ON THE NUMBER OF NON-REAL ZEROES OF A HOMOGENEOUS DIFFERENTIAL POLYNOMIAL AND A GENERALIZATION OF THE LAGUERRE INEQUALITIES

Mikhail Tyaglov and Mohamed J. Atia

Abstract. Given a real polynomial $p$ with only real zeroes, we find upper and lower bounds for the number of non-real zeroes of the differential polynomial

$$F_\kappa[p](z) \overset{df}{=} p(z)p''(z) - \kappa|p'(z)|^2,$$

where $\kappa$ is a real number.

We also construct a counterexample to a conjecture by B. Shapiro [26] on the number of real zeroes of the polynomial $F_{\frac{n-1}{n}}[p](z)$ in the case when the real polynomial $p$ of degree $n$ has non-real zeroes. Instead we formulate some new conjectures generalizing the Hawaii conjecture.

1. Introduction

Let $p$ be a real polynomial with only real zeroes. Then

$$p(x)p''(x) - |p'(x)|^2 \leq 0, \quad x \in \mathbb{R}.\quad (1.1)$$

This inequality is called the Laguerre inequality. It is well known that the entire functions of the Laguerre-Pólya class satisfy this inequality as well. The Laguerre inequality plays an important role in the study of distribution of zeroes of real entire functions and in understanding the nature of the Riemann $\xi$-function and trigonometric integrals, see [4, 9], [8–12] and references there for the generalizations of the Laguerre inequality, as well. The Laguerre inequality is sharp for entire functions of the Laguerre-Pólya class in the sense that for any entire function $f(z)$ in this class which is not a polynomial, the inequality

$$f(x)f''(x) - \kappa|f'(x)|^2 \leq 0, \quad x \in \mathbb{R},\quad (1.2)$$

holds for $\kappa \geq 1$, and for any $\kappa < 1$ there exists a function in Laguerre-Pólya class which is not a polynomial such that (1.2) is not true for this function. The function $e^{-x^2}$ is such an example for any $\kappa < 1$. In [21], there was studied a lower bound for the number of non-real zeroes of the function

$$F_\kappa[f](z) = f(z)f''(z) - \kappa|f'(z)|^2$$

in the case when the entire function $f(z)$ has only finitely many non-real zeroes. The zeroes of the function $F_\kappa[f](z)$ when $f(z)$ is a meromorphic function were studied in [2] [17] [18] [19] [27].

Date: December 24, 2019.
1991 Mathematics Subject Classification. Primary 12D10; Secondary 26C10; 26C15; 30C15; 30C10.
Key words and phrases. Zeroes of polynomials; non-real zeroes; Laguerre inequalities.

The class of entire functions that are uniform limits on compact sets of sequences of polynomials with only real zeroes, see [20] and references there.
For polynomials with only real zeroes, inequality (1.1) is not sharp. In fact, if \( p \) is a real polynomial of degree \( n \) with only real zeroes, then the following inequality holds \([21]\) [22]:

\[
p(x)p''(x) - \frac{n-1}{n}[p'(x)]^2 \leq 0, \quad x \in \mathbb{R}.
\]

This inequality is called the differential form of the Newton inequality [22]. According to [26], this inequality (together with some other ones) was found by G. Pólya while he studied unpublished notes of J. Jensen.

For an arbitrary real polynomial \( p \), the Laguerre inequality (1.1) does not hold anymore, generally speaking. In [6], it was conjectured that in this case, the number of real zeroes of the function

\[
Q_1[p](z) = \frac{p(z)p''(z) - [p'(z)]^2}{p^2(z)},
\]

does not exceed the number of non-real zeroes of the polynomial \( p \). This conjecture was nicknamed the Hawaii conjecture by T. Sheil-Small [29]. It was also noticed in [7] that the conjecture can be extended to entire functions. The zeroes of the function \( Q_1[p](z) \) of polynomials were also studied in [13, 14] as well as in [8, 15]. It was believed that the Hawaii conjecture (if true) follows from some geometric properties of level curves of logarithmic derivatives, see e.g. [3]. However, it turned out [28] that the fact claiming by the conjecture is a non-trivial consequence of Rolle’s theorem. Indeed, the long and sophisticated proof is based on laborious calculations of the number of zeroes of \( Q_1[p] \) on certain intervals. Some researches still hope to find another proof, more simple than the one from [28].

Inspired by the Hawaii conjecture and the Newton inequalities, in [26] it was conjectured that the number of real zeroes of the rational function

\[
Q_{n-1}[n-n] p(z) = \frac{p(z)p''(z) - [n-n][p'(z)]^2}{p^2(z)}.
\]

does not exceed the number of non-real zeroes of the polynomial \( p \) of degree \( n \). The present work was initially motivated by this conjecture. We disprove it by a counter-example (see Section 5), and estimate the number of non-real zeroes of the differential polynomial

\[
F_\kappa[p](z) \overset{def}{=} p(z)p''(z) - \kappa[p'(z)]^2
\]

for \( \kappa \in \mathbb{R} \) and \( p \) a real polynomial with only real zeroes. Any multiple zero of \( p \) is a zero of \( F_\kappa[p](z) \). The zeroes of the polynomial \( F_\kappa[p](z) \) which are zeroes of \( p \) are called trivial while all other zeroes are called non-trivial. So if \( p \) has only real zeroes, then all the non-real zeroes of \( F_\kappa[p](z) \) are non-trivial. In the present work, we find lower and upper bounds on the number of non-real zeroes of \( F_\kappa[p](z) \) for arbitrary real \( \kappa \). Note that if \( \kappa \) is a non-real number, then \( F_\kappa[p](z) \) has no non-trivial zeroes at all by de Gua’s rule [24], since in this case any zero of \( F_\kappa[p](z) \) must be a zero of \( p'(z) \). Thus, the case of non-real \( \kappa \) is trivial and is out of the scope of the present work.

The paper is organized as follows. In Section 2 we state our main results on the number of non-real zeroes of the differential polynomial \( F_\kappa[p](z) \) in the case when \( \kappa \) is real and the polynomial \( p \) has only real zeroes. Section 3 is devoted to the calculation of the total number of non-trivial zeroes of the polynomial \( F_\kappa[p](z) \) for arbitrary complex polynomial \( p \). In Section 4 we prove our main results, inequalities (2.2)–(2.7) stated in Section 2. In Section 5 we consider the differential polynomial \( F_\kappa[p](z) \) for \( p \) to be an arbitrary real polynomial, and disprove a conjecture of B. Shapiro [26] by a counterexample. We also provide a conjecture generalizing the Hawaii conjecture. In Appendix, we prove a generalization of an auxiliary fact established in [28, Lemma 2.5].

Throughout the paper we use the following notation.

**Notation.** If \( f(z) \) is a real rational function or a real polynomial, by \( Z_C(f) \) we denote the number of non-real zeroes of \( f(z) \), counting multiplicities, by \( Z_R(f) \) the number of real zeroes of \( f(z) \), counting multiplicities. In the sequel, we also denote the number of zeroes of \( f(z) \) in an interval \((a, b)\) and at a point \( \alpha \in \mathbb{R} \) by \( Z_{(a, b)}(f) \) and \( Z_{(\alpha)}(f) \), respectively, thus \( Z_E(f) = Z_{(-\infty, +\infty)}(f) \). Generally, the number of zeroes of \( f(z) \) on a set \( X \) where \( X \) is a subset of \( \mathbb{R} \) will be denoted by \( Z_X(f) \).
2. Main results

Let \( p \) be a real polynomial with only real zeroes

\[
p(z) = a_0 \prod_{k=1}^{d} (z - \lambda_k)^{n_k}
\]

where \( d, d \geq 2 \), is the number of distinct zeroes of \( p \), \( n_k \in \mathbb{N} \) is the multiplicity of the zero \( \lambda_k \) of \( p \), \( k = 1, \ldots, d \), so \( n = n_1 + \ldots + n_d \), and we set

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_d.
\]

**Theorem 2.1.** Let \( d \geq 4 \). Suppose that among the zeroes \( \lambda_2, \ldots, \lambda_{d-1} \) there are \( d_j \) zeroes of multiplicity \( m_j \), \( j = 1, \ldots, r \), \( r \geq 2 \), so that

\[
m_1 < m_2 < \cdots < m_r,
\]

where \( m_1 = \min\{n_2, \ldots, n_{d-1}\} \) and \( m_r = \max\{n_2, \ldots, n_{d-1}\} \), and

\[
\sum_{j=1}^{r} d_j = d - 2, \quad \sum_{j=1}^{r} m_j = n - n_1 - n_d.
\]

Then the following inequalities hold:

- if \( \kappa \leq \frac{m_1 - 1}{m_1} \), then
  \[
  Z_C(F_\kappa) = 0;
  \]
- if \( \frac{m_j - 1}{m_j} < \kappa \leq \frac{m_{j+1} - 1}{m_{j+1}}, j = 1, \ldots, r - 1 \), then
  \[
  0 \leq Z_C(F_\kappa) \leq 2 \sum_{i=1}^{j} d_i;
  \]
- if \( \frac{m_r - 1}{m_r} < \kappa < \frac{n - d + 1}{n - d + 2} \), then
  \[
  0 \leq Z_C(F_\kappa) \leq 2d - 4;
  \]
- if \( \frac{n - d + k - 1}{n - d + k} \leq \kappa < \frac{n - d + k}{n - d + k + 1}, k = 2, \ldots, d - 1, \) then
  \[
  2k - 2 \leq Z_C(F_\kappa) \leq 2d - 4;
  \]
- if \( \kappa = \frac{n - 1}{n} \), then
  \[
  Z_C(F_\kappa) = 2d - 4;
  \]
- if \( \kappa > \frac{n - 1}{n} \), then
  \[
  Z_C(F_\kappa) = 2d - 2.
  \]

Theorem follows from Theorems 4.5, 4.9, and 4.15 established in Section 3.

Moreover, if \( d \geq 3 \), and the zeroes \( \lambda_2, \ldots, \lambda_{d-1} \) of \( p \) defined in (2.1) are all of multiplicity \( m \), then in the results above we can formally set \( m_1 = m_r = m \), so for \( \kappa \leq \frac{m - 1}{m} \) the identity (2.2) holds, while for \( \kappa > \frac{m - 1}{m} \) we have inequalities (2.4)–(2.7) (see Theorems 4.5, 4.10, and 4.15).

Thus, the following theorem holds.
Theorem 2.2. Let \( d \geq 3 \), and the zeroes \( \lambda_2, \ldots, \lambda_{d-1} \) of the polynomial \( p \) defined in (2.1) are all of multiplicity \( m \), for \( \varkappa \leq \frac{m-1}{m} \) one has \( Z_C(F_\varkappa) = 0 \), while for \( \varkappa > \frac{m-1}{m} \) inequalities (2.4)–(2.7) hold (with \( m_r = m \)).

Finally, if \( d = 2 \), that is, if the polynomial \( p \) has only two distinct zeroes, then Theorems 4.5, 4.6, and 4.8 imply the following fact.

Theorem 2.3. If the polynomial \( p \) has exactly two distinct zeroes, then \( Z_C(F_\varkappa) = 0 \) for \( \varkappa \leq \frac{n-1}{n} \), and \( Z_C(F_\varkappa) = 2 \) for \( \varkappa > \frac{n-1}{n} \).

Remark 2.4. If the polynomial \( p \) has a unique zero, then the polynomial \( F_\varkappa[p](z) \) has no non-trivial zeroes, see (3.3)–(3.5), so this case is trivial.

Note that the non-trivial zeroes of the polynomial \( F_\varkappa[p](z) \) are (not vice versa!) the solutions of the equation

\[
R(z) = \varkappa,
\]

where the function \( R(z) \) is defined as follows

\[
R(z) \overset{\text{def}}{=} \frac{p(z)p''(z)}{[p'(z)]^2}.
\]

If \( p \) has only real zeroes and at least two of them are distinct, then the function \( R(z) \) is concave between its poles (Theorem 4.1), and inequalities (2.2)–(2.7), in fact, show that \( R(z) \) has no maximum values over \( \frac{n-2}{n-1} \) (inclusive), and can have at most one maximum value between \( \frac{n-3}{n-2} \) (inclusive) and \( \frac{n-2}{n-1} \) (exclusive), at most two maximum values between \( \frac{n-4}{n-3} \) (inclusive) and \( \frac{n-3}{n-2} \) (exclusive), etc., see Fig. 2.

Figure 1.

Figure 2. The function \( R(z) \) for the polynomial \( p(z) = (z - 15)(z + 13)^4(x - 20)^3(x + 10) \).

At the same time, not all the solutions of equation (2.8) are non-trivial zeroes of the polynomial \( F_\varkappa[p](z) \). But the non-real zeroes of this polynomial coincide with the non-real solutions of equation (2.8), see Theorem 4.3 and Corollary 4.4.

3. THE TOTAL NUMBER OF NON-TRIVIAL ZEROES OF THE POLYNOMIAL \( F_\varkappa[p](z) \)

Let \( p \) be an arbitrary complex polynomial of degree \( n, n \geq 2 \). Consider the differential polynomial

\[
F_\varkappa[p](z) = p(z)p''(z) - \varkappa[p'(z)]^2.
\]

defined in (4.6). As we mentioned in Introduction, all the zeroes of \( F_\varkappa[p](z) \) that are not common with the polynomial \( p \) are called non-trivial.
Notation. We denote the total number of the non-trivial zeroes of \( F_\kappa[p](z) \) as \( Z_{nt}(F_\kappa) \).

Suppose first that \( \kappa \neq \frac{n-1}{n} \). Then the polynomial \( F_\kappa[p](z) \) has exactly \( 2n - 2 \) zeroes, since

\[
F_\kappa[p](z) = \left[ \frac{p^{(n)}(0)}{(n-1)!} \right]^2 \left( \frac{n-1}{n} - \kappa \right) z^{2n-2} + \ldots
\]

so the leading coefficient of \( F_\kappa[p](z) \) is non-zero for \( \kappa \neq \frac{n-1}{n} \).

If the polynomial \( p \) has a unique zero \( \lambda_1 \) of multiplicity \( n \):

\[
p(z) = a_0(z - \lambda_1)^n,
\]

where

\[
\lambda_1 = -\frac{a_1}{na_0},
\]

then we have

\[
F_\kappa[p](z) = a_0^2n^2 \left( \frac{n-1}{n} - \kappa \right) (z - \lambda_1)^{2n-2},
\]

and \( F_\kappa[p](z) \) has no non-trivial zeroes.

