MAX CUT IN DEGENERATE $H$-FREE GRAPHS

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Abstract. We obtain several lower bounds on the Max-Cut of $d$-degenerate $H$-free graphs. Let $f(m, d, H)$ denote the smallest Max-Cut of an $H$-free $d$-degenerate graph on $m$ edges. We show that $f(m, d, K_r) \geq \left( \frac{1}{2} + d^{-1+\Omega(r^{-1})} \right) m$, generalizing a recent work of Carlson, Kolla, and Trevisan. We give bounds on $f(m, d, H)$ when $H$ is a cycle, odd wheel, or a complete bipartite graph with at most 4 vertices on one side. We also show stronger bounds on $f(m, d, K_r)$ assuming a conjecture of Alon, Bollabas, Krivelevich, and Sudakov (2003). We conjecture that $f(m, d, K_r) = \left( \frac{1}{2} + \Theta(d^{-1/2}) \right) m$ for every $r \geq 3$, and show that this conjecture implies the ABKS conjecture.

1. Introduction

Given a graph $G$, let $n = n(G)$ denote the number of vertices and $m = m(G)$ denote the number of edges. A cut of $G = (V, E)$ is a bipartition of the vertices $V = A \sqcup B$, and the size of a cut $A \sqcup B$ is the number of edges between $A$ and $B$. The Max-Cut of a graph $G$, denoted $\text{Max-Cut}(G)$, is the size of largest cut of $G$.

There has been extensive work understanding the Max-Cut from an extremal perspective. Most simply, by taking a random cut, we see that every graph with Max-Cut denoted $f$ of an $H$-free $d$-degenerate graph on $m$ edges satisfies $\text{Max-Cut}(G) \geq \frac{m}{2} + c\sqrt{m}$ for some $c > 0$, which is tight up to a choice of $c$. In this article, we adopt a similar perspective but study families of graphs equipped with additional structure, namely having a fixed forbidden subgraph $H$, and parametrize our bounds by a measure of sparseness called degeneracy.

We say a graph $G$ is $H$-free if it does not contain $H$ as a subgraph. Let $f(m, H)$ be the minimum Max-Cut of an $H$-free graph on $m$ edges. The quantity $f(m, H)$ has been studied extensively. Alon [1,2] studied the Max-Cut of triangle-free graphs, showing that $f(m, K_3) = \frac{m}{2} + \Omega(m^{4/5})$. Alon, Krivelevich and Sudakov [4] showed tight bounds on $f(m, H)$ for a number of sparse graphs $H$. When $H$ is a tree on $r$ vertices, they showed that $f(m, H) \geq \left( \frac{1}{2} + \frac{1}{2r^2} \right) m$ where $c$ is 2 if $m$ is even and 0 if $m$ is odd; they further observed that equality holds for infinitely many $m$. For even cycles $C_{r-1}$, they showed that $f(m, C_{r-1}) \geq \frac{m}{2} + \Omega(m^{r/(r+1)})$ and that this bound is tight up to a choice of constant in the lower order term for 4-cycles, 6-cycles, and 10-cycles. They also considered complete bipartite graphs, showing that $f(m, K_{2,s}) = \frac{m}{2} + \Theta(m^{5/6})$ and that $f(m, K_{3,s}) = \frac{m}{2} + \Theta(m^{4/5})$. They also showed that, for $H$ obtained by connecting a vertex to a nontrivial forest, $f(m, H) = \frac{m}{2} + \Theta(m^{4/5})$. Recently, Zeng and Hou [24] considered complete graphs, showing that $f(m, K_r) \geq \frac{m}{2} + \Omega(m^{r/(2r-1)})$ for all $r \geq 3$.

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1 In this paper, all graphs we consider are finite, undirected and simple, unless otherwise specified.
Previous work has also considered the complement of the Max-Cut problem, i.e. the minimum number of edges that must be removed to make a graph bipartite, or more generally $k$-partite. A longstanding conjecture of Erdős (he wrote in 1975 [14] that it was already old) states that every triangle $K_3$-free graph on $n$ vertices can be made bipartite by deleting at most $n^2/25$ edges. If true, this conjectured bound is the best possible: this can be seen by considering a balanced blow-up of a cycle on five vertices. While this problem has been seriously investigated, the best known upper bound [15] is approximately $n^2/18$, and thus Erdős’s conjecture remains open. Solving a different conjecture of Erdős, Sudakov [22] showed that any $K_4$-free graph on $n$ vertices can be made bipartite by removing at most $n^2/9$ edges. This bound is tight, which can be seen by considering a balanced blow-up of a triangle. Sudakov further conjectured for $r > 4$ that the balanced complete $(r-1)$-partite graph on $n$ vertices is the furthest from being bipartite over all $K_r$-free graphs. A recent result [17] showed that any $K_r$-free graph on $n$ vertices is at most $5 \cdot 8^{r-2} n^2/3e \cdot 2^{(r-1)/(r-2)}$ edges from bipartite.

The results in the previous two paragraphs are more useful for graphs with many edges, and give much weaker bounds for sparse graphs. One might hope to give bounds on the Max-Cut in terms of some sparseness property of the graph, like the maximum degree or degeneracy, and in this article, we address this question for a variety of choices of $H$.

A d-degenerate graph $G$ is a graph such that every induced subgraph has a vertex of degree at most $d$. Equivalently, $G$ is $d$-degenerate if there exists an ordering $1, \ldots, n$ of the vertices such that every vertex $i$ has at most $d$ neighbors with index $j < i$. Degeneracy is a broader notion of sparseness than maximum degree: all maximum degree $d$ graphs are $d$-degenerate, but the star graph is 1-degenerate while having maximum degree $n - 1$.

Let $f(m, d, H)$ be the minimum Max-Cut of a $d$-degenerate $H$-free graph with $m$ edges. We largely focus on the case $H = K_r$ is a clique on $r$ vertices, but also give bounds on $f(m, d, H)$ for several other families of forbidden subgraphs $H$, including odd wheels $W_{2r+1}$ (obtained by connecting a central vertex to each vertex of an even cycle $C_{2r}$), the complete bipartite graphs $K_{2,s}, K_{3,s}, K_{4,s}$, and cycles $C_r$.

Some bounds are known for the Max-Cut of $d$-degenerate graphs. The expected Max-Cut of a random $d$-regular graph $G$ is, with high probability, $\left(\frac{1}{2} + \Theta\left(\frac{1}{\sqrt{d}}\right)\right)m(G)$. For arbitrary $d$-degenerate graphs $G$,

$$\text{Max-Cut}(G) \geq \left(\frac{1}{2} + \frac{c}{d}\right)m(G),$$

a bound met up to the constant $c$ by the disjoint union of $K_{d+1}$’s. This bound can be obtained by randomly ordering the vertices of $G$ and greedily adding them to a constructed cut one at a time from that ordering. The expected number of vertices with an odd number of neighbors before it is $\frac{n}{2}$, which is at least $\frac{m(G)}{2d}$ (as $G$ is $d$-degenerate). Each such vertex increases the difference between the number of cut and uncut edges by at least 1, giving a cut of size $\left(\frac{1}{2} + \frac{1}{4d}\right)m(G)$. Shearer [21] gave a tight (up to a constant factor) bound on Max-Cut($G$) for $K_3$-free $d$-degenerate graphs, showing that there exists $c > 0$ such that, for all $m, d \geq 1$.

$$f(m, d, K_3) \geq \left(\frac{1}{2} + \frac{c}{\sqrt{d}}\right)m(G).$$
This bound is tight up to choice of $c$, seen by taking a random $d$-regular graph $G_{n,d}$ and removing an edge from every triangle (for details, see e.g. Proposition 1.3). In the case that graph $G$ has maximum degree $d$, Shearer’s bound was generalized by Carlson, Kolla, and Trevisan to $K_r$-free graphs, who showed the following result.

**Theorem 1 (8).** There exists a $c > 0$ such that for all $r \geq 3$ and all $m, d \geq 1$, every $K_r$-free graph $G$ on $m$ edges and maximum degree $d$ satisfies

$$\text{Max-Cut}(G) \geq \left(\frac{1}{2} + cd^{-1+2^{-r-2}}\right)m.$$ 

We improve on the above bound in two ways. First, we generalize from maximum degree $d$ to $d$-degenerate, and second, we improve the exponent of $d$ in the lower order term.

**Theorem 2.** There exists a $c > 0$ such that for all $r \geq 3$ and $m, d \geq 1$, we have

$$f(m, d, K_r) \geq \left(\frac{1}{2} + cd^{-1+1/(2r-4)}\right)m.$$ 

For maximum degree $d$ graphs, Theorem 2 matches Theorem 1 when $r = 3, 4$ and gives a strict improvement over Theorem 1 when $r \geq 5$ (up to the constant $c$). In the special case $r = 4$, we modify our method to improve the exponent in $d$ from $3/4$ to $2/3$.

**Theorem 3.** There exists a constant $c > 0$ such that for all $m, d \geq 1$, we have

$$f(m, d, K_4) \geq \left(\frac{1}{2} + cd^{-2/3}\right)m.$$ 

To prove Theorem 2 and Theorem 3, we make use of a more general framework for lower bounds described in Section 2.3. These methods also allow us to leverage the bounds in [4] to give nontrivial lower bounds on the Max-Cut of $d$-degenerate $H$-free graphs for several families of sparse forbidden subgraphs $H$.

