A New Relation between Lamb Shift Energies

Hiroaki Kubo,1 Takehisa Fujita,1 Naohiro Kanda,1 Hiroshi Kato,1 Yasunori Munakata,1 Sachiko Oshima,1† and Kazuhiro Tsuda1 †

1Department of Physics, Faculty of Science and Technology, Nihon University, Tokyo, Japan

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We derive a new relation between the observed Lamb shift energies of hydrogen and muonium atoms. The relation is based on the non-relativistic description of the Lamb shift, and the proper treatment of the reduced mass of electron and target particles (proton and muon) leads to the new formula which is expressed as

$$\frac{\Delta E_{2s_{1/2}}^{(H)}}{\Delta E_{2s_{1/2}}^{(\mu)}} = \left( \frac{1 + \frac{m_e}{m_\mu}}{1 + \frac{m_e}{M_p}} \right)^3.$$  

This relation achieves an excellent agreement with experiment and presents an important QED test free from the cutoff momentum Λ.

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I. INTRODUCTION

The Lamb shift energy in hydrogen atom is a symbol of the success in the QED renormalization scheme, and indeed the renormalization effect of the electron self-energy is responsible for the small deviation of the $2s_{1/2}$ energy from the prediction of the Dirac equation. In the theoretical calculation of the Lamb shift energy, there is some ambiguity which arises from the cutoff momentum Λ since the calculation is only possible for the non-relativistic treatment, at least, up to the present stage. In the non-relativistic evaluation, the Lamb shift energy has a logarithmic divergence, and people take the cutoff momentum Λ as electron mass, that is, Λ = $m_e$, which has, of course, no physically plausible reason. In addition, people normally consider the Coulomb propagator modification as

$$\frac{1}{q^2} \Rightarrow \frac{1}{q^2} \left( 1 - \frac{\alpha}{15 \pi m_e^2} \right) q^2$$

which is discussed in the textbook of Bjorken and Drell. However, the Coulomb field $A_0$ should not be quantized since it is a time independent field. Therefore, there should not be any change of the Coulomb propagator and the modified propagator discussed in Bjorken and Drell is not a correct treatment.

Here, we present a careful calculation of the Lamb shift energy in hydrogen and muonium atoms with the non-relativistic field equations. In this calculation, we find a new relation between Lamb shift energies of hydrogen and muonium atoms. The relation does not depend on the cutoff momentum Λ and achieves an excellent agreement with experimental observations of Lamb shifts, even though the experimental uncertainty of the Lamb shift energy in muonium is still too large to decide which of the theoretical model calculations should be preferred.

II. QUANTIZATION OF COULOMB FIELDS

Before going to the discussion of the Lamb shift energy, we should first clarify the quantization of the electromagnetic field $A_\mu$. After taking the Coulomb gauge fixing of $\nabla \cdot A = 0$, the field equation $\partial_\mu F^{\mu\nu} = e j^\nu$ can be written for the $A_0$ field as

$$\nabla^2 A^0 = -e j^0$$

*Electronic address: n-kubo@phys.cst.nihon-u.ac.jp
†Electronic address: fffujita@phys.cst.nihon-u.ac.jp
‡Electronic address: nkanda@phys.cst.nihon-u.ac.jp
§Electronic address: hhkato@phys.cst.nihon-u.ac.jp
¶Electronic address: munakata@phys.cst.nihon-u.ac.jp
∗ Electronic address: oshima@phys.cst.nihon-u.ac.jp
†† Electronic address: nobita@phys.cst.nihon-u.ac.jp
where \( j^0_e \) denotes the current density of electron. This is a constraint equation and therefore the \( A_0 \) field can be solved and written in terms of the electron current \( j^0_e \) as

\[
A^0(r) = \frac{e}{4\pi} \int \frac{j^0_e(r')}{|r - r'|} d^3 r'.
\]

(2.2)