Suppose that \( p \) has at least two distinct zeroes, and represent the polynomial \( p \) in the following form

\[
p(z) = a_0 \prod_{j=1}^{l_1} (z - \nu_j) \prod_{k=1}^{l_2} (z - \zeta_k)^{m_k},
\]

where \( n_j \geq 2, j = 1, \ldots, l_2 \), so \( p \) has \( l_1 \) simple zeroes and \( l_2 \) multiple zeroes. We denote by \( d \) the total number of distinct zeroes of the polynomial \( p \):

\[
d \overset{\text{def}}{=} l_1 + l_2,
\]

so

\[
n = d + \sum_{k=1}^{l_2} (m_k - 1).
\]

Theorem 3.1. Let \( p \) be a complex polynomial of degree \( n \), \( n \geq 2 \), with exactly \( d \) distinct zeroes, \( 2 \leq d \leq n \). Then

\[
Z_{nt}(F_\kappa) = 2d - 2,
\]

whenever \( \kappa \neq \frac{k-1}{k} \), \( k = 1, \ldots, n \).

Proof. If \( \lambda \) is a zero of the polynomial \( p \) of multiplicity \( m \), then

\[
p(z) = A(z - \lambda)^m + B(z - \lambda)^{m+1} + C(z - \lambda)^{m+2} + O((z - \lambda)^{m+3}), \quad A \neq 0, \quad \text{as} \quad z \to \lambda,
\]

therefore,

\[
F_\kappa[p]\lambda) = (m-1) \cdot m^2 \cdot A \cdot (z - \lambda)^{2m-2} + 2 \left( \frac{m}{m+1} - \kappa \right) \cdot m(m+1) \cdot AB \cdot (z - \lambda)^{2m-1} +
\]

\[
+ \left[ \left( \frac{m}{m+1} - \kappa \right) \cdot (m+1)^2 \cdot B^2 + 2 \left( \frac{m^2 + m + 1}{m(m+2)} - \kappa \right) \cdot m(m+2) \cdot AC \right] \cdot (z - \lambda)^{2m} + O((z - \lambda)^{2m+1}),
\]

as \( z \to \lambda \). Thus, if \( \kappa \neq \frac{k-1}{k} \), \( k = 1, \ldots, n \), then a zero \( \lambda \) of \( p \) of multiplicity \( m > 1 \) is a trivial zero of \( F_\kappa[p](z) \) of multiplicity \( 2m - 2 \), and is not a zero of \( F_\kappa[p](z) \) if \( m = 1 \).
Now from (3.6), we obtain that the total number of all the trivial zeroes of \( F_\kappa[p](z) \) is equal to
\[
\sum_{k=1}^{l_2} (2m_k - 2) = 2\sum_{k=1}^{l_2} (m_k - 1) = 2n - 2d.
\]
(3.9)

Therefore, the total number of all non-trivial zeroes of \( F_\kappa[p](z) \) equals 2d − 2, since \( \deg F_\kappa = 2n - 2 \), as we established above.

**Corollary 3.2.** If \( p \) is a complex polynomial of degree \( n, n \geq 2 \), with exactly \( d \) distinct zeroes, \( 2 \leq d \leq n \), and if \( \kappa = \frac{k-1}{k} \) for some \( k = 1, \ldots, n-1 \), then
\[
2d - 2 - 2\alpha_k \leq Z_m(F_\kappa) \leq 2d - 2 - \alpha_k,
\]
(3.10)
where \( \alpha_k \) is the number of zeroes of the polynomial \( p \) of multiplicity \( k \).

Moreover, if \( k = 1 \), i.e. \( \kappa = 0 \), then
\[
2d - 2\alpha_1 \leq Z_m(F_\kappa) \leq 2d - 2 - \alpha_1,
\]
(3.11)
where \( \alpha_1 \) is the number of simple zeroes of the polynomial \( p \).

To prove this corollary we need to remind De Gua’s rule.

**De Gua’s rule** ([24], see also [19]). If a real polynomial has only real zeroes, then its derivatives have no multiple zeroes but the zeroes of the polynomial itself. Additionally, if a number \( \xi \) is a (simple) real zero of \( l^{th} \) derivative of the polynomial \( p, l \geq 1 \), then
\[
p^{(l-1)}(\xi)p^{(l+1)}(\xi) < 0.
\]
(3.12)

**Proof of Corollary 3.2.** From formulæ (3.7)–(3.8) it follows that if \( \lambda \) is a zero of \( p \) of multiplicity \( m \), then it is a zero of \( F_\kappa[p](z) \) of multiplicity at least \( 2m - 2 \). However, if \( \kappa = \frac{m-1}{m} \), then \( \lambda \) is a zero of \( F_\kappa[p](z) \) of multiplicity at least \( 2m - 1 \). If \( B \neq 0 \) in (3.7), then its multiplicity is exactly \( 2m - 1 \), since in this case the coefficient at \( (z - \lambda)^{2m-1} \) in (3.8) is non-zero. However, if \( B = 0 \), which is equivalent to the identity \( p^{(m+1)}(\lambda) = 0 \), then \( \lambda \) is a zero of \( F_\kappa[p](z) \) of multiplicity at least \( 2m \). In fact, its multiplicity is exactly \( 2m \) in this case, since the coefficient at \( (z - \lambda)^{2m} \) can be zero only if additionally \( C = 0 \). But if it is so, then we have
\[
p^{(m)}(\lambda) = m!A \neq 0, \quad p^{(m+1)}(\lambda) = (m+1)!B = 0, \quad p^{(m+2)}(\lambda) = (m+2)!C = 0,
\]
that contradicts De Gua’s rule (see also (4.4)).

So, if \( \kappa = \frac{k-1}{k} \) for some \( k = 1, \ldots, n-1 \), then a zero of the polynomial \( p \) of multiplicity \( k \) is a trivial zero of \( F_\kappa[p](z) \) of multiplicity \( 2k - 1 \) or \( 2k \). Any other zero of \( p \) of multiplicity \( m \neq k \) is a trivial zero \( F_\kappa[p](z) \) of multiplicity \( 2m - 2 \). Summing all the trivial zeroes of \( F_\kappa[p](z) \) with their multiplicities and recalling that \( \deg F_\kappa = 2n - 2 \) we obtain the inequalities (3.10).

If \( \kappa = 0 \), then we can improve the lower bound in (3.10), since the \( p'' \) has at most \( \alpha_1 - 2 \) zeroes common with \( p \). \[\square\]

Consider now the exceptional case \( \kappa = \frac{n-1}{n} \). To calculate the total number of the non-trivial zeroes of the polynomial \( F_{n-1}^n(z) \), we should define its degree first.

Let the polynomial \( p \) has the form
\[
p(z) = a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n, \quad n \geq 2.
\]
(3.13)

If \( p \) has a unique multiple zero, then by (3.5) we have \( F_{n-1}^n[p](z) \equiv 0 \). Thus, to study the zeroes of the polynomial \( F(z) \) we exclude such a situation in the sequel.
Theorem 3.3. Let the polynomial \( p \) defined in (3.13) have at least two distinct zeroes. Then the polynomial \( F_{\frac{n-1}{n}}[p](z) \) has exactly \( 2n - 3 - l \) zeroes for some \( l \), \( 1 \leq l \leq n - 1 \), if and only if \( p \) has the form

\[
p(z) = a_0 \left( z + \frac{a_1}{a_0 n} \right)^n + q(z),
\]

where \( q(z) \) is a polynomial of degree \( n - l - 1 \).

Proof. Indeed, let the polynomial \( p \) be of the form (3.14). Denote by \( b \neq 0 \) the leading coefficient of the polynomial \( q(z) \), and

\[
\lambda := -\frac{a_1}{a_0 n}.
\]

Then we have

\[
F_{\frac{n-1}{n}}[p](z) = [a_0(z - \lambda)^n + q(z)] \cdot [a_0 n(n - 1)z(n - \lambda)^{n-2} + q''(z)] - \frac{n-1}{n} \left[ a_0 n(z - \lambda)^{n-1} + q'(z) \right]^2 =
\]

\[
a_0 b(l + 1)z^{2n - l - 3} + O(z^{2n - l - 4}) \quad \text{as} \quad |z| \to \infty,
\]

as required.

Conversely, for the polynomial \( p \) defined in (3.13) we have

\[
F_{\frac{n-1}{n}}[p](z) = 2 \left[ a_0 a_2 - \frac{n}{2} \left( \frac{a_1}{n} \right)^2 \right] z^{2n - 4} + \cdots,
\]

that is, the polynomial \( F_{\frac{n-1}{n}}[p](z) \) has at most \( 2n - 4 \) zeroes (unless it is not identically zero). In particular, \( F_{\frac{n-1}{n}}[p](z) \equiv \text{const} \) whenever \( n = 2 \).

The leading coefficient of the polynomial \( F_{\frac{n-1}{n}}[p](z) \) vanishes if and only if

\[
a_2 = \left( \frac{\binom{n}{3}}{a_0} \right) \left( \frac{a_1}{n} \right)^2.
\]

In this case, we have

\[
F(z) = 6 \left[ a_0 a_3 - \frac{1}{a_0} \cdot \frac{n}{3} \left( \frac{a_1}{n} \right)^3 \right] z^{2n-5} + \cdots.
\]

The coefficient at the power \( 2n - 5 \) can also be equal to zero.

Continuing in such a way, we obtain that the polynomial \( F_{\frac{n-1}{n}}[p](z) \) has the form

\[
F_{\frac{n-1}{n}}[p](z) = l(l+1) \left[ a_0 a_{l+1} - \frac{1}{a_0^{l+1}} \cdot \frac{n}{l+1} \left( \frac{a_1}{n} \right)^{l+1} \right] z^{2n-3-l} + \cdots,
\]

with \( 1 \leq l \leq n - 1 \) if and only if the first \( l \) coefficients of the polynomial \( p \) satisfy the following identities

\[
a_j = \frac{\binom{n}{j}}{a_0^{j-1}} \left( \frac{a_1}{n} \right)^j, \quad j = 1, 2, \ldots, l.
\]

It easy to see now that the polynomial \( p \) whose coefficients satisfy the identities (3.16) must have the form (3.14).

\[\square\]

Note that if \( l = n - 1 \), then by (3.15) the polynomial \( F_{\frac{n-1}{n}}[p](z) \) has at most \( n - 2 \) zeroes. Moreover, it has less than \( n - 2 \) zeroes if and only if the coefficient \( a_n \) satisfies the identity

\[
a_n = \frac{\binom{n}{n}}{a_0^{n-1}} \left( \frac{a_1}{n} \right)^n.
\]

\[\text{The case } j = 1 \text{ is trivial but we include it to formula (3.16) for generality.}\]
However, if the coefficients of the polynomial $p$ satisfy the identities (3.16) for $l = n - 1$ and the identity (3.17), then the polynomial $p$ has the form (3.3)–(3.4), that is, it has a unique zero of multiplicity $n$, so $F_{n-1}^\frac{n}{n}[p](z) \equiv 0$ in this case as we mentioned above.

Now we are in a position to find the number of non-trivial zeroes of the polynomial $F_{n-1}^\frac{n}{n}[p](z)$.

**Theorem 3.4.** Let $p$ be a complex polynomial of degree $n$, $n \geq 2$, and let $d$ be the number of distinct zeroes of $p$, $2 \leq d \leq n$. Suppose that the polynomial $p$ has the form (3.14). Then

$$Z_{nt} \left(F_{n-1}^\frac{n}{n}[p](z)\right) = 2^d - 3 - l \geq 0.$$  

If the polynomial $p$ has a unique multiple zero, then $F_{n-1}^\frac{n}{n}[p](z) \equiv 0$.

**Proof.** By Theorem 3.3, the degree of the polynomial $F_{n-1}^\frac{n}{n}[p](z)$ equals $2^d - 3 - l$, $1 \leq l \leq n - 1$, if and only if the polynomial $p$ has the form (3.14). In this case, the multiplicity of zeroes of $p$ is bounded by $n - l$. The number $2^d - 3 - l$ can be obtained in the same way as in the proof of Theorem 3.1, see (3.9). Note that this number must automatically be nonnegative, since the polynomial $F_{n-1}^\frac{n}{n}[p](z)$ cannot have negative number of zeroes.

If the polynomial $p$ has a unique multiple zero, then from (3.5) it follows that $F_{n-1}^\frac{n}{n}[p](z) \equiv 0$ as we mentioned above.

**Remark 3.5.** We note that by the aforementioned de Gua rule, Theorem 3.4 is not applicable for polynomials with only real zeroes for $l \geq 2$.

The proof of Theorem 3.4 implies the following curious fact on the lower bound for the number of distinct zeroes of a polynomial.

**Corollary 3.6.** Let $p$ be a complex polynomial of the form

$$p(z) = a_0(z - \lambda)^n + q(z), \quad n \geq 2,$$

where $\deg q = k - 1$ for some $k$, $1 \leq k \leq n - 1$. Then the number $d$ of distinct zeroes of $p$ satisfies the inequality

$$d \geq \left\lfloor \frac{n-k}{2} \right\rfloor + 2 \geq 2.$$  

**Proof.** According to (3.18), the number $2^d - 3 - (n-k)$ is nonnegative, so we have

$$d \geq \frac{n-k+3}{2}.$$  

Since $d$ is integer, inequality (3.19) holds.

In the sequel we use the following auxiliary rational function

$$Q_{\kappa}[p](z) = \frac{p(z)p''(z) - \kappa|p'(z)|^2}{p^2(z)}.$$  

It is easy to see that the set of all non-trivial zeroes of $F_{\kappa}[p](z)$ coincides with the set of all zeroes of $Q_{\kappa}[p](z)$.

**Remark 3.7.** If the polynomial $p$ has a unique zero $\lambda_1$ of multiplicity $n$, then $Q_{\kappa}[p](z) = \frac{C}{(z - \lambda_1)^2}$, where the constant $C$ equals zero if and only if $\kappa = \frac{n-1}{n}$. 

4. The number of non-real zeroes of the polynomial $F_κ[p](z)$ when $p$ has only real zeroes

In this section, we estimate the number of non-real zeroes of the polynomial $F_κ[p](z)$ defined in (3.1) provided $p$ is a real polynomial of degree $n, n \geq 2,$ with only real zeroes.

Let

\begin{equation}
\tag{4.1}
p(z) = a_0 \prod_{k=1}^{d} (z - \lambda_k)^{n_k}, \quad \sum_{k=1}^{d} n_k = n, \quad a_0 > 0, \quad d \leq n.
\end{equation}

Then

\begin{equation}
\tag{4.2}
p'(z) = a_0 n \prod_{k=1}^{d} (z - \lambda_k)^{n_k-1} \prod_{j=1}^{d-1} (z - \mu_j),
\end{equation}

where we fix the order of the zeroes indexing as follows

\begin{equation}
\tag{4.3}
\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{d-1} < \mu_{d-1} < \lambda_d.
\end{equation}

The simplicity of the zeroes $\mu_j, j = 1, \ldots, d - 1,$ of the polynomial $p'(z)$ is guaranteed by de Gua’s rule. By the same rule, we have

\begin{equation}
\tag{4.4}
p(\mu_k)p''(\mu_k) < 0, \quad k = 1, \ldots, d - 1.
\end{equation}

The following auxiliary lemma will be of use in the sequel.

