Reducing Sampling Ratios and Increasing Number of Estimates Improve Bagging in Sparse Regression

Luoluo Liu\textsuperscript{J}, Sang (Peter) Chin\textsuperscript{B,J}, and Trac D. Tran\textsuperscript{J}

\textit{J} Department of Electrical Engineering, Johns Hopkins University, Baltimore, MD, 21210

\textit{B} Department of Computer Science & Hariri Institute of Computing, Boston University, Boston, MA, 02215

Abstract—Classical sparse regression based on $\ell_1$ minimization solves the least squares problem with all available measurements via sparsity-promoting regularization. In challenging practical applications with high levels of noise, missing and adversarial samples, solving the problem using all measurements may fail. Bagging, a powerful ensemble method from machine learning, has shown its ability to improve the performance of unstable predictors, in difficult practical settings. Although the power of Bagging has been shown mostly in classification problems, we demonstrate the success of employing Bagging in sparse regression over the baseline method ($\ell_1$ minimization). The framework employs the generalized version of the original Bagging with various bootstrap ratios. The performance limits associated with different choices of bootstrap sampling ratio $L/m$ and number of estimates $K$ is analyzed theoretically. Simulation results show that the proposed method yields state-of-the-art recovery performance, outperforming $\ell_1$ minimization and Bolasso, another bootstrap-based technique, in the challenging case of low levels of measurements. While the original Bagging method uses bootstrap sampling ratio $L/m = 100\%$, we have shown that a lower $L/m$ ratio ($60\% - 90\%$) leads to better performance, especially with a small number of measurements. With the reduced sampling rate, SNR improves over the original Bagging by up to 24\%. With a properly chosen sampling ratio, a reasonably small number of estimates $K = 30$ gives satisfying result, even though increasing $K$ is discovered to always improve or at least maintain the performance.

Index Terms—Bagging, Bootstrapping, Sparse Regression, Sparse Recovery, $\ell_1$ minimization, LASSO

I. INTRODUCTION

Compressed Sensing (CS) and Sparse Regression studies solving the linear inverse problem in the form of least squares plus a sparsity-promoting penalty term. Formally speaking, a the measurements vector $y \in \mathbb{R}^m$ is generated by $y = Ax + z$, where $A \in \mathbb{R}^{m \times n}$ is the sensing matrix, $x \in \mathbb{R}^n$ is the sparse coefficient with very few non-zero entries and $z$ is a bounded noise vector. The problem of interest is finding the sparse vector $x$ given $A$ as well as $y$. Among various choices of sparse regularizers, the $\ell_1$ norm is the most commonly used. The noiseless case is referred to as Basis Pursuit (BP) whereas the noisy version is known as basis pursuit denoising [1], or least absolute shrinkage and selection operator (LASSO) [2].

$$p^{\lambda}_x : \min \lambda \|x\|_1 + \|y - Ax\|_2.$$ \hspace{1cm} (1)

The performance of $\ell_1$ minimization in recovering the true sparse solution has been thoroughly investigated in CS literature [3]–[6]. CS theory reveals that if the sensing matrix $A$ has good properties, then BP recovers the ground truth and the LASSO solution is close enough to the true solution with high probability [3].

Classical sparse regression recovery based on $\ell_1$ minimization solves the problem with all available measurements. In practice, it is often the case that not all measurements are available or required for recovery. Part of the measurements might be severely corrupted/missing or there are adversarial samples that break down the algorithm. Those issues could lead to the failure of the sparse regression problem.

Bagging procedure [7], an efficient parallel ensemble method, proposed by Leo Breiman, to improve the performance of unstable predictors. In the Bagging procedure, we firstly sample uniformly at random with replacement from all $m$ data points, termed bootstrap [8]; repeat the process and generate $K$ bootstrap samples with sizes same as $m$; solve each problem on the bootstrap samples using a baseline algorithm; and then combine multiple predictions to obtain the final prediction.

