Fast DecreaseKey Heaps with worst case variants

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Abstract
In the paper [1] we have described heaps with both Meld, DecreaseKey and DecreaseKey
interfaces, allowing operations with guaranteed worst case asymptotically optimal times. The
paper was intended to concentrate on DecreaseKey interface, but it could be hard to separate
the two described data structures without careful reading. Current paper’s goal is not to invent a
novel data structure, but to describe the rather easy DecreaseKey version in hopefully readable
form. The paper is intended not to require references to other papers.

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1 Introduction

Heap is a data structure maintaining set of elements with keys where keys belong to a linearly
ordered universe. Minimal heap interface supports Insert and ExtractMin methods. Insert
adds an element to the structure, ExtractMin returns and removes the element with the
smallest key from the structure. In the case the structure is empty, ExtractMin returns
null. In the case there are more elements with the minimal key in the heap, some of them is
chosen.

As we cannot distinguish n! possible permutations using just comparisons faster then by
Ω(log n!), at least one of the operation requires Ω(log n) time (we could sort using n Inserts
and ExtractMins). As on nonempty heap ExtractMin cannot be used often than Insert
the interface with O(1) running time for Insert, and O(log n) for ExtractMin, where n
denotes current heap size, is asymptotically optimal.

It could be advantageous to introduce FindMin method, which returns the minimal
heap element and it remains in the heap. ExtractMin used to be renamed to DeleteMin,
and it calls FindMin internally. Optimal interface declares O(1) for Insert and FindMin,
and O(log n) for DeleteMin.

DecreaseKey heap interface introduces DecreaseKey method, which allows decrease
of the key of a pointed heap element. To allow pointing, the Insert method should return
pointer to the element to be used for future references. Optimal DecreaseKey heap interface
declares O(1) for Insert, FindMin and DecreaseKey, and O(log n) for DeleteMin.

A lot of implementations of asymptotically optimal amortized DecreaseKey heap interface
are known for a long time. Recently several implementations of asymptotically
optimal worst case DecreaseKey heap interface were published. In this paper we present
relatively simple heap with asymptotically optimal worst case DecreaseKey heap interface.
Overview

We have an additional requirement, that all keys should be different. This could be achieved by lexicographical comparison with added second coordinate. Usually using binary representation of the pointer to the node as the second coordinate would work.

The base of our heap will be a tree of logarithmic arity which is maintained heap ordered, what means parent key is always smaller to the child key. Temporarily during execution of the heap methods the heap could consist of a list of such trees. FindMin method’s goal is to convert the forest to one tree, and return the root.

At most logarithmic arity is important for DeleteMin method as all root children become list of heap trees and internal FindMin require time proportional to the number of trees. To maintain the arity in bounds, we should carefully pick elements to compare. This is why ranks of nodes are introduced. Nodes are created with rank 0. Whenever two nodes of the same rank are compared, edge is added such that the node with smaller key becomes parent of the other and the rank of the parent is incremented. Most of comparisons we would make are between nodes of the same rank. Edges corresponding to such comparisons would be called rank edges. Unmaintanable ideal heap would have only rank edges. In reality there must be some violations to this. There will be nonrank edges in the tree created by comparisons of nodes of different ranks. We call children connected by nonrank edges nonrank roots, root is considered nonrank root as well.

To support DecreaseKeys some nodes could lose some of their children. To keep ranks in logarithmic bounds, we should be carefull with such losses. We define loss for a node. Rank roots have loss 0 by definition. When a rank edge is created the loss of the child is 0. Whenever a node loses a rank child, its rank is decremented. Whenever node other than nonrank root has rank decremented, its loss is incremented (loss violation created).

We introduce a strategy which keeps both loss violations (sum of all losses) and rank roots violations (number of nonrank roots) logarithmically bounded. To maintain the violations in bounds, we maintain pointers to affected nodes in arrays addressed by their ranks. Whenever we are ready to insert second pointer to the same place, we have two nodes of the same rank ready to make a violation reduction.

