GLOBAL SOLVABILITY AND ASYMPTOTICAL BEHAVIOR IN A TWO-SPECIES CHEMOTAXIS MODEL WITH SIGNAL ABSORPTION

GUOQIANG REN AND TIAN XIANG∗

Abstract. In this work, we study global existence, eventual smoothness and asymptotical behavior of positive solutions for the following two-species chemotaxis consumption model:

\[
\begin{aligned}
    u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - \chi_2 \nabla \cdot (u \nabla w), \quad x \in \Omega, t > 0, \\
    w_t &= \Delta w - (\alpha u + \beta v) w, \quad x \in \Omega, t > 0,
\end{aligned}
\]

in a bounded smooth but not necessarily convex domain \(\Omega \subset \mathbb{R}^n (n = 2, 3, 4, 5)\) with nonnegative initial data \(u_0, v_0, w_0\) and homogeneous Neumann boundary data. Here, the parameters \(\chi_1, \chi_2\) are positive and \(\alpha, \beta\) are nonnegative.

Under a smallness condition \(\max\{\chi_1, \chi_2\} \|w_0\|_{L^\infty} < \pi \sqrt{2/n}\), boundedness of classical solutions and stabilization to constant equilibrium have been shown in [47]. Here, without any smallness condition, we show global existence and uniform-in-time boundedness of classical solutions in 2D and global existence, eventual smoothness and asymptotical behavior (in convex domains) of weak solutions in \(nD (n=3,4,5)\). Our findings also extend and improve the one-species chemotaxis-consumption model studied in [23, 27].

1. Introduction and sketch of the main results

In this project, we investigate the following Neumann initial-boundary value problem for a two-species chemotaxis system with consumption of chemoattractant:

\[
\begin{aligned}
    u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - \chi_2 \nabla \cdot (v \nabla w), \quad x \in \Omega, t > 0, \\
    w_t &= \Delta w - (\alpha u + \beta v) w, \quad x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega.
\end{aligned}
\]

Hereafter, \(\Omega \subset \mathbb{R}^n (n \geq 1)\) is a bounded domain with a smooth boundary \(\partial \Omega\) and \(\frac{\partial}{\partial \nu}\) denotes the outer normal derivative; the unknown variables \(u = u(x, t)\) and \(v = v(x, t)\) denote the population densities of two species and \(w\) represents the concentration of the chemoattractant, \(\chi_1, \chi_2, \alpha\) and \(\beta\) are positive constants and

2000 Mathematics Subject Classification. Primary: 35K59, 35B65, 35B40, 35A09, 35K51; Secondary: 35A01, 92D25.

Key words and phrases. Two-species chemotaxis model, signal absorption, global existence, boundedness, asymptotics.

∗ Corresponding author.
the given initial data are conveniently assumed throughout this paper to satisfy, for some \( r > \max\{2, n\} \), that

\[
(u_0, v_0, w_0) \in C^0(\Omega) \times C^0(\Omega) \times W^{1,r}(\Omega), \quad u_0, v_0, w_0 \geq 0, \quad \forall 0.
\]

(1.2)

The model (1.1) is used in mathematical biology to account the biased movement of two populations in respond to the concentration gradient of one common chemical signal. It is an obvious extension of the well-known Keller-Segel second model:

\[
\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla w), & x \in \Omega, t > 0, \\
  w_t = \Delta w - uw, & x \in \Omega, t > 0,
\end{cases}
\]

(1.3)

which and its variants have been studied mathematically in various contexts. Speaking of classical solutions, if

either \( n \leq 2 \) or \( \chi \|w_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)} \),

then global boundedness of the solution \((u, w)\) to (1.3) is ensured and further any global bounded such solution converges uniformly according to

\[
\lim_{t \to \infty} \left( \|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \right) = 0, \quad \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} u_0.
\]

(1.4)

(1.5)

cf. [26, 41, 46]. Without smallness condition like (1.4), the problem possesses at least one global (certain) weak solution which is eventually smooth and enjoys the convergence property (1.5) in 3D bounded convex domains [27]. As a simple starting motivation, we are wondering whether such type weak solution continues to exist in 4D or higher and, if so, whether it also enjoys (1.5). So far, it is still widely open whether (1.3) possesses blow-ups in higher dimensions, only certain blow-up properties of the local classical solutions to (1.3) are recently known [10]. For studies on chemotaxis-consumption systems with different boundary conditions, we refer the interested reader to the very recent works [5, 15]. For properties of solutions in chemotaxis-consumption type models in more complex framework, for instance, with tensor-valued sensitivity, singular sensitivity, logistic source, predator-prey interaction or fluid interaction etc, we refer the interested reader to [1, 3, 11, 14, 36, 37, 38] and the references therein.

It is well-known that logistic type source has an effective role in enhancing global existence, and boundedness in chemotaxis-involving systems. Indeed, a lot of studies have been done to the IBVP (1.1) with Lotka-Volterra type competitive kinetics (the same boundary and initial conditions are suspended):

\[
\begin{cases}
  u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u (1 - u - a_1 v), & x \in \Omega, t > 0, \\
  v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v (1 - v - a_2 u), & x \in \Omega, t > 0, \\
  w_t = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, t > 0,
\end{cases}
\]

(1.6)

Global boundedness of classical solutions to (1.6) are guaranteed under

either \( n \leq 2 \) or \( \max\{\chi_1, \chi_2\} \|w_0\|_{L^\infty(\Omega)} < \frac{\pi}{\sqrt{n+1}} \).
Such bounded solutions are known \((8, 12, 20, 21, 31)\) to stabilize according to
\[
(u(\cdot, t), v(\cdot, t), w(\cdot, t)) \in L^\infty(T) \to \begin{cases} 
\left( \frac{1-a_1}{1-a_2}, \frac{1-a_1}{1-a_2}, 0 \right), & \text{if } a_1, a_2 \in (0, 1), \\
(0, 1, 0), & \text{if } a_1 \geq 1 > a_2 > 0, \\
(1, 0, 0), & \text{if } 0 < a_1 < 1 \leq a_2.
\end{cases}
\]

Recently, global existence of generalized weak solutions and their long time behaviors (similar to \((1.7)\)) to \((1.6)\) in \(nD\) are shown in \([23]\) under
\[
\max \{ \chi_1, \chi_2 \|w_0\|_{L^\infty(\Omega)} < \frac{1}{2}.
\]

Indeed, fluid interaction has been incorporated in \((1.6)\), cf. \([8, 12]\). We also mention that single or multiple-species signal-production type chemotaxis systems with/without fluid interaction have been widely investigated also e.g. in \([2, 4, 16, 17, 18, 29, 30, 35, 42]\) and the references therein.

Now, to formulate our main motivation of this project, we observe, without any damping source, global existence and boundedness of classical solutions to the IBVP \((1.1)\) in \(nD\) are obtained in \([47]\) under
\[
\max \{ \chi_1, \chi_2 \|w_0\|_{L^\infty(\Omega)} < \sqrt{\frac{2}{n}}.
\]

Under this smallness condition, stabilization of solutions is also naturally derived:
\[
\lim_{t \to \infty} \left( \|u(\cdot, t) - u_0\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_0\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \right) = 0.
\]

Comparing these existing results, we find, even in the presence of Lotka-Volterra type competitive kinetics, certain smallness condition on initial data still needs to be imposed to have global existence, boundedness and convergence. A natural question arises: whether and how far can we solve \((1.1)\) globally without any damping source? More specifically, inspired from \((1.4)\) and \((1.8)\), we are wondering, first,
\[
(Q1) \text{ without the smallness condition in } (1.8) \text{ with } n = 2, \text{ can we still have } 2D \text{ global existence and boundedness of classical solutions to the IBVP } (1.1)?
\]

Second, based on the existing knowledge about the one-species chemotaxis consumption model \((1.3)\), cf. \([25, 27, 38]\) and the references therein, we are wondering
\[
(Q2) \text{ without the smallness condition in } (1.8), \text{ how far can we solve the two-} \text{species chemotaxis-consumption model } (1.4) \text{ globally in a weak solution sense in } \geq 3D \text{ and, if so, how do such weak solutions behave after certain perhaps long waiting time?}
\]

In this work, we shall answer (Q1) and (Q2) in a positive way for the IBVP \((1.1)\): we show, without any smallness condition, global existence and uniform-in-time boundedness of classical solutions in 2D and global existence, eventual smoothness and asymptotical behavior (in convex domains) of weak solutions in \(nD\) \((n = 3, 4, 5)\).

**Theorem 1.1 (Global dynamics for \((1.1)\)).** Let \(\chi_1, \chi_2, \alpha, \beta > 0\) and \(\Omega \subset \mathbb{R}^n\) \((n \leq 5)\) be a bounded and smooth domain, and let \(u_0, v_0\) and \(w_0\) fulfill \((1.2)\).

(B1) **[Global boundedness and convergence in 2D]** When \(n = 2\), the IBVP \((1.1)\) has a unique global classical solution which is bounded on \(\Omega \times (0, \infty)\) in the sense there exists \(C > 0\) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C, \quad \forall t > 0.
\]

Naturally, such bounded solution converges according to \((1.9)\).
(B2) [Global existence of weak solutions in 3, 4, 5D] When $n = 3, 4, 5$, there exists at least one triple $(u, v, w)$ of nonnegative functions satisfying

\[
\begin{cases}
    u \in L^{n+2}_\text{loc}(\Omega \times [0, \infty)) \cap L^{n+2}_{2,1}(\Omega), \\
v \in L^{n+2}_\text{loc}(\Omega \times [0, \infty)) \cap L^{n+2}_{1,4}(\Omega), \\
w \in L^4_\text{loc}([0, \infty); W^{1,4}(\Omega)), 
\end{cases}
\]

which are a global weak solution of (1.1) in the sense of Definition 3.1 below.

(B3) [Eventual smoothness and convergence in convex domains] When $\Omega \subset \mathbb{R}^n (n = 3, 4, 5)$ is a smooth, bounded and convex domain, there exists $T^* > 0$ such that the global weak solution obtained in (B2) is bounded, belongs to $C^{2,1}(\Omega \times [T^*, \infty))$ and converges according to (1.9).

In 2D setting, our global boundedness of classical solutions in (B1) removes the smallness condition (1.8) with $n = 2$ as required in [47]. In 3D setting, our global existence weak solutions relaxes the commonly used convexity assumption on $\Omega$ in the literature, cf. [23, 27, 38]. Moreover, the known eventual smoothness and large time behavior of weak solutions in 3D convex domains has been extended to 4 and 5D and our findings also extend and improve the one-species chemotaxis-consumption model studied in [23, 27].

The layout of this paper is structured as follows: In Sect. 2, we combine and extend existing technique to study local and global well-posedness with focus on boundedness and convergence in 2D for (1.1), which relies on a crucial evolution identity (2.9). In Sect. 3, we formulate the approximating system of (1.1), introduce the concept of weak solutions and derive basic properties of approximating solutions. In Sect. 4, we motivate and extend arguments mainly from [27, 38] to derive global existence, eventual smoothness and convergence (in convex domains) of weak solutions in 3, 4 and 5D, as detailed in Subsect. 4.2 and Subsect. 4.3. The convexity of domain $\Omega$ could be removed mainly because the boundary integral emerging from the identity (2.9) can be properly controlled in a manner as in (2.15).

2. Boundedness and convergence in 2D

2.1. Basic facts and Local existence. In the subsequent analysis, we shall need the well-known Gagliardo-Nirenberg interpolation inequality, we list it here for convenience of reference.

Lemma 2.1. (Gagliardo-Nirenberg interpolation inequality [6, 19]) Let $p \geq 1$ and $q \in (0, p)$. Then there exists a positive constant $C_{\text{GN}} = C_{p,q}$ such that

\[
\|w\|_{L^p(\Omega)} \leq C_{\text{GN}} \left( \|\nabla w\|^2_{L^2(\Omega)} \|w\|^{(1-\delta)}_{L^q(\Omega)} + \|w\|_{L^r(\Omega)} \right), \quad \forall w \in H^1(\Omega) \cap L^r(\Omega),
\]

where $r > 0$ is arbitrary and $\delta$ is given by

\[
\frac{1}{p} = \delta \left( \frac{1}{2} - \frac{1}{n} \right) + \frac{1}{q} - \delta \iff \delta = \frac{1}{q} - \frac{1 - \frac{1}{p}}{\frac{1}{n} - \frac{1}{q}} \in (0, 1).
\]

Next, we state the following well-established local solvability, extendibility and basic estimates of solutions to the IBVP (1.1).

Lemma 2.2. Let $\chi_1, \chi_2, \alpha, \beta > 0$ and $\Omega \subset \mathbb{R}^n \ (n \geq 1)$ be a bounded and smooth domain, and let $u_0, v_0$ and $w_0$ fulfill (1.2) with $r > \max \{2, n\}$. Then there is a
unique, positive and classical maximal solution \((u,v,w)\) of the IBVP (1.1) on some maximal interval \([0,T_m)\) with \(0 < T_m \leq \infty\) such that
\[
(u,v) \in \left( C\left( \Omega \times [0,T_m)\right) \cap C^{2,1}(\Omega \times [0,T_m) ) \right)^2,
\]
and
\[
w \in C\left( \Omega \times [0,T_m)\right) \cap C^{2,1}(\Omega \times [0,T_m) ) \cap W^{1,\infty}(0,T_m), W^{1,r}(\Omega).
\]
If \(T_m < \infty\), then the following extensibility criterion holds:
\[
\limsup_{t \to T_m} \left( \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,r}(\Omega)} \right) = \infty.
\]
Furthermore, \(u\) and \(v\) have conservation of mass within \((0,T_m)\):
\[
\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad \|v(\cdot,t)\|_{L^1(\Omega)} = \|v_0\|_{L^1(\Omega)},
\]
and, for any \(p \in [1,\infty]\), the \(L^p\)-norm of \(w\) is non-increasing:
\[
t \mapsto \|w(\cdot,t)\|_{L^p(\Omega)} \text{ is non-increasing in } [0,T_m);
\]
in particular,
\[
\|w(\cdot,t)\|_{L^p(\Omega)} \leq \|w_0\|_{L^p(\Omega)}.
\]

Proof. The local existence, regularity and extendibility of classical solutions to the IBVP (1.1) are based on contraction mapping argument and parabolic regularity of parabolic systems, which can be found in [3, 25, 26, 32, 34, 39]. The conservations of \(u\) and \(v\) in (2.1) follows upon integration by parts due to the non-flux boundary conditions, and the positivity of solution, (2.2) and hence (2.3) with \(p = \infty\) follows from an application of the (strong) maximum principle. For the case \(p \in [1,\infty)\), testing the \(w\)-equation by \(w^{p-1}\) and integrating by parts, we get
\[
\frac{d}{dt} \int_\Omega w^p = -(p-1) \int_\Omega |\nabla w|^2 - \int_\Omega (\alpha u + \beta v) w \leq 0,
\]
which upon being integrated from \(s\) to \(t\) entails (2.2).

