2N qubit ”mirror states” for optimal quantum communication

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We introduce a new genuinely entangled 2N qubit ”mirror state”, with a high degree of connectedness and persistency. The well known Bell and the cluster states form a special case of these ”mirror states”, for N = 1 and N = 2 respectively. It can be experimentally realized using SWAP and controlled phase shift operations, for e.g., using the Dzyaloshinskii-Moriya interaction in a Heisenberg spin model. After establishing the general conditions for a state to be useful for various quantum communicational protocols, it is shown that the present state can optimally implement algorithms for the quantum teleportation of an arbitrary N qubit state and achieve quantum information splitting in more than one way. With regard to superdense coding, one can send 2N classical bits by sending only N qubits and consuming N ebits of entanglement.

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I. INTRODUCTION

Quantum communication protocols such as teleportation [1], secret sharing [2] and superdense coding [3] require entangled states. Apart from the regular measures like concurrence [4] and different types of entropies [5], one often needs to characterize entangled states keeping in mind, the nature of the quantum task at hand. It has been observed that the efficacy of a given state for a number of quantum tasks depend not only on the degree of entanglement, but also on ”connectedness” and ”persistency” [6]. Connectedness refers to the possibility of projecting two qubits of a state into the Bell basis by performing an appropriate local measurement on the other qubits, while the persistency of entanglement refers to the minimum number of local measurements needed to completely disentangle the given state.

In case of three qubits, the GHZ states are maximally connected, whereas W states [7] are not, although the latter has a higher persistency. GHZ states can be used for teleportation and secret sharing, whereas the symmetric W state fails to carry out this task. Based on LOCC, often used for quantum communicational tasks, entangled states have been classified only upto four qubits [8]. Higher dimensional entangled states, not belonging to the existing classifications, have been found through intense numerical search procedures [9], which becomes restrictive as the number of particles increases. Instead, approaches based on symmetry and making use of entangling operations, based on physical Hamiltonians, are often preferred since the same are experimentally feasible. Apart from the generalization of the well known GHZ and W states, an interesting new class of N qubit graph states, known as the cluster states [6] has been introduced into quantum information theory:

\[ |C_N \rangle = \frac{1}{2^{N/2}} \otimes_{i=1}^{N} (|0\rangle_{\alpha} \sigma_z^{i+1} + |1\rangle_{\alpha}) \],

(1)

with \( \sigma_z^{N+1} = 1 \). This state owes its origin to Ising type interactions and it simultaneously exhibits maximal connectedness, with a persistency of entanglement of \( \frac{N}{2} \).

Quantum teleportation of single and multiqubit states is a field of intense research. In a path breaking work, Bennett et al. [1], introduced the first scheme for the teleportation of an unknown single qubit state \( |\psi_{\alpha}\rangle = \alpha|0\rangle + \beta|1\rangle \) (\( \alpha, \beta \in C, |\alpha|^2 + |\beta|^2 = 1 \)), using a two qubit Einstein-Podolsky-Rosen (EPR) pair given by \( |\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{AB} \) as an entangled resource. The same can be achieved using the three qubit GHZ [10], the asymmetric W state [11, 12] and the cluster state [13]. Recently [14, 15, 16, 17], several schemes have been devised using different types of entangled channels for the teleportation of an arbitrary two qubit state given by \( |\psi_2\rangle = \sum_{i_1,i_2} \alpha_{i_1,i_2} |i_1i_2\rangle \), where \( \alpha_{i_1,i_2} \in C \) and \( \sum |\alpha_{i_1,i_2}|^2 = 1 \). While, there are experimentally feasible states that can teleport single and two qubit states, constructing genuine multiqubit entangled channels which can teleport an arbitrary N (N > 2) qubit state is a non-trivial task and is of obvious interest to experimentalists. Even the well known GHZ, W and the cluster states cannot be used for this purpose because there is no N ebits of entanglement between any bipartition of these states for N > 4. Recently, entangled 2N [18]
and \((2N + 1)\) qubit states have been introduced for this purpose. However, experimental feasibility of these states remains an open question.

