1. Introduction

Thermally driven turbulence is omnipresent in nature and technology. The thermal driving can be thanks to the temperature boundary conditions such as in Rayleigh-Bénard convection (RBC)—a flow in a container heated from below and cooled from above (Ahlers et al., 2009; Chilla & Schumacher, 2012; Lohse & Xia, 2010)—or in horizontal convection (HC) (Hughes & Griffiths, 2008; Shishkina & Wagner, 2016; Shishkina et al., 2016), where parts of the top, bottom, or sidewalls of the container are set at different temperatures. However, the thermal driving can also be thanks to internal heating, where the temperature field is driven by some forcing in the bulk. In many cases in nature, both ways of driving play a role at the same time. For example, this holds for the Earth's mantle due to the driving through the hot inner core of the Earth and an additional driving due to the decay of radioactive materials, producing heat (Bercovici et al., 1989; Bunge et al., 1996; Houseman, 1988; Lay et al., 2008; Mallard et al., 2016; Moore & Webb, 2013; Schubert et al., 2001; Tackley et al., 1993). Thus, in the Earth's mantle, about 10%–20% of the heat is transferred from the core, while the rest occurs due to the internal heating (Schubert et al., 2001). The internal heating dominates also in the atmosphere of Venus (Tritton, 1975; Tritton and Zarraga, 1967), which is heated up due to the absorption of solar light. One more example is the formation of Pluto's polygonal terrain, which is caused not only by convection of Rayleigh-Bénard type (McKinnon et al., 2016; Trowbridge et al., 2016), but also by internally heated convection (IHC) (Vilella & Deschamps, 2017). And, of course, IHC is relevant in many engineering applications, for example, liquid-metal batteries (Kim et al., 2013; Xiang & Zikanov, 2017).
and Spiegel (2012), Goluskin and van der Poel (2016), Goluskin (2016), and Vilella et al. (2018). For the former case, Grossmann and Lohse (GL) have developed a unifying theory (Grossmann & Lohse, 2000, 2001, 2002, 2004; Stevens et al., 2013), with which the heat transfer and the degree of turbulence can quantitatively be described as function of the control parameters, in excellent agreement with the experimental and numerical data over a range of more than 7 orders of magnitude in the control parameters $Ra$ and $Pr$. Later this theory was also extended to HC (Shishkina et al., 2016) and double diffusive convection (Y. Yang et al., 2018). GL arguments were also applied to IHC, to estimate the bulk temperature for small and moderate $Pr$ (Goluskin & Spiegel, 2012). A complete theory, however, does not yet exist for purely IHC.

The objective of the present work is to apply the reasoning of GL’s theory to the case of purely IHC and to develop a unifying theory for this case. In addition, we perform direct numerical simulations (DNS) of turbulent purely IHC over a large range of control parameters and compare the DNS results with the theoretical predictions. The DNS are conducted in two dimensions (2-D), as (i) the theory is based on Prandtl’s equations, which are also 2-D in spirit, as (ii) 2-D and 3-D thermally driven turbulence show very close analogies with respect to the integral quantities, in particular for large Prandtl numbers $Pr \geq 1$ (van der Poel et al., 2013), and as (iii) otherwise, due to unavoidable limitations in available CPU time, we could explore only a much smaller portion of the parameter space.

2. Control and Response Parameters and Governing Equations

In RBC, next to the geometric aspect ratio $\Gamma$ of the sample (the ratio between lateral and vertical extensions), the control parameters of the system are the temperature difference between top and bottom wall (in dimensionless form, the Rayleigh number) and the ratio between kinematic viscosity $\nu$ and thermal diffusivity $\kappa$, namely the Prandtl number $Pr = \nu / \kappa$. The response of the system consists of the heat flux from bottom to top (in dimensionless form, the Nusselt number $Nu$) and the degree of turbulence (in dimensionless form, the Reynolds number $Re$). In IHC, instead of the Rayleigh number, the dimensionless driving strength $Rr$ of the temperature field takes the role of the second control parameter, next to $Pr$. It is often called Rayleigh-Roberts number (and that is why we use the abbreviation $Rr$) and will be defined below. The main response parameter, next to $Re$, is the mean temperature which the bulk achieves thanks to the internal driving. This is related to the heat fluxes into the top and bottom plates; note that they are different from each other. So the objective of this paper is to explain how the mean temperature and the Reynolds number in turbulent IHC depend on $Rr$ and $Pr$, for large enough aspect ratio $\Gamma$ of the sample.

