NULLIFICATION WRTHE AND CHIRALITY OF ALTERNATING LINKS
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ABSTRACT

In this paper, we show how to split the writhe of reduced projections of oriented alternating links into two parts, called nullification writhe, or \( w_x \), and remaining writhe, or \( w_y \), such that the sum of these quantities equals the writhe \( w \), and each quantity remains an invariant of isotopy. The chirality of oriented alternating links can be detected by a non-zero \( w_x \) or \( w_y \), which constitutes an improvement compared to the detection of chirality by a non-zero \( w \). An interesting corollary is that all oriented alternating links with an even number of components are chiral, a result that also follows from properties of the Conway polynomial.

Keywords: Link, chirality, writhe, nullification writhe, remaining writhe.

1. Introduction

Determining whether knots and links are equivalent or not to their mirror images, i.e., whether they are achiral or chiral, has been a longstanding question in knot theory. Tait pioneered this field in the nineteenth century by developing empirical methods to this end. Subsequently, more and more sophisticated methods capable of detecting the chirality of links (essentially numerical and polynomial invariants) were elaborated by numerous mathematicians. A critical review of these methods can be found in [2, 3].

In 1991, Sola introduced a numerical invariant for oriented alternating links, called nullification number. This invariant is, however, insensitive to chirality. We show here that by a modification of the nullification number, a new numerical invariant capable of detecting chirality can be derived. It is called nullification writhe.
2. Nullification Number

Let us begin by explaining Sola’s definition of the nullification number \[\text{nullification number}\] and then restate it by use of another description. We first have to recall some definitions from Sola’s paper.

**Fig. 1**

**Definition 1.** The nullification of an oriented crossing of a link projected on \(S^2\) is the process described in Fig. 1, where means or .

**Fig. 2**

**Definition 2.** A crossing of a link projected on \(S^2\) is called isthmus (or nugatory crossing) if it can be depicted by Fig. 2 (left), where means or .

If a link projection contains an isthmus (i.e., it is a non-reduced projection), then the link can necessarily be represented by a projection where the isthmus has been removed by a movement of torsion (Fig. 2, right). This process is called reduction.

**Fig. 3**

**Definition 3.** The Seifert circles of a link projection on \(S^2\) are the disjoint closed curves into which the projection is transformed if all crossings are simultaneously nullified (Fig. 3).
Let us now follow the process of nullification of a link projection on the example of knot $4_1$ (Fig. 4). We consider a minimal (or reduced) projection of an alternating oriented link $K$ (for alternating links, minimal projections correspond to reduced projections $\bar{2}, \bar{3}, \bar{4}$). A first crossing is nullified. Notice that, as proved by Sola, the resulting projection is still alternating. If the projection is not reduced (i.e., it contains one or more isthmi), these crossings are suppressed by the process shown in Fig. 2. Then, another crossing is nullified, and so forth, until the unknot is reached. We thus have a sequence of crossing nullification-reduction-crossing nullification-reduction-...

**Definition 4.** The nullification number $o(K)$ of $K$ is the number of nullification steps in this sequence (e.g., $o(4_1) = 2$).

Sola proves that $o(K)$ is well-defined for a minimal projection of an alternating oriented link, and that it is an invariant of any minimal projection of those links. He shows that

$$o(K) = n(K) - s(K) + 1,$$

where $n(K)$ is the number of crossings of the projection and $s(K)$ is the number of Seifert circles of the projection.

Let us now introduce another description of the nullification of a link projection. Oriented link projections are depicted by the corresponding oriented Seifert circles connected by arms called connections, located at the position of the crossings (see Fig. 5, where means or ). Notice that the arrows on both sides of a connection always point in the same direction and that a connection necessarily joins two distinct circles. We call these diagrams Seifert diagrams. Using this description, the definition of an isthmus can be restated as:

**Definition 5.** An isthmus in a Seifert diagram is a connection that, if it is removed, makes the diagram disconnected.

The suppression of an isthmus (Fig. 2) transposed in this representation is shown in Fig. 3, where both connected Seifert circles are linked by no other connection (direct or indirect) than the depicted connection. Notice that the left-hand circle is rotated upside down while it is incorporated into the right-hand circle.

