HAMILTONIAN LOOP GROUP SPACES AND A THEOREM OF TELEMAN AND WOODWARD

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Abstract. We study a natural family of elliptic boundary problems on a compact surface \( \Sigma \) parametrized by a non-compact moduli space of flat \( G \)-connections with framings along \( \partial \Sigma \). We prove that the family has a well-defined equivariant analytic index and derive various cohomological formulas (non-abelian, delocalized, abelian). When \( \Sigma \) has a single boundary component and one takes the invariant part of the index, our abelian localization formula reproduces the Teleman-Woodward formula for the index of the Atiyah-Bott classes on the moduli stack of \( G_C \) bundles. We carry out the analysis more generally for suitable Fredholm families over Hamiltonian loop group spaces.

1. Introduction

Let \( G \) be a compact connected simply connected Lie group with Lie algebra \( \mathfrak{g} \) and maximal torus \( T \). Let \( \Sigma \) be a compact surface of genus \( g \) with \( b + 1 \geq 1 \) boundary components. As was observed by Atiyah and Bott [8], the space of \( G \)-connections \( \mathcal{A} \) on \( \Sigma \) carries a symplectic form and the action of the gauge group \( \mathcal{G} \) is Hamiltonian. The symplectic quotient at 0 is the moduli space of flat connections on the closed surface obtained by capping off the boundary of \( \Sigma \). Following Donaldson [27], Meinrenken-Woodward [55], and others, one can reduce in stages, first taking the symplectic quotient by the subgroup \( \mathcal{G}_{\partial \Sigma} \) of gauge transformations that are trivial along the boundary. The result is a smooth infinite dimensional manifold \( \mathcal{M} = \mathcal{A}_{fl}/\mathcal{G}_{\partial \Sigma} \), the moduli space of flat connections on \( \Sigma \) with framing along the boundary, and is an example of a Hamiltonian \( LG \)-space, where \( G = G^{b+1} \) and \( LG \) is the loop group.

Let \( V \) be a finite dimensional representation of \( G \) and \( S^\Sigma \) a spin structure on \( \Sigma \). In this article we study the \( \mathcal{G} \)-equivariant family of Dirac operators \( (D^+_A)_{A \in \mathcal{A}} \) acting on sections of \( S^\Sigma \otimes V \) obtained by varying the connection (or vector potential) used in the construction of the Dirac operator. Restriction to the zero fibre of the moment map and descending under the \( \mathcal{G}_{\partial \Sigma} \) action yields an \( LG \)-equivariant family of operators acting on the fibres of a Hilbert bundle over \( \mathcal{M} \). For closed manifolds such ‘tautological’ families of Dirac operators parametrized by moduli spaces of connections were considered for example by Atiyah and Singer [13]. Since in our case the manifold \( \Sigma \) is not closed, we impose a boundary condition (of generalized Atiyah-Patodi-Singer-type) to obtain a smooth \( G \)-equivariant Fredholm family \( (D^c_m)_{m \in \mathcal{M}} \), defining an element in the \( \mathcal{G} \)-equivariant K-theory of \( \mathcal{M} \). In previous work with E. Meinrenken and Y. Song [45], we constructed a finite dimensional non-compact \( Spin_c \) submanifold \( X \subset \mathcal{M} \), and studied the equivariant index of the Dirac operator [47]. The manifold \( X \) plays a similar role to the ‘extended moduli spaces’ used by Jeffrey-Kirwan [37] in their work on the cohomology ring of the moduli space of flat connections. In this article we prove that the Dirac operator on \( X \) coupled to the Fredholm family \( D^c \equiv (D^c_x)_{x \in X} \) has a well-defined distributional equivariant
analytic index. In fact we prove a more general result (Theorem 3.10) for suitable Fredholm families over Hamiltonian loop group spaces, and then specialize to the case above (Theorem 5.8). We refer to the Fredholm families \( Q_\bullet = (Q_x)_{x \in X} \) for which we are able to prove the analytic index is well-defined as ‘quasiperiodic cycles’ (Section 3.1); they include families pulled back from the compact quotient \( X/\Pi \) by the lattice \( \Pi \subset LG \) of 1-parameter subgroups of the maximal torus, as well as interesting families (such as \( D^\Sigma \)) for which II-equivariance fails rather seriously (for instance \( Q_\bullet, \eta \cdot (Q_\bullet) \) may have different domains when \( 0 \neq \eta \in \Pi \)).

We establish three types of formulas for the index. The first type is a non-abelian localization formula (Theorem 3.19), which shows that the index is expressible as a sum of contributions localized near the components of the critical set of the norm-square of the moment map \( \mu_\mathcal{M} : \mathcal{M} \to LG^* \). There is a large literature on non-abelian localization in various forms, for example [39, 65, 61, 72] amongst many others; in the more specific context of Hamiltonian loop group spaces references include [24, 76, 43, 48, 44]. We prove non-abelian localization by adapting a technique of Bismut-Lebeau [21, Chapter IX] to analyse the resolvents of a 1-parameter family of operators in the limit as the parameter goes to infinity.

The second type of formula we establish is a Kirillov-Berline-Vergne-type delocalized index formula (Theorem 4.5) expressing the equivariant index as a distributionally-convergent integral over the full manifold \( X \). An attractive feature of this formula is that the symmetries of the index under the action of the affine Weyl group of \( G \) are realized concretely as symmetries of the differential forms in the integrand. The final type of formula we establish is an abelian localization formula (Theorem 4.10 in general, and specialized to \( D^\Sigma \) and the other Atiyah-Bott classes in Theorems 5.17, 5.18). The result for the family \( D^\Sigma \) is as follows.

**Theorem 1.1.** The equivariant analytic index at level \( \ell = k + h^{\vee} \) of the K-theory class \( E^\Sigma V \) defined by the family \( (D^\Sigma)_{x \in X} \) is determined by the generating series

\[
\frac{1}{[T_\ell]^{k+1}} \sum_{g \in T_\ell^{\text{reg}}/W} \left( \frac{\Delta(g_x)^2}{[T_\ell] \det(1 + z \ell^{-1} H_V(g_x))} \right)^{-g} \delta_{g_x} \in D'(G)^E[z]
\]

where \( T_\ell = \ell^{-1} \Lambda/\Pi \subset T \), \( T_\ell^{\text{reg}} \) is an induction-type map, \( \Delta \) is the Weyl denominator, \( H_V \) is the Hessian of the character \( \text{Tr}_V \), and \( g_x \in J^\infty_C \) is the jet satisfying \( \ell g_x \exp(z \nabla \text{Tr}_V(g_x)) = g^\ell \).

When the level \( \ell \) is sufficiently high, the \([Q, R] = 0\) theorem holds (as a consequence of the non-abelian localization formula), and Theorem 1.1 implies formulas for the indices of the Atiyah-Bott K-theory classes on symplectic quotients, which include the moduli space of flat connections on the closed Riemann surface obtained by capping off the boundary, as well as on punctured Riemann surfaces with holonomies around punctures constrained to lie in certain fixed conjugacy classes (for context see for example [54]). In principle the result for low levels \( \ell \) can be extracted as well, using the quasi-polynomial behavior of the index. The result is a K-theoretic analogue of Witten’s formulas for intersection pairings on the moduli space [75, 37, 52].

In case \( b + 1 = 1 \) and (1) is paired with the constant function 1 on \( G \), the right hand side of (1) is the expression that appears in the index formula of Teleman and Woodward on the moduli stack of holomorphic \( G_C \)-bundles, a remarkable generalization of the Verlinde formula (obtained by setting \( z = 0 \)) that was conjectured by Teleman in [70] and proved in [71]. An interesting recent application of their formula was to the proof of the equivariant
Verlinde formula for Higgs bundles \([33, 4, 34]\). The work of Teleman and Woodward was an important motivation, and we would like to acknowledge our intellectual debt. Although the analytic/differential geometric approach taken in this article is rather different and neither result implies the other (to the best of our knowledge), it has been useful to know what shape to expect for the formula (1). Another point of contact worth mentioning is the (twisted) affine Weyl antisymmetric generating series that plays a key role in their proof (\([71, \text{Section } 3]\)). It seems likely to us that this series matches our equivariant analytic index, in which case the existence of a global object (the operator on \(X \times \Sigma\) that we construct) associated to this generating series provides a new perspective on its properties. For example the twisted affine Weyl equivariance of our index is shown to follow naturally from the behavior of the Atiyah-Patodi-Singer index under a change of boundary conditions.

The contents of the article are as follows. In Section 2 we introduce Hamiltonian loop group spaces and the motivating examples coming from moduli spaces of flat connections on a surface, briefly touching on its relationship with the moduli stack of holomorphic \(G_C\)-bundles (Remark 2.4). We describe the Spin\(_C\) submanifold \(X \subset \mathcal{M}\) that plays a fundamental role throughout the rest of the article. We also prove some special properties of the Kirwan vector field on \(X\). In Section 3 we introduce a subring of the (representable) K-theory \(\mathcal{RK}_0^T(X)\) whose elements we refer to as ‘quasiperiodic K-theory classes’, and we prove that such classes have an equivariant analytic index. We also prove a non-abelian localization formula for the index. In Section 4 we discuss desirable analytic properties of differential form representatives of the Chern character and prove a Kirillov-Berline-Vergne-type index formula. We also prove an abelian localization formula under a suitable further hypothesis on the Chern character. Section 5 returns to moduli spaces of flat connections. We introduce suitable representatives of the Atiyah-Bott K-theory classes, establish some of their properties, and then specialize the abelian localization formula to these classes, culminating in Theorems 5.17, 5.18. We also briefly discuss the generating series for the total \(\lambda\) operation, motivated by its appearance in the equivariant Verlinde formula.

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Notation. Throughout \(z\) is a formal variable. We often deal with vector spaces/bundles that are \(\mathbb{Z}_2\)-graded and in this context \([a, b]\) denotes the graded commutator of linear operators \(a, b\). Graded tensor products are denoted \(\hat{\otimes}\).

Let \(G\) be a compact connected and simply connected Lie group with Lie algebra \(\mathfrak{g}\). Then \(G\) is a product of its simple factors \(G = G_1 \times \cdots \times G_m\) and \(\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m\). The basic inner product on \(\mathfrak{g}_i\) is the unique invariant inner product such that the length squared of the short co-roots is 2. Using the basic inner product on each summand \(\mathfrak{g}_i\), we obtain an inner product \(\cdot\) on \(\mathfrak{g}\) that we use throughout the article to identify \(\mathfrak{g}\) with \(\mathfrak{g}^*\). An \(m\)-tuple \(\ell = (\ell_1, \ldots, \ell_m) \in \mathbb{Z}_{\geq 0}^m\) determines a map \(\ell: \mathfrak{g} \to \mathfrak{g}\) given by multiplication by \(\ell_i\) on the summand \(\mathfrak{g}_i\), and \(\ell^{-1}\) denotes the inverse map.

Fix a maximal torus \(T \subset G\) with Lie algebra \(\mathfrak{t}\), and a choice of positive chamber \(\mathfrak{t}_+ \subset \mathfrak{t}\). Let \(W\) be the Weyl group, and let \(\mathcal{R} \supset \mathcal{R}_+\) be the set of roots, positive roots respectively. Let \(n_+ = |\mathcal{R}_+|\) be the number of positive roots. The integer or co-root lattice is denoted
\[ \Pi = \ker(\exp: t \to T) \simeq \text{Hom}(U(1), T), \\] and the (real) weight lattice by \( \Lambda = \text{Hom}(\Pi, \mathbb{Z}) \simeq \text{Hom}(T, U(1)) \). The lattice \( \Pi \) in \( t \) determines a canonical normalization of Lebesgue measure, that we use to identify distributions with generalized functions. Usually \( t, g \) denote elements of \( T \) (or \( T_C \)), \( \xi \) an element of \( t \) (or \( t_C \)), \( \eta \) an element of \( \Pi \), and \( \lambda \) an element of \( \Lambda \). Given \( \lambda \in \Lambda \), the corresponding representation is denoted \( \mathbb{C}_\lambda \) and the corresponding homomorphism \( e_\lambda: T \to U(1) \) is \( e_\lambda(t) = e^{2\pi i \lambda \xi} \in U(1) \) where \( t = \exp(\xi) \). For \( \ell \in \mathbb{Z}_0^m \), the finite subgroup \( T_\ell := \ell^{-1}\Lambda/\Pi \subset t/\Pi = T \). The representation ring of \( T \) is \( R(T) = \mathbb{Z}[\Lambda] \), and \( \mathbb{C}R(T) = \mathbb{C} \otimes R(T) \). The formal completion \( \hat{R}^{-\infty}(T) = \mathbb{Z}^\Lambda \) is an \( R(T) \)-module. If \( T \) acts on a Hilbert space \( H \), the isotypical component labelled by \( \lambda \in \Lambda \) is denoted \( H_{[\lambda]} \). If \( a \) is a \( T \)-equivariant linear operator on \( H \), then the induced operator on \( H_{[\lambda]} \) is denoted \( a_{[\lambda]} \).

2. Background on Hamiltonian loop group spaces

In this section we review the definition and some basic properties of Hamiltonian loop group spaces \( \mathcal{M} \) (cf. [55]). We briefly recall from [45] the construction of a finite dimensional smooth \( \text{Spin}_n \) submanifold \( X \subset \mathcal{M} \) that plays a fundamental role throughout the rest of the paper, analogous to the role played by the ‘extended moduli spaces’ of Jeffrey-Kirwan [37]; one advantage of our \( X \) is that it is smooth; a disadvantage is that it is less explicit. We describe the motivating examples of Hamiltonian loop group spaces coming from moduli spaces of flat connections on a compact surface with framing along the boundary. In the last subsection we prove some special properties of the Kirwan vector field on \( X \).

2.1. Loop groups. For background on loop groups see [67]. Let \( LG \) denote the space of maps \( S^1 = \mathbb{R}/\mathbb{Z} \to G \) of fixed Sobolev class \( \varsigma > \frac{1}{2} \) (we usually suppress the Sobolev class from the notation). This is a Banach Lie group (the group operation is pointwise multiplication of loops) with Lie algebra \( Lg \) identified with \( g \)-valued 0-forms \( \Omega^0(S^1, g) \) of Sobolev class \( \varsigma \).

The \( U(1) \) central extensions of the loop group are classified by \( \ell \in \mathbb{Z}_0^m \), where the corresponding central extension \( \hat{LG}^{(\ell)} \) has Lie algebra cocycle

\[ (\xi_1, \xi_2) \mapsto 2\pi i \int_{S^1} \ell \xi_1 \cdot d\xi_2. \]

The group \( G \) sits inside \( LG \) as the subgroup of constant loops. The central extension \( \hat{LG}^{(\ell)} \) trivializes over \( G \), hence \( G \) can be regarded as a subgroup of \( \hat{LG}^{(\ell)} \). Another subgroup of \( LG \) that plays a key role for us is the integral lattice \( \Pi = \ker(\exp: t \to T) \) where \( \eta \in \Pi \) is identified with the 1-parameter subgroup \( s \in S^1 \mapsto \exp(-s\eta) \in T \). The restriction of the central extension \( \hat{LG}^{(\ell)} \) to the subgroup \( T \times \Pi \) is a semi-direct product \( T \times \hat{\Pi}^{(\ell)} \), where elements \( t \in T, \tilde{\eta} \in \tilde{\Pi}^{(\ell)} \) obey the commutation relation

\[ \tilde{\eta} t \tilde{\eta}^{-1} t^{-1} = e^{2\pi i \ell \xi \cdot \eta}, \quad t = \exp(\xi). \] (2)

Let \( Lg^* = \Omega_{-1}^1(S^1, g) \) denote the space of \( g \)-valued 1-forms (or connections) of Sobolev class \( \varsigma - 1 \). Since \( \varsigma - 1 > -\varsigma \), there is a non-degenerate pairing \( Lg \times Lg^* \to \mathbb{R} \) given in terms of the inner product \( \cdot \) on \( g \) followed by integration over \( S^1 \). The loop group \( LG \) acts smoothly and properly on \( Lg^* \) by gauge transformations:

\[ a \mapsto g \cdot a = \text{Ad}_g(a) - dg g^{-1}, \quad g \in LG, a \in Lg^*. \] (3)
(In fact the gauge action (3) coincides with the restriction of the coadjoint action of $\hat{LG} = LG^{(1,1,\cdots,1)}$ to the affine hyperplane $Lg^* \times \{1\} \subset \hat{Lg}^*$.) Given $\xi \in Lg$ the corresponding vector field for the infinitesimal action on $Lg^*$ is $\xi_{Lg^*}(a) = \partial_u|_{u=0} \exp(-u\xi) \cdot a = d_u\xi$ at $a \in Lg^*$, where $d_u = d + ad_u$ is the covariant derivative defined by the connection $a$. The subgroup $\Omega G \subset LG$ consisting of loops beginning at $1 \in G$ acts on $Lg^*$ freely with quotient $Lg^*/\Omega G \simeq G$. The holonomy map

$$\text{hol}: Lg^* \to G$$

is the quotient map for the $\Omega G$ action.

2.2. Hamiltonian loop group spaces.

**Definition 2.1** (cf. [55]). A proper Hamiltonian $LG$-space $(\mathcal{M}, \omega_\mathcal{M}, \mu_\mathcal{M})$ is a Banach manifold $\mathcal{M}$ equipped with a smooth action of $LG$, an $LG$-invariant weakly symplectic form $\omega_\mathcal{M}$ and an $LG$-equivariant proper moment map:

$$\mu_\mathcal{M}: \mathcal{M} \to Lg^*, \quad \iota_\mathcal{M}(\xi_\mathcal{M})\omega_\mathcal{M} = -d\langle \mu_\mathcal{M}, \xi \rangle, \quad \xi \in Lg.$$

(We will often drop the adjective ‘proper’ for brevity.)

Important examples are moduli spaces of flat connections on Riemann surfaces with $b+1 \geq 1$ boundary components (with framing along the boundary)—see Section 2.3.

Since $\Omega G$ acts freely and properly on $Lg^*$ and $\mu_\mathcal{M}$ is proper and equivariant, $\Omega G$ acts freely and properly on $\mathcal{M}$ with quotient $\mathcal{M} = \mathcal{M}/\Omega G$ a smooth compact manifold. Let $\text{hol}_\mathcal{M}: \mathcal{M} \to \mathcal{M}$ be the quotient map. The moment map $\mu_\mathcal{M}$ descends to an equivariant ‘group-valued moment map’ $\mu_\mathcal{M}: M \to G$ such that $\mu_\mathcal{M} \circ \text{hol}_\mathcal{M} = \text{hol} \circ \mu_\mathcal{M}$: see [1] for much more from this perspective. Under our assumptions on $G$, it is also known that $M$ is automatically even-dimensional.

**Remark 2.2.** In this article we will not use the symplectic form $\omega_\mathcal{M}$ directly, but only a Spin$_c$ structure derived from it (see Section 2.5). Thus for example the non-degeneracy (or even existence) of $\omega_\mathcal{M}$ could be relaxed, assuming instead the existence of a suitable Spin$_c$ structure.

2.3. Moduli spaces of flat connections on Riemann surfaces. For further background see [55, 56]. Let $\Sigma$ be a compact connected oriented surface with $b+1 \geq 1$ boundary components $\partial \Sigma = \partial b\Sigma \sqcup \partial_1\Sigma \sqcup \cdots \sqcup \partial_b\Sigma$. The orientation of $\Sigma$ induces an orientation of the boundary. We also choose a parametrization of the boundary, although following [55, p.427] and [56, p.143], we choose the parametrization to be orientation reverse. Let $G^*_\mathcal{G} = G^{b+1}$. As $G$ is simply connected, any principal $G$-bundle over $\Sigma$ is trivial. Let $\mathcal{G}$ be the gauge group, that is, the set of maps $\Sigma \to G$ of Sobolev class $\varsigma + \frac{1}{2}$. With this choice of Sobolev class there is a well-defined continuous group homomorphism $R: \mathcal{G} \to LG^*_\mathcal{G} = LG^{b+1}$ given by restriction (or trace) to the boundary. Since $G$ is simply connected, $R$ is surjective, hence we obtain a short exact sequence

$$1 \to \mathcal{G}_{\partial \Sigma} \to \mathcal{G} \to LG^*_\mathcal{G} \to 1.$$

Let

$$\mathcal{A} = \Omega^1_{\varsigma + \frac{1}{2}}(\Sigma, g) \supset \mathcal{A}_{fl}$$

be the space of connections, resp. the space of flat connections on the principal $G$-bundle $\Sigma \times G$, of Sobolev class $\varsigma - \frac{1}{2}$. The affine space $\mathcal{A}$ carries the well-known Atiyah-Bott symplectic
structure [8], and the action of $G_{\partial \Sigma}$ is Hamiltonian for the moment map $A \mapsto \text{curv}(A)$. The symplectic quotient

$$\mathcal{M} = \mathcal{A}_{fl}/G_{\partial \Sigma}$$

is a proper Hamiltonian $LG = LG^{h+1}$-space, where the moment map is given by the pullback of the connection to the boundary.

A sometimes useful fact that we note in passing is that a flat connection is determined up to gauge transformations by its holonomy data, or more precisely the quotient $\mathcal{A}_{fl}/G$ is homeomorphic to $\text{Hom}(\pi, G)/G$, where $\pi$ is the fundamental group of $\Sigma$ (for some choice of basepoint). Using this one can (for example) verify that $\mathcal{M}$ is also given by the quotient $\mathcal{A}_{fl,c}/G_{c}$, where $c$ is a collar neighborhood of $\partial \Sigma$. $\mathcal{A}_{fl,c}$ denotes flat connections that are constant on $c$ equal to the pullback of their pullback to the boundary, and $G_{c}$ denotes gauge transformations that are equal to 1 on $c$ (see the proof of [55, Theorem 3.2] for the technique).

Manifold charts for $\mathcal{M}$ are constructed using the implicit function theorem for Banach spaces (see for example [27, p.103], [55, Section 3.1], [76, Appendix C], [28, Chapters 2, 4], [10, 74]). We review this briefly here since we will use a couple of points later on.

Use a faithful representation $\mathbb{C}^{N}$ to view $G$ as a matrix group. Note that for $A \in \mathcal{A}, g \in G$

$$A - g \cdot A = (d_{Ag})g^{-1},$$

where $d_{Ag} = d_{g} + [A, g]$ is the covariant derivative for the adjoint representation of $g$ on $\text{End}(\mathbb{C}^{N})$. If $g$ fixes $A$ then (4) reads $d_{Ag} = 0$, and thus by parallel translation $g$ is uniquely determined by its value at a single point of the connected manifold $\Sigma$. In particular if $g \in G_{\partial \Sigma}$, then $g|_{\partial \Sigma} = 1$ and uniqueness implies $g = 1$ identically. This shows that the action of $G_{\partial \Sigma}$ on $\mathcal{A}$ is free. At the level of Lie algebras, this means that the map

$$\gamma \in \Omega^{0}_{\zeta = \frac{1}{2}}(\Sigma, g) \mapsto d_{A}\gamma \in \Omega^{1}_{\zeta = \frac{1}{2}}(\Sigma, g)$$

has trivial kernel, where the subscript ‘$\partial$’ indicates that we impose the boundary condition $\gamma|_{\partial \Sigma} = 0$. Equation (4) further shows that if $g \in G_{\partial \Sigma}$ and $\|A - g \cdot A\|_{\zeta = \frac{1}{2}}$ is small, then $g$ is close to 1 $\in G_{\partial \Sigma}$ in the Sobolev $\zeta + \frac{1}{2}$ topology (here we use ellipticity of $d_{A}$ on 0-forms and compactness of $\Sigma$).

Fix a Riemannian metric on $\Sigma$. Let $A \in \mathcal{A}$. We say that $A'$ is in Coulomb gauge relative to $A$ if

$$d_{A}^{*}(A' - A) = 0.$$ 

Define Coulomb gauge slices

$$C_{A, \epsilon} = \{A' \in \mathcal{A}|d_{A}^{*}(A' - A) = 0, \|A' - A\|_{\zeta = \frac{1}{2}} < \epsilon\}.$$

Note that these slices are equivariant for the full gauge group $G$ in the sense that

$$C_{g \cdot A, \epsilon} = g \cdot C_{A, \epsilon}, \quad g \in G.$$ (6)

For $A \in \mathcal{A}_{fl}$ let

$$U_{A, \epsilon} = \{A' \in \mathcal{A}|\text{curv}(A') = 0, d_{A}^{*}(A' - A) = 0, \|A' - A\|_{\zeta = \frac{1}{2}} < \epsilon\} \subset C_{A, \epsilon}.$$

**Proposition 2.3.** Let $A \in \mathcal{A}$. There is an $\epsilon > 0$ sufficiently small (depending on $A$) such that if $\|A' - A\|_{\zeta = \frac{1}{2}} < \epsilon$ then there is a unique $g_{A}(A') \in G_{\partial \Sigma}$ such that $g_{A}(A') \cdot A' \in C_{A, \epsilon}$, and moreover $g_{A}(A')$ depends smoothly on $A'$. Consequently $C_{A, \epsilon}$ is a slice for the $G_{\partial \Sigma}$ action and
q: $\mathcal{A} \to \mathcal{A}/\mathcal{G}_{\partial \Sigma}$ is a principal $\mathcal{G}_{\partial \Sigma}$-bundle. For $A \in \mathcal{A}_F$ and $\epsilon > 0$ sufficiently small, there is a diffeomorphism from $U_{A, \epsilon}$ to its tangent space at $A$, $T_A U_{A, \epsilon}$, and the composition of the latter with the quotient map $q$ provides a local chart for $\mathcal{M}$ near $q(A)$.

**Proof.** Let $\alpha = A' - A$. By assumption $\|\alpha\|_{\zeta - \frac{1}{2}} < \epsilon$. Consider the function of $g \in \mathcal{G}_{\partial \Sigma}$, $\alpha \in \Omega^1_{\zeta - \frac{1}{2}}(\Sigma, g)$ defined by

$$F(g, \alpha) = d_A^* (g \cdot (A + \alpha) - A).$$

Clearly $F(1, 0) = 0$. We claim that for $\epsilon$ sufficiently small, the equation $F(g, \alpha) = 0$ (with $\alpha$ fixed) has a unique solution $g \in \mathcal{G}_{\partial \Sigma}$ such that $\|g \cdot (A + \alpha) - A\|_{\zeta - \frac{1}{2}} < \epsilon$. Since $\alpha, g \cdot (A + \alpha) - A$ are small in the Sobolev $\zeta - \frac{1}{2}$ norm by assumption, the discussion following equation (4) implies that $g$ is close to 1 in $\mathcal{G}_{\partial \Sigma}$ in the Sobolev $\zeta + \frac{1}{2}$ topology. Thus taking $\epsilon$ sufficiently small we can assume that we are in the region where the Banach space implicit function theorem applies. The first partial derivative of $F$ with respect to $g$ at the point $(1, 0)$ is

$$(\mathbf{d} F)_{(1, 0)}(\gamma) = -d_A^* d_A \gamma,$$

$\gamma \in \Omega^0_{\zeta + \frac{1}{2}, 0}(\Sigma, g)$.

Since the Dirichlet problem for the Laplace-type operator $d_A^* d_A$ is uniquely soluble (this follows from, for example, the Lax-Milgram theorem, using injectivity of (5) to check coercivity), the implicit function theorem yields existence and uniqueness of $g$ and shows moreover that $g =: g_A(A')$ depends smoothly on $A' = A + \alpha$.

The flow out $\mathcal{G}_{\partial \Sigma} \cdot C_{A, \epsilon}$ is an open subset of $\mathcal{A}$. It follows from the uniqueness proved above that every $A' \in \mathcal{G}_{\partial \Sigma} \cdot C_{A, \epsilon}$ is in the $\mathcal{G}_{\partial \Sigma}$ orbit of a unique element of $C_{A, \epsilon}$. Thus $C_{A, \epsilon}$ is a slice for the free $\mathcal{G}_{\partial \Sigma}$ action, providing a local section for $q$. A similar argument using the implicit function theorem shows that given two local sections $C_{A_i, \epsilon}$, $i = 1, 2$ with $\epsilon$ sufficiently small, the transition function $q(C_{A_1, \epsilon}) \cap q(C_{A_2, \epsilon}) \to \mathcal{G}_{\partial \Sigma}$ is smooth.

Next let $A \in \mathcal{A}_F$. The derivative of $\text{curv}(\cdot)$ at $A$ is

$$(\mathbf{d} \text{curv})_A(\alpha) = d_A \alpha.$$

The map

$$\alpha \in \Omega^1_{\zeta - \frac{1}{2}}(\Sigma, g) \cap \text{ker}(d_A^*) \mapsto d_A \alpha \in \Omega^2_{\zeta - \frac{3}{2}}(\Sigma, g)$$

is surjective: $d_A \alpha = \beta$ has a solution $\alpha = -d_A \gamma$ where $\gamma$ is the unique solution of the Dirichlet problem

$$-d_A^* d_A \gamma = \beta, \quad \gamma|_{\partial \Sigma} = 0,$$

and moreover $d^*_A \alpha = 0$ since $d_A^2 = 0$ by flatness of $A$. This shows that $0$ is a regular value of $\text{curv}$. By the implicit function theorem, for $\epsilon$ sufficiently small, $U_{A, \epsilon}$ is diffeomorphic to its tangent space at $A$

$$T_A U_{A, \epsilon} = \{ \alpha \in \Omega^1_{\zeta - \frac{1}{2}}(\Sigma, g) | d_A \alpha = 0 = d_A^* \alpha \}.$$
a Kähler structure on \( M \), with which the \( LG \) action can be extended to an \( LG_\Sigma \) action. In [27] Donaldson proves that \( M \) can be identified with the moduli space of holomorphic \( G_\Sigma \)-bundles on \( \Sigma \) with framing along the boundary. Since it is known (the uniformisation theorem) that any holomorphic \( G_\Sigma \)-bundle over \( \Sigma \) is holomorphically trivial, it follows that \( M \simeq L^\Sigma G_\Sigma \backslash LG_\Sigma \), where \( L^\Sigma G_\Sigma \) is the subgroup consisting of loops in \( G_\Sigma \) with a holomorphic extension to \( \Sigma \). By the same reasoning, the moduli stack \( \mathcal{M}_{\Sigma, G_\Sigma} \) of holomorphic \( G_\Sigma \)-bundles on \( \Sigma \) is Fredholm. The orthogonal complement of the range is the kernel of the \( L^\Sigma G_\Sigma \backslash LG_\Sigma \). As \( L^\Sigma G_\Sigma \) is homotopic to \( G \), the homotopy type of \( \mathcal{M}_{\Sigma, G_\Sigma} \) is that of the homotopy quotient \( EG \times_G M \), whose K-theory is a completion of \( RK^0_\Sigma(M) \).

2.4. Transversals. The Lie algebra \( \mathfrak{t} \) can be viewed as the subspace of \( \mathfrak{Lg}^* = \Omega^1(S^1, \mathfrak{g}) \) consisting of constant \( t \)-valued 1-forms. Under the gauge action it is preserved by the subgroup \( N(T) \ltimes \Pi \subset LG \) where \( N(T) \) is the normalizer of \( T \) in \( G \). Since \( T \subset N(T) \) acts trivially on \( t \), there is an induced action of \( (N(T) \ltimes \Pi)/T = W \ltimes \Pi \), which one checks is the standard action of the affine Weyl group \( W_{aff} = W \ltimes \Pi \) on \( t \).

Recall from Section 2.1 that the based loop group \( \Omega G \) acts freely on \( \mathfrak{Lg}^* \). The subspace \( t \) is not transverse to the orbits of the \( \Omega G \) action. However it is possible to ‘thicken’ \( t \) to obtain a finite dimensional submanifold of \( \mathfrak{Lg}^* \) with this property. Let \( t^\perp \) be the orthogonal complement to \( t \) inside \( \mathfrak{g} \). Let \( N(T) \ltimes \Pi \) act on \( t \) as above, and on \( t^\perp \) via the adjoint action of \( N(T) \) (with \( \Pi \) acting trivially). Let \( \overline{B}_\epsilon(t^\perp) \) denote the closed ball in \( t^\perp \) of radius \( \epsilon \).

**Proposition 2.5** ([45], Section 6.4). For \( \epsilon > 0 \) sufficiently small, the inclusion \( t \hookrightarrow \mathfrak{Lg}^* \) extends to an \( N(T) \ltimes \Pi \)-equivariant embedding

\[
t \times \overline{B}_\epsilon(t^\perp) \to \mathfrak{Lg}^*
\]

with image \( R \subset \mathfrak{Lg}^* \) transverse to the \( \Omega G \) orbits and contained in the dense subspace of smooth loops. Each \( \Omega G \) orbit intersects \( R \) non-trivially in an orbit of \( \Pi \).

