Supersymmetric Calogero-Moser-Sutherland models and Jack superpolynomials

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Abstract

A new generalization of the Jack polynomials that incorporates fermionic variables is presented. These Jack superpolynomials are constructed as those eigenfunctions of the supersymmetric extension of the trigonometric Calogero-Moser-Sutherland (CMS) model that decomposes triangularly in terms of the symmetric monomial superfunctions. Many explicit examples are displayed. Furthermore, various new results have been obtained for the supersymmetric version of the CMS models: the Lax formulation, the construction of the Dunkl operators and the explicit expressions for the conserved charges. The reformulation of the models in terms of the exchange-operator formalism is a crucial aspect of our analysis.

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1 Introduction

Calogero-Moser-Sutherland (CMS) models \([1, 2, 3]\) have been studied extensively in the decade following their discovery. Apart from the pioneer works, devoted mainly to the study of the energy spectrum, the ground-state wave functions and their correlators, roughly speaking, the initial interest of this first wave of activity was mainly centered around their integrability and the formulation of their various (Lie algebraic) extensions (see e.g., \([1]\)). This subject was developed in parallel to the soliton theory. 1 Although the structure of both classes of models is quite different, they have some common properties, the existence of a Lax formalism being a good example. There are however deeper and curious connections: for instance, it has been observed that the time evolution of the movable poles of KdV rational solutions is governed by a dynamics of the CMS type (with a \(1/r^2\) potential) \([5]\).

The renewal in the interest for the CMS models that occurred in the nineties has many sources. One of it is rooted in the interest for systems having fractional statistics and the realization that the particles subject to CMS dynamics obey fractional statistics (see for instance \([6, 7, 8]\)). This motivation was triggered by two important problems in condensed matter. The first is the quantum Hall effect; it has been suggested in the mid-eighties that the quasiparticles obey fractional statistics (cf. for instance \([9]\) – and see \([10]\) for a relation between the quantum Hall effect and CMS models). The early nineties brought a second and somewhat stronger motivation in relation with high \(T_c\) superconductivity and its possible realization as a gas of anyons (see e.g., \([11]\) and \([12]\)). The formulation of Haldane’s generalized Pauli principle \([13]\) has also motivated further theoretical considerations on the issue of fractional statistics, this time for one-dimensional systems.

Another discovery of the late eighties which also partly accounts for the revival of the CMS models is that of a new integrable spin-chain model with long-range interaction, the Haldane-Shastry model \([14, 15]\). This model proves to have quite remarkable properties, among which a Yangian symmetry \([16]\). It turns out to be closely related to the trigonometric CMS (tCMS) model. Such a connection, already observed in the original papers, has been made precise by Polychronakos in the context of its seminal formulation of the exchange-operator formalism \([17]\). He showed that the Haldane-Shastry spin model can be recovered from the tCMS model augmented with spin degrees of freedom by freezing the dynamical degrees of freedom, thereby fixing the sites of the chains at the minima of the tCMS potential. This observation has led to the discovery of new integrable spin-chain models with long-range order (for instance \([18, 19]\)). It has also stimulated the interest for the study of symmetries of the Yangian-type in these CMS spin models. On the other hand, the physical motivations underlying the formulation of the Haldane-Shastry model, related to Anderson’s resonating-valence-bond model – at the time an alternative to the Néel state in high \(T_c\) superconductivity – and the 1D Hubbard models, have nourished the interest of CMS models in condensed matter physics.\(^2\) Many more physical applications have been found in the last decade, ranging from quantum chaos and matrix models \([21]\), mesoscopic systems \([22]\), black holes \([23]\) to supersymmetric integrable gauge theories \([24]\).

The return of CMS models on the hot seat in physics, mainly through the discoveries of many applications, has stimulated a wave of activities on the models’ intrinsic properties. Two objects, discovered in quite different contexts by mathematicians, have played a key role in that regard: these are the Dunkl operators \([25]\) and the Jack polynomials \([26]\).

Dunkl operators play a crucial role in various branches of mathematics, e.g., in the theory of affine Hecke algebras \([27]\) and for Schubert calculus (see e.g., \([28]\)). In the context of CMS models, they were a crucial ingredient in the formulation of the exchange-operator formalism, as they allowed a very simple and direct construction of the commuting charges \([29]\). They are also at the heart of the transfer matrix formulation of the CMS models \([30]\).

\(^1\)Curiously, soliton equations were first defined classically while mechanical models were initially formulated as quantum systems.

\(^2\)In that vein, we could also mention that the Jack polynomials, the eigenfunctions of the tCMS model – see below – have proved to be a rather convenient basis for performing some computations in specific field theoretical problems of condensed matter; see for instance \([20]\).
1 INTRODUCTION

The rebirth of CMS models coincides with a period of intense activity in mathematics regarding the theory of symmetric functions, centered mainly on the Jack and Macdonald polynomials, and in particular on their combinatorial applications. Jack polynomials are symmetric polynomials that were shown to be eigenfunctions of a simple differential operator (see e.g., sect. 5 in [31]). Then, Forrester pointed out that this operator is precisely the gauge transformation, by the ground-state wave function, of the tCMS hamiltonian [32], bringing the subject of Jack polynomials on the physicists’ desktabe.

At the time, there were still no explicit expressions for these polynomials. The search for an explicit description has led to two noteworthy discoveries by physicists. The first one is a Rodrigues’ type formula, namely, a recursive construction via the action of a differential operator built up from shifted products of Dunkl operators [33]. Independently, integral formulas for the Jack polynomials have been found in [34]; they led to the discovery of a fascinating but still mysterious connection between these polynomials and special singular vectors of the Virasoro and W algebras (see also [35]).

On the mathematical side, explicit expressions for the Jack polynomials have been obtained using a non-symmetric version of these polynomials [36]. And more recently, a rather simple determinant formula for the Jack polynomials has been presented [37].

Some of these results – in particular, the creation-operator formalism – have been extended to the construction of the hi-Jack polynomials [38], which are eigenfunctions of the CMS model with an inverse-square interaction augmented by a harmonic confining term, and of the Macdonald polynomials [39, 40], which are eigenfunctions of the trigonometric Ruijsenaars-Schneider model [41], a relativistic version of the tCMS model. In the latter case, integral formulas have also been obtained (see e.g., [42] and refs therein).

A quite natural extension of these studies is to consider their supersymmetric generalizations. The supersymmetric version of the rational CMS model has been considered by Freedman and Mende [43], with emphasis on the study of supersymmetric breaking, the very physical problem that has motivated the development of supersymmetric quantum mechanics [44]. The revival in the interest for the CMS models has stimulated a number of studies of their supersymmetric counterparts [45, 46, 47, 48, 49, 50]. However, there remain many open problems. For instance, there are no concise Lax formulation. Moreover, the suitable generalization of the Dunkl operators is known only for the rational model with confinement [46]. And more importantly, there are absolutely no known results concerning the proper superextension of the Jack polynomials. Indeed, it is only for the rational case with harmonic term that the solutions have been constructed in [43, 50] out of fermionic and bosonic creation operators related to those of a supersymmetric harmonic oscillator by a similarity transformation.

The initial goal of this work was to launch the study of the Jack superpolynomials. As an offshoot, we have obtained a number of new results on the supersymmetric tCMS (stCMS) models per se.

The first step in the construction of the supersymmetric model is the introduction of the fermionic variables $\theta_i$ and their conjugates $\theta_i^\dagger$. Out of these variables, two generic expressions for fermionic charges – the possible supersymmetric charges – can be constructed. The point we want to stress at this level is that this construction, rooted in the presence of two fermionic charges, leads necessarily to two supersymmetries. These charges are then used to build an hamiltonian. Explicitly, the hamiltonian is written as the anticommutator of these two charges, which thereby make the latter automatically conserved with respect to the dynamics generated by this hamiltonian.$^3$ We then adjust the precise expression of the charges in order to recover, when the fermionic variables are dropped, the bosonic hamiltonian to be supersymmetrized. The complete hamiltonian is thus the supersymmetric hamiltonian we are looking for. In our case, this is the stCMS hamiltonian [45]. This analysis is presented in sect. 3.1.

An observation that proves to be central for our subsequent analysis is that the part of the hamiltonian that contains the fermionic variables can be described in terms of a fermionic exchange operator – cf. sect. 3.1 (and we found afterwards that the same observation had been made before in [45]).

$^3$It is clear that this supersymmetrization process is quite different from the one used in classical field theory based on superspace techniques. There is no natural analogue of the superfield here, for instance.
This allows us to use the projection formalism developed in [17, 30] for the description of the CMS models with spin degrees of freedom. The key point of this projection technique is that by restricting the space of functions on which the operators act (namely, functions that are completely symmetric with respect to both the fermionic and the bosonic variables), we can trade the fermionic-exchange operator – hence, the fermionic degrees of freedom – for a standard position-exchange operator.

In particular, this method leads us to a novel but quite natural construction of the Dunkl operators in either their covariant or their commuting version – cf. sect. 3.3. This leads us to a direct proof of the integrability via the construction of commuting conserved bosonic charges. In sect. 3.2, another proof of the integrability is presented, this one based on the Lax formalism. Although we arrived at this Lax formulation independently, we realized that the same Lax operators, expressed in terms of exchange operators, had been presented in [35], albeit in a different context.

Before pursuing the presentation of the paper’s content, let us pause to discuss briefly the meaning of integrability for supersymmetric mechanical systems. In the non-supersymmetric case, this amounts to demonstrate the existence of \( N \) – the number of particles – commuting independent bosonic charges. A working criterion for an integrable supersymmetric extension of an integrable mechanical system could be the existence of \( N \) commuting independent bosonic charges that reduce to their non-supersymmetric version when the fermionic variables are dropped. For all the cases we can think of (including field theoretical models), this appears to be sufficiently restrictive. However, we could argue that having introduced \( N \) new degrees of freedom (the fermionic variables being split into a set of generalized variables, the \( \theta_i \)'s, and their conjugates, the \( \theta_i^* \)'s, \( i = 1, \ldots, N \)), we should expect, in the spirit of the Liouville theorem, that \( N \) additional conserved charges are required. Actually, the mere supersymmetry invariance appears to supply automatically further conserved charges. As already pointed out, the built in supersymmetry implies the existence of two conserved charges, denoted by \( Q \) and \( Q^\dagger \). Recall that the hamiltonian is given by their anticommutator. But this turns out to be true for all the higher-order hamiltonians of the system, i.e., they can all be expressed as anticommutators of higher-order fermionic charges. Indeed, for the stCMS model, we can construct rather directly (using the Dunkl operators, for instance) \( 2N \) conserved fermionic charges. However, it should be stressed that these do not anticommute among themselves. Moreover, by inspection, we readily find \( N \) additional bosonic conserved charges that commute with the bosonic ones previously constructed. Afterwards, this appears to be somewhat natural given that we have two supersymmetries, suggesting heuristically that the charges get organized in ‘multiplets’ of four, two bosonic and two fermionic.\(^4\)

In section 4 we turn to the main subject of this work: the formulation of the Jack superpolynomials. They are defined as eigenfunctions of the stCMS model. Notice that by a superpolynomial we refer to a polynomial in bosonic and fermionic variables without imposing a supersymmetric invariance constraint (i.e., these are not supersymmetric polynomials). We first unravel, in sect. 4.1, the mixed symmetry properties, with respect to the bosonic variables, that are induced by the presence of the fermionic variables on any symmetric superpolynomials. This leads us naturally to the central concept of superpartitions introduced in sect. 4.2 and which appears to be original. Superpartitions are used in turn to define the monomial superfunctions. Jack superpolynomials are then defined in sect. 4.3 as those stCMS eigenfunctions that are triangular with respect to a monomial superfunction decomposition. Many examples are presented.

Various straightforward extensions of the results presented in this paper, directions for future research and conclusions are collected in the final section. Some auxiliary sections complete the article. A brief review of the basic definitions pertaining to the usual Jack polynomials and some associated concepts is presented in sect. 2. The remaining complementary material is spread in three appendices. Excited states can be built from a vacuum state free of fermions, as it is done in the main body of the paper, or from a ‘vacuum’ filled by \( N \) fermions. This second option is considered in app.

\(^4\)This can be compared to the case of the classical \( \mathcal{N} = 1 \) supersymmetric Korteweg-de Vries equation [51], which is probably the best studied supersymmetric integrable system. In that case, we find that in addition to the bosonic supersymmetric extension of the usual KdV charges, there are (twice as many) nonlocal fermionic charges whose Poisson brackets yield a local bosonic charge - if dimensionally allowed - and vanish otherwise.
A. In app. B, we introduced creation operators analogous to those introduced in [33] for the standard Jack polynomials. Finally, a simple combinatorial expression counting the number of superpartitions of a given degree and a given fermionic number is presented in app. C.