Here we introduce a new experimentally realizable genuinely entangled 2N qubit "mirror state" \(|\zeta_{2N}\rangle\) for this purpose that is different from GHZ, W and cluster states under LOCC for \(N > 2\) and exhibits different entanglement properties. As is shown below, this state is well suited for a number of quantum communication purposes like teleportation, secret sharing and dense coding. To design entangled channels keeping the communication protocols in mind, we found it necessary to start with a SWAP operation between the second and the last qubit of \(N\) Bell pairs as \(|\psi_+)_{12}|\psi_+)_{13}\ldots|\psi_+)_{2N-1,2N}\rangle\xrightarrow{SWAP(2,2N)}|\zeta_{2N}\rangle\). The SWAP operation can be realized by switching on the Dzyaloshinskii-Moriya (DM) interaction \([20]\) in the Heisenberg model between the second and last qubits in \(N\) Bell pairs:

\[
H_{DM} = \frac{J}{2}(\sigma_{2x}\sigma_{2Nx} + \sigma_{2y}\sigma_{2Ny} + \sigma_{2z}\sigma_{2Nz} + D(\sigma_{2x}\sigma_{2N})),
\]

for time \(t = \frac{k\pi}{2J}\), where \(D\) is the vector coupling. \(|\zeta_{2N}\rangle\), can be obtained from \(|\zeta_{2N}\rangle\), by performing a controlled phase shift operation \([21]\) between the first \((N+1)\) qubits. These interactions together create \(N\) ebits of entanglement between the first \(N\) and the last \(N\) qubits. For \(N = 1\), these interactions take \(|\psi_+)\) to \(|\psi_+)\) and for \(N = 2\), the SWAP operation between the second and fourth qubits on two Bell pairs \(|\psi_+)_{12}\otimes|\psi_+)_{34}\) leads to

\[
|\zeta_4\rangle' = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle)_{1432}. \tag{3}
\]

After a controlled phase shift operation on the first three qubits, the state \(|\zeta_4\rangle\) reads

\[
|\zeta_4\rangle = \frac{1}{2}(|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle)_{1432}, \tag{4}
\]

which belongs to the class of cluster states. It is well known that the Bell and the cluster states, are well suited for teleportation of an arbitrary single and two qubit information respectively. For \(N \geq 3\), \(|\zeta_{2N}\rangle\) differs from the class of cluster states and exhibits different entanglement properties.

An arbitrary \(N\) qubit state that needs to be teleported is of the general form:

\[
|\psi_N\rangle = \sum_{i_1, i_2, \ldots, i_N = 0}^{1} \alpha_{i_1i_2\ldots i_N}|i_1i_2\ldots i_N\rangle = \sum_{m=1}^{2^N} \alpha_m|\psi_m\rangle,
\]

where \(\alpha_{i_1i_2\ldots i_N} \in C\) and \(\Sigma|\alpha_{i_1i_2\ldots i_N}|^2 = \Sigma|\alpha_m|^2 = 1\).

\(|\zeta_{2N}\rangle\) can be written in the convenient form

\[
|\zeta_{2N}\rangle = \frac{1}{\sqrt{N}} \sum_{i_1, i_2, \ldots, i_N = 0}^{1} (R|i_1i_2\ldots i_N\rangle \otimes |i_1i_2\ldots i_N\rangle - 2|1\rangle \otimes |2N\rangle). \tag{6}
\]

Here \(R\) is the unitary "Reflection operator", which yields the "mirror image" of the state, through the following transformation, \(|i_1i_2\ldots i_N\rangle R = |i_Ni_1i_2\ldots i_{N-1}\rangle\). Owing to this built in reflection symmetry in the state, we call it a "mirror state".

