Equal sums of like polynomials

T.D. Browning

Mathematical Institute, 24–29 St. Giles', Oxford OX1 3LB
browning@maths.ox.ac.uk

Abstract
Let \( f \in \mathbb{Z}[x] \) be a polynomial of degree \( d \). We establish the paucity of non-trivial positive integer solutions to the equation
\[
f(x_1) + f(x_2) = f(x_3) + f(x_4),
\]
provided that \( d \geq 7 \). We also investigate the corresponding situation for equal sums of three like polynomials.

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1 Introduction

Let \( f \in \mathbb{Z}[x] \) be a polynomial of degree \( d \geq 3 \), and let \( B \geq 1 \). Then for any \( s \geq 2 \) we define \( M_s(f; B) \) to be the number of positive integers \( x_1, \ldots, x_{2^s} \leq B \) such that
\[
f(x_1) + \cdots + f(x_s) = f(x_{s+1}) + \cdots + f(x_{2^s}).
\]
(1)

It is conjectured that \( M_s(f; B) \) is dominated by the \( s! B^s \) trivial solutions, in which \( x_1, \ldots, x_s \) are a permutation of \( x_{s+1}, \ldots, x_{2^s} \). We therefore define \( M_s^{(0)}(f; B) \) to be the number of non-trivial solutions to (1). It is in the special case \( f_0(x) = x^d \) that this quantity has received the most attention. Thanks to the work of numerous authors it is well known that
\[
M_2^{(0)}(f_0; B) = o(B^2)
\]
for any \( d \geq 3 \). Moreover, recent joint work of the author with Heath-Brown \cite{1} has established the bound
\[
M_3^{(0)}(f_0; B) = o(B^3)
\]
for \( d \geq 33 \).

Returning to the more general quantity \( M_s^{(0)}(f; B) \), it is only in the case \( s = 2 \) and \( d = 3 \) that the paucity of non-trivial solutions has been established. Indeed, using quite elementary means Wooley \cite{7} has shown that
\[
M_2^{(0)}(f; B) = O_{\varepsilon,f}(B^{5/3+\varepsilon}),
\]
for any cubic polynomial \( f \) and any choice of \( \varepsilon > 0 \). The best that is known for polynomials of higher degree is the estimate \( M_2^{(0)}(f; B) = O_{\varepsilon,f}(B^{2+\varepsilon}) \), that follows from the trivial estimate \( d(n) = O_{\varepsilon}(n^\varepsilon) \) for the divisor function. Our first result rectifies this situation somewhat for polynomials of sufficiently large degree.
Theorem 1. Let $f \in \mathbb{Z}[x]$ be a polynomial of degree $d \geq 4$. Then we have

$$M_2(f; B) \ll_{\varepsilon, f} B^{1+\varepsilon} \left( B^{1/3} + B^{2/\sqrt{d+1}/(d-1)} \right),$$

for any $\varepsilon > 0$. In particular we have

$$M_2(f; B) = 2B^2(1 + o(1))$$

for $d \geq 7$, and

$$M_2^{(0)}(f; B) \ll_{\varepsilon, f} B^{4/3+\varepsilon}$$

for $d \geq 42$.

Apart from in the special case $f = f_0$ and $s = 3$, nobody has yet been able to establish the paucity of non-trivial solutions to (1) for any value of $s \geq 3$. The best available estimates arise from applications of Hua’s inequality [3]. In the case $s = 3$ this provides the estimate

$$M_3(f; B) = O_{\varepsilon, f}(B^{7/2+\varepsilon}),$$

provided that $f$ has degree $d \geq 3$. Our second result improves upon this estimate for polynomials of sufficiently large degree.

Theorem 2. Let $f \in \mathbb{Z}[x]$ be a polynomial of degree $d \geq 4$. Then we have

$$M_3^{(0)}(f; B) \ll_{\varepsilon, f} B^{3+\varepsilon} \left( B^{1/3} + B^{2/\sqrt{d+1}/(d-1)} \right),$$

for any $\varepsilon > 0$. In particular we have

$$M_3(f; B) = o(B^{7/2})$$

for $d \geq 20$.

It would be of considerable interest to increase the range of $d$ in Theorem 2’s estimate for $M_3(f; B)$, for such progress would have applications to smooth Weyl differencing as employed by Vaughan and Wooley [6] to obtain further improvements in Waring’s problem. In order to facilitate future investigations, it will be convenient to state the following hypothesis for given $\delta \in \mathbb{N}$ and $\theta_3 \in \mathbb{R}$.

