DIRICHLET-TYPE SPACES ON THE UNIT BALL AND JOINT 2-ISOMETRIES

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Abstract. We obtain a formula that relates the spherical moments of the multiplication tuple on a Dirichlet-type space to a complex moment problem in several variables. This can be seen as the ball-analogue of a formula originally invented by Richter in [23]. We capitalize on this formula to study Dirichlet-type spaces on the unit ball and joint 2-isometries.

1. Introduction

The investigations in this paper are motivated by the theory of Dirichlet-type spaces first introduced and studied by Richter [23] in the context of the wandering subspace problem and the model theory of cyclic analytic 2-isometries. The present work may be seen as an attempt to understand the spherical counterpart of the theories of Dirichlet-type spaces and 2-isometries. It is well-known that every analytic 2-isometry admits the wandering subspace property (see [22, Theorem 1] and [28, Theorem 3.6]). In case $d > 1$, there are analytic joint 2-isometric $d$-tuples without the wandering subspace property. Indeed, by [6, Example 6.8], for every non-zero $a$ in the open unit ball $B^d$ in $\mathbb{C}^d$, the restriction of the Drury-Arveson $d$-shift to the invariant subspace $\mathcal{M}_a := \{ f \in H^2_d : f(a) = 0 \}$ admits the wandering subspace property if and only if $d = 1$, and one may infer from [15, Theorem 4.2] that the restriction of the Drury-Arveson $d$-shift to $\mathcal{M}_a$ is a joint $d$-isometry. Further, in the context of model theory for joint 2-isometric $d$-tuples, there is a disparity in the cases $d = 1$ and $d > 1$. In fact, by [23, Theorem 5.1], the operator of the multiplication by the coordinate function on a Dirichlet-type space provides a canonical model for any cyclic analytic 2-isometry. In case of $d > 1$, there exist cyclic analytic joint 2-isometric $d$-tuples which cannot be modeled in this way (see Example 5.5; see also [19, Theorem 3.2]).

We also find it necessary to comment on possible variants of the Dirichlet-type spaces in several variables. It is well-known [20] that there are two possible analogues of Poisson kernel for the open unit ball $B^d$ in $\mathbb{C}^d$ provided $d \geq 2$. One of which is the invariant Poisson kernel $P_t(z, \zeta)$ given by

$$P_t(z, \zeta) = \frac{(1 - \|z\|^2)^d}{|1 - \langle z, \zeta \rangle|^{2d}}, \quad z \in B^d, \ \zeta \in \partial B^d.$$
One may associate with any finite positive Borel measure $\mu$ on the unit sphere $\partial B^d$ the invariant Poisson integral

$$P_{i}[\mu](z) = \int_{\partial B^d} P(z, \zeta)d\mu(\zeta), \quad z \in B^d,$$

and define the invariant Dirichlet-type space $D_{i}(\mu)$ as the Hilbert space of those functions $f$ in the Hardy space $H^2(B^d)$ for which the weighted $L^2$-norm of the gradient of $f$, with weight being $P_{i}[\mu]$ is finite. Unlike the case of $d = 1$, the invariant Poisson integral need not define a harmonic function (see [26, Remark 3.3.10]). In particular, the tuple $M_z$ of the multiplication by the coordinate functions on an invariant Dirichlet-type space need not be a joint 2-isometry (see Remark 2.6). Thus the verbatim analogue of [22, Theorem 3.7] fails for invariant Dirichlet-type spaces. Rather unexpectedly, the Poisson kernel $P(z, \zeta)$ associated with the Euclidean ball of $\mathbb{R}^{2d}$ yields Dirichlet-type spaces that support joint 2-isometries:

$$P(z, \zeta) = \frac{1 - \|z\|^2}{\|z - \zeta\|^{2d}}, \quad z \in B^d, \; \zeta \in \partial B^d.$$

Throughout this article, we need the following properties of the Poisson kernel (see [31, Proposition 3.1.12]):

- For every $\zeta \in \partial B^d$,
  
  $$\int_{B^d} P(z, \zeta)dV(z) = 1,$$

  where $V$ is the normalized volume measure on $B^d$.

- For every $z \in B^d$,
  
  $$\int_{\partial B^d} P(z, \zeta)d\sigma(\zeta) = 1,$$

  where $\sigma$ is the normalized surface area measure on $\partial B^d$.

- $P(z, \zeta)$ is symmetric in the following sense:
  
  $$P(r\eta, \zeta) = P(r\zeta, \eta), \quad \eta, \zeta \in \partial B^d, \; 0 \leq r < 1.$$

- For any neighbourhood $N_\zeta$ of $\zeta \in \partial B^d$,
  
  $$\lim_{r \to 1^-} P(r\eta, \zeta) = 0, \quad \eta \in \partial B^d \setminus N_\zeta.$$

It is worth noting that (1.1) may be deduced from (1.2) and (1.3) in view the polar coordinates (see [26, Pg 13]). The analysis of this paper relies heavily on the existence and uniqueness of the solution of the Dirichlet problem for the unit ball (see [31, Theorem 3.1.13]). Recall that $C(X)$ denotes the vector space of complex-valued continuous functions on a topological space $X$.

**Theorem 1.1.** For every $f \in C(\partial B^d)$, there is a unique $u \in C(\overline{B^d})$, so that $u$ is harmonic on $B^d$ and $u|\partial B^d = f$. Moreover,

$$u(z) = \int_{\partial B^d} P(z, \zeta)f(\zeta)d\sigma(\zeta), \quad z \in B^d.$$
As in the case of invariant Poisson kernel, one may associate with a finite positive Borel measure $\mu$ on $\partial B^d$ the Poisson integral

$$P[\mu](z) = \int_{\partial B^d} P(z, \zeta)d\mu(\zeta), \quad z \in B^d,$$

and form the Dirichlet-type space $\mathcal{D}(\mu)$ (see Section 3 for a precise definition in a more general setting). It turns out that $P[\mu]$ can be defined for any complex Borel measure $\mu$, and in this case, $P[\mu]$ is a harmonic function on $B^d$. The Riesz-Herglotz Theorem asserts that every positive harmonic function on the unit ball $B^d$ is the Poisson integral of a finite positive Borel measure on $\partial B^d$ (see, for instance, [3 Corollary 6.15]).

The investigations in this paper are motivated by the following questions:

**Question 1.2.** Assume that $d$ is a positive integer and let $\mu$ be a finite positive Borel measure on $\partial B^d$.

(a) Is $z_j$, $j = 1, \ldots, d$ a multiplier of the Dirichlet-type space $\mathcal{D}(\mu)$?

(b) If $z_j$, $j = 1, \ldots, d$ is a multiplier of $\mathcal{D}(\mu)$, is the $d$-tuple $\mathcal{M}_z$ of multiplication operators $\mathcal{M}_z, \ldots, \mathcal{M}_d$ a joint 2-isometry?

It is worth noting that in dimension $d = 1$, the answers to the above questions are affirmative [23 Theorems 3.6 and 3.7]. Moreover, these answers are intimately related to the following formula obtained in [23 Proof of Theorem 4.1] in case of $k = 1$ (see Lemma 2.2 for the deduction of the general case from this one).

**Theorem 1.3** (Richter’s formula). For any finite positive Borel measure $\mu$ on the unit circle $\partial \mathbb{D}$, we have

$$\int_{\mathbb{D}} (z^kp(z))(z^kq(z))P[\mu](z)dA(z)$$

$$= \int_{\mathbb{D}} p'(z)\overline{q'(z)}P[\mu](z)dA(z) + k \int_{\partial \mathbb{D}} p(\zeta)\overline{q(\zeta)}d\mu(\zeta), \quad p, q \in \mathbb{C}[z], \quad k \geq 1,$$

where $dA$ denotes the normalized area measure on the unit disc $\mathbb{D}$, $\mathbb{C}[z]$ is the complex vector space of polynomials in $z$, and $f'$ denotes the complex derivative of $f \in \mathbb{C}[z]$.

**Remark 1.4.** By the formula above, $\|z^kp\|^2_{\mathcal{D}(\mu)}$ is a linear polynomial in $k$ for every $p \in \mathbb{C}[z]$. Further, if $p$ is a polynomial and $p(z) = p(0) + zq(z)$ for some polynomial $q$, then by a couple of applications of Richter’s formula,

$$\frac{1}{2} \left( \int_{\mathbb{D}} |zp(z)|^2 P[\mu](z)dA(z) - \int_{\mathbb{D}} |p'(z)|^2 P[\mu](z)dA(z) \right)$$

$$\leq \int_{\mathbb{D}} |q(\zeta)|^2d\mu(\zeta) + |p(0)|^2\mu(\partial \mathbb{D})$$

$$= \int_{\mathbb{D}} |zq(z)|^2 P[\mu](z)dA(z) - \int_{\mathbb{D}} |q'(z)|^2 P[\mu](z)dA(z) + |p(0)|^2\mu(\partial \mathbb{D})$$

$$\leq \int_{\mathbb{D}} |p'(z)|^2 P[\mu](z)dA(z) + \|p\|^2_{H^2(\mathbb{D})}\mu(\partial \mathbb{D}).$$

Since the polynomials are dense in $\mathcal{D}(\mu)$ (see [23 Corollary 3.8(d)]), $z$ is a multiplier for $\mathcal{D}(\mu)$ and the operator $\mathcal{M}_z$ of the multiplication by coordinate function $z$ is a 2-isometry. This recovers [23 Theorems 3.6 and 3.7].
One of the main results of this paper is the ball analogue of Richter’s formula (see Theorem 2.1). Our method of proof, that exploits the Dirichlet problem for the unit ball, Green’s Theorem and an approximation result, differs from the one employed in [23]. This allows us to answer Question 1.2(b) in the affirmative (see Theorem 5.3). We also answer Question 1.2(a) for weighted surface area measures with bounded measurable weight functions (see Corollary 5.7). In the remaining half of this paper, we discuss the ball-analogue of the trigonometric moment problem (see Theorem 5.1 and Appendix), and employ it to characterize joint $m$-isometries that admit the wandering subspace property (see Theorem 5.3). These results are then combined with the theory of Dirichlet-type spaces developed in the first half to model spherical moments of joint 2-isometries (see Corollary 6.4).

