Maxima of Weibull–like distributions and the Lambert W function

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Abstract. The Weibull–like distributions form a large class of probability distributions that belong to the domain of attraction for the maxima of the Gumbel law. Besides the Weibull distribution, it includes important distributions as the Gamma laws and, in particular, the $\chi^2$ distributions. In order to have explicit expressions of the norming constants for the maxima it is necessary to solve asymptotically a nonlinear equation; however, for some members of that family, numerical and simulation studies show that the constants that are usually suggested are inaccurate for moderate or even large sample sizes. In this paper we propose other norming constants computed with the asymptotics of the Lambert W function that significantly improve the accuracy of the approximation to the Gumbel law. These results are applied to the computation of the constants for the maxima of Gamma random variables that appear in some applied problems.

Keywords: Weibull–like distributions, gamma distributions, Extreme value theory, Lambert function

AMS classification: 60G70, 60F05, 62G32, 41A60

1 Introduction

The origin of this paper was when the authors tried to use Extreme Value Theory to the maxima of $\chi^2$ random variables in an applied problem of signal processing (see Turunen [12]). In particular, the goal was to accurately characterize the detection performance of a Global Navigation Satellite System (GNSS) receiver, whose main task is to provide positioning information by processing the signals emitted from Earth–orbiting satellites (see Seco–Granados et al. [11]). To do so, a GNSS receiver must first detect the presence of visible satellites, which is done by analyzing the signal that impinges onto the GNSS receiver antenna. This analysis requires a bi–dimensional search in order to determine the carrier frequency and the time–delay for each of the satellite signals of interest, in a similar manner to what occurs when tuning a radio into a specific radio station and a specific program, respectively. For each of the tentative carrier frequency and time-delay values, the GNSS receiver measures the received signal power, which can be modeled as a $\chi^2(m)$ random variable, and stores the resulting power measurement into a specific cell of a time-frequency matrix (see Seco–Granados et al. [11] Eq. (9)). When all tentative values have been tested, the GNSS receiver takes the maximum of all the entries within the time-frequency matrix. This leads to the so–called “parallel acquisition” approach, and the resulting maximum value is then compared to a threshold in order to determine whether the satellite being analyzed was actually present or not (see Seco–Granados et al. [11] Eq. (11)).

In practice, this parallel acquisition approach typically involves computing the maximum of $10$ to $10^6 \chi^2(m)$ random variables, for $m$ between $10$ and $20$, depending on whether assistance information is provided or not to the GNSS receiver. In this context, it is interesting to note that $\chi^2$ random variables are in the domain of attraction for maxima of the Gumbel law, however, the norming constants that are usually proposed give, for such sample sizes, inaccurate results (see Subsection 6.3). Then we realized that the computations needed to obtain these constants were related with the Lambert W function (see Corless et al. [2]), and that the asymptotic expansion for that function helps very much to improve the norming constants. As Resnick [8, Page 67] points out, Computing normalizing constants can be a brutal business, and any techniques which aid in this are welcome indeed. The purpose of this paper is to show how the asymptotics of the Lambert W function and its generalizations can be used in this problem.

We show that the centering constant for the maxima of $n$ i.i.d. random variables with distribution called Weibull–like (see Section 2) can be expressed in terms of the secondary branch of a real Lambert W function; the

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asymptotic expansion of that function is well known, and the centering constant that is deduced using standard methods of asymptotic analysis corresponds to the two dominant terms of that expansion, loosely speaking, of the form \( C_1 \log n + C_2 \log \log n \). However, for typical sample sizes the results are quite inaccurate, and we propose to add one more term of that asymptotic expansion, basically of the form \( \log \log n/\log n \); this term goes to zero when \( n \to \infty \), but so slowly that cannot be neglected. We pay special attention to the maxima of Gamma laws. In that case, we need a double enhancement of the standard technique: on the one hand, the usual distribution tail equivalent to a Gamma distribution needs to be improved; on the other hand, using the asymptotic expansion of a generalization of Lambert W function, we add an additional term, that, as before, goes to zero when the sample size increases, but also helps very much to get accurate approximations.

The contents of the paper are the following. In Section 2 we introduce the class of generalized Weibull distributions and its tail-equivalent distributions, called Weibull-like distributions; we also recall some essential facts about Extreme Value Theory. In Section 3 we comment the main results of the paper. In Section 4 we study a particular simple case of a generalized Weibull distribution and we describe the problem of the inaccuracy of the norming constants; further we introduce the Lambert W function and its asymptotics. In Section 5 we study the velocity of convergence to the maxima and show the importance of the election of the norming constants. In Section 6 we apply these results to the maxima of Gamma laws. Some technical matters are placed in the Appendix.

2 Weibull–like distributions and their maxima

To introduce notation and to describe the context of the paper we recall a few basic facts from Extreme Value Theory. Let \( X_1, \ldots, X_n \) be i.i.d. random variables with common cumulative distribution function \( F \), and denote by \( M_n \) its maximum,

\[
M_n = \max\{X_1, \ldots, X_n\}. 
\]

It is said that \( F \) is in the domain of attraction for maxima of the Gumbel law if there are sequences of real numbers \( \{a_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) (the norming—or normalizing—constants) with \( a_n > 0 \) such that

\[
\lim_{n} \frac{1}{a_n} (M_n - b_n) = H, \quad \text{in distribution,} 
\]

where \( H \) is a Gumbel random variable, with distribution function

\[
\Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}. 
\]

The norming constants can be taken (see, for example, Resnick [8 Proposition 1.11])

\[
b_n = F^{-1}(1 - n^{-1}) 
\]

and

\[
a_n = A(b_n) 
\]

where \( A(x) \) is an auxiliary function of the distribution function \( F \). Auxiliary functions are not unique though they are asymptotically equal. However, under certain conditions (in particular, \( F \) should have density, denoted by \( f \), for \( x > x_0 \), for some \( x_0 \)) an auxiliary function is (see again Resnick [8 Proposition 1.11])

\[
A(x) = \frac{1 - F(x)}{f(x)}. 
\]

We should remark that from the standard proof of the convergence (1) it is not deduced that these constants produce more accurate results than other constants computed with other auxiliary functions or other ways.

In order to obtain explicit expressions of \( b_n \) and \( a_n \) the following two results are used: the first one can be called simplification by tail equivalence (see Resnick [8 Proposition 1.19]):
Property 2.1. Let $F$ be a distribution function in the domain of attraction of a Gumbel law, and let $G$ be another distribution function right tail equivalent to $F$:

$$
\lim_{x \to \infty} \frac{1 - G(x)}{1 - F(x)} = 1.
$$

Then $G$ is also in the domain of attraction of the Gumbel law and the norming constants of $F$ and $G$ can be taken equal.

The second property is just a property of the convergence in law applied to our context.

Property 2.2. Let $F$ be a distribution function that belongs to the domain of attraction for maxima of the Gumbel law, with norming constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. If the sequences $\{a'_n, n \geq 1\}$ and $\{b'_n, n \geq 1\}$ satisfy

$$
\lim_{n} \frac{a_n}{a'_n} = 1 \quad \text{and} \quad \lim_{n} \frac{b_n - b'_n}{a_n} = 0,
$$

then

$$
\lim_{n} \frac{1}{a'_n} (M_n - b'_n) = H \text{ in distribution},
$$

that is, the sequences $\{a'_n, n \geq 1\}$ and $\{b'_n, n \geq 1\}$ are also norming constants for $F$.

We will study a rich family of distribution functions. To begin with, in this paper, a probability distribution function $F$ such that for some $x_0$ has the form

$$
F(x) = 1 - K x^a \exp\{-C x^\tau\}, \quad x \geq x_0,
$$

where $K$, $C$, $\tau > 0$, and $a \in \mathbb{R}$, will be called a generalized Weibull distribution; the standard Weibull law $W(\lambda, \nu)$, where $\lambda > 0$ is the scale parameter and $\nu > 0$ the shape parameter, is the case $\alpha = 0$, $\tau = \nu$, $C = 1/\lambda^\nu$, $K = 1$ and $x_0 = 0$; in particular an exponential law of parameter $\lambda > 0$ has $\alpha = 0$, $\tau = 1$ and $C = \lambda$, and a $\chi^2(2)$ law has $\alpha = 0$, $\tau = 1$, $C = 1/2$. It is easy to check that generalized Weibull distributions belong to the domain of attraction for maxima to the Gumbel law.

