On combinations of local theory extensions

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Abstract. In this paper we study possibilities of efficient reasoning in combinations of theories over possibly non-disjoint signatures. We first present a class of theory extensions (called local extensions) in which hierarchical reasoning is possible, and give several examples from computer science and mathematics in which such extensions occur in a natural way. We then identify situations in which combinations of local extensions of a theory are again local extensions of that theory. We thus obtain criteria both for recognizing wider classes of local theory extensions, and for modular reasoning in combinations of theories over non-disjoint signatures.

1 Introduction

Many problems in mathematics and computer science can be reduced to proving the satisfiability of conjunctions of literals in a background theory (which can be the extension of a base theory with additional functions – e.g., free, monotone, or recursively defined – or a combination of theories). It is therefore very important to identify situations where reasoning in complex theories can be done efficiently and accurately. Efficiency can be achieved for instance by:

(1) reducing the search space (preferably without losing completeness);
(2) modular reasoning, i.e., delegating some proof tasks which refer to a specific theory to provers specialized in handling formulae of that theory.

We are interested in identifying situations when both these goals can be achieved without loss of completeness.

Controlling the search space. The quest for identifying theories where the search space can be controlled without loss of completeness led McAllester and Givan to define local theories, that is sets \( N \) of Horn clauses with the property that for any ground clause \( G \), \( N \models G \) iff \( G \) can be proved already using those instances \( N[G] \) of \( N \) containing only ground terms occurring in \( G \) or in \( N \). For local theories, validity of ground Horn clauses can be checked in polynomial time. In [BG96, BG01], Ganzinger and Basin defined the more general notion of order locality and showed how to recognize (order-)local theories and how to use these results for automated complexity analysis.

Similar ideas also occurred in algebra, where the main interest was to identify classes of algebras for which the uniform word problem is decidable in polynomial
time. In [Bur95], Burris proved that if a quasi-variety axiomatized by a set $\mathcal{K}$ of Horn clauses has the property that *every finite partial algebra which is a partial model of the axioms in $\mathcal{K}$ can be extended to a total algebra model of $\mathcal{K}$* then the uniform word problem for $\mathcal{K}$ is decidable in polynomial time. In [Gan01], Ganzinger established a link between proof theoretic and semantic concepts for polynomial time decidability of uniform word problems. He defined two notions of locality for equational Horn theories, and established relationships between these notions of locality and corresponding semantic conditions, referring to embeddability of partial algebras into total algebras.

**Modular reasoning.** When reasoning in extensions or combinations of theories it is very important to find ways of delegating some proof tasks which refer to a specific theory to provers specialized in handling formulae of that theory. Of particular interest are situations when reasoning can be done:

- in a hierarchical way (that is, for reasoning in a theory extension a prover for the base theory can be used as a black-box), or
- in a modular way (that is, for reasoning in a combination of theories reasoning in the component theories is “decoupled”, i.e., the information about the component theories is never combined and only formulae in the joint signature are exchanged between provers for the components).

One of the first methods for modular reasoning in combinations of theories, due to Nelson and Oppen [NO79], can be applied for combining decision procedures of *stably infinite* theories over disjoint signatures. There were several attempts to extend the completeness results for modular inference systems for combinations of theories over non-disjoint signatures. In [Ghi04] the component theories need to satisfy a model theoretical compatibility condition with respect to the shared theory. In [Tin03], similar modularity results are achieved if the theories share all function symbols. Several modularity results using superposition were established for combinations of theories over disjoint signatures in [ARR03,Hil04,ABRS05]. In [GSSW04,GSSW06] we analyzed possibilities of modular reasoning (using special superposition calculi) in combination of first-order theories involving both total and partial functions. The calculi are shown to be complete provided that functions that are not in the intersection of the component signatures are declared as partial. Cases where the partial models can always be made total are identified: in such cases modular superposition is also complete with respect to the standard (total function) semantics of the theories. Inspired by the link between embeddability and locality established by Ganzinger in [Gan01], such extensions were called *local*.

**Reasoning in local theory extensions and their combinations.** In [GSSW04], [GSSW06] and, later, in [SS05] we showed that for *local theory extensions* efficient hierarchic reasoning is possible. For such extensions the two goals previously mentioned can be addressed at the same time: the locality of an extension allows to reduce the search space, but at the same time (as a by-product) it
allows to perform an easy reduction to a proof task in the base theory (for this, a specialized prover can be used as a black box).

Many theories important for computer science or mathematics are local extensions of a base theory: theories of data structures, theories of monotone functions or of functions satisfying the Lipschitz conditions. However, often it is necessary to consider complex extensions, with various types of functions (such as, for instance, extensions of the theory of real numbers with free, monotone and Lipschitz functions). It is important to have efficient methods for hierarchical and/or modular reasoning also for such combinations. Finding methods for reasoning in combinations of extensions of a base theory is far from trivial: as these are usually combinations of theories over non-disjoint signatures, classical combination results such as the Nelson-Oppen combination method [NO79] cannot be applied; methods for reasoning in combinations of theories over non-disjoint signatures – as studied by Ghilardi et al. [Ghi04,BG07] – may also not always be applicable (unless the base theory is universal and the extensions satisfy certain model-theoretic compatibility conditions required in [Ghi04,BG07]).

In this paper we identify situations in which a combination of local extensions of a base theory is guaranteed to be itself a local extension of the base theory. We thus obtain criteria for recognizing complex local theory extensions, and for efficient reasoning in such combinations of theories (over non-disjoint signatures) in a modular way.

Structure of the paper: The paper is structured as follows: Section 2 contains generalities on partial algebras, weak validity and embeddability of partial algebras into total algebras. In Section 3 the notion of local theory extension is introduced. In Section 4 links between embeddability and locality of an extension are established. In Section 5 examples of local theory extensions are given. In the following two sections we identify situations under which a combination of local extensions of a base theory is guaranteed to be itself a local extension of the base theory, under stronger (Section 6) or weaker (Section 7) embeddability conditions for the components. Some ideas on hierarchical and modular reasoning in such combinations are discussed in Section 8. Section 9 contains conclusions and plans for future work.

The results on combinations of local extensions of a base theory presented in this paper generalize results on combinations of local theories obtained in [GSS01].

2 Preliminaries

This section contains the main notions and definitions necessary in the paper.

2.1 Partial structures

Let $\Pi = (\Sigma, \text{Pred})$ be a signature where $\Sigma$ is a set of function symbols and $\text{Pred}$ a set of predicate symbols.
Definition 1 A partial \( \Pi \)-structure is a structure \( (A, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \text{Pred}}) \), where \( A \) is a non-empty set and for every \( f \in \Sigma \) with arity \( n \), \( f_A \) is a partial function from \( A^n \) to \( A \). The structure is a (total) structure if all functions \( f_A \) are total.

In what follows we usually denote both an algebra and its support with the same symbol. Details on partial algebras can be found in [Bur86].

The notion of evaluating a term \( t \) with respect to a variable assignment \( \beta : X \to A \) for its variables in a partial algebra \( A \) is the same as for total algebras, except that this evaluation is undefined if \( t = f(t_1, \ldots, t_n) \) and either one of \( \beta(t_i) \) is undefined, or else \( (\beta(t_1), \ldots, \beta(t_n)) \) is not in the domain of \( f_A \).

Definition 2 We define weak validity in structures \( (A, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \text{Pred}}) \), where \( \text{Pred} \) is a set of predicate symbols and \( (A, \{f_A\}_{f \in \Sigma}) \) is a partial \( \Sigma \)-algebra. Let \( \beta : X \to A \).

1. \((A, \beta) \models_w t = s \) if and only if one of the conditions below is fulfilled:
   - (a) \( \beta(t) \) and \( \beta(s) \) are both defined and equal; or
   - (b) at least one of \( \beta(s) \) and \( \beta(t) \) is undefined.

2. \((A, \beta) \models_w t \neq s \) if and only if one of the conditions below is fulfilled:
   - (a) \( \beta(t) \) and \( \beta(s) \) are both defined and different; or
   - (b) at least one of \( \beta(s) \) and \( \beta(t) \) is undefined.

3. \((A, \beta) \models_w P(t_1, \ldots, t_n) \) if and only if one of the conditions below is fulfilled:
   - (a) \( \beta(t_1), \ldots, \beta(t_n) \) are all defined and \( (\beta(t_1), \ldots, \beta(t_n)) \in P_A \); or
   - (b) at least one of \( \beta(t_1), \ldots, \beta(t_n) \) is undefined.

4. \((A, \beta) \models_w \neg P(t_1, \ldots, t_n) \) if and only if one of the conditions below is fulfilled:
   - (a) \( \beta(t_1), \ldots, \beta(t_n) \) are all defined and \( (\beta(t_1), \ldots, \beta(t_n)) \notin P_A \); or
   - (b) at least one of \( \beta(t_1), \ldots, \beta(t_n) \) is undefined.

\((A, \beta) \) weakly satisfies a clause \( C \) (notation: \( (A, \beta) \models_w C \)) if \( (A, \beta) \models_w L \) for at least one literal \( L \) in \( C \). \( A \) weakly satisfies \( C \) (notation: \( A \models_w C \)) if \( (A, \beta) \models_w C \) for all assignments \( \beta \). \( A \) weakly satisfies a set of clauses \( K \) (notation: \( A \models_w K \)) if \( A \models_w C \) for all \( C \in K \).

