Massive Coded-NOMA for Low-Capacity Channels: A Low-Complexity Recursive Approach

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Abstract—In this paper, we present a low-complexity recursive approach for massive and scalable code-domain nonorthogonal multiple access (NOMA) with applications to emerging low-capacity scenarios. The problem definition in this paper is inspired by three major requirements of the next generations of wireless networks. Firstly, the proposed scheme is particularly beneficial in low-capacity regimes which is important in practical scenarios of utmost interest such as the Internet-of-Things (IoT) and massive machine-type communication (mMTC). Secondly, we employ code-domain NOMA to efficiently share the scarce common resources among the users. Finally, the proposed recursive approach enables code-domain NOMA with low-complexity detection algorithms that are scalable with the number of users to satisfy the requirements of massive connectivity. To this end, we propose a novel encoding and decoding scheme for code-domain NOMA based on factorizing the pattern matrix, for assigning the available resource elements to the users, as the Kronecker product of several smaller factor matrices. As a result, both the pattern matrix design at the transmitter side and the mixed symbols’ detection at the receiver side can be performed over matrices with dimensions that are much smaller than the overall pattern matrix. Consequently, this leads to significant reduction in both the complexity and the latency of the detection. We present the detection algorithm for the general case of factor matrices. The proposed algorithm involves several recursions each involving certain sets of equations corresponding to a certain factor matrix. We then characterize the system performance in terms of average sum rate, latency, and detection complexity. Our latency and complexity analysis confirm the superiority of our proposed scheme in enabling large pattern matrices. Moreover, our numerical results for the average sum rate show that the proposed scheme provides better performance compared to straightforward code-domain NOMA with comparable complexity, especially at low-capacity regimes.

Index Terms—Code-domain NOMA, low-capacity channels, massive communication, low-complexity recursive detection, low-latency communication, IoT, mMTC.

I. INTRODUCTION

Low-capacity scenarios have become increasingly important in a variety of emerging applications such as the Internet-of-Things (IoT) and massive machine-type communication (mMTC) [2]. For example, the narrowband IoT (NB-IoT) feature, included in Release-13 of the 3rd generation partnership project (3GPP), is specifically meant for ultra-low-rate, wide-area, and low-power applications [3]. To ensure wide-area applications, NB-IoT is designed to support maximum coupling losses (MCLs) as large as 170 dB. Achieving such large MCLs requires reliable detection for signal-to-noise ratios (SNRs) as low as $-13$ dB [4]. Consequently, one needs to carefully design the communication protocols aimed for massive communication applications, such as IoT and mMTC, with respect to the low-SNR constraints.

Recently, nonorthogonal multiple access (NOMA) techniques, that borrow ideas from solutions to traditional problems in network information theory including multiple access and broadcast channels, have gained significant attention. In NOMA schemes, multiple users are served in the same orthogonal resource element (RE) or, more generally speaking, a set of users are served in a smaller set of REs. The goal is to significantly increase the system throughput and reliability, improve the users’ fairness, reduce the latency, and support massive connectivity [5], [6]. NOMA is a general setup and, in principle, any multiple access scheme that attempts to non-orthogonally share the REs among the users, e.g., random multiple access [7]–[9] and opportunistic approaches [10], can be formulated in this setting. In general, NOMA techniques in the literature can be classified into two categories: power-domain NOMA and code-domain NOMA.

Power-domain NOMA serves multiple users in the same orthogonal RE by properly allocating different power levels to the users [11]. Power-domain NOMA has attracted significant attention in recent years and several problems have been explored in this context. This includes cooperative communication [12]–[14], simultaneous wireless information and power transfer (SWIPT) [15], multiple-input multiple-output (MIMO) systems [16], mmWave communications [17], mixed radio frequency and free-space optics (RF-FSO) systems [18], [19], unmanned aerial vehicles (UAVs) communications [20], and cache-aided systems [21]. The detection procedure in power-domain NOMA highly relies on successive interference cancellation (SIC) which works well provided that the channel conditions of the users paired together are not close to each other.

Code-domain NOMA, on the other hand, aims at serving a set of users, say $K$, in a set of $M$ orthogonal REs, with $M \leq K$, using a certain code/pattern matrix. The pattern matrix comprises $K$ pattern vectors each assigned to a user specifying the set of available REs to that user. Unlike power-domain NOMA, code-domain NOMA works well even in power-balanced scenarios provided that each user is served by a unique pattern vector. Nevertheless, code-domain NOMA has received relatively less attention in the literature compared to the power-domain NOMA. This is mainly because its gain usually comes at the expense of complex multiuser detection (MUD) algorithms, such as maximum a posteriori (MAP) detection, message passing algorithm (MPA), and maximum likelihood (ML) detection. In this context, sparse code multiple
access (SCMA) is proposed in [22] based on directly mapping the incoming bits to multidimensional codewords of a certain SCMA codebook set. Also, a low-complexity SCMA decoding algorithm is proposed in [23] based on list sphere decoding. Another efficient SCMA decoder is discussed in [24] based on deterministic message passing algorithms. Moreover, lattice partition multiple access (LPMA) is proposed in [25–27] based on multilevel lattice codes for multiple users. Also, interleave-grid multiple access (IGMA) is proposed in [28], [29] in order to increase the user multiplexing capability and to improve the performance. In addition, pattern division multiple access (PDMA) is introduced in [30], [31], where the pattern vectors are designed with disparate orders of transmission diversity to mitigate the error propagation problem in SIC receivers. As opposed to most of the aforementioned prior works in code-domain NOMA, the pattern matrix in PDMA is not necessarily sparse. In other words, the number of REs allocated to a particular user can potentially be comparable to the total number of REs.

Design of the pattern matrix plays a critical role in code-domain NOMA to balance the trade-off between the system performance and the complexity. Impact of the pattern matrix on the average sum-rate of SCMA systems is explored in [32], where a low-complexity iterative algorithm is proposed to facilitate the design of the pattern matrix. Moreover, the total throughput of low-density code-domain (LDCD) NOMA is characterized in [33] for regular random pattern matrices with large dimensions. It is well understood that, for a given overload factor \( \beta \triangleq K/M \), expanding the dimension of the pattern matrix improves the system performance [31]. However, increasing the pattern matrix dimension significantly increases the detection complexity.

Inspired by the aforementioned trade-off between the system performance and the detection complexity, we propose a novel encoding and decoding approach toward code-domain NOMA which factorizes the pattern matrix as the Kronecker product of smaller factor matrices. Consequently, as we show, both the pattern matrix design at the transmitter side and the mixed symbols’ detection at the receiver side can be performed over much smaller dimensions and with significantly reduced complexity, through recursive detection. In other words, the system performance is significantly improved while keeping the detection complexity reasonably low.

The low complexity of the proposed detection algorithm for large dimension pattern matrices enables grouping a massive number of users together, referred to as “massive coded-NOMA” in this paper. Moreover, the proposed multiple access technique is particularly advantageous in the scenarios where there is a stringent constraint on the maximum power of the users’ symbols. Therefore, even full-power transmission of the users’ symbols (up to a predefined maximum allowed power) over only one RE (or few REs) does not meet the users’ desired data rates. In such circumstances, users are inevitable to transmit over several REs, possibly with the maximum allowed power per symbol, to achieve the desired data rates. For instance, a large number of repetitions are allowed in low-capacity scenarios such as NB-IoT and mMTC in order to enable reliable communications in these emerging applications.

Our proposed protocol, through facilitating coded-NOMA over large-dimension pattern matrices, enables spreading the symbols of the low-capacity users over several REs and then detecting them with a low complexity.

In this paper, we start by discussing the system model and the design principles. Then the traditional Gaussian multiple access channel (GMAC) model is considered. For GMAC, we first describe the detection algorithm over the Kronecker product of square matrices and rectangular matrices, and then characterize the detection algorithm for the general case with arbitrary factor matrices. Given the design procedure and the detection algorithm, we characterize the system performance in terms of average sum rate, latency, and detection complexity. Finally, we discuss how the results can be extended to several other major scenarios including uplink and downlink fading channels and joint power-and code-domain NOMA (JPC-NOMA) systems.

The rest of the paper is organized as follows. In Section II, we briefly describe the system and channel models. In Section III, we focus on the design procedure and detection algorithm of the proposed protocol over GMAC. We then characterize the average sum rate, latency, and detection complexity in Section IV. We devote Section V to more realistic scenarios and future directions. Finally, we provide the numerical results in Section VI, and conclude the paper in Section VII.

II. Preliminaries and System Model

In this section, we briefly review the basics of code-domain NOMA, specifically PDMA [31], which is relevant to the system model of our proposed scheme. We consider a collection of \( K \) users communicating using \( M \) REs, and define the overload factor as \( \beta \triangleq K/M \geq 1 \) implying the multiplexing gain of PDMA compared to the orthogonal multiple access (OMA). The modulation symbol \( x_k \) of the \( k \)-th user is spread over \( M \) orthogonal REs using the pattern vector \( g_k \) as \( y_k = g_k x_k \), \( 1 \leq k \leq K \), where \( g_k \in \mathbb{B}^{M \times 1} \), with \( \mathbb{B} \triangleq \{0,1\} \), is an \( M \times 1 \) binary vector defining the set of REs available to the \( k \)-th user; the \( k \)-th user can use the \( i \)-th RE, \( 1 \leq i \leq M \), if the \( i \)-th element of \( g_k \) is “1”, i.e., if \( g_{i,k} = 1 \). Otherwise, if \( g_{i,k} = 0 \), then the \( k \)-th user does not use the \( i \)-th RE. Then the overall \((M \times K)\)-dimensional pattern matrix \( G \in \mathbb{B}^{M \times K} \) is specified as follows [31]:

\[
G_{M \times K} \triangleq [g_1 g_2 \cdots g_K] = [g_{i,k}]_{M \times K}.
\]

For the uplink transmission each user transmits its spread symbol to the base station (BS). Therefore, assuming perfect synchronization at the BS, the vector \( y \) comprising the received signals at all \( M \) REs can be modeled as

\[
y = \sum_{k=1}^{K} \text{diag}(h_k) y_k + n,
\]

where \( h_k \) is the vector modeling the uplink channel response of the \( k \)-th user at all of \( M \) REs, and \( n \) is the noise vector at the BS with length \( M \). Furthermore, \( \text{diag}(h_k) \) is a diagonal matrix consisting of the elements of \( h_k \). Then the uplink transmission model can be reformulated as

\[
y = Hx + n,
\]
where \( x \triangleq [x_1, x_2, \ldots, x_K]^T \), \( H \triangleq \mathcal{H} \circ G_{M \times K} \) is the PDMA equivalent uplink channel response, \( \mathcal{H} \triangleq [h_1, h_2, \ldots, h_K] \), \( A^T \) is the transpose of the matrix \( A \), and \( \odot \) denotes the element-wise product [31].