In agreement with Embrechts et al. [4] page 155], a probability distribution function right tail equivalent to a generalized Weibull distribution is said to be a Weibull–like distribution. A main example is the Gamma law $G(\nu, \theta)$ (with $\alpha = \nu - 1$, $\tau = 1$), and, in particular, a $\chi^2(m)$ law, see Section 6. The normal law also is Weibull–like with $\alpha = -1$ and $\tau = 2$; however, this case has special properties: on the one hand, there is the remarkable result of Hall [5] where he proves that for some norming constants $a_n^*$ and $b_n^*$,

$$
\frac{C_1}{\log n} < \sup_{x \in \mathbb{R}} |\Phi^{*}(a_n^* x + b_n^*) - \Lambda(x)| < \frac{C_2}{\log n},
$$

($\Phi$ is the cumulative distribution function of the standard normal law) where $C_2$ can be taken equal to 3, and that the rate of convergence cannot be improved by choosing a different sequence of norming constants; on the other hand, the fact that $\alpha < 0$ introduces important changes in our approach; for some improvements on the norming constants for the normal case, see Gasull et al. [6]. Given that we are mainly interested in the Gamma law we will assume from now on that $\alpha > 0$ and $\tau \geq 1$.

Consider $X_1, \ldots, X_n$ be i.i.d. random variables with Weibull–like distribution function $G$, right tail equivalent to a generalized Weibull distribution function $F$ of the form (6). Thanks to Property 2.1, the norming constants can be taken

$$
b_n = F^{-1}(1 - n^{-1})
$$

$$
a_n = \frac{1}{C \tau b_n^{\tau - 1} - \alpha/b_n}
$$

(7)
where for the expression for $a_n$ we have used the auxiliary function (5) associated to $F$. From that expressions, by using Property 2.1 and asymptotic analysis, it is possible to find explicit expressions of the norming constants, and usually are suggested, see, for example, Embrechts et al. [4, page 155],

\[
\begin{align*}
    b_n' &= (C^{-1} \log n)^{1/\tau} + \frac{1}{\tau} (C^{-1} \log n)^{1/\tau - 1} \left( \frac{\alpha}{C\tau} \log (C^{-1} \log n) + \frac{1}{C} \log K \right) \\
    a_n' &= (C\tau)^{-1} (C^{-1} \log n)^{1/\tau - 1}
\end{align*}
\] (8)

These will be called the \textit{standard constants}.

3 \hspace{1em} \textbf{Main results}

Our purpose is to show that, for moderate or even quite large sample sizes, the election of the norming constants plays a major role in the velocity of convergence of (1), and that it is possible to choose constants that produce more accurate results than the standard ones. Our main finding is that for a generalized Weibull distribution (6), rather than the standard constants (8), it is more convenient to use other ones based on the asymptotic expansion of the Lambert W function (see Subsection 4.1). They are:

- If $\alpha > \tau$,
  \[
  b_n'' = \left( \frac{\alpha}{C\tau} \right)^{1/\tau} \left( -M_1 + M_2 - \frac{M_2}{M_1} \right)^{1/\tau}
  \] (9)
  and
  \[
  a_n'' = \frac{1}{C\tau (b_n'')^{\tau - 1} - \alpha / b_n''},
  \] (10)

  where
  \[
  M_1 = \log \left( \frac{C\tau}{\alpha(Kn)^{\tau/\alpha}} \right) \quad \text{and} \quad M_2 = \log(-M_1).
  \]

- If $\alpha \leq \tau$,
  \[
  b_n'' = \frac{1}{C^{1/\tau}} \left( N_1 + \frac{\alpha}{\tau} N_2 + \frac{\alpha^2}{\tau^2} \frac{N_2}{N_1} \right)^{1/\tau}
  \] (11)

  where
  \[
  N_1 = \log \left( \frac{Kn}{C\alpha/\tau} \right) \quad \text{and} \quad N_2 = \log(N_1),
  \]

and $a_n''$ the same as in (10).

Although $b_n'$ of (8) and these $b_n''$ look like quite different, $b_n'$ coincides, except some constants, with the first two terms of $b_n''$ (9) (related with $M_1$ and $M_2$) or (11), and the remaining part of $b_n''$ is a sequence that converges to zero, but so slowly that, jointly with the constants, it is important in typical sample sizes. See below the case of Gamma random variables, where the difference between the constants is more evident. To illustrate this point, in Section 4 we study the simplest case of a non–trivial generalized Weibull distribution:

\[
F(x) = \begin{cases} 
1 - e x e^{-x}, & \text{if } x \geq 1, \\
0, & \text{if } x < 1,
\end{cases}
\] (12)

where $K = e$ and $C = \tau = \alpha = 1$. The standard constants (8) are

\[
\begin{align*}
    b_n' &= \log n + \log \log n + 1, \\
    a_n' &= 1.
\end{align*}
\]
The proposed constants are

\[ b_n'' = \log n + \log(n + 1) + 1 + \frac{\log \log(n + 1)}{\log n + 1}, \]

\[ a_n'' = \frac{b_n''}{b_n' - 1}. \]

The difference between \( b_n' \) and \( b_n'' \) is essentially \( \frac{\log \log n}{\log n} \), that goes to zero, but so slowly that cannot be neglected in typical cases. This is illustrate in Figure 1. Random variables from the distribution (12) are easily simulated (see Subsection 4.4). In that figure there are the histogram of a simulation of \( 10^4 \) maxima of \( n = 100 \) random variables normalized by using the standard norming constants \( a_n' \) and \( b_n' \), and by using the proposed constants \( a_n'' \) and \( b_n'' \).

![Histograms of a simulation of 10^4 maxima with n = 100, of a generalized Weibull distribution of parameters C = \( \tau = \alpha = 1 \), with two different sets of norming constants.](image)

**Figure 1.** Solid line: Gumbel density. Histograms of a simulation of \( 10^4 \) maxima with \( n = 100 \), of a generalized Weibull distribution of parameters \( C = \tau = \alpha = 1 \), with two different sets of norming constants.

As we commented, our main interest is in Gamma laws \( G(\nu, \theta) \) (with \( \nu > 0 \) and \( \tau \geq 1 \)); the standard constants (3) are (see Embrechts et al. [4, page 156])

\[ b_n' = \theta \left( \log n + (\nu - 1) \log \log n - \log \Gamma(\nu) \right), \]

\[ a_n' = \theta. \] (13)

In the proposal of \( b_n'' \) for a generalized Weibull distribution we just added one more term to \( b_n' \) coming from the asymptotic expansion of the solution of the first equation of (7). However, here, that addition does not improves sufficiently the approximation to the Gumber law, and we need first to enhance the habitual tail equivalent distribution to Gamma law, and later to add an additional term to \( b_n' \). Our proposal is to use:

- If \( \nu \in (1, 2] \):
  \[ b_n'' = \theta \left( \log n + (\nu - 1) \log \log (n/\Gamma(\nu)) - \log \Gamma(\nu) + \frac{(\nu - 1)^2 \log \log (n/\Gamma(\nu)) + \nu - 1}{\log (n/\Gamma(\nu))} \right), \]

- If \( \nu > 2 \):
  \[ b_n'' = \theta \left( \log n + (\nu - 1) \log B_n - \log \Gamma(\nu) + \frac{(\nu - 1)^2 \log B_n - (\nu - 1)^2 \log(\nu - 1) + \nu - 1}{B_n} \right), \]

where

\[ B_n = \log n + (\nu - 1) \log(\nu - 1) - \log \Gamma(\nu). \]
In both cases, we propose
\[ a_n'' = \frac{b_n''}{b_n''/\theta - \nu + 1}. \]

Note that for \( \nu \in (1, 2] \),
\[ b_n'' - b_n' = \theta \left( (\nu - 1) \log \frac{n - \log \Gamma(\nu)}{\log n} + \frac{(\nu - 1)^2 \log \log (n/\Gamma(\nu)) + \nu - 1}{\log (n/\Gamma(\nu))} \right), \]
which goes to 0 as \( n \to \infty \). In the other case the difference is similar.