**Theorem 4.1.** If a polynomial $p$ of degree $n, d \geq 2,$ has only real zeroes, then the function

\begin{equation}
\tag{4.5}
R(z) \overset{\text{def}}{=} \frac{p(z)p''(z)}{|p'(z)|^2}
\end{equation}

has the form

\begin{equation}
\tag{4.6}
R(z) = \frac{n-1}{n} + \sum_{j=1}^{d-1} \frac{\beta_j}{(z - \mu_j)^2},
\end{equation}

where

\begin{equation}
\tag{4.7}
\beta_j = \frac{p(\mu_j)}{p''(\mu_j)} < 0, \quad j = 1, \ldots, d - 1.
\end{equation}

Here $d$ is the number of distinct zeroes of $p,$ and $\mu_j, j = 1, \ldots, d - 1,$ are simple distinct zeroes of $p'(z)$ such that $p(\mu_j) \neq 0, j = 1, \ldots, d - 1.$

In particular, the function $R$ is concave between its poles.

**Proof.** Consider a polynomial $p$ as in (4.1), and note that $\deg[pp''] = \deg[(p')^2], \text{ so } \lim_{|z| \to \infty} R(z)$ is equal to the ratio of the leading coefficients of the polynomials $pp''$ and $(p')^2.$ Since the leading coefficient of $pp''$ equals $a_0^2 n(n - 1),$ and the leading coefficient of $(p')^2$ is $a_0^2 n^2,$ we have

\[\lim_{|z| \to \infty} R(z) = \frac{a_0^2 n(n - 1)}{a_0^2 n^2} = \frac{n-1}{n}.
\]

Furthermore, it is clear that the numbers $\mu_j, j = 1, \ldots, d - 1$ are the only poles of the function $R(z),$ and $p(\mu_j)p''(\mu_j) \neq 0$ by (4.4). Thus, $R(z)$ has the form

\begin{equation}
\tag{4.8}
R(z) = \frac{n-1}{n} + \sum_{j=1}^{d-1} \left( \frac{\alpha_j}{z - \mu_j} + \frac{\beta_j}{(z - \mu_j)^2} \right).
\end{equation}
However, it is easy to see that \(\mu_j\) defined in (4.2)–(4.3) is a simple pole of the function \(\frac{p(z)}{p'(z)}\) with the residue \(p(\mu_j)/p''(\mu_j)\):
\[
\frac{p(z)}{p'(z)} = \frac{p(\mu_j)/p''(\mu_j)}{z - \mu_j} + O(1) \quad \text{as} \quad z \to \mu_j,
\]
so
\[
R(z) = 1 - \left(\frac{p(z)}{p'(z)}\right)' = \frac{p(\mu_j)/p''(\mu_j)}{(z - \mu_j)^2} + O(1) \quad \text{as} \quad z \to \mu_j.
\]
Consequently, in (4.8) the coefficients \(a_j\) are all zero, and the coefficients \(\beta_j\) are defined by formula (4.7), and their negativity follows from (4.4).

Finally, the function \(R''(z)\) has the form
\[
R''(z) = \sum_{j=1}^{d-1} 3! \frac{\beta_j}{(z - \mu_j)^4},
\]
so it is negative at any real point where it exists, hence \(R(x)\) is concave between its poles as \(x \in \mathbb{R}\).

From formulæ (4.5)–(4.6) it easy to describe the location of zeroes of the function \(R(z)\).

**Theorem 4.2.** Let a polynomial \(p\) of degree \(n \geq 2\) with only real zeroes and its derivative \(p'\) be defined as in (4.1)–(4.3). Then the function \(R(z)\) defined in (4.5) has exactly one zero (counting multiplicities) on each of the intervals \((-\infty, \mu_1)\) and \((\mu_{d-1}, +\infty)\), and exactly two zeroes (counting multiplicities) on each of the intervals \((\mu_j, \mu_{j+1})\), \(j = 1, \ldots, d - 2\).

**Proof.** From (4.6) it follows that
\[
R(x) \to \frac{n-1}{n} \quad \text{as} \quad x \to \pm \infty.
\]
Moreover, the function
\[
R'(z) = - \sum_{j=1}^{d-1} 2! \frac{\beta_j}{(z - \mu_j)^3}
\]
is decreasing between its poles (its derivative \(R''(x)\) is negative on \(\mathbb{R}\) where exists), and \(R'(x) \to 0\) as \(x \to \pm \infty\). Consequently, \(R'(x) < 0\) in the interval \((-\infty, \mu_1)\), and \(R'(x) > 0\) in the interval \((\mu_{d-1}, +\infty)\). (We remind the reader that the zeroes of \(p\) and \(p'(x)\) are indexed in the order (4.3).) Thus, the function \(R(x)\) decreases from \(\frac{n-1}{n}\) to \(-\infty\) in \((-\infty, \mu_1)\), and increases from \(-\infty\) to \(\frac{n}{n-1}\) in \((\mu_{d-1}, +\infty)\).

The monotone behaviour of the function \(R(z)\) on the intervals \((-\infty, \mu_1)\) and \((\mu_{d-1}, +\infty)\) shows that \(R(z)\) has exactly one zero, counting multiplicities, on each of these interval. Furthermore, since \(p\) has only real zeroes, the function \(R(z)\) has exactly \(2d - 2\) real zeroes (possibly multiple), since \((pp')(z)\) has exactly \(2n - 2\) zeroes, \(2n - 2d\) of which are common with \((p')^2(z)\). The concavity of \(R(z)\) between its poles implies that \(R(z)\) has exactly \(2\) zeroes, counting multiplicities, in each interval \((\mu_k, \mu_{k+1})\), \(k = 1, \ldots, d - 2\), so \(R(z)\) has \(2d - 4\) zeroes in the interval \((\mu_1, \mu_{d-1})\).

The most important property of the function \(R(z)\) is represented by the following theorem.

**Theorem 4.3.** Given a real polynomial \(p\) with only real zeroes and \(d \geq 2\), for any \(\alpha \in \mathbb{R}\), \(\alpha \neq \frac{n-1}{n}\), the equation
\[
R(z) = \alpha,
\]
has exactly \(2d - 2\) solutions. Moreover, if \(\alpha \neq \frac{m-1}{m}\), \(m = 1, \ldots, n\), then the set of solutions of equation (4.10) coincides with the set of non-trivial zeroes of \(F_{\alpha}[p](z)\).
If \( \varkappa = \frac{m-1}{m} \) for certain \( m, 1 \leq m \leq n \), then the set of solutions of equation (4.10) coincides with the set of non-trivial zeroes of \( F_\varkappa[p](z) \), except the zeroes of the polynomial \( p \) of multiplicity \( m \).

In particular, the set of non-real zeroes of the polynomial \( F_\varkappa[p](z) \) defined in (3.1) coincides with the set of non-real solutions of equation (4.10).

Proof. It is clear that equation (4.10) is equivalent to the equation

\[
\frac{F_\varkappa[p](z)}{|p'(z)|^2} = 0.
\]

If \( \lambda \) is a zero of \( p \) of multiplicity \( m \geq 2 \), then by (3.8) it is a zero of the polynomial \( F_\varkappa[p](z) \) of multiplicity at least \( 2m - 2 \) and is a zero of \([p'(z)]^2\) of multiplicity exactly \( 2m - 2 \). Thus, from (3.6) it follows that the total number of common zeroes of \( F_\varkappa[p](z) \) and \([p'(z)]^2\) equals \( 2n - 2d \) for any \( \varkappa \in \mathbb{R} \) (including \( \varkappa = \frac{n-1}{n} \)). Since for any \( \varkappa \in \mathbb{R}, \varkappa \neq \frac{n-1}{n} \), the total number of zeroes of \( F_\varkappa[p](z) \) is \( 2n - 2 \) by (3.2), we have that the total number of solutions of equation (4.10) equals \( 2d - 2 \).

Now let us notice the following simple fact. If \( \lambda_k \) is a zero of \( p \) of multiplicity \( n_k \geq 1 \), then from (3.7), it follows that

\[
R(z) = \frac{n_k - 1}{n_k} + \frac{2B}{An_k^2}(z - \lambda_k) + \frac{3[(n_k + 1)B^2 + 2A \cdot C \cdot n_k]}{A^2n_k^3}(z - \lambda_k)^2 + O((z - \lambda_k)^3) \quad \text{as} \quad z \to \lambda_k,
\]

therefore,

\[
R(\lambda_k) = \frac{n_k - 1}{n_k}.
\]

Consequently, if \( \varkappa \neq \frac{m-1}{m}, m = 1, \ldots, n \), then the zeroes of the polynomial \( p \) do not solve equation (4.10). So the set of zeroes of \( F_\varkappa[p](z) \), and the set of solutions of (4.10) coincide.

If \( \varkappa \neq \frac{m-1}{m} \) for certain \( m, 1 \leq m \leq n \), then any zero of \( p \) of multiplicity \( m \) is a solution to equation (4.10). Such solutions are trivial zeroes of the polynomial \( F_\varkappa[p](z) \). However, all other solutions (including all non-real solutions) of equation (4.10) are non-trivial zeroes of \( F_\varkappa[p](z) \), and only they.  

The exceptional case \( \varkappa = \frac{n-1}{n} \) can be treated in a similar way with use of Theorem 3.3 for \( l = 1 \).

**Corollary 4.4.** Given a real polynomial \( p \) with only real zeroes and \( d \geq 2 \), for \( \varkappa = \frac{n-1}{n} \), equation (4.10) has exactly \( 2d - 4 \) solutions. Moreover, the set of non-real zeroes of the polynomial \( F_{n-1}[p](z) \) defined in (3.1) coincides with the set of non-real solutions of equation (4.10).

Thus, in what follows we count the number of non-real solutions of equation (4.10) or the number of non-real zeroes of the function \( Q_\varkappa[p](z) \) that coincide with the number \( Z_C(F_\varkappa) \).

Now we are in a position to consider various intervals for the real parameter \( \varkappa \).

---

3When \( l > 1 \) in Theorem 3.3 the polynomial has non-real zeroes by de Gua’s rule, so we exclude such a case from our investigation, see Remark 3.5.
4.1. The cases \( \kappa \geq \frac{n-1}{n} \) and \( \kappa \leq 0 \). Formulae (4.6)–(4.7) imply that

\[
R(x) < \frac{n-1}{n}
\]

for any \( x \in \mathbb{R} \) (as \( x \) approaches a pole of \( R(x) \), it tends to \(-\infty\)), so equation (4.10) has no real solutions for

\[
\kappa \geq \frac{n-1}{n}.
\]

Thus, Theorem 4.3 and Corollary 4.4 imply the following fact.

**Theorem 4.5.** Let \( p \) be a real polynomial of degree \( n \) with only real zeroes, and \( d \geq 2 \). Then

\[
Z_C(F_\kappa) = 2d - 2 \quad \text{for} \quad \kappa > \frac{n-1}{n},
\]

and

\[
Z_C(F_\kappa) = 2d - 4 \quad \text{for} \quad \kappa = \frac{n-1}{n},
\]

where \( d \) is the number of distinct zeroes of the polynomial \( p \).

Note that inequality (4.13) is equivalent to the Newton inequality (1.3) for polynomials with only real zeroes. Moreover, it is clear that

\[
Q_\kappa[p](x) < 0, \quad \kappa \geq \frac{n-1}{n}
\]

for any \( x \in \mathbb{R} \) where \( Q_\kappa[p](x) \) is finite.

Let now \( \kappa \leq 0 \). Then the following theorem is true.

**Theorem 4.6.** Let a real polynomial \( p \) of degree \( n \) have only real zeroes, and let \( d \geq 2 \). If \( \kappa \leq 0 \), then

\[
Z_C(F_\kappa) = 0.
\]

Moreover, all the (real) non-trivial zeroes of \( F_\kappa[p](z) \) are simple for any \( \kappa < 0 \).

Here \( d \) is the number of distinct zeroes of the polynomial \( p \).

**Proof.** The theorem asserts that all the non-trivial zeroes of polynomial \( F_\kappa[p](z) \) are real if \( \kappa \leq 0 \) and \( p \) has only real zeroes. By de Gua’s rule, a number is a non-trivial zero of the polynomial \( F_\kappa[p](z) \) if and only if it is a solution of equation (4.10) provided \( \kappa \leq 0 \). Here \( R(z) \) is the rational function defined in (4.5).

From Theorem 4.2 the function \( R(z) \) has no non-real zeroes, so does the polynomial \( F_\kappa[p](z) \) for \( \kappa = 0 \) according to Theorem 4.3. Consequently,

\[
Z_C(F_0) = 0.
\]

Consider now \( \kappa < 0 \). By Theorem 4.2 the function \( R(z) \) has exactly one zero (counting multiplicities) in each of the intervals \((\infty, \mu_1)\) and \((\mu_{d-1}, \infty)\), and exactly two zeroes (counting multiplicities) in each of the intervals \((\mu_k, \mu_{k+1})\), \( k = 1, \ldots, d-2 \). Let us denote by \( \xi_1^{(k)} \) and \( \xi_2^{(k)} \), \( \xi_1^{(k)} \leq \xi_2^{(k)} \), the zeroes of \( R(z) \) in \((\mu_k, \mu_{k+1})\), \( k = 1, \ldots, d-2 \). Let \( \xi_0 \) and \( \xi_{d-1} \) be the zeroes of \( R(z) \) in the intervals \((\infty, \mu_1)\) and \((\mu_{d-1}, \infty)\), respectively. From (4.6)–(4.7) it follows that \( R(x) \) monotonously decreases to \(-\infty\) as \( x \to \pm \mu_j \). Therefore, for \( \kappa < 0 \), the equation \( R(x) = \kappa \) has exactly one solution, counting multiplicity, in each interval \((\xi_0, \mu_1)\), \((\mu_{d-1}, \xi_{d-1})\), \((\mu_k, \xi_1^{(k)})\), \((\xi_2^{(k)}, \mu_{k+1})\), \( k = 1, \ldots, d-2 \), and no solutions on the intervals \((\mu_k, \mu_{k+1})\). So it has exactly \( 2d - 2 \) real simple solutions.