The flow in IHC is confined between two parallel plates with distance $L$, with the gravitational acceleration $g \equiv -g\varepsilon$ acting orthogonally to these plates. The underlying dynamical equations within the Boussinesq approximation are the compressibility condition $\partial_i u_i = 0$, and

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_j^2 u_i + \beta g \delta_{ij} \theta,$$

$$\partial_t \theta + u_j \partial_j \theta = \kappa \partial_j^2 \theta + \Omega,$$

for the velocity field $u(x, t)$, the kinematic pressure field $p(x, t)$, and the reduced temperature field $\theta(x, t) \equiv T(x, t) - T_{\text{plate}}$. Here $T_{\text{plate}}$ is the temperature of both top and bottom plates, $\beta$ is the thermal expansion coefficient, $\delta_{ij}$ the Kronecker delta and $\Omega$ the constant bulk driving of the temperature field, which in nondimensional form is called Rayleigh-Roberts number

$$Rr = \beta g L^2 \Omega / (\kappa^2 \nu).$$

Equations 1 and 2 are supplemented by the boundary conditions (BCs) $u_i = 0$ and $\theta = 0$ at both plates. Periodic BCs are used in the horizontal direction.

The main responses of the system can be expressed in terms of the mean temperature $\Delta \equiv \langle \theta(x, t) \rangle_{V}$ achieved in the system, where the average $\langle \cdot \rangle_{V}$ is over volume and time. The nondimensional form of this response parameter is

$$\tilde{\Delta} = \kappa \Delta / (\Omega L^2).$$
The other main nondimensional response parameter is the Reynolds number \( Re = \frac{UL}{\nu} \), with \( U = \sqrt{\langle u_z^2 \rangle} \). There are different definitions of the Reynolds number (Ahlers et al., 2009), while these different \( Re \) usually have similar power-law dependence on \( Ra \). In DNS, one usually looks at the \( Re \) based on the global volume averaged root-mean-square velocity, as this \( Re \) reflects the flow strength of the whole flow field (Shishkina & Horn, 2016; Stevens et al., 2018; van der Poel et al., 2013; Wang et al., 2020a, 2020b; R. Yang et al., 2020).

Obviously, due to the internal heating, the heat flux \( Q(z) = \langle u_z \theta \rangle - \kappa \langle \partial_z \theta \rangle \) (or in dimensionless form \( \tilde{Q}(z) = Q(z) / (\Omega L) \)) in the system is not constant as in RBC, but depends on the height \( z \). Here, \( \langle \cdot \rangle \) means average in time and in a plane of constant \( z \). However, a simple time and plane average of Equation 2 yields that the quantity

\[
\tilde{Q}_0 = z / L - \bar{Q}(z)
\]

is constant for all \( z \) and equals \( \tilde{Q}_0 = -\bar{Q}(z = 0) = \frac{\kappa}{\Omega L} \langle \partial_z \theta \rangle \mid_{z=0} \geq 0 \).

\( \tilde{Q}_0 \) is thus a further dimensionless response parameter of the system. Equation 7 implies that the dimensionless heat flux \( \bar{Q} \) is nonpositive at \( z = 0 \). Applying Equation 6 at \( z = L \) gives the dimensionless flux at \( z = L \),

\[
\tilde{Q}(z = L) = -\frac{\kappa}{\Omega L} \langle \partial_z \theta \rangle \mid_{z=L} \geq 1 - \tilde{Q}_0 \geq 0,
\]

Relations 7 and 8 immediately show that \( 0 \leq \tilde{Q}_0 \leq 1 \).