4 The simplest case

As we commented, in order to show the problem at hand and the techniques that we use, we first consider the following particular case of a generalized Weibull distribution:
\[ F(x) = \begin{cases} 
1 - e^{xe^{-x}}, & \text{if } x \geq 1, \\
0, & \text{if } x < 1,
\end{cases} \tag{14} \]
where \( K = e \) and \( C = \tau = \alpha = 1 \). The standard constants (8) are
\[ b_n' = \log n + \log \log n + 1, \quad a_n' = 1. \tag{15} \]

Consider a sample size \( n = 100 \) from the distribution \( F \). Solving numerically the first equation of (7) we get \( b_n \approx 7.6384 \) (see next Subsection), and from the second equation we obtain \( a_n \approx 1.1506 \). The standard constants are \( b_n' \approx 7.1323 \) and \( a_n' = 1 \). In Figure 2 there is a plot of the density of the Gumbel law and the densities of the random variables
\[ Y_n = \frac{1}{a_n} (M_n - b_n) \quad \text{and} \quad Y_n' = \frac{1}{a_n'} (M_n - b_n'). \]

**Figure 2.** Solid line: Gumbel density. Red dotted line: Density of \( Y_n \). Blue dashed line: Density of \( Y_n' \).

In Figure 2 we observe that the densities of the Gumbel law and of \( Y_n \) are practically indistinguishable; on the contrary, the density of \( Y_n' \) is indeed quite far from the Gumbel density. As a consequence, this plot illustrates that for \( n = 100 \), the distribution of \( Y_n \) is very near to the limit, but \( Y_n' \) it is not, so the approximate norming constants are very important for such a sample size (and much bigger sample sizes, see Table 1). Then we try to improve the accuracy of \( Y_n' \) choosing other norming constants. To get some insight in this question we study the first equation of (7) using the Lambert W function.
4.1 The Lambert W function

In the real case the Lambert function $W$ is defined implicitly through the real solution of the equation

$$W(x) e^{W(x)} = x.$$ 

Equivalently, $W$ is the inverse of the function $f(t) = t e^t$. A plot of the function $f$ (see Figure 3) shows that the Lambert function has two branches (see Figure 4), the principal one, denoted by $W_0$, is defined on $(-1/e, \infty)$, and the secondary, denoted by $W_{-1}$, is defined on $(-1/e, 0)$ and it satisfies

$$\lim_{x \to 0^-} W_{-1}(x) = -\infty.$$ 

Figure 3. Function $f(t) = t e^t$

![Figure 3](image1.png)

Figure 4. Solid line: Principal branch of the real Lambert W function. Dashed line: Secondary branch.

We are interested in the secondary branch (see Subsection 4.1); its asymptotic expansion is (Corless et al. [2, pp. 22 and 23], see also Comtet [1] for the expression of the polynomials and De Bruijn [3, pp. 25–27])

$$W_{-1}(x) = L_1(-x) - L_2(-x) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{P_n(L_2(-x))}{L_1^n(-x)}, \ x \to 0^-,$$ 

(16)

where

$$L_1(x) = \log x \quad \text{and} \quad L_2(x) = \log |\log x|,$$ 

(17)
and $P_n(x)$ are polynomials related with the signed Stirling numbers of the first type; the first three polynomials are

$$P_1(x) = x, \quad P_2(x) = \frac{1}{2}x^2 - x \quad \text{and} \quad P_3(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + x.$$ 

A partial sum approximation to the series on the right hand side of (16) can be given in the following way

$$W_{-1}(x) = L_1(-x) - L_2(-x) + \sum_{n=1}^{N} (-1)^{n+1} \frac{P_n(L_2(-x))}{L_1^n(-x)} + O\left(\left(\frac{L_2(-x)}{L_1(-x)}\right)^{N+1}\right).$$

**Notations 4.1.** As usual, we write that $g(x) = O(h(x))$ when $x \to \infty$ if there is a point $x_0$ and a constant $C$ such that $|g(x)| \leq Ch(x)$, for all $x > x_0$ (it is assumed $h(x) > 0$ for $x > x_0$). We write $g(x) = o(h(x))$ if $\lim_{x \to \infty} \frac{g(x)}{h(x)} = 0$ (again, here, $h(x) > 0$ for $x > x_0$, for some $x_0$). Finally, we say that $g(x)$ and $f(x)$ are asymptotically equal and write $h \sim g$ if $\lim_{x \to \infty} \frac{g(x)}{h(x)} = 1$. Similar notations are used when we consider $x \to a$.

**Remark 4.2.** In this paper, all the computations related with the Lambert W function are done with the function `lambertW` of the package emdbook of the software R.

### 4.2 Computation of the norming constants via Lambert function

To compute the norming constants, the first equation of (7) for $F$ given in (14) is

$$eb_n e^{-bn} = \frac{1}{n},$$

and hence,

$$b_n = -W\left(-\frac{1}{en}\right).$$

By construction

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} F^{-1}_n(1 - n^{-1}) = \infty,$$

so in (18) it is needed to consider the secondary branch:

$$b_n = -W_{-1}\left(-\frac{1}{en}\right).$$

The asymptotic behaviour of $b_n$ can be deduced from (16) and gives

$$b_n = \log n + 1 + \log(\log n + 1) + \frac{\log(\log n + 1)}{\log n + 1} + O\left(\frac{\log(\log n + 1)}{\log n + 1}\right)^2.\quad (20)$$

From the second equation of (7) we deduce

$$a_n = \frac{b_n}{b_n - 1}.$$

### 4.3 Computation of the velocity of convergence

The convergence (1) is equivalent that for every $x \in \mathbb{R}$,

$$\lim_{n} P\left\{\frac{1}{a_n} (M_n - b_n) \leq x\right\} = \Lambda(x),$$

or

$$\lim_{n} F^a_n (a_n x + b_n) = \Lambda(x).$$
We will prove in Theorem 5.1 that

\[ F^n(a_n x + b_n) = \Lambda(x) \left( 1 + O(1/\log n) \right). \]

In this expression the constant implicit in \( O(1/\log n) \) may depend on \( x \).

Furthermore, see again Theorem 5.1 if \( \tilde{b}_n \) satisfies

\[ \lim_{n} \frac{b_n - \tilde{b}_n}{a_n} = 0, \]

and

\[ \tilde{a}_n = 1 + O(1/\tilde{b}_n), \]

then

\[ F^n(\tilde{a}_n x + \tilde{b}_n) = \Lambda(x) \left( 1 + O(1/\log n) + O(b_n/b_n - 1) + O(b_n - \tilde{b}_n) \right). \]

In particular, for the standard constants \(15\),

\[ b'_n = \log n + \log \log n + 1, \quad (21) \]

we have

\[ b_n - b'_n = O\left( \frac{\log \log n}{\log n} \right). \]

However, if we take one more term of the asymptotic expansion of Lambert W function in agreement with \(20\):

\[ b''_n = \log n + 1 + \log(\log n + 1) + \frac{\log(\log n + 1)}{\log n + 1}, \quad (22) \]

then

\[ b_n - b''_n = O\left( \left( \frac{\log \log n}{\log n} \right)^2 \right). \]

As it is shown in Table 1 the improvement is remarkable, specially for moderate sample sizes.

| n     | 10   | 10^2 | 10^3 | 10^4 | 10^5 | 10^6 |
|-------|------|------|------|------|------|------|
| \( b_n \) | 4.8897 | 7.6384 | 10.2334 | 12.7564 | 15.2366 | 17.6884 |
| \( b'_n \) | 4.1366 | 7.1323 | 9.8404 | 12.4307 | 14.9564 | 17.4413 |
| \( b''_n \) | 4.8590 | 7.6364 | 10.2371 | 12.7613 | 15.2416 | 17.6931 |

Table 1. Comparison of the constants for the distribution \(14\): \( b_n \) is computed numerically, \( b'_n \) is the standard constant \(21\), and \( b''_n \) the proposed constant \(22\).

The difference \( b_n - b'_n \) decreases very slowly, whereas \( b_n \) and \( b''_n \) are practically indistinguishable; it is worth noting that the difference between \( b'_n \) and \( b''_n \) is a sequence that converges to zero but so slowly that is important in typical cases. A plot of \( b_n, b'_n \) and \( b''_n \) for values of \( n \) from 10 to 1000 is given in Figure 5. The lines corresponding to \( b_n \) and \( b'_n \) are indistinguishable. Of course, we can add more terms to \( b''_n \), but Table 1 suggests that it is unnecessary.

