Higher Order Power Corrections in Inclusive $B$ Decays

TH. MANNEL, S. TURCZYK AND N. URALTSEV

Theoretische Physik 1, Fachbereich Physik, Universität Siegen. D-57068 Siegen, Germany

also Department of Physics, University of Notre Dame, Notre Dame, IN 46556 USA

We discuss order $1/m_b^4$ and $1/m_b^5$ corrections in inclusive semileptonic decay of a $B$ meson. We identify relevant hadronic matrix elements of dimension seven and eight and estimate them using the ground-state saturation approximation. Within this approach the effects on the integrated rate and on kinematic moments are estimated. The overall relative shift in $V_{cb}$ turns out about $+0.4\%$ as applied to the existing fits. Similar estimates are presented for $B \to X_s + \gamma$ decays.

* On leave of absence from Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia
1. Introduction

The operator product expansion (OPE) for heavy hadron decays has become the standard tool for the evaluation of differential decay rates. While the quality of the OPE may vary in different corners of the whole phase space, it has been established that for sufficiently inclusive observables the OPE yields an expansion in $\Lambda_{QCD}/m_Q$ ($Q$ is beauty or charm) which, in case of beauty weak decays, converges reasonably well, at least judging from the low orders calculated up to now. However, it has been argued that the OPE results in an asymptotic series with limitations paralleling those for the perturbative series. In particular, this implies a deteriorating behavior at sufficiently high orders; therefore, it is well motivated to investigate the higher orders of the power expansion.

The OPE for inclusive decays yields an expansion in $\Lambda_{QCD}/m_b$ with the coefficients which are themselves series in $\alpha_s$. Consequently we end up with a double expansion in the two parameters. Currently the leading power term, the partonic rate is known to order $\alpha_s^2$, including the differential distributions relevant for the calculation of moments \[1\]. The coefficients of even the first nonperturbative corrections are not completely known to order $\alpha_s$; the one for the chromomagnetic correction is only known at tree level, while the coefficient of the kinetic operator \[2\] can be related to the leading power term by considering the OPE for a moving $B$ meson\[3\]. Higher order terms in the OPE are only known at tree level where they can be

---

\[1\] This relation is sometimes referred to as a reparametrization invariance.
directly constructed by the method discussed in [3]. Certain enhanced terms to order $1/m_Q^4$ were calculated for the total decay rate to order $\mathcal{O}(\alpha_s)$ [4].

The present paper focuses on the higher orders in the $1/m_b$ expansion for $b \to c$ inclusive semileptonic decay at tree level. As expected, we observe a proliferation of nonperturbative expectation values starting from $1/m_b^4$. Nevertheless, we identify the set of hadronic parameters through order $1/m_b^5$, all those that appear at tree level.

Since at higher orders the number of nonperturbative parameters becomes too large, a straightforward fit to the data to extract them is not possible. To get around this obstacle, we suggest a simple way to estimate the required expectation values. This “ground state saturation assumption” evaluates higher-dimension matrix elements in terms of $\mu^2\pi$, $\mu^2 G$, $\rho^3 D$, and $\rho^3 LS$. Using the reasonably well known values for the latter from the data we are in the position to numerically assess the higher-order parameters in the power expansion.

In the next section we describe a general method to calculate the higher orders in the $1/m_Q$ expansion at tree level. In Sect. 2.1 we apply this to order $1/m_b^4$, including the set of the nonperturbative expectation values, and in section 2.2 we do the same for the terms at order $1/m_b^5$. In Sect. 3 we set up the factorization approximation for the heavy quark matrix elements and apply it to evaluate the $B$-meson expectation values of dimension seven and eight operators appearing at $1/m_b^4$ and $1/m_b^5$. Using these results we perform a pilot numerical study of the corrections at order $1/m_b^4$ and $1/m_b^5$ to the hadronic and leptonic moments and to $|V_{cb}|$ in Sect. 4. The impact of higher-order corrections turns out sizable in general; in particular, the extracted value of the Darwin term may shift significantly. Nevertheless, the value of $|V_{cb}|$ remains stable. Sect. 5 reports a brief summary of the similar analysis for the $B \to X_s + \gamma$ decays, including the analytic expressions for the corrections to the rate and the photon energy moments. Here the impact of $1/m_b^4$ and $1/m_b^5$ terms appears to be insignificant.

Sect. 6 summarizes our results and presents the main conclusions. Certain aspects of the derivation related to the factorization approximation are relegated to Appendices.

### 2. Calculational Scheme for $1/m_b^n$ at Tree Level

In this section we give a brief summary of our method to calculate power corrections. It follows the calculational scheme used in Ref. [3]. The starting point for the calculation is the differential decay rate

$$d\Gamma = \frac{G_F^2 |V_{cb}|^2}{2} W_{\mu\nu} dL^{\mu\nu},$$

where the leptonic tensor $dL^{\mu\nu}$ contains all information on the leptons including their phase space element. The nontrivial nonperturbative dynamics is encoded in the hadronic tensor $W_{\mu\nu}$ related, by the optical theorem, to the discontinuity of the time-ordered product of two weak currents across the cut. One then starts with a correlator of two hadronic currents

$$T_{\mu\nu}(q) = \frac{1}{2 M_B} \int d^4 x \ e^{-iqx} \langle B(P) | iT \bar{b}(x) \Gamma_{\mu}^c(x) \tau(0) \Gamma_{\nu} \ b(0) | B(P) \rangle .$$

Here

$$\Gamma_{\mu} = \gamma_\mu (1 - \gamma_5)$$

is the left-handed weak current and $q$ the momentum transfer to the leptons while $P$ denotes the $B$ meson momentum, and

$$W_{\mu\nu} = 2 \text{Im} T_{\mu\nu} .$$
Two-index amplitude $T_{\mu\nu}(q)$ or its discontinuity $W_{\mu\nu}(q)$ can be decomposed into five tensor structures with the coefficients given by scalar functions of $q_0 = v \cdot q$ and $q^2$:

$$W_{\mu\nu}(q) = -w_1(q_0, q^2)g_{\mu\nu} + w_2(q_0, q^2)v_\mu v_\nu - iw_3(q_0, q^2)\epsilon_{\mu\nu\rho\lambda}v^\rho q^\lambda$$

(5)

$$+ w_4(q_0, q^2)q_\mu q_\nu + w_5(q_0, q^2)(q_\mu v_\nu + v_\mu q_\nu),$$

where $v = P/M_B$ is the four-velocity of the decaying $B$ meson. The structure functions $w_4$ and $w_5$ do not affect decay amplitudes with massless leptons, and the fully differential semileptonic rate depending on the charge lepton energy $E_\ell$ is given by

$$\frac{d^3\Gamma}{dE_\ell dq^2 dq_0} = \frac{G_F^2 |V_{cb}|^2}{32\pi^4} \vartheta(q_0 - E_\ell) \vartheta(q_2) \times \{2q^2 w_1 + [4E_\ell(q_0 - E_\ell) - q^2] w_2 + 2q^2 (2E_\ell - q_0) w_3\}.$$ (6)

The tree-level expansion in $1/m_b$ is most easily set up by looking at the Feynman diagram in Fig. 1. The double line denotes the Green function of the charm quark propagating in the background field of soft gluons in the $B$ meson. We ‘rephase’ the $b$ fields according to

$$b(x) \rightarrow e^{-im_b(x)D}b(x);$$

this makes the actual $b$-quark momentum operator to become

$$p_b = m_b v + iD.$$ (7)

The phase factor from $\bar{b}(x)$ in Eq. (2) combines with the background-field $c$-quark propagator $1/(i\slashed{D} - m_c)$ to yield charm propagator

$$S_{\text{BGF}} = \frac{1}{\slashed{p} + m_c}, \quad p_\mu = m_b v_\mu + q_\mu$$

(8)

to be inserted between the nonrelativistic ‘rephased’ $b$ fields; this corresponds to the $b$ quark momentum Eq. (7).

Figure 1: Tree level Feynman diagram for the hadronic tensor in inclusive semileptonic $B$ decays

For semileptonic processes at tree level we only need to multiply by the appropriate Dirac matrices for the left handed current. A calculation of the OPE series to order $1/m_b^n$ requires to expand this expression to $n^{th}$ order in the covariant derivative $(iD)$ according to

$$S_{\text{BGF}} = \frac{1}{\slashed{p} - m_c} + \frac{1}{\slashed{p} - m_c}(-i\slashed{D}) \frac{1}{\slashed{p} - m_c} + \frac{1}{\slashed{p} - m_c}(-i\slashed{D}) \frac{1}{\slashed{p} - m_c}(-i\slashed{D}) \frac{1}{\slashed{p} - m_c} + \cdots$$ (9)

The covariant derivatives playing the role of the ‘residual’ momentum, do not commute in general; the above expansion takes care of their ordering. Calculating structure functions we take discontinuity of $S_{\text{BGF}}$; successive terms in the expansion yield higher derivatives of $\delta(m_b^2 - m_c^2 + q_2 - 2m_bq_0)$, together with powers of the spinor factor $m_b \not\! p - \not\! p + m_c$ ($m_c$ term can be
consistently dropped for purely left-handed weak vertices) producing polynomials in $q$. The cumbersome, for high-order terms, general product of many Dirac matrices is simplified using a computer routine.

Finally, the thus obtained operator expansion should be supplemented by the $B$-meson expectation values of the general form

$$\bar{b}_\alpha(iD_{\mu_1})...(iD_{\mu_n})b_\beta \quad (10)$$

where the spinor indices of the $b$ fields are shown explicitly. The field is the full four-component QCD field yet redefined by a phase factor. Note that performing the OPE this way yields only local operators; however, their expectation values still contain a nontrivial mass dependence from the small ‘lower’ bispinor components and from the explicit subleading terms in the equations of motions for $b(x)$ that will be discussed in the following, as well as from the fact that the $B$ meson states are the eigenstate of the finite-mass Hamiltonian.

Selecting the basis for independent expectation values to different orders in $1/m_b$ may be accomplished in different ways by partially reshuffling the power-suppressed pieces into the definition of the lower-dimension expectation values. The most familiar example through order $\mu^3_{\text{had}}$ is the chromomagnetic expectation value $\mu^2_G$, which is usually defined as $\langle \bar{b}(\bar{\sigma}B)b \rangle$, yet often the full Lorentz-scalar operator $\bar{b}\frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}b$ is used instead; these differ explicitly by $(\rho\bar{D}+\rho\bar{L})/2m_b$. One common strategy assumes that the lower components of the spinors are explicitly excluded (while passing to the genuine nonrelativistic $b$ fields, i.e. to the upper components corrected by power terms, still may or may not be performed). In this study we follow an alternative scheme used in Ref. [3] which may not necessary parallel such a separation; it generally keeps all four components of the $b$ field and only reduces the number of operators applying equations of motion. While in this scheme the generic operators are not expressed through the conventional nonrelativistic $b$ fields, it has an advantage of being more compact and more native to the tree-level local OPE calculations.

The enumeration and evaluation of these matrix elements is conveniently performed in a recursive fashion, starting with the operators of the highest dimension, i.e. those with the maximal number of covariant derivatives. There one can neglect all $1/m_b$-suppressed pieces and consider them in the static limit; to facilitate passing on to the next step we, however, still use the same full QCD fields. In the static limit one has two possible spinor structures of opposite parity [6]:

$$\langle B|\bar{b}_\alpha(iD_{\mu_1})...(iD_{\mu_n})b_\beta|B \rangle = \frac{1+\gamma^5}{2} A_{\mu_1\mu_2...\mu_n} + (s_\lambda)_{\beta\alpha} D^{\lambda}_{\mu_1\mu_2...\mu_n} \quad (11)$$

where $\frac{1+\gamma^5}{2}$ and $s_\lambda = \frac{1+\gamma^5}{2} \gamma^\lambda \gamma^5 \frac{1+\gamma^5}{2}$ are the generalizations, to the case of arbitrary velocity frame, of the unit and Pauli matrices projected onto the ‘upper’ components. Temporarily introducing an arbitrary restframe velocity vector $v_\mu$, the standard tensor decomposition technique expresses the tensor structures $A$ and $B$ through a minimal set of fundamental expectation values at a given order. This can be done by contracting the indices in all possible ways. In the following, the matrix elements of this set are referred to as basic parameters for a certain order in $1/m_b$. We emphasize that such relations are purely algebraic as long as a general parameterization like in Eq. (11) is adopted; they do not imply any assumption about heavy quark limit or subleading correction, or about dynamics – they are just the result of rotational symmetry of the $B$ expectation values. Consequently, the same way to relate the corresponding tensors to the basic parameters can be used for the expectation values of the
operators of any dimension. However, the basic parameters encountered at lower dimension generally include higher-order pieces which depend on the choice of the basic set.

