Shilnikov Lemma for a nondegenerate critical manifold of a Hamiltonian system.

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Abstract

We prove an analog of Shilnikov Lemma for a normally hyperbolic symplectic critical manifold \( M \subset H^{-1}(0) \) of a Hamiltonian system. Using this result, trajectories with small energy \( H = \mu > 0 \) shadowing chains of homoclinic orbits to \( M \) are represented as extremals of a discrete variational problem, and their existence is proved. This paper is motivated by applications to the Poincaré second species solutions of the 3 body problem with 2 masses small of order \( \mu \). As \( \mu \to 0 \), double collisions of small bodies correspond to a symplectic critical manifold of the regularized Hamiltonian system.

1 Introduction

Consider a smooth Hamiltonian system \((\mathcal{M}, \omega, H)\) with phase space \(\mathcal{M}\), symplectic form \(\omega\) and Hamiltonian \(H\). Let \(\mathbf{v} = \mathbf{v}_H\) be the Hamiltonian vector field: \(\omega(\mathbf{v}(x), \cdot) = -dH(x)\), and \(\phi^t = \phi^t_H\) the flow of the system. Suppose that \(H\) has a nondegenerate normally hyperbolic symplectic critical 2\(m\)-dimensional manifold \(M \subset \Sigma_0 = H^{-1}(0)\) with real eigenvalues. Thus for any \(z \in M\):

- \(\text{rank } d^2H(z) = 2k = \dim \mathcal{M} - 2m\);
- the restriction \(\omega|_{T_z \mathcal{M}}\) is nondegenerate;
- the eigenvalues of the linearization of \(\mathbf{v}\) at \(z\) are all real.

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Let $D\phi^t(z) = e^{tA(z)}$ be the linearized flow. Denote by

$$E_z = T_z^\perp M = \{\xi \in T_z M : \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_z M\}$$

the symplectic complement to $T_z M$. Since $M$ is symplectic, $T_z M = T_z M \oplus E_z$ and $\omega|_{E_z}$ is nondegenerate. Hence $E_z = E_z^+ \oplus E_z^-$, where $E_z^\pm$ are $k$-dimensional $A(z)$-invariant Lagrangian subspaces of $E_z$ corresponding to negative and positive eigenvalues respectively. We write $\xi \in E_z$ as $\xi = (\xi_+, \xi_-)$, where $\xi_+ \in E_z^+$ and $\xi_- \in E_z^-$. Then the linearized flow on $E_z$ is

$$D\phi^t(\xi_+, \xi_-) = (e^{-tA_+(z)}\xi_+, e^{tA_-(z)}\xi_-), \quad (1.1)$$

where the eigenvalues of $A_{\pm}(z) = \mp A(z)|_{E_z^\pm}$ are positive. Thus $E_z^+$ is the stable subspace, and $E_z^-$ the unstable subspace. The quadratic part of the Hamiltonian is

$$\frac{1}{2} \partial^2 H(z)(\xi) = -\omega(\xi_-, A_+(z)\xi_+) = -\omega(A_-(z)\xi_-, \xi_+). \quad (1.2)$$

The stable and unstable manifolds

$$W^\pm(z) = \{x \in M : \lim_{t \to \pm\infty} \phi^t(x) = z\}$$

of an equilibrium $z \in M$ have dimension $k$ and $T_z W^\pm(z) = E_z^\pm$. The stable and unstable manifolds

$$W^\pm = W^\pm(M) = \bigcup_{z \in M} W^\pm(z)$$

of $M$ have dimension $k + 2m$ and $T_z W^\pm = T_z M \oplus E_z^\pm$ for any $z \in M$. It is well known (see e.g. [12]) that $W^\pm(z)$ are isotropic: $\omega|_{W^\pm(z)} = 0$, and $W^\pm$ are coisotropic: for any $a \in W^\pm(z)$, we have $T_a W^\pm = T_a W^\pm(z)$. Thus $W^\pm(z)$ form a smooth isotropic foliation of $W^\pm$. Define projections $\pi^\pm : W^\pm \to M$ by $\pi^\pm(x) = z$ if $x \in W^\pm(z)$:

$$\pi^\pm(x) = \lim_{t \to \pm\infty} \phi^t(x).$$

Since $M \subset \Sigma_0 = H^{-1}(0)$, we have $W^\pm \subset \Sigma_0$. The intersection $\Gamma = (W^+ \cap W^-) \setminus M$ consists of orbits $\gamma : \mathbb{R} \to M$ homoclinic to $M$, i.e. heteroclinic from $z_- = \gamma(-\infty) \in M$ to $z_+ = \gamma(+\infty) \in M$. Define a scattering map $\mathcal{F} : \pi^-(\Gamma) \to \pi^+(\Gamma)$ setting $\mathcal{F}(z_-) = z_+$ if there is an orbit heteroclinic from $z_-$ to $z_+$, i.e. $W^-(z_-) \cap W^+(z_+) \neq \emptyset$.

**Remark 1.1.** Following [13], we call $\mathcal{F}$ the scattering map. However, our case is different from [13] because the manifold $M$ is critical. In particular, there is no straightforward cross section for the flow near $M$. The scattering map is also called the homoclinic map. In the applications to Celestial Mechanics [14, 15], we call $\mathcal{F}$ the collision map.\footnote{In what follows + corresponds to the stable manifold ($t \to +\infty$), and - to the unstable manifold ($t \to -\infty$).}
In general $\mathcal{F}$ is multivalued. To define a single valued smooth map, we need to consider local branches of $\mathcal{F}$. We call a heteroclinic orbit $\gamma(t) = \phi^t(a)$, $\gamma(\pm \infty) = c_\pm \in M$, transverse if the following conditions hold.

**Proposition 1.1.** The following conditions are equivalent:

1. $T_a W^-(c_-) \cap T_a W^+ = \mathbb{R}v(a)$,
2. $T_a W^+(c_+) \cap T_a W^- = \mathbb{R}v(a)$,
3. The symplectic form $\omega$ defines a nondegenerate modulo $\mathbb{R}v(a)$ bilinear form on $T_a W^-(c_-) \times T_a W^+(c_+)$.
4. There exist Lagrangian submanifolds $\Lambda^\pm \subset M$ containing $c_\pm$ such that the Lagrangian manifolds $W^\pm(\Lambda^\pm) = \cup_{z \in \Lambda^\pm} W^\pm(z)$ intersect transversely in $\Sigma_0$ along $\gamma$:
   $$T_a W^+(\Lambda^+) \cap T_a W^-(\Lambda^-) = \mathbb{R}v(a).$$

These conditions imply that $a$ is a point of transverse intersection of $W^+$ and $W^-$, i.e. $T_a W^+ + T_a W^- = T_a \Sigma_0$. We skip an elementary proof of Proposition 1.1.

If $\gamma$ is transverse, then $\mathcal{F}$ has a well defined smooth branch $f : V^- \to V^+$, where $V^\pm \subset M$ is a small neighborhood of $c_\pm$. Indeed, let $N \subset W^+$ be a local section at $a$ such that $T_a N \oplus \mathbb{R}v(a) = T_a W^+$. There exists a neighborhood $V^- \subset M$ of $c_-$ such that for any $z_- \in V^-$, the manifolds $W^-(z_-)$ and $N$ intersect transversely in $\Sigma_0$ at a point $b$ close to $a$. Set $z_+ = f(z_-) = \pi_+(b)$. Then $\sigma(t) = \phi^t(b)$ is a heteroclinic orbit joining $z_-$ with $z_+$. The map $f : V^- \to M$ is symplectic.

Indeed, let $(x_\pm, y_\pm)$ be local symplectic coordinates in $V^\pm$ and $\alpha$ a 1-form in a neighborhood of $\gamma(\mathbb{R})$ such that $d\alpha = \omega$ and $\alpha|_{V^\pm} = y_\pm dx_\pm$. Then by the first variation formula [1]

$$f(x_-, y_-) = (x_+, y_+) \Rightarrow y_+ dx_+ - y_- dx_- = dG, \quad G(z_-) = \int_\sigma \alpha.$$

We can choose symplectic coordinates $(x_\pm, y_\pm)$ in $V^\pm$ so that

$$\Lambda^+ = \{y_+ = b_+\} = B(a_+) \times \{b_+\}, \quad \Lambda^- = \{x_- = a_-\} = \{a_-\} \times B(b_-),$$

where $c_\pm = (a_\pm, b_\pm)$ and $B$ is a small ball in $\mathbb{R}^m$. Then for $(x_-, y_+) \in B(a_-) \times B(b_+)$, Lagrangian manifolds $W^-(\{x_-\} \times B(b_-))$ and $W^+(B(a_+) \times \{y_+\})$ intersect transversely in $\Sigma_0$ along a heteroclinic trajectory $\sigma(x_-, y_+)$ joining the points $(x_-, y_-)$ with $(x_+, y_+)$. Decreasing the sets $V^\pm \subset M$ if necessary, we represent $f : V^- \to V^+$ by a generating function $S(x_-, y_+) = \langle y_+, x_+ \rangle - G$ [1]:

$$f(x_-, y_-) = (x_+, y_+) \iff dS(x_-, y_+) = y_+ dx_+ + x_+ dy_+.$$

Introducing a local branch $f$ near any transverse heteroclinic orbit, we represent the scattering map by a countable collection $\mathcal{F}$ of smooth symplectic
diffeomorphisms \( f : V^- \to V^+ \) of open sets in \( M \). In general \( \mathcal{F} \) has infinitely many branches. For example, this is so in our application to Celestial Mechanics [8]. In fact \( \mathcal{F} \) being multivalued helps in constructing symbolic dynamics, see e.g. [4].

An orbit of \( \mathcal{F} \) is a pair of sequences \( f_i : V^- \to V^+ \) and \( z_i \in V_i = V_i^- \cap V_i^+ \) such that \( z_{i+1} = f_i(z_i) \). It defines a chain \( \sigma = (\sigma_i) \) of transverse heteroclinic orbits \( \sigma_i \) connecting \( z_i \) with \( z_{i+1} \).

Remark 1.2. The scattering map may be viewed as a single map – the skew product of the maps \( f \in \mathcal{F} \) which is a (partly defined) map of \( \mathbb{F}^2 \times M \). This is needed to study chaotic dynamics of \( \mathcal{F} \).

Let \( c_{i+1} = f_i(c_i) \) be a periodic orbit: \( f_{i+n} = f_i \), \( c_{i+n} = c_i \). Then \( c_0 \) is a fixed point of the composition \( F_n = f_{n-1} \circ \cdots \circ f_0 \). The periodic orbit is called nondegenerate if \( z_0 \) is a nondegenerate fixed point:

\[
\det(DF_n(c_0) - I) \neq 0. \tag{1.4}
\]

Then the corresponding periodic heteroclinic chain \( \sigma = (\sigma_i) \) will be called nondegenerate.

Let \( z_i = (x_i, y_i) \) be symplectic coordinates in \( V_i \) such that \( f_i \) is represented by a generating function as in (1.3):

\[
f_i(x_i, y_i) = (x_{i+1}, y_{i+1}) \iff dS_i(x_i, y_{i+1}) = y_i dx_i + x_{i+1} dy_{i+1}. \tag{1.5}
\]

A periodic orbit of \( \mathcal{F} \) corresponds to a critical point \( c = (c_i)_{i=0}^{n-1} \) of the discrete action functional

\[
\mathcal{A}(z) = \sum_{i=0}^{n-1} (S_i(x_i, y_{i+1}) - \langle x_i, y_i \rangle), \quad y_n = y_0. \tag{1.6}
\]

It is well known (see [18]) that the periodic orbit is nondegenerate iff \( c \) is a nondegenerate critical point of \( \mathcal{A} \).

To shadow a nondegenerate heteroclinic chain \( \sigma \) by a trajectory of the Hamiltonian system on \( \Sigma_\mu = H^{-1}(\mu) \) with small \( \mu \neq 0 \), we need extra conditions which depend on the sign of \( \mu \).

We assumed that the eigenvalues of equilibria in \( M \) are real. There are two main cases to consider:

- **Generic real eigenvalues**: for any \( z \in M \), eigenvalues of \( A_\pm(z) \) satisfy

\[
0 < \lambda(z) = \lambda_1(z) < \lambda_2(z) \leq \cdots \leq \lambda_k(z). \tag{1.7}
\]

- **Equal semisimple eigenvalues**: for any \( z \in M \),

\[
A_\pm(z) = \lambda(z)I, \quad \lambda(z) > 0. \tag{1.8}
\]
The last case is highly nongeneric. However, it appears in our main application to Celestial Mechanics which is briefly discussed in the next section. For this reason in this paper we assume \([1.8]\). Generic real case is similar, but the details will be published elsewhere. By \([1.11]-[1.2]\) and \([1.8]\),

\[
\begin{align*}
  d^2H(z)(\xi) &= -2\lambda(z)\omega(\xi,\xi_+), \\
  D\phi^t(z)(\xi) &= (e^{-\lambda(z)t}\xi_+, e^{\lambda(z)t}\xi_-).
\end{align*}
\]  

(1.9) (1.10)

Since the flow on \(W^\pm(z)\) is a node, for any \(a \in W^\pm(z)\) there exist tangent vectors

\[ v_\pm(a) = \mp\lambda(z) \lim_{t \to \pm\infty} e^{\pm\lambda(z)}v(\phi^t(a)) \in E^\pm_z. \]

The map \(v_\pm : W^\pm(z) \to E^\pm_z\) is smooth and \(v_\pm(z) = 0, Dv_\pm(z) = I_{E^\pm_z}\) (see Proposition \([5.1]\)).

