A note on minimum linear arrangement for BC graphs

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Abstract

A linear arrangement is a labeling or a numbering or a linear ordering of the vertices of a graph. In this paper we solve the minimum linear arrangement problem for bijective connection graphs (for short BC graphs) which include hypercubes, Möbius cubes, crossed cubes, twisted cubes, locally twisted cube, spined cube, Z-cubes, etc. as the subfamilies.

Keywords: Minimum linear arrangement, BC graphs

1 Introduction

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such a way that a certain objective function is optimized. In the literature, there are plenty of layout problems are discussed, such as Linear Arrangement, Bandwidth, Cutwidth, Modified Cut, Sum Cut, Edge Bisection and Vertex Bisection [1]. A large number of relevant problems in different domains formulated as graph layout problems include VLSI circuit design, network reliability, information retrieval, numerical analysis, computational biology, single machine job scheduling, automatic graph drawing and topology awareness of overlay networks [2, 3]. The problems are hard in general but known to be solvable in certain restricted classes of graphs [1].

A linear arrangement \( f \) of an undirected graph \( G = (V, E) \) with \( n \) nodes is a bijective function \( f : V \to \{1, 2, \ldots, n\} \). A linear arrangement is also called a labeling or a numbering or a linear ordering of the vertices of a graph. In [4], the Minimum Linear Arrangement (MinLA) problem is formulated as follows: Given a graph \( G = (V, E) \), find a linear arrangement \( f \) that minimizes \( \sum_{(u,v) \in E} |f(u) - f(v)| \). Linear arrangements are a particular case of embedding graphs in \( d \)-dimensional grids or other graphs. The case in which a graph with \( n \) vertices must be embedded into a path \( P_n \) is perhaps the simplest nontrivial embedding problem. The MinLA problem is NP-complete for bipartite graphs [5] and permutation graphs [6].

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2 Preliminaries

The following edge isoperimetric problems are used as tools to solve the MinLA problem. MinLA has been computed for regular graphs such as hypercubes \([4]\), circulant graphs \([8]\), folded hypercubes \([7]\), Petersen graphs \([9]\) chord graphs \([3]\) and locally twisted cubes \([10]\) using edge isoperimetric problem. In this paper, we compute the MinLA for certain families of regular graphs such as BC graphs.

**Problem 1:** For a given \(m\), if \(\theta_G(m) = \min_{A \subseteq V, |A| = m} |\theta_G(A)|\) where \(\theta_G(A) = \{(u, v) \in E : u \in A, v \notin A\}\), then the problem is to find \(A \subseteq V\) with \(|A| = m\) such that \(\theta_G(m) = |\theta_G(A)|\).

**Problem 2:** For a given \(m\), if \(I_G(m) = \max_{A \subseteq V, |A| = m} |I_G(A)|\) where \(I_G(A) = \{(u, v) \in E : u, v \in A\}\), then the problem is to find \(A \subseteq V\) with \(|A| = m\) such that \(I_G(m) = |I_G(A)|\). Such a set \(A\) is called an optimal set.

**Definition 2.1.** Let \(G\) and \(H\) be finite graphs. An embedding of \(G\) into \(H\) is a pair \((f, P_f)\) defined as follows:

1. \(f\) is a one-to-one map from \(V(G)\) to \(V(H)\)
2. \(P_f\) is a one-to-one map from \(E(G)\) to \(\{P_f(u, v) : P_f(u, v)\) is a path in \(H\) between \(f(u)\) and \(f(v)\), for \((u, v) \in E(G)\}\).

For brevity, we denote the pair \((f, P_f)\) as \(f\). The expansion of an embedding \(f\) is the ratio of the number of vertices of \(H\) to the number of vertices of \(G\). In this paper, we consider embeddings with expansion one.

