Massive stealth scalar fields from deformation method II: charged case

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Abstract

Extending the results of our previous work we construct an uniparametric class of action principles for complex scalar fields with the property that their energy momentum tensor and the electric current vanish for certain massive configurations. These stealth fields do not curve the spacetime and they do not source electromagnetic fields in spite their non-trivial degrees of freedom and $U(1)$ gauge invariance. We shall also show that the presence of these stealth fields can affect the strength of the gravity-matter and radiation-matter coupling of other massive configurations. Indeed, the energy momentum tensor of other massive (non-stealth) configurations and the electric current can be rescaled (with a stealth-mass depending factor) with respect to the predictions of the standard complex scalar field model. Hence we argue that stealth fields could be detected indirectly by means of their effects on standard configurations of matter.

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1 Introduction

It has been shown by many authors [1–14] that scalar matter fields may not always curve the spacetime, in spite of their non-trivial degrees of freedom. Similar results were obtained also in the context of massive gauge theories in three dimensions [15]. This type of fields are now dubbed stealth fields. These curious objects deserve some attention since, besides of being counter examples of the idea that matter always curues the space, they could also play a role in cosmology.

The existence of stealth field configurations suggest that the spacetime curvature may not reflect the presence of matter as expected from standard field theories in curved space. Indeed, it is well known that the gravitational Hilbert energy momentum tensor \( \text{i.e.} \) the variation of their action principle with respect to the spacetime metric) of the visible matter in galaxies fails to explain the observed spacetime curvature, which have lead to cosmologists to propose the existence of dark matter, which have not yet been discovered. Alternatively, one could argue that perhaps, there may be alternative field theories in which the curvature induced by the visible matter is modified with respect to standard field theories.

The goal of this paper is to construct a generic class of complex-scalar field theories containing stealth configurations. Our models suggest that the presence of these stealth configurations can also affect the strength of the energy-momentum tensor of other (non-stealth) matter fields, modifying their effects on their gravity backgrounds with respect to standard field theories. This is illustrated in the case of a massive and charged Klein-Gordon field.

In this work we extend the results of reference [16], for real scalar fields. This is, we consider a generic action principle for a complex scalar field coupled to gravity and electromagnetism, referred as to original action, and according to a determined procedure of “deformation”, a new action principle is produced and dubbed deformed action, which is an uni-parametric extension of the original action. The latter contains stealth field configurations which satisfy the minimally coupled Klein-Gordon equation in curved space. The stealth fields occur as result of the deformation procedure, independently of the details of the original action. The respective equations of motion, which are in general of higher
order in derivatives, contain stealth and non-stealth solutions. In some cases, the solutions of the undeformed theory are preserved in the deformed model, so that only stealth solutions are added to the spectrum of the theory. For the stealth configurations both, the energy momentum tensor and the electric current vector vanish. Surprisingly enough, the Hilbert energy momentum tensor of the non-stealth configurations in the deformed theory consists of the original energy momentum tensor times a constant factor depending on the deformation parameter.

For example, for the minimally coupled Klein-Gordon complex field, of mass $M$, our procedure yields a 6th order in derivatives action principle, which contains as solution the stealth configurations of mass $\theta^{-1}$ (being $\theta$ the deformation parameter) and it preserves the solution of the original model of mass $M$. When the Hilbert energy momentum tensor is calculated, we observe that it vanishes for the stealth configuration, and the same happens for the electric current (so that the magnetic field equations are the same than in the vacuum). However for the mass $M$ configuration the energy momentum tensor and the electric current are rescaled with respect to the original Klein-Gordon action by a factor $1 - (M\theta)^2$. Hence stealth field are invisible to gravity and electromagnetism but they affect the non-stealth configurations with respect to their original theories. This is, the effects of massive scalar fields on their gravity backgrounds can be fine-tuned using the $\theta$ parameter.

This paper is organized as follows. In section 2 we introduce our notation and define what a stealth field is. In section 3 we define the deformation of the action principle and obtain the respective equations of motion, as deformations of the original equations of motion. We prove that the deformed theories contain massive fields with mass inversely proportional to the deformation parameter. In section 4 we construct some examples and characterize their solutions, and in section 5 we present some conclusions. Though this paper is self-contained, it is meant to be considered the second part of reference [16], where the reader can find more details and references.

## 2 Stealth fields

In this section we shall follow closely reference [16]. This is, in $D$-dimensional spacetime dimensions, with background metric $g_{\mu \nu}$ with signature $(-, +, +, ..., +)$, consider the action principle,

$$S[g, A, \phi, \bar{\phi}] = S_G[g] + S_A[g, A] + S_M[g, A, \phi, \bar{\phi}], \quad (2.1)$$

where $S_G[g]$, $S_A[g, A]$ and $S_M[g, A, \phi, \bar{\phi}]$ represent respectively the gravitational sector, the $U(1)$ gauge field sector, and the complex scalar matter field sector. Here $\bar{\phi} = \phi^*$ is the complex conjugated of $\phi$. We do not specify the details of each sector.

