COMPLETENESS FOR RESTALL’S LOGIC AND AN APPLICATION TO TRUTH THEORY

BEN MIDDLETON
UNIVERSITY OF NOTRE DAME
BMIDDLET@ND.EDU

Abstract. I provide a sound and complete natural deduction system for a logic first defined by Restall [6]. I show that adding a transparent truth predicate to the system does not introduce paradoxes. The proof of non-paradoxicality uses a construction recently introduced by Field, Lederman and Øgaard [1] to generate a class of semantically closed models for Restall’s logic.

1. Introduction

Constant domain basic first-order logic \( (\text{BQL}_\text{CD}) \) is the constant domain first-order extension of Visser’s basic propositional logic [7]. The Kripke models for \( \text{BQL}_\text{CD} \) treat \( \rightarrow \) as a strict conditional but do not require that accessibility is reflexive. Consequently, modus ponens is invalid in \( \text{BQL}_\text{CD} \). Restall [6] defined an extension \( \text{BQL}_\text{CD}^R \) of \( \text{BQL}_\text{CD} \) in which modus ponens is valid by only allowing reflexive worlds to serve as counterexamples to logical consequence. However, Restall was unable to prove completeness for \( \text{BQL}_\text{CD}^R \). In this paper, I provide a natural deduction system \( \mathcal{N}\text{BQL}_\text{CD}^R \) for \( \text{BQL}_\text{CD}^R \) and prove soundness and completeness. I then show that adding the rules

\[
[ t = t ] \quad \frac{ t_1 = t_2 \quad \phi(t_1) }{ \phi(t_2) } \quad [ t_1 = t_2 \lor ( t_1 = t_2 \rightarrow \bot) ] \quad \frac{ \Gamma \phi \equiv \Gamma \psi \equiv }{ \bot } \phi \neq \psi
\]

\[
\frac{ \phi }{ T^\tau \phi \land } \quad \frac{ T^\tau \phi \land }{ \phi }\]

to \( \mathcal{N}\text{BQL}_\text{CD}^R \) results in a logic of truth \( \text{BQL}_\text{CD}^{RT} \) which is transparent, in the sense that \( \phi \) and \( T^\tau \phi \land \) are intersubstitutable everywhere in \( \text{BQL}_\text{CD}^{RT} \), and non-paradoxical, in
the sense that if \((\Gamma, \phi)\) is valid in \(BQL_{CP}^{RT}\) and neither \(\rightarrow\) nor \(T\) occurs in \(\Gamma \cup \{\phi\}\) then every classical model of \(\Gamma\) which interprets \(=\) and \(\lceil\phi\rceil\) correctly is also a model of \(\phi\). The conservativity proof uses a construction recently introduced by Field, Lederman and Øgaard [1] to generate a suitable class of semantically closed \(BQL_{CD}\) models.

2. Restall’s Logic

2.1. Model Theory. Let \(L\) be a first-order language with primitive operators \(\top, \bot, \land, \lor, \rightarrow, \forall, \exists\). The definition of an \(L\)-model \(M\) is given in [4]. We write \(\Gamma \models \phi\) iff for every \(L\)-model \(M\) and every world \(w \in M\): \(w \models \Gamma\) only if \(w \models \phi\). Constant domain basic first-order logic (\(BQL_{CD}\)) is the logic defined by \(\models\). A world \(w \in M\) is called reflexive if \(w < w\). We write \(\Gamma \models_R \phi\) iff for every \(L\)-model \(M\) and every reflexive world \(w \in M\): \(w \models \Gamma\) only if \(w \models \phi\). Restall’s logic (\(BQL_{CD}^R\)) is the logic defined by \(\models_R\). There are two notable differences between \(BQL_{CD}\) and \(BQL_{CD}^R\): (1) unlike in \(BQL_{CD}\), modus ponens is valid in \(BQL_{CD}^R\) and (2) unlike in \(BQL_{CD}^R\), conditional proof is valid in \(BQL_{CD}\). Conditional proof fails in \(BQL_{CD}^R\) due to the fact that a reflexive world may see an irreflexive world. Thus, even though \(\phi \land (\phi \rightarrow \psi) \models_R \psi\), \(\not\models_R \phi \land (\phi \rightarrow \psi) \rightarrow \psi\).

2.2. Proof Theory. A natural deduction system \(\mathcal{N}BQL_{CD}\) for \(BQL_{CD}\) is given in [4]. In order to use methods and results from [4], we suppose \(L\) is countable and work in the extension \(L^+\) of \(L\) obtained by adding \(\omega\)-many new constant symbols \(\{a_i\}_{i \in \omega}\) to \(L\) (\(a_i\) is treated in proofs as the name of an arbitrarily chosen object). Let \(\Pi\) be a proof-tree with root labelled \(\psi\). Suppose \(\Pi\) has a leaf labelled by an undischarged occurrence \(\phi^i\) of \(\phi\). We say \(\phi^i\) is unsafe iff \(\phi^i\) occupies the following position in \(\Pi\):

\[
\begin{array}{c}
\vdots \\
\phi^i \\
\vdots \\
\alpha \rightarrow \beta \\
\vdots \\
\beta \\
\vdots \\
\psi \\
\vdots
\end{array}
\]

The natural deduction system \(\mathcal{N}BQL_{CD}^R\) for \(BQL_{CD}^R\) consists of all trees of (possibly discharged) \(L^+\)-sentences constructed in accordance with the following inference
rules, where no unsafe occurrence of an open assumption may be discharged:

\[
\begin{array}{c}
\frac{\top}{\phi} \quad (\bot\text{-Elim}) \\
\frac{\phi \land \psi}{\phi \land \psi} \quad (\land\text{-Elim}) \\
\frac{[\phi]}{\phi \lor \psi} \quad (\lor\text{-Elim}) \\
\frac{[\phi]}{\phi \lor \psi} \quad (\lor\text{-Elim}) \\
\frac{\phi \lor \psi}{\phi \lor \psi} \quad (\lor\text{-Elim}) \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \quad (\rightarrow\text{-Elim}) \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \quad (\rightarrow\text{-Elim}) \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \quad (\rightarrow\text{-Elim}) \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \quad (\rightarrow\text{-Elim}) \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \quad (\rightarrow\text{-Elim}) \\
\end{array}
\]

In \(\forall\text{-Int}\), \(a_i\) does not occur in \(\phi\) or in any open assumption in the main subproof. In \(\exists\text{-Elim}\), \(a_i\) does not occur in \(\phi, \psi\) or in any open assumption besides \(\phi(a_i)\) in the right main subproof. We write \(\Gamma \vdash \phi\) iff there exists a proof of \(\phi\) from \(\Gamma\) in \(\mathcal{N}BQL_{\text{CD}}^R\). \(\mathcal{N}BQL_{\text{CD}}\) is the system which results from removing \(\rightarrow\text{-Elim}\) from \(\mathcal{N}BQL_{\text{CD}}^R\). We write \(\Gamma \vdash \phi\) iff there exists a proof of \(\phi\) from \(\Gamma\) in \(\mathcal{N}BQL_{\text{CD}}\). Since \(\mathcal{N}BQL_{\text{CD}}\) does not contain \(\rightarrow\text{-Elim}\), proofs in \(\mathcal{N}BQL_{\text{CD}}\) do not contain unsafe occurrences of open assumptions. Consequently, the discharge rules are unrestricted in \(\mathcal{N}BQL_{\text{CD}}\). The strategy for proving that \(\vdash_{\mathcal{R}}\) is sound and complete with respect to \(\models_{\mathcal{R}}\) is to first

\footnote{By stipulating that only \textit{sentences} are linked by inference rules, we determine which free variables, if any, a subformula may contain (e.g. \(\phi\) may not contain free variables in Internal \(\forall\text{-Int}\)).}
“reduce” proofs in $\mathcal{N}BQL^R_{CD}$ to proofs in $\mathcal{N}BQL_{CD}$ and then exploit the fact, proved in [4], that $\vdash$ is sound and complete with respect to $|=$. We first list some facts about $\vdash$ needed in the reduction proof.

**Lemma 1 (Distribution).** $\phi \land (\psi \lor \chi) \vdash (\phi \land \psi) \lor (\phi \land \chi)$.

**Lemma 2 (Infinite Distribution).** $\phi \land \exists v \psi \vdash \exists v (\phi \land \psi)$.

**Lemma 3 ($\land$-Release).** $\phi \land \psi \rightarrow \chi \vdash \phi \rightarrow (\psi \rightarrow \chi)$.

Let $\square \phi$ abbreviate $\top \rightarrow \phi$.

**Lemma 4 ($\forall$-Embedding).** $\forall v \square^n \phi \vdash \square^n \forall v \phi$.

**Proof.** By induction on $n$. The base case $n = 0$ is trivial. For the induction step we have

\[
\begin{align*}
\forall v \square^{n+1} \phi & \quad \text{(induction hyp.)} \\
\top \rightarrow \forall v \square^n \phi & \quad \text{(Internal $\forall$-Int)} \\
\square^n \forall v \phi & \quad \text{($\forall$-Ints)} \\
\square^{n+1} \forall v \phi & \quad \text{($\forall$-Ints)}
\end{align*}
\]

$\square^n \forall v \phi$.

**Lemma 5 ($\square \land$-Int).** $\square^n \phi_1, ..., \square^n \phi_m \vdash \square^n \bigwedge_i \phi_i$.

**Proof.** By induction on $n$. The base case $n = 0$ is trivial. For the induction step we have

\[
\begin{align*}
\bigwedge_i \square^n \phi_i & \quad \text{($\land$-Elims)} \\
\square^n \phi_1 & \quad \text{($\land$-Elims)} \\
\square^n \phi_m & \quad \text{($\land$-Elims)} \\
\square^{n+1} \phi_1, ..., \square^{n+1} \phi_m & \quad \text{(induction hyp.)} \\
\bigwedge_i \square^n \phi_i & \quad \text{($\land$-Ints)} \\
\bigwedge_i \square^n \phi_i \rightarrow \square^n \bigwedge_i \phi_i & \quad \text{($\land$-Ints)} \\
\square^{n+1} \bigwedge_i \phi_i & \quad \text{(induction hyp.)}
\end{align*}
\]

$\square^{n+1} \bigwedge_i \phi_i$.

