Commutativity of Prime Ring with Orthogonal Symmetric Biderivations

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1. Introduction

More than a few authors, investigated the structure prime ring & semiprime rings (commutativity) accepting the derivations, generalized derivations etc. The notion of derivations of prime rings was originated by (Posner 1957), jordan derivations of prime rings was originated by (cusack 1975). These derivations was extended by (Bell and Daif 1995) for commutativity of prime rings. Later on (Bresar 1993) used centralizing concept using derivations. These generalizations was done in the article derivations using semiprime rings with results are commutative by (Daif 1998). The concept of symmetric biderivations on prime and semiprime rings was introduced by (Vukman 1989). The notation and terminology in this paper follows (Vukman 1990 and Oukhtite 2011). Many authors have their contribution to orthogonality of derivations on semiprime as well as prime rings. The idea of orthogonality of derivations on semiprime as well as prime rings was developed by (Vukman and Bresar 1989). (Argac 2004) studied orthogonality conditions for generalized derivations. (Ashraf 2010) obtained the orthogonality conditions for a pair of derivations in gamma rings. with their results (Jaya Subba Reddy et. al 2016) obtained the essential and sufficient conditions of biderivations to be orthogonal. (Oukhtite et. al. 2014) proved the commutativity results of prime rings with derivations using jordan ideals. In this current study it was extended the results of commutativity of prime rings with orthogonal biderivations using Jordan ideals.

In the present article we studied some theorems related to commutativity of prime rings using commutator identities satisfied by biderivations with Jordan ideals. We established the following theorems as follows.

Theorem 1: Any two biderivations B and B2 satisfies the condition [B1(u,v),B2(v,w)] = [u,w], B1(u,v)B1(v,u) - B2(w,u)B2(v,w) = [u,v], for all u,v,w ∈ J then that ring is commutative.

Theorem 2: Any three nonzero biderivations B1, B2 and B3 of R satisfies one of the following
(i) B1(v,w)B1(w,u) = B1(w,u)B1(v,w),
(ii) B2(v,w)B2(w,u) - B3(u,v)B3(v,w) = [u,v], for all u,v,w ∈ J then is commutative and B1 = B2.

2. Preliminaries

In each part of this article all rings assumed to be associative and possesses an identity. As a well-known the commutator (uv − vu) will be symbolized as [u,v]. We are wellknown that R is a prime ring if uRu = 0 ⇒ u = 0 or v = 0 and is semiprime if uRu = 0 ⇒ u = 0. If D(u,v) = D(u)v + vD(u), for any u,v ∈ R then we call this additive map D: R → R is a derivation. We Defined, biadditive mapping B(.,.): R × R → R as a symmetric biderivation if B(u,v) = B(u,v) + uB(v,t), for any u,v,t ∈ R. Clearly, in next case also B(u,v) = B(u,v) + vB(u,v), for every u,v,t ∈ R. Any pair (d, g) of derivations

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are orthogonal if \(d(u)Rg(v) = 0 = g(v)Rd(u)\) for any \(u,v \in R\) (Vukman and Bresar 1989). Likewise, any pair \((B, D)\) of biderivations are said to be orthogonal if \(B(u,v)RD(v,r) = 0 = D(v,r)RB(u,v)\) for all \(u,v,r \in R\). If \(u \cdot x \in J\), for any \(u \in J, x \in R\), then we say \(J\) is a Jordan ideal of \(R\). Note that \(B(x)\) means \(B(x,m)\) means for some \(m \in J\).

In the entire paper \(R\) act as a prime ring with 2-torsion free & \(J \neq 0\) is a jordan ideal of \(R\)

Following are known results to the readers

**Res 1:** If \([a,a^2]\) = 0 for any \(u \in J\), then is in center of \(R\).

**Res 2:** If an additive subgroup is a subset of \(Z(R)\), then \(R\) is commutative ring.

**Res 3:** a non commutative ring \(R\) satisfies \(a[u,wv]b = 0\), for every \(v,w \in J\), \(u \in R\), then \(a = 0\) or \(b = 0\).

We studied the following lemmas for proving the main theorems

**Lemma 2.1 (Reddy C.J.S and Reddy B.R 2016)**

A semiprime ring \(R\) of characteristic not two, a pair of biderivations \(B_1\) and \(B_2\) are to be orthogonal iff the following results are equivalent:

1. \(B_1B_2 = 0\)
2. \(B_1(u,v)B_2(v,w) = 0\) or \(B_1(u,v)B(v,w) = 0\).
3. \(B_1B_2\) is a Biderivation.
4. \(B_1(u,v)B_2(v,w) + B_2(v,w)B_1(u,v) = 0\), for every \(u,v,w \in R\).

