FINITE MORPHISMS FROM CURVES OVER DEDEKIND RINGS TO $\mathbb{P}^1$.

T. CHINBURG*, G. PAPPAS†, AND M. J. TAYLOR‡

Abstract. A theorem of B. Green states that if $A$ is a Dedekind ring whose fraction field is a local or global field, every normal projective curve over $\text{Spec}(A)$ has a finite morphism to $\mathbb{P}^1_A$. We give a different proof of a variant of this result using intersection theory and work of Moret-Bailly.

1. Introduction

In this paper we will use intersection theory to prove a variant of a theorem first proved by B. Green. We will make the following hypotheses:

Hypothesis 1.1. Let $A$ be an excellent Dedekind ring such that:

i. The residue field of each maximal ideal of $A$ is an algebraic extension of a finite field.

ii. If $A'$ is the normalization of $A$ in a finite extension $K'$ of the fraction field $K$ of $A$, then $\text{Pic}(A')$ is a torsion group.

Theorem 1.2. Suppose that $\mathcal{Y}$ is a normal scheme over $\text{Spec}(A)$ whose structure morphism is flat and projective with fibers of dimension 1. Then there is a finite flat morphism $\pi : \mathcal{Y} \to \mathbb{P}^1_A$ over $\text{Spec}(A)$.

This theorem was proved by Green, see [3, Theorem 2], when $K$ is a local or global field. (See also [4].) The proof of Theorem 2 in [3] is written in the language of valuation theory, and follows from a more general result giving sufficient conditions for a family of valuations on the function field of $\mathcal{Y}$ to be principal.

Since Theorem 1.2 is a geometric result, it is natural to seek an entirely geometric proof. In this paper shall provide such a proof using intersection theory and the work of Moret-Bailly in [7]. To give a little more insight into the structure of the proof, we remark that first step is to show in [2] that the result follows from the existence of effective horizontal linearly equivalent ample divisors $D_1$ and $D_2$ on $\mathcal{Y}$ which do not intersect. Note that if there is
a finite morphism \( \mathcal{Y} \to \mathbb{P}^1_A \), then pulling back the divisors on \( \mathbb{P}^1_A \) associated to homogeneous coordinates \( x_0 \) and \( x_1 \) results in such \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \).

We then use results of Moret-Bailly \([7]\) to produce an element \( f \) of the function field \( K(\mathcal{Y}) \) for which \( \text{div}_\mathcal{Y}(f) = \mathcal{D}_1 - \mathcal{D}_2 \) for some \( \mathcal{D}_i \) of the above kind. Moret-Bailly’s method does not lead directly to \( \mathcal{D} \) having no vertical components. Instead we produce a finite set \( \{ f_i \} \) of functions for which the horizontal parts of \( \text{div}_\mathcal{Y}(f_i) \) are of the desired kind, and for which the vertical parts of the \( \text{div}_\mathcal{Y}(f_i) \) have the following property. The vertical parts as \( i \) varies generate a subgroup of finite index in the subgroup of the divisor class group of \( \mathcal{Y} \) generated by divisors contained in the reducible fibers of \( \mathcal{Y} \) over \( \text{Spec}(A) \). We then use the negative semi-definiteness of the intersection pairing in fibers to show that a constant times a product of positive integral powers of the \( f_i \) has a divisor \( \mathcal{D}_1 - \mathcal{D}_2 \) of the required kind.

In conclusion, we note that Theorem 1.2 and B. Green’s results in \([8]\) or \([4]\) are in fact slightly different. Each covers cases that the other does not. His results apply, for example, to the ring \( A = \mathbb{Z}[\mu_{p^n}]\lbrack 1/p \rbrack \) obtained by adjoining to \( \mathbb{Z} \) all \( p \)-power roots of unity in an algebraic closure of \( \mathbb{Q} \) and by then inverting the prime number \( p \).

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2. Horizontal divisors

Lemma 2.1. To prove Theorem 1.2, it will suffice to show that when \( \mathcal{Y} \) is connected, there are ample, effective linearly equivalent horizontal divisors \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) on \( \mathcal{Y} \) such that \( \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset \).

