EXTREME POINTS OF THE SET OF DENSITY MEASURES

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ABSTRACT. We study finitely additive measures on the set \( \mathbb{N} \) which extend
the asymptotic density (density measures). We show that there is a one-to-
one correspondence between density measures and positive functionals in \( \ell^*_\infty \)
which extend Cesàro mean. Then we study maximal possible value attained
by a density measure for a given set \( A \) and the corresponding question for
the positive functionals extending Cesàro mean. Using the obtained results,
we can find a set of functionals such that their closed convex hull in \( \ell^*_\infty \) with
weak* topology is precisely the set of all positive functionals extending Cesàro
mean. Since we have a one-to-one correspondence between such functionals
and density measures, this also gives a set of density measures, from which all
density measures can be obtained as the closed convex hull.

1. Introduction

Asymptotic density is a very natural way to measure size of subsets of \( \mathbb{N} \). One
of the drawbacks of this concept is that there are sets that do not have asymptotic
density. This problem leads us to studying finitely additive measures which extend
the asymptotic density and are defined for all subsets of \( \mathbb{N} \). Such measures are
called density measures and they were studied by several authors, for example [2, 17, 21, 25]. Density measures (and other types of densities on \( \mathbb{N} \)) have applications,
for example, in theory of social choice [8, 16, 24].

If a set \( A \subseteq \mathbb{N} \) has asymptotic density, then clearly \( \mu(A) = d(A) \). But for sets
not having asymptotic density, it might be interesting to find the maximal and
minimal possible values of \( \mu(A) \). This problem was posed in [9]. Some questions
concerning the possible values of density measures were also stated in [25]. Several
expressions of extreme values of density measures are known, see [23]. In this paper
we continue in the study of these extremal values and we find new possibilities how
to express them.

Every finitely additive measure on \( \mathbb{N} \) induces a positive continuous linear func-
tional on \( \ell_\infty \) and by restricting such functional to characteristic sequences \( \chi_A \) we
obtain a finitely additive measure. This correspondence between measures and
functionals is described in more detail in Section 3. Often it is easier to study
the corresponding functionals instead of measures. We show that the functionals
corresponding to density measures are precisely the positive functionals extending
Cesàro mean. Thus we are also interested in the extremal values of these function-
als.

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Another natural question pertaining to the density measures (and the corresponding functionals) is whether we can obtain them in some way from some set of simpler measures (simpler functionals). One possibility how to do this was proposed in [16]. Unfortunately, as shown in [22], the expression given in [16, p.49] does not entail all density measures.

Using the fact that we know the range of possible values of density measures together with results of [14] we obtain some sets of positive functionals which give the set of positive functionals extending Cesàro mean as their closed convex hulls in $\ell^*_\infty$ with weak*-topology. (In the other words, we find a set of functionals such that taking all pointwise limits of their convex combinations yields the set of all positive functionals extending Cesàro mean.) This also gives a corresponding results for density measures.

The paper is organized as follows: Section 2 contains some basic results on $\mathcal{F}$-limits, a few facts from functional analysis and some inequalities, which are needed later. In Section 3 we briefly describe the correspondence between the finitely additive measures and the positive functionals from $\ell^*_\infty$. We show that in this way density measures can be viewed as the positive functionals extending Cesàro mean.

In Section 4 we give four different expressions of the range of density measures. We start by stating the result that these four values are equal to each other, recapitulating which of the equalities are already known and show the remaining equality.

In Section 5 we study analogous results with functionals. First we give the functions corresponding to various expression of extreme values of density measures and state the result saying that they are in fact the same. This result is then shown in a series of lemmas. Some of the results follow from the corresponding results for density measures, but sometimes different techniques are needed.

In Section 6 we describe a set of functionals such that the closed convex hull of this set is the set of all positive functionals extending the Cesàro mean. We give consequences of this fact for the density measures. To get such set of functionals we can use the expressions of the extreme values obtained in Section 5. It suffices to find some functionals, which attain these values. In this way it is relatively easy to find a set generating all functionals with this property (see Remark 6.6). But we then show that the description of the density measures and the corresponding functionals can be further simplified and we show that there is a smaller set of functionals, which have simpler form, and they still generate all positive functionals extending Cesàro mean.

2. Preliminaries

By $\mathbb{N}$ we denote the set $\mathbb{N} = \{1, 2, 3, \ldots \}$ of all positive integers. The symbols $\beta\mathbb{N}$ and $\beta\mathbb{N}^*$ denote the set of all ultrafilters and the set of all free ultrafilters on $\mathbb{N}$, respectively.

2.1. $\mathcal{F}$-limits. We will use the notion of the limit of a real sequence along an ultrafilter. We will only briefly mention the definition and some basic properties. More information about them can be found, for example, in [11 8.23–8.26], [6 Sections 2.3 and 4.5], [13 Section 11.2], [15 Problem 17.19].
If \((x_n)_{n=1}^{\infty}\) is a bounded real sequence and \(F\) is an ultrafilter on \(\mathbb{N}\), then there exists a unique real number \(L\) such that
\[
\{ n \in \mathbb{N} ; |x_n - L| < \varepsilon \} \in F
\]
for each \(\varepsilon > 0\). The number \(L\) is called the \(F\)-limit of the sequence \((x_n)_{n=1}^{\infty}\) and it is denoted by \(L = \lim_F x_n\).

Some basic facts about \(F\)-limits are collected in the following proposition.

**Proposition 2.1.** Let \((x_n)_{n=1}^{\infty}\), \((y_n)_{n=1}^{\infty}\) be bounded real sequences, \(c \in \mathbb{R}\) and \(F\) be an ultrafilter on \(\mathbb{N}\).

1. \(\lim_F (x_n + y_n) = \lim_F x_n + \lim_F y_n\) and \(\lim_F (cx_n) = c \lim_F x_n\), in the other words \(\lim_F\) is a linear function on \(\ell_\infty\).
2. \(\lim_F (x_n y_n) = \lim_F x_n \cdot \lim_F y_n\).
3. If \(A \in F\) is an infinite set and the limit \(\lim_{n \in A} x_n = L\) exists, then the \(\lim_F\) has the same value \(\lim_F x_n = \lim_{n \in A} x_n\).
4. If \(F\) is a free ultrafilter and \((x_n)_{n=1}^{\infty}\) is a convergent sequence then \(\lim_F x_n = \lim_{n \to \infty} x_n\).
5. If \(F\) is a free ultrafilter, then \(\lim_F x_n\) is a limit point of the sequence \((x_n)_{n=1}^{\infty}\). Conversely, for each limit point \(L\) of the sequence \((x_n)_{n=1}^{\infty}\) there exists a free ultrafilter such that \(\lim_F x_n = L\).
6. If \(x_n \geq y_n\) for each \(n\), then \(\lim_F x_n \geq \lim_F y_n\). In particular, \(x_n \geq 0\) implies \(\lim_F x_n \geq 0\).
7. The function \((x_n)_{n=1}^{\infty} \mapsto \lim_F x_n\) belongs to \(\ell^*_\infty\) and has norm 1.

