BAROCLINIC INSTABILITY
IN DIFFERENTIALLY ROTATING STARS

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Abstract. A linear analysis of baroclinic instability in a stellar radiation zone with radial differential rotation is performed. The instability onsets at a very small rotation inhomogeneity, $\Delta \Omega \sim 10^{-3}\Omega$. There are two families of unstable disturbances corresponding to Rossby waves and internal gravity waves. The instability is dynamical: its growth time of several thousand rotation periods is short compared to the stellar evolution time. A decrease in thermal conductivity amplifies the instability. Unstable disturbances possess kinetic helicity thus indicating the possibility of magnetic field generation by the turbulence resulting from the instability.

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INTRODUCTION

Upon arrival on the main sequence, young stars rotate rapidly, with periods of about one day. Solar-type stars spin down with age due to the loss of angular momentum through a stellar wind (Skumanich 1972; Barnes 2003). The braking torque acts on the stellar surface, but the spin-down extends rapidly deep into the convective envelope due to the eddy viscosity existing here. In deeper layers of the radiation zone, the viscosity is low ($\sim 10\, \text{cm}^2/\text{s}$) and insufficient to smooth out the radial rotation inhomogeneity. Therefore, before the advent of helioseismology, it had been thought very likely that the solar radiation zone rotates much faster than the surface (see, e.g., Dicke 1970). Subsequently, it transpired that the radiation zone rotates nearly uniformly (Shou et al. 1998). In other words, there is a coupling between the convective envelope and deep layers

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of the radiation zone that is efficient enough to smooth out the rotation inhomogeneity in a short time compared to the Sun’s age. Observations of stellar rotation show that the characteristic time of the coupling is $\lesssim 10^8$ yr (Hartmann & Noyes 1987; Denissenkov et al. 2010).

One possible explanation for the smoothing of rotation inhomogeneities in stars is the instability of differential rotation: the turbulence resulting from the instability transports angular momentum in such a way that the rotation approaches uniformity. The difficulty of this explanation stems from the fact that the threshold rotation inhomogeneity

$$q = -\frac{r \, d\Omega}{\Omega \, dr}$$

for the appearance of hydrodynamic instabilities is not small, $q = O(0.1)$ (here, $\Omega$ is the angular velocity, $r$ is the radius). One might expect that such instabilities could reduce the rotation inhomogeneity to its threshold value but could not remove it completely. The baroclinic instability that is related to the rotation inhomogeneity indirectly may constitute an exception (Tassoul & Tassoul 1983). In the equilibrium state of a differentially rotating radiation zone, the surfaces of constant pressure and constant density do not coincide. Such a “baroclinic” equilibrium can be unstable (Spruit & Knobloch 1984). In this paper, we consider the baroclinic instability in a stellar radiation zone with radial differential rotation. As we will see, the instability occurs at a very small rotation inhomogeneity, $q \ll 1$.

Instabilities are also important for the mixing of chemical species in stars (see, e.g., Pinsonneault 1997). The study of the stability of differential rotation in stellar radiation zones has a long history (Goldreich & Schubert 1967; Acheson 1978; Spruit & Knobloch 1984; Korycansky 1991). This paper differs in that we consider the stability against global disturbances. The horizontal disturbance scale is not assumed to be small compared to the stellar radius. At the same time, a stable stratification of the radiation zone rules out mixing on a large radial scale. Therefore, the radial disturbance scale is assumed to be small. Such an approach was applied to analyze the stability of latitudinal differential rotation (Charbonneau et al. 1999; Gilman et al. 2007; Kitchatinov 2010) in connection with the problem of the solar tachocline. It showed that the horizontally-global modes are actually the dominant ones. As we will see, the same is true for the baroclinic instability.
The most unstable disturbances correspond to global Rossby waves \((r\)-modes) and internal gravity waves \((g\)-modes), which grow exponentially with time in the presence of a radial rotation inhomogeneity. Therefore, the baroclinic instability may be considered as the loss of stability by a differentially rotating star with respect to the excitation of \(r\)- and \(g\)-modes of global oscillations.