This means that there is no way to quantize the \( A_0 \) field, even though, in the literatures [1], this field is often quantized in the same way as the vector field \( A \). The important point is that fields should be quantized only when they are time dependent. The creation and annihilation operators depend, of course, on time since they occur at some fixed point of time in the reaction processes. In terms of the Hamiltonian, the Coulomb interaction \( H_C \) can be written as

\[
H_C = -\frac{e^2}{4\pi} \int \frac{j^0_p(r')j^0_e(r)d^3 r d^3 r'}{|r' - r|}
\]

(2.3)

where \( j^0_p \) denotes the proton current density. This expression is independent of the gauge choice and it clearly states that the Coulomb interaction is not influenced by the higher order corrections since eq. (2.3) is exact.

A. Uehling Potential

Here, we should comment on the Uehling potential which is essentially the same as the finite term of the vacuum polarization in the Coulomb potential [5]. Uehling obtained the induced charge distribution due to the creation of electron and positron pairs in the vacuum as

\[
\delta \rho(r) = -\frac{\alpha}{15\pi m_e^2} \nabla^2 \rho(r)
\]

(2.4)

which can generate the Uehling potential. However, it is well established by now that the creation of fermion pairs can be possible only when the fermion fields can couple to the vector field \( A \). Therefore, when the fermions interact with the Coulomb field \( A^0 \), there is no physical process which can create the fermion pairs. As an intuitive picture, one may say that the static Coulomb field cannot make any polarizations in the vacuum since the pair creations are physical processes which involve time dependent fluctuations of the vector field \( A \).

B. Classical Picture of Polarization

This classical picture of the fermion pair creations (Uehling potential) should come from the misunderstanding of the structure of the vacuum state. In the medium of solid state physics, the polarization can take place when there is an electric field present. In this case, the electric field can indeed induce the electric dipole moments in the medium, and this corresponds to the change of the charge density. However, this is a physical process which can happen in the real space (configuration space). On the other hand, the fermion pair creation in the vacuum in field theory is completely different in that the negative energy states are all filled in momentum space, and the time independent field of \( A^0 \) which is only a function of coordinates cannot induce any changes on the vacuum state. Therefore, unless some time dependent field is present in the reaction process, the pair creation of fermions cannot take place in physical processes.

Therefore, in contrast to the common belief, there is, unfortunately, no change of the charge distribution in QED vacuum, even at the presence of two charges, and this is basically because the Coulomb field is not time dependent.

C. Higher Order Corrections

Therefore, there are no higher order corrections of the vertex corrections and vacuum polarization effects to the Lamb shift energy. As a possible effect on the Lamb shift energy, there may be two loop self-energy corrections to the Lamb shift energy [6]. However, before examining the higher order corrections, we may have to understand how to control the cutoff \( \Lambda \) effects in the non-relativistic treatment of the Lamb shift energy.
III. LAMB SHIFT IN HYDROGEN ATOM

Here, we present a standard scheme of the Lamb shift in hydrogen atom which is based on the non-relativistic treatment.

A. Non-relativistic Treatment

We start from the Hamiltonian for electron in hydrogen atom with the electromagnetic interaction

\[ H = \frac{\hat{p}^2}{2m_0} - \frac{e^2}{r} - \frac{e}{m_0} \hat{p} \cdot \hat{A} \]  

(3.1)

where the \( \hat{A}^2 \) term is ignored in the Hamiltonian. In this calculation, the electromagnetic field \( \hat{A} \) should be quantized

\[ \hat{A}(x) = \sum_k \sum_{\lambda=1}^{2} \frac{1}{\sqrt{2V \omega_k}} e(k, \lambda) \left[ c_{k,\lambda} e^{-ikx} + c_{k,\lambda}^\dagger e^{ikx} \right] \]  

(3.2)

where \( c_{k,\lambda} \) and \( c_{k,\lambda}^\dagger \) denote the creation and annihilation operators which satisfy the following commutation relations

\[ [c_{k,\lambda}, c_{k',\lambda'}^\dagger] = \delta_{k,k'} \delta_{\lambda,\lambda'} \]  

(3.3)

and all other commutation relations vanish.

