Nambu-Goto Action and Qubit Theory

In Any Signature and in Higher Dimensions

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Abstract

We perform an extension of the relation between the Nambu-Goto action and qubit theory. Of course, the Cayley hyperdeterminant is the key mathematical tool in such generalization. Using the Wick rotation we find that in four dimensions such a relation can be established not only in (2+2)-dimensions but also in any signature. We generalize our result to a curved space-time of (2^{2n}+2^{2n})-dimensions and (2^{2n+1}+2^{2n+1})-dimensions.

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Some years ago, Duff [1] discovers hidden new symmetries in the Nambu-Goto action [2]-[3]. It turns out that the key mathematical tool in such a discovery is the Cayley hyperdeterminant [4]. In this pioneer work, however, the target space-time turns out to have an associated (2 + 2)-signature, corresponding to two time and two space dimensions. It was proved in Ref. [5]-[6] that the Duff’s formalism can also be generalized to (4 + 4)-dimensions and (8 + 8)-dimensions. Here, we shall prove that if one introduces a Wick rotations for various coordinates then one can actually extend the Duff’s procedure to any signature in 4-dimensions. Moreover, we also prove that our method can be extended to curved space-time in \((2^{2n} + 2^{2n})\)-dimensions and \((2^{2n+1} + 2^{2n+1})\)-dimensions.

There are a number of physical reasons to be interested on these developments, but perhaps the most important is that eventually our work may be useful on a possible generalization of the remarkable correspondence between black-holes and quantum information theory (see Refs. [7]-[10] and references therein).

Let us start recalling the Duff’s approach on the relation between the Nambu-Goto action and the (2 + 2)-signature. Consider the Nambu-Goto action [2]-[3],

\[
S = \int d\xi^2 \sqrt{\epsilon \det(\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu})}.
\] (1)

Here, the space-time coordinates \(x^\mu\) are real function of two parameters \((\tau, \sigma) = \xi^a\) and \(\eta_{\mu\nu}\) is a flat metric, determining the signature of the target space-time. Moreover, the parameter \(\epsilon\) takes the values +1 or −1, depending whether the signature of \(\eta_{\mu\nu}\) is Euclidean or Lorenziana, respectively.

It turns out that by introducing the world-sheet metric \(g^{ab}\) one can prove that (1) is equivalent to the action [11] (see also Ref. [12] and references therein)

\[
S = \int d\xi^2 \sqrt{-\epsilon \det g g^{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu},}
\] (2)

which is, of course, the Polyakov action (see Ref. [12] and references therein). In fact, from the expression

\[
\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} - \frac{1}{2} g_{ab} g^{cd} \partial_c x^\mu \partial_d x^\nu \eta_{\mu\nu} = 0,
\] (3)

obtained by varying the action (2) with respect to \(g^{ab}\), it is straightforward to show that from (2) one obtains (1) and \textit{vice versa}. Hence, the actions (1) and (2) are equivalents.

It is convenient to define the induced world-sheet metric
Using this definition, the Nambu-Goto action (1) becomes

$$S = \int d\xi^2 \sqrt{\epsilon \det(h_{ab})}. \quad (5)$$

It is not difficult to see that in (2 + 2)-dimensions the expression (4) can be written as

$$h_{ab} = \partial_a x^{ij} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl}, \quad (6)$$

where \(x^{ij}\) denotes a the 2\(\times\)2- matrix

$$x^{ij} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -x^2 + x^4 & x^1 - x^3 \end{pmatrix}. \quad (7)$$

It is important to observe that (7) corresponds to the set \(M(2, R)\) of any 2\(\times\)2-matrix. In fact, by introducing the fundamental base matrices

$$\delta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon^{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (8)$$

$$\eta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda^{ij} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

one observes that (7) can be rewritten as the linear combination

$$x^{ij} = x^1 \delta^{ij} + x^2 \varepsilon^{ij} + x^3 \eta^{ij} + x^4 \lambda^{ij}. \quad (9)$$

Let us now introduce the expression

$$h = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} h_{ac} h_{bd}. \quad (10)$$

If one uses (4) one gets

$$h = \det(h_{ab}). \quad (11)$$

However, if one considers (6) one obtains

$$h = \mathcal{Det}(h_{ab}), \quad (12)$$

where \(\mathcal{Det}(h_{ab})\) denotes the Cayley hyperdeterminant of \(h_{ab}\), namely