Sketching the proof, first note that via a tubular neighborhood embedding, it is enough to find an \( N(T) \ltimes \Pi \)-invariant complementary subbundle to the \( \Omega G \)-orbit directions of the normal bundle \( \nu(Lg^*, t) = t \times (\mathfrak{Lg}^*/t) \) (hereafter referred to as a ‘splitting’). Under the quotient map \( \text{hol}: \mathfrak{Lg}^* \to G \), the subspace \( t \) maps to the maximal torus \( T = t/\Pi \). Since the normal bundle \( \nu(G, T) \simeq T \times t^\perp \), at any point \( a \in t \) we can find a subspace of \( \nu(a)(\mathfrak{Lg}^*, t) \) mapped by the normal derivative \( \nu(a)(\text{hol}) : \nu(a)(\mathfrak{Lg}^*, t) \to \nu_{\exp(a)}(G, T) \) isomorphically to \( \nu_{\exp(a)}(G, T) \simeq t^\perp \). These splittings can be chosen to depend smoothly on \( a \in t \) and, since \( N(T) \ltimes \Pi \) acts properly on \( t \), they can be chosen \( N(T) \ltimes \Pi \)-equivariantly. One can arrange that \( R \) is contained in the space of smooth loops as well. The result is canonical up to homotopy, the only choices involved being a splitting (preserved under convex combinations) and a ‘thickness’ \( \epsilon \).

**Example 2.6.** We make the splitting of \( \nu(Lg^*, t) \) more explicit in the case \( \mathfrak{g} = \mathfrak{su}(2) \). In this case \( t \subset Lg^* \) is 1-dimensional. The tangent space to the \( LG \) orbit through the point \( a \in t \subset Lg^* \) is the range of the covariant derivative \( d_a : Lg \to Lg^* \), and is closed of finite codimension since \( d_a \) is Fredholm. The orthogonal complement of the range is the kernel of the \( L^2 \) adjoint \( d_a^* = -d_a \). This subspace is

\[
\ker(d_a^*) = \{ \zeta(s) = \text{Ad}_{\exp(-sa)}(\zeta_0) \in \mathfrak{g}_{\exp(a)} \} \supset t
\]

where \( \mathfrak{g}_{\exp(a)} \) is the Lie algebra of the centralizer \( G_{\exp(a)} \) in \( G \) of \( \exp(a) \in T \). Let \( \Gamma = \exp^{-1}(\{1, -1\}) = \frac{1}{2} \Pi \) be the lattice of elements in \( t \) that exponentiate to an element of the
center of $G$. When $a = c \in \Gamma$ then $\ker(d_\ast^c) \simeq \mathfrak{g}$ and \{c\} \times \ker(d_\ast^c)/t \subset \nu_c(L\mathfrak{g}^*, t) = \{c\} \times L\mathfrak{g}^*/t$ provides the desired splitting at the point $c \in t$.

When $\exp(a)$ is not central then $\ker(d_\ast^a) = t$ is too small. We claim that for all $c \in \Gamma$, \{a\} \times \ker(d_\ast^c) is a complement to the tangent space to the $\Omega G$ orbit through $a$. Since the codimension of the $\Omega G$ orbit through $a$ is $\dim(\mathfrak{g})$, it suffices to check that $\ker(d_\ast^a) \cap d_\ast(a)(\Omega \mathfrak{g}) = 0$. Thus we check for solutions to the equation

$$d_a \zeta = \xi, \quad \zeta(0) = 0, \quad \xi(s) = \text{Ad}_{\exp(-sc)}(\xi_0), \quad \xi_0 \in g_{\exp(c)} = \mathfrak{g}.$$  \hspace{1cm} (8)

Applying $\text{Ad}_{\exp(sc)}$ to both sides and setting $\tilde{\zeta}(s) = \text{Ad}_{\exp(sc)}(\zeta)$, $\tilde{a} = \text{Ad}_{\exp(sc)}(a) - c = a - c \neq 0$ yields

$$d_{\tilde{a}} \tilde{\zeta} = \xi_0 \Rightarrow d_{\tilde{a}}(\tilde{\zeta} - \text{ad}^{-1}_{\tilde{a}} \xi_0) = 0.$$  

Since $\tilde{a} \notin \Gamma$, (7) shows $\tilde{\zeta} - \text{ad}^{-1}_{\tilde{a}} \xi_0 \in t$, and in particular is constant. Since $\tilde{\zeta}(0) = 0$, we obtain $\tilde{\zeta} = 0$. Thus the only solution to (8) is $\zeta = 0$, verifying the claim.

Using the claim, we obtain the desired splitting of $\nu(L\mathfrak{g}^*, t)$ by using the subspace \{c\} \times \ker(d_\ast^c)/t at lattice points $c \in \Gamma \subset t$, and then simply interpolating between these splittings for $a \notin \Gamma$ (recall that convex combinations of splittings are again splittings). Interpolation can be done $N(T) \ltimes \Pi$-equivariantly by choosing a bump function $\chi \in C_c^\infty(t)$ such that (i) $\text{supp}(\chi) \cap \Gamma = \{0\}$, and (ii) the collection of translates of $\chi$ under elements of $\Gamma$ is a partition of unity.

The next definition plays an essential role in what follows.

**Definition 2.7 ([45]).** Let $(\mathcal{M}, \omega_\mathcal{M}, \mu_\mathcal{M})$ be a proper Hamiltonian $L\mathcal{G}$-space, and let $R \subset L\mathfrak{g}^*$ be as constructed in Proposition 2.5. As $\mu_\mathcal{M}$ is $L\mathcal{G}$-equivariant, the inverse image

$$X = \mu_\mathcal{M}^{-1}(R)$$

is a finite dimensional $N(T) \ltimes \Pi$-invariant submanifold (with boundary) of $\mathcal{M}$ that we will refer to as the (global) transversal of $\mathcal{M}$. The interior of $X$ (resp. $R$) is denoted $X^\circ$ (resp. $R^\circ$). The moment map $\mu_\mathcal{M}$ restricts to a map $\mu_X : X \to R$.

We introduce some further notation and properties related to $X$:

**Riemannian metrics.** Under the map $\text{hol}_\mathcal{M}$, the quotient $X/\Pi$ identifies with a compact $N(T)$-invariant submanifold with boundary of the compact manifold $M = \mathcal{M}/\Omega \mathcal{G}$ (moreover $X^\circ/\Pi$ identifies with an open subset of $M$). Choose an $N(T)$-invariant boundary defining function $r$ on $X/\Pi$. Let $g_{X/\Pi}$ be an $N(T)$-invariant Riemannian metric on $X/\Pi$ that takes a product form $dr^2 + g_{\partial X/\Pi}$ in a collar neighborhood of the boundary. The pullback is an $N(T) \ltimes \Pi$-invariant Riemannian metric $g_X$ on $X$. We also introduce a $b$-metric (in the sense of Melrose [57]) on $X/\Pi$, equal to $g_{X/\Pi}$ outside the collar, and given by $dr^2 + r^2 + g_{\partial X/\Pi}$ on the collar. The pullback of the $b$-metric to $X$ will be denoted $g$. The restriction of $g$ to the interior $X^\circ$ is an Riemannian metric, which is complete with a cylindrical end. We sometimes identify $TX^\circ \simeq T^*X^\circ$ using $g$.

**Proper maps.** Let $(\mu, \nu_0)$ be the two components of the composition $\mu_X : X \to R \simeq t \times B_\epsilon(t^\perp)$. Then $\mu$ is proper as the composition of the proper maps $\mu_X = \mu_\mathcal{M}|X$ and $R \to t$. Moreover $\mu|_{X^\circ}$ is the moment map for the $T$-action on $X^\circ$ (recalling that we identify $t \simeq t^\perp$ using the inner product $\langle \cdot, \cdot \rangle$). Similarly the restricted map $(\mu, \nu_0) : X^\circ \to R^\circ \simeq t \times B_\epsilon(t^\perp)$ is proper.
Composing $\nu_0$ with an $N(T)$-equivariant diffeomorphism $B_\ell(t^\perp) \simeq t^\perp$, we obtain a proper $N(T) \times \Pi$-equivariant map
\begin{equation}
(\mu, \nu) : X^\circ \to t \times t^\perp.
\end{equation}
The diffeomorphism can be chosen such that (see for example \cite[Section 4.7]{section} or \cite[Section 3.3]{section} for further discussion)
\begin{equation}
(1 + |\nu(x)|^2)^{-1} |d\nu(x)| \to 0 \quad \text{as} \quad x \to \partial X,
\end{equation}
and we assume this below.

**Remark 2.8.** In case the moment map $\mu_\mathcal M$ happens to be transverse to $t \subset L\mathfrak g^*$, one can work with the simpler choice $\mu_\mathcal M^{-1}(t)$ instead of $\mu_\mathcal M^{-1}(R)$. We will shortly introduce an operator on $X^\circ = \mu_\mathcal M^{-1}(R^\circ)$ that plays the role of a K-homological ‘virtual fundamental class’ for $\mu_\mathcal M^{-1}(t)$.

2.5. Levels and spinor bundles.

**Definition 2.9.** Let $\mathcal M$ be a Hamiltonian $LG$-space. A vector bundle $E \to \mathcal M$ is said to be at level $\ell \in \mathbb Z^m$ if it carries an action of $\widetilde{LG}^{(\ell)}$ compatible with the action of $LG$ on $\mathcal M$.

Fix a $G$-invariant Riemannian metric on $M = \mathcal M/\Omega G$ extending the metric $g_{X/\Pi}$ on $X/\Pi \subset M$. The pullback $\text{hol}_\mathcal M TM$ is a finite rank $LG$-equivariant Euclidean vector bundle over $\mathcal M$, hence we may define the Clifford algebra bundle $\mathcal C(\text{hol}_\mathcal M TM)$. We regard the fibres $\mathcal C(\text{hol}_\mathcal M TM) \to \mathbb Z_2$-graded complex *-algebras (isomorphic to complex square matrices of size $2^{\dim(M)/2}$).

**Definition 2.10.** A level $\ell$ spinor bundle for $\text{hol}_\mathcal M TM$ is a level $\ell \mathbb Z_2$-graded Hermitian vector bundle $\mathcal S = \mathcal S^+ \oplus \mathcal S^- \to \mathcal M$ which is also a bundle of irreducible modules for $\mathcal C(\text{hol}_\mathcal M TM)$ and such that the module structure map $c : \mathcal C(\text{hol}_\mathcal M TM) \to \text{End}(\mathcal S)$ is an $LG$-equivariant isomorphism of bundles of $\mathbb Z_2$-graded *-algebras.

We make the fundamental assumption throughout the rest of the article that the level $\ell = (\ell_1, \ldots, \ell_m)$ of the spinor bundle $\mathcal S$ satisfies
\begin{equation*}
\ell_1 > 0, \ldots, \ell_m > 0,
\end{equation*}
to be abbreviated $\ell > 0$ below.

Let $h^\vee = (h^\vee_1, \ldots, h^\vee_m) \in \mathbb Z^m_{>0}$ denote the $m$-tuple of dual Coxeter numbers of the simple Lie algebras $\mathfrak g_i$ in the direct sum decomposition $\mathfrak g = \mathfrak g_1 \oplus \cdots \oplus \mathfrak g_m$.

**Theorem 2.11 ([45]).** Let $\mathcal M$ be a proper Hamiltonian $LG$-space with quotient $M = \mathcal M/\Omega G$. There is a canonical level $h^\vee$ spinor bundle $\mathcal S_\text{can}$ for $\text{hol}_\mathcal M TM$.

The heuristic idea behind the construction in \cite{45} is as follows. In finite dimensions, spinor modules satisfy 2-out-of-3: given a short exact sequence of even rank Euclidean vector bundles $0 \to V_1 \to V_2 \to V_3 \to 0$ and given spinor modules for any two of the $V_i$, one obtains a spinor module for the third. The manifold $\mathcal M$ is symplectic, hence has a spinor module coming from a choice of compatible almost complex structure. The kernel of the quotient map $\text{Thol}_\mathcal M : TM \to \text{hol}_\mathcal M TM$ is trivial with fibres isomorphic to $\Omega \mathfrak g \simeq L\mathfrak g/\mathfrak g$. The latter has a canonical complex structure with $+i$-eigenspace given by the positive Fourier modes, and hence a canonical spinor module as well. Assuming the 2-out-of-3 property still holds in this infinite dimensional context, one obtains a spinor module for $\text{hol}_\mathcal M TM$. With care this strategy can be carried out rigorously.
The dual Coxeter number appears because this is the level of the spin representation of the loop group.

Further important examples of spinor bundles come from tensoring with line bundles: given a level \( k > -h^\vee \) line bundle \( L \) on \( \mathcal{M} \), one obtains a new spinor module \( \mathcal{F} = \mathcal{F}_{\text{can}} \otimes L \) at level \( \ell = k + h^\vee > 0 \). Line bundles appear for example in the context of geometric quantization: \( \mathcal{M} \) is prequantizable at level \( k \) if there is a level \( k \) line bundle \( L \) with \( c_1(L) = [k \omega_{\mathcal{M}}] \).

**Remark 2.12.** For the moduli space of flat connections \( \mathcal{M} \) considered in Section 2.3, prequantum line bundles are discussed in [55, Section 3.3], following others [58, 68, 75]. In short \( \mathcal{M} \) has a canonical \( LG \)-equivariant prequantum line bundle, obtained by reduction of a prequantum line bundle for the Atiyah-Bott symplectic structure on \( \mathcal{A} \).

**Definition 2.13.** Let \( \mathcal{F} \to \mathcal{M} \) be a level \( \ell > 0 \) spinor bundle for \( \text{hol}^*_\mathcal{A} TM \). The **anti-canonical line bundle** of \( \mathcal{F} \) is the level \( 2\ell \) line bundle

\[
\mathcal{L} = \text{Hom}_{\text{Cl}(\text{hol}^*_\mathcal{A} TM)}(\mathcal{F}^*, \mathcal{F}).
\]

Choose an invariant Hermitian connection \( \nabla_\mathcal{F} \) on \( \mathcal{L} \). Let

\[
\varpi = \frac{1}{2} c_1(\mathcal{L}, \nabla_\mathcal{F})
\]

be half the first Chern form associated to the connection. The **spin\text{c} moment map** is the map \( \phi_\mathcal{M} : \mathcal{M} \to Lg^* \) defined by

\[
2\pi\imath \langle \phi_\mathcal{M}, \xi \rangle = \frac{1}{2} (\mathcal{L}_\xi - \nabla_\xi \mathcal{F}), \quad \xi \in Lg.
\]

Define \( \phi : X \to \mathfrak{t} \) to be \( \phi_\mathcal{M}|_X \) composed with projection to \( \mathfrak{t} \).

Note that the pullback of \( \varpi \) to \( X \) is \( N(T) \times \Pi \)-invariant and the map \( \phi \) is \( N(T) \times \Pi \)-equivariant for the level \( \ell \) action of \( \Pi \) on \( \mathfrak{t} \), i.e.

\[
\phi(\eta \cdot x) = \phi(x) + \ell \eta \quad (11)
\]

for all \( \eta \in \Pi \). The assumption \( \ell > 0 \) implies that \( \phi : X \to \mathfrak{t} \) is a proper map.

**2.6. Dirac operators on the transversal.** Let \( X \) be the transversal of a Hamiltonian \( LG \)-space \( \mathcal{M} \) as introduced in Section 2.2. Let \( \mathcal{F} \) be a level \( \ell > 0 \) spinor bundle for \( \text{hol}^*_\mathcal{A} TM \) (Definition 2.10). As explained in Section 2.4, \( X^\circ / \Pi \) identifies with an open subset of the closed \( G \)-manifold \( M = \mathcal{M} / \Omega G \). It follows that the pullback of \( \mathcal{F} \) to \( X^\circ \) is a spinor module for the tangent bundle \( (TX^\circ, g_X|_{X^\circ}) \). Recall that we also defined a complete Riemannian metric \( g \) on \( X^\circ \) (see after Definition 2.7). Given a spinor bundle \( c_1 : \text{Cl}(TY, g_1) \to \text{End}(\mathcal{F}) \) on a Riemannian manifold \( (Y, g_1) \), and given a new Riemannian metric \( g_2 \) on \( Y \), one obtains a new Clifford action \( c_2 = c_1 \circ (g_1^{-1} g_2)^{1/2} : \text{Cl}(TY, g_2) \to \text{End}(\mathcal{F}) \). In this way \( \mathcal{F} \) becomes a spinor module for \( \text{Cl}(TX^\circ, g) \), and we will consider it as such from now on. Choose an \( N(T) \times \tilde{\Pi}^{(\mathfrak{t})} \)-invariant Clifford connection \( \nabla \) on \( \mathcal{F} \), and let \( \mathcal{D}_\mathcal{F} \) be the corresponding Dirac operator, defined as the composition

\[
C^\infty(X^\circ, \mathcal{F}) \xrightarrow{\nabla} C^\infty(X^\circ, T^*X^\circ \otimes \mathcal{F}) \xrightarrow{\xi \cdot} C^\infty(X^\circ, \mathcal{F}).
\]

Completeness of the Riemannian metric \( g \) on \( X^\circ \) guarantees that the operator \( \mathcal{D}_\mathcal{F} \) with initial domain \( C^\infty_0(X^\circ, \mathcal{F}) \) is essentially self-adjoint in \( L^2(X^\circ, \mathcal{F}) \). Following Atiyah [6] and Kasparov [38] (see also [36]), \( \mathcal{D}_\mathcal{F} \) may be thought of as representing a choice of fundamental class for \( X^\circ \).
in the analytic approach to K-homology (the K-homology group that we have in mind here is more analogous to Borel-Moore homology).

As mentioned in Remark 2.8, we would prefer to work with the ‘fundamental class’ of the subset \( \mu^{-1}_{\mathcal{M}}(t) \), except that this subset is often singular. To handle this we introduce a new operator \( \mathcal{D} \) below, obtained by twisting \( \mathcal{D}_0 \) by the pullback under \( \nu: X^0 \to \mathfrak{t}^\perp \) of a suitable Thom-Bott class \( \mathcal{B} \in K^0_{\mathfrak{t}}(\mathfrak{t}^\perp) \). The resulting operator can be thought of as representing the ‘virtual fundamental class’ of \( \mu^{-1}_{\mathcal{M}}(t) \) in K-homology.

Let \( (\mathcal{S}_{\mathcal{M}}, c_{\perp}) \) denote the \( T \)-equivariant \( \mathbb{Z}_2 \)-graded spin representation for \( \text{Cl}(\mathfrak{t}^\perp) \):

\[
\mathcal{S}_{\perp} = \wedge \mathfrak{n} \otimes \mathbb{C}_{-\rho}, \quad c_{\perp}(\xi) = \sqrt{2}(\epsilon(\xi^{1,0}) - \iota(\xi^{0,1})) \otimes 1
\]

where \( \mathfrak{n} \subset \mathfrak{g}_{\mathbb{C}} \) is the direct sum of the positive root spaces, \( \rho \) is the half sum of the positive roots, and we endow \( \mathfrak{t}^\perp \) with the complex structure \( (\mathfrak{t}^\perp)^{1,0} = \mathfrak{n} \). The corresponding Bott-Thom element for \( \mathfrak{t}^\perp \) is the element \( \mathcal{B} \in K^0_{\mathfrak{t}}(\mathfrak{t}^\perp) \) represented by the morphism of vector bundles \( \mathfrak{t}^\perp \times \mathcal{S}_{\mathcal{M}}^+ \to \mathfrak{t}^\perp \times \mathcal{S}_{\mathcal{M}}^+ \) given by \( c_{\perp}(\xi) \) over the point \( \xi \in \mathfrak{t}^\perp \). This element is Weyl antisymmetric: under the transformation of \( K^0_{\mathfrak{t}}(\mathfrak{t}^\perp) \) induced by \( w \in \mathcal{W} \), \( \mathcal{B} \) is multiplied by \( (-1)^{\ell(w)} \) (see for example [47, Proposition 4.8], except that in the latter we preferred not to include the \( \mathbb{C}_{-\rho} \) shift of (12)).

**Definition 2.14.** Let \( \mathcal{S} = \mathcal{S}_{\mathcal{M}} \tilde{\otimes} \mathcal{S}^* \). The operator \( \mathcal{D}_0 \) extends to act on sections of \( \mathcal{S} \). Define a new Dirac-type operator \( \mathcal{D} \) acting on sections of \( \mathcal{S} \) by

\[
\mathcal{D} = ic_{\perp}(\nu) \tilde{\otimes} 1 + \mathcal{D}_0.
\]

In terms of a local orthonormal frame \( e_1, \ldots, e_n \) of \( TX \) near \( x \in X \), the operator \( \mathcal{D} \) is

\[
\mathcal{D}(\beta \otimes s)(x) = ic_{\perp}(\nu(x))\tau(x) \tilde{\otimes} s(x) + (-1)^{\text{deg}(\tau)} \sum_{j=1}^n \partial_e_j \tau(x) \tilde{\nabla} c(e_j)s(x) + \tau(x) \tilde{\nabla} c(e_j)\nabla_e_j s(x).
\]

The term involving \( c_{\perp}(\nu) \) in Definition 2.14 plays the role of a potential in the \( \mathfrak{t}^\perp \)-directions, with \( -c_{\perp}(\nu(x))^2 = |\nu(x)|^2 \to \infty \) as \( x \to X^0 \) approaches \( \partial X \).

2.7. The Spin\(_c\) Kirwan vector field. Throughout this section \( \mathcal{S} \) denotes a level \( \ell > 0 \) spinor bundle on \( \mathcal{M} \), and \( \phi: X \to \mathfrak{t} \) denotes the (projection to \( \mathfrak{t} \)) of the Spin\(_c\) moment map for some choice of connection, as in Definition 2.13. In this section we prove some special properties of the Kirwan vector field associated to \( \phi \). Similar properties also hold for the Kirwan vector field associated to \( \mu \). But it is the corresponding properties of \( \phi \) that we will use later on.

**Definition 2.15.** The Spin\(_c\) Kirwan vector field is the \( N(T) \)-invariant vector field \( \kappa \) on \( X \) given at the point \( x \in X \) by

\[
\kappa(x) = \phi(x)_X(x),
\]

where \( \phi(x)_X \) denotes the vector field on \( X \) generated by \( \phi(x) \in \mathfrak{t} \). We also define

\[
\bar{\phi} = (1 + \phi^2)^{-1/2} \phi, \quad \bar{\kappa}(x) = \bar{\phi}(x)_X(x).
\]

Note that \( \bar{\phi} \) is a bounded map and \( \bar{\kappa} \) is a bounded vector field.

Let

\[
\mathcal{Z} = \kappa^{-1}(0_X) = \bigcup_{\beta \in \mathcal{B}} \mathcal{Z}_\beta, \quad \mathcal{Z}_\beta = X^\beta \cap \phi^{-1}(\beta)
\]
where $\mathcal{B}$ is the infinite, discrete, $W$-invariant subset of $\beta \in t$ such that $X^\beta \cap \phi^{-1}(\beta) \neq \emptyset$. By properness of $\phi$, each $Z_\beta$ is compact. Let

$$Z = Z \cap \nu^{-1}(0), \quad Z_\beta = Z_\beta \cap \nu^{-1}(0)$$

denote the intersection of $Z$, $Z_\beta$ with $\nu^{-1}(0)$. (Note that some of the $Z_\beta$ could be empty.) Although $\kappa$ is not $\Pi$-invariant, the subset $Z$ has, nevertheless, the following property:

**Proposition 2.16.** There is a minimal finite subset $\mathcal{B}_s \subset \mathcal{B}$ such that for each $\beta \in \mathcal{B}$ there is a $\beta_s \in \mathcal{B}_s$ and $\eta \in \Pi$ such that $Z_\beta = \eta \cdot Z_{\beta_s}$.

Before giving the proof, we introduce the notation

$$\{t_i \subset t\}_{i \in \mathcal{I}}, \quad (13)$$

for the list of rational subspaces that arise as infinitesimal stabilizers of subsets of $X$. Since the actions of $T$, $\Pi$ on $X$ commute and since $X/\Pi$ is compact, the set $\mathcal{I}$ is finite.

**Proof.** For $i \in \mathcal{I}$, let $\mathcal{B}_i \subset \mathcal{B}$ be the subset of $\beta$ such that $X^\beta = X^t_i$. Thus $\cup \mathcal{B}_i = \mathcal{B}$, and as $\beta$ ranges over $\mathcal{B}_i$, $X^\beta = X^t_i$ does not vary. Therefore it suffices to show that the image $\exp(\mathcal{B}_i)$ of each $\mathcal{B}_i$ under the quotient map $\exp: t \to t/\Pi = T$ is finite.

For an affine subspace $\Delta \subset t$, let $\Delta_0$ be the unique subspace parallel to $\Delta$. Note that since the inner product $\cdot$ on $t$ is integral, i.e. $\Pi \cdot \Pi \subset \mathbb{Z}$, a subspace $\Delta_0$ is rational if and only if its orthogonal complement $\Delta_0^\perp$ is.

The map $\phi$ satisfies $d(\phi, \xi) = -\iota(\xi \pi)\pi$, and it follows from this equation that the image under $\phi$ of each connected component of the fixed-point set $X_i^t$ is contained in an affine subspace $\Delta$ with $\Delta_0^\perp = t_i$. Let $\mathcal{S}_i$ denote the collection of affine subspaces $\Delta$ (each a translate of $t_i^\perp$) that arise in this way from a connected component of $X_i^t$. Note that $\Pi$ acts naturally on $\mathcal{S}_i$, and since $X/\Pi$ is compact, the set of cosets of the $\Pi$ action is finite; let $s_i$ be its cardinality.

If $\beta \in \mathcal{B}_i$ then there exists a $\Delta \in \mathcal{S}_i$ such that $\beta = \text{pr}_\Delta(0)$, the orthogonal projection of $0$ onto $\Delta$. Now suppose $\Delta \in \mathcal{S}_i$ and $\Delta' = \Delta + \eta$ for some $\eta \in \Pi$. Then $\Delta_0^\perp = (\Delta_0')^\perp = t_i$ and

$$\text{pr}_\Delta'(0) = \text{pr}_\Delta(0) + \text{pr}_t(\eta),$$

where $\text{pr}_t$ denotes orthogonal projection to $t_i$. Thus when $\Delta$ is translated by some $\eta \in \Pi$, the projection $\text{pr}_\Delta(0)$ changes by an element of $\text{pr}_t(\Pi)$. By integrality of the inner product, $\text{pr}_t(\Pi)$ is a lattice in $t_i$. On the other hand, since $t_i$ is integral, $t_i \cap \Pi$ is a full-rank lattice in $t_i$, hence has some finite index $n_i$ in $\text{pr}_t(\Pi)$. It follows that $\# \exp(\mathcal{B}_i) \leq s_i \cdot n_i$. (Thus in fact we have the bound $\# \mathcal{B}_s \leq \sum_{i \in \mathcal{I}} s_i \cdot n_i$.)

For each $\beta_s \in \mathcal{B}_s$, let $\mathcal{U}_{\beta_s}$ be an open $T$-invariant neighborhood of $Z_{\beta_s}$ in the manifold with boundary $X$. Using translations by elements of $\Pi$, we obtain open neighborhoods $\mathcal{U}_\beta$ of $Z_\beta$ for each $\beta \in \mathcal{B}$. Making the $\mathcal{U}_{\beta_s}$ smaller if necessary, we can ensure that closure of the images $\phi(\mathcal{U}_\beta)$ are pair-wise disjoint. Let

$$\mathcal{U} = \bigcup_{\beta \in \mathcal{B}} \mathcal{U}_\beta.$$

Let $d(\cdot, \cdot)$ be the topological metric on $X$ determined by the Riemannian metric $g_X$. Let $d_t(\cdot, \cdot)$ denote the metric induced by the norm on $t$.

**Lemma 2.17.** There is an $\epsilon > 0$ such that for all $i \in \mathcal{I}$ (see $(13)$),

$$d(x, X^t_i) < \epsilon, \quad d_t(\phi(x), t_i) < \epsilon \quad \Rightarrow \quad x \in \mathcal{U}.$$
Proof. First note that there is an \( \epsilon' > 0 \) such that for all \( \beta \in \mathcal{B} \),
\[
  d(x, X^\beta) < \epsilon', \quad d_i(\phi(x), \beta) < \epsilon' \implies x \in \mathcal{U}.
\] (14)
Indeed it is clear that we can find an \( \epsilon' > 0 \) that works for \( \beta \) in the finite set \( \mathcal{B}_\ast \), and it then works for all \( \beta \in \mathcal{B} \) by \( \Pi \)-invariance.

Suppose the statement is false. Then there exists an \( i \in \mathcal{I} \) and a sequence \( \{x_n\} \) in \( X/\mathcal{U} \) such that \( d(x_n, X^b), d_i(\phi(x_n), h) \to 0 \), where \( h = t_i \). Passing to a subsequence, we can assume \( x_n \Pi \to x \Pi \) in the compact space \( X/\Pi \). Necessarily \( x \Pi \in (X/\Pi)^b = X^b/\Pi \). Let \( y_n \in X \) be such that \( y_n \Pi = x \Pi \) and \( d(x_n, y_n) \) is minimal. Then \( d(x_n, y_n) \to 0 \) and \( y_n \in X^b \). Note also that by \( \Pi \)-equivariance, \( \phi \) is Lipschitz for some constant \( C \geq 1 \). Therefore
\[
d_i(\phi(y_n), h) \leq d_i(\phi(y_n), \phi(x_n)) + d_i(\phi(x_n), h) \leq C \cdot d(y_n, x_n) + d_i(\phi(x_n), h) \to 0.
\] (15)

Let \( X^b_j, j \in J \) be the connected components of \( X^b \). There is a finite subset \( J_\ast \subset J \) such that each \( X^b_j \) is of the form \( \eta \cdot X^b_j \) for some \( j \in J_\ast \) and \( \eta \in \Pi \). The moment map sends each \( X^b_j \) into a closed subset of an affine hyperplane parallel to the orthogonal complement \( h^\perp \). By \( \Pi \)-periodicity and because \( h \) is rational, there is a constant \( c > 0 \) such that for all \( j \in J \), either \( \phi(X^b_j) \cap h \neq \emptyset \), or else \( d_i(\phi(X^b_j), h) > c \). In particular (15) implies that \( \{y_n\} \) is eventually contained in the union of those components \( X^b_j \) such that \( \phi(X^b_j) \cap h \neq \emptyset \). Then by (15), there exists \( n_0 \) and \( \beta \in \phi(X^b) \cap h \subset \mathcal{B} \cap h \) such that
\[
d_i(\phi(y_{n_0}), \beta) < \epsilon' \quad \text{and} \quad d(x_{n_0}, y_{n_0}) < \epsilon'/2C.
\]
These inequalities, together with \( y_{n_0} \in X^b \subset X^\beta \), imply \( d(x_{n_0}, X^\beta) < \epsilon'/2C < \epsilon' \).
\[
d_i(\phi(x_{n_0}), \beta) < \epsilon', \quad \text{and hence (14) yields} \quad x_{n_0} \in \mathcal{U}, \text{a contradiction.} \tag*{\Box}
\]

Proposition 2.18. There is a constant \( c > 0 \) such that \( |\kappa(x)| \geq c \) for all \( x \in X/\mathcal{U} \).

Proof. Suppose the statement is false. Then there is a sequence \( \{x_n\} \) in \( X/\mathcal{U} \) such that \( |\kappa(x_n)| \to 0 \). Passing to a subsequence, we can assume that \( x_n \Pi \to x \Pi \) in the compact space \( X/\Pi \). Let \( h \subset t \) be maximal such that \( x \Pi \in (X/\Pi)^h = X^h/\Pi \) (thus \( h = t_i \) for some \( i \in \mathcal{I} \), see (13)). Let \( \phi = \phi_h + \phi_{h^\perp} \) be the decomposition of \( \phi \) into its \( h, h^\perp \) components.

Since \( d(x_n, X^h) \to 0 \), to obtain a contradiction it suffices, by Lemma 2.17, to show that \( d_i(\phi(x_n), h) = |\phi_{h^\perp}(x_n)| \to 0 \).