Applying Bagging to obtaining a sparse vector with a specific symmetric pattern, was shown empirically to reduce estimation error when the sparsity level $s$ is high [7]. This experiment shows the possibility of using Bagging to improve other sparse regression methods on general sparse signals. Although the well-known conventional Bagging method uses bootstrap ratio 1, some follow-up works have shown empirically that lower ratios improves Bagging in some classic classifiers: Nearest Neighbour Classifier [9], CART Trees [10], Linear SVM, LDA and Logistic Linear Classifier [11]. Based on this success, the hypothesize that reducing the bootstrap ratio will also improve performance in the sparse regression problem. Therefore, we set up the framework with a generic bootstrap ratio and study its behavior with various bootstrap ratios.

In this paper, (i) we demonstrate the generalized Bagging framework with bootstrap ratio $L/m$ and number of estimates $K$ as parameters. (ii) We explore the theoretical properties associated with finite $L/m$ and $K$. (iii) We present simulation results of various $L/m$, $K$ are explored and compared to $\ell_1$ minimization, conventional Bagging and Bolasso [12], another modern technique that incorporates Bagging into sparse recovery. Bolasso is a two-step process which recovers the support of the signal first and then the amplitudes. An important discovery is that: when $m$ is small, Bagging with a ratio $L/m$ that is smaller than the conventional ratio 1 can leads to better performance in challenging cases with small $m$. 

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II. PROPOSED METHOD

A. Bagging in Sparse Regression

Our proposed method is a generalized Bagging procedure for sparse recovery. It can be accomplished in three steps. First, we generate bootstrap samples: The multiple bootstrap process generates \( K \) multi-sets of the original data, each of size \( L \), which contains \( K \) measurements and sensing matrices pairs: \( \{ y[\mathbf{z}_1], A[\mathbf{z}_1] \}, \{ y[\mathbf{z}_2], A[\mathbf{z}_2] \}, \ldots, \{ y[\mathbf{z}_K], A[\mathbf{z}_K] \} \). In this paper, we use notation \( \mathbf{z} \) on matrices or vectors to take rows supported on \( \mathbf{z} \) and throw away all other rows in the complement \( \mathbf{z}^c \). Second, we solve the sparse recovery in parallel on those sets, for all \( j \), find

\[
x^B_j = \arg \min_{x \in \mathbb{R}^n} \lambda ||x||_1 + || y[\mathbf{z}_j] - A[\mathbf{z}_j]x ||_2^2.
\]

The proposed approach is in the form of LASSO and numerous optimization methods can solve it such as [13]–[16].

Finally, the Bagging solution is obtained through averaging all \( K \) solutions from solving (2):

\[
\text{Bagging: } x^B = \frac{1}{K} \sum_{j=1}^{K} x^B_j.
\]

Compared to the \( \ell_1 \) minimization solution which is solved from all the measurements, the bagged solution \( x^B \) is obtained by resampling without increasing the number of original measurements. We will show that in some cases, the bagged solution outperforms the base solution.

III. PRELIMINARIES

We summarize the theoretical results of CS theory which we need to analyze our algorithm mathematically. We are going to introduce Null Space Property (NSP), as well as Restricted Isometry Property (RIP).

A. Null Space Property (NSP)

The NSP [17] for standard sparse recovery characterizes the necessary and sufficient condition of successful sparse recovery for \( \ell_1 \) minimization.

**Theorem 1 (NSP).** Every \( s \)-sparse signal \( x \in \mathbb{R}^n \) is a unique solution to \( P_1: \min ||x||_1 \) s.t. \( y = Ax \) if and only if \( A \) satisfies NSP of order \( s \). Namely, if for all \( v \in \text{Null}(A) \backslash \{0\} \), such that for any set \( S \) of cardinality less than equals to \( s \), \( S \subset \{1, 2, ..., n\}, |\text{card}(S)| \leq s \), the following is satisfied:

\[
||v[S]||_1 < ||v[S^c]||_1,
\]

where \( v[S] \) only has the vector values on an index set \( S \) and zero elsewhere.

B. Restricted Isometry Property (RIP)

Although NSP directly characterizes the ability of success for sparse recovery, checking the NSP condition is computationally intractable. It is also not suitable to use NSP for quantifying performance in noisy conditions since it is a binary (True or False) metric instead of a continuous range. The Restricted isometry property (RIP) [3] is introduced for those purposes.