The congestion of an embedding \(f\) of \(G\) into \(H\) is the maximum number of edges of the graph \(G\) that are embedded on any single edge \(e\) of \(H\). Let \(EC_f(e)\) denote the number of edges \((u, v)\) of \(G\) such that \(e\) is in the path \(P_f(u, v)\) between \(f(u)\) and \(f(v)\) in \(H\). In other words,

\[
EC_f(e) = |\{(u, v) \in E(G) : e \in P_f(u, v)\}|
\]

where \(P_f(u, v)\) denotes the path between \(f(u)\) and \(f(v)\) in \(H\) with respect to \(f\). Further, if \(S\) is any subset of \(E(H)\), then we define \(EC_f(S) = \sum_{e \in S} EC_f(e)\).

**Definition 2.2.** The wirelength of an embedding \(f\) of \(G\) into \(H\) is given by

\[
WL_f(G, H) = \sum_{e \in E(H)} EC_f(e)
\]

The wirelength of \(G\) into \(H\) is defined as

\[
WL(G, H) = \min WL_f(G, H)
\]

where the minimum is taken over all embeddings \(f\) of \(G\) into \(H\).

When \(H\) is a path, we represent \(WL_f(G, H)\) by \(LA_f(G)\) and represent \(WL(G, H)\) by \(MinLA(G)\).

**Lemma 2.3.** The MinLA of a graph \(G\) of order \(n\) is given by

\[
MinLA(G) \geq \sum_{i=1}^{n-1} \theta_G(i).
\]
Proof. For \( 1 \leq i < n \), let \( S_i = (i, i + 1) \), then for any embedding \( f \), we have

\[
EC_f(S_i) \geq \theta_G(i)
\]

\[
\therefore \min_f \sum_{i=1}^{n-1} EC_f(S_i) \geq \sum_{i=1}^{n-1} \theta_G(i)
\]

\[i.e., \text{MinLA}(G) \geq \sum_{i=1}^{n-1} \theta_G(i). \quad \Box\]

3 Main Results

BC networks have received a great deal of attention in the past \[12, 13, 14, 15\]. Fan et al. \[12\] proposed a family of interconnection networks called BC graphs. BC networks are a class of networks which include several well-known interconnection networks like hypercubes, Möbius cubes \[16\], crossed cubes \[17\], twisted cubes \[18\], locally twisted cube \[19\], spined cube \[20\] and Z-cube \[21\]. These variations of hypercubes generally possess certain superior properties over the hypercubes and are recognized as attractive alternatives to the hypercubes.

Definition 3.1. \[12, 15\] Let \( G_1 = (V_1, E_1) \), \( G_2 = (V_2, E_2) \) be two vertex disjoint graph of the same order. A bijective connection between \( G_1 \) and \( G_2 \) is defined as an edge set \( E = \{(v, \phi(v))\} \), where \( \phi : V_1 \rightarrow V_2 \) is a bijection. Define \( G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E) \).

An \( n \)-dimensional BC graph, denoted by \( X_n \), is an \( n \)-regular graph with \( 2^n \) nodes and \( n \cdot 2^{n-1} \) edges. The set of all the \( n \)-dimensional BC graphs is called the family of the \( n \)-dimensional BC graphs, denoted by \( \ell_n \). We now define \( X_n \) mathematically as follows:

Definition 3.2. \[12, 15\] The one-dimensional BC network \( X_1 \) is a complete graph with two vertices, \( K_2 \). The family of the one-dimensional BC network is defined as \( \ell_1 = \{K_2\} \). When \( n \geq 2 \), \( G = X_n \in \ell_n \) if and only if \( G = G_1 \oplus G_2 \) for some \( G_1, G_2 \in \ell_{n-1} \).

Lemma 3.3. \[15\] Let \( G \) be a \( n \)-dimensional BC graph. For an integer \( m \), which can be uniquely written as \( m = \sum_{i=0}^{r-1} 2^i \) for some nonnegative integers \( r \) and \( l_0 > l_1 > \ldots > l_{r-1} \), then the maximum number of edges joining vertices from a set of \( m \) vertices is \( I_G(m) = \sum_{i=0}^{r-1} (l_i/2 + i)2^i \), where \( 1 \leq m \leq 2^n, n \geq 1 \).

