Weak topologies for Linear Logic

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May 22, 2014

Abstract

We construct a denotational model of Linear Logic, whose objects are all the locally convex and separated topological vector spaces endowed with their weak topology. Linear proofs are interpreted as continuous linear functions, and non-linear proofs as tuples of monomials. The duality in this very general interpretation of Linear Logic does not come from an orthogonality relation, thus we do not complete our constructions by a double-orthogonality operation. This yields an interpretation of the negative connectives of Linear Logic as those preserving the weak topology, while positive connectives must be applied a shift, that is must be endowed with their weak topology.

Introduction

Linear Logic \cite{Gir88} can be seen as an analysis of classical logic, through polarities and involutive linear negation \cite{LR03}. The linearity hypothesis has been made by Girard \cite{Gir87} after a semantical investigation of Intuitionistic Logic. Semantics has in turn led to various discoveries around Linear Logic, as in Game Semantics or Differential \(\lambda\)-calculus \cite{ER06}.

However, the linear negation is often modelized with an orthogonality relation \cite{Ehr02,Ehr05,Gir04} or with a Chu construction \cite{Gir99}. Many of these models also go towards algebra, by using the theory of vector spaces. It allows us to get closer to the algebraic intuitions of Linear Logic, and to reach analogies with functional analysis.

We would like to make the analogy between Linear Logic and algebra as little specific as we could. So we try to interpret formulas as vector spaces. Topology on these vector spaces help us to interpret the classical duality, and to construct the exponential. It allows to get far from the combinatorial point of view, and introduce basis-free vectorial interpretation of the connectives of Linear Logic. As in Scott Domains, we interpret our functions by continuous functions, and especially our linear proofs will be interpreted by linear continuous functions between topological vector spaces. As the topological dual of a space \(E\) is not constructed from \(E\) with an orthogonality relation, we have the opportunity to construct a new kind of negation.
We do not satisfy ourselves with a model of ILL, nor with a model of LL obtained by a Chu construction. We want the classical duality to be an intrinsic property of our objects. This lead us to the only real restricting choice of this paper: we endow our spaces with their weak topology. Every other construction in the interpretation of LL will stem from this first choice.

We are curious about the construction of the tensor product in denotational model of LL. The interpretation of the $\otimes$ connective is interpreted by a tensor product, but this one is practically always completed in some way: so as to obtain Cauchy-completeness [BET12, Gir99], or so as to obtain a self orthogonal object [Ehr02, Ehr05]. On the contrary, we manage here to define the tensor product as the algebraic tensor product, endowed with some specific topology. We proceed similarly with the exponential. Note that as topological tensor products do not preserve Cauchy-completeness, we can’t ask for our space to be Cauchy-complete. This reduces drastically the possibilities for the theory of non-linear functions on our spaces, as convergence will be more difficult to obtain. It explains the form of our interpretation of non-linear proofs, as sequences of monomials.

Thus, the reflexivity condition constructed here via topology is very different to what happens in other models of LL, where the interpretation of classical duality is constructed via an orthogonality. As explained by Girard in [Gir04], orthogonality in models of Linear Logic behaves like polarity in functional analysis. That is, spaces which equals their bi-orthogonal are subspaces of a unique vector space. This is not the way the duality is constructed here: reflexivity is not based on an orthogonality, as the dual $E'$ of a topological vector space is not defined as the polar of $E$ with respect to some relation. We didn’t find any orthogonality relation on which our spaces could be constructed.

The refusal to complete our objects with a double-orthogonality operation will lead to a distinction between the interpretation of positive connectives of LL and the one of the negative connectives. The negative connectives are those who preserve the fact to be endowed with the weak topology, while the positive connectives are those who need to be applied a shift, that is to be endowed with their weak topology.

Synthesis of the constructions  Our constructions are very simple, as they only use well-known tools of the theory of topological vector spaces. Formulas of Linear Logic are interpreted by any locally convex and separated topological vector space, endowed with its weak topology. The negation of a formula is interpreted by the dual of the interpretation of this formula, endowed with its weak* topology.

$\otimes$ is interpreted by a specific topological tensor product endowed with its weak topology: choosing the strong topology of the algebraic tensor product is indeed on of the determining steps in the construction of this model. The $\forall$ is interpreted as the topological dual of $\otimes$. As a result from these constructions, the type of linear proofs between two formulas is interpreted as the space of linear continuous functions between the interpretation of these formulas, endowed with the topology of simple convergence.

As for additive connectives, $\&$ is interpreted by a the topological product, and $\oplus$ by the topological co-product endowed once again with its weak topology. They coincide on finite indexes.

Finally, the exponential is constructed so that non-linear proofs between two spaces are interpreted by the tuples of monomials between these two spaces. This construction follows the idea of quantitative semantics, which is at the heart of
Related works  The construction presented here is very general, as any locally convex and separated topological vector space is an object of our category. Our approach differs from the one presented in the finiteness spaces [Ehr05], or in the Hopf algebras as model of Multiplicative Linear Logic [Blu96]: the topologies used there are Lefschetz topologies, that is topologies where neighbourhoods of 0 are sub-vector spaces, opposed to the intuitive idea of unit ball coming from normed spaces.

This generality allows us to define our tensor product as an algebraic tensor product, and not as its bidual or biorthogonal, contrasting with what happens in finiteness spaces [Ehr02] or Köthe spaces [Ehr05]. On the contrary to what happens in Ehrhard’s model, our constructions are basis-free. Moreover, the interpretation of the classical duality is internalized, and not obtained as the result of a Chu construction as in Coherent Banach spaces [Gir99] or as in works by M. Barr [Bar00, Bar79, Bar91]. With an adjunction between Chu spaces and the category of Topological vector spaces, Barr obtains a -autonomous category of spaces endowed with their weak topology, where the spaces of linear functions are the same than ours. However, the tensor product is completed, as \( E \otimes F \) is defined as \( L(E, F')' \). Here, our tensor product is not completed, as \( E \otimes F \) is the algebraic tensor product endowed with some topology, and our constructions avoid the digression through the Chu category. This work can therefore be seen as an extension of Barr’s work to a Seely category.

One could think of the interpretation of LLP in a control-category by its negative connectives, described by O. Laurent in its thesis [Lau02]. However, this is not what is used here, as positive connectives are not primarily interpreted as the dual of the interpretation of their negation. This model neither corresponds to the interpretation of LLP in a co-control category, as positive connectives do not preserve the property of being endowed with one’s weak topology.

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1 Topologies on vector spaces and spaces of functions

Remember that a topological vector space is a vector space endowed with a vector topology. A vector topology on a vector space is a topology making the sum and multiplication by any scalar continuous. The vector space is said to be Hausdorff when the topology is so, that is when given any two distinct point in the vector space, they belong respectively to two open set with empty intersection.

In a topological space, a basis $U$ of open sets is a collection of open sets such that every open set is the union of elements of $U$. In a topological vector space, we only need to know the neighbourhoods of 0 to retrieve the entire topology, as the addition is continuous. Therefore, we will most of the time describe the topology on our spaces by giving a basis of 0-neighbourhoods, or some time by giving a subbasis of 0-neighbourhoods. A subbasis $U$ of open sets is a collection of open sets such that every open set is the union of finite intersections of elements of $U$. See the definition of the weak* topology for an example of basis and pre-basis of an topological vector space.

**Definition 1.1.** [Jar81, 6.7] A Hausdorff topological locally convex vector space is a Hausdorff topological vector space with a pre-basis of neighbourhoods of 0 consisting of convex subsets. We write lctvs to denote such vector spaces.

From now on, $E$, $F$ and $G$ are lctvs. All our lctvs are vector spaces over $K = \mathbb{R}$ or $K = \mathbb{C}$.

1.1 Weak and weak* topologies

We will endow our spaces with their weak topology, that is with the topology generated by their continuous duals. To do so, they are first endowed with a strong topology on them, a topology which will allow us to define the continuous dual. Practically all the definitions presented here are well-known definition of functional analysis, and demonstrations can be found for example in [Jar81] or [Kö69].

Let us begin with the heart of our construction, that is the dual $E'$ of a vector space $E$. $E'$ is fundamental, as it defines the weak topology on $E$, and as the topology on itself allows to interpret the classical duality.

**Definition 1.2.** If $E$ is a lctvs, we will note $E'$ the space of all continuous linear forms $l : E \to K$ on $E$, endowed with its weak* topology.

**Definition 1.3.** The weak* topology on $E'$ is the topology of pointwise convergence on $E$.

A basis for the weak* topology on $E'$ is the collection of all

\[ W_{x_1, \ldots, x_n, \varepsilon} = \{ l \in E' | \|l(x_1)\| < \varepsilon, \ldots, \|l(x_n)\| < \varepsilon \} \]
where \( n \in \mathbb{N} \), \( x_i \in E \) and \( \epsilon > 0 \). A subbasis for the weak* topology is the collection of all
\[
W_{x,\epsilon} = \{ l \in E' \mid |l(x)| < \epsilon \}.
\]

**Definition 1.4.** The weak topology on a lctvs \( E \) is the inductive topology generated by \( E' \), that is the coarsest locally convex topology on \( E \) making all the functions \( l \in E' \) continuous.

A basis for the weak topology on \( E \) is the collection of all
\[
\mathcal{W}_{1,\ldots,n,\epsilon} = \{ x \in E \mid |l_1(x)| < \epsilon, \ldots, |l_n(x)| < \epsilon \}
\]
where \( n \in \mathbb{N} \), \( l_i \in E' \) and \( \epsilon > 0 \).

Let us generalize the description of the basis of the weak and weak* topology.

**Notation 1.5.** When \( \mathcal{F}(E, F) \) is a vector space of functions between \( E \) and \( F \), when \( B \subset E \) and \( U \subset F \), then
\[
W_{B,U} = \{ f \in \mathcal{F}(E, F) \mid f(B) \subset U \}.
\]
When \( B = \{ x_1, \ldots, x_n \} \) is finite then \( W_{B,U} \) is written \( W_{x_1,\ldots,x_n,U} \). When \( F = \mathbb{K} \) and \( U = \{ y \in F \mid |y| < \epsilon \} \), then \( W_{B,U} \) is written \( W_{B,\epsilon} \). Note that algebraically \( E \) can be considered as a sub-vector space of \( E'' \) through the application \( ev : x \mapsto (ev_x : l \mapsto l(x)) \). The notation used in the definition 1.2 is then coherent with the one described above.

