K3 SURFACES WITH AN AUTOMORPHISM OF ORDER 66, THE MAXIMUM POSSIBLE

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Abstract. In each characteristic \( p \neq 2, 3 \), it was shown in a previous work that the order of an automorphism of a K3 surface is bounded by 66, if finite. Here, it is shown that in each characteristic \( p \neq 2, 3 \) a K3 surface with a cyclic action of order 66 is unique up to isomorphism. The equation of the unique surface is given explicitly in the tame case \((p \nmid 66)\) and in the wild case \((p = 11)\).

An automorphism of finite order is called tame if its order is prime to the characteristic, and wild otherwise. Let \( X \) be a K3 surface over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). An automorphism \( g \) of \( X \) is called symplectic if it preserves a non-zero regular 2-form \( \omega_X \), and purely non-symplectic if no power of \( g \) is symplectic except the identity. An automorphism of order a power of \( p \) in characteristic \( p > 0 \) is symplectic, as there is no \( p \)-th root of unity.

Over \( \mathbb{C} \), Kondo \([7]\) gave an example of a complex K3 surface with an automorphism of order 66:

\[
(0.1) \quad X_{66} : y^2 + x^3 + t^{12} - t = 0,
\]

\[
(0.2) \quad g_{66}(t, x, y) = (\zeta_{66}^6 t, \zeta_{66}^2 x, \zeta_{66}^3 y)
\]

where \( \zeta_{66} \) is a primitive 66th root of unity. The K3 surface \( X_{66} \) is defined over the integers and both the surface and the automorphism have a good reduction mod \( p \) unless \( p \) divides 66.

In each characteristic \( p \neq 2, 3 \), it was shown in \([5]\) that the order of any automorphism of a K3 surface is bounded by 66, if finite. In this paper we characterize K3 surfaces admitting a cyclic action of order 66.

For an automorphism \( g \), tame or wild, of order \( mn \) of a K3 surface \( X \), we write

\[
\text{ord}(g) = m.n
\]

if the homomorphism \( \langle g \rangle \to \text{GL}(H^0(X, \Omega^2_X)) \) has kernel of order \( m \).

By \([5]\) (Lemma 4.2 and 4.4) a tame automorphism \( g \) of order 66 is purely non-symplectic, i.e., \( \text{ord}(g) = 1.66 \).

The first result deals the tame case.

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Theorem 0.1. Let $k$ be the field $\mathbb{C}$ of complex numbers or an algebraically closed field of characteristic $p > 3$, $p \neq 11$. Let $X$ be a K3 surface defined over $k$ with an automorphism $g$ of order 66. Then

$$(X, \langle g \rangle) \cong (X_{66}, \langle g_{66} \rangle),$$

i.e. there is an isomorphism $f : X \to X_{66}$ such that $f(g)f^{-1} = \langle g_{66} \rangle$.

Over $k = \mathbb{C}$, Theorem 0.1 was proved by Machida and Oguiso [8], under the assumption that the automorphism is purely non-symplectic. Our proof is characteristic free and does not use the tools in the complex case such as transcendental lattice and the holomorphic Lefschetz formula.

The surface $X_{66}$ is a weighted Delsarte surface. Using the algorithm for determining the Artin invariant of such a surface whose minimal resolution is a K3 surface ([11], [3]), one can show that in characteristic $p \equiv -1 \pmod{66}$ the surface $X_{66}$ is a supersingular K3 surface with Artin invariant 1.

In characteristic $p = 11$, there is an example of a K3 surface with a wild automorphism of order 66 ([2], [5]):

(0.3) $Y_{66}: y^2 + x^3 + t^{11} - t = 0,$

(0.4) $h_{66}(t, x, y) = (t + 1, \zeta_6^2 x, \zeta_6^3 y)$

where $\zeta_6 \in k$ is a primitive 6th root of unity. The surface $Y_{66}$ is a supersingular K3 surface with Artin invariant 1 in characteristic $p = 11 \pmod{12}$.

Our second result is the following.

Theorem 0.2. Let $k$ be an algebraically closed field of characteristic $p = 11$. Let $X$ be a K3 surface defined over $k$ with an automorphism $g$ of order 66. Then $\text{ord}(g) = 11.6$ and

$$(X, \langle g \rangle) \cong (Y_{66}, \langle h_{66} \rangle),$$

i.e. there is an isomorphism $f : X \to Y_{66}$ such that $f(g)f^{-1} = \langle h_{66} \rangle$.

