \in B_j} c_j f_{\alpha_j}^n(y)e_n\|
\]

Estimate the second term as follows, using (3.9) and (F6):

\[
\sum_{j=j_0}^{\infty} \|\sum_{n \in B_j} (f_n - c_j f_{\alpha_j}^n)(y)h(e_n)| \leq \sum_{j=j_0}^{\infty} \|\sum_{n \in D_j} (f_n - c_j f_{\alpha_j}^n)(y)h(e_n)|
\]

\[
+ \sum_{j=j_0}^{\infty} \|\sum_{n \in B_j \setminus D_j} (f_n - c_j f_{\alpha_j}^n)(y)h(e_n)|
\]

\[
\leq \sum_{j=j_0}^{\infty} (j+2)c_j\|y\| + 2 \sum_{j=j_0}^{\infty} c_j \sum_{t < j} m_t \|y\|
\]

\[
\leq \frac{6}{2^{j_0}} \|y\|
\]

Estimate the third term as follows

\[
\|\sum_{j=j_0}^{\infty} \sum_{n \in B_j} c_j f_{\alpha_j}^n(y)e_n\| \leq \sum_{j=j_0}^{\infty} c_j \|\sum_{n \in B_j} f_{\alpha_j}^n(y)e_n\| \leq \sum_{j=j_0}^{\infty} \frac{1}{2^{j+1}} \|y\| \leq \frac{1}{2^{j_0}} \|y\|
\]
using (F7) and the fact that for any block sequence \((h_k) \subset K\) and \(z \in X_S\) we have \(\| \sum_k h_k(z)e_k \| \leq \| z \|\) (cf. Fact 1.3 [MP2]).

The following theorem summarizes previous estimates setting the ground for the general criterion (Theorem 2.3) in the Schlumprecht space.

**Theorem 3.15.** Take a core tree \(T\) with parameters \((m_j, q_j)_j\), satisfying (F1)-(F7) and the operator \(T\) defined by sequences \((f_n, x_n)_n\) associated to \(T\). Then we have the following

1. the operator \(T\) is bounded, strictly singular and non-compact.

2. for any \(N \in \mathbb{N}\) and any \(q_{jN-1} \leq k \leq m_{jN}\) (where \(\alpha_{jN}\) is minimal in \(\{ \alpha \in T : |\alpha| = N\}\) with respect to the lexicographical order \(\leq_{lex}\)) we have

\[
e_k(T) \leq 2 \sum_{j < J} m_j, \quad d_k(T) \leq 14k \frac{\log_2(m_{jN-1} + 1)}{m_{jN-1}}.
\]

3. \(T \in \mathcal{S}_\infty(X_S) = \mathcal{S}_\infty(X_S, X_S)\) (where \(X_S\) is considered with the canonical basis \((e_n)_n\)).

**Proof.** (1) follows immediately from Proposition 3.14 as \(X_S\) does not contain \(c_0\). The non-compactness is guaranteed by the fact that \(Tx_n = e_n, n \in \mathbb{N}\), with (F1) \(((x_n)_n\) is seminormalized).

(2) follows by Propositions 3.10 and 3.12 and the definition of parameters \((d_j)_j\).

(3) Take a seminormalized sequence \((y_n)_n \subset X_S\) with \(\|Ty_l\|_{X_{S, \infty}} \to 0, l \to \infty\). As \(\|T\|_{X_{S, \infty}} \geq \frac{1}{2}\|y\|\) any \(j \in \mathbb{N}\), it follows that for any \(j \in \mathbb{N}\) \(\|y_l\|_{m_j} \to 0, l \to \infty\).

Let \(C = \sup_{l} \|y_l\|\). We show that any subsequence \((y_l)_l \in M\) contains a further subsequence \((y_l)_l \in L\) with \(\|Ty_l\|_{L_{\infty}} \leq \frac{C}{2l} \to 0\), which ends the proof.

