Data-Driven Gain Scheduling Control of Linear Parameter-Varying Systems Using Quadratic Matrix Inequalities

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Abstract—This letter synthesizes a gain-scheduled controller to stabilize all possible Linear Parameter-Varying (LPV) plants that are consistent with measured input/state data records. Inspired by prior work in data informativity and LTI stabilization, a set of Quadratic Matrix Inequalities is developed to represent the noise set, the class of consistent LPV plants, and the class of stabilizable plants. The bilinearity between unknown plants and ‘for all’ parameters is avoided by vertex enumeration of the parameter set. Effectiveness and computational tractability of this method is demonstrated on example systems.

Index Terms—Data-driven control, linear parameter-varying systems, LMIs, numerical algorithms.

I. INTRODUCTION

This letter performs Data Driven Control (DDC) of discrete-time Linear Parameter-Varying (LPV) systems using Quadratic Matrix Inequalities (QMIs). The problem setting involves parameter-affine LPV systems in which the parameter may vary arbitrarily within a polytope and the measured data admits a quadratic description in its noise. When the system has $n$ states, $m$ inputs, $L$ parameters, and $N_v$ vertices in the parameter polytope, we propose a non-conservative Linear Matrix Inequality (LMI) to find a quadratically stabilizing gain-scheduled controller for all consistent LPV plants involving $N_v$ Positive Semidefinite (PSD) constraints of size $(n(L+1)+m)$ (continuous-time) or $(n(L+2)+m)$ (discrete-time) and a single Positive Definite (PD) constraint of size $n$.

LPV systems are a class of linear systems whose plant dynamics depend on externally measured parameters. LPV systems have been employed to model and control nonlinear dynamics such as in vehicle control [1], missile control [2], and chemical processes [3]. Gain-scheduling control sets the input to be a function of the state and measured parameter [4]. Examples of quadratically stabilizing gain-scheduling through a common Lyapunov function include backsubstitution [5], interpolated vertex-controllers when the LPV dynamics are parameter-affine [2], and the use of a dynamic compensator when the plant dynamics are a Linear Fractional Transformation of the applied parameter [6]. The work in [7] applied different QMIs for robust control of a single given continuous-time LPV plant.

DDC is a methodology of formulating controllers for all possible plants that are consistent with measured input/output relations (data) [8]. Such algorithms avoid an expensive system-identification step to construct a generalized plant model. A survey of data-driven techniques is provided in [9]. One class of DDC methods applies Willem’s Fundamental Lemma, which parameterizes all possible system responses by linear combinations of a single trajectory’s Hankel matrices if a rank condition is satisfied (persistency of excitation) [10]. This Lemma can be used for stabilization/regulation [11] and Model Predictive Control [12], [13] with optional regularization to reduce sensitivity to noise.

When the noise corrupting the recorded data admits a quadratic description, QMIs may be used in a non-conservative manner to describe the noise set and the set of consistent plants [14]. Their work forms a matrix S-Lemma [15], providing conditions under which the satisfaction of one QMI implies another QMI [16], in order to perform quadratic stabilization and robust control ($H_2$ and $H_\infty$). The QMI-with-S-Lemma approach has also been used to stabilize nonlinear systems with state-dependent representations [17], to form a robust-control framework incorporating prior knowledge [18], to analyze and control continuous-time systems [19], to iteratively stabilize networked systems with block-structured controllers [20], and to impose LMI-region performance constraints on robust controllers [21].

DDC has been previously applied to LPV systems, as surveyed by [3]. Other instances of DDC for LPV include using Support Vector Machines [22], hierarchical control [23], and Willem’s Fundamental Lemma [24]. The related problem of DDC of switched systems was studied in [25] using polynomial optimization. To the best of our knowledge, QMIs and the matrix S-Lemma have not been used for DDC of LPV systems.

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The contributions of our work are

- A presentation of the Data-Driven LPV quadratic stabilization problem parameterized by QMIs.
- An LMI to achieve quadratic stabilization via gain-scheduling vertex-QMIs with Kronecker structure in continuous-time and discrete-time.
- An accounting of computational complexity.

This letter has the following structure: Section II reviews preliminaries such as notation, LPV stabilization, and the use of QMIs in forming stabilizing controllers. Section III applies this QMI method for LPV stabilization. Section IV performs worst-case suboptimal $H_2$ control on LPV plants consistent with the noise structure. Section V demonstrates this stabilization approach on example systems. Section VI concludes the letter.