Example 3 Let \( A \) be a partial \( \Sigma \)-algebra, where \( \Sigma = \{\text{car}/1, \text{nil}/0\} \). Assume that \( \text{nil}_A \) is defined and \( \text{car}_A(\text{nil}_A) \) is not defined. Then \( A \models_w \text{car}(\text{nil}) \approx \text{nil} \) and \( A \models_w \text{car}(\text{nil}) \not\approx \text{nil} \) (because one term is not defined in \( A \)).

Definition 4 A weak \( \Pi \)-embedding between the partial structures \( (A, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \text{Pred}}) \) and \( (B, \{f_B\}_{f \in \Sigma}, \{P_B\}_{P \in \text{Pred}}) \) is a total map \( i : A \to B \) such that

- whenever \( f_A(a_1, \ldots, a_n) \) is defined then \( f_B(i(a_1), \ldots, i(a_n)) \) is defined and \( i(f_A(a_1, \ldots, a_n)) = f_B(i(a_1), \ldots, i(a_n)) \);
- \( i \) is injective;
- \( i \) is an embedding w.r.t. \( \text{Pred} \), i.e. for every \( P \in \text{Pred} \) with arity \( n \) and every \( a_1, \ldots, a_n \in A \), \( P_A(a_1, \ldots, a_n) \) if and only if \( P_B(i(a_1), \ldots, i(a_n)) \).

In this case we say that \( A \) weakly embeds into \( B \).
2.2 Theories and extensions of theories

Theories can be regarded as sets of formulae or as sets of models. Let $T$ be a $\Pi$-theory and $\phi, \psi$ be $\Pi$-formulae. We say that $T \land \phi \models \psi$ (written also $\phi \models_T \psi$) is $\psi$ is true in all models of $T$ which satisfy $\phi$.

In what follows we consider extensions of theories, in which the signature is extended by new function symbols (i.e. we assume that the set of predicate symbols remains unchanged in the extension). If a theory is regarded as a set of formulae, then its extension with a set of formulae is set union. If $T$ is regarded as a collection of models then its extension with a set $K$ of formulae consists of all structures (in the extended signature) which are models of $K$ and whose reduct to the signature of $T_0$ is in $T_0$. In this paper we regard theories as sets of formulae. All the results of this paper can easily be reformulated to a setting in which $T_0$ is a collection of models.

Let $T_0$ be an arbitrary theory with signature $\Pi_0 = (\Sigma_0, \text{Pred})$, where the set of function symbols is $\Sigma_0$. We consider extensions $T_1$ of $T_0$ with signature $\Pi = (\Sigma, \text{Pred})$, where the set of function symbols is $\Sigma = \Sigma_0 \cup \Sigma_1$. We assume that $T_1$ is obtained from $T_0$ by adding a set $K$ of (universally quantified) clauses.

Definition 5 (Weak partial model) A partial $\Pi$-algebra $A$ is a weak partial model of $T_1$ with totally defined $\Sigma_0$-function symbols if (i) $A_{\Pi_0}$ is a model of $T_0$ and (ii) $A$ weakly satisfies all clauses in $K$.

If the base theory $T_0$ and its signature are clear from the context, we will refer to weak partial models of $T_1$. We will use the following notation:

- $\text{PMod}_w(\Sigma_1, T_1)$ is the class of all weak partial models of $T_1$ in which the $\Sigma_1$-functions are partial and all the other function symbols are total;
- $\text{PMod}_f^w(\Sigma_1, T_1)$ is the class of all finite weak partial models of $T_1$ in which the $\Sigma_1$-functions are partial and all the other function symbols are total;
- $\text{PMod}_fd^w(\Sigma_1, T_1)$ is the class of all weak partial models of $T_1$ in which the $\Sigma_1$-functions are partial and their definition domain is a finite set, and all the other function symbols are total;
- $\text{Mod}(T_1)$ denotes the class of all models of $T_1$ in which all functions in $\Sigma_0 \cup \Sigma_1$ are totally defined.

2.3 Embeddability

For theory extensions $T_0 \subseteq T_1 = T_0 \cup K$, where $K$ is a set of clauses, we consider the following condition:

(Emb$_w$) Every $A \in \text{PMod}_w(\Sigma_1, T_1)$ weakly embeds into a total model of $T_1$.

We also define a stronger notion of embeddability, which we call completability:

(Comp$_w$) Every $A \in \text{PMod}_w(\Sigma_1, T_1)$ weakly embeds into a total model $B$ of $T_1$ such that $A_{|\Pi_0}$ and $B_{|\Pi_0}$ are isomorphic.

Weaker conditions, which only refer to embeddability of finite partial models, will be denoted by (Emb$_f^w$), resp. (Comp$_f^w$). Conditions which refer to embeddability of partial models in $\text{PMod}_d^w(\Sigma_1, T_1)$ will be denoted by (Emb$_d^w$), resp. (Comp$_d^w$).
3 Locality

The notion of local theory was introduced by Givan and McAllester [GM92,McA93].

Definition 6 (Local theory) A local theory is a set of Horn clauses $\mathcal{K}$ such that, for any ground Horn clause $C$, $\mathcal{K} = C$ only if already $\mathcal{K}[C] = C$ (where $\mathcal{K}[C]$ is the set of instances of $\mathcal{K}$ in which all terms are subterms of ground terms in either $\mathcal{K}$ or $C$).

The notion of locality in equational theories was studied by Ganzinger [Gan01], who also related it to a semantical property, namely embeddability of partial algebras into total algebras. In [GSSW04,GSSW06,SS05] the notion of locality for Horn clauses is extended to the notion of local extension of a base theory.

Let $\mathcal{K}$ be a set of clauses in the signature $\Pi = (\Sigma_0 \cup \Sigma_1, \text{Pred})$. In what follows, when we refer to sets $G$ of ground clauses we assume that they are in the signature $\Pi^c = (\Sigma \cup \Sigma_c, \text{Pred})$, where $\Sigma_c$ is a set of new constants. If $\Psi$ is a set of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$-terms, we denote by $\mathcal{K}_\Psi$ the set of all instances of $\mathcal{K}$ in which all terms starting with a $\Sigma_1$-function symbol are ground terms in the set $\Psi$. If $G$ is a set of ground clauses and $\Psi = \text{st}(\mathcal{K},G)$ is the set of ground subterms occurring in either $\mathcal{K}$ or $G$ then we write $\mathcal{K}[G] := \mathcal{K}_\Psi$.

We will focus on the following type of locality of a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$, where $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ with $\mathcal{K}$ a set of (universally quantified) clauses:

$$(\text{Loc}) \quad \text{For every set } G \text{ of ground clauses } \mathcal{T}_1 \cup G \models \bot \text{ iff } \mathcal{T}_0 \cup \mathcal{K}[G] \cup G \text{ has no weak partial model in which all terms in } \text{st}(\mathcal{K},G) \text{ are defined.}$$

A weaker notion $(\text{Loc}^f)$ can be defined if we require that the respective conditions hold only for finite sets $G$ of ground clauses. An intermediate notion of locality $(\text{Loc}^{fd})$ can be defined if we require that the respective conditions hold only for sets $G$ of ground clauses containing only a finite set of terms starting with a function symbol in $\Sigma_1$.

Definition 7 (Local theory extension) An extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is local if it satisfies condition $(\text{Loc}^f)$.

A local theory [Gan01] is a local extension of the empty theory.

4 Locality and embeddability

There is a strong link between locality of a theory extension and embeddability of partial models into total ones. Links between locality of a theory and embeddability were established by Ganzinger in [Gan01]. We show that similar results can be obtained also for local theory extensions.

In what follows we say that a non-ground clause is $\Sigma_1$-flat if function symbols (including constants) do not occur as arguments of function symbols in $\Sigma_1$. A
4.1 Locality implies embeddability

We first show that for sets of $\Sigma_1$-flat clauses locality implies embeddability. This generalizes results presented in the case of local theories in [Gan01].

Theorem 8 Assume that $\mathcal{K}$ is a family of $\Sigma_1$-flat clauses in the signature $\Pi$.

1. If the extension $T_0 \subseteq T_1 := T_0 \cup \mathcal{K}$ satisfies (Loc) then it satisfies (Emb$_w$).
2. If the extension $T_0 \subseteq T_1 := T_0 \cup \mathcal{K}$ satisfies (Loc$^1$) then it satisfies (Emb$^1_w$).
3. If the extension $T_0 \subseteq T_1 := T_0 \cup \mathcal{K}$ satisfies (Loc$^{fd}$) then it satisfies (Emb$^{fd}_w$).
4. If $T_0$ is compact and the extension $T_0 \subseteq T_1$ satisfies (Loc$^1$), then $T_0 \subseteq T_1$ satisfies (Emb$_w$).