**Lemma 6 ($\square \land$-Elim).** $\square^n \bigwedge_i \phi_i \vdash \square^n \phi_j$. 

Proof. By induction on \( n \). The base case \( n = 0 \) is trivial. For the induction step we have

\[
\frac{\Box^n \bigwedge_i \phi_i \quad \Box^n \phi_j}{\Box^{n+1} \phi_j} \quad \text{(Internal Transitivity)}
\]

(\( \Box \wedge \)-Elim)

\[
\frac{\Box^n \bigwedge_i \phi_i \quad \Box^n \phi_j}{\Box^n \psi} \quad \text{(Internal Transitivity)}
\]

Lemma 7 (Boxing). If \( \phi_1, \ldots, \phi_m \vdash \psi \) then \( \Box^n \phi_1, \ldots, \Box^n \phi_m \vdash \Box^n \psi \).

Proof. Suppose \( \phi_1, \ldots, \phi_m \vdash \psi \). We prove the lemma by induction on \( n \). The base case \( n = 0 \) is trivial. For the induction step we have

\[
\frac{\Box^n \bigwedge_i \phi_i \quad \Box^n \bigwedge_i \phi_i \rightarrow \Box^n \phi_j}{\Box^{n+1} \psi} \quad \text{(Internal Transitivity)}
\]

Lemma 8 (Partial Reduction). If \( \Pi \in \mathcal{NBQL}_{CD}(\Sigma[\cdot]) \) has open assumptions in \( \Sigma \) then \( \Pi \in \mathcal{NBQL}_{CD}(\Sigma[n]) \) for some \( n \).

For sentences \( \Sigma \subseteq \mathcal{L}^+ \), let \( \mathcal{NBQL}_{CD}(\Sigma[\cdot]) = \mathcal{NBQL}_{CD} \) and, for \( n \geq 0 \), let \( \mathcal{NBQL}_{CD}(\Sigma[n]) \) denote the natural deduction system obtained by adding the rule

\[
\frac{\Sigma}{\phi \rightarrow \psi}
\]

\( \mathcal{NBQL}_{CD}(\Sigma[\cdot] - 1) \) to \( \mathcal{NBQL}_{CD} \) and keeping the discharge rules unrestricted. An easy induction on \( n \) verifies that \( \mathcal{NBQL}_{CD}(\Sigma[n]) \subseteq \mathcal{NBQL}_{CD}(\Sigma[n+1]) \) for all \( n \). We write \( \Gamma \vdash_{\Sigma[n]} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{NBQL}_{CD}(\Sigma[n]) \).

Lemma 8 (Partial Reduction). If \( \Pi \in \mathcal{NBQL}_{CD}^R \) has open assumptions in \( \Sigma \) then \( \Pi \in \mathcal{NBQL}_{CD}(\Sigma[n]) \) for some \( n \).
Proof. By induction on the construction of proofs in $\mathcal{NBQL}_{CD}$. The base case is easy. The inductions steps are also easy except for $\rightarrow$-Int, $\vee$-Elim and $\exists$-Elim. We do $\exists$-Elim. $\rightarrow$-Int and $\vee$-elim are similar.

$\exists$-Elim Let $\Pi$ be a proof of the form

$$
\begin{array}{c}
\Sigma \\
\vdots \\
\vdots \\
\exists v \phi \\
\psi
\end{array}
$$

in $\mathcal{NBQL}_{CD}$, with open assumptions in $\Sigma$. Let $\Pi_L$ denote the left main subproof and $\Pi_R$ denote the right main subproof. By the induction hypothesis, there exist $n, m$ such that $\Pi_L \in \mathcal{NBQL}_{CD}(\Sigma[\{n\}])$ and $\Pi_R \in \mathcal{NBQL}_{CD}(\Sigma \cup \{\phi(a_i)\}[m])$. There are two cases to consider.

Case 1 $\phi(a_i) \in \Sigma$. Then $\Pi \in \mathcal{NBQL}_{CD}(\Sigma[\{m\}])$.

Case 2 $\phi(a_i) \notin \Sigma$. Then $\phi(a_i)$ never occurred unsafely in the construction of $\Pi_R$. So $\Pi_R \in \mathcal{NBQL}_{CD}(\Sigma[n])$. Hence $\Pi \in \mathcal{NBQL}_{CD}(\Sigma[\{m\}])$. $\square$

To state the next lemma concisely, let $\bigwedge \emptyset = \top$.

Lemma 9 (Relative Deduction). For $\Sigma' \subseteq \Sigma$, $|\Gamma| < \omega$, $n \geq 0$: if $\Sigma', \Gamma \vdash_{\Sigma[n]} \phi$ then $\Sigma' \vdash \square^n(\bigwedge \Gamma \rightarrow \phi)$.

Proof. Suppose the lemma holds for all $m < n$. We prove by induction on the construction of proofs in $\mathcal{NBQL}_{CD}(\Sigma[n])$ that the lemma holds for $n$.

Base Cases Suppose we have a one-line proof in $\mathcal{NBQL}_{CD}(\Sigma[n])$ of $\phi$ from $\Sigma' \cup \Gamma$, where $\Sigma' \subseteq \Sigma$. There are three cases to consider.

Case 1 $\phi = \top$. Then

$$
\begin{array}{c}
[\phi] \\
\vdash_{\text{Ints}} \\
\square^n(\bigwedge \Gamma \rightarrow \phi)
\end{array}
$$

is a proof of $\square^n(\bigwedge \Gamma \rightarrow \phi)$ from $\Sigma'$ in $\mathcal{NBQL}_{CD}$.

Case 2 $\phi \in \Sigma'$. Then

$$
\begin{array}{c}
\phi \\
\vdash_{\text{Ints}} \\
\square^n(\bigwedge \Gamma \rightarrow \phi)
\end{array}
$$

is a proof of $\square^n(\bigwedge \Gamma \rightarrow \phi)$ from $\Sigma'$ in $\mathcal{NBQL}_{CD}$.
Case 3  $\phi \in \Gamma$. Then

\[
\begin{array}{c}
\frac{[\land \Gamma]}{\land\text{-Elims}} \\
\phi \\
\frac{\land \Gamma \rightarrow \phi}{\rightarrow\text{-Ints}} \\
\square^n(\land \Gamma \rightarrow \phi)
\end{array}
\]

is a proof of $\square^n(\land \Gamma \rightarrow \phi)$ from $\Sigma'$ in $\mathcal{NBQL}_{CD}$.

**Induction Steps** There are seven cases to consider.

**Case 1** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma', \Gamma \\
\vdots \\
\hat{\alpha} \\
\phi
\end{array}
\]

in $\mathcal{NBQL}_{CD}(\Sigma[n])$, where $\Sigma' \subseteq \Sigma$ and the final inference is $\bot\text{-Elim}$, $\land\text{-Elim}$, $\lor\text{-Int}$, Internal $\forall\text{-Int}$, Internal $\exists\text{-Elim}$, $\forall\text{-Elim}$, CD or $\exists\text{-Int}$. Then, by the induction hypothesis and Boxing applied to Internal Transitivity, we can find a proof of the form

\[
\begin{array}{c}
\Sigma' \\
\vdots \\
\hat{\alpha} \\
\phi
\end{array}
\]

in $\mathcal{NBQL}_{CD}$.

**Case 2** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma', \Gamma \\
\vdots \\
\hat{\alpha} \\
\beta \\
\phi
\end{array}
\]

in $\mathcal{NBQL}_{CD}(\Sigma[n])$, where $\Sigma' \subseteq \Sigma$ and the final inference is $\land\text{-Int}$, Internal Transitivity, Internal $\land\text{-Int}$ or Internal $\lor\text{-Elim}$. Then, by the induction hypothesis and Boxing
applied to Internal ∧-Int and Internal Transitivity, we can find a proof of the form

\[
\begin{array}{c}
\Sigma' \\
\vdots \\
\Box^n(\bigwedge \Gamma \rightarrow \alpha) \\
\vdots \\
\Box^n(\bigwedge \Gamma \rightarrow \alpha \land \beta) \\
\vdots \\
\Box^n(\bigwedge \Gamma \rightarrow \phi)
\end{array}
\]

in \( \mathcal{N} \mathcal{B} \mathcal{Q} \mathcal{L}_{\text{CD}} \).

**Case 3** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma', \Gamma \\
\vdots \\
\phi \lor \psi \\
\vdots \\
\chi \\
\vdots \\
\chi
\end{array}
\]

in \( \mathcal{N} \mathcal{B} \mathcal{Q} \mathcal{L}_{\text{CD}}(\Sigma[n]) \), where \( \Sigma' \subseteq \Sigma \). There are two subcases.

