On the effectiveness of the protocol creating the maximum entanglement of two charge-phase qubits by cavity field

C. Li
Institute of Theoretical Physics, The Chinese Academy of Science, Beijing, 100080, China

Y. B. Gao
Institute of Theoretical Physics, The Chinese Academy of Science, Beijing, 100080, China and Applied Physics Department, Beijing University of Technology, Beijing, 100022, China
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We revisit the protocols to create maximally entangled states between two Josephson junction (JJ) charge phase qubits coupled to a microwave field in a cavity as a quantum data bus. We devote to analyze a novel mechanism of quantum decoherence due to the adiabatic entanglement between qubits and the data bus, the off-resonance microwave field. We show that even through the variable of the data bus can be adiabatically eliminated, the entanglement between the qubits and data bus remains and can decoher the superposition of two-particle state. Fortunately we can construct a decoherence-free subspace of two-dimension to against this adiabatic decoherence. To carry out the analytic study for this decoherence problem, we develop Fröhlich transformation to re-derive the effective Hamiltonian of these system, which is equivalent to that obtained from the adiabatic elimination approach.

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I. INTRODUCTION

As a useful quantum resource, entanglement can not only be used to test fundamental principles in quantum mechanics, such as Bell’s inequalities, but also play a central role in quantum information processing including quantum computation, quantum teleportation and quantum cryptography. Therefore how to create a stable and controllable entangled state in a quantum bits (qubit) system is very important for quantum information protocols. A number of protocols have been proposed to produce quantum entanglement in different qubit systems, such as NMR, polarization photon, quantum dots, Josephson junction. Due to the prompt progresses in preparing various solid state qubits, these schemes become very promising to realize the practical quantum computing. Actually, according to the DiVincenzo criteria the couplings JJ qubits for quantum computation, the solid system is one of the best candidates for quantum computation, since qubit should be scalable, controllable and with longer decoherence time. Actually it seems difficult to fulfill all the requirements by quantum information processing. Recently several groups have demonstrated the macroscopic quantum coherence of Josephson junction (JJ) qubits with long decoherence time in experiments.

Quantum entanglement plays the central role in integrating multi-qubit to form a scalable quantum computing. We notice that, in most of the protocols to produce such JJ qubit entanglement, and correspondingly to carry out two qubit logic gate operations, each qubit interacts with a common quantum object as a data bus, which may be an electromagnetic cavity field, a quantum transmission line coplanar cavity or an nano-mechanical resonator. If the characterized frequency of the quantum data bus is off-resonate to the energy spacing of the qubit, the degree of freedom of the quantum data bus and the variables of the quantum object can be separated adiabatically form that of two qubit system. Then the induced inter-qubit interactions can create an efficient quantum entanglement of two qubits.

However, as we have investigated, there usually exists quantum entanglement between the states of data bus and those of the two qubit system even after removing the data bus. This adiabatic quantum entanglement has been studied according to the generalized Born-Oppenheimer (BO) approximation where the slow variables can be driven by different effective potentials provided by the fast internal states and then the entanglement between fast and slow variables forms. Recently, Averin et al. similarly considered the adiabatic entanglement of two JJ charge qubits. In this investigation, two JJ charge qubits are assumed to be coupled with a large junction which works as a faster data bus. With the BO adiabatic approximation, the energy of the lowest band of the latter junction can be considered as the effective interaction between the two JJ charge qubits. But the quantum decoherence induced by the adiabatic entanglement has not been considered here though it may occur in the case with higher excitation of large junction.

In this paper, we are also specific to the JJ qubit system coupling the cavity and show that the adiabatic entanglement may cause the extra errors of the logic gate operation for this two qubit system with high-excitation. We will consider decoherence of the JJ qubit caused by...
the thermal excitation of large junction through this adiabatic entanglement mechanism. Actually, without considering thermal excitation, we are not clear if the created entanglement between two JJ qubits is stable since it can be produced according to an effective Hamiltonian, which is obtained in usual by ’ignoring’ intermediate variables of data bus [12].

To carry out a totally analytic study, we utilize the generalized Fröhlich transformation to re-derive the effective Hamiltonian of this system. In this way we can study in details this novel decoherence phenomenon for the entanglement of two JJ-qubits. There exist four entangled states for two JJ qubit system, including two maximally entangled states that can be obtained by controllable the microwave field. Though the superposition of some two qubit states can decoher due to the adiabatic entanglement, there exist a decoherence-free subspace, against to the decoherence induced by the adiabatic separation process. Therefore, we found that only two of four maximal entangled states are stable in this scheme.