2. Richter’s formula in several variables

Before we state the main result of this section, let us set some standard notations. Let $\mathbb{Z}_+$ denote the set of non-negative integers. For a positive integer $d$, the $d$-fold Cartesian product of $\mathbb{Z}_+$ is denoted by $\mathbb{Z}^d_+$. For $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d_+$, set $|p| = p_1 + \cdots + p_d$. If $q = (q_1, \ldots, q_d) \in \mathbb{Z}^d_+$, then we set $q! = \prod_{j=1}^d q_j!$. We write $p \leq q$ if $p_j \leq q_j$ for every $j = 1, \ldots, d$. Let $\mathbb{C}$ denote the set of complex numbers and let $\mathbb{C}^d$ denote the $d$-fold Cartesian product of $\mathbb{C}$. We reserve the notations $\mathcal{H}(z), \mathcal{P}$ and $|z|$ for the real part, the complex conjugate and the modulus of the complex number $z$, respectively. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and let $\mathbb{T}^d$ be the $d$-fold Cartesian product of $\mathbb{T}$. For $z = (z_1, \ldots, z_d), w = (w_1, \ldots, w_d) \in \mathbb{C}^d$, let $\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w}_j$ and $|z| = \langle z, z \rangle^{1/2}$. For $j = 1, \ldots, d$, let $\varepsilon_j$ denote the $d$-tuple in $\mathbb{C}^d$ with 1 in the $j$-th entry and zeros elsewhere. For $z \in \mathbb{C}^d$ and $\alpha \in \mathbb{Z}^d_+$, let $z^\alpha$ denote the complex number $\prod_{j=1}^d z_j^{\alpha_j}$. For $R > 0$, $\mathbb{B}^d$ denotes the open ball in $\mathbb{C}^d$ centered at the origin and of radius $R$. In case $R = 1$, we denote $\mathbb{B}^d$ simply by $\mathbb{B}$.

The vector space of complex polynomials in $z_1, \ldots, z_d$ is denoted by $\mathbb{C}[z_1, \ldots, z_d]$. By abuse of notation, the space $\mathbb{C}[z_1, \ldots, z_d]|_\Omega$ of restriction of complex polynomials to a subset $\Omega$ of $\mathbb{C}^d$ is also denoted by $\mathbb{C}[z_1, \ldots, z_d]$. For a set $A$, let $\ell^2(A)$ denote the Hilbert space of complex-valued functions on $A$, which are square integrable with respect to the counting measure. Let $M_+(\partial \mathbb{B}^d)$ denote the cone of finite positive Borel measures on the unit sphere $\partial \mathbb{B}^d$ in $\mathbb{C}^d$. Let $\nabla$ denote the gradient $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d})$ and $\Delta$ denotes the complex Laplacian $\sum_{j=1}^d \frac{\partial^2}{\partial z_j^2}$ in dimension $d$.

We now state the main result of this section.

Theorem 2.1. Let $\mu \in M_+(\partial \mathbb{B}^d)$. Then

$$\sum_{|\gamma| = k} \frac{\gamma!}{\gamma!} \int_{\mathbb{B}^d} \langle \nabla z^\gamma p(z), \nabla z^\gamma q(z) \rangle P[\mu](z) dV(z)$$

$$= \int_{\mathbb{B}^d} \langle \nabla p(z), \nabla q(z) \rangle P[\mu](z) dV(z) + kd \int_{\partial \mathbb{B}^d} p(\zeta) \overline{q(\zeta)} d\mu(\zeta),$$

$$p, q \in \mathbb{C}[z_1, \ldots, z_d], \ k \in \mathbb{Z}_+.$$
The proof of Theorem 2.1, as presented below, consists of several lemmas.

**Lemma 2.2.** Let \( \mu \in M_+(\partial B^d) \). Then (2.1) holds if and only if (2.1) holds for \( k = 1 \), that is,

\[
\sum_{j=1}^{d} \int_{B^d} \langle \nabla z^{\alpha+\epsilon_j}, \nabla z^{\beta+\epsilon_j} \rangle P[\mu](z) dV(z)
\]

(2.2)

\[
= \int_{B^d} \langle \nabla z^{\alpha}, \nabla z^{\beta} \rangle P[\mu](z) dV(z) + d \int_{\partial B^d} \zeta^{\alpha} \overline{\zeta}^{\beta} d\mu(\zeta), \quad \alpha, \beta \in \mathbb{Z}_+^d.
\]

**Proof.** The necessity part is clear. To see the sufficiency part, note that by the linearity of the integral, (2.2) implies (2.1) in case of \( k = 1 \). To see the general case, for fixed \( \alpha, \beta \in \mathbb{Z}_+^d \), let

\[
Q_k(z) = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \langle \nabla z^{\alpha+\gamma}, \nabla z^{\beta+\gamma} \rangle, \quad z \in B^d, \quad k \in \mathbb{Z}_+.
\]

Let \( k \) be a positive integer and note that

\[
Q_k - Q_0 = \sum_{j=1}^{k} \left( Q_j - Q_{j-1} \right)
\]

and

\[
Q_j(z) = \sum_{i_1, \ldots, i_j=1}^{d} \langle \nabla z^{\alpha+\epsilon_{i_1} + \cdots + \epsilon_{i_j}}, \nabla z^{\beta + \epsilon_{i_1} + \cdots + \epsilon_{i_j}} \rangle
\]

\[
= \sum_{l=1}^{d} \sum_{|\gamma|=j-1} \frac{|\gamma|!}{\gamma!} \langle \nabla z^{\alpha+\gamma+\epsilon_l}, \nabla z^{\beta+\gamma+\epsilon_l} \rangle, \quad z \in B^d, \quad j = 1, \ldots, k.
\]

Thus, by (2.2) and an application of the multinomial theorem,

\[
\int_{B^d} \left( Q_k(z) - Q_0(z) \right) P[\mu](z) dV(z)
\]

\[
= \int_{B^d} \sum_{j=1}^{k} \left( Q_j(z) - Q_{j-1}(z) \right) P[\mu](z) dV(z)
\]

\[
= d \sum_{j=1}^{k} \sum_{|\gamma|=j-1} \frac{|\gamma|!}{\gamma!} \int_{\partial B^d} \zeta^{\alpha+\gamma} \overline{\zeta}^{\beta+\gamma} d\mu(\zeta)
\]

\[
= d \sum_{j=1}^{k} \int_{\partial B^d} \left( \sum_{|\gamma|=j-1} \frac{|\gamma|!}{\gamma!} \zeta^{\alpha} \overline{\zeta}^{\beta} \right) \zeta^{\alpha} \overline{\zeta}^{\beta} d\mu(\zeta)
\]

\[
= kd \int_{\partial B^d} \zeta^{\alpha} \overline{\zeta}^{\beta} d\mu(\zeta).
\]

This completes the verification. \( \square \)

The following identity relates the gradient and the complex Laplacian.

**Lemma 2.3.** For holomorphic functions \( f, g : B^d \to \mathbb{C} \) and constant \( C \),

\[
\sum_{j=1}^{d} \langle \nabla z_j f, \nabla z_j g \rangle - C \langle \nabla f, \nabla g \rangle = \Delta ((|z|^2 - C) f g).
\]
Lemma 2.3 (with \(C\)). Recall the well-known fact that the ratio of the volume of \(\Delta\) that
\[\sum_{j=1}^{d} \langle \nabla z_j f, \nabla z_j g \rangle - C \langle \nabla f, \nabla g \rangle = \sum_{j=1}^{d} \Delta(z_j f \overline{z_j} g) - C \Delta(f g)\]
\[= \Delta((||z||^2 - C) f g),\]
where we used the linearity of \(\Delta\). \(\square\)

We next state a special case of Green’s Theorem required in the proof of Theorem 2.1.

**Lemma 2.4.** Let \(f, g\) be \(C^2\)-functions on \(\mathbb{B}^d\) so that \(f, g\) and \(D^\alpha f, D^\alpha g\) have continuous extensions to \(\overline{\mathbb{B}^d}\) for all multi-indices, \(\alpha\), with \(|\alpha| \leq 2\). Then
\[\int_{\mathbb{B}^d} \left( (\Delta f)(z)g(z) - (\Delta g)(z)f(z) \right) dV(z) = \frac{d}{2} \int_{\partial \mathbb{B}^d} \left( \frac{\partial f}{\partial r} r \zeta \right) g(\zeta) - \left( \frac{\partial g}{\partial r} r \zeta \right) f(\zeta) \right|_{r=1} d\sigma(\zeta). \tag{2.3}\]

**Proof.** Recall the well-known fact that the ratio of the volume of \(\mathbb{B}^d\) and the surface area of \(\partial \mathbb{B}^d\) equals \(1/2d\) (see [30, Pg 291]). This combined with the facts that the (Euclidean) Laplacian is 4 times the complex Laplacian \(\Delta\) and the outward pointing derivative is the radial derivative, the desired conclusion may be deduced from [31, Theorem 1.4.1] (where non-normalized volume measure and surface area measure have been used). \(\square\)

The following is the ball analogue of [24, Lemma 3.4].

**Proposition 2.5.** Let \(\mu\) be a complex Borel measure on \(\partial \mathbb{B}^d\) and \(0 \leq R < 1\). Then, for holomorphic functions \(f, g : \mathbb{B}^d \to \mathbb{C}\),
\[\int_{\mathbb{B}^d} \langle \nabla z_j f, \nabla z_j g \rangle P[\mu](z) dV(z) = \int_{\mathbb{B}^d} \langle \nabla f, \nabla g \rangle P[\mu](z) dV(z) - R^2 \int_{\mathbb{B}^d} \langle \nabla f, \nabla g \rangle P[\mu](z) dV(z)\]
\[= dR^{2d} \int_{\partial \mathbb{B}^d} f(R \zeta) \overline{g(R \zeta)} P[\mu](R \zeta) d\sigma(\zeta).\]

**Proof.** For \(h : \mathbb{B}^d \to \mathbb{C}\) and \(0 \leq R < 1\), let \(h_R(z) = h(Rz), z \in \mathbb{B}^d\), and note that \(\Delta h_R(z) = R^2(\Delta h)(Rz)\). Since \(P[\mu]\) is a harmonic function on \(\mathbb{B}^d\), by Lemma 2.4 (with \(C = R^2\)),
\[\int_{\mathbb{B}^d} \langle \nabla z_j f, \nabla z_j g \rangle P[\mu](z) dV(z) - R^2 \int_{\mathbb{B}^d} \langle \nabla f, \nabla g \rangle P[\mu](z) dV(z)\]
\[= \int_{\mathbb{B}^d} \Delta(||z||^2 - R^2) f(R \zeta) \overline{g(R \zeta)} P[\mu](R \zeta) dV(z)\]
\[= R^{2d} \int_{\mathbb{B}^d} \Delta(||w||^2 - 1) f(w) \overline{g_R(w)} P[\mu](Rw) dV(w)\]
\[\leq \frac{dR^{2d}}{2} \int_{\partial \mathbb{B}^d} \frac{\partial}{\partial r} \left( (r^2 - 1) f_R(r \zeta) \overline{g_R(r \zeta)} \right) \bigg|_{r=1} P[\mu](R \zeta) d\sigma(\zeta)\]
\[= dR^{2d} \int_{\partial \mathbb{B}^d} f_R(\zeta) \overline{g_R}(\zeta) P[\mu](R \zeta) d\sigma(\zeta).\]
This completes the proof.