Hypothesis $[\delta, \theta_3]$. Let $S \subset \mathbb{A}^3$ be a non-singular affine surface of degree $\delta \geq 2$, and let $S_\delta \subseteq S$ denote the subset formed by deleting all of the curves of degree at most $\delta - 2$ from $S$, that are defined over $\mathbb{Q}$. Then we have

$$\#\{(x_1, x_2, x_3) \in S_\delta \cap \mathbb{Z}^3 : \max\{|x_1|, |x_2|, |x_3|\} \leq B\} \ll_{\varepsilon, \delta} B^{\theta_3+\varepsilon}.$$ 

It is worth underlining that the implied constant in this estimate is assumed to be independent of the coefficients of the polynomial defining $S$. It follows from an easy induction argument that Hypothesis $[\delta, 2]$ always holds, although we can actually do rather better than this. Indeed one can apply a result of Heath-Brown [2, Theorem 14] along much the same lines as in the proof of [2, Equation (1.15)], to deduce that Hypothesis $[\delta, 2/\sqrt{\delta + 1}/(\delta - 1)]$ holds. In order to prove Theorems 1 and 2 it will therefore suffice to establish the following result.
Theorem 3. Assume that Hypothesis $[d, \theta_d]$ holds, and let $s \geq 2$. Then for any polynomial $f \in \mathbb{Z}[x]$ of degree $d \geq 4$, we have

$$M_s^{(0)}(f; B) \ll_{\varepsilon, f} B^{2^{3\varepsilon} + \varepsilon} \left(B^{1/3} + B^{\theta_d}\right).$$

It should be highlighted that Theorem 3 is only interesting for $s = 2$ and $s = 3$, since for larger values of $s$ it is beaten by Hua’s inequality. We end this section by discussing the value of the implied constant in Theorem 3. As it stands the constant is clearly allowed to depend upon $f$ in some way. Suppose once and for all that $f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d$, (2) with $a_0, \ldots, a_d \in \mathbb{Z}$ and $a_0 > 0$. Then it is in fact possible to prove that the implied constant is independent of the coefficients of $f$ whenever $a_1 = 0$, and that it depends at most upon $d$ and the choice of $\varepsilon > 0$. We shall content ourselves with merely indicating at which points of the argument this sort of finer inequality can be retrieved.

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2 Proof of Theorem 3

Suppose that $f$ is given by (2) and that $d \geq 4$. We begin the proof of Theorem 3 by noticing that

$$a_0^{d-1} d f(x) = (a_0 dx + a_1)^d + b_2(a_0 dx + a_1)^{d-2} + \cdots + b_{d-1}(a_0 dx + a_1) + b_d,$$

for appropriate $b_i \in \mathbb{Z}$ depending upon $a_i$ and $d$. Upon making the change of variables $y_i = a_0 dx_i + a_1$ for $1 \leq i \leq 2s$, we therefore obtain the equation

$$g(y_1) + \cdots + g(y_s) = g(y_{s+1}) + \cdots + g(y_{2s})$$

from (1), where

$$g(y) = y^d + b_2 y^{d-2} + \cdots + b_{d-1} y.$$  

(4)

This transformation has the effect of taking positive integer points not exceeding $B$ to positive integer points not exceeding $O_f(B)$. We stress that this is the main point of the argument at which a necessary dependence upon the coefficients of $f$ appears. Such a transformation is in fact unnecessary if $a_1 = 0$ in (2). We have therefore shown that

$$M_s^{(0)}(f; B) \leq M_s^{(0)}(g; cB),$$

for some constant $c > 0$ depending only upon $f$.

During the course of our argument, it will be necessary to handle the contribution from certain “almost trivial” solutions to (3) separately. Let $S_s(B)$ denote the contribution to $M_s^{(0)}(g; cB)$ from those $y_1, \ldots, y_{2s}$ for which

$$\{y_1, \ldots, y_s\} \cap \{y_{s+1}, \ldots, y_{2s}\} = \emptyset,$$
and let $T_s(B)$ denote the remaining contribution. It follows that

$$M_s^{(0)}(g; cB) = S_s(B) + T_s(B).$$

(6)

Moreover, whenever the vector $(y_1, \ldots, y_{2s})$ is counted by $T_s(B)$, we must have $y_i = y_j$ for some $1 \leq i \leq s < j \leq 2s$.

