A SKEIN RELATION FOR SINGULAR SOERGEL BIMODULES

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Abstract. We study the skein relation that governs the HOMFLYPT invariant of links colored by one-column Young diagrams. Our main result is a categorification of this colored skein relation. This takes the form of a homotopy equivalence between two one-sided twisted complexes constructed from Rickard complexes of singular Soergel bimodules associated to braided webs. Along the way, we prove a conjecture of Beliakova–Habiro relating the colored 2-strand full twist complex with the categorical ribbon element for quantum $\mathfrak{sl}_2$.

1. Introduction

The HOMFLYPT polynomial is an invariant of framed oriented links that is determined by the skein relation

\[ \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{untwist}} \right] - \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{twist}} \right] = (q - q^{-1}) \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{unknot}} \right] \]

together with its behavior under framing change and disjoint union, and its value on the unknot. Algebraically, the HOMFLYPT polynomial can be obtained from the following two-step process. First, one considers the type $A$ Hecke algebra $H_n$, i.e. the quotient of the (group algebra of the) $n$-strand braid group $Br_n$ by the relation (1). As such, any $n$-strand braid $\beta$ determines a well-defined element $[\beta] \in H_n$. Second, there exists a linear map $H_n \to \mathbb{Z}[q, q^{-1}, \frac{a-a^{-1}}{q-q^{-1}}]$, known as the Jones-Ocneanu trace, which gives a Markov trace on the braid group. Applying the latter to the element of $H_n$ assigned to a braid gives the HOMFLYPT polynomial of the braid closure.

The triply-graded Khovanov–Rozansky homology [KR08, Kho07] is a categorification of the HOMFLYPT polynomial, which can be constructed using a similar framework. First, the category $SBim_n$ of type $A_{n-1}$ Soergel bimodules provides a categorical analogue of the Hecke algebras $H_n$. Paralleling the relation between $Br_n$ and $H_n$ is Rouquier’s construction [Rou04, Rou06], which associates to each braid (word) $\beta$ a complex $[\beta]$ of Soergel bimodules. In particular, the skein relation (1) is promoted to a homotopy equivalence:

\[ \text{cone} \left( \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{untwist}} \right] \xrightarrow{f} \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{twist}} \right] \right) \simeq \text{cone} \left( q \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{unknot}} \right] \xrightarrow{g} q^{-1} \left[ \raisebox{-1ex}{\includegraphics[scale=0.1]{unknot}} \right] \right) \]

for appropriate chain maps $f$ and $g$. Finally a categorical analogue of the Jones–Ocneanu trace is provided by the Hochschild (co)homology functor.

In recent years, it has proven to be increasingly important to consider not just categorifications of the HOMFLYPT polynomial, but also its colored variants, especially those where the coloring consists of 1-column Young diagrams. The two relevant algebraic structures in the decategorified story are the colored braid groupoid and the Hecke algebroid. Both can be considered as categories whose objects are finite sequences colors, i.e. natural numbers encoding the numbers of boxes in one-column Young diagrams, such as $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$.

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1The specialization of the thus colored HOMFLY polynomial at $a = q^m$ recovers the $\mathfrak{sl}_m$ Reshetikhin–Turaev invariant with colorings by fundamental representations, a.k.a. exterior powers of the defining representation.
In the colored braid groupoid $\text{Br}$, morphisms from $a$ to $b$ exist only if $r = s$, in which case they are braids $\beta \in \text{Br}_r$ whose strands connect equal colors $b_{\beta(i)} = a_i$. In the Hecke algebroid $H$, morphisms from $a$ to $b$ exist only if $|a| = |b| = n$, in which case they are given by $e_b H_n e_a$, where $e_a \in H_n$ (and similarly $e_b$) is a certain partially antisymmetrizing idempotent, modeled on the Young antisymmetrizer for $S_{a_1} \times \cdots \times S_{a_r}$. The maps $[-]: \text{Br}_n \to H_n$ now induce a functor $[-]: \text{Br} \to H$ given by sending a colored braid $\beta$ to the Hecke algebra element obtained by cabling the strands of $\beta$ with multiplicities specified by $a$, and then composing with the idempotent $e_a$.

Computations in the Hecke algebroid are facilitated by a diagrammatic calculus of braided webs that goes back to Murakami–Ohtsuki–Yamada [MOY98], and can be understood as the $m \to \infty$ limit of the web calculus from [CKM14]. For example, the decategorification of Theorem 1.1 below gives the following identity in $H_n$:

$$\sum_{s=0}^b (-q^{b-1})^s \left[ \begin{array}{c} a \\ b \end{array} \right] \left[ \begin{array}{c} s \\ a \end{array} \right] b = (-1)^b q^{-b} \prod_{i=1}^b (1 - q^{2i}) \left[ \begin{array}{c} a \\ b \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right].$$

For technical reasons we will actually be mostly interested in the following (equivalent) relation:

$$\sum_{s=0}^b (-q^{b-1})^s \left[ \begin{array}{c} b \\ a \end{array} \right] \left[ \begin{array}{c} s \\ b \end{array} \right] a = (-1)^b q^{b(a-b-1)} \prod_{i=1}^b (1 - q^{2i}) \left[ \begin{array}{c} a \\ b \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right].$$

The goal of this paper is to prove a categorical analog of the colored skein relation (3), which takes the form of a homotopy equivalence of complexes constructed from Rickard complexes of singular Soergel bimodules. We now discuss these ingredients in turn.

Singular Soergel bimodules [Wil08] in type $A$ form a monoidal 2-category $\text{SSBim}$, which provides a categorification of the Hecke algebroid $H$ in the same sense in which $\text{SBim}_n$ categorifies the Hecke algebra $H_n$. Moreover, $\text{SSBim}$ is obtained as the idempotent completion of a monoidal 2-category of so-called singular Bott–Samelson bimodules—composites of induction and restriction bimodules between partially symmetric polynomial rings, modeled on planar webs as drawn above; see Section 2.2 for details.

Rickard complexes can be considered as generalizations of the Rouquier complexes for Artin generators to the colored setting. They entered higher representation theory in the seminal work of Chuang–Rouquier [CR08] in the context of $\mathfrak{s}_2$-actions on categories. Closer to our setting, Rickard complexes of singular Soergel bimodules were proposed as the basic ingredient for a colored version of triply-graded Khovanov–Rozansky homology by Mackaay–Stošić–Vaz [MSV11], a proposal that was subsequently implemented by Webster–Williamson [WW17]. We will describe these in detail in Section 2.5. Just like Rouquier complexes, we denote the Rickard complexes of colored braids $\beta$ by $[\beta]$. The “right-hand” side of our categorified colored skein relation involves the following complex:

$$\text{MCS}_{a,b} := \left( \begin{array}{cccc} b & 0 & b & \cdots \\ a & a \\ b & 1 & a \\ a & \cdots \end{array} \right).$$

We will also adopt this notation for certain complexes of singular Bott–Samelson bimodules that are most-easily described as braided webs.
Here the webs represent certain singular Bott-Samelson bimodules, and (for now) we omit all degree shifts. In Proposition 2.31, we show that

\[ \text{MCS}_{a,b} \simeq \begin{bmatrix} b & b \\ a - b & a \end{bmatrix} \cdot \]

This holds for all integers \( a, b \) provided we interpret the right-hand side as zero when \( a < b \). Note that \( \text{MCS}_{a,b} \) (like any complex of singular Soergel bimodules) is a complex of modules over an appropriate ring of partially symmetric functions \( \text{Sym}(X_1|X_2|X_1'|X_2') \), so we may form the tensor product

\[ K(\text{MCS}_{a,b}) := \text{MCS}_{a,b} \otimes \text{Sym}(X_2|X_2') K, \]

where \( K \) is the Koszul complex

\[ K := \text{Sym}(X_2|X_2') \otimes \wedge [\xi_1, \ldots, \xi_b], \quad \delta(\xi_k) = \sum_{i+j=k} (-1)^j h_i(X_2)e_j(X_2'), \]

(see Definition 3.1). The colored skein relation then takes the following form.

**Theorem 1.1.** The complex \( K(\text{MCS}_{a,b}) \) is homotopy equivalent to the following one-sided twisted complexes:

\[ \begin{pmatrix} \begin{bmatrix} b & 0 \\ a & b \end{bmatrix} \end{pmatrix} \to \begin{pmatrix} \begin{bmatrix} b & 1 \\ a & b \end{bmatrix} \end{pmatrix} \to \ldots \to \begin{pmatrix} \begin{bmatrix} b & b \\ a & a \end{bmatrix} \end{pmatrix} \simeq K \begin{pmatrix} \begin{bmatrix} b & b \\ a - b & a \end{bmatrix} \end{pmatrix}. \]

Here, we have omitted all degree shifts as well as potentially longer arrows pointing to the right. For the precise statement, see Theorem 3.4.

**Remark 1.2.** We prove Theorem 3.4 essentially by showing that \( K(\text{MCS}_{a,b}) \) has a filtration whose subquotients are homotopy equivalent to the complexes associated to “threaded digons” as shown on the left-hand side of (4).

Composing with a negative crossing on the left (say) yields the following consequence.

**Corollary 1.3.** Let \( K' \) denote the Koszul complex

\[ K' := \text{Sym}(X_1|X_2') \otimes \wedge [\xi_1', \ldots, \xi_b'], \quad \delta(\xi_k') = \sum_{i+j=k} (-1)^j h_i(X_1)e_j(X_2'), \]

then we have

\[ \begin{pmatrix} \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} \end{pmatrix} \to \begin{pmatrix} \begin{bmatrix} a & 1 \\ b & a \end{bmatrix} \end{pmatrix} \to \ldots \to \begin{pmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \end{pmatrix} \simeq \begin{pmatrix} \begin{bmatrix} a & b \\ a - b & a \end{bmatrix} \end{pmatrix} \otimes \text{Sym}(X_1|X_2') K'. \]

In the course of proving Theorem 1.1, we obtain explicit descriptions of the chain complexes involved above. Of particular interest, we compute the complex assigned to a colored full twist braid on two strands and identify it with the image of the Beliakova–Habiro categorical ribbon element [BH21]. This verifies a version\(^3\) of [BH21, Conjecture 1.3]; see Theorem 3.24.

**Example 1.4.** (1-colored case) By composing the skein relation (2) with a positive crossing, we obtain the following homotopy equivalence:

\[ \begin{pmatrix} \begin{bmatrix} & & \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotis
in which the map on the right is multiplication by $h_1(X_2 - X_2')$. This is the special case of (4) corresponding to $a = b = 1$. More explicitly, the right-hand side of (5) is a complex of the form

\[
\begin{array}{c}
q \\
\xleftarrow{x_2-x_2'} \\
q^{-1}t
\end{array}
\xrightarrow{\chi_0^+} 
\begin{array}{c}
t \\
\xleftarrow{x_2-x_2' (=0)} \\
q^{-2}t^2
\end{array}
\]

(6)

On the other hand, there is a well-known homotopy equivalence

\[
\begin{array}{c}
\bigotimes \\
\xrightarrow{\chi_0^+} \\
\bigotimes
\end{array}
\xrightarrow{\chi_0^+} 
\begin{array}{c}
\bigotimes \\
\xleftarrow{x_2-x_2' (=0)} \\
\bigotimes
\end{array}
\]

(7)

thus (7) can be extracted as a quotient of (6). We show that this remarkable fact extends to arbitrary colors.

**Example 1.5.** (2-colored case) The Rickard complex for a crossing between two 2-colored strands has the form

\[
C_{2,2} := \begin{array}{c}
\bigotimes \\
\xrightarrow{\chi_0^+} \\
\bigotimes
\end{array}
= \text{MCS}_{2,2} = \begin{array}{c}
\bigotimes \\
\xrightarrow{\chi_0^+} \\
\bigotimes
\end{array}
\]

We denote the webs appearing in this complex as $W_2$, $W_1$ and $W_0$ respectively. After basis change in the exterior algebras, the twisted complex on the right-hand side of (4) has the following schematic form:

The subquotients with respect to the filtration indicated by the dotted lines are homotopy equivalent to the complexes on the left-hand side of (4). Additional details appear in Example 3.14.

**Remark 1.6.** In this paper we focus on the objects associated to braids, and not closed link diagrams. Paralleling the uncolored case, one obtains colored Khovanov-Rozansky homology by taking Hochschild (co)homology of the complex $[\tilde{\beta}]$ assigned to a colored braid $\beta$, and then taking homology. As such, our results have implications for (colored) Khovanov-Rozansky homology, but we do not explore them here. However, in the companion paper [HRW21], we use curved deformations of Theorem 3.4 to explore colored link splitting phenomena. Indeed, the results in this paper grew from the considerations in
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[HRW21]. We believe they are of general interest/utility, so we have independently packaged them here.

Remark 1.7. An expression for complexes associated to colored full twist braids on two strands, similar to the one implicit in Theorem 1.1, was obtained in [Wed16, Section 4] and described in terms of certain winding diagrams inspired by Heegaard–Floer theory. It would be interesting to find an interpretation of the entire colored skein relation from Theorem 1.1 in terms of suitable Fukaya categories (depending on the colors) associated with the 4-punctured sphere.

Convention 1.8. Throughout, we work over the field $\mathbb{Q}$ of rational numbers for simplicity (e.g. in treating the background on symmetric functions); however, our results hold over an arbitrary field. We further expect our results to hold over the integers, but certain statements (e.g. Lemma 2.32 and Proposition 2.33) will require additional arguments in this setting.

Acknowledgements. This project was conceived during the conference “Categorification and Higher Representation Theory” at the Institute Mittag-Leffler, and began in earnest during the workshop “Categorified Hecke algebras, link homology, and Hilbert schemes” at the American Institute for Mathematics. We thank the organizers and hosts for a productive working atmosphere. We would also thank Eugene Gorsky and Lev Rozansky for many useful discussions.

Funding. M.H. was supported by NSF grant DMS-2034516. D.R. and P.W. were supported in part by the National Science Foundation under Grant No. NSF PHY-1748958 during a visit to the program “Quantum Knot Invariants and Supersymmetric Gauge Theories” at the Kavli Institute for Theoretical Physics. D.R. was partially supported by Simons Collaboration Grant 523992: “Research on knot invariants, representation theory, and categorification.” P.W. was partially supported by the Australian Research Council grants ‘Braid groups and higher representation theory' DP140103821 and ‘Low dimensional categories’ DP160103479 while at the Australian National University during early stages of this project. P.W. was also supported by the National Science Foundation under Grant No. DMS-1440140, while in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester.

2. Webs, bimodules, and categorified quantum $\mathfrak{gl}_n$

In this section, we review background on singular Soergel bimodules and Rickard complexes.

2.1. Symmetric functions. We begin with some preliminaries on symmetric functions, which play a substantial role throughout.

Definition 2.1. If $\mathcal{X} = \{x_1, \ldots, x_N\}$ is a finite alphabet with $N$ letters, we let $\text{Sym}(\mathcal{X}) = \mathbb{Q}[\mathcal{X}]^{\mathbb{N}^N}$ denote the ring of symmetric polynomials. The elementary symmetric polynomials $e_j(\mathcal{X})$, complete symmetric polynomials $h_j(\mathcal{X})$, and power sum symmetric polynomials $p_j(\mathcal{X})$ are each defined via their generating functions as follows:

$$E(\mathcal{X}, t) := \prod_{x \in \mathcal{X}} (1 + xt) =: \sum_{j \geq 0} e_j(\mathcal{X}) t^j,$$
$$H(\mathcal{X}, t) := \prod_{x \in \mathcal{X}} (1 - xt)^{-1} =: \sum_{j \geq 0} h_j(\mathcal{X}) t^j,$$
$$P(\mathcal{X}, t) := \sum_{x \in \mathcal{X}} \frac{xt}{1 - xt} =: \sum_{j \geq 1} p_j(\mathcal{X}) t^j.$$

By convention, $e_0(\mathcal{X}) = h_0(\mathcal{X}) = 1$ and $p_0(\mathcal{X})$ is undefined. For pairwise disjoint alphabets $\mathcal{X}_1, \ldots, \mathcal{X}_r$, we write

$$\text{Sym}(\mathcal{X}_1 | \cdots | \mathcal{X}_r) \cong \text{Sym}(\mathcal{X}_1) \otimes \cdots \otimes \text{Sym}(\mathcal{X}_r)$$
for the ring of polynomials in $X_1 \cup \cdots \cup X_r$ that are separately symmetric in the alphabets $X_i$.