Let \( \varrho : X \times t \to TX \) be the infinitesimal action. We will use the same symbol for the infinitesimal action on \( X/\Pi \). By the slice theorem for actions of compact Lie groups, there is a compact neighborhood \( V \) of \( x \Pi \) in \( X/\Pi \), a constant \( \delta > 0 \) and a smooth subbundle \( A \) such that
\[
T(X/\Pi)|_V = \varrho(V \times h^\perp) \oplus A, \quad \varrho(V \times h) \subset A,
\]
and
\[
|\varrho_{x\Pi}(\xi)| > \delta |\xi|, \quad x \Pi \in V, \quad \xi \in h^\perp.
\] (16)

Let \( \theta \in (0, \pi/2] \) be the minimal angle (with respect to the Riemannian metric \( g_{X/\Pi} \) on \( X/\Pi \)) between \( \varrho(V \times h^\perp) \), \( A \) over the compact set \( V \).

The Kirwan vector field
\[
\kappa(x_n) = \varrho_{x_n}(\phi(x_n)) = \varrho_{x_n}(\phi_h(x_n)) + \varrho_{x_n}(\phi_{h^\perp}(x_n)).
\]
For $n$ sufficiently large, $x_n \Pi \in V$, and then by elementary geometry

$$|\kappa(x_n)| \geq |g_{x_n}(\phi_{h^n}(x_n))| \sin(\theta).$$

Since $|\kappa(x_n)| \to 0$, we get $|g_{x_n}(\phi_{h^n}(x_n))| \to 0$. By (16), $|\phi_{h^n}(x_n)| \to 0$, as desired. \hfill \Box

3. Quasiperiodic K-theory classes and the analytic index

In this section we introduce a subring of the representable K-theory ring $\mathcal{R}K^0(X)$ consisting of K-theory classes admitting a representative cycle with good analytic properties; we will call these ‘quasiperiodic K-theory classes’ and ‘quasiperiodic cycles’ respectively. Examples of quasiperiodic K-theory classes are K-theory classes pulled back from the quotient $X/\Pi$ (genuinely periodic), but the most interesting examples are those which are not of this kind. We describe an analytic index map for quasiperiodic cycles. We prove a non-abelian localization formula for the index, and use it to deduce that the index map descends to K-theory. The non-abelian localization formula expresses the index as a sum of contributions from the components of the vanishing locus of the Kirwan vector field (Section 2.7). This general type of formula originated in the work of many researchers; perhaps closest to the setting here are works by Tian-Zhang \cite{72}, Paradan \cite{62}, and Ma-Zhang \cite{51}.

The $T$-equivariant representable K-theory of $X$, denoted $\mathcal{R}K^0_T(X)$ is the non-compactly supported $T$-equivariant K-theory of $X$, cf. \cite[Section 5]{69}. For topologists $\mathcal{R}K^0_T(X)$ consists of homotopy classes of $T$-equivariant continuous maps from $X$ to the space of Fredholm operators on $L^2(T) \otimes l^2(\mathbb{Z})$ (equipped with the norm topology). When $X$ is locally compact, an equivalent (and closely related) description appears in Kasparov theory: $\mathcal{R}K^0_T(X) = \mathcal{R}KK_T(X; C_0(X), C_0(\mathbb{X}))$ can be realized as equivalence classes of pairs $(\mathcal{E}, F)$ where $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^−$ is a $T$-equivariant $\mathbb{Z}_2$-graded Hilbert bundle over $X$ and $F = \{F_x\}_{x \in X}$ is a bounded continuous section of the bundle $\mathcal{B}^{\text{odd}}(\mathcal{E})$ of bounded odd operators on the fibres of $\mathcal{E}$ such that for all $x \in X$, $F_x^* = F_x$, and $1 - F_x^2 \in \mathcal{K}(\mathcal{E}_x)$ (compact operators); see for example \cite{35, 23, 38} for background. Given such a pair $(\mathcal{E}, F)$, one recovers the topologist’s definition by trivializing $\mathcal{E}$ (after stabilization if necessary) and taking the component $F^+$ of $F$ that maps $\mathcal{E}^+$ to $\mathcal{E}^−$. In some cases it is more convenient to work with pairs $(\mathcal{E}, Q)$ where $Q$ is a family of possibly unbounded odd self-adjoint operators on the fibres of $\mathcal{E}$, where the corresponding bounded family is $F = Q(1 + Q^2)^{-1/2}$. Such ‘unbounded cycles’ have been studied extensively and in great generality in the KK-theory literature. We will formulate the definition of quasiperiodic cycles in those terms, as this is convenient for the motivating example (Atiyah-Bott classes) to be treated in Section 5.

3.1. Quasiperiodic cycles. By a Hilbert bundle over $X$ we will always mean a smooth bundle of Hilbert spaces $\mathcal{E} \to X$, with structure group the unitary group carrying the norm topology.

Definition 3.1. Let $\mathcal{E}$ be a $T$-equivariant Hilbert bundle over $X$. We say that $\mathcal{E}$ is $T$-finite if $\mathcal{E}$ can be realized as a subbundle of a trivial $T$-equivariant Hilbert bundle $X \times \mathcal{E}_0$ such that $\mathcal{E}_0$ has finitely many $T$-isotypical components.

Remark 3.2. A Hilbert bundle $\mathcal{E}$ is $T$-finite if and only if the map $T \to \text{Aut}(\mathcal{E})$ is continuous in the norm topology.

We will specialize to $T \times \Pi$-equivariant $T$-finite Hilbert bundles $\mathcal{E} \to X$. Since $T \times \Pi$ acts properly on $X$, it is always possible to construct smooth $T \times \Pi$-invariant Hermitian connections
on $\mathcal{E}$ (since the $\Pi$ action is free this is especially straight-forward: construct a $T$-invariant connection on $\mathcal{E}/\Pi \rightarrow X/\Pi$ and pull it back). Any two such connections differ by a smooth $T \times \Pi$-invariant section of $u(\mathcal{E}) \subset \mathcal{B}(\mathcal{E})$, the bundle of bounded skew-adjoint endomorphisms of $\mathcal{E}$.

**Definition 3.3.** Let $(\mathcal{E}, \nabla^\mathcal{E})$ be a $T \times \Pi$-equivariant $T$-finite Hilbert bundle with $T \times \Pi$-invariant Hermitian connection. Define the moment map $\phi_\mathcal{E}$ of the pair $(\mathcal{E}, \nabla^\mathcal{E})$ (cf. [16] in the finite dimensional case) by

$$2\pi i \langle \phi_\mathcal{E}, \xi \rangle = L^\mathcal{E}_\xi - \nabla^\mathcal{E}_{\xi X}, \quad \xi \in \mathfrak{t}. \quad (17)$$

Note that $\langle \phi_\mathcal{E}, \xi \rangle \in C^\infty(X, \mathcal{B}(\mathcal{E}))$ is bounded because of the $T$-finiteness condition, hence $\phi_\mathcal{E} \in \mathfrak{t}^* \otimes C^\infty(X, \mathcal{B}(\mathcal{E}))$ and is $T \times \Pi$-invariant.

**Definition 3.4.** Let $\mathcal{E} \rightarrow X$ be a $T$-equivariant $\mathbb{Z}_2$-graded Hilbert bundle with $T$-invariant Hermitian connection $\nabla^\mathcal{E}$, and let $Q = \{Q_x\}_{x \in X}$ be a $T$-equivariant family of odd unbounded self-adjoint operators on the fibres $\{\mathcal{E}_x\}_{x \in X}$ of $\mathcal{E}$. The triple $(\mathcal{E}, \nabla^\mathcal{E}, Q)$ will be called a *quasiperiodic cycle* if

(a) For each $x \in X$, $(1 + Q^2_x)^{-1}$ is a compact operator.
(b) $(\mathcal{E}, \nabla^\mathcal{E})$ is $T \times \Pi$-equivariant and $\mathcal{E}$ is $T$-finite.
(c) The family $Q$ is smooth in the sense that for any smooth compactly supported section $s$ of $\mathcal{E}$ such that $s_x \in \text{dom}(Q_x)$ for all $x \in X$, the compactly supported section $Qs$ of $\mathcal{E}$ is again smooth.
(d) $\nabla^\mathcal{E}$-parallel transport maps the subset $\text{dom}(Q_\bullet) = \bigsqcup_{x \in X} \text{dom}(Q_x) \subset \mathcal{E}$ to itself, and the total $\nabla^\mathcal{E}$-covariant derivative of $Q$ is a smooth bounded section of $T^*X \otimes \mathcal{B}(\mathcal{E})$.

**Remark 3.5.** Condition (d) requires further explanation. The condition that $\nabla^\mathcal{E}$-parallel transport maps $\text{dom}(Q_\bullet)$ into itself means that we may simultaneously trivialize $\mathcal{E}$, $\text{dom}(Q_\bullet)$ on geodesic balls $B_{x_0}$ in $X$ by parallel translation along radial geodesics (in particular observe that $\text{dom}(Q_\bullet)$ is a subbundle of $\mathcal{E}$). Thus $Q|_{B_{x_0}}$ can be viewed as a family of unbounded self-adjoint operators on a fixed Hilbert space $\mathcal{E}_{x_0}$ having the same domain $\text{dom}(Q_{x_0})$. A functional analytic argument [40, p.549] shows that the seemingly weak smoothness condition (c) already implies that the corresponding map

$$Q|_{B_{x_0}} : B_{x_0} \rightarrow \mathcal{B}(\text{dom}(Q_{x_0}), \mathcal{E}_{x_0}) \quad (18)$$

is smooth, where $\text{dom}(Q_{x_0})$ is equipped with the graph norm. Consequences of this include that the resolvents $(Q \pm i)^{-1}$ vary smoothly, and also that a section $s$ of $\text{dom}(Q_\bullet)|_{B_{x_0}}$ (topologized with the graph norm of $Q_{x_0}$) is smooth if and only if $s$ is smooth as a section of $\mathcal{E}|_{B_{x_0}}$ (because $s = (Q + i)^{-1}(Q + i)s$). The total $\nabla^\mathcal{E}$-covariant derivative of $Q$ at the point $x_0$ can be defined to be the derivative of the map (18) at the point $x_0$. The result is a priori an element of $T^*X \otimes \mathcal{B}(\text{dom}(Q_{x_0}), \mathcal{E}_{x_0})$, and part of condition (d) is that we require it to extend continuously to an element of $T^*X \otimes \mathcal{B}(\mathcal{E}_{x_0})$. The second part of condition (d) is that the section of $T^*X \otimes \mathcal{B}(\mathcal{E})$ obtained by assembling the total derivative at all points $x_0 \in X$ should be smooth and bounded.

**Remark 3.6.** Notice that we do not assume $Q$ is $\Pi$-equivariant. The adjective *quasiperiodic* is meant to indicate that part of the data $(\mathcal{E}, \nabla^\mathcal{E})$ is $\Pi$-equivariant, and although $Q$ might not be, it at least varies in a controlled way relative to the $\Pi$-invariant connection $\nabla^\mathcal{E}$. Another
rationale for the name will appear in Section 5.5, where we will see an example where the K-theory class of the cycle behaves like a quasiperiodic function. We should add the caution that for our principal example (Atiyah-Bott classes) to be considered in Section 5, even the domain bundle dom(\(Q_\ast\)) will not be \(\Pi\)-invariant, so the failure of \(\Pi\)-equivariance can still be rather severe.

It is not difficult to show that the smoothness assumption on \(Q\) implies (a fortiori) that the bounded transform \(Q(1 + Q^2)^{-1/2}\) varies continuously in the norm topology, and hence a quasiperiodic cycle as above determines a Kasparov cycle. Let \(\varrho\) be the obvious representation of \(C_0(\mathbb{X})\) on \(C_0(\mathbb{X}, \mathcal{E})\) by multiplication operators.

**Definition 3.7.** A class \(\mathcal{E} \in RK^0_T(\mathbb{X})\) will be called quasiperiodic if there is a quasiperiodic cycle \((\mathcal{E}, \nabla^\mathcal{E}, Q)\) such that \((C_0(\mathbb{X}, \mathcal{E}), \varrho, Q(1 + Q^2)^{-1/2})\) is a Kasparov cycle representing \(\mathcal{E}\).

The simplest examples of such classes come from \(T \times \Pi\)-equivariant vector bundles on \(\mathbb{X}\).

### 3.2. The analytic index of quasiperiodic cycles.

Given a Hermitian connection \(\nabla^\mathcal{E}\) on a Hilbert bundle \(\mathcal{E} \to \mathbb{X}\), let \(\mathcal{D}^\mathcal{E} = 1 \otimes_{\mathcal{E}} \mathcal{D}\) be the unbounded self-adjoint operator on \(L^2(\mathbb{X}, \mathcal{E} \otimes S)\) obtained by coupling \(\mathcal{D}\) to \(\mathcal{E}\) using \(\nabla^\mathcal{E}\) (for self-adjointness of Dirac operators on complete Riemannian manifolds coupled to Hilbert bundles, cf. [31, Proposition 1.16]).

**Lemma 3.8.** Let \((\mathcal{E}, \nabla^\mathcal{E})\) be a \(T \times \Pi\)-equivariant Hilbert bundle with Hermitian connection over \(\mathbb{X}\), where \(\mathcal{E}\) is \(T\)-finite. Let \(\chi \in C_\infty^{\text{c}}(\mathbb{X})^T\) and let \(f \in C_0(\mathbb{R})\). Then \(\|\chi f(\mathcal{D}^\mathcal{E})\| \to 0\) as \(|\lambda| \to \infty\), \(\lambda \in \Lambda\).

**Proof.** A function \(f \in C_0(\mathbb{R})\) can be approximated uniformly in norm by functions of the form \(g(s)(i + s)^{-1}\), \(g \in C_0(\mathbb{R})\). The properties of the potential term \(c_\perp(\nu)\) and in particular equation (10) (see [47, Section 4.7] for detailed discussion) imply that the operator \(\chi f(\mathcal{D}^\mathcal{E})\) can be approximated in norm by operators of the form \(\chi'(\mathcal{D}^\mathcal{E})\), where \(\chi' \in C_\infty^{\text{c}}(\mathbb{X}^\circ)^\perp\). Therefore we may reduce to the case where \(f(s) = (i + s)^{-1}\) and \(\chi \in C_\infty(\mathbb{X}^\circ)^T\). Then \(f(\mathcal{D}^\mathcal{E})\) is the resolvent of \(\mathcal{D}^\mathcal{E}\), which has range contained in \(L^2(\mathbb{X}, \mathcal{E} \otimes S)\) by ellipticity, and \(\chi f(\mathcal{D}^\mathcal{E})\) has range contained in \(L^2(\mathbb{K}, \mathcal{E} \otimes \tilde{S})\) where \(\mathbb{K} = \text{supp}(\chi) \subset \mathbb{X}^\circ\) is compact. Let 

\[
\iota_\lambda^\mathcal{E} : L^2(\mathbb{K}, \mathcal{E} \otimes \tilde{S})|_{\lambda} \to L^2(\mathbb{K}, \mathcal{E} \otimes \tilde{S})|_{\lambda}, \quad \iota_\lambda : L^2(\mathbb{K}, S)|_{\lambda} \to L^2(\mathbb{K}, S)|_{\lambda}\]

be the inclusions. Note that the norms

\[
\|\iota_\lambda\| \to 0 \quad \text{as} \quad |\lambda| \to \infty
\]

by the Rellich lemma, because \(\iota_\lambda\) are the isotypical components of the compact embedding \(\iota : L^2(\mathbb{K}, S) \hookrightarrow L^2(\mathbb{K}, S)\).

Likewise the \(\iota_\lambda^\mathcal{E}\) are the isotypical components of an inclusion \(\iota^\mathcal{E}\), but unlike \(\iota\), \(\iota^\mathcal{E}\) need not be a compact embedding since \(\mathcal{E}\) is allowed to have infinite rank. We claim that nevertheless the norm \(\|\iota_\lambda^\mathcal{E}\| \to 0\) as \(|\lambda| \to \infty\). Indeed we can argue locally, and we may use any connection, since the Sobolev space \(L^2(\mathbb{K}, \mathcal{E} \otimes S)\) does not depend on the choice. Therefore assume \(\mathcal{E} = \mathbb{X} \times \mathcal{E}_0\) is the trivial \(T\)-equivariant Hilbert bundle with the trivial connection, and \(\mathcal{D}^\mathcal{E} = \text{id}_{\mathcal{E}} \otimes \mathcal{D}\). By \(T\)-finiteness, \(\mathcal{E}_0\) has finitely many non-zero \(T\)-isotypical components \(\mathcal{E}_{0, \lambda'}\), \(\lambda' \in \Lambda'\). Then

\[
L^2(\mathbb{K}, \mathcal{E} \otimes \tilde{S})|_{\lambda} = \bigoplus_{\lambda' \in \Lambda'} \mathcal{E}_{0, \lambda'} \otimes L^2(\mathbb{K}, S)|_{\lambda - \lambda'}
\]
and hence
\[ \| \mathcal{E}_\Lambda \| = \sup_{\lambda' \in \Lambda'} \| \iota_{\lambda'} \lambda \|. \]
The result follows from (19) and the finiteness of $\Lambda'$. \hfill \Box

A family of operators $Q$ as in Definition 3.4 determines an odd essentially self-adjoint operator on $L^2(X^0, \mathcal{E} \otimes S)$ with domain $C_c^\infty(X^0, \text{dom}(Q) \otimes S)$. Its closure is a self-adjoint operator, also denoted $Q$ when there is no risk of confusion, with domain dom$(Q)$.

**Proposition 3.9.** Let $(\mathcal{E}, \nabla^\mathcal{E}, Q)$ be a quasiperiodic cycle. Then the sum
\[ \mathcal{P}^Q = Q + \mathcal{P}^\mathcal{E} \]
is self-adjoint on dom$(Q) \cap$ dom$(\mathcal{P}^\mathcal{E}) = \text{dom}(Q) \cap L^{2,1}(X^0, \mathcal{E} \otimes S)$.

**Proof.** We will deduce this as a very special case of the abstract functional analytic result [42, Theorem 1.1]. Let
\[ \mathcal{D}(Q, \mathcal{P}^\mathcal{E}) = \{ v \in \text{dom}(Q) \cap \text{dom}(\mathcal{P}^\mathcal{E}) | Qv \in \text{dom}(\mathcal{P}^\mathcal{E}), \mathcal{P}^\mathcal{E}v \in \text{dom}(Q) \} \subset L^2(X^0, \mathcal{E} \otimes S) \]
be the initial domain of the graded commutator $[Q, \mathcal{P}^\mathcal{E}]$. Then according to loc. cit., and since $C_c^\infty(X^0, \mathcal{E} \otimes S)$ is a core for $\mathcal{P}^\mathcal{E}$, it suffices to verify that: $C_c^\infty(X^0, \text{dom}(Q) \otimes S) \subset \mathcal{D}(Q, \mathcal{P}^\mathcal{E})$, the graded commutator $[Q, \mathcal{P}^\mathcal{E}]$ is bounded, and $(Q \pm i)^{-1}(C_c^\infty(X^0, \mathcal{E} \otimes S)) \subset C_c^\infty(X^0, \text{dom}(Q) \otimes S)$.

By condition (c) in the definition of quasiperiodic cycles $Q(C_c^\infty(X^0, \text{dom}(Q)) \subset C_c^\infty(X^0, \mathcal{E})$, while by condition (d), $\nabla^\mathcal{E}(C_c^\infty(X^0, \text{dom}(Q))) \subset C_c^\infty(X^0, T^*X \otimes \text{dom}(Q)).$ It follows that $C_c^\infty(X^0, \text{dom}(Q)) \subset \mathcal{D}(Q, \mathcal{P}^\mathcal{E})$. Boundedness of $[Q, \mathcal{P}^\mathcal{E}]$ is a consequence of boundedness of $[\nabla^\mathcal{E}, Q]$, which in turn follows from condition (d) in the definition. The resolvent $(Q \pm i)^{-1} : \mathcal{E} \to \text{dom}(Q)$ is smooth as we noted in Remark 3.5, hence $(Q \pm i)^{-1}(C_c^\infty(X^0, \mathcal{E} \otimes S)) \subset C_c^\infty(X^0, \text{dom}(Q) \otimes S)$.

**Theorem 3.10.** Let $(\mathcal{E}, \nabla^\mathcal{E}, Q)$ be a quasiperiodic cycle, and let $\mathcal{P}^Q = \mathcal{P}^\mathcal{E} + Q$ be the corresponding unbounded self-adjoint operator on $L^2(X^0, \mathcal{E} \otimes S)$. For each $\lambda \in \Lambda$ the operator $\mathcal{P}^Q_{[\lambda]}$ has compact resolvent.

**Proof.** We must show that $(1 + \mathcal{P}^{Q,2})_{[\lambda]}^{-1}$ is compact. Since
\[ (1 + Q^2 + \mathcal{P}^{\mathcal{E},2})^{-1} - (1 + \mathcal{P}^{\mathcal{E},2})^{-1} = (1 + \mathcal{P}^{Q,2})^{-1}(Q, \mathcal{P}^\mathcal{E})(1 + Q^2 + \mathcal{P}^{\mathcal{E},2})^{-1} \]
and $[Q, \mathcal{P}^\mathcal{E}]$ is bounded, it suffices to show that $(1 + Q^2 + \mathcal{P}^{\mathcal{E},2})^{-1}$ is compact. By positivity (see [66, Proposition 1.4.5]); we learned of this trick from [59]) there is a factorization
\[ (1 + Q^2 + \mathcal{P}^{\mathcal{E},2})^{-1} = b(1 + Q^2)^{-1/4}(1 + \mathcal{P}^{\mathcal{E},2})^{-1/4}b' \]
for some bounded operators $b, b'$. The operator $(1 + Q^2)^{-1/4}$ is a bounded $T$-invariant section of the bundle of compact operators $\mathcal{H}(\mathcal{E} \otimes S)$. The result therefore follows if we prove the following: for any continuous bounded $T$-invariant section $a$ of $\mathcal{H}(\mathcal{E} \otimes S)$, the operator $a(1 + \mathcal{P}^{\mathcal{E},2})_{[\lambda]}^{-1/4}$ on $L^2(X^0, \mathcal{E} \otimes S)_{[\lambda]}$ is compact.

The function $s \in \mathbb{R} \mapsto (1 + s^2)^{-1/4} \in (0, \infty)$ is a uniform limit of Schwartz functions having compactly supported Fourier transform. Since the compact operators form a closed ideal, it suffices to show that $ah(\mathcal{P}^\mathcal{E})_{[\lambda]}$ is compact for any Schwartz function $h \in C_c^\infty(\mathbb{R})$ having compactly support Fourier transform. For such functions $h(\mathcal{P}^\mathcal{E})$ is an operator of
finite propagation (cf. [36]): if \( s \in C_c^\infty(X^0, \mathcal{E} \otimes S) \) then \( h(\mathcal{D}^s) \) has support contained in a neighborhood of \( \text{supp}(s) \) of size \( r \), where \( \text{supp}(\hat{h}) \subset [-r, r] \).

Let \( \chi_t \in C_c^\infty(t) \) be a compactly supported bump function such that \( \{ \eta \cdot \chi_t | \eta \in \Pi\} \) is a partition of unity. Let \( \chi = \mu^*\chi_t \) be the pullback. The set \( \{ \chi_\eta = \eta \cdot \chi | \eta \in \Pi \} \) is a partition of unity on \( X \), and therefore

\[
ah(\mathcal{D}^s) = \sum_{\eta \in \Pi} \chi_\eta ah(\mathcal{D}^s),
\]

the sum converging in the weak topology. Using the properness of \( (\mu, \nu) : X^0 \to t \times t^1 \), the properties of the potential term \( c_{\perp \perp} (\nu) \) (in particular (10); see [47, Section 4.7] for details), and the compactness of \( a \) on the fibres, the Rellich lemma implies that each of the summands in (21) is compact. Thus the result follows if we show that (21) converges in norm when restricted to the \( \lambda \)-isotypical subspace \( L^2(X^0, \mathcal{E} \otimes S)_{[\lambda]} \). Since a finite sum of compact operators is compact, it is enough to prove this for the sum over a finite index sublattice \( \Pi' \subset \Pi \). Since \( h(\mathcal{D}^s) \) has finite propagation, we may choose \( \Pi' \) such that the operators \( \chi_\eta ah(\mathcal{D}^s) \) have disjoint supports for \( \eta \in \Pi' \). Then proving convergence of the sum in norm on \( L^2(X^0, \mathcal{E} \otimes S)_{[\lambda]} \) is the same as proving that \( \chi_\eta (ah(\mathcal{D}^s))_{[\lambda]} \to 0 \) in norm as \( |\eta| \to \infty \). Choose lifts \( \hat{\eta} \in \hat{\Pi}(t^\ell) \) and let \( a_\eta = \hat{\eta}^{-1}a\hat{\eta} \). Then using the \( \bar{\Pi}(t^\ell)\)-equivariance of \( \mathcal{D}^s \),

\[
(\chi_\eta ah(\mathcal{D}^s))_{[\lambda]} = (\hat{\eta}(\chi a_\eta h(\mathcal{D}^s))\hat{\eta}^{-1})_{[\lambda]} = \hat{\eta}(\chi a_\eta h(\mathcal{D}^s))_{[\lambda + \ell \eta]} \hat{\eta}^{-1}
\]

where in the second equality we used the commutation relation (2). The norm of the operator in the last expression is at most

\[
\|a\|_\infty \cdot \|\chi h(\mathcal{D}^s)_{[\lambda + \ell \eta]}\|.
\]

Since \( \ell > 0 \), \( |\lambda + \ell \eta| \to \infty \) as \( |\eta| \to \infty \). By Lemma 3.8, \( \|\chi h(\mathcal{D}^s)_{[\lambda + \ell \eta]}\| \to 0 \).

In particular \( \mathcal{P}_{[\lambda]}^Q \) is Fredholm, allowing us to make the following definition.

**Definition 3.11.** The equivariant analytic index of the quasiperiodic cycle \((\mathcal{E}, \nabla^\mathcal{E}, Q)\) is the element of \( R^{-\infty}(T) \) given by

\[
\text{index}_T(\mathcal{P}^Q) = \sum_{\lambda \in \Lambda} \text{index}(\mathcal{P}^Q_{[\lambda]}) \cdot e_\lambda.
\]

### 3.3. Non-abelian localization

In Section 2.7 we introduced the Spin\(_c\) Kirwan vector field \( \kappa \) and its bounded version \( \bar{\kappa} \) associated to \( \phi, \hat{\phi} = \phi(1 + \phi^2)^{-1/2} \) respectively. In Section 2.4 we introduced a map \( \nu : X^0 \to t^1 \) with the property that \( (\mu, \nu) : X^0 \to t \times t^1 \) is a proper map. Let \( \mathcal{P}^Q = \mathcal{D}^s + Q \) be the operator on \( L^2(X^0, \mathcal{E} \otimes S) \) associated to a quasiperiodic cycle \((\mathcal{E}, \nabla^\mathcal{E}, Q)\) as in Section 3.2. We will study the deformation

\[
\mathcal{P}^Q_s = \mathcal{P}^Q + s\bar{\Phi}, \quad \bar{\Phi} = 1(c_{\perp \perp} (\nu)\hat{\otimes} 1 - 1\hat{\otimes} c(\bar{\kappa})),
\]

where \( s \in \mathbb{R} \) is a parameter. Related deformations have been studied extensively in various contexts by Tian-Zhang [72], Ma-Zhang [49], Braverman [25] and many others.

**Lemma 3.12 ([48]).** Let \((\mathcal{E}, \nabla^\mathcal{E}, Q)\) be a quasiperiodic cycle. The square \( \mathcal{P}^Q_s^2 \) is given by

\[
\mathcal{P}^Q_s^2 = \mathcal{P}^Q + s^2\bar{\Phi}^2 + s[\mathcal{P}^Q, \bar{\Phi}].
\]
where $\Phi^2 = \nu^2 + \kappa^2$,

$$[\mathcal{P}^\mathcal{E}, \Phi] = 2\nu^2 + 4\pi(\phi + \phi^*) \cdot \Phi + i[\mathcal{P}^\mathcal{E}, c_1^\perp(\nu)] - i \sum_{j=1}^{\dim(t)} 2\bar{\phi}_j c(\nabla \cdot \xi^j_X) + c(d\bar{\phi}_j)c(\xi^j_X) - 2\bar{\phi}_j L^j,$$

and $\xi^1, ..., \xi^{\dim(t)}$ is a basis of $t$. For each $\lambda \in \Lambda$ there is a proper and bounded below function $f_\lambda$ on $X^\circ$ such that the operator inequality $[\mathcal{P}^\mathcal{E}, \Phi]_\lambda \geq f_\lambda$ holds. For each $s \geq 0$ the operator $(\mathcal{P}^{\mathcal{E},2}_\lambda)[\lambda]$ is Fredholm, and its index does not depend on $s$.

The proof is essentially the same as the proof of a similar result in [48, Section 4] for the special case where $\mathcal{E}$ is a finite dimensional $T \times \Pi$-equivariant vector bundle and $Q = 0$, and so we will not repeat it here. One can take $f_\lambda = f - c_\lambda$ where $c_\lambda$ is a constant only depending on $\lambda$ (its appearance comes from the Lie derivative term in the expression for the cross-term $[\mathcal{P}^\mathcal{E}, \Phi]$). The lemma suggests that as $s \to \infty$, the term that is quadratic in $s$ dominates and the kernel of $\mathcal{P}^{\mathcal{E}}_Q$ localizes near the subset $Z \subset X$ where $\Phi$ vanishes, the fact that $f_\lambda$ is proper and bounded below meaning that the cross-term is under control. Making this statement precise involves some analysis to which the rest of this section is devoted.

For each $\beta_* \in B$, let $U'_{\beta_*} \subseteq U_{\beta_*} \subset U_{\beta_*} \cap (|\nu|)^{-1}[0, 1)$ be $T$-invariant open neighborhoods of $Z_{\beta_*}$, where $U_{\beta_*}$ is the neighborhood of $Z_{\beta_*}$ introduced in Section 2.7. We may assume the closure $\overline{U}_{\beta_*}$ is a smooth manifold with boundary. For $\beta = \beta_* + \eta \in B$ where $\eta \in \Pi$, let $U_{\beta} = \eta \cdot U_{\beta_*}$, $U'_{\beta} = \eta \cdot U'_{\beta_*}$, and put

$$U = \bigcup_{\beta \in B} U_{\beta}, \quad U' = \bigcup_{\beta \in B} U'_{\beta}.$$ 

By Proposition 2.18 we may arrange that $U, U'$ have the property that

$$\kappa^2(x) + \nu^2(x) > c^2 > 0 \quad \text{for} \quad x \in U \setminus U'.$$

Let $\rho^{1/2} : X \to [0, 1]$ be a smooth $T$-invariant function equal to 1 on $U'$ and satisfying $\rho(U) \subset (0, 1], \supp(\rho) = \overline{U}$, $\|d\rho\|_\infty < \infty$ (the latter property can be arranged by Proposition 2.16). Define operators

$$\mathcal{P}^{\mathcal{E}}_{U,s} = \mathcal{P}^\mathcal{E}_U + s\rho^{-1}\Phi, \quad \mathcal{P}^{\mathcal{E}}_U = \rho^{1/2}\mathcal{P}^\mathcal{E}\rho^{1/2} + Q$$  \hspace{1cm} (23)

where

$$\Phi = i(c_{1^\perp}(\nu)\tilde{\gamma}1 - 1\tilde{\gamma}c(\kappa)).$$  \hspace{1cm} (24)

Note that $\Phi$ differs from $\Phi$ in that $\kappa$ has been replaced with $\kappa$. We regard $\mathcal{P}^{\mathcal{E}}_U$ as an unbounded operator in the Hilbert space $L^2(U, \mathcal{E} \otimes S|_U)$ where the measure is the restriction of the Riemannian measure on $X$ (although $U$ is not complete, $\mathcal{P}^{\mathcal{E}}_U$ is still essentially self-adjoint thanks to the bump function $\rho^{1/2}$).

**Lemma 3.13.** Let $(\mathcal{E}, \nabla \mathcal{E}, Q)$ be a quasiperiodic cycle. The square $\mathcal{P}^{\mathcal{E},2}_{U,s}$ is given by

$$\mathcal{P}^{\mathcal{E},2}_{U,s} = \mathcal{P}^{\mathcal{E},2}_U + s^2\rho^{-2}\Phi^2 - s\rho^{-1}c(d\rho)\Phi + s[\mathcal{P}^\mathcal{E}_{U}, \Phi],$$
where $\Phi^2 = \nu^2 + \kappa^2$,

$$[\mathcal{P}^\xi, \Phi] = 2\nu^2 + 4\pi \phi^2 + 4\pi \phi \cdot \phi + i[\mathcal{P}^\xi, c_{\Lambda}(\nu)] - i \sum_{j=1}^{\dim(t)} 2\phi_j c(\nabla \cdot \xi_j^U) + c(d\phi_j) c(\xi_j^U) - 2\phi_j L\xi_j,$$

and $\xi^1, ..., \xi^{\dim(t)}$ is a basis of $\mathfrak{t}$. There is a constant $s_0$ such that for $s > s_0$ and for each $\lambda \in \Lambda$ there is a proper and bounded below function $f_{U,\lambda}$ on $U$ such that there is an operator inequality

$$s \rho^{-2} \Phi^2 - \rho^{-1} c(d\rho) \Phi + [\mathcal{P}^\xi, \Phi] |_{[\lambda]} \geq (s - 2s_0) \rho^{-2} \Phi^2 + f_{U,\lambda}.$$

For $s > 2s_0$ the operator $(\mathcal{P}^{\xi}_{U,s}) |_{[\lambda]}$ is Fredholm and its index does not depend on $s$.