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2.2. **Functional analysis.** We will recall a few results from functional analysis needed in this paper. We work only with vector spaces and normed spaces over the field \(\mathbb{R}\).

We include a formulation of Hahn-Banach Theorem, since in addition to the usual formulation we also use the facts about possible values of the extensions summarized below.

Recall that a function \(p : X \to \mathbb{R}\) defined on a vector space \(X\) is called *sublinear* if it is subadditive and positively homogeneous, i.e., if it fulfills
\[
p(x + y) \leq p(x) + p(y)
\]
\[
p(cx) = cp(x)
\]
for any \(x, y \in X\) and \(c > 0\).

**Theorem 2.2** (Hahn-Banach theorem). Let \(X\) be a vector space and let \(p : X \to \mathbb{R}\) be any sublinear function. Let \(M\) be a vector subspace of \(X\) and let \(f : M \to \mathbb{R}\) be a linear functional dominated by \(p\) on \(M\). Then there is a linear extension \(\hat{f}\) of \(f\) to \(X\) that is dominated by \(p\) on \(X\).

Moreover, for any given \(v \in X\), there exists an extension \(\hat{f}\) such that \(\hat{f}(v) = c\) if and only if
\[
\sup_{x \in M} |f(x) - p(x - v)| \leq c \leq \inf_{y \in M} [p(y + v) - f(y)].
\]

In case the \(p\) and \(f\) have the additional property that
\[
(\forall x \in X)(\forall y \in M)p(x + y) = p(x) + f(y)
\]
then the above interval can be simplified to
\[-p(-v) \leq c \leq p(v).\]

Usually Hahn-Banach theorem is formulated without the conditions about the possible values of the extensions, but they can be deduced from the usual proof of this result; see, for example, [7, Theorem 2.1].

We will use the following formulation of Krein-Milman Theorem:

**Theorem 2.3 ([14, Theorem 1]).** Let \( E \) be a locally convex topological vector space and \( C \) be a compact convex subset of \( E \). Let \( S \subseteq C \). The following assertions are equivalent:

(i) For every linear continuous function \( f : E \to \mathbb{R} \) the equality
\[ \sup_{x \in S} f(x) = \sup_{x \in C} f(x) \]
holds;

(ii) \( C = \text{cl}(S) \), i.e., \( C \) is the closed convex hull of \( S \);

(iii) the closure \( \overline{S} \) of the set \( S \) contains all extreme points of \( C \).

For the definition of extreme points see, for example, [7, Section 3.8].

We will use Theorem 2.3 in the special case where \( E = X^* \) with the weak*-topology. It is known that in this case continuous linear functionals on \( E \) are precisely the maps \( x^* \) for \( x \in X \), where \( x^* \) denotes the evaluation at the point \( x \) (see [7, Proposition 3.22]). Hence in this case we get:

**Proposition 2.4.** Let \( X \) be a linear normed space and \( C \) be a subset of \( X^* \) which is convex and compact in the weak*-topology. Let \( S \subseteq C \). The following conditions are equivalent:

(i) \( (2.1) \sup_{\varphi \in S} \varphi(x) = \sup_{\varphi \in C} \varphi(x) \)
holds for each \( x \in X \);

(ii) \( C = \text{cl}(S) \), i.e., \( C \) is the closed convex hull of \( S \);

(iii) the closure \( \overline{S} \) of the set \( S \) contains all extreme points of \( C \).

2.3. Inequalities. The following result summarizes the parts of [20, Theorem 3, Corollary 1] which we will need:

**Theorem 2.5.** Let \( (x_n)_{n=1}^\infty \), \( (a_n)_{n=1}^\infty \), \( (d_n)_{n=1}^\infty \) be real sequences such that \( a_n, d_n > 0 \) and \( \lim_{n \to \infty} A_n = \lim_{n \to \infty} D_n = \infty \) holds for the partial sums \( \sum_{k=1}^n a_k = A_n \) and \( \sum_{k=1}^n d_k = D_n \).

(i) If \( a_n/d_n \) is decreasing and \( x_n \geq 0 \) then
\[ \liminf_{n \to \infty} \frac{\sum_{k=1}^n d_k x_k}{D_n} \leq \liminf_{n \to \infty} \frac{\sum_{k=1}^n a_k x_k}{A_n} \leq \limsup_{n \to \infty} \frac{\sum_{k=1}^n a_k x_k}{A_n} \leq \limsup_{n \to \infty} \frac{\sum_{k=1}^n d_k x_k}{D_n}. \]

(ii) If \( a_n/d_n \) is increasing and there exists a constant \( C \) such that \( \frac{a_n}{d_n} < C \frac{A_n}{D_n} \) for each \( n \in \mathbb{N} \) then the limit \( \lim_{n \to \infty} \frac{\sum_{k=1}^n a_k x_k}{A_n} \) exists for each sequence such that \( \lim_{n \to \infty} \frac{\sum_{k=1}^n d_k x_k}{D_n} \) exists and
\[ \lim_{n \to \infty} \frac{\sum_{k=1}^n d_k x_k}{D_n} = \lim_{n \to \infty} \frac{\sum_{k=1}^n a_k x_k}{A_n}. \]
To be more precise, we will only need some special cases, where the sequences $a_n$ and $d_n$ are of the form $n^\alpha$.

**Corollary 2.6.** Let $(x_n)_{n=1}^\infty$ be a real sequence.

(i) If $-1 \leq \alpha \leq \beta$ and $x_n \geq 0$ then

\[
\lim\inf_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n} \leq \lim\inf_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n} \leq \lim\sup_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n} \leq \lim\sup_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n}.
\]

(ii) If $\alpha > 0$ and the limit $\lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n}$ exists then

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n} = \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n}.
\]

3. **Finitely additive measures and $\ell_\infty^*$**

We say that a function $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is a finitely additive measure on $\mathbb{N}$ if $\mu(A + B) = \mu(A) + \mu(B)$ for any disjoint sets $A, B \subseteq \mathbb{N}$. (Notice that, at this point, we allow the values of $\mu$ to be negative.)

There is a very natural correspondence between the finitely additive measures and linear continuous functionals $f : \ell_\infty \to \mathbb{R}$. For $f \in \ell_\infty^*$ we can obtain a measure by putting $\mu(A) = f(\chi_A)$.

The process of obtaining a functional from a given measure is similar to definition of Riemann integral. It uses the fact that any bounded sequence can be uniformly approximated by step sequences. (By a step sequence we mean a sequence of the form $\sum_{i=1}^n c_i \chi_{A_i}$ for some $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \subseteq \mathbb{N}$, i.e., a finite linear combination of characteristic sequences.)

More details about this construction can be found, for example, in [5, Theorem 16.7], [13, p.50, Example 1.19], [25, Section 3]. Many texts in functional analysis provide also a more general version of this result dealing with dual of $L_\infty(X, \mu)$.

