CENTRALIZERS OF CERTAIN QUADRATIC ELEMENTS IN POISSON–LIE ALGEBRAS AND ARGUMENT SHIFT METHOD

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1. Introduction.

Let \( g \) be a semisimple complex Lie algebra. The universal enveloping algebra \( U(g) \) bears a natural filtration by the degree with respect to the generators. The associated graded algebra \( \text{gr} U(g) \) is naturally isomorphic to the symmetric algebra \( S(g) = \mathbb{C}[g^*] \) by the Poincaré-Birkhoff-Witt theorem. The commutator operation on \( U(g) \) defines a Poisson bracket on \( S(g) \), which we call the Poisson–Lie bracket.

The argument shift method gives a way to construct Poisson-commutative subalgebras in \( S(g) \). The method is as follows. Let \( ZS(g) = S(g)^g \) be the center of \( S(g) \) with respect to the Poisson bracket, and let \( \mu \in g^* \) be a regular semisimple element. Then the algebra \( A_\mu \subset S(g) \) generated by the elements \( \partial^\mu \Phi \), where \( \Phi \in ZS(g) \), (or, equivalently, generated by central elements of \( S(g) = \mathbb{C}[g^*] \) shifted by \( t\mu \) for all \( t \in \mathbb{C} \)) is Poisson-commutative and has maximal possible transcendence degree equal to \( \frac{1}{2}(\dim g + \text{rk} g) \) (see [3]). Moreover, the subalgebras \( A_\mu \) are maximal Poisson-commutative subalgebras in \( S(g) \) (see [7]). In [9], the subalgebras \( A_\mu \subset S(g) \) are named the Mischenko–Fomenko subalgebras.

Let \( h \subset g \) be a Cartan subalgebra of the Lie algebra \( g \). We denote by \( \Delta \) and \( \Delta_+ \) the root system of \( g \) and the set of positive roots, respectively. Let \( \alpha_1, \ldots, \alpha_l \) be the simple roots. Fix a non-degenerate invariant scalar product \( (\cdot, \cdot) \) on \( g^* \) and choose from each root space \( g_\alpha \), \( \alpha \in \Delta \), a nonzero element \( e_\alpha \) such that \( (e_\alpha, e_{-\alpha}) = 1 \). Set \( h_\alpha := [e_\alpha, e_{-\alpha}] \), then for any \( h \in h \) we have \( (h_\alpha, h) = (\alpha, h) \).

The elements \( e_\alpha \) (\( \alpha \in \Delta \)) together with \( h_1, \ldots, h_l \in h \) form a basis of \( g \).

We identify \( g \) with \( g^* \) via the scalar product \( (\cdot, \cdot) \) and assume that \( \mu \) is a regular semisimple element of the fixed Cartan subalgebra \( h \subset g = g^* \). The linear and quadratic part of the Mischenko–Fomenko subalgebras can be described as follows [2]:

\[
A_\mu \cap g = h,
\]

\[
A_\mu \cap S^2(g) = S^2(h) \oplus Q_\mu, \quad \text{where} \quad Q_\mu = \left\{ \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} e_{\alpha} e_{-\alpha} | h \in h \right\}.
\]

The main result of the present paper is the following

Theorem 1. For generic \( \mu \in h \) (i.e. for \( \mu \) in the complement to a certain countable union of Zariski-closed subsets in \( h \)), the algebra \( A_\mu \) is the Poisson centralizer of the subspace \( Q_\mu \) in \( S(g) \).

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In [1, 4, 6] the Mischenko–Fomenko subalgebras were lifted (quantized) to the universal enveloping algebra, i.e. the family of commutative subalgebras $A_\mu \subset U(g)$ such that $\text{gr} A_\mu = A_\mu$ was constructed for any classical Lie algebra $g$ (i.e. $sl_r$, $so_r$, $sp_{2r}$). In [5] we do this (by different methods) for any semisimple $g$.

We deduce the following assertion from Theorem 1.

**Theorem 2.** For generic $\mu \in h$ there exist no more than one commutative subalgebra $A_\mu \subset U(g)$ satisfying $\text{gr} A_\mu = A_\mu$.

This means that there is a unique quantization of Mischenko–Fomenko subalgebras. In particular, the methods of [1, 4, 6] and [5] give the same for classical Lie algebras. In the case $g = gl_n$ the assertion of Theorem 2 was proved by A. Tarasov [8] for any regular $\mu \in h$.

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2. Proof of Theorem 1

Note that the set $E_n \subset h$ of such $\mu \in h$ that the Poisson centralizer of the space $Q_\mu$ in $S^n(g)$ has the dimension greater than $\dim A_\mu \cap S^n(g)$ is Zariski-closed in $h$ for any $n$. Therefore it suffices to prove that $E_n \neq h$ for any $n$. Thus, it suffices to prove the existence of $\mu \in h$ satisfying the conditions of the Theorem.

**Lemma 1.** There exist $\mu, h \in h$ such that numbers $\frac{\langle h, \alpha \rangle}{\langle \alpha, \mu \rangle}$ ($\alpha \in \Delta_+$) are linearly independent over $Q$.

