Potential Algebra Approach to Quantum Mechanics with Generalized Uncertainty Principle

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Abstract

In this note we study the potential algebra for several models arising out of quantum mechanics with generalized uncertainty principle. We first show that the eigenvalue equation corresponding to the momentum-space Hamiltonian

\[ H = -(1 + \beta p^2) \frac{d}{dp}(1 + \beta p^2) \frac{d}{dp} + g(1 + \beta p^2) \beta^2 p^2 - g \beta, \]

which is associated with some one-dimensional models with minimal length uncertainty, can be solved by the unitary representations of the Lie algebra \( su(2) \) if \( g \in \{ \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots \} \). We then apply this result to spectral problems for the non-relativistic harmonic oscillator as well as the relativistic Dirac oscillator in the presence of a minimal length and show that these problems can be solved solely in terms of \( su(2) \).
1. Introduction

Study of symmetry structure of quantum mechanical models is interesting as well as useful as one may determine various observables using symmetry properties [1]. In this respect, potential algebra is a powerful tool to obtain the spectrum and scattering amplitude of quantum mechanical models in a purely algebraic fashion [2]. On the other hand, in supersymmetric quantum mechanics, which is closely related to the factorization method [3], the concept of shape invariance [4] plays a very important role in obtaining solutions without solving differential equations: shape-invariant potentials allow complete determination of the spectrum and eigenfunctions in a purely algebraic fashion. In fact, it has been discussed that these two approaches are essentially equivalent [5] (see also [6]).

The purpose of this note is to examine energy spectrum of some quantum mechanical models arising out of the minimal length uncertainty formalism [7] from the viewpoint of potential algebra. To be more specific, in this note we shall focus on the following two-parameter family of momentum-space Hamiltonian:

\[ H = -(1 + \beta p^2) \frac{d}{dp}(1 + \beta p^2) \frac{d}{dp} + g(g-1)\beta^2 p^2 - g \beta, \]  

(1.1)

where \( \beta \) and \( g \) are parameters of the model. As discussed in [8], this Hamiltonian is known to be shape invariant. We shall show that the potential algebra associated with (1.1) is nothing but the Lie algebra \( \mathfrak{su}(2) \) and determine the spectrum only through the unitary representations of \( \mathfrak{su}(2) \).

Subsequently, we shall consider a number of minimal length uncertainty models and identify them with the Hamiltonian (1.1) by choosing the parameter \( g \) suitably. As illustrative examples, we shall focus on the non-relativistic harmonic oscillator and the relativistic Dirac oscillator subject to the minimal length uncertainty principle. The Hamiltonians of these models are respectively given by

\[ H = \frac{1}{2m} p^2 + \frac{1}{2} mc^2 x^2, \]  

(1.2a)

\[ H = c \sigma_y (p - i \sigma_z mc), \]  

(1.2b)

whose energy eigenvalues are known to be of the following forms [7, 9]:

\[ E_n = h \omega \left( n + \frac{1}{2} \right) \left[ \frac{m h \omega \beta}{2} + \frac{1}{2} \left( \frac{m h \omega \beta}{2} \right)^2 \right] + \frac{1}{2} m h^2 \omega^2 \beta n^2, \]  

(1.3a)

\[ E_n = \pm mc^2 \sqrt{1 + \frac{h^2 \omega^2 \beta}{c^2} n^2 + \frac{2 h \omega}{mc^2} n}, \]  

(1.3b)

where \( n \in \{0, 1, 2, \ldots \} \) and \( h \sqrt{\beta} \) is the minimal length. The goal of this note is to show that the energy spectrum (1.3a) and (1.3b), though look quite different, can be completely determined through the unitary representations of the Lie algebra \( \mathfrak{su}(2) \). Before going into this, however, let us first briefly recall the basics of the minimal length uncertainty principle and the shape invariance of the Hamiltonian (1.1).

