ON CONVERGENCE TO A FOOTBALL

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Abstract. We show that spheres of positive constant curvature with \( n \) \((n \geq 3)\) conic points converge to a sphere of positive constant curvature with two conic points, or an (American) football in Gromov-Hausdorff topology when the conic angles of the sequence pass from the subcritical case in the sense of Troyanov to the critical case in the limit.

We prove this convergence in two different ways. Geometrically, the convergence follows from Luo-Tian’s explicit description of conic spheres as boundaries of convex polytopes in \( S^3 \). Analytically, considering the conformal factors as the singular solutions to the corresponding PDE, we derive the required a priori estimates and convergence result after proper reparametrization.

1. Introduction

We study convergence properties of metrics on Riemann surface with conical singularities. It has been an old topic and been first studied as early as in 1905 by Picard [P], when he was considering the uniformization problem for Riemann surfaces with branched points. In 1990s, it was systematically studied by several authors [LT, CL1, CL2]. Recent progress on the study of canonical metrics on Fano manifolds has brought up renewed interests in this field [D, JMR].

We first start with some definitions.

For a compact closed Riemann surface \( S \), a metric \( g \) is said to have a conic singularity of order \( \beta \in (-1, \infty) \) at \( p \), if under a local holomorphic coordinate centered at \( p \),

\[
g = e^{f(z)}|z|^{2\beta}|dz|^2,
\]

where \( f(z) \) is continuous and \( C^2 \) away from \( p \).

The singularity is modeled on the Euclidean cone: \( \mathbb{C} \) with the metric \( |z|^{2\beta}|dz|^2 \) is isometric to a Euclidean cone of cone angle \( 2\pi(\beta + 1) \). In general, we use the triple \((S, g, \beta)\), where \( \beta = \sum_{i=1}^{n} \beta_i p_i \), to denote a Riemann surface \( S \) endowed with a metric \( g \) with conic singularities at each \( p_i \) of order \( \beta_i \).

H.F.’s work is partially supported by Simons Foundation and NSF DMS-100829.
Let $K = K(g)$ be the Gaussian curvature of $g$ where it is smooth. The Gauss-Bonnet formula for the conic surface $(S, g, \beta)$ becomes

$$\int_S Kds^2 = 2\pi \chi(S, \beta),$$

where

$$\chi(S, \beta) := \chi + \sum_{i=1}^{n} \beta_i$$

is the Euler characteristic for the conic surface $(S, \beta)$.

Troyanov [Tr] systematically studied the prescribing curvature problem on conic surfaces. The problem is divided into three main cases according to the sign of the Euler characteristic; while the positive Euler characteristic case is divided further into three sub-cases. Namely, we have

1. **negative case**: $\chi(S, \beta) < 0$;
2. **zero case**: $\chi(S, \beta) = 0$;
3. **positive case**: $\chi(S, \beta) > 0$;
   3.a **subcritical case**: $\chi(S, \beta) < \min\{2, 2 + 2 \min_i \beta_i\}$;
   3.b **critical case**: $\chi(S, \beta) = \min\{2, 2 + 2 \min_i \beta_i\}$;
   3.c **supercritical case**: $\chi(S, \beta) > \min\{2, 2 + 2 \min_i \beta_i\}$.

It turns out that cases 1, 2, and 3.a are parallel to the corresponding cases in the prescribing curvature problem on a smooth surface as the corresponding functionals are coercive; while cases 3.b and 3.c are more delicate. We refer readers to, e.g. [BDM, CL1, E, LZ] for details.

For the Yamabe problem on surfaces with the conic singularity, where we prescribe the constant curvature, the answer is more complete. Without loss of generality, we assume that $S$ is orientable. If $\chi(S, \beta) \leq 0$, it has been shown [Tr] that there always admits a conic metric with constant curvature. While for $\chi(S, \beta) > 0$, $S$ would necessarily be $S^2$, if in addition $\beta_i \in (-1, 0)$, then $S$ admits a conic metric of positive constant curvature if and only if:

- $n = 2, \beta_1 = \beta_2$;
- $n \geq 3, \chi(S, \beta) < \min\{2, 2 + 2 \min_i \beta_i\}$.

Note that surfaces in the first class are often called (American) footballs, and they belong to the critical case. In [CL2], it has been shown a conic metric of positive constant curvature on $S^2$ in the critical case is necessarily a football. For the second class, the sufficiency is proved by Troyanov [Tr], the necessity and uniqueness argument is due to Luo-Tian [LT].
Recently, there is renewed interest on metrics with conic singularities in the study of Kähler geometry, namely the Kähler-Einstein metrics with cone singularities along a divisor \([D, JMR]\). Conic metrics of constant curvature on Riemann surfaces are just one-dimensional examples of Kähler-Einstein metrics with cone singularities along a divisor. In higher dimension, for the existence of Kähler-Einstein metrics with cone singularities, Troyanov’s condition can be generalized to the coercivity of twisted Mabuchi \(K\)-energy functional \([JMR]\), which can be reinterpreted as the pair \((S, \beta)\) being logarithmically K-stable \([RT]\).

The smooth version of this connection between algebraic stability and existence of Kähler-Einstein metrics on Fano manifolds is the essence of the recently solved Yau-Tian-Donaldson conjecture \([CDS1, CDS2, CDS3, Ti]\).

From this point of view, Troyanov’s classification can be understood as the stability conditions: the subcritical pairs and critical pairs can be viewed as being stable and semi-stable respectively; while the supercritical pairs are considered being unstable. While there exist metrics with constant curvature on stable and semi-stable pairs; the canonical metric on unstable pairs should be conical gradient shrinking Ricci solitons instead.

Ricci flow method has been considered for conical surfaces \([Y1, Y2]\). Yin has proved long time existence as well as the convergence of the flow when \(\chi(S, \beta) \leq 0\). Another approach was given by Mazzeo, Rubinstein and Sesum \([MRS]\) using extensive machinery of polyhomogeneous expansions. They have also studied flows which have cone angles varying in a prescribed way.

In a recent preprint \([PSSW]\), the Ricci flow on conic spheres is shown to converge in stable, semi-stable and unstable cases. In particular, it was shown in \([PSSW]\) that conic metrics on sphere with \(n (n \geq 3)\) conic points in the semi-stable case converges in Gromov-Hausdorff topology to a football along the Ricci flow.

