Vertex identification in trees

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Abstract

A red-white coloring of a nontrivial connected graph $G$ of diameter $d$ is an assignment of red and white colors to the vertices of $G$ where at least one vertex is colored red. Associated with each vertex $v$ of $G$ is a $d$-vector, called the code of $v$, whose $i$th coordinate is the number of red vertices at distance $i$ from $v$. A red-white coloring of $G$ for which distinct vertices have distinct codes is called an identification coloring or ID-coloring of $G$. A graph $G$ possessing an ID-coloring is an ID-graph. The minimum number of red vertices among all ID-colorings of an ID-graph $G$ is the identification number or ID-number of $G$. Necessary conditions are established for those trees that are ID-graphs. A tree $T$ is starlike if $T$ is obtained by subdividing the edges of a star of order 4 or more. It is shown that for every positive integer $r$ different from 2, there exist starlike trees satisfying some prescribed properties having ID-number $r$.

Keywords: distance; vertex identification; identification coloring; tree.

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1. Introduction

Over the years, many methods have been introduced with the goal of uniquely identifying the vertices of a connected graph. Often these approaches have employed distance and coloring. The oldest of these methods deal with what is referred to as the metric dimension of a connected graph. For a nontrivial connected graph $G$ of order $n$, the goal is to locate an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of $k$ vertices in $G$, $1 \leq k \leq n$, and associate with each vertex $v$ of $G$ the $k$-vector $(a_1, a_2, \ldots, a_k)$, where $a_i$ is the distance $d(v, w_i)$ between $v$ and $w_i$, $1 \leq i \leq k$. If the $k$-vectors produced in this manners are distinct, then the vertices of $G$ have been uniquely identified. For each connected graph $G$, such a set $W$ can always be found since we can always choose $W = V(G)$. The primary problem here is to determine the minimum size of such a set $W$. This is referred to as the metric dimension of $G$. Equivalently, the metric dimension of a connected graph $G$ can be defined as the minimum number of vertices of $G$ that can be assigned the same color, say red, such that for every two vertices $u$ and $v$ of $G$, there exists a red vertex $w$ such that $d(u, w) \neq d(v, w)$. This parameter is defined for every connected graph.

Another method that has been studied to uniquely identify the vertices of a connected graph $G$ has been referred to as the partition dimension of $G$. For a nontrivial connected graph $G$ of order $n$, the goal is to obtain a $k$-coloring, $1 \leq k \leq n$, of the vertices of $G$, where the coloring is not required (or expected) to be a proper coloring. This results in $k$ color classes $V_1, V_2, \ldots, V_k$ of $V(G)$. For each vertex $v$ of $G$, we once again associate a vector, here a $k$-vector $(a_1, a_2, \ldots, a_k)$ where $a_i$ denotes the distance from $v$ to a nearest vertex in $V_i$ for $1 \leq i \leq k$. If the vertices of $G$ have distinct $k$-vectors, then the vertices of $G$ have been uniquely identified. Such a coloring always exists since we can always assign distinct colors to the vertices of $G$, thereby obtaining a procedure that has similarity to metric dimension. The minimum number of colors that accomplishes this goal is referred to as the partition dimension of $G$. The partition dimension of a connected graph $G$ can also be defined as the minimum number $k$ of colors (denoted by $1, 2, \ldots, k$) that can be assigned to the vertices of $G$, one color to each vertex, so that for every two vertices $u$ and $v$ of $G$, there exists a color $i$ such that the distance between $u$ and a nearest vertex colored $i$ is distinct from the distance between $v$ and a nearest vertex colored $i$. This parameter is also defined for every connected graph.

Another method that has been introduced for the purpose of uniquely identifying the vertices of a connected graph is referred to as an identification coloring. Let $G$ be a connected graph of diameter $d \geq 2$ and let there be given a red-white vertex coloring $c$ of the graph $G$ where at least one vertex is colored red. That is, the color $c(v)$ of a vertex $v$ in $G$ is either red or white and $c(v)$ is red for at least one vertex $v$ of $G$. With each vertex $v$ of $G$, there is associated a $d$-vector $d(v) = (a_1, a_2, \ldots, a_d)$ called the code of $v$ corresponding to $c$, where the $i$th coordinate $a_i$ is the number of red vertices at
distance $i$ from $v$ for $1 \leq i \leq d$. If distinct vertices of $G$ have distinct codes, then $c$ is called an identification coloring or ID-coloring. Equivalently, an identification coloring of a connected graph $G$ is an assignment of the color red to a nonempty subset of $V(G)$ (with the color white assigned to the remaining vertices of $G$) such that for every two vertices $u$ and $v$ of $G$, there is an integer $k$ with $1 \leq k \leq d$ such that the number of red vertices at distance $k$ from $u$ is different from the number of red vertices at distance $k$ from $v$. A graph possessing an identification coloring is an ID-graph. A major difference here from the two methods described above is that not all connected graphs are ID-graphs.

The concept of metric dimension was introduced independently by Slater [15] and by Harary and Melter [10] and has been studied by many (see [4, 7], for example). Slater described the usefulness of these ideas when working with U.S. Coast Guard Loran (long range aids to navigation) stations in [15, 16]. Johnson [13, 14] of the former Upjohn Pharmaceutical Company applied this in attempts to develop the capability of large datasets of chemical graphs. The concept of partition dimension was introduced in [6]. These concepts as well as other methods of vertex identifications in graphs have been studied by many with various applications (see [2, 3, 8, 9, 11, 12, 17, 18] for example). The concepts of ID-colorings and ID-graphs were introduced and studied in [5].

All of the methods mentioned above involve constructing a vertex coloring of a connected graph $G$ with the goal of producing a vertex labeling of $G$ (using vectors of the same size as labels) so that distinct vertices of $G$ have the distinct labels. Consequently, the goal of each of these methods is to obtain an irregular labeling of $G$. The general topic of irregularity in graphs is described in [2]. There is the related topic of obtaining a labeling of $G$ by means of colorings where exactly two vertices of $G$ have the same label. These are called articular labels, a topic discussed in [1].