Moreover, for the downlink transmission, the BS first encodes the data symbol of each user according to its pattern vector and then transmits the superimposed encoded symbols \( \sum_{j} v_j \) through the channel. Therefore, the received signal \( y_k \) at the \( k \)-th user can be expressed as

\[
y_k = \text{diag}(h_k) \sum_j H_k x_j + n_k = H_k x + n_k, \quad (4)
\]

where \( H_k \triangleq \text{diag}(h_k)G_{M \times K} \) is the PDMA equivalent downlink channel response of the \( k \)-th user. Moreover, \( h_k \) and \( n_k \) are the downlink channel response and the noise vector, both with length \( M \), at the \( k \)-th user, respectively [31].

As a standard MUD algorithm in NOMA systems SIC provides a proper trade-off between the system performance and the complexity. However, SIC receivers often suffer from error propagation problems as the system performance is highly dependent on the correctness of early-detected symbols. In order to resolve this issue, one can either improve the reliability of the initially-detected users or employ more advanced detection algorithms including ML and MAP. In [31], disparate diversity orders are adopted for different users by assigning patterns with heavier weights to those early-detected users in order to increase their transmission reliability. Furthermore, employing more advanced detection algorithms severely increases the system complexity especially for larger pattern matrices; this may hinder their practical implementation particularly for downlink transmission where the users are supposed to have a lower computational resources than the BS.

In the next section, we elaborate how the proposed scheme scales with the dimensions of the pattern matrix, even with rather heavy pattern weights, to boost massive connectivity without a significant increase on the overall system complexity. In Sections III and IV, we consider the conventional model of GMAC, i.e., \( y = Gx + n \), that corresponds to the case where \( \mathcal{H} (h_k) \), specified in (3) (4), is replaced by an all-one matrix (vector). We then demonstrate in Section V how the results can be extended to the cases of uplink and downlink channels, as in (3) and (4), respectively.

### III. Design and Detection over Gaussian MAC

#### A. Design Principles

Depending on the number of users and the available REs to them, we propose to consider pattern matrices that are factorized as the Kronecker product of a certain number, denoted by \( L \), of smaller factor matrices as follows:

\[
G_{M \times K} = G^{(1)}_{m_1 \times k_1} \odot G^{(2)}_{m_2 \times k_2} \odot \cdots \odot G^{(L)}_{m_L \times k_L}, \quad (5)
\]

where \( L \) is a design parameter and \( \odot \) denotes the Kronecker product defined as

\[
A_{m \times k} \odot B_{m' \times k'} \triangleq \begin{bmatrix}
    a_{11}B & \cdots & a_{1k}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \cdots & a_{mk}B
\end{bmatrix}, \quad (6)
\]

for any two matrices \( A \) and \( B \). The dimensions of the resulting pattern matrix in (5) relate to the dimensions of the factor matrices as \( M = \prod_{l=1}^{L} m_l \) and \( K = \prod_{k=1}^{k} k_l \). In general, if at least one of \( M \) or \( K \) is not a prime number, we can find some \( m_l > 1 \) or \( k_l > 1 \) to construct the pattern matrix as the Kronecker product of some smaller factor matrices. On the other hand, if both \( M \) and \( K \) are prime numbers, \( L \) is equal to one and the design procedure simplifies to that of conventional pattern matrix design (such as PDMA [31]). Note that both \( M \) and \( K \) are design parameters and we can always properly group a certain number of users over a desired number of REs to optimize the system performance and the complexity.

The proposed structure not only alleviates the detection complexity and the latency at the receiver side (through a recursive detection discussed in the next subsections) but also significantly reduces the search space at the transmitting party, enabling the usage of large pattern matrices with a reasonable complexity for massive connectivity. In fact, a regular pattern matrix design requires a comprehensive search over all \((2^M-1) \) possible \((M \times K)\)-dimensional matrices with distinct nonzero columns (patterns assigned to each user) to find an optimal pattern matrix [31]. On the other hand, it is easy to show that satisfying distinct nonzero columns for the overall pattern matrix \( G_{M \times K} \) of the form given by (5) requires distinct nonzero columns for all of the factor matrices \( G^{(l)}_{m_l \times k_l}, l = 1, 2, \ldots, L \). Therefore, the search space for our proposed design method reduces to \( \prod_{l=1}^{L} (2^{m_l} - 1) \) which is significantly smaller than \((2^M-1) \). For example, while \( M = 6 \) and \( K = 9 \), the regular pattern matrix design requires searching over \((2^6-1) \approx 2.36 \times 10^{10} \) possible matrices, while our design method with the factorization of \( G^{(1)}_{2 \times 3} \otimes G^{(2)}_{3 \times 3} \) only needs to search over \((2^2-1)(2^3-1) = 35 \) matrices.

That recursive construction of large matrices based on the Kronecker product of some smaller matrices has been used in different contexts such as polar coding [34] and its extended versions such as compound polar coding [35, 36].

#### B. Square Factor Matrices

In this subsection, we explore the design of square factor matrices and describe the corresponding detection algorithm over GMAC. In particular, we assume that the overall pattern matrix is represented as \( G_{M \times K} = P^{(1)}_{m_1 \times m_1} \otimes P^{(2)}_{m_2 \times m_2} \otimes \cdots \otimes P^{(L)}_{m_L \times m_L} \), where \( P^{(l)}_{m \times m}, l = 1, 2, \ldots, L \), is an \( m \times m \) binary square matrix. In this case, \( M = K = \prod_{l=1}^{L} m_l \) and the overload factor \( \beta \) is equal to 1. However, we will observe that with a careful design of the square factor matrices one can improve the effective SNR of individual data symbols to a desired level that can guarantee a predetermined data rate. As it will be clarified in the next example, we define the effective SNR as the SNR of the individual data symbols at the end of the detection algorithm after several rounds of recursive combining. The aforementioned gain is obtained by a low-complexity detection algorithm involving only few linear operations (additions/subtractions) at each recursion, as detailed in Section III.B2. Therefore, this scheme is useful especially when the transmission of each symbol over each
RE is constrained by a maximum power limit which is often the case in low-capacity scenarios as discussed in Section I. To proceed, we begin with the following illustrative example that helps clarifying the design procedure and the recursive detection algorithm provided afterward.

**Example 1.** Consider the transmission of $K = 12$ users (symbols) over $M = 12$ REs realized using the Kronecker product of the following two square factor matrices

$$
P_{3 \times 3}^{(1)} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_{4 \times 4}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.
$$

(7)

Let us further define the following square matrices $\alpha_{i,j}^{(l)}$, $l = 1, 2$, such that the matrix product $\alpha_{i,j}^{(l)} P_{i,j}$ is diagonal.

$$
\alpha_{3 \times 3}^{(1)} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \alpha_{4 \times 4}^{(2)} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
$$

(8)

Then $y_{12 \times 1} = G_{12 \times 12} x_{12 \times 1} + n_{12 \times 1}$, with $G_{12 \times 12} = P_{3 \times 3}^{(1)} \otimes P_{4 \times 4}^{(2)}$, defines the received signals vector.

The resulting set of 12 equations for the received signals $y_i$’s over all REs can be analyzed using different MUD methods to detect the data symbols $x_i$’s, $i = 1, 2, \ldots, 12$. However, the proposed design method in this paper enables recursive detection of the data symbols with a significantly lower complexity. Here, for the sake of brevity, we focus on the detection of $x_1$, $x_5$, and $x_9$. The rest of the symbols can be detected in a similar fashion. Let us define $y_{1}^{(1)}$, $y_{5}^{(1)}$, and $y_{9}^{(1)}$ as each the linear combination of four consecutive received signals according to the first row of $\alpha_{i,j}^{(2)}$, i.e.,

$$
y_{1}^{(1)} = y_3 + y_4 - y_2 = 2x_1 + 2x_5 + n_{1}^{(1)}, \\
y_{5}^{(1)} = y_7 + y_8 - y_6 = 2x_1 + 2x_9 + n_{5}^{(1)}, \\
y_{9}^{(1)} = y_{11} + y_{12} - y_{10} = 2x_5 + 2x_9 + n_{9}^{(1)},
$$

(9)

where $n_{1}^{(1)} = n_3 + n_4 - n_2$, $n_{5}^{(1)} = n_7 + n_8 - n_6$, and $n_{9}^{(1)} = n_{11} + n_{12} - n_{10}$. In a given equation involving a certain number of data symbols and a noise term, we refer to the ratio of the power of each data symbol to the noise variance as the effective SNR of that data symbol. For instance, the effective SNR of each of the symbols $x_1$, $x_5$, and $x_9$ in (9) is 4/3 times the original SNR. This is because the noise terms’ in different REs are independent, i.e., each of $n_{j}^{(1)}$’s, for $j = 1, 5, 9$, has a mean equal to zero and variance $\sigma^2 = 3\sigma^2$, where $\sigma^2$ is the variance of the original noise terms $n_{j}$’s. Hence, the effective SNR is increased by a factor of 4/3 through this recursion.

Note that the set of three equations in (9) is defined according to the factor matrix $P_{3 \times 3}^{(1)}$ as follows:

$$
\begin{bmatrix}
y_{1}^{(1)} \\
y_{5}^{(1)} \\
y_{9}^{(1)}
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_5 \\ 2x_9 \end{bmatrix} + \begin{bmatrix} n_{1}^{(1)} \\ n_{5}^{(1)} \\ n_{9}^{(1)} \end{bmatrix}.
$$

(10)

Therefore, combining the three new symbols according to the rows of $\alpha_{i,j}^{(1)}$ results in the following set of equations involving only one data symbol, also referred to as singleton equations,

$$
y_{1}^{(2)} \triangleq y_{1}^{(1)} + y_{5}^{(1)} - y_{9}^{(1)} = 4x_1 + n_{1}^{(2)}, \\
y_{5}^{(2)} \triangleq y_{1}^{(1)} + y_{9}^{(1)} - y_{5}^{(1)} = 4x_5 + n_{5}^{(2)}, \\
y_{9}^{(2)} \triangleq y_{5}^{(1)} + y_{9}^{(1)} - y_{1}^{(1)} = 4x_9 + n_{9}^{(2)},
$$

(11)

in which $n_{1}^{(2)} = n_{1}^{(1)} + n_{5}^{(1)} - n_{9}^{(1)}$, $n_{5}^{(2)} = n_{1}^{(1)} + n_{9}^{(1)} - n_{5}^{(1)}$, and $n_{9}^{(2)} = n_{5}^{(1)} + n_{9}^{(1)} - n_{1}^{(1)}$. Hence, the effective SNR is increased again by a factor of 4/3 since the noise components $n_{j}^{(2)}$’s have mean zero and variance $\sigma^2 = 3\sigma^2 = 3^2\sigma^2$ due to the independence of $n_{j}^{(1)}$’s. Finally, (11) can be used to decode the original data symbols $x_j$’s through with the effective SNRs increased by a factor of $(4/3)^2$.