Note that \( W_{B,U} \) is convex as soon as \( U \) is convex.

**Notation 1.6.** From now on \( E_w \) denote a lctvs endowed with its weak topology. The original topology on \( E \) is called its strong topology.

**Proposition 1.7.** When \( E \) is a lctvs, then so are \( E_w \) and \( E' \).

**Proof.** As a consequence of Hahn-Banach separation theorem [Jar81, 7.2.2.a], we have that \( E' \) separates the points of \( E \): if \( x, y \in E \) are distinct, then there is \( l \in E' \) such that \( l(x) \neq l(y) \). This makes \( E \) endowed with its weak topology a Haussdorff topological vector space. The fact that it is locally convex follows from the aspect of the subbasis \( \{ W_{x,\epsilon} \} \) explained above. The same arguments make \( E' \) a lctvs.

**Notation 1.8.** We work in the category of lctvs endowed with their weak topology and continuous linear maps. Let us denote \( Weak \) this category. The relation \( \simeq \) thus denotes an isomorphism in \( Weak \) between two lctvs. When we need to speak about an isomorphism in the category of vector spaces and linear maps, we will use the sign \( \sim \).

In fact, in this category it is not necessary to distinguish \( (E_w)' \) from \( E' \).

**Proposition 1.9.** \( (E_w)' \) is linearly homomorphic to \( E' \). That is, \( (E_w)' \simeq E' \)

To show this fact, we first need to recall a classical result from algebra:

**Lemma 1.10.** Consider \( E \) a vector space and \( l, l_1, \ldots, l_n \) linear forms on \( E \). Then \( l \in Vect(l_1, \ldots, l_n) \) if and only if \( \cap_{k=1}^n Ker(l_k) \subset Ker(l) \).
Proof. If \( l \in Vect(l_1, \ldots, l_n) \) then clearly \( \bigcap_{k=1}^n Ker(l_k) \subset Ker(l) \). Conversely, suppose \( \bigcap_{k=1}^n Ker(l_k) \subset Ker(l) \). Without loss of generality, we can suppose the family \( \{l_k\} \) free. Let us show the result by induction on \( n \). If \( n = 1 \), then \( Ker(l) = Ker(l_1) \) as they have the same codimension, and one has \( l = \frac{f(x_0)}{l_1(x_0)}l_1 \) for any fixed \( x_0 \notin Ker(l) \).

Consider now \( l, l_1, \ldots, l_n \) linear forms on \( E \) such that \( \bigcap_{k=1}^n Ker(l_k) \subset Ker(l) \). Then by restricting \( l \) to \( Ker(l_n) \) we obtain scalars \( \lambda_1, \ldots, \lambda_{n-1} \) such that

\[
l|_{Ker(l_n)} = \sum_{k=1}^{n-1} \lambda_k l_k|_{Ker(l)}.
\]

Then \( Ker(l_n) \subset Ker(l - \sum_{k=1}^{n-1} \lambda_k l_k) \), and we have our result.

Proof. Proof of proposition 1.9 Let us show first that \( (E_w)' \sim E' \). As the weak topology on \( E \) is coarser than the initial topology on \( E \), we have \( E' \subset (E_w)' \). Consider now a continuous linear form \( l \) on \( E_w \). Then by continuity of \( l \), and with the description of the weak topology given in [1.3], there is \( l_1, \ldots, l_n \in E' \) and \( \epsilon > 0 \) such that

\[
l(W_{l_1, \ldots, l_n, \epsilon}) \subset \{ \lambda \in \mathbb{K} | \|\lambda\| < 1 \}.
\]

By homogeneity, we have \( \bigcap_{k=1}^n Ker(l_k) \subset Ker(l) \) and the preceding lemma gives us \( l \in E' \). Thus \( (E_w)' \sim E' \). Their respective topology being both the weak* topology induced by points of \( E \), we have \( (E_w)' \approx E' \).

We can then continue to write \( E' \) for the dual of a space \( E \), regardless of the fact he may be endowed with its weak topology. We will write \( E' \) for \( (E_w)' \) and \( E_w' \) for \( (E')_w \).

1.2 The evaluation function and reflexivity

Let us define

\[
ev : \begin{cases} E \to E'' \\
x \mapsto ev_x = (l \in E' \mapsto l(x))
\end{cases}
\]

\( ev \) is linear, injective as \( E' \) separates \( E \), continuous and open as both \( E \) (resp. \( E'' \)) are endowed with the weak (resp. weak*) topology induced by \( E' \).

The starting point of this paper is the fact that when \( E' \) is endowed with its weak* topology, \( E \) can be considered as a reflexive space, that is \( E_w' \approx E''_w \). This equality models the double-negation of classical logic, and will make our category of topological vector spaces and linear maps *-autonomous.

Proposition 1.11. \( E'' \) is algebraically isomorphic through \( ev \) to \( E \).

Proof. The proof is done as in the proposition [1.9] using Lemma [1.10]. The key to this proof is the fact that \( E' \) is endowed with the weak* topology, and thus for every \( l \in E'' \) there is \( x_1, \ldots, x_n \in E \) such that

\[
l(W_{x_1, \ldots, x_n, 1}) \subset \{ x \in \mathbb{K} | \|x\| < 1 \}.
\]

This implies \( \bigcap_i Ker(ev_{x_i}) \subset Ker(l) \), and thus through lemma [1.10] \( l \in Vect(ev_{x_i}). \) That is, there is \( \lambda_1, \ldots, \lambda_n \in \mathbb{K} \) such that \( l = ev_{\sum_{i=1}^n \lambda_i x_i} \).
As $ev$ is bicontinuous, we have:

**Proposition 1.12.** $E_w$ is linearly homeomorphic through $ev$ to $E''_w$.

As a consequence of this result, our decision to put the weak* topology on $E'$ makes it an object of Weak, without any further operation on its topology:

**Corollary 1.13.** $E' \simeq (E')_{w}$. That is, $E'$ is linearly homeomorphic to $(E')_{w}$ through the identity function $Id: E' \rightarrow (E')_{w}$.

**Proof.** The topology of $E'$ is the weak* topology induced by $E$, and the topology of $(E')_{w}$ is the weak topology induced by $E'' \sim E$.

This theory of weak and weak* topology fits in the more general theory of dual pairs, see for example chapter 8 of [Jar81].

### 2 Multiplicative connectives

We will now make use of the evaluation function in other contexts, as it can be defined as $ev : E \rightarrow \mathcal{F}(E, C)'$, where $\mathcal{F}(E, C)$ is some vector space of functions between $E$ and $C$. When $\mathcal{F}(E, C)$ contains only linear functions, $ev$ is linear. When $E' \subset \mathcal{F}(E, C)$, $ev$ is injective, as $E'$ separates the points of $E$.

#### 2.1 Spaces of linear maps

**Definition 2.1.** Let us denote $\mathcal{L}(E, F)$ the space of all continuous linear maps between $E$ and $F$, endowed with the topology of simple convergence on points of $E$.

A basis for the topology of simple convergence on $\mathcal{L}(E, F)$ is the collection of all

$$W_{x_1, \ldots, x_n, V} = \{ l \in \mathcal{L}(E, F)| l(x_1) \in V, \ldots, l(x_n) \in V \}$$

where $n \in \mathbb{N}$, $x_i \in E$ and $V$ is a neighbourhood of 0 in $F$.

The weak* topology on $E'$ is exactly the topology of simple convergence on points of $E$, thus:

**Fact 2.2.** $E' \simeq \mathcal{L}(E, \mathbb{K})$

We will now begin to speak about tensor product, but in an algebraic way. For the moment, we do not suppose any topology on the algebraic tensor product $E \otimes F$.

**Proposition 2.3.** $\mathcal{L}(E_w, F_w)'$ is algebraically isomorphic to $E \otimes F$.

**Proof.** See [Jar81] 15.3.5 or [Kot79] 39.7 for references in the literature. We will sketch here the proof done in [Kot79], as the proof by Jarchow uses the projective tensor product\(^1\). Consider first the space $L(E, F)$ of all linear maps between $E$ and $F$, endowed with the topology of simple convergence on points of $E$. Then, if we choose an algebraic basis $X = \{ x \in X \}$ of $E$, we have $L(E, F) \simeq \prod_{x \in X} F_x$ where $F_x$ is a copy of $F$. Thus $L(E, F)' \sim (\prod_{x \in X} F_x)' \sim \bigoplus_{x \in X} F_x'^*$ (the dual of a cartesian product is the direct sum of the duals, see Proposition 3.4). Linear forms in $\bigoplus_{x \in X} F_x'^*$ are exactly finite sums of linear forms in $F_x'$, each one applying on a different $F_x$, that is applying on the space $L(E, F)(\{ x \})$. Thus, when we consider linear forms on

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\(^1\)The definition of the projective tensor product is recalled after the definition 2.15 of the cotensor.
\( \oplus_X F'_x \) as elements of \( L(E,F)' \), we write them as finite sums \( \sum l_x \circ ev_x \) with \( x \in X \) and \( l_x \in F' \). Thus the linear application

\[
\begin{cases}
E \otimes F' \to L(E,F)'
\sum (x \otimes l_x) \mapsto \sum l_x \circ ev_x
\end{cases}
\]

is well-defined and surjective. Köthe shows in detail in his proof why this morphism is injective, proving that \( L(E,F)' \) is algebraically isomorphic to \( E \otimes F' \).

Now let us get back to \( L(E,F) \). This one is dense in \( L(E,F)' \) when it is endowed with the topology of simple convergence on \( E \), as for every \( x_1,...,x_n \in E \) distinct and open set \( V \) in \( F \) we can find a continuous linear map \( f \) such that \( f(x_i) \in V \). Without loss of restriction, we suppose the family \( \{x_i\} \) free. Select \( y \neq 0 \in V \), and \( l_i \in E' \) such that \( l_i(x_j) = \delta_{i,j} \). The function \( f : x \mapsto \sum l_i(x)y \) is linear continuous, and sends \( x_i \) on \( y \).