**Subcase 1** \( \Gamma = \emptyset \). Then, by the induction hypothesis and Boxing applied to Internal \( \lor \)-Elim and Internal Transitivity, we can find a proof of the form

\[
\begin{array}{c}
\Sigma' \\
\vdots \\
\Box^n(\phi \rightarrow \chi) \\
\vdots \\
\Box^n(\phi \lor \psi \rightarrow \chi) \\
\vdots \\
\Box^n(\top \rightarrow \chi)
\end{array}
\]

in \( \mathcal{N} \mathcal{B} \mathcal{Q} \mathcal{L}_{\text{CD}} \).
Subcase 2 \( \Gamma \neq \emptyset \). Then, by Distribution, the induction hypothesis and Boxing applied to Internal \( \lor \) -Elim and Internal Transitivity, we can find a proof of the form
\[
\Sigma' \quad \Sigma'
\]
\[
\Box^n(\bigwedge \Gamma \land \phi \to \chi) \quad \Box^n(\bigwedge \Gamma \land \psi \to \chi)
\]
\[
\Box^n(\bigwedge \Gamma \land (\phi \lor \psi) \to (\bigwedge \Gamma \land \phi) \lor (\bigwedge \Gamma \land \psi)) \quad \Box^n((\bigwedge \Gamma \land \phi) \lor (\bigwedge \Gamma \land \psi) \to \chi)
\]
\[
\Box^n(\bigwedge \Gamma \land (\phi \lor \psi) \to \chi)
\]
in \( \land \mathbf{BQL}_{CD} \). So, by the induction hypothesis and Boxing applied to Internal \( \land \) -Int and Internal Transitivity, we can find a proof of the form
\[
\Sigma' \quad \Sigma'
\]
\[
\Box^n(\bigwedge \Gamma \to \bigwedge \Gamma) \quad \Box^n(\bigwedge \Gamma \to \phi \lor \psi)
\]
\[
\Box^n(\bigwedge \Gamma \to \bigwedge \Gamma \land (\phi \lor \psi)) \quad \Box^n(\bigwedge \Gamma \land (\phi \lor \psi) \to \chi)
\]
\[
\Box^n(\bigwedge \Gamma \to \chi)
\]
in \( \land \mathbf{BQL}_{CD} \).

Case 4 Suppose we have a proof of the form
\[
\Sigma', \Gamma, [\phi] \quad \psi \quad \phi \to \psi
\]
in \( \land \mathbf{BQL}_{CD}(\Sigma[n]) \), where \( \Sigma' \subseteq \Sigma \). There are two subcases.

Subcase 1 \( \Gamma = \emptyset \). Then, by the induction hypothesis, we can find a proof of the form
\[
\Sigma' \quad \Sigma'
\]
\[
\Box^n(\phi \to \psi)
\]
\[
\Box^n(\top \to (\phi \to \psi)) \quad (\to \text{-Int})
\]
in \( \land \mathbf{BQL}_{CD} \).
Subcase 2 $\Gamma \neq \emptyset$. Then, by the induction hypothesis and Boxing applied to $\land$-Release, we can find a proof of the form

$$\Sigma'$$

$$\Box^n(\land \Gamma \land \phi \rightarrow \psi)$$

$$\Box^n(\land \Gamma \rightarrow (\phi \rightarrow \psi))$$

in $\mathcal{N}^\mathsf{BQL}_{CD}$.

Case 5 Suppose we have a proof of the form

$$\Sigma', \Gamma \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}(\Sigma[n-1])}^\mathcal{N}^\mathsf{BQL}_{CD}$$

$$\phi \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}(\Sigma[n])}^\mathcal{N}^\mathsf{BQL}_{CD}$$

in $\mathcal{N}^\mathsf{BQL}_{CD}(\Sigma[n])$, where $\Sigma' \subseteq \Sigma$. There are two subcases to consider.

Subcase 1 $n = 0$. Then, by the induction hypothesis, we can find a proof of the form

$$\Sigma', \Gamma \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}}^\mathcal{N}^\mathsf{BQL}_{CD}$$

$$\land \Gamma \rightarrow \phi \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}}^\mathcal{N}^\mathsf{BQL}_{CD}$$

in $\mathcal{N}^\mathsf{BQL}_{CD}$.

Subcase 2 $n > 0$. Then, by the outer induction hypothesis, the induction hypothesis and Boxing applied to Internal Transitivity, we can find a proof of the form

$$\Sigma', \Gamma \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}}^\mathcal{N}^\mathsf{BQL}_{CD}$$

$$\square^n(\land \Gamma \rightarrow \phi) \vdash_{\mathcal{N}^\mathsf{BQL}_{CD}}^\mathcal{N}^\mathsf{BQL}_{CD}$$

in $\mathcal{N}^\mathsf{BQL}_{CD}$. 
**Case 6** Suppose we have a proof of the form

\[
\Sigma', \Gamma \quad \phi(a_i) \quad \forall v \phi
\]

in \( \mathbf{N'BQL}_{\text{CD}}(\Sigma[n]) \), where \( \Sigma' \subseteq \Sigma \). Let \( \Sigma^* \subseteq \Sigma', \Gamma^* \subseteq \Gamma \) contain exactly the open assumptions in the main subproof. Then \( a_i \) does not occur in \( \Sigma^* \cup \Gamma^* \cup \{\phi\} \). So, by the induction hypothesis, \( \forall \)-Embedding and Boxing applied to Internal \( \forall \)-Int and Internal Transitivity, we can find a proof of the form

\[
\begin{align*}
\Sigma^* \\
\Box^n(\wedge \Gamma^* \rightarrow \phi(a_i)) \\
\forall v \Box^n(\wedge \Gamma^* \rightarrow \phi) \\
\Box^n \forall v(\wedge \Gamma^* \rightarrow \phi) \\
\Box^n(\wedge \Gamma \rightarrow \wedge \Gamma^*) \\
\Box^n(\wedge \Gamma^* \rightarrow \forall v \phi) \\
\Box^n(\wedge \Gamma \rightarrow \forall v \phi)
\end{align*}
\]

in \( \mathbf{N'BQL}_{\text{CD}} \).

**Case 7** Suppose we have a proof of the form

\[
\Sigma', \Gamma \quad \Sigma', [\phi(a_i)] \\
\exists v \phi \quad \psi
\]

in \( \mathbf{N'BQL}_{\text{CD}}(\Sigma[n]) \), where \( \Sigma' \subseteq \Sigma \). Let \( \Sigma^* \subseteq \Sigma', \Gamma^* \subseteq \Gamma \) contain exactly the open assumptions in the right main subproof besides \( \phi(a_i) \). Then \( a_i \) does not occur in \( \Sigma^* \cup \Gamma^* \cup \{\phi, \psi\} \). There are two subcases.
Subcase 1. \( \Gamma = \emptyset \). Then, by the induction hypothesis, \( \forall \)-Embedding and Boxing applied to Internal \( \exists \)-Elim and Internal Transitivity, we can find a proof of the form

\[
\begin{align*}
\Sigma^* \\
\Box^n (\phi(a_i) \rightarrow \psi) \\
\forall v \Box^n (\phi \rightarrow \psi)
\end{align*}
\]

\[
\begin{align*}
\Sigma' \\
\Box^n (\exists v \phi) \\
\Box^n (\forall v (\phi \rightarrow \psi))
\end{align*}
\]

in \( \mathcal{N}BQL_{\text{CD}} \).

Subcase 2. \( \Gamma \neq \emptyset \). Then, by the induction hypothesis, Infinite Distribution, \( \forall \)-Embedding and Boxing applied to Internal \( \exists \)-Elim and Internal Transitivity, we can find a proof of the form

\[
\begin{align*}
\Sigma^* \\
\Box^n (\bigwedge^* \land \phi(a_i) \rightarrow \psi) \\
\forall v \Box^n (\bigwedge^* \land \phi \rightarrow \psi) \\
\Box^n (\forall v (\bigwedge^* \land \phi) \rightarrow \psi) \\
\Box^n (\exists v (\bigwedge^* \land \exists v \phi) \rightarrow \psi)
\end{align*}
\]

in \( \mathcal{N}BQL_{\text{CD}} \). So, by Boxing applied to Internal Transitivity, we can find a proof of the form

\[
\begin{align*}
\Sigma^* \\
\Box^n (\bigwedge \Gamma \land \exists v \phi \rightarrow \bigwedge^* \land \exists v \phi) \\
\Box^n (\bigwedge^* \land \exists v \phi \rightarrow \psi)
\end{align*}
\]
in $\mathcal{N}\text{BQL}_{\text{CD}}$. Hence, by the induction hypothesis and Boxing applied to Internal ∀-Int and Internal Transitivity, we can find a proof of the form

$$
\Sigma' \\vdash_n (\bigwedge \Gamma \rightarrow \bigwedge \Gamma) \quad \Sigma' \vdash_n (\bigwedge \Gamma \rightarrow \exists v \phi) \\
\ldots \quad \Sigma' \vdash_n (\bigwedge \Gamma \rightarrow \bigwedge \Gamma \land \exists v \phi) \quad \Sigma' \vdash_n (\bigwedge \Gamma \land \exists v \phi \rightarrow \psi) \\
\ldots \quad \Sigma' \vdash_n (\bigwedge \Gamma \rightarrow \psi)
$$

in $\mathcal{N}\text{BQL}_{\text{CD}}$. □

**Lemma 10** (Reduction). If $\Gamma \vdash_R \phi$ then $\Gamma \vdash \Box^n \phi$ for some $n$.

*Proof.* Suppose $\Gamma \vdash_R \phi$. By Partial Reduction, $\Gamma \vdash_{[n]} \phi$ for some $n$. If $n = -1$ then $\Gamma \vdash \phi$ and we're done. Suppose $n \geq 0$. Then, by Relative Deduction, $\Gamma \vdash \Box^{n+1} \phi$. □

So any occurrences of $\rightarrow$-Elim in a $\mathcal{N}\text{BQL}_{\text{CD}}^R$-proof can be “pushed down” to the bottom of the proof (indeed, the proof of Relative Deduction provides us with an algorithm for doing this).