II. THE MODEL OF TWO JJ QUBITS IN CAVITY

Without loss of generality, we investigate a simplified model, which consisting of two JJ qubits in a cavity with a single mode micro-wave fields (FIG. 1). The Hamiltonian of the coupled system $H$ can be described as a sum of that of the junctions, the cavity field and a interaction term between the cavity and the junction $\mathbb{II}$, i.e,

$$ H = \hbar \omega a^\dagger a + 4E_{C1} (n_1 - n_{g1})^2 - E_{J1} (\Phi) \cos \varphi_1 + 4E_{C2} (n_2 - n_{g2})^2 - E_{J2} (\Phi) \cos \varphi_2, \tag{1} $$

where

$$ E_{C} = e^2 / 2 (C_g + 2C_j) \tag{2} $$

is the single-particle charging energy of the island, $C_j$ the capacitance of the junction, $C_g$ the capacitance of gate and $\varphi_i$ the phase difference between points on the opposite sides of the $i$-th junction. The Josephson coupling energy

$$ E_{J} (\Phi) = 2E_{J0} \cos \left( \frac{2\pi \Phi}{\Phi_0} \right) \tag{3} $$

depends on the the total flux $\Phi$ and the maximal coupling energy $E_{J0} = I_c \frac{\Phi_0}{2\pi}$. Here, $I_c$ is the critical current of the junction, and $\Phi_0$ the total flux and flux quanta.

When a nonclassical microwave field with the vector potential

$$ \vec{A}(r) = -u_{\lambda}(r) a + u_{\lambda}^\dagger(r) a^\dagger. \tag{4} $$

is applied, where $a^\dagger$ and $a$ are the creation and annihilation operators of the cavity fields, the total flux $\Phi$ is divided into two part

$$ \Phi = \Phi_e + \Phi_f. \tag{5} $$

$\Phi_e$ is static magnetic flux through the SQUIDs and

$$ \Phi_f = |\Phi_\lambda| (e^{-i\theta} a + e^{i\theta} a^\dagger), \tag{6} $$

the microwave-filed-induced flux through the SQUIDs where

$$ \Phi_\lambda = \oint \vec{u}_{\lambda}(r) \cdot d\vec{l}. \tag{7} $$

We take $E_{C1} = E_{C2}$ and $E_{J1}(\Phi) = E_{J2}(\Phi)$ and then the Hamiltonian of the system becomes

$$ H = \varepsilon (V_g) (\sigma_1^x + \sigma_2^x) + \hbar \omega a^\dagger a - 2E_{J0} \cos \left( \frac{\pi \Phi_e + \Phi_f}{\Phi_0} \right) (\sigma_1^x + \sigma_2^x), \tag{8} $$

where the quasi-spin operators $\sigma_x, \sigma_y$, and $\sigma_z$ are defined with respect to the the states $|0\rangle$ and $|1\rangle$ of no (one) excess cooper pair on the island. To form a qubit or a two-level system, one need to tune the gate voltage $V_g$ so that $n_g$ is approximately a half-integer. In this case the charge eigen-states are $|0\rangle$ and $|1\rangle$. We assume $|\Phi_\lambda| < \Phi_0$, and focus on the charging regime $E_{C} \gg E_{J}$.

Then, the Hamiltonian can be approximated as

$$ H = H_0 + H_I: \tag{9} $$

$$ H_0 = \hbar \omega_{I} (\sigma_1^x + \sigma_2^x) + \hbar \omega a^\dagger a \tag{9a} $$

$$ H_I = g (a + a^\dagger) (\sigma_1^x + \sigma_2^x) \tag{9b} $$

where

$$ \hbar \omega_{I} = 2E_{C} \left( \frac{C_g V_g}{e} - (2n + 1) \right) \tag{9c} $$

and the coupling constant between qubit and the cavity field is

$$ g = -2I_c \frac{\Phi_0}{2\pi} \sqrt{\frac{\hbar \nu}{2\mu_0}} \int_{\mathcal{S}} \vec{e} \cdot d\mathcal{S} \sin \frac{\phi}{\phi_0} \pi. \tag{9d} $$

FIG. 1: SQUID S1 and SQUID S2 in a cavity coupled to a microwave field.
In practice we take the volume of the cavity and the wavelength of microwave respectively as \(1\) cm and \(1\) cm, the dimension of the Josephson junction as \(1\) \(\mu\)m, the critical current of the junction as \(I_c \approx 10^{-5}\) A. Due to Eq.(4), we have \(\frac{\theta}{\hbar \omega} \ll 1\), which means \(H_I \ll H_0\). So we can perform perturbation theory represented by a generalized Fröhlich transformation\[13\] on the Hamiltonian(2). Then we can obtain the effective Hamiltonian of two JJ qubit by removing the variables of the microwave field approximately.