**Remark 1.3.** In the case \([1.7]\) of generic real eigenvalues, \(v_\pm(a) = 0\) for \(a\) in the strong stable (unstable) manifold of \(z\). Otherwise, \(v_\pm(a)\) is collinear to the eigenvector \(u_\pm(z)\) of \(A\) \((z)\) associated to the eigenvalue \(\lambda(z)\).

For a heteroclinic orbit \(\gamma(t) = \phi^t(a)\) with \(\gamma(\pm\infty) = z_\pm \in M\), let \(v_\pm(\gamma) = v_\pm(a) \in E^\pm_z\) be the vectors \([1.11]\). They depend on the choice of the initial point \(a\) on \(\gamma\), but the directions are well defined.

If \(\sigma = (\sigma_i)\) is a heteroclinic chain, so that \(\sigma_{i-1}(+\infty) = \sigma_i(-\infty) = c_i \in M\), we set

\[
  a_i(\sigma) = \omega(v_i^+(\sigma), v_i^-(\sigma)), \quad v_i^+(\sigma) = v_+(\sigma_{i-1}), \quad v_i^-(\sigma) = v_-(\sigma_i).
\]

**Definition 1.1.** We call a heteroclinic chain positive (negative) if \(a_i(\sigma) > 0\) \((a_i(\sigma) < 0)\) for all \(i\).

**Remark 1.4.** This definition makes sense also for generic real eigenvalues. Then \(v_i^+(\sigma) = k_i^+u_k(c_i)\). If we choose the eigenvectors \(u_\pm\) so that \(\omega(u_+, u_-) > 0\), then the positivity condition means \(k_i^+ k_i^- > 0\) for all \(i\).

Geometrically the chain \(\sigma\) is a piece wise smooth curve

\[ C = \cup_{i \in \sigma}(R) \]

with “reflections” from \(M\) at the points \(c_i\). Then \(a_i(\sigma)\) measures symplectic angles at these reflections.

Positive heteroclinic chains can be shadowed by orbits with small positive energy, and negative chains with small negative energy. It is not possible to shadow chains of mixed type.

**Theorem 1.1.** Let \(\sigma\) be a positive nondegenerate periodic heteroclinic chain. Then there is \(\mu_0 > 0\) such that for any \(\mu \in (0, \mu_0)\):

- There exists a periodic orbit \(\gamma_\mu\) on \(\Sigma_\mu = H^{-1}(\mu)\), smoothly depending on \(\mu\), which is \(O(\sqrt{\mu})\)-shadowing the chain \(\sigma\):

\[ d(\gamma_\mu(t), C) \leq \text{const} \sqrt{\mu}. \]
• Except for a small neighborhood $U$ of $M$ in $M$, $\gamma_\mu$ is $O(\ln |\mu|)$-shadowing $\sigma$:
\[
d(\gamma_\mu(t), C) \leq \text{const} |\ln \mu| \quad \text{for} \quad \gamma_\mu(t) \in M \setminus U.
\] (1.13)

• The period of $\gamma_\mu$ is of order $\mathbb{P}$
\[
T_\mu \sim \sum_{i=0}^{n-1} \frac{|\ln \mu|}{\lambda(c_i)}.
\] (1.14)

Remark 1.5. The periodic orbit $\gamma_\mu$ has $m$ pairs of multipliers (eigenvalues of the linear Poincaré map) close to the eigenvalues of $DF_n(c_0)$, and $k - 1$ pairs of hyperbolic multipliers $\rho, \rho^{-1}$ with $|\rho|$ large of order $\mu^{-1}$. Thus $\gamma_\mu$ is always strongly unstable. If $DF_n(c_0)$ is hyperbolic, then $\gamma_\mu$ is a hyperbolic periodic orbit.

The set $\bigcup_{0 < \mu \leq \mu_0} \gamma_\mu(\mathbb{R})$ is a smooth invariant cylinder with piece-wise smooth boundary $C \cup \gamma_{\mu_0}(\mathbb{R})$.

If the chain $\sigma$ is negative, then shadowing orbits exist on $\Sigma_\mu$ with $\mu \in (-\mu_0, 0)$.

A result similar to Theorem 1.1 holds for orbits shadowing nonperiodic heteroclinic chains. Consider the skew product of a finite subcollection $K$ of maps $f \in F$.

Theorem 1.2. Let $\Lambda \subset K^Z \times M$ be a compact hyperbolic invariant set. Take any orbit in $\Lambda$ and let $\sigma = (\sigma_i)_{i \in \mathbb{Z}}$ be the corresponding heteroclinic chain. Suppose that $\sigma$ is uniformly positive: there is $\delta > 0$ such that $a_0(\sigma, i) \geq \delta$ for all $i$. There exists $\mu_0 = \mu_0(\Lambda, \delta)$ such that for any $\mu \in (0, \mu_0]$ there exists an orbit on $\Sigma_\mu$ which $O(\sqrt{\mu})$-shadows the chain $\sigma$.

When $M = \{z_0\}$ is a single hyperbolic equilibrium, a version of Theorem 1.2 was proved in [7] and used to study Poincaré second species solutions of the restricted circular 3 body problem. Then the scattering map is trivial, and so the nondegeneracy condition for the heteroclinic chain does not appear. For $M = \{z_0\}$ and generic real eigenvalues, an analog of Theorem 1.2 was announced in [21]. The proof appeared in [9]. In [21] systems with discrete symmetries were studied. In [16], regularity at $\mu = 0$ of the cylinder formed by periodic orbits was investigated in relation to the problem of Arnold’s diffusion.

In [9] also global results on the existence of chaotic shadowing orbits were obtained by variational methods. For a hyperbolic equilibrium with complex eigenvalues, shadowing via variational methods was done in [10]. We are not able to use global variational methods in the current setting, although the proof of Theorem 1.1 has variational flavor.

The proof of Theorem 1.2 is similar to that of Theorem 1.1 but needs more work. In order not to make the paper too long, we postpone this to a subsequent publication. Also the existence of “diffusion” shadowing orbits with average speed along $M$ of order $|\ln \mu|^{-1}$ can be proved. Note that this is much faster

\footnote{The notation means that the difference is bounded as $\mu \to 0$.}
than in the problem of Arnold’s diffusion, where (in the initially hyperbolic case) the speed is of order $O(|\ln \mu|)$ [21]. The reason is that we do not have the resonance gap problem.

Recently shadowing chains of homoclinic orbits to a symplectic normally hyperbolic invariant manifold was studied in [12] by the windows method. However, our situation is very different since the manifold $M$ is critical. In particular, in [12] the positivity condition does not appear.

As a corollary of Theorem 1.1, we obtain a seemingly more general bifurcation result. Consider a Hamiltonian

$$H_\mu = H_0 + \mu h + O(\mu^2) \quad (1.15)$$

smoothly depending on the parameter $\mu$. Suppose $H_0$ satisfies the conditions above, so it has a critical hyperbolic manifold $M \subset \Sigma_0 = H_0^{-1}(0)$ with real eigenvalues and (1.8) holds. Let $F$ be the corresponding scattering map.

**Theorem 1.3.** Suppose $c_{i+1} = f_i(c_i)$ is a nondegenerate periodic orbit of $F$ and let $\sigma = (\sigma_i)$ be the corresponding periodic heteroclinic chain of the flow $\phi^t_{H_0}$. Suppose that $a_i(\sigma)h(c_i) < 0$ for all $i$. There exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ there exists a periodic orbit of the flow $\phi^t_{H_\mu}$ on $\Sigma_\mu = H_\mu^{-1}(0)$ which $O(\sqrt{\mu})$-shadows the chain $\sigma$. Moreover (1.13)–(1.14) hold.

A similar generalization of Theorem 1.2 also holds.

If $h$ has constant sign on $\Sigma_0$, for example $h|_{\Sigma_0} < 0$, then Theorem 1.3 immediately follows from Theorem 1.1. Indeed, in a compact subset of a neighborhood of $\Sigma_0$ we can solve the equation $H_\mu(x) = 0$ for

$$\mu = \mathcal{H}(x) = -\frac{H_0(x)}{h(x)} + \cdots$$

and obtain a Hamiltonian $\mathcal{H}$ such that $\mathcal{H}^{-1}(\mu) = \Sigma_\mu$. Then the flows $\phi^t_{H_\mu}|_{\Sigma_\mu}$ and $\phi^t_{H_0}|_{\Sigma_\mu}$ have the same trajectories, but with different time parametrizations. Theorem 1.1 can be applied to the flow $\phi^t_{H_\mu}$ which yields Theorem 1.3.

When $h$ changes sign, one can define $\mathcal{H}$ in the domains $h > 0$ and $h < 0$, but not for $h = 0$. Thus, in this case, Theorem 1.3 does not follow from Theorem 1.1. However, the only place where there appear trajectories crossing the surface $h = 0$ is in Corollary 5.1 whose proof does not require introduction of the Hamiltonian $\mathcal{H}$. Thus the proof of Theorem 1.1 works for Theorem 1.3. □

The idea of the proof of Theorem 1.1 is variational. We will construct a discrete action functional $A_\mu$, $\mu \in (0, \mu_0]$, whose critical points correspond to trajectories $\gamma_\mu$ on $\Sigma_\mu$ shadowing the heteroclinic chain $\sigma$. The functional $A_\mu$ has a limit $A_0$ as $\mu \to 0$ and $A_\mu = A_0 + O(\mu |\ln \mu|)$. A nondegenerate critical point of the functional (1.16) gives a nondegenerate critical point of $A_0$ and hence a nondegenerate critical point of $A_\mu$ for small $\mu$.

Construction of a functional $A_\mu$ continuous at $\mu = 0$ is not evident, because $\gamma_\mu$ spends a long time of order $|\ln \mu|$ near $M$ and so, in some sense, the perturbation is singular at $\mu = 0$. The way out was found by Shilnikov [19] in the
proof of the Shilnikov Lemma, which is a version of the well known $\lambda$-lemma [17]. Shilnikov’s method was used in [13] to prove the strong $\lambda$-lemma.

The main result of the present paper is Theorem 4.3 (generalization of the Shilnikov Lemma) which describes solutions of a boundary value problem for trajectories on $\Sigma_\mu$ near $M$. It makes possible to construct a functional $A_\mu = A_0 + O(\mu |\ln \mu|)$ and then prove Theorem 1.1. A weaker analog of Theorem 4.3 was proved in [4].

Theorem 4.3 was already used without proof in [8] to establish the existence of Poincaré second species solutions of the (nonrestricted) plane 3 body problem. So now the proof in [8] is finally complete. Application to the 3 body problem is briefly discussed in the next section.

2 Critical manifolds via Levi-Civita regularization in the 3 body problem

Consider the plane 3-body problem with masses $m_1, m_2, m_3$. Suppose that $m_3$ is much larger than $m_1, m_2$:

$$\frac{m_1}{m_3} = \mu \alpha_1, \quad \frac{m_2}{m_3} = \mu \alpha_2, \quad \alpha_1 + \alpha_2 = 1, \quad \mu \ll 1.$$ 

Let $q_1, q_2 \in \mathbb{R}^2$ be positions of $m_1, m_2$ relative to $m_3$, and $p_1, p_2, p_3 \in \mathbb{R}^2$ the momenta. Setting $p_1 + p_2 + p_3 = 0$, we obtain the Hamiltonian

$$H_\mu(q, p) = H_0(q, p) + \mu \left( \frac{|p_1 + p_2|^2}{2} - \frac{\alpha_1 \alpha_2}{|q_1 - q_2|} \right), \quad (2.1)$$

where $q = (q_1, q_2), \ p = (p_1, p_2)$. The unperturbed Hamiltonian

$$H_0(q, p) = \frac{|p_1|^2}{2\alpha_1} + \frac{|p_2|^2}{2\alpha_2} - \frac{\alpha_1}{|q_1|} - \frac{\alpha_2}{|q_2|},$$

describes 2 uncoupled Kepler problems.

To regularize double collisions of $m_1, m_2$ at $\Delta = \{q_1 = q_2 \neq 0\}$, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and perform the Levi-Civita symplectic transformation $g(x, y, \xi, \eta) = (q_1, q_2, p_1, p_2)$,

$$q_1 = x - \alpha_2 \xi^2, \quad q_1 = x + \alpha_1 \xi^2, \quad p_1 = \alpha_1 y - \frac{\eta}{2\xi}, \quad p_2 = \alpha_2 y + \frac{\eta}{2\xi}.$$ 

The map $g$ is a double covering undefined at $\xi = 0$ which corresponds to double collisions at $\Delta$. We fix energy $E$ and set

$$H^E_\mu(x, y, \xi, \eta) = |\xi|^2 (H_\mu \circ g - E) \quad (2.2)$$

$$= \frac{|\eta|^2}{\alpha_1 \alpha_2} - |\xi|^2 \left( E + \frac{\alpha_1}{|\alpha_2 \xi^2 - x|} + \frac{\alpha_2}{|\alpha_1 \xi^2 + x|} - \frac{(1+\mu)|\eta|^2}{2} \right) + \mu \alpha_1 \alpha_2.$$
Denote \( \Sigma^E_\mu = H^{-1}_\mu(E) \) and \( \Gamma^E_\mu = (H^E_\mu)^{-1}(0) \). Since \( g(\Gamma^E_\mu) = \Sigma^E_\mu \), the map \( g \) takes orbits of the flow \( \phi^E_{H_\mu} \) on \( \Gamma^E_\mu \) to orbits of the flow \( \phi^E_{H_\mu} \) on \( \Sigma^E_\mu \). The time parametrization is changed: the new time is given by \( d\tau = |\xi|^2 dt \).