Let us look at the nullification of a reduced projection of the knot $4_1$ in this representation (Fig. 5). Each step consists in the suppression of a connection. If the
Fig. 6

Fig. 7
connection is an isthmus, then the step is the one illustrated in Fig. 6 and is called a reduction. If the connection is not an isthmus, then it is simply removed and the step is called a crossing nullification. The nullification number is the number of nullification steps. It is clear from this that we may as well consider only the nullification steps, by forbidding the suppression of the isthmi. The process is then stopped when all remaining connections are isthmi (see Fig. 8). This process is called type II-nullification of a link projection. Among the $n(K)$ connections, there are two sets: the connections that will be removed, and those that will be kept (the isthmi). By definition, the number of connections that will be removed is the nullification number $o(K)$. Sola proved that “all the nullification sequences of a minimal alternating projection have the same length”, so $o(K)$ is well-defined: it is the same for any order of crossings chosen. We suggest here to denote the nullification number by $n_x(K)$, because it is the number of connections that have been extracted. The number of remaining connections is then denoted by $n_y(K)$. $n_y(K)$ is the smallest possible number of connections required to connect all Seifert circles. We have:

$$n_x(K) + n_y(K) = n(K).$$ (2.2)
3. Nullification Writhe and Remaining Writhe

From now on, we use the same type of Seifert diagrams, but keep track of the crossing signs. Positive and negative connections are represented by a solid bar and a dashed bar, respectively (Fig. 9).

**Definition 6.** The nullification writhe of a reduced projection of an oriented alternating link $K$, denoted $w_x(K)$, is the number of positive connections minus the number of negative connections that are removed during a type II-nullification of the link projection.

Let us call these numbers $n_{x+}(K)$ and $n_{x-}(K)$. We have by definition:

\[ n_x(K) = n_{x+}(K) + n_{x-}(K), \]  
\[ w_x(K) = n_{x+}(K) - n_{x-}(K). \]  

**Definition 7.** The remaining writhe of a reduced projection of an oriented alternating link $K$, denoted $w_y(K)$, is the number of positive connections minus the number of negative connections that remain at the end of a type II-nullification of the link projection.

Let us call these numbers $n_{y+}(K)$ and $n_{y-}(K)$. We have by definition:

\[ n_y(K) = n_{y+}(K) + n_{y-}(K), \]  
\[ w_y(K) = n_{y+}(K) - n_{y-}(K). \]

As its name suggests, $w_y(K)$ is the writhe of the non-reduced unknot projection obtained at the end of a type II-nullification of the link projection, as can
be understood by the correspondence between connections in Seifert diagrams and crossings in link projections. For the same reason, \( w_x(K) + w_y(K) \) is the total number of positive connections of the Seifert diagram of \( K \), minus the total number of negative connections of the Seifert diagram of \( K \), and is thus equal to the writhe of the link projection, \( w(K) \):

\[
w_x(K) + w_y(K) = w(K). \tag{3.5}
\]

We now have to prove that \( w_x(K) \) and \( w_y(K) \) are well-defined for a reduced alternating link projection, i.e., that they are independent of the choice of the set of connections removed during nullification of the link projection. Let us first examine Seifert diagrams. There are two types of Seifert circles:

**Definition 8.** An *external Seifert circle* is a Seifert circle that defines two regions of \( S^2 \) such that all other circles are contained in one of these regions.

**Definition 9.** An *internal Seifert circle* is a Seifert circle that defines two regions of \( S^2 \) such that the other circles are contained in both regions.

External circles only have connections on one side, whereas internal circles have connections on both sides. At the end of a nullification of the link projection, the smallest number of connections have to be kept such that all circles are connected (all remaining connections are then isthmi). The only problem that could occur, leading to a bad definition of \( w_x(K) \) or \( w_y(K) \), is that in the Seifert diagram of the link before nullification, one given circle would be connected to the rest of the diagram simultaneously by a positive and a negative connection. Then, depending on the connection we choose to remove, the value of \( w_x(K) \) and \( w_y(K) \) would be different. We thus have to prove that:

**Lemma 10.** *In the Seifert diagram corresponding to a reduced projection of an oriented alternating link, all connections located on one side of any Seifert circle have the same sign.*

**Proof.** Let us first consider the case of an external Seifert circle with connections on one side only. Two adjacent connections can never have opposite signs, because the corresponding link projection would have to be non-alternating (see Fig. [10]). The connections are thus all positive or all negative. This proves the lemma for external Seifert circles.

Let us now consider the case of an internal Seifert circle which has connections on both sides. Two adjacent connections on the same side of the Seifert circle can never have opposite signs, because the corresponding link projection would have to be non-alternating (see Fig. [10]). Two adjacent connections, one on one side and one on the other side of the Seifert circle, can never have the same sign, because the corresponding link projection would have to be non-alternating (see Fig. [1]). Thus, all connections on one side of an internal Seifert circles are positive, and
all connections on the other side are negative. This completes the proof of lemma 10. \qed

**Proposition 11.** \( w_x(K) \) and \( w_y(K) \) are independent of the choice of the set of connections removed during a type II-nullification of a reduced projection of an oriented alternating link \( K \).