In this note, we investigate from another perspective. We consider the moduli space of all conic spheres with constant curvature. According to numbers of the conic points on the sphere, the moduli has complicated topology with components with varying dimensions. Algebraically, points in the moduli can be separated as stable and semi-stable points. We show that any sequence of spheres of positive constant curvature with \(n (n \geq 3)\) conic points passing from stable case to semi-stable case converges to a football in Gromov-Hausdorff topology. Geometrically, all but one conic points will merge into a single conic point of the limit football. Following method of Luo-Tian \([LT]\), we have the following
Theorem 1.1. Let \((S^2, g, \beta^{(l)})\) be a sequence of Riemann spheres with conic metric of positive constant curvature 1. Suppose the order of conic points \(\beta^{(l)} = (\beta_1^{(l)}, \ldots, \beta_n^{(l)}) \in \mathbb{R}^n\) converges to \(\beta^{(\infty)} = (\beta_1^{(\infty)}, \ldots, \beta_n^{(\infty)})\) which is in the critical case. Then \((S^2, ds^2(l))\) converges in the Gromov-Hausdorff topology to \((S^2, g, \beta)\), the unique football of constant curvature 1 with \(\beta = \min_i \beta_i^{(\infty)} p + \min_i \beta_i^{(\infty)} q\). Moreover, suppose \(\beta_1^{(l)} \leq \cdots \leq \beta_n^{(l)}\), then the corresponding conic points converge in the following fashion:

\[
\lim_{l \to \infty} p_1^{(l)} = p \quad \text{and} \quad \lim_{l \to \infty} p_i^{(l)} = q, \quad \text{for } i \geq 2.
\]

To understand this convergence phenomenon in an analytical way, we investigate the problem in the conformal geometrical setting.

Let \(g_0\) be the standard Euclidean metric. Under the stereographic projection, a conformal metric \(g = e^{2u}g_0\) is of constant curvature 1 and represents \((S^2, g, \beta = \sum_i \beta_i p_i)\) if and only if \(u\) satisfies the equation

\[
\Delta u = -e^{2u}, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_n\}. \tag{1.1}
\]

The asymptotic behavior of \(u\) near \(z_i\) is:

- \(u \sim \beta_i \ln |z - z_i|\) as \(z \to z_i\);
- \(u \sim -2 \ln |z|\) as \(|z| \to \infty\).

Notice that \(u\) is uniquely associated to a conic metric only up to a conformal normalization. Our main result is the following

Theorem 1.2. Let \(u_l\) be functions on \(\mathbb{C}^*\) representing \(g^{(l)}\) given in Theorem 1.1. Under proper normalization, there exist two distinct points \(p, q \in \mathbb{C}^*\) such that a subsequence of \(u_l\) converges to \(u_\infty\) on any compact set \(K \subset \mathbb{C}^* \setminus \{p, q\}\). Furthermore, \(e^{2u_\infty} g_0\) on \(\mathbb{C}^* - \{p, q\}\) represents a football.

The standard stereographic projection gives a unique correspondence of metrics on sphere and a conformal factor function on \(\mathbb{C}^*\) up to a Möbius transformation. This large conformal transformation group poses analytical difficulty, namely a proper conformal gauge must be chosen with care so that the resulting limit for the conformal factors exists and is non-trivial. This requires precise choices of a shifting factor and a scaling factor for each configuration that we consider.

Our main approach is to explore the rotational symmetry of the football solution. This was examined in earlier works (e.g. [CL2]) when a single manifold with conic singularities is considered. Especially, level sets of the conformal factor are considered and isoperimetric inequalities play crucial roles. For our set-up, we follow a similar but more
delicate approach. We would like to consider the level set of each conformal factor $u_i$ and analyze the isoperimetric defect more carefully. Consequently, our choice of normalization is also connected to the level sets. See Section 3 for more details.

While the method applied in this article heavily relies on the rotational symmetry of football solutions, it is a perfect example highlighting the equivalence of different convergence concepts on the moduli space of constant curvature metrics with conic singularities. Several set-up for higher dimensions can be considered from both Kähler geometry and conformal geometry points of views.

From a more analytical prospect, it would also be interesting to understand the convergence described in Theorem 1.2 in more details. By Theorem 1.2, all but one singular points will merge into one end of the football solution. It would be interesting to understand the blow-up limit of the sequence near this end. Flat metrics with conical singularities are to be expected while exact formation is unknown. Similar problems can also be considered for the Ricci flow limit.

This paper is organized as the following: In Section 2, we provide a proof of Theorem 1.1 following [LT]. In Section 3, we prove Theorem 1.2.

Acknowledgements: Both authors would like to thank Jian Song and Lihe Wang for discussion. Part of the work was done when both authors were visiting Beijing International Center for Mathematical Research. We are thankful for its hospitality.

2. Proof of Theorem 1.1

We adopt the geometric setting of [LT] and notations therein. By a theorem of Alexandrov, each spherical conic metric of constant curvature 1 is isometric to the boundary of a convex polytope in $S^3$. There are two degenerate cases: one is the metric doubling of a "lens", which is a degenerate spherical triangle with length of three sides being $\pi, \pi, 0$; the other one is the metric doubling of a usual spherical triangle. Clearly the former one corresponds to a football (2-conic points) and the latter one corresponds to a sphere with three conic points.

For a convex polytope of $n$ vertices, we denote its angles at vertices by $(\alpha_1, \cdots, \alpha_n)$ with each $\alpha_i \in (0, 2\pi)$. Let $P_n$ be the space of all boundaries of labeled $n$-vertex convex polytopes in $S^3$ modulo isometry, with the topology induced by the Hausdorff metric. For each convex polytope $P$, construct a totally geodesic triangulation, then there are exactly $3(n-2)$ edges and $2(n-2)$ triangles. Variation of the length of each edge gives rise to distinct convex polytopes (up to isometry). Therefore, the
dimension of $P_n$ is $3(n-2)$. Meanwhile, denote the conformal structure of $n$-labeled Riemannian sphere by $M_n$. Since Möbius transformations are 3-transitive, it follows that $\dim M_n = 2(n - 3)$.

