LOGARITHMIC MODULI SPACES FOR SURFACES OF
CLASS VII

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Abstract. In this paper we describe logarithmic moduli spaces of pairs
(S, D) consisting of a minimal surface S of class VII with second Betti
number $b_2 > 0$ together with a reduced maximal divisor D of $b_2$ rational
curves. The special case of Enoki surfaces has already been considered
by Dloussky and Kohler. We use normal forms for the action of the fun-
damental group of $S \setminus D$ and for the associated holomorphic contraction
$(\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

1. Introduction

Compact complex surfaces with first Betti number $b_1 = 1$ form the class VII
in Kodaira’s classification, see [11]. Minimal such surfaces are said to belong
to class VII$_0$. When their second Betti number vanishes these surfaces have
been completely classified, see [9], [1], [12] and [15]. Among them one finds
the Hopf surfaces for which moduli spaces have been constructed in [2].

In this paper we shall restrict our attention to minimal compact complex
surfaces with $b_1 = 1$ and $b_2 > 0$. All known surfaces in this subclass contain
global spherical shells and can be obtained by a construction due to Ma.
Kato, see [10]. A finer subdivision of these surfaces may be done by looking
at the dual graph of the maximal reduced divisor $D$ of rational curves. One
gets:

1. Enoki surfaces, i.e. the graph of $D$ is a cycle and $D$ is homologically
trivial,

2. Inoue-Hirzebruch surfaces, i.e. the graph of $D$ consists of one or two
cycles and $D$ is not homologically trivial, (in fact $D$ is an exceptional
divisor),

3. intermediate surfaces, i.e. the graph consists of a cycle with at least
one non-empty tree appended, (in this case too, $D$ is exceptional).

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surfaces of class VII.
Our point of interest is to study logarithmic moduli spaces, i.e. such that the maximal reduced divisor \(D\) is preserved. Logarithmic moduli spaces of Enoki surfaces were described in [4], whereas Inoue-Hirzebruch surfaces are logarithmically rigid.

In this paper we introduce logarithmic moduli spaces for intermediate surfaces. This will be done by examining normal forms of germs of analytic mappings which can be associated to these surfaces and of the fundamental group of the complement of the maximal reduced divisor, respectively.

The article is organized as follows. General facts on surfaces with global spherical shells are recalled in Section 2. Then we give a description of the fundamental group of the complement of the rational curves of an intermediate surface. This description will allow us a better understanding of the isomorphisms between the different parameter spaces we will get for logarithmic deformations. In Section 4 we introduce the main ingredients of our proof: the normal form of a contracting germ associated to an intermediate surface and its type. The next two sections are devoted to the decomposition of such normal forms and to the associated blowing up sequences which are needed in order to recover the surface from its contracting germ in a canonical way. Using these techniques we come in Section 7 to the description of logarithmically versal families and moduli spaces of intermediate surfaces, see Theorems 7.13 and 7.14. In the Appendix we show how the universal covers of Section 3 are organized into families. Theorem 8.16 stresses again the importance of the logarithmic type.

2. Preliminary facts on surfaces with global spherical shells

In this section we recall some facts on surfaces with global spherical shells. We refer to [10] and [3] for more details.

By a surface we always mean a compact connected complex manifold of dimension 2. A global spherical shell (GSS) in a surface \(S\) is the image \(\Sigma\) of \(S^3\) through a holomorphic imbedding of an open neighbourhood of \(S^3 \subset \mathbb{C}^2 \setminus \{0\}\) such that \(S \setminus \Sigma\) is connected.

A minimal surface \(S\) with \(b_2 := b_2(S) > 0\) admits a GSS if it can be obtained by the following method: Blow up the origin of the unit ball \(B\) in \(\mathbb{C}^2\) and choose a point \(p_1\) on the exceptional curve \(C_1\). Continue by blowing up this point and by choosing a further point \(p_2\) on the new exceptional \((-1)\)-curve \(C_2\). Repeat this process \(b_2\) times, i.e. until you reach the curve \(C_{b_2}\) and choose a point \(p_{b_2}\) on this curve. Call \(\hat{B}\) the manifold thus obtained and \(\pi: \hat{B} \to B\) the blowing down map. Choose now a biholomorphic map \(\sigma: \hat{B} \to \sigma(\hat{B})\) onto a compact neighbourhood of \(p_{b_2}\). Use finally \(\sigma \circ \pi\) to glue together the two components of the boundary of \(\hat{B} \setminus \sigma(B)\). It is not difficult to see that in this way one gets a minimal surface \(S\) with \(\pi_1(S) \simeq \mathbb{Z}\)
There are exactly $b_2$ rational curves on $S$ which form a divisor the dual graph of which has one of the three types described in the introduction. Besides these rational curves at most one further curve might appear on some Enoki surfaces and this curve will be elliptic. One can see a GSS as the image of $\sigma(S^3)$ in $S$ via the above identification. The universal covering $\tilde{S}$ of $S$ is obtained by gluing an infinite number, indexed by $\mathbb{Z}$, of copies of $\tilde{B} \setminus \sigma(B)$ through maps analogous to $\sigma \circ \pi$.

Conversely, let $S$ be a minimal surface with a fixed GSS $\Sigma$. One can fill the pseudoconcave end of $S \setminus \Sigma$ holomorphically by $\overline{B}$ and one obtains a holomorphically convex complex manifold $M$. The maximal compact complex analytic set in this manifold is exceptional and the blow-down of $M$ is isomorphic to $B$. One shows that this is the inverse of the above construction.

In [3], Dloussky remarked an important object associated to this construction, which is the germ $\phi$ of the holomorphic mapping $\pi \circ \sigma : B \to B$ around the origin of $B$. It is a germ of a contracting mapping and it is shown to completely determine the surface $S$ up to isomorphism. Conjugating $\phi$ by an automorphisms of $(B, 0)$ does not change the isomorphy class of $S$, but to the same surface $S$ several conjugacy classes of germs may correspond. In fact the conjugacy class of the contracting germ depends on the homotopy class of the GSS $\Sigma \subset S$.

In [7], Favre gave polynomial normal forms for the contracting germs and classified them up to biholomorphic conjugacy. In order to obtain logarithmic moduli spaces, we shall choose one of these normal forms and describe it more closely as well as its relation to the maximal reduced divisor $D$ of rational curves on the surface. A choice of a normal form will be suggested by an examination of the fundamental group of $S \setminus D$. (Since $D$ is contractible and does not contain $(-1)$-curves, $S$ is completely determined by $S \setminus D$.)

3. The fundamental group of $S \setminus D$

For a surface $S$ admitting a GSS, one finds using the exponential sequences

$$Pic^0(S) \simeq H^1(S, \mathbb{C}^*) \simeq \mathbb{C}^* \simeq Hom(\pi_1(S), \mathbb{C}^*).$$

Therefore for each $\lambda \in \mathbb{C}^*$ there is a unique associated flat line bundle which we denote by $L_\lambda$. Note that the above isomorphisms depend on the choice of a positive generator of $\pi_1(S) \simeq \mathbb{Z}$. We recall also that for the intermediate surface $S$ there exists a positive integer $m$, a flat line bundle $L$ and an effective divisor $D_m$ such that $(K_S \otimes L)^{\otimes m} = \mathcal{O}_S(-D_m)$, cf. [6], Lemma 1.1. The smallest possible $m$ in the above formula is called the index of the surface $S$ and is denoted by $m(S)$. By Proposition 1.3 of [6], for each intermediate surface $S$ of index $m$ there is a unique intermediate surface $S'$.
together with a proper map $S' \to S$ which is generically finite of degree $m$ such that $S'$ is of index 1. Moreover, on the complements of the maximal reduced divisors $D'$ and $D$ the induced mapping $S' \setminus D' \to S \setminus D$ is a cyclic unramified covering of degree $m$.

3.1. Surfaces of index 1. In this subsection we assume that $S$ is an intermediate surface of index 1. Later on we will show how to recover normal forms of contracting maps for surfaces of higher index. In [5] and [6] normal forms for the generators of the fundamental group of $S \setminus D$ were given. We will now recall and further develop these computations.