2. Minimal length uncertainty principle and shape invariance

To begin with, let us first recall that position \( (x) \) and momentum \( (p) \) of particles whose length cannot be measured below a minimum value satisfy the following commutation relation [7]:

\[ [x, p] = i h(1 + \beta p^2), \]  

(2.1)

where \( h \sqrt{\beta} (\beta > 0) \) gives the minimal length. The corresponding uncertainty relation reads

\[ \Delta x \Delta p \geq \frac{\hbar}{2} [1 + \beta (\Delta p)^2 + \beta \langle p^2 \rangle]. \]  

(2.2)
A realization of the operators \( x \) and \( p \) which satisfy the commutation relation (2.1) can be taken as

\[
x = i\hbar(1 + \beta p^2) \frac{d}{dp} \quad \text{and} \quad p = p.
\] (2.3)

One of the most important features of this class of models is that the inner product should be given by

\[
\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} \psi^*(p) \phi(p),
\] (2.4)

under which the operators \( x \) and \( p \) in (2.3) become (formally) hermitian.

Now, following ref. [8] let us consider the following first-order differential operators in momentum space:

\[
A(g) = +(1 + \beta p^2) \frac{d}{dp} + g \beta p = +(1 + \beta p^2)^{1-%}\frac{d}{dp}(1 + \beta p^2)^{%}, (2.5a)
\]

\[
\tilde{A}(g) = -(1 + \beta p^2) \frac{d}{dp} + g \beta p = -(1 + \beta p^2)^{1-%}\frac{d}{dp}(1 + \beta p^2)^{%}, (2.5b)
\]

where \( g \) is a dimensionless real parameter. Notice that these operators are (formally) hermitian conjugate with each other with respect to the inner product (2.4). Let us next introduce the following factorized Hamiltonians in momentum space:

\[
H(g) = \tilde{A}(g)A(g) = -(1 + \beta p^2) \frac{d}{dp}(1 + \beta p^2) \frac{d}{dp} + g(g - 1)\beta^2 p^2 - g \beta, (2.6a)
\]

\[
\tilde{H}(g) = A(g)\tilde{A}(g) = -(1 + \beta p^2) \frac{d}{dp}(1 + \beta p^2) \frac{d}{dp} + g(g + 1)\beta^2 p^2 + g \beta, (2.6b)
\]

both of which are (formally) hermitian with respect to the inner product (2.4). It should be noted that these Hamiltonians satisfy the following identity (translational shape invariance):

\[
\tilde{H}(g) = H(g + 1) + [2(g + 1) - 1]\beta. (2.7)
\]

Now we wish to solve the following eigenvalue equation for the Hamiltonian \( H(g) \):

\[
H(g)\psi(p) = E\psi(p). (2.8)
\]

By using the relation (2.7) the energy eigenvalues are readily found to be of the form [8]

\[
E_n = \sum_{k=1}^{n} [2(g + n) - 1]\beta = (n^2 + 2ng)\beta, \quad n \in \{0, 1, 2, \ldots\}. (2.9)
\]

In what follows we shall show that the eigenvalue problem (2.8) can be solved solely in terms of the Lie algebra \( su(2) \).

3. Potential algebra

The purpose of this note is to understand the (Lie-)algebraic structure behind the spectral problem for the Hamiltonian (1.1). To this end, we would like to translate the shape invariance (2.7) into the language of potential algebra. To the best of our knowledge, there have been proposed two seemingly different approaches to the (Lie-)algebraic description of shape invariance. The first is due to Balantekin [6], and the second is due to Gangopadhyaya et al. [5]. As briefly discussed in appendix A, however, these two approaches are essentially equivalent and give the same result in

