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FLAT COMPACT HERMITE-LORENTZ MANIFOLDS IN DIMENSION 4

BIANCA BARUCCHIERI

Abstract. We give a classification, up to finite cover, of flat compact complete Hermite-Lorentz manifolds up to complex dimension 4.

1. Introduction

For a vector space $V$, subgroups $\Gamma$ of the affine group $\text{Aff}(V) = \text{GL}(V) \times V$ that act properly discontinuously and cocompactly on $V$ are called crystallographic groups and have a long history. The first results date back to Bieberbach around the years 1911-12, when he studied the case $\Gamma \leq O(n) \ltimes \mathbb{R}^n$. His results show that such groups are, up to finite index, abelian groups made of translations. Later the Lorentzian case, $\Gamma \leq O(n, 1) \ltimes \mathbb{R}^{n+1}$, was studied. For this case the first results were obtained by Auslander and Markus in dimension 3, [4], and, more generally, Fried and Goldman studied all 3-dimensional crystallographic groups in [12]. Still for the Lorentzian case, in the years 1987-88 Fried investigated the dimension 4, [11], and afterwards results for all dimensions were given by Grunewald and Margulis, [17].

In this article we are interested in the Hermite-Lorentz case, namely $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$. Following the strategy used by Grunewald and Margulis in [17] we give a classification, up to finite index, of these crystallographic groups for $n \leq 3$. These groups are the fundamental groups of flat compact complete Hermite-Lorentz manifolds and they determine the manifold. Indeed, after a classical result of Mostow [18], if two such groups are isomorphic the corresponding manifolds are diffeomorphic. Since we are interested in classifying the manifolds we will only be concerned with isomorphism classes of these groups and not with the different ways one can realise them as subgroups of $U(n, 1) \ltimes \mathbb{C}^{n+1}$. Therefore, according to Mostow’s Theorem, the classification, up to finite index, of crystallographic subgroups $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$ given in this article corresponds to a classification, up to finite covering, and for complex dimension at most 4, of flat compact complete Hermite-Lorentz manifolds. Indeed, let $a(\mathbb{C}^{n+1})$ be the affine space associated to the complex vector space $\mathbb{C}^{n+1}$ endowed with a Hermitian form of signature $(n, 1)$ and $U(n, 1)$ be the subgroup of the general linear group preserving this Hermitian form. Flat compact complete Hermite-Lorentz manifolds can then be presented as the quotient $\Gamma \backslash a(\mathbb{C}^{n+1})$ where the group $\Gamma$ is a subgroup of $\mathcal{H}(n, 1) = U(n, 1) \ltimes \mathbb{C}^{n+1}$ acting properly discontinuously and cocompactly on $a(\mathbb{C}^{n+1})$. These manifolds can be thought as analogue of both Hermitian manifolds (definite positive) in complex geometry and Lorentzian manifolds in real differential geometry, see [5]. Let us notice that Scheuneman in [21] already studied the case of compact complete affine complex surfaces.

About crystallographic groups there is a long-standing conjecture due to Auslander that says that such groups are virtually solvable. Otherwise said, from the manifold point of view, the fundamental group of every compact complete flat affine manifold is virtually solvable. Sometimes this statement is formulated using the term polycyclic instead of solvable.
Remark 1.1. A polycyclic group is solvable. The converse is not true in general. But every discrete solvable subgroup of $\text{GL}(n, \mathbb{R})$ is polycyclic, [19, Proposition 3.10]. Hence we will use the word polycyclic or solvable interchangeably.

In some special cases the Auslander conjecture is known to be true. One such case is the Riemannian case as we have seen in Bieberbach’s result. Another such case is the Lorentzian case, which was proved in [14] by Goldman and Kamishima, see also [8]. Finally, after a result of Grunewald and Margulis, [17], also the case that interests us, namely the Hermite-Lorentz case $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$, is known to satisfy the Auslander conjecture.

For virtually solvable crystallographic groups Fried and Goldman gave the following generalisation of Bieberbach’s result.

**Theorem 1.2** ([12, Corollary 1.5.]). Let $V$ be a finite dimensional real vector space and $\Gamma \leq \text{Aff}(V)$ be a virtually solvable group acting properly discontinuously and cocompactly on $a(V)$ then there exists a subgroup $H \leq \text{Aff}(V)$ such that

1. $H$ acts simply transitively on $a(V)$;
2. $\Gamma \cap H$ has finite index in $\Gamma$;
3. $\Gamma \cap H$ is a lattice in $H$.

Indeed, when the linear part of $\Gamma$ preserves a positive definite bilinear form, Bieberbach’s results tell us that the simply transitive group of the theorem is the group of pure translations. The simply transitive groups appearing in this theorem are sometimes called connected crystallographic hulls. This theorem will be our starting point for the investigation of the Hermite-Lorentz manifolds $\Gamma \backslash \mathbb{C}^{n+1}$. More precisely, in order to study crystallographic subgroups $\Gamma \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$ we will study lattices in simply transitive affine groups $H \leq U(n, 1) \ltimes \mathbb{C}^{n+1}$. Hence, even though the Hermite-Lorentz manifolds are complex manifolds, we see them as real objects with an integrable complex structure. More precisely we will look at this manifolds, up to finite index, as $\Gamma \backslash H$ where $H$ is a simply transitive subgroup of $U(n, 1) \ltimes \mathbb{C}^{n+1}$ and $\Gamma$ is a lattice in it. The Lie groups $H$ that act simply transitively on $\mathbb{C}^{n+1}$ are real Lie groups which inherit an integrable left invariant complex structure from $\mathbb{C}^{n+1}$. The integrability condition, see for example [20], is more general than the condition for a Lie group to be a complex Lie group. Indeed the Lie groups we are looking at are not complex Lie groups.

This article is organised as follows. In Section 2 we will study unipotent groups $H$ acting simply transitively on $a(\mathbb{C}^{n+1})$. Let us remember the following definition.

**Definition 1.3.** A subgroup $U$ of the affine group $\text{Aff}(V)$ is unipotent if its linear part consists of endomorphisms whose eigenvalues are all 1.

**Remark 1.4.** A unipotent group is, in particular, a nilpotent group. The converse in general is not true. But every simply transitive nilpotent affine group of motion that appears as a connected crystallographic hull is unipotent, [13, Theorem A].

Thus in Section 2 we give a presentation, valid in any dimension, of unipotent simply transitive groups of Hermite-Lorentz affine motion and prove a proposition about their conjugacy classes. Let us remark that, although we haven’t adopted this point of view at all, Lie groups acting simply transitively on $a(\mathbb{C}^{n+1})$ can be seen as real Lie groups that are endowed with a left invariant flat Hermite-Lorentz metric. There is a vast literature that treats similar questions with the point of view of metric Lie algebras. For example, [1] and [6] give construction results for Lorentzian and pseudo-Riemannian nilpotent Lie algebras of signature $(2, n-2)$. 
But in these articles they are not interested in the classification problem. The Lie algebras that appear in this article are particular cases of pseudo-Riemannian nilpotent Lie algebras of signature $(2, n-2)$. A posteriori we can show, but we will not do it in this article, that the Lie algebras we have found can be obtained with a double extension process as in [6].

In Section 3 we reduce, as for the Lorentzian case, the study of crystallographic groups to the study of lattices in simply transitive unipotent Lie groups. Let us define some terminology.

**Definition 1.5.** A group $\Gamma$ is said to be *abelian by cyclic* if it has a normal abelian subgroup $\Gamma_1$ such that $\Gamma/\Gamma_1$ is cyclic.

We then prove the following.

**Theorem 1.6.** Let $\Gamma$ be a subgroup of $\mathcal{H}(n, 1)$ that acts properly discontinuously and cocompactly on $\mathfrak{a}(V)$ then $\Gamma$ is either virtually nilpotent or contains a subgroup of finite index that is abelian by cyclic.

According to this theorem, the unipotent hypothesis we made in Section 2 only leaves out the easy abelian by cyclic case. We end this section with the classification of the latter case.

In Section 4 we start the classification, up to isomorphism, of those $H$ found in Section 2. We complete their classification for the dimensions 2 and 3. In dimension 4 we give the classification for some particular cases that we have called degenerate cases.

In Section 5 we finish the classification of the unipotent simply transitive groups $H$ in dimension 4. For this more general case we are left with a classification problem of 8-dimensional nilpotent Lie algebras defined by three parameters. Since there are no complete classifications of nilpotent Lie algebras in dimension bigger than 7 we introduce an ad hoc method to study our particular family of Lie algebras. Using the fact that these Lie algebras are Carnot, see Subsection 5.2, we can identify their isomorphism classes with the orbits of the $\text{SL}(3, \mathbb{R})$-action on the Grassmannian of 2-dimensional subspaces of $\mathfrak{sl}(3, \mathbb{R})$ induced by the adjoint action. Hence studying the orbits for this action is the same as studying the isomorphism classes for the family of Lie algebras. In contrast with the Lorentzian case where, for every fixed dimension, there are finitely many isomorphism classes of unipotent subgroups of $O(n, 1) \ltimes \mathbb{R}^{n+1}$ acting simply transitively on $\mathfrak{a}(\mathbb{R}^{n+1})$ we find that these Lie algebras constitute an infinite family of pairwise non isomorphic Lie algebras of Lie groups acting simply transitively on $\mathfrak{a}(\mathbb{C}^{3+1})$. In order to classify unipotent simply transitive subgroups of $U(3, 1) \ltimes \mathbb{C}^{3+1}$ up to isomorphism it is sufficient to classify their Lie algebras up to Lie algebra isomorphism. Thus in this section we complete the proof of the following.

**Theorem 1.7.** The list given in Appendix A with $K = \mathbb{R}$ is a complete non redundant list of isomorphism classes of unipotent simply transitive subgroups of $U(3, 1) \ltimes \mathbb{C}^{3+1}$.

In Section 6 we study the Hermite-Lorentz crystallographic groups in the nilpotent case. Indeed we have reduced it to the study of lattices in simply transitive unipotent groups. Since in Sections 4 and 5 we have given the complete list of those simply transitive groups we just need to understand which of them admit lattices and what are their abstract commensurability classes. Let us give the following definition.

**Definition 1.8.** Two groups $\Gamma_1$ and $\Gamma_2$ are said to be *abstractly commensurable* if there exist two subgroups $\Delta_1 \leq \Gamma_1$ and $\Delta_2 \leq \Gamma_2$ of finite index such that $\Delta_1$ is isomorphic to $\Delta_2$. 
Lattices in nilpotent Lie groups are described by Malcev’s theorems, see Section 6 and for more details see [10] or [19, Section II]. These theorems say that a nilpotent Lie group $G$ admits a lattice if and only if its Lie algebra $\mathfrak{g}$ admits a $\mathbb{Q}$-form, that is a rational subalgebra $\mathfrak{g}_0$ such that $\mathfrak{g}_0 \otimes \mathbb{R}$ is isomorphic to $\mathfrak{g}$. Furthermore abstract commensurability classes of lattices in $G$ are in correspondence with $\mathbb{Q}$-isomorphism classes of $\mathbb{Q}$-forms of $\mathfrak{g}$. To be more precise let us remember that for a simply connected nilpotent Lie group the exponential map from its Lie algebra is a diffeomorphism and we will call log its inverse. Then from a lattice $\Gamma$ in $G$, $\text{span}_\mathbb{Q}\{\text{log} \Gamma\}$ provides a $\mathbb{Q}$-form of $\mathfrak{g}$. On the other hand let $\mathfrak{g}$ be a Lie algebra with a basis with respect to which the structure constants are rational and let $\mathfrak{g}_0$ be the $\mathbb{Q}$-span of this basis. If $\mathcal{L}$ is any lattice of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ the group generated by $\exp \mathcal{L}$ is a lattice in $G$. The main result of this section is then the following.

**Theorem 1.9.** The list given in Appendix A with $K = \mathbb{Q}$ is a complete list without repetition of abstract commensurability classes of nilpotent crystallographic subgroups of $U(3,1) \ltimes \mathbb{C}^{3+1}$.

Finally in Section 7 we give some topological considerations about the manifolds $\Gamma \setminus a(\mathbb{C}^{n+1})$ that are virtually nilpotent, namely that they are finitely covered by torus bundles over tori.

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2. **Unipotent simply transitive groups of Hermite-Lorentz affine motion**

In this section we look at unipotent subgroups of $U(n,1) \ltimes \mathbb{C}^{n+1}$ acting simply transitively on $a(\mathbb{C}^{n+1})$. The reason why we have the unipotent hypothesis, that simplifies our study, is that, as we will see in the following section, the study of crystallographic groups will be reduced to the study of lattices in unipotent simply transitive Lie groups.

Let $V$ be a complex vector space of dimension $n+1$ endowed with a Hermitian form $h$ of signature $(n,1)$, such an Hermitian form will be called Hermite-Lorentz. Also we denote by $a(V)$ the affine space associated to $V$ and by $\text{Aff}(V)$ the group of affine transformations of $a(V)$. Remember that the affine group can be seen as a linear group as follows

$$\text{Aff}(V) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in \text{GL}(V), v \in V \right\}.$$  

Denote furthermore by $L: \text{Aff}(V) \to \text{GL}(V)$ the homomorphism that associates to each affine transformation its linear part, then its kernel consists of pure translations and let us denote it by

$$T = \left\{ \begin{pmatrix} \text{Id} & v \\ 0 & 1 \end{pmatrix} \mid v \in V \right\}.$$  

For a subgroup $G \leq \text{Aff}(V)$, let us denote $T_G = G \cap T$ the subgroup of pure translations in $G$. Let furthermore $U(n,1)$ be the group of linear transformations of $V$ that preserve $h$ and $\mathcal{H}(n,1)$ the group of affine transformations of $a(V)$ whose linear part is in $U(n,1)$. 


Remark 2.1. Let us notice that the pair \((V, h)\) where \(V\) is a complex vector space and \(h\) is a Hermite-Lorentz form defined on it is equivalent to the triple \((V(\mathbb{R}), J, \langle \cdot, \cdot \rangle)\). Here \(V(\mathbb{R})\) is the associated real vector space, \(J : V(\mathbb{R}) \to V(\mathbb{R})\) is a complex structure on \(V(\mathbb{R})\), i.e. a linear endomorphism such that \(J^2 = -\text{Id}\) and \(\langle \cdot, \cdot \rangle\) is a pseudo-Riemannian metric on \(V(\mathbb{R})\) of signature \((2n, 2)\) such that, if \(\omega(u, v) := \langle u, Jv \rangle = -\langle Ju, v \rangle\), we have

\[
h(u, v) = \langle u, v \rangle + i\omega(u, v).
\]

Let us also fix the following notation: for \(z \in \mathbb{C}\) we denote by \(\Re(z)\) and \(\Im(z)\) its real and imaginary part respectively.

Lemma 2.2. A unipotent subgroup of \(U(n, 1)\) fixes a null vector for \(h\).

Proof. Let \(U\) be a unipotent subgroup of \(U(n, 1)\). From Engel’s theorem we know that \(U\) fixes a vector \(v_0 \in V\). If \(v_0\) is timelike then \(U\) fixes \(v_0\) that is spacelike, hence it is contained in \(U(n)\), but a unipotent unitary matrix is trivial hence \(U\) would be trivial. If instead \(v_0\) were spacelike then, if \(U\) does not fix a null vector in \(v_0\), \(U\) would have to fix a spacelike vector \(v_1\) in \(v_0\). This implies that \(U\) preserves the Hermite-Lorentz space \(\text{span}\{v_0, v_1\} \perp\). By induction we get that then \(U\) is contained in \(U(1, 1)\) and we can see that a unipotent element in \(U(1, 1)\) is trivial. □

Lemma 2.3. We can choose a basis for \(V\) with respect to which a maximal unipotent subgroup of \(U(n, 1)\) is written as

\[
\mathcal{U} = \left\{ \begin{pmatrix} 1 & -\bar{\sigma}^t & -\frac{1}{2}\bar{\sigma}^t v + ib \\ 0 & \text{Id}_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \mid v \in \mathbb{C}^{n-1}, b \in \mathbb{R} \right\}.
\]

Proof. Let \(\mathcal{U}\) be a maximal unipotent subgroup of \(U(n, 1)\). We might choose the fixed null vector \(v_0\) of \(\mathcal{U}\) from Lemma 2.2 to be the first vector of a base of \(V\). Let \((r, u, s) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}\) be the coordinates with respect to this basis. Then we write the quadratic form \(q\) associated to the Hermitian form \(h\) as

\[
q(r, u, s) = r\bar{\sigma} + rs + u^t\bar{\sigma}u.
\]

With respect to this basis we can write an element of \(\mathcal{U}\) as \(A = \begin{pmatrix} 1 & v^t & a \\ 0 & M & w \\ 0 & 0 & 1 \end{pmatrix}\) with \(v, w \in \mathbb{C}^{n-1}\), \(a \in \mathbb{C}\) and \(M\) an upper triangular matrix with 1’s on the diagonal. Imposing the condition

\[
\bar{A}^tHA = H,
\]

we get that \(A\) has the desired form. □

Lemma 2.4. We can choose a basis for \(V\) with respect to which a parabolic subgroup of \(U(n, 1)\) is written as

\[
\mathcal{P} = \left\{ \begin{pmatrix} \bar{\lambda} & -\sigma^t & a \\ 0 & \sigma & \lambda^{-1}\sigma v \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^*, \sigma \in U(n-1), v \in \mathbb{C}^{n-1}, \Re(\bar{\lambda}a) = -\frac{1}{2}\bar{\sigma}^t v \right\}.
\]

A minimal parabolic subgroup of \(U(n, 1)\), that is a Borel subgroup \(\mathcal{B}\), is given by elements \(A \in \mathcal{P}\) with \(\sigma = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}})\).

Proof. By definition a Borel subgroup is a solvable subgroup of \(U(n, 1)\). By Lie’s theorem this implies that all the elements of \(\mathcal{B}\) have a common eigenvector. As in Lemma 2.2 we can see that this common vector is isotropic and hence we may choose it to be the first vector of a basis. With respect to this basis we write the quadratic form associated to \(h\) as \(q(r, u, s) = r\bar{\sigma} + rs + u^t\bar{\sigma}u\).
with \((r, u, s) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}\). Let us apply again Lie’s theorem to the orthogonal space of the first vector of the basis. After noticing that a solvable subgroup of \(U(n-1)\) is abelian and imposing the equation \(A^t H A = H\) where \(H\) is the matrix associated to \(q\), we find the desired form. Finally an element \(A\) in a parabolic subgroup should have a block upper triangular form. □

**Proposition 2.5.** Let \(U\) be a unipotent subgroup of \(H(n, 1)\) that is acting simply transitively and affinely on \(\mathfrak{a}(V)\). Then \(U = \exp(\mathfrak{u})\) where \(\mathfrak{u}\) is a nilpotent Lie algebra. In suitable coordinates \((r, u, s) \in V = \mathbb{C} \times W \times \mathbb{C}\), where \(W = \mathbb{C}^{n-1}\), the quadratic form \(q\) associated to the Hermitian form \(h\) is given by

\[
q(r, u, s) = r\overline{s} + \overline{r}s + u^t \overline{u}
\]

and \(u = u(\gamma_2, \gamma_3, b_2, b_3)\) has the expression

\[
\begin{pmatrix}
0 & -\gamma_2(u)^t + \gamma_3(s)^t & ib_2(u) + b_3(s)
0 & 0 & \gamma_2(u) + \gamma_3(s)
0 & 0 & 0
\end{pmatrix}
\]

where

1. \(\gamma_2 : W \to W\) is an \(\mathbb{R}\)-linear map such that
   \(\text{Im} \gamma_2 \oplus J \text{Im} \gamma_2 \subseteq \ker \gamma_2\) and \(\omega(\text{Im} \gamma_2, \text{Im} \gamma_2) = 0\),

2. \(\gamma_3 : \mathbb{C} \to W\) is an \(\mathbb{R}\)-linear map such that \(\gamma_3 is \in \ker \gamma_2\) for all \(s \in \mathbb{C}\),

3. \(b_2 : W \to \mathbb{R}\) is an \(\mathbb{R}\)-linear map such that
   \[b_2(s\gamma_2(u)) = 2\omega(\gamma_2(u), \gamma_3(s))\] and
   \[b_2(\gamma_3(is) - J\gamma_3(s)) = 2\omega(\gamma_3(is), \gamma_3(s))\] for all \(u \in W, s \in \mathbb{C}\),

4. \(b_3 : \mathbb{C} \to \mathbb{R}\) is an \(\mathbb{R}\)-linear map.

