Lemniscates as Trajectories of Quadratic Differentials (I)

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Abstract

In this note, we study polynomial and rational lemniscates as trajectories of related quadratic differentials. Many classic results can be then proved easily...

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1 A quadratic differential

Given a rational function \( r(z) = \frac{p(z)}{q(z)} \), where \( p(z) \) and \( q(z) \) are two co-prime complex polynomials, we consider the quadratic differential on the Riemann sphere \( \hat{\mathbb{C}} \):

\[
\varpi_r(z) = -\left( \frac{r'(z)}{r(z)} \right)^2 dz^2 = -\left( \frac{p'(z) q(z) - p(z) q'(z)}{p(z) q(z)} \right)^2 dz^2. \tag{1}
\]

Finite critical points and infinite critical points of \( \varpi_r \) are respectively its zero’s and poles; all other points of \( \hat{\mathbb{C}} \) are called regular points of \( \varpi_r \).

It is obvious that the partial fraction decomposition of \( \frac{r'(z)}{r(z)} \) is as follows:

\[
\frac{r'(z)}{r(z)} = \sum_{\substack{a \in \mathbb{C} \setminus \{0\} \setminus \{a \} \cap \{a \}}} \frac{m_a}{z - a}, \tag{2}
\]
where \( m_a \in \mathbb{N}^* \) is the multiplicity of the zero \( a \) of \( p(z)q(z) \). We deduce that
\[
\varpi_r(z) = -\frac{m_a^2}{(z-a)^2} (1 + O(z-a)) \, dz^2, \quad z \to a.
\]
In other words, the zero’s of \( p \) and \( q \) are poles of order 2 of \( \varpi_r \) with negative residue.

If
\[
\deg (p'q - pq') = \deg (pq) - 1,
\]
(in particular, if \( \deg (p) \neq \deg (q) \)), then, with the parametrization \( u = 1/z \), we get
\[
\varpi_r(u) = -\left( \frac{(\deg (p) - \deg (q))^2}{u^2} \right) (1 + O(u)) \, du^2, \quad u \to 0;
\]
thus, \( \infty \) is another double pole of \( \varpi_r \) with negative residue. If
\[
\deg (p'q - pq') < \deg (pq) - 2,
\]
then \( \infty \) is zero of \( \varpi_r \) with multiplicity greater than 1. In the case
\[
\deg (p'q - pq') = \deg (pq) - 2,
\]
\( \infty \) is a regular point.

*Horizontal trajectories* (or just trajectories) of the quadratic differential \( \varpi_r \) are the zero loci of the equation
\[
\varpi_r(z) > 0,
\]
or equivalently
\[
\Re \int^z \frac{r'(t)}{r(t)} \, dt = \log |r(z)| = \text{const}.
\tag{3}
\]
If \( z(t), t \in \mathbb{R} \) is a horizontal trajectory, then the function
\[
t \mapsto \Im \int_0^t \frac{r'(z(u))}{r(z(u))} \, z'(u) \, du = \arg (r(z(t))) - \arg (r(z(0)))
\]
is monotone.

The *vertical* (or, *orthogonal*) trajectories are obtained by replacing \( \Im \) by \( \Re \) in equation (3). The horizontal and vertical trajectories of the quadratic differential \( \varpi_r \) produce two pairwise orthogonal foliations of the Riemann sphere \( \hat{\mathbb{C}} \).
A trajectory passing through a critical point of $\varpi_r$ is called critical trajectory. In particular, if it starts and ends at a finite critical point, it is called finite critical trajectory, otherwise, we call it an infinite critical trajectory. If two different trajectories are not disjoint, then their intersection must be a zero of the quadratic differential.

The closure of the set of finite and infinite critical trajectories is called the critical graph of $\varpi_r$, we denote it by $\Gamma_r$.