Proof. Since \( \mathcal{Y} \) is normal it is the disjoint union of its connected components, so we can reduce to the case in which \( \mathcal{Y} \) is connected. Given \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) as in the Lemma, we can replace each of these divisors by a high integral multiple of themselves to be able to assume that there is a projective embedding \( \mathcal{Y} \to \mathbb{P}^n_A \) and hyperplanes \( H_1 \) and \( H_2 \) in \( \mathbb{P}^n_A \) such that \( H_i \cap \mathcal{Y} = \mathcal{D}_i \) for \( i = 1, 2 \).

The fact that \( \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset \) implies that \( \mathcal{Y} \) is contained in the open set \( U = \mathbb{P}^n_A - (H_1 \cap H_2) \). Let \( h_i(x) \) be a linear form in the homogenous coordinates \( x = (x_0; \ldots; x_n) \) of \( \mathbb{P}^n_A \) such that \( H_i \) is the zero locus of \( h_i(x) \). There is a morphism \( f : U \to \mathbb{P}^1_A \) defined by \( x = (x_0; \ldots; x_n) \to (h_1(x); h_2(x)) \).

We first show that the restriction \( f_\mathcal{Y} : \mathcal{Y} \to \mathbb{P}^1_A \) of \( f \) to \( \mathcal{Y} \) is quasi-finite. Let \( c \) be a point of \( \text{Spec}(A) \), and let \( Z \) be the reduction of an irreducible component of the fiber \( \mathcal{Y}_c \) of \( \mathcal{Y} \) over \( c \). Since \( \mathcal{Y}_c \) has finitely many irreducible components, it will suffice to show that the restriction \( f_Z : Z \to \mathbb{P}^1_A \) of \( f \) is quasi-finite. If \( H_1 \) contains \( Z \), then \( Z - (Z \cap U) = Z \cap H_1 \cap H_2 = Z \cap H_2 \neq \emptyset \) since \( Z \) is projective and \( H_2 \) is a hyperplane in \( \mathbb{P}^n_A \). This is a contradiction.
and similarly $H_2$ cannot contain $Z$. Thus $h(x) = h_2(x)/h_1(x)$ defines a non-zero rational function on $Z$, and it will suffice to show that $h(x)$ is not in the field of constants $\ell(Z)$ of the function field $k(Z)$ of $Z$.

Suppose first that $c$ is the generic point of $\text{Spec}(A)$. Since $\mathcal{Y}$ is normal and connected, its generic fiber $Y$ is regular and irreducible. Thus $Z = Y$. If $h(x) \in \ell(Z)$, the ample divisors which $D_1$ and $D_2$ determine on $Y$ are equal, contradicting $D_1 \cap D_2 = \emptyset$.

Suppose now that $c$ is a closed point of $\text{Spec}(A)$. Then $\ell(Z)$ is a finite extension of the residue field $k(c)$ of $c$. By Hypothesis 1.1(i), $k(c)$ is an algebraic extension of a finite field, so $\ell(Z)$ is also such an extension. Therefore if $h(x) \in \ell(Z)$, the fact that $h(x)$ is non-zero implies that there is an integer $m > 0$ such that $h(x)^m = (h_2(x)/h_1(x))^m = 1$ in $\ell(Z)$. Hence the zero locus of $h_2(x)^m - h_1(x)^m$ contains a dense open subset of $Z$, and thus all of $Z$. Since $Z$ is projective over $A$, there is a point $z$ in $Z \cap H_2$. Then $h_2(z) = 0$ and $h_2(z)^m - h_1(z)^m = 0$, so $h_1(z)^m = 0$ and $h_1(z) = 0$. This implies $z \in Z \cap H_0 \cap H_1$, which contradicts $D_1 \cap D_2 = \emptyset$. Thus $f_Y$ is quasi-finite.