**Proof.** Since \(U\) is acting simply transitively on \(\mathfrak{a}(V)\) it is simply connected, hence \(U = \exp(\mathfrak{u})\) where \(\mathfrak{u}\) is its Lie algebra. The linear part of \(U\) can be conjugated to be as in Lemma 2.3. Equivalently we can find coordinates \((r, u, s) \in V = \mathbb{C} \times W \times \mathbb{C}\), where \(W = \mathbb{C}^{n-1}\), with respect to which the quadratic form \(q\) associated to the Hermitian form \(h\) reads as

\[
q(r, u, s) = r\overline{s} + \overline{r}s + u^t \overline{u}
\]

and the Lie algebra of the linear part of \(U\) is

\[
L(\mathfrak{u}) = \left\{ \begin{pmatrix} 0 & -\gamma^t & ib \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \middle| \gamma \in W, b \in \mathbb{R} \right\} .
\]

Since \(\mathfrak{u}\) is the Lie algebra of \(U\) that is acting simply transitively on \(\mathfrak{a}(V)\) there exist two \(\mathbb{R}\)-linear maps

\[
\gamma : \mathbb{C} \times W \times \mathbb{C} \to \mathbb{C}^{n-1} \quad \text{and} \quad b : \mathbb{C} \times W \times \mathbb{C} \to \mathbb{R},
\]
such that

\[
\begin{pmatrix}
0 & -\gamma(v) & i\beta(v) & r \\
0 & 0 & \gamma(v) & u \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\bigg| v = (r, u, s) \in \mathbb{C} \times W \times \mathbb{C}
\]

Let us write \((A(v), v)\) for an element in \(u\) where \(A(v)\) denotes the linear part and \(v\) the translation part. If we compute the commutator bracket of two elements in \(u\) we get

\[
[(A(v), v), (A(v'), v')] = \begin{pmatrix}
0 & 0 & \gamma(v') \gamma(v) - \gamma(v) \gamma(v') & i \gamma(v') u - \gamma(v) u' + i(s' \beta(v) - s \beta(v')) \\
0 & 0 & 0 & i(s' \beta(v) - s \beta(v')) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \]

Since \(u\) is a Lie algebra we must have \(\gamma(v') = 0\) and \(b(v') = 23(\gamma(v') \gamma(v))\) where

\[
v'(t) = (\gamma(v') u - \gamma(v) u' + i(s' \beta(v) - s \beta(v')), s' \gamma(v) - s \gamma(v'), 0).
\]

So let us write \(\gamma(v) = \gamma_1(r) + \gamma_2(u) + \gamma_3(s)\) and \(b(v) = b_1(r) + b_2(u) + b_3(s)\).

**Lemma 2.6.** We claim that \(\gamma_1 = 0\), \(\text{Im} \gamma_2 \oplus J \text{Im} \gamma_2 \subseteq \ker \gamma_2\) and that property 2 holds.

**Proof.** Let \(s = s' = 0\) then \(\gamma(v') = \gamma_1(\gamma(v') u - \gamma(v) u') = 0\). Now let \(u' = 0\) we get \(\gamma(v') = \gamma_1(\gamma(v') u) = 0\). Letting \(r'\) and \(u\) vary we see that \(\gamma_1 = 0\). Now take \(s = 0\) we have \(\gamma(v'(s)) = \gamma_2(s' \gamma(v)) = \gamma_2(s' \gamma_2(u)) = 0\) hence if we let \(s'\) be real we obtain \((\gamma_2)^2 = 0\) and if we let \(s'\) be purely imaginary we obtain \(\gamma_2(\gamma_{2}) = 0\). Now the equation \(\gamma(v') = 0\) becomes \(\gamma_2(s' \gamma_3(s)) - \gamma_2(s \gamma_3(s')) = 0\). Let \(s = is'\) and the lemma follows.

**Lemma 2.7.** We claim that \(b_1 = 0\), \(\omega(\text{Im} \gamma_2, \text{Im} \gamma_2) = 0\) and that property 3 holds.

**Proof.** Let \(u = u' = 0\) then \(b(v'') = b_1(i(s' \beta(v) - s \beta(v'))) + b_2(s' \gamma_3(s) - s \gamma_3(s')) = 23(\gamma_3(s') \gamma_3(s))\). Now let \(s = 0\) we get \(b(v'') = b_1(i(s' \beta(v) - s \beta(v'))) = 0\) hence \(b_1 = 0\). Looking again at the equation \(b(v'') = 23(\gamma_3(s') \gamma_3(s))\) and letting \(s = s' = 0\) we get \(3(\gamma_2(u^t) \gamma_2(u)) = 0\). Now the equation \(b(v'') = 23(\gamma_3(s') \gamma_3(s))\) becomes

\[
b_2(s' \gamma_2(u) - s \gamma_2(u') + s' \gamma_3(s) - s \gamma_3(s')) = 23(\gamma_2(u^t) \gamma_3(s) + \gamma_3(s') \gamma_2(\gamma_2(u') + \gamma_3(s') \gamma_3(s)).
\]

Now take \(s = 0\) we get \(b_2(s' \gamma_2(u) = 23(\gamma_3(s') \gamma_2(u))\). Finally if we consider \(u = u' = 0\) we get \(b_2(s' \gamma_3(s) - s \gamma_3(s')) = 23(\gamma_3(s') \gamma_3(s))\) now if we let \(s = is'\) the lemma follows.

We can notice that these conditions are also sufficient in order to have \(\gamma(v'') = 0\) and \(b(v'') = 23(\gamma(v') \gamma(v))\).

Let us decompose \(W\) as a real vector space as follows

\[(1.2) \quad W = \text{Im} \gamma_2 \oplus J \text{Im} \gamma_2 \oplus S \oplus T\]

where \(S \oplus T\) is orthogonal to \(\text{Im} \gamma_2 \oplus J \text{Im} \gamma_2\) with respect to \(h_W\) and \(T\) is orthogonal to \(S\) with respect to \(\langle \cdot, \cdot \rangle_W\) and

\[
\ker \gamma_2 = \text{Im} \gamma_2 \oplus J \text{Im} \gamma_2 \oplus S.
\]

Write \(\pi_i\) for the projection of \(\ker \gamma_2\) on the \(i\)th factor with \(i = 1, 2, 3\).

**Proposition 2.8.** If we think \(\gamma_2\) as a real linear map we have an upper bound \(\text{rank}(\gamma_2) \leq \frac{2n-2}{3}\).
Proof. We have \( \text{Im} \gamma_2 \cap J \text{Im} \gamma_2 = \{0\} \). This is because if \( v_1 = Jv_2 \) with \( v_1, v_2 \in \text{Im} \gamma_2 \) then \( (v_1, v_1) = (v_1, Jv_2) = \omega(v_1, v_2) = 0 \) then \( v_1 = 0 \). So finally, since both \( \text{Im} \gamma_2 \) and \( J \text{Im} \gamma_2 \) are contained in \( \ker \gamma_2 \), we have \( 2\text{rank}(\gamma_2) \leq \dim \ker \gamma_2 \) then \( 3\text{rank}(\gamma_2) \leq \text{rank}(\gamma_2) + \dim \ker \gamma_2 = 2n - 2 \). \qed

Remark 2.9. We can notice that \( \pi_3(\gamma_3(is) - J\gamma_3(s)) = 0 \) for some \( s \in \mathbb{C} \), \( s \neq 0 \), if and only if \( \pi_3(\gamma_3(is) - J\gamma_3(s)) = 0 \) for all \( s \in \mathbb{C} \), \( s \neq 0 \). Indeed it suffices to notice that \( \gamma_3(is) - J\gamma_3(s) = s(\gamma_3(i) - J\gamma_3(1)) \) for all \( s \in \mathbb{C} \). Hence \( \gamma_3(is) - J\gamma_3(s) \in \text{Im} \gamma_2 + J \text{Im} \gamma_2 \) if and only if \( \gamma_3(i) - J\gamma_3(1) \) does.

**Proposition 2.10.** If there exists \( s \in \mathbb{C} \), \( s \neq 0 \), such that \( \pi_3(\gamma_3(is) - J\gamma_3(s)) = 0 \) then \( \text{Im} \gamma_3 \subseteq \text{Im} \gamma_2 \) and hence \( b_3 \) is 0 on \( \text{Im} \gamma_2 + J \text{Im} \gamma_2 \).

Proof. From Remark 2.9 if for some \( s \in \mathbb{C} \), \( s \neq 0 \) we have \( \pi_3(\gamma_3(is) - J\gamma_3(s)) = 0 \) then also \( \pi_3(\gamma_3(i) - J\gamma_3(1)) = 0 \). Hence let \( w_0 = \gamma_3(i) - J\gamma_3(1) \in \ker \gamma_2 \). If \( \pi_3(w_0) = 0 \) then write \( w_0 = w_1 + Jw_2 \) with \( w_1, w_2 \in \text{Im} \gamma_2 \). Condition (3) of Proposition 2.5 implies that \( 2\omega(\gamma_3(i), \gamma_3(1)) = b_3(w_0) = b_2(w_1 + Jw_2) = 2\omega(w_1, \gamma_3(1)) + 2\omega(w_2, \gamma_3(i)) \) hence \( \omega(w_1 - \gamma_3(i), \gamma_3(1)) + \omega(w_2, \gamma_3(i)) = 0 \) since \( w_1 - \gamma_3(i) = -J(w_2 + \gamma_3(1)) \) and \( \gamma_3(i) = w_1 + Jw_2 + J\gamma_3(1) \) substituting them in the previous expression we obtain \( \omega(J\gamma_3(1), \gamma_3(1)) + \omega(Jw_2, w_2) + 2\omega(Jw_2, \gamma_3(1)) = 0 \), i.e. \( \|\gamma_3(1)\|^2 + 2\|w_2, \gamma_3(1)\| + \|w_2\|^2 = 0 \). This implies \( \gamma_3(1) = -w_2 \), in other words \( \text{Im} \gamma_3 \subseteq \text{Im} \gamma_2 \) hence \( b_2 \) is 0 on \( \text{Im} \gamma_2 + J \text{Im} \gamma_2 \) \qed

As an abstract algebra \( u(\gamma_2, \gamma_3, b_2, b_3) \) can be described as the real vector space \( \mathbb{C} \times W \times \mathbb{C} \), with \( W = \mathbb{C}^{n-1} \), with Lie brackets

\[
[(r, u, s), (r', u', s')] = (h(u, \gamma(u', s')) - h(u', \gamma(u, s))) + i(b(u, s)s' - b(u', s')s), \gamma(u, s)s' - \gamma(u', s')s, 0)
\]

and with \( \gamma : W \times \mathbb{C} \to W \) and \( b : W \times \mathbb{C} \to \mathbb{R} \) linear maps, \( \gamma = \gamma_2 + \gamma_3 \) and \( b = b_2 + b_3 \) satisfying the conditions of Proposition 2.5.

Remark 2.11. From Remark 2.9 we saw that the condition \( \pi_3(\gamma_3(is) - J\gamma_3(s)) = 0 \) does not depend on the basis we have chosen for \( \mathbb{C} \) as a real vector space. Hence we will write this condition as \( \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0 \) where \( \{\xi, i\xi\} \) is any basis of \( \mathbb{C} \) as a real vector space.

**Definition 2.12.** We say that a Lie algebra \( g \) is a \( k \)-step nilpotent Lie algebra if \( C^{k+1}u = \{0\} \) and \( C^k u \neq \{0\} \) where \( C^k u = [u, C^{k-1}u] \) and \( C^1 u = u \).

**Proposition 2.13.** The Lie algebras \( u(\gamma_2, \gamma_3, b_2, b_3) \) are at most 3-step nilpotent. The lower central series looks like

\[
u(\gamma_2, \gamma_3, b_2, b_3) \supseteq C^2 u(\gamma_2, \gamma_3, b_2, b_3) \supseteq C^3 u(\gamma_2, \gamma_3, b_2, b_3) \supseteq \{0\}
\]

where \( C^2 u(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C} \otimes \mathbb{C} \text{Im}(\gamma_2) \oplus \mathbb{R} \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) \), \( C^3 u(\gamma_2, \gamma_3, b_2, b_3) \subseteq \mathbb{C} \).

Remark 2.14. More precisely if \( \gamma_2 \neq 0 \) then the Lie algebras \( u(\gamma_2, \gamma_3, b_2, b_3) \) are 3-step nilpotent and indeed we have equalities \( C^2 u(\gamma_2, \gamma_3, b_2, b_3) = \mathbb{C} \otimes \mathbb{C} \text{Im}(\gamma_2) \oplus \mathbb{R} \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) \) and \( C^3 u(\gamma_2, \gamma_3, b_2, b_3) = \mathbb{C} \). Instead if \( \gamma_2 = 0 \) then all possibilities can occur, see Appendix A for the case \( n = 3 \).

2.1. Classification up to conjugation. We now write the classification of the unipotent subgroup \( U(\gamma_2, \gamma_3, b_2, b_3) = \exp u(\gamma_2, \gamma_3, b_2, b_3) \) of \( H(n, 1) \) up to conjugation. This will be useful for the proof of Proposition 3.4.

**Proposition 2.15.** There exists \( g \in H(n, 1) \) such that \( g U(\gamma_2, \gamma_3, b_2, b_3) g^{-1} = U(\gamma_2', \gamma_3', b_2', b_3') \) if and only if there exist \( \lambda \in \mathbb{C}^*, \sigma \in U(n-1), v \in W, s_1 \in \mathbb{C} \) such that
Notice that the inverse of the linear part of $g$ and the translation part of $\tilde{g}$.

Since a parabolic subgroup is self normalizing if there exist $L(g) \in U(n,1)$ and $L(h) \in U$ such that $L(ghg^{-1}) \in U$ then $L(g) \in P$. Hence we may assume that

$$g = \begin{pmatrix} \lambda & -\sigma t & a & r_1 \\ 0 & \sigma & \lambda^{-1} \sigma v & u_1 \\ 0 & 0 & \lambda^{-1} & s_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\lambda \in \mathbb{C}^*$, $\sigma \in U(n-1)$, $v \in W$, $a \in \mathbb{C}$ and $\overline{\lambda a} + \lambda a = -\sigma t v$.

Notice that the inverse of the linear part of $g$ is of the form

$$L(g)^{-1} = \begin{pmatrix} \lambda^{-1} & \lambda^{-1} \sigma t & \bar{a} \\ 0 & \sigma t & -v \\ 0 & 0 & \lambda \end{pmatrix}.$$ 

Since $U(\gamma_2, \gamma_3, b_2, b_3) = \exp(u(\gamma_2, \gamma_3, b_2, b_3))$ and

$$g \exp(X) g^{-1} = \exp(gXg^{-1})$$

we might work on the level of the Lie algebra. We let

$$X = \begin{pmatrix} 0 & -\lambda(\gamma_2(u) + \gamma_3(s))^t & i(b_2(u) + b_3(s)) & r \\ 0 & 0 & \gamma_2(u) + \gamma_3(s) & u \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix} \in u(\gamma_2, \gamma_3, b_2, b_3),$$

then the linear part of $gXg^{-1}$ is

$$\begin{pmatrix} 0 & -\lambda(\gamma_2(u) + \gamma_3(s))^t \sigma t & \bar{X}(\gamma_2(\bar{u}) + \gamma_3(\bar{s})) v + i |\lambda|^2 (b_2(u) + b_3(s)) - \lambda \sigma t (\gamma_2(u) + \gamma_3(s)) \\ 0 & 0 & \lambda \sigma (\gamma_2(u) + \gamma_3(s)) & 0 \end{pmatrix}$$

and the translation part of $gXg^{-1}$ is

$$\begin{pmatrix} -\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v \\ \lambda^{-1} s \end{pmatrix}.$$ 

Let

$$(r', u', s') = (\ast, -\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v, \lambda^{-1} s).$$

When $s' = 0$ then $s = 0$ and we get $\gamma_2'(s) = \lambda \sigma (\gamma_2(u)) = \lambda \sigma \gamma_2(u)$. Take $u = \gamma_2(\bar{u})$ for some $\bar{u} \in W$ then we get $\gamma_2'(\sigma \gamma_2(\bar{u})) = 0$ hence $\gamma_2'(\sigma u) = \lambda \sigma \gamma_2(u)$ that is $\gamma_2' = \lambda \sigma \gamma_2^{-1}$.

When $u' = 0$, $u = \lambda s_1 \gamma_2(u) + \lambda s_3 \gamma_3(s) - \lambda^{-1} s \sigma v$ and we get $\gamma_3'(\lambda^{-1} s) = \lambda \sigma \gamma_2(u) + \lambda \sigma \gamma_3(s)$, hence $\gamma_3(\lambda^{-1} s) = \lambda \sigma \gamma_3(s) + \lambda \sigma \gamma_2(\lambda s_1 \gamma_3(s) - \lambda^{-1} s \sigma v)$ that is $\gamma_3'(s) = \lambda \sigma \gamma_3(s) + \lambda \sigma \gamma_2(\lambda s_1 \gamma_3(s)) - \lambda^{-1} s \sigma v$.

Now consider the equation

$$|\lambda|^2 b_2(u) + |\lambda|^2 b_3(s) - 2 \Re(\lambda \sigma t (\gamma_2(u) + \gamma_3(s))) = b_2'(-\lambda s_1 \sigma (\gamma_2(u) + \gamma_3(s)) + \sigma u + \lambda^{-1} s \sigma v) + b_3'(\lambda^{-1} s).$$
Let $s = 0$ then we are left with
$$|\lambda|^2 b_2(u) - 2\Im(\lambda^t \gamma_2(u)) = b_2^\prime(-\lambda s_1 \sigma \gamma_2(u) + \sigma u).$$

Take $u = \gamma_2(\bar{u})$ then we have $|\lambda|^2 b_2(\gamma_2(\bar{u})) = b_2^\prime(\sigma \gamma_2(\bar{u}))$. Hence the equation becomes $b_2^\prime(\sigma u) = |\lambda|^2 b_2(u - \lambda s_1 \gamma_2(u)) - 2\Im(\lambda^t \gamma_2(u))$.

Let $u = 0$ then the equation becomes
$$|\lambda|^2 b_3(s) - 2\Im(\lambda^t \gamma_3(s)) = b_2^\prime(-\lambda s_1 \sigma \gamma_3(s) + \lambda^{-1}s \sigma v) + b_3^\prime(\lambda^{-1}s),$$

substituting $b_2^\prime$ we get
$$|\lambda|^2 b_3(s) - 2\Im(\lambda^t \gamma_3(s)) = |\lambda|^2 b_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1}s v - \lambda s_1 \gamma_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1}s v))$$
$$- 2\Im(\lambda^t \gamma_2(-\lambda s_1 \gamma_3(s) + \lambda^{-1}s v)) + b_3^\prime(\lambda^{-1}s).$$

Hence the proposition follows. \hfill \qed

**Remark 2.16.** We saw in Proposition 2.10 that if $\pi_3(\gamma_3(i \xi) - J \gamma_3(\xi)) = 0$ then $b_2$ is 0 on $\Im \gamma_2 \oplus J \Im \gamma_2$. On the other hand when $\pi_3(\gamma_3(i \xi) - J \gamma_3(\xi)) \neq 0,$ then $\gamma_2 \neq 0$, using Proposition 2.15 we can conjugate the Lie group $U(\gamma_2, \gamma_3, b_2, b_3)$ so that $\pi_2(\gamma_3(\xi)) = 0$. In other words we can always suppose that $b_2$ is 0 on $\Im \gamma_2$.

### 3. Properly discontinuous and cocompact groups of Hermite-Lorentz affine motions

In this section we go back to our original question about crystallographic subgroups of $U(n, 1) \ltimes \mathbb{C}^{n+1}$ and prove the main result of this section, namely Theorem 1.6.

**Definition 3.1.** Consider the following subgroups of the Borel subgroup $B$ of $U(n, 1)$
$$\mathcal{D} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \mid \sigma = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}) \right\}$$

and
$$\tilde{B} = \left\{ \begin{pmatrix} \lambda & -\pi^t & a \\ 0 & \sigma & \lambda \sigma v \\ 0 & 0 & \lambda \end{pmatrix} \mid |\lambda|^2 = 1, \sigma = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{n-1}}), \Re(\lambda a) = -\frac{1}{2}\pi^t v \right\}.$$  

**Definition 3.2.** The **unipotent radical** of a linear group $G$ is the set of all unipotent elements in the largest connected solvable normal subgroup of $G$. It may be characterised as the largest connected unipotent normal subgroup of $G$.

**Proposition 3.3.** Let $H \leq \mathcal{H}(n, 1)$ be a subgroup that acts simply transitively on $a(V)$ then $H$ is $\mathcal{H}(n, 1)$-conjugate to a subgroup whose linear part is in $\tilde{B}$.

**Proof.** Since $H$ acts simply transitively by [2, Theorem I.1.] $H$ is solvable. Then its linear part $L(H) \leq U(n, 1)$ is solvable and the Zariski closure of $L(H)$ is solvable as well. Thus $H$ is conjugate in $\mathcal{H}(n, 1)$ to a group whose linear part is a subgroup of the Borel group $\tilde{B}$. \hfill \qed

**Proposition 3.4.** Let $H \leq \mathcal{H}(n, 1)$ be a group acting simply transitively on $a(V)$ such that $L(H) \leq B$. Then either $L(H) \leq \tilde{B}$ or $L(H)$ is $\tilde{B}$-conjugate to a subgroup of $\mathcal{D}$.  


