A complete Heyting algebra whose Scott space is non-sober

by

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Abstract. We prove that (1) for any complete lattice $L$, the set $D(L)$ of all non-empty saturated compact subsets of the Scott space of $L$ is a complete Heyting algebra (with the reverse inclusion order); and (2) if the Scott space of a complete lattice $L$ is non-sober, then the Scott space of $D(L)$ is non-sober. Using these results and Isbell’s example of a non-sober complete lattice, we deduce that there is a complete Heyting algebra whose Scott space is non-sober, thus giving an affirmative answer to a problem posed by Achim Jung. We also prove that a $T_0$ space is well-filtered iff its upper space (the set $D(X)$ of all non-empty saturated compact subsets of $X$ equipped with the upper Vietoris topology) is well-filtered, which answers another open problem.

Sobriety and well-filteredness are two of the most important properties for non-Hausdorff topological spaces. The Scott space of every domain (continuous directed complete poset) is sober. Johnstone [8] gave the first example of a dcpo whose Scott space is non-sober. Soon after that, Isbell [6] constructed a complete lattice whose Scott space is non-sober. The general problem in this line is whether each object in a classic class of lattices has a sober Scott space. Isbell’s non-sober complete lattice is not distributive. Thus Achim Jung asked whether there is a distributive complete lattice whose Scott space is non-sober. In this paper we shall give a positive answer to Jung’s problem. The main structure we shall use is the poset $D(X)$ of all non-empty saturated compact subsets of a topological space $X$ equipped with the reverse inclusion order. We first show that for any complete lattice $L$, the poset $D(L)$ of all non-empty saturated compact subsets of the Scott space of $L$ is a complete Heyting algebra. Then we prove that for a certain type of $T_0$ spaces $X$, if $X$ is non-sober, then the Scott space of $D(X)$ is non-sober. An immediate conclusion is that for any complete lattice $L$, if...
the Scott space of $L$ is non-sober, then the Scott space of the complete Heyting algebra $D(L)$ is non-sober. Taking $L$ to be Isbell’s example, we obtain a complete Heyting algebra whose Scott space is non-sober, thus answering Jung’s problem.

Heckmann and Keimel [4] proved that a space $X$ is sober if and only if the upper space $D(X)$ of $X$ is sober. In [12], Xi and Zhao proved that a space $X$ is well-filtered iff its upper space $D(X)$ is a d-space. They also asked whether it is true that a space is well-filtered if and only if the upper space of $X$ is well-filtered. In the last part of this paper we will give an affirmative answer to this problem.

1. Preliminaries. A complete Heyting algebra is a complete lattice $L$ satisfying the following infinite distributive law:

\[ a \land \bigvee \{a_i : i \in I\} = \bigvee \{a \land a_i : i \in I\} \]

for any $a \in L$ and $\{a_i : i \in I\} \subseteq L$. Such a complete lattice is also called a frame. Clearly, every complete Heyting algebra is distributive when viewed as a lattice.

An element $p$ of a meet-semilattice $S$ is a prime element if for any $a, b \in S$, $a \land b \leq p$ implies $a \leq p$ or $b \leq p$. A frame $A$ is called spatial if every element of $A$ can be expressed as a meet of prime elements. It is well-known that a complete lattice $L$ is a spatial frame iff it is isomorphic to the lattice of all open subsets of some topological space (cf. [9]).

A subset $U$ of a poset $(P, \leq)$ is Scott open if (i) $U$ is an upper set (that is, $U = \uparrow U = \{x \in P : y \leq x \text{ for some } y \in U\}$), and (ii) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cup U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets of a poset $P$ form a topology on $P$, denoted by $\sigma(P)$ and called the Scott topology on $P$. The space $(P, \sigma(P))$ is denoted by $\Sigma P$ and called the Scott space of $P$.

A poset is called a directed complete poset (dcpo, for short) if every directed subset of the poset has a supremum. For more about the Scott topology and dcpo, see [1], [2].