The state is genuinely entangled according to many measures of entanglement. The von-Neumann entropy between the subsystems \(E(p_1,2, \ldots, k|p_k, \ldots, 2N) = k\); hence, for teleporting an arbitrary \(k\) \((k \leq N)\) qubit state, Alice can have the first \(k\) particles and Bob the last \((2N - k)\) particles in \(|\zeta_{2N}\rangle\). The monogamy inequality for entanglement \([4]\),

\[
\sum_{i=2}^{N} C^2_{A_iA_i} \leq C^2_{A_i|A_2A_3A_n}; \tag{7}
\]

holds for the present state. Here, \(C_{A|B}\), represents the concurrence between the subsystems \(A\) and \(B\). Moreover, it is always possible to project two qubits into a Bell state by performing an appropriate local operations on the other qubits. Hence, the present state is "maximally connected" like the GHZ and the cluster states. Further, the entanglement of this state prevails even after we perform local measurements on the other qubits, making the state "highly persistent". In general one needs to perform a minimum of \(N/2\) local measurements to break the entanglement of \(|\zeta_{2N}\rangle\) which makes it behave like the cluster state under particle loss. In this paper, we construct suitable protocols for the teleportation of an unknown \(N\) qubit state, the information splitting of an unknown \((N - k)\) state \((k < N)\) and superdense coding using \(|\zeta_{2N}\rangle\), as a shared entangled channel.

Let us now investigate the usefulness of this state for the teleportation of \(|\psi_N\rangle\). The general condition for an entangled channel \(|\psi_{AB}\rangle\), where \(A\) and \(B\) refer to the subsystems of Alice and Bob respectively, to be used for teleportation of an arbitrary \(k\) \((k \leq N)\) qubit state, is that there has to be at least \(k\) ebits of entanglement between them. As mentioned earlier, this property is not satisfied by \(|C_N\rangle\) for \(N > 4\) while it is satisfied by \(|\zeta_{2N}\rangle\), owing to the fact that after we trace out \((2N - k)\) particles, the resulting density matrix is completely mixed. Hence, it can be used for the perfect teleportation of an arbitrary \(k\) qubit state.

We let Alice possess, particles 1 to \(N\) and Bob, the last \(N\) particles. Alice has an arbitrary \(N\) qubit state \(|\psi_N\rangle\), which she needs to teleport to Bob. Alice can perform a \(2N\) partite joint measurement on her particles as:

\[
|\psi_C\rangle = \sum_{m=1}^{2^N} \alpha_m|\psi_m\rangle \otimes |\zeta_{2N}\rangle = \frac{1}{2^{N/2}} \sum_{s} |\phi_s\rangle U_x(\Sigma \alpha_m|\psi_m\rangle), \tag{8}
\]

where \(|\phi_s\rangle\) form the orthogonal outcomes of the measurements which can be rewritten by making use of the
reflection operator as $|\phi_x\rangle = \sum_k \sum_l (|\psi_l\rangle R |\psi_k\rangle)$ ($k \neq l$) or $|\phi_x\rangle = \sum_m (|\psi_m\rangle R |\psi_m\rangle)$ ($k = l = m$). Alice can convey the outcome of her measurement to Bob via $2N$ qubits of information. Bob’s state collapses to

$$
\sum_k \sum_l (\alpha_k |\psi_l\rangle + \alpha_l |\psi_k\rangle) \text{ (for } k \neq l),
$$

$$
\sum_m \alpha_m |\psi_m\rangle \text{ (for } k = l = m).  \tag{9}
$$

Bob can obtain $|\psi_N\rangle$, by performing an appropriate unitary operation on his particles. We now illustrate the working of this protocol for the teleportation of an arbitrary three qubit state using $|\phi_6\rangle$. The unknown three qubit state, that is to be teleported is given by:

$$
|\psi_3\rangle_{abc} = (\alpha_1|000\rangle + \alpha_2|001\rangle + \alpha_3|011\rangle + \alpha_4|111\rangle + \alpha_5|110\rangle + \alpha_6|010\rangle + \alpha_7|100\rangle + \alpha_8|001\rangle)_{abc}. \tag{10}
$$

The circuit diagram that generates $|\phi_6\rangle$ that is used to teleport $|\psi_3\rangle$, is shown in Fig 1:

Fig 1 : Circuit diagram for the construction of $|\phi_6\rangle$.