Henceforth, we will denote by \(C_i\) various constants which may vary line by line.

2.2. Uniform boundedness and global existence in 2D. In this subsection, we shall extend the idea in [31] to show 2D boundedness and global existence of classical solutions to the IBVP (1.1) without any smallness condition, thus proving (B1). To gain our goal, we first establish a series of important a-priori estimates; we state them in \(n\) dimensional setting since they are valid in arbitrary space dimensions and they are quite convenient in subsequent sections.

To move from \(L^1\)-boundness obtained in (2.1) and (2.2) to higher order \(L^p\)-regularity, we first compute from the \(u\)- and \(v\)-equations in (1.1) that
\[
\begin{aligned}
\frac{d}{dt} \int_\Omega u \ln u + \int_\Omega \frac{\nabla u \cdot \nabla w}{w} &= \chi_1 \int_\Omega \nabla u \nabla w, \quad \forall t \in (0,T_m), \\
\frac{d}{dt} \int_\Omega v \ln v + \int_\Omega \frac{\nabla v \cdot \nabla w}{w} &= \chi_2 \int_\Omega \nabla v \nabla w, \quad \forall t \in (0,T_m).
\end{aligned}
\]

To cancel out exactly the chemotaxis involving terms on the right-hand side of (2.4), inspired from [51, 11], we compute the following time evolution:
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{w^2}{w^2} = \int_\Omega \nabla w \nabla w = \frac{1}{2} \int_\Omega \frac{w^2}{w^2}.
\]
For the first term, an integration by parts along with the $w$-equation entails that
\[
\int_\Omega \frac{\nabla w \nabla t}{w} = \int_\Omega \frac{1}{w} \nabla w \cdot \nabla \Delta w - \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2 - \int_\Omega \left( \alpha \nabla u + \beta \nabla v \right) \nabla w
\]
\[
= \frac{1}{2} \int_{\partial \Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \nu} + \frac{1}{2} \int_\Omega \frac{\nabla w \cdot \nabla |\nabla w|^2}{w^2} - \int_\Omega \frac{|D^2 w|^2}{w}
\]
\[
- \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2 - \alpha \int_\Omega \nabla u \nabla w - \beta \int_\Omega \nabla v \nabla w,
\]
where we have applied the following point-wise identity
\[
2\nabla w \cdot \nabla \Delta w = \Delta |\nabla w|^2 - 2|D^2 w|^2, \quad |D^2 w|^2 = \sum_{i,j=1}^n |w_{x_i x_j}|^2.
\]
In the same spirit, for the second term, one has
\[
\frac{1}{2} \int_\Omega \frac{|\nabla w|^2}{w} + \frac{1}{2} \int_\Omega \frac{\nabla w \cdot \nabla |\nabla w|^2}{w^2} = \frac{1}{2} \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2 - \frac{\partial |\nabla w|^2}{\partial \nu} - \int_\Omega \frac{|\nabla w|^4}{w^3}.
\]
Collecting these equalities together, we end up with
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{|\nabla w|^2}{w} + \left( \int_\Omega \frac{|D^2 w|^2}{w} - \int_\Omega \frac{\nabla w \cdot \nabla |\nabla w|^2}{w^2} + \int_\Omega \frac{|\nabla w|^4}{w^3} \right) = \frac{1}{2} \int_{\partial \Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \nu} - \frac{1}{2} \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2 - \alpha \int_\Omega \nabla u \nabla w - \beta \int_\Omega \nabla v \nabla w. \tag{2.5}
\]
For later use, cf. Lemma 2.3, it is a good place to notice (cf. [11] (3.9)) that
\[
\int_\Omega w |D^2 \ln w|^2 = \frac{1}{2} \int_\Omega |D^2 w|^2 - \frac{1}{2} \int_\Omega \frac{\nabla w \cdot D^2 w \cdot \nabla w}{w^2} + \int_\Omega \frac{|\nabla w|^4}{w^3}. \tag{2.6}
\]
Then the elementary inequality $-2ab \geq -\frac{1}{2}a^2 - 2b^2$ implies
\[
\int_\Omega w |D^2 \ln w|^2 \geq \frac{1}{2} \int_\Omega \frac{|D^2 w|^2}{w} - \int_\Omega \frac{|\nabla w|^4}{w^3}. \tag{2.7}
\]
Plugging (2.4) into (2.7) and noticing $\nabla w \nabla |\nabla w|^2 = 2 \nabla w \cdot D^2 w \cdot \nabla w$, we get that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla w|^2 + \int_\Omega \frac{w}{w} |D^2 \ln w|^2 + \frac{1}{2} \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2
\]
\[
= \frac{1}{2} \int_{\partial \Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \nu} - \alpha \int_\Omega \nabla u \nabla w - \beta \int_\Omega \nabla v \nabla w. \tag{2.8}
\]
Combining (2.4) with (2.8), we conclude an important identity for (1.1) as follows:
\[
\frac{d}{dt} \left( \int_\Omega \frac{\alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \chi_1 \chi_2 |\nabla w|^2}{2} \right) + \alpha \chi_2 \int_\Omega \frac{|\nabla u|^2}{w}
\]
\[
+ \beta \chi_1 \int_\Omega \frac{|\nabla v|^2}{w} + \chi_1 \chi_2 \frac{1}{2} \int_\Omega \frac{(\alpha u + \beta v)}{w} |\nabla w|^2 + \chi_1 \chi_2 \int_\Omega w |D^2 \ln w|^2 \tag{2.9}
\]
\[
= \chi_1 \chi_2 \frac{1}{2} \int_{\partial \Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \nu}.
\]
Thanks to this crucial evolution identity, we then improve the basic estimates in (2.1) and (2.2) to ($L^1$, $L^1$, $L^2$)-boundedness of ($u \ln u$, $v \ln v$, $\nabla w$) as follows.
Lemma 2.3. Under the basic conditions in Lemma 2.2, for any \( \tau \in (0, T_m) \), there exists \( C = C(u_0, v_0, w_0, \tau, |\Omega|) > 0 \) such that the solution \((u, v, w)\) of (1.1) verifies

\[
\int_{\Omega} \left( |u \ln u| + |v \ln v| + |\nabla w|^2 \right) \, (\cdot, t) \leq C, \quad \forall t \in (\tau, T_m). \tag{2.10}
\]

Proof. In the case that \( \Omega \) is convex (as the case in [27]), notice that due to \( \frac{\partial w}{\partial \nu} \leq 0 \), and so the boundary integral on the right-hand side of (2.9) is non-positive. Then a simple integration of (2.9) from \( \tau \) to \( t \) gives rise to

\[
\int_{\Omega} \left( \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \frac{\chi_1 \chi_2}{2} \frac{\nabla w}{w} \right) \, (\cdot, t)
\]

\[
\leq \int_{\Omega} \left( \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \frac{\chi_1 \chi_2}{2} \frac{\nabla w}{w} \right) \, (\cdot, \tau), \quad \forall t \in (\tau, T_m). \tag{2.11}
\]

On the other hand, the algebraic fact that \(-z \ln z \leq e^{-1}\) for all \( z > 0 \) implies

\[
\int_{\Omega} \left( \alpha \chi_2 |u \ln u| + \beta \chi_1 |v \ln v| \right) \, (\cdot, t)
\]

\[
= \alpha \chi_2 \left( \int_{\Omega} u \ln u - 2 \int_{\{0 < u < 1\}} u \ln u \right)
\]

\[
+ \beta \chi_1 \left( \int_{\Omega} v \ln v - 2 \int_{\{0 < v < 1\}} v \ln v \right)
\]

\[
\leq \int_{\Omega} \left( \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v \right) \, (\cdot, t) + 2 (\alpha \chi_2 + \beta \chi_1) e^{-1} |\Omega|,
\]

which along with (2.11) and the fact \( w \leq \|w_0\|_{L^\infty(\Omega)} \) immediately yields (2.10).

In the general case that \( \Omega \) is non-convex, the idea used to control the boundary integral in (2.9) has been detailed in [31] (3.24)-(3.28). Here, we provide another version which does not involve \( L^\infty \)-norm of \( w \) and thus somehow refine the outcome in [31] (3.28). Indeed, thanks to [33] Lemma 3.3, we have

\[
\int_{\Omega} \frac{|\nabla w|^4}{w^3} \leq (2 + \sqrt{n})^2 \int_{\Omega} |D^2 \ln w|^2, \tag{2.13}
\]

which together with (2.7) further shows

\[
\int_{\Omega} \frac{|D^2 w|^2}{w} \leq 2 [(2 + \sqrt{n})^2 + 1] \int_{\Omega} |D^2 \ln w|^2. \tag{2.14}
\]

Now, repeating the arguments in [31] (3.26)-(3.28), using (2.13) and (2.14), we conclude, for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that

\[
\frac{\chi_1 \chi_2}{2} \int_{\partial \Omega} \frac{1}{w} \frac{\partial |\nabla w|^2}{\partial \nu}
\]

\[
\leq \frac{\epsilon}{2} \int_{\Omega} \left( \frac{2 |D^2 w|^2}{w} + \frac{|\nabla w|^4}{2w^3} \right) + C_\epsilon \int_{\Omega} \frac{|\nabla w|^2}{w}
\]

\[
\leq \frac{\epsilon}{2} \int_{\Omega} \left( |D^2 w|^2 + \frac{|\nabla w|^4}{w^3} \right) + C_\epsilon \int_{\Omega} w
\]

\[
\leq \epsilon \left( 2 \left[ (2 + \sqrt{n})^2 + 1 \right] + (2 + \sqrt{n})^2 \right) \int_{\Omega} |D^2 \ln w|^2 + C_\epsilon \int_{\Omega} w_0,
\]
where, from the first to the second inequality, we have used Hölder’s inequality to estimate: for any \(\eta > 0\), there exists \(C_\eta > 0\) such that

\[
\int_\Omega \frac{|\nabla w|^2}{w} \leq \eta \int_\Omega \frac{|\nabla w|^4}{w^3} + C_\eta \int_\Omega w.
\] (2.16)

Now, choosing \(\epsilon = \min \left\{1, \frac{\chi_1\chi_2}{2[(2 + \sqrt{n})^2 + 1]} \right\}\) in (2.15), and then, substituting (2.15) into (2.9), we obtain a key ordinary differential inequality (ODI) as follows:

\[
\frac{d}{dt} \left( \int_\Omega \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \frac{\chi_1 \chi_2}{2} \left| \nabla w \right|^2 \right) + \int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega \frac{|\nabla v|^2}{v} + \chi_1 \chi_2 \int_\Omega w D^2 \ln |w|^2 \leq C_1.
\] (2.17)

By the Gagliardo-Nirenberg inequality (cf. Lemma 2.1) and the mass conservation of \(u\) and \(v\) in (2.1), one can easily show (cf. [41, (3.36)], for any \(\eta > 0\),

\[
\int_\Omega u \ln u \leq \eta \int_\Omega \frac{|\nabla u|^2}{u} + C_\eta, \quad \int_\Omega v \ln v \leq \eta \int_\Omega \frac{|\nabla v|^2}{v} + C_\eta.
\] (2.18)

On the other hand, by (2.13) and (2.16), we readily infer that

\[
\int_\Omega \frac{|\nabla w|^2}{w} \leq \eta \int_\Omega w D^2 \ln |w|^2 + C_\eta.
\] (2.19)

Combining (2.13) and (2.19) with (2.17) and choosing sufficiently small \(\eta > 0\), we establish a key final ODI of the form:

\[
\frac{d}{dt} \left( \int_\Omega \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \frac{\chi_1 \chi_2}{2} \left| \nabla w \right|^2 \right) + C_2 \left( \int_\Omega \alpha \chi_2 u \ln u + \beta \chi_1 v \ln v + \frac{\chi_1 \chi_2}{2} \left| \nabla w \right|^2 \right) \leq C_3.
\] (2.20)

Solving the Gronwall inequality (2.20) using the trick as in (2.12) and noticing the fact \(w \leq ||w||_{L^\infty(\Omega)}\), we finally end up with the desired estimate (2.10). \(\square\)

In signal production single species chemotaxis models, the boundedness information provided in (2.10) is quite known to allow one to infer 2D global boundedness, cf. [3, 40, 44]. Here, in signal consumption multi-species cases, instead of using the technique used in [27], we shall also show that the compound boundedness in (2.10) enable us to derive first the \((L^2, L^2, L^4)\)-boundedness of \((u, v, \nabla w)\) and then \((L^\infty, L^\infty, W^{1,\infty})\)-boundedness of \((u, v, w)\), which clarifies the boundedness proof for the case of \(a_2 \leq 0\) in [41] the right-hand side of (3.34) therein indeed depends on \(t\).