In [LT], Luo-Tian show that there admits a conic metric on $S^2$ of positive constant curvature representing $\beta = \sum_{i=1}^{k} \beta_i p_i$, $(k \geq 3)$, if and only if the corresponding cone angles $\alpha_i = 2\pi(1 + \beta_i)$ satisfy

$$\sum_{i=1}^{n} \alpha_i > 2(n - 2)\pi, \quad \sum_{i=1}^{n} \alpha_i < 2(n - 2)\pi + 2 \min_i \alpha_i.$$  \hspace{1cm} (2.1)

This condition is exactly same as the subcritical condition of Troyanov. It defines a convex open set in $\mathbb{R}^n$, which we denote by $A$. The critical case corresponds to the equality

$$\sum_{i=1}^{n} \alpha_i = 2\pi(n - 2) + 2 \min_i \alpha_i.$$

In addition, Luo-Tian [LT] have proved the following

**Theorem 2.1** (Luo-Tian). The map

$$\Pi : P_n \rightarrow M_n \times A,$$  \hspace{1cm} (2.2)

$$P \rightarrow (\text{conformal structure of } P, \text{angles of } P \text{ at vertices})$$

is a homeomorphism.

We are now ready to give a proof of Theorem 1.1.

**Proof of Theorem 1.1.** As mentioned above, each $(S^2, g_l, \beta(l))$ is isometric to the boundary of a convex polytope $P(l)$ in $S^3$, and it satisfies the subcritical condition (2.1). By compactness of compact sets in $S^3$ with respect to the Hausdorff metric, we may assume a subsequence, still denoted by $P(l)$, converges to a convex polytope $P(\infty)$. $P(\infty)$ represents a conic sphere of positive constant curvature, which is either in the subcritical case or the critical case. For the latter, $P(\infty)$ is necessarily the so-called lens or lune, which is defined as a region on the 2-sphere bounded between two geodesics lines connecting two antipodal points. By Theorem 2.1 if $P(\infty)$ does not degenerate to a lens, then

$$\beta^{(\infty)} = \lim_{l \rightarrow \infty} \beta(l) = \lim_{l \rightarrow \infty} \Pi_2(P(l)) = \Pi_2(P(\infty))$$

must be in the subcritical case as well, a contradiction. Thus, we conclude that $P(\infty)$ is a lens.

Thus to prove our result, we are left to show that the conic angle of the corresponding football is $2\pi(\min_i \beta_i^{(\infty)} + 1)$. Let $\alpha_i(l) = 2\pi(\beta_i(l) + 1)$
be angles of $P(l)$, with corresponding vertices denoted by $V^i_l$, $i = 1, 2, \cdots, n$. It was shown [LT] under this situation that there exists a $k$ such that
\[
\lim_{l \to \infty} d(V^k_l, V^i_l) > 0, \quad \text{for } i \neq k,
\]
and
\[
\lim_{l \to \infty} d(V^i_l, V^j_l) = 0, \quad \text{for } i, j \neq k.
\]

Now consider the triangulation of $P(l)$ which consists of $2(n-2)$ triangles $\{\Delta^i_l\}_{i=1}^{2(n-2)}$. For each $\Delta^i_l$ we denote its inner angles as $\alpha^i_1, \alpha^i_2, \alpha^i_3$. For simplicity, if $\Delta^i_l$ is incident to $V^k_l$, assume that $\alpha^i_1$ is the angle at the point $V^k_l$. We denote, for $k = 1, 2, 3$ and $i = 1, 2, \cdots, 2(n-2),$
\[
\alpha^i_0 = \lim_{l \to \infty} \alpha^i_l.
\]

When $l$ is converging to $\infty$, we have the following two cases. If $\Delta^i_l$ is incident to $V^k_l$, we conclude that $\Delta^i_l$ converges to a lens whose angles satisfy
\[
\alpha_1^i(\infty) = \alpha_2^i(\infty) + \alpha_3^i(\infty) - \pi. \tag{2.3}
\]
If $\Delta^j_l$ is not incident to $V^k_l$, all its three vertices merge in the limit. Hence its inner angles has the following limit behavior
\[
\alpha_1^j(\infty) + \alpha_2^j(\infty) + \alpha_3^j(\infty) = \pi \tag{2.4}
\]
Summing relations (2.3) and (2.4) for all $2(n-2)$ triangles, we have
\[
\alpha_k^\infty + 2(n-2)\pi = \sum_{i \neq k} \alpha_i^\infty.
\]
Comparing with the critical condition, it follows that
\[
\alpha_k^\infty = \min_i \alpha_i^\infty.
\]
We have thus finished the proof. \qed

3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Under the stereographic projection, the conformal factors of positive constant curvature metrics are solutions to a semi-linear elliptic equation in the complex plane $\mathbb{C}$. We shall study the limit behavior of these solutions.

More precisely, given an $(S^2, g, \beta)$, under the stereographic projection we assume $z_i$'s are the corresponding projection of $p_i$ in the complex plane. Let $g_0$ be the standard Euclidean metric, then $\frac{4}{(1+|z|^2)^2} g_0$ is
the standard metric on $S^2$. A conic metric $g = e^{2u}g_0$ is of constant curvature 1 representing $\beta = \sum_{i=1}^n \beta_i p_i$ if and only if $u$ satisfies the equation
\[
\Delta u = -e^{2u}, \quad z \in \mathbb{C} \setminus \{z_1, \cdots, z_n\}. \tag{3.1}
\]
The asymptotic behavior of $u$ near $z_i$ is:
- $u \sim \beta_i \ln |z - z_i|$ as $z \to z_i$;
- $u \sim -2 \ln |z|$ as $|z| \to \infty$.

Let $v = u - \sum_{i=1}^n \beta_i \ln |z - z_i|$, then $v$ is bounded near each singular point $z_i$ and satisfies
\[
\Delta v = -e^{2v} \prod_{i=1}^n |z - z_i|^{2\beta_i}. \tag{3.2}
\]

**Remark 3.1.** One can equally work out this receipt in the sense of complex geometry. For a given $(S^2, g, \beta)$ with
\[
\beta = \sum_{i=1}^n \beta_i p_i,
\]
choose the background metric to be the Kähler metric
\[
\omega = (1 + \sum_{i=1}^n \frac{\beta_i}{2}) \omega_{FS} \in 2\pi(1 + \sum_{i=1}^n \frac{\beta_i}{2}) c_1(S^2),
\]
where $\omega_{FS}$ is the standard Fubini-Study metric on $S^2$. Let $s_i$ be the defining section of the line bundle $[2p_i] = -K_{S^2}$, let $h$ be the hermitian metric on $-K_{S^2}$ such that its curvature form $\Theta_h = \omega_{FS}$. A conic metric $\omega_\varphi$ on $S^2$ with constant curvature 1 represents the divisor $\beta$ if and only if $\varphi$ satisfies
\[
(\omega + \partial \bar{\partial} \varphi) = e^{-\varphi} \prod_{i=1}^n |s_i| \frac{\beta_i}{h} \omega, \tag{3.3}
\]
or equivalently
\[
Ric(\omega_\varphi) = \omega_\varphi - 2\pi \sum_{i=1}^n \beta_i [p_i], \tag{3.4}
\]
where $[p_i]$ is current of integration or Dirac-measure at $p_i$.