The above example illustrates the idea behind the proposed structure for the square pattern matrix and shows its potential advantages by enabling a low-complexity recursive detection process. However, this is merely an illustrative example which we are going to address throughout the rest of the section. In particular, what is the criteria for selecting the factor matrices? How can the received signals in different REs be combined to get smaller dimensions and simpler sets of equations (e.g., converting the original equations for $y_i$’s to (9) and then (11))? What exactly will be the gain of such a combining in terms of increasing the effective SNRs? And, how many equations and with what dimensions will be left at the end to perform the advanced detection algorithms? Next, we aim at properly answering these questions with respect to a square pattern matrix factorized as the Kronecker product of several smaller square factor matrices.

1) **Pattern Matrix Design and Recursive Combining:** With the considered pattern matrix structure, $G_{M \times K} = P_{m_1 \times m_1}^{(1)} \otimes P_{m_2 \times m_2}^{(2)}$, $\cdots \otimes P_{m_L \times m_L}^{(L)}$, both the combining procedure of the received signals over different REs and the resulting SNR gains, in each recursion, directly relate to the underlying square factor matrices $P_{l}^{(i)}$’s. As it will be elaborated in Section III-B2 our proposed detection algorithm starts from the rightmost factor matrix $P_{l}^{(L)}$ and involves $L$ recursions.

We will further establish that the $l$-th recursion, for $l' = 1, 2, \ldots, L$, involves equations defined according to the factor matrix $P_{l'}^{(i)}$ with $l = L - l' + 1$. Now, let $P_{l}^{(v)}$, for $v = 1, 2, \ldots, (2^{m_i-1})$, denote the $v$-th possible matrix for $P_{l}^{(i)}$. Then at the beginning of the $l$'-th recursion we have certain sets of auxiliary equations of the following form (see, e.g., (10)):

$$
\begin{bmatrix}
y_{i_1}^{(l'-1)} \\
y_{i_2}^{(l'-1)} \\
y_{i_3}^{(l'-1)} \\
\end{bmatrix} = \begin{bmatrix} P_{l_1,1}^{(v)} & P_{l_1,2}^{(v)} & \cdots & P_{l_1,m_1}^{(v)} \\ P_{l_2,1}^{(v)} & P_{l_2,2}^{(v)} & \cdots & P_{l_2,m_1}^{(v)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{l_{m_i},1}^{(v)} & P_{l_{m_i},2}^{(v)} & \cdots & P_{l_{m_i},m_1}^{(v)} \end{bmatrix} \begin{bmatrix} x_{i_1}^{(l'-1)} \\ x_{i_2}^{(l'-1)} \\ \vdots \\ x_{i_{m_i}}^{(l'-1)} \end{bmatrix} + \begin{bmatrix} n_{i_1}^{(l'-1)} \\ n_{i_2}^{(l'-1)} \\ \vdots \\ n_{i_{m_i}}^{(l'-1)} \end{bmatrix},
$$

(12)

where $P_{l_{i,j}}^{(v)}$ is the $(i,j)$-th element of the factor matrix $P_{l}^{(v)}$, $\{i_1, i_2, \ldots, i_{m_i}\}$ is a length-$m_i$ subset of $\{1, 2, \ldots, M\}$, and
Algorithm 1 Calculation of the combining coefficients and the corresponding SNR gains.

1: Input: dimension of the square factor matrix $m_l$
2: Output: combining matrices $\alpha^{(l,v)} = \left[\alpha_{i,j}^{(l,v)}\right]_{m_l \times m_l}$ and the corresponding sets $\{\gamma_{i,v}^{(l,v)}\}_{i=1}^{m_l}$ of SNR gains for all $v$’s
3: for $v = 1 : \left(2^m_l - 1\right)$ do
4: \[ P^{(l,v)} = \left[ P_{i,j}^{(l,v)} \right]_{m_l \times m_l} \]
5: for $i = 1 : m_l$ do
6: Find all $T$ possible sets of coefficients $\left\{\alpha_{i,j}^{(l,v)}\right\}_{j=1}^{m_l}$ such that:
7: C1: $\alpha_{i,j}^{(l,v)} \in \{-1, 0, 1\}$
8: C2: $\sum_{j=1}^{m_l} \alpha_{i,j}^{(l,v)} P_{j,i}^{(l,v)} = w_i^{(l,v)} \neq 0$
9: C3: $\sum_{j=1}^{m_l} \alpha_{i,j}^{(l,v)} P_{j,i'}^{(l,v)} = 0$, $i' \neq i, 1, 2, \ldots, m_l$ (12)
10: Output $\left\{\alpha_{i,j}^{(l,v)}\right\}_{j=1}^{m_l}$
11: Output $\gamma_{i,v}^{(l,v)} = \gamma_{i,(v,v^*)}$
12: \end{for}
13: \end{for}

Output 13:

Proof: Please refer to Appendix A.
2) Recursive Detection Algorithm: Suppose that the communication protocol is established between the transmitters and receivers, i.e., the design parameters such as the optimal square factor matrices $P_{m_l}^{(i)}$, for $l = 1, 2, \ldots, L$, and their corresponding combining matrices $\alpha_{m_l}^{(i)}$'s from Algorithm 1 (also recall Remark 1) are known to the receiver. Next, we describe the low-complexity recursive detection algorithm thanks to the underlying structure of the overall pattern matrix. The proposed detection algorithm proceeds by combining the received signals according to the combining matrix corresponding to the rightmost factor matrix and involves $L$ recursions.

First Recursion: The receiver in the first recursion takes the vector of received signals over $M = \prod_{l=1}^{L} m_l$ REs, i.e., $y_i$, for $i = 1, 2, \ldots, M$, and divides them into $M_{L-1} = M/m_l$ groups of $m_l$ received signals; therefore, the $i_{th}$ group, for $i_{th} = 1, 2, \ldots, M_{L-1}$, includes $\{y(i_{L-1} - 1)m_l + 1, y(i_{L-1} - 1)m_l + 2, \ldots, y(i_{L-1})m_l\}$. Each of these $y_i$'s contains at most $K = \sum_{l=1}^{L} m_l$ different transmitted symbols $x_{l,k}$, $k = 1, 2, \ldots, K$, since each one is constructed as a linear combination of $x_{l,k}$, defined with respect to the $i$-th row of the overall pattern matrix $G_{M \times K}$, and the noise component $n_i$ over the $i$-th row. Note that our recursive detection algorithm attempts to reduce the maximum number of different symbols by a factor of $m_l$, $L + 1$, after each $L$-th recursion, such that after all $L$ recursions we have $M$ equations each containing only a single data symbol. The receiver now combines the elements of each group using the combining matrix $\alpha_{m_l}^{(L)}$ to form new $m_l$ symbols at each group; the first new symbol is constructed by combining the previous symbols using the first row of $\alpha^{(L)}$. etc. Note that the impact of such a combining on the involved data symbols can be expressed in a matrix form through multiplying $\alpha^{(L)}$ by a new matrix comprising the $m_l$ rows of $G$ corresponding to the indices of each group. Then it is easy to verify that the $L$-th new equation of each group (which is constructed through the $L$-th row of $\alpha^{(L)}$), for $j_L = 1, 2, \ldots, m_l$, contains at most $K_{L-1} = K/m_l = \prod_{l=1}^{L} m_l$ different symbols from the set $\{x_{j_L}, x_{j_L + m_l}, \ldots, x_{j_L + (K_{L-1}-1)m_l}\}$ (recall that the product of $\alpha^{(L)}P^{(L)}$ is a diagonal matrix), i.e., the number of unknown variables is reduced by a factor of $m_l$ from $K$ to $K_{L-1}$. Also, based on the detailed discussion in Section 3-B1 (see, e.g., Eq. (13)), the effective SNR of each symbol in the $L$-th new equation is increased by a factor of $\gamma^{(L)}$.  

Second Recursion: Note that after the first recursion the $L$-th new equation of each group can only contain data symbols from the set $\{x_{j_L}, x_{j_L + m_l}, \ldots, x_{j_L + (K_{L-1}-1)m_l}\}$. This means different equations of a given group contain disjoint sets of data symbols while equations with the same index of different groups (e.g., the $L$-th equation of all groups) contain symbols from the same set. Therefore, the receiver in the second recursion forms $m_l$ super-groups of $M_{L-1}$ equations/signals by, consecutively, placing the $L$-th new equation of each of those $M_{L-1}$ groups of the first recursion into the $L$-th super-group. Note that the combination of the original data symbols over each super-group (in Section 3-B1) we denoted each of those combinations by an auxiliary data symbol obtained after the first recursion is defined according to the Kronecker product $P^{(1)}_{m_1 \times m_{1}} \otimes P^{(2)}_{m_2 \times m_{2}} \otimes \cdots \otimes P^{(L-1)}_{m_{L-1} \times m_{L-1}}$. Now, the receiver follows exactly the same procedure as the first recursion over each of these disjoint $m_l$ super-groups of the size $M_{L-1}$. In other words, it divides the equations in each of the super-groups into $M_{L-2} = M_{L-1}/m_l = \prod_{l=1}^{L} m_l$ groups of $m_l$ equations each and combines the signals within each group using the combining matrix $\alpha_{m_l}^{(L-1)}$. Following the same logic, we argue that the maximum number of unknown variables at each of the new equations is reduced by a factor of $m_l$ from $K_{L-1}$ to $K_{L-2} = K_{L-1}/m_l = \prod_{l=1}^{L} m_l$, and the SNR of the symbols in the $L$-th super-group is increased by a factor of $\gamma^{(L-1)}$ from $\gamma^{(L)}$ in the first recursion to $\gamma^{(L-1)}$.

Final Recursion: It can be verified by induction that in the $L$-th recursion we have $\prod_{l=2}^{L} m_l$ super-groups of size $m_1$. The sets of $m_1$ equations in each of these $\prod_{l=2}^{L} m_l$ super-groups is formed according to the square factor matrix $P^{(L)}_{m_1 \times m_{1}}$. Therefore, the receiver in the $L$-th recursion combines the $m_1$ symbols of each super-group using $\alpha_{m_1 \times m_{1}}^{(L)}$ to get equations containing a single unknown variable. Finally, the receiver detects the original $K = M$ data symbols according to the $M$ singleton equations in the form of a single-user additive white Gaussian noise (AWGN) channel. However, the effective SNR of each of these $K$ data symbols is increased to a desired level that can guarantee a predetermined data rate, particularly for low-capacity applications. The recursive detection procedure is schematically shown through a tree diagram in Fig. 1.

