A New Two Derivative FSAL Runge-Kutta Method of Order Five in Four Stages

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Abstract: A new efficient Two Derivative Runge-Kutta method (TDRK) of order five is developed for the numerical solution of the special first order ordinary differential equations (ODEs). The new method is derived using the property of First Same As Last (FSAL). We analyzed the stability of our method. The numerical results are presented to illustrate the efficiency of the new method in comparison with some well-known RK methods.

Key words: Explicit methods, FSAL property, Initial value problems, Two Derivative Runge-Kutta method.

Introduction: In the last years, several methods have been proposed to solve numerically first order ODEs of the form
\[ y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 = a \leq x \leq b. \tag{1} \]

where \( y \in \mathbb{R}^d \) and \( f: \mathbb{R}^d \to \mathbb{R}^d \).

Butcher (1) derived the order conditions of Runge-Kutta method using the theory of trees. Franco (2) constructed the embedded EFRKN4(3) methods relying on FSAL technique. Van de Vyver (3) constructed embedded phase-fitted modified Runge-Kutta method of order five and four based on the FSAL technique for solving the radial Schrödinger equation. Fang et al. (4) developed a new fifth-order Runge-Kutta method and embedded RK5(4) pair based on FSAL technique adapted for solving oscillatory problems. Chan and Tsai (5) constructed an explicit Two Derivative Runge-Kutta (TDRK) methods of order up to seven that include one function evaluation of \( f \) and a minimal number of function evaluations of \( g \). They derived the order conditions of TDRK method based on Butcher’s theory of trees in (1). Fang et al. (6) proposed fourth-order extended Runge-Kutta Nystrom (EKRN) methods and then they derived embedded EKRN4(3) pairs based on FSAL property to solve second-order ODEs with perturbed oscillators solutions. Very recently, Ahmad and Senu (7) have proposed a new explicit TDRK method of order four based on the FSAL property for solving first order ODEs.

Here in this paper, motivated by Chan and Tsai (5) and Ahmad and Senu (7), we developed a new four-stage fifth-order TDRK method designed utilizing the FSAL technique. The preliminaries of TDRK methods are presented in Section 2. In Section 3, a new TDRK method with FSAL property is derived. In Section 4, we analyzed the stability of the proposed method. In Section 5, numerical tests are given to demonstrate the efficiency of our TDRK method when it is compared with other Runge-Kutta methods in the scientific literature. Finally, in Section 6, we give some conclusion.

Preliminaries: In this work, we are interested in the efficient numerical method for solving first-order ordinary differential equations (ODEs) (1). We consider the special explicit TDRK methods studied in (5)...
\[
\begin{align*}
Y_i &= y_n + c_i h f(x_n, y_n) + h^2 \sum_{j=1}^{i-1} a_{ij} g(x_n + c_j h, Y_j), \quad i = 2, \ldots, s \\
y_{n+1} &= y_n + h f(x_n, y_n) + h^2 \sum_{i=1}^{s} b_i g(x_n + c_i h, Y_i),
\end{align*}
\]  

where \( y^{\prime\prime}(x) = g(x, y) = f_x(x, y) + f_y(x, y) f(x, y) \). The special explicit TDRK method (2) includes only one function evaluation of \( f \) and \( s \) function evaluations of \( g \) per step. The coefficients of the special explicit TDRK method (2) can be expressed in Butcher tableau as follows:

| \( i \) | \( \sum b_i = \frac{1}{2^i} \) |
|---|---|
| 0 | 0 |
| 1 | \( a_{21} \) |
| 2 | \( a_{21} \) |
| 3 | \( a_{31} \) |
| 4 | \( a_{32} \) |
| \( s \) | \( c_s \) |

According to Chan and Tsai (5), the order conditions for new TDRK method up to five are presented as follows:

order 2:

\[
\sum_{i=1}^{s} b_i = \frac{1}{2^i},
\]

order 3:

\[
\sum_{i=2}^{s} b_i c_i = \frac{1}{6},
\]

order 4:

\[
\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{12^i},
\]

order 5:

\[
\sum_{i=2}^{s} b_i c_i^3 = \frac{1}{20}, \quad \sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i a_{ij} c_j = \frac{1}{120},
\]

In practice, the following Nystrom row assumption is helpful in (8)

\[
\sum_{i=1}^{s} a_{ij} = \frac{1}{2^i c_i^2}, \quad i = 2, \ldots, s.
\]

A Fifth Order TDRK with FSAL Property:

A new TDRK method with “First Same As Last” (FSAL) property will be derived where

\[
b_i = a_{si}, \quad i = 1, \ldots, s
\]

- \( 1 \), and \( b_s = 0 \).