Proof: We prove (4) and show how the proof can be changed to provide proofs for (1), (2) and (3). Let $A$ be a partial $\Pi$-algebra with totally defined $\Sigma_0$-functions, which is a model of $T_0$ and weakly satisfies $\mathcal{K}$. Let

$$\Delta(A) = \{ f(a_1, \ldots, a_n) \equiv a \mid \text{if } f_A(a_1, \ldots, a_n) \text{ is defined and equal to } a \}$$

$$\cup \{ f(a_1, \ldots, a_n) \not\equiv a \mid \text{if } f_A(a_1, \ldots, a_n) \text{ is defined and not equal to } a \}$$

$$\cup \{ P(a_1, \ldots, a_n) \mid P \in \text{Pred and } (a_1, \ldots, a_n) \in P_A \}$$

$$\cup \{ \neg P(a_1, \ldots, a_n) \mid P \in \text{Pred and } (a_1, \ldots, a_n) \not\in P_A \} \cup \bigwedge_{a \neq a', a, a' \in A} \neg a$$

We prove that $T_0 \cup \mathcal{K} \cup \Delta(A)$ is consistent, where the elements of $A$ are regarded as new constants. Assume $T_0 \cup \mathcal{K} \cup \Delta(A) \models \bot$. By compactness of $T_0$, $T_0 \cup \mathcal{K} \cup \Gamma \models \bot$, for some finite subset $\Gamma$ of $\Delta(A)$. We know that $A$ is a model of $T_0$. Every term starting with a function symbol in $\Sigma_1$ contained in the clauses in $\mathcal{K}[\Gamma]$ is either a ground (subterm of a) term occurring in $\Gamma$ (and, hence, a constant $a \in A$, or a term $f(a_1, \ldots, a_n)$, where $f_A(a_1, \ldots, a_n)$ is defined), or is a ground subterm in $\mathcal{K}$, i.e. a constant, and hence, again defined in $A$. Therefore, all terms occurring in the clauses in $\mathcal{K}[\Gamma]$ are defined in $A$, so $A$ satisfies all these clauses, i.e. $A$ is a model of $T_0 \cup \mathcal{K} [\Gamma]$. Since $\Delta(A)$ is obviously true in $A$ and $\Gamma \subseteq \Delta(A)$, $A$ is a partial model of $T_0 \cup \mathcal{K} [\Gamma] \cup \Gamma$, in which all ground terms occurring in $\mathcal{K}$ or $\Gamma$ are defined. This contradicts the fact that $T_1$ is a local extension of $T_0$. Hence, the assumption that $T_0 \cup \mathcal{K} \cup \Delta(A) \models \bot$ was false, so $T_0 \cup \mathcal{K} \cup \Delta(A)$ has a model $A'$ in which, therefore, $A$ weakly embeds.

1. If (Loc) holds then we can choose $\Gamma = \Delta(A)$.
2. If $A$ is finite we can choose $\Gamma = \Delta(T_0)$, so the compactness of $T_0$ is not needed.
3. If all functions in $\Sigma_1$ have a finite domain of definition in $A$, then $\Delta(A)$ contains only finitely many terms starting with a $\Sigma_1$-function. Therefore also in this case we can choose $\Gamma = \Delta(A)$.

$\square$
4.2 Embeddability implies locality

Conversely, embeddability implies locality. The following results appear in [SS05] and [SSI07]. This result allows to give several examples of local theory extensions.

**Theorem 9** ([SS05,SSI07]) Let \( K \) be a set of \( \Sigma \)-flat and \( \Sigma \)-linear clauses.

1. If the extension \( T_0 \subseteq T_1 \) satisfies \((\text{Emb}_w)\) then it satisfies \((\text{Loc})\).
2. Assume that \( T_0 \) is a locally finite universal theory, and that \( K \) contains only finitely many ground subterms. If the extension \( T_0 \subseteq T_1 \) satisfies \((\text{Emb}_f^w)\), then \( T_0 \subseteq T_1 \) satisfies \((\text{Loc}_f^w)\).
3. \( T_0 \subseteq T_1 \) satisfies \((\text{Emb}_{fd}^w)\). Then \( T_0 \subseteq T_1 \) satisfies \((\text{Loc}_{fd}^w)\).

5 Examples of local theory extensions

We present several examples of theory extensions for which embedding conditions among those mentioned above hold and are thus local. For details cf. [SS05,SS06a,SSI07].

**Extensions with free functions.** Any extension \( T_0 \cup \text{Free}(\Sigma) \) of a theory \( T_0 \) with a set \( \Sigma \) of free function symbols satisfies condition \((\text{Comp}_w)\).

**Extensions with selector functions.** Let \( T_0 \) be a theory with signature \( \Pi_0 = (\Sigma_0, \text{Pred}) \), let \( c \in \Sigma_0 \) with arity \( n \), and let \( \Sigma_1 = \{s_1, \ldots, s_n\} \) consist of \( n \) unary function symbols. Let \( T_1 = T_0 \cup \text{Sel}_c \) (a theory with signature \( \Pi = (\Sigma_0 \cup \Sigma_1, \text{Pred}) \)) be the extension of \( T_0 \) with the set \( \text{Sel}_c \) of clauses below. Assume that \( T_0 \) satisfies the (universally quantified) formula \( \text{Inj}_c \) (i.e. \( c \) is injective in \( T_0 \)) then the extension \( T_0 \subseteq T_1 \) satisfies condition \((\text{Comp}_w)\) [SS05].

\[
(\text{Sel}_c) \quad \begin{align*}
    s_1(c(x_1, \ldots, x_n)) & \approx x_1 \\
    \vdots \\
    s_n(c(x_1, \ldots, x_n)) & \approx x_n \\
    x & \approx c(x_1, \ldots, x_n) \rightarrow c(s_1(x), \ldots, s_n(x)) \approx x
\end{align*}
\]

\[
(\text{Inj}_c) \quad c(x_1, \ldots, x_n) \approx c(y_1, \ldots, y_n) \rightarrow (\bigwedge_{i=1}^n x_i \approx y_i)
\]

**Extensions with functions satisfying general monotonicity conditions.**

In [SS05] and [SSI07] we analyzed extensions with monotonicity conditions for an \( n \)-ary function \( f \) w.r.t. a subset \( I \subseteq \{1, \ldots, n\} \) of its arguments:

\[
(\text{Mon}_f^I) \quad \bigwedge_{i \in I} x_i \leq y_i \land \bigwedge_{i \notin I} x_i = y_i \rightarrow f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n).
\]

Here, \( \text{Mon}_f^I \) is equivalent to the congruence axiom for \( f \). If \( I = \{1, \ldots, n\} \) we speak of monotonicity in all arguments; we denote \( \text{Mon}_f^{\{1,\ldots,n\}} \) by \( \text{Mon}_f \).
Monotonicity in some arguments and antitonicity in other arguments is modeled by considering functions $f : \prod_{i \in I} P_i^+ \times \prod_{j \notin I} P_j \to P$ with $\sigma_i \in \{-, +\}$, where $P_i^+ = P_i$ and $P_i^- = P_i^\partial$, the order dual of the poset $P_i$. The corresponding axioms are denoted by $\text{Mon}_i^\sigma$, where for $i \in I$, $\sigma(i) = \sigma_i \in \{-, +\}$, and for $i \notin I$, $\sigma(i) = 0$. The following hold [SS05, SSI07]:

1. Let $T_0$ be a class of (many-sorted) bounded semilattice-ordered $\Sigma_0$-structures. Let $\Sigma_1$ be disjoint from $\Sigma_0$ and $T_1 = T_0 \cup \{\text{Mon}_i^\sigma(f) | f \in \Sigma_1\}$. Then the extension $T_0 \subseteq T_1$ satisfies ($\text{Comp}^{fd}_w$), hence is local.

2. Any extension of the theory of posets with functions in a set $\Sigma_1$ satisfying $\{\text{Mon}_i^\sigma(f) | f \in \Sigma_1\}$ satisfies condition ($\text{Emb}_w$), hence is local.

This provides us with a large number of concrete examples. For instance the extensions with functions satisfying monotonicity axioms $\text{Mon}_i^\sigma$ of the following (possibly many-sorted) classes of algebras are local:

- any class of algebras with a bounded (semi)lattice reduct, a bounded distributive lattice reduct, or a Boolean algebra reduct ($\text{Comp}^{fd}_w$ holds);
- any extension of a class of algebras with a semilattice reduct, a (distributive) lattice reduct, or a Boolean algebra reduct, with monotone functions into an infinite numeric domain ($\text{Comp}^{fd}_w$ holds);
- $\mathcal{T}$, the class of totally-ordered sets; $\mathcal{DO}$, the theory of dense totally-ordered sets ($\text{Comp}^{fd}_w$ holds);
- the class $\mathcal{P}$ of partially-ordered sets ($\text{Emb}_w$ holds).

Similarly, it can be proved that any extension of the theory of reals (integers) with functions satisfying $\text{Mon}_i^\sigma$ into a fixed infinite numerical domain is local (condition ($\text{Comp}^{fd}_w$) holds).

**Boundedness conditions.** Any extension of a theory for which $\leq$ is reflexive with functions satisfying ($\text{Mon}_i^\sigma$) and boundedness ($\text{Bound}_i^\sigma$) conditions is local [SS06a, SSI07].

($\text{Bound}_i^\sigma$) $\forall x_1, \ldots, x_n(f(x_1, \ldots, x_n) \leq t(x_1, \ldots, x_n))$

where $t(x_1, \ldots, x_n)$ is a term in the base signature $\Pi_0$ with variables among $x_1, \ldots, x_n$ (such that in any model the associated function has the same monotonicity as $f$).

Similar results can be given for guarded monotonicity conditions with mutually disjoint guards [SS06a].

**Extensions with Lipschitz functions.** The extension $\mathbb{R} \subseteq \mathbb{R} \cup (\mathcal{L}_\lambda)$ of $\mathbb{R}$ with a unary function which is $\lambda$-Lipschitz in a point $x_0$ (for $\lambda > 0$) satisfies condition ($\text{Comp}_w$).