3. **Soundness**

**Theorem 1** (Soundness). If $\Gamma \vdash_R \phi$ then $\Gamma \models_R \phi$.

*Proof.* Suppose $\Gamma \vdash_R \phi$. By Reduction, $\Gamma \vdash \Box^n \phi$ for some $n$. Then, since $\vdash$ is sound with respect to $\models [4]$, $\Gamma \models \Box^n \phi$. So for every reflexive $w \in \mathfrak{M}$: $w \models \Gamma$ only if $w \models \phi$. Hence $\Gamma \models_R \phi$. □

4. **Completeness**

The canonical model for $\text{BQL}_{\text{CD}}^R$ is identical to the canonical model $\mathcal{C}$ for $\text{BQL}_{\text{CD}}$ (see [4] for the definition of $\mathcal{C}$). $\text{Sat}(\text{BQL}_{\text{CD}}^R)$, the set of prime saturated $\text{BQL}_{\text{CD}}^R$-theories, is defined analogously to $\text{Sat}(\text{BQL})$ (see [4] again).

**Lemma 11** (Extension). For $\Gamma$ such that $|\{i : a_i \not\in \Gamma\}| = \omega$: if $\Gamma \not\vdash_R \phi$ then there exists $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \text{Sat}(\text{BQL}_{\text{CD}}^R)$ and $\phi \not\in \Gamma^*$. 
Proof. Similar to the proof of the Belnap Extension Lemma (see e.g Priest [5] §6.2) except we draw witnesses from \( \{a_i\}_{i \in \omega} \) (the assumption that \(|\{i : a_i \not\in \Gamma\}| = \omega\) ensures we never run out of witnesses) and require three additional steps due to the restriction on discharges.

**Subclaim 1.** Suppose \( \Gamma, \exists v \phi, \phi(a_i) \vdash_R \psi \), where \( a_i \) does not occur in \( \Gamma \cup \{\phi, \psi\} \). Then \( \Gamma, \exists v \phi \vdash_R \psi \).

**Proof.** By Reduction: \( \Gamma, \exists v \phi, \phi(a_i) \vdash \Box^n \psi \) for some \( n \). So we can construct the following proof in \( \mathcal{N}\text{BQL}^R_{CD} \):

\[
\begin{array}{c}
\Gamma, \exists v \phi, [\phi(a_i)] \\
\mathcal{N}\text{BQL}^R_{CD} \\
\exists v \phi \\
\Box^n \psi \\
\to\text{-Elims} \\
\psi
\end{array}
\]

**Subclaim 2.** Suppose \( \Gamma \vdash_R \psi \vee \phi \) and \( \Gamma, \phi \vdash_R \psi \). Then \( \Gamma \vdash_R \psi \).

**Proof.** By Reduction: \( \Gamma, \phi \vdash \Box^n \psi \) for some \( n \). So we can construct the following proof in \( \mathcal{N}\text{BQL}^R_{CD} \):

\[
\begin{array}{c}
\Gamma \\
\psi \vee \phi \\
\to\text{-Ints} \\
\Box^n \psi \\
\to\text{-Elims} \\
\psi
\end{array}
\]

**Subclaim 3.** Suppose \( \Gamma, \forall v \phi \vdash_R \psi \) and \( \Gamma \vdash_R (\psi \vee \forall v \phi) \vee \phi(a_i) \), where \( a_i \) does not occur in \( \Gamma \cup \{\phi, \psi\} \). Then \( \Gamma \vdash_R \psi \).
Proof. By Reduction: $\Gamma, \forall v \phi \vdash \Box^n \psi$ for some $n$. So we can construct the following proof in $\mathcal{N}BQL_{CD}^R$:

$$
\begin{array}{c}
\Gamma \\
(\psi \vee \forall v \phi) \vee \phi(a_i) \\
\forall v((\psi \vee \forall v \phi) \vee \phi) \\
(\psi \vee \forall v \phi) \vee \forall v \phi \\
\psi \vee \forall v \phi \\
\end{array}
\Rightarrow
\begin{array}{c}
[\psi] \\
\Gamma, [\forall v \phi] \\
\mathcal{N}BQL_{CD} \\
\square^m \psi \\
\square^n \psi \\
\psi
\end{array}
$$

\hfill \Box

Lemma 12 ($\mathcal{L}$-Completeness). For $\Gamma \subseteq \mathcal{L}$: if $\Gamma \models \mathcal{L} \phi$ then $\Gamma \vdash \mathcal{L} \phi$.

Proof. Suppose $\Gamma \not\models \mathcal{L} \phi$. Since $\{i : a_i \in \Gamma\} = \emptyset$, Extension gives us $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \text{Sat}(\mathcal{BQL}_{CD}^R)$ and $\phi \notin \Gamma^*$. Since $\Gamma^*$ is closed under modus ponens, we have by the definition of $\prec$ in $\mathcal{C}$ that $\Gamma^* \prec \Gamma^\star$. Also, by Truth [4]: $\mathcal{C}, \Gamma^\star \models \Gamma$ and $\mathcal{C}, \Gamma^\star \not\models \phi$. So restricting $\mathcal{C}$ to $\mathcal{L}$ gives $\Gamma \not\models \mathcal{L} \phi$. \hfill \Box

Theorem 2 ($\mathcal{L}^+\text{-Completeness}$). If $\Gamma \models \mathcal{L}^+ \phi$ then $\Gamma \vdash \mathcal{L}^+ \phi$.

Proof. Suppose $\Gamma \models \mathcal{L}^+ \phi$. By $\mathcal{L}$-completeness, if we had started with $\mathcal{L}^+$ as our base language then we could have constructed a proof of $\phi$ from some finite $\Gamma_0 \subseteq \Gamma$ in the version of $\mathcal{N}BQL_{CD}^R$ obtained by adding $\omega$-many fresh constant symbols $\{b_i\}_{i \in \omega}$ to $\mathcal{L}^+$. But then, by soundness, $\Gamma_0 \models \mathcal{L}^+ \phi$. Since $|\{i : a_i \notin \Gamma_0\}| = \omega$, a similar argument to $\mathcal{L}$-completeness gives $\Gamma_0 \vdash \mathcal{L}^+ \phi$. \hfill \Box

5. Disjunction and Existence Properties

We can use the canonical model to show that, like $\mathcal{BQL}_{CD}$, $\mathcal{BQL}_{CD}^R$ satisfies the disjunction and existence properties over the base language $\mathcal{L}$.

Theorem 3 (Disjunction Property). For $\Gamma \subseteq \mathcal{L} \setminus \{\rightarrow, \vee, \exists\}$: if $\models \mathcal{L} \phi \vee \psi$ then $\models \mathcal{L} \phi$ or $\models \mathcal{L} \psi$. 

Proof. Similar to the proof of the corresponding theorem in [4]. □

**Theorem 4** (Existence Property). Suppose \( \mathcal{L} \) contains at least one constant symbol. Then, for \( \Gamma \subseteq \mathcal{L} \setminus \{\rightarrow, \vee, \exists\} \): \( \Gamma \models \exists \forall \phi \) only if \( \Gamma \models \phi(t) \) for some \( t \in \mathcal{L} \).

Proof. Similar to the proof of the corresponding theorem in [4]. □

6. Classical Logic

In this section we show that we obtain classical logic by adding the rule of excluded middle

\[
[\phi \vee (\phi \rightarrow \bot)] \quad (X)
\]

to \( \mathcal{N}BQL^{RX}_{CD} \). Call the resulting system \( \mathcal{N}BQL^{RX}_{CD} \) and write \( \Gamma \vdash_{RX} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{N}BQL^{RX}_{CD} \). Let \( \mathcal{N}BQL^{X}_{CD} \) denote the natural deduction system obtained by removing \( \rightarrow \)-Elim from \( \mathcal{N}BQL^{RX}_{CD} \).

**Lemma 13** (Reduction). If \( \Gamma \vdash_{RX} \phi \) then \( \Gamma \vdash_{X} \Box^n \phi \) for some \( n \).

Proof. Similar to the proof of the corresponding lemma for \( \vdash_{R} \). □

**Lemma 14** (Box Elimination). \( \phi \vee (\phi \rightarrow \bot), \Box^n \bot \vee (\Box^n \bot \rightarrow \bot) \vdash_{R} \Box^n \phi \rightarrow \phi \).

Proof. Let \( w \in \mathfrak{M} \) be reflexive. Suppose for a reductio that \( w \Vdash \phi \vee (\phi \rightarrow \bot) \), \( w \Vdash \Box^n \bot \vee (\Box^n \bot \rightarrow \bot) \) and \( w \nVdash \Box^n \phi \rightarrow \phi \). Then there exists \( u > w \) such that \( u \Vdash \Box^n \phi \) and \( u \nVdash \phi \). By Persistence [4], \( u \Vdash \phi \vee (\phi \rightarrow \bot) \). So \( u \Vdash \phi \rightarrow \bot \). But then \( u \Vdash \Box^n \bot \). So \( w \nVdash \Box^n \bot \rightarrow \bot \). Hence \( w \Vdash \Box^n \bot \), which contradicts the fact that \( w \prec w \). □

We write \( \Gamma \vdash_{C} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in the natural deduction system \( \mathcal{N}CQL \) for classical logic (the system obtained by adding the rule of excluded middle to the intuitionistic introduction and elimination rules).

**Theorem 5** (Classical Collapse). \( \Gamma \vdash_{RX} \phi \) iff \( \Gamma \vdash_{C} \phi \).

Proof. \( \implies \) An easy induction on the construction of proofs in \( \mathcal{N}BQL^{RX}_{CD} \).

\( \impliedby \) By induction on the construction of proofs in \( \mathcal{N}CQL \). The base cases are easy. The induction steps are also easy except for \( \vee \)-Elim, \( \rightarrow \)-Int and \( \exists \)-Elim. We do \( \rightarrow \)-Int. \( \vee \)-Elim and \( \exists \)-Elim are similar.
Suppose we have a proof of the form
\[
\frac{\Gamma, [\phi] \quad \psi}{\phi \to \psi}
\]
in $N^\text{CQL}$. By the induction hypothesis: $\Gamma, \phi \vdash_{RX} \psi$. So, by Reduction: $\Gamma, \phi \vdash_X \Box^n \psi$ for some $n$. But then, by Box Elimination and completeness, we can find a proof of the form
\[
\frac{\Gamma, [\phi] \quad \psi}{\phi \to \Box^n \psi}
\]
in $N^\text{BQL}_{\text{CD}}^{RX}$.