3. Application of Grossmann and Lohse’s Unifying Theory

As it is well known (Ahlers et al., 2009; Shraiman & Siggia, 1990), in RBC exact relations between the time and volume averaged thermal and kinetic dissipation rates, \( \epsilon_\theta = \kappa \langle \partial_t \theta(x,t)^2 \rangle_v \) and \( \epsilon_u = \nu \langle \partial_t u(x,t)^2 \rangle_v \), and the dimensionless control and response parameters \( Nu, Ra, \) and \( Pr \) can be obtained from multiplying the thermal advection equation with \( \partial \theta(x,t) \) and the Navier-Stokes equation with \( u_i(x,t) \) and subsequent Gauss integration and time and space averaging. Here, we apply the same procedure to Equations 2 and 1 and obtain

\[
\epsilon_\theta = \frac{\Omega \Delta L^2}{\kappa \Omega^2 \Delta^2} = \frac{\kappa \Delta^2}{L^2} \Delta^{-1},
\]

\[
\epsilon_u = \frac{\nu^3}{L^3} R r P r^{-2} \left( \frac{1}{2} - \tilde{Q}_b \right).
\]

As \( \epsilon_u \geq 0 \) is nonnegative by definition, we can now further restrain the magnitude of the dimensionless heat flux through the bottom plate: \( 0 \leq \tilde{Q}_b \leq 1 / 2 \). Just as the corresponding relations in RBC, also here, Equations 9 and 10 relate the averaged thermal and kinetic dissipation rates with the dimensionless control (\( Rr, Pr \)) and response (\( \Lambda, \tilde{Q}_b \)) parameters.

The key idea of the GL theory (Grossmann & Lohse, 2000, 2001) is to split the kinetic and thermal dissipation rates into contributions from the corresponding boundary layers (BLs) and bulks,

\[
\epsilon_u = \epsilon_{u,BL} + \epsilon_{u,Bulk}, \quad \epsilon_\theta = \epsilon_{\theta,BL} + \epsilon_{\theta,Bulk},
\]

and to apply the respective scaling relations for those (i.e., for \( \epsilon_{u,BL}, \epsilon_{u,Bulk}, \epsilon_{\theta,BL}, \) and \( \epsilon_{\theta,Bulk} \)), based on BL theory and Kolmogorov’s theory for fully developed turbulence in the bulk. The introduced scaling regimes I, II, III, and IV correspond to BL–BL, bulk–BL, BL–bulk, and bulk–bulk dominance in \( \epsilon_u \) and \( \epsilon_\theta \), respectively. Here, one should also take into account mean thicknesses of the thermal BLs (\( \lambda_\theta \)) and viscous BLs (\( \lambda_u \)).
The cases $\lambda_0 < \lambda_u$ (large $Pr$) and $\lambda_0 > \lambda_u$ (small $Pr$) correspond to different scaling regimes, and therefore we assign the subscripts $u$ and $\ell$ to regimes I, II, III and IV, which indicate the upper-$Pr$ and lower-$Pr$ cases, respectively. Equating $\epsilon_u$ and $\epsilon_\ell$ to their estimated either bulk or BL contributions and employing the classical Prandtl scaling relations for the BL thicknesses $\lambda_0$ and $\lambda_u$ (Schlichting, 1979), one in principle obtains eight theoretically possible scaling regimes. The fractions of the phase space occupied by regimes II, III and IV are rather small, because, for example, in II, it is expected that $\lambda_0 \geq \lambda_u$ due to the BL-dominance in $\epsilon_u$, but on the other hand, $\lambda_0 \leq \lambda_u$ should hold due to the large $Pr$. By similar arguments, regime III is also small.

The mean thicknesses of the BLs are estimated as follows: $\lambda_u \sim L / \sqrt{Re}$, as in RBC (Ching et al., 2019; Grossmann & Lohse, 2000, 2001; Shishkina et al., 2015), and $\lambda_\ell = 2 / (\lambda_\ell^\text{top} + \lambda_\ell^\text{bottom})$. Approximating $\epsilon_u$ and $\ell$ at $z = 0$ and $z = H$ with the ratio of $\Delta$ and the top and bottom thermal BL thicknesses, $\lambda_\ell^\text{top}$ and $\lambda_\ell^\text{bottom}$, from Equations 4, 7, and 8 we obtain $\lambda_0 \sim L\Delta$.