3) Detection and SNR Evolution Trees: For the sake of clarity, we have separately explained the second and the final recursions in Section 3-B2. Note that the same procedure is applied in all of the recursion steps for $L' = 2, \ldots, L$. In particular, the receiver at the beginning of the $L'$-th recursion, first, forms $\prod_{l=2}^{L'} m_l$ super-groups each containing $M_{L'} = \prod_{l=1}^{L'} m_l$ equations/signals defined according to the pattern matrix $P^{(1)}_{m_1 \times m_{1}} \otimes P^{(2)}_{m_2 \times m_{2}} \otimes \cdots \otimes P^{(L')}_{m_{L'} \times m_{L'}}$, where $L' = L - L' + 1$. The receiver then applies similar steps to the first recursion, i.e., combines the equations/signals inside each super-group according to the rightmost combining matrix $\alpha_{m_{L'} \times m_{L'}}^{(L')}$ to reduce the (maximum) number of unknown data symbols included in each equation/signal by a factor of $m_{L'}$.
from \( K_{i'j'} = \prod_{l=1}^{m_{i'}} m_l \) to \( K_{i'j'-1} \). Finally, as a result of this combining, the effective SNR of each data symbol involved in the \( j^{th} \)-th combined signal, \( j^{th} = 1, 2, \ldots, m_{i'} \), is increased by a factor of \( \gamma_{j^{th}} \). The whole process is, schematically, shown in Fig. 1. Each node in the tree diagram of Fig. 1 represents a super-group while the weights of the edges characterize the SNR gains after combining the signals inside each super-group. Moreover, the pattern governing the way data symbols involved in each super-group, at the beginning and end of each recursion, are merged together is also specified in Fig. 1 which further clarifies how after \( L \) recursions we end up with \( M \) singleton equations over single-user AWGN channels with increased effective SNRs.

It is worth mentioning that the \( M \) overall SNR gains obtained by multiplying the edge weights involved from the topmost node to each of the \( M \) bottommost nodes in Fig. 1 specify the set of \( M \) overall SNR gains for the \( M \) data symbols in an unsorted fashion. In order to get the overall SNR gains sorted, we need to shuffle the edge weights, i.e., the SNR gains, according to Fig. 2. This way we can assure that the overall SNR gain obtained by multiplying the edge weights involved from the topmost node to each \( i \)-th bottommost node in Fig. 2 for \( i = 1, 2, \ldots, M \), exactly specifies the overall SNR gain on the \( i \)-th data symbol \( x_i \) after \( L \) layers of combining. This can be understood from the detection algorithm elaborated in Section III-B2, especially the parts emphasizing on the data symbols that remain in each new combined equation/signal and the gain those data symbols attain after each combining. Now, according to Fig. 2 we have the following lemma for the overall SNR gain on the \( i \)-th data symbol, denoted by \( \gamma_{i} \).

**Lemma 2.** The overall SNR gain \( \gamma_{i} \) on the \( i \)-th data symbol, for \( i = 1, 2, \ldots, M \), can be obtained as

\[
\gamma_{i} = \prod_{l=1}^{L} \gamma_{s_{l}^{(i)}},
\]

where the integers \( s_{l} \in \{1, 2, \ldots, m_{l}\} \), for \( l = 1, 2, \ldots, L \), constitute a unique representation of \( i \) as follows:

\[
i = s_{L} + \sum_{l=1}^{L-1} (s_{l-1} - 1) \times \prod_{l_{1}=L-l+1}^{L} m_{l_{1}}.
\]

Equivalently, \( x_{i} \) is the single data symbol remained in the singleton equation of the final recursion indexed by the path \( (s_{L}, s_{L-1}, \ldots, s_{2}, s_{1}) \) of super-groups in Fig. 2.

**Example 2.** For the factor matrices considered in Example 1 and their corresponding combining matrices, according to the detailed discussions in Section III-B1 we have \( \gamma_{1} = \gamma_{2} = \gamma_{3} = \gamma_{4} = 1 \). Then, according to Lemma 2 we have, for instance, \( s_{1} = 2 \) and \( s_{2} = 4 \) for \( i = 8 \). Hence, \( \gamma_{1,8} = \gamma_{2,8} = \gamma_{3,8} = \gamma_{4,8} = 4/3 \). Similarly, \( \gamma_{1,4} = \gamma_{1,12} = 4/3 \), and \( \gamma_{7,i} = (4/3)^{2}, \ i \notin \{4, 8, 12\} \). □

### C. Rectangular Factor Matrices

In this subsection, we describe the detection algorithm when the data symbols are mixed using the Kronecker product of rectangular factor matrices over GMAC. In particular, we assume that the overall pattern matrix is given by \( G_{M \times K} = F_{m_{1} \times k_{1}} \otimes F_{m_{2} \times k_{2}} \otimes \cdots \otimes F_{m_{L} \times k_{L}} \), where \( F_{m_{i} \times k_{i}} \), for \( l = 1, 2, \ldots, L \), is an \( m_{i} \times k_{i} \) binary matrix with \( k_{l} > m_{i} \).

In this case, the overload factor \( \beta = K/M \) is greater than 1 resulting in an improved spectral efficiency compared to the case of square factor matrices considered in Section III-B.

We begin with the following illustrative example that helps better understand the recursive detection algorithm provided afterward.

**Example 3.** Consider the transmission of \( K = 18 \) users (symbols) over \( M = 4 \) REs realized using the Kronecker product of the following rectangular factor matrices

\[
F_{1 \times 2}^{(1)} = \begin{bmatrix} 1 & 1 \\ \end{bmatrix}, \quad F_{2 \times 3}^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ \end{bmatrix}.
\]

Hence, \( y_{4 \times 1} = G_{4 \times 18} x_{18 \times 1} + n_{4 \times 1} \), with \( G_{4 \times 18} = F_{1 \times 2}^{(1)} \otimes F_{2 \times 3}^{(2)} \otimes F_{2 \times 3}^{(2)} \), represents the received signals vector.

The detection starts by defining 6 auxiliary symbols \( Z_{i+j}^{(3)} \), for \( i = 0, 1 \) and \( j = 1, 2, 3 \), each obtained by spreading the \( j \)-th data symbol of \( F_{i}^{(3)} \) according to the \( (i+1) \)-st row of \( R_{i}^{(2)} \) defined according to \( F_{i}^{(2)} \) as

\[
\begin{bmatrix} y_{1} \\ y_{2} \end{bmatrix} = F_{2 \times 3}^{(2)} \begin{bmatrix} Z_{1}^{(3)} \\ Z_{2}^{(3)} \\ Z_{3}^{(3)} \end{bmatrix} + \begin{bmatrix} n_{1} \\ n_{2} \end{bmatrix},
\]

\[
\begin{bmatrix} y_{3} \\ y_{4} \end{bmatrix} = F_{3 \times 3}^{(2)} \begin{bmatrix} Z_{4}^{(3)} \\ Z_{5}^{(3)} \\ Z_{6}^{(3)} \end{bmatrix} + \begin{bmatrix} n_{3} \\ n_{4} \end{bmatrix}.
\]

Therefore, at the beginning of the first recursion in the receiver, the sets of equations are formed according to \( F_{2 \times 3}^{(2)} \) as (18).

The receiver detects \( Z_{i}^{(3)} \), \( Z_{i+1}^{(3)} \), \ldots, \( Z_{6}^{(3)} \), and then groups \( Z_{i}^{(3)} \) with \( Z_{i+1}^{(3)} \) to obtain three sets of equations each defined according to \( R_{i}^{(2)} \). Let us consider the first group containing \( Z_{1}^{(3)} \) and \( Z_{4}^{(3)} \). According to (17),

\[
\begin{bmatrix} Z_{1}^{(3)} \\ Z_{4}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Z_{2}^{(2)} \\ Z_{5}^{(2)} \\ Z_{6}^{(2)} \end{bmatrix},
\]

Fig. 2. Tree diagram representation of the progressive development of SNR gains.
after setting $Z_1^{(2)} \triangleq x_1 + x_{10}$, $Z_2^{(2)} \triangleq x_4 + x_{13}$, $Z_3^{(2)} \triangleq x_7 + x_{16}$ each according to $R^{(1)} \triangleq F^{(1)}$. Now, the receiver first sets $Z_1^{(2)}$, $Z_2^{(2)}$, and $Z_3^{(2)}$ from the system of equations in (19) defined according to $F^{(2)}_{3 \times 3}$, and then detects the corresponding 6 original data symbols from the three subsequent sets of equations each defined according to $F^{(1)}$. Finally, by applying the same procedure to the other two groups, we detect all of the 18 original data symbols.

The above example illustrates how the detection proceeds recursively by processing, at each recursion, sets of equations with smaller number of variables than the formed according to one of the rectangular factor matrices. In the following, we present the detailed description of the recursive detection algorithm over rectangular factor matrices. Same as in Section III-B2, we assume that the rectangular factor matrices $R_l^{(i)}$, for $l = 1, 2, \ldots, L$, are known to the receiver. The proposed detection algorithm entails $L$ recursions while the $l'$-th recursion, for $l' = 1, 2, \ldots, L$, involves sets of equations formed by the factor matrix $F_{m_{l'} \times k_{l'}}$ with $l' \leq L - l' + 1$. Note that when $L = 1$ we directly detect the $K$ data symbols mixed over $M$ REs using the single factor matrix. Hence, in the algorithm it is assumed that $L > 1$. Also, no specific structure is imposed on the rectangular factor matrices. Instead, we assume a generic form for the factor matrices such that a given advanced MUD algorithm can properly work to detect the unknown variables of the $l'$-th recursion defined according to $F_{m_{l'} \times k_{l'}}$. However, one can impose further constraints on the rectangular factor matrices to improve the detection performance of the scheme at each recursion. Throughout this subsection, we define $R_l^{(l)} = M_l^{(l)} \otimes F_{m_{l} \times k_{l}}$ with $M_l = \prod_{i=1}^{L} m_i$ and $K_l = \prod_{i=1}^{L} k_i$, $l = 1, 2, \ldots, L$.