The elementary and complete symmetric polynomials are related by the identity

$$H(X, t) E(X, -t) = 1,$$

i.e.

$$\sum_{i+j=k} (-1)^j h_i(X) e_j(X) = 0 \quad \forall k \geq 1,$$

and each are related to the power sum symmetric polynomials by the Newton identity:

$$\frac{t \frac{d}{dt} H(X, t)}{H(X, t)} = P(X, t), \quad i.e. \quad H(X, t) = \exp \int P(X, t) \frac{dt}{t}.$$

We will establish identities involving symmetric polynomials via the manipulation of generating functions. For example, for disjoint alphabets $X$ and $X'$, the identity

$$\sum_{i+j=k} (-1)^j h_i(X \cup X') e_j(X) = h_k(X')$$

follows from the generating function identity

$$H(X \cup X', t) E(X, -t) = H(X, t) H(X', t) = H(X', t).$$

In the following, when the parameter $t$ is understood, we shall omit it from the notation.

Let us now consider an alphabet $X^N = \{x_1, \ldots, x_N\}$ on $N$ letters. There is a map of graded algebras $\text{Sym}(X^{N+1}) \to \text{Sym}(X^N)$ sending $x_{N+1} \mapsto 0$. By definition, the \textit{ring of symmetric functions} in infinitely many variables $X^\infty = \{x_1, x_2, \ldots\}$ is the inverse limit

$$\text{Sym}(X^\infty) := \lim_{\leftarrow} \text{Sym}(X^N).$$

The symmetric functions $e_i(X^N), h_i(X^N), p_i(X^N) \in \text{Sym}(X^N)$ are stable with respect to the projections $\text{Sym}(X^N) \to \text{Sym}(X^{N-1})$, hence determine well-defined elements of $\text{Sym}(X^\infty)$. When we do not wish to commit ourselves to a particular alphabet, we will utilize the following notation.

**Definition 2.2.** Let $\Lambda$ denote the ring $\text{Sym}(X^\infty)$ of symmetric functions. The elementary, complete, and power sum symmetric functions $e_k(X^\infty), h_k(X^\infty), p_k(X^\infty)$ are denoted as $e_k, h_k, p_k \in \Lambda$, respectively. As an algebra, we have $\Lambda \cong \mathbb{Q}[e_1, e_2, \ldots] \cong \mathbb{Q}[h_1, h_2, \ldots] \cong \mathbb{Q}[p_1, p_2, \ldots]$.

Our considerations necessitate working with unions of disjoint alphabets, as well as differences of alphabets. These operations can be placed on equal footing by considering \textit{formal linear combinations of alphabets}.

**Definition 2.3.** Let $X_1, \ldots, X_r$ be alphabets and let $a_1, \ldots, a_r \in \mathbb{Q}$ be scalars. For $f \in \Lambda$, define

$$f(a_1 X_1 + \cdots + a_r X_r) \in \text{Sym}(X_1) \otimes \cdots \otimes \text{Sym}(X_r)$$

as follows. If $f = p_k$ is a power sum symmetric function, then set

$$p_k(a_1 X_1 + \cdots + a_r X_r) = a_1 p_k(X_1) + \cdots + a_r p_k(X_r).$$

This extends to all of $\Lambda$ by linearity:

$$(f + g)(a_1 X_1 + \cdots + a_r X_r) = f(a_1 X_1 + \cdots + a_r X_r) + g(a_1 X_1 + \cdots + a_r X_r)$$

and multiplicativity:

$$(fg)(a_1 X_1 + \cdots + a_r X_r) = f(a_1 X_1 + \cdots + a_r X_r) g(a_1 X_1 + \cdots + a_r X_r).$$
If \( X_1 \) and \( X_2 \) are disjoint alphabets, then \( p_k(X_1 + X_2) = p_k(X_1) + p_k(X_2) = p_k(X_1 \cup X_2) \), thus Definition 2.3 implies that

\[
f(X_1 + X_2) = f(X_1 \cup X_2)
\]

for every symmetric functions \( f \). Similarly, formal subtraction of alphabets behaves as expected: if \( X_1, X_2 \) are alphabets and \( X_0 \) is disjoint from both, then

\[
f((X_1 \cup X_0) - (X_2 \cup X_0)) = f(X_1 - X_2)
\]

Again, this identity need only be checked in the special case that \( f = p_k \) and there it is immediate.

Next, we evaluate elementary and complete symmetric functions on formal linear combinations of alphabets. For a power series \( F(t) \in A[[t]] \) with coefficients in a \( \mathbb{Q} \)-algebra \( A \) and \( a \in \mathbb{Q} \), we write

\[
F(t)^a := \exp(a \ln(F(t))),
\]

where \( \exp \) and \( \ln \) are the obvious operators acting on power series.

**Lemma 2.4.** On the level of generating functions, we have

\[
P(a_1X_1 + a_2X_2, t) = a_1P(X_1, t) + a_2P(X_2, t),
\]

\[
H(a_1X_1 + a_2X_2, t) = H(X_1, t)^a_1 H(X_2, t)^a_2,
\]

\[
E(a_1X_1 + a_2X_2, t) = E(X_1, t)^a_1 E(X_2, t)^a_2
\]

for all \( a_1, a_2 \in \mathbb{Q} \).

**Proof.** The statement for \( P(X, t) \) is immediate from Definition 2.3. The remaining statements follow via equation (9). For example,

\[
H(a_1X_1 + a_2X_2, t) = \exp \int P(a_1X_1 + a_2X_2, t) \frac{dt}{t}
\]

\[
= \exp \int (a_1P(X_1, t) + a_2P(X_2, t)) \frac{dt}{t}
\]

\[
= \exp(a_1 \int P(X_1, t) \frac{dt}{t}) \exp(a_2 \int P(X_2, t) \frac{dt}{t})
\]

\[
= \exp(a_1 \ln(H(X_1, t))) \exp(a_2 \ln(H(X_2, t)))
\]

\[
= H(X_1, t)^a_1 H(X_2, t)^a_2.
\]

It follows that this notational convention for formal addition and subtraction of alphabets is consistent with that in [Las]. Useful special cases of Lemma 2.4 include

\[
H(-X, t) = H(X, t)^{-1} = E(X, -t),
\]

and

\[
H(X_1 + X_2) = H(X_1) H(X_2), \quad H(X_1 - X_2) = \frac{H(X_1)}{H(X_2)}
\]

\[
E(X_1 + X_2) = E(X_1) E(X_2), \quad E(X_1 - X_2) = \frac{E(X_1)}{E(X_2)}
\]

(in the latter we we have omitted the parameter \( t \)). In particular, this gives the following generalization of (8):

\[
h_r(X_1 - X_2) = \sum_{j=0}^{r} (-1)^j h_{r-j}(X_1) e_j(X_2)
\]

(10)

We will need an alternative formulation of this identity, in which the lower index of summation starts at \( j = 1 \).
Lemma 2.5. Let $X, X'$ be alphabets, then we have the following identities for all $r \geq 1$:

$$e_r(X) - e_r(X') = \sum_{j=1}^r (-1)^{j-1} e_{r-j}(X) h_j(X - X'),$$

$$h_r(X - X') = \sum_{j=1}^r (-1)^{j-1} h_{r-j}(X) \left(e_j(X) - e_j(X')\right).$$

Proof. This follows immediately from the generating function identities

$$E(X, t) - E(X', t) = -E(X, t) \left(\frac{H(X, -t)}{H(X', -t)} - 1\right)$$

and

$$\left(\frac{H(X, t)}{H(X', t)} - 1\right) = -H(X, t) \left(E(X, -t) - E(X', -t)\right).$$

Remark 2.6. The ring of symmetric functions is a Hopf algebra. The antipode corresponds to the substitution of alphabets $X \mapsto -X$, which is to say that

$$(S f)(X) = f(-X) \in \text{Sym}(X).$$

The comultiplication corresponds to the substitution $X \mapsto X_1 + X_2$, i.e.

$$\sum f^{(1)}(X_1)f^{(2)}(X_2) = f(X_1 + X_2) \in \text{Sym}(X_1|X_2) \cong \text{Sym}(X_1) \otimes \text{Sym}(X_2)$$

where we have used the Sweedler notation $\Delta(f) = \sum f^{(1)} \otimes f^{(2)} \in \Lambda \otimes \Lambda$.

2.2. Singular Soergel bimodules and webs. Recall from the introduction that a categorification of the Hecke algebroid (and the natural setting for colored, triply-graded link homology) is the monoidal 2-category of type $A$ singular Soergel bimodules. Fix $N > 0$, and let $R := \mathbb{Q}[x_1, \ldots, x_N]$ be the polynomial ring in variables $x_i$, graded by declaring $\deg(x_i) = 2$. Given a parabolic subgroup $J_a = \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_m}$ of the symmetric group $\mathfrak{S}_N$, we let $R^a \subseteq R$ denote the ring of polynomials invariant under the action of $J_a$. Note that $R^b \subset R^a$ if and only if $J_b \supset J_a$.

Consider the 2-category $\text{Bim}_N$ given as follows:

- Objects are tuples $a = (a_1, \ldots, a_m)$ with $a_i \geq 1$ and $\sum_{i=1}^m a_i = N$.
- 1-morphisms $a \to b$ are graded $(R^b, R^a)$-bimodules.
- 2-morphisms are homomorphisms of graded bimodules.

Horizontal composition is given by tensor product over the rings $R^a$, and will be denoted by $\ast$. Vertical composition is the usual composition of bimodule homomorphisms. We will write $1_a := R^a$ for the identity bimodule, saving the notation $R^a$ for the rings themselves.

A singular Bott-Samelson bimodule is, by definition, any $(R^a, R^{a'})$-bimodule of the form

$$B = R^{a_0} \otimes_{R^{b_1}} R^{a_1} \otimes_{R^{b_2}} \cdots \otimes_{R^{b_r}} R^{a_r}$$

for some sequence of rings and subrings $R^{a_0} \supset R^{b_1} \supset \cdots \supset R^{b_r} \subset R^{a_r}$, or a grading shift thereof. In particular, whenever $R^b \subset R^a$ (equivalently $J_b \supset J_a$), we have the merge and split bimodules (terminology explained below) given by

$$bM_a := q^{\ell(a) - \ell(b)}R^a \otimes_{R^b} R^a, \quad aS_b := R^a \otimes_{R^b} R^b.$$

Here, $q^k$ denotes a shift up in degree by $k$, and $\ell(a)$ denotes the length of the longest element in $J_a$.

Definition 2.7. The 2-category $\text{SSBim}_N$ of singular Soergel bimodules is the smallest full 2-subcategory of $\text{Bim}_N$ containing the singular Bott-Samelson bimodules that is closed under taking shifts, direct sums, and direct summands. We denote the Hom-category from $a \to b$ by $\text{bSSBim}_a$. 
There is an external tensor product \( \boxtimes : \text{SSBim}_{N_1} \times \text{SSBim}_{N_2} \to \text{SSBim}_{N_1+N_2} \) given on objects by concatenation of tuples:
\[
(a_1, \ldots, a_{m_1}) \boxtimes (b_1, \ldots, b_{m_2}) := (a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2})
\]
and on 1- and 2-morphisms by tensor product over \( \mathbb{Q} \). This implies that the collection \( \{\text{SSBim}_N\}_{N \geq 0} \) assemble to form a monoidal 2-category, that we denote \( \text{SSBim} \).

There are a number of combinatorial/diagrammatic models for the 2-category generated by the singular Bott-Samelson modules, e.g. (the singular analogue of) Elias-Williamson’s graphical calculus [EW16, ESW17], or the \( k \to \infty \) (inverse) limit of the \( \mathfrak{sl}_k \) foam 2-category [QR16]; see e.g. [QRS18, Section 5.2] and [Wed19, Proposition 3.4]. (This \( k \) is independent/unrelated to \( N \).)

We will use aspects of the latter, as the graphical description of the 1-morphisms therein is directly related to braid and link diagrams.

To wit, in this description, singular Bott-Samelson bimodules are denoted using MOY webs, certain labeled, trivalent graphs, e.g. for \( a = (a, b) \) and \( a' = (a + b) \), we have
\[
(a') M_a = a + b \\ and \quad a S_{a'} = b \\
\]
All other singular Bott-Samelson bimodules can be obtained from these using direct sum and grading shift, together with the horizontal composition \( \star \) and tensor product \( \boxtimes \). Graphically, \( \star \) corresponds to gluing of diagrams along a common boundary and \( \boxtimes \) corresponds to disjoint union of diagrams, as depicted in the following.

**Example 2.8.** For \( a' M_a \) and \( a S_{a'} \) as in (12), we have:
\[
a' M_a \star a S_{a'} = a + b \\ and \quad a' M_a \boxtimes a S_{a'} = b \\
\]
For the duration, we will refer to the graphs built from the diagrams in (12) via \( \star \) and \( \boxtimes \) as webs, which we always understand\(^4\) as mapping from the labels at their right endpoints to those at their left.

Let \( W \) be a web and let \( B(W) \) be the associated singular Bott-Samelson bimodule. We now given an alternate description of \( B(W) \), following [Ras15]. For each edge \( e \) of \( W \), choose an alphabet \( \mathcal{X}_e \) of cardinality equal to the label on the edge and define the edge ring associated to \( W \):
\[
R(W) := \bigotimes_{e \in \text{Edges}(W)} \text{Sym}(\mathcal{X}_e) .
\]
For each symmetric function \( f \), expressions such as \( f(\mathcal{X}_e) \) and \( f(\mathcal{X}_{e_1} + \mathcal{X}_{e_2} - \mathcal{X}_{e_3}) \) represent well-defined elements of \( R(W) \). An edge \( e \) of \( W \) is called an **exterior edge** if it meets the boundary \( \partial W \). More specifically, if \( e \) meets the left boundary we call it **outgoing**, and if it meets the right we call it **incoming**. We define the outgoing (respectively incoming) edge rings by
\[
R^\text{out}(W) := \bigotimes_{e \text{ is outgoing}} \text{Sym}(\mathcal{X}_e) , \quad R^\text{in}(W) := \bigotimes_{e \text{ is incoming}} \text{Sym}(\mathcal{X}_e) .
\]
The following is immediate.

