REMARKS ON GRADIENT RICCI SOLITONS

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Abstract. In this paper, we study the gradient Ricci soliton equation on a complete Riemannian manifold. We show that under a natural decay condition on the Ricci curvature, the Ricci soliton is Ricci-flat and ALE.

1. Introduction

In the study of Ricci flow on a Riemannian manifold, we meet Ricci solitons (see [5]). Ricci solitons and their sisters Kähler-Ricci solitons are important objects by their own right (see also [3], [4], and [6]).

Given a Riemannian manifold \((X, g)\) of dimension \(n\). Let \(Rc\) be the Ricci tensor of the metric \(g\). The equation for a homothetic Ricci soliton is

\[Rc = cg + LVg\]

where \(c\) is a homothetic constant, \(V\) is a smooth vector field on \(X\), and \(LVg\) is the Lie derivative of the metric \(g\). When \(c = 0\), the soliton is steady. For \(c > 0\) the soliton is shrinking, and one can consider the Ricci flow on the sphere as such an example. For \(c < 0\) the soliton is expanding. When \(V\) is the gradient of a smooth function, we call such solitons Gradient Homothetic Ricci Solitons. Let \(R\) be the scalar curvature of \(g\). An important equation in Riemannian geometry and general relativity theory is the so called Einstein equation:

\[E_{ij} = T_{ij},\]

where \(E = Rc - \frac{R}{2}g\), and \(T\) is the energy momentum tensor in the space. The tensor \(T\) sometimes is also an unknown being. So it is interesting to know whether \(T\) is the Hessian matrix of a smooth function, and to explore more properties about this function.

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In this short paper we constrict ourselves to consider the following question about the gradient Ricci soliton on $X$. We deduce a Liouville type theorem for a smooth solution $f$ for the equation

$$Rc = D^2 f$$
on $X$, where $D^2 f$ is the Hessian matrix of the function $f$.

Taking the trace of the equation above, we get that

$$R = \Delta f.$$ 

Here we denote by $R$ the scalar curvature of the metric $g$ and $\Delta f$ the Laplacian of $f$. If such a smooth function exists, we call $(X, g)$ a gradient Ricci soliton.

In the following, we write $D$ the covariant derivative of $g$.

As in [5], we can derive (see the next section) that there is a constant $M$ such that

$$|Df|^2 + R = M$$
on $X$. Then we have the following simple observation.

**Proposition 1.** Assume $X$ is compact. Then $(X, g)$ is Ricci-flat.

**Proof.** In fact, we have

$$\Delta f + |Df|^2 = M.$$ 

Set

$$u = e^f.$$ 

Then we have that $u > 0$ and

$$\Delta u = Mu.$$ 

Integrating we find

$$M \int_X u = 0.$$ 

Then we have $M = 0$. Hence, $u$ is a harmonic function. By the Maximum principle, we know that $u$ is a constant, so $f$ is a constant. This implies that $Rc = 0$ and $(X, g)$ is Ricci-flat. \hfill $\Box$

We want to generalize the above result. Observe that if $(X, g)$ is non-compact and complete, the matter is not simple. However, if $R(x) \to 0$ as $r := r(x) \to \infty$, then we can get some conclusions. Here $r(x) = \text{dist}(x, o)$ is the distance of the point $x$ from a fixed point $o$.

Since $M \geq R$ and $R$ can be very small at infinity, we have $M \geq 0$. If we also assume that $|Df(x)| \to 0$ as $r(x) \to \infty$, then we must have that $M = 0$. Hence $u$ is again a positive harmonic function. If we further assume that $Rc \geq 0$, then we must have by the Liouville theorem of Yau [7] that $u$ is a constant. So we have obtained the following result.
Proposition 2. Assume $X$ is a non-compact complete Riemannian manifold with non-negative Ricci curvature. If either case 1) $\int_X |R|^p < +\infty$ and $\int_X |f|^p < +\infty$ or $\int_X |Df|^p < +\infty$ (for some $p \geq 1$) or case 2) both $R$ and $f$ decay to zero at infinity, then $f$ is a constant and $(X, g)$ is Ricci-flat.

Proof. In both cases we can easily conclude that $M = 0$. Then we have that

$$\Delta u = M = 0.$$ 

By the Liouville theorem of Yau [7] we have that $u$ is a constant.

Now the nontrivial matter is to treat the case when $Rc$ has no sign assumption. The natural consideration is sing the cut-off function trick. In fact, it works for some cases. We have the following theorem, which is the main result of this paper.

Theorem 3. Assume $X$ is a non-compact complete Riemannian manifold which is quasi-isometric to Euclidean space at infinity. If $n \geq 3$ and $Rc$ satisfies that

$$\int_X |Rc|^{n/2} < +\infty,$$

then $f$ is a constant and $(X, g)$ is Ricci-flat and ALE of order $n - 1$. If we further assume that $n = 4$, then $(X, g)$ is ALE of order $4$.

The definition about ALE space will be given in the next section. In our proof of the result above, we will use the Bochner formula, Moser iteration method, and the interesting result of Bando-Kasue-Nakajima [1]. Our idea can also be used to study Ricci-Kahler soliton and related question for Einstein equations.