**Theorem 4.** For a graph $H$, let

$$\alpha_H(d) = \begin{cases} 
  d^{-(r+1)/(2r-1)} & \text{if } H = W_r \text{ and } r \text{ is odd} \\
  d^{-7/11} & \text{if } H = K_{3,s} \\
  d^{-2/3} & \text{if } H = K_{4,s} \\
  d^{-1/2} & \text{if deleting some vertex from } H \text{ gives a forest (forest+1)} \\
  d^{-2/3} & \text{if deleting two vertices from } H \text{ gives a forest (forest+2)}
\end{cases}.$$

When $H$ is one of the above, there exists $c = c(H) > 0$ such that, for all $m, d \geq 1$

$$f(m, d, H) \geq \left(\frac{1}{2} + c \cdot \alpha_H(d)\right)m.$$ 

Note that the case forest+1 includes the cases when $H$ is a cycle and when $H = K_{2,s}$. While Theorems 2 and 3 improve on Theorem 1, we show, using the same methods, that a stronger lower bound is true assuming the following conjecture of Alon, Bollobás, Krivelevich, and Sudakov (3).

**Conjecture 1.1 (3).** For any graph $H$, there exists constants $\varepsilon = \varepsilon(H) > 0$ and $c = c(H) > 0$ such that for all $m \geq 1$,

$$f(m, H) \geq \frac{m}{2} + cm^{3/4+\varepsilon}.$$
Assuming Conjecture 1.1 we show that the exponent of $d$ in the lower order term of $f(m, d, H)$ is bounded away from $-1$ for every graph $H$. Qualitatively, this contrasts with Theorems 1 and 2 where the exponents approach $-1$ as $r$ increases.

Theorem 5. Assuming Conjecture 1.1, for any graph $H$, there exist constants $\varepsilon = \varepsilon(H) > 0$ and $c = c(H) > 0$ such that, for all $m, d \geq 1$, we have

$$f(m, d, H) \geq \left( \frac{1}{2} + cd^{-5/7+\varepsilon} \right) m.$$  

With these results in mind, we pose the following conjecture.

Conjecture 1.2. For any graph $H$, there exists a constant $c = c(H) > 0$ such that, for all $m, d \geq 1$, we have

$$f(m, d, H) \geq \left( \frac{1}{2} + \frac{c}{\sqrt{d}} \right) m.$$  

Theorem 4 shows that Conjecture 1.2 is true when $H$ is a forest with a common neighbor (up to logarithmic factors in the lower order term). To disprove Conjecture 1.2, one would need to construct $d$-degenerate graphs on $m$ vertices with $\text{Max-Cut}$ at most $(\frac{1}{2} + o(\frac{1}{\sqrt{d}}))m$. However, Turán’s theorem implies that a $K_r$-free graph with $\frac{d}{2}$ edges has at least $(1 + \varepsilon_r)d$ vertices. For $n \geq (1+\varepsilon)d$, the Erdős-Rényi graph $G_{n, d/n}$ with high probability (as $d, n \to \infty$) satisfies

$$\text{Max-Cut}(G_{n, d/n}) \geq \left( \frac{1}{2} + \frac{c}{\sqrt{d}} \right) m.$$  

As an additional remark, in all of the tight constructions in [4] that are not random graphs or the disjoint union of cliques, the $\text{Max-Cut}$ is upper bounded by $\frac{m}{2} - \frac{\lambda_n n}{4}$ (see e.g. Lemma 4.1 of [4]), where $\lambda_n$ is the smallest eigenvalue of the graph. In $d$-regular graphs with $n \geq (1+\varepsilon)d$, by Alon-Boppana [5] theorem, this bound cannot be smaller than $(\frac{1}{2} + \frac{c}{\sqrt{d}})m$.

Note that Conjecture 1.2 immediately implies a weaker form of Conjecture 1.1, showing that $\text{Max-Cut}(G) \geq \frac{m}{2} + cm^{3/4}$ for all $H$-free graphs $G$ on $m$ edges, as $\frac{m}{\sqrt{d}} \geq m^{3/4}$. We show that Conjecture 1.2 in fact implies Conjecture 1.1 in full.

Theorem 6. Conjecture 1.2 implies Conjecture 1.1.

Alon, Krivelevich, and Sudakov [4] showed that, when $H$ is a forest, the $\text{Max-Cut}$ of $H$-free graphs is $(\frac{1}{2} + c) \cdot m$ for some $c > 0$ independent of $m$. This result holds independently of the density of the graph, and in particular also applies to $d$-degenerate graphs, where the constant in the lower order term is independent of $d$. For $d$-degenerate graphs, we observe that forests are the only graphs for which this is true: whenever $H$ contains a cycle, there exist infinitely many $H$-free $d$-degenerate (and, in fact, maximum degree $d$) graphs $G$ on $n$ vertices with $\text{Max-Cut}$ no larger than $(\frac{1}{2} + \frac{c}{\sqrt{d}}) \cdot m(G)$. In particular, Conjecture 1.2 is optimal (up to a constant depending on $H$ in the lower order term) if it is true when $H$ is not a forest.

Proposition 1.3. For all $r \geq 3$ and $d \geq 1$, there exist $c = c(r) > 0$ and $n_0 = n_0(r, d)$ such that for all $n \geq n_0$, there exists a $C_r$-free graph $G$ on $n$ vertices with maximum degree $d$ and

$$\text{Max-Cut}(G) \leq \left( \frac{1}{2} + \frac{c}{\sqrt{d}} \right) \cdot m(G).$$
### Forbidden subgraph

| Forbidden subgraph | Prior work | This work | Tight? |
|--------------------|------------|-----------|--------|
| None               | $cd^{-1}$  |           | Y      |
| $K_3$              | $cd^{-1/2}$ |           | Y      |
| $K_4$              | $cd^{-3/4}$ | $\frac{21}{8}$ |        |
| $K_r$              | $cd^{-1+2^{-r}}$ | $\frac{8}{8}$ |        |
| $C_r$              | $cd^{-1/2}$ |           | Y      |
| $W_r$, for odd $r$ | $cd^{-(r+1)/(2r-1)}$ | Thm 4 |        |
| $K_{2,s}$          | $cd^{-1/2}$ |           | Y      |
| $K_{3,s}$          | $cd^{-7/11}$ | Thm 4 |        |
| $K_{4,s}$          | $cd^{-2/3}$ |           | Y      |
| forest             | $\frac{1}{2^r}$ | $\frac{4}{4}$ |        |
| forest+1           | $cd^{-1/2}$ |           | Y      |
| forest+2           | $cd^{-2/3}$ |           | Y      |

**Table 1.** Lower bounds for $f(m,d,H)$ in the literature and our work. Here, we provide the lower order term $cd^{-a}$ in the expression $(\frac{1}{2} + cd^{-a})m$. They are noted as tight if there is a construction that achieves the lower bound on $f(m,d,H)$ up to a constant function of $H$ in the lower order term.

In Table 1, we summarize our lower bounds on Max-Cut($G$) for $H$-free graphs $G$ and how they compare to those in the literature.

**Concurrent work by Sudakov.** In concurrent and independent work, Sudakov obtained results similar to Theorems 2 and 5 for graphs with maximum degree $d$.

**Organization of paper.** In Section 2, we present a general framework to convert lower bounds on the Max-Cut in general (denser) graphs to the Max-Cut of $d$-degenerate graphs. In particular, we show how to convert bounds on $f(m,H)$ to bounds on $f(m,d,H)$. In Section 3, we apply the results in Section 2 to obtain improved bounds on $f(m,d,K_r)$ for $r \geq 4$, proving Theorems 2, 3, and 5. In Section 4, we apply the results in Section 2 to obtain bounds on $f(m,d,H)$ for a variety of forbidden subgraphs $H$, proving Theorem 4. In Section 5, we prove Theorem 6 showing that Conjecture 1.2 implies Conjecture 1.1. In Section 6, we construct cycle-free graphs from random $d$-regular graphs with small Max-Cut, proving Proposition 1.3. We conclude with some remarks and further directions in Section 7.

### 2. Max-Cut in $d$-degenerate graphs

To prove Theorems 2, 3, 4, and 5, we adapt methods historically used to give Max-Cut bounds in general graphs to give meaningful lower bounds on the Max-Cut of $d$-degenerate graphs. In other words, we are converting bounds on $f(m,H)$ to bounds on $f(m,d,H)$ (except in Theorem 2 where we do something slightly better).

To generalize from the setting of degree bounded graphs, we make use of some helpful notation. Give a graph $G$ and a subset of vertices $X$, we let $G[X]$ denote the subgraph...
induced by vertices $X$, and we let $m(X)$ be shorthand for $m(G[X])$, the number of edges in $G[X]$. We also let $t(G)$ denote the number of triangles of $G$.

**Definition 2.1.** Given a graph $G = (V, E)$ with $|V| = n$, an ordering is a bijection $\rho : V \to [n]$. With respect to some ordering $\rho$, let $d_<(i)$ be the number of neighbors $w$ of $v = \rho^{-1}(i) \in V$ such that $\rho(w) < i$.

By definition, every $d$-degenerate graph has an ordering for which $d_<(i) \leq d$ for all $i$.