Under gauge transformation, with parameter $\alpha(x)$, the fields must transform as,

$$g = \exp(i\alpha(x)) \in U(1): \quad \phi \rightarrow g\phi, \quad \bar{\phi} \rightarrow \bar{\phi}g^*, \quad A_\mu \rightarrow A_\mu + \partial_\mu g. \quad (2.2)$$

The variation of this action with respect to the (inverse) metric tensor $g^{\mu \nu}$ yields,

$$\frac{\delta S[g, A, \phi, \bar{\phi}]}{\delta g^{\mu \nu}} = \sqrt{-g} \left( H_{\mu \nu}[g] - \Theta_{\mu \nu}[g, A] - \Xi_{\mu \nu}[g, A, \phi, \bar{\phi}] \right), \quad (2.3)$$

where

$$H_{\mu \nu}[g] := \frac{1}{\sqrt{-g}} \frac{\delta S_G[g]}{\delta g^{\mu \nu}}, \quad (2.4)$$

$$\Theta_{\mu \nu}[g, A] := - \frac{1}{\sqrt{-g}} \frac{\delta S_A[g, A]}{\delta g^{\mu \nu}}, \quad (2.5)$$

$$\Xi_{\mu \nu}[g, A, \phi, \bar{\phi}] := - \frac{1}{\sqrt{-g}} \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta g^{\mu \nu}}. \quad (2.6)$$
are respectively the (generalized) Einstein tensor, and the $U(1)$ gauge field and the scalar field Hilbert energy-momentum tensors.

From the variation of the action with respect to the scalar fields $\phi$ and $\bar{\phi}$ we define,

$$\Upsilon[g, A, \phi, \bar{\phi}] := \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta \phi},$$

(2.7)

$$\bar{\Upsilon}[g, A, \phi, \bar{\phi}] := \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta \bar{\phi}},$$

(2.8)

which are functionals of the fields $\phi$ and $\bar{\phi}$ containing differential operators. Note that, since $\phi$ ($\bar{\phi}$) transforms under $U(1)$ from the left (right) the operator (2.7) (operator (2.8)) transforms from the opposite right (left) side:

$$g = \exp(ia(x)) \in U(1): \quad \Upsilon[g, A, \phi, \bar{\phi}] \to g\Upsilon[g, A, \phi, \bar{\phi}], \quad \Upsilon[g, A, \phi, \bar{\phi}] \to \Upsilon[g, A, \phi, \bar{\phi}]g^*.$$  

(2.9)

The variation of the action with respect to the gauge field $A_\mu$ yields,

$$\frac{\delta S[g, A, \phi, \bar{\phi}]}{\delta A_\mu} = \sqrt{-g}(\bar{\Theta}[A, \phi, \bar{\phi}]) = \sqrt{-g}\left(-E^\mu[g, A] + J^\mu[g, A, \phi, \bar{\phi}]\right),$$

(2.10)

where we have defined the term proportional to the variation of the pure gauge sector as

$$E^\mu[g, A] := -\frac{1}{\sqrt{-g}} \frac{\delta S_A[g, A]}{\delta A_\mu},$$

(2.11)

and the $U(1)$ gauge current as

$$J^\mu[g, A, \phi, \bar{\phi}] := \frac{1}{\sqrt{-g}} \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta A_\mu}.$$  

(2.12)

Using these definitions, the field equations of the theory (2.1) read,

$$H_{\mu\nu}[g] - \Theta_{\mu\nu}[g, A] - \Xi_{\mu\nu}[g, A, \phi, \bar{\phi}] = 0,$$

(2.13)

$$\Upsilon[g, A, \phi, \bar{\phi}] = 0, \quad \bar{\Upsilon}[g, A, \phi, \bar{\phi}] = 0,$$

(2.14)

$$E^\mu[g, A] - J^\mu[g, A, \phi, \bar{\phi}] = 0,$$

(2.15)

obtained respectively from the variation with respect to the metric ($\sqrt{-g} \neq 0$), the matter fields, and the gauge field.

A stealth complex scalar field is a non trivial solution of the field equations which in addition satisfies the vanishing Hilbert energy-momentum tensor equations. This is,

$$\Upsilon[g, A, \phi, \bar{\phi}] = 0, \quad \bar{\Upsilon}[g, A, \phi, \bar{\phi}] = 0, \quad \Xi_{\mu\nu}[g, A, \phi, \bar{\phi}] = 0.$$  

(2.16)

Hence, from (2.13), the generalized Einstein tensor is sourced only by the $U(1)$ gauge energy-momentum tensor,

$$H_{\mu\nu}[g] = \Theta_{\mu\nu}[g, A],$$

(2.17)

so that in presence of the stealth fields $\phi$ and $\bar{\phi}$ the metric tensor must satisfy identical equations of motion than in the vacuum $\phi = \bar{\phi} = 0$ of the scalar fields.
3 \ θ-deformation of complex scalar field theory

Consider now the following map:

\[ \phi^\theta[g, A, \phi] = (1 - \theta^2 D^2)\phi, \quad \bar{\phi}^\theta[g, A, \bar{\phi}] = (1 - \theta^2 \bar{D}^2)\bar{\phi} \]  (3.18)

where \( \theta \) is a real-valued parameter and

\[ D_\mu := \nabla_\mu + iq A_\mu, \quad \bar{D}_\mu := \nabla_\mu - iq A_\mu, \]  (3.19)

are the covariant derivatives, acting upon the right-module and the left-module of the \( U(1) \) representation respectively, \( \nabla_\mu \) is the Levi-Civita derivative and \( q \) is the electromagnetic coupling constant. Hence,

\[ D^2 \phi = g^{\mu\nu} D_\mu D_\nu \phi, \quad \bar{D}^2 \bar{\phi} := g^{\mu\nu} \bar{D}_\mu \bar{D}_\nu \bar{\phi}. \]  (3.20)

Note that \( \bar{\phi}^\theta = (\phi^\theta)^* \).