The singularity at \( \Delta \) disappeared: the regularized Hamiltonian \( H^E_\mu \) is smooth on

\[
\mathcal{M} = \{ (x, y, \xi, \eta) : x \neq \alpha_2 \xi^2, \ x \neq -\alpha_1 \xi^2 \}
\]

which means excluding collisions of \( m_1 \) and \( m_2 \) with \( m_3 \). Double collisions of \( m_1 \) and \( m_2 \) correspond to \( \xi = \eta = 0 \). For \( \mu = 0 \), the Hamiltonian

\[
H^E_0 (x, y, \xi, \eta) = \frac{|\eta|^2}{8\alpha_1 \alpha_2} - |\xi|^2 \left( E + \frac{1}{|x|} - \frac{|y|^2}{2} \right) + O(|\xi|^4)
\]

has a normally hyperbolic symplectic critical manifold

\[
M_E = \{ (x, y, 0, 0) : \frac{1}{2} |y|^2 - \frac{1}{|x|} < E \}
\]

with real semisimple eigenvalues

\[
\pm \sqrt{\frac{1}{2\alpha_1 \alpha_2} \left( E + \frac{1}{|x|} - \frac{|y|^2}{2} \right)}.
\]

For \( \mu = 0 \), collision orbits of \( m_1, m_2 \) (pairs of arcs of Kepler orbits starting and ending at \( \Delta \)) with energy \( E \) correspond to trajectories of \( \phi^E_{H_\mu_E} \) asymptotic to \( M_E \), and chains of collision orbits with continuous total momentum \( y = p_1 + p_2 \) correspond to chains of heteroclinic orbits. For small \( \mu > 0 \), orbits of the 3 body problem with energy \( E \) passing \( O(\mu) \)-close to the singular set \( \Delta \) correspond to orbits of the flow \( \phi^E_{H_\mu} \) on the level \( \Gamma^E_\mu \) passing \( O(\sqrt{\mu}) \)-close to \( M_E \).

The Hamiltonian \( (2.2) \) has the form \( (1.15) \):

\[
H^E_\mu = H^E_0 + \mu h,
\]

where \( h|_{M_E} = \alpha_1 \alpha_2 > 0 \). Thus we are in the situation of Theorem 1.3. In [8] many nondegenerate periodic collision chains to \( M_E \) were obtained. Then for small \( \mu > 0 \) Theorem 1.3 implies the existence of many periodic almost collision solutions of the 3 body problem. Such solutions were named by Poincaré second species solutions. See [8] for details.

The plan of the paper is as follows. In section 3 we represent the stable and unstable manifolds by generating functions. In section 4 different versions of local connection theorems are formulated. The proofs are given in section 5. In section 6 relations between the generating functions of the scattering map and of the stable and unstable manifolds are discussed. In section 7 trajectories shadowing heteroclinic chains are represented by critical points of a discrete action functional, and then Theorem 1.1 is proved.
3 Generating functions of the stable and unstable manifolds

In this section it does not matter if the eigenvalues of critical points in $M$ are real or complex: we only need the critical manifold $M$ to be symplectic and normally hyperbolic.

Take an open set $V \subset M$ with symplectic coordinates $z = (x, y) \in \mathbb{R}^{2m}$ and identify $V$ with a domain in $\mathbb{R}^{2m}$. If $V$ is small enough, the stable and unstable bundles $E^\pm|_V$ are trivial over $V$. Hence a tubular neighborhood $U$ of $V$ in $M$ can be identified with

\[ U \cong V \times B_r \times B_r = \{ (z, q, p) : z \in V, q, p \in B_r \}, \quad B_r = \{ q \in \mathbb{R}^k : |q| \leq r \}, \]

in such a way that $V \cong V \times (0, 0)$ and for $z \in M$,

\[ E_z \cong \mathbb{R}^k \times \mathbb{R}^k, \quad E^+_z \cong \mathbb{R}^k \times \{ 0 \}, \quad E^-_z \cong \{ 0 \} \times \mathbb{R}^k. \]

By the generalized Darboux Theorem (see [13]), we can assume that the coordinates in $U$ are symplectic:

\[ \omega|_U = dy \wedge dx + dp \wedge dq. \]

Then for $\xi = (\xi_+, \xi_-)$ and $\eta = (\eta_+, \eta_-)$ in $E_z$,

\[ \omega(\xi, \eta) = (\xi_-, \eta_+) \cdot (\eta_-, \xi_+). \quad (3.1) \]

Since the local stable and unstable manifolds $W^\pm_{\text{loc}}(V)$ are tangent to $E^\pm|_V$, they are graphs

\[ W^+_\text{loc}(V) = \{ (z, q, p) : z \in V, q \in B_r, p = f_+(z, q) \}, \quad W^-\text{loc}(V) = \{ (z, q, p) : z \in V, p \in B_r, q = f_-(z, p) \}, \quad (3.2) \]

where

\[ f_+(z, q) = O_2(q), \quad f_-(z, p) = O_2(p). \]

**Remark 3.1.** $O_2(q)$ means a function of the form $\sum_{|i|=2} a_i(z, q)q^i$ with smooth coefficients. For $i \in \mathbb{Z}^k_+$ we write $|i| = i_1 + \cdots + i_k$.

Take a smaller open set $V_0 \subset V$. For any $z_0 \in V_0$ the local stable and unstable manifolds are given by $W^\pm_{\text{loc}}(z_0) = \psi_{\pm}(z_0, B_r)$, where

\[ \psi_+(z_0, q) = (g_+(z_0, q), q, h_+(z_0, q)) = (z_0, q, 0) + O_2(q), \]

\[ \psi_-(z_0, p) = (g_-(z_0, p), h_-(z_0, p), p) = (z_0, 0, p) + O_2(p). \]

and

\[ h_+(z_0, q) = f_+(g_+(z_0, q), q), \quad h_-(z_0, p) = f_-(g_-(z_0, p), p). \]

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For \(z_0 \in V_0\) and \(q_+, p_- \in B_r\), let
\[
\gamma_+ : [0, +\infty) \to W^+_0(z_0), \quad \gamma_+(t) = \phi^t \circ \psi_+(z_0, q_+),
\]
\[
\gamma_- : (-\infty, 0] \to W^-_0(z_0), \quad \gamma_-(t) = \phi^t \circ \psi_-(z_0, p_-),
\]
be the trajectories asymptotic to \(z_0\) as \(t \to \pm \infty\). Then
\[
\gamma_+(0) = \psi_+(z_0, q_+) = (z_+, q_+, p_+),
\]
\[
\gamma_-(0) = \psi_-(z_0, p_-) = (z_-, q_-, p_-).
\]

We will represent \(W^\pm_0(z_0)\) by generating functions as follows.

**Proposition 3.1.** There exist smooth functions
\[
S_+(x_+, y_0, q_+) = \langle x_+, y_0 \rangle + O_2(q_+),
\]
\[
S_-(x_0, y_-, p_-) = \langle x_0, y_- \rangle + O_2(p_-),
\]
on open sets in \(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k\) such that for any \(z_0 = (x_0, y_0) \in V_0\) and \(A_\pm = (x_\pm, y_\pm, q_\pm, p_\pm) \in U\),
\[
A_+ \in W^+_0(z_0) \iff p_+ = \frac{\partial S_+}{\partial q_+}, \quad y_+ = \frac{\partial S_+}{\partial x_+}, \quad x_0 = \frac{\partial S_+}{\partial y_0},
\]
\[
A_- \in W^-_0(z_0) \iff q_- = \frac{\partial S_-}{\partial p_-}, \quad x_- = \frac{\partial S_-}{\partial y_-}, \quad y_0 = \frac{\partial S_-}{\partial x_0}.
\]

Equivalently,
\[
dS_+(x_+, y_0, q_+) = p_+ dq_+ + y_+ dx_+ + x_0 dy_0,
\]
\[
dS_-(x_0, y_-, p_-) = q_- dp_- + x_- dy_- + y_0 dx_0.
\]

In particular,
\[
(x_+, q_+) \to S_+(x_+, y_0, q_+), \quad (y_-, p_-) \to S_-(x_0, y_-, p_-)
\]
are the generating functions of the Lagrangian manifolds \(W^+_0(y = y_0)\) and \(W^-_0(x = x_0)\).

**Proof.** Let
\[
J_+(z_0, q_+) = \int_{\gamma_+} \alpha, \quad J_-(z_0, p_-) = \int_{\gamma_-} \alpha, \quad \alpha = y \, dx + p \, dq.
\]
be the Maupertuis actions of the asymptotic trajectories \(\gamma_\pm\). The first variation formula \(\Pi\) gives
\[
dJ_+(z_0, q_+) = y_0 \, dx_0 - y_+ \, dx_+ - p_+ \, dq_+,
\]
\[
dJ_-(z_0, p_-) = y_- \, dx_- + p_- \, dq_- - y_0 \, dx_0.
\]
Equations (3.11)–(3.12) imply that
\[
\begin{align*}
    z_0 \to z_+ &= g_+(z_0, q_+) = z_0 + O_2(q_+), \\
    z_0 \to z_- &= g_-(z_0, p_-) = z_0 + O_2(p_-),
\end{align*}
\]
are symplectic maps which are close to identity. We represent them by appropriate generating functions \([1]\). Let
\[
\begin{align*}
g_+(z_0, q_+) &= (X_+(z_0, q_+), Y_+(z_0, q_+)), \\
g_-(z_0, p_-) &= (X_-(z_0, p_-), Y_-(z_0, p_-)).
\end{align*}
\]
Set
\[
S_+(x_+, y_0, q_+) = \langle y_0, x_0 \rangle - J_+(z_0, q_+),
\]
where \(x_0(x_+, y_0, q_+)\) is a solution of the equation
\[
x_+ = X_+(x_0, y_0, q_+) = x_0 + O_2(q_+).
\]
Similarly, set
\[
S_-(x_-, y_-, p_-) = \langle y_-, x_- \rangle + \langle p_-, q_- \rangle - J_-(z_0, p_-),
\]
where \((z_-, q_-, p_-) = \psi_-(z_0, p_-)\) and \(y_0(x_0, y_-, p_-)\) is a solution of the equation
\[
y_- = Y_-(x_0, y_0, p_-) = y_0 + O_2(p_-).
\]
By (3.19)–(3.10), the functions \(S_\pm\) satisfy (3.7)–(3.8).

Next we combine asymptotic orbits \(\gamma_\pm\) in one curve \(\gamma_+ \cdot \gamma_-\) with reflection from \(M\) at \(z_0\). If \(r > 0\) is small enough, for any \((x_+, y_-) \in V_0\) and \(q_+, p_- \in B_r\), we can solve equations (3.13)–(3.14) for
\[
\begin{align*}
z_0 &= \zeta(Z) = (x_+, y_-) + O_2(q_+, p_-), \\
Z &= (x_+, y_-, q_+, p_-). \tag{3.15}
\end{align*}
\]

**Proposition 3.2.** Suppose \(r > 0\) is sufficiently small. Then for any \(Z = (x_+, y_-, q_+, p_-) \in V_0 \times B_r \times B_r\):

- **There exist** \(z_0 \in V, x_-, y_+ \in \mathbb{R}^m, \) and \(q_-, p_+ \in \mathbb{R}^k\) **such that**
  \[
  \begin{align*}
  A_+ &= (x_+, y_+, q_+, p_+) = \psi_+(z_0, q_+) \in W^+_{loc}(z_0), \\
  A_- &= (x_-, y_-, q_-, p_-) = \psi_-(z_0, p_-) \in W^-_{loc}(z_0). \tag{3.16}
  \end{align*}
  \]

- **The relation** \(A_+ \to A_-\) **is symplectic:** there is a smooth generating function
  \[
  L(Z) = \langle x_+, y_- \rangle + O_2(q_+, p_-). \tag{3.17}
  \]
  **such that** (3.10) **is equivalent to**
  \[
  dL(Z) = y_+ \, dx_+ + x_- \, dy_- + p_+ \, dq_+ + q_- \, dp_. \tag{3.18}
  \]

\[^3\]The notation \((x_+, y_-) \in V_0\) makes sense because we identified \(V_0\) with a domain in \(\mathbb{R}^{2m}\).
Proof. Consider the function

\[ F(z_0, Z) = S_+(x_0, y_0, q_0) + S_- (x_0, y_0, p_0) - \langle x_0, y_0 \rangle. \]

Then (3.5)–(3.6) imply that \( A_+ \in W_{loc}^+(z_0) \) and \( A_- \in W_{loc}^-(z_0) \) iff

\[ \frac{\partial F}{\partial z_0} = 0, \quad y_+ = \frac{\partial F}{\partial x_+}, \quad x_- = \frac{\partial F}{\partial y_-}, \quad p_+ = \frac{\partial F}{\partial q_+}, \quad q_- = \frac{\partial F}{\partial p_-}. \tag{3.19} \]

We have

\[ z_0 = \zeta(Z) \iff \frac{\partial F}{\partial z_0} = 0. \]

Define the generating function \( L \) by

\[ L(Z) = F(\zeta(Z), Z) = \text{Crit}_{z_0} F(z_0, Z) \tag{3.20} \]

which means taking the nondegenerate critical value with respect to \( z_0 \). Then (3.19) implies (3.18).