Thus the dual of \( L(E,F) \) is algebraically isomorphic to the dual of \( L(E,F)' \), that is to \( E \otimes F' \).

This proposition allows us to write every linear function \( f \in L(E,F)' \) as a unique finite sum

\[ f = \sum_{i=1}^n l_i \circ ev_{x_i} \]

where \( l_i \in F' \) and \( x_i \in E \).

Let us now describe how linear functions behave with respect to weak topologies.

**Lemma 2.4.** Functions in \( \mathcal{L}(E,F_w) \) are exactly the linear maps from \( E \) to \( F \) which results when postcomposed with any map from \( F' \) into a map in \( E' \).

**Proof.** By definition of the weak topology on \( F \), a function \( f : E_w \to F_w \) is continuous if and only if for every \( l \in F' \), \( f \circ l : E \to \mathbb{K} \) is continuous. If \( f \) is linear, this means that \( f \circ l \in E' \).

**Proposition 2.5.** For all \( E, F \) lctvs, we have \( \mathcal{L}(E,F_w) \approx \mathcal{L}(E_w,F_w) \).

**Proof.** A continuous linear map from \( E_w \) to \( F_w \) is continuous from \( E \) to \( F_w \), as the weak topology is coarser than the initial topology on \( E \). Consider now \( f \in \mathcal{L}(E,F_w) \). For every \( l \in F' \) we have \( f \circ l \in E' \) by proposition 1.9. Thus \( f \in \mathcal{L}(E_w,F_w) \).

### 2.2 Tensor and cotensor

Various ways exists to create a lctvs from the tensor product of two lctvs \( E \) and \( F \). That is, several topologies exist on the vector space \( E \otimes F \), the most used of them being the projective topology [Jar81, III.15] and the injective topology [Jar81, III.16]. These topologies behave particularly well with respect to the completion of the tensor product, and were originally studied in Grothendieck’s thesis [Gro66].

However, we would like a topology on \( E \otimes F \) that would endow \( \text{Weak} \) with a monoidal closed category structure. This is mainly why we use the inductive tensor product, defined in [Gro66] 1.3.1. So as to define this topology, we need to mention the topological product of two lctvs.

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\[ \text{Proof.} \]
The tensor product

**Definition 2.6.** Consider $E$ and $F$ two lctvs. $E \times F$ is the algebraic cartesian product of the two vector space, endowed with the product topology, that is the coarsest topology such that the projections $p_E: E \times F \to E$ and $p_F: E \times F \to F$ are continuous.

Neighbourhoods of 0 in $E \times F$ are generated by the set $U \times V$, where $U$ is a 0-neighbourhood in $E$ and $V$ is a 0-neighbourhood in $F$.

**Definition 2.7.** Let us denote by $B(E; F; G)$ the space of all bilinear and separately continuous functions from $E \times F$ to $G$, and by $B(E, F)$ the space of all bilinear and separately continuous functions from $E \times F$ to $\mathbb{K}$. We endow it with the topology of simple convergence on $E \times F$. $B(E; F; G)$ is then a lctvs.

**Fact 2.8.** Consider $E$, $F$ and $G$ three lctvs, and $f$ a bilinear map from $E \times F$ to $G$. Then $f \in B(E; F; G)$ if and only if for every $l \in G'$, $l \circ f \in B(E; F)$.

**Definition 2.9.** Consider $E$ and $F$ two lctvs. $E \otimes F$ is the algebraic tensor product, endowed with the finest topology making the canonical bilinear map $E \times F \to E \otimes F$ separately continuous.

**Proposition 2.10.** \cite[I.3.1.13]{Gro65} For every lctvs $G$, we have $\mathcal{L}(E \otimes F, G) \sim B(E; F; G)$. Especially, $(E \otimes F)' \sim B(E; F)$.

**Proof.** Let us write $B(E; F, G)$ for the vector space of all bilinear maps from $E \times F$ to $G$. As $E \times F \to E \otimes F$ is separately continuous, the canonical isomorphism $L(E \otimes F, G) \sim B(E; F, G)$ induces an injection from $\mathcal{L}(E \otimes F, G)$ to $B(E; F, G)$. Let us show by contradiction that this injection is onto. Consider $f \in B(E; F; G)$ such that its linearisation $\tilde{f} \in L(E \otimes F, G)$ is not continuous. Let us denote $E \otimes_{\tau} F$ the vector space $E \otimes F$ endowed with the topology $\tau$ induced by $\tilde{f}$. Then, because $f$ is separately continuous, $E \times F \to E \otimes_{\tau} F$ is separately continuous. Thus $\tau$ is coarser than the inductive topology. However this would implies that $\tilde{f}: E \otimes F \to G$ would be continuous. We have a contradiction.

**Proposition 2.11.** (associativity of $\otimes$ in Weak). Consider $E$, $F$, and $G$ three lctvs. Then $(E \otimes (F \otimes G_w)_w)_w \simeq ((E \otimes F_w)_w \otimes G_w)_w$

**Proof.** As the algebraic tensor product is associative we have $(E \otimes (F_w \otimes G_w)_w)_w \simeq ((E_w \otimes F_w)_w \otimes G_w)_w$. Let us show that the two space bear the same topology. The dual of the first space is $(E \otimes (F \otimes G_w)_w)' \simeq B(E_w; (F_w \otimes G_w)_w)$ according to proposition \cite[11]{Top}. One can show as above that $B(E_w; (F_w \otimes G_w)_w)$ corresponds to the space of all trilinear separately continuous functions on $E_w \times F_w \times G_w$. Likewise, the dual of the second space is $(E \otimes (F_w \otimes G_w)_w) \simeq B((E_w \otimes F_w)_w; G_w)$, which corresponds also to the space of all trilinear separately continuous functions on $E_w \times F_w \times G_w$. Then $(E_w \otimes (F_w \otimes G_w)_w)_w$ and $((E \otimes F_w)_w \otimes G_w)_w$ are algebraically isomorphic and have the same dual, thus the same weak topology.

**Monoidal closedness**

**Proposition 2.12.** Consider $E$, $F$ and $G$ three lctvs. Then we have $\mathcal{L}((E_w \otimes F_w)_w; G_w) \sim B(E_w, F_w; G_w)$. 

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Proof. A map \( f \) lies in \( \mathcal{L}((E_w \otimes F_w)_w, G_w) \) if and only if for every \( l \in G_l \), \( l \circ f \in (E_w \otimes F_w)' \). But according to proposition 2.14 we have \( (E_w \otimes F_w)' \sim B(E_w, F_w) \). Thus \( f \in \mathcal{L}((E_w \otimes F_w)_w, G_w) \) if and only if \( f \) as a bilinear map is in \( B(E_w; F_w, G_w) \).

**Proposition 2.13.** Consider \( E, F \) and \( G \) three lctvs. Then we have

\[
B(E_w; F_w, G_w) \sim \mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w).
\]

Proof. Remember from Proposition 2.3 that \( \mathcal{L}(F_w, G_w)' \sim F \otimes G' \). Consider \( g \) a continuous linear function from \( E_w \) to \( \mathcal{L}(F_w, G_w)_w \). So as to be well defined must verify that for \( x \in E \) fixed, for all \( l \in G_l \), \( y \mapsto l(g(x)(y)) \) is continuous. So as to be continuous \( g \) must verify that for \( y \) and \( l \in G' \) both fixed, \( x \mapsto l(g(x)(y)) \) is continuous. \( l \in G' \) being fixed, we see that \( l \circ g \) transforms into a separately continuous map in \( B(E_w, F_w) \). Thus \( g \) can be seen as a function \( \tilde{g} \) of \( B(E_w; F_w, G_w) \). The transformation of a map in \( B(E_w, F_w; G_w) \) into a map of \( \mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w) \) is done likewise.

Thus we have an algebraic isomorphism between \( \mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w) \) and \( \mathcal{L}((E_w \otimes F_w)_w, G_w) \). To show that they bear the same weak topology, we just have to show that they have the same dual. But according to proposition 2.3 \( \mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w)' = E_w \otimes \mathcal{L}(F_w, G_w)' = E_w \otimes F_w \otimes G'_w = \mathcal{L}((E_w \otimes F_w)_w, G_w)' \).

**Theorem 2.14.** The category \( \text{Weak} \) in monoidal closed, as we have for each lctvs \( E_w, F_w, G_w \) in \( \text{Weak} \):

\[
\mathcal{L}(E_w, \mathcal{L}(F_w, G_w)_w) \simeq \mathcal{L}((E_w \otimes F_w)_w, G_w)_w.
\]

**The co-tensor product**

**Definition 2.15.** The co-tensor \( \mathcal{N} \) of linear logic is interpreted by \( E \mathcal{N} F \sim B(E', F') \).

**Proposition 2.16.** The \( \mathcal{N} \) connective preserves the weak topology: indeed, for every lctvs \( E \) and \( F \), \( (E \mathcal{N} F)_w \simeq E_w \mathcal{N} F_w \).

Proof. As \( E \mathcal{N} F \simeq (E' \otimes F')' \), the result follows immediately from proposition 2.14.

Let us explain that if we had chosen to work with the projective tensor product, whose dual is the space of all bilinear continuous forms, then \( \mathcal{N} \) would have been a tensor product. However, the projective tensor product would not give a monoidal closed category in our setting, as bilinear maps can be separately continuous and not continuous. See [Jar81, 15.1] for details on the projective tensor product.

Let us denote \( B(E, F) \) the space of all continuous, and not only separately continuous, bilinear forms on \( E \times F \), and \( E \otimes \# F \) the algebraic tensor product \( E \otimes F \) endowed with the finest locally convex topology making \( E \times F \to E \otimes F \) continuous.