It could happen the array becomes almost full and invocation of one method could allow big reduction of the violations even when worst case bound does not allow us to perform all the allowed reductions. This is why we maintain stacks of pending pointer insertions and we have to define strategy when to stop processing the stacks. Amortized strategy would just empty the stacks. We would propose two worst case strategies, both would maintain both types of violations on at least maximal rank plus one (corresponding to empty stacks and maximally filled arrays).

We will show maximal ranks are logarithmic in the chapter 3. Description of violation reductions would follow in chapter 4, stack reduction strategies analysis will start there and end in chapter 8. Chapters 5, 6, and 7 would concentrate on implementation details.

Maximal rank

Let us study relathionship between number of nodes, maximal rank and maximal total loss. Let $R$ be maximal rank and $L$ maximal total loss, let us try to find minimal possible number of nodes $n(R, L)$, where it could appear. As we would maintain $L \leq R + 1$, we are especially interested on $n(R, R + 1)$. Inverse of this function will give use maximal possible rank $R(n)$ for heap of $n$ nodes. Binomial tree with root of rank $R$ is a tree obtained by adding edge among roots of two binomial trees of rank $R - 1$. Such tree has $2^R$ nodes and could be
depicted as root having roots of binomial trees of ranks 0 up to $R - 1$ as children. All edges of a binomial tree are rank edges. Let us call binomial tree of rank $R$ with losses any tree which could be obtained from binomial tree of rank $R$ by cutting edges where none cuts a root child and cutting a rank child of a successor is accompanied with rank decrement of the successor and loss increase there (then need not be cut at all).

\textbf{Lemma 1.} During history of the heap for any node $x$ of rank $R$ holds that there is a subtree $B_x$ rooted at $x$ of the full subtree $T_x$ rooted at $x$ such that $B_x$ is binomial tree of rank $R$ with losses.

\textbf{Proof.} Let us call the stated subtree $B_x$ binomial guarantee under $x$. By induction of the heap history. There are two cases. Either the last operation creates an edge or destroys an edge. Start with the latter case.

If the destroyed edge was nonrank edge, it was not part of a binomial guarantee, no rank has changed so the lemma remains valid.

If the destroyed edge was a rank edge $(x, y)$ with $y$ closer to root, just all ancestors of $x$ by rank path are affected. Let $x$ had rank $R_x$. Prior to the operation there was a binomial guarantee $B_x$ under $x$. Let subtree under vertex $y$ corresponded to binomial subtree of $B_x$ of rank $i$ with losses. After the cut the rank of $x$ changes to $R_x - 1$. A children $c$ of $x$ which corresponded to binomial subtree of $B_x$ of rank bigger than $i$ (with losses) contain binomial subtree with losses of one smaller rank (just ignore the youngest child of $c$ in $B_x$ prior the cut). This gives us new binomial subtree with losses of rank $R_x - 1$ so binomial guarantee under $x$ after the cut. For an ancestor $a$ by rank path different from $x$ let $R_a$ be its rank. There existed binomial guarantee $B_a$ under $a$. If the $(x, y)$ edge was not part of the subtree, the same subtree $B_a$ remains binomial guarantee under $a$. Otherwise cutting subtree under $(x, y)$ from $B_a$ creates binomial guarantee under $a$ after the cut (the rank and loss is updated).

Similarly adding nonrank edge does not change ranks and guarantees remain valid.

Adding rank edge from parent $x$ to nonrank root $y$ of the same rank $R$ increases rank of $x$ to $R + 1$. This operation is allowed only for nonrank root $x$ or node $x$ with loss equal 1. Binomial guarantee $B_x$ under $x$ connected by $(x, y)$ to binomial guarantee $B_y$ of $y$ creates binomial guarantee under $x$ after the link. For an ancestor $a$ of $x$ by rank path different from $x$ the binomial guarantee under $a$ is extended by $B_y$ as well (the rank and loss of $x$ is updated).