**Lemma 2.2**

Any two biderivations \(B_1\) and \(B_2\) satisfies the condition \(B_1(B_2(u,v) - u) = 0\), for every pair \(u,v \in J\), then orthogonality of \(B_1\) and \(B_2\) are satisfied, also either \(B_1 = 0\) or \(B_2 = 0\).

**Proof:** Consider \(B_2(u,v) = 0\).

We have \(B_1B_2(u,v) = 0\), for any \(u,v \in J\) (1)

By using lemma 2.1 \(B_1, B_2\) are orthogonal, that is

\(B_1(u,m)B_2(v,w) + B_2(u,w)B_1(v,m) = 0\), for any \(m \in J\) (2)

Put \([s,pq]v\) instead of \(v\), for any \(p,q \in J, s \in R\) in the equation (2) and use (2), to get

\(B_1(u,m)B_2([s,pq]v,w) + B_2(u,w)B_1([s,pq]v,m) = 0\) (3)

Replace \(v\) by \(v\) for some \(t \in J\) in the equation (3), to obtain

\(B_1(u,m)[s,pq]B_2(t,w) + B_2(u,w)[s,pq]B_1(t,m) = 0\) (4)

Writing \(B_2(u,v)\) instead of \(t\) in the equation (4), to get the following

\(B_1(u,m)[s,pq]vB_2(t, w) + B_1(u,w)[s,pq]qB_1(t,m) = 0\) (5)

With the use of Res 3, since \(B_2(u,v) = 0\), either \(R\) is commutative or \(B_1(u,m) = 0\).

Assume that \(R\) is commutative, then replacing \(u\) by \(u^2\) in the equation (1), we get

\(2B_2(u,v)B_1(u,w) = 0\). (6)

Thus either \(B_2(u,v) = 0\) or \(B_1(u,w) = 0\). Therefore in all the cases \(B_2 = 0\).

**Lemma 2.3**

Any two biderivations \(B_1\) and \(B_2\) satisfies the condition \(B_1(B_2(u,v) - u) = 0\), for every pair \(u,v \in J\), then orthogonality of \(B_1\) and \(B_2\) are satisfied, also \(B_2 = 0\).

**Proof:** consider for every \(u,v \in J\), \(B_1(B_2(u,v) - u) = 0\). (7)

If \(R\) is commutative, substitute \(u\) by \(u^2\) in equation (7), we get

\(B_1(u,m)B_2(u,v) = 0\). (8)

Already a well known result, from the definition, a prime ring itself an integral domain, so the equation (8) reduces to

\(B_1(u,m) = 0\) or \(B_2(u,v) = 0\).

If \(B_2(u,v) \neq 0\) then \(B_1(u,m) = 0\) and let \(R\) is non commutative and \(B_2(u,v) = 0\). Put \(u\) by \(up\) in the equation (7), where \(p \in J\), find that

\(B_1(u,w)B_2(p,v) + B_2(u,v)B_1(p,w) = 0\). (9)

From lemma 2.1, \(B_1\) and \(B_2\) are orthogonal.

One can see that the equation (9) is same as compared with equation (2) so using the above lemma, we conclude

\(B_1(u,v) = 0\).

**Lemma 2.4**

Any two biderivations \(B_1\) and \(B_2\) satisfies the condition \(B_1(B_2(u,v) - u) = 0\), for every pair \(u,v \in J\), then either \(B_1 = 0\) or \(B_2 = 0\), and also \(R\) is commutative.

**Proof:** Consider for every \(u,v,w \in J\), \(B_1(B_2(u,v) - u) = 0\), we have

\(B_1(u,v)B_2(v,w) = [u,w]\). (10)

Replacing \(w\) by \(wm\), for any \(m \in J\), in the equation (10), to get

\(B_1(u,v)wB_2(v,m) = w[u,m]\). (11)
Replacing  by  for some  where  in the equation (11), to get,
\[ B_i (u, v)[s, t]wB_i (v, m) = [s, t]w[u, m] \]  \hspace{1cm} (12)
Left multiplication of (11) by  to get
\[ [s, t]B_i (u, v)wB_i (v, m) = [s, t]w[u, m]. \]  \hspace{1cm} (13)
From equation (12) and equation (13), we get
\[ B_i (u, v)[s, t]wB_i (v, m) = 0. \]  \hspace{1cm} (14)
So  is commuting, then Bresar (Bresar 1993), gives that  is commutative and equation (10) becomes
\[ B_i (u, v)B_i (v, w) = 0. \]  \hspace{1cm} (15)
Because of  leads to
\[ B_i (u, v) = 0. \]  \hspace{1cm} (16)
Using (10), we conclude that  for all  therefore  is commutative.