Proof. Let $U$ be the unipotent radical of the Zariski closure $\tilde{H}$ of $H$, then by [2, Theorem III.1.] $U$ also acts simply transitively on $a(V)$. Hence by Proposition 2.5 we have that $U = U(\gamma_2, \gamma_3, b_2, b_3)$ for some $\gamma_2, \gamma_3, b_2, b_3$. By definition $U$ is a normal subgroup of $\tilde{H}$ hence $H$ normalizes $U$. That is for all $h \in H$ we have $hU(\gamma_2, \gamma_3, b_2, b_3)h^{-1} = U(\gamma_2, \gamma_3, b_2, b_3)$. Now for every fixed $h \in H$ its linear part is of the form

$$\begin{pmatrix}
\lambda & -\sigma^t & a \\
0 & \sigma & \lambda^{-1}\sigma v \\
0 & 0 & \lambda^{-1}
\end{pmatrix}$$

and it follows from Proposition 2.15 that $\gamma_2(u) = \lambda\sigma_2(\sigma^{-1}u)$, hence if $\gamma_2 \neq 0$ then $|\lambda|^2 = 1$. If $\gamma_2 = 0$ then $\gamma_3(s) = \lambda\sigma_3(\lambda s)$ and $b_2(u) = |\lambda|^2b_2(\sigma^{-1}u)$ so if $b_2$ is non zero then $|\lambda|^2 = 1$ if instead $\gamma_3$ is not 0 then $\lambda^2 = 1$. Finally if all of $\gamma_2, \gamma_3, b_2$ are identically 0 then $b_3(s) = |\lambda|^2b_3(\lambda s)$ and hence if it is non zero $\lambda = 1$. Hence if at least one of $\gamma_2, \gamma_3, b_2, b_3$ is non zero then $|\lambda|^2 = 1$ and $L(H) \leq \tilde{B}$, otherwise all of $\gamma_2, \gamma_3, b_2, b_3$ are 0 and hence $L(U) = \{Id\}$. Since $L(U)$ is the unipotent radical of $L(H)$ then this implies that $L(H)$ is reductive and being solvable and connected it is a torus. Hence $L(H)$ is $B$-conjugate to a subgroup of $D$. □

Definition 3.5. The nilradical, $N$, of a Lie group $G$ is the largest connected normal nilpotent subgroup of $G$. Analogously the nilradical, $n$, of a Lie algebra $g$ is the largest nilpotent ideal of $g$. Then we have that $N = \exp(n)$.

Lemma 3.6. Let $H \leq H(n, 1)$ be a group acting simply transitively on affine space and let $N$ be its nilradical. Consider furthermore its Zariski closure, $\tilde{H}$, inside $H(n, 1)$ and let $U$ be the unipotent radical of $\tilde{H}$, then $H \cap U = N$.

Proof. Clearly $H \cap U \subseteq N$. On the other side from [2, Corollary III.3] we have that $N \subseteq U$, hence the lemma follows. □

Lemma 3.7. Let $H$ be a simply connected connected solvable Lie group and $N$ its nilradical. Let $\Gamma$ be a lattice in $H$ then $\Gamma N/N \subseteq H/N$ is discrete.

Proof. From [3, Mostow Theorem] we know that $\Gamma N^0 = N$ hence $N$ is open in $\Gamma N$ and the lemma follows. □

We are now ready to prove the main result of this section.

Proof of Theorem 1.6. Since $\Gamma \leq H(n, 1)$ acts properly discontinuously and cocompactly on $a(V)$ then $\Gamma$ is virtually polycyclic, see [17, Theorem 3.1.]. Let $H$ be a subgroup of $Aff(V)$ acting simply transitively on $a(V)$ coming from Theorem 1.2. Then we know that $\Gamma \cap H$ has finite index in $\Gamma$ and is a lattice in $H$. Hence, after replacing $\Gamma$ with $\Gamma \cap H$, we can assume that $\Gamma$ is a lattice in $H$. Notice that actually $H$ is the connected component of the identity of a crystallographic hull of $\Gamma$ and after [12, Theorem 1.4.] the Zariski closure of $\Gamma$ is the same as the one of its crystallographic hull. Hence $H$ is contained in the Zariski closure of $\Gamma$ that lies inside $H(n, 1)$, it follows that $H \leq H(n, 1)$.

From Proposition 3.3 $H$ can be conjugated to a subgroup whose linear part is in $B$ and after Proposition 3.4 either $L(H) \leq \tilde{B}$ or $L(H)$ is conjugated to a subgroup of $D$.

If $L(H) \leq D$ then the unipotent radical of the Zariski closure of $H$, $\tilde{H}$, is contained in the group $T$ of pure translations of $H(n, 1)$. Then from Lemma 3.6 the nilradical of $H$ is contained in $TH$, the subgroup of pure translations that lies in $H$. From Lemma 3.7 the group
\[\Gamma/\Gamma \cap N \cong \Gamma N/N\] is discrete in \(H/T\) that is contained in \(\mathbb{C}^* \times U(n-1)\). Hence we can replace \(\Gamma N/N\) with a finite index subgroup which has trivial intersection with \(S^1 \times U(n-1)\). Doing this we obtain that \(\Gamma N/N\) is discrete in \(\mathbb{R}^*\) hence it is cyclic. Finally \(\Gamma \cap N\) is discrete in \(N\) which is some group of translations so \(\Gamma /\Gamma \cap N\) is isomorphic to \(\mathbb{Z}^m\) for some \(m\) and then for dimension reasons we have \(m = 2n + 1\). So finally in this case \(\Gamma\) has a finite index subgroup that is abelian by cyclic.

If \(L(H) \leq B\) then \(U\), the unipotent radical of \(\tilde{H}\), has linear part in \(U\). Hence from Lemma 3.6 the nilpotent radical of \(H\) is such that \(H/N\) is contained in the compact group \(S^1 \times U(n-1)\). Then, as before, being \(\Gamma /\Gamma \cap N\) discrete in \(H/N\), it is finite, hence up to finite index we can replace \(\Gamma\) by the group \(\Gamma \cap N\) inside \(N\). Now clearly \(\Gamma \cap N\) is a discrete subgroup of \(N\), let us see that it is also cocompact. Since both \(H\) and \(U\) act simply transitively on \(a(V)\) they are contractible. Hence the cohomology of the manifolds \(U/U \cap \Gamma\) and \(H/U \cap \Gamma\) is the same and the same as the cohomology of the group \(U \cap \Gamma\). But, having \(U \cap \Gamma\) finite index in \(\Gamma\) the manifold \(H/U \cap \Gamma\) is still compact, hence the cohomological dimension of \(\Gamma \cap U\) is \(2n + 2\). Finally from [7, Proposition 8.1 of Chapter VIII] the variety \(U/\Gamma \cap U\) is compact as well. Hence, up to finite index, \(\Gamma\) is a lattice inside a nilpotent group. \(\square\)

Let us now focus on the abelian by cyclic case.

**Definition 3.8.** For \(n \geq 1\) and \(A \in \text{GL}(2n+1, \mathbb{Z})\) diagonalisable with eigenvalues \(1\)'s and \(\lambda, \lambda^{-1}\) with multiplicity 2 let \(\Gamma(2n+2, A) = \mathbb{Z} \rtimes_A \mathbb{Z}^{2n+1}\).

Then the following propositions provide a classification in the abelian by cyclic case.

**Proposition 3.9.** Let \(n \geq 1\) and \(A, A' \in \text{GL}(2n+1, \mathbb{Z})\) then

- \(\Gamma(2n+2, A) \cong \Gamma(2n+2, A')\) if and only if \(A\) is \(\text{GL}(2n+1, \mathbb{Z})\)-conjugated to either \(A'\) or \(A'^{-1}\).
- \(\Gamma(2n+2, A)\) is commensurable with \(\Gamma(2n+2, A')\) if and only if \(A'\) is \(\text{GL}(2n+1, \mathbb{Q})\)-conjugated to \(A^s\) for some \(s \in \mathbb{Z} \setminus \{0\}\).

**Proof.** For the first claim let us notice that \(\mathbb{Z}^{2n+1}\) is a maximal abelian subgroup of \(\Gamma(2n+2, A)\). Indeed if \((r, u) \in \Gamma(2n+2, A)\), with \(r \neq 0\), commutes with \((0, u')\) for all \(u' \in \mathbb{Z}^{2n+1}\) then \((A' - \text{Id})u' = 0\) but then \(A'\) would be nilpotent and being diagonalisable it is the identity which is a contradiction. Hence we can write an isomorphism between \(\Gamma(2n+2, A)\) and \(\Gamma(2n+2, A')\) as \(\varphi(r, u) = (\varphi_1(r), \varphi_1^1(r) + \varphi_2^2(u))\). Then in order for \(\varphi\) to be a group homomorphism we need

\[\varphi_2^1(r + r') = A^{\varphi_1(r)}\varphi_2^1(r') + \varphi_2^2(r') \text{ and } \varphi_2^2(A'u' + u) = A^{\varphi_1(r)}\varphi_2^2(u') + \varphi_2^2(u).\]

Taking then \(u = 0\) we get \(\varphi_2^2(A'u') = A^{\varphi_1(r)}\varphi_2^2(u')\). Since \(\varphi_2^2 \in \text{GL}(2n+1, \mathbb{Z})\) and, being \(\varphi_1 : Z \to Z\) an isomorphism, \(\varphi_1\) sends 1 to \(\pm 1\), this means that \(A\) is \(\text{GL}(2n+1, \mathbb{Z})\)-conjugated to \(A'\) or \(A'^{-1}\). For the second claim just notice that a finite index subgroup \(\Gamma_0\) of \(\Gamma(2n+2, A)\) is of the form \(\mathbb{Z} \rtimes_B \mathbb{Z}^{2n+1}\) for some \(B \in \text{GL}(2n+1, \mathbb{Z})\). In order to see it as a finite index subgroup of \(\Gamma(2n+2, A)\) we have to give an injective morphism \(\varphi : \mathbb{Z} \rtimes_B \mathbb{Z}^{2n+1} \to \Gamma(2n+2, A)\). Using the same notation as before this means that \(\varphi_1(r) = mr\) with \(m \in \mathbb{Z} \setminus \{0\}\) and if we denote by \(\{v_i\}\) a basis for \(\mathbb{Z}^{2n+1}\) we have \(\varphi_2^2(v_i) = m_i v_i\) i.e. \(\varphi_2^2 \in \text{GL}(2n+1, \mathbb{Q})\). As before being \(\varphi\) a morphism we have \(\varphi_2^2(B'u') = A^{mr}\varphi_2^2(u')\). That means that \(B\) is \(\text{GL}(2n+1, \mathbb{Q})\)-conjugated to \(A^m\) for some \(m \in \mathbb{Z} \setminus \{0\}\). Apply then the first part of the lemma. \(\square\)

**Proposition 3.10.** Let \(n \geq 1\) then we have the following.
If $\Gamma \leq \mathcal{H}(n,1)$ is a crystallographic group and $\Gamma$ is not virtually nilpotent then $\Gamma$ contains a subgroup of finite index $\Gamma_0$ that is isomorphic to $\Gamma(2n+2,A)$ for some $A \in \text{GL}(2n+1,\mathbb{Z})$ diagonalisable with eigenvalues $1$’s and $\lambda, \lambda^{-1}$ with multiplicity $2$.

Every $\Gamma(2n+2,A)$ as before can be realised as a crystallographic group.

Proof. The first claim follows from the proof of Theorem 1.6. Indeed we are in the case where $\Gamma$ is a lattice in $H$ with $L(H) \leq \mathcal{D}$. For the second claim we can just realise the group $\Gamma(2n+2,A)$ as an affine group as follows

$$\begin{cases} A^{u_1} v \\ 0 \\ 1 \end{cases} | v = (r_1, r_2, u_2, \ldots, u_{2n-2}, s_1, s_2) \in \mathbb{Z}^2 \times \mathbb{Z}^{2n-3} \times \mathbb{Z}^2, u_1 \in \mathbb{Z}.$$}

Then since $A$ is diagonalisable with eigenvalues $1$’s and $\lambda, \lambda^{-1}$ of multiplicity $2$ this means that we can conjugate $A$ to a matrix belonging to $U(n,1)$ and hence conjugate the whole group to a subgroup of $\mathcal{H}(n,1)$.

\qed

4. Classification up to isomorphism, dimension 2, 3 and 4 the degenerate cases

The classification of the groups $U(\gamma_2, \gamma_3, b_2, b_3)$ up to isomorphism translates to the classification of the Lie algebras $u(\gamma_2, \gamma_3, b_2, b_3)$ up to isomorphism. For the Lie algebras that appear in dimension 2 and 3 we will use the terminology of [16]. All the other Lie algebras that appear in this section are defined in Appendix A where we also elucidate the correspondence with the terminology of [16] for isomorphism classes of nilpotent Lie algebras up to dimension 6 and of [15] for dimension 7. Let us start with the classification in small dimensions.

Example 4.1. For $n + 1 = 2$ we see that, up to isomorphism, we have just one non abelian Lie algebra. In fact with respect to the basis $\{\tau, i\tau, \xi, i\xi\}$ of $\mathbb{C} \times \mathbb{C}$ the Lie brackets are given by $[\xi, i\xi] = -b_3(\xi)\tau - b_3(i\xi)i\tau$. And we can see that if $b_3 \neq 0$ these Lie algebras are isomorphic to a Lie algebra that is the direct sum of the $3$-dimensional Heisenberg Lie algebra, $L_{3,2}$, and a one dimensional abelian ideal.

Example 4.2. For $n + 1 = 3$ let us notice that from Lemma 2.8 we have $\gamma_2 = 0$. Let us recall, from Section 2, the following notations: $W = S = \mathbb{C}$, $b_2 : W \to \mathbb{R}$, $\gamma_3 : \mathbb{C} \to W$ and $b_3 : \mathbb{C} \to \mathbb{R}$. Let $\{\tau, i\tau, g_1, g_2, \xi, i\xi\}$ be a basis of the Lie algebra. Let us denote by $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$.

1. Assume $w_0 = 0$.
2. Assume $w_0 \neq 0$.

(a) Assume $b_2 = 0$. Then it is clear that the Lie algebra is either abelian if $b_3 = 0$ or isomorphic to $L_{3,2} \oplus \mathbb{R}^3$.

(b) Assume $b_2 \neq 0$. Then using Proposition 2.15 we can conjugate the group in order to replace $b_2$ by $b'_2(u) = |\lambda|^2 b_2(\sigma^{-1} u)$ with $\sigma \in U(1)$ and $\lambda \in \mathbb{C}^*$ so that $b_2(u) = \omega(u, g_1)$ for $u \in W$. Furthermore we can modify $b_3$ by $b'_3(s) = b_3(s) + b_2(-sx)$ for some $x \in \mathbb{C}$, hence we may assume $b_3 = 0$. Hence the only non zero brackets are $[g_2, \xi] = i\tau$, $[g_2, i\xi] = -\tau$. Then in this case the Lie algebra is isomorphic to $L_{5,8} \oplus \mathbb{R}$.

2. Assume $w_0 \neq 0$.

(a) Assume that the real rank of $\gamma_3$ is $1$ and $b_2 = 0$. Then we might change $\gamma_3$ by $\gamma'_3(s) = \lambda \sigma \gamma_3(\lambda s)$ with $\lambda \in \mathbb{C}^*$ and $\sigma \in U(1)$ so that $\gamma_3(1) = \varepsilon g_1$ and $\gamma_3(i) = 0$, notice that since $\gamma_3 \neq 0$ we have $\varepsilon \neq 0$. Then the non zero brackets are $[g_1, \xi] = e\varepsilon \tau$, $[g_2, \xi] = i\varepsilon \tau$, $[\xi, i\xi] = -b_3(\xi) \tau - b_3(i\xi) i\tau - \varepsilon g_2$. We can then see that in this case the Lie algebra is isomorphic to $L_{6,25}$.
(b) Assume that the real rank of $\gamma_3$ is 1 and $b_2 \neq 0$. As before we conjugate the group so that $\gamma_3(1) = e_{g_1}$ and $\gamma_3(i) = 0$ with $\varepsilon \neq 0$. Then since $b_2(w_0) = 2\omega(\gamma_3(1), \gamma_3(1)) = 0$ we have that $b_2(g_2) = 0$ and since $b_2 \neq 0$ then $b_2(g_1) \neq 0$. Hence the Lie brackets are $[g_1, \xi] = e_{\xi} + b_2(g_1)i\tau, [g_2, \xi] = \varepsilon i\tau, [g_1, i\xi] = -b_2(g_1)\tau, [\xi, i\xi] = -b_3(\xi)\tau - b_3(i\xi)i\tau - e_{g_2}$. We can see they are all isomorphic to $L_{6,27}$.

(c) Assume that the real rank of $\gamma_3$ is 2. Notice that nevertheless $\gamma_3(\xi)$ and $\gamma_3(i\xi)$ are linearly dependent over $\mathbb{C}$ hence $\gamma_3(i\xi) = \lambda_1 + i\lambda_2 \in \mathbb{C}$. Let $v_1 = \frac{\gamma_3(\xi)}{||\gamma_3(\xi)||} \in W$ so that $h(v_1, \gamma_3(\xi)) = 1$ and $h(v_1, \gamma_3(\xi)) = \lambda$. Let $\alpha = ||\lambda(\xi)||^2$ and $\{x_1, y_1\}$ be a basis for $W$ as a real vector space, we have $w_0 = \alpha_1 x_1 - \alpha(\lambda_2 + 1)y_1$, then the Lie brackets are

\[
[x_1, x_1] = \tau_1 + b_2(x_1)\tau_2, \quad [y_1, \xi] = (1 + b_2(y_1))\tau_2,
\]

\[
[x_1, \xi] = (\lambda_1 - b_2(x_1))\tau_1 + \lambda_2 \tau_2, \quad [y_1, \xi] = (-\lambda_2 - b_2(y_1))\tau_1 + \lambda_2 \tau_2,
\]

\[
[\xi_1, \xi_2] = -b_3(1)\tau_1 - b_3(i)\tau_2 - w_0.
\]

Notice that we have

\[
[w_0, \xi_1] = \alpha_1 \tau_1 - \alpha(1 + 3\lambda_2)\tau_2,
\]

\[
[w_0, \xi_2] = \alpha(\lambda_1^2 + \lambda_2^2 + 3\lambda_2)\tau_1 - \alpha_1 \lambda_1 \tau_2.
\]

Now if the transformation $\tau_1 = \alpha_1 \tau_1 + (2\alpha \lambda_2 - \alpha(\lambda_2 + 1))\tau_2, \tau'_2 = (\alpha(\lambda_1^2 + \lambda_2^2 + \lambda_2) - 2\alpha \lambda_2)\tau_1 - \alpha_1 \lambda_1 \tau_2$ has non zero determinant we can bring the Lie algebra to the following form

\[
[x_1, x_2] = x_3, \quad [x_1, x_3] = x_5, \quad [x_2, x_3] = x_6,
\]

\[
[x_1, x_4] = ax_5 + bx_6, \quad [x_2, x_4] = cx_5 + dx_6
\]

that furthermore can be brought to $L_{6,24}(\varepsilon)$. If instead the determinant is 0 we can bring it to the standard form $L_{6,23}$.

Now we want to give representatives of the isomorphism classes of the groups $U(\gamma_2, \gamma_3, b_2, b_3)$ when either $\tau_3(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$ or $\text{rank}(\gamma_2) = 0$ in dimension 4. The dimension 4 is the smallest dimension where $\gamma_2$ can be non trivial.

Proposition 4.3. The list of Lie algebras $L_j$ for $j = 1, \ldots, 6$ given in Appendix A with $K = \mathbb{R}$ is a complete non redundant list of isomorphism classes of the Lie algebras $u(\gamma_2, \gamma_3, b_2, b_3)$ when $\tau_3(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$ and $n + 1 = 4$.

Proof. Let us call $w_0 := \gamma_3(i\xi) - J\gamma_3(\xi)$. If $\tau_3(w_0) = 0$ we saw in Proposition 2.10 that $\text{Im} \gamma_3 \subseteq \text{Im} \gamma_2$ and $b_2 = 0$ on $\text{Im} \gamma_2 + J \text{Im} \gamma_2$. If $\gamma_2 = 0$ the result follows easily and we find the Lie algebras $L_i$ with $i = 1, 2, 3$ that are 2-step nilpotent. Hence let us suppose $\text{rank}(\gamma_2) = 1$ and let $\{\tau, i\tau, e, i\xi, f, \xi, i\xi\}$ be a basis for $\mathbb{C} \times W \times \mathbb{C}$ that respects the decomposition (2.1). Then $\gamma_2(f) = \delta e, \gamma_3(\xi) = \nu e$ and $\gamma_3(i\xi) = \mu e$ with $\delta, \nu, \mu \in \mathbb{R}$. If $f' = \delta^{-1} f$ the Lie brackets become

\[
[e, f'] = \tau, \quad [e, \xi] = \nu \tau, \quad [e, i\xi] = \mu \tau,
\]

\[
[J e, f'] = i\tau, \quad [J e, \xi] = \nu i\tau, \quad [J e, i\xi] = \mu i\tau,
\]

\[
[g, \xi] = b_2(g) i\tau, \quad [g, i\xi] = -b_2(g) \tau,
\]

\[
[f', \xi] = b_2(f') i\tau + e, \quad [f', i\xi] = -b_2(f') \tau + J e,
\]

\[
[\xi, i\xi] = -b_3(\xi) \tau - b_3(i\xi)i\tau - w_0.
\]
After replacing \(e\) by \(b_2(f')(\omega + e) + Je\) we can assume that \([f', \xi] = e\) and \([f', i\xi] = Je\). Furthermore, letting \(\xi_1 = \xi - \nu f'\) and \(\xi_2 = i\xi - \mu f'\), we have \([e, \xi_1] = [e, \xi_2] = [J e, \xi] = 0\) and \([\xi_1, \xi_2] = br + c i \tau\) for some \(b, c \in \mathbb{R}\). Now, if \(b_2(g) = 0\) and \(b_3 = 0\), then we can find an isomorphism between the Lie algebra \(\mathfrak{u}(\gamma_2, \tau, b_2, b_3)\) and \(L_4\). If instead \(b_2(g) = 0\) but \(b \neq 0\) then defining \(x_1 = f', x_2 = -(b_2(c) + c_2), x_3 = -b^{-1}c_2, x_4 = -(b e + c J e), x_5 = -br + c i \tau, x_7 = b^{-1}i \tau\) we see that the Lie algebra \(\mathfrak{u}(\gamma_2, \tau, b_2, b_3)\) is isomorphic to \(L_5\). Finally, assuming \(b_2(g) \neq 0\), we can define \(x_1 = f', x_2 = \xi + \frac{b}{b_2(g)} g, x_3 = i\xi + \frac{b}{b_2(g)} g, x_4 = e, x_5 = J e, x_6 = \frac{1}{b_2(g)} g, x_7 = -\tau\) and \(x_8 = -i \tau\) in order to see that \(\mathfrak{u}(\gamma_2, \tau, b_2, b_3)\) is isomorphic to \(L_6(1)\).