Thanks to the lemma $n(R, L)$ must be a binomial tree of rank $R$ with losses. Our goal is to cut $L$ edges from the binomial tree of rank $R$, not decreasing the rank such that the tree becomes as small as possible. Cutting children is forbidden as it would decrease rank. Cutting deeper successors than grandchildren is ineffective as cutting the grandchild on the path decreases the tree size more. The most effective strategy is to cut the grandchildren whose subtrees are the largest. There are $k$ grandchildren of rank $R - 1 - k$ so of the size $2^{R-1-k}$. There are $\sum_{i=1}^{k} i = \binom{k+1}{2}$ grandchildren of rank at most $R - 1 - k$. If $L < \binom{k+1}{2}$, we know no grandchild of rank $R - 2 - k$ was cut in the smallest tree of rank $R$ and total loss $L$ and therefore $n(R, L) > (k + 1) \cdot 2^{R-2-k}$.

We are especially interested in the case $n(R, R + 1)$, so $R + 1 < (k^2 + k)/2$. This gives us $k^2 + k - 2R - 2 > 0$. Higher root of this polynomial is $\sqrt{2R + 9/4} - 1/2$ so for $k \geq \sqrt{2R + 9/4} + 1$ we have $n(R, R + 1) > (k + 1) \cdot 2^{R-2-k} \geq (2 + \sqrt{2R + 9/4}) \cdot 2^{R-2} - \frac{2^{R-3} - \sqrt{2R+9/4}}{2^{R-2}} \in \Omega(2^{R(1-\varepsilon)})$. Our goal is to find a nice function what will still be lower bound for $n(R, R + 1)$. Numerical evaluation clearly shows $2^{(R-4)/2}$ is for all integers smaller than the shown bound and therefore $R(n) < 4 + 1.2 \log_2 n$. 

\[\]
1:4 Fast DecreaseKey Heaps with worst case variants

Table 1 Effect of stacks reductions $|R_A| + 2|C_A| = \Phi_A$, $3|R_L| + 4|C_L| = \Phi_L$, $\Phi = \Phi_A + \Phi_L$

| Reduction               | $|R_A|$ | $|C_A|$ | $\Phi_A$ | $|R_L|$ | $|C_L|$ | $\Phi_L$ | $\Phi$ | $p$ |
|-------------------------|---------|---------|----------|---------|---------|----------|--------|-----|
| $|C_A|$ type $\neq A$    | 0       | -1      | -2       | 0       | 0       | -2       | 0      | 0   |
| $|C_A|$ type $A$ no match| +1      | -1      | -1       | 0       | 0       | 0        | -1     | 1   |
| $|C_A|$ type $A$ matched  | -1      | 0       | -1       | 0       | 0       | -1       | 2      |     |
| $|C_L|$ type $\neq L$    | 0       | 0       | 0        | 0       | -1      | -4       | -4     | 0   |
| $|C_L|$ subtype $L_2$    | $\leq 0$| $\leq +2$| $\leq +3$| $\leq 0$| $\leq -3$| $\leq -1$| $\leq 3$|     |
| - parent $A$ in $R_A$   | -1      | +2      | +3       | 0       | $\leq -2$| $\leq -8$| $\leq -5$| 3   |
| - parent $A$ in $C_A$   | 0       | +1      | +2       | 0       | $\leq -2$| $\leq -8$| $\leq -6$| 1   |
| - parent $L_1$ in $R_L$ | 0       | +1      | +2       | -1      | $\leq 0$ | $\leq -3$| $\leq -1$| 3   |
| - parent $L$ in $C_L$   | 0       | +1      | +2       | 0       | $\leq -1$| $\leq -4$| $\leq -2$| 1   |
| - parent $N$            | 0       | +1      | +2       | 0       | $\leq -1$| $\leq -4$| $\leq -2$| 2   |
| $|C_L|$ subtype $L_1$ no match | 0 | 0 | 0 | +1 | -1 | -1 | -1 | 1 |
| $|C_L|$ subtype $L_1$ matched | $\leq 0$| $\leq +1$| $\leq +1$| $\leq 0$| $\leq +1$| $\leq -2$| $\leq -2$| $\leq 3$ |
| - parent of $h$ in $R_A$ | -1      | +1      | +1       | -1      | -1      | -7       | -6     | 3   |
| - parent of $h$ in $C_A$ | 0       | 0       | 0        | -1      | -1      | -7       | -7     | 1   |
| - parent of $h$ in $R_L$ | 0       | 0       | 0        | -2      | +1      | -2       | -2     | 3   |
| - parent of $h$ in $C_L$ | 0       | 0       | 0        | -1      | 0       | -3       | -3     | 1   |
| - parent of $h$ in $N$  | 0       | 0       | 0        | -1      | 0       | -3       | -3     | 2   |