For the operators of dimensions lower than the highest, therefore, we have to include in consideration originally all possible 16 bilinear bispinor structures including power-suppressed ones when generalizing Eq. (11). Likewise, we use equations of motions for the \( b \) field to relate certain structures to higher-dimension expectation values; we do this at the second step, at the level of invariants thus reducing directly the set of basic parameters. In particular, we successively replace the rightmost and leftmost time derivatives \( (ivD) \) acting on \( b (\bar{b}) \) according to the QCD equation of motion (see Ref. \[7\], Sect. 2); the same is done to reduce certain bispinor structures:

\[
(ivD)b = -\frac{(iD)^2 + \frac{1}{2} \sigma_{\mu\nu} G^{\mu\nu}}{2m_b} b, \quad \frac{1 - \frac{1}{2}}{2} b = \frac{iD}{2m_b} b, \quad \gamma_\mu \gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu}
\]

where we have also specified the convention used for \( \sigma_{\mu\nu} \). Using various identities for the Dirac matrices combined with the second of the above equations one can reduce the number of spin structures to one spin-singlet and one spin-triplet, cf. Eq. (11); for the latter we choose the dual spin matrix \(-i\sigma_{\mu\nu}\). The spin reduction is less automated than eliminating time derivatives of the \( b \) fields, nevertheless to the order we are interested in, through operators with five derivatives, this is straightforward. The reduction algorithm includes effectively eliminating the terms which would mix lower and upper components of the spinors; then the leading order of a given expectation value is directly given by the number of covariant derivatives in the operator. Spin-triplet operators contain four-dimensional \(-i\sigma_{\mu\nu}\) (reduced to three-dimensional \( \sigma_{mn} \) to the leading order), and spin-singlet ones are free from any Dirac structures. The added convenience of such bases is that the corrections from the presence of lower components are automatically \(1/m_b^2\) suppressed. However, we bear in mind that the classification itself upon spin properties is valid only to the leading order for a given operator. Upon completing the reduction we end up with the expression in terms of basic set at each order in \( \Lambda_{\text{QCD}} \); the expectation values at a given order generally add \(1/m_b\)-suppressed terms proportional to the basic expectation values of higher orders.

For the meson expectation values Lorentz covariance of the operators proper is undermined by presence of the external \( b \)-states selecting the preferred frame; only rotation invariance remains. Therefore, the time and the spatial derivatives (and indices in general) are physically distinct and should be treated differently. Nevertheless, we write the operators in a superficially Lorentz-invariant form employing the \( b \)-hadron velocity, as a remnant of the way the relations to the basic set have been derived. We denote the projector onto the spatial components by

\[
\Pi_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu.
\]

Then the ‘usual’ basic parameters, those appearing through order \( \Lambda_{\text{QCD}}^3 \) can be written as

\[
2M_B \mu_\pi^2 = -(B|\bar{b}_v iD_\rho iD_\sigma b_v|B) \Pi^{\rho\sigma}, \quad 2M_B \mu_G^2 = \frac{1}{2} (B|\bar{b}_v [iD_\rho, iD_\lambda] (-i\sigma_{\alpha\beta}) b_v|B) \Pi^{\rho\alpha} \Pi^{\beta\lambda}
\]

\(^2\)Note that this convention differs from the one adopted in a number of earlier papers on the subject.
index contractions of a matrix element of the form which can form the set of basic parameters to this order. The starting point are all possible Passing to higher-order power correction requires first to find independent expectation values. The difference for \( \mu_2^2 \) and \( \mu_2^3 \) is only \( 1/m_b^2 \).

Performing the calculation along the lines described above and contracting the indices obtained in the OPE series Eq. (9) with the tensor structures derived for the expectation values generalizing Eq. (11) we obtain the formal 1/\( m_b \) expansion of the \( B \)-decay structure functions, in this case for semileptonic decays. Using the basic parameters in Eq. (13) we readily reproduce the known results for the structure functions, lepton spectrum and the total decay rate through order 1/\( m_b^3 \) \[ , \Pi \].

2.1. Expectation values at 1/\( m_b^4 \)

Passing to higher-order power correction requires first to find independent expectation values which can form the set of basic parameters to this order. The starting point are all possible index contractions of a matrix element of the form

\[
⟨B| b_v \ i D_\rho i D_σ i D_λ i D_δ \Gamma b_v | B⟩ ,
\]

where \( \Gamma \) is either the unit matrix or \(-iσ_{\mu\nu}\). However, not all of them can appear independently; \( T \)-invariance of the correlator \[ \Pi \] ensures that only Hermitian operators are present. This also automatically yields a real differential rate, since the coefficient functions are real.

Taking this into account, we end up with nine dimension-7 operators, four spin-singlet and five spin-triplet:

\[
2M_B m_1 = ⟨B| b_v \ i D_\rho i D_σ i D_λ i D_δ \Gamma b_v | B⟩ \frac{1}{3} \left( Π^{σσ} Π^{λδ} + Π^{σλ} Π^{δσ} + Π^{δλ} Π^{σδ} \right)
\]

\[
2M_B m_2 = ⟨B| b_v \ [iD_ρ, iD_σ] [iD_λ, iD_δ] b_v | B⟩ \ Π^{σλ} Π^{δσ} Π^{λδ}
\]

\[
2M_B m_3 = ⟨B| b_v \ [iD_ρ, iD_σ] [iD_λ, iD_δ] b_v | B⟩ \ Π^{σδ} Π^{λδ}
\]

\[
2M_B m_4 = ⟨B| b_v \ \{iD_ρ, iD_σ\} [iD_λ, iD_δ] (−iσ_{αβ}) b_v | B⟩ Π^{σρ} Π^{δλ} Π^{λδ}
\]

\[
2M_B m_5 = ⟨B| b_v \ \{iD_ρ, iD_σ\} [iD_λ, iD_δ] (−iσ_{αβ}) b_v | B⟩ Π^{σλ} Π^{δα} Π^{δβ}
\]

\[
2M_B m_6 = ⟨B| b_v \ \{iD_ρ, iD_σ\} [iD_λ, iD_δ] (−iσ_{αβ}) b_v | B⟩ Π^{σρ} Π^{λα} Π^{δβ}
\]

\[
2M_B m_7 = ⟨B| b_v \ [iD_ρ, iD_σ] [iD_λ, iD_δ] (−iσ_{αβ}) b_v | B⟩ Π^{δρ} Π^{λα} Π^{δβ} .
\]

The commutator of covariant derivatives equals field strength, and all operators above but the first one include as a factor an explicit gluon field operator. Therefore, their expectation values without at least one gluon would vanish, whether the hadronic state has zero momentum or not. Boosting the \( b \)-hadron would not generate per se such expectation values. Only
$m_1$ can emerge from the boost; it can be identified with average $(\bar{p}^2)^2 \equiv \bar{p}^4$ in the nonrelativistic picture. However, a subtlety must be born in mind when pursuing such an interpretation, for $\mu_\pi^2$ as defined in Eqs. (13) contains, along with $(\bar{p}^2)$, an extra piece $\propto (\bar{p}^4)/m_b^2$ which has to be included when checking the effect of the boost through the OPE. This $1/m_b^2$ piece comes both from the presence of the lower components of the $b$ fields in $\mu_\pi^2$ in Eqs. (13) and from the fact that the upper components proper should be corrected at order $1/m_b^2$ by the additional operator $1 + (\bar{p}^2)^2 = 1 + (\bar{p}^4)/8m_b^2$ to represent the true nonrelativistic fields $\mathbf{7}$.

The corresponding correction most simply can be recovered using two identities

\[
\frac{1}{2M_B} \langle B|\bar{b}\gamma_0 b|B\rangle = 1, \quad \frac{1}{2M_B} \langle B|\bar{b}\bar{b}|B\rangle = 1 - \frac{1}{2m_b^2}(\mu_\pi^2 - \mu_e^2) + \ldots; \tag{16}
\]

this yields

\[
\mu_\pi^2 \iff (\bar{p}^2) = \frac{1}{2m_b^2}(\bar{p}^4) + O(\frac{G_{\alpha\beta}}{m_b^2}, \frac{p^6}{m_b^4}). \tag{17}
\]

The other operators can also be interpreted in a simple manner. The expectation value $m_2$ is proportional to $(\bar{b}\tilde{E}\vec{E}b)$; likewise is $m_3$ proportional to $(\bar{b}\tilde{B}\vec{B}b)$, where $\vec{E}$ and $\vec{B}$ are the chromoelectric and chromomagnetic fields. The parameter $m_4$ is related to $(\bar{p}\cdot \vec{B})$ or, by the gluon field equations of motion, to a combination of $m_2$ and $\langle g_2^2\bar{b}(\bar{p}\vec{E})b \rangle$ where $\vec{E}$ is the non-Abelian version of the rotator and $J_\mu$ the color octet flavor-singlet vector current of light quarks.

The spin-dependent operators have similar interpretation: $m_5 \propto \langle \bar{b}\vec{E}\cdot \vec{E}\bar{b} \rangle$ and $m_6 \propto \langle \bar{b}\vec{B}\times \vec{B}\bar{b} \rangle$ (different components of the chromoelectric and chromomagnetic fields do not commute) while $m_7 \propto \langle (\bar{p}^2)(\vec{s}\cdot \vec{B}) - (\vec{s}\cdot \bar{p})(\vec{p}\cdot \vec{B}) \rangle$, $m_8 \propto \langle (\bar{p}^2)(\vec{s}\cdot \vec{B}) \rangle$ and $m_9$ is a combination of $\langle \Delta(\vec{s}\cdot \vec{B}) \rangle$ and $\langle \vec{s}\cdot \vec{B}\bar{b} \rangle$.

### 2.2. Expectation values at $1/m_b^5$

The number of possible operators with five covariant derivatives and of their expectation values further increases. Requiring T-invariance we get 18 new parameters in total, 7 of which are spin singlet and 11 are spin triplet:

\[
\begin{align*}
2MB_{r1} &= \langle B|\bar{b}iD_\mu (iv\cdot D)^3 iD^\rho b|B\rangle \\
2MB_{r2} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD^\mu iD_\rho iD^\sigma b|B\rangle \\
2MB_{r3} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu iD^\sigma b|B\rangle \\
2MB_{r4} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu iD^\sigma b|B\rangle \\
2MB_{r5} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu iD^\sigma b|B\rangle \\
2MB_{r6} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu iD^\sigma b|B\rangle \\
2MB_{r7} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu iD^\sigma b|B\rangle \\
2MB_{r8} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho (\sigma_{\mu\nu}) b|B\rangle \\
2MB_{r9} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu (\sigma_{\mu\nu}) b|B\rangle \\
2MB_{r10} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD^\mu iD_\rho (-\sigma_{\mu\nu}) b|B\rangle \\
2MB_{r11} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu (-\sigma_{\mu\nu}) b|B\rangle \\
2MB_{r12} &= \langle B|\bar{b}iD_\mu (iv\cdot D) iD_\rho iD^\mu (-\sigma_{\mu\nu}) b|B\rangle \\
\end{align*}
\]
Note that at this dimension we have an odd number of derivatives, therefore at least one time derivative \((ivD)\) must be present. This derivative can be commuted through the rest until it acts on one of the \(b\) quark operators, which yields zero to this order. The remaining commutators can be expressed in terms of chromoelectric field, so all these matrix elements are necessarily proportional to the gluon field. Unlike the case of an even number of derivatives, none of such expectation values can be generated by a \(B\) boost alone.

### 2.3. OPE to orders \(1/m_b^4\) and \(1/m_b^5\)

Using the above set of expectation values and the expansion procedure described earlier in this section, we derive all five semileptonic decay structure functions of \(B\) mesons through order \(1/m_b^5\). They contain \(\delta(m_b^2-m_q^2+q^2-2m_bq_0)\) and its derivatives from first through fifth. In view of extremely large number of lengthy terms due to polynomial coefficients in \(q_0\) and \(q^2\), to different multiple derivatives of the \(\delta\)-function for each of the expectation values and to algebraic expressions resulting from reduction of various Lorentz structures in the generic matrix elements to the basic parameters, we do not attempt to present them explicitly. They are handled in Mathematica, with the typical size of the file with their definition around 0.4 MB.

The actual inclusive observables we are interested in involve kinematic integrations which eliminate the \(\delta\)-function and its derivatives and yield literal \(1/m\) expansion for the effects associated with higher-dimension expectation values. We obtain the particular observable using Eq. (6) and include the cut on lepton energy if imposed. This is likewise done in Mathematica; for the moments with a cut we evaluate the expressions numerically assuming certain values for the \(D=8\) expectation values. Where hadronic mass moments are considered, we use the expression for \(M_X^2\)

\[
M_X^2 = (m_b^2 - m_c^2 + q^2 - 2m_bq_0) + 2(M_B - m_b)(m_b - q_0) + (M_B - m_b)^2 = m_x^2 + 2(M_B - m_b)e_x + (M_B - m_b)^2;
\]

the corresponding factor then multiplies the right hand side of Eq. (6). For integer \(M_X^2\) moments the factor is a polynomial in \(M_B - m_b\) with the coefficients that are polynomials in \(q_0\) and \(q^2\). The quantities \(m_x^2\) and \(e_x\) have a meaning of hadron invariant mass squared and hadron energy, respectively, at the partonic level in the decay of an isolated \(b\) quark.

### 3. Estimate of the hadronic expectation values

As we have seen in the previous section, the independent hadronic matrix elements start to proliferate at order \(1/m_b^4\). This would make their complete extraction from data difficult if not impossible. However, not all the corrections may be expected equally important, and...
neglecting numerically insignificant effects reduces the number of new parameters, possibly even to a manageable level. Yet this requires estimating the scale of the emerging expectation values well beyond naive dimensional guessing.

In the present paper we are concerned with the heavy quark expectation values at tree level, appearing through dimension 8, and in the following we present a method for estimating them based on the saturation by intermediate states.

Our approach starts with representing the matrix elements of the composite operators of interest as a sum over the full set of zero-recoil intermediate single-heavy–quark hadronic states. While this is still exact, the approximation scheme we shall employ is to retain in the sum the contribution of only the multiplet of the allowed lowest-energy intermediate states.

