Research Article

A Net with Applications for Continuity in a Fuzzy Soft Topological Space

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In this paper, the concept of a fuzzy soft point is redefined, and the definition of a fuzzy soft net in a fuzzy soft topological space is given. On this basis, the convergence of a fuzzy soft net is defined by using the Q-neighborhood theory, and the continuity of fuzzy soft mappings is characterized by the net. The obtained results demonstrate that the concepts proposed in this paper are very suitable and will provide powerful research tools for further research in this field.

1. Introduction

In 1965, Zadeh introduced the concept of fuzzy sets in his classic work [1]. In 1999, Molodtsov [2] introduced the theory of soft sets, which have been applied in several fields, including the smoothness of functions, game theory, Riemann integration, and the theory of probability [3]. In 2001, Maji et al. [4] combined fuzzy sets with soft sets and proposed the concept of fuzzy soft sets. Since then, many researchers have applied fuzzy soft sets to group theory [5], decision making, and medical diagnosis [6,7]. In 2009, Kharal and Ahmad [8] studied the properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets.

In 2011, Tanay and Kandemir [9] proposed the concept of a fuzzy soft set in a fuzzy soft topological space and explained some of its structural properties. They also claimed that fuzzy soft topological spaces may be used in the theory of information systems. In 2012, Mahanta and Das [10] introduced the definition of a fuzzy soft point and its neighborhood. They also studied the interior and closure of a fuzzy soft set and investigated the separation axioms and connectedness. Varol and Aygün [11] introduced the fuzzy soft continuity of fuzzy soft mappings. In 2013, Gunduz (Aras) and Bayramov [3] presented fuzzy soft continuous mappings, fuzzy soft open and fuzzy soft closed mappings, and fuzzy soft homeomorphism. In 2014, Ping et al. [12] proposed the sum of fuzzy soft topological spaces. In 2016, Mishra and Srivastava [13] studied compactness in fuzzy soft topological spaces. In 2017, Kandil et al. [14] discussed the connectedness of fuzzy soft sets. Riaz and Hashmi [15] proposed the concept of fuzzifying soft sets, called fuzzy parameterized fuzzy soft sets (FPFS-sets). Mahanta and Das [16] studied fuzzy soft closure and the fuzzy soft interior. In 2018, Abbas et al. [17] explored connectedness in fuzzy soft topological spaces. In 2019, Riaz and Tehrim [18] proved some properties of bipolar fuzzy soft topology (BFS-topology) via the use of the concept of the Q-neighborhood.

In 2012, Roy and Samanta [19] redefined the concept of fuzzy soft topology and obtained some basic results. In a subsequent work [20], they adopted a new definition of a fuzzy soft point, proposed the concepts of quasi-coincidence and Q-neighborhoods, and demonstrated the relationship between the limit point and the closure of a fuzzy soft set.

As pointed out in Example 1 in this paper, the existing concept of fuzzy soft points does not satisfy selectivity, which makes it difficult for a fuzzy soft point to play its expected role. Therefore, a more suitable definition of a fuzzy soft point must be given. Moreover, as is commonly known, the net plays an important role in classical topology theory; however, the concept of the net has not been introduced into fuzzy soft topological spaces.
In view of these considerations, this paper first redefines the concept of a fuzzy soft point and introduces the net into fuzzy soft topological spaces. The continuity of a mapping in fuzzy soft topological spaces with the use of a net is then studied. The remainder of this article is organized as follows. In Section 2, some necessary concepts of fuzzy soft sets and fuzzy soft topological spaces are recalled. In Section 3, fuzzy soft points are refined and the notion of a fuzzy soft net consisting of fuzzy soft points is introduced. By using the theory of the Q-neighborhood, the concept of the convergence of a fuzzy soft net is introduced. In Section 4, the net is applied to characterize the continuity of fuzzy soft mappings. Section 5 presents a discussion of convergence for a net of fuzzy soft mappings. In Section 6, an example of the application of the fuzzy soft set theory to medical diagnosis is provided. Finally, the conclusion of this paper is given in Section 7.

2. Preliminaries

Throughout this paper, $U$ refers to an initial universe and $E$ is the set of all parameters of $U$. In this case, $U$ is also denoted by $(U, E)$. $I^U$ is the set of all fuzzy subsets over $U$, where $I = [0, 1]$. The elements $0, 1 \in I^U$, respectively, refer to the functions $0(x) = 0$ and $1(x) = 1$ for all $x \in U$. For an element $A \in I^U$, if there exists an $x \in U$ such that $A(x) = \lambda > 0$ and $A(y) = 0$, $\forall y \in U \setminus \{x\}$, then $A$ is called a fuzzy point over $U$ and is denoted by $x_1$, and $x$ and $\lambda$ are called the support and height of $x_1$, respectively. The set of all fuzzy points over $U$ is denoted by $FP(U)$.