\footnote{Eqs. (2.5a) and (2.5b) correspond to the choice \( a = 1, b = \beta, \) and \( c = g\beta \) in [8].}
Indeed, by redefining the operators as one immediately finds that \((3.1a)-(3.1c)\) and introduce the potential algebra as follows. We first introduce an auxiliary periodic variable \(\theta \in [0, 2\pi]\) and consider the following first-order differential operators:

\[
\begin{align*}
J_z &= -i \partial_\theta, \quad (3.1a) \\
J_+ &= e^{i\theta} A(J_z) = e^{i\theta} \left[ + (1 + \beta p^2) \partial_p + \beta p J_z \right], \quad (3.1b) \\
J_- &= \bar{A}(J_z) e^{-i\theta} = \left[ - (1 + \beta p^2) \partial_p + \beta p J_z \right] e^{-i\theta}. \quad (3.1c)
\end{align*}
\]

Note that \(J_z\) is (formally) hermitian and \(J_\pm\) are (formally) hermitian conjugate with each other with respect to the inner product

\[
\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} \int_0^{2\pi} d\theta \, \psi^*(p, \theta) \phi(p, \theta). \quad (3.2)
\]

Let us next study the commutation relations of the operators \((3.1a)-(3.1c)\). By using the identities \(J_z = e^{i\theta} A(J_z) = A(J_z - 1) e^{i\theta}\) and \(J_- = \bar{A}(J_z) e^{-i\theta} = e^{-i\theta} \bar{A}(J_z - 1)\), which follow from the relation \(e^{i\theta} J_z e^{-i\theta} = J_z - 1\), we get

\[
J_+ J_- - A(J_z - 1) \bar{A}(J_z - 1) = \bar{H}(J_z - 1) = H(J_z) + [2(J_z - 1) + 1] \beta \\
= \bar{A}(J_z) A(J_z) + 2 \beta \left( J_z - \frac{1}{2} \right) = J_+ J_- + 2 \beta \left( J_z - \frac{1}{2} \right), \quad (3.3)
\]

where we have used the relations \(H(J_z) = \bar{A}(J_z) A(J_z)\) and \(\bar{H}(J_z) = A(J_z) \bar{A}(J_z)\). The third equality follows from the shape invariance condition \((2.7)\). Similarly, a straightforward calculation gives

\[
J_z J_\pm = J_\pm J_z \pm J_\pm. \quad (3.4)
\]

We thus find the following commutation relations:

\[
\begin{align*}
[J_+, J_-] &= 2 \beta \left( J_z - \frac{1}{2} \right), \quad (3.5a) \\
[J_z, J_\pm] &= \pm J_\pm. \quad (3.5b)
\end{align*}
\]

Note that these commutation relations are essentially equivalent to those of the Lie algebra \(su(2)\). Indeed, by redefining the operators as

\[
\begin{align*}
\tilde{J}_z &= J_z - \frac{1}{2}, \quad (3.6a) \\
\tilde{J}_\pm &= \frac{1}{\sqrt{\beta}} J_\pm, \quad (3.6b)
\end{align*}
\]

one immediately finds that \((3.6a)\) and \((3.6b)\) reduce to the standard commutation relations of the Lie algebra \(su(2)\):

\[
\begin{align*}
[\tilde{J}_+, \tilde{J}_-] &= 2 \tilde{J}_z, \quad (3.7a) \\
[\tilde{J}_z, \tilde{J}_\pm] &= \pm \tilde{J}_\pm. \quad (3.7b)
\end{align*}
\]

The quadratic Casimir operator is therefore given by

\[
\begin{align*}
C &= \tilde{J}_- \tilde{J}_+ + \tilde{J}_z (\tilde{J}_z + 1) = \frac{1}{\beta} J_z J_+ + J_z^2 - \frac{1}{4} = \frac{1}{\beta} H(J_z) + J_z^2 - \frac{1}{4} \\
&= \tilde{J}_+ \tilde{J}_- + \tilde{J}_z (\tilde{J}_z - 1) = \frac{1}{\beta} J_z J_- + (J_z - 1)^2 - \frac{1}{4} = \frac{1}{\beta} \bar{H}(J_z) + (J_z - 1)^2 - \frac{1}{4}, \quad (3.8a)
\end{align*}
\]

