Interaction representation method for Markov master equations in quantum optics

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ABSTRACT Conditions sufficient for a quantum dynamical semigroup (QDS) to be unital are proved for a class of problems in quantum optics with Hamiltonians which are self-adjoint polynomials of any finite order in creation and annihilation operators. The order of the Hamiltonian may be higher than the order of completely positive part of the formal generator of a QDS.

The unital property of a minimal quantum dynamical semigroup implies the uniqueness of the solution of the corresponding Markov master equation in the class of quantum dynamical semigroups and, in the corresponding representation, it ensures preservation of the trace or unit operator. We recall that only in the unital case the formal generator of MME determines uniquely the corresponding QDS.

Introduction

Numerical experiments with the remarkable quantum trajectories algorithm [3], which solves the Markov master equation (MME) by the Monte-Carlo method, show that the numerical code forces a solution to be unital even in cases, when the exact minimal solution does not preserves the trace of initial state or unit initial operator, and the corresponding Poisson process explodes on finite time intervals. Unfortunately, the mathematical theory of MME still does not work for many basic equations in quantum physics. In this paper we extend the approach developed in [2] to MMEs in quantum optics.

The formal generator $L(\cdot)$ of a quantum dynamical semigroup on von Neumann algebra $B(\mathcal{H})$ of all bounded operators in a separable Hilbert space $\mathcal{H}$ is called regular if it defines the semigroup in an unambiguous way. Following [3], we assume that the coefficients of the generator $L(\cdot)$ are a densely defined symmetric Hamiltonian operator $H$ and a completely positive map $\Phi(\cdot)$. The structure of the generator $L(\cdot)$ is similar to the structure of the classical Kolmogorov–Feller equation, and similarly to the classical case, under rather general assumptions, MME has the minimal solution called a minimal quantum dynamical semigroup (QDS). Moreover, if the minimal QDS is unital, it is the unique solution of the corresponding MME; this implies regularity of the formal generator $L(\cdot)$.

In the present paper we suggest a new test for regularity of MME with the formal generator $L(\cdot)$. The idea consists in a suitable choice of some $\Lambda$-pair consisting of a “reference” operator $\Lambda$ and “interaction” part $H_{\text{int}} = \Lambda + H_{\text{s.a.}}$, which generates the interaction representation for the Hamiltonian $H = H_s + H_{\text{s.a.}} = (H_s - \Lambda) + (\Lambda + H_{\text{s.a.}})$, where $H_s$ and $H_{\text{s.a.}}$ are symmetric and self-adjoint components of $H$.

We call the pair $(\Lambda, H_{\text{s.a.}})$ a $\Lambda$-pair for the generator $L(\cdot)$, if the positive self-adjoint $\Lambda$ is a reference operator for the reduced generator $L_0(\cdot)$, i.e.

$$L_0(X) = \Phi(X) - G_0^* X - X G_0, \quad G_0 = \frac{1}{2} \Phi(I) + i(H_s - \Lambda),$$

$$L_0(\Lambda) \leq c \Lambda, \quad 1 \leq \Phi(I) \leq \Lambda,$$

and $H_{\text{s.a.}}$ and $H_s$ are such that

$$-\mu H_{\text{s.a.}}^\varepsilon \leq H_{\text{s.a.}} \leq \mu H_{\text{s.a.}}^\varepsilon, \quad 0 \leq H_{\text{int}} \leq \nu \Lambda,$$

$$\Phi(H_{\text{s.a.}}) \leq c_1 \Lambda, \quad \Phi(H_{\text{int}}) \leq c_2 \Lambda,$$

for some $c, \mu, \nu \geq 0$ and $\varepsilon \in (0, 1)$. Under these algebraic assumptions, together with some domain and continuity conditions which will be specified in Sections 1 and 2, we prove that the minimal dynamical semigroup with the formal generator $L(\cdot)$ is unital.

The paper is organized as follows. In Section 1, we discuss the construction of a minimal solution for the Markov master equation with time-dependent coefficients and prove new rather general conditions sufficient for the conservativity of a minimal solution. In Section 2, we discuss the properties of the interaction representation of MME.

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and introduce definitions of a reference family and $\Lambda$-pair. In Section 3, we describe the properties of generators of MME in quantum optics, and present basic examples of violation of the unital property. In Section 4 several physical examples of nontrivial MME are considered which in the most cases are regular. The main conclusion of the paper is rather simple: for the class of CP-maps $\Phi(\cdot)$ in quantum optics, the minimal QDS is unital if the Hamiltonian $H$ is a s.a. operator.

1 Conditions sufficient for conservativity

Consider the Markov master equation with time-dependent generator

$$\frac{\partial}{\partial t} P_{\tau,t}(B) = L_t(P_{\tau,t}(B)), \quad P_{\tau,t}(B)|_{\tau=t} = B,$$

(1.1)

$$L_t(B) = \Phi_t(B) - G_t^*X - XG_t, \quad G_t = \frac{1}{2}\Phi_t(I) + iH_t.$$ 

and assume that there exists strongly continuous contractive evolution system $W_{s,t} \in \mathcal{B}(\mathcal{H})$ $(t \geq s)$ such that

$$W_{s,t}W_{\tau,t} = W_{s,t}, \quad \frac{\partial}{\partial t} W_{s,t} = -W_{s,t}G_t, \quad \frac{\partial}{\partial s} W_{s,t} = G_sW_{s,t}.$$ 

For simplicity, we assume that all the generators $G_t$ have a joint core $\mathcal{D}_N \subseteq \text{dom} G_t^N$ for some $N \geq 1$ such that the following preliminary domain assumptions are fulfilled

$$W_{s,t} : \mathcal{D}_N \to \mathcal{D}_N \subseteq \text{dom} \Phi_{s,t}(I)[\cdot], \quad G_s^* + G_s = \Phi_s(I) \text{ on } \mathcal{D}_N,$$

(1.2)

and that there exists some positive self-adjoint operator $\Lambda : \Lambda \geq \Phi_t(I) \geq I$ such that the preliminary continuity conditions are fulfilled:

the family of CP-maps $\Lambda^{-1/2}\Phi_t(\cdot)\Lambda^{-1/2}$ is bounded, normal and ultraweakly continuous in $t$, and on the other hand, for any $\psi \in \mathcal{D}_N$, the family of vectors $\psi(s,t) = \Lambda^{1/2}W_{s,t}\psi$ belongs to $L^1_{\text{loc}}(\mathbb{R}_+, \mathcal{H})$ in variable $s$ and is norm-continuous in variable $t$. Under these domain and continuity conditions, for any bounded strongly continuous family of operators $X_s$, the family of quadratic forms

$$\Phi_s(X_s)[W_{s,t}\psi] \quad \forall \psi \in \mathcal{D}_N$$

(1.3)

belongs to $L^1_{\text{loc}}(\mathbb{R}_+)$ in variable $s$ and is continuous in $t$.