The following definitions in this section were obtained from the existing literature [19,20].

**Definition 1.** Let $A \subseteq E$. A mapping $F_A: E \rightarrow I^U$, defined by $F_A(e) = \mu_{eA}$ (a fuzzy subset of $U$), is called a fuzzy soft set over $(U, E)$, where $\mu_{eA} = 0$ if $e \in E \setminus A$ and $\mu_{eA} \neq 0$ if $e \in A$.

If $A = E$, then $F_A$ is shortened to $F$.

The set of all fuzzy soft sets over $(U, E)$ is denoted by $FS(U, E)$.

The fuzzy soft set $F_A \subseteq FS(U, E)$ is called the null fuzzy soft set and is denoted by $\Phi$. Here, $F_A(e) = 0$ for every $e \in E$.

For $F_A \subseteq FS(U, E)$, if $F_A(e) = 1$ for all $e \in E$, then $F_A$ is called the absolute fuzzy soft set and is denoted by $\widetilde{F}$.

Let $F_A, F_B \subseteq FS(U, E)$. If $F_A(e) \subseteq F_B(e)$ for all $e \in E$, then $F_A$ is said to be a fuzzy soft subset of $F_B$ and is denoted by $F_A \subseteq F_B$ or $F_B \supseteq F_A$. In addition, $F_A \not\subseteq F_B$ means that $F_A$ is not a fuzzy soft subset of $F_B$. If $F_A \not\subseteq F_B$ and $F_B \not\subseteq F_A$, then $F_A$ and $F_B$ are said to be equivalent, which is denoted by $F_A \equiv F_B$.

**Remark 1.** If $F_A \subseteq F_B$, then $A \subseteq B$.

**Definition 2.** Let $F_A, F_B \in FS(U, E)$.

(1) The complement of $F_A$, denoted by $F_A^c$, is defined as

$$F_A^c(e) = \begin{cases} 1 - F_A(e), & \text{for } e \in A, \\ 1, & \text{otherwise.} \end{cases}$$

(2) The union of $F_A$ and $F_B$, denoted by $F_C = F_A \cup F_B$, is defined as $F_C(e) = F_A(e) \cup F_B(e)$ for all $e \in E$, where $C = A \cup B$.

(3) The intersection of $F_A$ and $F_B$, denoted by $F_C = F_A \cap F_B$, is defined as $F_C(e) = F_A(e) \cap F_B(e)$ for all $e \in E$, where $C = A \cap B$.

Similarly, the union (intersection) of a family of fuzzy soft sets $\{F_{C_a} : a \in A \}$ can be defined and is denoted by $\bigcup_{a \in A} F_{C_a} \bigcup_{a \in A} F_{C_a}$, where $A$ is an arbitrary index set.

**Remark 2.** It is clear that

(1) $\widetilde{\Phi} = \widetilde{E}, \widetilde{\Phi} = \widetilde{\Phi}$

(2) $(\bigcup_{a \in A} F_{A_a})^c = \bigcap_{a \in A} F_{A_a}^c$ for all $a \in A$ (any index set), then $\bigcup_{a \in A} F_{A_a}^c \subseteq \bigcap_{a \in A} F_{A_a}$

**Definition 3.** A fuzzy soft topology $\tau$ over $(U, E)$ is a family of fuzzy soft sets over $(U, E)$ satisfying the following properties:

(1) $\Phi, \widetilde{E} \in \tau$.

(2) If $F_A, F_B \in \tau$, then $F_A \cap F_B \in \tau$.

(3) If $F_{A_a} \in \tau$ for all $a \in A$ (any index set), then $\bigcup_{a \in A} F_{A_a} \in \tau$.

If $\tau$ is a fuzzy soft topology over $(U, E)$, the triple $(U, E, \tau)$ is said to be a fuzzy soft topological space. Each element of $\tau$ is called an open set. If $F_{A_a}$ is an open set, then $F_A$ is called a closed set.

**Definition 4.** Let $(U, E, \tau)$ be a fuzzy soft topological space, $F_A \in FS(U, E)$:

(1) The intersection of all closed sets $F_{A_a} \subseteq F_A$ is called the closure of $F_A$ and is denoted by $\overline{F_A}$.

(2) The union of all open subsets of $F_A$ over $(U, E, \tau)$ is called the interior of $F_A$ and is denoted by $int F_A$.

**Remark 3.** Let $F_A \in FS(U, E)$. It is evident that

(1) $\overline{F_A}$ is closed and $int F_A$ is open.

