A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

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Introduction

Motivated by basic questions in Minkowski geometry, H. Busemann and C. M. Petty posed ten problems about convex bodies in 1956 (see [BP]). The first problem, now known as the Busemann-Petty problem, states:

If $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and for each hyperplane $H$ through the origin the volumes of their central slices satisfy

$$\text{vol}_{n-1}(K \cap H) < \text{vol}_{n-1}(L \cap H),$$

does it follow that the volumes of the bodies themselves satisfy

$$\text{vol}_n(K) < \text{vol}_n(L)?$$

The problem is trivially positive in $\mathbb{R}^2$. However, a surprising negative answer for $n \geq 12$ was given by Larman and Rogers [LR] in 1975. Subsequently, a series of contributions were made to reduce the dimensions to $n \geq 5$ by a number of authors (see [Ba], [Bo], [G2], [Gi], [Pa], and [Z1]). That is, the problem has a negative answer for $n \geq 5$. See [G3] for a detailed description. It was proved by Gardner [G1] that the problem has a positive answer for $n = 3$. The case of $n = 4$ was considered in [Z1]. But the answer to this case in [Z1] is not correct. This paper presents the correct solution, namely, the Busemann-Petty problem has a positive solution in $\mathbb{R}^4$, which, together with results of other cases, brings the Busemann-Petty problem to a conclusion.

A key step to the solution of the Busemann-Petty problem is the discovery of the relation of the problem to intersection bodies by Lutwak [Lu]. An origin-symmetric convex body $K$ in $\mathbb{R}^n$ is called an intersection body if its radial function $\rho_K$ is the spherical Radon transform of a nonnegative measure $\mu$ on the unit sphere $S^{n-1}$. The value of the radial function of $K$, $\rho_K(u)$, in the direction $u \in S^{n-1}$, is defined as the distance from the center of $K$ to its boundary in that direction. When $\mu$ is a positive continuous function, $K$ is

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called the intersection body of a star body. The notion of intersection body was introduced by Lutwak [Lu] who proved that the Busemann-Petty problem has a positive answer if $K$ is an intersection body in $\mathbb{R}^n$. Based on this relation, a positive answer to the Busemann-Petty problem in $\mathbb{R}^3$ was given by Gardner [G1] who showed that all origin-symmetric convex bodies in $\mathbb{R}^3$ are intersection bodies.

The relation of the Busemann-Petty problem to intersection bodies proved by Lutwak can be formulated as: A negative answer to the Busemann-Petty problem is equivalent to the existence of convex nonintersection bodies (see [G2] and [Z2]). The author attempted in [Z1] to give a negative answer for all dimensions $\geq 4$ by trying to show that cubes in $\mathbb{R}^n (n \geq 4)$ are not intersection bodies (see Theorem 5.3 in [Z1]). However, there is an error in Lemma 5.1 of [Z1]. It affects only Theorems 5.3 and 5.4 there. The correct version of Theorem 5.3 is that no cube in $\mathbb{R}^n (n > 4)$ is an intersection body. This follows immediately from Theorem 6.1 of [Z1] which says that no generalized cylinder in $\mathbb{R}^n (n > 4)$ is an intersection body. Note that the proof of Theorem 6.1 in [Z1] holds for intersection bodies, although the definition of intersection body of a star body was the one used in [Z1]. Therefore, Theorem 5.4 in [Z1] should have stated: The Busemann-Petty problem has a negative solution in $\mathbb{R}^n$ for $n > 4$.

In his important work [K1], Koldobsky applied the Fourier transform to the study of intersection bodies. In [K2], he showed that cubes in $\mathbb{R}^4$ are intersection bodies. It was this result that exposed the error mentioned above and led to the present paper, which presents the correct solution to the Busemann-Petty problem in $\mathbb{R}^4$. One of the key ideas in the proof, previously employed by Gardner [G1], is the use of cylindrical coordinates in computing the inverse spherical Radon transform.