As for proposition 2.17 we can show that:

**Proposition 2.17.** [Jar81, 15.1.2] The dual of \( E \otimes \# F \) is \( B(E, F)' \).

**Proposition 2.18.** The space of continuous bilinear forms on \( E' \times F' \) behaves like a tensor product of \( E \) and \( F \). That is: \( B(E', F') \sim E \otimes F \).
Proof. Remember that $E'$ and $F'$ are endowed by the weak* topology. Consider the linear mapping
\[
\begin{aligned}
E \otimes F &\to B(E',F') \\
x \otimes y \mapsto ev_{x \otimes y} : (l_1, l_2) \in E' \times F' \mapsto l_1(x)l_2(y).
\end{aligned}
\]
It is injective, as $E'$ (resp. $F'$) separates $E$ (resp. $F$). Let us show it is surjective. Consider $\phi \in B(E',F')$. As $\phi$ is continuous, there are $x_1, \ldots, x_n \in E$ and $y_1, \ldots, y_m \in F$ such that
\[
if |l_1(x_i)| < 1 \text{ and } |l_2(y_j)| < 1 \text{ for all } i, j, \text{ then } |\phi(l_1, l_2)| < 1.
\]
Let us fix $l_1 \in E'$. We show as in proposition 1.12 that
\[
\bigcap_{i=1}^n Ker( ev_{x_i \otimes y_j} ) \cap (\{l_1\} \times F') \subset Ker(\phi) \cap (\{l_1\} \times F')
\]
and thus by lemma 1.10 we have a collection of scalars $\lambda_{i,j}$ such that
\[
\phi(l_1 \times F') = \sum_{i,j} \lambda_{i,j}(ev_{x_i \otimes y_j}).
\]
Likewise, for all $l_2 \in F'$, we obtain a collection of scalars $\lambda'_{i,j}$ such that
\[
\phi(l_1 \times F') = \sum_{i,j} \lambda'_{i,j}(ev_{x_i \otimes y_j}).
\]
Thus for all $(l_1, l_2) \in E' \times F'$,
\[
\phi(l_1, l_2) = \sum_{i,j} \lambda_{i,j}l_1(x_i)l_2(y_j) = \sum_{i,j} \lambda'_{i,j}l_1(x_i)l_2(y_j).
\]
If we extract from the family $\{ev_{x_i \otimes y_j}\}$ a free family, we obtain $\lambda_{i,j} = \lambda'_{i,j}$, and thus $\phi = \sum_{i,j} \lambda_{i,j}ev_{x_i \otimes y_j}$. The linear mapping consider above is thus bijective, and $E \otimes F \cong B(E',F')$.

One can find another proof of this result in the book by F. Trèves [Trè06], 42.4.

2.3 A star-autonomous category

According to Theorem 2.14 $Weak$ is a monoidal closed category, with $ev_E : E \to E'' = \mathcal{L}(E, \mathcal{L}(E, \mathbb{K}))$ being an isomorphism in this category for every object $E$.

**Theorem 2.19.** $Weak$ is a star-autonomous category, with dualizing object $\mathbb{K}$.

**Proof.** Let us take $\mathbb{K} = 1 = 1$ the dualizing object. Then the evaluation map
\[
(A \to 1) \otimes A \to 1
\]
leads by symmetry of $\otimes$ and closure exactly to $ev : A \to ((A \to 1) \to 1)$, that is $ev : A \to A''$. As shown in proposition 1.12 $ev : A \to A''$ is an isomorphism in the category $Weak$, and $Weak$ is then $*$-autonomous. \qed
3 Additive connectives

The additive connectives of Linear Logic are of course interpreted by the product and co-product between lctvs. Sadly, the two coincide on finite indexes. However, they behave differently with respect to weak topology: the product preserves the weak topology, while the coproduct doesn’t. See proposition 3.6 and section 5 for an interpretation of this phenomenon in terms of polarities.

Practically all the results in this section are classical results from functional analysis. We nonetheless detail their proofs, which can also be found in the literature [Jar81, Sch71, Kot69].

Definition 3.1. When $I$ is a set, when for all $i \in I$ $E_i$ is a lctvs, then we can define $\prod_{i \in I} E_i$ as the vector space product over $I$ of the $E_i$, endowed with the coarsest topology on $E$ such that all $p_i$ are continuous.

If $U_j$ is a 0-basis in $E_j$, then a subbasis for the topology on $\prod_i E_i$ consists of all the

$$U \sim \prod_{i \in I, i \in U_i} E_i \times U_i \cap U_i$$

with $U_i \in U_i$.

Definition 3.2. We define $E \sim \bigoplus_{i \in I} E_i$ as the algebraic direct sum of the vector spaces $E_i$, endowed with the finest locally convex topology such that every injection $I_j : E_j \to E$ is continuous. Remember that the algebraic direct sum $E$ is the subspace of $\prod_j E_j$ consisting of elements $(x_j)$ having finitely many non-zero $x_j$.

If $U_j$ is a 0-basis in $E_j$, then a 0-basis for $\bigoplus_j E_j$ is described by all the sets:

$$U = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \bigcup_{j \in J, k \in \mathbb{N}} U_{j,k}$$

where $J \subseteq \mathbb{N}$. See [Jar81] 4.3 for an explanation of this result. Note that this topology is finer than the topology induced by $\prod E_i$ on $\bigoplus E_i$.

Proposition 3.3. [Jar81] 4.3.2. $I$ is finite if and only if the canonical injection from $\bigoplus_{i \in I} E_i$ to $\prod_{i \in I} E_i$ is surjective.

Proof. By the definition of the algebraic co-product, as the sets of elements of the product with finitely many non-zero composants, we have that $\bigoplus_{i \in I} E_i \sim \prod_{i \in I} E_i$ when $I$ is finite. The description of the subbasis of the inductive and projective topologies gives us the result. The converse proposition is trivial. ☐

Proposition 3.4. Algebraically we have $(\bigoplus_{i \in I} E_i)' \sim (\prod_{i \in I} E_i)'$ and $(\prod_{i \in I} E_i)' \sim (\bigoplus_{i \in I} E_i)'$.

Proof. Consider $l \in \prod_{i \in I} E_i'$. Then the function $x \in \bigoplus_{i \in I} E_i \mapsto \sum_i l_i(x_i)$ is well defined, linear and continuous. Reciprocally, to any $l \in (\bigoplus_{i \in I} E_i)'$ corresponds the tuple $(l_i) \in \prod_{i \in I} E_i'$ with $l_i = l \circ I_i$.

Consider now $l \in (\prod_{i \in I} E_i)'$. Then by definition of the product topology, $l_i = l_{i|E_i} \in E_i'$. As $l$ is continuous, there is $H \subset I$ finite, and 0-neighbourhoods $U_i$ for $i \in H$ such that

$$l(\prod_{i \in H} U_i \times \prod_{i \in H} E_i) \subset \{ \lambda \in \mathbb{K} \mid \lambda < 1 \}.$$ 

By homogeneity, $l_i = 0$ for $i \not\in H$, and $l$ corresponds to an element of $\bigoplus E_i'$. Conversely, an element of $\bigoplus E_i'$ acts on $\prod_i E_i$ as a continuous linear form. ☐
Proposition 3.5. [Jar81] II.8.8, Theorem 5 and Theorem 10. We have always $(\prod_{i \in I} E_i)_w \simeq \prod_{i \in I} (E_i)_w$, but $(\bigoplus_{i \in I} E_i)_w \simeq \bigoplus_{i \in I} (E_i)_w$ holds only when $I$ is finite.

Proof. Let us show first that $(\prod_{i \in I} E_i)_w \simeq \prod_{i \in I} (E_i)_w$. The topology of $(\prod_{i \in I} E_i)_w$ is the coarsest one of $\prod E_i$ making all $l \in (\prod E_i)_w$ continuous. But as $(\prod E_i)_w' \sim \bigoplus E_i'$, every $l \in (\prod E_i)_w$ is continuous from $\prod (E_i)_w$ to $\mathbb{K}$. Thus the topology of $(\prod (E_i)_w)$ is finer than the topology of $(\prod_{i \in I} E_i)_w$. Now the topology of $(\prod_{i \in I} E_i)_w$ is the coarsest one making all the projections $p_i : \prod (E_i)_w \to (E_i)_w$ continuous. But as the $p_i : \prod E_i \to E_i$ are continuous, all the $p_i : (\prod_{i \in I} E_i)_w \to (E_i)_w$ are continuous. Thus the topology of $(\prod_{i \in I} E_i)_w$ is finer than the topology of $\prod (E_i)_w$, and the two are equal.

Now when $I$ is finite, proposition 3.3 and the result above tells us that $(\bigoplus_{i \in I} E_i)_w \simeq \bigoplus_{i \in I} (E_i)_w$. Suppose now that $I$ is not finite, and that $E_i \neq \{0\}$ for all $i$. To follow the proof by Jarchow we introduce the notion of equicontinuity: a set $B$ of linear continuous functions from $E$ to $F$ is equicontinuous if for every $0$-neighbourhood $V$ in $F$ there is a $0$-neighbourhood $U$ in $E$ such that $B(U) \subset V$. One can check as in proposition 1.12 and thanks to the lemma 1.10 that equicontinuous subsets of $(F_w)'$ are the finite ones. Thus if $B_i$ is a finite but nonempty subset of $E_i'$, $\prod B_i$ is equicontinuous in $(\bigoplus_{i \in I} (E_i)_w)'$, but not in $((\bigoplus_{i \in I} E_i)_w)'$ as it is not finite dimensional. Thus $(\bigoplus_{i \in I} E_i)_w \simeq \bigoplus_{i \in I} (E_i)_w$. 

We can now characterize the dual of a product and of a coproduct in the category $Wcak$.

Proposition 3.6. We have always $(\bigoplus_{i \in I} E_i)' \simeq \prod_{i \in I} E_i'$ but $(\prod_{i \in I} E_i)' \simeq \bigoplus_{i \in I} E_i'$ holds only when $I$ is finite.