(2) $F_A$ is closed if and only if $F_A = \overline{F_A}$.

(3) $F_A$ is open if and only if $F_A = int F_A$.

3. Fuzzy Soft Point and Fuzzy Soft Net

Roy and Samanta [20] defined a fuzzy soft point $F_e$ over $(U, E)$ as a special fuzzy soft set such that $F_e(a) \neq 0$ if $a = e$, and $F_e(a) = 0$ if $a \neq e$. For $F_A \subseteq FS(U, E)$, they stated that a fuzzy soft point $F_e$ belongs in $F_A$, denoted by $F_e \subseteq F_A$, if and only if $(F_e(x))(x) \subseteq (F_A(x))(x)$ for all $x \in U$.

**Example 1.** Let $U = \{x, y\}$ and $E = \{e, f\}$. A fuzzy soft point $F_e$ is defined as
The sense of Definition 5.

In the remainder of this paper, a fuzzy soft point is always referred to as given by Definition 5 and is called a point for short.

For $P_\alpha^x$, $P_\beta^y \in P(F(U, E))$, if $x \neq y$ or $\alpha \neq \beta$ or $x \neq f$, then it is said that $P_\alpha^x$ and $P_\beta^y$ are different, which is written as $P_\alpha^x \neq P_\beta^y$.

**Theorem 1.** Let $P_\alpha^x \in P(F(U, E))$, $F_A, F_B \in P(F(U, E))$, and $F_{A_\alpha} \in P(F(U, E))$ ($\alpha \in \Lambda$):

1. If $P_\alpha^x \in F_A \cap F_B$, then $P_\alpha^x \in F_A$ or $P_\alpha^x \in F_B$.
2. If $P_\alpha^x \in F_{A_\alpha}$ (for any $\alpha \in \Lambda$) and only if $P_\alpha^x \in \bigcap_{\alpha \in \Lambda} F_{A_\alpha}$.

**Proof**

1. Because $P_\alpha^x \in F_A \cap F_B$, then $\lambda \leq \mu_\alpha^x$ ($x$); that is, $\lambda \leq \mu_\alpha^F_A (x) \cup \mu_\alpha^F_B (x)$. Therefore, $\lambda \leq \mu_\alpha^F_A (x)$ or $\lambda \leq \mu_\alpha^F_B (x)$. Hence, $P_\alpha^x \in F_A$ or $P_\alpha^x \in F_B$.
2. $P_\alpha^x \in F_{A_\alpha}$ (for any $\alpha \in \Lambda$) is equal to $\lambda \leq \mu_{A_\alpha}^\alpha (x)$ (for any $\alpha \in \Lambda$).

Equivalently, $\lambda \leq \bigcap_{\alpha \in \Lambda} \mu_{A_\alpha}^\alpha (x)$; that is, $P_\alpha^x \in \bigcap_{\alpha \in \Lambda} F_{A_\alpha}$.

However, it is clear that Theorem 1(1) does not hold for the union of infinite fuzzy soft sets. For this reason, the concept of quasi-coincidence is subsequently introduced.

**Definition 6.** A point $P_\alpha^x$ is said to be quasi-coincident with $F_A \in P(F(U, E))$, which is denoted by $P_\alpha^x \in F_A$, if $\lambda + \mu_\alpha^F_A (x) > 1$.

That $P_\alpha^x$ is not quasi-coincident with $F_A$ is denoted by $P_\alpha^x \notin F_A$.

**Remark 4.** In the work by Roy and Samanta [20], a fuzzy soft point $P_\alpha^x$ over $(U, E)$ was said to be quasi-coincident with $F_A \in P(F(U, E))$ if $\mu_{A_\alpha}^\alpha (x) > 1$ for some $x \in U$. If $P_\alpha^x$ is a fuzzy soft point in the sense of Definition 5, then the definition of quasi-coincidence in this paper is equivalent to that in the work by Roy and Samanta [20].

**Remark 5.** It is evident that $P_\alpha^x \in F_A$ if and only if $P_\alpha^x \notin F_A$. Additionally, if $P_\alpha^x \in F_A$, then there exists $0 < \mu < \lambda$ such that $P_\alpha^{1\mu} \in F_A$.

**Theorem 2.** Let $F_A, F_B, F_{A_\alpha} \in P(F(U, E))$ (for any $\alpha \in \Lambda$), and $P_\alpha^x \in P(F(U, E))$:

1. $F_A \subseteq F_B$ if and only if $P_\alpha^x \subseteq F_B$ for any $P_\alpha^x \in F_A$.
2. $P_\alpha^x \subseteq \bigcap_{\alpha \in \Lambda} F_{A_\alpha}$ implies $P_\alpha^x \subseteq F_{A_\alpha}$ (for any $\alpha \in \Lambda$).
3. $P_\alpha^x \subseteq \bigcup_{\alpha \in \Lambda} F_{A_\alpha}$ if and only if there exists an $\alpha_0 \in \Lambda$ such that $P_\alpha^x \subseteq F_{A_\alpha}$.