Remark 0.3. In characteristic $p = 2, 3$, there is an example of a K3 surface with an automorphism of order 66, as was noticed by Matthias Schütt:

$p = 2$

(0.5) $X : y^2 - y = x^3 + t^{11} - 1,$

(0.6) $g(t, x, y) = (\zeta_{33}^3 t, \zeta_{33}^{11} x, y + 1)$

$p = 3$

(0.7) $X : y^2 = x^3 - x + t^{11} - 1,$

(0.8) $g(t, x, y) = (\zeta_{22}^2 t, x + 1, \zeta_{22}^{11} y)$

where $\zeta_{22}$ and $\zeta_{33} \in k$ are primitive 22nd and 33rd roots of unity, respectively. In characteristic 2 and 3, it seems that 66 is the maximum finite order and is realized only by the above surface up to isomorphism.
Notation

For an automorphism $g$ of a K3 surface $X$, we use the following notation:

- $X^g = \text{Fix}(g)$: the fixed locus of $g$
- $e(g) := e(\text{Fix}(g))$, the Euler characteristic of $\text{Fix}(g)$;
- $\text{Tr}(g^*|H^*(X)) := \sum_{j=0}^{2\dim X} (-1)^j \text{Tr}(g^*|H^j_{\text{et}}(X, \Q_l))$.
- $[g^*] = [\lambda_1, \ldots, \lambda_{22}]$ : the eigenvalues of $g^*|H^2_{\text{et}}(X, \Q_l)$
- $\zeta_a$ : a primitive $a$-th root of unity in $\Q_l$
- $\zeta_a : \phi(a)$ : all primitive $a$-th roots of unity in $\Q_l$ where $\phi$ is the Euler function and $\phi(a)$ the number of conjugates of $\zeta_a$
- $[\lambda.r] \subset [g^*] : \lambda$ repeats $r$ times in $[g^*]$
- $[(\zeta_a : \phi(a)).r] \subset [g^*] :$ the list $\zeta_a : \phi(a)$ repeats $r$ times in $[g^*]$

1. Preliminaries

The following basic results can be found in the previous paper [5].

Proposition 1.1. (Proposition 2.1 [5]) Let $g$ be an automorphism of a projective variety $X$ over an algebraically closed field $k$ of characteristic $p > 0$. Let $l$ be a prime $\neq p$. Then the following hold true.

1. (3.7.3 [4]) The characteristic polynomial of $g^*|H^j_{\text{et}}(X, \Q_l)$ has integer coefficients for each $j$. The characteristic polynomial does not depend on the choice of cohomology, $l$-adic or crystalline. In particular, if a primitive $m$-th root of unity appears with multiplicity $r$ as an eigenvalue of $g^*|H^j_{\text{et}}(X, \Q_l)$, then so does each of its conjugates.

2. If $g$ is of finite order, then $g$ has an invariant ample divisor, and $g^*|H^2_{\text{et}}(X, \Q_l)$ has $1$ as an eigenvalue.

3. If $X$ is a K3 surface, $g$ is tame and $g^*|H^0(X, \Omega_X^2)$ has $\zeta_n \in k$ as an eigenvalue, then $g^*|H^2_{\text{et}}(X, \Q_l)$ has $\zeta_n \in \Q_l$ as an eigenvalue.

Proposition 1.2. (Topological Lefschetz formula, cf. [1] Theorem 3.2) Let $X$ be a smooth projective variety over an algebraically closed field $k$ of characteristic $p > 0$ and let $g$ be a tame automorphism of $X$. Then $X^g = \text{Fix}(g)$ is smooth and

$$e(g) := e(X^g) = \text{Tr}(g^*|H^*(X)).$$

Lemma 1.3. (Lemma 1.6 [3]) Let $X$ be a K3 surface in characteristic $p \neq 2$, admitting an automorphism $h$ of order $2$ with $\dim H^2_{\text{et}}(X, \Q_l)^h = 2$. Then $h$ is non-symplectic and has an $h$-invariant elliptic fibration $\psi : X \to \P^1$, $X/\langle h \rangle \cong \F_e$

