Dynamo effect beyond the kinematic approximation: suppression of linear instabilities by non-linear feedback

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The turbulent dynamo effect, which describes the generation of magnetic fields in astrophysical objects, is described by the dynamo equation. This, in the kinematic (linear) approximation gives an unbounded exponential growth of the long wavelength part of the magnetic fields. Here we, in a systematic diagrammatic, perturbation theory, show how non-linear effects suppress the linear instability and bring down the growth rate to zero in the large time limit. We work with different background velocity spectrum and initial magnetic field correlations. Our results indicate the robustness and very general nature of dynamo growth: It is qualitatively independent of the background velticy and initial magnetic field spectra. We also argue that our results can be justified within the framework of the first order smoothing approximation, as applicable for the full non-linear problem. We discuss our results from the view points of renormalisation group analysis.

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I. INTRODUCTION

Magnetic fields are ubiquitous. All astrophysical objects are known to have magnetic fields of different magnitudes, e.g., 1 gauss at the stellar scale to $10^{-6}$ gauss at the galactic scale. The origin of this field (primordial field) is not very clear - there are several competing theories which attempt to describe this. However, a finite magnetic field in any physical system undergoes a temporal decay due to the finite conductivities of the medium. So, for steady magnetic fields to remain in astrophysical bodies, there has to be regeneration of the magnetic fields which takes place in the form of dynamo process. Typically astrophysical bodies are thought to have fast dynamo operating within themselves, resulting into exponential growth of the magnetic fields. This mechanism requires a turbulent velocity background [though non-turbulent velocity fields too can make a seed (initial) magnetic field to grow (for details see)] we will not consider such cases here]. Since the dynamo equations, in the linear approximation (see below) gives unbounded exponentially growing solutions for the long wavelength (large scale) part of the magnetic fields, it is linear unstable in the low wavenumber limit. An intriguing question, that arises very naturally is, whether magnetic fields continue to grow even after a long time. This obviously does not happen as we do not see ever growing magnetic fields in the core of the earth or in the sun. For example, geomagnetic fields ($\sim 1$ gauss) are known to be stable for about $10^6$ years. Secondly, continuously growing magnetic fields violate energy conservation. In other words, if the dynamo equation correctly describes the problem, then its physically realisable solutions must not be linearly unstable in the long time limit. So there must be a counter mechanism to stabilise it, which we investigate here.

There have been numerous works in this field in the past by many groups. For examples Pouquet, Frisch and Léorat in a eddy damped quasi-normal Markovian approximation studied the connections between the dynamo process and the inverse cascade of magnetic and kinetic energies. Moffatt, by linearising the equations of motion of three-dimensional (3d) magnetohydrodynamics (MHD) examined the back reactions due to the Lorentz force for magnetic Prandtl number $P_m \gg 1$. Vainshtein and Cattaneo discussed several nonlinear restrictions on the generations of magnetic fields. Field et al. discussed nonlinear $\alpha$-effects within a two-scale approach. Rogachevskii and Kleerin studied the effects of an anisotropic background turbulence on the dynamo process. Brandenburg examined nonlinear $\alpha$-effects in numerical simulation of helical MHD turbulence. He particularly examined the dependences of dynamo growth and the saturation field on $P_m$. Bhattacharjee and Yuan studied the problem in a two-scale approach by linearising the equations of motion. However these issues are not yet fully closed. We examine the following questions: i) instead of a two-scale approach (which is rather adhoc) whether we can employ a diagrammatic perturbation theory, which has been highly successful in the context of critical dynamics, driven systems etc., can be easily extended to higher orders in perturbation expansion and provides natural connections with standard
renormalisation group framework, and ii) if a turbulent background is essential for the dynamo mechanism. To put it differently we ask if dynamo process can take place with velocity fields with arbitrary statistics. We explicitly demonstrate that the nonlinear feedback of the magnetic fields on the velocity fields in the form of the Lorentz force stabilises the instability. We show it for a very general velocity and initial magnetic field correlations - thus our results demonstrate the very generality of the dynamo process. The plan of the rest of the paper is as follows: In Section II we discuss the general dynamo mechanism within the standard linear approximation. In Sec. III B we show that one needs to go beyond the linear approximation, i.e., include the non-linear effects to see eventual saturation of magnetic field growth. In Sec. IV we conclude.