($\mathcal{L}_\lambda$) $\forall x \ |f(x) - f(x_0)| \leq \lambda \cdot |x - x_0|$

The results described before can easily be extended to a many-sorted framework. Therefore various additional examples of (many-sorted) theory extensions related to data structures can be given cf. e.g. [SS06b].
6 Combinations of local extensions satisfying \( \text{Comp}_w \)

In this and the following sections we study the locality of combinations of local theory extensions. In the light of the results in Section 4 we concentrate on studying which embeddability properties are preserved under combinations of theories. For the sake of simplicity, in what follows we consider only conditions \( \text{Emb}_w \) and \( \text{Comp}_w \). Analogous results can be given for conditions \( \text{Emb}_w' \), \( \text{Comp}_w' \), resp. \( \text{Emb}_w^{(d)} \), \( \text{Comp}_w^{(d)} \) and combinations thereof.

We start with a simple case of combinations of local extensions of a base theory: we consider the situation when both components satisfy the embeddability condition \( \text{Comp}_w \). We first analyze the simple case of combinations of local extensions of a base theory \( T_0 \) by means of sets of mutually disjoint function symbols. Then some results on combining extensions with non-disjoint sets of function symbols are discussed.

**Theorem 10** Let \( T_0 \) be a first-order theory with signature \( \Pi_0 = (\Sigma_0, \text{Pred}) \) and \( T_1 = T_0 \cup K_1 \) and \( T_2 = T_0 \cup K_2 \) two extensions of \( T_0 \) with signatures \( \Pi_1 = (\Sigma_0 \cup \Sigma_1, \text{Pred}) \) and \( \Pi_2 = (\Sigma_0 \cup \Sigma_2, \text{Pred}) \), respectively. Assume that both extensions \( T_0 \subseteq T_1 \) and \( T_0 \subseteq T_2 \) satisfy condition \( \text{Comp}_w \), and that \( \Sigma_1 \cap \Sigma_2 = \emptyset \). Then the extension \( T_0 \subseteq T = T_0 \cup K_1 \cup K_2 \) satisfies condition \( \text{Comp}_w \). If, additionally, in \( K_i \) all terms starting with a function symbol in \( \Sigma_i \) are flat and linear, for \( i = 1, 2 \), then the extension is local.

**Proof:** Let \( P \in \text{PMod}_w(\Sigma_1 \cup \Sigma_2, T) \). Then \( P|_{\Pi_1} \in \text{PMod}_w(\Sigma_1, T_1) \), hence \( P|_{\Pi_1} \) weakly embeds into a total model \( B \) of \( T_1 \), such that \( P|_{\Pi_0} \) and \( B|_{\Pi_0} \) are isomorphic. Let \( i : P|_{\Pi_0} \rightarrow B|_{\Pi_0} \) be the isomorphism between these two \( \Pi_0 \)-structures. We use the isomorphism \( i \) to transfer also the \( \Sigma_2 \)-structure from \( P \) to \( B \). That is, for every \( f \in \Sigma_2 \) with arity \( n \), and every \( b_1, \ldots, b_n \in B \), we define:

\[
  f_B(b_1, \ldots, b_n) = \begin{cases} 
    i(f_P(i^{-1}(b_1), \ldots, i^{-1}(b_n))) & \text{if } f_P(i^{-1}(b_1), \ldots, i^{-1}(b_n)) \text{ is defined in } P \\
    \text{undefined} & \text{otherwise}
  \end{cases}
\]

With these definitions of \( \Sigma_2 \)-functions, \( B|_{\Pi_2} \in \text{PMod}_w(\Sigma_2, T_2) \). Therefore, \( B|_{\Pi_2} \) weakly embeds into a total model \( C \) of \( T_1 \), such that \( B|_{\Pi_0} \) and \( C|_{\Pi_0} \) are isomorphic. Let \( j : B|_{\Pi_0} \rightarrow C|_{\Pi_0} \) be the isomorphism between these two structures. We use this isomorphism to transfer, as explained above, the (total) \( \Sigma_1 \)-structure from \( B \) to \( C \). The algebra \( A \) obtained this way from \( C \) is a total model of \( T \), and \( j \circ i : P|_{\Pi_0} \rightarrow A|_{\Pi_0} \) is an isomorphism. Thus, the extension \( T_0 \subseteq T = T_0 \cup K_1 \cup K_2 \) satisfies condition \( \text{Comp}_w \). The last claim is an immediate consequence of Theorem 9. \( \square \)

**Example 11** The following combinations of theories (seen as extensions of a first-order theory \( T_0 \)) satisfy condition \( \text{Comp}_w \) (or in case (4) condition \( \text{Comp}_w^{(d)} \)):
(1) \( T_0 \cup \text{Free}(\Sigma_1) \) and \( T_0 \cup \text{Sel}_c \) if \( T_0 \) is a theory and \( c \in \Sigma_0 \) is injective in \( T_0 \).

(2) \( \mathbb{R} \cup \text{Free}(\Sigma_1) \) and \( \mathbb{R} \cup \text{Lip}_1^\lambda(f) \), where \( f \notin \Sigma_1 \).

(3) \( \mathbb{R} \cup \text{Lip}_1^\lambda(f) \) and \( \mathbb{R} \cup \text{Lip}_2^\lambda(g) \), where \( f \neq g \).

(4) \( T_0 \cup \text{Free}(\Sigma_1) \) and \( T_0 \cup \text{Mon}_f^\lambda \), where \( f \notin \Sigma_1 \) has arity \( n \), \( \sigma : \{1, \ldots, n\} \rightarrow \{-1, 1, 0\} \), if \( T_0 \) is, e.g., a theory of algebras with a bounded semilattice reduct.

A more general result holds, which allows to prove locality also for extensions which share non-base function symbols.

**Theorem 12** Let \( T_0 \) be an arbitrary first-order theory, and \( T_1 = T_0 \cup \mathcal{K}_1 \) and \( T_2 = T_0 \cup \mathcal{K}_2 \) two extensions of \( T_0 \) with functions in \( \Sigma_1 \) and \( \Sigma_2 \) respectively, which satisfy condition \((\text{Comp}_\omega)\). Assume that there exists a set \( \mathcal{K} \) of clauses in signature \( \Sigma_0 \cup \Sigma \), where \( \Sigma = \Sigma_1 \cap \Sigma_2 \subseteq \Sigma_i \), \( i = 1, 2 \), such that every model of \( T_0 \cup \mathcal{K}_i \) is a model of \( T_0 \cup \mathcal{K} \) for \( i = 1, 2 \). Then the extension \( T_0 \cup \mathcal{K} \subseteq (T_0 \cup \mathcal{K}_1) \cup \mathcal{K}_2 \cup \mathcal{K} \) again satisfies condition \((\text{Comp}_\omega)\) and hence is a local extension.

**Proof:** Note that if \( T_0 \subseteq T_0 \cup \mathcal{K}_1 \) satisfies condition \((\text{Comp}_\omega)\) then the extension \( T_0 \cup \mathcal{K} \subseteq (T_0 \cup \mathcal{K}_1) \cup \mathcal{K}_2 \) also satisfies condition \((\text{Comp}_\omega)\). The conclusion now follows from Theorem 10, taking into account the fact that the signatures \((\Sigma_1 \setminus \Sigma)\) and \((\Sigma_2 \setminus \Sigma)\) are disjoint. \( \square \)

**Example 13** The following theory extensions satisfy condition \((\text{Comp}_\omega)\):

(1) \( T_0 \cup \text{Free}(\Sigma) \subseteq (T_0 \cup \text{Free}(\Sigma \cup \Sigma_1)) \cup (T_0 \cup \text{Free}(\Sigma) \cup \text{Sel}_c) \), provided that \( T_0 \) is a theory containing an injective function \( c \).

(2) \( \mathbb{R} \cup \text{Free}(f) \subseteq (\mathbb{R} \cup \text{Mon}_f \cup \text{Mon}_g) \cup (\mathbb{R} \cup \text{Free}(f) \cup \text{Lip}_1^\lambda(h)) \), where \( f, g, h \) are different function symbols.

(3) \( \mathbb{R} \cup \text{Lip}_1^\lambda(f) \subseteq (\mathbb{R} \cup \text{Lip}_1^\lambda(f) \cup \text{Mon}(g)) \cup (\mathbb{R} \cup \text{Lip}_2^\lambda(f) \cup \text{Free}(h)) \), where \( f, g, h \) are different function symbols and \( \lambda_1 \leq \lambda_2 \).

**Proof:** Immediate consequences of Theorem 12 (1) is obvious; for (2) note that every model of \( \mathbb{R} \cup \text{Mon}_f \cup \text{Mon}_g \) is a model of \( \mathbb{R} \cup \text{Free}(f) \); for (3) note that, as \( \lambda_1 \leq \lambda_2 \), every model of \( \mathbb{R} \cup \text{Lip}_1^\lambda(f) \cup \text{Mon}(g) \) is a model of \( \mathbb{R} \cup \text{Lip}_2^\lambda(f) \). \( \square \)

## 7 More general combinations of local theory extensions

The result above can be extended to the more general situation in which one of the extensions, say \( T_0 \subseteq T_1 = T_0 \cup \mathcal{K}_1 \), satisfies condition \((\text{Emb}_\omega)\) and the other extension \( T_0 \subseteq T_2 = T_0 \cup \mathcal{K}_2 \) satisfies condition \((\text{Comp}_\omega)\), or if both extensions satisfy condition \((\text{Emb}_\omega)\). The natural analogon of the proof of Theorem 10 would be the following: Start with a partial model \( P \) of \( T_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \); extend it, using property \((\text{Emb}_\omega)\), to a total model \( A \) of \( T_1 \). The technical problem which occurs when we now try to use the embedding property for \( T_2 \) is that we need to be sure that \( A \) remains also a partial model of \( T_2 \), with the operations inherited from \( P \). Unfortunately this may not always be the case, as shown below.
Example 14 Let $\Pi_0 = \{\{f\}, \text{Pred}\}$ and let $T_0$ be a $\Pi_0$-theory. Let $T_1 = T_0 \cup K_1$, and $T_2 = T_0 \cup K_2$ be two theories over extensions of $\Pi_0$ with function symbols in $\Sigma_1, \Sigma_2$. Assume that $\Sigma_2 = \{g\}$, $\Sigma_1 \cap \Sigma_2 = \emptyset$, and $K_2 = \{x = f(x) \rightarrow g(y) = y\}$ ($f$ and $g$ are unary function symbols).