Since the projective morphism $\mathcal{Y} \to \text{Spec}(A)$ factors through $f_Y$, we see that $f_Y$ must be projective. By [4, Ex. III.11.2] a quasi-finite projective morphism is finite. By [6, Thm. 38, p. 124], $\mathcal{Y}$ is Cohen-Macaulay because it noetherian and normal of dimension two. By [6, Thm. 46, p. 140], a finite morphism from a Cohen-Macaulay scheme to a regular scheme is flat. Hence $f_Y : \mathcal{Y} \to \mathbb{P}^1_A$ is finite and flat, so Lemma 2.1 is proved.

\[ \square \]

3. Intersection Numbers and Ample Divisors

In this section we define some notation and we recall some well known results about intersection numbers and ample divisors. We will assume throughout that $\mathcal{Y}$ is connected.

**Definition 3.1.** Let $S_v = S_v(\mathcal{Y})$ be the set of irreducible components of the fiber $\mathcal{Y}_v = k(v) \otimes_A \mathcal{Y}$ of $\mathcal{Y}$ over $v \in \text{Spec}(A)$. Define $\mathcal{Y}_v^{\text{red}}$ to be the reduction of $\mathcal{Y}_v$. Let $Y = K \otimes_A \mathcal{Y}$ be the general fiber of $\mathcal{Y}$.

**Definition 3.2.** Suppose $E$ is a Cartier divisor on $\mathcal{Y}$ and that $C \in S_v$ for some maximal ideal $v$ of $A$. Let $C^\#$ be the normalization of $C$, and let $i : C^\# \to \mathcal{Y}$ be the composition of the natural morphism $C^\# \to C$ with the closed immersion $C \to \mathcal{Y}$. Define

\[ \langle E, C \rangle_v = \deg_{k(v)} i^*(O_Y(E)) \]

where $i^*(O_Y(E))$ is a line bundle on the regular curve $C^\#$ over the residue field $k(v)$ of $v$. This pairing may be extended by bilinearity to all Cartier divisors $E$ and to all Weil divisors $C$ in the free abelian group $W_v$ generated by $S_v$.

The value of $\langle E, C \rangle$ clearly depends only on the linear equivalence class of $E$. We will need the following result.
Lemma 3.3. (Moret-Bailly) A non-zero integral multiple of a Weil divisor on \( Y \) is a Cartier divisor. One may thus extend \( \langle E, C \rangle_v \) to all Weil divisors \( E \) and all \( C \in W_v \) by linearity in both arguments. Define \( QW_v = Q \otimes \mathbb{Z} W_v \) and let \( QY_v \) be the subspace spanned by the Weil divisor \( Y_v \). Then \( \langle \cdot, \cdot \rangle_v \) gives rise to a negative definite pairing
\[
\langle \cdot, \cdot \rangle_v : \frac{QW_v}{QY_v} \times \frac{QW_v}{QY_v} \to \mathbb{Q}.
\] (3.1)

Let \( T \) be a horizontal Cartier divisor on \( Y \), and let \( T \) be the general fiber of \( T \). Then
\[
\langle T, Y_v \rangle_v = \deg_K(T)
\] (3.2)
for all maximal ideals \( v \in \text{Spec}(A) \).

Proof. The first assertion is shown in [7, Lemme 3.3]. Since \( \langle E, C \rangle_v \) is bi-linear over Cartier divisors \( E \), it follows that we can extend this pairing to all Weil divisors \( E \). The proof of the second assertion concerning (3.1) is indicated immediately after [7, eq. (3.5.4)]. For further details, see [8, exp. 1, Prop. 2.6] and [2, §2.4, Appendices A.1 and A.2]. The last assertion is from [7, §3.5]. \( \square \)

Note that if \( Y \) is not regular, the extension of \( \langle \cdot, \cdot \rangle_v \) described in 3.3 is different in general from the proper intersection pairing considered by Artin in [1, §2].

Lemma 3.4. Suppose \( D \) is an ample Cartier divisor on \( Y \) and that \( E \) is an effective horizontal Cartier divisor. Then \( D + E \) is ample.

