AN INTEGRAL TYPE BREZIS-NIRENBERG PROBLEM ON
THE HEISENBERG GROUP

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Abstract This paper is devoted to study a class of integral type Brezis-Nirenberg problem on the Heisenberg group. It is a class of new nonlinear integral equations on the bounded domains of Heisenberg group and related to the CR Yamabe problems on the CR manifold. Based on the sharp Hardy-Littlewood-Sobolev inequalities, the nonexistence and existence results are obtained by Pohozaev type identity, variational method and blow-up analysis, respectively.

Keywords Heisenberg group, Brezis-Nirenberg problem, Integral equations, Existence

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1. Introduction

CR manifold is a class of noncommutative geometry and arises from the study of the real hypersurface of complex manifold (see [12, 14] and the references therein). The complex structure of the real hypersurface, induced from the complex manifold, inspire many interesting geometric property and bring some new difficulties. Particularly, in the study of CR manifold, Heisenberg group \( H^n \) plays a similar role as \( \mathbb{R}^n \) to Riemannian manifold. So, this paper is devoted to study the integral type Brezis-Nirenberg problem on the Heisenberg group.

Let us recall the Sobolev inequality, Hardy-Littlewood-Sobolev (HLS) inequality and their corresponding equations on the Heisenberg group. Then, we will give our integral equations and our results. Involved notations can be seen in the Section 2.1

1.1. Sobolev inequality on the Heisenberg group. In 1980s, Jerison and Lee studied the CR Yamabe problem on CR manifolds in their series papers [20, 21, 22, 23]. As the idea of Yamabe, Trudinger and Aubin (see [32, 30, 1, 24]), the study of CR Yamabe problem is closely related to the sharp Sobolev inequality on the Heisenberg group, which can stated as:

\[
S_{n,2} \left( \int_{\mathbb{H}^n} |f|^{2^*} \, d\xi \right)^{2/2^*} \leq \int_{\mathbb{H}^n} |\nabla_H f|^2 \, d\xi,
\]

where \( 2^* = \frac{2Q}{Q-2} \) is Sobolev critical exponent and \( S_{n,2} = \frac{4n^2 \pi^2}{(2^{2n+1}/(n+1))^n} \) is the best constant. In [22], Jerison and Lee used the Obata’s idea and classified all extremal function, up to group translations, dilations and multiplication by a constant, as

\[
U(\xi) = U(z,t) = ((1 + |z|^2)^{2} + t^2)^{-(Q-2)/4}.
\]

Recently, Frank and Lieb [16] gave a new proof to the extremal function by a rearrangement-free method.

Obviously, Sobolev inequality holds on any subset \( \Omega \subset \mathbb{H}^n \). But, because of the classification of extremal functions on the above, we know that the best constant can
not be attained if \( \Omega \neq \mathbb{H}^n \). Namely, if \( \Omega \neq \mathbb{H}^n \), there is not an energy minimizing solution to the Euler-Lagrange equation of (1.1)

\[
\begin{cases}
-\Delta_H f = f^{\frac{Q + 2}{2}} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(1.3)

Inspired by the above fact, on bounded domain \( \Omega \subset \mathbb{H}^n \), the Brezis-Nireberg problems

\[
\begin{cases}
-\Delta_H f = f^{2^* - 1} + g(\xi, f), & u > 0, \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(1.4)

were studied extensively, such as the results of [3, 17, 19, 28, 29, 31], etc.

1.2. Hardy-Littlewood-Sobolev (HLS) inequality on the Heisenberg group.

In [14], Folland and Stein studied the singular integral operator and obtain the following HLS inequalities

\[
\left| \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f(\xi)g(\eta)|\eta|^{-\alpha}d\eta d\xi} {\left| \xi - \eta \right|^{n+1}} \right| \leq D(n, \alpha, p) \|f\|_{L^q(\mathbb{H}^n)} \|g\|_{L^p(\mathbb{H}^n)}
\]  

(1.5)

where \( f \in L^q \), \( g \in L^p \), \( 0 < \alpha < Q \) and \( \frac{1}{q} + \frac{1}{p} + \frac{\alpha}{Q} = 2 \). In fact, the result is followed from the Proposition 8.7 of [14] (see the Proposition 2.9 and its Remark in Section 2).

Recently, for the diagonal case \( p = q = \frac{2Q}{Q + \alpha} \), Frank and Lieb [16] identified the sharp constant \( D(n, \alpha, p) \) and classified all extremal functions. We can summarize their results as

**Theorem 1.1** (Sharp HLS inequality on \( \mathbb{H}^n \)). For \( 0 < \alpha < Q \) and \( p = \frac{2Q}{Q + \alpha} \). Then for any \( f, g \in L^p(\mathbb{H}^n) \),

\[
\left| \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f(\xi)g(\eta)|\eta|^{-(Q - \alpha)}d\eta d\xi} {\left| \xi - \eta \right|^{n+1}} \right| \leq D_{n, \alpha} \|f\|_{L^p(\mathbb{H}^n)} \|g\|_{L^p(\mathbb{H}^n)}
\]  

(1.6)

where

\[
D_{n, \alpha} := \left( \frac{\pi^{n+1}}{2^{n+1}n!} \right)^\frac{(Q - \alpha)/Q}{\Gamma(\frac{Q}{2})}  
\]  

(1.7)

And equality holds if and only if

\[
f(\xi) = c_1 g(\xi) = c_2 H(\delta_r(\xi^{-1} \xi)),
\]  

(1.8)

for some \( c_1, c_2 \in \mathbb{C} \), \( r > 0 \) and \( \xi \in \mathbb{H}^n \) (unless \( f \equiv 0 \) or \( g \equiv 0 \)). Here \( H \) is defined as

\[
H(\xi) = H(z, t) = ((1 + |z|^2)^2 + t^2)^{-(Q + \alpha)/4}.
\]  

(1.9)

By a duality argument, based on the fundamental solution of sub-Laplace \(-\Delta_H\) (see [13] or [14]), we see that the case \( \alpha = 2 \) of Theorem 1.1 is equivalent to the sharp Sobolev inequality (1.4). Hence, it is worth to study the integral equation related to HLS inequality on the Heisenberg group.

On the other hand, the integral form curvature problems was introduced and studied by Prof. Zhu in [33], which gives an idea of global analysis to curvature problems. Dou and Zhu [11] discussed the existence and nonexistence of positive solutions for an integral equation related to HLS inequality on the bounded domain of \( \mathbb{R}^n \), and found some new phenomena which is different with partial differential equations. This also implies that integral equations have the independent research interests except using as tools for the study of differential equations. Hence, in this
work we will discuss the following integral equation related to HLS inequality on the Heisenberg group.

For any smooth domain $\Omega \subset \mathbb{H}^n$ (for example, say, the boundary is $C^2$), we consider

$$D_{n,\alpha}(\Omega) = \sup_{f \in L_{2Q+\alpha}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \int_{\Omega} f(\xi) |\eta - \xi|^{-(Q-\alpha)} f(\eta) d\xi d\eta}{\|f\|^2_{L_{2Q+\alpha}(\Omega)}}.$$ 

Without loss of generality, we only need to consider non-negative functions.

Similar to CR Yamabe problem, we also can investigate the fact

$$D_{n,\alpha}(\Omega) = D_{n,\alpha},$$

and $D_{n,\alpha}(\Omega)$ is not attained by any functions if $\Omega \neq \mathbb{H}^n$ (see Proposition 3.1 below).

Notice that the corresponding Euler-Lagrange equation for the maximizer (if the supremum is attained) is the following integral equation:

$$f^{Q+\alpha}(\xi) = \int_{\Omega} \frac{f(\eta)}{|\eta - \xi|^{n-\alpha}} d\eta, \quad \xi \in \overline{\Omega}.$$ (1.11)

We thus know that there is not an energy maximizing solution to above integral equation.