Let $P = \{\{a, b\}, f_P, g_P, \{\sigma_P\}_{\sigma \in \Sigma_1}\}$ be a partial algebra, where $f_P$ is total with $f_P(a) = b$ and $f_P(b) = a$; $g_P(a) = b$ and $g_P(b)$ is undefined. $P$ weakly satisfies $K_2$ because the premise of the clause in $K_2$ is always false in $P$. Assume that $P$ weakly embeds into a total model $A$ of $T_1$ via a $\Pi_1$-embedding $h : P \hookrightarrow A$, and that $A$ contains an element $c \notin \{h(a), h(b)\}$, such that $f_A(c) = c$. $A$ “inherits” the $\Sigma_2$-operation $g$ from $P$ via $h$, in the sense that we can define $g_A(h(a)) := h(g_P(a)) = h(b)$ and assume that $g_A$ is undefined in rest. However, with the $\Sigma_2$-operation defined this way $A$ does not weakly satisfy $K_2$. Let $\beta : X \rightarrow A$ with $\beta(x) = c$ and $\beta(y) = h(a)$. $(A, \beta)$ does not weakly satisfy the clause in $K_2$, since:

\[\beta(f(x)) = f_A(\beta(x)) = f_A(c) = c, \text{ whereas} \]
\[\beta(g(y)) = g_A(\beta(y)) = g_A(h(a)) = h(g_P(a)) = h(b) \neq h(a) = \beta(y).\]

This happens because the variable $x$ in the clause in $K_2$ does not occur below any function symbol in $\Sigma_2$.

In what follows we identify conditions which ensure that an extension $A$ of a partial algebra $P$ which weakly satisfies $K_2$ remains a partial model of $K_2$ with the $\Sigma_2$-operations inherited from $P$.

7.1 Preservation of truth under extensions

Lemma 15 Let $T_0$ be a theory with signature $\Pi_0 = (\Sigma_0, \text{Pred})$, and let $T_0 \subseteq T := T_0 \cup K$ be a theory extension by means of a set $K$ of $\Sigma$-flat clauses over the signature $\Pi = (\Sigma_0 \cup \Sigma, \text{Pred})$. Assume that for each clause $C$ of $K$ all variables in $C$ occur below some $\Sigma$-function symbol.

Let $P \in \text{PMod}_\Sigma(\Sigma, T)$, $A \in \text{Mod}(T_0)$, and $h : P \hookrightarrow A$ be a $\Pi_0$-embedding. Then a partial $\Sigma$-structure can be defined on $A$ such that $A$ weakly satisfies $K$, and $h$ is a weak $\Pi$-embedding.

Proof: For every $a_1, \ldots, a_n \in A$ and every $f \in \Sigma$ define

\[f_A(a_1, \ldots, a_n) := \begin{cases} a & \text{if } \exists p_1, \ldots, p_n \in P \text{ such that } a = h(p_i), \\ f_P(p_1, \ldots, p_n) & \text{is defined in } P, \\ \text{undefined} & \text{otherwise.} \end{cases}\]

As $h$ is injective, $f_A$ is well-defined. By hypothesis, $h$ is a $\Pi_0$-embedding. With the definition of operations in $\Sigma$ given above, $h$ is also a weak $\Sigma$-homomorphism. Let $p_1, \ldots, p_n \in P$ and $f \in \Sigma$ be such that $f_P(p_1, \ldots, p_n)$ is defined. Then, by the definition of $f_A$, $f_A(h(p_1), \ldots, h(p_n))$ is defined and equal to $h(f_P(p_1, \ldots, p_n))$. Citing the original source is appreciated for the LaTex style and the content structure.
We now prove that with the operations defined as shown before $A$ weakly satisfies $K$. Let $C \in K$ and let $\beta : X \to A$ be an assignment of elements in $A$ to the variables in $C$. Assume that for every term $t$ occurring in $C$, $\beta(t)$ is defined in $A$ (otherwise, due to the definition of weak satisfiability, $(A, \beta) \models_w C$ trivially). In order to show that $(A, \beta) \models_w C$, we construct an assignment $\alpha$ of elements in $P$ to the variables in $C$, and use the fact that $(P,\alpha) \models_w C$.

Let $t = f(t_1, \ldots, t_k)$ be an arbitrary term occurring in $C$, with $f \in \Sigma$. As $\beta(t)$ is defined, $f_A(\beta(t_1), \ldots, \beta(t_k))$ is defined in $A$, hence there exist $p_1, \ldots, p_k \in P$ such that $h(p_i) = \beta(t_i)$, $f_P(p_1, \ldots, p_k)$ is defined, and $f_A(\beta(t_1), \ldots, \beta(t_k)) = h(f_P(p_1, \ldots, p_k))$. As all clauses in $K$ are $\Sigma$-flat, all terms $t_i$ are variables. In this way we can associate with every variable $x$ occurring as argument in a term $f(t_1, \ldots, t_n)$ of $C$ with $f \in \Sigma$ an element $p_x \in P$ such that $h(p_x) = \beta(x)$. Assume that for some such (variable) subterm $x$, two elements of $P$, say $p_x$ and $q_x$, can be associated in this way. Then $h(p_x) = \beta(x) = h(q_x)$, and the injectivity of $h$ guarantees that $p_x = q_x$. This shows that an assignment $\alpha : X \to P$ can be defined, such that for all variables in $C$ occurring below a function symbol in $\Sigma$ (hence for all variables in $C$) $\alpha(x) := p_x$. It is easy to see that for every term $t$ occurring in $C$, $h(\alpha(t)) = \beta(t)$. As $(P,\alpha) \models C$ and $h$ is a weak $\Pi$-embedding it follows that $(A, \beta) \models C$. □

The result above will be applied in Theorems 17 and 19 in the following form:

**Corollary 16** Let $T_0$ be a first-order theory with signature $\Pi_0 = (\Sigma_0, \text{Pred})$. Let $\Sigma_1, \Sigma_2$ be two disjoint sets of function symbols, and let $\Pi_i = (\Sigma_0 \cup \Sigma_i, \text{Pred})$, $i = 1, 2$, and $\Pi = (\Sigma_0 \cup \Sigma_1 \cup \Sigma_2, \text{Pred})$. Let $K_2$ be a set of $\Sigma_2$-flat clauses over $\Pi_2$. Assume that for each clause $C$ of $K_2$ all variables in $C$ occur below some function symbol in $\Sigma_2$.

Let $P$ be a partial $\Pi$-structure such that $P|_{\Pi_0}$ is a total model of $T_0$, and $P$ weakly satisfies $K_2$. Let $A$ be a total $\Pi_1$-structure, and let $h : P \hookrightarrow A$ be a weak $\Pi_1$-embedding. Then a partial $\Sigma_2$-structure can be defined on $A$ such that $A$ weakly satisfies $K_2$, and $h$ is a weak $\Pi$-embedding.

### 7.2 Combining local extensions, one of which satisfies (\text{Comp}_w)

We now analyze the case of combinations of theories in which one component satisfies condition (\text{Comp}_w) and the other component satisfies condition (\text{Emb}_w).

**Theorem 17** Let $T_0$ be a first-order theory with signature $\Pi_0 = (\Sigma_0, \text{Pred})$, and let $T_1 = T_0 \cup K_1$ and $T_2 = T_0 \cup K_2$ be two extensions of $T_0$ with signatures $\Pi_1 = (\Sigma_0 \cup \Sigma_1, \text{Pred})$ and $\Pi_2 = (\Sigma_0 \cup \Sigma_2, \text{Pred})$, respectively. Assume that:

1. $T_0 \subseteq T_1$ satisfies condition (\text{Comp}_w),
2. $T_0 \subseteq T_2$ satisfies condition (\text{Emb}_w),
3. $K_1$ is a set of $\Sigma_1$-flat clauses in which all variables occur below a $\Sigma_1$-function.

Then the extension $T_0 \subseteq T_0 \cup K_1 \cup K_2$ satisfies (\text{Emb}_w). If, additionally, in $K_i$ all terms starting with a function symbol in $\Sigma_i$ are flat and linear, for $i = 1, 2$, then the extension is local.
Let $P \in \text{PMod}_w(\Sigma_1 \cup \Sigma_2, T_0 \cup K_1 \cup K_2)$. Then $P_{|T_2} \in \text{PMod}_w(\Sigma_2, T_2)$, hence $P_{|T_2}$ weakly embeds into a total model $B$ of $T_2$. By (3), in $K_1$ all variables occur below some function symbol in $\Sigma_1$, and all clauses in $K_1$ are $\Sigma_1$-flat. Then, by Lemma 15, we can transform $B$ into a weak partial model $B'$ of $T_1$ (with the $\Sigma_2$-structure inherited from $B$ and the $\Sigma_1$-structure inherited from $P$). But then $B'$ weakly embeds into a total model $C$ of $T_1$ such that $B'|_{T_0}$ and $C|_{T_0}$ are $T_0$-isomorphic. We can use this isomorphism to transfer the (total) $\Sigma_2$-structure from $B$ to $C$. This way, we obtain a total model $A$ of $T_0 \cup K_1 \cup K_2$ in which $P$ weakly embeds. The last claim is an immediate consequence of Theorem 9. \qed

**Example 18** The following theory extensions satisfy ($\text{Emb}_w$), hence are local:

1. $\mathcal{E}_q \subseteq \text{Free}(\Sigma_1) \cup L$, where $\mathcal{E}_q$ is the pure theory of equality, without function symbols, and $L$ the theory of lattices.
2. $T_0 \subseteq (T_0 \cup \text{Free}(\Sigma_1)) \cup (T_0 \cup \text{Mon}(\Sigma_2))$, where $\Sigma_1 \cap \Sigma_2 = \emptyset$, and $T_0$ is, e.g., the theory of posets.