II. DYNAMO GROWTH: THE LINEAR APPROXIMATION

In the kinematic approximation, i.e., in the early time when magnetic energy is much smaller than the kinetic energy \( \int u^2 \, d^3r >> \int b^2 \, d^3r \), where \( u(r, t) \) and \( b(r, t) \) are the velocity and magnetic fields respectively, the Lorentz force term of the Navier Stokes equation is neglected. In that weak magnetic field limit, which is reasonable at an early time, the time evolution problem of the magnetic fields is a linear problem as the Induction equation is linear in magnetic fields:

\[
\frac{\partial b}{\partial t} = \nabla \times (u \times b) + \mu \nabla^2 b, \tag{1}
\]

where \( \mu \) is the magnetic viscosity. The velocity field is governed by the Navier-Stokes equation (dropping the Lorentz force)

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = - \frac{\nabla p}{\rho} + \nu \nabla^2 u + f. \tag{2}
\]

Here \( \nu \) is the fluid viscosity, \( f \) an external forcing function, \( p \) the pressure and \( \rho \) the density of the fluid. We take \( f \) to be a zero mean, Gaussian stochastic force with a specified variance (see below).

In a two-scale approach one can then write an effective equation for \( B \), the long-wavelength part of the magnetic fields:

\[
\frac{\partial B}{\partial t} = \nabla \times (U \times B) + \nabla \times E + \mu \nabla^2 B, \tag{3}
\]

where the Electromotive force \( E = \langle v \times b \rangle \). \( U \) is the large scale component of the velocity field \( u \). An Operator Product Expansion (OPE) is shown to hold which provides a gradient expansion in terms of \( B \) for the product \( E = \langle u \times b \rangle \)

\[
E_i = \alpha_{ij} B_j + \beta_{ijk} \frac{\partial B_j}{\partial x_k} + .... \tag{4}
\]

For homogenous and isotropic flows \( (\alpha_{ij} = \alpha \delta_{ij}) \) Eq.(4) gives,

\[
\frac{\partial B}{\partial t} = \nabla \times (U \times B) + \alpha \nabla \times B + \mu \nabla^2 B, \tag{5}
\]

which is the standard turbulent dynamo equation. Here \( \mu \) now is the effective magnetic viscosity which includes turbulent diffusion, represented by \( \beta_{ijk} \) in Eq.(5). \( \alpha \) depends upon the statistics of the velocity field (or, equivalently, the correlations of \( f \)). Retaining only the \( \alpha \)-term and dropping all others from the RHS of Eq.(5), the equations for the cartesian components of \( B \) become (we neglect the dissipative terms proportional to \( k^2 \) as we are interested only in the long wavelength properties)

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1By a turbulent background we do not mean any kind of fluctuating state but a fluctuating state with K41 spectrum for the kinetic and magnetic energies and cascades of appropriate quantities; if there is no mean magnetic field then the energy spectra is expected to be K41-type - see Ref. [3].
in the rotating frame. The bare propagators $G$ which does not contribute to the dynamics for incompressible flows. The correlations Eqs.(8) and (9) do not change.

The eigenvalues of the matrix is proportional to $\Omega$ and initial conditions on $b_i$. We choose $f_i(k, t)$ and $b_i(k, t=0)$ to be zero mean Gaussian distributed with the following variances:

$$
\langle f_i(k, t)f_j(-k, 0) \rangle = P_{ij}D_1(k)\delta(t),
$$

$$
\langle b_i(k, t)b_j(-k, 0) \rangle = P_{ij}D_2(k)\delta(t).
$$

$D_1$ and $D_2$ are some functions of $k$ (to be specified later).

In a rotating frame with a rotation velocity $\Omega = \Omega \hat{z}$ the Eqs. (8) and (9) take the form

$$
\frac{d}{dt} \begin{pmatrix} B_x(k, t) \\ B_y(k, t) \\ B_z(k, t) \end{pmatrix} = i\alpha \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \begin{pmatrix} B_x(k, t) \\ B_y(k, t) \\ B_z(k, t) \end{pmatrix}.
$$

The eigenvalues of the matrix is $\lambda = \pm i\alpha$, 0. Thus depending on the sign of the product $\alpha k$, one mode grows and the other decays. The third mode stays constant in time. Since growth rate is proportional to $|k|$ and dissipation proportional to $k^2$, large scale fields continue to grow leading to low wavelength instability. Thus in the long time limit effectively only the growing mode remains. Since the cartesian components of $B$ are just linear combinations of the eigenmodes of the matrix above, they also grow exponentially. Growth rate $\alpha$ is a pseudo-scalar quantity, i.e., under parity transformation $r \rightarrow -r, \alpha \rightarrow -\alpha$. Since $\alpha$ depends upon the statistical properties of the velocity field, its statistics should not be parity invariant. This can happen in a rotating frame, where the angular velocity explicitly breaks parity.