**Proof.** Any allowed set of connections removed during a type II-nullification of a reduced projection of an oriented alternating link \( K \) will lead to a Seifert diagram that consists of Seifert circles connected together with the smallest possible number of connections. The total number of such remaining connections is well-defined for the projection, because it is equal to \( n_y(K) \). In order to prove that \( w_y(K) \) is well-defined, it remains to show that the number of positive and negative remaining connections does not depend on the choice of the set of removed connections.

Let us consider a given external Seifert circle, \( C_1 \), of a Seifert diagram before nullification. If \( C_1 \) is connected to one other Seifert circle only (external or internal), then this connection must be kept during nullification. If \( C_1 \) is connected to two other Seifert circles, \( C_0 \) and \( C_2 \) (external or internal), two cases arise. \( C_0 \) and \( C_2 \) can be connected through \( C_1 \) only. Then both connections (\( C_0-C_1 \) and \( C_1-C_2 \)) must be kept during nullification. Alternatively, \( C_0 \) and \( C_2 \) can be connected through \( C_1 \) and through another chain of connected circles, \( C_3, C_4, \ldots, C_m \) (each one may be external or internal). There is thus a “cycle” of connected circles (Fig. 12). During nullification, any one of the connections connecting those circles has to be removed. Because of lemma 10, all of the connections have the same sign, so the choice of the connection to be removed will not affect the number of positive and negative
remaining connections. If $C_1$ is connected to more than two other Seifert circles, any two of them are treated as above.

Let us now consider a given internal Seifert circle of a Seifert diagram before nullification. Exactly the same reasoning can be used, but this time both sides of the Seifert circle are considered separately.

We have thus proved that $w_y(K)$ is well-defined for the projection. Because of Eq. (3.5), it follows that $w_x(K)$ is well-defined too. This completes the proof of proposition 11.

We are now ready to prove the invariance of $w_x(K)$ and $w_y(K)$ for any reduced alternating projection of the link.

**Proposition 12.** Any two reduced projections of an oriented alternating link $K$ have the same $w_x(K)$ and the same $w_y(K)$.

**Proof.** As asserted by the Tait’s flyping conjecture [1], and as proved by Menasco and Thistlethwaite [8], any two reduced projections of an alternating link are related by a sequence of moves called “flypes”. It remains to prove that two oriented reduced alternating projections related by a flype have the same $w_x(K)$ and the same $w_y(K)$.

Two projections related by a flype can be represented as in Fig. [3], where $R$ and $S$ are tangles, i.e., parts of a link projection with four emerging arcs. The depicted crossing may be positive or negative. The sign of the crossing is unchanged by the flyping operation. Let us now represent this operation in the description of Seifert diagrams. Depending on the orientation, two situations occur, shown in Figs. [4] and [5]. The highlighted connection is represented as a positive connection, but it could also have been a negative connection, for the same reason as above.
Let us analyze the first situation (Fig. 14). The Seifert diagram on the left is composed of at least two Seifert circles, denoted \( A \) and \( B \), where \( A \) and \( B \) are connected by the highlighted connection, plus other possible connections located in tangle \( R \) and/or in tangle \( S \). The Seifert diagram on the right is the same, except that the highlighted connection is now between tangles \( R \) and \( S \), and that tangle \( R \) has been rotated upside down. Any circle that was connected to circle \( A \) in tangle \( R \) is now connected to circle \( B \) in tangle \( R \), but there still exists a one-to-one correspondence between connections of both Seifert diagrams. The diagrams are then nullified, selecting the same set of connections to remove in both diagrams (which is legitimate because of proposition \([1]\)). The aim is to prove that if the highlighted connection is kept (resp. removed) after nullification in the left-hand diagram, it has to be kept (resp. removed) in the right-hand diagram. Then \( w_x \) (the sum of the signs of the removed connections) will be the same for both projections, and so will \( w_y \). At the end of nullification, circles \( A \) and \( B \) will remain connected by only one connection (or chain of connections). This could be the highlighted connection, or it could be another connection (or chain of connections) located in tangle \( R \) or \( S \).

