Existence and Large Time Behavior of Entropy Solutions to One-Dimensional Unipolar Hydrodynamic Model for Semiconductor Devices with Variable Coefficient Damping

Yan Li, Yanqiu Cheng, and Huimin Yu

Department of Mathematics, Shandong Normal University, Jinan 250014, China

Correspondence should be addressed to Huimin Yu; hmyu@sdnu.edu.cn

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In this paper, we investigate the global existence and large time behavior of entropy solutions to one-dimensional unipolar hydrodynamic model for semiconductors in the form of Euler-Possion equations with time and space dependent damping in a bounded interval. Firstly, we prove the existence of entropy solutions through vanishing viscosity method and compensated compactness framework. Based on the uniform estimates of density, we then prove the entropy solutions converge to the corresponding unique stationary solution exponentially with time. We generalize the existing results to the variable coefficient damping case.

1. Introduction

The present paper is concerned with the one-dimensional isentropic Euler-Possion model for semiconductor devices with damping:

\[
\begin{align*}
\rho_t + m_x &= 0, \\
m_t + \left( \frac{m^2}{\rho} + P(\rho) \right)_x &= \rho E + H(x, t)m, \\
E_x &= \rho - b(x),
\end{align*}
\]

where space variable \( x \in [L_1, L_2] \) (\( L_1 \) and \( L_2 \) are two positive constants) and time variable \( t \in [0, T) \) \((T > 0)\). Here, \( \rho \geq 0 \), \( m \), \( H(x, t) \), \( P(\rho) \), and \( E \) stand for electron density, electron current density, damping coefficient, pressure, and electric field, respectively. We assume the damping function \( H(x, t) \) is bounded, and the pressure function is given by \( P(\rho) = p_0 \rho^\gamma \), where \( p_0 = \theta^2/\gamma \) and \( \theta = (\gamma - 1)/2 \). Here, \( \gamma \) presents the adiabatic coefficient, and \( \gamma > 1 \) corresponds to the isentropic case. The doping profile \( b(x) \geq 0 \) stands for the density of fixed, positively charged background ions. In this paper, we assume

\[
b(x) \in C[L_1, L_2], \quad 0 < b_* \leq b(x) \leq b^*,
\]

where \( b_* \) and \( b^* \) are two positive constants. The initial-boundary value conditions of system (1) are

\[
\begin{align*}
\rho(x, 0) &= \rho_0(x), \quad L_1 < x < L_2, \\
m(L_1, t) &= m(L_2, t) = 0, \quad t \geq 0, \\
E(L_1, t) &= E_-, \quad t \geq 0,
\end{align*}
\]

where \( \rho_0(x) \) satisfies

\[
\int_{L_1}^{L_2} (\rho_0(x) - b(x)) dx = 0.
\]

Firstly, let us survey the related mathematical results. In 1990, Degond and Markowich [1] firstly proved the existence and uniqueness of the steady-state to (1) in subsonic case, which is characterized by a smallness assumption on the current flowing through the device. It was proved that the
The existence of local smooth solution to the time-dependent problem by using Lagrangian mass coordinates in [2]. However, Chen-Wang in [3] had studied the smooth solution would blow up in finite time; therefore, it is worthwhile considering the existence and other properties of weak solutions. As for weak solutions, Zhang [4] and Marcati-Natalini [5] proved the global existence of entropy solutions to the initial-boundary value and Cauchy problems for $\gamma > 1$, respectively. Li [6] and Huang et al. [7] proved the existence of $L^\infty$ entropy solution of (1) with $\gamma = 1$ on a bounded interval and the whole space by using a fractional Lax-Friedrichs scheme. It is worth noting that the $L^\infty$ estimates of entropy solution, especially the estimate of density, in all of the above works [4–7] depend on time $t$, which restricted us to consider their large time behavior further. We refer [8–10] for more results on this model and topic. In this paper, for $1 < \gamma \leq 3$ and variable coefficient damping, we shall first verify the assumption in [11], where the density is assumed to be uniformly bounded with respect to space $x$ and time $t$ and then use the entropy inequality to consider the large time behavior of the obtained solutions.