Thus, for \( \kappa < 0 \) all solutions of the equation \( R(x) = \kappa \) are real and simple. By Theorem 4.3, the polynomial \( F_\kappa[p](z) \) has no non-real zeroes for \( \kappa < 0 \), as required. \( \square \)
So we found out that if $\kappa \leq 0$, then the polynomial $F_\kappa[p](z)$ has no non-real non-trivial zeroes, but it has only non-real non-trivial zeroes for $\kappa \geq \frac{n-1}{n}$ whenever the polynomial $p$ of degree $n$ has only real zeroes. Thus, the number of non-real zeroes of $Q_\kappa[p](z)$ must increase as $\kappa$ changes continually from 0 to $\frac{n-1}{n}$.

Additionally, we found out that the function $R(z)$ has exactly one local maximum on each interval $(\mu_k, \mu_{k+1})$, $k = 1, \ldots, d-2$, and the values of these maxima are on the interval $[0, \frac{n-1}{n})$. Thus, if $\kappa$ increases from 0 to $\frac{n-1}{n}$ and becomes larger than some local maximum of $R(z)$, then equation (4.10) loses a pair of real solutions.

4.2. The case $0 < \kappa < \frac{n-1}{n}$. Now we are in a position to estimate the number of non-real zeroes of the polynomial $F_\kappa[p](z)$ for $0 < \kappa < \frac{n-1}{n}$.

First, we prove the following simple auxiliary fact.

**Lemma 4.7.** Let $p$ be a real polynomial of degree $n \geq 2$, $d \geq 2$. Then equation (4.10) exactly one zero (counting multiplicities) on each of the intervals $(-\infty, \mu_1)$ and $(\mu_{d-1}, +\infty)$, provided $p$ is real-rooted and $0 < \kappa < \frac{n-1}{n}$.

**Proof.** In the proof of Theorem 4.2, we established that the function $R(z)$ decreases from $\frac{n-1}{n}$ to $-\infty$ in $(-\infty, \mu_1)$, and increases from $-\infty$ to $\frac{n-1}{n}$ in $(\mu_{d-1}, +\infty)$ whenever $p$ has only real zeroes. So the equation $R(z) = \kappa$ has exactly one root, counting multiplicities, in each interval $(-\infty, \mu_1)$ and $(\mu_{d-1}, +\infty)$ for any $0 < \kappa < \frac{n-1}{n}$.

Let us now consider the case $d = 2$. Recall that by $d$ we denote the number of distinct zeroes of $p$, see (4.1).

**Theorem 4.8.** Let the polynomial $p$ have two distinct zeroes, and let all zeroes of $p$ are real. Then

$$Z_C(F_\kappa) = 0,$$

whenever $0 < \kappa < \frac{n-1}{n}$.

**Proof.** According to Theorem 4.3, all the non-real zeroes of the polynomial $F_\kappa[p](z)$ are solutions to equation (4.10). By the same theorem, this equation has exactly $2d - 2$ solutions if $\kappa \neq \frac{n-1}{n}$. Since $d = 2$, we obtain that equation (4.10) has exactly 2 solutions. At the same time, Lemma 4.7 guarantees that equation (4.10) has at least 2 real solutions. Consequently, if $d = 2$, equation (4.10) has no non-real solutions. Therefore, $F_\kappa[p](z)$ has no non-real zeroes in this case.

Thus, the case $d = 2$ is completely covered by Theorems 4.5, 4.6 and 4.8 and we deal with the case $d \geq 3$ in the rest of the present section.

Let again the polynomial $p$ and its derivative be given by (4.1)–(4.3). We will distinguish the following two cases.

1) $d \geq 4$, and among the zeroes $\lambda_2, \ldots, \lambda_{d-1}$ we have $d_j$ zeroes of multiplicity $m_j$, $j = 1, \ldots, r$, $r \geq 2$, such that

$$m_1 < m_2 < \cdots < m_r.$$
where \( m_1 = \min\{n_2, \ldots, n_{d-1}\} \) and \( m_r = \max\{n_2, \ldots, n_{d-1}\} \), and

\[
(4.19) \quad \sum_{j=1}^r d_j = d - 2, \quad \sum_{j=1}^r m_j = n - n_1 - n_2,
\]

2) \( d \geq 3 \), and all the zeroes \( \lambda_2, \ldots, \lambda_{d-1} \) are of multiplicity \( m = \max\{n_2, \ldots, n_{d-1}\} = \min\{n_2, \ldots, n_{d-1}\} \).

The last case we can treat as the case when \( m = m_1 = m_r \), or as the case of \( r = 1 \).

Now we are in a position to find the upper bound of the number of non-real roots of the polynomial \( F_{\kappa}[p](z) \) for \( 0 < \kappa < \frac{n-1}{n} \).

**Theorem 4.9.** In the case 1) above, the following inequalities hold:

If \( 0 < \kappa \leq \frac{m_1 - 1}{m_1} \), then

\[
(4.20) \quad Z_C(F_{\kappa}) = 0;
\]

If \( \frac{m_j - 1}{m_j} < \kappa \leq \frac{m_{j+1} - 1}{m_{j+1}} \), \( j = 1, \ldots, r - 1 \), then

\[
(4.21) \quad Z_C(F_{\kappa}) \leq 2 \sum_{i=1}^j d_i;
\]

If \( \frac{m_r - 1}{m_r} < \kappa < \frac{n - 1}{n} \), then

\[
(4.22) \quad Z_C(F_{\kappa}) \leq 2d - 4.
\]

**Proof.** By Theorem 4.3, equation (4.10) has exactly \( 2d - 2 \), and the set all the non-real solutions to this equation coincides with the set of all non-real zeroes of the polynomials \( F_{\kappa}[p](z) \) whose number of non-trivial zeroes is at most \( 2d - 2 \) according to Theorem 3.1 and Corollary 3.2. Since equation (4.10) has at least 2 real solutions by Lemma 4.7, we have

\[
(4.23) \quad Z_C(F_{\kappa}) \leq 2d - 4 \quad \text{for} \quad \kappa < \frac{n - 1}{n}.
\]

However, this inequality can be improved for some values of \( \kappa \).

Indeed, from (4.12) it follows that if \( \lambda \) is a zero of the polynomial \( p \) of multiplicity \( m_i \geq 2 \), then for any \( \kappa \leq \frac{m_i - 1}{m_i} \), the equation \( R(z) = \kappa \) has exactly two solutions (counting multiplicities), on the interval \((\mu_{k-1}, \mu_k)\) containing \( \lambda \), since \( R(z) \) is concave between its poles by Theorem 4.1. So if \( \kappa \leq \frac{m_i - 1}{m_i} \) for certain \( m_i \), \( i = 2, \ldots, r \), defined in (4.18), then the equation \( R(z) = \kappa \) has at least

\[
2 + \sum_{j=i}^r d_j
\]

real solutions (counting multiplicities). Therefore, equation (4.10) has at most

\[
2d - 2 - 2 - \sum_{j=i}^{i-1} d_j = \sum_{j=1}^{i-1} d_j
\]

non-real solutions by (4.19). So inequalities (4.21) are true, since the set of non-real solution of equation (4.10) coincides with the set of non-real zeroes of the polynomial \( F_{\kappa}[p](z) \) according to Theorem 4.3.

If \( 0 < \kappa \leq \frac{m_1 - 1}{m_1} \), then the equation \( R(z) = \kappa \) has exactly two zeroes in every interval \((\mu_{k-1}, \mu_k)\), \( k = 1, \ldots, d - 1 \), due to concavity of \( R \) and by (4.12). Consequently, all the solutions of this equation are real, so the identity (4.20) is true. \( \square \)
In the same way, one can prove the corresponding result for the case 2).

**Theorem 4.10.** In the case 2) above, the following holds:

If $0 < \kappa \leq \frac{m-1}{m}$, then
$$Z_C(F_\kappa) = 0;$$

if $\frac{m-1}{m} < \kappa < \frac{n-1}{n}$, then
$$Z_C(F_\kappa) \leq 2d - 4.$$

Thus, the upper bound for the number of non-real zeroes of the polynomial $F_\kappa p(z)$ is established for any $\kappa \in \mathbb{R}$.

In what follows, we find the lower bound for the number of non-real zeroes of the polynomial $F_\kappa p(z)$. To do this, we estimate from above the number of real zeroes of the auxiliary rational function $Q_\kappa p(z)$ defined in (3.20). As we mentioned above, the set of zeroes of $Q_\kappa p(z)$ coincides with the set of all non-trivial zeroes of $F_\kappa p(z)$.

Together with $Q_\kappa p(z)$, let us consider the function:

$$Q_\kappa p(z) = Q_{2, \frac{1}{\kappa}}(p')(z) = \frac{p'(z)p'''(z) - (2 - \frac{1}{\kappa}) |p''(z)|^2}{|p'(z)|^2}.

Relation between the number of zeroes of the functions $Q_\kappa p(z)$ and $\hat{Q}_\kappa p(z)$ on an interval is provided by the following proposition.

**Proposition 4.11.** Let $p$ be a real polynomial, and let $a$ and $b$ be real, and $\kappa > 0$. If $p(z) \neq 0$, $p'(z) \neq 0$ and $p''(z) \neq 0$ for $z \in (a, b)$, then

$$Z_{(a,b)}(Q_\kappa) \leq 1 + Z_{(a,b)}(\hat{Q}_\kappa).$$

For the case $\kappa = 1$, this fact was proved in [28, Lemma 2.5]. The proof of Proposition 4.11 is the same as the proof of Lemma 2.5 in [28], so we skip the proof here but provide it in Appendix for completeness (see Theorem 4.3).

If $p$ has only real zeroes, then the following consequence of inequality (4.25) is true.

**Theorem 4.12.** Let $p$ be a real polynomial of degree $n$ with real zeroes, and $d \geq 3$. Then

$$Z_{\mathbb{R}}(Q_\kappa) \leq Z_{\mathbb{R}}(\hat{Q}_\kappa),$$

for any $\frac{1}{2} \leq \kappa \leq \frac{n-1}{n}$.

We split the proof of this theorem into a few lemmata.

Let the polynomial $p$ and its derivative be defined in (4.1)–(4.2). We fix the order of the zeroes indexing as in (4.3):

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{d-1} < \mu_{d-1} < \lambda_d.$$

Consider the following auxiliary functions

$$R[p](z) \overset{\text{def}}{=} \frac{p(z)p''(z)}{|p'(z)|^2}, \quad R[p'](z) \overset{\text{def}}{=} \frac{p'(z)p'''(z)}{|p''(z)|^2}.$$

By Theorem 4.3, the sets of non-real zeroes of the functions $Q_\kappa p(z)$ and $\hat{Q}_\kappa p(z)$ coincide with the sets of non-real solutions of the equations

$$R[p](z) = \kappa, \quad \text{and} \quad R[p'](z) = 2 - \frac{1}{\kappa},$$

respectively, since the sets of all zeroes of $Q_\kappa p(z)$ and $\hat{Q}_\kappa p(z)$ coincides the sets of all non-trivial zeroes of the polynomials $F_\kappa p(z)$ and $F_{2-\frac{1}{\kappa}} p'(z)$, respectively.
If \( \lambda \) is a (real) zero of the polynomial \( p \) of multiplicity \( m \geq 2 \), then it is a zero of \( p' \) of multiplicity \( m - 1 \). Moreover, if \( \nu = \frac{m - 1}{m} \), then \( 2 - \frac{1}{\nu} = \frac{m - 2}{m - 1} \). Thus, if \( \nu \neq \frac{m - 1}{m} \), then the sets of zeroes of the functions \( Q_\nu[p](z) \) and \( \tilde{Q}_\nu[p](z) \) coincide with the sets of all solutions of the equations (4.28), respectively.

**Lemma 4.13.** For the intervals \((-\infty, \mu_1)\) and \([\mu_{d-1}, +\infty)\), the following inequalities hold

\[
Z_{(-\infty, \mu_1)}(Q_\nu) \leq Z_{(-\infty, \mu_1)}(\tilde{Q}_\nu) \quad \text{and} \quad Z_{[\mu_{d-1}, +\infty)}(Q_\nu) \leq Z_{[\mu_{d-1}, +\infty)}(\tilde{Q}_\nu)
\]

for any \( \frac{1}{2} < \nu < \frac{n - 1}{n} \).

**Proof.** Consider the interval \((-\infty, \mu_1)\). If \( \lambda_1 \) is a simple zero of \( p \), then \( p'(\lambda_1) \neq 0 \), and \( p''(z) \neq 0 \) for all \( z \in (-\infty, \mu_1) \), and \( \mu_1 \) is a simple zero of \( p' \). By Theorems 4.2 and 4.3, the function \( Q_\nu[p](z) \) has a unique simple zero on \((-\infty, \mu_1)\) for any \( \frac{1}{2} < \nu < \frac{n - 1}{n} \), that is,

\[
Z_{(-\infty, \mu_1)}(Q_\nu) = 1.
\]

Moreover, since \( R[p](z) \) is monotone decreasing on \((-\infty, \mu_1)\) and \( R[p](\lambda_1) = 0 \), the function \( Q_\nu[p](z) \) has a unique simple zero on the interval \((-\infty, \lambda_1)\) and no zeroes on \((\lambda_1, \mu_1)\). Analogously, we conclude that \( \tilde{Q}_\nu[p](z) \) has a simple real zero on the interval \((-\infty, \mu_1)\):

\[
Z_{(-\infty, \mu_1)}(\tilde{Q}_\nu) = 1
\]

since \( \mu_1 \) is a zero of \( R[p'](z) \).

Suppose now that \( \lambda_1 \) is a zero of \( p \) of multiplicity \( m \), \( 2 \leq m \leq n - 1 \). Then the polynomial \( p'' \) has a simple zero \( \gamma_0 \) on the interval \((\lambda_1, \mu_1)\) which is a pole of the function \( R[p'](z) \). If \( \frac{m - 1}{m} < \nu < \frac{n - 1}{n} \), so that, \( \frac{m - 2}{m - 1} < \frac{2 - \frac{1}{\nu}}{\frac{n - 1}{n}} < \frac{m - 2}{m - 1} \), then the function \( Q_\nu[p](z) \) has a unique simple zero on the interval \((-\infty, \lambda_1)\) and no zeroes on \([\lambda_1, \mu_1]\) by Theorems 4.2 and 4.3 and by (4.12). The same argument proves that \( \tilde{Q}_\nu[p](z) \) has a unique simple zero on \((-\infty, \lambda_1)\) and no zeroes on \([\lambda_1, \gamma_0]\).

If \( \frac{1}{2} < \nu < \frac{m - 1}{m} \), so that, \( 0 < \frac{2 - \frac{1}{\nu}}{\frac{n - 1}{n}} < \frac{m - 2}{m - 1} \), then the function \( Q_\nu[p](z) \) has a unique simple zero on the interval \([\lambda_1, \mu_1]\) and no zeroes on \((-\infty, \lambda_1)\) by Theorems 4.2 and 4.3 and by (4.12). Analogously, we have that \( \tilde{Q}_\nu[p](z) \) has a unique simple zero on \([\lambda_1, \gamma_0]\) and no zeroes on \((-\infty, \lambda_1)\).