It is shown loc. cit. that the universal cover $\tilde{S} \setminus D$ of $S \setminus D$ is isomorphic to $\mathbb{C} \times \mathbb{H}_l$, where $\mathbb{H}_l := \{ w \in \mathbb{C} \mid \Re(w) < 0 \}$ is the left half plane. We take $(z, w)$ to be a system of holomorphic coordinates on $\mathbb{C} \times \mathbb{H}_l$. The fundamental group of $S \setminus D$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}[1/k]$ and is generated by the following two automorphisms of $\mathbb{C} \times \mathbb{H}_l$:

\[
\begin{align*}
(FG1) \quad g_\gamma(z, w) &= (z, w + 2\pi i) \\
g(z, w) &= (\lambda z + a_0 + H(e^{-w}), kw),
\end{align*}
\]

where $\lambda \in \mathbb{C}^*$, $H = H(\zeta) = \sum_{m=1}^{s} a_m \zeta^m$ is a complex polynomial of positive degree and $k := 1 + \sqrt{|\det M(S)|} \in \mathbb{Z}, k \geq 2$. Here $M(S)$ denotes the intersection matrix of the $b_2(S)$ rational curves of $S$. Moreover the complex parameters $\lambda, a_i$ are further subject to the following conditions:

$$(\lambda - 1)a_0 = 0 \text{ and } a_m = 0 \text{ for all } m > 0 \text{ with } k|m,$$

see Theorem 3.1 in [5] and Theorems 4.1 and 4.2 in [6]. The case $\lambda = 1$ occurs precisely when $S$ admits a nontrivial holomorphic vector field. Note also that the generator $g_\gamma$ corresponds to some loop $\gamma$ around $D$, whereas $g$ induces a generator of $\pi_1(S)$. It is claimed in [5] and [6] that this normal form is unique if one preserves the generator $g_\gamma$. We shall see later that this claim is valid only up to the action of some finite group on the space of parameters given by the coefficients of $H$.

We first show how the normal form changes if one allows a different choice of the loop $\gamma$.

We start by replacing the normal form (FG1) as in [5], p. 659 and [6], p. 307, in order to obtain a new normal form which will lead to the contracting germ:

\[
(FG2) \quad \begin{align*}
g_\gamma(z, w) &= (z, w + 2\pi i) \\
g(z, w) &= (\lambda z + a_0 + Q(e^{-w}), kw),
\end{align*}
\]

where $Q = Q(\zeta) := \sum_{m=1}^{s} b_m \zeta^m$ is a new polynomial such that $b_\sigma = 1, k \nmid \sigma, l := [\sigma/k] + 1$ and $\gcd\{k, m \mid b_m \neq 0\} = 1$. As before $(\lambda - 1)a_0 = 0$. The normalisation $b_\sigma = 1$ is obtained by a conjugation $(z, w) \mapsto (\alpha z, w)$ with a suitable $\alpha$. 
At the end of this process the loop has to go back by the inverse conjugation procedure to the initial (FG2).

If \( k \) is fixed and let \( Q \) be a polynomial as in (FG2). Define inductively the following finite sequences of integers \( \sigma := n_1 > ... > n_t \geq 1 \) and \( k > j_1 > j_2 > ... > j_t = 1 \) by:

i) \( n_1 := \sigma, j_1 := \gcd(k, n_1) \);

ii) \( n_\alpha := \max\{i < n_{\alpha-1} \mid b_i \neq 0, \gcd(j_{\alpha-1}, i) < j_{\alpha-1}\}, j_\alpha := \gcd(k, n_1, ..., n_\alpha) = \gcd(j_{\alpha-1}, n_\alpha) \);

iii) \( 1 = j_t := \gcd(k, n_1, ..., n_{t-1}, n_t) < \gcd(k, n_1, ..., n_{t-1}). \)

We call \((n_1, ..., n_t)\) the type of (FG2) and \( t \) the length of the type. If \( t = 1 \) we say that (FG2) is of simple type.

We remark now that for any divisor \( d > 1 \) of \( k \) the following are also generators of \( \pi_1(S \setminus D) \subset \text{Aut}_\mathcal{O}(\mathbb{C} \times \mathbb{H}) \):

\[
\begin{align*}
(fg3) & \quad \begin{cases} g^d(z, w) = (z, w + 2\pi id) \\
g(z, w) = (\lambda z + a_0 + Q(e^{-w}), kw),
\end{cases}
\end{align*}
\]

since \((g^{-1} \circ g^d \circ g)^\frac{k}{2} = g\).

In order to reestablish (FG2) we perform the following conjugations on (FG3). First conjugate with \( (z, w) \mapsto (z, dw) \). The generators become

\[
\begin{align*}
(z, w) & \mapsto (z, w + 2\pi i) \\
(z, w) & \mapsto (\lambda z + a_0 + Q(e^{-dw}), kw),
\end{align*}
\]

This is not in normal form (FG2); in fact there are two cases. If \( k \nmid d\sigma \) one has to go back by the inverse conjugation procedure to the initial (FG2).

At the end of this process the loop \( \gamma \) remains unchanged.

If \( k \mid d\sigma \) take \( p := \max\{i \in \mathbb{N} \mid i > 1, k \mid j_{i-1}d\} \). Now let

\[
\Phi_1(z, w) = (z + \sum_{m=\sigma+1} b_m e^{-\frac{mdw}{k}}, w)
\]

and apply the conjugation \( \Phi_1 \circ (1) \circ \Phi_1^{-1} \). We obtain the new generators

\[
\begin{align*}
(z, w) & \mapsto (z, w + 2\pi i) \\
(z, w) & \mapsto (\lambda z + a_0 + b_n\hat{Q}_d(e^{-w}), kw)
\end{align*}
\]

where

\[
\hat{Q}_d(\zeta) := b_n^{-1}\left(Q(\zeta^d) + \lambda \sum_{m=\sigma+1} b_m\zeta^{\frac{md}{k}} - \sum_{m=\sigma+1} b_m\zeta^{md}\right).
\]

The conjugation \( \Phi_2 \circ (2) \circ \Phi_2^{-1} \) by \( \Phi_2(z, w) = (b_nz, w) \) leads us to

\[
\begin{align*}
(z, w) & \mapsto (z, w + 2\pi i) \\
(z, w) & \mapsto (\lambda z + b_n^{-1}a_0 + \hat{Q}_d(e^{-w}), kw).
\end{align*}
\]

The idea is to bring (3) in normal form (FG2) presumably with the new type \((n_1, ..., n_t) = (dn_1, ..., dn_t, \frac{dn_1}{k}, ..., \frac{dn_t}{k}) \). We have \( \gcd(dn, k) = \)

We introduce now a numerical invariant of (FG2) which will be shown to be equivalent to the dual graph of \( D \), see Section 4.
\gcd(d_{n_p}, d_{j_{p-1}}, k)$ because \( \gcd(d_{j_p-1}, k) = k \) and further \( \tilde{j}_1 = \gcd(d_{n_p}, k) = \gcd(d_{n_p}, d_{j_p-1}, k) = \gcd(d_{j_p}, k) < k \). In the same way one obtains \( \gcd(d_{m}, k) = \gcd(d_{j_p}, k) = \tilde{j}_1 \) for \( n_{p+1} < m < n_p \), \( \tilde{j}_2 = \gcd(d_{n_{p+1}}, d_{n_p}, k) = \gcd(d_{n_{p+1}}, d_{j_p}, k) \) and so on. The last in this series is \( \gcd(d_{m}, k) = \gcd(d_{j_1}, k) = d \). The translated exponents \( \frac{d_{n_p}}{k}, ..., \frac{d_{n_{p-1}}}{k} \) now give

\[ \tilde{j}_{t-p+2} = \gcd(d, \tilde{n}_{t-p}) = \gcd(\frac{d_{n_1}}{k}, d, k) = \gcd(\frac{d_{n_1}}{k}, \frac{d_{j_1}}{k}, d) = \frac{d_{j_1}}{k} < d \]

because \( k \nmid n_1 \),

\[ \tilde{j}_{t-p+3} = \gcd(\frac{d_{n_2}}{k}, \frac{d_{n_1}}{k}, k) = \gcd(\frac{d_{n_2}}{k}, \frac{d_{j_1}}{k}, d) = \frac{d_{j_2}}{k}, \]

and so on, down to \( \tilde{j}_t = \frac{d_{j_{p-1}}}{k} \). Set \( d' := \frac{d_{j_{p-1}}}{k} \). If \( d' = 1 \), then \( \tilde{Q}_d \) is in normal form and we are ready. If not all its ”active” exponents \( m \), i.e. with \( b_m \neq 0 \), are divisible by \( d' \). Then we perform the same conjugations which lead us from (FG3) to (1) in inverse order and with \( d' \) instead of \( d \). We get

\[ \left\{ \begin{array}{c} (z, w) \mapsto (z, w + 2\pi i) \\ (z, w) \mapsto (\lambda z + b_{m}^{-1}a_0 + \tilde{Q}_d(e^{-\frac{w}{m}}, kw)) \end{array} \right. \]

The type of this normal form is

\[ \frac{1}{d'}(d_{n_p}, ..., d_{n_t}, \frac{d_{n_1}}{k}, ..., \frac{d_{n_{p-1}}}{k}). \]

It is clear that we get exactly \( t \) (FG2)-normal forms through this kind of transformation corresponding to the following choices for \( d' = \frac{k}{n_1}, ..., \frac{k}{n_p} \), the last one giving our starting type again. In these cases we also have \( d' = 1 \), hence the type transformations are of the form:

\[ (n_1, ..., n_t) \mapsto \left( \frac{k n_{p+1}}{j_0}, ..., \frac{k n_t}{j_p}, \frac{n_1}{j_0}, ..., \frac{n_p}{j_p} \right). \]

This amounts to an action of \( \mathbb{Z}/(t) \) on the set of types of length \( t \).