It is relatively easy to see that positive functionals correspond to positive measures and positive functionals such that $\|f\| = 1$ correspond to probabilistic positive measures.

From now on we will say briefly *measure* instead of finitely additive positive probabilistic measure.

We will study the measures which extend asymptotic density.

**Definition 3.1.** For a set $A \subseteq \mathbb{N}$ the upper and lower asymptotic density is defined by

\[
d(A) = \lim\sup_{n \to \infty} \frac{A(n)}{n} \quad \text{and} \quad \tilde{d}(A) = \lim\inf_{n \to \infty} \frac{A(n)}{n}.
\]

If $\tilde{d}(A) = \tilde{d}(A)$ then this common value is denoted by $d(A)$ and it is called the asymptotic density of the set $A$.

We will denote the class of all subsets of $\mathbb{N}$ possessing asymptotic density by $\mathcal{D}$. A finitely additive measure $\mu : \mathcal{P}(\mathbb{N}) \to [0, 1]$ is called a density measure if

\[
\mu(A) = d(A)
\]

for every $A \in \mathcal{D}$.

The set of all density measures will be denoted by $\hat{\mathcal{D}}$. 

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In the other words, density measures are precisely the measures which extend asymptotic density. Density measures have been studied, for example, in [2, 17, 21, 25, 22, 23].

Sometimes it is easier to work with the corresponding functionals instead of measures. Since we work with density measures, we need to know what class of functionals corresponds to such measures. We will show in Theorem 3.3 that they are precisely the positive functionals which extend Cesàro mean.

**Definition 3.2.** Let \( x \in \ell_\infty \). If the limit \( C(x) = \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} \) exists, then it is called the Cesàro mean of the sequence \( x \). We will denote the set of all bounded sequences that have Cesàro mean by \( \mathcal{C} \).

The set of all positive functionals \( f \in \ell_\infty^* \) which extend Cesàro mean, i.e., which fulfill
\[
|f|_C = C
\]
will be denoted by \( \hat{\mathcal{C}} \).

**Theorem 3.3.** Let \( \mu \) be a measure and \( f \in \ell_\infty^* \) be the corresponding functional. The measure \( \mu \) is a density measure if and only if \( f \) extends Cesàro mean.

**Proof.** It is obvious that if \( f \) extends Cesàro mean then \( \mu \) is a density measure, since we have \( d(A) = C(\chi_A) \) whenever \( A \in \mathcal{D} \).

To prove the opposite direction we need a characterization of \( \hat{\mathcal{D}} \) and \( \hat{\mathcal{C}} \) using invariance under permutations from the Lévy group \( \mathcal{G} \). (We will not include the definition of Lévy group, an interested reader can find the definition, for example, in [3], [4] or [22]. For this proof it suffices to know that it is a set of permutations of \( \mathbb{N} \).)

It is shown in [22, Theorem 2.6] that density measures are precisely the \( \mathcal{G} \)-invariant measures. By [4, Theorem 2] a positive functional with \( \|f\| = 1 \) extends Cesàro mean if and only if it is \( \mathcal{G} \)-invariant.

So if \( \mu \) is a density measure, then the corresponding functional is positive and \( \|f\| = 1 \). Since \( \mu \) is \( \mathcal{G} \)-invariant, we also get that \( f \) is \( \mathcal{G} \)-invariant for step sequences. Using the fact that step sequences are dense in \( \ell_\infty \) we can extend the \( \mathcal{G} \)-invariance to all sequences. \( \square \)

### 4. Extremal values of density measures

For a given subset \( A \subseteq \mathbb{N} \), we are interested in the values
\[
g(A) = \inf \{ \mu(A); \mu \in \hat{\mathcal{D}} \} \quad \text{and} \quad \overline{d}(A) = \sup \{ \mu(A); \mu \in \hat{\mathcal{D}} \}.
\]

In this section we will mention several equivalent expressions of \( \overline{d}(A) \) and \( g(A) \).

For any \( A \subseteq \mathbb{N} \) and \( \alpha \in \mathbb{R} \) we define
\[
A_\alpha(n) = \sum_{\begin{smallmatrix} k \in A \\&\\ k \leq n \end{smallmatrix}} k^\alpha.
\]

Notice that \( A_0(n) = A(n) \).

It is relatively easy to see that \( A_\alpha(n) \sim \frac{n^{\alpha+1}}{\alpha+1} \) for \( \alpha > -1 \). This observation is often useful when working with functions similar to \( \overline{d}_\infty(A) \), \( d_\infty(A) \) defined below.
In addition to \( \overline{d}(A) \) and \( \underline{d}(A) \) we will also use the following quantities.

\[
\begin{align*}
\overline{d}(A) & = \inf \{d(B); B \supseteq A; B \in \mathcal{D}\} \\
\overline{d}_\infty(A) & = \lim_{\alpha \to \infty} \limsup_{n \to \infty} \frac{A_\alpha(n)}{N_\alpha(n)} \\
\overline{d}_\infty(A) & = \lim_{\theta \to 1^-} \limsup_{n \to \infty} \frac{A(n) - A(\theta n)}{n - \theta n}
\end{align*}
\]

We should show that the limits in the definitions of \( \overline{d}_\infty(A) \) and \( \overline{d}_\infty(A) \) really exist. The existence of the limits appearing in the definition of \( \overline{d}_\infty(A) \) is shown in \cite{19}. In the next section we will define function \( s(x) \) such that \( s(\chi_A) = \overline{d}_\infty(A) \), see \cite{5.1}. In Remark \cite{5.3} we show that the limit used in the definition of \( s(x) \) exists. (This is the same as the above limit if \( x = \chi_A \).)

We also define the lower versions of the last three densities \( d_\ast(A) = \sup \{d(B); B \subseteq A; B \in \mathcal{D}\} \), \( d_\infty(A) = \lim_{\alpha \to \infty} \liminf_{n \to \infty} \frac{A_\alpha(n)}{N_\alpha(n)} \) and \( d_\ast(A) = \lim_{\theta \to 1^-} \liminf_{n \to \infty} \frac{A(n) - A(\theta n)}{n - \theta n} \).

There is an obvious correspondence between the lower and upper densities, \( \overline{d}(\mathbb{N} \setminus A) = 1 - \underline{d}(A) \) and the analogous equalities are true for \( \overline{d}(A), \overline{d}(A) \) and \( \overline{d}(A) \).

The values \( \overline{d}(A) \) and \( \underline{d}(A) \) were studied in a slightly more general setting in \cite{19}. They are sometimes called (upper and lower) Polya density, see e.g. \cite{11}. The densities \( \overline{d}(A) \) and \( \underline{d}(A) \) are related to \( \alpha \)-densities, which were studied in \cite{10}.

Now we can formulate the main result of this section, in which we obtain several expressions for the extremal values of density measures.

**Theorem 4.1.** For any \( A \subseteq \mathbb{N} \) we have

\[
\overline{d}(A) = d_\ast(A) = \overline{d}(A) = \overline{d}(A)
\]

and \( \overline{d}(A) = d_\ast(A) = \overline{d}(A) = \overline{d}(A) \).