Similar to Brezis-Nirenberg problem on $\mathbb{R}^n$ (see [7]), we will study the existence or non-existence of positive solutions to the above integral equation. To this end, we consider the following general equation:

$$f^{q-1}(\xi) = \int_{\Omega} \frac{f(\eta)}{|\eta - \xi|^{Q-\alpha}} d\eta + \lambda \int_{\Omega} \frac{f(\eta)}{|\eta - \xi|^{Q+\alpha-1}} d\eta, \quad \xi \in \overline{\Omega}. $$ (1.12)

For simplicity, we denote $p_\alpha = \frac{2Q}{Q+\alpha}$, $q_\alpha = \frac{2Q}{Q+\alpha}$ throughout this paper.

Our main results is as follows.

**Theorem 1.2.** Assume $\alpha \in (0, Q)$ and $\Omega \subset \mathbb{H}^n$ is a smooth bounded domain.

1. For $\frac{2Q}{Q+\alpha} < q < 2$ (subcritical case), there is a positive solution $f \in \Gamma^\alpha(\Omega) \subset C^{\alpha/2}(\Omega)$ to equation (1.12) for any given $\lambda \in \mathbb{R}$;
2. For $q = \frac{2Q}{Q+\alpha}$ (critical case) and $\lambda > 0$, there is a positive solution $f \in \Gamma^\alpha(\Omega)$ to equation (1.12);
3. For $1 < q \leq \frac{2Q}{Q+\alpha}$ (critical and supercritical case) and $\lambda \leq 0$, if $\Omega$ is a $\delta$-starshaped domain, then there is only trivial non-negative $C^1$ continuous (up to the boundary) solution to (1.12).

**Remark 1.3.** In [11], Dou and Zhu discussed the integral equations (1.12) on the bounded domain $\Omega \subset \mathbb{R}^n$ and proved the results similar to Theorem 1.2 with a constraint $\alpha > 1$. We give a different proof of compactness and regularity, it can extend $0 < \alpha < 1$.

We organize the paper as follows: In section 2 we introduce some notations and some known facts about Heisenberg group. In Section 3, based on Frank and Lieb’s result, namely Theorem 1.1, we show the estimate (1.10) and prove that $D_{n,\alpha}(\Omega)$ can not be attained. Then, by establishing a class of Pohozaev identity related to integral equations (1.12), we can prove the nonexistence result (part (3) of Theorem 1.2). Section 4 is devoted to the part (1) of Theorem 1.2. This is completed by two steps: existence result in $L^q(\Omega)$ (Lemma 4.2) and regularity (Lemma 4.3). In section 5 we will give the existence for the critical exponent case (part (2) of Theorem 1.2).
by the approximation method from subcritical to critical. To complete the proof, we need the uniform bound about the solutions of subcritical equations, which is obtained by the blow-up analysis (see Lemma 5.3).

2. Preliminaries of $\mathbb{H}^n$

In this section, we will state some notations and some known facts about the Heisenberg group $\mathbb{H}^n$. More details can be found in [13, 14, 15] and the references therein.

The Heisenberg group $\mathbb{H}^n$ consists of the set
\[ \mathbb{C}^n \times \mathbb{R} = \{(z, t) : z = (z_1, \cdots, z_n) \in \mathbb{C}^n, t \in \mathbb{R}\} \]
with the multiplication law
\[ (z, t)(z', t') = (z + z', t + t' + 2Im(z \cdot \overline{z}')), \]
where \( z \cdot \overline{z}' = \sum_{j=1}^{n} z_j \overline{z}_j' \). As usual, we write \( z_j = x_j + \sqrt{-1}y_j \). In the sequel, we always denote the point of $\mathbb{H}^n$ by lowercase Greek characters such as \( \xi = (z, t) = (x, y, t) \), \( \eta = (w, s) = (u, v, s) \), etc.

The Lie algebra is spanned by the left invariant vector fields
\[ T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n. \]
The horizontal gradient and the sub-Laplacian are defined by
\[ \nabla_H = (X_1, \cdots, X_n, Y_1, \cdots, Y_n) \]
and
\[ \triangle_H = \sum_{j=1}^{n} (X_j^2 + Y_j^2), \]
respectively.

For any points \( \xi = (z, t) \) and \( \eta = (w, s) \), the norm function \( |\xi| \) is defined as
\[ |\xi| = (|z|^4 + t^2)^{1/4}, \]
and, correspondingly, the distance between \( \xi \) and \( \eta \) is defined as \( |\eta^{-1}\xi| \). A family of dilations is defined as
\[ \delta_r(z, t) = (r z, r^2 t), \quad \forall r > 0, \]
and the homogeneous dimension with respect to the dilations is \( Q = 2n + 2 \).

Now, we state some basic facts on Heisenberg group as follows.

Proposition 2.1 ((8.8) of [14]). For any \( \xi \in \mathbb{H}^n \) with \( |\xi| \leq 1 \), then
\[ \|\xi\| \leq |\xi| \leq \|\xi\|^{1/2}, \]
where \( \|\cdot\| \) is the Euclidean norm.

Proposition 2.2 (Lemma 8.9 of [14]). There exists a constant \( C \geq 1 \) such that, for all \( \xi, \eta \in \mathbb{H}^n \),
\[ |\xi + \eta| \leq C(|\xi| + |\eta|), \quad |\xi\eta| \leq C(|\xi| + |\eta|), \]
where \( \xi + \eta \) represents the common vector adding.

We say that function \( f \) is homogeneous of degree \( \lambda \) if \( f(\delta_r(z, t)) = r^\lambda f(z, t) \), and that a distribution \( F \in \mathcal{D}' \) is homogeneous of degree \( \lambda \) if
\[ F(r^{-Q} g(\delta_{r^{-1}}(z, t))) = r^\lambda F(g). \]
Proposition 2.3 (Proposition 8.1 of [14]). If $F \in \mathcal{D}'$ is homogeneous of degree $\lambda$, then $X_j F$ and $Y_j F$ are homogeneous of $\lambda - 1$ for $1 \leq j \leq n$.

Similar to Lemma 8.10 of [14] and Proposition 1.15 of [15], we have the following result:

Proposition 2.4. Let $f$ be a homogeneous function of degree $\lambda$ ($\lambda \in \mathbb{R}$) which is $C^2$ away from 0. There exists a constant $C > 0$ such that

$$|f(\eta) - f(\xi)| \leq C|\eta||\xi|^{\lambda - 1}, \quad \text{whenever} \quad |\eta| \leq \frac{1}{2}|\xi|,$$

$$|f(\eta) + f(\eta^{-1}) - 2f(\xi)| \leq C|\eta|^2|\xi|^{\lambda - 2}, \quad \text{whenever} \quad |\eta| \leq \frac{1}{2}|\xi|.$$

Proof. The first result can be found in Lemma 8.10 of [14], which had be generalized to nilpotent Lie groups, i.e., Proposition 1.15 of [15]. For completeness, we will give the proof of the second result.

By homogeneity, we can assume that $|\xi| = 1$ and $|\eta| \leq 1/2$. Since $\xi - (\xi \eta - \xi) = \xi \eta^{-1}$, $f \in C^2(\mathbb{H}^n \setminus \{0\})$ and the smooth property of the mapping $\eta \mapsto \xi \eta$, we have

$$\begin{aligned}
|f(\xi \eta) + f(\xi \eta^{-1}) - 2f(\xi)| &= |f(\xi + (\xi \eta - \xi)) + f(\xi - (\xi \eta - \xi)) - 2f(\xi)| \\
&\leq C\|\xi \eta - \xi\| \leq C|\eta|, \\
&\leq C|\eta| |\xi|^{\lambda - 2},
\end{aligned}$$

where the last inequality is deduced by Proposition 2.1. \qed

Similar to [17], we introduce the $\delta$-starshaped domain as follows:

Definition 2.5 (Definition 2.1 of [17]). Given a piecewise $C^1$ open set $\Omega \neq \mathbb{H}^n$ and $(0,0) \in \Omega$, we say that it is $\delta$-starshaped with respect to origin if and only if

$$E \cdot \nu > 0 \quad (2.1)$$

holds at every point of the boundary $\partial \Omega$, where $\nu$ is the outer normal to the boundary $\partial \Omega$ and the vector field $E$ is defined as

$$E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t}. \quad (2.2)$$

Remark 2.6. It is easy to verify that condition (2.1) is equivalent to the following condition: if the point $\xi = (z,t) \in \Omega$, then

$$\delta_\lambda(\xi) = (\lambda z, \lambda^2 t) \in \Omega \quad \forall \lambda \in [0,1]. \quad (2.3)$$

Following, we introduce the convolution on the $\mathbb{H}^n$ and their properties. More details can be found in [14] and the references therein.