### 2.1. Max-Cut in Triangle-Deficient Graphs

We first show a lower bound that arises from the SDP relaxation of Max-Cut, formulated below for a graph $G = (V, E)$:

$$
\text{maximize} \quad \sum_{(i,j) \in E} \frac{1}{2}(1 - \langle v(i), v(j) \rangle)
$$

subject to $\|v(i)\|^2 = 1 \forall i \in V$.

The Goemans-Williamson \cite{19} rounding algorithm is a classical rounding algorithm for Max-Cut that gives an integral solution from a vector solution. This rounding was used in \cite{8} to lower bound the Max-Cut of a maximum degree $d$ graph with few triangles, and we extend their approach to $d$-degenerate graphs.

**Lemma 2.2.** Let $\varepsilon \leq \frac{1}{\sqrt{d}}$. Let $G$ be a $d$-degenerate graph with $m$ edges and $t$ triangles. Then

$$
\text{Max-Cut}(G) \geq \frac{m}{2} + \frac{\varepsilon m}{4\pi} - \frac{\varepsilon^2 t}{2}.
$$

**Proof.** Since $G$ is $d$-degenerate, there exists an ordering $1, \ldots, n$ of the vertices such that for all $i \in [n]$, we have $d_<(i) \leq d$. For $i \in [n]$ define $\tilde{v}(i) \in \mathbb{R}^n$ by

$$
\tilde{v}_j^{(i)} = \begin{cases} 
1 & i = j \\
-\varepsilon & j < i \text{ and } (i,j) \in E \\
0 & \text{otherwise}
\end{cases}.
$$

For $i \in [n]$, let $v^{(i)} \overset{\text{def}}{=} \frac{\tilde{v}(i)}{\|\tilde{v}(i)\|} \in \mathbb{R}^n$. By the definition of the $d$-degenerate ordering, we have $1 \leq \|\tilde{v}(i)\| \leq 1 + \varepsilon^2 d \leq 2$ for all $i$. For edges $(i,j)$ with $i < j$, we have

$$
v^{(i)}_i v^{(j)}_j = \frac{1}{\|\tilde{v}(i)\| \|\tilde{v}(j)\|} \leq \frac{-\varepsilon}{4}.
$$

For $k < i$, we observe that $v^{(i)}_k v^{(j)}_k$ is at most $\varepsilon^2$ if vertices $i,j,k$ form a triangle in $G$ and 0 otherwise. For $k \geq i + 1$, we have $v^{(i)}_k v^{(j)}_k = 0$ as $v^{(i)}_i = 0$. Thus, for all edges $(i,j)$ with $i < j$,

$$
\langle v^{(i)}_i, v^{(j)}_j \rangle \leq \frac{-\varepsilon}{4} + \varepsilon^2 t_<(i,j).
$$

where $t_<(i,j)$ denotes the number of indices $k$ with $k < i < j$ such that $i,j,k$ form a triangle.

Vectors $v^{(1)}, \ldots, v^{(n)}$ form a vector solution to the SDP (2.1). We now round this solution using the Goemans-Williamson \cite{19} rounding algorithm. Let $w$ denote a uniformly random unit vector, $A = \{i \in [n] : \langle v^{(i)}, w \rangle \geq 0\}$, and $B = [n] \setminus A$. Note that the angle between
vectors \(v^{(i)}, v^{(j)}\) is equal to \(\cos^{-1}(\langle v^{(i)}, v^{(j)} \rangle)\), so the probability an edge \((i, j)\) is cut is
\[
\Pr[(i, j) \text{ cut}] = \frac{\cos^{-1}(\langle v^{(i)}, v^{(j)} \rangle)}{\pi}
\]
\[
= \frac{1}{2} - \frac{\sin^{-1}(\langle v^{(i)}, v^{(j)} \rangle)}{\pi}
\]
\[
\geq \frac{1}{2} - \frac{\sin^{-1}(\varepsilon^2 t_{<}(i, j) - \frac{\varepsilon}{2})}{\pi}
\]
\[
\geq \frac{1}{2} - \frac{\varepsilon}{2} \cdot \frac{\varepsilon^2 t_{<}(i, j)}{4\pi} - \frac{\varepsilon}{2}
\]
\[
= \frac{1}{2} + \frac{\varepsilon}{4\pi} - \frac{\varepsilon^2}{2} \cdot t_{<}(i, j).
\]

In the last inequality, we used that, for \(a, b \in [0, 1]\), we have \(\sin^{-1}(a - b) \leq \frac{a}{2} - \frac{a^2}{2}\). This is true as \(\sin^{-1}(x) \leq \frac{\pi}{2} x\) when \(x\) is positive and \(\sin^{-1}(x) \leq x\) when \(x\) is negative. Thus, the expected size of the cut given by \(A \sqcup B\) is, by linearity of expectation,
\[
\sum_{(i, j) \in E} \Pr[(i, j) \text{ cut}] \geq \sum_{i < j, (i, j) \in E} \left( \frac{1}{2} + \frac{\varepsilon}{4\pi} - \frac{\varepsilon^2}{2} \cdot t_{<}(i, j) \right)
\]
\[
= \frac{m}{2} + \frac{\varepsilon m}{4\pi} - \frac{\varepsilon^2 t}{2}.
\]
The equality holds because \(\sum_{i < j, (i, j) \in E} t_{<}(i, j)\) counts each triangle of \(G\) exactly once. ■

Lemma 2.2 gives the following immediate corollary.

**Corollary 2.3.** There exists an absolute constant \(c_1 > 0\) such that the following holds. For all \(d \geq 1\) and \(\varepsilon \leq \frac{1}{\sqrt{d}}\), if a \(d\)-degenerate graph \(G = (V, E)\) has \(m\) edges and at most \(\frac{m}{8\varepsilon}\) triangles then
\[
\text{Max-Cut}(G) \geq \left( \frac{1}{2} + c_1 \varepsilon \right) \cdot m.
\]

In Corollary 2.3, taking \(\varepsilon = \frac{1}{\sqrt{d}}\), matches Shearer’s bound in [21] on the Max-Cut of triangle-free graphs up to a constant factor in the lower order term.

2.2. Decomposing degenerate graphs. Graphs that are \(K_r\)-free have fewer than the expected number of triangles of a random graph of similar density. Carlson, Kolla, and Trevisan (Claim 4.3 of [8]) noted that maximum-degree \(d\) graphs with few triangles must have small subsets of neighborhoods with many edges. We give a \(d\)-degenerate generalization of this lemma.

**Lemma 2.4.** Let \(d \geq 1\) and \(\varepsilon > 0\), and let \(G = (V, E)\) be a \(d\)-degenerate graph with at least \(\frac{m(G)}{\varepsilon}\) triangles. Then there exists a subset \(V'\) of at most \(d\) vertices with a common neighbor in \(G\) such that the induced subgraph \(G[V']\) has at least \(\frac{|V'|}{\varepsilon}\) edges.

**Proof.** Since \(G\) is \(d\)-degenerate, we fix an ordering \(1, \ldots, n\) of the vertices such that \(d_{<}(i) \leq d\) for all \(i \in [n]\). Then, if \(t_{<}(i)\) denotes the number of triangles \(\{i, j, k\}\) of \(G\) where \(j, k < i\), we
have
\[\sum_{i} t_{<}(i) = t(G) \geq \frac{m(G)}{\varepsilon} = \sum_{i=1}^{n \varepsilon} d_{<}(i).\]
Hence, there must exist some \(i\) such that \(t_{<}(i) \geq \frac{d_{<}(i)}{\varepsilon}\). Let \(V'\) denote the neighbors of \(i\) with index less than \(i\). By definition, the vertices of \(V'\) have common neighbor \(i\). Additionally, \(G[V']\) has at least \(\frac{d_{<}(i)}{\varepsilon}\) edges and \(d_{<}(i) \leq d\) vertices, proving the lemma.

We can use this bound to describe \(G = (V, E)\) as the union of a collection of subgraphs with helpful properties. The following lemma was proven implicitly in [8] for graphs with maximum degree \(d\), and we generalize it to \(d\)-degenerate graphs.

**Lemma 2.5.** Let \(\varepsilon > 0\). Let \(G = (V, E)\) be a \(d\)-degenerate graph on \(n\) vertices with \(m\) edges. Then there exists a partition \(V_1, \ldots, V_{k+1}\) of the vertex set \(V\) with the following properties.

1. For \(i = 1, \ldots, k\), the vertex subset \(V_i\) has at most \(d\) vertices and has a common neighbor, and the induced subgraph \(G[V_i]\) has at least \(\frac{|V_i|}{\varepsilon}\) edges.
2. The induced subgraph \(G[V_{k+1}]\) has at most \(\frac{m(V_{k+1})}{\varepsilon}\) triangles.

**Proof.** We construct the partition iteratively. Let \(V_0' = V\). For \(i \geq 1\), we partition the vertex subset \(V_{i-1}'\) into \(V_i \sqcup V_i'\) as follows. If \(G[V_{i-1}']\) has at least \(\frac{m(V_{i-1}')}{\varepsilon}\) triangles, then by applying Lemma 2.4 to the induced subgraph \(G[V_{i-1}']\), there exists a vertex subset \(V_i\) with a common neighbor in \(V_{i-1}'\) such that \(|V_i| \leq d\) and the induced subgraph \(G[V_i]\) has at most \(\frac{|V_i|}{\varepsilon}\) edges. In this case, let \(V_i' = V_i' \setminus V_i\). Let \(k\) denote the maximum index such that \(V_k'\) is defined, and let \(V_{k+1} = V_k'\setminus V_k\). By construction, \(V_1, \ldots, V_k\) satisfy the desired conditions. By definition of \(k\), the induced subgraph \(G[V_k]\) has at most \(\frac{m(V_k)}{\varepsilon}\) triangles, so for \(V_{k+1} = V_k'\), we obtain the desired result.