**Definition 2 (RIP).** A matrix \( A \) with \( \ell_2 \)-normalized columns satisfies RIP of order \( s \) if there exists a constant \( \delta_s(A) \in [0, 1) \) such that for every \( s \)-sparse \( v \in \mathbb{R}^n \), we have:

\[
(1 - \delta_s(A)) ||v||_2^2 \leq ||Av||_2^2 \leq (1 + \delta_s(A)) ||v||_2^2.
\]

C. Noisy Recovery bounds based on RIP constants

It is known that RIP conditions imply NSP conditions satisfied for sparse recovery [3]. More specifically, if the RIP constant in the order \( 2s \) is strictly less than \( \sqrt{2} - 1 \), then it implies that NSP is satisfied in the order of \( s \). The noisy recovery performance for \( \ell_1 \) minimization is bounded based on the RIP constant is stated in the following theorem.

**Theorem 3** (Noisy recovery for \( \ell_1 \) minimization, Theorem 1.2 in [3]). Let \( y = Ax^* + z \), \( ||z||_2 \leq \epsilon \). \( x^0 \) is \( s \)-sparse that minimizes \( ||x - x^*||_2 \) over all \( s \)-sparse signals. If \( \delta_{2s}(A) < \sqrt{2} - 1 \), \( x^{\ell_1} \) be the solution of \( \ell_1 \) minimization, then

\[
||x^{\ell_1} - x^*||_2 \leq C_0(\delta_{2s}(A))s^{-1/2}||x_0 - x^*||_1 + C_1(\delta_{2s}(A))\epsilon,
\]

where \( C_0(\cdot), C_1(\cdot) \) are some constants, which are determined by RIP constant \( \delta_{2s} \). The form of these two constants terms are \( C_0(\delta) = \frac{2(1-\sqrt{2-\delta})}{1-(1+\sqrt{2-\delta})} \) and \( C_1(\delta) = \frac{4\sqrt{\pi \delta}}{1-(1+\sqrt{2-\delta})} \).

IV. THEORETICAL RESULTS FOR BAGGING ASSOCIATED WITH SAMPLING RATIO \( L/m \) AND THE NUMBER OF ESTIMATES \( K \)

A. Noisy Recovery for Employing Bagging in Sparse Recovery

We derive the performance bound for employing Bagging in sparse recovery, in which the final estimate is the average over multiple estimates solved individually from bootstrap samples. We give the theoretical results for the case that true signal \( x^* \) is exactly \( s \)-sparse and the general case with no assumption of the sparsity level of the ground truth signal.

To prove these two theorems, we combine Theorem 3 and the tail bounds of independent bounded random variables; more details of the proof can be found in [18].

**Theorem 4** (Bagging: Error bound for \( \|x^*\|_0 = s \)). Let \( y = Ax^* + z \), \( ||z||_2 < \infty \). If under the assumption that, for \( \mathcal{I}_j \) that generates a set of sensing matrices \( A[\mathbf{z}_1], A[\mathbf{z}_2], ..., A[\mathbf{z}_K] \), there exists a constant that is relates to \( L \) and \( K: \delta_{s(L,K)} \) such that for all \( j \in \{1, 2, ..., K\} \), \( \delta_{2s}(A[\mathbf{z}_j]) < \sqrt{2} - 1 \). Let \( x^B \) be the solution of Bagging, then for any \( \tau > 0 \), \( x^B \) satisfies

\[
P\{ ||x^B - x^*||_2 \leq C_1(\delta_{s(L,K)}) \left( \sqrt{\frac{L}{m}} ||z||_2 + \tau \right) \}
\geq 1 - \exp \left( \frac{-2K\tau^4}{L^2 ||z||_2^2} \right).
\]