Take a subsequence \((y_l)_l \in L\). Diagonalizing pass to a subsequence \((y_l)_l \in L\), such that \(\|y_l\|_{m_j} \leq \frac{C}{2l}\) for any \(j \leq l, l \in L\). Then by Prop. 3.14 we have for any \(l \in L\)

\[
\|Ty_l\| \leq \sum_{j=1}^{l} \|y_l\|_{m_j} + \frac{7}{2} \|y_l\| \leq \frac{l}{2} + \frac{7C}{2} \to 0, \quad L \ni \delta \to \infty
\]

\[\Box\]

### 3.4. Small operator ideals

Fix two core trees: \(T\) with parameters \((m_j, q_j)_j\) and \(R\) with parameters \((k_j, p_j)_j\), with both sets of parameters satisfying (F1)-(F7). Pick block sequences of vectors \((x_n)_n\) and functionals \((f_n)_n\) associated to \(T\) and block sequences of vectors \((y_n)_n\) and functionals \((g_n)_n\) associated to \(R\) and associated operators \(T, R \in \mathcal{B}_{seq}(X_S)\): \(T : X_S \ni x \mapsto \sum_n f_n(x)e_n \in X_S, \quad R : X_S \ni x \mapsto \sum_n g_n(x)e_n \in X_S\). (3.10)

We define now conditions concerning only two fixed consecutive levels of the trees \(T\) and \(R\).

For any \(N \in \mathbb{N}\) let \(J_N\) (resp. \(i_N\)) be such that \(\alpha_{jN}\) is minimal in \(\{ \alpha \in T : |\alpha| = N\}\) (resp. \(\gamma_{iN}\) is minimal in \(\{ \gamma \in R : |\gamma| = N\}\)) with respect to the lexicographical order \(\leq_{lex}\).

We define the following property: we write that \((m_0, q_0)_{|\alpha|=N-1, N} \succ (k_0, p_0)_{|\gamma|=N-1, N}\) for \(0 < N \in \mathbb{N}\), provided

\[
(L1) \quad \frac{\log_2(m_{jN-1} + 1)}{m_{jN-1}} \sum_{|\gamma|<N} k_\gamma \leq \frac{1}{N}
\]

\[
(L2) \quad q_{jN-1} \leq k_{iN} \leq m_{jN}.
\]

**Remark 3.16.** Notice that it is possible to build core trees with parameters \((m_j, q_j)_{j=N-1, N} \succ (k_j, p_j)_{j=N-1, N}\) for any \(N \in \mathbb{N}\), with both sets of parameters satisfying (F1)-(F7) by induction on \(N\). Indeed, choose \((m_0, q_0, k_0, p_0)\), to ensure (F1), (F2), (L1), (L2). For a fixed \(N \in \mathbb{N}\), having chosen parameters of the trees \(T, R\) up to level \(N-1\), i.e. \((m_0)_{|\alpha|<N}, (q_0)_{|\alpha|<N}\) (in
particular \( q_{jN-1} \), since \( |\alpha_{jN-1}| = N - 1 \) and \( (k_\gamma)|\gamma|<\infty \), \( (p_\gamma)|\gamma|<\infty \), we pick \( k_{iN} \geq q_{jN-1} \) and the rest of parameters on the \( N \)-th level of the tree \( \mathcal{R} \), i.e. \( (p_\gamma)|\gamma|=N \) and \( (k_\gamma)|\gamma|=N, \gamma > i_{\gamma_{iN}} \) so that (F1)-(F7) in the case of \( \mathcal{R} \) are satisfied. Then we pick \( m_{jN} \geq k_{iN} \) big enough to ensure (F1)-(F7) in the case of \( \mathcal{T} \) and moreover so that

\[
\frac{\log_2(m_{jN} + 1)}{m_{jN}} \sum_{|\gamma| \leq N} k_{\gamma} \leq \frac{1}{N + 1}
\]

Then we choose \( (q_\alpha)|\alpha|=N \) and \( (m_\alpha)|\alpha|=N, \alpha > i_{\gamma_{iN}} \) to ensure (F1)-(F7) in the case of \( \mathcal{T} \) and thus we finish the inductive procedure.

Theorem 3.15 (2) implies the following

**Corollary 3.17.** Let \( T \) and \( R \) be bounded strictly singular operators defined as in (3.10) by core trees \( \mathcal{T} \) and \( \mathcal{R} \) with parameters \( (m_\alpha, q_\alpha), (k_\gamma, p_\gamma) \). If there is an infinite \( J \subset \mathbb{N} \) such that \( (m_\alpha, q_\alpha)|\alpha|=N-1,N \geq (k_\gamma, p_\gamma)|\gamma|=N-1,N \) for any \( N \in J \), then

\[
\liminf_{k \in \mathbb{N}} \frac{1}{k} \omega_k(R) d_k(T) = 0
\]

**Lemma 3.18.** There is a family \( (T_r)_{r \in \mathbb{R}} \) of bounded strictly singular operators on \( X_S \) such that for any \( r, s \in \mathbb{R} \), \( r \neq s \) we have

\[
\liminf_{k \in \mathbb{N}} \frac{1}{k} \omega_k(T_r) d_k(T_s) = 0
\]