### 2.3. Large Max-Cut from smaller forbidden subgraphs.
In [8], the authors obtain a partition \(V_1, \ldots, V_{k+1}\) of \(V(G)\) similar to that Lemma 2.5, such that the induced subgraphs \(G[V_1], \ldots, G[V_k]\) are all \(K_{r-1}\)-free. They recursively bound the Max-Cut of these smaller \(K_{r-1}\)-free induced subgraphs \(G[V_1], \ldots, G[V_k]\), applying a version of Corollary 2.3 to bound the Max-Cut of \(G[V_{k+1}]\). Finally they combine the cuts randomly to obtain a cut of \(G\).

While we follow a similar approach at the outset, we observe that in the partition, graphs \(G[V_1], \ldots, G[V_k]\) all have at most \(d\) vertices. Thus we can obtain a stronger bound on the Max-Cut of these induced subgraphs by applying known results about the Max-Cut of more general, dense graphs.

Towards our goal of obtaining tighter bounds on \(f(m, d, H)\), we show how to leverage existing bounds on the Max-Cut in general graphs to obtain bounds in the \(d\)-degenerate setting by finding subgraphs of \(G\) that are either small and dense or triangle-deficient, and combining maximal cuts of these subgraphs.

**Lemma 2.6.** There exists an absolute constant \(c_2 > 0\) such that the following holds. Let \(H\) be a graph and \(H'\) be obtained by deleting any vertex of \(H\). Let \(0 < \varepsilon < \frac{1}{\sqrt{d}}\). For any \(H\)-free \(d\)-degenerate graph \(G = (V, E)\), one of the following holds:
We have

\[ \text{Max-Cut}(G) \geq \left( \frac{1}{2} + c_2 \varepsilon \right) m. \tag{2.2} \]

There exist graphs \( G_1, \ldots, G_k \) such that five conditions hold: (i) graphs \( G_i \) are \( H' \)-free for all \( i \), (ii) \( n(G_i) \leq d \) for all \( i \), (iii) \( m(G_i) \geq \frac{n(G_i)}{8 \varepsilon} \) for all \( i \), (iv) \( n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d} \), and (v)

\[ \text{Max-Cut}(G) \geq \frac{m(G)}{2} + \sum_{i=1}^{k} \left( \text{Max-Cut}(G_i) - \frac{m(G_i)}{2} \right). \tag{2.3} \]

**Proof.** Let \( c_1 < 1 \) be the parameter given by Corollary 2.3. Let \( c_2 = \frac{m}{6d} \).

Let \( G = (V, E) \) be a \( d \)-degenerate \( H \)-free graph. Applying Lemma 2.5 with parameter \( 8 \varepsilon \), we can find a partition \( V_1, \ldots, V_{k+1} \) of the vertex set \( V \) with the following properties.

1. For \( i = 1, \ldots, k \), the vertex subset \( V_i \) has at most \( d \) vertices and has a common neighbor, and the induced subgraph \( G[V_i] \) at least \( \frac{V_i}{8 \varepsilon} \) edges.

2. The subgraph \( G[V_{k+1}] \) has at most \( \frac{m(V_{k+1})}{8 \varepsilon} \) triangles.

For \( i = 1, \ldots, k+1 \), let \( G_i = G[V_i] \) and let \( m_i = m(G_i) \). For \( i = 1, \ldots, k \), since \( G \) is \( H \)-free and each \( V_i \) is a subset of some vertex neighborhood in \( G \), the graphs \( G_i \) are \( H' \)-free. For \( i = 1, \ldots, k \), fix a maximal cut of \( G_i \) with associated vertex partition \( V_i = A_i \cup B_i \). By the second property above, the graph \( G_{k+1} \) has at most \( \frac{m_{k+1}}{8 \varepsilon} \) triangles. Applying Corollary 2.3 with parameter \( \varepsilon \), we can find a cut of \( G_{k+1} \) of size at least \( \left( \frac{1}{2} + c_1 \varepsilon \right)m_{k+1} \) with associated vertex partition \( V_{k+1} = A_{k+1} \cup B_{k+1} \).

We now construct a cut of \( G \) by randomly combining the cuts obtained above for each \( G_i \) as in [8]. Independently, for each \( i = 1, \ldots, k+1 \), we add either \( A_i \) or \( B_i \) to vertex set \( A \), each with probability \( \frac{1}{2} \). Setting \( B = V \setminus A \), gives a cut of \( G \). As \( V_1, \ldots, V_{k+1} \) partition \( V \), each of the \( m - (m_1 + \cdots + m_{k+1}) \) edges that is not in one of the induced graphs \( G_1, \ldots, G_{k+1} \) has exactly one endpoint in each of \( A, B \) with probability \( 1/2 \). This allows us to compute the expected size of the cut (a lower bound on \( \text{Max-Cut}(G) \) as there is some instantiation of this random process that achieves this expected size).

\[
\text{Max-Cut}(G) \geq \frac{1}{2} (m - (m_1 + \cdots + m_{k+1})) + \left( \frac{1}{2} + c_1 \varepsilon \right) \cdot m_{k+1} + \sum_{i=1}^{k} \text{Max-Cut}(G_i)
\]

\[
= \frac{m}{2} + c_1 \varepsilon m_{k+1} + \sum_{i=1}^{k} \left( \text{Max-Cut}(G_i) - \frac{m_i}{2} \right). \tag{2.4}
\]

We bound (2.4) based on the distribution of edges in \( G \) in 3 cases:

1. If \( m_{k+1} \geq \frac{m}{6} \). Then, (2.2) holds, as

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + c_1 \varepsilon m_{k+1} \geq \left( \frac{1}{2} + \frac{c_1 \varepsilon}{6} \right) \cdot m.
\]

2. The number of edges between \( V_1 \cup \cdots \cup V_k \) and \( V_{k+1} \) is at least \( \frac{2m}{3} \). Then, the cut given by vertex partition \( V = A' \cup B' \) with \( A' = V_1 \cup \cdots \cup V_k \) and \( B' = V_{k+1} \) has at least \( \frac{2m}{3} \) edges, in which case \( \text{Max-Cut}(G) \geq \frac{2m}{3} > \left( \frac{1}{2} + \frac{c_1 \varepsilon}{6} \right) \cdot m \), so (2.3) holds.
• Else, \( G' = G[V_1 \cup \cdots \cup V_k] \) must have at least \( \frac{m}{6} \) edges. Note that for all \( i \), the graph \( G_i \) is \( H' \) free, has at most \( d \) vertices, and at least \( \frac{m_i}{8d} \) edges by construction. Since \( G \) is \( d \)-degenerate, \( G' \) is as well, so

\[
\frac{m}{6} \leq m(G') \leq d \cdot \sum_{i=1}^{k} n(G_i),
\]

Hence \( n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d} \). Lastly, by (2.4), we have

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \left( \text{Max-Cut}(G_i) - \frac{m_i}{2} \right).
\]

This covers all possible cases, and in each possible case we showed either (2.2) or (2.3) hold.

Remark 2.7. In Corollary 2.3 and Lemma 2.6, we can take \( c_1 = \frac{1}{60} \) and \( c_2 = \frac{1}{360} \), respectively.

Lemma 2.6 allows us to convert bounds on \( f(m, H) \) to bounds on \( f(m, d, H) \).

**Lemma 2.8.** Let \( H \) be a graph and \( H' \) be obtained by deleting any vertex of \( H \). Suppose that there exists constants \( a = a(H') \in \left[ \frac{1}{2}, 1 \right] \) and \( c' = c'(H') > 0 \) such that, for all positive integers \( m' \), we have \( f(m', H') \geq m' \frac{1}{2} + c' \cdot (m')^{1-a} \). Then there exists a constant \( c_3 = c_3(H) > 0 \) such that for all \( m, d \geq 1 \),

\[
f(m, d, H) \geq \left( \frac{1}{2} + c_3 d^{-\frac{2-a}{1+a}} \right) m.
\]

**Proof.** Let \( c_2 \) be the parameter in Lemma 2.6. We may assume without loss of generality that \( c' \leq 1 \). Let \( G \) be a \( d \)-degenerate \( H \)-free graph and \( \varepsilon = c'd^{-\frac{2-a}{1+a}} < d^{-1/2} \). Let \( c_3 \overset{\text{def}}{=} \min(c'c_2, \frac{c'}{48}) \).