The value of $\epsilon_u\text{bulk}$ is estimated as

$$\epsilon_u\text{bulk} \sim U^2 \left( \frac{L - \lambda_u}{L} \right) \approx U^3 \frac{v^3}{L^2} Re^3,$$

which is relevant in the $\epsilon_u\text{bulk}$ dominating regimes $\ell, \text{IV}_u$, and $\text{IV}_w$, while the value of $\epsilon_\ell\text{bulk}$ is estimated as

$$\epsilon_\ell\text{bulk} \sim U^2 \left( \frac{L - \lambda_\ell}{L} \right) \approx U^3 \frac{k\Delta^2}{L^2} Re,$$

which is relevant in the $\epsilon_\ell\text{bulk}$ dominating regime $\text{III}_c$. For large $Pr$ (regimes $\ell, \text{III}_u$, and $\text{IV}_u$), the thermal BL is embedded into the kinetic one and therefore in Equation 10, the magnitude of the velocity of the flow, which carries the temperature in the bulk, is reduced from $U$ to $(\lambda_0/\lambda_u)U$, leading to

$$\epsilon_\ell\text{bulk} \sim \frac{\lambda_\ell U^3}{\lambda_u} \left( \frac{L - \lambda_\ell}{L} \right) \approx \frac{k\Delta^2}{L^2} Re^{3/2} \Delta.$$ (11)

The kinetic dissipation rate in the BL is $\sim v(U / \lambda_u)^2$. Hence,

$$\epsilon_u\text{BL} \sim \frac{v^3}{L^2} Re^{3/2},$$ (12)

which is relevant in the $\epsilon_u\text{BL}$ dominating regimes $\ell, \text{I}_u$, and $\text{III}_u$. As in Grossmann and Lohse (2000, 2001), the factor $\lambda_u/L$ accounts for the volume fraction of the kinetic BL. With increasing $Pr$, $\lambda_u$ saturates to $\sim L$, so this factor becomes one (just as argued in Grossmann and Lohse [2001]), which yields

$$\epsilon_u\text{BL} \sim \frac{v^3}{L^2} Re^2.$$ (13)

For small $Ra$ or very large $Pr$, this leads to special regimes $\ell, \text{III}_u$, and $\text{III}_c$ on top of, respectively, $\text{I}_u$ and $\text{III}_u$. In $\text{III}_u$, also $\epsilon_\ell\text{bulk}$ scales differently to (11), namely as

$$\epsilon_\ell\text{bulk} \sim \frac{\lambda_\ell U^3}{L} \left( \frac{L - \lambda_\ell}{L} \right) \approx \frac{k\Delta^2}{L^2} Re^{3/2} \Delta.$$ (14)

The thermal dissipation rate in the BL scales as $\sim k(\Delta / \lambda_\ell)^2$, which is relevant in the $\epsilon_\ell\text{BL}$ dominating regimes $\ell, \text{I}_u$, $\text{I}_c$, and $\text{III}_c$. This (again with the volume fraction factor) leads to

$$\epsilon_\ell\text{BL} \sim k \left( \frac{\lambda_\ell}{\lambda_u} \right)^2 \frac{\Delta^2}{L} \approx k \frac{\lambda_\ell^2}{L^2} \lambda_u Re^3.$$ (14)

In the limiting regimes $\ell, \text{I}_u$, and $\text{I}_c$, it holds $\lambda_\ell/\lambda_0 \sim Pr^{3/2}$ (Schlichting & Gersten, 2000; Shishkina et al., 2017), while in regime $\text{I}_u$ it holds $\lambda_u/\lambda_0 \sim Pr^{3/2}$, all just as in the classical Prandtl-Blasius-Pohlhausen theory (Schlichting, 1979).