Proof. The formula for the square $\mathcal{P}^{\xi}_{U,s}$ involves a calculation similar to that in Lemma 3.12. Consider the expression for the graded commutator $[\mathcal{P}^\xi, \Phi]$. The terms involving $[\mathcal{P}^\xi, c_{\Lambda}(\nu)]$, $c(d\phi_j)$ are bounded on $U$ and unimportant. The total Lie derivative $L\xi_j$ becomes bounded after restricting to the $\lambda$-isotypical component. The dominant term in the expression is therefore $4\pi \phi^2$, as all the other terms are at most linear in $\phi$. It follows that there is a proper and bounded below function $f_{U}^\prime$ on $X$ such that the operator inequality $[\mathcal{P}^\xi, \Phi] |_{[\lambda]} \geq f_{U}^\prime$ holds. Therefore

$$s \rho^{-2} \Phi^2 - \rho^{-1} c(d\rho) \Phi + [\mathcal{P}^\xi, \Phi] |_{[\lambda]} \geq s \rho^{-2} \Phi^2 - \rho^{-1} |d\rho| |\Phi| + f_{U}^\prime. \quad (25)$$

For any $s_0 > 0$ we may rearrange the right hand side of (25) as

$$(s - 2s_0) \rho^{-2} \Phi^2 + (s_0 \rho^{-2} \Phi^2 + f_{U}^\prime) + (s_0 \rho^{-1} |\Phi| - |d\rho|) \rho^{-1} |\Phi|.$$ \(\text{Note that } \rho^{-1}(x) \geq 1 \text{ for all } x \in U.\) On the other hand $d\rho|_{U'} = 0$ and $|\Phi(x)| < c > 0$ for $x \in U \setminus U'$. Therefore choosing $s_0 > c^{-1} ||d\rho||_{\infty}$ ensures $(s_0 \rho^{-1} |\Phi| - |d\rho|) \geq 0$. Dropping this term (25) becomes

$$s \rho^{-2} \Phi^2 - \rho^{-1} c(d\rho) \Phi + [\mathcal{P}^\xi, \Phi] |_{[\lambda]} \geq (s - 2s_0) \rho^{-2} \Phi^2 + (s_0 \rho^{-2} \Phi^2 + f_{U}^\prime).$$

The function $f_{U,\lambda} = s_0 \rho^{-2} \Phi^2 + f_{U}^\prime$ is proper and bounded below on $U$, because $\Phi^2 > c^2 > 0$ on $U \setminus U'$, $\rho^{-2}(x) \to \infty$ as $x \to \partial U$, and $f_{U}^\prime$ is proper and bounded below on the closure of $U$. This proves the desired inequality. The remaining statements follow from the inequality. \(\square\)

**Theorem 3.14.** Let $(\mathcal{E}, \nabla^\xi, Q)$ be a quasiperiodic cycle. For $s$ sufficiently large $\text{index}_T(\mathcal{P}^\xi) = \text{index}_T(\mathcal{P}^\xi_{U,s}).$

Proof. The difference of the indices equals the index of the block diagonal operator $\mathcal{P}^\xi \oplus \mathcal{P}^\xi_{U,s}$ acting on $\mathcal{H} \oplus \mathcal{H}^\text{op}_U$, where $\mathcal{H} = L^2(X^\circ, \mathcal{E} \otimes S)_U$, $\mathcal{H}^\text{op}_U = L^2(U, \mathcal{E} \otimes S|_U)$ and $\mathcal{H}^\text{op}_U$ denotes $\mathcal{H}^\text{op}_U$ equipped with the opposite $\mathbb{Z}_2$-grading. Let $\chi: X \to [0,1]$ be a smooth $T$-invariant bump function equal to 1 on a neighborhood of $Z$ and such that $\text{supp}(\chi) \subset \rho^{-1}(1)$ (hence $\rho \chi = \chi$) and $||d\chi||_{\infty} < \infty$ (possible because of Proposition 2.16). Let $m_{\chi}$ denote multiplication by $\chi$ followed by restriction to $U$, viewed as an odd operator $\mathcal{H} \to \mathcal{H}^\text{op}_U$. Its adjoint $m_{\chi}^*: \mathcal{H}^\text{op}_U \to \mathcal{H}$ is multiplication by $\chi|_U$ followed by extension to $X^\circ$ by 0. Let $\gamma$ be the grading operator for $\mathcal{H}$. Define an odd unbounded self-adjoint operator on $\mathcal{H} \oplus \mathcal{H}^\text{op}_U$:

$$D_s = \begin{pmatrix} \mathcal{P}^\xi_{U,s} & s\gamma m_{\chi}^* \\ sm_{\chi} \gamma & \mathcal{P}^\xi_{U,s} \end{pmatrix}.$$
Fix \( \lambda \in \Lambda \). The operator \((D_s)_{[\lambda]}\) is a bounded perturbation of the compact resolvent operator \((\mathcal{P}^Q_s \oplus \mathcal{P}^Q_{U,s})_{[\lambda]}\) and therefore has the same index. We claim that for \( s \gg 0 \), \((D_s)_{[\lambda]}\) does not contain 0 in its spectrum and hence has index 0. In fact we prove the stronger result that \((1 + D^2_s)_{[\lambda]} \to 0 \) in norm as \( s \to \infty \).

Using \( [\mathcal{P}^Q, \chi] = c(d\chi), \gamma \mathcal{P}^Q = -\mathcal{P}^Q \gamma \) and \( \chi \rho = \chi \), one finds

\[
1 + (D^2_s)_{[\lambda]} = \left( 1 + \frac{(\mathcal{P}^Q_s)^2_{[\lambda]} + s^2 \chi^2}{s c(d\chi) \gamma} - s \gamma c(d\chi) 1 + (\mathcal{P}^Q_{U,s})_{[\lambda]} + s^2 \chi^2 \right) =: \begin{pmatrix} w_s & x_s \\ y_s & z_s \end{pmatrix}. \tag{26}
\]

Note that \( w_s, z_s \) are invertible for all \( s \). By Lemma 3.12,

\[
s^{-1}w_s \geq s(\Phi^2 + \chi^2) + f_s, \tag{27}
\]

for a function \( f_s \) that is proper and bounded below. Since \( \Phi^2 + \chi^2 > 0 \) everywhere and \( f_s \) is proper and bounded below, the infimum of \( s(\Phi^2 + \chi^2) + f_s \) goes to infinity as \( s \to \infty \). Hence by taking inverses, (27) shows that \( \|w_s^{-1}\| = o(s^{-1}) \) as \( s \to \infty \). By Lemma 3.13,

\[
s^{-1}z_s \geq (s - 2s_0)(\rho^{-2}\Phi^2 + \chi^2) + f_{U,\lambda} \tag{28}
\]

for a function \( f_{U,\lambda} \) that is proper and bounded below on \( U \). Since \( \Phi^2 + \chi^2 > 0 \) everywhere and \( f_{U,\lambda} \) is proper and bounded below on \( U \), the infimum of \( (s - 2s_0)(\rho^{-2}\Phi^2 + \chi^2) + f_{U,\lambda} \) goes to infinity as \( s \to \infty \). Hence by taking inverses, (28) shows that \( \|z_s^{-1}\| = o(s^{-1}) \) as \( s \to \infty \). On the other hand \( \|x_s\|, \|y_s\| \) are both \( O(s) \) because \( \|d\chi\|_{\infty} < \infty \).

Define

\[
q_s = w_s - x_s z_s^{-1} y_s = w_s(1 - w_s^{-1} x_s z_s^{-1} y_s).
\]

By the order estimates above, \( \|w_s^{-1} x_s z_s^{-1} y_s\| \to 0 \) as \( s \to \infty \). Hence for \( s \gg 0 \), \( q_s \) is invertible and moreover \( \|q_s^{-1}\| = o(s^{-1}) \). This in turn means we can apply the following explicit formula for the inverse of a block 2 \( \times \) 2 matrix:

\[
(1 + D^2_s)_{[\lambda]}^{-1} = \left( \begin{array}{cc} w_s & x_s \\ y_s & z_s \end{array} \right)^{-1} = \left( \begin{array}{cc} q_s^{-1} & q_s^{-1} x_s z_s^{-1} \\ z_s^{-1} y_s q_s^{-1} & z_s^{-1} y_s q_s^{-1} x_s z_s^{-1} \end{array} \right). \tag{29}
\]

By the order estimates above, each entry converges to 0 in norm as \( s \to \infty \). \( \square \)

Remark 3.15. An analytic localization argument employing the explicit formula (29) for the inverse of a block 2 \( \times \) 2 matrix appeared in the work of Bismut and Lebeau [21, Chapter IX].

Since \( U = \sqcup_{\beta \in \mathcal{B}} U_\beta \), the theorem yields an expression for \( \text{index}_T(\mathcal{P}^Q) \) as a sum of contributions from components \( Z_\beta, \beta \in \mathcal{B} \) of \( Z \).

Corollary 3.16. Let \( (\mathcal{E}, \nabla^\mathcal{E}, Q) \) be a quasiperiodic cycle. For each \( \beta \in \mathcal{B}, \) let \( \mathcal{P}^Q_{U,\beta,s} \) be the restriction of the operator \( \mathcal{P}^Q_{U,s} \) to \( U_\beta \subseteq U \). For \( s \gg 0 \) the equation

\[
\text{index}_T(\mathcal{P}^Q) = \sum_{\beta \in \mathcal{B}} \text{index}_T(\mathcal{P}^Q_{U,\beta,s})
\]

holds in \( R^{-\infty}(T) \).
3.4. Non-abelian localization and transversely elliptic symbols. The contributions \( \text{index}_T(\mathcal{P}^Q_{U_0,\beta}) \), \( \beta \in \mathcal{B} \) to the index formula proved in Corollary 3.16 are conveniently described in terms of indices of transversely elliptic operators \([7]\).

**Definition 3.17.** Let \( \sigma : T^*X^0 \to \text{End}(S) \) be the product symbol

\[
\sigma(x, p) = 1 \otimes \text{ic}(p) + \text{ic}_T(\nu(x)) \otimes 1, \quad (x, p) \in T^*X.
\]

Using the Kirwan vector field, define the deformed symbol

\[
\sigma_\kappa(x, p) = 1 \otimes \text{ic}(p - \kappa(x)) + \text{ic}_T(\nu(x)) \otimes 1, \quad (x, p) \in T^*X^0
\]

and let \( \sigma_\beta = \sigma_\kappa \mid T^*U_\beta \).

If \( H \) is a compact Lie group acting on a manifold \( M \), one says that an \( H \)-invariant symbol \( \sigma \) is \( H \)-transversely elliptic \((\text{[7]})) \) if \( \sigma \upharpoonright T^*_HM \) is invertible outside a compact subset, where \( T^*_HM \subset T^*M \) is the conormal space to the \( H \) orbits. An \( H \)-transversely elliptic symbol \( \sigma \) defines an element of \( K^0_H(T^*_HM) \) that we also denote by \( \sigma \) when there is no risk of confusion.

**Proposition 3.18.** The symbol \( \sigma_\beta \) is \( T \)-transversely elliptic.

**Proof.** Use \( g \) to identify \( TX^0 = T^*X^0 \). Let \( \Gamma(\kappa) \subset TX \) be the graph of \( \kappa \) (restricted to \( X^0 \)), and let \( \pi : T^*X^0 \to X^0 \) be the projection. The subset of \( T^*U_\beta \) where \( \sigma_\beta \) fails to be invertible is \( T^*U_\beta \cap \Gamma(\kappa) \cap \pi^{-1}(\nu^{-1}(0)) \). As \( \kappa \) is tangent to the group orbit directions, the intersection of \( \Gamma(\kappa) \) with \( T^*_UX^0 \) is the vanishing locus of \( \kappa \), viewed as a subset of the 0 section. Thus

\[
T^*_U\beta \cap \Gamma(\kappa) \cap \pi^{-1}(\nu^{-1}(0)) = Z_\beta
\]

and is compact. \( \square \)

On a compact \( H \)-manifold Atiyah \([7]\) proved that the \( H \)-equivariant index of an \( H \)-transversely elliptic operator makes sense as a distribution on \( H \), and depends only on the \( K \)-theory class defined by the symbol. Atiyah defined the analytic index of a transversely elliptic symbol \( \tilde{\sigma} \in K^0_H(T^*_HM) \), denoted \( \text{index}_H(\tilde{\sigma}) \in R^{-\infty}(H) \), as the index of any transversely elliptic operator with symbol \( \tilde{\sigma} \). More generally, on a non-compact manifold (such as \( U_\beta \)) Atiyah defines the index by embedding in a compact manifold and extending the symbol appropriately.

One can twist transversely elliptic symbols by representable \( K \)-theory classes: given an \( H \)-transversely elliptic symbol \( \tilde{\sigma} \in K^0_H(T^*_HM) \) and given \( E \in \mathcal{R}K^0_H(M) \), the product \( \tilde{\sigma} \otimes E \in K^0_H(T^*_HM) \) is defined. For example, represent \( E \) by a difference of vector bundles \( E_1, E_2 \) near the projection to \( M \) of the compact subset where \( \tilde{\sigma} \) fails to be invertible, and extend \( E_1, E_2 \) arbitrarily to \( M \). Then \( \tilde{\sigma} \otimes E \) is the difference of the \( K \)-theory classes represented by the symbols \( \tilde{\sigma} \otimes \text{id}_{E_1}, \tilde{\sigma} \otimes \text{id}_{E_2} \).

**Theorem 3.19.** Let \( E \in \mathcal{R}K^0_T(X) \) be a quasiperiodic class represented by the quasiperiodic cycle \( (\mathcal{E}, \nabla^\mathcal{E}, Q) \). Let \( E_\beta \in \mathcal{R}K^0_T(U_\beta) \) be the pullback of \( E \) to \( U_\beta \). Then

\[
\text{index}_T(\mathcal{P}^Q) = \sum_{\beta \in \mathcal{B}} \text{index}_T(\sigma_\beta \otimes E_\beta). \quad (30)
\]

In particular the index depends only on the \( K \)-theory class of the quasiperiodic cycle.
Proof. By Corollary 3.16 one needs to show \( \text{index}_T(\mathcal{D}^Q_{U_\beta,\lambda}) = \text{index}_T(\sigma_\beta \otimes E_\beta) \). Recall \( U_\beta \) is a small neighborhood of the compact set \( Z_\beta = X^\beta \cap \phi^{-1}(\beta) \cap \nu^{-1}(0) \) where the 0-th order deformation \( \rho^{-1}\Phi|_{U_\beta} = ip^{-1}(c_1(\nu)\otimes 1 - 1\otimes c(\kappa))|_{U_\beta} \) vanishes. Since \( Z_\beta \) is compact, the result follows from known results relating 0-th order deformations of this type to indices of transversely elliptic symbols: details and various approaches may be found in [25, 50, 48, 46]. \( \square \)

**Definition 3.20.** Let \( E \in \mathcal{RK}_T^0(X) \) be a quasiperiodic class. The analytic index of \( E \) is \( \text{index}_T^D(E) = \text{index}_T(\mathcal{D}^Q) \) where \( (\mathcal{E}, \nabla^\mathcal{E}, Q) \) is a quasiperiodic cycle representing \( E \).

In Theorem 3.19, the contribution labelled by \( \beta \in \mathcal{B} \) can be expressed in terms of objects defined on the fixed-point set \( (U_\beta)^\beta \). The element \( \beta \) determines a complex structure on the normal bundle \( \nu(\beta) \) to \( (U_\beta)^\beta \) with the property that the eigenvalues of the action of \( i^{-1}\beta \) on \( \nu(\beta)^{1,0} \) are positive. There is a Clifford module \( S(\beta) \) on \( (U_\beta)^\beta \) with the property
\[
S(\beta) \otimes \nu(\beta)^{0,1} \simeq S|_{(U_\beta)^\beta},
\]
and a corresponding induced transversely elliptic symbol \( \sigma_{S(\beta)} \). Then
\[
\text{index}_T(\sigma_\beta \otimes E_\beta) = \text{index}_T(\sigma_{S(\beta)} \otimes \text{Sym}(\nu(\beta)^{1,0}) \otimes E_\beta),
\]
where \( \text{Sym}(\nu(\beta)^{1,0}) \) denotes the symmetric algebra bundle. See [62, 65] and [48, Theorem 6.14] for further details.

The local finiteness of the sum (in \( \mathbb{Z}^A = R^{-\infty}(T) \)) in Theorem 3.19 can be verified directly using (31). Because of the \( \text{Sym}(\nu(\beta)^{1,0}) \) factor, the contribution of \( 0 \neq \beta \in \mathcal{B} \) has support contained in a half space of the form \( H(\beta, c_\beta) = \{ \xi \in t^*| \beta/|\beta| \geq c_\beta \} \). We claim that the constants \( c_\beta \to \infty \) as \( |\beta| \to \infty \), from which the result follows. The constant \( c_\beta \) depends on the eigenvalues for the action of \( i^{-1}\beta/|\beta| \) on \( S(\beta) \) and \( E|_{Z_\beta} \). If \( \mathcal{E} = \mathbb{C} \), then the claim is true and follows from our assumption \( \ell > 0 \), equation (2), and Proposition 2.16, which imply that the eigenvalues for the action of \( i^{-1}\beta/|\beta| \) on \( S(\beta) \) go to \( +\infty \) as \( |\beta| \to \infty \); see for example [48] for further details. In the general case it remains true because \( \mathcal{E} \) is \( T \)-finite, meaning the eigenvalues for the action of \( \beta/|\beta| \) on \( E|_{Z_\beta} \) can be bounded by a constant that is independent of \( \beta \).

4. Cohomological Formulas

In this section we derive some cohomological formulas for the index of a quasiperiodic class \( E \in \mathcal{RK}_T^0(X) \) under additional hypotheses on the existence of a well-behaved equivariant Chern character form for \( E \). Using a cohomological index formula for transversely elliptic operators due to Paradan and Vergne [63] (building on work of Berline-Vergne [20]) and a non-abelian localization formula in cohomology, we obtain a delocalized cohomological formula for the index. Under an additional twisted equivariance assumption on the Chern character form we obtain an abelian localization formula involving integration over a compact manifold.

4.1. Index formula for transversely elliptic operators. A cohomological index formula for transversely elliptic operators due to Paradan and Vergne [63] can be applied to each term on the right hand side of Theorem 3.19. The result is expressed in terms of \( T \)-equivariant characteristic forms: these are differential forms \( \alpha(\xi) \) on \( M \) depending smoothly on a parameter \( \xi \in t \) (sometimes only defined on a neighborhood of \( 0 \in t \)), defined by replacing the curvature.
with the equivariant curvature in the formula for the Chern-Weil representative, cf. [16, 53, 63]. For example the equivariant Todd form for a $T$-equivariant vector bundle with connection $(E, \nabla^E)$ is

$$\text{Td}(E, \xi) = \det_{\xi} \left( \frac{(i/2\pi)F^E(\xi)}{1 - \exp(-i/2\pi)F^E(\xi)} \right), \quad F^E(\xi) = (\nabla^E)^2 - 2\pi i(\mathcal{L}^E_\xi - \nabla^E_\xi),$$

and is closed for the differential $d_\xi = d + 2\pi i(\xi_X)$.

The class of the symbol $\sigma$ in K-theory is the product of the class defined by the symbol $\sigma_\mathcal{X}$ of the Dirac operator for $\mathcal{X}$, and of the pullback under $\nu: X^0 \to t^\perp$ of the Bott-Thom element $\mathcal{B}$ for $0 \in t^\perp$ (see Definition 3.17). Consequently the equivariant Chern character of $\sigma$ is a product

$$\text{Ch}^t(\sigma, \xi) = \text{Ch}^t(\sigma_\mathcal{X}, \xi) \nu^* \text{Ch}^t(\mathcal{B}, \xi).$$

The Bott-Thom element $\mathcal{B}$ has equivariant Chern character

$$\text{Ch}^t(\mathcal{B}, \xi) = (t \exp(\xi))^{-\rho} \det_{\xi}(1 - t \exp(\xi)) \det_{\xi} \left( \frac{1 - \exp(\xi)}{\xi} \right) \mathcal{J}_{(t^\perp)^\perp}(\xi)$$

where $\mathcal{J}_{(t^\perp)^\perp}(\xi) \in \Omega_c((t^\perp)^\perp)$ is an equivariant Thom form (whose pullback to $0 \in (t^\perp)^t$ is $\det_{\xi}(-\xi)$). Hence one can arrange that the support of $\text{Ch}^t(\sigma, \xi)$ is compact in the vertical direction in $T^n'X^t$, and lies over a $\Pi$-invariant subset $X' \subset X^0 \subset X$ such that $\nu(X')$ is a compact subset of $t^\perp$ (and the latter subset of $t^\perp$ can be chosen as small as one wishes by choosing $\mathcal{J}_{(t^\perp)^\perp}(\xi)$ appropriately); this also has the consequence that $X'/\Pi$ is a compact subset of $X^0/\Pi$.

Another important ingredient is the equivariant differential form with generalized coefficients $P_\beta(\xi)$ (i.e. a generalized function of $\xi$ taking values in smooth differential forms on $X$—see [41] for general background) introduced by Paradan [60, 61], given by the expression

$$P_\beta(\xi) = \chi_\beta - d\chi_\beta \cdot \Theta \int_0^\infty e^{s\partial_\xi} \Theta ds, \quad \Theta = g(\kappa) \in \Omega^1(X)^T$$

where $\chi_\beta$ is a $T$-invariant bump function equal to 1 on a small neighborhood of $Z_\beta = X^\beta \cap \phi^{-1}(\beta)$ (see [61, 63] or the proof of Theorem 4.5 below for further details). The integral in equation (34) can be thought of as a distributional replacement for the expression $"-(d_\xi \Theta)^{-1}"$.

By construction the product $P_\beta(\xi) \text{Ch}^t(\sigma, \xi)$ has compact support near $Z^\beta_\beta \subset T^\ast U^\beta_\beta$.

The Paradan-Vergne formula [63, Theorem 5.6] applied to $\sigma_\beta \otimes E_\beta$ yields that for $t \in T$ and $\xi \in t$ sufficiently small one has

$$\text{index}_t(\sigma_\beta \otimes E_\beta)(t \exp(\xi)) = \int_{U^\beta_\beta} P_\beta(\xi) \mathcal{A}S^t(\sigma, \xi) \text{Ch}^t(E_\beta, \xi),$$

where

$$\mathcal{A}S^t(\sigma, \xi) = (-1)^{\dim(X^t)/2} 2\pi \frac{\text{Td}(T^C_{X^t}, \xi) \text{Ch}^t(\sigma, \xi)}{\text{Ch}^t(\lambda_{-1} \nu_C(\pi), X^t)},$$

is the integral over the fibres $\pi: T^\ast X^t \to X^t$ of the equivariant Atiyah-Segal-Singer integrand. The integrand on the right hand side of equation (35) should be paired with a test function supported sufficiently near 0 \in t before the integral over $U^t_\beta$ is performed.
To make the Atiyah-Singer integrand more concrete, note that in case \( t = 1 \), carrying out the integral over the fibres leads to
\[
\text{AS}(\sigma, \xi) = \tilde{A}(X, \xi) \text{Ch}(\mathcal{L}, \xi) \frac{1}{2} \nu^* \text{Ch}(\mathcal{B}, \xi)
\]
(37)
where \( \mathcal{L} \) is the anti-canonical line bundle (Definition 2.13). Moreover
\[
\text{Ch}(\mathcal{L}, \xi)^{1/2} = e^{\omega + 2\pi i(\phi, \xi)},
\]
(38)
where \( \omega, \phi \) were introduced in Definition 2.13. A similar formula exists for general \( t \in T \), although it involves a phase factor that is somewhat cumbersome to describe (cf. [16, 30, 2, 44]). Equation (35) holds for \( \xi \) contained in a ball of some radius \( r_{\beta, t} > 0 \) in \( t \). The maximal \( r_{\beta, t} \) depends on the geometry of the \( T \)-action near \( Z_{\beta} \), and in general it can happen that \( \{ r_{\beta, t} \mid \beta \in \mathcal{B} \} \) has no positive lower bound. An important feature of our setting is that Proposition 2.16 guarantees that this does not occur, i.e. one can find a uniform radius \( r_{t} > 0 \) such that (35) holds for \( \xi \) in a ball \( B_{r_{t}} \) of radius \( r_{t} \) in \( t \) for all \( \beta \in \mathcal{B} \). Since \( T \) is compact, we may find a finite set \( t_{1}, ..., t_{N} \) such that the sets \( t_{i} \exp B_{r_{t_{i}}} \) cover \( T \). Let \( \rho_{i} \in C^{\infty}(T), i = 1, ..., N \) be a partition of unity subordinate to the cover. By (35),
\[
\text{index}_{T}(\sigma_{\beta} \otimes E_{\beta})(t) = \sum_{i=1}^{N} \rho_{i}(t) \int_{U_{\beta}^{i}} P_{\beta}(\xi_{i}) \text{AS}^{h}(\sigma, \xi_{i}) \text{Ch}^{t_{i}}(E_{\beta}, \xi_{i}),
\]
(39)
where \( \xi_{i} = \log(t_{i}^{-1} t) \).

Our next aim is to replace the contributions (35) localized around the components \( X^{t} \cap Z_{\beta} \) with a global expression involving integration over \( X^{t} \). In brief the strategy (which we learned from [64]) is to use the non-abelian localization formula in cohomology ([61]) in reverse. Since \( X^{t} \) is non-compact and the integral over \( X^{t} \) will only converge in the sense of generalized functions, the argument requires some care in choosing well-behaved differential form representatives.

We have already written the Atiyah-Segal-Singer integrand (36), (37) in a way that suggests it makes sense globally on \( X^{t} \). Recall that the \( \text{Spin}_{c} \) structure \( \mathcal{S} \) is \( N(T) \times \hat{\Pi}^{(t)} \)-equivariant. Using invariant connections in the equivariant Chern-Weil construction yields a differential form representing \( \text{AS}^{t}(\sigma, \xi) \) that is \( \Pi \)-periodic up to a phase factor:
\[
\eta \cdot \text{AS}^{t}(\sigma, \xi) = t^{-\ell \eta} e^{-2\pi i\ell \eta \xi} \text{AS}^{t}(\sigma, \xi)
\]
(40)
for \( \eta \in \Pi \); for example, in the case \( t = 1 \), this is clear from (37): \( \tilde{A}(X, \xi) \text{Ch}(\mathcal{B}, \xi) \) is \( \Pi \)-periodic, while \( \text{Ch}(\mathcal{L}, \xi)^{1/2} \) changes by the phase factor \( e^{-2\pi i\ell \eta \xi} \) due to (11).

4.2. Chern character forms. The next definition lists desirable properties of a globally defined differential form on \( X \) whose pullback to each relatively compact subset \( U_{\beta} \) represents the Chern character of \( E_{\beta} \).

**Definition 4.1.** Let \( E \in \mathcal{R}K_{0}^{T}(X) \) be a quasiperiodic class. Let \( \text{Ch}^{t}(E, \xi), t \in T \) denote any collection of smooth \( T \)-equivariant forms on \( X^{t} \), depending smoothly on \( \xi \), closed for the differential \( d + 2\pi i\text{Im}(\xi) \), and having the following properties:

(a) As \( t \) varies, the family forms a *bouquet* in the sense of [29, Definition 60], [73, p. 150]. In the abelian case this simply means: for \( t \in T \) and \( \xi \in t \) sufficiently small, the pullback of \( \text{Ch}^{t}(E, \xi + \xi') \) to \( X^{t} \exp(\xi) \) is \( \text{Ch}^{t \exp(\xi)}(E, \xi') \).
(b) The $T$-equivariant cohomology class of the pullback of $\text{Ch}^t(E, \xi)$ to $U^t_\beta$ is $\text{Ch}^t(E_\beta, \xi)$.

(c) There is a compact subset $K \subset \mathfrak{t}^*$ such that for all $t \in T$ and $x \in X^t$, the Fourier transform $\mathcal{F}_t \text{Ch}^t(E, \cdot)_x$ of the smooth function

$$\xi \in \mathfrak{t} \mapsto \text{Ch}^t(E, \xi)_x \in \wedge T^*_x X_C$$

has support contained in $K$.

(d) There are integers $m, m' \geq 0$ and a constant $C > 0$ such that for all $t \in T$, $\xi \in \mathfrak{t}$ and $x \in X^t$,

$$|\text{Ch}^t(E, \xi)_x| \leq C(1 + |\phi(x)|^2)^{m'/2}(1 + |\xi|^2)^{m/2}. \quad (41)$$

**Remark 4.2.** The assumption (d) could be expressed in terms of the Fourier transform: there exist $C, m, m'$ such that

$$\langle \mathcal{F}_t \text{Ch}^t(E, \cdot)_x, f \rangle \leq C(1 + |\phi(x)|^2)^{m'/2} ||f||_\infty + ||\nabla f||_\infty + \cdots + ||\nabla^m f||_\infty, \quad (42)$$

for all $f \in C^\infty_c(\mathfrak{t})$.

**Example 4.3.** Suppose $E$ can be represented by a smooth $T$-equivariant Hermitian vector bundle $E \to X$ of rank $r$ with compatible connection $\nabla^E$ having bounded curvature $||F_E||_{\infty} < C_1$ and such that the moment map determined by the connection $\phi_E \in \mathfrak{t}^* \otimes C^\infty(X, \text{End}(E))$ satisfies $||\phi_E \cdot \xi||_{\infty} < C_2|\xi|$ (or more generally by a formal difference of such). In particular any $T \times \Pi$-equivariant vector bundle has such a connection. Then the corresponding equivariant Chern-Weil form on $X^t$ (cf. [16, Chapter 7], [53]),

$$\text{Tr}_E(te^{i/2\pi}F_E(\xi)),$$

where

$$\frac{i}{2\pi}F_E(\xi) = \frac{i}{2\pi}F_E + 2\pi i \phi_E \cdot \xi, \quad F_E = (\nabla^E)^2, \quad 2\pi i \phi_E \cdot \xi = L^E_\xi - \nabla^E_\xi,$$

satisfies the conditions in Definition 4.1. Conditions (a),(b) are immediate. Conditions (c),(d) are straight-forward consequences of the bounds $||F_E||_{\infty} < C_1$, $||\phi_E \cdot \xi||_{\infty} < C_2|\xi|$. Indeed to keep the notation simple, consider the case where the rank of $E$ is 1 (the general case is similar). Then $\phi_E$ can be viewed as a smooth $\mathfrak{t}^*$-valued function with norm bounded by $C_2$. The Fourier transform of $\text{Tr}_E(te^{i/2\pi}F_E(\xi))$ at the point $x \in X^t$ is the generalized function

$$\nu \mapsto t_x e^{i/2\pi}F_{E,x} \int t \mathbb{C}^{dim(X)/2} e^{-2\pi i (\nu - \phi_E(x), \xi)} d\xi$$

which is $t_x e^{i/2\pi}F_{E,x} \in \wedge T^*_x X_C$ times the delta distribution at $\phi_E(x) \in \mathfrak{t}^*$. Thus we can take $K$ to be the closed ball in $\mathfrak{t}^*$ of radius $C_2$, $m = m' = 0$, and $C = C^d_1(X)$. Under suitable analytic hypotheses Example 4.3 could be generalized to construct a Chern character form for a quasiperiodic cycle $(\mathcal{E}, \nabla^\mathcal{E}, Q)$ via the equivariant Chern-Weil construction for the superconnection built from $\nabla^\mathcal{E}$ and $Q$ ([16]). We will not pursue this here as when we turn to the Atiyah-Bott classes in Section 5, it will be more convenient to simply guess a candidate differential form and verify conditions (a)–(d) of Definition 4.1 by other means.