The feature of “First Same As Last” (FSAL) technique is that the fourth stage can be reused as the first stage of the next step. Therefore, the efficient number of function evaluations is three per step. According to the Nystrom row assumption (7), we have

\[
a_{21} = \frac{e^2}{2},
\]

\[
a_{31} = \frac{e^3}{2} - a_{32},
\]

In order to construct four-stage fifth-order TDRK method by using FSAL technique, solving the order conditions (3)–(6) simultaneously results in a solution with one free parameter \( c_3 \) as follows:

\[
b_1 = \frac{1}{42} 10 c_3^2 - 8 c_3 + 1
\]

\[
b_2 = \frac{-200 c_3^2 + 300 c_3^2 - 150 c_3 + 25}{600 c_3^2 - 960 c_3 + 540 c_3 - 108}
\]

\[
b_3 = \frac{1}{120 c_3^2 - 120 c_3 + 36 c_3}
\]

\[
a_{32} = \frac{-20 c_3^2 + 10 c_3^2 - 16 c_3^2 + 3 c_3}{-6 + 10 c_3}
\]

\[
c_2 = \frac{3 - 5 c_3}{10 c_3 - 5}
\]

Choosing \( c_3 = \frac{4}{5} \) yields a fifth order TDRK method with FSAL property denoted as TDRK5F, which is given in the following Butcher tableau;

| \( i \) | \( b_i \) |
|---|---|
| 0 | 0 |
| 1/5 | \( \frac{1}{18} \) |
| 4/5 | \( \frac{-2}{125} \) |
| 1 | \( \frac{5}{48} \) |

| \( j \) | \( a_{ij} \) |
|---|---|
| 0 | 0 |
| 1/5 | \( \frac{1}{28} \) |
| 4/5 | \( \frac{25}{336} \) |
| 1 | \( \frac{5}{48} \) |

Stability of TDRK5F method:

In this section, we discuss the stability of TDRK5F method. We consider the following test equation:

\[
y^{\prime} = i \lambda y, \quad \lambda > 0.
\]

By applying TDRK method (2) to (9) produces the following difference equation

\[
y_{n+1} = M(v) y_n, \quad v = \lambda h,
\]

where

| \( v \) | \( M(v) \) |
|---|---|
| 0 | 0 |
| 1/5 | \( \frac{1}{28} \) |
| 4/5 | \( \frac{25}{336} \) |
| 1 | \( \frac{5}{48} \) |
The problems are integrated in the interval [0,10]. The accuracy criteria calculated by taking \( \log_{10} \) of the maximum absolute error as follows:

The accuracy = \( \log_{10}(\max |y(x_n) - y_n|) \).

The numerical results and the efficiency curves of the methods are presented in Tables 2, 3, 4, 5, 6 and Figures 2, 3, 4, 5, 6 respectively.

**Problem 1:** (7)

\( y' = -2xy, \quad y(0) = 1. \)

The analytic solution is: \( y(x) = e^{-x^2}. \)

**Problem 2:** (11)

\[
\begin{align*}
    y_1' &= y_2, \\
    y_2' &= -13y_1 + 12y_3 + 9\cos(2x) \\
    &\quad - 12\sin(2x), \\
    y_3' &= y_4, \\
    y_4' &= 12y_1 - 13y_3 - 12\cos(2x) + 9\sin(2x),
\end{align*}
\]

\( y_1(0) = 1, \quad y_2(0) = -4, \quad y_3(0) = 0, \quad y_4(0) = 8. \)

The analytic solution is:

\[
\begin{align*}
    y_1(x) &= \sin(x) - \sin(5x) + \cos(2x), \\
    y_2(x) &= \cos(x) - 5\cos(5x) - 2\sin(2x), \\
    y_3(x) &= \sin(x) - \sin(5x) + \sin(2x), \\
    y_4(x) &= \cos(x) - 5\cos(5x) + 2\cos(2x).
\end{align*}
\]