If circle \( A \) and circle \( B \) remain connected by the highlighted connection in the left-hand diagram, which means that all other connections between \( A \) and \( B \) have been removed, then circles \( A \) and \( B \) in the right-hand diagram also have to remain connected by the highlighted connection because there is no other connection between \( A \) and \( B \). \( w_x \) and \( w_y \) will thus be the same for both diagrams.

If the highlighted connection is removed during nullification of the left-hand diagram, which means that there exists a connection (direct or indirect) between \( A \) and \( B \) through tangle \( R \) or tangle \( S \), then the highlighted connection in the right-hand diagram also has to be removed because \( A \) and \( B \) are still connected through tangle \( R \) or \( S \). \( w_x \) and \( w_y \) will thus be the same for both diagrams.

Let us now analyze the second situation (Fig. 15). The Seifert diagram on the left is composed of at least three Seifert circles, denoted \( A \), \( B \), and \( C \), where \( A \) and \( B \) are connected by one and only one connection, which is highlighted, and all other circles and connections, if any, are located inside the tangles \( R \) and \( S \). Under the flype, the diagram is transformed as follows: the circle \( B \) is incorporated into the circle \( A \), and the circle \( C \) is split into two circles, \( C \) and \( C' \). The connection located between \( A \) and \( B \) in the left-hand diagram is transposed between \( C \) and \( C' \) in the
right-hand diagram. Again, the aim is to prove that if the highlighted connection is kept (resp. removed) after nullification in the left-hand diagram, it has to be kept (resp. removed) in the right-hand diagram.

In the left-hand diagram after nullification, if \( A \) and \( B \) are connected by the highlighted connection, then \( C \) is connected to \( B \) in tangle \( R \) (directly or not) or it is connected to \( A \) in tangle \( S \) (directly or not). It is not connected to both \( B \) and \( A \), because then the diagram would not have been completely nullified. In the right-hand diagram, as the same set of connections have been removed during nullification, \( C \) is connected to \( A \) in tangle \( R \) (directly or not), or \( C' \) is connected to \( A \) in tangle \( S \) (directly or not). \( C \) has to be connected to \( C' \) by the highlighted connection in order for \( C' \) to be connected to the rest of the diagram. If \( C' \) is connected to \( A \) in tangle \( S \), then \( C \) has to be connected to \( C' \) by the highlighted connection in order for \( C \) to be connected to the rest of the diagram. So the highlighted connection has to be kept, and \( w_x \) and \( w_y \) will be the same for both diagrams.

This completes the proof of proposition 12. \( \square \)

4. Chirality of Alternating Links

A simple way to detect the chirality of an oriented alternating link \( K \) is to look at the writhe \( w \) of a reduced projection of \( K \). The writhe satisfies the following implication: \( w(K) \neq 0 \implies K \) is chiral. A proof of this implication for knots can be found in \( \text{[2]} \), p. 220, and is readily extendable to links. Unfortunately, the converse of this implication is not true. That is, there exist many chiral links whose writhe is equal to zero.

The same problem of zero-writhe chiral links has been encountered in a previous study aimed at partitioning chiral knots and links into \( D \) and \( L \) classes, and was overcome by defining a property called writhe profile \( \text{[9, 10]} \). This property is of no help here, because it is a “chirality-classifier” and not a “chirality-detector”. The chirality of the link is a prerequisite in order to apply the method. In contrast, the new invariants introduced in this paper, \( w_x \) and \( w_y \), are capable of detecting chirality, as is proved hereafter.

**Proposition 13.** If \( K \) is an oriented achiral alternating link represented by a reduced projection, then \( w_x(K) = 0 \) and \( w_y(K) = 0 \).
Proof. Let $K$ be an oriented achiral alternating link, represented by a reduced projection, denoted $P(K)$. $P(K^*)$ is the projection with all crossings reversed. $K^*$ is the mirror image of $K$. The corresponding Seifert diagrams, denoted $S(K)$ and $S(K^*)$, are identical except that all positive connections in $S(K)$ are replaced with negative connections in $S(K^*)$, and vice versa. $K^*$ is isotopic to $K$ because $K$ is achiral.