3. Main Theorems

Theorem 3.1
Any two biderivations  and  satisfies the condition
\[ B_i (u, v)B_i (v, w) = [u, w], \]  \hspace{1cm} (17)
for every  in  then  is commutative.

Proof: If  or  then the given condition becomes  for any  and  in  shows that  and  are nonzero biderivations, implies
\[ B_0 (u, v)B_0 (v, w) = [u, w]. \]  \hspace{1cm} (18)
Next consider  and  are nonzero biderivations, for every
\[ B_0 (u, v)[s, pq]B_1 (v, m) = 0, \]  \hspace{1cm} (19)
By replacing  with  in the equation (17), to get
\[ B_2 (u, v)[s, pq][B_1 (u, v), m] = 0; \]  \hspace{1cm} (20)
Using the primeness of  and  satisfies using lemma 2.1. Now from the Res 3 and using lemma 2.2, either  or  then  is commutative. Otherwise
\[ [s, pq][B_1 (u, v), m] = 0. \]  \hspace{1cm} (21)
In such case also is commutative.

Corollary 3.1
Any two biderivations  and  satisfies the condition
\[ B_i (u, v)B_i (v, w) = 0, \]  \hspace{1cm} (22)
for every  in  then  is commutative.

Proof: Given condition that \( B_i (u, v)B_i (v, w) = 0 \) for every \( u, v, w \in J \).

Replacing  with  in the equation (21), we get
\[ B_0 (u, v)B_0 (v, w) = B_0 (u, v)B_0 (v, w) = 0. \]  \hspace{1cm} (23)
We observed that the equation (22) and equation (17) are identical, continuing the procedure as done in the theorem 3.1, it is clear that  is commutative.

Theorem 3.2
For any three nonzero biderivations  and  of R satisfies one of the conditions
\( i \) \( B_i (v, w)B_i (w, u) = B_i (w, u)B_i (v, w) \),
\( ii \) \( B_i (v, w)B_i (w, u) = B_i (v, w)B_i (v, w) = [u, v] \),
for every  in  then  is commutative and  becomes
\[ B_0 (u, v)[s, pq]B_0 (v, m) = 0. \]  \hspace{1cm} (24)
Replacing  in place of  in the Eq. (23), to get
\[ B_0 (v, w)[v, w]m^2 + B_0 (v, w)u[w, m^2] = 0. \]  \hspace{1cm} (25)
Replacing  by  in the equation (24), we get
\[ B_0 (u, v)[s, pq]B_0 (v, m) = 0. \]  \hspace{1cm} (26)
Replacing  by  in the equation (25), to get
\[ B_0 (v, w)[v, w]u[w, m^2] = 0. \]  \hspace{1cm} (27)
it is clear that (27) and (20) are identical, hence we conclude that  is commutative then equation (23) becomes
\[B_2(v, w)R \left[ B_1(w, u) - B_2(w, u) \right] = 0 \]  \hspace{1cm} (28)

Since \( B_3(q, r) \) is non zero, \( B_3(w, u) = B_3(w, u) \).

(ii) Consider \( B_1 \) and \( B_2 \) are non zero biderivations such that
\[B_3(v, w)B_1(w, u) - B_2(w, u)B_3(v, w) = [u, v]. \]  \hspace{1cm} (29)

Replacing \( u \) with \( um^2 \) in the equation (29), we get
\[B_3(v, w)B_1(w, um^2) - B_2(r, um^2)B_3(v, w) = [um^2, v] \]
\[B_2(w, u)[B_1(v, w), m^2] + [B_2(v, w), u]B_1(w, m^2) = 0. \]  \hspace{1cm} (30)

It is clear that the equation (30) and the equation (24) are identical, proceeding the same procedure as in (i) it is clear that \( R \) is commutative, then condition (ii) becomes
\[B_3(v, w)R \left[ B_1(w, u) - B_2(w, u) \right] = 0 \]

Since \( B_3(v, w) \neq 0 \), leads to \( B_1 = B_2 \).  \hspace{1cm} (31)

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