4.4 The simulation approach

Random variables with distribution function given by \(14\) can be simulated by inversion method because the quantile function corresponding to \( F \) is explicit in terms of the Lambert W function:

\[ F^{-1}(u) = -W_{-1}((u - 1)/e), \quad u \in (0, 1). \]

The random variables used to construct the histograms in Figure 1 were simulated in this way.
5 Generalized Weibull distribution

In this section we deal with a sample $X_1, \ldots, X_n$ of i.i.d. random variables with generalized Weibull distribution as presented in Section 2, with distribution function

$$F(x) = 1 - Kx^\alpha \exp\{-Cx^\tau\}, \ x \geq x_0,$$

with $\alpha > 0$ and $\tau \geq 1$. Let $a_n$ and $b_n$ be given by

$$b_n = F^{-1}(1 - n^{-1}),$$
$$a_n = \frac{1}{C \tau b_n^{\tau - 1} - \alpha/b_n}.$$

(23)

Theorem 5.1. With the previous notations,

1. $$F^n(a_n x + b_n) = \Lambda(x) \left(1 + O\left(\frac{1}{b_n^{\tau - 1}}\right)\right) = \Lambda(x) \left(1 + O\left(\frac{1}{\log n}\right)\right),$$

(24)

where $\Lambda(x) = \exp\{-e^{-x}\}$ is the distribution function of the Gumbel law.

2. If $\tilde{b}_n$ satisfies

$$\lim_{n} \frac{b_n - \tilde{b}_n}{a_n} = 0,$$

(25)

and

$$\tilde{a}_n = \frac{1}{C \tau b_n^{\tau - 1}} + O\left(\frac{1}{b_n^{\tau - 1}}\right),$$

(26)

then

$$F^n(\tilde{a}_n x + \tilde{b}_n) = \Lambda(x) \left(1 + O\left(b_n^{\tau - 1} - \tilde{b}_n^{\tau - 1}\right) + O\left((\frac{\tilde{b}_n}{b_n})^\alpha - 1\right) + O\left(\frac{1}{b_n^{\tau - 1}}\right)\right).$$

3. For every sequence of norming constant $\{\tilde{a}_n, n \geq 1\}$ of the form

$$\tilde{a}_n = \frac{1}{C \tau b_n^{\tau - 1}} + O\left(\frac{1}{b_n^{\tau - 1}}\right),$$

(27)

we have

$$F^n(\tilde{a}_n x + b_n) = \Lambda(x) \left(1 + (2(\alpha - C^2 \tau^2 \delta)x - (\tau - 1)x^2) O\left(\frac{1}{b_n^{\tau - 1}}\right) + O\left(\frac{1}{b_n^{\tau + 1}}\right)\right).$$

As a consequence, when $\tau = 1$ and $\delta = \alpha/C^2$, the sequence $\{a_n, n \geq 1\}$ is optimal between all sequences $\{\tilde{a}_n, n \geq 1\}$ of the above form. [27]
Remarks 5.2.

(i) The constant \(a_n\) given in (23) satisfies (26) with \(b_n\); specifically,

\[
a_n = \frac{1}{C\tau b_n^{-1}} \left(1 - \frac{\alpha}{C\tau b_n^\tau}\right)^{-1} = \frac{1}{C\tau b_n^{-1}} + \frac{\alpha}{C^2\tau^2 b_n^{2\tau-1}} + O\left(\frac{1}{b_n^{3\tau-1}}\right).
\]

Note that it has the form (27) with \(\delta = \alpha/(C\tau)^2\).

(ii) It holds that \(\lim_n (b_n^\tau - \tilde{b}_n^\tau) = 0\). This is proved in the following way: from (25) and the expression of \(a_n\) given in (23)

\[
\lim_n \tilde{b}_n - b_n = 0,
\]

that is,

\[
\tilde{b}_n = b_n + o\left(\frac{1}{b_n^{\tau-1}}\right) = b_n \left(1 + o\left(\frac{1}{b_n^{\tau}}\right)\right).
\]

Thus

\[
\tilde{b}_n^\tau = b_n^\tau \left(1 + o\left(\frac{1}{b_n^{\tau}}\right)\right),
\]

and

\[
\tilde{b}_n^\tau - b_n^\tau = o(1).
\]

(iii) The case \(\tau = 1\) is important because a Gamma law is Weibull-like with such a \(\tau\) (see Section 6). In this case, in agreement with Remark (i), the sequence \(\{a_n, n \geq 1\}\) is optimal.

To prove this theorem we need the following lemma:

**Lemma 5.3.** Let \(A \neq 0\) and consider two sequences \(\{c_n, n \geq 1\}\) and \(\{d_n, n \geq 1\}\) such that \(\lim_n c_n = 0\) and \(\lim_n d_n = 1\) Then, when \(n \to \infty\),

\[
\left(1 + \frac{Ad_n}{n} (1 + c_n)\right)^n = e^{A} \left(1 + Ac_n + A(d_n - 1) + O(1/n)\right).
\]

**Proof of the lemma.**

From the asymptotic approximation

\[
\log(1 + x) = x + O(x^2), \text{ when } x \to 0,
\]

it follows that

\[
n \log \left(1 + \frac{Ad_n}{n} (1 + c_n)\right) = Ad_n (1 + c_n) + nO\left(d_n^2 (1 + c_n)^2/n^2\right) = Ad_n (1 + c_n) + O(1/n).
\]

Then

\[
\left(1 + \frac{Ad_n}{n} (1 + c_n)\right)^n = e^{A} e^{Ac_n + A(d_n - 1) + O(1/n)} = e^{A} \left(1 + Ac_n + A(d_n - 1) + O(1/n)\right). \quad \blacksquare
\]

**Proof of Theorem 5.1.**

1. In this proof we will use that \(b_n\) is asymptotically equivalent to \(\log^{1/\tau} n\), that is proved in Subsection 5.1. First, note that from (23),

\[
a_n = \frac{1}{C\tau b_n^{\tau-1} - \alpha/b_n} = O\left(\frac{1}{b_n^{\tau-1}}\right),
\]
and
\[ \frac{a_n}{b_n} = O\left(\frac{1}{b_n}\right). \]

Then
\[
K(a_n x + b_n)^\alpha \exp\{-C(a_n x + b_n)^\gamma\} = K b_n^\alpha \exp\{-C b_n^\gamma\} \left(\frac{a_n}{b_n} x + 1\right)^\alpha \exp\{-C(a_n x + b_n)^\gamma + C b_n^\gamma\}
\]
\[
= \frac{e^{-x}}{n} \left(1 + O\left(\frac{1}{b_n}\right)\right) \exp\{-C b_n^\gamma\left(\frac{a_n}{b_n} x + 1\right)^\gamma - 1\} + x
\]
\[
= \frac{e^{-x}}{n} \left(1 + O\left(\frac{1}{b_n}\right)\right) \exp\{-C b_n^\gamma\left(1 + \tau \frac{a_n}{b_n} x + O\left(\frac{a_n}{b_n}\right)^2\right) - 1\} + x
\]
\[
= \frac{e^{-x}}{n} \left(1 + O\left(\frac{1}{b_n}\right)\right) \exp\{-C \tau b_n^{\gamma-1} a_n x + O\left(\frac{1}{b_n}\right) + x\}
\]
\[
= \frac{e^{-x}}{n} \left(1 + O\left(\frac{1}{b_n}\right)\right) \exp\left\{O\left(\frac{1}{b_n}\right)\right\} = \frac{e^{-x}}{n} \left(1 + O\left(\frac{1}{b_n}\right)\right),
\]
where the equality (*) follows from the definition of $b_n$ given in (23). Since $\lim_{n} 1/b_n = 0$, we can apply Lemma 6.3 (with $d_n = 1$):
\[
F^n(a_n x + b) = \left(1 - K(a_n x + b_n)^\alpha \exp\{-C(a_n x + b_n)^\gamma\}\right)^n = \exp\{-e^{-x}\} \left(1 + O\left(\frac{1}{b_n}\right) + O\left(\frac{1}{b_n}\right)\right),
\]
and from the estimation $b_n \sim \log^{1/\gamma} n$, the term $O(1/n)$ can be eliminated.

