Spatially localized attacks on interdependent networks: the existence of a finite critical attack size

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Many real world complex systems such as infrastructure, communication and transportation networks are embedded in space, where entities of one system may depend on entities of other systems. These systems are subject to geographically localized failures due to malicious attacks or natural disasters. Here we study the resilience of a system composed of two interdependent spatially embedded networks to localized geographical attacks. We find that if an attack is larger than a finite (zero fraction of the system) critical size, it will spread through the entire system and lead to its complete collapse. If the attack is below the critical size, it will remain localized. In contrast, under random attack a finite fraction of the system needs to be removed to initiate system collapse. We present both numerical simulations and a theoretical approach to analyze and predict the effect of local attacks and the critical attack size. Our results demonstrate the high risk of local attacks on interdependent spatially embedded infrastructures and can be useful for designing more resilient systems.

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Modern critical infrastructures are embedded in space and have extensive interdependencies. Entities in one network (e.g., power generation/distribution, communications, transportation etc.) are dependent upon entities in another and failures in one network can trigger failures in another. It has been shown that these dependencies lead to substantially decreased robustness and even abrupt first order transitions which are absent in isolated networks [1–16]. For spatially embedded interdependent networks under random attack, it was shown that if the maximal dependency link length is above a critical value a new kind of abrupt collapse occurs, characterized by a uniform spreading process [1–17]. However, a purely random failure of a finite fraction of nodes in a very large network can be unrealistic. A more realistic scenario is a failure of a group of neighboring nodes due to a natural disaster like the 2011 Tohoku earthquake and tsunami or due to a malicious attack affecting all networks in a given region (e.g., a nuclear strike) or only certain infrastructures (e.g., an electromagnetic pulse or chemical/biological attack). The resilience of a system of interdependent networks to an attack of this sort, which we call “localized attack,” has not been addressed before.

We show here that there exists a critical damage size with radius $r_c^h$, above which localized geographical damage will spread and destroy the whole system and below which it will remain localized (see Fig. 1). This critical size is determined solely by intensive system quantities and thus, in contrast to random failures, constitutes a zero-fraction of the system in the large system limit, $N \to \infty$.

The resilience of single complex networks to random attacks [18–20] and malicious attacks targeting nodes with special topological properties [21, 22] has been studied on specific single networks [23, 24], but a general theoretical approach of such attacks is currently missing. In particular, the effects of cascading failures due to interactions between networks has not been evaluated with respect to localized attacks even though the positive feedback caused by interdependencies has been shown to have catastrophic consequences such as the 2003 Italian blackout which resulted from a localized failure in a system of interdependent networks [25].

The introduction of a percolation framework for random coupled networks [4, 6] brought attention to many
other properties of coupled networks. Examples include the study of transport [27], epidemic spreading [28], diffusion [29], suppressing cascading loads [11], designing robust coupled networks [30] and dynamical transitions in coupled networks [31].

In this Letter, we study the new phenomenon of localized attacks on interdependent spatially embedded networks. We find that even though the damage, connectivity and dependency links are all highly localized, a small local attack (independent of system size) can spread and destroy the entire network. We show that the system will fail if a geographically local attack is greater than a critical size which is a zero-fraction of the system size. These results have profound implications for the role of network topology in the design of resilient infrastructures.

We model spatially embedded networks by assuming two square lattices $A$ and $B$ with periodic boundary conditions and overlaying them both on the same Cartesian plane. Each node in network $A$ is dependent upon a node in network $B$ (and vice versa) which is chosen at random from all of the nodes within a radius $r$. If a node in $A$ is dependent on a node in $B$, the failure of the node in $B$ will cause the node in $A$ to fail immediately and vice versa. These dependency relationships are taken to be mutual to prevent a single failure from propagating through the entire system [12].