Note that this implies \( \theta_G(m) = nm - 2I_G(m) = \sum_{i=0}^{r-1} (n - l_i - 2i)2^i \). And hence

\[
\sum_{m=1}^{2^n-1} \theta_G(m) = \sum_{k=0}^{n-1} 2^k \sum_{i=0}^{n-k-1} \left( (n-k-2i)2^k \binom{n-k-1}{i} \right),
\]

where for a sequence \( n > l_0 > l_1 > \ldots > l_{r-1} \geq 0 \) with \( l_i = k \) for some \( 0 \leq i < n \), there are \( 2^k \) choices for \( l_{i+1}, \ldots, l_{r-1} \) and \( \binom{n-k-1}{i} \) choices for \( l_0, \ldots, l_{i-1} \). Then
Definition 3.4. For any $G$ any $v \in \ell(X)$ is linear arrangement. If $n \geq 1$, then there exist $(n-1)$-dimensional BC structures linear arrangements. Define $f : V(G) \to P_n$ as follows. For any $v \in V(G_1)$, let $f(v) = f_1(v)$ and for any $v \in V(G_2)$, let $f(v) = f_2(v) + 2^{n-1}$. We call it a $n$-dimensional BC structure linear arrangement.

Theorem 3.5. The MinLA of BC graph $X_n$ is

$$\text{MinLA}(X_n) = \sum_{i=1}^{2^{n-1}} \theta_{X_n}(i) = 2^{n-1}(2^n - 1).$$

Proof. By Lemma 2.3 and analysis above, we have $\text{MinLA}(X_n) \geq \sum_{i=1}^{2^{n-1}} \theta_{X_n}(i) = 2^{n-1}(2^n - 1)$. To prove the equality, we need to show that for any $X_n \in \ell_n$, for any BC structure linear arrangement $f_n$,

$$LA_{f_n}(X_n) = 2^{n-1}(2^n - 1).$$

It is direct to show that $LA_{f_1}(X_1) = 1 = 2^0(2^1 - 1)$.

Suppose for $n \geq 2$, any $X_{n-1} \in \ell_{n-1}$ and any BC structure linear arrangement $f_{n-1}$,

$$LA_{f_{n-1}}(X_{n-1}) = 2^{n-2}(2^{n-1} - 1).$$

Take any $X_n \in \ell_n$ and any $n$-dimensional BC linear arrangement $f_n : V(X_n) \to P_n$. Then there is $G_1, G_2 \in \ell_{n-1}$ such that $X_n = G_1 \oplus G_2$ and there are $(n-1)$-dimensional BC structure linear arrangements $f' : V(G_1) \to P_{n-1}$, $f'' : V(G_2) \to P_{n-1}$ such that for any $v \in V(G_1)$, $f_n(v) = f'(v)$, for any $v \in V(G_2)$, $f_n(v) = f''(v) + 2^{n-1}$. Then

$$LA_{f_n}(X_n) = LA_{f'}(G_1) + LA_{f''}(G_2) + \sum_{a \in G_1, \ b \in G_2} |P_{f_n}(a, b)|.$$

By induction hypothesis, $LA_{f'}(G_1) = LA_{f''}(G_2) = 2^{n-2}(2^{n-1} - 1)$.

By direct computation,

$$\sum_{a \in G_1, \ b \in G_2} |P_{f_n}(a, b)| = 2^{n-1} + 2 \sum_{i=1}^{2^{n-1}-1} i = 2^{2n-2}.$$

Thus for any $n$-dimensional BC structure linear arrangement $f_n$, we have

$$LA_{f_n}(X_n) = 2^{n-1}(2^n - 1).$$

Hence, the theorem is proved by induction.
4 Concluding Remarks

In this paper, we computed the MinLA for BC graphs. Finding the other parameters, such as bandwidth, cutwidth, edge bisection and vertex bisection for BC graphs are under investigation.

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