Thus, we get that if \( \lambda_1 \) is a zero of \( p \) of multiplicity \( m \), \( 2 \leq m \leq n - 1 \) and \( \nu \neq \frac{m - 1}{m} \), then the following holds

\[
1 = Z_{(-\infty, \mu_1)}(Q_\nu) \leq Z_{(-\infty, \mu_1)}(\tilde{Q}_\nu),
\]

since the function \( \tilde{Q}_\nu[p](z) \) can have zeroes on the interval \([\gamma_0, \mu_1]\).

Finally, if \( \lambda_1 \) is a zero of \( p \) of multiplicity \( m \), \( 3 \leq m \leq n - 1 \) and \( \nu = \frac{m - 1}{m} \), so that \( 2 - \frac{1}{\nu} = \frac{m - 2}{m - 1} \), then \( \lambda_1 \) is a solution to both equations (4.28), and it is not a zero of the functions \( Q_\nu[p](z) \) and \( \tilde{Q}_\nu[p](z) \).

So we have

\[
Z_{(-\infty, \mu_1)}(Q_\nu) = Z_{(-\infty, \gamma_0)}(\tilde{Q}_\nu) = 0, \quad \text{and} \quad Z_{(-\infty, \mu_1)}(\tilde{Q}_\nu) \geq 0.
\]

Thus, from (4.30)–(4.33) we have

\[
1 = Z_{(-\infty, \mu_1)}(Q_\nu) \leq Z_{(-\infty, \mu_1)}(\tilde{Q}_\nu)
\]

for any \( \frac{1}{2} < \nu < \frac{n - 1}{n} \).

In the same way, one can prove that

\[
Z_{[\mu_{d-1}, +\infty)}(Q_\nu) \leq Z_{[\mu_{d-1}, +\infty)}(\tilde{Q}_\nu)
\]
for any \( \frac{1}{2} < \alpha < \frac{n-1}{n} \), as required.

The next lemma deals with the intervals \((\mu_k, \mu_{k+1})\).

**Lemma 4.14.** For any interval \((\mu_k, \mu_{k+1})\), \(k = 1, \ldots, d - 2\), the following inequality holds

\[
Z_{(\mu_k, \mu_{k+1})}(Q_{\alpha}) \leq Z_{(\mu_k, \mu_{k+1})}(\tilde{Q}_{\alpha})
\]

for any \( \frac{1}{2} < \alpha < \frac{n-1}{n} \).

**Proof.** Since \( \frac{1}{2} < \alpha < \frac{n-1}{n} \), one has \( 0 < 2 - \frac{1}{\alpha} < \frac{n-2}{n-1} \). By (4.3), the polynomial \( p \) has a unique zero \( \lambda_{k+1} \) on the interval \((\mu_k, \mu_{k+1})\).

Suppose that \( \lambda_{k+1} \) is a simple zero of \( p \). Then \( p'' \) has a unique simple zero \( \gamma_k \) on the interval \((\mu_k, \mu_{k+1})\). Without loss of generality we may assume \( \gamma_k \leq \lambda_{k+1} \). Since \( \lambda_{k+1} \) is a simple zero of \( p \), by Theorem 4.3 the zeroes of \( Q_{\alpha} \) and \( \tilde{Q}_{\alpha} \) on the interval \((\mu_k, \mu_{k+1})\) coincide with solutions of equations (4.28),

The numbers \( \gamma_k \) and \( \lambda_{k+1} \) are zeroes of the function \( R[p](z) \) on the interval \((\mu_k, \mu_{k+1})\), so Theorems 4.1 and 4.3 imply

\[
0 = Z_{(\mu_k, \gamma_k)}(Q_{\alpha}) \leq Z_{(\mu_k, \gamma_k)}(\tilde{Q}_{\alpha}),
\]

and

\[
Z_{(\lambda_{k+1}, \mu_{k+1})}(Q_{\alpha}) = 0
\]

for any \( \frac{1}{2} < \alpha < \frac{n-1}{n} \).

Suppose first that \( \gamma_k < \lambda_{k+1} \). Then \( Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) \) equals 0 or 2, since \( R[p](z) \) is concave on \((\mu_k, \mu_{k+1})\) by Theorem 4.1. Moreover, by Theorems 4.1 and 4.3 the function \( \tilde{Q}_{\alpha} \) has an even number of zeroes (at most two) on each interval \((\mu_k, \gamma_k)\) and \((\gamma_k, \mu_{k+1})\), since \( R[p]^1(\mu_k) = R[p]^1(\mu_{k+1}) = 0 \) according to (4.27).

If \( Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 1 \) and \( Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 1 \), then from (4.38) we get

\[
0 = Z_{(\lambda_{k+1}, \mu_{k+1})}(Q_{\alpha}) \leq -1 + Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 0,
\]

and from (4.25)

\[
Z_{(\lambda_{k+1}, \gamma_k)}(Q_{\alpha}) \leq 1 + Z_{(\lambda_{k+1}, \gamma_k)}(\tilde{Q}_{\alpha}).
\]

If \( Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 0 \) and \( Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 2 \), then from (4.38) it follows that

\[
0 = Z_{(\lambda_{k+1}, \mu_{k+1})}(Q_{\alpha}) \leq Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 2,
\]

and by (4.25) we have

\[
0 = Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) \leq Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 0,
\]

since \( Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) \) is an even number as we mentioned above.

If \( Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 2 \) and \( Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 0 \), then from (4.25) we obtain

\[
Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) \leq Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 2,
\]

since \( Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) \) is an even number. We also have

\[
0 = Z_{(\lambda_{k+1}, \mu_{k+1})}(Q_{\alpha}) \leq Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 0.
\]

Finally, if \( Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}) = 0 \) and \( Z_{(\lambda_{k+1}, \mu_{k+1})}(\tilde{Q}_{\alpha}) = 0 \), then inequality (4.25) and identity (4.38) imply

\[
0 = Z_{(\gamma_k, \lambda_{k+1})}(Q_{\alpha}) = Z_{(\gamma_k, \lambda_{k+1})}(\tilde{Q}_{\alpha}).
\]

\footnote{The case \( \gamma_k > \lambda_{k+1} \) can be considered in the same way.}
Thus, from \((4.37)\) and \((4.39)-(4.45)\) we obtain inequality \((4.36)\) for \(\gamma_k < \lambda_{k+1}\) and \(\frac{1}{2} < \kappa < \frac{n-1}{n}\).

If \(\lambda_{k+1} = \gamma_k\), then the equation \(R(z) = \kappa\) has no solutions on the interval \((\mu_k, \mu_{k+1})\) for any \(\kappa > 0\), so inequality \((4.36)\) is true in this case. Consequently, if \(\lambda_{k+1}\) is a simple zero of the polynomial \(p\), then inequality \((4.36)\) holds for any \(\frac{1}{2} < \kappa < \frac{n-1}{n}\).

Let now \(\lambda_{k+1}\) be a zero of \(p\) of multiplicity \(n_{k+1} \geq 2\). Then \(p'(\lambda_{k+1}) = 0\), so \(p''\) has a unique simple zero at each of the intervals \((\mu_k, \lambda_{k+1})\) and \((\lambda_{k+1}, \mu_{k+1})\). We denote these zeroes as \(\gamma'_k\) and \(\gamma''_k\), respectively, so
\[
\mu_k < \gamma'_k < \lambda_{k+1} < \gamma''_k < \mu_{k+1}.
\]

It is clear that \(R[p](\gamma'_k) = R[p](\gamma''_k) = 0\). By Theorem 4.1, the function \(R[p](z)\) is concave (as well as the function \(R[p'](z)\)), so the equation \(R[p](\kappa) = \kappa\) has no solutions on the intervals \((\mu_k, \gamma'_k)\) and \((\gamma''_k, \mu_{k+1})\), and an even number of solution (at most two) on the interval \((\gamma'_k, \gamma''_k)\) for any \(\frac{1}{2} < \kappa < \frac{n-1}{n}\). Consequently, by Theorem 4.3 we have
\[(4.46)\]
\[
0 = Z_{(\mu_k, \gamma'_k)}(Q_\kappa) \leq Z_{(\mu_k, \gamma'_k)}(\hat{Q}_\kappa), \quad 0 = Z_{(\gamma''_k, \mu_{k+1})}(Q_\kappa) \leq Z_{(\gamma''_k, \mu_{k+1})}(\hat{Q}_\kappa).
\]

On the interval \((\gamma'_k, \gamma''_k)\), the function \(Q_\kappa[p](z)\) has 0 or 2 zeroes if \(\kappa \neq \frac{n_{k+1} - 1}{n_{k+1}}\) by Theorem 4.3. However, if \(\kappa = \frac{n_{k+1} - 1}{n_{k+1}}\), then the equation \(R[p](z) = \kappa\) has two solutions (counting multiplicities) on the interval \((\gamma'_k, \gamma''_k)\), and at least one of the solutions is always \(\lambda_{k+1}\) by \((4.12)\). But \(\lambda_{k+1}\) is not a zero of the function \(Q_\kappa[p](z)\). Thus, \(Q_\kappa[p](z)\) has at most one (counting multiplicities) zero on the interval \((\gamma'_k, \gamma''_k)\) in this case.

Let \(n_{k+1} \geq 2\), and \(\frac{n_{k+1} - 1}{n_{k+1}} < \kappa < \frac{n-1}{n}\), so that \(\frac{n_{k+1} - 2}{n_{k+1} - 1} < 2 - \frac{1}{\kappa} < \frac{n-2}{n-1}\). Then the zeroes of \(Q_\kappa[p](z)\) and \(Q_\kappa[p'](z)\) on the interval \((\gamma'_k, \gamma''_k)\) coincide with the solutions of equations \((4.28)\), respectively. Due to concavity of the functions \(R[p](z)\) and \(R[p'](z)\) on the interval \((\gamma'_k, \gamma''_k)\), the functions \(Q_\kappa[p](z)\) and \(Q_\kappa[p'](z)\) have even number of zeroes on this interval and at most two. Moreover, each of these functions has no zeroes on one of the intervals \((\gamma'_k, \lambda_{k+1})\) and \([\lambda_{k+1}, \gamma''_k)\).

There are possible two situations:

1) \(0 = Z_{(\gamma'_k, \lambda_{k+1})}(\hat{Q}_\kappa)\). Then by Proposition 4.11 we have
\[(4.47)\]
\[
0 = Z_{(\gamma'_k, \lambda_{k+1})}(Q_\kappa) \leq Z_{(\lambda_{k+1}, \gamma''_k)}(Q_\kappa) = 0,
\]
since the number \(Z_{(\gamma'_k, \lambda_{k+1})}(Q_\kappa)\) can be 0 or 2 only. It is clear now that the numbers \(Z_{(\lambda_{k+1}, \gamma''_k)}(Q_\kappa)\) and \(Z_{(\lambda_{k+1}, \gamma''_k)}(\hat{Q}_\kappa)\) are even (at most two), so by Proposition 4.11 one has
\[(4.48)\]
\[
Z_{(\lambda_{k+1}, \gamma''_k)}(Q_\kappa) \leq Z_{(\lambda_{k+1}, \gamma''_k)}(\hat{Q}_\kappa),
\]
so from \((4.47)-(4.48)\) it follows that
\[(4.49)\]
\[
Z_{(\gamma'_k, \gamma''_k)}(Q_\kappa) \leq Z_{(\gamma'_k, \gamma''_k)}(\hat{Q}_\kappa).
\]

2) \(0 = Z_{(\lambda_{k+1}, \gamma''_k)}(\hat{Q}_\kappa)\). Then analogously, Proposition 4.11 implies inequality \((4.49)\).

Let now \(n_{k+1} \geq 3\), and \(\frac{1}{2} < \kappa < \frac{n_{k+1} - 1}{n_{k+1}}\), so that \(0 < 2 - \frac{1}{\kappa} < \frac{n_{k+1} - 2}{n_{k+1} - 1}\). Then the roots of \(Q_\kappa[p](z)\) and \(Q_\kappa[p'](z)\) on the interval \((\gamma'_k, \gamma''_k)\) coincide with the solutions of equations \((4.28)\), respectively. By concavity of functions \(R[p](z)\) and \(R[p'](z)\) and by \((4.12)\), we obtain that both equations \((4.28)\) have exactly two solutions (of multiplicity one) on \((\gamma'_k, \gamma''_k)\), therefore,
\[(4.50)\]
\[
2 = Z_{(\gamma'_k, \gamma''_k)}(Q_\kappa) \leq Z_{(\gamma'_k, \gamma''_k)}(\hat{Q}_\kappa) = 2.
\]
Finally, suppose that \( n_{k+1} \geq 3 \), and \( \varkappa = \frac{n_{k+1} - 1}{n_{k+1}} \), so \( \varkappa = \frac{n_{k+1} - 2}{n_{k+1} - 1} \). In this case, from (4.7) and (4.11) we have

\[
R(z) = \frac{n_{k+1} - 1}{n_{k+1}} + \frac{2B}{An_{k+1}^2}(z - \lambda_{k+1}) + \\
3 \left[ (n_{k+1} + 1)B^2 + 2A \cdot C \cdot n_{k+1} \right] \frac{A^2 n_{k+1}^3}{(z - \lambda_{k+1})^2} + O \left( (z - \lambda_{k+1})^2 \right) \quad \text{as} \quad z \to \lambda_{k+1},
\]

(4.51)

and

\[
R[p'](z) = \frac{n_{k+1} - 2}{n_{k+1} - 1} + \frac{2B(n_{k+1} + 1)}{An_{k+1}(n_{k+1} - 1)^2}(z - \lambda_{k+1}) + \\
3 \left[ (n_{k+1} + 1)^2B^2 + 2A \cdot C \cdot (n_{k+1} - 1)(n_{k+1} + 2) \right] \frac{A^2 n_{k+1}^3}{(z - \lambda_{k+1})^2} + O \left( (z - \lambda_{k+1})^3 \right),
\]

(4.52)

as \( z \to \lambda_{k+1} \).

From (4.51) and (4.52) it follows that the number \( \lambda_{k+1} \) is a solution of equations (4.28) of the same multiplicity. That is, it is simultaneously a simple or a multiple (of multiplicity 2) solution of both equations on the interval \((\gamma_1', \gamma_{k+1}'')\) (even for \( n_{k+1} = 2 \)). Therefore, by Theorem 4.3, the functions \( Q_{\varkappa}[p](z) \) and \( \hat{Q}_{\varkappa}[p](z) \) have simultaneously one or no zeroes (counting multiplicities) on the interval \((\gamma_1', \gamma_{k+1}'')\). Consequently, we obtain

\[
Z_{(\gamma_1', \gamma_{k+1}'')}(Q_{\varkappa}) \leq Z_{(\gamma_1', \gamma_{k+1}'')}(\hat{Q}_{\varkappa}),
\]

(4.53)

in this case.