We first present five results obtained in [5] on ID-colorings. For an integer $t \geq 2$, the members of a set $S$ of $t$ vertices of a graph $G$ are called $t$-tuplets (twins if $t = 2$ and triplets if $t = 3$) if either (1) $S$ is an independent set in $G$ and every two vertices in $S$ have the same neighborhood or (2) $S$ is a clique, that is the subgraph $G[S]$ induced by $S$ is complete and every two vertices in $S$ have the same closed neighborhood.

**Proposition 1.1.** Let $c$ be an ID-coloring of a connected graph $G$. If $u$ and $v$ are twins of $G$, then $c(u) \neq c(v)$. Consequently, if $G$ is an ID-graph, then $G$ is triplet-free.

**Proposition 1.2.** Let $c$ be a red-white coloring of a connected graph $G$ where there is at least one vertex of each color. If $x$ is a red vertex and $y$ is a white vertex, then $\bar{d}(x) \neq \bar{d}(y)$.

**Theorem 1.1.** A nontrivial connected graph $G$ has $\text{ID}(G) = 1$ if and only if $G$ is a path.

**Theorem 1.2.** A connected graph $G$ of diameter 2 is an ID-graph if and only if $G = P_3$.

**Theorem 1.3.** For a positive integer $r$, there exists a connected graph $G$ with $\text{ID}(G) = r$ if and only if $r \neq 2$.

The following result describes a property of ID-colorings.

**Proposition 1.3.** Let $G$ be a connected graph with an ID-coloring $c$. If $H$ is a connected subgraph of $G$ such that (i) $H$ contains all red vertices in $G$ and (ii) $d_G(x, y) = d_H(x, y)$ for every two vertices $x$ and $y$ of $H$, then the restriction of $c$ to $H$ is an ID-coloring of $H$.

**Proof.** Let $\text{diam}(H) = d$ and let $c_H$ be the restriction of $c$ to $H$. For a vertex $v$ of $H$, let $\bar{d}_H(v) = (a'_1, a'_2, \ldots, a'_d)$ and let $d_c(v) = (a_1, a_2, \ldots, a_d)$. Notice that if $d = \text{diam}(G)$, then $\bar{d}_c(v) = (a_1, a_2, \ldots, a_d)$ while if $d < \text{diam}(G)$, then $a_i = 0$ for each integer $i$ with $d + 1 \leq t \leq \text{diam}(G)$. Since $H$ contains all red vertices in $G$ and $d_G(v, w) = d_H(v, w)$ for every vertex $w$ of $G$, it follows that $a_i = a'_i$ for $1 \leq i \leq d$ and so the restriction of $c$ to $H$ is an ID-coloring of $H$.

Both conditions stated in the hypothesis of Proposition 1.3 for a connected subgraph $H$ of a graph $G$ are needed. For example, consider the ID-graph $G$ in Figure 1. The subgraph $H_1$ of $G$ does not contain all red vertices of $G$ while the subgraph $H_2$ is not distance-preserving. For $i = 1, 2$, the restriction of the ID-coloring $c$ to the subgraph $H_i$ of $G$ is not an ID-coloring of $H_i$ (since there are twins both of which are colored white).

Here, our emphasis turns to trees that are ID-graphs, namely ID-trees. We investigate structural problems of ID-trees, provide necessary conditions for trees to be ID-trees, and establish a realization result on ID-numbers of ID-trees satisfying some prescribed conditions.

### 2. ID-colorings of trees

The only $t$-tuplets, $t \geq 2$, in a tree $T$ are end-vertices of $T$, all with the same neighbor. As we saw, if $T$ contains triplets, then $T$ is not an ID-tree. If $T$ contains twins and possesses an ID-coloring, then the twins must be colored differently in every ID-coloring. We now see that for trees, the concepts of twins and triplets are special cases of something more general.
If \( T \) is a tree with a vertex \( v \) possessing two isomorphic branches \( B_1 \) and \( B_2 \), then \( B_1 \) and \( B_2 \) are twin branches at \( v \) if there is an isomorphism from \( B_1 \) to \( B_2 \) fixing \( v \). If \( T \) contains a vertex \( v \) possessing three isomorphic branches \( B_1, B_2, \) and \( B_3 \) such that every two of them are twin branches, then \( B_1, B_2, \) and \( B_3 \) are triplet branches at \( v \). If the size of each branch at \( v \) is 1, then \( T \) contains twins or triplets. For example, there are three isomorphic branches of size 5 at the vertex \( v \) of the tree \( T \) of Figure 2. However, \( T \) has twin branches at \( v \) but no triplet branches at \( v \).

![Figure 1: Two subgraphs of an ID-graph \( G \).](image)

Let \( T_1 \) and \( T_2 \) be two rooted trees whose roots are \( v_1 \) and \( v_2 \), respectively. Then \( T_1 \) and \( T_2 \) are considered to be isomorphic rooted trees, denoted \( T_1 \cong T_2 \), if there is an isomorphism \( \alpha : V(T_1) \to V(T_2) \) such that \( \alpha(v_1) = v_2 \). For \( i = 1, 2 \), let \( c_i \) be a red-white coloring of a tree \( T_i \) rooted at \( v_i \) where \( T_1 \cong T_2 \). Then \( c_1 \) and \( c_2 \) are considered to be isomorphic colorings, denoted \( c_1 \cong c_2 \), if there is an isomorphism \( \alpha : V(T_1) \to V(T_2) \) such that \( \alpha(v_1) = v_2 \) and \( c_1(x) = c_2(\alpha(x)) \) for every vertex \( x \) of \( T_1 \). In particular, \( c_1(v_1) = c_2(v_2) \).

**Observation 2.1.** Suppose that a tree \( T \) has twin branches \( B_1 \) and \( B_2 \) at a vertex \( v \) and \( c \) is a red-white coloring of \( T \). For \( i = 1, 2 \), let \( c_i \) be the restriction of \( c \) to \( B_i \), rooted at \( v \). If \( c_1 \cong c_2 \), then \( c \) is not an ID-coloring of \( T \).