### III. FORMULATION OF THE DYNAMO PROBLEM IN A ROTATING FRAME

The NS and the Induction equation in the inertial (lab) frame in $(k, t)$ space become

$$
\frac{\partial u_i(k, t)}{\partial t} + \frac{1}{2}P_{ijp}(k) \sum_q u_j(q, t)u_p(k - q, t) = \frac{1}{2}P_{ijp}(k) \sum_q b_j(q, t)b_p(k - q, t) + \nu \nabla^2 u_i + f_i(k, t),
$$

$$
\frac{\partial b_i(k, t)}{\partial t} = \tilde{P}_{ijp}(k) \sum_q u_j(q, t)b_p(k - q, t) + \mu \nabla^2 b_i.
$$

Here, $u_i(k, t)$ and $b_i(k, t)$ are the fourier transforms of $u_i(r, t)$ and $b_i(r, t)$ respectively, $P_{ijp}(k) = P_{ij}(k)k_p + P_{ip}(k)k_j, \tilde{P}_{ijp}(k) = P_{ij}(k)k_p - P_{ip}(k)k_j, P_{ij}$ is the projection operator, which appears due to the divergence-free conditions on the velocity and magnetic fields. The Eqs.(8) and (9) are to be supplemented by appropriate correlations for $f_i$ and initial conditions on $b_i$. We choose $f_i(k, t)$ and $b_i(k, t=0)$ to be zero mean Gaussian distributed with the following variances:

$$
\langle f_i(k, t)f_j(-k, 0) \rangle = P_{ij}D_1(k)\delta(t),
$$

$$
\langle b_i(k, t)b_j(-k, 0) \rangle = P_{ij}D_2(k)\delta(t).
$$

$D_1$ and $D_2$ are some functions of $k$ (to be specified later).

In a rotating frame with a rotation velocity $\Omega = \Omega \hat{z}$ the Eqs. (8) and (9) take the form

$$
\frac{\partial u_i(k, t)}{\partial t} + 2(\Omega \times u)_i + \frac{1}{2}P_{ijp}(k) \sum_q u_j(q, t)u_p(k - q, t) = \frac{1}{2}P_{ijp}(k) \sum_q b_j(q, t)b_p(k - q, t) + \nu \nabla^2 u_i + f_i(k, t),
$$

$$
\frac{\partial b_i(k, t)}{\partial t} + (\Omega \times b)_i = \tilde{P}_{ijp}(k) \sum_q u_j(q, t)b_p(k - q, t) + \mu \nabla^2 b_i.
$$

$\Omega \times u$ is the coriolis force. The centrifugal force $\Omega \times (\Omega \times r)$ is put in as a part of the effective pressure $p + \frac{1}{2} |\Omega \times r|^2$ which does not contribute to the dynamics for incompressible flows. The correlations Eqs.(8) and (9) do not change in the rotating frame. The bare propagators $G_u$ and $G_b$ of $u_i$ and $b_i$ are

$$
G_u = \begin{pmatrix}
\frac{i\omega + \mu k^2}{(i\omega + \mu k^2)^2 + \Omega^2} & 0 & \frac{2i}{(i\omega + \mu k^2)^2 + \Omega^2} \\
0 & 0 & 0 \\
\frac{2i}{(i\omega + \mu k^2)^2 + \Omega^2} & 0 & \frac{i\omega + \mu k^2}{(i\omega + \mu k^2)^2 + \Omega^2}
\end{pmatrix},
$$

$$
G_b = \begin{pmatrix}
\frac{i\omega + \mu k^2}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} \\
\frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} \\
\frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2} & \frac{0}{(i\omega + \mu k^2)^2 + \Omega^2}
\end{pmatrix}
$$

such that $u = G_u f$ and $b(t) = G_b b(t=0)$ where
magnetic field correlations grow in time due to the dynamo effects but the inertial range scale dependence does not

\[ \langle \alpha \rangle = \text{term - responsible for growth} \] below (see Fig. 1a):

where \( \alpha = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \beta = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}. \)

