The limit distribution of the largest interpoint distance for distributions supported by an ellipse and generalizations

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Abstract

We study the asymptotic behaviour of the maximum interpoint distance of random points in a planar bounded set with an unique major axis and a boundary behaving like an ellipse at the endpoints. Our main result covers the case of uniformly distributed points in an ellipse.

Keywords: Maximum interpoint distance, geometric extreme value theory, Poisson process, uniform distribution in an ellipse

1. Introduction

For some fixed integer \( d \geq 2 \), let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed (i.i.d.) \( d \)-dimensional random vectors, defined on a common probability space \((\Omega, \mathcal{A}, \mathbb{P})\). Writing \( |\cdot| \) for the Euclidean norm on \( \mathbb{R}^d \), the convergence in distribution of the suitably normalized maximum interpoint distance

\[
M_n := \max_{1 \leq i < j \leq n} |X_i - X_j|
\]

has been a topic of interest for more than 20 years. Results obtained so far are mostly for the case that the distribution \( \mathbb{P}_{X_1} \) of \( X_1 \) is spherically symmetric, and they may roughly be classified according to whether \( \mathbb{P}_{X_1} \) has an unbounded or a bounded support. If \( X_1 \) has a spherically symmetric normal distribution, Matthews and Rukhin (1993) obtained a Gumbel limit distribution for \( M_n \). Henze and Klein (1996) generalized this result to the case that \( X_1 \) has a spherically symmetric Kotz distribution. An even more general spherically symmetric setting has recently been studied by Jammalamadaka and Janson (2015). In the unbounded case, Henze and Lao (2015) obtained a (non-Gumbel) limit distribution of \( M_n \) if the distribution of \( X_1 \) is power-tailed spherically decomposable. This case covers certain long-tailed spherically symmetric distributions for \( X_1 \). Finally, Demichel et al. (2014) proved several results for the diameter of an elliptical cloud.

If \( \mathbb{P}_{X_1} \) has a bounded support, Appel et al. (2002) obtained a convolution of two Weibull distributions as limit law of \( M_n \) if \( X_1 \) has uniform distribution in a planar set with unique major axis and sub-\( \sqrt{x} \) decay of its boundary at the endpoints. Moreover, they derived bounds for the limit law of \( M_n \) if \( X_1 \) has a uniform distribution in an ellipse. Lao (2010), and Mayer and Molchanov (2007) obtained Weibull limit distributions for \( M_n \) under very general settings if the distribution of \( X_1 \) is supported by the \( d \)-dimensional unit ball \( \mathbb{B}^d \) for \( d \geq 2 \) (including the case of a uniform distribution). Lao (2010) obtained limit laws for \( M_n \) if \( \mathbb{P}_{X_1} \) is uniform or non-uniform in the unit square, uniform in regular polygons, or uniform in the unit \( d \)-cube, \( d \geq 2 \). Moreover, if \( \mathbb{P}_{X_1} \) is uniform in a proper ellipse, she improved the lower bound on the limit distribution of \( M_n \) given in Appel et al. (2002). The limit behaviour of \( M_n \) if \( \mathbb{P}_{X_1} \) is uniform in a proper ellipse has been an open problem for many years. Without giving a proof, Jammalamadaka and Janson (2015) state that \( n^{2/3}(2 - M_n) \) has a limit distribution (involving two independent Poisson processes) if \( X_1 \) has a uniform distribution in a proper ellipse with major axis 2. We generalize this result to the case that the distribution is uniform or non-uniform over a planar bounded set satisfying certain regularity conditions. Furthermore, the limit distribution of \( M_n \) will be given in a different way.

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In what follows, let \( d = 2 \) and write \( \lambda^2 \) for Lebesgue measure in the plane. Throughout this work we consider distributions \( \mathbb{P}_{X_1} \) with a \( \lambda^2 \)-density \( f \) and compact support \( E \subset \mathbb{R}^2 \). By our main assumptions (see A1 to A7 in Section 2), \( E \) has a unique major axis, and its boundary decays as fast as \( \sqrt{r} \) at the endpoints. In addition, the density \( f \) is continuous and bounded away from \( 0 \) at the endpoints. Since the boundary of the unit disk \( \mathbb{B}^2 \) also decays as fast as \( \sqrt{r} \) at the points \((1,0)\) and \((-1,0)\), but \( \mathbb{B}^2 \) has no unique major axis, this paper can be interpreted as a missing link between the results of Appel et al. (2002) and Lao (2010).

We also consider a related setting in which the number of points follows a Poisson distribution. To this end, let \( \Phi_n \) be a Poisson process in \( \mathbb{R}^2 \) with intensity measure \( n \mathbb{P}_{X_1} \). The diameter of the support of \( \Phi_n \) is denoted by \( \text{diam}(\Phi_n) \). With a few assumptions and \( \text{diam}(E) := \sup \{ |x - y| : x, y \in E \} \), it follows that \( \text{diam}(\Phi_n) \) converges almost surely to \( \text{diam}(E) \) as \( n \to \infty \). We will show that the assumptions on \( E \) and \( f \) stated in Section 2 imply the weak convergence of \( n^{2/3} (\text{diam}(E) - \text{diam}(\Phi_n)) \) by use of the De-Poissonization technique we then obtain the weak convergence of \( n^{2/3} \) towards the same limit distribution.

The rest of the paper is organized as follows: Section 2 contains the assumptions on \( \Phi_n \) and \( f \), respectively, and \( U(I) \) denotes the uniform distribution in the interval \( I \). Each unspecified limit refers to \( n \to \infty \). Convergence in probability, convergence in distribution and equality in distribution will be denoted by \( \overset{P}{\to}, \overset{D}{\to} \) and \( = \), respectively. By \( A_n \overset{P}{=}(1) \) we mean \( A_n \overset{P}{\to} 0 \). Finally, \( f(x) \sim g(x) \) as \( x \to x^* \) stands for \( f(x)/g(x) \to 1 \) as \( x \to x^* \).

2. Assumptions and preliminaries

We first state the basic assumptions on the set \( E \) that supports the distribution of \( X_1 \). As a bit of notation, we write \( B(h) := \{(x,y) \in \mathbb{R}^2 : |(x,y)| \leq a-h \} \) for the closed circle centered at the origin with radius \( a-h \), where \( 0 \leq h < a \).

A1) There is a constant \( a > 0 \) with \( \text{diam}(E) = 2a \), and \( E \) is oriented in the plane so that \( \inf \{ x : (x,y) \in E \} = -a \) and \( \sup \{ x : (x,y) \in E \} = a \).

A2) Putting \( U(x,\varepsilon) := \{(r,s) \in \mathbb{R}^2 : |(r,s) - (x,0)| < \varepsilon \} \), we have \( \text{diam}(E \setminus (U(-a,\varepsilon) \cup U(a,\varepsilon))) < 2a \) for each \( \varepsilon > 0 \).

A3) Writing \( Q_i \) for the \( i \)-th open quadrant in \( \mathbb{R}^2 \), \( i \in \{1,2,3,4\} \), where \( Q_1 = \{(x,y) : x > 0, y > 0 \} \) and the numbering is anti-clockwise, and putting \( E_i := E \cap Q_i \), \( i \in \{1,2,3,4\} \), we have for \( i \in \{1,4\} \) and \( j \in \{2,3\} \), \( \lambda^2(E_i \cap U(a,\varepsilon)) > 0 \) and \( \lambda^2(E_j \cap U(-a,\varepsilon)) > 0 \).