Now from (4.46), (4.49), (4.50), and (4.53), we get that inequality (4.36) holds for any \( \frac{1}{2} < \varkappa < \frac{n - 1}{n} \) in the case when \( \lambda_{k+1} \) is a multiple zero of \( p \) as well, as required.

Now Theorem 4.12 follows from Lemmata 4.13, 4.14 since both functions \( Q_{\varkappa}[p](z) \) and \( \hat{Q}_{\varkappa}[p](z) \) do not vanish at the points \( \mu_k \), \( k = 1, \ldots, d - 1 \).

Now we are in a position to find the lower bound for the number of non-real zeroes of the polynomial \( F_{\varkappa}[p](z) \).

**Theorem 4.15.** Let \( p \) be a real polynomial with pure real zeroes given in (4.1), and let its zeroes be indexed in the following order

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_d,
\]

where \( d \) is the number of distinct zeroes of \( p \). Then the following inequalities hold:

if \( \frac{m_1 - 1}{m_1} < \varkappa < \frac{n - d + 1}{n - d + 2} \), then \( F_{\varkappa}[p](z) \) can have only real zeroes

\[
Z_{C}(F_{\varkappa}) \geq 0,
\]

(4.54)

if \( \frac{n - d + k - 1}{n - d + k} \leq \varkappa < \frac{n - d + k}{n - d + k + 1}, \) \( k = 2, \ldots, d - 1 \), then

\[
Z_{C}(F_{\varkappa}) \geq 2k - 2,
\]

(4.55)

where \( m_1 \) is defined in (4.18).

**Proof.** We prove this theorem by induction. Suppose first that all zeroes of the polynomial \( p \) are simple, that is, \( d = n \) and \( m_1 = 1 \). Then inequalities (4.54)–(4.55) have the following form, in this case:

if \( 0 < \varkappa < \frac{1}{2} \), then

\[
Z_{C}(F_{\varkappa}) \geq 0,
\]

(4.56)
\[
\text{if } \frac{k - 1}{k} \leq \varkappa < \frac{k}{k + 1}, \quad k = 2, \ldots, n - 1, \text{ then}
\]
\[
(4.57) \quad Z_C(F_\varkappa) \geq 2k - 2,
\]

Since all zeros of \( p \) are simple, the set of all non-trivial zeroes of \( F_\varkappa[p] \) (the set of all zeroes of \( Q_\varkappa[p] \)) coincides with the set of all solutions of the equation \( R[p](z) = \varkappa \) by Theorem 4.3. Moreover, if \( 0 < \varkappa < \frac{1}{2} \), then the \( R[p](z) = \varkappa \) may have all real solutions. Indeed, if the polynomial \( p \) is such that \( p \) and \( p'' \) have no common zeroes, then the function \( R[p](z) \) has exactly two distinct zeroes on every interval \((\mu_k, \mu_{k+1})\), \( k = 1, \ldots, n - 2 \), and exactly one (counting multiplicities) zero on each interval \((\infty, \mu_1)\) and \((\mu_{n-1}, +\infty)\) by Theorem 4.2. Thus, on each interval \((\mu_k, \mu_{k+1})\), \( k = 1, \ldots, n - 2 \), the function \( R[p](z) \) has the maximum \( M_k \). Consequently, for any \( 0 < \varkappa < \min\{M_1, \ldots, M_{n-1}\} \) all solutions of the equation \( R[p](z) = \varkappa \) are real. At the same time, for any \( n \geq 3 \) each zero of the second derivative of the polynomial \( q_n(z) = (z^2 - 1)F^{(1,1)}_{n-2}(z) \), where \( P^{(\alpha,\beta)}_n(z) \) is the Jacobi polynomial, is a zero of the polynomial \( q_n \) itself, see, e.g. [1]. Therefore, in this case, the equation \( R[p](z) = \varkappa \) has exactly 2 real solutions for any \( 0 < \varkappa < \frac{n - 1}{n} \). Thus, we obtain that
\[
(4.58) \quad 2 \leq Z_\mathbb{R}(Q_\varkappa) \leq 2n - 2,
\]
for any \( 0 < \varkappa < \frac{1}{2} \).

Let \( \varkappa \in \left(\frac{k - 1}{k}, \frac{k}{k + 1}\right) \) for some number \( k, \ k = 2, \ldots, n - 1 \). Introduce the following sequence of numbers
\[
(4.59) \quad \varkappa_0 \xdef= \varkappa, \quad \varkappa_i \xdef= 2 - \frac{1}{\varkappa_{i-1}}, \quad i = 1, \ldots, k - 1,
\]
and note that \( \varkappa_{k-1} \in \left(0, \frac{1}{2}\right) \). Since \( \tilde{Q}_{\varkappa_0}[p](z) = Q_{\varkappa_1}[p](z) \) and \( \deg p^{(k-1)} = n - k + 1 \), we have from (4.26) and (4.58)
\[
(4.60) \quad Z_\mathbb{R}(Q_{\varkappa_0}[p]) = Z_\mathbb{R}(Q_{\varkappa_0}[p]) \leq Z_\mathbb{R}(Q_{\varkappa_1}[p']) \leq \ldots \leq Z_\mathbb{R}(Q_{\varkappa_{k-1}}[p^{(k-1)}]) \leq 2(n - k + 1) - 2 = 2n - 2k
\]
for any \( \frac{k - 1}{k} < \varkappa < \frac{k}{k + 1}, \ k = 2, \ldots, n - 1 \).

Suppose now that \( \varkappa = \frac{1}{2} \). On each interval \((\mu_k, \mu_{k+1})\), \( k = 1, \ldots, n - 2 \), there lies a zero \( \lambda_{k+1} \) of \( p \) and a zero \( \gamma_k \) of \( p'' \). By Theorem 4.3, the function \( Q_{\frac{1}{2}}[p](z) \) has at most 2 zeroes on \((\mu_k, \mu_{k+1})\), counting multiplicities.

If \( \lambda_{k+1} = \gamma_k \), then the function \( Q_{\frac{1}{2}}[p](z) \) has no zeroes on \((\mu_k, \mu_{k+1})\) due to concavity of \( R[p](z) \).

Let \( \lambda_{k+1} < \gamma_k \) (the case \( \lambda_{k+1} > \gamma_k \) can be considered similarly). Then by concavity of \( R[p](z) \), the function \( Q_{\frac{1}{2}}[p](z) \) has no zeroes on the intervals \((\mu_k, \lambda_{k+1})\) and \([\lambda_{k+1}, \mu_{k+1})\) and has an even number (at most 2) on the interval \((\lambda_{k+1}, \gamma_k)\). If \( \tilde{Q}_{\frac{1}{2}}[p](z) \neq 0 \) on \((\lambda_{k+1}, \gamma_k)\), then by Proposition 4.11 one has \( Z(\lambda_{k+1}, \gamma_k)(Q_{\varkappa}) = 0 \). If there is a (simple) zero of \( \tilde{Q}_{\frac{1}{2}}[p](z) \) on \((\lambda_{k+1}, \gamma_k)\), then \( Z(\lambda_{k+1}, \gamma_k)(Q_{\varkappa}) = 0 \) or 2 according to (4.25). Since \( \tilde{Q}_{\frac{1}{2}}[p](z) = \frac{p''(z)}{p'(z)} \), we have that there at most \( n - 3 \) intervals \((\mu_k, \mu_{k+1})\) where \( Q_{\frac{1}{2}}[p](z) \) has exactly two zeroes (counting multiplicities). Moreover, it has at most one zero (counting multiplicities) on each interval \((\infty, \mu_1)\) and \((\mu_{n-1}, +\infty)\) by Theorems 4.2, 4.3. Thus, we obtain that
\[
(4.61) \quad 0 \leq Z_\mathbb{R}(Q_{\frac{1}{2}}) \leq 2n - 4,
\]
since \( \mu_k, k = 1, \ldots, n - 1 \), are not zeroes of \( Q_{\frac{1}{2}}[p](z) \).

Let now \( \varkappa = \frac{k - 1}{k} \) for some \( k = 3, \ldots, n - 1 \). Then with the numbers (4.59), we have

\[
\varkappa_0 = \varkappa = \frac{k - 1}{k}, \quad \varkappa_i = 2 - \frac{1}{\varkappa_{i-1}}, \quad i = 1, \ldots, k - 2, 
\]

where \( \varkappa_{k-2} = \frac{1}{2} \). Now from (4.26) and (4.61) it follows

\[
(4.63) \quad Z_{\mathbb{R}}(Q_{\varkappa}[p]) = Z_{\mathbb{R}}(Q_{\varkappa_0}[p]) \leq Z_{\mathbb{R}}(Q_{\varkappa_1}[p]) \leq \ldots \leq Z_{\mathbb{R}}(Q_{\varkappa_{k-2}}[p^{(k-2)}]) \leq 2(n - k + 2) - 4 = 2n - 2k, 
\]

since \( \hat{Q}_{\varkappa}[p](z) = Q_{\varkappa_{i+1}}[p'][z] \) and \( \deg p^{(k-2)} = n - k + 2 \). Thus, inequalities (4.58), (4.60) and (4.63) hold for polynomials with only simple zeroes.

For the sequel, it is more convenient to rewrite (4.60) and (4.63) as follows.

\[
(4.64) \quad Z_{\mathbb{R}}(Q_{\varkappa}[p]) \leq 2j, 
\]

for \( \frac{n - j - 1}{n - j} \leq \varkappa \leq \frac{n - j}{n - j + 1}, \ j = 1, \ldots, n - 2. \)

Suppose now that the polynomial has roots of multiplicity at most 2. Then its derivative has only simple zeroes. Then by Theorem 4.12 (4.58), and (4.64), one has

\[
(4.65) \quad 2 \leq Z_{\mathbb{R}}(Q_{\varkappa}) \leq Z_{\mathbb{R}}(\hat{Q}_{\varkappa}) \leq 2d - 2, 
\]

for any \( \frac{n - d}{n - d + 1} < \varkappa < \frac{n - d + 1}{n - d + 2} \) and

\[
(4.66) \quad Z_{\mathbb{R}}(Q_{\varkappa}[p]) \leq Z_{\mathbb{R}}(\hat{Q}_{\varkappa}[p]) \leq 2j, 
\]

for \( \frac{n - j - 1}{n - j} \leq \varkappa \leq \frac{n - j}{n - j + 1}, \ j = 1, \ldots, d - 2. \)

It is clear now that if inequalities (4.65)–(4.66) are true for polynomials whose zeroes have multiplicity at most \( M, 1 \leq M \leq n - 2 \), then they hold for polynomials whose zeroes have multiplicity at most \( M + 1 \) according to Theorem 4.12, since for \( p' \) they are true by assumption. Consequently, inequalities (4.65)–(4.66) hold for any polynomial with pure real zeroes.

Note now that for \( \frac{m_1 - 1}{m_1} < \varkappa \leq \frac{n - d}{n - d + 1} \), where \( m_1 \) is defined in (4.18), we can conclude that \( Q_{\varkappa} \) may have only real zeroes as well as in (4.65). Therefore,

\[
(4.67) \quad 2 \leq Z_{\mathbb{R}}(Q_{\varkappa}) \leq Z_{\mathbb{R}}(\hat{Q}_{\varkappa}) \leq 2d - 2, 
\]

for any \( \frac{m_1 - 1}{m_1} < \varkappa \leq \frac{n - d}{n - d + 1}. \)

Let us now denote by \( N_{\mathbb{R}}^{(\varkappa)} \) and \( \hat{N}_{\mathbb{R}}^{(\varkappa)} \) the number of real solutions of equations (4.28), respectively. From (4.51)–(4.52) it follows that if a zero \( \lambda_k \) of the polynomial \( p \) is multiple, then it is a solution to both equations (4.28) of the same multiplicity (1 or 2). Since all other solutions to these equations (and only they) are zeroes of the functions \( Q_{\varkappa}[p](z) \) and \( \hat{Q}_{\varkappa}[p](z) \) for \( \frac{1}{2} < \varkappa < \frac{n - 1}{n} \), we obtain from Theorem 4.12 the following inequality

\[
N_{\mathbb{R}}^{(\varkappa)} \leq \hat{N}_{\mathbb{R}}^{(\varkappa)} 
\]

for \( \frac{1}{2} < \varkappa < \frac{n - 1}{n} \).

Moreover, \( N_{\mathbb{R}}^{(\frac{1}{2})} = Z_{\mathbb{R}}(Q_{\frac{1}{2}}) \), so if \( p \) has only simple zeroes, then

\[
N_{\mathbb{R}}^{(\frac{1}{2})} \leq 2n - 4 
\]
Consequently, for $N_R^{(\kappa)}$ the following inequalities hold

$$N_R^{(\kappa)} \leq 2j,$$

for $\frac{n-j-1}{n-j} \leq \kappa \leq \frac{n-j}{n-j+1}$, $j = 1, \ldots, d-2$ and

$$N_R^{(\kappa)} \leq 2d-2,$$

for $\frac{m_1-1}{m_1} \leq \kappa \leq \frac{n-d+1}{n-d+2}$.

Recall now that by Theorem 4.3 the number of solutions of the equation $R[p](z) = \kappa$ equals $2d-2$ where $d$ is the number of distinct zeroes of $p$, and the set of non-real zeroes of the polynomial $F_{\kappa}[p](z)$ coincides with the set of non-real solutions of the equation $R[p](z) = \kappa$. Now inequalities (4.54)–(4.55) follow from (4.67)–(4.70), as required. \qed

5. Polynomials with non-real zeroes

In this section, we disprove a conjecture by B. Shapiro [26] and discuss possible extensions of our results from Section 4 for arbitrary real polynomials.

Let $p$ be an arbitrary real polynomial. In this case, the polynomial $F_{\kappa}[p](z)$ can have both real and non-real non-trivial zeroes. So, to study non-trivial zeroes of $F_{\kappa}[p](z)$ it is more convenient to consider the function $Q_{\kappa}[p](z)$ defined in (3.20).

In [26, Conjecture 11], the following analogue of the Hawaii conjecture [6, 28] was conjectured.

**Conjecture 1** (B. Shapiro). Let $p$ be an arbitrary real polynomial of degree $n$, $n \geq 2$, then

$$Z_R(Q_{n-1}/n) \leq Z_C(p).$$

The Hawaii conjecture posed in [6] and proved in [28] states that inequality (5.1) is true for the function $Q_1[p](z)$. As it was shown in Sections 3–4, the value $\kappa = \frac{n-1}{n}$ is important, and for polynomials with real zeroes the properties of $Q_{n-1}/n[p](z)$ are close to the ones of $Q_1[p](z)$. However, Conjecture 1 is not true.