**Proof.** Let \( \mathcal{D} \) be a dyadic tree with the root \( \emptyset \), the order \( \leq \), the lexicographic order \( \leq_{\text{lex}} \) and levels \( |d|, d \in \mathcal{D} \). For any \( d \in \mathcal{D}, d \neq \emptyset \) let \( d^- \) be the immediate predecessor of \( d \) in \( (\mathcal{D}, \leq) \), i.e. \( d \in \text{succ}(d^-) \). Recall that a branch of a tree is a maximal linearly ordered subset of \( \mathcal{D} \) with respect to \( \leq \).

We attach to every node \( d \in \mathcal{D} \) a set of parameters \( m^d = (m^d_j, q^d_j)_{j \in A_d} \) for some finite \( A_d \subset \mathbb{N} \) so that

1. for any branch \( I \subset \mathcal{D} \) we have \( \mathbb{N} = \bigcup_{d \in I} A_d \), and for any \( d, d' \in I \) with \( d \leq d' \) we have \( \max A_d < \min A_{d'} \),
2. for any branch \( I \subset \mathcal{D} \) parameters \( (m^d_j)_{j \in A_d, d \in I} \) define a core tree \( \mathcal{T}_I \), such that for any \( d \in I \) we have \( \{ j \in \mathbb{N} : |\alpha_j| = |d| \} = A_d \) (i.e. \( m^d \) is the set of parameters on the \( |d| \)-level of \( \mathcal{T}_I \)),
3. for any branch \( I \subset \mathcal{D} \) parameters \( u_{d \in I} m^d = (m^d_j, q^d_j)_{j \in A_d, d \in I} \) satisfy (F1)-(F7),
4. for any \( 0 < N \in \mathbb{N} \), any \( d, d' \in \mathcal{D} \) with \( |d| = |d'| = 4N, d \leq_{\text{lex}} d' \) we have \( m^d \geq m^{d'} \cup m^{d'} \),
5. for any \( N \in \mathbb{N} \), any \( d, d' \in \mathcal{D} \) with \( |d| = |d'| = 4N + 2, d \leq_{\text{lex}} d' \) we have \( m^{d-} \cup m^d \leq m^{d'-} \cup m^{d'} \).

Conditions (1)-(3) above describe the way the parameters defining trees \( \mathcal{T}_I \), with \( I \) - a branch of \( \mathcal{D} \), are represented in the tree \( \mathcal{D} \). Then for any branch \( I \) of \( \mathcal{D} \) we let \( T_I \in \mathcal{B}_{\text{seg}}(X_S) \) be the operator defined by sequences associated to the tree \( \mathcal{T}_I \) with parameters \( u_{d \in I} m^d = (m^d_j, q^d_j)_{j \in A_d, d \in I} \). Conditions (4) and (5) above imply, by Corollary 3.17 that for any different branches \( I, I' \) of \( \mathcal{D} \) we have

\[
\liminf_{k \in \mathbb{N}} \frac{1}{k} \omega_k(T_I) d_k(T_{I'}) = 0
\]

In order to finish the proof notice that we can choose sets \( m^d = (m^d_j, q^d_j)_{j \in A_d} \), \( d \in \mathcal{D} \) satisfying (1)-(5) by induction on the level of \( \mathcal{D} \), more precisely choosing for every \( 0 < N \in \mathbb{N} \) parameters \( (m^d_j, q^d_j)_{j \in A_d, |d| = 4N-1,2N} \) satisfying (1)-(3) and either (4) or (5) similarly to the way it was done in Remark 3.16 profiting from the fact that on each level of \( \mathcal{D} \) we have only finitely many \( d \)'s. 

\[\square\]
Theorem 3.19. There is a family of small closed operator ideals \((I_A)_{A \subset \mathbb{R}}\) on the Schlumprecht space such that \(I_A \subset I_B \iff A \subset B\), for any \(A, B \subset \mathbb{R}\). In particular there are exactly \(2^\kappa\) distinct small closed ideals on the Schlumprecht space.