2. Proceeding as before,
\[
K(\tilde{a}_n x + \tilde{b}_n)^\alpha \exp\{-C(\tilde{a}_n x + \tilde{b}_n)^\gamma\}
\]
\[
= K b_n^\alpha \exp\{-C b_n^\gamma\} e^{-x} \left(\frac{\tilde{a}_n x + \tilde{b}_n}{b_n}\right)^\alpha \exp\{-C(\tilde{a}_n x + \tilde{b}_n)^\gamma + C b_n^\gamma + x\}
\]
\[
= \frac{e^{-x}}{n} \left(\frac{\tilde{a}_n x + \tilde{b}_n}{b_n}\right)^\alpha \exp\{-C(\tilde{a}_n x + \tilde{b}_n)^\gamma + C b_n^\gamma + x\}
\]
\[
\text{(29)}
\]
The term in the exponential is
\[
-C(\tilde{a}_n x + \tilde{b}_n)^\gamma + C b_n^\gamma + x = -C b_n^\gamma \left(1 + \frac{\tilde{a}_n x + \tilde{b}_n}{b_n}\right)^\gamma + C b_n^\gamma + x
\]
\[
= -C b_n^\gamma \left(1 + \tau \frac{\tilde{a}_n x + \tilde{b}_n}{b_n} + O\left(\frac{\tilde{a}_n x + \tilde{b}_n}{b_n}\right)^2\right) + C b_n^\gamma + x
\]
\[
= -C(b_n^\gamma - b_n^\gamma) - C\tau \tilde{a}_n b_n^{\gamma-1} x + x + O\left(\frac{1}{b_n}\right)
\]
\[
= -C(b_n^\gamma - b_n^\gamma) + O\left(\frac{1}{b_n}\right),
\]
where in the last equality we used (26).

Then
\[
\text{(29)} = \frac{e^{-x}}{n} \left(\frac{\tilde{b}_n}{b_n}\right)^\alpha \left(1 + \frac{\tilde{a}_n x + \tilde{b}_n}{b_n}\right)^\alpha \left(1 - C(b_n^\gamma - b_n^\gamma) + O\left(\frac{1}{b_n}\right)\right)
\]
\[
= \frac{e^{-x}}{n} \left(\frac{\tilde{b}_n}{b_n}\right)^\alpha \left(1 - C(b_n^\gamma - b_n^\gamma) + O\left(\frac{1}{b_n}\right)\right)
\]
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and we apply again Lemma 5.3.

3. Developing as in 1, but now taking one more term in the expansions

\[
K(\hat{a}_n x + b_n)\alpha \exp\{-C(\hat{a}_n x + b_n)^\tau\} = \frac{e^{-x}}{n} \left(\frac{\hat{a}_n}{b_n} x + 1\right)^\alpha \exp\left\{-C b_n^\tau \left(\frac{\hat{a}_n}{b_n} x + 1\right)^\tau + C b_n^\tau + x\right\}
\]

\[
= \frac{e^{-x}}{n} \left(1 + \alpha \frac{\hat{a}_n}{b_n} x + O\left((\frac{\hat{a}_n}{b_n})^2\right)\right) \times \exp\left\{-C b_n^\tau \left(1 + \tau \frac{\hat{a}_n}{b_n} x + \frac{\tau(\tau - 1)}{2} x^2 \frac{\hat{a}_n^2}{b_n^2} + O\left((\frac{\hat{a}_n}{b_n})^3\right) - 1\right) + x\right\}
\]

\[
= \frac{e^{-x}}{n} \left(1 + \frac{\alpha}{C\tau b_n^\tau} x + O\left(\frac{1}{b_n^\tau}\right)\right) \exp\left\{-C \tau \delta \frac{b_n^\tau}{b_n^\tau} x - \frac{\tau - 1}{2C\tau b_n^\tau} x^2 + O\left(\frac{1}{b_n^{\tau+1}}\right)\right\}
\]

\[
= \frac{e^{-x}}{n} \left(1 + \frac{2(\alpha - C^2\tau^2\delta)x - (\tau - 1)x^2}{2C\tau} \frac{1}{b_n^\tau} + O\left(\frac{1}{b_n^{\tau+1}}\right)\right).
\]

By [28], when \(\hat{a}_n = a_n\), then \(\delta = \frac{\alpha}{(C\tau)^2}\), and if, moreover, \(\tau = 1\), all terms of order \(b_n^{-\tau}\) cancel.

5.1 Computation of the constants using Lambert W function

The first equation of (23) in this case is

\[
K b_n^\alpha \exp\{-C b_n^\tau\} = \frac{1}{n}. \tag{30}
\]

In terms of the Lambert W function the solution is

\[
b_n = \left(-\frac{\alpha}{C\tau} W_{-1}\left(-\frac{C\tau}{\alpha\left(K n\right)^{\tau/\alpha}}\right)\right)^{1/\tau}.
\]

From the asymptotic expansion of the Lambert function [16], we have that

\[
b_n = \left(\frac{\alpha}{C\tau}\right)^{1/\tau} \left(-M_1 + M_2 - \frac{M_2}{M_1} + \cdots\right)^{1/\tau} \tag{31}
\]

where,

\[
M_1 = L_1\left(\frac{C\tau}{\alpha\left(K n\right)^{\tau/\alpha}}\right) = \log\left(\frac{C\tau}{\alpha\left(K n\right)^{\tau/\alpha}}\right),
\]

\[
M_2 = L_2\left(\frac{C\tau}{\alpha\left(K n\right)^{\tau/\alpha}}\right) = \log(-M_1).
\]

5.2 Computation of the constants using Comtet expansion

The solution of equation (30) can be expressed in an alternative way: Fix \(\gamma \neq 0\) and denote by \(U_{\gamma}(x)\) the (unique) solution \(t\) of the equation

\[
t^\gamma e^t = x
\]

such that \(t \to \infty\) when \(x \to \infty\). Equation (30) is equivalent to

\[
\frac{1}{K} b_n^{-\alpha} e^{C b_n^\tau} = n,
\]

or

\[
(C b_n^\tau)^{-\alpha/\tau} e^{C b_n^\tau} = C^{-\alpha/\tau} K n.
\]
Hence,

\[ b_n = \left( \frac{1}{C} U_{-\alpha/\tau} \left( \frac{Kn}{C\alpha/\tau} \right) \right)^{1/\tau}. \]  

(32)

Comtet [11] extended De Bruijn expansion of the principal branch of the Lambert \( W \) function to \( U \), obtaining

\[ U_\gamma(x) = L_1(x) - \gamma L_2(x) + \sum_{n=1}^{\infty} (-\gamma)^{n+1} \frac{P_n(L_2(x))}{L_1^n(x)}, \quad x \to \infty, \]  

(33)

where, \( L_1(x), L_2(x) \) and \( P_n(x) \) are as in (16) and (17). So applying that expansion to (32) we get a new asymptotic expansion for \( b_n \). The finite expansion obtained by truncation of (33) and the one corresponding to (31) are not equal; the difference is originated from the fact that some constants appearing early in (31) are delayed in (32): the whole sum of both series is the same, but the truncated series are slightly different. It turns out that when \( \alpha > \tau \) then it is better the truncation from (31), when \( \alpha < \tau \) it is better the approximation given by Comtet expansion (33), and when \( \alpha = \tau \) both expansion coincide. See Appendix A1 for a proof.

5.3 Return to the norming constants

Returning to the norming constants of the general Weibull maxima, when \( \alpha > \tau \), the better finite asymptotic expansion is given by the truncation of (31), and we get the constants given in (9). When \( \alpha \leq \tau \), we use the asymptotic expansion (33) and we deduce the constant given in (11).

6 Maxima of Gamma random variables

In this section we study the case when the sample comes from a Gamma distribution. That case appears (for \( \chi^2(m) \) laws) in some practical problems of signal analysis and, as we commented in the Introduction, it is in the origin of this work. Since a \( \chi^2(m) \) law is a Gamma(\( \nu, \theta \)) with \( \nu = m/2 \) and \( \theta = 2 \), we study directly the latter case. A Gamma law is Weibull–like with \( \alpha = \nu - 1 \) and \( \tau = 1 \) (see below); we will also assume that \( \nu > 1 \) (in agreement with our general restriction \( \alpha > 0 \)).