**Remark 4.4.** Existence of a Chern character form for a quasiperiodic cycle $(\mathcal{E}, \nabla^\mathcal{E}, \phi_\mathcal{E})$ satisfying the conditions in Definition 4.1 is not automatic, and will appear as an additional assumption below. For a somewhat artificial example, if $\mathcal{M} = LG \cdot \xi$ is the coadjoint orbit of $\xi \in \mathfrak{t} \subset L\mathfrak{g}^*\mathfrak{t}$, then the space $X$ is a small thickening of a discrete space $W_{aff} \cdot \xi$. Let $\mathcal{E} = X \times \ell^2(\mathbb{Z})$ where the
Theorem 4.5. Let \( E \in \mathcal{RK}_T^0(X) \) be a quasiperiodic class that admits \( T \)-equivariant Chern character forms \( \text{Ch}^t(E, \xi) \) satisfying the conditions in Definition 4.1. Then the non-abelian localization formula (30) converges in the sense of distributions to a distribution on \( T \), and for \( t \in T \) and \( \xi \in t \) sufficiently small

\[
\text{index}^P_{\mathcal{R}}(E)(t \exp(\xi)) = \int_{X^t} \text{Ch}^t(E, \xi)AS^t(\sigma, \xi).
\] (43)

The integral in (43) is meant in the distributional sense, i.e. the integrand should be paired with a test function supported sufficiently near \( 0 \in t \) before the integral over \( X^t \) is performed. Moreover the integration can be taken instead over the manifold without boundary \( (X^\circ)^t \), since the support of the integrand is contained in a subset \( (X^t)^t \subset (X^\circ)^t \) (which moreover has the property that \( X^t/\Pi \) is a compact subset of \( X^\circ/\Pi \)).

Proof. Let \( f \in C^\infty_c(t) \) with support contained in the region where \( AS^t(\sigma, \xi) \) is defined. Recall \( \text{Ch}(\mathcal{L}, \xi)^{1/2} = e^{\omega + 2\pi i \phi \cdot \xi} \) and that \( \phi(\eta \cdot x) = \phi(x) + t\eta \) for all \( \eta \in \Pi \). Let

\[
\alpha(\xi) = \text{Ch}^t(E, \xi)AS^t(\sigma, \xi)e^{-\omega - 2\pi i \phi \cdot \xi} f(\xi).
\]

The pairing \( \langle \text{Ch}^t(E, \cdot)AS^t(\sigma, \cdot), f \rangle \) of the integrand of (43) with \( f \), evaluated at \( x \in X \) is

\[
\int \alpha_x(\xi)e^{\omega_x + 2\pi i \phi_x \cdot \xi} d\xi = e^{\omega_x}(\mathcal{F}_t\alpha_x)(-\phi_x) \in \wedge T^*_xX.
\] (44)

The Fourier transform of a product is the convolution of the Fourier transforms. The function \( \gamma_x(\xi) = AS^t(\sigma, \xi)e^{-\omega - 2\pi i \phi \cdot \xi} f(\xi) \in \wedge T^*_xX^\circ \) is smooth and compactly supported (since \( f \) is), hence its Fourier transform \( \mathcal{F}_t\gamma_x \) is a Schwartz function. Under pullback by an element of the lattice \( \eta \in \Pi \), \( AS^t(\sigma, \xi)e^{-\omega - 2\pi i \phi \cdot \xi} \) transforms with a phase \( t^{-\ell} \eta \) that does not depend on \( \xi \).

Since the support of \( AS^t(\sigma, \xi) \) is contained in the inverse image of a compact subset \( X^t/\Pi \) of \( X^\circ/\Pi \), it follows that the Schwartz semi-norms of \( \mathcal{F}_t\gamma_x \) are uniformly bounded as functions of \( x \). The properties of \( \text{Ch}^t(E, \xi) \) (in particular equation (42)) imply that the Schwartz semi-norms of the convolution \( \mathcal{F}_t\alpha_x \) are bounded (as a function of \( x \)) by a constant times \( (1 + |\phi_x|^2)^{m'/2} \). In particular the pointwise norm \( |\mathcal{F}_t\alpha_x(\nu)| \) (taken in \( \wedge T^*_xX^t \)) satisfies an estimate

\[
|\mathcal{F}_t\alpha_x(\nu)| \leq (1 + |\phi_x|^2)^{m'/2} p(|\nu|)
\] (45)

where \( p: [0, \infty) \rightarrow [0, \infty) \) is a rapidly decreasing monotone function that can be chosen independently of \( x \in X^t \). Equations (44), (45) show that the pairing \( \langle \text{Ch}^t(E, \cdot)AS^t(\sigma, \cdot), f \rangle \in \Omega(X^t) \) is integrable, and hence the right hand side of (43) is well-defined.
To prove the result, we carry out non-abelian localization for the right hand side of (43) and verify that the result coincides with (36). (The result is not an immediate consequence of [61], because the latter treats equivariant forms with polynomial coefficients, and the results typically do not extend to smooth coefficients needed here.) Let \( d_\xi = d + 2\pi i e(\xi) \) denote the equivariant differential. The localization works more generally, for a \( d_\xi \)-closed equivariant differential form \( \alpha(\xi) \) with \( C^\infty(t) \)-coefficients such that (i) the support of \( \alpha(\xi) \) is contained in a \( \Pi \)-invariant subset \( X' \subset X^o \) such that \( X' / \Pi \subset X^o / \Pi \) is compact (in particular, \( \phi \) is proper on the support of \( \alpha(\xi) \)), and (ii) \( P_\alpha(\xi) \) satisfies an estimate as in (45) where \( p \) is rapidly decreasing and independent of \( x \in X'\).

Recall Paradan’s closed equivariant form with generalized coefficients, which appears in the index formula (35):

\[
P_\beta(\xi) = \chi_\beta - d\chi_\beta \cdot \Theta \int_0^\infty e^{sd_\xi} \Theta ds, \quad \Theta = g(\kappa) \in \Omega^1(X)^T,
\]

where \( \chi_\beta : \mathcal{X} \rightarrow [0,1] \) is a smooth \( T \)-invariant bump function with compact support near \( \mathcal{Z}_\beta \), equal to 1 on a neighborhood of \( \mathcal{Z}_\beta \), and such that \( \{\chi_\beta | \beta \in \mathcal{B} \} \) have pairwise disjoint supports. Let

\[
\chi = \sum_{\beta \in \mathcal{B}} \chi_\beta, \quad P(\xi) = \sum_{\beta \in \mathcal{B}} P_\beta(\xi) = \chi - d\chi \cdot \Theta \int_0^\infty e^{sd_\xi} \Theta ds.
\]

Thanks to Propositions 2.16 and 2.18, the \( \chi_\beta \) can be chosen such that: (i) the \( \chi_\beta \) are all translates by elements of \( \Pi \) of a finite collection \( \chi_\beta, \beta^* \in \mathcal{B}^* \) (in particular \( d\chi \) is bounded), (ii) there is a constant \( c > 0 \) such that \( |\kappa(x)| \geq c \) for all \( x \in \text{supp}(1 - \chi) \). Property (ii) plays a particularly important role below.

The form \( P(\xi) \) satisfies ([61, Proposition 2.1])

\[
1 = P(\xi) - d_\xi \delta(\xi), \quad \delta(\xi) = (1 - \chi) \Theta \int_0^\infty e^{sd_\xi} \Theta ds,
\]

where as above, the meaning of the integral in the definition of \( \delta(\xi) \) is that the integrand should be paired with a test function on \( t \), before the integral with respect to \( s \) is carried out. The desired result amounts to proving that

\[
\int_{X'} \int_t d_\xi \alpha(\xi)e^{\circ} + 2\pi i e(\xi) d_\xi \delta(\xi) = 0.
\]

Let \( r_j > 0 \) be a sequence of regular values of \( |\phi| : X \rightarrow [0, \infty) \) such that \( r_j \rightarrow \infty \) as \( j \rightarrow \infty \), and let \( X_j = |\phi|^{-1}([0, r_j]) \cap X^o \), a smooth manifold with boundary. By Stokes’ theorem, and using the definition of \( \delta(\xi) \), (46) is equivalent to

\[
\lim_{j \rightarrow \infty} \int_{\partial X_j} (1 - \chi) \Theta e^{\circ} \int_0^\infty ds e^{sd_\Theta} \int_t d_\xi \alpha(\xi)e^{2\pi i(\phi + s\phi_\Theta) \cdot \xi} = 0,
\]

where \( \phi_\Theta \cdot \xi := \iota(\xi X) \Theta = g(\kappa, \xi X) \). Performing the integral over \( t \) on the left hand side of (47) yields

\[
\lim_{j \rightarrow \infty} \int_{\partial X_j} (1 - \chi) \Theta e^{\circ} \int_0^\infty ds e^{sd_\Theta} (\Omega_\alpha)(-\phi - s\phi_\Theta).
\]
On supp(1 − χ) ∩ ∂X^t_j, we have
\[ |φ + sφ|_2^2 = |φ|^2 + s^2|φ|_2^2 + 2s|κ|^2 ≥ r_j^2 + 2c^2s, \]  
with \( c > 0 \). For \( j \) large there are estimates, valid on ∂X^t_j, of the form
\[ \text{vol}(∂X^t_j) ≤ C|r_j|^\dim(t)−1, \quad |Θ| ≤ C|r_j|, \quad |e^{sdΘ}| ≤ C(1 + s)^n/2|r_j|^{n/2}, \quad |e^σ| ≤ C. \] (50)

Therefore for \( j \) large the expression (48) is bounded by a constant times
\[ \lim_{j \to ∞} |r_j|^{\dim(t)+n/2+m'/2} \int_0^∞ ds \left( (1 + s)^{n/2} p\left( \sqrt{r_j^2 + 2c^2s} \right) \right), \]
which vanishes in the limit as \( j \to ∞ \) because \( p \) is rapidly decreasing. \( \square \)

4.4. **Formal series, formal change of variables and Poisson summation.** Before turning to the abelian localization formula, we make a brief detour to discuss aspects of a formal deformation that will arise. The formal deformation is controlled by a formal 1-parameter family of vector fields \( χ_j \) defined by composition of jets. Let \( d \)
\[ \text{vol}(∂X^t_j) \]
in \( C \) be the isogeny given by taking the \( ℓ \)-th power on the maximal torus of the \( j \)-th simple factor of \( G \). Let
\[ \chi_j ∈ t_C ⊗ CR(T)[z]. \]
The latter determines an \( ∞ \)-jet \( \chi_{j,z} \) of maps deforming \( X_t \), by
\[ \chi_{j,z}(t) = t^j \exp(zχ_j(t)). \] (51)

The corresponding \( ∞ \)-jet \( \chi_{j,z} \) of maps \( t_C → t_C \) is \( ξ → ℓξ + zχ_j(\exp(ξ)) \).

The collection of \( ∞ \)-jets of curves in the complexified torus \( T_C \) forms a group \( J^{∞}T_C \). For algebraic geometries these are the \( C[z]- \)points of \( T_C \). \( An ∞ \)-jet with base point \( g ∈ T_C \) will typically be denoted \( g_z ∈ J^{∞}_gT_C \) below, where the parameter \( z ∈ R \) or \( C \) (in the latter case we think of \( J^{∞}_gT_C \) as holomorphic jets). An element \( g_z ∈ J^{∞}_gT_C \) can be factored as
\[ g_z = g \exp(g_z^+), \quad g_z^+ = zξ_1 + \frac{1}{2}z^2ξ_2 + · · · ∈ zt_C[z]. \] (52)

The exponential map \( \exp \): \( t_C[z] = J^{∞}t_C → J^{∞}T_C \) sends \( ξ_z = ξ_0 + zξ_1 + \frac{1}{2}z^2ξ_2 + · · · \) to \( g_z \) as in (52) with \( g = \exp(ξ_0) \). If \( g_z \) is based at \( g ∈ T_C \) and \( X_{t,z} \) is as (51), then \( X_{t,z}(g_z) ∈ J^{∞}_gT_C \) is defined by composition of jets. Let \( dχ_z ∈ \text{End}(t_C ⊗ CR(T)[z]) \) be the differential (in the \( T_C \) variables), and let \( \det(1 + z^2χ_z) ∈ 1 + zCR(T)[z] \) denote the complex determinant.

Jets of maps \( T → C \) based at \( g ∈ T \) are in 1-1 correspondence with holomorphic jets of maps \( T_C → C \) based at \( g \). If \( f ∈ C^{∞}(T) \) and \( g ∈ T \), let \( J_{hol,g} f \) denote the holomorphic jet corresponding to the Taylor series of \( f \) at the point \( g \) (put more simply, we treat the variables in the Taylor series of \( f \) at \( g \) as complex variables). For \( g_z ∈ J^{∞}_gT_C \) we define \( f(g_z) ∈ C[z] \) to be the composition of \( ∞ \)-jets \( (J^{∞}_{hol,g} f) \circ g_z ∈ J^{∞}C = C[z] \); in other words \( f(g_z) \) is the result
of substituting $g^+_z \in \mathfrak{t}_\mathbb{C}[z]$ into the Taylor series of $f$ at the point $g$ and expanding in powers of $z$. The map

$$f \in C^\infty(T) \mapsto f(g_z) \in \mathbb{C}[z]$$

defines a formal series of distributions on $T$ supported at $g \in T$; we denote this formal series by $\delta g_z \in \mathcal{D}'(T)[z]$. More generally if $f_z \in C^\infty(T)[z]$, say $f_z = f_0 + z f_1 + \cdots$, then $f_z(g_z) = f_0(g_z) + z f_1(g_z) + \cdots \in \mathbb{C}[z]$.

Let $g_z \in \mathcal{J}_g^\infty(T \mathbb{C})$ and $\chi_z \in \mathfrak{t}_\mathbb{C} \otimes \mathcal{C}R(T)[z]$ be as above, and expand $\chi_z$ in powers of $z$:

$$\chi_z = \sum_{j \geq 0} \frac{z^j}{j!} \chi^{(j)}, \quad \chi^{(j)} \in \mathfrak{t}_{\mathbb{C}} \otimes \mathcal{C}R(T).$$

The equation $\mathcal{X}_{1,z}(g_z) = g$, $g_0 = g$ is equivalent to

$$g_z^+ + z \chi_z (g \exp(g_z^+)) = 0. \quad (53)$$

for the formal series $g_z^+ \in \mathfrak{t}_\mathbb{C}[z]$. By expanding each $\chi^{(j)}$ in Taylor series at $g$,

$$\chi^{(j)}(g \exp(\xi)) = \sum_{\alpha \geq 0} \frac{\xi^\alpha}{\alpha!} \partial^\alpha \chi^{(j)}(g),$$

equation (53) becomes a sequence of algebraic equations for the coefficients $\xi_i$ from (52) that can be solved recursively. For example the first two terms are

$$\xi_1 = -\chi^{(0)}(g), \quad \xi_2 = -2(\chi^{(1)}(g) + \partial_{\xi_1} \chi^{(0)}(g)),$$

The equations for the higher terms quickly become more complicated, but in any case (53) has a unique solution. Define $\psi_z \in \mathfrak{t}_\mathbb{C} \otimes \mathcal{C}R(T)[z]$ by $z \psi_z(g) = g_z^+$ where $g_z^+$ is the unique solution of (53). Let $\Psi_{\ell,z}(g) = g \exp(z \ell^{-1} \psi_z(g))$, an $\infty$-jet of diffeomorphisms $T_{\mathbb{C}} \to T_{\mathbb{C}}$ deforming the identity. By construction, the composition of jets

$$\mathcal{X}_{\ell,z} \circ \Psi_{\ell,z} = \mathcal{X}_{\ell}.$$ 

Recall the finite subgroup $T_\ell = \ell^{-1} \Lambda / \Pi \subset \mathfrak{t} / \Pi = T$

**Theorem 4.6.** One has the following equality of formal series (in $z$) of distributions on $T$

$$\sum_{\eta \in \Pi} \mathcal{X}_{\ell,z}(h)^\eta = \sum_{g \in T_\ell} \frac{\delta g_z(h)}{\det(1 + z \ell^{-1} d\chi_z(g_z))} \quad (54)$$

where $g_z \in \mathcal{J}_g^\infty T_{\mathbb{C}}$ is the unique solution to the equation $\mathcal{X}_{\ell,z}(g_z) = g^\ell$.

**Proof.** By expanding the left hand side of (54) in powers of $z$ and using the Poisson summation formula, one checks easily that it is a well-defined formal series of distributions with support contained in $T_\ell$. It suffices to check both sides of (54) agree when paired with test functions $f \in \mathcal{C}R(T)$. The function $f$ extends uniquely to a holomorphic function on $T_{\mathbb{C}}$ that we also denote by $f$. Using Borel summation (i.e. replacing $z^j$ by $z^j \rho_j(z)$, where $\rho_j$ is a bump function on $\mathbb{R}$ with $\rho_j = 1$ near $z = 0$, and supports shrinking to 0 as $j \to \infty$ at a rate depending on the properties of the series $\psi_z$) one constructs a smooth function $\psi : \mathbb{R} \times T_{\mathbb{C}} \to t_{\mathbb{C}}$, denoted $\psi_z(g)$ where $(z,g) \in \mathbb{R} \times T_{\mathbb{C}}$ (holomorphic in $g$), whose Taylor series at $z = 0$ is $\psi_z$. Let $\Psi_{\ell,z}(g) = g \exp(z \ell^{-1} \psi_z(g))$ be the corresponding map $\mathbb{R} \times T_{\mathbb{C}} \to T_{\mathbb{C}}$. Since $\Psi_{\ell,0}$ is the identity, smoothness in $z$ implies that, for $z$ small, $\Psi_{\ell,z}$ restricts to a biholomorphism from a relatively
compact annulus $\mathcal{A} \subset T_C$ containing $T$ to its image $\Psi_{\ell,z}(\mathcal{A})$. Let $\Psi_{\ell,z}^{-1}$ denote its inverse and let $\mathbf{X}_{\ell,z} = \mathbf{X}_{\ell} \circ \Psi_{\ell,z}^{-1}$. Using the chain rule,

$$\det(d\Psi_{\ell,z}^{-1}) = \det(\ell^{-1}d\mathbf{X}_{\ell,z}).$$

(55)

Note that the Taylor series of $\mathbf{X}_{\ell,z}$ at $z = 0$ is $\mathbf{X}_{\ell,0}$, and consequently the Taylor series of $\det(\ell^{-1}d\mathbf{X}_{\ell,z})$ at $z = 0$ is $\det(\ell^{-1}d\mathbf{X}_{\ell,0}) = \det(1 + z \ell^{-1}d\chi_z)$. We will prove that for $z$ sufficiently small,

$$\sum_{\eta \in \Pi} \int_T \mathbf{X}_{\ell,z}(h)^\eta f(h)dh = \sum_{g \in T_{\ell}} \frac{f(g(z))}{\det(\ell^{-1}d\mathbf{X}_{\ell,z}(g(z)))}$$

(56)

where $g(z) = \Psi_{\ell,z}(g)$, and then deduce (54) by taking Taylor series in $z$ of both sides at $z = 0$. For $t \in T$ define

$$F(z, t) = \sum_{g \in T_{\ell}} \frac{f(\Psi_{\ell,z}(gt))}{\det(\ell^{-1}d\mathbf{X}_{\ell,z}(\Psi_{\ell,z}(gt)))},$$

hence the desired sum (56) is $F(z, 1)$. By construction $\xi \mapsto F(z, \exp(\xi))$ is not only $\Pi$-periodic but $\ell^{-1}\Lambda$-periodic, hence has a Fourier expansion (over the dual lattice $(\ell^{-1}\Lambda)^* = \ell\Pi)$,

$$F(z, t) = \sum_{\eta \in \Pi} \hat{F}(z, \eta) t^{-\ell\eta}$$

(57)

where

$$\hat{F}(z, \eta) = \int_{t/\ell^{-1}\Lambda} F(z, h) h^{\ell\eta} dh = \int_T \frac{f(\Psi_{\ell,z}(h))}{\det(\ell^{-1}d\mathbf{X}_{\ell,z}(\Psi_{\ell,z}(h)))} h^{\ell\eta} dh.$$

For the second equality, the sum over $T_{\ell}$ and integral over $t/\ell^{-1}\Lambda$ were combined to yield the integral over $T = t/\Pi$ (note $g^{\ell\eta} = 1$ for $g \in T_{\ell} = \ell^{-1}\Lambda/\Pi$ and $\eta \in \Pi$). This is a contour integral over the contour $T \subset T_C$. Making the change of variables $\Psi_{\ell,z}(h) \mapsto h$ and using (55), the Jacobian factor is absorbed,\(^1\) and the integral becomes

$$\hat{F}(z, \eta) = \int_{\Psi_{\ell,z}(T)} f(h)(\Psi_{\ell,z}^{-1}(h))^{\ell\eta} dh = \int_{\Psi_{\ell,z}(T)} f(h)\mathbf{X}_{\ell,z}(h)^{\eta}$$

with $\Psi_{\ell,z}(T) \subset T_C$ the new contour, and where we used $\mathbf{X}_{\ell,z} = \mathbf{X}_{\ell} \circ \Psi_{\ell,z}^{-1}$ in the second equality. The integrand is a holomorphic function of $h$, so the integral does not change if the contour is deformed back to $T$ through the family of contours $\Psi_{\ell,sz}(T)$, $s \in [0, 1]$, hence

$$\hat{F}(z, \eta) = \int_T f(h)\mathbf{X}_{\ell,z}(h)^{\eta}$$

Substituting in (57) and setting $t = 1$ yields the result.

\(^1\)When $\dim(t) = 1$ for example, the change of variables here is of the kind

$$\int_C G(\Psi(w)) \Psi'(w) dw = \int_{\Psi(C)} G(w) dw.$$
4.5. **Abelian localization in cohomology.** In this section we use Theorem 4.5 to derive an abelian localization formula for the index, under an additional ‘twisted’ II-equivariance assumption on the equivariant Chern character form, formulated in terms of a formal series of vector fields \( \chi_z \in t_C \otimes CR(T)[z] \) as in the previous subsection. The Atiyah-Bott classes to be discussed in the next section provide interesting examples.

Let \( E_z \in CRK^0_T(X)[z] \) be a formal series of quasiperiodic classes. By applying the index map term-by-term in \( z \), the index

\[
\text{index}^P_T(E_z) \in CR^{-\infty}(T)[z]
\]

is defined. For example, if \( E \in RK^0_T(X) \) is quasiperiodic, then \( E_z = \exp(zE) \in CRK^0_T(X)[z] \) is such a formal series, hence

\[
\text{index}^P_T(\exp(zE)) \in CR^{-\infty}(T)[z]
\]

is defined. Without any additional difficulty one could consider series involving multiple formal variables.

**Definition 4.7.** Let \( \chi_z \in t_C \otimes CR(T)[z] \) and let \( E_z \in CRK^0_T(X)[z] \) be a series of quasiperiodic classes. We say that \( E_z \) admits \( \chi_z \)-twisted equivariant Chern character forms if there is a formal series (in \( z \)) of Chern character forms \( Ch^t(E_z, \xi) \) as in Definition 4.1 such that for all \( \eta \in \Pi \),

\[
\eta \cdot Ch^t(E_z, \xi) = e^{-2\pi i \chi_z(t \exp(\xi))} \eta Ch^t(E_z, \xi).
\]

**Definition 4.8.** Let \( \chi_z \in t_C \otimes R(T)[z] \) and let \( \mathcal{X}_{t, z}(t) = t^\ell \exp(z \chi_z(t)) \). The \( \mathcal{X}_{t, z} \)-twisted action of \( \eta \in \Pi \) on \( \varphi_z(t) \in CR^{-\infty}(T)[z] \) is

\[
\eta \cdot \mathcal{X}_{t, z} \varphi_z(t) = \mathcal{X}_{t, z}(t)^\eta \varphi_z(t).
\]

**Corollary 4.9.** Let \( \chi_z \in t_C \otimes R(T)[z] \) and let \( \mathcal{X}_{t, z}(t) = t^\ell \exp(z \chi_z(t)) \). Let \( E_z \in CRK^0_T(X)[z] \) be quasiperiodic and admit \( \chi_z \)-twisted equivariant Chern character forms \( Ch^t(E_z, \xi) \). Then \( \text{index}^P_T(E_z) \in CR^{-\infty}(T)[z] \) is invariant under the \( \mathcal{X}_{t, z} \)-twisted II-action.

**Proof.** By Theorem 4.5, for \( \xi \) sufficiently small

\[
\text{index}^P_T(E_z)(t \exp(\xi)) = \int_{X^t} Ch^t(E_z, \xi) AS^t(\sigma, \xi).
\]

Let \( g = t \exp(\xi) \). Multiplying both sides by \( \mathcal{X}_{t, z}(g)^\eta \), using Definition 4.7 as well as equation (40), yields

\[
\mathcal{X}_{t, z}(g)^\eta \text{index}^P_T(E_z)(g) = \int_{X^t} \eta^{-1} \cdot (Ch^t(E_z, \xi) AS^t(\sigma, \xi)) = \text{index}^P_T(E_z)(g)
\]

since \( X^t \) is II-invariant. \( \square \)

For a \( T \)-equivariant form \( \alpha^g(\xi) \in \Omega(X^g) \) depending smoothly on \( \xi \) (such as the Chern character form or Atiyah-Singer form considered above), we define the formal series of differential forms

\[
\alpha^{g_z} = \alpha^g(g_z^+)[z] \in \Omega(X^g)[z]
\]

where, similar to our discussion of functions in Section 4.4, the right hand side is the result of substituting \( g_z^+ \in zt_C[z] \) into the Taylor series at 0 of the function \( \xi \in t \mapsto \alpha^g(\xi) \in \Omega(X^g) \) and expanding in powers of \( z \).
Theorem 4.10. Let \( \chi_z \in t_C \otimes \mathcal{CR}(T)[z] \) and let \( \mathcal{X}_{\ell,z}(t) = t^\ell \exp(z \chi_z(t)) \). Let \( E_z \in \mathcal{CR}^{(h)}(X)[z] \) be a series of quasiperiodic classes admitting \( \chi_z \)-twisted equivariant Chern character forms \( \text{Ch}^t(E_z, \xi) \). Then \( \text{index}^T(E_z) \) has support contained in the finite set \( T_\ell = \ell^{-1} \Lambda/\Pi \subset T \). Let \( g_z \in J^\infty T_C \) be the unique \( \infty \)-jet at \( g \in T_\ell \) satisfying \( \mathcal{X}_{\ell,z}(g_z) = g^\ell \). Then \( \text{Ch}^{\eta_t}(E_z) \mathcal{A} \mathcal{S}^{\eta_t}(\sigma) \in \Omega(X^g)[z] \) is \( \Pi \)-invariant, and

\[
\text{index}^T(E_z)(h) = \sum_{g \in T_\ell} \frac{\delta_{g_z}(h)}{\det(1 + z\ell^{-1}d\chi_z(g_z))} \int_{X^g/\Pi} \text{Ch}^{\eta_t}(E_z) \mathcal{A} \mathcal{S}^{\eta_t}(\sigma).
\]

Proof. For \( t \in T \) fixed and \( \xi \in t \) sufficiently small, let

\[ \alpha^t(\xi) = \text{Ch}^t(E_z, \xi) \mathcal{A} \mathcal{S}^t(\sigma, \xi) \]

denote the integrand in Theorem 4.5, and let \( h = t \exp(\xi) \). Let \( X_0 \subset X \) denote a fundamental domain for the \( \Pi \) action. By Definition 4.7 and equation (40),

\[ \text{index}^T(E_z)(h) = \sum_{\eta \in \Pi} \mathcal{X}_{\ell,z}(h)^\eta \int_{X^g_\ell} \alpha^t(\xi). \]

The sum over \( \Pi \) can be evaluated using Theorem 4.6. The result is

\[
\text{index}^T(E_z)(h) = \sum_{g \in T_\ell} \frac{\delta_{g_z}(h)}{\det(1 + z\ell^{-1}d\chi_z(g_z))} \int_{X^g_\ell} \alpha^t(\xi), \tag{58}
\]

where \( g_z \in J^\infty T_C \) is the unique \( \infty \)-jet satisfying \( \mathcal{X}_{\ell,z}(g_z) = g^\ell \).

Fix \( g \in T_\ell \) and consider the corresponding term in (58). Recall \( h = t \exp(\xi) \). Since \( t \) is fixed, while \( \xi \) is small and variable, the Dirac delta distribution \( \delta_{g_z}(h) \) leads to some constraint on \( \xi \). Solving the constraint yields the series

\[ \xi_z = \xi_0 + z\xi_1 + \cdots = \log(t^{-1} g) + g_\xi^+ \in t_C[z] \]

where \( g_z = g \exp(g_\xi^+), \ g_\xi^+ \in z t_C[z] \), and \( \log(-) \) denotes the preferred logarithm when the argument is near \( 1 \in T_C \). The Dirac delta distribution in (58) means we may replace \( \alpha^t(\xi) \) by \( \alpha^t(\xi_z) \). We claim that \( \alpha^t(\xi_z) \) is \( \Pi \)-invariant, hence descends to \( X^t/\Pi \). Indeed using Definition 4.7 and equation (40) once again, we have

\[ \eta^{-1} \cdot \alpha^t(\xi) = \mathcal{X}_{\ell,z}(g_z)^\eta \alpha^t(\xi) = g^{\eta \eta} \alpha^t(\xi_z) = \alpha^t(\xi_z), \]

using the definition of \( g_z \) and the equation \( g^{\eta \eta} = 1 \) which holds because \( g \in T_\ell = \ell^{-1} \Lambda/\Pi \). Thus (58) becomes

\[
\text{index}^T(E_z)(h) = \sum_{g \in T_\ell} \frac{\delta_{g_z}(h)}{\det(1 + z\ell^{-1}d\chi_z(g_z))} \int_{X^g_\ell} \alpha^t(\xi_z).
\]

The integrand is closed for the differential \( d\xi_z = d+2\pi i \nu(\xi_z)_M \) (where \( (\xi_z)_M \in C^\infty(M, TM_C)[z] \) denotes the formal series of vector fields \( (\xi_z)_M + z(\xi_1)_M + \cdots \) and localizes to the fixed point set of \( \xi_0 = \log(t^{-1} g) \) in \( X^t \), which is \( X^{t \exp(\xi_0)} = X_g \) (this holds when \( \xi_0 \) is sufficiently small, as we are assuming, cf. [73]). A small addendum to the usual abelian localization formula (see Lemma 4.11 below) yields

\[
\text{index}^T(E_z)(h) = \sum_{g \in T_\ell} \frac{\delta_{g_z}(h)}{\det(1 + z\ell^{-1}d\chi_z(g_z))} \int_{X^g_\ell \Pi} t^\nu \alpha^t(\xi_z),
\]

where \( \nu = \Theta(\xi, \chi). \)
where \( \nu \) is the normal bundle to \( X^g \) in \( X^t \) and \( \iota : X^g/\Pi \hookrightarrow X^t/\Pi \) is the inclusion. Using the properties of the Atiyah-Segal-Singer integrand and the bouquet property of the equivariant Chern forms, one finds (cf. [19])

\[
\frac{\iota^* \alpha^t(\xi_z)}{\text{Eul}(\nu, \xi_z)} = \alpha^g(g^+_z) = \alpha^g_z.
\]

This completes the proof. \( \square \)

In the proof of the result above we used the following addendum to the standard abelian localization formula.

**Lemma 4.11.** Let \( M \) be a \( T \)-manifold where \( T \) is a compact torus with Lie algebra \( t \). Let \( \xi_z = \sum_{j \geq 0} \frac{z^j}{j!} \xi_j \in t_C[z] \). Let \( \beta \in \Omega_c(M)[z]^T \) be a compactly-supported \( d_{\xi_z} \)-closed \( T \)-invariant \( C \)-valued differential form. Then

\[
\int_M \beta = \int_{M^{\xi_0}} \frac{\beta}{\text{Eul}(\nu, \xi_z)}.
\]

**Proof.** We start by reviewing the standard Atiyah-Bott-Berline-Vergne abelian localization formula [9, 17, 18]. For \( \xi \in t_C \) define the differential

\[
d_{\xi} = d + 2\pi i(\xi_M).
\]

It squares to zero on the subspace \( \Omega(M)^T \) of \( T \)-invariant forms. One has

\[
\int_M d_{\xi} \beta = 0 \quad (59)
\]

for any \( \beta \in \Omega_c(M) \) (the subscript ‘c’ meaning compactly supported), by Stokes’ theorem and because \( \iota(\xi_M)\beta \) has exterior degree less than \( \dim(M) \).