Proof. Take a family \((T_r)_{r \in \mathbb{R}} \subset \mathcal{S}(X_S)\) as in Lemma 3.18 and for any \(A \subset \mathbb{R}\) define \(I_A\) to be the closed operator ideal generated by \((T_r)_{r \in A}\). Then obviously \(A \subset B\) implies \(I_A \subset I_B\). On the other hand, fix \(A \subset \mathbb{R}\) and \(r \in \mathbb{R} \setminus A\). Then \(T_r\) and \((T_s)_{s \in A}\) satisfy the assumptions of Theorem 2.3, thus in particular \(T_r \not\in I_A\). In other words \(T_r \in I_A\) implies \(r \in A\), which finishes the proof. \(\square\)

Remark 3.20. (1) It follows easily that there is also a chain of cardinality \(\kappa\) of small closed ideals and an antichain of cardinality \(2^\kappa\) of small closed ideals on the Schlumprecht space.

(2) Theorem 3.19 holds true in the case of Banach spaces with a basis whose spreading model is equivalent to the basis \((e_n)_n\) of the Schlumprecht space, like Gowers-Maurey space (cf. [MP2]).

(3) The last step of construction (creating \((I_A)_{A \subset \mathbb{R}}\) from a family of bounded operators \((T_r)_{r \in \mathbb{R}}\) with certain separation property described in Theorem 2.3) works as in [JS], however, the constructed operators are of a different type.

4. Schreier Space

We recall the definition of Schreier spaces \(X[S_N], N \in \mathbb{N}\), given in [AA]. Define the Schreier families \(S_N, N \in \mathbb{N}\), by induction. Let

\[ S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\} \]

and for any \(N \in \mathbb{N}\) let

\[ S_{N+1} = \left\{ \bigcup_{i=1}^{k} E_i : k \in \mathbb{N}, E_1, \ldots, E_k \in S_N, k \leq \min E_1, E_1 < \cdots < E_k \right\} \cup \{\emptyset\} \]

We recall here one of the basic properties of Schreier families we shall need later, namely the spreading property: for any \(N \in \mathbb{N}\) and \(\{n_1, \ldots, n_k\} \in S_N\), and \(m_1 < \cdots < m_k \in \mathbb{N}\) with \(n_1 \leq m_1, \ldots, n_k \leq m_k\) also \(\{m_1, \ldots, m_k\} \in S_N\).

The Schreier space \(X[S_N], N \in \mathbb{N}\), is the completion of \(c_{00}\) with the norm

\[ \|x_n\|_{S_N} = \sup_{A \in S_N \setminus A \neq \emptyset} \sum_{n \in A} |x_n|, \quad (x_n)_{n \in c_{00}} \]

The unit vector basis of \(c_{00}\), which we denote in this section by \((\tilde{e}_n)_{n}\), was proved in [AA] to be a shrinking unconditional basis for any \(X[S_N], N \in \mathbb{N}\). By the spreading property of Schreier families any subsequence \((\tilde{e}_{i_n})_n\), 1-dominates the basis \((\tilde{e}_n)_{n}\) in \(X[S_N]\).

For any \(I \subset \mathbb{N}\) by \(P_I\) denote the canonical projection on \([\tilde{e}_n : n \in I]\).

We shall need the following observations. Repeating the reasoning of [BL] we obtain the following (one needs only to observe that \(S_1 \subset S_N, N \in \mathbb{N}, N \geq 1\))

Lemma 4.1. [BL] Corollary 3.17(ii) Fix \(N \in \mathbb{N}, N \geq 1\), and infinite \(I = (i_n)_n, J = (j_n)_n \subset \mathbb{N}\) and assume the mapping \(\tilde{e}_{i_n} \to \tilde{e}_{j_n}, n \in \mathbb{N}\), extends to a bounded operator \(R_{I,J} : X[S_N] \to [\tilde{e}_i : i \in I] \to [\tilde{e}_j : j \in J] \subset X[S_N]\). Then there is a constant \(C > 0\) with \(j_n \leq C i_n\) for all \(n \in \mathbb{N}\).

The following lemma is proved in [CS] Prop. 0.7 in the case of \(N = 1\) (recall that \(X[S_0] = c_{00}\)) and with domination replaced by 1-equivalence. In the proof we use this case (\(N = 1\)).