Indeed, consider the polynomial

$$p(z) = (z^2 + a^2)(z + a^2)(z - 1), \quad a \in \mathbb{R}\{−1, 0, 1\},$$

of degree 4. It has two distinct real zeroes, $-a^2$ and 1, and two non-real zeroes $\pm ia$, so $Z_C(p) = 2$. For this polynomial, the function $Q_{n-1}/n[p](z)$ has the form

$$Q_{\frac{3}{4}}[p](z) = -\frac{3}{4} \cdot \frac{(a^2 - 1)^2 z^3 - 8a^2(a^2 - 1)z^2 - 2a^2(a^4 - 10a^2 + 1)z + 8a^4(a^2 - 1)z + a^4(a^2 - 1)^2}{(z^2 + a^2)^2(z + a^2)^2(z - 1)^2}.$$ 

This rational function has four zeroes

$$\lambda_1 = \lambda_2 = \frac{a(a + 1)}{a - 1}, \quad \lambda_3 = \lambda_4 = \frac{-a(a - 1)}{a + 1},$$

all of which are real whenever $a \in \mathbb{R}\{−1, 1\}$, so

$$Z_R(Q_{\frac{3}{4}}) = 4 > 2 = Z_C(p),$$

and Conjecture 1 fails.
Remark 5.1. Conjecture 11 in [26], in fact, looks as follows

\[ Z_C(F_n - 1) \leq Z_C(p) \]

with additional condition that all real zeroes of \( p \) are simple. It is easy to see that this conjecture is equivalent to Conjecture 1 in the considered case, since \( Z_R(F_n - 1) = Z_R(Q_n - 1) \) whenever real zeroes of \( p \) are simple.

Let us look at inequality \( \frac{5.3}{5.3} \) from the point of view of the function \( R(z) \) defined in \( (5.5) \). For the polynomial \( (5.2) \), the rational function \( R(z) \) has the form

\[ R(z) = \frac{p(z)p''(z)}{[p'(z)]^2} = \frac{6z(2z-1+a^2)(2z^2+a^2)(z+a^2)(z-1)}{(4z^4-3z^2+3z^2a^2-a^2+a^4)^2}. \]

From \( (5.3) \) it follows that the equation

\[ R(z) = \frac{3}{4} \]

has 4 real solutions. The function \( R(z) \) is drawn at Fig. 3 for \( a = 10 \). It is easy to see that \( R(z) \) has two maxima with maximum values \( \frac{3}{4} \). In fact, simple calculations show that \( R(z) \) has two maximum points at \( -\frac{a(a-1)}{a+1} \) and \( \frac{a(a+1)}{a-1} \) with the value \( \frac{3}{4} \).

Thus, according to this counterexample we conjecture another inequality generalizing the Hawaii conjecture.

**Conjecture 2.** Let \( p \) be a real polynomial of degree \( n, n \geq 2 \). Then

\[ Z_C(p) - Z_C(p') \leq Z_R(Q_\zeta) \leq Z_C(p) \]

for \( \zeta > \frac{n-1}{n} \).

This conjecture is proved only for the case \( \zeta = 1 \), see [28].

For \( \zeta \leq \frac{n-1}{n} \), it is not easy to predict the estimates for the number of real zeroes of \( Q_\zeta[p](z) \). However, calculations show that the following conjecture for a special case of polynomials may be true.

**Conjecture 3.** Let \( p \) be a real polynomial of degree \( n = 2m \) with only non-real zeroes, and let \( p' \) have only real zeroes. Then

\[ Z_R(Q_\zeta) = Z_C(p) \quad \text{for} \quad \zeta > \frac{n-1}{n}, \]

\[ Z_C(p) - 2 \leq Z_R(Q_\zeta) \leq Z_C(p) \quad \text{for} \quad \frac{1}{2} < \zeta \leq \frac{n-1}{n}. \]
and
\[ Z_R(Q_\kappa) = Z_C(p) - 2 \quad \text{for} \quad \kappa \leq \frac{1}{2}. \]

In [6, 28] it was established that this conjecture is true for \( \kappa = 1 \).

Note that the polynomial \( p \) defined in (5.2) does not satisfy the conditions of Conjecture 3 since its derivative always has non-real zeroes whenever \( a \neq \pm 1, 0 \).

6. Conclusion and open problems

In the present work, we find the upper and lower bounds for the number of non-real zeroes of the differential polynomial
\[ F_\kappa[p](z) = p(z)p''(z) - \kappa[p'(z)]^2, \]
whenever \( p \) has only real zeroes. We also disprove a conjecture by B. Shapiro [26] on the number of real zeroes of \( F_\kappa[p](z) \) for arbitrary real polynomial \( p \). Instead, we provide two new conjectures that generalise the Hawaii conjecture [6] proved in [28]. We believe that our method of combining Proposition 4.11 and some properties of the function \( R(z) \) defined in (4.5) can be useful for proof of these conjectures and might provide a new, more simple, proof to the Hawaii conjecture.

Finally, we note that it is intuitively clear that our results of Section 4 can be extended to the case when \( p \) is an entire function in a subclass of the Laguerre-Pólya class \( \mathcal{L} - \mathcal{P} \). For example, it must be true for entire functions whose supremum of multiplicities of their zeroes is finite, in particular, if they have finitely many zeroes.

Acknowledgement

The work of M. Tyaglov was partially supported by The Program for Professor of Special Appointment (Oriental Scholar) at Shanghai Institutions of Higher Learning and by National Natural Science Foundation of China under Grant No. 11871336.

M. Tyaglov thanks Anna Vishnyakova and Alexander Dyachenko for helpful discussions.

M.J. Atia acknowledges the hospitality of the School of Mathematical Sciences of Shanghai Jiao Tong University in June 2015 and June 2016 partially supported by The Program for Professor of Special Appointment (Oriental Scholar) at Shanghai Institutions of Higher Learning and by National Natural Science Foundation of China.

Appendix A. Appendix: Proof of Proposition 4.11

Let \( p \) be a real polynomial of degree \( n \), \( n \geq 2 \). Recall that by \( F_\kappa[p](z) \) we denote the following polynomial
\[ F_\kappa[p](z) = p(z)p''(z) - \kappa[p'(z)]^2, \]
defined in (1.6). By \( \tilde{F}_\kappa[p](z) \) we denote the polynomial
\[ \tilde{F}_\kappa[p](z) = F_{2-\frac{1}{\kappa}}[p'(z)] = p'(z)p'''(z) - \left(2 - \frac{1}{\kappa}\right)[p''(z)]^2. \]

The functions \( Q_\kappa[p](z) \) and \( \tilde{Q}_\kappa[p](z) \) are defined in (3.20) and (4.24), respectively.

Our first result is about the number of zeroes of \( Q_\kappa[p](z) \) on a finite interval free of zeroes of the functions \( Q_\kappa[p](z) \), \( p' \), \( p'' \) and \( \tilde{Q}_\kappa[p](z) \).

Lemma A.1. Let \( p \) be a real polynomial, \( 0 < \kappa < \frac{n-1}{n} \), \( a, b \in \mathbb{R} \), and let \( p(z) \neq 0 \), \( p'(z) \neq 0 \), \( p''(z) \neq 0 \), \( Q_\kappa(z) \neq 0 \) in the interval \((a, b)\). Suppose additionally that if \( p(b) \neq 0 \) then \( p'(b) \neq 0 \) as well.

1. If, for all sufficiently small \( \delta > 0 \),
\[ p'(a+\delta)p''(a+\delta)Q_\kappa(a+\delta)\tilde{Q}_\kappa(a+\delta) > 0, \]
then \( Q_\kappa[p](z) \) has no zeroes in \((a, b)\).
II. If, for all sufficiently small \( \delta > 0 \),

\[
p'(a + \delta)p''(a + \delta)Q_\iota(a + \delta)\hat{Q}_\iota(a + \delta) < 0,
\]

then \( Q_\iota(z) \) has at most one zero in \((a, b)\), counting multiplicities. Moreover, if \( Q_\iota(\zeta) = 0 \) for some \( \zeta \in (a, b) \), then \( Q_\iota(b) \neq 0 \) (if \( Q_\iota(z) \) is finite at \( b \)).

Proof. The condition \( p(z) \neq 0 \) for \( z \in (a, b) \) means that \( Q_\iota(z) \) is finite at every point of \((a, b)\). Note that from (A.1) it follows that

\[
p'(z) = \frac{p(z)p''(z) - F_\iota(z)}{\varepsilon p'(z)},
\]

since \( p'(z) \neq 0 \) for \( z \in (a, b) \) by assumption. Substituting this expression into formula (A.2), we obtain

\[
\hat{F}_\iota(z) = \frac{p(z)p''(z)p'''(z)}{\varepsilon p'(z)} - (2\varepsilon - 1) \cdot \frac{p'(z)p''(z)p'''(z)}{\varepsilon p'(z)} - \frac{F_\iota(z)p'''(z)}{\varepsilon p'(z)} =
\]

(A.5)

\[
= \frac{p''(z)}{\varepsilon p'(z)} \cdot F'_\iota(z) - \frac{F_\iota(z)p'''(z)}{\varepsilon p'(z)}.
\]

If \( \zeta \in (a, b) \) and \( Q_\iota(\zeta) = 0 \), then \( F_\iota(\zeta) = 0 \) and (A.5) implies

(A.6)

\[
\hat{F}_\iota(\zeta) = \frac{p''(\zeta)}{\varepsilon \varepsilon'(\zeta)} F'_\iota(\zeta).
\]

Since \( p'(z) \neq 0, p''(z) \neq 0, \hat{Q}_\iota(z) \neq 0 \) (and therefore \( \hat{F}_\iota(z) \neq 0 \)) in \((a, b)\) by assumption, from (A.6) it follows that \( \zeta \) is a simple zero of \( Q_\iota(z) \). That is, all zeroes of \( Q_\iota(z) \) in \((a, b)\) are simple.

I. Let inequality (A.3) hold. Assume that, for all sufficiently small \( \delta > 0 \),

(A.7)

\[
p'(a + \delta)p''(a + \delta)\hat{Q}_\iota(a + \delta) > 0,
\]

then \( Q_\iota(a + \delta) > 0 \), that is, \( F_\iota(a + \delta) > 0 \). Therefore, if \( \zeta \) is the leftmost zero of \( Q_\iota(z) \) in \((a, b)\), then \( F'_\iota(\zeta) < 0 \). This inequality contradicts (A.6), since

\[
p'(z)p''(z)\hat{Q}_\iota(z) > 0
\]

for \( z \in (a, b) \), which follows from (A.7) and from the assumption of the lemma. Consequently, \( Q_\iota(z) \) cannot have zeroes in the interval \((a, b)\) if the inequalities (A.3) and (A.7) hold. In the same way, one can prove that if \( \varepsilon p'(a + \delta)p''(a + \delta)\hat{Q}_\iota(a + \delta) < 0 \) for all sufficiently small \( \delta > 0 \) and if the inequality (A.3) holds, then \( Q_\iota(z) \neq 0 \) for \( z \in (a, b) \).

Thus, \( Q_\iota(z) \) has no zeroes in the interval \((a, b)\) if the inequality (A.3) holds. Moreover, it is easy to show that \( Q_\iota(b) \neq 0 \) as well. Indeed, let, on the contrary, \( Q_\iota(b) = 0 \). Then \( p(b) \neq 0 \), therefore, \( p'(b) \neq 0 \) by assumption. So we obtain \( F_\iota(b) = 0 \) and, from (A.1), \( p''(b) \neq 0 \). Thus, we have \((pp'p'')(b) \neq 0 \). From (A.5) it follows that the functions \( F'_\iota \) and \( \hat{F}_\iota \) have a zero of the same order at \( b \). In particular, \( F'_\iota(b) \neq 0 \) if and only if \( \hat{F}_\iota(b) \neq 0 \). Furthermore, it is clear that the order of the zero of \( F'_\iota \) (and \( \hat{F}_\iota \)) at \( b \) is strictly smaller than the order of the zero of \( p'''F \) at \( b \). Consequently, from (A.5) we obtain, for all sufficiently small \( \varepsilon > 0 \),

(A.8)

\[
\text{sign} \left( \frac{p'(b - \varepsilon)}{p''(b - \varepsilon)} \hat{F}_\iota(b - \varepsilon) \right) = \text{sign}(F'_\iota(b - \varepsilon)).
\]

But if the inequality (A.3) holds, then

(A.9)

\[
\text{sign} \left( \frac{p'(b - \varepsilon)}{p''(b - \varepsilon)} \hat{F}_\iota(b - \varepsilon) \right) = \text{sign}(F_\iota(b - \varepsilon))
\]
for all sufficiently small \( \varepsilon > 0 \), since \( p'(z) \neq 0 \) and \( p''(z) \neq 0 \) in the interval \((a, b)\) by assumption and since \( Q_{\kappa}(z) \neq 0 \) in \((a, b)\), which was proved above. So, if the inequality \((A.3)\) holds and if \( F_{\kappa}(b) = 0 \), then from \((A.8)\) and \((A.9)\) we obtain that
\[
F_{\kappa}(b - \varepsilon) F'_{\kappa}(b - \varepsilon) > 0
\]
for all sufficiently small \( \varepsilon > 0 \). This inequality contradicts the analyticity of the polynomial \( F_{\kappa}(z) \).

Lemma A.2. Let \( p \) be a real polynomial, \( 0 < \kappa < \frac{n - 1}{n} \), \( a, b \in \mathbb{R} \) be real and let \( p(z) \neq 0 \), \( p'(z) \neq 0 \), \( p''(z) \neq 0 \) in the interval \((a, b)\). Suppose that \( \hat{Q}_{\kappa} \) has a unique zero \( \xi \in (a, b) \) of multiplicity \( M \) in \((a, b)\), and suppose additionally that \( p'(b) \neq 0 \) if \( p(b) \neq 0 \).

If \( Q_{\kappa}(\xi) = 0 \), then \( \xi \) is a zero of \( Q_{\kappa}(z) \) of multiplicity \( M + 1 \), and \( Q_{\kappa}(z) \neq 0 \) for \( z \in (a, \xi) \cup (\xi, b) \).

Proof. The condition \( p(z) \neq 0 \) for \( z \in (a, b) \) means that \( Q_{\kappa}(z) \) is finite at every point of \((a, b)\).

By assumption, \( \xi \) is a zero of \( \hat{F}_{\kappa} \) of multiplicity \( M \) and \( F_{\kappa}(\xi) = 0 \). First, we prove that \( \xi \) is a zero of \( F_{\kappa}(z) \) of multiplicity \( M + 1 \).