Definition 2.7 (Convolution). The convolution of two functions $f,g$ on $\mathbb{H}^n$ is defined by

$$f * g(\xi) = \int f(\eta)g(\eta^{-1}\xi)d\eta = \int f(\xi \eta^{-1})g(\eta)d\eta.$$

If $f \in C^\infty_0$ and $G \in \mathcal{D}'$, we define the $C^\infty$ function $G * f$ and $f * G$ by

$$G * f(\xi) = G(f(\eta^{-1}\xi)), \quad f * G(\xi) = G(f(\xi \eta^{-1})).$$

Definition 2.8 (Regular distribution). A distribution $F$ is said to be regular if there exists a function $f$ which is $C^\infty$ on $\mathbb{H}^n \setminus \{0\}$ such that $F(g) = \int fg dV$ for all $g \in C^\infty_0(\mathbb{H}^n \setminus \{0\}).$
Proposition 2.9 (Proposition 8.7 of [13]). If $F$ is a regular homogeneous distribution of degree $\lambda$, $-Q < \lambda < 0$, then the mapping $g \to g \ast F$ extends to a bounded mapping from $L^p$ to $L^q$, where $q^{-1} = p^{-1} - \lambda/Q - 1$ provided $1 < p < q < \infty$, and from $L^1$ to $L^{-Q/\lambda-\epsilon}(\text{loc})$ for any $\epsilon > 0$.

Remark 2.10. If the distribution $F$ is taken as $|\xi|^{\alpha-Q}(0 < \alpha < Q)$, then for $1 < p < q < +\infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, the above result can be specified as

$$\|g \ast |\xi|^{\alpha-Q}\|_{L^q} \leq C\|g\|_{L^p}.$$  

It is just a dual form of HLS inequalities [150].

On $\mathbb{H}^n$, let $C^\beta$, $0 < \beta < \infty$ be the classical Lipschitz spaces of order $\beta$, namely, they are defined in terms of the Euclidean norm. In [13], they introduced the following family $\Gamma_\beta$ of Lipschitz spaces with respect to the norm $|\cdot|$ on the Heisenberg group.

Definition 2.11 (Lipschitz spaces). i) For $0 < \beta < 1$,

$$\Gamma_\beta = \{f \in L^\infty \cup C : \sup_{\xi, \eta} \frac{|f(\xi \eta) - f(\xi)|}{|\eta|^{\beta}} < \infty\};$$

ii) For $\beta = 1$,

$$\Gamma_1 = \{f \in L^\infty \cup C : \sup_{\xi, \eta} \frac{|f(\xi \eta) + f(\xi^{-1}) - 2f(\xi)|}{|\eta|} < \infty\};$$

iii) For $\beta = k + \beta'$, where $k$ is a positive integer and $0 < \beta' \leq 1$,

$$\Gamma_\beta = \{f \in L^\infty \cup C : f \in \Gamma_{\beta'} \text{ and } Df \in \Gamma_{\beta'} \text{ for all } D \in \mathbb{B}_k\}$$

where

$$\mathbb{B}_k = \{L_{a_1}L_{a_2} \cdots L_{a_j} : 1 \leq a_i \leq 2n, i = 1, 2, \cdots, j, j \leq k\}$$

with $L_j = X_j$ and $L_{j+n} = Y_j$ for $j = 1, 2, \cdots, n$.

Proposition 2.12 (Theorem 20.1 of [14]). $\Gamma_\beta \subset C^{\beta/2}(\text{loc})$ for $0 < \beta < \infty$.

Notation: for any function $f(\xi)$ defined on $\Omega$, we always use $\tilde{f}(\xi)$ to represent its trivial extension in $\mathbb{H}^n$, namely,

$$\tilde{f}(\xi) = \begin{cases} f(\xi) & \xi \in \Omega, \\ 0 & \xi \in \mathbb{H}^n \setminus \Omega. \end{cases}$$

And

$$I_\alpha f(\xi) = \int_{\mathbb{H}^n} \frac{f(\eta)}{|\eta|^{\alpha+Q-\alpha}} d\eta, \quad I_{\alpha, \Omega} f(\xi) = \int_{\mathbb{H}^n} \frac{f(\eta)}{|\eta|^{\alpha+Q-\alpha}} d\eta.$$

We also denote that $c, C$ different positive constant.

3. Nonexistence for Critical and Supcritical Case

In this section, we mainly devote to discuss the nonexistence result for the critical case. Firstly, we derive energy estimate [110], and show that the supremum $D_{n,\alpha}(\Omega)$ is not achieved by any function on any domain $\Omega \neq \mathbb{H}^n$. Then, we establish a Pohozaev type identity for integral equation, which deduce the third part of Theorem 1.12.

Proposition 3.1. For any domain $\Omega \subset \mathbb{H}^n$, $D_{n,\alpha}(\Omega) = D_{n,\alpha}$; further the supremum $D_{n,\alpha}(\Omega)$ is not achieved by any function in $L^{q_0}(\Omega)$ on any domain $\Omega \neq \mathbb{H}^n$. 


Proof. If \( f \in L^{q_0}(\Omega) \), then \( \tilde{f} \in L^{q_0}(\mathbb{H}^n) \). It follows that
\[
D_{n,\alpha}(\Omega) = \sup_{f \in L^{q_0}(\Omega), (\eta)} \frac{\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \tilde{f}(\xi) \eta^{-1} |\eta|^{-Q} \tilde{f}(\eta) d\eta d\xi}{\|\tilde{f}\|^2_{L^{q_0}(\mathbb{H}^n)}} \leq \sup_{g \in L^{q_0}(\mathbb{H}^n), (\eta)} \frac{\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} g(\xi) \eta^{-1} |\eta|^{-Q} g(\eta) d\eta d\xi}{\|g\|^2_{L^{q_0}(\mathbb{H}^n)}} = D_{n,\alpha}.
\]

On the other hand, recall that \( f(\xi) = H(\xi^{-1}) \) with \( \zeta \in \mathbb{H}^n \) is an extremal function to the sharp HLS inequality in Theorem 1.1, as well as its conformal equivalent class:
\[
f_\epsilon(\xi) = e^{-\frac{\epsilon}{3(n-1)}} H(\delta_{\epsilon^{-1}}(\xi^{-1})), \quad \forall \epsilon > 0.
\]
It is easy to verify
\[
\|I_{\alpha}f\|_{L^{p_0}(\mathbb{H}^n)} = \|I_{\alpha}f_\epsilon\|_{L^{p_0}(\mathbb{H}^n)}, \quad \|f\|_{L^{q_0}(\mathbb{H}^n)} = \|f_\epsilon\|_{L^{q_0}(\mathbb{H}^n)},
\]
and \( f_\epsilon(\xi) \) satisfies integral equation
\[
f_\epsilon^{\frac{Q-\alpha}{Q}}(\xi) = B \int_{\mathbb{H}^n} \frac{f_\epsilon(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\eta,
\]
where \( B \) is a positive constant. Following, based on the extremal function \( f_\epsilon \), we will choose a specific test function to prove that the reverse inequality holds too.