Let \( T_0 \) be a tree of size \( k \geq 1 \) rooted at a vertex \( v \). If the color of \( v \) is fixed, say \( v \) is white, then there are at most \( 2^k \) distinct (non-isomorphic) red-white colorings of \( T_0 \) in which \( v \) is colored white. Consequently, if there are more than \( 2^k \) copies of a particular branch of size \( k \) at \( v \), then \( T \) is not an ID-tree by Observation 2.1. In the case when \( k = 1 \), this simply says that no ID-tree can contain a triplet.

Let \( T \) be a tree rooted at a vertex \( v \) and let \( c \) be an ID-coloring of \( T \). If \( T_0 \) is a subtree of \( T \) of minimum order rooted at \( v \) such that \( T_0 \) contains all red vertices in \( T \), then the restriction of \( c \) to \( T_0 \) is an ID-coloring of \( T_0 \) by Proposition 1.3. Necessarily, all end-vertices of \( T_0 \) are red. In the case when \( T_0 \) is a path \( P_{k+1} \) of size \( k \) whose end-vertices are \( v \) and \( w \), there are at most \( 2^{k-1} \) distinct red-white colorings of \( P_{k+1} \) in which \( v \) is white and \( w \) is red.

A tree \( T \) is starlike if \( T \) is obtained by subdividing the edges of a star of order 4 or more. Equivalently, a tree \( T \) is starlike if and only if \( T \) has exactly one vertex whose degree is greater than 2. This vertex is referred to as the central vertex of \( T \). If the degree of the central vertex \( v \) of a starlike \( T \) is \( k \geq 3 \), then \( T \) has \( k \) branches (paths) at \( v \), each branch containing \( v \) as an end-vertex of \( T \). For example, the starlike tree \( T \) in Figure 3 has four branches at its central vertex. This tree is twin-free but does contain twin branches at its central vertex. This starlike tree is an ID-graph and an ID-coloring having exactly four red vertices is shown in Figure 3. In fact, \( \text{ID}(T) = 4 \).

**Proposition 2.1.** Let \( T \) be a starlike tree whose largest branch at its central vertex \( v \) has size \( k \). If \( T \) is an ID-tree, then for each integer \( i \) with \( 1 \leq i \leq k \), there are at most \( 2^i \) branches of size \( i \) or less at \( v \). Consequently, if \( T \) is an ID-tree, then \( T \) has at most \( 2^k \) branches at \( v \).

**Proof.** In view of Proposition 1.3, it suffices to determine the maximum number of distinct red-white colorings of all branches (paths) of \( T \) such that \( v \) is white and all end-vertices are red. For each integer \( i \) with \( 1 \leq i \leq k \), there are \( 2^{i-1} \) distinct red-white colorings of branches of size \( i \) at \( v \) in which \( v \) is colored white and the other end-vertex of each
branch is colored red. Thus, the minimum number of branches of size \( i \) at \( v \) without duplicating a red-white coloring of these branches is \( 2^{i-1} \). Therefore, the maximum number of all such red-white colorings of branches of all possibilities sizes at \( v \) is \( \sum_{i=1}^{k} 2^{i-1} = 2^{k} - 1 \). Since there can be one branch of size \( k \) or less at \( v \) all of whose vertices are colored white, it follows that there can be \( 2^{k} \) branches at \( v \) such that the red-white colorings of every two isomorphic branches at \( v \) are different. 

\( \square \)

**Corollary 2.1.** Let \( T \) be a starlike tree whose largest branch at its central vertex \( v \) has size \( k \). If \( T \) has more than \( 2^{k} \) branches at \( v \), then \( T \) is not an ID-tree.

For example, if \( T \) is a starlike ID-tree whose largest branch at its central vertex \( v \) has size 3, then (1) there are at most two branches of size 1 at \( v \), (2) there are at most four branches of size 2 or less at \( v \), and (3) there are at most eight branches of size 3 or less at \( v \). As an illustration, the three starlike trees of Figure 4 satisfy all conditions (1)–(3).

![Figure 4: Three starlike trees whose largest branch at its central vertex has size 3.](image)

The tree of Figure 4(a) has eight branches of size 3 at its central vertex and no branches of size less than 3 at its central vertex. The tree of Figure 4(b) has four branches of size 3, four branches of size 2, and no branches of size 1 at its central vertex. The tree of Figure 4(c) has four branches of size 3, two branches of size 2, and two branches of size 1 at its central vertex. In each case, there are eight branches at the central vertex of the tree. The red-white colorings of the three trees in Figure 4 are essentially the same coloring. It can be shown that this coloring is an ID-coloring. For the red-white coloring of the tree \( T \) of Figure 4(c), partial codes of the vertices of \( T \) containing the initial coordinates of each code are shown in Figure 5. (These partial codes are sufficient to show that all codes are distinct.) Since every two distinct vertices of \( T \) have distinct codes, this red-white coloring is an ID-coloring of \( T \).

![Figure 5: An ID-coloring of a starlike tree.](image)

**Theorem 2.1.** If \( T \) is a starlike tree with central vertex \( v \) whose branches at \( v \) have distinct sizes, then \( T \) is an ID-tree.

**Proof.** Suppose that \( \deg v = k \geq 3 \) and let \( B_1, B_2, \ldots, B_k \) be the branches (paths) of \( T \) at \( v \), where \( B_i \) has size \( m_i \) and \( m_i < m_{i+1} \) for \( 1 \leq i < k \). Define a red-white coloring \( c \) of \( T \) that assigns the color white to \( v \) and the color red to all other vertices of \( T \). We show that \( c \) is an ID-coloring of \( T \). By Proposition 1.2, it suffices to show that every two red vertices have
distinct codes. Let \( x, y \in V(T) - \{v\} \) and let \( \vec{d}(x) = (a_1, a_2, \ldots, a_d) \) and \( \vec{d}(y) = (b_1, b_1, \ldots, b_d) \) where \( d = \text{diam}(T) = m_{k-1} + m_k \).

Suppose that \( \vec{d}(x, v) = s \) and \( \vec{d}(y, v) = t \). We consider two cases, according to whether \( s \neq t \) or \( s = t \).