**Proof**

1. The necessity is evident. To prove the sufficiency, it is supposed that $P_\alpha^x \subseteq F_B$ for any $P_\alpha^x \in F_A$. If $P_\alpha^x \subseteq F_B$, then there exist an $e \in E$ and $x \in U$ such that $\mu_{A_\alpha}^\alpha (x) > \mu_{A_\alpha}^\alpha (x)$. For $\lambda \in (0, 1)$ with $\mu_{A_\alpha}^\alpha (x) > 1 - \lambda > \mu_{A_\alpha}^\alpha (x)$, the point $P_\alpha^x \in F_A$ and...
\( P^*_e \in F_B \), which contradicts the assumption. Thus, \( F_A \nsubseteq F_B \).

(2) If \( P_e^* \in \bar{F}_A \), then \( \lambda_{ \wedge \mu \bar{F}_A } (x) + \lambda > 1 \). Hence, \( \mu_{\bar{F}_A}(x) + \lambda > 1 \). That is, \( P_e^* \in F_{\bar{F}_A}(\lambda_{ \wedge \mu \bar{F}_A } (x) + \lambda > 1 \).

From Definition 6, it is clear that \( P_e^* \in F_{\bar{F}_B} \) if and only if \( \mu_{\bar{F}_B}(x) + \lambda > 1 \). Equivalently, \( \mu_{\bar{F}_B}(x) + \lambda > 1 \) and \( \mu_{\bar{F}_B}(x) + \lambda > 1 \). That is, \( P_e^* \in F_A \) and \( P_e^* \in F_B \).

(3) From Definition 6, it can be seen that \( P_e^* \in \bigcup_{ \lambda \in \Lambda } F_{A_{\lambda}} \) if and only if \( \bigcup_{ \lambda \in \Lambda } F_{A_{\lambda}} \subseteq \bigcup_{ \lambda \in \Lambda } F_{A_{\lambda}} \). Equivalently, there exists an \( \alpha \in \Lambda \) such that \( \mu_{A_{\alpha}}(x) + \lambda > 1 \); that is, \( P_e^* \in F_{A_{\alpha}} \).

**Definition 7.** Let \( F_A, F_B \in F(S(U, E)) \) and \( \xi \in F(S(U, E)) \). If \( \xi \notin F_A \cap F_B \), then it is said that \( F_A \) and \( F_B \) are quasi-coincident at \( \xi \).

**Remark 6.** It is clear that \( F_A \cap F_B \neq \emptyset \) if and only if \( F_A \) and \( F_B \) are quasi-coincident at a point \( \xi \in F(S(U, E)) \).

**Definition 8.** (see the work by Roy and Samanta [20]). A fuzzy soft set \( F_A \) is said to be quasi-coincident with \( F_B \), which is denoted by \( F_A \equiv F_B \), if \( \mu_{F_A}(x) + \mu_{F_B}(x) > 1 \) for some \( x \in U \) and \( e \in A \cap B \).

That \( F_A \) is not quasi-coincident with \( F_B \) is denoted by \( F_A \not\equiv F_B \).

**Theorem 3.** Let \( F_A, F_B \in F(S(U, E)) \). If \( F_A \equiv F_B \), then \( F_A \cap F_B \neq \emptyset \).

**Proof.** Suppose that \( F_A \equiv F_B \); then, there exist an \( x \in U \) and \( e \in A \cap B \) such that \( \mu_{F_A}(x) + \mu_{F_B}(x) > 1 \). Set \( \lambda = \max \{ \mu_{F_A}(x), \mu_{F_B}(x) \} \). Then, \( P_e^* \in F_A \cap F_B \). Hence, \( F_A \cap F_B \neq \emptyset \).

**Remark 7.** The converse of Theorem 3 does not hold. Indeed, in Example 1,

\[
\begin{align*}
((F_A \cap F_B)(e))(t) & = \begin{cases} 
2/5 & t = x, \\
1/3 & t = y,
\end{cases} \\
((F_A \cap F_B)(f))(t) & = \begin{cases} 
1/2 & t = x, \\
1/2 & t = y.
\end{cases}
\end{align*}
\]

Therefore, \( F_A \cap F_B \neq \emptyset \).