Lemma 2.4. Let \(\Omega \subset \mathbb{R}^2\) be bounded and smooth. Then for any \(\tau \in (0, T_m)\), there exists \(C = C(u_0, v_0, w_0, \tau, |\Omega|) > 0\) such that the solution \((u, v, w)\) of (1.1) verifies

\[
\int_\Omega \left( u^2 + v^2 + |\nabla w|^4 \right) \leq C, \quad \forall t \in (\tau, T_m).
\] (2.21)
Proof. Integrating by parts from (1.1) and using Cauchy-Schwarz inequality, we get
\begin{equation}
\frac{d}{dt} \int_\Omega u^2 + \int_\Omega |\nabla u|^2 \leq \chi_1^2 \int_\Omega u^2 |\nabla w|^2, \quad \forall t \in (0, T_m),
\end{equation}
(2.22)

Similarly, taking gradient of the $w$-equation and then multiplying it by $|\nabla w|^2 \nabla w$ and finally integrating over $\Omega$ by parts, we derive that
\begin{align*}
\frac{d}{dt} \int_\Omega |\nabla w|^4 &+ 2 \int_\Omega |\nabla |\nabla w|^2|^2 + 4 \int_\Omega |\nabla w|^2 |D^2 w|^2 \\
&= 2 \int_\Omega |\nabla w|^2 \frac{\partial}{\partial t} |\nabla w|^2 + 4 \int_\Omega (\alpha u + \beta v) w |\nabla w|^2 \nabla w
\end{align*}
(2.23)

Next, based on (2.22) and (2.23), we estimate the terms on the right-hand side of (2.23). For the boundary integral, one can use (cf. [42, 43, 44]) the boundary trace embedding to bound it in terms of the boundedness of $\|w\|_{L^2}$ in (2.10) to infer, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that
\begin{equation}
2 \int_{\partial \Omega} |\nabla w|^2 \frac{\partial}{\partial t} |\nabla w|^2 \leq \epsilon \int_\Omega |\nabla w|^2 |\nabla w|^2 + C_\epsilon \left( \int_\Omega |\nabla w|^2 \right)^2
\end{equation}
(2.24)

Noticing $\|w\|_{L^\infty} \leq \|w_0\|_{L^\infty}$, we deduce that
\begin{align*}
4 \int_\Omega (\alpha u + \beta v) w |\nabla w|^2 \nabla w \\
&\leq \frac{1}{2} \int_\Omega |\nabla w|^2|^2 + 8 \|w_0\|_{L^\infty}^2 \int_\Omega (\alpha u + \beta v)^2 |\nabla w|^2 \\
&\leq \frac{1}{2} \int_\Omega |\nabla w|^2|^2 + 16 \alpha^2 \|w_0\|_{L^\infty}^2 \int_\Omega u^2 |\nabla w|^2 + 16 \beta^2 \|w_0\|_{L^\infty} \int_\Omega v^2 |\nabla w|^2
\end{align*}
(2.25)

and, similarly, since $|\Delta w|^2 \leq 2|D^2 w|^2$, we obtain that
\begin{align*}
4 \int_\Omega (\alpha u + \beta v) w |\nabla w|^2 |\Delta w| \\
&\leq 4 \int_\Omega |\nabla w|^2 |D^2 w|^2 + 4 \alpha^2 \|w_0\|_{L^\infty}^2 \int_\Omega u^2 |\nabla w|^2 + 4 \beta^2 \|w_0\|_{L^\infty} \int_\Omega v^2 |\nabla w|^2.
\end{align*}
(2.26)

Combining the estimates (2.22), (2.23), (2.24) with $\epsilon = \frac{1}{2}$, (2.25) and (2.26), we conclude a Key ODI as follows, for $t \in (0, T_m)$,
\begin{equation}
\frac{d}{dt} \int_\Omega (u^2 + v^2 + |\nabla w|^4) + \int_\Omega (u^2 + v^2 + |\nabla |\nabla w|^2|^2) \\
\leq (\chi_1^2 + 20 \alpha^2 \|w_0\|_{L^\infty}^2) \int_\Omega u^2 |\nabla w|^2 + (\chi_3^2 + 20 \beta^2 \|w_0\|_{L^\infty}^2) \int_\Omega v^2 |\nabla w|^2.
\end{equation}
(2.27)

Now, the Young’s inequality with epsilon shows, for any $\epsilon_1 > 0$, that
\begin{equation}
\int_\Omega u^2 |\nabla w|^2 + \int_\Omega v^2 |\nabla w|^2 \leq 2\epsilon_1 \int_\Omega |\nabla w|^6 + \frac{2}{3\sqrt{5\epsilon_1}} \int_\Omega u^3 + \frac{2}{3\sqrt{5\epsilon_1}} \int_\Omega v^3.
\end{equation}
(2.28)

Now, thanks to the compound boundedness information in (2.10), using the usual 2D GN inequality as in Lemma (2.1) and its extended version involving logarithmic
functions (cf. [28] Lemma A. 5 or [45] Lemma 3.4), we can easily deduce, for any \( \epsilon_2, \epsilon_3 \), there exist \( C_{\epsilon_2}, C_{\epsilon_3} > 0 \) and \( C > 0 \) such that

\[
\begin{aligned}
&f_0 u^2 + f_0 u^3 \leq \epsilon_2 f_0 |\nabla u|^2 + C_{\epsilon_2}, \\
&f_0 v^2 + f_0 v^3 \leq \epsilon_3 f_0 |\nabla v|^2 + C_{\epsilon_3}, \\
&f_0 |\nabla w|^6 \leq C f_0 |\nabla |\nabla w|^2|^2 + C.
\end{aligned}
\] (2.29)

Substituting (2.28) and (2.29) into (2.27) and then choosing sufficiently small \( \epsilon_i > 0 \), we finally obtain a simple yet important ODE as follows:

\[
\frac{d}{dt} \int_\Omega (u^2 + v^2 + |\nabla w|^4) + \int_\Omega (u^2 + v^2 + |\nabla w|^4) \leq C,
\]

which immediately entails (2.21). \( \square \)

**Proof of 2D global existence, boundedness and convergence.** In light of the gained \( (L^2, L^2, L^4) \)-boundedness of \( (u, v, \nabla w) \) in (2.21), and the \( L^\infty \)-boundedness of \( w \) in (2.23), using semigroup type arguments to \( w \)-equation, one can easily derive first \( W^{1,q} \)-boundedness of \( w \) for any finite \( q \), and then, testing the \( u, v \)-equations to derive \( (L^3, L^3) \)-boundedness of \( (u, v) \), and then, using semigroup type arguments to \( u \)-equation again to derive \( W^{1,\infty} \)-boundedness of \( w \), and finally, applying semigroup type arguments to \( u, v \)-equations to derive \( (L^\infty, L^\infty, W^{1,\infty}) \)-boundedness of \( (u, v, w) \) as in (1.10), see details in e.g., [3, 45, 46, 47], for instance. The convergence in (1.9) goes in the same way as [27, 47]. \( \square \)

3. Preliminaries on weak solutions in 3D or higher dimensions

In this section, we first introduce the concept of weak solutions, and then, we state some useful lemmas for later use.

**Definition 3.1.** By a global weak solution of (1.1), we mean a triple \( (u, v, w) \) of nonnegative functions

\[
\begin{aligned}
n &= 3: \quad \begin{cases} u \in L^1_{\text{loc}}([0, \infty); L^1(\Omega)), \\ v \in L^1_{\text{loc}}([0, \infty); L^1(\Omega)) \text{ and} \\ w \in L^1_{\text{loc}}([0, \infty); W^{1,3}(\Omega)), \end{cases} \\
n &= 4, 5: \quad \begin{cases} u \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)), \\ v \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)) \text{ and} \\ w \in L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)) \cap \Omega \times (0, \infty)) \cap L^1_{\text{loc}}([0, \infty); W^{1,1}(\Omega)), \end{cases}
\end{aligned}
\]

such that

\[
\begin{aligned}
&u, v, w, u\nabla w \text{ and } v\nabla w \text{ belong to } L^1_{\text{loc}}([0, \infty); L^1(\Omega)) \text{ and that the following identities motivated from integration by parts hold for all } \varphi \in C_0^\infty(\Omega \times [0, \infty)).
\end{aligned}
\]
In order to achieve global solvability within this framework through an appropriate regularization process, for \( \epsilon \in (0, 1) \), let us define \( F_\epsilon : [0, \infty) \mapsto \mathbb{R}^+ \) by

\[
F_\epsilon(s) := \begin{cases} 
\frac{1}{\epsilon} \ln(1 + \epsilon s), & \text{if } n = 3, \\
\frac{s}{1 + \epsilon s}, & \text{if } n = 4, 5. 
\end{cases}
\]

It then follows easily that the \( C^\infty([0, \infty)) \)-family \( (F_\epsilon)_{\epsilon \in (0, 1)} \) has the properties that

\[
F_\epsilon(0) = 0, \quad F_\epsilon(s) \to s \text{ as } \epsilon \searrow 0 \quad \text{and} \quad 0 < F_\epsilon'(s) \leq 1 \quad \text{for all } s \geq 0,
\]

and that, for any \( s \geq 0 \),

\[
0 \leq F_\epsilon(s) \nearrow 1 \quad \text{as } \epsilon \searrow 0, \quad 0 \leq sF_\epsilon'(s) \leq \frac{1}{\epsilon}, \quad 0 \leq -sF_\epsilon''(s) \leq 2, \quad \forall \epsilon \in (0, 1).
\]

Then, for \( \epsilon \in (0, 1) \), we consider the following regularized problem:

\[
\begin{aligned}
&u_{\epsilon t} = \Delta u_\epsilon - \chi_1 \nabla \cdot (u_\epsilon F_\epsilon'(u_\epsilon) \nabla w_\epsilon), \quad x \in \Omega, t > 0, \\
v_{\epsilon t} = \Delta v_\epsilon - \chi_2 \nabla \cdot (v_\epsilon F_\epsilon'(v_\epsilon) \nabla w_\epsilon), \quad x \in \Omega, t > 0, \\
w_{\epsilon t} = \Delta w_\epsilon - (\alpha F_\epsilon(u_\epsilon) + \beta F_\epsilon(v_\epsilon)) w_\epsilon, \quad x \in \Omega, t > 0, \\
\frac{\partial u_\epsilon}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} = \frac{\partial w_\epsilon}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u_\epsilon(x, 0) = u_0(x), v_\epsilon(x, 0) = v_0(x), w_\epsilon(x, 0) = w_0(x), \quad x \in \Omega.
\end{aligned}
\]

The framework of contraction mapping argument first allows one to conclude local well-posedness on \((0, T_{m, \epsilon})\), and then, given the basic estimates in Lemmas\[136\] and \[137\] on \((0, T_{m, \epsilon})\) and the choice of \( F_\epsilon \) in \[136\], using Neumann semigroup estimates to the \( u \)-equation in \[137\], one can easily see that \( \|\nabla w_\epsilon\|_{L^\infty} \) is uniformly bounded on \((0, T_{m, \epsilon})\), and this allows one to conclude finally \( T_{m, \epsilon} = \infty \), namely, the global existence of classical solution to the approximating system \[137\]; see quite detailed display of similar reasonings in related circumstances\[9, 20, 27, 34, 35\].

**Lemma 3.2.** Let \( \chi_1, \chi_2 > 0 \) and \( \alpha, \beta > 0 \) and \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) be a bounded and smooth domain, and, let \( F_\epsilon \) be defined by \[3.4\] and initial data \((u_0, v_0, w_0)\) satisfy \[1.2\]. Then for each \( \epsilon \in (0, 1) \), the system \[3.7\] admits a global classical solution \((u_\epsilon, v_\epsilon, w_\epsilon)\) such that \( u_\epsilon > 0, v_\epsilon > 0 \) and \( w_\epsilon > 0 \) on \( \Omega \times (0, \infty) \).

**Lemma 3.3.** For all \( \epsilon \in (0, 1) \), the solution of \[3.7\] satisfies, for \( t > 0 \),

\[
\|u_\epsilon(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}, \quad \|v_\epsilon(\cdot, t)\|_{L^1(\Omega)} = \|v_0\|_{L^1(\Omega)}.
\]

**Proof.** Integrating the first and second equations in \[3.7\] and the no-flux boundary conditions, we immediately obtain \[3.8\]. \( \square \)

**Lemma 3.4.** Let \( \epsilon \in (0, 1) \) and \( p \in [1, \infty] \). Then the solution of \[3.7\] verifies

\[
t \mapsto \|w_\epsilon(\cdot, t)\|_{L^p(\Omega)} \text{ is nonincreasing in } [0, \infty).
\]

In particular,

\[
\|w_\epsilon(\cdot, t)\|_{L^p(\Omega)} \leq \|w_0\|_{L^p(\Omega)}.
\]

**Proof.** Notice that \( F_\epsilon \) and \( \alpha, \beta, u_\epsilon, v_\epsilon, w_\epsilon \) are nonnegative, we have \( w_{\epsilon t} \leq \Delta w_\epsilon \), and thus, using the maximum principle and energy estimate, we readily derive \[3.9\] and \[3.10\], see details in Lemma \[2.2\]. \( \square \)
Lemma 3.5. For all \( \varepsilon \in (0, 1) \), we have

\[
\alpha \int_0^\infty \int_\Omega F_\varepsilon (u_\varepsilon) w_\varepsilon + \beta \int_0^\infty \int_\Omega F_\varepsilon (v_\varepsilon) w_\varepsilon \leq \int_\Omega w_0. \tag{3.11}
\]

In particular, the limit triple \((u, v, w)\) defined by Lemma 4.8 satisfies

\[
\alpha \int_0^\infty \int_\Omega w + \beta \int_0^\infty \int_\Omega v \leq \int_\Omega w_0. \tag{3.12}
\]

Proof. Integrating the third equation in (3.7) we get

\[
\int_\Omega w_\varepsilon (\cdot, t) + \alpha \int_0^t \int_\Omega F_\varepsilon (u_\varepsilon) w_\varepsilon + \beta \int_0^t \int_\Omega F_\varepsilon (v_\varepsilon) w_\varepsilon = \int_\Omega w_0
\]

for all \( t \geq 0 \). Due to \( w_\varepsilon \geq 0 \), it immediately derives (3.11). Applying the Fatou’s lemma, we also readily conclude (3.12). This completes the proof. □

4. Global dynamics of weak solutions in 3, 4 and 5D

In this section, we first establish important a-priori \( \varepsilon \)-independent estimates for classical solutions to (3.7) and then we pass to the limit as \( \varepsilon \to 0 \) to show the global existence and convergence of weak solutions in the sense of Definition 3.1 for the IBVP (1.1) in 3, 4, 5-D, and thus accomplishing (B2) and (B3) in Theorem 1.1.

Lemma 4.1. There exists positive constant \( K_1 = K_1(\|u_0\|_{L^1}, \|v_0\|_{L^1}, \|w_0\|_{L^1}) > 0 \) such that the global solution of (3.7) fulfills, for all \( \varepsilon \in (0, 1) \) and \( t > 0 \),

\[
\frac{d}{dt} \int_\Omega \left( \alpha \chi_2 u_\varepsilon \ln u_\varepsilon + \beta \chi_1 v_\varepsilon \ln v_\varepsilon + \frac{\chi_1 \chi_2}{2} \frac{\nabla w_\varepsilon}{w_\varepsilon} \right)
+ \int_\Omega \left( \alpha \chi_2 u_\varepsilon \ln u_\varepsilon + \beta \chi_1 v_\varepsilon \ln v_\varepsilon + \frac{\chi_1 \chi_2}{2} \frac{\nabla w_\varepsilon}{w_\varepsilon} \right)
+ \frac{\alpha \chi_2}{2} \int_\Omega \frac{\nabla u_\varepsilon}{u_\varepsilon} + \frac{\beta \chi_1}{2} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{\chi_1 \chi_2}{2} \int_\Omega \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon} \right) \leq \begin{cases} 0, & \text{if } \Omega \text{ is convex,} \\ K_1, & \text{if } \Omega \text{ is non-convex}. \end{cases} \tag{4.1}
\]