Both dominant modes possess kinetic helicity, \(\mathbf{u} \cdot (\nabla \times \mathbf{u}) \neq 0\). Helicity is indicative the ability of a flow to generate magnetic fields (see, e.g., Vainshtein et al. 1980). Here, our instability analysis joins with another possible explanation for the uniform rotation of the solar radiation zone, the magnetic field effect.

**FORMULATION OF THE PROBLEM**

**Background Equilibrium and the Origin of Instability**

When analyzing the stability, we will assume the initial equilibrium state to be stationary and cylinder symmetric about the rotation axis. We consider the hydrodynamic stability, i.e., the magnetic field is disregarded. We will proceed from the stationary hydrodynamic equation

\[
(V \cdot \nabla)V = -\frac{1}{\rho}\nabla P - \nabla \psi,
\]

where \(\psi\) is the gravitational potential, the standard notation is used, the influence of viscosity on the global flow is neglected. The main flow component in the radiation zone is rotation,

\[
V = e_\phi r \sin \theta \Omega,
\]

where the usual spherical coordinates \((r, \theta, \phi)\) are used and \(e_\phi\) is the azimuthal unit vector. We assume the rotation to be sufficiently slow, \(\Omega^2 \ll GM/R^3\), for the deviation of stratification from spherical symmetry to be small. The stratification in stellar radiation zones is stable, i.e., the specific entropy \(s = c_v \ln(P) - c_p \ln(\rho)\), increases with radius \(r\),

\[
N^2 = \frac{g}{c_p} \frac{\partial s}{\partial r} > 0.
\]

The buoyancy forces counteract the radial displacements. Therefore, the meridional circulation is small and the main flow component is rotation (3). The characteristic time of meridional circulation in the radiation zone exceeds the Sun’s age (Tassoul 1982).
Nevertheless, the most important force balance condition follows from the equation for a meridional flow. This condition can be derived by calculating the azimuthal component of the curl of the equation of motion (2). This gives

\[ r \sin \theta \frac{\partial \Omega^2}{\partial z} = -\frac{1}{\rho^2} (\nabla \rho \times \nabla P)_\phi, \]  

(5)

where \( \partial/\partial z = \cos \theta \partial/\partial r - r^{-1} \sin \theta \partial/\partial \theta \) is the spatial derivative along the rotation axis, the subscript \( \phi \) denotes the azimuthal component of the vector. The centrifugal force is conservative only if the angular velocity does not vary with distance \( z \) from the equatorial plane. The left part of Eq. (5) allows for the non-conservative part of the centrifugal force, which by itself produces a vortical meridional flow. In a stellar radiation zone, this non-conservative force is balanced by the buoyancy force included in the right part of Eq. (5).

In the case of \( z \)-dependent differential rotation, the equilibrium is baroclinic: the surfaces of constant pressure and constant density do not coincide. One might expect such an equilibrium to be unstable. This can be seen after the following transformations of the right part of Eq. (5):

\[ -\frac{1}{\rho^2} \nabla \rho \times \nabla P = \frac{1}{c_p \rho} \nabla s \times \nabla P = \frac{1}{c_p} \nabla s \times g^*, \]  

(6)

where \( g^* = -\nabla \psi + r \sin \theta \Omega e_\phi \times \Omega \) is the “effective” gravity. It can be seen from Eq. (6) that the isobaric and isentropic surfaces do not coincide either. Figure 1 explains why an instability is possible in this situation (Shibahashi 1980). For displacements in the narrow cone between the isobaric and isentropic surfaces, the gravitational forces increase the energy of the fluid particles being displaced. The relatively light particles with a positive entropy (temperature) perturbation are displaced opposite to the gravity, while the colder and relatively dense particles are displaced in the direction of gravity. One might expect the disturbances with such displacements to be amplified due to the release of (gravitational) energy of the equilibrium state. Remarkably, the instability arises from the buoyancy forces that usually exhibit a stabilizing effect in stellar radiation zones.