Hence the sequence of CP-maps

$$P_{\tau,t}^{(0)}(B) = W_{s,t}^*BW_{s,t} \quad \text{def} \quad V_{\tau,t}(B),$$

$$P_{\tau,t}^{(n+1)}(B) \quad \text{def} \quad V_{\tau,t}(B) + \int_\tau^t ds V_{s,t}\Phi_sV_{s,t}P_{s,t}^{(n)}(B)$$

is well-defined as a sequence of bounded operators corresponding to the sequence of densely defined and uniformly bounded quadratic forms. Indeed, $V_{\tau,t}(I) \leq I$, and the identity

$$V_{\tau,t}(I) = I - \int_\tau^t ds V_{s,t}\Phi_s(I) = I - \int_\tau^t ds V_{s,t}\Phi_s(I)$$

(1.4)

readily shows that the sequence $P_{\tau,t}^{(n)}(B)$ is uniformly bounded:

$$||P_{\tau,t}^{(n)}(B)|| \leq ||B||.$$ 

Moreover, it increases monotonically if $B \in \mathcal{B}_+(\mathcal{H})$, and defines the least upper bound:

$$P_{s,t}^{\text{min}}(B) = \text{l.u.b.} P_{s,t}^{(n)}(B).$$

This construction is analogous to the construction of the minimal solution for the Markov master equation with constant coefficients [4]–[5]; it was discussed in details in [6]. The identity (1.4) implies that

$$P_{\tau,t}^{(1)}(I) = V_{\tau,t}(I) + \int_\tau^t ds V_{s,t}\Phi_sV_{s,t}(I)$$

$$= V_{\tau,t}(I) + \int_\tau^t ds V_{s,t}\Phi_s(I) - \int_\tau^t ds_1 V_{s_1,t}\Phi_{s_1}\int_\tau^{s_1} ds_2 V_{s_2,s_1}\Phi_{s_2}(I)$$

$$= I - \int_\tau^t ds_1 V_{s_1,t}\Phi_{s_1}\int_\tau^{s_1} ds_2 V_{s_2,s_1}\Phi_{s_2}(I).$$
Similarly, by using sequentially the identity (1.4), we obtain

$$P_{s,t}^{(n)}(I) = I - \Delta^{(n+1)}(\tau, t),$$

$$\Delta^{(n)}(\tau, t) \overset{def}{=} \int_{\tau}^{t} ds V_{s,t} \Phi_{s_n} \cdots \int_{\tau}^{s_{n-1}} ds V_{s_n,s_{n-1}} \Phi_{s_n}(I).$$

(1.5)

Hence, $P_{s,t}^{(n)}(I) \to I$ strongly as $n \to \infty$, if and only if $\Delta_n(s,t) \to 0$ weakly.

To prove a condition sufficient for the minimal solution of the Markov master equation (1.1) to be unital, let us consider an estimate for the integral of the operator $P_{s,t}^{(n)}\Phi_s(I)$:

$$\int_{\tau}^{t} ds P_{s,t}^{(n)}\Phi_s(I) = \int_{\tau}^{t} ds V_s \Phi_s(I) + \ldots$$

$$+ \int_{\tau}^{t} ds \int_{s}^{s_{n-1}} ds V_{s_n,s_{n-1}} \Phi_{s_n} \cdots \int_{s}^{s_n} ds V_{s_n,s_n} \Phi_s(I).$$

In the last multiple integral, the variables $s_k$ take greater values then $s$, i.e. $\tau \leq s \leq s_n \leq \cdots \leq s_1$. Hence by changing the order of integration, we have

$$\int_{\tau}^{t} ds P_{s,t}^{(n)}\Phi_s(I) = \int_{\tau}^{t} ds V_s \Phi_s(I) + \ldots$$

$$+ \int_{\tau}^{t} ds V_{s_1,s_1} \Phi_{s_1} \cdots \int_{s}^{s_{n-1}} ds V_{s_n,s_n} \Phi_{s_n} V_{s_n,s_n} \Phi_s(I).$$

(1.6)

The last integral in (1.6) can be rewritten as the last integral in (1.5) in notation $s \to s_{n+1}$. By comparing Eqs. (1.5) and (1.6) and passing to the least upper bound in $n$, we obtain the following important equality:

$$\int_{\tau}^{t} ds P_{s,t}^{(n)}\Phi_s(I) = \sum_{n=1}^{\infty} \Delta^{(n)}(\tau, t),$$

(1.7)

where the monotone sequence of bounded positive operators $\Delta^{(n)}(\tau, t)$ (see (1.4)) converges to 0 if and only if the integral in the left-hand side is a densely defined operator. To make rigorous the above algebraic considerations of integrals, we must impose additional assumptions on domains and continuity.

In the sequel we assume that the CP-map $\Phi_t(\cdot)$ is such that for each $t \in \mathbb{R}_+$ the map $A_t(\cdot) = \Lambda^{-1/2} \Phi_t(\cdot) \Lambda^{-1/2}$ is bounded and normal. In fact, the boundedness follows from the inequality $\Lambda \geq \Phi_t(I)$. The Kraus theorem \footnote{We refer to Chapter 1 of Ref. [7].} implies that any normal bounded CP-map $A_t(X)$ on $B(\mathcal{H})$ ($\mathcal{H}$ is a separable Hilbert space) can be represented as the sum $A(X) = \sum_k A_k(t)XA_k(t)$, $\sum_k A^\dagger_k(t)A_k(t) \in B(\mathcal{H})$. This ensures a canonical representation of unbounded CP-map $\Phi_t(\cdot)$ \footnote{We refer to Chapter 1 of Ref. [7].}

$$\Phi_t(X) = \sum_k \Phi^\dagger_k(t)X\Phi_k(t), \quad \Phi_k(t) = A_k(t)\Lambda_{t}^{1/2},$$

where $\sum_k A_k(t)A_k(t) \in B(\mathcal{H})$. To study conditions sufficient for the minimal solution to be unital, we must extend the domain and continuity assumptions. We assume that for some $N \geq 2$ the operators $\Lambda^{1/2}\Phi_k(t)$ are densely defined,

$$\mathcal{D}_N \subset \text{dom} \Lambda^{1/2}\Phi_k(t),$$

(1.8)

and $\Lambda^{1/2}\Phi_k(s)W_{s,t}\psi \in L^2_{loc}(\mathbb{R}_+, \mathcal{H})$ in variable $s$ and norm-continuous in $t$. Thus the inequality (1.7) justifies the following assertion.

**Theorem 1.1.** Under the domain and continuity assumptions, if the domain of the operator $\int_{\tau}^{t} ds P_{s,t}^{(n)}\Phi_s(I)$ is dense in $\mathcal{H}$, then the minimal solution of the Markov master equation (1.1) is unital.

