HELIX SURFACES IN THE BERGER SPHERE

BY

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ABSTRACT

We characterize helix surfaces in the Berger sphere, that is surfaces which form a constant angle with the Hopf vector field. In particular, we show that, locally, a helix surface is determined by a suitable 1-parameter family of isometries of the Berger sphere and by a geodesic of a 2-torus in the 3-dimensional sphere.

1. Introduction

In the Euclidean space \( \mathbb{R}^3 \) a helix surface or a constant angle surface is an oriented surface such that its normal vector field forms a constant angle with a fixed direction in the space. These surfaces have an important role in the physics of interfaces in liquid crystals and of layered fluids, as shown by Cermelli and

* Supported by PRIN 2010–11, Varietà reali e complesse: geometria, topologia e analisi armonica, Italy, N. 2010NNBZ78_003.

** Supported by CNPq, Brazil.

Received September 1, 2012 and in revised form April 11, 2013
Di Scala in [1], where they have also obtained a remarkable relation with a Hamilton–Jacobi type equation.

A constant direction in the Euclidean space can be regarded as a Killing vector field with constant norm. Thus, if we want to generalize the notion of helix surfaces in a 3-dimensional Riemannian manifold, a natural problem is to study surfaces such that their normal vector field forms a constant angle with a Killing vector field of constant norm.

Among the three-dimensional manifolds, beside the space forms, probably the most important are the 3-dimensional homogeneous manifolds. Most of these spaces admit a Killing vector field of constant norm and thus it is natural to study the corresponding helix surfaces. In fact, the study and classification of helix surfaces in 3-dimensional homogeneous manifolds was done: for surfaces in $\mathbb{S}^2 \times \mathbb{R}$ by Dillen–Fastenakels–Van der Veken–Vrancken ([4]); for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by Dillen–Munteanu ([3]); for surfaces in the Heisenberg group by Fastenakels–Munteanu–Van Der Veken ([7]); for surfaces in $\text{Sol}_3$ by López–Munteanu ([8]).

We also would like to point out that, in a similar way, we can define helix submanifolds in higher-dimensional Euclidean spaces and we advise the interested reader to have a look at [6, 5, 11].

In this paper, in order to take a step towards the classification of helix surfaces in 3-dimensional homogeneous manifolds, we consider surfaces in the 3-dimensional Berger sphere which is defined, using the Hopf fibration, as follows.

Let

$$
\mathbb{S}^2(1/2) = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = 1/4\} \subset \mathbb{C} \times \mathbb{R}
$$

be the usual 2-sphere and let

$$
\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4
$$

be the usual 3-sphere. Then the Hopf map $\psi : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$, defined by

$$
\psi(z, w) = \frac{1}{2} (2zw, |z|^2 - |w|^2),
$$

is a Riemannian submersion and the vector fields

$$
X_1(z, w) = (iz, iw), \quad X_2(z, w) = (-i\bar{w}, i\bar{z}), \quad X_3(z, w) = (-\bar{w}, \bar{z})
$$

parallelize $\mathbb{S}^3$ with $X_1$ being vertical and $X_2, X_3$ horizontal. The vector field $X_1$ is called the **Hopf vector field**. The **Berger sphere** $\mathbb{S}_\varepsilon^3$, $\varepsilon > 0$, is the