Equating $\epsilon_u$ (Equation 9) and $\epsilon_\ell$ (Equation 10) to their estimated bulk or BL contributions, we obtain the scalings of $\Delta$ and $Re$ in IHC, which are summarized in Table 1 and sketched in Figure 1.
As already mentioned above, the very same idea was already applied to HC (Shishkina et al., 2016). Interestingly enough, even a formal analogy between IHC and HC exists, out of which we could have already derived the scaling relations of Table 1 and Figure 1. The reason for this formal analogy is that the relations obtained for \( \epsilon_0 \) and \( \epsilon_\phi \) (see Equations 5 and 6 of Shishkina et al. [2016]) formally resemble the corresponding relations 9 and 10 here. For the first equation this becomes particular obvious when writing \( \epsilon_0 = \frac{L^2}{\kappa} \Omega^2 \Delta^2 = \frac{K}{L^2} R^2 \Delta^{-1} \) and for the second when realizing that \( \left( \frac{1}{2} - \bar{\Theta}_0 \right) \) is only a dimensionless factor between 0 and 1/2. Then one sees immediately that the role of the control parameter \( Ra \) in HC is taken by that of the control parameter \( Rr \) in IHC and the role of the response parameter \( Nu \) in HC is taken by that of the (inverse) response parameter \( \Delta^{-1} \) in IHC. All derived scaling relations in the different limiting regimes of HC can directly be taken over. The corresponding values for IHC give the same results as obtained above and have already been shown in Table 1 and Figure 1.

4. Comparison With Direct Numerical Simulations

To check these predictions of the GL theory generalized to IHC, we have performed 2-D DNS according to Equations 1 and 2 with the corresponding BCs. We chose an aspect ratio of \( \Gamma = 2 \) for the laterally periodic box. The numerics have been validated by making sure that the exact relations 9 and 10 are fulfilled. Simulations were performed using the second-order staggered finite difference code AFiD (van der Poel et al., 2015; Verzicco & Orlandi, 1996). This code has already been extensively used to study RBC (see Wang et al., 2020a, 2020b).

The parameter combinations \( (Rr, Pr) \) for which we performed simulations are shown in the parameter space of Figure 2a. A typical snapshot of the temperature field together with the mean temperature profile for one parameter combination are displayed in Figure 2b. One can see the stably stratified layer near the bottom plate. The interaction of the upper convection zone and the lower stably stratified region leads to the so-called penetrative convection (Veronis, 1963; Wang et al., 2019). The mean temperature profile, which, as expected and typical for IHC, displays top-down asymmetry.

The results for the response parameters \( \bar{\Delta} \) and \( Re \) as functions of the control parameters \( Rr \) and \( Pr \) are shown in Figures 3 and 4. As can be seen, in general, there are no pure scaling laws over the simulated range, but smooth crossovers from one regime to the other, very similarly as in RBC (Stevens et al., 2013), reflecting the key idea of the unifying theory by Grossmann and Lohse (2000, 2001). We first discuss the dependences for the dimensionless mean temperature \( \bar{\Delta} \), see Figures 3a and 3b. As a function of \( Pr \) (Figure 3b), for all \( Rr \) the transition from \( \bar{\Delta} \sim Pr^{-1/10} \) of regime I\(_I\) to the \( Pr^{-1/5} \) of regime I\(_\infty\) can clearly be seen. The more turbulent regimes IV\(_{u,c}\) are not yet realized, as the driving is not strong enough. This is also reflected in the \( Rr \) dependence \( \bar{\Delta} \sim Rr^{-1/5} \) reflecting that of regimes IV\(_{u,c}\). No indication to a stronger dependence as typical for the more turbulent regimes IV\(_{u,c}\) can yet be seen. This is also seen in the dependences of the Reynolds number (Figures 3c and 3d): For small \( Pr \leq 1 \), it goes like \( Re \sim Rr^{2/3} \) as in regimes IV\(_{u,c}\). For large \( Pr \geq 10 \) the results are consistent with \( Re \sim Rr^{1/4} \) as in regime I\(_\infty\). This scaling should go hand in hand with the scaling \( \bar{\Delta} \sim Rr^{-1/4} \) for the dimensionless mean...
temperature, but as seen from Figure 3a, those data are presumably better described by $\Delta \sim Rr^{-1/5}$. Finally, on the $Pr$-dependence of $Re$: As seen from Figure 3d, for all $Rr$ the data show a transition from the $Re \sim Pr^{-4/5}$ scaling of regimes $I_u, \ell$ to the $Re \sim Pr^{-1}$ scaling of regime $I_{\infty}$, consistent with the corresponding transition for $\hat{\Delta}$ in Figure 3b.