Choose some point \( \zeta \in \Omega \) and \( R \) small enough so that \( \Sigma_R(\zeta) \subset \Omega \), where \( \Sigma_R(0) = \{\xi = (z,t) \in \mathbb{H}^n : |z| < R, |t| < R^2\} \) is a cylindrical set and \( \Sigma_R(\zeta) = \zeta \circ \Sigma_R(0) \), and choose test function \( g(\xi) \in L^{q_0}(\mathbb{H}^n) \) as
\[
g(\xi) = \begin{cases} f_\epsilon(\xi) & \xi \in \Sigma_R(\zeta) \subset \Omega, \\ 0 & \xi \in \mathbb{H}^n \setminus \Sigma_R(\zeta). \end{cases}
\]
Then,
\[
\int_{\Omega} \int_{\Omega} \frac{g(\xi) g(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\xi d\eta = \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\xi d\eta - 2 \int_{\mathbb{H}^n} \int_{\Sigma^c_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\xi d\eta + \int_{\Sigma^c_R(\zeta)} \int_{\Sigma^c_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\xi d\eta,
\]
where \( \Sigma^c_R(\zeta) = \mathbb{H}^n \setminus \Sigma_R(\zeta) \) and
\[
I_1 := 2 \int_{\Sigma^c_R(\zeta)} \int_{\Sigma^c_R(\zeta)} f_\epsilon(\xi) f_\epsilon(\eta) \frac{Q-\alpha}{Q} d\xi d\eta, \quad I_2 := \int_{\Sigma^c_R(\zeta)} \int_{\Sigma^c_R(\zeta)} f_\epsilon(\xi) f_\epsilon(\eta) \frac{Q-\alpha}{Q} d\xi d\eta.
\]
It follows from \((3.2)\) that
\[
I_1 = C \int_{\Sigma^c_R(\zeta)} f_\epsilon^{\frac{Q-\alpha}{Q}}(\xi) d\xi = O\left(\frac{R}{\epsilon}\right)^{-Q}, \quad \text{as } \epsilon \to 0.
\]
By HLS inequality \((1.6)\), we have
\[
I_2 \leq D_{n,\alpha} \|f_\epsilon\|^2_{L^{q_0}(\Sigma^c_R(\zeta))} = O\left(\frac{R}{\epsilon}\right)^{-Q-\alpha} \quad \text{as } \epsilon \to 0.
\]
Combining the above, we arrive at
\[
D_{n,\alpha}(\Omega) \geq \frac{\int_{\Omega} \int_{\Omega} \frac{g(\xi) g(\eta)}{|\eta|^{-1} \xi^{-1}Q^-} d\xi d\eta}{\|g\|^2_{L^{q_0}(\mathbb{H}^n)}}
\]
Lemma 3.2. If \( f \in C^1(\Omega) \) is a non-negative solution to

\[
f(\xi) = \int_{\Omega} \frac{f^{p-1}(\eta)}{|\eta^{-1}\xi|^Q-\alpha} d\eta + \lambda \int_{\Omega} \frac{f^{p-1}(\eta)}{|\eta^{-1}\xi|^Q-\alpha} d\eta, \quad \xi \in \overline{\Omega},
\]

where \( p \neq 0, \lambda \in \mathbb{R} \), then

\[
\frac{Q}{p} + \frac{\alpha - Q}{2} \int_{\Omega} f^p(\xi) d\xi = -\frac{\lambda}{2} \int_{\Omega} f^{p-1}(\xi) f^{p-1}(\eta) |\eta^{-1}\xi|^Q-\alpha d\eta, \quad \xi \in \overline{\Omega}.
\]

where \( \nu \) is the outward normal to \( \partial \Omega \).

Proof. Denote by \( \xi = (z, t) = (x, y, t), \eta = (w, s) = (u, v, s) \) and then

\[
\eta^{-1}\xi = (z - w, t - s - 2 \text{Im}(w \cdot \xi) = (z - w, t - s + 2(yu - xv)).
\]

Noting that \( \Omega \) is a \( \delta \)-starshaped domain with respect to the origin and \( E_f = \frac{\partial}{\partial r} f(\delta_r(\xi)) |_{r=1} \), we have that, by (3.3),

\[
E_f(\xi) = \frac{\partial}{\partial r} \left( \int_{\Omega} \frac{f^{p-1}(\eta)}{|\eta^{-1}\delta_r(\xi)|^Q-\alpha} d\eta + \lambda \int_{\Omega} \frac{f^{p-1}(\eta)}{|\eta^{-1}\delta_r(\xi)|^Q-\alpha} d\eta \right) |_{r=1}.
\]

A direct calculation leads to

\[
\left. \frac{\partial}{\partial r} |\eta^{-1}\delta_r(\xi)|^4 \right|_{r=1} = 4|z - w|^2|x(x - u) + y(y - v)| + 4(t - s + 2(yu - xv))(t + (yu - xv)).
\]

By (3.3), we have

\[
\int_{\Omega} f^{p-1}(\xi) E_f(\xi) d\xi = (\alpha - Q) \int_{\Omega} \frac{f^{p-1}(\xi) f^{p-1}(\eta)}{|\eta^{-1}\delta_r(\xi)|^{Q-\alpha+4}} \left( \frac{1}{4} \frac{\partial}{\partial r} |\eta^{-1}\delta_r(\xi)|^4 \right) r \ d\eta d\xi
\]

\[
+ (\alpha - Q + 1) \lambda \int_{\Omega} \frac{f^{p-1}(\xi) f^{p-1}(\eta)}{|\eta^{-1}\delta_r(\xi)|^{Q-\alpha+3}} \left( \frac{1}{4} \frac{\partial}{\partial r} |\eta^{-1}\delta_r(\xi)|^4 \right) r \ d\eta d\xi
\]

\[= I + II.
\]

Finally, we show that \( D_{n, \alpha}(\Omega) \) is not achieved if \( \Omega \neq \mathbb{H}^n \). In fact, if \( D_{n, \alpha}(\Omega) \) is attained by some function \( u \in L^p(\Omega) \), then \( \tilde{u} \in L^p(\mathbb{H}^n) \) would be an extremal function to the sharp HLS inequality on \( \mathbb{H}^n \), which is impossible due to Theorem B. \( \square \)

Proposition 3.2 indicates that there is no maximizing energy solution to (1.11). We shall show that there is not any non-trivial positive continuous solution to (1.11) on any \( \delta \)-starshaped domain using the following Pohozaev identity. Without loss of generality, in the rest of this section we always assume that the origin is in \( \Omega \) and the domain is \( \delta \)-starshaped with respect to the origin.
Estimating $I$ by (3.6), we have

$$2 \times I = (\alpha - Q) \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha}|} \left\{ |z - w|^2 \left[ x(x-u) + y(y-v) \right] + (t - s + 2(yu - xv))(t + (yu - xv)) \right\} \, d\eta d\xi$$

$$+ (\alpha - Q) \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\xi^{-1}\eta^{Q-\alpha}|} \left\{ |w - z|^2 \left[ u(u-x) + v(v-y) \right] + (s - t + 2(vx - uy))(s + (vx - uy)) \right\} \, d\xi d\eta$$

$$= (\alpha - Q) \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha}|} \, d\eta d\xi.$$ 

That is

$$I = \frac{\alpha - Q}{2} \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha}|} \, d\eta d\xi.$$ 

Similarly,

$$II = \frac{\alpha - Q + 1}{2} \lambda \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha-1}|} \, d\eta d\xi.$$ 

Combining the above into (3.5), we arrive at

$$\int_{\Omega} f^{p-1}(\xi)Ef(\xi) \, d\xi$$

$$= \frac{\alpha - Q}{2} \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha}|} \, d\eta d\xi$$

$$+ \frac{\alpha - Q + 1}{2} \lambda \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha-1}|} \, d\eta d\xi$$

$$= \frac{\alpha - Q}{2} \int_{\Omega} f^{p}(\xi) \, d\xi + \frac{\lambda}{2} \int_{\Omega} \int_{\Omega} \frac{f^{p-1}(\xi)f^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha-1}|} \, d\eta d\xi.$$ 

(3.7)

On the other hand, by integration by part, we have

$$\int_{\Omega} f^{p-1}(\xi)Ef(\xi) \, d\xi = \int_{\Omega} E\left( \frac{u^{p}(\xi)}{p} \right) \, d\xi = \frac{1}{p} \int_{\partial \Omega} u^{p}(\xi)E \cdot \nu dS - \frac{Q}{p} \int_{\Omega} u^{p}(\xi) \, d\xi.$$ 

(3.8)

Hence, we deduce (3.6) by combining (3.7) and (3.8).