A4) For some \( \nu \in (0,a) \) and continuous functions \( g_1 : [\nu,a] \to \mathbb{R}_{\geq 0} \) and \( g_4 : [\nu,a] \to \mathbb{R}_{\leq 0} \) satisfying \( g_1(a) = g_4(a) = 0 \), we have \( E^0 \cap \{(x,y) \in \mathbb{R}^2 : x > \nu \} = \{(x,y) \in \mathbb{R}^2 : \nu < x < a \} \) and \( g_4(x) < g_1(x) \). Likewise, for continuous functions \( g_2 : [-a,-\nu] \to \mathbb{R}_{\geq 0}, g_3 : [-a,-\nu] \to \mathbb{R}_{\leq 0} \) with \( g_2(-a) = g_3(-a) = 0 \), we have \( E^0 \cap \{(x,y) \in \mathbb{R}^2 : x < -\nu \} = \{(x,y) \in \mathbb{R}^2 : -a < x < -\nu \} \) and \( g_3(x) < g_2(x) \).\)

A5) Writing \( f_a(x) := \sqrt{a^2 - x^2}/2 \) for the ‘upper boundary function’ of an ellipse with major axis \( 2a \) and minor axis \( a \), we assume that for constants \( q_1, q_2, q_3, q_4 \) satisfying \( 0 < q_i < 2, i \in \{1,2,3,4\} \),

\[
\begin{align*}
g_1(x) &\to q_1, \quad -g_4(x) \to q_4 \quad \text{as} \quad x \to a, \\
g_2(x) &\to q_2, \quad -g_3(x) \to q_3 \quad \text{as} \quad x \to -a.
\end{align*}
\]

A6) For \( i \in \{1,2,3,4\} \) and sufficiently small \( h \), we have \( E^0 \cap \{(x,y) \in \mathbb{R}^2 : |x| < \nu \} \subset B(h) \). Moreover, \( g_i \) has only one point of intersection with \( \partial B(h) \). The abscissa of this point is denoted by \( \tau_i(h) \).

A7) For sufficiently small \( \varepsilon \), the density \( f \) is continuous on \( E \cap U(a,\varepsilon) \) and \( E \cap U(-a,\varepsilon) \) and we have \( p_1 := p_4 := f((a,0)) > 0 \) and \( p_2 := p_3 := f((-a,0)) > 0 \).
Assumption A1 entails no loss of generality since the problem is invariant under rigid motions. A2 means that \((-a, 0)\) and \((a, 0)\) (henceforth called the ‘poles’) are the endpoints of the unique major axis of \(E\). By A3, the area near the poles is positive in each quadrant. Assumption A4 means that \(E^0\) is vertically convex near the poles. In the sequel, \(q_1, \ldots, q_4\) will be called the ‘boundary functions’ of \(E\). By A5, these functions decay as fast as \(\sqrt{a - x}\) at the poles. For example, the condition in the first quadrant is equivalent to \(q_1(x)/\sqrt{a - x} \to q_1\sqrt{a}/\sqrt{a} > 0\) as \(x \to a\), which means that \(q_1\) actually decays like a square root. The reason why A5 is formulated in terms of \(a\) instead of \(\sqrt{a - x}\) is to facilitate many calculations in Subsection 4.2. The choice of the factor 1/2 in \(f_a(x)\) is arbitrary but necessary (in fact, it can be any number in the open interval \((0, 1)\)) in order to have points of intersection of \(f_a\) and \(\partial B(h)\) in the proofs of Lemma 4.1 and Lemma 4.2. Notice that (4) ensures the existence of at least one point of intersection of the boundary functions \(g_i\) and the boundary of \(B(h)\) for small \(h > 0\). If \(q_i = 2\) the set \(E\) would behave like a circle in the pole in the \(i\)-th quadrant. This case is explicitly excluded in this work. If \(E\) is a circle, we have \(q_1 = \ldots = q_4 = 2\). For a circle, the limit distribution of \(M_n\) is well-known, see Lao (2010) and Mayer and Molchanov (2007). Because of A6, the set \(\{ z \in E^2 : |z| > a - h\}\) consists only of points lying close to a pole for sufficiently small \(h\). The notations \(p_1 = p_4\) and \(p_2 = p_3\) in A7 are redundant but useful, since we hereby avoid a distinction of several cases.

3. Main results

To state the main result, put
\[
c_i := \frac{2q_i\sqrt{2a}}{3\sqrt{4 - q_i^2}}, \quad \sigma_i := (p_i c_i)^{-2/3}, \quad \tau_i := \frac{3}{2a} c_i, \quad i \in \{1, 2, 3, 4\}. \tag{2}
\]
Let \(Y_1, Y_2, \ldots\) and \(U_1, U_2, \ldots\) be independent random variables, where \(Y_1, Y_2, \ldots\) are i.i.d. with a unit exponential distribution, and \(U_1, U_2, \ldots\) are i.i.d. with the uniform distribution \(U([0, 1])\). For \(m \geq 1\), set \(S_m := Y_1 + \ldots + Y_m\). Let \(\sigma, \tau > 0\), and put
\[
Z_{1,m} := \sigma S_m^{2/3}, \quad Z_{2,m} := U_m \tau \sigma^{1/2} S_m^{1/3} (= U_m \tau Z_{1,m}^{1/2}).
\]
The sequence \(Z := (Z_{1,1}, Z_{1,2}, Z_{1,3}, Z_{1,4}, \ldots, Z_{1,m}, Z_{2,m}, \ldots)\) defines a \(\mathbb{R}^3\)-valued random element, the distribution of which will be denoted by \(\text{NA}_\infty(\sigma, \tau)\). Here, the coining \(\text{NA}\) stands for ‘norm-angle distribution’. Notice that for each \(m\) the conditional distribution of \(Z_{2,m}\) given \(Z_{1,m}\) is uniform on \([0, \tau Z_{1,m}^{1/2}]\). In what follows, we will write \(Z := (Z_{1,k}, Z_{2,k})_{k \geq 1}\). Our main result is as follows.

**Theorem 3.1.** Let \((Z_{1,k}, Z_{2,k})_{k \geq 1} \sim \text{NA}_\infty(\sigma, \tau), i \in \{1, 2, 3, 4\}, \) be independent random elements of \(\mathbb{R}^3\), and put
\[
S^{i,j} := \min_{k, l} \left\{ \left( Z_{1,k}^{i} + Z_{1,l}^{j} + \frac{a}{4} \left( Z_{2,k}^{i} - Z_{2,l}^{j} \right) \right)^2 \right\}, \quad (i, j) \in \{(1, 3), (2, 4)\},
\]
\[
S^{i,j} := \min_{k, l} \left\{ \left( Z_{1,k}^{i} + Z_{1,l}^{j} + \frac{a}{4} \left( Z_{2,k}^{i} + Z_{2,l}^{j} \right) \right)^2 \right\}, \quad (i, j) \in \{(1, 2), (3, 4)\}.
\]
Then, under the assumptions A1 - A7, we have
\[
n^{2/3}(2a - \text{diam}(\Phi_n)) \overset{D}{\longrightarrow} \min \left\{ S^{1,2}, S^{1,3}, S^{2,4}, S^{3,4} \right\}. \tag{3}
\]

The proof of Theorem 3.1 is given in Section 4. By use of a De-Poissonization theorem by Mayer and Molchanov (2007), we can restate this result for the maximum interpoint distance \(M_n\) of independent and identically distributed random points.

**Theorem 3.2.** Under A1 to A7 we have \(n^{2/3}(2a - M_n) \overset{D}{\longrightarrow} \min \left\{ S^{1,2}, S^{1,3}, S^{2,4}, S^{3,4} \right\}\).