a rational ruled surface, and $X^h$ is either a curve of genus $9$ which is a $4$-section of $\psi$ or the union of a section and a curve of genus $10$ which is a $3$-section. In the first case $e = 0, 1$ or $2$, and in the second $e = 4$. Each
singular fibre of \( \psi \) is of type \( I_1 \) (nodal), \( I_2 \) (cuspidal) or \( III \), and is intersected by \( X^h \) at the node and two smooth points if of type \( I_1 \), at the two singular points if of type \( I_2 \), at the cusp with multiplicity 3 and a smooth point if of type \( II \), at the singular point tangentially to both components if of type \( III \). If \( X^h \) contains a section, then each singular fibre is of type \( I_1 \) or \( II \).

**Remark 1.4.** If \( e \neq 0 \), the \( h \)-invariant elliptic fibration \( \psi \) is the pull-back of the unique ruling of \( F_e \). If \( e = 0 \), either ruling of \( F_0 \) lifts to an \( h \)-invariant elliptic fibration.

**Lemma 1.5.** (Lemma 2.10 [5]) Let \( S \) be a set and \( \text{Aut}(S) \) be the group of bijections of \( S \). For any \( g \in \text{Aut}(S) \) and positive integers \( a \) and \( b \),

1. \( \text{Fix}(g) \subset \text{Fix}(g^a) \);
2. \( \text{Fix}(g^a) \cap \text{Fix}(g^b) = \text{Fix}(g^{\text{gcd}(a, b)}) \);
3. \( \text{Fix}(g) = \text{Fix}(g^a) \) if \( \text{ord}(g) \) is finite and prime to \( a \).

**Lemma 1.6.** (Lemma 2.11 [5]) Let \( R(n) \) be the sum of all primitive \( n \)-th root of unity in \( \mathbb{Q} \) or in \( \mathbb{Q}_l \) where \( (l, n) = 1 \). Then

\[
R(n) = \begin{cases} 
0 & \text{if } n \text{ has a square factor,} \\
(-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes.}
\end{cases}
\]

2. INVARIANT ELLIPTIC FIBRATION

The following two lemmas will play a key role in our proof.

**Lemma 2.1.** Let \( g \) be an automorphism of order 66 of a K3 surface \( X \) in characteristic \( p \neq 2, 3, 11 \). If the eigenvalues of \( g^* \) on the second cohomology is given by

\[
[g^*] = [1, \zeta_{66} : 20, 1],
\]

then

1. there is a \( g \)-invariant elliptic fibration \( \psi : X \to \mathbb{P}^1 \) with 12 cuspidal fibres, say \( F_0, F_1, \ldots, F_{11} \);
2. \( \text{Fix}(g^{33}) \) consists of a section \( R \) of \( \psi \) and a curve \( C_{10} \) of genus 10 which is a 3-section passing through each cusp with multiplicity 3;
3. the action of \( g \) on the base \( \mathbb{P}^1 \) is of order 11, fixes 2 points, say \( \infty \) and 0, and makes the 11 points \( t_1, \ldots, t_{11} \) form a single orbit, where \( F_{\infty} \) is a smooth fibre;
4. \( \text{Fix}(g^{11}) = R \cup \{ \text{the cusps of the 12 cuspidal fibres} \} \);
5. \( \text{Fix}(g) \) consists of the 3 points,

\[
R \cap F_{\infty}, R \cap F_0, C_{10} \cap F_0.
\]

**Proof.** Note that \([g^{33*}] = [1, -1.20, 1]\). Thus, we can apply Lemma [1.3] to \( h = g^{33} \). We compute \( e(g) = 3 \) and