1. The inverse Radon transform on $S^3$ and intersection bodies in $\mathbb{R}^4$

The radial function $\rho_L$ of a star body $L$ is defined by

$$\rho_L(u) = \max\{r \geq 0 : ru \in L\}, \quad u \in S^{n-1}.$$ 

It is required in this paper that the radial function is continuous and even. For basic facts about star bodies and convex bodies, see [G3] and [S].

For a continuous function $f$ on $S^{n-1}$, the spherical Radon transform $Rf$ of $f$ is defined by

$$(Rf)(u) = \int_{S^{n-1} \cap u^\perp} f(v)dv, \quad u \in S^{n-1},$$
where \( u^\perp \) is the \((n-1)\)-dimensional subspace orthogonal to the unit vector \( u \).

Since the spherical Radon transform is self-adjoint, one can define the Radon transform \( \mathbf{R}\mu \) for a measure \( \mu \) on \( S^{n-1} \) by

\[
\langle \mathbf{R}\mu, f \rangle = \langle \mu, \mathbf{R}f \rangle.
\]

The intersection body \( \mathbf{I}\mathbf{L} \) of star body \( \mathbf{L} \) is defined by

\[
\rho_{\mathbf{I}\mathbf{L}}(u) = \text{vol}_{n-1}(\mathbf{L} \cap u^\perp) = \mathbf{R}\left( \frac{1}{n-1}\rho_{\mathbf{L}}^{n-1} \right)(u), \quad u \in S^{n-1}.
\]

An origin-symmetric convex body \( \mathbf{K} \) is called the intersection body of a star body if there exists a star body \( \mathbf{L} \) so that \( \mathbf{K} = \mathbf{I}\mathbf{L} \). That is, the inverse spherical Radon transform \( \mathbf{R}^{-1}\rho_{\mathbf{K}} \) is a positive continuous function. A slight extension of this definition is that an origin-symmetric convex body \( \mathbf{K} \) is called an intersection body if the inverse spherical Radon transform \( \mathbf{R}^{-1}\rho_{\mathbf{K}} \) is a non-negative measure.

Let \( \Delta \) be the Laplacian on the unit sphere \( S^3 \). Helgason’s inversion formula for the Radon transform \( \mathbf{R} \) on \( S^3 \) is (see [H, p. 161])

\[
\frac{1}{16\pi^2}(1 - \Delta)\mathbf{R}\mathbf{R} = 1.
\]

It implies that

\[
\mathbf{R}^{-1}\rho_{\mathbf{K}} = \frac{1}{16\pi^2}\mathbf{R}(1 - \Delta)\rho_{\mathbf{K}}
\]

for an origin-symmetric convex body \( \mathbf{K} \) in \( \mathbb{R}^4 \). This formula shows that \( \mathbf{R}^{-1}\rho_{\mathbf{K}} \) is continuous when \( \rho_{\mathbf{K}} \) is of class \( C^2 \). The following lemma provides an inversion formula which gives the positivity of \( \mathbf{R}^{-1}\rho_{\mathbf{K}} \).

Let \( \mathbf{K} \) be an origin-symmetric convex body in \( \mathbb{R}^4 \), and let \( A_u(z) \) be the volume of \( \mathbf{K} \cap (zu + u^\perp) \), where \( z \) is real and \( u \in S^3 \).