Now we can state our result for the uniform distribution in an ellipse:

**Corollary 3.3.** Consider uniformly distributed points inside an ellipse \(E\) with major axis a = 2 and minor axis \(2b < 2\). If the major axis is placed between \((-1, 0)\) and \((1, 0)\), \(E\) satisfies A1 - A6. Since the border functions of \(E\) are given by \( \pm b \sqrt{1 - x^2}\), we get by symmetry \(q_1 = \ldots = q_4 = 2b\). Because of \(X^2(E) = \pi b\), we have \(p_1 = \ldots = p_4 = 1/(\pi b)\). Hence, Theorem 3.1 (and thus also Theorem 3.2) is applicable with i.i.d. random elements \(Z_1, \ldots, Z_4 \sim \text{NA}_\infty(\sigma_1, \tau_1)\).
Jammalamadaka and Janson (2015) described the limit distribution as that of

\[ \pi^{2/3} \min_{i,j \in \mathbb{N}} \left\{ x'_i + x''_j - \frac{b^2}{4} (y'_i - y''_j)^2 \right\}, \]

where \((x'_i, y'_i)\) and \((x''_j, y''_j)\) are two independent Poisson processes in the parabola \(\{(x, y) \in \mathbb{R}^2 : y^2 \leq 2x\}\) with intensity 1. For simulations, the representation of the limit distribution given in (3) is much more useful, since the latter can easily be approximated. For the latter purpose, fix \(m \geq 1\) and replace \(\min_{k,l \in \mathbb{N}}\) in the definition of \(S_{k,l}\) in Theorem 3.1 by \(\min_{1 \leq k,l \leq m}\). This approximation is a consequence of Lemma 4.9. The bigger the minor half-axis \(b\) is (i.e. the more \(E\) becomes 'circlelike'), the bigger \(m\) has to be chosen in order to have a good approximation of the distributional limit in (3) (we omit details), see Figure 1.

![Figure 1: Empirical distribution function in the setting of Corollary 3.3 with \(b = 1/2\), \(n = 1000\) (solid, 5000 replications). The limit distribution is approximated as described in Corollary 3.3 for \(m = 8\) (dashed, 5000 replications).](image)

4. Proof of Theorem 3.1

4.1. The stochastic model

Let \(\bar{N}\) have a Poisson distribution \(\text{Po}(n)\) and, independently of \(N\), let \(Y_1, Y_2, \ldots\) be i.i.d. random variables with the same distribution as \(X_1\). Then \(\Phi_n \equiv \bar{\Xi}_N := \{Y_1, Y_2, \ldots, Y_N\}\) and hence

\[ \text{diam}(\Phi_n) \overset{D}{=} \text{diam}(\bar{\Xi}_N). \] (4)

It will be useful to discriminate the points of \(\bar{\Xi}_N\) according to the quadrant in which they realize. To this end, put \(l_i := \int_{E_i} f(z)dz, \ i \in \{1, 2, 3, 4\}\). For \(n \geq 1\), let \(N_i \sim \text{Po}(nl_i)\) and, independently of \(N_i\), let \(X^i_1, X^i_2, \ldots\) be i.i.d. \(\sim l_i^{-1}\mathbb{P}|_{E_i}\), where \(\mathbb{P}|_{E_i}\) is the restriction of \(\mathbb{P}_{X_1}\) to the set \(E_i\). The densities of these distributions are given by \(f_i := l_i^{-1}f|_{E_i}\). With \(\Xi_{N_i} := \{X^i_1, X^i_2, \ldots, X^i_{N_i}\}, N := N_1 + \ldots + N_4\) and \(\Xi_N := \Xi^1_{N_1} \cup \Xi^2_{N_2} \cup \Xi^3_{N_3} \cup \Xi^4_{N_4}\) we get \(\bar{\Xi}_N \overset{D}{=} \Xi_N\) and therefore

\[ \text{diam}(\bar{\Xi}_N) \overset{D}{=} \text{diam}(\Xi_N). \] (5)

Because of this equality in distribution and (4), it is sufficient to investigate \(\text{diam}(\Xi_N)\). With the notation

\[ M_{N}^{j_1,j_2} := \max \left\{ |X^{j_1}_{k_1} - X^{j_2}_{k_2}| : 1 \leq k_1 \leq N_{j_1}, 1 \leq k_2 \leq N_{j_2} \right\}, \quad 1 \leq j_1 \leq j_2 \leq 4, \]

we have \(\text{diam}(\Xi_N) = \max_{1 \leq j_1 \leq j_2 \leq 4} M_{N}^{j_1,j_2}\). The conditions A1 to A3 and A7 in mind, it is obvious that for sufficiently large \(n\) only the pairs \((1, 2), (1, 3), (2, 4)\) and \((3, 4)\) can be relevant for \((j_1, j_2)\). We obtain the important convergence

\[ \mathbb{P}\left( \text{diam}(\Xi_N) \neq \max \left\{ M_{N}^{1,2}, M_{N}^{1,3}, M_{N}^{2,4}, M_{N}^{3,4} \right\} \right) \rightarrow 0. \] (6)
For example, $M_{N}^{1,2}$ will be determined by points inside the first (resp. second) quadrant, which are located close to the right (resp. left) pole of $E$ (cf. A1 to A3 and A7). By A6, these points can easily be characterized by their distance to the origin: the norms of these points are close to the largest possible value $a$. In Subsection 4.3 we thus study the asymptotic behaviour of those points with the largest norms. To this end, we have norms inside one quadrant for fixed $k$ x close to the right (resp. left) pole of $E$.

For example, $M$ will be determined by points inside the first (resp. second) quadrant, which are located by their distance to the origin: the norms of these points are close to the largest possible value $a$. If we denote by $\gamma$ the polar angle of the point $\pm \left( \frac{a(1,1)}{\sqrt{2(a-h)^2-x^2}} \right)$ for the abscissae of the points of intersection between $(q_1 \pm \varepsilon)f_0(x)$ and $B_h(x)$, some algebra gives

$$x^\pm(h) = \sqrt{a^2 - \frac{4h(2a-h)}{4-(q_1 \pm \varepsilon)^2}}.$$  

Putting $F^\pm(\varepsilon) := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a \text{ and } 0 \leq y \leq (q_1 \pm \varepsilon)f_0(x)\}$ and using 4.1. we obtain

$$\lambda^2 \left( F^- (\varepsilon) \setminus B(h) \right) \leq \lambda^2 \left( A_1(h) \right) \leq \lambda^2 \left( F^+ (\varepsilon) \setminus B(h) \right).$$  

(10)

See figure 2 for an illustration. To calculate the values on the left- and on the right-hand side of (10) we use polar coordinates. The upper boundary of $F^\pm(\varepsilon)$ is given by $\{r^\pm(\varphi)(\cos \varphi, \sin \varphi) : 0 \leq \varphi \leq \pi/2\}$ with

$$r^\pm(\varphi) := \frac{a(q_1 \pm \varepsilon)}{2 \sqrt{1 + \frac{(q_1 \pm \varepsilon)^2}{4} \cos^2 \varphi}}.$$  

If we denote by $\gamma^\pm_h$ the polar angle of the point $\left( x^\pm(h), B_h(x^\pm(h)) \right)$, it follows that

$$\tan(\gamma^\pm_h) = \frac{B_h(x^\pm(h))}{x^\pm(h)}.$$  

(11)
This leads to
\[
\lambda^2 \left( F^\pm(\varepsilon) \setminus B(h) \right) = \int_0^{\gamma_h^\pm} r^\pm d\varphi
\]
\[
= \frac{1}{2} \cdot \int_0^{\gamma_h^+} \left( \frac{a^2 \cdot (q_1 \pm \varepsilon)^2}{4 + 4 \left( \frac{(q_1 \pm \varepsilon)^2}{4} - 1 \right) \cos^2 \varphi} - (a - h)^2 \right) d\varphi
\]
\[
= a^2 \cdot (q_1 \pm \varepsilon) \cdot \arctan \left( \frac{2}{q_1 \pm \varepsilon} \cdot \frac{B_h(x^\pm(h))}{x^\pm(h)} \right) - \frac{(a - h)^2}{2} \arctan \left( \frac{B_h(x^\pm(h))}{x^\pm(h)} \right).
\]