In order to estimate $S_s(B)$ and $T_s(B)$ we shall employ the following result due to Pila [4, Theorem A].

**Lemma 1.** Let $C \subset \mathbb{A}^3$ be an absolutely irreducible affine curve of degree $\delta$. Then we have

$$\#\{(x_1, x_2, x_3) \in C \cap \mathbb{Z}^3 : \max\{|x_1|, |x_2|, |x_3|\} \leq B\} \ll_{\varepsilon, \delta} B^{1/\delta + \varepsilon}.$$

As in the statement of Hypothesis $[\delta, \theta]$ the implied constant in Lemma 1 is understood to be independent of the coefficients of the polynomials defining $C$.

### 2.1 Estimating $S_s(B)$

In this section we provide an estimate for $S_s(B)$. This constitutes the main part of our argument. The idea will simply be to count points on the affine surfaces obtained by fixing values of $y_4, \ldots, y_{2s}$ in (3). Let

$$\epsilon_i = \begin{cases} -1, & i \leq s, \\ +1, & i > s, \end{cases}$$

and write $N$ for the set of vectors $n = (n_4, \ldots, n_{2s}) \in (\mathbb{N} \cap [1, cB])^{2s-3}$. For any $n \in N$ we define the surface

$$\Gamma_n : g(y_1) + g(y_2) - \epsilon_3 g(y_3) = \sum_{i=4}^{2s} \epsilon_i g(n_i).$$

Let $N_1$ be the set of $n \in N$ for which $\Gamma_n$ is singular, and let $N_2 = N \setminus N_1$. Clearly $\Gamma_n$ is non-singular, and so absolutely irreducible, for $n \in N_2$. Our first task is to establish the following result, which ensures that the same is true for $n \in N_1$.

**Lemma 2.** The surface $\Gamma_n$ is absolutely irreducible for any $n \in N_1$, and we have $\#N_1 = O_f(B^{2s-4})$.

**Proof.** Suppose that $(\xi_1, \xi_2, \xi_3)$ is a singular point of the surface $\Gamma_n$, for any $n \in N_1$. Then it follows that $\frac{dg}{dx}$ vanishes at $\xi_i$ for $1 \leq i \leq 3$, and that

$$g(\xi_1) + g(\xi_2) - \epsilon_3 g(\xi_3) = \sum_{i=4}^{2s} \epsilon_i g(n_i).$$

(7)

Since $\frac{dg}{dx}$ is a polynomial of degree $d-1$, it follows that there are at most $(d-1)^3$ possible singular points $(\xi_1, \xi_2, \xi_3) \in C^3$ on the surface $\Gamma_n$. This establishes that $\Gamma_n$ is absolutely irreducible. Indeed, if we had a non-trivial decomposition of the form

$$\Gamma_n = \Gamma_1 \cup \Gamma_2 \subset \mathbb{A}^3,$$
then $\Gamma_1$ and $\Gamma_2$ would necessarily intersect in a variety of dimension at least 1. Since every point of this set would produce a singular point in the surface $\Gamma_n$, this would contradict the fact that it has finite singular locus.

It therefore remains to count the number of $n \in \mathcal{N}$ for which (4) holds, for $O_d(1)$ values of $(\xi_1, \xi_2, \xi_3)$. But if we fix a choice of $(\xi_1, \xi_2, \xi_3)$ and $(n_5, \ldots, n_{2s})$, then there can clearly only be $O_d(1)$ values of $n_4$ such that (7) holds. Hence it follows that there are $O_f(B^{2s-4})$ values of $n \in \mathcal{N}$ such that $\Gamma_n$ has singularity $(\xi_1, \xi_2, \xi_3)$. This suffices to establish the second part of the lemma.

Let $S_s(B; n)$ denote the number of positive integers $y_1, y_2, y_3 \ll_{s} B$ that lie on the surface $\Gamma_n$, with the constraint that $y_3 \notin \{y_1, y_2\}$ whenever $s = 2$. Then in order to estimate $S_s(B; n)$, it will suffice to estimate $S_s(B; n)$ for each $n \in \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, since we clearly have

$$S_s(B) \leq \sum_{n \in \mathcal{N}_1} S_s(B; n) + \sum_{n \in \mathcal{N}_2} S_s(B; n).$$

In estimating $S_s(B; n)$ it will prove necessary to pay special attention to the points lying on curves of low degree contained in $\Gamma_n$.