Remark 3.2. The generating function \( L \) does not satisfy the twist condition. Indeed, a computation gives

\[ \left( \begin{array}{cc} \frac{\partial^2 L}{\partial x_+ \partial y_-} & \frac{\partial^2 L}{\partial x_+ \partial p_-} \\ \frac{\partial^2 L}{\partial q_+ \partial y_-} & \frac{\partial^2 L}{\partial q_+ \partial p_-} \end{array} \right) = \left( \begin{array}{cc} \frac{\partial x_0}{\partial x_+} & \frac{\partial y_0}{\partial x_+} \\ \frac{\partial y_0}{\partial q_+} & \frac{\partial y_0}{\partial p_-} \end{array} \right) \left( \begin{array}{cc} \frac{\partial y_0}{\partial x_-} & \frac{\partial p_-}{\partial x_-} \\ \frac{\partial y_0}{\partial p_-} & \frac{\partial p_-}{\partial p_-} \end{array} \right) \]

Hence the rank of this matrix is \( m \). Equations (3.18) do not define a map \( A_+ \to A_- \). The correspondence \( A_+ \to A_- \) is a symplectic relation, i.e. a Lagrangian submanifold in \( M \times M \).

4 Local connection

In this section we formulate several connection theorems describing the behavior of trajectories of the Hamiltonian system near the critical manifold \( M \). In the rest of the paper we assume (1.8). In the generic case (1.7) the results are similar, but they will be published elsewhere.

By (1.9), in the coordinates \((z, q, p)\) in a tubular neighborhood \( U \cong V \times B_r \times B_r \) of \( V \in M \), the Hamiltonian has the form

\[ H|_U = H(z, q, p) = -\lambda(z) \langle p, q \rangle + O_3(p, q). \tag{4.1} \]

The corresponding Hamiltonian system is

\[ \dot{z} = O_2(p, q), \]
\[ \dot{q} = \frac{\partial H}{\partial p} = -\lambda(z)q + O_2(p, q), \]
\[ \dot{p} = -\frac{\partial H}{\partial q} = \lambda(z)p + O_2(p, q). \]
The limit directions (1.11) of the asymptotic orbits (3.3) are
\[
v_+(\gamma_+) = (0, v_+, 0), \quad v_+(z_0, q_+) = \lim_{t \to +\infty} e^{\lambda(z_0)t} q(t) = q_+ + O_Z(q_+),
\]
\[
v_-(\gamma_-) = (0, 0, v_-), \quad v_-(z_0, p_-) = \lim_{t \to -\infty} e^{-\lambda(z_0)t} p(t) = p_- + O_Z(p_-).
\]

By (3.3), the symplectic angle of the concatenation \(\gamma_+ \cdot \gamma_-\) at \(z_0\) is
\[
\omega(v_+(\gamma_+), v_-(\gamma_-)) = -\langle v_+(z_0, q_+), v_-(z_0, p_-) \rangle = -\langle q_+, p_- \rangle + O_z(q_+, p_-).
\]

There are two main versions of connection theorems: for fixed time and for fixed energy.

**Theorem 4.1** (Fixed time connection). Suppose that \(r > 0\) is small enough. For any \(Y = (z_0, q_+, p_-) \in V_0 \times B_r \times B_r\) and \(T \geq 1:\)

- There exists a unique solution
  \[
  \gamma(t) = (z(t), q(t), p(t)) \in V \times B_r \times B_r, \quad t \in [-T, T],
  \]
  satisfying the initial–boundary conditions
  \[
  z(0) = z_0, \quad p(T) = p_-, \quad q(-T) = q_+.
  \]
- \(\gamma\) smoothly depends on \((Y, T) \in V_0 \times B_r \times B_r \times [1, +\infty)\).
- \(\gamma(t)\) converges to \(\gamma_+(t+T)\) on \([-T, 0]\) and to \(\gamma_-(t-T)\) on \([0, T]\) as \(T \to \infty:\)
  \[
  \gamma(t) = \gamma_+(t+T) + \gamma_-(t-T) - (z_0, 0, 0) + e^{-\lambda(z_0)T} O(r^2).
  \]
  Thus \(\gamma([-T, T])\) converges to the concatenation \(\gamma_+ \cdot \gamma_-\).
- Let
  \[
  \gamma(\mp T) = A_{\pm} = (z_{\pm}, q_{\pm}, p_{\pm}), \quad \gamma(0) = (z_0, q_0, p_0).
  \]
  Then
  \[
  z_+ = g_+(z_0, p_+) + T e^{-2\lambda(z_0)T} O(r^2), \quad \text{(4.7)}
  \]
  \[
  z_- = g_-(z_0, p_-) + T e^{-2\lambda(z_0)T} O(r^2), \quad \text{(4.8)}
  \]
  \[
  p_+ = h_+(z_0, q_+) + e^{-2\lambda(z_0)T} (p_+ + O(r^2)), \quad \text{(4.9)}
  \]
  \[
  q_- = h_-(z_0, p_-) + e^{-2\lambda(z_0)T} (q_+ + O(r^2)), \quad \text{(4.10)}
  \]
  \[
  q_0 = e^{-\lambda(z_0)T} v_+(z_0, q_+) + e^{-2\lambda(z_0)T} O(r^2), \quad \text{(4.11)}
  \]
  \[
  p_0 = e^{-\lambda(z_0)T} v_-(z_0, p_-) + e^{-2\lambda(z_0)T} O(r^2). \quad \text{(4.12)}
  \]

**Remark 4.1.** Here \(O(r^2)\) means a function \(f(z_0, q_+, p_-, T)\) on \(V \times B_r \times B_r\), depending also on \(T \geq 1\), such that

\[
\|f\|_{C^1(V \times B_r \times B_r)} = \sup_{V \times B_r \times B_r} \max \left\{ |f|, \left| \frac{\partial f}{\partial z_0} \right| r, \left| \frac{\partial f}{\partial q_+} \right| r, \left| \frac{\partial f}{\partial p_-} \right| r \right\} \leq C r^2 \quad \text{(4.13)}
\]
with $C$ independent of $r$ and $T$. Thus the norms of the derivatives with respect to $q_+ , p_- \in B_r$ are taken with weight $r$. Equivalently, (4.13) is the $C^1$ norm of the function $f(z_0 , r\hat{q}_+ , r\hat{p}_-, T)$ on $V \times B_1 \times B_1$.

When $M = \{ z_0 \}$ is a single equilibrium with equal eigenvalues, Theorem 4.1 was proved in [7]. When $M$ is a single equilibrium with generic real eigenvalues, an analog of Theorem 4.1 can be deduced from the strong $\lambda$-lemma [13], see [9].

With minor modifications Theorem 4.1 holds also for non-Hamiltonian and non-autonomous systems. Now we will use the Hamiltonian structure. For large $T$, we solve (4.7)–(4.8) for $z_0 = \zeta_T(Z) = \zeta(Z) + Te^{2\lambda(\zeta)T}O(r^2)$, $Z = (x_+, y_- , q_+ , p_-)$, (4.14)

where $\zeta$ is the function (3.15). We obtain

Theorem 4.2. Let $T_0 > 0$ be sufficiently large. For every $T \geq T_0$ and $Z = (x_+, y_- , q_+ , p_-) \in V_0 \times B_r \times B_r$:

- There exists a solution (4.3) satisfying the boundary conditions
  
  \[ x(-T) = x_+ , \quad y(T) = y_- , \quad q(-T) = q_+ , \quad p(T) = p_- . \]  
  (4.15)

- The relation $A_+ \rightarrow A_-$ between the points (4.6) is symplectic: there exists a smooth generating function $L_T(Z)$ such that
  
  \[ dL_T(Z) = y_+ dx_+ + x_- dy_- + p_+ dq_+ + q_- dp_- . \]

- As $T \rightarrow +\infty$, the generating function has the asymptotics
  
  \[ L_T(Z) = L(Z) + e^{-2\lambda(\zeta)T}(\langle q_+ , p_- \rangle + TO(r^3)) \]  
  (4.16)

  where $L$ is the generating function (3.18).

Since generating functions are defined up to a constant, the equality (4.16) is modulo a constant. The symplectic relation $A_+ \rightarrow A_-$ has a smooth limit as $T \rightarrow +\infty$. This is true because of a right choice of the boundary conditions which is motivated by the Shilnikov Lemma [19]. For small $r$, the generating function $L_T$ satisfies the twist condition, but the twist is exponentially small for $T \rightarrow +\infty$.

Next we formulate the fixed energy version of the connection theorem. Fix arbitrary $\nu , \kappa \in (0,1)$ and let

\[ D_r = B_r \setminus B_{\nu r} , \quad Q_r = \{(q_+ , p_-) \in D_r \times D_r : \langle q_+ , p_- \rangle \leq -\kappa r^2 \} . \]  
(4.17)

Theorem 4.3 (Fixed energy connection). Let $r > 0$ and $\mu_0 > 0$ be sufficiently small. Then for any $\mu \in (0, \mu_0]$ and $Y = (z_0 , q_+ , p_-) \in V_0 \times Q_r$:
Remark 4.2. Here $O(\mu)$ or $O(\mu |\ln \mu|)$ means a function $f$ on $V \times Q_r$, depending also on $\mu \in (0, \mu_0]$, such that

$$\|f\|_{C^1(V \times Q_r)} \leq C\mu \quad \text{or} \quad \|f\|_{C^1(V \times Q_r)} \leq C\mu |\ln \mu|,$$

where the constant is independent of $r$ and $\mu$. The $C^1$ norm is weighted as in [4.15]. Hence the second terms in (4.22)–(4.23) are not $O(\mu)$. They provide nontrivial twist in the Poincaré map, see Remark 4.3.

Remark 4.3. By [4.2], for small $r$, $(q_+, p_-) \in Q_r$ implies $\omega(v_+(\gamma_+), v_-(\gamma_-)) > 0$. Thus the concatenation of $\gamma_+$ and $\gamma_-$ at $z_0$ is positive (see Definition 1.1). This explains how the positivity condition appears in Theorem 1.1. If we replace the set $Q_r$ by

$$\{(q_+, p_-) \in D_r \times D_r : \langle q_+, p_- \rangle \geq kr^2\},$$

then the concatenation of $\gamma_+$ and $\gamma_-$ at $z_0$ is negative, and the connecting solution $\gamma$ exists for $\mu \in [-\mu_0, 0)$.

Remark 4.4. For simplicity we fixed $\kappa > 0$ in (4.17). In fact Theorem 4.3 can be improved to include $\kappa = C\mu^{1/3}$ with $C > 0$ sufficiently large constant. However, we do not need this for our purposes.
Let us deduce Theorem 4.3 from Theorem 4.1. By (4.11)–(4.12) and (4.1), on the connecting trajectory $\gamma$ in Theorem 4.1,

$$H|_{\gamma} = H(\gamma(0)) = -\lambda(z_0)e^{-2\lambda(z_0)T}\langle v_+(z_0, q_+), v_-(z_0, p_-) \rangle + e^{-3\lambda(z_0)T}O(r^3).$$

To find $\gamma$ on $\Sigma_\mu$, we solve the equation $H|_{\gamma} = \mu$ for $T$. For $(q_+, p_-) \in Q_r$ and small $\mu > 0$ we obtain

$$e^{-2\lambda(z_0)T} = -\frac{\mu + O(\mu^{3/2})}{\lambda(z_0)\langle v_+(z_0, q_+), v_-(z_0, p_-) \rangle} > 0.$$  

This implies (4.18) and Theorem 4.3 follows easily. In the next section we give an independent proof of Theorem 4.3.

Solving (4.20)–(4.21) for $z_0$, we obtain a symplectic version of the fixed energy connection theorem.

**Theorem 4.4.** Let $r > 0$ and $\mu_0 > 0$ be sufficiently small. Then for any $\mu \in (0, \mu_0]$ and $Z = (x_+, y_-, q_+, p_-) \in V_0 \times Q_r$:

- There exist $z_0 = \zeta_\mu(Z) = \zeta(Z) + O(\mu|\ln \mu|)$ and a solution (4.3) on $\Sigma_\mu$ satisfying boundary conditions (4.15) with $T = T_\mu(Z)$ as in (4.18).
- $\gamma$ and $T$ smoothly depend on $(Z, \mu) \in V_0 \times Q_r \times (0, \mu_0]$.
- The relation $A_+ \rightarrow A_-$ between the points (4.6) is given by
  
  $$y_+ = \frac{\partial L}{\partial x_+} + O(\mu|\ln \mu|),$$

  $$x_- = \frac{\partial L}{\partial y_-} + O(\mu|\ln \mu|),$$

  $$p_+ = \frac{\partial L}{\partial q_+} - \frac{\mu p_-}{\lambda(\zeta)\langle q_+, p_- \rangle} + O(\mu),$$

  $$q_- = \frac{\partial L}{\partial p_-} - \frac{\mu q_+}{\lambda(\zeta)\langle q_+, p_- \rangle} + O(\mu).$$

- The generating function of the symplectic relation $A_+ \rightarrow A_-$ has the form
  
  $$R_\mu(Z) = L(Z) - \frac{\mu\ln |\langle q_+, p_- \rangle|}{\lambda(x_-, y_+)} + O(\mu|\ln \mu|).$$

Here $O(\mu|\ln \mu|)$ means a function with $\|f\|_{C^2(V_0 \times Q_r)} \leq C \mu|\ln \mu|$, where $C$ is independent of $r, \mu$, and the norm is weighted as in (4.13).