Next we give the classification in dimension 4 for the case \(\pi_3(\gamma_3(\xi) - J\gamma_3(\xi)) \neq 0\) and \(\gamma_2 = 0\).

**Proposition 4.4.** The list of Lie algebras \(N_j\) for \(j = 1, \ldots, 19\) given in Appendix A with \(k = \mathbb{R}\) is a complete non redundant list of isomorphism classes of the Lie algebras \(\mathfrak{u}(\gamma_2, \tau, b_2, b_3)\) when \(\pi_3(\gamma_3(\xi) - J\gamma_3(\xi)) \neq 0\), \(\gamma_2 = 0\) and \(n + 1 = 4\).

**Proof.** Let \(\{\tau, i \tau, g_1, g_2, g_3, g_4, i \xi, a_i\}\) be a basis for the real vector space \(\mathbb{C} \times W \times \mathbb{C}\). Let us call \(w_0 := \gamma_3(\xi) - J\gamma_3(\xi)\) then the Lie brackets of \(u(0, \gamma_3, \tau, b_2, b_3)\) read as

\[
[g_j, \xi] = (g_j, \gamma_3(\xi))\tau + (\omega(g_j, \gamma_3(\xi)) + b_2(g_j)) i \tau,
\]

\[
[g_j, i \xi] = ((g_j, \gamma_3(i \xi) - b_2(g_j))\tau + \omega(g_j, \gamma_3(i \xi)) i \tau,
\]

\[
[\xi, i \xi] = -b_3(\xi)\tau - b_3(i \xi) i \tau - w_0.
\]

First of all by redefining \(w_0\) we might assume that \(\{\xi, i \xi\} = w_0\). Let \(b_2(u) = \langle u, v_0 \rangle\), for some vector \(v_0 \in W\). Then we can notice that the restrictions of \(ad(\xi)\) and \(ad(i \xi)\) to \(W\) define two linear maps \(W \to \mathbb{C}\) and that taking their real and imaginary part we get four linear forms over the reals. Call these linear forms \(\alpha_i\), explicitly \(\alpha_1 = \langle \cdot, \gamma_3(\xi) \rangle, \alpha_2 = \langle \cdot, J\gamma_3(\xi) + v_0 \rangle, \alpha_3 = \langle \cdot, \gamma_3(i \xi) - v_0 \rangle, \alpha_4 = \langle \cdot, J\gamma_3(\xi) \rangle\). For \(i = 1, \ldots, 4\) let us denote by \(v_i\) the vector associated with the linear form \(\alpha_i\). Notice that, if \(Z\) is the center of \(u(0, \gamma_3, \tau, b_2, b_3)\), the center of \(u(0, \gamma_3, \tau, b_2, b_3)/\mathbb{C}\tau\) is \(W\), hence \(\dim(W \cap Z)\) is an invariant and we have

\[
\dim(Z \cap W) = \dim \left(\bigcap \ker \alpha_i\right) = \dim \left(\text{span}\{v_i\}\right).
\]

Since \(w_0 \neq 0\) we have \(\dim(\text{span}\{v_i\}) \geq 1\) hence \(0 \leq \dim(Z \cap W) \leq 3\).

Let us first assume \(\dim(\text{span}\{v_i\}) = 4\), take then as basis for \(W\) the dual basis of the \(\alpha_i\), call it \(\{z_i\}\), with respect to which we have

\[
[z_1, \xi] = \tau, \quad [z_2, \xi] = i \tau, \quad [z_3, \xi] = \tau, \quad [z_4, i \xi] = i \tau, \quad [\xi, i \xi] = w_0.
\]

If \(w_0 = \sum a_i z_i\), then \([\xi, i \xi], [\xi, \xi], [\xi, i \xi], [\xi, \xi], [\xi, i \xi], [\xi, \xi] = a_1 \tau + a_2 i \tau, [\xi, i \xi], i \xi] = a_3 \tau + a_4 i \tau\). Let us write the constants of structure in a matrix \(A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\) and let us represent a change of basis in \(\text{span}\{\xi, i \xi\}\) with \(P \in \text{GL}(2, \mathbb{R})\). Then the matrix that represents the constants of structure in the new basis is just \(P A\). Hence, depending on the rank of the matrix \(A\), using the left action of \(\text{GL}(2, \mathbb{R})\) just defined, we can bring \(A\) to one of the three normal forms \(\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\).

The first matrix, if we call \(z_1' = z_1 + a z_2\), leads to the following Lie brackets

\[
[\xi, i \xi] = z_1', \quad [z_1', i \xi] = \tau + a i \tau
\]

\[
[z_2, \xi] = i \tau, \quad [z_3, i \xi] = \tau, \quad [z_3, i \xi] = i \tau.
\]
and one can easily see that this Lie algebra is isomorphic to \( N_{16} \). The second normal form brings as well to the class \( N_{16} \). For the third, after calling \( z'_1 = z_1 + z_4 \), we have the following Lie brackets

\[
\begin{align*}
\{\xi, i\xi\} &= z'_1, \\
\{z'_1, \xi\} &= \tau, \\
\{z'_1, i\xi\} &= i\tau, \\
\{z_2, \xi\} &= i\tau, \\
\{z_3, i\xi\} &= \tau, \\
\{z_4, i\xi\} &= i\tau
\end{align*}
\]

and this Lie algebra is isomorphic to the Lie algebra \( N_{17} \).

Let us now assume that \( \dim(\langle v_1 \rangle) = 3 \). If \( w_0 \in Z \cap W \) suppose then that \( \{v_1, v_2, v_3\} \) are linearly independent, indeed the other combinations can be reduced to this case by a change of variables. Complete \( \{\alpha_1, \alpha_2, \alpha_3\} \) to \( \{\alpha_1, \alpha_2, \alpha_3, \beta\} \) in order to have a basis for \( W^* \). Let us take the dual basis, call it \( \{z_i\}_{i=1}^3 \), as basis for \( W \). The Lie brackets expressed in this basis become

\[
\begin{align*}
\{z_1, \xi\} &= \tau, \\
\{z_1, i\xi\} &= \alpha_4(z_1)i\tau, \\
\{z_2, \xi\} &= i\tau, \\
\{z_2, i\xi\} &= \alpha_4(z_2)i\tau, \\
\{z_3, i\xi\} &= \tau + \alpha_4(z_3)i\tau, \\
\{\xi, i\xi\} &= w_0 = z_4.
\end{align*}
\]

After a change of basis we can arrive to the following form

\[
\begin{align*}
\{z_1, \xi\} &= \tau, \\
\{z_2, \xi\} &= \tau_2, \\
\{z_1, \xi_2\} &= \mu\tau_2, \\
\{z_3, \xi_2\} &= \tau_1, \\
\{\xi_1, \xi_2\} &= z_4
\end{align*}
\]

with \( \mu = \alpha_4(z_1) + \alpha_4(z_2)\alpha_4(z_3) \) and, depending on whether \( \mu \) is 0 or not, we find the normal forms \( N_{18} \) or \( N_{19} \). If instead \( w_0 \notin Z \cap W \) there exists \( x \in Z \cap W \) such that \( u(0, \gamma_3, b_2, b_3) \cong Rx \oplus \mathfrak{g}' \) where \( \mathfrak{g}' \) is a 7-dimensional 3-nilpotent Lie algebra with a 2-dimensional center, hence in Gong’s list \( \mathfrak{g}' \) can be one of \((257\delta)\) with \( \delta \in \{A, \ldots, L\} \) but some of them can be excluded since we know that \( W \cong \mathbb{C}^3 \) is an abelian subalgebra, so we get that the only possibilities are \((257\hat{\delta})\) with \( \delta \in \{A, B, C, D, I, J, H\} \). One can see that each of them can be realised hence \( u(0, \gamma_3, b_2, b_3) \) is isomorphic to one of \( N_j \) with \( j = 11, \ldots, 14 \) or \( N_{15}(\varepsilon) \) with \( \varepsilon \in \{0, 1, -1\} \).

Let us now assume that \( \dim(\langle v_1 \rangle) = 3 \). If \( w_0 \notin Z \cap W \) then we exist \( x \in Z \cap W \) such that \( u(0, \gamma_3, b_2, b_3) \cong Rx \oplus \mathfrak{g}' \) where \( \mathfrak{g}' \) is a 7-dimensional 2-nilpotent Lie algebra with a 3-dimensional center. Hence if \( \mathfrak{g}' \) is indecomposable it can be one of \((37\delta)\) with \( \delta \in \{A, \ldots, D, B_1, D_1\} \) in Gong’s list, otherwise it can be \( L_{6,22}(\varepsilon) \oplus \mathbb{R} \) as in de Graaf’s list. We can exclude \((37D), (37D_1) \) and \( L_{6,22}(\varepsilon) \) with \( \varepsilon \neq 0 \) since in the Lie algebras \( u(0, \gamma_3, b_2, b_3) \) we are considering there exists an abelian ideal of dimension 4 that has a 2-dimensional intersection with the derived algebra, namely \((W \setminus \langle x, w_0 \rangle) \oplus \mathfrak{c} \) that does not exists in the just aforementioned Lie algebras. Finally we can see that all the others can be realised. Then \( u(0, \gamma_3, b_2, b_3) \) is one of \( N_8, N_9(\varepsilon) \) with \( \varepsilon \in \{0, 1, -1\} \) or \( N_{10} \). If instead \( w_0 \notin Z \cap W \) then there exist \( x, y \in Z \cap W \) such that \( u(0, \gamma_3, b_2, b_3) \cong \mathfrak{span}\{x, y\} \oplus \mathfrak{g}' \) where \( \mathfrak{g}' \) is a 6-dimensional 3-nilpotent Lie algebra with a center of dimension 2, hence one of \( L_{6,23}, L_{6,24}(\varepsilon), L_{6,25}, L_{6,27} \) or \( L_{5,5} \oplus \mathbb{R} \) in de Graaf’s list. One can see that all of these Lie algebras can be realised, hence \( u(0, \gamma_3, b_2, b_3) \) is isomorphic to one of \( N_j \) with \( j = 3, \ldots, 7 \).

Finally assume that \( \dim(\langle v_1 \rangle) = 1 \). The case \( w_0 \in Z \cap W \) can happen only if \( \gamma_3(\xi) \) and \( \gamma_3(\xi) \) are linearly independent over \( C \). Indeed, since \( w_0 \) is in the center, we must have \( \langle \gamma_3(\xi), \gamma_3(\xi) \rangle = 0 \) and \( \|\gamma_3(\xi)\|^2 = 3\omega(\gamma_3(\xi), \gamma_3(\xi)) = \|\gamma_3(\xi)\|^2 \). Hence, if \( \gamma_3(\xi) = \lambda \gamma_3(\xi) \) with \( \lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} \), we get a contradiction. Then \( v_1 \) and \( v_4 \) are linearly independent. Hence, if \( \dim(\langle v_1 \rangle) = 1 \) then \( w_0 \notin Z \cap W \) and \( u(0, \gamma_3, b_2, b_3) \cong \mathbb{R}^3 \oplus \mathfrak{g}' \) with \( \mathfrak{g}' \) a 5-dimensional 3-nilpotent Lie algebra with a 2-dimensional center, hence either \( L_{5,9} \) or \( L_{4,3} \oplus \mathbb{R} \). So finally in this case \( u(0, \gamma_3, b_2, b_3) \) is isomorphic to one of \( N_j \) with \( j = 1, 2 \).
5. Classification up to isomorphism, dimension 4 the general case

In this section we will treat the most general case \( \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0 \) and \( \gamma_2 \neq 0 \) for dimension 4. We will first see how we can bring these Lie algebras to a simpler form. Then we will introduce the concept of Carnot Lie algebras and see how the classification up to isomorphism is translated to the classification of the orbits of the SL(3, \( \mathbb{R} \))-adjoint action on \( \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \). Finally we will study separately a particular case, namely the case \( \alpha = 0 \) before studying the most general case.

5.1. Reduction. Let us introduce the following family of Lie algebras to which the Lie algebras \( u(\gamma_2, \gamma_3, b_2, b_3) \) are isomorphic in the case \( \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0 \) and rank(\( \gamma_2 \)) = 1 in dimension 4.

**Definition 5.1.** Let \( \alpha, a, b, c \in \mathbb{R} \) and define the family of Lie algebras \( \{g(\alpha, a, b, c)\}_{\alpha, a, b, c} \) by the following non zero Lie brackets, expressed in the basis \( \{x_1, \ldots, x_8\} \) of \( \mathbb{R}^8 \):

\[
\begin{align*}
[x_1, x_2] &= x_4, & [x_1, x_3] &= x_5, & [x_2, x_3] &= x_6, \\
[x_1, x_4] &= x_7, & [x_1, x_5] &= x_8, \\
[x_2, x_4] &= ax_7, & [x_2, x_5] &= -ax_8, \\
[x_3, x_4] &= -ax_8, & [x_3, x_5] &= 3ax_7, \\
[x_2, x_6] &= ax_7 + bx_8, & [x_3, x_6] &= cx_7 - ax_8.
\end{align*}
\]

**Proposition 5.2.** If \( \pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) \neq 0, n + 1 = 4 \) and rank(\( \gamma_2 \)) = 1 then the Lie algebras \( u(\gamma_2, \gamma_3, b_2, b_3) \) are isomorphic to \( g(\alpha, a, b, c) \) for some \( \alpha, a, b, c \).

**Proof.** Let \( \{\tau, i\tau, e, Je, g, f, \xi, i\xi\} \) be a basis for \( \mathbb{C} \times W \times \mathbb{C} \) adapted to the decomposition (2.1) and let \( w_0 := \gamma_3(i\xi) - J\gamma_3(\xi) \). The Lie brackets of \( u(\gamma_2, \gamma_3, b_2, b_3) \) expressed in this basis are as follows

\[
\begin{align*}
[e, f] &= h(e, \gamma_2(f))\tau, & [Je, f] &= h(Je, \gamma_2(f))\tau, \\
[e, \xi] &= (h(e, \gamma_3(\xi)) + ib_2(e))\tau, & [e, i\xi] &= (h(e, \gamma_3(i\xi)) - b_2(e))\tau, \\
[Je, \xi] &= (h(Je, \gamma_3(\xi)) + ib_2(\xi))\tau, & [Je, i\xi] &= (h(\xi, \gamma_3(i\xi)) - b_2(\xi))\tau, \\
[g, \xi] &= (h(g, \gamma_3(\xi)) + ib_2(\xi))\tau, & [g, i\xi] &= (h(g, \gamma_3(i\xi)) - b_2(\xi))\tau, \\
[f, \xi] &= (h(f, \gamma_3(\xi)) + ib_2(f))\tau + \gamma_2(\xi), & [f, i\xi] &= (h(f, \gamma_3(i\xi)) - b_2(f))\tau + J\gamma_2(f), \\
[\xi, i\xi] &= (-b_3(\xi) - ib_3(i\xi))\tau - w_0.
\end{align*}
\]

First of all, up to conjugating the group, we can assume that \( b_2(e) = 0 \) i.e. \( \langle e, J\gamma_3(\xi) \rangle = 0 \) and \( b_2(f) = 0 \). If \( \gamma_2(f) = \delta e \), with \( \delta \in \mathbb{R}^* \), making the following change of variables

\[
\begin{align*}
\xi_1 &= \xi - \frac{1}{\delta}((e, \gamma_3(\xi)) - \langle e, \gamma_3(i\xi) \rangle) f \\
\xi_2 &= i\xi - \frac{1}{\delta}(e, \gamma_3(i\xi)) f
\end{align*}
\]

we have that \( \gamma_3(\xi_1) = \gamma_3(\xi) - ((e, \gamma_3(\xi)) - \langle e, \gamma_3(i\xi) \rangle) e \) and \( \gamma_3(\xi_2) = \gamma_3(\xi) - e, \gamma_3(i\xi) e \), in other words \( \langle \gamma_3(\xi_2), e \rangle = 0 \) and \( \langle \gamma_3(\xi_1), e \rangle = \langle \gamma_3(\xi_2), Je \rangle \). Then \( \xi_1, \xi_2 = -b_3(\xi_1)\tau - b_3(\xi_2)i\tau - w_0^\prime \) with \( w_0^\prime = \gamma_3(\xi_2) - J\gamma_3(\xi_1) \) and \( \langle w_0^\prime, e \rangle = \langle w_0^\prime, Je \rangle = 0 \), hence \( w_0^\prime = \beta g \) for some \( \beta \in \mathbb{R}^* \).

Defining now \( x_1 = \delta^{-1}f = f', x_2 = \xi_1 \) and \( x_3 = \xi_2 \) we have

\[
\begin{align*}
[x_1, x_2] &= h(f', \gamma_3(\xi_1))\tau + e, \\
[x_1, x_3] &= h(f', \gamma_3(\xi_2))\tau + Je, \\
[x_2, x_3] &= -b_3(\xi_1)\tau - b_3(\xi_2)i\tau - w_0^\prime.
\end{align*}
\]
Hence let us define $x_4 = h(f', \gamma_3(\xi_1)) \tau + e$, $x_5 = h(f', \gamma_3(\xi_2)) \tau + Je$, $x_6 = -b_3(\xi_1) \tau - b_3(\xi_2) i \tau - u'_0$, $x_7 = -\tau$ and $x_8 = -i \tau$. Then, remembering that $b_2(Je) = 2(e, J\gamma_3(\xi_2)) = -2(\gamma_3(\xi_2), Je)$, we have

\[
\begin{align*}
[x_1, x_4] &= x_7, \quad [x_2, x_4] = \langle \gamma_3(\xi_2), Je \rangle x_7, \quad [x_3, x_4] = -\langle \gamma_3(\xi_2), Je \rangle x_8, \\
[x_1, x_5] &= x_8, \quad [x_2, x_5] = -\langle \gamma_3(\xi_2), Je \rangle x_8, \quad [x_3, x_5] = 3\langle \gamma_3(\xi_2), Je \rangle x_7, \\
[x_2, x_6] &= -(\langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle x_7 + 3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle - \|\gamma_3(\xi_1)\|^2)x_8, \\
[x_3, x_6] &= -(3\langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle x_7 - \langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle x_8).
\end{align*}
\]

Calling $\alpha = \langle \gamma_3(\xi_2), Je \rangle$,

\[
\begin{align*}
a &= -\langle \gamma_3(\xi_2), \gamma_3(\xi_1) \rangle, \\
b &= -(3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle - \|\gamma_3(\xi_1)\|^2), \\
c &= -(\|\gamma_3(\xi_2)\|^2 - 3\langle \gamma_3(\xi_2), J\gamma_3(\xi_1) \rangle)
\end{align*}
\]

we can see the isomorphism with $g(\alpha, a, b, c)$. □

**Remark 5.3.** If $a = b = c = 0$ the center of the corresponding Lie algebras has dimension 3 and is generated by $\{x_6, x_7, x_8\}$, otherwise it is of dimension 2 and is generated by $\{x_7, x_8\}$.

### 5.2. Carnot Lie algebras.

**Definition 5.4.** A Carnot grading on a Lie algebra $g$ is an algebra grading of $g$, $g = \oplus_{i \geq 1} g_i$, such that $g$ is generated by $g_1$. A Lie algebra is Carnot graded if it is endowed with a Carnot grading and Carnot if it admits a Carnot grading.

**Definition 5.5.** Given a Lie algebra $g$ the direct sum $\text{Car}(g) = \bigoplus_{i \geq 1} v_i$, where $v_i = g^i / g^{i+1}$, endowed with the Lie brackets induced on each quotient by the ones of the Lie algebra $g$ is called the associated Carnot-graded Lie algebra to the Lie algebra $g$.