By expanding this term about \( h = 0 \), we get
\[
\lambda^2 \left( F^\pm(\varepsilon) \setminus B(h) \right) = c_1^+(\varepsilon) h^{3/2} + O \left( h^{5/2} \right)
\]
with
\[
c_1^+(\varepsilon) := \frac{2(q_1 \pm \varepsilon) \sqrt{2a}}{3 \sqrt{4 - (q_1 \pm \varepsilon)^2}}.
\]
Hence, by (10) and (12) the inequality \( \lambda^2(A_1(h)) \leq c_1^+(\varepsilon) h^{3/2} + O(h^{5/2}) \) holds as \( h \to 0 \), and we have
\[
\frac{\lambda^2(A_1(h)) - c_1 h^{3/2}}{c_1 h^{3/2}} \leq \frac{c_1^+(\varepsilon) h^{3/2} + O(h^{5/2}) - c_1 h^{3/2}}{c_1 h^{3/2}} = \frac{c_1^+(\varepsilon) - c_1}{c_1} + O(h).
\]

Now fix \( \delta > 0 \). Since the function
\[
c_1^+(x) := \frac{2(q_1 + x) \sqrt{2a}}{3 \sqrt{4 - (q_1 + x)^2}}
\]
is continuous from the right at \( x = 0 \) for all valid \( a \) and \( q_1 \), we can choose \( \varepsilon > 0 \) in such a way that \( (c_1^+(\varepsilon) - c_1)/c_1 \leq \delta/2 \). By (13) we get \( \lambda^2(A_1(h)) - c_1 h^{3/2} \leq \delta c_1 h^{3/2} \) as \( h \to 0 \). In the same way one can show \( -\delta c_1 h^{3/2} \leq \lambda^2(A_1(h)) - c_1 h^{3/2} \) as \( h \to 0 \), and the proof is finished.

Now, for sufficiently small \( h \) (see A6), let \( \eta_i(h) \) be the polar angle of the point \( (\pi_i(h), B_h(\pi_i(h))) \), \( i \in \{1, 2\} \),
and let $\eta_i(h)$ be the polar angle of $(\overline{r}_i(h), -B_h(\overline{r}_i(h)))$ if $i \in \{3, 4\}$. Furthermore, put

$$
\gamma_i(h) := \begin{cases} 
\eta_i(h), & i = 1, \\
\pi - \eta_i(h), & i = 2, \\
\eta_i(h) - \pi, & i = 3, \\
2\pi - \eta_i(h), & i = 4.
\end{cases}
$$

(14)

**Lemma 4.2.** For $i \in \{1, 2, 3, 4\}$ we have $\gamma_i(h) \sim \tau_i h^{1/2}$ as $h \to 0$, with $\tau_i$ given in (2).

**Proof.** W.l.o.g. let $i = 1$. Fix an arbitrary $\delta > 0$. We have to show that the inequalities

$$
-\delta \leq \gamma_1(h) - \tau_1 h^{1/2} \leq \delta
$$

hold for sufficiently small $h$. With the same notation as in the proof of Lemma 4.1 and (11), it follows that

$$
\arctan \left( \frac{B_h(x^{-}(h))}{x^{-}(h)} \right) \leq \gamma_1(h) \leq \arctan \left( \frac{B_h(x^{+}(h))}{x^{+}(h)} \right).
$$

Power series expansions about $h = 0$ show

$$
\arctan \left( \frac{B_h(x^{\pm}(h))}{x^{\pm}(h)} \right) = \tau_1^{\pm}(\epsilon) h^{1/2} + O(h^{3/2})
$$

with

$$
\tau_1^{\pm}(\epsilon) = \frac{(q_1 \pm \epsilon)\sqrt{2a}}{a \sqrt{4 - (q_1 \pm \epsilon)^2}}.
$$

The rest of the proof is by complete analogy with the proof of Lemma 4.1.

For the sake of readability, we change our notation until the end of this subsection by using capitals for deterministic sequences. Moreover, we denote the underlying quadrant by a subscript instead of a superscript. For $i \in \{1, 2, 3, 4\}$ the value $N_{i,n}$ and $V_{i,n}$ denote the norm and the polar angle of the $n$-th deterministic point, respectively. As in (14) we set

$$
W_{i,n} := \begin{cases} 
V_{i,n}, & i = 1, \\
\pi - V_{i,n}, & i = 2, \\
V_{i,n} - \pi, & i = 3, \\
2\pi - V_{i,n}, & i = 4.
\end{cases}
$$

(15)

and define a function $p : \mathbb{R}_{\geq 0} \times [0, 2\pi) \to \mathbb{R}^2$ by $p(r, \varphi) := (r \cos \varphi, r \sin \varphi)$. For $i \in \{1, 2, 3, 4\}$ we write $(N_{i,n}, W_{i,n}) \to \text{Pole}_i$, if $N_{i,n} \geq 0, W_{i,n} \geq 0$ for each $n$, the point $p(N_{i,n}, V_{i,n})$ lies in $E_i$ and for $i \in \{1, 4\}$ (resp. $i \in \{2, 3\}$) the points $p(N_{i,n}, V_{i,n})$ converge to $(a, 0)$ (resp. $(-a, 0)$). Notice that $(N_{i,n}, W_{i,n}) \to \text{Pole}_i$ implies $W_{i,n} \to 0$.

**Lemma 4.3.** Let $(N_{1,n}, W_{1,n})$ and $(N_{3,n}, W_{3,n})$ be deterministic sequences satisfying $(N_{i,n}, W_{i,n}) \to \text{Pole}_i$, $i \in \{1, 3\}$. Then

$$
|p(N_{1,n}, V_{1,n}) - p(N_{3,n}, V_{3,n})| = N_{1,n} + N_{3,n} - \frac{a}{4} E_n^2 + \tilde{R}_n,
$$

(16)

where

$$
E_n := W_{1,n} - W_{3,n}, \quad \tilde{R}_n := O(E_n^4) + A_n + B_n + C_n + D_n
$$

and

$$
A_n := \frac{1}{4} \left( \frac{1}{2} E_n^2 + O(E_n^4) \right) (a - N_{1,n}), \quad B_n := \frac{1}{4} \left( \frac{1}{2} E_n^2 + O(E_n^4) \right) (a - N_{3,n}),
$$

$$
C_n := -\frac{a}{16} \left( \frac{1}{2} E_n^2 + O(E_n^4) \right)^2, \quad D_n = O \left( (a - N_{1,n})^2 + (a - N_{3,n})^2 + \frac{1}{2} E_n^2 + O(E_n^4) \right)^2.
$$
Proof. By the law of cosines and $V_{3,n} = \pi + W_{3,n}$ we get

\[
\begin{align*}
|p(N_{1,n}, V_{1,n}) - p(N_{3,n}, V_{3,n})|^2 &= |p(N_{1,n}, W_{1,n}) - p(N_{3,n}, \pi + W_{3,n})|^2 \\
&= N_{1,n}^2 + N_{3,n}^2 - 2N_{1,n}N_{3,n}\cos(\pi + W_{3,n} - W_{1,n}) \\
&= N_{1,n}^2 + N_{3,n}^2 + 2N_{1,n}N_{3,n}\cos(W_{1,n} - W_{3,n}).
\end{align*}
\]