**Proof of part (3) in Theorem 1.2 (nonexistence part).** If \( f(\xi) \) is a non-negative \( C^{1}(\Omega) \) solution to (1.12) for \( \lambda \leq 0 \), then by Lemma 3.2, we know that \( g(\xi) = f^{q-1}(\xi) \) satisfies

$$-\frac{\lambda}{2} \int_{\Omega} \int_{\Omega} \frac{g^{p-1}(\xi)g^{p-1}(\eta)}{|\eta^{-1}\xi^{Q-\alpha-1}|} \, d\eta d\xi + \frac{1}{p} \int_{\partial \Omega} (E \cdot \nu)g^{p-1}(\xi) \, d\sigma \leq 0.$$ 

(3.9)

Since \( \Omega \) is \( \delta \)-starshaped domain about the origin, we have \( E \cdot \nu > 0 \) on \( \partial \Omega \). If \( \lambda < 0 \), then \( g(\xi) \equiv 0 \) on \( \Omega \). If \( \lambda = 0 \), it follows immediately from (3.9) that \( g \equiv 0 \) on \( \partial \Omega \). Therefore, from (1.12), we conclude that \( g \equiv 0 \) on \( \Omega \). Part (3) in Theorem 1.1 is proved.

\( \square \)
4. Existence result for subcritical case

To obtain the existence to equation (1.12) with subcritical powers, we need the following compactness lemma.

**Lemma 4.1** (Compactness). For any compact domain \( \Omega \), operator \( I_\alpha : L^{\frac{2q}{q-\alpha}}(\Omega) \to L'(\Omega) \), \( r < \frac{2q}{Q-\alpha} \), is compact. Namely, for any bounded sequence \( \{f_j\}_{j=1}^{\infty} \subset L^{\frac{2q}{Q-\alpha}}(\Omega) \), there exist a function \( f \in L^{\frac{2q}{Q-\alpha}}(\Omega) \) and a subsequence of \( \{I_\alpha f_j(\xi)\}_{j=1}^{\infty} \) which converges to \( I_\alpha f \) in \( L^r(\Omega) \).

**Proof.** Since the sequence \( \{f_j\} \) is bounded in \( L^{\frac{2q}{Q-\alpha}}(\Omega) \), then there exist a subsequence (still denoted by \( \{f_j\} \)) and a function \( f \in L^{\frac{2q}{Q-\alpha}}(\Omega) \) such that

\[
 f_j \rightharpoonup f \quad \text{weakly in} \quad L^{\frac{2q}{Q-\alpha}}(\Omega).
\]

Decompose \( |\xi|^{\alpha-Q} = |\xi|^{\alpha-Q} \chi_{\{|\xi|>\rho\}} + |\xi|^{\alpha-Q} \chi_{\{|\xi|<\rho\}} \), where \( \rho > 0 \) will be chosen later. Then,

\[
 I_\alpha f_j(\xi) = I^1_\alpha f_j(\xi) + I^2_\alpha f_j(\xi) = f_j * |\xi|^{\alpha-Q} \chi_{\{|\xi|>\rho\}} + f_j * |\xi|^{\alpha-Q} \chi_{\{|\xi|<\rho\}}.
\]

Noting \( |\xi|^{\alpha-Q} \chi_{\{|\xi|>\rho\}} \in L^{\frac{2q}{Q-\alpha}} \), then \( I^1_\alpha f_j(\xi) \) converges pointwise to \( I^1_\alpha f(\xi) \) by the weak convergence. On the other hand, since

\[
 |I^1_\alpha f_j(\xi)| \leq \|f_j\|_{L^{\frac{2q}{Q-\alpha}}} \left\| |\xi|^{\alpha-Q} \chi_{\{|\xi|>\rho\}} \right\|_{L^{\frac{2q}{Q-\alpha}}} \leq C(\rho),
\]

where \( C(\rho) \) is independent of \( f_j \), then the dominated convergence deduce that

\[
 I^1_\alpha f_j \to I^1_\alpha f \quad \text{strongly in} \quad L^{r}(\Omega).
\]

Next, we analyze the convergence of \( \{I^2_\alpha f_j\} \) by the Young inequality. Take

\[
 s = \left(1 + \frac{Q-\alpha}{2Q} \right)^{-1}.
\]

Then, \( s < \frac{Q-\alpha}{Q} \) and \( \left\| |\xi|^{\alpha-Q} \chi_{\{|\xi|>\rho\}} \right\|_{L^s} < C\rho^s \) with \( \beta = Q(\frac{2Q}{Q-\alpha} - \frac{Q-\alpha}{Q}) \). By the Young inequality, we have

\[
 \|I^2_\alpha (f_j - f)\|_{L^s} \leq C\|f_j - f\|_{L^{\frac{2q}{Q-\alpha}}} \|\xi|^{\alpha-Q} \chi_{\{|\xi|<\rho\}}\|_{L^s} \leq C\rho^s.
\]

By now, through choosing first \( \rho \) small and then \( j \) large, we deduce by (4.1) and (4.2) that

\[
 I_\alpha f_j \to I_\alpha f \quad \text{strongly in} \quad L^{r}(\Omega).
\]

The Lemma is proved. \( \square \)

Based on the Lemma 4.1, we can obtain the existence result (part (1) in Theorem 1.2). For simplicity, we only present the proof for \( \lambda = 0 \).

**Lemma 4.2.** For \( q > q_\alpha \), supremum

\[
 D_{\alpha,q}(\Omega) := \sup_{f \in L^q(\Omega) \setminus \{0\}} \frac{\int_\Omega \int_\Omega f(\xi)|\eta|^{-\frac{Q-\alpha}{Q}} f(\eta)d\eta d\xi}{\|f\|_{L^q(\Omega)}^2}
\]

is attained by some nonnegative function in \( L^q(\Omega) \).

**Proof.** First, by HLS inequality (1.5) or (1.9), we know that

\[
 D_{\alpha,q}(\Omega) \leq \sup_{f \in L^q(\Omega) \setminus \{0\}} \frac{\|I_\alpha f\|_{L^{q'}(\Omega)}}{\|f\|_{L^q(\Omega)}} < +\infty.
\]
Choosing a nonnegative maximizing sequence \( \{ f_j \}_{j=1}^{\infty} \subset L^q(\Omega) \) satisfying \( \| f_j \|_{L^q(\Omega)} = 1 \) and
\[
\lim_{j \to +\infty} \int_{\Omega} f_j(\xi)|\eta^{-1}\xi|^{-(Q-\alpha)}f_j(\eta)d\eta d\xi = D_{\alpha,q}(\Omega).
\]
Combining the boundedness of \( \{ f_j \} \) in \( L^q(\Omega) \) and the compactness of the operator \( I_{\alpha,\Omega} \), we deduce by Lemma 4.1 that there exists a subsequence (still denoted as \( \{ f_j \} \)) and \( f_* \in L^q(\Omega) \) such that

\[
f_j \rightharpoonup f_* \quad \text{weakly in } L^q(\Omega),
\]

\[
I_{\alpha,\Omega}f_j \to I_{\alpha,\Omega}f_* \quad \text{strongly in } L^q(\Omega).
\]

Thus, \( \| f_* \|_{L^q(\Omega)} \leq \liminf_{j \to +\infty} \| f_j \|_{L^q(\Omega)} \) and

\[
\lim_{j \to +\infty} \langle I_{\alpha,\Omega}f_j, f_j \rangle = \langle I_{\alpha,\Omega}f, f \rangle.
\]

Then,

\[
D_{\alpha,q}(\Omega) = \lim_{j \to +\infty} \frac{\langle I_{\alpha,\Omega}f_j, f_j \rangle}{\| f_j \|_{L^q(\Omega)}} \leq \frac{\langle I_{\alpha,\Omega}f, f \rangle}{\| f \|_{L^q(\Omega)}},
\]

namely, \( f_* \) is a maximizer.