**Problem 3:** (Periodic Orbit problem (12))

\[
\begin{align*}
    y_1' &= y_2, \\
    y_2' &= -y_1 + 0.001\cos(x), \\
    y_3' &= y_4, \\
    y_4' &= y_3 + 0.001\sin(x),
\end{align*}
\]

\( y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0.9995. \)

The analytic solution is:

\[
\begin{align*}
    y_1(x) &= \cos(x) + 0.0005\sin(x), \\
    y_2(x) &= -0.9995\sin(x) + 0.0005\cos(x), \\
    y_3(x) &= \sin(x) - 0.0005\cos(x), \\
    y_4(x) &= 0.9995\cos(x) + 0.0005\sin(x).
\end{align*}
\]

**Problem 4:** (Kepler’s problem (13))

\[
\begin{align*}
    y_1' &= y_2, \\
    y_2' &= -\frac{y_1}{\sqrt{y_1^2 + y_2^2}}, \\
    y_3' &= y_4, \\
    y_4' &= -\frac{y_3}{\sqrt{y_1^2 + y_2^2}},
\end{align*}
\]

\( y_1(0) = 1 - e, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = \frac{1 + e}{\sqrt{1 - e^2}}. \)

The analytic solution is:

\[
\begin{align*}
    y_1(x) &= \cos(x) - e, \quad y_2(x) = -\sin(x), \\
    y_3(x) &= \sqrt{1 - e^2}\sin(x), \\
    y_4(x) &= \sqrt{1 - e^2}\cos(x).
\end{align*}
\]
where \( e \) (0 ≤ \( e \) < 1) is the eccentricity of the orbit, we choose \( e = 0 \).

**Problem 5:** (14)

\[
\begin{align*}
y_1' &= y_2, \\
y_2' &= -\frac{101}{2} y_1 + \frac{99}{2} y_3 + \frac{93}{2} \cos (2x) \\
&\quad - \frac{99}{2} \sin (2x), \\
y_3' &= y_4, \\
y_4' &= \frac{99}{2} y_1 - \frac{101}{2} y_3 + \frac{93}{2} \sin (2x) \\
&\quad - \frac{2}{2} \cos (2x), \\
y_1(0) &= 0, \quad y_2(0) = -10, \quad y_3(0) = 1, \\
y_4(0) &= 12.
\end{align*}
\]

The analytic solution is:

\[
\begin{align*}
y_1(x) &= -\cos (10x) - \sin (10x) + \cos (2x), \\
y_2(x) &= 10 \sin (10x) - 10 \cos (10x) \\
&\quad - 2 \sin (2x), \\
y_3(x) &= \cos (10x) + \sin (10x) + \sin (2x), \\
y_4(x) &= -10 \sin (10x) + 10 \cos (10x) \\
&\quad + 2 \cos (2x).
\end{align*}
\]

**Table 2. The Numerical Results for Problems 1**

| \( h \) | Method | MAXERR | FC |
|---|---|---|---|
| 0.1 | TDRK5F | 8.260301764817513e-08 | 50 |
|      | RK5W | 8.070167282561713e-07 | 50 |
|      | RK5N | 4.584971776584734e-07 | 60 |
|      | RK5B | 6.938911018763189e-07 | 60 |
| 0.05 | TDRK5F | 2.42693419776960e-08 | 500 |
|      | RK5W | 4.021845102580627e-08 | 1000 |
|      | RK5N | 1.258029772369107e-08 | 1200 |
|      | RK5B | 1.85275294689509e-08 | 1200 |
| 0.025 | TDRK5F | 7.354195030728761e-11 | 1600 |
|      | RK5W | 2.24633069092632e-09 | 2000 |
|      | RK5N | 3.684996774019697e-10 | 2400 |
|      | RK5B | 5.354157794207337e-10 | 2400 |
| 0.0125 | TDRK5F | 2.262079412673757e-12 | 3000 |
|      | RK5W | 1.328272214440318e-10 | 4000 |
|      | RK5N | 1.11461118129988e-11 | 4800 |
|      | RK5B | 1.608168979927438e-11 | 4800 |
| 0.00625 | TDRK5F | 6.900036098045348e-14 | 6400 |
|      | RK5W | 8.076289637060866e-12 | 6000 |
|      | RK5N | 3.426946226792182e-13 | 9600 |
|      | RK5B | 4.9274443942601e-13 | 9600 |

**Figure 2. The performance curves with step size**

\( h = 0.1^{2i}, i = 0, 1, 2, 3, 4 \) for Problem 1.