Of course, it suffices to show this only for the upper densities or only for lower densities, because of the aforementioned correspondence.

In fact, with the exception of the part about \( \overline{d}_\infty(A) \) and \( \overline{d}_\infty(A) \) the above theorem is only a summarization of known results. The equality \( \overline{d}(A) = d_\ast(A) \) was shown in \cite{23} Theorem 3. The equality \( \overline{d}(A) = \overline{d}(A) \) can be obtained from (more general) result \cite{19} Satz VIII.

The missing equality between \( \overline{d}_\infty(A) \) and the remaining expressions will be shown in Lemma \cite{4.3}. This answers \cite{23} Problem 1.

Let us start by summarizing some results on \( \overline{d}(A) \) and \( \underline{d}(A) \) which will be needed in the proof.

**Lemma 4.2.** Let \( A, B \subseteq \mathbb{N} \).

(i) \( 0 \leq \underline{d}(A) \leq d_\infty(A) \leq \overline{d}_\infty(A) \leq \overline{d}(A) \leq 1 \)

(ii) For any \( B \) there exists \( A \subseteq B \) such that \( A \in \mathcal{D} \) and \( d(A) = \underline{d}(B) \).

(iii) If \( A \cap B = \emptyset \) and \( A \in \mathcal{D} \), then

\[
\underline{d}(A \cup B) = \underline{d}(A) + \underline{d}(B).
\]
(iv) If $A \cap B = \emptyset$ and $A \in \mathcal{D}$ then
\[
d_{\infty}(A \cup B) = d(A) + d_{\infty}(B).
\]

**Proof.** The part (i) is shown in [23, Corollary 6]. The second part is [23, Proposition 2] and the third part is [23, Proposition 1].

(iv) From the existence of the limit $d(A) = \lim_{n \to \infty} \frac{A(n)}{n} = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \chi_{A}(n)}{n}$ we get, for any $\alpha > 0$, $d_{\alpha}(A) = \lim_{n \to \infty} \frac{A_{\alpha}(n)}{N_{\alpha}(n)} = d(A)$ from Corollary 2.6. Therefore
\[
d_{\alpha}(A \cup B) = \lim_{n \to \infty} \frac{A_{\alpha}(n) + B_{\alpha}(n)}{N_{\alpha}(n)} = \lim_{n \to \infty} \frac{A_{\alpha}(n)}{N_{\alpha}(n)} + \lim_{n \to \infty} \frac{B_{\alpha}(n)}{N_{\alpha}(n)} = d(A) + d_{\alpha}(B).
\]
Taking limits $\alpha \to \infty$ on both sides gives $d_{\infty}(A \cup B) = d(A) + d_{\infty}(B)$. \qed

**Lemma 4.3.** For any set $A \subseteq \mathbb{N}$ we have
\[
d(A) = d_{\infty}(A).
\]

**Proof.** We already know $0 \leq d(A) \leq d_{\infty}(A)$ from Lemma 4.2 so it suffices to show the inequality $d_{\infty}(A) \leq d(A)$ for all $A \subseteq \mathbb{N}$.

First let us assume that $d(A) = 0$. We want to show that $d_{\infty}(A) = 0$.

We have
\[
\lim_{\theta \to 1} \liminf_{n \to \infty} \frac{A(n) - A(\theta n)}{n - \theta n} = 0.
\]

Fix $\varepsilon > 0$. Then there exists $\theta_0 < 1$ such that
\[
\liminf_{n \to \infty} \frac{A(n) - A(\theta n)}{n - \theta n} < \varepsilon
\]
whenever $\theta_0 < \theta < 1$. This implies that
\[
A(n) - A(\theta n) < 2\varepsilon
\]
for infinitely many $n$’s.

If $n$ fulfills (4.1) then
\[
A_{\alpha}(n) \leq A_{\alpha}(\theta n) + n^\alpha (A(n) - A(\theta n)) < A_{\alpha}(\theta n) + n^{\alpha+1}(1 - \theta)2\varepsilon
\]
\[
\frac{A_{\alpha}(n)}{n^{\alpha+1}} \leq \frac{A_{\alpha}(\theta n)}{(\theta n)^{\alpha+1}} \cdot \theta^{\alpha+1} + (1 - \theta)2\varepsilon
\]
\[
A_{\alpha}(n) \leq N_{\alpha}(\theta n)(\alpha + 1) \cdot \frac{\theta^{\alpha+1}}{\alpha + 1} + (1 - \theta)2\varepsilon
\]

Since the above inequality holds for infinitely many $n$’s we get
\[
d_{\alpha}(A) = \liminf_{n \to \infty} \frac{A_{\alpha}(n)(\alpha + 1)}{n^{\alpha+1}} \leq \theta^{\alpha+1} + (1 - \theta)2\varepsilon(\alpha + 1)
\]
for any $\theta \in (\theta_0, 1)$ and any $\alpha > 0$.

Let us assume that, moreover,
\[
(1 - \theta)(\alpha + 1) = \frac{1}{\sqrt{\varepsilon}}
\]

Using (4.3) we get from (4.2)
\[
d_{\alpha}(A) \leq \left(1 - \frac{1}{(\alpha + 1)\sqrt{\varepsilon}}\right)^{\alpha+1} + 2\sqrt{\varepsilon}.
\]

This inequality is valid for any $\alpha$ and $\theta \in (\theta_0, 1)$ that fulfill (4.3).
Now, if \( \alpha \to \infty \) (note that this means \( \theta \to 1^- \), hence we can always find \( \theta \in (\theta_0, 1) \) such that (4.3) holds), we get

\[
d_\infty(A) \leq e^{-\frac{1}{\sqrt{\varepsilon}}} + 2\sqrt{\varepsilon}.
\]

As the RHS tends to 0 for \( \varepsilon \to 0^+ \) and \( \varepsilon > 0 \) can be chosen arbitrarily, we finally get

\[
d_\infty(A) = 0.
\]

If we combine the above with the inequality \( 0 \leq d(A) \leq d_\infty(A) \), we have so far proved \( d(A) = d_\infty(A) \) for the case that some of these values is zero.

By Lemma 4.2(ii) we know that \( d(A) = d(B) \) for some \( B \subseteq A, B \in \mathcal{D} \). For this set we have \( d(A \setminus B) = 0 \). Using other parts of Lemma 4.2 we get that

\[
d(A) = d(B \cup A \setminus B) = d(B) + d(A \setminus B) = d(B) + d_\infty(A \setminus B) = d_\infty(A).
\]

\[\square\]

5. Extremal values of positive functionals extending Cesàro mean

Next we turn to study the set \( \hat{\mathcal{C}} \) of all positive functionals extending Cesàro mean. From Theorem 3.3 we already know that they correspond to density measures. Sometimes working with functionals instead of measures can be more convenient.