The interdependent networks are then diluted from degree $k = 4$ to a lower average degree. This is accomplished by removing a random fraction $1 - p$ of the nodes from the system, along with the links that are attached to them. This removal triggers a cascade which leaves the average degree $\langle k \rangle$ lower than its value after site dilution of $1 - p$ on a single lattice. Our motivation in reducing the degree from 4 is based on empirical studies of the power grid which have shown a characteristic degree of $\langle k \rangle \approx 3$ [32].

We examine the effects of localized geographical damage of characteristic size $r_h$ for systems with different values of $r$ and $\langle k \rangle$. We model this damage by removing a hole of radius $r_h$ from a random location in network $A$. This triggers a cascade in which the nodes in $B$ which depend on the removed nodes fail, triggering further losses as more nodes in $B$ get cut off from the largest connected component. The percolative damage in $B$ triggers further damage in $A$ due to the dependencies between the networks. This process is continued iteratively until no more nodes fail. At the end of this cascade, the system is categorized as functional or non-functional depending on whether a finite-fraction largest connected component remains or not.

For every system with a given $r$ (maximal dependency link length) and $\langle k \rangle$ we find that there is a critical damage size $r_h^c$ below which the system remains intact and above which the damage propagates throughout the system and destroys it. Furthermore, we discovered three distinct phases in this system according to which the system will remain intact or collapse even with $r_h^c = 0$.

We model the metastable region in the design of resilient infrastructures.

**FIG. 2:** Phase diagram of the problem. (a) Depending on $\langle k \rangle$ and $r$, the system is either stable, unstable or metastable. The circles shown in the metastable region illustrate the sensitivity of the critical attack size that leads to system collapse in the metastable region. For precise critical sizes for the whole metastable region see Fig. 5 below. (b) Demonstration of the theoretical considerations. Near the edge of a hole, the survival probability of a node increases with the distance from the edge. The parameter $p_c$ denotes the distance from the edge of the hole at which the occupation probability is equal to $p_c \approx 0.5927$. In the case illustrated here, the clusters have ample room to fall off and the damage will propagate and destroy the whole system, even though the relative size of the hole is small.

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In the stable region, no matter how large $r_h$ is (as long as it is finite) the damage will remain localized and the system will stay intact. In the unstable region, the system spontaneously collapses even with $r_h = 0$ (no lo-
This is because in this region large holes develop spontaneously due to percolation and overwhelm the system \[17\]. The intermediate region is metastable. Without the removal of a hole or the removal of a hole smaller than \(r_h^c\), the system remains intact. However, if a hole of size \(\geq r_h^c\) is removed, it will begin a cascade that will eventually destroy the entire system. This metastability is analogous to the well known supercooling property of water in which water can be cooled well below its freezing point and remain in the liquid phase until a disturbance of some sort triggers crystallization and it turns to ice \[23\].

In Figs. 3 and 4 we show, based on numerical simulations, how the critical damage size \(r_h^c\) changes with respect to \(r\) and \(\langle k \rangle\). In Fig. 3 we see that for low \(\langle k \rangle\), \(r_h^c\) is very small. For larger \(\langle k \rangle\), we see that \(r_h^c\) increases dramatically at a certain \(\langle k \rangle\) value. The jump occurs at larger \(\langle k \rangle\) values for larger \(r\) values. In Fig. 4 we find that the minimum of \(r_h^c\) is found near the lowest \(r\) of the metastable region, making it most susceptible to local damage.