Notice moreover that we also get a biholomorphic polynomial map \( \tau_d \) between the parameter spaces for the coefficients of \( Q \) in (FG2) and those of \( \tilde{Q}_d \) in (4).

### 3.2. Surfaces of higher index.

We consider now the normal form (FG2) for a surface \( S \) of index 1 and a positive integer \( q \). We say that the set of parameters \( a_0, b_1, ..., b_{\sigma} \) of the normal form has the property \( (I_q) \) if

\[ a_0 = 0 \]

and

\[ q | \gcd(k - 1; m - m', b_m b_{m'} \neq 0) \].
Suppose now that this is the case and conjugate $g_{\gamma}$ and $g$ by 
\[ \varphi(z, w) := (e^{\frac{2\pi i}{q}} z, w - \frac{2\pi i}{q}). \]
We get
\[ \varphi \circ g_{\gamma} \circ \varphi^{-1} = g_{\gamma}, \]
\[ \varphi \circ g \circ \varphi^{-1} = g^{k-1} \circ g, \]
showing that $\varphi$ lies in the normalizer of the fundamental group $\pi_1(S \setminus D)$. This leads to an action of $\mathbb{Z}/(q)$ on $S \setminus D$ and to a quotient $S' \setminus D'$, where $S'$ is a surface of possibly higher index. We will see later that every surface of index $q$ arises in this way, see Remark 4.4. Looking now at a divisor $d$ of $k$ we notice that the transformation $Q : \tau d \mapsto \tilde{Q} d$ of the preceding subsection preserves the property $(I_q)$. This happens because $d$ and $q$ are relatively prime. Our next step will be to translate these properties for the associated contracting germs of holomorphic maps.

4. Contracting germs of holomorphic mappings associated to intermediate surfaces

In [7] Favre gave normal forms for contracting rigid germs of holomorphic maps $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and completely characterized those germs which give rise to surfaces with global spherical shells. We retain the following result which will be essential for our study.

\textbf{Theorem 4.2 (Favre [7], see also [8]).} Every intermediate surface is associated to a polynomial germ in the following normal form:

\[ (CG) \quad \varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^{\frac{k-1}{k}} \zeta^k), \]
where $k, s \in \mathbb{Z}$, $k > 1$, $s > 0$, $\lambda \in \mathbb{C}^*$, $P(\zeta) := c_j \zeta^j + c_{j+1} \zeta^{j+1} + \ldots + c_s \zeta^s$ is a complex polynomial, $0 < j < k$, $j \leq s$, $c_j = 1$, $c_{\frac{k-1}{k}} := c \in \mathbb{C}$ with $c = 0$ whenever $\frac{k-1}{k} \notin \mathbb{Z}$ or $\lambda \neq 1$ and \( \gcd\{k, m \mid c_m \neq 0\} = 1 \). Moreover, two polynomial germs in normal form $(CG)$ $\varphi$ and 
\[ \tilde{\varphi} := (\tilde{\lambda} \tilde{\zeta}^s \tilde{z} + \tilde{P}(\zeta) + \tilde{c} \tilde{\zeta}^{\frac{k-1}{k}} \tilde{\zeta}^k) \]
are conjugated if and only if there exists $\epsilon \in \mathbb{C}$ with $\epsilon^{k-1} = 1$ and $\tilde{k} = k$, $\tilde{s} = s$, $\tilde{\lambda} = \epsilon^s \lambda$, $\tilde{P}(\zeta) = \epsilon^{-j} P(\epsilon \zeta)$, $\tilde{c} = \epsilon^{\frac{k-1}{k}} - 1 \epsilon^c$.

\textbf{Remark 4.3.} Intermediate surfaces of index one correspond precisely to germs $\varphi$ in normal form $(CG)$ such that $(k-1)|s$.

In order to see this we look at a surface $S$ of index one and at the normal form $(FG2)$ of the fundamental group of $S \setminus D$. We conjugate $(FG2)$ by $(z, w) \mapsto (e^{lw} z, w)$ and get the following form for the generators of $\pi_1(S \setminus D)$:
\[
\begin{align*}
(z, w) &\mapsto (z, w + 2\pi i) \\
(z, w) &\mapsto (\lambda e^{l(k-1)} w z + e^{lw} a_0 + P(e^w), kw)
\end{align*}
\]
where $P$ is the polynomial defined by $P(\zeta) = \zeta^{lk}Q(\zeta^{-1})$. Thus $P(\zeta) = \sum_{m=\sigma}^{l(k-1)-\sigma} c_m \zeta^m$, with $c_m = b_{lk-m}$. As shown in \cite{5} and \cite{6} the surface $S$ is then associated to the polynomial germ

$$(z, \zeta) \mapsto (\lambda\zeta^s z + P(\zeta) + c\zeta^{s/k}, \zeta^k)$$

which is in normal form (CG). Here we have set $s = l(k-1)$ and $c = a_0$.

**Remark 4.4.** Every intermediate surface of index higher than one may be constructed as in section 3.2.

Indeed, let $\varphi'$ be a polynomial germ in normal form (CG) associated to a surface $S'$ of higher index,

$$\varphi'(z, \zeta) = (\lambda\zeta^{s'} z + \sum_{m=j'} c_m' \zeta^m, \zeta^k),$$

with $(k-1) \nmid s'$ and let $q$ be some positive divisor of $k-1$ such that $(k-1)$ divides $qs'$. Set $r := \left\lfloor q \frac{j'}{k} \right\rfloor$, $s := qs' - r(k-1)$, $j := qj' - rk$, $P(\zeta) := \sum_{m=j'} c_m' \zeta^{qm-rk}$ and

$$\varphi(z, \zeta) := (\lambda\zeta^s + P(\zeta), \zeta^k).$$

One checks now easily that the polynomial germ $\varphi$ is in normal form (CG), corresponds to a surface $S$ of index one and admits an automorphism of the form

$$(z, \zeta) \mapsto (\epsilon^{-r} z, \epsilon \zeta),$$

where $\epsilon$ is a primitive root of unity of order $q$. This automorphism lifts to a conjugation of a corresponding normal form (FG2) for $S$, which has property $(I_q)$ as in section 3.2. In fact the covering map $S \setminus D \to S' \setminus D'$ is induced by $(z, \zeta) \mapsto (\zeta^r z, \zeta^q)$.

One also sees that the index of $S'$ is the least possible $q$ allowed in the above construction; more precisely, one gets

**Remark 4.5.** The index of an intermediate surface associated to a polynomial germ $\varphi(z, \zeta) = (\lambda\zeta^s z + P(\zeta) + c\zeta^{s/k}, \zeta^k)$ in normal form (CG) equals

$$\frac{k-1}{\gcd(k-1, s)}.$$

**Remark 4.6.** (Cf. \cite{5}.)

The intermediate surfaces admitting non-trivial holomorphic vector fields are precisely those associated to polynomial germs

$$\varphi(z, \zeta) := (\lambda\zeta^s z + P(\zeta) + c\zeta^{s/k}, \zeta^k)$$

in normal form (CG) with $(k-1)|s$ and $\lambda = 1$.

We now define the type of a polynomial germ in normal form (CG) in a similar way we did it for (FG2).
Definition 4.7. For fixed $k$ and $s$ and for a polynomial germ $\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^{\frac{t}{i_1}}, \zeta^k)$ in normal form (CG) with $P(\zeta) := \zeta^j + c_{j+1} \zeta^{j+1} + \ldots + c_s \zeta^s$ we define inductively the following finite sequences of integers $j =: m_1 < \ldots < m_t \leq s$ and $k > i_1 > j_2 \ldots > i_t = 1$ by:

i) $m_1 := j$, $i_1 := \gcd(k, m_1)$;

ii) $m_\alpha := \min\{m > m_{\alpha-1} \mid c_m \neq 0, \gcd(i_{\alpha-1}, m) < i_{\alpha-1}\}, i_\alpha := \gcd(k, m_1, \ldots, m_\alpha)$;

iii) $1 = i_t := \gcd(k, m_1, \ldots, m_{t-1}, m_t) < \gcd(k, m_1, \ldots, m_{t-1})$.

We call $(m_1, \ldots, m_t)$ the type of $\varphi$ and $t$ the length of the type. If $t = 1$ we say that $\varphi$ is of simple type.