2 Background

2.1 Calogero-Moser-Sutherland models

The CMS-models describe systems of \( N \) particles interacting pairwise through long-range potentials. The classical and quantum versions of these models are integrable. In this article, we focus on the supersymmetric extension of the quantum tCMS model in which the identical particles of mass \( m \) lie on a circle of circumference \( L \). If we set \( m = \hbar = 1 \), the hamiltonian of the tCMS model is the following [3]:

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \left( \frac{\pi}{L} \right)^2 \beta(\beta - 1) \sum_{1 \leq i < j \leq N} \frac{1}{\sin^2(\pi x_{ij}/L)}.
\]  

(1)

where \( \beta \) is a dimensionless real coupling constant. In this equation, and for the remainder of the article, double indexing stands for the difference between two variables, i.e.,

\[
x_{ij} \equiv x_i - x_j.
\]  

(2)

Position and momentum variables obey the usual commutation relations:

\[
[x_j, p_k] = i\delta_{jk}.
\]  

(3)

Two other models can be obtained from the tCMS model: the replacement \( L \to iL \) yields the hyperbolic model, whereas the limit \( L \to \infty \) gives the rational model (on an infinite line).

The hamiltonian (1) is semi-positive i.e.,

\[
H = \frac{1}{2} \sum_j A_j^\dagger A_j + E_0,
\]  

(4)

with

\[
A_j = p_j - i \sum_{k \neq j} X_{jk}.
\]  

(5)

Hence, the minimal value in the spectrum of \( H \) is given by

\[
E_0 = \left( \frac{\pi \beta}{L} \right)^2 \frac{N(N^2 - 1)}{6}.
\]  

(6)

The ground state, which is annihilated by every operator \( A_j \), corresponds to the following Jastrow-type function:

\[
\psi_0(x) = e^{\sum_{j<k} \int dx_j x_{jk}} = \prod_{j<k} \sin^\beta \left( \frac{\pi x_{jk}}{L} \right) \equiv \Delta^\beta(x).
\]  

(7)

The simplest way of showing the integrability of the quantum CMS models is by displaying a Lax pair, namely two \( N \times N \) Hermitian matrices, denoted by \( L \) and \( M \), satisfying the relations:

\[
\dot{L}_{jk} = -i[L_{jk}, H] = -i[L, M]_{jk} \quad \text{and} \quad \sum_j M_{jk} = \sum_k M_{jk} = 0.
\]  

(8)

The constraint on \( M \) is essential for the following \( N \) independent quantities to be conserved:

\[
H_{(n)} = \frac{1}{n} \text{tr} L^n \equiv \frac{1}{n} \text{tr}(L^n \Delta),
\]  

(9)
where $\Delta$ is the matrix whose entries are all 1’s. Therefore, $\text{tr}\Delta A$ denotes the ‘total trace’ of $A$, that is, the sum of all the entries of $A$. For the tCMS model, the Lax pair reads \cite{2}:

$$
\begin{align*}
L_{jk} &= p_j \delta_{jk} + i(1 - \delta_{jk}) X_{jk}, \\
M_{jk} &= \delta_{jk} \sum_{l \neq j} X'_{jl} - (1 - \delta_{jk}) X'_{jk},
\end{align*}
$$

(10)

where $X'_{jk} = dX(x_{jk})/dx_{jk}$. From eq. (9), we see that the first and second conserved quantities correspond respectively to the momentum $P = \sum_i p_i$ and the hamiltonian $H$ of the system.

In order to solve the Schrödinger equation associated to the CMS model, it is convenient to set:

$$
z_j = e^{2\pi i x_j/L}.
$$

(11)

The variable $z_j$ thus gives the position of the $j$th particle on a circle of circumference $L$ in the complex plane. In this notation, $H$ becomes:

$$
H = 2 \left( \frac{\pi}{L} \right)^2 \left[ \sum_i \left( z_i \frac{\partial}{\partial z_i} \right)^2 - 2\beta(\beta - 1) \sum_{i<j} \frac{z_i z_j}{z_{ij}} \right].
$$

(12)

The eigenfunctions of the excited states of the hamiltonian \cite{12} are written in the form $\psi(x) = \phi(x) \psi_0(x)$ where $\phi(x)$ is required to be symmetric in order for $\psi$ to behave like $\psi_0$ under the exchange of particles. It is thus natural to conjugate the hamiltonian with the ground-state wave function $\psi_0(x)$:

$$
\bar{H} = \frac{\beta}{2} \left( \frac{\pi}{L} \right)^2 \Delta^{-\beta}(H - E_0)\Delta^{\beta},
$$

$$
= \sum_i (z_i \partial_i)^2 + \beta \sum_{i<j} \frac{z_i z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j),
$$

(13)

and look for the eigenfunctions $\phi(x)$ of this conjugated hamiltonian. In the following, ‘bar’ operators will stand for operators that have been obtained by a similar conjugation of the ground state.

The symmetric eigenfunctions $\phi(x)$ of \cite{13} are known as the Jack polynomials.

2.2 Symmetric functions and Jack polynomials

We now summarize some basic results concerning symmetric functions \cite{39}. This will allow us to define properly the Jack polynomials \cite{26,31}, and thereby, to present the solutions of the tCMS model \cite{4}.

1- Symmetric functions and exchange operators. Symmetric functions are invariant under the action of the symmetric (or permutation) group $S_N$. If $z = (z_1, \ldots, z_N)$ denotes the set of variables, then a function $F(z)$ is symmetric if it remains invariant under the exchange of its variables:

$$
K_{ij} F(z) = F(z) \quad \forall i, j,
$$

(14)

where $K_{ij}$ is a transposition of $S_N$, i.e., an exchange operator, whose action is defined as follows:

$$
K_{ij} f(z_i, z_j) = f(z_j, z_i) K_{ij},
$$

(15)

with $f(z_i, z_j)$ standing for a function or an operator. The fundamental properties of the exchange operators are

$$
K_{ij} = K_{ji}, \quad K_{ij}^+ = K_{ij}, \quad K_{ij} K_{jk} = K_{ik} K_{ij}, \quad K_{ij} K_{ki} = K_{ij}, \quad K_{ij}^2 = 1.
$$

(16)

Let $\Sigma_N$ denote the ring of symmetric polynomials in the variables $z_1, \ldots, z_N$, and let $\Sigma_N^{(n)}$ be the subspace of symmetric polynomials of degree $n$. Before introducing various bases for $\Sigma_N$, we need to introduce the notion of partitions.
2- Partitions. A partition is a weakly-decreasing sequence of non-negative integers. More precisely, a partition \( \lambda \) of weight (or degree) \( n \) is defined as follows:

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \\
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq 1, \\
n = \lambda_1 + \lambda_2 + \ldots \lambda_l = |\lambda|, 
\]

where \( l = l(\lambda) \) is the length of the partition, that is, the number of its non-zero parts. We use \( p(n) \) for the number of partitions of \( n \), e.g., the number of partitions of 4 is \( p(4) = 5 \). We can also represent a partition \( \lambda \) as:

\[
\lambda = (1^{m_1}, 2^{m_2}, \ldots, i^{m_i}, \ldots), 
\]

where \( m_i \) is the number of parts of \( \lambda \) equal to \( i \). This allows us to define the following constant

\[
z_\lambda = 1^{m_1} m_1! \cdot 2^{m_2} m_2! \cdot \ldots,
\]

which enters in the definition of the Jack polynomials. There exists a natural partial order on partitions called the dominance ordering. It is defined in the following way:

\[
\lambda \geq \mu \text{ if } \lambda_1 + \lambda_2 + \ldots + \lambda_i \geq \mu_1 + \mu_2 + \ldots + \mu_i, \quad \forall i.
\]

The dominance ordering is a total order only for weights up to \( n = 5 \). Note finally that to \( \lambda \) we can associate a Young tableau having \( \lambda_i \) boxes in the \( i \)-th row. The conjugate partition, denoted \( \lambda' \) corresponds to the partition resulting from the interchange of the rows and columns in the Young tableau associated to \( \lambda \).

3- Power sums. The symmetric polynomials

\[
p_n = \sum_i z_i^n,
\]

where the sum extends over the \( N \) variables, are called power sums. The set of all products of power sums, i.e.,

\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l},
\]

forms a basis of \( \Sigma_N \).

4- Elementary symmetric functions. The elementary symmetric functions are:

\[
e_n = \sum_{i_1 < i_2 < \ldots < i_n} z_{i_1} \cdots z_{i_n}.
\]

Again, the set of all products of elementary functions

\[
e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l},
\]

is a basis of \( \Sigma_N \).

5- Monomial symmetric functions. The monomial symmetric functions are defined as follows:

\[
m_\lambda = \sum_{\mathcal{P} \in \mathcal{S}_N} \mathcal{P}(\lambda) = \sum_{\mathcal{P} \in \mathcal{S}_N} \lambda_{\mathcal{P}(1)} \lambda_{\mathcal{P}(2)} \cdots \lambda_{\mathcal{P}(N)},
\]

where here and below, the prime on the sum is used to indicate that it is done only over distinct permutations, which means that no monomial is repeated. The \( p(n) \) possible monomial functions of degree \( n \) constitute another basis of \( \Sigma_N^{(n)} \). Also, the monomial symmetric functions generalize \( e_n \) and \( p_n \):

\[
m_{(n)} = p_n \quad \text{and} \quad m_{(1^n)} = e_n.
\]
Table 1: Monomial functions of weight $|\lambda| \leq 3$ for $N = 4$ variables

| Weight $|\lambda|$ | Partition $\lambda$ | Monomial function $m_\lambda(z)$ |
|-----------------|-----------------|-------------------------------|
| 0               | (0)             | $z_1 + z_2 + z_3 + z_4$      |
| 1               | (1)             | $z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3$ |
| 2               | (11)            | $z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_1^2 + z_2^2 + z_3^2 + z_4^2$ |
| 3               | (111)           | $z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4 + z_1^2z_2 + z_1^2z_3 + z_1^2z_4 + z_2^2z_3 + z_2^2z_4 + z_3^2z_4 + z_1^3 + z_2^3 + z_3^3 + z_4^3$ |

The simplest monomial functions are given in Table 1.

6- Jack polynomials. The Jack polynomials, $J_\lambda(z_1, \ldots, z_N; \alpha)$, are symmetric polynomials depending on a parameter $\alpha$ that also form a basis of $\Sigma_N$. They belong to the ring $\mathbb{Q}(\alpha)[z_1, \ldots, z_N]$ of symmetric polynomials with rational coefficients in $\alpha$. They are uniquely characterized by the following two conditions (see e.g., [39]):

$$\langle J_\lambda, J_\mu \rangle_\alpha = 0 \quad \text{and} \quad \lambda \neq \mu \quad (\text{orthogonality}),$$

$$J_\lambda(z; \alpha) = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu}(\alpha)m_\mu \quad (\text{unitriangularity}),$$

where the scalar product is defined in the following way, with respect to the power sums:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu}z_\lambda^{(\lambda)},$$

where $z_\lambda$ has been introduced in eq. (19). The Jack polynomials generalize several types of symmetric polynomials:

$$J_\lambda(z; \alpha) \rightarrow \begin{cases} s_\lambda(z) & (\text{Schur functions}) \quad , \quad \alpha \rightarrow 1 \\ m_\lambda(z) & (\text{monomial functions}) \quad , \quad \alpha \rightarrow \infty \\ e_\lambda(z) & (\text{elementary functions}) \quad , \quad \alpha \rightarrow 0 \end{cases}$$

(recall that $\lambda'$ refers to the conjugate partition).

A few examples of Jack polynomials expanded in terms of monomial symmetric functions are shown in Table 2. In this notation, the number of variables is irrelevant as long as it is not smaller than the degree of the polynomial.

7- Jack polynomials and the tCSM model. Jack polynomials are related to the tCMS model for:

$$\alpha \equiv 1/\beta$$

where $\beta$ is the model’s coupling constant. In this context, we have to replace the scalar product $\langle A, B \rangle_\alpha$ by the physical one:

$$\langle A(x), B(x) \rangle = \int \frac{dz_1}{2\pi i} \cdots \frac{dz_N}{2\pi i} \prod_{i \neq j} \left(1 - \frac{z_i}{z_j}\right)^\beta A(1/z)B(z),$$

$$\propto \int_0^{2\pi} dx_1 \cdots dx_N |\psi_0|^2 A(x)^* B(x),$$

(32)
Table 2: Jack polynomials of weight $|\lambda| \leq 4$

| Weight $|\lambda|$ | Partition $\lambda$ | Eigenvalue $\varepsilon_{\lambda}(\beta, N)$ | Jack polynomials $J_{\lambda}(z; 1/\beta)$ |
|------------------|------------------|------------------|------------------|
| 0                | (0)              | 0                | $m_{(0)}$        |
| 1                | (1)              | $1 + \beta N - \beta$ | $m_{(1)}$        |
| 2                | (1$^2$)          | $2 + 2\beta N - 4\beta$ | $m_{(2)} + m_{(1)}$ |
| (2)              |                  |                  |                  |
| 3                | (1$^3$)          | $3 + 3\beta N - 9\beta$ | $m_{(3)} + m_{(1)}$ |
| (21)             |                  |                  |                  |
| (3)              |                  |                  |                  |
| 4                | (1$^4$)          | $4 + 4\beta N - 16\beta$ | $m_{(4)} + m_{(1)}$ |
| (21$^2$)         |                  |                  |                  |
| (2$^2$)          |                  |                  |                  |
| (31)             |                  |                  |                  |
| (4)              |                  |                  |                  |

where $\psi_0$ is the ground state of the trigonometric model. The Hamiltonian $\tilde{H}$ defined in (13) is self-adjoint with respect to the scalar product (32). This physical scalar product can equally be used to characterize the Jack polynomials as the unique polynomials satisfying (28) that are orthogonal with respect to (32).