It can be seen from Fig. 1 that not the deviation of stratification from spherical symmetry related to rotation but the baroclinicity caused by the rotation inhomogeneity is responsible for the instability. For simplicity, we will neglect the deviation of the
Fig. 1. If the isobaric and isentropic surfaces do not coincide, then the displacements in the cone between these surfaces (indicated by the arrows) can be unstable.

pressure distribution from spherical symmetry but will take into account the latitudinal entropy inhomogeneity. For the special case of rotation dependent only on the radius, from Eqs. (5) and (6) we find

$$\frac{\partial s}{\partial \theta} = -2qc_r \Omega^2 g^{-1} \sin \theta \cos \theta,$$

(7)

where $q$ is the rotation inhomogeneity parameter (1).

**Linear Stability Equations**

The main approximations and methods of deriving the equations for small disturbances were discussed in detail previously (Kitchatinov 2008; Kitchatinov & Rüdiger 2008). This paper differs only in allowance for the deviation of stratification from barotropy. Therefore, the equations of the linear stability problem will be written without repeating their derivation. We repeat, however, the main approximations and assumptions used in deriving these equations.

The initial equilibrium state does not depend on time and longitude. Therefore, the dependence of the disturbances on longitude and time in the linear stability problem can be written as $\exp(i m \phi - i \omega t)$, where $m$ is the azimuthal wave number. A positive imaginary part of the eigenvalue, $\Im(\omega) > 0$, means an instability.

Stable stratification of the radiation zone prevents mixing on large radial scales. Therefore, the radial scale of disturbances is assumed to be small and the stability analysis is local in radius: perturbations of the velocity, $u$, and entropy, $s'$, depend on radius as $\exp(ikr)$ with $kr \gg 1$. At the same time, the mixing in horizontal directions encounters
no counteraction and the stability analysis is global in these directions. As we will see, the most unstable disturbances actually have large horizontal scales.

We use the incompressibility approximation, \( \text{div} \mathbf{u} = 0 \). It is justified for disturbances whose wavelength in the radial direction is small compared to the pressure scale height. The magnetic fields are disregarded. The angular velocity is assumed to be dependent on radius only but not on latitude.

The equations are written for the scalar potentials \( P_u \) and \( T_u \) of the the poloidal and toroidal components of the velocity perturbations:

\[
\mathbf{u} = \frac{e_r}{r^2} \hat{L} P_u - \frac{e_\theta}{r} \left( \frac{im}{\sin \theta} T_u + ik \frac{\partial P_u}{\partial \theta} \right) + \frac{e_\phi}{r} \left( \frac{\partial T_u}{\partial \theta} + \frac{km}{\sin \theta} P_u \right)
\]

(Chandrasekhar 1961), where

\[
\hat{L} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{m^2}{\sin^2 \theta}
\]

is the angular part of the Laplacian. We use non-dimensional variables. The physical quantities can be restored from the normalized perturbations of entropy \( S \) and the poloidal \( V \) and the toroidal \( W \) flow potentials using Eq. (8) and the relations

\[
s' = -i c_p N^2 \frac{S}{g k}, \quad P_u = \left( \frac{\Omega r^2}{k} \right) V, \quad T_u = \Omega r^2 W.
\]

(10)

The equation for the entropy perturbations is

\[
\hat{\omega} S = -i \frac{\epsilon_x}{\lambda^2} S + \hat{L} V + i \frac{Q}{\lambda} \mu \left( m W - \left( 1 - \mu^2 \right) \frac{\partial V}{\partial \mu} \right),
\]

(11)

where \( \hat{\omega} = \omega / \Omega - m \) is the dimensionless eigenvalue in the co-rotating frame of reference, \( \mu = \cos \theta \), \( \hat{\lambda} \) and \( Q \) are the two basic parameters controlling the influence of fluid stratification and differential rotation

\[
\hat{\lambda} = \frac{N}{\Omega kr}, \quad Q = 2q \frac{\Omega}{N}.
\]