The distribution function \( G(x) \) of a Gamma(\( \nu, \theta \)) law can be written in terms of the incomplete Gamma function as

\[ G(x) = 1 - \frac{\Gamma(\nu, x/\theta)}{\Gamma(\nu)}, \quad x > 0, \]  

(34)

where \( \Gamma(a) \) is the Gamma function,

\[ \Gamma(a) = \int_0^\infty t^{a-1}e^{-t} \, dt, \quad a > 0, \]

and \( \Gamma(a, y) \) is the upper incomplete Gamma function

\[ \Gamma(a, y) = \int_y^\infty t^{a-1}e^{-t} \, dt, \quad a, y > 0. \]

Then, from the asymptotics for the incomplete Gamma function deduced from Olver et al. [17, formula 8.11.2]

\[ \lim_{y \to \infty} \frac{\Gamma(a, y)}{y^{a-1}e^{-y}} = 1, \]

it follows

\[ \lim_{x \to \infty} \frac{1 - G(x)}{Kx^\alpha \exp\{-Cx^\tau\}} = 1, \]  

(35)

with

\[ K = \frac{1}{\theta^{\nu-1}\Gamma(\nu)}, \quad \alpha = \nu - 1, \quad C = \frac{1}{\theta}, \quad \text{and} \quad \tau = 1. \]
So we can consider the tail equivalent function

\[ F_1(x) = 1 - \frac{1}{\theta^{\nu-1}\Gamma(\nu)} x^{\nu-1} \exp\{-x/\theta\}, \ x \geq x_0. \]

The auxiliary function \( A_1(x) \) corresponding to \( F_1 \) is

\[ A_1(x) = \frac{x}{x/\theta - \nu + 1}. \]  

This means that the Gamma(\( \nu, \theta \)) law belongs to the Weibull–like distributions. In agreement with our comments on Section 2, the standard constants (see (8)) are

\[ b'_n = \theta \left( \log n + (\nu - 1) \log(\log n) - \log \Gamma(\nu) \right), \]

\[ a'_n = \theta. \]  

(37)

Numerical computations show that these constants produce very inaccurate results (see Subsection 6.3). However, in this case, the simple addition of more terms using Comtet or Lambert expansion as in previous section does not improve the results and we need to consider a right tail equivalent distribution function more accurate than \( F_1 \). We will use that the incomplete Gamma function \( \Gamma(\nu, y) \) \((\nu > 0)\) admits the following asymptotic expansion for \( y \to \infty \)(see Olver et al. [7, formula 8.11.2]):

\[ \Gamma(\nu, y) \sim y^{\nu-1} e^{-y} \left( 1 + \frac{\nu - 1}{y} + \frac{(\nu - 1)(\nu - 2)}{y^2} + \cdots \right). \]

Observe that when \( a \) is an integer, the series in the right hand side terminates, and the expression is not only asymptotic but exact for all \( y > 0 \); this is what happens with the distribution function of a \( \chi^2(m) \) random variable with \( m \) even. So, we will consider a distribution function of the form

\[ F_2(x) = 1 - \frac{1}{\theta^{\nu-1}\Gamma(\nu)} x^{\nu-1} \exp\{-x/\theta\} \left( 1 + \frac{\theta(\nu - 1)}{x} \right), \ x \geq x_0. \]  

(38)

Now we also need an extension of Lambert and Comtet asymptotic expansions to this new context.

### 6.1 Extension of Comtet expansion

Robin [9] and Salvi [10] extended Comtet [11] results in order to deduce an asymptotic expansion of the solution of the equation

\[ t^\gamma e^D\left(\frac{1}{t}\right) = x, \]  

(39)

such that \( t \to \infty \) when \( x \to \infty \), where \( \gamma \neq 0 \) and

\[ D(t) = \sum_{n=0}^{\infty} d_n t^n, \ \text{with} \ d_0 \neq 0, \]

is a power series convergent in a neighborhood of the origin. Denote by \( U_{\gamma,D}(x) \) that solution. Robin [9] and Salvi [10] prove that for every \( N \), for \( x \to \infty \),

\[ U_{\gamma,D}(x) = L_1(x) + \sum_{n=0}^{N} \frac{Q_n(L_2(x))}{L_1^n(x)} + o\left( \frac{1}{L_1^N(x)} \right), \]  

(40)

where \( L_1 \) and \( L_2 \) are the same as in [16], and \( Q_n(x) \) are polynomials now depending on \( \gamma \) and \( D \):

\[ Q_n = Q_n(\gamma, d_0, \ldots, d_n), \]

with degree \( n \) for \( n \geq 1 \), and \( Q_0 \) has degree 1. The first two polynomials (fortunately, the only ones that we need), see Appendix A2, are

\[ Q_0(x) = -\gamma x - \log d_0 \quad \text{and} \quad Q_1(x) = \gamma^2 x + \gamma \log d_0 - \frac{d_1}{d_0}. \]  

(41)

When \( D(x) = 1 \), then \( Q_0(x) = -\gamma x \), and for \( n \geq 1 \), \( Q_n(x) = (-\gamma)^{n+1} P_n(x) \), where \( P_n \) are the Comtet polynomials in Subsection 4.1.
6.2 Extension of Lambert expansion

Consider the equation (39) for $\gamma = 1$. The inverse of the function $f(t) = te^t D\left(\frac{1}{t}\right)$ (out of a neighborhood of the origen) has a secondary branch, denoted by $W_{-1,D}(x)$, that goes to $-\infty$ when $x \to 0^-$. The asymptotic expansion of this branch (see Appendix A3) is

$$W_{-1,D}(x) = L_1(-x) + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{R_n(L_2(-x))}{L_1^n(-x)}.$$  \hspace{1cm} (42)

where the first two polynomials are

$R_0(x) = x + \log d_0$ and $R_1(x) = x + \log d_0 - \frac{d_1}{d_0}.$ \hspace{1cm} (43)

The relationship between the polynomials $Q_n$ of section [6.1] and $R_n$ is studied in Appendix A4.

6.3 New norming constants for the maxima of Gamma random variables

We apply the principle of simplification by tail equivalence given in Property [2.1] and from [7] for $F_2$ in (38), $b_n$ verifies

$$\left(\frac{b_n}{\theta}\right)^{\nu-1} \exp\{-b_n/\theta\} \left(1 + \frac{\theta(\nu - 1)}{b_n}\right) = \frac{\Gamma(\nu)}{n}.$$  \hspace{1cm} (44)

As a consequence of the previous two subsections, we have two ways to express the solution of this equation. For the first one, write $y = b_n/\theta$: we need to solve

$$y^{\nu-1} e^{-y} \left(1 + \frac{\nu - 1}{y}\right) = \frac{\Gamma(\nu)}{n},$$  \hspace{1cm} (45)

or equivalently,

$$y^{1-\nu} e^{-y} \left(1 + \frac{\nu - 1}{y}\right)^{-1} = \frac{n}{\Gamma(\nu)},$$

Hence, with the notation of Subsection [6.1]

$$b_n = \theta U_{1-\nu,D}\left(\frac{n}{\Gamma(\nu)}\right),$$  \hspace{1cm} (46)

where, $D(t)$ is the series

$$D(t) = (1 + (\nu - 1)t)^{-1} = 1 - (\nu - 1)t + O(t^2).$$  \hspace{1cm} (47)

In a similar way, we can transform equation (45):

$$ye^{-y/(\nu-1)} \left(1 + \frac{\nu - 1}{y}\right)^{1/(\nu-1)} = \left(\frac{\Gamma(\nu)}{n}\right)^{1/(\nu-1)},$$

or

$$\left(-\frac{y}{\nu - 1}\right) e^{-y/(\nu-1)} \left(1 - \frac{1}{-y/(\nu - 1)}\right)^{1/(\nu-1)} = -\frac{1}{\nu - 1} \left(\frac{\Gamma(\nu)}{n}\right)^{1/(\nu-1)}.$$  \hspace{1cm} (48)

Thus, from Subsection [6.2]

$$b_n = -(\nu - 1)\theta W_{-1,E}\left(-\frac{1}{\nu - 1} \left(\frac{\Gamma(\nu)}{n}\right)^{1/(\nu-1)}\right),$$
where $E(t)$ is

$$E(t) = (1 - t)^{1/2} = 1 - \frac{1}{\nu - 1} t + O(t^2).$$

(49)