### II. PRELIMINARIES

#### A. Notation

The double dots in $1..L$ represent the sequence of natural numbers between 1 and $L$. The $n$-dimensional real vector space is $\mathbb{R}^n$. The nonnegative real orthant is $\mathbb{R}_{\geq 0}^n$ and the cone of positive vectors is $\mathbb{R}_{> 0}^n$. The set of real-valued $m \times n$ matrices is $\mathbb{R}^{m \times n}$. The transpose of a matrix $M$ is $M^T$. The kernel (nullspace) of a matrix $M$ is $\ker(M)$. The set of symmetric matrices of size $n$ is $\mathbb{S}^n$, and its subsets of PSD and PD matrices are $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$. The vertical concatenation of matrices $A$ and $B$ of compatible dimensions is $[A; B]$ and their horizontal concatenation is $[A, B]$. The symmetrization operator applied to $M \in \mathbb{R}^{n \times n}$ is $\text{sym}(M) = (M + M^T)/2$. The pseudoinverse of a matrix $M$ is $M^+$. The notation $\text{ker}(\cdot)$ is used.

The matrices $I_n$, $0_{m \times n}$, $I_{m \times n}$ are respectively the identity, zeros, and ones matrices of appropriate dimensions. The dimension subscripts will be dropped when the matrix sizes are unambiguous. The * marking will be used in block matrices to refer to the canonical transpose of oppositely-indexed elements. The Kronecker product of matrices $P$ and $Q$ is $P \otimes Q$. The Hadamard (elementwise) product of matrices is $P \circ Q$. The symbol $\otimes_{\text{col}}$ will denote the column-wise Khatri-Rao product for matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times n}$ [26]

$$A \otimes_{\text{col}} B = (1_p \times 1) \odot (B \otimes 1_{m \times 1}).$$

(1)

The convex hull of a set of points $P = \{p_i\}_{i=1}^N$ is $\text{conv}(P)$. The notation $\delta x$ will mean the derivative $\dot{x}$ in continuous-time or the next state $x_+$ in discrete-time.

#### B. LPV Stabilization

LPV dynamics with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and measurable parameter $\theta \in \Theta \subset \mathbb{R}^L$ are

$$\dot{x} = A(\theta)x + B(\theta)u. \quad (2)$$

The LPV A-affine (LPVA) structure [27] has $B$ constant and $A \theta$-affine for some set of matrices $\forall \ell : A_\ell \in \mathbb{R}^{n \times n}$ if

$$\dot{x} = \left(\sum_{\ell=1}^L A_\ell \theta_{\ell}\right)x + Bu. \quad (3)$$

This preliminary subsection will deliver exposition on the case where $\{A_\ell, B\}$ are known and fixed while $\theta$ is unknown and measured on-line. The main body of this letter will focus on the setting where the plant $\{A_\ell, B\}$ is unknown but consistent with observed data.

#### Remark 1: LPVA structure may be rendered affine in the parameter by adjoining a new constant $\theta_0 = 1$ to $\theta$.

Let $\Omega = \{\omega_\ell\}_{\ell=1}^N$ be a finite set of $N_\ell$ points in $\mathbb{R}^L$. In this letter, the parameter set $\Theta$ will be chosen to be the compact convex polytope $\Theta = \text{conv}(\Omega)$. We will refer to $\Omega$ as the vertices of $\Theta$ (or as vertices more generally).

A vertex-controller $K_\ell \in \mathbb{R}^{m \times n}$ is defined at each vertex $\omega_\ell$ in $\Omega$, yielding the state-feedback law $u = K_\ell x$. Given a parameter $\theta \in \Omega$, a gain-scheduled controller $u = K(\theta)x$ may be found by first solving for a feasible $c \in \mathbb{R}^{N_\ell}$ using Linear Programming

$$\text{find } c \in \mathbb{R}^{N_\ell} \quad \sum_{\ell=1}^{N_\ell} c_{\ell} \omega_\ell = \theta, \quad (4a)$$

and then returning the control policy,

$$K(\theta) = \sum_{\ell=1}^{N_\ell} c_{\ell} K_\ell \quad u = K(\theta)x. \quad (4b)$$

Any feasible point $c$ of (4a) will serve: uniqueness of $K(\theta)$ is not required.