\[
[g^{11*}] = [1, (\zeta_6 : 2).10, 1], \ e(g^{11}) = 14.
\]
Note that
\[ \text{Fix}(g^d) \subset \text{Fix}(g^{33}) \]
for any \( d \) dividing 33. If \( \text{Fix}(g^{33}) \) is a curve \( C_9 \) of genus 9, then \( g^{11} \) acts on \( C_9 \) with 14 fixed points, too many for an order 3 automorphism on a curve of genus 9. Thus
\[ X/(g^{33}) \cong F_4, \]
there is a \( g^{33} \)-invariant elliptic fibration
\[ \psi : X \to \mathbb{P}^1 \]
and \( \text{Fix}(g^{33}) \) consists of a section \( R \) of \( \psi \) and a curve \( C_{10} \) of genus 10 which is a 3-section. The automorphism \( \bar{g} \) of \( F_4 \) induced by \( g \) preserves the unique ruling, so \( g \) preserves the elliptic fibration. Since \( g^{33} \) acts trivially on \( F_4 \), \( g^{33} \) acts trivially on the base \( \mathbb{P}^1 \), and hence the orbit of a fibre under the action of \( g^i \) has length 1, 3, 11 or 33. By Lemma \( \boxed{1.3} \) a fibre of \( \psi \) is of type \( I_0 \) (smooth), \( I_1, I_2, I_3 \) or \( III \). Claim that \( \psi \) has no fibre of type \( I_2 \) or \( III \). If a fibre \( F \) is of type \( III \), then its orbit under \( g^i \) has length 1 or 3, then \( g^3 \) preserves \( F \), hence \( g^6 \) preserves both components of \( F \) and, together with an invariant ample class, preserves 3 linearly independent classes, hence \( [g^6] \supset [1, 1, 1] \), impossible. If a fibre \( F \) is of type \( I_2 \), then its orbit under \( g^i \) has length 1 or 3 or 11, then \( g^6 \) or \( g^{22} \) would have 3 linearly independent invariant classes, again impossible. Next, claim that there is an orbit of singular fibres of length 11. Otherwise, all orbits of singular fibres would have length 1 or 3, then \( g^3 \) would preserve all fibres, hence fix the curve \( R \) and induces on a general smooth fibre an automorphism of order 22. But in any characteristic no elliptic curve admits an automorphism of such high order that fixes a point. If there are two orbits of length 11 of singular fibres of type \( I_1 \), then \( g^{11} \) would preserve all fibres, hence fix \( R \) and the singular points of singular fibres, then \( e(g^{11}) > 14 \). Thus there is one orbit of length 11 of singular fibres of type \( II \). If \( g \) preserves two fibres of type \( I_1 \), then the same argument as above would yield \( e(g^{11}) > 14 \). Thus \( g \) preserves one fibre of type \( II \) and a smooth fibre. This proves (1), (2) and (3).

The statement (4) follows from (3) and the fact that \( \text{Fix}(g^{11}) \) has Euler number 14 and is contained in \( R \cup C_{10} \).

To see (5), take \( R \) as the 0-section of \( \psi \). Then on each smooth fibre \( F \), \( g^{11} \) induces an order 6 automorphism, fixing the point \( F \cap R \) and rotating the three 2-torsions \( C_{10} \cap F \).

**Lemma 2.2.** Let \( g \) be an automorphism of order 66 of a K3 surface \( X \) in characteristic \( p = 11 \). If
\[ [g^*] = [1, \zeta_{66} : 20, 1], \]
then
\begin{enumerate}
  \item there is a \( g \)-invariant elliptic fibration \( \psi : X \to \mathbb{P}^1 \) with 12 cuspidal fibres, say \( F_\infty, F_{t_1}, \ldots, F_{t_{11}} \);
\end{enumerate}
(2) \(\text{Fix}(g^{33})\) consists of a section \(R\) of \(\psi\) and a curve \(C_{10}\) of genus 10 which is a 3-section passing through each cusp with multiplicity 3;

(3) the action of \(g\) on the base \(\mathbb{P}^1\) is of order 11, fixes 1 point, say \(\infty\), and makes the 11 points \(t_1, \ldots, t_{11}\) form a single orbit;

(4) \(\text{Fix}(g^{11}) = R \cup \{\text{the cusps of the 12 cuspidal fibres}\}\);

(5) \(\text{Fix}(g)\) consists of the 2 points,

\[ R \cap F_\infty, \ C_{10} \cap F_\infty. \]

Proof. The proof of Lemma 2.1, with a modification, will work here. Note first that the automorphisms \(g^{33}, g^{11}\) are tame in characteristic \(p = 11\), so the Lefschetz fixed point formula holds for them and the argument using their fixed loci is valid.

We compute \([g^{33}] = [1, -1.20, 1]\) and apply Lemma 1.3 to \(h = g^{33}\). We also compute \([g^{11}] = [1, (\zeta_6 : 2).10, 1]\), \(e(g^{11}) = 14\).