(We need only the first two terms of this series). We prove in Appendix A4 that the finite expansions deduced from (46) and (48) when $\nu = 2$ are equal, when $\nu > 2$ then (48) gives a more accurate value, and when $1 < \nu < 2$ it is better to use the expansion deduced from (46). In both cases, the results on Section 4 suggest to add just one term of the asymptotic expansion to the standard constants, and hence it suffices to consider the polynomials $Q_0$ and $Q_1$, that depend only on the terms $d_0$ and $d_1$ of the series $D(t)$ or $B(t)$. The formulas are the following:

**For $\nu \in (1, 2]$**: From (46), (40) and (41), we propose

$$b''_n = \theta \left( \log (n/\Gamma(\nu)) + (\nu - 1) \log \log (n/\Gamma(\nu)) + \frac{(\nu - 1)^2 \log \log (n/\Gamma(\nu)) + \nu - 1}{\log (n/\Gamma(\nu))} \right).$$

and

$$a''_n = \frac{b''_n}{b''_n / \theta - \nu + 1}.$$  

(50)

**For $\nu \geq 2$**: From (48), (42) and (43),

$$b''_n = \theta \left( \log n + (\nu - 1) \log B_n - \log \Gamma (\nu) + \frac{(\nu - 1)^2 \log n - (\nu - 1)^2 \log (\nu - 1) + \nu - 1}{B_n} \right),$$

(51)

where

$$B_n = \log n + (\nu - 1) \log (\nu - 1) - \log \Gamma (\nu),$$

and $a''_n$ the same as (50).

In Table 2 there are some numerical results for a $\chi^2(10)$ distribution; the numeric value of $b_n$ is computed using the quantile function of a $\chi^2$ distribution implemented in R program. Similar results are obtained for the case $\nu \in (1, 2)$, for example, for a $\chi^2(3)$ distribution; however, in this case the discrepancy between the approximation using the standard constants and the Gumbel distribution is not as grave as in the case $\nu > 2$.

| $n$   | 10  | 10^2 | 10^3 | 10^4 | 10^5 | 10^6 |
|-------|-----|------|------|------|------|------|
| $b_n$ | 15.9872 | 23.2093 | 29.5883 | 35.5640 | 41.2962 | 46.8630 |
| $b'_n$ | 4.9213 | 15.0717 | 22.9606 | 29.8272 | 36.2175 | 42.2812 |
| $b''_n$ | 13.3518 | 22.0874 | 29.0421 | 35.2855 | 41.1581 | 46.8045 |

Table 2. Comparison of the constants $b_n$ for the $\chi^2(10)$ distribution: $b_n$ is the numeric value, $b'_n$ is the standard value, $b''_n$ is the value given in (51).

For the norming constant $a_n$ there are also important differences, see Table 3.

| $n$   | 10  | 10^2 | 10^3 | 10^4 | 10^5 | 10^6 |
|-------|-----|------|------|------|------|------|
| $a_n$ | 4.0032 | 3.0520 | 2.7411 | 2.5805 | 2.4805 | 2.4117 |
| $a'_n$ | 2   | 2    | 2    | 2    | 2    | 2    |
| $a''_n$ | 4.9896 | 3.1358 | 2.7604 | 2.5864 | 2.4825 | 2.4123 |

Table 3. Comparison of the constants $a_n$ for the $\chi^2(10)$ distribution: $a_n$ is the numeric value, $a'_n$ is the standard value, $a''_n$ is the value given in (50).
In Figure 6 there is a plot of the density functions of the random variables

\[ Y_n = \frac{1}{a_n} (M_n - b_n), \quad Y'_n = \frac{1}{a'_n} (M_n - b'_n), \quad \text{and} \quad Y''_n = \frac{1}{a''_n} (M_n - b''_n) \]

from a sample of size \( n = 100 \) of \( \chi^2(10) \) random variables, where \( b_n \) and \( a_n \) are the numeric solutions of equations (7), \( b'_n \) and \( a'_n \) are the standard solution given in (37), \( b''_n \) and \( a''_n \) are the constants (51) and (50).

Figure 6. Maximum of 100 \( \chi^2(10) \) random variables. Solid line: Gumbel density. Dashed Blue line: Density of \( Y_n \). Loosely dashed red line: density of \( Y'_n \). Dotted green line: Density of \( Y''_n \).

Appendix

Appendix A1. Lambert versus Comtet asymptotic expansions

Fix \( \beta > 0 \) and consider the solution \( t \) of the equation

\[ t^\beta e^{-t} = x, \quad (52) \]

such that \( t \to \infty \) when \( x \to 0^+ \). With the notations of subsections 5.1 and 5.2, the solution can be written in two ways:

\[ t = -\beta W_{-1}(-x^{1/\beta}) = U_{-\beta}(1/x). \quad (53) \]

From (16), the central term of (53) is

\[ t = -\beta L_1(x^{1/\beta}) + \beta L_2(x^{1/\beta}) - \beta \sum_{n=1}^{\infty} (-1)^n+1 \frac{P_n(L_2(x^{1/\beta}))}{L_1^n(x^{1/\beta})}, \quad (54) \]

where \( L_1 \) and \( L_2 \) are defined in (17). From (33), the term of the right side of (53) is

\[ t = L_1(1/x) + \beta L_2(1/x) + \sum_{n=1}^{\infty} \beta^{n+1} \frac{P_n(L_2(1/x))}{L_1^n(1/x)}, \quad (55) \]
Comparing (54) and (55) we realize that both asymptotic expansion are the same function applied to different points; specifically, define

\[ h(y, z) = -y + \beta z - \sum_{n=1}^{\infty} (-\beta)^{n+1} \frac{P_n(z)}{y^n}, \]  

(56)

Then (54) is the function \( h(y, z) \) applied to

\[ y = L_1\left(\frac{x}{\beta}\right) \quad \text{and} \quad z = L_2\left(\frac{x^{1/\beta}}{\beta}\right), \]

whereas (55) is the function \( h(y, z) \) applied to \( y = L_1(x) \) and \( z = L_2(x) \). The argument of De Bruijn [3, p. 25–27] in this case shows that there exists constants \( a \) and \( b \) such that if \( y > a \) and \( 0 < z/y < b \), then the series in the right hand side of (56) is absolutely convergent and

\[ |h(y, z) + y - \beta z + \sum_{n=1}^{N} (-\beta)^{n+1} \frac{P_n(z)}{y^n}| \leq C \left(\frac{z}{y}\right)^{N+1}. \]

So we deduce

\[ \left| t + L_1\left(\frac{x}{\beta}\right) - \beta L_2\left(\frac{x^{1/\beta}}{\beta}\right) + \sum_{n=1}^{N} (-\beta)^{n+1} \frac{P_n(L_2(x^{1/\beta}/\beta))}{L_1(x/\beta^3)} \right| \leq C \left(\frac{L_2\left(\frac{x^{1/\beta}}{\beta}\right)}{L_1\left(\frac{x}{\beta}\right)}\right)^{N+1}, \]  

(57)

and

\[ \left| t + L_1(x) - \beta L_2(x) + \sum_{n=1}^{N} (-\beta)^{n+1} \frac{P_n(L_2(x))}{L_1(x)} \right| \leq C \left(\frac{L_2(x)}{L_1(x)}\right)^{N+1}. \]  

(58)

Now,

\[ \frac{L_2\left(\frac{x^{1/\beta}}{\beta}\right)}{L_1\left(\frac{x}{\beta}\right)} = \frac{L_2\left(\frac{x^{1/\beta}}{\beta}\right)}{L_2(x)} \cdot \frac{L_1(x)}{L_1\left(\frac{x}{\beta}\right)} = \left(1 - \frac{\log \beta}{L_2(x)} - \frac{\beta \log \beta}{L_1(x)} + \cdots\right) \left(1 + \frac{\beta \log \beta}{L_1(x)} + \cdots\right) \]

\[ = \left(1 - \frac{\log \beta}{L_2(x)} + \cdots\right) \]  

(59)

and, for \( x > 0 \) small enough, (remember \( L_2(x) = \log(-\log x) < 0 \), when \( x \to 0^+ \),

\[
\text{(59) is} \quad \begin{cases} > 1, & \text{if } 0 < \beta < 1, \\ = 1, & \text{if } \beta = 1, \\ < 1, & \text{if } \beta > 1. \end{cases}
\]

This indicates that

- If \( 0 < \beta < 1 \) then the finite expansion deduced from (55) seems to produce a more accurate approximation.
- If \( \beta > 1 \), the finite expansion from (54) seems better.
- If \( \beta = 1 \), both expansions are equal.