An analogon of Theorem 12 holds also in this case.

### 7.3 Combinations of theory extensions satisfying ($\text{Emb}_w$)

We identify conditions under which embeddability conditions for the component theories imply embeddability conditions for the theory combination.

*Theorem 19* Let $T_0$ be an arbitrary theory in signature $\Pi_0 = (\Sigma_0, \text{Pred})$. Let $K_1$ and $K_2$ be two sets of clauses over signatures $\Pi_i = (\Sigma_0 \cup \Sigma_i, \text{Pred})$, where $\Sigma_1$ and $\Sigma_2$ are disjoint. We make the following assumptions:

- (A1) The class of models of $T_0$ is closed under direct limits of diagrams in which all maps are embeddings (or, equivalently, $T_0$ is a $\forall \exists$ theory).
- (A2) $K_i$ is $\Sigma_i$-flat and $\Sigma_i$-linear for $i = 1, 2$, and $T_0 \subseteq T_0 \cup K_i$, $i = 1, 2$ are both local extensions of $T_0$.
- (A3) For all clauses in $K_1$ and $K_2$, every variable occurs below some extension function.

Then $T_0 \cup K_1 \cup K_2$ is a local extension of $T_0$.

*Proof:* The proof uses the semantical characterization of locality in Theorems 8 and 9. Assumption (A2) guarantees that the extensions $T_0 \subseteq T_0 \cup K_i$, $i = 1, 2$ are both local and that, by Theorem 8 they satisfy condition ($\text{Emb}_w$). We show that $T_0 \subseteq T_0 \cup K_1 \cup K_2$ satisfies condition ($\text{Emb}_w$), hence, by Theorem 9, is local.

Let $\Pi = (\Sigma_0 \cup \Sigma_1 \cup \Sigma_2, \text{Pred})$ and let $P$ be a partial $\Pi$-algebra which weakly satisfies $K_1 \cup K_2$ and whose $\Pi_0$-reduct is a total model of $T_0$. By the locality of the extension $T_0 \subseteq T_0 \cup K_1$, there exists a total $\Pi_1$-model of $T_0 \cup K_1$, which we denote $P_1^1$, and a weak embedding $\pi_1^1 : P \leftrightarrow P_1^1$. By Lemma 15 and Corollary 16 a partial $\Sigma_2$-structure can be defined on $P_1^1$ such that $P_1^1$ weakly satisfies $K_2$ and $\pi_1^1$ is a weak $\Pi$-embedding.
Thus, $P_1^1$ becomes a partial $II_2$-algebra which weakly satisfies $K_2$, and is a total $II_0$-model of $T_0$. By the locality of the extension $T_0 \subseteq T_0 \cup K_2$, there exists a total $II_2$-model of $T_0 \cup K_2$, which we denote $P_2^1$, and a weak embedding $\pi_1^1 : P_1^1 \hookrightarrow P_2^1$. Again, a partial $\Sigma_1$-structure can be defined on $P_2^1$ such that $P_2^1$ weakly satisfies $K_1$ and $\pi_1^1$ is a weak $II$-embedding.

By iterating this process we obtain a sequence of partial $II$-structures $P_i^i$, $i \geq 1$, all of whose reducts to $II_0$ are total models of $T_0$, which weakly satisfy $K_1 \cup K_2$, and have the property that, for every $i \geq 1$, $P_i^i$ is a total $\Sigma_1$-algebra, $P_i^2$ is a total $\Sigma_2$-algebra, and there are weak $II$-embeddings $\pi_i^1 : P_i^i \rightarrow P_2^2$ and $\pi_i^2 : P_i^2 \rightarrow P_i^{i+1}$.

If $P_i^j$ precedes $P_k^l$ in the chain above (where $k, l \in \{1, 2\}$ and $i, j \geq 1$), let $g_{ij}^{kl} : P_i^j \rightarrow P_k^l$ be the composition of the corresponding weak embeddings from $P_i^j$ to $P_k^l$. Being a composition of weak embeddings, $g_{ij}^{kl}$ is itself a weak embedding.

Let $P \prod (\bigcup_{i \geq 1} (P_i^1 \bigcup P_i^2))$ be the disjoint union of all partial $II$-structures constructed this way. In this disjoint union we identify all elements that are images of the same element in some $P_i^k$. This is, we define an equivalence relation $\equiv$ on this disjoint union by $x \equiv y$ if $x \in P_i^j$, $y \in P_k^l$ and either (i) $P_i^j$ precedes $P_k^l$ in the chain above and $g_{ij}^{kl}(x) = y$, or (ii) $P_k^l$ precedes $P_i^j$ in the chain above and $g_{ij}^{kl}(y) = x$. As for every $l \in \{1, 2\}, i \geq 1$, $g_{ij}^{kl}$ is the identity map, if $x \equiv y$ for $x, y \in P_i^j$ then $x = y$. It is easy to see that $\equiv$ is an equivalence relation.

Let $A_0 := P \prod (\bigcup_{i \geq 1} (P_i^1 \bigcup P_i^2))/\equiv$. We show that total functions in $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ and predicates in $\text{Pred}$ can be defined on $A_0$ such that the expansion $A$ of $A_0$ obtained this way is a (total) model of $T_0 \cup K_1 \cup K_2$, and that the map $g : P \rightarrow A$ defined by $g(p) = [p]$ (the equivalence class of $p$ in $A$) is a weak $II$-embedding.

A $II$-structure on $A$ can be defined as follows:

**Interpretation of signature $II_0$.** We first define the $\Sigma_0$-functions. Let $f \in \Sigma_0$ with arity $n$, and let $[a_1], \ldots, [a_n] \in A$. Then, for every $1 \leq j \leq n$, there exist $i_j \geq 1$ such that $a_j \in P_i^j \prod P_i^j$. Let $m = \max \{i_j \mid 1 \leq j \leq n\}$. Let $b_1, \ldots, b_n$ be the images of $a_1, \ldots, a_n$ in $P_i^{m+1}$. By the definition of $\equiv$, $[b_j] = [a_j]$ for every $1 \leq j \leq n$. $P_i^{m+1}$ is a total $\Sigma_0$-algebra, so $b = f_{P_i^{m+1}}(b_1, \ldots, b_n)$ exists in $P_i^{m+1}$. The fact that the definition does not depend on the representatives follows from the fact that all embeddings in the diagram are $\Sigma_0$-homomorphisms.

The predicates in $\text{Pred}$ are defined in a similar way. The fact that the definitions do not depend on the choice of representatives in the equivalence classes follows from the fact that all the maps in the diagram are $II_0$-embeddings.

**Interpretation of the signature $\Sigma_1 \cup \Sigma_2$.** We define the $\Sigma_1$-functions (the $\Sigma_2$-functions can be defined similarly). Let $f \in \Sigma_1$ with arity $n$, and let $[a_1], \ldots, [a_n] \in$
A. Then, for every $1 \leq j \leq n$, there exist $i_j \geq 1$ such that $a_j \in P^i_j \bigcup P^j_i$.

Let $m = \max \{i_j \mid 1 \leq j \leq n\}$. Let $b_1, \ldots, b_n$ be the images of $a_1, \ldots, a_n$ in $P^{m+1}_1$. By the definition of $\equiv$, $[b_j] = [a_j]$ for every $1 \leq j \leq n$. $P^{m+1}_1$ is a total $\Sigma_1$-algebra, so $b = f_{P^{m+1}_1}(b_1, \ldots, b_n)$ exists in $P^{m+1}_1$. The equivalence class of $b$ does not depend on the choice of representatives of the equivalence classes $[a_1], \ldots, [a_n]$. Indeed, assume that $c_1, \ldots, c_n$ are images of $a_1, \ldots, a_n$ in $P^{k+1}_1$, with e.g. $k > m$. By the definition of $g^{1,k+1}_{P^{m+1}_1} : P^{m+1}_1 \rightarrow P^{k+1}_1$, $c_j = g^{1,k+1}_{P^{m+1}_1}(b_j)$. As $f_{P^{m+1}_1}(b_1, \ldots, b_n)$ is defined in $P^{m+1}_1$, we know that $g^{1,k+1}_{P^{m+1}_1}(f_{P^{m+1}_1}(b_1, \ldots, b_n)) = f^{k+1}_{P^{m+1}_1}(g^{1,k+1}_{P^{m+1}_1}(b_1), \ldots, g^{1,k+1}_{P^{m+1}_1}(b_n)) = f^{k+1}_{P^{m+1}_1}(c_1, \ldots, c_n)$. It follows therefore that $b = f^{k+1}_{P^{m+1}_1}(c_1, \ldots, c_n)$, so the equivalence class of $b$ does not depend on the choice of the representatives of $[a_1], \ldots, [a_n]$. We can define $f_A([a_1], \ldots, [a_n]) = [b]$. $f_A$ is well-defined for every $f \in \Sigma_1$.