**Proposition 5.6** ([9, Proposition 3.5.]). A Lie algebra is Carnot if and only if it is isomorphic, as a Lie algebra, to its associated Carnot-graded Lie algebra. Furthermore if these conditions hold, then:

- for any Carnot grading on $g$, the graded Lie algebras $g$ and $\text{Car}(g)$ are isomorphic,
- for any two Carnot gradings on $g$, there is a unique automorphism mapping the first to the second and inducing the identity modulo $[g, g]$.

**Corollary 5.7** ([9, Corollary 3.6.]). Let $g$ be a Carnot graded Lie algebra. Denote by $\text{Aut}(g)$ its automorphism group as a Lie algebra and $\text{Aut}(g)_0$ its automorphism group as a graded Lie algebra. Let $\text{Aut}(g)_{\geq 1}$ be the group of automorphism of the Lie algebra $g$ inducing the identity on $g/[g, g]$. Then $\text{Aut}(g)_{\geq 1}$ is a normal subgroup and

\[
\text{Aut}(g) = \text{Aut}(g)_0 \rtimes \text{Aut}(g)_{\geq 1}
\]

**Example 5.8.** Here is an example of a nilpotent Lie algebra that is not Carnot. It is given by the following non zero Lie brackets on the basis $\{x_1, \ldots, x_5\}$

\[
\begin{align*}
[x_1, x_3] &= x_4, \\
[x_1, x_4] &= [x_2, x_3] = x_5.
\end{align*}
\]

**Remark 5.9.** Each Lie algebra $g(\alpha, a, b, c)$ is Carnot with grading $v_1 = \text{span}_\mathbb{R}\{x_1, x_2, x_3\}$, $v_2 = \text{span}_\mathbb{R}\{x_4, x_5, x_6\}$ and $v_3 = \text{span}_\mathbb{R}\{x_7, x_8\}$.

**Example 5.10.** Every free $k$-step nilpotent Lie algebra of rank $n$ is Carnot.
Proposition 5.11. Every quotient of the free k-step nilpotent Lie algebra of rank n by a graded ideal is a Carnot nilpotent Lie algebra.

Proof. Denote by $F_{k,n}$ the free k-step nilpotent Lie algebra of rank n and let $\mathfrak{h}$ be an homogeneous ideal of $F_{k,n}$, if $F_{k,n} = \bigoplus_{i=1}^{k} F_i$ we have $\mathfrak{h} = \bigoplus_{i} (\mathfrak{h} \cap F_i) = \bigoplus_{i} \mathfrak{h}_i$. Then the quotient $\mathfrak{g} = F_{k,n}/\mathfrak{h}$ inherits the grading and it will be generated by $F_1/\mathfrak{h}_1$ since $\mathfrak{h}_1$ is an ideal. Indeed if an element $x \in \mathfrak{g}$ is obtained as the brackets of elements in $F_1$ with some of them in $\mathfrak{h}_1$ then $x$ is itself in $\mathfrak{h}_1$. Finally the quotient will be a nilpotent Lie algebra. □

Let us introduce a general setting in which we will see our family of Lie algebras $\mathfrak{g}(\alpha, a, b, c)$.

Definition 5.12. Let $F$ be the free 3-step nilpotent Lie algebra on 3 generators. Then with respect to the basis $\{y_1, \ldots, y_{14}\}$ of $\mathbb{R}^{14}$ the Lie brackets are as follows

\[
\begin{align*}
[y_1, y_2] &= y_4, & [y_1, y_3] &= y_5, & [y_2, y_3] &= y_6, \\
[y_1, y_4] &= y_7, & [y_1, y_5] &= y_8, & [y_1, y_6] &= y_9, \\
[y_2, y_4] &= y_{10}, & [y_2, y_5] &= y_{11}, & [y_2, y_6] &= y_{12}, \\
[y_3, y_4] &= y_{11} - y_9, & [y_3, y_5] &= y_{13}, & [y_3, y_6] &= y_{14}.
\end{align*}
\]

We also have a grading of $F$ as $F \cong \bigoplus_{i=1}^{3} F_i$ where $F_i = F^i/F^{i+1}$. Notice that $F_1 = \text{span}_{\mathbb{R}}\{y_1, y_2, y_3\}$, $F_2 = \text{span}_{\mathbb{R}}\{y_4, y_5, y_6\}$ and $F_3 = \text{span}_{\mathbb{R}}\{y_7, \ldots, y_{14}\}$, where we are using an abuse of notation, thinking the elements of the basis of $F_i$ as equivalence classes.

Proposition 5.13. We have an action of $\text{GL}(3, \mathbb{R})$ on $F_3$.

Proof. By definition each element of the basis of $F_3$ is uniquely determined as the Lie brackets of an element of $F$ and an element of the derived algebra of $F$, $F^2$. Then we can define the action of $\text{GL}(3, \mathbb{R})$ on $F_3$ as follows, let $M \in \text{GL}(3, \mathbb{R})$ and $v$ an element of the basis of $F_3$ then $M * v := [Mu_1, [Mu_2, Mu_3]]$ where $u_1, u_2, u_3 \in F_1$ are the unique vectors such that $v = [u_1, [u_2, u_3]]$, then we extend this action on $F_3$ by linearity. □

Remark 5.14. We have hence a representation $\rho : \text{GL}(3, \mathbb{R}) \to \text{GL}(F_3)$, and by abuse of notation let us call $\rho$ also the induced representation $\rho : \text{SL}(3, \mathbb{R}) \to \text{GL}(F_3)$.

Proposition 5.15. There exists an $\text{SL}(3, \mathbb{R})$-equivariant isomorphism $\varphi : \mathfrak{sl}(3, \mathbb{R}) \to F_3$, where $\text{SL}(3, \mathbb{R})$ acts on $\mathfrak{sl}(3, \mathbb{R})$ via the adjoint action and on $F_3$ via $\rho$.

Proof. Let us denote by $E_{ij}$ the $3 \times 3$ matrix whose $(i,j)$-th entry is 1 and all the rest is 0. Then the matrices $E_i, i = 1, \ldots, 8$, where $E_1 = E_{11} - E_{22}, E_2 = E_{22} - E_{33}, E_3 = E_{12}, E_4 = E_{13}, E_5 = E_{21}, E_6 = E_{23}, E_7 = E_{31}, E_8 = E_{32}$, form a basis of $\mathfrak{sl}(3, \mathbb{R})$ and the isomorphism $\varphi : \mathfrak{sl}(3, \mathbb{R}) \to F_3$ is given by $\varphi(E_1) = y_9 + y_{11}, \varphi(E_2) = y_9 - 2y_{11}, \varphi(E_3) = -y_8, \varphi(E_4) = y_7, \varphi(E_5) = y_{12}, \varphi(E_6) = y_{10}, \varphi(E_7) = y_{14}, \varphi(E_8) = -y_{13}$. In order to check the equivariance of $\varphi$ it is sufficient to look at the level of the Lie algebras. If we let $\rho^\ast$ be the induced representation $\rho^\ast : \mathfrak{sl}(3, \mathbb{R}) \to \text{End}(F_3)$ defined by $\rho^\ast(E_1)v = \frac{d}{dt}\big|_{t=0}(\exp(tE_1))v$ then the equivariant condition becomes $\varphi(\text{ad}(E_i)E_j) = \rho^\ast(E_i)\varphi(E_j)$ for all $i, j = 1, \ldots, 8$. This can be verified by easy calculations, for example $\varphi(\text{ad}(E_1)E_4) = \varphi(E_4) = y_7$ and $\rho^\ast(E_1)y_7 = \frac{d}{dt}\big|_{t=0}[e^t y_1, [e^t y_1, e^{-t} y_2]] = y_7$. □

Definition 5.16. For any $V \in \text{Gr}(6, F_3)$ let us define the Lie algebra $\mathfrak{g}(V)$ to be the vector space $F_1 \oplus F_2 \oplus F_3/V$ with the Lie algebra structure induced by the one of $F$. Let us also define $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ by $P := \varphi^{-1}(V)^\perp$ where the orthogonal space is taken with respect to the Killing form on $\mathfrak{sl}(3, \mathbb{R})$. □
Remark 5.17. The Lie algebra \( \mathfrak{g}(V) \) is a 8-dimensional 3-step nilpotent Lie algebra that is Carnot.

Proposition 5.18. We have a bijection

\[
\{ \text{Isomorphism classes of the Lie algebras } \mathfrak{g}(V) \} \leftrightarrow \{ \text{Orbits of elements } P := \varphi^{-1}(V)^+ \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \} \quad \text{under the adjoint action}.
\]

Proof. As we have seen, being the Lie algebras \( \mathfrak{g}(V) \) Carnot, each isomorphism between them is just induced by a linear isomorphism between their homogeneous parts of degree 1. So, after fixing a basis for the two Lie algebras, an isomorphism between \( \mathfrak{g}(V) \) and \( \mathfrak{g}(V') \) is induced by an element of \( \text{GL}(3, \mathbb{R}) \) that sends the \( V \) to \( V' \). Since scalar multiples of the identity act trivially on \( \text{Gr}(6, F_3) \) if there exists an element of \( \text{GL}(3, \mathbb{R}) \) that sends \( V \) to \( V' \) then there exists an element of \( \text{SL}(3, \mathbb{R}) \) doing the same. Hence, after having identified the 6-dimensional subspaces of \( F_3 \) with elements \( P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \) and the action of \( \text{SL}(3, \mathbb{R}) \) with the adjoint action, the orbit of each \( P \) under the \( \text{SL}(3, \mathbb{R}) \)-adjoint action represents the isomorphism class of the associated Lie algebra. \( \square \)

5.3. The case \( \alpha = 0 \). In this case the family of Lie algebras \( \mathfrak{g}(\alpha, a, b, c) \) is quite easy to classify. Indeed we have the following.

Proposition 5.19. The isomorphism classes of \( \mathfrak{g}(0, a, b, c) \) are represented by \( \mathfrak{g}(0, 1, 0, 0) \) if \( a^2 + bc > 0 \), \( \mathfrak{g}(0, 0, 1, -1) \) if \( a^2 + bc < 0 \), \( \mathfrak{g}(0, 0, 0, 1) \) if \( a^2 + bc = 0 \) but \( b \neq 0 \) or \( c \neq 0 \) and \( \mathfrak{g}(0, 0, 0, 0) \) if \( a = b = c = 0 \).

Proof. Since we know that the Lie algebras \( \mathfrak{g}(0, a, b, c) \) are Carnot an isomorphism between them is induced by an element of \( \text{SL}(3, \mathbb{R}) \). Hence consider the injection \( \iota : \text{GL}(2, \mathbb{R}) \to \text{SL}(3, \mathbb{R}) \) that associates to \( \tilde{g} \in \text{GL}(2, \mathbb{R}) \) the matrix

\[
\left( \begin{array}{cc} \frac{1}{\det \tilde{g}} & 0 \\ 0 & \tilde{g} \end{array} \right)
\]

then the map

\[
\psi : \mathfrak{sl}(2, \mathbb{R}) \to \{ \mathfrak{g}(0, a, b, c) \}
\]

\[
\left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) \mapsto \mathfrak{g}(0, a, b, c)
\]

intertwines the two actions, i.e for \( A \in \mathfrak{sl}(2, \mathbb{R}) \) we have \( \Delta^3 \psi(\tilde{g}A\tilde{g}^{-1}) = \iota(\tilde{g}) \psi(A) \). Explicitly if \( g \in \text{SL}(3, \mathbb{R}) \) is such that \( g \cdot \mathfrak{g}(0, a, b, c) = \mathfrak{g}(0, a', b', c') \) we have, letting \( \tilde{g} = \left( \begin{array}{ccc} \lambda & \mu \\ \delta & \rho \end{array} \right) \) and \( \Delta = \det \tilde{g} \),

\[
a' = \Delta^2 (\rho(a\lambda + c\delta) - \mu(b\lambda - a\delta)),
\]

\[
b' = \Delta^2 (b\lambda^2 - 2a\delta \lambda - c\delta^2),
\]

\[
c' = \Delta^2 (-b\mu^2 + 2a\rho \mu + c\rho^2).
\]

Depending on the sign of \( a^2 + bc \) we can bring the matrix in \( \mathfrak{sl}(2, \mathbb{R}) \) to one of the following normal form

\[
\left( \begin{array}{cc} \sqrt{a^2 + bc} & 0 \\ 0 & -\sqrt{a^2 + bc} \end{array} \right), \quad \left( \begin{array}{cc} 0 & 0 \\ \sqrt{-(a^2 + bc)} & 0 \end{array} \right), \quad \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \text{ or } 0.
\]
Definition 5.21. When \( \alpha \varepsilon, \in \) Definition 5.16. We will now find explicitly the submanifold of \( \text{Gr}(2, F) \gmenthesis \alpha,a,b,c \) the isomorphism classes of \( \text{Gr}(2, F) \gmenthesis \alpha,a,b,c \) is defined as \( \text{ad}(x) \), where the \( g \) is defined as \( g(x) \), the characteristic sequence is \((3, 2, 1, 1, 1)\) if either \( a = b = 0 \) or \( a = c = 0 \) and \((3, 2, 1, 1, 1)\) otherwise.

Remark 5.2. Notice that no Lie algebra \( g(\alpha, a, b, c) \) with \( \alpha \neq 0 \) is isomorphic to one of \( g(0, a, b, c) \). To see this let use define an invariant, called the characteristic sequence, of isomorphism classes of nilpotent Lie algebras. The characteristic sequence of a nilpotent Lie algebra \( g \) is defined as \( c(g) = \inf\{c(x) \in g \mid x \in g \setminus \mathfrak{g}^1\} \), where \( c(x) \) is the decreasing sequence of dimensions of Jordan blocks of \( \text{ad}(x) \). Now for the family of Lie algebras \( \{g(a, b, c)\} \) the characteristic sequence is \((3, 2, 1, 1)\) if one of \( a \) or \( b \) is 0 and \((3, 3, 1, 1)\) otherwise. While the family \( \{g(0, a, b, c)\} \) has characteristic sequence \((2, 2, 1, 1, 1)\) if either \( a = b = 0 \) or \( a = c = 0 \) and \((3, 2, 1, 1, 1)\) otherwise.

Our family of Lie algebras \( g(a, b, c) \) is a particular case of the Lie algebras \( g(V) \) defined in Definition 5.16. We will now find explicitly the submanifold of \( \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \) to which it corresponds.

Definition 5.23. Let us consider the following 6-dimensional subspace of \( F_3 \)
\[
W(a, b, c) = \text{span}_\mathbb{R}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7, y_{12} - ay_7 - by_8, y_{14} - cy_7 + ay_8\} \subseteq F_3.
\]

Proposition 5.24. For all \( a, b, c \in \mathbb{R} \) we have an isomorphism of Lie algebras between \( g(a, b, c) \) and the Lie algebra whose stratification is given by \( F_3, F_2 \) and \( F_3/W(a, b, c) \) and whose structure of Lie algebra is induced by the one of \( F \).

Proof. By imposing the conditions
\[
[x_1, x_4] = x_7 = y_7, \quad [x_1, x_5] = x_8 = y_8, \quad [x_1, x_6] = 0 = y_9,
[x_2, x_4] = x_7 = y_{10}, \quad [x_2, x_5] = -x_8 = y_{11}, \quad [x_2, x_6] = ax_7 + bx_8 = y_{12},
[x_3, x_5] = 3x_7 = y_{13}, \quad [x_3, x_6] = cx_7 - ax_8 = y_{14}
\]
we can find the generators for \( W(a, b, c) \) and hence define a natural isomorphism.

Remark 5.25. With this point of view we see our family of Lie algebras as a submanifold of the Grassmannian of \( \text{Gr}(6, F_3) \). Furthermore we can see that this submanifold is contained in \( \text{Gr}(2, F_3/W_0) \cong \text{Gr}(2, 4) \) where \( W_0 = \text{span}_\mathbb{R}\{y_9, y_{10} - y_7, y_{11} + y_8, y_{13} - 3y_7\} \) and it corresponds actually to just one chart of \( \text{Gr}(2, 4) \).
Remark 5.26. Under \( \varphi \), the isomorphism between \( \mathfrak{sl}(3, \mathbb{R}) \) and \( F_3 \), the subspace \( W(a, b, c) \) of \( F_3 \) corresponds to the subspace, that we denote by \( V(a, b, c) \), of \( \mathfrak{sl}(3, \mathbb{R}) \), whose basis is \( \{ 2E_1 + E_2, -E_4 + E_6, E_1 - E_2 - 3E_3, bE_4 - aE_4 + E_5, -3E_4 - E_8, -aE_3 - cE_4 + E_7 \} \). Using the Killing form on \( \mathfrak{sl}(3, \mathbb{R}) \) we can identify \( V(a, b, c) \) with a 2-dimensional subspace \( P(a, b, c) \) of \( \mathfrak{sl}(3, \mathbb{R}) \) spanned by \( \{ -E_2 + bE_3 + aE_4 + E_5, aE_3 + cE_4 - 3E_6 + E_7 + E_8 \} \).

Definition 5.27. Let us define \( \varphi := P(a, b, c) \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \) the 2-dimensional subspace spanned by \( u := u(a, b, c) = -E_2 - bE_3 + aE_4 + E_5 \) and \( v := v(a, b, c) = aE_3 + cE_4 - 3E_6 + E_7 + E_8 \).

The following is just a reformulation of Proposition 5.18 in our particular case.

Proposition 5.28. We have a bijection

\[
\begin{align*}
\{ & \text{Isomorphism classes of} \\
& \text{the Lie algebras} \, \mathfrak{g}(a, b, c) \} \\
\leftrightarrow & \{ \text{Orbits of elements} \, P(a, b, c) \\
& \text{under the adjoint action} \}
\end{align*}
\]

5.4.1. The \( \text{SL}(3, \mathbb{R}) \)-action on \( \{ \mathfrak{g}(a, b, c) \} \subseteq \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \). In order to better understand the action we will embed the Grassmannian into a projective space and decompose it in \( \text{SL}(3, \mathbb{R}) \)-invariant subspaces. Let us fix as embedding of \( \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \) in projective space the Plücker embedding:

\[ \iota : \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \to \mathbb{P} \left( \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \right). \]

Fixing \( \{ E_i \}_{i=1}^8 \) as basis for \( \mathfrak{sl}(3, \mathbb{R}) \) we can represent a generic element \( V \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R})) \) as a \( 2 \times 8 \) matrix whose lines are the vectors spanning it, then we have that \( \iota(V) = [m_{ij}] \) where the Plücker coordinates \( m_{ij} \) are the minors of the \( 2 \times 2 \) submatrix of \( V \) obtained taking the \( i \)-th and \( j \)-th columns.

From classical representation theory, or from what we will show later, we have the following decomposition of \( \mathfrak{sl}(3, \mathbb{R}) \)-representations

\[
\bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) = \mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^4(\mathbb{R}^3^*),
\]

where \( S^3(\mathbb{R}^3) \) is the 3-rd symmetric power of \( \mathbb{R}^3 \). Let us call \( \pi_1 \) the projection to the first factor of the decomposition,

\[
\pi_1 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \to \mathfrak{sl}(3, \mathbb{R})
\]

\[ u_1 \wedge u_2 \to [u_1, u_2]. \]

Lemma 5.29. The projection just defined, \( \pi_1 \), is a morphism of \( \mathfrak{sl}(3, \mathbb{R}) \)-representations.

Proof. Take \( u_1, u_2 \in \mathfrak{sl}(3, \mathbb{R}) \) and \( x \in \mathfrak{sl}(3, \mathbb{R}) \) we have \( \pi_1(x \cdot (u_1 \wedge u_2)) = \pi_1([x, u_1] \wedge u_2 + u_1 \wedge [x, u_2]) = ([x, u_1], u_2) + [u_1, [x, u_2]] = [x, [u_1, u_2]] = x \cdot \pi_1(u_1 \wedge u_2). \)

In order to define the projection to the second and third factor of the decomposition let us denote by \( \times \) the standard cross product on \( \mathbb{R}^3 \), by \( \{ e_i \}_{i=1}^3 \) the standard basis for \( \mathbb{R}^3 \) and by \( \odot \)
the symmetric tensor product on \( \mathbb{R}^3 \). Let us then define \( \pi_2 \) as
\[
\pi_2 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \longrightarrow S^3(\mathbb{R}^3)
\]
\[
u_1 \wedge \nu_2 \rightarrow \sum_{i,j=1}^{3} u_1 e_i \otimes u_2 e_j \otimes (e_i \times e_j).
\]

**Lemma 5.30.** The second projection, \( \pi_2 \), is a morphism of \( \mathfrak{sl}(3, \mathbb{R}) \)-representations.

**Proof.** Indeed it is sufficient to check it for \( u_1 = e_i^* \otimes u_1(e_i) \) and \( u_2 = e_j^* \otimes u_2(e_j) \in \mathfrak{sl}(3, \mathbb{R}) \).