Using $\cos(x - y) = 1 - (x - y)^2/2 + O((x - y)^4)$ as $(x, y) \to 0$ yields

\[
|p(N_{1,n}, V_{1,n}) - p(N_{3,n}, V_{3,n})| = \sqrt{N_{1,n}^2 + N_{3,n}^2 + 2N_{1,n}N_{3,n}\left(1 - \frac{1}{2}E_n^2 + O(E_n^4)\right)}.
\]

Taylor’s theorem for multivariate functions then gives

\[
\sqrt{x^2 + y^2 + 2xy(1 - z)} = x + y - \frac{a}{2}z + \frac{1}{4}z(a - x) + \frac{1}{4}z(a - y) - \frac{a}{16}z^2 + O((a - x)^2 + (a - y)^2 + z^2)
\]

as $(x, y, z) \to (a, a, 0)$. Putting $x = N_{1,n}, y = N_{3,n}$ and $z = \frac{1}{2}E_n^2 + O(E_n^4)$ leads to

\[
|p(N_{1,n}, V_{1,n}) - p(N_{3,n}, V_{3,n})| = N_{1,n} + N_{3,n} - \frac{a}{2} \left(\frac{1}{2}E_n^2 + O(E_n^4)\right) + R_n = N_{1,n} + N_{3,n} - \frac{a}{4}E_n^2 + \tilde{R}_n.
\]

By the same reasoning, we have:

**Lemma 4.4.** Let $(N_{1,n}, W_{1,n})$ and $(N_{2,n}, W_{2,n})$ be deterministic sequences satisfying $(N_{i,n}, W_{i,n}) \to \text{Poi}(1), i \in \{1, 2\}$. Then $|p(N_{1,n}, V_{1,n}) - p(N_{2,n}, V_{2,n})| = N_{1,n} + N_{2,n} - aE_n^2/4 + \tilde{R}_n$, where $F_n := W_{1,n} + W_{2,n}$ instead of $E_n$ and $\tilde{R}_n$ adjusted accordingly.

Because of symmetry, Lemma 4.3 (resp. 4.4) can be applied to sequences in the second and fourth (resp. third and fourth) quadrant.

### 4.3. A single quadrant

We now study the joint asymptotic behaviour of those points inside a fixed quadrant $Q_i$ that have the $k$ largest norms, where $k \geq 1$ is fixed. Since the number $N_i$ of points follows a Poisson distribution, which means that $\mathbb{P}(N_i < k) > 0$ for every $n \in \mathbb{N}$, we put

\[
X_{i,j} := 0 \text{ for } j \in \{N_i + 1, \ldots, k\} \text{ provided that } N_i < k.
\]

We start with the following lemma, the proof of which is omitted.

**Lemma 4.5.** Let $N \sim \text{Po}(\mu)$ and, independently of $N$, let $U_1, U_2, \ldots$ be i.i.d. $\sim U([0, 1])$. Writing $U_{(1)} \leq U_{(2)} \leq \ldots \leq U_{(N)}$ for the order statistics of $U_1, \ldots, U_N$, we have for fixed $k$

\[
\mu(U_{(1)}, U_{(2)}, \ldots, U_{(k)}) \overset{\mathcal{D}}{\to} (S_1, S_2, \ldots, S_k)
\]

as $\mu \to \infty$. Here, $S_m = \sum_{j=1}^{m} Y_j$, and $Y_1, \ldots, Y_n$ are i.i.d. unit exponential random variables.

**Lemma 4.6.** Let $k \geq 1$ be fixed. Based on $Y_1, \ldots, Y_k$ and $S_m$ as above, we have

\[
n^{2/3}(a - X_{i(1)}^2, \ldots, a - X_{i(k)}^2) \overset{\mathcal{D}}{\to} \sigma_i \left(S_1^{2/3}, S_2^{2/3}, \ldots, S_k^{2/3}\right),
\]

where $\sigma_i$ is given in (2).
Proof. Since $X_i^1$ has a $\lambda^2$-density, the distribution function $F$ of $|X_i^1|$ is continuous and hence $1 - F(|X_i^1|) \sim U([0, 1])$. Independently of $N_i$ and $X_i^1, \ldots, X_i^{N_i}$, let $U_1, U_2, \ldots$ be i.i.d. $\sim U([0, 1])$ and write $U_{(1)} \leq U_{(2)} \leq \ldots \leq U_{(N_i)}$ for the order statistics of $U_1, \ldots, U_{N_i}$. Because of $P(N_i < k) \to 0$, the case $N_i < k$ is negligible in the following. We get

$$(1 - F(|X_{(1)}^1|), \ldots, 1 - F(|X_{(k)}^i|)) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(k)})$$

and, due to $N_i \sim \text{Po}(nl)$, Lemma 4.6 yields $nl_i(1 - F(|X_{(1)}^i|), \ldots, 1 - F(|X_{(k)}^i|)) \overset{D}{\to} (S_1, \ldots, S_k)$. Assumption A6, the equations (7) and (8) and Lemma 4.1 show

$$1 - F(x) = P(|X_i^1| > x) = \int_{A_i(a-x)} f(z) \frac{dz}{l_i} \sim \frac{P_i}{l_i} \lambda^2(A_i(a-x)) \sim \frac{P_i}{l_i} c_i(a-x)^{3/2}$$

as $x \to a$. Therefore, $nl_i(1 - F(|X_{(1)}^i|), \ldots, 1 - F(|X_{(k)}^i|))$ and $np_i c_i ((a - |X_{(1)}^i|)^{3/2}, \ldots, (a - |X_{(k)}^i|)^{3/2})$ are asymptotically equivalent and thus have the same limit $(S_1, \ldots, S_k)$ in distribution. With the function $g(x_1, \ldots, x_k) := (x_1^{2/3}, \ldots, x_k^{2/3})$ and the continuous mapping theorem we obtain

$$n^{2/3}(a - |X_{(1)}^i|, \ldots, a - |X_{(k)}^i|) = \left(\frac{1}{np_i c_i}\right)^{2/3} (np_i c_i)^{2/3} (a - |X_{(1)}^i|, \ldots, a - |X_{(k)}^i|)$$

$$= \sigma_i g((np_i c_i (a - |X_{(1)}^i|)^{3/2}, \ldots, np_i c_i (a - |X_{(k)}^i|)^{3/2})$$

$$\overset{D}{\to} \sigma_i g(S_1, S_2, \ldots, S_k)$$

$$= \sigma_i (S_1^{2/3}, S_2^{2/3}, \ldots, S_k^{2/3}).$$

In what follows, observe that $k$ is fixed in this subsection. If $k \leq N_i$, let $\eta_{(l)}^j$ the polar angle of $X_{(l)}^i$ for $l = 1, \ldots, k$, and set $\eta_{(l)}^0 := 0$ for $l \in \{N_i + 1, \ldots, k\}$ if $N_i < k$. Furthermore, put

$$\gamma_{(l)}^i := \begin{cases} \eta_{(l)}^i, & i = 1, \\ \pi - \eta_{(l)}^i, & i = 2, \\ \eta_{(l)}^i - \pi, & i = 3, \\ 2\pi - \eta_{(l)}^i, & i = 4, \\ \end{cases} \quad l \in \{1, \ldots, k\}. \quad (18)$$