Remark 2.6. Let $\mu \in M_+(\partial B^d)$. Then the invariant Poisson integral $P_\mu$ is an $\mathcal{M}$-harmonic function (see [26, Section 4.3]), which is not necessarily harmonic. The argument above shows that if Proposition 2.5 holds for $P_\mu$, then for $0 < R < 1$ and every holomorphic functions $f, g : B^d \to \mathbb{C}$,

$$\int_{B^d} \left( \|w\|^2 - 1 \right) f_R(w) \overline{g_R}(w) \Delta P_\mu(Rw) dV(w) = 0.$$ 

Now, it may be concluded from Stone-Weierstrass and Riesz representation theorems that Proposition 2.5 fails for the invariant Poisson integral $P_\mu$, unless $P_\mu$ is harmonic. Similarly, one may see that Richter’s formula fails for the invariant Poisson integral in general. Since the invariant Poisson kernel is naturally linked to the theory of $\mathcal{M}$-harmonic functions arising from the notion of the invariant Laplacian (refer to [26]), one may tempt to replace the gradient in the definition of invariant Dirichlet-type spaces by the invariant gradient. However, this does not yield a successful analogue of Dirichlet-type spaces even in dimension $d = 1$ in the sense that the associated multiplication operator is not a 2-isometry.

Proof of Theorem 2.1. In view of Lemma 2.2, it suffices to derive (2.2). For $0 < R < 1$, let $\mu_R$ be the weighted surface area measure with weight $w_R(\zeta) = P_\mu(R\zeta), \zeta \in \partial B^d$. Note that the proof of [26, Theorem 3.3.4(c)] (asserted for invariant Poisson kernel $P(z, \zeta)$) essentially relies on variants of (1.2), (1.3) and (1.4) for $P(z, \zeta)$. Hence [26, Theorem 3.3.4(c)] extends to the Euclidean Poisson kernel $P(z, \zeta)$ (with the same proof), and we may conclude that

$$\lim_{R \to 1^-} \mu_R = \mu \text{ in the weak*-topology of the dual of } C(\partial B^d). \quad (2.4)$$

Applying Proposition 2.5 to $f(z) = z^\alpha$ and $g(z) = z^\beta$ for $\alpha, \beta \in \mathbb{Z}^d_+$ together with the dominated convergence theorem (twice) and (2.4) yields (2.2). □

Here is a special case of Theorem 2.1 (the case in which $\mu \in M_+(\partial B^d)$ is the Dirac delta measure with point mass) of independent interest.

Corollary 2.7. For every $\zeta \in \partial B^d$, we have

$$\sum_{|\gamma|=k} \frac{\gamma!}{\gamma!} \int_{B^d} \langle \nabla z^\gamma p(z), \nabla z^\gamma q(z) \rangle P(z, \zeta) dV(z)$$

$$= \int_{B^d} \langle \nabla p(z), \nabla q(z) \rangle P(z, \zeta) dV(z) + kd p(\zeta) q(\zeta),$$

$p, q \in \mathcal{C}[z_1, \ldots, z_d], \quad k \in \mathbb{Z}^d_+.$

In the following section, we see the role of Richter’s formula (including Proposition 2.5) in the study of Dirichlet-type spaces. In particular, we answer major half of Question 1.2.

3. Joint $m$-isometries and Dirichlet-type spaces

Let $\mathcal{H}$ be a complex Hilbert space. Let $\langle \cdot, \cdot \rangle_\mathcal{H}$ and $\| \cdot \|_\mathcal{H}$ denote the inner-product and norm on $\mathcal{H}$, respectively. If no confusion occurs, we omit the subscript $\mathcal{H}$ from these notations. Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of
bounded linear operators on $\mathcal{H}$. By a commuting $d$-tuple $T$ on $\mathcal{H}$, we understand the $d$-tuple $(T_1, \ldots, T_d)$ consisting of commuting operators $T_1, \ldots, T_d$ in $\mathcal{B}(\mathcal{H})$. Given a commuting $d$-tuple $T$ on $\mathcal{H}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, let $T^\alpha$ denote the commuting $d$-tuple $(T_1^\alpha, \ldots, T_d^\alpha)$ and $T^\alpha$ denote the operator $T_1^\alpha \cdots T_d^\alpha$ in $\mathcal{B}(\mathcal{H})$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$ such that $T_j\mathcal{M} \subseteq \mathcal{M}$ for every $j = 1, \ldots, d$, then we use the notation $T|_{\mathcal{M}}$ to denote the commuting $d$-tuple $(T_1|_{\mathcal{M}}, \ldots, T_d|_{\mathcal{M}})$ on $\mathcal{M}$. With every commuting $d$-tuple $T$ on $\mathcal{H}$, we associate the positive map $Q_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given by

$$Q_T(X) := \sum_{j=1}^d T_j^* X T_j, \quad X \in \mathcal{B}(\mathcal{H}).$$

The operator $Q^n_T$ is inductively defined for all integers $n \geq 0$ through the relations $Q^n_1(X) = X$ and $Q^n_T(X) = Q_T(Q^{n-1}_T(X))$, $n \geq 1$, $X \in \mathcal{B}(\mathcal{H})$. It is easy to see that

$$Q^n_T(I) = \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} T^{\alpha\alpha} X^n, \quad n \in \mathbb{Z}_+. \quad (3.1)$$

For $m \in \mathbb{Z}_+$, let

$$B_m(T) := \sum_{n=0}^m (-1)^n \binom{m}{n} Q^n_T(I). \quad (3.2)$$

If $B_m(T) = 0$, then $T$ is said to be a joint $m$-isometry. In $d = 1$, this notion first appeared in [1], and in the general case, it was formally introduced in [15]. The reader is referred to [8, 13, 15, 9, 17, 12, 16, 19] for examples and basic properties of joint $m$-isometries. We refer to joint 1-isometry simply as joint isometry or spherical isometry. Following [28], we say that $T$ is joint $m$-concave (resp. joint $m$-convex) if $(-1)^m B_m(T) \preceq 0$ (resp. $(-1)^m B_m(T) \succeq 0$).

Before we define Dirichlet-type spaces, we discuss some examples of joint $m$-isometries arising from a family of Hardy-Besov spaces.

**Example 3.1.** For a real number $p > 0$, let

$$\kappa_p(z, w) = \frac{1}{(1 - \langle z, w \rangle)^p}, \quad z, w \in \mathbb{B}^d.\quad \text{Note that } \kappa_p \text{ defines a positive definite kernel on } \mathbb{B}^d, \text{ which is holomorphic in } z \text{ and conjugate holomorphic in } w. \text{ Thus we may associate a reproducing kernel Hilbert space } \mathcal{H}_p \text{ of holomorphic functions on the unit ball } \mathbb{B}^d \text{ with the reproducing kernel } \kappa_p \text{ (refer to [21] for this construction). Note that the monomials are orthogonal in } \mathcal{H}_p \text{ (see, for instance, [9, Lemma 2.14]). Indeed, for any } \alpha, \beta \in \mathbb{Z}_+^d, \text{ we have}

$$

$$\langle z^\alpha, z^\beta \rangle_{\mathcal{H}_p} = \begin{cases} \frac{\alpha! \Gamma(p)}{\Gamma(|\alpha|+p)} & \text{if } \alpha = \beta, \\ 0 & \text{otherwise}, \end{cases} \quad (3.3)$$

where $\Gamma$ denotes the gamma function (see [15, Equation (4.1)]). In particular, for every $p > 0$, $\mathcal{H}_p$ is contractively embedded into $\mathcal{H}_{p+1}$. The reproducing kernel Hilbert spaces $\mathcal{H}_p$ and $\mathcal{H}_1$ are the Hardy space $H^2(\mathbb{B}^d)$ and the Drury-Arveson space $H^2_F$, respectively. It may be deduced from (3.3) that...
z_j, j = 1, \ldots, d, is a multiplier for \mathcal{H}_p. Let \mathcal{M}_{z,p} denote the d-tuple of multiplication operators \mathcal{M}_{z_1}, \ldots, \mathcal{M}_{z_d} in \mathcal{B}(\mathcal{H}_p). It turns out that \mathcal{M}_{z,p} is a joint m-isometry if and only if p is a positive integer, p \leq d, and m \geq d - p + 1 (see [15, Theorem 4.2]; see also the discussion following [9, Theorem 5.3]). In particular, the multiplication tuple \mathcal{M}_{z,d} (Szego d-shift) turns out to be a joint isometry, while \mathcal{M}_{z,1} (Drury-Arveson d-shift) is a joint d-isometry. ■

Following [20] and considering the role of vector-valued Dirichlet-type spaces in the model theory for arbitrary 2-isometries, we have chosen to work in the vector-valued set-up.

For a complex separable Hilbert space \mathcal{M}, let \mathcal{M}_+(\partial \mathbb{B}^d, \mathcal{B}(\mathcal{M})) denote the cone of \mathcal{B}(\mathcal{M})-valued semispectral measures F on the unit sphere \partial \mathbb{B}^d in \mathbb{C}^d. In case \mathcal{M} = \mathbb{C}, following our earlier notation, we denote \mathcal{M}_+(\partial \mathbb{B}^d, \mathcal{B}(\mathcal{M})) simply by \mathcal{M}_+(\partial \mathbb{B}^d). Here by a \mathcal{B}(\mathcal{M})-valued semispectral measure F on \partial \mathbb{B}^d, we understand a set function on \partial \mathbb{B}^d which is finitely additive with values being positive operators in \mathcal{B}(\mathcal{M}) such that \langle F(\cdot)x, y \rangle defines a complex regular Borel measure on \partial \mathbb{B}^d for every x, y \in \mathcal{M}. For F \in \mathcal{M}_+(\partial \mathbb{B}^d, \mathcal{B}(\mathcal{M})), consider now the Poisson integral \text{P}[F] of F given by

\[ P[F](z) = \int_{\partial \mathbb{B}^d} P(z, \zeta)dF(\zeta), \quad z \in \mathbb{B}^d. \]

Let \text{H}_M^2(\mathbb{B}^d) denote the Hardy space of \mathcal{M}-valued holomorphic functions f(z) = \sum_{\alpha \in \mathbb{Z}_+^d} a_\alpha z^\alpha on \mathbb{B}^d endowed with the norm governed by

\[ \|f\|^2_{\text{H}_M^2(\mathbb{B}^d)} = \sum_{\alpha \in \mathbb{Z}_+^d} \|a_\alpha\|^2_{\mathcal{M}} \|z^\alpha\|^2_{\text{H}_M^2(\mathbb{B}^d)}, \]

where \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^d} \subseteq \mathcal{M}. The Dirichlet-type space \mathcal{Q}(F) associated with F is defined as the complex vector space of \mathcal{M}-valued holomorphic functions f : \mathbb{B}^d \to \mathcal{M} such that \|f\|_{\mathcal{Q}(F)} is finite, where

\[ \|f\|^2_{\mathcal{Q}(F)} := \|f\|^2_{\text{H}_M^2(\mathbb{B}^d)} + \frac{1}{d} \int_{\mathbb{B}^d} \sum_{j=1}^d \left\langle P[F](z) \frac{\partial f}{\partial z_j}, \frac{\partial f}{\partial z_j} \right\rangle_\mathcal{M} dV(z), \]

where, by abuse of notation, \frac{\partial f}{\partial z_j} evaluated at the point z is denoted by \frac{\partial f}{\partial z_j}.