Application of the gain-scheduled $u = K(\theta)x$ to the LPVA system (3) leads to the decomposed dynamics

$$\dot{x} = A(\theta)x + BK(\theta)x \quad (5a)$$

$$= \sum_{\ell=1}^L \theta_{\ell} A_\ell x + \sum_{\ell=1}^{N_\ell} c_{\ell} B K_\ell x \quad (5b)$$

$$= \sum_{\ell=1}^{N_\ell} c_{\ell} \left(\sum_{\ell=1}^L \omega_{\ell} A_\ell \right) + c_{B} B K_\ell x \quad (5c)$$

The open-loop system $A_\ell$ for each vertex $\omega_\ell$ (multiplied in (5) by $c_{\ell}$) may be defined as

$$A_\ell = \sum_{\ell=1}^L \omega_{\ell} A_\ell. \quad (6)$$

**Lemma 1:** If $C$ is a convex cone with elements $z$ and $\Theta = \text{conv}(\Omega)$, then the following statements are equivalent:

$$\sum_{\ell=1}^L \theta_{\ell} z_{\ell} \in C \quad \forall \theta \in \Theta \quad (7a)$$

$$\sum_{\ell=1}^L \omega_{\ell} z_{\ell} \in C \quad \forall \ell \in 1..N_\ell \quad (7b)$$

**Proof:** Statement (7a) implies (7b) because each vertex $\omega_\ell$ is an element of $\Theta$. Every point $\theta \in \Theta$ may be represented by a possibly non-unique convex combination of vertices with coordinates $\theta_{\ell} = \sum_{\ell=1}^{N_\ell} c_{\ell} \omega_{\ell}$ given that $\Theta = \text{conv}(\Omega)$ ([4a] and [28, Sec. 2.1.4]). Eq. (7b) implies (7a), because $\sum_{\ell=1}^L \theta_{\ell} z_{\ell}$ may be expressed as the convex combination of $C$-elements $\sum_{\ell=1}^L \sum_{\ell=1}^{N_\ell} (c_{\ell} \omega_{\ell}) z_{\ell}$.

**Definition 1:** The controller $u = K(\theta)x$ from Eq. (4) quadratically stabilizes the LPVA system (3) if there exists a $\theta$-independent $Y \in \mathbb{S}^n_{++}$ (for continuous-time) or a $P \in \mathbb{S}^n_{++}$ (for discrete-time)

$$-2 \text{sym}(Y(A(\theta) + BK(\theta))) \in \mathbb{S}^n_{++} \quad \forall \theta \in \Theta \quad (8a)$$

$$[P(A(\theta) + BK(\theta))] \in \mathbb{S}^{2n}_{++} \quad \forall \theta \in \Theta \quad (8b)$$

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Lemma 2: Equations (8a) and (8b) are equivalent to the following respective conditions,
\[-2 \text{sym}(Y(A_v + B K_v)) \in \mathbb{S}_{++}^n \quad \forall v = 1..N_v \] (9a)
\[P(A_v + B K_v)P \in \mathbb{S}^n_+ \quad \forall v = 1..N_v \] (9b)

Proof: Equivalence of the respective pairs (8a), (9a) and (8b), (9b) holds by Lemma 1 with regard to the cones \(\mathbb{S}^n_{++}\) and \(\mathbb{S}^{2n}_{++}\) [2].

Pre- and post-multiplying (9a) by \(Y^{-1}\) yields
\[-2 \text{sym}((A_v + B K_v)Y^{-1}) \in \mathbb{S}^n_{++} \quad \forall v = 1..N_v.\] (10)

Problems (9a) and (9b) are convex after substituting \(S_v = K_v Y^{-1}\) (using (10)) and \(S_v = K_v P\) respectively [29].

C. Quadratic Matrix Inequalities

This section reviews QMIs and the matrix S-Lemma approach proposed by [14], [16].

Definition 2: Given a matrix \(M \in \mathbb{S}^n\), a QMI is the quadratic statement in \(X \in \mathbb{R}^{n \times k}\) that \(X^T M X \in \mathbb{S}^{n}_+\).

QMIs can also be strict with \(X^T M X \in \mathbb{S}^n_+\). The works in [14], [16] present conditions under which one QMI implies another QMI, with specific attention on the scenario where the set \(X\) can be partitioned as \(X = [I, Z^T]\) for some \(Z\). In this case, the variable \(Z\) is referred to as satisfying a QMI constraint.