It is also known that the Jack polynomials are eigenfunctions of the transformed Hamiltonian $\tilde{H}$ (31) (32):

$$\tilde{H} J_{\lambda}(z; 1/\beta) = \varepsilon_{\lambda} J_{\lambda}(z; 1/\beta)$$  (33)

with eigenvalues:

$$\varepsilon_{\lambda} = \sum_j [\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j].$$  (34)

The wave functions of the original trigonometric model are now simply $\psi_{\lambda}(z) = J_{\lambda}(z; 1/\beta) \Delta^\beta$, with eigenvalues $E_{\lambda} = 2(\pi/L)^2 \varepsilon_{\lambda} + E_0$. Therefore, if we introduce the quasi-momenta

$$\kappa_i = \left(\frac{2\pi}{L}\right) \left[\lambda_i + \beta(N + 1 - 2i)\right],$$  (35)

we observe that the spectrum of the model is that of a system of $N$ free quasi-particles, each of these with quasi-momentum $\kappa_i$ :

$$E_{\lambda} = \sum_i \frac{\kappa_i^2}{2}.$$  (36)

The quasi-momenta of two neighboring quasi-particles satisfies:

$$\kappa_i - \kappa_{i+1} \geq \frac{4\pi \beta}{L}.$$  (37)

The excited states of the tCMS model thus obey a generalized exclusion principle (13) (7) (8). In particular, we recover free bosons if $\beta = 0$ and free fermions if $\beta = 1$. 

3 Supersymmetric Calogero-Moser-Sutherland models

3.1 Supersymmetric quantum mechanics

Consider a quantum model that contains both bosonic and fermionic variables and whose hamiltonian is denoted $H$. Following the usual methods of supersymmetric quantum mechanics [52, 44, 53], we consider, in addition to the $2N$ bosonic variables $(x, p)$, the $2N$ fermionic variables $(\theta, \theta^\dagger)$.

The bosonic and fermionic variables satisfy respectively a Heisenberg and a Clifford algebra:

$$[x_j, p_k] = i \delta_{jk}, \quad \{\theta_j, \theta_k^\dagger\} = \delta_{jk},$$

with all other commutators or anticommutators equal to zero. We will usually work with a differential realization of these algebras:

$$p_j = -i \frac{\partial}{\partial x_j}, \quad \theta_j^\dagger = \frac{\partial}{\partial \theta_i}.$$  \hspace{1cm} (39)

To construct a supersymmetric hamiltonian, we will first construct two supersymmetric charges, denoted $Q$ and $Q^\dagger$, and define the hamiltonian as their anticommutator:

$$H = \frac{1}{2} \{Q, Q^\dagger\}.$$  \hspace{1cm} (40)

By construction, the hamiltonian’s eigenvalues are non-negative. The hamiltonian is invariant under a supersymmetric transformation if:

$$Q^2 = (Q^\dagger)^2 = 0.$$  \hspace{1cm} (41)

By writing the charges under the form

$$Q = \sum_{i=1}^N \theta_i^\dagger A_i(x, p), \quad Q^\dagger = \sum_{i=1}^N \theta_i A_i^\dagger(x, p),$$

we find that eq. (41) requires:

$$[A_i, A_j] = 0 = [A_i^\dagger, A_j^\dagger], \quad \forall i, j.$$  \hspace{1cm} (43)

The generic supersymmetric hamiltonian is thus:

$$H = \frac{1}{2} \left( \sum_i A_i^\dagger A_i + \sum_{i,j} \theta_i^\dagger \theta_j [A_i, A_j^\dagger] \right).$$  \hspace{1cm} (44)

For non-relativistic models, the hamiltonian is proportional to the square of the particles’ speed. We therefore write $A_i$ as a linear function of the momentum $p_i$:

$$Q = \sum_j \theta_j^\dagger (p_j - i \Phi_j(x)), \quad Q^\dagger = \sum_j \theta_j (p_j + i \Phi_j(x))$$

From eq. (44), the potential $\Phi_j(x)$ must be of the form

$$\Phi_j(x) = \partial_{x_j} W(x),$$

where $W(x)$ (called the prepotential), is an arbitrary function of the variables $x_1, \ldots, x_N$. The supersymmetric hamiltonian now takes the form [43]:

$$H = \frac{1}{2} \sum_i (p_i^2 + (\partial_{x_i} W)^2 + \partial_{x_i}^2 W) - \sum_{i,j} \theta_i^\dagger \theta_j \partial_{x_i} \partial_{x_j} W.$$  \hspace{1cm} (47)

5This amounts to considering $N' = 2$ supersymmetries – i.e., there will be two conserved supersymmetric charges. Extensions to more supersymmetries are discussed in the conclusion.
This hamiltonian is an extension of the purely bosonic model whose potential is \( \sum_i [(\partial_x W)^2 + \partial^2_{x_i} W] \).

Since the hamiltonian is semi-positive, any state annihilated by the charges \( Q \) and \( Q^\dagger \) is a ground state (vacuum). Obviously, only the vacuum is supersymmetric since an excited state cannot be simultaneously annihilated by both charges. The charges defined in (43) naturally lead to two ground states:

\[
\psi_0 = e^W |0\rangle \quad \text{and} \quad \tilde{\psi}_0 = e^{-W} |\tilde{0}\rangle,
\]

where the ground states \(|0\rangle\) and \(|\tilde{0}\rangle\) belong to the fermionic Fock space and are defined as follows:

\[
\theta_i |0\rangle = 0, \quad \theta_i^\dagger |0\rangle = 0, \quad \forall i.
\]

If we interpret the \( \theta_i \)'s as fermionic creation operators, then \(|0\rangle\) and \(|\tilde{0}\rangle\) correspond to the \( N \)-fermion and the \( 0 \)-fermion states respectively. In the realization (39), the ground states must then be of the form:

\[
|0\rangle \rightarrow 1, \quad |\tilde{0}\rangle \rightarrow \theta_1 \ldots \theta_N.
\]

To be physically meaningful, the functions \( \psi_0 \) and/or \( \tilde{\psi}_0 \) must be normalizable. If this is not the case, the supersymmetry is said to be broken. It should be noted that eq. (48) provides a natural way to supersymmetrize a model. Knowing the ground state \( \psi_0 \) of that model, it suffices to let \( W = \ln \psi_0 \) to get its supersymmetric extension (39).

We now specialize to a prepotential of the form:

\[
W(x) = \sum_{i<j} w(x_{ij}).
\]

The comparison of eqs (16) and (48) immediately gives the right choice of \( W \) for the CMS models:

\[
w'(x_{ij}) = X_{ij}.
\]

Consequently, the stCMS hamiltonian reads:

\[
\mathcal{H} = \frac{1}{2} \sum_i m_i^2 + \sum_{i<j} [X_{ij}^2 + X_{ij}'(1 - \theta_{ij}\theta_{ij}^\dagger)] - N(N - 1)(N - 2) \left( \frac{\pi \beta}{L} \right)^2 + \frac{1}{2} \sum_{i=1}^N m_i^2 + \left( \frac{\pi}{L} \right)^2 \sum_{i<j} \frac{\beta(\beta - 1 + \theta_{ij}\theta_{ij}^\dagger)}{\sin^2(\pi x_{ij}/L)} = E_0,
\]

where \( E_0 \) is as given in (14). The two ground states

\[
\psi_0(x) = \Delta^\beta(x) \quad \text{and} \quad \tilde{\psi}_0(x, \theta) = \Delta^{-\beta}(x)\theta_1 \ldots \theta_N
\]

are invariant under supersymmetric transformations and are normalizable for any value of \( \beta \).

The supersymmetric model can be solved much more easily if we notice that the term

\[
\kappa_{ij} = 1 - \theta_{ij}\theta_{ij}^\dagger = 1 - (\theta_i - \theta_j)(\partial_{x_i} - \partial_{x_j}).
\]

is a fermionic-exchange operator (45), that is,

\[
\kappa_{ij} f(\theta_i, \theta_j, \theta_{ij}^\dagger, \theta_{ij}^\dagger) = f(\theta_j, \theta_i, \theta_{ij}^\dagger, \theta_{ij}^\dagger) \kappa_{ij}
\]

for any monomial function \( f \). Moreover, the \( \kappa_{ij} \)'s satisfy the usual properties (16) of exchange operators.

As in the non-supersymmetric case, it is convenient to use \( z_j = e^{2\pi i x_j}/L \), in terms of which the hamiltonian reads:

\[
\mathcal{H} = 2 \left( \frac{\pi}{L} \right)^2 \left[ \sum_i (z_i\partial_i)^2 - 2 \sum_{i<j} \frac{z_i z_j}{z_{ij}} \beta(\beta - \kappa_{ij}) \right] - E_0.
\]
Removing the ground-state contribution leads to:

\[
\tilde{H} \equiv \frac{1}{2} \left( \frac{L}{\pi} \right)^2 \Delta - \beta H \Delta - \beta \sum_{i<j} \frac{z_i z_j}{z_{ij}}(1 - \kappa_{ij}) ,
\]

which is still supersymmetric because it is invariant under the action of the transformed fermionic charges:

\[
\tilde{Q} = \Delta - \beta Q \Delta \quad \text{and} \quad \tilde{Q}^\dagger = \Delta - \beta Q^\dagger \Delta .
\]

A complete set of eigenfunctions of the hamiltonian \((58)\) is given in sect. 4. Excited states built from the second ground state are considered in appendix A.

### 3.2 Lax formalism in the supersymmetric CMS models

Quite remarkably, knowing the CMS models’ Lax pair is enough to guarantee the existence of their supersymmetric extensions. Moreover, the supersymmetric Lax pair is a simple extension of the non-supersymmetric one.

The first statement is proved as follows. Recall that the supersymmetric hamiltonian is the anti-commutator of the two supersymmetric charges. Comparing eqs (10), (45), (46) and (52), we see that the supersymmetric charges, hence the supersymmetric hamiltonian, can easily be built from the Lax matrices \((60)\):

\[
Q = \sum_{i,j} \theta^\dagger_j L_{ij}, \quad Q^\dagger = \sum_{i,j} \theta_j L_{ij} .
\]

Therefore, the Lax matrices of the quantum CMS models guarantee the existence of their supersymmetric extensions!

In order to prove the integrability of the supersymmetric models, we introduce the four matrices that will provide the Lax formulation of the supersymmetric system:

\[
\mathcal{L}_{jk} = p_j \delta_{jk} + i(1 - \delta_{jk})X_{jk}\kappa_{jk} , \\
\mathcal{M}_{jk} = \delta_{jk} \sum_{l \neq j} X_{jl}' \kappa_{jl} - (1 - \delta_{jk})X_{jk}' \kappa_{jk} . \\
\Theta^\dagger_{jk} = \theta^\dagger_j \delta_{jk} , \\
\Theta_{jk} = \theta_j \delta_{jk} .
\]

These matrices obey the relations:

\[
\dot{\mathcal{L}}_{jk} = -i[\mathcal{L}_{jk}, \mathcal{H}] = -i[\mathcal{L}, \mathcal{M}]_{jk} , \\
\dot{\Theta}^\dagger_{jk} = -i[\Theta^\dagger_{jk}, \mathcal{H}] = -i[\Theta^\dagger, \mathcal{M}]_{jk} , \\
\dot{\Theta}_{jk} = -i[\Theta_{jk}, \mathcal{H}] = -i[\Theta, \mathcal{M}]_{jk} .
\]

To verify the equivalence between these relations and the equations of motion, we use the properties

\[
\mathcal{L}_{jk} \theta_k = \theta_j \mathcal{L}_{jk} , \quad \mathcal{L}_{jk} \theta^\dagger_k = \theta^\dagger_j \mathcal{L}_{jk}
\]

and

\[
\sum_i \mathcal{M}_{ij} = \sum_j \mathcal{M}_{ij} = 0 \Leftrightarrow \Delta \mathcal{M} = \mathcal{M} \Delta = 0 ,
\]

where \(\Delta_{ij} = 1\).

We see that \(\mathcal{L}\) and \(\mathcal{M}\) are obtained from \(L\) and \(M\) by simply multiplying each \(X_{jk}\) or \(X'_{jk}\) factor by the exchange operator \(\kappa_{jk}\).
Using the matrices $L$ and $\Theta$, one can construct the following independent quantities that can easily be shown to be conserved:

\[
\begin{align*}
H_n &= \frac{1}{n} \text{tr} \Delta L^n = \frac{1}{n} \sum_{jk} L^n_{jk}, \quad n = 1, 2, \ldots, N, \\
Q_n &= \frac{1}{n} \text{tr} \Delta (\Theta L^n) = \frac{1}{n} \sum_{jk} \theta_j L^n_{jk}, \quad n = 0, 1, \ldots, N - 1, \\
Q^\dagger_n &= \frac{1}{n} \text{tr} \Delta (\Theta^\dagger L^n) = \frac{1}{n} \sum_{jk} \theta^\dagger_j L^n_{jk}, \quad n = 0, 1, \ldots, N - 1, \\
I_n &= \frac{1}{n} \text{tr} \Delta (\Theta \Theta^\dagger L^n) = \frac{1}{n} \sum_{jk} \theta_j \theta^\dagger_j L^n_{jk}, \quad n = 0, 1, \ldots, N - 1.
\end{align*}
\]

More generally, any operator that can be written as the total trace of a polynomial function $F$ only depending on the matrices $\Theta$, $\Theta^\dagger$, and $L$ is conserved:

\[
\frac{d}{dt} \text{tr} \Delta F(\Theta, \Theta^\dagger, L) = 0.
\]

The quantities $Q(1)$ and $Q^\dagger(1)$ are simply the generators of the supersymmetric transformation $s$. However, the fermionic charges are not in involution: their anticommutators generate the hamiltonians $H(n)$ (see below).