(12)

The finite diffusion is taken into account in the parameters

\[
\epsilon_x = \frac{\lambda N^2}{Q^2 \gamma^2}, \quad \epsilon_\nu = \frac{\nu N^2}{Q^2 \gamma^2},
\]

(13)

where \( \chi \) and \( \nu \) are the thermal diffusivity and viscosity, respectively.
The complete system consists of three equations. In addition to Eq. (11) for the entropy perturbations, it includes the equations for the poloidal flow,

\[ \hat{\omega}(\hat{L}V) = -i\frac{\epsilon_\nu}{\lambda^2}(\hat{L}V) - \hat{\lambda}^2(\hat{L}S) + 2mV - 2\mu(\hat{L}W) - 2(1 - \mu^2)\frac{\partial W}{\partial \mu}, \]  

(14)

and the toroidal flow,

\[ \hat{\omega}(\hat{L}W) = -i\frac{\epsilon_\nu}{\lambda^2}(\hat{L}W) + 2mW - 2\mu(\hat{L}V) - 2(1 - \mu^2)\frac{\partial V}{\partial \mu}. \]  

(15)

The eigenvalue problem for the system of equations (11), (14), and (15) was solved numerically. The independent variables were expanded in a series of the associated Legendre polynomials, for example,

\[ S = \sum_{l=\max(|m|,1)}^K S_l P_{l|m|}^m(\mu), \]  

(16)

and similarly for \( W \) and \( V \). This leads to a system of linear algebraic equations for the expansion amplitudes \( S_l, W_l \) and \( V_l \). The number of equations in the system is not about \( 3K \) but a factor of 2 smaller, because the complete system splits into two independent subsystems governing the eigenmodes symmetric and antisymmetric relative to the equator.

Most of the calculations were performed for the following values of dissipation parameters of Eq. (13),

\[ \epsilon_\chi = 10^{-4}, \quad \epsilon_\nu = 2 \times 10^{-10}, \]  

(17)

typical of the upper part of the solar radiation zone (Kitchatinov & Rüdiger 2008). In the cases where we used other values, this is stipulated.

**Symmetry Properties**

Two types of equatorial symmetry are possible: symmetric modes for which \( S(\mu) = S(-\mu), V(\mu) = V(-\mu) \) and \( W(\mu) = -W(-\mu), \) and antisymmetric modes with symmetric \( W \) and antisymmetric \( S \) and \( V \). For the symmetric and antisymmetric modes, we will use the notations \( S_m \) and \( A_m \), respectively, where \( m \) is the azimuthal wave number. These notations correspond to the symmetry relative to the mirror-reflection about the equatorial plane. For example, for the \( S_m \)-modes, \( u_r \) and \( u_\phi \) are symmetric relative to the equator, while \( u_\theta \) is antisymmetric.
A more significant property of the system of equations (11), (14), and (15) consists in its symmetry relative to the transformation

\[(q, m, \hat{\omega}, W, V, S) \rightarrow (-q, -m, -\hat{\omega}^*, -W^*, V^*, -S^*),\]  
(18)

where the asterisk denotes complex conjugation. This means that if the mode with some \(m\) is unstable at a certain rotation inhomogeneity \(q\), then at a rotation inhomogeneity of opposite sense \((-q)\) there is an unstable mode with the same growth rate and azimuthal wave number \(-m\). Therefore, it will suffice to consider the stability, for example, only for \(q > 0\); the stability properties for \(q < 0\) will then be known. Below, we consider only the case where the rotation rate increases with depth, i.e., \(q > 0\).