The corresponding six qubit "mirror state" reads:

$$
|\phi_6\rangle = \frac{1}{2\sqrt{2}}(|000\rangle |\psi_+\rangle|00\rangle + |01\rangle |\psi_+\rangle|10\rangle + |11\rangle |\psi_+\rangle|11\rangle + |10\rangle |\psi_+\rangle|01\rangle)_{163452} \tag{11}
$$

Alice can perform a joint six partite measurement on "abc163" and classically communicate the outcome of her measurement to Bob via six qubits of information. For instance, if the outcome of Alice’s measurement is,

$$
|\phi_{x2}\rangle = \frac{1}{2\sqrt{2}}(|000010\rangle + |000100\rangle + |011111\rangle + |111110\rangle + |100001\rangle + |100011\rangle + |101010\rangle + |010101\rangle)_{abc163}. \tag{12}
$$

the corresponding state obtained by Bob is:

$$
|\phi_{x2}\rangle = (\alpha_1|001\rangle + \alpha_2|000\rangle - \alpha_3|111\rangle + \alpha_4|011\rangle + \alpha_5|100\rangle + \alpha_6|110\rangle + \alpha_7|010\rangle + \alpha_8|001\rangle)_{452}. \tag{13}
$$

Bob can perform a suitable controlled phase shift gate and a unitary transformation to obtain $|\psi_3\rangle$. This completes the teleportation protocol of an arbitrary three qubit state using $|\phi_6\rangle$. Entanglement is also crucial to Quantum information splitting (QIS) $\cite{13,22}$, which refers to the technique of splitting and sharing of quantum information among two or more parties such that none of them can retrieve the information fully by operating on their own qubits. For a given entangled channel, one can construct different protocols for QIS by distributing the qubits among the parties in different ways. However, some protocols are preferred over others, owing to their robustness and experimental feasibility. It has been conjectured $\cite{17}$ that through a genuinely entangled channel of $N$ qubits, a maximum of $(N - 2n)$ protocols can be devised for the QIS of an arbitrary $n$ qubit state among two parties in the case where they need not meet. According to this, one can devise a maximum of $k$ protocols for QIS of an arbitrary $(N - k)$ state $(k < N)$ using $|\zeta_{2N}\rangle$. A protocol can be considered successful only if, after Alice performs the measurement, Bob-Charlie system collapses into a partially entangled state. In general any $P$ qubit entangled state having $P \geq (2N + 1)$ qubits can be used for QIS of an arbitrary $N$ qubit state, if it is possible to project any $(2N - 1)$ qubits into the form

$$
\sum_1^{2^{(N-1)}} |\Omega\rangle_{N-1} U_1 \otimes U_2 \otimes ... \otimes U_N |\rho_i\rangle_N \tag{14}
$$

by performing local measurements on the other qubits. Here $|\rho_i\rangle_N$ refers to the Bell and the GHZ states for $N = 2$ and $N > 2$ respectively, $|\Omega\rangle$ refers to the computational basis which contains one qubit less than $|\rho_i\rangle_N$ and $U_i \in (\sigma_1, I)$ represents a bit flip or an identity operation which acts on $|\rho_i\rangle_N$ rendering another orthogonal Bell or GHZ state. While there are a total of $2^N$ orthogonal GHZ states for a $N$ qubit system, we consider only half of them, namely the ones with a positive phase difference between the superposition terms. This condition is satisfied for $\zeta_{2N}$ and it can be used for the QIS of an arbitrary $k$ qubit state for $k < N$. Let us consider the example of $|\phi_6\rangle$ for splitting up of $|\psi_2\rangle$, in the case where Alice possesses the first three qubits, Bob possesses the fourth qubit and Charlie possesses the last two qubits. Alice can perform a joint five qubit measurement on her qubits and convey the outcome of her measurement to Charlie. For instance, if the outcome of Alice’s measurement reads $
\frac{1}{2\sqrt{2}}(|0\rangle |\psi_+\rangle|00\rangle + |1\rangle |\psi_+\rangle|11\rangle)$ then the Bob-Charlie system collapses to the entangled state, $(\alpha_{00}|000\rangle - \alpha_{01}|111\rangle + \alpha_{10}|001\rangle + \alpha_{11}|110\rangle)$. Bob can now perform a measurement, in the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and communicate the outcome of his measurement to Charlie. Having known the outcomes of both their measurements, Charlie obtains the state by performing an appropriate controlled phase shift gate followed by an unitary operation. Hence, this protocol succeeds. All the protocols succeed for the QIS of $|\psi_3\rangle$ and $|\psi_6\rangle$, using $|\phi_6\rangle$ as an entangled channel. The security of these protocols against eavesdropping attacks will require further investigation. Let us now turn our attention towards superdense coding.