Since the sequence $\Delta^{(n)}(\tau, t)$ is positive and decreases monotonically, the sum

$$C = \sum_{n=1}^{\infty} a_n \Delta^{(n)}(\tau, t), \quad a_n \geq 0, \quad \sum_{n} a_n = \infty$$

converges to a densely defined operator only if $\Delta^{(n)}(\tau, t)$ converges to 0. The series which correspond to this sum with $a_n = n^{-1}$ can be represented as an integral of the minimal solutions of MMEs with the generators regularized as in \footnote{We refer to Chapter 1 of Ref. [7].}

$$\mathcal{L}_{t,\lambda}(B) = \lambda \Phi_t(B) - G_t^*X - XG_t, \quad \lambda \in (0, 1].$$

More precise, the series, representing the minimal solution of the equation

$$\frac{\partial}{\partial t} P_{s,t}^{(\lambda)}(B) = \mathcal{L}_{t,\lambda}(P_{s,t}^{(\lambda)}(B)), \quad P_{s,t}^{(\lambda)}(B)|_{s=t} = B$$
is the following:

$$P_{\tau,t}^{(\lambda)}(B) = V_{\tau,t}(B) + \sum_{n=1}^{\infty} \lambda^n \int_{\tau}^{t} ds_1 V_{s_1,t}\Phi_{s_1} \cdots \int_{\tau}^{s_{n-1}} ds_n \Phi_{s_n} V_{s_n,s_n}(B).$$

This identity and definition (1.5) imply

$$\int_{0}^{1} d\lambda \int_{\tau}^{t} ds P_{\tau,t}^{(\lambda)}\Phi_s(I) = \sum_{n=1}^{\infty} \frac{1}{n} \Delta^{(n)}(\tau,t).$$

Therefore, the following assertion is true.

**Theorem 1.2.** Assume that the domain and continuity conditions are fulfilled. If the operator

$$\tilde{\mathcal{C}} = \int_{0}^{1} d\lambda \int_{\tau}^{t} ds P_{\tau,t}^{(\lambda)}\Phi_s(I)$$

is densely defined in \(\mathcal{H}\), the minimal solution of the Markov master equation (1.1) is unital.

Let us consider a priori bounds for the operator \(P_{\tau,t}\Phi_s(I)\).

### 2 A priori bounds

Assume that there exists a smooth family of positive self-adjoint operators \(\Lambda_t \geq \Phi_t(I) \geq I\) and \(N \geq 2\) such that for any \(\psi \in D_N \subseteq \text{dom} \Lambda_t\) and

$$\Phi_{t,t}(\Lambda_t)\lvert \psi \rvert - 2\Re (G_t\psi, \Lambda_t\psi) - (\psi, \Lambda_t\psi) \leq c_t ||\Lambda_t^{1/2}\psi||,$$

(cf. [8] and [10]), where \(c_t \in L^1_1(\mathbb{R}_+), c_t \geq 0\). Such operator family is called a family of reference operators. We assume that the family of operators \(\Lambda^{-1/2}\Lambda^{1/2}\), with the previously defined operator \(\Lambda_t\), is densely defined on \(\mathcal{H}\) and admits a continuation on the whole space \(\mathcal{H}\) which is uniformly bounded and strongly continuous. Let us prove that condition (2.1) ensures the a priori estimate

$$P_{\tau,t}^{\text{min}}(\Lambda_r) \leq \Lambda_t e^{\int_{\tau}^{t} ds c_s}. \tag{2.2}$$

This estimate can easily be proved by induction. Indeed, for all \(0 \leq \tau \leq t\) we have

$$\frac{\partial}{\partial \tau} V_{\tau,t}\Lambda_r = \frac{\partial}{\partial \tau} W_{\tau,t}\Lambda_r W_{\tau,t} = W_{\tau,t}(G_t^*\Lambda_r + \Lambda_r G_t + \Lambda_t)W_{\tau,t} \geq W_{\tau,t}(\Phi_t(\Lambda_r) + c_r \Lambda_r)W_{\tau,t} \geq c_r W_{\tau,t}\Lambda_r W_{\tau,t} = c_r V_{\tau,t}\Lambda_r,$$

Hence, by solving this terminal differential inequality, we obtain

$$P_{\tau,t}^{(0)}(\Lambda_r) = V_{\tau,t}\Lambda_r \leq \Lambda_t e^{\int_{\tau}^{t} ds c_s}. \tag{2.3}$$

Assume that \(P_{\tau,t}^{(n)}(\Lambda_r) \leq \Lambda_t e^{\int_{\tau}^{t} ds c_s}\) and let us prove this inequality for \(P_{\tau,t}^{(n+1)}(\Lambda_r)\).

From the recurrent definition of \(P_{\tau,t}^{(n+1)}(\Lambda_r)\) and assumption (2.1) we have

$$P_{\tau,t}^{(n+1)}(\Lambda_r) = W_{\tau,t}\Lambda_r W_{\tau,t} + \int_{\tau}^{t} ds V_{s,t}\Phi_s P_{\tau,s}^{(n)}(\Lambda_r) \leq W_{\tau,t}\Lambda_r W_{\tau,t} + \int_{\tau}^{t} ds e^{\int_{\tau}^{s} dr c_r V_{s,t}\Phi_s(\Lambda_s) \leq W_{\tau,t}\Lambda_r W_{\tau,t} + \int_{\tau}^{t} ds e^{\int_{\tau}^{s} dr c_r V_{s,t}(c_s\Lambda_s + G_s^*\Lambda_s + \Lambda_s G_s + \Lambda_s)} = V_{\tau,t}\Lambda_r + \int_{\tau}^{t} ds \frac{\partial}{\partial s} \left( e^{\int_{\tau}^{s} dr c_r V_{s,t}\Lambda_s} \right) = \Lambda_t e^{\int_{\tau}^{t} ds c_s}.$$

This estimate readily implies that the operator \(P_{\tau,t}^{\text{min}}(\Lambda_r)\) is densely defined, \(\text{dom} \Lambda \subseteq \text{dom} P_{\tau,t}^{\text{min}}(\Lambda_r)\), and hence the unital property holds. Therefore, the following assertion holds true.

**Theorem 2.1.** Let the domain and continuity assumptions be fulfilled and there exist a reference family \(\Lambda_t\). Then the minimal solution of Eq. (1.1) is unital.
Assume that for a formal generator $\mathcal{L}(\cdot)$ with constant coefficients there exists some constant reference operator $\Lambda$, i.e.

$$\mathcal{L}(\Lambda) \leq c\Lambda,$$

where $G = \frac{1}{2}\Phi(I) + iH$, $H = H_s + H_s.a.$, and let $H_{\text{int}} = H_s.a. + \Lambda$ be a self-adjoint operator. Then inequality (2.1) holds for $\mathcal{L}_t(\cdot) = \Phi_t(B) - G^*B - BG$, and the reference family $\Lambda_t = U_t^*\Lambda U_t$, where $U_t = e^{itH_{\text{int}}}$.

Thus under the above assumptions (2.3) there exists a constant $\varepsilon$ such that

$$\int_0^t \Phi_t(B) - G^*B - BG \leq \varepsilon e^{\lambda t},$$

for any constant $\Lambda$, where $\lambda$ is positive.