Case 1. \( s \neq t \), say \( s < t \). Then \( a_s \in \{0, 1\} \) and \( b_t \in \{1, 2\} \). If \( a_s \neq b_t \), then \( \vec{d}(x) \neq \vec{d}(y) \). Thus, we may assume that \( a_s = b_t = 1 \). Thus, \( a_{s+1} \in \{k - 1, k\} \) and \( b_{t+1} \in \{0, 1\} \). Since \( k \geq 3 \), it follows that \( a_{s+1} \geq 2 \) and so \( a_{s+1} \neq b_{t+1} \), implying that \( \vec{d}(x) \neq \vec{d}(y) \).

Case 2. \( s = t \). Then \( x \) and \( y \) belong to different branches of \( T \) at \( v \), say \( x \in V(B_i) \) and \( y \in V(B_j) \) where \( 1 \leq i < j \leq k \). Let \( B_i = (v = v_0, v_1, \ldots, v_{m_i}) \) and \( B_j = (v = v_0, u_1, \ldots, u_{m_j}) \), where then \( x = v_i \) and \( y = u_t \). If \( m_i - s + 1 = s_t \) then \( a_{m_i - s + 1} = 0 \) and \( b_{m_j - s + 1} \geq 1 \). If \( m_i - s + 1 \neq s_t \), then \( b_{m_j - s + 1} = a_{m_i - s + 1} + 1 \). In either case, \( \vec{d}(x) \neq \vec{d}(y) \). Therefore, \( c \) is an ID-coloring of \( T \).

3. Starlike trees with prescribed ID-number

We saw in Theorem 1.3 that for every integer \( r \geq 3 \), there exists a connected graph \( G \) with \( \text{ID}(G) = r \). For each given integer \( r \), the graph \( G \) described in the proof of Theorem 1.3 contains \( r \) pairwise disjoint twins from which it follows that \( \text{ID}(G) \geq r \). It was therefore only necessary to show that \( \text{ID}(G) \leq r \). We now show that for every integer \( r \geq 3 \), there exists a tree \( T \) with no twins at all such that \( \text{ID}(T) = r \). In addition, we show that there is a tree without twin branches having ID-number \( r \). In particular, we show that for every odd integer \( r \geq 5 \) there is a twin-free tree \( T \) whose automorphism group contains \((r+1)!\) elements such that \( \text{ID}(T) = r \). We also show that there is a red-white coloring \( c \) of the same class of trees \( T \) where exactly \( r - 1 \) vertices are colored red such that \( \vec{d}(x) = \vec{d}(y) \) for exactly one pair \( x, y \) of vertices of \( T \). Consequently, there is a red-white coloring of these trees \( T \) with exactly two vertices having the same code. As we metioned earlier, such a (red-white) coloring results in an antiregular labeling (see [1,2], for example.)

For each integer \( r \geq 3 \), let \( T = S_{r-1}(K_1, r+1) \) be the starlike tree obtained from the star \( K_1, r+1 \) of order \( r+2 \) by subdividing each edge of the \( r+1 \) edges in \( K_1, r+1 \) exactly \( r-1 \) times. Let \( v \) be the central vertex of \( T \). Then the degree of \( v \) is \( r+1 \) and each of the \( r+1 \) branches of \( T \) at \( v \) has length \( r \). For each integer \( i \) with \( 0 \leq i \leq r \), let \( B_i = (v, v_{i,1}, v_{i,2}, \ldots, v_{i,r}) \) be a branch of \( T \) at \( v \). Then \( \text{diam}(T) = 2r \) and \( T \) is twin-free.

**Theorem 3.1.** For each odd integer \( r \geq 3 \), \( \text{ID}(S_{r-1}(K_1, r+1)) = r \).

**Proof.** For an odd integer \( r \geq 3 \), let \( T = S_{r-1}(K_1, r+1) \), where \( v \) is the central vertex of \( T \) and \( B_i = (v, v_{i,1}, v_{i,2}, \ldots, v_{i,r}) \) is a branch of \( T \) at \( v \) for \( 0 \leq i \leq r \). First, we show that \( \text{ID}(T) \geq r \). For any red-white coloring of \( T \) that assigns the color red to at most \( r - 1 \) vertices of \( T \), there are at least two branches, say \( B_{01} \) and \( B_{11} \), of \( T \) at \( v \) such that the paths \( B_0 - v \) and \( B_1 - v \) contain no red vertices of \( T \). However then, \( \vec{d}(v_{0,1}) = \vec{d}(v_{1,1}) \), for example, and so this red-white coloring is not an ID-coloring of \( T \). Therefore, \( \text{ID}(T) \geq r \).

Next, we show that \( T \) has an ID-coloring with exactly \( r \) red vertices. Define a red-white coloring \( c \) of \( T \) by assigning the color red to each vertex \( v_{i,j} \) for \( 1 \leq i \leq r \) and white to the remaining vertices of \( T \). Thus, \( T \) has exactly \( r \) red vertices. It remains to show that \( c \) is an ID-coloring of \( T \). Since \( \text{diam}(T) = 2r \), the code of each vertex of \( T \) is a \((2r)\)-vector. Let \( x \) and \( y \) be two distinct vertices of \( T \). We consider two cases, according to whether \( x \) and \( y \) are both red or both white. Let \( \vec{d}(x) = (a_1, a_2, \ldots, a_{2r}) \) and \( \vec{d}(y) = (b_1, b_2, \ldots, b_{2r}) \).

Case 1. \( x \) and \( y \) are both red. Let \( x = v_{i,j} \) and \( y = v_{j,k} \) where \( 1 \leq i < j \leq r \).