The quality of this approximation depends on the degree of saturation of the sum by the lowest states. The related accuracy in certain cases can be estimated and the whole approximation refined.

3.1. Intermediate state representation for the relevant matrix elements

Our goal are forward matrix elements of the form

\[ \langle B | \bar{b} i D_{\mu_1} i D_{\mu_2} \cdots i D_{\mu_n} \Gamma b(0) | B \rangle, \]  

(20)

where \( \Gamma \) denotes an arbitrary Dirac matrix. The representation is obtained by splitting the full chain \( i D_{\mu_1} i D_{\mu_2} \cdots i D_{\mu_n} \) into \( A = i D_{\mu_1} i D_{\mu_2} \cdots i D_{\mu_k} \) and \( C = i D_{\mu_{k+1}} i D_{\mu_{k+2}} \cdots i D_{\mu_n} \).

The intermediate states representation says that

\[ \langle B | \bar{b} A C \Gamma b(0) | B \rangle = \frac{1}{2M_B} \sum_n \langle B | \bar{b} A b(0) | n \rangle \cdot \langle n | \bar{b} C \Gamma b(0) | B \rangle, \]  

(21)

where we have assumed the \( B \) mesons to be static and at rest, \( |B\rangle = |B(p = (M_B, \vec{0}))\rangle \), and \( |n\rangle \) are the single-\( b \) hadronic states with vanishing spatial momentum.

We present an OPE-based proof of Eq. (21); a more conventional effective–field-theory derivation is given in Appendix A. In either way we introduce a fictitious heavy quark \( Q \) which will be treated as static (i.e., \( m_Q \to \infty \) and normalization point is much lower than \( m_Q \)), and in the OPE approach consider a correlator of the form

\[ T_{AC}(q_0) = \int d^4x \, e^{iq_0 x_0} \langle B | iT \{ \bar{b} A Q(x) \bar{Q} C b(0) \} | B \rangle; \]  

(22)

note that spatial momentum transfer \( \vec{q} \) has been explicitly set to vanish.

We shall use the static limit for both \( b \) and \( Q \), and hence introduce the ‘rephased’ fields \( \bar{Q}(x) = e^{im_Q q_0 x_0} Q(x) \) and likewise for \( b \), and omit tilde in them in what follows. The form of the resulting exponent suggests to define \( \omega = q_0 - m_b + m_Q \) as the natural variable for \( T_{AC} \), and \( \frac{1}{2M_B} T_{AC}(\omega) \) is assumed to have a heavy quark limit.

With large \( m_Q \) we can perform the OPE for \( T_{AC}(\omega) \) at \( |\omega| \gg \Lambda_{QCD} \) still assuming that \( |\omega| \ll m_Q \) and neglecting thereby all powers of \( 1/m_Q \). In this case the propagator of \( Q \) becomes static,

\[ iT\{Q(x)\bar{Q}(0)\} = \frac{1 + \gamma_0}{2} \delta^3(x) \theta(x_0) P \exp \left( i \int_0^{x_0} A_0 dx_0 \right), \]  

(23)
and yields
\[ T_{AC}(\omega) = \langle B| \bar{b} A \frac{1}{-\omega - \pi_0 - i0} C \frac{1+\gamma_0}{2} \Gamma \bar{b}|B \rangle, \] (24)
where \( \pi_0 = i D_0 \) is the time component of the covariant derivative. This representation allows immediate expansion of \( T_{AC}(\omega) \) in a series in \( 1/\omega \) at large \( |\omega| \):
\[ T_{AC}(\omega) = - \sum_{k=0}^{\infty} \langle B| \bar{b} A \frac{(-\pi_0)^k}{\omega^{k+1}} C \frac{1+\gamma_0}{2} \Gamma \bar{b}|B \rangle. \] (25)

Alternatively the scattering amplitude can be written through its dispersion relation
\[ T_{AC}(\omega) = \frac{1}{2\pi i} \int_0^{\infty} d\epsilon \frac{1}{\epsilon - \omega - i0} \text{disc} T_{AC}(\epsilon), \] (26)
where we have used the fact that in the static theory the scattering amplitude has only one, ‘physical’ cut corresponding to positive \( \omega \). The discontinuity is given by
\[ i \int d^4x \, e^{i\epsilon x_0} \langle B| \bar{b}AQ(0)|n\rangle \langle n|\bar{QC}Γb(0)|B\rangle, \] (27)
and amounts to
\[ \text{disc} T_{AC}(\epsilon) = \sum_{n\mathcal{Q}} i \int d^4x \, e^{-i\vec{p}_n \cdot \vec{x}} \, e^{i(\epsilon - E_n) x_0} \langle B|\bar{b}AQ(0)|n\rangle \langle n|\bar{QC}Γb(0)|B\rangle, \] (28)
where the sum runs over the complete set of the intermediate states \( |n\mathcal{Q}\rangle \); their overall spatial momentum is denoted by \( \vec{p}_n \) and energy by \( E_n \).

The spatial integration over \( d^3x \) and integration over time \( dx_0 \) in Eq. (28) yield \((2\pi)^3 \delta^3(\vec{p}_n)\) and \( 2\pi \delta(E_n - \epsilon) \), respectively. Therefore only the states with vanishing spatial momentum are projected out, and we denote them as \( |n\rangle \):
\[ \text{disc} T_{AC}(\epsilon) = \sum_{n\mathcal{Q}} 2\pi i \delta(\epsilon - E_n) \langle B|\bar{b}AQ(0)|n\rangle \langle n|\bar{QC}Γb(0)|B\rangle. \] (29)

Inserting the optical theorem relation (29) into the dispersion integral (26) we get
\[ T_{AC}(\omega) = \sum_n \frac{\langle B|\bar{b}AQ(0)|n\rangle \langle n|\bar{QC}Γb(0)|B\rangle}{E_n - \omega + i0}, \] (30)
and the large-\( \omega \) expansion takes the form
\[ T_{AC}(\omega) = - \sum_{k=0}^{\infty} \frac{1}{\omega^{k+1}} \sum_n E_n^k \langle B|\bar{b}AQ(0)|n\rangle \langle n|\bar{QC}Γb(0)|B\rangle. \] (31)

Equating the leading terms in \( 1/\omega \) of \( T_{AC}(\omega) \) in Eq. (25) and in Eq. (31) we arrive at the relation
\[ \langle B|\bar{b} A \frac{1+\gamma_0}{2} \Gamma b(0)|B\rangle = \sum_n \langle B|\bar{b} A Q(0)|n\rangle \cdot \langle n|\bar{Q} C \Gamma b(0)|B\rangle \] (32)
which is the intermediate state representation (21). Note that the projector \((1+\gamma_0)/2\) in the left hand side can be omitted since the \( \bar{b} \) field satisfies \( \bar{b} = \bar{b}(1+\gamma_0)/2 \) in the static limit.
Considering higher values of \( k \) in Eqs. (25) and (31) which describe the subleading in \( 1/\omega \) terms in the asymptotics of \( T_{AC}(\omega) \), we readily generalize the saturation relation (21):

\[
\langle B| \bar{b} A \pi_0^k C \frac{1+\gamma_5}{2} \Gamma \frac{1+\gamma_5}{2} b(0)|B \rangle = \sum_n (E_B - E_n)^k \langle B| \bar{b} A Q(0)|n \rangle \cdot \langle n| \bar{Q} C \frac{1+\gamma_5}{2} \Gamma \frac{1+\gamma_5}{2} b(0)|B \rangle.
\]

(33)

Thus, each insertion of operator \((-\pi_0)\) inside a composite operator acts as a factor of the intermediate state excitation energy. This is expected, for equation of motion of the static quark field \( Q \) allows to equate

\[
i\partial_0 \bar{Q}Cb(x) = \bar{Q}\pi_0 Cb(x)
\]

for any color-singlet operator \( \bar{Q}Cb(x) \). At the same time

\[
i\partial_0 \langle n|\bar{Q}Cb(x)|B \rangle = -(E_n - M_B) \langle n|\bar{Q}Cb(x)|B \rangle.
\]

This reasoning is presented in more detail in Appendix A dedicated to a conventional derivation of the saturation relations.

A couple of comments are in order before closing this subsection. Although we phrased consideration for the case of expectation values in \( B \) mesons at rest, these assumptions are not mandatory. The very same saturation by complete set of heavy quark intermediate states of a given spatial momentum holds for matrix elements where initial and final states may be different, and may have nonvanishing momenta. They neither have to be the ground pseudoscalar states, but with arbitrary spin flavor content.

Likewise, it is worth noting that even the static approximation for \( b \) quarks is actually superfluous; the saturation by physical intermediate states relies solely upon large mass of their \( Q \) quarks. The only modification required for finite \( m_b \) is taking care of projector \( \frac{1+\gamma_5}{2} \) introduced by the \( Q \)-quark propagator. Using the identity

\[
1 = \frac{1+\gamma_5}{2} + \gamma_5 \frac{1+\gamma_5}{2} \gamma_5
\]

we arrive at the following generalization:

\[
\langle B| \bar{b} A C \Gamma b(0)|B \rangle = \sum_n \left( \langle B| \bar{b} A Q(0)|n \rangle \cdot \langle n| \bar{Q} C \Gamma b(0)|B \rangle + \langle B| \bar{b} A \gamma_5 Q(0)|n \rangle \cdot \langle n| \bar{Q} C \gamma_5 \Gamma b(0)|B \rangle \right).
\]

(34)

The similar relation between \( \pi_0 \) and the excitation energy is only modified by the mass shift of the finite-mass \( B \) meson:

\[
\langle B| \bar{b} A \pi_0^k C \Gamma b(0)|B \rangle = \sum_n \left( (M_B - m_b) - (M_n - m_Q) \right)^k \times
\]

\[
\left( \langle B| \bar{b} A Q(0)|n \rangle \cdot \langle n| \bar{Q} C \Gamma b(0)|B \rangle + \langle B| \bar{b} A \gamma_5 Q(0)|n \rangle \cdot \langle n| \bar{Q} C \gamma_5 \Gamma b(0)|B \rangle \right).
\]

(35)

### 3.2. Lowest state saturation ansatz

The intermediate state representation (21) still does not assume any approximation aside from the static limit for the \( b \) quark, yet it may be used to apply a dynamic QCD approximation. The one we employ here uses as an input the \( B \)-meson heavy quark expectation values (20) of dimension 5 and 6, which are expressed through \( \mu_\pi^2, \mu_\gamma^2, \rho_D^3 \) and \( \rho_{LS}^3 \).
All operators with four and more derivatives must have an even number of spatial derivatives due to rotational invariance. Thus the operators with four derivatives have either four spatial derivatives, or two time and two spatial derivatives. Likewise, the $D = 8$ operators with five derivatives may have four spatial and a single derivative, or two spatial and three time derivatives.

We start with the $D = 7$ operators with four spatial derivatives, and apply $^{[32]}$:

$$
\langle B|\bar{b} i D_j i D_k i D_l i D_m \Gamma b|B\rangle = \sum_n \langle B|\bar{b} i D_j i D_k b|n\rangle \langle n|\bar{b} i D_l i D_m \Gamma b|B\rangle.
$$

(36)

The intermediate states $|n\rangle$ in the sum are either the ground-state multiplet $B, B^*$, or excited states with the suitable parity of light degrees of freedom. The ground-state factorization approximation assumes that the sum in (36) is to a large extent saturated by the ground state spin-symmetry doublet. Hence we retain only the contribution of the ground state and discard the contribution of higher excitations. In the case of dimension seven operators the result is expressed in terms of the expectation values with two derivatives, i.e. $\mu^2_G$ and $\mu^2_\pi$; matrix elements involving $B^*$ are related to them by spin symmetry. We illustrate below the compact result of summation over the multiplet of states. The method we use is most economic and turns out particularly transparent when generalizing the ground-state approximation. A derivation using the more conventional Lorentz-covariant trace formalism is given in Appendix $^{[3]}$

Abstracting first from the heavy quark spin we focus on the indices associated with the light degrees of freedom $^{[10]}$: the corresponding ground state is denoted by $|\Omega_0\rangle$ and its (spinor) wavefunction by $\Psi_0$. The ground-state saturation approximation then reads as

$$
\langle \Omega_0|\bar{Q} i D_j i D_k i D_l i D_m Q|\Omega_0\rangle = \langle \Omega_0|\bar{Q} i D_j i D_k Q|\Omega_0\rangle \langle \Omega_0|\bar{Q} i D_l i D_m Q|\Omega_0\rangle
$$

(37)

where the summation over the polarizations of intermediate $|\Omega_0\rangle$ is assumed; the spin states of the initial and final $\Omega_0$ may be arbitrary. Using $^{[4]}$

$$
\langle \Omega_0|\bar{Q} i D_j i D_k Q|\Omega_0\rangle = \frac{\mu^2_\pi}{3} \Psi_0^j \delta_{jk} \Psi_0 - \frac{\mu^2_G}{6} \Psi_0^j \sigma_{jk} \Psi_0,
$$

where $\sigma_j \sigma_k = \delta_{jk} + \sigma_{jk}$, (38)

we get (from now on explicit spinors $\Psi_0^{(1)}$ will be omitted)

$$
\langle \Omega_0|\bar{Q} i D_j i D_k i D_l i D_m Q|\Omega_0\rangle = \left(\frac{\mu^2_\pi}{2} \delta_{jk} \delta_{lm} - \frac{\mu^2_G}{18} (\delta_{jk} \sigma_{lm} + \sigma_{jk} \delta_{lm}) + \frac{\mu^2_G}{36} (\delta_{jm} \delta_{kl} - \delta_{jl} \delta_{km} + \delta_{jm} \sigma_{kl} - \delta_{jl} \sigma_{km} + \sigma_{jm} \delta_{kl} - \sigma_{jl} \delta_{km})
$$

(39)

where spin matrices $\sigma$ act on the spinor indices of the hadron wavefunctions (heavy quark field $Q$ can be considered spinless). On the contrary, $\Gamma$ in Eq. (36) acts on the spin of real $b$ quarks of QCD; for this reason the corresponding $\sigma$-matrices are denoted by $\sigma^Q$ where the confusion is possible.