The length of the type of $\varphi$ is 1, i.e. $\varphi$ is of simple type, if and only if $k$ and $j$ are relatively prime, $\gcd(k, j) = 1$.

We also set $\varepsilon(k, m_1, \ldots, m_t, s) := [\frac{m_2 - m_1}{i_2}] + [\frac{m_1 - m_2}{i_1}] + \ldots + [\frac{m_{t-1} - m_t}{i_{t-1}}] + s - m_t$.

It is the number of coefficients of $P$ whose vanishing or non-vanishing does not affect the type.

The type is obviously preserved by the conjugations appearing in Theorem 4.2.

For fixed $k$, $s$ and fixed type $(m_1, \ldots, m_t)$ we consider the following obvious parameter spaces for the coefficients $(\lambda, c_{j+1}, \ldots, c_s, c)$ appearing in (CG):

- when $(k - 1)$ does not divide $s$ we take
  $$U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^s \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$

- in case $(k - 1) \mid s$
  $$U_{k,s,m_1,\ldots,m_t}^\lambda = \mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)},$$
  $$U_{k,s,m_1,\ldots,m_t}^{\lambda = 1} = (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{\varepsilon(k,m_1,\ldots,m_t,s)} \times \mathbb{C}.$$

In the second case we have considered separate parameter spaces for surfaces without, respectively with, holomorphic vector fields. We will come back later to this point.

By the discussion of the previous section we see that for each $p \in \{1, \ldots, t\}$ choosing $d = \frac{k}{i_p}$ gives us a transformation on types and a biholomorphic map $\tau_d$ between corresponding parameter spaces. Surfaces associated to germs of one type correspond via $\tau_d$ to isomorphic surfaces associated to germs of the transformed type.

5. Decomposition of Germs

Definition 5.8. A polynomial germ

$$\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^{\frac{t}{i_1}}, \zeta^k)$$

in normal form (CG) is said to be in pure normal form if $c = 0$. We say that a polynomial germ $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is in modified normal form if

$$\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^{kn}, \zeta^k)$$
is such that the germ

\[ p\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta), \zeta^k) \]

is in pure normal form \((CG)\), \(n \in \mathbb{Z}, \frac{c}{k} < n \leq \frac{s}{k-1}\) and \(c \in \mathbb{C}\). In this case the type of \(\varphi\) is by definition the same as the type of the purified germ \(p\varphi\).

**Remark 5.9.** The pure normal form is a special case of the normal form \((CG)\). The normal form \((CG)\) is a special case of the modified normal form. A germ \(\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c\zeta^{kn}, \zeta^k)\) in modified normal form is in normal form \((CG)\) if and only if either \(n = \frac{s}{k-1}\) and \(\lambda = 1\) or \(c = 0\). When this is not the case \(\varphi\) is conjugated to some germ \(\tilde{\varphi} := (\lambda \zeta^s z + \tilde{P}(\zeta), \zeta^k)\) in pure normal form and of the same type as \(\varphi\).

For suppose \(c \neq 0\). We have to consider two cases.

When \(\frac{s}{k} < n \leq \frac{s}{k-1}\) we conjugate \(\varphi\) by \((z, \zeta) \mapsto (z - \lambda^{-1}c\zeta^{kn-s}, \zeta)\) to get the new germ

\[ (z, \zeta) \mapsto (\lambda \zeta^s z + P(\zeta) + \lambda^{-1}c\zeta^{k(kn-s)}, \zeta^k). \]

By assumption we have \(kn - s < n\). If \(k(kn - s) \leq s\) we are ready since \((z, \zeta) \mapsto (\lambda \zeta^s z + P(\zeta) + \lambda^{-1}c\zeta^{k(kn-s)}, \zeta^k)\) is in pure normal form. If not we can work with \(kn - n\) instead of \(n\) and continue to conjugate until the exponent of the supplementary \(\zeta\)-term doesn’t exceed \(s\) any more. Remark that this exponent remains a multiple of \(k\) and thus leaves the type of \(\varphi\) intact.

When \(n = \frac{s}{k-1}\) and \(\lambda \neq 1\) a conjugation by \((z, \zeta) \mapsto (z - \frac{c}{k-1}\zeta^n, \zeta)\) leads \(\varphi\) directly to the pure normal form

\[ (z, \zeta) \mapsto (\lambda \zeta^s z + P(\zeta), \zeta^k). \]

**Proposition 5.10.** Let \(\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c\zeta^{kn}, \zeta^k)\) be a germ in modified normal form and whose type has length \(t\). If \(\varphi\) is in pure normal form, then \(\varphi\) admits a canonical decomposition \(\varphi = \varphi_1 \circ \ldots \circ \varphi_t\) into \(t\) polynomial germs in pure normal form and of simple type. If \(\varphi\) is not in pure normal form, then \(\varphi\) admits a canonical decomposition \(\varphi = \varphi_1 \circ \ldots \circ \varphi_t\) into \(t\) polynomial germs in modified normal form and of simple type out of which \(t-1\) are pure. Moreover \(p\varphi = p\varphi_1 \circ \ldots \circ p\varphi_t\). The canonical choice of the decomposition \(\varphi = \varphi_1 \circ \ldots \circ \varphi_t\) and the types of the factors \(\varphi_1, \ldots, \varphi_t\) are determined by the type of \(\varphi\).

**Proof.** We first deal with the pure case.

Let

\[ \varphi_1(z, \zeta) := (\lambda_1 \zeta^{j_1} z + P_1(\zeta), \zeta^{k_1}) \]

and

\[ \varphi_2(z, \zeta) := (\lambda_2 \zeta^{j_2} z + P_2(\zeta), \zeta^{k_2}) \]

be two germs in pure normal form with invariants \((k_1, j_1, s_1)\) resp. \((k_2, j_2, s_2)\) of types

\[ (j_1 = m_1^{(1)}, m_2^{(1)}, \ldots, m_t^{(1)}) \]
and 
\[(j_2 = m_1^{(2)}, m_2^{(2)}, \ldots, m_t^{(2)})\]
respectively. Their composition
\[\varphi_1 \circ \varphi_2(z, \zeta) = (\lambda_1 \lambda_2 \zeta^{s_1 k_2 + s_2} z + P_1(z^{k_2}) + \lambda_1 \zeta^{s_1 k_2} P_2(z), \zeta^{k_1 k_2})\]
is again in pure normal form with invariants \((k = k_1 k_2, j = j_1 k_2, s = s_1 k_2 + s_2)\) and has type
\[(m_1^{(1)} k_2, m_2^{(1)} k_2, \ldots, m_t^{(1)} k_2, s_1 k_2 + m_1^{(2)}, s_1 k_2 + m_2^{(2)}, \ldots, s_1 k_2 + m_t^{(2)})\]
with length \(t = t_1 + t_2\).

Conversely, let
\[\varphi(z, \zeta) := (\lambda \zeta^s z + P(z), \zeta^k),\]
where
\[P(z) := \zeta^j + c_{j+1} \zeta^{j+1} + \cdots + c_s \zeta^s\]
be a germ in pure normal form and suppose that
\[d := \gcd(j, k) > 1.\]
Define \(k_1 := k/d\), \(j_1 := j/d\) and
\[s_1 := \max\{m/d | \tilde{m}/d \in \mathbb{Z} \text{ or } c_{\tilde{m}} = 0 \text{ for all } \tilde{m} = j, \ldots, m\}.\]
It is clear that \(0 < j_1 < k_1\) and \(j_1 \leq s_1\). Furthermore let \(k_2 := d\), 
\[j_2 := -s_1 k_2 + \min\{m | m > s_1 k_2 \text{ and } c_m \neq 0\}\]
and \(s_2 := s - s_1 k_2\). It is clear that \(0 < j_2 < k_2\) and \(j_2 \leq s_2\). We also put \(\lambda_1 := c_{s_1 k_2+j_2} \neq 0\) and \(\lambda_2 := \lambda/\lambda_1\). Finally let
\[\varphi_1(z, \zeta) := (\lambda_1 \zeta^{s_1 z} + \sum_{m=j}^{s_1 k_2} c_m \zeta^{m/d}, \zeta^{k_1}),\]
\[\varphi_2(z, \zeta) := (\lambda_2 \zeta^{s_2 z} + \sum_{m=s_1 k_2+j_2}^{s} \lambda_1^{-1} c_m \zeta^{m-s_1 k_2}, \zeta^{k_2}).\]
One verifies directly that \(\varphi_1 \circ \varphi_2 = \varphi\), that \(\varphi_1, \varphi_2\) are in pure normal form
and that \(\varphi\) is of simple type. Note also that the types of \(\varphi_1\) and \(\varphi_2\) are determined by the type of \(\varphi\). Repeating this procedure if necessary with \(\varphi_2\) one gets a decomposition of \(\varphi\) into germs in pure normal form and of simple type in a canonical way.