Again we want to find out the maximal and minimal possible value of such functionals for a given bounded sequence \( x \). We will obtain several expressions similar to the expressions obtained for densities in Theorem 4.1.

We will use the following sublinear functions defined on \( \ell_\infty \)

\[
f_C(x) = \sup\{f(x); f \in \hat{\mathcal{C}}\}
\]

\[
p(x) = \inf\{C(a); a \in \mathcal{C}; a \geq x\}
\]

\[
t(x) = \lim_{\theta \to 1^-} \limsup_{n \to \infty} \frac{\sum x_i \chi(\theta n, n \setminus i)}{n(1 - \theta)}
\]

\[
s(x) = \lim_{\alpha \to \infty} \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} x_i \alpha^\alpha}{N_\alpha(n)}
\]

Again, we should show the existence of the above limits. For the function \( s(x) \), see Remark 5.3. The existence of the limit used in the definition of \( t(x) \) is shown in Remark 5.12.

Since \( N_\alpha(n) \sim \frac{\alpha^{\alpha+1}}{\alpha+1} \) for \( \alpha > -1 \), we could use either of these two expressions in the definition of \( s(x) \).

**Remark 5.1.** Note that for \( A \subseteq \mathbb{N} \) we have \( f_C(\chi_A) = d(A) \), \( t(\chi_A) = d_F(A) \) and \( s(\chi_A) = d_\infty(A) \). The definitions of \( p(\chi_A) \) and \( d^*(A) \) are slightly different, but \( p(x) \) seems to be a natural counterpart of \( d^*(A) \) if we are working with arbitrary bounded sequences.

Now we are ready to formulate the main result of this section, which is the fact, that all four functions we have defined above are in fact equal. The rest of this section is devoted to the proof of this theorem.

**Theorem 5.2.** For any \( x \in \ell_\infty \) we have

\[
f_C(x) = p(x) = t(x) = s(x).
\]
Proof. We will show this result in a series of auxiliary results. Namely, we will show in Proposition 5.7 that \( f_C(x) = p(x) \), in Proposition 5.11 that \( s(x) = f_C(x) \) and in Proposition 5.13 that \( t(x) = f_C(x) \). □

We will introduce some notation which will be useful when proving some facts about these functions.

Let us denote
\[
T_{\theta,n}(x) = \frac{\sum x_i \chi(\theta n, n)(i)}{n(1 - \theta)} \\
T_{\theta}(x) = \limsup_{n \to \infty} T_{\theta,n}(x)
\]

When the sequence \( x \) will be clear from the context, we will just write \( T_{\theta,n} \) and \( T_{\theta} \) instead.

This means that \( t(x) = \lim_{\theta \to 1^-} T_{\theta}(x) \).

Similarly we denote
\[
S_{\alpha,n} = \frac{\sum_{i=1}^{n} x_i \alpha_i}{N_{\alpha}(n)} \\
S_{\alpha} = \limsup_{n \to \infty} S_{\alpha,n}
\]

With this notation we have \( s(x) = \lim_{\alpha \to \infty} S_{\alpha}(x) \).

Remark 5.3. Note that directly from Corollary 2.6 we have \( S_{\alpha} \leq S_{\beta} \leq 1 \) for \(-1 < \alpha < \beta\). This implies the existence of the limit in the definition of \( s(x) \).

Now we can formulate and prove the following lemma, which will be useful in several proofs.

Lemma 5.4. Let \( x \in [0, 1]^\mathbb{N} \). Then there exists \( \tilde{x} \in \{0, 1\}^\mathbb{N} \) such that \( \tilde{x} - x \in \mathcal{C} \) and
\[
C(\tilde{x} - x) = 0 \\
f_C(x) = f_C(\tilde{x}) \\
p(x) = p(\tilde{x}) \\
T_{\theta}(x) = T_{\theta}(\tilde{x}) \text{ and } t(x) = t(\tilde{x})
\]

Proof. Define \( \tilde{x}_n \) by
\[
\tilde{x}_1 + \cdots + \tilde{x}_n = \lfloor x_1 + \cdots + x_n \rfloor.
\]
Then we have
\[
\tilde{x}_1 + \cdots + \tilde{x}_n \leq x_1 + \cdots + x_n \leq \tilde{x}_1 + \cdots + \tilde{x}_n + 1,
\]
which implies \( C(\tilde{x} - x) = 0 \). From this we immediately get \( f_C(x) = f_C(\tilde{x}) \) and \( p(x) = p(\tilde{x}) \).

We also have
\[
\sum x_i \chi(\theta n, n)(i) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{[\theta n]} x_i,
\]
which implies \( |\sum x_i \chi(\theta n, n)(i) - \sum \tilde{x}_i \chi(\theta n, n)(i)| \leq 2 \) and, consequently, \( T_{\theta}(x) = T_{\theta}(\tilde{x}) \) and \( t(x) = t(\tilde{x}) \). □
5.1. Basic properties of $f_C(x)$. Let us mention some basic properties of the function $f_C(x)$. Proofs of these facts are easy, so they will be omitted.

**Lemma 5.5.**

\[(\forall x, y \in \ell_\infty) f_C(x + y) \leq f_C(x) + f_C(y)\]

\[(\forall x \in \ell_\infty)(\forall c > 0) f_C(cx) = cf_C(x)\]

\[(\forall x \in \ell_\infty) f_C(x + y) = f_C(x) + C(y)\]

\[(\forall x \in C) f_C(x) = C(x)\]

\[(\forall x, y \in \ell_\infty) x - y \in C \Rightarrow f_C(x - y) = 0 \Rightarrow f_C(x) = f_C(y)\]

After proving Theorem 5.2, it will be clear that the functions $p(x)$, $s(x)$ and $t(x)$ have the same properties. So we will formulate only those properties of functions $p(x)$, $s(x)$ and $t(x)$, which are needed in our proofs.

5.2. Function $p(x)$. We first show that $f_C(x) = p(x)$.

It is relatively easy to see that the function $p(x)$ defined above is sublinear and that

\[(5.1) \quad p(x + y) = p(x) + C(y)\]

whenever $y \in C$.

The following lemma shows how this function is related to the positive functionals extending Cesàro mean.

**Lemma 5.6.** Let $f: \ell_\infty \to \mathbb{R}$ be a linear function. Then $f \in \widehat{C}$ (i.e., $f$ extends Cesàro mean and it is positive) if and only if

\[(\forall x \in \ell_\infty) f(x) \leq p(x).\]

**Proof.** $\Leftarrow$ From the linearity of $f$ we get $f(x) = -f(-x) \geq -p(-x)$.

If $x \in C$ then $p(x) = C(x) = -C(-x) = -p(-x)$, thus $C(x) \leq f(x) \leq C(x)$.

If $x \geq 0$ then $f(x) \geq -p(-x) \geq 0$.

$\Rightarrow$ For $x \leq a$ and $a \in C$ we get $f(x) \leq f(a) = C(a)$, which implies $f(x) \leq p(x)$.