It is easy to see that the maximizer for energy \( D_{\alpha,q}(\Omega) \), up to a constant multiplier, satisfies following equation:

\[
f^{\alpha-1}(\xi) = \int_{\Omega} \frac{f(\eta)}{|\eta^{-1}\xi|^{(Q-\alpha)}}d\eta, \quad \xi \in \Omega.
\]

(4.3)

Let \( g(\xi) = f^{\alpha-1}(\xi) \), and then (4.3) is changed into the form

\[
g(\xi) = \int_{\Omega} \frac{g^{\alpha-1}(\eta)}{|\eta^{-1}\xi|^{(Q-\alpha)}}d\eta, \quad \xi \in \Omega
\]

(4.4)

for \( q' < \frac{2Q}{Q-\alpha} = p_\alpha \). To complete the proof of Part (1) in Theorem 1.2 we need to show that \( g \in \Gamma^\alpha(\Omega) \).

**Lemma 4.3 (Regularity).** Suppose that \( g \in L^{q'}(\Omega) \) is a positive solution to (4.4). If \( q' < p_\alpha \), then \( g \in \Gamma^{\alpha}(\Omega) \subset C^{\alpha/2}(\Omega) \).

**Proof.** **Step 1.** We show \( g \in L^{\infty}(\Omega) \cup C(\Omega) \).

For proving \( g \in L^{\infty}(\Omega) \), it is necessary to prove that there exists some constant \( s^* > 0 \) such that \( g \in L^{s^*}(\Omega) \) and \( \frac{s^*}{q' - 1} > \frac{Q}{\alpha} \). In fact, if there exists such \( s^* \), then

\[
g(\xi) \leq \| g \|_{L^{s^*}(\Omega)}^{q'-1}(\int_{\Omega} |\eta^{-1}\xi|^{(Q-\alpha)(s^*/(q' - 1))'}d\eta)^{1/(s^*/(q' - 1))'} \leq C\| g \|_{L^{s^*}(\Omega)}^{q'-1},
\]

where \( (s^*/(q' - 1))' \) is the conjugate number of \( (s^*/(q' - 1)) \). Hence, \( g \in L^{\infty}(\Omega) \), which leads to \( g \in C(\Omega) \) by the dominant convergence theorem.

i) If \( q' < \frac{Q}{Q-\alpha} \), we can take \( s^* = q' \).

ii) If \( q' = \frac{Q}{Q-\alpha} \), then \( g \in L^{q'}(\Omega) \) with \( q_1 = (1 - \frac{1}{q'})\frac{Q}{Q-\alpha} \) and \( k = \left[ \frac{Q}{Q-\alpha} \right] + 1 \). By (4.4) and HLS inequality (1.5), we know that \( g \in L^{s^*}(\Omega) \) with \( \frac{1}{s^*} = \frac{q'-1}{q_1} - \frac{Q}{Q-\alpha} = \frac{k}{(k-1)q_1} \) and \( s^* > \frac{Q}{\alpha} \).

iii) For the case \( \frac{Q}{Q-\alpha} < q' < \frac{2Q}{Q-\alpha} \), we will find the constant \( s^* \) by the following iteration process. By (1.2) and HLS inequality (1.5), we have

\[
\| g \|_{L^{s^*}(\Omega)} = \| I_{\alpha,\Omega}(g^{q'-1}) \|_{L^{s^*}(\Omega)} \leq C\| g^{q'-1} \|_{L^1(\Omega)} \tag{4.5}
\]

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for \(1/s = 1/t - \alpha/Q\). We use the above inequality to do iteration. First, choose 
\[ t = q'/q' - 1 := t_1/q' - 1, \]
and let \(t_2 = s\). Since \(q' < 2Q/(Q - \alpha)\), it is easy to check that 
\[ t_2 > 2Q/(Q - \alpha). \]

Iterate the above: let \(t = t_i/(q' - 1)\), then \(t_{i+1} = s\) for \(i = 1, 2, \cdots\). Note that 
\[ q' - 2 < 2\alpha/(Q - \alpha), \]
and \(1/t_i < (Q - \alpha)/(2Q)\). Thus, one can check that \(t_{i+1} > t_i\) 
when \(t_{i+1}\) is positive. So, after iterating certain times, say, \(k_0\) times, we shall have: 
\[ \frac{q' - 1}{t_{k_0}} > \frac{\alpha}{Q} \quad \text{and} \quad \frac{q' - 1}{t_{k_0 + 1}} \leq \frac{\alpha}{Q}. \]

If \(\frac{q' - 1}{t_{k_0 + 1}} > \frac{\alpha}{Q}\), we can take \(s^* = t_{k_0 + 1}\).

If \(\frac{q' - 1}{t_{k_0 + 1}} = \frac{\alpha}{Q}\), then \(u \in L^{q_2}((\Omega) = L^{q' - 1})\). By the boundedness of \(\Omega\) and 
the Hölder inequality, we know that \(u \in L^{q_2}(\Omega)\) with 
\[ q_2 = (1 - \frac{1}{2})(q' - 1)\] 
and \(k = \left[\frac{2Q}{Q - \alpha}\right] + 1\). Using (4.4) and HLS inequality (17), we get 
\(u \in L^{s^*}(\Omega)\) with \(\frac{1}{s^*} > \frac{q' - 1}{q_2} - \frac{\alpha}{Q}\) and \(s^* > \frac{q'}{Q}\).

**Step 2.** We show \(g \in \Gamma^{(\Omega)}\). We divide it into four cases.

**Case i)** \(0 < \alpha < 1\).

Since \(g \in L^{\infty}(\Omega)\), we have 
\[ |g(\xi\gamma) - g(\xi)| = \left| \int_{\Omega} (g(\xi\eta^{-1}))^{q' - 1} (|\eta\gamma|^{\alpha - Q} - |\eta|^{\alpha - Q})\, d\eta \right| \leq \|g\|_{L^{\infty}(\Omega)}^{q' - 1} \int_{\Omega} (|\eta\gamma|^{\alpha - Q} - |\eta|^{\alpha - Q})\, d\eta. \tag{4.6} \]

By the Proposition 2.4 we can estimate 
\[ \int_{\Omega \cap \{|\eta| \geq 2|\gamma|\}} (|\eta\gamma|^{\alpha - Q} - |\eta|^{\alpha - Q})\, d\eta \leq C \int_{\Omega \cap \{|\eta| \geq 2|\gamma|\}} |\gamma||\eta|^{\alpha - Q - 1}\, d\eta \leq C|\gamma|^\alpha. \tag{4.7} \]

On the other hand, there exists some \(B \geq 2\) such that \(|\eta\gamma| \leq B|\gamma|\) since \(|\eta| \leq 2|\gamma|\).

By Proposition 2.2, 
\[ \int_{\Omega \cap \{|\eta| \leq 2|\gamma|\}} (|\eta\gamma|^{\alpha - Q} - |\eta|^{\alpha - Q})\, d\eta \leq \int_{\Omega \cap \{|\eta| \leq 2|\gamma|\}} |\eta\gamma|^{\alpha - Q}\, d\eta + \int_{\Omega \cap \{|\eta| \\leq 2|\gamma|\}} |\eta|^{\alpha - Q}\, d\eta \leq C|\gamma|^\alpha. \tag{4.8} \]

Substituting (4.7) and (4.8) into (4.6), we have 
\[ |g(\xi\gamma) - g(\xi)| \leq C\|g\|_{L^{\infty}(\Omega)}^{q' - 1}|\gamma|^\alpha, \]
i.e., \(g \in \Gamma^{(\Omega)}(\Omega)\).