The matrix elements for $B$ and $B^*$ mesons in actual QCD, with arbitrary heavy quark spin matrices $\Gamma^Q$, are directly expressed through those for the $\Omega_0$-states with spinless heavy quarks. To obtain them one would take the trace over the spin indices of both the heavy quark and of light degrees of freedom, convoluted with the corresponding meson spin wavefunctions

$$
\mathcal{M} = B + B^* \tilde{\sigma},
$$

(40)

$^3$Note a difference in the definition of three-dimensional $\sigma_{mn}$ compared to four-dimensional $\sigma_{\mu\nu}$, cf. Eq. $^{[12]}$. 

12
where $B$ and $B_k^*$ are the $B$- and $B^*$-meson fields, respectively. Namely, the generic matrix element of an operator $b \Omega Q b$ takes the form
\[
\langle M | b \Omega Q b | M' \rangle = \frac{1}{2} \text{Tr} \left[ M^\dagger \Gamma^Q M' \bar{\Sigma} \right], \quad \bar{\Sigma} = \Sigma^T \sigma_2 \tag{41}
\]
if the matrix $\Sigma$ describes the spinor part of the matrix element of $Q^\dagger \Omega Q$ between the $\Omega_0$ states, cf. Eqs. (37)-(39). The charge-conjugate spinor matrix $\bar{\Sigma}$ is evidently $\Sigma$ itself for $\Sigma = 1$ and $-\Sigma$ for $\Sigma = \sigma_k$.

The above rule becomes especially transparent where matrix elements over spinless $B$ mesons are considered: then $M$, $M'$ are unit matrices and one simply takes half the trace over spin indices; the structures with an odd overall number of spin matrices vanish (i.e., spin of the light cloud must multiply spin of heavy quark in $B$), whereas products of an even number are reduced to a numeric factor by assuming that $\sigma_Q = -\sigma$ (total angular momentum of $B$ meson vanishes!), in agreement with Eqs. (41).

Using Eqs. (39), (41) we get the master equations for spin-singlet and spin-triplet $B$ expectation values of $D=7$:
\[
\frac{1}{2M_B} \langle B | b i D_j i D_k i D_l i D_m b | B \rangle = \frac{(\mu_\Sigma^2)^2}{9} \delta_{jk} \delta_{lm} + \frac{(\mu_{2G}^2)^2}{36} (\delta_{jm} \delta_{kl} - \delta_{jl} \delta_{km}) \tag{42}
\]
\[
\frac{1}{2M_B} \langle B | b i D_j i D_k i D_l i D_m \sigma_{ab} b | B \rangle = -\frac{\mu_{2G}^2}{18} \delta_{jm} (\delta_{lb} \delta_{ka} - \delta_{la} \delta_{kb}) + \frac{\mu_{2G}^2}{36} \left[ \delta_{jm} (\delta_{lb} \delta_{ka} - \delta_{la} \delta_{kb}) - \delta_{jl} (\delta_{ka} \delta_{mb} - \delta_{kb} \delta_{ma}) + \delta_{kl} (\delta_{ja} \delta_{lb} - \delta_{jb} \delta_{la}) \right]. \tag{43}
\]

The case of the $D=8$ operators with four spatial and one time derivative requires minimal modification. Time derivative can occupy the second or fourth position, or stay in the center, at the third position. In the former case the evaluation proceeds in the same way, one only needs to complement Eq. (38) by the similar relation for Darwin and $LS$ operators:
\[
\langle \Omega_0 | Q i D_j i D_k i D_l b | \Omega_0 \rangle = -\frac{\rho_D^3}{3} \delta_{jk} - \frac{\rho_{LS}^3}{6} \sigma_{jk}. \tag{44}
\]

The two related master equations for $D=8$ then become
\[
\frac{1}{2M_B} \langle B | b i D_j i D_k i D_l i D_m b | B \rangle = -\frac{\rho_D^3}{9} \delta_{jk} \delta_{lm} + \frac{\rho_{LS}^3}{9} \delta_{jm} \delta_{kl} - \delta_{jl} \delta_{km}) \tag{45}
\]
\[
\frac{1}{2M_B} \langle B | b i D_j i D_k i D_l i D_m \sigma_{ab} b | B \rangle = \frac{\rho_D^3}{9} \delta_{jk} (\delta_{la} \delta_{mb} - \delta_{lb} \delta_{ma}) - \frac{\rho_{LS}^3}{9} \delta_{lm} (\delta_{ja} \delta_{kb} - \delta_{jb} \delta_{ka}) - \frac{\rho_D^3}{9} \rho_{LS}^3 \delta_{jm} (\delta_{lb} \delta_{ka} - \delta_{la} \delta_{kb}) - \delta_{jl} (\delta_{ka} \delta_{mb} - \delta_{kb} \delta_{ma}) + \delta_{kl} (\delta_{ja} \delta_{lb} - \delta_{jb} \delta_{la}) \tag{46}
\]

For the expectation values with time derivative in the middle position the corresponding ground-state contribution vanishes; they appear only due to the ‘radially’ excited states. Therefore, in the ground-state factorization we set these to zero:
\[
\langle B | b i D_j i D_k i D_l i D_m \sigma | b | B \rangle \overset{\text{GSP}}{=} 0.
\]
Table 1: Expressions and values for the dimension seven matrix elements

| expression | $m_k$, GeV$^4$ | expression | $m_k$, GeV$^4$ | expression | $m_k$, GeV$^4$ | expression | $m_k$, GeV$^4$ |
|------------|---------------|------------|---------------|------------|---------------|------------|---------------|
| $m_1$      | $\frac{5}{2} (\mu_2^2)^2$ | 0.11       | $m_2$        | $-\bar{\epsilon}\rho_D^3$ | -0.072      | $m_3$      | $-\frac{2}{3} (\mu_G^2)^2$ | -0.082 |
| $m_4$      | $\frac{1}{2} (\mu_2^2)^2 + (\mu_G^2)^2$ | 0.39       | $m_5$        | $-\bar{\epsilon}\rho_{LS}^3$ | 0.048       | $m_6$      | $\frac{4}{3} (\mu_G^2)^2$ | 0.082 |
| $m_7$      | $-\frac{4}{3} \mu_2^2 \mu_G^2$ | -0.42      | $m_8$        | $-8\mu_2^2 \mu_G^2$ | -1.26       | $m_9$      | $(\mu_G^2)^2 - \frac{10}{9} \mu_2^2 \mu_G^2$ | -0.40 |

Finally, we need to consider the expectation values of the form $\langle B|b \bar{D}_j iD_0^{k+1}iD_1|b\rangle$ for $k = 2, 3$ which evidently belong to the tower of $\ell_{3, G}^2$ and $\rho_{D, LS}^3$. Likewise, their values could be considered as the input describing strong dynamics, along with the latter; yet they have not been constrained experimentally. The intermediate states saturating such expectation values have opposite parity to the ground state ($P$-wave states) regardless of number of time derivatives. The counterpart of the ground-state saturation approximation here is retaining the contribution of the lowest $P$-wave resonance in the sum; then each power of time derivative amounts to the extra power of $-\bar{\epsilon}$, where $\bar{\epsilon} = M_p - M_B \approx 0.4$ GeV.

In fact, there are two families of the $P$-wave excitations of $B$ mesons corresponding to spin of light degrees of freedom $\frac{3}{2}$ or $\frac{1}{2}$. The combinations $\mu_2^2 - \mu_2^2$, $\rho_D^3 + \rho_{LS}^3$, ... receive contributions only from the $\frac{1}{2}$-family, whereas the $\frac{3}{2}$-family gives rise to $\mu_2^2 + 2\bar{\epsilon}\mu_2^2$, $\bar{\epsilon}\rho_D^3 - \rho_{LS}^3$, etc. [10] (the transition amplitude into the lowest $\frac{1}{2}$-state appears to be suppressed). Therefore, it makes sense to consider these two structures separately and approximate

$$\langle B|b \bar{D}_j (-iD_0)^{k+1}iD_1|b\rangle = \left(\bar{\epsilon}_{3/2}^k \frac{2\rho_D^3 - \rho_{LS}^3}{9} + \bar{\epsilon}_{1/2}^k \frac{\rho_D^3 + \rho_{LS}^3}{9}\right) \delta_{jl}$$

(47)

$$\langle B|b \bar{D}_j (-iD_0)^{k+1}iD_1\sigma_{jl}|b\rangle = -\bar{\epsilon}_{1/2}^k \frac{2\rho_D^3 - \rho_{LS}^3}{3} + \bar{\epsilon}_{1/2}^k \frac{2\rho_D^3 + 2\rho_{LS}^3}{3}.$$

(48)

Note that assuming $\bar{\epsilon}_{1/2} = \bar{\epsilon}_{3/2} = \bar{\epsilon}$ implies $\rho_D^3 \simeq \bar{\epsilon}\mu_2^2$ and $-\rho_{LS}^3 \simeq \bar{\epsilon}\mu_G^2$; the first relation seems to be satisfied by the preliminary values of $\mu_2^2$ and $\rho_D^3$ extracted from experiment.

3.3. Summary for $\mathcal{O}(\Lambda_{QCD}^4)$ and $\mathcal{O}(\Lambda_{QCD}^5)$ expectation values

Combining the above relations we can evaluate all the required nonperturbative parameters at order $1/m_b^4$ and $1/m_b^5$ in terms of a few quantities $\mu_2^2$, $\mu_G^2$, $\rho_D^3$, $\rho_{LS}^3$ and $\bar{\epsilon}$ taken as an input. Tables 1 and 2 list the resulting expressions for the expectation values (at $\bar{\epsilon}_{1/2} = \bar{\epsilon}_{3/2} = \bar{\epsilon}$), and give the corresponding numerical estimates. For the latter we assume $\bar{\epsilon} \simeq 0.4$ GeV, $\mu_2^2 = 0.45$ GeV$^2$, $\mu_G^2 = 0.35$ GeV$^2$, $\rho_D^3 = 0.18$ GeV$^3$, $\rho_{LS}^3 = -0.12$ GeV$^3$.

The values in the tables are not precision predictions, of course, for a number of reasons. First, they depend on $\mu_2^2$, $\mu_G^2$, $\rho_D^3$, $\rho_{LS}^3$ which are themselves only known with limited accuracy; the same holds true for the value of $\bar{\epsilon}$. This aspect, however, is easy to quantify using the expression in Tables 1 and 2.

Secondly, the estimates are formulated in the infinite mass limit for the $b$ quarks, while the parameters of the heavy quark expansion should actually include the full mass dependence. Generally the finite-mass corrections are governed by the parameter $\mu_{had}/2m_b$ and can be sizable in $b$ hadrons [11], up to 15% However, there is a specific suppression of such preasymptotic correction in the ground-state pseudoscalar mesons related to the observed
proximity of these states to the so-called ‘BPS’ regime \[12\]. Since we deal here exclusively with the pseudoscalar ground state, we expect the finite mass corrections to be substantially smaller.

The major issue is thus the validity of the employed approximation for the matrix elements, in particular retaining only the relevant lowest states. The degree to which this ansatz is applicable depends on the operator in question, and is expected to deteriorate when the number of derivatives (i.e. the operator dimension) increases.

The dominance of the lowest state and suppression of transitions into highly excited states typically holds for the bound states with a smooth potential. In field theory for heavy-light mesons this question was studied \[13\] in two-dimensional QCD, the so called ’t Hooft model, which is exactly solvable in the limit of a large number of colors. In particular, the ground-state expectation values for operators with two spatial derivatives were found to be saturated by the first ‘P-wave’ state to an amazing degree of accuracy (we should remind that there is no spin in 1 + 1 dimensions, therefore only one, not two P-wave families). In case this also applies to real QCD we would expect a good accuracy of the employed factorization ansatz for \(m_2\) and \(m_3\) at order \(1/m_3^2\) and a reasonable one for \(r_1, r_6, r_8, r_{17}\) (and some other related combinations) at order \(1/m_3^4\). All such expectation values are simply given by moments of the combinations of two small-velocity structure functions \(w_{3/2}(c)\) and \(w_{1/2}(c)\) which are positive, which strongly constrains the expectation values.

The situation is a priori less clear with the operators containing four spatial derivatives. In the ’t Hooft model the effects related to deviations from the ground-state factorization were studied and were found to be nearly saturated, again to a very good degree, by the first radial excitation \[13\]. For some of the expectation values at order \(1/m_3^4\) they can be estimated following the reasoning of Refs. \[11, 14\]. The nonfactorizable contributions taken at face value appear to be about 50% of the ground-state one \[15\].