which commutes with all the generators.
Now, let \(|j, g\rangle\) be a simultaneous eigenstate of \(C\) and \(J_z\) that satisfies the eigenvalue equations
\[
C|j, g\rangle = \left(j^2 - \frac{1}{4}\right)|j, g\rangle, \tag{3.9a}
\]
\[
J_z|j, g\rangle = g|j, g\rangle, \tag{3.9b}
\]
where \(j \geq \frac{1}{2}\). We assume that the eigenstate \(|j, g\rangle\) satisfies the normalization condition \(|||j, g\rangle|| = 1\), where the norm \(|||\cdot\rangle|| = \sqrt{\langle \cdot | \cdot \rangle}\) is defined through the inner product (3.2). Note that the commutation relations (3.5b) imply the following ladder equations:
\[
J_\pm|j, g\rangle \propto |j, g \pm 1\rangle. \tag{3.10}
\]
The coefficients of proportionality can be determined by computing the norms \(||J_\pm|j, g\rangle||\). By using the relations \(J_-J_+ = \beta(C - J_z^2 + \frac{1}{4})\) and \(J_+J_- = \beta(C - (J_z - 1)^2 + \frac{1}{4})\) we find
\[
||J_+|j, g\rangle||^2 = \beta(j^2 - g^2) \geq 0, \tag{3.11a}
\]
\[
||J_-|j, g\rangle||^2 = \beta(j^2 - (g - 1)^2) \geq 0, \tag{3.11b}
\]
where the inequalities follow from the positivity of the norms. These equations not only fix the coefficients of proportionality but also determine the possible values of \(j\) and \(g\). In fact, it is easy to see that the constraints \(j^2 - g^2 \geq 0\) and \(j^2 - (g - 1)^2 \geq 0\) together with the ladder equations (3.10) are compatible with each other if and only if \(j\) is quantized as follows:
\[
j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}. \tag{3.12}
\]
Once such \(j\) is given the eigenvalue \(g\) takes the following values:
\[
g \in \{j, j - 1, j - 2, \cdots, 1 - j\}, \tag{3.13}
\]
and the normalized eigenstate \(|j, g\rangle\) is found to be of the form
\[
|j, g\rangle = \sqrt{\frac{\Gamma(j + g)}{\beta^{j-g}\Gamma(2j)\Gamma(j - g + 1)}}(J_-)^{j-g}|j, j\rangle, \tag{3.14}
\]
where \(|j, j\rangle\) is the highest weight state that satisfies the condition \(J_+|j, j\rangle = 0\).

Now it is easy to solve the original spectral problem. To see this, let us first note that, thanks to the relation \(H(g) = \beta(C - J_z^2 + \frac{1}{4})\), the eigenvalue equation (3.9a) can be recast into the following form:
\[
H(g)|j, g\rangle = \beta(j^2 - g^2)|j, g\rangle. \tag{3.15}
\]
Now let \(g \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}\) be fixed. Then the ground state of the Hamiltonian \(H(g)\) corresponds to the state \(|g, g\rangle\) and the \(n\)th excited state of \(H(g)\) corresponds to the state \(|g + n, g\rangle\); see figure 1. The energy eigenvalue \(E_n\) of the \(n\)th excited state can therefore be read by just substituting \(j = g + n\) in (3.15). Thus we find
\[
E_n = \beta [(g + n)^2 - g^2] = (n^2 + 2ng)\beta, \quad n \in \{0, 1, 2, \cdots\}, \tag{3.16}
\]
which exactly coincides with (2.9).