7. Truth

Suppose $L$ contains the identity predicate $\equiv$, the truth predicate $T$ and a designated constant symbol $\Downarrow \phi$ for every $L$-sentence $\phi$. $|=\text{RT}$ is the consequence relation obtained by modifying the definition of $|=R$ to range over all and only those $L^+$-models $\mathfrak{M}$ such that (i) for sentences $\phi, \psi \in L$: if $\phi \neq \psi$ then $|\Downarrow \phi| \neq |\Downarrow \psi|$, (ii) $|=(w) = \{\langle a, a \rangle : a \in \text{dom}(\mathfrak{M})\}$ and (iii) for every sentence $\phi \in L$: $|\Downarrow \phi| \in |T|(w)$ iff $\mathfrak{M}, w \models \phi$. We refer to the logic defined by $|=\text{RT}$ as $\text{BQL}^{RT}_{\text{CD}}$. In this section I prove that $\text{BQL}^{RT}_{\text{CD}}$ is completely axiomatized by the natural deduction system $N^\text{BQL}^{RT}_{\text{CD}}$ obtained by adding the rules
\[
[t = t] \quad t_1 = t_2 \quad \phi(t_1) \quad \phi(t_2) \quad \phi(t_2)
\]

---

Two points of clarification. (1) We do not require that (i) and (iii) hold for all $L^+$-sentences because (a) we do not require that $L$ contains names for all $L^+$-sentences and (b) more importantly, $\forall$-Int and $\exists$-Elim would become unsound if we insisted that (iii) — and hence the rules $T$-Int and $T$-Elim — apply to sentences containing occurrences of $a_i$ (for example, $T^+ F(a_i)$ would prove $\forall x F(x)$ and $\exists x F(x)$ would prove $T^+ F(a_i)$). (2) It would be preferable to only allow objects denoted by names for $L$-sentences into $|T|(w)$, since really we are thinking of $T$ as a truth predicate for $L$. Unfortunately, as pointed out by Kremer [2], this would result in non-compactness, since $\{c = \Downarrow \phi \to \bot : \phi \in L\}$ would entail $T(c) \to \bot$ even though no finite subset of $\{c = \Downarrow \phi \to \bot : \phi \in L\}$ would entail $T(c) \to \bot$. 

\[\square\]
\[
[t_1 = t_2 \lor (t_1 = t_2 \rightarrow \bot)] \quad \frac{\phi \land \psi \rightarrow \bot}{\bot} \quad (=-X) \\
\frac{\phi}{T \phi \land} (T\text{-Int}) \\
\frac{T \phi \land}{\phi} (T\text{-Elim})
\]
to \( \mathcal{NBQL}_{CD}^R \), where the sentences \( \phi, \psi \) in \( S \), \( T\text{-Int} \) and \( T\text{-Elim} \) belong to \( \mathcal{L} \). So, in particular, \( \mathcal{BQL}_{CD}^{RT} \) does not satisfy the disjunction property. We write \( \Gamma \vdash_{RT} \phi \) if there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{NBQL}_{CD}^{RT} \). We write \( \Gamma \vdash_{RT, \Sigma} \phi \) if there exists a proof of \( \phi \) from \( \Gamma \cup \Sigma \) in \( \mathcal{NBQL}_{CD}^{RT} \) whose open assumptions with unsafe occurrences all belong to \( \Sigma \).

**Lemma 15** (Substitution). If \( \phi \vdash_{RT, \Gamma} \psi \) then \( \chi[\phi] \vdash_{RT, \Gamma} \chi[\psi] \).

*Proof.* An easy induction on the construction of \( \chi[p] \). \( \square \)

**Theorem 6** (Transparency). \( \chi[\phi] \vdash_{RT} \chi[T \phi] \)

*Proof.* Immediate from \( T\text{-Int} \), \( T\text{-Elim} \) and Substitution. \( \square \)

### 7.1. Expressive Functions of \( T \)

In ordinary English, “true” is often used to indirectly express disjunctions and conjunctions. For example, “something Shirley said last night is true” is used to express “\( \phi_1 \) or...or \( \phi_n \)”, where \( \phi_1, ..., \phi_n \) are the sentences Shirley said last night. Similarly, “everything Shirley said last night is true” is used to express “\( \phi_1 \) and...and \( \phi_n \)”. The following results demonstrate that \( T \) almost fulfills these functions in \( \mathcal{BQL}_{CD}^{RT} \).

**Lemma 16** (Weak \( \exists / \lor \)). \( \exists v (\theta(v) \land T(v)) \vdash_{RT, \forall v(\theta(v) \leftrightarrow \lor_i v \phi_i)} \lor_i \phi_i \)

*Proof.* The left-right direction:

\[
\frac{\theta(a_k) \land T(a_k)}{\theta(a_k)} \quad \frac{\forall v(\theta(v) \leftrightarrow \lor_i v = \phi_i)}{\lor_i a_k = \phi_i} \\
\frac{\theta(a_k) \rightarrow \lor_i a_k = \phi_i}{\lor_i \phi_i} \quad \frac{T \lor_i \phi_i}{\lor_i \phi_i}
\]

\( \exists v(\theta(v) \land T(v)) \)

\( \lor_i \phi_i \)
The right-left direction:

\[
\begin{align*}
\left[ \neg \phi_j \rightarrow \phi_j \right] & \quad \forall v(\theta(v) \leftrightarrow \bigvee_i v = \phi_i) \\
\vdots & \\
\bigvee_i \neg \phi_j = \phi_i & \quad (\bigvee_i \neg \phi_j = \phi_i) \rightarrow \theta^\neg \phi_j \equiv \neg \phi_i \\
\theta^\neg \phi_j \equiv \neg \phi_i & \quad \exists v(\theta(v) \land T(v)) \\

\bigvee_i \phi_i & \\
\exists v(\theta(v) \land T(v)) & \\
\end{align*}
\]

\[\begin{array}{c}
\square
\end{array}\]
So we have:
\[
\forall v (\theta(v) \leftrightarrow \bigvee_i \phi_i) \quad \top \rightarrow \bigwedge_i \phi_i
\]
\[
\vdash \theta(a_k) \rightarrow \bigvee_i \phi_i \quad \theta(a_k) \rightarrow \bigwedge_i \phi_i
\]
\[
\frac{\vdash \theta(a_k) \rightarrow \bigvee_i \phi_i \land \bigwedge_i \phi_i}{\vdash \theta(a_k) \rightarrow \top \rightarrow \bigwedge_i \phi_i}
\]
\[
\frac{\vdash \theta(a_k) \rightarrow \bigvee_i \phi_i \land \bigwedge_i \phi_i}{\vdash \forall v (\theta(v) \rightarrow T(v))}
\]

**Theorem 8** \((\forall/\land)\). \(\chi[\forall v (\theta(v) \rightarrow T(v))] \vdash_{\text{RT}} \forall v (\theta(v) \leftrightarrow \bigvee_i \phi_i) \chi[\top \rightarrow \bigwedge_i \phi_i].\)

**Proof.** Immediate from Weak \(\forall/\land\) and Substitution. \(\square\)

Note that by Weak \(\forall/\land\) we have:

(i) \(\forall v (\theta(v) \rightarrow T(v)), \forall v (\theta(v) \leftrightarrow \bigvee_i \phi_i) \vdash_{\text{RT}} \bigwedge_i \phi_i\)

(ii) \(\bigwedge_i \phi_i, \forall v (\theta(v) \leftrightarrow \bigvee_i \phi_i) \vdash_{\text{RT}} \forall v (\theta(v) \rightarrow T(v))\).

So there is a weaker sense in which “everything Shirley said last night is true” expresses “\(\phi_1\) and...and \(\phi_n\)” in \(\text{BQL}_{CD}^{\text{RT}}\).

### 7.2. Soundness

\(\mathcal{N}\text{BQL}_{CD}^T\) is the natural deduction system which results from removing \(\rightarrow\)-Elim from \(\mathcal{N}\text{BQL}_{CD}^{\text{RT}}\). We write \(\Gamma \vdash_T \phi\) iff there exists a proof of \(\phi\) from \(\Gamma\) in \(\mathcal{N}\text{BQL}_{CD}^T\). \(\models_T\) is the consequence relation obtained by modifying the definition of \(\models\) to range over the same class of models as \(\models_{\text{RT}}\).

**Lemma 18** \((\vdash_T\text{-Soundness})\). If \(\Gamma \vdash_T \phi\) then \(\Gamma \models_T \phi\).

**Proof.** Similar to the proof in [4] that \(\vdash\) is sound with respect to \(\models\) (in particular, the new inference rules are clearly sound with respect to \(\models_T\)). \(\square\)

**Theorem 9** \((\text{Soundness})\). If \(\Gamma \vdash_{\text{RT}} \phi\) then \(\Gamma \models_{\text{RT}} \phi\).

**Proof.** Similar to the proof that \(\vdash_R\) is sound with respect to \(\models_R\) (in particular, the proof of Relative Deduction for \(\vdash_{\text{RT}}\) introduces no new cases). \(\square\)
7.3. Completeness. Sat(BQL\(^T_{CD}\)), the set of prime saturated BQL\(^T_{CD}\)-theories, is defined analogously to Sat(BQL\(_{CD}\)) (see [4]). For \(\Gamma, \Sigma \in \text{Sat}(\text{BQL}_{CD})\), we let \(\Gamma \prec \Sigma\) iff for all \(\phi, \psi\) defined analogously to \(\text{Sat}\)\(\SIM\) = \(-\text{X}, \text{t}\)\(\text{Lemma 19}\) (Identity Invariance) for all \(\phi, \psi\) defined analogously to \(\text{Sat}\), then \(\psi \in \Sigma\). Let \([t]_\Gamma = \{t' \in L^+: t = t' \in \Gamma\}\).