Definition 3: Let \(\Phi \in \mathbb{S}^{n \times k}\) be a partitioned matrix,
\[
\Phi_{11} \in \mathbb{S}^n, -\Phi_{22} \in \mathbb{S}^k_+.
\] (11a)

A matrix \(Z \in \mathbb{R}^{n \times k}\) satisfies the Quadratic Boundedness Property with respect to \(\Phi\) \((Z \in \text{QBQP}(\Phi))\) if
\[
[I_n \quad Z^T] [\Phi_{11} \Phi_{12} \Phi_{21} \Phi_{22}] [I_n \quad Z^T] \in \mathbb{S}^k_+.
\] (11b)

Lemma 3 (Theorem 3.2b of [16]): Assuming that \(\Phi\) satisfies (11a), let \(\Phi_{12} \in \mathbb{S}^{n \times k}\) be the Generalized Schur complement \(\Phi_{11} - \Phi_{12} \Phi_{22} \Phi_{12}^\top\), \(\|\cdot\|\) be the Frobenius norm, and \(\lambda_{\min}(\lambda_{\max})\) be the maximum (minimum) matrix eigenvalue. Then for all matrices \(Z \in \text{QBQP}(\Phi)\):
\[
\|Z + \Phi_{22}^{-1}\Phi_{12}\|_F^2 < k\lambda_{\max}(\Phi) / \lambda_{\min}(-\Phi_{22}).
\]

Z is therefore bounded if \(-\Phi_{22} \in \mathbb{S}^k_+\).

Definition 4: The Strict Quadratic Boundedness Property \((Z \in \text{QBQP}(\Phi))\) holds if the matrix in (11b) is in \(\mathbb{S}^{n \times k}_+\).

Structures of \(\Phi\) are listed in [16, Sec. 2]. Particular instances include energy bounds \(\Phi_{11} - ZZ^\top \in \mathbb{S}^n_+\) (with \(\Phi_{12} = 0\), \(\Phi_{22} = -I_k\)) and individual sample \(L_2\) bounds.

Theorem 1 (Strict Matrix S-Lemma, [16, Cor. 4.13]): Let \(M, N \in \mathbb{S}^{m \times k}\) be matrices satisfying (11a) with the same partition scheme and let \(Z \in \mathbb{R}^{n \times k}\). The following conditions are equivalent under the assumptions that \(\ker N_{22} \subseteq \ker N_{12}, N \mid N_{22} \in \mathbb{S}^n_+\), and \(-M_{22} \in \mathbb{S}^k_+\):
\[
Z \in \text{QBQP}(M), \quad \forall Z \in \text{QBQP}(N) \quad (12a)
\]
\[
\exists \alpha, \beta > 0: \quad M - \alpha N - \beta I_m \begin{bmatrix} 0 & 0 \\ 0 & k \times k \end{bmatrix} \in \mathbb{S}^{m \times k}_+. \quad (12b)
\]

III. LPV STABILIZATION WITH QMIS

A. Problem Description

A sampling process records a set of \(T\) observations from an unknown LPV system (3) under a bounded noise process \(w(\cdot)\) (discrepancy) for \(t = 0..T\)
\[
\delta x(t) = \left(\sum_{i=1}^L A_i \delta \theta_i \right) x(t) + B u(t) + w(t). \quad (13)
\]

This data is collected into matrices \((X_-, U, \Theta)\)
\[
X_- := [x(0) \; x(1) \ldots \; x(T-1)]
\]
\[
U := [u(0) \; u(1) \ldots \; u(T-1)]
\]
\[
\Theta := [\theta(0) \; \theta(1) \ldots \; \theta(T-1)].
\] (14)

The derivative observations \(\dot{X}\) (continuous-time) and one-step-ahead records \(X_+\) (discrete-time) are
\[
\dot{X} := [\dot{x}(0) \; \dot{x}(1) \ldots \; \dot{x}(T-1)]
\]
\[
X_+ := [x(1) \; x(2) \ldots \; x(T)].
\] (15)

The symbol \(X_\delta\) will refer to \(\dot{X}\) or \(X_+\) as appropriate. The data \(D\) will denote the tuple \((X_-, U, \Theta, X_\delta)\).

Let \(\Theta_{t} \in \mathbb{R}^{1 \times N}\) be the row of \(\Theta\) associated with parameter \(\theta_t\). The discrepancy \(W\) collected from (13) (mathematically equivalent to process noise for discrete-time) associated with the observations in \(D\) for a given LPV \((A(\theta), B)\) is
\[
W = X_\delta - \sum_{t=1}^L \Theta_{t} \otimes_{\text{col}} A_{t} X_- - BU. \quad (16)
\]

The following assumptions will be imposed,
A1 \(n, \ell, m, L, T\) are all finite and known.
A2 The set \(\Theta\) is a known compact non-empty polytope with vertices \(\Omega\).
A3 The ground truth system has LPV structure (3).
A4 There exists a known \(\Phi \in \mathbb{S}^{m \times k}_+\) satisfying (11a) such that \(W \in \text{QBQP}(\Phi)\) for the ground-truth system.