Applying Lemma 2.6 with parameter \( \varepsilon \), either (2.2) or (2.3) holds. If (2.2) holds, then, as desired,

\[
\text{Max-Cut}(G) \geq \left( \frac{1}{2} + c_2 \varepsilon \right) m \geq \left( \frac{1}{2} + c_3 d^{-\frac{2-a}{1+a}} \right) m.
\]

Else (2.3) holds. Let \( G_1, \ldots, G_{k+1} \) be the \( H' \)-free induced subgraphs satisfying the properties in Lemma 2.6 so that

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \left( \text{Max-Cut}(G_i) - \frac{m(G_i)}{2} \right)
\]

\[
\geq \frac{m}{2} + \sum_{i=1}^{k} c' \cdot m(G_i)^a.
\]

For all \( i \), we have

\[
c' \cdot m(G_i)^a \overset{(\ast)}{=} \frac{c' \varepsilon}{8(1+a)} \cdot n(G_i)^a \overset{(\ast \ast)}{=} \frac{\varepsilon d}{8(c')^a} \cdot n(G_i) \overset{(\ast \ast \ast)}{=} \frac{\varepsilon d}{8} \cdot n(G_i),
\]
where (*) follows since \( m(G_1) \geq \frac{n(G_i)}{8} \), (**) follows since \( n(G_i)^{a-1} \geq d^{r-1} \) and \( c^{1+a} = (c')^{1+a}d^{a-2} \), and (+) follows since \( c' \leq 1 \). Hence, as \( n(G_1) + \cdots + n(G_k) \geq \frac{m}{6d} \), we have

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + \varepsilon d \sum_{i=1}^{k} \frac{n(G_i)}{8} \geq \frac{m}{2} + \varepsilon m \geq \left( \frac{1}{2} + c_3 d^{\frac{2}{4 + a}} \right) m,
\]
as desired. \( \blacksquare \)

3. **Max-Cut of \( K_r \)-free sparse graphs**

We specialize Lemmas 2.6 and 2.8 to the case that \( H = K_r \) to obtain both a lower bound and conditional lower bound on the Max-Cut of a \( K_r \)-free graph. Let \( \chi(G) \) denote the chromatic number of a graph \( G \), the minimum number of colors needed to properly color the vertices of the graph so that no two adjacent vertices receive the same color.

3.1. **\( K_r \)-free graphs.** We obtain a nontrivial upper bound on the chromatic number of a \( K_r \)-free graph \( G \), giving an lower bound (Lemma 3.4) on the Max-Cut of \( K_r \)-free graphs. This lower bound was implicit in [3], but we provide a proof for completeness. The lower bound on the Max-Cut of general \( K_r \)-free graphs enables us to apply Lemma 2.6 to give a lower bound on the Max-Cut of \( d \)-degenerate \( K_r \)-free graphs per Theorem 2. The following well known lemma gives a lower bound on the Max-Cut using the chromatic number.

**Lemma 3.1** (see e.g. Lemma 2.1 of [3]). Given a graph \( G = (V,E) \) with \( m \) edges and chromatic number \( \chi(G) \leq t \), we have \( \text{Max-Cut}(G) \geq \left( \frac{1}{2} + \frac{1}{2t} \right) m \).

**Proof.** Since \( \chi(G) \leq t \), we can decompose \( V \) into independent subsets \( V = V_1, \ldots, V_t \). Partition the subsets randomly into two parts containing \( \lfloor \frac{t}{2} \rfloor \) and \( \lceil \frac{t}{2} \rceil \) subsets \( V_i \), respectively, to obtain a cut. The probability any edge is cut is \( \frac{\lfloor t/2 \rfloor - \lceil t/2 \rceil}{t} \geq \frac{t+1}{2t} \), so the result follows from linearity of expectation. \( \blacksquare \)

**Lemma 3.2.** Let \( r \geq 3 \) and \( G = (V,E) \) be a \( K_r \)-free graph on \( n \) vertices. Then,

\[
\chi(G) \leq 4n^{(r-2)/(r-1)}.
\]

**Proof.** We proceed by induction on \( n \). For \( n \leq 4r^{-1} \), the statement is trivial as the chromatic number is always at most the number of vertices. Now assume \( G = (V,E) \) has \( n > 4r^{-1} \) vertices and that \( \chi(G) \leq 4n_0^{(r-2)/(r-1)} \) for all \( K_r \)-free graphs on \( n_0 \leq n - 1 \) vertices. The off-diagonal Ramsey number \( R(r,s) \) satisfies \( R(r,s) \leq (r+s-2) \leq s^{-1} \). Hence, \( G \) has an independent set \( I \) of size \( s = \lfloor n^{1/(r-1)} \rfloor \). The induced subgraph \( G[V \setminus I] \) is \( K_r \)-free and has fewer than \( n \) vertices, so its chromatic number is at most \( 4(n-s)^{(r-2)/(r-1)} \). Hence, \( G \) has chromatic number at most

\[
1 + 4(n-s)^{(r-2)/(r-1)} = 1 + 4n^{(r-2)/(r-1)} \left( 1 - \frac{s}{n} \right)^{(r-2)/(r-1)} \\
\leq 1 + 4n^{(r-2)/(r-1)} - 4n^{(r-2)/(r-1)} \cdot \frac{s}{3n} < 4n^{(r-2)/(r-1)}
\]

In (*), we used that \( \frac{r-2}{r-1} \geq \frac{1}{2} \), that \( \frac{s}{n} \leq \frac{1}{4} \), and that \( (1-x)^a \leq 1 - \frac{x}{3} \) for \( a \geq \frac{1}{2} \) and \( x \leq \frac{1}{4} \). In (**), we used that \( s \geq 4 \) and hence \( \frac{3s}{4} < n^{1/(r-1)} \). This completes the induction, completing the proof. \( \blacksquare \)
Remark 3.3. Note that the upper bound on the off-diagonal Ramsey number \( R(r, k^{1/(r-1)}) \) has an extra logarithmic factor which suggests that the upper bound on \( \chi(G) \) of Lemma 3.2 can be improved by a logarithmic factor with a more careful analysis.

**Lemma 3.4.** If \( G \) is a \( K_r \)-free graph with at most \( n \) vertices and \( m \) edges, then

\[
\text{Max-Cut}(G) \geq \left( \frac{1}{2} + \frac{1}{8n^{(r-2)/(r-1)}} \right) m
\]

**Proof.** This follows immediately via Lemma 3.1 and Lemma 3.2. ■

The above bounds allow us to show that \( f(m, d, K_r) \geq \left( \frac{1}{2} + \tilde{\Omega}(d^{-1+\frac{1}{2r-1}}) \right) m \).

**Proof of Theorem 2.** Let \( G \) be a \( d \)-degenerate \( K_r \)-free graph and \( \varepsilon = d^{-1+\frac{1}{2r-1}} \). Let \( c_2 \) be the parameter given by Lemma 2.6. Let \( c = \min(c_2, \frac{1}{388}) \).

Applying Lemma 2.6 with parameter \( \varepsilon \), one of two properties hold. If (2.2) holds, then

\[
\text{Max-Cut}(G) \geq \left( \frac{1}{2} + c_2 \varepsilon \right) m \geq \left( \frac{1}{2} + cd^{-1+\frac{1}{2r-1}} \right) m
\]
as desired. If (2.3) holds, there exist graphs \( G_1, \ldots, G_k \) that are \( K_{r-1} \)-free with at most \( d \) vertices such that \( G_i \) has at least \( \frac{n(G_i)}{8\varepsilon} \) edges, \( n(G_1) + \cdots + n(G_k) \geq \frac{m}{64d} \), and

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \left( \text{Max-Cut}(G_i) - \frac{m(G_i)}{2} \right).
\]

For all \( i \), we have

\[
\text{Max-Cut}(G_i) - \frac{m(G_i)}{2} \geq \frac{m(G_i)}{8n(G_i)^{r-3)/(r-2)} \geq \frac{n(G_i)}{64\varepsilon n(G_i)^{r-3)/(r-2)} \geq \frac{n(G_i)}{64d^{r-3)/(r-2)} = \frac{\varepsilon dn(G_i)}{64}.
\]

In the first inequality, we used Lemma 3.4. In the second inequality, we used that \( m(G_i) \geq \frac{n(G_i)}{8\varepsilon} \). In the third inequality, we used that \( n(G_i) \leq d \). Hence, as \( d(n(G_1) + \cdots + n(G_k)) \geq \frac{m}{6} \), we have

\[
\text{Max-Cut}(G) \geq \frac{m}{2} + \sum_{i=1}^{k} \frac{\varepsilon dn(G_i)}{64} \geq \frac{m}{2} + \frac{\varepsilon m}{388} \geq \left( \frac{1}{2} + cd^{-1+\frac{1}{2r-1}} \right) \cdot m
\]
as desired. ■

3.2. \( K_4 \)-free graphs. Theorem 2 gives a lower bound on the Max-Cut of a \( K_r \)-free degree bounded graph. We can improve this bound in the case that \( r = 4 \) using Lemma 2.8.

**Proof of Theorem 3.** Let \( H = K_4 \), and \( H' = K_3 \). By a result of [2], there exists a constant \( c' > 0 \) such that, for all \( m' \geq 1 \), we have \( f(m', H') \geq \frac{m'}{2} + c'(m')^{4/5} \). By Lemma 2.8 with \( H \) and \( H' \) and \( a = 4/5 \), there exists a constant \( c_3 > 0 \) such that any \( K_4 \)-free \( d \)-degenerate graph \( G \) with \( m \) edges satisfies

\[
\text{Max-Cut} \geq \left( \frac{1}{2} + c_3d^{-2/(4+15/5)} \right) \cdot m = \left( \frac{1}{2} + c_3d^{-2/3} \right) \cdot m
\]
as desired. ■
3.3. \( K_r \)-free graphs from Conjecture \(^{[1.1]} \). Finally, assuming Conjecture \(^{[1.1]} \) we can improve our lower bound on the \textsc{Max-Cut} for \( d \)-degenerate \( K_r \)-free graphs.