Here $p$ denotes number of pointer changes not reflected in heap trees during reduction. We can see each stack reduction decrements $\Phi$ by at least 1.

4 Violation reductions

![Nontrivial cases of reductions to maintain the heap shape](image)

Figure 1: Nontrivial cases of reductions to maintain the heap shape

Each node remembers its violation type, which is either $A$, $L$ or $N$ otherwise.

Rank roots would be maintained as violations of type $A$. Violations of type $A$ would be pushed to stack $C_A$ and from the stack popped to the array $R_A$ addressed by ranks.

During $|C_A|$ reduction if popped stack node $x$ is already not of type $A$, the stack item is just discarded. The stack item is just discarded as well if $R_A$ for $x$’s rank points to $x$. Other case is $R_A$ for $x$’s rank contains null, than pointer $x$ is stored there, and $|C_A|$ reduction step ends. Last, and the most important case is when $R_A$ for $x$’s rank pointed to other violation $y$ of the same rank and actual violation type $A$, reduction step could be applied after putting null to $R_A$ of $x$’s rank. Violation reduction step of type $A$ links nodes $x$, $y$ of the same rank.
(Their keys are compared, let node $s$ be the one with smaller key while $h$ the other. We cut $h$ from its parent (if there exists nonrank edge) and put it as a rank child of $s$. This increases rank of $s$. Node $h$ violation type is changed to $\mathbb{N}$, node $s$ is added to $C_A$.)

All nodes with nonzero loss would be maintained as violations of type $L$, this type has subtype $L_1$ for nodes with loss exactly 1 and subtype $L_2$ for nodes with loss at least 2. Violations of type $L$ would be pushed to stack $C_L$, from which popped violations of subtype $L_1$ will be inserted to the array $R_L$ adressed by ranks. Symbol $|C_L|$ has weighted meaning. Weight of nodes of subtype $L_2$ corresponds to the loss of the node, while weights of other nodes are 1 (including nodes of other violation type than $L$).

Similarly as for $|C_A|$ reduction, when during $|C_L|$ reduction popped stack node $x$ is already not of type $L$ or $R_L$ of $x$’s rank points to $x$, the stack item is just discarded. Different is the second case when $x$’s subtype is $L_2$. It invokes one node loss reduction, which takes node $x$ with loss at least 2, it makes it nonrank child of it’s parent $p$. This creates new rank root $x$, so $x$ is put to $C_A$ and violation type of $x$ is changed to $\mathbb{A}$. The rank of $p$ is decremented. Unless violation type of $p$ is $\mathbb{N}$, $p$ should be removed from the array identified by its type (If there was null in the place, we know $p$ resists on the stack). If $p$ is a rank child it should be inserted to $C_L$ (if it does not resist there), and type changed to $L$, its subtype should be set to reflect the loss increase. Total loss was reduced by at least 1. If $p$ is a rank root, it should be pushed to the stack $C_A$ unless it already resists there. Third case of $|C_L|$ reduction is for $L_1$ subtype when the array $R_L$ of $x$’s rank contains null. As for $|C_A|$ reduction the pointer to $x$ is stored in $R_L$ and the $|C_L|$ reduction step ends. Last case is when $R_L$ for $x$’s rank (for node of subtype $L_1$) pointed to other violation $y$ of the same rank and actual violation type $L$ reduction step could be applied after putting null to $R_L$ of $x$’s rank. Violation reduction step of type $L$ for nodes $x$, $y$ of equal rank and loss 1 links the two nodes. (Their keys are compared, let $h$ and $s$ be the nodes with higher and smaller keys respectively. Remove $h$ from it’s parent and link it under $s$ by a rank edge. This reduces loss of $s$ to 0 and sets loss of $h$ to 0, so both $s$ and $h$ violation types are changed to $\mathbb{N}$. Original parent $p$ of $h$ decrements rank by 1. Unless violation type of $p$ is $\mathbb{N}$, $p$ should be removed from the array identified by its type (if there was null in the place, we know $p$ resists on the stack). If $p$ is a rank child its type should be changed to $L$ and $p$ inserted to $C_L$ (if it does not resist there), and its subtype should be set to reflect loss increase. Total loss was reduced by at least 1. If $p$ was rank root, it should be inserted to $C_A$ (if it does not resist there).)