**Lemma 1.** If \( \mathbf{K} \) is an origin-symmetric convex body in \( \mathbb{R}^4 \) whose boundary is of class \( C^2 \), then

\[
\mathbf{R}^{-1}\rho_{\mathbf{K}}(u) = -\frac{1}{16\pi^2}A_u''(0), \quad u \in S^3.
\]

**Proof.** By rotation, it suffices to prove (2) for the north pole of \( S^3 \). From Helgason’s inversion formula (1), the inverse spherical Radon transform of \( \rho_{\mathbf{K}} \), \( f = \mathbf{R}^{-1}\rho_{\mathbf{K}} \), is a continuous function when \( \rho_{\mathbf{K}} \) is of class \( C^2 \). Let

\[
u = u(v, \phi) = (v \sin \phi, \cos \phi), \quad u \in S^3, \quad v \in S^2, \quad 0 \leq \phi \leq \pi,
\]
and let \( \rho_K(v, \phi) = \rho_K(u) \) be the radial function of \( K \). Define

\[
\tilde{\rho}_K(\phi) = \int_{S^2} \rho_K(v, \phi) dv,
\]

\[
\tilde{f}(\phi) = \int_{S^2} f(u) dv,
\]

\[
r(v, \phi) = \rho_K(v, \phi) \sin \phi,
\]

\[
\tilde{r}(\phi) = \tilde{\rho}_K(\phi) \sin \phi.
\]

Consider \( \tilde{\rho}_K \) and \( \tilde{f} \) as functions on \( S^3 \) which are \( \text{SO}(3) \) invariant. Since the spherical Radon transform is intertwining, we have \( \tilde{\rho}_K = R \tilde{f} \) (for a simple proof, see [G3, Th C.2.8]). From this and Lemma 2.1 in [Z1], or Theorem C.2.9 in [G3], we obtain

\[
\tilde{\rho}_K(\phi) = \frac{4\pi}{\sin \phi} \int_{\phi}^{\frac{\pi}{2}} \tilde{f}(\psi) \sin \psi d\psi.
\]

Taking the derivative on both sides of this equation gives

\[
(\tilde{\rho}_K(\phi) \sin \phi)' = 4\pi \tilde{f}(\frac{\pi}{2} - \phi) \sin(\frac{\pi}{2} - \phi).
\]

It follows that

\[
4\pi \tilde{f}(0) = \lim_{\phi \to \frac{\pi}{2}} \frac{(\tilde{\rho}_K(\phi) \sin \phi)'}{\cos \phi} = -\tilde{r}''(\frac{\pi}{2}).
\]

Since \( \frac{1}{4\pi} \tilde{f}(0) \) is the value of \( f \) at the north pole, we obtain

\[
f(u_0) = -\frac{1}{16\pi^2} r''(\frac{\pi}{2}),
\]

where \( u_0 \) is the north pole of \( S^3 \).

Consider the variable \( z \) defined by \( z = \rho_K \cos \phi \). Then \( \tan \phi = \frac{r}{z} \). Differentiating this equation and using \( \frac{1}{\cos^2 \phi} = 1 + \tan^2 \phi = 1 + \frac{r^2}{z^2} \) give

\[
z^2 + r^2 = z \frac{dr}{d\phi} - r \frac{dz}{d\phi}
\]

This yields

\[
\frac{dz}{d\phi} \bigg|_{\phi = \frac{\pi}{2}} = -r(v, \frac{\pi}{2}).
\]

Differentiating (4) gives

\[
2z \frac{dz}{d\phi} + 2r \frac{dr}{d\phi} = z \frac{d^2r}{d\phi^2} - r \frac{d^2z}{d\phi^2}.
\]
From (5),

\[
\frac{dr}{d\phi} \bigg|_{\phi = \frac{\pi}{2}} = \frac{dr}{dz} \bigg|_{\phi = \frac{\pi}{2}} = -r \frac{dr}{dz} \bigg|_{z=0}.
\]

From (6) and (7),

\[
\frac{d^2z}{d\phi^2} \bigg|_{\phi = \frac{\pi}{2}} = 2r \frac{dr}{dz} \bigg|_{z=0}.
\]