**First Recursion:** Given that $G_{M \times K} = R_{M_{L-1} \times K_{L-1}} \otimes F_{m_{L} \times k_{L}}$, in the first recursion, the receiver forms sets of equations each defined according to $F_{m_{L} \times k_{L}}$. To this end, it detects $Q_L \triangleq k_{L}M_{L-1}$ new/auxiliary symbols $Z_{kl_{L}+jL}$, for $i_{L} = 0, 1, \ldots, M_{L-1} - 1$ and $j_{L} = 1, 2, \ldots, k_{L}$, each defined as the expansion of the $j_{L}$-th data symbol of $F_{m_{L} \times k_{L}}$ according to the $(i_{L} + 1)$-st row of $R^{(L-1)}$ as

$$Z_{kl_{L}+jL}^{(L)} = r_{i_{L}+1}^{(L-1)} \cdot y_{j_{L}+1}^{m_{L} \times k_{L}},$$

where $r_{i_{L}+1}^{(L-1)}$ is the $(i_{L} + 1)$-st row of $R^{(L-1)}_{M_{L-1} \times K_{L-1}}$. Using (20), the original set of equations $y_{M \times 1} = G_{M \times K} x_{K \times 1} + n_{M \times 1}$ can be decomposed into $M_{L-1}$ sets of equations each having $m_{L}$ consecutive $y_{i_{L}}$'s as the input and $k_{L}$ consecutive auxiliary symbols $Z_{kl_{L}+jL}$'s as the unknown variables. The $(i_{L} + 1)$-st such set of equations is expressed as

$$\begin{bmatrix}
y_{i_{L}+1}^{m_{L} \times 1} 
y_{i_{L}+1}^{m_{L} \times 2} 
\vdots 
y_{(i_{L}+1)m_{L}}^{m_{L} \times 1}
\end{bmatrix} = F_{m_{L} \times k_{L}}^{(L)} \begin{bmatrix}
Z_{i_{L}k_{L}+1}^{(L)} 
Z_{i_{L}k_{L}+2}^{(L)} 
\vdots 
Z_{(i_{L}+1)k_{L}}^{(L)}
\end{bmatrix} + \begin{bmatrix}
n_{i_{L}+1}^{m_{L} \times 1} 
n_{i_{L}+1}^{m_{L} \times 2} 
\vdots 
n_{(i_{L}+1)m_{L}}^{m_{L} \times 1}
\end{bmatrix}.$$  

(21)

Therefore, the first recursion involves $M_{L-1}$ separate sets of equations, each defined according to the rectangular factor matrix $F_{m_{L} \times k_{L}}$, as specified in (21), such that $Q_L$ auxiliary symbols are detected. Next, the receiver forms $k_{L}$ disjoint super-groups, given that all of the $M_{L-1}$ auxiliary symbols $Z_{kl_{L}+jL}^{(L)}$, for $i_{L} = 0, 1, \ldots, M_{L-1} - 1$, contain the same disjoint set of data symbols for a fixed $j_{L}$ (see (20)). The $j_{L}$-th super-group contains $M_{L-1}$ auxiliary symbols $Z_{kl_{L}+jL}^{(L)}$, $Z_{kl_{L}+jL+1}^{(L)}$, \ldots, $Z_{(M_{L-1} - 1)k_{L}+jL}^{(L)}$. Therefore, the set of equations in the $j_{L}$-th super-group can be expressed as

$$\begin{bmatrix}
Z_{jL}^{(L)} 
Z_{kL+jL}^{(L)} 
\vdots 
Z_{(M_{L-1} - 1)kL+jL}^{(L)}
\end{bmatrix} = R_{M_{L-1} \times K_{L-1}}^{(L-1)} \begin{bmatrix}
x_{jL} 
x_{kL+jL} 
\vdots 
x_{(K_{L-1} - 1)kL+jL}
\end{bmatrix}.$$  

(22)

**Middle Recursions** ($1 < l' < L$): When $L > 2$, the detection algorithm involves $L - 2$ middle recursions excluding the first and the last recursions. Each of these middle recursions involves a procedure that is somewhat similar to the first recursion. For example, in the second recursion we start with $k_{L}$ disjoint super-groups each containing a set of equations as (22) and then we apply similar steps as the first recession to this set of equations. However, since the indices of the data and the auxiliary symbols matter in the detection done at each recursion, here, we carefully describe these middle recursions. In general, at the $l'$-th recursion, for $1 < l' < L$, the algorithm processes the $k_{L'}$ starting super-groups of the $(l' - 1)$-s recursion separately. The algorithm then comes up with $k_{L-L' + 1}$ smaller size super-groups for each of those $k_{L'}$ starting super-groups, resulting in $k_{L-L' + 1} = k_{L'}$ super-groups at the end. In particular, let us consider the path $(j_{L}, j_{L-1}, \ldots, j_{L-L'+2})$ of the super-groups from the previous recursions, with $j_{l} = 1, 2, \ldots, k_{l}$ for $l = 1, 2, \ldots, L$, (note that a tree diagram somewhat similar to Fig. III can be considered for the recursive detection here). The indices in the path denote the indices of the super-groups considered from the previous recursions while there are total of $k_{L'}$ such paths, i.e., starting super-groups. Using induction the set of equations in the super-group indexed by the path $(j_{L}, j_{L-1}, \ldots, j_{L-L'+2})$ can be expressed as follows:

$$\begin{bmatrix}
Z_{\psi_{l'-1}+1}^{(l'+1)} 
Z_{\psi_{l'-1}+2}^{(l'+1)} 
\vdots 
Z_{\psi_{l'-1}+M_{l'}}^{(l'+1)}
\end{bmatrix} = R_{M_{l'} \times K_{l'}}^{(l'')} \begin{bmatrix}
x_{\tau_{l'-1}+1} 
x_{\tau_{l'-1}+2} 
\vdots 
x_{(K_{l'}-1)\psi_{l'-1}+\tau_{l'-1}}
\end{bmatrix},$$  

(23)

where $\psi_{l'-1} \triangleq j_{L-L'+2} + \sum_{i=1}^{l'-2} (J_{L-1} - 1) \prod_{i=1}^{L-2-l'+3} k_{l'}$, $\tau_{l'-1} \triangleq j_{L} + \sum_{i=1}^{l'-2} (J_{L-1} - 1) k_{l'}$ is the index of the super-group represented by the path $(j_{L}, j_{L-1}, \ldots, j_{L-L'+2})$, and $\psi_{l'-1} \triangleq j_{L} + \sum_{i=1}^{l'-2} (J_{L-1} - 1) k_{l'}$. Moreover, $Z_{\psi_{l'-1}+1}^{(l'+1)}, \ldots, Z_{\psi_{l'-1}+M_{l'}}^{(l'+1)}$ are $M_{l'}$ auxiliary symbols defined in the $(l'-1)$-st recursion, similar to (22), corresponding to the $\psi_{l'-1}$-s super-group. Note that tracking the indices of the auxiliary symbols is not needed since the algorithm processes each super-group separately. More specifically, these symbols are defined at the beginning of a recursion and they are known when the detection is done at the end of that recursion. However, the indices of the data symbols matters since we need to know the data symbols in each super-group/path and the order they appeared in the corresponding equations.
The algorithm involves \( \kappa_{L-1}^S \) sets of equations at the beginning of the \( l' \)-th recursion. Each such a set of equations is processed separately and simpler sets of equations defined according to \( F_{m'_i \times k_{i'}} \) are derived. Let us consider the \( \psi_i \)-st super-group with the set of equations given by (23). Similar to the first recursion, the receiver assumes \( Q_{i'} \oplus k_{L_i} M_{L_i} \) auxiliary symbols \( \psi_{i'}^{(l')} \), for \( i' = 0, 1, \ldots, M_{L_i} - 1 \) and \( j_i = 1, 2, \ldots, k_{i'} \), each defined as the expansion of the \( j_i \)-th data symbol according to \( r_{i'}^{(l')} \), i.e., the \( (i' + 1) \)-st row of \( R_{M_{L_i} \times K_{L_i}} \), as

\[
Z_{k_{i'}^{j_i} l', j_i}^{(l')} = r_{i'}^{(l')} = x_{\tau_i} x_{\tau_i + s_p} \ldots x_{\tau_i + (K_{L_i} - 1) s_p} \equiv \tau_i.
\]  

(24)

Note that the \( j_i \)-th data symbol of (23) is \( x_{\tau_i} \) since \( \tau_i \equiv \tau_i \). Now, instead of detecting the original data symbols from rather complex equations such as (23), the receiver forms \( M_{L_i} \) simpler sets of equations each defined according to the rectangular factor matrix \( F_{m'_i \times k_{i'}} \). The \( (i' + 1) \)-st such set of equations can be expressed as

\[
\begin{bmatrix}
Z_{\psi_{i' - 1}, l', m_{i' + 1} + 1}^{(l')} \\
Z_{\psi_{i' - 1}, l', m_{i' + 2} + 1}^{(l')} \\
\vdots \\
Z_{\psi_{i' - 1}, l', M_{L_i} - 1}^{(l')}
\end{bmatrix} = \begin{bmatrix}
F_{m'_{i} \times k_i} \\
F_{m'_{i} \times k_i} \\
\vdots \\
F_{m'_{i} \times k_i}
\end{bmatrix} \begin{bmatrix}
x_{\tau_i} \\
x_{\tau_i + s_p} \\
\vdots \\
x_{(K_{L_i} - 1) s_p + \tau_i}
\end{bmatrix}.
\]  

(25)

After detecting the \( Q_{i'} \)-auxiliary symbols \( \psi_{i'}^{(l')} \)'s from the sets of equations as (25), the receiver then forms \( k_{L_i} \) disjoint super-groups, given that all of the \( M_{L_i} \) auxiliary symbols \( \psi_{i'}^{(l')} \), \( i' = 0, 1, \ldots, M_{L_i} - 1 \), contain the same disjoint set of data symbols for a fixed \( j_i \). The set of equations in the \( j_i \)-th super-group can be expressed as

\[
\begin{bmatrix}
Z_{\psi_{i'}, 1}^{(l')} \\
Z_{\psi_{i'}, 2}^{(l')} \\
\vdots \\
Z_{\psi_{i'}, M_{L_i} - 1}^{(l')}
\end{bmatrix} = \begin{bmatrix}
R_{M_{L_i} \times K_{L_i} - 1}^{(l' - 1)} \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix} \begin{bmatrix}
x_{\tau_i} \\
x_{\tau_i + s_p} \\
\vdots \\
x_{(K_{L_i} - 1) s_p + \tau_i}
\end{bmatrix}.
\]  

(26)

given that \( l' - 1 = L - l' \). Note that in (26) \( Z_{\psi_{i'}, j_i}^{(l')} \) is denoted by \( Z_{k_{i'}^{j_i} l', j_i}^{(l')} \) in order to be consistent with (23) given that the index of the super-group represented by the path \( (j_1, j_{L_1}, \ldots, j_{L_{l'} - 1}, j_{L_{l'} - 1}) \) is \( \psi_i \). Hence, \( l' \equiv j_{L_{l'} - 1} + \sum_{i=1}^{l'} (j_{L_{l'} - i + 1} - 1) \prod_{i=2}^{l'} k_{i'} \). Finally, repeating the same procedure for all of the \( \kappa_{L-1}^S \) starting super-groups results in \( k_{L_{l'} - 1} \kappa_{L-1}^S \) super-groups, each of the form given by (26), at the end of the \( l' \)-th recursion.