\(^4\)Strictly speaking, web edges should be equipped with an orientation. In this paper, we only consider webs with edges that are oriented towards the left, so we omit orientation arrows from all figures.
Lemma 2.9. Up to the shifts coming from (11), there is an isomorphism

$$B(W) \cong R(W)/I(W)$$

of \((R^{out}(W), R^{in}(W))\)-bimodules, where \(I(W) \subset R(W)\) is the ideal generated by all elements of the form \(f(X_{e_1} + X_{e_2} - X_{e_3})\), where \(f \in \Lambda\) and \(e_1, e_2, e_3\) are edges of \(W\) that meet at a trivalent vertex as in:

\[
\begin{align*}
\text{or} & \\
& \begin{array}{c}
\quad \quad a+b\\
\quad \quad X_{e_3} \\
\quad \quad X_{e_1} \\
\quad \quad X_{e_2} \\
\quad \quad b \quad a+b
\end{array}
\end{align*}
\]

\[
\quad a+b
\]

\[
\quad \quad X_{e_3} \\
\quad \quad X_{e_1} \\
\quad \quad X_{e_2} \\
\quad \quad a+b
\]

\[
\quad b \quad a+b
\]

\[\square\]

Despite this result, it is at times helpful to distinguish the bimodule \(B(W)\) from the ring \(R(W)/I(W)\). Our primary use for the latter will be in specifying bimodule endomorphisms of \(B(W)\). Indeed, in the web-and-foam formalism for SSBim, morphisms between singular Bott-Samelson bimodules \(B(W)\) are described by (linear combinations of) foams, certain 2-dimensional CW complexes with facets labeled by non-negative integers that are embedded in \([0,1]^3\) and carry decorations by symmetric polynomials on their facets. Such foams should be viewed as embedded singular cobordisms with corners between the domain and codomain webs. In particular, elements of \(R(W)/I(W)\) correspond to the singular cobordism \(W \times [0,1]\), with facets appropriately decorated.

However, almost all of the morphisms between singular Bott-Samelson bimodules needed for the present work fall into two classes:

1. endomorphisms of \(B(W)\) given by multiplication by elements in \(R(W)/I(W)\), or
2. those in the image of a 2-functor from categorified quantum \(gl_m\) (see §2.4).

As such, we will rarely use the language of foams, but see Appendix A for a short dictionary.

Convention 2.10. In many places in the present work, we will consider endomorphisms of Bott-Samelson bimodules corresponding to webs appearing in equation (13) below, for various edge labels. As shorthand, we assign alphabets of variables to each web edge with cardinality equal to the label on the edge as follows:

\[
\begin{align*}
X_1 & : B \\
M & : X_2 \\
M' & : X_3 \\
F & : X_1' \\
B & : X_2' \\
F' & : X_3'
\end{align*}
\]

Example 2.11. For the web \(W\) from Convention 2.10, we have

\[R(W) := \text{Sym}(X_1|X_2|X_3|F|B|M'|X_1', X_2')\]

and \(I(W)\) is the ideal generated by elements of the form

\[f(X_2 - B - M), \quad f(X_1 + M - F), \quad f(B + M' - X_2'), \quad f(F - M' - X_1'),\]

or equivalently

\[f(X_2) - f(B + M), \quad f(X_1 + M) - f(F), \quad f(B + M') - f(X_2'), \quad f(F) - f(M' + X_1'),\]

as \(f\) ranges over all symmetric functions.

Remark 2.12. For every 1-morphism \(bM_a\) in \(SSBim_N\), we have embeddings \(R^{\otimes N} \hookrightarrow R^a\) and \(R^{\otimes N} \hookrightarrow R^b\) and the endomorphisms of \(bM_a\) induced by \(f \in R^{\otimes N}\) on the left and on the right agree.
2.3. The dg category of complexes. In order to consider the braid group representation on singular Soergel bimodules, we must first discuss the monoidal dg 2-category of complexes of singular Soergel bimodules. We being by recalling the basic framework of dg categories of complexes.

**Definition 2.13.** Let $A$ be a $\mathbb{Q}$-linear category, then $\mathcal{C}(A)$ denotes the dg category of bounded complexes over $A$. Objects of this category are complexes

$$(X, \delta_X) = \ldots \xrightarrow{\delta_X} X^k \xrightarrow{\delta_X} X^{k+1} \xrightarrow{\delta_X} \ldots$$

in $A$ with $X^k = 0$ for all but finitely many $k$. Morphism spaces in this category are complexes $(\text{Hom}_{\mathcal{C}(A)}(X,Y), d)$ where

$$\text{Hom}_{\mathcal{C}(A)}(X,Y) := \prod_{i \in \mathbb{Z}} \text{Hom}_A(X^i, Y^{i+k})$$

and the component of the differential $d : \text{Hom}_{\mathcal{C}(A)}^k(X,Y) \to \text{Hom}_{\mathcal{C}(A)}^{k+1}(X,Y)$ is given by

$$d(f) := [\delta, f] = \delta_Y \circ f - (-1)^{|f|} f \circ \delta_X.$$

The notation $|f| = k$ means that $f$ is homogeneous of (homological) degree $k$, i.e. that $f \in \text{Hom}_{\mathcal{C}(A)}^k(X,Y)$. We say that such $f$ is closed if $[\delta, f] = 0$ and exact (or null-homotopic) if $f = [\delta, h]$ for some $h \in \text{Hom}_{\mathcal{C}(A)}^{k-1}(X,Y)$. The category $\mathcal{C}(A)$ is endowed with an autoequivalence (homological) shift functor, that we denote by $t$. By convention, $t^k$ denotes a shift up in homological degree.

We will use the following to build certain complexes (in particular, to construct the left-hand side of the colored skein relation).

**Definition 2.14.** If $(X, \delta_X)$ is a complex and $\alpha \in \text{End}_{\mathcal{C}(A)}^1(X)$ satisfies $(\delta_X + \alpha)^2 = 0$, then we denote the complex $(X, \delta_X + \alpha)$ by $\text{tw}_\alpha(X)$. We will refer to $\text{tw}_\alpha(X)$ as a twist of the complex $(X, \delta_X)$. Further, we call $\text{tw}_\alpha(X)$ a one-sided twisted complex, if $X$ takes the form

$$(X, \delta) = \bigoplus_{i \in \mathbb{Z}} (X_i, \delta_i)$$

where the components $\alpha_{i,j} : X_j \to X_i$ of $\alpha$ satisfy $\alpha_{i,j} = 0$ for $i \leq j$.

Note that any complex $(X, \delta_X)$ can itself be written as a one-sided twisted complex

$$X = \text{tw}_{\delta_X} \left( \bigoplus_k t^k X^k \right)$$

where we view each $X^k$ as a complex concentrated in homological degree zero with differential.

**Remark 2.15.** If $A$ is enriched in a symmetric monoidal category $\mathcal{K}$, then $\mathcal{C}(A)$ is enriched in the category of complexes $\mathcal{C}(\mathcal{K})$. In particular, if Hom-spaces in $\mathcal{A}$ are (already) $\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces, then Hom-spaces in $\mathcal{C}(A)$ are $\mathbb{Z} \times \mathbb{Z}$-graded complexes of $\mathbb{Q}$-vector spaces. In this context, we will decorate the grading group by subscripts, e.g. $\mathbb{Z}_q \times \mathbb{Z}_t$ to distinguish the internal $\mathbb{Z}_q = \mathbb{Z}$-grading from the homological $\mathbb{Z}_t = \mathbb{Z}$-grading.

We are interested in complexes of singular Soergel bimodules.

**Definition 2.16.** Let $\mathcal{C}(\text{SSBim})$ be the monoidal 2-category with the same objects as SSBim, and wherein the Hom-category $a \to b$ equals $\mathcal{C}(b \text{SSBim} a)$ and the composition operations and monoidal structure are inherited from SSBim and described below.

In other words, 1-morphisms in $\mathcal{C}(\text{SSBim})$ are complexes of Soergel bimodules, and 2-morphism spaces in $\mathcal{C}(\text{SSBim})$ are complexes of bimodule maps.
Convention 2.17. In the notation of Remark 2.15, the 1-morphism categories of \( \text{SSBim} \) are enriched in \( \mathbb{Z}_q \)-graded \( \mathbb{Q} \)-vector spaces, so the 1-morphism category \( \text{Bim}_\mathcal{C}(\text{SSBim})_1 \) is enriched in \( \mathbb{Z}_q \times \mathbb{Z}_t \)-graded complexes of \( \mathbb{Q} \)-vector spaces. We will use the convention that \( \text{deg}(f) = (i, j) \) means \( f \) has \( q \)-degree (or “Soergel degree”) \( i \) and homological degree \( j \). Further, the singly-indexed Hom-space \( \text{Hom}^k_{\mathcal{C}(\text{SSBim})}(X, Y) \) always refers to homological degree, while the doubly-indexed \( \text{Hom}^{i,j}_{\mathcal{C}(\text{SSBim})}(X, Y) \) consists of \( f \) with \( \text{deg}(f) = (i, j) \). We will typically indicate these degrees multiplicatively by writing \( \text{wt}(f) = q^i t^j \), and will also use the notation \( q, t \) to denote the corresponding shift functors.

The (horizontal) composition of 1-morphisms is defined as usual:

\[
(X \star Y)^k = \bigoplus_{i+j=k} X^i \star Y^j, \quad \delta_{X \star Y} = \delta_X \star \text{id}_Y + \text{id}_X \star \delta_Y.
\]

Here, the components of a horizontal composition of 2-morphisms are defined using the Koszul sign rule. Explicitly, if \( f \in \text{Hom}_{\mathcal{C}(\text{SSBim})}(X, X') \) and \( g \in \text{Hom}_{\mathcal{C}(\text{SSBim})}(Y, Y') \) are given, then \( f \star g \) is defined component-wise by:

\[
(f \star g)|_{X' \star Y'} = (-1)^{|g||f|} f|_{X'} \star g|_{Y'}.
\]

A direct computation shows that the (graded) middle interchange law is satisfied:

\[
(f_1 \star g_1) \circ (f_2 \star g_2) = (-1)^{|g_1||f_2|} (f_1 \circ f_2) \star (g_1 \circ g_2).
\]

The monoidal structure on \( \mathcal{C}(\text{SSBim}) \) is given by extending the external tensor product \( \boxtimes: \text{SSBim} \to \text{SSBim} \) to complexes, again following standard conventions. Explicitly, the external tensor product of 1-morphisms \( X, Y \in \mathcal{C}(\text{SSBim}) \) is defined by

\[
(X \boxtimes Y)^k := \bigoplus_{i+j=k} X^i \boxtimes Y^j, \quad \delta_{X \boxtimes Y} = \delta_X \boxtimes \text{id}_Y + \text{id}_X \boxtimes \delta_Y
\]

where, as before, the external tensor product of 2-morphisms in \( \mathcal{C}(\text{SSBim}) \) is defined component-wise using the Koszul sign rule:

\[
(f \boxtimes g)|_{X' \boxtimes Y'} = (-1)^{|g||f|} f|_{X'} \boxtimes g|_{Y'}.
\]

It is straightforward to see that \( \mathcal{C}(\text{SSBim}) \) is a monoidal 2-category in which the 2-morphism spaces are \( \mathbb{Z}_q \times \mathbb{Z}_t \)-graded complexes, and all three of vertical composition \( \circ \), horizontal composition \( \star \), and external tensor product \( \boxtimes \) of 2-morphisms satisfy appropriate versions of the Leibniz rule; i.e. \( \mathcal{C}(\text{SSBim}) \) is a differential \( \mathbb{Z}_q \times \mathbb{Z}_t \)-graded monoidal 2-category. Henceforth, we will slightly abuse terminology and simply refer to \( \mathcal{C}(\text{SSBim}) \) as a dg monoidal 2-category (the additional grading on 2-morphism complexes will be understood throughout).

We let \( \mathcal{K}(\text{SSBim}) = H^0(\mathcal{C}(\text{SSBim})) \) be the cohomology category of \( \mathcal{C}(\text{SSBim}) \). Its objects and 1-morphisms are the same as in \( \mathcal{C}(\text{SSBim}) \), but its 2-morphisms are now given by degree-zero cohomology classes in \( \text{Hom}_{\mathcal{C}(\text{SSBim})}(-, -) \), i.e. degree-zero chain maps modulo homotopy. In other words, \( \mathcal{K}(\text{SSBim}) \) is the usual homotopy category of (bounded) complexes over \( \text{SSBim} \). The horizontal composition and external tensor product descend to \( \mathcal{K}(\text{SSBim}) \), making the latter into a triangulated monoidal 2-category.

2.4. Categorified quantum \( \mathfrak{gl}_n \). Let \( \mathcal{U}(\mathfrak{gl}_n) \) denote the \( \mathfrak{gl}_n \) analogue of the Khovanov-Lauda-Rouquier categorified quantum group [KL09, KL11, KL10, Rou08] associated to the Lie algebra \( \mathfrak{sl}_m \). This 2-category is the Karoubi completion of the graded, additive 2-category \( \mathcal{U}(\mathfrak{gl}_m) \) in which objects are \( \mathfrak{gl}_m \) weights \( a = (a_1, \ldots, a_m) \), 1-morphisms are generated by

\[
E_i \mathbb{1}_a : a \to a + \varepsilon_i, \quad F_i \mathbb{1}_a : a \to a - \varepsilon_i
\]
for $i = 1, \ldots, m - 1$ (here $\epsilon_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$), and 2-morphisms are given using $\mathfrak{sl}_m$ Khovanov-Lauda string diagrams. We will assume some familiarity with the diagrammatic presentation of $\mathcal{U}(\mathfrak{gl}_m)$; in fact, only categorified $\mathfrak{gl}_2$ computations will be used in this paper, so knowledge of the latter will suffice.

Of particular importance are the “divided power” 1-morphisms $E_i^{(k)} \mathbb{1}_a$ and $F_i^{(k)} \mathbb{1}_a$ in $\mathcal{U}(\mathfrak{gl}_m)$. These are indecomposable 1-morphisms that satisfy

$$E_i^{k} \mathbb{1}_a \cong \bigoplus_{[k]!} E_i^{(k)} \mathbb{1}_a, \quad F_i^{k} \mathbb{1}_a \cong \bigoplus_{[k]!} F_i^{(k)} \mathbb{1}_a$$

We will use $\mathcal{U}(\mathfrak{gl}_m)$ as a technical tool for studying SSBim$_N$ via the following result. This essentially appears in [KL10], but can also be deduced from the main result of [QR16] and the correspondence between foams and SSBim.

**Proposition 2.18.** For $m \leq N$, there is a 2-functor $\Phi: \mathcal{U}(\mathfrak{gl}_m) \to$ SSBim$_N$ that extends to the full 2-subcategory generated by the divided powers, that sends objects $a \mapsto R^a$ and 1-morphisms:

$$\mathbb{1}_a \mapsto 1_a$$

$$E_i^{(k)} \mathbb{1}_a \mapsto 1_{(a_1, \ldots, a_{i-1})} \boxtimes \frac{a_{i+1} - k}{a_i + k} a_i \boxtimes 1_{(a_{i+2}, \ldots, a_m)}$$

$$F_i^{(k)} \mathbb{1}_a \mapsto 1_{(a_1, \ldots, a_{i-1})} \boxtimes \frac{a_{i+1} + k}{a_i - k} a_i \boxtimes 1_{(a_{i+2}, \ldots, a_m)}$$

The value of $\Phi$ on 2-morphisms can be deduced from [QR16, Lemma 3.7, Theorem 3.9, and Corollary 3.10] and the correspondence between foams and singular Soergel bimodules. However, we caution the reader that the 2-functor $\Phi$ appearing in Proposition 2.18 does not agree on the nose with the one appearing in [QR16], since our current conventions for where $\Phi$ sends the 1-morphisms $E_i^{(k)} \mathbb{1}_a$ and $F_i^{(k)} \mathbb{1}_a$ are opposite. Indeed, it is obtained from the 2-functor in [QR16] by further composing with an autoequivalence that reflects foams in the direction perpendicular to the page (and rescales certain generators by $\pm 1$).