Because

\[
\begin{align*}
(F_A(e))(x) + (F_B(e))(x) & = \frac{3}{5} + \frac{2}{5} = 1, \\
(F_A(f))(x) + (F_B(f))(x) & = \frac{1}{2} + \frac{1}{2} = 1, \\
(F_A(e))(y) + (F_B(e))(y) & = \frac{1}{3} + \frac{2}{3} = 1, \\
(F_A(f))(y) + (F_B(f))(y) & = \frac{1}{2} + \frac{1}{2} = 1,
\end{align*}
\]

then \( F_A \neq F_B \). However, \( F_A \cap F_B \neq \emptyset \).

**Theorem 4.** Let \( F_A, F_B \in F(S(U, E)) \). If \( \lambda \in [0, 1/2] \) such that \( P_e^* \in F_A \cap F_B \), then \( F_A \equiv F_B \).

**Proof.** Because \( P_e^* \in F_A \cap F_B \), via Theorem 2(2), \( \mu_{F_A}(x) + \lambda > 1 \) and \( \mu_{F_B}(x) + \lambda > 1 \); that is, \( \mu_{F_A}(x) + \mu_{F_B}(x) > 2 - 2\lambda \). From \( \lambda \in [0, 1/2] \), \( 1 \leq 2 - 2\lambda \leq 2 \). Therefore, \( \mu_{F_A}(x) + \mu_{F_B}(x) > 1 \); that is, \( F_A \equiv F_B \).

**Definition 9.** Let \( \xi \in F(S(U, E)) \) and \( F_A, F_B \in F(S(U, E)) \):

1. \( F_A \) is said to be a neighborhood of \( \xi \) if there exists an \( F_B \in \tau \) such that \( \xi \in F_B \subseteq F_A \).
2. \( F_A \) is called a Q-neighborhood of \( \xi \) if there exists an \( F_B \in \tau \) such that \( \xi \notin F_B \subseteq F_A \).

The set of all Q-neighborhoods of \( \xi \) is denoted by \( \mathcal{Q}(\xi) \).

In the remainder of this paper, \( \Delta \) is a directed set with the partial order \( \prec \).

**Definition 8.** It is clear that \( \mathcal{Q}(\xi) \) is a directed set with the partial order \( \subseteq \).

**Definition 10.** The mapping \( S : \Delta \rightarrow F(S(U, E)) \) is called a fuzzy soft net in \( (U, E) \) and is denoted by \( \{ S(\delta), \delta \in \Delta \} \), or \( S \) for simplicity.

In particular, if there exists an \( F_A \in F(S(U, E)) \) such that \( S(\delta) \in F_A \) for any \( \delta \in \Delta \), then \( S \) is said to be a fuzzy soft net in \( F_A \), or a net for simplicity.

**Definition 11.** Let \( F_A \in F(S(U, E)) \) and \( S = \{ S(\delta), \delta \in \Delta \} \) be a net in \( (U, E) \). If there exists a \( \delta_0 \in \Delta \) such that \( S(\delta_0) \subseteq F_A \) whenever \( \delta_0 \prec \delta \), then \( S \) is said to be eventually quasi-coincident with \( F_A \). If for each \( \delta \in \Delta \) there exists a \( \delta_0 \in \Delta \) with \( \delta \prec \delta_0 \) such that \( S(\delta_0) \subseteq F_A \), then \( S \) is said to be frequently quasi-coincident with \( F_A \).

**Definition 12.** A net \( \{ S(\delta), \delta \in \Delta \} \) in \( (U, E, \tau) \) is said to be convergent to a point \( \xi \) if \( S \) is eventually quasi-coincident with each \( Q \)-neighborhood of \( \xi \). In this case, \( \xi \) is called the limit of \( S \) and is denoted by \( \lim S(\delta) \).

**4. Fuzzy Soft Continuous Mapping**

In this section, the definition of fuzzy soft continuous mapping is first recalled, and the net is then applied to characterize the continuity.

**Definition 13.** (see the work of Aygın [5]). Let \( F_A, F_B \in F(S(U_1, E_1)) \) and \( F_A, F_B \in F(S(U_2, E_2)) \) be the families of all fuzzy soft sets over \( U_1 \) and \( U_2 \), respectively. Let \( \psi : U_1 \rightarrow U_2 \) and \( \varphi : E_1 \rightarrow E_2 \) be two functions. Then, the pair \( (\psi, \varphi) \) is called a fuzzy soft mapping from \( U_1 \) to \( U_2 \) and is denoted by \( (\psi, \varphi) : F(S(U_1, E_1)) \rightarrow F(S(U_2, E_2)) \):