**Remark 3.2.** We note the following:

(i) The complex vector space \mathcal{Q}(F) contains all \mathcal{M}-valued polynomials in z_1, \ldots, z_d, and it is a subspace of the Hardy space \text{H}_M^2(\mathbb{B}^d). If for j = 1, \ldots, d, \mathcal{M}_j denotes the linear operator of multiplication by the coordinate function z_j, then the d-tuple \mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_d) is defined and commuting on the space of \mathcal{M}-valued polynomials.

(ii) As in the one variable scalar case (refer to [11, Chapter 1]), it may be deduced from the Fatou’s Lemma and the reproducing property of \text{H}_M^2(\mathbb{B}^d) that \mathcal{Q}(F) is a reproducing kernel Hilbert space. Further, every z \in \mathbb{B}^d is a bounded point evaluation for \mathcal{Q}(F), and if \kappa : \mathbb{B}^d \times \mathbb{B}^d \to \mathcal{B}(\mathcal{M}) denotes the reproducing kernel for \mathcal{Q}(F), then

\[ \kappa(\cdot, 0) = I_{\mathcal{M}}, \]

where \kappa denotes the identity operator on \mathcal{M}. The latter one follows from \langle f, x \rangle_{\mathcal{Q}(F)} = \langle f, x \rangle_{\text{H}_M^2(\mathbb{B}^d)} = \langle f(0), x \rangle_\mathcal{M}, f, x \in \mathcal{Q}(F), x \in \mathcal{M}.
Proposition 3.5. Let \( \text{reader is referred to } [34] \) for the definition of Carleson measure. \( \text{Dirichlet-type spaces can be related to the notion of Carleson measure (the which is easily seen to be equal to zero.} \)

\[ \square \]

The following vector-valued analogue of Proposition 2.5 is immediate from its scalar counterpart.

Proposition 3.3. Let \( F \in M_+(\partial \mathbb{B}^d, \mathcal{M}) \) and \( 0 \leq R < 1 \). Then, for every \( \mathcal{M} \)-valued holomorphic functions \( f, g : \mathbb{B}^d \to \mathcal{M} \), we have

\[
\sum_{i,j=1}^{d} \int_{\partial \mathbb{B}^d} \left( P[F](z) \frac{\partial f}{\partial z_i} \cdot \frac{\partial g}{\partial z_j} \right) dV(z) - R^2 \sum_{j=1}^{d} \int_{\partial \mathbb{B}^d} \left( P[F](z) \frac{\partial f}{\partial z_j} \cdot \frac{\partial g}{\partial z_j} \right) dV(z)
\]

\[
= dR^2 \int_{\partial \mathbb{B}^d} \langle P[F](R\zeta), f(R\zeta), g(R\zeta) \rangle d\sigma(\zeta).
\]

Proof. This follows from the scalar case (see Proposition 2.5) applied to the complex measures \( \langle F(\cdot)x, y \rangle, x, y \in \mathcal{M} \).

We can now answer Question 1.2(b).

Theorem 3.4. Let \( F \in M_+(\partial \mathbb{B}^d, \mathcal{B}(\mathcal{M})) \). Assume that for \( k = 1, \ldots, d \), \( z_k \) is a multiplier for \( \mathcal{D}(F) \). Then the \( d \)-tuple \( \mathcal{M}_z \) on \( \mathcal{D}(F) \) is a joint 2-isometry.

Proof. Let \( f \in \mathcal{D}(F) \). Thus, \( z_j f \in \mathcal{D}(F) \), \( j = 1, \ldots, d \), and hence by Proposition 3.3 and the dominated convergence theorem,

\[
\sum_{j=1}^{d} \|z_j f\|^2_{\mathcal{D}(F)} \leq \|f\|^2_{\mathcal{D}(F)} = \lim_{R \to 1^-} \int_{\partial \mathbb{B}^d} \langle P[F](R\zeta), f(R\zeta), g(R\zeta) \rangle d\sigma(\zeta).
\]

It follows that

\[
\|f\|^2_{\mathcal{D}(F)} - 2 \sum_{j=1}^{d} \|z_j f\|^2_{\mathcal{D}(F)} + \sum_{j,k=1}^{d} \|z_j z_k f\|^2_{\mathcal{D}(F)}
\]

\[
= \left( \|f\|^2_{\mathcal{D}(F)} - \sum_{j=1}^{d} \|z_j f\|^2_{\mathcal{D}(F)} \right) - \sum_{j=1}^{d} \left( \|z_j f\|^2_{\mathcal{D}(F)} - \sum_{k=1}^{d} \|z_k (z_j f)\|^2_{\mathcal{D}(F)} \right)
\]

\[
= \lim_{R \to 1^-} \int_{\partial \mathbb{B}^d} \langle P[F](R\zeta), f(R\zeta), g(R\zeta) \rangle d\sigma(\zeta)
\]

\[
- \sum_{j=1}^{d} \lim_{R \to 1^-} \int_{\partial \mathbb{B}^d} R^2 \langle P[F](R\zeta)\zeta_j f(R\zeta), \zeta_j f(R\zeta) \rangle d\sigma(\zeta),
\]

which is easily seen to be equal to zero. \( \square \)

The question of when the coordinate functions are multipliers for the Dirichlet-type spaces can be related to the notion of Carleson measure (the reader is referred to [34] for the definition of Carleson measure).

Proposition 3.5. Let \( F \in M_+(\partial \mathbb{B}^d, \mathcal{B}(\mathcal{M})) \). Then the following statements are equivalent:

(iii) Unlike the one-dimensional case, there are various possibilities for the definition of a Dirichlet-type space in general. Indeed, in the definition of \( \mathcal{D}(F) \), one may replace \( H^2_\mathcal{M}(\mathbb{B}^d) \) by any Hilbert space \( \mathcal{H} \) of \( \mathcal{M} \)-valued holomorphic functions for which the \( d \)-tuple \( \mathcal{M}_z \) of multiplication operators on \( \mathcal{H} \) is a joint isometry (cf. [27, Equation (1.4)]). However, we do not take up this notion here.
(i) For \( k = 1, \ldots, d \), \( z_k \) is a multiplier for \( \mathcal{D}(F) \).

(ii) There exists a positive constant \( C \) such that

\[
\int_{\mathbb{B}^d} \left\langle P[F](z)f(z), f(z) \right\rangle_{\mathcal{M}} dV(z) \leq C \| f \|_{\mathcal{D}(F)}^2, \quad f \in \mathcal{D}(F).
\]

(iii) There exists a positive constant \( \tilde{C} \) such that

\[
\sup_{0 < R < 1} \int_{\partial \mathbb{B}^d} \left\langle P[F](R \zeta)f(R \zeta), f(R \zeta) \right\rangle d\sigma(\zeta) \leq \tilde{C} \| f \|_{\mathcal{D}(F)}^2, \quad f \in \mathcal{D}(F).
\]

Proof. Let \( \text{Hol}(\mathbb{B}^d, \mathcal{M}) \) denote the vector space of \( \mathcal{M} \)-valued holomorphic functions on \( \mathbb{B}^d \). For \( f \in \text{Hol}(\mathbb{B}^d, \mathcal{M}) \), define

\[
\| f \|_*^2 = \int_{\mathbb{B}^d} \left\langle P[F](z)f(z), f(z) \right\rangle_{\mathcal{M}} dV(z),
\]

and note that \( \| \cdot \|_* \) defines (possibly an extended real-valued) semi-norm on \( \text{Hol}(\mathbb{B}^d, \mathcal{M}) \). In particular, for \( g, h \in \text{Hol}(\mathbb{B}^d, \mathcal{M}) \),

\[
\text{if } \| g \|_* < \infty, \text{ then } \| g + h \|_* < \infty \text{ if and only if } \| h \|_* < \infty. \quad (3.5)
\]

Note that \( f \in \mathcal{D}(F) \) if and only if \( f \in H^2_{\mathcal{M}}(\mathbb{B}^d) \) and \( \| \partial f / \partial z_j \|_* < \infty \) for every \( j = 1, \ldots, d \). This yields in particular that for any \( f \in \mathcal{D}(F) \),

\[
\left\| z_k \frac{\partial f}{\partial z_j} \right\|_* < \infty, \quad j, k = 1, \ldots, d. \quad (3.6)
\]

Further, for any \( k = 1, \ldots, d \), we have

\[
\int_{\mathbb{B}^d} \sum_{j=1}^{d} \left\langle P[F](z)\frac{\partial(z_k f)}{\partial z_j}, \frac{\partial(z_k f)}{\partial z_j} \right\rangle_{\mathcal{M}} dV(z) = \sum_{j=1}^{d} \left\| z_k \frac{\partial f}{\partial z_j} \right\|_*^2 + \left\| z_k \frac{\partial f}{\partial z_k} + f \right\|_*^2.
\]

It now follows from (3.5) and (3.6) that for any \( f \in \mathcal{D}(F) \),

\[
z_k f \in \mathcal{D}(F) \text{ for every } k = 1, \ldots, d \text{ if and only if } \| f \|_* < \infty. \quad (3.7)
\]

This yields (ii) \( \Rightarrow \) (i).

To see (i) \( \Rightarrow \) (ii), note first that by Remark 5.2(i), \( \mathcal{D}(F) \subseteq H^2_{\mathcal{M}}(\mathbb{B}^d) \), and hence we may consider the normed linear space \( \mathcal{H} = \mathcal{D}(F) \) endowed with the norm max\( \{ \| \cdot \|_*, \| \cdot \|_{H^2_{\mathcal{M}}(\mathbb{B}^d)} \} \). In view of (3.7), the map \( f \mapsto f \) from \( \mathcal{D}(F) \) into the completion of \( \mathcal{H} \) is well-defined, and hence one may apply the closed graph theorem to obtain (ii).

The implication (i) \( \Rightarrow \) (iii) is immediate from (3.6). The other implication follows from Proposition 3.3 and the monotone convergence theorem. \( \square \)

Remark 3.6. In case of \( d = 1 \), (i) above always holds (see [23] Theorem 3.6) for \( \dim \mathcal{M} = 1 \) and [20] Theorem 3.1 for the general case. In case \( \mathcal{M} = \mathbb{C} \), the condition (ii) above says that the Dirichlet space \( \mathcal{D}(F) \) is boundedly embedded into the weighted Bergman space with weight being \( P[F] \). In turn, this is equivalent to the assertion that the weighted volume measure with weight being \( P[F] \) is a Carleson measure for \( \mathcal{D}(F) \) (see [34] Theorem 2) for a characterization of Carleson measures for Besov-Sobolev spaces on \( \mathbb{B}^d \).
The following provides a partial answer to Question 1.2(a).