Proof. By [5, Prop. III.5.3], \( D + E \) is ample if and only if for each coherent sheaf \( F \) on \( Y \), there is an integer \( n_0(F) > 0 \) such that
\[
H^i(Y, F \otimes O_Y(D + E)^{\otimes n}) = 0
\]
for all \( n \geq n_0 \) and all \( i > 0 \). Consider the long exact cohomology sequence associated to the exact sequence of sheaves
\[
0 \to F \otimes O_Y(D)^{\otimes n} \to F \otimes O_Y(D + E)^{\otimes n} \to (F \otimes O_Y(D + E)^{\otimes n})|_{nE} \to 0.
\]
Because \( nE \) is horizontal, it is affine, and the higher cohomology of coherent sheaves on \( nE \) is trivial. It follows that \( D + E \) is ample because \( D \) is ample. \( \square \)

Note that if \( E \) is allowed to have vertical components, then \( D + E \) might have negative degree on some irreducible vertical component of \( Y \). Thus the conclusion of Lemma 3.4 need not hold for arbitrary effective Cartier divisors \( E \).

We will leave the proof of the following Lemma to the reader.

Lemma 3.5. Suppose \( T \) is a finite set of closed points of \( \mathbb{P}_A^m \) for some integer \( m \geq 1 \). Then there is an integer \( n \geq 1 \) and a homogenous polynomial \( f = f(x_0, \ldots, x_m) \) of degree \( n \) in homogenous coordinates \( (x_0; \ldots; x_m) \) for \( \mathbb{P}_A^m \) such that \( f \) does not vanish at any point of \( T \).
Lemma 3.6. There is an effective horizontal divisor $D$ on $Y$ which is very ample relative to the structure morphism $Y \to \text{Spec}(A)$. Each such $D$ intersects every irreducible component of a fiber of $Y$ over $\text{Spec}(A)$.

Proof. Since we have assumed $Y$ is projective, there is an effective very ample Cartier divisor $D$ on $Y$. Let $T$ be a finite set of closed points of $Y$ which contains a point on every irreducible component of every reducible fiber of $Y$ over $\text{Spec}(A)$. Lemma 3.5 implies that there is an effective very ample Cartier divisor which is linearly equivalent to $nD$ for some $n > 0$ and which contains no point of $T$; we replace $D$ by this divisor. Now $D$ can contain no irreducible component of a reducible vertical fiber of $Y$ over $\text{Spec}(A)$. Thus the vertical part of $D$ is an integral combination of fibers of $Y$. Since Pic$(A)$ is finite by assumption, we can now replace $D$ by $dD + \text{div}_Y(g)$ for some $g$ in the fraction field $K$ of $A$ and some $d > 0$ to be able to assume that $D$ is horizontal, effective and very ample. Since $D$ is effective and ample when restricted to every irreducible component of a fiber of $Y$, it must intersect each such component. □

4. An application of work of Moret-Bailly

Proposition 4.1. Suppose $Y$ is connected. Let $D$ be a divisor with the properties stated in Lemma 3.6. Let $M$ be the finite set of maximal ideals $v \in \text{Spec}(A)$ such that $Y_v$ has more than one irreducible component. For each $v \in M$, choose an element $C(v)$ of $S_v$. Suppose

$$Y_v = \sum_{C \in S_v} n_C C$$

as Weil divisors for some integers $n_C > 0$. There is a non-constant function $f$ in the function field $K(Y)$ having the following properties. The divisor of $f$ on $Y$ has the form

$$\text{div}_Y(f) = D_1(f) - nD + \sum_{v \in M} E_v$$

(4.1)

where $0 < n \in \mathbb{Z}$, $E_v$ is a Cartier divisor supported on $Y_v$ for $v \in M$ and the following is true:

i. $D_1(f)$ is an effective, horizontal Cartier divisor and is equal to the Zariski closure of its general fiber $D_1(f)$. The intersection $D_1(f) \cap D$ is empty.