Based on the related results in [12–16], we are convinced that the method developed in this paper can be used to bi-polar Euler-Poisson system with time depended damping. We will investigate this problem in next papers.

To start our main theorem, we define the entropy solution of system (1) as.

**Definition 1.** For every $T > 0$, a pair of bounded measurable functions $v(x, t) = (\rho(x, t), m(x, t), E(x, t))$ is called a $L^\infty$ weak solution of (1) with initial-boundary condition (3) if

$$
\begin{align*}
\int_0^T \int_{L_1} (\rho \varphi_t + m \varphi_x) dx dt + \int_0^T \rho_0 \varphi(x, 0) dx & = 0, \\
\int_0^T \int_{L_1} \left( m \varphi_t + \left( \frac{m^2}{\rho} + P(\rho) \right) \varphi_x \right) dx dt + \int_0^T \rho E + H(x, t) m \varphi_0 dx dt & = 0, \\
+ \int_{L_1} m_0 \varphi(x, 0) dx & = 0, \\
E(x, t) = \int_{L_1} (\rho - h(s)) ds + E_0.
\end{align*}
$$

holds for any test function $\varphi \in C^0_0([L_1, L_2] \times [0, T])$, and the boundary condition is satisfied in the sense of divergence-measure field [17]. Furthermore, we call the weak solution $(\rho, m, E)(x, t)$ to be an entropy solution if the entropy inequality

$$
\eta_t + q_x \leq \eta_m (\rho E + H(x, t)),
$$

satisfies in the sense of distribution for any weak convex entropy pairs $(\eta(\rho, m), q(\rho, m))$.

**Definition 2.** The stationary solution of problems (1) and (3) is the smooth solution of

$$
\begin{align*}
P(\bar{\rho})_t = \bar{\rho} \bar{E}, \\
\bar{E}_x = \bar{\rho} - b(x),
\end{align*}
$$

with the boundary condition

$$
\bar{E}(L_1) = \bar{E}(L_2) = 0.
$$

Our main results in this paper are as follows.

**Theorem 3 (Existence).** Let $1 < \gamma \leq 3$, we assume that the initial data and the damping coefficient satisfy

$$
0 \leq \rho_0(x) \leq M_0, \quad |m_0(x)| \leq M_0, \quad |H(x, t)| \leq M_1,
$$

for some positive constants $M_0$ and $M_1$. Then, there exists a global entropy solution $(\rho, m, E)(x, t)$ of the initial-boundary value problems (1) and (3) satisfying

$$
0 \leq \rho(x, t) \leq C, \quad |m(x, t)| \leq C, \quad |E(x, t)| \leq C, \quad (x, t) \in [L_1, L_2] \times [0, T),
$$

where $C$ is independent of $t$.

**Remark 4.** To get the global existence of the $L^\infty$ weak solution, we only need $H(x, t)$ is bounded. However, to get the large time behavior of the obtained solution, the uniform negative upper bound is necessary.

**Theorem 5 (Large time behavior).** Suppose there exists a positive constant $\delta_0 > 0$, such that the damping coefficient

$$
H(x, t) < -\delta_0 \text{ and } H_t(x, t) > -2b_*,
$$

for any $[L_1, L_2] \times \mathbb{R}^+$. Denote $(\rho, m, E)(x, t)$ is the global entropy solution of (1) and (3) obtained in Theorem 3, and $(\bar{\rho}, \bar{E})$ is the stationary solution; then, it holds that

$$
\int_{L_1} \left( (\rho - \bar{\rho})^2 (E - \bar{E})^2 + m^2 \right) (x, t) dx \leq Ce^{-Ct},
$$

for some positive constant $C$.