Note that

\[ \text{Fix}(g^d) \subset \text{Fix}(g^{33}) \]

for any \(d\) dividing 33. If \(\text{Fix}(g^{33})\) is a curve \(C_9\) of genus 9, then \(g^{11}\) acts on \(C_9\) with 14 fixed points, too many for an order 3 automorphism on a curve of genus 9. Thus

\[ X/\langle g^{33} \rangle \cong F_4, \]

there is a \(g^{33}\)-invariant elliptic fibration

\[ \psi: X \to \mathbb{P}^1 \]

and \(\text{Fix}(g^{33})\) consists of a section \(R\) of \(\psi\) and a curve \(C_{10}\) of genus 10 which is a 3-section. The automorphism \(g\) of \(\mathbb{P}^1\) induced by \(g\) preserves the unique ruling, so \(g\) preserves the elliptic fibration. Note that \(g^{33}\) acts trivially on the base \(\mathbb{P}^1\). By Lemma 1.3, a fibre of \(\psi\) is of type \(I_0\) (smooth), \(I_1, I_2, II\) or \(III\).

By the same argument as in Lemma 2.1, \(\psi\) has no fibre of type \(I_2\) or \(III\) and there is an orbit of singular fibres of length 11. If there are two orbits of length 11 of singular fibres of type \(I_1\), then \(g^{11}\) would preserve all fibres, hence fix \(R\) and the singular points of singular fibres, then \(e(g^{11}) > 14\). Thus there is one orbit of length 11 of singular fibres of type \(II\). If \(g\) preserves two fibres of type \(I_1\), then the same argument as above would yield \(e(g^{11}) > 14\). Thus \(g\) preserves one fibre of type \(II\). Since \(g|\mathbb{P}^1\) is of order 11 and an wild automorphism on \(\mathbb{P}^1\) fixes only one point, we see that \(g\) preserves no other fibre. This proves (1), (2) and (3).

The statement (4) follows from (3). Since \(g^{11}\) is tame, \(\text{Fix}(g^{11})\) has Euler number 14 and is contained in \(R \cup C_{10}\).

To see (5), note that \(\text{Fix}(g)\) is contained not only in the cuspidal fibre \(F_\infty\) but also in \(\text{Fix}(g^{33}) = C_{10} \cup R\). \(\square\)
3. the Tame Case

Throughout this section, we assume that the characteristic \( p > 0, p \nmid 66, \) and \( g \) is an automorphism of order 66 of a K3 surface. By [5] Lemma 4.2 and 4.4, \( g \) is purely non-symplectic, i.e. \( \text{ord}(g) = 1.66 \).

**Lemma 3.1.** The eigenvalues of \( g^* \) on the second cohomology is given by

\[
[g^*] = [1, \zeta_{66} : 20, 1].
\]

**Proof.** By Proposition 1.1 the action of \( g^* \) on \( H^2_{\et}(X, \mathbb{Q}_l) \) has \( \zeta_{66} \in \overline{\mathbb{Q}_l} \) as an eigenvalue. Thus \( [\zeta_{66} : 20] \subset [g^*] \). Suppose that

\[
[g^*] = [1, \zeta_{66} : 20, -1].
\]

Then

\[
[g^{33*}] = [1, -1 : 20, -1], \quad e(g^{33}) = -18.
\]

Since the \( g^{33*} \)-invariant subspace of \( H^2_{\et}(X, \mathbb{Q}_l) \) has dimension 1, we see that

\[
\text{Fix}(g^{33}) = C_{10},
\]

a smooth curve of genus 10, and the quotient surface

\[
X/(g^{33}) \cong \mathbb{P}^2.
\]

The image \( C'_{10} \subset \mathbb{P}^2 \) is a smooth sextic curve. Since \( e(g^{11}) = 12 \) and

\[
\text{Fix}(g^{11}) \subset \text{Fix}(g^{33}) = C_{10},
\]

we see that \( \text{Fix}(g^{11}) \) consists of 12 points. On the other hand, \( g^{22} \) has

\[
[g^{22*}] = [1, (\zeta_3 : 2).10, 1], \quad e(g^{22}) = -6.
\]