In the Figure 1 you can see the reductions and in Table 1 you can see the effect of reductions. The situation is simplified as $|C_A|$ reductions do not change $\Phi_L$, so we can make $|C_L|$ reductions first and finish with $|C_A|$ reductions.

Both worst case stack reduction strategies would empty the stacks after each $\text{DeleteMin}$ operation (amortized version of $\text{DeleteMin}$ is always used). First worst case strategy calculates differences $\Delta \Phi_A$ and $\Delta \Phi_L$ happening from the start of the invoked method (for method other than $\text{DeleteMin}$ they will be bounded by a constant). While any of them is positive and corresponding stack nonempty, corresponding stack size reduction is invoked. As each reduction decreases $\Phi$ by at least 1 and each coordinate change is bounded by a constant, number of reduction steps is therefore bounded by a constant. Second worst case strategy calculates maximal number of reduction steps required by first strategy for each method, plans this number of reduction steps and do planned reduction steps unless all stacks with unfinished plan are empty. The situation is simplified as $|C_A|$ reductions do not change $\Phi_L$, so we can make $|C_L|$ reductions first and finish with $|C_A|$ reductions.

Both strategies has in common that either corresponding stack $C_X$ is empty or corresponding $\Phi_X$ is at most as big as the last time the stack was empty and current $n$ did not decreased from $n'$ at that time. In the former case all violations of given type
Fast DecreaseKey Heaps with worst case variants

X are addressed in the array by at most one pointer per rank of ranks 0 up to R(n), so there are at most R(n) + 1 violations. From the form of $\Phi_X$ ($x > 0$) we got in the latter case $x(R(n') + 1) \geq x|R_X| + (x + 1)|C_X| \geq x|R_X| + x|C_X|$ bounding $|R_X| + |C_X|$ by $R(n') + 1 \leq R(n) + 1$ as well. Actually the violation size could be smaller than $|R_X| + |C_X| \leq R(n) + 1$ in the case a violation is pointed more times in $R_X \cup C_X$ or if there is a pointer in $C_X$ to a node which is not violation of the type X.

We will return to the second strategy after presenting implementation details.

5 Heap structure

Heap information contains pointer to list of heap tree roots, array of pointers to four arrays $R[A]$, $R[L]$, $C[A]$, and $C[L]$ and stack pointer indices $P[A]$, $P[L]$ initialised to 0. The pointed arrays are expected to have sufficient size, but standard worst case array doubling could be used to solve a problem with maximal rank exceedig the planned value. The pointed arrays are filled with nulls, pointer to list of heap tree roots is initialised to null.

Heap node contains element with key, integer rank, violation subtype (either $A$, $L_1$, $L_2$ or $N$), pointer to parent (which is null for a heap tree root), and pointers left and right to siblings.

List of heap tree roots uses sibling pointers maintained in the heap nodes. Parent edge from node $x$ is rank edge whenever violation type of $x$ is not $A$. The violation type is maintained implicitly using the violation subtype.

Left pointers in sibling lists are maintained cyclic (left of leftmost points to rightmost), while right pointers acyclic (right of rightmost contains null). Except for the first node of a sibling list $x \to left \to right = x$. This allows access of both ends in constant time as well as adding or removing of a given node.