**Remark 6.** Theorems 3 and 5 are generalizations of the corresponding theorem of [18], in which the damping coefficient $H(x, t) = 1$. Suppose $\alpha$, $\beta$, and $\lambda$ are three positive constants, then $H(x, t) = -\alpha(1 + t)^{\beta} - \beta$ satisfies all the assumptions of $H(x, t)$ in Theorems 3 and 5.
2. Preliminary and Formulation

We consider the homogeneous system

\[
\begin{aligned}
\rho_t + m_x &= 0, \\
\rho_t + \left(\frac{\rho^2}{\rho} + P(\rho)\right)_x &= 0.
\end{aligned}
\] (13)

Firstly, we use \(r_1\) and \(r_2\) to denote the right eigenvectors corresponding to the eigenvalues \(\lambda_1\) and \(\lambda_2\). After simple calculation, we have

\[
\lambda_1 = \frac{m}{\rho} - \theta\rho^\theta, \quad \lambda_2 = \frac{m}{\rho} + \theta\rho^\theta, \quad \theta = \frac{y - 1}{2}.
\] (14)

The Riemann invariants \((w, z)\) are given by

\[
w = \frac{m}{\rho} + \rho^\theta, \quad z = \frac{m}{\rho} - \rho^\theta,
\] (15)

satisfying \(\nabla w \cdot r_1 = 0\) and \(\nabla z \cdot r_2 = 0\), where \(\nabla = (\partial_{\rho}, \partial_m)\) is the gradient with respect to \(U = (\rho, m)\).

A pair of functions \((\eta, \xi)\): \(\mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}^2\) is called an entropy-entropy flux of system (13) if it satisfies

\[
\nabla q(U) = \nabla \eta(U) \nabla \begin{pmatrix} m \\ \frac{m^2}{\rho} + P(\rho) \end{pmatrix}.
\] (16)

Furthermore, if for any fixed \(m/\rho \epsilon (0, +\infty)\), \(\eta\) vanishes on the vacuum \(\rho = 0\); then, \(\eta\) is called a weak entropy. For example, the mechanical energy-energy flux pair

\[
\eta_e = \frac{m^2}{2\rho} + \frac{\rho \rho' \rho'}{\gamma - 1}, \quad q_e = \frac{m^3}{2\rho^2} + \frac{\rho \rho' \rho'}{\gamma - 1},
\] (17)

should be a strictly convex entropy pair. We approximate the equations in (1) by adding artificial viscosity to get the smooth approximate solutions \((\rho^\epsilon, m^\epsilon)\), that is,

\[
\begin{aligned}
\rho_t + m_x &= \epsilon \rho_{xx}, \\
m_t + \left(\frac{m^2}{\rho} + P(\rho)\right)_x &= \epsilon m_{xx} + \rho E - 2M\epsilon \rho_x + H(x, t)m, \\
E(x, t) &= \int_{L'} (\rho - b(s))ds + E_0,
\end{aligned}
\] (18)

with initial-boundary value conditions

\[
(\rho, m)(x, 0) = (\rho_0(x), m_0(x)) = (\rho_0(x) + \epsilon, m_0(x)),
\]

\[
m(L_1, t) = m(L_2, t) = 0, \quad \rho(L_1, t) = \rho_0(L_1), \quad \rho(L_2, t) = \rho_0(L_2), \quad t \geq 0,
\] (19)

where \(M\) in (18) is a big enough constant to be determined later and \(m^\epsilon\) in (19) is the standard mollifier with small parameter \(\epsilon\). We shall prove that the viscosity solutions of (18) and (19) are uniformly bounded with respect to time \(t\).

3. Viscosity Solutions and A Priori Estimates

For any fixed \(\epsilon > 0\), we denote the solution of (18) and (19) by \((\rho^\epsilon, m^\epsilon, E^\epsilon)\), since \(E^\epsilon(x, t)\) is uniquely determined by \(\rho^\epsilon(x, t), b(x),\) and \(E_0\); then, the system (18) may be seen as one system with the unknowns \(\rho^\epsilon\) and \(m^\epsilon\). Regarding the proof of local existence of approximate solution, the techniques used in this article are similar to those used in [19]. To extend the local solution to global one, the key point is to obtain the uniform upper bound of \(|\rho^\epsilon|^2, |m^\epsilon|^2\) and the lower bound of density \(\rho^\epsilon\). The following theorem gives the uniform bound of \((\rho^\epsilon, m^\epsilon)\).