In reality, the effective interaction in full QCD is singular at short distances due to perturbative physics. Hard gluon corrections lead to a slow decrease of the transitions to the highly excited states – yet they are dual to perturbation theory. This is taken care of in

\[4\] The contribution itself turned out quite significant if normalized literally to \((\mu_+^2)^2\), apparently since \(\mu_+^2\) was anomalously small there lacking the factor of 3, the number of space dimensions. If normalized to \(N^3\) it was about 3/4. In actual \(B\) mesons \(\mu_+^2\) is close to \(N\).

| expression | \(r_k, \text{GeV}^2\) | expression | \(r_k, \text{GeV}^2\) |
|------------|-----------------|------------|-----------------|
| \(r_1\)    | \(c_1^2\rho_D^1\) | 0.029      | \(r_2\)    | \(-\mu_+^2\rho_D^1\) | -0.081 |
| \(r_3\)    | \(-\frac{1}{3}\mu_+^2\rho_D^3 - \frac{1}{6}\mu_+^2\rho_{LS}^3\) | -0.020    | \(r_4\)    | \(-\frac{1}{3}\mu_+^2\rho_D^3 + \frac{1}{6}\mu_+^2\rho_{LS}^3 + c_1^2\rho_D^1\) | -0.005 |
| \(r_5\)    | 0                | 0          | \(r_6\)    | \(c_1^2\rho_D^1\)    | 0.029  |
| \(r_7\)    | 0                | 0          | \(r_8\)    | \(\bar{c}_1^2\rho_{LS}^1\) | -0.019 |
| \(r_9\)    | \(-\mu_+^2\rho_{LS}^1\) | 0.054     | \(r_{10}\) | \(\mu_+^2\rho_D^1\) | 0.063  |
| \(r_{11}\) | \(\frac{1}{3} (\mu_+^2\rho_D^3 + \mu_+^2\rho_{LS}^3) - \frac{1}{6}\mu_+^2\rho_{LS}^3\) | 0.010     | \(r_{12}\) | \(-\frac{1}{3} (\mu_+^2\rho_D^3 + \mu_+^2\rho_{LS}^3) - \frac{1}{5}\mu_+^2\rho_{LS}^3\) | 0.004  |
| \(r_{13}\) | \(\frac{1}{3} (\mu_+^2\rho_D^3 + \mu_+^2\rho_{LS}^3) + \frac{1}{6}\mu_+^2\rho_{LS}^3\) | -0.046    | \(r_{14}\) | \(\frac{1}{3} (\mu_+^2\rho_D^3 - \mu_+^2\rho_{LS}^3) + \frac{1}{6}\mu_+^2\rho_{LS}^3 + \bar{c}_1^2\rho_{LS}^1\) | 0.013  |
| \(r_{15}\) | 0                | 0          | \(r_{16}\) | 0                | 0                  |
| \(r_{17}\) | \(\bar{c}_1^2\rho_{LS}^1\) | -0.019    | \(r_{18}\) | 0                | 0                  |

Table 2: Expressions and values for the dimension eight matrix elements
the Wilsonian renormalization procedure which is assumed in the kinetic scheme. From this perspective, one can say that the factorization ansatz yields the expectation values at a low normalization point $\mu < \epsilon_{\text{rad}} \approx 0.6 \text{ GeV}$, before the channels to radially excited states open up; the excitation energy for such lowest resonance states is probably around $700 \text{ MeV}$. Of course, the actual $\mu$-dependence at such low scale does not coincide with the one derived from perturbation theory; at intermediate excitation energies it must be in some respect dual to the perturbative one.

Keeping this in mind it should be appreciated that even for definitely positive correlators, or those expectation values where all intermediate states contribute with the same sign, simply adding the first excitation to the ground state contribution may already constitute some overshooting. Indeed, a resonance state residing at mass $\epsilon_{\text{rad}}$ may be dual to the perturbative contribution over the domain of masses

$$\frac{\epsilon_{\text{rad}}}{2} < \epsilon < \frac{\epsilon_{\text{rad}} + \epsilon''_{\text{rad}}}{2}$$

where $\epsilon''_{\text{rad}}$ is the mass of the second excitation, and likewise for higher resonances. Then a better approximation for the expectation value normalized at $\mu = \epsilon_{\text{rad}}$ would be to add only a half of the first excitation contribution. In practical terms, as long as account for the power mixing in the perturbative corrections to the conventional Wilson coefficients (most notably, of the unit operator) has not been extended to order $1/m^4_b$ and $1/m^5_b$, the effective nonfactorizable piece may turn out even less.

An additional feature of actual QCD is existence of the low-mass continuum contribution beyond pure resonances, most notably states like $B^{(*)}\pi$ and their $SU(3)$ siblings. Their contribution is typically $1/N_c$ suppressed and usually does not produce a prominent effect in quantities which are finite in the chiral limit (and the corrections to factorization for higher-dimension expectation values are). They can be expected to contribute up to 25% of the ground state, yet this may be partially offset once the actual QCD conventional expectation values are used in the factorization ansatz, that likewise incorporate such states [14].

Considering all these arguments, we tentatively assign the uncertainty in the factorization estimate to be at the scale of 50%. Yet this should be understood to apply to the ‘positive’ operators where the lowest-state contribution does not vanish and the excited state multiplets yield the same-sign contribution. The corrections to factorization will be addressed in more detail in the forthcoming paper [15].

4. Numerical Estimates for the Higher Order Corrections in the Rate and Moments

Armed with the numerical estimates of all the required expectation values we are in the position to evaluate the higher-order power corrections to inclusive $B$ decays. The primary quantity of interest is the total semileptonic width $\Gamma_{\text{sl}}(b \rightarrow c)$ used for the precision extraction of $|V_{cb}|$. 
4.1. $\Gamma(B \to X_c \ell \nu)$

Assuming the fixed values of $m_b$ and $m_c$ we find the following power corrections at different orders in $1/m_b$:

\[
\frac{\delta \Gamma_{1/m^2}}{\Gamma_{\text{tree}}} = -0.043, \quad \frac{\delta \Gamma_{1/m^3}}{\Gamma_{\text{tree}}} = -0.030, \quad \frac{\delta \Gamma_{1/m^4}}{\Gamma_{\text{tree}}} = 0.0075, \quad \frac{\delta \Gamma_{\text{IC}}}{\Gamma_{\text{tree}}} = 0.007, \quad \frac{\delta \Gamma_{1/m^5}}{\Gamma_{\text{tree}}} = 0.006, \quad (49)
\]

where $\Gamma_{\text{tree}}$ includes the phase space suppression factor of approximately 0.63. We have shown separately the contribution scaling like $1/m_b^3$ and denoted it by $\delta \Gamma_{\text{IC}}$. As anticipated [16], it dominates the high-order effects and may even exceed the $1/m_b^4$ correction, yet it is to some extent offset by the regular $1/m_b^5$ terms.

The numerical results (49) suggest that the power series for $\Gamma_{\text{sl}}(B \to c)$ is well behaved, and is under good numerical control provided the nonperturbative expectation values are known. Higher order terms induce decreasing corrections except where anticipated on theoretical grounds. The estimated overall shift due to higher-order terms

\[
\frac{\delta \Gamma_{1/m^4} + \delta \Gamma_{1/m^5}}{\Gamma_{\text{tree}}} \simeq 0.013
\]

is well within the interval assessed in [16] and, taken at face value, would yield a 0.65% direct reduction in $|V_{cb}|$.

This would not be the whole story, however, for the quark masses determining the partonic width are not known beforehand with an accuracy required to extract $|V_{cb}|$ with the percent precision. Rather, their relevant combination is extracted from the fit to the data on kinematic moments of the $B \to X_c \ell \nu$ decay distributions, that in turn are affected by power corrections.

4.2. Moments

The key in the OPE evaluation of $\Gamma_{\text{sl}}(b \to c)$ and, therefore, in extraction of $|V_{cb}|$ are the first moments of lepton energy $\langle E_\ell \rangle$ and of hadron invariant mass squared $\langle M_{X_c}^2 \rangle$, which pinpoint the precision value of the combination of $m_b$ and $m_c$ that drives the total decay probability. Moreover, analyzed through order $1/m_b^3$ these two moments turned out to depend on nearly the same combination of the heavy quark parameters. This allowed for a nontrivial cross check of the OPE-based theory prediction [17]: essentially, $\langle M_{X_c}^2 \rangle$ could be predicted in terms of $\langle E_\ell \rangle$ and vice versa, once the heavy quark parameters were allowed to vary within theoretically acceptable range.

Therefore, besides the practical question about the shift in the fitted heavy quark parameters, another important issue emerges of whether the consistency between measured values of $\langle E_\ell \rangle$ and of $\langle M_{X_c}^2 \rangle$ persists once higher-order corrections are accounted for.

Numerically we find

\[
\delta \langle E_\ell \rangle = 0.013 \text{ GeV}, \quad \delta \langle M_{X_c}^2 \rangle = -0.086 \text{ GeV}^2
\]

where the changes shown are a combined contribution of corrections to order $1/m_b^4$ and $1/m_b^5$. In our analysis we follow Ref. [18] and evaluate the moments as the literal ratios

\[
\frac{\int dE_\ell \ dq_0 \ dq^2 \ K(E_\ell, q_0, q^2) \ C(E_\ell, q_0, q^2) \ d^2 \Gamma_a \ dE_\ell dq_0 dq^2}{\int dE_\ell \ dq_0 \ dq^2 \ C(E_\ell, q_0, q^2) \ d^2 \Gamma_a \ dE_\ell dq_0 dq^2} \quad (52)
\]
without prior expanding the ratio itself in $1/m_b$. Here $K$ is the corresponding kinematic observable in question (powers of $E_\ell$ or $M_\ell^2$) and $C$ is an explicit kinematic cut if imposed. The integrals both in denominator and in numerator are taken directly as obtained in the OPE through the corresponding terms in $1/m_b$; in particular, powers of $(M_B-m_b)$ for hadronic mass moments effectively are not treated as power suppressed. Throughout this paper we perform numeric evaluation of higher-order–induced changes in observables discarding perturbative corrections altogether; this turned out a good approximation in the kinetic scheme.

To assess practical significance of the numerical changes in the moments we gauge them considering the amount of commensurate shifts required in the values of 'conventional' heavy quark parameters used in the OPE so far, to make up for the new effects. Specifically, we consider the amount of commensurate shifts required in the values of 'conventional' heavy quark parameters simultaneously. For visualization purposes we, however, quote for each $\delta m$, the stated significance of variation in $m$ values of $m$ and $m_c$ separately is much wider than 10 MeV, therefore in practice the required variation in $m_c$ is not independent and is derived from the corresponding variation in $m_b$. The stated significance of $\delta m_b \sim 10$ MeV refers, in fact, to the above combination of masses. In actual fits the effect of higher-order corrections is compensated by a change in all heavy quark parameters simultaneously. For visualization purposes we, however, quote for each moment $\mathcal{M}$ the separate values

$$\delta m_b = -\frac{\delta \mathcal{M}}{\partial m_b}, \quad \delta \mu^2 = -\frac{\delta \mathcal{M}}{\partial \mu^2}, \quad \delta \rho^3_D = -\frac{\delta \mathcal{M}}{\partial \rho^3_D}$$

as if only one of them were responsible for the adjustment and if the higher-order effect had been made up for completely. Clearly, if a particular shift in a heavy quark parameter comes out abnormally large, it should simply be discarded: this only signals that the moment in question is insensitive to this parameter and the moment rather constrains other OPE parameters. On the contrary, if a shift is small, this generally means that the parameter is well-constrained and, typically, should not be adjusted. In a sense, the values in Eq. (53) would assume that the quality of the fit before including the calculated corrections has been perfect, and this clearly is oversimplification. Yet this is suitable to gauge the significance of the corrections we study.

The dependence of the considered moments upon heavy quark parameters, entering denominators in Eq. (53) is given in a ready-to-use form in Ref. [18]. From that we obtain

$$\langle E_\ell \rangle : \quad \delta m_b = -33 \text{ MeV} (0.022); \quad \delta \mu^2 = -0.39 \text{ GeV}^2 (-0.005); \quad \delta \rho^3_D = 0.15 \text{ GeV}^3 (0.014)$$

$$\langle M_\ell^2 \rangle : \quad \delta m_b = -17 \text{ MeV} (0.011); \quad \delta \mu^2 = -0.12 \text{ GeV}^2 (-0.0015); \quad \delta \rho^3_D = 0.086 \text{ GeV}^3 (0.008)$$

The dependence of $\Gamma_{sl}(B)$ on heavy quark parameters has also been carefully studied [19] [16] [18], and above we have supplemented each variation by the corresponding relative shift in $|V_{cb}|$,

$$\frac{\delta |V_{cb}|}{|V_{cb}|} = -\frac{1}{2} \frac{1}{\Gamma_{sl} \partial \Gamma_{sl} \partial \text{ HQP}} \frac{\delta \Gamma_{sl} \partial \text{ HQP}}{\partial \text{ HQP}}, \quad \frac{1}{\Gamma_{sl} \partial \text{ HQP}} = \begin{cases} 0.0013 \text{ MeV}^{-1} & \text{ HQP } = m_b \\ -0.026 \text{ GeV}^{-2} & \text{ HQP } = \mu^2 \\ -0.18 \text{ GeV}^{-3} & \text{ HQP } = \rho^3_D \end{cases}$$

(55)
the numbers shown in the parenthesis in Eqs. (54).