**Lemma 3.** For any $n \in \mathcal{N}$, there is no contribution to $S_s(B; n)$ from any lines or conics contained in $\Gamma_n$ that are defined over $\mathbb{Q}$.

**Proof.** We begin by considering the possibility that $\Gamma_n$ contains a line defined over $\mathbb{Q}$, and we write $c_n = \sum_{i=4}^{2s} \epsilon_i n_i$ for convenience. Thus there exist $\lambda_i, \mu_i \in \mathbb{Q}$ such that the polynomial

$$g(\lambda_1 t + \mu_1) + g(\lambda_2 t + \mu_2) - \epsilon_3 g(\lambda_3 t + \mu_3) - c_n$$

vanishes identically. We may clearly assume that at most one $\lambda_i$ is zero, since otherwise it is easy to see that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Suppose first that $\lambda_1 \neq 0$. Then after a possible change of variables we may assume that $\lambda_1 = 1$ and $\mu_1 = 0$. Upon recalling the shape (4) that $g$ takes, it therefore follows that

$$1 + \lambda_2^d = \epsilon_3 \lambda_3^d.$$ 

Wiles’ proof [5] Theorem 0.5] of Fermat’s Last Theorem shows that $\lambda_2 \lambda_3 = 0$. If $\lambda_3 = 0$, then $d$ must be odd and $\lambda_2 = -1$. We must now consider the possibility that we have an identity of the shape $g(t) - g(-t + \mu_2) = k_n$, for some constant $k_n$. Upon examining the coefficient of $t^{d-1}$, we conclude from (4) that $\mu_2 = 0$. In terms of the original coordinates we have shown that this case produces the affine line $y_1 = -y_2, y_3 = \mu_3$ provided that $d$ is odd. Although this line may be contained in $\Gamma_n$ for certain choices of $g$, such solutions actually contribute nothing to $S_s(B; n)$ since we are only interested in positive integer points on $\Gamma_n$. Next we suppose that $\lambda_2 = 0$ and $\lambda_3 = \epsilon_3$. If $\epsilon_3 = -1$ then the previous argument can be repeated to yield the line $y_1 = -y_3, y_2 = \mu_2$, which shows that this case also contributes nothing to $S_s(B; n)$. If $\epsilon_3 = 1$ however, so that $s = 2$, then either $\lambda_3 = -1$ and $d$ is even, or else $\lambda_3 = 1$. In the former case we obtain the line $y_1 = -y_3, y_2 = \mu_2$, and in the latter case we obtain the line $y_1 = y_3, y_2 = \mu_2$. Neither of these cases contribute anything to $S_2(B; n)$, since we must have $y_3 \notin \{y_1, y_2\}$ whenever $s = 2$. Upon treating the case corresponding to $\lambda_1 = 0$ in a similar fashion, one is led to the pair of lines $y_1 = \mu_1, y_2 = \pm y_3$. Neither of these contribute anything to $S_s(B; n)$. 