**Lemma 7.** For any \(T > 0\), let \((\rho^\epsilon, m^\epsilon)(x, t) \in C^1([0, T], C^2[L_1, L_2])\) to be the smooth solution of (18) and (19). Then

\[
0 \leq \rho^\epsilon(x, t) \leq C, \quad |m^\epsilon(x, t)| \leq C |\rho^\epsilon(x, t)|,
\] (20)

where \(C\) is a positive constant independent of time \(t\).

**Proof.** (For simplicity of notation, the superscript of \(\rho^\epsilon\) and \(m^\epsilon\) will be omitted as \((\rho, m)\).) By the formulas of Riemann invariants (15), we can decouple the viscous perturbation equation (18) as

\[
\begin{aligned}
\rho_t + m_x &= \epsilon w_x + 2\epsilon(w_x - M)\frac{\rho_x}{\rho} - \epsilon \theta(\theta + 1)\rho^\theta - \rho_{xx} + E + H(x, t)\frac{m}{\rho}, \\
\rho_t + \frac{m^2}{\rho} + P(\rho)_x &= \epsilon m_{xx} + \rho E - 2M\epsilon \rho_x + H(x, t)m, \\
E(x, t) &= \int_{L'} (\rho - b(s))ds + E_0,
\end{aligned}
\] (21)

We set the control functions \((\varphi, \psi)\) as

\[
\begin{aligned}
\varphi &= M(M + x), \\
\psi &= M(M - x).
\end{aligned}
\] (22)

A direct calculation tells us

\[
\begin{aligned}
\varphi_t &= 0, \quad \varphi_x = M, \quad \varphi_{xx} = 0, \\
\psi_t &= 0, \quad \psi_x = -M, \quad \psi_{xx} = 0.
\end{aligned}
\] (23)
Define the modified Riemann invariants $\bar{w}, \bar{z}$ as:
\[
\bar{w} = w - \varphi, \quad \bar{z} = z + \psi.
\] (24)

Then, inserting the above formulas into (21) yields the decoupled equations for $\bar{w}$ and $\bar{z}$:
\[
\begin{aligned}
\bar{w}_t + \left( \lambda_2 - 2\epsilon \frac{\rho_a}{\rho} \right) \bar{w}_x &= \epsilon \bar{w}_{xx} + a_{11} \bar{w} + a_{12} \bar{z} + R_1, \\
\bar{z}_t + \left( \lambda_1 - 2\epsilon \frac{\rho_a}{\rho} \right) \bar{z}_x &= \epsilon \bar{z}_{xx} + a_{21} \bar{w} + a_{22} \bar{z} + R_2,
\end{aligned}
\] (25)

We rewrite (25) into
\[
\begin{aligned}
&\begin{cases}
\bar{w}_t + \left( \lambda_2 - 2\epsilon \frac{\rho_a}{\rho} \right) \bar{w}_x = \epsilon \bar{w}_{xx} + a_{11} \bar{w} + a_{12} \bar{z} + R_1, \\
\bar{z}_t + \left( \lambda_1 - 2\epsilon \frac{\rho_a}{\rho} \right) \bar{z}_x = \epsilon \bar{z}_{xx} + a_{21} \bar{w} + a_{22} \bar{z} + R_2,
\end{cases}
\end{aligned}
\] (26)

with
\[
\begin{aligned}
a_{11} &= -\left( \frac{1 + \theta}{2} \right) M + \frac{1}{2} H(x, t), \\
a_{12} &= \frac{1}{2} M + \frac{1}{2} H(x, t), \\
a_{21} &= \frac{1 + \theta}{2} M + \frac{1}{2} H(x, t), \\
a_{22} &= -\left( \frac{1 + \theta}{2} \right) M + \frac{1}{2} H(x, t), \\
R_1 &= -\epsilon \theta + (1 + \theta) \rho^2 M^3 + E - \theta M^3 - M^2 x + MH(x, t)x, \\
R_2 &= \epsilon \theta (1 + \theta) \rho^2 M^2 + E + \theta M^3 - M^2 x + MH(x, t)x.
\end{aligned}
\] (27)

In above calculation, we have used the relations:
\[
\begin{aligned}
\lambda_1 &= \frac{w + z}{2} - \theta \frac{w - z}{2}, \\
\lambda_2 &= \frac{w + z}{2} + \theta \frac{w - z}{2}, \\
m &= \frac{w + z}{2}, \\
\rho &= \frac{w + z}{2}.
\end{aligned}
\] (28)