Final Recursion: The receiver in the \( L \)-th recursion starts with \( \kappa_{L-1}^S = \prod_{i=2}^{L} k_i \) super-groups each containing a disjoint subset of the original data symbols of size \( k_1 \). The set of equations in the \( \psi_{L-1} \)-st such super-group, where \( \psi_{L-1} \equiv j_2 \), \( \sum_{i=2}^{L-1} (j_{L_{L-1} - i + 1} - 1) \prod_{i=2}^{L-1} k_{i'} \), indexed by the path \( (j_L, j_{L-1}, \ldots, j_2) \) can be expressed as (see, e.g., (23):

\[
\begin{bmatrix}
Z_{\psi_{L-1}, 1}^{(2)} \\
Z_{\psi_{L-1}, 2}^{(2)} \\
\vdots \\
Z_{\psi_{L-1}, m_{L-1}}^{(2)}
\end{bmatrix} = \begin{bmatrix}
F_{m_1 \times k_1}^{(1)} \\
F_{m_2 \times k_2}^{(1)} \\
\vdots \\
F_{m_{L-1} \times k_{L-1}}^{(1)}
\end{bmatrix} \begin{bmatrix}
x_{\tau_{L-1}} \\
x_{\tau_{L-1} + s_p} \\
\vdots \\
x_{(k_1 - 1) \kappa_{L-1}^S + \tau_{L-1}}
\end{bmatrix},
\]  

(27)

where \( \tau_{L-1} \equiv j_L + \sum_{i=1}^{L-2} (j_{L_{L-1} - i} - 1) \kappa_{L-1}^S \), and \( (2^{(2)}_{\psi_{L-1}}, \ldots, 2^{(2)}_{\psi_{L-1}, m_{L-1}} \) are \( m_1 \) auxiliary symbols defined in the \( (L - 1) \)-st recursion and their values are known at the beginning of the \( L \)-th recursion. Finally, after detecting the \( k_1 \) disjoint data symbols of each of the \( \kappa_{L-1}^S \) super-groups of the form (27), the receiver obtains the values of all \( K \) data symbols. Note that the indices of the data symbols contained in each super-group is fully known according to (27).

Remark 2. The recursive algorithm proposed in this subsection can readily be applied to the Kronecker product of any general square factor matrices by inserting \( k_l = m_l \), \( l = 1, 2, \ldots, L \). In that case, any given MUD scheme can be applied to sets of equations defined according to the corresponding factor matrix at each recursion, while the algorithm in Section III-B2 results in low-complexity detection.

D. General Pattern Matrices

In this subsection, we turn our attention to the general case of pattern matrices by building upon the analysis in Sections III-B and III-C specifically on square and rectangular factor matrices, respectively. In particular, the general case of the pattern matrix is considered as

\[
G_{M \times K} = F_{m_1 \times k_1}^{(1)} \otimes F_{m_2 \times k_2}^{(2)} \otimes \cdots \otimes F_{m_L \times k_{L-1}}^{(L-1)} \otimes P_{m'_1 \times m'_1}^{(1)} \otimes P_{m'_2 \times m'_2}^{(2)} \otimes \cdots \otimes P_{m'_L \times m'_L},
\]  

(28)

where \( L_r \) is the number of rectangular factor matrices \( F_{m'_i \times m_i}^{(l_i)} \)'s, for \( i_r = 1, 2, \ldots, L_r \), and \( L_s \) is the number of square factor matrices \( P_{m'_i \times m'_i}^{(l_i)} \)'s, for \( l_s = 1, 2, \ldots, L_s \), in the scheme. Moreover, let \( M_r = \prod_{l_s=1}^{L_s} m_{l_s} \), \( K_r = \prod_{l_s=1}^{L_s} k_{l_s} \), and \( M_s = \prod_{l_s=1}^{L_s} m_{l_s}' \). Then \( M = M_r M_s \), \( K = K_r M_s \), and the overall load factor \( \beta \) is \( K_r M_s / M_r > 1 \). This is because \( k_{l_r} > m_{l_s} \) for \( l_r = 1, 2, \ldots, L_r \). In addition to the structure of the overall pattern matrix given by (28), it is assumed that the combining matrices \( \alpha_{m'_i \times m'_i}^{(l_i)} \)'s corresponding to each \( F_{m'_i \times m'_i}^{(l_i)} \) are also known to the receiver. As discussed in Section III-B2, our proposed detection algorithm with only the Kronecker product of \( L_s \) square factor matrices as the pattern matrix results in \( M_s \) singleton equations with the SNR gain on each equation obtained using Lemma 2. The detection algorithm for the general case of the pattern matrix is then summarized as the following two phases each incurring \( L_s \) and \( L_r \) recursions, respectively.

Phase 1: In the first phase, the receiver applies \( L_s \) recursions, by following combining and \( (super-) \) grouping as described in Section III-B2 to the system of equation

\[
y_{M \times 1} = G_{M \times K} x_{K \times 1} + n_{M \times 1},
\]  

with \( G_{M \times K} \) defined in (28).

Then the receiver obtains \( M_s \) super-groups each containing a set of equations defined according to the pattern matrix
posed code-domain NOMA with the general pattern matrix
form given by (29). According to the detailed description of
\( \gamma \)

combining, and their values are fully known to the receiver at
time

where

Moreover, \( \tilde{y}_1', \tilde{y}_2', \ldots, \tilde{y}_L \) are the combined versions
of the channel outputs in \( y_{M \times 1} \), after \( L_s \) rounds of
combining, and their values are fully known to the receiver at
the end of Phase I. Furthermore, \( \tilde{n}_1', \tilde{n}_2', \ldots, \tilde{n}_L \) are
independent noise terms each having a variance equal to
\( \sigma^2/\gamma_t, \gamma_{t_1}' \). Note that, as a result of \( L_s \) square
factor matrices, the effective SNR of the equations in the \( \psi'_1 \)-th super-group
in (29) is increased by a factor of \( \gamma_t, \gamma_{t_1}' \), where, by Lemma 2,
\( \gamma_t, \gamma_{t_1}' \) defined the same way as in
Section III-B for \( P_{l_j}'(l_1) \times m_i \).

Phase II: The receiver in the second phase separately
processes each of the \( M_s \) disjoint sets of equations of the
form given by (29). According to the detailed description
of the detection algorithm in Section III-C the receiver, after
applying \( L_r \) recursions, detects the \( K_r \) symbols of each of
the aforementioned \( M_s \) sets of equations resulting in \( K = K_r M_s \)
data symbols in total.

IV. PERFORMANCE CHARACTERIZATION

A. Average Sum Rate

Based on the analysis provided in Section III we have the
following theorem on the average sum-rate of the proposed
scheme over GMAC.

Theorem 3. The per-RE average sum-rate \( C_M \) of the
proposed code-domain NOMA with the general pattern matrix
\( G_{M \times K} \), specified in (25), and the proposed recursive detection
algorithm is expressed as

\[
C_M = \frac{1}{2M} \log_2 \det \left( I_{M_r} + \rho \sum_{l_1=1}^{L_s} \gamma_{l_2}(l_1) F F^T \right),
\]

where \( I_{M_r} \) is an \( M_r \) \times \( M_r \) identity matrix, \( F_{M \times K} \) is
\( F_{M \times K} = F_{m_1 \times k_1} \otimes F_{m_2 \times k_2} \otimes \cdots \otimes F_{m_{L_s} \times k_{L_s}} \), \( \rho \) is \( P_z/\sigma^2 \) with
\( P_z = \mathbb{E} \left[ x_j^2 \right] \), for \( j = 1, 2, \ldots, K \), and \( \sigma^2 = \mathbb{E} \left[ n_i^2 \right] \), for
\( i = 1, 2, \ldots, M_s \).

Proof: It is well known that for a PDMA system with the
pattern matrix \( A_{M \times K} \) and the average SNR equal to \( \rho \) for all
of the received original data symbols, the per-RE average sum
rate is given by \( C^{PDMA}_M = \frac{1}{2M} \log_2 \det \left( I_{M_r} + \rho \mathbf{A A}^T \right) \) for
the optimal MAP detection (33). As elaborated in Phase I of the
detection algorithm in Section III-D we end up with \( M_s \) sets
of equations each defined according to (29). Therefore, the
per-RE average sum-rate \( C_M(\psi'_1) \) of the \( \psi'_1 \)-th super-group,
indexed by the path \((j_1, j_1, \ldots, j_2, j_1)\), is obtained as

\[
C_M(\psi'_1) = \frac{1}{2M} \log_2 \det \left( I_{M_r} + \rho \gamma_t, \gamma_{t_1}' F F^T \right).
\]

Averaging (31) over all \( M_s \) paths completes the proof.

Corollary 4. The per-RE average sum-rate of the proposed
code-domain NOMA with the pattern matrix \( G_{M \times K} = F_{M \times K} \otimes F_{m_1 \times m_2} \), where \( F_{\oplus} \) denotes the \( L_{s-t} \) time
Knörrer product of \( F \) with itself, is given by

\[
C_M = \frac{1}{2M} \log_2 \det \left( I_{M_r} + \rho \gamma_{l_1}', \gamma_{l_2}', \ldots, \gamma_{l_m} F F^T \right),
\]

where the summation is over all disjoint sets \( \{r_1, r_2, \ldots, r_{m_p} : r_{i+1} \in \{0, 1, \ldots, L_s \}, \sum r_{i+1} r_{i} = L_s \} \).

Remark 3 (Optimal Factor Matrices). When the design
constraint is to maximize the average sum rate, the design of the optimal
pattern matrix can be formulated as the selection of an appropriate
rectangular matrix \( F \) (that can properly trade off between
the performance and complexity) together with the \( v_{l_2}' \)-th square
matrices, \( l_s = 1, 2, \ldots, L_s \) from Algorithm 1 with the corresponding set of
SNRs \( \{\gamma_{l_1}', \gamma_{l_2}', \ldots, \gamma_{l_m} \} \), such that they result in the maximum sum rate in (30).

Remark 4 (Rate-Reliability Trade-off). After the detection of a
properly chosen subset of data symbols, one can successfully
cancel the interference on some detection equations such that
higher SNR gains are achieved for the detection of some
remained symbols (see Example 4). This, in addition to
increasing the average sum rate in (30), may also increase
the error probability due to the error propagation issue
in SIC detection, resulting in a trade-off between the rate and
reliability. This mechanism also increases the latency because
we should wait for the detection of some symbols before
starting the detection of the others.

Example 4. For the system parameters in Example 1, by
using Theorem 3 and Example 2 the average sum rate is

\[
C_{12} = (1/8) \log_2 (1 + (4/3)\rho) + (3/8) \log_2 (1 + (4/3)^2 \rho).
\]

However, after detecting \( x_1 \) and \( x_2 \) from the first two equations in (11), we can apply SIC to form \( \tilde{y}_9 \) \( \triangleq \tilde{y}_9 \otimes y_9^{(1)} \) \( y_9^{(1)} - 2x_1 - 2x_5 = 4x_9 + n_9^{(1)} + n_9 \), instead of
the third equation in (11). Applying the same procedure to all of
the four sets of equations defined according to \( P_{l_1}^{(1)} \times 3 \times 3 \),
increases the SNR gain \( \gamma_3 \) on the third data symbols from
\( 4/3 \) to 2. Therefore, the average sum rate is increased to

\[
C_{12} = (1/12) \log_2 (1 + (4/3)\rho) + (1/4) \log_2 (1 + (4/3)^2 \rho) + (1/8) \log_2 (1 + (8/3)\rho) + (1/24) \log_2 (1 + 2\rho).
\]

\( \square \)
B. Latency Analysis

Given our proposed fully parallel detection algorithm, we have the following theorem for the system latency.