**Corollary 3.7.** For a bounded measurable function \( w : \partial \mathbb{B}^d \to [0, \infty) \), consider the weighted surface area measure \( \mu_w \) with weight function \( w \). Then the \( d \)-tuple \( \mathcal{M}_w \) on \( \mathcal{D}(\mu_w) \) is a joint 2-isometry.

**Proof.** In view of Proposition 3.5 and Theorem 3.4, it suffices to check that there exists a positive constant \( C_w \) such that

\[
\int_{\mathbb{B}^d} |f(z)|^2 P[\mu_w](z) dV(z) \leq C_w \|f\|^2_{\mathcal{D}(\mu_w)}, \quad f \in \mathcal{D}(\mu_w).
\]

Since \( \mathcal{D}(\mu_w) \subseteq H^2(\mathbb{B}^d) = \mathcal{M}_d \) and \( \mathcal{M}_d \) is contractively embedded into \( \mathcal{M}_{d+1} \) (see Example 3.1 and Remark 3.2(i)) and \( P[\mu_w] \) is bounded above by \( \sup w \), we obtain the desired inequality with \( C_w = \sup w \). \( \square \)

We do not know whether the weighted volume measure with weight being the Poisson integral of an arbitrary finite positive Borel measure \( \mu \) on \( \partial \mathbb{B}^d \), \( d \geq 2 \) is a Carleson measure for \( \mathcal{D}(\mu) \).

### 4. Rotation invariant and pluriharmonic measures

In this section, we study Dirichlet-type spaces arising from two families of positive finite Borel measures on the unit sphere. Before we discuss the case of rotation invariant measures, we recall the notion of \( \mathbb{T}^d \)-invariance.

A subset \( \Omega \) of \( \mathbb{C}^d \) is \( \mathbb{T}^d \)-invariant if for every \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d \),

\[
e^{i\theta} \cdot \Omega := \{(e^{i\theta_1}z_1, \ldots, e^{i\theta_d}z_d) : (z_1, \ldots, z_d) \in \Omega \} \subseteq \Omega.
\]

A positive finite Borel measure \( \mu \) on a \( \mathbb{T}^d \)-invariant subset \( \Omega \) of \( \mathbb{C}^d \) is said to be \( \mathbb{T}^d \)-invariant if for every Borel subset \( A \) of \( \Omega \),

\[
\mu(e^{i\theta} \cdot A) = \mu(A), \quad \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d.
\]

**Lemma 4.1.** Let \( \mu \in M_+ (\partial \mathbb{B}^d) \). Then \( \mu \) is \( \mathbb{T}^d \)-invariant if and only if the monomials are orthogonal in \( \mathcal{D}(\mu) \).

**Proof.** By Theorem 2.1 the orthogonality of monomials in \( \mathcal{D}(\mu) \) implies the orthogonality of monomials in the space \( L^2(\mu) \) of square-integrable functions on \( \partial \mathbb{B}^d \), and hence by [8] Lemma 2.3, \( \mu \) is \( \mathbb{T}^d \)-invariant. Conversely, if \( \mu \) is \( \mathbb{T}^d \)-invariant, then a routine verification shows that \( P[\mu]dV \) is \( \mathbb{T}^d \)-invariant, and hence the monomials are orthogonal in \( \mathcal{D}(\mu) \). \( \square \)

To show that Dirichlet-type spaces associated with rotation-invariant measures support joint 2-isometries, we need another lemma.

**Lemma 4.2.** Let \( \mu \in M_+ (\partial \mathbb{B}^d) \) and let \( \mathcal{D}(\mu) \) be the associated Dirichlet-type space. For \( f \in \mathcal{D}(\mu) \), set

\[
\|f\|^2 := |f(0)|^2 + \frac{1}{d} \int_{\mathbb{B}^d} \|\nabla f(z)\|^2 P[\mu](z) dV(z).
\]

Then we have

\[
\sum_{k=1}^d \|z^{\alpha + e_k}\|_\infty^2 \leq \max \left\{ 2(1 + d), \frac{\mu(\partial \mathbb{B}^d)}{d} \right\} \|z^\alpha\|_\infty^2, \quad \alpha \in \mathbb{Z}_+^d. \quad (4.1)
\]
Proof. By the mean value property for harmonic functions (see, for instance, [31, (3.1.13)]), we have

$$\sum_{k=1}^{d} \|z_k\|^2 = \frac{1}{d} \int_{\mathbb{B}^d} P[\mu](z) dV(z) = \frac{P[\mu][0]}{d} = \frac{\mu(\partial \mathbb{B}^d)}{d}. \quad (4.2)$$

Let $\alpha \in \mathbb{Z}^d_+ \setminus \{0\}$ and note that

$$|z^\alpha|^2 \leq \sum_{j=1}^{d} \alpha_j^2 |z^{\alpha-\varepsilon_j}|^2, \quad z \in \mathbb{B}^d. \quad (4.3)$$

Let $\delta_{jk}$ denote the Kronecker delta function, and note that

$$\sum_{k=1}^{d} \|z^{\alpha+\varepsilon_k}\|_o^2 = \frac{1}{d} \sum_{k=1}^{d} \int_{\mathbb{B}^d} \sum_{j=1}^{d} (\alpha_j + \delta_{jk})^2 |z^{\alpha-\varepsilon_j+\varepsilon_k}|^2 P[\mu](z) dV(z)
 \leq \frac{2}{d} \sum_{k=1}^{d} \sum_{j=1}^{d} (\alpha_j^2 + \delta_{jk}) |z^{\alpha-\varepsilon_j+\varepsilon_k}|^2 P[\mu](z) dV(z)
 = \frac{2}{d} \int_{\mathbb{B}^d} |z|^2 \sum_{j=1}^{d} \alpha_j^2 |z^{\alpha-\varepsilon_j}|^2 P[\mu](z) dV(z)
 + 2 \int_{\mathbb{B}^d} |z|^2 P[\mu](z) dV(z)
 \leq 2(1 + d) \|z^\alpha\|_o^2,$$

where we used the estimate (4.3) in the last step. Combining this with (4.2), we obtain (4.1). □

The following lists several properties of Dirichlet-type spaces associated with $T^d$-invariant measures.

**Proposition 4.3.** Let $\mu \in M_+(\partial \mathbb{B}^d)$ and let $\mathcal{D}(\mu)$ be the associated Dirichlet-type space. If $\mu$ is $T^d$-invariant, then the following statements are true:

(i) the monomials $z^\alpha$, $\alpha \in \mathbb{Z}^d_+$ form an orthogonal basis for $\mathcal{D}(\mu)$,
(ii) for every $j = 1, \ldots, d$, $z_j$ is a multiplier for $\mathcal{D}(\mu)$,
(iii) the $d$-tuple $\mathcal{M}_z$ on $\mathcal{D}(\mu)$ is a joint $2$-isometry,
(iv) if $\langle f \circ U, g \circ U \rangle_{\mathcal{D}(\mu)} = \langle f, g \rangle_{\mathcal{D}(\mu)}$ for every unitary $d \times d$ matrix $U$ (considered as a function from $\mathbb{B}^d$ onto $\mathbb{B}^d$), then $\mu$, up to a scalar multiple, is the surface area measure $\sigma$.

**Proof.** Assume now that $\mu$ is $T^d$-invariant. Let $f(z) = \sum_{\alpha \in \mathbb{Z}^d_+} \hat{f}(\alpha) z^\alpha \in \mathcal{D}(\mu)$ for some $\hat{f}(\alpha) \in \mathbb{C}$ and let $0 < R < 1$. Since $f$ is uniformly convergent
on $R \mathbb{B}^d$, by Lemma 4.1
\[
\int_{R \mathbb{B}^d} \|\nabla f(z)\|^2 P[\mu](z) dV(z)
= \int_{R \mathbb{B}^d} \sum_{j=1}^{d} \alpha_j \beta_j \sum_{\alpha, \beta \in \mathbb{Z}_+^d} \hat{f}(\alpha) \hat{f}(\beta) z^{(\alpha-\epsilon_j) \cdot (\beta-\epsilon_j)} P[\mu](z) dV(z)
= \sum_{\alpha \in \mathbb{Z}_+^d} |\hat{f}(\alpha)|^2 \int_{R \mathbb{B}^d} \sum_{j=1}^{d} \alpha_j^2 |z^{\alpha-j}|^2 P[\mu](z) dV(z)
= \sum_{\alpha \in \mathbb{Z}_+^d} |\hat{f}(\alpha)|^2 \int_{R \mathbb{B}^d} \|\nabla z^{\alpha}\|^2 P[\mu](z) dV(z).
\]
Letting $R \to 1^-$ on both sides, we obtain
\[
\|f\|_{\mathcal{D}(\mu)}^2 = \sum_{\alpha \in \mathbb{Z}_+^d} |\hat{f}(\alpha)|^2 \|z^{\alpha}\|_{\mathcal{D}(\mu)}^2.
\] (4.4)

In particular, the sequence of partial sums $\sum_{|\alpha| \leq n} \hat{f}(\alpha) z^\alpha$ of $f$ converges to $f$ in $\mathcal{D}(\mu)$. Further, (4.4) combined with (1.2) yields that each $z_j, j = 1, \ldots, d$ is a multiplier for $\mathcal{D}(\mu)$, and hence the multiplication operator $\mathcal{M}_z$ is bounded on $\mathcal{D}(\mu)$. Thus, by Theorem 3.4, $\mathcal{M}_z$ on $\mathcal{D}(\mu)$ is a joint 2-isometry. Finally, if $(f \circ U, g \circ U) \mathcal{D}(\mu) = (f, g) \mathcal{D}(\mu)$ for every unitary $d \times d$ matrix $U$, then by Theorem 2.1
\[
\int_{\mathbb{B}^d} p \circ U(z) q \circ U(z) d\mu(z) = \int_{\mathbb{B}^d} p(z) q(z) d\mu(z), \quad p, q \in \mathbb{C}[z_1, \ldots, z_d].
\]
By Stone-Weierstrass and Riesz representation theorems, $\mu$ is invariant under the action of the unitary group. Hence, by [26, Remark, Pg 16], $\mu$ is a scalar multiple of $\sigma$.

We now exhibit a family of rotation invariant measures for which the Dirichlet-type spaces and the associated multiplication tuples can be described explicitly (it is worth noting that, up to a scalar multiple, the only rotation invariant measure on the unit circle is the arc-length measure).