We now prove that for every $k, i$, the map $g^k_i : P^i_k \rightarrow A$ defined by $g(x) := [x]$ is a weak $H$-embedding.

The fact that $g^k_i$ is a $\Sigma_0$-homomorphism is obvious.

We show that $g^k_i$ is a weak $\Sigma_1$-homomorphism. Let $f \in \Sigma_1$ of arity $n$ and $x_1, \ldots, x_n \in P^i_k$ be such that $f_{P^i_k}(x_1, \ldots, x_n)$ is defined. Then, by the definition of $f_A$, $f_A([x_1], \ldots, [x_n]) = [f_{P^i_k}(x_1, \ldots, x_n)] = g^k_i(f_{P^i_k}(x_1, \ldots, x_n))$.

The fact that $g^k_i$ is a $\Sigma_2$-homomorphism can be proved analogously.

We prove that $g^k_i$ is injective. Assume that $g^k_i(x) = g^k_i(y)$ for $x, y \in P^i_k$. Then $x \equiv y$, hence $g^k_i(x) = y$, i.e. $x = y$ (since $g^k_i$ is the identity map). This also shows that $g : P \rightarrow A$, $g(p) = [p]$ is an injective weak homomorphism.

We prove that $g^k_i$ is an embedding w.r.t. Pred. Let $Q \in \text{Pred}$ be an $n$-ary predicate symbol, and let $x_1, \ldots, x_n \in P^i_k$. We show that $Q_{P^i_k}(x_1, \ldots, x_n)$ if and only if $Q_A(g^k_i(x_1), \ldots, g^k_i(x_n))$. By the way $Q_A$ is constructed it is obvious that if $Q_{P^i_k}(x_1, \ldots, x_n)$ then $Q_A([x_1], \ldots, [x_n])$. Conversely, assume that $Q_A([x_1], \ldots, [x_n])$. By definition, there exists $m$ and $b_1, \ldots, b_n \in P^{m+1}_1$ such that $[x_1] = [b_1], \ldots, [x_n] = [b_n]$ and $Q_{P^{m+1}_1}(b_1, \ldots, b_n)$. The conclusion now follows from the fact that the composition of all maps in the diagram leading from $P^i_k$ to $P^{m+1}_1$ (or vice versa) is a weak $H$-embedding, and hence also $Q_{P^i_k}(x_1, \ldots, x_n)$.

The reduct to $\Pi_0$ of $A$ is the direct limit of a diagram of models of $\mathcal{T}_0$, in which all maps are embeddings. Therefore, if $\mathcal{T}_0$ is closed under such direct limits (i.e. it is a $\forall \exists$ theory) then $A$ is a model of $\mathcal{T}_0$.

Finally, we show that $A$ satisfies all clauses in $\mathcal{K}_1 \cup \mathcal{K}_2$. Let $C \in \mathcal{K}_1$ (the case $C \in \mathcal{K}_2$ is similar). Let $\beta : X \rightarrow A$. We know that every variable of $C$ occurs below a function symbol in $\Sigma_1$, and that all terms of $C$ containing a function symbol in $\Sigma_1$ are of the form $f(x_1, \ldots, x_n)$. For every variable $x$ occurring in $C$, $\beta(x) = [a_x]$, where $a_x \in P^x_k$ for some $x \geq 1$. Let $m = \max \{j_x \mid x \text{ variable of } C\}$, and let $b_x$ be the image of $a_x$ in $P^{m+1}_1$ for each variable $x$ of $C$. Then $\beta(f(x_1, \ldots, x_n))$ is defined in $P^{m+1}_1$ for every term of $C$ of the form $f(x_1, \ldots, x_n)$. In fact, it is easy to see that for every term occurring in $C$, $\beta(t) = [b_t]$ for some $b_t \in P^{m+1}_1$. Let
\[\alpha : X \rightarrow P_{m+1} \text{ with } \alpha(x) := b_x \text{ for every variable } x \text{ of } C.\]

It can be seen that \(g_{1m+1}(\alpha(t)) = \beta(t)\) for every subterm \(t\) of \(C\). As \(P_{m+1}\) satisfies \(C\) and all terms in \(C\) are defined under the assignment \(\alpha\) it follows that there exists a literal \(L\) in \(C\) such that \((P_{m+1}, \alpha) \models L\). We know that \(g_{1m+1} : P_{m+1} \hookrightarrow A\) is a weak embedding w.r.t. \(\Pi_1\). It therefore preserves the truth of positive and negative \(\Pi_1\)-literals. Therefore, as \(g_{1m+1}(\alpha(t)) = \beta(t)\) for every term \(t\) of \(C\), \((A, \beta) \models L\).

Example 20 The following combinations of theories (seen as extensions of the theory \(T_0\)) satisfy condition \((\text{Emb}_w)\):

1. The combination of the theory of lattices and the theory of integers with injective successor and predecessor is local (local extension of the theory of pure equality).
2. \(T_0 \subseteq T_0 \cup \text{Mon}(\Sigma)\), where \(\text{Mon}(\Sigma) = \bigwedge_{f \in \Sigma} \text{Mon}_{\sigma(f)}\), and \(T_0\) is one of the theories of posets, (dense) totally-ordered sets, (semi)lattices, distributive lattices, Boolean algebras, \(\mathbb{R}\).

8 Hierarchical and modular reasoning

In what follows we discuss some issues related to modular reasoning in combinations of local theory extensions. By results in [SS05], hierarchical reasoning is always possible in local theory extensions. In this section we analyze possibilities of modular reasoning, and, in particular, the form of information which needs to be exchanged between provers for the component theories when reasoning in combinations of local theory extensions.

8.1 Hierarchical reasoning in local theory extensions

Consider a local theory extension \(T_0 \subseteq T_0 \cup K\), where \(K\) is a set of clauses in the signature \(\Pi = (\Sigma_0 \cup \Sigma_1, \text{Pred})\). The locality condition requires that, for every set \(G\) of ground clauses, \(T_1 \cup G\) is satisfiable if and only if \(T_0 \cup K[G] \cup G\) has a weak partial model with additional properties. All clauses in \(K[G] \cup G\) have the property that the function symbols in \(\Sigma_1\) only occur at the root of ground terms. Therefore, \(K[G] \cup G\) can be flattened and purified (i.e., the function symbols in \(\Sigma_1\) are separated from the other symbols) by introducing, in a bottom-up manner, new constants \(c_t\) for subterms \(t = f(g_1, \ldots, g_n)\) with \(f \in \Sigma_1\), \(g_i\) ground \(\Sigma_0 \cup \Sigma_c\)-terms (where \(\Sigma_c\) is a set of constants which contains the constants introduced by flattening, resp. purification), together with corresponding definitions \(c_t \approx t\).

The set of clauses thus obtained has the form \(K_0 \cup G_0 \cup D\), where \(D\) is a set of ground unit clauses of the form \(f(g_1, \ldots, g_n) \approx c\), where \(f \in \Sigma_1\), \(c\) is a constant, \(g_1, \ldots, g_n\) are ground terms without function symbols in \(\Sigma_1\), and \(K_0\) and \(G_0\) are clauses without function symbols in \(\Sigma_1\). These flattening and purification transformations preserve both satisfiability and unsatisfiability with respect to total algebras, and also with respect to partial algebras in which all ground subterms which are flattened are defined [SS05].
For the sake of simplicity in what follows we will always flatten and then purify $\mathcal{K}[G] \cup G$. Thus we ensure that $D$ consists of ground unit clauses of the form $f(c_1, \ldots, c_n) \approx c$, where $f \in \Sigma_1$, and $c_1, \ldots, c_n, c$ are constants.

**Lemma 21** ([SS05]) Let $\mathcal{K}$ be a set of clauses and $G$ a set of ground clauses, and let $\mathcal{K}_0 \cup G_0 \cup D$ be obtained from $\mathcal{K}[G] \cup G$ by flattening and purification, as explained above. Assume that $T_0 \subseteq T_0 \cup \mathcal{K}$ is a local theory extension. Then the following are equivalent:

1. $T_0 \cup \mathcal{K}[G] \cup G$ has a partial model in which all terms in $st(\mathcal{K}, G)$ are defined.
2. $T_0 \cup \mathcal{K}_0 \cup G_0 \cup D$ has a partial model with all terms in $st(\mathcal{K}_0, G_0, D)$ defined.
3. $T_0 \cup \mathcal{K}_0 \cup G_0 \cup N_0$ has a (total) model, where

$$N_0 = \{ \bigwedge_{i=1}^n c_i \approx d_i \rightarrow c = d \mid f(c_1, \ldots, c_n) \approx c, f(d_1, \ldots, d_n) \approx d \in D \}.$$ 

### 8.2 Modular reasoning in local combinations of theory extensions

Let $T_1$ and $T_2$ be theories with signatures $\Pi_1 = (\Sigma_1, \text{Pred})$ and $\Pi_2 = (\Sigma_2, \text{Pred})$, and $G$ a set of ground clauses in the joint signature with additional constants $\Pi^c = (\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_c, \text{Pred})$. We want to decide whether $T_1 \cup T_2 \cup G \models \bot$.

The set $G$ of ground clauses can be flattened and purified as explained above. For the sake of simplicity, everywhere in what follows we will assume w.l.o.g. that $G = G_1 \land G_2$, where $G_1, G_2$ are flat and linear sets of clauses in the signatures $\Pi_1, \Pi_2$ respectively, i.e. for $i = 1, 2$, $G_i = G_i^0 \land G_0 \land D_i$, where $G_i^0$ and $G_0$ are clauses in the base theory and $D_i$ a conjunction of unit clauses of the form $f(c_1, \ldots, c_n) = c, f \in \Sigma_i$.