**Theorem 5.** The maximum execution time $T_{\text{exec}}^\text{max}$ for the detection of $K$ data symbols in the proposed code-domain NOMA with the general pattern matrix $G_{M \times K}$, specified in (28), using the detection algorithm in Section II is as follows:

$$
T_{\text{exec}}^\text{max} = t_a \sum_{l_t=1}^{L_t} (m_{l_t} - 1) + T_{\text{MUD}}^{\text{noisy}}(F_{m_{l_t} \times k_{l_t}}, C(K_{l_t-1}, C_0)) + \sum_{l_t=2}^{L_t} T_{\text{MUD}}^{\text{noisy}}(F(l_t), C(K_{l_t-1}, C_0)) + T_{\text{MUD}}^{\text{noisy}}(F(1), C_0),
$$

(33)

where $t_a$ is the required time for an addition/subtraction, $C(K_{l_t}, C_0)$ denotes the constellation space of the sum of (at most) $K_{l_t}$ original data symbols each from a modulation with the constellation space $C_0$, $T_{\text{MUD}}^{\text{noisy}}(F_{m_{l_t} \times k_{l_t}}, C)$ is the required time for a given MUD algorithm to detect the $k_{l_t}$ symbols with the constellation space $C$ from a set of $m_{l_t}$ equations defined according to the pattern matrix $F_{m_{l_t} \times k_{l_t}}$, in the presence of additive noise, and $T_{\text{MUD}}^{\text{noisy}}(F(l_t), C(K_{l_t-1}, C_0))$ is defined similar to $T_{\text{MUD}}^{\text{noisy}}(F_{m_{l_t} \times k_{l_t}}, C)$ in the absence of noise.

**Proof:** The detection algorithm processes $L_s$ square factor matrices through $L_s$ recursions. The $t_r'$-th recursion, for $t'_r = 1, 2, \ldots, L_s$, involves combining sets of $m_{t'_r}$ equations, where $m_{t'_r} \geq L_s - t'_r + 1$, according to the rows of $L_r$-dimensional matrices $\alpha_{m_{t'_r} \times m_{t'_r}}$ requiring at most $m_{t'_r} - 1$ additions/subtractions. Thus, assuming all such operations are done in parallel, the $t_r'$-th recursion requires at most $t_{a}(m_{t'_r} - 1)$ units of time. Now, after at most $t_{a}(m_{t'_r} - 1)$ units of time, we get sets of equations defined according to the Kronecker product of $L_r$ rectangular factor matrices, as (29), requiring $L_r$ recursions. According to the analysis in Section III-C the first recursion involves processing sets of equations in the form of (21) each requiring at most $T_{\text{MUD}}^{\text{noisy}}(F(l_r), m_{l_r} \times k_{l_r}, C(K_{l_r-1}, C_0))$ units of time. Furthermore, the middle recursions require processing sets of equations in the form of (25) each incurring at most $T_{\text{MUD}}^{\text{noisy}}(F(l_r'), m_{l_r'} \times k_{l_r'}, C(K_{l_r'}-1, C_0))$ time, where $l_r' \geq L_s - t'_r + 1$. Finally, the last recursion contains sets of equations defined according to (27) each necessitating $T_{\text{MUD}}^{\text{noisy}}(F(l_r'), m_{l_r'} \times k_{l_r'}, C(K_{l_r'}-1, C_0))$ time. Summing up the execution times of all $L_s + L_r$ recursions completes the proof.

**Remark 5.** Theorem 5 characterizes the latency in the worst-case scenario. However, the actual execution time might be smaller. For example, the unknown variables of the middle recursions of rectangular factor matrices are obtained according to (24). Therefore, if, for instance, half of the elements of $\gamma_{l_r'-1}^{l_r'-1}$ are zero, the constellation space is $\mathcal{C}(1/2K_{l_r'-1}, C_0)$. Hence, the $t_r'$-th recursion will require $T_{\text{MUD}}^{\text{noisy}}(F(l_r'), m_{l_r'} \times k_{l_r'}, C(1/2K_{l_r'-1}, C_0))$ time which is, in general, smaller than $T_{\text{MUD}}^{\text{noisy}}(F(l_r'), m_{l_r'} \times k_{l_r'}, C(K_{l_r'}-1, C_0))$. Similar arguments hold for all other recursions.

**Remark 6.** Note that (33) holds for $L_r > 2$. It can be observed that with $L_r = 1$ the last three terms of (33) should be replaced by $T_{\text{MUD}}^{\text{noisy}}(F_{m_{l_t} \times k_{l_t}}, C_0)$. Moreover, for $L_r = 2$, the third term in (33) becomes zero.

**Remark 7.** As shown throughout this section, the proposed detection algorithm leads to a latency that grows logarithmically with the total number of users/REs. Thanks to the proposed parallel structure of the detection algorithm, a powerful BS can simultaneously process all the branches of the detection tree in uplink. Also, a given user with a typical processing power during the downlink phase can focus only on the detection of the branch containing its data symbol to avoid unnecessary detections.

C. Detection Complexity

Recall that the proposed recursive approach results in a relatively small size for each factor matrix, even for a large $M$ and $K$; hence, it is highly desired to exactly characterize the maximum number of required operations.

**Theorem 6.** The maximum number of additions/subtractions $N_{\text{add}}^\text{max}$ and multiplications $N_{\text{mul}}^\text{max}$ for the detection of $K$ data symbols in the proposed code-domain NOMA system with the general pattern matrix $G_{M \times K}$, specified in (28), using the recursive detection algorithm in Section II is as follows:

$$
N_{\text{add}}^\text{max} = M \sum_{l_t=1}^{L_s} (m_{l_t} - 1) + M_s N_{\text{add}}^\text{max}(F_M \times K_s),
$$

(34)

$$
N_{\text{mul}}^\text{max} = M_s N_{\text{mul}}(F_M \times K_s),
$$

(35)

where $N_{\text{add}}(F_M \times K_s)$ ($N_{\text{mul}}(F_M \times K_s)$) is the maximum number of additions/subtractions (multiplications) for a NOMA system with the pattern matrix $F_M \times K_s = F(1) \otimes \cdots \otimes F(L_r)$ and the recursive detection algorithm in Section III-C

$$
\begin{align*}
N_{\text{add}}^\text{max}(F_M \times K_s) &= M_{L_r-1} N_{\text{add}}^\text{noisy}(F_{m_{l_r} \times k_{l_r}}, C(K_{l_r-1}, C_0)) \\
&+ \sum_{l_r'=2}^{L_r} \kappa_{l_r'-1}^{S} N_{\text{add}}^\text{noisy}(F_{l_r'-1}, C(K_{l_r'-1}, C_0)) \\
&+ \kappa_{L_r-1}^{S} N_{\text{add}}^\text{noisy}(F(1), C_0),
\end{align*}
$$

(36)

$$
\begin{align*}
N_{\text{mul}}^\text{max}(F_M \times K_s) &= M_{L_r-1} N_{\text{mul}}^\text{noisy}(F_{m_{l_r} \times k_{l_r}}, C(K_{l_r-1}, C_0)) \\
&+ \sum_{l_r'=2}^{L_r} \kappa_{l_r'-1}^{S} N_{\text{mul}}^\text{noisy}(F_{l_r'-1}, C(K_{l_r'-1}, C_0)) \\
&+ \kappa_{L_r-1}^{S} N_{\text{mul}}^\text{noisy}(F(1), C_0).
\end{align*}
$$

(37)

where $M_r \equiv \prod_{l_t=1}^{L_s} m_{l_t}$, $K_r \equiv \prod_{l_t=1}^{L_s} k_{l_t}$, $N_{\text{add}}^\text{noisy}(F_{m_{l_r} \times k_{l_r}}, C)$ is the required number of operations for a given MUD algorithm to detect the $k_{l_r}$ symbols with the constellation space $C$ from a set of $m_{l_r}$ equations defined according to the pattern matrix $F_{m_{l_r} \times k_{l_r}}$ in the presence of additive noise, and $N_{\text{add}}^\text{noisy}(F(1), C)$ is defined similarly in the absence of noise.

**Proof:** The proof is based on the details of the detection algorithm presented in Section III and following similar steps to the proof of Theorem 5 while counting the maximum number of operations at each recursion.

**Remark 8.** Note that (35) is obtained assuming $L_r > 2$. For $L_r = 1$, (35) should be replaced by $N_{\text{add}}^\text{noisy}(F_{m_{l_t} \times k_{l_t}}, C_0)$. Moreover, the second term in (35) becomes zero for $L_r = 2$. 
V. EXTENSIONS AND DISCUSSIONS

In this section, we discuss how the results of the paper can be applied to several other practical scenarios.

A. Downlink and Uplink Fading Channels

Throughout the paper, we focused on the case of GMAC where unit/equal gains are assumed for the channels between the users and the BS in different REs. In practice, the channel gain \( h_{mk} \) between the \( k \)-th user, for \( k = 1, 2, \ldots, K \), and the BS at the \( m \)-th RE, for \( m = 1, 2, \ldots, M \), can be different. In this subsection we discuss how the results of the paper can be readily extended to downlink and uplink fading channels.

1) Downlink Transmission: According to (4), the received signal vector at the \( k \)-th user is given by \( y_k = \text{diag}(h_{1k}, h_{2k}, \ldots, h_{MK})G_{M \times K}x + n_k \). By dividing the \( m \)-th equation in \( y_k \) by \( h_{mk} \) we get a new set of equations as \( \tilde{y}_k = G_{M \times K}x + \tilde{n}_k \), where \( \tilde{n}_k(m) \) is Gaussian with mean zero and variance \( \sigma^2_N \triangleq \sigma^2/h_{mk}^2 \). Therefore, all the results can be extended to the case of downlink fading channels with the slight change that the noise components over different REs do not have identical variances though they are still independent.

2) Uplink Transmission: Here, we assume in this paper that for a given user the channel gains over all \( M \) REs are equal, i.e., \( h_{mk} = h_k, \forall m, k \). This is a reasonable (and common) assumption, e.g., if the \( M \) REs are the adjacent sub-channels of a slowly changing channel, they have almost equal gains. However, if the channel gains are different, the users can adjust their power at each RE in such a way that equal gain is observed on the received signals over different REs. Note that this assumption for the case of downlink transmission yields exactly the same channel model \( y_k/h_k = G_{M \times K}x + n_k \) as GMAC with identical noise variances over different REs.

Recall, based on (3), that the uplink signal received by the BS can be expressed as \( y = Hx + n \), where \( H \triangleq \mathcal{H} \circ G_{M \times K} \) and \( \mathcal{H} \triangleq [h_1, h_2, \ldots, h_K] \). With the above assumption, we have \( h_k = h_k \mathbf{1}_{M \times 1}, \forall k \). The received signal vector can then be rewritten as \( y = G_{M \times K}x + n \), where \( \tilde{x}(k) = h_k x_k \). Therefore, all of our earlier results can be readily extended to the case of uplink fading channels with the slight change that each \( k \)-th data symbol is scaled by the channel gain \( h_k \).