Theorem 4.4 follows from Theorem 4.3 and the implicit function theorem. Conversely, Theorem 4.3 can be deduced from Theorem 4.4. We prove Theorems
and \(4.3\) in the next section. The proof of Theorem \(4.1\) is similar and we skip it.

The relation \(A_+ \to A_-\) is restricted to the contact manifold \(\Sigma_\mu\). To get a symplectic map, we take symplectic cross sections

\[
\begin{align*}
N^+_\mu &= \{(z_+, q_+, p_+) \in U \cap \Sigma_\mu : q_+ \in S_t\}, \\
N^-_\mu &= \{(z_-, q_-, p_-) \in U \cap \Sigma_\mu : p_- \in S_t\},
\end{align*}
\]

(4.31)

where \(S_t = \partial B_r\) is a sphere.

**Corollary 4.1.** The restriction of the function \(R_\mu\) to the set

\[
E_r = \{Z = (x_+, y_-, q_+, p_-) \in V \times Q_r : q_+, p_- \in S_r\}
\]

is the generating function of the local Poincaré map \(P_\mu : N^+_\mu \cap O^+ \to N^-_\mu \cap O^-:\)

\[
dR_\mu(Z) = y_+ \, dx_+ + x_- \, dy_- + p_+ \, dq_+ + q_- \, dp_-.
\]

Here \(O^\pm\) are open sets in \(U\). We introduce local symplectic coordinates \(x_\pm, y_\pm, \xi_\pm, \eta_\pm\) on \(N^\pm_\mu \cap O^\pm\) such that

\[
\begin{align*}
(y_+ \, dx_+ + p_+ \, dq_+)|_{N^+_\mu} &= y_+ \, dx_+ + \eta_+ \, d\xi_+ , \\
(x_- \, dy_- + q_- \, dp_-)|_{N^-_\mu} &= x_- \, dy_- + \xi_- \, d\eta_- .
\end{align*}
\]

(4.32)

Then

\[
R_\mu(x_+, y_-, \xi_+, \eta_-) = R_\mu(x_+, y_-, q_+(\xi_+), p_-(\xi_-))
\]

is the generating function of the coordinate representation of the Poincaré map \(P_\mu : (x_+, y_+, \xi_+, \eta_+) \to (x_-, y_-, \xi_-, \eta_-)\):

\[
\begin{align*}
R_\mu(x_+, y_-, \xi_+, \eta_-) &= y_+ \, dx_+ + x_- \, dy_- + \eta_+ \, d\xi_+ + \xi_- \, d\eta_-.
\end{align*}
\]

The coordinates \(x_\pm, y_\pm, \xi_\pm, \eta_\pm\) on \(N^\pm_\mu\) are defined as follows. Choose local coordinates on the sphere \(S_r\), for example given by a stereographic projection. Then \(q_+ = q_+(\xi_+) \in S_r\) and \(p_- = p_- (\eta_-) \in S_r\), where \(\xi_+, \eta_- \in \mathbb{R}^{k-1}\). Set

\[
\eta_+ = p_+ \cdot Dq_+ (\xi_+), \quad \xi_- = q_- \cdot Dp_- (\eta_-).
\]

(4.33)

Then \((z_\pm, \xi_\pm, \eta_\pm)\) determine \((z_\pm, q_\pm, p_\pm)\) and so they are local coordinates on \(N^\pm_\mu\).

Indeed, let \((z_+, q_+(\xi_+), p_+) \in N^+_\mu\). The orthogonal projection \(\tilde{p}_+ \perp q_+\) of \(p_+\) to \(T_{q_+}S_r\) is determined by \(\eta_+ = \tilde{p}_+ \cdot Dq_+ (\xi_+)\). Then \(p_+ = cq_+ + \tilde{p}_+\), where the scalar \(c\) is the solution of the equation

\[
H(z_+, q_+, p_+) = \lambda(z_+) (q_+, p_+) + O(r^3) = \lambda(z_+) r^2 c + O(r^3) = \mu.
\]

**Remark 4.5.** The generating function \(L|_{E_r}\) does not satisfy the twist condition, but the function \(R_\mu|_{E_r}\) does, with the twist in \(q_+, p_-\) of order \(\mu\). Thus we are in the situation of the so called anti-integrable limit [3].
5 Proof of local connection theorems

Following Shilnikov [19], we will rewrite the boundary value problem (4.4) as a fixed point problem. First it is convenient to make a change of variable $s$.

**Proposition 5.1.** There is a diffeomorphism $\Phi$ of a neighborhood of $V \times (0, 0)$ in $V \times \mathbb{R}^k \times \mathbb{R}^k$ such that:

- **$\Phi$ is almost identity near $V$:**
  \[
  \Phi(z, q, p) = (w, u, v) = (z, q, p) + O_2(q, p). 
  \]

- **For any $z_0 \in V_0 \subset V$,**
  \[
  \Phi W^+_{loc}(z_0) = \{(z_0, u, 0) : u \in B_r\}, \\
  \Phi W^-_{loc}(z_0) = \{(z_0, 0, v) : v \in B_r\}. 
  \]

- **The flow $\Phi \circ \phi^t \circ \Phi^{-1}$ on $\Phi W_{loc}^\pm(z_0)$ is linear:**
  \[
  \Phi \circ \phi^t \circ \Phi^{-1}(z_0, u, 0) = (z_0, e^{-\lambda(z_0)t}u, 0), \\
  \Phi \circ \phi^t \circ \Phi^{-1}(z_0, 0, v) = (z_0, 0, e^{\lambda(z_0)t}v). 
  \]

In general $\Phi$ is not symplectic.

**Proof.** We modify the coordinates $(z, q, p)$ in $U$ by setting
\[
u = q - f_-(z, p), \quad v = p - f_+(z, q),
\]
where $f_\pm$ are as in (3.2). In the variables $(z, u, v)$, the local stable and unstable manifolds $W^\pm_{loc}(V)$ are given by $v = 0$ and $u = 0$ respectively. Hence for $z_0 \in V_0$, the manifold $W^+_{loc}(z_0)$ is given by the equations
\[v = 0, \quad z_0 = z + \eta_+(z, u),\]
and $W^-_{loc}(z_0)$ by the equations
\[u = 0, \quad z_0 = z + \eta_+(z, v),\]
where
\[\eta_+(z, u) = O_2(u), \quad \eta_-(z, v) = O_2(v).\]
The projection $\pi_+$ is given by $z_0 = z + \eta_+(z, u)$, and the projection $\pi_-$ by $z_0 = z + \eta_-(z, v)$.

We change the variable $z$ to
\[w = z + \eta_+(z, u) + \eta_-(z, v).\]

Then
\[w|_{W^+_{loc}(z_0)} = z + \eta_+(z, u) + \eta_-(z, 0) = z + \eta_+(z, u) = z_0, \]
\[w|_{W^-_{loc}(z_0)} = z + \eta_+(z, 0) + \eta_-(z, v) = z + \eta_-(z, v) = z_0.\]
Thus the diffeomorphism $\Phi(z, q, p) = (w, u, v)$ satisfies (5.2).

The restriction of the Hamiltonian system to $W^+_{loc}(w)$ is now
\[ \dot{u} = -\lambda(w)u + O_2(u). \] (5.4)

Since there are no resonances of order $\geq 2$, by Sternberg’s theorem [22], there is a smooth normalizing transformation $u \rightarrow \bar{u} = \phi(u, w) = u + O_2(u)$, smoothly depending on $w$ and transforming system (5.4) to its linear part $\dot{\bar{u}} = -\lambda(w)\bar{u}$. Similarly, we can transform the system on $W^-_{loc}(w)$ to $\dot{\bar{v}} = \lambda(w)\bar{v}$ via the change $v \rightarrow \bar{v} = \psi(v, w) = v + O_2(v)$. Then the map $\Phi(z, q, p) = (w, \bar{u}, \bar{v})$ satisfies (5.3).

We will use the same notation $u, v$ for the new variables $\bar{u}, \bar{v}$. Proposition 5.1 is proved.

**Remark 5.1.** The last part of the proof is the main place in the paper where the equal eigenvalues case (1.8) differs from the generic case (1.7). Then there may be resonances, and the normal form is more complicated.

The variables $u, v$ are closely related to the limit directions: for $a_\pm = (z_\pm, q_\pm, p_\pm) \in W^\pm_{loc}(z_0)$, we have
\[ u(a_+) = v_+(z_0, q_+), \quad v(a_-) = v_-(z_0, p_-). \] (5.5)

In the variables $w, u, v$, the Hamiltonian system takes the form
\[ \begin{align*}
\dot{w} &= O(u; v), \\
\dot{u} &= -\lambda(w)u + O(u; v), \\
\dot{v} &= \lambda(w)v + O(u; v).
\end{align*} \] (5.6)

**Remark 5.2.** Here $O(u; v)$ means a function of the form
\[ \sum_{|i|=|j|=1} a_{ij}(w, u, v)u^iv^j \]
with smooth coefficients. Thus it vanishes on $W^+ \cup W^-$. The Hamiltonian is transformed to
\[ H(w, u, v) = H \circ \Phi^{-1}(w, u, v) = -\lambda(w)\langle u, v \rangle + O_3(u, v). \] (5.7)

However, since $\Phi$ is non-symplectic, system (5.6) does not have a standard Hamiltonian form.

Finally we make a time change $d\tau = \lambda(w)dt$ and obtain the system
\[ \begin{align*}
w' &= O(u; v), \\
u' &= -u + O(u; v), \\
v' &= v + O(u; v).
\end{align*} \] (5.8)

Once a solution of system (5.8) is known, the time $t$ is determined by
\[ t = \theta(\tau) = \int_0^\tau \frac{ds}{\lambda(w(s))}. \] (5.9)

Next we reformulate Theorem 4.1 in the new variables.
Proposition 5.2. Suppose \( r > 0 \) is sufficiently small and \( T \geq 1 \). Let \( w_0 \in V_0 \) and \( u_+, v_- \in B_r \). Then:

- There exists a unique solution
\[
\sigma(t) = (w(\tau), u(\tau), v(\tau)) \in V \times B_r \times B_r, \quad |\tau| \leq T,
\]
of (5.8) satisfying the initial-boundary conditions
\[
w(0) = w_0, \quad u(-T) = u_+, \quad v(T) = v_-.
\]

- \( \sigma \) smoothly depends on \((w_0, u_+, v_-, T) \in V_0 \times B_r \times B_r \times [1, +\infty)\).

- Set \((u_\pm, v_\pm) = \sigma(\mp T), \quad (w_0, u_0, v_0) = \sigma(0)\).

As \( T \to +\infty \), we have
\[
u_0 = u_+ e^{-T} + e^{-2T} O(r^2),
\]
\[
v_0 = v_- e^{-T} + e^{-2T} O(r^2),
\]
\[
w_+ = w_0 + T e^{-2T} O(r^2),
\]
\[
w_- = w_0 + T e^{-2T} O(r^2),
\]
\[
u_+ = e^{-2T} (u_+ + O(r^2)),
\]
\[
u_- = e^{-2T} (v_- + O(r^2)).
\]

- The initial and final time moments are
\[
T_\pm = \theta(\mp T) = \mp \lambda(w_0) T + T^2 e^{-2T} O(r^2).
\]

Remark 5.3. The meaning of \( O(r^2) \) is as in (4.13): this is a function \( f(w_0, u_+, v_-, T) \) with \( \|f\|_{C^1(V \times B_r \times B_r)} \leq C r^2 \), where the constant is independent of \( r \) and \( T \), and the weighted norm (4.13) is used for the derivatives in \( u_+ \) and \( v_- \).