ii. Suppose $v \in M$ and $C \in S_v$. Then

$$\langle nD, C \rangle_v - \langle D_1(f), C \rangle_v = \langle E_v, C \rangle_v \in \mathbb{Z}.$$  (4.2)

iii. Let $m$ be the degree of $f$ on the general fiber $Y$. If $v \in M$ and $C \in S_v$ then

$$0 < \langle nD, n_C C \rangle_v < m.$$  (4.3)

iv. If $v \in M$, the unique component of the special fiber $Y_v$ which $D_1(f)$ intersects is the $C(v)$ we have chosen. For $C \in S_v$ one has

$$\langle D_1(f), C \rangle_v = 0 \quad \text{if} \quad C \neq C(v) \quad \text{and} \quad \langle D_1(f), n_{C(v)} C(v) \rangle_v = m.$$  (4.4)
v. For \( v \in M \) one has
\[
\langle E_v, n_C C \rangle_v > 0 \quad \text{if} \quad C(v) \neq C \in S_v \quad \text{and} \quad \langle E_v, n_C C(v) \rangle_v < 0. \quad (4.5)
\]

**Proof.** We use the construction given in the proof of [7, Prop. 3.8]. To match the notation used in [7], let \( \mathcal{X} = \mathcal{Y} \). Define \( Z \) to be the union of \( \mathcal{D} \) with
\[
V = \cup \{ C : v \in M \quad \text{and} \quad C(v) \neq C \in S_v \}. \quad (4.6)
\]
In the proof of [7, Prop. 3.8], Moret-Bailly shows there is an integer \( n > 0 \) and rational linear combination \( \Delta \) of vertical divisors with the following properties:

i. The rational divisor \( n(\mathcal{D} + \Delta) \) is a Cartier divisor.

ii. Let \( \mathcal{L} = O_\mathcal{Y}(n(\mathcal{D} + \Delta)) \). There is a non-zero global section \( t \in H^0(\mathcal{Y}, \mathcal{L}) \) such that \( t \) generates the stalk of \( \mathcal{L} \) at all points of \( Z \).

On viewing \( \mathcal{L} \) as a subsheaf of the function field \( K(\mathcal{Y}) \) we may identify \( t \) with a function \( f \in K(\mathcal{Y}) \). Then
\[
\text{ord}_w(f) = -\text{ord}_w(n(\mathcal{D} + \Delta)) \quad (4.7)
\]
at all codimension 1 points \( w \) of \( \mathcal{Y} \) lying in \( Z = \mathcal{D} + V \). Since \( V \) and \( n\Delta \) are fibral, and \( Z \) contains \( \mathcal{D} \), we conclude that
\[
\text{div}_\mathcal{Y}(f) = \mathcal{D}_1(f) - n\mathcal{D} + \mathcal{T}
\]
where \( \mathcal{T} \) is a fibral divisor and \( \mathcal{D}_1(f) \) is an effective, horizontal divisor having no irreducible components in common with \( \mathcal{D} \). It follows that \( \mathcal{D}_1(f) \) is the Zariski closure of its general fiber \( \mathcal{D}_1(f) \).

Write \( \mathcal{T} \) as a finite integral combination of irreducible components of the \( \mathcal{Y}_v \) as \( v \) ranges over \( \text{Spec}(A) \). If \( v \not\in M \), then \( \mathcal{Y}_v^{\text{red}} \) is irreducible, and a non-zero integral multiple of \( \mathcal{Y}_v^{\text{red}} \) is the divisor of a non-zero element of \( K \) since \( \text{Pic}(A) \) is torsion by Hypothesis [11]. By Lemma [8.3] a non-zero integral multiple of a Weil divisor on \( \mathcal{Y} \) is a Cartier divisor. Therefore on replacing \( f \) by \( f^m\alpha \) for some \( 0 \neq \alpha \in K \) and some sufficiently divisible integer \( m > 0 \), we will have an equality of the form in [11], with \( n\mathcal{D} \) and \( n\Delta \) being Cartier divisors.

