Solving the problem of the evolutionary tasks of economics and organization of production in spatial networks

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Abstract. The complex structure of modern production involves the development of innovative algorithms for servers control. It is necessary to take into account the structure and organization of the enterprise for a long planning horizon. In addition to the intellectual processing of data about the external environment, economic calculation is required. Currently the ways used for solving problems by methods of mathematical programming are insufficient, since production and commerce over the past decades have been combined into branched transnational networks. In this paper, an approximation approach to the analysis of evolutionary problems is proposed. The modeling of industrial-commercial networks using graph theory is applied. Spatial variables that strike the entire spectrum of indicators of the current activity of the enterprise have been introduced.

1. Introduction
Classical methods of numerical analysis involve the use of areas of the variation of a spatial variable that do not have structural features. These features are generated by the presence of a finite number of joints (joints) of classical regions. Such areas include the spatial graph (spatial network), whose ribs are connected at the end points - nodes (internal vertices). The differential equation at such points is replaced by the matching conditions (generalized Kirchhoff conditions) and it is required to find an applied justification for such conditions. There are the approaches below [1-5], which make it possible to describe the mathematical formalisms of evolutionary equations of the physical and economic transport processes (including diffusion processes) quite completely [6-8].

2. The main formalisms
Let the operator $A^0$ be defined by a differential expression

$$A^0 \varphi = -\frac{d^2}{dx^2} \varphi + q(x)\varphi, \ x \in (\mathfrak{A} \setminus \partial \mathfrak{A}) \setminus J(\mathfrak{A}),$$

(1)
on the set $\mathcal{R}$ of functions $\varphi(x) \in C(\mathcal{I}) \cap C^2[\mathcal{I}]$ satisfying the following conditions in an arbitrary node $\xi$ of the set $J(\mathcal{I})$ of internal nodes of the tree $\mathcal{I}$:

$$\sum_{i=1}^{m_i-1} y'(a_i^i)_{\gamma_i^i} = y'(a_i_{\gamma_i^i})_{\gamma_{i+1}^i},$$

where $m_i$ – the number of ribs $\gamma_i^i$ adjacent to the node $\xi$, $a_i$ – is a fixed number corresponding to the node $\xi$ due to the chosen parameterization of the tree $\mathcal{I}$ (for the general notation on the graph, see [2, 3, 9-13]); $q(x) \in C[\mathcal{I}]$ ($q(x) > 0$) – real function. The domain of the definition $\Phi^0$ of an operator $\Lambda^0$ is a set of functions $\varphi(x) \in \mathcal{R}$ and which satisfy the boundary conditions

$$\varphi(b_i) = 0, \xi \in \partial J,$$  

the number $b_i$ corresponds to the node $\xi$ due to the selected parameterization on $\mathcal{I}$. Consider the problem in the operator form

$$\Lambda^0 \varphi = f, \varphi \in \Phi^0,$$

where $f \in F$, $\Phi^0$ and $F$ – subspaces $L^2(\mathcal{I})$.

Along with the problem (4), we consider the problem in a finite-dimensional space of grid functions $\varphi^h$

$$\Lambda^0 \varphi^h = f^h, \varphi^h \in \Phi^{0h},$$

where $\Lambda^{0h}$, – the linear operator depends on the grid step $h$, $f^h \in F^h$, and $\Phi^{0h}$ and $F^h$ – the spaces of grid functions corresponding to the spaces $\Phi^0$ and $F$ (notation is used here [12-15]). We introduce in the grid spaces $\Phi^{0h}$ and $F^h$ the grid norms $\| \cdot \|_{\Phi^{0h}}$ and $\| \cdot \|_{F^h}$. Let $(\cdot)^h$ – a linear operator associates an element $(\varphi)^h \in \Phi^{0h}$ with an element $\varphi \in \Phi^0$ such that

$$\| (\varphi)^h \|_{\Phi^{0h}} \to \| \varphi \|_{\Phi^0}, \quad h \to 0.$$  

### 3. Definition

Problem (5) approximates problem (4) with order $s$ relative to the step $h$ on the solutions $\varphi \in \Phi^0$, if there exist positive constants $h^*, M$ such that for all $h < h^*$ the inequalities