From (5), (8), and

\[
\frac{d^2r}{d\phi^2} = \frac{d^2r}{dz^2} \left( \frac{dz}{d\phi} \right)^2 + \frac{dr}{dz} \frac{d^2z}{d\phi^2},
\]

we have

\[
\frac{d^2r}{d\phi^2} \bigg|_{\phi = \frac{\pi}{2}} = \frac{d^2r}{dz^2} \bigg|_{z=0} r(v, \frac{\pi}{2})^2 + 2r(v, \frac{\pi}{2}) \left( \frac{dr}{dz} \right)^2 \bigg|_{z=0}
\]

\[
= \left( r^2 \frac{d^2r}{dz^2} \bigg|_{z=0} \right) + \left( 2r \left( \frac{dr}{dz} \right)^2 \bigg|_{z=0} \right)
\]

\[
= \frac{1}{3} \frac{d^2r}{dz^2} \bigg|_{z=0}.
\]

Integrating both sides of (9) over \(S^2\) with respect to \(v\) gives

\[
\int_{S^2} \frac{d^2r}{d\phi^2}(v, \phi) \bigg|_{\phi = \frac{\pi}{2}} \frac{dv}{2} = \frac{1}{3} \int_{S^2} \frac{d^2r}{dz^2}(v, z) \bigg|_{z=0} dv.
\]

Since \(K\) has \(C^2\) boundary, one can interchange the second order derivative and the integral on each side of the last equation. We obtain

\[
\frac{d^2}{d\phi^2} r''(\phi) \bigg|_{\phi = \frac{\pi}{2}} = \frac{d^2}{dz^2} \left( \frac{1}{3} \int_{S^2} r^3(v, z) dv \right) \bigg|_{z=0}.
\]

Note that the 3-dimensional volume of the intersection of the hyperplane \(x_4 = z\) with the convex body \(K\), denoted by \(A_{u_0}(z)\), is given by

\[
A_{u_0}(z) = \frac{1}{3} \int_{S^2} r^3(v, z) dv.
\]

Therefore, we have

\[
r''(\frac{\pi}{2}) = A''_{u_0}(0).
\]

Formula (2) follows from (3) and (10).
Recently, Gardner, Koldobsky and Schlumprecht [GKS] have generalized the formula (2) to \( n \) dimensions by using techniques of the Fourier transform. A different proof of their formulas is given by Barthe, Fradelizi and Maurey [BFM].

**Theorem 2.** If \( K \) is an origin-symmetric convex body in \( \mathbb{R}^4 \) whose boundary is of class \( C^2 \) and has positive curvature, then \( K \) is an intersection body of a star body.

**Proof.** By the Brunn-Minkowski inequality and the strict convexity of \( K \), \( A(z)^{\frac{1}{3}} \) is strictly concave. When one slices a symmetric convex body by parallel hyperplanes, the central section has maximal volume. Hence, \( A'(0) = 0 \). It follows that

\[
A''(0) = 3A(0)^{\frac{2}{3}}(A(z)^{\frac{1}{3}})''_{z=0} < 0.
\]

By Lemma 1, \( R^{-1}\rho_K \) is a positive continuous function. Therefore, \( K \) is the intersection body of a star body.

When a convex body is identified with its radial function, the class of intersection bodies is closed under the uniform topology. Since every origin-symmetric convex body can be approximated by origin-symmetric convex bodies whose boundaries are of class \( C^2 \) and have positive curvatures, we obtain:

**Theorem 3.** All origin-symmetric convex bodies in \( \mathbb{R}^4 \) are intersection bodies.

Theorem 3 is proved for convex bodies of revolution by Gardner [G2] and by Zhang [Z1], and is proved for cubes and other special cases by Koldobsky [K2]. In higher dimensions, the situation is different. For example, it is proved by Zhang [Z1] that generalized cylinders in \( \mathbb{R}^n \) (\( n > 4 \)) are not intersection bodies, and is proved by Koldobsky [K1] that the unit balls of finite dimensional subspaces of an \( L_p \) space, \( 1 \leq p \leq 2 \), are intersection bodies. In three dimensions, Gardner [G1] proved that all origin-symmetric convex bodies in \( \mathbb{R}^3 \) are intersection bodies. One can also prove this by Theorem 3 and a result of Fallert, Goodey and Weil [FGW] which says that central sections of intersection bodies are again intersection bodies. An intersection body may not be the intersection body of a star body. It is shown by Zhang [Z4] that no polytope in \( \mathbb{R}^n \) (\( n > 3 \)) is an intersection body of a star body. Campi [C] is able to prove a complete result which says that no polytope in \( \mathbb{R}^n \) (\( n > 2 \)) is an intersection body of a star body.
2. A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