The $m = 2$ case will be particularly important. In this case,

$$F_i^{(l)} E_i^{(k)} \mathbb{1}_{a,b} \xrightarrow{\Phi} \frac{b-k+l}{a+k-l} \frac{b-k}{a-k} \frac{b}{a}$$

and all 2-morphisms in $\mathcal{U}(\mathfrak{gl}_2)$ can be described using the extended graphical calculus from [KLMS12]. For example, the following give 2-morphisms in SSBim that will appear throughout this paper:

$$\chi^+_{r} := \Phi \begin{pmatrix} (-1)^{b-k} & (a, b) \end{pmatrix} \xrightarrow{(14)} \begin{pmatrix} \frac{b-k+l}{a+k-l} \frac{b-k}{a-k} \frac{b}{a} \end{pmatrix}$$

$$\chi^-_{r} := \Phi \begin{pmatrix} (-1)^{a+b+k+l-1} & (a, b) \end{pmatrix} \xrightarrow{(15)} \begin{pmatrix} \frac{b-k+l}{a+k-l} \frac{b-k}{a-k} \frac{b}{a} \end{pmatrix}$$
Both of these 2-morphisms have $q$-degree equal to $1+2r+(a-b)+(k-l)$. The green signs appearing here (and in some places below) are conventional, and guarantee that the image is the bimodule morphism corresponding to an unsigned foam. See Appendix A for the translation between foams and bimodule morphisms.

**Remark 2.19.** If $f,g$ are symmetric functions, then, by Convention 2.10, $f(M) \otimes g(M')$ is a well-defined endomorphisms of $\Phi(F^{(k)}E^{(k)}I_{a,b})$. In fact, this endomorphism is in the image of $\Phi$, and is described in extended graphical calculus as:

$$\Phi = \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}$$

**Remark 2.20.** The graphical calculus for $\hat{U}(\mathfrak{gl}_2)$ contains cap and cup morphisms between the identity morphisms $I_a$ and the (horizontal) compositions $F^{(k)}E^{(k)}I_a$ and $E^{(k)}F^{(k)}I_a$. Vertical composition of these cap and cup morphisms with endomorphisms (as in Remark 2.19) give so-called bubble endomorphisms of $I_a$. To record the images of these endomorphisms under $\Phi$, let us denote the alphabets associated to $I_a = I_{a,b}$ by $F$ with $|F| = a$ and $B$ with $|B| = b$. This is compatible with Convention 2.10, since in the case of no rungs we have $X_1 = F = X_1'$ and $X_2 = B = X_2'$.

In the case of a thin bubble (the $k = 1$ case), [QR16, (3.10) and (3.14)] imply that

$$\Phi\left(\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}\right) = h_r(B - F), \quad \Phi\left(\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}\right) = h_r(F - B)$$

Here the $\text{\textbullet}$ is a placeholder for a minimal decoration required to obtain a non-trivial evaluation (the precise value, which depends on the weight $a$, will not be relevant here). The values of thick bubbles ($k > 1$) are then

$$\Phi\left(\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}\right) = (-1)^{k(k-1)/2}s_\alpha(B - F), \quad \Phi\left(\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}\right) = (-1)^{k(k-1)/2}s_\alpha(F - B)$$

which can be deduced from (16), e.g. using [KLMS12, (4.33) and (4.34)] and the Jacobi-Trudi formula.

**Convention 2.21.** In the following, we will almost exclusively be interested in the images of 1- and 2-morphisms of $\hat{U}(\mathfrak{gl}_m)$ under $\Phi$, rather than the elements in the categorified quantum group itself. As such, we will omit $\Phi$ from our notation and use the notation in $\hat{U}(\mathfrak{gl}_m)$ (but with the identity 1-morphisms $I_a$ in $\hat{U}(\mathfrak{gl}_m)$ replaced by the identity 1-morphisms $1_a$ in SSBim) to denote the corresponding 1- and 2-morphisms in SSBim.

2.5. **Rickard complexes.** In this section, we recall the complexes of singular Soergel bimodules assigned to colored braids. To begin, fix a set of colors $S$, which will be $\mathbb{Z}_{\geq 1}$ in this paper. Let $Br_m$ denote the $m$-strand braid group, which acts on $S^m$ by permuting coordinates (this action factors through the symmetric group $S_m$).

**Definition 2.22.** The $S$-colored braid groupoid $\mathcal{B}r(S)$ is the category wherein objects are sequences $(a_1, \ldots, a_m)$ with $a_i \in S$, $m \geq 0$, and morphisms given by

$$\text{Hom}_{\mathcal{B}r(S)}(a, b) = \{ \beta \in Br_m \mid a_i = b_{\beta(i)} \text{ for } 1 \leq i \leq m \}$$

with $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$.

Morphisms in $\mathcal{B}r(S)$ are called colored braids, and elements in $\text{Hom}_{\mathcal{B}r(S)}(a, b)$ will be denoted by $\beta_{ab}$, or occasionally by $a\beta$ or $\beta a$ since the domain/codomain are determined by one another.
Given a braid \( \beta \in \text{Br}_m \), a \textit{strand} of \( \beta \) is a pair of indices \((i, j)\) \( \in \{1, \ldots, m\}^2 \) with \( i = \beta(j) \). In the
topological interpretation of \( \text{Br}_m \), a strand of \( \beta \) corresponds to a connected component. Denote
the set of strands of \( \beta \) by strands(\( \beta \)). A colored braid \( b \beta \sigma \) gives rise to a well-defined function
\[(18) \quad \varphi : \text{strands}(\beta) \to \mathbb{Z}_{\geq 1} \]
defined by declaring \( \varphi(s) = b_i = a_j \), where \( s \) is the strand \( s = (i, j) \) (with \( i = \beta(j) \)). Conversely, given
\( \beta \in \text{Br}_m \), we can associate to it a colored braid \( b \beta \sigma \) by specifying a function as in \((18)\).

The colored braid groupoid is generated by the colored Artin generators
\[
\sigma_i : (a_1, \ldots, a_i, a_{i+1}, \ldots, a_m) \to (a_1, \ldots, a_{i+1}, a_i, \ldots, a_m)
\]
which, when composable, satisfy relations analogous to the usual (type \(A\)) braid relations. A colored \textit{braid word}
is a sequence of colored Artin generators and their inverses. We say that a colored braid word \( (\beta)_a \) \textit{represents}
the corresponding product of colored Artin generators in \( \mathcal{Bt}(\mathbb{S}) \).

We now use the colored Artin generators to associate complexes \( C(\beta)_a \) in \( SSBim \) to colored braids
\( b \beta \sigma \). Strictly speaking \( C(\beta)_a \) depends on a choice of colored braid word \( \beta \) representing \( \beta \), but two
different choices are (canonically) homotopy equivalent; see Proposition 2.25 below. We often abuse
notation by writing:
\[
C(\beta)_a = 1_b C(\beta) 1_a = 1_b C(\beta) = C(\beta) 1_a .
\]
(Not that \( C(\beta) \) alone does not denote a well-defined complex.) We will define \( C(\beta)_a \) by first defining it
for the colored Artin generators \( \sigma^\pm_i \), and then extending to arbitrary braid words using horizontal
composition \( \ast \). In turn, to define \( C(\sigma^\pm_i)_a \) it suffices to consider the \( m = 2 \) case and extend to arbitrary
\( m \) using the external tensor product.

**Definition 2.23.** Let \( a, b \geq 0 \). The 2-strand \textit{Rickard complex} \( C_{a,b} \) is the (bounded) complex
\[
C_{a,b} := \left[ \begin{array}{c}
\vdots \\
\pi_a \rightarrow q^{-k} t^k a \leftrightarrow b \\
\pi_a \rightarrow q^{-k-1} t^{k+1} a \rightarrow b
\end{array} \right] := \left( \begin{array}{c}
\cdots \pi_a \rightarrow q^{-k} t^k a \leftrightarrow b \\
\pi_a \rightarrow q^{-k-1} t^{k+1} a \rightarrow b
\end{array} \right)
\]
of singular Soergel bimodules. The rightmost non-zero term is either \( q^{-b} t^k \mathcal{E}^{(a-b)} 1_{a,b} \) or \( q^{-a} t^k \mathcal{E}^{(b-a)} 1_{a,b} \)
via Convention 2.21) depending on whether \( a \geq b \) or \( a \leq b \), respectively. As a graded object, we identify
\( C_{a,b} = \bigoplus_{k=0}^{\min(a,b)} q^{-k} t^k C_{a,b} \), where \( C_{a,b} = \mathcal{E}^{(a-k)} \mathcal{E}^{(b-k)} 1_{a,b} \).

**Remark 2.24.** In some works, the complex in Definition 2.23 is used only in the case that \( a \geq b \), and
is instead replaced by an analogously defined complex
\[
\left( \begin{array}{c}
\cdots \pi_a \rightarrow q^{-k} t^k a \leftrightarrow b \\
\pi_a \rightarrow q^{-k-1} t^{k+1} a \rightarrow b
\end{array} \right)
\]
when \( a < b \). However, it follows e.g. from [KLMS12, Corollary 5.5] that these complexes are isomorphic
for all \( a, b \geq 0 \).

For \( \beta = \sigma_i \) and \( a = (a_1, \ldots, a_m) \), we then set
\[
C(\sigma_i)_a := 1_{(a_1, \ldots, a_i-1)} \boxtimes C_{a_i, a_{i+1}} \boxtimes 1_{(a_{i+2}, \ldots, a_m)}
\]
(19)
\[
C(\sigma^{-1}_i)_a := 1_{(a_1, \ldots, a_i-1)} \boxtimes C^\vee_{a_i, a_{i+1}} \boxtimes 1_{(a_{i+2}, \ldots, a_m)}
\]
where \( C_{a_i, a_{i+1}} \) is the 2-strand Rickard complex from Definition 2.23 and \( C^\vee_{a_i, a_{i+1}} \) is its inverse. The latter
is obtained from \( C_{a_i, a_{i+1}} \) by applying the contravariant duality functor \((-)^\vee : \text{Hom}_{\mathcal{R}(a_i, a_{i+1})}(\_., \mathcal{R}(a_i, a_{i+1})) \)
and is explicitly given by
\[
C^\vee_{a,b} := \left[ \begin{array}{c}
\vdots \\
\pi_a \rightarrow q^{k+1} t^{-k-1} a \leftrightarrow b \\
\pi_a \rightarrow q^{k} t^{-k} a \rightarrow b
\end{array} \right] := \left( \begin{array}{c}
\cdots \pi_a \rightarrow q^{k+1} t^{-k-1} a \leftrightarrow b \\
\pi_a \rightarrow q^{k} t^{-k} a \rightarrow b
\end{array} \right)
\]
The assignment \( (19) \) extends to arbitrary colored braid words using horizontal composition:
\[
(C(\sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}))_a := C(\sigma_{i_1}^{\varepsilon_1}) \ast \cdots \ast C(\sigma_{i_r}^{\varepsilon_r})_a
\]
for \( \varepsilon_1, \ldots, \varepsilon_r \in \{+1, -1\} \) and \( 1 \leq i_1, \ldots, i_r \leq m - 1 \).

**Proposition 2.25.** The complexes \( C(\sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r})_a \) satisfy the (colored) braid relations, up to canonical homotopy equivalence.

This is well-known in the uncolored case, i.e. when \( a \) has \( a_i = 1 \) for all \( 1 \leq i \leq m \); see e.g. [EK10].

**Proof.** The existence of such homotopy equivalences was conjectured in [MSV11] and proven in the geometric setting in [WW17]. In the singular Soergel bimodule setting, the braid relations follow from [CKL10] and Proposition 2.18. As in the uncolored case, these homotopy equivalences live in \( \text{SSBim} \), and canonicity amounts to a coherent choice of scaling. The latter can be obtained from the corresponding coherent scaling in the framework of \( \mathfrak{g}l_N \) foams for \( N \gg 0 \) that was constructed in [ETW18]. \( \square \)

**Convention 2.26.** If \( \beta = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r} \), then we call \( C(\beta)_a = C(\sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r})_a \) the Rickard complex assigned to the colored braid \( \beta \).

Rickard complexes of colored braids extend to invariants of braided webs (using horizontal composition and external tensor product), since they satisfy the following fork-slide and twist-zipper relations

**Proposition 2.27.** We have homotopy equivalences
\[
(C_{a, b}) := \begin{pmatrix}
\begin{array}{c}
\ldots \\
\overset{c}{a+b} \\
\overset{c}{a+b} \\
\end{array}
\end{pmatrix}
\overset{2}{\underset{1}{\rightarrow}}
\begin{pmatrix}
\begin{array}{c}
\ldots \\
\overset{c}{a+b} \\
\overset{c}{a+b} \\
\end{array}
\end{pmatrix}
\overset{2}{\underset{1}{\rightarrow}}
\begin{pmatrix}
\begin{array}{c}
\ldots \\
\overset{c}{a+b} \\
\overset{c}{a+b} \\
\end{array}
\end{pmatrix}
\]

as well as reflections thereof.

**Proof.** See [QR16, (4.3) and (4.16)] and [Cau12, Lemma 5.2]. \( \square \)

2.6. **Shifted Rickard complexes.** We now define the shifted Rickard complexes, which previously appeared in [Cau12, equations (12) and (13)] in the setting of the categorified quantum group \( \hat{\mathfrak{sl}}_2 \).

In passing to \( \text{SSBim} \), we show that these complexes possess a topological interpretation.

**Definition 2.28.** Fix integers \( a, b, c, d \) with \( a + b = c + d \), and consider the complex
\[
_{(c, d)}C_{(a, b)} := \bigoplus_{k \geq 0} q^{-(a-d+1)k} t^k \mathbb{E}(d-k) \mathbb{F}(b-k), \delta_C
\]
for
\[
\delta_C := \bigoplus_{k} \chi^+_a q^{-k(a-d+1)} t^k \mathbb{F}(d-k) \mathbb{E}(b-k) \rightarrow q^{-(k+1)(d-a+1)} t^k \mathbb{F}(d-k) \mathbb{E}(b-k-1).
\]

We refer to \( _{(c, d)}C_{(a, b)} \) as an \( \ell \)-shifted Rickard complex, where \( \ell = a - d = c - b \).
The right-most term in the complex \((c,d)C_{(a,b)}\) is either:

\[
q^{-b(a-d+1)}t^b c \phi_d b a \quad (\text{if } b \leq d), \quad \text{or} \quad q^{-d(a-d+1)}t^d c \phi_d b a \quad (\text{if } d \leq b).
\]

**Remark 2.29.** The usual Rickard complex is the unshifted case \((b,a)C_{(a,b)}\). In subsequent sections, we will be especially interested in the case \((a,b)C_{(a,b)}\).

Via Convention 2.10, there is an algebra homomorphism

\[
\text{Sym}(X_1|X_2|X_1') \to Z(\text{End}_{C_{(SSBim)}}((c,d)C_{(a,b)}))
\]

for all \(a, b, c, d \geq 0\). In the special case of the (unshifted) Rickard complex \(C_{a,b} = (b,a)C_{(a,b)}\), [RW16, Proposition 5.7] shows that, for any symmetric function \(f \in \Lambda\), \(f(X_2) \sim f(X_1')\). Equivalently, by Lemma 2.5, the action of \(h_{r+1}(X_2 - X_1')\) is null-homotopic for all \(r \geq 0\). We now generalize this fact to the shifted Rickard complexes.

**Lemma 2.30.** The action of \(h_{a-d+r+1}(X_2 - X_1')\) on the complex \((c,d)C_{(a,b)}\) is null-homotopic for all \(r \geq 0\). In particular, if \(a < d\) then \((c,d)C_{(a,b)} \simeq 0\).