### III. THE EFFECTIVE HAMILTONIAN FROM THE GENERALIZED FRÖHLICH TRANSFORMATION

In its original approach for superconductivity BCS theory, the Fröhlich transformation\[13\] is utilized to get the effective Hamiltonian for electron-electron interaction from electron-phonon interaction. In general we can consider a interaction system described by a sum of free Hamiltonian and interaction Hamiltonian,

\[
H = H_0 + H_I. \tag{11}
\]

Comparing with the free part \(H_0\), the interaction part \(H_I\) can be regard as a perturbation. Let us define an anti-Hermitian operator \(S\), and a corresponding unitary operator \(U = \exp\{-S\}\). We perform an unitary transformation on the Hamiltonian(11) by this unitary operator, and then get the equivalent Hamiltonian as

\[
H = U\dagger H U
= H_0 + \sum_{n=1} \frac{(-1)^n}{(n+1)!} [S, \ldots [S, S, H_I] \ldots]. \tag{12}
\]

Since the unitary transformation \(U\) is time-independent, Hamiltonians (11) and (12) describe the same physical process. We can take both the interaction \(H_I\) and operator \(S\) in the first order terms in the right hand side. At the same time, we require the operator \(S\) to satisfy the following condition

\[
H_I + [H_0, S] = 0. \tag{13}
\]

In Eq.(12), if we discard the higher order terms and only keep the second-order term, the effective Hamiltonian can be achieved approximately as

\[
H_{\text{eff}} \approx H_0 + \frac{1}{2} [H_I, S]. \tag{14}
\]

From the Eq.(13) we certainly know how to construct the anti-Hermitian operator \(S\), which has the following form

\[
S = \sum_{m \neq n} \frac{(H_I)_{mn}}{E_m - E_n} |m\rangle \langle n|, \tag{15}
\]

where \(|m\rangle\) and \(E_m\) are the eigenvectors and eigenvalues of \(H_0\) respectively. The transformation, by which one can draw out effective Hamiltonian(14) from the Hamiltonian(11), is the so-called general Fröhlich transformation. It has been proved in Ref\[13\] that this generalized Fröhlich transformation is just equivalent to the second-order perturbative theory.

Now we use the above approximation method to derive the effective Hamiltonian for the two-JJ qubit entanglement. Under the condition \(g \ll \hbar \omega\), we explicitly construct the anti-Hermitian operator \(S\) of Eq.(3) as following

\[
S = \frac{g}{2} \{\Delta_+ (a - a^\dagger) (\sigma_x^1 + \sigma_x^2)
+ i \Delta_- (a + a^\dagger) (\sigma_y^1 + \sigma_y^2)\} \tag{16}
\]

where the coefficients

\[
\Delta_{\pm} = \left(\frac{1}{\hbar \omega - 2 \hbar \omega_1} \pm \frac{1}{\hbar \omega + 2 \hbar \omega_1}\right) \tag{17}
\]

Using the above explicit expression for anti-Hermitian operator \(S\), we can finish the generalized Fröhlich transformation and then obtain effective Hamiltonian obviously.

\[
H_{\text{eff}} = \hbar \omega_1 \left(\sigma_x^1 + \sigma_x^2\right) - \frac{g^2}{2} \Delta_- (a + a^\dagger)^2 (\sigma_z^1 + \sigma_z^2)
+ \hbar \omega_1 a - \frac{g^2}{2} \Delta_+ \sigma_z^1 \sigma_x^2. \tag{18}
\]

If the micro-wave field is very weak, we can discard the second terms of \(a^2\) and \(a^4\) in the effective Hamiltonian under the rotating wave approximation. Then the effective Hamiltonian Eq.(18) reads

\[
H_{\text{eff}} = \left(\hbar \omega_1 + g^2 \Delta_- a_a^\dagger\right) (\sigma_x^1 + \sigma_x^2)
+ \hbar \omega_1 a + \frac{g^2}{2} \Delta_+ \sigma_z^1 \sigma_x^2. \tag{19}
\]