The consistency set of plants \((A(\theta), B)\) compatible with \(\Phi\) given \(\Phi\) is
\[
\Sigma_D(\Phi) = \{(A_{t})_{t=1}^L, B\} \mid W \text{ from (16) } \in \text{QBQP}(\Phi). \]

Remark 2: Data matrices arising from multiple trajectories may be horizontally concatenated if the noise structure in \(\Phi\) is compatible with the arrangement ([30, Example 2]).

Our goal is to solve the following problem,

Problem 1: Find a gain-scheduled (Eq. (4)) control policy \(u = K(\theta)x\) such that \(x_+ = (A(\theta) + BK(\theta))x\) is quadratically stable for all \((A_t, B) \in \Sigma_D\).

Remark 3: Problem (1) will be solved by enforcing that (9) holds for all \((A_t, B) \in \Sigma_D\) (Lemma 2).

B. Data Consistency QMI

The set \(\Sigma_D(\Phi)\) may be represented as a QMI.

Using the convention that \([A_{t}] = [A_1, A_2, \ldots, A_L]\) and \([A_{t}^T] = [A_1^T, A_2^T, \ldots, A_L^T]\), the discrepancy matrix \(W\) from (16) may be represented as
\[
\begin{bmatrix} I_n \\ 0_{n \times L} \end{bmatrix} X_\delta - \Theta \otimes_{\text{col}} X_- - U \begin{bmatrix} A_{t}^T \end{bmatrix}. \quad (17)
\]
Defining the matrix \( \Psi \in \mathbb{S}_{n+ (L_n + m)} \) as
\[
\Psi = \begin{bmatrix}
I_n & X_3 \\
0 & - \Theta \boxtimes \col X_2 \\
0 & - U
\end{bmatrix} \Phi \begin{bmatrix}
I_n & X_3 \\
0 & - \Theta \boxtimes \col X_2 \\
0 & - U
\end{bmatrix}^T,
\] (18)
it holds that the following two descriptions are identical:
\[
([A], B) \in \Sigma_{D}(\Phi) \iff ([A], B) \in \text{QBP}(\Psi).
\] (19)

C. Stabilization QMI

This section will form a QMI for stabilization of the subsystem \( A_v \in \mathbb{R}^{n \times m} \) at vertex \( v \) from (6) by a controller \( K_v \in \mathbb{R}^{m \times n} \). The continuous-time LMI criterion in (10) is equivalent to the following QMI
\[
([A], B) \in \text{SQBP}\left( \begin{bmatrix}
0 & \ast & \ast \\
- \alpha_v \boxtimes \col Y_1 & 0 & \ast \\
- K_v Y_1 & 0 & 0
\end{bmatrix} \right),
\] (20)
as obtained by pre- and post-multiplying (9a) by the invertible \( Y_1 \in \mathbb{S}_{n+}^{n} \). The discrete-time LMI criterion in (9b) is equivalent to the following QMI by collecting terms
\[
([A], B) \in \text{SQBP}\left( \begin{bmatrix}
P & \ast & \ast \\
- (\alpha_v \omega_v^T) \boxtimes P & \ast & \ast \\
- (\omega_v^T) \boxtimes (K_v P) & - K_v P \Psi
\end{bmatrix} \right).
\] (21)

Theorem 2 (Continuous-Time): Under assumptions A1-A4, QMI (20) holds for all \( ([A], B) \in \Sigma_{D}(\Phi) \) if and only if \( \forall \alpha_v \geq 0, \beta_v > 0 \) such that
\[
\begin{bmatrix}
- \beta_v I_n & \ast & \ast \\
- \alpha_v \boxtimes \col Y_1 & 0 & \ast \\
- K_v Y_1 & 0 & 0
\end{bmatrix}
\] (22)