$$\mathcal{Det}(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a x^{ij} \partial_c x^{kl} \partial_b x^{mn} \partial_d x^{rs}. \quad (13)$$
Of course, (11) and (12) imply that
\[ \text{det}(h_{ab}) = \mathcal{D} \text{et}(h_{ab}). \] (14)
In turn, (14) means that in (2 + 2)-dimensions the Nambu-Goto action (5) can also be written as
\[ S = \int d\xi^2 \sqrt{\mathcal{D} \text{et}(h_{ab})}. \] (15)
Note that, since in this case one is considering the (2 + 2)-signature one must set \( \epsilon = +1 \) in (5).
In (4 + 4)-dimensions the key formula (6) can be generalized as
\[ h_{ab} = \partial_a x^{ijm} \partial_b x^{kl} \delta_{ik} \delta_{jl} \eta_{ms}. \] (16)
While in (8 + 8)-dimensions one has
\[ h_{ab} = \partial_a x^{ijmn} \partial_b x^{kl} \delta_{ik} \delta_{jl} \eta_{ms} \eta_{nr}. \] (17)
(see Refs. [5] and [6] for details). So by considering the real variables \( x^{i_1...i_n} \) and properly considering the matrices \( \delta_{ij} \) and \( \eta_{ij} \) the previous formalism can be generalized to higher dimensions. Of course, in such cases the Cayley hyperdeterminant \( \mathcal{D} \text{et}(h_{ab}) \) must be modified accordingly.
Observing (7) one wonders whether one can consider in (6) other signatures in 4-dimensions besides the (2+2)-signature. It is not difficult to see that using the Wick rotation in any of the coordinates \( x^1, x^2, x^3 \) or \( x^4 \) one can modify the signature. For instance, one can achieve the (1 + 3)-signature if one uses the prescription \( x^2 \rightarrow ix^2 \) in (6). This method lead us inevitable to generalize our method to a complex structure. One simple introduce the complex matrix
\[ z^{ij} = z^1 \delta^{ij} + z^2 \epsilon^{ij} + z^3 \eta^{ij} + z^4 \lambda^{ij}, \] (18)
where the variables \( z^1, z^2, z^3 \) and \( z^4 \) are complex numbers. The expression (6) is generalized accordingly as [13]
\[ h_{ab} = \partial_a z^{ij} \partial_b z^{kl} \delta_{ik} \delta_{jl}. \] (19)
Thus, in this case, the Cayley hyperdeterminant becomes
\[ \mathcal{D} \text{et}(h_{ab}) = \frac{1}{2!} \epsilon^{ab} \epsilon^{cd} \epsilon_{ik} \epsilon_{jl} \epsilon_{ms} \epsilon_{nr} \partial_a z^{ij} \partial_b z^{kl} \partial_a z^{mn} \partial_b z^{rs}. \] (20)
and consequently the Nambu-Goto action must be written using (20). Of course, the Nambu-Goto action, or the Polyakov action, must be real and
therefore one must choose any of the coordinates $z^1, z^2, z^3$ and $z^4$ in (20) either as pure real or pure imaginary.

Similarly, the generalization to a complex structure can be made by introducing the complex variables $z^{i_1...i_n}$ and writing

$$
\text{Det}(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{i_1j_1...i_n-j_n} \eta_{i_1...i_n-1} \eta_{j_1...j_n} \cdot \partial_a z^{i_1...i_n} \partial_b z^{j_1...j_n} \partial_c z^{k_1...k_n} \partial_d z^{l_1...l_n}.
$$

(21)

or

$$
\text{Det}(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{i_1j_1...i_n-j_n} \eta_{i_1...i_n-1} \eta_{j_1...j_n} \cdot \partial_a z^{i_1...i_n} \partial_b z^{j_1...j_n} \partial_c z^{k_1...k_n} \partial_d z^{l_1...l_n},
$$

(22)

depending whether the signature is $(2^{2n} + 2^{2n})$ or $(2^{2n+1} + 2^{2n+1})$, respectively.

One can further generalize our procedure by considering a target curved space-time. For this purpose let us introduce the curved space-time metric

$$
g_{\mu\nu} = e^A_{\mu} e^B_{\nu} \eta_{AB}.
$$

(23)

Here, $e^A_{\mu}$ denotes a vielbein field and $\eta_{AB}$ is a flat metric. The Polyakov action in a curved target space-time becomes

$$
S = \int d\xi^2 \sqrt{-\epsilon} \text{det} g^{\mu\nu} \partial_\mu x^a \partial_\nu x^b g_{ab}.
$$

(24)

Using (23), one sees that this action can be written as

$$
S = \int d\xi^2 \sqrt{-\epsilon} \text{det} g^{\mu\nu}(\partial_\mu x^a e^A_{\mu})(\partial_\nu x^b e^B_{\nu}) \eta_{AB}.
$$

(25)

So, by defining the quantity

$$
E^A_a \equiv \partial_\mu x^a e^A_{\mu},
$$

(26)

the action in (25) reads as

$$
S = \int d\xi^2 \sqrt{-\epsilon} \text{det} g^{ab} E^A_a E^B_b \eta_{AB}.
$$

(27)