Take now \(\varphi(z, \zeta) = (\lambda \zeta^s z + P(\zeta) + c \zeta^{kn}, \zeta^k)\) a germ in modified normal form, 
\[p(\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta), \zeta^k)\]
and \(\varphi_1(z, \zeta) := (\lambda_1 \zeta^{s_1 z} + P_1(\zeta), \zeta^{k_1}),\)
\(\varphi_2(z, \zeta) := (\lambda_2 \zeta^{s_2 z} + P_2(\zeta), \zeta^{k_2})\) in pure normal form such that \(p(\varphi) = \varphi_1 \circ \varphi_2\).

We shall modify \(\varphi_1\) or \(\varphi_2\) in order to get \(\varphi\) by composition. If \(n \leq \frac{s_1}{k_1-1}\), then \(\frac{n}{k_1} < n\) and the germ
\[\tilde{\varphi}_1(z, \zeta) := (\lambda_1 \zeta^{s_1 z} + P_1(\zeta) + c \zeta^{k_1 n}, \zeta^{k_1})\]
is in modified normal form and \(\tilde{\varphi}_1 \circ \varphi_2 = \varphi\). If not, take \(n_2 := k_1 n - s_1\) and
\[\tilde{\varphi}_2(z, \zeta) := (\lambda_2 \zeta^{s_2 z} + P_2(\zeta) + c \lambda_1 \zeta^{k_2 n_2}, \zeta^{k_2}).\]
Then \( \varphi_1 \circ \tilde{\varphi}_2 = \varphi \). The inequalities \( \frac{s_1}{k_1-1} < n \leq \frac{s_2}{k_2-1} \) imply \( n_2 < \frac{s_2}{k_2-1} \) and thus \( \tilde{\varphi}_2 \) will be in modified normal form.

Note that in case \( n = \frac{s_1}{k_1-1} = \frac{s_2}{k_2-1} = \frac{s_3}{k_3-1} \) both decompositions \( \varphi = \tilde{\varphi}_1 \circ \varphi_2 = \varphi_1 \circ \tilde{\varphi}_2 \) are possible but we chose the first \( \tilde{\varphi}_1 \circ \varphi_2 \) as the canonical one. 

\[ \square \]

6. The blow-up sequence and the Dloussky sequence

In this section we calculate for a given germ in modified normal form the configuration of the rational curves on the associated intermediate surface. This is equivalent to giving the Dloussky sequence of the surface, see [3], pp. 37, which in turn will be computed by the sequence of blow-ups for the given germ. We recall that in the Dloussky sequence each entry represents a rational curve with the negative of its self-intersection and the order in the sequence is given by the order of creation of the curves in the blow-up process. A Dloussky sequence for an intermediate surface is called simple if it is of the form

\[
[DlS] = [\alpha_1 + 2, 2, \ldots, 2, \ldots, \alpha_q + 2, 2, \ldots, 2, \alpha_{q+1} + 1, 2, \ldots, 2, 2, \ldots, 2] = [s_{\alpha_1}, \ldots, s_{\alpha_q}, s_{\alpha_{q+1}-1}, r_m],
\]

with \( q \geq 0, \alpha_i \geq 1 \) for \( 1 \leq i \leq q, \alpha_{q+1} \geq 2 \) and \( m \geq 1 \). So the shortest possible sequence here appears for \( q = 0 \) and \( m = 1 \) and is of the form [3, 2]. A general Dloussky sequence is of the form \([DlS_1, \ldots, DlS_N]\), where \([DlS_j], j = 1, \ldots, N\) are simple Dloussky sequences. The dual graph of the divisor \( D \) on an intermediate surface with known Dloussky sequence is now constructed in the following way: The entries of the sequence represent the knots of the graph. An entry with value \( \alpha \) is connected with the entry following \( \alpha - 1 \) places after it at the right hand (with the entries in cyclic order!).

**Example 6.11.** a) The Dloussky sequence \([3, 4, 2, 2]\) produces the graph

\[
\begin{array}{c}
0
\end{array}
\]

\[
\begin{array}{c}
0
\end{array}
\]

b) The Dloussky sequence \([3, 2, 4, 2, 2, 2]\) produces the graph
Note that the Dloussky sequence induces an orientation on the cycle of the dual graph. We will call such a graph a directed dual graph. Note also that the dual graph of the cycle of rational curves on a surface with a GSS has a natural orientation given by the order of creation of the curves in the blowing-up process. After gluing as described in Section 2 this orientation may be recovered by looking at the pseudoconvex side of a GSS in the compact surface $S$.

We start with the description of the blow-up sequence in the case of a germ $\varphi$ of simple type. This is the sequence of blow-ups which leads to the maps $\pi : \hat{B} \to B$, $\sigma : \bar{B} \to \sigma(\bar{B})$ such that $\varphi = \pi \circ \sigma$ as in Section 2.

Let

$$\varphi(z, \zeta) := (\lambda \zeta^s z + \zeta^j + c_{j+1} \zeta^{j+1} + \ldots + c_s \zeta^s + c \zeta^{kn}, \zeta^k) =$$

$$= (\lambda \zeta^s z + P(\zeta) + c \zeta^{kn}, \zeta^k) = (\zeta^j A(z, \zeta), \zeta^k) = (\zeta^j A, \zeta^k),$$

where

$$P(\zeta) := \zeta^j + c_{j+1} \zeta^{j+1} + \ldots + c_s \zeta^s$$

and

$$A(z, \zeta) := \lambda \zeta^{s-j} z + 1 + c_{j+1} \zeta + \ldots + c_s \zeta^{s-j} + c \zeta^{kn-j}, \ A(0, 0) = 1.$$
In the above sequence of blow-ups all the powers of $A$ indicated by $\cdots$ are non-zero integers. The map $\sigma$ is a germ of a biholomorphism at the origin of $(\mathbb{C}^2, 0)$ with the property that the inverse image $\sigma^{-1}(C \cap U)$ of the intersection of the last created rational curve $C$ with an open neighbourhood $U$ of $\sigma(0,0)$ is given by $\{\zeta = 0\}$. Furthermore $\tilde{\eta}$ is the composition of $s - j$ blowups in the form $(u, v) \mapsto ((u + \text{const.})v, v)$.

The construction shows that the Dloussky sequence of the associated surface is simple and given by

$$[\text{DIS}] = [\alpha_1 + 2, 2, \ldots, 2, \ldots, \alpha_q + 2, 2, \ldots, 2, \alpha_{q+1} + 1, 2, \ldots, 2, 2, \ldots, 2] =$$

$$= [s_{\alpha_1}, \ldots, s_{\alpha_q}, s_{\alpha_{q+1} - 1}, r_{s - j + 1}],$$
In particular, the second Betti number of the surface is
\[ b_2 = (-1 + \sum_{i=1}^{q+1} \alpha_i) + (s - j + 1) = (\sum_{i=1}^{q+1} \alpha_i) + (s - j) \]
which is the length of [DIS].

**Remark 6.12.** The above construction shows that the obtained Dloussky sequence is independent of the germ being in normal form, pure normal form or modified normal form.

For the general case let \( \varphi = \varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_N \) be the decomposition of a germ \( \varphi \) into germs of simple type and DIS resp. DIS\(_i\) the associated Dloussky sequences of \( \varphi \) resp. \( \varphi_i, i = 1, \ldots, N \). An easy calculation similar to the above one shows that
\[ \text{[DIS]} = \text{[DIS}_1, \ldots, \text{DIS}_N], \]
i.e. the operations of composition of germs and concatenations of Dloussky sequences are compatible.

We note in conclusion that the following three objects associated to an intermediate surface are algorithmically computable from one another: the directed dual graph of the rational curves, the Dloussky sequence, the type of a contracting germ in modified normal form.