In general this is a typical effective Hamiltonian leading the two-qubit quantum logic gate. In usual it is obtained by adiabatically eliminating the variable data bus with various methods\[12\]. However, in most of previous works, this crucial terms

\[
g^2 \Delta_- a_a^\dagger (\sigma_x^1 + \sigma_x^2) \tag{21}
\]
referred to the ac Stark effect (a dispersive frequency shift effects) has been either irrationally ignored or passed over in silence. This is unsatisfactory even though we can now prove that it can run the logic gate of two qubit system in next section.

IV. A NOVEL DECOHERENCE MECHANISM: THE INVERSE STERN-GERLACH EFFECT

Having gotten the effective Hamiltonian with a perfect inter-qubit interaction, we can show how to create the quantum entanglement by controllable the coupling between photon and qubit. As a data bus, the role of cavity field is to introduce extra controllable parameters. In the next section we will show the details to implement an ideal two qubit logical gate operations in the decoherence free subspace (DFS). However, for those states outside the DFS, we can demonstrate a novel decoherence phenomenon related to the so-called inverse Stern-Gerlach effect from the adiabatic variable separation based on the BO approximation.

Let us generally consider the adiabatic evolution of two identical charge qubits 1 and 2, coupled to a single-mode field in the microwave cavity. In the off-resonance case, the motion of the qubits does not excite the transitions from a cavity mode to another, and then the photon number is conserved, i.e.,

\[ [H_{\text{eff}}, a^\dagger a] = 0 \]  

This shows that the total wave function will adiabatically keep the factorized structure

\[ |\Psi(t)\rangle = |\phi_n(t)\rangle \otimes |n\rangle \]  

during evolution only if the cavity is exactly prepared initially in a single number state with definite phonon number, namely, a Fock state \(|n\rangle\). In this case the qubit part is just governed by the effective Hamiltonian \(|n\rangle H_{\text{eff}} |n\rangle = H(n)\) and then we can manipulate the qubit system according to the \(n\)-dependent Hamiltonian to form maximal entanglement. However, if one cannot prepare the cavity in a single Fock state, the part \(|\phi(t)\rangle\) of qubit must depend on the different phonon number \(n\) and then we cannot make an exact manipulation for qubit part due to this correlation to cavity field. This kind feature of quantum adiabatic entanglement is just a novel physical source of the quantum decoherence in the process of two qubit logical gate operations.

We remark that this phenomenon is an analog of "inverse Stern Gerlach effect" in atomic optics, in which discrete atomic trajectories are correlated to different photon numbers in the cavity. When the atom is non-resonant with the cavity modes, there appears a dispersive frequency shift effects affecting both the atomic transition and the field mode. It can be interpreted as single atom and single photon index effects. These effects lead to various interesting potential applications, which have been investigated in cavity QED for atoms, both theoretically and experimentally, e.g., the interference schemes to measure matter-wave phase shifts produced by the non resonant interaction. This schemes performs a quantum non-demolition measurement of photon numbers in a cavity, at the single photon level. Its experimental demonstration is based on the detection of Ramsey resonances on circular Rydberg atoms crossing a very high Q cavity. For this kind of "inverse Stern Gerlach effect", we even presented an extensive generalization based on the Born-Openheimer approximation to analyzed the the adiabatic separation induced quantum entanglements. Thus it defines the adiabatic quantum decoherence in general case. We can discuss this effect for a solid state based system with two charge qubit.

To see the quantum decoherence due to the generalized "inverse Stern Gerlach effect", we assume the two JJ qubits and the cavity field are initially prepared in a factorizable state:

\[ |\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle \]  

where \(|\phi\rangle\) is the initial state of the two JJ qubits and \(|\varphi\rangle\) the state of the field. In general, if the cavity is prepared initially in a superposition state of Fock state

\[ |\varphi\rangle = \sum_n c_n |n\rangle \]  

rather than a single Fock states, the total system will evolve according to

\[ |\Psi(t)\rangle = \sum_n c_n |\phi_n(t)\rangle \otimes |n\rangle \]  

where

\[ |\phi_n(t)\rangle = U_n(t) |\phi(0)\rangle . \]  

The effective evolution matrix \(U_n(t) = \exp[-iH(n)t]\) is governed by \(H(n)\). This "inverse Stern Gerlach effect" result from the dependence \(|\phi_n(t)\rangle\) to different \(n\).