**Case ii)** \(\alpha = 1\).

By an argument similar to Case i), we can get 
\[ |g(\xi\gamma) + g(\xi\gamma^{-1}) - 2g(\xi)| \leq C\|g\|_{L^{\infty}(\Omega)}^{q' - 1}|\gamma|\alpha, \]
i.e., \(g \in \Gamma^1(\Omega)\).

**Case iii)** \(\alpha = 1 + \alpha'\) with \(0 < \alpha' \leq 1\).
To this end, we consider Lemma 5.1. Notice that the corresponding Euler-Lagrange equation for extremal functions, up to a constant multiplier, is integral equation (1.12) for $q$. Following, we will prove that $g \in \Gamma^\alpha (\Omega)$. Since $g \in L^\infty (\Omega)$ and $\Omega$ is bounded, we have $g^{\alpha - 1} \in L^2 (\Omega)$. If $0 < \alpha' < 1$,

$$
|g(\xi) - g(\xi)| \\
\leq ||g^{\alpha - 1}||_{L^q(\Omega)} \left( \int_{\Omega \cap \{|\eta| \geq 2|\gamma|\}} |\eta|^{\alpha - Q} - |\eta|^{Q'} d\eta \right)^{1/Q'} \\
+ ||g^{\alpha - 1}||_{L^q(\Omega)} \left( \int_{\Omega \cap \{|\eta| \leq 2|\gamma|\}} |\eta|^{\alpha - Q} - |\eta|^{Q'} d\eta \right)^{1/Q'},
$$

where $Q' = \frac{Q}{Q - 1}$. Similar to Case $0 < \alpha < 1$, we have

$$
\left( \int_{\Omega \cap \{|\eta| \geq 2|\gamma|\}} |\eta|^{\alpha - Q} - |\eta|^{Q'} d\eta \right)^{1/Q'} \\
\leq C \left( \int_{\{|\eta| \geq 2|\gamma|\}} (|\eta|^{\alpha - Q - 1})^{Q'} d\eta \right)^{1/Q'} \\
\leq C |\gamma| \cdot |\gamma|^{-Q - 1 + \frac{Q}{Q'}} = C |\gamma|^{\alpha'},
$$

and

$$
\left( \int_{\Omega \cap \{|\eta| \leq 2|\gamma|\}} |\eta|^{\alpha - Q} - |\eta|^{Q'} d\eta \right)^{1/Q'} \\
\leq C \left( \int_{\{|\eta| \leq B|\gamma|\}} |\eta|^{(\alpha - Q)Q'} d\eta \right)^{1/Q'} \\
\leq C |\gamma|^{-Q + \frac{Q}{Q'}} = C |\gamma|^{\alpha'}.
$$

Combining (4.9), (4.10) and (4.11) leads to $g \in \Gamma^\alpha (\Omega)$.

If $\alpha' = 1$, the proof can be completed similarly.

**Case iv)** $\alpha = k + \alpha'$ with $0 < \alpha' \leq 1$ and $k = 2, 3, \ldots$.

This case can be discussed with a similar argument with Case iii). □

### 5. Existence result for critical case

Now we shall establish the existence results for (1.12) with $\lambda > 0$ and $q = q_\alpha$. To this end, we consider

$$
Q_\lambda (\Omega) := \sup_{f \in L^{n,\alpha}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \int_{\Omega} f(\xi)(|\eta^{-1} \xi|^{-(Q-\alpha)} + \lambda |\eta^{-1} \xi|^{-(Q-\alpha-1)}) f(\eta) d\eta d\xi}{\|f\|_{L^{n,\alpha}(\Omega)}^2}.
$$

Notice that the corresponding Euler-Lagrange equation for extremal functions, up to a constant multiplier, is integral equation (1.12) for $q = q_\alpha$.

First, we show

**Lemma 5.1.** $Q_\lambda (\Omega) > D_{n,\alpha}$ for all $\lambda > 0$. 
Proof. Let $\zeta \in \Omega$. For small positive $\epsilon$ and a fixed $R > 0$ so that $\Sigma_R(\zeta) \subset \Omega$, we define

$$\tilde{f}_\epsilon(\xi) = \begin{cases} f_\epsilon(\xi) & \xi \in \Sigma_R(\zeta) \subset \Omega, \\ 0 & \xi \in \mathbb{R}^n \setminus \Sigma_R(\zeta), \end{cases}$$

where $f_\epsilon$ is given by Proposition 3.1. Obviously, $\tilde{f}_\epsilon \in L^{q_0}(\mathbb{R}^n)$. Thus, similar to the proof of Proposition 3.1 we have

$$\int_{\Omega} \int_{\Omega} \frac{1}{|\eta^{-1}\xi|^{Q-\alpha}} + \frac{\lambda}{|\eta^{-1}\xi|^{Q-\alpha-1}} \tilde{f}_\epsilon(\xi) \tilde{f}_\epsilon(\eta) d\eta d\xi,$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|\eta^{-1}\xi|^{Q-\alpha}} f_\epsilon(\xi) f_\epsilon(\eta) d\eta d\xi$$

$$- 2 \int_{\mathbb{R}^n} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha}} d\xi d\eta + \int_{\Sigma_R(\zeta)} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha}} d\xi d\eta$$

$$+ \lambda \int_{\Sigma_R(\zeta)} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha-1}} d\xi d\eta,$$

where

$$I_1 = 2 \int_{\mathbb{R}^n} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha}} d\eta d\xi,$$

$$I_2 = \int_{\Sigma_R(\zeta)} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha}} d\xi d\eta,$$

$$I_3 = \lambda \int_{\Sigma_R(\zeta)} \int_{\Sigma_R(\zeta)} \frac{f_\epsilon(\xi) f_\epsilon(\eta)}{|\eta^{-1}\xi|^{Q-\alpha-1}} d\xi d\eta.$$

For $I_3$, we have

$$I_3 = \lambda \int_{\Sigma_R(0)} \int_{\Sigma_R(0)} \frac{\epsilon^{\frac{Q-\alpha}{2}} H(\delta^{-1}(\xi)) \epsilon^{\frac{Q-\alpha}{2}} H(\delta^{-1}(\eta))}{|\eta^{-1}\xi|^{Q-\alpha-1}} d\xi d\eta$$

$$= \lambda \int_{\Sigma_{R/\epsilon}(0)} \int_{\Sigma_{R/\epsilon}(0)} \frac{H(\zeta) H(\eta)}{|\eta^{-1}\xi|^{Q-\alpha-1}} d\xi d\eta \geq C_0 \lambda \epsilon.$$

So, for $\lambda > 0$, and small enough $\epsilon$, we have

$$-I_1 + I_2 + I_3 \geq -C_1 \left( \frac{R}{\epsilon} \right)^{-Q} + C_0 \lambda \epsilon = \epsilon \left( -\frac{C_1}{R} \left( \frac{\epsilon}{R} \right)^{Q-1} + C_0 \lambda \right) > 0.$$

□

To show the existence of weak solution, we first establish the following criterion for the existence of maximizer for energy $Q_\lambda(\Omega)$.

**Proposition 5.2.** If $Q_\lambda(\Omega) > D_{n,\alpha}$ for a given $\lambda > 0$, $Q_\lambda(\Omega)$ is achieved by a positive function $f_\epsilon \in L^{q_0}(\Omega)$.

To complete the proof of Proposition 5.2 we will adapt the method of blow-up analysis. Namely, we will prove firstly the existence of (1.12) with subcritical exponent problem and then get the existence of (1.12) with critical exponent by compactness.
Consider

\[ Q_{\lambda,q}(\Omega) = \sup_{f \in L^q(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \int_{\Omega} f(\xi) (|\eta^{-1} \xi|^{-Q-\alpha} + \lambda |\eta^{-1} \xi|^{-Q-\alpha-1}) f(\eta) d\xi d\eta}{\|f\|_{L^q(\Omega)}^2} \]

for \( q > q_\alpha \). Similar to the proof of Lemma 5.2, we easily show that the supreme is attained by a positive function \( f_q \), which satisfies the subcritical equation

\[ Q_{\lambda,q}(\Omega) f^{q-1}(\xi) = \int_{\Omega} \frac{f(\eta)}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta + \lambda \int_{\Omega} \frac{f(\eta)}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta, \quad \xi \in \Omega \quad (5.1) \]

with the constraint \( \|f_q\|_q = 1 \). Further, we can show easily that \( f_q \in \Gamma^\alpha(\Omega) \) and \( Q_{\lambda,q}(\Omega) \to Q_{\lambda}(\Omega) \) for \( q \to q_\alpha^+ \).