**Proof.** Consider the homotopies \(\Theta_{r+1} \in \text{End}_{C_{(SSBim)}}((c,d)C_{(a,b)})\) that are given as the direct sum of the maps

\[
(-1)^{a-d+k} \chi_r^\ast : q^{-k(a-d+1)}t^k F^{(d-k)}E^{(b-k)} \to q^{(1-k)(a-d+1)}t^{k-1} F^{(d-k+1)}E^{(b-k+1)}.
\]

Note that \(\text{wt}(\Theta_{r+1}) = q^{2(a-d+r+1)}t^{-1}\). The component of \(\delta_C \Theta_{r+1}\) in \(t\)-degree \(k\) is

\[
(-1)^{a-d+k} \chi_0^+ \circ \chi_r^- + (-1)^{a-d+k+1} \chi_r^- \circ \chi_0^+ = \sum_{p+q+s = a-d+r+1} h_p(M)h_s(M)h_q(M').
\]

Here we have used (a reflection of) the “square flop” relation in [KLMS12, Lemma 4.6.4]. By (16), the bubble on the right-hand side above is equal to the endomorphism \(h_{a-d+r+1}(B - F)\); here we use Convention 2.10. The result now follows since this gives

\[
(\chi_0^+ \circ (-1)^{a-d+k} \chi_r^- + (-1)^{a-d+k+1} \chi_r^- \circ \chi_0^+)|_{F^{(d-k)}E^{(b-k)}} = \sum_{p+q+s = a-d+r+1} h_p(M)h_s(M)h_q(M')
\]

\[
= h_{a-d+r+1}((M + B) - (F - M'))
\]

\[
= h_{a-d+r+1}(X_2 - X_1').
\]

We now arrive at the topological interpretation of \((b,a)C_{(a,b)}\).

**Proposition 2.31.** For all integers \(a, b, c, d \geq 0\) with \(a + b = c + d\) we have a homotopy equivalence

\[
(c,d)C_{(a,b)} \simeq \left[ \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{array} \right].
\]

This remains valid even when \(a < d\), provided we interpret the right-hand side as zero.
Proof. If \(a < d\), then contractibility of \((c,d)C_{(a,b)}\) was established in Lemma 2.30. If \(a \geq d\), then using Reidemeister II, fork-sliding (21), and twist-zipper (22) moves, we have
\[
\begin{bmatrix}
 d & e \\
 c & a - d \\
\end{bmatrix} \simeq q^{-b(a-d)} \begin{bmatrix}
 d & e \\
 c & a - d \\
\end{bmatrix} = q^{-b(a-d)}E^{(a-d)} \ast C_{a,b}.
\]
The homotopy equivalence \(E^{(a-d)} \ast C_{a,b} \simeq q^{b(a-d)}(c,d)C_{(a,b)}\) is proved in Lemma 2.32 below.

**Lemma 2.32.** We have
\[
E^{(\ell)} \ast C_{a,b} \simeq q^{bd_{(b+\ell,a-\ell)}(c,d)C_{(a,b)}}
\]
for all integers \(a, b, \ell \geq 0\).

If \(a \leq b\) or \(a \geq b+\ell\), this follows from [Cau12, Proposition 4.5]. In our setting of SSBim (as opposed to the setting of an arbitrary integrable \(U(\mathfrak{sl}_2)\) representation from [Cau12]), the proof strategy of [Cau12, Proposition 4.5] carries over to give a uniform proof with no assumptions other than \(a, b, \ell \geq 0\). Note that exactly one (additional) step here (the observation that \(X_{-1} = 0\) below) uses that we are working in SSBim.

Proof. We proceed by induction on \(\ell\). The case \(\ell = 0\) case holds trivially. Thus, suppose we have established the result for some fixed \(\ell \geq 0\). Set \(c := b + \ell\) and \(d := a - \ell\), so \(\ell = a - d = c - b\). We begin by computing \(E \ast (c,d)C_{(a,b)}\) on the level of chain groups. Note that
\[
(c,d)C_{(a,b)} = \bigoplus_{k \geq 0} q^{-k(\ell+1)}t^k E^{(d-k)}E^{(b-k)}1_{a,b}
\]
where we interpret \(F^{(m)} = 0 = E^{(m)}\) when \(m < 0\). For \(k \geq 0\), we thus have
\[
q^{-k(\ell+1)}EF^{(d-k)}E^{(b-k)}1_{a,b} \simeq q^{-k(\ell+1)} \left( F^{(d-k)}E^{(b-k)}1_{a,b} \oplus [b - k + \ell + 1]F^{(d-k-1)}E^{(b-k)} \right)
\]
\[
\simeq q^{-k(\ell+1)} \left( [b - k + 1]F^{(d-k)}E^{(b-k+1)}1_{a,b} \oplus [b - k + 1 + \ell]F^{(d-k-1)}E^{(b-k)} \right)
\]
\[
\simeq X_{k-1} \oplus X_k \oplus Y_k
\]
where\(^5\) we set
\[
X_k := q^{-(k+1)(1+\ell)}[b - k]F^{(d-k-1)}E^{(b-k)}, \quad Y_k := q^{b-k(2+\ell)}[1+\ell]F^{(d-k-1)}E^{(b-k)}1_{a,b}.
\]
Note that \(X_{-1} = 0\) since \(F^{(b+1)}1_{a,b} = 0\). We thus have an isomorphism
\[
E \ast (c,d)C_{(a,b)} \simeq \bigoplus_{k \geq 0} t^k(X_{k-1} \oplus X_k \oplus Y_k) =: M
\]
for some differential \(\delta_M\) on \(M\).

Applying the \(p = 0, 1, 2\) cases of Corollary B.3, we find that the components \(\delta_M : X_{k-1} \oplus X_k \oplus Y_k \rightarrow X_k \oplus X_{k+1} \oplus Y_{k+1}\) take the form
\[
\begin{pmatrix}
 * & \varphi & * \\
 * & * & * \\
 0 & 0 & *
\end{pmatrix}
\]
where \(\varphi\) is upper triangular with multiples of the identity on the diagonal. The zeros in the bottom left tell us that
\[
\bigoplus_{k \geq 0} t^k(X_{k-1} \oplus X_k)
\]
is a subcomplex of \(M\), with differential \((\ast \varphi \ast)\). Moreover, an explicit computation (e.g. using the extended graphical calculus from [KLMS12]) shows that the diagonal entries of \(\varphi\) are non-zero, hence
\(^5\)Here we use the quantum integer identity \([b - k + 1 + \ell] = q^{b-k}[1+\ell] + q^{-(1+\ell)}[b - k].\)
\( \varphi \) is an isomorphism. Successive Gaussian elimination homotopies show that this subcomplex is contractible, hence

\[
\mathcal{E}^+_{(c,d)}C'_{(a,b)} \simeq \bigoplus_{k \geq 0} q^{b-k(2+\ell)}[\ell + 1]t^{k}[\ell + 1]d \mathcal{E}^{(b-k)}1_{a,b}
\]

for some differential. The “trick” used in the proof of [Can12, Proposition 4.5] now applies mutatis mutandis, showing that \( \mathcal{E}^+_{(c,d)}C'_{(a,b)} \) is homotopy equivalent to \([\ell + 1] \) copies of a complex of the form

\[
N := \bigoplus_{k \geq 0} q^{b-k(\ell+2)}t^k \mathcal{E}^{(d-k)}1_{a,b}
\]

for some differential.

Now, by induction, we have that \( (c,d)C_{(a,b)} \simeq q^{-\ell} \mathcal{E}^{(\ell)}C_{(a,b)} \), hence \( \mathcal{E}^+_{(c,d)}C'_{(a,b)} \simeq q^{-\ell}[\ell + 1] \mathcal{E}^{(\ell+1)}C_{(a,b)} \). Since \( \mathcal{E}^{(\ell+1)} \) is indecomposable and \( C_{(a,b)} \) is invertible, the complex \( \mathcal{E}^{(\ell+1)}C_{(a,b)} \) is indecomposable. The equivalence \( [\ell + 1] \mathcal{E}^{(\ell+1)}C_{(a,b)} \simeq q^{[\ell + 1]}N \) now implies that \( \mathcal{E}^{(\ell+1)}C_{(a,b)} \simeq q^{\ell}N \), so the latter is indecomposable. In particular, all differentials in \( N \) are non-zero. Corollary B.2 implies that the space of \((q\text{-degree zero})\) maps between consecutive terms in \( N \) is one-dimensional, thus \( N \simeq q^{h_{(c+1,d-1)}}C'_{(a,b)} \), which completes the proof.

We conclude this section by establishing a technical result that is needed below. It shows that Lemma 2.30 uniquely characterizes the Rickard complex \( C_{(a,b)} \) (and its inverse \( C'_{(a,b)} \)) amongst complexes having the same underlying bigraded bimodule.

**Proposition 2.33.** Let \( X := \bigoplus_k q^{-k}t^kC_{(a,b)} \). Suppose \( X \) is equipped with a differential \( \delta_X \) with respect to which \( \text{ad}_{t^2} \) is null-homotopic for some \( r \geq 0 \), then \( (X,\delta_X) \) is isomorphic to \( C_{(a,b)} \). The analogous statement for \( C'_{(a,b)} \) holds as well.

**Proof.** We only consider \( C_{(a,b)} \) and assume that \( a \geq b \), since the other cases are similar. Further, suppose that \( b > 0 \) since otherwise the result holds trivially. Proposition B.2 implies that

\[
\delta_X|_{C^k_{(a,b)}} = c_k \cdot \chi_0^+.
\]

for some scalars \( c_k \in \mathbb{Q} \), and that \( (X,\delta_X) \simeq C_{(a,b)} \) if and only if \( c_k \neq 0 \) for all \( 0 \leq k \leq b-1 \). Let \( f = h_{r+1} \) and observe that \( 0 \neq f(X_2 - X_1') \in Z(\text{End}_{C\text{-SSBim}}(X)) \). By hypothesis, there exists \( \eta \in \text{End}_{C\text{-SSBim}}(X) \) so that \( [\delta_X,\eta] = f(X_2 - X_1') \).

Now suppose that \( (X,\delta_X) \not\simeq C_{(a,b)} \), thus \( \delta_X|_{C^k_{(a,b)}} = 0 \) for some \( 0 \leq k \leq b-1 \). Since \( [\delta_X,\eta] = f(X_2 - X_1') \), this implies that \( f(X_2 - X_1')|_{C^k_{(a,b)}} = c_k - 1 \cdot \chi_0^+ \cdot \eta|_{C^k_{(a,b)}} \). The equality \( (\chi_0^+)^2 = 0 \) then implies that \( \chi_0^+ \circ f(X_2 - X_1') = 0 \) on \( C^k_{(a,b)} \). Since \( f(X_2 - X_1') \) is central, the composition

\[
\begin{array}{ccc}
C^k_{(a,b)} & \xrightarrow{a-b} & C^k_{(a,b)} \\
\downarrow & & \downarrow \\
C^k_{(a,b)} & \xrightarrow{\chi_0^+} & C^{k+1}_{(a,b)} \\
\downarrow & & \downarrow \\
C^k_{(a,b)} & \xrightarrow{b-k} & C^k_{(a,b)}
\end{array}
\]

is thus annihilated by \( f(X_2 - X_1') \) as well. On the other hand this composition equals

\[
(23) \quad (-1)^k \begin{array}{ccc}
\begin{array}{c}
\downarrow \\
\downarrow \end{array} & \xrightarrow{a-b} & \begin{array}{c}
\downarrow \\
\downarrow \end{array} \\
\begin{array}{c}
\downarrow \\
\downarrow \end{array} & \xrightarrow{a-b} & \begin{array}{c}
\downarrow \\
\downarrow \end{array}
\end{array} = (-1)^{b-1} \begin{array}{ccc}
\begin{array}{c}
\downarrow \end{array} & \xrightarrow{a-b} & \begin{array}{c}
\downarrow \end{array} \\
\begin{array}{c}
\downarrow \end{array} & \xrightarrow{a-b} & \begin{array}{c}
\downarrow \end{array}
\end{array} = (-1)^{b-1} \sum_{s+t+g = b-k} c_{s,t,g} b_{s-t} \cdot \chi_0^+ \cdot \eta|_{C^k_{(a,b)}}.
\]
By (17), the thick bubble evaluates to \((-1)^{(b-k)(b-k-1)/2} s_a (X'_2 - X'_1)\) thus
\[(23) = \pm \sum_{\alpha, \gamma} c_{(b-k)}^{(b-k)} s_\alpha (X'_2 - X'_1) s_\gamma (X'_2 - X'_1) = \pm s_{(b-k)(b-k-1)} (X'_2 - X'_1) \neq 0.\]
This endomorphism of \(C_{a,b}^b\) is annihilated by \(f(X'_2 - X'_1)\), contradicting the fact that \(\text{End}_{SS\text{Bim}}(C_{a,b}^b)\) contains no zero divisors. To see the latter, note that
\[
\begin{array}{c}
n
\end{array}
\]
is a quotient of its incoming and outgoing edge rings. Thus, \(C_{a,b}^b\) is a cyclic bimodule and its algebra of endomorphism is isomorphic to \(C_{a,b}^b \cong \text{Sym}(X_2 | M | X'_1)\), which has no zero divisors. \(\square\)

3. THE COLORED SKEIN RELATION

The colored skein relation (Theorem 3.4 below) asserts that there exists a one-sided twisted complex constructed from the complexes of “threaded digons”

\[
\begin{array}{c}
n
\end{array}
\]
for \(0 \leq s \leq b\) that is homotopy equivalent to a certain Koszul complex constructed from the complexes

\[
\begin{array}{c}
n
\end{array}
\]

This section is organized as follows. In §3.1, we develop just enough background to precisely state the colored skein relation and construct a filtration thereof. The subquotients with respect to this filtration will be denoted by \(\text{MCCS}^s_{a,b}\) for the duration. In §3.3, we show that

\[
\text{MCCS}^0_{a,b} \simeq \begin{array}{c}
n
\end{array}
\]

This equivalence proves (a version of) [BH21, Conjecture 1.3]. The proof of our colored skein relation is completed in §3.4; the main ingredient is an isomorphism

\[
\text{MCCS}^s_{a,b} \cong \begin{array}{c}
n
\end{array}
\]

3.1. Statement of the colored skein relation. For the duration, fix integers \(a, b \geq 0\) and let \(\mathcal{C}_{a,b} := \mathcal{C}_{(a,b,SS\text{Bim},a,b)}\). For \(X \in \mathcal{C}_{a,b}\), we will use the following conventions for the boundary alphabets

\[
\begin{array}{c}
n
\end{array}
\]

Note that we have an algebra homomorphism \(\text{Sym}(X_1 | X_2 | X'_1 | X'_2) \to Z(\text{End}_{\mathcal{C}_{a,b}}(X))\).

The “right-hand side” of our skein relation involves the construction of Koszul complexes, which we now recall.

Definition 3.1. For each \(X \in \mathcal{C}_{a,b}\), let \(K(X)\) denote the Koszul complex associated to the action of \(h_1(X_2 - X'_2), \ldots, h_b(X_2 - X'_2)\) on \(X\). Explicitly, we consider the bigraded \(\mathbb{Q}\)-vector space \(\wedge [\xi_1, \ldots, \xi_b]\) in which the \(\xi_i\) are exterior variables with \(\text{wt}(\xi_i) = q^i t^{-i}\) and define bimodules

\[
K(X) := \text{tw}_{\sum_{i=1}^b h_i (X_2 - X'_2) \otimes \xi_i} (X \otimes \wedge [\xi_1, \ldots, \xi_b]).
\]
Here, $\xi^*_i$ is the endomorphism (in fact, derivation) of $\wedge[\xi_1, \ldots, \xi_b]$ with $\text{wt}(\xi^*_i) = q^{-2i}t^1$ defined by
\[ \xi^*_i(\xi_j) = 1, \quad \xi^*_i(\eta) = \xi^*_i(\eta) + (-1)^{|\eta|}\xi^*_i(\nu). \]

**Remark 3.2.** Before turning on the Koszul differential we have
\[ X \otimes \wedge[\xi_1, \ldots, \xi_b] = \bigoplus_{l=0}^{b} \bigoplus_{i_1 < \cdots < i_l} X \otimes \xi_{i_1} \cdots \xi_{i_l}, \]
where each $X \otimes \xi_{i_1} \cdots \xi_{i_l}$ denotes a copy of $X$ (appropriately shifted). The usual Koszul sign conventions tell us that the differential on $X \otimes \xi_{i_1} \cdots \xi_{i_l}$ coincides with $\delta_X$ with no sign, since the monomial in $\xi$'s appears on the right.