Note that (3.2) is equivalent to (3.3). Indeed, the defining section $s_i$ on the complex plane is $(z - z_i)^2$, and the conformal change of metric amounts to scale the hermitian metric $h$ on $-K_{S^2}$ by the same conformal factor.
Remark 3.2. For $0 < \alpha < 1$, notice that $z \to z^\alpha$ maps $\mathbb{C}^*$ to a football with conic angles $2\pi\alpha$ at 0 and $\infty$. By the standard stereographic projection, this constant scalar curvature metric on this football is represented as $e^{2u_\alpha} g_0$, where

$$e^{2u_\alpha} = 4\alpha^2 \frac{|z|^{2\alpha - 2}}{(1 + |z|^{2\alpha})^2}. \quad (3.5)$$

Recall we are given a sequence of conic metrics of constant curvature 1 on $S^2$ representing $\beta^{(l)} = (\beta_1^{(l)}, \beta_2^{(l)}, \ldots, \beta_n^{(l)})$, and $\lim_{l \to \infty} \beta^{(l)} = \beta^{(\infty)}$, where $\beta^{(\infty)}$ is in the critical case. Without loss of generality, we may assume $\min \beta_i^{(l)} = \beta_1^{(l)}$, thus the condition $\beta^{(\infty)}$ being in the critical case means $\beta_1^{(\infty)} = \sum_{i=2}^n \beta_i^{(\infty)}$.

For each $l$, by the conformal description above, and assuming that we fix $z_1^{(l)}$ at $\infty$, we have $g_l = e^{2u_l} g_0$ where $u_l$ is the solution of

$$\Delta u_l = -e^{2u_l} \quad \text{in} \quad \mathbb{C} \setminus \{z_2^{(l)}, \ldots, z_n^{(l)}\}, \quad (3.6)$$

subject to the asymptotic behavior

- $u_l \sim \beta_i^{(l)} \ln |z - z_i^{(l)}|$ as $z \to z_i^{(l)}$ for $i = 2, \ldots, n$;
- $u_l \sim -(2 + \beta_1^{(l)}) \ln |z|$ as $|z| \to \infty$.

The difficulty of the conformal geometry on sphere lies in the fact there exists a large conformal transformation group. In our set-up, this indicates $u_l$ in (3.6) is not unique. In particular, for

- scaling $u_l^{\lambda,0}(z) := u_l(\lambda z) + \ln \lambda$; \hfill (†)
- translation $u_0^{0,\kappa}(z) := u(z - \kappa)$, \hfill (‡)

$e^{2u_\lambda} g_0$ and $e^{2u_\kappa} g_0$ all represent the same conic metric on the punctured sphere as $e^{2u_l} g_0$.

To clearly state the normalization we shall choose, we first present the main tools of the proof: to study the level set of $u_l$ and apply the isoperimetric inequality. While these ideas have been explored before (cf. [CL2]), our problem required more delicate analysis. We would examine the defect of isoperimetric inequality carefully under the limit procedure. With the help of the proper normalizations we prove the Hausdorff convergence of level sets. This convergence leads to a uniform bound of $u_l$ on compact sets, which allows to extract a limit function $u_\infty$. Then the isoperimetric inequality is applied again to prove that $u_\infty$ must be radially symmetric about some point $z_0$. 


For each $u_l$, let $\Omega^{u_l}(t) := \{u_l > t\}$, 

$$A^{u_l}(t) := \int_{\Omega^{u_l}(t)} e^{2u_l} \quad \text{and} \quad B^{u_l}(t) = \int_{\Omega^{u_l}(t)} 1 = |\Omega(t)|.$$ 

Thus $A^{u_l}$ is monotone decreasing and the Gauss-Bonnet formula gives the following 

$$\int_{\mathbb{R}^2} e^{2u_l} = 2\pi(2 + \beta^{(l)}_1 + \cdots + \beta^{(l)}_n). \quad (3.7)$$ 

It follows that 

$$\lim_{t \to -\infty} A^{u_l}(t) = 2\pi(2 + \beta^{(l)}_1 + \cdots + \beta^{(l)}_n) \quad \text{and} \quad \lim_{t \to \infty} A^{u_l}(t) = 0.$$ 

Under the scaling and translation, we have 

$$A^{u_l_\lambda, \kappa}_l(t) = A^{u_l}(t - \ln \lambda).$$ 

We can now state our normalization for all $u_l$. For each $l$, we pick a suitable $\lambda_l$ and $\kappa_l$ such that 

$$A^{u_l_\lambda, \kappa}_l(\ln(1 + \beta^{(\infty)}_1)) = \pi(2 + \beta^{(l)}_1 + \cdots + \beta^{(l)}_n) = \frac{1}{2} A^{u_l_\lambda, \kappa}_l(-\infty). \quad (3.8)$$ 

The centroid of $\Omega^{u_l_\lambda, \kappa}_l(\ln(1 + \beta^{(\infty)}_1))$ is at 0. \quad (3.9) 

From now on, without confusion we write $u_l$ for $u_l_\lambda, \kappa_l$. For simplicity, we denote 

$$\Omega_l(t) = \Omega^{u_l}(t), \quad A_l(t) = A^{u_l}(t), \quad B_l(t) = B^{u_l}(t).$$ 

We can thus restate Theorem 1.2 into the following

**Theorem 3.3.** For a sequence of functions $\{u_l\}$ satisfying (3.6), assume that 

$$A_l(\ln(1 + \beta^{(\infty)}_1)) = \pi(2 + \beta^{(l)}_1 + \cdots + \beta^{(l)}_n) = \frac{1}{2} A_l(-\infty), \quad (3.10)$$

The centroid of $\Omega_l(t^*)$ is at 0, \quad (3.11) 

where $t^* \in \mathbb{R}$ is a fixed generic point, then $u_l$ sub-converges to $u_\infty$ in $C^\infty_\text{loc}(\mathbb{C} \setminus \{0\})$, where $u_\infty$ is given by 

$$e^{2u_\infty(z)} = 4(1 + \beta^{(\infty)}_1)^2 \frac{|z|^{2\beta^{(\infty)}_1}}{(1 + |z|^{2 + 2\beta^{(\infty)}_1})^2}. \quad (3.12)$$

Moreover, 

$$\lim_{l \to \infty} z^{(l)}_i = 0, \quad \text{for } i \geq 2.$$


In view of (3.5), Theorem 1.1 is an immediate consequence of Theorem 3.3.