Since the \( g^{22*} \)-invariant subspace of \( H^2_{\et}(X, \mathbb{Q}_l) \) has dimension 2, \( \text{Fix}(g^{22}) \) consists of a curve \( C \) of genus 1, at most one \( \mathbb{P}^1 \) and some isolated points. If \( \text{Fix}(g^{22}) \) contains a \( \mathbb{P}^1 \), then the action of \( g \) on \( \text{Fix}(g^{22}) \) preserves \( C \) and the \( \mathbb{P}^1 \), so the \( g^* \)-invariant subspace of \( H^2_{\et}(X, \mathbb{Q}_l) \) has dimension at least 2, a contradiction. Here we use the fact that the Chern class map

\[
c_1 : \text{Pic}(X) \to H^2_{\text{crys}}(X/W)
\]

is injective and the fact that the characteristic polynomial of \( g^* \) does not depends on the choice of cohomology. Thus \( \text{Fix}(g^{22}) \) contains no \( \mathbb{P}^1 \) and

\[
\text{Fix}(g^{22}) = C_{k+4} \cup \{ 2k \text{ points} \}
\]

for a smooth curve \( C_{k+4} \) of genus \( k + 4 \). Note that

\[
C_{k+4} \cap C_{10} \subset \text{Fix}(g^{22}) \cap \text{Fix}(g^{33}) = \text{Fix}(g^{11}),
\]

thus the intersection number

\[
C_{k+4}.C_{10} \leq 12.
\]

Then the Hodge Index Theorem gives

\[
(C_{k+4}^2)(C_{10}^2) = 18(2k + 6) \leq (C_{k+4}.C_{10})^2 \leq 12^2,
\]

thus \( k \leq 1 \).
Suppose that \( k = 0 \) and \( \text{Fix}(g^{22}) = C_4 \). Since \( C_4^2/C_{10}^2 \) is not a square of a rational number, the two curves \( C_4 \) and \( C_{10} \) are linearly independent in \( \text{Pic}(X) \otimes \mathbb{Q} \), giving two linearly independent \( g^* \)-invariant vectors of \( H^2_{et}(X, \mathbb{Q}_l) \), a contradiction.

Suppose that \( k = 1 \) and \( \text{Fix}(g^{22}) = C_5 \cup \{2 \text{ points}\} \). Since \( (C_5^2)(C_{10}^2) = 144 \), we have the equality in the Hodge Index Theorem and \( C_5.C_{10} = 12 \).

Since \( g^{33}|C_5 = g^{11}|C_5 \), the action of \( g^{33} \) on \( C_5 \) has 12 fixed points, hence the image
\[
C_5' \subset X/\langle g^{33} \rangle \cong \mathbb{P}^2
\]
has genus 0. Since \( C_5.C_{10}' = 12 \), \( C_5' \) must be a smooth conic. Consider the automorphism \( \tilde{g}^{11} \) of \( X/\langle g^{33} \rangle \) induced by \( g^{11} \). It has order 3 and its fixed locus \( \text{Fix}(\tilde{g}^{11}) \subset \mathbb{P}^2 \) is the image of the locus
\[
\text{Fix}(g^{11}) \cup \text{Fix}(g^{22}) = \text{Fix}(g^{22}),
\]
hence \( \text{Fix}(\tilde{g}^{11}) \) consists of the conic \( C_5' \) and the point which is the image of the two points in \( \text{Fix}(g^{22}) \). But the fixed locus of any order 3 automorphism of \( \mathbb{P}^2 \) is either 3 isolated points or the union of a point and a line, a contradiction. \( \square \)