Let \( g \) be a \( T \)-invariant Riemannian metric, which we extend \( C \)-bilinearly to \( TM_C \). For \( \xi \in t_C \), let \( \bar{\xi} \) be the complex conjugate, and define \( \theta_{\xi} \in \Omega^1(M) \) by

\[
\theta_{\xi}(\cdot) = g(\bar{\xi}_M, \cdot). \quad (60)
\]

Then

\[
d_{\xi} \theta_{\xi} = d\theta_{\xi} + 2\pi i g(\bar{\xi}_M, \xi_M).
\]

Since the component of \( d_{\xi} \theta_{\xi} \) of exterior degree 0 is non-zero away from \( M^\xi = \{\xi_M = 0\} \), \( d_{\xi} \theta_{\xi} \) is invertible away from this submanifold.

Now let \( \xi_0 \in t_C \) and let \( \chi \) be a \( T \)-invariant bump function equal to 1 near \( M^{\xi_0} \) and with support contained in a tubular neighborhood \( U \) of the latter. For \( \xi \in t_C \) sufficiently close to \( \xi_0 \) one has \( M^\xi \subset M^{\xi_0} \), and consequently \( \chi \) is also equal to 1 on a neighborhood of \( M^\xi \). Thus for \( \xi \) sufficiently close to \( \xi_0 \) it makes sense to define

\[
P(\xi) = \chi + d\chi \frac{\theta_{\xi}}{d_{\xi} \theta_{\xi}} \in \Omega(M), \quad (61)
\]

since \( d\chi \) vanishes on a neighborhood of the subset \( M^\xi \) where \( d_{\xi} \theta_{\xi} \) fails to be invertible. By construction \( P(\xi) \in \Omega(M) \) has support contained in a tubular neighborhood of \( M^{\xi_0} \). Note that \( (1 - \chi) \) vanishes near \( M^\xi \), so \( (1 - \chi)(d_{\xi} \theta_{\xi})^{-1} \) makes sense, which implies that the difference

\[
1 - P(\xi) = d_{\xi} \left( (1 - \chi) \frac{\theta_{\xi}}{d_{\xi} \theta_{\xi}} \right) \quad (62)
\]
is $d_\xi$-exact. Consequently if $\beta \in \Omega_c(M)^T$ is $d_\xi$-closed then the integral
$$\int_M \beta = \int_M \beta \cdot P(\xi) = \int_U \beta \cdot P(\xi)$$
localizes near $M^{S_0}$. Using the Thom isomorphism and the inverse of the equivariant Euler form $\text{Eul}(\nu, \xi)$ of the normal bundle $\nu = \nu(M, M^{S_0})$, one easily reduces further to an integral over $M^{S_0}$, giving
$$\int_M \beta = \int_{M^{S_0}} \frac{\beta}{\text{Eul}(\nu, \xi)}. \quad (63)$$
This is the usual abelian localization formula.

Since (63) depends smoothly on $\xi$ sufficiently close to $\xi_0$, it easily adapts to ‘formal deformations’ $\xi_z = \xi_0 + z\xi_1 + \frac{z^2}{2}\xi_2 + \cdots$ where $z$ is a formal variable. Just as above, $d_{\xi_z}$ restricts to a differential on $\Omega(M)[z]^T$, and has the property in (59) for any $\beta \in \Omega_c(M)[z]$. We define $\theta_{\xi_z} \in \Omega^1(M)[z]^T$ by substituting $\xi_z$ for $\xi$ in (60). Similar to before, the $z^0 \cdot \Omega^0(M)$-component of $d_{\xi_z}\theta_{\xi_z}$ is invertible away from $M^{S_0}$, hence we may define $P(\xi_z)$ by substituting $\xi_z$ for $\xi$ in (61), and the obvious analogue of (62) holds. The remaining discussion also goes through, and proves the lemma.

4.6. **Affine Weyl symmetry.** If a quasiperiodic K-theory class $E \in \text{RK}_T^0(X)$ is in fact $N(T)$-equivariant, then its index has additional Weyl group symmetry. For an element $w \in W$, let $l(w)$ denote the length. We say that an element $\varphi \in R^{-\infty}(T)$ is $W$-antisymmetric if $w \cdot \varphi = (-1)^{l(w)} \varphi$ for all $w \in W$. Definition 3.4 has an obvious $N(T)$-equivariant analogue where we require $(E, \nabla^E)$ to be $N(T) \ltimes W$-equivariant and $Q$ to be $N(T)$ equivariant, and we take this as the definition of a quasiperiodic class in $\text{RK}_{N(T)}^0(X)$.

**Proposition 4.12.** *If $E \in \text{RK}_{N(T)}^0(X)$ is quasiperiodic then $\text{index}_T^W(E)$ is $W$-antisymmetric.*

**Proof.** The Dirac operator $\mathcal{D}_\varphi$ is $N(T)$-invariant. The operator $\mathcal{D}$ represents the KK-product of a K-homology class defined by $\mathcal{D}_\varphi$ and the pullback of the $T$-equivariant Bott-Thom element $B$ for $T^1$. As mentioned in Section 2.6, the latter element is $W$-antisymmetric, and the result follows from this; see [47, Section 4.5] for further explanation.

In case $E \in \text{RK}_{N(T)}^0(X)$ is quasiperiodic, it makes sense to ask for Chern character forms $\text{Ch}^i(E, \xi) \in \Omega(X^i)$ satisfying the bouquet condition for $N(T)$ in addition to the conditions in Definition 4.1. In particular $n_w \cdot \text{Ch}^i(E, \xi) = \text{Ch}^{wt}(E, w\xi)$ in $\Omega(X^{wt})$ where $n_w \in N(T)$ with image $w \in W = N(T)/T$. We take this extra symmetry property as an additional requirement in the definition of an equivariant Chern character form in case $E \in \text{RK}_{N(T)}^0(X)$.

Let $\chi_z \in t_C \otimes \mathcal{C}R(T)$ as in Section 4.5. Recall the $\chi_{t_z}$-twisted action of $\Pi$ on $\mathcal{C}R^{-\infty}(T)[z]$ from Definition 4.8. If the element $\chi_z$ is $W$-invariant, then the twisted action of $\Pi$ and the action of $W$ fit together into an action (denoted $\bullet_{\chi_{t_z}}$) of the affine Weyl group $W_{\text{aff}} = W \ltimes \Pi$ on $\mathcal{C}R^{-\infty}(T)[z]$, that we will refer to as the $\chi_{t_z}$-twisted action of $W_{\text{aff}}$. An element $\chi_z \in \mathcal{C}R^{-\infty}(T)[z]$ is said to be antisymmetric for this action if $w \bullet_{\chi_{t_z}} \chi_z = (-1)^{l(w)} \chi_z$ for all $w \in W_{\text{aff}}$.

**Corollary 4.13.** *Let $\chi_z \in (t_C \otimes \mathcal{C}R(T)[z])^W$ and let $\chi_{t_z}(t) = t^t \exp(z\chi_z(t))$. Let $E_z \in \mathcal{C}R_{N(T)}^0(X)[z]$ be quasiperiodic and admit $\chi_z$-twisted equivariant Chern character forms.*
\[ \text{Ch}^i(E_z, \xi). \] Then \( \text{index}^p_T(E_z) \in \mathbb{C}R^{-\infty}(T)[z] \) is antisymmetric under the \( \mathcal{X}_{t,z} \)-twisted \( W \)-action.

**Proof.** This follows immediately from Proposition 4.12 and Corollary 4.9. \( \square \)

Recall the finite subgroup \( T_\ell = \ell^{-1} \Lambda/\Pi \subset T \). Let \( T_\ell^{\text{reg}} = T_\ell \cap T^{\text{reg}} \), where \( T^{\text{reg}} \) is the set of regular elements.

**Proposition 4.14.** Let \( \chi_z \in (t_C \otimes \mathcal{C}R(T)[z])^W \) and let \( E_z \in \mathcal{C}R_K^0(t_C)(X)[z] \) be quasiperiodic and admit \( \chi_z \)-twisted equivariant Chern character forms \( \text{Ch}^i(E_z, \xi). \) Then \( \text{index}^p_T(E_z) \in \mathbb{C}R^{-\infty}(T)[z] \) has support contained in \( T_\ell^{\text{reg}} \) and

\[ \text{index}^p_T(E_z)(h) = \sum_{g \in T_\ell^{\text{reg}}/W} \sum_{w \in W} (-1)^{|w|} \delta_{w,g_z}(h) \int_{X^g/\Pi} \text{Ch}^{g_z}(E_z)\mathcal{A}S^{g_z}(\sigma). \tag{64} \]

**Proof.** For \( g \in T_\ell \), let

\[ f(g_z) = \frac{1}{\det(1 + z\ell^{-1}t\chi_z(g_z))} \int_{X^g/\Pi} \text{Ch}^{g_z}(E_z)\mathcal{A}S^{g_z}(\sigma), \]

so that by Theorem 4.10,

\[ \text{index}^p_T(E_z)(h) = \sum_{g \in T_\ell} \delta_{g_z}(h)f(g_z). \]

Let \( w \in W \) and let \( n_w \in N(T) \) be a lift. By the bouquet property

\[ \text{Ch}^{w,g}(E_z, w\xi) = n_w \cdot \text{Ch}^g(E_z, \xi) \]

hence \( \text{Ch}^{w,g_z}(E_z) = n_w \cdot \text{Ch}^{g_z}(E_z) \). On the other hand, because of the \( W \)-antisymmetry of the Thom-Bott element \( \mathcal{B} \), the Atiyah-Singer integrand obeys

\[ \mathcal{A}S^{w,t}(\sigma, w\xi) = (-1)^{|w|}n_w \cdot \mathcal{A}S^t(\sigma, \xi) \]

in cohomology, hence \( \mathcal{A}S^{w,g_z}(\sigma) = (-1)^{|w|}n_w \cdot \mathcal{A}S^{g_z}(\sigma) \). Consequently

\[ f(wg_z) = (-1)^{|w|}f(g_z). \tag{65} \]

Suppose \( g \in T_\ell \) is non-regular. Then there is an element \( w \in W \) of order 2 such that \( wg = g \). Since \( \chi_z \) is \( W \)-invariant, the element \( g_z \in \mathcal{J}_g^\infty(T_C) \) satisfies \( wg_z = g_z \). Therefore (65) becomes

\[ f(g_z) = f(wg_z) = -f(g_z) \Rightarrow f(g_z) = 0. \]

This shows that the support consists of regular elements. Choosing any set of representatives for the elements of \( T_\ell^{\text{reg}}/W \), the expression (64) follows from (65). \( \square \)

Let \( n_+ = |\mathcal{R}_+| \) be the number of positive roots and let

\[ \Delta = i^{-n_+} \sum_{w \in W} (-1)^{|w|} e_{w\rho} = i^{-n_+} e_\rho \prod_{\alpha \in \mathcal{R}_+} (1 - e_{-\alpha}) = \prod_{\alpha \in \mathcal{R}_+} 2\sin(\pi\alpha) \]

be the Weyl denominator (up to the factor \( i^{-n_+} \)): it is a \( W \)-antisymmetric holomorphic function on \( T_C \) which is real-valued on \( T \). Given a Weyl antisymmetric distribution \( \psi \in \mathcal{D}'(T) \), define a \( G \)-invariant distribution \( I^G_T \psi \in \mathcal{D}'(G)^G \) by

\[ \langle I^G_T \psi, \chi \rangle = \frac{1}{|W|} \langle \psi, \chi|T \cdot i^{n_+} \Delta \rangle. \]
If \( g \in T_{\text{reg}} \), then up to an overall normalization, this induction-type operation sends 
\[ \sum_{w \in W} (-1)^{\ell(w)} \delta_{uw} \] 
to the G-invariant distribution given by integration over the conjugacy class through \( g \) using the quasi-Hamiltonian volume form (the analogue of the symplectic volume form of a coadjoint orbit in \( g^* \); see [3]). By equation (64),
\[ I^*_T \text{index} \, \mathcal{F}^*_T (E_z) = \sum_{g \in T_{\text{reg}}/W} \frac{i^{n+1} \Delta(g_z) \delta_{g_z}}{\det(1 + z \ell^{-1} dx_z(g_z))} \int_{X^g/W} \text{Ch}^{g_z}(E_z) A^{S^g_z} (\sigma). \] (66)

5. INDEX OF THE ATIYAH-BOTT CLASSES

In this section we return to the moduli space examples from Section 2.3 and introduce Fredholm families representing K-theory classes on \( M \) that are closely related to the Atiyah-Bott K-theory classes [8]; we will refer to these simply as ‘the Atiyah-Bott classes’. We prove that the restriction of these families to \( X \subset M \) are quasiperiodic in the sense of Definition 3.4, and describe their behavior under the affine Weyl group. Finally we specialize Theorem 4.10 to these examples.

Throughout this section \( \Sigma \) denotes a compact connected oriented genus \( g \) surface with \( b+1 \geq 1 \) boundary components \( \partial_0 \Sigma \sqcup \cdots \sqcup \partial_b \Sigma \). Recall that we take the boundary components to be parametrized with the orientation opposite the orientation induced from orientation of \( \Sigma \). Let \( \mathbb{D} \) denote the unit disk, and let \( \mathbb{D}_0, \ldots, \mathbb{D}_b \) denote \( b+1 \) copies of \( \mathbb{D} \). We denote by \( \Sigma \) the closed surface obtained by capping off the boundary:
\[ \Sigma = \Sigma \cup_{\partial \Sigma} \bigcup_{j=0}^b \mathbb{D}_j. \] (67)

We use an underline \( G \) to denote the product group \( G^{b+1} \), as well as similar notation for objects associated with \( G \), for example \( g = (g_0, \ldots, g_b) \) denotes an element of \( G \). Let \( M \) be the Hamiltonian \( LG \)-space (loops of Sobolev class \( \varsigma \)) obtained as the quotient \( A_{fl}/G_{\partial \Sigma} \), where \( A_{fl} \) denotes the space of flat \( G \)-connections (Sobolev class \( \varsigma - \frac{1}{2} \)) on \( \Sigma \times G \) and \( G_{\partial \Sigma} \) is the normal subgroup of the gauge group consisting of maps \( \Sigma \to G \) whose restriction to \( \partial \Sigma \) is the identity. The quotient map \( A_{fl} \to M \) is denoted by \( q \). For the considerations of this section we will need the connections on the surface \( \Sigma \) to be at least continuous, hence we take \( \varsigma > \frac{1}{2} \).

5.1. The Atiyah-Bott classes. Fix a finite dimensional representation \((V, \pi_V)\) of \( G \). The gauge group \( G \) acts on the trivial bundle \( \Sigma \times V : g \in G \) acts on the fibre \( \{p\} \times V \) by \( \pi_V(g(p)) \).

**Definition 5.1.** The quotient of \( A_{fl} \times (\Sigma \times V) \) by the diagonal action of \( G_{\partial \Sigma} \) is an \( LG \)-equivariant vector bundle
\[ EV = (A_{fl} \times (\Sigma \times V))/G_{\partial \Sigma} \to (A_{fl} \times \Sigma)/G_{\partial \Sigma} = M \times \Sigma. \]
It defines a class \( EV \in \mathcal{RK}_{\Sigma}^0 (M \times \Sigma) \).

The idea is to construct K-theory classes on \( M \) by pulling back \( EV \) to \( M \times N \) where \( \iota : N \hookrightarrow \Sigma \) is a submanifold, and then push forward along \( M \times N \to M \). The class so obtained only depends on the homotopy class of the map \( \iota \). This works directly when \( \dim(N) = 0, 1 \) and, suitably interpreted, when \( \dim(N) = 2 \) (i.e. \( N = \Sigma \)).

The simplest case is when \( N = \text{pt} \) is a point. Since \( \Sigma \) is connected, any two points in \( \Sigma \) are connected by a path, and hence we obtain only one distinct K-theory class \( E^{\text{pt}} V \in \mathcal{RK}_{\Sigma}^0 (M) \).
Choosing the point to lie on ∂Σ, we see that E pt V is represented by the trivial vector bundle \( M \times V \), where all but one of the copies of \( G \) in \( G = G^{b+1} \) acts trivially on the fibres (in particular, choosing different factors in \( G^{b+1} \) to act on the fibre \( V \) just leads to homotopic \( G^{b+1} \)-equivariant vector bundles). The restriction of this vector bundle to \( X \subset M \) defines a (rather trivial, quasiperiodic) class \( E^{pt} V \in \mathcal{R} K_{N(\Sigma)}(X) \).

Moving to dimension 1, let \( N = C \) be a closed curve in \( \Sigma \). Pulling \( E V \) back to \( M \times C \) and taking \( L^2 \)-sections along \( C \) yields an \( LG \)-equivariant Hilbert bundle:

\[
\mathcal{E}_C = (A_{fl} \times L^2(C, V))/G_{\partial \Sigma} \to A_{fl}/G_{\partial \Sigma} = M
\]

where \( G \) acts diagonally, with the action on \( L^2(C, V) \) being through evaluation at points along \( C \). (Note that the gauge action preserves the \( L^2 \)-inner product.) Fix a parametrization \( C \simeq S^1 \) and let \( \mathcal{P}^C_0 = -i\frac{d}{ds} \) a Dirac operator acting on smooth sections of the trivial bundle \( C \times V \) over \( C \). More generally for \( A \in A_{fl} \), let \( \mathcal{P}^C_A = \mathcal{P}^C_0 + \iota^* A \). We therefore obtain a tautological family of Dirac operators \( \mathcal{P}^C = \{ \mathcal{P}^C_A | A \in A_{fl} \} \) over \( A_{fl} \), which is equivariant for the diagonal action of \( G \) on \( A_{fl} \times L^2(C, V) \). Descending the family along the fibres of \( q: A_{fl} \to A_{fl}/G_{\partial \Sigma} = M \) we obtain an \( LG \)-equivariant family of Dirac operators, denoted \( \mathcal{P}^C_{(\bullet)} \), acting on the fibres of the Hilbert bundle \( \mathcal{E}_C \). This family defines an odd K-theory class \( E^C V \in \mathcal{R} K_{1}_{N(\Sigma)}(M) \). Restricting the family to \( X \subset M \) yields a class \( E^C V \in \mathcal{R} K_{1}_{N(\Sigma)}(X) \). Even classes can be obtained by taking products of even numbers of these classes. For example if \( C_1, C_2 \) are two closed curves, then \( E^{C_1 V_1, E^{C_2 V_2}} \) is represented by a family of Dirac operators on the 2-torus \( C_1 \times C_2 \subset \Sigma \times \Sigma \). These classes are all rather trivially quasiperiodic, since the family of Fredholm operators defining \( E^C V \) is \( \Pi \)-equivariant.

**Proposition 5.2.** \( E^C V \in \mathcal{R} K_{1}_{N(\Sigma)}(X) \) only depends on the free homotopy class of \( C \) in \( \Sigma \).

**Proof.** By our remarks above, it is clear that \( E^C V \) only depends on the free homotopy class of \( C \) in \( \Sigma \). Suppose \( C = \partial \Sigma \) is one of the boundary components of \( \Sigma \). Then \( E^C V \in \mathcal{R} K_{1}_{N(\Sigma)}(M) \) is the pullback of a tautological \( G \)-equivariant class over \( LG^* \). But \( \mathcal{R} K_{1}_{G}(LG^*) = 0 \) because \( LG^* \) is \( G \)-equivariantly contractible (by holonomy, \( LG^* \) is the same as the space \( P_G G \) of paths of Sobolev class \( \varsigma \) in \( G \) beginning at 1). It follows from this that the K-theory class only depends on the free homotopy class of \( C \) in \( \Sigma \). \( \square \)

The 2-dimensional case \( N = \Sigma \) is less immediate. Since \( \partial \Sigma \neq \emptyset \), the analogous \( G \)-equivariant family of Dirac operators on \( \Sigma \) is not a Fredholm family. There are several different approaches one might take to construct a Fredholm family. The method that we pursue below is to impose a boundary condition. We will see that this leads to a quasiperiodic cycle.

5.2. A family of Atiyah-Patodi-Singer boundary problems. Fix a Riemannian metric on \( \Sigma \) of product form in a collar neighborhood of the boundary, and choose a spin structure. Let \( c^\Sigma: \mathcal{O}(T \Sigma) \to \text{End}(S^\Sigma), S^\Sigma = S^{\Sigma,+} \oplus S^{\Sigma,-} \) be the corresponding \( \mathbb{Z}_{2} \)-graded \( \mathcal{O}(T \Sigma) \)-module. Let \( \mathcal{D}^\Sigma_{0} \) be the Dirac operator acting on sections of \( S^\Sigma \boxtimes V \) (coupled to \( \Sigma \times V \) using the trivial connection). For \( A \in A_{fl} \), let

\[
\mathcal{D}^\Sigma_{A} = \mathcal{D}^\Sigma_{0} + (c^\Sigma \boxtimes \pi_V)(A) = \begin{pmatrix} 0 & \mathcal{P}^\Sigma_{A,-} \\ \mathcal{P}^\Sigma_{A,+} & 0 \end{pmatrix},
\]

(68)
be the Dirac operator obtained by coupling the Dirac operator to $\Sigma \times V$ using the connection (or vector potential) $A$. This construction produces a tautological family of Dirac operators $\Phi^\Sigma_\bullet = (\Phi^\Sigma_A)_A \in A_{fl}$ parametrized by $A_{fl}$.

Owing to the non-empty boundary $\partial \Sigma$, the operators $\Phi^\Sigma_A$ are not Fredholm. To obtain a Fredholm family we impose a boundary condition. The spin structure $S^\Sigma$ on $\Sigma$ induces a spin structure $S^{\partial \Sigma} \simeq S^{\Sigma, +}|_{\partial \Sigma}$ on $\partial \Sigma$. Let $\tilde{\partial} = \partial_0$ denote the spin Dirac operator on the boundary twisted by $V$, which acts on sections of $S^{\partial \Sigma} \otimes V$. Let $a \in L^1_{g*} = \Omega^1_{g, -1}(\partial \Sigma, g)$ be a connection on the boundary. The corresponding spin Dirac operator on the boundary is

$$\partial_a = \partial + (c^{\partial \Sigma} \otimes \pi_V)(a).$$

(69)

For $v \in \mathbb{R}$ let

$$B_{\leq v}(\partial_a) \subset L^2(\partial \Sigma, S^{\Sigma, +} \otimes V|_{\partial \Sigma})$$

be the closure (in the Sobolev $\frac{1}{2}$ norm) of the sum of the eigenvalues of $\partial_a$ with eigenvalue $< v$. We similarly define $B_{> v}(\partial_a)$, $B_{\leq v}(\partial_a)$. Let

$$\mathcal{A} : L^{2,1}(\Sigma, S^\Sigma \otimes V) \to L^{2,1}(\partial \Sigma, S^{\Sigma} \otimes V|_{\partial \Sigma})$$

denote the trace operator, the continuous extension of the operator given on smooth sections by restriction to the boundary.

Imposing the boundary condition $B_{< 0}(\partial)$ on the operators of the family $\Phi^\Sigma_\bullet$, we obtain a family of unbounded Fredholm operators:

**Definition 5.3.** For $A \in A_{fl}$, let $(\Phi^\Sigma_A, B_{< 0}(\partial))$ denote the unbounded Hilbert space operator defined by the differential operator $\Phi^\Sigma_A$ with domain

$$\text{dom}(\Phi^\Sigma_A, B_{< 0}(\partial)) = L^{2,1}(\Sigma, S^{\Sigma, +} \otimes V) \cap \mathcal{A}^{-1}(B_{< 0}(\partial)).$$

The corresponding odd self-adjoint operator is denoted

$$D_A = \begin{pmatrix} 0 & (\Phi^\Sigma_A, B_{< 0}(\partial))^* \\ (\Phi^\Sigma_A, B_{< 0}(\partial)) & 0 \end{pmatrix}.$$

Let $D_\bullet = \{D_A|A \in A_{fl}\}$ denote the tautological family. The bounded transforms are denoted

$$F_A = (1 + D_A^2)^{-1/2} D_A, \quad F_\bullet = (F_A)_A \in A_{fl}.$$

The condition $\mathcal{A}(s) \in B_{< 0}(\partial)$ is the Atiyah-Patodi-Singer (APS) boundary condition [11]. The operator $D^\perp_A = (\Phi^\Sigma_A, B_{< 0}(\partial))$ is Fredholm, and in fact elliptic in the strong sense that there is an estimate $\|s\| \leq C(\|D^\perp_A s\| + \|s\|)$ for the $L^{2,1}$-norm when $s \in \text{dom}(D^\perp_A)$; by the Rellich lemma $D_A$ has compact resolvent. A detailed recent reference for the APS and other more general elliptic boundary problems is [14, 15]. More generally one can define Fredholm operators $(\Phi^\Sigma_A, B_{< v}(\partial_0))$ for any $v \in \mathbb{R}$, $a \in L^1_{g*}$. The index is independent of $A$, but is highly sensitive to the choice of $v, a$.

The index of each operator in the family of Definition 5.3 is 0. We may check this using the Atiyah-Patodi-Singer theorem. The spin structure on $S^3$ induced by that on $\Sigma$ is the non-trivial one (cf. [32, Exercise 2.2.18]). The corresponding Dirac operator is (after making
a non-canonical choice of trivialization of $S^{\Sigma +}$ the operator $-i\frac{d}{dz} + \frac{1}{2}$. The latter has trivial kernel and trivial $\eta$-invariant. Since $\hat{A}(\Sigma) = 1$, the Atiyah-Patodi-Singer theorem reads

$$\text{index}(\mathcal{P}_A^{\Sigma +}, B_{<0}(\partial)) = 0 - \frac{0 + 0}{2} = 0.$$ 

See also [11, pp.61–62] for general discussion of the spin case, and also for the discussion of the $\overline{\partial}$-operator on Riemann surfaces.

5.3. Equivariance properties of the family and the class $E^\Sigma V$. The gauge group $G$ acts on the Hilbert space $E_0 = L^2(\Sigma, S^\Sigma \otimes V)$ point-wise via the representation $\pi_V$. Taking the quotient by the diagonal action produces a smooth $LG$-equivariant Hilbert bundle

$$E = (A_{fl} \times E_0)/G_{0\Sigma} \rightarrow A_{fl}/G_{0\Sigma} = \mathcal{M}.$$ 

Equivalently $E$ is obtained from the vector bundle $(\mathbb{C} \otimes S^\Sigma) \otimes EV$ over $\mathcal{M} \times \Sigma$ by taking $L^2$-sections along the fibres $\mathcal{M} \times \Sigma \rightarrow \mathcal{M}$.

The family of differential operators $\mathcal{D}_\Sigma$ is equivariant for the full $G$-action, but the boundary condition is only equivariant for the action of a subgroup.

**Proposition 5.4.** Under the diagonal action of $g \in G$ on $A_{fl} \times L^2(\Sigma, S^\Sigma \otimes V)$, the family $D_\bullet = (\mathcal{D}_\Sigma^{\Sigma}, B_{<0}(\partial))$ is sent to the family

$$(\mathcal{D}_\Sigma^{\Sigma}, B_{<0}(\partial \hat{\mathcal{R}}(g)0)), $$

where $\mathcal{R}(g)$ is the restriction (or trace) to the boundary. Equivalently $\partial \hat{\mathcal{R}}(g)0 = \pi_V(\mathcal{R}(g)) \circ \partial \circ \pi_V(\mathcal{R}(g^{-1}))$. Consequently $D_\bullet$ (or equivalently the bounded transform $F_\bullet$) is equivariant for the subgroup $G_{lc,0\Sigma}$ consisting of $g \in G$ such that $\mathcal{R}(g)$ is locally constant on $\partial \Sigma$; equivalently $\mathcal{R}(g) \in LG_0 = G$, the stabilizer of $0 \in LG^\Sigma$ under the gauge action.

**Proof.** Under the action of $g \in G$ on $E_0$, the operator $(\mathcal{D}_A^{\Sigma +}, B_{<0}(\partial))$ is sent to the operator $(\mathcal{D}_g^{\Sigma +}, B_{<0}(\partial \hat{\mathcal{R}}(g)0))$, where the action of $g$ on $A$ (resp. $\mathcal{R}(g)$ on $0$) is by gauge transformations. If $g \in G_{lc}$ then $\mathcal{R}(g)0 = -d\mathcal{R}(g)\mathcal{R}(g^{-1}) = 0$ by definition, hence $g$ maps the family into itself. □

Since $G_{0\Sigma} \subset G_{lc,0\Sigma}$, passing to $G_{0\Sigma}$-equivalence classes yields a $G_{lc,0\Sigma}/G_{0\Sigma} \simeq G$-equivariant family of Fredholm operators over $\mathcal{M}$. Recall $q: A_{fl} \rightarrow A_{fl}/G_{0\Sigma} = \mathcal{M}$ is the quotient map.

**Definition 5.5.** Let $D_{q(\bullet)}$ denote the $LG_0 = G$-equivariant family of unbounded Fredholm operators acting on the fibres of $E$ and parametrized by $\mathcal{M}$, which is obtained by descending the family $D_\bullet$ along the fibres of $q$. Likewise let $F_{q(\bullet)}$ denote the family of bounded transforms.

The family $F_{q(\bullet)}$ varies norm-continuously. Indeed the family $F_\bullet$ on $A_{fl}$ is norm-continuous, on account of the fact that the boundary condition $B_{<0}(\partial)$ is fixed, and the differential operators $\mathcal{D}_\Sigma$ all have the same symbol. As explained in Proposition 2.3, $A_{fl} \rightarrow \mathcal{M}$ is a principal $G_{0\Sigma}$-bundle. Elements of $G_{0\Sigma}$ are of Sobolev class $s + \frac{1}{2} > 2$, and as the latter Sobolev norm dominates the supremum norm, the map $G_{0\Sigma} \rightarrow \mathcal{W}(E_0)$ is continuous with respect to the norm topology. Thus the structure group of the Hilbert bundle $E$ can be taken to be the unitary group $\mathcal{W}(E_0)$ with the norm topology, and continuity of $F_{q(\bullet)}$ follows.

**Definition 5.6.** The pair $(E, F_{q(\bullet)})$ defines a class $E^\Sigma V \in \mathcal{R}K^0_\Sigma(\mathcal{M})$. 

HAMILTONIAN LOOP GROUP SPACES AND A THEOREM OF TELEMAN AND WOODWARD 41
5.4. Quasiperiodicity of $\mathcal{E}^2 V$. The Hilbert bundle $\mathcal{E} \to M$, the family of operators $F_{q(\bullet)}$ (or its unbounded version $D_{q(\bullet)}$), and the class $\mathcal{E}^2 V$ restrict to corresponding objects over the finite dimensional submanifold $X \subset M$ that we denote with the same symbols when there is no risk of confusion. Thus there is a class $\mathcal{E}^2 V \in \mathcal{R}K_0 N(T)(X) = \mathcal{R}K_0 N(T)(X; C_0(X), C_0(X))$, which is represented by the Kasparov triple $(C_0(X, \mathcal{E}), \varrho, \mathcal{F})$, where $\varrho$ is the obvious multiplication action of $C_0(X)$ on $C_0(X, \mathcal{E})$, and $\mathcal{F}$ is the $C_0(X)$-linear operator on $C_0(X, \mathcal{E})$ induced by the family $F_{q(\bullet)}$. In this section we will show that $\mathcal{E}^2 V$ is quasiperiodic.

The Hilbert bundle $\mathcal{E} \to X$ is $N(T) \times \Pi$-equivariant. It will be useful to have a somewhat explicit description of an invariant Hermitian connection $\nabla^\mathcal{E}$ built using Coulomb gauge charts (see Section 2.3). Choose a compact fundamental domain $X_0$ for the $\Pi$ action, and choose a finite number $i = 1,...,N$ of Coulomb gauge charts whose restrictions $U_{i,0}$ to $X$ cover $X_0$. By making the charts smaller if necessary, we may assume that $\eta \cdot U_{i,0}$, $\eta' \cdot U_{i,0}$ are disjoint when $\eta \neq \eta'$, and moreover that the closures $\overline{U}_{i,0}$ are still contained in the domain of some slightly larger Coulomb chart. Let

$$U_i = \bigcup_{\eta \in \Pi} \eta \cdot U_{i,0} \simeq \Pi \times U_{i,0}.$$ 

Thus $U = \{U_1, ..., U_N\}$ is a finite open cover of $X$ by $\Pi$-invariant subsets. The covering is obviously uniformly finite, in the sense that there is a $\delta > 0$ such that for every $x \in X$, there is an $i \in \{1,...,N\}$ such that the ball of radius $\delta$ around $x$ is contained in $U_i$. Let us also fix a partition of unity $\{\rho_1, ..., \rho_N\}$ subordinate to the cover, with each $\rho_i$ being $\Pi$-invariant.