- **First, suppose that \( j \neq r \).** Since (1) the last nonzero coordinate in \( \vec{d}(v_{i,j}) \) is the \((i + r)\)th coordinate where \( i + r = d(v_{i,j}, v_{r,r}) \) and the last nonzero coordinate in \( \vec{d}(v_{j,k}) \) is the \((j + r)\)th coordinate where \( j + r = d(v_{j,k}, v_{r,r}) \) and (2) \( i < j \), it follows that \( a_{j+r} = 0 \) and \( b_{j+r} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

- **Next, suppose that \( j = r \).** We saw that the last nonzero coordinate in \( \vec{d}(v_{i,i}) \) where \( 1 \leq i \leq r - 1 \) is the \((i + r)\)th coordinate. Since the last nonzero coordinate in \( \vec{d}(v_{r,r}) \) is the \((2r - 1)\)th coordinate where \( 2r - 1 = d(v_{r-1,r-1}, v_{r,r}) \), it follows that if \( i \neq r - 1 \), then \( \vec{d}(x) \neq \vec{d}(y) \). Thus, we may assume that \( x = v_{r-1,r-1} \). Because the first nonzero coordinate in \( \vec{d}(v_{r-1,r-1}) \) is the \( r \)th coordinate where \( r = d(v_{1,1}, v_{r-1,r-1}) \) and the first nonzero coordinate in \( \vec{d}(v_{r,r}) \) is the \((r + 1)\)th coordinate where \( r + 1 = d(v_{1,1}, v_{r,r}) \), it follows that \( a_r = 1 \) and \( b_r \neq 0 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Case 2. \( x \) and \( y \) are both white. First, we make some observations on the codes of vertices on \( B_0 \).

- The vertices on \( B_0 \) are the only white vertices of \( T \) whose codes contain the \( r \)-tuple \((1, 1, \ldots, 1) = 1^r \) as a subsequence. The vertex \( v \) is the only white vertex of \( T \) such that the first \( r \) coordinates of its code are 1 (that is, \( \vec{d}(v) = (1^r, 0^r) \)). For \( 1 \leq t \leq r \), the vertex \( v_0,t \) is the only white vertex such that in \( \vec{d}(v_0,t) \) the first \( t \) coordinates and the last \( r - t \) coordinates are 0 while the remaining coordinates are 1 (that is, \( \vec{d}(v_0,t) = (0^t, 1^r, 0^{r-t}) \) for \( 1 \leq t \leq r \)). Thus, all codes of the vertices of \( B_0 \) are distinct and they are also distinct from the codes of those white vertices that are not in \( B_0 \).
Hence, we may assume that neither $x$ nor $y$ belongs to $B_0$. Let $Q_i = B_i - v = (v_{i,1}, v_{i,2}, \ldots, v_{i,r})$ be the subpath of $B_i$ for $1 \leq i \leq r$. We consider two subcases, according to the location of $x$ and $y$.

**Subcase 2.1.** $x, y \in V(Q_i)$ where $1 \leq i \leq r$. Let $x = v_{i,p}$ and $y = v_{i,q}$ where $1 \leq p < q \leq r$ and $p, q \neq i$.

- First, suppose that $i \neq r$. Since (1) the last nonzero coordinate in $d(v_{i,p})$ is the $(p + r)$th coordinate where $p + r = d(v_{i,p}, v_{i,r})$ and the last nonzero coordinate in $d(v_{i,q})$ is the $(q + r)$th coordinate where $q + r = d(v_{i,q}, v_{i,r})$ and (2) $p < q$, it follows that $a_{q+r-1} = 0$ and $b_{q+r-1} = 1$ and so $d(x) \neq d(y)$.

- Next, suppose that $i = r$. Since (1) the last nonzero coordinate in $d(v_{i,p})$ is the $(p + r - 1)$th coordinate where $p + r - 1 = d(v_{i,p}, v_{i-1,r-1})$ and the last nonzero coordinate in $d(v_{i,q})$ is the $(q + r - 1)$th coordinate where $q + r - 1 = d(v_{i,q}, v_{i-1,r-1})$ and (2) $p < q$, it follows that $a_{q+r-1} = 0$ and $b_{q+r-1} = 1$ and so $d(x) \neq d(y)$.

**Subcase 2.2.** $x \in V(Q_i)$ and $y \in V(Q_j)$ where $1 \leq i < j \leq r$. Let $x = v_{i,p}$ and $y = v_{j,q}$ where $1 \leq p, q \leq r$, $p \neq i$, and $q \neq j$.

We consider two subcases, according to whether $p = q$ or $p \neq q$.

**Subcase 2.2.1.** $p = q$. Then $x = v_{i,p}$ and $y = v_{i,p}$ where $1 \leq i < j \leq r$ and $p \notin \{i, j\}$.

- First, suppose that $j + 1 \leq p \leq r$. Since (1) the first nonzero coordinate in $d(v_{i,p})$ is the $(p - i)$th coordinate where $p - i = d(v_{i,p}, v_{i,i})$ and the first nonzero coordinate in $d(v_{j,p})$ is the $(p - j)$th coordinate where $p - j = d(v_{j,p}, v_{j,j})$ and (2) $i < j$, it follows that $a_{j-p} = 0$ and $b_{j-p} = 1$ and so $d(x) \neq d(y)$.

- Next suppose that $1 \leq p \leq i - 1$. Let $c_0$ be the red-white coloring of $T$ obtained by recoloring $v_{i,i}$ and $v_{j,j}$ white and all other vertices of $T$ remain the same colors as in $c$. Since $d(v_{i,p}, w) = d(v_{j,p}, w)$ for every red vertex $w$ such that $w \notin \{v_{i,i}, v_{j,j}\}$, it follows that $\overrightarrow{d}(c_0)(x) = \overrightarrow{d}(c_0)(y) = (f_1, f_2, \ldots, f_{p})$. Observe that $d(v_{i,p}, v_{i,i}) = i - p$, $d(v_{i,p}, v_{i,j}) = j + p$, $d(v_{j,p}, v_{j,i}) = i + p$, and $d(v_{j,p}, v_{j,j}) = j - p$. Since $i - p < \min\{i + p, j - p, j + p\}$, it follows that $a_{i-p} = f_{i-p} + 1$ and $b_{i-p} = f_{j-p}$ and so $d(x) \neq d(y)$.

- Finally, suppose that $i + 1 \leq p \leq j - 1$. Let $c_0$ be the red-white coloring of $T$ obtained by recoloring $v_{i,i}$ and $v_{j,j}$ white and all other vertices of $T$ remain the same colors as in $c$. Since $d(v_{i,p}, w) = d(v_{j,p}, w)$ for every red vertex $w$ such that $w \notin \{v_{i,i}, v_{j,j}\}$, it follows that $\overrightarrow{d}(c_0)(x) = \overrightarrow{d}(c_0)(y) = (f_1, f_2, \ldots, f_{p})$. Observe that $d(v_{i,p}, v_{i,i}) = p - i$, $d(v_{i,p}, v_{i,j}) = j + p$, $d(v_{j,p}, v_{j,i}) = i + p$, and $d(v_{j,p}, v_{j,j}) = j - p$. Since $j + p > \max\{i + p, j - p, j + p\}$, it follows that $a_{j+p} = f_{j+p} + 1$ and $b_{j+p} = f_{j+p}$ and so $d(x) \neq d(y)$.