Transformation (18) also shows that stability properties depend on the sign of the azimuthal wave number \(m\). This dependence usually implies that unstable modes possess a finite helicity (Rüdiger et al. 2012). The absolute helicity in the linear problem is indefinite, but the relative helicity

\[H_{rel} = \langle u \cdot (\nabla \times u) \rangle / (ku^2),\]  
(19)

can be defined. The angular brackets here denote the azimuthal averaging:

\[\langle X \rangle = \frac{1}{2\pi} \int_0^{2\pi} X \, d\phi.\]  
(20)

For axisymmetric modes \((m = 0)\), this corresponds to averaging over the oscillation phase \(\phi\) (the linear solutions are determined to within phase factor \(e^{i\phi}\)). The overline in (19) and below denotes averaging over a spherical surface:

\[\overline{u^2} = \frac{1}{2} \int_{-1}^{1} \langle u^2 \rangle \, d\mu.\]  
(21)

For barotropic fluids, the total (volume-integrated) kinetic helicity is an integral of motion. For baroclinic fluids, this is not the case. As we will see, the unstable modes of baroclinic instability are indeed helical.

\[\text{\textsuperscript{†}}\text{Only the real components of the disturbances are used to calculate the relative helicity (19), along with any other nonlinear characteristics of the disturbances. The imaginary parts are omitted.}\]
Two Modes of Stable Oscillations

The solutions for special limiting cases are helpful in discussing the results to follow. In this Section, we consider uniform rotation \((q = 0)\) in the absence of dissipation \((\chi = \nu = 0)\).

In the limiting case of a “very stable” stratification, \(\hat{\lambda} \gg 1\), or \(N \gg \Omega kr\), two modes of stable oscillations can be revealed:

1. For one of them, the frequency is low, \(\hat{\omega} \ll \hat{\lambda}\). It then follows from Eq. (14) that \(S = 0\) and Eq. (11) gives \(V = 0\). The flow possesses no poloidal component and the spectrum of purely toroidal oscillations can be found from Eq. (15):

\[
\hat{\omega} = \frac{2m}{l(l + 1)}.
\]

These are the \(r\)-modes of global oscillations also known as Rossby waves.

2. There is another solution for which the frequency is not low, \(\hat{\omega} \sim \hat{\lambda}\). In this case, Eq. (15) in the highest order in \(\hat{\lambda}\) gives \(W = 0\). The flow does not contain a toroidal part. We write Eqs. (14) and (11), also in the highest order in \(\hat{\lambda}\), as \(\hat{\omega}(\hat{L}V) = -\hat{\lambda}^2(\hat{L}S)\) and \(\hat{\omega}S = \hat{L}V\), respectively. Poloidal oscillations with the following spectrum are found:

\[
\hat{\omega} = \pm \hat{\lambda}\sqrt{l(l + 1)}.
\]

As can be seen from the expression for the frequency

\[
\omega = \pm \frac{N}{kr}\sqrt{l(l + 1)},
\]

rotation does not affect these high-frequency oscillations. These are the internal gravity waves or \(g\)-modes.

As we will see, in a differentially rotating fluid with baroclinic stratification, both modes of global oscillations acquire positive growth rates, i.e., become unstable.

RESULTS AND DISCUSSION

Stability Borders and Growth Rates

The lines separating the regions of stability and instability for disturbances with different equatorial and axial symmetries are shown in Fig. 2. The instability appears at a small rotation inhomogeneity. In the upper part of the solar radiation zone, \(N/\Omega \approx 400\). Even
a weakly inhomogeneous rotation \( q \sim 10^{-4} \) is unstable. In this way the instability under consideration differs from the barotropic instabilities that appear at a relatively large rotation inhomogeneity. Another important difference is that baroclinic instability exists both for axisymmetric disturbances and for various azimuthal wave numbers \( m \neq 0 \). However, the greater the number \( |m| \), the larger rotation inhomogeneity is required for the onset of instability. This trend is confirmed by our calculations for \( |m| \leq 10 \). The disturbances that are global in horizontal dimensions are most unstable.