Given the original action principle, say \( S[g, A, \phi, \bar{\phi}] \), we obtain its \( \theta \)-deformed action principle, say \( S^\theta[g, A, \phi, \bar{\phi}] \), by means of the replacement,

\[ \phi \rightarrow \phi^\theta, \quad \bar{\phi} \rightarrow \bar{\phi}^\theta \]  (3.21)

in the action, such that,

\[ S[g, A, \phi, \bar{\phi}] \rightarrow S[g, A, \phi^\theta, \bar{\phi}^\theta]. \]  (3.22)

Then, the deformed matter field action principle corresponding to the original theory \( S_M[g, A, \phi, \bar{\phi}] \) consists of the formal replacement of \( \phi \) by \( \phi^\theta \) (3.18) and the correspondent conjugated,

\[ S^\theta_M[g, A, \phi, \bar{\phi}] := S_M[g, A, \phi^\theta[g, A, \phi], \bar{\phi}^\theta[g, A, \bar{\phi}]], \]  (3.23)

which leads to the following decomposition,

\[ S^\theta[g, A, \phi, \bar{\phi}] := S[g, A, \phi^\theta[g, A, \phi], \bar{\phi}^\theta[g, A, \bar{\phi}]] = S_G[g] + S_A[g, A] + S^\theta_M[g, A, \phi, \bar{\phi}], \]  (3.24)

where we notice that only the pure-matter sector \( S_M[g] \) is affected by the deformation, because \( S_G[g] \) and \( S_A[g, A] \) are not functionals of \( \phi \) and \( \bar{\phi} \). We note also that the resulting action principle will be in general of higher order, since the map (3.21) increases the order of the derivatives with respect to the original action.

Observe that the deformed action principle takes the same value

\[ S_M[g, A, \phi^\theta[g, A, \phi], \bar{\phi}^\theta[g, A, \bar{\phi}]]|_{\phi = 0} = S_M[g, A, \phi^\theta[g, A, \phi], \bar{\phi}^\theta[g, A, \bar{\phi}]|_{\phi^\theta = 0}, \]  (3.25)

for \( \phi = 0 \) and for the non-trivial \( \phi = \phi_m \) satisfying \( \bar{\phi}^\theta = 0 \), i.e. for the solution of the minimally coupled Klein-Gordon equation,

\[ \phi^\theta[g, A, \phi_m] = -\theta^2(D^2 - \theta^{-2})\phi_m = 0, \]  (3.26)

(analogously for the complex conjugated \( \bar{\phi} \)). Assuming that the trivial vacuum \( \phi = 0 \) is a solution of the system of equations produced by the deformed action principle, which is therefore a saddle point of the action, implies (cf. [16]) that the non-trivial solution of (3.26) must be also a saddle point, where the action takes the same value. This suggest that the massive charged scalar field \( \phi = \phi_m \) (\( \bar{\phi} = (\phi_m)^* \)) of mass \( m = \theta^{-1} \) is at the same foot than the vacuum \( \phi = 0 \). We can ask whether these two configurations have also the same effects in their gravitational/electromagnetic backgrounds, i.e. null effects in their correspondent curvatures. Though this is less straightforward, we shall see that this is indeed the case.
### 3.1 Field equations

The equations of motion of deformed theory, \emph{i.e.} of the class (3.24) are given by:

\[
\frac{\delta S^\theta}{\delta g^{\mu\nu}}[g, A, \phi, \bar{\phi}] = 0 \quad \frac{\delta S^\theta}{\delta A_\mu}[g, A, \phi, \bar{\phi}] = 0, \quad \frac{\delta S^\theta}{\delta \phi}[g, A, \phi, \bar{\phi}] = 0 \quad \frac{\delta S^\theta}{\delta \bar{\phi}}[g, A, \phi, \bar{\phi}] = 0
\]  

which yield respectively,

\[
H_{\mu\nu}[g] - E_{\mu}^{\nu}[g, A] - \tilde{\Xi}_{\mu\nu}[g, A, \phi, \bar{\phi}] = 0, \quad E^{\mu}[g, A] - \tilde{J}^{\mu}[g, A, \phi, \bar{\phi}] = 0.
\]  

(3.28)

\[
\bar{\Upsilon}[g, A, \phi, \bar{\phi}] = 0, \quad \bar{\Upsilon}[g, A, \phi, \bar{\phi}] = 0,
\]  

(3.29)

with the definitions (2.4), (2.5), (2.11), and

\[
\tilde{\Xi}_{\mu\nu}[g, A, \phi, \bar{\phi}] := - \frac{1}{\sqrt{-g}} \frac{\delta S^\theta_M[g, A, \phi, \bar{\phi}]}{\delta A_\mu},
\]  