The local and global structures of the trajectories is well known (more details about the theory of quadratic differentials can be found in [5], [3], or [6]), in particular:

• At any regular point, horizontal (resp. vertical) trajectories look locally as simple analytic arcs passing through this point, and through every regular point of $\varpi_p$ passes a uniquely determined horizontal (resp. vertical) trajectory of $\varpi_p$; these horizontal and vertical trajectories are locally orthogonal at this point.

• From each zero with multiplicity $m$ of $\varpi_r$, there emanate $m + 2$ critical trajectories spacing under equal angle $2\pi/(m + 2)$.

• Any double pole has a neighborhood such that, all trajectories inside it take a loop-shape encircling the pole or a radial form diverging to the pole, respectively if the residue is negative or positive.

• A trajectory in the large can be, either a closed curve not passing through any critical point (closed trajectory), or an arc connecting two critical points, or an arc that has no limit along at least one of its directions (recurrent trajectory).

The set $\hat{C} \setminus \Gamma_r$ consists of a finite number of domains called the domain configurations of $\varpi_r$. For a general quadratic differential on a $\hat{C}$, there are five kind of domain configuration, see [3, Theorem 3.5]. Since all the infinite critical points of $\varpi_r$ are poles of order 2 with negative residues, then there are two possible domain configurations:

• the Circle domain: It is swept by closed trajectories and contains exactly one double pole. Its boundary is a closed critical trajectory;

• the Dense domain: It is swept by recurrent critical trajectory i.e., the interior of its closure is non-empty. Jenkins Three-pole Theorem (see [5, Theorem 15.2]) asserts that a quadratic differential on the Riemann
sphere with at most three poles cannot have recurrent trajectories. In general, the non-existence of such trajectories is not guaranteed, but here, following the idea of *level function* of Baryshnikov and Shapiro (see [1]), the quadratic differential \( \varpi_r \) excludes the dense domain, as we will see in Proposition 5.

A very helpful tool that will be used in our investigation is the Teichmüller lemma (see [5, Theorem 14.1]).

**Definition 1** A domain in \( \widehat{\mathbb{C}} \) bounded only by segments of horizontal and/or vertical trajectories of \( \varpi_r \) (and their endpoints) is called \( \varpi_r \)-polygon.

**Lemma 2** (Teichmüller) Let \( \Omega \) be a \( \varpi_r \)-polygon, and let \( z_j \) be the critical points on the boundary \( \partial \Omega \) of \( \Omega \), and let \( t_j \) be the corresponding interior angles with vertices at \( z_j \), respectively. Then

\[
\sum \left( 1 - \frac{(m_j + 2)t_j}{2\pi} \right) = 2 + \sum n_i, \tag{4}
\]

where \( m_j \) are the multiplicities of \( z_j \), and \( n_i \) are the multiplicities of critical points of \( \varpi_r \) inside \( \Omega \).

## 2 Lemniscates

We use the notations of [2]. Let us denote \( n = \deg r = \max (\deg p, \deg q) > 0 \). For \( c > 0 \), the set

\[
\Gamma_{r,c} = \{ z \in \mathbb{C} : |r(z)| = c \} \tag{5}
\]

is called rational lemniscate of degree \( n \). It is a real algebraic curve of degree \( 2n \); indeed, its defining equation can be seen as

\[
p(x, y)p(x, y) - c^2 q(x, y)q(x, y) = 0,
\]