**Example 4.4.** Let $\lambda \in \mathbb{R}$ and $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ be such that $\lambda > \max_{j=1}^{d} |c_j|$ and $\sum_{j=1}^{d} c_j = 0$. Consider the weighted surface area measure $\mu_{\lambda, c}$ on $\partial \mathbb{B}^d$ with weight function $w_{\lambda, c}$ given by
\[
w_{\lambda, c}(z) := \lambda + \sum_{j=1}^{d} c_j |z^j|^2, \quad z \in \mathbb{C}^d.
\]
Clearly, $w_{\lambda, c}|_{\partial \mathbb{B}^d}$ is positive and $\mu_{\lambda, c}$ is $\mathbb{T}^d$-invariant. Consider the Dirichlet-type space $\mathcal{D}(\mu_{\lambda, c})$ associated with $\mu_{\lambda, c}$. It follows from Proposition 4.3 that $\{e_\alpha := z^\alpha / \|z^\alpha\|_{\mathcal{D}(\mu_{\lambda, c})} : \alpha \in \mathbb{Z}_+^d\}$ is an orthonormal basis for $\mathcal{D}(\mu_{\lambda, c})$ and the $d$-tuple $\mathcal{M}_z$ on $\mathcal{D}(\mu_{\lambda, c})$ defines a joint 2-isometry. Moreover, $\mathcal{M}_z$ is a weighted multishift for some weights $w = \{w^{(j)}_\alpha : \alpha \in \mathbb{Z}_+^d, j = 1, \ldots, d\} \subseteq (0, \infty)$, that is,
\[
\mathcal{M}_z e_\alpha = w^{(j)}_\alpha e_{\alpha+\epsilon_j}, \quad \alpha \in \mathbb{Z}_+^d, j = 1, \ldots, d.
\]
To compute the weights $w$ of $\mathcal{M}_z$, note first that $w_{\lambda,c}$ defines a harmonic function on $\mathbb{C}^d$, and hence by the uniqueness part of Theorem 1.1, $P[\mu_{\lambda,c}] = w_{\lambda,c}$ on $\mathbb{B}^d$. It now follows from (3.3) and the assumption $\sum_{j=1}^{d} c_k = 0$ that

$$
\|z^\alpha\|^2_{\mathcal{D}(\mu_{\lambda,c})} = \|z^\alpha\|^2_{\mathcal{H}(\mathbb{B}^d)} + \frac{1}{d} \int_{\mathbb{B}^d} \|\nabla z^\alpha\|^2 w_{\lambda,c}(z) dV(z)
$$

Furthermore, by Proposition 4.5, $(\mu_{\lambda,c} = w_{\lambda,c})$ is a joint 2-isometry. Finally, we note that in case $d = 2$, the choices $\lambda = 1$ and $c = 0$, the Dirichlet-type space $\mathcal{D}(\mu_{\lambda,c})$ is nothing but the Drury-Arveson space $H^2_2$.

Following [2, Section 1], we say that a measure $\mu \in M_+(\partial \mathbb{B}^d)$ is pluriharmonic if $P[\mu]$ is a pluriharmonic function in the open unit ball $\mathbb{B}^d$. We now exhibit Dirichlet-type spaces associated with a family of pluriharmonic measures (see [25, Pg 44]).

**Proposition 4.5.** For $h$ belonging to the ball algebra $A(\mathbb{B}^d)$ such that $w(z) := \Re(h(z)) \geq 0$ for every $z \in \partial \mathbb{B}^d$, let $\mu_w$ be the weighted surface area measure with weight function $w|_{\partial \mathbb{B}^d}$. Then the $d$-tuple $\mathcal{M}_z$ on $\mathcal{D}(\mu_w)$ defines a joint 2-isometry.

**Proof.** This is immediate from Corollary 3.7

We exhibit below Dirichlet-type spaces associated with a family of pluriharmonic measures, which are not necessarily rotation-invariant.

**Example 4.6.** Let $\lambda \in \mathbb{R}$ and $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$ be such that $\lambda^2 > 2 \sum_{j=1}^{d} |b_j|^2$. Consider the weighted surface area measure $\mu_{b,\lambda}$ on $\partial \mathbb{B}^d$ with weight function $w_{b,\lambda}$ given by

$$
w_{b,\lambda}(z) := \lambda + 2 \sum_{j=1}^{d} \Re(b_j z_j), \quad z \in \mathbb{C}^d.
$$

Since $w_{b,\lambda}$ is positive on $\partial \mathbb{B}^d$, we may consider the Dirichlet-type space $\mathcal{D}(\mu_{b,\lambda})$ associated with $\mu_{b,\lambda}$. Further, since $\mu_{b,\lambda}$ is a pluriharmonic measure, by Proposition 4.5 the $d$-tuple $\mathcal{M}_z$ on $\mathcal{D}(\mu_{b,\lambda})$ is a joint 2-isometry.
We claim that
\[
\langle z^\alpha, z^\beta \rangle = \begin{cases} 
\frac{\lambda |\alpha|!(d-1)!}{(|\alpha|+d-1)!} b_l & \text{if } \alpha = \beta, \\
\frac{|\alpha+(\epsilon_l)!(d-1)!}{(|\alpha|+d)!} b_l & \text{if } \alpha + \epsilon_l = \beta, \ l = 1, \ldots, d, \\
\frac{|\alpha-1|!(d-1)!}{(|\alpha|+d-1)!} b_l & \text{if } \beta + \epsilon_l = \alpha, \ l = 1, \ldots, d, \\
0 & \text{otherwise.}
\end{cases}
\] (4.5)

To see this, let \(\alpha, \beta \in \mathbb{Z}^d_+\) and let \(\delta_{\alpha, \beta}\) denote the Kronecker delta function. Note that for \(l = 1, \ldots, d\), by (3.3),
\[
\int_{B^d} \langle \nabla z^\alpha, \nabla z^\beta \rangle z_l dV(z) = \delta_{\alpha, \beta + \epsilon_l} d! (|\alpha| + d - 1)! \sum_{k=1}^d \alpha_k \beta_k (\alpha + \epsilon_l - \epsilon_k)!
\]
\[
= \delta_{\alpha, \beta + \epsilon_l} \frac{|\alpha|!(\alpha + \epsilon_l)! d!}{(|\alpha| + d)!}.
\]

Similarly, one may see that
\[
\int_{B^d} \langle \nabla z^\alpha, \nabla z^\beta \rangle z_l dV(z) = \delta_{\alpha, \beta} d! (|\alpha| - 1)! \sum_{k=1}^d \alpha_k \beta_k (\alpha - \epsilon_l - \epsilon_k)!
\]
\[
= \delta_{\alpha, \beta} \frac{(|\alpha| - 1)! d!}{(|\alpha| + d - 1)!}.
\]
Combining these two identities with (3.3), we obtain (4.5). It follows that \(M_z\) is a joint 2-isometry that is not a weighted multishift, unless \(b = 0\).

5. A complex moment problem and a Gramian matrix associated with joint m-isometries

In the first half of this section, we discuss a complex moment problem in several variables that arises naturally in the study of joint m-isometries. We have already encountered such a problem in Theorem 2.1. The solution to this moment problem can be considered as a spherical analog of the solution of the trigonometric moment problem. In the scalar one-dimensional case, this is commonly attributed to Akhiezer and Krein (see [29, Theorem 1.4]). In the general case, this can be derived from [33, Proposition 34] and [32, Proposition 2.1]. Although this result has been known to experts in the moment theory (refer to [14, 32, 33, 10]), we could not locate it in the form we need in this paper. Hence we include the statement and relegate its proof to the appendix.

**Theorem 5.1.** For a complex Hilbert space \(\mathcal{M}\), let \(\phi : \mathbb{Z}^d_+ \times \mathbb{Z}^d_+ \rightarrow \mathcal{B}(\mathcal{M})\) be a \(\mathcal{B}(\mathcal{M})\)-valued kernel function. Then the following are equivalent:

(i) There exists \(F \in M_+(\partial B^d, \mathcal{B}(\mathcal{M}))\) such that
\[
\phi(\alpha, \beta) = \int_{\partial B^d} \zeta^\alpha \overline{\zeta}^\beta dF(\zeta), \ \alpha, \beta \in \mathbb{Z}^d_+. \tag{5.1}
\]

(ii) The kernel function \(\phi\) is positive definite and
\[
\sum_{j=1}^d \phi(\alpha + \epsilon_j, \beta + \epsilon_j) = \phi(\alpha, \beta), \ \alpha, \beta \in \mathbb{Z}^d_+. \tag{5.2}
\]

The semispectral measure \(F\) is uniquely determined by (5.1).

Here is an application of Theorem 5.1 to joint m-isometries.
Corollary 5.2. Let $T$ be a joint $m$-isometric $d$-tuple on $\mathcal{H}$. Then there exists $F \in M_+(\partial B^d, \mathcal{B}(\mathcal{H}))$ such that

$$
\sum_{j=0}^{m-1} (-1)^{j+m-1} \binom{m-1}{j} T^{\ast \beta} Q_T^j(I) T^\alpha = \int_{\partial B^d} \zeta^\alpha \bar{\zeta}^\beta dF(\zeta), \quad \alpha, \beta \in \mathbb{Z}_+^d.
$$

Proof. Define a $\mathcal{B}(\mathcal{H})$-valued kernel function $\phi : \mathbb{Z}_+^d \times \mathbb{Z}_+^d \to \mathcal{B}(\mathcal{H})$ by

$$
\phi(\alpha, \beta) = \sum_{j=0}^{m-1} (-1)^{j+m-1} \binom{m-1}{j} T^{\ast \beta} Q_T^j(I) T^\alpha, \quad \alpha, \beta \in \mathbb{Z}_+^d.
$$

By [15] Proposition 2.3, $(-1)^{m-1} B_{m-1}(T) \succeq 0$ (see (5.2)), and hence

$$
\phi(\alpha, \beta) = T^{\ast \beta} ((-1)^{m-1} B_{m-1}(T)) T^\alpha \quad (5.3)
$$

is positive definite. In view of Theorem 5.1 it now suffices to check that $\phi$ satisfies (5.2). Since $T$ is an $m$-isometry, it follows from the identity $B_m(T) = B_{m-1}(T) - Q_T(B_{m-1}(T))$ that $B_{m-1}(T) = Q_T(B_{m-1}(T))$ (see [15] Equation (2.1)). Hence, by (5.3), for every $\alpha, \beta \in \mathbb{Z}_+^d$,

$$
\sum_{j=1}^d \phi(\alpha + \varepsilon_j, \beta + \varepsilon_j) = \sum_{j=1}^d (-1)^{m-1} T^{\ast \beta + \varepsilon_j} B_{m-1}(T) T^{\alpha + \varepsilon_j}
$$

$$
= (-1)^{m-1} T^{\ast \beta} Q_T(B_{m-1}(T)) T^\alpha
$$

$$
= (-1)^{m-1} T^{\ast \beta} B_{m-1}(T) T^\alpha
$$

$$
= \phi(\alpha, \beta).
$$

This completes the proof.