**Corollary 22** Assume that $T_1 = T_0 \cup K_1$ and $T_2 = T_0 \cup K_2$ are local extensions of a theory $T_0$ with signature $\Pi_0 = (\Sigma_0, \text{Pred})$, where $\Sigma_0 = \Sigma_1 \cap \Sigma_2$, and that the extension $T_0 \subseteq T_0 \cup K_1 \cup K_2$ is local. Let $G = G_1 \land G_2$ be a set of flat, linear are purified ground clauses, such that $G_i = G_i^0 \land G_0 \land D_i$ as explained above. Then the following are equivalent:

1. $T_1 \cup T_2 \cup (G_1 \land G_2) \models \bot$,
2. $T_0 \cup (K_1 \cup K_2) \cup (G_1 \land G_2) \cup (G_1^0 \land G_0 \land D_1) \land (G_2^0 \land G_0 \land D_2) \models \bot$,
3. $T_0 \cup K_1 \cup K_2 \cup (G_1 \land G_2) \cup (G_1^0 \land G_0 \land D_1) \land (G_2^0 \land G_0 \land D_2) \models \bot$,
4. $T_0 \cup K_1 \cup K_2 \cup (G_1^0 \cup G_0) \cup (G_2^0 \cup G_0) \cup N_1 \cup N_2 \models \bot$, where

$$N_1 = \{ \bigwedge_{i=1}^n c_i \approx d_i \rightarrow c = d \mid f(c_1, \ldots, c_n) \approx c, f(d_1, \ldots, d_n) \approx d \in D_1 \}$$

$$N_2 = \{ \bigwedge_{i=1}^n c_i \approx d_i \rightarrow c = d \mid f(c_1, \ldots, c_n) \approx c, f(d_1, \ldots, d_n) \approx d \in D_2 \}$$

and $K_i^0$ is the formula obtained from $K_i[G_i]$ after purification and flattening, taking into account the definitions from $D_i$. 

All variables in clauses in \( \text{Iorem 23} \) to construct an interpolant

If the goal is not flattened, then we can flatten and purify it first and use Theorem \[24\] for reasoning in the combined theory one can proceed as follows:

- Purify (and flatten) the goal \( \mathcal{G} \), and thus transform it into an equisatisfiable conjunction \( \mathcal{G}_1 \land \mathcal{G}_2 \), where \( \mathcal{G}_i \) consists of clauses in the signature \( \Pi_i \), for \( i = 1, 2 \), and \( \mathcal{G}_i = \mathcal{G}_i^0 \land \mathcal{G}_0 \land D_i \), as above.

- The formulae containing extension functions in the signature \( \Sigma_i \), \( \mathcal{K}_i[\mathcal{G}_i] \land \mathcal{G}_i \) are “reduced” (using the equivalence of (3) and (6)) to the formula \( \mathcal{K}_i^0 \land \mathcal{G}_i^0 \land \mathcal{G}_0 \land \mathcal{N}_i \) in the base theory.

- The conjunction of all the formulae obtained this way, for all component theories, is used as input for a decision procedure for the base theory.

**Remark 23** Let \( \mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i \) be local extensions for \( i = 1, 2 \). Assume that \( \mathcal{K}_i \) are \( \Sigma_i \)-flat and \( \Sigma_i \)-linear and all variables in clauses in \( \mathcal{K}_i \) occur below a \( \Sigma_i \)-symbol, and that the extension \( \mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \) is local. Let \( \mathcal{G} = \mathcal{G}_1 \land \mathcal{G}_2 \) be as constructed before. Assume that \( \mathcal{T}_0 \cup (\mathcal{K}_1 \land \mathcal{G}_1) \land (\mathcal{K}_2 \land \mathcal{G}_2) \models \bot \). Then we can construct a ground formula \( \mathcal{I} \) which contains only function symbols in \( \Sigma_0 = \Sigma_1 \land \Sigma_2 \) such that

\[
(\mathcal{T}_0 \cup \mathcal{K}_1) \land \mathcal{G}_1 \models \mathcal{I} \quad (\mathcal{T}_0 \cup \mathcal{K}_2) \land \mathcal{G}_2 \land \mathcal{I} \models \bot
\]

**Proof:** We assumed that the goal is flat and linear, i.e. \( \mathcal{G}_i = \mathcal{G}_i^0 \land \mathcal{G}_0 \land D_i \) where \( \mathcal{G}_i^0, \mathcal{G}_0 \) contains only function symbols in \( \Sigma_0 \) and \( D_i \) is a set of definitions of the form \( c \approx f(c_1, \ldots, c_n) \) with \( f \in \Sigma_i \). If \( \mathcal{T}_0 \cup (\mathcal{K}_1 \land \mathcal{G}_1) \land (\mathcal{K}_2 \land \mathcal{G}_2) \models \bot \) then, by Corollary \[24\] (with the notations used there):

\[
\mathcal{T}_0 \cup \mathcal{K}_1^0 \cup \mathcal{K}_2^0 \cup (\mathcal{G}_1^0 \land \mathcal{G}_0) \cup (\mathcal{G}_2^0 \land \mathcal{G}_0) \cup \mathcal{N}_1 \cup \mathcal{N}_2 \models \bot.
\]

Obviously, every model of \( \mathcal{T}_0 \) which satisfies \( \mathcal{K}_1 \land \mathcal{G}_1^0 \land \mathcal{G}_0 \land D_1 \) is also a model of \( \mathcal{T}_0 \cup \mathcal{K}_1^0 \cup \mathcal{G}_1^0 \land \mathcal{G}_0 \cup \mathcal{N}_1 \), and every model of \( \mathcal{T}_0 \) which satisfies \( \mathcal{K}_2 \land \mathcal{G}_2^0 \land \mathcal{G}_0 \land D_2 \) is also a model of \( \mathcal{T}_0 \cup \mathcal{K}_2^0 \cup \mathcal{G}_2^0 \land \mathcal{G}_0 \cup \mathcal{N}_2 \). Let \( \mathcal{I} = \mathcal{K}_1^0 \cup \mathcal{G}_1^0 \cup \mathcal{G}_0 \cup \mathcal{N}_1 \). Then

\[
\mathcal{T}_1 \land \mathcal{G}_1^0 \land \mathcal{G}_0 \land D_1 \models \mathcal{I},
\]

\[
\mathcal{I} \land \mathcal{T}_2 \land \mathcal{G}_2^0 \land \mathcal{G}_0 \land D_2 \models \mathcal{T}_0 \cup (\mathcal{K}_1^0 \cup \mathcal{G}_1^0 \land \mathcal{G}_0 \land \mathcal{N}_1) \cup (\mathcal{K}_2^0 \cup \mathcal{G}_2^0 \land \mathcal{G}_0 \land \mathcal{N}_2) \models \bot.
\]

All variables in clauses in \( \mathcal{K}_i \) occur below a \( \Sigma_i \)-symbol, so \( \mathcal{K}_i[\mathcal{G}_i] \) (hence also \( \mathcal{K}_i^0 \)) is ground for \( i = 1, 2 \), i.e. \( \mathcal{I} \) is quantifier-free.

If the goal is not flattened, then we can flatten and purify it first and use Theorem \[23\] to construct an interpolant \( \mathcal{I} \). We can now construct \( \mathcal{I} \) from \( \mathcal{I}_1 \) by
replacing each constant $c_t$ introduced in the purification process (and therefore contained in a definition $c_t \approx t$ in $D_1 \cup D_2$) with the term $t$. It is easy to see that $I$ satisfies the required conditions. We can, in fact prove that only information over the shared signature (i.e. shared functions and constants) is necessary.

**Theorem 24 (SS06a)** With the notations above, assume that $G_1 \land G_2 \models T_1 \cup T_2 \bot$. Then there exists a ground formula $I$, containing only constants shared by $G_1$ and $G_2$, with $G_1 \models T_1 \cup T_2 \bot$ and $I \land G_2 \models T_1 \cup T_2 \bot$.

9 Conclusions

We presented criteria for recognizing situations when combinations of theory extensions of a base theory are again local extensions of the base theory. We showed, for instance, that if both component theories satisfy the embeddability condition ($\text{Comp}_w$), which guarantees that we can always embed a partial model into one with isomorphic support, then the combinations of the two theories satisfies again condition ($\text{Comp}_w$). The main problem which we needed to overcome when considering more general combinations of local theory extensions was the preservation of truth of clauses when extending partial operations to total operations in a partial algebra. We identified some conditions which guarantee that this is the case. These results allow to recognize wider classes of local theory extensions, and open the way for studying possibilities of modular reasoning in such extensions. From the point of view of modular reasoning in such combinations of local extensions of a base theory, it is interesting to analyze the exact amount of information which needs to be exchanged between provers for the component theories. We showed that if we start with a goal in purified form $G = G_1 \land G_2$, it is sufficient to exchange only ground formulae containing only constants and function symbols common to $G_1 \land T_1$ and $G_2 \land T_2$. We would like to understand whether there are any links between the results described in this paper and other methods for reasoning in combinations of theories over non-disjoint signatures e.g. by Ghilardi [Ghi04].

Acknowledgments. This work was partly supported by the German Research Council (DFG) as part of the Transregional Collaborative Research Center “Automatic Verification and Analysis of Complex Systems” (SFB/TR 14 AVACS). See [www.avacs.org](http://www.avacs.org) for more information.

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