with \( z = x + iy \). For more details, see [4]. From the point of view of the theory of quadratic differentials, each connected component of the lemniscate \( \Gamma_{r,c} \) coincides with a horizontal trajectory of \( \varpi_r = - \left( \frac{r'(z)}{r(z)} \right)^2 \, dz^2 \), as we have seen in equation \( \text{[3]} \). The lemniscate \( \Gamma_{r,c} \) is entirely determined by the knowledge of the critical graph \( \Gamma_r \) (which is the union of the lemniscates \( \Gamma_{r,|r(a)|} \), for
all zero’s \( a \) of \( \varpi_r \) of the quadratic differential of \( \varpi_r \). In particular, if we denote by \( n_z \) and \( n_p \) respectively the number of zero’s and poles \( r(z) \) in \( \hat{\mathbb{C}} \), then, from the local behavior of the trajectories, we see that, for \( c \to 0^+ \), the lemniscate \( \Gamma_{r,c} \) is formed by exactly \( n_z \) disjoint closed curves each of them encircles a zero of \( r(z) \), while for \( c \to +\infty \), \( \Gamma_{r,c} \) is formed by exactly \( n_p \) disjoint closed curves each of them encircles a pole of \( r(z) \). If \( \deg (p'q - pq') < \deg (pq) - 2 \), then, \( \infty \) is a zero of \( \varpi_r \) of multiplicity \( m \geq 2 \), and there are \( m + 2 \) critical trajectories emerging from \( \infty \) dividing in a symmetric way the complement of some zero centred ball into \( m + 2 \) connected components. See Figure 1. Notice that since the structure of the critical graph of a quadratic differential is invariant by a Möbius Transform, we can suppose (which we will assume in the rest of this note) that \( \infty \) is a double pole, i.e., \( \deg (p'q - pq') = \deg (pq) - 1 \).

![Figure 1: Critical graphs of \( \varpi_r \), \( r = \frac{z^2-1}{z^2+1} \) (left), and \( r = \frac{z^2-4}{z^2+1} \) (right).](image)

**Historical Example 3** Astronomer Giovanni Domenico Cassini was the first who met lemniscates (1680) when he was studying the relative motions of the earth and the sun. He investigated the set of points in the plane such that the product of the distances to two fixed points is constant. This set (known as Cassini oval) is the special case of lemniscate when \( p(z) = z^2 - 1 \) and \( q = 1 \), (the two fixed points here are \( \pm 1 \)). It is more known by the cartesian equation

\[
(x^2 + y^2)^2 - 2(x^2 - y^2) + 1 = c.
\]

The associated quadratic differential is \( \varpi_p = -\left( \frac{z}{z^2-1} \right)^2 dz^2 \); it has 3 double poles: \(-1, 1, \infty\) and the origin as a double zero. An immediate investigation gives the following structure of the lemniscates:
• If $c = 1$, $\Gamma_{p,1}$ is the critical graph of $\varpi_p$. It is formed by two closed curves intersecting at the origin, and encircling respectively $\pm 1$. $\Gamma_{p,1}$ is also known as Bernouilli’s lemniscate;

• if $c < 1$, $\Gamma_{p,c}$ is formed by two disjoined closed curves encircling respectively $\pm 1$.

• if $c > 1$, $\Gamma_{p,c}$ is a closed curve encircling $\Gamma_{p,1}$. See Figure 2.

More general, the lemniscates of the polynomials $z^m - 1, m > 2$ can be illustrated easily via the quadratic differentials $\left(\frac{z^{m-1}}{z^m - 1}\right)^2 dz^2$. See Figure 3.

Definition 4 A quadratic differential on $\hat{\mathbb{C}}$ is called Strebel if the complement to the union of its closed trajectories has vanishing area.

Proposition 5 The quadratic differential $\varpi_r$ is Strebel.

Proof. Since the critical points of $\varpi_r$ are only zero’s and double poles with negative residues, it is sufficient to prove that $\varpi_r$ has no recurrent trajectory. The function

$$f : \mathbb{C} \setminus \{\text{poles of } r(z)\} \rightarrow \mathbb{R} \quad z \mapsto |r(z)|$$
is continuous, and constant on each horizontal trajectory of $\varpi_r$. If $\varpi_r$ has a recurrent trajectory, then, its domain configuration contains a dense domain $D$. Thus, the function $f$ must be constant on $D$, which is clearly impossible by analyticity of the rational function $z \mapsto r(z)$. ■