**Subcase 2.2.2.** $p \neq q$. First, suppose that $j \neq r$. Since (1) the last nonzero coordinate in $d(v_{i,p})$ is the $(p + r)$th coordinate where $p + r = d(v_{i,p}, v_{i,r})$ and the last nonzero coordinate in $d(v_{i,q})$ is the $(q + r)$th coordinate where $q + r = d(v_{i,q}, v_{i,r})$ and (2) $p \neq q$, it follows that either $a_{p+r} \neq b_{p+r}$ or $a_{q+r} \neq b_{q+r}$, implying that $d(x) \neq d(y)$.

For simplification, we now introduce notation where a code is expressed when no 0 coordinate is given after the final nonzero coordinate of a code. For example, if a code of a vertex is a 7-tuple $(1, 0, 2, 1, 0, 0, 0)$, we simply write this code as the 4-tuple $(1, 0, 2, 1)$.

Next, suppose that $j = r$. Thus, $x = v_{i,p}$ where $1 \leq i \leq r - 1$ and $p \neq i$ and $y = v_{r,q}$ where $1 \leq q \leq r - 1$ and $p \neq q$. We consider two possibilities.

**Subcase 2.2.2.1.** $2 \leq i \leq r - 1$. First, suppose that $p \geq i + 1$. Then $d(x) = d(v_{i,p}) = (0^{r-p-1}, 1, 0^{q-1}, 0, 1^{r-i})$. If $d(y) = d(v_{r,q})$ contains a coordinate $2$, then $d(x) \neq d(y)$. Thus, we may assume that $d(v_{r,q}, v_{r,r}) = r - q \neq q + t$ for $1 \leq t \leq r - 1$ and so $d(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{r-q}, 1^{r-i})$. Since $1^{r-i}$ is a subsequence in $d(y)$ and is not a subsequence of $d(x)$, it follows that $d(x) \neq d(y)$.

Next, suppose that $1 \leq p \leq i - 1$. If $i - p \neq p + t$ for some $t \in [r] - \{i\}$, then $(1^{i-1}, 0, 1^{r-i})$ is a subsequence of $d(x)$ and so there is no 2 as a coordinate of $d(x)$. If $d(y) = d(v_{r,q})$ contains a coordinate 2, then $d(x) \neq d(y)$ and so $d(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{r-q}, 1^{r-i})$. Thus, $d(x) \neq d(y)$. Hence, we may assume that $i - p = p + t$ for some $t \in [r] - \{i\}$ and so $i = 2t + p$. This implies that there is exactly one coordinate 2 of $d(x)$, namely $a_{i-p} = 2$. If $d(y)$ has no coordinate 2, then $d(x) \neq d(y)$. Hence, we assume that $d(y)$ has coordinate 2. This implies that $d(v_{r,q}, v_{r,r}) = r - q = q + t$ for some $t \in [r - 1]$ and $b_{r-q} = 2$ is the only coordinate 2 in $d(y)$. Hence, $i - p = r - q$ or $r - i = q + p$ and so $q > p$. There are two possibilities here. If $i - p = r - q < p + q$, then the second nonzero coordinate in $d(x)$ is $a_{p+1}$ while the second nonzero coordinate in $d(y)$ is $b_{q+1}$. If $i - p = r - q > p + 1$, then the first nonzero coordinate in $d(x)$ is $a_{q+1}$ while the the first nonzero coordinate in $d(y)$ is $b_{q+1}$. In either case, $d(x) \neq d(y)$. Therefore, $c$ is an ID-coloring and so ID$(T) = r$.

**Subcase 2.2.2.2.** $i = 1$. Then $d(x) = d(v_{1,p}) = (0^{p-2}, 1, 0, 1^{r-i-1})$. If $d(y) = d(v_{r,q})$ contains a coordinate 2, then $d(x) \neq d(y)$. Thus, we may assume that $d(v_{r,q}, v_{r,r}) = r - q \neq q + t$ for $1 \leq t \leq r - 1$. Since $r - q < r - q + 1$, it follows that $r - q \leq q$ or $q \geq r/2$. Thus, $d(y) = d(v_{r,q}) = (0^{r-q-1}, 1, 0^{r-q}, 1^{r-i-1})$. Thus, $p - 2 = r - q - 1$ (or $p + q = r + 1), 2 = 2q - r$ (or $r = 2q - 2$ is even) and so $p = q - 1$ (or $q = p + 1$). Since $r$ is odd, it follows that $d(x) \neq d(y)$. $\Box$
Corollary 3.1. For each odd integer \( r \geq 3 \), there exist a twin-free starlike tree \( T \) such that \( \text{ID}(T) = r \).

In the statement of Theorem 3.1, the condition that \( r \geq 3 \) is an odd integer is only required in Subcase 2.2.2.2. In fact, if \( r \geq 4 \) is an even integer, then there are exactly two white vertices in the red-white coloring described in the proof of Theorem 3.1, namely \( v_{i,p} \) and \( v_{r,q} \) where \( q = p + 1 \) and \( p + q = r + 1 \), that have the same code. That is, this red-white coloring is antiregular. Therefore, we have the following.

Proposition 3.1. For each even integer \( r \geq 4 \), there is an antiregular red-white coloring of the starlike tree \( S_{r-1}(K_{1,r+1}) \) having exactly \( r \) red vertices.

By the technique used in the proof of Theorem 3.1, the following result can be verified.

Proposition 3.2. Let \( r \geq 4 \) be an even integer. If \( T \) is the starlike tree obtained by subdividing exactly one edge of \( S_{r-1}(K_{1,r+1}) \), then \( \text{ID}(T) = r \).