![Fig. 2. Lines of neutral stability for symmetric (solid lines) and antisymmetric (dotted lines) disturbances about the equator. The lines are marked by the corresponding symmetry notations. The instability regions are above the lines.](image)

The lines for modes S-1 and S-2 in Fig. 2 have kinks. This implies that different line segments correspond to disturbances of different nature. Unstable disturbances close to the \( r \)- and \( g \)-modes of global oscillations are revealed. This can be seen from the Table, where the characteristics of unstable disturbances are given. The kinetic energy of the disturbances is the sum of the energies of their poloidal and toroidal components.
(Chandrasekhar 1961):

\[ \bar{u}^2 = u_p^2 + u_t^2 = \frac{1}{4} \sum l(l+1) \left( |V_l|^2 + |W_l|^2 \right). \tag{25} \]

The Table lists the closest frequencies of the poloidal \( g \)-modes (23) for unstable poloidal disturbances \( (u_p^2/u_t^2 > 1) \) and the closest frequencies of the toroidal \( r \)-modes (22) for toroidal disturbances \( (u_p^2/u_t^2 < 1) \). The frequencies of the unstable disturbances and the corresponding oscillation modes differ little; there is a correspondence to the largest-scale oscillations. For example, the frequency of the unstable poloidal disturbance \( A_3 \) is 13.6. The lowest value of \( l \) in expansion (16) for the poloidal potential of this disturbance is \( l = 4 \). For \( l = 4 \) we find a frequency of 13.4 close to that of the unstable disturbance from (23). The cases where there is no correspondence to the largest-scale oscillation mode are marked with an asterisk in the Table. For example, the expansion of the toroidal potential for the toroidal mode \( A-3 \) with a frequency of 0.199 begins from \( l = 3 \), but the \( r \)-mode (22) with the next \( l = 5 \) has the closest frequency \( \hat{\omega}^r = 0.2 \) (the summation in (16) for disturbances with a certain equatorial symmetry is over either even or odd \( l \)). The correspondence of the frequencies and the poloidal or toroidal character of growing disturbances to the \( r \)- and \( g \)-modes of global oscillations allows us to interpret the instability as the loss of stability against the excitation of these global oscillations. The question of what determines the transport of angular momentum in differentially rotating stars, the instability or the \( g \)-modes (Spruit 1987; Charbonnel & Talon 2005), may find an unexpected answer: the instability excites the \( g \)-modes.

The Table also gives the correlation of the entropy and radial velocity perturbations, to which the power supplied by buoyancy forces is proportional. This correlation is positive for all unstable modes. Calculations show that this correlation can be negative only for damped disturbances. The energy of the growing disturbances increases due to the work of buoyancy forces, as it should be for baroclinic instability (Fig. 1).

The Table gives the disturbance growth rate

\[ \dot{\gamma} = 2\pi \Im(\hat{\omega}), \tag{26} \]

normalized to the rotation period, i.e., the disturbances grow by a factor of \( e^{\dot{\gamma}} \) in one stellar rotation. The growth rates are small. The star makes about 10 000 rotations in a
Table 1: Parameters of unstable disturbances for $\hat{\lambda} = 3$ and $Q = 10^{-3}$. $\hat{\gamma}$ is the disturbance growth rate (26), $\Re(\hat{\omega})$ is the oscillation frequency, $\hat{\omega}^r$ and $\hat{\omega}^g$ are the closest frequencies of the $r$- or $g$-modes (22) or (23), respectively, $\frac{u_p^2}{u_t^2}$ is the ratio of the energies of the poloidal and toroidal flow components, and $Su_r/\sqrt{u_r^2 S^2}$ is the relative correlation of the entropy and radial velocity perturbations.