Hence, there exists \( K_2 := K_2(u_0, v_0, w_0) > 0 \) such that

\[
\int_\Omega \left( |u_\varepsilon \ln u_\varepsilon| + |v_\varepsilon \ln v_\varepsilon| + |\nabla w_\varepsilon|^2 + \frac{|\nabla w_\varepsilon|^2}{w_\varepsilon} \right)(\cdot, t) \leq K_2, \quad \forall t \in (0, \infty). \tag{4.2}
\]
Proof. Conducting similar computations leading to (2.18), we calculate that
\[
\frac{d}{dt} \int_\Omega \left( \alpha \chi_2 u_t \ln u_t + \beta \chi_1 v_t \ln v_t + \frac{\chi_1 \chi_2}{2} |\nabla w_t|^2 \right) + \int_\Omega \left( \alpha \chi_2 u_t \ln u_t + \beta \chi_1 v_t \ln v_t + \frac{\chi_1 \chi_2}{2} |\nabla w_t|^2 \right) \\
+ \alpha \chi_2 \int_\Omega \left( \frac{|\nabla u_t|^2}{u_t} + \beta \chi_1 \int_\Omega \frac{|\nabla v_t|^2}{v_t} + \chi_1 \chi_2 \int_\Omega w_t |D^2 \ln w_t|^2 \right) \\
+ \frac{\chi_1 \chi_2}{2} \int_\Omega (\alpha F_\varepsilon(u_t) + \beta F_\varepsilon(v_t)) \frac{|\nabla w_t|^2}{w_t} \\
eq \int_\Omega \left( \alpha \chi_2 u_t \ln u_t + \beta \chi_1 v_t \ln v_t + \frac{\chi_1 \chi_2}{2} |\nabla w_t|^2 \right) + \chi_1 \chi_2 \int_{\partial \Omega} \frac{1}{w_t} |\nabla w_t|^2.
\]
By the $L^1$-boundedness of $u_t$ and $v_t$ in (3.8), an straightforward application of (2.18) shows that
\[
\alpha \chi_2 \int_\Omega u_t \ln u_t \leq \frac{\alpha \chi_2}{2} \int_\Omega \frac{|\nabla u_t|^2}{u_t} + C_1(\|u_0\|_{L^1})
\]
and
\[
\beta \chi_1 \int_\Omega v_t \ln v_t \leq \frac{\beta \chi_1}{2} \int_\Omega \frac{|\nabla v_t|^2}{v_t} + C_2(\|v_0\|_{L^1}).
\]
Also, thanks to (3.10), a simple use of (2.19) entails
\[
\frac{\chi_1 \chi_2}{2} \int_\Omega \frac{|\nabla w_t|^2}{w_t} \leq \frac{\chi_1 \chi_2}{4} \int_\Omega w_t |D^2 \ln w_t|^2 + C_3(\|w_0\|_{L^1})
\]
and, in the case that $\Omega$ is non-convex, an easy use of (2.15) shows that
\[
\frac{\chi_1 \chi_2}{2} \int_{\partial \Omega} \frac{1}{w_t} |\nabla w_t|^2 \\
\leq \begin{cases} \\
0, & \text{if } \Omega \text{ is convex,} \\
\frac{\chi_1 \chi_2}{4} \int_\Omega w_t |D^2 \ln w_t|^2 + C_3(\|w_0\|_{L^1}), & \text{if } \Omega \text{ is non-convex.}
\end{cases}
\]
Substituting (4.4), (4.5), (4.6) and (4.7) into (4.3), we derive (4.4). Since $(u_t, v_t, w_t)$ satisfies an ODI of the form (2.20), and so (4.2) follows similarly as (4.10). \qed

4.1. $\varepsilon$-independent estimates for the regularized problem.

Lemma 4.2. There exists $K_3 = K_3(u_0, v_0, w_0) > 0$ such that the global solution of (3.7) fulfills, for all $\varepsilon \in (0, 1)$,
\[
\int_0^\infty \int_\Omega \left( \frac{|\nabla u_t|^2}{u_t} + \frac{|\nabla v_t|^2}{v_t} + |D^2 w_t|^2 + |\nabla w_t|^4 \right) \\
+ \int_0^\infty \int_\Omega \left( F_\varepsilon(u_t)|\nabla w_t|^2 + F_\varepsilon(v_t)|\nabla w_t|^2 \right) \leq K_3, \quad \text{if } \Omega \text{ is convex.}
\]

There exists $K_4 = K_4(u_0, v_0, w_0) > 0$ such that the global solution of (3.7) fulfills, for any $\varepsilon \in (0, 1)$ and $t \in [0, \infty)$,
\[
\int_t^{t+1} \int_\Omega \left( \frac{|\nabla u_t|^2}{u_t} + \frac{|\nabla v_t|^2}{v_t} + |D^2 w_t|^2 + |\nabla w_t|^4 \right) \\
+ \int_t^{t+1} \int_\Omega \left( F_\varepsilon(u_t)|\nabla w_t|^2 + F_\varepsilon(v_t)|\nabla w_t|^2 \right) \leq K_4, \quad \text{if } \Omega \text{ is non-convex.}
Proof. In the case that $\Omega$ is convex, integrating (4.3) with respect to $t \in (0, \infty)$ and using the fact that $-\ln z \leq e^{-1}$ for all $z > 0$, we have
\[
\int_0^t \int_{\Omega} \left( \alpha \chi_2 \frac{\nabla u_{\varepsilon}}{u_{\varepsilon}} + \beta \chi_1 \frac{\nabla v_{\varepsilon}}{v_{\varepsilon}} + \chi_1 \chi_2 w_{\varepsilon} |D^2 \ln w_{\varepsilon}|^2 \right) \, dx \, dt + \frac{\chi_1 \chi_2}{2} \int_0^t \int_{\Omega} (\alpha F_{\varepsilon}(u_{\varepsilon}) + \beta F_{\varepsilon}(v_{\varepsilon})) \frac{|\nabla w_{\varepsilon}|^2}{w_{\varepsilon}} \, dx \, dt
\leq \int_0^t \left( \alpha \chi_2 u_0 \ln u_0 + \beta \chi_1 v_0 \ln v_0 + \frac{\chi_1 \chi_2}{2} \frac{|\nabla w_0|^2}{w_0} \right) \, dt
- \int_0^t (\alpha \chi_2 u_{\varepsilon} \ln u_{\varepsilon} + \beta \chi_1 v_{\varepsilon} \ln v_{\varepsilon}) \, dt
\leq \int_0^t \left( \alpha \chi_2 u_0 \ln u_0 + \beta \chi_1 v_0 \ln v_0 + \frac{\chi_1 \chi_2}{2} \frac{|\nabla w_0|^2}{w_0} \right) + (\alpha \chi_2 + \beta \chi_1) e^{-1} |\Omega|.
\] (4.10)

In light of (2.13) and (2.14) and the fact $w_{\varepsilon} \leq \|w_0\|_{L^\infty}$, we infer that
\[
\frac{1}{\|w_0\|_{L^\infty}} \int_0^t \int_{\Omega} |\nabla w_{\varepsilon}|^4 \, dx \, dt \leq \int_0^t \int_{\Omega} \frac{|\nabla w_{\varepsilon}|^4}{w_{\varepsilon}^2} \, dx \, dt \leq (2 + \sqrt{n})^2 \int_0^t \int_{\Omega} w_{\varepsilon} |D^2 \ln w_{\varepsilon}|^2 \, dx \, dt.
\] (4.11)

and
\[
\frac{1}{\|w_0\|_{L^\infty}} \int_0^t \int_{\Omega} |D^2 w_{\varepsilon}|^2 \, dx \, dt \leq 2 [(2 + \sqrt{n})^2 + 1] \int_0^t \int_{\Omega} w_{\varepsilon} |D^2 \ln w_{\varepsilon}|^2 \, dx \, dt.
\] (4.12)

Combining (4.11) and (4.12) into (4.10) and sending $t \to \infty$, we derive (4.8).

Similarly, when $\Omega$ is nonconvex, substituting (4.7) into (4.3), we obtain that
\[
\frac{d}{dt} \int \left( \alpha \chi_2 u_{\varepsilon} \ln u_{\varepsilon} + \beta \chi_1 v_{\varepsilon} \ln v_{\varepsilon} + \frac{\chi_1 \chi_2}{2} \frac{|\nabla w_{\varepsilon}|^2}{w_{\varepsilon}} \right) + \frac{\chi_1 \chi_2}{2} \int (\alpha F_{\varepsilon}(u_{\varepsilon}) + \beta F_{\varepsilon}(v_{\varepsilon})) \frac{|\nabla w_{\varepsilon}|^2}{w_{\varepsilon}} \, dx \, dt
\leq C_4 (\|w_0\|_{L^\infty}).
\] (4.13)

Integrating (4.13) from $t$ to $t + 1$ and using (4.20), we achieve (4.9). \qed

Lemma 4.3. There exists $K_5 = K_5(u_0, v_0, w_0) > 0$ such that the global solution of (4.7) fulfills, for all $t \in (0, 1)$ and for any $t \in [0, \infty)$,
\[
\int_t^{t+1} \int_{\Omega} \left( \frac{n+2}{2} w_{\varepsilon}^{n+2} + v_{\varepsilon}^{n+2} + |\nabla u_{\varepsilon}|^{n+2}_{\frac{n+2}{n}} + |\nabla v_{\varepsilon}|^{n+2}_{\frac{n+2}{n}} \right) \, dx \, dt \leq K_5.
\] (4.14)

Proof. Given the mass conservation of $u_{\varepsilon}$ in (4.8), the Gagliardo-Nirenberg inequality (cf. Lemma 2.11) allows us to deduce that
\[
\int_{\Omega} u_{\varepsilon}^{\frac{n+2}{2}} = \|u_{\varepsilon}\|_{L^{\frac{2(n+2)}{n}}}^{\frac{2(n+2)}{n}} \leq C_1 \|\nabla u_{\varepsilon}\|_{L^2} \|u_{\varepsilon}\|_{L^\infty}^{\frac{n}{2}} + C_1 \|u_{\varepsilon}\|_{L^2}^{\frac{2(n+2)}{n}}
= C_1 \frac{4}{\|u_0\|_{L^\infty}^{\frac{n}{2}}} \int_{\Omega} \frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} + C_1 \|u_0\|_{L^\infty}^{\frac{n+2}{n}}.
\] (4.15)

Likewise, one also that
\[
\int_{\Omega} v_{\varepsilon}^{\frac{n+2}{2}} \leq C_2 \|v_0\|_{L^\infty} \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon}} + C_2 \|v_0\|_{L^\infty}^{\frac{n+2}{n}}.
\] (4.16)
Applying the Young inequality, we obtain
\[
\int_{\Omega} |\nabla u_\varepsilon|^\frac{n+2}{n+1} = \int_{\Omega} \left( \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \right)^{\frac{n+2}{n+1}} u_\varepsilon^{\frac{n+2}{n+1}} \leq \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_{\Omega} u_\varepsilon^{\frac{n+2}{n+1}}. \tag{4.17}
\]
Similarly,
\[
\int_{\Omega} |\nabla v_\varepsilon|^\frac{n+2}{n+1} \leq \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_{\Omega} v_\varepsilon^{\frac{n+2}{n+1}}. \tag{4.18}
\]

For any $t \geq 0$, integrating (4.15), (4.16), (4.17) and (4.18) from $t$ to $t+1$, then using (4.8) or (4.9), we readily conclude (4.14). □

**Lemma 4.4.** There exists $K_6 = K_6(u_0, v_0, w_0) > 0$ such that for all $\varepsilon \in (0, 1)$, the global solution of (3.7) fulfills
\[
\int_{0}^{\infty} \left( \|u_\varepsilon - \bar{u}_0\|_{L^{\frac{n+2}{n}}}^2 + \|v_\varepsilon - \bar{v}_0\|_{L^{\frac{n+2}{n}}}^2 \right) \leq K_6, \quad \text{if } \Omega \text{ is convex}; \tag{4.19}
\]
and, there exists $K_7 = K_7(u_0, v_0, w_0) > 0$ such that, for any $\varepsilon \in (0, 1)$ and $t \geq 0$,
\[
\int_{t}^{t+1} \left( \|u_\varepsilon - \bar{u}_0\|_{L^{\frac{n+2}{n}}}^2 + \|v_\varepsilon - \bar{v}_0\|_{L^{\frac{n+2}{n}}}^2 \right) \leq K_7, \quad \text{if } \Omega \text{ is non-convex}. \tag{4.20}
\]

**Proof.** The Cauchy-Schwarz inequality entails that
\[
\left( \int_{\Omega} |\nabla u_\varepsilon| \right)^2 + \left( \int_{\Omega} |\nabla v_\varepsilon| \right)^2 \leq \|u_0\|_{L^1} \int_{\Omega} |\nabla u_\varepsilon|^2 + \|v_0\|_{L^1} \int_{\Omega} |\nabla v_\varepsilon|^2. \]

Notice from (3.8) that $\bar{u}_\varepsilon = \bar{u}_0$ and $\bar{v}_\varepsilon = \bar{v}_0$; we then we infer from the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^{\frac{n+2}{n}}(\Omega)$ and the Poincare inequality that
\[
\left( \|u_\varepsilon - \bar{u}_0\|_{L^{\frac{n+2}{n}}}^2 + \|v_\varepsilon - \bar{v}_0\|_{L^{\frac{n+2}{n}}}^2 \right) \leq C \left( \int_{\Omega} |\nabla u_\varepsilon| \right)^2 + C \left( \int_{\Omega} |\nabla v_\varepsilon| \right)^2.
\]

Integrating those inequalities from 0 to $t$ (if $\Omega$ is convex) or from $t$ and $t+1$ (if $\Omega$ is non-convex) and making use of (4.8) or (4.9), we readily infer (4.19) and (4.20). □

In the sequel, for our subsequent compactness argument, we study the space-time regularity of the time derivatives of solutions to the regularized system (3.7).