1. Let \( F_A \in F(S(U_1, E_1)) \). Then, the image of \( F_A \) under the fuzzy soft mapping \( (\psi, \varphi) \) is the fuzzy soft set over \( U_2 \) defined by \( (\psi, \varphi)(F_A) \), where \( (\psi, \varphi)(F_A)(k)(y) \)
Lemma 2. \( P_x \) following is then obtained:
\[
\forall k \in \mathcal{E}(E_1), \forall y \in U_2, \quad \mathcal{F}_A(e) = \mathcal{F}_B(e) = \mathcal{F}_A(e) \big( \mathcal{F}_B(e) \big) = \mathcal{F}_A(e) \big( \mathcal{F}_B(e) \big) = \mathcal{F}_A(e) \big( \mathcal{F}_B(e) \big)
\]
\[\forall e \in \mathcal{V}^{-1}(E_2) \forall x \in U_1, \tag{9}\]

If both \( \phi \) and \( \psi \) are injective (surjective), then the fuzzy soft mapping \( \mathcal{F}_A(e) \) is said to be injective (surjective).

The composition of two fuzzy soft mappings \( \mathcal{F}_A(e) \) from \( (U_1, E_1) \) to \( (U_2, E_2) \) and \( \mathcal{F}_B(e) \) from \( (U_2, E_2) \) to \( (U_3, E_3) \) is defined as \( (\mathcal{F}_A \circ \mathcal{F}_B)(e) \) from \( (U_1, E_1) \) to \( (U_3, E_3) \).

Lemma 1 (see the work of Kharal and Ahmad [8]). Let \( \mathcal{F}_A(e) \), \( \mathcal{F}_B(e) \) be fuzzy soft mappings. \( P_x \) follows from \( \mathcal{F}_A(e) \). From Lemma 2, it can be found that \( \mathcal{F}_A(e) \) is fuzzy soft continuous. In fact, for a fuzzy soft set \( \mathcal{F}_A \) on \( U_1, \)
\[
(\phi, \psi)^{-1}(\mathcal{F}_A(e)) = (\phi, \psi)^{-1}(\mathcal{F}_A(e)) = (\phi, \psi)^{-1}(\mathcal{F}_A(e))
\]
\[
(\phi, \psi)^{-1}(\mathcal{F}_A(e)) = (\phi, \psi)^{-1}(\mathcal{F}_A(e)), \tag{10}\]

which implies that \( (\phi, \psi)((\phi, \psi)(e)) \) is fuzzy soft continuous. In fact, for a fuzzy soft set \( \mathcal{F}_A \) on \( U_1, \)
\[
((\phi, \psi)^{-1}(\mathcal{F}_A(e)) = ((\phi, \psi)^{-1}(\mathcal{F}_A(e)) = ((\phi, \psi)^{-1}(\mathcal{F}_A(e))
\]
\[
((\phi, \psi)^{-1}(\mathcal{F}_A(e)) = ((\phi, \psi)^{-1}(\mathcal{F}_A(e)), \tag{11}\]

Hence,
\[
(\phi, \psi)^{-1}(\mathcal{F}_A(e)) = (\phi, \psi)^{-1}(\mathcal{F}_A(e)), \tag{12}\]

Theorem 6. Let \( (U_1, E_1, \tau_1) \) and \( (U_2, E_2, \tau_2) \) be two fuzzy soft topological spaces and \( (\phi, \psi): (U_1, E_1, \tau_1) \rightarrow (U_2, E_2, \tau_2) \) be a fuzzy soft mapping. The following are then equivalent:
\[
(1) \quad (\phi, \psi): (U_1, E_1, \tau_1) \rightarrow (U_2, E_2, \tau_2) \text{ is fuzzy soft continuous}
\]
\[
(2) \quad \text{For each } \xi \in \mathcal{F}_A(e) \text{ and each neighborhood } \mathcal{F}_B(e) \text{ of } (\phi, \psi)(\xi), \text{ there exists a neighborhood } \mathcal{F}_A(e) \text{ such that } (\phi, \psi)(\xi) \subseteq \mathcal{F}_B(e)
\]
\[
(3) \quad \text{For each } P_x \in \mathcal{F}_A(e) \text{ and each Q-neighborhood } \mathcal{F}_B(e) \text{ of } (\phi, \psi)(P_x) \text{ such that } (\phi, \psi)(P_x) \subseteq \mathcal{F}_B(e)
\]
\[
(4) \quad \text{For each net } S = \{S(\delta), \delta \in \Delta\} \text{ in } (U_1, E_1), \text{ if } S \text{ converges to } P_x, \text{ then } \mathcal{F}_A(e) \text{ converges to } (\phi, \psi)(P_x)
\]