We also study the behavior of Bagging for a general signal \( x^* \), \( \|x^*\|_0 \geq s \), in which the performance involves the \( s \)-sparse approximation error. We use the vector \( e \) to denote this error, \( e = x^* - x_0 \), where \( x_0 \) is the top \( s \) approximation of the ground truth signal (containing \( s \) entries with the largest amplitudes of \( x \)).
Theorem 5 (Bagging: Error bound for general signal recovery). Let $y = Ax^\star + z$, $\|z\|_2 < \infty$. If under the assumption that, for $\{I_j\}_s$ that generates a set of sensing matrices $A[I_1], A[I_2], \ldots, A[I_K]$, there exists $\delta(L,K)$ such that for all $j \in \{1,2,\ldots,K\}$, $\delta_{2\sigma}(A[I_j]) \leq \delta(L,K) < \sqrt{2} - 1$. Let $x^B$ be the solution of Bagging, then for any $\tau > 0$, $x^B$ satisfies

$$\mathbb{P}\{\|x^B - x^\star\|_2 \leq (C_0\delta(L,K))s^{-1/2}\|e\|_1 + C_1(\delta(L,K))(\sqrt{\frac{L}{m}}\|z\|_2 + \tau)\} \geq 1 - \exp\left(-\frac{2KC_1^4(\delta(L,K))\tau^4}{(b')^2}\right),$$

where $b' = (C_0(\delta(L,K)))s^{-1/2}\|e\|_1 + C_1(\delta(L,K))\sqrt{L}\|z\|_\infty^2$.

Theorem 5 gives the performance bound for Bagging for general signal recovery without the $s$-sparse assumption, and it reduces to Theorem 4 when the $s$-sparse approximation error is zero $\|e\|_1 = 0$. Theorem 5 can be used to analyze the cases with small $m$.

Both Theorem 4 and 5 above show that increasing the number of estimates $K$ improves the result, by increasing the lower bound of certainty of the same performance. As for the sampling ratio $L/m$, because the RIP constant in general decreases with increasing $L$ (proof with Gaussian assumption in [19]) and $C_1(\delta)$ is a non-decreasing function of $\delta$, a larger $L$ in general results in a smaller $C_1(\delta)$. The second factor associated with the noise power term, $\sqrt{L/m}$, suggests a smaller $L$.

Combining two factors indicates that the best $L/m$ ratio is in between a small and a large number. In the experiment results, we will show that when $m$ is small, varying $L/m$ from 0 – 1 creates peaks with the largest value at $L/m < 1$. The first factor is dominating in the stable case when there are enough measurements, in which a larger $L$ leads to better performance.

V. SIMULATIONS

In this section, we perform sparse recovery on simulated data to study the performance of our algorithm. In our experiment, all entries of $A \in \mathbb{R}^{m \times n}$ are i.i.d. samples from the standard normal distribution $\mathcal{N}(0,1)$. The signal dimension $n = 200$ and various numbers of measurements from 50 to 2000 are explored. For the ground truth signals, their sparsity levels are all $s = 50$, and the non-zeros entries are sampled from the standard Gaussian with their locations being generated uniformly at random. For the noise processes $z$, which entries are sampled i.i.d. from $\mathcal{N}(0,\sigma^2)$, with variance $\sigma^2 = 10^{-SNR/10}\|Ax\|_2^2$, where SNR represents the Signal to Noise Ratio. In our experiment, we add white Gaussian noise to make the SNR = 0 dB. We use the ADMM [13] implementation of LASSO to solve all sparse regression problems, in which the parameter $\lambda(K,L)$ balances the least squares fit and the sparsity penalty.

We study how the number of estimates $K$ as well as the bootstrapping ratio $L/m$ affects the result. In our experiment, we take $K = 30, 50, 100$, while the bootstrap ratio $L/m$ varies from 0.1 to 1. We report the Signal to Noise Ratio (SNR) as the error measure for recovery: $\text{SNR}(x,x^\star) = 10\log_{10}\frac{\|x-x^\star\|_2^2}{\|x^\star\|_2^2}$ averaged over 20 independent trials. For all algorithms, we evaluate $\lambda(K,L)$ at different values from .01 to 200 and then select optimal values that gives the maximum averaged SNR over all trials.