A subset $A$ of a topological space is called saturated if $A$ equals the intersection of all open sets containing it. A $T_0$ space $X$ is well-filtered if for any open set $U$ and filtered family $\mathcal{F}$ of saturated compact subsets of $X$, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$.

A non-empty subset $A$ of a space $X$ is irreducible if for any closed subsets $F_1, F_2$ of $X$, $F \subseteq F_1 \cup F_2$ implies $F \subseteq F_1$ or $F \subseteq F_2$. Obviously, the closure of every singleton is irreducible. A space $X$ is called sober if every irreducible closed subset of $X$ is the closure of a unique singleton set. It is well-known that every sober space is well-filtered. Johnstone [8] constructed the first example of a dcpo whose Scott space is non-sober. Isbell [6] constructed a
complete lattice whose Scott space is non-sober. Kou [5] constructed the first example of a dcpo whose Scott space is well-filtered but non-sober, giving a negative answer to a problem posed by Heckmann [3].

The specialization order $\leq_\tau$ on a $T_0$ space $(X, \tau)$ is defined by $x \leq_\tau y$ iff $x \in \text{cl}\{y\}$, where $\text{cl}\{y\}$ is the closure of the set $\{y\}$. A space $(X, \tau)$ is called a d-space (or monotone convergence space) if $(X, \leq_\tau)$ is a dcpo and $\tau \subseteq \sigma((X, \leq_\tau))$ (cf. [1]).

2. The existence of a complete Heyting algebra whose Scott space is non-sober. For any topological space $X$, following Heckmann and Keimel [4], we shall use $\mathcal{D}(X)$ to denote the set of all non-empty compact saturated subsets of $X$. The upper Vietoris topology on $\mathcal{D}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{D}(X) : K \subseteq U\}$. The set $\mathcal{D}(X)$ equipped with the upper Vietoris topology is called the Smyth power space or upper space of $X$ (cf. [1], [10]).

The specialization order on the upper space $\mathcal{D}(X)$ is the reverse inclusion order $\supseteq$. This is the order we will be concerned with in this section.

For a poset $P$, we shall use $\mathcal{D}(P)$ to denote the poset of all non-empty compact saturated subsets of the Scott space $(P, \sigma(P))$.

A space $X$ is called coherent if the intersection of any two compact saturated subsets in $X$ is compact.

**Lemma 1.** For any complete lattice $L$, $\mathcal{D}(L)$ is a complete Heyting algebra.

**Proof.** The Scott space $\Sigma L$ of $L$ is well-filtered by Xi and Lawson [11], and is coherent by Jia and Jung [7]. We now show that the poset $\mathcal{D}(L)$ is a complete Heyting algebra.

(i) Since $(L, \sigma(L))$ is well-filtered, $\mathcal{D}(L)$ is closed under filtered intersections [12, Remark 3(3)], thus it is a dcpo, in which the infimum of a directed subset $\mathcal{K}$ of $\mathcal{D}(L)$ is its intersection.

(ii) Also, $\Sigma L$ is coherent and every member of $\mathcal{D}(L)$ contains the top element $1_L$ of $L$, so the intersection $K_1 \cap K_2$ of any two members $K_1, K_2$ of $\mathcal{D}(L)$ is again a member of $\mathcal{D}(L)$, which equals their join $K_1 \lor K_2$. Also, $\mathcal{D}(L)$ has $L$ as the least element, and $\{1_L\}$ as the top element. It follows that $\mathcal{D}(L)$ is both a dcpo and a join semilattice, and has a least element. Therefore $\mathcal{D}(L)$ is a complete lattice. In addition, for any $K_1, K_2 \in \mathcal{D}(L)$, the meet $K_1 \land K_2$ of $K_1, K_2$ in $\mathcal{D}(L)$ clearly equals $K_1 \cap K_2$.