**Proof of Theorem 0.1**

By Lemma [3.1] \([g^*] = [1, \zeta_{66} : 20, 1] \). We can apply Lemma [2.1] and will use the elliptic structure and the notation. Let
\[
y^2 + x^3 + A(t_0, t_1)x + B(t_0, t_1) = 0
\]
be the Weierstrass equation of the \( g \)-invariant elliptic pencil, where \( A \) (resp. \( B \)) is a binary form of degree 8 (resp. 12). By Lemma [2.1], \( g \) leaves invariant the section \( R \) and the action of \( g \) on the base of the fibration \( \psi : X \to \mathbb{P}^1 \) is of order 11. After a linear change of the coordinates \((t_0, t_1)\) we may assume that \( g \) acts on the base by
\[
g : (t_0, t_1) \mapsto (t_0, \zeta_{11} t_1)
\]
for some primitive 11th root of unity \( \zeta_{11} \). We know that \( g \) preserves one cuspidal fibre \( F_0 \) and makes the remaining 11 cuspidal fibres form one orbit. Thus the discriminant polynomial
\[
\Delta = -4A^3 - 27B^2 = c_1^2(t_{11}^1 - t_{011}^1)^2
\]
for some constant \( c \in k \), as it must have one double root (corresponding to the fibres \( F_0 \)) and one orbit of double roots. From the equality (3.1), it is easy to see that \( A \) is not a non-zero constant. If \( \deg(A) > 0 \), then the zeros of \( A \) correspond to either cuspidal fibres (which may contain a singular point of \( X \), i.e. yield a reducible fibre) or nonsingular fibres with “complex multiplication” of order 6. This set has cardinality at most 8, but invariant
with respect to the order 11 action of \( g|\mathbb{P}^1 \), impossible. Thus \( A = 0 \). Then the above Weierstrass equation can be written in the form
\[
(3.2) \quad y^2 + x^3 + at_1(t_1^{11} - t_0^{11}) = 0
\]
for some constant \( a \). A suitable linear change of variables makes \( a = 1 \) without changing the action of \( g \) on the base. Thus
\[
X \cong X_{66}
\]
as an elliptic surface. Let
\[
t = t_1/t_0.
\]
Choose a primitive 66th root of unity \( \zeta_{66} \) such that
\[
(3.3) \quad g^*\left(\frac{dx \wedge dt}{y}\right) = \zeta_{66}^5 \frac{dx \wedge dt}{y}, \quad g^{11*}\left(\frac{dx \wedge dt}{y}\right) = \zeta_{66}^{55} \frac{dx \wedge dt}{y}.
\]
Since \( g^{11} \) is of order 6, acts trivially on the base and fixes the section \( R \), it is a complex multiplication of order 6 on a general fibre, so
\[
g^{11}(x, y, t) = (\zeta_{66}^{22}x, -y, t).
\]
Here, the other primitive 3rd root of unity \( \zeta_{66}^{44} \) cannot appear as the coefficient of \( x \) by (3.3). We will analyse the local action of \( g \) at the fixed point \((x, y, t) = (0, 0, 0)\), the cusp of \( F_0 \). We first determine the linear terms of \( g \), then infer that the higher degree terms must vanish. Write the linear terms of \( g \) as follows:
\[
g(x, y, t) = (\zeta_{66}^a x, \zeta_{66}^b y, \zeta_{66}^c t).
\]
Since the Weierstrass equation (3.2) is invariant under \( g \), we have the following system of congruence equation modulo 66:
\[
3a \equiv 2b \equiv 12c \equiv c
\]
\[
11a \equiv 22
\]
\[
11b \equiv 33.
\]
The solutions are
\[
a \equiv 2 + 6a' \]
\[
b \equiv 3 + 9a' \quad (a' \text{ even}) \text{ or } 36 + 9a' \quad (a' \text{ odd})
\]
\[
c \equiv 6 + 18a'
\]
for some integer \( a' \). On the other hand, by (3.3)
\[
5 \equiv a + c - b \pmod{66}.
\]
This congruence equation is satisfied by the solution \( a \equiv 2, b \equiv 3, c \equiv 6 \), but by no other solution among the above solutions. This completes the proof of Theorem 0.1 in the tame case.
4. the Complex Case

We may assume that $X$ is projective, since a non-projective complex K3 surface cannot admit a non-symplectic automorphism of finite order (see [12], [9]) and its automorphisms of finite order are symplectic, hence of order $\leq 8$. Now the same proof goes, once $H^2_{\text{et}}(X, \mathbb{Q}_l)$ is replaced by $H^2(X, \mathbb{Z})$.

5. in characteristic $p = 11$

Throughout this section, we assume that the characteristic $p = 11$ and $g$ is an automorphism of order 66 of a K3 surface. By [2] we know that $\text{ord}(g) = 11.6$.