Now, we nullify both Seifert diagrams, selecting the same set of connections to remove (which is legitimate because of proposition 11). $w_x(K)$ is the sum of the removed connections of $S(K)$, $w_x(K^*)$ is the sum of the removed connections of $S(K^*)$, so we have

$$w_x(K) = -w_x(K^*). \tag{4.1}$$

But we know that $K^*$ is isotopic to $K$ and, since $w_x$ is an invariant of isotopy (proposition 13), it implies that

$$w_x(K) = w_x(K^*). \tag{4.2}$$

Putting together Eqs. (4.1) and (4.2), we get $w_x(K) = 0$. And because the link is achiral ($w(K) = 0$), we also have by eq. (3.5), $w_y(K) = 0$. □

The proposition can be summarized by the implication: $w_x(K) \neq 0$ or $w_y(K) \neq 0 \implies K$ is chiral. What are the consequences of this? For each oriented alternating link represented by a reduced projection, the writhe $w$ is split into two parts, $w_x$ and $w_y$. If $w$ is different from zero, $w_x$ and/or $w_y$ are different from zero and the link is chiral. In this case, $w_x$ and $w_y$ are thus no better than $w$ in detecting chirality. In contradistinction, if $w$ equals zero but the link is chiral, in some cases, $w_x$ and $w_y$ could be different from zero ($w_y = -w_x$), which would constitute a way to detect the chirality of the link even if it is not detectable by $w$.

We applied this procedure to all oriented chiral alternating “classical” knots and links, i.e., prime knots with up to ten crossings and prime links with up to nine crossings and four components, that have a writhe of zero. The results are shown in Tables 1 and 2 and confirm that chirality is detected in several cases where it is not detectable by the writhe.

Let us end up with a very interesting corollary of proposition 13.

Lemma 14. At the end of a type II-nullification of a reduced projection of an oriented alternating link, the remaining link has one component.

Proof. This is a direct consequence of the definition of a type II-nullification, given in section 2. All crossings of the remaining link at the end of a type II-nullification are isthmi, i.e. nuguatory crossings. The link is thus an unknot and has one component. □
Table 1. Oriented chiral alternating prime knots with up to ten crossings, and $w = 0$ ($w_y = -w_x$).

| knot $^{1,2}$ | 84 | 10_{13} | 10_{19} | 10_{31} | 10_{42} | 10_{48} | 10_{52} | 10_{54} | 10_{71} | 10_{91} | 10_{93} | 10_{104} | 10_{107} | 10_{108} |
|---------------|----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $w_x$         | −2 | +2      | −2      | 0       | 0       | +2      | +2      | 0       | 0       | −2      | 0       | 0       | 0       | +2      |

$^1$The enantiomorphs are those represented in [11], oriented either way.
$^2$This list should also contain the noninvertible oriented knots whose non-oriented version is achiral. Because of noninvertibility, these knots are chiral when oriented. They have been omitted since the nullification writhe is insensitive to noninvertibility.

Table 2. Oriented chiral alternating prime links with up to nine crossings and four components, and $w = 0$ ($w_y = -w_x$).

| link $^3$ | 8_{10}^2++ | 8_{14}^2++ | 8_{14}^2++ | 8_{14}++ | 8_{14}++ | 8_{14}++ | 8_{14}++ | 8_{14}++ | 8_{14}++ |
|----------|------------|------------|------------|----------|----------|----------|----------|----------|----------|
| $w_x$    | −1         | +1         | −1         | +1       | −1       | −2       | 0        | −1       | −1       |

$^3$The enantiomorphs are those represented in [12], using the same convention of labeling and orientation of components.
Lemma 15. During the process of type II-nullification of an oriented alternating link projection, each step changes the parity of the number of components of the link.

Proof. This lemma is easily understood by looking at Fig. 16, where means or . If the two arrows of a crossing are part of the same component, nullification of the crossing increases by one the number of components. If the two arrows belong to two different components, nullification of the crossing decreases by one the number of components. □

We may now express our last proposition:

Corollary 16. All oriented alternating links with an even number of components are chiral.

Proof. Let us consider an oriented alternating link \( K \) with an even number of components. Using lemmas 14 and 15, we deduce that the type II-nullification process of a reduced projection of \( K \) contains an odd number of steps. Since each step contributes \( \pm 1 \) to \( w_x, w_e \) can never be equal to zero, which implies that \( K \) is chiral. □

John Conway points out that this also follows from properties of his \( \nabla \) polynomial [13], which satisfies

\[
\nabla(K^*) = (-1)^{c+1}\nabla(K), \tag{4.3}
\]

where \( K^* \) is the mirror image of \( K \) and \( c \) is the number of components of \( K \). This implies that an oriented link with an even number of components can only be achiral if its \( \nabla \) polynomial vanishes identically. This includes our corollary, since it is known that all oriented alternating links have non-zero Alexander polynomials [14] and thus non-zero \( \nabla \) polynomials (\( \nabla \) polynomials are normalized Alexander polynomials).

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crossing nullification

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$c$ components $\rightarrow$ $c+1$ components

c components $\rightarrow$ $c-1$ components