Let $\Lambda$ be a self-adjoint operator. Then inequality (2.1) holds for $\mathcal{L}_t(B) = \Phi_t(B) - G^*B - BG$, and the reference family $\Lambda_t = U_t^*\Lambda U_t$, where $U_t = e^{itH_{\text{int}}}$.

Let us discuss an opportunity to use some fixed reference operator $\Lambda$ for problems with time-dependent coefficients, which arise in the interaction representation.

Let $\Lambda \geq 0$ and $H_{s.a.}$ be self-adjoint operators such that the sum $H_{\text{int}} = H_{s.a.} + \Lambda$ is positive and self-adjoint, and there exist $\mu, \nu \geq 0$ and $\varepsilon \in (0, 1)$ such that

$$-\mu H^2_{\text{int}} \leq H_{s.a.} \leq \mu H^2_{\text{int}}, \quad 0 \leq H_{\text{int}} \leq \nu\Lambda.$$

Note that for any positive self-adjoint operator $X$ and $\varepsilon \in (0, 1]$, we have $X^\varepsilon \leq I + X$, since $X^\varepsilon \leq 1 + \lambda$ for any positive $\lambda$, and hence

$$X^\varepsilon = \int X^\varepsilon E_X(d\lambda) \leq \int (1 + \lambda) E_X(d\lambda),$$

where $E_X(d\lambda)$ is the spectral family of the operator $X$. Then

$$\Lambda_t = U_t\Lambda U_t^* = U_t(\int_0^t (1 + \lambda) E_X(d\lambda)) U_t^*$$

$$= H^{1/2}_{\text{int}} \left\{ I - U_t H^{1/2}_{\text{int}} H^{1/2}_{s.a} H^{1/2}_{\text{int}} \right\} H^{1/2}_{\text{int}}$$

$$\leq H^{1/2}_{\text{int}} \left\{ I + \mu H^{1/2}_{\text{int}} H^{1/2}_{s.a} H^{1/2}_{\text{int}} \right\} H^{1/2}_{\text{int}}$$

$$\leq H_{\text{int}} + \mu (H_{\text{int}} + I) \leq (1 + \mu)\nu\Lambda + 1 \mu \leq c_0\Lambda.$$

Thus under the above assumptions (2.3) there exists a constant $c_0 = \mu + (1 + \mu)\nu$ such that $U_t\Lambda U_t^* \leq c_0\Lambda.$

Assume that $\Lambda$ is a reference operator for some formal generator $\mathcal{L}(\cdot)$. Then we have $\Phi(\Lambda) - G^*\Lambda - AG \leq c\Lambda.$

Assume that

$$\Phi(H_{s.a.}) \leq c_1\Lambda, \quad \Phi(H^2_{\text{int}}) \leq c_2\Lambda,$$

and consider an estimate the action of the formal generator $\mathcal{L}(\cdot)$ in the interaction representation on the element $\Lambda$:

$$\mathcal{L}_t(\Lambda) = U_t^* \left( \Phi(U_t\Lambda U_t^*) - G^*\Lambda - AG \right) U_t$$

$$= U_t^* \left( \Phi(U_t(H_{\text{int}} - H_{s.a.}) U_t^*) - G^*\Lambda - AG \right) U_t$$

$$= U_t^* \left( \Phi(H_{s.a.} - U_t H_{s.a.} U_t^*) + \Phi(\Lambda) - G^*\Lambda - AG \right) U_t$$

$$\leq U_t^* \left( \Phi(H_{s.a.} - U_t H_{s.a.} U_t^*) + c\Lambda \right) U_t$$

$$\leq U_t^* \left( \Phi(H_{s.a.}) + \mu \Phi(H^2_{\text{int}}) + c\Lambda \right) U_t$$

$$\leq (c_1 + \mu c_2 + c) U_t^* \Lambda U_t \leq c_0(c_1 + \mu c_2 + c).$$

Thus, under the above assumptions

$$\mathcal{L}_t(\Lambda) \leq \lambda\Lambda,$$

$\lambda = c_0 (c + c_1 + \mu c_2).$

By Theorem 2.1, this estimate implies that the formal generator $\mathcal{L}_t(X) = U_t^* \mathcal{L}(U_t X U_t^*) U_t$ is regular. On the other hand, the minimal quantum dynamical semigroup generated by $\mathcal{L}_t(\cdot)$ is unitary equivalent to the minimal dynamical semigroup generated by

$$\mathcal{E}(\cdot) = \mathcal{L}(\cdot) + i[H_{\text{int}}, \cdot].$$

Hence $\mathcal{E}(\cdot)$ also generates a unital minimal dynamical semigroup, and it is regular too. Its coefficients are $\Phi(\cdot)$ (the same CP-map), and $H = H_s - \Lambda + H_{\text{int}} = H_s + H_{s.a.}$ Thus one can add a self-adjoint operator $H_{s.a.}$ to any regular generator $\mathcal{L}(\cdot)$ which possesses a reference operator $\Lambda$ if conditions (2.3)-(2.4) are fulfilled. In this case we call $(\Lambda, H_{\text{int}})$ a $\Lambda$-pair for the generator $\mathcal{L}(\cdot)$.

**Theorem 2.2.** Assume that the domain and continuity conditions are fulfilled. If for a formal generator $\mathcal{L}(\cdot)$ there exists a $\Lambda$-pair, the generator $\mathcal{L}(\cdot)$ is regular.
3 Structure of generators of MME in quantum optics

The typical formal generator $\mathcal{L}(\cdot)$ of a Markov master equation in quantum optics (see [11] (Schack & Brun ’96), [12] (Brun & Gisin ’96), [13] (Ariano & Sacchi ’97), [14] (Zoller & Gardiner ’97)) acts in $\mathcal{B}(\mathcal{H})$, $\mathcal{H} = (\ell_2)^{\otimes N} \otimes \mathbb{C}^M$; its Lindbladian form reads as follows: $\mathcal{L}(B) = \Phi(B) = G^*B - BG$, where

$$\Phi(B) = \sum_{k=1}^{N} \Phi_k(B), \quad G = \frac{1}{2}\Phi(I) + \hat{H},$$

$$\Phi_k(B) = \lambda_k a_k^\dagger B a_k \quad \text{or} \quad \Phi_k(B) = \mu_k a_k B a_k^\dagger, \quad (3.1)$$

$\lambda_k, \mu_k \geq 0$ are positive operators in $\mathbb{C}^M$, $a_k$ and $a_k^\dagger$ are adjoint creation and annihilation operators acting on $k$-th factor of the tensor product $(\ell_2)^{\otimes N}$, i.e. $([a_k, a_n] = 0, \quad [a_n, a_k^\dagger] = \delta_{kn}$), and

$$H = \sum_j \left( h_j \prod_{k=1}^{N} (a_k^\dagger)^{\gamma_{jk}} (a_k)^{\gamma_{jk}} + \text{h.a.} \right) \quad (3.2)$$

is an operator in $\mathcal{H}$ represented by a symmetric polynomial of a finite degree in creation and annihilation operators with matrix coefficients $h_j \in \mathbb{C}^M \otimes \mathbb{C}^M$ (see [13] (Wiseman & Vaccaro ‘98), [14] (Kist, Orszag, Brun & Davidovich ‘99).

