Farey determinants matrix

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Abstract

A new matrix operation based on inserting columns and rows, similarly to the mediant operation between fractions, gives rise to the Farey determinants matrix or, equivalently, the matrix of the numerators of the differences of Farey fractions. This matrix allows to visualize established properties and theorems of Farey fractions allowing for more intuitive demonstrations and easier understanding. Furthermore it is shown how some Farey determinants matrices contain other lower order Farey determinants matrices as block matrices around the main diagonal.

1 The Farey determinants matrix

The Farey sequence $F_N$ of order $N$ is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed $N$ [1]. Let $h_i/k_i < h_{i+1}/k_{i+1}$ be two Farey neighbors then, $h_{i+1}k_i - h_i k_{i+1} = 1$. The next Farey fraction to appear between two Farey neighbors is given by the mediant as

$$\frac{h_i}{k_i} < \frac{h_i + h_j}{k_i + k_j} < \frac{h_j}{k_j}.$$ 

We define the determinant of any two Farey fractions $h_i/k_i$ and $h_j/k_j$, both in $F_N$, as

$$d_{ij}(N) = \begin{vmatrix} h_j & h_i \\ k_j & k_i \end{vmatrix} = h_j k_i - h_i k_j .$$

$d_{ij}(N)$ is also the numerator of the difference $h_j/k_j - h_i/k_i$, so that $d_{ii} = 0$ and $d_{ij(\pm 1)} = \pm 1$. Note that $(N)$ is omitted from $d(N)$ when it is not necessary.

In the following a matrix operation is introduced resembling the mediant that allows to iteratively compute $d(N)$ for increasing $N$. Starting from $F_1 = \{0/1, 1/1\}$, the skew-symmetric unitary matrix $d(1)$ is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we insert one row and one column in the middle with values given by the sum of the neighboring horizontal or vertical entries. This is illustrated as follows,
where \(d(2)\) and \(d(3)\) have been generated as the reader can verify. Between \(d(N)\) and \(d(N+1)\) new rows and columns should be inserted at the same positions as the new fractions appearing between \(F_N\) and \(F_{N+1}\). By construction the top row consists of the numerators of Farey fractions with opposite sign. The bottom row consists of numerators of Farey fractions in reverse order. Similarly happens for the first and last columns. The denominators corresponding to the numerators in the bottom row can be obtained by subtracting the top row to the bottom row. This is illustrated with \(d(4)\) together with the Farey fractions corresponding to the first and last elements in rows and columns,

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
0 & -1 & -1 & 0 & 1 & 1 \\
1 & -1 & -1 & 0 & 1 & 1 \\
0 & -1 & -2 & -1 & 0 & 1 \\
1 & 0 & -3 & -2 & 0 & 1 \\
1 & 0 & -1 & -1 & 0 & 1 \\
2 & 3 & 1 & 0 & -1 & 1 \\
1 & 2 & 1 & 1 & 0 & 1 \\
\end{array}
\]

However it is not yet demonstrated that this process actually generates \(d(N)\). Let \(\delta\) be a \(n \times n\) matrix built following the above procedure. The element \(\delta_{ij}\) is a linear combination of the corresponding top and bottom elements \(\delta_{i0}\) and \(\delta_{in}\). This linear combination is the same for all the elements in row \(j\), so we can use the elements \(\delta_{0j}\) and \(\delta_{nj}\) to reconstruct the linear combination as follows:

\[
\delta_{ij} = \frac{\delta_{i0} \delta_{nj}}{\delta_{n0}} + \frac{\delta_{in} \delta_{0j}}{\delta_{0n}} = -\delta_{i0} \delta_{nj} + \delta_{in} \delta_{0j} ,
\]

where we have used \(\delta_{n0} = -1\) and \(\delta_{0n} = 1\), which are true by construction. The first row and column correspond to the numerators of Farey fractions as \(\delta_{i0} = -h_i\) and \(\delta_{0j} = h_j\), respectively. The last row and column correspond to the numerators of Farey fractions in reverse order as \(\delta_{in} = h_{n-i+1} = k_i - h_i\).
and \( \delta_{nj} = -h_{n-j+1} = -(k_j - h_j) \), respectively. Therefore,

\[
\delta_{ij} = -h_i(k_j - h_j) + (k_i - h_i)h_j = -h_ik_j + k_ih_j = \begin{vmatrix} h_j & h_i \\ k_j & k_i \end{vmatrix},
\]

and \( \delta_{ij} = d_{ij} \). By construction \( \delta \), or \( d \), is a square, skew-symmetric matrix with rank equal 2 as the new rows are a linear combination of the neighboring rows for \( N > 2 \). \( d \) is not only skew-symmetric but also symmetric around the secondary diagonal and therefore one could keep only one quarter of the matrix still being able to generate higher order \( d \)'s. Starting from \( d(3) \),

\[
\begin{array}{ccc}
0 & -1 & -1 \\
1 & 0 & -1 \\
1 & 1 & 0
\end{array}
\]

we can proceed as before but only taking the quarter of the matrix highlighted with the blue triangle,

\[
\begin{array}{c}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 3 \\
0 & 1 & 1 & 2 & 1
\end{array}
\]

Note that to obtain the 3 we recall the symmetries of the original matrix and we add the 2 and the 1 below and left of the 3, respectively.

In Section 2 \( d \) is related to the index of Farey fractions as defined in [2, 3]. Section 3 illustrates an equality among the greatest common divisors between elements in \( d \) as presented in [4]. Section 4 shows how some \( d(N) \) contain other smaller \( d(i) \) thanks to maps in [5] that preserve the determinant of two Farey fractions.