None of the trees appearing in the results just above have the identity automorphism group. We next describe a class of trees having the identity automorphism group, where each such tree is necessarily twin-free, which can be used to show that for every integer \( r \geq 3 \), there is a tree \( T \) with the identity automorphism group such that \( \text{ID}(T) = r \).

Theorem 3.2. For each integer \( r \geq 3 \), there is a starlike tree \( T \) of order \( 1 + \binom{r+2}{2} \) having the identity automorphism group such that \( \text{ID}(T) = r \).

Proof: For each integer \( r \geq 3 \), let \( K_{1,r+1} \) be the star of order \( r + 2 \) with central vertex \( v \) that is adjacent to the \( r + 1 \) end-vertices \( v_{1}, v_{2}, \ldots, v_{r+1} \). Let \( T \) be the starlike tree obtained from the star \( K_{1,r+1} \) by subdividing the edge \( v_{i}v_{j} \) of \( K_{1,r+1} \) exactly \( i - 1 \) times for \( 1 \leq i \leq r + 1 \). In particular, \( v_{1} \) is not subdivided and \( v_{r+1} \) is subdivided exactly \( r \) times. Thus, \( T \) is twin-free, the order of \( T \) is \( 1 + \binom{r+2}{2} \) and \( \text{diam}(T) = 2r + 1 \). Since no two vertices of \( T \) are similar, it follows that \( T \) has the identity automorphism group. For each integer \( i \) with \( 1 \leq i \leq r + 1 \), let \( B_{i} = (v_{i1}, v_{i2}, \ldots, v_{ii}) \) be a branch of \( T \) at \( v \).

First, we show that \( \text{ID}(T) \geq r \). For any red-white coloring of \( T \) that assigns the color red to at most \( r - 1 \) vertices of \( T \), there are at least two branches \( B_{i} \) and \( B_{j} \) of \( T \) at \( v \) such that the paths \( B_{i} - v \) and \( B_{j} - v \) contain no red vertices of \( T \). However then, \( \vec{d}(v_{i1}) = \vec{d}(v_{j1}) \), for example, and so this red-white coloring is not an ID-coloring of \( T \). Therefore, \( \text{ID}(T) \geq r \).

Next, we show that \( T \) has an ID-coloring with exactly \( r \) red vertices. Define a red-white coloring \( c \) of \( T \) by assigning the color red to each vertex \( v_{ij} \) for \( 1 \leq i \leq r \) and white to the remaining vertices of \( T \). Thus, \( T \) has exactly \( r \) red vertices. It remains to show that \( c \) is an ID-coloring of \( T \). Since \( \text{diam}(T) = 2r + 1 \), the code of each vertex of \( T \) is a \((2r + 1)\)-vector. Let \( x \) and \( y \) be two distinct vertices of \( T \). We consider two cases, according to whether \( x \) and \( y \) are both red or both white. Let \( \vec{d}(x) = (a_{1}, a_{2}, \ldots, a_{2r+1}) \) and \( \vec{d}(y) = (b_{1}, b_{2}, \ldots, b_{2r+1}) \).

Case 1. \( x \) and \( y \) are both red. Let \( x = v_{ij} \) and \( y = v_{ij} \) where \( 1 \leq i < j \leq r \).

* First, suppose that \( j \neq r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_{ij}) \) is the \((i + r)\)th coordinate where \( i + r = d(v_{ij}, v_{r}) \) and the last nonzero coordinate in \( \vec{d}(v_{ij}) \) is the \((j + r)\)th coordinate where \( j + r = d(v_{ij}, v_{r}) \) and (2) \( i < j \), it follows that \( a_{i+j+r} = 0 \) and \( b_{i+j+r} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

* Next, suppose that \( j = r \). We saw that the last nonzero coordinate in \( \vec{d}(v_{ij}) \) where \( 1 \leq i \leq r - 1 \) is the \((i + r)\)th coordinate. Since the last nonzero coordinate in \( \vec{d}(v_{ir}) \) is the \((2r - 1)\)th coordinate where \( 2r - 1 = d(v_{r-1,r-1}, v_{r}) \), it follows that if \( i \neq r - 1 \), then \( \vec{d}(x) \neq \vec{d}(y) \). Thus, we may assume that \( x = v_{r-1,r-1} \). Because the first nonzero coordinate in \( \vec{d}(v_{r-1,r-1}) \) is the \( r \)th coordinate where \( r = d(v_{11}, v_{r-1,r-1}) \) and the first nonzero coordinate in \( \vec{d}(v_{11}) \) is the \((r + 1)\)th coordinate where \( r + 1 = d(v_{11}, v_{r}) \), it follows that \( a_{r} = 1 \) and \( b_{r} = 0 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Case 2. \( x \) and \( y \) are both white. First, we verify the following claim.

Claim. If \( x \in V(B_{r+1}) \) or \( y \in V(B_{r+1}) \), then \( \vec{d}(x) \neq \vec{d}(y) \).

The vertices on \( B_{r+1} \) are only the white vertices of \( T \) whose codes contain the \( r \)-tuple \((1, 1, \ldots, 1) = \vec{v}^{r+1} \) as a subsequence. The vertex \( v \) is the only white vertex of \( T \) such that the first \( r \) coordinates of its code are 1 (that is, \( \vec{d}(v) = (1, 0^{r+1}) \)). For \( 1 \leq t \leq r + 1 \), the vertex \( v_{1,t+1} \) is the only white vertex such that in \( \vec{d}(v_{1,t+1}) \) the first \( t \) coordinates and the last \( r + 1 - t \) coordinates are 0 while the remaining coordinates are 1 (that is, \( \vec{d}(v_{1,t+1}) = (0^{t}, \vec{v}^{r+1-t}) \) for \( 1 \leq t \leq r \)). Thus, all codes of the vertices of \( B_{r+1} \) are distinct and they are also distinct from the codes of those white vertices that are not in \( B_{r+1} \). Hence, the claim holds.

By the claim, we may assume that \( x \notin V(B_{r+1}) \) and \( y \notin V(B_{r+1}) \). Let \( Q_{i} = B_{i} - v = (v_{i1}, v_{i2}, \ldots, v_{ii}) \) be the subpath of \( B_{i} \), for \( 2 \leq i \leq r \). We consider two subcases, according to the location of \( x \) and \( y \).