**Lemma 4.5.** There exists $K_8 = K_8(u_0, v_0, w_0) > 0$ such that for all $\varepsilon \in (0, 1)$, the global solution of (3.7) fulfills
\[
\int_{0}^{\infty} \int_{\Omega} w_{\varepsilon t}^2 \leq K_8, \quad \text{if } \Omega \text{ is convex}; \tag{4.21}
\]
and, there exists $K_9 = K_9(u_0, v_0, w_0) > 0$ such that, for any $\varepsilon \in (0, 1)$ and $t \geq 0$,
\[
\int_{t}^{t+1} \int_{\Omega} w_{\varepsilon t}^2 \leq K_9, \quad \text{if } \Omega \text{ is non-convex}. \tag{4.22}
\]
Proof. Multiplying the third equation in (3.7) by $2w_{zt}$ and then integrating over $\Omega$ by parts, we obtain that
\[
2 \int_\Omega w_{zt}^2 + \frac{d}{dt} \int_\Omega |\nabla w_z|^2 \\
= -\alpha \int_\Omega F_z(u_z)(w_z^2)_t - \beta \int_\Omega F_z(v_z)(w_z^2)_t \\
= -\alpha \frac{d}{dt} \int_\Omega F_z(u_z)w_z^2 + \alpha \int_\Omega F_z'(u_z)w_z^2 w_{zt} \\
- \beta \frac{d}{dt} \int_\Omega F_z(v_z)w_z^2 + \beta \int_\Omega F_z'(v_z)w_z^2 w_{zt};
\]
that is,
\[
2 \int_\Omega w_{zt}^2 + \frac{d}{dt} \int_\Omega (|\nabla w_z|^2 + \alpha F_z(u_z)w_z^2 + \beta F_z(v_z)w_z^2) \\
= \alpha \int_\Omega F_z'(u_z) w_z^2 w_{zt} + \beta \int_\Omega F_z'(v_z) w_z^2 w_{zt}. \tag{4.23}
\]
Using the first equation in (3.7) and integrating by parts, we get
\[
\int_\Omega F_z'(u_z) w_z^2 w_{zt} \\
= - \int_\Omega F_z''(u_z)w_z^2 |\nabla w_z|^2 - 2 \int_\Omega F_z'(u_z)w_z \nabla w_z \cdot \nabla w_z \\
+ \chi_1 \int_\Omega F_z'(u_z)F_z''(u_z)w_z^3 |\nabla w_z|^2 + 2 \chi_1 \int_\Omega (F_z'(u_z))^2 u_z w_z |\nabla w_z|^2 \\
=: H_1 + H_2 + H_3 + H_4.
\]Since $0 \leq -s F_z''(s) \leq 2$ and $w_z \leq \|w_0\|_{L^\infty}$ due to (3.6) and (3.10) we estimate
\[
H_1 \leq \|w_0\|_{L^\infty}^2 \int_\Omega u_z |F_z''(u_z)| \cdot \frac{|\nabla u_z|^2}{u_z} \leq 2 \|w_0\|_{L^\infty} \int_\Omega \frac{|\nabla u_z|^2}{u_z}. \tag{4.25}
\]
Similarly, by Young’s inequality, we estimate $H_2$ as follows:
\[
H_2 \leq \int_\Omega F_z(u_z) |\nabla w_z|^2 + \|w_0\|_{L^\infty}^2 \int_\Omega \frac{u_z (F_z'(u_z))^2 |\nabla u_z|^2}{F_z(u_z)} \\
\leq \int_\Omega F_z(u_z) |\nabla w_z|^2 + \|w_0\|_{L^\infty}^2 \int_\Omega \frac{|\nabla u_z|^2}{u_z}, \tag{4.26}
\]
where we used the following fact due to the definition of $F_z$ in (3.4):
\[
0 \leq \frac{s (F_z(s))^2}{F_z'(s)} = \begin{cases} \frac{s s}{(1+es)^{n+1}}, & \text{if } n = 3, \\ \frac{s}{(1+es)} & \text{if } n = 4, 5 \end{cases} \leq 1. \tag{4.27}
\]
Analogously, the term $H_3$ is bounded according to
\[
H_3 \leq \|w_0\|_{L^\infty}^2 \int_\Omega F_z(u_z) |\nabla w_z|^2 \\
+ \chi_1^2 \|w_0\|_{L^\infty}^2 \int_\Omega \frac{u_z^3 (F_z'(u_z))^2 F_z''(u_z)}{F_z(u_z)} |\nabla u_z|^2 \\
\leq \|w_0\|_{L^\infty}^2 \int_\Omega F_z(u_z) |\nabla w_z|^2 + \chi_1^2 \|w_0\|_{L^\infty} \int_\Omega \frac{|\nabla u_z|^2}{u_z}. \tag{4.28}
\]
where we used the following fact due to the definition of $F_\varepsilon$ in (4.4):

$$0 \leq \frac{s^3 (F_\varepsilon'(s) F''_\varepsilon(s))^2}{F_\varepsilon(s)} = \begin{cases} \frac{(1+\varepsilon s)^3}{(1+\varepsilon s)^3}, & \text{if } n = 3, \\ \frac{4\varepsilon s}{(1+\varepsilon s)^2}, & \text{if } n = 4, 5, \leq 4. \end{cases}$$

Finally, we use (4.27) to bound $H_4$ as

$$H_4 \leq 2\chi_1 \|w_0\|_{L^\infty} \int_\Omega \frac{u_\varepsilon (F_\varepsilon'(u_\varepsilon))^2}{F_\varepsilon(u_\varepsilon)} \cdot F_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2$$

$$\leq 2\chi_1 \|w_0\|_{L^\infty} \int_\Omega F_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2. \quad (4.29)$$

Collecting (4.24), (4.25), (4.26), (4.28) and (4.29), we obtain that

$$\alpha \int_\Omega F_\varepsilon'(u_\varepsilon) w_\varepsilon^2 u_\varepsilon \varepsilon t \leq (3 + \chi_1^2) \alpha \|w_0\|_{L^\infty}^2 \int_\Omega |\nabla u_\varepsilon|^2 u_\varepsilon$$

$$+ (1 + 2\chi_1 \|w_0\|_{L^\infty} + \|w_0\|_{L^\infty}^2) \alpha \int_\Omega F_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2. \quad (4.30)$$

In a similar manner, one can show that

$$\beta \int_\Omega F_\varepsilon'(v_\varepsilon) w_\varepsilon^2 v_\varepsilon \varepsilon t \leq (3 + \chi_2^2) \beta \|w_0\|_{L^\infty}^2 \int_\Omega |\nabla v_\varepsilon|^2 v_\varepsilon$$

$$+ (1 + 2\chi_2 \|w_0\|_{L^\infty} + \|w_0\|_{L^\infty}^2) \beta \int_\Omega F_\varepsilon(v_\varepsilon) |\nabla v_\varepsilon|^2. \quad (4.31)$$

For any $0 \leq s \leq t$, integrating (4.28) from $s$ to $t$ and combining (4.30) with (4.31), we end up with

$$2 \int_s^t \int_\Omega w_\varepsilon^2 \leq \int_\Omega |\nabla u_\varepsilon|^2 + \alpha F_\varepsilon(u_\varepsilon) w_\varepsilon^2 + \beta F_\varepsilon(v_\varepsilon) w_\varepsilon^2 \cdot (s, s)$$

$$+ (3 + \chi_1^2) \alpha \|w_0\|_{L^\infty}^2 \int_\Omega |\nabla u_\varepsilon|^2$$

$$+ (3 + \chi_2^2) \beta \|w_0\|_{L^\infty}^2 \int_\Omega |\nabla v_\varepsilon|^2 + (1 + 2\chi_1 \|w_0\|_{L^\infty} + \|w_0\|_{L^\infty}^2) \alpha \int_\Omega F_\varepsilon(u_\varepsilon) |\nabla u_\varepsilon|^2$$

$$+ (1 + 2\chi_2 \|w_0\|_{L^\infty} + \|w_0\|_{L^\infty}^2) \beta \int_\Omega F_\varepsilon(v_\varepsilon) |\nabla v_\varepsilon|^2. \quad (4.32)$$

Using the boundedness of $|\nabla u_\varepsilon|_{L^2}$ in (4.24) and the conservations of $u_\varepsilon$ and $v_\varepsilon$, $0 \leq F_\varepsilon(s) \leq s$ and $w_\varepsilon \leq \|w_0\|_{L^\infty}$, cf. (3.6), (4.13) and (4.8), we see that

$$\int_\Omega (|\nabla w_\varepsilon|^2 + \alpha F_\varepsilon(u_\varepsilon) w_\varepsilon^2 + \beta F_\varepsilon(v_\varepsilon) w_\varepsilon^2) \cdot (s, s) \leq K_2 + (\alpha \|w_0\|_{L^1} + \beta \|v_0\|_{L^1}) \|w_0\|_{L^\infty}^2.$$

Inserting this into (4.32) and using (4.8) or (4.9), we accomplish (4.21) or (4.22). □

**Lemma 4.6.** For $m > \frac{n}{2} + 1$, there exists $K_{10} = K_{10}(u_0, v_0, w_0) > 0$ such that the global solution of (4.7) fulfills, for all $\varepsilon \in (0, 1),$

$$\int_0^\infty \left( \|u_\varepsilon(\cdot, t)\|_{(W^{m, 2})^*}^2 + \|v_\varepsilon(\cdot, t)\|_{(W^{m, 2})^*}^2, \right) \leq K_{10}, \quad \text{if } \Omega \text{ is convex}; \quad (4.33)$$
and, there exists $K_{11} = K_{11}(u_0, v_0, w_0) > 0$ such that, for any $\varepsilon \in (0, 1)$ and $t \geq 0$,
\[
\int_t^{t+1} \left(\|u_{\varepsilon t}(\cdot, s)\|_{(W^{m,2})^*}^2 + \|v_{\varepsilon t}(\cdot, s)\|_{(W^{m,2})^*}^2\right) \leq K_{11}, \text{ if } \Omega \text{ is non-convex.} \tag{4.34}
\]

**Proof.** For given $\varphi \in W^{m,2}$, we multiply the first equation in (3.7) by $\varphi$, and integrate over $\Omega$ by parts and use (4.27) to get
\[
\left| \int_\Omega u_{\varepsilon t} \varphi \right| = - \int_\Omega \nabla u_{\varepsilon t} \cdot \nabla \varphi + \chi_1 \int_\Omega u_{\varepsilon t} F'_\varepsilon(u_{\varepsilon t}) \nabla w_{\varepsilon t} \cdot \nabla \varphi \leq \left( \int_\Omega \frac{|\nabla u_{\varepsilon t}|^2}{u_{\varepsilon t}} \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla \varphi|^2 \right)^{\frac{1}{2}}
\]
\[
+ \chi_1 \left( \int_\Omega F'_\varepsilon(u_{\varepsilon t}) |\nabla w_{\varepsilon t}|^2 \right)^{\frac{1}{2}} \left( \int_\Omega \frac{u_{\varepsilon t}^2 F'_\varepsilon(u_{\varepsilon t})^2}{F'_\varepsilon(u_{\varepsilon t})} |\nabla \varphi|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_\Omega \frac{|\nabla u_{\varepsilon t}|^2}{u_{\varepsilon t}} \right)^{\frac{1}{2}} \|u_0\|_{L^1}^{\frac{1}{2}} \|\nabla \varphi\|_{L^\infty}
\]
\[
+ \chi_1 \left( \int_\Omega F'_\varepsilon(u_{\varepsilon t}) |\nabla w_{\varepsilon t}|^2 \right)^{\frac{1}{2}} \|u_0\|_{L^1}^{\frac{1}{2}} \|\nabla \varphi\|_{L^\infty}.
\]
Hence, the Sobolev embedding $W^{m,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ due to $m > 1 + \frac{2}{n}$ shows that
\[
\|u_{\varepsilon t}(\cdot, t)\|_{(W^{m,2})^*}^2 \leq C_1 \int_\Omega \frac{|\nabla u_{\varepsilon t}|^2}{u_{\varepsilon t}} + C_1 \int_\Omega F'_\varepsilon(u_{\varepsilon t}) |\nabla w_{\varepsilon t}|^2, \quad \forall t > 0.
\]
Likewise,
\[
\|v_{\varepsilon t}(\cdot, t)\|_{(W^{m,2})^*} \leq C_2 \int_\Omega \frac{|\nabla v_{\varepsilon t}|^2}{v_{\varepsilon t}} + C_2 \int_\Omega F'_\varepsilon(v_{\varepsilon t}) |\nabla w_{\varepsilon t}|^2, \quad \forall t > 0.
\]
By these two inequalities, we readily conclude (4.33) or (4.34) from (4.8) or (4.9). □

**Lemma 4.7.** For $n \leq 6$, there exists $K_{12} = K_{12}(u_0, v_0, w_0) > 0$ such that the global solution of (3.7) satisfies, for any $\varepsilon \in (0, 1)$ and $t > 0$,
\[
\int_t^{t+1} \left(\|u_{\varepsilon t}(\cdot, s)\|_{(W^{1,\infty})^*} + \|v_{\varepsilon t}(\cdot, s)\|_{(W^{1,\infty})^*} + \|w_{\varepsilon t}(\cdot, s)\|_{(W^{1,\infty})^*}\right) \leq K_{12}. \tag{4.35}
\]

**Proof.** For any given $\psi \in W^{1,\infty}$ with $\|\psi\|_{W^{1,\infty}} \leq 1$, notice that $0 \leq F' \leq 1$ by (3.35) and $\frac{n+2}{n+4} \leq 4$ since $n \leq 6$, and so by Young’s inequality, we get that
\[
\left| \int_\Omega u_{\varepsilon t} \psi \right| = \left| \int_\Omega \nabla u_{\varepsilon t} \cdot \nabla \psi + \chi_1 \int_\Omega u_{\varepsilon t} F'_\varepsilon(u_{\varepsilon t}) \nabla w_{\varepsilon t} \cdot \nabla \psi \right|
\]
\[
\leq \left| \int_\Omega \frac{|\nabla u_{\varepsilon t}|^2}{u_{\varepsilon t}} + |\Omega| \int_\Omega u_{\varepsilon t} |\nabla \psi|^2 \right| + \chi_1 \int_\Omega \frac{u_{\varepsilon t}^2 F'_\varepsilon(u_{\varepsilon t})^2}{F'_\varepsilon(u_{\varepsilon t})} |\nabla \psi|^2
\]
\[
\leq \left| \int_\Omega \frac{|\nabla u_{\varepsilon t}|^2}{u_{\varepsilon t}} + |\Omega| + \chi_1 \int_\Omega \frac{u_{\varepsilon t}^2 F'_\varepsilon(u_{\varepsilon t})^2}{F'_\varepsilon(u_{\varepsilon t})} |\nabla \psi|^2 \right|
\]
\[
+ \chi_1 \int_\Omega \frac{u_{\varepsilon t}^2 F'_\varepsilon(u_{\varepsilon t})^2}{F'_\varepsilon(u_{\varepsilon t})} + |\nabla \psi|^2
\]
\[
+ \chi_1 \int_\Omega \frac{u_{\varepsilon t}^2 F'_\varepsilon(u_{\varepsilon t})^2}{F'_\varepsilon(u_{\varepsilon t})} + |\nabla \psi|^2
\]
which, upon being integrated from $t$ to $t+1$, gives rise to
\[
\int_t^{t+1} \|u_{\varepsilon t}\|_{(W^{1,\infty})^*} \leq C_1 \left( 1 + \int_t^{t+1} \int_\Omega \frac{u_{\varepsilon t}^2}{u_{\varepsilon t}^2 + |\nabla u_{\varepsilon t}|^2 + |\nabla w_{\varepsilon t}|^4} \right). \tag{4.36}
\]
Moreover, and using the third equation in (3.7) and the facts that \( \|u_0\|_{L^1}, \|v_0\|_{L^1} = \|v_0\|_{L^\infty} \leq F_\varepsilon(s) \leq s \) and \( w_\varepsilon \leq \|w_0\|_{L^\infty} \), cf. (5.6), (5.10) and (5.8), we deduce that

\[
\left| \int_\Omega w_\varepsilon \psi \right| = - \int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi - \alpha \int_\Omega F_\varepsilon(u_\varepsilon) w_\varepsilon \psi - \beta \int_\Omega F_\varepsilon(v_\varepsilon) w_\varepsilon \psi \leq \int_\Omega |\nabla w_\varepsilon| + \alpha \int_\Omega u_\varepsilon w_\varepsilon + \beta \int_\Omega v_\varepsilon
\]

entailing

\[
\int_t^{t+1} \|
\]

\[
\|
\]

The desired estimate (4.36) follows from a combination of (4.30), (4.37) and (4.38) with Lemmas 4.2 and 4.3. \( \square \)

4.2. Global existence of weak solutions in 3,4,5 D. With the estimates gained in last subsection, we can first derive strong compactness properties by means of the Aubin-Lions and then obtain the existence of weak solutions in in 3, 4 and 5D via extraction procedure, cf. [27, 34, 38].