Now for any subfamily $\{K_i : i \in I\} \subseteq \mathcal{D}(L)$, (i) and (ii) yield $\bigvee\{K_i : i \in I\} = \bigcap\{K_i : i \in I\}$. Then for any $K \in \mathcal{D}(L)$ and $\{K_i : i \in I\} \subseteq \mathcal{D}(L)$, we have

$$K \land \bigvee_{i \in I} K_i = K \cup \bigcap_{i \in I} K_i = \bigcap_{i \in I} (K \cup K_i) = \bigvee_{i \in I} (K \land K_i).$$

Hence $\mathcal{D}(L)$ is a complete Heyting algebra. ■
For any $T_0$ space $(X, \tau)$, let $\xi_X : X \to \mathcal{D}(X)$ be the canonical mapping given by

$$\xi_X(x) = \uparrow x = \{y \in X : x \leq \tau y\}.$$  

It is easy to see that $\xi_X : (X, \leq \tau) \to (\mathcal{D}(X), \supseteq)$ is an order embedding.

To emphasize the codomain, we shall use $\xi_X^\sigma$ to denote the corresponding mapping $\xi_X^\sigma : (X, \tau) \to (\mathcal{D}(X), \sigma(\mathcal{D}(X)))$, where $\xi_X^\sigma(x) = \uparrow x$ for each $x \in X$.

**Theorem 1.** Let $X$ be a $T_0$ space such that

(i) the upper Vietoris topology on $\mathcal{D}(X)$ is contained in $\sigma(\mathcal{D}(X))$ (that is, the upper Vietoris topology is weaker than the Scott topology);

(ii) the mapping $\xi_X^\sigma : (X, \tau) \to (\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is continuous; and

(iii) $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$ is sober.

Then $X$ is sober.

**Proof.** Let $F$ be a closed irreducible subset of $X$. Then, as $\xi_X^\sigma$ is continuous, $\xi_X^\sigma(F)$ is an irreducible subset of $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$. Therefore, there exists $K \in \mathcal{D}(X)$ such that

$$\text{cl}_{\sigma(\mathcal{D}(X))}(\xi_X^\sigma(F)) = \downarrow_{\mathcal{D}(X)}K \ (= \{A \in \mathcal{D}(X) : K \subseteq A\}),$$

where $\text{cl}_{\sigma(\mathcal{D}(X))}$ is the closure operator in the Scott space $(\mathcal{D}(X), \sigma(\mathcal{D}(X)))$.

**Claim 1.** Every element of $K$ is an upper bound of $F$ in the poset $(X, \leq \tau)$.

In fact, let $k \in K$ and $x \in F$. Then $\uparrow k \in \mathcal{D}(X)$ and $\uparrow k \subseteq K$. In addition, $\uparrow x = \xi_X^\sigma(x) \in \downarrow_{\mathcal{D}(X)}K$, so $\uparrow x \supseteq K$. Hence $\uparrow x \supseteq K \supseteq \uparrow k$, which implies $x \leq \tau k$.

**Claim 2.** $K$ has a least element.

If, on the contrary, for each $k \in K$, there is $s(k) \in K$ such that $k \not\leq \tau s(k)$, then

$$K \subseteq \bigcup\{X \downarrow s(k) : k \in K\}.$$

Thus $K \in \square \bigcup\{X \downarrow s(k) : k \in K\} \in \sigma(\mathcal{D}(X))$ (by assumption (i)), implying

$$K \in \text{cl}_{\sigma(\mathcal{D}(X))}(\xi_X^\sigma(F)) \cap \square \bigcup\{X \downarrow s(k) : k \in K\}.$$

Hence

$$\text{cl}_{\sigma(\mathcal{D}(X))}(\xi_X^\sigma(F)) \cap \square \bigcup\{X \downarrow s(k) : k \in K\} \neq \emptyset.$$

Therefore $\xi_X^\sigma(F) \cap \square \bigcup\{X \downarrow s(k) : k \in K\} \neq \emptyset$. Hence there exists $y \in F$ with $\uparrow y \subseteq \bigcup\{X \downarrow s(k) : k \in K\}$. It follows that $y \not\leq \tau s(k)$ for some $s(k) \in K$. But this contradicts Claim 1 (every element of $K$ is an upper bound of $F$).

Therefore $K$ has a least element, say $s$. Then $K = \uparrow s$. 
Claim 3. \( F = \text{cl}_X(\{s\}) \).