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Next we suppose that $\Gamma_n$ contains a conic defined over $\mathbb{Q}$. Thus there exist $\kappa_i, \lambda_i, \mu_i \in \mathbb{Q}$ such that the polynomial
\[
g(\kappa_1 t^2 + \lambda_1 t + \mu_1) + g(\kappa_2 t^2 + \lambda_2 t + \mu_2) - \epsilon_3 g(\kappa_3 t^2 + \lambda_3 t + \mu_3) - c_n
\]
vanishes identically. Suppose that $\kappa_1 \neq 0$, say, and let
\[
h(y) = \kappa_1^d g(y/\kappa_1) = y^d + b_2 \kappa_1^2 y^{d-2} + \cdots + b_{d-1} \kappa_1^{d-1} y.
\]
Then after a possible relabling of variables it suffices to consider the vanishing of the polynomial
\[
h(t^2 + \mu_1) + h(\kappa_2 t^2 + \lambda_2 t + \mu_2) - \epsilon_3 h(\kappa_3 t^2 + \lambda_3 t + \mu_3) - \kappa_1^d c_n.
\]
Upon examining the coefficient of the leading monomial $t^2$, we deduce that
\[
1 + \kappa_2^d = \epsilon_3 \kappa_3^d,
\]
and so $\kappa_2 \kappa_3 = 0$. If $\kappa_3 = 0$ then $d$ must be odd and $\kappa_2 = -1$. Using the fact that the coefficient of $t^{2d-1}$ must also vanish, we further deduce that $\lambda_2 = 0$. Similarly, the coefficient of $t^{2d-2}$ is equal to $d(\mu_1 + \mu_2)$ since $d$ is odd, from which it follows that $\mu_2 = -\mu_1$. Again appealing to the fact that $d$ is odd, we finally deduce that $\lambda_3 = 0$ by considering the coefficient of $t^d$. In terms of the original coordinates we therefore have
\[
y_1 = t^2 + \mu_1 = -y_2, \quad y_3 = \mu_3,
\]
and it is clear that such solutions contribute nothing to $S_s(B; n)$. Alternatively, if $\kappa_2 = 0$ then $\kappa_3^d = \epsilon_3$. Arguing as above it suffices to assume that $\epsilon_3 = 1$, and so that $s = 2$. But then the same analysis ultimately leads to solutions of the form
\[
y_1 = t^2 + \mu_1 = y_3, \quad y_2 = \mu_2,
\]
which are not permissible. The case $\kappa_1 = 0, \kappa_2 \neq 0$ is handled similarly. This completes the proof of Lemma 3.

The remainder of this section is taken up with establishing the following result.

**Lemma 4.** Assume that Hypothesis $[d, \theta_d]$ holds. Then we have
\[
S_s(B; n) \ll_{\epsilon, f} \begin{cases} 
B^{2/\sqrt{3}+1/3+\epsilon}, & n \in N_1, \\
B^{1/3+\epsilon} + B^{\theta_d+\epsilon}, & n \in N_2.
\end{cases}
\]

**Proof.** The proof of Lemma 4 will hinge upon work of Heath-Brown [2] Theorem 14. Let $\|\Gamma_n\|$ denote the maximum modulus of the coefficients of the polynomial defining $\Gamma_n$, so that in particular $\log \|\Gamma_n\| = O_f(\log B)$ for any $n \in N_1 \cup N_2$. Now let $(y_1, y_2, y_3)$ be any point counted by $S_s(B; n)$, and note that $\Gamma_n$ is absolutely irreducible for $n \in N_1 \cup N_2$ by Lemma 2. Therefore an application of [2] Theorem 14] implies that $(y_1, y_2, y_3)$ lies on one of at most
\[
\ll_{\epsilon, f} B^{2/\sqrt{3}+\epsilon} \|\Gamma_n\| \ll_{\epsilon, f} B^{2/\sqrt{3}+\epsilon}
\]
proper subvarieties of $\Gamma_n$, each of degree $O_{\varepsilon,d}(1)$. We remark that it is actually possible to make the assumption $\log \|\Gamma_n\| = O_{\varepsilon,d}(\log B)$ at this stage, by employing the argument of \cite[Theorem 4]{2}. When $a_1 = 0$ in \cite{2}, this would lead to the uniformity result mentioned after the statement of Theorem \cite{3}.

It remains to estimate the number of points of bounded height lying on $O_{\varepsilon,f}(B^{2/\sqrt{d}+1/3+\varepsilon})$ absolutely irreducible curves of degree $O_{\varepsilon,d}(1)$ that are contained in $\Gamma_n$. For this we use Lemma \ref{lem1}. Suppose first that $n \in N_1$. Then Lemma \ref{lem3} implies that we may ignore points lying on any curves of degree at most 2 that are defined over $\mathbb{Q}$. Any curve of degree at most 2 that is not defined over $\mathbb{Q}$ clearly contains only $O(1)$ points. Hence it follows from Lemma \ref{lem1} that

$$S_s(B; n) \ll_{\varepsilon,f} \frac{B}{3} + \frac{B^{1/3}}{3} + \varepsilon,$$

whenever $n \in N_1$.