We stress that the results of this subsection apply to all types of supersymmetric CMS models and not just the stCMS one.

### 3.3 Dunkl operator formalism and the supersymmetric Calogero-Moser-Sutherland models

In this section we construct the Dunkl operators of the stCMS model. With these operators in hands, the integrability of the model can be very easily (re)established.

We first introduce a new exchange operator that acts on the bosonic and fermionic variables:

\[
K_{ij} \equiv \kappa_{ij} K_{ij}, \quad \text{where} \quad [\kappa_{ij}, K_{ij}] = 0.
\]

A function of the variables $z_i$ and $\theta_i$ is said to be a symmetric superfunction if it is invariant under the action of the $K_{ij}$’s. It is worth noticing that the action of $\kappa_{ij}$ on symmetric superfunctions is equivalent to the action of $K_{ij}$ on those functions. For instance, if $F_K$ is a symmetric superfunction, that is,

\[
\kappa_{ij} F_K = K_{ij} F_K,
\]

we can rewrite the hamiltonian $H \equiv H_K$ as:

\[
H_K = 2 \left( \frac{\pi}{L} \right)^2 \left[ \sum_i (z_i \partial_i)^2 - 2 \sum_{i<j} \frac{z_i z_j}{z_{ij}^2} \beta (\beta - K_{ij}) \right] - E_0.
\]

This remark holds for any supersymmetric model whose hamiltonian contains a fermionic-exchange term.

Let us denote by $\Pi_K(O)$ the projection of an operator $O$ on a vector space invariant under the action of $K$, cf. the conclusion.

For instance,

\[
\Pi_K(K_{ij}) = 1, \quad \Pi_K(K_{ij}) = \kappa_{ij}, \quad \Pi_K(\kappa_{ij} \kappa_{kl}) = K_{kl} K_{ij}.
\]

We can consider the Hamiltonian with exchange term $K_{ij}$ as the most fundamental one. Indeed, by an appropriate choice of projection, various models can be generated from it. For instance, the tCMS model and its supersymmetric generalization are obtained respectively from:

\[
\Pi_K(H_K) = H, \quad \Pi_K(H_K) = H_K.
\]

We could also choose projections on antisymmetric spaces, e.g., $K_{ij} = -\kappa_{ij}$. This is considered in app. A. In a similar vein, spin degrees of freedom can be introduced in that way – cf. the conclusion.
We now present a simple way to derive the various types of Dunkl operators for the CMS models found in the literature out of the Lax operator (cf. [17, 30, 8]). The 'covariant' Dunkl operator is simply given by

$$D_j = \sum_k L_{jk}(X_{jk} \rightarrow X_{jk}K_{jk}) = p_j + i \sum_{k \neq j} X_{jk}K_{jk}. \quad (73)$$

This Dunkl operator satisfies the following properties:

$$K_{ij}D_i = D_jK_{ij} \quad \text{(covariance)},$$

$$[D_i, D_j] = - (\beta \pi / L)^2 \sum_{k \neq i,j} (K_{ik} - K_{jk})K_{ij} \quad \text{(non-commutativity)},$$

$$[D_i, H_K] = 0 \quad \text{(conservation)}. \quad (74)$$

where 'covariance' means that $D_i$ behaves like the variable $z_i$ under the action of the symmetric group $S_N$. It should be noted that this Dunkl operator can be viewed as the 'square' of fermionic derivatives, i.e.,

$$D_i = C_i^2 = (C_i^†)^2, \quad (75)$$

where

$$C_i = \frac{\partial}{\partial \theta_i} + \theta_i D_i \quad \text{and} \quad C_i^† = \theta_i + D_i \frac{\partial}{\partial \theta_i}. \quad (76)$$

It is useful to introduce another Dunkl operator:

$$D_i = D_i \pm \frac{\pi \beta}{L} (\sum_{j<i} K_{ij} - \sum_{j>i} K_{ij}). \quad (77)$$

that satisfies

$$K_{i,i+1}D_{i+1} - D_iK_{i,i+1} = \mp \frac{2\pi \beta}{L} (\text{degenerated Hecke algebra}),$$

$$[D_i, D_j] = 0 \quad \text{(commutativity)},$$

$$[D_i, H_K] = 0 \quad \text{(conservation)}. \quad (78)$$

In addition to commuting among themselves, the $D_i$ have the nice property that the hamiltonian with exchange term $K$ lies in their universal algebra:

$$\frac{1}{2} \sum_i (D_i)^2 = H_K + E_0. \quad (79)$$

The supersymmetric hamiltonian is recovered by a simple projection.

Using these two versions of the Dunkl operators, it is now fairly easy to prove the integrability of the supersymmetric trigonometric model by constructing explicitly conserved charges from sums of powers of the Dunkl operators. First, the $N$ commuting conserved bosonic quantities which generalize those of the non-supersymmetric model are simply:

$$H_n = \Pi_K(\sum_i D_i^n), \quad n = 1, 2, \ldots, N \quad (80)$$

The proof of the commutativity relies on a simple property of the projections [53]:

$$\Pi_K([A, B]) = [\Pi_K(A), \Pi_K(B)] \quad \text{if} \quad [K_{ij}, A] = [K_{ij}, B] = 0. \quad (81)$$

Another simple 'covariant' Dunkl operator is $\hat{D}_j = D_j \pm (\beta \pi / L) \sum_{k \neq i} K_{ij}$. It has the following, somewhat more natural, commutation property: $[\hat{D}_i, \hat{D}_j] = \mp (2\beta \pi / L)(\hat{D}_i - \hat{D}_j)K_{ij}$. 

We use the same notation for the charges constructed from the Lax operators and from the Dunkl operators. Although the lowest order charges calculated from both expressions agree, this may not be so for the higher-order ones. Nevertheless, they are equivalent sets of independent charges.
The operators $\mathcal{H}_{(n)}$ meet this requirement since $[K_{ij}, \{\sum_i D_i^n\}] = 0$. The latter property implies also $[D_i, \{\sum_j D_j^n\}] = 0$. In addition to these bosonic conserved quantities, charges with fermions can be constructed as:

\[
\begin{align*}
Q_{(n)} &= \Pi_\mathcal{K}(\sum_i \theta_i D_i^n) , \quad n = 0, 1, \ldots, N - 1 , \\
Q^\dagger_{(n)} &= \Pi_\mathcal{K}(\sum_i \theta_i^\dagger D_i^n) , \quad n = 0, 1, \ldots, N - 1 , \\
I_{(n)} &= \Pi_\mathcal{K}(\sum_i \theta_i \theta_i^\dagger D_i^n) , \quad n = 0, 1, \ldots, N - 1 .
\end{align*}
\]

Note that here we use the covariant Dunkl operators rather than the commuting ones. This is imposed by the presence of the $\theta$ factors. Indeed, proving that those quantities are conserved still requires $\mathcal{K}$ and the covariant character of the Dunkl operators ensures the commutativity of the operators $\sum_i \theta_i D_i^n$ with an arbitrary exchange operator $K_{ij}$ (which would not be true if the commuting Dunkl operators were used instead).

The fermionic charges are not in involution: their anticommutators generate bosonic quantities, e.g., $\mathcal{H}_{(n)}$. Take for instance the rational case (on the infinite line) where $D_i \equiv D_i$; the conserved quantities constructed from the Dunkl operators (like those defined from the Lax matrices) satisfy the following algebra:

\[
\begin{align*}
\{Q_{(n)}, Q_{(m)}^\dagger\} &= \mathcal{H}_{(n+m)} , \\
\left[Q^\dagger_{(n)}, I_{(m)}\right] &= Q_{(n+m)} , \quad \left[Q_{(n)}, I_{(m)}\right] = -Q^\dagger_{(n+m)} , \\
\{Q_{(n)}, Q_{(m)}\} &= \{Q_{(n)}^\dagger, Q_{(m)}^\dagger\} = [I_{(n)}, I_{(m)}] = 0 ,
\end{align*}
\]

Only the last line remains true for the trigonometric and the hyperbolic models. In fact, it seems that the algebra of $\{\mathcal{H}, Q, Q^\dagger, I\}$ does not close linearly for those models.

Moreover, we could also replace the $N$ independent hamiltonians $\mathcal{H}_{(n)}$ by the following conserved quantities:

\[
J_{(n)} = \Pi_\mathcal{K}(\sum_i D_i^n) , \quad n = 0, 1, \ldots, N - 1 .
\]

However, the supersymmetric hamiltonian $\mathcal{H}$ would not belong to this set. There is thus some freedom in the way we choose a set of independent conserved charges. But quite generally, the projection $\Pi_\mathcal{K}$ of any quantity, made out of either the $D_i$’s or the $D_i$’s as well as out of the fermionic quantities $\theta_i$ and $\theta_i^\dagger$, invariant under the action of the exchange operator $K_{ij}$, is always conserved in the supersymmetric model.

We end this section by mentioning that the Dunkl operators of the transformed hamiltonian $\mathcal{H}$ are obtained as follows:

\[
\begin{align*}
\bar{D}_i &= \frac{1}{2\pi} \sum_j L_{ij} (X_{ij} \to X_{ij} K_{ij}) , \\
\bar{D}_i &= z_i \partial_t + \frac{\beta}{4} \sum_{j \neq i} z_i z_j (1 - K_{ij}) , \\
\bar{D}_i &= \bar{D}_i \pm \frac{\beta}{4} (\sum_{j < i} K_{ij} - \sum_{j > i} K_{ij}) ,
\end{align*}
\]

and we verify that:

\[
\Pi_\mathcal{K}(\sum_i \bar{D}_i^2) = \mathcal{H} + \frac{1}{2} \left(\frac{L}{\pi}\right)^2 E_0 .
\]

All the quantities constructed in this section can thus be directly transposed to the case where the ground-state wave function is factored out.

4 Jack superpolynomials

4.1 Symmetry of the tCMS model’s eigenfunctions

It is known that, when defining the Jack polynomials, we can replace condition $27$ by the condition that the Jack polynomials be eigenfunctions of the hamiltonian $\mathcal{H}$ of the tCMS model. Similarly, one
of the conditions entering the definition of their superanalogues will be that they be eigenfunctions of the stCMS model. We thus begin by making general observations regarding the symmetry properties of the eigenfunctions of the stCMS model.

We are looking for functions of \( \theta \) and \( z \) that are invariant under the transpositions \( K_{ij} \) and that are eigenfunctions of the hamiltonian:

\[
\bar{H} = \sum_i (z_i \partial_i)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j) - 2\beta \sum_{i<j} \frac{z_i z_j}{x_{ij}^2} (1 - \kappa_{ij}).
\] (87)

Since the hamiltonian is of degree 0 in both \( \theta \) and \( z \), the eigenfunctions have to be homogeneous in both variables. Moreover, since the underlying mechanical problem describes the dynamics of a system of particles on a circle, the solutions must be invariant under the transformation \( x_i \rightarrow x_i + 2\pi \); therefore, only integral powers of the variables \( z \) must be considered. Moreover, because the product of an eigenfunction of degree \( n \) by a Galilean ‘boost’,

\[
G^q = \prod_i z_i^q, \quad q \in \mathbb{Z},
\] (88)

gives another eigenfunction, now of degree \( n + Nq \), we can restrict ourselves to non-negative powers of \( z \).

We are thus seeking polynomial eigenfunctions that are invariant under the action of \( K_{ij} \). This operator commutes with the superhamiltonian, which is not the case with the operators \( K_{ij} \) and \( \kappa_{ij} \) taken separately. As already pointed out, the polynomials need to be homogeneous in \( \theta \) and \( z \), let’s say with degree \( m \) and \( n \) respectively. These degrees are good quantum numbers. Indeed, the total ‘momentum’

\[
\bar{P} = \sum_i z_i \partial_i
\] (89)

commutes with \( \bar{H} \) and its eigenvalue is the degree in \( z \) of the monomial on which it acts:

\[
\bar{P}(z_i^{n_i} \cdots z_N^{n_N}) = (\sum_i n_i)(z_i^{n_i} \cdots z_N^{n_N}).
\] (90)

Likewise, the quantity

\[
\eta = \sum_i \theta_i \theta_i^\dagger = \sum_i \theta_i \frac{\partial}{\partial \theta_i}
\] (91)

commutes with the supersymmetric hamiltonian and counts the number of fermions in a monomial:

\[
\eta(\theta_{i_1} \cdots \theta_{i_m}) = m(\theta_{i_1} \cdots \theta_{i_m}).
\] (92)

We say that the above monomial belongs to the \( m \)-fermion sector.