Lemma 4.2. For any \(N \in \mathbb{N}, N \geq 1\), and any sequence \((E_m)_{m \in \mathbb{N}}\) of finite subsets of \(\mathbb{N}\) with \(#E_m = \min E_m, \#E_{m+1} = 2\#E_m \max E_m, m \in \mathbb{N}\) the sequence of averages \((y_m)_{m \in \mathbb{N}} \subset X[S_N]\) defined as \(y_m = \frac{1}{s_m} \sum_{n \in E_m} \pm \tilde{e}_n, m \in \mathbb{N}, is 2-dominated by the unit vector basis \((\tilde{e}_{s_m})_m\) of \(X[S_{N-1}]\), where \(s_m = \max E_m, m \in \mathbb{N}\).
Proof. Take a maximal, with respect to the inclusion, set $F \in S_N$ and consider $\| \sum_m a_m y_m | F \|_{l_1}$, $(a_m)_m \subset \mathbb{R}$. By [GL] Lemma 3.8 $F = \cup_{i=1}^k F_i$, where $F_1 \subset \cdots \subset F_k$ are maximal members of $S_1$ and $(\min F_i)_{i=1}^k \in S_{N-1}$. Let $G_i = \{ m \in \mathbb{N} : F_i \cap E_m \neq \emptyset \}$, pick $m_i \in G_i$ with $|a_{m_i}| = \max \{ |a_m| : m \in G_i \}$ for each $i = 1, \ldots, k$. Estimate, using the fact that $(y_m)_m$ in $X[S_1]$ is 1-equivalent to the unit vector basis of $c_0$,

$$\| \sum_m a_m y_m | F \|_{l_1} = \sum_{i=1}^k \| \sum_{m \in G_i} a_m y_m | F_i \|_{l_1} \leq \sum_{i=1}^k \| a_m y_m \|_{S_1} \leq \sum_{i=1}^k \| a_m \sum_n e_{m_n} \| \leq \cdots \leq 2 \| \sum_m a_m \sum_n e_{m_n} \|_{S_{N-1}}$$

Notice that for any $m \in \mathbb{N}$ by maximality of $F_i$’s there can be at most 2 sets from the family $(F_i)_i$ intersecting $E_m$, thus each $m_i$ can appear in the above sum at most twice. Thus, as $s_{m_i} \geq \min F_i, i = 1, \ldots, k$, and $(\min F_i)_{i=1}^k \in S_{N-1}$, we continue

$$\cdots \leq 2 \| \sum_m a_m \sum_n e_{m_n} \|_{S_{N-1}}$$

which ends the proof. \hfill \square

The following lemma is well known, for the sake of completeness we present a proof.

**Lemma 4.3.** The basis $(\tilde{e}_n)_n \subset X[S_N]$ is not dominated by any subsequence of the basis $(\hat{e}_n)_n \subset X[S_{N-1}]$, for any $N \in \mathbb{N}, N \geq 1$.

**Proof.** Take $(\tilde{e}_m)_m \subset X[S_{N-1}]$. Pick $(m_n)_n \subset \mathbb{N}$ with

$$m_n \leq k_{m_n} < m_{n+1} \leq k_{m_{n+1}}, \quad n \in \mathbb{N}$$

Fix $\epsilon > 0$ and take $(a_n)_{n \in F} \subset [0, 1]$ so that $\{ m_n : n \in F \} \subset S_N$, with $\sum_{n \in F} a_n = 1$ and $\sum_{n \in G} a_n < \epsilon$ for any $G$ with $\{ m_n : n \in G \} \subset S_{N-1}$. An average of this type is called a $(N, \epsilon)$-special convex combination, its existence was proved in [AD].

Then $\| \sum_{n \in F} a_n \tilde{e}_m \|_{S_N} = 1$. On the other side for any $G \in S_{N-1}$ with $(k_{m_n})_n \subset G \subset S_{N-1}$, we have $\{ m_n : n \in G \} = \{ m_{\min G} \} \cup \{ m_n : n \in G \setminus \{ \min G \} \}$ and each of these sets is a member of $S_{N-1}$. It follows that $\sum_{n \in G} a_n \tilde{e}_m \leq 2\epsilon$ and hence $\| \sum_{n \in F} a_n \tilde{e}_m \|_{S_{N-1}} \leq 2\epsilon$. As $\epsilon > 0$ was arbitrarily small the basic sequence $(\tilde{e}_m)_m \subset X[S_{N-1}]$ does not dominate the basis $(\hat{e}_m)_m \subset X[S_N]$. \hfill \square

Recall that by [O] any Schreier space $X[S_N], N \in \mathbb{N}$, is saturated by isomorphic copies of $c_0$. Alternatively we can apply Lemma 4.2 inductively in any $X[S_N]$ obtaining normalized repeated averages of the basis dominated by (and thus equivalent to) the unit vector basis of $c_0$.