Since \( t \) generates the stalk of \( \mathcal{L} = O_\mathcal{Y}(n(\mathcal{D} + \Delta)) \) at each point of \( Z \), \( f^{-1} \) is a local equation for the Cartier divisor \( n(\mathcal{D} + \Delta) \) at each point of \( Z \). From [11], \( f^{-1} \) is also a local equation for \( -\mathcal{D}_1(f) + n\mathcal{D} + \sum_{v \in M} E_v \) at all points of \( \mathcal{Y} \). Recall that the \( E_v \) and \( n\Delta \) are vertical Cartier divisors, and \( \mathcal{D}_1(f) \) is a horizontal Cartier divisor with no irreducible components in common with \( \mathcal{D} \subset Z \). Therefore if \( c \in \mathcal{D}_1(f) \cap \mathcal{D} \subset Z \), a local equation for \( \mathcal{D}_1(f) \) in a neighborhood of \( c \) would have to be a local equation for \( \sum_{v \in M} E_v - n\Delta \). These Cartier divisors would then agree in an open neighborhood of \( c \) in \( \mathcal{Y} \), which is impossible since one is vertical and the other is horizontal. Therefore \( \mathcal{D}_1(f) \cap \mathcal{D} = \emptyset \), which completes the proof of (i).

Statement (ii) is clear from [11] and the fact that since \( \text{div}_\mathcal{Y}(f) \) is principal, \( \langle \text{div}_\mathcal{Y}(f), C \rangle_v = 0 \).
Concerning (iii), we know by (3.2) that
\[ \sum_{C \in S_v} \langle nD, nC \rangle_v = \langle nD, Y_v \rangle_v = \deg(nD) = m \] (4.8)
since \( nD \) is the polar part of the divisor \( \text{div}_Y(f) \) on the general fiber \( Y \) of \( \mathcal{Y} \). We have \( \langle D, C \rangle_v > 0 \) for all \( C \in S_v \) because \( D \) intersects each such \( C \) by Lemma 3.6. Therefore (4.3) follows from (4.8) and the fact that \( S_v \) has more than one element if \( v \in M \).

To show (iv), suppose \( v \in M \) and that \( C(v) \neq C \in S_v \). Since \( C \subset Z \) and \( C \) is fibral, (4.1) and (4.7) imply that the multiplicities of \( C \) in \( -n(D + \Delta) \), \( -n\Delta \), \( \text{div}_Y(f) \) and \( E_v \) must be equal. We conclude from (4.1) and (4.7) that if \( D_1(f) \) intersects a point \( c \in C \subset Z \), then \( t \) would not be a local generator of the stalk of \( \mathcal{L} \) at \( c \). This contradicts condition (ii) in the definition of \( t \) following equation (4.6). Thus \( D_1(f) \) can only intersect the component \( C(v) \) of \( Y_v \). Now (3.2) shows
\[ \langle D_1(f), nC(v)C(v) \rangle_v = \langle D_1(f), Y_v \rangle_v = \deg(D_1(f)) = m. \]
This shows (4.4) and completes the proof of (iv).

Finally, the inequalities in (4.5) of part (v) are a consequence of (4.2), (4.3) and (4.4). \( \square \)

5. Controlling vertical divisors

Lemma 5.1. Let \( D, M \) and \( S_v \) be as in Proposition 4.1. Suppose \( M' \subset M \) and that \( C_0(v) \in S_v \) for \( v \in M' \). Then there is a function \( h \in K(\mathcal{Y}) \) such that
\[ \text{div}_Y(h) = D_1 - D_2 + \sum_{v \in M'} E_v \] (5.1)
where \( D_1 \) and \( D_2 \) are horizontal effective divisors which do not intersect, \( D_2 \) has the same support as \( D \), \( E_v \) is supported on \( Y_v \), and for \( v \in M' \) and \( C' \in S_v \) we have
\[ \langle E_v, C' \rangle_v > 0 \quad \text{if} \quad C_0(v) \neq C \in S_v \quad \text{and} \quad \langle E_v, C_0(v) \rangle_v < 0. \] (5.2)