In case the joint 2-isometric $d$-tuple $T$ admits the wandering subspace property, the semispectral measure $F$ as ensured in the above corollary may be replaced by a $\mathcal{B}(\ker T^\ast)$-valued semispectral measure, where $\ker T^\ast$ denotes the joint kernel $\cap_{j=1}^d \ker T_j^\ast$ of $T^\ast$. As we will see in the remaining part of this section, this leads to a model theorem for the spherical moments of joint $m$-isometries that admit the wandering subspace property.

Following [28], we define a $d$-tuple $\sigma$ of matrix backward shifts $\sigma_1, \ldots, \sigma_d$ on the matrix array $\mathcal{A} = [\mathcal{A}_{\alpha, \beta}]_{\alpha, \beta \in \mathbb{Z}_+^d}$ as follows:

$$
\sigma_j \mathcal{A} := [\mathcal{A}_{\alpha + \varepsilon_j, \beta + \varepsilon_j}]_{\alpha, \beta \in \mathbb{Z}_+^d}, \quad j = 1, \ldots, d.
$$

Since the actions of $\sigma_1, \ldots, \sigma_d$ are mutually commuting, we get the following:

$$
\sigma^\gamma \mathcal{A} = [\mathcal{A}_{\alpha + \gamma, \beta + \gamma}]_{\alpha, \beta \in \mathbb{Z}_+^d}, \quad \gamma \in \mathbb{Z}_+^d. \quad (5.4)
$$

For an integer $n \in \mathbb{Z}_+$, let $\Delta_{\mathcal{A}, n}$ denote the matrix given by

$$
\Delta_{\mathcal{A}, n} := \sum_{j=0}^n (-1)^{j+n} \binom{n}{j} \sum_{|\gamma| = j} \frac{|\gamma|!}{\gamma!} \sigma^\gamma \mathcal{A}.
$$

(5.5)

To state the main result of this section, we need another notion. A commuting $d$-tuple $T$ on $\mathcal{H}$ admits the wandering subspace property if

$$
\mathcal{H} = \bigvee \{ T^\alpha h : h \in \ker T^\ast, \alpha \in \mathbb{Z}_+^d \}.
$$
Note that the joint kernel \( \ker T^* \) of \( T^* \) is a wandering subspace in the sense that \( T^\alpha(\ker T^*) \) is orthogonal to \( \ker T^* \) for every non-zero \( \alpha \in \mathbb{Z}_+^d \).

We are now ready to state the main result of this section.

**Theorem 5.3.** Let \( \mathcal{H} \) be a separable Hilbert space and let \( T \) be a commuting \( d \)-tuple on \( \mathcal{H} \). Let \( \{f_j\}_{j \in I} \) be an orthonormal basis for the joint kernel of \( T^* \) for some nonempty directed set \( I \). Consider the Gramian matrix \( G = [\langle G_{\alpha,\beta} \rangle]_{\alpha,\beta \in \mathbb{Z}_+^d} \) associated with \( T \) given by

\[
G_{\alpha,\beta} := \left[ \langle T^\beta f_j, T^\alpha f_i \rangle \right]_{i,j \in I}, \quad \alpha, \beta \in \mathbb{Z}_+^d,
\]

and let \( \Delta_{\alpha,j} \) be as defined in (5.3) (with \( \mathcal{A} \) replaced by \( \mathcal{G} \)). Assume that \( T \) admits the wandering subspace property. Then the following are equivalent:

(i) \( T \) is a joint \( m \)-isometry,

(ii) \( \Delta_{\alpha,m} = 0 \),

(iii) \( \sum_{|\gamma|=k} \|\gamma\!\!\!| \sigma^{\gamma}G = \sum_{j=0}^{m-1} \binom{k}{j} \Delta_{\alpha,j} \) for every \( k \in \mathbb{Z}_+ \), where we used the convention that \( \binom{k}{j} = 0 \) if \( k < j \),

(iv) \( \sum_{j=1}^{d} \sigma_j \Delta_{\alpha,m-1} = \Delta_{\alpha,m-1} \).

Moreover, if (i) holds, then \( \Delta_{\alpha,m-1} \geq 0 \) and there exists a semi-spectral measure \( F \in M_+(\partial \mathcal{B}; \mathcal{B}(l^2(I))) \) such that for every \( \alpha, \beta \in \mathbb{Z}_+^d \),

\[
\sum_{j=1}^{d} \sigma_j \Delta_{\alpha,m-1} = \Delta_{\alpha,m-1}.
\]

In the proof of Theorem 5.3, we need a lemma that relates certain operator identities to the respective matrix identities involving the Gramian matrix (cf. [28, Theorem 2.15]).

**Lemma 5.4.** Let \( T, G, \Delta_{\alpha,m} \) be as in the statement of Theorem 5.3. If \( T \) admits the wandering subspace property, then the following are true:

(i) \( T \) is a joint \( m \)-concave \( d \)-tuple if and only if \( \Delta_{\alpha,m} \leq 0 \),

(ii) \( T \) is a joint \( m \)-convex \( d \)-tuple if and only if \( \Delta_{\alpha,m} \geq 0 \).

(iii) \( T \) is a joint \( m \)-isometry if and only if

\[
\sum_{|\gamma|=k} \|\gamma\!\!\!| \sigma^{\gamma}G = \sum_{j=0}^{m-1} \binom{k}{j} \Delta_{\alpha,j}, \quad k \in \mathbb{Z}_+,
\]

(iv) \( T \) is a joint \( m \)-isometry if and only if

\[
\sum_{j=1}^{d} \sigma_j \Delta_{\alpha,m-1} = \Delta_{\alpha,m-1}.
\]

**Proof.** Assume that \( T \) admits the wandering subspace property. Since the verifications of (i) and (ii) are similar, we only verify (i). For finite subsets \( A \subseteq I \) and \( B \subseteq \mathbb{Z}_+^d \), set

\[
f := \sum_{j \in A} \sum_{\beta \in B} c_{j,\beta}(f) T^\beta f_j, \quad C_\beta(f) = (c_{j,\beta}(f))_{j \in A} \in l^2(A), \quad \beta \in B,
\]
and let \( \mathcal{N} \) denote the subspace of \( \mathcal{H} \) consisting of all such vectors \( f \) in \( \mathcal{H} \).

Note that the wandering subspace property for \( T \) is equivalent to the density of the subspace \( \mathcal{N} \) in \( \mathcal{H} \). Fix \( f \in \mathcal{N} \) as given in (5.8) and observe that

\[
\|f\|^2 = \sum_{\alpha, \beta \in B} \|c_{\alpha, \beta}(f)T_{\alpha}f\|^2 = \sum_{\alpha, \beta \in B} \langle \mathcal{G}_{\alpha, \beta}C_{\alpha}(f), C_{\alpha}(f) \rangle.
\]

Letting \( C(f) = (C_{\alpha}(f))_{\alpha \in B} \), we obtain

\[
\|T\gamma f\|^2 = \sum_{\alpha, \beta \in B} \langle \mathcal{G}_{\alpha + \gamma, \beta + \gamma}C_{\alpha}(f), C_{\alpha}(f) \rangle
\]

Thus, by (5.1), for any positive integer \( k \), we have

\[
\langle Q^k_T(I)f, f \rangle = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \|T\gamma f\|^2 = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \langle (\sigma^\gamma \mathcal{G})C(f), C(f) \rangle.
\]

Since \( f \) is varying over the dense subspace \( \mathcal{N} \) of \( \mathcal{H} \), we deduce from (5.9) that \( T \) is a joint \( m \)-concave \( d \)-tuple if and only if

\[
\langle \Delta_{\mathcal{G},m}C(f), C(f) \rangle \leq 0, \quad f \in \mathcal{N}.
\]

Moreover, if we vary \( f \) over \( \mathcal{N} \), \( C(f) \) varies over scalar column vectors of all sizes. Since \( \sigma^\gamma \mathcal{G} \) is a self-adjoint matrix, (5.11) is equivalent to \( \Delta_{\mathcal{G},m} \leq 0 \). This completes the verification of (i).

To see (iii), recall the fact that any joint \( m \)-isometry \( d \)-tuple \( T \) satisfies the following operator identity:

\[
Q^k_T(I) = \sum_{j=0}^{m-1} (-1)^j \binom{k}{j} B_j(T), \quad k \in \mathbb{Z}_+.
\]

This is a consequence of [15 Lemma 2.2] and the fact that for any joint \( m \)-isometry \( T \), \( B_j(T) = 0 \) for every \( j \geq m \). Let \( k \) be a positive integer. Note that by (5.10) and the density of \( \mathcal{N} \) in \( \mathcal{H} \), (5.12) holds if and only if

\[
\sum_{j=0}^{m-1} (-1)^j \binom{k}{j} \langle B_j(T)f, f \rangle = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \langle (\sigma^\gamma \mathcal{G})C(f), C(f) \rangle, \quad f \in \mathcal{N},
\]

which, in view of (3.2), (5.9) and (5.11), is equivalent to

\[
\sum_{j=0}^{m-1} \binom{k}{j} \langle \Delta_{\mathcal{G},j}C(f), C(f) \rangle = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \langle (\sigma^\gamma \mathcal{G})C(f), C(f) \rangle, \quad f \in \mathcal{N}.
\]

This is, as in the last paragraph, seen to be equivalent to (5.7). The necessary part in (iii) now follows from (5.12). To see the sufficiency part, in view of the discussion above, we may assume that (5.12) holds. In particular, we have the polynomial \( p_{x,y}(k) = \langle Q^k_T(I)x, y \rangle \) of degree at most \( m-1 \) for every \( x, y \in \mathcal{H} \). However, for any complex polynomial \( p \) in one variable of degree at most \( m-1 \), \( \sum_{j=0}^{m} (-1)^j \binom{m}{j} p(j) = 0 \) (see [18 Proposition 2.1]). We may now apply the above fact to each \( p_{x,y} \) to get (iii).

Finally, to see (iv), note that by [14 Equation (2.1)], \( T \) is a joint \( m \)-isometry if and only if \( B_{m-1}(T) = Q_T(B_{m-1}(T)) \), and argue as above. □
It is worth noting that the idea of Gramian may be employed to give an alternate proof of [28, Theorem 2.15]. We leave the details to the reader.