As \( s \) is an upper bound of \( F \) and \( F \) is closed, we only need to confirm that \( s \in F \).

Assume that, on the contrary, \( s \notin F \). Then \( K \subseteq X \setminus F \), so
\[
K \in \text{cl}_{\sigma(\mathcal{D}(X))}(\xi_X^g(F)) \cap \Box(X \setminus F)
\]
and \( \Box(X \setminus F) \in \sigma(\mathcal{D}(X)) \).

Therefore \( \xi_X^g(F) \cap \Box(X \setminus F) \neq \emptyset \), which is impossible. Hence \( s \in F \), thus
\[
F = \downarrow s = \text{cl}_X(\{s\})
\]
and all these together show that \((X, \tau)\) is sober.

Proposition 1.

1. For every \( T_0 \) space \( X \), the mapping \( \xi_X : X \to \mathcal{D}(X) \) (the upper space of \( X \)) is a topological embedding.
2. For any poset \( P \), the mapping
\[
\xi_P^\sigma : (P, \sigma(P)) \to (\mathcal{D}(P), \sigma(\mathcal{D}(P)))
\]
is continuous, i.e., it preserves all existing directed suprema.
3. Every well-filtered space is a \( d \)-space.
4. A \( T_0 \) space \( X \) is well-filtered iff \( \mathcal{D}(X) \) is a dcpo and the upper Vietoris topology on \( \mathcal{D}(X) \) is contained in \( \sigma(\mathcal{D}(X)) \) (equivalently, the upper space \( \mathcal{D}(X) \) is a \( d \)-space).

Proof. (1) See [4].
(2) This follows from a straightforward verification.
(3) See [13, Corollary 3.3].
(4) See [12, Proposition 3].

In general, the well-filteredness of \( X \) is stronger than the condition that the upper Vietoris topology on \( \mathcal{D}(X) \) is contained in \( \sigma(\mathcal{D}(X)) \). For example, consider the poset \( \mathbb{N} \) of natural numbers with the usual order. Then every element in \( \mathbb{N} \) is compact and so \( \mathbb{N} \) is an algebraic poset. Hence \( \Sigma \mathbb{N} \) (\( \sigma(\mathbb{N}) \) equals the Aleksandrov topology on \( \mathbb{N} \)) is locally compact and \( \mathcal{D}(\mathbb{N}) = \{ \uparrow n : n \in \mathbb{N} \} \), which is isomorphic to \( \mathbb{N} \). Therefore \( \mathcal{D}(\mathbb{N}) \) is not a dcpo. Now the upper Vietoris topology on \( \mathcal{D}(\mathbb{N}) \) equals the Scott topology \( \sigma(\mathcal{D}(\mathbb{N})) \) (and also equals the Aleksandrov topology on \( \mathcal{D}(\mathbb{N}) \)). But \( \Sigma \mathbb{N} \) is not well-filtered.

Example 1. Let \( X \) be any non-countable set and \( \tau \) be the co-countable topology on \( X \). Then \((X, \tau)\) is a \( T_1 \) space. Clearly, the non-empty compact (saturated) subsets of \((X, \tau)\) are exactly the non-empty finite subsets of \( X \), that is, \( \mathcal{D}(X) = \{ F : F \text{ is a non-empty finite subset of } X \} \). Every directed subset \( \mathcal{E} \) of \( \mathcal{D}(X) \) has a largest element (which is the intersection of \( \mathcal{E} \)), so \((X, \tau)\) is well-filtered but non-sober (\( X \) is an irreducible closed set but not the closure of any singleton set). Clearly \( \mathcal{D}(X) \) is a dcpo and every element
in \(D(X)\) is compact. Hence \(D(X)\) is an algebraic domain and \(\sigma(D(X))\) (which equals the Aleksandrov topology) is sober. For \((X, \tau)\), the conditions (i) and (iii) in Theorem 1 are satisfied, but the assumption (ii) does not hold. In this case, the sobriety of \((D(X), \sigma(D(X)))\) does not imply the sobriety of \((X, \tau)\).

By Proposition 1(4), Theorem 1 can be restated as follows.