Note that the Hermitian structure (3.2) of the operator $H$ does not imply its self-adjointness. For example, the Hamiltonian of the third order

$$H = -\frac{i}{\sqrt{2}} \left( (1 + \frac{1}{2}(a + a^\dagger)^2)(a - a^\dagger) + (a - a^\dagger)(1 + \frac{1}{2}(a + a^\dagger)^2) \right)$$

is not a s.a. operator in $L_2$, because it is unitarily equivalent to the symmetric operator $\hat{H} = i(1+x^2)\partial_x + x(1+x^2)$ in $L_2(\mathbb{R})$, which has the nontrivial eigenvector

$$\psi(x) = \frac{\psi_0}{\sqrt{1+x^2}} e^{-\frac{1}{2}\text{arctg}(x)} \in L_2(\mathbb{R}), \quad \psi_0 \in \mathbb{C} \quad (3.3)$$

such that $\hat{H}\psi = -i\psi$. Hence the symmetric operators $H$ and $\hat{H}$ have the same nontrivial deficiency index, and $\pi = |\psi\rangle\langle\psi|$ is a projector to the deficiency subspace $X = \mathcal{H}_d$.

Consider conditions on the projector $\pi$ which ensure the violation of the unital property for equations with constant operator coefficients. We recall that the condition necessary and sufficient for the minimal solution to be unital is the weak convergence to 0 of the monotone sequence of bounded positive operators

$$Q^\varepsilon_\pi(I) \to 0, \quad Q^\varepsilon_\pi(X) \overset{d.f.}{=} \int_0^\infty dt e^{-\varepsilon t} V_{0,t} \Phi(X) \quad (3.4)$$

(see [11]). Hence the existence of a positive bounded operator $X$, $\|X\| \leq 1$, such that $Q^\varepsilon_\pi(X) \geq X$ for some $\varepsilon > 0$ is sufficient for the violation of the unital property, since the sequence

$$Q^\varepsilon_\pi(I) \geq Q^\varepsilon_\pi(X) \geq X \geq 0 \quad (3.5)$$

clearly does not converge to 0.

**Theorem 3.1.** *If the coefficients of a formal generator $\mathcal{L}(\cdot)$ satisfy domain and continuity assumptions and there exist a positive bounded operator $X$, $\|X\| \leq 1$, and $\varepsilon > 0$ such that

$$\mathcal{L}_\pi^*(X)[\psi] \geq \varepsilon X_\pi[\psi] \quad \forall \psi \in \text{dom} G^N = \mathcal{D}_N, \quad (3.6)$$

then the corresponding minimal quantum dynamical semigroup does not preserve the unit operator.*

**Proof.** Let us derive the inequality $Q^\varepsilon_\pi(X) \geq X$ from (3.6). Inequality (3.6) implies that $\Phi(X)_\pi \geq (\varepsilon X + G^*X + XG)_\pi$ on $\mathcal{D}_N$. Then for $\psi \in \mathcal{D}_N$, we have $\psi_t = W_t\psi \in \mathcal{D}_N$ and

$$e^{-\varepsilon t} \Phi(X)[\psi_t] \geq e^{-\varepsilon t}(\varepsilon \psi_t, X \psi_t) + (G\psi_t, X \psi_t) + (X \psi_t, G\psi_t)$$

$$= -\frac{d}{dt}(e^{-\varepsilon t}\|X \psi_t\|^2).$$

Therefore,

$$\int_0^t e^{-\varepsilon \tau} \Phi(X)[\psi_\tau] d\tau \geq (\psi, X \psi) - e^{-\varepsilon t}(\psi_t, X \psi_t).$$
The limit as \( t \to \infty \) yields an inequality for \( X \): \( (\psi, Q_c(X)\psi) \geq (\psi, X\psi) \) for any \( \psi \in \mathcal{D}_N \) by definition of the map \( Q_c(\cdot) \). Since \( \mathcal{D}_N \) is dense in \( \mathcal{H} \) and the map \( Q_c(\cdot) \) is bounded, this inequality is equivalent to
\[
Q_c(X) \geq X, \quad (3.7)
\]
which contradicts the necessary unitality condition. \( \square \)

A natural candidate to be used as \( X \) in inequality (3.6) is the projector to the deficiency subspace of the operator \( H \), if such a subspace exists.

**Theorem 3.2.** \( \square \) (Chebotarev & Slushtikov '00)

Let \( H \) be a densely defined symmetric operator. Assume that it has a nontrivial deficiency subspace \( \mathcal{H}_d = \{ \psi : H^*\psi = -i\psi \} \) and \( \pi_d \) is the projection onto \( \mathcal{H}_d \). If moreover, there exists \( \varepsilon > 0 \) such that
\[
\Phi(\pi_d)_*[\psi] - \Re (\Phi(I)\psi, \pi_d\psi) \geq -2(\varepsilon)\|\pi_d\psi\|^2
\]
for all \( \psi \in \mathcal{D}_N \), then inequality (3.6) holds and the necessary unitality condition (3.4) is violated.

**Proof.** Let us prove that (3.8) implies (3.6) in the following form:
\[
\mathcal{L}(\pi_d)_*[\psi] \geq \varepsilon\|\pi_d\psi\|^2 \quad \forall \psi \in \mathcal{D}_N.
\]
Indeed, since \( H^*\pi_d = -i\pi_d \) and \( \pi_d = \pi_d^2 \), it follows from (3.8) that
\[
\mathcal{L}(\pi_d)_*[\psi] = \Phi(\pi_d)_*[\psi] - \Re (\Phi(I)\psi, \pi_d\psi) + i((H\psi, \pi_d\psi) - (\pi_d\psi, H\psi))
\]
\[
= \Phi(\pi_d)_*[\psi] - \Re (\Phi(I)\psi, \pi_d\psi) + 2\|\pi_d\psi\|^2 \geq \varepsilon\|\pi_d\psi\|^2
\]
and inequality (3.6) is true. \( \square \)

Note that for any finite polynomial \( H = H_2 + H_{s,a} \), in creation and annihilation operators \( a_k^\dagger \) and \( a_k \), there exists a diagonal operator
\[
\Lambda = c_\Lambda \left( 1 + \sum_{k=1}^{N} (a_k^\dagger a_k)^{m_k} \right), \quad c_\Lambda > 0
\]
(3.10)
such that \( H_2 \) and \( H_{s,a} \) are relatively bounded by \( \Lambda \) with the relative upper bound \( O(c_\Lambda^{-1}) \). One can use \( \Lambda \) as the reference operator. In any case we assume that \( \Phi(I) \geq I \) is a s.a. operator and
\[a \text{ dom } \Lambda \subseteq \text{ dom } \Lambda^{1/2}, \quad a^\dagger \text{ dom } \Lambda \subseteq \text{ dom } \Lambda^{1/2}.
\]
The last two assumptions readily hold if \( m_k \geq 2 \) in (3.10).