**Proof of Theorem \(^{[5]} \)** Fix a graph \( H \), and let \( H' \) be any graph obtained by removing one vertex from \( H \). Assuming Conjecture \(^{[1.1]} \) there exist constants \( c' = c'(H'), \varepsilon' = \varepsilon'(H') > 0 \) such that any \( H' \)-free graph with \( m \) edges satisfies \( \text{Max-Cut}(G) \geq \frac{m}{2} + c'm^{3/4+\varepsilon'} \). By Lemma \(^{[2.8]} \) with \( H \) and \( H' \) and \( a = 3/4 + c' \), there exists constants \( c_3 = c_3(H), \varepsilon = \varepsilon(H) > 0 \) such that any \( H \)-free \( d \)-degenerate graph \( G \) with \( m \) edges satisfies

\[
\text{Max-Cut}(G) \geq \left( \frac{1}{2} + c_3d^{5/7+\varepsilon} \right) m \geq \left( \frac{1}{2} + c_3d^{5/7+\varepsilon} \right) m.
\]

\[\blacksquare\]

4. **\textsc{Max-Cut} in \( H \)-free graphs**

In this section, we apply Lemma \(^{[2.8]} \) to families of forbidden subgraphs \( H \).

**Proof of Theorem \(^{[4]} \)** We repeatedly apply Lemma \(^{[2.8]} \) by combining it with results from \(^{[4]} \). Table \(^{2}\) shows the choices of \( H, H' \), and \( a \) used in the applications of Lemma \(^{[2.8]} \) along with the associated bounds on \( f(m, H) \) from \(^{[4]} \) and the resulting bounds on \( f(m, d, H) \).

| \( H \)   | \( H' \)   | \( f(m, H') - \frac{m}{2} \) | \( a \) | \( \frac{2-a}{1+a} \) | Lower Bound on \( f(m, d, H) \) |
|---------|---------|-----------------|-----|------------|-------------------|
| forest+1 | forest  | \( c'm \)      | 1   | \( \frac{1}{2} \) | \( \frac{1}{2} + cd^{-1/2}m \) |
| forest+2 | forest+1| \( c'm^{4/5} \) | \( \frac{4}{5} \) | \( \frac{2}{3} \) | \( \frac{1}{2} + cd^{-2/3}m \) |
| \( W_r \) \( (r \text{ odd}) \) | \( C_{r-1} \) | \( c'm^{r/(r+1)} \) | \( \frac{r}{r+1} \) | \( \frac{r+2}{2r+1} \) | \( \frac{1}{2} + cd^{-(r+2)/(2r+1)}m \) |
| \( K_{3,s} \) | \( K_{2,s} \) | \( c'm^{5/6} \) | \( \frac{5}{6} \) | \( \frac{7}{11} \) | \( \frac{1}{2} + cd^{-7/11}m \) |
| \( K_{4,s} \) | \( K_{3,s} \) | \( c'm^{4/5} \) | \( \frac{4}{5} \) | \( \frac{2}{3} \) | \( \frac{1}{2} + cd^{-2/3}m \) |

**Table 2.** We apply Lemma \(^{[2.8]} \) to the above given \( H \) using the listed values of \( H' \) and \( a \) to obtain the given lower bound.

Here, forest+1 means that \( H \) is some forbidden subgraph such that removing one vertex from \( H \) gives a forest, and forest+2 means that removing two vertices from \( H \) gives a forest.

\[\blacksquare\]

5. **Proof of Theorem \(^{[6]} \)**

In this section, we prove Theorem \(^{[6]} \). The next lemma shows that large cuts in induced subgraphs can be extended to large cuts in the overall graph.

**Lemma 5.1.** Let \( G \) be a graph and \( U \) be a subset of the vertices. If the induced subgraph \( G[U] \) has a cut of size at least \( \frac{m(U)}{2} + C \) for some \( C > 0 \), then \( \text{Max-Cut}(G) \geq \frac{m}{2} + C \).

**Proof.** Fix a cut of \( G[U] \) into vertex sets \( U_1 \sqcup U_2 = U \) of size at least \( m(U)/2 \). Then, for all \( v \in V \setminus U \), uniformly at random add \( v \) to either \( U_1 \) or \( U_2 \) (cutting any internal edges) to grow \( U_1 \sqcup U_2 \) into a partition of \( V \) that induces a cut of expected size at least

\[
\frac{m - m(U)}{2} + \frac{m(U)}{2} + C = \frac{m}{2} + C.
\]

Thus, there exists a cut of \( G \) with at least this size, as desired.

\[\blacksquare\]
In the next lemma, we show that a graph with few $K_{r+1}$’s and with every vertex participating in many $K_i$’s has a cut with large advantage over a random cut. To do this, we adapt an argument of [2] to show that such a graph has a large subgraph with small chromatic number. Hence, this large subgraph has a cut with a significant advantage over a random cut. This cut can then be extended (using Lemma 5.1) to a cut over the original graph with large advantage.

Lemma 5.2. Let $r$ be an integer at least 2. For any $\delta \in (0,1)$, Then, for all graphs $G = (V,E)$ on $n$ vertices and $m$ edges with $n$ sufficiently large, if $G$ contains at most $n^{r+1-\delta}$ copies of $K_{r+1}$ and each $v \in V$ is part of at least $n^{r-1-\left(\delta/3r\right)}$ many copies of $K_r$, then

$$\text{Max-Cut}(G) \geq \frac{m}{2} + m^{1-\delta/3}.$$ 

Proof. Let $G = (V,E)$ be as above and let $\varepsilon = \delta/3r$. Since each $v \in V$ is part of at least $n^{r-1-\varepsilon}$ many copies of $K_r$, the graph $G$ has at least $\frac{1}{r}n^{r-\varepsilon}$ copies of $K_r$. Since each edge is in at most $n^{r-2}$ many copies of $K_r$, we have

$$m \geq \left(\frac{r}{2}\right) \cdot \frac{1}{r} \cdot \frac{n^{r-\varepsilon}}{n^{r-2}} > \frac{n^{2-\varepsilon}}{2}.$$ 

Let $t = 64n^{\varepsilon}$, so $m > n^2/t$ and choose a set $T$ of exactly $t$ distinct vertices of $V$ uniformly at random. Let $X \subset V$ be the set of vertices that, along with some collection of $r - 1$ elements of $T$, form a copy of $K_r$ in $G$.

We next show that we expect most vertices to lie in $X$. Fix some vertex $v \in V$. Let $A_1, \ldots, A_\ell$ denote the subsets of $r - 1$ vertices that form a $K_r$ with $v$, where $\ell \geq n^{r-1-\varepsilon}$. For $i = 1, \ldots, \ell$, let $Z_i$ be the indicator random variable $1\{A_i \subseteq T\}$. Let random variable $Z := Z_1 + \cdots + Z_\ell$. Note that

$$\mathbb{P}(Z_i = 1) = \mathbb{P}(A_i \subset T) = \left(\begin{array}{c} n-(r-1) \\ t-(r-1) \end{array}\right) = \frac{t^{r-1}}{2n^{r-1}},$$

where the inequality holds if $n$ is sufficiently large. Thus,

$$\mathbb{E}[Z] = \sum_{i=1}^{\ell} \mathbb{E}[Z_i] \geq n^{r-1-\varepsilon} \cdot \frac{t^{r-1}}{2n^{r-1}} = \frac{t^{r-1}}{2n^{\varepsilon}}.$$ 

If $A_i$ and $A_j$ are disjoint, $Z_i$ and $Z_j$ are negatively correlated, so $\mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \leq 0$. If $|A_i \cup A_j| = s$ for $r \leq s \leq 2r - 3$, then we have $\mathbb{E}[Z_i Z_j] = \left(\begin{array}{c} t-s \\ t \end{array}\right) \leq \frac{t^s}{n^s}$. Furthermore, for $r \leq s \leq 2r - 3$, there are at most $n^{r-1-\varepsilon} \cdot n^{s-(r-1)} = n^{s-\varepsilon}$ pairs $(A_i, A_j)$ such that $|A_i \cap A_j| = s$. Thus,

$$\text{Var}[Z] = \sum_{i,j} \mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i] \mathbb{E}[Z_j] \leq \sum_{s=r}^{2r-3} \sum_{i,j | |A_i \cap A_j| = s} \mathbb{E}[Z_i Z_j] \leq \sum_{s=r}^{2r-3} \frac{t^s}{n^s} \cdot \#\{i,j : |A_i \cup A_j| = s\} \leq \sum_{s=r}^{2r-3} t^s n^{-\varepsilon} < 2t^{2r-3} n^{-\varepsilon}.$$
For all random variables, we have $\Pr[Z = 0] \leq \frac{\text{Var}[Z]}{E[Z]^2}$ (see, e.g. Theorem 4.3.1 of \cite{6}). Hence,

$$\Pr[v \notin X] = \Pr[Z = 0] \leq \frac{\text{Var}[Z]}{E[Z]^2} < \frac{2t^{2r-3}/n^\varepsilon}{(t^{r-1}/2n^\varepsilon)^2} = \frac{1}{8}.$$  

Thus, the probability an edge has at least one vertex not in $X$ is less than $\frac{1}{4}$, so the expected number of edges not in $X$ is less than $\frac{m}{4}$. Thus, by Markov’s inequality, with probability less than $\frac{1}{2}$, at most $\frac{m}{2}$ edges are in $X$.