Proof of Proposition 5.2. We follow Shilnikov [19]. Set
\[
u = e^{-\tau - T} \xi, \quad v = e^{\tau - T} \eta.
\]

In the variables \( w, \xi, \eta \), system (5.8) takes the form
\[
\begin{align*}
w' &= O(e^{-\tau - T} \xi; e^{\tau - T} \eta) = e^{-2T} O(\xi; \eta), \\
\xi' &= e^{\tau + T} O(e^{-\tau - T} \xi; e^{\tau - T} \eta) = e^{\tau - T} O(\xi; \eta), \\
\eta' &= e^{\tau - T} O(e^{-\tau - T} \xi; e^{\tau - T} \eta) = e^{\tau - T} O(\xi; \eta).
\end{align*}
\]

Here \( O(\xi; \eta) \) is a function of the form
\[
\sum_{|i| = |j| = 1} a_{ij}(w, \xi, \eta, \tau, T) \xi^i \eta^j,
\]
where the coefficients are smooth and uniformly bounded for \( T \geq 1 \) and \( |\tau| \leq T \).
We obtain (5.17)–(5.19) are defined for \((w, \xi, \eta)\) if \(C > r > 0\) be a ball in \(L^p\), where \(F\) is a map \(F: X \to X\),

**Proof.**

**Lemma 5.1.** Let \((w, \xi, \eta)\) be the Banach space with the norm \(\|w, \xi, \eta\| = \max\{\|w\|_{C^0}, \|\xi\|_{C^0}, \|\eta\|_{C^0}\}\), and let

\[ Y = \{(w, \xi, \eta) : \|w - w_0, \xi, \eta\| \leq 2r\} \]

be a ball in \(X\). We take \(r > 0\) so the small that the right hand sides of equations (5.17)–(5.19) are defined for \((w, \xi, \eta) \in Y\). Then the right hand sides define a map \(F: Y \to Y\).

**Lemma 5.1.** Let \(r > 0\) be sufficiently small. Then \(F(Y) \subset Y\) and \(F: Y \to Y\) is a contraction.

**Proof.** If \((w, \xi, \eta) \in Y\), then \(|\xi(\tau)|, |\eta(\tau)| \leq 2r\) for \(|\tau| \leq \tau\). There is a constant \(C > 0\), independent of \(r\) and \(\tau\), such that \(|O(\xi; \eta)| \leq Cr^2\). Set \(F(w, \xi, \eta) = (w_1, \xi_1, \eta_1)\), then by (5.17)–(5.19),

\[
\begin{align*}
|\xi(\tau) - u_+| & \leq \int_{-\tau}^{\tau} e^{-s-\tau}|O(\xi(s); \eta(s))| ds \leq C r^2 (e^{2\tau} - e^{-2\tau}) \leq r, \\
|\eta(\tau) - v_-| & \leq \int_{-\tau}^{\tau} e^{-s-\tau}|O(\xi(s); \eta(s))| ds \leq C r^2 (e^{2\tau} - e^{-2\tau}) \leq r, \\
|w_1(\tau) - w_0| & \leq e^{-2\tau} \left| \int_{-\tau}^{\tau} Cr^2 ds \right| = C r^2 e^{-2\tau} \leq r,
\end{align*}
\]

if \(r < C^{-1}\). Hence \(F(w, \xi, \eta) \in Y\). Similarly we show that for small \(r > 0\) the Lipschitz constant for \(F\) is less than 1, so \(F\) is a contraction.

Let \((w, \xi, \eta) \in Y\) be the fixed point for \(F\). Then by (5.15),

\[
\begin{align*}
|u(\tau) - u_+ e^{-\tau}| & = e^{-\tau} |\xi(\tau) - u_+| \leq Cr^2 e^{-2\tau} (1 - e^{-\tau}), \\
|v(\tau) - v_- e^{-\tau}| & = e^{-\tau} |\eta(\tau) - v_+| \leq Cr^2 e^{-2\tau} (1 - e^{-\tau}).
\end{align*}
\]

We obtain

\[
\begin{align*}
|w(\tau) - w_0| & \leq C r^2 e^{-2\tau}, \\
|u(\tau) - u_+ e^{-\tau}| & \leq C r^2 e^{-2\tau}, \\
|v(\tau) - v_- e^{-\tau}| & \leq C r^2 e^{-2\tau}.
\end{align*}
\]
Then by (5.9),
\[ |\theta(\tau) - \lambda(w_0)\tau| \leq C \tau^2 T^2 e^{-2T}. \]

It remains to estimate the derivatives of solution \( \sigma \) with respect to \( w_0, u_+, v_- \). Then we use integral equations for the corresponding variational system and get e.g.
\[ \left| \frac{\partial}{\partial u_+} \theta(\tau) - e^{-\tau - T} I \right| \leq C r e^{-2T}. \]

Similar estimates hold for other variables. This gives (5.13) and then (5.14) follows from (5.9).

Last we check that \( u(\tau), v(\tau) \in B_r \) for \( |\tau| \leq T \). Equation (5.20) gives
\[ |u(\tau)| \leq e^{-\tau - T} |u_+| + |u(\tau) - u_+ e^{-\tau - T}| \]
\[ \leq r - (1 - e^{-\tau - T})(r - C \tau^2 e^{-2T}) \leq r \]
if \( r < C^{-1} \). Thus \( u(\tau) \in B_r \) for \( |\tau| \leq T \). Similarly (5.21) implies \( v(\tau) \in B_r \) for \( |\tau| \leq T \).

Proposition 5.2 is proved.

Next we prove an analog of Theorem 4.3 in the variables \( w, u, v \).

**Proposition 5.3.** Suppose \( r > 0 \) and \( \mu_0 > 0 \) are sufficiently small. Let \( w_0 \in V_0 \) and \( (u_+, v_-) \in Q_r \). Then for every \( \mu \in (0, \mu_0) \):

- There exists \( T > 0 \) and a unique solution (5.10) with \( H = \mu \) satisfying (5.11).
- \( T \) and \( \sigma \) smoothly depend on \( (w_0, u_+, v_-, \mu) \in V_0 \times Q_r \times (0, \mu_0) \). Moreover
  \[ T = -\frac{1}{2\lambda(w_0)} \ln \left( -\frac{\mu}{\lambda(w_0)(u_+, v_-)} \right) + O(\sqrt{\mu}) \]  
  (5.22)
- The boundary points (5.12) satisfy
  \[
  \begin{align*}
  w_+ &= w_0 + O(\mu|\ln \mu|), \\
  w_- &= w_0 + O(\mu|\ln \mu|), \\
  u_- &= -\frac{\lambda(w_0)(u_+, v_-)}{\lambda(w_0)(u_+, v_-)} + O(\mu), \\
  v_+ &= -\frac{\lambda(w_0)(u_+, v_-)}{\lambda(w_0)(u_+, v_-)} + O(\mu).
  \end{align*}
  \]
  (5.23)
- The initial and final time moments are
  \[ T_\pm = \theta(\pm T) = \pm T + O(\mu|\ln \mu|^2). \]  
  (5.24)

As before, \( O(\mu) \) or \( O(\mu|\ln \mu|) \) means a smooth function with weighted \( C^1 \) norm bounded by \( C \mu \) or \( C \mu|\ln \mu| \), where \( C \) is independent of \( r \) and \( \mu \).
Proof. Equations (5.7) and (5.13) imply that on the solution (5.10),

\[ H|_\sigma = -e^{-2T} \lambda(w_0) \langle u_+, v_- \rangle + e^{-3T} O(r^3). \]

For \( H|_\sigma = \mu \), the implicit function theorem gives

\[ e^{-2T} = \frac{\mu + O(\mu^{3/2})}{\lambda(w_0) \langle u_+, v_- \rangle}, \]

which implies (5.22). Hence

\[ e^{-2T} O(r^2) = O(\mu), \quad T e^{-2T} O(r^2) = O(\mu \ln \mu). \]

Then (5.23) follow from (5.13), and (5.24) from (5.9).

Proof of Theorem 4.4. We rewrite Proposition 5.3 in the variables \((x, y, q, p)\) via the change (5.1), where \(z = (x, y)\). Let

\[ x = X(w, u, v), \quad y = Y(w, u, v), \quad q = Q(w, u, v), \quad p = P(w, u, v) \]

be the components of \( \Phi^{-1} \). According to (5.23), to find a solution satisfying boundary conditions (4.15), for given \( x_+, y_-, q_+, p_-, \mu \) we need to find \( w_0, u_+, v_- \) such that

\[
\begin{align*}
X(w_0 + O(\mu \ln \mu)), u_+, &- \frac{\mu v_+}{\lambda(w_0) \langle u_+, v_- \rangle} + O(\mu) = x_+, \\
Y(w_0 + O(\mu \ln \mu)), &- \frac{\mu v_+}{\lambda(w_0) \langle u_+, v_- \rangle} + O(\mu), v_- = y_-, \\
Q(w_0 + O(\mu \ln \mu)), u_+, &- \frac{\mu v_-}{\lambda(w_0) \langle u_+, v_- \rangle} + O(\mu) = q_+, \\
P(w_0 + O(\mu \ln \mu)), &- \frac{\mu v_-}{\lambda(w_0) \langle u_+, v_- \rangle} + O(\mu), v_- = p_.
\end{align*}
\]

Hence

\[
\begin{align*}
X(w_0, u_+, 0) + O(\mu \ln \mu) &= x_+, \quad (5.25) \\
Y(w_0, 0, v_-) + O(\mu \ln \mu) &= y_-, \quad (5.26) \\
Q(w_0, u_+, 0) + O(\mu) &= q_+, \quad (5.27) \\
P(w_0, 0, v_-) + O(\mu) &= p_. \quad (5.28)
\end{align*}
\]

Equations (5.27)–(5.28) and (5.14) imply

\[
\begin{align*}
u_+ &= v_+(w_0, q_+) + O(\mu), \\
v_- &= v_-(w_0, p_-) + O(\mu).
\end{align*}
\]

Then by (5.26)–(5.28),

\[ w_0 = \zeta(x_+, y_-, q_+, p_-) + O(\mu \ln \mu). \]
Let $\sigma(\tau)$ be the trajectory in Proposition 5.3 corresponding to $w_0, u_+, v_-$ and let $t = \theta(\tau)$ be the corresponding time. Set

$$
t_0 = \frac{1}{2}(T_+ + T_-), \quad T = \frac{1}{2}(T_--T_+).
$$

Then

$$
\gamma(t) = \Phi^{-1}(\sigma(\theta(t + t_0))), \quad -T \leq t \leq T, \quad (5.29)
$$

satisfies the conditions of Theorem 4.4.

**Proof of Theorem 4.2.** Now we use Proposition 5.2. For given $x_+, y_-, q_+, p_-$, $T$ we need to find $w_0, u_+, v_-$, $T$ such that

$$
X(w_0 + T e^{-2T} O(r^2), u_+, e^{-2T} v_- + e^{-2T} O(r^2)) = x_+,
$$

$$
Y(w_0 + T e^{-2T} O(r^2), e^{-2T} u_+ + e^{-2T} O(r^2), v_-) = y_-,
$$

$$
Q(w_0 + T e^{-2T} O(r^2), u_+, e^{-2T} v_- + e^{-2T} O(r^2)) = q_+,
$$

$$
P(w_0 + T e^{-2T} O(r^2), e^{-2T} u_+ + e^{-2T} O(r^2), v_-) = p_-,
$$

$$
\lambda(w_0) T + T e^{-2T} O(r^2) = T.
$$

One can check that for large $T$, this is possible by the implicit function theorem. Let $\sigma(t)$ be the trajectory (5.10). Define $\gamma(t)$ as in (5.29). Theorem 4.2 follows easily.

The proof of Theorem 4.1 is similar, and we skip it.

### 6 Generating functions of the scattering map

In this section we relate the generating functions of the stable and unstable manifolds $W^\pm$ and of the scattering map $F$.

Let $f : V^- \to V^+$ be a local branch of $F$ represented by a generating function (1.3) in symplectic coordinates $z^\pm = (x^\pm, y^\pm)$ in $V^\pm$. Let $(x^\pm, y^\pm, q^\pm, p^\pm)$ be the symplectic coordinates in a tubular neighborhood

$$
U^\pm \cong V^\pm \times B_r \times B_r
$$

of $V^\pm$ such that the stable and unstable manifolds $W^\pm_{loc}(V^\pm)$ are graphs (5.2). As in (4.31), take the cross sections

$$
N^+ = \{(z_+, q_+, p_+) \in U^+ \cap \Sigma_0 : q_+ \in S_r\},
\quad N^- = \{(z_-, q_-, p_-) \in U^- \cap \Sigma_0 : p_- \in S_r\}. \quad (6.1)
$$

Let $\sigma$ be the transverse heteroclinic joining a point $c_0 = (a_0, b_0) \in V^-$ with $c_1 = f(c_0) = (a_1, b_1) \in V^+$. Let

$$
\sigma(t^\pm) = (a^\pm, b^\pm, c^\pm, d^\pm) = A^\pm \in N^\pm
$$
be the intersection points of \(\sigma\) with \(N^\pm\) such that \(\sigma(t) \in U^-\) for \(t \leq t^-\) and \(\sigma(t) \in U^+\) for \(t \geq t^+\). Since \(\sigma\) crosses \(N^\pm\) transversely in \(\Sigma_0\), there exist neighborhoods \(O^\pm\) of \(A^\pm\) such that the Poincaré map

\[
P : O^- \cap N^- \to O^+ \cap N^+,
\]

\[
P(B) = \phi^r(B),
\] (6.2)
is a smooth symplectic diffeomorphism. We have \(\tau(A^-) = t^+ - t^-\) and \(\mathcal{P}(A^-) = A^+\). We will locally represent \(\mathcal{P}\) by a generating function.

Suppose the neighborhoods \(O^\pm\) are sufficiently small. Let \(D\) be a small neighborhood of \(C = (b_-, d_-, a_+, c_+)\) and \(K = \{X = (y_-, p_-, x_+, q_+) \in D : q_+, p_- \in S_r\}\).