2. Notations, definitions, and basic facts

In local coordinates $(x^i)$ of the Riemannian manifold $(X, g)$, we write the metric $g$ as $(g_{ij})$. The corresponding Riemannian curvature tensor and Ricci tensor are denoted by $Rm = (R_{ijkl})$ and $Rc = (R_{ij})$ respectively. Hence,

$$R_{ij} = g^{kl}R_{ijkl},$$

and

$$R = g^{ij}R_{ij}.$$ 

We write the covariant derivative of a smooth function $f$ by $Df = (f_i)$, and denote the Hessian matrix of the function $f$ by $D^2f = (f_{ij})$, where $D$ the covariant derivative of $g$ on $X$. The higher order covariant derivatives are denoted by $f_{ijk}$, etc. Similarly, we use the $T_{ij,k}$ to denote
the covariant derivative of the tensor \((T_{ij})\). We write \(T^i_j = g^{ik}T_{jk}\). Then the Ricci soliton equation is

\[ R_{ij} = f_{ij}. \]

Taking covariant derivative we get

\[ f_{ijk} = R_{ij,k}. \]

So we have

\[ f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}. \]

By the Ricci formula we have that

\[ f_{ijk} - f_{ikj} = R^l_{ijk}f_l. \]

Hence we obtain that

\[ R_{ij,k} - R_{ik,j} = R^l_{ijk}f_l. \]

Recall that the contracted Bianchi identity is

\[ R_{ij,j} = \frac{1}{2}R_{i}. \]

Upon taking the trace of the previous equation we get that

\[ \frac{1}{2}R_{i} + R^k_{i}f_k = 0, \]

i.e.,

\[ R_k = -2R^l_{k}f_l. \]

Then

\[ D_k(|Df|^2 + R) = 2f_j(f_{jk} - R_{jk}) = 0. \]

So, \(|Df|^2 + R\) is a constant, which is denoted by \(M\).

To analyze the global behavior of the geometry of the manifold \(X\), we need the following definition of quasi-isometry.

**Definition 4.** We say that \((X, g)\) is quasi-isometric to the Euclidean space \(R^n\) if there is a compact subset \(K\) in \(X\) such that the metric \(X - K\) is uniformly equivalent to \(R^n - B_1(0)\).

It is well-known that for such a Riemannian manifold, the Sobolev inequality is true. We shall use this fact to make the Moser iteration method work in \(X\). A special case of such a manifold with one end is called asymptotically locally Euclidean (ALE) of order \(\tau > 0\) (see \([2]\)); namely there are constant \(\rho > 0\), \(\alpha \in (0, 1)\) and a \(C^\infty\)-diffeomorphism

\[ \Xi : X - K \to R^n - B_\rho(0) \]

such that \(\phi = \Xi^{-1}\) satisfies

\[ (\phi^*g)_{ij}(x) = \delta_{ij} + O(|x|^{-\tau}), \]
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\[ \partial_k (\phi^* g)_{ij}(x) = 0(|x|^{-1-\tau}), \]

and

\[ \frac{|\partial_k (\phi^* g)_{ij}(x) - \partial_k (\phi^* g)_{ij}(z)|}{|x-z|^\alpha} = 0(|x|^{-1-\alpha-\tau}, |z|^{-1-\alpha-\tau}) \]

for any \( x, z \in \mathbb{R}^n - B_\rho(0) \).

3. PROOF OF THEOREM 3

We now recall the Bochner formula for any smooth function \( w \) on \( X \):

\[ \frac{1}{2} \Delta |Dw|^2 = |D^2 w|^2 + < D\Delta w, Dw > + Rc(Dw, Dw). \]

For our function \( f \) in the Ricci soliton equation, we have that

\[ \frac{1}{2} \Delta |Df|^2 = |D^2 f|^2 + < DR, Df > + Rc(Df, Df) \]

Recall that

\[ R_k = -2R_{kj}f_j. \]

Then we have

\[ < DR, Df > = -2Rc(Df, Df). \]

So we have

\[ \frac{1}{2} \Delta |Df|^2 = |Rc|^2 - Rc(Df, Df). \]

This equation can also be written as

\[ -\frac{1}{2} \Delta R = |Rc|^2 - \frac{1}{2} (DR, Df), \]

from which we know the following inequality

\[ \frac{1}{2} \Delta |Df|^2 \geq -|Rc||Df|^2. \]

We shall use this differential inequality to study the behavior of \( |Df| \) at infinity. Let \( w = |Df|^2 \). Then \( w = |R| \). Using the assumption that

\[ \int |Rc|^{n/2} < +\infty \]

we get that

\[ \int |w|^{n/2} < +\infty. \]

Using Moser's iteration method we know (see Lemmas 4.1,4.2,4.6 in [III]) that there is a positive constant \( \alpha > 0 \) such that

\[ w = 0(r^{-\alpha}). \]
That is to say that
\[ |R(x)| = |Df|^2(x) = 0(r^{-\alpha}). \]
This implies that \( M = 0 \), and \( R \leq 0 \) on \( X \). Using the maximum principle to the equation
\[ -\frac{1}{2} \Delta R = |Rc|^2 - \frac{1}{2}(DR, Df) \]
and the fact that \( |Rc|^2 \geq \frac{1}{n} R^2 \), we find that the minimum of \( R \) on \( X \) can not be negative, and it must be zero. Hence \( R = 0 \) on \( X \). This implies from (1) that \( Rc = 0 \). Using Theorem 1.5 in [1] we know that \((X, g)\) is ALE of order \( n-1 \). If further \( n = 4 \), \((X, g)\) is ALE of order 4.

This proves our Theorem 3.

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