**Lemma 19** (Identity Invariance). For \(\Gamma, \Sigma \in \text{Sat}(\text{BQL}_{CD})\) if \(\Gamma \prec \Sigma\) then \([t]_\Gamma = [t]_\Sigma\).

*Proof.* Suppose \(\Gamma \prec \Sigma\).

First suppose \(t' \in [t]_\Gamma\). Then \(t = t' \in \Gamma\). By Subset [4], \(\Gamma \subseteq \Sigma\). So \(t = t' \in \Sigma\). Hence, \(t' \in [t]_\Sigma\).

Conversely, suppose \(t' \in [t]_\Sigma\). Then \(t = t' \in \Sigma\). Since \(\bot \notin \Sigma\), \(t = t' \rightarrow \bot \notin \Gamma\). By \(-\text{X}, t = t' \lor (t = t' \rightarrow \bot) \in \Gamma\). So \(t = t' \in \Gamma\). Hence \(t' \in [t]_\Gamma\). \(\square\)

For \(\Gamma \in \text{Sat}(\text{BQL}_{CD})\), let \(\text{Sat}(\text{BQL}_{CD})[\Gamma] = \{\Sigma \in \text{Sat}(\text{BQL}_{CD}) : \Gamma \prec \Sigma\}\). Suppose \(\text{Sat}(\text{BQL}_{CD})[\Gamma] \neq \emptyset\). Then the canonical model \(\mathcal{C}[\Gamma]\) generated by \(\Gamma\) is the \(L^+:\) model

\[
\langle \text{Sat}(\text{BQL}_{CD})[\Gamma], \prec \cap \text{Sat}(\text{BQL}_{CD})[\Gamma], \{[t]_r : t \in L^+\}, \cdot | r \rangle
\]

where

\[
|c|_r = [c]_\Gamma
\]

\[
|f^n|_r([t_1]_\Gamma, \ldots, [t_n]_\Gamma) = [f^n(t_1, \ldots, t_n)]_\Gamma
\]

\[
|R^n|_r(\Sigma) = \{([t_1]_\Gamma, \ldots, [t_n]_\Gamma) : R^n(t_1, \ldots, t_n) \in \Sigma\}.
\]

\(-\text{Int}, \text{=}-\text{Elim} and Identity Invariance ensure that \(\mathcal{C}[\Gamma]\) is well-defined. Furthermore, by Subset [4], if \(\Sigma \prec \Delta\) then \(\Sigma \subseteq \Delta\). Hence \(|R^n|_r(\Sigma) \subseteq |R^n|_r(\Delta)|\) for all \(\Sigma, \Delta \in \text{Sat}(\text{BQL}_{CD})[\Gamma]\) such that \(\Sigma \prec \Delta\). So \(\mathcal{C}[\Gamma]\) is in fact an \(L^+:\) model.

We next verify that \(\Gamma \phi = \text{and} T\) behaves correctly in \(\mathcal{C}[\Gamma]\). The inference rule \(S\) ensures that \(|\Gamma \phi|_r \neq |\Gamma \psi|_r|\) for \(L\)-sentences \(\phi \neq \psi\). An easy induction on the construction of \(L^+:\)formulas gives \(|t|_r = [t]_r\). So we have

\[
([t_1]_r, [t_2]_r) \in |\Sigma| \iff t_1 = t_2 \in \Sigma
\]

\(\iff |t_1|_\Sigma = |t_2|_\Sigma\) \(\text{(-Int, =-Elim)}\)

\(\iff |t_1|_r = |t_2|_r\) \(\text{(Identity Invariance)}\).

In order to show that \(T\) behaves correctly in \(\mathcal{C}[\Gamma]\) we need an additional lemma.

**Lemma 20** (Truth). \(\mathcal{C}[\Gamma], \Sigma \models \phi\) iff \(\phi \in \Sigma\).
Proof. Similar to the proof of the corresponding lemma in [4].

So we have

\[ |\Gamma \phi \rangle_\Gamma \in |T|_\Gamma(\Sigma) \iff T^\Gamma \phi \in \Sigma \]
\[ \iff \phi \in \Sigma \quad (T\text{-Int}, T\text{-Elim}) \]
\[ \iff C[\Gamma], \Sigma \models_0 \phi \quad \text{(Truth)}. \]

\(Sat(BQL_{CD}^{RT})\) is defined analogously to \(Sat(BQL_{CD}^T)\).

**Lemma 21** (Extension). For \(\Gamma\) such that \(|\{i : a_i \notin \Gamma\}| = \omega\): if \(\Gamma \not\models_{RT} \phi\) then there exists \(\Gamma^* \supseteq \Gamma\) such that \(\Gamma^* \in Sat(BQL_{CD}^{RT})\) and \(\phi \notin \Gamma^*\).

Proof. Similar to the proof of the corresponding lemma for \(\models_R\).

**Lemma 22** (\(\mathcal{L}\)-Completeness). For \(\Gamma \cup \{\phi\} \subseteq \mathcal{L}\): if \(\Gamma \models_{RT} \phi\) then \(\Gamma \models_{RT} \phi\).

Proof. Suppose \(\Gamma \not\models_{RT} \phi\). Since \(\{i : a_i \in \Gamma\} = \emptyset\), Extension gives us \(\Gamma^* \supseteq \Gamma\) such that \(\Gamma^* \in Sat(BQL_{CD}^{RT})\) and \(\phi \notin \Gamma^*\). Since \(\Gamma^*\) is closed under modus ponens, we have \(\Gamma^* \prec \Gamma^*\). So \(Sat(BQL_{CD}^{RT})[\Gamma^*] \neq \emptyset\) and hence \(C[\Gamma^*]\) exists. By Truth: \(C[\Gamma^*], \Gamma^* \models \Gamma\) and \(C[\Gamma^*], \Gamma^* \not\models \phi\). So, given that \(\Gamma \phi \rangle, =\) and \(T\) behave correctly in \(C[\Gamma^*]\), restricting \(C[\Gamma^*]\) to \(\mathcal{L}\) gives \(\Gamma \not\models_{RT} \phi\).

**Theorem 10** (\(\mathcal{L}^+\)-Completeness). For \(\Gamma \cup \{\phi\} \subseteq \mathcal{L}^+\): if \(\Gamma \models_{RT} \phi\) then \(\Gamma \models_{RT} \phi\).

Proof. Similar to the proof of the corresponding theorem for \(\models_R\).

8. Non-Paradoxicality

Say that a classical \(\mathcal{L}^+ \setminus \{T\}\)-model \(\mathcal{M}\) is a ground model iff (i) \(=^\mathcal{M} = \{(a, a) : a \in \text{dom}(\mathcal{M})\}\) and (ii) for all sentences \(\phi, \psi \in \mathcal{L}\): if \(\phi \neq \psi\) then \(\Gamma \phi \rangle^\mathcal{M} \neq \Gamma \psi \rangle^\mathcal{M}\). For \(\Gamma \cup \{\phi\} \subseteq \mathcal{L}^+ \setminus \{T, \rightarrow\}\), we write \(\Gamma \models_{G} \phi\) iff for every ground model \(\mathcal{M}\): \(\mathcal{M} \models \Gamma\) only if \(\mathcal{M} \models \phi\). In this section we show that \(BQL_{CD}^{RT}\) is non-paradoxical, in the sense that \(BQL_{CD}^{RT}\) conservatively extends \(\models_G\). So, in particular, \(\forall BQL_{CD}^{RT}\) does not contain the liar paradox. This is most easily be seen by observing that the liar paradox contains
the discharge of an unsafe occurrence of an open assumption:

\[
\frac{\vdash T(c) \rightarrow \bot}{c = \vdash T(c) \rightarrow \bot}
\]

It can be shown by a standard Belnap-Henkin argument (see e.g. Priest [5] §6.2) that \( \models G \) is completely axiomatized by the restriction of \( \mathcal{N}BQL_{CD}^{RT} \) to \( \mathcal{L}^+ \setminus \{ \rightarrow, T \} \). Consequently, proving that \( \mathcal{BQL}_{CD}^{RT} \) is conservative over \( \models G \) also establishes that \( \mathcal{N}BQL_{CD}^{RT} \) conservatively extends the restriction of \( \mathcal{N}BQL_{CD}^{RT} \) to \( \mathcal{L}^+ \setminus \{ \rightarrow, T \} \).

8.1. The Conservativity Proof. Let \( \mathcal{M} \) be a ground model. An \( \mathcal{L}^+ \)-model \( \mathfrak{M} \) is called an \( \mathcal{L}^+ \)-expansion of \( \mathcal{M} \) iff (i) \( \text{dom}(\mathfrak{M}) = \text{dom}(\mathcal{M}) \), (ii) \( |c| = c^\mathcal{M} \), (iii) \( |f^n| = (f^n)^\mathcal{M} \) and (iv) \( |R^n|(w) = (R^n)^\mathcal{M} \) for \( R^n \neq T \). To show that \( \models RT \) is conservative over \( \models G \) it suffices to show that every ground model has an \( \mathcal{L}^+ \)-expansion in which \( \models RT \) assigns to each world \( w \) the set of formulas \( \{ |\Gamma \phi \boxdot | : \phi \in \mathcal{L} \text{ and } w \models \phi \} \) and at least one world is reflexive. We do this using a construction recently introduced by Field, Lederman and Øgaard [1] in the context of naive class theory.