**Proof of Theorem 5.3.** The equivalence of (i)-(iv) is immediate from Lemma 5.4. To see the remaining part, assume that $T$ is a joint $m$-isometric $d$-tuple. By [15, Proposition 2.3], $\Delta_{T, m-1} \geq 0$, and hence by Lemma 5.4(ii),

$$\Delta_{\mathcal{D}, m-1} \geq 0. \quad (5.13)$$

To see the remaining part, set

$$\phi(\alpha, \beta) := \sum_{j=0}^{m-1} (-1)^{j+m-1} \binom{m-1}{j} \sum_{|\gamma|=j} \frac{|\gamma|!}{\gamma!} G_{\alpha+\gamma, \beta+\gamma}, \quad \alpha, \beta \in \mathbb{Z}_+^d,$$

and note that by (5.13) and Lemma 5.4(iv) together with (5.5), $\phi$ is positive definite and satisfies (5.2). Further, a simple application of the Cauchy-Schwarz inequality shows that for every $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\langle G_{\alpha, \beta}X, Y \rangle_{\ell^2(I)} \leq \|T\alpha\| \|T^\beta\| \|X\|_{\ell^2(I)} \|Y\|_{\ell^2(I)}, \quad X, Y \in \ell^2(I).$$

This shows that $\phi(\alpha, \beta) \in \mathcal{B}(\ell^2(I))$ for every $\alpha, \beta \in \mathbb{Z}_+^d$. Now applying Theorem 5.1 (with $\mathcal{M} = \ell^2(I)$) completes the proof. □

Before we see an analytic model for the spherical moments of joint 2-isometries, it is worth noting that analytic cyclic joint 2-isometries can not be modeled as the multiplication tuples on a Dirichlet-type space in dimension bigger than 1.

**Example 5.5.** Let $\nu$ be a positive finite Borel measure on the unit circle $\mathbb{T}$ and let $\mathcal{M}_z$ be the operator of the multiplication by $z$ acting on the Dirichlet-type space $\mathcal{D}(\nu)$. By [23, Theorem 3.6, 3.7 and Corollary 3.8], $\mathcal{M}_z$ defines an analytic cyclic 2-isometry. Consider the commuting pair $T = (T_1, T_2)$, where $T_j = \mathcal{M}_z / \sqrt{2}$ for $j = 1, 2$. Since $\mathcal{M}_z$ is an analytic cyclic operator with cyclic vector 1, so is $T$. Also, since $\mathcal{M}_z$ is a 2-isometry, by [7, Proposition 3.7], $T$ is a joint 2-isometry. Since $z_1 \neq z_2$ on $\mathbb{D}$, there is no $\mu \in M_+ (\partial \mathbb{D})$ and a unitary $U : \mathcal{D} \rightarrow \mathcal{D}(\mu)$ such that $U1 = 1$ and $\mathcal{M}_z U = UT_j$ for $j = 1, 2$. □

We conclude the paper with a model theorem for the spherical moments of joint 2-isometries (cf. [24, Pg 30]).

**Corollary 5.6.** Let $T$ be a commuting $d$-tuple on a separable Hilbert space $\mathcal{H}$ and let $\{f_j\}_{j \in I}$ be an orthonormal basis for the joint kernel $\ker T^* \circ T^*$ of $T^*$ for some nonempty directed set $I$. If $T$ is a joint 2-isometry, then there exists $F_T \in M_+ (\partial \mathbb{D}, \mathcal{B}(\ell^2(I)))$ such that for every $\alpha, \beta \in \mathbb{Z}_+^d$, $k \in \mathbb{Z}_+$ and $x, y \in \ker T^*$,

$$\langle Q_k^T(I)T^\alpha x, T^\beta y \rangle - \langle T^\alpha x, T^\beta y \rangle = \sum_{|\gamma|=k} \frac{|\gamma|!}{\gamma!} \langle z^{\alpha+\gamma} x, z^{\beta+\gamma} y \rangle_{\mathcal{D}(F_T)} - \langle z^\alpha x, z^\beta y \rangle_{\mathcal{D}(F_T)}. \quad (5.14)$$

**Proof.** Assume that $T$ is a joint 2-isometry. Since the restriction of a joint 2-isometry to an invariant subspace is again a joint 2-isometry, after replacing $\mathcal{H}$ by the invariant subspace $\bigvee \{T^\alpha h : h \in \ker T^*, \alpha \in \mathbb{Z}_+^d \} \circ T$, if necessary, we may assume that $T$ admits the wandering subspace property. It suffices to
check the above formula for \( x, y \in \{ f_j \}_{j \in I} \). The existence of the semispectral measure \( F_T \) is ensured by Theorem 5.3 (with \( m = 2 \)). A simple calculation using (5.13) shows that
\[
\left[ \left[ (Q_T(I) - I)T^\beta f_j, T^\alpha f_i \right] \right]_{i,j \in I} = \int_{\mathbb{R}_+^d} \zeta^\alpha \zeta^\beta dF(\zeta), \quad \alpha, \beta \in \mathbb{Z}_+^d. \tag{5.15}
\]

One may now consider the Dirichlet-type space \( D(T) \) associated with \( T \). Since \( Q_T^k(I) = I + k(Q_T(I) - I) \), \( k \geq 1 \), the desired conclusion now follows from Theorem 2.1 and (5.15). \( \square \)

Remark 5.7. In case of \( d = 1 \), it may be concluded from the formula (5.14) that the Gramian matrices associated with \( T \) and the multiplication \( d \)-tuple \( M_z \) on \( D(T) \) coincide. In general, this fails in case \( d \geq 2 \) (see Example 5.3). However, for a 2-variable weighted shift \( T \), it can be seen by induction that these Gramian matrices are same provided they coincide for indices \( \alpha = k \varepsilon_j = \beta, k \in \mathbb{Z}_+, j = 1, 2 \).

As far as the classification of joint 2-isometries is concerned, the only known case, due to Richter-Sundberg, is of finite dimensional joint 2-isometries (see [24, Pg 17] for the statement and [19, Theorem 3.2] for a proof). Nevertheless, the results in this paper set the ground for the problem of classifying all analytic joint 2-isometries that can be modeled as the multiplication tuples \( M_z \) on a Dirichlet-type space \( D(T) \).

**Appendix: Proof of Theorem 5.1**

In this appendix, we present a proof of Theorem 5.1 that exploits the joint subnormality of spherical isometries (see [4, Proposition 2]).

**Proof of Theorem 5.1.** The implication (i) \( \Rightarrow \) (ii) is a routine verification.

(ii) \( \Rightarrow \) (i): The essential idea of the proof of this implication is due to Fuglede (see [11, Section 4]). Consider the complex vector space \( \mathcal{M}[z] := \mathbb{C}[z_1, \ldots, z_d] \otimes \mathcal{M} \) of \( \mathcal{M} \)-valued polynomials in \( z_1, \ldots, z_d \). For finite subsets \( A, B \) of \( \mathbb{Z}_+^d \) and \( x_\alpha, y_\beta \in \mathcal{M}, \alpha \in A \) and \( \beta \in B \), define
\[
\left\langle \sum_{\alpha \in A} z^\alpha \otimes x_\alpha, \sum_{\beta \in B} z^\beta \otimes y_\beta \right\rangle := \sum_{\alpha \in A} \sum_{\beta \in B} (\phi(\alpha, \beta)x_\alpha, y_\beta)_{\mathcal{M}}. \tag{5.16}
\]

Since \( \phi \) is positive definite, this defines a semi-inner-product on \( \mathcal{M}[z] \). Consider the quotient space \( \mathcal{M}[z]/\mathcal{N} \), where \( \mathcal{N} := \{ f \in \mathcal{M}[z] : \langle f, f \rangle = 0 \} \).

Let \( \mathcal{H} \) be the completion of \( \mathcal{M}[z]/\mathcal{N} \). Since \( \langle f, g \rangle \leq \| f \| \| g \| \) for every \( f, g \in \mathcal{M}[z], \langle f, g \rangle = 0 \) provided \( f \) or \( g \) belongs to \( \mathcal{N} \), and hence for any \( [g] = g + \mathcal{N} \in \mathcal{M}[z]/\mathcal{N} \),
\[
\| [g] \|_{\mathcal{H}}^2 = \inf_{f \in \mathcal{N}} \| g + f \|^2 = \| g \|^2. \tag{5.17}
\]

Thus the inner-product on \( \mathcal{M}[z] \) induces an inner-product on \( \mathcal{M}[z]/\mathcal{N} \). Consider the commuting \( d \)-tuple \( M_z \) of multiplication operators \( M_{z_1}, \ldots, M_{z_d} \) defined on \( \mathcal{M}[z] \). Note that for any finite subset \( A \) of \( \mathbb{Z}_+^d \) and \( x_\alpha \in \mathcal{M}, \)}
\(\alpha \in A\), by (5.16) and (5.2), we have

\[
\sum_{j=1}^{d} \left\| M_{z_j} \left( \sum_{\alpha \in A} z^\alpha \otimes x_\alpha \right) \right\|^2 = \sum_{j=1}^{d} \left\| \sum_{\alpha \in A} z^{\alpha+\epsilon_j} \otimes x_\alpha \right\|^2 \\
= \sum_{j=1}^{d} \sum_{\alpha,\beta \in A} \langle \phi(\alpha + \epsilon_j, \beta + \epsilon_j) x_\alpha, x_\beta \rangle_M \\
= \sum_{\alpha,\beta \in A} \langle \phi(\alpha, \beta) x_\alpha, x_\beta \rangle_M \\
= \left\| \sum_{\alpha \in A} z^\alpha \otimes x_\alpha \right\|^2.
\]

This together with (5.17) shows that the \(d\)-tuple \(M_z\) induces a joint isometry on \(H\), say, \(S = (S_1, \ldots, S_d)\), given by

\[
S_j([f]) = [z_j f], \quad [f] \in \mathcal{M}[z]/\mathcal{N}, \quad j = 1, \ldots, d.
\]  

(5.18)

By [3, Proposition 2] and the multivariate Bram’s characterization of joint subnormality ([11, Equation (0)]), there exists \(E \in M_+(\partial B^d, \mathcal{B}(H))\) such that

\[
S^{\alpha \beta} S^{\alpha} = \int_{\partial B^d} \zeta^{\alpha \beta} dE(\zeta), \quad \alpha, \beta \in \mathbb{Z}_+^d.
\]  

(5.19)

To complete the proof, we define a bounded linear map \(V : \mathcal{M} \to H\) by

\[
V(x) = [1 \otimes x], \quad x \in \mathcal{M},
\]

and let \(F(\cdot) = V^* E(\cdot)V\). Clearly, \(F\) defines a \(\mathcal{B}(\mathcal{M})\)-valued semispectral measure on \(\partial B^d\). Moreover, for any \(\alpha, \beta \in \mathbb{Z}_+^d\) and \(x, y \in \mathcal{M}\), we may infer from (5.17) and the polarization identity that

\[
\langle \phi(\alpha, \beta) x, y \rangle_{\mathcal{M}} = \langle [z^\alpha \otimes x], [z^\beta \otimes y] \rangle_H \\
= \langle [S^\alpha ([1 \otimes x]), S^\beta ([1 \otimes y]) \rangle_H \\
= \int_{\partial B^d} \zeta^\alpha \zeta^\beta d(E(\zeta)V x, V y) \\
= \int_{\partial B^d} \zeta^\alpha \zeta^\beta d(F(\zeta)x, y).
\]

This completes the proof of the equivalence (i) \(\Leftrightarrow\) (ii). The uniqueness part may be deduced from Riesz representation and Stone-Weierstrass approximation theorems. \(\square\)

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