Noting $0 < \theta \leq 1$, $|H(x, t)| \leq M_1$, and choosing $M \geq (2/ (1 - \theta)) M_1$, we have
\[
a_{12} \leq 0, a_{21} \leq 0.
\] (29)

On the other hand, (27) tells us
\[
\begin{aligned}
R_1 &\leq E - \theta M^3 - M^2 x + MH(x, t)x, \\
R_2 &\geq E + \theta M^3 - M^2 x + MH(x, t)x.
\end{aligned}
\] (30)

And use the same calculations in [18], we estimate the approximate electric fields and obtain
\[
|E(x, t)| \leq M_2,
\] (31)

where $M_2$ depends only on initial data. Thus, taking $M$ big enough, we have
\[
\begin{aligned}
R_1 &\leq M_2 - \theta M^3 + MM_1 L_2 \leq 0, \\
R_2 &\geq -M_2 + \theta M^3 - M^2 L_2 - MM_1 L_1 \geq 0,
\end{aligned}
\] (32)

and the initial-boundary value conditions satisfy
\[
\begin{aligned}
\bar{w}(x, 0) &= w(x, 0) - \varphi(x, 0) = \frac{m_0}{\rho_0} + \rho_0 - M_2 - Mx \leq 0, \\
\bar{z}(x, 0) &= z(x, 0) + \psi(x, 0) = \frac{m_0}{\rho_0} - \rho_0^2 + M_2 - Mx \geq 0, \\
\bar{w}(L_1, t) &\leq 0, \quad \bar{z}(L_1, t) \geq 0, \quad \bar{w}(L_2, t) \leq 0, \quad \bar{z}(L_2, t) \geq 0.
\end{aligned}
\] (33)

Basing on the above discussion, using Lemma 7 of [18], we have
\[
\bar{w}(x, t) \leq 0, \quad \bar{z}(x, t) \geq 0, \quad \Psi(x, t) \in [L_1, L_2] \times [0, T).
\] (34)

Therefore,
\[
\begin{aligned}
&\bar{w}(x, t) \leq \varphi(x, t) \leq M^2 + Mx \leq M^2 + ML_2, \\
&\bar{z}(x, t) \geq -\psi(x, t) \geq -M^2 + ML_2.
\end{aligned}
\] (35)

By (35), we have
\[
\rho \leq \left( \frac{w - z}{2} \right)^{1/\theta} \leq \left( \frac{3}{2} M^2 \right)^{1/\theta},
\] (36)

and Lemma 7 is completed.

From (20), the velocity $u = m/\rho$ is uniformly bounded, i.e., $|u| < C$. Then, following the same way of [20], we could obtain
\[
\rho(x, t) \geq \delta(t, \epsilon) > 0.
\] (37)

Based on the local existence of smooth solution, the uniform upper estimates (Lemma 7) and the lower bound estimate of density (37), we derive the following lemma.

\textbf{Lemma 8.} For any time $T > 0$, there exists a unique global classical solution $(\rho^*, m^*) (x, t) \in C^1([0, T], C^2[L_1, L_2])$ to the initial-boundary value problems (18) and (19) satisfying
\[
0 \leq \delta(t, \epsilon) \leq \rho^*(x, t) \leq C, \quad |m^*(x, t)| \leq C\rho^*(x, t),
\] (38)

where $C$ is independent of $\epsilon$ and $T$.

Through Lemma 8 and the compensated compactness framework theory established in [19, 21–23], we can prove that there has a subsequence of $(\rho^*, m^*)$ (still denoted by
\((\rho^*, m^*)\), so that

\[
\left(\rho^*, m^*\right) \longrightarrow (\rho, m), \text{ in } L^p_{\text{loc}}([L_1, L_2] \times [0, T]) \quad (39)
\]

Furthermore, it is clear for us that \((\rho, m)\) is an entropy solution of initial-boundary value problems (1) and (3). We complete the proof of Theorem 3.