$$\| \Lambda^{0h}(\varphi)^h - f^h \| \leq Mh^s,$$

in other words, the operator $\Lambda^{0h}$ approximates the operator $\Lambda^0$ on the solutions $\varphi$ of problem (4) if relation (7) holds. In the case when the solution $\varphi$ to problem (4) is sufficiently smooth, it is convenient to find the approximation order using the norm that is natural for the space of continuous and differentiable functions. We introduce the difference expressions $(\nabla^h y^h)_k = 1/h((y^h)'_{k+1} - (y^h)'_{k-1})$, $(\nabla^h y^h)'_k = 1/h((y^h)'_{k+1} - (y^h)'_k)$, for the grid function $y^h$. Let us consider the approximations of the operator $\Lambda^0$ on the simplest graph, a chain of $L$ stars and an arbitrary tree, as well as the approximations of the evolution equations generated by the operator $\Lambda^0$.

We denote by $\mathcal{I}_0$ the simplest graph and consider the set $\mathcal{R}_{\mathcal{I}_0}$ of functions $y(x) \in C(\mathcal{I}_0) \cap C^2[\mathcal{I}_0]$ whose first derivative at each node $\xi_i (i = 1, M - 1)$ satisfies the conditions:
\[ y'(i\pi / M)_{\gamma_i} - y'(i\pi / M)_{\gamma_i} = \alpha_i y'(i\pi / M)_{\gamma_i}, \quad i = 1, M - 1. \] (8)

Let further \( \Lambda^0 = \Lambda^0_{\mathcal{N}_0}, \Phi^0 = \Phi^0_{\mathcal{N}_0} \) and let \( \varphi \in \Phi^0_{\mathcal{N}_0} \) - the solution to the problem

\[ \Lambda^0_{\mathcal{N}_0} \varphi = f, \quad f \in F_{\mathcal{N}_0}. \] (9)

Denote by \( \mathcal{R}^h_{\mathcal{N}_0} \) the set of functions \( y^h \), satisfying the conditions \((i=1,M-1)\)

\[ (y^h)''_{i} = (y^h)''_{i+1}, \quad 1 / h (y^h)'''_{i+1} - (y^h)'''_{i} - 1 / h((y^h)_i^h - (y^h)_i^{h-1}) = \alpha_i (y^h)_i^{h+1}, \]

approximating conditions \((8)\). The operator \( \Lambda^{0h}_{\mathcal{N}_0} \) has the representation

\[ (k = 1, n - 1, i=1, M) \quad (\Lambda^{0h}_{\mathcal{N}_0} y^h)_i = -1 / h^2 ((y^h)_{i+1}^h - 2(y^h)_i^h + (y^h)_{i-1}^h) + (q^h)'(y^h)'_i; \]

the domain \( \Phi^{0h}_{\mathcal{N}_0} \) of this operator is the set of grid functions \( y^h \in \mathcal{R}^h_{\mathcal{N}_0} \) that vanish at the \( \partial \Phi^{0h}_{\mathcal{N}_0} \) grid boundary \( \Phi^{0h}_{\mathcal{N}_0} \), i.e. functions \( y^h \) satisfy the conditions \((y^h)_i^h = (y^h)_M^h = 0\), approximating boundary conditions \((3)\).