Let \( x \in \mathfrak{sl}(3, \mathbb{R}) \) then on one side we have
\[
x \cdot \pi_1(u_1 \wedge u_2) = x \cdot (u_1(e_i) \otimes u_2(e_j) \otimes (e_i \times e_j)) =
\]
\[
x(u_1(e_i)) \otimes u_2(e_j) \otimes (e_i \times e_j) + u_1(e_i) \otimes x(u_2(e_j)) \otimes (e_i \times e_j) + u_1(e_i) \otimes u_2(e_j) \otimes (x(e_i) \times e_j).
\]

On the other hand
\[
\pi_2(x \cdot (u_1 \wedge u_2)) = \pi_2((x \cdot u_1) \wedge u_2 + u_1 \wedge (x \cdot u_2))
\]
but \( x \cdot u_1 = x \cdot (e_i^* \otimes u_1(e_i)) = -e_i^* x \otimes u_1(e_i) + e_i^* \otimes xu_1(e_i) \), hence
\[
\pi_2(-e_i^* x \otimes u_1(e_i) \wedge e_j^* \otimes u_2(e_j) + e_i^* \otimes xu_1(e_i) \wedge e_j^* \otimes u_2(e_j)) - e_i^* \otimes u_1(e_i) \wedge e_j^* \otimes xu_2(e_j) + x(u_1(e_i)) \otimes u_2(e_j) \otimes (e_i \times e_j) + u_1(e_i) \otimes u_2(e_j) \otimes (x(e_i) \times e_j) + u_1(e_i) \otimes u_2(e_j) \otimes (e_i \times e_j) + u_1(e_i) \otimes u_2(e_j) \otimes (x(e_i) \times e_j) + u_1(e_i) \otimes u_2(e_j) \otimes (e_i \times e_j).
\]

So it is left to prove that \( x(e_i) \times e_j = -x^t e_i \times e_j - e_i \times x^t e_j \). For this consider a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^3 \) and then by definition of cross product we have \( \det(e_i, e_j, v) = (e_i \times e_j, v) \) for any \( e_i, e_j, v \in \mathbb{R}^3 \). Taking \( g \in \text{SL}(3, \mathbb{R}) \) we have \( \det(g^t e_i, g^t e_j, g^t v) = c \) for some constant \( c \in \mathbb{R} \) that does not depend on \( g \). Hence differentiating this expression at the identity we get for \( x \in \mathfrak{sl}(3, \mathbb{R}) \) that \( \det(x^t e_i, e_j, v) + \det(e_i, x^t e_j, v) + \det(e_i, e_j, x^t v) = 0 \) hence \( (x^t e_i \times e_j, v) + (e_i \times x^t e_j, v) + (e_i \times e_j, x^t v) = 0 \) for any \( v \in \mathbb{R}^3 \) and the result follows. \( \square \)

We also define
\[
\pi_3 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \longrightarrow S^3(\mathbb{R}^3^*)
\]
\[
u_1 \wedge \nu_2 \rightarrow \sum_{i,j=1}^{3} e_i^* \otimes e_j^* \otimes (u_1 e_i \times u_2 e_j)^*.
\]

**Remark 5.31.** Notice that \( \pi_3(u_1 \wedge u_2) = \pi_2(tu_1 \wedge tu_2) \). Hence since \( \pi_2 \) is \( \mathfrak{sl}(3, \mathbb{R}) \)-equivariant we get that \( \pi_3 \) is \( \mathfrak{sl}(3, \mathbb{R}) \)-equivariant as well since \( \pi_3(gu_1 g^{-1} \wedge gu_2 g^{-1}) = \pi_2(t^{-1} tu_1^t g \wedge t^{-1} tu_2^t g) = t^{-1} \pi_2(tu_1 \wedge tu_2) = t^{-1} \pi_3(u_1 \wedge u_2) \).

**Proposition 5.32.** The morphism \( \pi_1, \pi_2, \pi_3 : \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \rightarrow \mathfrak{sl}(3, \mathbb{R}) \oplus S^3(\mathbb{R}^3) \oplus S^3(\mathbb{R}^3^*) \) is an \( \mathfrak{sl}(3, \mathbb{R}) \)-equivariant isomorphism.

**Proof.** In the previous commentaries and lemmas we have already seen that the morphism is \( \mathfrak{sl}(3, \mathbb{R}) \)-equivariant. Let us show that it is an isomorphism. Since the dimensions coincide we can just show that the morphism is injective. Consider then an element \( u_1 \wedge u_2 \in \bigwedge^2 \mathfrak{sl}(3, \mathbb{R}) \) with all the projections equal to 0. Assume \( u_1 \) is diagonalisable then since \( \det(u_1, u_2) = 0 \) we have that \( u_2 \) is diagonalisable as well, hence let us write, up to conjugation, \( u_1 = \text{diag}(\lambda_1, \lambda_2 - \lambda_1, -\lambda_2) \) and \( u_2 = \text{diag}(\mu_1, \mu_2 - \mu_1, -\mu_2) \) then \( \pi_2(u_1 \wedge u_2) = 3(\lambda_1 \mu_2 - \lambda_2 \mu_1) e_1 \otimes e_2 \otimes e_3 \). Since
we have supposed the projections to be 0 we have $u_1 = \frac{\lambda_1}{\mu_1}u_2$ and then $u_1 \wedge u_2 = 0$. The case when $u_1$ is not diagonalisable can be treated similarly and one can arrive at the conclusion that $u_1$ is a scalar multiple of $u_2$ hence $u_1 \wedge u_2 = 0$. \hfill \square

**Proposition 5.33.** The Lie algebras $g(a, b, c)$ and $g(a', b', c')$ are isomorphic if and only if $a' = \pm a, b' = b, c' = c$.

**Proof.** Under the identification $\mathbb{P}(\mathfrak{A}^2 \mathfrak{sl}(3, \mathbb{C})) = \mathbb{P}(\mathfrak{sl}(3, \mathbb{C}) \oplus S^3(\mathbb{C}^3) \oplus S^3(\mathbb{C}^3^*))$ we write the elements $P(a, b, c)$ as $[(M(a, b, c), f_1(a, b, c), f_2(a, b, c))]$ where

$$M(a, b, c) = \begin{pmatrix} 0 & 2a & -c + 3b \\ 0 & a & c + 6 \\ 0 & b + 2 & -a \end{pmatrix},$$

$$f_1(a, b, c) = (-a^2 - bc)e_1 \odot e_1 \odot e_1 + (3b - c)e_1 \odot e_1 \odot e_2 - 2ae_1 \odot e_1 \odot e_3$$

$$+ (3 - c)e_1 \odot e_2 \odot e_2 + 2ae_1 \odot e_2 \odot e_3 + (b - 1)e_1 \odot e_3 \odot e_3 + 3e_2 \odot e_2 \odot e_2 + 3e_2 \odot e_3 \odot e_3$$

and

$$f_2(a, b, c) = e_1^* \odot e_1^* \odot e_1^* + (b - 1)e_1^* \odot e_2^* \odot e_3^* - 2ae_1^* \odot e_2^* \odot e_3^*$$

$$+ (3 - c)e_1^* \odot e_3^* \odot e_3^* + be_2^* \odot e_2^* \odot e_2^* + ae_2^* \odot e_2^* \odot e_3^* + (2c + 3b)e_2^* \odot e_3^* \odot e_3^* - 3ae_2^* \odot e_3^* \odot e_3^*.$$

Let us identify furthermore the elements of both $S^3(\mathbb{C}^3)$ and $S^3(\mathbb{C}^3^*)$ with ternary cubics in the variables $x, y, z$ for convenience. Now an element $g$ of $\text{SL}(3, \mathbb{R})$ that sends $P(a, b, c)$ to $P(a', b', c')$ should preserve the kernel of $M(a, b, c)$. If $M(a, b, c)$ is of rank 2 then $\ker M(a, b, c) = \mathbb{R}e_1$, hence $g$ has the form

$$\begin{pmatrix} (\sigma_1\sigma_4 - \sigma_2\sigma_3)^{-1} & \mu & \nu \\ 0 & \sigma_1 & \sigma_2 \\ 0 & \sigma_3 & \sigma_4 \end{pmatrix}$$

and we are left with an action of the group $\text{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$. Let us notice that this group induces an action of $\text{GL}(2, \mathbb{R})$ on $S^3(\mathbb{C}^2)$, the ternary cubics in two variables $y$ and $z$. More precisely, if $P \in \text{Gr}(2, \mathfrak{sl}(3, \mathbb{R}))$ and $\pi_2(P) = p_0x^3 + p_1(y, z)x^2 + p_2(y, z)x + p_3(y, z)$ let us call $\text{pr}(P) = p_3(y, z)$. Then if $g = (h, v) \in \text{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ we have $[\pi_2(g \cdot P)] = [h \cdot \text{pr}(P)]$. Let us notice that for all $a, b, c$ we have $\text{pr}(P(a, b, c)) = [3(y^3 + y^2z)]$. Since we search for the $g \in \text{GL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ such that $g \cdot f_1(a, b, c) = f_1(a', b', c')$ then in particular we want $g$ such that $[h \cdot \text{pr}(P(a, b, c))] = [3(y^3 + y^2z)]$. Hence since

$$h \cdot \text{pr}(P(a, b, c)) = 3\sigma_1(\sigma_1^2 + \sigma_2^2)y^3 + 3(3\sigma_1^2\sigma_3 + 2\sigma_1\sigma_2\sigma_4 + \sigma_3\sigma_2^2)y^2z$$

$$+ 3(2\sigma_1\sigma_3^2 + 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2)yz^2 + 3\sigma_3(\sigma_3^2 + \sigma_4^2)z^3$$

then we should have $3\sigma_3(\sigma_3^2 + \sigma_4^2) = 0$, i.e. $\sigma_3 = 0$ and $3(3\sigma_1^2\sigma_3 + 2\sigma_1\sigma_2\sigma_4 + \sigma_3\sigma_2^2) = 6\sigma_1\sigma_2\sigma_4 = 0$, i.e. $\sigma_2 = 0$, also $3\sigma_1(\sigma_1^2 + \sigma_2^2) = 3(3\sigma_1^2 + 2\sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_4^2)$ that is $\sigma_1^2 = \sigma_4^2$. In a similar way we want $g$ such that $g \cdot f_2(a, b, c) = f_2(a', b', c')$. Hence we can compare the terms with at least one $x^2$ in $f_2(a, b, c)$, that is just $x^3$, and the ones in $g \cdot f_2(a, b, c)$ that are as follows

$$(\sigma_1\sigma_4)^3x^3 - 3(\sigma_1\sigma_4)^2\mu_4x^2y - 3(\sigma_1\sigma_4)^2\nu_4x^2z.$$
Hence $g \in \text{SL}(3, \mathbb{R})$ is such that $g \cdot P(a, b, c) = P(a', b', c')$ if and only if $g = \text{Id}$ or $g = g_1$ where

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The element $g_1$ sends $P(a, b, c)$ to $P(-a, b, c)$.

Finally let us treat the case where the rank of $M(a, b, c)$ is less than 2, i.e. where $a = 0$ and $b = -2$. When $c = -6$ the point $P(0, -2, -6)$ is the only point with a stabiliser of dimension 2 then of course it is not in the orbits of any other point. If instead $c \neq -6$ moving the point $P(0, -2, c)$ inside its orbit let us write it as

$$[(N, p_1, p_2)] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{3}{(c + 6)^2} x^3 + \frac{12 - c}{c + 6} x^2 z + 3 x y^2 - 6 x y z - 3(c - 4) x z^2,$$

$$3(c - 4) x y^2 - 6 x y z - 3 x z^2 + \frac{4 - c}{c + 6} y^3 + \frac{6 - c}{c + 6} y^2 z + \frac{3}{c + 6} y z^2 + \frac{1}{c + 6} z^3 \right]$$

then a point in the same orbit should have the same coefficients of the cubics modulo the actions of the stabiliser of $N$ that is

$$\text{Stab}_{\text{SL}(3, \mathbb{R})} \left( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = \left\{ \left( \begin{array}{ccc} \sigma_1 & \sigma_2 & \sigma_3 \\ 0 & \sigma_1 & 0 \\ 0 & \sigma_4 & \sigma_1^{-1} \end{array} \right) \mid \sigma_1 \in \mathbb{R}^*, \sigma_2, \sigma_3, \sigma_4 \in \mathbb{R} \right\}$$

Taking $\sigma_1 = 3^{-1/3}$ and $\sigma_4 = \frac{7}{6} \sqrt{3}$ so that $\text{coeff}(p_1, x y^2) = \text{coeff}(p_1, x y z) = 1$ we get that $\text{coeff}(p_1, x z^2) = -9c + \frac{109}{4}$. Since this coefficient is left invariant under the action of the stabiliser of $N$, the parameter $c$ is an invariant and all the points $P(0, -2, c)$ are pairwise not isomorphic.

Putting together Proposition 4.3, 4.4, 5.19 and 5.33 we see that the proof of Theorem 1.7 is achieved.

6. LATTICES IN UNIPOTENT GROUPS

In Section 3 we looked at crystallographic groups $\Gamma$ and studied the abelian by cyclic case. We are then left with the virtually nilpotent case. We know that they are lattices in unipotent simply transitive subgroups of $\mathcal{H}(3, 1)$. Since we have listed all the possible unipotent Lie subgroups of $\mathcal{H}(3, 1)$ that act simply transitively on $\mathfrak{a}(V)$ we are interested in studying their lattices, in particular the question of existence of lattices and of their classification up to finite index. For lattices in unipotent groups the theory is illustrated by Malcev’s theorems. The first one concerns conditions of existence of lattices in nilpotent Lie groups and the second a criterion of classification of lattices up to finite index.

**Theorem 6.1** ([19, Section II, Theorem 2.12.]). Let $U$ be a simply connected nilpotent Lie group and $\mathfrak{u}$ its Lie algebra then $U$ admits a lattice if and only if $\mathfrak{u}$ admits a basis with respect to which the constant of structure are rational.

**Remark 6.2.** The statement is equivalent to say that $\mathfrak{u}$ admits a $\mathbb{Q}$-form, i.e. a rational Lie subalgebra $\mathfrak{u}_\mathbb{Q}$ such that $\mathfrak{u}_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{R} \cong \mathfrak{u}$. 
Theorem 6.3 ([10, Theorem 5.1.12]). Let \( \Gamma_1, \Gamma_2 \) be lattices in a nilpotent Lie group \( G \). Then \( \Gamma_1 \) and \( \Gamma_2 \) determine the same rational structure, \( \text{span}_Q\{\log \Gamma_1\} \), in \( g \) if and only if they are commensurable: \( \Gamma_1 \cap \Gamma_2 \) has finite index in both \( \Gamma_1 \) and \( \Gamma_2 \).

Remark 6.4. A part from the family \( g(a,b,c) \), it is clear from the presentation we have given of representatives of the isomorphism classes of the other nilpotent Lie algebras, that all the others admit \( Q \)-forms.

**Proposition 6.5.** The Lie algebra \( g(a,b,c) \) has a \( Q \)-form if and only if \( a^2 \in Q \) and \( b,c \in Q \).

**Proof.** If \( a^2, b,c \in Q \) then we can modify the basis taking \( x_3' = ax_3 \) in order to get rational constant of structure. On the contrary suppose \( g(a,b,c) \) is Lie subalgebra defined over \( Q \) such that \( h \otimes Q \cong g(a,b,c) \). We first want to show that we can find a basis for \( h \) that has simple relations with the basis \( \{x_1, \ldots, x_8\} \) of the Lie algebra \( g(a,b,c) \).

We define \( \hat{z}_1 \) and \( \hat{z}_2 \) to be a basis for the center of \( h \). Then we define \( \hat{x}_2 \) as a basis for the kernel of the map induce by \( [x_2, \cdot] \) on \( h/Q/\{\hat{z}_1, \hat{z}_2\} \). So \( \hat{x}_2 = \pi_2 x_2 \mod Q/\{\hat{z}_1, \hat{z}_2\} \) and similarly \( \hat{x}_3 = \pi_3 x_3 \mod Q/\{\hat{z}_1, \hat{z}_2\} \). Then \( \hat{x}_1 = \pi_1 x_1 + \pi_2 x_6 \mod Q/\{\hat{z}_1, \hat{z}_2\} \) with \( \pi_1, \pi_2, \pi_3, \pi_6 \in R^* \). Since \( h \) is a subalgebra let \( \hat{x}_4 = [\hat{x}_1, \hat{x}_2] = \pi_1 \pi_2 x_4 \mod Q/\{\hat{z}_1, \hat{z}_2\} \) and \( \hat{x}_5 = [\hat{x}_1, \hat{x}_3] = \pi_1 \pi_3 x_5 \mod Q/\{\hat{z}_1, \hat{z}_2\} \) in \( h \) and \( \hat{x}_6 = [\hat{x}_2, \hat{x}_3] = \pi_2 \pi_3 x_6 \). Then \( \{\hat{x}_1, \ldots, \hat{x}_6, \hat{z}_1, \hat{z}_2\} \) is a basis for \( h \).

Since
\[
[\hat{x}_1, \hat{x}_4] = \pi_1^2 \pi_2 x_7 \quad \text{and} \quad [\hat{x}_1, \hat{x}_5] = \pi_1^2 \pi_3 x_8,
\]
we can rename \( \hat{z}_1, \hat{z}_2 \) so that \( [\hat{x}_1, \hat{x}_4] = \hat{z}_1 \) and \( [\hat{x}_1, \hat{x}_5] = \hat{z}_2 \). Then we have
\[
[\hat{x}_2, \hat{x}_4] = \pi_1 \pi_2^2 x_7 = \frac{\pi_2}{\pi_1} \hat{z}_1, \quad [\hat{x}_2, \hat{x}_5] = -\pi_1 \pi_2 \pi_3 x_8 = -\frac{\pi_2}{\pi_1} \hat{z}_2,
\]
\[
[\hat{x}_3, \hat{x}_4] = -\pi_1 \pi_2 \pi_3 x_8, \quad [\hat{x}_3, \hat{x}_5] = 3\pi_1 \pi_3^2 x_7 = \frac{\pi_3^2}{\pi_1^2} \hat{z}_1,
\]
\[
[\hat{x}_2, \hat{x}_6] = \pi_2^2 \pi_3 a x_7 + \pi_2^2 \pi_3 b x_8 = a \frac{\pi_2}{\pi_1} \hat{z}_1 + b \frac{\pi_2}{\pi_1} \hat{z}_2,
\]
\[
[\hat{x}_3, \hat{x}_6] = \pi_2 \pi_3^2 c x_7 - \pi_3^2 \pi_2 a x_8 = c \frac{\pi_2}{\pi_1} \hat{z}_1 - a \frac{\pi_2}{\pi_1} \hat{z}_2.
\]

So
\[
\begin{cases}
\pi_2 \in Q \\
\frac{\pi_2}{\pi_1} \in Q \\
\pi_3 \pi_2 \hat{z}_1 \\
\pi_3 \pi_2 \hat{z}_2 \\
\pi_2 \pi_3 \hat{z}_1 \\
\pi_2 \pi_3 \hat{z}_2
\end{cases}
\]

hence \( \frac{\pi_2}{\pi_1} \in Q \) so that \( b,c \in Q \) and \( \pi_2 \pi_3 \hat{z}_1 \in Q \) hence \( a^2 \in Q \). \( \square \)

From the proof of the previous proposition and Proposition 5.33 we also get the following.

**Corollary 6.6.** The Lie algebras \( g(a,b,c) \) that admit a \( Q \)-form, have a unique \( Q \)-form modulo \( Q \)-isomorphism.

The following proposition answers the question of \( Q \)-isomorphism classes of \( Q \)-forms in the Lie algebras \( g(0,a,b,c) \).

**Proposition 6.7.** The \( Q \)-isomorphism classes of the family \( g(0,a,b,c) \) are \( g(0,0,0,0) \) and \( g(0,0,\varepsilon,1) \) with \( \varepsilon \in Q \). Furthermore \( g(0,0,\varepsilon',1) \cong g(0,0,\varepsilon,1) \) if and only if there exists \( \alpha \in Q^* \) such that \( \varepsilon' = \alpha^2 \varepsilon \).
that the coboundaries form a subspace of the cocycles and finally the quotient space.