(3.30)

\[
\tilde{J}^{\mu}[g, A, \phi, \bar{\phi}] := \frac{1}{\sqrt{-g}} \frac{\delta S^\theta_M[g, A, \phi, \bar{\phi}]}{\delta A_\mu},
\]  

(3.31)

\[
\bar{\Upsilon}_{\mu\nu}[g, A, \phi, \bar{\phi}] := \frac{\delta S^\theta_M[g, A, \phi, \bar{\phi}]}{\delta \phi}, \quad \bar{\Upsilon}_{\mu
\nu}[g, A, \phi, \bar{\phi}] := \frac{\delta S^\theta_M[g, A, \phi, \bar{\phi}]}{\delta \bar{\phi}}.
\]  

(3.32)

Note that in (3.28) \(H_{\mu\nu}[g], \Theta_{\mu\nu}[g, A]\) and \(E^{\mu}[g, A]\) remain undeformed with respect to the definitions (2.4), (2.5) and (2.11), since field redefinitions (3.18) do not affect neither the metric tensor nor the \(U(1)\) gauge field. The tildes on \(\tilde{\Xi}, \tilde{J}\) and \(\bar{\Upsilon}\) indicate that these magnitudes are analogous to the original variables (2.6), (2.7), (2.8) and (2.12), in the deformed theory (3.23).

The results of the variation of the deformed action principle (3.24) with respect to the matter fields and the spacetime metric can be related to the results of the undeformed action principle (2.1) using the chain-rule of functionals, \emph{i.e.} reminding that the deformed action depends implicitly on \(g_{\mu\nu}\) and \(\phi\) by means of \(\phi^\theta[g, \phi]\) and analogously for the conjugated field \(\bar{\phi}\). The chain rule for functional derivation reads,

\[
\frac{\delta F[G[f]]}{\delta f(y)} = \int d^D z \frac{\delta F[G[f]]}{\delta G[f](z)} \frac{\delta G[f](z)}{\delta f(y)}
\]  

(3.33)

where \(F[f]\) and \(G[f]\) are two functionals of the function class element \(f\) and \(F[G[f]]\) is the composition of functionals. We shall declare the dependence on of functionals, say \(F[f]\), on the point \(x\) by attaching the symbol \(\langle x \rangle\), say \(F[f](x)\), whenever is necessary. Applying this in the computation of the equations of motion (3.32), in the point \(y\), we obtain from the deformed action,

\[
\frac{\delta S^\theta_M[g, A, \phi, \bar{\phi}]}{\delta \phi(y)} = \int d^D z \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta \phi(z)} \frac{\delta \phi^\theta(z)}{\delta \phi(y)}
\]  

(3.34)

in account of (3.23). The latter expression is equivalent to,

\[
\frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta \phi^\theta(z)} = \int d^D z \Upsilon^\theta[g, A, \phi, \bar{\phi}](z) \frac{\delta \phi^\theta(z)}{\delta \phi(y)}
\]  

(3.35)

where

\[
\Upsilon^\theta[g, A, \phi, \bar{\phi}] := \Upsilon[g, A, \phi^\theta, \bar{\phi}^\theta],
\]  

(3.36)

and

\[
\frac{\delta S_M[g, A, \phi^\theta, \bar{\phi}^\theta]}{\delta \phi^\theta(z)} = \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta \phi(z)} \bigg|_{\phi \rightarrow \phi^\theta[g, A, \phi]},
\]  

(3.37)
which is equivalent to the original operator (2.7) valued in $\phi^\theta[g, A, \phi, \bar{\phi}]$ and $\bar{\phi}^\theta[g, A, \phi, \bar{\phi}]$. Note that in (3.35),

$$\frac{\delta \phi^\theta(z)}{\delta \phi(y)} = (1 - \theta^2 D^2) \delta^D(z - y). \quad (3.38)$$

Integrating by parts and with the definitions (3.32) and (3.20) we obtain the equation of motion for the matter field in the deformed theory (3.28),

$$\hat{\Upsilon}[g, A, \phi, \bar{\phi}] := (1 - \theta^2 D^2) \Upsilon^\theta[g, A, \phi, \bar{\phi}] = 0. \quad (3.39)$$

Note that $\Upsilon^\theta[g, A, \phi, \bar{\phi}]$ transforms as a left-module of the $U(1)$ group. Analogously, the result of the variation with respect to $\bar{\phi}$ yields,

$$\hat{\Upsilon}[g, A, \phi, \bar{\phi}] := (1 - \theta^2 D^2) \hat{\Upsilon}^\theta[g, A, \phi, \bar{\phi}] = 0, \quad (3.40)$$

where

$$\hat{\Upsilon}^\theta[g, A, \phi, \bar{\phi}] := \Upsilon[g, A, \phi^\theta, \bar{\phi}^\theta]. \quad (3.41)$$

The variation of the action with respect to $g^{\mu\nu}$ should be carried out taking similar considerations. For this purpose, let us introduce the Lagrangian density $\mathcal{L}_M[g, A, \phi, \bar{\phi}]$, such that,

$$S_M[g, A, \phi, \bar{\phi}] = \int d^D x \sqrt{-g} \mathcal{L}_M[g, A, \phi, \bar{\phi}], \quad (3.42)$$

which with the substitution (3.23) yields,

$$S^\theta_M[g, A, \phi, \bar{\phi}] = \int d^D x \sqrt{-g} \mathcal{L}_M[g, A, \phi^\theta, \bar{\phi}^\theta]. \quad (3.43)$$