Using Hahn-Banach Theorem 2.2 we can show:

**Proposition 5.7.** For any $x \in \ell_\infty$ the equality

\[f_C(x) = p(x)\]

holds.

**Proof.** Fix $x \in \ell_\infty$.

We know that $f_C(x) \leq p(x)$, so it suffices to show that there exists an extension $f$ of $C$ such that $f(x) = p(x)$. Using Hahn-Banach Theorem 2.2 for $C: \mathcal{C} \to \mathbb{R}$ and $p: \ell_\infty \to \mathbb{R}$ we get, that there exists an extension fulfilling $f(x) = c$ for any choice $c \in [-p(-x), p(x)]$. (Note that the equality [5.1] helps us to simplify the range of all possible values of extensions of $C$.)
5.3. Function \( s(x) \). In the proof of the equality \( s(x) = f_C(x) \) we can use the fact, that we already know this in the case that \( x = \chi_A \) for some \( A \subseteq \mathbb{N} \).

**Lemma 5.8.** If \( x \in C \) then \( S_\alpha(x) = C(x) \) for every \( \alpha > 0 \) and, consequently, \( s(x) = C(x) \).

**Proof.** This follows from Corollary 2.6. \( \square \)

**Lemma 5.9.** For each \( x, y \in \ell_\infty \) and any \( \alpha > 0 \) we have
\[
S_\alpha(x + y) \leq S_\alpha(x) + S_\alpha(y)
\]
\[
s(x + y) \leq s(x) + s(y)
\]

**Proof.** Using Lemma 5.8 and Lemma 5.10 we get
\[
S_\alpha(x) = S_\alpha(y)
\]

**Lemma 5.10.** Let \( x, y \in \ell_\infty \). If \( x - y \in C \) and \( C(x - y) = 0 \), then we have for any \( \alpha > 0 \)
\[
S_\alpha(x) = S_\alpha(y)
\]

**Proof.** Using Lemma 5.9 and Lemma 5.8 we get
\[
S_\alpha(y) = S_\alpha(x + y - x) \leq S_\alpha(x) + S_\alpha(y - x) = S_\alpha(x) + C(y - x) = S_\alpha(x).
\]
Similarly we get \( S_\alpha(x) \leq S_\alpha(y) \). \( \square \)

**Proposition 5.11.** For any \( x \in \ell_\infty \) the equality
\[
s(x) = f_C(x)
\]
holds.

**Proof.** It suffices to show this for \( x \in [0,1]^\mathbb{N} \), since \( s(x + \overline{c}) = s(x) + c \) for any constant sequence \( \overline{c} \), \( s(cx) = cs(x) \) for any \( c > 0 \) and the same is true for the function \( f_C \).

If \( x \in [0,1]^\mathbb{N} \) then by Lemma 5.4 there is a sequence \( \tilde{x} \in \{0,1\}^\mathbb{N} \) such that \( C(x - \tilde{x}) = 0 \) and \( f_C(x) = f_C(\tilde{x}) \). From Theorem 4.1 we know that \( s(\tilde{x}) = f_C(\tilde{x}) \) (see Remark 5.1). From Lemma 5.10 we get
\[
s(x) = s(\tilde{x}) = f_C(\tilde{x}) = f_C(x).
\]

5.4. Function \( t(x) \). For the function \( t(x) \) we can again use that the equality is true for \( x \in \{0,1\}^\mathbb{N} \).

**Remark 5.12.** From [19] we know that the limit in the definition of \( t(x) \) exists for sequences from \( \{0,1\}^\mathbb{N} \). Using Lemma 5.4 we get the existence of this limit for any sequences in \( [0,1]^\mathbb{N} \).

It is relatively easy to show that \( T_0(cx) = cT_0(x) \) for \( c > 0 \) and \( T_0(x + \overline{c}) = c + T_0(x) \) for any constant \( c \). Once we know this, we get the existence of this limit for all bounded sequences.

In fact, the proof from [19] could be modified so that it works for any bounded sequence.

The following result shows the remaining equality between the functions we are studying.
Proposition 5.13. For any \( x \in \ell_\infty \) we have
\[
t(x) = f_C(x).
\]

Proof. Again, we know from Theorem 4.11 that \( t(\chi_A) = f_C(\chi_A) \) and using Lemma 5.4 we get the validity of \( t(x) = f_C(x) \) for each \( x \in [0, 1]^N \). Then we can continue in a similar way as in the proof of Proposition 5.11. \( \square \)

5.5. Sets for which all density measures have the same value. Using Theorem 5.2 we can answer the following natural question: How can be sequences such that the value \( f(x) \) is the same for each \( f \in \hat{C} \) characterized? It can be shown that only sequences having Cesàro mean have this property.

From this we also get answer to the analogous problem for density measures. In this case we get precisely the sets which have asymptotic density.

Corollary 5.14. A sequence \( x = (x_n)_{n=1}^\infty \) has the property that \( f(x) \) is equal to the same number \( L \) for each \( f \in \hat{C} \) if and only if \( x \in \mathcal{C} \) (and \( C(x) = L \)).

Proof. Clearly, if \( x \) has Cesàro mean \( C(x) = L \), then \( f(x) = L \) for each \( f \in \hat{C} \).

Now if \( f(x) = L \) for each \( f \in \hat{C} \), then we have \( p(x) = L = -p(-x) \), i.e.,
\[
\inf \{ C(a); a \in \mathcal{C}; a \geq x \} = \sup \{ C(b); b \in \mathcal{C}; b \leq x \} = L.
\]
From this we get for any \( \varepsilon > 0 \) that there exist \( a, b \in \mathcal{C} \) such that \( b \leq x \leq a \) and
\[
L - \varepsilon = C(b) \leq C(a) = L + \varepsilon.
\]
Then we get
\[
\frac{b_1 + \cdots + b_n}{n} \leq \frac{x_1 + \cdots + x_n}{n} \leq \frac{a_1 + \cdots + a_n}{n},
\]
\[
L - \varepsilon \leq \lim_{n \to \infty} \frac{b_1 + \cdots + b_n}{n} \leq \liminf_{n \to \infty} \frac{x_1 + \cdots + x_n}{n},
\]
\[
\limsup_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} \leq \lim_{n \to \infty} \frac{a_1 + \cdots + a_n}{n} = L + \varepsilon.
\]
Since this is true for arbitrary \( \varepsilon > 0 \), we get \( C(x) = \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} = L \).

Using the first part of this result for \( x = \chi_A \) we get the second part. \( \square \)

6. Extreme points

We have found the maximal value of \( f(x) \) for functionals \( f \in \hat{C} \). We will use Proposition 2.4 to obtain some sets of functionals such that their closed convex hull is \( \hat{C} \). This gives, in a sense, a description of the whole space \( \hat{C} \) using some simpler functionals. Namely we can obtain all functionals by taking the convex hull of this set and then the closure in the weak* topology. (Closure in the weak* topology means taking all limits of nets of functionals from this set, which converge pointwise.)