We can thus solve the supersymmetric Schrödinger equation in a fixed fermionic sector at a time. The independent eigenfunctions in a given fermionic sector will be denoted \( A^{(m)}(z, \theta; 1/\beta) \); they are indexed by a set of integers \( \Lambda \), called a superpartition, whose ‘norm’ refers to the degree in \( z \): \( |\Lambda| = n \) (see the following subsection for their actual definition).

We now clarify the symmetry properties, with respect to the \( z \) variables, of any symmetric superpolynomials and, in particular, of the eigenfunctions \( A^{(m)}_\Lambda \). The key observation is that the solutions \( A^{(m)}_\Lambda \) must necessarily be of the form:

\[
A^{(m)}_\Lambda(z, \theta; 1/\beta) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} \theta^{i_1 \cdots i_m} A^{(m)}_{\Lambda(i_1 \cdots i_m)}(z; 1/\beta),
\] (93)

where

\[
\theta^{i_1 \cdots i_m} = \theta_{i_1} \cdots \theta_{i_m}.
\] (94)
Indeed, the various terms in the $m$-fermion sector can always be rearranged as sums of $z$ polynomials with a monomial prefactor in the $\theta_i$'s. $A^{\theta_1\cdots\theta_m}_\Lambda$ is a homogeneous polynomial in $z$ indexed by a superpartition $\Lambda$. The solutions $\mathcal{A}^{(m)}_\Lambda$ being symmetric superpolynomials, must be invariant under the action of the exchange operators $K_{ij}$. Given that the $\theta$ products are antisymmetric, i.e.,

$$K_{ia} \theta^{i_1\cdots i_m} = -\theta^{i_1\cdots i_m}$$

the superpolynomials $A^{\theta_1\cdots\theta_m}_\Lambda$ must be partially antisymmetric to ensure the complete symmetry of $\mathcal{A}^{(m)}_\Lambda$. More precisely, the functions $A^{\theta_1\cdots\theta_m}_\Lambda$ must satisfy the following relations:

$$K_{ij} A^{\theta_1\cdots\theta_m}_\Lambda(z;1/\beta) = \begin{cases} -A^{\theta_1\cdots\theta_m}_\Lambda(z;1/\beta) & \forall \ i \neq j, \ i \in \{i_1,\ldots,i_m\}, \\ A^{\theta_1\cdots\theta_m}_\Lambda(z;1/\beta) & \forall \ i \neq j, \ j \notin \{i_1,\ldots,i_m\}. \end{cases}$$

Note that the case $m = 1$ is special:

$$\mathcal{A}^{(1)} = \sum_i \theta_i A^{\theta_i}_\Lambda(z;1/\beta),$$

$$K_{ij} A^{\theta_i}_\Lambda = A^{\theta_k}_\Lambda \text{ if and only if } i,j \neq k. \quad (97)$$

We have thus established that any symmetric eigenfunction of the stCMS model contains terms of mixed symmetry in $z$: each polynomial $A^{\theta_1\cdots\theta_m}_\Lambda$ is completely antisymmetric in the variables $\{z_{i_1},\ldots,z_{i_m}\}$, and totally symmetric in the remaining variables $z\setminus\{z_{i_1},\ldots,z_{i_m}\}$. Appendix B presents a simple way of generating such eigenfunctions by acting with appropriate operators on the Jack polynomials. However, this method does not lead to a unique characterization of the eigenfunctions. Later in this section will be presented an approach free from this drawback. But first, we need to define properly the superpartitions and some related concepts.

### 4.2 Symmetric superpolynomials

1. **The ring of symmetric superfunctions.** The symmetric superpolynomials are polynomials in $z$ and $\theta$ that commute with the generators $K_{ij}$ of the symmetric group $S_N$ of all possible permutations of the $N$ variables $\zeta_i = (z_i, \theta_i)$. As in the symmetric polynomial case, the set of all symmetric superpolynomials in the $N$ variables $\zeta_i$ forms a ring over the field of integers:

$$\tilde{\Sigma}_N = \mathbb{Z}[\zeta_1, \ldots, \zeta_N]|_{S_N} = \mathbb{Z}[z_1, \ldots, z_N; \theta_1, \ldots, \theta_N]|_{S_N}. \quad (98)$$

It is clear that the set of superpolynomials of degrees $m$ in $\theta$ and $n$ in $z$ is a $\mathbb{Z}$-module, which we will denote in the following way:

$$\tilde{\Sigma}_N^{(m;n)} = \mathbb{Z}[\zeta_1, \ldots, \zeta_N]^{(m;n)}|_{S_N} \quad (99)$$

The ring of symmetric superpolynomials is thus bigraded:

$$\tilde{\Sigma}_N = \bigoplus_{m,n} \tilde{\Sigma}_N^{(m;n)} \quad (100)$$

2. **Superpartitions.** In the case of symmetric polynomials, the basis elements of $\Sigma_N$ are indexed by partitions. In the same manner, basis elements of $\tilde{\Sigma}_N$ can be indexed by superpartitions. To motivate the following definition of superpartitions, recall that the symmetric superpolynomials in the $m$-fermion sector are antisymmetric in the $m$ variables $\{z_{i_1},\ldots,z_{i_m}\}$ and symmetric in the remaining ones. We thus define a superpartition of a $m$-fermion sector as a sequence of integers that generates two partitions separated by a semicolon:

$$\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_L) = (\lambda^a; \lambda^s), \quad (101)$$

the first one being associated to an antisymmetric function

$$\lambda^a = (\Lambda_1, \ldots, \Lambda_m),$$

$$\Lambda_i > \Lambda_{i+1} \quad \forall i = 1, \ldots, m - 1,$$

$$\Lambda_i \geq 0 \quad \forall 1 \leq i \leq m, \quad (102)$$
4  JACK SUPERPOLYNOMIALS

and the second one, to a symmetric function:

\[
\begin{align*}
\lambda^s &= (\Lambda_{m+1}, \ldots, \Lambda_L), \\
\Lambda_i &\geq \Lambda_{i+1} \quad \forall i > m, \\
\Lambda_i &\geq 0 \text{ if } i = m + 1; \quad \Lambda_i > 0 \quad \forall i > m + 1.
\end{align*}
\] (103)

In the zero-fermion sector \((m = 0)\), the semicolon disappears and we recover the partition \(\lambda^s\).

The length \(L \leq N\) of a superpartition corresponds to the total number of its parts, of which at most two can be zero: one on the antisymmetric side and one on the symmetric side. The weight (or degree) of a superpartition is simply the sum of its parts:

\[
|\Lambda| = \sum_{i=1}^{L} \Lambda_i = |\lambda^a| + |\lambda^s|.
\] (104)

For instance, the only possible superpartitions of weight 2 in the one-fermion sector are:

\[(2; 0), \quad (0; 2), \quad (1; 1), \quad (0; 1, 1).\] (105)

For 2 fermions, we have instead:

\[(2, 0; 0) \text{ and } (1, 0; 1).\] (106)

In order to specify explicitly the fermionic sector, we will sometimes denote the degree of a superpartition as:

\[
\text{degree (} m; n \text{)} \leftrightarrow (\text{fermionic sector } m; \text{weight } n = |\Lambda|).
\] (107)

Summation formulas giving the number of superpartitions of degree \((m; n)\) are presented in appendix C.

We mention finally that to any superpartition \(\Lambda\) there corresponds a single standard partition \(\lambda\) obtained by rearranging the parts of the superpartition in decreasing order:

\[
\lambda = \{\lambda_i | \lambda_i \in \{\Lambda_1, \ldots, \Lambda_L\}, \lambda_i \geq \lambda_{i+1}\}. \tag{108}
\]

3. Monomial symmetric superpolynomials. We can now introduce a basis of \(\Sigma_{N}^{(m; n)}\) that generalizes the symmetric monomial basis of \(\Sigma_{N}^{(n)}:\)

\[
m_{\Lambda}^{(m)}(z, \theta) = m_{(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_L)}(z, \theta) = \sum_{P \in S_N} \theta^{P(\Lambda)} z^{P(\Lambda)}, \tag{109}
\]

(recall that the prime indicates that the summation is restricted to distinct terms). It is understood that the action of the permutations on a superpartition is not affected by the semicolon:

\[P(\Lambda) = (\Lambda_{P(1)}, \ldots, \Lambda_{P(m)}; \Lambda_{P(m+1)}, \ldots, \Lambda_{P(L)}). \tag{110}\]

The functions \(m_{\Lambda}^{(m)}\) are called monomial symmetric superfunctions and \(m_{\Lambda}^{(0)} = m_{\lambda}(z)\). More explicitly, the monomial superfunctions can be written in the following way:

\[
m_{(\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_L)} = m_{(\lambda^a; \lambda^s)} = \sum_{i_1 < i_2 < \ldots < i_m} a_\lambda(z_{i_1}, \ldots, z_{i_m}) m_{\lambda^s}(z/\{z_{i_1}, \ldots, z_{i_m}\}), \tag{111}
\]

where we have introduced the antisymmetric monomial function

\[
a_\lambda(z_1, \ldots, z_N) = \sum_{P \in S_N} \theta_{P(\Lambda)} z^{P(\lambda)} = \sum_{P \in S_N} \theta_{P(\Lambda)} z_{\Lambda_{P(1)}} \ldots z_{\Lambda_{P(m)}} \ldots z_{\Lambda_{P(m+1)}} \ldots z_{\Lambda_{P(L)}}, \tag{112}
\]

Many examples of monomial superfunctions are shown in Table 4.
Table 3: List of all the monomial superfunctions of weight $|\Lambda| \leq 3$ for $N = \max(3, L)$ variables

| Weight $|\Lambda|$ | Sector $m$ | Superpartition $\Lambda$ | Monomial superfunction $m^{(m)}(z, \theta)$ |
|-------------------|-----------|--------------------------|-----------------------------------------------|
| 0                 | 1         | (0;0)                    | $\theta_1 + \theta_2 + \theta_3$             |
| 1                 | 1         | (1;0)                    | $\theta_1 z_1 + \theta_2 z_2 + \theta_3 z_3$ |
| 2                 | (0;1)     |                          | $\theta_1(z_2 + z_3) + \theta_2(z_1 + z_3) + \theta_3(z_1 + z_2)$ |
|                   | 2         | (1;0;0)                  | $\theta_1 \theta_2(z_1 - z_2) + \theta_1 \theta_3(z_1 - z_3) + \theta_2 \theta_3(z_2 - z_3)$ |
| 2                 | 1         | (1;1)                    | $\theta_1 z_1 (z_2 + z_3) + \theta_2 z_2 (z_1 + z_3) + \theta_3 z_3 (z_1 + z_2)$ |
|                   | 2         | (0;1;1)                  | $\theta_1 \theta_2 (z_1 - z_2) z_3 + \theta_1 \theta_3 (z_1 - z_3) z_2 + \theta_2 \theta_3 (z_2 - z_3) z_1$ |
| 1                 | 1         | (2;0)                    | $\theta_1 (z_2^2 + z_3^2) + \theta_2 (z_1^2 + z_3^2) + \theta_3 (z_1^2 + z_2^2)$ |
| 2                 | (2;0;0)   |                          | $\theta_1 \theta_2 (z_1^2 - z_2^2) + \theta_1 \theta_3 (z_1^2 - z_3^2) + \theta_2 \theta_3 (z_2^2 - z_3^2)$ |
| 3                 | 1         | (1;1;1)                  | $\theta_1 z_1(z_2 z_3 + z_2 z_4 + z_3 z_4) + \theta_2 z_2(z_1 z_3 + z_1 z_4 + z_2 z_4)$ |
|                   | 2         | (0;1;1;1)                | $\theta_1 \theta_2(z_1 - z_2)(z_3 z_4) + \theta_1 \theta_3(z_1 - z_3)(z_2 z_4) + \theta_2 \theta_3(z_2 - z_3)(z_1 z_4)$ |
|                   |           | (1;0;1;1)                | $\theta_1 \theta_2(z_1 - z_2)(z_3 z_4) + \theta_1 \theta_3(z_1 - z_3)(z_2 z_4) + \theta_2 \theta_3(z_2 - z_3)(z_1 z_4)$ |
| 1                 | 1         | (2;1)                    | $\theta_1 z_1^2(z_2 + z_3) + \theta_2 z_2^2(z_1 + z_3) + \theta_3 z_3^2(z_1 + z_2)$ |
|                   | 2         | (1;2)                    | $\theta_1 z_1(z_2^2 + z_3^2) + \theta_2 z_2(z_1^2 + z_3^2) + \theta_3 z_3(z_1^2 + z_2^2)$ |
|                   |           | (0;2;1)                  | $\theta_1 z_1^2 z_3 + z_2 z_3 z_4 + \theta_2(z_1^2 z_3 + z_1 z_4) + \theta_3(z_2 z_3 z_4 + z_2 z_4)$ |
| 2                 | (2;1;0)   |                          | $\theta_1 \theta_2(z_1^2 z_2 + z_1 z_2^2) + \theta_1 \theta_3(z_1^2 z_3 + z_1 z_3^2) + \theta_2 \theta_3(z_1^2 z_4 + z_1 z_4^2)$ |
|                   | (2;0;1)   |                          | $\theta_1 \theta_2(z_2^2 - z_1^2 z_2) + \theta_1 \theta_3(z_2^2 - z_1^2 z_3) + \theta_2 \theta_3(z_2^2 - z_1^2 z_4) + \theta_3(z_2^2 z_2 - z_2 z_4^2)$ |
|                   | (1;0;2)   |                          | $\theta_1 \theta_2(z_1 - z_2)(z_2 z_3) + \theta_1 \theta_3(z_1 - z_3)(z_2 z_4) + \theta_2 \theta_3(z_2 - z_3)(z_1 z_4)$ |
| 1                 | 1         | (3;0)                    | $\theta_1 z_1^2 + \theta_2 z_2^2 + \theta_3 z_3^2$ |
|                   | 2         | (3;0;0)                  | $\theta_1(z_2^2 + z_3^2) + \theta_2(z_1^2 + z_3^2) + \theta_3(z_1^2 + z_2^2)$ |
|                   |           |                          | $\theta_1 \theta_2(z_1^2 - z_2^2) + \theta_1 \theta_3(z_1^2 - z_3^2) + \theta_2 \theta_3(z_2^2 - z_3^2)$ |
4.3 Jack superpolynomials: monomial expansion

We now define the Jack superpolynomials in the $m$-fermion sector as the unique eigenfunctions of the supersymmetric hamiltonian $\tilde{H}$ that can be decomposed in terms of monomial superfunctions in the following way:

$$\mathcal{J}_\lambda^{(m)}(z, \theta; 1/\beta) = m_\lambda^{(m)}(z, \theta) + \sum_{\omega < \lambda} c_{\lambda, \Omega}(\beta)m_{\Omega}^{(m)}(z, \theta),$$

(113)

where $\omega$ and $\lambda$ are the partitions associated to the rearrangements of $\Omega$ and $\lambda$ respectively 9.