We fix for the rest of the section a normalized block sequence $(e_n)_n$ equivalent to the unit vector basis of $c_0$ and let $Y \subset X[S_N]$ be the closed subspace spanned by $(e_n)_n$.

**Lemma 4.4.** [BKL] Lemma 4.14 For any $N \in \mathbb{N}, N \geq 1$, and any infinite $I \subset \mathbb{N}$ the formal identity operator $X[S_N] \ni [\hat{e}_i : i \in I] \mapsto c_0$ is non-compact strictly singular with norm 1.

For any infinite $I = (i_n)_n \subset \mathbb{N}$ let $T_I \in B_{\text{seq}}(X[S_N], Y)$ be the operator defined by sequences $(\tilde{e}_{i_n}, e_{i_n})_{n \in \mathbb{N}}$, i.e.

$$T_I : X[S_N] \ni (a_n)_n \mapsto \sum_n a_n e_n \in Y \subset X[S_N]$$

Notice that $T_I \in B_{\infty}(X, Y) \cap B(X[S_N])$ by Lemma 4.3 and the choice of $(e_n)_n$.

**Theorem 4.5.** Fix $N \in \mathbb{N}, N \geq 1$, and infinite $I = (i_n)_n, J = (j_n)_n \subset \mathbb{N}$ such that for any $K \in \mathbb{N}$ we have

$$\lim_{n \to \infty} \frac{j_n}{i_n} = \infty$$

Then $\| T_I - Q \| \geq 1$ for any $Q$ in the operator ideal $I(T_J)$ generated by $T_J$ in $B(X[S_N])$. 
Proof. The proof goes along the lines of the proof of Theorem 2.3, however, different reasoning is needed as \( \inf_k \mathbb{E}\chi(T_J)dk(T_j) = 1 \).

Take operators as in the theorem. Assume there is \( Q \in I(T_J) \) with \( \|T_J - Q\| < 1 \). Then there are \( Q_t, S_t \in \mathcal{B}(X), t = 1, \ldots, k \), such that \( \|T_J - \sum_{t=1}^k Q_t \circ S_t\| < 1 \). As \( T(J) = e_n \), \( n \in \mathbb{N} \), the sequence \( \{\sum_{t=1}^k Q_t \circ S_t \circ J\} \) is semi-normalized. It follows that for any \( i \in I \) there is \( t_i \in \{1, \ldots, k\} \) such that the sequence \( \{T(J) \circ S_t\} \) is semi-normalized. For any \( t = 1, \ldots, k \) let \( I_t = \{i \in I : t = t_i\} \). Then for the operator \( S = \sum_{t=1}^k S_t \circ P_t \in \mathcal{B}(X|S_N) \) the sequence \( \{(T(J) \circ S)\} \) is semi-normalized. As the sequence \( (e_n)_n \) is equivalent to the unit vector basis of \( c_0 \), there is universal \( c > 0 \) such that

\[
\sup_{t \in \mathbb{N}} |e_n^\sharp(T(J)\circ S)| = \sup_{j \in J} |S(e_j)(j)| > c, \quad i \in I
\]

Let \( A_j = \{i \in I : |S(e_i)(j)| > c\} \) for any \( j \in J \). We prove here a stronger property of \( A_j \)'s than in the proof of Theorem 2.3, namely

**Claim.** There is universal \( K \in \mathbb{N} \) such that \( \#A_j \leq K, j \in J \).

**Proof of Claim.** Assume otherwise, then we can pick a sequence \( (E_m)_m \) of subsets of \( \mathbb{N} \) as in Lemma 1.2 such that each \( E_m \) is contained in some \( A_m \). For each \( m \in \mathbb{N} \) let \( y_m = \frac{1}{\#A_m} \sum_{n \in E_m} (\text{sign}(e_n)(l_m))e_n \). Then by Lemma 1.2 the block sequence \( (y_m)_m \) is 2-dominated by some subsequence of the basis \( (e_n)_m \) in \( X|S_N-1 \), whereas \( (y_m)_m \) for each \( m \in \mathbb{N} \). Notice that \( S(y_m)(l) \xrightarrow{m \to \infty} 0 \) for each \( l \in \mathbb{N} \). Otherwise there would be \( l \in \mathbb{N} \) and \( \delta > 0 \) with \( |S(y_m)(l)| > \delta \) for infinitely many \( m \)'s, without loss of generality \( S(y_m)(l) > \delta \). Taking the infinite set \( M \) of such \( m \)'s for any finite \( A \subset M \) we obtain

\[
\#A \delta < \sum_{m \in A} S(y_m)(l) \leq \| \sum_{m \in A} S(y_m) \|_{S_N} \leq \|S\| \sum_{m \in A} y_m \|_{S_N}
\]

which, as the block sequence \( (y_m)_{m \in A} \) is normalized and unconditional, yields that \( (y_m)_{m \in A} \) is equivalent to the unit vector basis of \( \ell_1 \), contradicting the fact that \( (e_n)_n \) is a shrinking basis of \( X|S_N \) (\([AA]\)).