**Lemma 3.3.** The assignment $X \mapsto K(X)$ is a dg functor.

**Proof.** This follows since we can describe $K(X) \cong X \otimes \text{Sym}(X_2 | X_2') \otimes \xi^*_i \left( \text{Sym}(X_2 | X_2') \otimes \wedge[\xi_1, \ldots, \xi_b] \right)$.

The following shorthand will often be useful.

**Definition 3.5.** We will use the following notation for the complexes appearing in (24)
\[ \text{MCCS}_{a,b} := \left[ \begin{array}{c} \begin{array}{c} b \cdots b \\ a \cdots a \end{array} \end{array} \right], \quad \text{MCS}_{a,b} := \left[ \begin{array}{c} b \cdots b \\ a \cdots a \end{array} \right] \]
(read as “Merge-Crossing-Crossing-Split” and “Merge-Crossing-Split”). Additionally, set $\text{KMCS}_{a,b} := K(\text{MCS}_{a,b})$.

Using Proposition 2.31, we can give a precise algebraic model for $\text{KMCS}_{a,b}$.

**Definition 3.6.** Set $\text{MCS}_{a,b} := (a,b)C_{(a,b)}$, i.e. diagrammatically:
\[ \text{MCS}_{a,b} := \left( \begin{array}{c} b \cdots b \\ a \cdots a \end{array} \right). \]

Let $\text{KMCS}_{a,b} := K(\text{MCS}_{a,b})$.

The $d = b$ case of Proposition 2.31 gives that
\[ \text{MCS}_{a,b} \cong \text{MCS}_{a,b}, \quad \text{KMCS}_{a,b} \cong \text{KMCS}_{a,b}, \]
where the second homotopy equivalence follows from the first by Lemma 3.3.

We now establish language for discussing $\text{KMCS}_{a,b}$ and its chain groups. Fix $a, b \geq 0$ and consider the bimodules
\[ W_k := \left[ \begin{array}{c} b \cdots b \\ a \cdots a \end{array} \right] = F^{(k)}E^{(k)}1_{a,b}. \]

\[ (26) \]
\[ \text{MCS}_{a,b} \cong \text{MCS}_{a,b}, \quad \text{KMCS}_{a,b} \cong \text{KMCS}_{a,b}, \]

where the second homotopy equivalence follows from the first by Lemma 3.3.
for $0 \leq k \leq b$. We follow Convention 2.10 in assigning alphabets to each of the edges in the web depicting these bimodules, namely:

![Diagram](image)

If we wish to emphasize the index $k$, we will write $M^{(k)}, M'^{(k)}, etc.$ In particular, we note that

$$|M^{(k)}| = k = |M'^{(k)}|, \quad |B^{(k)}| = b - k, \quad |F^{(k)}| = a + k$$

while

$$|X_1| = a = |X'_1|, \quad |X_2| = b = |X'_2|$$

for all $k$. The Koszul complex $\text{KMCS}_{a,b}$ can be efficiently described as follows.

**Proposition 3.7.** We have

$$\text{KMCS}_{a,b} = \left( K(W_b) \xrightarrow{\delta^H} a^{a+b+1} t K(W_{b-1}) \xrightarrow{\delta^H} \cdots \xrightarrow{\delta^H} a^{b(a-b+1)} t b K(W_0) \right),$$

where $\delta^H = K(\chi^H_0): K(W_k) \rightarrow K(W_{k-1}).$ 

The differential internal to each $K(W_k)$ will be denoted $\delta^c$, and referred to as the *vertical differential*. The differential $\delta^H$ will be referred to as the *total horizontal differential*. In §3.2 below, we introduce an additional “$s$-grading” on $K(\text{KMCS}_{a,b})$ and decompose $\delta^H$ further as $\delta^H = \delta^h + \delta^c$ where $\delta^h$ respects the $s$-grading and $\delta^c$ strictly increases it. These differentials $\delta^h$ and $\delta^c$ will be called the *horizontal differential* and the *connecting differential*, respectively.

### 3.2. The $\zeta$-filtration

We now aim to filter the complex $\text{KMCS}_{a,b}$ and explicitly identify the associated graded complex. To do so, we perform a change of basis within the exterior algebra tensor factor of each $K(W_k)$, i.e., we replace each column complex $W_k \otimes \wedge[\xi_1, \ldots, \xi_b]$ by an isomorphic Koszul complex.

**Definition 3.8.** Let $\zeta_1^{(k)}, \ldots, \zeta_b^{(k)}$ be odd variables given by the formula

$$\zeta_j^{(k)} := \sum_{i=1}^j (-1)^{i-1} e_{j-i}(M^{(k)}) \otimes \xi_i.$$

Given this, equation (10) implies that the formula

$$\xi_i = \sum_{j=1}^i (-1)^{j-1} h_{i-j}(M^{(k)}) \otimes \zeta_j^{(k)}$$

recovers the variables $\xi_i$ from the $\zeta_j^{(k)}$. We now wish to describe $\text{KMCS}_{a,b}$ in terms of the $\zeta$-basis.

**Lemma 3.9.** Consider the dg algebra $\text{Sym}(M[M']) \otimes \wedge[\xi_1, \ldots, \xi_b]$ with $\text{Sym}(M[M'])$-linear derivation defined by $d(\xi_i) = h_i(M - M')$ for all $1 \leq i \leq b$. The elements $\zeta_i := \sum_{j=1}^i (-1)^{j-1} e_{j-i}(M) \otimes \xi_i$ satisfy

$$d(\zeta_j) = e_j(M) - e_j(M').$$

**Proof.** This is an immediate consequence of Lemma 2.5. 

**Proposition 3.10.** We have that $K(W_k) \cong tw_{\delta^c}(W_k \otimes \wedge[\zeta_1^{(k)}, \ldots, \zeta_b^{(k)}])$ where

$$\delta^c = \sum_{i=1}^k (e_{ij}(M^{(k)}) - e_{ij}(M'^{(k)})) \otimes (\zeta_i^{(k)})^*.$$
With respect to this isomorphism, the differential $\delta^H: K(W_k) \to K(W_{k-1})$ has a nonzero component

$$W_k \otimes \xi_{i_1}^{(k)} \cdot \cdot \cdot \xi_{i_r}^{(k)} \xrightarrow{\delta^H} W_{k-1} \otimes \xi_{j_1}^{(k-1)} \cdot \cdot \cdot \xi_{j_r}^{(k-1)}$$

if and only if $i_p - j_p \in \{0, 1\}$ for all $1 \leq p \leq r$. In that case, it equals $\chi_m^+$ where $m = \sum_{p=1}^r (i_p - j_p)$.

**Proof.** The first statement is immediate from Lemma 3.9.

For the second, recall that the components of $\delta^H$ are described in the $\xi$-basis by

$$\chi_0^+ | W_k \otimes \text{id} = (-1)^{b-k} \otimes \text{id}: W_k \otimes \bigwedge [\xi_1, \ldots, \xi_b] \to W_{k-1} \otimes \bigwedge [\xi_1, \ldots, \xi_b].$$

We now compute these components under a basis change to monomials in the variables $\xi_i^{(k)}$ and $\xi_i^{(k-1)}$ in the domain and co-domain, respectively. In the domain, the requisite basis change is given by maps

$$W_k \otimes \xi_{i_1}^{(k)} \cdot \cdot \cdot \xi_{i_r}^{(k)} \to \bigoplus_{l_1, \ldots, l_r} W_k \otimes \xi_{l_1} \cdot \cdot \cdot \xi_{l_r}$$

with components

$$(-1)^{l_1 + \cdots + l_r - r} \prod_{p=1}^r e_{i_p - l_p} (\mathbb{M}^{(k)}).$$

Note that these are non-zero only if $1 \leq l_p \leq i_p$ for all $1 \leq p \leq r$. Next, each $W_k \otimes \xi_{l_1} \cdot \cdot \cdot \xi_{l_r}$ maps to $W_{k-1} \otimes \xi_{j_1} \cdot \cdot \cdot \xi_{j_r}$ via $\chi_0^+ \otimes \text{id}$. Finally, the basis change in the codomain is given by maps

$$W_{k-1} \otimes \xi_{j_1} \cdot \cdot \cdot \xi_{j_r} \to \bigoplus_{j_1, \ldots, j_r} W_{k-1} \otimes \xi_{j_1}^{(k-1)} \cdot \cdot \cdot \xi_{j_r}^{(k-1)}$$

with components

$$(-1)^{j_1 + \cdots + j_r - r} \prod_{p=1}^r h_{i_p - j_p} (\mathbb{M}^{(k-1)}).$$

As before, this is non-zero only if $1 \leq j_p \leq l_p$ for all $1 \leq p \leq r$. Thus, the component of $\delta^H$ from $W_k \otimes \xi_{i_1}^{(k)} \cdot \cdot \cdot \xi_{i_r}^{(k)}$ to $W_{k-1} \otimes \xi_{j_1}^{(k-1)} \cdot \cdot \cdot \xi_{j_r}^{(k-1)}$ is:

$$(-1)^{b-k} \sum_{l_1, \ldots, l_r} (-1)^{\sum_{p=1}^r l_p - j_p} \prod_{p=1}^r e_{i_p - l_p} = (-1)^{b-k} \begin{cases} \chi_m^+ & \text{if } i_p - j_p \in \{0, 1\} \text{ for all } 1 \leq p \leq r \\ 0 & \text{else} \end{cases}$$

where here $m = \sum_{p=1}^r (i_p - j_p)$. This gives the description of $\delta^H$ from the statement. \qed

The fact that $e_i(\mathbb{M}^{(k)}) - e_i(\mathbb{M}^{(k)})$ is zero when $i > k$ suggests that we should treat the variables $\xi_i^{(k)}$ differently according to whether $i \leq k$ or $i > k$. The following definition emphasizes this distinction.

**Definition 3.11.** Set $P_{k,l,s} := q^{k(1-a-b+1)-2b+2k} W_k \otimes \bigwedge [\xi_1^{(k)}, \ldots, \xi_k^{(k)}] \otimes \bigwedge [\xi_{k+1}^{(k)}, \ldots, \xi_h^{(k)}]$. 
Proposition 3.12. We have

\[ q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \cong tw_{\delta^v, \delta^h, \delta^c} \left( \bigoplus_{0 \leq l \leq b-s} P_{k,l,s} \right), \]

where \( \delta^v, \delta^h, \delta^c \) are pairwise anti-commuting differentials given as follows:

- the vertical differential \( \delta^v : P_{k,l,s} \to P_{k,l-1,s} \) is the direct sum of the Koszul differentials, up to sign \((-1)^k\); its component
  
  \[ W_k \otimes \zeta^{(k)}_{i_1} \cdots \zeta^{(k)}_{i_r} \xrightarrow{\delta^v} W_k \otimes \zeta^{(k)}_{i_1} \cdots \zeta^{(k)}_{i_r} \]

  is \((-1)^{-k+j-1}(e_{ij}(M^{(k)}) - e_{ij}(M'^{(k)}))\) if \(1 \leq i_j \leq k\) (and all other components are zero).

- the horizontal differential \( \delta^h \) and the connecting differential \( \delta^c \) are uniquely characterized by \( \delta^h + \delta^c = \delta^H \) from Proposition 3.10, together with

  \[ \delta^h(P_{k,l,s}) \subset P_{k-1,l,s}, \quad \delta^c(P_{k,l,s}) \subset P_{k-1,l-1,s+1}. \]

That is, \( \delta^h \) is the part of \( \delta^H \) which preserves the \( s \)-degree and \( \delta^c \) is the part of \( \delta^H \) which increases \( s \)-degree by 1.

Remark 3.13. Since each \( \zeta^{(k)}_i \) carries cohomological degree \(-1\), the object \( P_{k,l,s} \) contributes to the cohomological degree \( 2b - k - l - s \) part of \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \).

Proof of Proposition 3.12. By construction, the complex \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \) from Definition 3.5 is isomorphic to \( \bigoplus_{k,l,s} P_{k,l,s} \) with differential \( \delta^v + \delta^h \) as in Proposition 3.10. It is immediate from (28) that \( \delta^c \) maps \( P_{k,l,s} \) to \( P_{k,l-1,s} \). It follows from Definition 3.11 and the characterization of the non-zero components of \( \delta^H \) in Proposition 3.10 that \( \delta^H \) maps \( P_{k,l,s} \) to \( P_{k-1,l,s} \oplus P_{k-1,l-1,s+1} \). Hence \( \delta^h \) and \( \delta^c \) are well-defined.

The desired relations concerning \( \delta^v, \delta^h, \delta^c \) follow from taking components of \((\delta^v + \delta^h + \delta^c)^2 = 0\) under the trigrading \((l+s, k+s, -s)\). (This uses the fact that \( \delta^v, \delta^h, \) and \( \delta^c \) have tridegrees \((-1, 0, 0), (0, -1, 0), \) and \((0, 0, -1)\) with respect to this trigrading.)

An instructive example of the complex \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \) showing the three types of differentials is given in the following.

Example 3.14. We illustrate the complex \( q^{2t^2} \text{KMCS}_{2,2} \), as well as the subquotients \( P_{\bullet, \bullet,s} = q^{t^s} \text{MCS}_{2,2} \) for \( 0 \leq s \leq 2 \). We use the symbol \( \cdot \) instead of \( \otimes \) to declutter the diagram. We also suppress the homological shifts \( t^k \), which are determined by placing the underlined term in the top left
in homological degree zero (and noting that all arrows increase homological degree by one).

\[
P_{1,0} \xrightarrow{q^{-2}W_2 \cdot \zeta_1(2) \cdot 1} P_{0,1} \xrightarrow{q^{-3}W_1 \cdot \zeta_1(1) \cdot 1} P_{-1,1} \xrightarrow{q^{-4}W_0 \cdot 1 \cdot \zeta_2(0) \cdot 1} P_{-2,0}
\]

Black and blue horizontal arrows correspond to components of \( \delta^h \). All other black and blue arrows indicate non-zero components of \( \delta^v \). The connecting differential \( \delta^v \) is depicted by the grey horizontal arrows.

We may regard \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \) as filtered by \( s \)-degree, since the differentials \( \delta^v \) and \( \delta^h \) preserve \( s \)-degree, while \( \delta^v \) increases \( s \)-degree by one. The following gives names to the subquotients with respect to this filtration.

**Definition 3.15.** For each \( 0 \leq s \leq b \), let

\[
\text{MCCS}^s_{a,b} := q^{-s(b-1)} t^{-s} t_{W_0+s,b} \left( \bigoplus_{0 \leq k \leq b-s} P_{k,t,s} \right).
\]

Given this, the complex \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \) from Definition 3.5 can be described as the one-sided twisted complex

\[
q^{b(a-b-1)} t^b \text{KMCS}_{a,b} = \left( \text{MCCS}^0_{a,b} \xrightarrow{\delta^v} q^{b-1} t \text{MCCS}^1_{a,b} \xrightarrow{\delta^v} \ldots \xrightarrow{\delta^v} q^{b(b-1)} t^b \text{MCCS}^b_{a,b} \right)
\]

(30)

Our ultimate goal is to show that \( \text{MCCS}^s_{a,b} \) is homotopy equivalent to the complex \( \text{MCCS}^s_{a,b} \) from Definition 3.5. For this, we need one more technical result, namely that any partially symmetric function of the form \( f(\mathcal{X}_2) - f(\mathcal{X}_2') \) acts null-homotopically on \( q^{b(a-b-1)} t^b \text{KMCS}_{a,b} \) and its subquotients \( \text{MCCS}^s_{a,b} \).