The relation between Jack superpolynomials and the usual Jack polynomials can now be stated precisely: $\mathcal{J}_\lambda^{(0)} = J_\lambda$.

The coefficients $c_{\lambda, \Omega}(\beta)$ in (113) are rational functions in $\beta$. We can easily verify, from the leading terms, that the spectrum of the supersymmetric hamiltonian $\tilde{H}$ is the same as that of $\tilde{H}$, i.e.,

$$\tilde{H}\mathcal{J}_\lambda^{(m)}(z, \theta; 1/\beta) = \varepsilon_\lambda \mathcal{J}_\lambda^{(m)}(z, \theta; 1/\beta) = \varepsilon_\lambda \mathcal{J}_\lambda^{(m)}(z, \theta; 1/\beta),$$

(114)

where the eigenvalues are given by:

$$\varepsilon_\lambda = \sum_j [\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j],$$

(115)

$\lambda$ being the rearrangement of $\Lambda$. Observe that the eigenvalue $\varepsilon_\lambda$ is independent of the fermionic sector, i.e., independent of the value of $m$. 10

Tables 4 and 5 present simple examples whose degrees $(m; n)$ are not larger than $(3; 4)$. The coefficients $c_{\lambda, \Omega}$ are obtained by simply diagonalizing the hamiltonian. Polynomials with the same partition have the same eigenvalue. Given that the eigenvalues are independent of $m$, they can be read in Table 2.

We should stress that the decomposition is not simply a straightforward extension of the triangular decomposition of the Jack polynomials, where the ordering is on partitions. Here the ‘ordering’ that allows a triangular decomposition is on superpartitions, albeit rearranged. The existence of such an ‘ordering’ seems to us quite remarkable (even though it is not a genuine ordering, as it is shown in [55]).

A closer look at those results shows that expression (113) is not restrictive enough: certain monomials allowed by the dominance ordering of the rearranged superpartitions do not appear in the actual expansion of the Jack superpolynomials. For instance, no monomial superfunction associated to a superpartition with a 0 on the antisymmetric side appears in the expansion of a Jack superpolynomial whose superpartition does not contain any 0 to the left of the semicolon. This information is not however encoded in eq. (113). A partial ordering formulated directly among superpartitions would lead to a more precise formulation of the monomial expansion of the Jack superpolynomials. Such

9 The reader is referred to [55] for a proof that this definition does in fact characterize a family of polynomials that forms a basis of $\hat{\Sigma}_N^{(m,n)}$.
10 Therefore, the most general eigenfunction having energy $\varepsilon_\Lambda$ is a linear combination of all the eigenfunctions $\mathcal{J}_{\lambda}^{(m)}$ whose eigenvalue is $\varepsilon_\lambda = \varepsilon_\Lambda$:

$$\mathcal{J}_{\lambda}(z, \theta; 1/\beta) = \sum_{m=0}^N \sum_{\lambda} \tau_{\lambda}^{(m)} \mathcal{J}_{\lambda}^{(m)}(z, \theta; 1/\beta) = \mathcal{J}_{\lambda}(z; 1/\beta) + \sum_{\lambda} \theta^0 \mathcal{J}_{\lambda}^{(1)}(z; 1/\beta) + \tau_1^1 \mathcal{J}_{\lambda}^{(2)}(z; 1/\beta) + \cdots + \tau_N^{1\cdots N} \mathcal{J}_{\lambda}^{(N)}(z; 1/\beta),$$

(116)

where an ordered summation on repeated indices is understood:

$$\tau_{\lambda}^{(m)} \mathcal{J}_{\lambda}^{(m)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} \theta_{i_1} \cdots \theta_{i_m} \mathcal{J}_{\lambda}^{(i_1 \cdots i_m)}.$$
Table 4: The Jack superpolynomials of weight $|\Lambda| \leq 3$

| Weight $|\Lambda|$ | Partition $\lambda$ | Sector $m$ | Superpartition $\Lambda$ | Jack superpolynomial $\mathcal{J}_A^{(m)}(z, \theta; 1/\beta)$ |
|------------------|-----------------|-----------|----------------|----------------------------------|
| 0                | (0)             | 1         | (0; 0)         | $m_{(0;0)}$                       |
| 1                | (1)             | 1         | (1; 0)         | $m_{(1;0)}$                       |
|                   |                 | 2         | (0; 1)         | $m_{(0;1)}$                       |
|                   |                 |           | (1, 0; 0)      | $m_{(1,0;0)}$                     |
| 2                | (1^2)           | 1         | (1; 1)         | $m_{(1;1)}$                       |
|                   |                 |           | (0; 1^2)       | $m_{(0;1^2)}$                     |
|                   |                 |           | (1, 0; 1)      | $m_{(1,0;1)}$                     |
| 2                |                 | 2         | (2; 0)         | $m_{(2;0)} + \frac{\beta}{1+\beta} m_{(1;1)}$ |
|                   |                 |           | (0; 2)         | $m_{(0;2)} + \frac{\beta}{1+\beta} m_{(0;1^2)} + \frac{2\beta}{1+\beta} m_{(0;1^2)}$ |
| 3                | (1^3)           | 1         | (1; 1^2)       | $m_{(1;1^2)}$                     |
|                   |                 |           | (0; 1^3)       | $m_{(0;1^3)}$                     |
|                   |                 |           | (1, 0; 1^2)    | $m_{(1,0;1^2)}$                   |
| 2                |                 | 2         | (2; 1)         | $m_{(2;1)} + \frac{2\beta}{2+\beta} m_{(1;1^2)}$ |
|                   |                 |           | (1; 2)         | $m_{(1;2)} + \frac{2\beta}{2+\beta} m_{(1;1^2)}$ |
|                   |                 |           | (0; 2; 1)      | $m_{(0;2;1)} + \frac{2\beta}{2+2\beta} m_{(0;1^2)} + \frac{6\beta}{1+2\beta} m_{(0;1^2)}$ |
|                   |                 |           | (2; 1; 0)      | $m_{(2;1;0)} + \frac{2\beta}{2+2\beta} m_{(1;0;1^2)}$ |
|                   |                 |           | (2; 0; 1)      | $m_{(2;0;1)} + \frac{2\beta}{2+2\beta} m_{(1;0;1^2)}$ |
|                   |                 |           | (1; 0; 2)      | $m_{(1;0;2)} + \frac{2\beta}{2+2\beta} m_{(1;0;1^2)}$ |
| 3                |                 | 1         | (3; 0)         | $m_{(3;0)} + \frac{2\beta}{2+\beta} m_{(2;1)} + \frac{\beta}{2+\beta} m_{(1;2)} + \frac{2\beta}{(1+\beta)(2+\beta)} m_{(1;1^2)}$ |
|                   |                 |           | (0; 3)         | $m_{(0;3)} + \frac{2\beta}{2+\beta} m_{(2;1)} + \frac{\beta}{2+\beta} m_{(1;2)} + \frac{3\beta}{(1+\beta)(2+\beta)} m_{(0;1^3)}$ |
|                   |                 |           | (3; 0; 0)      | $m_{(3;0;0)} + \frac{2\beta}{2+\beta} m_{(2;1;0)} + \frac{\beta}{2+\beta} m_{(2;0;1)} + \frac{2\beta}{2+\beta} m_{(1;0;2)} + \frac{3\beta}{(1+\beta)(2+\beta)} m_{(1;0;1^2)}$ |

an ordering has indeed been found \cite{55}. But because its formulation is somewhat technical, it will be presented elsewhere.

5 Conclusion

In this work, we have presented a number of results concerning the stCMS model: its reformulation in terms of the exchange-operator formalism, the Lax formalism, the Dunkl operators and an explicit construction for the conserved charges. In fact, $4N$ conserved charges have been constructed, $2N$ bosonic and $2N$ fermionic ones.

However, our most important results pertain to the construction of the stCMS eigenfunctions, with particular emphasis on the subclass which we call the Jack superpolynomials and which is a natural generalization of the Jack polynomials. In view of defining them properly, we have introduced the pivotal concept of superpartitions providing the natural labelling of the Jack superpolynomials. The Jack superpolynomials are then naturally defined by further imposing that they decompose in a specific manner in terms of monomial superfunctions, a procedure that extends the standard way of defining the Jack polynomials.
Table 5: Jack superpolynomials of weight $|\Lambda| = 4$

| $\lambda$ | Sector $m$ | Superpartition $\Lambda$ | $J_\Lambda^{(m)}(z, \theta; 1/\beta)$ |
|----------|------------|---------------------------|----------------------------------|
| (1$^4$)  | 1          | (1; 1$^3$)                | $m_{(1;1^3)}$                    |
|          |            | (0; 1$^4$)                | $m_{(0;1^4)}$                    |
|          |            | (1, 0; 1$^3$)             | $m_{(1,0;1^3)}$                  |
| (2, 1$^2$)| 1          | (2; 1$^2$)                | $m_{(2;1^2)} + \frac{2\beta}{1+\beta}m_{(1;1^3)} + \frac{\beta}{1+\beta}m_{(1;1^2)}$ |
|          |            | (1; 2, 1)                 | $m_{(1;2,1)} + \frac{2\beta}{1+\beta}m_{(1;1^2)}$ |
|          |            | (0; 2, 1$^2$)             | $m_{(0;2,1^2)} + \frac{1+3\beta}{1+\beta}m_{(1;1^3)} + \frac{12\beta}{1+\beta}m_{(0;1^4)}$ |
|          |            | (2, 1; 1)                 | $m_{(2,1;1)} + \frac{3\beta}{1+\beta}m_{(1;1^2)} + \frac{6\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (2; 0; 1$^2$)             | $m_{(2;0,1^2)} + \frac{2\beta}{1+\beta}m_{(2,1;1)}$ |
|          |            | (1, 0; 2, 1)              | $m_{(1,0;2,1)} + \frac{6\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (2, 1; 0$^2$)             | $m_{(2,1;0,1)}$                  |
| (2$^2$)  | 1          | (2; 2)                    | $m_{(2;2)} + \frac{2\beta}{1+\beta}m_{(2;1^2)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{6\beta}{1+\beta}m_{(1;1^3)}$ |
|          |            | (0; 2$^2$)                | $m_{(0;2^2)} + \frac{2\beta}{1+\beta}m_{(1;1^3)} + \frac{6\beta}{1+\beta}m_{(0;1^4)}$ |
|          |            | (2; 0; 2)                 | $m_{(2;0,2)} + \frac{2\beta}{1+\beta}m_{(2;1^2)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{6\beta}{1+\beta}m_{(1,0;1^3)}$ |
| (3$^1$)  | 1          | (3; 1)                    | $m_{(3;1)} + \frac{\beta}{1+\beta}m_{(2;2)} + \frac{\beta}{1+\beta}m_{(1;1^3)} + \frac{\beta}{1+\beta}m_{(1;1^2)}$ |
|          |            | (1; 3)                    | $m_{(1;3)} + \frac{\beta}{1+\beta}m_{(2;2)} + \frac{\beta}{1+\beta}m_{(1;1^3)} + \frac{\beta}{1+\beta}m_{(1;1^2)}$ |
|          |            | (0; 3, 1)                 | $m_{(0;3,1)} + \frac{2\beta}{1+\beta}m_{(0;2,2)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (3; 1, 0$^2$)             | $m_{(3;1,0^2)} + \frac{\beta}{1+\beta}m_{(2,1;1)}$ |
| (3$^1$)  | 2          | (3; 0, 1$^2$)             | $m_{(3;0,1^2)} + \frac{\beta}{1+\beta}m_{(2,1;1)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (1, 0$^3$)                | $m_{(1,0^3)} + \frac{\beta}{1+\beta}m_{(2,0,2)} + \frac{\beta}{1+\beta}m_{(2,1;1)} + \frac{\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (3, 1, 0$^2$)             | $m_{(3,1,0^2)} + \frac{\beta}{1+\beta}m_{(2,1;1)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{\beta}{1+\beta}m_{(1,0;1^3)}$ |
| (4$^1$)  | 1          | (4; 0$^2$)                | $m_{(4;0^2)} + \frac{3\beta}{1+\beta}m_{(3;1)} + \frac{\beta}{1+\beta}m_{(1;1^3)} + \frac{\beta}{1+\beta}m_{(2;1^2)}$ |
|          |            | (0; 0$^4$)                | $m_{(0;0^4)} + \frac{\beta}{1+\beta}m_{(3;1)} + \frac{3\beta}{1+\beta}m_{(1;1^3)} + \frac{6\beta}{1+\beta}m_{(0;1^4)}$ |
|          |            | (0; 2$^3$)                | $m_{(0;2^3)} + \frac{1+3\beta}{1+\beta}m_{(1;1^3)} + \frac{12\beta}{1+\beta}m_{(0;1^4)}$ |
|          |            | (2; 0$^3$)                | $m_{(2;0^3)} + \frac{2\beta}{1+\beta}m_{(2,1;1)} + \frac{3\beta}{1+\beta}m_{(1;1^2)} + \frac{6\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (2; 1$^2$)                | $m_{(2;1^2)} + \frac{3\beta}{1+\beta}m_{(1;1^3)} + \frac{6\beta}{1+\beta}m_{(0;1^4)}$ |
|          |            | (2; 0; 1$^2$)             | $m_{(2;0,1^2)} + \frac{2\beta}{1+\beta}m_{(2,1;1)}$ |
|          |            | (1; 0; 2$^2$)             | $m_{(1;0;2^2)} + \frac{\beta}{1+\beta}m_{(1;1^2)} + \frac{\beta}{1+\beta}m_{(1,0;1^3)}$ |
|          |            | (2; 1; 0$^2$)             | $m_{(2;1;0,1)} + \frac{3\beta}{1+\beta}m_{(1;1^3)} + \frac{6\beta}{1+\beta}m_{(0;1^4)}$ |
CONCLUSION