**Proposition 6.1.** The coordinates \(x_+, q_+\) can be slightly modified in \(O^+\) in a way which does not invalidate the results of sections 3-4 and so that

- For any \(X = (x_-, y_+, y_+, p_+)^T \in K\) there exist points \(B^\pm = (x_\pm, y_\pm, q_\pm, p_\pm) \in N^\pm \cap O^\pm\) such that \(\mathcal{P}(B^-) = B^+\).
- \(B^\pm = B^\pm(X)\) are smooth functions and \(B^\pm(C) = A^\pm\).
- The Poincaré map \(\mathcal{P}\) is locally represented by a smooth generating function \(F(X)\) on \(K\) for \(B^\pm \in N^\pm \cap O^\pm\),

\[
\mathcal{P}(B^-) = B^+ \iff dF(X) = p_+ dq_+ + y_+ dx_+ + x_- dy_- + q_- dp_.
\] (6.3)

**Proof.** Consider the Lagrangian manifolds

\[
L^+ = \{(x_+, y_+, q_+, p_+) \in U^+ : x_+ = a_+, \ q_+ = c_+\},
\]

\[
L^- = \{(x_-, y_-, q_-, p_-) \in U^- : y_- = b_-, \ p_- = d_-\}.
\]

Since \(d(H|_{L^\pm})(A^\pm) \neq 0\), \(\Pi^\pm = L^\pm \cap N^\pm \cap O^\pm\) are smooth Lagrangian manifolds in \(N^\pm\). We need to show that the Lagrangian manifolds \(\mathcal{P}(\Pi^-)\) and \(\Pi^+\) are transverse in \(N^\pm\) at \(A^+\), i.e.

\[
T_{A^+} \mathcal{P}(\Pi^-) \cap T_{A^+} \Pi^+ = \{0\}.
\] (6.4)

Since \(\nu(A^+)\) is transverse to \(N^+\), the symplectic space \(T_{A^+} N^+\) is identified with the quotient space \(W = T_{A^+} \Sigma_0/\mathbb{R}v(A^+)\). The Lagrangian subspace \(T_{A^+} \Pi^+\) is identified with \(\nu^+ = (T_{A^+} L^+ \cap T_{A^+} \Sigma_0)/\mathbb{R}v(A^+) \subset W\), and \(T_{A^+} \mathcal{P}(\Pi^-)\) with a Lagrangian subspace \(\nu^- \subset W\).

The transversality condition \(\nu^- \cap \nu^+ = \{0\}\) can be achieved by a slight perturbation of the manifold \(L^+\) via local modification of the coordinates \(x_+, q_+\) in a neighborhood of the point \(A^+\). Set

\[
\tilde{x}_+ = x_+ + \frac{\partial}{\partial y_+} \phi(y_+, p_+), \quad \tilde{q}_+ = q_+ + \frac{\partial}{\partial p_+} \phi(y_+, p_+),
\]
where $\phi$ is a small smooth function supported near $(b_+, d_+)$ such that $d\phi(b_+, d_+) = 0$. Let $H$ be the Hessian matrix of $\phi$ at $(b_+, d_+)$. If we use the coordinates $\tilde{x}_+, y_+, \tilde{q}_+, p_+$, the manifold $L^+$ is replaced by

$$L^+ = \{(x_+, y_+, q_+, p_+) : x_+ + \frac{\partial}{\partial y_+} \phi(y_+, p_+) = a_+, q_+ + \frac{\partial}{\partial p_+} \phi(y_+, p_+) = c_+ \}. $$

Then $T_A L^+$ is replaced by a Lagrangian subspace $L^+_H = T_A L^+_\phi$ depending on $H$. Changing $H$, we get an open set $\{L^+_H\}$ of Lagrangian subspaces in $T_A \Sigma_0$. Hence we obtain an open set $\{V^+_H\}$ of Lagrangian subspaces $V^+_H = (L^+_H \cap T_A \Sigma_0)/\mathbb{R}v(A^+)$ in $W$. Thus for almost all $H$, the Lagrangian subspaces $V^+_H$ and $V^-$ are transverse. 

Proposition 6.1 is more clear in local symplectic coordinates $x_\pm, y_\pm, \xi_\pm, \eta_\pm$ on $N^\pm \cap O^\pm$ defined as in [1383]. Then

$$B^\pm \leftrightarrow (x_\pm, y_\pm, \xi_\pm, \eta_\pm), \quad X \leftrightarrow (y_-, \eta_-, x_+, \xi_+). \quad (6.5)$$

Let

$$x_+ = x_+(x_-, y_-, \xi_-, \eta_-), \quad \xi_+ = \xi_+(x_-, y_-, \xi_-, \eta_-). \quad (6.6)$$

be the components of the Poincaré map

$$(x_-, y_-, \xi_-, \eta_-) \rightarrow (x_+, y_+, \xi_+, \eta_+). \quad (6.7)$$

Then the transversality condition (6.3) is

$$\det \left. \frac{\partial (x_+, \xi_+)}{\partial (x_-, \xi_-)} \right|_{A^-} \neq 0. \quad (6.8)$$

Under condition (6.8), equations (6.6) can be solved for

$$x_- = x_-(y_-, \eta_-, x_+, \xi_+), \quad \xi_- = \xi_-(y_-, \eta_-, x_+, \xi_+),$$

which gives the point $B^-(X)$ and then $B^+(X) = \mathcal{P}(B^-)$. The Poincaré map (6.7) is represented by the generating function

$$\varphi(y_-, \eta_-, x_+, \xi_+) = F(y_-, \eta_-, x_+, q_+(\xi_+))$$

as follows:

$$\mathcal{P}(B^-) = B^+ \Leftrightarrow d\varphi = \eta_+ d\xi_+ + y_+ dx_+ + x_- dy_- + \xi_- d\eta_-.$$(6.9)

**Remark 6.1.** Transversality of $\sigma$ implies, without any modification of the coordinates, that $\mathcal{P}$ can be represented by a generating function of the variables $x_-, q_-, y_+, p_+$. However, for the proof of Theorem 1.1 the generating function of the variables $y_-, p_-, x_+, q_+$ is more convenient.

Let $S_\pm$ be the generating functions (6.4) of the local stable and unstable manifolds $W^\pm$. Set

$$G_{x_0, y_1}(X) = S_-(x_0, y_-, p_-) - F(X) + S_+(x_+, y_1, q_+), \quad X \in K. \quad (6.10)$$
Proposition 6.2. • \( X \in K \) is a critical point of \( G_{x_0,y_1} \) iff \( B^- (X) \in W^-(x = x_0) \) and \( B^+ (X) \in W^+(y = y_1) \), i.e. the points \( B^\pm \) lie on a heteroclinic orbit.

• If the heteroclinic orbit \( \sigma \) is transverse, then \( C \) is a nondegenerate critical point of \( G_{a_0,b_1} \) on \( K \).

• For \( (x_0, y_1) \) close to \( (a_0, b_1) \), the function \( G_{x_0,y_1} \) has a nondegenerate critical point \( X(x_0,y_1) \in K \) such that \( X(a_0,b_1) = C \). The critical value is the generating function of the scattering map:

\[
S(x_0,y_1) = \text{Crit}_{X \in K} G_{x_0,y_1}(X) = G_{x_0,y_1}(X(x_0,y_1)). \quad (6.11)
\]

Proof. We represent \( X \in K \) and the corresponding points \( B^\pm (X) \) in local coordinates as in (6.5). Set

\[
R_- (x_0,y_-,\eta_-) = S_- (x_0,y_-,p_- (\eta_-)), \quad R_+ (x_+,y_1,\xi_+) = S_+ (x_+,y_1,q_+ (\xi_+)),
\]

\[
R(x_0,y_1,y_-,\eta_,x_+,\xi_+) = G_{x_0,y_1}(y-,p_- (\eta_-),x_+,q_+ (\xi_+)).
\]

Then by (6.5),

\[
dR = (\dot{x}_- - x_-) dy_- + (\dot{\xi}_- - \xi_-) d\eta_- + (\dot{y}_- - y_-) dx_+ + (\dot{\eta}_+ - \eta_-) d\xi_+ + y_0 dx_0 + x_1 dy_1,
\]

where

\[
\dot{x}_- = \frac{\partial}{\partial y_-} R_-(x_0,y_-,\eta_-), \quad \dot{\xi}_- = \frac{\partial}{\partial \eta_-} R_-(x_0,y_-,\eta_-), \quad \\
\dot{y}_+ = \frac{\partial}{\partial x_+} R_+(y_1,x_+\xi_+), \quad \dot{\eta}_+ = \frac{\partial}{\partial \xi_+} R_+(y_1,x_+\xi_+).
\]

Let

\[
\dot{B}^- \leftrightarrow (\dot{x}_-, y_-, \dot{\xi}_-, \eta_-), \quad \dot{B}^+ \leftrightarrow (x_+, \dot{y}_+, \xi_+, \dot{\eta}_+). \quad (6.13)
\]

By (6.5)–(6.6), \( \dot{B}^- \in W^-(x = x_0) \) and \( \dot{B}^+ \in W^+(y = y_1) \). If \( X \) is a critical point of \( G_{x_0,y_1} \), then \( B^\pm = B^\pm \). Hence \( B^\pm \) lie on a heteroclinic orbit which proves the first item of Proposition 6.2. Then by (6.12),

\[
dR = y_0 dx_0 + x_1 dy_1. \quad (6.14)
\]

Suppose that \( C \) is a degenerate critical point of \( G_{a_0,b_1} \) on \( K \). Then there is a family of nearly critical points

\[
X(\varepsilon) \leftrightarrow (y_-(\varepsilon), \eta_- (\varepsilon), x_+ (\varepsilon), \xi_+ (\varepsilon))
\]

such that \( X(0) = C \), \( X'(0) \neq 0 \) and

\[
dG_{a_0,b_1}(X(\varepsilon)) = O(\varepsilon^2). \quad (6.15)
\]
Let

\[ B^\pm(\varepsilon) \leftrightarrow (x_\pm(\varepsilon), y_\pm(\varepsilon), \xi_\pm(\varepsilon), \eta_\pm(\varepsilon)) \]

be the points corresponding to \( X(\varepsilon) \) by Proposition 6.1 and let

\[ \hat{B}^-(\varepsilon) \leftrightarrow (\hat{x}_-(\varepsilon), \hat{y}_-(\varepsilon), \hat{\xi}_-(\varepsilon), \hat{\eta}_-(\varepsilon)), \quad \hat{B}^+(\varepsilon) \leftrightarrow (x_+(\varepsilon), \hat{y}_+(\varepsilon), \hat{\xi}_+(\varepsilon), \hat{\eta}_+(\varepsilon)) \]

be the points defined in (6.13). Then (6.12) and (6.15) imply \( \hat{B}^\pm(\varepsilon) = B^\pm(\varepsilon) + O(\varepsilon^2) \). Applying the Poincaré map, we obtain

\[ \mathcal{P}(\hat{B}^-(\varepsilon)) = \mathcal{P}(B^-(\varepsilon)) + O(\varepsilon^2) = B^+(\varepsilon) + O(\varepsilon^2) \in W^-(x = a_0). \]

Thus the curve \( B^+(\varepsilon) \in W^+(y = b_1) \) is tangent to \( W^-(x = a_0) \). This contradicts the assumption that the heteroclinic \( \sigma \) is transverse.

The last item follows from the first two and (6.14).

Suppose now that \( \mu_0 > 0 \) is sufficiently small and let \( \mu \in [-\mu_0, \mu_0] \). We introduce cross sections \( N_\mu^+ \subset U_\mu \cap U^+ \) as in (4.31). Then \( N_0^+ = N^+ \). By the implicit function theorem, the Poincaré map \( \mathcal{P}_\mu : O^- \cap N^-_\mu \to O^+ \cap N^+_\mu \) is well defined and coincides with \( \mathcal{P} \) for \( \mu = 0 \). Proposition 6.1 implies

**Corollary 6.1.** For any \( \mu \in [-\mu_0, \mu_0] \) and \( X = (y_-, p_-, x_+, q_+) \in K \):

- There exist \( x_-, p_-, y_+, q_+ \) such that \( B^\pm(X, \mu) = (x_\pm, y_\pm, q_\pm, p_\pm) \in \Sigma_\mu \) and \( \mathcal{P}_\mu(B^-) = B^+ \).
- The Poincaré map \( \mathcal{P}_\mu : N^-_\mu \cap O^- \to N^+_\mu \cap O^+ \) has a smooth generating function \( F_\mu(X) = F(X) + O(\mu), X \in K \), smoothly depending on \( \mu \in [-\mu_0, \mu_0] \):

\[ \mathcal{P}_\mu(B^-) = B^+ \quad \iff \quad dF_\mu(X) = p_+ dq_+ + y_+ dx_+ + x_- dy_- + q_- dp_- . \]

### 7 Variational problem

In this section we define 2 functionals: one whose critical points correspond to periodic heteroclinic chains and another whose critical points correspond to shadowing orbits on \( \Sigma_\mu \). Then Theorem 6.1 follows easily.