**4. Large Time Behavior of Weak Solutions**

This section is devoted to the proof of Theorem 5. Firstly, for stationary solution, from the result in [24], we have the following argument:

**Lemma 9.** Under the assumption (2) of \(b(x)\), there exists a unique solution \((\tilde{\rho}, \tilde{E})\) to problems (7) and (8) satisfying

\[
0 < b_0 \leq \tilde{b}(x) \leq b^*, \quad |\tilde{\rho}'(x)|, |\tilde{\rho}'(x)| \leq C, x \in [L_1, L_2],
\]

where \(C\) only depends on \(\gamma, b^*\) and \(b_0\).

Now, we shall derive that the entropy solution \((\rho, m, E)\) acquired in Theorem 3 converges strongly to the corresponding stationary solution \((\tilde{\rho}, \tilde{E})\) in the norm of \(L^2\) with exponential decay rate. From (7) and (8), we see that

\[
\int_{L_1}^{L_2} (\tilde{\rho}(x) - b(x)) dx = \int_{L_1}^{L_2} \tilde{E}_t dx + \tilde{E}(L_2) - \tilde{E}(L_1) = 0. \quad (41)
\]

Give the definition of the new function as follows

\[
y(x, t) = -\int_{L_1}^{x} (\rho(s, t) - \tilde{\rho}(s)) ds = -(E - \tilde{E}), \quad (x, t) \in [L_1, L_2] \times [0, \infty).
\]

Obviously, we observe that

\[
y_x = -(\rho - \tilde{\rho}), \quad y_1 = m, \quad y(L_1) = y(L_2). \quad (43)
\]

From (1) and (7), we have

\[
y_{tt} + \left(\frac{m^2}{\rho}\right)_x + (p(\rho) - p(\tilde{\rho}))_x - H(x, t)y_1 = -\tilde{p}y - \tilde{E}y_x + yy_x.
\]

Multiplying \(y\) with (44) and integrating from \(L_1\) to \(L_2\), we have

\[
\frac{d}{dt} \int_{L_1}^{L_2} \left( \frac{1}{2} y^2 - \frac{1}{2} y^2 \right) dx + \int_{L_1}^{L_2} \frac{1}{2} y^2 H_t dx
\]

\[
+ \int_{L_1}^{L_2} (\rho(\rho - P(\tilde{\rho}))(\rho - \tilde{\rho}) + \left(\rho - \tilde{E}\right) y^2) dx \leq \int_{L_1}^{L_2} \tilde{p} y^2 dx + \int_{L_1}^{L_2} \rho y^2 dx = \int_{L_1}^{L_2} \rho y^2 dx. \quad (45)
\]

Lemma 7 of [25] tells us there exist two nonnegative constants \(\tilde{C}_1\) and \(\tilde{C}_2\) such that

\[
\tilde{C}_2 (\rho - \tilde{\rho})^2 \geq (P(\rho) - P(\tilde{\rho}))(\rho - \tilde{\rho}) \geq \tilde{C}_1 (\rho - \tilde{\rho})^2 = \tilde{C}_1 y^2. \quad (46)
\]

Putting (46) into (45), we have

\[
\frac{d}{dt} \int_{L_1}^{L_2} \left( \frac{1}{2} y^2 - \frac{1}{2} y^2 \right) dx + \int_{L_1}^{L_2} \frac{1}{2} y^2 H_t dx
\]

\[
+ \tilde{C}_1 \int_{L_1}^{L_2} y^2 dx + \int_{L_1}^{L_2} b_2 y^2 dx \leq \int_{L_1}^{L_2} \tilde{p} y^2 dx. \quad (47)
\]

Additionally, denote the relative entropy-entropy flux by

\[
\eta_x = \eta_x - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1}(\rho - \tilde{\rho}),
\]

\[
q_x = q_x - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1} m. \quad (48)
\]

From the entropy inequality (16), we have the following inequality holds in the sense of distribution:

\[
\eta^*_x + q^*_x = \eta^*_x + q^*_x - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1}(\rho - \tilde{\rho}) - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1} m_x
\]

\[
\leq mE + \frac{H m^2}{\rho} - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1}(\rho - \tilde{\rho}) - \frac{p_0 \rho}{\rho - 1} \tilde{p} \tilde{\rho}^{-1} m_x
\]

\[
= mE + \frac{H m^2}{\rho} - p_0 \rho \tilde{p} y^2 y_1 \tilde{p}_x. \quad (49)
\]