**Theorem 2.** Let \((X, \tau)\) be a \(T_0\) space such that

1. \(X\) is well-filtered;
2. the mapping \(\xi_X : (X, \tau) \to (D(X), \sigma(D(X)))\) is continuous; and
3. \((D(X), \sigma(D(X)))\) is sober.

Then \(X\) is sober.

By Theorem 2 and Proposition 1 we deduce the following.

**Corollary 1.** For a dcpo \(P\), if \((P, \sigma(P))\) is well-filtered and \((D(P), \sigma(D(P)))\) is sober (equivalently, the upper space \(D(P)\) is a d-space and \((D(P), \sigma(D(P)))\) is sober), then \((P, \sigma(P))\) is sober.

By Xi and Lawson [11], for any complete lattice \(L\), \((L, \sigma(L))\) is well-filtered. Now applying Corollary 1 we obtain the following.

**Theorem 3.** For any complete lattice \(L\), if \((L, \sigma(L))\) is non-sober, then \((D(L), \sigma(D(L)))\) is non-sober.

Now we are ready to answer Jung’s problem mentioned in the introduction.

**Example 2.** In [6], Isbell constructed a complete lattice whose Scott topology is non-sober, thus answered a question posed by Johnstone in [8]. Isbell’s complete lattice is not distributive. In one of his recent talks in Singapore, Achim Jung asked whether there is a distributive complete lattice whose Scott topology is non-sober. We now can give an affirmative answer. Let \(M\) be the complete lattice constructed by Isbell and let \(L = D(M)\). Then by Lemma 1, \(L\) is a complete Heyting algebra. Since the Scott space of \(M\) is non-sober, by Theorem 3 the Scott space of \(L\) is non-sober. Hence \(L\) is a complete Heyting algebra whose Scott space is non-sober.

**Remark 1.** For any \(T_0\) space \((X, \tau)\), the poset \((D(X), \supseteq)\) is a meet-semilattice, where the meet of \(K_1, K_2 \in D(X)\) equals \(K_1 \cup K_2\). Then clearly every principal filter \(\uparrow x = \{y \in X : x \leq \tau y\}\) is a prime element of \(D(X)\). In addition, for any \(K \in D(X)\),

\[
K = \bigwedge \{\uparrow x : x \in K\},
\]

showing that every element of \(D(X)\) can be expressed as a meet of prime elements. Hence by Lemma 1 \((D(L), \supseteq)\) is actually a spatial frame for any complete lattice \(L\) (see [9] for more about spatial frames). Thus the non-sober complete Heyting algebra \(L\) obtained in Example 2 is also a spatial frame.
3. Well-filteredness of upper spaces. In this section, the symbol \( \mathcal{D}(X) \) will denote the upper space of the topological space \( X \).

In [4], it is proved that a space \( X \) is sober iff the upper space \( \mathcal{D}(X) \) is sober. In [12], it is proved that a \( T_0 \) space \( X \) is well-filtered if and only if its upper space is a d-space. In that paper, the question was asked whether \( \mathcal{D}(X) \) will be well-filtered if \( X \) is well-filtered.

We now give an affirmative answer to the above problem.

**Lemma 2** ([4]). Let \( X \) be a topological space and \( A \) an irreducible subset of \( \mathcal{D}(X) \). Then every closed set \( C \subseteq X \) that meets all members of \( A \) contains a minimal irreducible closed subset \( A \) that still meets all members of \( A \).

The following result can be verified straightforwardly (see e.g. [9, p. 128] or [7, proof of Lemma 3.1]).

**Lemma 3.** If \( K \subseteq \mathcal{D}(X) \) is a non-empty compact subset of \( \mathcal{D}(X) \), then \( \bigcup K \in \mathcal{D}(X) \).