Proof: This will follow a similar proof strategy as [14, Sec. IV] and [16, Sec. VI]. The \( \alpha_v, \beta_v \) structure follows from Theorem 1. It remains to affirm the assumptions under which this theorem is valid. Given that \( Y \in \mathbb{S}_{n+}^{n} \) and \( - \Phi_{22} \in \mathbb{S}_{n+}^{n} \), the lower-right corner of the matrix in (20) and \( \Psi \) may each be expressed as
\[
\begin{bmatrix}
0 & \ast & \ast \\
0 & \ast & \ast \\
0 & \ast & \ast
\end{bmatrix} \in \mathbb{S}_{n+}^{n+ (L_1) n + m + n + m}
\] (23a)
\[
\begin{bmatrix}
0 & \ast & \ast \\
0 & \ast & \ast \\
0 & \ast & \ast
\end{bmatrix} \in \mathbb{S}_{n+}^{n+ (L_2) m + m}
\] (23b)
The final condition is that \( \ker \Psi_{22} \subseteq \ker \Psi_{12} \) with
\[
\ker \Psi_{22} = \ker \begin{bmatrix}
\Theta \boxtimes \col X_2 \\
U
\end{bmatrix}
\] (24a)
\[
\ker \Psi_{12} = \ker \begin{bmatrix}
\Theta \boxtimes \col U \\
X_2
\end{bmatrix}
\] (24b)
All conditions are satisfied, so Theorem 2 is proven. ■

Theorem 3 (Discrete-Time): Under assumptions A1-A4, QMI (21) is satisfied \( \forall ([A], B) \in \Sigma_{D}(\Phi) \) if and only if \( \forall \alpha_v \geq 0, \beta_v > 0 \) such that
\[
\begin{bmatrix}
P - \beta_v I_n & \ast & \ast \\
0 & - (\alpha_v \omega_v^T) \boxtimes P & - (\omega_v^T) \boxtimes (K_v P) & - K_v P \Psi
\end{bmatrix}
\] (25)
\[
\in \mathbb{S}_{n+}^{n+ (L_2) m + m + n + m}
\]

Proof: This proof follows the same pattern as the above Theorem 2. The only modification required is demonstrating that the negative of the lower right-corner matrix in (21) is PSD, which holds by
\[
\begin{bmatrix}
\alpha_v \otimes I_n & \ast & \ast \\
\omega_v \otimes I_n & \ast & \ast
\end{bmatrix}
\] (26)

D. Controller Generation Program

This subsection will pose a pair of Semidefinite Programs (SDPs) to solve data-driven LPV stabilization under continuous-time and discrete-time, as introduced by Remark 3 under assumptions A1-A4. In the language of [30], the tuple \((D, \Phi, \Omega)\) is informative for LPV quadratic stabilization if the respective LMI is feasible.

1) Continuous-Time: The first matrix of (22) admits the substitution \( P = Y^{-1}, S_v = K_v P \) to form the LMI
\[
\begin{bmatrix}
- \beta_v I_n & \ast & \ast \\
- \omega_v \boxtimes \col P & 0 & 0 \\
- S \ & 0 & 0
\end{bmatrix}
\] (27)
The continuous-time stabilization SDP with gain-scheduled control matrices \( \{K_v = S_v P^{-1} \}_{v=1}^{N_v} \) is
\[
\begin{bmatrix}
P - \beta_v I_n & \ast & \ast \\
0 & - (\omega_v \otimes \col P) & - \omega_v \otimes (S_v^T) & 0 \\
0 & - (S_v P^{-1} S_v^T) & S_v P^{-1} S_v & P
\end{bmatrix}
\] (29)
followed by a Schur Complement
\[
\begin{bmatrix}
P - \beta_v I_n & 0 & 0 & 0 \\
0 & - (\omega_v \otimes \col P) & - \omega_v \otimes (S_v^T) & 0 \\
0 & - (S_v P^{-1} S_v^T) & S_v P^{-1} S_v & P
\end{bmatrix}
\] (30)
Letting \( \Gamma_v(\beta_v) \) be the matrix in (30), the LMI (25) from Theorem 3 may be restated as,
\[
\begin{bmatrix}
P - \beta_v I_n & 0 & 0 & 0 \\
0 & - (\omega_v \otimes \col P) & - \omega_v \otimes (S_v^T) & 0 \\
0 & - (S_v P^{-1} S_v^T) & S_v P^{-1} S_v & P
\end{bmatrix}
\] (31)
The discrete-time stabilization SDP with gain-scheduled control matrices \( \{K_v = S_v P^{-1} \}_{v=1}^{N_v} \) is
\[
\begin{bmatrix}
P - \beta_v I_n & 0 & 0 & 0 \\
0 & - (\omega_v \otimes \col P) & - \omega_v \otimes (S_v^T) & 0 \\
0 & - (S_v P^{-1} S_v^T) & S_v P^{-1} S_v & P
\end{bmatrix}
\] (32)

Remark 4: In the specific discrete-time case where \( L = 1 \) and \( \Theta = \{ \theta = 1 \} \), Eq. (32) is identical to [14, Th. 14].