From the chain rule (3.33), considering that the deformed action functional depends on $g^{\mu\nu}$ explicitly and also implicitly in the deformed fields $\phi^\theta$ and $\bar{\phi}^\theta$, the variation of (3.23) is equivalent to,

$$\frac{\delta S_M[g, A, \phi^\theta, \bar{\phi}^\theta]}{\delta g^{\mu\nu}(y)} = \left. \frac{\delta S_M[g, A, \phi, \bar{\phi}]}{\delta g^{\mu\nu}(y)} \right|_{\phi \to \phi^\theta} + \int d^D z \frac{\delta S_M[g, A, \phi^\theta, \bar{\phi}^\theta]}{\delta \phi^\theta(z)} \frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} + \int d^D z \frac{\delta S_M[g, A, \phi^\theta, \bar{\phi}^\theta]}{\delta \bar{\phi}^\theta(z)} \frac{\delta \bar{\phi}^\theta(z)}{\delta g^{\mu\nu}(y)}. \quad (3.44)$$

Here the first term on the r.h.s. is provided by the explicit dependence of the action on the metric tensor, while the second and third terms are those coming from the variation with respect to the metric through the deformed scalar fields defined in (3.18).

Considering the definitions (3.36), (3.41) and (3.43), (3.44) can be written also as,

$$\frac{\delta S^\theta_M[g, A, \phi^\theta, \bar{\phi}^\theta]}{\delta g^{\mu\nu}(y)} = -\sqrt{-g} \Xi^\theta_{\mu\nu}(y) + \int d^D z \hat{\Upsilon}^\theta(z) \frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} + \int d^D z \hat{\Upsilon}^\theta(z) \frac{\delta \bar{\phi}^\theta(z)}{\delta g^{\mu\nu}(y)}, \quad (3.45)$$

where we have defined

$$\Xi^\theta_{\mu\nu} := \Xi_{\mu\nu}[g, A, \phi^\theta, \bar{\phi}^\theta],$$

and we have omitted the arguments of the functionals $\Upsilon^\theta$ and $\hat{\Upsilon}^\theta$ for simplicity, though we declare the points where they are valued, i.e. $y$ or $z$.

From (3.45) the Hilbert energy-momentum tensor of the scalar field in deformed theory is given by,

$$\Xi^\theta_{\mu\nu}[g, \phi](y) = \Xi^\theta_{\mu\nu}(y) - \frac{1}{\sqrt{-g}} \int d^D z \hat{\Upsilon}^\theta(z) \frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} - \frac{1}{\sqrt{-g}} \int d^D z \hat{\Upsilon}^\theta(z) \frac{\delta \bar{\phi}^\theta(z)}{\delta g^{\mu\nu}(y)}. \quad (3.46)$$
Here,
\[
\frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} = -\frac{1}{2} \left( g_{\mu\nu} g^{\sigma\rho} (D_\sigma \phi) (\tilde{\partial}_\rho \delta^D(z - y)) + \frac{1}{\sqrt{-g}} (\tilde{\partial}_\mu + iq A_{(\mu}) \left( \sqrt{-g} \delta^D(z - y) D_\nu \phi \right) \right), \tag{3.47}
\]
\[
\frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} = \left( \frac{\delta \phi^\theta(z)}{\delta g^{\mu\nu}(y)} \right)^*. \tag{3.48}
\]

Here symmetrized indices (with factor 1/2) have been enclosed in parenthesis. Replaced in (3.46) and followed by an integration by parts and boundary condition,
\[
\Upsilon^\theta[g, A, \phi, \bar{\phi}]|_{\partial M} = \bar{\Upsilon}^\theta[g, A, \phi, \bar{\phi}]|_{\partial M} = 0, \tag{3.49}
\]
we obtain the energy momentum tensor,
\[
\tilde{\Xi}_{\mu\nu} = \Xi^\theta_{\mu\nu} + g^2 \frac{1}{2} \sqrt{-g} g_{\mu\nu} \left( (D^2 \phi) \Upsilon^\theta + (\tilde{D}^2 \bar{\phi}) \bar{\Upsilon}^\theta \right) \tag{3.50}
\]
\[- \frac{g^2}{2} (\delta^\mu_\mu \delta^\nu_\nu + \delta^\mu_\nu \delta^\nu_\mu - g_{\mu\nu} g^{\sigma\rho}) \left( (D_\rho \phi) \tilde{D}_\sigma \left( \frac{\Upsilon^\theta}{\sqrt{-g}} \right) + (\tilde{D}_\rho \bar{\phi}) D_\sigma \left( \frac{\bar{\Upsilon}^\theta}{\sqrt{-g}} \right) \right). \tag{3.51}
\]