A necessary condition for the existence of a finite critical trajectory connecting two finite critical points of $\varpi_r$ is the existence of a Jordan arc $\gamma$ connecting them, such that

$$
\Re \int_{\gamma} \frac{r'(t)}{r(t)} dt = 0.
$$

Unfortunately, this condition is not sufficient in general, as it can be shown easily for the case of $r(z) = (z^2 - 1) (z^2 - 4)$; see Figure 4. However, a more sufficient condition will be shown by the following Proposition

**Proposition 6** If $|w_i| = |w_j| = \max \{|w_k|; k = 1, \ldots, n - 1\}$ for some $1 \leq i < j \leq n - 1$, then, there exists a finite critical trajectory joining $z_i$ and $z_j$. In particular, the critical graph $\Gamma_r$ is connected, if and only if $|w_1| = \cdots = |w_{n-1}|$.

**Proof.** If no finite critical trajectory joins $z_i$ and $z_j$, then a lemniscate $\Gamma_{r,c}$, for some $c > |w_i|$, is not connected : $\Gamma_{r,c}$ is a disjoint union of $m \geq 2$ loops $L_1, \ldots, L_p$, each of them encircles a part of the critical graph $\Gamma_r$. Looking at each of these loops as a $\varpi_r$-polygon and applying Lemma \[2\] we get :

$$
0 = 2 + \sum n_k, k = 1, \ldots, m.
$$

Figure 4: Critical graph of $\varpi_r$, $r = (z^2 - 1) (z^2 - 4)$.
Making the sum of all equalities in (7), and taking into account our assumption that \((\deg (p'q - pq') = \deg (pq)) - 1\), we get

\[ 0 = 2m + 2 \left( \deg (p'q - pq') - \deg (pq) \right) = 2m - 2; \]

a contradiction. The second point is a mere consequence. Observe that the complement of the interior of the critical graph \(\Gamma_r\) is a doubly-connected domain in \(\mathbb{C}\).

Here following a brief mention of the case of polynomial lemniscates \(\Gamma_{p,1}\).

Let us denote by

\[ \Omega_- := \{ z \in \mathbb{C} : |p(z)| < 1 \}, \]
\[ \Omega_+ := \{ z \in \mathbb{C} : |p(z)| > 1 \}. \]

The maximum modulus theorem asserts that \(\Omega_+\) is a connected open subset containing a neighborhood of \(\infty\) in \(\hat{\mathbb{C}}\). We shall say that \(\Gamma_{p,1}\) is a proper lemniscate of degree \(n\) if \(\Gamma_{p,1}\) is smooth \((p'(z) \neq 0\) on \(\Gamma_{p,1}\)) and is connected. Note that the interior \(\Omega_-\) of a proper lemniscate of degree \(n\) (or, for a general smooth lemniscate, each component of ) is also simply connected, since its complement is connected.

Let \(z_1, ..., z_s, s \leq n - 1\) be the zero’s (repeated according to their multiplicity) of \(\varpi_p\). The non-vanishing critical values for \(p(z)\) are the values \(w_1 = p(z_1), ..., w_s = p(z_s)\). For a smooth lemniscate \(\Gamma_{p,1}\) of degree \(n\), the following characterizes the property of being proper through the critical values:

**Proposition 7** Assume that the lemniscate \(\Gamma_{p,1}\) is smooth. Then, \(\Gamma_{p,1}\) is proper if and only if all the critical values \(w_1, ..., w_s\) satisfy \(|w_k| < 1\).

**Proof.** Proof of this Proposition can be found in [2]. We provide here a more evident proof relying on quadratic differentials theory. The smoothness of \(\Gamma_{p,1}\) implies that it is not a critical trajectory. Suppose that \(|w_k| > 1\) for some \(k \in \{1, ..., s\}\), and consider two critical trajectories emerging from \(z_k\) that form a loop \(\gamma\). This loop cannot intersect \(\Gamma_{p,1}\), and \(\gamma \cap \Omega_- \neq \emptyset\) since \(\gamma\) contains a pole in its interior; a contradiction. The other point is clear.

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