We notice that

\[
mE = m \tilde{E} + m(E - \tilde{E}) = y \tilde{E} - yy_t,
\]

\[
= \frac{p(\tilde{\rho})}{\rho} y_t - yy_t = \rho \rho \tilde{p} y^2 y_1 \tilde{p}_x - yy_t,
\]

and use the theory of divergence-measure fields [17] to arrive at

\[
\frac{d}{dt} \int_{L_1}^{L_2} \left( \eta_x + \frac{1}{2} y^2 \right) dx - \int_{L_1}^{L_2} H y dx \leq 0. \quad (51)
\]

Let \(A\) sufficiently big so that \(A > b^*/\delta_0 + \|\rho\|_{L^p} + 1\).
Multiply (51) by $A$ and add the result to (47), we have
\[
\frac{d}{dt} \int_{L_1} \left( A\eta_* + \frac{\Lambda y^2}{2} + y y_i - \frac{y^2}{2} H \right) dx \\
+ \int_{L_1} \left( \frac{1}{2} H + b_\ast \right) y^2 + \frac{C_\lambda y_x^2 + \frac{-\Lambda H - \tilde{\rho}}{\rho} y^2}{\frac{\rho}{\partial}} \right) dx \leq 0.
\]
(52)

Since
\[
-\Lambda H - \tilde{\rho} > \left( b^* + \frac{\rho\|\|_{L^{\infty}}}{} \right) \delta_0 - b^* > \|\rho\|_{L^{\infty}} = \delta_0,
\]
(53)

then there exists $\tilde{C}_\lambda > 0$ such that
\[
\frac{d}{dt} \int_{L_1} \left( A\eta_* + \frac{\Lambda y^2}{2} + y y_i - \frac{y^2}{2} H \right) dx \\
+ \tilde{C}_\lambda \int_{L_1} \left( y^2 + y_x^2 + \frac{y^2}{\rho} \right) dx \leq 0.
\]
(54)

Since $\|\rho(x, t)\|_{L^{\infty}} \leq C$ and
\[
\eta_* \sim y^2 + \frac{y_x^2}{\rho},
\]
(55)

we can directly conclude that
\[
A\eta_* + \frac{\Lambda y^2}{2} + y y_i - \frac{y^2}{2} H \sim y^2 + y_x^2 + \frac{y^2}{\rho}.
\]
(56)

Now from (54), the Gronwall inequality implies Theorem 5.