Eqs. (54) suggest that one of the possible ‘solutions’ is an increase in $\rho_{2}^{\delta}$ by about 0.1 GeV$^3$; this value may be affected by a variation in $\mu_{2}^{\delta}$ of the scale of 0.05 GeV$^2$ or in $m_{b}$ about 10 MeV. The relevance of this solution essentially depends on how precisely the presently fitted standard heavy quark parameters accommodate both $\langle E_{\ell} \rangle$ and $\langle M_{X}^{2} \rangle$.

In actuality, the most precise measurements coming from the threshold production of $B$-mesons at $B$-factories require a lower cut on lepton energy. Therefore, the proper analysis of the moments should include a $E_{\ell}^{\text{cut}}$-cut around 1 GeV.

Figs. 2a–f show the size of the nonperturbative OPE terms for the first three central lepton energy and hadron mass squared moments depending on the lower cut $E_{\ell}^{\text{cut}}$ on the charged lepton energy, at different orders in $1/m_{b}$.\(^5\)

![Figure 2: Power corrections to the first three (central) moments of charged lepton energy (upper row) or hadron invariant mass squared (lower row), at different orders in the $1/m_{Q}$ expansion, in units of GeV in the corresponding power. Blue is the effect of $\mu_{2}^{\delta}$ and $\rho_{2}^{\delta}$ (order $1/m_{b}$), green at order $1/m_{b}^{3}$, red at $1/m_{b}^{4}$ and magenta finally shows the shift upon including $D=8$ expectation values at order $1/m_{b}^{5}$.

Keeping in mind that higher moments must generally be sensitive to higher-order OPE expectation values, we conclude that at moderate cuts $E_{\ell}^{\text{cut}} \lesssim 1.5$ GeV preserving sufficient ‘hardness’ of the inclusive probability, the power expansion is well behaved; the $1/m_{b}^{5}$ effects are small compared to the $1/m_{b}^{4}$ corrections. This is expected since the IC effects are not parametrically enhanced in the higher moments [16, 4, 20].

At the same time, it is clear that the estimated effects from higher powers in the $1/m_{b}$ expansion are not negligible, in particular in the second and higher moments. Those are sensitive to $\mu_{2}^{\delta}$ and $\rho_{2}^{\delta}$, and high-order terms may produce their sizable shift.

To visualize the potential effect, we have plotted the analogies of the shifts in Eqs. (54), $\delta m_{b}(E_{\ell}^{\text{cut}})$, $\delta \mu_{2}^{\delta}(E_{\ell}^{\text{cut}})$, $\delta \rho_{2}^{\delta}(E_{\ell}^{\text{cut}})$ for all six moments as functions of $E_{\ell}^{\text{cut}}$ and the corresponding $\delta [V_{cb}]/|V_{cb}|$. Here we present these dependences only for $\langle E_{\ell} \rangle$ and for $\langle M_{X}^{2} \rangle$, on single plots, separately for $\delta m_{b}$, $\delta \mu_{2}^{\delta}$ and $\delta \rho_{2}^{\delta}$. Figs. 3a–c; the special role of these two moments has been

\(^5\)As before, the explicit factor $M_{B} - m_{b}$ in hadronic mass moments does not count as a power suppression.
Figure 3: Changes in \( m_b \), in \( \mu^2 \), and in \( \rho^3_D \), respectively required alone to literally offset the effect of higher-order power terms in \( \langle E_\ell \rangle \) (blue) and in \( \langle M^2_X \rangle \) (magenta), at a given \( E^\text{cut}_\ell \). In units of GeV in the corresponding power

| \( \delta m_b \), MeV | \( \langle E_\ell \rangle \) | \( \langle (E_\ell - \langle E_\ell \rangle)^2 \rangle \) | \( \langle (E_\ell - \langle E_\ell \rangle)^3 \rangle \) | \( \langle M^2_X \rangle \) | \( \langle (M^2_X - \langle M^2_X \rangle)^2 \rangle \) | \( \langle (M^2_X - \langle M^2_X \rangle)^3 \rangle \) |
|-----------------|---------|-----------------|-----------------|---------|-----------------|-----------------|
| (\( \delta |V_{cb}|/|V_{ub}| \)) | -39 | -00 | --- | -21 | --- | --- |
| (\( \delta |V_{ub}|/|V_{cb}| \)) | -0140 | -0040 | --- | -0014 | --- | --- |
| (\( \delta \mu^2 \), GeV\(^2 \)) | -0130 | -0140 | -0040 | -0010 | -0080 | 033 |
| (\( \delta |V_{ub}|/|V_{cb}| \)) | -0004 | -0016 | -0005 | -0017 | -0010 | (0043) |
| (\( \delta \rho^3_D \), GeV\(^3 \)) | 0160 | 0090 | 0020 | 0008 | 0050 | 010 |
| (\( \delta |V_{ub}|/|V_{cb}| \)) | 0014 | 0008 | 0020 | 0008 | 0005 | 0009 |

Table 3: Higher order power corrections to the moments with \( E^\text{cut}_\ell = 1 \) GeV translated into the required conventional heavy quark parameter shifts to offset them; also shown are relative shifts in \( |V_{cb}| \) these would induce assuming the fixed value of \( \Gamma_{sl}(B) \). Entries where \( \delta m_b \) would exceed 100 MeV were left blank

mentioned earlier in this subsection. The corresponding values [53] for all six moments at a representative mild cut \( E^\text{cut}_\ell = 1 \) GeV are shown in Table 3 together with the commensurate relative shift \( \delta |V_{cb}|/|V_{cb}| \). The latter is easily estimated using Ref. [10], cf. Eq. (55):

\[
\frac{\delta |V_{cb}|}{|V_{cb}|} \approx \begin{cases} 
-0.0066 \quad \text{at} \quad \delta m_b = 10 \text{MeV} \\
0.0013 \quad \text{at} \quad \delta \mu^2 = 0.1 \text{GeV}^2 \\
0.009 \quad \text{at} \quad \delta \rho^3_D = 0.1 \text{GeV}^3 
\end{cases}
\] (56)

As seen from the plots Figs. 2 the cut-dependence of higher-order corrections shows a generally expected behavior which qualitatively follows the behavior already found in the \( 1/m_b^2 \) and \( 1/m_b^3 \) effects. The new corrections likewise show mild cut dependence at \( E^\text{cut}_\ell \lesssim 1 \) GeV and, typically, sharply change above \( E^\text{cut}_\ell \approx 1.5 \) GeV, in line with the overall deterioration of the process hardness with raising cut on \( E_\ell \).

Based on the numeric pattern of the corrections to the moments we anticipate that inclusion of higher-order power-suppressed effects will mostly amount to increase in the fitted value of \( \rho^3_D \) by about 0.1 GeV\(^3 \) compared to the fit where only \( D = 5 \) and \( D = 6 \) nonperturbative expectation values are retained, with a possible shift in \( \mu^2 \) by about \( \pm 0.05 \) GeV\(^2 \). Figs. 3a–f illustrate this assertion showing the combined effect of the new power corrections for the six moments together with the effect of decreasing \( \rho^3_D \) by 0.12 GeV\(^3 \); the similarity of the shifts suggests that the lack of higher-power corrections in the theoretical expressions used so far could be to some extent faked by a lower value of the Darwin expectation value. (The
third lepton moment is highly sensitive to the Darwin expectation value, and uncalculated \(\alpha_s\)-corrections to the latter may be blamed for the mismatch apparent in the plot Figs. [4c. Besides, lepton energy moments are to a large extent saturated by the parton expressions; therefore their high precision allowing to discuss nonperturbative effects in their value relies on a high degree of cancellation of conventional perturbative corrections. The extent of such cancellation at higher loops is not known beforehand, which warrants a cautious attitude towards theoretical precision of higher lepton moments at the required level, and places more emphasis in this respect on higher moments of the hadronic mass.)

A good way to experimentally extract information on the Darwin expectation value is the third central hadronic mass squared moment. So far \(B\)-factories did not attempt to measure it; it has been extracted with an informative accuracy in the DELPHI analysis [21], however for unspecified reasons this moment was not included into the global fit, according to HFAG. We find that the higher-order power corrections (predominantly \(1/m_s^4\)) tend to shift this moment by about twice the DELPHI error bar.

Figure 4: Effect of including higher-order power corrections (blue) to the first three moments of charged lepton energy, upper row, and of hadron invariant mass squared, lower row, and effect of decreasing the Darwin expectation value by 0.12 GeV^{3} (green), in GeV to the corresponding power

Combining the increase in \(\rho_{D}^{2}\) with the direct effect on \(\Gamma_{al}\), Eq. [50], we expect an overall increase in \(|V_{cb}|\) by something like 0.4 percentage points:

\[
\frac{\delta |V_{cb}|}{|V_{cb}|} \approx + (0.003 \div 0.005) .
\]

This estimate is based on the expectation values in the ground-state factorization approximation and would scale with their magnitude; the actual number may be up to a factor of 1.5 larger. We emphasize that this would remain only an educated expectation, for the result strongly depends on the details of the existing fit to the data, on the precision of different data points and on their correlations. The final conclusions should be drawn through incorporating the new corrections in the actual fit to the data.
5. Power corrections in $b \to s + \gamma$

As another application of the technique described in Sect. 2 we have considered higher-order power corrections to the decay rate and to the photon energy moments in radiative $b \to s + \gamma$ decays. They have been treated in the approximation of the local weak vertex; no operators with charm quarks or chromomagnetic $b \to s$ vertex were considered. The analysis parallels that of the semileptonic case, except that the simplicity of the kinematics (corresponding to $q^2 = 0$ in the latter) leads to reasonably compact analytic expressions even in higher orders. We have performed the calculations for an arbitrary mass ratio $m_s/m_b$, however quote here the results only at $m_s = 0$.

Therefore, we assume the $b \to s + \gamma$ transition to be mediated by the effective vertex

$$\frac{\lambda}{2} \bar{s} \sigma_{\mu\nu} (1 - \gamma_5) b F^{\mu\nu}.$$  \hspace{1cm} (57)

In this approximation power corrections to the integrated decay rate become

$$\Gamma_{bsg}(B) = \frac{\lambda^2 m_b^3}{4 \pi} \left[ 1 - \frac{r_1^2 + 3 m_b^2}{2 m_b^2} - \frac{11 \rho_D^3 - 9 \rho_{LS}^3}{6 m_b^2} \right. \right.$$

$$+ \frac{1}{m_b^4} \left( \frac{1}{8} m_1 + \frac{7}{12} m_2 + \frac{1}{3} m_3 + \frac{7}{8} m_4 - \frac{17}{12} m_5 + \frac{11}{24} m_6 + \frac{1}{32} m_8 - \frac{1}{6} m_9 \right) \right.$$

$$+ \frac{1}{m_b^5} \left( - \frac{28}{15} r_1 - \frac{69}{20} r_2 + \frac{21}{20} r_3 - \frac{107}{60} r_4 + \frac{173}{120} r_5 + \frac{17}{8} r_7 + \frac{31}{24} r_9 \right. \right.$$

$$+ \frac{7}{24} r_{10} - \frac{13}{8} r_{11} + \frac{1}{24} r_{12} - \frac{1}{6} r_{13} - \frac{17}{12} r_{14} + \frac{17}{24} r_{15} - \frac{1}{4} r_{16} - \frac{3}{8} r_{17} + \frac{7}{24} r_{18} \left. \left. \right) \right]. \hspace{1cm} (58)$$

The first moment, the average photon energy in the decay, is given by

$$2 \langle E_\gamma \rangle = m_b + \frac{\mu_s^2 - \mu_{D*}^2}{2 m_b} - \frac{5 \rho_D^3 - 7 \rho_{LS}^3}{6 m_b^2} \right.$$

$$+ \frac{1}{m_b^4} \left( - \frac{1}{8} m_1 + \frac{23}{12} m_2 - \frac{5}{6} m_3 + \frac{7}{24} m_4 - \frac{17}{12} m_5 + \frac{13}{24} m_6 + \frac{1}{32} m_8 - \frac{1}{3} m_9 \right) \right.$$

$$+ \left( \frac{1}{4} (\mu_s^2)^2 + \frac{1}{2} \mu_s^2 \mu_{D*}^2 - \frac{3}{4} (\mu_{D*}^2)^2 \right) \right.$$

$$+ \frac{1}{m_b^4} \left( - \frac{5}{4} r_2 + \frac{23}{12} r_3 - \frac{7}{4} r_4 - \frac{5}{8} r_5 + \frac{5}{8} r_6 - \frac{17}{24} r_7 + \frac{25}{24} r_9 - \frac{1}{8} r_{10} \right. \right.$$

$$+ \frac{19}{24} r_{11} + \frac{7}{24} r_{12} - \frac{7}{6} r_{13} + \frac{19}{12} r_{14} - \frac{25}{24} r_{15} + \frac{7}{12} r_{16} + \frac{61}{24} r_{17} - \frac{17}{8} r_{18} \right.$$ 

$$+ \frac{1}{2} \mu_s^2 \rho_D^3 - \frac{13}{6} \mu_s^2 \rho_{LS}^3 + \frac{5}{2} \mu_s^2 \rho_{LS} + \frac{1}{6} \mu_s^2 \rho_{LS}^3 \right]. \hspace{1cm} (59)$$