Since the metastable region spreads over a wide range of realistic values of \(r\) and \(\langle k \rangle\), it is of great interest to understand how this transition takes place and to predict the value of \(r_h^c(r, k)\). To present a theoretical understanding of this phenomenon, we first consider in detail the chain of events triggered by the localized geographical damage. When a hole of \(r_h\) is removed from \(A\), it can directly disable nodes in \(B\) up to a distance \(r\) from its edge. The probability that a given node in \(B\) was dependent on one of the removed nodes is highest at the edge of the hole and decreases linearly until it equals zero at distance \(r\). This creates a lattice concentration gradient in the form of an annulus of width \(r\) surrounding the removed hole, see Fig. 2b. Taking \(\rho\) as the distance from the edge of the hole, the gradient of occupation probability following an attack can be evaluated as

\[
p(\rho, r, r_h, \langle k \rangle) = p_s(\langle k \rangle) \frac{I(r_h, r, \rho)}{\pi r^2}
\]

where \(p_s(\langle k \rangle)\) is the system-wide occupation concentration and \(I(r_h, r, \rho)\) is the standard formula for the area of intersection of two circles of radius \(r\) and \(r_h\), with centers located a distance \(\rho + r_h\) from each other. For a given set of system parameters \((r, r_h, \langle k \rangle)\) we can set \(p = p_c\) in Eq. (1) and solve for \(\rho\). If a solution in the region of interest \((0 < \rho < r)\) exists, it corresponds to a distance \(\rho_c\) at which the lattice concentration will be equal to its critical value. The existence of such a point is a necessary but not sufficient condition for the hole to propagate. Below \(\rho_c\), the lattice does not spontaneously disintegrate but rather forms clusters of characteristic size \(\xi_c(p)\), which diverges at \(\rho_c\) \[34, 35\].

Hence the critical region \(0 < \rho < \rho_c\) needs to be wide enough for clusters of size \(\xi_c(p)\) to form and break away. The value of \(\xi_c(p)\) is determined by the underlying space topology and can thus be measured from a standard lattice using an appropriate estimation for \(p\) in the \(0 < \rho < \rho_c\) region. From Eq. (1), \(p\) is not constant and an exact solution for \(\xi_c\) would require treating the full gradient percolation problem \[36\]. In this work, for simplicity we take \(\bar{p}\) which is the average of \(p\) over the region of interest. Additionally, the removal of the hole causes secondary damage due to dependencies in the annulus and the concentration of the gradient is decreased by a factor of \(g\) which we measure numerically and which varies monotonically from 0.85 to 0.89 as a function of \(r\). We can thereby estimate \(\bar{p} \approx g(r) \int_0^{\rho_c} p(\rho) d\rho\). We evaluate \(\xi_c\) following \[34\] as:

\[
\xi_c^2 = \frac{1}{N_p \sum_{(i,j)} |r_i - r_j|^2}
\]
where \((i, j)\) refers to nodes \(i\) and \(j\) which are in the same connected component and \(N_p\) is the total number of such pairs of nodes. This leads to a self-consistent condition for hole propagation:

\[
    \xi_c < \rho_c
\]

both sides are functions of \(r, r_h\) and \(p_a\). Using these considerations, we can predict \(r_h^\ast\) for every set of \((k, r)\) parameters with close agreement to the numerical results, see Fig. 5.

Everything about the scenario described above is local: nodes in \(A\) and \(B\) can have dependency links only up to length \(r\), the connectivity links in \(A\) and \(B\) are tied to an underlying lattice structure with characteristic length of one and the attack is restricted to a hole of radius \(r_h\). However, for a wide range of system parameters, this leads to a catastrophic cascade which destroys the entire system. In fact, the localization of dependency opens the door for the spreading phenomenon that characterizes such a collapse. When a hole of radius \(r_h\) is removed from \(A\), the nodes in \(B\) that depended on them must be within a distance \(r\) of the hole. Thus the secondary damage is highly concentrated around the edge of the hole, leading to the creation of a damage front which propagates outwards from step to step. If \(r \to \infty\) or \(r \to 0\), this weakness would not exist because the secondary damage would be spread out uniformly or remain in place, respectively.

In Summary, we find that paradoxically, the highly localized topology of embedded interdependent networks enables relatively small attacks to cause global damage. Given the low average degree of the power grid [32, 37] and its evidence of self-organizing criticality [38, 39], we anticipate high susceptibility even to relatively small local attacks.

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