**Theorem 3.3.** If the Hamiltonian \( H \) can be represented as \( H = H_2 + H_{s,a} \), where \( H_{s,a} \) is a self-adjoint polynomial of a finite order \( M \) in creation and annihilation operators and \( H_2 = a_k^\dagger a_k + \Lambda \) is a polynomial of the second order, then there exist \( c_\Lambda > 0 \) and \( \{ m_k \} \geq M \) such that \( (\Lambda, H_{s,a} = H_{s,a} + \Lambda) \) is a \( \Lambda \)-pair for the generator (3.1)–(3.2).

**Proof.** We recall that any finite polynomial in creation and annihilation operators of order \( M \) can be dominated by the diagonal operator (3.10) of higher order, provided the constant \( c_D \) is sufficiently large and \( N = \min \{ m_k \} \geq M \). Hence for sufficiently large \( c_D \), and \( N \) by the classical perturbation theory \( \square \), \( H_{s,a} = H_{s,a} + \Lambda \) is a positive s.a. operator such that \( \text{dom } H_{s,a} = \text{dom } \Lambda \), and \( G_0 = i[H_2 - \Lambda] + \Phi(I)/2 \) is an accretive operator, \( \text{dom } G_0 = \text{dom } \Lambda \).

Since \( \Phi(\cdot) \) and \( H_{s,a} \) are operators of a finite (second) order, the property (2.3) and (2.4) of \( \Lambda \)-pair can readily be fulfilled by choosing \( M, N \) and \( c_\Lambda \) sufficiently large.

The commutator of a polynomial of the second order in creation and annihilation operators with arbitrary polynomial of order \( M < \infty \) has the order \( M \) or less. Hence, the commutator
\[i[H_0, \Lambda] = i[H_2 - \Lambda, \Lambda] = i[H_2, \Lambda]
\]
is an operator of the same order as \( \Lambda \), and hence there exists a constant \( c \in \mathbb{R} \) such that \( i[H_0, \Lambda] \leq c\Lambda \).

A simple algebra shows that for CP-map (3.1), the operator \( \Phi(\Lambda) - (\Lambda\Phi(I) + \Phi(I)\Lambda)/2 \) is also a polynomial of the same order as \( \Lambda \). More precise, the following two estimates hold:
\[
(a_k^\dagger)^l a_k - \frac{1}{2} \left( (a_k^\dagger)^l a_k^\dagger \Lambda + \Lambda (a_k^\dagger)^l a_k \right) \leq 0,
\]
for any \( l \geq 0 \), and on the other hand there exists \( c \geq 0 \) such that
\[
a_k \Lambda a_k^\dagger - (a_k a_k^\dagger \Lambda + \Lambda a_k a_k^\dagger)/2 \leq c\Lambda,
\]
for the operator \( \Lambda \) (3.10). Therefore, there exists \( c > 0 \) such that \( \mathcal{L}_0(\Lambda) \leq c\Lambda \) on \( \text{dom } \Lambda \), and hence \( \Lambda \) is a reference operator for
\[
\Phi_{k,l}(B) = \lambda_{k,l}(a_k^\dagger B) a_k, \quad \Phi_k(B) = \lambda_k a_k B a_k^\dagger.
\]
This proves the theorem. \( \square \)
4 Examples

In this section we consider some classes of Hamiltonians and completely positive maps for which our Theorem 2.2 is applicable.

1. Let $\lambda$ be a complex number and $m, n \geq 0$. Set

$$H = \lambda(a_1^\dagger a_1^2 + \sum a_2^\dagger a_2^m \cdot \lambda \in \mathbb{C}. \quad (4.1)$$

Let us prove that all Hamiltonians of such form are essentially self-adjoint in $H_2 = l_2 \otimes l_2$. It suffices to prove that there does not exist a vector

$$\psi = \{\psi_{k,j}, k, j \geq 1, \sum_{k,j} |\psi_{k,j}|^2 = ||\psi||_{H_2}^2 \}$$

such that $H \psi = \pm i \psi$ \cite{20}. We set $\psi_{k,j} = 0$ if $\min\{k, j\} \leq 0$.

Let us rewrite these equations for components $\psi_{k,j}$ as follows:

$$\pm i \psi_{k,j} = \lambda A_{k,j}^m \psi_{k-m,j+n} + \sum B_{k,j}^m \psi_{k+m,j-n}, \quad A_{k,j}^m, B_{k,j}^m \geq 0. \quad (4.2)$$

We set $\psi_{k,j} = 0$ if $\min\{k, j\} \leq 0$ and skip exact expressions for the functions $A_{k,j}^m$ and $B_{k,j}^m$ because they are irrelevant for the proof. The important property of this system is that it splits into a set of independent finite subsystems of linear algebraic equations with respect to values of one of the components of the set

$$X_k = \{x_j = \psi_{k-m,j+n}, k - jm \geq 1, \quad jn \geq 1, \quad j = 0, 1, \ldots, [k/m] - 1\},$$

where $x_j = 0$ for all $j < 0$. For each $k, m, n$ fixed, the system of linear algebraic equations corresponding to (4.2) has the three-diagonal form

$$\pm ix_j = \lambda A_j x_{j+1} + \sum B_j x_{j-1}$$

with some positive $A_j$ and $B_j$. But it is a well-known fact (see \cite{19}) that

$$D_N = \det \begin{pmatrix}
\pm iI & \lambda A_1 & 0 & \ldots & 0 & 0 \\
& \lambda \ast B_1 & \pm iI & \lambda A_2 & \ldots & 0 & 0 \\
& & \ddots & \ddots & \ddots & \ddots \ddots \\
0 & 0 & 0 & \ldots & \lambda \ast B_{N-1} & \pm iI
\end{pmatrix} \neq 0 \quad (4.3)$$

where the entries of the matrix are $(k \times k)$-blocks, $\lambda$ and $\lambda \ast$ are Hermitian adjoint $(k \times k)$-matrices, and $I$ is the unit matrix in $\mathbb{C}^M$.

By the Gershgorin theorem \cite{9}, Hamiltonians (4.1) are relatively bounded by the diagonal matrix $\Lambda$ (2.3) of order $M \geq m + n$, and the relative upper bound decreases as $c_0 \to \infty$. Hence all formal generators with the completely positive parts (3.1) and Hamiltonian part (4.1) are regular.