For the second and third moments we, to simplify the expressions, quote the corrections to
the moments with respect to $m_b/2$ rather than for the usually considered central moments:

$$3\langle (2E_\gamma-m_b)^2 \rangle = \mu_x^2 - \frac{2\rho_D^2-\rho_L^2}{m_b} + \frac{1}{m_b^2} \left( \frac{3}{4} m_1 - \frac{7}{2} m_2 - m_3 + \frac{5}{2} m_5 + \frac{3}{4} m_6 - \frac{1}{2} m_7 + \frac{11}{16} m_8 - \frac{1}{4} m_9 \right)$$

$$+ \frac{1}{2} \left( \frac{\mu_x^2}{2} \right)^2 + \frac{3}{2} \mu_x^2 \rho_G$$

$$+ \frac{1}{m_b^3} \left( -4 r_1 + 5 r_2 + \frac{3}{2} r_3 - 5 r_4 - \frac{21}{4} r_5 + \frac{63}{4} r_6 - \frac{23}{4} r_7 - \frac{13}{4} r_9 + \frac{9}{4} r_{10} \right)$$

$$+ \frac{25}{4} r_{11} + \frac{21}{4} r_{12} - 8 r_{13} + r_{14} + \frac{1}{4} r_{15} + \frac{9}{2} r_{16} + \frac{27}{4} r_{17} - \frac{37}{4} r_{18}$$

$$+ \frac{5}{6} \mu_x^2 \rho_D - 3 \mu_G^2 \rho_D - \mu_x^2 \rho_L^3 + \frac{3}{2} \mu_G^2 \rho_L$$, \hspace{1cm} (60)

$$3\langle (2E_\gamma-m_b)^3 \rangle = - \rho_D^3 + \frac{1}{m_b^3} \left( \frac{3}{2} m_1 - 2 m_2 + \frac{1}{4} m_3 + \frac{8}{9} m_5 + \frac{1}{10} m_6 - \frac{9}{20} m_7 + \frac{27}{80} m_8 + \frac{1}{10} m_9 \right)$$

$$+ \frac{1}{m_b^3} \left( - \frac{16}{5} r_1 + \frac{51}{10} r_2 + \frac{9}{10} r_3 - \frac{17}{5} r_4 - \frac{27}{10} r_5 + \frac{63}{10} r_6 - \frac{8}{5} r_7 + 4 r_8 - \frac{11}{5} r_9 \right)$$

$$+ \frac{71}{10} r_{10} + \frac{9}{10} r_{11} + \frac{13}{5} r_{12} - \frac{7}{5} r_{13} - \frac{11}{5} r_{14} + \frac{23}{5} r_{15} + \frac{27}{10} r_{16} + \frac{9}{10} r_{17} - \frac{37}{10} r_{18}$$

$$- \frac{1}{2} (\mu_x^2 + 3 \mu_G^2) \rho_D$$ \hspace{1cm} (61)

The terms through $D=6$ in Eqs. (59) and (60) coincide with the known ones, cf. Ref. [22].

Numerical aspects have been analyzed similarly to the semileptonic moments, employing the factorization approximation of Sect. 3 for the expectation values. The corrections turn out rather small not only in the integrated width and in the average photon energy, but also in the second and even in the third moments corresponding, in the heavy quark limit, to the kinetic and Darwin expectation values, respectively. Moreover, accounting for the $D=8$ expectation values yields small effect compared to the $D=7$ one, except for the second moment where both are quite suppressed. Direct evaluation results in

$$\frac{\delta \Gamma(B \to X_s + \gamma)}{\Gamma(B \to X_s + \gamma)} = -0.0361/m_b^2 - 0.00531/m_b^2 + 0.000641/m_b^2 + 0.000151/m_b^2$$

$$\delta \langle 2E_\gamma \rangle = 11 \text{ MeV} / m_b^2 - 14 \text{ MeV} / m_b^3 + 3 \text{ MeV} / m_b^4 + 0.7 \text{ MeV} / m_b^5$$

$$12 \delta \langle (E_\gamma - \langle E_\gamma \rangle)^2 \rangle = -0.106 \text{ GeV}^2 / m_b^3 + 0.002 \text{ GeV}^2 / m_b^4 + 0.0025 \text{ GeV}^2 / m_b^5$$

$$24 \delta \langle (E_\gamma - \langle E_\gamma \rangle)^3 \rangle = 0.02 \text{ GeV}^3 / m_b^4 - 0.0025 \text{ GeV}^3 / m_b^5$$

(62)

the shifts in the second, third and fourth lines here may be interpreted as an apparent change, up to the sign, in $m_b$, $\mu_x^2$ and $-\rho_D^2$, respectively.

The small effect on $\mu_x^2$ and $\rho_D^2$, far below the level anticipated in Ref. [22], is somewhat surprising. The above numbers probably are smaller than the corrections due to the non-valence four-quark operators of the form $\bar{b}s\bar{s}b$ appearing at order $O(\alpha_s)$; they were discussed.
in Ref. [22], Sect. 4. It is conceivable that the numerical suppression we obtain is partially accidental or is an artifact of the factorization approximation for the expectation values. We do not dwell further on this here and plan to look into the issue in the subsequent studies.

6. Conclusions and Outlook

In this paper we report a detailed study of higher-order power correction in inclusive weak decays of heavy flavor hadrons, focused on the semileptonic $B$ meson decays. The calculations existed since the mid 1990s included $1/m_b^2$ and $1/m_b^3$ corrections. We extended the analysis to order $1/m_b^4$ and $1/m_b^5$: this is still done at tree level, without computing $\alpha_s$-corrections to the power suppressed Wilson coefficients.

The new calculations contain two elements: deriving the OPE terms proper through the order in question, and relating the $B$-meson expectation values of the new operators to a set of hadronic heavy quark parameters of a given dimension. This has been done in the most general setting, by calculating the weak decay structure functions of $B$ mesons through order $1/m_b^5$. They are expressed in terms of nine new expectation values at order $1/m_b^4$ and of eighteen independent expectation values at order $1/m_b^5$ (more operators appear with $\mathcal{O}(\alpha_s)$ corrections, cf. Ref. [4]). Structure functions allow one to calculate any inclusive differential distribution incorporating arbitrary lepton kinematic constraints.

The results for the structure functions are rather lengthy, especially so to order $1/m_b^5$, and we do not quote them; they are generated in a computer program and saved as Mathematica definition files. They are used in this form to derive the power corrections to various moments of the distributions, both without and with a cut on the charged lepton energy.

Since the fast growing number of new heavy quark parameters in successive orders in $1/m$, such an analysis would be of little practical value without means to estimate the emerging expectation values with a reasonable confidence. We have presented a formal derivation of the saturation representation for the heavy quark matrix elements of operators encountered in the tree-level OPE which had been used earlier [4], yet often regarded skeptically. Based on this exact representation a ground-state factorization has been formulated and we have presented the resulting expressions explicitly for all the encountered $B$-meson expectation values.

These two elements of the higher-order analysis allowed us to arrive at meaningful numerical estimates and in this way to analyze such important questions as the accuracy of the OPE related to the practical truncation of the $1/m_b$ expansion and the numerical convergence of the OPE series.

We conclude that the power expansion in inclusive $B$ decays derived from the OPE is numerically in a good shape as long as the lower cut on lepton energy remains soft. We find that, generally, the $1/m_b^4$ corrections are somewhat smaller than those at $1/m_b^3$ and typically tend to partially offset the latter; the situation is expected to be different for high moments of hadronic invariant mass. The $1/m_b^5$ terms in usually considered moments are already noticeably smaller and this suggests that the truncation error at this order is largely negligible in practice. The notable exception is the effect of ‘Intrinsic Charm’ (IC) on the total decay rate which had been argued [16] to be a potentially significant effect being driven, to higher orders, by an expansion in $1/m_c$ rather than in $1/m_b$.

We have also presented the analytic expressions for the corresponding higher-order corrections in the $B \to X_s + \gamma$ decays. A pilot evaluation suggested that here the effects are
numerically insignificant. A more reliable conclusion, however should include estimates for a wider range of hadronic parameters.

Based on the pattern of the numerical corrections found for various moments important in the fit to inclusive semileptonic data, we expect an overall moderate upward shift in the extracted value of $|V_{cb}|$ about a half percentage point. This comes as an interplay of the direct downward shift about 0.65% from the explicit corrections to the semileptonic width and a larger upward shift due to the change in the heavy quark parameters extracted from the fit to the moments.

The overall scale of the calculated higher-order power corrections shows that, as a rule, they are not negligible at the attained level of precision and are in line with the expectations laid down in Ref. [18]. Practically speaking, we expect the $1/m_b^4$ corrections to be of the same scale as the terms from not yet calculated $\alpha_s$-corrections to the Wilson coefficients of the chromomagnetic and of the Darwin operators.

We expect that the main effect of the estimated power corrections in the fit to the semileptonic data will be an increase in the Darwin expectation value $\rho_D^3$ by about 0.1 GeV$^3$, while $\mu_2^2$ would not change significantly; neither the main combination of quark masses $m_b - 0.7 m_c$ shaping the mass dependence will change essentially. This does not apply to $m_b$ or $m_c$ separately; the residual dependence on the absolute values is subtle and the changes in the central value of $m_b$ as large as 50 MeV may not have real significance.

An increase in the Darwin expectation value would be welcomed theoretically; based on the small velocity sum rules and assuming their early saturation we expect

$$\rho_D^3(0.6 \text{ GeV}) \approx 0.45 \text{ GeV} \cdot \mu_2^2(0.6 \text{ GeV}) \approx 0.15 \text{ GeV}^3; \quad (63)$$

The subtraction heavy quark parameters from the $E_\ell^{\text{cut}}$-dependence of the moments, a procedure effectively introduced in the fits to the data by assuming a strong correlation of theoretical uncertainties at different $E_\ell^{\text{cut}}$ for a given moment, looks an unsafe option. The cut-dependence of the moments is rather mild until the cut is placed relatively high, and there all corrections start to inflate degrading the theoretical accuracy of the OPE predictions. This is seen in the power expansion, and the similar behavior is expected from the uncalculated perturbative effects. The correction begin to blow up apparently somewhere near $E_\ell^{\text{cut}} \gtrsim 1.65 \text{ GeV}$; to remain on the safe side we suggest to rely on theory at $E_\ell^{\text{cut}} \leq 1.5 \text{ GeV}$.

**Note added.** When the present write-up was being finalized, P. Gambino informed us about a pilot fit to the semileptonic data including the higher-order corrections to the moments quoted here. The preliminary result did not follow our expectation that the bulk of the effects reduces to the increase in $\rho_D^3$, although the suppression of the shift in the extracted value of $|V_{cb}|$ was observed. We note, however, that the reported outcome was largely dependent on the assumed strong correlations in the theoretical uncertainties at different $E_\ell^{\text{cut}}$. More reliable conclusions can be drawn once a more general fit analysis is performed.

**Acknowledgments**

We thank Ikaros Bigi and Paolo Gambino for discussions. This work is supported by the German research foundation DFG under contract MA1187/10-1 and by the German Ministry of Research (BMBF), contracts 05H09PSF; partial support from the RSGSS-65751.2010.2 grant is gratefully acknowledged.
7. Appendices

A. Conventional derivation of the saturation representation

Here we present a conventional derivation of the representation of Eqs. (32), (33) as a sum over intermediate states, based on the usual field-theoretic language in the framework of the effective field theory. We start with the case of operator \( A \) without time derivatives:

\[
\langle B | \bar{b} A C b(0) | B \rangle = \int d^3 \vec{x} \, \delta^3(\vec{x}) \, \langle B | \bar{b} A(0, \vec{x}) C b(0) | B \rangle; \quad (A.1)
\]

time and space coordinates are explicitly shown separately where necessary. \( A, C \) typically contain derivatives; to avoid related ambiguities one may assume that \( A \) acts on the left, on the \( \bar{b} \) field.

Using the property of the infinitely heavy quark fields \( Q \) in the sector with a single heavy quark \( \bar{Q}(0, \vec{x}) Q(0) = \delta^3(\vec{x}) \frac{1 + \gamma_0}{2} \) (which is nothing but to say they are static) we represent the matrix element as

\[
\langle B | \bar{b} A C b(0) | B \rangle = \int d^3 \vec{x} \, \sum_n \langle B | \bar{b} A(0, \vec{x}) | n \rangle | n \rangle | Q C b(0) | B \rangle. \quad (A.3)
\]

The summation over intermediate states includes both summing over internal degrees of freedom for a particular heavy hadron state (this is traditionally assumed by quantum-mechanical summation over states where the center-of-mass motion is factored out), and also integration over the momentum \( \vec{p} \) of the hadronic system as a whole:

\[
\sum_n = \int \frac{d^3 \vec{p}}{(2\pi)^3} \sum_{\text{states}}, \quad |n\rangle = |n_{QM}, \vec{p}\rangle; \quad (A.5)
\]

below we omit the subscript QM when refer to the conventional states with vanishing total spatial momentum (>quantum-mechanical< states).