Proof. According to proposition 3.5, we have $(\bigoplus_{i \in I} E_i)' \sim \prod_{i \in I} E_i'$. The first bears the weak topology induced by $\bigoplus_{i \in I} E_i$, that is $(\bigoplus_{i \in I} E_i)' \simeq (\prod_{i \in I} E_i)'_w$ and the second bears the product topology induced by all the $E_i$, that is $\prod_{i \in I} E_i' = \prod_{i \in I} (E_i)_w$. The previous proposition 3.5 gives us a linear homeomorphism between the two. Likewise, we have $(\bigoplus_{i \in I} E_i)' \sim \bigoplus_{i \in I} E_i'$, $(\prod_{i \in I} E_i)' = (\bigoplus_{i \in I} E_i)'_w$ and $\bigoplus_{i \in I} E_i' \simeq \bigoplus_{i \in I} (E_i)'_w$. Proposition 3.5 tells us then that $(\prod_{i \in I} E_i)' \simeq \bigoplus_{i \in I} E_i'$ if and only if $I$ is finite.

4 A quantitative model of Linear Logic

4.1 Quantitative semantics

Introduced by Girard in [Gir88], quantitative semantics refines the analogy between linear functions and linear programs (consuming exactly once its input). Indeed, programs consuming exactly $n$-times their resources are seen as monomials of degree $n$. General programs are seen as the disjunction of their executions consuming $n$-times their resources. Mathematically, one can try to agree with this semantic by interpreting non-linear proofs as sums of $n$-monomials. The structure presented here is very simple, our spaces provide us with practically no tools except the Hahn-Banach theorem. This is why we simply choose to represent non-linear maps as finite tuples over $\mathbb{N}$ of $n$-monomials.

We have no difficulties in defining $n$-linear mappings $f$ on topological vector spaces. From them, we define $n$-monomials as the function matching $x$ to $f(x, \ldots, x)$. 

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Not however that to know the existence of such a function, we need a ring structure on our topological vector space. We don’t restrict ourselves to topological algebras, but they are the levs where we are sure of the existence of an exponential.

The exponential we define here has a lot of similarities with the one defined in Fock spaces [BPS94], or the free symmetric algebra studied by Mellies, Tabareau and Tasson [MTT09]. The difference here is that we considered sequences of monomials in the co-Kleisli category and not n-linear symmetric maps. Therefore our exponential is the direct sum over n \in \mathbb{N} of dual space of the space of n-monomials, and not a direct sum of symmetric n-tensor product of A.

4.2 The exponential

Monomials

Definition 4.1. \( L^n(E, F) \) is the space of n-linear separately continuous functions from E to F.

A n-monomial from E to F is a function \( f : E \to F \) such that there is \( \hat{f} \in L^n(E, F) \) verifying that for all \( x \in E \) \( f(x) = \hat{f}(x, ..., x) \).

Proposition 4.2 (Polarization formula, see [KM97] 7.13). Consider \( f \) a n-monomial from E to F. Then we have \( f(x) = \hat{f}(x, ..., x) \) where \( \hat{f} \) is a symmetric n-linear function from E to F defined by:

For every \( x_1, ..., x_n \in E \) \( \hat{f}(x_1, ..., x_n) = \frac{1}{n!} \sum_{\epsilon_1, ..., \epsilon_n = 0} (-1)^{\sum \epsilon_j} f(\sum_j \epsilon_j x_j) \).

Proof. Let us write

\[
\begin{align*}
\binom{n}{k_1, k_2, ..., k_m} &= \frac{n!}{k_1! k_2! ... k_m!} = \left(\frac{k_1}{k_1}\right) \left(\frac{k_1 + k_2}{k_2}\right) ... \left(\frac{k_1 + k_2 + ... + k_m}{k_m}\right)
\end{align*}
\]

for the multinomial coefficient. For every \( x_1, ..., x_n \), we have

\[
\begin{align*}
\hat{f}(\sum_{j=1}^n x_j) &= \sum_{j_1 + ... + j_n = n} \frac{n!}{k_1! k_2! ... k_n!} (k_1 + k_2 + ... + k_n) \sum_{\epsilon_1, ..., \epsilon_n = 0} \frac{1}{\epsilon_1! \epsilon_2! ... \epsilon_n!} \sum_{j_1 + ... + j_n = n} (-1)^{\sum \epsilon_j} \epsilon_j x_{\epsilon_j} \\
&= \sum_{j_1 + ... + j_n = n} \frac{1}{j_1! j_2! ... j_n!} \hat{f}(\sum_{j=1}^n x_j) \sum_{\epsilon_1, ..., \epsilon_n = 0} \frac{1}{\epsilon_1! \epsilon_2! ... \epsilon_n!} (-1)^{\sum \epsilon_j} \epsilon_j x_{\epsilon_j}
\end{align*}
\]

Thus

\[
\begin{align*}
\hat{f}(\sum_j \epsilon_j x_j) &= \sum_{j_1 + ... + j_n = n} \epsilon_1^{k_1} ... \epsilon_n^{k_n} \frac{1}{k_1! k_2! ... k_n!} (k_1 + k_2 + ... + k_n) \sum_{\epsilon_1, ..., \epsilon_n = 0} \frac{1}{\epsilon_1! \epsilon_2! ... \epsilon_n!} (-1)^{\sum \epsilon_j} \epsilon_j x_{\epsilon_j}
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{n!} \sum_{\epsilon_1, ..., \epsilon_n = 0} (-1)^{\sum \epsilon_j} f(\sum_j \epsilon_j x_j) &= \sum_{j_1 + ... + j_n = n} \frac{1}{j_1! j_2! ... j_n!} \hat{f}(\sum_{j=1}^n x_j) \sum_{\epsilon_1, ..., \epsilon_n = 0} \frac{1}{\epsilon_1! \epsilon_2! ... \epsilon_n!} (-1)^{\sum \epsilon_j} \epsilon_j x_{\epsilon_j}
\end{align*}
\]
Let us show that \( \sum_{i_1, \ldots, i_n=0}^1 (-1)^{n-\sum_{j} i_j} \epsilon_{i_1}^1 \ldots \epsilon_{i_n}^n \) is non-zero if and only if \( k_1 = \ldots = k_n = 1 \). Indeed, if there is \( i \) such that \( k_i = 0 \), then there is \( j \) such that \( k_j = 0 \), as \( k_1 + \ldots + k_n = n \). Let us suppose \( k_1 = 0 \). Then

\[
\sum_{\epsilon_1, \ldots, \epsilon_n=0}^1 (-1)^{n-\sum_{j} \epsilon_j} \epsilon_1^1 \ldots \epsilon_n^n = \sum_{\epsilon_2, \ldots, \epsilon_n=0}^1 (-1)^{n-1-\sum_{j} \epsilon_j} \epsilon_2^2 \ldots \epsilon_n^n \\
+ \sum_{\epsilon_2, \ldots, \epsilon_n=0}^1 (-1)^{n-\sum_{j} \epsilon_j} \epsilon_2^2 \ldots \epsilon_n^n \\
= 0
\]

Thus \( \frac{1}{m} \sum_{i_1, \ldots, i_n=0}^1 (-1)^{n-\sum_{j} i_j} f(\sum_{j} i_j x_j) = f(x_1, \ldots, x_n) \).

**Corollary 4.3.** There is a unique symmetric \( n \)-linear map \( \hat{f} \) associated to a \( n \)-monomial \( f \).

**Definition 4.4.** Let us write \( \mathcal{H}^n(E, F) \) for the space of \( n \)-monomials over \( E \), and endow it with the topology of simple convergence on points of \( E \). We write \( \mathcal{L}_n(E, F) \) for the space of all symmetric \( n \)-linear maps from \( E \) to \( F \), endowed with the topology of simple convergence on points of \( E \).

**Fact 4.5.** \( \mathcal{H}^n(E, F) \) is a lcvs.

**Proposition 4.6.** For every lcvs \( E \) and \( F \), for every \( n \in \mathbb{N} \), we have \( \mathcal{H}^n(E, F) \cong \mathcal{L}_n(E, F) \).

**Proof.** The equality between the two vector spaces follows from the previous corollary. As they are respectively endowed with the topology of pointwise convergence of points of \( E \) (resp \( E \times \ldots \times E \)), the bijection between the two spaces is bicontinuous.

If we write \( E_{\text{sym}}^n \) for the symmetrized \( n \)-tensor product of \( E \) with himself, we have \( \mathcal{L}_n^s(E, F) = \mathcal{L}(E_{\text{sym}}^n, F) \).

As \( \mathcal{H}^n(E, F) \cong \mathcal{L}_n^s(E, F) \) by proposition 4.6, we have the following information on the space \( \mathcal{H}^n(E, F) \).

**Proposition 4.7.** For every lcvs \( E \) and \( F \), \( \mathcal{H}^n(E, \mathbb{K}) = E_{\text{sym}}^n \otimes F' \). That is, every continuous linear form \( \theta \) on \( \mathcal{H}^n(E, F) \) can be written as a finite sum of functions of the type \( l \circ ev_{x_1} \otimes \ldots \otimes ev_{x_n} \) with \( l \in F' \) and \( x_1, \ldots, x_n \in E \).

From this, we can deduce that \( \mathcal{H}^n(E_w, F_w) \) is a weak space: it is already endowed with its weak topology.

**Corollary 4.8.** For every lcvs \( E \) and \( F \), we have that \( \mathcal{H}^n(E_w, F_w) \cong \mathcal{H}^n(E, F) \).

**Proof.** The topology on \( \mathcal{H}^n(E_w, F_w) \) is the topology of simple convergence on \( E_{\text{sym}}^n \), with weak convergence on \( F \). This is exactly the topology induced by its dual \( E_{\text{sym}}^n \otimes F' \).
The exponential

Definition 4.9. Let us define !E as the lctvs $\bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(E, \mathbb{K})'$.

Proposition 4.10. We have $(!E)' \sim \prod_n \mathcal{H}^n(E, \mathbb{K})$, and thus $(!E)_w \simeq (\prod_n \mathcal{H}^n(E, \mathbb{K}))'$.

Proof. According to proposition 3.4, we have that $(!E)' \sim \prod_n \mathcal{H}^n(E, \mathbb{K})'$, as both spaces in this equality are endowed by the topology induced by $!E$. Then $(!E)' \simeq \prod_n \mathcal{H}^n(E, \mathbb{K})_w \simeq \prod_n \mathcal{H}^n(E, \mathbb{K})$. Taking the dual of these spaces, we get $!E_w \simeq (\prod_n \mathcal{H}^n(E, \mathbb{K}))'$.