We need the joint asymptotic behaviour of the angles $\gamma_{(1)}^i, \ldots, \gamma_{(k)}^i$. As before, the case $N_i < k$ is negligible. In Lemma 4.6 we have shown the weak convergence of $n^{2/3}(a - |X_{(j)}^i|)$ for $j \in \{1, \ldots, k\}$. Let $0 < x_1 < \ldots < x_k$ and consider the conditions $n^{2/3}(a - |X_{(j)}^i|) = x_j > 0$, $j = 1, \ldots, k$. Under these conditions, the angles $\gamma_{(1)}^i, \ldots, \gamma_{(k)}^i$ are asymptotically independent. Some calculations and Lemma 4.2 show that the conditional density of $n^{1/3} \gamma_{(j)}^i$ converges pointwise to the density of a $U([0, \pi \sqrt{x_j}])$-distributed random variable (with $\pi_i$ given in (2)). We thus have the following result:

**Lemma 4.7.** For fixed $k \geq 1$ let $Y_1, \ldots, Y_k$ and $U_1, \ldots, U_k$ be independent random variables, where $Y_1, \ldots, Y_k$ are i.i.d. with a unit exponential distribution, and $U_1, \ldots, U_k$ are i.i.d. with the uniform distribution $U([0, 1])$. For $i \in \{1, 2, 3, 4\}$, let $Z_{1,m}^i := \sigma_i \left(\sum_{j=1}^m Y_j \right)^{2/3}$ and $Z_{2,m}^i := U_m \tau_i (Z_{1,m}^i)^{1/2}$ for $m = 1, \ldots, k$. Then

$$\left(n^{2/3}(a - |X_{(1)}^i|), n^{1/3} \gamma_{(1)}^i, \ldots, n^{2/3}(a - |X_{(k)}^i|), n^{1/3} \gamma_{(k)}^i\right) \overset{D}{\to} \left(Z_{1,1}^i, Z_{2,1}^i, \ldots, Z_{1,k}^i, Z_{2,k}^i\right).$$

The limit distribution in Lemma 4.7 shall be denoted by $\text{NA}_k(\sigma_i, \tau_i)$ (‘norm-angle distribution of order $k$’).
4.4. Different quadrants

Instead of studying diam($\Xi$), we make two restrictions at this point: On the one hand we examine the behaviour of the diameters $M^{1,2}_N$ separately for each pair $\{j_1, j_2\} \in \{(1, 2), (1, 3), (2, 4), (3, 4)\}$. On the other hand, we approximate these random variables by $M^{1,2}_N$. In a first step we fix $k \geq 1$, and then we let $k$ tend to infinity. We will see, that the difference between $M^{1,2}_N$ and $M^{2,3}_N$ asymptotically negligible. For fixed $k$ and $i \in \{1, 2, 3, 4\}$ we have $\{(X^i_1), \gamma^i_1\} \to \text{Pole}$, almost surely for each $l \in \{1, \ldots, k\}$. In other words, the series expansions of Lemmas 4.3 and 4.4 can be applied to each pair $\{(X^i_1), \gamma^i_1\}$.

**Proposition 4.8.** For fixed $k \geq 1$ we have $n^{2/3}(2a - M^{1,3}_N) \xrightarrow{d} \min_{1 \leq i,j \leq k} \{Z^1_1 + Z^1_3 + \frac{a}{2} (Z^2_1 - Z^3_2)^2\}$ with two independent random elements $(Z^1_1, Z^1_3)_{j \geq 1} \sim \mathcal{N}_k(\gamma_1, \sigma_1)$ and $(Z^2_1, Z^3_2)_{j \geq 1} \sim \mathcal{N}_k(\sigma_3, \tau_3)$.

**Proof.** As $n$ tends to infinity, the probabilities $P(N_1 < k)$ and $P(N_3 < k)$ converge to 0. Because of this asymptotic negligibility, we assume $k < N_1$ and $k < N_3$. Since the points in $Q_1$ and $Q_3$ are independent, Lemma 4.7 implies
\[
\left(\frac{n^{2/3}(a - |X^i_1|)}{n^{1/3} \gamma^i_1}, \ldots, \frac{n^{2/3}(a - |X^i_1|)}{n^{1/3} \gamma^i_1}\right) \xrightarrow{d} \left(\frac{Z_1}{Z_3}\right),
\]
with two independent random elements $Z_1 \sim \mathcal{N}_k(\gamma_1, \sigma_1)$ and $Z_3 \sim \mathcal{N}_k(\sigma_3, \tau_3)$. Since
\[
n^{2/3}(2a - M^{1,3}_N) = n^{2/3}(2a - \max_{1 \leq i,j \leq k} |X^i_1 - X^3_j|) = \min_{1 \leq i,j \leq k} \left\{n^{2/3}(2a - |X^i_1 - X^3_j|)\right\},
\]
we define $h^{-}(y_1, y_2, z_1, z_2) := y_1 + z_1 + a(y_2 - z_2)^2/4$ and, using Lemma 4.7 for $(i, j) \in \{1, \ldots, k\}^2$, obtain
\[
n^{2/3}(2a - |X^i_1 - X^3_j|) = n^{2/3}(2a - (|X^i_1| + |X^3_j|) - \frac{a}{4} (\gamma^i_1 - \gamma^3_j)^2 + \bar{R}_n)
\]
\[
= n^{2/3}(2a - |X^i_1|) + n^{2/3}(a - |X^3_j|) + \frac{a}{4} (\gamma^i_1 - \gamma^3_j)^2 + n^{2/3}\bar{R}_n
\]
\[
= h^{-}(n^{2/3}(2a - |X^i_1|), n^{1/3} \gamma^i_1, n^{2/3}(a - |X^3_j|), n^{1/3} \gamma^3_j) + n^{2/3}\bar{R}_n.
\]
To show that $n^{2/3}\bar{R}_n = o_p(1)$ put $E_n := \gamma^i_1 - \gamma^3_j$. From (16) we have $\bar{R}_n = O(E^4_n) + A_n + B_n + C_n + D_n$, where
\[
A_n := \frac{1}{4} \left(\frac{1}{2} E^2_n + O(E^4_n)\right) \cdot (a - |X^i_1|), \quad B_n := \frac{1}{4} \left(\frac{1}{2} E^2_n + O(E^4_n)\right) \cdot (a - |X^3_j|),
\]
\[
C_n := -\frac{a}{16} \left(\frac{1}{2} E^2_n + O(E^4_n)\right)^2, \quad D_n = O\left((a - |X^i_1|)^2 + (a - |X^3_j|)^2 + \left(\frac{1}{2} E^2_n + O(E^4_n)\right)^2\right).
\]
Since $E_n = o_p(1)$ and $(n^{1/3}E_n)$ is a tight sequence, we get $n^{2/3}O(E^4_n) = (n^{1/3}E_n)^2 O(E^2_n) = o_p(1)$. From $n^{2/3}A_n = \frac{1}{4} \left(E^2_n + O(E^4_n)\right) \cdot (n^{2/3}(a - |X^i_1|))$, Lemma 4.0 and $E_n = o_p(1)$ we obtain $n^{2/3}A_n = o_p(1)$. The same reasoning gives $n^{2/3}B_n = o_p(1)$. Now,
\[
n^{2/3}C_n = -\frac{a}{16} \left(\frac{1}{2} n^{1/3} E^2_n + n^{1/3} O(E^4_n)\right)^2 = -\frac{a}{16} \left(\frac{1}{2} n^{1/3} E_n\right)^2 E_n \left(\frac{1}{2} O(E^2_n)\right)^2
\]
teans $n^{2/3}C_n = o_p(1)$, since $E_n = o_p(1)$ and $(n^{1/3}E_n)$ is tight. Lemma 1.6 yields $n^{2/3}(a - |X^m_1|)^2 = o_p(1)$ for $m \in \{1, 3\}$ and, together with $n^{2/3}C_n = o_p(1)$, we get $n^{2/3}D_n = o_p(1)$, i.e $n^{2/3}\bar{R}_n = o_p(1)$. We thus can rewrite (20) as
\[
n^{2/3}(2a - M^{1,3}_N) = \min_{1 \leq i,j \leq k} \left\{h^{-}\left(n^{2/3}(a - |X^i_1|), n^{1/3} \gamma^i_1, n^{2/3}(a - |X^3_j|), n^{1/3} \gamma^3_j\right) + o_p(1)\right\},
\]
and the assertion follows from the continuous mapping theorem and (19).
Lemma 4.9. There exists a null sequence \((\varepsilon_k)_{k \geq 1}\) with \(P(M_{N,k}^{1,3} \neq M_{N,k}^{1,3}) \leq \varepsilon_k\) for every \(k \geq 1\) and sufficiently large \(n\).