**Theorem 4.** A topological space \( X \) is well-filtered iff its upper space \( \mathcal{D}(X) \) is well-filtered.

**Proof.** By Proposition [13, 4], we only need to show that if \( X \) is well-filtered, then so is \( \mathcal{D}(X) \). Let \( \{ K_t : t \in T \} \) be a filtered family of saturated compact subsets of \( \mathcal{D}(X) \), and \( U = \bigcup \{ \square U_i : i \in I \} \) an open subset of \( \mathcal{D}(X) \) such that

\[
\bigcap \{ K_t : t \in T \} \subseteq U.
\]

Suppose that \( K_t \not\subseteq U \) for all \( t \), that is, \( K_t \cap (\mathcal{D}(X) \setminus U) \neq \emptyset \). Then as \( \{ K_t : t \in T \} \) is an irreducible subset of the space \( \mathcal{D}(X) \), by Lemma 2 there is a minimal closed irreducible subset \( C \subseteq \mathcal{D}(X) \setminus U \) that meets every \( K_t(t \in T) \).

For each \( t \in T \), let \( K_t = \bigcup (K_t \cap C) \). As \( K_t \cap C \) is non-empty and compact in \( \mathcal{D}(X) \), by Lemma 3 we have \( K_t \in \mathcal{D}(X) \). Also \( \{ K_t : t \in T \} \) is a filtered family of members of \( \mathcal{D}(X) \), thus \( K = \bigcap \{ K_t : t \in T \} \) is a member of \( \mathcal{D}(X) \) because \( X \) is well-filtered.

**Claim 1.** \( K \not\subseteq U \).

Assume, on the contrary, that \( K \in U \). Then \( K \in \square U_i \) for some \( i \in I \), so \( K = \bigcap \{ K_t : t \in T \} \subseteq U_i \). Thus, as \( X \) is well-filtered, \( K_t \subseteq U_i \) for some \( t \in T \). Then \( \emptyset \neq K_t \cap C \subseteq \square U_i \subseteq U \), contradicting \( C \subseteq \mathcal{D}(X) \setminus U \).

**Claim 2.** \( K \in \bigcap \{ \uparrow_{\mathcal{D}(X)}(K_t \cap C) : t \in T \} \).

Suppose, on the contrary, that \( K \not\in \bigcap \{ \uparrow_{\mathcal{D}(X)}(K_t \cap C) : t \in T \} \). Then there is \( t_0 \in T \) such that \( K \not\in \uparrow_{\mathcal{D}(X)}(K_{t_0} \cap C) \). Thus, for any \( G \in K_{t_0} \cap C \), there exists \( e(G) \in K \setminus G \). Then \( G \cap \downarrow e(G) = \emptyset \) (note that \( G \) is a saturated compact set). Now for any \( G \in K_{t_0} \cap C \) and any \( t \in T \), since \( e(G) \in K \) (so...
We have $e(G) \in K_t \cap C$. Thus there exists $H_t \in K_t \cap C$ such that $e(G) \in H_t$, implying

$$H_t \in K_t \cap C \cap \downarrow e(G).$$

It follows that

$$K_t \cap C \cap \downarrow e(G) \neq \emptyset \quad \text{for all } t \in T.$$ 

By the minimality of $C$, we have $C \cap \downarrow e(G) = C$, which implies that $C \subseteq \downarrow e(G)$.

Thus $C \subseteq \bigcap \{\downarrow e(G) : G \in K_{t_0} \cap C\}$. Note that for any $G \in K_{t_0} \cap C$, $G \notin \downarrow e(G)$. Hence

$$\emptyset \neq K_{t_0} \cap C = K_{t_0} \cap C \cap \bigcap \{\downarrow e(G) : G \in K_{t_0} \cap C\} = \emptyset.$$ 

This contradiction confirms Claim 2.

Now $K \in \bigcap \{\uparrow D(X)(K \cap C) : t \in T\} \subseteq \bigcap \{K_t : t \in T\} \subseteq U$, which implies $K \in U$. But this contradicts Claim 1.

All these together show that there must be some $t_0 \in T$ such that $K_{t_0} \subseteq U$. Hence $D(X)$ is well-filtered.

**Theorem 5.** For any $T_0$ space $X$, the following are equivalent:

1. $X$ is well-filtered.
2. $D(X)$ is a d-space.
3. $D(X)$ is well-filtered.

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