2 \( d_{ij} \) and \( k \)-indexes

The index \( \nu(x_i) \) and the \( k \)-index \( \nu_k(x_i) \) of the \( i^{th} \) Farey fraction in \( F_N \) are introduced in [2] and [3], respectively. We can relate them to the determinant matrix as

\[
\nu(x_i) = d_{i(i+1)}(i-1), \text{ for } 2 \leq i \leq |F_N| - 1,
\]

\[
\nu_k(x_i) = d_{i(i+k-1)}(i-1), \text{ for } 2 \leq i \leq |F_N| - k + 1.
\]

Being \( \nu_k(x_i) \) a generalized definition of the index: \( \nu_2(x_i) = \nu(x_i) \). The highlighted diagonals in the following \( d(3) \) matrix contain part of the \( k \)-indexes of fractions in \( F_3 \).
The sum of the numbers in the red diagonal for $d(N)$ is easily obtained from Theorem 1 in [2] as

$$\sum_{i=2}^{\lvert F_N \rvert - 1} d_{(i-1)(i+1)} = 3(\lvert F_N \rvert - 1) - 2N - 1 .$$

Any two adjacent rows (or columns) of $d$ correspond to ordered lists of numerators and denominators of Farey neighbours, illustrated as follows.

\[
\begin{array}{cccccc}
0 & -1 & -1 & -2 & -1 \\
1 & 0 & -1 & -3 & -2 \\
1 & 1 & 0 & -1 & -1 \\
2 & 3 & 1 & 0 & -1 \\
1 & 2 & 1 & 1 & 0
\end{array}
\rightarrow \begin{cases}
1 & 0 & -1 & -3 & -2 \\
1 & 1 & 0 & -1 & -1 \end{cases}
\]

This is easy to demonstrate by realizing that the first and last elements of the rows fulfill

$$d_0 d_{n(i+1)} - d_0 d_{n(i)} = 1 ,$$

as $d_0 = h_i$ and $d_{n(i)} = -(k_i - h_i)$. Hence $d_0 / d_0 d_{n(i+1)}$ and $d_{n(i)} / d_{n(i+1)}$ are Farey neighbors. Therefore, the numbers within the rows constitute Farey fractions as they are computed as mediants. Adjacent fractions are also Farey neighbors. This is equivalent to Lemma 1 in [3] expressed here as

$$\left| \begin{array}{cc}
d_{ij} & d_{(i+1)j} \\
d_{i(j+1)} & d_{(i+1)(j+1)}
\end{array} \right| = 1 , \text{ for } i \leq \lvert F_N \rvert - 1 , \ j \leq \lvert F_N \rvert - 1 .$$

### 3 Great common divisors among $d_{ij}$

According to Theorem 1 in [4] the following equality holds between the great common divisors of elements $d_{pq}$ with $p$ and $q$ in $\{i > j > k\}$,

$$\gcd (d_{k,j}, d_{k,i}) = \gcd (d_{k,j}, d_{j,i}) = \gcd (d_{k,i}, d_{j,i}) .$$

This property is illustrated using $d(6)$ as follows,
where the number in the blue circle together with another 2 numbers connected by any blue line define a triplet of numbers for which the gcd’s computed for all possible combinations within the triplet are equal, e.g.,

\[ \gcd(14, 2) = \gcd(2, 8) = \gcd(14, 8) = 2. \]

Let the blue circed number be on the antidiagonal \( k = n - j + 1 \), assuming \( d \) is a \( n \times n \) matrix. This is illustrated for \( d(5) \) as follows,

The symmetry of \( d \) around the antidiagonal \( d_{(n-j)+i} = d_{j(n-i+1)} \) implies

\[ \gcd(d_{(n-j)+i}, d_{(n-j)i}) = \gcd(d_{(n-j)i}, d_{(n-j)(n-i+1)}) = \gcd(d_{(n-j)i}, d_{(n-j)(n-i)}) , \]

so the previous property can be seen as applied to a single column (or row), as shown in the following illustration.
Let \( F_1^{1/a, 1/b} \) be the subsequence of \( F_N \) defined as all the fractions of \( F_N \) in \([1/a, 1/b]\) with \( 1 \leq b \leq a \leq N \).

In [5] it is demonstrated that the map

\[
F_i \rightarrow F_N^{1/q, 1/(q-1)}, \quad \frac{h}{k} \mapsto \frac{kq - h}{k - h'}, \quad (1)
\]

is bijective between \( F_i \) and \( F_N^{1/q, 1/(q-1)} \) when \( N \) is a multiple of \( i(i+1) \) and \( N/(i+1) < q \leq N/i \). It is straightforward to show that this map preserves the determinant, meaning that

\[
\begin{vmatrix} h & h' \\ k & k' \end{vmatrix} = \begin{vmatrix} k & k' \\ kq - h & k'q - h' \end{vmatrix},
\]

for \( h/k \) and \( h'/k' \) belonging to \( F_i \). Therefore \( d(N) \) contains \( d(i) \) as a matrix block. In other words, \( d(i) \) is contained \( p \) times in \( d(i(i+1)p) \). As an illustration a portion of \( d(30) \) is shown containing \( d(5) \).
the left columns show the corresponding Farey fraction in $F_5$ and $F_{30}$ according to the map in Eq. (1) with $q = 6$. The blue triangle highlights $d(5)$.

References

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