Assuming \( B \)-meson is at rest, the first transition matrix element in the r.h.s. of Eq. (A.4) has the following dependence on \( \vec{x} \):

\[
\langle B | \bar{b} A b(0, \vec{x}) | n, \vec{p} \rangle = e^{i \vec{p} \vec{x}} (B | \bar{b} A Q(0) | n) ; \quad (A.6)
\]

the integration over \( d^3 \vec{x} \) then yields \((2\pi)^3 \delta^3(\vec{p})\) which removes the integration over the center of mass momentum \( \vec{p} \) of the intermediate state \(|n\rangle\), and we arrive at

\[
\langle B | \bar{b} A C b(0) | B \rangle = \frac{1}{2M_n} \sum_n \langle B | \bar{b} A Q(0) | n \rangle | Q C b(0) | B \rangle, \quad (A.7)
\]

where \(|n\rangle\) include only the states at rest; the factor \( \frac{1}{2M_n} \) appears if one uses their standard relativistic normalization.
To cover the case of time derivatives we apply the above relation to the following product:

\[
\langle B | \bar{b} A \pi_x^0 C b(0) | B \rangle = \frac{1}{2M_n} \sum_n \langle B | \bar{b} A Q(0) | n \rangle \langle n | \pi_x^0 C b(0) | B \rangle,
\]

(A.8)

where time derivative acts on the right. Now, for any local operator \( O \) the relation holds

\[
i \partial_0 \langle n | O(x_0) | B \rangle = (E_n - E_0) \langle n | O(x_0) | B \rangle,
\]

(A.9)

which expresses the fact that the time derivative is given by the commutator with Hamiltonian.

In the case of heavy quark operators one can additionally apply equation of motion for the heavy quark field \( \bar{Q} \):

\[
i \partial_0 \langle n | \bar{Q} C b(x_0, \vec{0}) | B \rangle = \langle n | \bar{Q} \pi_0 C b(0) | B \rangle + \langle n | \bar{Q} \pi_0 C b(0) | B \rangle = \langle n | \bar{Q} \pi_0 C b(0) | B \rangle.
\]

(A.10)

Combining this with Eq. (A.9) and repeating, if necessary, this reduction we obtain

\[
\langle n | \bar{Q} \pi_x^0 C b(0) | B \rangle = (M_B - M_n)^k \langle n | \bar{Q} \pi_0 C b(0) | B \rangle.
\]

(A.11)

Therefore, Eq. (A.8) generalizes relation (A.7) in the following way:

\[
\langle B | \bar{b} A \pi_x^0 C b(0) | B \rangle = \frac{1}{2M_n} \sum_n \langle B | \bar{b} A Q(0) | n \rangle (M_B - M_n)^k \langle n | Q C b(0) | B \rangle,
\]

(A.12)

in agreement with the OPE result Eq. (33).

B. Ground-state contribution in conventional Lorentz-covariant trace formalism

Here we derive the expressions for the sum over ground-state heavy quark symmetry multiplet of meson states, encountered in the factorization approximation for the \( D = 7 \) and 8 expectation values, in the framework of conventional trace formalism. It was developed in particular to describe the heavy quark spin multiplets at different velocities. Since only heavy states at rest enter our problem, full Lorentz covariance is superfluous; of four spinor components used in conventional formalism only two are independent, corresponding to the spin indices of \( \Omega_0 \)-states considered in Sect. 3.2.

With the spin of the heavy quark dynamically decoupled in the heavy quark limit, the \( B \) and \( B^* \) meson wavefunctions at rest \( M \) and \( M^{(s)}_\lambda \) can be represented as matrices

\[
M = \sqrt{M_B} \frac{1 + \gamma_0}{2} i \gamma_5 \quad M^{(s)}_\lambda = \sqrt{M_B} \frac{1 + \gamma_0}{2} (\vec{\gamma} \vec{\epsilon}_\lambda),
\]

(A.13)

where we have equated the \( B \) and \( B^* \) masses in the heavy quark limit. One of the indices in these matrices corresponds to the heavy quark spin and another to that of the light degrees of freedom. Spin symmetry yields the well known trace formula for the matrix elements:

\[
\langle H | \bar{b} D_{\mu} i D_{\nu} \Gamma b | H \rangle = -\text{Tr} \left[ \tilde{M}_H \Gamma M_H A^{\text{light}}_{\mu \nu} \right]
\]

(A.14)

where \( H \) and \( H' \) are either \( B \) or \( B^* \), \( M_{H,H'} \) are the corresponding meson wavefunctions and \( A^{\text{light}}_{\mu \nu} \) encodes dynamics associated with light degrees of freedom. While a complicated
unknown hadronic tensor in the general situation, $\Lambda_{\mu\nu}^{\text{light}}$ takes a simple form for mesons at rest:

$$\Lambda_{\mu\nu}^{\text{light}} \big|_{(w)\gamma=1} = -\frac{\mu_2^2}{3} \Pi_{\mu\nu} + \frac{\mu_2^2}{6} \epsilon_{\mu\nu},$$  \hspace{1cm} (A.15)

components other than spatial vanish. In fact, $\Lambda_{jk}^{\text{light}}$ amounts to the corresponding matrix elements of the heavy quark states $\Omega_0$ of Sect. 3.2.

In order to evaluate the ground-state contribution for the matrix element (36) we must sum over the states of the $(B, B^*)$ spin-symmetry doublet. To do this we employ the relation valid for arbitrary $R$

$$-\frac{1}{2M_B} \text{Tr} \left[ R M \right] M - \sum_{\lambda} \frac{1}{2M_B} \text{Tr} \left[ R M_{\lambda}^{(*)} \right] M_{\lambda}^{(*)} = \frac{1+\gamma_0}{2} R \frac{1-\gamma_0}{2}$$  \hspace{1cm} (A.16)

expressing the completeness of the $B, B^*$ states in the spin space. Evaluating the right hand side of (36) in the ground-state saturation with the trace formula (A.14) and using relation (A.16) we find

$$\frac{1}{2M_B} \langle B | b D_j i D_k b | B \rangle \langle b D_j i D_k b | B \rangle - \frac{1}{2M_B} \langle B | b D_j i D_k b | B \rangle \langle b D_j i D_k b | B \rangle^* \langle B | b D_j i D_k b | B \rangle =$$

$$\frac{1}{2M_B} \left[ \text{Tr} \left[ M M \Lambda_{jk}^{\text{light}} \right] \text{Tr} \left[ \text{Tr} \left[ M M M \Lambda_{lm}^{\text{light}} \right] \right] + \sum_{\lambda} \frac{1}{2M_B} \text{Tr} \left[ M M_{\lambda}^{(*)} \Lambda_{jk}^{\text{light}} \right] \text{Tr} \left[ M_{\lambda}^{(*)} \Gamma M \Lambda_{lm}^{\text{light}} \right] =$$

$$-\text{Tr} \left[ -\frac{1}{2} \Gamma M \Lambda_{lm}^{\text{light}} \Lambda_{jk}^{\text{light}} \right] = -\text{Tr} \left[ M M \Lambda_{lm}^{\text{light}} \Lambda_{jk}^{\text{light}} \right].$$  \hspace{1cm} (A.17)

The product of the two hadronic tensors $\Lambda_{\mu\nu}$ in (A.15) is given by

$$A_{\mu\nu}^{\text{light}} \ A_{\rho\sigma}^{\text{light}} = \frac{(\mu_2^2)^2}{9} g_{jk} g_{lm} - \frac{\mu_2^2 \epsilon_{G}^2}{18} (g_{jk} i \sigma_{lm} + i \sigma_{jk} g_{lm}) +$$

$$\frac{(\mu_2^2)^2}{36} (g_{jm} g_{kl} - g_{jl} g_{km} + g_{jm} i \sigma_{kl} - g_{jl} i \sigma_{km} + i \sigma_{jm} g_{kl} - i \sigma_{jl} g_{km}).$$  \hspace{1cm} (A.18)

In the static limit we therefore obtain for spin-singlet and spin-triplet $B$ expectation values

$$\frac{1}{2M_B} \langle B | b D_j i D_k D_l b | B \rangle = \frac{(\mu_2^2)^2}{9} g_{jk} g_{lm} + \frac{(\mu_2^2)^2}{36} (g_{jm} g_{kl} - g_{jl} g_{km})$$  \hspace{1cm} (A.19)

$$\frac{1}{2M_B} \langle B | b D_j i D_k D_l D_m b | B \rangle - i \sigma_{ab} | b b | B \rangle = -\frac{(\mu_2^2)^2}{18} (g_{jk} g_{la} g_{mb} - g_{jk} g_{lb} g_{ma} +$$

$$g_{lm} g_{ja} g_{kb} - g_{lm} g_{jb} g_{ka}) + \frac{(\mu_2^2)^2}{36} \left[ g_{jm} (g_{lb} g_{ka} - g_{la} g_{kb}) - g_{jl} (g_{ka} g_{mb} - g_{kb} g_{ma}) +$$

$$g_{kl} (g_{ja} g_{mb} - g_{jb} g_{ma}) - g_{km} (g_{ja} g_{lb} - g_{ja} g_{lb}) \right],$$  \hspace{1cm} (A.20)

reproducing Eqs. (42) and (43).

For the $D=8$ operators with four spatial and one time derivative we follow the same route, yet one of the two $\Lambda_{\mu\nu}$ now describes the operator of the form $b D_j i D_0 i D_k \Gamma b$. Denoting the corresponding hadronic tensor of light degrees of freedom by $R_{\mu\nu}^{\text{light}}$, we express it through the Darwin and Spin-Orbit expectation values:

$$R_{\mu\nu}^{\text{light}} \big|_{(w)\gamma=1} = \frac{\rho_D^2}{3} \Pi_{\mu\nu} + \frac{\rho_{LS}^2}{6} i \sigma_{\mu\nu}.$$  \hspace{1cm} (A.21)
Repeating the same steps we end up with

\[
\frac{1}{2M_B} \langle B | \bar{b} i D_j i D_0 i D_k i D_l i D_m b | B \rangle = - \frac{\rho^3 D}{9} \mu^2 \pi g_{jm} g_{kl} + \frac{\rho^3 L S \mu^2}{36} (g_{jm} g_{kl} - g_{jl} g_{km}) \quad (A.22)
\]

\[
\frac{1}{2M_B} \langle B | \bar{b} i D_j i D_0 i D_k i D_l i D_m (-i \sigma_{ab}) b | B \rangle = \frac{\rho^3 D}{18} \mu^2 (g_{ja} g_{mb} - g_{ja} g_{ma}) - \frac{\rho^3 L S}{18} \mu^2 g_{lm} (g_{ja} g_{kb} - g_{j b} g_{ka}) + \frac{\rho^3 L S}{36} [g_{jm} (g_{ka} g_{mb} - g_{ka} g_{mb}) - g_{jl} (g_{ka} g_{mb} - g_{kb} g_{ma}) + g_{kl} (g_{ja} g_{mb} - g_{ja} g_{mb}) - g_{km} (g_{ja} g_{mb} - g_{ja} g_{mb})], \quad (A.23)
\]

identical to Eqs. (45) and (46).

References

[1] S. Biswas and K. Melnikov, JHEP 1002 (2010) 089 arXiv:0911.4142 [hep-ph], and references therein to earlier two-loop calculations.

[2] T. Becher, H. Boos and E. Lunghi, JHEP 0712 (2007) 062 arXiv:0708.0855 [hep-ph].

[3] B. M. Dassinger, T. Mannel and S. Turczyk, JHEP 0703 (2007) 087 arXiv:hep-ph/0611168.

[4] I.I. Bigi, N. Uraltsev and R. Zwicky, Eur. Phys. J. C 50 (2007) 539.

[5] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Fortsch. Phys. 32 (1984) 585.

[6] T. Mannel, Phys. Rev. D 50 (1994) 428 arXiv:hep-ph/9403249.

[7] I.I. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. D 52 (1995) 196.

[8] B. Blok, L. Koyrakh, M. A. Shifman and A. I. Vainshtein, Phys. Rev. D 49 (1994) 3356 [Erratum-ibid. D 50 (1994) 3572] arXiv:hep-ph/9307247.

[9] M. Gremm and A. Kapustin, Phys. Rev. D 55 (1997) 6924 arXiv:hep-ph/9603448.

[10] I.I. Bigi, M. Shifman and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. 47 (1997) 591.

[11] N. Uraltsev, Phys. Lett. B 545 (2002) 337.

[12] N. Uraltsev, Phys. Lett. B 585 (2004) 253.

[13] R. Lebed and N. Uraltsev, Phys. Rev. D62 (2000) 094011.

[14] P. Gambino, T. Mannel and N. Uraltsev, Phys. Rev. D 81 (2010) 113002 arXiv:1004.2859 [hep-ph].

[15] P. Gambino, T. Mannel and N. Uraltsev, paper in preparation.

[16] D. Benson, I.I. Bigi, T. Mannel and N. Uraltsev, Nucl. Phys. B 665 (2003) 367.
[17] N. Uraltsev, Proc. of the 31st Int. Conference on High Energy Physics, Amsterdam, The Netherlands, 25-31 July 2002 (North-Holland – Elsevier, The Netherlands, 2003), S. Bentvelsen, P. de Jong, J. Koch and E. Laenen Eds., p. 554; arXiv:hep-ph/0210044.

[18] P. Gambino and N. Uraltsev, Eur. Phys. J. C34 (2004) 181.

[19] M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. D 51 (1995) 2217.

[20] I. Bigi, T. Mannel, S. Turczyk and N. Uraltsev, JHEP 1004, (2010) 073.

[21] J. Abdallah et al. [DELPHI Collaboration], Eur. Phys. J. C 45 (2006) 35

[22] D. Benson, I. Bigi and N. Uraltsev, Nucl. Phys. B710 (2005) 371.