Let \( \mathcal{M} \) be an arbitrarily chosen ground model with domain \( M \). We first define a transfinite sequence \( \{ \mathfrak{M}_\alpha \}_{\alpha \in \text{Ord}} \) of \( \mathcal{L}^+ \)-expansions of \( \mathcal{M} \). Each \( \mathfrak{M}_\alpha \) has the form \( \langle \alpha + 1, >, M, | \cdot |_\alpha \rangle \) (so an ordinal \( \beta \) sees all and only those ordinals strictly smaller than \( \beta \)), where \( |c|_\alpha = c^\mathcal{M} \), \( |f^n|_\alpha = (f^n)^\mathcal{M} \) and \( |R^n|_\alpha(\beta) = (R^n)^\mathcal{M} \) for \( R^n \neq T \). Since \( |t|_\alpha = |t|_\beta \) for all \( \beta \), we drop the subscript on \( |t|_\alpha \). We define \( |T|_\alpha \) by transfinite induction on \( \alpha \). Suppose \( |T|_\beta \) has already been defined for every \( \beta < \alpha \). Then, for \( \beta < \alpha \), we let \( |T|_\alpha(\beta) = |T|_\beta(\beta) \). It remains to specify \( |T|_\alpha(\alpha) \). For arbitrary \( X \subseteq M \), let \( \mathfrak{M}_\alpha[X] \) denote the object which would be obtained were we to set \( |T|_\alpha(\alpha) = X \). \( \mathfrak{M}_\alpha[X] \) is not necessarily an \( \mathcal{L}^+ \)-model, since we need not have \( X \subseteq |T|_\alpha(\beta) \) for all \( \beta < \alpha \). Nevertheless, we can still define satisfaction on \( \mathfrak{M}_\alpha[X] \) in the same way as a real \( \mathcal{L}^+ \)-model. Let \( \Phi_\alpha(X) = \{ |\Gamma \phi \boxdot | : \phi \in \mathcal{L} \text{ and } \mathfrak{M}_\alpha[X], \alpha \models \phi \} \).
Lemma 23 (Monotonicity). If $X \subseteq Y$ then $\Phi_\alpha(X) \subseteq \Phi_\alpha(Y)$.

Proof. Suppose $X \subseteq Y$. We show by induction on the construction of $\mathcal{L}^+$-formulas that $\mathcal{M}_\alpha[X], \alpha \models \phi(\overline{v})$ only if $\mathcal{M}_\alpha[Y], \alpha \models \phi(\overline{v})$.

Base Cases The claim holds trivially for atomic $\phi \neq T(t)$. For $\phi = T(t)$ we have

$$\mathcal{M}_\alpha[X], \alpha \models T(t)(\overline{a}) \implies |t(\overline{v})| \in X \implies |t(\overline{v})| \in Y \implies \mathcal{M}_\alpha[Y], \alpha \models T(t)(\overline{a}).$$

Induction Steps The induction steps are standard except for $\rightarrow$.

$\rightarrow$ Suppose $\mathcal{M}_\alpha[X], \alpha \models (\phi \rightarrow \psi)(\overline{v})$. Let $\beta < \alpha$. Since $\mathcal{M}_\alpha[X] \upharpoonright \beta = \mathcal{M}_\alpha[Y] \upharpoonright \beta$,

$$\mathcal{M}_\alpha[Y], \beta \models \phi(\overline{v}) \implies \mathcal{M}_\alpha[X], \beta \models \phi(\overline{v}) \implies \mathcal{M}_\alpha[X], \beta \models \psi(\overline{v}) \implies \mathcal{M}_\alpha[Y], \beta \models \psi(\overline{v}).$$

So $\mathcal{M}_\alpha[Y], \alpha \models (\phi \rightarrow \psi)(\overline{v})$. \qed

We can now define an increasing sequence of increasingly better extensions for $T$ at $\alpha$ in the style of Kripke [3]:

$$X_\alpha(0) = \emptyset$$

$$X_\alpha(\beta + 1) = \Phi_\alpha(X_\alpha(\beta))$$

$$X_\alpha(\gamma) = \bigcup_{\beta < \gamma} X_\alpha(\beta) \quad \text{for } \gamma \text{ a limit.}$$

Lemma 24 (Locally Increasing). If $\beta \leq \beta'$ then $X_\alpha(\beta) \subseteq X_\alpha(\beta')$.

Proof. By transfinite induction on $\beta$. The base case $\beta = 0$ holds trivially.

Successor Step Suppose $\beta + 1 \leq \beta'$. There are two cases.

Case 1 $\beta'$ is a successor. Then we have

$$\beta \leq \beta' - 1 \implies X_\alpha(\beta) \subseteq X_\alpha(\beta' - 1) \quad \text{(induction hypothesis)}$$

$$\implies \Phi_\alpha(X_\alpha(\beta)) \subseteq \Phi_\alpha(X_\alpha(\beta' - 1)) \quad \text{(Monotonicity)}$$

$$\implies X_\alpha(\beta + 1) \subseteq X_\alpha(\beta').$$
Case 2 $\beta'$ is a limit. Then, trivially, $X_\alpha(\beta + 1) \subseteq X_\alpha(\beta')$.

Limit Step Suppose $\beta \leq \beta'$ for $\beta$ a limit. Suppose $a \in X_\alpha(\beta)$. Then $a \in X_\alpha(\beta_0)$ for some $\beta_0 < \beta$. By the induction hypothesis, $X_\alpha(\beta_0) \subseteq X_\alpha(\beta')$. So $a \in X_\alpha(\beta')$. □

Lemma 25 (Locally Convergent). There exists $\beta$ such that $X_\alpha(\beta) = X_\alpha(\beta')$ for all $\beta' \geq \beta$.

Proof. Suppose not. Then, by Locally Increasing, for every $\beta$ there exists $\beta' > \beta$ such that $X_\alpha(\beta) \subset X_\alpha(\beta')$, which contradicts the fact that $\bigcup_{\beta \in \text{Ord}} X_\alpha(\beta)$ is a set. □

We then let $|T|_\alpha(\alpha) = X_\alpha(\alpha^+)$, where $\alpha^+$ is the least $\beta$ such that $X_\alpha(\beta) = X_\alpha(\beta')$ for all $\beta' \geq \beta$. This completes the definition of $\mathcal{M}_\alpha$.

Lemma 26 (Semantic Closure). For $\phi \in \mathcal{L}$: $\mathcal{M}_\alpha, \beta \models \phi$ if and only if $|\phi| \in |T|_\alpha(\beta)$.

Proof. By transfinite induction on $\alpha$. Suppose the claim holds for all $\alpha_0 < \alpha$. There are two cases.

Case 1 $\beta < \alpha$. Then, since $\mathcal{M}_\alpha \upharpoonright \beta = \mathcal{M}_\beta$, we have

$$
\mathcal{M}_\alpha, \beta \models \phi \iff \mathcal{M}_\beta, \beta \models \phi
\iff \phi \in |T|_\beta(\beta) \quad \text{(induction hypothesis)}
\iff \phi \in |T|_\alpha(\beta).
$$

Case 2 $\beta = \alpha$. Then we have

$$
\mathcal{M}_\alpha, \alpha \models \phi \iff |\phi| \in \Phi_\alpha(|T|_\alpha(\alpha))
\iff |\phi| \in \Phi_\alpha(X_\alpha(\alpha^+))
\iff |\phi| \in X_\alpha(\alpha^+ + 1)
\iff |\phi| \in X_\alpha(\alpha^+)
\iff |\phi| \in |T|_\alpha(\alpha).
$$

□

Lemma 27 (Globally Decreasing). If $\beta_0 \leq \beta \leq \alpha$ then $|T|_\alpha(\beta) \subseteq |T|_\alpha(\beta_0)$.

Proof. Suppose $\beta_0 \leq \beta \leq \alpha$.

Subclaim 4 (Layers). For all $\xi : X_\beta(\xi) \subseteq X_{\beta_0}(\xi)$. 
Proof. By transfinite induction on \( \xi \). The base case \( \xi = 0 \) holds trivially.

**Successor Step** Suppose \( X_\beta(\xi) \subseteq X_{\beta_0}(\xi) \). Then, by a similar argument to Monotonicity, \( M_{\beta}[X_\beta(\xi)], \beta \notmodels \phi(\overline{v}) \) only if \( M_{\beta_0}[X_{\beta_0}(\xi)], \beta_0 \notmodels \phi(\overline{v}) \). So \( X_\beta(\xi + 1) \subseteq X_{\beta_0}(\xi + 1) \).

**Limit Step** Let \( a \in X_\beta(\gamma) \) for \( \gamma \) a limit. Then \( a \in X_\beta(\xi) \) for some \( \xi < \gamma \). By the induction hypothesis, \( X_\beta(\xi) \subseteq X_{\beta_0}(\xi) \). So \( a \in X_{\beta_0}(\xi) \subseteq X_{\beta_0}(\gamma) \). ■

There are now two cases to consider.

**Case 1** \( \beta^+ = \beta_0^+ + \xi \) for some ordinal \( \xi \). Then

\[
|T|_\alpha(\beta) = X_\beta(\beta^+)
\]
\[
= X_\beta(\beta_0^+ + \xi)
\]
\[
\subseteq X_{\beta_0}(\beta_0^+ + \xi) \quad \text{(Layers)}
\]
\[
= X_{\beta_0}(\beta_0^+)
\]
\[
= |T|_\alpha(\beta_0^+).
\]

**Case 2** \( \beta_0^+ = \beta^+ + \xi \) for some ordinal \( \xi \). Then

\[
|T|_\alpha(\beta) = X_\beta(\beta^+)
\]
\[
= X_\beta(\beta^+ + \xi)
\]
\[
= X_{\beta_0}(\beta_0^+)
\]
\[
\subseteq X_{\beta_0}(\beta_0^+ + \xi) \quad \text{(Layers)}
\]
\[
= |T|_\alpha(\beta_0).
\]

\( \square \)

It follows from Globally Decreasing that \( \mathcal{M}_\alpha \) is in fact an \( \mathcal{L}^+ \)-model and so \( \mathcal{M}_\alpha \) is in fact an \( \mathcal{L}^+ \)-expansion of \( \mathcal{M} \). Accordingly, \( \mathcal{M}_\alpha \) obeys Persistence [4], which allows us to prove that we eventually reach an ordinal \( \alpha \) such that for all \( \alpha' \geq \alpha \): \( \mathcal{M}_{\alpha'}, \alpha' \models \phi(\overline{v}) \) iff \( \mathcal{M}_\alpha, \alpha \models \phi(\overline{v}) \). Let \( S(\alpha) = \{ \langle \phi(\overline{v}), \langle \overline{v} \rangle \rangle : \mathcal{M}_\alpha, \alpha \models \phi(\overline{v}) \} \).