**Definition 3.16.** For each \( r \in \{1, \ldots, b\} \), let \( \Theta_r \in \text{End}^{2r-1} (q^{b(a-b-1)} t^b \text{KMCS}_{a,b}) \) be given by

\[
\Theta_r := \bigoplus_{k=0}^b (-1)^{b-k} \text{id}_{W_k} \otimes \xi_r.
\]

Since \( \Theta_r : P_{k,t,s} \to P_{k,t+1,s} \oplus P_{k,t,s+1} \), we have the decomposition \( \Theta_r = \Theta_r^v + \Theta_r^c \), where \( \Theta_r^v \) and \( \Theta_r^c \) are uniquely characterized by
(1) $\Theta^v_r$ restricts to morphisms $P_{k,l,s} \to P_{k,l+1,s}$, and  
(2) $\Theta^c_r$ restricts to morphisms $P_{k,l,s} \to P_{k,l,s+1}$.

**Proposition 3.17.** The element $\Theta^v_r \in \text{End}^{2r-1}(\text{MCCS}^s_{a,b})$ satisfies $[\delta^v + \delta^h, \Theta^v_r] = h_r(X_2 - X'_2)$ for all $0 \leq s \leq b$.

**Proof.** Definition 3.1 and Definition 3.16 directly imply that $[\delta^v, \Theta^v_r + \Theta^c_r] = h_r(X_2 - X'_2)$ and $[\delta^h + \delta^c, \Theta^v_r + \Theta^c_r] = 0$. Taking the components that preserve $s$-degree gives $[\delta^v, \Theta^v_r] = h_r(X_2 - X'_2)$ and $[\delta^h, \Theta^v_r] = 0$. □

3.3. **The colored 2-strand full twist.** In this section, we prove that $\text{MCCS}^0_{a,b} \simeq \text{MCCS}^0_{a,b}$. This gives an explicit model for the Rickard complex of the $(a,b)$-colored 2-strand full twist. This result is of independent interest, but will also serve as an ingredient in proving that $\text{MCCS}^s_{a,b} \simeq \text{MCCS}^s_{a,b}$ for all $0 \leq s \leq b$ below in Corollary 3.29.

We visualize the main object of study $\text{MCCS}^0_{a,b} = \bigoplus_{0 \leq l \leq k \leq b} P_{k,l,0}$, with its two anti-commuting differentials $\delta^v, \delta^h$, as the following double complex

$$
\begin{array}{ccccccc}
\cdots & \delta^h & \to & P_{3,3,0} & \delta^v & \to & P_{3,2,0} \\
\cdots & \delta^h & \to & P_{3,2,0} & \delta^h & \to & P_{2,2,0} \\
& \delta^v & \downarrow & & \delta^v & \downarrow & \\
\cdots & \delta^h & \to & P_{3,1,0} & \delta^h & \to & P_{2,1,0} \\
& \delta^v & \downarrow & & \delta^v & \downarrow & \\
\cdots & \delta^h & \to & P_{3,0,0} & \delta^h & \to & P_{2,0,0} \\
& \delta^v & \downarrow & & \delta^v & \downarrow & \\
\cdots & \delta^h & \to & P_{1,1,0} & \delta^h & \to & P_{1,0,0} \\
& \delta^v & \downarrow & & \delta^v & \downarrow & \\
\cdots & \delta^h & \to & P_{0,0,0} & \delta^h & \to & P_{0,0,0} \\
\end{array}
$$

(31)

**Remark 3.18.** Up to grading shift, this double complex is isomorphic to the image of the categorical inverse ribbon element $r^{-1}1_{a-b}$ of quantum $\mathfrak{sl}_2$, as defined by Beliakova–Habiro [BH21], under the 2-functor $\Phi$ to singular Soergel bimodules. More precisely, the version of the double complex considered here has vertical differentials modeled on differences of elementary symmetric polynomials, corresponding to the version $\tilde{r}^{-1}1_{a-b}$ from [BH21, Section 11]. The original version $r^{-1}1_{a-b}$ of the inverse ribbon complex defined in [BH21, Section 4] uses differentials modeled on complete symmetric polynomials in a difference of alphabets and is closer to $\text{MCCS}^0$ expressed in terms of the exterior algebra generators $\xi_i$. Also note that the notions of horizontal and vertical differentials are interchanged between this paper and [BH21].

It will be convenient to give special notation to the rows of the double complex $\text{MCCS}^0$.

**Definition 3.19.** Let $R_l$ denote the complex $(\bigoplus_{k=l} P_{k,l,0}, \delta^h)$.

By construction, we have $\text{MCCS}^0 = \text{tw}_{\delta^h}(\bigoplus_{l=0}^b R_l)$. The key to proving $\text{MCCS}^0 \simeq \text{MCCS}^0$ is the following topological interpretation of the rows $R_l$. 
Proposition 3.20. For $0 \leq l \leq b$ we have

$$R_l \simeq q^{-(b-l)} t^{b-l} F(l) E(a-b+l) * C_{a,b} = q^{-(b-l)} t^{b-l} \left[ \begin{array}{c} b \\ a \end{array} \right].$$

The proof requires the following preparatory results.

Definition 3.21. For each pair of integers $r, s \geq 0$, let $P(r, s)$ denote the set of partitions $\alpha$ with $\alpha_1 \leq s$ and at most $r$ parts (i.e. the Young diagram for $\alpha$ fits in an $r \times s$ rectangle). For each $\alpha \in P(r, s)$, let $\hat{\alpha} \in P(s, r)$ denote the dual complementary partition. Let $\mathbb{Z}^P(r, s)$ denote the graded abelian group that is free on the partitions $\alpha \in P(r, s)$, graded by declaring that $\deg(\alpha) = 2|\alpha| - rs$.

Lemma 3.22 ([KLMS12, Theorem 5.1.1]). For $r, s \geq 0$, there is an isomorphism

$$F(s) F(r) \cong \bigoplus_{\alpha \in P(r, s)} q^{2|\alpha| - rs} F(r+s)$$

with components given by

$$(−1)^{|\hat{\alpha}|} \uparrow:\ F(s) F(r) \rightarrow q^{2|\alpha| - rs} F(r+s), \quad \downarrow:\ q^{2|\alpha| - rs} F(r+s) \rightarrow F(s) F(r).$$

Lemma 3.23. Fix $r, s \geq 0$ and let $\zeta_j$ for $1 \leq j \leq r + s$ be variables of degree $q^{2j} t^{-1}$. The bijections between the following:

1. the set $B(r, s)$ of binary sequences $\varepsilon \in \{0, 1\}^{r+s}$ with exactly $r$ 0’s in positions $i_1 < \cdots < i_r$ and $s$ 1’s in positions $j_1 < \cdots < j_s$,
2. the set of of non-zero monomial basis elements $\zeta_\varepsilon := \zeta_{i_1} \cdots \zeta_{i_r}$ in $q^{-s(r+s+1)} t^s \Lambda^s [\zeta_1, \ldots, \zeta_{r+s}]$,
3. the set of partitions $P(r, s)$

given by $\varepsilon \leftrightarrow \zeta_\varepsilon \leftrightarrow \alpha(\varepsilon)$ with $\alpha(\varepsilon)_m := \# \{ \varepsilon \in \{1, \ldots, s\} \mid \varepsilon j > j_m \}$ determine an isomorphism of (bi)graded abelian groups

$$(32) \quad \psi: q^{-s(r+s+1)} t^s \Lambda^s [\zeta_1, \ldots, \zeta_{r+s}] \xrightarrow{\cong} \mathbb{Z}^P(r, s), \quad \psi(\zeta_\varepsilon) := \alpha(\varepsilon)$$

Proof. The bijections are standard, thus clearly induce an isomorphism $\psi$ of abelian groups. To verify that $\psi$ preserves the bigrading, note that, prior to any shifts, the monomial $\zeta_\varepsilon$ is of degree $q^{2|\alpha(\varepsilon)| + s(s+1)} t^{-s}$ in $\Lambda^s [\zeta_1, \ldots, \zeta_{r+s}]$. To see this, observe that it holds for the sequence $1, \ldots, 1, 0, \ldots, 0$, and that if a sequence $\varepsilon$ is obtained from a sequence $\varepsilon'$ by replacing 1, 0 by 0, 1, then $2|\alpha(\varepsilon)| - 2|\alpha(\varepsilon')| = 2 = \deg_q(\zeta_\varepsilon) - \deg_q(\zeta_{\varepsilon'})$.

Proof of Proposition 3.20. Let $0 \leq l \leq b$. Lemma 2.32 implies that

$$(33) \quad q^{-(b-l)} t^{b-l} F(l) E(a-b+l) C_{a,b} \simeq \text{tw}(\delta_{h'}) \left( \bigoplus_{k=l}^{b} q^{k d - 2b + l} t^{2b - k - l} F(l) F(k-l) E(k) \right)$$

where $d = a - b + l + 1$ and

$$(\delta^h)' := \bigoplus_{i=0}^{b} (-1)^{b-k} \left[ \begin{array}{c} b \\ l \end{array} \right].$$
Using Lemmata 3.22 and 3.23, we deduce that

\[
\text{Right-hand side of (33) } \cong tw(\delta h)\cdot \left( \bigoplus_{k=0}^{b} \bigoplus_{\alpha \in P(k-l,l)} q^{kd-2b+l-\ell(k-l)+2|\alpha|} t^{2b-k-l} W_k \right) \\
\cong tw(\delta h)\cdot \left( \bigoplus_{k=1}^{b} q^{kd-2b+l} t^{2b-k-l} W_k \otimes \mathbb{Z} P(k-l,l) \right) \\
\cong tw(\delta h)\cdot \left( \bigoplus_{k=1}^{b} q^{k(a-b-1)-2b} t^{2b-k} W_k \otimes \Lambda^{[s_1^{(k)}, \ldots, s_l^{(k)}]} \right).
\]

We conclude that the latter chain complex has the same chain groups as the complex \((R_l, \delta h)\), so it suffices to equate their differentials.

The component of the differential \((\delta h)'\) in the first line of (34) from the \((k, \alpha)\) summand to the \((k-1, \gamma)\) summand is

\[
(-1)^{|\gamma|} \cdot (-1)^{b-k} \left( \sum_{\alpha/\gamma} \right)
\]

so we must show that, with respect to the isomorphism (32), we have \(\psi \circ \delta h = (\delta h)' \circ \psi\). (Recall that the differential \(\delta h\) on \(R_l\) was characterized in Proposition 3.10.) For this we use the following symmetric function identity:

\[
s_\alpha(\chi + z) = \sum_\lambda s_\lambda(\chi) z^{m_\lambda},
\]

where \(m_\lambda = |\alpha| - |\lambda|\) and the sum on the right is over all Young diagrams \(\lambda \subset \alpha\) for which the skew diagram \(\alpha/\lambda\) does not contain two boxes in the same column. Such a skew diagram is called a horizontal strip. Thus,

\[
(-1)^{|\gamma|} \cdot (-1)^{b-k} \sum_{\lambda} (-1)^{|\lambda|} \cdot s_\lambda(\chi) z^{m_\lambda},
\]

where the sum on the right is over partitions \(\lambda \in P(k-l-1,l)\) such that \(\alpha/\lambda\) is a horizontal strip. By Lemma 3.22, all terms in this sum vanish, unless \(\lambda = \gamma\). The latter holds precisely when \(\alpha/\gamma\) is a horizontal strip, in which case the only surviving term in the sum evaluates to \(\chi^{m_\lambda}\) with \(m := |\alpha| - |\gamma|\).

Now, suppose \(\varepsilon, \varepsilon' \in \{0,1\}^k\) are binary sequences with \(l\) occurrences of 1. Let \(j_1 < \cdots < j_l\) be the indices for which \(\varepsilon_{j_p} = 1\), and similarly for \(j_{l+1} < \cdots < j_{l'}\). Let \(\alpha(\varepsilon), \alpha(\varepsilon') \in P(k-l,l)\) be the associated partitions, then Lemma 3.23 implies this is a horizontal strip if and only if \(j_p - j'_p \in \{0,1\}\) for all \(p = 1, \ldots, l\). Indeed, the bijection therein gives that \(\alpha(\varepsilon') \subset \alpha(\varepsilon)\) if and only if \(\varepsilon'\) can be obtained from \(\varepsilon\) by a sequence of operations on binary sequences that replace the (adjacent) symbols 0, 1 with 1, 0. We hence can pass from \(\varepsilon\) to \(\varepsilon'\) by permuting the initial 1 in \(\varepsilon\) left through some 0’s to its position in \(\varepsilon'\), then do the same for the second 1 in \(\varepsilon\), and so on. If \(j_p - j'_p > 1\), then at the \(p^{th}\) step of this
procedure, we move a 1 past more than one 0, which produces two or more boxes in a column of \( \alpha(\varepsilon) \) that are not in \( \alpha(\varepsilon') \). Given this, the result now follows from Proposition 3.10. \( \square \)

**Theorem 3.24.** We have \( \text{MCCS}_{a,b}^0 \simeq \text{MCCS}_{a,b}^0 \). In alternative notation:

\[
\begin{align*}
\left[ \begin{array}{c}
\vcenter{\hbox{\includegraphics[width=1cm]{fig1}}} \\
\end{array} \right] & \simeq \text{tw}_{\delta^v + \delta^h} \left( \bigoplus_{0 \leq l \leq k \leq b} q^{k(a-b+1)-2b} t^{2b-k} \right) \left[ \begin{array}{c}
\vcenter{\hbox{\includegraphics[width=1cm]{fig2}}} \\
\end{array} \right] \\
& \otimes \wedge^p \left[ \zeta_1^{(k)}, \ldots, \zeta_k^{(k)} \right]
\end{align*}
\]

where the anticommuting differentials \( \delta^v \) and \( \delta^h \) are as described in Proposition 3.10.

This shows that [BH21, Conjecture 1.3] holds in the singular Soergel bimodule 2-representation of categorified quantum groups, and hence in any integrable quotient of \( \mathfrak{u}(\mathfrak{sl}_2) \). See Remark 3.18.

**Proof.** Recall that \( C_{a,b}^v \) denotes the inverse to the Rickard complex \( C_{a,b} \). Using Proposition 3.20, we compute

\[
\text{MCCS}^0 \ast C_{a,b}^v \cong \left( \bigoplus_{l=0}^b R_l \ast C_{a,b}^v, \delta^v \ast \text{id}_{C_{a,b}^v} \right) \cong \text{tw}_\delta \left( \bigoplus_{l=0}^b q^{-(b-l)} t^{b-l} E \right)
\]

for some differential \( \delta \). Note that this agrees with \( \text{MCCS}^0 \) as a graded bimodule.

Proposition 3.17 shows that the action of \( h_r(\mathcal{X}_2 - \mathcal{X}_1) \) on \( \text{MCCS}^0 \) is null-homotopic for all \( r > 0 \). Further, by Proposition 2.33, the action of \( h_r(\mathcal{X}_2 - \mathcal{X}_1) \) on \( C_{a,b}^v \) is null-homotopic for all \( r > 0 \). Together, these facts imply that the action of \( h_r(\mathcal{X}_2 - \mathcal{X}_1) \) on \( \text{MCCS}^0 \ast C_{a,b}^v \) is null-homotopic. Proposition 2.33 then implies that \( \text{MCCS}^0 \ast C_{a,b}^v \simeq C_{b,a} \), and thus \( \text{MCCS}^0 \simeq C_{b,a} \ast C_{a,b} = \text{MCCS}^0 \). \( \square \)

### 3.4. Proof of the colored skein relation

In this section, we prove Theorem 3.4. The key step is to show that \( \text{MCCS}_{a,b}^s \) is related to \( \text{MCCS}_{a,b}^0 \ast \text{MCCS}_{a,b}^0 \) in precisely the same way that \( \text{MCCS}_{a,b}^s \) is related to \( \text{MCCS}_{a,b}^0 \).