| Mode | $\hat{\gamma}$, $10^{-4}$ | $\Re(\hat{\omega})$ | $\hat{\omega}^r$ | $\hat{\omega}^g$ | $\frac{u_p^2}{u_t^2}$ | $Su_r/\sqrt{u_r^2 S^2}$ |
|------|-----------------|-----------------|--------------|---------|-----------------|-----------------|
| A0   | 8.41            | 4.34            | 4.24         | 24.0    | $3.35 \times 10^{-5}$ |
| A1   | 2.91            | 7.27            | 7.35         | 43.0    | $6.38 \times 10^{-6}$ |
| A3   | 1.14            | $-13.6$         | $-13.4$      | 171     | $1.38 \times 10^{-6}$ |
| A10  | 0.709           | $-34.6$         | $-34.5$      | 2487    | $3.32 \times 10^{-7}$ |
| A-1  | 2.04            | 0.989           | 1            | $1.99 \times 10^{-4}$ | $2.99 \times 10^{-3}$ |
| A-3  | 1.46            | 0.199           | 0.2*         | $9.62 \times 10^{-7}$ | $2.17 \times 10^{-2}$ |
| A-10 | 0.507           | 0.0952          | 0.0952*      | $2.79 \times 10^{-9}$ | 0.155            |
| S0   | 3.29            | 7.46            | 7.35         | 32.7    | $7.46 \times 10^{-6}$ |
| S1   | 4.93            | $-4.85$         | $-4.24$      | 34.6    | $2.17 \times 10^{-5}$ |
| S3   | 3.10            | $-10.7$         | $-10.4$      | 260     | $4.90 \times 10^{-6}$ |
| S10  | 1.03            | $-31.5$         | $-31.5$      | 5727    | $5.24 \times 10^{-7}$ |
| S-1  | 2.94            | 0.320           | 0.333        | $1.93 \times 10^{-4}$ | $3.67 \times 10^{-3}$ |
| S-3  | 2.22            | $-10.2$         | $-10.4$      | 265     | $3.35 \times 10^{-6}$ |
| S-10 | 0.874           | $-31.4$         | $-31.5$      | 5772    | $4.44 \times 10^{-7}$ |

*The asterisk marks the frequencies that do not correspond to the largest-scale oscillations, i.e., not to the smallest $l$ in Eq. (22) for a given mode.

The disturbance e-folding time. Even for slowly rotating stars, however, this time ($\sim 1000$ yr) is short compared to evolutionary time scales. In Fig. 3, the growth rate of disturbances is plotted against their dimensionless wavelength $\hat{\lambda}$ (12). The equatorial symmetry does not determine the properties of unstable modes uniquely. For example, there is a discrete spectrum of modes S1. Figure 3 shows the highest growth rates. The kinks in the lines for modes with negative $m$ correspond to a change in the type of the most rapidly growing disturbance. The highest growth rates belong to the $r$-modes for relatively small $\hat{\lambda}$ and to the $g$-modes for large $\hat{\lambda}$.

In Fig. 4, the highest growth rates are plotted against the rotation inhomogeneity parameter $Q$ (12). For relatively large $Q$, these dependencies are nearly linear, $\hat{\gamma} \sim Q$.

**Dependence on Thermal Conductivity**

The dependence on thermal conductivity is of interest in connection with the possible influence of chemical composition inhomogeneity. Such inhomogeneity is important for
stability. The increase in mean molecular weight $\mu$ with depth makes the stratification “more stable”. This can be taken into account by replacing the frequency $N$ with its effective value $N_e$, 

$$N_e^2 = N^2 + N^2_{\mu}, \quad N^2_{\mu} = -\frac{g}{\mu} \frac{d\mu}{dr}$$

(Kippenhahn & Weigert 1990). This, however, is not the only effect of the compositional gradient. The diffusivity for chemical inhomogeneities in stellar radiation zones is much smaller than the thermal diffusivity. Therefore, the inhomogeneity of $\mu$ reduces the dissipation rate of density inhomogeneities in unstable disturbances.

Here, we do not account for composition inhomogeneity, but the character of its influence can be seen by analyzing dependence of the instability on thermal conductivity. Figure 5 shows the growth rates of the unstable $g$-mode A0 for three values of the dimensionless thermal diffusivity $\epsilon_\chi$. Similar results are also obtained for other unstable
modes. An increase in $\epsilon \chi$ suppresses the instability.

It is generally believed that conduction of heat *amplifies* the instabilities in stellar radiation zones. Radial displacements produce the temperature and density disturbances and are, therefore, opposed by buoyancy. The dissipation of temperature inhomogeneities reduces the stabilizing buoyancy effect, thereby amplifying the instabilities.