In a forthcoming publication, we will present a dominance ordering on the superpartitions that suggests an exact expression for the coefficients $c_{\Omega,\Lambda}$ in and a related determinantal formula. These results can in turn be used to demonstrate the actual existence of the Jack superpolynomials.

Numerous extensions of this work can be contemplated. The most immediate one concerns the generalization from 2 to an arbitrary even number $2M$ of supersymmetries. This is rather straightforward in the exchange-operator formalism: it suffices to set

$$K_{ij} \equiv \kappa_{ij}K_{ij},$$
$$\kappa_{ij} \equiv \kappa^1_{ij} \cdots \kappa^M_{ij},$$

where $\kappa^a_{ij}$ is the operator that exchanges the Grassmannian variables $\theta^a$ and $\theta^a\dagger$, where $a = 1, \cdots, M$.

The $2M$ generators of the supersymmetric transformations are then:

$$Q^a = \Pi_K(\sum_i \theta^a_i D_i),$$
$$Q^a\dagger = \Pi_K(\sum_i \theta^{a\dagger}_i D_i).$$

The construction of the conserved quantities is analogous to the one we have presented in the case with $2M = 2$ supersymmetries. The construction of the eigenfunctions is also rather direct: the Jack superpolynomials are then indexed by $M$ fermionic sectors and a superpartition involving $M$ antisymmetric partitions:

$$\mathcal{J}_{\Lambda}^{(m)} \rightarrow \mathcal{J}_{\Lambda}^{(m_1, \cdots, m_M)},$$
$$\rightarrow \mathcal{J}(\Lambda_{1, \cdots, \Lambda_{m_1}; \Lambda_{m_1+1, \cdots, \Lambda_{m_1+m_2}; \cdots; \Lambda_{m_1+\cdots+m_M+1, \cdots, \Lambda_L}}).$$

Another simple way of extending the model is by adding spin degrees of freedom. Again, this is very simple in the framework of the exchange-operator formalism, where we only need to add an extra piece in the total exchange operator, i.e., set:

$$K_{ij} = \kappa_{ij}K_{ij}\sigma_{ij}.$$
A ANTISYMMETRIC EIGENFUNCTIONS

We also expect that there exists creation operators that provide the super analogues of the operators constructed in \cite{???}. A first trial in that direction has been presented in app. B. This indeed maps simple stCMS eigenfunctions – in fact, Jack polynomials, hence Jack superpolynomials specialized to the zero-fermion sector – to other stCMS eigenfunctions. However, as we have already pointed out, the action of these creation operators does not close within the set of Jack superpolynomials labelled by superpartitions.

Two other issues, both rooted in our initial motivations for studying sCMS models, are worth mentioning. Given the rather intriguing relation that exists between the Virasoro singular vectors and the Jack polynomials, one can naturally ask whether there is a similar relation at the supersymmetric level. But this would presumably require the definition of a proper super-Jack polynomial (i.e., an appropriate sum over the different fermionic sectors of the Jack superpolynomials). Note that the potentially related singular vectors would be those of the \( \mathcal{N} = 2 \) superconformal algebra.

On the other hand, we have recalled in the introduction the remarkable connection relating the dynamics of the rational CMS models and the time evolution of the rational solutions of the Korteweg-de Vries equation. A quite natural quest would be to try to find a supersymmetric counterpart to this phenomenon.

We expect to report elsewhere on some of these other issues.

A Antisymmetric eigenfunctions

In this appendix, we construct excited states related to the second ground state \( \tilde{\psi}_0 = \Delta^{-\beta} |\tilde{0}\rangle \). The differential representation \( |\tilde{0}\rangle \rightarrow \theta_1 \cdots \theta_N \) makes clear the complete antisymmetry of the fermionic ground state \( |\tilde{0}\rangle \). Thus, the exited states that behave like \( \tilde{\psi}_0 \) under the exchange of bosonic and fermionic variables take the following form:

\[
\tilde{\psi}(z, \theta) = \phi(z, \theta^t)\tilde{\psi}_0 = \phi(z, \partial/\partial \theta)\Delta^{-\beta}\theta_1 \cdots \theta_N \quad \text{where} \quad K_{ij}\phi = \phi K_{ij}, \tag{124}
\]

Equivalently, we can define \( \tilde{\psi}_\Lambda \) as:

\[
\tilde{\psi}(z, \theta) = \tilde{\phi}(z, \theta)\Delta^{-\beta} \quad \text{where} \quad K_{ij}\tilde{\phi} = -\tilde{\phi} K_{ij}. \tag{125}
\]

The antisymmetric states \( \tilde{\phi} \) are eigenfunctions of the supersymmetric hamiltonian:

\[
\tilde{\mathcal{H}} = \frac{1}{2} \left( \frac{\lambda^2}{\pi} \right)^2 \Delta^2 \mathcal{H} \Delta^{-\beta},
\]

\[
= \sum_i (z_i \partial_i)^2 - \beta \sum_{i<j} \frac{z_i + z_j}{\pi_{ij}}(z_i \partial_i - z_j \partial_j) + 2\beta \sum_{i<j} \frac{z_i z_j}{\pi_{ij}}(1 + \kappa_{ij}),
\]

\[
= \mathcal{H}(\beta \rightarrow -\beta, \kappa_{ij} \rightarrow -\kappa_{ij}). \tag{126}
\]

Since \( \kappa_{ij}\tilde{\phi} = -K_{ij}\tilde{\phi} \), the superfunctions are also eigenfunctions of the (modified) hamiltonian with an exchange term:

\[
\mathcal{H}\tilde{\phi} = \mathcal{H}_{k}\tilde{\phi}, \tag{127}
\]

where the prime symbol indicates that the sign of the coupling constant \( \beta \) is reversed: \( f'(\beta) = f(-\beta) \).

Like the symmetric superfunctions, the antisymmetric superfunctions \( \tilde{\phi} \) of degree \((m; n)\) can be indexed by a superpartition \( \Lambda \) and denoted \( \tilde{J}_\Lambda^{(m)} \). These solutions will be called the antisymmetric Jack superpolynomials. They can be defined in terms of a triangular decomposition in antisymmetric monomial superfunctions:

\[
\tilde{J}_\Lambda^{(m)}(z, \theta; 1/\beta) = a_\Lambda^{(m)}(z, \theta) + \sum_{\omega \subset \Lambda} c_{\Lambda,\Omega}(\beta)a_\Omega^{(m)}(z, \theta), \tag{128}
\]

where

\[
a_\Lambda^{(m)} = \frac{a_{(\lambda'; \lambda'')}}{a_{(\lambda; \lambda)'}} = \sum_{i_1 < i_2 < \cdots < i_m} \gamma^{i_1, \ldots, i_m} m_{\lambda'}(z_{i_1}, \ldots, z_{i_m})a_{\lambda'}(z/\{z_{i_1}, \ldots, z_{i_m}\}) \tag{129}
\]
where $a_\lambda$ is the antisymmetric monomial function defined in \eqref{eq:antisymmetric_monomial}. The eigenvalues are given by:

$$
\varepsilon'_\lambda = \varepsilon'_{-\lambda} = \sum_j [\lambda_j^2 - \beta(N + 1 - 2j)\lambda_j].
$$

\section*{B Eigenfunction generators}

It is possible to generate eigenfunctions of $\tilde{H}$ by applying some differential operators directly on Jack polynomials. Let us consider the operator $B^{(m)}_\gamma$, indexed by a positive integer $m$ and a partition $\gamma$, given by:

$$
B^{(m)}_\gamma = \sum_{p \in S_N} \theta^{p(1, \ldots, m)} \bar{D}^{\gamma}_{p(1, \ldots, N)}.
$$

In this formula, $\bar{D}_i$ is the Dunkl operator defined in \eqref{eq:Dunkl_operator}. The partition $\gamma$ is used to identify the independent polynomials in $\bar{D}_i$. The $B^{(m)}_\gamma$'s commute with the hamiltonian with an exchange term and are invariant under the action of the exchange operators $K_{ij}$:

$$
[B^{(m)}_\gamma, H_K] = 0 = [B^{(m)}_\gamma, K_{ij}].
$$

These properties ensure that the application of $B^{(m)}_\gamma$ on the Jack polynomial $J_\lambda(z; \theta)$ gives a solution to the supersymmetric model, the solution being indexed by two partitions $\gamma$ and $\lambda$:

$$
\begin{align*}
A^{(m)}_{\gamma, \lambda}(z; \theta; 1/\beta) &\equiv B^{(m)}_\gamma J_\lambda(z; 1/\beta), \\
\mathcal{H}_\kappa A^{(m)}_{\gamma, \lambda}(z, \theta; 1/\beta) &\equiv \mathcal{H}^{(m)}_\kappa J_\lambda(z; 1/\beta).
\end{align*}
$$

In the previous equation, we have used the equivalence between the supersymmetric hamiltonian $\mathcal{H} \equiv \mathcal{H}_\kappa$ and the hamiltonian with an exchange term $\mathcal{H}_K$ when they act on a function which is invariant under the operators $K_{ij}$. The solutions $A^{(m)}_{\gamma, \lambda}$ are of degree $(m, |\lambda|)$ in $z$. We can easily check that these symmetric superfunctions are non-zero only if the partitions $\gamma$ are strictly decreasing with respect to their $m$ first parts.

The operators $B^{(m)}_\gamma$ can thus generate any $\mathcal{H}_\kappa$ eigenfunctions. They play the double role of fermionic creation operators and of symmetrizer in $\theta$ and in $z$. Moreover, they generalize the supersymmetric charges introduced in section \ref{sec:symmetrizer}. For example:

$$
\Pi_\kappa (B^{(1)}_{(n)}) = \bar{Q}_{(n)}.
$$

Given the infinite number of partitions $\gamma$, there exists an infinite number of such operators, even in a specific fermionic sector. Obviously, since the number of Jack superpolynomials is finite for a given degree $m$, most of the solutions $A^{(m)}_{\gamma, \lambda}$ are linearly dependent. There is however no precise relation between a given eigenfunction $A^{(m)}_{\gamma, \lambda}$ and the Jack superpolynomials defined in \eqref{eq:jack_superpolynomials} and indexed by a superpartition.