We get for the correlation function \( \langle u_i(k, \omega)u_j(-k, -\omega) \rangle \equiv \langle uu^T \rangle \) (\( u^T \) is the transpose of \( u \)), when \( i \neq j \) there are terms proportional to \( \Omega \) (for \( i = j \), there are no \( O(\Omega) \) terms). Under parity transformation these \( i \neq j \) terms change sign. Thus the \( i \neq j \) terms have both even parity and odd parity parts. Similarly in \( \langle b_ib_j \rangle \), for \( i \neq j \) both have odd and even parity parts. A direct consequence of these parity breaking parts is that fluid helicity (\( \equiv \int_x (u.(\nabla \times u)) \)) is non-zero: \( \int_x (u.(\nabla \times u)) \propto \Omega \). In the same way \( \int_x (b.(\nabla \times b)) \) is non-zero and proportional to \( \Omega \). Both \( \int_x (u.(\nabla \times u)) \) and \( \int_x (b.(\nabla \times b)) \) have same signs - a fact of great importance for the results discussed here. Notice that \( G_{xx}^{u,b} \) are different from \( G_{xx,yy}^{u,b} \) - this is just the consequence of the fact that \( \Omega \) distinguishes the \( z \)-direction from others, making the system anisotropic. However for frequencies \( \omega \gg \Omega \) or length scales \( k^2 \gg \Omega \), isotropy is restored. In that regime the role of the global rotation is only to introduce parity breaking contributions proportional to \( \Omega \) to \( \langle u_iu_j \rangle \) and \( \langle b_ib_j \rangle \) for \( i \neq j \). These can be modeled by introducing parity breaking parts in Eqs. (8) and (6):

\[
\langle f_i(k, t)f_j(-k, 0) \rangle = P_{ij}D_1(k)\delta(t) + 2i\epsilon_{ijlp}k_p\bar{D}_1(k)\delta(t),
\]

\[
\langle b_i(k, t)b_j(-k, 0) \rangle = P_{ij}D_2(k)\delta(t) + 2i\epsilon_{ijlp}k_p\bar{D}_2(k)\delta(t),
\]

in conjunction with the inertial frame Eqs. (8) and (6). The parity breaking parts in the noise correlations or initial conditions ensure that \( \int_x (u.(\nabla \times u)) \) and \( \int_x (b.(\nabla \times b)) \) are non-zero as is the case with Eqs. (9) and Eqs. (8) along with Eqs. (8) and (6). What is the relative sign between \( \bar{D}_1 \) and \( \bar{D}_2 \)? Since \( \int_x (u.(\nabla \times u)) \) and \( \int_x (b.(\nabla \times b)) \) are proportional to \( \bar{D}_1 \) and \( \bar{D}_2 \) respectively, and they have same signs, \( \bar{D}_1 \) and \( \bar{D}_2 \) must have same signs.

### A. Calculation of \( \alpha \) in the kinematic approximation

In the kinematic approximation neglecting the Lorentz force term of the Navier-Stokes equation, the time evolution of the magnetic fields follows from the linear Induction Equation (8). We assume, for convenience of calculations, that the velocity field \( (u) \) statistics has reached a steady state. This is acceptable as long as the loss due to the transfer of kinetic energy by the dynamo process is compensated by the external drive. In the kinematic (i.e., linear) approximation, we work with Eqs. (8) (without the Lorentz force) and (8). We choose \( f_i(k, t) \) to be a zero-mean, Gaussian random field with correlations

\[
\langle f_i(k, t)f_m(k, 0) \rangle = 2P_{im}D_1(k)\delta(t) + 2i\epsilon_{imnp}D_1(k)\eta_n\delta(t).
\]

Our initial conditions for the magnetic fields are

\[
\langle b_\alpha(k, t = 0)b_\beta(-k, t = 0) \rangle = P_{\alpha\beta}2D_2(k) + 2i\epsilon_{\alpha\beta\gamma}k_\gamma\bar{D}(k),
\]