In the rest of this section, we present a proof of Theorem 3.3.

Proof. we take a careful look of $\Omega_l(t)$. In view of asymptotic behavior of $u_l$, $\Omega_l(t)$ is a bounded region for each $t$, and

$$A_l(t) = \int_{\Omega_l(t)} e^{2u_l} = \int_{\Omega_l(t)} -\Delta u_l = \int_{\partial \Omega_l(t)} |\nabla u_l| + 2\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}).$$

(3.13)

In general, there are multiple connected components for $\Omega_l(t)$. For a regular value $t$ of $u_l$, $\Omega_l(t)$ consists of finitely many disjoint regions bounded by Jordan curves. Each component is simply connected due to the maximum principle.

We now present the following estimate relating the size of level sets and the upper bound of the function.

**Lemma 3.4.** For a fixed $t_0 \in \mathbb{R}$, let $\Gamma_l(t_0)$ be a connected component of $\Omega_l(t_0)$ does not contain any singular point of $u_l$ and $H_l = \max_{\Gamma_l(t_0)} \{u_l\}$. For $t \in [t_0, H_l]$, let $\Gamma_l(t) = \Omega_l(t) \cap \Gamma_l(t_0)$, $a_l(t) = \int_{\Gamma_l(t)} e^{2u_l}$ and $b_l(t) = |\Gamma_l(t)|$, then we have

$$a_l(t) \geq 4\pi (1 - e^{t - H_l}).$$

(3.14)

Furthermore, for $a_l(t) \leq 2\pi$, we have

$$b_l(t) \geq 4\pi e^{-H_l} (e^{-t} - e^{-H_l}).$$

(3.15)

**Proof.** Since $\Gamma_l(t_0)$ does not contain any singularity, a similar computation of (3.13) shows

$$a_l(t) = \int_{\Gamma_l(t)} e^{2u_l} = \int_{\Gamma_l(t)} -\Delta u_l = \int_{\partial \Gamma_l(t)} |\nabla u_l|, \quad t \in [t_0, H_l],$$

(3.16)

and $a_l(H_l) = 0$.

By the co-area formula, we have

$$a'_l(t) = -e^{2t} \int_{\partial \Gamma_l(t)} \frac{1}{|\nabla u|},$$

(3.17)

and

$$b'_l(t) = -\int_{\partial \Gamma_l(t)} \frac{1}{|\nabla u|}. $$

(3.18)
Hence
\[
(a_l(t))^2' = 2 a_l(t) a_l'(t) = -2 e^{2t} \int_{\partial \Gamma_l(t)} |\nabla u_l| \int_{\partial \Gamma_l(t)} \frac{1}{|\nabla u_l|} \tag{3.19}
\]
\[
\leq -2 e^{2t} \left( \int_{\partial \Gamma_l(t)} 1 \right)^2 \leq -8 \pi e^{2t} |\Gamma(t)| = -8 \pi e^{2t} b_l(t).
\]
Here we have used the Hölder’s inequality and isoperimetric inequality for $\Gamma_l(t)$:
\[
\int_{\partial \Gamma_l(t)} |\nabla u_l| \int_{\partial \Gamma_l(t)} \frac{1}{|\nabla u_l|} \geq |\partial \Gamma_l(t)|^2 \geq 4 \pi |\Gamma_l(t)|. \tag{3.20}
\]
By Fubini’s theorem, we also have
\[
a_l(t) = \int_{\Gamma_l(t)} e^{2u_l} = \int_{0}^{\infty} |e^{2u_l} > \lambda| \lambda d \lambda \tag{3.21}
\]
\[
= \int_{-\infty}^{H} |u > t|2e^{2t} dt
\]
\[
= e^{2t} b_l(t) + \int_{t}^{H} 2e^{2t} b_l(t) dt.
\]
Integrating (3.19) from $t$ to $H_l$ and using (3.21), we obtain
\[
-a_l(t)^2 \leq -4 \pi a_l(t) + 4 \pi e^{2t} b_l(t). \tag{3.22}
\]
Combining (3.19) and (3.22), we have
\[
-a_l(t) \leq -4 \pi - a_l'(t). \tag{3.23}
\]
(3.14) then follows from (3.23) and the fact that $a_l(H) = 0$. When $a_l \leq 2 \pi$, (3.15) is thus a consequence of (3.14) and (3.22). \hfill \Box

Lemma 3.5. Let
\[
A_{\infty}(t) := 4 \pi (1 + \beta_1^{(\infty)}) \frac{\rho^{2+2\beta_1^{(\infty)}}}{1 + \rho^{2+2\beta_1^{(\infty)}}}, \tag{3.24}
\]
where $\rho$ is determined by $e^{2t} = 4(1 + \beta_1^{(\infty)})^2 \frac{\rho^{2\beta_1^{(\infty)}}}{(1 + \rho^{2+2\beta_1^{(\infty)})^2}$, then under the normalization (3.5), we have
\[
\lim_{l \to \infty} A_l(t) = A_{\infty}(t), \quad \forall t \in \mathbb{R},
\]
and
\[
\lim_{l \to \infty} B_l(t) = B_{\infty}(t) = \pi \rho^2.
\]
Proof. We now run a similar argument for level sets including singular points.