Hence, in a target space-time of $(2 + 2)$-dimensions one can write (27) in the form

$$
S = \int d\xi^2 \sqrt{-\epsilon} \text{det} g^{ab} E^{ij}_a E^{kl}_b \varepsilon_{ik} \varepsilon_{jl},
$$

(28)

where

$$
E^{ij}_a \equiv \partial_\mu x^a e^{ij}_{\mu}.
$$

(29)
Here, we considered the fact that one can always write
\[ e_{ij}^\mu = \varepsilon_1^\mu \delta_{ij} + e_2^\mu \varepsilon_{ij} + e_3^\mu \eta_{ij} + e_4^\mu \lambda_{ij}. \]  
(30)

Observe that in this development one can consider a generalization of (4) namely
\[ h_{ab} = E_a^A E_b^B \eta_{AB} \]  
(31)
and therefore in \((2 + 2)\)-dimensions this expression becomes
\[ h_{ab} = \varepsilon_{ik}^{ab} \varepsilon_{jl}^{kl}, \]  
(32)
while in \((4 + 4)\)-dimensions and \((8 + 8)\)-dimensions one obtains
\[ h_{ab} = \varepsilon_{ik}^{ijm} \varepsilon_{jl}^{kkr} \eta_{mr} \]  
(33)
and
\[ h_{ab} = \varepsilon_{ik}^{ijmn} \varepsilon_{jl}^{kls} \varepsilon_{mr} \varepsilon_{ns}, \]  
(34)
respectively.

At this stage, it is evident that if one wants to generalize the procedure to any signature in a curved space-time one simply substitute in the action \((27)\) either
\[ h_{ab} = \varepsilon_{i_1...i_n}^a \varepsilon_{j_1...j_n}^b \varepsilon_{ik...i_{n-1}j_{n-1}} \eta_{i_n j_n} \]  
(35)

or
\[ h_{ab} = \varepsilon_{i_1...i_n}^a \varepsilon_{j_1...j_n}^b \varepsilon_{ik...i_{n-1}j_{n-1}} \varepsilon_{i_n j_n}, \]  
(36)
depending whether the signature is \((2^{2n} + 2^n)\) or \((2^{2n+1} + 2^{2n+1})\), respectively. Here, we used the prescription \(E_{a_1...a_n} \rightarrow \varepsilon_{a_1...a_n}\), with \(\varepsilon_{a_1...a_n}\) a complex function.

In order to include \(p\)-branes in our formalism, one notes that the expression \((35)\) and \((36)\) can still be used. In such a case, one allows the indice \(a\) in \((35)\) and \((36)\) to run from 0 to \(p\). Braking such kind of indices as \(a = (\hat{a}_1, \hat{a}_2)\) for a 3-brane, as \(a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)\), for a 5-brane and so on one observes that \((35)\) and \((36)\) can be written as
\[ h_{a_1...a_2 b_1...b_2} = \varepsilon_{a_1...a_2}^{i_1...i_p} \varepsilon_{b_1...b_2}^{j_1...j_p} \varepsilon_{ik...i_{p-1}j_{p-1}} \eta_{i_p j_p} \]  
(37)
or
\[ h_{\hat{a}_1...\hat{a}_2 \hat{b}_1...\hat{b}_2} = \varepsilon_{\hat{a}_1...\hat{a}_2}^{i_1...i_p} \varepsilon_{\hat{b}_1...\hat{b}_2}^{j_1...j_p} \varepsilon_{ik...i_{p-1}j_{p-1}} \varepsilon_{i_p j_p}, \]  
(38)
respectively. The analogue of Cayley hyperdeterminant in this case will be

\[ \hat{\text{Det}}(\hat{h}_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2}) = \epsilon^{\hat{a}_1\hat{b}_1...\epsilon^{\hat{a}_p\hat{b}_p} \epsilon^{j_1...j_p} \epsilon_{i_1k...\epsilon_{i_{p-1}j_{p-1}}\epsilon_{i_pj_p}} \]

and therefore the corresponding Nambu-Goto action becomes

\[ S = \int d\xi^{p+1} \sqrt{\epsilon \hat{\text{Det}}(\hat{h}_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2})}. \] (40)

Summarizing, we have generalized the Duff’s procedure concerning the combination of the Nambu-Goto action and the Cayley hyperdeterminant in target space-time of (2+2)-dimensions. Such a generalization first corresponds to a curved worlds with \((2^{2n} + 2^{2n})\)-signature or \((2^{2n+1} + 2^{2n+1})\)-signature. Using complex structure we may be able to extend the procedure to any signature. Further, we generalize the method to \(p\)-branes.