Remark 5: Programs (28) and (32) can be normalized by constraining \( \text{Tr}(P) = 1 \).
E. Computational Considerations

The per-iteration complexity of solving an SDP using an interior point method up to (nonzero) accuracy with a single PSD variable of size $N$ with $M$ affine constraints is $O(N^2M + M^2N^2)$ [31]. The continuous-time SDP in (28) has 1 PSD constraint of size $n$ (28a) and $N_v$ PSD constraints of size $n(L + 1) + m$ (28d). The discrete-time SDP in (32) has 1 PSD constraint of size $n$ (32a) and $N_v$ PSD constraints of size $n(L + 2) + m$ (32d).

The performance of SDPs (28) and (32) therefore scales linearly in $N_v$, polynomially in $(n, L, m)$, and independently of $T$. Linear dependence on $N_v$ may result in an exponential scaling on $L$ (e.g., a hypercube with $N_v = 2^L$).

IV. H2 Optimal Control

A continuous-time LPV state-space system with external input $\xi \in \mathbb{R}^c$ and regulated output $z \in \mathbb{R}^r$ given matrices $C \in \mathbb{R}^{r \times n}, D \in \mathbb{R}^{r \times m}; F \in \mathbb{R}^{n \times e}$ is

$$\dot{x} = \sum_{i=1}^L \theta_i A_i x + B u + F \xi, \quad z = C x + D u. \quad (33)$$

The recorded data in $D$ has $\xi = 0$ while $W \in QBP(\Phi)$. The input $\xi$ is applied during system execution.

Define the $H_2$ norm of (33) as the worst-case (over all parameter trajectories) expected root-mean-square value of $\|z\|^2$ when the input $\xi$ is a white noise process with identity covariance. Then we have the following bound:

**Proposition 1:** There exists a gain-scheduled controller $u = K(\theta)x$ such that the closed-loop $H_2$ norm of the LPVA system (33) is bounded above by $\gamma \in \mathbb{R}_+$ if for all $v = 1..N_v$, the following LMI is feasible [32]

$$\begin{align*}
\text{find} & P, Z, S \quad 2 \text{ sym}(A_v P + B S_v) - FF^T \in S^+_n \quad \text{(34a)} \\
& Z \quad CP + DS_v \quad \in S^{n \times r}_+ \\
& \text{Tr}(Z) \leq \gamma^2 \\
& P \in S^q_{n \times n}, Z \in S^r_{n \times r}, S_v \in \mathbb{R}^{m \times n}. \quad \text{(34d)}
\end{align*}$$

The gained-scheduled controller $K(\theta)$ may be recovered from $\{v \mid K_v = S_v P^{-1}\}$ and Eq. (4). The variables $(Z, P)$ and given entries $(C, D, F)$ are independent of $(A, B) \in \Sigma_D$.

$$\begin{bmatrix}
-\beta_I R - FD^T & * & * \\
-\alpha_P \otimes \text{col} P & 0 & 0 \\
-\alpha_S & 0 & 0
\end{bmatrix} \in S^{(L+1)n+m}_+ \quad \text{(35)}$$

**Constraint** (35) is equal to (27) when $F = 0_{n \times e}$, given that conditions (34a) and (9a) are identical under this restriction.

Worst-case $H_2$ control of (33) for all $(A(\theta), B) \in \Sigma_D$ given $(C, D, F)$ may be conducted by solving

$$\gamma^2 = \inf_{P, Z, S} \text{Tr}(Z) \quad \text{(36a)}$$

$$P \in S^q_{n \times n}, Z \in S^r_{n \times r}, \quad \alpha \in \mathbb{R}^r_{\geq 0}, \quad \beta \in \mathbb{R}^r_{\geq 0}, \quad \text{LMIs} (34b) \text{ and } (35) \text{ hold } \forall v = 1..N_v. \quad (36d)$$

The resultant $H_2$ norm is upper-bounded by $\gamma = \sqrt{\text{Tr}(Z)}$ when using gain-scheduled control matrices $[K_v = S_v P^{-1}]_{v=1}^{N_v}$. All results in this section may be extended to discrete-time $H_2$ control with appropriate LMIs.

V. Numerical Examples

Experiments were written in MATLAB R2021a and are available at https://github.com/jarmill/lpv_qmi in the folder experiments. Dependencies include Mosek [33] and YALMIP [34]. For both examples, the problem of finding a $\theta$-independent controller $K' \in \mathbb{R}^{m \times n}$ with $\forall v: K_v = K'_{\theta}$ that stabilizes all plants $(A_v, B)$ in the consistency set $\Sigma_D$ is infeasible.