**7. Versal families and moduli spaces**

We have seen in section 4 that for fixed \( k \), \( s \) and fixed type \( (m_1, \ldots, m_t) \) we get parameter spaces for germs in normal form (CG)
\[ U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^* \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{e(k,m_1,\ldots,m_t,s)}, \]
when \((k-1) \nmid s\) and
\[ U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0} = \mathbb{C} \setminus \{0, 1\} \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{e(k,m_1,\ldots,m_t,s)}, \]
\[ U_{k,s,m_1,\ldots,m_t}^{\lambda = 1} = (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{e(k,m_1,\ldots,m_t,s)} \times \mathbb{C}, \]
when \((k-1) \mid s\). In case \((k-1) \mid s\) the spaces \( U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0} \) and \( U_{k,s,m_1,\ldots,m_t}^{\lambda = 1} \) appear as subspaces of
\[ U_{k,s,m_1,\ldots,m_t} = \mathbb{C}^* \times (\mathbb{C}^*)^{t-1} \times \mathbb{C}^{e(k,m_1,\ldots,m_t,s)} \times \mathbb{C}, \]
which parameterizes germs \((z, \zeta) \mapsto (\lambda \zeta^s z + P(\zeta) + c\zeta^{m_1}s + \zeta^{k})\) in modified normal type, i.e. the same directed dual graph of rational curves for the associated intermediate surfaces. Conversely, we have seen that every intermediate surface with such configuration of curves corresponds to a germ of this type. Moreover, one may perform the blow-ups and the glueing over the parameter space \( U_{k,s,m_1,\ldots,m_t} \) thus obtaining a family \( S_{k,s,m_1,\ldots,m_t} \to U_{k,s,m_1,\ldots,m_t} \) of intermediate surfaces over \( U_{k,s,m_1,\ldots,m_t} \). It is clear that \( S_{k,s,m_1,\ldots,m_t} \) is a complex manifold of dimension \( \dim U_{k,s,m_1,\ldots,m_t} + 2 = t + e(k,m_1,\ldots,m_t,s) + \delta + 2 \), where \( \delta = 1 \) if \((k-1) \mid s\) and otherwise \( \delta = 0 \). The projection \( S_{k,s,m_1,\ldots,m_t} \to U_{k,s,m_1,\ldots,m_t} \) is proper and smooth, since locally around each
point of $S_{k,s,m_1,...,m_t}$ it looks like the projection on $U_{k,s,m_1,...,m_t}$ of a small open subset of $\hat{B} \times U_{k,s,m_1,...,m_t}$. See also our Appendix for a discussion on the family of the open surfaces $S \setminus D$.

Consider now an intermediate surface $S$ with maximal effective reduced divisor $D$. We would like to consider families of logarithmic deformations as in [8]. However the definition of that paper requires that $D$ be a subspace with simple normal crossings. But when the cycle of curves of $D$ is reduced to only one curve $C$, this condition is not satisfied. In this case we blow up the singularity of $C$ on $S$ and work with the blown-up surface instead. We may then apply Theorem 2 of [8] to compare the logarithmic deformations.

In the sequel we shall work with the blown-up surface, without mentioning it explicitly again. In particular the new $D$ will have simple normal crossings. (Another way to avoid the singularity of $C$ would be to look at a non-ramified double cover of $S$ instead of $S$.)

A family of logarithmic deformations for the pair $(S,D)$ is a 6-tuple $(S, D, \pi, V, v, \psi)$, where $D$ is a divisor on $S$, $\pi : S \to V$ is a proper smooth morphism of complex spaces, which is locally a projection as well as its restriction to $D$, $v \in V$ and $\psi : S \to \pi^{-1}(v)$ is an isomorphism restricting to an isomorphism $S \setminus D \to \pi^{-1}(v) \setminus D$. Let $T_S(- \log D)$ be the logarithmic tangent sheaf of $(S,D)$. It is the dual of the sheaf $\Omega_S(\log D)$ of logarithmic differential 1-forms on $(S,D)$. By [8] versal logarithmic deformations of $(S,D)$ exist and their tangent space is $H^1(S, T_S(- \log D))$. The space $H^2(S, T_S(- \log D))$ of obstructions vanishes by Theorem 1.3 of [13]. In particular the basis of the versal logarithmic deformation of a pair $(S,D)$ is smooth. On the other side $H^0(S, T_S(- \log D)) = H^0(S, T_S)$ and this space is at most one dimensional.

**Theorem 7.13.** With the above notations we have:

- If $(k-1)$ does not divide $s$ the family $S_{k,s,m_1,...,m_t} \to U_{k,s,m_1,...,m_t}$ is logarithmically versal around every point of $U_{k,s,m_1,...,m_t}$.
- If $(k-1) \mid s$ the restriction of the family $S_{k,s,m_1,...,m_t} \to U_{k,s,m_1,...,m_t}$ to $U_{k,s,m_1,...,m_t}^{\lambda \neq 1,c=0}$ is logarithmically versal around every point of $U_{k,s,m_1,...,m_t}^{\lambda \neq 1,c=0}$.
- If $(k-1) \mid s$ the family $S_{k,s,m_1,...,m_t} \to U_{k,s,m_1,...,m_t}$ is logarithmically versal around every point of $U_{k,s,m_1,...,m_t}^{\lambda = 1,c=0}$.

**Proof.** We start with the case $(k-1) \nmid s$. Take $(S,D)$ a pair as above and $\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta), \zeta^k)$ an associated germ in normal form (CG). We see $\varphi$ as a point in $U_{k,s,m_1,...,m_t}$. Let $(S', D', \pi', v, \psi)$ be the logarithmically versal deformation of the pair $(S,D)$. Then the family $S_{k,s,m_1,...,m_t} \to$
$U_{k,s,m_1,...,m_t}$ is obtained from $S' \to V$ by base change by means of a map $F : (U_{k,s,m_1,...,m_t}, \varphi) \to (V,v)$. By Theorem 4.2 $F$ must have finite fibres. Thus $\dim U_{k,s,m_1,...,m_t} \leq \dim V$. Since every point $v'$ of $V$ is covered by a similar map $(U_{k,s,m_1,...,m_t}, \varphi') \to (V,v')$ one gets $\dim U_{k,s,m_1,...,m_t} = \dim V$.

Next we show that $F$ is injective near $\varphi$ hence locally biholomorphic. Assume the contrary. Then by Theorem 4.2 there exists a root of unity $\epsilon$ of order $q$ with $1 \neq q \mid (k-1)$ and a sequence $(\varphi_n)_{n \in \mathbb{N}}$, $\varphi_n(z, \zeta) := (\lambda_n \zeta^s z + P_n(\zeta), \zeta^k)$, converging to $\varphi$ in $U_{k,s,m_1,...,m_t}$ such that for $\tilde{\varphi}_n(z, \zeta) := (\epsilon^s \lambda_n \zeta^s z + \epsilon^{-m_1} P_n(\epsilon \zeta), \zeta^k)$ one has $\tilde{\varphi}_n \to \varphi$ and $F(\tilde{\varphi}_n) = F(\varphi_n)$ for all $n \in \mathbb{N}$. Let $r = -m_1 - \lceil \frac{m_k}{q} \rceil q$ and $\chi(z, \zeta) = (\epsilon^{-r} z, \epsilon \zeta)$. Then $\tilde{\varphi}_n = \chi^{-1} \circ \varphi_n \circ \chi$ for all $n \in \mathbb{N}$. Thus $\chi$ induces an automorphism of the germ $\varphi$ and in fact a non-trivial automorphism of the surface $S = S_\varphi$ as in the proof of Remark 4.1. On the other side $\chi$ induces for each $n \in \mathbb{N}$ the isomorphisms $S_{\varphi_n} \cong S'_{F(\varphi_n)} \cong S_{\tilde{\varphi}_n}$ which by construction converge to the identity on $S = S_\varphi$. This is a contradiction. The second assertion of the theorem has a completely analogous proof.

For the third we have $(k-1) \mid s$ and we consider the pair $(S, D)$ associated to a germ $\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^k, \zeta^k)$ in normal form ($\text{CG}$), $(S', D', \pi, V, v, \psi)$ its logarithmically versal deformation and $F : (U_{k,s,m_1,...,m_t}, \varphi) \to (V,v)$ the induced morphism as before. From the second part of the theorem it follows that $\dim U_{k,s,m_1,...,m_t} = \dim H^1(S, T_S(-\log D)) = \dim V$. Since the fiber over $v$ is finite we only need to check that $F$ is injective. If it was not, then by the discussion of the first case ramification should occur along the divisor $U_{k,s,m_1,...,m_t}$. But this is not possible, because one can define the function $\lambda$ on the base of every family of intermediate surfaces of this logarithmic type, in particular also on $V$. Indeed, for such a surface $S$ one defines $\lambda(S)$ as the unique twisting factor such that $H^0(S, T_S \otimes L_{\lambda(S)}) \neq 0$, see for instance [4].

We now fix a logarithmic type of intermediate surfaces, i.e. we fix the directed dual graph of the maximal reduced divisor of rational curves on such a surface, and want to describe corresponding logarithmic moduli spaces. By our previous discussions it is enough to fix one set $k, s, m_1, ..., m_t$ of adapted numerical invariants and look at polynomial germs parameterized by $U_{k,s,m_1,...,m_t}$, $U_{k,s,m_1,...,m_t}^{1,c=0}$, $U_{k,s,m_1,...,m_t}^{\lambda=1}$. Although several such sets of invariants may correspond to the desired moduli space, they all will be related by the transformations $\tau_d$ described in sections 3.2, 4. By Theorem 4.2 we get a natural action of $\mathbb{Z}/(k-1)$ on $U_{k,s,m_1,...,m_t}$ which permutes conjugated germs: through a generator of $\mathbb{Z}/(k-1)$ a germ

$\varphi(z, \zeta) := (\lambda \zeta^s z + P(\zeta) + c \zeta^{\frac{mk}{k-1}}, \zeta^k)$

is mapped to

$\varphi(z, \zeta) := (\epsilon^s \lambda \zeta^s z + \epsilon^{-m_1} P_n(\epsilon \zeta) + \epsilon^{\frac{mk}{k-1} - m_1} c \zeta^{\frac{mk}{k-1}}, \zeta^k)$,
The quotient spaces \( U \)

**Theorem 7.14.** Fix \( k, s \) and a type \((m_1, \ldots, m_t)\) for polynomial germs in normal form \((\text{CG})\). Set \( j = m_1 \) as before.