Since we are interested to investigate the dynamo process with arbitrary statistics for the velocity and magnetic fields we work with arbitrary \( D_1(k), D_1(k), D_2(k) \) and \( D_2(k) \). For K41 spectra, we require \( D_1(k) = D_1k^{-3}, D_1(k) = D_1k^{-4}, D_2(k) = D_2k^{-5/3} \) and \( D_2(k) = k^{-8/3} \). These choices ensure that under spatial rescaling \( x \rightarrow \xi, v, b \rightarrow l^{1/3}v, b \) which is the Kolmogorov scaling. Starting with such a correlations ensures that only the amplitudes of the magnetic field correlations grow in time due to the dynamo effects but the inertial range scale dependence does not change. However, this may not be the case always. In general, not only the amplitude grows in time due to dynamo actions, the scale dependence too at \( t = 0 \) and evolve in time. Note that both the force correlations in the Eq. (8) and the initial conditions on Eq. (8) have parts that are parity breaking, in conformity with our previous discussions. We now calculate the \( \alpha \)-term (at the tree level) in the kinematic approximation (which we call the ‘direct’ term - responsible for growth) below (see Fig. 1a):

\[
\langle (u \times b)_D \rangle = \left( \int q \epsilon_{\mu\beta\gamma}u_\beta(q, t)b_\gamma(k - q, t) \right) = \left( \int q \epsilon_{\alpha\beta\gamma}u_\beta(q, t)\epsilon_{\gamma\delta\lambda}i(k - q, \mu_\eta(q, t_1) b_\xi(k - q - q_1, t_1)G_0^\mu(k - q, t - t_1) \right)
\]

which gives the \( \alpha \)-term:
\[
\alpha_D B_\alpha(k, t) = \int_q i \tilde{D}_1(q) \frac{\epsilon_{\alpha\beta\gamma}}{\nu q^2} \epsilon_{\beta\gamma\delta} q_{\delta} b_p(k, t = 0) \left[ \frac{1}{q^2(\nu + \mu)} + \exp(-2\nu q^2) \right] \frac{1}{q^2(\nu - \mu)}
\]

(17)
giving \( \alpha_D = \frac{2S_3}{3} \frac{1}{\nu(\nu + \mu)} \int_q 2 \tilde{D}_1(q) \frac{\delta^3(q)}{(\nu + \mu)^3} \) for large \( t \). Thus self-consistently, \( \alpha_D = \frac{2S_3}{3} \int_q 2 \tilde{D}_1(q) \frac{\alpha_D}{(\nu + \mu)^3} \). The suffix \( D \) refers to growth or the direct term, as opposed to feedback which we discuss in the next Section IIIIB. The growth term is proportional to \( |k| \) and diffusive decay proportional to \( k^2 \). So large scale components grow and small scale components decay.

FIG. 1. Tree level diagrams for \( < u(q) \times b(k - q) > \). (a) A solid line indicates a bare magnetic field response function, a broken line indicates a bare velocity response function, a 'o' joined by two broken lines indicates a bare velocity correlation function (proportional to \( \tilde{ub} \)). A wavy line indicates a magnetic field, a solid triangle indicates a \( ub \) vertex. This contributes of \( \alpha_D \). (b) A solid line indicates a bare magnetic field response function, a broken line indicates a bare velocity response function, a 'o' joined by two broken lines indicates a bare magnetic field correlation function (proportional to \( \tilde{D}_2 \)), a wavy line indicates a magnetic field, a solid triangle indicates a \( ub \) vertex. This contributes to \( \alpha_F \).

B. Suppression of growth rate: Nonlinear feedback

When the magnetic fields become strong neglecting the feedback of the magnetic fields in the form of the Lorentz force is no longer justified. So we work with the full Eqs. (3) and (4). We here follow a diagrammatic perturbation approach. In presence of the Lorentz force there is an additional contribution to \( \alpha \) (Fig.1b).

\[
\langle (u \times b_\nu) \rangle_F = \left( \int_q \epsilon_{ijp} u_j(q, t) b_p(k - q, t) \right)
\]

(18)

\[
= \left( \frac{i}{2} \epsilon_{ijp} \int_q P_{jm\nu}(q) G_\nu^\circ(q, t - t_1) b_m(q_1, t_1) b_n(q_1 - q, t_1) G_\nu^b(k - q, t) b_p(k - q, t = 0) \right)
\]

(19)

which gives (\( F \) refers to feedback)

\[
\alpha B_\alpha(k, t) = i\epsilon_{ijp} \int_q P_{jm\nu}(q) e^{2\alpha_D |q| t} e^{-2\nu q^2} b_n(k, t) \frac{2i\tilde{D}_2(q) \epsilon_{mpq} q_p}{2\alpha_D |q| - 2\mu q^2}.
\]

(20)

This on simplification gives

\[
\alpha_F = \frac{2S_3}{3} \frac{4}{15} \int_q \frac{\tilde{D}_2(q, t)}{\alpha_D |q| - 2\mu q^2},
\]

(21)

where \( \tilde{D}_2(q, t) = \exp[2\alpha_D |q| t - 2\mu q^2 t] \tilde{D}_2(q) \) is a growing function of time for small wavenumbers. Thus \( \alpha_F \) grows in time.