Let \(|m\rangle\) be the single Fock state that we want to prepare and \(H(m)\) be the controlled Hamiltonian. Then we can characterize the difference between the real evolution and the ideal one \(\rho_m(t) = |\phi_m(t)\rangle \langle \phi_m(t)|\), by the fidelity

\[ F = Tr[\rho_m(t) \rho(t)] = \sum_n |c_n|^2 |\langle \phi_m(t)|\phi_n(t)\rangle|^2 \]  

where

\[ \rho(t) = Tr_C (|\Psi(t)\rangle \langle \Psi(t)|) = \sum_n |c_n|^2 |\phi_n(t)\rangle \langle \phi_n(t)|\]  

is the reduced density matrix of the two JJ qubits.

Usually it is difficult to prepare the Fock state \(|m\rangle\) and we can only use the coherent state \(|\alpha\rangle\) with average photon number \(\langle \alpha|a^\dagger a|\alpha\rangle = m\). Then we assume the
junctions are initially in the state $|0\rangle_1 |0\rangle_2$, and the initial state of the micro-wave field is the coherent state $|\alpha\rangle$. Then total system will evolve into

$$|\Psi\rangle = e^{-\frac{i}{2}|m|^2} \sum_n \frac{m^n}{n!} |\phi^{00}_n(t)\rangle |n\rangle,$$

where the time-dependent coefficients are

$$|\phi^{00}_n(t)\rangle = \{c_2 (n) |0\rangle_1 |0\rangle_2 + ic_1 (n) |1\rangle_1 |1\rangle_2\}.$$

The above equation shows that, when we prepare

$$\langle |\phi^{00}_n(t)\rangle |\phi^{00}_m(t)\rangle \rangle = e^{-\frac{i}{2}|m|^2 |m|^{2m}/m!}.$$

When $m \rightarrow \infty$, $P_m \rightarrow 0$ quickly.

The above discussion shows us that, when we prepare the controllable cavity field in different initial states, one can get different entangled states for the two qubit. This motivates us to explore the possibility to realize the perfect logic gate operation by initially preparing the micro-wave field in coherent state and Fock state. Let us consider the above mentioned problem in the following.

We aim to get a standard Bell state $|\phi^+\rangle$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}} ((|00\rangle + |11\rangle))$$

from the state $|0\rangle_1 |0\rangle_2$, by preparing the cavity field initially in Fock state $|k\rangle$. The evolution from $|0\rangle_1 |0\rangle_2$ to $|\phi^+\rangle$ naturally realize a perfect ideal logic gate.

The real evolution process governed by the effective adiabatic Hamiltonian $H(k)$ is

$$|\phi^{00}_k(t)\rangle = e^{-\frac{iH(k)t}{\hbar}} |00\rangle$$

$$= c_2^* (k) |00\rangle + ic_1 (k) |11\rangle,$$

where the Hamiltonian $H(k)$ corresponds to the Fock state $|k\rangle$ for fixed $k$. We can use the square of the norm of the inner product

$$f_k = \left| \langle \phi^{00}_k(t) | \phi^+ \rangle \right|^2$$

FIG.2 and FIG.3 illustrate that the fidelity decays sharply with the value of $m$, namely, coherent state $|\alpha\rangle$ leads to big deviation of $\rho_0$ and $\rho_m$ with big $\alpha$. This is because that coherent state $|\alpha\rangle$ is just in Fock state $|m\rangle$ with the probability $P_m$.

$$P_m |\alpha = m\rangle = |\langle m | \alpha \rangle|^2 |\alpha = m\rangle = e^{-|m|^2 |m|^{2m}/m!}.$$
It is shown from the FIG.4 and FIG.5, whatever the state of the microwave field is prepared in, the ideal logic gate operation can not be realized in this system. But this does not means that we can not obtain any maximal entangled state in this way. We will discuss this problem in next section.