Proof. Let \(B_{N_1}^1\) be the \(\mathbb{R}^N\)-valued random element with components \(n^{2/3}(a - |X_{(1)}^1|), n^{1/3} \gamma_{(1)}, \ldots, n^{2/3}(a - |X_{(N_1)}^1|), n^{1/3} \gamma_{(1)}\), followed by \(n^{2/3}(a - |X_{(N_1)}^1|)\) and \(n^{1/3} \gamma_{(1)}\), repeated infinitely often. Let \(Y_1^1, Y_2^1, \ldots\) and \(U_1^1, U_2^1, \ldots\) be independent random variables, where \(Y_1^1, Y_2^1, \ldots\) are i.i.d. with a uniform distribution, and \(U_1^1, U_2^1, \ldots\) are i.i.d. with the uniform distribution \(U([0,1])\). For every \(m \in \mathbb{N}\) we put

\[
S_m^1 := \sigma_1 \left( \sum_{j=1}^m Y_j^1 \right)^{2/3}, \quad T_m^1 := \frac{U_m}{\max \{S_m^1, 1\}}
\]

and finally we set \(R_1 := (S_j^1, T_j^1)_{j \geq 1}\). From Lemma 4.7 we obtain \(\pi_k (B_{N_1}^1) \xrightarrow{D} \pi_k (R_1)\), for each fixed \(k\), where \(\pi_k : \mathbb{R}^N \to \mathbb{R}^k\) denotes the projection onto the first \(k\) components. Since the class of finite-dimensional sets is a convergence-determining class for \(\mathbb{R}^N\) (see Example 2.4 in [Billingsley 1999]), we get

\[
B_{N_1}^1 \xrightarrow{D} R_1. \tag{21}
\]

With similar definitions, we also conclude that

\[
B_{N_3}^3 \xrightarrow{D} R_3. \tag{22}
\]

Since the points in the first and in the third quadrant are independent, the limit distributions \(R_1\) and \(R_3\) are also independent. We now assume \(k \leq \min \{N_1, N_3\}\), and for \(k \geq 1\) fixed and \(1 \leq i, j \leq k\) we define

\[
p_{i,j,n} := P \left( \left| X_{(i)}^1 - X_{(j)}^3 \right| = \max_{1 \leq l \leq N_1, 1 \leq m \leq N_3} \left| X_{(l)}^1 - X_{(m)}^3 \right| \right).
\]

With

\[
h_{i,j}^- : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad (x, y) \mapsto x_{2i-1} + y_{2j-1} - \frac{a}{4} (x_{2i} - y_{2j})^2
\]

and the same reasoning as in the proof of Proposition 1.8 we get

\[
p_{i,j,n} = P \left( \left| X_{(i)}^1 - X_{(j)}^3 \right| = \max_{1 \leq l \leq N_1, 1 \leq m \leq N_3} \left| X_{(l)}^1 - X_{(m)}^3 \right| \right) \\
= P \left( n^{2/3} \left( 2a - \left| X_{(i)}^1 - X_{(j)}^3 \right| \right) = n^{2/3} \left( 2a - \max_{1 \leq l \leq N_1, 1 \leq m \leq N_3} \left| X_{(l)}^1 - X_{(m)}^3 \right| \right) \right) \\
= P \left( n^{2/3} \left( 2a - \left| X_{(i)}^1 - X_{(j)}^3 \right| \right) = \min_{1 \leq l \leq N_1, 1 \leq m \leq N_3} \left\{ n^{2/3} \left( 2a - \left| X_{(l)}^1 - X_{(m)}^3 \right| \right) \right\} \right) \\
= P \left( h_{i,j}^- (B_{N_1}^1, B_{N_3}^3) + o_{i,j} = \min_{1 \leq l \leq N_1, 1 \leq m \leq N_3} \left\{ h_{l,m}^- (B_{N_1}^1, B_{N_3}^3) + o_{l,m} \right\} \right),
\]

with \(o_{i,j} = o_{ij}(1)\) and \(o_{l,m} = o_{lm}(1)\). These stochastic sequences are written explicitly, since it will be important that \(o_{i,j} = o_{l,m}\) for \((i, j) = (l, m)\). In view of the definition of \(B_{N_1}^1\) and \(B_{N_3}^3\), it is obvious that we can take the minimum over \(l, m \in \mathbb{N}\) instead of \(1 \leq l \leq N_1, 1 \leq m \leq N_3\). Equations (21) and (22) and the continuous mapping theorem yield

\[
p_{i,j,n} = P \left( h_{i,j}^- (B_{N_1}^1, B_{N_3}^3) + o_{i,j} = \min_{l, m \in \mathbb{N}} \left\{ h_{l,m}^- (B_{N_1}^1, B_{N_3}^3) + o_{l,m} \right\} \right) \to p_{i,j},
\]

with

\[
p_{i,j} := P \left( h_{i,j}^- (R_1, R_3) = \min_{l, m \in \mathbb{N}} h_{l,m}^- (R_1, R_3) \right).
\]
Putting \( B_n \) := \( \{ k \leq \min \{ N_1, N_3 \} \} \) for \( k \geq 1 \) fixed, we get \( \mathbb{P}(B_n) \to 1 \) and consequently

\[
\mathbb{P}\left( M_{N,k}^{1,3} \neq M_N^{1,3} \right) = \mathbb{P}\left( M_{N,k}^{1,3} \neq M_N^{1,3} | B_n \right) \cdot \mathbb{P}(B_n) + \mathbb{P}\left( M_{N,k}^{1,3} \neq M_N^{1,3} | B_n^c \right) \cdot \mathbb{P}(B_n^c)
\]

\[
= \mathbb{P}\left( M_{N,k}^{1,3} \neq M_N^{1,3} | B_n \right) (1 + o(1)) + o(1)
\]

\[
= \left( 1 - \mathbb{P}\left( M_{N,k}^{1,3} = M_N^{1,3} | B_n \right) \right) (1 + o(1)) + o(1)
\]

\[
= \left( 1 - \sum_{i,j=1}^{k} p_{i,j,n} \right) (1 + o(1)) + o(1)
\]

\[
\overset{n \to \infty}{\longrightarrow} 1 - \sum_{i,j=1}^{k} p_{i,j}.
\]

Since \( \sum_{i,j=1}^{k} p_{i,j} = 1 \) and \( p_{i,j} \geq 0 \), the probability above converges to 0 as \( k \to \infty \).

**Proposition 4.10.** We have

\[
n^{2/3}(2a - M_N^{1,3}) \overset{D}{\to} \min_{i,j \in \mathbb{N}} \left\{ Z_{1,i}^1 + Z_{1,j}^3 + \frac{a}{4} \left( Z_{2,i}^1 - Z_{2,j}^3 \right)^2 \right\}
\]

with two independent random elements \( (Z_{1,j}^1, Z_{2,j}^1)_{j \geq 1} \sim NA_\infty(\sigma_1, \tau_1) \) and \( (Z_{1,j}^3, Z_{2,j}^3)_{j \geq 1} \sim NA_\infty(\sigma_3, \tau_3) \).