B. Joint Power-and Code-Domain NOMA (JPC-NOMA)

The code-domain NOMA scheme proposed in this paper can be combined in many ways with power-domain NOMA. For instance, if, for some \( l \), \( 2^{m_l} - 1 < k_l \), then we cannot satisfy distinct nonzero columns for the factor matrix \( G^{(l)}_{m_l \times k_l} \). Consequently, the overall pattern matrix \( G_{M \times K} \) will have some repeated columns, i.e., same pattern vectors for some of the users (see, e.g., Example 1). Therefore, one cannot distinguish between the data of the users with an equal pattern vector. In other words, the data symbols of the users with an equal pattern vector will be paired together, i.e., they coexist in all sets of equations containing any of those symbols. In such a case, we propose to assign different power coefficients to the users having the same pattern vector in order to differentiate between them using power-domain NOMA techniques. Then, the receiver after detecting the paired symbols using our proposed detection algorithm, detects the individual data symbols contained in each paired symbol according to the principles of power-domain NOMA. In this case, users with the most disparate channel qualities should be paired together.

More importantly, we can incorporate power-domain NOMA principles to increase the reliability of the detection performed over sets of equations at each layer of the detection algorithm proposed in Section III-C. Recall that each of those sets of equations contains a disjoint subset of symbols. For example, considering the general case of pattern matrix, specified in (25), the final recursion involves \( M_k \prod_{l=2}^{L} k_l \) sets of equations defined similar to (27), each containing \( k_1 \) disjoint symbols while the indices of the symbols are known. As shown in (31), assigning different power scaling and phase shifting factors to the data symbols in a PDMA system improves the detection reliability. Therefore, the transmitter can group each subset of \( k_1 \) users together. Then different power levels (as well as, possibly, phase shifts) are assigned to the data symbols of the users in each group according to the principles of power-domain NOMA in order to increase the reliability of the detection over the aforementioned \( M_k \prod_{l=2}^{L} k_l \) sets of equations.

C. Future Directions

The proposed approach in this paper can be extended in various directions. Next, we mention a few of such potential future directions.

1) Optimal design of factor matrices: Theorem 3 and Corollary 4 naturally lead to a design strategy for optimal factor matrices that maximize the average sum rate (see Remark 3). The characterization of other performance metrics, such as individual rates [37], and then finding the optimal factor matrices with respect to these metrics is a direction for future research.

2) Rate-Reliability-Latency Trade-offs: Given the set of factor matrices and the overall recursive approach discussed in Section III, various detection schemes can be applied at each layer/recursion trading off between the rate, reliability, and latency, as exemplified in Remark 4 and Example 4. Detailed characterizations of such trade-offs is another direction for future research.

3) Joint Power-and Code-Domain NOMA (JPC-NOMA): As discussed in Section V-B, power-domain NOMA can be combined with the proposed code-domain NOMA in several ways to improve the system design in terms of various performance metrics including reliability, throughput, fairness, etc. Designing such joint networks and analytically characterizing their performance metrics to show the advantages of the joint design is another important future research direction.

4) Grant-Free Transmission: In many use cases of massive communication, it is essential to reduce the coordination overhead of the communication protocol. Grant-free protocols attempt to serve a massive number of users with minimal (or even without any) coordination. Incorporating the proposed low-complexity code-domain NOMA scheme in the context of grant-free transmission is a vital future direction.
In this section, we numerically compare the average sum-rate of various configurations, including the traditional OMA, regular PDMA, and our proposed recursive scheme. We considered the sum-rate of various multiple access techniques. Here, for our proposed method with the recursive detection performs significantly better than the traditional OMA.

By running Algorithm 1 for $m_f = 3$ and $r = 2$, we get the optimal square factor matrices $F_{3 \times 3}$ and $P_{4 \times 4}$ in (7) with the combining matrices as $\mathbf{F}$ and individual SNR gains reported in Example 2. Fig. 3 shows the per-RE average sum-rate of various multiple access techniques. Here, for our proposed scheme, we considered $G_{M \times K} = \mathbf{F} \otimes \mathbf{P}_{\mathcal{K}}^{\otimes r}$, with $\mathbf{F} = [1 \ 1]$, $\mathbf{P}$ being either $\mathbf{P}_{3 \times 3}$ or $\mathbf{P}_{4 \times 4}$, and $r = 1$ or 2. It is observed that both the regular PDMA and our proposed scheme significantly outperform the traditional OMA method. Moreover, for relatively similar system parameters, our proposed method with the recursive detection provides the largest sum-rate very close to the capacity of regular PDMA with the optimal pattern matrix $A_{M \times K}$ and optimal MAP detection, i.e., $C_{\text{PDMA}}^{\text{opt}} = \frac{1}{2^M} \log_2 \det \left( \mathbf{I}_M + \rho \mathbf{A} \mathbf{A}^T \right)$. However, due to the significantly lower complexity of our scheme, we can utilize pattern matrices with higher dimensions (by increasing $r$) and, also, apply SIC detection at some recursions similar to Example 4 (see also Example 4) to get larger sum rates.

In Fig. 4, the focus is specifically on low-SNR regimes. Here, we consider $G_{M \times K} = \mathbf{F}_{1 \times 2}^{(1)} \otimes F_{3 \times 6}^{(2)} \otimes F_{4 \times 8}^{(3)} \otimes \mathbf{P}_{3 \times 3}^{(1) \otimes r}$ for the proposed scheme with $\mathbf{F}_{1 \times 2} = [1 \ 1]$, and $F_{3 \times 6}$ and $F_{4 \times 8}$ being the $3 \times 6$ and $4 \times 8$ PDMA matrices with $d_f = 4$ from [31]. It can be observed in Fig. 4 that having large pattern matrices are essential at low SNRs to avoid getting close-to-zero sum rates. While it is not practically feasible to implement conventional code-domain NOMA schemes at such large dimensions considered in this setting, our low-complexity recursive approach makes it possible to spread the data symbols of the users using desirably large pattern vectors. As a consequence, 6.1 and 5.04 times larger average sum rates are achieved for our recursive scheme with $G_{36 \times 288}$ compared to $4 \times 8$ PDMA scheme at SNRs of $-16$ dB and $-13$ dB, respectively. These gains are as large as $23.07 \times$ and $18.2 \times$, respectively, when compared to OMA.

VI. NUMERICAL RESULTS

We proposed a low-complexity and scalable recursive approach toward code-domain NOMA by constructing the overall pattern matrix as the Kronecker product of several factor matrices. This way, we showed that not only the procedure for designing large pattern matrices is simplified at the transmitter side, but also the detection procedure over such large pattern matrices at the receiver is reduced to several layers/recursions each dealing with sets of equations corresponding to certain relatively small factor matrices. As a result, the overall detection complexity is reduced significantly, compared to a straightforward code-domain NOMA, allowing the incorporation of desirably large pattern matrices that are of particular interest for massive communication and low-capacity channels. For the Kronecker product of square factor matrices we proposed a systematic way of choosing the factor matrices that enables a remarkably low-complexity detection involving only few linear operations (additions/subtractions). Although expanding the dimension of the pattern matrix by adding square factor matrices to the Kronecker product does not improve the overload factor, it increases the effective SNRs of data symbols that is of particular importance for low-capacity channels. Furthermore, for the Kronecker product of rectangular factor matrices we provided a recursive detection algorithm that can work on the general case of factor matrices. Given the proposed schemes and the detection algorithm we characterized the system performance in terms of average sum rate, latency, and detection complexity. We further discussed possible extensions of the work to the case of fading channels and joint power-and code-domain NOMA while highlighting several directions for the future research. We showed that the proposed scheme has significantly lower complexity and latency compared to straightforward code-domain NOMA schemes. Moreover, it is numerically verified that by utilizing large pattern matrices the proposed scheme significantly improves the average sum rate. For instance, for SNR equal to $-13$ dB a 5.04 times larger rate is achieved.

VII. CONCLUSIONS
using the proposed scheme with a $36 \times 288$ pattern matrix compared to the $4 \times 8$ PDMA system.

**APPENDIX A**

**PROOF OF LEMMA 1**

Let $P = [p_{i,j}]_{m \times m}$. If there exists no such a combining matrix $\alpha = [\alpha_{i,j}]_{m \times m}$ with $\alpha_{i,j} \in \{-1, 0, +1\}$, then we are done. Otherwise, if such a matrix exists, then we prove the uniqueness of the $i$-th row of $\alpha$, for $i = 1, 2, \ldots, m$, by contradiction.

Assume to the contrary that there are $T = 2$ distinct sets of coefficients $\{\alpha_{i,j} \}_{j=1}^{m}$ and $\{\alpha'_{i,j} \}_{j=1}^{m}$ for the $i$-th row of $\alpha$ that result in singleton vectors with nonzero elements at the $i$-th position of the $i$-th row of $\alpha P$. In other words, we have

$$\sum_{j=1}^{m} \alpha_{i,j} p_{j,i} = w_t \neq 0, \quad (36)$$

$$\sum_{j=1}^{m} \alpha'_{i,j} p_{j,i} = 0, \quad (37)$$

for $t = 1, 2$, where $w_1$ and $w_2$ are two nonzero integers. Then we can define a new set of combining coefficients $\{\alpha''_{i,j} \}_{j=1}^{m}$ with $\alpha''_{i,j} = w_1 \alpha_{i,j} - w_2 \alpha'_{i,j}$. Note that

$$\sum_{j=1}^{m} \alpha''_{i,j} p_{j,i} = w_1 \sum_{j=1}^{m} \alpha_{i,j} p_{j,i} - w_2 \sum_{j=1}^{m} \alpha'_{i,j} p_{j,i} = w_1 w_2 - w_2 w_1 = 0, \quad (38)$$

$$\sum_{j=1}^{m} \alpha''_{i,j} p_{j,i} = 0, \quad (39)$$

Now, two cases are possible:

- $\alpha''_{i,j} = 0$, i.e., $\alpha''_{i,j} = (w_2/w_1) \times \alpha''_{i,j}$ for all $j = 1, 2, \ldots, m$. This implies that there is only one set of coefficients $\{\alpha''_{i,j} \}_{j=1}^{m}$ for the $i$-th row of $\alpha$ since scaling the combining coefficients by a constant factor (here, $w_2/w_1$) does not change the performance and the SNR gains as explained for the condition (1) in Section III-B1.
- At least for one $j$, $\alpha''_{i,j} \neq 0$. This is in contradiction with the linear independence of the rows of $P$ since the linear combination of the rows with nonzero combining coefficients is equal to zero as shown by (38) and (39).

This proves that the $i$-th row of $\alpha$ is unique. Repeating the same procedure for $i = 1, 2, \ldots, m$ completes the proof. 

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