Suppose now that $n \in N_2$. Then on the assumption that Hypothesis $[d, \theta_d]$ holds, we obtain the overall contribution $O_{\varepsilon,f}(B^{1/3+\varepsilon})$ from points not contained on any curve of degree at most $d-2$ that is defined over $\mathbb{Q}$. It remains to consider the contribution to $S_s(B; n)$ from the curves of degree at most $d-2$, that are defined over $\mathbb{Q}$ and are contained in $\Gamma_n$. Since $\Gamma_n$ is non-singular we may apply a result of Colliot-Thélène \cite[Appendix]{2}. We conclude that $\Gamma_n$ contains $O_d(1)$ curves of degree $\leq d-2$. Lemma \ref{lem3} implies that we may ignore points lying on those curves of degree at most 2. Hence Lemma \ref{lem1} yields the overall contribution $O_{\varepsilon,f}(B^{1/3+\varepsilon})$ from the curves of degree at most $d-2$ contained in $\Gamma_n$. This completes the proof of Lemma \ref{lem4}.

Recall the estimate in Lemma \ref{lem2} for $\# N_1$, and note that $\# N_2 = O_f(B^{2s-3})$. Then we may combine Lemma \ref{lem3} and \ref{lem4} to deduce that

$$S_s(B) \ll_{\varepsilon,f} B^{2s-3+\varepsilon} \left( B^{1/3} + B^{\theta_d} \right),$$

since $d \geq 4$.

\subsection*{2.2 Estimating $T_s(B)$}

In this section we shall study the quantity $T_s(B)$. Under the assumption that Hypothesis $[d, \theta_d]$ holds, our aim is to establish the inequality

$$T_s(B) \ll_{\varepsilon,f} B^{2s-3+\varepsilon} \left( B^{1/3} + B^{\theta_d} \right),$$

for any $s \geq 2$. We shall argue by induction on $s$.

In order to handle the base case $s = 2$, it will suffice to estimate the contribution to $T_2(B)$ from those $y_1, y_2, y_3, y_4$ for which $y_1 = y_3$, say. There are then $O_f(B)$ choices for $y_1, y_3$, and it remains to count the number of positive integers $y_2, y_4 \ll_f B$ such that $y_2 \neq y_4$ and

$$g(y_2) = g(y_4).$$

This equation defines a curve of degree $d$ in $\mathbb{A}^2$. We claim that those points lying on curves of degree at most 2, that form components of \eqref{11} and are defined over $\mathbb{Q}$, contribute nothing to $T_2(B)$. This is established along exactly the same lines as the proof of Lemma \ref{lem3} and so we will be brief. Suppose
first that there exist \( \lambda, \mu, \lambda', \mu' \in \mathbb{Q} \) such that \( g(\lambda t + \mu) = g(\lambda' t + \mu') \) vanishes identically. After a possible change of variables we may assume without loss of generality that \( \lambda' = 1 \) and \( \mu' = 0 \). By equating coefficients it therefore follows from (11) that \( \lambda = \pm 1 \) and \( \mu = 0 \). Neither of these cases contribute anything to \( T_2(B) \). The case in which (11) contains a conic defined over \( \mathbb{Q} \) is despatched in precisely the same way. Thus it remains to estimate the contribution from the remaining absolutely irreducible components of (11). An application of Lemma 1 therefore yields the overall contribution \( O_{\varepsilon,f}(B^{1/3+\varepsilon}) \) to \( T_2(B) \). This establishes that \( T_2(B) = O_{\varepsilon,f}(B^{4/3+\varepsilon}) \), which is satisfactory for (10).

Suppose now that \( s > 2 \). But then it is trivial to see that we have

\[
T_s(B) \ll \sum_{y \leq f B} \left( S_{s-1}(B) + T_{s-1}(B) \right).
\]

Applying the induction hypothesis, in conjunction with (9), therefore yields

\[
T_s(B) \ll_{\varepsilon,f} \sum_{y \leq f B} B^{2s-5+\varepsilon} \left( B^{1/3} + B^{\theta_d} \right) \ll_{\varepsilon,f} B^{2s-3+\varepsilon} \left( B^{1/3} + B^{\theta_d} \right).
\]

This completes the proof of (10).

### 2.3 Completion of the proof

Assume that Hypothesis \([d, \theta_d]\) holds. Then it remains to combine (9) and (10) in (5) and (6), to conclude that

\[
M_s^{(0)}(f; B) \leq S_s(B) + T_s(B) \ll_{\varepsilon,f} B^{2s-3+\varepsilon} \left( B^{1/3} + B^{\theta_d} \right).
\]

This completes the proof of Theorem 3.

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