*Proof.* Write \( F_n \) for the distribution function (df) of \( n^{2/3}(2a - M_N^{1,3}) \), and let \( G \) be the df of the limit occurring in (23). Furthermore, \( G_k \) denotes the df of the right-hand side of (23) with \( \min_{k,t \in \mathbb{N}} \) replaced by \( \min_{1 \leq i,j \leq k} \). For \( k \to \infty \) we have

\[
\min_{1 \leq i,j \leq k} \left\{ Z_{1,i}^1 + Z_{1,j}^3 + \frac{a}{4} \left( Z_{2,i}^1 - Z_{2,j}^3 \right)^2 \right\} \overset{D}{\to} \min_{i,j \in \mathbb{N}} \left\{ Z_{1,i}^1 + Z_{1,j}^3 + \frac{a}{4} \left( Z_{2,i}^1 - Z_{2,j}^3 \right)^2 \right\},
\]

and hence \( G_k \overset{D}{\to} G \). Fix \( t > 0 \). On the one hand, Proposition 4.8 and Lemma 4.9 shows

\[
F_n(t) = \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t) = \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t, M_{N,k}^{1,3} = M_N^{1,3}) + \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t, M_{N,k}^{1,3} \neq M_N^{1,3})
\]

\[
\leq \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t | M_{N,k}^{1,3} = M_N^{1,3}) \cdot \mathbb{P}(M_{N,k}^{1,3} = M_N^{1,3}) + \varepsilon_k
\]

\[
\leq \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t | M_{N,k}^{1,3} = M_N^{1,3}) + \varepsilon_k
\]

\[
= \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t) + \varepsilon_k
\]

\[
\overset{n \to \infty}{\longrightarrow} G_k(t) + \varepsilon_k.
\]

On the other hand we get

\[
F_n(t) = \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t, M_{N,k}^{1,3} = M_N^{1,3}) + \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t, M_{N,k}^{1,3} \neq M_N^{1,3})
\]

\[
\geq \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t | M_{N,k}^{1,3} = M_N^{1,3}) \cdot \mathbb{P}(M_{N,k}^{1,3} = M_N^{1,3})
\]

\[
\geq \mathbb{P}(n^{2/3}(2a - M_N^{1,3}) \leq t) \cdot (1 - \varepsilon_k)
\]

\[
\overset{n \to \infty}{\longrightarrow} G_k(t) \cdot (1 - \varepsilon_k).
\]

Since \( G_k(t) \to G(t) \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \) (see Lemma 4.9), the assertion follows.

For reasons of symmetry we get:

**Proposition 4.11.** We have \( n^{2/3}(2a - M_N^{2,4}) \overset{D}{\to} \min_{i,j \in \mathbb{N}} \left\{ Z_{1,i}^2 + Z_{1,j}^4 + \frac{a}{4} \left( Z_{2,i}^2 - Z_{2,j}^4 \right)^2 \right\} \) with two independent random elements \( (Z_{1,j}^2, Z_{2,j}^2)_{j \geq 1} \sim NA_\infty(\sigma_2, \tau_2) \) and \( (Z_{1,j}^4, Z_{2,j}^4)_{j \geq 1} \sim NA_\infty(\sigma_4, \tau_4) \).
The same steps as above and Lemma 4.4 yield:

**Proposition 4.12.** For \((i, j) \in \{(1, 2), (3, 4)\}\) we have

\[
n^{2/3}(2a - M_N^{ij}) \overset{D}{\to} \min_{k \in \mathbb{N}} \left\{ Z_{1,k}^i + Z_{1,k}^j + \frac{a}{4} \left( Z_{2,k}^i + Z_{2,k}^j \right)^2 \right\}
\]

with two independent random elements \((Z_{1,k}^i, Z_{1,k}^j)_{k \geq 1} \sim \mathcal{N}(\sigma_i, \tau_i)\) and \((Z_{2,k}^i, Z_{2,k}^j)_{k \geq 1} \sim \mathcal{N}(\sigma_j, \tau_j)\).

Now we are able to prove Theorem 3.1 for \(\Xi_N\) (and thus, by 4.1 and 4.2, also for \(\Phi_n\)).

**Proof.** (of Theorem 3.1) Let \(A_n := \{\text{diam}(\Xi_N) = \max\{M_{N,1}^{1,2}, M_{N,1}^{1,3}, M_{N,1}^{2,4}, M_{N,1}^{3,4}\}\}\) and fix \(t > 0\). From 4.1 we obtain

\[
\mathbb{P}\left(n^{2/3}(2a - \text{diam}(\Xi_N)) \leq t\right) = \mathbb{P}\left(n^{2/3}(2a - \text{diam}(\Xi_N)) \leq t\middle| A_n\right) \cdot \mathbb{P}(A_n) + \mathbb{P}\left(n^{2/3}(2a - \text{diam}(\Xi_N)) \leq t\middle| A_n^c\right) \cdot \mathbb{P}(A_n^c)
\]

\[
= \mathbb{P}\left(n^{2/3}(2a - \max\{M_{N,1}^{1,2}, M_{N,1}^{1,3}, M_{N,1}^{2,4}, M_{N,1}^{3,4}\}) \leq t\right)(1 + o(1)) + o(1).
\]

Since

\[
n^{2/3}(2a - \max\{M_{N,1}^{1,2}, M_{N,1}^{1,3}, M_{N,1}^{2,4}, M_{N,1}^{3,4}\}) = \min_{(i,j) \in \{(1,2), (1,3), (2,4), (3,4)\}} \left\{ n^{2/3}\left(2a - M_N^{ij}\right) \right\},
\]

the result follows from Propositions 4.10 to 4.12. \(\square\)

### 5. Generalisations and open questions

A very easy generalisation of our setting is given if we allow one boundary function to be equal to 0 close to the corresponding pole. Even the case \(g_i(x_1) \equiv g_i(x_2) = 0\) for \(i \in \{1, 4\}, j \in \{2, 3\}, x_1\) close to \(a\) and \(x_2\) close to \(-a\) is allowed. In these cases, the minimum in (3) has to be taken over fewer random variables \(S_i^{j, i}\), since certain combinations of quadrants do not contribute to the maximum interpoint distance for \(n\) large. Because of the unique major axis, A6 can be weakened to a certain extent without changing the asymptotic behaviour of \(M_n\). We omit the details. Instead of A7, we can demand that there are constants \(p_i > 0, i \in \{1, 2, 3, 4\}\), such that

\[
\sup\left\{|f(z) - p_i| : z \in E_1 \cap U(a_i, \varepsilon)\right\} \xrightarrow{\varepsilon \to 0} 0
\]

with \(a_1 := a_4 := a\) and \(a_2 := a_3 := -a\). Now it is possible that \(p_1 \neq p_4\) or \(p_2 \neq p_3\). In this situation, all results remain unchanged. Another obvious generalisation is to consider \(k \geq 2\) major axes of \(E\) with no common endpoints. If the boundary of \(E\) fulfils all assumptions at every endpoint (after suitable rotations), we obtain a minimum over \(k\) independent random variables \(\min\{S_1^{1,2}, S_1^{1,3}, S_1^{2,4}, S_1^{3,4}\}\) as limit distribution (we omit the details). A completely different setting is given if we weaken A7 by demanding \(f(z) > 0\) for \(z\) close to the poles but \(f(z) \to 0\) as \(z\) tends to \((a, 0)\) (resp. \((-a, 0)\)). Can \(2a - \text{diam}(\Phi_n)\) be scaled appropriately (depending on the speed of the convergence to 0) to obtain a limit distribution of \(M_n\) also in this case? We leave this as an open problem.

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