Call an $(r + 1)$-clique of $G$ bad if exactly $r − 1$ of the vertices are in $T$. Each $(r + 1)$-clique is bad with probability at most $(\frac{r+1}{r-1})\left(\frac{n-(r-1)}{n}\right) < \frac{r^2r-1}{n^{r-1}}$. As there are at most $n^{r+1-\delta}$ many $(r + 1)$-cliques, the expected number of bad cliques is at most $r^2t^{r-1}n^{2-\delta}$. By Markov’s inequality, with probability at least $1/2$, there are at most $2r^2t^{r-1}n^{2-\delta}$ bad cliques. This means that there exists some subset $T$ of $t$ vertices such that (1) the corresponding $X$ has $m(X) \geq \frac{m}{2}$ edges and (2) there are at most $2r^2t^{r-1}n^{2-\delta}$ bad cliques.

Fix this $T$, and let $G'$ be the graph on vertex set $X$ obtained by removing the edges from every $(r + 1)$-clique in the induced subgraph $G[X]$. The total number of edges in bad cliques is at most

$$\left(\frac{r+1}{2}\right) \cdot 2r^2t^{r-1}n^{2-\delta} \leq \frac{n^2}{r^{r+1}} \leq \frac{m}{2t^r} \leq t^{-r}m(X).$$

In (*), we used that $2r^4t^{2r} = c_r n^{2r} < n^\delta$ for $n$ sufficiently large. In (**), we used that $m > n^2/t$. Hence, $G'$ has at least $m(X) \cdot (1 - t^{-r})$ edges. Additionally, $\chi(G') \leq \left(\frac{t}{r-1}\right)$, seen by coloring each vertex $v \in X$ with an unordered $(r - 1)$-tuple corresponding to a subset of $(r - 1)$ vertices in $T$ that form a $K_r$ with $v$. By definition of $X$, such an $(r - 1)$-tuple exists. Since $G'$ has no edge forming a $K_{r+1}$ with $r - 1$ elements of $T$, the above coloring is a proper coloring of $X$. Hence, by Lemma 3.1,

$$\text{Max-Cut}(G') \geq \left(\frac{1}{2} + \frac{1}{2(r-1)}\right) \cdot m(X) \cdot (1 - t^{-r})$$

$$> \left(\frac{1}{2} + \frac{1}{4(r-1)}\right) m(X) \geq \frac{m(X)}{2} + m^{1-\delta/3},$$

where (*) follows since $\frac{m(X)}{4(r-1)} \geq \frac{m}{8t^{r-1}} > \frac{m}{n^{r\varepsilon}} > m^{1-r\varepsilon} = m^{1-\delta/3}$. Hence, the induced subgraph $G[X]$ has a cut of at least the same value. By Lemma 5.1, $G$ has a cut of size $\frac{m}{2} + m^{1-\delta/3}$.  

In the next lemma, we show that a graph with few $K_{r+1}$’s and many edges has a cut with large advantage over a random cut. To do this, we induct on $r$. We show there are two nontrivial cases: either (1) there is a subgraph with many edges and few $K_r$’s, in which case we apply the induction hypothesis or (2) there is some subgraph with many edges and every vertex is in many $K_r$’s, in which case we apply Lemma 5.2.

**Lemma 5.3.** Let $r \geq 1$. Let $\delta \in (0, 1)$. For $n$ sufficiently large, every graph $G$ on $n$ vertices with more than $n^{2-\delta/(2r^r)}$ edges and at most $n^{r+1-\delta}$ many $K_{r+1}$’s, has $\text{Max-Cut}(G) \geq \frac{m}{2} + m^{1-\delta}$.

**Proof.** We prove by induction on $r$. For $r = 1$, the statement is vacuous: no graph $G$ has more than $n^{2-\delta/2}$ edges while also having at most $n^{2-\delta}$ many $K_2$’s.
Assume the assertion is true for \( r - 1 \). For simplicity, let \( \varepsilon = \frac{\delta}{2r!} \). Let \( \delta' = \frac{11\delta}{20r} \) and \( \varepsilon' = \frac{\delta'}{2^{r-1}(r-1)!} \), so that \( \varepsilon' > \varepsilon \).

Suppose \( G \) has at most \( n^{r+1-\delta} \) many \( K_{r+1} \)'s and \( m \geq n^{2-\varepsilon} \) edges. Suppose we find a vertex of \( G \) contained in less than \( n^{r-1-(\delta/6r)} \) many \( K_r \)'s, delete it, and repeat on the resulting graph until no such vertex exists. Let \( W \) be the set of vertices that remain after this procedure, and let \( U \) be the set of vertices that are deleted. We have three cases.

Case 1 (Easy). If there are at least \( \frac{2m}{3} \) edges between \( U \) and \( W \), then \( (U, W) \) forms a cut of \( G \) with at least \( \frac{2m}{3} > \frac{m}{2} + m^{1-\delta} \) edges.

Case 2 (Few \( K_r \)'s). If there are at least \( \frac{m}{6} \) edges in the induced subgraph \( G[U] \), then the following two statements are true about \( G[U] \):

- The induced subgraph \( G[U] \) has at most \( |U|^{r-\delta'} \) many \( K_r \)'s.

  Since \( G[U] \) has at least \( m/6 \) edges,

  \[
  |U| \geq \sqrt{\frac{m}{3}} > \frac{n^{1-\varepsilon/2}}{2}.
  \]

  When each vertex in \( U \) was deleted, it was in at most \( n^{r-1-(\delta/6r)} \) many \( K_r \)'s. Thus, the total number of \( K_r \)'s of \( G \) that touch the vertex subset \( U \) is at most

  \[
  |U|n^{r-1-(\delta/6r)} < n^{r-(\delta/6r)}.
  \]

Hence, \( G[U] \) has at most \( n^{r-(\delta/6r)} \leq |U|^{r-\delta'} \) many \( K_r \)'s; the inequality follows since \( n \) is sufficiently large and \( r - \delta' < (r - \frac{\delta}{6r})(1 - \frac{\varepsilon}{2}) \).

- The induced subgraph \( G[U] \) has at least \( |U|^{2-\varepsilon'} \) edges. This follows since

  \[
  e(G[U]) = \frac{n^{2-\varepsilon}}{6} \geq \frac{|U|^{2-\varepsilon}}{6} \geq |U|^{2-\varepsilon'},
  \]

  which holds since \( \varepsilon' > \varepsilon \) and \( n \) is sufficiently large.

By the above two properties, the \( G[U] \) satisfies the setup of the inductive hypothesis, with parameters \( r - 1 \) and \( \delta' \). Hence, by the inductive hypothesis, we have that for sufficiently large \( n \)

\[
\text{Max-Cut}(G[U]) \geq \frac{m(U)}{2} + (m/6)^{1-\delta'} > \frac{m(U)}{2} + m^{1-\delta} \geq \frac{m}{2} + m^{1-\delta},
\]

since \( m(U) > m/6 \) and \( \delta' < \delta \), applying Lemma 5.1.

Case 3 (Many \( K_r \)'s). If there are at least \( \frac{m}{6} \) edges in the induced subgraph \( G[W] \), the following two statements are true about the induced subgraph \( G[W] \):

- Each vertex is in at least \( |W|^{r-1-(\delta/6r)} \) many \( K_r \)'s.

  By construction, each vertex is in at least \( n^{r-1-(\delta/6r)} \) many \( K_r \)'s, or else we would have deleted it in the above procedure. Furthermore \( n \geq |W| \), so each vertex is in at least \( |W|^{r-1-(\delta/6r)} \) many \( K_r \)'s.

- It has at most \( |W|^{r+1-\delta/2} \) many \( K_r \)'s.

Since \( G[W] \) has at least \( \frac{m}{6} \) edges, \( W \) has at least \( \sqrt{\frac{m}{3}} > \frac{n^{1-\varepsilon/2}}{2} \) vertices. In \( G[W] \), there are at most \( n^{r+1-\delta} \leq |W|^{r+1-\delta/2} \) many \( K_{r+1} \)'s, which holds since

\[
r + 1 - \delta < \left(1 - \frac{\varepsilon}{2}\right) \left(r + 1 - \frac{\delta}{2}\right).
\]
By the above two properties, \( G[W] \) satisfies the setup of the Lemma \ref{lem:technical} with parameters \( r \) and \( \delta \). Hence, by Lemma \ref{lem:technical}, we have for sufficiently large \( m \)
\[
\text{Max-Cut}(G[W]) \geq \frac{m(W)}{2} + (m/6)^{1-\delta/2} > \frac{m(W)}{2} + m^{1-\delta}.
\]
Hence, by Lemma \ref{lem:k}, we have
\[
\text{Max-Cut}(G) \geq \frac{m}{2} + m^{1-\delta}.
\]
This covers all the cases, and in each case, we have \( \text{Max-Cut}(G) \geq \frac{m}{2} + m^{1-\delta} \), as desired.

The above tools will enable us to show Theorem \ref{thm:main}

**Proof of Theorem \ref{thm:main}** Fix \( r \geq 2 \). Assume Conjecture \ref{conj:main} is true. It suffices to lower bound the \text{Max-Cut} of \( K_{r+1} \)-free graphs, since all graphs are subgraphs of a clique. Let \( \delta = \frac{1}{5} \) and \( \varepsilon = \frac{\delta}{2^{r+1}} \). Suppose \( G \) is a graph with \( m \) edges and \( n \) vertices. We show that \( G \) has a cut of size \( \Omega(m^{3/4} + \varepsilon) \) in two cases: Let \( d = \frac{m}{2} - \varepsilon^2 \) and assume \( m \) and \( n \) are sufficiently large.