Like what happens in spaces of linear functions, see proposition 2.5, we have always that $\mathcal{H}^n(E,F_w) \simeq \mathcal{H}^n(E,F_w)$. Thus $!(E_w) \simeq \bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(E,\mathbb{W})' \simeq \bigoplus_{n \in \mathbb{N}} \mathcal{H}^n(E,\mathbb{K})' \simeq !E$.

Notation 4.11. We will write without any ambiguity $!E$ for $!(E_w)$ and $!E_w$ for $(!E_w)$.

Definition 4.12. $!$ is a functor on $\text{Lin}$. For $f \in \mathcal{L}(E_w, F_w)$ we define

\[
!f : \begin{cases} !E_w \to !F_w \\ \phi \mapsto ((g_n) \in \prod_n \mathcal{H}^n(F,\mathbb{K}) \mapsto \phi((g_n \circ f)_n)\end{cases}
\]

So as to show that $!$ is in fact a co-monad, we need to understand better the elements of $!E$. $!E$ is defined as $\bigoplus_n \mathcal{H}^n(E,\mathbb{K})'$, so $\phi \in !E$ can be described as a finite sum $\phi = \sum_{n=0}^{N} \phi_n$, with $\phi_n \in \mathcal{H}^n(E,\mathbb{K})'$. The whole co-monad is thought so as to have a nice composition between $f \in \prod_n \mathcal{H}^n(E,F)$ and $g \in \prod_n \mathcal{H}^n(E,G)$, that is $f \circ g \in \prod_n \mathcal{H}^n(E,G)$ with $(f \circ g)_p = \sum_{k,l,p} g_k \circ f_{k,l}$.

Proposition 4.13. $!: \text{Lin} \to \text{Lin}$ is a co-monad. Its co-unit $\epsilon : !1 \to 1$ is defined by

\[
\epsilon_E : \begin{cases} !E_w \to E_w \\ \phi \mapsto \phi_1 \in E'' = E \end{cases}
\]

The co-unit is the operator extracting from $\phi \in !E$ its part operating on linear maps.

The co-multiplication $\delta : !1 \to !!1$ is defined by

\[
\delta_E : \begin{cases} !E_w \simeq (\prod_p \mathcal{H}^p(E,\mathbb{K}))' \to !!E_w \simeq (\prod_n (\prod_m \mathcal{H}^m(E,\mathbb{K})))' \\ \phi \in (\prod_p \mathcal{H}^p(E,\mathbb{K}))' \mapsto (g_n)_n \mapsto \phi \left((x \in E \mapsto \sum_{k,l,p} g_k(f_{m})_m \mapsto f_{p,k}(x))\right)_p
\end{cases}
\]

The proofs presented below are based more on the idea of non-linear continuation than on a combinatoric point of view. $!E_w = (\prod_p \mathcal{H}^p(E,\mathbb{K}))'$ can be thought as a space of quantitative-linear continuations, $\mathbb{K}$ being the basic space of the result of a computation.

Proof. We have to check the two equations of a co-monad, that is:

- $\delta \delta = \delta \delta$

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\[ \epsilon_i \delta = ! \epsilon \delta = Id \]

Let us detail the calculus of the last one. For every \( \phi = \sum \phi_p \epsilon ! E \), we have:

\[
e_i E \delta E(\phi) = e_i E \left( (g_n)_n \in \prod E^n (! E, K) \mapsto \phi \left( [x \in E \mapsto \sum_{k \mid p} g_k((f_m)_m \epsilon ! E \mapsto f_m^x(x))]_p \right) \right)
\]

\[
= e_i E \left( (g_n)_n \mapsto \phi \left( [x \in E \mapsto \sum_{k \mid p} g_k(ev_x^k(E, K'))]_p \right) \right)
\]

\[
= e_i E \left( (g_n)_n \mapsto \sum_p \phi_p(x \mapsto \sum_{k \mid p} g_k(ev_x^k(E, K'))) \right)
\]

As \( e_i E \) maps a function in \( ! E_w \cong (\prod_n \mathcal{H}^n(E, K))^\prime \) to its restriction to \( \mathcal{L}(E, K) \), and then to the corresponding element in \( ! E \), we have without the isomorphism \( ! E' \cong ! E \):

\[
e_i E \delta E(\phi) = g_1 \epsilon ! E' \sum_p \phi_p(x \mapsto \sum_{k \mid p} g_k(ev_x^k(E, K'))^0 \text{ if and only if } k=1
\]

\[
= g_1 \epsilon ! E' \sum_p \phi_p(x \mapsto g_1(ev_x^k(E, K'))
\]

As \( g_1 \) lives in \( ! E' \cong \prod_n \mathcal{H}^n(E, K) \), we can write \( g_1 \) as a tuple \((g_1, m)_m\) of \( m \)-monomials:

\[
e_i E \delta E(\phi) = g_1 \epsilon ! E' \sum_p \phi_p(x \mapsto ev_x(g_1, p))
\]

\[
= g_1 \epsilon ! E' \sum_p \phi_p(x \mapsto g_1, p(x))
\]

\[
= g_1 \epsilon ! E' \sum_p \phi_p(g_1, p)
\]

\[
= g_1 \epsilon ! E' \mapsto \phi(g_1)
\]

With the isomorphism \( ! E' \cong ! E \) we obtain \( \epsilon \delta = Id \).

The equation \( ! \delta = Id \) is proved likewise: consider \( \phi = \sum \phi_p \epsilon ! E \). Then

\[
! \delta(\phi) = ! \epsilon \left( (g_n)_n \in \prod E^n (! E, K) \mapsto \sum_p \phi_p(x \mapsto \sum_{k \mid p} g_k(ev_x^k(E, K'))) \right)
\]

\[
= (h_m)_m \in \prod E^m (E, K) \mapsto \sum_p \phi_p(x \mapsto h_k \circ \epsilon(ev_x^k(E, K')))^{0 \text{ if and only if } k=1}
\]

\[
= (h_m)_m \mapsto \phi((h_m)_m)
\]

So \( ! \delta = Id \). \( \square \)
Proposition 4.14. The \( ? \) connective of Linear Logic is interpreted by \( \? E \equiv (\! E')' \cong \Pi_n \mathcal{H}^n(E', K) \).

We will write \( \text{Weak}^{1} \) for the co-Kleisli category of \( \text{Weak} \) with \( ! \). We first show that morphisms of this category are easy to understand, as they are just tuples of \( n \)-monomials.

4.3 The co-Kleisli category

The exponential above is chosen because of its co-Kleisli category. Indeed, we want to decompose non-linear proofs as conjunctions of \( n \)-linear proofs, and the simplest way we found to do that is by interpreting non-linear maps from \( E \) to \( F \), that is linear maps from \( ! E \) to \( F \), as tuples of \( n \)-monoms from \( E \) to \( F \). Of course, other choices may have been possible, but as no Cauchy-completeness hypothesis is required on \( E \) and \( F \) it may be difficult to build power series. Power series are converging sums of monoms, and convergence in topological vector spaces is mainly possible thanks to completeness.\(^2\)

Theorem 4.15. For all lctvs \( E \) and \( F \), \( \mathcal{L}(!E_w, F_w) \sim \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \).

Proof. Consider \( f \in \mathcal{L}(!E_w, F_w) \). Define, for each \( n \in \mathbb{N} \), \( f_n : x \in E_w \mapsto f(ev_x \in \mathcal{H}^n(E_w, \mathbb{K})') \). Then \( f_n \) is clearly \( n \)-linear. Let us show that it is continuous from \( E_w \) to \( F_w \). Consider \( l \in F' \). Then \( x \mapsto ev_x \in \mathcal{H}^n(E_w, \mathbb{K})' \) is continuous from \( E_w \) to \( \mathcal{H}^n(E_w, \mathbb{K})' \), as the last space is endowed with the topology of simple convergence on \( \mathcal{H}^n(E_w, \mathbb{K}) \). \( \mathcal{H}^n(E_w, \mathbb{K})' \sim !E_w \) is continuous, as \( !E_w \sim (\bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})')_w \) according to proposition 3.8, and as \( \mathcal{H}^n(E_w, \mathbb{K})' \sim \bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})' \) and \( \bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})' \sim (\bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})')_w \) are continuous. Thus

\[
\begin{align*}
  f_n : E_w &\xrightarrow{ev} \mathcal{H}^n(E_w, \mathbb{K})' \xrightarrow{\bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})'} (\bigoplus_k \mathcal{H}^k(E_w, \mathbb{K})')_w \xrightarrow{f} F_w
\end{align*}
\]

is continuous, and \( f_n \in \mathcal{H}^n(E_w, F_w) \). To every \( f \in \mathcal{L}(!E_w, F_w) \) we associate this way \( f_n \in \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \).

Consider now \( (f_n) \in \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \) and define \( f : \! E_w \mapsto \{ l \in F' \mapsto \phi((l \circ f_n)_n) \} \). \( f \) is well-defined as \( l \circ f_n \in \mathcal{H}^n(E_w, \mathbb{K}) \) for every \( n \in \mathbb{N} \) and every \( l \in F' \). When \( \phi \) is fixed, let us denote \( \phi_f \) the function \( l \in F' \mapsto \phi((l \circ f_n)_n) \). Then:

- \( l \in E' \mapsto l \circ f_n \in \mathcal{H}^n(E, \mathbb{K}) \) is continuous as \( F' \) (resp. \( \mathcal{H}^n(E, \mathbb{K}) \)) is endowed with the topology of simple convergence on points of \( F \) (resp. on points of \( E \)).
- \( l \mapsto (l \circ f_n)_n \in \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, \mathbb{K}) \) is then continuous by definition of the product topology.
- \( \phi_f \) is then continuous.

Thus \( \phi_f \in F'' \cong F \). For each \( \phi \), there is \( y \in F \) such that \( \phi_f = ev_y \). We can now consider \( f : \phi \in !E \mapsto y \in F \). \( f \) is clearly linear in \( \phi \). It is continuous as \( !E_w \) is endowed with the topology of simple convergence on \( \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, \mathbb{K}) \).