A. Two-Parameter, Two-State

The experiment ground truth with $\Theta = [0, 2] \times [-1, 1]$ is

$$A_1^{\text{true}} = \begin{bmatrix} -0.2396 & -0.5845 \\ 0.5845 & -0.2396 \end{bmatrix}, \quad A_2^{\text{true}} = \begin{bmatrix} -0.1696 & 0.8434 \\ 0.4848 & 0 \end{bmatrix}. \quad (37)$$

The plant $A_2$ in (37) is open-loop unstable for both continuous-time and discrete-time with eigenvalues of $-0.7703, 1.0146$. Data $D$ with $T = 35$ was collected under an individual-sample noise bound of $\epsilon = 0.1$. Eq. (28) synthesizes the following continuous-time vertex-controllers

$$K_{(0,1)} = \begin{bmatrix} -5.3354 & -11.7774 \\ 10.3972 & 8.7615 \end{bmatrix}$$

$$K_{(0,-1)} = \begin{bmatrix} -6.1840 & -13.5543 \\ 12.3168 & 10.1554 \end{bmatrix}$$

$$K_{(2,1)} = \begin{bmatrix} -5.5081 & -11.3647 \\ 9.8734 & 8.2277 \end{bmatrix}$$

$$K_{(2,-1)} = \begin{bmatrix} -5.6840 & -11.6743 \\ 10.4987 & 8.5444 \end{bmatrix}. \quad (38)$$

The LMI parameters associated with $K$ in (38) are

$$P = \begin{bmatrix} 0.6952 & -0.1743 \\ -0.1743 & 0.3048 \end{bmatrix} \in S^2_{++} \quad \text{(Tr}(P) = 1)$$

$$\alpha = [0.4820, 0.5683, 0.4603, 0.4858] \in \mathbb{R}^4_{\geq 0}$$

$$\beta = [0.5859, 0.3130, 0.8071, 0.6056] \in \mathbb{R}^4_{\geq 0}. \quad (39)$$

The blue trajectories in Figure 1 are system executions from 15 plants in the set $(A_1, A_2, B) \in \Sigma_D$ starting from the point $x(0) = [2; 1.5]$. The parameter values $\theta$ are drawn uniformly from $[0, 2] \times [-1, 1]$ with exponentially distributed switching times (mean switching time is 0.05). The red dotted-line in the top plot is the ground truth system from (37) given the fixed parameter sequence. The bottom plot contains system trajectories for 30 parameter sequences on the ground truth and each of the 15 sampled plants.

System (37) may be stabilized in discrete-time (with the same data $X_1$ using (32), which produces a unit-trace $P$ matrix of $[0.3627, 0.0089; 0.0089, 0.6373]$. The continuous-time worst-case $H_2$ norm with $C = [I_2; 0_2], D = [0_2; \sqrt{2}I_2], F = I_2$ is upper-bounded by $\gamma = 5.2075$ using (36), and its discrete-time analogue with the same $(C, D, F, D)$ is upper-bounded by $\gamma = 9.334$.

B. Three-Parameter, Five-State

The second experiment involves a system with $n = 5, m = 3, L = 3$. The parameter set is $\Theta = [-0.3, 0.3] \times [0.2, 0.8] \times [0.5, 1.5]$ with $N_v = 8$. A trajectory is recorded

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with a time horizon of $T = 50$ and an individual-sample noise bound of $\epsilon = 0.1$. The controller’s unit-trace $P$ matrix is

$$
P = \begin{bmatrix}
0.0762 & 0.0389 & -0.0308 & 0.0233 & -0.0418 \\
0.0389 & 0.0958 & -0.0320 & 0.0833 & 0.0896 \\
-0.0308 & -0.0320 & 0.0822 & 0.0507 & -0.0341 \\
0.0233 & 0.0833 & 0.0507 & 0.3647 & -0.3521 \\
-0.0418 & -0.0896 & -0.0341 & -0.3521 & 0.3811
\end{bmatrix}
$$

Due to space constraints, we are unable to list the 8 associated vertex-controllers $\{K_0^i \in \mathbb{R}^{3\times5}\}$ in this subsection.

VI. CONCLUSION

This letter considered quadratic stabilization of all LPV systems $(A_\ell, B) \in \Sigma_\ell(\Phi)$. SDPs (28) and (32) perform this task by solving a set of $N_\ell + 1$ LMIs in order to recover a gain-scheduled controller. The unknown LPV plants may be regulated by using a worst-case gain-scheduled controller. The unknown LPV $A$ plants may be associated vertex-controllers reducing conservatism by letting $P$ depend on $\theta$.

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