**Lemma 5.1.** $\text{ord}(g) = 11.6$.

*Proof.* Any automorphism of order $p$ in characteristic $p$ is symplectic, as there is no $p$-th root of unity. Thus the symplectic order of $g$ must be a multiple of 11. In characteristic 11 it is known [2] that 11 is the maximum possible among all orders of symplectic automorphisms of finite order. □

**Lemma 5.2.** The eigenvalues of $g^*$ on the second cohomology is given by

$$[g^*] = [1, \zeta_{66} : 20, 1].$$

*Proof.* In characteristic $p = 11$ it was proved in [2] Proposition 4.2 that the representation on $H^2_{\text{et}}(X, \mathbb{Q}_l)$ of a finite group of symplectic automorphisms is Mathieu. It follows that the order 11 automorphism $g^6$ has

$$[g^6] = [1, (\zeta_{11} : 10).2, 1].$$

There is a $g$-invariant ample divisor class, so 1 appears in $[g^*]$. Since the representation of $\text{Aut}(X)$ on $H^2_{\text{et}}(X, \mathbb{Q}_l)$ is faithful ([10] Corollary 2.5, [5] Theorem 1.4), $g^*|H^2_{\text{et}}(X, \mathbb{Q}_l)$ has order 66 and we infer that $[g^*]$ is one of the following 3 cases:

$$[g^*] = [1, \zeta_{66} : 20, \pm 1], \quad [1, \zeta_{33} : 20, -1].$$

On the other hand, $g^{11}$ is tame and non-symplectic of order 6, hence $\zeta_6 \in [g^{11*}]$ by Proposition 1.1. This excludes the last case.

Suppose that $[g^*] = [1, \zeta_{66} : 20, -1]$. This case can be ruled out by the same proof as in Lemma 5.1. Indeed, the automorphisms $g^{33}, g^{22}, g^{11}$ are tame in characteristic $p = 11$, so the Lefschetz fixed point formula holds for them and the argument using their fixed loci is valid. □

**Proof of Theorem 0.2.**

The first statement follows from Lemma 5.1. It remains to prove the second. By Lemma 5.2 $[g^*] = [1, \zeta_{66} : 20, 1]$. We can apply Lemma 2.2 and will use the elliptic structure and the notation. Let

$$y^2 + x^3 + A(t_0, t_1)x + B(t_0, t_1) = 0.$$
be the Weierstrass equation of the $g$-invariant elliptic pencil, where $A$ (resp. $B$) is a binary form of degree 8 (resp. 12). By Lemma 2.2, $g$ leaves invariant the section $R$ and the action of $g$ on the base of the fibration $\psi : X \to \mathbb{P}^1$ is of order 11. Any wild automorphism of $\mathbb{P}^1$ is uni-potent, so after a linear change of the coordinates $(t_0, t_1)$ we may assume that $g$ acts on the base by $g : (t_0, t_1) \mapsto (t_0, t_1 + t_0)$.

Then $g$ preserves the cuspidal fibre $F_\infty$ and makes the remaining 11 cuspidal fibres form one orbit. Thus the discriminant polynomial

$$\Delta = -4A^3 - 27B^2 = c t_0^2 (t_1^{11} - t_0^{10} t_1)^2$$

for some constant $c \in k$, as it must have one double root (corresponding to the fibres $F_\infty$) and one orbit of double roots. The zeros of $A$ correspond to either cuspidal fibres (which may contain a singular point of $X$) or non-singular fibres with “complex multiplication” of order 6. Since this set is invariant with respect to the order 11 action of $g|\mathbb{P}^1$, we see that the only possibility is $A = 0$. Then the above Weierstrass equation can be written in the form

$$y^2 + x^3 + at_0 (t_1^{11} - t_0^{10} t_1) = 0$$

for some constant $a$. A suitable linear change of variables makes $a = 1$ without changing the action of $g$ on the base. Thus

$$X \cong Y_{66}$$

as an elliptic surface. Let $t = t_1/t_0$.

Since $g$ has non-symplectic order 6, one can choose a primitive 6th root of unity $\zeta_6$ such that

$$g^* \left( \frac{dx \wedge dt}{y} \right) = \zeta_6^{-1} \frac{dx \wedge dt}{y}, \quad g^{11} \left( \frac{dx \wedge dt}{y} \right) = \zeta_6 \frac{dx \wedge dt}{y}.$$  

Since $g^{11}$ is of order 6, acts trivially on the base and fixes the section $R$, it is a complex multiplication of order 6 on a general fibre, so

$$g^{11} (x, y, t) = (\zeta_6^4 x, \zeta_6^3 y, t).$$

We know that $g(t) = t + 1$, so infer that

$$g(x, y, t) = (\zeta_6^2 x, \zeta_6^3 y, t + 1).$$

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