- When \( j < \max(s, k - 1) \) the natural action of \( \mathbb{Z}/(k-1) \) on

\[
U_{k,s,m_1,\ldots,m_t}, U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0} \quad \text{and} \quad U_{k,s,m_1,\ldots,m_t}^{\lambda = 1}
\]

is effective.

- In the remaining case, i.e. when \( j = k - 1 = s \), the natural action of \( \mathbb{Z}/(k-1) \) is effective on \( U_{k,s,m_1,\ldots,m_t}^{\lambda = 1} = \mathbb{C} \) and trivial on \( U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0} = \mathbb{C} \setminus \{0, 1\} \).

The quotient spaces \( U_{k,s,m_1,\ldots,m_t}/(\mathbb{Z}/(k-1)) \) when \( (k-1) \nmid s \), and

\[
U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0}/(\mathbb{Z}/(k-1)), U_{k,s,m_1,\ldots,m_t}^{\lambda = 1}/(\mathbb{Z}/(k-1)), \text{ when } (k-1) \mid s,\n\]

are coarse logarithmic moduli spaces for intermediate surfaces of the given logarithmic type without, respectively with, non-trivial holomorphic vector fields.

These spaces are fine moduli spaces if and only if either the corresponding action of \( \mathbb{Z}/(k-1) \) is trivial or this action is free.

The natural action of \( \mathbb{Z}/(k-1) \) on either of the spaces

\[
U_{k,s,m_1,\ldots,m_t}, U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0} \quad \text{and} \quad U_{k,s,m_1,\ldots,m_t}^{\lambda = 1}
\]

is free if and only if \( \gcd(k-1, s, m_2 - j, \ldots, m_t - j) = 1 \).

**Proof.** The assertions on the effectiveness of the action are immediately verified.

When the action is trivial it is clear that the corresponding families over

\[
U_{k,s,m_1,\ldots,m_t}, U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0}, U_{k,s,m_1,\ldots,m_t}^{\lambda = 1}
\]

are universal.

When the action is free the families over \( U_{k,s,m_1,\ldots,m_t}, U_{k,s,m_1,\ldots,m_t}^{\lambda \neq 1, c=0}, U_{k,s,m_1,\ldots,m_t}^{\lambda = 1} \)
descend to the moduli spaces since we can extend the conjugation \( \chi(z, \zeta) = (e^{-\tau} z, e^\zeta) \) from the proof of Theorem 7.13 to the whole family \( \mathcal{S}_{k,s,m_1,\ldots,m_t} \to U_{k,s,m_1,\ldots,m_t} \).

When the action is not free but effective we see as in the proof of Theorem 7.13 that such a family around a fixed point for some non-trivial subgroup of \( \mathbb{Z}/(k-1) \) cannot descend to the quotient.

Suppose now that \( d \mid \gcd(k-1, s, m_2 - j, \ldots, m_t - j) \). It is easy to check that a germ of the form \( \varphi(z, \zeta) := (\lambda \zeta^{k-1} z + \zeta + c_{m_2} \zeta^{m_2} + \ldots + c_{m_t} \zeta^{m_t} \zeta^k) \) is a fixed point for the action of the subgroup of order \( d \) of \( \mathbb{Z}/(k-1) \).

Conversely, the existence of a fixed point for the action of the subgroup of order \( d \) of \( \mathbb{Z}/(k-1) \) implies \( d \mid \gcd(k-1, s, m_2 - j, \ldots, m_t - j) \).

We have treated surfaces with and surfaces without vector fields separately in order to avoid non-separation phenomena. If we look for example at the families

\[
\varphi_{1,\lambda}(z, \zeta) := (\lambda \zeta^{k-1} z + \zeta + c_1 \zeta^k, \zeta^k), \\
\varphi_{2,\lambda}(z, \zeta) := (\lambda \zeta^{k-1} z + \zeta + c_2 \zeta^k, \zeta^k),
\]

where \( \epsilon \) is a primitive \((k - 1)\)-th root of the unity.
for \( c_1 \neq c_2 \) fixed and \( \lambda \) varying in \( \mathbb{C}^* \), we see that \( \varphi_{1,\lambda}, \varphi_{2,\lambda} \) are conjugated to one another if and only if \( \lambda \neq 1 \).

8. Appendix: Deformation families for \( \tilde{S} \setminus D \)

In this Appendix we shall construct deformation families for the universal covers of the open surfaces \( S \setminus D \) where \( S \) is an intermediate surface and \( D \) its maximal, effective, reduced divisor. As a side product of our investigations we obtain examples of fibered analytic spaces together with groups acting holomorphically on them, such that the actions are free and properly discontinuous when restricted to each fiber but not on the total spaces. For simplicity we shall restrict ourselves to intermediate surfaces of index one.

Let \( S \) be an intermediate surface and \( D \) its maximal, effective, reduced divisor. We shall relax the conditions on the normal form (FG2) for the generators of the fundamental group of \( S \setminus D \) to get the following normal form:

\[
\begin{align*}
(FG4) & \\
\begin{cases}
g_\gamma(z, w) &= (z, w + 2\pi i) \\
g(z, w) &= (\lambda z + a_0 + Q(e^{-w}), kw),
\end{cases}
\end{align*}
\]

where \( k \geq 2, l \geq 1, Q = Q(\zeta) := \sum_{m=0}^{l-2} b_m \zeta^m \) is a polynomial such that \( (b_{lk}, \ldots, b_{lk-1}) \neq 0 \), gcd\{\( k, m \mid b_m \neq 0 \}\} = 1. \) As before \( (\lambda - 1)a_0 = 0 \). The difference from (FG2) is that we don’t fix the leading coefficient \( b_0 \) of \( Q \) to be 1 and require only that \( lk - k < \sigma < lk \). In fact a conjugation by \( (z, w) \mapsto (cz, w), \) with \( c \in \mathbb{C}^* \), leads us to the generators

\[
\begin{align*}
\begin{cases}
g_\gamma(z, w) &= (z, w + 2\pi i) \\
g(z, w) &= (\lambda z + ca_0 + cQ(e^{-w}), kw),
\end{cases}
\end{align*}
\]

In the following we shall fix \( k \) and \( l \) and move the parameters \( (\lambda, a_0, b_1, \ldots, b_{lk-1}) \in A \times B \) of the normal form (FG4). Here \( A := \{(\lambda, a_0) \in \mathbb{C}^* \times \mathbb{C} \mid (\lambda - 1)a_0 = 0 \}, B := \{(b_1, \ldots, b_{lk-1}) \in \mathbb{C}^{lk-1} \mid (b_{lk}, \ldots, b_{lk-1}) \neq 0 \}\). For each fixed point \( (\lambda, a, b) \in A \times B \) the operation of the group \( \mathbb{Z} \times \mathbb{Z}[1/k] \) generated by \( g_\gamma \) and \( g \) in the normal form (FG4) on \( \mathbb{C} \times H_\lambda \) is free and properly discontinuous and the quotient is a surface \( S_{\lambda, a_0, b} \setminus D_{\lambda, a_0, b} \).

We can define the type for (FG4) in the same way as for the normal form (FG2), see Definition 3[[N]] but note that in this case the type need not be constant on the \( B \)-component.

Lemma 8.15. Consider the coefficients \( b_1, \ldots, b_{lk-1} \) of the polynomial \( Q \) appearing in the normal form (FG4) of type \( (n_1, \ldots, n_t) \) and set \( n'_d := \max\{n \in \mathbb{N} \mid b_d \neq 0, \frac{k}{d} \nmid n\} \) for each \( d \in \{1, \ldots, k-1\} \) with \( d \mid k \). Then \( \{n_1, \ldots, n_t\} = \{n'_d \mid 1 \leq d < k, \ d \mid k\} \).