The normalized energy eigenfunction can also be obtained from the representation theory. To see this, let us first substitute \(j = g + n\) in (3.14):
\[
|g + n, g\rangle = \sqrt{\frac{\Gamma(2g + n)}{\beta^n n!\Gamma(2g + 2n)}}(J_-)^n|g + n, g + n\rangle. \tag{3.17}
\]
Let $\Psi_{g+n,g}(p, \theta)$ be a wavefunction corresponding to the state $|g+n,g\rangle$. Then, it follows from (3.1a) and (3.9b) that the $\theta$-dependence of $\Psi_{g+n,g}(p, \theta)$ is just the plane wave $e^{ig\theta}$. Thus one may write $\Psi_{g+n,g}(p, \theta) = \psi_{g+n,g}(p) e^{ig\theta}$, where $\psi_{g+n,g}(p)$ gives the normalized energy eigenfunction of the eigenvalue $E_n$ of the Hamiltonian $H(g)$. Noting that $\Psi_{g+n,g}(p, \theta) \propto (J_-)^n \Psi_{g+n,g+n}(p, \theta) = \tilde{A}(J_x)e^{-i\theta}\tilde{A}(J_y)e^{-i\theta}\cdots\tilde{A}(J_n)e^{-i\theta} \psi_{g+n,g+n}(p)$, we find the following normalized energy eigenfunction:

$$\psi_{g+n,g+n}(p) = (-1)^n \sqrt{\frac{\Gamma(2g+n)}{\beta^n n! \Gamma(2g+2n)}} (1 + \beta p^2)^{-\frac{g+n}{2}} (1 + \beta p^2)^\frac{d}{dp} \left[ (1 + \beta p^2)^{-\frac{g}{2}} \psi_{g+n,g+n}(p) \right],$$

(3.18)

where in the second equality we have used (2.5b). Here $\psi_{g+n,g+n}$ is the normalized solution to the first-order differential equation $A(g+n)p\psi_{g+n,g+n}(p) = 0$, which turns out to be given by

$$\psi_{g+n,g+n}(p) = \left( \frac{\beta}{\pi} \right)^{-\frac{g+n}{4}} \sqrt{\frac{\Gamma(g+n+2)}{\Gamma(g+n+1)}} (1 + \beta p^2)^{-\frac{g+n}{2}}.$$

(3.19)

To summarize, we have solved the spectral problem only through the unitary representations of the Lie algebra $su(2)$. The defect of this approach, however, is that the representation theory works only for $g \in \{ \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots \}$, though the original eigenvalue equation (2.8) can be solved for any real $g$.

4. Examples

**Example #1.** Having obtained the spectrum of the two-parameter family of momentum-space Hamiltonian (1.1), we now proceed to examine specific cases. The first example is that of the one-dimensional harmonic oscillator within the minimal length formalism. The Hamiltonian of this system is given by

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2.$$

(4.1)

---

The eigenfunction (3.18) can be written in terms of the Gegenbauer polynomial. Indeed, by introducing a new dimensionless variable $\xi = \frac{\sqrt{2}p}{m\omega}$ we find $\psi_{g+n,g} \propto (1 - \xi^2)^{-\frac{g+n}{2}} \frac{d^{g+n}}{d\xi^{g+n}} \propto (1 - \xi^2)^\frac{g}{2} C_n^g(\xi)$, where $C_n^g$ stands for the Gegenbauer polynomial of degree $n$. 

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\[ \text{Figure 1: Unitary representations of the potential algebra } su(2) \text{ and the energy spectrum. White circles represent the states } (|j, g\rangle). \]
It follows from (2.3) that this Hamiltonian can be expressed in momentum space as

\[ H = \frac{\hbar^2 \omega^2}{2} \left[ -(1 + \beta p^2) \frac{d}{dp} (1 + \beta p^2) \frac{d}{dp} + \frac{1}{m^2 \hbar^2 \omega^2 p^2} \right]. \tag{4.2} \]

Now comparing (4.2) with (2.6a) we find

\[ g = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{m^2 \hbar^2 \omega^2 \beta^2}}, \tag{4.3} \]

where we have assumed \( g \) is positive. The energy eigenvalue then reads

\[ E_n = \frac{\hbar^2 \omega^2}{2} \left[ (n^2 + 2ng) \beta + g \beta \right] = \hbar \omega \left( n + \frac{1}{2} \right) \left[ \frac{m \hbar \omega \beta}{2} + \sqrt{1 + \left( \frac{m \hbar \omega \beta}{2} \right)^2} \right] + \frac{1}{2} m \hbar^2 \omega^2 \beta n^2, \quad n \in \{0, 1, 2, \cdots\}, \tag{4.4} \]

which exactly coincides with the known result [7]. The normalized energy eigenfunctions can be given by (3.18) and (3.19) under the substitution (4.3).