4. Solution

Expanding the solution \( \varphi(x) \) by the Taylor formula in a neighbourhood of points \( x_i^j \in \gamma_i \)

\( (k = 1, n - 1, i=1, M) \) and assuming that the derivatives are bounded to fourth order inclusive at points

\( x \in (\mathcal{N} \setminus \partial \mathcal{N}) \cup J_{\mathcal{N}_0} \), we obtain

\[ \varphi(x)_{\gamma_i} = \sum_{\nu=0}^{3} \varphi^{(\nu)}(x_i^j)_{\gamma_i} (x - x_i^j)^\nu + 1 / 4! \varphi^{(4)}(x_i^j + \theta' h)_{\gamma_i} (x - x_i^j)^4, \quad i = 1, M, \] (10)

where \( x \in \Gamma_{x_{i-1}^j} \cap \Gamma_{x_{i+1}^j} \in \gamma_i, \quad |\theta'| < 1 \) \((i=1,M)\). We will have a similar expansion for the function

\( f(x) \in F_{\mathcal{N}_0}. \)

We introduce in space \( F_{\mathcal{N}_0} \) the quantity

\[ ||f^h||_{F_{\mathcal{N}_0}^h} = \max_{x-i \leq h, i \leq x+i} |(f^h)_i|. \] (11)

Let's take as \( (\varphi)^h \) a vector the components of which are the values of the function \( \varphi \) at the corresponding grid \( \mathcal{N}^h_0 \) points belonging to the set \( \mathcal{R}^h_{\mathcal{N}_0} \). Then, using the expansion of functions \( \varphi(x) \) and \( f(x) \) by the Taylor formula of the form \((10)\), we obtain

\[ ||\Lambda^{0h}(\varphi)^h - f^h||_{F_{\mathcal{N}_0}^h} \leq Mh^2, \] (12)

where \( M = 1 / 12 \max_{x_{i-1}^j, x_{i+1}^j \in \gamma_i} |(f^{(4)}(x))|. \) The approximation of conditions \((8)\) has an error of the order

of 1 with \( h \) respect to assuming that the derivatives are bounded to the second order inclusive in the

semicircle of the end points \( x_{i-1}^j, x_{i+1}^j \) of the ribs \( \gamma, i = 1, M - 1 \). The approximation of the boundary

conditions \((3)\) is accurate.

Further, let \( \mathcal{N} \) - the graph tree, \( \Lambda^0 = \Lambda^0_{\mathcal{N}}, \Phi^0 = \Phi^0_{\mathcal{N}} \) and \( \varphi \in \Phi^0_{\mathcal{N}} \) - the solution to the problem

\[ \Lambda^0_{\mathcal{N}} \varphi = f, \quad f \in F_{\mathcal{N}}. \] (13)

The set \( \mathcal{R}^h_{\mathcal{N}} \) of grid functions \( y^h \) with values on the grid \( \mathcal{N}^h \) is defined by relations that are difference analogues of conditions \((2)\):
\[(y^h)^\xi = (y^h)_0^m \quad (i = 1, m_i - 1), \quad \sum_{i=1}^{m_i} ((y^h)_i^\xi - (y^h)_{i-1}^\xi) = (y^h)_0^m - (y^h)_0^m,\]

for all nodes \( \xi \in J(\mathcal{Z}) \). The operator \( \Lambda^h \) on the grid functions \( y^h \in \mathcal{R}^h \) has the representation

\[\Lambda^h = -1 / h^2 \sum_{i=1}^{m_i} ((y^h)_i^\xi - 2(y^h)_i^\xi + (y^h)_{i-1}^\xi + (q^h)_{i-1}^\xi),\]

here \( \xi \) – an arbitrary node of the set \( J(\mathcal{Z}) \), \( q^h \) – a grid function on the grid \( \mathcal{Z}^h \), the corresponding function \( g(x) \in C[\mathcal{Z}] \). Similarly to the previous one, provided that there exist bounded derivatives of the solution \( \varphi \) of equation (18) up to the fourth order inclusive at points \( x \in (\mathcal{Z} \setminus \partial \mathcal{Z}) \setminus J(\mathcal{Z}) \), we obtain the estimate

\[\| \Lambda \varphi - f \|_{x, 2} \leq M h^2,\]  