Intuitively, (54) and (55) are asymptotic expansions when \( x \to 0^+ \), and, for example, when \( \beta > 1 \), the dominant part of both, \( \log(\ldots) \) is applied in (54) to a smaller number, \( x/\beta^3 \).
In Table 4 there is a numerical study to illustrate this point. We denote by \( t \) the numeric solution of equation (52), by \( t_W \) the approximation deduced from Lambert expansion (54),

\[
t_W = -L_1(x/\beta^\gamma) + \beta L_2(x^{1/\gamma}/\beta) - \beta^2 \frac{L_2(x^{1/\gamma}/\beta)}{L_1(x/\beta^\gamma)},
\]
and by \( t_C \) the approximation deduced from Comtet expansion (55),

\[
t_C = -L_1(x) + \beta L_2(x) - \beta^2 \frac{L_2(x)}{L_1(x)}.
\]

| \( x \)   | 10^{-1} | 10^{-2} | 10^{-3} | 10^{-4} | 10^{-5} | 10^{-6} |
|----------|---------|---------|---------|---------|---------|---------|
| \( \beta = 0.5 \) | \( t \)   | 2.8212  | 5.4533  | 7.9440  | 10.3803 | 12.7871 | 15.1753 |
|          | \( t_W \) | 2.8124  | 5.4554  | 7.9464  | 10.3824 | 12.7889 | 15.1768 |
|          | \( t_C \) | 2.8102  | 5.4517  | 7.9440  | 10.3808 | 12.7877 | 15.1759 |
| \( \beta = 4 \) | \( t \)   | 12.3607 | 15.5923 | 18.6005 | 21.4786 | 24.2699 | 26.9987 |
|          | \( t_W \) | 11.9175 | 15.3431 | 18.4547 | 21.3922 | 24.2198 | 26.9717 |
|          | \( t_C \) | 11.4342 | 16.0199 | 19.1148 | 21.9488 | 24.6826 | 27.3597 |

Table 4. Comparison of the approximations \( t_W \) and \( t_C \) given in (60) and (61) with the numeric solution \( t \) of equation (52).

A2. Extension of Comtet expansion

Following the notations of Subsection 6.1, the solution \( U_{\gamma,D}(x) \) of the equation

\[
t^\gamma e^t D\left(\frac{1}{t}\right) = x,
\]

such that \( t \to \infty \) when \( x \to \infty \), has an asymptotic (formal) expansion (see Robin [9])

\[
U_{\gamma,D}(x) = L_1(x) + \sum_{n=0}^{\infty} \frac{Q_n(L_2(x))}{L_1^n(x)},
\]

where the polynomials verify the recurrence relation

\[
Q'_{n+1} = -\gamma(Q'_n - nQ_n), \quad n \geq 0, \quad \text{and} \quad Q'_0 = -\gamma.
\]

Due that this recurrence does not determine the independent term of the polynomials, Salvi [10] considers the generating function of the independent terms of the polynomials, \( Q_0(0), Q_1(0), \ldots \),

\[
G(s) := \sum_{n=0}^{\infty} Q_0(0)s^n,
\]

and he proves that it satisfies

\[
G(s) = -\gamma \log(1 + s G(s)) - \log D\left(\frac{s}{1 + s G(s)}\right),
\]

which allows to compute the independent terms. Joining (64) and (65) the polynomials \( Q_k(x) \) can be deduced iteratively. Salvi [10] gives the code of a Maple program to compute recurrently those polynomials. The first two polynomials are given in (41).
A3. The secondary branch of the inverse function of $te^t D(1/t)$

As we commented in Subsection 6.2, the equation

$$te^t D\left(\frac{1}{t}\right) = x$$

has a unique solution $t < 0$ for $x < 0$, such that $t \to -\infty$ when $x \to 0^-$. Write $t = W_{-1,D}(x)$. Its asymptotic expansion can be deduced from the very general results of Robin [9] and Salvi [10] commented in Appendix A2: Indeed, equation (66) is equivalent to

$$( - t )^{-1} e^{-t} D^{-1} \left( - \frac{1}{-t} \right) = - \frac{1}{x},$$

where $D^{-1}(t) = 1/D(t)$. Since $-\frac{1}{x} \to \infty$, changing $-t$ by $u$, we have

$$u^{-1} e^u C(u) = -1/x,$$

where $C(t) = 1/D(-t)$. Hence, with the notations of Appendix A2, the solution is

$$t = -U_{-1,C}(-1/x).$$

Thus, from (63), the asymptotic expansion of $W_{-1,D}(x)$ is

$$W_{-1,D}(x) = -U_{-1,C}(-1/x) = -L_1(-1/x) - \sum_{n=0}^{\infty} \frac{R_n(L_2(-1/x))}{L_1^n(-1/x)},$$

where the polynomials $R_n$ satisfy

$$R'_{n+1} = R'_n - nR_n, \ n \geq 0, \ \text{and} \ \ R'_0 = 1,$$

(68)

and the generating function of the independent terms of the polynomials, $\mathcal{H}(s) := \sum_{n=0}^{\infty} R_n(0)s^n$, satisfies

$$\mathcal{H}(s) = \log(1 + s \mathcal{H}(s)) - \log C\left( \frac{s}{1 + s \mathcal{H}(s)} \right) = \log(1 + s \mathcal{H}(s)) + \log D\left( - \frac{s}{1 + s \mathcal{H}(s)} \right).$$

(69)

Again, from (68) and (69), the polynomials $R_k(x)$ can be computed iteratively. The first two polynomials are given in [43].

A4. Comparison of two asymptotic expansions

In a similar way that in Appendix A1, we are going to compare two asymptotic expansions for the solution of the equation

$$t^\beta e^{-t} A\left(\frac{1}{t}\right) = x,$$

where $\beta > 0$ and $A(x) = \sum_{n=0}^{\infty} a_n x^n$, such that $t \to \infty$ when $x \to 0^+$. The first way to get the solution is noting that

$$t = U_{-\beta,A^{-1}}(1/x).$$

The (formal) asymptotic expansion is (see Appendix A2)

$$t = L_1(1/x) + \sum_{n=0}^{\infty} \frac{Q_n(L_2(1/x))}{L_1^n(1/x)} = -L_1(x) - \sum_{n=0}^{\infty} (-1)^{n+1} \frac{Q_n(L_2(x))}{L_1^n(x)},$$

(70)
where
\[ Q'_{n+1} = \beta (Q'_n - nQ_n), \quad n \geq 0, \quad \text{and} \quad Q'_0 = \beta, \] (71)
and \( G(s) := \sum_{n=0}^{\infty} Q_n(0)s^n \), verifies
\[ G(s) = \beta \log(1 + s G(s)) - \log A \left( \frac{s}{1 + s G(s)} \right) \] (72)
The second asymptotic expansion is deduced from
\[ t = -\beta W_{-1,D}(\frac{x^{1/\beta}}{\beta}), \]
where
\[ D(t) = \frac{1}{\beta} \left( -\frac{1}{\beta} t \right), \]
and \( A^{1/\beta}(t) = (A(t))^{1/\beta} \). Then, from (67),
\[ t = -\beta \left( L_1(x^{1/\beta}/\beta) + \sum_{n=0}^{\infty} (-\beta)^{n+1} \frac{R_n(L_2(x^{1/\beta}/\beta))}{L_1^n(x^{1/\beta}/\beta)} \right) \] (73)
\[ = -L_1(x^{\beta}/\beta^\beta) - \sum_{n=0}^{\infty} (-\beta)^{n+1} \frac{R_n(L_2(x^{1/\beta}/\beta))}{L_1^n(x^{1/\beta}/\beta)}, \] (74)
where the polynomials \( R_n \) are determined by (68) and (69). With the notations of Appendix A3, it follows that \( Q_n(x) = \beta^{n+1} R_n(x) \): just define the polynomials \( S_n(x) = \beta^{n+1} R_n(x) \) and check that they satisfy (71), and that the corresponding generating function of the independent terms satisfy (72). So, as in Appendix A1, we deduce that the two asymptotic expansions are the same function applied to different points, and the analysis of Appendix A1 can be extended to this more general context.

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