**Example #2.** Let us next consider a problem of relativistic quantum mechanics, namely that of the minimum length Dirac oscillator [9]. The eigenvalue equation for the Dirac oscillator is given by [9]

\[ H \psi \equiv \left[ c \sigma_y (p - i \sigma_z m \omega x) + \sigma_z m c^2 \right] \psi = E \psi, \tag{4.5} \]

where \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), and \( x \) and \( p \) are given by (2.3). Writing \( \psi = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \), one finds that the eigenvalue equation for the upper component \( f_1 \) can be written as

\[ \left[ -(1 + \beta p^2) \frac{d}{dp} (1 + \beta p^2) \frac{d}{dp} + \frac{1 - m \hbar \omega \beta}{m^2 \hbar^2 \omega^2} p^2 \right] f_1 = \frac{1}{m^2 \hbar^2 \omega^2} \left( \frac{E^2 - m^2 c^4}{c^2} + m \hbar \omega \right) f_1. \tag{4.6} \]

Now again comparing (4.6) with (2.6a) we find

\[ g = \frac{1}{m \hbar \omega \beta}, \tag{4.7} \]

where we have again assumed \( g \) is positive. Setting \( \frac{1}{m \hbar^2 \omega^2} \left( \frac{E^2 - m^2 c^4}{c^2} + m \hbar \omega \right) = (n^2 + 2gn) \beta + g \beta \) and solving this with respect to \( E_n \) we get

\[ E_n = \pm mc^2 \sqrt{1 + \frac{\hbar^2 \omega^2 \beta}{c^2} n^2 + \frac{2 \hbar \omega}{mc^2} n}, \quad n \in \{0, 1, 2, \cdots\}, \tag{4.8} \]

which is in perfect agreement with the known result [9].

It should be noted that we could have also considered the eigenvalue equation for the lower component \( f_2 \), in which case the Hamiltonian should be identified with (2.6b). Clearly the potential algebra is the same as the Lie algebra \( su(2) \) once the identification of the parameter is made. Note also that the energy eigenfunctions are basically the same form as (3.18) and (3.19) but the normalization constant should be different, because in the Dirac oscillator the normalization condition should be \( \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} (|f_1|^2 + |f_2|^2) = 1 \).

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A. Two realizations of the potential algebra

There have been proposed two approaches to the algebraic description of shape invariance [5, 6]. However, these two approaches are essentially equivalent. Focusing on our specific example, in this appendix we shall discuss the equivalence between the algebraic descriptions of shape invariance proposed by Balantekin [6] and by Gangopadhyaya et al. [5]. To this end, let us first introduce two new operators \( \hat{\theta} \) and \( \hat{n} \) which are canonical conjugate with each other and satisfy the following commutation relations:

\[
[\hat{\theta}, \hat{n}] = i. \tag{A.1}
\]

We then introduce the following operators:

\[
J_z = \hat{n}, \tag{A.2a}
\]

\[
J_+ = e^{i\hat{\theta}}A(\hat{n}) = A(\hat{n}-1)e^{i\hat{\theta}}, \tag{A.2b}
\]

\[
J_- = \bar{A}(\hat{n})e^{-i\hat{\theta}} = e^{-i\hat{\theta}}\bar{A}(\hat{n}-1), \tag{A.2c}
\]

where \( A \) and \( \bar{A} \) are given in (2.5a) and (2.5b). We note that the second equalities in (A.2b) and (A.2c) follow from the identities \( e^{\pm i\hat{\theta}}\hat{n}e^{\mp i\hat{\theta}} = \hat{n} \mp 1 \). A straightforward calculation then gives