We could obtain some similar sets of functionals with this property from the results we have already shown so far, see Remark 6.6. The main purpose of this section is to obtain a smaller and simpler set which generates all functionals extending Cesàro mean.
In the proof of Theorem 6.1, which is one of the main results of this section, we will be working with some fixed sequence $x = (x_n)_{n=1}^\infty \in [0,1]^\mathbb{N}$. It suffices to show this result for sequences with values in the interval $[0,1]$, since the result can be easily extended to all bounded sequences by scaling and adding constant sequence.

We can use $T_{\theta,n}$ from the definition of the function $t(x)$ not only for integers, but for positive real numbers, too. For any real number $r > 0$ and $\theta \in (0,1)$ we define:

$$
\Sigma(\theta,r,r') = \sum_{i \in \mathbb{N}} x_i \chi_{(\theta r, r')}(i)
$$

$$
T_{\theta,r}(x) = \frac{\Sigma(\theta r,r)}{r(1-\theta)}
$$

$$
T_\theta(x) = \limsup_{r \to \infty} T_{\theta,r}(x)
$$

$$
t(x) = \lim_{\theta \to 1^-} T_\theta(x)
$$

We will often write just $\Sigma(\theta r,r)$, $T_{\theta,r}$ and $T_\theta$ instead.

The resulting function $t(x)$ will be the same in both cases, whether we use integers or real numbers. (We will need to use both possibilities.) To see that this is indeed the case, notice that if $|r - r'| < 1$ (e.g., if $r' = \lfloor r \rfloor$ or $r' = \lceil r \rceil$) then also $|\theta r - \theta r'| < 1$ and consequently

$$
|\Sigma(\theta r,r) - \Sigma(\theta r',r')| \leq 2.
$$

Hence

$$
\lim_{r \to \infty} \left( \frac{\Sigma(\theta r,r)}{r} - \frac{\Sigma(\theta r',r')}{r'} \right) = 0
$$

holds for any $\theta \in (0,1)$.

**Theorem 6.1.** For any sequence $\theta_k \in (0,1)$ such that $\theta_k \to 1^-$, any $x \in \ell_\infty$ and any free ultrafilter $\mathcal{F}$ we have

$$
t(x) = \sup_{G \in \beta \mathbb{N}} \mathcal{F}\text{-lim} \ G\text{-lim} n T_{\theta_k,n}(x).
$$

**Proof.** Fix a sequence $x = (x_k) \in [0,1]^\mathbb{N}$.

Choose any $L < H < t(x)$.

By the definition of $t(x)$ there exists an $\theta_0 \in (0,1)$ such that

$$
T_\theta = \limsup_{r \to \infty} T_{\theta,r} > H
$$

for each $\theta \geq \theta_0$.

This implies that the set

$$
\{n \in \mathbb{N}; T_{\theta_0,n} \geq H\}
$$

is infinite.

Inductively we can choose an infinite subset $A = \{n_1 < n_2 < \ldots\}$ such that $\frac{n_k}{n_{k+1}} < \theta_0$. We have

$$
n \in A \quad \Rightarrow \quad T_{\theta_0,n} \geq H.
$$

Using this set we can define a function $\varphi : [0,1) \to [0,1]$ by

$$
\varphi(\theta) = \liminf_{n \in A} T_{\theta,n}
$$

This function is continuous. (The proof is postponed to Lemma 6.2.)
Case 1. First we suppose that \((\forall \theta \in [\theta_0, 1))(\varphi(\theta) \geq H)\). This means that
\[
\lim \inf_{n \in A} T_{\theta,n} \geq H.
\]
For every such \(\theta\) and for any free ultrafilter \(\mathcal{G}\) containing the infinite set \(A\) we get
\[
\mathcal{G} \lim_{n} T_{\theta,n} \geq H \geq L.
\]

Case 2. If \(\varphi\) does not fulfill the above property, then there exists \(\theta' > \theta_0\) such that \(L < \varphi(\theta') < H\). Let us denote \(M = \varphi(\theta')\) and
\[
\theta'' = \inf\{\theta \in [\theta_0, 1); \varphi(\theta) \leq M\}.
\]
We have
\[
\varphi(\theta'') = M
\]
(by the continuity of \(\varphi\)) and the definition of \(M\) implies
\[
\varphi(\theta) > M
\]
whenever \(\theta_0 \leq \theta < \theta''\). (See Figure 1)
Since
\[
M = \varphi(\theta'') = \lim \inf_{n \in A} T_{\theta'',n},
\]
there exists an infinite set \(A' \subseteq A\) such that
\[
\lim_{n \in A'} T_{\theta'',n} = \lim_{n \in A'} \frac{\sum_{(\theta'',n,n)}}{n(1 - \theta'')} = M
\]
or equivalently
\[
\lim_{n \in A'} \frac{\sum_{(\theta'',n,n)}}{n} = M(1 - \theta'')
\]
For every $n \in A'$ and $\theta$ such that $\frac{\theta_0}{\theta_0'} \leq \theta < 1$ we get
\[
\Sigma_n(\theta''n, n) = \Sigma_n(\theta''n, \theta''n) + \Sigma_n(\theta''n, n),
\]
\[
\Sigma_n(\theta''n, n) = \Sigma_n(\theta''n, \theta''n) + \Sigma_n(\theta''n, n),
\]
(since $[\theta''n, n] = [\theta''n, \theta''n] \cup [\theta''n, n]$; see Figure 2.)

The condition $\theta_0 \leq \theta'' < \theta''$ implies $\varphi(\theta'') \geq M$, i.e.,
\[
\lim \inf_{n \in A'} T_{\theta''n, n} \geq \lim \inf_{n \in A'} T_{\theta''n, n} \geq M(1 - \theta'')
\]
From (6.1) we get
\[
\lim \inf_{n \in A'} \Sigma_n(\theta''n, \theta''n) = \Sigma_n(\theta''n, \theta''n) - \Sigma_n(\theta''n, n)
\]
and thus
\[
\lim \inf_{n \in A'} \Sigma_n(\theta''n, \theta''n) \geq M(1 - \theta'') - M(1 - \theta'') = M\theta''(1 - \theta)
\]

Now denote $B = \{\theta''a; a \in A'\}$.

(Note that $B \subseteq \mathbb{R}$.)

Since $B$ is precisely the set of all real numbers of the form $r = \theta''n$ for $n \in A'$, this is equivalent to
\[
\lim \inf_{r \in B} \frac{\Sigma_{\theta, r}}{r(1 - \theta)} \geq M,
\]
\[
\lim \inf_{r \in B} T_{\theta, r} \geq M.
\]

Now put $|B| = \{|b|; b \in B\}$.

For this set we have
\[
\lim \inf_{n \in [B]} T_{\theta, n} \geq M.
\]

(We have chosen the set $A$ in such way that for any two elements $m_1 < m_2$ in $A$ we have $\frac{m_1}{m_2} < \theta_0$. This implies that $\theta''m_1 < m_1 < \theta_0m_2 < \theta''m_2$. Hence for any two different elements $m_1, m_2 \in A'$, the corresponding elements $|\theta''m_1|$ and $|\theta''m_2|$ of $B$ will be different.)