Finally, one can check that the mapping \( \theta : f \in \mathcal{L}(!E_w, F_w) \mapsto (f_n) \in \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \) and \( \Delta : (f_n) \in \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \mapsto f \in \mathcal{L}(!E_w, F_w) \) just described are inverse one of each other. \( \square \)

\(^2\)It appears that the weakest completeness condition necessary to model quantitative Linear Logic should be Mackey completeness, see [KT14].
Let us show now that the equality described above is topological.

**Theorem 4.16.** For all lctvs $E$ and $F$,

$$\mathcal{L}(|E_w, F_w|) \cong \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w),$$

and so

$$\mathcal{L}(|E_w, F_w|)_w \cong \left( \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w) \right)_w \cong \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, F_w)_w.$$

**Proof.** Let us show first that the function $\theta : f \in \mathcal{L}(|E_w, F_w|) \mapsto (f_n) \in \prod_n \mathcal{H}^n(E_w, F_w)$ is continuous. It is enough to show that $f \mapsto f_n$ is continuous. But if $(f_n)_{n \in \mathbb{N}}$ is a net converging towards $f$ in $\mathcal{L}(|E_w, F_w|)$, that is a net such that for every $\phi \in \mathcal{L}(E_w, F_w)$, the function $f_n \mapsto f_n(\phi)$ converges towards $f(\phi)$ in $F_w$, we have that for every $x \in E f_n(\text{ev}_x \in \mathcal{H}^n(E_w, \mathbb{K}))$ converges towards $f(\text{ev}_x)$ in $F_w$. Thus $(f_n)_n$ converges towards $f$ in $\mathcal{L}(E_w, F_w)$.

The composition in $\text{Weak}^1$ is thus given by the definition of a co-Kleisli category: if $f \in \mathcal{L}(|E, F|)$ and $g \in \mathcal{L}(|F, G|)$ we define:

$$g \circ f : !E \xrightarrow{\delta_E} !E \xrightarrow{f} !F \xrightarrow{g} G.$$

**Notation 4.17.** For $f \in \mathcal{L}(|E, F|)$, we will write $\tilde{f}$ the corresponding tuple of monomials in $\prod_n \mathcal{H}^n(E, F)$.

**Proposition 4.18.** For every $f \in \mathcal{L}(|E, F|)$ and $g \in \mathcal{L}(|F, G|)$, we have

$$(g \circ f)_p = \sum_{k|p} \tilde{g}_k \circ \tilde{f}_k.$$ 

**Proof.** By definition, for $\phi \in !E$,

$$g \circ f(\phi) = g(f(\delta(\phi))) = g((g_n) \in \prod_n \mathcal{H}^n(F, \mathbb{K}) \mapsto \delta(\phi)((g_n \circ f)_n))$$

For every $p \in \mathbb{N}^*$, and $x \in E$, we have then:

$$(g \circ f)_p(x) = g \circ f(\text{ev}_x \in \mathcal{H}^p(E, \mathbb{K})) = g((g_n) \in \prod_n \mathcal{H}^n(F, \mathbb{K}) \mapsto \delta(\text{ev}_x \in \mathcal{H}^p(E, \mathbb{K}))((g_n \circ f)_n))$$

Now $\delta(\text{ev}_x \in \mathcal{H}^p(E, \mathbb{K})) = (h_j)_j \in \mathcal{H}^p(!E, \mathbb{K}) \mapsto \sum_{k|p} h_k(\text{ev}_x \in \mathcal{H}^{p/k}(E, \mathbb{K})).$ Thus

$$(g \circ f)_p(x) = g((g_n) \in \prod_n \mathcal{H}^n(F, \mathbb{K}) \mapsto \delta(\text{ev}_x \in \mathcal{H}^p(E, \mathbb{K}))((g_n \circ f)_n)$$

$$= \sum_{k|p} \tilde{g}_k \circ \tilde{f}_{p/k}$$

$$= \sum_{k|p} \tilde{g}_k \circ \tilde{f}_{p/k}.$$
4.4 Cartesian Closedness

Let us show that $Weak^l$ endowed with the cartesian product described in section 3 is cartesian closed.

**Theorem 4.19.** For every lctvs $E$, $F$ and $G$, we have:

$$
\prod_{p \in \mathbb{N}} \mathcal{H}^p(\prod_{w} E_w \times F_w, G_w) \approx \prod_{n \in \mathbb{N}} \mathcal{H}^n(E_w, \prod_{m \in \mathbb{N}} \mathcal{H}^m(F_w, G_w)).
$$

The equality above means also that

$$(\prod_{p} \mathcal{H}^p(\prod_{w} E_w \times F_w, G_w))_w \approx [\prod_{n} \mathcal{H}^n(E_w, \prod_{m} \mathcal{H}^m(F_w, G_w))]_w,$$

as $(\prod_{m} \mathcal{H}^m(F_w, G_w))_w \approx \prod_{m} \mathcal{H}^m(F_w, G_w)_w$ by proposition 3.24 and as $\mathcal{H}^p(E_w, F_w)$ is already endowed with its weak topology by proposition 4.8.

**Proof.** As for every $n \in \mathbb{N}$, for every lctvs $E$ and $F$ we have $\mathcal{H}^n(E, F) \approx \mathcal{L}^n(E, F)$, we are going to prove the following proposition: for every lctvs $E$, $F$ and $G$:

$$
\prod_{p} \mathcal{L}^p(E_w \times F_w, G_w) \approx \prod_{n} \mathcal{L}^n(E_w, \prod_{m} \mathcal{L}^m(F_w, G_w)_w).
$$

Let us fix $E$, $F$ and $G$. Define:

$$
\phi : \prod_{p} \mathcal{L}^p(E_w \times F_w, G_w) \to \prod_{n} \mathcal{L}^n(E_w, \prod_{m} \mathcal{L}^m(F_w, G_w)_w)
$$

$$
(f_p) \mapsto [(x_1, \ldots, x_n) \mapsto ((y_1, \ldots, y_m) \mapsto f_{n+m}((x_1,0),\ldots,(x_n,0),(0,y_1),\ldots,(0,y_m))))_n]
$$

and

$$
\psi : \prod_{n} \mathcal{L}^n(E_w, \prod_{m} \mathcal{L}^m(F_w, G_w)_w) \to \prod_{p} \mathcal{L}^p(E_w \times F_w, G_w)
$$

$$
[f_n : x_1, \ldots, x_n \mapsto (f_n(x_i), m)]_{n} \mapsto [(x_1, y_1), \ldots, (x_p, y_p)] \\
\mapsto \sum_{I,J \subseteq [1,p]} \frac{1}{\prod_{n \in I, m \in J} \text{card}(I)} f_{n,\{x_i\}_{i \in I}, m}(\{y_j\}_{j \in J})
$$

where in the index of the sum $I$ and $J$ are ordered subsets (that is tuples) of $[[1,n]]$. If $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_m\}$, $f_{n,\{x_i\}_{i \in I}, m}(\{y_j\}_{j \in J})$ is a shorten notation for $f_{n,(x_{i_1}, \ldots, x_{i_n})}(y_{j_1}, \ldots, y_{j_m})$.

Let us show that $\phi$ is well-defined.

- Consider $(f_p)_p \in \prod_{p} \mathcal{L}^p(E_w \times F_w, G_w)$, $n \in \mathbb{N}$, $x \in E$, and $m \in \mathbb{N}$. Then

  $$
y \in F \mapsto f_{n+m}((x_1,0),\ldots,(x_n,0),(0,y_1),\ldots,(0,y_m))
$$

  is clearly $m$-linear and symmetric, and continuous from $E_w \times F_w$ to $G_w$ as $f_{n+m} : E_w \times F_w \to G_w$ is continuous.
• Consider \((f_p)_p \in \prod_p \mathcal{L}_w^p(E_w \times F_w, G_w)\) and \(n \in \mathbb{N}\). Then

\[
x_1, \ldots, x_n \in E_w \mapsto (y_1, \ldots, y_m) \in F \mapsto f_{n+m}((x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_m))
\]

is clearly \(n\)-linear and symmetric. It is continuous from \(E_w\) on \(\mathcal{L}_w^n(F_w, G_w)\) as the last one bears the topology of simple convergence, and as \(f_{n+m}\) is continuous from \(E_w \times F_w\) to \(G_w\). Since the weak topology on \(\mathcal{L}_w^n(F_w, G_w)\) is coarser than the strong topology, the function considered is also continuous from \(E_w\) to \(\mathcal{L}_w^n(F_w, G_w)_w\).

Let us show that \(\psi\) is well defined. Consider

\[
[f_n : x_1, \ldots, x_n \mapsto (f_{n, (x_i), (m)})_n \in \prod_n \mathcal{L}_w^n(E_w, \prod_m \mathcal{L}_w^m(F_w, G_w)_w).
\]

The function mapping \(((x_1, y_1), \ldots, (x_p, y_p)) \in (E_w \times F_w)^p\) to \(f_{n,(x_i),m}(\{y_j\}_{j \in J})\) is \(n+m\)-linear and symmetric. It is continuous, as the restrictions to fixed terms in \(E_w\) or \(F_w\) are continuous. So \(\psi\) is well defined. Note that both \(\phi\) and \(\psi\) are continuous as the spaces \(\mathcal{L}_w^n(E, F)_w\) are endowed with the topology induced by their dual \(\mathcal{L}_w^m(E, F)\). Finally, one checks that \(\phi\) and \(\psi\) are each other’s inverse. Consider \(f \in \prod_p \mathcal{L}_w^p(E_w \times F_w, G_w)\). Then \(\psi(\phi(f))\) corresponds to the function mapping \(p\) to the function in \(\mathcal{L}_w^p(E_w \times F_w, G_w)\) mapping \(((x_1, y_1), \ldots, (x_p, y_p))\) to:

\[
\sum_{\substack{I, J \subseteq \{1, \ldots, p\} \\
\text{card}(I) = n \\
\text{card}(J) = m}} \frac{1}{\binom{n}{I}} f((x_i, 0), \ldots, (x_i, 0), (0, y_j), \ldots, (0, y_m))
\]

when we write \(I = \{i_1, \ldots, i_n\}\) and \(J = \{j_1, \ldots, j_m\}\). This sum equals

\[
f_p((x_1, y_1), \ldots, (x_p, y_p))
\]

by \(n\)-linearity of \(f_p\). Thus \(\psi \circ \phi = \text{Id}\). Consider now

\[
g = [g_n : x_1, \ldots, x_n \mapsto (g_{n,(x_i), (m)})_n \in \prod_n \mathcal{L}_w^n(E_w, \prod_m \mathcal{L}_w^m(F_w, G_w)_w).
\]