V. CREATING MAXIMAL ENTANGLEMENT IN THE DECOHERENCE FREE SUBSPACE

From the discussions in the above section, we find that the maximally entangled state can not be obtained only from the state $|0\rangle_1 |0\rangle_2$ and $|1\rangle_1 |1\rangle_2$, even though one can prepared external controlled microwave cavity field in an arbitrary state. While the other two states $|0\rangle_1 |1\rangle_2$ and $|1\rangle_1 |0\rangle_2$ can span a decoherence-free subspace (DFS) $\mathcal{M}$, it means that any superposition of the state $|0\rangle_1 |1\rangle_2$ and $|1\rangle_1 |0\rangle_2$ can evolve into this kind of DFS. Easily seen in Eq.(19), the effective interaction between the cavity and qubits $g^2 \Delta \sigma_z$ vanishes in the DFS and can not distinguish between any two states in this DFS. So we conclude that there is not a "which-way detection" to determine the "paths" in this case, i.e., there is not decoherence appearing in DFS.

Let us use a special example to demonstrate the above observation. When junctions are prepared initially in the state $|0\rangle_1 |1\rangle_2$ and the manipulative field prepared in the Fock state $|n\rangle$, then the evolution of the total system will evolve into

$$
|\psi_{01}\rangle |n\rangle = e^{-i\mathcal{H}t} |0\rangle_1 |1\rangle_2 |n\rangle = \cos \left( g^2 \Delta - t \right) |0\rangle_1 |1\rangle_2 |n\rangle - i \sin \left( g^2 \Delta - t \right) |1\rangle_1 |0\rangle_2 |n\rangle \quad (35)
$$

It is obvious that, when $t = \frac{\pi \hbar (\omega^2 - 4\omega^2)}{16g^2\Delta}$, the two qubit system reaches a maximally entangled state

$$
|\psi_{01}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2 \right) . \quad (36)
$$

By the same way, we can obtain the other maximal entangled state

$$
|\psi_{10}\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle_1 |1\rangle_2 + i |1\rangle_1 |0\rangle_2 \right) , \quad (37)
$$

when we take the $t = \frac{3\pi \hbar (\omega^2 - 4\omega^2)}{16g^2\Delta}$. Both $|\psi_{01}\rangle$ and $|\psi_{10}\rangle$ are independent of the controllable micro-wave field, they belong to the DFS $\mathcal{M}$. The basis $|0\rangle_1 |0\rangle_2$ and $|1\rangle_1 |1\rangle_2$ span the other subspace of the Hilbert space of the JJ qubits(\mathcal{M}), we denote $\mathcal{M} \perp$. Thus we have $\mathcal{M} = \mathcal{M} \oplus \mathcal{M} \perp$. 

FIG.4: The vertical axis represent the function $f_m$, the horizontal axis represent time $t$, $m$ is the eigenvalue of the Fock state, $m = 0$ (line), $m = 10$ (crosses) and $m = 20$ (circles).

FIG.5: The vertical axis represent the function $f_\alpha$, the horizontal axis represent time $t$, and $\alpha$ is the eigenvalue of the coherent state. $\alpha = 0.1$ (circles), $\alpha = 1.1$ (crosses) and $\alpha = 5$ (line).

Now we consider another case that the micro-wave field is prepared in coherent state $|\alpha\rangle$ initially, and the junctions is prepared initially in the state $|0\rangle_1 |0\rangle_2$. By a simple calculation, we get the final state depicted by the reduced density matrix (RDM)

$$
\rho_\alpha = tr_{mw} \{ |\Psi\rangle \langle \Psi | \} = e^{-|\alpha|^2} \sum \frac{|\alpha|^{2k}}{k! k!} |\psi_k\rangle \langle \psi_k |. \quad (33)
$$

We explicitly calculate the function

$$
f_\alpha = | \langle \phi^+ | \rho_\alpha | \phi^+ | \rangle |
$$

$$
= \frac{1}{2} e^{-|\alpha|^2} \sum \frac{|\alpha|^{2k}}{k! k!} \cos^2 (\omega_k t) + \sin^2 (\omega_k t) | \cos (\theta_k) + \sin (\theta_k) |^2 \quad (34)
$$

FIG.5 displays the evolution of the function $f_\alpha$ calculated from eq.(34) for the qubits with coherent microwave fields. When the eigenvalue $\alpha \to \infty$, the fidelity $F \to 0$. 


VI. OVERALL QUALITY OF CREATED ENTANGLEMENTS

In above section we have discussed that when junctions are prepared initially in the state $|0\rangle_1 |0\rangle_2$ or state $|1\rangle_1 |1\rangle_2$, we can not obtain maximally entangled state of the junctions with any controllable microwave field. But we can study the entanglement of these states evolved from the initial state $|0\rangle_1 |0\rangle_2$ or state $|1\rangle_1 |1\rangle_2$.