Lemma 4.8. For \( n \in \{3,4,5\} \), there exist \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1) \) and nonnegative functions \( u, v \) and \( w \) satisfying (4.11) such that \( \varepsilon_j \to 0 \) as \( j \to \infty \) and that the global solution of (3.4) satisfies, as \( \varepsilon = \varepsilon_j \to 0 \), that

\[
(u_\varepsilon, v_\varepsilon, w_\varepsilon) \to (u, v, w) \quad \text{a.e. in } \Omega \times (0,\infty),
\]

\[
(u_\varepsilon, F_\varepsilon(u_\varepsilon), u_\varepsilon F_\varepsilon'(u_\varepsilon)) \to (u, u, u) \quad \text{in } L_p^p(\Omega \times [0,\infty)), \quad \forall p \in [1, \frac{n+2}{n} - 1),
\]

\[
(v_\varepsilon, F_\varepsilon(v_\varepsilon), v_\varepsilon F_\varepsilon'(v_\varepsilon)) \to (v, v, v) \quad \text{in } L_p^p(\Omega \times [0,\infty)), \quad \forall p \in [1, \frac{n+2}{n} - 1).
\]

\[
(\nabla u_\varepsilon, \nabla v_\varepsilon) \to (\nabla u, \nabla v) \quad \text{in } L^{\frac{n+2}{n}}(\Omega \times [0,\infty)),
\]

\[
w_\varepsilon(\cdot, t) \rightharpoonup w(\cdot, t) \quad \text{in } L^q(\Omega) \quad \text{for a.e. } t \in (0,\infty),
\]

\[
w_\varepsilon \rightharpoonup w \quad \text{in } L^{\frac{nq}{n+2q}}(\Omega \times [0,\infty)), \quad \nabla w_\varepsilon \rightharpoonup \nabla w \quad \text{in } L_{-\text{loc}}^{\frac{n+2}{n+2q}}(\Omega \times [0,\infty)) \quad \text{and}
\]

\[
(\nabla F_\varepsilon(u_\varepsilon) \nabla w_\varepsilon, v_\varepsilon F_\varepsilon'(v_\varepsilon) \nabla w_\varepsilon) \to (u \nabla w, v \nabla w) \quad \text{in } L^{\frac{n+2}{n+2q}}(\Omega \times [0,\infty)).
\]

Here, the exponent \( q \) is related to the Sobolev conjugate number \( \frac{4n}{(n-4)^+} \) and satisfies

\[
q \in [1,\infty] \quad \text{if } n = 3; \quad q \in [1,\infty] \quad \text{if } n = 4; \quad q \in [1,20] \quad \text{if } n = 5.
\]

Moreover, \( (u, v, w) \) is a global weak solution of (1.1) in the sense of Definition 5.1.

Proof. By Lemmas 5.3, 5.4, 5.1, 5.2, 5.3 and 5.7 for each \( T > 0 \), we know that

\[
(u_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{and} \quad (v_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{are bounded in } L^{\frac{n+2}{n+2q}}(0,T; W^{1,\frac{n+2}{n+2q}}(\Omega))
\]

and

\[
(u_{\varepsilon t})_{\varepsilon \in (0,1)} \quad \text{and} \quad (v_{\varepsilon t})_{\varepsilon \in (0,1)} \quad \text{are bounded in } L^1(0,T; W^{1,\infty}(\Omega)^*)
\]

as well as

\[
(w_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{is bounded in } L^4(0,T; W^{1,4}(\Omega))
\]
and

\((w_{εt})_{ε∈(0,1)} \text{ is bounded in } L^1((0,T); (W^{1,∞}(Ω))^*)\).

By the compact embeddings \(W^{1,∞}(Ω) \hookrightarrow L^1(Ω)\) and \(W^{1,q}(Ω) \hookrightarrow L^q(Ω)\) for all \(q ∈ [1,∞]\) with \(1 - \frac{4}{n} > -\frac{q}{n}\), twice direct applications of the Aubin-Lions lemma [23], we see there exist \((ε_j)_{j∈N} ⊂ (0,1)\) and nonnegative functions \(u, v\) and \(w\) fulfilling (1.11) and such that, for any such \(q\) and \(ε = ε_j ∩ 0, 0\),

\[ u_ε → u \quad \text{and} \quad v_ε → v \quad \text{in } L^1(Ω × (0,T)) \quad \text{(4.47)} \]

as well as

\[ w_ε → w \quad \text{in } L^1((0,T); L^q(Ω)). \quad \text{(4.48)} \]

Now, based on (4.47) and (4.48), upon passing to a subsequence if necessary, we can infer that (4.39) and (4.43) hold, and also that (4.32) and (4.41) hold. Using the a.e. convergence in (4.39), the boundedness in (4.41) and the properties of \(F_ε\) in (4.30), we use the Vitali convergence theorem (roughly, a.e. convergence plus uniform integrability imply convergence) to infer (4.40) and (4.41). Using the convergence features in (4.40), (4.41) and (4.44) and noting the fact that \(\frac{n}{n+2} + \frac{1}{4} < 1\) since \(n ≤ 5\), we conclude (4.45), which together with (1.11) implies that the regularity conditions in Definition [3.1] are fulfilled. The remaining verifications of the integration by parts identities (5.1), (5.2) and (5.3) can be easily adapted from [22, 23, 35].

\[□\]

**Proof of 3.4, 5D global existence of weak solutions.** The statement on 3, 4, 5D global existence of weak solutions has been fully contained in Lemma 4.8.

\[□\]

4.3. **Large time behavior of global weak solutions in convex domains.** In this subsection, we focus on the eventual smoothness and stabilization of global weak solutions to (1.1), that is, the limiting functions \((u, v, w)\) of \((u_ε, v_ε, w_ε)\).

**Lemma 4.9.** For \(n ∈ \{3, 4, 5\}\), there exists \((ε_j)_{j∈N} ⊂ (0,1)\) of numbers \(ε_j ∩ 0, 0\) such that as \(ε = ε_j ∩ 0, 0\), the global solution of (1.7) fulfill the following properties:

\[ w ∈ C^0([0,∞); L^2(Ω)) \quad \text{(4.49)} \]

and

\[ w_ε → w \quad \text{in } L^∞_{loc}([0,∞); L^2(Ω)) \quad \text{(4.50)} \]

**Proof.** In light of the \(ε\)-independent estimates provided in Lemmas 3.3 and 4.5, we shall adapt the arguments in [27] Corollary 5.3 to derive (4.49) and (4.50). As a matter of fact, for any \(T > 0\), it follows from Lemmas 5.4 and 4.5 that \((w_ε)_{ε∈(0,1)}\) is bounded in \(L^∞((0,T); L^∞(Ω))\) and that \((w_{εt})_{ε∈(0,1)}\) is bounded in \(L^2((0,T); L^2(Ω))\). Then, arguing as [24] Corollary 5.3, we see that

\[ \sup_{ε∈(0,T)} \|w_ε(·,t)\|_{L^2} + \sup_{t≠s, t,s∈(0,T)} \frac{\|w_ε(·, t)\|_{L^2} - \|w_ε(·, s)\|_{L^2}}{|t - s|^{\frac{4}{n}}} \leq C_1, \]

that is, \((w_ε)_{ε∈(0,1)}\) is bounded in \(C^2(0,T; L^2(Ω))\), and thus is relatively compact in \(C(0,T; L^2(Ω))\) by the Arzela and Ascoli compactness theorem. This establishes (4.50) and thus (4.49).

\[□\]

In the sequel, we fix \((ε = ε_j)_{j∈N}\) so that Lemmas 1.3 and 4.9 hold. With the strong convergence provided in Lemma 4.14 below, the following lemma follows.

**Lemma 4.10.** With \((w_ε, v_ε, w_ε)\) replaced by \((u, v, w)\) constructed in Lemma 4.8, the conclusions of Lemmas 3.3, 3.4 and 4.7 through 4.7 still hold.
Lemma 4.11. The global weak solution of (1.1) constructed in Lemma 4.8 fulfills

\[
w(\cdot, t) \to 0 \text{ in } L^\infty(\Omega) \text{ as } t \to \infty.
\]  

(4.51)

Proof. We shall extend the arguments in [27, Lemma 5.4] (with a small flaw) for \( n = 3 \) to higher dimensional cases. Notice from Lemmas 3.4 and 4.2 that

\[
\|w\|_{L^\infty(\Omega \times (0, \infty))} + \int_{t_k}^{t_k+1} \int_\Omega |\nabla w|^4 \leq C_1, \quad \forall t \geq 0.
\]

This ensures the existence of a sequence times \( t_k \to \infty \) with \( 1 + t_k \leq t_k+1 \leq 2 + t_k \) such that \((w(\cdot, t_k))_{k \in \mathbb{N}}\) is bounded in \( W^{1,4}(\Omega) \). This together with the compact embedding from \( W^{1,4}(\Omega) \) into \( L^q(\Omega) \) for \( q \) satisfying (4.46) allows us to deduce, up to a subsequence, for some nonnegative function \( w_\infty \), that

\[
w(\cdot, t_k) \to w_\infty \text{ in } L^q(\Omega) \text{ as } k \to \infty.
\]  

(4.52)

Using Cauchy-Schwarz inequality, we infer that

\[
\int_{t_k}^{1+t_k} \int_\Omega |w(\cdot, t) - w(\cdot, t_k)|^2 \\
= \int_{t_k}^{1+t_k} \int_\Omega \left( \int_{t_k}^{t} w_\varepsilon(\cdot, s) \right)^2 \leq \int_{t_k}^{1+t_k} \int_\Omega w_\varepsilon^2(\cdot, t), \quad \forall \varepsilon \in (0, 1).
\]

In view of Lemma 4.5 upon passing to the limit \( \varepsilon = \varepsilon_j \to 0 \), this gives rise to

\[
\int_{t_k}^{1+t_k} \int_\Omega |w(\cdot, t) - w(\cdot, t_k)|^2 \leq \int_{t_k}^{1+t_k} \int_\Omega w^2(\cdot, t) \to 0 \text{ as } k \to \infty.
\]

This together with (4.52) with \( q = 2 \) implies that

\[
\frac{1}{2} \int_{t_k}^{1+t_k} \int_\Omega |w(\cdot, t) - w_\infty|^2 \\
\leq \|w(\cdot, t_k) - w_\infty\|_{L^2}^2 + \int_{t_k}^{1+t_k} \int_\Omega |w(\cdot, t) - w(\cdot, t_k)|^2 \to 0 \text{ as } k \to \infty.
\]  

(4.53)

Extracting Lemma 4.4 from Lemma 4.10, we have

\[
\int_{t_k}^{1+t_k} \left( \|u - \bar{u}_0\|_{L^{\frac{p}{p-r}}}^2 + \|v - \bar{v}_0\|_{L^{\frac{q}{q-r}}}^2 \right) \to 0 \text{ as } k \to \infty.
\]  

(4.54)
Since $w_{\varepsilon} \leq \|w_{0}\|_{L_{\infty}}$, we use Hölder’s inequality to estimate, for all $k \in \mathbb{N}$, that
\[
\int_{t_{k}}^{1+t_{k}} \int_{\Omega} |(\alpha u + \beta v) w - (\alpha \bar{u}_{0} + \beta \bar{v}_{0}) w_{\infty}| \leq \int_{t_{k}}^{1+t_{k}} \int_{\Omega} (|u - \bar{u}_{0}| + |v - \bar{v}_{0}|) w + (\alpha u_{0} + \beta \bar{v}_{0}) \int_{t_{k}}^{1+t_{k}} \int_{\Omega} w - w_{\infty} | \leq \left[ \alpha \left( \int_{t_{k}}^{1+t_{k}} \|u - \bar{u}_{0}\|^{2}_{L_{\infty}} \right)^{\frac{1}{2}} + \beta \left( \int_{t_{k}}^{1+t_{k}} \|v - \bar{v}_{0}\|^{2}_{L_{\infty}} \right)^{\frac{1}{2}} \right] \left( \int_{t_{k}}^{1+t_{k}} \|w\|^{\frac{1}{2}}_{L_{\infty}} \right)^{\frac{1}{2}} + (\alpha u_{0} + \beta \bar{v}_{0}) |\Omega| \left( \int_{t_{k}}^{1+t_{k}} \|w - w_{\infty}\|^{2}_{L_{\infty}} \right)^{\frac{1}{2}},
\]
and so, we obtain from (4.53) and (4.54) that
\[
\int_{t_{k}}^{1+t_{k}} \int_{\Omega} (\alpha u + \beta v) w \rightarrow (\alpha \bar{u}_{0} + \beta \bar{v}_{0}) \int_{\Omega} w_{\infty} \text{ as } k \rightarrow \infty.
\]
(4.55)

On the other hand, by (3.12) and the fact $1 + t_{k} \leq t_{k+1}$, we have
\[
\sum_{k=1}^{\infty} \int_{t_{k}}^{1+t_{k}} \int_{\Omega} (\alpha u + \beta v) w \leq \int_{0}^{\infty} \int_{\Omega} (\alpha u + \beta v) w < +\infty.
\]
Since $\bar{u}_{0}, \bar{v}_{0} > 0$, this couples with (4.55) implies $w_{\infty} \equiv 0$, and so (4.52) becomes
\[
w(\cdot, t_{k}) \rightarrow 0 \text{ in } L^{q}(\Omega) \text{ as } k \rightarrow \infty.
\]
This together with the fact that $t \mapsto \|w(\cdot, t)\|_{L^{q}}$ is non-increasing by Lemma 4.11 shows actually that
\[
w(\cdot, t) \rightarrow 0 \text{ in } L^{q}(\Omega) \text{ as } k \rightarrow \infty.
\]
(4.56)
Since $\|w(\cdot, t)\|_{L^{\infty}}$ is non-increasing in $t$ and is nonnegative, we see, as $t \rightarrow \infty$, that $\|w(\cdot, t)\|_{L^{\infty}}$ converges decreasingly to some $a \geq 0$. If $a > 0$, then, for any $\eta \in (0, a)$ and $t > 0$, we define $\Omega(t) = \{x \in \Omega : w(x, t) \geq a - \eta\}$, and thus we get
\[
\|w(\cdot, t)\|_{L^{\infty}} = \left( \int_{\Omega} w^{q}(\cdot, t) \right)^{\frac{1}{q}} \geq (a - \eta)|\Omega(t)|^{\frac{1}{q}},
\]
which couples with (4.55) immediately yields $\lim_{t \rightarrow \infty} |\Omega(t)| = 0$. While, this is incompatible with the fact that $\|w(\cdot, t)\|_{L^{\infty}} \geq a$ for all $t \geq 0$. Therefore, we must have $a = 0$, and so (4.51) follows.