Data Availability

This paper uses the method of theoretical analysis.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] P. Degond and P. A. Markowich, “On a one-dimensional steady-state hydrodynamic model for semiconductors,” *Applied Mathematics Letters*, vol. 3, no. 3, pp. 25–29, 1990.
[2] B. Zhang, “On a local existence theorem for a simplified one-dimensional hydrodynamic model for semiconductor devices,” *SIAM Journal on Mathematical Analysis*, vol. 25, no. 3, pp. 941–947, 1994.
[3] D. Wang and G. Q. Chen, “Formation of Singularities in compressible Euler–Poisson fluids with heat diffusion and damping relaxation,” *Journal of Differential Equations*, vol. 144, no. 1, pp. 44–65, 1998.
[4] B. Zhang, “Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices,” *Communications in Mathematical Physics*, vol. 157, no. 1, pp. 1–22, 1993.
[5] P. Marcati and R. Natalini, “Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem,” *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, vol. 125, no. 1, pp. 115–131, 1995.
[6] T. H. Li, “Convergence of the Lax-Friedrichs scheme for isothermal gas dynamics with semiconductor devices,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 57, no. 1, pp. 12–32, 2005.
[7] F. M. Huang, T. H. Li, and H. M. Yu, “Weak solutions to isothermal hydrodynamic model for semiconductor devices,” *Journal of Difference Equations*, vol. 247, no. 11, 3099 pages, 2009.
[8] W. Cao, F. Huang, T. Li, and H. Yu, “Global entropy solutions to an inhomogeneous isentropic compressible Euler system,” *Acta Mathematica Scientia*, vol. 36, no. 4, pp. 1215–1224, 2016.
[9] X. X. Fang and H. M. Yu, “Uniform boundedness in weak solutions to a specific dissipative system,” *Journal of Mathematical Analysis and Applications*, vol. 461, no. 2, pp. 1153–1164, 2018.
[10] H. M. Yu, “Large time behavior of entropy solutions to a unipolar hydrodynamic model of semiconductors,” *Communications in Mathematical Sciences*, vol. 14, no. 1, pp. 69–82, 2016.
[11] F. M. Huang, R. H. Pan, and H. M. Yu, “Large time behavior of Euler-Poisson system for semiconductor,” *Science in China Series A: Mathematics*, vol. 51, no. 5, pp. 965–972, 2008.
[12] H. X. Guo and H. M. Yu, “Existence and asymptotic behavior of smooth solutions to bipolar hydrodynamic model,” *Applicable Analysis*, vol. 97, no. 16, pp. 2880–2892, 2017.
[13] R. Guo and H. M. Yu, “Multi-dimensional bipolar hydrodynamic model of semiconductor with insulating boundary conditions and non-zero doping profile,” *Nonlinear Analysis: Real World Applications*, vol. 46, pp. 12–28, 2019.
[14] J. Li and H. M. Yu, “Large time behavior of solutions to a bipolar hydrodynamic model with big data and vacuum,” *Nonlinear Analysis: Real World Applications*, vol. 34, pp. 446–458, 2017.
[15] H. M. Yu and Y. L. Zhan, “Large time behavior of solutions to multi-dimensional bipolar hydrodynamic model of semiconductors with vacuum,” *Journal of Mathematical Analysis and Applications*, vol. 438, no. 2, pp. 856–874, 2016.
[16] H. M. Yu, “On the stationary solutions of multi-dimensional bipolar hydrodynamic model of semiconductors,” *Applied Mathematics Letters*, vol. 64, pp. 108–112, 2017.
[17] G. Q. Chen and H. Frid, “Divergence-measure fields and hyperbolic conservation laws,” *Archive for Rational Mechanics and Analysis*, vol. 147, no. 2, pp. 89–118, 1999.
[18] F. M. Huang, T. H. Li, H. M. Yu, and D. F. Yuan, “Large time behavior of entropy solutions to one-dimensional unipolar hydrodynamic model for semiconductor devices,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 69, no. 3, pp. 69–82, 2018.
[19] R. J. DiPerna, “Convergence of the viscosity method for isentropic gas dynamics,” *Communications in Mathematical Physics*, vol. 91, no. 1, pp. 1–30, 1983.
[20] F. M. Huang, T. H. Li, and D. F. Yuan, “Global entropy solutions to multi-dimensional isentropic gas dynamics with
spherical symmetry,” *Nonlinearity*, vol. 32, no. 11, pp. 4505–4523, 2019.

[21] X. Q. Ding, G. Q. Chen, and P. Z. Luo, “Convergence of the lax-friedrichs scheme for isentropic gas dynamics (II),” *Acta Mathematica Scientia*, vol. 5, no. 4, pp. 433–472, 1985.

[22] P. L. Lions, B. Perthame, and E. Tadmor, “Kinetic formulation of the isentropic gas dynamics and p-systems,” *Communications in Mathematical Physics*, vol. 163, no. 2, pp. 415–431, 1994.

[23] P. L. Lions, B. Perthame, and P. E. Souganidis, “Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates,” *Communications on Pure and Applied Mathematics*, vol. 49, no. 6, pp. 599–638, 1996.

[24] L. Hsiao and T. Yang, “Asymptotics of initial boundary value problems for hydrodynamic and drift diffusion models for semiconductors,” *Journal of Difference Equations*, vol. 170, no. 2, 493 pages, 2001.

[25] F. M. Huang and R. H. Pan, “Convergence rate for compressible Euler equations with damping and vacuum,” *Archive for Rational Mechanics and Analysis*, vol. 166, no. 4, pp. 359–376, 2003.