As well-known, for a bipartite system, composing of two subsystems $A$ and $B$, the bipartite entanglement can be measured by its concurrence\cite{17} which is defined by

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$  \hspace{1cm} (38)

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is the square root of non-Hermitian matrix $R$ in decreasing order, and

$$R = \rho (\sigma_1^y \otimes \sigma_1^y) \rho^* (\sigma_1^y \otimes \sigma_1^y).$$  \hspace{1cm} (39)

We consider that when the initial state of junctions is $|0\rangle_1 |0\rangle_2$ and the controllable microwave field is prepared in coherent state. Then, through a simple calculation, we obtain the concurrence of the states of JJ qubits subsystem Eq.(33) is

$$C = \sqrt{\sqrt{AB - |D|}}^2$$

$$A = e^{-|\alpha|^2} \sum_k \frac{|\alpha|^{2k}}{k!} \cos(\omega_k t) - i\sin(\omega_k t) \sin(\theta_k)|^2$$

$$B = e^{-|\alpha|^2} \sum_k \frac{|\alpha|^{2k}}{k!} \sin^2(\omega_k t) \cos^2(\theta_k)$$

$$D = e^{-|\alpha|^2} \sum_k \frac{|\alpha|^{2k}}{k!} \sin(\omega_k t) \cos(\theta_k) \cdot \{\cos(\theta_k) \cos(\omega_k t) - i\sin(\omega_k t) \sin(\theta_k)\}.$$  \hspace{1cm} (40)

In FIG.6, the concurrences of the qubits are plotted for different values of $\alpha$. It is seen that, with the increasing the eigenvalue of the controllable microwave field, the concurrence decrease sharply.

We study another case, when the initial state of the controllable micro-wave field is thermal state

$$\rho_{mw} = \frac{1}{Z} \sum_n e^{-n\beta E} |n\rangle \langle n|$$  \hspace{1cm} (41)

the state of the total system evolve into

$$\rho = \frac{1}{Z} \sum_n e^{-n\beta E} |\psi_n\rangle |n\rangle \langle n| \langle \psi_n|.$$  \hspace{1cm} (42)

By a simple calculation, we get the RDM of the JJ qubits

$$\rho_{jj} = \frac{1}{Z} \sum_n e^{-n\beta E} |\psi_n\rangle \langle \psi_n|. \hspace{1cm} (43)$$

By the same way, we calculate the concurrence of this state in the following

$$C = \sqrt{\sqrt{AB - |D|}}^2$$  \hspace{1cm} (44)

$$A = \frac{1}{Z} \sum_k e^{-k\beta E} \cos(\omega_k t) - i\sin(\omega_k t) \sin(\theta_k)|^2$$

$$B = \frac{1}{Z} \sum_k e^{-k\beta E} \sin^2(\omega_k t) \cos^2(\theta_k)$$

$$D = \frac{1}{Z} \sum_k e^{-k\beta E} \sin(\omega_k t) \sin(\theta_k) \cdot \{\cos(\theta_k) \cos(\omega_k t) - i\sin(\omega_k t) \sin(\theta_k)\}.$$  \hspace{1cm} (43)

In FIG.7, the concurrence of the state $\rho_{jj}$ is periodic function of time $t$ and the concurrence $C \to$ the maximal value, when $\beta E \to \infty$. This is because $\beta E \to \infty$, the thermal state $\rho_{mw} \to |0\rangle \langle 0|.$

FIG.6 and FIG.7 illustrate that any types of the controllable micro-wave field can not increase the entanglement of the JJ qubits when its initial state is superposition of $|00\rangle$ and $|11\rangle$, and the maximal entanglement is much smaller than 1. Only the initial state of the controllable micro-wave field is vacuum state, the entanglement can reach the maximal value.

VII. CONCLUSION

In summary, we study the protocols which can create maximally entangled states between two qubit coupled
to a controllable microwave field in a cavity. In order to obtain the analytic study for this decoherence problem, we generalized Fröhlich transformation to re-derive the effective Hamiltonian of these system, which is equivalent to that obtained from the adiabatic elimination approach. Because of nontrivial decoherence, we can not construct an ideal logic gate by this system. But we can construct a decoherence-free subspace of two-dimension to against this adiabatic decoherence in this system.

VIII. ACKNOWLEDGMENT

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