Proof. (1) \(\Rightarrow\) (2): Suppose that \( \xi \in \mathcal{F}_A(e) \) and \( \mathcal{F}_B(e) \) is a neighborhood of \( (\phi, \psi)(\xi) \). Then, there exists an \( F_C \subseteq \tau_1 \) such that \( (\phi, \psi)(\xi) \subseteq F_C \). Because \( (\phi, \psi) \) is fuzzy soft continuous, \( \mathcal{F}_A(e) = (\phi, \psi)^{-1}(F_C) \subseteq \tau_1 \). Then, \( \mathcal{F}_A(e) \) is a neighborhood of \( \xi \) and \( (\phi, \psi)(\mathcal{F}_A(e)) \subseteq \mathcal{F}_B(e) \) follows from Lemma 1(4).

(2) \(\Rightarrow\) (3): Suppose that \( P_x \in \mathcal{F}_A(e) \) and \( \mathcal{F}_B(e) \) is a Q-neighborhood of \( (\phi, \psi)(P_x) \). Then, there is an open fuzzy soft set \( \mathcal{F}_B(e) \subseteq \mathcal{U}(\mathcal{F}_A(e)) \) such that \( \mathcal{F}_B(e) \subseteq \mathcal{F}_B(e) \).

From Remark 5 and Lemma 2(1), it can be determined...
that there exists $0<\mu<\lambda$ such that $(\varphi, \psi)(P_{\mu, \lambda}^{x}) \in F_{\mu}$. Thus, $F_{\mu}$ is a neighborhood of $(\varphi, \psi)(P_{\mu, \lambda}^{x})$. Under condition (2), there exists an open neighborhood $F_{A}$ of $P_{\mu, \lambda}^{x}$ such that $(\varphi, \psi)(F_{A}) \subseteq F_{\mu}$. Because $0<\mu<\lambda$, $\mu_{F_{A}}(x) \geq 1 - \mu > 1 - \lambda$. Therefore, $F_{A}$ is a $\delta$-neighborhood of $P_{\mu, \lambda}^{x}$ and $(\varphi, \psi)(F_{A}) \subseteq F_{\mu}$.

(3)$\Rightarrow$(4): Suppose a net $S = \{S(\delta), \delta \in \Delta\}$ in $(U, E)$ converges to $P_{\mu, \lambda}^{x}$ in $F_{\mu}$. For any $F_{\mu} \in \mathcal{U}$, under condition (3), there exists an $F_{A} \in \mathcal{U}(P_{\mu, \lambda}^{x})$ such that $(\varphi, \psi)(F_{A}) \subseteq F_{\mu}$. From the supposition, there is a $\delta_{0} \in \Delta$ such that $S(\delta) \in F_{A}$ whenever $\delta < \delta_{0}$. Lemma 2 implies that $(\varphi, \psi)(S(\delta)) \subseteq F_{\mu}$. Therefore, $(\varphi, \psi)S = ((\varphi, \psi)(S(\delta)), \delta \in \Delta)$ converges to $(\varphi, \psi)(P_{\mu, \lambda}^{x})$.

(4)$\Rightarrow$(1): This is proven by contradiction. If $(\varphi, \psi)$ is not fuzzy soft continuous, then there is an $F_{B} \in \tau_{r}$ such that $(\varphi, \psi)^{-1}(F_{B})$ is not open. From Remark 3(3), it can be found that there exists a $P_{\mu, \lambda}^{x} \in \mathcal{U}(U_{\mu}, E_{\mu})$ such that $P_{\mu, \lambda}^{x} \notin (\varphi, \psi)^{-1}(F_{B})$. For any $F_{\mu} \in \mathcal{U}(P_{\mu, \lambda}^{x})$, by Definition 4(2), there is an $F_{A} \in \mathcal{U}(P_{\mu, \lambda}^{x})$ for any $F_{A} \in \mathcal{U}(P_{\mu, \lambda}^{x})$. Moreover, $F_{B}$ converges to $P_{\mu, \lambda}^{x}$. Under condition (4), the net $(\varphi, \psi)S$ converges to $(\varphi, \psi)(P_{\mu, \lambda}^{x})$. Recall that $P_{\mu, \lambda}^{x} \in (\varphi, \psi)^{-1}(F_{B})$, and $(\varphi, \psi)(P_{\mu, \lambda}^{x}) \in F_{\mu}$. It is easy to verify that $S$ converges to $P_{\mu, \lambda}^{x}$. Therefore, $(\varphi, \psi)$ is fuzzy soft continuous.