**Figure 3. The performance curves with step size**

\( h = 0.1^{2i}, i = 0, 1, 2, 3, 4 \) for Problem 2.
Table 4. The Numerical Results for Problems 3

| $h$  | Method   | MAXERR | FC  |
|------|----------|--------|-----|
| 0.125 | TDRK5F  | 6.76356246211652e-09 | 321 |
|      | RK5W    | 3.775046538700977e-07 | 400 |
|      | RK5N    | 3.78208407086795e-07  | 480 |
|      | RK5B    | 1.293847644801005e-06 | 480 |
| 0.0625| TDRK5F  | 1.027672391629153e-10 | 641 |
|      | RK5W    | 1.144161054780995e-08 | 800 |
|      | RK5N    | 1.148518058435855e-08 | 960 |
|      | RK5B    | 3.91847900110880e-08  | 960 |
| 0.03125| TDRK5F | 1.584399278442561e-12 | 1281 |
|       | RK5W    | 3.50589113295960e-10  | 1600 |
|       | RK5N    | 3.532993897437109e-10 | 1920 |
|       | RK5B    | 1.20335305889721e-09  | 1920 |
| 0.015625| TDRK5F | 2.50910435652854e-14  | 2561 |
|       | RK5W    | 1.078148681443736e-11 | 3200 |
|       | RK5N    | 1.094990764727299e-11 | 3840 |
|       | RK5B    | 3.726075040495719e-12 | 3840 |
| 0.0078125| TDRK5F| 1.221245372078676e-15 | 5121 |
|        | RK5W    | 3.31623617455343e-13  | 6400 |
|        | RK5N    | 3.415046023746982e-13  | 7680 |
|        | RK5B    | 1.159405904616051e-12  | 7680 |

Figure 4. The performance curves with step size $h = 1/2^i, i = 3, 4, 5, 6, 7, 8, 9$ for Problem 3.

Table 5. The Numerical Results for Problems 4

| $h$   | Method   | MAXERR | FC  |
|--------|----------|--------|-----|
| 0.1    | TDRK5F  | 2.385396100354898e-06 | 401 |
|        | RK5W    | 5.521056523183354e-06 | 500 |
|        | RK5N    | 6.441461286144090e-06 | 600 |
|        | RK5B    | 5.98142454195946e-05  | 600 |
| 0.05   | TDRK5F  | 1.074797493227919e-07 | 801 |
|        | RK5W    | 3.355656120751505e-07 | 1000 |
|        | RK5N    | 2.026292326151591e-07 | 1200 |
|        | RK5B    | 1.88879689977163e-06  | 1200 |
| 0.025  | TDRK5F  | 4.510416151681795e-09 | 1601 |
|        | RK5W    | 2.067595450405690e-08 | 2000 |
|        | RK5N    | 6.348468464913694e-09 | 2400 |
|        | RK5B    | 5.930724311653535e-08 | 2400 |
| 0.0125 | TDRK5F  | 1.656299541963335e-10 | 3201 |
|        | RK5W    | 1.282923878243025e-09 | 4000 |
|        | RK5N    | 1.986670828740792e-10 | 4800 |
|        | RK5B    | 1.857541898075965e-09 | 4800 |
| 0.00625| TDRK5F  | 5.857536677922326e-12 | 6401 |
|        | RK5W    | 7.962897008439995e-11 | 8000 |
|        | RK5N    | 6.44573286368734e-12  | 9600 |
|        | RK5B    | 5.78507241105042e-11  | 9600 |

Figure 5. The performance curves with step size $h = 0.1/2^i, i = 0, 1, 2, 3, 4$ for Problem 4.