Let \( c_{i+1} = f_i(c_i) \) be a \( n \)-periodic orbit of \( F \) and let \( \sigma = (\sigma_i) \) be the corresponding periodic heteroclinic chain: \( c_i = \sigma_i(-\infty) \) and \( c_{i+1} = \sigma_i(+\infty) \). In the symplectic coordinates \( z_i = (x_i, y_i) \) in a neighborhood \( V_i \) of \( c_i = (a_i, b_i) \), \( f_i \) is represented by a generating function \( S_i(x_i, y_{i+1}) \) as in (13.5). Then \( c = (c_i)_{i=0}^{n-1} \) is a critical point of the action functional (1.6).

In a neighborhood \( U_i \equiv V_i \times B_r \times B_r \) of \( c_i \) in \( \mathcal{M} \) we will use symplectic coordinates \( (x_i, y_i, q_i, p_i) \) as in (3.2). Define the cross sections as in (6.1):

\[
N_i^+ = \{(x_i, y_i, q_i, p_i) \in U_i \cap \Sigma_0 : q_i \in S_r \}, \quad N_i^- = \{(x_i, y_i, q_i, p_i) \in U_i \cap \Sigma_0 : p_i \in S_r \}.
\]
Let
\[ A_i^-(a_i^-, b_i^-, c_i^-, d_i^-) \in N_i^- \quad A_i^{i+1} = (a_{i+1}^+, b_{i+1}^+, c_{i+1}^+, d_{i+1}^+) \in N_{i+1}^+ \]
be the first and last intersection points of \( \sigma_i \) with \( N_i^- \) and \( N_{i+1}^+ \) respectively. Take small neighborhoods \( O_i^\pm \) of \( A_i^\pm \) and let \( \mathcal{P}_i : N_i^- \cap O_i^- \to N_{i+1}^+ \cap O_{i+1}^+ \) be the local Poincaré map. Then \( \mathcal{P}_i(A_i^-) = A_{i+1}^+ \).

Let \( D_i \) be a small neighborhood of \( C_i = (b_i^-, d_i^+, a_{i+1}^+, c_{i+1}^+) \) and
\[ K_i = \{ x_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in D_i : p_i^-, q_{i+1}^+ \in S \} \].

By Proposition 6.1 without loss of generality we may assume that for any \( X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in K_i \) there exist \( x_i^-, y_i^-, p_i^+, q_{i+1}^+ \), smoothly depending on \( X_i \), such that the points
\[ B_i^-(X_i) = (x_i^-, y_i^-, q_{i+1}^+) \in N_i^- \quad B_i^{i+1}(X_i) = (x_{i+1}^+, y_{i+1}^+, p_{i+1}^+, q_{i+1}^+ + 1) \in N_{i+1}^+ \]
satisfy \( \mathcal{P}_i(B_i^-) = B_i^{i+1} \). The Poincaré map \( \mathcal{P}_i \) is locally given by the generating function \( F_i(X_i) \) on \( K_i \):
\[ dF_i(X_i) = p_i^+ dq_{i+1}^+ + y_i^+ dx_{i+1}^+ + x_i^- dy_i^- + q_i^- dp_i^- \].

As in (6.13), let
\[ G_i(x_i, y_{i+1}, X_i) = S_i^-(x_i, y_i^-, p_i^-) - F_i(X_i) + S_{i+1}^+(x_{i+1}^+, y_{i+1}^+, q_{i+1}^+ + 1) \].

By Proposition 6.2, \( X_i \to G_i(x_i, y_{i+1}, X_i) \) has a nondegenerate critical value
\[ S_i(x_i, y_{i+1}) = \text{Crit}_{X_i \in K_i} G_i(x_i, y_{i+1}, X_i) = G_i(x_i, y_{i+1}, X_i(x_i, y_{i+1})) \quad (7.1) \]
which is the generating function of the symplectic map \( f_i \).

Let
\[ B(z, X) = \sum_{i=0}^{n-1} (G_i(x_i, y_{i+1}, X_i) - \langle x_i, y_i \rangle), \quad z = (z_i)_{i=0}^{n-1}, \quad X = (X_i)_{i=0}^{n-1} \]
where
\[ z_i = (x_i, y_i) \in V_i, \quad X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in K_i, \]
and
\[ y_n = y_0, \quad x_n^+ = x_0^+, \quad q_n^+ = q_0^+. \]

In fact \( B \) is a modified Maupertuis action of the concatenation of trajectories of the Hamiltonian system on \( \Sigma_0 \). It is a smooth function on
\[ \mathcal{N} = \mathcal{V} \times \mathcal{K}, \quad \mathcal{V} = \prod_{i=0}^{n-1} V_i, \quad \mathcal{K} = \prod_{i=0}^{n-1} K_i. \]
Proposition 7.1. For any \( z \in V \) close to \( c \), the function \( X \in \mathcal{K} \rightarrow B(z, X) \) has a nondegenerate critical point \( X(z) \). The critical value equals the action functional (1.4):

\[
\mathcal{A}(z) = \text{Crit}_{X \in \mathcal{K}} B(z, X) = B(z, X(z)).
\]

- Let \( (c, C), C = X(c) \), be the critical point of \( B \) corresponding to the periodic orbit \( c \). If \( c \) is nondegenerate, then \( (c, C) \) is nondegenerate.

The first statement follows from (7.1), and the second from the following elementary and well known

Lemma 7.1. Let \( f(x, y) \) be a smooth function and let let \( y = h(x) \) be a nondegenerate critical point of \( f(x, y) \) with respect to \( y \). Then \( (x_0, y_0) \) is a nondegenerate critical point of \( f(x, y) \) iff \( x_0 \) is a nondegenerate critical point of \( g(x) = f(x, h(x)) \).

Suppose now that the heteroclinic chain \( \sigma \) is positive. Let \( \kappa > 0 \) and \( r > 0 \) be so small that

\[
(c_i^+, d_i^-) < -\kappa r^2.
\]

Then \( (q_i^+, p_i^-) \in Q_r \) for \( (q_i^+, p_i^-) \) close to \( (c_i^+, d_i^-) \).

Take small \( \mu_0 > 0 \) and let \( \mu \in (0, \mu_0] \). Let \( \mathcal{R}_i^\mu(Z_i) \), \( Z_i = (x_i^+, y_i^+, q_i^+, p_i^-) \in V_i \times Q_r \), be the generating function in Theorem 4.4 corresponding to \( V_i \subset M \). It generates the Poincaré map \( P_i^\mu : N_i^+ \cap O_i^+ \rightarrow N_i^- \cap O_i^- \) of the cross sections \( N_i^\pm \subset U_i \cap \Sigma_\mu \) defined in (4.31).

Let \( F_i^\mu(X_i) = (y_i^+, p_i^-) \) be the generating function of the Poincaré map \( P_i^\mu : N_i^- \rightarrow N_i^+ \) in Corollary 6.1. Set

\[
\mathcal{A}_\mu(X) = \sum_{i=0}^{n-1} (F_i^\mu(X_i) + R_i^\mu(Z_i)), \quad X = (X_i)_{i=0}^{n-1}.
\]

We obtain

Proposition 7.2. \( X \) is a critical point of \( \mathcal{A}_\mu \) iff the corresponding points \( B_i^\pm = B_i^\pm(X_i, \mu) \in N_i^\pm \) in Corollary 6.1 lie on a periodic orbit \( \gamma_\mu \) in \( \Sigma_\mu \). Equivalently, \( B_0^- \) is a fixed point of the total Poincaré map

\[
P_{n-1} \circ \mathcal{P}_{n-2} \circ \cdots \circ \mathcal{P}_1 \circ P_1^\mu \circ \mathcal{P}_1 \circ P_1^\mu : N_{0, \mu}^- \rightarrow N_{0, \mu}^-.
\]

For \( \mu = 0 \) we have

\[
\mathcal{A}_0(X) = \sum_{i=0}^{n-1} (F_i(X_i) + L_i(Z_i)),
\]

where \( F_i(X_i) \) is the generating function of the Poincaré map \( P_i \), and \( L_i(Z_i) \) the generating function of the symplectic relation in (3.18):

\[
dL_i(Z_i) = y_i^+ dx_i^+ + y_i^- dx_i^- + p_i^+ dq_i^+ + q_i^- dp_i^-.
\]
Proposition 3.2 implies that to \( X \in K \) there corresponds \( z(X) \in V \) such that
\[
A_0(X) = \text{Crit}_z B(z(X), X).
\]
By Lemma 7.1 if \((c, C)\) is a nondegenerate critical point of \( B \) on \( N \), then \( c \) is a nondegenerate critical point of \( A \), and \( C \) is a nondegenerate critical point of \( A_0 \).

Now we can prove Theorem 1.1. Let \( c \) be a nondegenerate periodic orbit of \( F \) corresponding to a positive heteroclinic chain \( \sigma \). By Proposition 7.1 it defines a nondegenerate critical point \((c, C)\) of \( B \) which gives a nondegenerate critical point \( C \) of \( A_0 \). By Theorem 4.3
\[
\|A_\mu - A_0\|_{C^2} \leq \text{const} |\ln \mu|.
\] (7.3)
Hence for small \( \mu > 0 \), \( A_\mu \) has a nondegenerate critical point \( C_\mu = C + O(\mu |\ln \mu|) \) which gives a periodic shadowing trajectory \( \gamma_\mu \). Theorem 1.1 is proved.

**Remark 7.1.** The constant in (7.3) may depend on \( n \), so in this proof we are unable to pass to the limit as \( n \to +\infty \). To get chaotic shadowing trajectories and prove Theorem 1.2, we need to use the \( L_\infty \) norm on the space of sequences. This will be done in a subsequent publication.

**References**

[1] V.I. Arnold, *Mathematical Methods of Classical Mechanics*. Springer Verlag, 1989.

[2] V.I. Arnold, V.V. Kozlov, and A.I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*. Encyclopedia of Math. Sciences, 3, Springer-Verlag, 1989.

[3] S. Aubry, Anti-integrability in dynamical and variational problems. *Phys. D*, 86 (1995), 284–296.

[4] S. Bolotin, Shadowing chains of collision orbits. *Discr. & Conts. Dynam. Syst.*, 14 (2006), 235–260.

[5] S. Bolotin, Second species periodic orbits of the elliptic 3 body problem. *Celest. & Mech. Dynam. Astron.*, 93 (2006), 345–373.

[6] S. Bolotin, Symbolic dynamics of almost collision orbits and skew products of symplectic maps. *Nonlinearity*, 19 (2006), 2041–2063.

[7] S. Bolotin and R.S. MacKay, Periodic and chaotic trajectories of the second species for the \( n \)-centre problem. *Celest. Mech. & Dynam. Astron.*, 77 (2000), 49–75.
[8] S. Bolotin and P. Negrini, Variational approach to second species periodic solutions of Poincaré of the 3 body problem. *Discrete Contin. Dyn. Syst.* **33** (2013), 1009–1032.

[9] S. Bolotin and P.H. Rabinowitz, A variational construction of chaotic trajectories for a reversible Hamiltonian system. *J. Differ. Equat.*, **48** (1998), 365–387.

[10] B. Buffoni and E. Séré, A global condition for quasi-random behavior in a class of conservative systems. *Comm. in Pure and Appl. Math.* **49** (1996), 285–305.

[11] A. Delshams, R. de la Llave, and T. Seara, Geometric properties of the scattering map of a normally hyperbolic invariant manifold. *Adv. Math.* **217** (2008), 1096–1153.

[12] A. Delshams, M. Gidea, and P. Roldan, Transition map and shadowing lemma for normally hyperbolic invariant manifolds. *Discr. & Conts. Dynam. Syst.*, **33** (2013), 1089–1112.

[13] B. Deng, The Shilnikov problem, exponential expansion, strong $\lambda$-lemma, $C^1$-linearization and homoclinic bifurcation. *J. Differ. Equat.*, **79** (1989), 189–231.

[14] N. Fenichel, Asymptotic Stability with Rate Conditions for Dynamical Systems. *Bull. Am. Math.Soc.*, **80**, (1974), 346–349.

[15] V. Gelfreich and D. Turaev, Unbounded energy growth in Hamiltonian systems with a slowly varying parameter. *Comm. Math. Phys.* **283** (2008), 769–794.

[16] V. Kaloshin and K. Zhang, Normally normally hyperbolic invariant manifolds near strong double resonance. *Preprint* (2012).

[17] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.

[18] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[19] L.P. Shilnikov, On a Poincaré-Birkhoff problem. *Math. USSR Sbornik* **3** (1967), 353–371.

[20] D.V. Turaev and L.P. Shilnikov, Hamiltonian systems with homoclinic saddle curves. *Soviet Math. Dokl.*, **39** (1989), 165–168.

[21] D.V. Turaev and L.P. Shilnikov, Super-homoclinic orbits and multipulse homoclinic loops in Hamiltonian systems with discrete symmetries. *Regular and Chaotic Dynamics*, **2** (1997), 126–138.
[22] S. Sternberg, Local contraction and a theorem of Poincaré. *Amer. J. Math.* 80 (1957), 809–824.

[23] G.N. Piftankin and D.V. Treschev, Separatrix maps in Hamiltonian systems. *Russian Math. Surveys*, 62 (2007), 219–322.

[24] D. Treschev, Trajectories in a neighborhood of asymptotic surfaces of a priori unstable Hamiltonian systems. *Nonlinearity*, 15 (2002), 2033–2052.