Therefore passing to a subsequence we can assume that \( l_m \nearrow \infty \) and \( |S(y_m)(l_m)| < \frac{\|S\|}{2m + m^2} \) for any \( m \neq m' \). Now take any scalars \( (a_m)_m \) with \( \| \sum_m a_m y_m \|_{S_N} = 1 \) (recall that \( \sup_m \|a_m\| \leq 2 \)) and estimate by unconditionality of \( (e_n)_n \), as follows

\[
\|S\| = \|S\| \| \sum_m a_m y_m \|_{S_N} \geq \| \sum_m a_m (P_J \circ S)(y_m) \|_{S_N}
\]

\[
\geq \| \sum_m a_m S(y_m)(l_m) e_{l_m} \|_{S_N} - \sum_{k \neq m} |a_k S_m(l_k)|
\]

\[
\geq c \| \sum_m a_m e_{l_m} \|_{S_N} - \frac{\|S\|}{2}
\]

\[
\geq c \| \sum_m a_m e_{l_m} \|_{S_N} - \frac{\|S\|}{2}
\]

which yields a contradiction by Lemma 1.3. \( \square \)

**Claim.** Let \( (B_n)_{n \in \mathbb{N}} \), be a family of subsets of \( \mathbb{N} \) of cardinality at most \( K \in \mathbb{N} \) such that \( \bigcup_n B_n = \mathbb{N} \). Then there is infinite \( M \subset \mathbb{N} \) and an increasing mapping \( \phi : M \to \mathbb{N} \) such that \( m \in B_{\phi(m)} \) and \( m \leq K \phi(m) \) for any \( m \in M \).

**Proof of Claim.** We pick \( M \subset \mathbb{N} \) by induction. Let \( m_1 = 1 \) and \( \phi(m_1) \in \mathbb{N} \) arbitrary with \( m_1 \in B_{\phi(m_1)} \). Then obviously \( K \phi(m_1) > 1 \). Fix \( k \in \mathbb{N} \) and assume we picked first \( m_1, \ldots, m_k \) as desired. Let \( B = \bigcup_{n \leq \phi(m_k)} B_n \) and notice that \( \#B \leq K \phi(m_k) \). Let \( m_{k+1} = \min((\mathbb{N} \setminus B) \cap (m_k, \infty)) \) and \( \phi(m_{k+1}) \in \mathbb{N} \) arbitrary with \( m_{k+1} \in B_{\phi(m_{k+1})} \). Then

\[
m_{k+1} \leq \max\{K \phi(m_k) + 1, m_k + 1\} = K \phi(m_k) + 1 \leq K(\phi(m_{k+1}) - 1) + 1 \leq K \phi(m_{k+1})
\]
which ends the inductive step. Taking $M = \{m_k : k \in \mathbb{N}\}$ ends the proof of Claim. 

Now we continue the proof of the theorem. We apply the Claim above for $B_k = \{n \in \mathbb{N} : i_n \in A_{j_k}\}$, where $J = (j_k)_{k \in \mathbb{N}}$, obtaining a suitable infinite $M \subset \mathbb{N}$ and a mapping $\phi : M \to \mathbb{N}$ with $m \in B_{\phi(m)}$ (i.e., $i_m \in A_{j_{\phi(m)}}$) and $m \leq K \phi(m)$ for any $m \in M$.

As in the proof of the first claim passing to infinite $M'$ of $M$ we can assume that $S(\bar{e}_{im})(j_{\phi(m')}) < \frac{1}{2^{m+m'}}$ for any $m \neq m'$ in $M'$ and thus the sequence $(\bar{e}_{im})_{m \in M'}$ dominates the sequence $(\bar{e}_{j_{\phi(m)}})_{m \in M'}$. By Lemma 4.4 there is universal $C > 0$ with $\bar{j}_{\phi(m)} \leq CI_m$ for any $m \in M'$. It follows that $j_{m/K}/i_m \leq C$ for any $m \in M'$ which contradicts the assumptions on $I, J$.