Figure 5 shows that the opposite is true of the baroclinic instability. This instability is peculiar in that it emerges precisely due to special features of the radiation zone stratification and is produced by buoyancy forces (Fig. 1). Therefore, an increase in thermal conductivity suppresses this instability. Assertions in the literature that the compositional gradient in stellar radiation zones switches off baroclinic instability seem questionable.
Helicity and the Possibility of Dynamo

Figure 6 shows the distributions of the relative helicity for three $g$-modes of the instability under consideration. Positive and negative helicities dominate in the northern and southern hemispheres, respectively. In the dynamo theory, the helicity is known to be important for magnetic field generation.

The origin of magnetic fields in stellar radiation zones presents a problem. Solar-type stars at early evolutionary stages are fully convective for more than a million years, which is approximately a factor of $10^4$ longer than the turbulent diffusion time. A hydromagnetic dynamo can operate in such fully convective stars (Dudorov et al. 1989). Subsequently, a radiative core emerges and grows in the central part of the star. During its growth, it can capture the magnetic field from the surrounding convective envelope. However, this field is weak ($<1$ G), because the convective dynamo field is oscillating and the frequency of its
oscillations is much higher than the growth rate of the radiation zone (Kitchatinov et al. 2001). The helicity of the eigenmodes of baroclinic instability (Fig. 6) points to another possibility — the dynamo action in a differentially rotating unstable radiation zone.

The radiation zones of solar-type stars are deep beneath the surface and are inaccessible to direct observations. Higher-mass stars have outer radiative envelopes. Differential rotation can be present in such stars as they approach the main sequence due to radially nonuniform contraction. Recently, Alecian et al. (2013) detected rapid (in several years) changes of the global magnetic field on one of such Herbig Ae/Be stars with an extended outer radiation zone. They interpreted these changes as a manifestation of a deep dynamo in the newly-born convective core. However, an alternative explanation is also possible — the dynamo action due to baroclinic instability in a differentially rotating radiative envelope.

CONCLUDING REMARKS

Linear analysis does not permit determination of the final state to which instability growth will lead. However, one might expect fully developed turbulence in view of
the great variety of baroclinic instability modes. Turbulence in the radiation zone, irrespectively of its source, is highly anisotropic with a predominance of horizontal flows, \( \frac{u^2}{u^2} \sim \frac{\Omega^2}{(\tau^2 N^4)} \ll 1 \), where \( \tau \) is the eddy turnover time (Kitchatinov & Brandenburg 2012). Such turbulence efficiently transports angular momentum, removing rotation inhomogeneity. Note that the transport of angular momentum by anisotropic turbulence is not reduced to the action of eddy viscosity (Lebedinskii 1941). There are nondissipative angular momentum flows; as a result, the smoothing of rotation inhomogeneities in stellar radiation zones is much faster than the diffusion of chemical species. Since the threshold value of differential rotation for the onset of baroclinic instability is very low (Fig. 2), this instability can lead to an essentially uniform rotation of the radiation zone, which is revealed by helioseismology.

Baroclinic instability can also have a bearing on the origin of magnetic fields in stellar radiation zones. The convective instability in rotating stars is known to be capable of generating magnetic fields. The helicity of convective flows plays the most important role in this process (see, e.g., Vainshtein et al. 1980). The growing global modes of baroclinic instability also possess helicity and may be capable of generating magnetic fields.

Baroclinic instability has a bearing not only on stars. Already Tassoul & Tassoul (1983) pointed to this instability as a possible cause of turbulence in accretion disks. Subsequently, Klahr & Bodenhaimer (2003) analyzed this possibility. The so-called stratorotational instability of a Couette flow (Shalybkov & Rüdiger 2005) is also most likely of the baroclinic type.

Figure 2 shows that the threshold rotation inhomogeneity for the onset of instability decreases with increasing radial scale of the \( g \)-modes. Therefore, a stability analysis for disturbances that are global not only horizontally but also radially can be a perspective for further study of baroclinic instability.

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