A. Performance of Bagging, Bolasso and $\ell_1$ minimization

Bagging and Bolasso with the various parameters $K, L$ and $\ell_1$ minimization are studied. The results are plotted in Figure 1. The colored curves show the cases of Bagging with various number of estimates $K$. The intersections of colored curves and the purple solid vertical lines at $L/m = 1$ illustrates conventional Bagging with full bootstrap rate. The grey circle highlights the best performance and the grey area highlights the optimal bootstrap ratio $L/m$. The performance of $\ell_1$ minimization is depicted by the black dashed lines, while the best Bolasso performance is plotted using light green dashed lines. In those figures, for each condition with a choice of $K, L$, the information available to Bagging and Bolasso algorithms is identical, and $\ell_1$ minimization always has access to all $m$ measurements.

From Figure 1, we see that when $m$ is small, Bagging can outperform $\ell_1$ minimization. As $m$ decreases, the margin increases. The important observation is that with a low number of measurements ($m$ is between $s$ to $2s$: 50 – 100, $s$ is the sparsity level), and a reduced bootstrap ratio $L/m$ (60% – 90%), Bagging beats the conventional choice of full bootstrap ratio 100% for all different choices of $K$. Also with a reduced ratio and a small $K$ our algorithm is already quite robust.
and outperforms $\ell_1$ minimization by a large margin. When the number of measurements is moderate $m = 3s = 150$, Bagging still beats the baseline; however, the peaks take at full bootstrapping ratio and reduced bootstrap ratios does not gain more benefits. Increasing the level measurement makes the base algorithm more stable and the advantage of Bagging starts decaying.

We perform the same experiments with more number of measurements $m$ and Table I illustrates the best performance for various schemes: $\ell_1$ minimization, the original Bagging scheme with full bootstrap ratio, Bagging, and Bolasso with SNR = 0 dB. For Bagging, the peak values are found among different choices of parameters $K$ and $L$ that we explored. We see that when the number of measurements $m$ is small (50 – 100), Bagging outperforms $\ell_1$ minimization. The reduced bootstrap rate also improves the conventional Bagging: the improvement is significant: 24% on SNR when $m = 50$. When $m$ is moderate (125 – 200), the reduced rate does not improve the performance compared to conventional Bagging. Bagging still outperform $\ell_1$ minimization with smaller margins than the cases with small $m$. While $m$ is large ($\geq 500$), Bagging starts losing its advantage over $\ell_1$ minimization.

Bolasso only performs similarly to other algorithms for an extremely large $m$ ($= 2000$) where it slightly outperforms all other algorithms. Bolasso only behaves in easy cases. When the noise level is high, it may easily lead to too sparse solutions. The reason is that the supports of different estimators may vary too much and then the size of the common support among all estimators, used in Bolasso algorithm as the recovered support, can shrink dramatically.

| The number of measurements $m$ | Small $m$ | Moderate $m$ | Large $m$ | Very large $m$ |
|-------------------------------|-----------|--------------|-----------|---------------|
| $\ell_1$ min.                |           |              |           |               |
| Original Bagging ($L/m=1$)   | 0.12      | 0.94         | 0.56      | 0.28          |
| Bagging                      | 0.02      | 0.09         | 0.08      | 0.28          |
| Bolasso                      |           |              |           |               |

### VI. CONCLUSION

We extend the conventional Bagging scheme in sparse recovery with the bootstrap sampling ratio $L/m$ as adjustable parameters, and derive error bounds for the algorithm associated with $L/m$ and the number of estimates $K$. Bagging is particularly powerful when the number of measurements $m$ is small. This condition is notoriously difficult, both in terms of improving sparse recovery results and obtaining tight bounds of theoretical properties. Despite these challenges, Bagging outperforms $\ell_1$ minimization by a large margin and the reduced sampling rate has a larger margin over the conventional Bagging algorithm $L/m = 100\%$. When the number of measurements $m$ is $s - 2s$, where $s$ is the sparsity level, the conventional Bagging achieves 270% – 29% and the generalized Bagging achieves 300% – 32% SNR improvement over the original $\ell_1$ minimization with reduced sampling rate. Our Bagging scheme achieves acceptable performance even with very small $L/m$ (around 60%) and relative small $K$ (around 30 in our experimental study). The error bounds for Bagging show that increasing $K$ will improve the certainty of the bound, which is validated in the simulation. For a parallel system that allows a large number of processes to be run at the same time, a large $K$ is preferred since it in general gives a better result.

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