**Definition 3.25.** Let \( I^{(s)} : \mathcal{C}_{a,b} \rightarrow \mathcal{C}_{a,b+s} \) denote the functor defined by

\[
I^{(s)}(X) := \begin{array}{c}
\vcenter{\hbox{\includegraphics[width=1cm]{fig3}}} \\
\end{array}.
\]

In other words, \( I^{(s)}(X) = (1_a \boxtimes (\ell+s) M_{(t,s)}) \ast (X \boxtimes 1_a) \ast (1_a \boxtimes (\ell,s) S_{(t,s)}) \). We will write \( I := I^{(1)} \).

**Remark 3.26.** We have\(^7\) \( I^{(s)} \circ I^{(s)}(X) \cong \bigoplus \right\}_s \implies I^{(s)}(X) = \bigoplus \right\}_s \implies \). Thus \( I^{(s)} \) may be thought of as the \( s \)-th divided power of \( I \), in the same way that \( E^{(s)} \) and \( F^{(s)} \) are the divided powers of \( E \) and \( F \) in the setting of categorified quantum groups.

Theorem 3.4 will follow almost immediately from the following result.

**Proposition 3.27.** We have \( \text{MCCS}_{a,b}^s \cong I^{(s)}(\text{MCCS}_{a,b}^0) \).

This proposition requires careful bookkeeping, taken care of by the following.

**Lemma 3.28.** For each \( 0 \leq s \leq b \) and each \( 0 \leq k \leq b - s \), we have an isomorphism of weight \( q^{s(b+k+1)} t - s \):

\[
\mu_k : \begin{array}{c}
\vcenter{\hbox{\includegraphics[width=1cm]{fig4}}} \\
\end{array} \cong W_k \otimes \wedge^s [\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}].
\]

\(^7\)The isomorphism is given by applying the "associativity" relation for webs/bimodules, and then "removing the digon."
For each integer $m \geq 0$, these isomorphisms fit into a commutative diagram

\[
\begin{array}{cccc}
\overset{\mu_k}{\longrightarrow} & W_k \otimes \bigwedge^s [\psi_{k+1}^{(k)}, \ldots, \psi_b^{(k)}] \\
\overset{f}{\longrightarrow} & W_{k-1} \otimes \bigwedge^s [\psi_{k}^{(k-1)}, \ldots, \psi_b^{(k-1)}] \\
\end{array}
\]

where, for $k + 1 \leq i_1 < \cdots < i_s \leq b$ and $k + 1 \leq j_1 < \cdots < j_s \leq b$, the component

\[
W_k \otimes \psi_{i_1}^{(k)} \cdots \psi_{i_s}^{(k)} \to W_{k-1} \otimes \psi_{j_1}^{(k-1)} \cdots \psi_{j_s}^{(k-1)}
\]

is zero unless $i_p - j_p \in \{0, 1\}$ for all $k + 1 \leq p \leq b$. In this case, it equals $\chi_{m+n}$ where $n = \sum_p (i_p - j_p)$.

**Proof.** The isomorphism $\mu_k$ is defined to be the composition of

\[
\begin{array}{cccc}
\overset{\approx}{\longrightarrow} & \overset{\approx}{\longrightarrow} & \overset{\approx}{\longrightarrow} \\
\end{array}
\]

followed by the “digon removal” isomorphism described as follows. Let $S \subset \{k+1, \ldots, b\}$ with $|S| = s$, and set $S^c := \{k+1, \ldots, b\} \setminus S$. We may write $S = \{i_1 < \cdots < i_s\}$ and $S^c = \{j_1 < \cdots < j_{b-k-s}\}$. With this notation in place, define $\psi_S^{(k)} := \psi_{i_1}^{(k)} \cdots \psi_{i_s}^{(k)}$ and $\alpha(S)_{b-k-s-m+1} := \# \{ e \in \{1, \ldots, s\} | i_e < j_m \}$. Using this setup, and the alphabet labeling conventions for the digon:

\[
\begin{array}{ccc}
D & \overset{E}{\longrightarrow} & D' \\
\end{array}
\]

we have the isomorphism

\[
\begin{array}{ccc}
\overset{\Theta_S}{\longrightarrow} & \overset{\Theta_S}{\longrightarrow} & \overset{\Theta_S}{\longrightarrow} \\
\end{array}
\]

Here, the bimodule morphisms $\text{col}$ and $\text{cr}$ are given in Appendix A. Note that the correspondence between degree-$s$ monomials $\psi_S$ and partitions $\alpha(S) \in P(b-k-s, s)$ used here differs from the standard bijection\(^8\) from Lemma 3.23 by the symmetry $S \mapsto S^c$ that reverses the order of a binary sequence. Nonetheless, [QR16, Equations (3.10) and (3.11)] imply that (37) define inverse isomorphisms. The degree of the map $\mu_k$ obtained in this way can be deduced by comparing minimal degree summands.

Finally, the statement concerning the components of $f$ holds since the map $f := \mu_{k-1} \circ I^{(s)}(\chi_{m}^+) \circ \mu_k^{-1}$ can be simplified in a manner analogous to the computation that simplifies (35) in the proof of Proposition 3.20. (Alternatively, this can be computed explicitly using foams.)

**Proof of Proposition 3.27.** By definition,

\[
I^{(s)}(\text{MCCS}_{a,b}) = \left( \bigoplus_{0 \leq i \leq b-s} P_{k,i,s}^r (\delta^i)' + (\delta^h)' \right),
\]

\(^{8}\)Note also that the roles of $i$ and $j$ are opposite to Lemma 3.23; the indices $i_k$ here index the terms in the monomial $\psi_S$, thus correspond to 1’s in the corresponding binary sequence.
where

\begin{equation}
\label{eqn:skew}
P'_{k,l,s} := q^{k(a-b+s+1)-2(b-s)} t^{2(b-s)-k} \otimes \land^t [\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}].
\end{equation}

Here, \((\delta^v)^' = I^s(\delta^v)\) and \((\delta^h)^' = I^s(\delta^h)\) where \(\delta^v, \delta^h\) in this instance are the differentials on \(\text{MCCS}_{a,b,s}^0\) from Proposition \ref{prop:skew}. Moreover, recall from Definition \ref{def:skew} that

\[ \text{MCCS}^s_{a,b} = \left( \bigoplus_{0 \leq i \leq b-s} q^{-s(b-1)} t^{-s} P_{k,l,s}, \delta^v + \delta^h \right), \]

where

\[ q^{-s(b-1)} t^{-s} P_{k,l,s} = q^{k(a-b+s+1)-sb-2b+s} t^{s-2b} \otimes \land^s [\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}]. \]

Lemma \ref{lem:skew} implies that \(q^{-s(b-1)} t^{-s} P_{k,l,s}\) and \(P'_{k,l,s}\) are isomorphic. This isomorphism involves the natural isomorphism which swaps the order of tensor factors \(\land^s [\zeta_1^{(k)}, \ldots, \zeta_k^{(k)}] \otimes \land^s [\zeta_{k+1}^{(k)}, \ldots, \zeta_b^{(k)}] \cong \land^s [\zeta_1^{(k)}, \ldots, \zeta_b^{(k)}] \otimes \land^s [\zeta_{k+1}^{(k)}, \ldots, \zeta_b^{(k)}].\) By slight abuse of the notation from Lemma \ref{lem:skew}, we also denote this isomorphism by \(\mu_k : P'_{k,l,s} \to q^{-s(b-1)} t^{-s} P_{k,l,s}.\)

It remains to show that the isomorphisms \(\mu_k\) intertwine the differentials \(\delta^v, \delta^h\) with the differentials \((\delta^v)^', (\delta^h)^',\) i.e. that \(\mu_{k-1} \circ (\delta^h)^' \circ \mu_k = \delta^v \) and \(\mu_{k-1} \circ (\delta^v)^' \circ \mu_k = \delta^h.\) For the vertical differentials, this is immediate since these differentials are of Koszul type in both complexes, acting by differences of elementary symmetric polynomials on the \(k\)-labeled “rungs” of the web. Such endomorphisms commute with the digon removal isomorphism \((36).\)

To compare the horizontal differentials, we explicitly match the components of \(\mu_{k-1} \circ (\delta^h)^' \circ \mu_k^{-1}\) with those of \(\delta^h\) using Lemma \ref{lem:skew}. Suppose we have subsets

\( S = \{i_1 < \cdots < i_t\} \subset \{1, \ldots, k\}, \quad T = \{j_1 < \cdots < j_s\} \subset \{k+1, \ldots, b\} \)

\( S' = \{i'_1 < \cdots < i'_t\} \subset \{1, \ldots, k-1\}, \quad T' = \{j'_1 < \cdots < j'_s\} \subset \{k, \ldots, b\} \)

with \(i_p - i'_p \in \{0, 1\}\) and \(j_p - j'_p \in \{0, 1\}\) for all \(p\). The corresponding component

\[ W_k \otimes \chi^{(k)}_{ST} \xrightarrow{\delta^h} W_{k-1} \otimes \chi^{(k-1)}_{S'T'}, \]

is \(\chi^{+}_{m+n}\) where \(m = \sum p(i_p - i'_p)\) and \(n = \sum p(j_p - j'_p)\) (and all nonzero components of \(\delta^h\) are of this form). Now, Lemma \ref{lem:skew} gives us commutative squares

\[ \begin{array}{ccc}
W_k \otimes \chi^{(k)}_{ST} & \xrightarrow{(\delta^h)^' = I^s(\chi^+_m)} & W_{k-1} \otimes \chi^{(k-1)}_{S'T'} \\
\mu_k^{-1} & & \mu_k^{-1}
\end{array} \]

in which the vertical arrows are restrictions of \(\mu_k^{-1}\) and \(\mu_{k-1}^{-1}\) from Lemma \ref{lem:skew} to the indicated direct summands. Taking the direct sum over all such \(S, T, S', T'\) shows that the isomorphisms \(\mu_k\) intertwine the horizontal differentials.

\[ \square \]

\textbf{Corollary 3.29.} \(\text{MCCS}^s_{a,b} \simeq \text{MCCS}^s_{a,b}.\)
Proof. We have
\[
\text{MCCS}_{a,b}^s \cong I^s(\text{MCCS}_{a,b-s}^0) \cong I^s\left( \begin{array}{c} b \\ a \end{array} \right) \cong \begin{array}{c} b \\ a \end{array}
\]
where the first isomorphism holds by Proposition 3.27 and the homotopy equivalence follows from Theorem 3.24. □

Proof of Theorem 3.4. We have that
\[
q^{b(a-b-1)} \cdot K\left( \begin{array}{c} b \\ a \end{array} \right) \cong q^{b(a-b-1)} \cdot KMCS_{a,b} \cong q^{b(a-b-1)} \cdot KMCS_{a,b}
\]
\[
\cong \text{tw}_{\delta^c} \left( \bigoplus_{s=0}^b q s(b-1) \cdot \text{MCCS}_{a,b}^s \right)
\]
The latter is a one-sided twisted complex (see Definition 2.14) since \(\delta^c\) strictly increases the index \(s\). Corollary 3.29, together with standard homological perturbation techniques (see [Mar01, Crude Perturbation Lemma] or [Hog, Corollary 4.10]), gives us a homotopy equivalence
\[
\text{tw}_{\delta^c} \left( \bigoplus_{s=0}^b q s(b-1) \cdot \text{MCCS}_{a,b}^s \right) \cong \text{tw}_{D^c} \left( \bigoplus_{s=0}^b q s(b-1) \cdot \text{MCCS}_{a,b}^s \right)
\]
for some twist \(D^c\), which also strictly increases the index \(s\). □

Appendix A. Foams and singular Soergel bimodules

As is well-known in certain circles, the main results of [Web17] and [QR16] taken together imply that the \(k \to \infty\) limit of the monoidal 2-category of “enhanced \(\mathfrak{sl}_k\) foams” (i.e. \(\mathfrak{gl}_k\) foams) from [QR16] is equivalent to the monoidal 2-category of singular Bott-Samelson bimodules.

We record the bimodule morphisms corresponding to the (non-isomorphism) generating foams in [QR16, Definition 3.1]. Let \(\partial_i : \mathbb{Q}[x_1, \ldots, x_N] \to \mathbb{Q}[x_1, \ldots, x_N]\) be the \(i^{th}\) Demazure operator
\[
\partial_i(f) := f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)
\]
and let \(\partial_{a,b} : R^{a,b} \to R^{a+b}\) be the Sylvester operator
\[
\partial_{a,b} := (\partial_b \cdots \partial_1)(\partial_{a+1} \cdots \partial_2) \cdots (\partial_{a+b-1} \cdots \partial_a).
\]

We now record
\[
\text{un} := \begin{array}{c} a+b \\ a \end{array}, \quad \text{col} := \begin{array}{c} a+b \\ b \\ a \end{array}, \quad \text{zip} := \begin{array}{c} a+b \\ a \end{array}, \quad \text{cr} := \begin{array}{c} a+b \\ b \end{array}
\]
\[
\begin{array}{c} a+b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad 1
\]

\[
\begin{array}{c} a+b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad 1
\]

\[
\begin{array}{c} a+b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad 1
\]

\[
\begin{array}{c} a+b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad \begin{array}{c} b \\ a \end{array}, \quad 1
\]
Appendix B. Some Hom-space computations

In [KLMS12, Section 5.4], a basis is computed for certain Hom-spaces in the categorified quantum group $\hat{U}(\mathfrak{sl}_2)$. This implies the following result, by computing the degree of basis elements.

**Proposition B.1.** Let $x, y, p \in \mathbb{N}$ and suppose that $\lambda + y - x + p \geq 0$. Up to scalar multiple, there is a unique lowest degree 2-morphism in $\text{Hom}_{\hat{U}(\mathfrak{sl}_2)}(F^{x+p}E^{y+p}1_\lambda, F^xE^y1_\lambda)$ of degree $p(\lambda + y - x + p)$.

It is known, e.g. from [Web17, Theorem 9], that the 2-functor $\Phi: \hat{U}(\mathfrak{sl}_2) \to \text{SSBim}$ is full\(^9\) in lowest degree. Thus, Proposition B.1 has the following implications for Hom-spaces between singular Soergel bimodules.

**Corollary B.2.** Let $a, b, d \in \mathbb{N}$, then up to scalar
\[
\chi^+_0 \in \text{Hom}_{\text{SSBim}}(F^{(d-k)}E^{b-k}1_{a,b}, F^{(d-k-1)}E^{(b-k-1)}1_{a,b})
\]
is the unique map of lowest degree. (It has degree $a - d + 1$.)

**Corollary B.3.** Let $a, b, c, d, k, p \in \mathbb{N}$. Suppose that $k + p \leq \min(b, d - 1)$, then
\[
\text{Hom}(q^TE^{(d-k-1)}1_{a,b}, q^TE^{(d-k-p-1)}E^{(b-k-p)}1_{a,b}) \cong \begin{cases} 
\mathbb{Q} & \text{if } r - s = p(a - d + p + 1) \\
0 & \text{if } r - s < p(a - d + p + 1).
\end{cases}
\]

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\(^9\)In fact, the failure of this 2-functor to be full is due to the fact that $\text{End}_{\hat{U}(\mathfrak{sl}_2)}(1_\lambda) \cong \Lambda \xrightarrow{\Phi} \text{Sym}(B - F)$. See e.g. (16). It becomes full after extending scalars in $\text{End}_{\hat{U}(\mathfrak{sl}_2)}(1_\lambda) \cong \Lambda \otimes \Lambda$. 

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