B. Second Order Perturbation Energy

Now, the second order perturbation energy due to the electromagnetic interaction for a free electron state can be written as

\[ \delta E = - \sum_\lambda \sum_k \sum_{p'} \left( \frac{e}{m_0} \right)^2 \frac{1}{2V \omega_k} \frac{|\langle p' | e(k, \lambda) \cdot \hat{p} | p \rangle|^2}{E_{p'} + k - E_p} \]  

(3.4)

where \( |p\rangle \) and \( |p'\rangle \) denote the free electron state with its momentum. Since the photon energy (\( \omega_k = k \)) is much larger than the energy difference of the electron states (\( E_{p'} - E_p \)), one obtains

\[ \delta E = - \frac{1}{6\pi^2 \Lambda} \left( \frac{e}{m_0} \right)^2 p^2 \]  

(3.5)

where \( \Lambda \) is the cutoff momentum of photon. This divergence is proportional to the cutoff \( \Lambda \) which is not the logarithmic divergence. However, this is essentially due to the non-relativistic treatment, and if one carries out the relativistic calculation of quantum field theory, then the divergence becomes logarithmic.

C. Mass Renormalization and New Hamiltonian

Defining the effective mass \( \delta m \) as

\[ \delta m = \frac{1}{3\pi^2} \frac{\Lambda e^2}{m_0} \]  

(3.6)

the free energy of electron can be written as

\[ E_F = \frac{p^2}{2m_0} - \frac{p^2}{2m_0^2} \delta m \approx \frac{p^2}{2(m_0 + \delta m)} \]  

(3.7)

where one should keep only the term up to order of \( e^2 \) because of the perturbative expansion. Now, one defines the renormalized (physical) electron mass \( m_e \) by

\[ m_e = m_0 + \delta m \]  

(3.8)
and rewrites the Hamiltonian $H$ in terms of the renormalized electron mass $m$

$$H = \frac{p^2}{2m_e} - \frac{e^2}{r} + \frac{\hat{p}^2}{2m_e^2} \delta m - \frac{e}{m_e} \hat{p} \cdot \hat{A}.$$ 

(3.9)

Here, the third term ($\frac{\hat{p}^2}{2m_e^2} \delta m$) corresponds to the counter term which cancels out the second order perturbation energy [eq.(3.4)].

### D. Lamb Shift Energy in Hydrogen Atom

Now, we consider hydrogen atom, and its Hamiltonian can be written as

$$H_0 = \frac{\hat{p}^2}{2m_r} - \frac{e^2}{r}$$

(3.10)

where $m_r$ denotes the reduced mass of the electron and proton system. Using eq.(3.9), one can calculate the first and the second order perturbation energies due to the electromagnetic interaction for the $2s_{1/2}$ state in hydrogen atom

$$\Delta E_{2s_{1/2}} = \frac{1}{6\pi^2} \Lambda \left( \frac{e}{m_e} \right)^2 \langle 2s_{1/2} | \hat{p}^2 | 2s_{1/2} \rangle - \sum_{\lambda} \sum_k \sum_{n,\ell} \left( \frac{e}{m_e} \right)^2 \frac{1}{2V\omega_k} | \langle n, \ell | \epsilon(k, \lambda) : \hat{p} | 2s_{1/2} \rangle |^2$$

(3.11)

where the first term comes from the counter term. This Lamb shift energy for the $2s_{1/2}$ state can be rewritten as

$$\Delta E_{2s_{1/2}} = \frac{1}{6\pi^2} \left( \frac{e}{m_e} \right)^2 \sum_{n,\ell} | \langle n, \ell | \hat{p} | 2s_{1/2} \rangle |^2$$

$$\times \int_0^\Lambda dk \frac{E_{n,\ell} - E_{2s_{1/2}}}{E_{n,\ell} + k - E_{2s_{1/2}}}.$$  