The choice of a Coulomb gauge chart $U_{i,0}$ determines a trivialization $\Phi_{i,0}: \mathcal{E} | U_{i,0} \to E_0 \times U_{i,0}$, and hence a Hermitian connection $\nabla^{\mathcal{E}, U_{i,0}}$. Since the bundle $\mathcal{E}$ is $\Pi$-equivariant, we may use the $\Pi$ action to obtain a connection $\nabla^{\mathcal{E}, U_{i}}$ defined over all of $U_i = \Pi \cdot U_{i,0}$, and we define

$$\nabla^{\mathcal{E}, \mathcal{U}} = \sum_{i=1}^m \rho_i \nabla^{\mathcal{E}, U_{i}}.$$ 

Equivalently the trivializations and partition of unity determine a $\Pi$-equivariant embedding of $\mathcal{E}$ into the trivial bundle $X \times (E_0 \otimes \mathbb{C}^N)$, and $\nabla^{\mathcal{E}, \mathcal{U}}$ is the Grassmann connection associated to this embedding. To obtain an $N(T) \times \Pi$-invariant connection $\nabla^\mathcal{E}$, average $\nabla^{\mathcal{E}, \mathcal{U}}$ with the action of the compact group $N(T)$.

We first check that the Hilbert bundle $\mathcal{E}$ is $T^2$-finite. Roughly speaking this is because $V$ is a finite dimensional representation of $T$ and $T$ acts trivially on $\Sigma$. Slightly more care should be taken when $b + 1 > 1$ as $T \neq T$.

**Proposition 5.7.** The Hilbert bundle $\mathcal{E}$ is $T$-finite.

**Proof.** The Riemannian metric $g_X$ on $X$, the Hermitian structure on $\mathcal{E}$, and the connection $\nabla^{\mathcal{E}, \mathcal{U}}$ determine a Riemannian metric on the unit ball bundle $B$ in $\mathcal{E}$. It will suffice to show that the vector fields $\xi_B$, $\xi = (\xi^0, ..., \xi^p) \in \mathfrak{t} = \mathfrak{t}^{b+1}$ for the infinitesimal action on the unit ball bundle have bounded norm (compare Remark 3.2). The horizontal component (relative to $\nabla^{\mathcal{E}, \mathcal{U}}$) of $\xi_B$ is the horizontal lift of $\xi_X$, which clearly has bounded norm (it is $\Pi$-invariant). By $\Pi$-invariance of the norms and connections, together with the fact that the $T$, $\Pi$ actions commute, it is enough to check that the $\nabla^{\mathcal{E}, U_{i,0}}$-vertical component of $\xi_B$ has bounded norm over each of the compact closures $\overline{U}_{i,0} \subset X$, $i = 1,...,N$ (recall we assumed that the closures...
$U_{i,\phi}$ is contained in some slightly larger Coulomb chart, so $\nabla^{E,U_{i,\phi}}$ extends over this slightly larger subset.

The unit ball bundle $B = (A_{fl} \times B_{0})/G_{\partial \Sigma}$, where $B_{0} \subseteq E_{0}$ is the unit ball. The vector field $\xi_{B}$ can be described as follows: let $\xi$ be any smooth $g$-valued function on $\Sigma$ whose restriction to the $j$-th boundary component $\partial_{j}\Sigma$ of $\Sigma$ is $\xi_{j} \in t$. Then $\xi$ induces a vector field $(\xi_{A_{fl}}, \xi_{B_{0}})$ on $A_{fl} \times B_{0}$ which descends to $\xi_{B}$. The Coulomb gauge condition uniquely determines a continuous (because $\varsigma + \frac{1}{2} > 1$) map $\xi : \overline{U}_{i,\phi} \times \Sigma \to g$ such that the (partially-defined) vector field $\xi_{A_{fl}}$ is tangent to the Coulomb slice, where $\xi_{A_{fl}} : \overline{U}_{i,\phi} \times \Sigma \to g$ is the map $\xi_{A_{fl}}(x,p) = \xi(p) - \zeta(x,p)$, for $x \in \overline{U}_{i,\phi}$, $p \in \Sigma$. The norm of the $\nabla^{E,U_{i,\phi}}$-vertical component of $\xi_{B}$ over $B \setminus \overline{U}_{i,\phi}$ is bounded by

$$\sup_{(x,p)} \pi_{V}(\xi_{A_{fl}}(x,p))|_{\text{End}(V)},$$

where the supremum is taken over $(x,p) \in \overline{U}_{i,\phi} \times \Sigma$. This is finite because $\xi_{A_{fl}} : \overline{U}_{i,\phi} \times \Sigma \to g$ is bounded and $(V, \pi_{V})$ is a finite dimensional representation.

**Theorem 5.8.** The triple $(E, \nabla^{E}, D_{q}(\bullet))$ is quasiperiodic, hence the class $\mathcal{E}^{V}V$ is quasiperiodic.

**Proof.** Property (a), compactness of $(1 + D_{2})^{-1}$ for each $x \in X$, holds because of the APS boundary condition. Property (b), $T$-finiteness of $E$, was shown in Proposition 5.7. The smoothness condition of Property (c) is clear from the construction. It remains to show Property (d). Each of the Coulomb gauge connections $\nabla^{E,U_{i,\phi}}$ preserves the domain of $D_{q}(\bullet)$ since the boundary condition is constant. The extension of $\nabla^{E,U_{i,\phi}}$ to a connection $\nabla^{E,U_{i}}$ over $U_{i} = \Pi \cdot U_{i,\phi}$ using the $\Pi$ action also has this property, because the image of a Coulomb chart under the action of an element of the gauge group is again a Coulomb chart. Since $\nabla^{E}$ is built from the $\nabla^{E,U_{i}}$ by patching together with a partition of unity and then averaging over $N(T)$, $\nabla^{E}$ parallel translation also preserves $\text{dom}(D_{q}(\bullet))$.

Next we argue that the $\nabla^{E}$-covariant derivative of $D$ is a bounded section of $T^{*}X \otimes \mathcal{B}(E)$. By the construction of $\nabla^{E}$, it is enough to prove instead that the $\nabla^{E,U_{i}}$-covariant derivative of $D |_{U_{i}}$ is a bounded section of $T^{*}U_{i} \otimes \mathcal{B}(E |_{U_{i}})$, for each $i = 1, ..., N$ ($\rho_{1}, ..., \rho_{N}$ are periodic so have globally bounded gradient). Fix $i$ and let $U_{\phi} = U_{i,\phi}$, $U = U_{i} \simeq \Pi \cdot U_{\phi}$. Recall that $U_{\phi} \subseteq X$ is the restriction of a Coulomb gauge chart for some connection $A_{\phi}$ on $\Sigma$. Let $\psi : U_{\phi} \to A_{fl}$ be the smooth local section of the bundle $A_{fl} \to \mathcal{M}$ over $U_{\phi}$ that is determined by the chart. By smoothness of $\psi$, and since $U_{\phi}$ is relatively compact, there is an estimate of the form

$$\|B\|_{C^{0}} \leq C|T\psi(U_{\phi})|$$

for $B \in T\psi(U_{\phi})$, where $T\psi$ is the tangent map and $|T\psi(B)|$ is computed using the Riemannian metric $g_{X}$. Since $\varsigma > \frac{3}{2}$, Sobolev embedding yields an estimate

$$\|B\|_{C^{0}} \leq C|T\psi(U_{\phi})|,$$

(70)

for $B \in T\psi(U_{\phi})$.

For each $\eta \in \Pi$, let $D_{B,q}(\bullet)$ denote the family of operators $\eta^{-1} \circ (D_{q}(\bullet) | \eta \cdot U_{\phi}) \circ \eta$ on $E | U_{\phi}$ obtained by restricting the family $D_{q}(\bullet)$ to $\eta \cdot U_{\phi}$ and pulling back to $U_{\phi}$ using the $\Pi$-action on $E$. By construction of $\nabla^{E,U}$, boundedness of the derivative of $D_{q}(\bullet)$ over $U$ amounts to showing that $\nabla^{E,U}D_{B,q}(\bullet)$ is a bounded operator for each $\eta$ and moreover the norm is bounded by a constant that is independent of $\eta$. 

By Proposition 5.4, the family $D_{q,q}(\bullet)$ over $U_\circ$ is obtained by descending a family $D_{q,\bullet}$ along the fibres of $q: A_{fl} \to M$ (over $U_\circ$), where $D_{2,A}$ is the odd self-adjoint operator constructed from the family $(\mathcal{D}_{A}^{\Sigma,+}, B_{<0}(\partial_{n}))$ as in Definition 5.3. Fix $A \in \psi(U_\circ)$, and let $B \in T_{A}\psi(U_\circ)$. Then by definition
\[
\nabla_{Tq(B)}^{E,U_{\circ}}D_{2,q}(\bullet) = \lim_{t \to 0} t^{-1}(D_{2,A+tB} - D_{2,A}) = (c^{\Sigma} \otimes \pi_{V})(B),
\]
a bounded operator, with operator norm at most $\|B\|_{\infty}$ independent of $\eta$. We conclude that the derivative satisfies
\[
\|\nabla_{Tq(B)}^{E,U_{\circ}}D_{2,q}(\bullet)\| \leq \|B\|_{\infty} \leq C|Tq(B)|,
\]
where the second inequality is (70). This verifies the required boundedness property. The smoothness of the $\nabla^{E}$-covariant derivative is clear from (71), and this completes the proof. □

As a consequence of Theorem 5.8 the index of $E^{\Sigma}V$ can be defined. Let $\mathcal{S}_{\text{can}} \to M$ be the canonical level $h^{\mathcal{S}} = (h^{\mathcal{S}}, \ldots, h^{\mathcal{S}})$ spinor module for $\text{hol}_{\mathcal{M}}TM$. Let $\mathcal{S} \to M$ be a level $k > -h^{\mathcal{S}}$ line bundle and let $\mathcal{S} = \mathcal{S}_{\text{can}} \otimes \mathcal{L}$, which has level $\ell = k + h^{\mathcal{S}}$. Then specializing Definition 3.20, the equivariant analytic index
\[
\text{ind}_{\mathcal{T}}^{P}(E^{\Sigma}V) \in R^{-\infty}(\mathcal{T})
\]
is defined.

5.5. The action of $W_{\text{aff}}$. Let $\Gamma$ be a topological group and let $\Gamma' \subset \Gamma$ a compact normal subgroup. If $Y$ is a $\Gamma$-space, then the component group $\pi_{0}(\Gamma/\Gamma')$ acts on $RK^{0}_{\mathcal{T}}(Y)$. In particular for $Y = X$ we deduce that there is an action of $W_{\text{aff}} = W_{\text{aff}}^{b+1}$ on $RK^{0}_{\mathcal{T}}(X)$.

Let $\text{Tr}_{\mathcal{T}}: T \to \mathbb{C}$ be the restriction of the character of the finite dimensional representation $V$ to the maximal torus $T \subset G$. It admits a decomposition
\[
\text{Tr}_{\mathcal{T}} = \sum_{\lambda \in \Lambda} \dim(V_{[\lambda]})e_{\lambda}
\]
where $V_{[\lambda]} = \text{Hom}(\mathbb{C}_{\lambda}, V)$ is the $\lambda$ weight space. Let $\text{Tr}_{\mathcal{T}V}$ be the character of $V^{\oplus(b+1)}$ viewed as a representation of $T = T^{b+1}$, or in other words
\[
\text{Tr}_{\mathcal{T}V} : \lambda = (t_{0}, \ldots, t_{b}) \in T = T^{b+1} \mapsto \text{Tr}_{V}(t_{0}) + \cdots + \text{Tr}_{V}(t_{b}) \in \mathbb{C}.
\]
If $\eta \in \Pi$ let
\[
\nabla_{\eta}\text{Tr}_{\mathcal{T}V} = \sum_{\lambda \in \Lambda} \dim(V_{[\lambda]})\langle \lambda, \eta \rangle e_{\lambda}.
\]
Since $\langle \lambda, \eta \rangle \in \mathbb{Z}$, $\nabla_{\eta}\text{Tr}_{\mathcal{T}V} \in R(T)$. For $\eta = (\eta_{0}, \ldots, \eta_{b}) \in \Pi$, $\nabla_{\eta}\text{Tr}_{\mathcal{T}V} \in R(T)$ denotes $(t_{0}, \ldots, t_{b}) \mapsto \nabla_{\eta_{0}}\text{Tr}_{V}(t_{0}) + \cdots + \nabla_{\eta_{b}}\text{Tr}_{V}(t_{b})$. There is an obvious map $R(T) \to RK^{0}_{\mathcal{T}}(X)$ defined using the unit and $R(T)$-module structure of $RK^{0}_{\mathcal{T}}(X)$.

Proposition 5.9. The Weyl group $W = W^{b+1}$ acts on $E^{\Sigma}V \in RK^{0}_{\mathcal{T}}(X)$ trivially, while $\eta \in \Pi$ acts as $\eta \cdot E^{\Sigma}V = E^{\Sigma}V - \nabla_{\eta}\text{Tr}_{\mathcal{T}V}$.
Proof. The class $E^\Sigma V$ comes from a class in $RK^\mathfrak{g}_0(M)$, hence $W$ acts trivially. Recall $\eta = (\eta_0, ..., \eta_b) \in \prod$ embeds in $LG$ as the exponential loop $s \mapsto (\exp(-s\eta_0), ..., \exp(-s\eta_b))$, $s \in [0, 1]$. However we chose the parametrization of the boundary components to be orientation reversing. Hence if $s_j$ is the opposite parametrization of $\partial_j \Sigma$ (which therefore is consistent with the orientation of $\Sigma$), then the corresponding gauge transformation along $\partial_j \Sigma$ is given by $s_j \mapsto \exp(s_j \eta_j)$, which maps the trivial connection along $\partial_j \Sigma$ to the constant connection $-\eta_j$. Therefore by Proposition 5.4, $\eta \cdot E^\Sigma V$ is represented by the family of boundary problems obtained by descending the family $(\mathcal{P}_A^{\Sigma^+}, B_{<0}(\partial-\eta))_{\mathcal{A} \in \mathcal{A}_{fl}}$, where $\partial-\eta$ is as in equation (69).

(Progress this family is $T$-equivariant because $\partial_{-\eta}$ is $T$-invariant, the adjoint action of $T$ on $\mathfrak{g} = (\eta_0, ..., \eta_b) \in \prod \subset \mathfrak{g}$ being trivial.) We will deduce the result from a families version of the formula for the change of index under a change of boundary conditions (cf. [14, Corollary 8.8, Theorem 8.15] for the standard version).

Let $P_1$ (resp. $P_2, P_0$) denote $L^2$-orthogonal projection onto $B_1 = B_{\geq 0}(\partial)$ (resp. $B_2 = B_{\geq 0}(\partial-\eta)$, $B_0 = B_1 + B_2$). For $i = 0, 1, 2$ consider the families $\mathcal{D}_i$ of operators over $\mathcal{A}_{fl}$:

$$(\mathcal{P}_A^{\Sigma^+}, P_i \circ \mathcal{R}) : L^2(\Sigma, S^{\Sigma^+, \mathbb{C}} V) \to L^2(\Sigma, S^{\Sigma^-, \mathbb{C}} V) \oplus B_i.$$ 

Upon descending to $M$, the families $\mathcal{D}_1, \mathcal{D}_2$ represent $E^\Sigma V$, $\eta \cdot E^\Sigma V$ respectively.

By construction

$$\mathcal{D}_i = (\text{id}, p_i) \circ \mathcal{D}_0$$

holds for $i = 1, 2$, where $p_i = P_i|_{B_0}$ and $\circ$ denotes (pointwise) composition of families of operators over $\mathcal{A}_{fl}$. By additivity of the index for compositions (families version) we obtain the following equations in $RK^\mathfrak{g}_0(M)$,

$$E^\Sigma V = [\mathcal{D}_0] + [p_1], \quad \eta \cdot E^\Sigma V = [\mathcal{D}_0] + [p_2]$$

hence

$$\eta \cdot E^\Sigma V = E^\Sigma V + [p_2] - [p_1].$$

But $[p_1] = [B_1 \cap B_2^\perp]$, $[p_2] = [B_2 \cap B_1^\perp]$, and these subspaces are easy to determine. Consider $B_2 \cap B_1^\perp = B_{\geq 0}(\partial-\eta)^\perp \cap B_{<0}(\partial)$. Let $\lambda \in \Lambda$. Restricted to sections of $V[\lambda]$ over the $j$-th component of $\partial \Sigma$, the operator $\partial_{-\eta}$ simplifies to $\partial - \langle \lambda, \eta_j \rangle$, hence for the $\lambda$-isotypical component

$$\dim (B_{\geq 0}(\partial - \langle \lambda, \eta_j \rangle)^\perp \cap B_{<0}(\partial)[\lambda]) = \begin{cases} -\langle \lambda, \eta_j \rangle, & \text{if } -\langle \lambda, \eta_j \rangle > 0 \\ 0, & \text{else.} \end{cases}$$

One finds a similar result for $B_1 \cap B_2^\perp$, except with $-\langle \lambda, \eta_j \rangle$ replaced with $\langle \lambda, \eta_j \rangle$. Comparing with (73) gives the result. \qed

Remark 5.10. The action of $W_{aff}$ cannot change the virtual dimension of the K-theory class. Hence the virtual dimension of $\nabla_{\eta_j} Tr_V$ should be 0 for consistency. The virtual dimension is the sum over $j$ of $\nabla_{\eta_j} Tr_V$ evaluated at $1 \in G$, which is necessarily 0 by $G$-invariance of the gradient.

Remark 5.11. If $x \in X$ is a $T$-fixed point, then the pullback of $E^\Sigma V$ to $x$ has a $T$-equivariant virtual dimension: $\text{index}_T(D_x) \in R(T)$. Proposition 5.9 shows that

$$\text{index}_T(D_{px}) = \text{index}_T(D_x) + \nabla_{px} Tr_V.$$
Thus the $T$-equivariant virtual dimension is a $\prod$-quasiperiodic function $X^T \to R(T)$. In particular this implies that the dimension of the kernel of the operators in the family $D_\bullet$ is unbounded. It also rules out the possibility that the class $E^2V \in RK^2_T(X)$ is pulled back from any $T$-equivariant compactification of $X$, because in the latter case the $T$-equivariant index of $D_x$ would be bounded as $x$ ranges over the fixed point set $X^T$.

5.6. Chern character forms. In this section we introduce another description of the class $E^2V$ based on capping off the boundary of $\Sigma$ and use it to construct a Chern character form with good properties, in view of applying Theorem 4.10. As a side comment, one might also hope that a Chern character form could be constructed using the local families Atiyah-Patodi-Singer index theorems of Bismut-Cheeger and Melrose-Piazza.

For the purpose of this section, it is convenient to work with the presentation $\mathcal{M} = A_{fl,\mathcal{C}}/\mathcal{G}_C$ mentioned briefly in Section 2.3 and reviewed here; this presentation leads, along the same lines as Section 5.2, to a canonically isomorphic tautological family of Dirac operators parametrized by $\mathcal{M}$, hence nothing is lost in passing between presentations.

Let $\mathcal{C} \simeq \partial \Sigma \times [1, 2]$ denote a collar neighborhood of the boundary, where the boundary sits at $r = 1$ in terms of the coordinate $r \in [1, 2]$. The boundary $\partial \Sigma$ has $b + 1 \geq 1$ parametrized components, denoted $\partial_0 \Sigma, \ldots, \partial_b \Sigma$. Let $\mathcal{C}_j \simeq \partial_j \Sigma \times [1, 2]$ denote the $j$-th component of the collar. Let $r_j = r|_{\mathcal{C}_j}$. Let $s_j : \partial_j \Sigma \to S^1 = \mathbb{R}/\mathbb{Z}$ denote the opposite parameterization of $\partial_j \Sigma$ as in the proof of Proposition 5.9 (hence this parametrization is consistent with the orientation on $\partial_j \Sigma$ induced from the interior). The pullback of $A \in A_{fl}$ to $\partial_j \Sigma$ is

$$A_j(s_j)ds_j,$$  \hspace{1cm} (74)

where $A_j \in \Omega_{b-1}^0(S^1, g)$.

The subspace $A_{fl,\mathcal{C}} \subset A_{fl}$ denotes flat connections $A$ on $\Sigma$ such that for $j = 0, \ldots, b$,

$$A|_{\mathcal{C}_j} = A_j(s_j)ds_j.$$ \hspace{1cm} (75)

In other words $A|_{\mathcal{C}_j}$ has no component normal to $\partial_j \Sigma$ and the coefficient of $ds_j$ only depends on $s_j$. The subgroup $\mathcal{G}_C \subset \mathcal{G}_\partial \Sigma$ denotes gauge transformations $g$ such that $g|_{\mathcal{C}} = 1$. As mentioned in Section 2.3, the loop group space $\mathcal{M}$ can be presented as the quotient $A_{fl,\mathcal{C}}/\mathcal{G}_C$.

Let $\mathbb{D}$ denote the unit disk in $\mathbb{R}^2$. Let $\mathbb{D}_0, \ldots, \mathbb{D}_b$ denote $b + 1$ copies of $\mathbb{D}$, with polar coordinates $(r_j, s_j) \in [0, 1] \times \mathbb{R}/\mathbb{Z}$ on the $j$-th copy. Define the closed surface $\Sigma$ by capping off the boundary:

$$\Sigma = \Sigma \cup_{\partial \Sigma} \bigcup_{j=0}^{b} \mathbb{D}_j.$$  \hspace{1cm} (76)

The coordinates $r_j, s_j$ are thus defined on the subset $\mathcal{C}_j \cup_{\partial_j \Sigma} \mathbb{D}_j$ of $\Sigma$.

Elements of $A_{fl,\mathcal{C}}$ can be extended to (non-flat) connections on $\Sigma$, such that smooth connections are extended to smooth connections, as follows. Fix a bump function $\chi(r)$ equal to 1 for $r > \frac{1}{2}$ and equal to 0 for $r < \frac{1}{2}$. For $A \in A_{fl,\mathcal{C}}$, let $\mathcal{A}$ be the connection obtained by extending $A$ constantly over each punctured disk $\mathbb{D}_j\setminus\{0\}$ and multiplying by $\chi(r_j)$ in order that the product extends by 0 to $\mathbb{D}_j$:

$$\mathcal{A}|_{\mathcal{C}_j \cup \mathbb{D}_j}(r_j, s_j) = A_j(s_j)\chi(r_j)ds_j, \quad r_j \in [0, 2], \quad s_j \in \partial_j \Sigma \simeq S^1.$$  \hspace{1cm} (77)
Let $G_{lc,C} \subset G$ be the subgroup consisting of gauge transformations that are locally constant on $C$. Elements of $G_{lc,C}$ have a canonical extension to $\Sigma$ (constant on each of the caps). In particular, elements of $G_C \subset G_{lc,C}$ are extended to $\Sigma$ by the identity.

Extend the Riemannian metric and spinor module to $\Sigma$. Let $D^\Sigma_h$ denote the Dirac operator acting in $L^2(\Sigma, S^2 \mathbb{R} V)$ constructed using the connection $\Lambda$. The family $D^\Sigma_{h} = (D^\Sigma_{h})_{\Lambda \in A_{fl,C}}$ is $G_{lc,C}$-equivariant, hence descending to $M$ we obtain a $G_{lc,C}/G_C \simeq G$-equivariant family of Fredholm operators $D^\Sigma_{h}$ over $M$.

**Proposition 5.12.** The class in $\mathcal{R}K_G^0(\Sigma)$ defined by the family $D^\Sigma_{h}$ is $E^\Sigma V$.

**Proof.** This is a consequence of a families version of the splitting theorem (cf. [14, Theorem 8.17], and see for example [26] for a families version which is more sophisticated than is really needed here): the $K$-theory class defined by the family of Dirac operators $D^\Sigma_{h}$ is the sum of the class $E^\Sigma V$ defined by the family of boundary problems $(D^\Sigma_{A_{lc}}, B_{0}(\partial))$, and the classes defined by the family of boundary problems $(D^\Sigma_{A_{b}}, B_{0}(\partial))$, $j=0,\ldots,b$. Each of the latter boundary problems on the disk is the pullback under the moment map of a tautological family of boundary problems parametrized by $L_{G}^*$. But $\mathcal{R}K_G^0(L_{G}^*) = R(G)$ because $L_{G}^*$ is $G$-equivariantly contractible. To determine which element of $R(G)$, it suffices to compute the index of a single operator in the family, which is 0 (see for example the discussion at the end of Section 5.2). □

Thus using the family $D^\Sigma_{h}$ leads to the same class $E^\Sigma V \in \mathcal{R}K_G^0(\Sigma)$. The loop group space $M$ has a finite dimensional (singular) symplectic quotient (or GIT moduli space) $M_g^1(0)/G$, and Proposition 5.12 also shows that the class indexed by $E^\Sigma V$ on this finite dimensional moduli space coincides with the corresponding classes introduced by Atiyah and Bott [8].

Recall that we defined (Definition 5.1) an $L_{G}$-equivariant vector bundle $E V \to M \times \Sigma$.

**Definition 5.13.** We define a vector bundle $E V \to M \times \Sigma$ extending $E V$ as the quotient

$$
E V = (A_{fl,C} \times \Sigma \times V)/G_C \to M \times \Sigma.
$$

Unlike $E V$, the vector bundle $E V$ is only $G_{lc,C}/G_C = G$-equivariant. We will use the same symbol $E V$ (resp. $E V$) to denote the $N(\Sigma)$-equivariant (resp. $N(\Sigma) \times \Pi$-equivariant) vector bundle obtained by restricting $E V$ to $X \times \Sigma$ (resp. restricting $E V$ to $X \times \Sigma$).

The restriction of $E V$ to $X \times (C \cup D_0 \cup \cdots \cup D_b)$ is canonically trivial since $G_C$ acts trivially over this subset of $\Sigma$. Let $\nabla^E V$ be a $N(\Sigma) \times \Pi$-invariant connection on the vector bundle $E V = E V|_{X \times \Sigma}$. Over the collar and relative to the trivialization, $\nabla^E V|_{X \times C} = d + \alpha_j$ for some connection $1$-form $\alpha_j$ on $X \times C_j$. The trivialization of $E V|_{X \times C_j}$ is not compatible with the $\Pi$-action: under the action of $\Sigma = (\eta_0,\ldots,\eta_b) \in \Pi$, the de Rham differential $d = d_X + d_{r_j} + d_{s_j}$ on the collar transforms as

$$
\exp(s_j \eta_j)(d_X + d_{r_j} + d_{s_j}) \exp(-s_j \eta_j) = d_X + d_{r_j} + d_{s_j} - \eta_j.
$$

Therefore to obtain a $\Pi$-invariant connection, the $1$-form $\alpha_j$ cannot be 0. Perhaps the simplest choice is to take $\alpha_j$ of the form

$$
\alpha_j(x, r_j, s_j) = \mu_j(x)f(s_j)ds_j,
$$

where $\mu_j(x) = \mu_j(x_0)$ is a smooth function on $X$.
where \( f(s_j) \) is any smooth function on the circle with total integral 1 (we could choose \( f = 1 \) but the small additional flexibility will be convenient later on). Since \( \mu_j(\eta \cdot x) = \mu_j(x) + \eta_j \),
equation (78) shows that the connection \( d + \alpha_j \) on \( EV|_{X \times C_j} \) is \( \Pi \)-invariant.

Let \( \nabla^{EV} \) be the \( N(\Gamma) \)-invariant connection on \( EV \) extending \( \nabla^{EV} \) given by \( d + \chi(r_j)\alpha_j \) over \( X \times (C_j \cup D_j) \), where from now on we assume \( \alpha_j \) is as in (79). The Chern-Weil construction yields equivariant Chern character forms \( \text{Ch}^\ell(E^jV, \xi) \in \Omega(X^j \times \Sigma), \xi \in \Gamma, \xi \in \Pi \). Define \( \text{Ch}^\ell(E^jV, \xi) \in \Omega(X^j) \) by integrating over the fibres:

\[
\text{Ch}^\ell(E^jV, \xi) = \int_{\Sigma} \text{Ch}^\ell(EV, \xi).
\]

**Proposition 5.14.** The differential forms \( \text{Ch}^\ell(E^jV, \xi) \in \Omega(X^j) \) defined in (80) satisfy conditions (a)–(d) in Definition 4.1. The contribution of the caps \( D_0, \ldots, D_b \) to \( \text{Ch}^\ell(E^jV, \xi) \) is the 0-form

\[
\sum_{j=0}^{b} \nabla_{\mu_j} \text{Tr}_V(t_j \exp(\xi_j))
\]

where \( t = (t_0, \ldots, t_b) \), \( \xi = (\xi_0, \ldots, \xi_b) \) and \( \mu = (\mu_0, \ldots, \mu_b) \). Under the action of \( \eta \in \Pi \),

\[
\eta \cdot \text{Ch}^\ell(E^jV, \xi) = \text{Ch}^\ell(E^jV, \xi) - \nabla_{\eta} \text{Tr}_V(t_j \exp(\xi))
\]

**Proof.** The bouquet property (a) follows from the Chern-Weil construction. Property (b) is a consequence of the equivariant Atiyah-Singer families index theorem over a compact base \([12, 16, \text{Chapters 8, 10}] \) (since each \( U^j \) is relatively compact and \( \hat{A}(\Sigma) = 1 \)) and Proposition 5.12.

The fibre integral over \( \Sigma \) can be split into integrals over \( \Sigma \), and \( D_0, \ldots, D_b \). Since \( \nabla^{EV} \restriction X \times \Sigma \) is \( \Pi \)-invariant, the contribution of the integral over \( \Sigma \) is \( \Pi \)-invariant hence easily satisfies properties (c) and (d). On the other hand, over \( X \times D_j \) we have an explicit formula for the connection 1-form \( (\chi(r_j)\mu_j(x)f(s_j)ds_j) \), with curvature

\[
F_j(x, r_j, s_j) = \chi'(r_j)\mu_j(x)f(s_j)dr_jds_j + \chi(r_j)f(s_j)\mu_j(x)ds_j.
\]

At the point \((x, r_j, s_j) \in X^j \times D_j\), the form \( \text{Ch}^\ell(EV, \xi) \) is \( \text{Tr}_V(t_j \exp(\xi_j)) \exp(\frac{i}{2\pi} F_j(x, r_j, s_j)) \).

Integrating over \( D_j \), the term involving \( d\mu_j(x) \) does not contribute since it does not contain a factor of \( dr_j \), while the integral of \( \chi'(r_j)\mu_j(x)f(s_j)dr_jds_j \) is \( -\mu_j(x) \). Hence the contribution of \( D_j \) to the integral over the fibre at \( x \in X^j \) is

\[
- \text{Tr}_V(t_j \exp(\xi_j) \frac{i}{2\pi} \mu_j(x)) = \nabla_{\mu_j(x)} \text{Tr}_V(t_j \exp(\xi_j)).
\]

Properties (c) and (d) are consequences of this expression: the t-Fourier transform of (83) (i.e. the Fourier transform in the variable \( \xi_j \)) is contained in \( K \), where \( K \subset \eta^* \) is any compact set that contains the weights of the finite dimensional \( T \)-representation \( V \), and since \( \ell > 0 \) there is an estimate \( |\mu_j(x)| \leq C(1 + |x|^2)\ell/2 \) so that the constant \( m' \) can be taken to equal 1. Equation (82) is an immediate consequence of (83) and \( \mu_j(\eta \cdot x) = \mu_j(x) + \eta_j \), for \( \eta = (\eta_0, \ldots, \eta_b) \in \Pi \). □

**Corollary 5.15.** The class \( E^jV = \exp(zE^jV) \in \mathcal{C} \mathcal{R} \mathcal{K}^R_0(\mathcal{X}(\Sigma)) \) admits \( \nabla^{TV} \)-twisted Chern character forms \( \text{Ch}^\ell(E^jV, \xi) = \exp(z\text{Ch}^\ell(E^jV, \xi)) \).