**Definition 6.8.**
Let us introduce some terminology.

**Proof.** As for the above case we can construct a basis of the Lie \( \mathfrak{q} \)-subalgebra \( \mathfrak{h} \) of \( \mathfrak{g}(0, a, b, c) \) for which the structure of the Lie brackets is the same. Namely, let \( \mathcal{Z} \) be the center of \( \mathfrak{h} \), we can have \( \hat{x}_1 = \pi_1 x_1 + \pi_6 x_6 \mod \mathcal{Z} \), \( \hat{x}_2 = \pi_2 x_2 + \nu_4 x_4 + \nu_5 x_5 + \nu_6 x_6 \mod \mathcal{Z} \) and \( \hat{x}_3 = \pi_3 x_3 + \mu_4 x_4 + \mu_5 x_5 + \mu_6 x_6 \mod \mathcal{Z} \) with \( \pi_i, \nu_i, \mu_i \in \mathbb{R} \). Then \( [\hat{x}_1, \hat{x}_2] = \pi_1 \pi_2 x_4 \mod \mathcal{Z}, [\hat{x}_1, \hat{x}_3] = \pi_1 \pi_3 x_5 \mod \mathcal{Z} \) and \( [\hat{x}_2, \hat{x}_3] = \pi_2 \pi_3 x_6 \mod \mathcal{Z} \). Call them respectively \( \hat{x}_4, \hat{x}_5, \hat{x}_6 \). Then

\[
[\hat{x}_1, \hat{x}_4] = \pi_1^2 \pi_2 x_7 := \hat{x}_7 \quad [\hat{x}_1, \hat{x}_5] = \pi_1^2 \pi_3 x_8 := \hat{x}_8
\]

\[
[\hat{x}_2, \hat{x}_6] = \pi_2^2 \pi_3 (ax_7 + bx_8) = a \frac{\pi_2 \pi_3}{\pi_1} \hat{x}_7 + b \frac{\pi_2^2}{\pi_1} \hat{x}_8
\]

\[
[\hat{x}_3, \hat{x}_6] = \pi_2 \pi_3^2 (cx_7 - ax_8) = c \frac{\pi_2^3}{\pi_1} \hat{x}_7 - a \frac{\pi_2 \pi_3}{\pi_1} \hat{x}_8.
\]

Hence, as we have seen in Proposition 5.19, now finding representatives for the isomorphism classes corresponds to find normal forms for the adjoint action of \( \text{GL}_2(\mathbb{Q}) \) over \( \mathfrak{P}(\text{sl}(2, \mathbb{Q})) \). Using the theory of rational canonical forms we can see that the normal forms are

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & \varepsilon \\
1 & 0
\end{pmatrix}
\]

with \( \varepsilon \notin \mathbb{Q}^2 \) or the 0 matrix. Hence as representatives of the \( \mathbb{Q} \)-isomorphism classes of the family \( \mathfrak{g}(0, a, b, c) \) we have \( \mathfrak{g}(0, 0, 0, 0) \) and \( \mathfrak{g}(0, 0, \varepsilon, 1) \), noticing that \( \mathfrak{g}(0, 0, \varepsilon, 1) \cong \mathfrak{g}(0, 1, 0, 0) \) if \( \varepsilon \in \mathbb{Q}^2 \). Finally it is clear that \( \mathfrak{g}(0, 0, \varepsilon', 1) \cong \mathfrak{g}(0, 0, \varepsilon, 1) \) if and only if the ratio of \( \varepsilon \) and \( \varepsilon' \) is the square of a rational number.

**6.0.1. Skjelbred and Sund method.** Now that the questions of existence of lattices and their commensurability classes in the Lie groups associated with the Lie algebras \( \mathfrak{g}(\alpha, a, b, c) \) is settled we want to study the commensurability question for the Lie groups associated to the Lie algebras \( L_j \) and \( N_j \). We have seen that in order to study this question we have to understand the \( \mathbb{Q} \)-isomorphism classes of rational subalgebras of these Lie algebras. To do so we introduce a method that was developed by Skjelbred and Sund in [22] and applied by Gong in his thesis [15] in order to classify 7-dimensional real nilpotent Lie algebras. To state the theorem we need to introduce some terminology.

**Definition 6.8.** Let \( \mathbb{K} \) be a field and \( \mathfrak{g} \) a Lie algebra over \( \mathbb{K} \). A map \( B : \bigwedge^2 \mathfrak{g} \to \mathbb{K} \) such that

\[
B([x_1, x_2], x_3) + B([x_2, x_3], x_1) + B([x_3, x_1], x_2) = 0
\]

is called a **cocycle** and the space of cocycles is denoted by \( Z^2(\mathfrak{g}, \mathbb{K}) \). A map \( B : \bigwedge^2 \mathfrak{g} \to \mathbb{K} \) such that there exists \( g \in \text{Hom}(\mathfrak{g}, \mathbb{K}) \) and

\[
B(x, y) = g([x, y])
\]

is called a **coboundary** and the space of coboundaries is denoted by \( B^2(\mathfrak{g}, \mathbb{K}) \). It can be noticed that the coboundaries form a subspace of the cocycles and finally the quotient space

\[
H^2(\mathfrak{g}, \mathbb{K}) = Z^2(\mathfrak{g}, \mathbb{K})/B^2(\mathfrak{g}, \mathbb{K})
\]

is called the **2-nd cohomology group** of \( \mathfrak{g} \) with coefficients in \( \mathbb{K} \).

**Definition 6.9.** For \( B : \bigwedge^2 \mathfrak{g} \to \mathbb{K} \) we can define the set

\[
\mathfrak{g}^B = \{ x \in \mathfrak{g} \mid B(x, \mathfrak{g}) = 0 \}.
\]

And if \( B = (B_1, \ldots, B_k) : \bigwedge^2 \mathfrak{g} \to \mathbb{K}^k \) we can define \( \mathfrak{g}^B = \mathfrak{g}^B_1 \cap \ldots \cap \mathfrak{g}^B_k \).
**Definition 6.10.** Let us denote by \( G_k(H^2(\mathfrak{g}, \mathbb{K})) \) the \( k \)-th Grassmannian of \( H^2(\mathfrak{g}, \mathbb{K}) \). Furthermore if \( B \) is a cocycle let us denote by \( B \) its image in cohomology. Thus we define a subspace of the \( k \)-th Grassmannian of the cohomology of \( \mathfrak{g} \) as follows

\[
U_k(\mathfrak{g}) = \{ \tilde{B}_1 \mathbb{K} \oplus \cdots \oplus \tilde{B}_k \mathbb{K} \in G_k(H^2(\mathfrak{g}, \mathbb{K})) \mid \mathfrak{g}^{B_1,...,B_k} \cap \mathbb{Z}(\mathfrak{g}) = 0 \}
\]

where \( \mathbb{Z}(\mathfrak{g}) \) is the center of \( \mathfrak{g} \).

If we call \( \text{Aut}(\mathfrak{g}) \) the automorphism group of the Lie algebra \( \mathfrak{g} \) we can notice that we have an action of \( \text{Aut}(\mathfrak{g}) \) on \( H^2(\mathfrak{g}, \mathbb{K}) \) induce by the following action on cocycles:

\[
\text{if } \varphi \in \text{Aut}(\mathfrak{g}) \text{ and } B \in \mathbb{Z}^2(\mathfrak{g}, \mathbb{K}) \text{ then } \varphi \cdot B(x, y) = B(\varphi(x), \varphi(y)).
\]

Furthermore it can be proved that this action induces an action on \( U_k(\mathfrak{g}) \). We are now ready to state the theorem of Skjelbred and Sund.

**Theorem 6.11** ([22, Theorem 3.5.]). Let \( \mathfrak{g} \) be a Lie algebra over a field \( \mathbb{K} \). The isomorphism classes of Lie algebras \( \tilde{\mathfrak{g}} \), with center \( \tilde{\mathfrak{g}}/\mathbb{K} \) of dimension \( k \), with \( \tilde{\mathfrak{g}}/\mathbb{K} \cong \mathfrak{g} \) and without abelian factors are in bijective correspondence with elements in \( U_k(\mathfrak{g})/\text{Aut}(\mathfrak{g}) \).

**Remark 6.12.** If \( B \in U_k(\mathfrak{g})/\text{Aut}(\mathfrak{g}) \) is a representative of one orbit then the corresponding \( k \)-dimensional central extension of \( \mathfrak{g} \) is defined as the direct sum of the vector spaces \( \mathfrak{g}(B) = \mathfrak{g} \oplus \mathbb{K}^k \) and with Lie brackets \([x,y],B(x,y)] = ([x,y],B(x,y))\).

We will now apply this method to the aforementioned Lie algebras.

**Proposition 6.13.** All the Lie algebras \( L_i \) and \( N_i \) have just one \( \mathbb{Q} \)-form, up to \( \mathbb{Q} \)-isomorphism, except \( L_6^R(\varepsilon), L_6^R(\varepsilon), N_9^R(\varepsilon) \) and \( N_{15}^R(\varepsilon) \). For these Lie algebras their \( \mathbb{Q} \)-forms are \( L_6^Q(\varepsilon), L_6^Q(\varepsilon), N_9^Q(\varepsilon) \) and \( N_{15}^Q(\varepsilon) \) respectively, as defined in Appendix A.

**Proof.** For the 6-dimensional Lie algebras the analysis in [16] is done over any field of characteristic different from 2 hence the result follows. For almost all the Lie algebras in dimension 7 the analysis that is done for \( \mathbb{R} \) in [15] works for \( \mathbb{Q} \) without any problem so we will point out only the cases where there is a difference. Let again fix some terminology. If \( \{e_i\} \) is a basis for \( \mathfrak{g} \) we let \( \Delta_{ij} = (e_i \wedge e_j)^* \) be the elements of the basis for \( (\wedge^2 \mathfrak{g})^* \). We will write the elements of the cohomology group \( H^2(\mathfrak{g}, \mathbb{K}) \) simply as cocycles thinking them as equivalence classes. Finally for the action of \( \text{Aut}(\mathfrak{g}) \) on \( H^2(\mathfrak{g}, \mathbb{K}) \) if \( \text{g} \cdot (\sum \alpha_{ij} \Delta_{ij}) = \alpha'_{ij} \Delta_{ij} \) we will write \( \alpha_{ij} \mapsto \alpha'_{ij} \).

The Lie algebra \( N_9^R(\varepsilon) \) is a 3-dimensional central extension of \( \mathbb{R}^4 \), the abelian Lie algebra of dimension 4. Elements of the cohomology group of \( \mathbb{R}^4 \) can be represented as \( B = a \Delta_{12} + b \Delta_{13} + c \Delta_{14} + d \Delta_{23} + e \Delta_{24} + f \Delta_{34} \) and we will just write \( B = [a, b, c, d, e, f] \). If we denote by \( g = (a_{ij}) \) an element of the automorphism group of \( \mathbb{R}^4 \), that in this case is just \( GL(4, \mathbb{R}) \), then its action on the cohomology group is as follows:

\[
\begin{align*}
a &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}, \\
b &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}, \\
c &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}, \\
d &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}, \\
e &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}, \\
f &\mapsto a \Sigma_{12} + b \Sigma_{13} + c \Sigma_{14} + d \Sigma_{23} + e \Sigma_{24} + f \Sigma_{34}
\end{align*}
\]
where $\Sigma^j_i = a_{ij}a_{ij} - a_{ij}a_{jk}$. The family of Lie algebras $N^\mathbb{R}_0(\varepsilon)$ corresponds to the three dimensional subspace of the cohomology group of $\mathbb{R}^4$ represented by $(\Delta_{12} + \Delta_{34}) \wedge \Delta_{13} \wedge (\Delta_{14} + \varepsilon \Delta_{23})$ with $\varepsilon \in \{0, 1, -1\}$. We can notice that one of the generators of this three dimensional space is a degenerate 2-form. In the following lemma we prove that this is also the case for a $\mathbb{Q}$-form of $N^\mathbb{R}_0(\varepsilon)$. Indeed let $\mathfrak{h}$ be a $\mathbb{Q}$-form of $N^\mathbb{R}_0(\varepsilon)$. Then $\mathfrak{h}$ is a three dimensional central extension of the four dimensional abelian Lie algebra over $\mathbb{Q}$. Let then $P$ be the three dimensional subspace of $H^2(\mathbb{Q}^4, \mathbb{Q})$ corresponding to this extension.

**Lemma 6.14.** We can choose one of the generators of $P$ to be a degenerate 2-form.

**Proof.** We will follow the work of Gong, [15, p.71], and use the same notations. If the first generator of $P$ is a degenerate 2-form then the lemma follows, otherwise there exist an element of $\text{GL}(4, \mathbb{Q})$ that brings the first generator to $A = \{0, 1, 0, 0, 1, 0\}$. Then we can assume the second generator has the form $B = [a, 0, c, d, e, f]$. Using the action and subtracting scalar multiples of $A$ we see, as is done by Gong, we can bring $B$ to either $[0, 0, 0, 0, 1]$ and the lemma follows, or to the form $[1, 0, 0, 0, 0, \varepsilon]$ with $\varepsilon \neq 0$. Assume then $B = [a, 1, 0, 0, 0, \varepsilon]$, we can consequently choose the third generator to be $C = [a, b, c, d, e, 0]$. In order to fix $A$ and $B$ we can choose $a_{11} = a_{21} = a_{31} = a_{22} = a_{32} = a_{33} = a_{34} = a_{44} = 0$, $a_{12} = -a_{43}, a_{13} = -\varepsilon a_{42}, a_{14} = \varepsilon a_{41}, a_{23} = -\frac{1}{a_{44}}, a_{32} = -\frac{1}{\varepsilon a_{41}}$. Then the action on $C$ is as follows

$$
a \mapsto ca_{41}a_{13}; b \mapsto eca_{41}a_{42}; c \mapsto -\varepsilon ca_{41}^2; d \mapsto a_{44}^2a_{12} + b\varepsilon a_{42} + c(\varepsilon a_{32} - a_{43}) - \frac{d}{\varepsilon a_{41}}; e \mapsto b - eca_{41}a_{42}; f \mapsto \varepsilon(a - ca_{41}a_{43}).$$

If $c \neq 0$ we can make $b = e$ solving for $a_{42}$ and $f = \varepsilon a$ solving for $a_{43}$ and then subtracting a multiple of $A$ and $B$ respectively put them equal 0. Taking then $a_{42} = a_{43} = 0$ we obtain

$$a \mapsto 0; b \mapsto 0; c \mapsto -\varepsilon c a_{41}^2; d \mapsto -\frac{d}{\varepsilon a_{41}}; e \mapsto 0; f \mapsto 0.$$  

Hence finally if $d = 0$ then $C$ represent a degenerate 2-form, if instead $d \neq 0$ then the extension given by $A, B$ and $C$ corresponds to a $\mathbb{Q}$-form of the Lie algebras $(37D)$ and $(37D_1)$ in which we are not interested. If instead $c$ were 0 then $C$ is immediately a degenerate 2-form and the lemma follows. \qed

Then we might assume that $A$, the first generator of $P$, is degenerate and using the action bring it to the form $A = [0, 1, 0, 0, 0, 0]$, so that we are in case 1 of Gong’s proof. Using an element of $\text{GL}(4, \mathbb{Q})$ that fixes $A$ we can see, following Gong’s proof which works the same over $\mathbb{Q}$, that we can bring the second generator of $P$ to one of the following forms $B = [0, 0, 0, 0, 1, 0], B = [0, 0, 0, 0, 1, 1]$ or $B = [0, 0, 0, 0, 0, 1]$. Again following Gong’s proof the first two cases lead to the same representatives of the orbits over $\mathbb{Q}$ that he has found over $\mathbb{R}$, hence we will not write the details. Assume then that $A = [0, 0, 0, 0, 0, 0]$ and $B = [0, 0, 0, 0, 0, 1]$. Subtracting a multiple of $A$ and $B$ let us take $C = [a, 0, c, d, e, f]$. In order to fix $A$ and $B$ up to scalar we can take $a_{12} = a_{14} = a_{24} = a_{31} = a_{32} = a_{34} = 0$, $a_{11} = \frac{a_{33}a_{44}}{a_{22}}, a_{13} = \frac{a_{33}a_{42}}{a_{22}}$, call the group defined by these condition $H$. Then the group action on $C$ is as follows

$$a \mapsto aa_{33}a_{44} + e\frac{a_{33}a_{42}}{a_{22}} + e\frac{a_{33}a_{44}}{a_{22}} + e\Sigma^2_{23}b \mapsto 0(\text{by subtracting a multiple of } A \text{ from } C); c \mapsto e\frac{a_{33}a_{42}^2}{a_{22}} + ea_{21}a_{44}; d \mapsto aa_{33}a_{42} + e\frac{a_{33}a_{41}}{a_{22}} + da_{22}a_{33} + e\Sigma^3_{23}; e \mapsto e\Sigma^2_{24}; f \mapsto e\Sigma^3_{24}. $$
If $e \neq 0$ or if $e = c = 0$ the analysis of Gong works the same over $\mathbb{Q}$ leading to the same representatives of the $\mathbb{Q}$-form as for the Lie algebra. Hence we will omit the details. If $e = 0$ and $c \neq 0$, by solving for $a_{42}$ we can make $a = f$ and then subtract a multiple of $B$ to make them equal 0. Then taking $a_{42} = 0$ we are left with

$$a \mapsto 0; b \mapsto 0; c \mapsto c \frac{a_{33}a_{42}}{a_{22}}; d \mapsto da_{22}a_{33}; e \mapsto 0; f \mapsto 0.$$

Hence a representative of the $\mathbb{Q}$-orbit is $C_\varepsilon = [0, 0, 1, \varepsilon, 0, 0]$ with $C_{\varepsilon'}$ in the same orbit of $C_\varepsilon$ under $H$ if and only if there exists $\alpha \in \mathbb{Q}^*$ such that $\varepsilon' = \alpha^2 \varepsilon$.

**Lemma 6.15.** The Lie algebras $N_9^{\mathbb{Q}}(\varepsilon) \cong N_9^{\mathbb{Q}}(\varepsilon')$ if and only if there exists $\alpha \in \mathbb{Q}^*$ such that $\varepsilon' = \alpha^2 \varepsilon$.

**Proof.** Indeed the Lie algebras $N_9^{\mathbb{Q}}(\varepsilon)$ can be characterised, after our previous analysis, by saying that $B$ is a symplectic 2-form, i.e. non degenerate, and that the kernel of $A$ is a Lagrangian subspace of $H$. Hence, from general theory of symplectic forms, it is obvious that $B$ can be brought to the form $\Delta_{12} + \Delta_{34}$. Furthermore, since symplectic transformations act transitively on Lagrangian subspace, and this works over any field of characteristic 0, see [23, Proposition 2.2.13], $A$ can be brought to the form $\Delta_{13}$. Then, if we consider two Lie algebras $N_9^{\mathbb{Q}}(\varepsilon)$ and $N_9^{\mathbb{Q}}(\varepsilon')$ and the 3-dimensional subspaces of the cohomology group associated to them, we can bring their first two generators to the form $A = [0, 1, 0, 0, 0, 0]$ and $B = [1, 0, 0, 0, 0, 1]$. Therefore $N_9^{\mathbb{Q}}(\varepsilon)$ and $N_9^{\mathbb{Q}}(\varepsilon')$ are isomorphic if and only if there exists an element that stabilise $A$ and $B$ and that brings $C_\varepsilon$ to $C_{\varepsilon'}$. From classical theory the stabiliser, up to scalar multiples, of $A$ and $B$ is of the form

$$\left\{ \begin{pmatrix} a_{11} & 0 & -a_{42} \\ a_{21} & a_{22} & a_{23} \\ -a_{24} & 0 & a_{22} \\ a_{41} & a_{42} & a_{43} \\ a_{41} & a_{42} & a_{41} \end{pmatrix} \mid a_{11}a_{22} - a_{24}a_{42} \neq 0, -a_{11}a_{23} - a_{42}a_{21} + a_{24}a_{43} + a_{22}a_{41} = 0 \right\}.$$

One can then check that $C_\varepsilon$ is sent to $C_{\varepsilon'}$ under the stabiliser if and only if $\varepsilon$ and $\varepsilon'$ differ by a square. \hfill $\Box$

The Lie algebras $N_9^{\mathbb{C}}(\varepsilon)$ are 2-dimensional central extensions of the Lie algebra $\mathfrak{g}$ defined on the basis $\{x_1, \ldots, x_5\}$ by $[x_1, x_2] = x_3$. An element of the cohomology group of $\mathfrak{g}$ reads as $a\Delta_{13} + b\Delta_{14} + c\Delta_{15} + d\Delta_{23} + e\Delta_{24} + f\Delta_{25} + g\Delta_{45}$. The action of the automorphism group of $\mathfrak{g}$ on it is as follows:

$$a \mapsto aa_{11}\delta + da_{21}\delta;$$
$$b \mapsto a_{11}(aa_{34} + ba_{44} + ca_{44}) + a_{21}(da_{34} + ea_{44} + fa_{54}) + g(a_{41}a_{54} - a_{51}a_{44});$$
$$c \mapsto a_{11}(aa_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + ea_{45} + fa_{55}) + g(a_{41}a_{55} - a_{51}a_{45});$$
$$d \mapsto aa_{12}\delta + da_{22}\delta;$$
$$e \mapsto a_{12}(aa_{34} + ba_{44} + ca_{44}) + a_{22}(da_{34} + ea_{44} + fa_{54}) + g(a_{42}a_{54} - a_{52}a_{44});$$
$$f \mapsto a_{12}(aa_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45} + fa_{55}) + g(a_{42}a_{55} - a_{52}a_{45});$$
$$g \mapsto g(a_{44}a_{55} - a_{54}a_{45}).$$
where $\delta = a_{11}a_{22} - a_{12}a_{21}$. Looking at the definition of the family of Lie algebras $N^R_{15}(\varepsilon)$ in Appendix A we can see that it corresponds to the 2-dimensional subspace of the cohomology of $\mathfrak{g}$ generated by $(\Delta_{13} + \Delta_{14} + \varepsilon\Delta_{25})$ and $(\Delta_{15} + \Delta_{23})$ with $\varepsilon \in \{0, 1, -1\}$. Let $V_1$ be the subspace of $H^2(\mathfrak{g}, \mathbb{R})$ generated by $\{\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}\}$, $V_2$ the subspace generated by $\{\Delta_{14}, \Delta_{15}, \Delta_{24}, \Delta_{25}, \Delta_{45}\}$ and $V_3 = V_1 \cap V_2$. If $L$ is a 2-dimensional subspace of $H^2(\mathfrak{g}, \mathbb{R})$ associated to $N^R_{15}(\varepsilon)$ then in Gong’s analysis $L$ is characterised by $L \subseteq V_1$ and $L \cap V_3 = 0$. Let us now consider a $\mathbb{Q}$-form $\mathfrak{h}$ of $N^R_{15}(\varepsilon)$. Then $\mathfrak{h}$ is a 2-dimensional central extension of $\mathfrak{g}$ and let $P$ be the 2-dimensional subspace of $H^2(\mathfrak{g}, \mathbb{Q})$ associated to it. We can then assume $P \subseteq V_1$ and $P \cap V_3 = 0$ so that the elements of a basis for $P$ are $A = [1, b, c, d, e, f, 0]$. Considering then the action on $P$, we can make $b = 0$ otherwise there will be a non trivial element in the intersection $\mathfrak{g}_1^1 \cap \mathbb{Z}(\mathfrak{g})$. Now assuming $b \neq 0$ we can make it 1 taking $a_{44} = \frac{1}{16a_{11}}$ and solving for $a_{12}$ make $e$ equal 0. Solving for $a_{12}$ we can make $e = 0$ and then subtracting a multiple of $b$ make them equal 0. Then taking $a_{12} = a_{35} = a_{45} = 0$ we obtain

$$a = 1 \mapsto a_{11}\delta; \quad b \mapsto a_{11}ba_{44}; \quad c \mapsto a_{11}(a_{35} + ba_{45} + ca_{55}); \quad d \mapsto a_{12}\delta; \quad e \mapsto a_{12}ba_{44} + a_{22}ca_{44}; \quad f \mapsto a_{12}(a_{35} + ba_{45} + ca_{55}) + a_{22}(ca_{45} + fa_{55}); \quad g \mapsto 0.$$

Notice that $b$ and $e$ cannot be both 0 otherwise there will be a non trivial element in the intersection $\mathfrak{g}_1^1 \cap \mathbb{Z}(\mathfrak{g})$. Now assuming $b \neq 0$ we can make it 1 taking $a_{44} = \frac{1}{16a_{11}}$ and solving for $a_{12}$ make $e$ equal 0. Solving for $a_{12}$ we can make $e = 0$ and then subtracting a multiple of $b$ make them equal 0. Then taking $a_{12} = a_{35} = a_{45} = 0$ we obtain

$$a = 1 \mapsto a_{11}\delta; \quad b = 1 \mapsto a_{11}a_{44}; \quad c = 0 \mapsto 0; \quad d = 0 \mapsto 0; \quad e = 0 \mapsto 0; \quad f \mapsto a_{22}fa_{55}; \quad g \mapsto 0.$$

The representatives of the orbits are then $A_\varepsilon = [1, 1, 0, 0, 0, \varepsilon, 0]$ with $\varepsilon \in \mathbb{Q}$ and $A_{\varepsilon'}$ is in the same orbit of $A_\varepsilon$ if and only if there exists $\alpha \in \mathbb{Q}^*$ such that $\varepsilon' = \alpha \varepsilon$. If instead $b = 0$, we can assume $e = 1$ then one find the representative $A = [1, 0, 0, 0, 1, 0, 0]$. But then this case is in the same orbit as $[1, 1, 0, 0, 0, -1, 0] \wedge B$.