Let

\[ f_i(t) := A_i^2(t) - (4\pi + 4\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}))A_i(t) + 4\pi e^{2t} B_i(t). \quad (3.25) \]

Then

\[
\begin{align*}
  f_i'(t) &= -2e^{2t} \int_{\Omega_i(t)} e^{2u} \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} + (4\pi + 4\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}))e^{2t} \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} \\
  &\quad + 8\pi e^{2t} B_i - 4\pi e^{2t} \int_{\Omega_i(t)} |\nabla u_i| + 2\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}) \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} \\
  &\quad + (4\pi + 4\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}))e^{2t} \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} + 8\pi e^{2t} B_i - 4\pi e^{2t} \int_{\Omega_i(t)} \frac{1}{|\nabla u_i|} \\
  &= 2e^{2t}(4\pi B_i - \int_{\partial \Omega_i(t)} |\nabla u_i| \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} ) \leq 0. \quad (3.26)
\end{align*}
\]

Since
\[
\int_{\mathbb{R}^2} e^{2u} dx = \int_{-\infty}^{\infty} 2e^{2t} B_i(t) dt < \infty,
\]

it follows
\[
e^{2t} B_i(t) \to 0, \quad \text{as } t \to \pm \infty.
\]

Let \( C_i = 2\pi(2 + \sum_{i=1}^{n} \beta_i^{(l)}) = \lim_{t \to -\infty} A_i(t) \), we obtain
\[
0 \leq f_i(t) \leq C_i^2 - (4\pi + 4\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}))C_i.
\]

\[
\lim_{t \to -\infty} \beta_i^{(l)} = \beta^{(\infty)} \quad \text{implies} \quad \lim_{t \to -\infty} C_i^2 - (4\pi + 4\pi (\beta_2^{(l)} + \cdots + \beta_n^{(l)}))C_i = 0,
\]

thus \( f_i \) converges to 0 uniformly. Moreover \( f_i' \) is integrable with
\[
\|f_i'\|_{L^1} = f_i(-\infty) - f_i(\infty) \quad \text{and} \quad \lim_{t \to \infty} ||f_i'||_{L^1} = 0.
\]

Combining (3.25) and (3.26) we find that \( A_i \) satisfy
\[
A_i A_i' - (2\pi \sum_{i=2}^{n} \beta_i^{(l)}) A_i' - (A_i^2 - (4\pi + 4\pi \sum_{i=2}^{n} \beta_i^{(l)}) A_i) = \frac{1}{2}(f_i' - 2f_i).
\quad (3.27)
\]
For simplicity, we denote \( \sum_{i=2}^{n} \beta_i(l) \) by \( \tilde{\beta}(l) \). Thus (3.27) can be rearranged as
\[
\frac{a}{A_l} + \frac{b}{4\pi(1 + \tilde{\beta}(l) - A_l)} A_l' = 1 + \frac{\frac{1}{2} f'_l - f_l}{A_l^2 - 4\pi(1 + \tilde{\beta}(l)) A_l},
\]
(3.28)
where \( a = \frac{\tilde{\beta}(l)}{2\tilde{\beta}(l) + 2} \) and \( b = -\frac{\tilde{\beta}(l) + 2}{2\tilde{\beta}(l) + 2} \).

Let \( t_l = \sup \{ t \mid \lim \inf_{l \to \infty} 4\pi(1 + \tilde{\beta}(l) - A_l(t) = 0) \} \) and \( t_r = \inf \{ t \mid \lim \inf_{l \to \infty} A_l(t) = 0 \} \).

Due to (3.10), we have \( t_l < \ln(1 + \beta_1(\infty)) < t_r \). It follows from the definition of \( t_l \) and \( t_r \) that \( \frac{1}{A_l^2 - 4\pi(1 + \tilde{\beta}(l)) A_l} \) is uniformly bounded on any finite closed interval \([r, s] \subset (t_l, t_r)\).

Recalling that \( \lim_{l \to \infty} \| f'_l \|_{L^1} = 0 \), thus \( f'_l \) converges to 0 in \( L^1 \) and we also have \( f_l \) converges uniformly to 0. Hence integrating (3.28) and taking the limit, we obtain
\[
\lim_{l \to \infty} \ln(A_l^a(4\pi(1 + \tilde{\beta}(l) - A_l)^{-b})) = \lim_{l \to \infty} \int_{r}^{s} 1 + \frac{\frac{1}{2} f'_l - f_l}{A_l^2 - 4\pi(1 + \tilde{\beta}(l)) A_l} dt = s - r
\]
(3.29)

Notice that \( \lim_{l \to \infty} A_l(\ln(1 + \beta_1(\infty))) = 2\pi(1 + \beta_1(\infty)) \), from this single point convergence and (3.29) we conclude that \( A_l \) has a pointwise limit \( A_\infty \) on \((t_l, t_r)\), which satisfies
\[
\left( \frac{a}{A_\infty} + \frac{b}{4\pi(1 + \tilde{\beta}(l) - A_\infty)} \right) A_\infty' = 1
\]
(3.30)
with \( A_\infty(\ln(1 + \beta_1(\infty))) = 2\pi(1 + \beta_1(\infty)) \).

By separation of variables, we have the solution of (3.30)
\[
A_\infty(t) = 4\pi(1 + \beta_1(\infty)) \frac{\rho^{2+2\beta_1(\infty)}}{1 + \rho^{2+2\beta_1(\infty)}},
\]
(3.31)
where \( \rho \) is such that \( e^{2t} = 4(1 + \beta_1(\infty))^2 \frac{\rho^{2\beta_1(\infty)}}{(1 + \rho^{2+2\beta_1(\infty)})^2} \). It is easy to see that
\[
\lim_{t \to -\infty} A_\infty(t) = 4\pi(1 + \beta_1(\infty)) \quad \text{and} \quad \lim_{t \to \infty} A_\infty(t) = 0.
\]
(3.32)
Combining (3.28) and (3.32), it is easy to see that $(t_l, t_r) = (-\infty, \infty)$. A simple computation which we will skip here gives the corresponding result for $B_l(t)$.

We now study the isoperimetric defect. Define, for any region $\Omega \subset \mathbb{R}^2$ with boundary a Jordan curve $\partial \Omega$, the isoperimetric defect is

$$D(\Omega) := |\partial \Omega|^2 - 4\pi |\Omega|.$$  \hfill (3.33)

It is easy to show that $D(\Omega)$ is super additive. This means, if $\Omega_1$ and $\Omega_2$ are two disjoint sets in $\mathbb{R}^2$, $\Omega = \Omega_1 \cup \Omega_2$, we have

$$D(\Omega) \geq D(\Omega_1) + D(\Omega_2).$$  \hfill (3.34)

Furthermore, we have the following

**Lemma 3.6** (Bonnesen’s inequality). For a bounded region $\Omega \subset \mathbb{R}^2$, let $r$ and $R$ be the radii of incircle and circumcircle of $\Omega$, then

$$D(\Omega) \geq \pi^2 (R - r)^2;$$

The equality holds if and only if $\Omega$ is a round disk.