**Lemma 5.3.** Let \( f_q \geq 0 \) being maximum energy solutions to \( (5.1) \) for \( q \in (q_\alpha, 2) \). If there exists some \( q_0 \in (q_\alpha, 2) \) such that \( Q_{\lambda,q}(\Omega) \geq D_{n,\alpha} + \epsilon \) for any \( q \in (q_\alpha, q_0) \), then the sequence \( \{f_q\}_{q_\alpha < q < q_0} \) is uniformly bounded in \( \Omega \).

**Proof.** We only need to show \( \lim_{q \to q_\alpha^+} \|f_q\|_{C^\alpha(\Omega)} \leq C \). We prove this by contradiction. Suppose not. Let \( f_q(\xi) = \maxf_q(\xi) \). Then \( f_q(\xi) \to \infty \) for \( q \to q_\alpha^+ \).

Let

\[ \mu_q = f_\frac{2-q}{q}(\xi), \quad \text{and} \quad \Omega_q = \delta_{\mu_q}(\xi^{-1} \Omega) := \{ \xi \in \Omega \} \]

Define

\[ g_q(\xi) = \mu_q^{-1} f_q(\xi, \delta_{\mu_q}(\xi)), \quad \text{for} \ \xi \in \Omega_q. \quad (5.2) \]

Then, \( g_q \) satisfies

\[ Q_{\lambda,q}(\Omega) g^{q-1} (\xi) = \int_{\Omega_q} \frac{g_q(\eta)}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta + \lambda \int_{\Omega_q} \frac{f_q(\xi) - q - |f_q|}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta \quad (5.3) \]

and \( g_q(0) = 1, \ g_q(\xi) \in (0, 1] \).

Choose \( q_\delta = q - \delta \) for some small \( \delta > 0 \). We can check that

\[ \int_{\Omega} \frac{f_q(\xi) - q - |f_q|}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta = f_q(\xi) - \delta \int_{\Omega_q} \frac{f_q(\xi) - q - |f_q|}{|\eta^{-1} \xi|^{Q-\alpha-1}} d\eta \]

\[ \leq C f_q(\xi) - \delta \|f_q\|_{q-q_\delta} \to 0, \quad \text{as} \quad q \to q_\alpha^+. \quad (5.4) \]

For \( \xi \in \mathbb{H}^n \) and \( R > B|\xi| \), where the constant \( B \) is larger than the constant in Proposition 2.2 we have

\[ \int_{\Omega_q \setminus B_R} \frac{g_q(\eta)}{|\eta^{-1} \xi|^{Q-\alpha}} d\eta \leq C \int_{\Omega_q \setminus B_R} \frac{g_q(\eta)}{|\eta|^{Q-\alpha}} d\eta \]

\[ = C \mu_q^{-\frac{2-q}{q}} \int_{\Omega_q \setminus B_{R\mu_q}(\xi)} \frac{f_q(\eta)}{|\xi^{-1} \eta|^{Q-\alpha}} d\eta \]

\[ \leq C \mu_q^{-\frac{2-q}{q}} \|f_q\|_q \left( \int_{\Omega_q \setminus B_{R\mu_q}(\xi)} |\xi^{-1} \eta|^{(\alpha-Q)\frac{2-q}{q}} d\eta \right) \]

\[ \leq C \mu_q^{-\frac{2-q}{q}} (R \mu_q)^{\alpha - \frac{Q}{q}} \to 0, \quad (5.5) \]

as \( q \to q_\alpha^+ \) and \( R \to +\infty \).
As $q \to q^+_0$, there are two cases: Case 1. $\Omega_q \to \mathbb{H}^n$, and $g_q(z) \to g(z)$ pointwise in $\mathbb{H}^n$, where $g(z)$ satisfies (from (5.3), and estimates (5.4) and (5.5)):

$$Q_\lambda(\Omega)g^{a_0-1}(\xi) = \int_{\mathbb{H}^n} \frac{g(\eta)}{|\eta - \xi|^{Q-a}} d\eta, \quad g(0) = 1. \quad (5.6)$$

Also, a direct computation yields:

$$1 = \int_\Omega f^q(\xi) d\xi = \mu_q^{-\frac{2(n+\alpha)}{n}} \int_{\Omega_q} g^q(\xi) d\xi \geq \int_{\Omega_q} g^q d\xi.$$

Thus $\int_{\mathbb{H}^n} g^{a_0} d\xi \leq 1$. Combining this with (5.6), we have

$$D_{n,\alpha} + \epsilon \leq Q_\lambda(\Omega) = \frac{\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{g(\xi)g(\eta)}{|\eta - \xi|^{Q-a}} d\xi d\eta}{\|g\|_{L^{a_0}(\mathbb{H}^n)}^{a_0}} \leq \frac{\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{g(\xi)g(\eta)}{|\eta - \xi|^{Q-a}} d\xi d\eta}{\|\tilde{g}\|_{L^{a_0}(\mathbb{H}^n)}^{2}} \leq D_{n,\alpha}. \quad (5.7)$$

Contradiction!

Case 2. $\Omega_q \to \Omega_{q_0}$, where $\Omega_{q_0}$ is some subset of $\mathbb{H}^n$ satisfying $\omega_{q_0} \neq \mathbb{H}^n$, $g_q(z) \to g(z)$ pointwise in $\Omega_{q_0}$, where $g(z)$ satisfies (from (5.3), and estimates (5.4) and (5.5)):

$$Q_\lambda(\omega)g^{a_0-1}(\xi) = \int_{\Omega_{q_0}} \frac{g(\eta)}{|\eta - \xi|^{Q-a}} d\eta, \quad g(0) = 1. \quad (5.8)$$

Similarly, we know $\int_{\Omega_{q_0}} g^{a_0} d\eta \leq 1$. Combining this with (5.7), we have

$$D_{n,\alpha} + \epsilon \leq Q_\lambda = \frac{\int_{\Omega_{q_0}} \int_{\Omega_{q_0}} \frac{g(\xi)g(\eta)}{|\eta - \xi|^{Q-a}} d\xi d\eta}{\|g\|_{L^{a_0}(\Omega_{q_0})}^{a_0}} \leq \frac{\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{g(\xi)g(\eta)}{|\eta - \xi|^{Q-a}} d\xi d\eta}{\|\tilde{g}\|_{L^{a_0}(\mathbb{H}^n)}^{2}} \leq D_{n,\alpha}. \quad (5.9)$$

Contradiction! □

Proof of Proposition 5.2 Let $f_q > 0$ be solutions to (5.1) for $q \in (q_0,2)$, which are also the maximal functions to energy $Q_{\lambda,q}(\Omega)$. Then $\|f_q\|_{L^\infty(\mathbb{H}^n)} \leq C$ by Lemma 5.3 which yields $f_q$ is uniformly bounded and equi-continuous due to equation (5.1). Thus $f_q \to f_*$ as $q \to q_0$ in $C^0(\Omega)$, and $f_*$ is the energy maximizer for $Q_\lambda(\Omega)$.

Completion of the Proof of Theorem 1.2 (existence for critical case). From Lemma 5.1 and Proposition 5.2 we know that there exists a positive solution $f_* \in C^0(\Omega)$ of (1.12) with critical exponent. Moreover, by a similar argument of the second part of Lemma 5.3 we have $f_* \in \Gamma^\alpha(\Omega)$.

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