Thus, at a late time \( t \), when the non-linear feedback on the velocity field due to the Lorentz force is no longer negligible, i.e., for a finite \( \alpha_D \) and \( \alpha_F \) we find self-consistently,

\[
\alpha_D = -\frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1(q)}{\nu[(\alpha_D + \alpha_F)|q| - (\nu + \mu)q^2]}.
\]

(22)

\[
\alpha_F = \frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(q, t)}{[(\alpha_D + \alpha_F)|q| - 2\mu q^2].}
\]

(23)
Thus net growth rate ∝ |(α_D + α_F)|k| for the mode B_i(k, t).

Let us now consider various k dependences of D_1(k) and D_2(k). For the case when the background velocity field is driven by the Navier-Stokes equation with a conserved noise (thermal noise) one requires to have D_1(k) = D_1 k^2, D_1 = D_1 k, giving (v_i(k, t) v_i(-k, t)) = constant. If we assume similar k-dependences for \( b_i(k, 0) b_i(-k, 0) \) then we require D_2(k) ∼ constant and D_2(k) = \( \Delta_2 \). These choices give

\[
\alpha_D = -\frac{2s_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1 q}{q |(\alpha_D + \alpha_F)|q - (\nu + \mu)q^2},
\]

\[
\alpha_F = \frac{2s_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(t) q^{-5/3}}{|(\alpha_D + \alpha_F)|q - 2\mu q^2},
\]

which remain finite even as the system size diverges.

Fully developed turbulent state, characterised by K41 energy spectra, is generated by D_1(k) ∼ k^{-3} and \( \tilde{D}_1(k) = \tilde{D}_1 k^{-4} \). In addition we assume that the initial magnetic field correlations also have k41 scaling then D_2(k) ∼ k^{-5/3} and \( \tilde{D}_2(k) = \tilde{D}_2 k^{-8/3} \). If one starts with a K41-type initial correlations for the magnetic fields, then at a later time the scale dependence for the magnetic field correlations are likely to remain same; only the amplitudes grow. Notice that the spectra diverge as k → 0, i.e., as the system size diverges. This is a typical characteristic of fully developed turbulence. For such a system self-consistently,

\[
\alpha_D = -\frac{2s_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1 q^{-4}}{q |(\alpha_D + \alpha_F)|q - (\nu + \mu)q^2},
\]

\[
\alpha_F = \frac{2s_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(t) q^{-8/3}}{|(\alpha_D + \alpha_F)|q - 2\mu q^2},
\]

The notable difference between the expressions Eqs.(25) and Eqs.(27) for the α coefficients is that the α coefficients diverge with the system size when the energy spectra are singular in the infrared limit (for fully developed turbulence).

In general, at early times (small \( \alpha_F \)), \( \alpha_D \) increases exponentially in time. For \( t \lesssim \frac{\nu}{\alpha_D} \ln |\alpha_D|, \alpha_D \) and \( \alpha_F \) are comparable. The growth rate of \( \alpha_F \) comes down and \( \alpha_D \) decays. Since \( \alpha_D \) and \( \alpha_F \) have different signs, \( |(\alpha_D + \alpha_F)| \) → 0 as \( t \to \) large. Thus the net growth rate comes down to zero. Hence, Eqs.(25) and Eqs.(27) suggest that the early time growth and late time saturation of magnetic fields take place for different kind of background velocity correlations and initial magnetic field correlations. Thus dynamo instability and its saturation are rather intrinsic properties of the 3dMHD equations in a rotating frame. Our results also suggest that these processes may take place for varying magnetic Prandtl number \( \mu/\nu \). The above analysis crucially depends on the fact that \( \alpha_D \) and \( \alpha_D \) have opposite signs, which, in turn, imply that \( D_1 \) and \( D_2 \) have same signs. We have already seen that in a physically realisable situation where parity is broken entirely due to the global rotation, \( D_1 \) and \( D_2 \) indeed have the same sign. Our results also suggest that the net growth rate is proportional to the difference between the fluid and the magnetic helicities at any time \( t \).