**Proof.** This follows from (82). □
5.7. Application of Theorem 4.10. By Corollary 5.15, we may apply Theorem 4.10 to $E_z = \exp(z E^2 V)$. In this case $\chi_z = \nabla \Tr_V$ does not depend on $z$, is $W$-invariant, and $d\chi_z = H_V$ where $H_V$ is the Hessian of the character $\Tr_V$. By Theorem 4.14, the index has support contained in $(T\ell)^{\reg}$. By equation (66), $[T\ell]^{-1} z^{\ell} \chi_z \cdot \delta \cdot \partial \cdot \chi_z \cdot \delta \cdot \partial \cdot \chi_z \cdot \delta \cdot \partial$ is $z^{\ell} \chi_z \cdot \delta \cdot \partial \cdot \chi_z \cdot \delta \cdot \partial$. Under the isomorphism $M = M/\Omega G$, $M$ being the quasi-Hamiltonian $G$-space (or holonomy manifold) associated to $M$. We assumed the boundary components of $\Sigma$ are parametrized, so let $p_j \in \partial \Sigma$ be the point corresponding to $1 \in S^1$ for each $j = 0, 1, \ldots, b$. Let $\alpha_1, \beta_1, \ldots, \alpha_b, \beta_b$ be a standard set of generators for $\pi_1(\Sigma, p_0)$ with images contained in $\Sigma; \gamma_1, \ldots, \gamma_b$ a set of paths connecting $p_0$ to each of $p_1, \ldots, p_b$; and $\delta_1 = \delta_1 \Sigma, \ldots, \delta_b = \delta_b \Sigma$ the boundary circles. We may arrange that all of these curves are smooth, simple and disjoint except for intersections of the $\alpha_i, \beta_i, \gamma_j, \delta_j$ at $p_0$ and the intersections $\gamma_j \cup \delta_j = \{p_j\}$. The set $\{\alpha_i, \beta_i, \gamma_j, \delta_j\}$ generates the fundamental groupoid $\Pi_1(\Sigma, J)$ of paths in $\Sigma$ with endpoints contained in $\Sigma = \{p_0, \ldots, p_b\}$.

Let $\mathcal{G}_J \subset \mathcal{G}$ denote gauge transformations equal to the identity over the subset $J$. The quasi-Hamiltonian space $M = M/\Omega G = A_M/\mathcal{G}_J \simeq \text{Hom}(\Pi_1(\Sigma, J), G)$. The system of generators $\{\alpha_i, \beta_i, \gamma_j, \delta_j\}$ determines a diffeomorphism $h: M \cong G^{2(g+b)}$ given by

$$[A] \mapsto h([A]) = (h_{\alpha_1}(A), h_{\beta_1}(A), \ldots, h_{\alpha_b}(A), h_{\beta_b}(A), h_{\gamma_1}(A), h_{\delta_1}(A), \ldots, h_{\gamma_b}(A), h_{\delta_b}(A)) \quad (85)$$

where $h_{\nu}(A)$ is the holonomy of the connection $A$ along the path $\nu$. The action of $A = (g_0, g_1, \ldots, g_b) \in \mathcal{G}$ is $[1, \text{Theorem 9.3}],$

$$h_{\alpha_i} \mapsto g_0 h_{\alpha_i} g_0^{-1}, \quad h_{\beta_i} \mapsto g_0 h_{\beta_i} g_0^{-1}, \quad h_{\gamma_j} \mapsto g_0 h_{\gamma_j} g_0^{-1}, \quad h_{\delta_j} \mapsto g_0 h_{\delta_j} g_0^{-1}.$$ It is convenient to identify $G$ with the diagonal in $\mathcal{G}$. For $g \in T_{\ell}^{\reg} \cap \exp(\mathcal{A})$, one verifies easily that $M^g$ is empty unless $g_0 = g_1 = \cdots = g_b = g$, and in this case $M^g = M^g = M^g$ does not depend on the choice of $g \in T_{\ell}^{\reg}$ by regularity. Since this space will appear frequently below, we make the following definition.

**Definition 5.16.** Let $F = M^T$ be the fixed-point set of $T$ in $M$ (also the fixed point set of any regular element in $T$). Equivalently $F = \text{Hom}(\Pi_1(\Sigma, J), T)$ is the moduli space of $T$-connections with framing over $J$. Under the isomorphism $h: M \cong G^{2(g+b)}$, $F$ is identified with the torus $T^{2(g+b)}$.

Equation (84) becomes

$$\frac{1}{|T\ell|^{b+1}} \sum_{g \in T_{\ell}^{\reg} \cap W} \frac{\Delta(g_z)^{b+1} \delta \cdot \partial \cdot \Delta(g_z)^{b+1} \delta \cdot \partial}{\det(1 + z \ell^{-1} H_V(g_z))^{b+1}} \int_F \exp \left( z \text{Ch}_{g_z}(E^2 V) \right) A S^{g_z}(\sigma), \quad (86)$$

where $g_z^\ell \exp(z \nabla \Tr_V(g_z)) = g_z^\ell$. 


In terms of the non-compact manifold $X$, $F \simeq X^T/\Pi$ and $X^T$ can be identified with the fibre product

$$X^T = \mathfrak{t} \times_T F = \mathfrak{t} \times_T \text{Hom}(\Pi_1(\Sigma, J), T),$$

with the moment map $\mu$ being the first projection and the map to $T$ being the map that associates to an equivalence class of a connection its list of holonomies around the boundary circles. For an abelian flat connection on $\Sigma$, the product of the holonomies around the boundary circles is 1, and it follows that the sum

$$\bar{\mu} := \mu_0 + \mu_1 + \cdots + \mu_b : X \to t$$

takes values in $\Pi$. In fact the connected components of $X^T$ are the fibres $\bar{\mu}^{-1}(\eta)$ as $\eta$ ranges over the lattice $\Pi$.

On the fixed point set $X^T$, the Atiyah-Segal-Singer integrand $\mathcal{A}\Sigma g_\mu(\sigma)$ simplifies to

$$\frac{i^{-(b+1)n_+}\Delta(g_z)^{b+1}}{\Delta(g_z)^{2g+b}}g_z^{\ell \bar{\mu}}. \quad (88)$$

The main ingredients involved in the calculation (88) (we refer the reader to [2] for certain details) are (i) $\bar{\gamma}(F) = 1$; (ii) the fixed-point set $X^T$ is contained in $\nu^{-1}(0)$, so the Chern character $\text{Ch}_g^\eta(\mathbb{B})$ of the Bott-Thom element may be replaced by its pullback to $0 \in (t^\perp)^{b+1}$ which is $i^{-(b+1)n_+}\Delta(g_z)^{b+1}$; (iii) the normal bundle to the fixed-point manifold is the trivial bundle with fibre $(g/\ell)^{2g+b}$ and contributes a factor $(-1)^{(g+b)n_+}\Delta(g_z)^{2g+b}$ in the denominator; (iv) over the component $\bar{\mu}^{-1}(0) \subset X^T$ there is an additional phase factor of $(-1)^{(g+b)n_+}$, a square root of the action of $g$ on the fibres of the anticanonical line bundle $\mathcal{L}|_{X^T}$ of $\mathcal{L}|_{X^T}$ ([2, Proposition 5.3]), and this phase factor transforms at level $\ell$ with respect to the diagonal action of $\Pi$ (notice as well that $g^{\ell \bar{\mu}} = 1$ because $g \in T_\ell$ and $\bar{\mu} \in \Pi$; hence $g_z^{\ell \bar{\mu}} = (g_z^{+\bar{\mu}})$. With these simplifications (86) for $|T_\ell|^{-1}\sum \sum \det(1 + z\Delta(g_z)\delta_{g_z})^{b+1}$ becomes

$$\frac{1}{|T_\ell|^{b+1}} \sum \sum \Delta(g_z)^{2(1-g)}\delta_{g_z} \int_{\mathcal{L}} g_z^{\ell \bar{\mu}} \exp \left( \frac{1}{2} c_1(\mathcal{L}) + z \text{Ch}_g^\eta(\mathbb{E}^2 V) \right), \quad (89)$$

where $g_z^{\ell \bar{\mu}} \exp(z\nabla \text{Tr}_V(g_z)) = g^{\ell}$.

The integrand involves quantities defined on the covering space $X^T$ of $F = X^T/\Pi$, but as proved in Section 4.5, the integrand is $\Pi$-invariant, so descends to the quotient. We evaluate the integral over $F$ in the next section.

5.8. **The Teleman-Woodward formula for $\mathbb{E}^2 V$.** In Section 5.6 we defined the equivariant Chern character form of $\mathbb{E}^2 V$ by integration over the fibres $X \times \Sigma \to X$ of an equivariant Chern character form for $\mathbb{E}V$. We calculated the contribution from the caps $D_0, ..., D_b$ in (81); when restricted to the diagonal in $T$ (as in (89)), it simplifies to

$$\nabla_{\bar{\mu}} \text{Tr}_V(t \exp(\xi)).$$

Hence the product

$$g_z^{\ell \bar{\mu}} \exp(z\nabla_{\bar{\mu}} \text{Tr}_V(g_z)) = g^{\ell \bar{\mu}} = 1$$

since $g_z^{\ell} \exp(z\nabla \text{Tr}_V(g_z)) = g^{\ell}$, $\bar{\mu} \in \Pi$ and $g \in T_\ell$. This cancellation verifies for this special case the $\Pi$-invariance of the integrand of (89) that we established more generally in Section 4.5.
What remains is the $\Pi$-invariant contribution from the integral of the Chern character form for the vector bundle $EV|_{X^T \times \Sigma} = EV|_{X^T \times \Sigma}$ over the fibres $X^T \times \Sigma \to X$. Since $EV$ is $\Pi$-equivariant, we can work with the vector bundle

$$EV = EV|_{X^T \times \Sigma}/\Pi$$

(90)

over the compact manifold with boundary $F \times \Sigma = (X^T/\Pi) \times \Sigma$.

In the discussion below, homology/cohomology groups are taken with $\mathbb{C}$ coefficients. Recall $J = \{p_0, ..., p_b\}$ and note that $\text{Hom}(\Pi_1(\Sigma, J), T) \simeq \text{Hom}(\pi_1(\Sigma/J, J), T)$. Let $(\Sigma/J)^\sim$ be the universal covering space of $\Sigma/J$. Given a character $\lambda \in \Lambda = \text{Hom}(T, U(1))$, let $h^\lambda : F \to \text{Hom}(\pi_1(\Sigma/J, J), U(1))$ be the composition of $\lambda$ with the map $F \xrightarrow{\sim} \text{Hom}(\pi_1(\Sigma/J, J), T)$. Define a complex line bundle

$$EC_\lambda = (F \times (\Sigma/J)^\sim \times \mathbb{C})/\pi_1(\Sigma/J, J) \to F \times \Sigma/J$$

where $\varepsilon \in \pi_1(\Sigma/J, J)$ acts on the product by

$$\varepsilon \cdot (y, p, z) = (y, \varepsilon \cdot p, h^\lambda(y, \varepsilon) \cdot z).$$

Let $\{\varepsilon_s\}_{s \in S}$ be a set of integral generators of $H_1(\Sigma/J)$ and let $e^s$ be the dual basis of $H^1(\Sigma/J)$ with respect to the canonical pairing. Pairing $h^\lambda$ with $\varepsilon_s$ yields a map $\langle h^\lambda, \varepsilon_s \rangle : F \to U(1)$ and we let $e^\lambda_s \in H^1(F)$ be the pullback of the integral generator of $H^1(U(1))$ under this map. The 1-st Chern class of the line bundle $EC_\lambda$ is

$$c_1(EC_\lambda) = \sum_s e^\lambda_s e^s \in H^1(F) \otimes H^1(\Sigma/J) \subset H^2(F \times \Sigma/J).$$

(91)

We may take the set of generators $\{\varepsilon_s\}_{s \in S}$ of $\pi_1(\Sigma/J, J)$ to be the images of the elements $\{\alpha_i, \beta_i, \gamma_j, \delta_j\}$ introduced in the previous section under the quotient map $\Sigma \to \Sigma/J$. The set $S$ thus contains $2(g + b)$ elements. Let $a^i, b^i, c^j, d^j \in H^1(\Sigma/J)$ be the corresponding cohomology classes. To obtain differential form representatives, recall that $H^1(\Sigma/J) \simeq H^1(\Sigma, J)$, so the relevant de Rham cohomology group is the relative group

$$H^1_{dR}(\Sigma, J) = \Omega^1_{dR}(\Sigma)/\{df | f \in \Omega^0(\Sigma), \ f|_J \text{ is constant}\}.$$

Represent the classes $a^i, b^j$ by Thom forms supported in tubular neighborhoods of smooth 1-cycles contained in $\Sigma \setminus \mathcal{C}$ which are homotopic to $\beta_i, \alpha_i$ respectively. Represent the classes $c^j, d^j$ by Thom forms supported in tubular neighborhoods of the curves $\delta_j, \gamma_j$ respectively. We will abuse notation and use the same symbols $e^s = a^i, b^j, c^j, d^j$ for the differential form representatives. Using the pullback of the standard 1-form on $U(1)$, we also obtain differential form representatives for the $e^\lambda_s = a^\lambda_i, b^\lambda_i, c^\lambda_j, d^\lambda_j$, and hence a 1-st Chern form given by the same formula (91) and denoted by the same symbol $c_1(EC_\lambda)$.

The corresponding Chern character form is

$$\text{Ch}^1(EC_\lambda) = t^\lambda e^{c_1(EC_\lambda)} \in \Omega(X^T/\Pi).$$

(92)

If

$$V = \bigoplus_\lambda n_\lambda EC_\lambda$$

is a representation of $T$ then

$$EV = \bigoplus_\lambda n_\lambda EC_\lambda$$
identifies with the vector bundle $EV|_{X^T \times \Sigma / \Pi}$ in (90). Let
\[ \text{Ch}^t(EV) = \sum_{\lambda} n_{\lambda} \text{Ch}^t(E\mathcal{C}_\lambda). \]  
This is a differential form representative of the $T$-equivariant Chern class of $EV$.

Shortly we will use (93) to calculate the contribution of $\Sigma \subset \Sigma$ to the Chern character form of $E\Sigma V$ and the integral in (89). Justification that (93) can be used for this purpose is called for because $F \times \Sigma$ has non-empty boundary (having the correct de Rham cohomology class is not quite enough): based on the construction of Section 5.6, it suffices to show that there is a $T \times \Pi$-invariant connection on the pullback $\pi^*E\mathcal{C}_\lambda$ of $E\mathcal{C}_\lambda$ under the quotient map $\pi: X^T \times \Sigma \to (X^T / \Pi) \times \Sigma = F \times \Sigma$ that takes the form (relative to the trivialization)
\[ d + \langle \alpha_j(x, r_j, s_j), \lambda \rangle = d + \langle \mu_j(x), \lambda \rangle f(s_j) ds_j, \quad \text{where} \quad \int_{S^1} f(s_j) ds_j = 1 \]  
over the collar $X^T \times \mathcal{C}_j$, and such that the corresponding Chern-Weil form is the pullback $\pi^*c_1(E\mathcal{C}_\lambda)$ of the form constructed above. This follows from the following simple observations:

(a) For $j = 1, \ldots, b$, we chose the form $d^j$ to be a Thom form supported in a tubular neighborhood of $\gamma_j$ (a smooth curve connecting $p_0 \in \partial_0 \Sigma$ to $p_j \in \partial_j \Sigma$). We may choose $d^j$ to take a product form on $\mathcal{C}_j$, and then $d^j|_{\mathcal{C}_j} = f(s_j) ds_j$ for a suitable smooth function $f(s_j)$ whose integral over $S^1$ is 1.

(b) The exponential $\exp(\mu_j(x)) \in T$ is the holonomy of the $\mathcal{G}_{\mathcal{C}_j}$-equivalence class of connections $x = [A]$ around the boundary circle $\partial_j \Sigma$. Hence for $j = 1, \ldots, b$ we have
\[ d(\mu_j, \lambda) = \pi^*c_1, \quad \text{and so} \]
\[ d(\alpha_j, \lambda) = \pi^*c_1^\lambda d^j|_{X^T \times \mathcal{C}_j}. \]

(c) We may choose $d^j$ to take a product form on $\mathcal{C}_0$ as well (near the other end of $\gamma_j$), say $-f(s_0) ds_0$ (the minus sign appears because the orientations of the boundary components $\partial_j \Sigma$ is induced from that on $\Sigma$). Then
\[ \sum_{j=1}^b \pi^*c_1^\lambda d^j|_{X^T \times \mathcal{C}_0} = - \sum_{j=1}^b d(\mu_j, \lambda) f(s_0) ds_0 \]
but $\sum_{j=0}^b \mu_j$ is locally constant on $X^T$ as we explained above. Thus
\[ \sum_{j=1}^b \pi^*c_1^\lambda d^j|_{X^T \times \mathcal{C}_0} = d(\mu_0, \lambda) f(s_0) ds_0. \]

(d) The three items above show that $\sum_j \pi^*e^*|_{X^T \times \mathcal{C}_j} = d(\alpha_j, \lambda)$ where $\alpha_j$ is as in (94). It follows from the standard construction of a prequantum connection for a closed integral 2-form that there is a $T \times \Pi$-invariant connection on $\pi^*E\mathcal{C}_\lambda \to X^T \times \Sigma$ with 1-st Chern form $\sum_j \pi^*e^*|_{\mathcal{C}_j}$. Recall that the construction involves choosing trivializations and primitives over the open subsets of a good cover, thus choosing the specific primitives $\alpha_j$ over $X^T \times \mathcal{C}_j$ ensures that the prequantum connection matches (94) on $X^T \times \mathcal{C}_j$. 

Proceeding to the calculation, the contribution of $\Sigma$ to $\operatorname{Ch}^i(EV)$ is the pullback under $\pi: X^T \to X^T/\Pi = F$ of
\[
\int_{\Sigma} \operatorname{Ch}^i(EV) = \sum_{\lambda} n_\lambda t^\lambda \int_{\Sigma} \exp \left( \sum_s e_s^\lambda e^s \right).
\] (95)
Since $\dim(\Sigma) = 2$ and the $e_s^\lambda$ are pullbacks of forms on $X^T$, the only term that contributes in the integral over $\Sigma$ comes from the quadratic term in the exponential. Thus
\[
\int_{\Sigma} \operatorname{Ch}^i(EV) = \sum_{\lambda} n_\lambda t^\lambda \frac{1}{2} \int_{\Sigma} \left( \sum_s e_s^\lambda e^s \right)^2.
\]
From our description of the forms, it is clear that
\[
\int_{\Sigma} a_i^b i = 1 = -\int_{\Sigma} b_i^a i, \quad \int_{\Sigma} c_i^j d_j = 1 = -\int_{\Sigma} d_i^c j,
\]
while all other intersection pairings are trivial. Thus (95) yields
\[
\int_{\Sigma} \operatorname{Ch}^i(EV) = \sum_{\lambda} n_\lambda t^\lambda \left( \sum_i a_i^p b_i^q + \sum_j c_j^p d_j^q \right).
\] (96)
Let $\nu_1, \ldots, \nu_{\dim(\Sigma)}$ be a lattice basis of $\Lambda$. Equation (96) is bilinear in $\lambda$, and therefore if we decompose $\lambda = \sum \lambda_p \nu_p$ in terms of the basis, then
\[
\int_{\Sigma} \operatorname{Ch}^i(EV) = \sum_{\lambda} \sum_{p,q} n_\lambda t^\lambda \lambda_p \lambda_q \left( \sum_i a_i^p b_i^q + \sum_j c_j^p d_j^q \right),
\]
where $a_i^p = a_i^\lambda_p$ and likewise for the other forms. The sum
\[
\sum_{\lambda} n_\lambda t^\lambda \lambda_p \lambda_q = H_V(t)_{pq}
\]
is the $(p, q)$-entry of the Hessian matrix of $\operatorname{Tr}_V$ at the point $t$ relative to the basis. Therefore (96) may be written
\[
\int_{\Sigma} \operatorname{Ch}^i(EV) = \sum_{p,q} H_V(t)_{pq} \left( \sum_i a_i^p b_i^q + \sum_j c_j^p d_j^q \right).
\]
The spinor bundle is at level $\ell$, and one shows (cf. [56, Corollary 3.12], [2, Proposition 5.2])
\[
\frac{1}{2} c_1(\mathcal{L}) = \ell \sum_p \left( \sum_i a_i^p b_i^p + \sum_j c_j^p d_j^p \right)
\]
in $H^2(F)$. Thus the integral from (89) that we need to compute is
\[
\int_F \exp \left( \ell \sum_{p,q} (\delta_{pq} + z^{-1} H_V(t)_{pq}) \left( \sum_i a_i^p b_i^q + \sum_j c_j^p d_j^q \right) \right).
\] (97)
The product
\[
\prod_{p,i} a_i^p b_i^p \prod_{p,j} c_j^p d_j^p
\]
is a top degree form on the \( 2(g + b) \text{dim}(T) \)-dimensional torus \( F \simeq T^{2(g+b)} \), whose integral is \( |T_1| \) (cf. [2, Proposition 5.2], [22]). Therefore a short calculation with (97) leads to
\[
\det(1 + \ell^{-1} H_V(t))^{g+b}|T_1|^{g+b}.
\]

Substituting the result of these calculations into (89) gives the following.

**Theorem 5.17.** The index of the Atiyah-Bott class \( E^2V \) is determined by the generating series:
\[
\frac{1}{|T_1|} \text{index}_F^P(\exp(z E^2V)) = \sum_{g \in T_1^{reg}/W} \left( \frac{\Delta(g_z)^2}{|T_1| \det(1 + z \ell^{-1} H_V(g_z))} \right)^{1-g} \delta_{g_z} \in \mathcal{D}'(\mathcal{G}[\mathbb{Z}])
\]
where \( g_z^k \exp(z \nabla \text{Tr}_V(g_z)) = g^k \) and \( H_V \) is the Hessian of \( \text{Tr}_V \).

### 5.9. The Teleman-Woodward formula for general classes.

For completeness, in this section we briefly explain how to extend Theorem 5.17 to include the other more elementary Atiyah-Bott classes introduced in Section 5.1. For simplicity we restrict to the case of a product of two odd classes.

Let \( C_1, C_2 \) be smooth simple closed curves in \( \Sigma \), and let \( U_1, U_2 \) be representations of \( G \). We consider the product \( E^{C_1}U_1 \cdot E^{C_2}U_2 \in K^2(X) \), which is represented by a family of Dirac operators on the 2-torus \( C_1 \times C_2 \subset \Sigma \times \Sigma \). The index of \( \exp(z E^2V) \cdot E^{C_1}U_1 \cdot E^{C_2}U_2 \) is computed using the fixed-point formula as in (89). The only change is an additional factor of \( \text{Ch}^g_z(\exp(E^{C_1}U_1 \cdot E^{C_2}U_2)) \), and the latter Chern character can be expressed using the families index theorem. The calculations are similar to those carried out for \( E^2V \), and in fact much simpler, since the class \( E^{C_1}U_1 \cdot E^{C_2}U_2 \) is simply the pullback of a K-theory class on the quotient \( X/\Pi \).

Decompose \( U_1, U_2 \) into weight spaces:
\[
U_i = \bigoplus \lambda n_{i,\lambda} C_\lambda.
\]

For \( \lambda_1, \lambda_2 \in \Lambda \), and let \( E C_{\lambda_1} \to F \times \Sigma \) be the corresponding line bundles (using the same notation as in Section 5.8). Let \( EC_{\lambda_1, \lambda_2} \to F \times \Sigma(1) \times \Sigma(2) \) (subscripts (1), (2) denoting two copies of \( \Sigma \)) be the pullback of the exterior product \( EC_{\lambda_1} \otimes EC_{\lambda_2} \) to \( F \times \Sigma(1) \times \Sigma(2) \), where \( F \to F \times F \) is identified with the diagonal. The 1-st Chern class of this line bundle is
\[
c_1(EC_{\lambda_1, \lambda_2}) = \sum_s (\lambda_1^s e_s^{(1)} + \lambda_2^s e_s^{(2)}),
\]
and the Chern character
\[
\text{Ch}^t(EC_{\lambda_1, \lambda_2}) = t^{\lambda_1 + \lambda_2} \exp \left( \sum_s (\lambda_1^s e_s^{(1)} + \lambda_2^s e_s^{(2)}) \right).
\]

We then compute the slant product with \( C_1 \times C_2 \subset \Sigma(1) \times \Sigma(2) \). The only term in the exponential that contributes is the quadratic term. At the same time we sum over weight space decompositions of \( U_1, U_2 \), which yields:
\[
\text{Ch}^t(E^{C_1}U_1 \cdot E^{C_2}U_2) = -\sum_{\lambda_1, \lambda_2} n_{1,\lambda_1} n_{2,\lambda_2} t^{\lambda_1 + \lambda_2} \sum_s (\lambda_1^s e_s^{(1)} \lambda_2^s e_s^{(2)}), \quad (98)
\]
where the angled brackets denote pairings between cohomology and homology classes. To slightly simplify the formulas we will assume from now on that \( C_1 \) is dual to \( a^r \) and \( C_2 \) is dual to \( b^r \) for some \( 1 \leq r \leq g \). There is no essential loss of generality, because (98) only depends
on the homology classes of $C_1, C_2$ in $\Sigma$, each of which is a sum of classes dual to the $a^i, b^j, d^i$, and moreover we will see shortly that the contribution of any other combination $(a^r, b^r$ with $r \neq r'$; $a^r, a^{r'}; b^r, b^{r'}; a^r, d^r; b^r, d^{r'})$ vanishes (in particular this will provide a consistency check of Proposition 5.2). With this assumption

$$\text{Ch}^t(E^1U_1 \cdot E^2U_2) = - \sum_{\lambda_1, \lambda_2} n_{1, \lambda_1} n_{2, \lambda_2} t^{\lambda_1 + \lambda_2} a_{\lambda_1} b_{\lambda_2}.$$ 

Decompose $\lambda_1, \lambda_2$ in terms of a lattice basis and note that

$$\sum_{\lambda_1, \lambda_2} n_{1, \lambda_1} n_{2, \lambda_2} t^{\lambda_1 + \lambda_2} a_{\lambda_1} b_{\lambda_2}$$

is the $(p', q')$ component of $\nabla \text{Tr}U_1 \otimes \nabla \text{Tr}U_2$ at the point $t \in T$. Thus

$$\text{Ch}^t(E^1U_1 \cdot E^2U_2) = - \sum_{p', q'} (\nabla \text{Tr}U_1(t) \otimes \nabla \text{Tr}U_2(t))_{p'q'} a_{p'} b_{q'}.$$ 

Including the Chern character of $E^1U_1 \cdot E^2U_2$ in (97) therefore results in

$$- \int_F \exp \left( \sum_{p,q} A_{pq}(t) \left( \sum_i a_i b_i^+ + \sum_j c_j^+ d_j^+ \right) \right) \sum_{p', q'} (\nabla \text{Tr}U_1(t) \otimes \nabla \text{Tr}U_2(t))_{p'q'} a_{p'} b_{q'},$$

(99)

where $A_{pq}(t) = \ell \delta_{pq} + z H_V(t)_{pq}$. Inspection of (99) now explains why other combinations of generators of $H^1(\Sigma)$ give a vanishing contribution, for then the integrand has vanishing top degree part. Equation (99) can be calculated like a Gaussian integral, and results in

$$\det(1 + \ell^{-1}H_V(t))^{\delta_{pq} \epsilon} \left| T_\ell \right|^{\epsilon \delta_{pq} \ell^+ H_V(t)} \alpha_{U_1, U_2}^A,$$

where for a matrix $A_{pq}$ with inverse $A^{pq}$ and representations $U_1, U_2$, we have set

$$\alpha_{U_1, U_2}^A(t) = - \sum_{p,q} A_{pq}(t)(\nabla \text{Tr}U_1(t) \otimes \nabla \text{Tr}U_2(t))_{pq}.$$ (100)

The following theorem summarizes these calculations. We include $E^1U_0$ now as well.

**Theorem 5.18.** Let $C_1, C_2$ be smooth closed curves in $\Sigma$. Let $U_0, U_1, U_2, V$ be representations of $G$. Then $\left| T_\ell \right|^{-1} T_\ell^{C_1^2} \text{index}_T^D (E^1U_0 \cdot E^1U_1 \cdot E^2U_2 \cdot \exp(z \Sigma V))$ is given by

$$\sum_{g \in T_\ell^{C_1^2}/W} \#(C_1 \cap C_2) \text{Tr}U_0(g_2) \alpha_{U_1, U_2}^{\ell z H_V(g_2)} (g_2) \left( \frac{\Delta(g_2)^2}{\left| T_\ell \right| \det(1 + z \ell^{-1} H_V(g_2))} \right)^{1 - \epsilon} \delta_{g_2},$$

where $g_2^\ell \exp(z \nabla \text{Tr}V(g_2)) = g^\ell$, $H_V$ is the Hessian of $\text{Tr}V$, $\#(C_1 \cap C_2)$ is the intersection pairing, and $\alpha$ is defined in (100).

5.10. **The total $\lambda$-operation.** In previous sections we worked with the exponential generating series $\exp(z \Sigma V) \in \mathbb{C} \text{RK}^0_T(X)[z]$, and found that the appropriate twist $\chi_z = \nabla \text{Tr}V$ was independent of $z$. Another interesting generating series is the total $\lambda$-operation:

$$\lambda_z \Sigma V = \sum_{p \geq 0} z^p \lambda^p \Sigma V \in \mathbb{R} \text{RK}^0_T(X)[z].$$ (101)

This series appears for example in the application to the equivariant Verlinde formula for the moduli space of Higgs bundles, with $V = g_C$ the adjoint representation [33, 4, 34].
We should first note more generally that if $E \in \mathcal{R}K^0_{T}(X)$ is quasiperiodic, then the classes $\lambda^pE$ (and likewise the Adams operations $\psi^jE$) are also quasiperiodic. Indeed if $(E, \nabla^E, Q)$ is a quasiperiodic cycle representing $E$, then the various cohomology operations applied to $E$ are represented by cycles built by decomposing graded tensor powers $E^\otimes j$ under the action of the symmetric group (the action having appropriate signs inserted according to the Koszul sign rule; see [5]), and it is clear that the resulting cycles are again quasiperiodic.

We briefly explain why (101) leads to a twist $\chi_z$ that depends on $z$ following [71, Section 6].

**Proposition 5.19.** $E_z = \lambda_z E^\Sigma V \in \mathcal{R}K^0_{T}(X)[[z]]$ admits $\chi_z$-twisted equivariant Chern character forms with

$$\chi_z = -\sum_{j>0} \frac{(-z)^{j-1}}{j^2} \nabla \text{Tr} \psi^j V.$$ 

**Proof.** One has the formula

$$\lambda_z E = \exp \left( -\sum_{j>0} \frac{(-z)^j}{j} \psi^j E \right)$$

in terms of the Adams operations $\psi^j$. By the splitting principle

$$\text{Ch}^z(\psi^j E, \xi)[2p] = j^p \text{Ch}^j(\psi^j E, j\xi)[2p]$$

where $\alpha_{[2p]}$ is the component of $\alpha$ in degree $2p$. Using the families index formula (80) for the Chern character form, (and the relation $\psi^j EV = E(\psi^j V)$),

$$\text{Ch}^L(\psi^j E^\Sigma V, \xi)[2p] = j^p \text{Ch}^j(\psi^j E^\Sigma V, j\xi)[2p] = \frac{1}{j} \int \frac{1}{2} j^{p+1} \text{Ch}^{j+1}(EV, j\xi)[2p+2] = \frac{1}{j} \text{Ch}^L(\psi^j E^\Sigma V, \xi)[2p].$$

Hence

$$\text{Ch}^L(\lambda_z E^\Sigma V, \xi) = \exp \left( -\sum_{j>0} \frac{(-z)^j}{j^2} \text{Ch}^L(\psi^j E^\Sigma V, \xi) \right)$$

Recall that for $\eta \in \Pi$,

$$\eta \cdot \text{Ch}^L(E^\Sigma V, \xi) = \text{Ch}^L(\eta E^\Sigma V, \xi) - \nabla \text{Tr} V(\xi \exp(\xi)).$$

It follows that

$$\eta \cdot \text{Ch}^L(\lambda_z E^\Sigma V, \xi) = \exp \left( \sum_{j>0} \frac{(-z)^j}{j^2} \nabla \text{Tr} \psi^j V(\xi \exp(\xi)) \right) \text{Ch}^L(\lambda_z E^\Sigma V, \xi).$$

□

In the special case $V = g_C$ relevant to the equivariant Verlinde formula, one has

$$\nabla \text{Tr} \psi^j g_C = \sum_{\alpha \in \mathfrak{g}^+} j e_{\alpha} \alpha = \sum_{\alpha \in \mathfrak{g}^+} j(\epsilon_{\alpha} - e_{-\alpha}) \alpha,$$

from which we find (taking $b + 1 = 1$),

$$z \chi_z = \sum_{\alpha \in \mathfrak{g}^+_+} \sum_{j>0} \frac{(-z)^j}{j}(e_{\alpha} - e_{-\alpha}) \alpha = \sum_{\alpha \in \mathfrak{g}^+_+} \log \left( \frac{1 + ze_{\alpha}}{1 + ze_{-\alpha}} \right) \alpha.$$
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