2. The same assertion is true for Hamiltonians from $\mathcal{C}(l_2^N \otimes \mathbb{C}^M)$ of the following form: $H = H_{\text{int}} + H_0$,

$$H_{\text{int}} = \lambda(a_1^\dagger a_1^m a_1^{n_1} \cdots a_2^\dagger a_2^m a_2^{n_2} + \lambda \ast a_1^\dagger a_1^m a_1^{n_1} \cdots a_2^\dagger a_2^m a_2^{n_2}), \quad \lambda, \lambda \ast \in \mathbb{C}$$

where $\lambda$ and $\lambda \ast$ are Hermitian adjoint $(M \times M)$-matrices, $\sum m_k + n_k = K$, and $H_0$ is any symmetric operator dominated by $\Lambda$ and such that

$$\exists c \in \mathbb{R} : \quad i[H_0, \Lambda] \leq c \Lambda.$$ 

The proof of self-adjointness of $H_{\text{int}}$ is based on a similar factorization of the set of block-matrices $\{\psi_{k_1 \ldots k_N}\} \in l_2^N \otimes \mathbb{C}^M$ and on the reduction of the homogeneous system of linear algebraic equations to the set of finite-dimensional linear equations with nondegenerate three-diagonal $(M \times M)$-block matrix (4.3). As in the previous case, the interaction representation is generated by the self-adjoint operator $H_{\text{int}}$ dominated by the diagonal operator $D$ for $M \geq \sum (m_k + n_k)$, and the Hamiltonian $H_0$ of $\mathcal{L}_0(\cdot)$, because it satisfies the conservativity and compatibility conditions. For generators (1.1), any symmetric operator on $H_0$ of the second order in creation and annihilation satisfies the above assumptions.

3. Consider the physical example \cite{12} (Schack, Brun & Pecival ’96) of a formal generator $\mathcal{L}(\cdot)$ in $B(l_2 \otimes l_2 \otimes \mathbb{C}^2)$ with CP-part (1.1) and the Hamiltonian

$$\hat{H} = E i(a_1^\dagger - a_1) + \frac{\chi}{2} (a_1^2 a_2 - a_2^2 a_1^2) + \omega \sigma_+ \sigma_- + \eta (a_2 \sigma_+ - a_2^2 \sigma_-), \quad (4.5)$$

where $E$ is the strength of an external pump field, $\chi$ is the strength of the interaction, $\omega$ is the detuning between the frequency of the field mode $a_2$ and the spin transition frequency, and $\eta$ is the strength of the coupling of the spin to the field mode $a_2$. The completely positive part of the generator reads as follows

$$\Phi(B) = 2 \gamma_1 a_1^\dagger B a_1 + 2 \gamma_2 a_2^\dagger B a_2 + 2 \kappa \sigma_+ B \sigma_- \quad (4.6)$$
It describes the dissipation of the field modes and the spin with coefficients $\gamma_1$, $\gamma_2$, and $\kappa$, respectively; $\sigma_\pm$ are two by two matrices. The Hamiltonian (4.5) can be readily represented in the form (4.3) with $k = 2$, $K = 3$, $\lambda = I$, $H_{\text{int}} = \frac{1}{2}\bar{i}(a^2 a_2 - a_2^2 a)$ and $H_0 = H - H_{\text{int}}$. The completely positive part has the form (3.1). Hence the formal generator (4.5)–(4.6) is regular.

4. The kinetic stage of the evolution of a quantum system interacting with environment is described in [21] (Kilin & Schreiber ‘97) by the following Markov master equation:

$$
\frac{\partial \sigma}{\partial t} = -i\omega [H(a^\dagger, a), \sigma] + \Gamma_2(n_2 + 1) \left\{ \left[ a^2 \sigma, (a^\dagger)^2 \right] + \left[ (a^\dagger)^2, \sigma a^2 \right] \right\} + \Gamma_2 n_2 \left\{ \left[ (a^\dagger)^2 \sigma, a^2 \right] + \left[ a^2, (a^\dagger)^2 \sigma \right] \right\},
$$

where s.a. operator $H = H(a^\dagger, a)$ is a finite symmetric polynomial in $a^\dagger$ and $a$ of order no greater 4, $\Gamma_2 = \pi K^2 g_2$ is the decay rate of the vibrational amplitude. Here, the number of quanta in the bath mode $n_2 = n(2\omega)$, the coupling function $K = K(2\omega)$, and the density of bath states $g_2 = g(2\omega)$ are evaluated at the double frequency of the selected oscillator. The corresponding dual CP-map $\Phi(\cdot)$ acts as follows

$$
\Phi(B) = 2\Gamma_2 ((n_2 + 2)(a^\dagger)^2 Ba^2 + n_2 a^2 B(a^\dagger)^2).
$$

This case is rather simple: $\Lambda = c(a^\dagger)^2 a^2$ and $H_{\text{int}} = H + \Lambda$, where $c$ is sufficiently large: $\lambda \geq \Phi(I)$, and $||\Delta h|| \geq 2||Hh||$, so that $H_{\text{int}}$ is a s.a. operator, provided $H$ is s.a. operator. Hence the generator of the above master equation is regular.

5. The previous example can be generalized as follows. Set

$$
\Phi_m^+(B) = (a^\dagger)^n B a^m, \quad \Phi_m^-(B) = a^m B(a^\dagger)^n, \quad \Lambda_n = ((a^\dagger a)^n + I)\lambda, \quad \lambda > 0.
$$

Then we set

$$
\Lambda_n \psi_N = \lambda (N^n + 1) \psi_N, \quad \Phi_m^\pm(I) \psi_N = \left( \frac{(N \mp m)!}{N!} \right)^{\pm 1} \psi_N
$$

for $N$-particle component of $\psi_N$ of the vector $\psi = \{\psi_0, \psi_1, \ldots \} \in l_2$. Hence there exists $n \geq m$ and $\lambda = \lambda(m, n) > 0$ such that $\Lambda_n \geq \Phi_m^\pm(I)$.

Similarly we obtain

$$
\left( \Phi_m^\pm(\Lambda_n) - \left( \Lambda_n \Phi_m^\pm(I) + \Phi_m^ \pm(I) \Lambda_n \right)/2 \right) \psi_N = \lambda \left( \frac{(N \mp m)!}{N!} \right)^{\mp 1} \left( (N \mp m)^n - N^n \right) \psi_N.
$$

Therefore, for any formal generator with completely positive part

$$
\Phi(B) = \sum_k \left\{ c_k^+ \Phi_m^+(B) + c_k^- \Phi_m^-(B) \right\}
$$

with positive matrix coefficients $c_k^+ \in \mathbb{C}^M \otimes \mathbb{C}^M$, the third $\Lambda$-pair assumption is fulfilled if the balance condition is true:

$$
\sup_{N \geq 1} \sum_k \left\{ c_k^+ \frac{N!}{(N - m_k)!} \left[ (1 - m_k/N)^n - 1 \right] + c_k^- \frac{(N + m_k)!}{N!} \left[ (1 + m_k/N)^n - 1 \right] \right\} < c I.
$$