5. Convergence for a Net of Fuzzy Soft Mappings

Let $(U_{1}, E_{1}, \tau_{r})$ and $(U_{2}, E_{2}, \tau_{r})$ be two fuzzy soft topological spaces. $FSM(U_{1}, U_{2})$ denotes the set of all mappings of $(U_{1}, E_{1}, \tau_{r})$ into $(U_{2}, E_{2}, \tau_{r})$.

Definition 15. A net $\{(\varphi, \psi)_{\mu}, \mu \in M\}$ in $FSM(U_{1}, U_{2})$ is said to be convergent to $(\varphi, \psi) \in FSM(U_{1}, U_{2})$ if, for every point $\xi \in FSM(U_{1}, E_{1})$ and for every $F_{\mu} \in \mathcal{U}(\varphi, \psi)(\xi)$, there exists a $\mu_{0} \in M$ and $F_{A} \in \mathcal{V}(\xi)$ such that $(\varphi, \psi)_{\mu_{0}}(F_{A}) \subseteq F_{\mu}$ for every $\mu \in M$ with $\mu_{0} \leq \mu$.

Theorem 7. A net $\{(\varphi, \psi)_{\mu}, \mu \in M\}$ in $FSM(U_{1}, U_{2})$ converges to $(\varphi, \psi) \in FSM(U_{1}, U_{2})$ if and only if, for every net $\{S(\delta), \delta \in \Delta\}$ in $(U_{1}, E_{1}, \tau_{r})$ converging to a point $\xi \in FSM(U_{1}, E_{1})$, the net $\{(\varphi, \psi)_{\mu}(S(\delta)), (\delta, \mu) \in \Delta \times M\}$ converges to the point $(\varphi, \psi)(\xi) \in FSM(U_{2}, E_{2})$, where $\Delta \times M$ is the product of $\Delta$ and $M$.

Proof. (Necessity) Let $\{S(\delta), \delta \in \Delta\}$ be a net in $FSM(U_{1}, U_{2})$ that converges to $\xi \in FSM(U_{1}, E_{1})$ and let $F_{\mu} \in \mathcal{U}(\varphi, \psi)(\xi)$ arbitrarily. By assumption, there exist an $F_{A} \in \mathcal{U}(\xi)$ and $\mu_{0} \in M$ such that $(\varphi, \psi)_{\mu}(F_{A}) \subseteq F_{\mu}$ for every $\mu \in M$ with $\mu_{0} \leq \mu$. Because $S(\delta), \delta \in \Delta$ converges to $\xi$, there is a $\delta_{0} \in \Delta$ such that $S(\delta) \in F_{A}$ for every $\delta \in \Delta$ with $\delta_{0} < \delta$. Let $\{F_{\mu}, (\delta, \mu) \in \Delta \times M\}$ be a net in $FSM(U_{1}, U_{2})$, and let $\mu, \mu_{0} \in M$.

6. Application of Fuzzy Soft Set Theory to Medical Diagnosis

In a hospital, some doctors usually decide what disease a patient is suffering from by observing the patient’s symptoms. However, due to the complexity of symptoms, it is difficult to find the precise relationship between diseases and symptoms. The concept of fuzzy soft sets partially resolves this difficulty.

Suppose that the initial universe $U = \{x_{1}, x_{2}, \ldots, x_{n}\}$ is the set of all the disease objects that the patient may be infected with, and the set of parameters $E = \{e_{1}, e_{2}, \ldots, e_{n}\}$ is all of the patient’s symptoms. Generally speaking, from a symptom $e \in E$, one cannot completely determine the corresponding disease $x \in U$; however, one can determine the membership degree in which object $x \in U$ holds parameter $e \in E$, which is denoted by $\mu_{e}(x)$; that is, for every $e \in E$, there is a fuzzy subset $F(e)$ of $U$. Obviously, the mapping $F: E \rightarrow 2^{U}$ is a fuzzy soft set over $(U, E)$.

Let

$$\mu(x_{i}) = \sum_{k=1}^{n} \mu_{ik}(x_{i}).$$

(15)

If $\mu(x_{i}) = \max_{1 \leq i \leq n} \mu_{x_{i}}$, then it may be claimed that the patient has disease $x_{i}$.

7. Conclusions

In this paper, the new concepts of fuzzy soft points and fuzzy soft nets were introduced to fuzzy soft topological spaces. On these bases, the fuzzy soft net was used to accurately describe the convergence, which was used to characterize the continuity. Moreover, the convergence for a net of fuzzy soft mappings was investigated. The obtained results demonstrate that the concepts proposed in this paper are very useful and will provide powerful research tools for further research in this field. Particularly, the convergence of fuzzy soft nets may be used to characterize some important properties of fuzzy soft topological spaces, such as closure, separation, compactness, etc.
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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