Remark 4.6. Careful examination of the proof shows that the idea behind it is the same as in Theorem 2.4, however, it works in a different technical setting. We make sure in the proof first that there is infinite $M \subset \mathbb{N}$ so that the set of indices $(l_n)_{n \in \mathbb{N}}$ of functionals defining $T_I$, i.e. $\{g_{l_n} = (\bar{e}_{\phi(l_n)})_{n \in \mathbb{N}}\}$ is a spread of $\{\lfloor n/K \rfloor : n \in M\}$ for some $K \in \mathbb{N}$. Unboundedness of the operator carrying each $\bar{e}_{l_n}$ to $\bar{e}_{\phi(l_n)}$, $n \in M$ implied by the assumptions on $I, J$ and the choice of $M$ is witnessed by convex combinations $\sum_{n \in A} a_n x_n$ of $(x_n)_{n \in \mathbb{N}} = (\bar{e}_{l_n})_{n \in \mathbb{N}}$ with the Schreier norm tending to zero (cf. [BL]), whereas $\|\sum_{n \in A} g_{l_n} x_n\| = 1$. Thus the proof relies on the fact that for any infinite $M \subset \mathbb{N}$ we have

$$\inf_{A \subset M} \left( \inf_{n \in \mathbb{N}} \| \sum_{n \in A} a_n x_n \| S_N \right) \left( \sup_{n \in \mathbb{N}} \| \sum_{n \in A} g_{l_n} x_n \| S_N \right) = 0,$$

where the supremum is taken over all $\{l_n\}_{n \in \mathbb{N}}$ that are spreads of $\{\lfloor n/K \rfloor : n \in M\}$ for some $K \in \mathbb{N}$. In the above formula instead of averages defining $\frac{1}{k!} d_k(T_I)$ we use all convex combinations (describing the "distance" of finite subsequences of $(\bar{e}_{l_n})$ to the unit vector basis of $\ell_1$ of corresponding length), and instead of all sums of functionals in $c_k(T_J)$ we consider only those with indices from a certain restricted family that is sufficient in our case, which desires a separate proof.

We notice also that - in spite of involving Schreier families in the definition of the space - the presented reasoning reduces the to comparing behaviour of sequences defining operators from the family $R_{\text{seq}}(X_{S_N})$ on the sets of equal cardinality - on the families of sets of low complexity when compared to the complexity of Schreier families.

Remark 4.7. Notice that if either of the conditions in Lemma 4.4 is satisfied, the operator $T_I$ belongs to the operator ideal generated by $T_J$, as $T_I = T_J \circ R_{l,J} \circ P_I$.

It follows that for any $I$ and $J$ satisfying assumptions of Theorem 4.6, the operator $T_J$ belongs to the ideal generated by $T_I$.

Theorem 4.8. For any $N \in \mathbb{N}$, $N \geq 1$, there is a chain of cardinality $\mathfrak{c}$ of closed small ideals in $\mathcal{B}(X[S_N])$.

Proof. Let $D$ be a dyadic tree with the root $\emptyset$. Enumerate the nodes of $D$ according to the lexicographic order $\leq_{\text{lex}}$ as $(d_k)_{k \in \mathbb{N}}$. For any branch $B$ of $D$ let $I_B = (k!)_{d_k \in B}$. We extend the lexicographic order on the family of branches of $D$: we write $B \leq_{\text{lex}} B'$ provided for any $d \in B$ and $d' \in B'$ with $|d| = |d'|$ we have $d \leq_{\text{lex}} d'$.

Notice that for any branches $B \leq_{\text{lex}} B'$ the sets of indices $I_B = (i_n)_{n \in \mathbb{N}}$, $I_{B'} = (j_n)_{n \in \mathbb{N}}$ satisfy $\lim_{n \to \infty} j_{n/K}/i_n = \infty$ for any $K \in \mathbb{N}$. By Theorems 4.3 and Remark 4.6, the mapping carrying each branch $B$ of $D$ to the closed operator ideal generated by the operator $T_{i_B}$ is strictly increasing with respect to the order $\leq_{\text{lex}}$ on the family of branches of $D$ and inclusion in $\mathcal{B}(X[S_N])$. As the operators $T_{i_B}$ are strictly singular (by Lemma 4.4) we finish the proof.

The result of [BKL] and the one presented above suggest the following Question. Do the Schreier spaces $X[S_N]$, $N \in \mathbb{N}$, $N \geq 1$, admit exactly continuum many distinct closed operator ideals?
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