This implies that for every free ultrafilter $G \ni [B]$ the inequalities
\[
G\text{-lim} T_{\theta, n} \geq M \geq L
\]
hold for each $\theta \in [\theta_0', 1]$.

Conclusion. In both cases we have shown that there exists a free ultrafilter $G$ such that
\[
G\text{-lim} T_{\theta, n} \geq L
\]
for each $\theta$ close enough to 1. Thus for any choice of $(\theta_k)_{k=1}^\infty$ such that $\theta_k \to 1^-$ and any free ultrafilter $F$

$$F\lim_k G\lim_n T_{\theta_k,n} \geq L.$$ 

Since $L$ can be chosen arbitrarily close to $t(x)$, we get

$$F\lim_k G\lim_n T_{\theta_k,n} \geq t(x).$$

\[\square\]

**Lemma 6.2.** The function $\varphi(\theta) = \liminf_{n \in A} T_{\theta,n}$ is continuous for any infinite set $A \subseteq \mathbb{N}$.

**Proof.** Note that we have for any $\theta \in (0, 1)$, $r > 0$ and $0 < \delta < 1 - \theta$.

$$\sum_{(\theta r, r)} = \sum_{(\theta + \delta, r)} + \sum_{(\theta, (\theta + \delta) r)}$$

$$T_{\theta,r} = \frac{\sum_{(\theta r, r)}}{r(1 - \theta)} = \frac{\sum_{(\theta + \delta, r)} + \sum_{(\theta, (\theta + \delta) r)}}{r(1 - \theta)}$$

$$T_{\theta,r} = T_{\theta + \delta,r} - \frac{\delta}{1 - \theta} T_{\theta + \delta,r} + \frac{\sum_{(\theta r, (\theta + \delta) r)}}{r(1 - \theta)}$$

From this we get

$$\left| \liminf_{n \in A} T_{\theta + \delta,n} - \liminf_{n \in A} T_{\theta,n} \right| \leq \frac{2\delta}{1 - \theta} \|x\|,$$

which implies the continuity of $\varphi$. \[\square\]

**Lemma 6.3.** Let $F, G \in \beta\mathbb{N}^*$, and $\theta_k \to 1^-$. Let us define $f : \ell_\infty \to \mathbb{R}$ by

$$f(x) = F\lim_k G\lim_n T_{\theta_k,n}(x).$$

Then $f \in \mathring{C}$, i.e., $f$ is a positive linear functional on $\ell_\infty$ which extends Cesàro mean.

**Proof.** It is clear that $f(x)$ defined above is a positive linear functional on $\ell_\infty$.

Now let $x \in \mathring{C}$. Then we have $t(x) = \lim_{\theta \to 1^-} \limsup_{n \to \infty} T_{\theta,n}(x) = C(x)$ and also $C(x) = -C(-x) = -t(-x) = \lim_{\theta \to 1^-} \liminf_{n \to \infty} T_{\theta,n}(x)$. For each $k$ we get

$$\liminf_{n \to \infty} T_{\theta_k,n}(x) \leq G\lim T_{\theta_k,n}(x) \leq \limsup_{n \to \infty} T_{\theta_k,n}(x)$$

which yields

$$C(x) \leq F\lim_k G\lim T_{\theta_k,n}(x) \leq C(x).$$

So we have $f(x) = C(x)$. \[\square\]
**Theorem 6.4.** Let \( \theta_k \in (0, 1) \) be a sequence such that \( \theta_k \to 1^- \) and \( \mathcal{F} \) be any free ultrafilter \( \mathcal{F} \). Let us define
\[
S = \{ f_G; G \in \beta N^* \}
\]
where \( f_G: \ell_\infty \to \mathbb{R} \) is defined by
\[
f_G(x) = \mathcal{F}\text{-}\lim_k \mathcal{G}\text{-}\lim_n T_{\theta_k,n}(x).
\]
We consider \( S \) as a subset of the space \( \ell_\infty^* \) endowed with the weak* topology.

Then
\[
\overline{\overline{S}} = \hat{C},
\]
i.e., the closed convex hull of \( S \) is the set of all positive functionals on \( \ell_\infty \) extending Cesàro mean. Or, equivalently, \( \overline{S} \) contains all extreme points of \( \hat{C} \).

**Proof.** The set \( \hat{C} \) is a subset of the unit ball of \( \ell_\infty^* \). It is closed in the weak* topology. Thus by Banach-Alaoglu theorem it is compact.

Lemma 6.3 implies that each \( \varphi \in S \) is indeed an element of \( \hat{C} \) and by Theorem 6.1 we have
\[
\sup_{\varphi \in S} \varphi(x) = \sup_{\varphi \in \hat{C}} \varphi(x).
\]
So from Proposition 2.4 we get that \( \overline{\overline{S}} = \hat{C} \) and \( S \) contains all extreme points of \( \hat{C} \). \( \square \)

Of course, we can work with finitely additive measures instead of the corresponding functionals. The weak* topology is then determined by the condition that a net \( \mu_\sigma \) of measures converges to \( \mu \) if and only if \( \mu_\sigma(A) \) converges to \( \mu(A) \) for each \( A \subseteq \mathbb{N} \).

**Corollary 6.5.** Let \( \theta_k \in (0, 1) \) be a sequence such that \( \theta_k \to 1^- \) and \( \mathcal{F} \) be any free ultrafilter \( \mathcal{F} \). Let us define
\[
S = \{ \mu_G; G \in \beta N^* \}
\]
where \( \mu_G \) is defined by
\[
\mu_G(A) = \mathcal{F}\text{-}\lim_k \mathcal{G}\text{-}\lim_n T_{\theta_k,n}(\chi_A).
\]

Then
\[
\overline{\overline{S}} = \hat{D},
\]
i.e., the closed convex hull of \( S \) is the set of all density measures. Or, equivalently, \( \overline{S} \) contains all extreme points of \( \hat{D} \).

**Remark 6.6.** It is easy to show that any functional of the form
\[
f(x) = \mathcal{F}\text{-}\lim_k \mathcal{G}\text{-}\lim_n T_{\theta_k,n}(x)
\]
for \( \mathcal{F}, G_k \in \beta N^* \) is also a positive functional extending Cesàro mean.

For any given \( x \) and \( (\theta_k) \) there exists a sequence \( G_k \) of free ultrafilters such that
\[
\limsup_{n \to \infty} T_{\theta_k,n} = \mathcal{G_k}\text{-}\lim_n T_{\theta_k,n}.
\]

From this we can immediately see that the set of all functionals of this form has the same properties as the set \( S \) from Theorem 6.4. But Theorem 6.4 is a stronger result than this, we do not need a sequence of ultrafilters, we have the same ultrafilter \( \mathcal{G} \) for each \( k \).
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