The computation of the electric current (3.31) of the deformed theory (3.23) yields,
\[
\tilde{J}^\mu[g, A, \phi, \bar{\phi}] = J^\mu[g, A, \phi^\theta, \bar{\phi}^\theta] \tag{3.52}
\]
\[-iq \frac{g^2}{2} \sqrt{-g} \left( \Upsilon^\theta D^\mu \phi - \bar{\Upsilon}^\theta \tilde{D}^\mu \bar{\phi} - \phi \bar{D}^\mu \Upsilon^\theta + \bar{\phi} \tilde{D}^\mu \bar{\Upsilon}^\theta \right). \tag{3.53}
\]

The (3.51) term appears from the variation of the part of the action which depends explicitly on the gauge field $A_{\mu}$ and it is equivalent to electric current functional of the original theory (2.12) but valued in $\phi^\theta$ (3.18) instead of $\phi$, and their complex conjugated. The terms (3.52) appear from the implicit dependence, i.e. where the functional chain rule has to be used, of $\phi^\theta$ and $\bar{\phi}^\theta$ in the $A$-field dependence from the definitions (3.18).

### 3.2 Stealth theorem

Consider the solutions $\phi = \phi_m$ and $\bar{\phi} = \bar{\phi}_m$ of mass $m = \theta^{-1}$ the Klein-Gordon equations (3.26), such that
\[
\phi^\theta[g, A, \phi_m] = \bar{\phi}^\theta[g, A, \bar{\phi}_m] = 0. \tag{3.54}
\]
Using this in the field equations of the deformed theory (3.29),
\[
\tilde{\Upsilon}[g, A, \phi_m, \bar{\phi}_m] = (1 - \theta^2 D^2) \Upsilon^\theta[g, A, \phi_m, \bar{\phi}_m] = (1 - \theta^2 \tilde{D}^2) \tilde{\Upsilon}[g, A, 0, 0] = 0, \tag{3.55}
\]
\[
\tilde{\Upsilon}[g, A, \phi_m, \bar{\phi}_m] = (1 - \theta^2 \tilde{D}^2) \bar{\Upsilon}^\theta[g, A, \phi_m, \bar{\phi}_m] = (1 - \theta^2 \tilde{D}^2) \bar{\Upsilon}[g, A, 0, 0] = 0, \tag{3.56}
\]
we observe that they are satisfied, under the assumption that the field equations of the original theory (2.14) admit the trivial solutions $\phi = \bar{\phi} = 0$, i.e. such that
\[
\Upsilon^\theta[g, A, 0, 0] = 0, \quad \bar{\Upsilon}^\theta[g, A, 0, 0] = 0. \tag{3.57}
\]
Now, let us evaluate the energy-momentum tensor (3.50) for the same solutions,

\[ \Xi_{\mu\nu}[g, A, \phi_0, \bar{\phi}_0] = \Xi_{\mu\nu}[g, A, \phi_0, \bar{\phi}_0] + \frac{\theta^2}{2} \frac{g_{\mu\nu}}{\sqrt{-g}} \left\{ (D^2 \phi_0) \Gamma^\theta[g, A, \phi_0, \bar{\phi}_0] \right\} + (\bar{D}^2 \bar{\phi}_0) \bar{\Gamma}^\theta[g, A, \phi_0, \bar{\phi}_0] \]

(3.57)

\[ \Xi_{\mu\nu}^\theta[g, A, \phi_0, \bar{\phi}_0] = \Xi_{\mu\nu}[g, A, \phi_0, \bar{\phi}_0] + \frac{\theta^2}{2} \frac{g_{\mu\nu}}{\sqrt{-g}} \left\{ (D^2 \phi_0) \Gamma^\theta[g, A, \phi_0, \bar{\phi}_0] \right\} + (\bar{D}^2 \bar{\phi}_0) \bar{\Gamma}^\theta[g, A, \phi_0, \bar{\phi}_0] \]

From (3.56) all the terms containing these operators vanish. The only potentially surviving term is the first in the right hand side,

\[ \Xi_{\mu\nu}^\theta[g, A, \phi_0, \bar{\phi}_0] = \Xi_{\mu\nu}[g, A, \phi_0, \bar{\phi}_0] = \Xi_{\mu\nu}[g, A, 0, 0] . \]

(3.58)

The expression in the middle is the energy momentum tensor of the original theory valued \( \phi^0 \) and \( \bar{\phi}^0 \), but from (3.56) they vanish for stealth configurations. Hence we obtain the last term, \( \Xi_{\mu\nu}[g, 0, 0] = 0 \), which must vanish assuming that the the energy momentum of trivial scalar fields must vanish in the original theory.

Let us analyze now the electric current (3.51) of the deformed theory valued at stealth configurations. The terms (3.52) vanish as consequence of (3.56). Hence we are left with,

\[ \tilde{J}^\mu[g, A, \phi_0, \bar{\phi}_0] = J^\mu[g, A, \phi_0, \phi_0], \tilde{\phi}^\mu[g, A, \phi_0, \bar{\phi}_0] = J^\mu[g, A, 0, 0] = 0 , \]

(3.59)

which is just the current functional of the original theory valued at the trivial vacuum of the scalar field. Of course, this expression must vanish since the trivial vacuum can not carry electric current or, equivalently, as a functional the current must be at least of second order in the fields.