\[
J_-J_+ = \bar{A}(\hat{n})A(\hat{n}) = -(1 + \beta p^2)\frac{d}{dp}(1 + \beta p^2)\frac{d}{dp} + \hat{n}(\hat{n}-1)\beta^2 p^2 - \hat{n}\beta, \tag{A.3a}
\]

\[
J_+J_- = A(\hat{n}-1)\bar{A}(\hat{n}-1) = -(1 + \beta p^2)\frac{d}{dp}(1 + \beta p^2)\frac{d}{dp} + \hat{n}(\hat{n}-1)\beta^2 p^2 + (\hat{n}-1)\beta, \tag{A.3b}
\]

from which we find

\[
[J_+, J_-] = (\hat{n}-1)\beta + \hat{n}\beta = 2\beta \left( \hat{n} - \frac{1}{2} \right) = 2\beta \left( J_z - \frac{1}{2} \right). \tag{A.4}
\]

Similarly, we have

\[
J_+J_z = \hat{n}e^{i\hat{\theta}}A(\hat{n}) = e^{i\hat{\theta}}A(\hat{n})(\hat{n} + 1), \tag{A.5a}
\]

\[
J_+J_- = e^{i\hat{\theta}}A(\hat{n})\hat{n}, \tag{A.5b}
\]

\[
J_+J_- = \hat{n}\bar{A}(\hat{n})e^{-i\hat{\theta}} = \bar{A}(\hat{n})e^{-i\hat{\theta}}(\hat{n} - 1), \tag{A.5c}
\]

\[
J_+J_+ = \bar{A}(\hat{n})e^{-i\hat{\theta}}. \tag{A.5d}
\]

Thus we get the following commutation relations:

\[
[J_z, J_{\pm}] = \pm J_{\pm}. \tag{A.6}
\]

By redefining the operators as \( J_+ = J_z - \frac{1}{2} \) and \( J_{\pm} = \frac{1}{\sqrt{\beta}}J_{\pm} \), one can easily find that the set of operators \( \{J_z, J_+, J_-\} \) satisfies the standard commutation relations of the Lie algebra \( su(2) \).

Now let us specialize to the following two realizations of the operators \( \hat{n} \) and \( \hat{\theta} \):

\[
\text{(Case A)} \quad \hat{\theta} = i\frac{\partial}{\partial \hat{n}} \quad & \quad \hat{n} = n, \tag{A.7a}
\]

\[
\text{(Case B)} \quad \hat{\theta} = \theta \quad & \quad \hat{n} = -i\frac{\partial}{\partial \theta}. \tag{A.7b}
\]

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It is obvious that these two realizations satisfy the commutation relations $[\hat{\theta}, \hat{n}] = i$. Correspondingly, we have the following two realizations of the $SU(2)$ generators:

\[
\begin{align*}
\text{(Case A)} \\
J_z &= n, \\
J_+ &= e^{-\frac{i\pi}{2}}A(n), \\
J_- &= \bar{A}(n)e^{\frac{i\pi}{2}},
\end{align*}
\]

\[
\begin{align*}
\text{(Case B)} \\
J_z &= -i\frac{\partial}{\partial \theta}, \\
J_+ &= e^{i\theta}A(-i\frac{\partial}{\partial \theta}), \\
J_- &= \bar{A}(-i\frac{\partial}{\partial \theta})e^{-i\theta}.
\end{align*}
\]

Note that eqs. (A.7a) and (A.8a) correspond to the operators discussed by Balantekin [6]. Eqs. (A.7b) and (A.8b), on the other hand, correspond to the operators discussed by Gangopadhyaya et al. [5].\(^3\) Now it is obvious that these two descriptions are merely the choice of the realizations for the operators $\hat{n}$ and $\hat{\theta}$ and hence essentially equivalent.

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\(^3\)Note that eqs. (A.7a)–(A.8b) do not exactly coincide with the notations and prescriptions of [6] and [5]. But the essential ideas are the same as those proposed in [6] and [5].