All these results are consistent with our unifying theory, which however goes much beyond the simulated parameter range into the regimes in which the kinetic and thermal energy dissipation rates are dominated by the turbulent bulk contributions. These regimes are inaccessible with our present numerical simulations, even in 2-D.

As an additional check of our unifying theory we also plot the kinetic energy dissipation rate as function of $Re$, see Figure 5. Indeed, we find $\varepsilon_u / (L^{-3} \nu^{4}) \sim Re^{5/2}$ and $\sim Re^{2}$ as characteristic for the kinetic BL dominated regimes $I_u, \ell$ and $I_{\infty}$, consistent with what we have seen in Table 1 and Figure 3.

Another (less important) response parameter of IHC is the magnitude of the dimensionless heat flux $\bar{Q}$ through the bottom plate. The numerical results for $\bar{Q}$ are shown in Figure 6. One sees from Figure 6a that $\bar{Q}$ only weakly depends on $Rr$ in the present parameter range; this behavior has also been found before in Goluskin and van der Poel (2016). Figure 6b illustrates that much less heat is transported outwards from the bottom plate with increasing $Pr$. The small $\bar{Q}$ for large $Pr$ is due to the less efficient shear-driven mixing of the fluid near the bottom plate.

5. Conclusions

In conclusion, in the spirit of the prior unifying theories for RBC (Grossmann & Lohse, 2000, 2001) and for HC (Shishkina et al., 2016), in this paper we have developed a unifying theory of IHC for the scaling of the mean temperature and the Reynolds number as functions of the control parameters $Rr$ and $Pr$. The main

Figure 2. (a) $Rr$ versus $Pr$ parameter space of the simulated cases. Symbols denote the different grid resolutions used in DNS: 512 × 256 (△), 1024 × 512 (□), 2048 × 1024 (○). (b) Instantaneous temperature field (color coded) and mean temperature profile for $Rr = 10^{10}$ and $Pr = 1$.

Figure 3. Response parameters $\hat{\Delta}$ (the dimensionless mean temperature of the bulk) and $Re$ as function of the control parameters $Rr$ and $Pr$: (a) Compensated $\Delta$ as function of $Rr$ for fixed $Pr = 10^{-1}, 1, 10$. (b) Compensated $\Delta$ as function of $Pr$ for fixed $Rr = 10^{8}, 10^{9}, 10^{10}$. (c) Compensated $Re$ as function of $Rr$ for fixed $Pr = 10^{-1}, 1, 10$. (d) $RePr$ as function of $Pr$ for fixed $Rr = 10^{8}, 10^{9}, 10^{10}$. The straight lines with the corresponding scaling laws are added as guide to the eye.
result is visualized in Figure 1. We have shown that the 2-D DNS results are consistent with this theory, though the numerically accessible regimes are still dominated by the BLs, and not all predictions of the theory can already be tested at this point. Also 3-D DNS over a large fraction of the control parameter space are presently too demanding from the viewpoint of computational cost and have not yet been done. We have furthermore pointed toward the formal analogy between IHC and HC and it will be illuminating to explore this analogy also numerically.

Data Availability Statement

The data used in this paper are available for download at http://doi.org/10.5281/zenodo.4081485.
Acknowledgments
R. Verzicco, D. Goluskin, and K. L. Ching are gratefully acknowledged for discussions and support. The authors also acknowledge the Twente Max-Planck Center, the Deutsche Forschungsgemeinschaft (Priority Programme SP 1881 "Turbulent Superstructures"), PRACE for awarding us access to MareNostrum 4 based in Spain at the Barcelona Computing Center (BSC) under PRACE project 2020245335. The simulations were partly carried out on the national e-infrastructure of SURFsara, a subsidiary of SURF cooperation, the collaborative ICT organization for Dutch education and research. Q. Wang acknowledges financial support from China Scholarship Council (CSC) and Natural Science Foundation of China under grant no. 11621202.

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