The following relation of the Busemann-Petty problem to intersection bodies was proved by Lutwak [Lu].

**Theorem 4 (Lutwak).** The Busemann-Petty problem has a positive solution if the convex body with smaller cross sections is an intersection body.

From Theorems 3 and 4, we conclude:

**Theorem 5.** The Busemann-Petty problem in $\mathbb{R}^4$ has a positive solution.

From Theorem 3 and Corollary 2.19 in [Z2], we have the following corollary about the maximal cross section of a convex body.

**Corollary 6.** If $K$ is an origin-symmetric convex body in $\mathbb{R}^4$, then

$$\text{vol}_4(K)^{\frac{3}{4}} \leq \frac{3}{8}(\sqrt{2\pi})^{\frac{3}{2}} \max_{u \in S^3} \text{vol}_3(K \cap u^\perp)$$

with equality if and only if $K$ is a ball.

Inequality (11) implies that, in $\mathbb{R}^4$, balls attain the minmax of the volume of central hyperplane sections of origin-symmetric convex bodies with fixed volume. The corresponding inequality in $\mathbb{R}^3$ to inequality (11) was proved by Gardner (see [G3, Th. 9.4.11]). However, it is no longer the case in higher dimensions at least for $n \geq 7$. Ball [Ba] showed that cubes are counterexamples for $n \geq 10$. Giannopoulos [Gi] showed that certain cylinders are counterexamples for $n \geq 7$. The following question, known as the slicing problem, has been of interest (see [MP] for details):

Does there exist a positive constant $c$ independent of the dimension $n$ so that

$$\text{vol}_n(K)^{\frac{n-1}{n}} \leq c \max_{u \in S^{n-1}} \text{vol}_{n-1}(K \cap u^\perp)$$

for every origin-symmetric convex body $K$ in $\mathbb{R}^n$?

3. The generalized Busemann-Petty problem

Besides considering hyperplane sections, one can also consider intermediate sections of convex bodies. For a fixed integer $1 < i < n$, the Busemann-Petty problem has the following generalization (see Problem 8.2 in [G3]):

If $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and for every $i$-dimensional subspace $H$ the volumes of sections satisfy

$$\text{vol}_i(K \cap H) < \text{vol}_i(L \cap H),$$
does it follow that the volumes of the bodies themselves satisfy
\[ \operatorname{vol}_n(K) < \operatorname{vol}_n(L) \, ? \]

When \( i = n - 1 \), this is the Busemann-Petty problem. It turns out that the solution to the generalized Busemann-Petty problem depends strongly on the dimension \( i \) of the sections of convex bodies. It is proved by Bourgain and Zhang [BoZ] that the solution is negative when \( 3 < i < n \). The generalized Busemann-Petty problem has a positive solution when \( K \) belongs to a certain class of convex bodies, called \( i \)-intersection bodies, which contains all intersection bodies (see Theorem 5 in [Z3] and Lemma 6.1 in [GrZ]). In particular, when \( K \) is an intersection body, the generalized Busemann-Petty problem has a positive solution. From this fact and Theorem 3, we have:

**Theorem 7.** The generalized Busemann-Petty problem in \( \mathbb{R}^4 \) has a positive solution.

It might be still true that the generalized Busemann-Petty problem has a positive solution when \( i = 2, 3, \) and \( n \geq 5 \). This remains open.

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