Finally, in the deformed theory (3.24) the system of equations (3.28)-(3.29), are reduced to the Einstein equations in the presence of a electromagnetic fields and the Maxwell equations in a curved background with no electric-charge sources,

\[ H_{\mu\nu}[g] - \Theta_{\mu\nu}[g, A] = 0 , \quad E^\mu[g, A] = 0 , \]

(3.60)

i.e. the same than the equations of motion of the original theory (2.1) without matter field \( S_M[g, A, \phi, \bar{\phi}] = 0 \). Therefore, the stealth configuration of mass \( m = \theta^{-1} \), i.e. which satisfy the minimally coupled Klein-Gordon equation (3.26), neither source space-time curvature nor electromagnetism. However, the space of functions on the spacetime manifold has nontrivial content and it may affect the topology of the spacetime (for example if singular regions in the domain of \( \phi_m \) need to be removed).

4 Example: Deformation of the massive field action

The results presented so far are valid for any theory (accepting the trivial vacuum solution \( \phi = 0 \)). But to see how our theory works, let us analyze one of the simplest cases.

The minimally coupled Klein-Gordon action of a complex field of mass \( M \) is given by,

\[ S_M[g, A, \phi, \bar{\phi}] = - \int d^D x \sqrt{-g} \left( g^{\mu\nu} D_\mu \phi D_\nu \bar{\phi} + M^2 \phi \bar{\phi} \right) , \]

(4.61)

with the covariant derivatives defined as in (3.19). The field equations are given by,

\[ \Upsilon[g, A, \phi, \bar{\phi}] = \sqrt{-g} \left( \tilde{D}^2 - M^2 \right) \phi = 0 , \quad \tilde{\Upsilon}[g, A, \phi, \bar{\phi}] = \sqrt{-g} \left( D^2 - M^2 \right) \phi = 0 . \]

(4.62)
The electric current (2.12) takes the form,

\[ J^\mu = i q \left( \bar{\phi} D^\mu \phi - \phi \bar{D}^\mu \bar{\phi} \right) \]  

and the energy-momentum tensor \( \Xi_{\mu\nu} \) of the scalar field is

\[ \Xi_{\mu\nu} = \frac{1}{2} \left\{ g_{\mu\nu} \left( D^\rho \bar{\phi} \bar{D}_\rho \phi + M^2 \phi \bar{\phi} \right) - \left( D_\mu \phi \bar{D}_\nu \bar{\phi} + D_\nu \phi \bar{D}_\mu \bar{\phi} \right) \right\} . \]  

### 4.1 Deformed matter action

When applied to the theory (4.61) the deformation procedure (3.23) produces the new action principle,

\[ S_M^\theta [g, A, \phi, \bar{\phi}] = - \int d^D x \sqrt{-g} \left( g^{\mu\nu} D_\mu \phi^0 \bar{D}_{\nu} \bar{\phi}^0 + M^2 \phi^0 \bar{\phi}^0 \phi^0 \bar{\phi}^0 \right) . \]  

We can expand this action in terms of the fields \( \phi \) and \( \bar{\phi} \) using the definitions (3.18). The equations of motion obtained by varying with respect to \( \bar{\phi} \) is,

\[ \tilde{\Gamma}[g, A, \phi, \bar{\phi}] = \sqrt{-g} \left( 1 - \theta^2 D^2 \right) \left( D^2 - M^2 \right) \left( 1 - \theta^2 D^2 \right) \phi = 0 , \]  

which is of the form (3.39). Alternatively,

\[ \tilde{\Gamma}[g, A, \phi, \bar{\phi}] = \theta^4 \sqrt{-g} \left( D^2 - m^2 \right)^2 \left( D^2 - M^2 \right) \phi = 0 , \]  

where \( m = \theta^{-1} \) is the stealth field mass. Analogously, the equations of motion of the conjugated field can be written as,

\[ \tilde{\Gamma}[g, A, \phi, \bar{\phi}] = \theta^4 \sqrt{-g} \left( D^2 - m^2 \right)^2 \left( D^2 - M^2 \right) \bar{\phi} = 0 . \]  

The electric current, as defined in (3.51), is given by,

\[ \tilde{J}^\mu[g, A, \phi, \bar{\phi}] = i q \left( \bar{\phi}^0 D^\mu \phi^0 - \phi^0 \bar{D}^\mu \bar{\phi}^0 \right) - i q \theta^2 \left\{ D^\mu \phi \left( \bar{D}^2 - M^2 \right) \bar{\phi}^0 - \left( D^2 - M^2 \right) \phi^0 \bar{D}^\mu \bar{\phi} \right\} . \]  

The energy-momentum tensor (3.30) of deformed matter yields in this case,

\[ \tilde{\Xi}_{\mu\nu}[g, A, \phi, \bar{\phi}] = \frac{1}{2} \left\{ g_{\mu\nu} \left( D^\rho \phi^0 \bar{D}_\rho \bar{\phi}^0 + M^2 \phi^0 \bar{\phi}^0 \right) - \left( D_\mu \phi^0 \bar{D}_\nu \bar{\phi}^0 + D_\nu \phi^0 \bar{D}_\mu \bar{\phi}^0 \right) \right\} + \frac{\theta^2}{2} g_{\mu\nu} \left\{ D^2 \phi \left( \bar{D}^2 - M^2 \right) \bar{\phi}^0 + \left( D^2 - M^2 \right) \phi^0 \bar{D}^2 \bar{\phi} \right\} . \]  