**Lemma 4.12.** For $n \in \{3, 4, 5\}$ and for any $\delta > 0$, there exist $t_{0}(\delta) > 0$ and $\varepsilon_{0}(\delta) \in (0, 1)$ such that for all $\varepsilon \in (\varepsilon_{j})_{j \in \mathbb{N}}$ satisfying $\varepsilon < \varepsilon_{0}(\delta)$, the $w$-component of the global classical solution of (3.47) fulfills
\[
0 \leq w_{\varepsilon} \leq \delta \text{ on } \Omega \times (t_{0}(\delta), \infty).
\]
(4.57)
Proof. For given $\delta > 0$ and $q$ satisfying (4.59), it follows from Lemma 4.11 there exists $t_0 > 0$ such that the limit $w$ defined by Lemma 4.13 fulfills $\|w(\cdot, t)\|_{L^q} \leq \delta/2$ for $t > t_0$. Thanks to (4.3), there exists $t_0 \in (\hat{t}_0, \hat{t}_0 + 1)$ such that $w_\varepsilon(\cdot, t_0) \to w(\cdot, t_0)$ in $L^q(\Omega)$ as $\varepsilon = \epsilon_j \to 0$. This joined with the fact from Lemma 5.31 that $t \mapsto \|w(\cdot, t)\|_{L^q}$ is non-increasing ensures there exists $\varepsilon_0(\delta) > 0$ such that, for $t \geq t_0$ and $\varepsilon \in (\epsilon_j)_{j \in \mathbb{N}}$ with $\varepsilon < \varepsilon_0(\delta)$,

$$
\left\| w_\varepsilon(\cdot, t) \right\|_{L^q} \leq \left\| w(\cdot, t_0) \right\|_{L^q} \leq \left\| w(\cdot, t_0) - w(\cdot, t_0) \right\|_{L^q} + \left\| w(\cdot, t_0) \right\|_{L^q}
\leq \frac{\delta}{2} + \|w(\cdot, t_0)\|_{L^q} \leq \delta.
$$

(4.58)

Now, let $a = \limsup_{(x, t) \to (0, \infty)} \|w_\varepsilon(\cdot, t)\|_{L^q}$; if $a > 0$, then, for any $\eta \in (0, a)$, we define $E_\varepsilon^\eta = \{x \in \Omega : w_\varepsilon(x, t) \geq a - \eta\}$, and then we derive

$$
\left\| w_\varepsilon(\cdot, t) \right\|_{L^q} = \left( \int_{\Omega} w_\varepsilon^q(\cdot, t) \right)^{\frac{1}{q}} \geq (a - \eta)|E_\varepsilon^\eta|^\frac{1}{q},
$$

which couples with (4.60) immediately yields $\limsup_{(x, t) \to (0, \infty)} |E_\varepsilon^\eta| = 0$. While, this is contradictory to the definitions of $a$ and $E_\varepsilon^\eta$, and therefore, we must have $a = 0$, and so the decay estimate (4.59) follows.

\[ \square \]

Lemma 4.13. For $n \in \{3, 4, 5\}$ and $p \in (1, \infty)$, there exist $t_1(p) > 0$, $\varepsilon_1(p) \in (0, 1)$ and $K_1(p) = K_1(t_0, \varepsilon_0, \omega, t_0, p) > 0$ such that for $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ with $\varepsilon < \varepsilon_1(p)$, the global classical solution of (3.7) satisfies

$$
\int_{\Omega} w_\varepsilon^p(\cdot, t) + \int_{\Omega} v_\varepsilon^p(\cdot, t) \leq K_1(p), \quad \forall t \geq t_1(p).
$$

(4.59)

Proof. With Lemmas 4.11 and 4.12 at hand, we can extend the arguments in [27, Lemma 6.2] for $n = 3$ to its higher dimensional cases (4.59). To this end, for $p \in (1, \infty)$ and for $r \in (0, p - 1)$, we first see that

$$
\left\{ 0 \leq z < \frac{r + 1}{p}, \quad p - 1 - \frac{p}{4r} \cdot \frac{4r^2 + (p - 1)^2z^2}{r + 1 - pz} > 0 \right\}
$$

$$
\iff 0 \leq z < \frac{2(p - 1 - r)\sqrt{p^r}}{(p - 1)p \left( \sqrt{(p - 1)(r + 1)} + \sqrt{p^r} \right)}
$$

and

$$
\frac{2(p - 1 - r)\sqrt{p^r}}{(p - 1)p \left( \sqrt{(p - 1)(r + 1)} + \sqrt{p^r} \right)} > \frac{2(p - 1 - r)r}{p^2(p + p)} = \frac{(p - 1 - r)r}{p^3}.
$$

and thus, we see, for

$$
\delta = \min \left\{ 1, \frac{1}{\chi_1}, \frac{1}{\chi_2} \right\} \min \left\{ \frac{(p - 1 - r)r}{2p^3}, \frac{r + 1}{2p} \right\}
$$

$$
\min \left\{ 1, \frac{1}{\chi_1}, \frac{1}{\chi_2} \right\} \frac{(p - 1 - r)r}{2p^3} > 0,
$$

(4.60)

we have that

$$
A_i := p(2\delta)^{-r} \left\{ p - 1 - \frac{p}{4r} \cdot \frac{4r^2 + (p - 1)^2(2\delta \chi_1)^2}{r + 1 - 2p(\chi_1)^2} \right\} > 0, \quad i = 1, 2.
$$

(4.61)

With these preparations, we now specify $\phi$ as follows:

$$
\phi(s) = (2\delta - s)^{-r}, \quad s \in [0, 2\delta).
$$
Hence, we employ Young’s inequality to estimate

\[
\frac{d}{dt} \int_{\Omega} (u_\epsilon^p + v_\epsilon^p) \phi(w_\epsilon)
\]

\[
= -(p-1)p \int_{\Omega} |u_\epsilon^{p-2}| \nabla u_\epsilon^2 + v_\epsilon^{p-2}| \nabla v_\epsilon^2| \phi(w_\epsilon)
\]

\[
- \int_{\Omega} \phi''(w_\epsilon) - p\chi_1 F'_\epsilon(u_\epsilon) \phi'(w_\epsilon) |u_\epsilon|^2 \nabla w_\epsilon^2
\]

\[
- \int_{\Omega} \phi''(w_\epsilon) - p\chi_2 F'_\epsilon(v_\epsilon) \phi'(w_\epsilon) |v_\epsilon|^2 \nabla w_\epsilon^2
\]

\[
+ p \int_{\Omega} u_\epsilon^{p-1} [-2\phi'(w_\epsilon) + (p-1)\chi_1 F''_\epsilon(u_\epsilon) \phi(w_\epsilon)] \nabla u_\epsilon \nabla w_\epsilon
\]

\[
+ p \int_{\Omega} v_\epsilon^{p-1} [-2\phi'(w_\epsilon) + (p-1)\chi_2 F''_\epsilon(v_\epsilon) \phi(w_\epsilon)] \nabla v_\epsilon \nabla w_\epsilon
\]

\[
- \int_{\Omega} \alpha F_\epsilon(u_\epsilon) + \beta F_\epsilon(v_\epsilon) \phi(w_\epsilon) \phi(w_\epsilon) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

By the nonnegativity of \( u_\epsilon, v_\epsilon, w_\epsilon, F_\epsilon, \phi' \), we first see obviously that \( I_6 \leq 0 \), and then, by the facts \( 0 \leq w_\epsilon \leq \delta \) in (4.57), \( 0 \leq F''_\epsilon(s) \leq 1 \) in (4.56) and the choice of \( \delta \) in (4.60), we infer that

\[
\phi''(w_\epsilon) - p\chi_1 F''_\epsilon(u_\epsilon) \phi'(w_\epsilon) \geq \phi''(w_\epsilon) - p\chi_2 F''_\epsilon(v_\epsilon) \phi'(w_\epsilon) \geq (2\delta)^{-r-2} (\chi_1 \chi_2)^{-1} (r + 1 - 2p\chi_1 \delta) > 0,
\]

\[
\phi''(w_\epsilon) - p\chi_2 F''_\epsilon(v_\epsilon) \phi'(w_\epsilon) \geq \phi''(w_\epsilon) - p\chi_2 F''_\epsilon(v_\epsilon) \phi'(w_\epsilon) \geq (2\delta)^{-r-2} (\chi_1 \chi_2)^{-1} (r + 1 - 2p\chi_2 \delta) > 0.
\]

Hence, we employ Young’s inequality to estimate \( I_4 \) as

\[
I_4 \leq -I_2 + \frac{p^2}{4} \int_{\Omega} \frac{-2\phi'(w_\epsilon) + (p-1)\chi_1 F''_\epsilon(u_\epsilon) \phi(w_\epsilon)}{\phi''(w_\epsilon) - p\chi_1 \phi'(w_\epsilon)} u_\epsilon^{p-2}| \nabla u_\epsilon^2|, \tag{4.63}
\]

and then, based on (4.62), we further use the facts \( 0 \leq F''_\epsilon(s) \leq 1 \), the choices of \( \delta \) and \( A_1 \) in (4.60) and (4.61) to estimate, for \( (x, t, s) \in \Omega \times (t_0(\delta), \infty) \times [0, \delta] \),

\[
B_1(x, t, s) := p(p-1) \phi(s) - \frac{p^2}{4} \frac{-2\phi'(s) + (p-1)\chi_1 F''_\epsilon(u_\epsilon) \phi(s)}{\phi''(s) - p\chi_1 \phi'(s)} \frac{-2\phi'(s) + (p-1)\chi_1 F''_\epsilon(u_\epsilon) \phi(s)}{\phi''(s) - p\chi_1 \phi'(s)}
\]

\[
\geq p(p-1) \phi(s) - \frac{p^2}{4} \frac{4\phi''(s)}{\phi''(s) - p\chi_1 \phi'(s)} \left\{ p - 1 - \frac{p^2}{4r} \frac{4r^2 + (p-1)^2(2\delta - s)^2 \chi_1^2}{r + 1 - (2\delta - s)p\chi_1} \right\}
\]

\[
\geq p(2\delta - s)^{-r} \left\{ p - 1 - \frac{p^2}{4r} \frac{4r^2 + (p-1)^2(2\delta - s)^2 \chi_1^2}{r + 1 - 2p\chi_1} \right\} = A_1 > 0.
\]

Combining this with (4.63), we conclude that

\[
I_4 \leq -I_2 + (p-1)p \int_{\Omega} \phi(w_\epsilon) u_\epsilon^{p-2}| \nabla u_\epsilon^2| - A_1 \int_{\Omega} u_\epsilon^{p-2}| \nabla u_\epsilon^2|. \quad (4.64)
\]

In the same reasoning, we readily estimate the term \( I_5 \) as

\[
I_5 \leq -I_3 + (p-1)p \int_{\Omega} \phi(w_\epsilon) v_\epsilon^{p-2}| \nabla v_\epsilon^2| - A_2 \int_{\Omega} v_\epsilon^{p-2}| \nabla v_\epsilon^2|. \quad (4.65)
\]
Substituting (4.62), (4.64) and (4.65) into (4.62) and recalling \( I_6 \leq 0 \), we infer that
\[
\frac{d}{dt} \int_{\Omega} (u^p_t + v^p_t) \phi(w_t) \leq -A_1 \int_{\Omega} u_t^{p-2} |\nabla u|_t^2 - A_2 \int_{\Omega} v_t^{p-2} |\nabla v|_t^2
= -\frac{4A_1}{p^2} \int_{\Omega} |\nabla u^\frac{p}{2}|_t^2 - \frac{4A_2}{p^2} \int_{\Omega} |\nabla v^\frac{p}{2}|_t^2.
\]
(4.66)

Next, due to the mass conservations of \( u_t \) and \( v_t \) in (3.38), by the GN inequality (cf. Lemma 2.11) and the fact that \( \phi(w_t) \leq \delta^{-r} \), we infer that
\[
\int_{\Omega} u_t^p \phi(w_t) \leq \delta^{-r} \| u^\frac{p}{2} \|_{L^2} \leq C_1 \| \nabla u^\frac{p}{2} \|_{L^2} + C_1 \| u^\frac{p}{2} \|_{L^2}^2
\]
and, similarly, that
\[
\int_{\Omega} v_t^p \phi(w_t) \leq C_3 \| \nabla v^\frac{p}{2} \|_{L^2}^2 + C_3.
\]

Setting \( y_t(t) = \int_{\Omega} (u_t^p + v_t^p) \phi(w_t)(\cdot, t) \), we derive from (4.66) an ODI as follows:
\[
y_t(t) \leq -C_4 (y_t(t) - 1)_{t}^{-\frac{2}{p-1}} \leq \left[ (y_t(t) - 1)_{t}^{-\frac{2}{p-1}} \right] \geq \frac{2C_4}{(p-1)n}, \quad t > t_0(\delta).
\]

An integration enables us to deduce that
\[
y_t(t) \leq 1 + \left( \frac{2C_4}{(p-1)n} \right) (t - t_0(\delta))^{-\frac{2}{p-1}}, \quad t > t_0(\delta).
\]

Recalling the definition of \( y_t \) and \( \phi(w_t) \geq (2\delta)^{-r} \), we immediately arrive at
\[
\int_{\Omega} (u_t^p + v_t^p) \leq (2\delta)^r y_t(t) \leq (2\delta)^r \left[ 1 + \left( \frac{2C_4}{(p-1)n} \right) (t - t_0(\delta))^{-\frac{2}{p-1}} \right], \quad t \geq 1 + t_0(\delta),
\]
yielding our desired estimate (4.50) upon setting \( t_1(\delta) = 1 + t_0(\delta) \). \( \square \)

With the uniform eventual \( L^p \)-boundedness of \( u_t \) and \( v_t \) in Lemma 4.13 it is quite standard via bootstrap argument or semigroup technique (cf. [9, 27]) to obtain the uniform eventual \( C^2 \)-boundedness of weak solutions.