Note that the expression \((A.5)\) can be rewritten in the form
\[
\kappa \frac{p'(z)}{[p''(z)]^2} \hat{F}_{\kappa}(z) = \left( \frac{F_{\kappa}(z)}{p''(z)} \right)'
\]
\((A.10)\)
since \( p''(z) \neq 0 \) for \( z \in (a, b) \) by assumption. Differentiating this equality \( j \) times with respect to \( z \), we get
\[
\kappa \left( \frac{p'(z)}{[p''(z)]^2} \hat{F}_{\kappa}(z) \right)^{(j)} = \left( \frac{F_{\kappa}(z)}{p''(z)} \right)^{(j+1)}.
\]
From \((A.10)\) it follows that \( F_{\kappa}^{(j+1)}(\xi) = 0 \) if \( p'(\xi) \neq 0 \), \( p''(\xi) \neq 0 \), \( \hat{F}_{\kappa}^{(i)}(\xi) = 0 \) and \( F_{\kappa}^{(i)}(\xi) = 0 \), \( i = 0, 1, \ldots, j \). Consequently, \( \xi \) is a zero of \( F_{\kappa} \) of multiplicity at least \( M + 1 \). But by assumptions, \((A.10)\) implies the following formula
\[
0 \neq p'(\xi) \hat{F}_{\kappa}^{(M)}(\xi) = p''(\xi) F_{\kappa}^{(M+1)}(\xi).
\]
Hence, \( \xi \) is a zero of \( F_{\kappa}(z) \) of multiplicity exactly \( M + 1 \). But \( p(\xi) \neq 0 \) by assumption, therefore, \( \xi \) is a zero of \( Q_{\kappa}(z) \) of multiplicity \( M + 1 \).

\[5\] If a real function \( f \) is analytic at some neighbourhood of a real point \( a \) and equals zero at this point, then, for all sufficiently small \( \varepsilon > 0 \),
\[
f(a - \varepsilon)f'(a - \varepsilon) < 0.
\]
ON THE NUMBER OF NON-REAL ZEROES OF A HOMOGENEOUS DIFFERENTIAL POLYNOMIAL AND A GENERALIZATION OF THE LAGUERRE INEQUALITIES

It remains to prove that \( Q_\kappa(z) \) has no zeroes in \((a, b)\) except \( \xi \). In fact, consider the interval \((a, \xi)\). According to Lemma A.1, \( Q_\kappa(z) \) can have a zero at \( \xi \) only if the inequality \( A.4 \) holds and \( Q_\kappa(z) \neq 0 \) for \( z \in (a, \xi) \). Furthermore, the polynomial \( p'(z)p''(z) \) does not change its sign at \( \xi \) but the function \( Q_\kappa(z)Q_\kappa(z) \) does, since \( \xi \) is a zero of \( Q_\kappa(z)Q_\kappa(z) \) of multiplicity \( 2M + 1 \). Thus, for all sufficiently small \( \delta > 0 \),

\[
y'(z + \delta)p''(z + \delta)Q_\kappa(z + \delta) > 0,
\]

since inequality \( A.4 \) must hold in the interval \((a, \xi)\) by Lemma A.1. From \( A.11 \) it follows that Case I of Lemma A.1 holds in the interval \((\xi, b)\), so \( Q_\kappa(z) \neq 0 \) for \( z \in (\xi, b) \).

Now combining the two last lemmata, we provide a general bound on the number of real zeroes of \( Q_\kappa \) in terms of the number of real zeroes of \( \hat{Q}_\kappa \) in a given interval.

**Theorem A.3.** Let \( p \) be a real polynomial and let \( a \) and \( b \) be real. If \( p(z) \neq 0 \), \( p'(z) \neq 0 \) and \( p''(z) \neq 0 \) for \( z \in (a, b) \), then

\[
Z_{(a, b)}(Q_\kappa) \leq 1 + Z_{(a, b)}(\hat{Q}_\kappa).
\]

**Proof.** If \( p(z)p''(z) < 0 \) in \((a, b)\), then \( Q_\kappa(z) < 0 \) for \( z \in (a, b) \) by (3.20), that is, \( Z_{(a, b)}(Q_\kappa) = 0 \). Therefore, the inequality \( A.12 \) holds automatically in this case.

Let now \( p(z)p''(z) > 0 \) for \( z \in (a, b) \). If \( \hat{Q}_\kappa(z) \neq 0 \) in \((a, b)\), that is, \( Z_{(a, b)}(\hat{Q}_\kappa) = 0 \), then by Lemma A.1, \( Q_\kappa(z) \) has at most one real zero, counting multiplicity, in \((a, b)\). Therefore, \( A.12 \) holds in this case.

If \( \hat{Q}_\kappa(z) \) has a unique zero \( \xi \) in \((a, b)\) and \( Q_\kappa(\xi) \neq 0 \), then by Lemma A.1, \( Q_\kappa(z) \) has at most one real zero, counting multiplicity, in each interval \((a, \xi)\) and \((\xi, b)\):

\[
Z_{(a, \xi)}(Q_\kappa) \leq 1 + Z_{(a, \xi)}(\hat{Q}_\kappa),
\]

where \( Z_{(a, \xi)}(\hat{Q}_\kappa) = 0 \), and

\[
Z_{(\xi, b)}(Q_\kappa) \leq 1 + Z_{(\xi, b)}(\hat{Q}_\kappa),
\]

where \( Z_{(\xi, b)}(\hat{Q}_\kappa) = 0 \). Since \( Q_\kappa(\xi) \neq 0 \) and \( \hat{Q}_\kappa(\xi) = 0 \), we have

\[
0 = Z_{(\xi)}(Q_\kappa) \leq -1 + Z_{(\xi)}(\hat{Q}_\kappa).
\]

Thus, summing the inequalities \( A.13 \) to \( A.15 \), we obtain \( A.12 \).

If \( \hat{Q}_\kappa(z) \) has a unique zero \( \xi \) in \((a, b)\) and \( Q_\kappa(\xi) = 0 \), then, by Lemma A.2, we have

\[
Z_{(\xi)}(Q_\kappa) = 1 + Z_{(\xi)}(\hat{Q}_\kappa),
\]

and \( Q_\kappa(z) \neq 0 \) for \( z \in (a, \xi) \cup (\xi, b) \). Therefore, the inequality \( A.12 \) is also true in this case.

Now, let \( \hat{Q}_\kappa \) have exactly \( r \geq 2 \) distinct real zeroes, say \( \xi_1 < \xi_2 < \ldots < \xi_r \), in the interval \((a, b)\). These zeroes divide \((a, b)\) into \( r + 1 \) subintervals. If, for some number \( i \), \( 1 \leq i \leq r \), \( Q_\kappa(\xi_i) \neq 0 \), then by Lemma A.1, \( Q_\kappa(z) \) has at most one real zero, counting multiplicity, in \((\xi_{i-1}, \xi_i)\) \( (\xi_0 \overset{def}{=} a) \). But \( \hat{Q}_\kappa \) has at least one real zero in \((\xi_{i-1}, \xi_i)\), counting multiplicities (at the point \( \xi_i \)). Consequently,

\[
Z_{(\xi_{i-1}, \xi_i)}(Q_\kappa) \leq Z_{(\xi_{i-1}, \xi_i)}(\hat{Q}_\kappa)
\]

If, for some number \( i \), \( 1 \leq i \leq r - 1 \), \( Q_\kappa(\xi_i) = 0 \) and \( \xi_i \) is a zero of \( \hat{Q}_\kappa(z) \) of multiplicity \( M \), then by Lemma A.2, \( Q_\kappa(z) \) has only one zero \( \xi_i \) of multiplicity \( M + 1 \) in \((\xi_{i-1}, \xi_{i+1})\). But in the interval \((\xi_{i-1}, \xi_{i+1})\), \( \hat{Q}_\kappa(z) \) has at least \( M + 1 \) real zeroes, counting multiplicities (namely, \( \xi_i \) which is a zero of multiplicity \( M \), and \( \xi_{i+1} \)). Therefore, in this case, the following inequality holds

\[
Z_{(\xi_{i-1}, \xi_{i+1})}(Q_\kappa) \leq Z_{(\xi_{i-1}, \xi_{i+1})}(\hat{Q}_\kappa)
\]
Thus, if \( Q_\kappa(\xi_r) \neq 0 \), then from (A.16)–(A.17) it follows that
\[
Z_{(\alpha, \xi^1)}(Q_\kappa) \leq Z_{(\alpha, \xi^1)}(Q_\kappa).
\]
But by Lemma A.1 \( Q_\kappa(z) \) has at most one real zero, counting multiplicity, in the interval \((\xi_r, b)\). Consequently, if \( Q_\kappa(\xi_r) \neq 0 \), then the inequality (A.12) is valid.

If \( Q_\kappa(\xi_r) = 0 \), then by Lemma A.2 \( Q_\kappa(\xi_{r-1}) \neq 0 \) (otherwise, \( \xi_r \) cannot be a zero of \( Q_\kappa(z) \)) and from (A.16)–(A.17) it follows that
\[
Z_{(\alpha, \xi_{r-1})}(Q_\kappa) \leq Z_{(\alpha, \xi_{r-1})}(\hat{Q}_\kappa).
\]
Now Lemma A.2 implies
\[
Z_{(\xi_{r-1}, b)}(Q_\kappa) = 1 + Z_{(\xi_{r-1}, b)}(\hat{Q}_\kappa),
\]
therefore, inequality (A.12) follows from (A.19)–(A.20).

\[\square\]

References

[1] G. Andrews, R. Askey, and R. Roy, *Special functions*, Cambridge University Press, 1999.
[2] W. Bergweiler, On the zeros of certain homogeneous differential polynomials, *Arch. Math.*, 64, 1995, pp. 199–202.
[3] J. Borcea and B. Shapiro, Classifying real polynomial pencils, *Int. Math. Res. Not.*, no. 69, 2004, pp. 3689–3708.
[4] T. Craven and G. Csordas, Jensen polynomials and the Turán and Laguerre inequalities, *Pacific J. Math.*, 136, no. 2, 1990, pp. 241–260.
[5] T. Craven and G. Csordas, Iterated Laguerre and Turán inequalities, *J. Inequal. Pure Appl. Math.*, 3, no. 3, 2002, art. 39, 14 pp. (electronic).
[6] G. Csordas, T. Craven, and W. Smith, The zeros of derivatives of entire functions and the Wiman–Polya conjecture, *Ann. of Math.*, 125, no. 2, 1987, pp. 405–431.
[7] G. Csordas, Linear operators and the distribution of zeros of entire functions, *Complex Var. Elliptic Equ.*, 51, no. 7, 2006, pp. 625–632.
[8] G. Csordas and A. Escassut, The Laguerre inequality and the distribution of zeros of entire functions, *Ann. Math. Blaise Pascal*, 12, no. 2, 2005, pp. 331–345.
[9] G. Csordas, A. Ruttan, and R. Varga, The Laguerre inequalities with applications to a problem associated with the Riemann hypothesis, *Numer. Algorithms*, 1, no. 3, 1991, pp. 305–329.
[10] G. Csordas, W. Smith, and R. Varga, Level sets of real entire functions and the Laguerre inequalities, *Analysis*, 12, no. 3–4, 1992, pp. 377–402.
[11] G. Csordas and R. Varga, Necessary and sufficient conditions and the Riemann hypothesis, *Adv. in Appl. Math.*, 11, no. 3, 1990, pp. 328–357.
[12] G. Csordas and A. Vishnyakova, The generalized Laguerre inequalities and functions in the Laguerre–Polya class, *Cent. Eur. J. Math.*, 11, no. 9, 2013, pp. 1643–1650.
[13] K. Dilcher, Real Wronskian zeros of polynomials with nonreal zeros, *J. Math. Anal. Appl.*, 154, 1991, pp. 164–183.
[14] K. Dilcher and K. Stolarsky, Zeros of the Wronskian of a polynomial, *J. Math. Anal. Appl.*, 162, 1991, pp. 430–451.
[15] S. Edwards and A. Hinkkanen, Level sets, a Gauss-Fourier conjecture, and a counter-example to a conjecture of Borcea and Shapiro, *Comput. Methods Funct. Theory*, 11, no. 1, 2011, pp. 1–12.
[16] H. Ki and Y.-O. Kim, On the number of nonreal zeros of real entire functions and the Fourier–Polya conjecture, *Duke Math. J.*, 104, no. 1, 2000, pp. 45–73.
[17] J.K. Langley, A lower bound for the number of zeros of a meromorphic function and its second derivative, *Proc. Edinburgh Math. Soc.*, 39, 1996, pp. 171–185.
[18] J.K. Langley, Non-real zeros of real differential polynomials, *Proc. Roy. Soc. Edinburgh Sect. A*, 141, 2011, pp. 631–639.
[19] J.K. Langley, Non-real zeros of derivatives of real meromorphic functions of infinite order, *Math. Proc. Cambridge Philos. Soc.*, 150, 2011, pp. 343–351.
[20] J.B. Love, Problem E1532, *Amer. Math. Monthly*, 69, 1962, p. 668.
[21] D. Nicks, Non-real zeros of real entire derivatives, *J. Anal. Math.*, 117, 2012, pp. 87–118.
[22] C. Niculescu, A new look at Newton’s inequalities, *J. Inequal. Pure and Appl. Math.*, 1, no.2, 2000, art. 17 (electronic).
[23] N. Ohreschkoff, *Verteilung und Berechnung der Nullstellen Reelse Polynome*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
[24] G. Pólya, Some problems connected with Fourier’s work on transcendental equations, *Quart. J. Math. Oxford Ser.*, 1, no. 1, 1930, pp. 21–34.
[25] G. Pólya and J. Schur, Über zwei Arten von Faktorensenken in der Theorie der algebraischen Gleichungen, *J. Reine Angew. Math.* 144, 1914, pp. 89–113.
[26] B. Shapiro, Problems around polynomials: the good, the bad, and the ugly..., *Arnold Math. J.*, 1, no. 1, 2015, pp. 91–99.
[27] K. Tohge, On zeros of a homogeneous differential polynomial, *Kodai Math. J.*, 16, 1993, pp. 398–415.
[28] M. Tyaglov, On the number of critical points of logarithmic derivatives and the Hawaii conjecture, *J. Anal. Math.*, **114**, 2011, pp. 1–62.

[29] T. Sheil-Small, *Complex polynomials*, Cambridge Studies in Advanced Mathematics, **75**, Cambridge University Press, 2002.

School of Mathematical Sciences, Shanghai Jiao Tong University
*E-mail address*: tyaglov@sjtu.edu.cn

Qassim University, College of Sciences, Buraydah, Saudi Arabia
*E-mail address*: jalel.atia@gmail.com