Proof. Set \( i_0 = 1 \). Then \( n_i = n'_{i-1} \) for all \( 1 \leq i \leq t \). Hence \( \{n_1, \ldots, n_t\} \subset \{n'_d \mid 1 \leq d < k, \ d \mid k\} \).
Conversely, suppose \( \{n_d' \mid 1 \leq d < k, d \mid k\} \setminus \{n_1, \ldots, n_t\} \neq \emptyset \) and let 
\( n'_t = \min\{n_d' \mid 1 \leq d < k, d \mid k\} \setminus \{n_1, \ldots, n_t\}, i = \max\{j \in \mathbb{N} \mid n_j > n'_t\}. \)
Suppose first that \( i < t \). Then \( n_t > n'_t > n_{i+1} \) and \( j_i \mid n'_t \) by the definition of \( n_{i+1} \). On the other side \( \frac{k}{j_i} \mid n_i \) and thus \( \frac{k}{j_i} \mid j_i \) by the definition of \( n'_t \).
Hence \( \frac{k}{j_i} \mid j_i \) for all \( 1 \leq j \leq t \), which implies \( \frac{k}{j_i} \mid \gcd(n_1, \ldots, n_t) = 1 \): again a contradiction.

When \( i = t \), we get \( \frac{k}{j_i} \mid n_j \) for all \( 1 \leq j \leq t \), which implies \( \frac{k}{j_i} \mid \gcd(n_1, \ldots, n_t) = 1 \): again a contradiction.

**Theorem 8.16.** There is an action of \( \mathbb{Z} \times \mathbb{Z}[1/k] \) generated by the holomorphic automorphisms \( g_r \) and \( g \) in the normal form \((FG4)\) on \( \mathbb{C} \times \mathbb{H}_t \times \mathbb{A} \times \mathbb{B} \) which is compatible with the projection \( \mathbb{C} \times \mathbb{H}_t \times \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{A} \times \mathbb{B} \). Let \( T \subset \mathbb{A} \times \mathbb{B} \) be a connected analytic subspace. The restriction of the above action to \( \mathbb{C} \times \mathbb{H}_t \times T \) is properly discontinuous if and only if the type is constant on \( T \). In this case the quotient space fibers smoothly over \( T \).

**Proof.** We consider the subgroup \( \Gamma < \mathbb{Z} \times \mathbb{Z}[1/k], \)
\( \Gamma := \{g_{r,m} := g^{-m} \circ g_r \circ g^m \mid m \in \mathbb{Z}, r \in \mathbb{Z}\} \)
We have \( \Gamma \cong \mathbb{Z}[1/k] \). Actually
\[
gr_{r,m}(z,w) = \left(z + \sum_{j=0}^{m-1} \lambda^{-j-1} \left( \sum_{n=1}^{l-1} b_n e^{-nkjw} \left(1 - \exp\left(\frac{2\pi irn}{k^m-j}\right)\right) \right), w + \frac{2\pi ir}{k^m}\right)
\]
for each point \((z,w) \in \mathbb{C} \times \mathbb{H}_t\). Note that in the above formula for \( g_{r,m} \) we may always reduce ourselves to the situation when \( k \nmid r \).
One can show as in \([5]\) p. 659, that the action of \( \mathbb{Z} \times \mathbb{Z}[1/k] \) on \( \mathbb{C} \times \mathbb{H}_t \times T \) is properly discontinuous if and only the induced action of the subgroup \( \Gamma \) is properly discontinuous.
Suppose now that the type is constant on \( T \), that the action of \( \Gamma \) on \( \mathbb{C} \times \mathbb{H}_t \times T \) is not properly discontinuous and let \( (g_{r,m_\nu})_{\nu \in \mathbb{N}} \) be a sequence in \( \Gamma \) which contradicts the proper discontinuity of the action. Then \( \left(\frac{2\pi ir}{k^m}\right)_{\nu} \) will be a bounded sequence, so by passing to some subsequence we may assume that \( (m_\nu)_{\nu} \) is a strictly increasing sequence. Again by passing to some subsequence if necessary we may assume that \( \gcd(k, r_\nu) =: d \) is constant for \( \nu \in \mathbb{N} \). Let \( n'_d := \max\{n \in \mathbb{N} \mid b_n \neq 0, \frac{k}{d} \nmid m\}. \)
Since the type is constant on \( T \) and by the above lemma \( n'_d \) is well-defined. Since \( (g_{r,m_\nu})_{\nu \in \mathbb{N}} \) contradicts the proper discontinuity of the action, for \((\lambda, a_0)\) bounded on the \( \mathbb{A} \)-component of \( T \) and \( w \) bounded in \( \mathbb{H}_t \) the quantity
\[
\sum_{j=0}^{m_\nu-1} \lambda^{-j-1} \left( \sum_{n=1}^{l-1} b_n e^{-nkjw} \left(1 - \exp\left(\frac{2\pi ir_\nu n}{k^m-j}\right)\right) \right)
\]
must be equally bounded independently of \( \nu \in \mathbb{N} \). It suffices now to remark that the term
\[
\lambda^{-m_\nu} b_{n'_d} e^{-n'_d k^{m_\nu-1}w} \left(1 - \exp\left(\frac{2\pi ir_\nu n'_d}{k}\right)\right)
\]
is dominant in this expression. Indeed the factor \(|1 - \exp(\frac{2\pi i r_n' n_d'}{k})|\) is bounded from below by \(|1 - \exp(\frac{2\pi i}{k})|\). On the other side the exponent \(n_d' k^m\nu - 1\) is the highest appearing in a non-vanishing term of this sum, since \(kn_d' \geq kl > kl - 1\) and thus \(b_n = 0\) for \(n > kn_d'\). Hence the term \(\lambda^{-m\nu} b_{n_d'} e^{-n_d' k^m\nu - 1}(1 - \exp(\frac{2\pi i r_n' n_d'}{k}))\) of the above sum goes to infinity more rapidly than the rest and the sum cannot be bounded: a contradiction.

Conversely suppose that the type is not constant on \(T\) and consider the first \(n_i\) appearing in the decreasing sequence \(n_1, n_2, \ldots\) of type, such that \(b_{n_i}\) takes both zero and non-zero values on \(T\). It is clear that one can find an analytic arc \(\gamma : \Delta \to T\) such that \(b_{n_i}\) vanishes at \(\gamma(0)\) but such that the type is constant on \(\gamma(\Delta \setminus \{0\})\) and contains \(n_i\). Let \(d\) be such that \(n_i = n_d'\) for the generic type as in Lemma 8.18. For the non-generic type the new \(n_d'\), call it \(n_d''\) will assume a strictly lower value, by definition. Take \(w = -1\), \(z\) arbitrary, \(r_\nu = d\) for all \(\nu \in \mathbb{N}\) and \(m_\nu = \nu\). By the previous argument, for \(\nu >> 0\), the dominant term at \(\gamma(0)\)

\[
\lambda^{-\nu} b_{n_d''} e^{n_d'' k^\nu - 1}(1 - \exp(\frac{2\pi i d_\nu n_d''}{k}))
\]

will be lower than

\[
\lambda^{-\nu} b_{n_d''} e^{n_d'' k^\nu - 1}(1 - \exp(\frac{2\pi i d_\nu n_d'}{k}))
\]

in case \(|b_{n_d'}|\) is uniformly bounded from below by some positive constant.

We can now choose a sequence converging to \(\gamma(0)\) in \(\gamma(\Delta)\) in such a way that the corresponding \(b_{n_d'}(\nu)\) converge to zero at such a rate that the whole sum

\[
\sigma(\nu) := \sum_{j=0}^{\nu-1} \lambda^{-j-1} \sum_{n=1}^{lk-1} b_n(\nu) e^{nkj} (1 - \exp(\frac{2\pi i d_\nu}{k^{\nu-j}}))
\]

converges to zero. In order to see this we solve the equation

\[
b_{n_d'} + \frac{\sum_{j=0}^{\nu-1} \lambda^{-j-1} (\sum_{n=1}^{lk-1} b_n e^{nkj} (1 - \exp(\frac{2\pi i d_\nu}{k^{\nu-j}})))}{\sum_{j=0}^{\nu-1} \lambda^{-j-1} e^{n_d' klj} (1 - \exp(\frac{2\pi i d_\nu}{k^{\nu-j}}))} = 0
\]

in \(b_{n_d'}\) on \(\gamma(\Delta)\) for each \(\nu >> 0\). This is certainly possible if the \(b_n\)-functions are supposed to be constant for \(n \neq n_d'\). The solutions \((b_{n_d'}(\nu))_\nu\) form in this case the desired sequence and the sums \(\sigma(\nu)\) vanish. In general we rewrite the equation (5) as

\[
b_{n_d'} + \sum_{n=1}^{n_d'-1} C_n(\nu) b_n = 0.
\]

It is easy to see that the coefficients \(C_n(\nu)\) converge to zero as \(\nu\) tends to infinity and thus (6) will have a solution on \(\gamma(\Delta)\) for \(\nu >> 0\). This closes the proof. \(\square\)
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