The general condition for an entangled channel $|\Gamma_{AB}\rangle$, where $A$ and $B$ refer to the subsystems of Alice and Bob
respectively, to be used for superdense coding of 2\(k\) qubits \((k \leq N)\) in \(k\) qubits is that there has to be at least \(k\) qubits of entanglement between \(A\) and \(B\). The channel capacities of higher qubit GHZ and the cluster states does not reach the “Holevo bound”, while that of \(|\zeta_{2N}\rangle\) does. It is interesting to notice that it is always possible to generate a set of \(4^k\) orthogonal states by locally unitary operations on the first \(k\) qubits of \(|\zeta_{2N}\rangle\). Hence, if we let, Alice have first \(k\) particles and Bob, the remaining \((2N-k)\) particles in \(|\zeta_{2N}\rangle\), then Alice can perform unitary operations on her particles and convert it into a set of orthogonal states. After performing the unitary operations, Alice can send her particles to Bob. Bob can then, perform a measurement and retrieve the classical information. The given entangled channel can be used to send \(2k\) qubits by sending \(k\) qubits while consuming \(k\) qubits of entanglement. The channel capacity \(|\zeta_{2N}\rangle\rangle\) of \(|\zeta_{2N}\rangle\rangle\) reaches the “Holevo bound”, which is given by, \(X(\rho^{AB}) = N + N - 0 = 2N\) allowing \(2N\) classical bits to be transmitted through \(N\) quantum bits consuming only \(N\) qubits of entanglement. This is impossible using a GHZ, W or the cluster states of more than five qubits. Thus, this state can also be used instead of the Bell or the cluster states. A detailed investigation of Quantum secure direct communication schemes using \(|\zeta_{2N}\rangle\rangle\) with super dense coding is under current investigation.

II. CONCLUSION

In conclusion, we introduced a new genuinely entangled \(N\) partite analogue of the Bell state known as the “mirror state” with spectacular properties. The proposed state is experimentally realizable by performing a SWAP operation between the second and the last qubit followed by a controlled phase shift operation between the first \((N+1)\) qubits in \(N\) Bell pairs. Since SWAP and controlled phase shift operations have been experimentally realized in different systems, the production of the state is experimentally accessible. This state turns out to be an important resource for quantum communication purposes like teleportation, information splitting and superdense coding. The introduced “mirror state” equals the well known Bell and the cluster states for \(N = 1\) and \(N = 2\) and differs from the class of cluster states for \(N \geq 3\). It is shown that, the proposed state can be used for the teleportation of an arbitrary \(N\) qubit state and information splitting of an arbitrary \((N-k)\) qubit state. The state is found to be an excellent resource for superdense coding as well. The given entangled channel can also be used to send \(2k\) qubits by sending \(k\) qubits by utilizing \(k\) qubits of entanglement \((k \leq N)\), making the superdense coding capacity reach the "Holevo bound". The experimental creation of these states, is still a challenge although its experimentally feasible in condensed matter and NMR systems. A detailed analysis of the decoherence properties of this state is under current investigation. The robustness of the protocols considered in this paper against particle loss and eavesdropping attacks also needs to be investigated. Apart from the quantum communication protocols discussed here, we hope that the present state with useful entanglement properties will find applications in other aspects of quantum information science.