In view of Bonnesen’s inequality, we can prove

**Lemma 3.7.** Let $D_l(t) := D(\Omega_l(t))$ be the isoperimetric defect of the level set $\Omega_l(t)$. Then there exists a subset $V \subset \mathbb{R}$ such that $|\mathbb{R} \setminus V| = 0$ and after passing to a subsequence,

$$\lim_{l \to \infty} D_l(t) = 0, \ \forall t \in V.$$  \hfill (3.35)

**Proof.** By (3.26) and (3.20), we have

$$2e^{2t} D_l(t) \leq 2e^{2t} \left( \int_{\partial \Omega_l(t)} |\nabla u_l| \int_{\partial \Omega_l(t)} \frac{1}{|\nabla u_l|} - 4\pi |\Omega_l(t)| \right) \leq -f_l'(t).$$  \hfill (3.36)

For each fixed $t_0$, we then conclude

$$2e^{2t_0} \int_{t_0}^{\infty} D_l(t) dt \leq \int_{t_0}^{\infty} 2e^{2t} B_l(t) dt \leq \int_{t_0}^{\infty} -f_l'(t) dt = f_l(t_0).$$

Since $f_l(t_0) \to 0$ as $l \to \infty$, we have $D_l(t)$ converges to 0 in $L^1_{(t_0, \infty)}$ norm, thus after passing to a subsequence $D_l(t)$ converges to 0 almost everywhere for $t \geq t_0$. Repeating the same argument for a sequence of $t_i \to -\infty$ and using a diagonal argument, it follows $D_l(t)$ converges to 0 almost everywhere on $\mathbb{R}$. Thanks to Sard’s theorem, after disregarding critical values for all of $u_l$, we still get the convergence (3.35) almost everywhere. \qed
Lemma 3.8. For each \( t \in V \), where \( V \) is given as in Lemma 3.7, let \( \Sigma_i(t) \) be the connected component of \( \Omega_i(t) \) with largest area. Then
\[
|\Sigma_i(t)| \to B_\infty(t), \quad \text{as} \quad l \to \infty,
\]
\[
|\Omega_i(t) \setminus \Sigma_i(t)| \to 0 \quad \text{and} \quad |\partial(\Omega_i(t) \setminus \Sigma_i(t))| \to 0, \quad \text{as} \quad l \to \infty.
\]

Proof. We prove by contradiction. If for \( \Omega_i^1(t) = \Sigma_i(t) \) and \( \Omega_i^2(t) = \Omega_i(t) \setminus \Sigma_i(t) \) we have
\[
\liminf_{l \to \infty} |\Omega_i^1(t)| \geq \delta_1 > 0 \quad \text{and} \quad \liminf_{l \to \infty} |\Omega_i^2(t)| \geq \delta_2 > 0.
\]
Then
\[
D_i(t) = (|\partial \Omega_i^1(t)| + |\partial \Omega_i^2(t)|)^2 - 4\pi(|\Omega_i^1(t)| + |\Omega_i^2(t)|) \geq 2\sqrt{4\pi |\Omega_i^1(t)|} \sqrt{4\pi |\Omega_i^2(t)|} \geq 8\pi \sqrt{\delta_1 \delta_2} > 0,
\]
a contradiction to Lemma 3.7.

Hence we conclude that there is exactly one component whose area tends to \( B_\infty(t) \), which we denote by \( \Sigma_i(t) \). Moreover, both the area and boundary length of the remaining components must go to zero. \( \square \)

Now take a monotone sequence \( \{t_i\}_{i \in \mathbb{Z}} \subset V \), where \( V \) is obtained in Lemma 3.7. By Lemma 3.5 and Lemma 3.8, for any \( 0 < \lambda << 1/2 \), there exists a positive integer \( L_i \) such that for all \( l > L_i \),
\[
\frac{|\Sigma_l(t)|}{|\Omega_l(t)|} \geq 1 - \lambda. \tag{3.38}
\]
Such \( \Sigma_l(t) \) is thus unique. By a diagonal argument, we may pick a subsequence of \( u_l \) (which we still call \( u_l \)) and assume that (3.38) holds for all \( l \). Notice that for \( t_i > t_j \), we have
\[
\Sigma_l(t_i) \subset \Sigma_l(t_j). \tag{3.39}
\]

We now explain our choice of \( t^* \) in (3.11). Without loss of generality, let \( t^* = t_0 \in V \). We have thus the following

Lemma 3.9. There exists a sequence of descending balls
\[
\cdots \supset B_{r_{i-1}}(p_{i-1}) \supset B_{r_i}(p_i) \supset B_{r_{i+1}}(p_{i+1}) \supset \cdots
\]
with \( \lim_{i \to \infty} r_i = \infty \) and \( \lim_{i \to \infty} r_i = 0 \), such that \( \Sigma_l(t_i) \) converges in Hausdorff distance to \( B_{r_i}(p_i) \).

Proof. Without loss of generality, assume \( t^* = t_0 \). By our choice of normalization, the centroid of \( \Omega_l(t_0) \) is the origin. It thus follows from
\[
\lim_{l \to \infty} D_l(t_0) = 0 \quad \text{that a subsequence of } \Sigma_l(t_0) \text{ converges in Hausdorff distance to } B_{r_0}(0) \quad \text{with } r_0 = \sqrt{\frac{B_\infty(t_0)}{\pi}}.
\]

By (3.39),
\[
\Sigma_l(t_i) \supset \Sigma_l(t_0) \supset \Sigma_l(t_j), \quad \text{for } t_i < t_0 < t_j.
\]

It thus follows that for each fixed \( t_i \), the centroid of \( \Sigma_l(t) \) is contained in a bounded set. Hence the conclusion follows. \( \Box \)