(14)

where \( M = \sup_{x \in \mathcal{Z} \setminus \partial \mathcal{Z}} | f^{(4)}(x) | \) the norm in space \( F_{x_3}^h \) is defined by the relation

\[\| f^h \|_{x_3} = \max_{x \in \mathcal{Z}, t \in \gamma \in \mathcal{Z}} | (f^h)^{x,t} |,\]

here \( (f^h)^{x,t} \) – the restriction of the grid function \( f^h \) to the grid \( \gamma \) of the tree \( \mathcal{Z} \). The approximation of conditions (2) has an error of the order of 1 with respect \( h \) to assuming that the derivatives are bounded to the second order inclusive in the semi-neighbourhoods of the points corresponding to the nodes \( \xi \in J(\mathcal{Z}) \), the approximation \( (y^h)_0^m = 0, (y^h)^\xi = 0 \), where \( \xi \in \partial \mathcal{Z} \), of the boundary conditions (3) is accurate.

Next, we consider approximations of evolution equations on an arbitrary tree \( \mathcal{Z} \). Let \( \mathcal{R}(t) \) – a linear variety of functions \( \varphi(x,t) \) for each fixed \( t \in [0,T] \) belong to the set \( \mathcal{R} \), which may be one of the sets \( \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2 \) considered above. Consider the problem

\[\frac{\partial \varphi}{\partial t} + \Lambda \varphi = f, \quad x, t \in (\mathcal{Z} \setminus \partial \mathcal{Z}) \setminus J(\mathcal{Z}) \times (0,T), \quad \varphi |_{t=0} = \theta, \quad x \in \mathcal{Z} \setminus \partial \mathcal{Z},\]  

(15)

on functions \( \varphi(x,t) \in \Phi \) where \( \Phi \) – the set of functions \( \varphi \in \mathcal{R}(t) \) satisfying the condition

\[\varphi = g, \quad x, t \in \partial \mathcal{Z} \times [0,T];\]  

(16)

\( \theta \in G \), the operator \( \Lambda^0 \) - is one of the operators considered above \( \Lambda^0_{\mathcal{Z}_0}, \Lambda^0_{\mathcal{Z}_1}, \Lambda^0_{\mathcal{Z}_2} \). We assume that problem (15) has a unique solution \( \varphi \), and this solution is continuous in \( \mathcal{Z} \times [0,T] \), derivatives \( \frac{\partial \varphi}{\partial t} (\nu = 1, 2), \frac{\partial \varphi}{\partial \xi} (\zeta = 1, 2, 3, 4) \) continuous in \( ((\mathcal{Z} \setminus \partial \mathcal{Z}) \setminus J(\mathcal{Z})) \times (0,T) \).

5. Results

We approximate the problem (15) in two stages. First, we approximate this problem in the domain \( \mathcal{Z}^h \times [0,T] \) with respect to the spatial variable. As a result, we arrive at a differential equation with respect to time and a difference equation with respect to the spatial variable. In the obtained differential-difference problem, it is easy to exclude the solution values at the boundary points of the domain \( \mathcal{Z}^h \times [0,T] \) in accordance with the difference analogue of conditions (22) in problem (15), (16). Assume that this is done, we arrive at a problem of the form

\[\frac{d \varphi^h}{dt} + \Lambda \varphi^h = f^h, \quad \varphi^h = \varphi^h,\]  

(17)
where $\varphi^b, f^b$ – the grid functions of time $t$; a finite-difference operator $\Lambda^0$ can be one of the operators $\Lambda^0_{\varphi^b}, \Lambda^0_{f^b}$. The difference scheme with an order error $O(\tau + h^s)$ ($s$ – the approximation order of the operator $\Lambda^0$ by the difference operator $\Lambda^0_{\varphi^b}$) has the form:

$$(\varphi^b_{j+1} - \varphi^b_j) / \tau + \Lambda^0_{\varphi^b} \varphi^b_j = f^b_{j+1}, \quad \varphi^b_0 = \varphi^b, \quad f^b_{j+1} = f^b_j,$$

where $f^b_j$ – are the projection components of the function $f^b_{j+1}$ onto the grid {$t_j : t_j = j \tau (j = 0, M), \tau = \frac{s}{a}$.}

6. Conclusion

The obtained results can be used in the numerical analysis of the problems of the stabilization [16–18] and the determination of solutions of differential systems with the delay [19–21].

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