Subcase 2.1. \( x, y \in V(Q_{i}) \) where \( 2 \leq i \leq r \). Let \( x = v_{i,p} \) and \( y = v_{i,q} \) where \( 1 \leq p < q \leq i - 1 \).
∗ First, suppose that \( i \neq r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_i,p) \) is the \((p+r)\)th coordinate where \( p+r = d(v_i,p, v_{r,r}) \) and the last nonzero coordinate in \( \vec{d}(v_q,q) \) is the \((q+r)\)th coordinate where \( q+r = d(v_q,q, v_{r,r}) \), and (2) \( p < q \), it follows that \( a_{q+r-1} = 0 \) and \( b_{q+r-1} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

∗ Next, suppose that \( i = r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_r,p) \) is the \((p+r-1)\)th coordinate where \( p+r-1 = d(v_r,p, v_{r-1,r-1}) \) and the last nonzero coordinate in \( \vec{d}(v_q,q) \) is the \((q+r-1)\)th coordinate where \( q+r-1 = d(v_q,q, v_{r-1,r-1}) \), and (2) \( p < q \), it follows that \( a_{q+r-1} = 0 \) and \( b_{q+r-1} = 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.2. \( x \in V(Q_i) \) and \( y \in V(Q_j) \) where \( 2 < i < j \leq r \). Let \( x \neq v_i,p \) and \( y = v_j,q \) where \( 1 \leq p \leq i-1 \) and \( 1 \leq q \leq j-1 \).

We consider two subcases, according to \( p = q \) or \( p \neq q \).

Subcase 2.2.1. \( p = q \). Then \( x = v_i,p \) and \( y = v_j,p \) where \( 2 \leq i < j \leq r \) and \( p \notin \{ i,j \} \). Let \( c_0 \) be the red-white coloring of \( T \) obtained by recoloring \( v_i,i \) and \( v_j,j \) white and all other vertices of \( T \) remain the same colors as in \( c \). Since \( d(v_i,p, w) = d(v_j,p, w) \) for every red vertex \( w \) such that \( w \notin \{ v_i,i,v_j,j \} \), it follows that \( \vec{d}_c(x) = \vec{d}_c(y) = (f_1,f_2,\ldots,f_r) \). Observe that \( d(v_{i,q}, v_{i,i}) = i - p, d(v_{j,q}, v_{j,j}) = j + p, d(v_{i,p}, v_{i,i}) = i + p, \) and \( d(v_{j,p}, v_{j,j}) = j - p \). Since \( i - p < \min \{ i + p, j - p, j + p \} \), it follows that \( a_{i-p} = f_1 - i + 1 \) and \( b_{i-p} = f_i - p + 1 \) and so \( \vec{d}(x) \neq \vec{d}(y) \).

Subcase 2.2.2. \( p \neq q \). First, suppose that \( j \neq r \). Since (1) the last nonzero coordinate in \( \vec{d}(v_i,p) \) is the \((p+r)\)th coordinate where \( p+r = d(v_i,p, v_{r,r}) \) and the last nonzero coordinate in \( \vec{d}(v_q,q) \) is the \((q+r)\)th coordinate where \( q+r = d(v_q,q, v_{r,r}) \), and (2) \( p \neq q \), it follows that either \( a_{p+r} \neq b_{p+r} \) or \( a_{q+r} \neq b_{q+r} \), implying that \( \vec{d}(x) \neq \vec{d}(y) \).

Next, suppose that \( j = r \). Thus, \( x = v_i,p \) where \( 2 \leq i \leq r-1 \) and \( 1 \leq p \leq i-1 \) and \( y = v_r,q \) where \( 1 \leq q \leq r-1 \) and \( p \neq q \). If \( i - p \neq p + \ell \) for some \( \ell \in [r-i] \), then \((1^{r-i},0,1^{r-\ell})\) is a subsequence of \( \vec{d}(x) \) and so there is no coordinate 2 in \( \vec{d}(x) \). If \( \vec{d}(y) = \vec{d}(v_{r,q}) \) contains 2 as a coordinate, then \( \vec{d}(x) \neq \vec{d}(y) \) and so \( \vec{d}(y) = \vec{d}(v_{r,q}) = (0,\ldots,0,1^{r-\ell},0,\ldots,0) \). Thus, \( \vec{d}(x) \neq \vec{d}(y) \). Hence, we may assume that \( i - p = p + \ell \) for some \( \ell \in [r-i] \) and so \( i = 2p + \ell \). This implies that there is exactly one coordinate of \( \vec{d}(x) \) which is 2, namely \( a_{i-p} = 2 \). If \( \vec{d}(y) \) has no coordinate 2, then \( \vec{d}(x) \neq \vec{d}(y) \). Hence, we assume that \( \vec{d}(y) \) has 2 as a coordinate. This implies that \( d(v_{r,q}, v_{r,r}) = r - q = q + t \) for some \( t \in [r-1] \) and \( b_{r-q} = 2 \) is the only coordinate 2 in \( \vec{d}(y) \). Hence, \( i - p = r - q \) or \( r - i = q - p \) and so \( q > p \). There are two possibilities here. If \( i - p = r - q \) and \( p + 1 \), then the second nonzero coordinate in \( \vec{d}(x) \) is \( a_{p+1} \) while the the second nonzero coordinate in \( \vec{d}(y) \) is \( b_{q+1} \). If \( i - p = r - q \) and \( q + 1 \), then the first nonzero coordinate in \( \vec{d}(x) \) is \( a_{p+1} \) while the the first nonzero coordinate in \( \vec{d}(y) \) is \( b_{q+1} \). In either case, \( \vec{d}(x) \neq \vec{d}(y) \).

Therefore, \( c \) is an ID-coloring and so \( \text{ID}(T) = r \).

Several problems are suggested by the results presented here.

(1) For a given integer \( r \geq 3 \), what is the smallest order of a tree \( T \) such that \( \text{ID}(T) = r \)?

(2) For a given integer \( r \geq 3 \), what is the smallest order of a twin-free tree \( T \) such that \( \text{ID}(T) = r \)?

For (2), we have seen that this smallest order is no more than \( 1 + \binom{r+1}{2} \).

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