Proof. We use induction on the number of elements of \( M - M' \). If \( M = M' \), the Lemma is shown by Proposition 4.1. Suppose now that Lemma 5.1 holds when \( M' \) is replaced by \( M' \cup \{v_0\} \) for some \( v_0 \in M - M' \). For each \( C \in S_{v_0} \), we thus can find a function \( h_C \) with the following properties:

i. The divisor of \( h_C \) is
\[ \text{div}_Y(h_C) = D_{C,1} - D_{C,2} + \sum_{v \in M'} E_{C,v} + E_{C,v_0} \] (5.3)
where \( D_{C,1} \) and \( D_{C,2} \) are horizontal effective divisors which do not intersect, \( D_{C,2} \) has the same support as \( D \), and \( E_{C,v} \) is supported on \( Y_v \) for \( v \in M' \cup \{v_0\} \).

ii. For \( v \in M' \) and \( C' \in S_v \) we have
\[ \langle E_{C,v}, C' \rangle_v > 0 \quad \text{if} \quad C_0(v) \neq C' \quad \text{and} \quad \langle E_{C,v}, C_0(v) \rangle_v < 0. \] (5.4)
iii. For $C' \in S_{v_0}$ we have
\[ \langle E_{C,v_0}, C' \rangle_{v_0} > 0 \quad \text{if} \quad C \neq C' \quad \text{and} \quad \langle E_{C,v_0}, C \rangle_{v_0} < 0. \] (5.5)

We claim that there are positive integers $\{a_C\}_{C \in S_{v_0}}$ such that the divisor
\[ E_{v_0} = \sum_{C \in S_{v_0}} a_C E_{C,v_0} \]
has the property that
\[ \langle E_{v_0}, C' \rangle_{v_0} = 0 \quad \text{for all} \quad C' \in S_{v_0}. \] (5.6)

Before showing this, let us first show how it can be used to complete the proof of Lemma 5.1.

By Lemma 3.3, the intersection pairing $\langle , \rangle_{v_0}$ is negative semi-definite on the vector space spanned by $S_{v_0}$. Hence (5.6) implies that $E_{v_0}$ is a rational multiple of the fiber $\mathcal{Y}_{v_0}$. Since $\text{Pic}(A)$ is finite by Hypothesis ii), there is a positive integer $d$ such that $d \cdot E_{v_0}$ is the principal (vertical) divisor of a constant $a \in K^*$. We have
\[ \sum_{C \in S_{v_0}} a_C \cdot \text{div}_Y(h_C) = D_1 - D_2 + \sum_{v \in M'} E_v + E_{v_0} \] (5.7)

where
\[ D_i = \sum_{C \in S_{v_0}} a_C D_{C,i} \] (5.8)

and
\[ E_v = \sum_{C \in S_{v_0}} a_C E_{C,v} \] (5.9)

for $v \in M' \cup \{v_0\}$. The support of each $D_{C,2}$ equals that of $D$, and this does not intersect the support of any of the $D_{C,1}$ by our induction hypothesis. It follows that $D_1$ and $D_2$ are effective horizontal divisors which do not intersect, and $D_2$ and $D$ have the same support. Because of the induction hypotheses (5.4) and (5.5), the fact that all of the $a_C$ associated to $C \in S_{v_0}$ are positive integers implies that condition (5.4) holds if we replace $E_{C,v}$ in that condition by $E_v$. Now since
\[ \text{div}_Y(a) = d \cdot E_{v_0} \]
and $d > 0$ we conclude from (5.7) that the function
\[ h = a^{-1} \cdot \left( \prod_{C \in S_{v_0}} h_{C}^{a_C} \right)^d \] (5.10)

will have all the properties required to show the induction step for Lemma 5.1 and complete the proof.