Table 6. The Numerical Results for Problems 5

| $h$   | Method   | MAXERR | FC  |
|--------|----------|--------|-----|
| 0.1    | TDRK5F  | 2.2957667437399e-02 | 401 |
|        | RK5W    | 1.24193069260350e-01 | 500 |
|        | RK5N    | 1.24724140552514e-01 | 600 |
|        | RK5B    | 5.947253574392014e-01 | 600 |
| 0.05   | TDRK5F  | 4.304830287424968e-04 | 801 |
|        | RK5W    | 1.06889798817984e-03 | 1000 |
|        | RK5N    | 1.068254081341641e-03 | 1200 |
|        | RK5B    | 4.739714669978024e-03 | 1200 |
| 0.025  | TDRK5F  | 6.843461654172656e-06 | 1601 |
|        | RK5W    | 7.701960732076074e-06 | 2000 |
|        | RK5N    | 7.717269661755566e-06 | 2400 |
|        | RK5B    | 9.773522581246752e-06 | 2400 |
| 0.0125 | TDRK5F  | 1.059042478157579e-07 | 3201 |
|        | RK5W    | 8.795918041246253e-04 | 4800 |
|        | RK5N    | 8.802996185330869e-07 | 4800 |
|        | RK5B    | 2.738380274536212e-06 | 4800 |
| 0.00625| TDRK5F  | 1.643346070273376e-09 | 6401 |
|        | RK5W    | 3.735857315168012e-08 | 8000 |
|        | RK5N    | 3.739636024457926e-08 | 9600 |
|        | RK5B    | 1.231891475494962e-07 | 9600 |

Figure 6. The performance curves with step size $h = 0.1/2^i, i = 0, 1, 2, 3, 4$ for Problem 5.

It can be observed from Tables 2–6 and Figures 2–6 that the new TDRK5F method is more efficient than the Runge-Kutta methods chosen.
from the scientific literature in terms of accuracy and the number of function evaluations per each step.

**Conclusion:** A new explicit two derivative Runge-Kutta method of order five with FSAL property is developed in this paper. Also, the linear stability of the new method is analyzed. From numerical results, we conclude that the new TDRK5F method is more efficient compared with the existing RK methods of the same order in the literature in terms of the number of function evaluations and the accuracy per step. The computations were implemented on a DELL PC with i3-3227U CPU, 4.0GB memory.

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**Conflicts of Interest:** None.

**References:**
1. Butcher JC. An algebraic theory of integration methods. Math. Comput. 1972; 26(117): 79-106.
2. Franco JM. Exponentially fitted explicit Runge–Kutta Nyström methods. J. Comput. Appl. Math. 2004; 167(1): 1-19.
3. Van de Vyver H. An embedded phase-fitted modified Runge-Kutta method for the numerical integration of the radial Schrödinger equation. Phys. Lett. A. 2006; 352(4-5): 278-285.
4. Fang Y, Song Y, Wu X. New embedded pairs of explicit Runge-Kutta methods with FSAL properties adapted to the numerical integration of oscillatory problems. Phys. Lett. A. 2008; 372(44): 6551-6559.
5. Chan RPK, Tsai AYJ. On explicit two-derivative Runge-Kutta methods. Numer. Algorithms. 2010; 53(2-3): 171-194.
6. Fang Y, Li Q, Wu X. Extended RKN methods with FSAL property for oscillatory systems. Comput. Phys. Commun. 2010; 181(9): 1538-1548.
7. Ahmad NA, Senu N. Two Derivative Runge-Kutta Method with FSAL Property for the Solution of First Order Initial Value Problems. Indian J. Sci. Technol. 2016; 9(28): 1-8.
8. Butcher JC. The numerical analysis of ordinary differential equations. Chichester: Wiley; 1987.
9. Hairer E, Norsett SP, Wanner G. Solving ordinary differential equations I: Nonstiff problems. Berlin: Springer-Verlag; 1993.
10. Butcher JC. Numerical methods for ordinary differential equations. 3rd ed. Chichester: Wiley; 2016.
11. Li J. Trigonometrically fitted three-derivative Runge-Kutta methods for solving oscillatory initial value problems. Appl. Math. Comput. 2018; 330: 103-117.
12. Hussain K, Ismail F, Senu N, Rabiei, F. Optimized fourth-order Runge-Kutta method for solving oscillatory problems. AIP Conference Proceedings, 2016; 1739: 020032.
13. Franco JM, Gomez I. Trigonometrically fitted nonlinear two-step methods for solving second order oscillatory IVPs. Appl. Math. Comput. 2014; 232: 643-657.
14. Hussain K, Ismail F, Senu N. A new optimized Runge-Kutta method for solving oscillation problems. Int. J. Pure Appl. Math. 2016; 106(3): 715-723.