Then the regularity conditions of Theorem 2.2 are fulfilled for $\lambda$ sufficiently large if $H = H_2 + H_{s.a.}$, where $H_2(a, a^\dagger)$ is any Hermitian quadratic polynomial, and $H_{s.a.}(a, a^\dagger)$ is any self-adjoint Hamiltonian of order less or equal 2n. The balance condition is readily fulfilled if

$$
S = \sum_k \left\{ c_k^+ m_k - c_k^- n_k \right\} > 0
$$

is a strictly positive operator in $\mathbb{C}^M \otimes \mathbb{C}^M$. 
The generators of MME with completely positive component of the fourth order was used in \(\text{[22]}\) (Schneider & Milburn ’97):

\[
\mathcal{H} = l_2 \otimes \mathbb{C}^2, \quad \Phi(B) = (a^2\sigma_+ + (a^\dagger)^2\sigma_-)B(a^2\sigma_+ + (a^\dagger)^2\sigma_-),
\]

where

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Hence, for diagonal operators \(B \in \mathcal{B}(\mathcal{H})\)

\[
B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad B_{1,2} \in \mathcal{B}(l_2)
\]

we have \(\sigma^2_{\pm} = 0\) and

\[
\Phi(B) = \begin{pmatrix} a^2B_1(a^\dagger)^2 & 0 \\ 0 & (a^\dagger)^2B_2a^2 \end{pmatrix}.
\]

Therefore, the generator \(\mathcal{L}(\cdot)\) has the component

\[
\mathcal{L}_{11}(X) = a^2X(a^\dagger)^2 - (a^\dagger)^2X + X(a^\dagger)^2)/2 + i[H_{11}, X]
\]

which is unregular for any first order operator \(H_{11}\).

6. The paper \(\text{[23]}\) (Lanz, Melsheimer & Vaccini ’97) presents examples of formal generators in \(\mathcal{H} = l_2 \otimes l_2\) with the CP-component \(\Phi(B) = a_1a_1^\dagger B a_2a_2^\dagger\). Let us prove that for the generators with coefficients

\[
\Phi(B) = a_1^\dagger(\sigma_1^1)MBa_2^\dagger(\sigma_2^1)^L, \quad L, M \geq 1, \quad H = H_2 + H_{s.a.}(a^\dagger, a)
\]

there exists a \(\Lambda\)-pair. Consider the generator

\[
\mathcal{L}_0(B) = a_1^\dagger(\sigma_1^1)MBa_2^\dagger(\sigma_2^1)^L - B \circ a_1^\dagger(\sigma_1^1)La_2^\dagger(\sigma_2^1)^LMa_2^\dagger.
\]

Straightforward computation proves that \(\Phi(I)\) is not a reference operator for \(\mathcal{L}_0(\cdot)\).

**Lemma 4.1.** For any \(N \geq 0\), there exists a polynomial

\[
\Lambda_N = \lambda^{(N)}I + \sum_{k=0}^{N} \lambda_k^{(N)}(a_1^\dagger)^{N-k}a_1^{-k}(a_2^\dagger)^ka_2^k, \quad \lambda^{(N)}, \lambda_k^{(N)} \geq 1
\]

and a constant \(c_N\) such that \(\mathcal{L}_0(\lambda_N) \leq c_N\Lambda_N\).

**Proof.** Note that

\[
\mathcal{L}_0((\sigma_1^1)^n(\sigma_2^1)^m) = mL(\sigma_1^1)^{L+m-1}a_1^\dagger(\sigma_2^1)^{N-M}a_2^N
\]

\[
- nM(\sigma_1^1)^{L+m}a_1^\dagger(\sigma_2^1)^{N-M-1}a_2^{N+1} + \text{l.o.t.}
\]

where the lower order terms (l.o.t.) can be dominated by the main terms and \(\lambda^{(N)}I\) for all \(\lambda^{(N)}\) sufficiently large. Hence,

\[
\mathcal{L}_0(\lambda_N) = \sum_{k=0}^{N-1} [L(N-k)\lambda_k^{(N)} - M(k+1)\lambda_{k+1}^{(N)}]
\]

\[
\times (a_1^\dagger)^{L+N-k-1}a_1^{-1}(a_2^\dagger)^{k+M}a_2^M + \text{l.o.t.}
\]

Therefore, all main terms have negative coefficients if

\[
\lambda_{k+1}^{(N)} > \lambda_k^{(N)} \frac{(N-k)L}{(k+1)M}
\]

For the fixed \(N\), the lower order terms can be dominated by the main terms plus \(\lambda^{(N)}I\) for all \(\lambda^{(N)}\) sufficiently large, that is

\[
\exists c > 0: \quad \mathcal{L}_0(\lambda_N) \leq c\lambda^{(N)}I \leq c\Lambda_N
\]

if (4.10) holds. Since

\[
\Lambda_N \geq \lambda^{(N)}I + \lambda_0^{(N)}(a_1^\dagger)^Na_1^N + \lambda_{N}^{(N)}(a_2^\dagger)^Na_2^N,
\]

where the coefficients \(\lambda^{(N)}, \lambda_0^{(N)}, \lambda_{N}^{(N)}\) can be chosen greater than any constant \(c \geq 0\). In particular, they can be chosen such that the diagonal s.a. operator \(\Lambda_N\) dominates with arbitrary small upper relative bound a given polynomial \(H_{s.a.}\). Note that for any quadratic operator \(H_2 = H_2(a^\dagger, a)\), \(\mathcal{L}_0(H_2)\) is a symmetric polynomial of order \(2(L + M)\) in creation and annihilation operators. Hence it can be dominated by \(\Lambda_N\) for any \(N \geq 2(L + M)\). In this case, \(\{\Lambda_N, H_{int} = \Lambda_N + H_{s.a.}\}\) is a \(\Lambda\)-pair for the formal generator \(\mathcal{L}(\cdot)\) with coefficients (4.8). This proves that \(\mathcal{L}(\cdot)\) is regular. \(\square\)
5 Discussion

By using the concept of Λ-pair, we have analyzed the regularity property for a wide class of generators of MME in quantum optics which are available for authors. We proved that the generators of the form

\[ L(B) = \Phi(B) - (\Phi(I)B + B\Phi(I))/2 + i[H_2 + H_{s.a.}, B] \]

are regular for CP-maps (3.1), (4.6), and (4.8) if \( H_{s.a.} \) is a self-adjoint polynomial of a finite order in creation and annihilation operators, and \( H_2 \) is a symmetric operator of the second order. To conclude the paper, we recall the most important open problems.

Generator of MME can be irregular if \( \Phi(\cdot) \) is as in (4.7). From mathematical viewpoint, to select a unique solution, one must introduce a kind of boundary condition as was done in [24], where all unital extensions of the minimal quantum dynamical semigroup are described in terms of extension of its resolvent. In analogous classical cases, the boundary conditions for stochastic processes follow from Dynkin’s formula [25], [26] for infinitesimal operator of the Markov semigroup. The physical sense of boundary conditions for quantum systems should be related to conservation laws, but physical examples of MME with boundary conditions still are not known.

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