**Lemma 4.14.** For \( n \in \{3, 4, 5\} \), there exist \( T > 0 \) and \( K_{14} = K_{14}(u_0, v_0, w_0) > 0 \) such that for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) of \( (\varepsilon_j)_{j \in \mathbb{N}} \), the global classical solution of (3.7) fulfills
\[
\| u_\varepsilon(\cdot, t) \|_{C^2(\Omega)} + \| v_\varepsilon(\cdot, t) \|_{C^2(\Omega)} \leq K_{14}, \quad \forall t \geq T,
\]
and such that
\[
u_\varepsilon \to u, \quad v_\varepsilon \to v \quad \text{and} \quad w_\varepsilon \to w \quad \text{in} \quad C^2_{\text{loc}}(\Omega \times [T, \infty)) \quad \text{as} \quad \varepsilon = \varepsilon_j \searrow 0.
\]
(4.68)

**Proof.** By Lemma 4.13, for \( p > 2n \), there exist \( t_1 = t_1(p) > 0 \) and \( \varepsilon_1(p) \in (0, 1) \) such that for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) with \( \varepsilon < \varepsilon_1(p) \), the solution of (3.7) satisfies
\[
\| u_\varepsilon(\cdot, t) \|_{L^p(\Omega)} \leq C_1, \quad \forall t \geq t_1.
\]
(4.69)

We use the variation-of-constants formula for \( w_\varepsilon \) to write
\[
w_\varepsilon(\cdot, t) = e^{(\Delta-1) t} w_\varepsilon(\cdot, t_1) - \int_{t_1}^{t} e^{(t-s)(\Delta-1)} (\alpha F_\varepsilon(u_\varepsilon) + \beta F_\varepsilon(v_\varepsilon) - 1) w_\varepsilon(\cdot, s) ds, \quad t \geq t_1.
\]
Since \(|F_\varepsilon(s)| \leq |s|\) and \(w_\varepsilon \leq \|w_0\|_{L^\infty(\Omega)}\), we employ the well-known smoothing \(L^p-L^q\)-estimates for the Neumann heat semigroup \(\{e^{t\Delta}\}_{t \geq 0}\) (c.f. \([33]\)) to infer
\[
\|\nabla w_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_2 \|w_\varepsilon(\cdot,t_1)\|_{L^\infty(\Omega)} + C_2 \int_{t_1}^t \left[1 + (t-s)^{-\frac{1}{2}}\right] e^{-(t-s)}
\times (\alpha\|u_\varepsilon(\cdot,s)\|_{L^p} + \beta\|v_\varepsilon(\cdot,s)\|_{L^p} + 1) \, ds
\leq C_3, \quad t \geq t_1 + 1.
\]

Given \((4.69)\) and \((4.70)\), H"older's inequality enables us to infer that
\[
\|u_\varepsilon(\cdot,t)\nabla w_\varepsilon(\cdot,t)\|_{L^q(\Omega)} + \|v_\varepsilon(\cdot,t)\nabla w_\varepsilon(\cdot,t)\|_{L^q(\Omega)}
\leq \left(\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\right) \|\nabla w_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C_4, \quad t \geq t_2 := t_1 + 1.
\]

Applying the variation-of-constants formula for \(u_\varepsilon\), we have
\[
\begin{align*}
\quad u_\varepsilon(\cdot,t) &= e^{t\Delta} u_\varepsilon(\cdot,t_2) - \chi \int_{t_2}^t e^{(t-\tau)\Delta} \nabla \cdot (u_\varepsilon F_\varepsilon(u_\varepsilon)\nabla w_\varepsilon(\cdot,\tau)) \, d\tau, \quad t \geq t_2.
\end{align*}
\]
This together with \((4.69)\) and \((4.71)\) allows us to find \(\theta \in (0,1), \gamma \in (0,1)\) and \(q > 1\) suitably large such that \(2\theta - \frac{\gamma}{q} > 0\),
\[
\|A^\theta u_\varepsilon(\cdot,t)\|_{L^q} \leq C_5, \quad \forall t > t_3 := t_2 + 1
\]
and, for any \(t, s \geq t_3\) such that \(|t-s| \leq 1\),
\[
\|A^\theta u_\varepsilon(\cdot,t) - A^\theta u_\varepsilon(\cdot,s)\|_{L^q} \leq C_6 |t-s|^{\gamma},
\]
where \(A^\theta\) denotes the fractional power of the realization of \(-\Delta + 1\) in \(L^q(\Omega)\) under homogeneous Neumann boundary conditions. By the continuous embedding \(D(A^\theta) \hookrightarrow C^\sigma\) for all \(\sigma \in (0, 2\theta - \frac{\gamma}{q})\) (c.f. \([17, 33]\)), \((4.72)\) and \((4.73)\) we know that \((u_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}\) is bounded in both \(L^\infty(\Omega \times (t_3, \infty))\) and \(C_{loc}^{\alpha, \frac{\gamma}{q}}(\Omega \times [t_3, \infty))\) for some \(\sigma \in (0,1)\). Analogously, we also have \((v_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}\) bounded in both \(L^\infty(\Omega \times (t_3, \infty))\) and \(C_{loc}^{\alpha, \frac{\gamma}{q}}(\Omega \times [t_3, \infty))\) for some \(\sigma \in (0,1)\). Thus, the standard parabolic Schauder estimates \([13]\) applied to the third equation in \((3.7)\) yield boundedness of \((v_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}\) in both \(L^\infty((t_4, \infty); C^{2+\sigma, \frac{\gamma}{q}}(\Omega))\) and in \(C_{loc}^{2+\sigma, \frac{\gamma}{q}}(\Omega \times [t_4, \infty))\) with \(t_4 = t_3 + 1\). This, in turn, by a similar argument, we also obtain the boundedness of \((u_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}\) and \((v_\varepsilon)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}}\) in both \(L^\infty((t_5, \infty); C^{2+\sigma, \frac{\gamma}{q}}(\Omega))\) and in \(C_{loc}^{2+\sigma, \frac{\gamma}{q}}(\Omega \times [t_5, \infty))\) for some \(\sigma' > 0\) and \(t_5 = t_4 + 1\) and thus \((4.67)\) follows. Finally, an application of the Arzel"a-Ascoli theorem implies \((4.68)\). \hfill \Box

**Lemma 4.15.** For \(n \in \{3, 4, 5\}\), the global weak solution of \((1.1)\) constructed from Lemma 4.3 satisfies
\[
u(\cdot,t) \to \bar{u}_0 \quad \text{and} \quad v(\cdot,t) \to \bar{v}_0 \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \quad t \to \infty,
\]
where \(\bar{u}_0\) and \(\bar{v}_0\) are the average of \(u_0\) and \(v_0\) over \(\Omega\), respectively.

**Proof.** With the information provided by Lemmas 4.3, 4.4 and 4.11 we can readily extend and adapt the arguments in \([27]\) Lemma 7.2 to derive \((4.72)\). Indeed, let us assume to the contrary there exists a sequence of \(t_k \to \infty\) such that
\[
d := \inf_{k \in \mathbb{N}} \|u(\cdot, t_k) - \bar{u}_0\|_{L^\infty} > 0,
\]
where with no loss of generality can assume that all \(t_k > T\) and \(1 + t_k \leq t_{k+1}\) as \(T\) provided by Lemma 4.11. In light of \((4.67)\) and \((4.68)\), \((u(\cdot, t_k))_{k \in \mathbb{N}}\) is relatively
compact, and then by the Arzelà-Ascoli theorem, we may assume for convenience, for some nonnegative function $u_\infty$, that

$$u(\cdot, t_k) \to u_\infty \text{ in } L^\infty(\Omega) \text{ as } k \to \infty. \quad (4.76)$$

In the spirit of Lemma 4.11, we use the Cauchy-Schwarz inequality to estimate that

$$\int_{t_k}^{1+t_k} \|u_\varepsilon(\cdot, t) - u_\varepsilon(\cdot, t_k)\|_{(W^{m,2})^*}^2 dt \leq \int_{t_k}^{1+t_k} \|u_\varepsilon(\cdot, s)\|_{(W^{m,2})^*}^2, \quad \forall \varepsilon \in (0, 1).$$

In view of (4.33) in Lemma 4.6, upon passing to the limit $\varepsilon = \varepsilon_j \to 0$, this yields

$$\int_{t_k}^{1+t_k} \|u(\cdot, t) - u(\cdot, t_k)\|_{(W^{m,2})^*}^2 \to 0 \text{ as } k \to \infty.$$  

Hence, we use triangle inequality to deduce from the above two estimates that

$$\int_{t_k}^{1+t_k} \|u(\cdot, t) - u_\infty\|_{(W^{m,2})^*}^2 \to 0 \text{ as } k \to \infty. \quad (4.77)$$

On the other hand, notice also that $L^{\infty}(\Omega) \hookrightarrow (W^{m,2}(\Omega))^*$ due to $m > \frac{n}{2} + 1$; The content of Lemma 4.3 from Lemma 4.10 ensures that

$$\int_{t_k}^{1+t_k} \|u(\cdot, t) - \bar{u}_0\|_{(W^{m,2})^*}^2 \to 0 \text{ as } k \to \infty. \quad (4.78)$$

By uniqueness, it follows from (4.77) and (4.78) that $u_\infty \equiv \bar{u}_0$, which is impossible by (4.75) and (4.76). This contradiction says that the $u$-limit in (4.74) is true; similarly, the $v$-limits follows in the same way. \hfill \Box

**Proof of eventual smoothness and convergence in convex domains.** The eventual smoothness and boundedness of weak solutions result immediately from (4.67) and (4.68) in Lemma 4.13. The convergence of weak solutions as in (1.9) follows from Lemmas 4.13 and 4.15. \hfill \Box

**Acknowledgments** G. Ren was supported by the National Natural Science Foundation of China (No.12001214) and the Postdoctoral Science Foundation (Nos. 2020M672319, 2020TQ0111). T. Xiang was funded by the National Natural Science Foundation of China (Nos. 12071476 and 11871226).

**References**

[1] K. Baghaei, A. Khelghati, Global existence and boundedness of classical solutions for a chemotaxis model with consumption of chemoattractant and logistic source, Math. Methods Appl. Sci., 40 (2017) 3799–3807.

[2] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, Indiana Univ. Math. J., 65 (2016), 553–583.

[3] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[4] X. Cao, S. Kurima and M. Mizukami, Global existence and asymptotic behavior of classical solutions for a 3D two-species Keller–Segel-Stokes system with competitive kinetics, Math. Methods Appl. Sci. 41 (2018), 3138–3154.

[5] M. Fuest, J. Lankeit and M. Mizukami, Long-term behaviour in a parabolic-elliptic chemotaxis-consumption model, J. Differential Equations 271, (2021), 254–279.

[6] A. Friedman, Partial differential equations. Holt, Rinehart and Winston, New York-Montreal, Que.-London, 1969.

[7] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin/New York, 1981.

[8] M. Hirata, S. Kurima, M. Mizukami and T. Yokota, Boundedness and stabilization in a two-dimensional two-species chemotaxis-Navier-Stokes system with competitive kinetics, J. Differential Equations, 263 (2017), 470–490.

[9] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005) 52–107.

[10] J. Jiang, H. Wu and S. Zheng Blow-up for a three dimensional Keller–Segel model with consumption of chemotactant, J. Differential Equations 264 (2018), 5432–5464.

[11] H. Jin and Z. Wang, Global stability of prey-taxis systems, J. Differential Equations, 262 (2017), 1257–1290.

[12] H. Jin and T. Xiang, Convergence rates of solutions for a two-species chemotaxis-Navier-Stokes system with competitive kinetics, Discrete Contin. Dyn. Syst. Ser. B 24 (2019), 1919–1942.

[13] O. Ladyzhenskaya, V. Solonnikov and N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, AMS, Providence, RI, 1968.

[14] J. Lankeit, Y. Wang, Global existence, boundedness and stabilization in a high-dimensional chemotaxis system with consumption, Discrete Contin. Dyn. Syst., 37 (2017) 6099–6121.

[15] C. Lee, Z. Wang and Wen Yang, Boundary-layer profile of a singularly perturbed nonlocal semilinear problem arising in chemotaxis, Nonlinearity 33 (2020), 5111–5141.

[16] X. Li and Y. Wang, On a fully parabolic chemotaxis system with Lotka-Volterra competitive kinetics, J. Math. Anal. Appl., 381 (2011), 521–529.

[17] Y. Tao, Global existence of classical solutions to a predator-prey model with nonlinear prey-taxis, Nonlinear Anal. Real World Appl., 11(3) (2010), 2056–2064.

[18] Y. Tao and M. Winkler, Energy-type estimates and global solvability in a two-dimensional chemotaxis–haptotaxis model with remodeling of non-diffusible attractant, J. Diff. Eqns., 257 (2014), 784–815.

[19] J. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, Nonlinearity, 25 (2012), 1413–1425.
Two-Species Chemotaxis Model with Signal Absorption

[30] X. Tu, C. Mu and S. Qiu, Boundedness and convergence of constant equilibria in a two-species chemotaxis-competition system with loop, Nonlinear Anal. 198 (2020), 111923, 19 pp.

[31] L. Wang, C. Mu, X. Hu and P. Zheng, Boundedness and asymptotic stability of solutions to a two-species chemotaxis system with consumption of chemoattractant, J. Differential Equations, 264 (2018), 3389–3401.

[32] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, Comm. Partial Differential Equations, 35 (2010), 1516–1537.

[33] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, J. Diff. Eqns., 248 (2010), 2889–2905.

[34] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Differential Equations, 37 (2012), 319–351.

[35] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, J. Math. Pures Appl., 100 (2013), 748–767.

[36] M. Winkler, Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities, SIAM J. Math. Anal., 47 (2015) 3092–3115.

[37] M. Winkler, The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: global large-data solutions and their relaxation properties, Math. Models Methods Appl. Sci., 26 (2016), 987–1024.

[38] M. Winkler, Asymptotic homogenization in a three-dimensional nutrient taxis system involving food-supported proliferation, J. Differential Equations, 263 (2017), 4826–4869.

[39] S. Wu, J. Shi and B. Wu, Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis, J. Differential Equations, 7(2016), 5847–5874.

[40] T. Xiang, Global dynamics for a diffusive predator-prey model with prey-taxis and classical Lotka-Volterra kinetics, Nonlinear Anal. Real World Appl., 39 (2018), 278–299.

[41] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? J. Math. Anal. Appl., 459 (2018), 1172–1200.

[42] T. Xiang, Chemotactic aggregation versus logistic damping on boundedness in the 3D minimal Keller-Segel model, SIAM J. Appl. Math., 78 (2018), 2420–2438.

[43] T. Xiang, Sub-logistic source can prevent blow-up in the 2D minimal Keller-Segel chemotaxis system, J. Math. Phys., 59 (2018), 081502, 11 pp.

[44] T. Xiang and J. Zheng, A new result for 2D boundedness of solutions to a chemotaxis–haptotaxis model with/without sub-logistic source, Nonlinearity, 32 (2019), 4890–4911.

[45] Q. Zhang and Y. Li, Stabilization and convergence rate in a chemotaxis system with consumption of chemoattractant, J. Math. Phys. 56 (2015), 081506, 10 pp.

[46] Y. Zhang and W. Tao, Boundedness and stabilization in a two-species chemotaxis system with signal absorption, Comput. Math. Appl., 78 (2019), 2672–2681.

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, 430074, Hubei, P. R. China; Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan, 430074, Hubei, P. R. China

Email address: 597746385@qq.com

Institute for Mathematical Sciences, Renmin University of China, Beijing, 100872, China

Email address: txiang@ruc.edu.cn