**Proof of Theorem 3.3.** By passing to a subsequence, we shall assume that
\[
\lim_{l \to \infty} z_{l_i}^l = z_i^\infty, \quad i \geq 2,
\]
where \( z_i^\infty \) is possibly at \( \infty \). Also let \( z_0 = \cap_{i \in \mathbb{Z}} B_{r_i}(p_i) \) be the point given by Lemma 3.9. We will show that \( u_l \) is uniformly bounded on any compact subset \( K \subset \mathbb{C} \setminus \{z_0^\infty, \ldots, z_n^\infty, z_0\} \). For any given \( \varepsilon > 0 \), we have thus a uniform \( L_{1} \in \mathbb{N} \) such that for any \( l > L_1 \),
\[
d(z_i^l, K) < \varepsilon.
\]
Clearly, for such a compact set \( K \), there exist \( r_i > r_j \) such that
\[
K \subset B_{r_i}(p_i) \setminus B_{r_j}(p_j).
\]
Hence for a \( \delta > 0 \) small enough, we have
\[
K \subset N_\delta(B_{r_i}(p_i)) \setminus N_\delta(B_{r_j}(p_j)),
\]
where \( N_\delta(\cdot) \) stands for the \( \delta \)-neighborhood. By Lemma 3.9, there exists a \( L_2 > 0 \) such that for \( l > L_2 \),
\[
K \subset \Sigma_l(t_j) \setminus \Sigma_l(t_i).
\]
It follows that
\[
u_l(x) \geq t_j, \quad \text{for } x \in K \text{ and } l > L_2.
\]
It remains to show a uniform upper bound for \( u_l \). By Lemma 3.8, for the chosen \( \varepsilon \), there exists a \( L_3 \in \mathbb{N} \) such that for all \( l > L_3 \), any connected component \( \Omega \) of \( \Omega_l(t_i) \setminus \Sigma_l(t_i) \) satisfies
\[
|\Omega| \leq \varepsilon/2, \quad |\partial \Omega| \leq \varepsilon/2.
\]
Now for \( l > \max\{L_1, L_2, L_3\} \) large enough, any component of \( \Omega_l(t_i) \) containing any singular point will not intersect \( K \). Thus, if \( \Sigma' \) is a connected component of \( \Omega_l(t_i) \) such that \( K \cap \Sigma' \neq \emptyset \), it contains no singular point. By Lemma 3.4, we conclude that
\[
\max_{\Sigma'} u_l \leq -\ln\left(1 - \frac{a_l(t_i)}{4\pi}\right) + t_i. \quad (3.40)
\]
Since \( a_l(t_i) \leq C_l \) and \( \lim_{l \to \infty} C_l = 4\pi(1 + \beta_1^{(\infty)}) < 4\pi \), there exists an \( L_4 \in \mathbb{N} \) such that for all \( l > L_4 \),
\[
C_l \leq 4\pi + \pi \beta_1^{(\infty)} < 4\pi. \tag{3.41}
\]
Combining (3.40) and (3.41), we thus get a uniform upper bound for \( u_l(z) \) for all \( z \in K \) and \( l > \max\{L_1, L_2, L_3, L_4\} \).

In summary, a subsequence of \( \{u_l\} \) is uniformly bounded in any compact subset \( K \subset \mathbb{C} \setminus \{z_\infty^1, \ldots, z_\infty^n, z_0\} \). In particular, by the standard \( L^p \) estimates of the Poisson equation (3.6), we have
\[
||u_l||_{W^{2,p}(K)} \leq C, \quad \forall p > 0.
\]
For \( p > n \), we apply Sobolev’s embedding and classical Schauder’s estimate to get
\[
||u_l||_{C^\infty(K)} \leq C.
\]
Let \( u_\infty := \lim_{l \to \infty} u_l \), then \( u_\infty \) satisfies
\[
\Delta u_\infty(z) = -e^{2u_\infty(z)}, \quad \text{for } z \in \mathbb{C} \setminus \{z_\infty^1, \ldots, z_\infty^n, z_0\}.
\]
It follows the corresponding \( A \) and \( B \) for \( u_\infty \) are just \( A_\infty \) and \( B_\infty \). In particular, any level set of \( u_\infty \) has vanishing isoperimetric deficit, which means each level set must be a circle. In addition, (3.19) being identity shows that \( |\nabla u_\infty| \) are constants on the round circles \( \{x; u_\infty(x) = c\} \).

Hence \( u_\infty \) has to be radially symmetric with center \( z_0 \), which in turn has to be the origin by our normalization (3.11). Since \( u_\infty \) is unbounded in view of \( B_\infty, z_0 = 0 \) has to be a singular point of \( u_\infty \).

Henceforth, \( u_\infty \) satisfies
\[
\Delta u_\infty(z) = -e^{2u_\infty(z)}, \quad \text{for } z \in \mathbb{C} \setminus 0. \tag{3.42}
\]
All solutions to (3.42) are classified in [CL2]. By direct computation, having \( A_\infty \) and \( B_\infty \) match with \( u_\infty \)’s, \( u_\infty \) is necessarily given by (3.12).

Finally, we show that
\[
\lim_{l \to \infty} z_i^{(l)} = z_0, \quad \text{for } i \geq 2. \tag{3.43}
\]
Suppose on the contrary, there are some singular points going to \( \infty \), then there exists \( L \) and \( T \), such that for all \( l \geq L, t \geq T \), \( \Sigma_l(t) \) contains only parts of singular points, say \( z_\infty^{(l)}_1, \ldots, z_\infty^{(l)}_k \). Let
\[
\beta^{(l)} = \beta^{(l)}_1 + \cdots + \beta^{(l)}_k.
\]
Applying the analysis of Lemma 3.5 only for the quantity \( a_l(t) := \int_{\Sigma_l(t)} e^{2u_l} \), we would have
\[
a_l a'_l - (2\pi \beta^{(l)}) a'_l - (a_l^2 - (4\pi + 4\pi \beta^{(l)}) a_l) \leq 0 \tag{3.44}
\]
hold for $t \geq T$ and $l \geq L$. Since $\hat{\beta}(l) > \tilde{\beta}(l)$, we deduce, by direct computation, that
\[
\limsup_{l \to \infty} a_l(t) < A_\infty(t), \quad t \geq T.
\]
While for any other component $\Sigma'$ of $\Omega(t)$, since $|\Sigma'| \to 0$ and $e^{2u_l}$ is uniformly integrable, we get
\[
\lim_{l \to \infty} \int_{\Sigma'} e^{2u_l} = 0.
\]
Hence the total contribution to $A_l(t)$ does not converge to $A_\infty(t)$, a contradiction. □

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