**Lemma 28** (Globally Convergent). Then there exists \( \alpha \) such that for all \( \alpha' \geq \alpha \): \( S(\alpha) = S(\alpha') \).
Lemma 29. \( \neg \). Lemma 30. \( \neg \).

Proof. Suppose not. Then, by Persistence, for every \( \alpha \) there exists \( \alpha' > \alpha \) such that \( S(\alpha') \subset S(\alpha) \), which contradicts the fact that \( S(0) \) is a set.

Let \( \alpha^* \) denote the least \( \alpha \) such that \( S(\alpha) = S(\alpha') \) for all \( \alpha' \geq \alpha \). Let \( \mathcal{M} \) denote the \( \mathcal{L}^{+} \)-model obtained from \( \mathcal{M}_{\alpha^*} \) by letting \( \alpha^* \) but no other ordinal see itself. To establish that \( \mathcal{M} \) is the desired \( \mathcal{L}^{+} \)-expansion of the ground model \( \mathcal{M} \), we need to verify that \( |T|_{\alpha^*}(\alpha) = \{ |\Gamma \phi| : \phi \in \mathcal{L} \text{ and } \mathcal{M}, \alpha \models \phi \} \). By Semantic Closure, it suffices to show that \( \mathcal{M}, \alpha \models \phi \) iff \( \mathcal{M}_{\alpha^*}, \alpha \models \phi \). This holds trivially for \( \alpha < \alpha^* \). The case where \( \alpha = \alpha^* \) is handled by the following lemma.

**Lemma 29.** \( \mathcal{M}, \alpha^* \models \phi(\overline{a}) \) iff \( \mathcal{M}_{\alpha^*}, \alpha^* \models \phi(\overline{a}) \).

**Proof.** By induction on the construction of \( \mathcal{L}^{+} \)-formulas. The base cases are easy. The induction steps are also easy except for \( \rightarrow \).

\[ \rightarrow \implies \text{Easy.} \]

\[ \iff \text{Suppose } \mathcal{M}, \alpha^* \not\models (\phi \rightarrow \psi)(\overline{a}). \text{ Then } \mathcal{M}, \alpha \models \phi(\overline{a}) \text{ and } \mathcal{M}, \alpha \not\models \psi(\overline{a}) \text{ for some } \alpha \leq \alpha^*. \text{ If } \alpha < \alpha^* \text{ then we’re done. Suppose } \alpha = \alpha^*. \text{ Then, by the induction hypothesis, } \mathcal{M}_{\alpha^*}, \alpha^* \models \phi(\overline{a}) \text{ and } \mathcal{M}_{\alpha^*}, \alpha^* \not\models \psi(\overline{a}). \text{ So } \mathcal{M}_{\alpha^*+1}, \alpha^* + 1 \not\models (\phi \rightarrow \psi)(\overline{a}). \text{ Since } S(\alpha^* + 1) = S(\alpha^*), \mathcal{M}_{\alpha^*}, \alpha^* \not\models (\phi \rightarrow \psi)(\overline{a}). \]

**Theorem 11 (Conservativity).** For \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L}^{+} \setminus \{ \rightarrow \} \): if \( \Gamma \models_{BQL} \phi \) then \( \Gamma \models_{G} \phi \).

**Proof.** Suppose \( \Gamma \not\models_{G} \phi \). Then there exists a ground model \( \mathcal{M} \) such that \( \mathcal{M} \models \Gamma \) and \( \mathcal{M} \not\models \phi \). Let \( \mathcal{M} \) be the Field-Lederman-Ogaard \( \mathcal{L}^{+} \)-expansion of \( \mathcal{M} \). Then \( \mathcal{M}, \alpha^* \models \Gamma \) and \( \mathcal{M}, \alpha^* \not\models \phi \). Since \( \alpha^* \) is reflexive in \( \mathcal{M} \) and \( \Gamma \phi, =, T \) behave correctly in \( \mathcal{M} \): \( \Gamma \not\models_{RT} \phi \).

9. Failure of Conservativity

The logic \( \mathcal{L}^{+} \) ("quotation logic") is obtained by restricting \( \mathcal{L}^{+} \) to \( \mathcal{L}^{+} \setminus \{ T \} \) (call the resulting system \( \mathcal{L}^{+} \)). We write \( \Gamma \vdash_{BQL} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{L}^{+} \). In this section we show that \( \Gamma \vdash_{BQL} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{L}^{+} \). Hence, adding \( T \)-Int and \( T \)-Elim to \( \mathcal{L}^{+} \) could force us to revise which non-truth-involving sentences we accept and reject, which suggests \( \mathcal{L}^{+} \) does not support a deflationary conception of truth.

**Lemma 30.** \( c = T(c) \rightarrow \bot \land (T \rightarrow \bot) \rightarrow \bot \vdash_{RT} \bot \).
Proof. First note:

\[
\frac{
\frac{\top \rightarrow T(c)}{T(c) \rightarrow \bot}
}{\top \rightarrow \bot}
\]

So we have:

\[
c = \top\neg (\top \rightarrow \bot) \rightarrow \bot
\]

Say that an \(\mathcal{L}^+ \setminus \{T\}\)-model \(\mathfrak{M}\) is quotation-acceptable iff (i) \(\top\neg \phi \neq \top\neg \psi\) for \(\mathcal{L}\)-sentences \(\phi \neq \psi\) and (ii) \(\models = \models (w) = \{\langle a, a \rangle : a \in dom(\mathfrak{M})\}\). For \(\Gamma \cup \{\phi\} \subseteq \mathcal{L}^+ \setminus \{T\}\), we write \(\Gamma \models_{RQ} \phi\) iff for every reflexive world \(w\) in every quotation-acceptable \(\mathcal{L}^+ \setminus \{T\}\)-model \(\mathfrak{M}\): \(w \models \Gamma\) only if \(w \models \phi\).

**Lemma 31.** If \(\Gamma \vdash_{RQ} \phi\) then \(\Gamma \models_{RQ} \phi\).

**Proof.** Similar to the proof that \(\vdash_{RT}\) is sound with respect to \(\models_{RT}\). \(\square\)

**Lemma 32.** \(c = \top\neg (\top \rightarrow \bot) \rightarrow \bot \neq_{RQ} \bot\).

**Proof.** Take an \(\mathcal{L}^+ \setminus \{T\}\)-model \(\mathfrak{M}\) with a single reflexive world \(w\) such that (i) the domain of \(\mathfrak{M}\) is the set of \(\mathcal{L}\)-sentences, (ii) \(\models = \models (w) = \{\langle a, a \rangle : a \in dom(\mathfrak{M})\}\), (iii) \(\top\neg \phi = \phi\) for every \(\phi \in \mathcal{L}\) and (iv) \(\models c = T(c) \rightarrow \bot\).

**Theorem 12.** \(\vdash_{RT}\) is not conservative over \(\vdash_{RQ}\).

**Proof.** Immediate from the preceding lemmas. \(\square\)
10. Compositional Laws

Suppose $\mathcal{L}$ contains a unary relation symbol $\text{Sent}(\cdot)$ and binary function symbols $\neg, \lor, \to$. Suppose $\Gamma \subseteq \mathcal{L} \setminus \{\to, T\}$ has a ground model $\mathcal{M}$ such that (1) $a \in \text{Sent}^\mathcal{M}$ iff $a = \Gamma \phi^{\mathcal{M}}$ for some $\phi \in \mathcal{L}$ and (2) for $\circ \in \{\land, \lor, \to\}$: (i) $\Gamma \phi^{\mathcal{M}} \Gamma \psi^{\mathcal{M}} = \Gamma \phi \circ \psi^{\mathcal{M}}$, (ii) $a \circ^\mathcal{M} b \in \text{Sent}^\mathcal{M}$ only if $a,b \in \text{Sent}^\mathcal{M}$. Let $\mathfrak{M}$ be the Field-Lederman-Øgaard $\mathcal{L}^+$-expansion of $\mathcal{M}$. Then both $\Gamma$ and the following compositional laws are true at $\alpha^*$ in $\mathfrak{M}$:

\begin{align*}
&\quad \land-C \quad \forall x \forall y (T(x \land y) \iff T(x) \land T(y)) \\
&\quad \lor-C \quad \forall x \forall y (T(x \lor y) \iff \text{Sent}(x) \land \text{Sent}(y) \land (T(x) \lor T(y))) \\
&\quad \to-C \quad \forall x \forall y (T(x \to y) \iff \text{Sent}(x) \land \text{Sent}(y) \land (T(x) \to T(y))).
\end{align*}

So for all $\phi \in \mathcal{L} \setminus \{\to, T\}$: $\Gamma, \land-C, \lor-C, \to-C \vdash_{RT} \phi$ only if $\mathcal{M} \models \phi$. This shows we can safely add the compositional laws to a true theory $\Gamma$ of the syntax of $\mathcal{L}$ formulated in $\mathcal{L} \setminus \{\to, T\}$.

11. Works Cited

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