**Case 1** (Sparse: \( G \) has no induced subgraph of minimum degree \( d \)). This implies that \( G \) is \( d \)-degenerate, in which case Conjecture \ref{conj:main} implies that for some \( c > 0 \)
\[
\text{Max-Cut}(G) \geq \frac{m}{2} + cm^{3/4 + \varepsilon/8}.
\]
**Case 2** (Dense: there exists an induced subgraph \( G[U] \) of minimum degree \( d \)). Then
\[
m(U) \geq |U|d = \frac{|U| \cdot m^{1/2-\varepsilon/4}}{2} \geq \frac{|U| \cdot m(U)^{1/2-\varepsilon/4}}{2}.
\]
Rearranging, and using that \( |U| \) and \( m(U) \) are sufficiently large, gives that
\[
m(U) > |U|^{2-\varepsilon}.
\]
Since \( G[U] \) has \( 0 < n^{r+1-\delta} \) many \( K_{r+1} \)'s, we may apply Lemma \ref{lem:technical} to obtain that \( G[U] \) has a cut of size
\[
\text{Max-Cut}(G[U]) \geq \frac{m(U)}{2} + m(U)^{1-\delta}
\]
We know that \( m(U) \geq \frac{d^2}{2} \geq \frac{m^{1-\varepsilon/2}}{2} \), so
\[
\text{Max-Cut}(G[U]) > \frac{m(U)}{2} + \frac{m^{1-\varepsilon/2)(1-\delta)}}{2^{1-\delta}} > \frac{m(U)}{2} + m^{3/4 + \varepsilon/8}.
\]
In the last inequality, we used that \( \delta = \frac{1}{5} \) and \( \varepsilon \leq \frac{\delta}{2^{r+1}} = \frac{1}{40} \). By Lemma \ref{lem:k} we have
\[
\text{Max-Cut}(G) \geq \frac{m}{2} + m^{3/4 + \varepsilon/8}.
\]
This completes the proof.

**Remark 5.4.** The above argument proves that, assuming Conjecture \ref{conj:main}, a \( K_r \)-free graph with \( m \) edges has \text{Max-Cut} value at least \( \frac{m}{2} + c_r m^{3/4 + \varepsilon_r} \) for \( \varepsilon_r = 2^{-\Theta(r \log r)} \). For clarity, we did not optimize the value of \( \varepsilon_r \).
6. Upper bounds on sparse Max-Cut

Let $G_{n,d}$ denote a random $d$-regular graph on $n$ vertices. In this section, we prove Proposition 1.3. That is, for all $r \geq 3$, there exist $C_r$-free regular graphs with Max-Cut that matches the bound in Conjecture 1.2 up to a constant factor in the lower order term. In particular, we show that a random regular graph $G_{n,d}$ (with a few alterations to make it $C_r$-free) gives the desired bound. The following result of Bollobás [7] implies that a random regular graph has few $r$-cycles with high probability.

**Proposition 6.1** (Theorem 2 of [7]). For $r \geq 3$ and $d \geq 1$ fixed, as $n \to \infty$, the distribution of the number of copies of $C_r$ in a random $d$-regular graph $G_{n,d}$ converges to $\text{Poi}(\lambda)$ for $\lambda = (d-1)^r/2r$.

The following result (e.g. in [10]) shows that with high probability, random regular graphs have Max-Cut within a constant factor of the bound in Conjecture 1.2.

**Proposition 6.2** ([10]). There exists an absolute constant $c > 0$ such that, for any $d \geq 1$, there exists $n_d$ such that for all $n \geq n_d$, with probability at least 0.99, the random regular graph $G_{n,d}$ has Max-Cut at most $(1 + \frac{c}{\sqrt{d}})m$, where $m = \frac{dn}{2}$.

Combining the above two results gives Proposition 1.3.

**Proof of Proposition 1.3**. Let $\lambda = \frac{(d-1)^r}{2r}$. By Proposition 6.1, there exists an $n'_0$ such that, for all $n \geq n'_0$, the probability that a random regular graph $G_{n,d}$ has at least $d^r > 6\lambda$ copies of $C_r$ is at most $e^{-6} < 0.01$. Let $c > 0$ and $n_d$ be given by Proposition 6.2 and let $n_0 = \max(n'_0, n_d, 2d')$. For all $n \geq n_0$, with probability at least 0.98, a random regular graph $G_{n,d}$ has at most $d^r$ copies of $C_r$ and Max-Cut at most $(1 + \frac{\sqrt{d}}{c}) \cdot \frac{dn}{2}$. Let $G'$ be such a graph, and let $G$ be the graph obtained by removing (at least) one edge from each $C_r$, so that at most $d^r$ edges are removed, and $G$ has $m(G) \geq \frac{dn}{2} - d^r \geq \frac{dn}{2} (1 - \frac{1}{2^r})$ edges. Then, the Max-Cut of $G$ is at most

$$\text{Max-Cut}(G) \leq \left(\frac{1}{2} + \frac{c}{\sqrt{d}}\right) \cdot \frac{dn}{2} \leq \left(\frac{1}{2} + \frac{c}{\sqrt{d}}\right) \frac{m(G)}{1 - (1/2d)} < \left(\frac{1}{2} + \frac{c'}{\sqrt{d}}\right) m(G)$$

for some $c' > 0$. \hfill \blacksquare

7. Concluding Remarks

We conclude by briefly discussing some practical applications and generalizations of our work.

7.1. Efficient computation of Max-Cut. It is always possible to find a cut close to optimal value in polynomial time by (approximately) solving the SDP (2.1) (see, e.g. [18, 23]) and rounding it. Precisely, if the optimal cut has size $(\frac{1}{2} + W)m$, we observe below that we can efficiently find a cut of size $(\frac{1}{2} + \frac{cW}{\log(d+1)})m$ for some absolute $c > 0$. In this subsection, all asymptotics are as $W \to 0$.

Feige and Langberg [16] present an alternative to the Goemans-Williamson rounding method to round the SDP (2.1). This was analyzed in a relevant parameter regime by Charikar and Wirth [9].

**Proposition 7.1** (Lemmas 5 and 8 of [9]). Given a vector solution to the SDP (2.1) of a graph on $m$ vertices with optimal value $(\frac{1}{2} + W)m$, there is an efficient, randomized algorithm to find a cut of $S$ in $G$ that cuts at least $(\frac{1}{2} + \Omega(\frac{W}{\log(1/W)}))m$ edges.
Remark 7.2. By a work of O’Donnell and Wu [20], this is tight in the dependence on $W$ (there exist Max-Cut instances with matching SDP integrality gaps), and the optimal constant in the $\Omega(\cdot)$ is $\frac{1}{2} + o(1)$.

Since we can always take $W \geq \Omega(1/d)$ when $G$ is $d$-degenerate, and since the SDP can be solved within error, say, $W^2$, in time polynomial in the instance [18,23], we have the following corollary.

**Corollary 7.3.** Let $m,d \geq 1$. For any $d$-degenerate graph with $m$ edges with $\text{Max-Cut}(G) \geq \left(\frac{1}{2} + W\right)m$, there is an efficient, randomized algorithm to find a cut of $G$ that cuts at least $\left(\frac{1}{2} + \Omega\left(\frac{W}{\log (d+1)}\right)\right)m$ edges.

**Proof.** First solve the SDP within error $\frac{W^2}{2}$. This gives a vector solution of value at least $\left(\frac{1}{2} + \frac{W}{2}\right)m$. By Lemma 7.1, we can efficiently find a cut of $S$ cutting at least $\left(\frac{1}{2} + \Omega\left(\frac{W}{\log (2/W)}\right)\right)m$, which, as $W \geq \frac{c}{d}$ for an absolute constant $c > 0$, is at least $\left(\frac{1}{2} + \Omega\left(\frac{W}{\log (d+1)}\right)\right)m$. ■

7.2. **Max-$t$-Cuts.** The above article is written in the context of computing the Max-Cut of a graph. A related problem of interest is computing the Max-$t$-Cut of a graph, i.e. the largest $t$-colorable ($t$-partite) subgraph of a given graph. The methods we leverage above can be readily adapted to give lower bounds on the Max-$t$-Cut of a $d$-degenerate $H$-free graph. Notably, by randomly combining the cuts of induced subgraphs $G_i$ from Lemma 2.6 into $t$ groups rather than 2. We obtain that for $d$-degenerate $H$-free graphs $G$ with $m$ edges where $H$ satisfies the conditions of Lemma 2.8, that for some $c > 0$

$$\text{Max-$t$-Cut}(G) \geq \left(\frac{t}{t} - 1 + c \cdot \alpha_H(d)\right) \cdot m.$$  

where $\alpha_H(d)$ is as in Theorem 4.

In the case that $H = K_r$, by considering the bound on $\chi(G)$ for $K_r$-free graphs proved in Lemma 3.2, we observe that we can obtain a bound on the Max-$t$-Cut of the graph, finding that for a $K_r$-free graph $G$ with $n$ vertices and $m$ edges that

$$\text{Max-$t$-Cut}(G) \geq \left(1 - \frac{1}{t}\right) \left(1 + d^{-1+1/(2r-4)}\right) m.$$  

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