(3.12)

After some calculations, we obtain

$$\Delta E_{2s_{1/2}} = \frac{2m_r^2 \alpha^5}{3m_e^2} \ln \left( \frac{\Lambda}{< E_{n,\ell} >} \right)$$

(3.13)

where we have neglected the $(n, \ell)$ dependence in the denominator when summing up $(n, \ell)$, and $< E_{n,\ell} >$ is defined as some average value of the excitation energies with respect to the $2s_{1/2}$ state. For the cutoff $\Lambda$, people normally take $\Lambda \approx m_e$, but there is no special reason why one should take the $\Lambda$ as electron mass.

### E. Cutoff $\Lambda$ Dependence

Here, we should note that there is, of course, no way to get rid of the cutoff $\Lambda$ in the Lamb shift energy in the non-relativistic treatment. The divergence of the Lamb shift energy is inevitably a linear one in the free state, and it becomes the logarithmic divergence. If one can obtain some results which are convergent, then this means that he must have made some mistakes.

### IV. PHYSICAL MEANING OF CUTOFF $\Lambda$

As one sees, the calculated result of the Lamb shift energy depends on the cutoff $\Lambda$, which is not satisfactory at all. However, it should be noted that there is no way to avoid the presence of the cutoff $\Lambda$ as long as we treat the Lamb shift in the non-relativistic field equations.

The important point is that we should understand the origin of the value of the cutoff $\Lambda$. This should, of course, be understood if one treats it relativistically.
A. Relativistic Treatment of Lamb Shift

The correct scenario of the relativistic treatment of Lamb shift must be as follows. In the non-relativistic treatment, the mass counter term is linear divergent. However, if one treats it relativistically, the divergence is logarithmic. This reason of the one rank down of the divergence is originated from the fact that the relativistic treatment considers the negative energy states which in fact reduce the divergence rank due to the cancellation. Now, we consider the renormalization effect in hydrogen atom, and if we calculate the Lamb shift energy in the non-relativistic treatment, then it has the logarithmic divergence as we saw above, and this is the one rank down of the divergence. This is due to the fact that the evaluation of the Lamb shift energy is based on the cancellation between the counter term and the perturbation energy in hydrogen atom. In the same way, if one can calculate the Lamb shift energy relativistically, then one should obtain the one rank down of the divergence, and this means that it should be finite.

Unfortunately, one cannot carry out the relativistic calculation of the Lamb shift energy because of the conceptual difficulty related to the negative energy states of the bound systems like hydrogen atom. However, we may understand the physical meaning of the cutoff \( \Lambda \) in the non-relativistic treatment. The value of the \( \Lambda \) must be chosen such that the inclusion of the negative energy states are properly simulated in the non-relativistic calculation of the Lamb shift. This indicates that we should take the \( \Lambda \) value so as to reproduce the experimental observation when we evaluate eq.(3.13), and this is the physical reason of the \( \Lambda \) value why we can take a finite value of \( \Lambda \).

B. Lamb Shift in Anti-hydrogen Atom

The structure of the negative energy state should be examined if one can measure the Lamb shift of anti-hydrogen atom. In this case, positron should feel the effect of the vacuum state which should be somewhat different from the case in which electron may feel in the same situation. The Lamb shift energy of the \( 2s_{1/2} \) state in anti-hydrogen atom can be written as

\[
\Delta E_{2s_{1/2}} = \frac{2m_e^3\alpha^5}{3m_e^2} \ln \left( \frac{\bar{\Lambda}}{\langle E_{n,\ell} \rangle} \right)
\]

(4.1)

where \( \bar{\Lambda} \) denotes the effective cutoff value of the anti-hydrogen atom. If the observed value of the \( \bar{\Lambda} \) differs from the \( \Lambda \) value, then it should mean that there is some chance of understanding the structure of the negative energy state in the interacting field theory model.