We are now left to consider the Lie algebras $L^R_{6}(1), N_{16}, N_{17}, N_{18}$ and $N_{19}$ of Appendix A. The one that we have called $N_{16}$ and $N_{17}$ are 2-dimensional central extensions of the Lie algebra defined on the basis $\{x_1, \ldots, x_6\}$ by $[x_1, x_2] = x_3$ with $x_3$ and extensions of the abelian Lie algebra of dimension 5. With an analysis that follows the one that we have just done we can see that the Lie algebras $N_{16}, N_{17}, N_{18}$ and $N_{19}$ have just one $\mathbb{Q}$-form up to $\mathbb{Q}$-isomorphism. Hence we are led with the family of Lie algebras $L^R_{6}(1)$ that is a 2-dimensional central extension of the Lie algebra defined on the basis $\{x_1, \ldots, x_6\}$ by $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6$. The elements in its cohomology group can be represented by $a\Delta_{14} + b\Delta_{15} + c\Delta_{16} + d\Delta_{23} + e\Delta_{24} + f(\Delta_{25} + \Delta_{34}) + g\Delta_{26} + h\Delta_{35} + i\Delta_{36}$. The action of the automorphism group is as follows:

$$a \mapsto a(\Delta_{14} = ba_{22} + ba_{12}a_{33} + ea_{11}a_{22}a_{21} + f(a_{12}(a_{21}a_{32} + a_{31}a_{22}) + ha_{11}a_{31}a_{32});$$
$$b \mapsto a(\Delta_{14} = ba_{12}a_{23} + ba_{11}a_{33} + ea_{11}a_{22}a_{21} + f(a_{12}(a_{21}a_{32} + a_{31}a_{22}) + ha_{11}a_{31}a_{32});$$
$$c \mapsto a(a_{11}a_{46} + ba_{11}a_{56} + ca_{11}a_{66} + ea_{21}a_{46} + f(a_{21}a_{56} + a_{31}a_{46}) + ga_{21}a_{66} + ha_{31}a_{56} + ia_{31}a_{66};$$
$$d \mapsto d(a_{22}a_{33} - a_{23}a_{43} + e(a_{22}a_{43} - a_{32}a_{42}) + f(a_{22}a_{53} - a_{23}a_{53} + a_{32}a_{53} - a_{33}a_{42}) + g(a_{22}a_{63} - a_{23}a_{62}) + h(a_{22}a_{53} - a_{32}a_{52} + i(a_{23}a_{63} - a_{33}a_{52};$$

$$e \mapsto e(a_{11}a_{46} + ba_{11}a_{56} + ca_{11}a_{66} + ea_{21}a_{46} + f(a_{21}a_{56} + a_{31}a_{46}) + ga_{21}a_{66} + ha_{31}a_{56} + ia_{31}a_{66};$$

$$f \mapsto f(a_{22}a_{33} - a_{23}a_{43} + e(a_{22}a_{43} - a_{32}a_{42}) + f(a_{22}a_{53} - a_{23}a_{53} + a_{32}a_{53} - a_{33}a_{42}) + g(a_{22}a_{63} - a_{23}a_{62}) + h(a_{22}a_{53} - a_{32}a_{52} + i(a_{23}a_{63} - a_{33}a_{52});$$

$$g \mapsto g(a_{22}a_{33} - a_{23}a_{43} + e(a_{22}a_{43} - a_{32}a_{42}) + f(a_{22}a_{53} - a_{23}a_{53} + a_{32}a_{53} - a_{33}a_{42}) + g(a_{22}a_{63} - a_{23}a_{62}) + h(a_{22}a_{53} - a_{32}a_{52} + i(a_{23}a_{63} - a_{33}a_{52};$$

$$h \mapsto h(a_{22}a_{33} - a_{23}a_{43} + e(a_{22}a_{43} - a_{32}a_{42}) + f(a_{22}a_{53} - a_{23}a_{53} + a_{32}a_{53} - a_{33}a_{42}) + g(a_{22}a_{63} - a_{23}a_{62}) + h(a_{22}a_{53} - a_{32}a_{52} + i(a_{23}a_{63} - a_{33}a_{52});$$

$$i \mapsto i(a_{22}a_{33} - a_{23}a_{43} + e(a_{22}a_{43} - a_{32}a_{42}) + f(a_{22}a_{53} - a_{23}a_{53} + a_{32}a_{53} - a_{33}a_{42}) + g(a_{22}a_{63} - a_{23}a_{62}) + h(a_{22}a_{53} - a_{32}a_{52} + i(a_{23}a_{63} - a_{33}a_{52}).$$
The family $L^\mathbb{R}(1)$ corresponds to $(\Delta_{12} - \Delta_{36}) \wedge (\Delta_{15} + \Delta_{26})$. Then we can bring the first generator of a 2-dimensional subspace of $H^2(g, \mathbb{Q})$ related to a $\mathbb{Q}$-form of $L^\mathbb{R}(1)$ to $A = [1, 0, 0, 0, 0, 0, 0, -1]$. In order to leave $A$ stable we need $a_{23} = a_{32} = a_{62} = 0, a_{46} = a_{31}a_{66}, a_{11}a_{22} = a_{33}a_{66} = 1$. Then the action on $B = [0, b, c, d, e, f, g, h, i]$ is as follows

$$a \mapsto ea_{11}a_{21}a_{22} + fa_{11}a_{22}a_{31}; b \mapsto ba_{11}a_{33} + fa_{11}a_{21}a_{33} + ha_{11}a_{31}a_{33}; c \mapsto ba_{11}a_{56} + ca_{11}a_{66} + ea_{21}a_{46} + f(a_{21}a_{56} + a_{31}a_{46}) + ga_{22}a_{66} + ha_{31}a_{56} + ia_{31}a_{66};$$

$$d \mapsto da_{22}a_{33} + ea_{22}a_{43} + f(a_{22}a_{53} - a_{33}a_{42}) + ga_{22}a_{63} - ha_{33}a_{52}; e \mapsto ea_{11}a_{22}; f \mapsto fa_{11}a_{22}a_{33}; g \mapsto ea_{22}a_{46} + fa_{22}a_{56} + ga_{22}a_{66}; h \mapsto ha_{11}a_{33}; i \mapsto fa_{33}a_{46} + ha_{33}a_{56} + ia_{33}a_{66}.$$

Since $e, f, h$ cannot be put to 0 they should be 0, and for the same reason $i = 0$. Then solving for $a_{63}, a_{66}$ put $d = c = 0$. We are left with

$$a \mapsto 0; b \mapsto ba_{11}a_{33}; c \mapsto 0; d \mapsto 0; e \mapsto 0; f \mapsto 0; g \mapsto ga_{22}a_{66}; h \mapsto 0; i \mapsto 0.$$

Then a representative of the orbit is $B_\varepsilon = [0, 1, 0, 0, 0, 0, 0, \varepsilon, 0, 0]$ and $B_{\varepsilon'}$ is in the same orbit as $B_\varepsilon$ if and only if there exists $\alpha \in \mathbb{Q}^*$ such that $\varepsilon' = \alpha^2 \varepsilon$. 

Putting together Proposition 6.5, 6.7 and 6.13 we see that the Proof of Theorem 1.9 is achieved.

**Remark 6.16.** A priori the method just presented could have been applied also to classify the family of Lie algebras that we found in the non degenerate case, namely $g(\alpha, a, b, c)$. Nevertheless, the method was not that easy to apply for that case hence we decided to use an ad hoc method. Indeed also in [15] for the case of central extensions of the free Lie algebra of rank 2 over 3 generators the author uses another method.

### 7. Topological Considerations

Let $\Gamma$ be a subgroup of $\mathcal{H}(n, 1)$ acting properly discontinuously and cocompactly on $a(\mathbb{C}^{n+1})$. Then from [17, Theorem 1.3.] $\Gamma$ is virtually polycyclic. From a theorem of Selberg every finitely generated linear group contains a torsion free subgroup of finite index. Hence, up to replacing $\Gamma$ by a finite index subgroup, we can consider $M = \Gamma \setminus a(\mathbb{C}^{n+1})$ to be a compact flat Hermite-Lorentz manifold. From Theorem 1.2 there exists a subgroup $H \leq \mathcal{H}(n, 1)$ that acts simply transitively on $a(\mathbb{C}^{n+1})$ and $\Gamma \cap H$ has finite index in $\Gamma$ and it is a lattice in $H$. Hence, up to finite cover, $M$ is diffeomorphic to $(\Gamma \cap H) \setminus H$.

Let us suppose that $H = U(\gamma_2, \gamma_3, b_2, b_3)$ is a unipotent group so that it has the form of Proposition 2.5.

**Proposition 7.1.** Let $\Gamma$ be a lattice in the group $U := U(\gamma_2, \gamma_3, b_2, b_3)$. The manifold $\Gamma \setminus U$ is a fiber bundle over a real torus $\Gamma/(\Gamma \cap \mathbb{C}^2 U) \setminus \mathbb{C}^2 U$ of dimension $\frac{2n+1}{2} < p < 2n+2$ with fibers
that are real tori of dimension $q = 2n + 2 - p$. Furthermore this fibration split into two fiber bundles as follows

$$\Gamma \backslash U \rightarrow \Gamma / (\Gamma \cap C^3U) \backslash U / C^3U \rightarrow \Gamma / (\Gamma \cap C^2U) \backslash U / C^2U.$$ 

Proof. Let $C^i U$, with $i \leq 3$, be the elements of the lower central series, we have the following commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & C^3U & \longrightarrow & U & \overset{\pi_1}{\longrightarrow} & U/C^3U & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \pi_2 & & \downarrow \pi & & \downarrow 0 \\
0 & \longrightarrow & C^2U & \longrightarrow & U & \overset{\pi}{\longrightarrow} & U/C^2U & \longrightarrow & 0
\end{array}
$$

From [19, Corollary 1 of Theorem 2.3.] if $\Gamma$ is a lattice in $U$ nilpotent simply connected Lie group then $\Gamma \cap C^i U$ is a lattice in $C^i U$. Hence considering the induces maps by $\pi_1$ and $\pi_2$ on the quotients by $\Gamma$ we have that $\Gamma \backslash U$ can be seen as a sequence of fiber bundles. Furthermore the derived group $C^i U$ is abelian of dimension $q$ with $0 \leq q \leq 2 + 2\text{rank}(\gamma_2) + 1$. From Lemma 2.8 we then have $q \leq 4n+5$. Hence the quotient $U/C^2U$ is an abelian Lie group of dimension $p$ with $2n+3 \leq p \leq 2n+2$. Then $C^i U$ is isomorphic to $\mathbb{R}^q$ and $U/C^2U$ is isomorphic to $\mathbb{R}^p$. Since, as we have seen, $\Gamma$ intersects $C^2 U$ in a lattice we have that $\Gamma \cap C^2 U$ is isomorphic to $\mathbb{Z}^q$ and $\pi(\Gamma)$ is isomorphic to $\mathbb{Z}^p$. Then finally considering the fiber bundle induce by $\pi$ the manifold $\Gamma / (\Gamma \cap C^2U) \backslash U / C^2U$ is a torus and the fibers $\Gamma \cap C^2U \backslash U / C^2U$ are real tori. \hfill $\Box$

Remark 7.2. Notice that the above proposition is just a translation of the fact that the group $U(\gamma_2, \gamma_3, b_2, b_3)$ is 3-step nilpotent.

APPENDIX A.

For $K = \mathbb{R}$ the following is a non redundant list, up to isomorphism, of the 8-dimensional nilpotent Lie algebras that appear as Lie algebras of unipotent simply transitive subgroups of $U(3,1) \ltimes C^{3+1}$. They are found putting together Proposition 4.3, 4.4, 5.19 and 5.33. Furthermore taking $K = \mathbb{Q}$ this is also the complete non redundant list of the $\mathbb{Q}$-isomorphism classes of $\mathbb{Q}$-forms in the aforementioned Lie algebras and hence of the abstract commensurability classes of nilpotent crystallographic subgroups of $U(3,1) \ltimes C^{3+1}$. They are found putting together Proposition 6.5, 6.7 and 6.13. We present these Lie algebras defined over the field $K$ in a compact version that is valid for both $K = \mathbb{R}$ or $K = \mathbb{Q}$. The presentation is given in the basis $\{x_1, \ldots, x_8\}$ and we will write only the non zero Lie brackets. For the Lie algebras that decompose as a direct sum of an abelian ideal and a smaller dimensional Lie algebra we have written in brackets the corresponding names in the lists of de Graaf [16] for dimension up to 6 and of Gong [15] for dimension 7.

For the case $\pi_3(\gamma_3(i\xi) - J\gamma_3(\xi)) = 0$, see Proposition 4.3, we have

- $L_1$: abelian,
- $L_2$: $[x_1, x_2] = x_3$ (L3,2),
- $L_3$: $[x_1, x_2] = x_4, [x_1, x_3] = x_5$ (L5,8),
- $L_4$: $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_1, x_5] = x_7$ (247A),
- $L_5$: $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_1, x_5] = x_7, [x_2, x_3] = x_6$ (247L),
- $L_6^{(\varepsilon)}(\varepsilon)$: $[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_3] = \varepsilon x_8, [x_3, x_6] = -x_7$ with $\varepsilon \in K_{>0}$ and $L_6(\varepsilon) \cong L_6(\varepsilon')$ if and only if there exists $\alpha \in K^*$ such that $\varepsilon' = \alpha^2 \varepsilon$. 

APPENDIX A.
For the case \( \tau_3(\gamma_3(\xi)) - J\gamma_3(\xi) \neq 0 \) and \( \gamma_2 = 0 \), see Proposition 4.4, we have

- \( N_1: [x_1, x_2] = x_3, [x_1, x_3] = x_4 (L_{4,3}) \),
- \( N_2: [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5 (L_{5,9}) \),
- \( N_3: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_3] = x_5 (L_{5,5}) \),
- \( N_4: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_6 (L_{6,27}) \),
- \( N_5: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6 (L_{6,25}) \),
- \( N_6^\varepsilon(\varepsilon): [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_3] = x_5 \) with \( \varepsilon \in \mathbb{K} \) and \( N_6(\varepsilon) \cong N_6(\varepsilon') \) if and only if there exists \( \alpha \in \mathbb{K}^* \) such that \( \varepsilon' = \alpha^2 \varepsilon \) (L_{6,24}(\varepsilon))
- \( N_7: [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_1, x_4] = x_6, [x_2, x_4] = x_5 (L_{6,23}) \),
- \( N_8: [x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7 (37A) \),
- \( N_9^\varepsilon(\varepsilon): [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_3] = \varepsilon x_7, [x_3, x_4] = x_5 \) with \( \varepsilon \in \mathbb{K} \) and \( N_9(\varepsilon) \cong N_9(\varepsilon') \) if and only if there exists \( \alpha \in \mathbb{K}^* \) such that \( \varepsilon' = \alpha^2 \varepsilon \). Over \( \mathbb{R} \) to classify equivalence classes the parameter \( \varepsilon \) can then take three values \( \varepsilon \in \{0, 1, -1\} \) that correspond to (37C), (37B1) and (37B) respectively.
- \( N_{10}: [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_1, x_4] = x_5 (L_{6,22}(0)) \),
- \( N_{11}: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_5] = x_7, [x_2, x_4] = x_6 (257A) \),
- \( N_{12}: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_5] = x_7 (257B) \),
- \( N_{13}: [x_1, x_3] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_6, [x_2, x_5] = x_7 (257C) \),
- \( N_{14}: [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_4] = x_6, [x_2, x_5] = x_7 (257D) \),
- \( N_{15}^\varepsilon(\varepsilon): [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_3] = \varepsilon x_7, [x_3, x_4] = x_7 \) with \( \varepsilon \in \mathbb{K} \) and \( N_{19}(\varepsilon) \cong N_{19}(\varepsilon') \) if and only if there exists \( \alpha \in \mathbb{K}^* \) such that \( \varepsilon' = \alpha^4 \varepsilon \). Over \( \mathbb{R} \) to classify equivalence classes the parameter \( \varepsilon \) can then take three values \( \varepsilon \in \{0, 1, -1\} \) that correspond to (257I), (257J1) and (257J) respectively.
- \( N_{16}: [x_1, x_2] = x_3, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_8 (257) \),
- \( N_{17}: [x_1, x_2] = x_3, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_7, [x_2, x_6] = x_8 \),
- \( N_{18}: [x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_5] = x_8 \),
- \( N_{19}: [x_1, x_2] = x_6, [x_1, x_3] = x_7, [x_1, x_4] = x_8, [x_2, x_4] = x_7, [x_2, x_5] = x_8 \).

For the case \( \tau_3(\gamma_3(\xi)) - J\gamma_3(\xi) \neq 0 \) and \( \gamma_2 \neq 0 \), see Proposition 5.19 and 5.33, we have

- \( gK(0, 0, 0, 0): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8 \),
- \( gK(0, 0, 0, 1): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_6] = \varepsilon x_8, [x_3, x_6] = x_7 \) with \( \varepsilon \in \mathbb{K} \) such that \( gK(0, 0, 0, 1) \cong gK(0, 0, 0, 1) \) if and only if there exists \( \alpha \in \mathbb{K}^* \) such that \( \varepsilon' = \alpha^2 \varepsilon \)
- \( gK(a, b, c): [x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_1, x_4] = x_7, [x_1, x_5] = x_8, [x_2, x_4] = x_7, [x_2, x_5] = \varepsilon x_8, [x_3, x_4] = \varepsilon x_8, [x_3, x_5] = x_7, [x_2, x_6] = x_8, [x_3, x_6] = c x_7 - a x_8 \) with \( a^2, b, c \in \mathbb{K} \) and \( gK(a, b, c) \cong gK(-a, b, c) \).

**Remark A.1.** Since it is not written in Gong’s thesis we point out that the isomorphism between \( N_6^\varepsilon(-1) \) and (37B) is given by \( x'_1 = x_2 - x_4, x'_2 = x_1 - x_3, x'_3 = x_1 + x_3, x'_4 = x_2 + x_4 x_5 = 2(x_7 - x_5), x'_6 = 2x_6, x'_7 = 2(x_5 + x_7) \).

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