How can we understand our results in a simple way? To do that we resort to first order smoothing approximation. In the kinematic limit, in this smoothing approximation to calculate \( \langle v \times b \rangle \), one considers only the Induction equation as \( v \) is supposed to be given. However when one goes beyond the kinematic approximation, one has to consider the Navier-Stokes equation as well. Thus in the first-order smoothing approximation one writes the equations for the fluctuations \( v \) and \( b \) as (to the first order)

\[
\frac{\partial b}{\partial t} \approx \nabla \times (v \times B) + \nabla \times (\nabla \times b),
\]

and

\[
\frac{\partial v}{\partial t} \approx \ldots + (\nabla v) b,
\]

where the ellipsis refer to all other terms in the Navier-Stokes equation and \( \nabla \) and \( \nabla \) are the large scale (mean field) part of the velocity and magnetic fields. With these we can write

\[
\langle v \times b \rangle_t = \langle \epsilon_{ijk} v_j b_k \rangle = \langle \epsilon_{ijk} v_j B_m \frac{\partial}{\partial x_m} v_i \rangle + \langle \epsilon_{ijk} b_p B_m \frac{\partial}{\partial x_m} b_i \rangle \equiv \alpha_{im} B_m + \ldots
\]

Here the ellipsis refer to non-α terms in the expansion of \( \langle v \times b \rangle \) (see Eq.(4)). Thus for isotropic situations \( \alpha = \frac{1}{3} [-\langle v, (\nabla \times v) \rangle + \langle b, (\nabla \times b) \rangle] \). Thus \( \alpha \) is proportional to the difference in the fluid and magnetic helicities, a result we have already obtained through a more detailed calculation above. In our model fluid helicity is statistically constant in time, but magnetic helicity grows in time and hence \( \alpha \to 0 \) in the long time limit.
IV. SUMMARY

Thus in conclusions we have shown how the initial exponential growth of the magnetic fields in a turbulent dynamo can be arrested by the action of the magnetic fields on the velocity fields. Our mechanism required that the parity breaking parts of the velocity and magnetic field variances must have the same sign, which must be the case in any physical system. It is also worth noting the role of the symmetries of the velocity and magnetic field correlation functions. The antisymmetric part, which is also an irreducible part of the $\langle v_i v_j \rangle$ tensor is responsible for the growth. Again the antisymmetric (irreducible) part, part of the $\langle b_i b_j \rangle$ tensor is responsible for stabilisation. Even though our explicit calculations were done by using simple initial conditions for the calculational convenience, the results that we obtain are general enough and it is obvious that the feedback mechanism is independent of the details of the initial conditions. Thus our results should be valid for more realistic initial conditions also. We have also demonstrated that our results can be easily understood within the first-order smoothing approximation. From the point of view of nonlinear systems, our results can be interpreted as 'non-linear stabilisation of linear instabilities', qualitatively similar to the well-known example of the Kuramoto-Shivanisky (KS) equation in one and two dimensions. This is linearly unstable for small wavenumbers, but the nonlinear term stabilises it. In fact, the long wavelength properties of the KS equation is same as that of the stochastically driven Kardar-Parisi-Zhang equation (KPZ) in one and two dimensions. However we must add words of caution while making the comparison between the KS-KPZ problem and the dynamo problem. In the KS-KPZ case the KS equation is not stochastically driven. The long wavelength instability serves as drive. However, in the present case of dynamo, the velocity field (or the NS equation) is stochastically driven. Recently it has been shown that the stochastically and determinitically driven NS equations belong to the same multiscaling universality class (in the inertial frames). Even though the same has not been shown for 3dMHD, it is probably true there too. So with some confidence we can draw the analogy between the KS-KPZ problem and the dynamo problem which we discussed here. An important issue is still however left open. In fully developed 3dMHD, in the steady state, correlation and response functions exhibit dynamical scaling and the dynamic exponent $z = 2/3$, which means renormalise dissipations (kinetic as well as magnetic) diverge $\sim k^{-4/3}$ for a wavenumber $k$ belonging to the inertial range. Even for decaying MHD with initial K41-type correlations this turns out to be true where equal time correlations exhibit dynamical scaling with $z = 2/3$. The question is, what it is in the initial transient of dynamo growth? K41-type of spectra suggest the roughness exponent for both the velocity and the magnetic is $1/3$. This, together with Galilean invariance for the 3dMHD equations give $z = 2/3$. So, unless $\alpha$ coefficients too pick up divergent corrections dynamo growth will be subdominant to dissipative decay in the long wavelength limit. However, as our results (Eqs.27) suggest, the alpha coefficients diverge in the long wavelength limit, indicating that they pick up $k$-dependent singular corrections. Simple minded calculations suggest $\alpha \sim k^{-1/3}$. If this is really the case then neither growth nor dissipative decay dominate in the inertial range; the sign of $(\alpha - \mu)$ determines it. However this kind of RG calculations suffer from few technical problems. Thus to settle it conclusively one requires more sophisticated technique and/or numerical simulations. This issue of divergent effective viscosities in the inertial range assumes importance as it may help to overcome some of the non-linear restrictions as discussed by Vainshtein and Cattaneo. A system of magnetohydrodynamic turbulence in a rotating frame, after the saturation time (i.e., after which there is no net dynamo growth) belongs to the universality class of usual three-dimensional magnetohydrodynamic turbulence in a laboratory. This can be seen easily as both in the lab and rotating frames, the roughness and the dynamic exponents can be calculated exactly by using the Galilean invariance and noise-nonrenormalisation conditions. However, the same cannot be immediately said about the multiscaling exponents of the higher order structure functions. Further investigation is required in this direction. It will also be very interesting to find out the detailed quantitative dependences of $\alpha$ on the magnetic Prandtl number $(P_m)$ in view of the recent results that $P_m$ is connected to other dimensionless numbers like the ratio of the total cross helicity to the kinetic energy in the steady state.