It remains to produce positive integers $\{a_C\}_{C \in S_{v_0}}$ such that
\[ E_{v_0} = \sum_{C \in S_{v_0}} a_C E_{C,v_0} \]
has property (5.6), i.e. is perpendicular to every irreducible component $C'$ of $\mathcal{Y}_{v_0}$. It will suffice to show that we can do this using positive rational
numbers \( a_C \) since the intersection pairing is well defined for all rational linear combinations of fibral divisors.

Consider the square matrix \( W = (W_{C,C'})_{C,C' \in S_v} \) with integral entries

\[
W_{C,C'} = \langle E_{C,v_0}, n(C')C' \rangle
\]

where \( n(C') > 0 \) is the multiplicity of \( C' \) in the fiber \( Y_v \). The sum of all the entries in the row indexed by \( C \) is

\[
\sum_{C' \in S_v} \langle E_{C,v_0}, n(C')C' \rangle v_0 = \langle E_{C,v_0}, \sum_{C' \in S_v} n(C')C' \rangle v_0 = \langle E_{C,v_0}, Y_v \rangle v_0 = 0
\]

where the last equality is from Lemma 3.3. Condition (5.5) of the induction hypothesis now says that \( W \) satisfies the hypotheses of the following Lemma, and this Lemma completes the proof of Lemma 5.1.

\[
\text{Lemma 5.2. Suppose } W = (w_{i,j})_{1 \leq i,j \leq t} \text{ is a square matrix of rational numbers such that the diagonal (resp. off-diagonal) entries are negative (resp. positive) and that the sum of the entries in any row is 0. Then there is a positive rational linear combination of the rows which is the zero vector.}
\]

\[
\text{Proof. We prove this assertion by ascending induction on the size } t \text{ of } W. \text{ If } t = 1 \text{ then } W \text{ has to be the zero matrix since the sum of the entries in any row of } W \text{ is trivial. If } t = 2 \text{ then the rows of } W \text{ have the form } (-a, a) \text{ and } (b, -b) \text{ for some positive rationals } a \text{ and } b, \text{ so } b \text{ times the first row plus } a \text{ times the second is } (0, 0). \text{ We now suppose the statement is true for matrices of smaller size than } t \geq 3. \text{ Without loss of generality, we can multiply the } i\text{-th row of } W \text{ by } -1/w_{i,i} > 0 \text{ to be able to assume that the diagonal entries are all equal to } -1. \text{ Since every off diagonal entry is positive, every off diagonal entry has to be a rational number in the open interval } (0, 1) \text{ because the sum of the entries in each row is 0 and } t \geq 3. \text{ Thus when we add } w_{i,t} \text{ times the last row to the } i\text{-th row for } i = 1, \ldots, t-1, \text{ we arrive at a matrix } W' = (w'_{i,j})_{i,j=1}^t \text{ such that } w'_{i,t} = 0 \text{ for } i = 1, \ldots, t-1. \text{ It is elementary to check that the } (t-1) \times (t-1) \text{ matrix } W'' = (w'_{i,j})_{i,j=1}^{t-1} \text{ which results from dropping the last row and the last column of } W' \text{ satisfies our induction hypotheses. We now conclude by induction that there is a positive rational linear combination of the rows of } W'' \text{ which equals 0. The corresponding linear combination of the rows of } W' \text{ is then also 0. Since each of the first } t-1 \text{ rows of } W' \text{ is the sum of the corresponding row of } W \text{ with a positive multiple of the last row of } W, \text{ we arrive in this way at the a positive linear combination of the rows of } W \text{ which is the zero vector.}
\]

\[
\text{Completion of the proof of Theorem 1.2}
\]

Let \( M' \) be the empty set in Lemma 5.1. This Lemma now produces divisors \( D_1 \) and \( D_2 \) which are horizontal, effective, disjoint, linearly equivalent, and for which \( D_2 \) has the same support as the horizontal effective ample divisor \( D \). Lemma 3.4 shows there is an integer \( m > 0 \) such that \( mD_2 \) is
ample. So on replacing $D_i$ by $mD_i$ for $i = 1, 2$ we arrive at divisors of the kind needed in Lemma 2.1 which finishes the proof.

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