\section*{C Combinatorics of superpartitions}

We now determine the number $p(m; n)$ of independent states of degree $(m; n)$. This amounts to counting the number of superpartitions of degree $(m; n)$. As we will show, this number is simply given by the following sum:

$$
p(m; n) = \sum_{n_1 + n_2 = n} p_D^{(m)}(n_1)p(n_2),
$$

where $p_D^{(m)}(n_1)$ is the antisymmetric monomial function defined in \eqref{eq:antisymmetric_monomial}. The eigenvalues are given by:

$$
\varepsilon'_\lambda = \varepsilon'_{-\lambda} = \sum_j [\lambda_j^2 - \beta(N + 1 - 2j)\lambda_j].
$$
where \( n_a = |\lambda^a| \) and \( n_s = |\lambda^s| \). As usual \( p(n) \) is the number of partitions of \( n \) so that \( p(n_a) \) counts the number of partitions of \( \lambda^a \). The quantity \( p^{(m)}_0(n_a) \) is a restricted partition whose ‘restriction symbols’ agree with the usual definitions of combinatorial analysis:

\[
p^{(m)}(n) = \text{number of partitions of } n \text{ of length less or equal to } m, \\
p^{(m)}_0(n) = \text{number of partitions of } n \text{ whose parts are all distinct}.
\]

For instance, \( p^{(2)}(4) = 3 \) since there are 3 partitions of 4 with at most 2 parts: \( (4) \), \( (3,1) \) and \( (2,2) \). Hence, \( p^{(m)}_0(n) \) gives the number of partitions (without zero) of \( n \) whose parts are strictly decreasing and whose length is smaller or equal to \( m \). This quantity counts the number of antisymmetric partitions \( \lambda^a \). The summation takes into account the various ways of splitting \( n \) into the two integers \( n_a \) and \( n_s \). Note that, since the superpartitions of the form \((; \lambda^s) = (\lambda^s)\) are acceptable, we have to adopt the following conventions:

\[
p^{(0)}(0) = 1 \quad \text{and} \quad p^{(0)}(n \geq 1) = 0.
\]

For example, we find 5 superpartitions of weight 5 in the 2-fermion sector:

\[
(3,0;0), \quad (2,1;0), \quad (2,0;1), \quad (1,0;2), \quad (1,0;1,1)
\]

This agrees with eq. (135):

\[
p(2;3) = \sum_{n_a+n_s=3} p^{(2)}_0(n_a)p(n_s),
\]

\[
= p^{(2)}_0(3)p(0) + p^{(2)}_0(2)p(1) + p^{(2)}_0(1)p(2) + p^{(2)}_0(0)p(3),
\]

\[
= 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 5.
\]

If we compare the number of superpartitions of degree \( (m;n) \) for \( n \) fixed but for different values of \( m \), we see that this number is maximum in the one-fermion sector and minimum in the \( N \)-fermion sector (since \( N \) corresponds to the maximal length of the superpartition). Since the antisymmetric partitions are strictly decreasing, we have:

\[
p(N;n) = 0 \quad \text{if} \quad n \leq \frac{(N-1)(N-2)}{2}.
\]

For the superpolynomials of degree 3 in \( z \) and in \( \theta \), for instance, only one state exists because there is only one possible superpartition, \( (2,1,0) \). This is less than the \( p(3) = 3 \) possible partitions of 3. On the other hand, if the fermionic sector is \( m = 1 \), we have \( p(1;3) \) independent states, each associated to one of the 7 possible superpartitions:

\[
(3;0), \quad (0;3), \quad (2;1), \quad (1;2), \quad (0;2,1), \quad (1;1,1), \quad (0;1,1,1).
\]

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References

[1] F. Calogero, *Ground State Of One-Dimensional N Body System*, J. Math. Phys. 10, 2197 (1969); *Solution Of A Three Body Problem In One-Dimension*, J. Math. Phys. 10, 2191 (1969); *Solution Of The One-Dimensional N Body Problems With Quadratic And/Or Inversely Quadratic Pair Potentials*, J. Math. Phys. 12, 419 (1971).
REFERENCES

[2] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16**, 197 (1975).

[3] B. Sutherland, *Quantum Many Body Problem In One-Dimension: Ground State*, J. Math. Phys. **12**, 246 (1971); *Exact Results For A Quantum Many Body Problem In One-Dimension*, Phys. Rev. **A4**, 209 (1971); *Exact Results For A Quantum Many Body Problem In One-Dimension. II*, Phys. Rev. **A5**, 1372 (1972).

[4] M. A. Olshanetsky and A. M. Perelomov, *Classical Integrable Finite Dimensional Systems Related To Lie Algebras*, Phys. Rept. **71**, 313 (1981); *Quantum Integrable Systems Related To Lie Algebras*, Phys. Rept. **94**, (1983) 313.

[5] H. Airault, H.P. McKean and J. Moser, *Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem*, Comm. Pure Appl. Math., **30**, 1, 95(1977).

[6] M. V. N. Murthy, R. Shankar, *Thermodynamics of a One-Dimensional Ideal Gas with Fractional Exclusion Statistics*, Phys. Rev. Lett. **73**, 3331 (1994).

[7] D. Bernard, *Some simple (integrable) models of fractional statistics*, hep-th/9411017.

[8] A. P. Polychronakos, *Generalized statistics in one dimension*, hep-th/9902157.

[9] R. E. Prange and S. M. Girvin, *The quantum Hall effect*, 2nd ed. Springer Verlag, 1990.

[10] H. Azuma and S. Iso, *Explicit relation of the quantum Hall effect and the Calogero-Sutherland model*, Phys. Lett. **B331**, 107 (1994).

[11] E. Fradkin, *Field theory of condensed matter systems*, Addison-Wesley, 1991.

[12] S. Ouvry, *On the relation between anyon and Calogero Models*, cond-mat/9907239.

[13] F. D. Haldane, ‘Fractional statistics’ in arbitrary dimensions: A Generalization of the Pauli principle, Phys. Rev. Lett. **67**, 937 (1991).

[14] F. D. Haldane, *Exact Jastrow-Gutzwiller Resonating Valence Bond Ground State Of The Spin 1/2 Antiferromagnetic Heisenberg Chain With 1/R² Exchange*, Phys. Rev. Lett. **60**, 635 (1988).

[15] B. Sriram Shastry, *Exact Solution Of An S = 1/2 Heisenberg Antiferromagnetic Chain With Long Ranged Interactions*, Phys. Rev. Lett. **60**, 639 (1988).

[16] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard and V. Pasquier, *Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory*, Phys. Rev. Lett. **69**, 2021 (1992).

[17] A. P. Polychronakos, *Exchange operator formalism for integrable systems of particles*, Phys. Rev. Lett. **69**, 703 (1992); J. A. Minahan and A. P. Polychronakos, *Integrable systems for particles with internal degrees of freedom*, Phys. Lett. **B302** 299 (1993), hep-th/9206046; hep-th/9202057.

[18] H. Frahm, *Spectrum of a spin chain with inverse square exchange*, J. Phys. **A26**, L473 (1993).

[19] A. P. Polychronakos, *Exact spectrum of SU(n) spin chain with inverse square exchange*, Nucl. Phys. **B419**, 553 (1994) hep-th/9310095.

[20] P. Fendley, F. Lesage and H. Saleur, *Solving the 1D log gas using Jack polynomials and functional relations*, J. Stat. Phys. **79**, 799 (1995) hep-th/9405008.

[21] B. D. Simons, P. A. Lee and B. L. Altshuler, *Matrix Models, One-Dimensional Fermions, and Quantum Chaos*, Phys. Rev. Lett., **72**, 64 (1994).

[22] M. Caselle, *Distribution of Transmission Eigenvalues in Disodored Wires*, Phys. Rev. Lett. **74**, 2776 (1995).
REFERENCES

[23] G. W. Gibbons and P. K. Townsend, Black holes and Calogero models, Phys. Lett. B 454, 187 (1999).

[24] E. D’Hoker and D. H. Phong, Seiberg-Witten theory and Calogero-Moser systems, Prog. Theor. Phys. Suppl. 135, 75 (1999) [hep-th/9906027]

[25] C.F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc., 311, 1 (1989).

[26] H. Jack, A class of symmetric polynomials with a parameter, Proc. Roy. Soc. Edinburgh Sect. A, 69, 1–18 (1970/1971); A surface integral and symmetric functions, Proc. Roy. Soc. Edinburgh Sect. A, 69, part 4, 347–364 (1972).

[27] I. Cherednik, Double affine Hecke algebras and Macdonald’s conjectures, Ann. Math. 141, 191 (1995); I. G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Séminaire Bourbaki vol 1994/95, Astérisque 247, 189 (1996).

[28] S. Fomin and A. N. Kirillov, Quadratic algebras, Dunkl elements and Schubert calculus, Prog. Math. 172, 147 (1999).

[29] M. Flower, J. A. Minahan, Invariants of the Haldane-Shastry SU(N) chain, Phys. Rev. Lett. 70, 2325 (1993) [cond-mat/9208016]

[30] D. Bernard, M. Gaudin, F. D. Haldane and V. Pasquier, Yang-Baxter equation in long range interacting system, J. Phys. A A26, 5219 (1993) [hep-th/9301084]

[31] R. P. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math., 77, 76-115 (1988).

[32] P. J. Forrester, Selberg correlation integrals and the $1/r^2$ quantum many-body system, Nucl. Phys. B388, 671 (1992).

[33] L. Lapointe and L. Vinet, A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture, Internat. Math. Res. Notices, 1995, 9, 419–424 (1995); Exact operator solution of the Calogero-Sutherland model, Comm. Math. Phys., 178, 2, 425–452 (1996).

[34] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, A Note on Calogero-Sutherland model, W(n) singular vectors and generalized matrix models, [hep-th/9503028]. Excited states of Calogero-Sutherland model and singular vectors of the W(N) algebra, Nucl. Phys. B449, 347 (1995) [hep-th/9503043]

[35] K. Hikami and M. Wadati, Infinite symmetry of the spin systems with inverse square interactions, J. Phys. Soc. Jap., 62, 12, 4203–4217 (1993); Integrable systems with long range interactions, $w$-infinity algebra and energy spectrum, Phys. Rev. Lett. 73, 1191 (1994).

[36] F. Knop and S. Sahi, A recursion and a combinatorial formula for the Jack polynomials, Invent. Math. 128 9 (1997).

[37] L. Lapointe and L. Lascoux and J. Morse, Determinantal formula and recursion for Jack polynomials, Electro. J. Comb. 7 467 (2000).

[38] H. Ujino and M. Wadati, Rodrigues formula for the hi-Jack symmetric polynomials associated with the quantum Calogero model, J. Phys. Soc. Japna 65 2423 (1996).

[39] I. G. Macdonald, Symmetric functions and Hall polynomials, 2ieme edition, The Clarendon Press Oxford University Press, (1995); Symmetric functions and orthogonal polynomials, Dean Jacqueline B. Lewis Memorial Lectures presented at Rutgers University, American Mathematical Society, (1998).
REFERENCES

[40] L. Lapointe and L. Vinet, *Rodrigues formula for the Macdonald polynomials*, Adv. Math. **130**, 261 (1997).

[41] S. N. Ruijsenaars and H. Schneider, *A New Class of Integrable Systems and Its Relation to Solitons*, Ann. Phys. **170**, 370 (1986); S. N. Ruijsenaars, *Complete Integrability of Relativistic Calogero-Moser Systems and Elliptic Function Identities*, Commun. Math. Phys. **110**, 191 (1987).

[42] H. Awata, *Hidden algebraic structure of the Calogero-Sutherland model, integral formula for Jack polynomial and their relativistic analog*, in *Calogero-Moser-Sutherland Models*, ed. by J. F. van Diejen and L. Vinet, Springer (2000), 23.

[43] D. Z. Freedman and P. F. Mende, *An Exactly Solvable N Particle System In Supersymmetric Quantum Mechanics*, Nucl. Phys. **B344**, 317 (1990).

[44] E. Witten, *Dynamical Breaking Of Supersymmetry*, Nucl. Phys. **B188**, 513 (1981); *Constraints On Supersymmetry Breaking*, Nucl. Phys. **B202**, 253 (1982).

[45] B. Sriram Shastry and B. Sutherland, *Superlax pairs and infinite symmetries in the 1/r^2 system*, Phys. Rev. Lett. **70**, 4029 (1993) cond-mat/9212029.

[46] L. Brink, T. H. Hansson, S. Konstein and M. A. Vasiliev, *The Calogero model: Anyonic representation, fermionic extension and supersymmetry*, Nucl. Phys. **B401**, 591 (1993) hep-th/9302023.

[47] L. Brink, A. Turbiner and N. Wyllard, *Hidden Algebras of the (super) Calogero and Sutherland models*, J. Math. Phys. **39**, 1285 (1998) hep-th/9705219.

[48] N. Wyllard, *Superconformal many-body quantum mechanics with extended supersymmetry*, J. Math. Phys. **41**, 2826 (2000), hep-th/9910160.

[49] A. J. Bordner, N. S. Manton and R. Sasaki, *Calogero-Moser models. V: Supersymmetry and quantum lax pair*, Prog. Theor. Phys. **103**, 463 (2000) hep-th/9910033.

[50] P. K. Ghosh, *Super-Calogero-Moser-Sutherland systems and free super-oscillators: A mapping*, hep-th/0007208.

[51] P. Dargis and P. Mathieu, *Nonlocal conservation laws for supersymmetric KdV equation*, Phys. Lett. **A176**, 67 (1993) hep-th/9301080.

[52] H. Nicolai, *Supersymmetry And Spin Systems*, J. Phys. **A9**, 1497 (1976); *Extensions Of Supersymmetric Spin Systems*, J. Phys. **AA10**, 2143 (1977).

[53] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, Phys. Rept. **251**, 267 (1995) hep-th/9405029.

[54] P. Mathieu and Y. Xudous, *Conserved charges of non-yangian type for the Frahm-Polychronakos spin chain*, hep-th/0008036.

[55] P. Desrosiers, L. Lapointe and P. Mathieu, *Jack superpolynomials, superpartition ordering and determinantal formulas*, hep-th/0105107.