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H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids*, Cambridge University Press, Cambridge (1978).

P. P. Kronberg, *Rep. Prog. Phys.* **57**, 325 (1994); R. Beck, A. Brandenburg, D. Moss, A. Shokorov, and D. Sokoloff, *Annual Review of Astronomy and Astrophysics*, **34**, 155 (1996); K. Enqvist, *Int. J. Mod. Phys.* **D7**, 331 (1998).

A. Basu, *Phys Rev. E* **56**, 2869 (1997).

A. Pouquet, U. Frisch and J. Léorat, *J. Fluid Mech.* **77**, 321 (1976).

H. K. Moffatt, *J. Fluid Mech.* **53**, 385 (1972).

S. L. Vainshtein and F. Cattaneo, *Astrophysical Jl.* **393**, 165 (1992).

G. B. Field, E. G. Blackman and H. Chou, *ApJ* **513**, 638 (1999).

I. Rogachevskii and N. Kleeorin, *Phys. Rev. E*, **64**, 056307 (2001).

A. Brandenburg, *ApJ*, **550**, 824 (2001).

A. Bhattacharjee and Y. Yuan, *ApJ*, **449**, 739 (1995).

P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.*, **49**, 435 (1977).

D. Forster, D. R. Nelson, and M. J. Stephen, *Phys. Rev A*, **16**, 732 (1977).

A. Basu, A. Sain, S. K. Dhar and R. Pandit, *Phys. Rev. Lett.*, **81**, 2687 (1998).

A. Basu and J. K. Bhattacharjee, *Europhys. Lett.*, **46**, 183 (1997).

J. D. Jackson, *Classical Electrodynamics*, 2nd edn., Wiley Eastern, New Delhi, (1975).

U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

V. Yakhot and S. A. Orzag, *J. Sci. Comput.*, **1**, 1 (1986).

A. Raichudhuri, *The Physics of Fluids and Plasmas*, Cambridge University Press, Cambridge (1998).

C. Jayaprakash, F. Hayot and R. Pandit, *Phys. Rev. Lett.*, **71**, 12 (1993).

A. Basu, J. K. Bhattacharjee and S. Ramaswamy, *Eur. Phys. J B*, **9**, 725 (1999).

A. Sain, Manu and R. Pandit *Phys. Rev. Lett.*, **81**, 4377 (1998).

J-D Fournier, P-L Sulem and A. Pouquet, *J. Phys. A* **15**, 1393 (1982).

A. Basu, J. K. Bhattacharjee and S. Ramaswamy, *Eur. Phys. J B*, **9**, 425 (1999).

A. Basu, *Phys. Rev. E*, **61**, 1407 (2000).

A. Basu, unpublished.