It turns out that these generalization may be useful in a number of physical scenario beyond string theory and \(p\)-branes. In fact, since the quantity \(z^{j_1...j_n}\) can be identified with a \(n\)-qubit one may be interested in the route leading to oriented matroid theory [14] (see also Ref. [15]-[16]). In this direction, using the phirotpe concept (see Ref. [17] and references therein), which is a complex generalization of the concept of chirotpe in oriented matroid theory, a link between super \(p\)-branes and qubit theory has already been established [17]. Thus, it may be interesting for further developments to explore the connection between the results of the present work and supersymmetry via the Grassmann-Plücker relations (see Refs. [8]-[9] and references therein). It is worth mentioning that such relations are natural mathematical notions in information theory linked to \(n\)-qubit entanglement. Indeed, in such a case, the Hilbert space can be broken in the form \(C^{2n} = C^L \otimes C^l\) with \(L = 2n - 1\) and \(l = 2\). This allows a geometric interpretation in terms of the complex Grassmannian variety \(Gr(L, l)\) of 2-planes in \(C^{2n}\) via the Plücker embedding. In this context, the Plücker coordinates of Grassmannians \(Gr(L, l)\) are natural invariants of the theory (see Ref. [9] for details). However, it has been mentioned in Ref. [18], and proved in Refs. [19] and [20], that for normalized qubits the complex 1-qubit, 2-qubit and the 3-qubit are deeply related to division algebras via the Hopf maps, \(S^3 \xrightarrow{S^1} S^2, S^7 \xrightarrow{S^3} S^4\) and \(S^{15} \xrightarrow{S^7} S^8\), respectively. In order to clarify the possible application of these observations in the context of our formalism let us consider the general complex state \(\mid \psi \rangle \in C^{2n}\),

\[ \mid \psi \rangle = \sum_{i_1 i_2...i_n=0}^1 \ z^{i_1i_2...i_n} \mid i_1 i_2...i_n \rangle, \] (41)
where $|i_1 i_2 \ldots i_n> = |i_1> \otimes |i_2> \otimes \ldots \otimes |i_n>$ correspond to a standard basis of the $n$-qubit. It is interesting to make the following observations. First, let us denote a $n$-rebit system (real $n$-qubit) by $x^{i_1 i_2 \ldots i_n}$. So, one finds that a 3-rebit and 4-rebit have 8 and 16 real degrees of freedom, respectively. Thus, one learns that the 4-rebit can be associated with the 16 degrees of freedom of a 3-qubit. It turns out that this is the kind of embedding discussed in Ref. [9]. In this context, one sees that in the Nambu-Goto context one may consider the 16-dimensions target space-time as the maximum dimension required by division algebras via the Hopf map $S^{15} \rightarrow S^8$.

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References

[1] M. J. Duff, Phys. Lett. B 641 (2006) 335; hep-th/0602160.
[2] Y. Nambu, Lectures at the Copenhagen Symposium, 1970, unpublished.
[3] T. Goto, Progr. Theor. Phys. 46 (1971) 1560.
[4] A. Cayley, J. Camb. Math. 4 (1845) 193.
[5] J. A. Nieto, Phys. Lett. B 692 (2010) 43; arXiv:1004.5372 [hep-th].
[6] J. A. Nieto, Phys. Lett. B 718 (2013) 1543; arXiv:1210.0928 [hep-th].
[7] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim and W. Rubens, Phys. Rept. 471 (2009) 113; arXiv: hep-th/0809.4685.
[8] P. Levay, Phys. Rev. D 74 (2006) 024030; arXiv: hep-th/0603136.
[9] P. Levay, J. Phys. A 38 (2005) 9075.
[10] P. Levay, Phys. Rev. D 82 (2010) 026002; arXiv:1004.2346 [hep-th].
[11] A. Polyakov Phys. Lett. B 103 (1981) 207.

[12] M. Green, J. Schwarz, E. Witten, *Superstring Theory* (Cambridge U. Press, Cambridge, UK, 1987).

[13] H. Larraguível, G. V. López and J. A. Nieto, work in progress (2015); Bachelor Thesis ”Antropic Gravity, Black-Holes and Q-bits”, H. Larraguível, Physics Deparment, Guadalajara University, December (2015).

[14] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, Oriented Matroids, (Cambridge University Press, Cambridge, 1993).

[15] J. A. Nieto, Adv. Theor. Math. Phys. 10 (2006) 747; hep-th/0506106.

[16] J. A. Nieto, Adv. Theor. Math. Phys. 8 (2004) 177; hep-th/0310071.

[17] J. A. Nieto, Nucl. Phys. B 883 (2014) 350; arXiv:1402.6998 [hep-th].

[18] R. Mosseri and R. Dandoloff, J. Phys. A: Math. Gen. 34, (2001) 10243.

[19] R. Mosseri, “Two and Three Qubits Geometry and Hopf Fibrations”; arXiv:quant-ph/0310053.

[20] B. A. Bernevig and H. D. Chen, J. Phys. A; Math. Gen. 36, (2003) 8325.