Let us show that \(\phi(\psi(g)) = g\). \(\psi(g)\) is the function mapping \(p\) to \((z_1, w_1), \ldots, (z_p, w_p)\) to

\[
\sum_{\substack{I, J \subseteq \{1, \ldots, p\} \\
\text{card}(I) = a \\
\text{card}(J) = b}} \frac{1}{\binom{p}{I}} g_{n,(z_i), (m)}(\{w_j\}_{j \in J}).
\]

\(\phi(\psi(g))\) maps \(n \in \mathbb{N}\), \(x_1, \ldots, x_n \in E\), \(m \in \mathbb{N}\) and \(y_1, \ldots, y_m \in F\) to this function applied to \(n+m\) and \(((x_1, 0), \ldots, (x_n, 0), (0, y_1), \ldots, (0, y_m))\). But note that \(g_{n,(z_i), (m)}(\{w_j\}_{j \in J})\) is null as soon as one of the \(z_i\) or one of the \(w_i\) is null. So \(\phi(\psi(g))\) applied \(n \in \mathbb{N}\), \(x_1, \ldots, x_n \in E\), \(m \in \mathbb{N}\) and \(y_1, \ldots, y_m \in F\) results in

\[
\frac{1}{\binom{n+m}{n}} \sum_{\substack{I, J \subseteq \{1, \ldots, n+m\} \\
\text{card}(I) = n \\
\text{card}(J) = m}} g_{n,(z_i), (m)}(\{w_j\}_{j \in J})
\]

which is exactly \(g_{n,(z_i), (m)}(y_1, \ldots, y_m)\).
The Seely isomorphism

**Theorem 4.20.** For all locs $E$ and $F$ we have

$$!(E_w \times F_w) \simeq !E_w \otimes !F_w$$

**Proof.** This follows from the cartesian closedness of $Weak^l$, the monoidal closedness of $Weak$, and the description of $Weak^l$ obtained in theorem 4.15. Indeed

$$!(E_w \times F_w) \simeq \prod_p \mathcal{H}^p(E_w \times F_w, \mathbb{K})'$$

$$\simeq \prod_n \mathcal{H}^n(E_w, \prod_m \mathcal{H}^m(F_w, \mathbb{K}_w))'$$

$$\simeq \mathcal{L}(!(E_w, \prod_m \mathcal{H}^m(F_w, \mathbb{K}_w)))'$$

$$\simeq \mathcal{L}(!(E_w, !(F_w, \mathbb{K}_w)))$$

$$\simeq !(E_w \otimes !F_w)''$$

$$\simeq !E_w \otimes !F_w.$$

4.5 Derivation and integration

As a quantitative model of Linear Logic, this model of linear logic interprets Differential Linear Logic [Ehr11]. However, the interpretation of derivation remains combinatorial, and not as close to the usual differentiation operation as one would want. See [BET12] for an interpretation of Intuitionistic Differential Linear Logic with a satisfactory differentiation.

**Definition 4.21.** The co-dereliction rule of Differential Linear Logic is interpreted by:

$$coder_E : \begin{cases} E_w \rightarrow !E_w \\ x \mapsto ((f_n) \in \prod_n \mathcal{H}^n(E, \mathbb{K}) \mapsto f_1(x)) \end{cases}$$

**Proposition 4.22.** For every space $E$, $coder_E$ is linear continuous from $E_w$ to $!E_w$.

**Proof.** Let us fix $\phi = (\phi_n)_n \in \prod_n \mathcal{H}^n(E, \mathbb{K})$. Then $\phi \circ coder_E$ maps $x \in E$ to $\phi_1(x)$. As $\phi_1 \in E'$, $coder_E$ is continuous from $E_w$ to $!E_w$. □

We do not have an interpretation of a syntactic integration in this category. Indeed, Ehrhard’s anti-derivative operator [Ehr11 2.3] would implies some sort of integration. We do not have a way to integrate in our spaces, as no completeness condition is verified. It is noticeable that if our spaces would be reflexive, that is isomorphic to their bidual when the dual is endowed with the topology of uniform convergence over bounded sets, a weak integration would be available.
5 Weak topologies and polarities

A shift operator We choose to use weak topologies here as to obtain a model of classical linear logic. Indeed, if $E$ is not endowed with its weak topologies, we have $E \sim E''$ but not $E \simeq E''$.

As a consequence, we observe that negative and positive connectives of linear logic behave differently with respect to weak topologies. Interpretation of positive connectives must be endowed with their weak topology, whereas interpretations of negative are already endowed with their weak topology. Let us explain that idea.

For negative connectives, we have: $E'_w \simeq E'$ (proposition 1.9), $(E_w \times F_w)_w \simeq E_w \times F_w$ (proposition 3.5), $(E_w \lnot F_w)_w \simeq E_w \lnot F_w$ (proposition 2.15) and $(?E)_w \simeq ?E$ (proposition 1.9).

On the contrary, we do not have $(E_w \otimes F_w)_w \simeq (E_w \otimes F_w)_w$, nor $(!E)_w \simeq !E$. The case of the coproduct is more delicate, as when it is indexed by a finite set it corresponds with the product. We have $\Theta_{\alpha I}(E_i)_w \simeq (\Theta_{\alpha I}(E_i)_w)_w$ if and only if $I$ is finite (see proposition 3.5).

What we can conclude here is that we in fact constructed a model of the negative connectives of Linear Logic. Positive connectives are translated into negative connectives through a shift, and the shift is interpreted as fact to put its weak topology on some lctvs.

Definition 5.1. Formally, let us write $\llbracket A \rrbracket$ the interpretation of a formula $A$ of $LL_{pol}$ as a lctvs, and $\llbracket A \rrbracket_w$ the interpretation of $A$ as an object of our model, that is a lctvs endowed with its weak topology.

Let us recall the definition of $LL_{pol}$ (see [Lau02]).

Negative formulas $N := X^\downarrow | N \& N | N \& ?N \uparrow P$

Positive formulas $P := X | P \otimes P | P \oplus P | !P | P \downarrow N$.

The first interpretation $\llbracket . \rrbracket$ of a formula as an lctvs is easy, as notations of Linear Logic are inspired by mathematics: $\llbracket A \times B \rrbracket \simeq \llbracket A \rrbracket \times \llbracket B \rrbracket$, $\llbracket A \otimes B \rrbracket \simeq \llbracket A \rrbracket \otimes \llbracket B \rrbracket$, ...

Things differ when we interpret them as object of our model.

From the previous explanations it follows that:

**Proposition 5.2.** When $P$ is a positive formulas $\llbracket P \rrbracket \simeq \llbracket \uparrow P \rrbracket \simeq \llbracket \llbracket P \rrbracket_w \rrbracket$.

**Proposition 5.3.** For negative formulas, the interpretation in our model is straightforward: when $N$ is a negative formula $\llbracket [N] \rrbracket = [N]$.

**Proposition 5.4.** When $P$ is a positive formula, then $\llbracket P \rrbracket \simeq \llbracket \uparrow P \rrbracket \simeq \llbracket \llbracket P \rrbracket_w \rrbracket$. In general, we do not have $\llbracket [P] \rrbracket \simeq [P]$, except for finite coproduct.

We have no mathematical interpretation of $\downarrow$, as the interpretations of negative formulas bear no other strong topology than their weak topology. All lctvs can be seen as positive, but not all of them can be seen as negative, that is not all of them are endowed with their weak topology. Our model consists of negative lctvs.
Double orthogonalities and polarities  The interpretation of $A^*$ in our setting is $[A]'$, which is in no way an orthogonal of $A$. This is the reason why this model was constructed in the first place, so as to find a topological denotational model with intern completeness.

From this, it follows that $E \otimes F$ is not defined as $(E \rightarrow F')'$, as its usual in denotational semantics $(E \rightarrow F')$ (see [Ehr02], [Ehr05], [Bar00]) nor completed (see [Gir99], [BET12]), but as an algebraic tensor product endowed with some polarity.

The fact that $E \otimes F$ is not constructed as being the double-dual of some other lcts is responsible for the fact that we can see it as positive, that is $(E_w \otimes F_w)' \neq (E_w \otimes F_w)_{w'}$.

The same phenomenon happens for the exponential: we could have defined $!E$ as $(\prod_n H_n(E, K))'$, and this would have made it a negative connective, that is a connective which is already endowed with its weak topology.

Conclusion

We obtain a very general model of Linear Logic, using spaces which are commonly used in mathematics. It can trigger studies on computational interpretations of various theories used within the theory of topological vector spaces. The algebraic structure allows to interpret the connectives of Linear Logic, whereas the topology on our spaces interprets the duality of classical logic and the polarities of the connectives. This paper appeals to a further work on the relations of weak spaces with polarised linear logic and focalized proofs.

One could also try to define weaker exponential on this model, such as $!E = (\oplus_n H_n(E, K))'$ for an interpretation of light linear logic. The understanding of the shift as an operation of weak topology can also help to understand the decomposition of the exponential under polarities, as explained in section 4 of [Lau04].

As suggested by Barr’s work [Bar00], we could try to construct a similar model of Linear Logic with Mackey spaces, that is spaces endowed with their Mackey topology. It would not interpret polarities, but could have other interesting properties.

Another direction of research would be to construct a model with reflexive spaces as used in the literature, that is lcts which are isomorphic to their bidual when the dual is endowed with the topology of uniform convergence on bounded sets. These spaces are endowed with a weak integral, and would allow for a step towards the understanding of the computational meaning of differential equations.
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