**Proof.** Choose $\mu$ such that the values $\alpha_i(\mu)$ are algebraically independent over $Q$ for simple roots $\alpha_i$. Since there are no proportional positive roots, the numbers $\frac{1}{\langle \alpha, \mu \rangle}$, $\alpha \in \Delta_+$, are linearly independent over $Q$. Choose $h$ such that the values $\langle \alpha, h \rangle$ are nonzero rational numbers. Then the numbers $\frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle}$, $\alpha \in \Delta_+$, are linearly independent over $Q$. $\square$

Choose $\gamma \in g^*$ such that $\gamma(\alpha_i) = 1$ for any simple root $\alpha_i$ and $\gamma(e_\alpha) = 0$ for $\alpha \in \Delta$. We define a new Poisson bracket $\{\cdot, \cdot\}_\gamma$ on $S(g)$ by setting $\{x, y\}_\gamma = \gamma([x, y])$ for $x, y \in g$. This bracket is compatible with the Poisson–Lie bracket, i.e. the linear combination $t\{\cdot, \cdot\} + (1 - t)\{\cdot, \cdot\}_\gamma$ is a Poisson bracket on $S(g)$ (i.e. satisfies the Jacobi identity) for any $t \in \mathbb{C}$. Moreover, for $t \neq 0$, the corresponding Poisson algebras are isomorphic. Namely, denote by $S(g)_t$ the algebra $S(g)$ equipped with the Poisson bracket $t\{\cdot, \cdot\} + (1 - t)\{\cdot, \cdot\}_\gamma$; then for $t \neq 0$ the Poisson algebra isomorphism $\psi_t : S(g)_t \to S(g)_t$ is defined on the generators $x \in g$ as follows: $\psi_t(x) = t^{-1}x + t^{-2}(1 - t)\gamma(x)$. Clearly, we have $\psi_t(Q_\mu) = Q_\mu$.

**Lemma 2.** The transcendence degree of the Poisson centralizer of the subspace $Q_\mu$ in $S(g)_0$ is not greater than $\frac{1}{2}(\dim g + \text{rk } g)$ for some $\mu \in h$. 

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Proof. Choose \( \mu \) and \( h \) as in Lemma 1 and set \( q = \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} e_\alpha e_{-\alpha} \in Q_\mu \). For any \( f \in S(\mathfrak{g}) \), we have \( \{q, f\}_\gamma = \sum_{\alpha \in \Delta_+} \gamma(h_\alpha) \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} (e_\alpha e_{-\alpha} - e_{-\alpha} e_\alpha) \). In particular,

\[
(1) \quad \{q, \prod_{i=1}^l h_{\alpha_i}^{m_i} \prod_{\alpha \in \Delta_+} e_\alpha^{n_\alpha} e_{-\alpha}^{n_{-\alpha}} \}_\gamma = \sum_{\alpha \in \Delta_+} \gamma(h_\alpha) \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} \left( n_{-\alpha} - n_\alpha \right) \prod_{i=1}^l h_{\alpha_i}^{m_i} \prod_{\alpha \in \Delta_+} e_\alpha^{n_\alpha} e_{-\alpha}^{n_{-\alpha}}.
\]

For any \( \alpha = \sum_{i=1}^l k_i \alpha_i \in \Delta_+ \), we have \( \gamma(h_\alpha) = \sum_{i=1}^l k_i \in Q \setminus \{0\} \). Since the numbers \( \frac{\langle \alpha, h \rangle}{\langle \alpha, \mu \rangle} \) are linearly independent over \( Q \), the right hand part of (1) is zero iff \( n_\alpha - n_{-\alpha} = 0 \) for any \( \alpha \in \Delta_+ \). This means that the Poisson centralizer of \( q \) in \( S(\mathfrak{g})_0 \) is linearly generated by monomials having equal degrees in \( e_\alpha \) and \( e_{-\alpha} \) for any \( \alpha \in \Delta_+ \), i.e. the Poisson centralizer of \( q \) in \( S(\mathfrak{g})_0 \) is generated (as a commutative algebra) by the elements \( h_{\alpha_i} \) \( (i = 1, \ldots , l) \) and \( e_\alpha e_{-\alpha} \) \( (\alpha \in \Delta_+) \). Therefore, the transcendence degree of the Poisson centralizer of \( q \) in \( S(\mathfrak{g})_0 \) is equal to \( \frac{1}{2} (\dim \mathfrak{g} + \text{rk} \mathfrak{g}) \).

By Lemma 2, the transcendence degree of the Poisson centralizer of the subspace \( Q_\mu \) in \( S(\mathfrak{g})_t \) is not greater than \( \frac{1}{2} (\dim \mathfrak{g} + \text{rk} \mathfrak{g}) \) for generic \( t \). Since the Poisson algebras \( S(\mathfrak{g})_t \) are isomorphic to each other for \( t \neq 0 \), this lower bound of the transcendence degree holds for any \( t \in \mathbb{C} \). Let \( Z \subset S(\mathfrak{g}) \) be the Poisson centralizer of \( Q_\mu \) in \( S(\mathfrak{g})_1 \). Since \( \text{tr deg}(Z) \leq \text{tr deg}(A_\mu) \) and \( A_\mu \subset Z \), we see that each element of \( Z \) is algebraic over \( A_\mu \). By Tarasov’s results \( [7] \), the subalgebra \( A_\mu \) is algebraically closed in \( S(\mathfrak{g})_1 \), hence, \( Z = A_\mu \). Theorem 1 is proved.

3. Proof of Theorem 2

By \( [3] \), the subspace \( A_\mu^{(2)} = \mathbb{C} + h + S^2(h) + Q_\mu \subset S(\mathfrak{g})^{(2)} \) can be uniquely lifted to a commutative subspace \( A_\mu^{(2)} \subset U(\mathfrak{g})^{(2)} \) (this subspace is the image of \( A_\mu^{(2)} \) under the symmetrization map). By Theorem 1, any lifting \( A_\mu \subset U(\mathfrak{g}) \) of \( A_\mu \) is the centralizer of the subspace \( A_\mu^{(2)} \) in \( U(\mathfrak{g}) \) for generic \( \mu \). Theorem 2 is proved.

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