C. Higher Order Corrections

Up to now, there are many calculations which treat the higher order corrections to the Lamb shift energy. However, as we discuss in section 2, the Coulomb interaction or Coulomb propagator cannot be influenced by the higher order effects. The only possible higher order effects must be due to the self-energy of electron at the order of \( e^4 \) level \([6]\). However, the Lamb shift energy depends on the cutoff \( \Lambda \), and therefore there is no chance that we can predict the Lamb shift energy to a high accuracy. In this respect, we should give up reproducing the absolute value of the Lamb shift energy in hydrogen atom with sufficient accuracy.

V. LAMB SHIFT IN MUONIUM

There are very interesting measurements of the Lamb shift energy of \( 2s_{1/2} \) state in muonium (\( \mu^+e^- \) system) \([3, 7–9]\). This presents an important QED test, and we will compare the experimental value with the predictions.

A. Calculations with Vacuum Polarizations

The Lamb shift energy of \( 2s_{1/2} \) state in muonium has been calculated \([10, 12]\), and the predicted value becomes

\[
\Delta E_{2s_{1/2}}^{(\mu)} = 1047.5 \text{ MHz}
\]

where the main deviation from the hydrogen atom case is due to the vacuum polarization contributions.
B. Prediction with New Relation

On the other hand, we see that the Lamb shift of muonium can be related to that of hydrogen atom as

\[ \Delta E_{2s_{1/2}}^{(\mu)} = \left( \frac{m_{\tau}^{(\mu)}}{m_{\tau}^{(H)}} \right)^3 \Delta E_{2s_{1/2}}^{(H)} \]  

(5.1)

where \( m_{\tau}^{(\mu)} \) and \( m_{\tau}^{(H)} \) are given as

\[ m_{\tau}^{(\mu)} = \frac{m_e}{1 + \frac{m_e}{m_{\mu}}}, \quad m_{\tau}^{(H)} = \frac{m_e}{1 + \frac{m_e}{M_p}}. \]  

(5.2)

Here, \( m_\mu \) and \( M_p \) denote the masses of muon and proton, respectively. Using the experimental value of the hydrogen Lamb shift energy

\[ \Delta E_{2s_{1/2}}^{(H)} (exp) = 1057.862 \pm 0.020 \text{ MHz} \]  

(5.3)

we can predict the Lamb shift energy of muonium

\[ \Delta E_{2s_{1/2}}^{(\mu)} (th) = 1044 \text{ MHz}. \]  

(5.4)

This value should be compared to the observed value

\[ \Delta E_{2s_{1/2}}^{(\mu)} (exp) = 1042 \pm 22 \text{ MHz}. \]  

(5.5)

As can be seen, the agreement is remarkable, and the important point of the prediction by the new relation is that it does not depend on the cutoff \( \Lambda \). However, the experimental accuracy is not yet sufficient to decide which of the model calculations is preferred. In this respect, it should be very important to improve the observed accuracy of the Lamb shift energy in muonium since this should be a very good test of the QED renormalization scheme.

VI. CONCLUSIONS

We have presented a new relation of the Lamb shift energies between the hydrogen and muonium systems, which gives a good test of the QED renormalization scheme. This is quite important since the relation does not depend on the cutoff \( \Lambda \). This clearly shows that the renormalization scheme is indeed a correct theoretical framework even though the absolute magnitude of the Lamb shift energy cannot be properly predicted by the non-relativistic treatment.

The new relation can predict the Lamb shift energy of muonium, and the observed Lamb shift energy of muonium is consistent with the prediction, though, at present, the experimental uncertainty of the measurement in muonium is still a bit too large. However, if the error bar of the observed value can be improved, then there is a good chance that the muonium Lamb shift energy can give a stringent test of QED renormalization scheme.

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