### 4.2 Stealth solutions

The stealth field satisfies,

\[ \phi^0 = 0 = (1 - \theta^2 D^2) \phi_m = - \frac{1}{m^2} \left( D^2 - m^2 \right) \phi_m , \]  

and their respective complex conjugated condition. Therefore the stealth field has mass \( m = \theta^{-1} \). Let us denote these solutions as \( \phi_m \) and \( \bar{\phi}_m \). It is straightforward to show that the field equations (4.67)-(4.68) are satisfied and that the electric current (4.69) and energy-momentum tensor (4.70) vanish for \( \phi = \phi_m \). Hence, though non-trivial, \( \phi_m \) does not curve the spacetime and it does not source electromagnetic fields.
4.3 Non-stealth solutions

The field equation (4.67) admits other solutions besides stealth solutions. Those satisfy,

\[ (D^2 - M^2) \phi = 0, \]  

(4.72)
as one can see from the factorization of the operators in (4.67)-(4.68). Note that (4.72) coincides with the equation of motion of the original theory (4.62). So, after deformation of the theory, this configuration still persists.

Let \( \phi_M \) be a solution of (4.72), which has mass \( M \), so that the operator \( D^2 \) can be replaced by its value

\[ D^2 \phi_M = M^2 \phi_M. \]  

(4.73)

Hence the functional \( \phi^\theta \) (3.18) takes the value,

\[ \phi^\theta[g, A, \phi_M, \bar{\phi}_M] = -\frac{1}{m^2} (D^2 - m^2) \phi_M = \lambda \phi_M, \quad \lambda := \left( 1 - \frac{M^2}{m^2} \right), \]  

(4.74)

and analogously for the conjugated field. The electric current (4.69) takes now the form,

\[ \tilde{J}^\mu[g, A, \phi_M, \bar{\phi}_M] = i q \lambda^2 (D^\mu \phi_M \bar{\phi}_M - \phi_M D^\mu \bar{\phi}_M) = \lambda^2 J^\mu[g, A, \phi_M, \bar{\phi}_M]. \]  

(4.75)

As we see, in the deformed theory the electric current is rescaled with respect to its counterpart in the original theory (4.63). The same behavior is observed in the energy-momentum tensor of deformed theory, it is also rescaled with respect to the undeformed theory, as we can see replacing \( \phi_M (\bar{\phi}_M) \) in (4.70),

\[ \tilde{\Xi}_{\mu\nu}[g, A, \phi_M, \bar{\phi}_M] = \lambda^2 \Xi_{\mu\nu}[g, A, \phi_M, \bar{\phi}_M]. \]  

(4.76)

Note that a general solution of the deformed theory consists of a linear combination of the solution of the original theory and the stealth field, since the differential operators \( (D^2 - M^2) \) and \( (D^2 - m^2) \) commute each other, so that each operator annihilates its respective field. Hence the charged Klein-Gordon field is preserved in the deformed theory, but its effects on the electromagnetic and gravity backgrounds is modified by the \( \lambda^2 \) factor. As we see, though stealth fields do not have direct gravitational and electromagnetic effects, they can modify the strength of the effects of regular massive configurations.

5 Overview and remarks

In this paper we observed that there exist a wide class of complex scalar field theories, coupled to gravity and to electromagnetism, that admit stealth configurations. These theories are constructed by means of a “deformation method”, which can be regarded as a \( \theta \)-parametric extension of some “original theory”. We show that the details of the original theory are not important for the existence of stealth configurations in the deformed theories, except that it must accept the trivial vacuum solution.

As a novel aspect, the stealth fields does not source neither gravity nor electromagnetic fields, in spite of their non-trivial coupling to electromagnetism at the level of the action principle. They can, however, modify the strength of the effects of regular (non-stealth) scalar field configurations on their electromagnetic/gravity background.

We provide one example, the deformation of the action principle of a complex Klein-Gordon field of mass \( M \), and we observe that the deformed theory predicts a modification of the strength of the sources of gravity and electromagnetism, respectively its energy momentum tensor and its electric current, by a constant factor \( \lambda^2 = 1 - M^2/m^2 \), constructed from the stealth field mass \( m \) and the non-stealth field.
mass $M$. At the level of the equations of motion, this is equivalent to a renormalization of the Newton constant and the electric charge,

$$G_N \rightarrow \lambda^2 G_N, \quad q \rightarrow \lambda^2 q.$$  

Hence, the mass of the stealth field $m = \theta^{-1}$ (equivalently the inverse of the deformation parameter $\theta$) can be used to smooth or amplify the effects of the massive field of mass $M$ on the gravitational background and the electromagnetic field.

We expect that our results may be useful to reduce the discrepancies between the observations and the predictions of regular scalar field theories, which have lead for example to the proposal of dark matter, i.e. a discrepancy between the amount of observed matter and their effects in their gravity backgrounds. For example, one can determine the mass $m$ of the stealth field such that the observed matter (non-stealth) will feedback the correct curvature to the background geometry, i.e. adjusting the $\lambda^2$ parameter in the correspondent field equations.

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