6 Private methods

We will describe the public methods using private methods. Their use could be slightly optimized. Decomposition into private methods makes the description easier.

Type($S$): Tabulated conversion of subtype to type by description ($A \to A$), ($N \to N$), ($L_1 \to L_1$), ($L_2 \to L_2$).

SetViolationSubtype($S$) for a node $x$: Unless original violation subtype $O$ of $x$ is $N$, pointer to $x$ is removed from corresponding array $R[Type(O)]$ (addressed by $x$’s rank) if there was not pointer to $x$, we know $x$ remains in $C[Type(O)]$. Then we change the violation subtype of $X$ to $S$ and insert $x$ to the stack $C[Type(S)]$ (unless $S = N$ or we know the node is already there).

DecrementRank() for a node $x$: Unless violation subtype $O$ of $x$ is $L_2$, let $S$ be obtained from $O$ by tabulated conversion by description ($A \to A$), ($N \to L_1$), ($L_1 \to L_2$), and SetViolationSubtype($S$) will be called for $x$. Then in all cases the rank of the node $x$ is decremented.

CutFromParent() for a node $c$: Unless original violation subtype $O$ of $c$ is $A$, DecrementRank() is called for parent of $c$. In all cases {parent pointer of $c$ would be set to null and} $c$ would be removed from it’s sibling list. Calling method should afterwards add $c$ to another sibling list. The method could be called even when $c$ has no parent.

\[1\] We need loss of a node only in the analysis, the subtype is all we need in the implementation.

\[2\] On few places '{}' mark duplicated work.
### Table 2 Effect of private methods to violations

| Method                          | $\Phi_A$ | $\Phi_L$ | $\Phi$ | $p$ |
|---------------------------------|----------|----------|--------|-----|
| SetViolationSubtype $\Lambda$   | $\leq +2$ | $\leq 0$ | $\leq +2$ | $\leq 2$ |
| | from $\Lambda$                 | $\leq +1$ | $0$     | $\leq +1$ | $\leq 2$ |
| | from $L$                       | $+2$     | $\leq 0$ | $\leq +2$ | $\leq 2$ |
| | from $N$                       | $+2$     | $0$     | $+2$   | $1$  |
| SetViolationSubtype $L_1$ or $L_2$ | $\leq 0$ | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| | from $\Lambda$                 | $\leq 0$ | $+4$    | $\leq 4$ | $\leq 2$ |
| | from $L$                       | $0$      | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| | from $N$                       | $0$      | $+4$    | $+4$   | $1$  |
| SetViolationSubtype $N$         | $\leq 0$ | $\leq 0$ | $\leq 0$ | $\leq 1$ |
| | from $\Lambda$                 | $\leq 0$ | $0$     | $\leq 0$ | $\leq 1$ |
| | from $L$                       | $0$      | $\leq 0$ | $\leq 0$ | $\leq 1$ |
| | from $N$                       | $0$      | $0$     | $0$   | $0$  |
| DecrementRank                   | $\leq +1$ | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| | $A$                            | $\leq +1$ | $0$     | $\leq +1$ | $\leq 2$ |
| | $N$                            | $0$      | $+4$    | $+4$   | $1$  |
| | $L_1$                          | $0$      | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| | $L_2$                          | $0$      | $+4$    | $+4$   | $0$  |
| CutFromParent                   | $\leq +1$ | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| Link                            | $\leq +4$ | $\leq +5$ | $\leq +8$ | $\leq 6$ |
| + CutFromParent for $h$         | $\leq +1$ | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| + SetViolationSubtype($H$) of $h$ | $\leq +2$ | $\leq 0$ | $\leq +2$ | $\leq 2$ |
| + SetViolationSubtype($A$) of $s$ | $\leq +1$ | $0$     | $\leq +1$ | $\leq 2$ |
| Link of rank roots of the same rank | $\leq +1$ | $0$     | $\leq +1$ | $\leq 3$ |
| Link of rank roots of different rank | $0$     | $0$     | $0$   | $0$  |
| Link of $L_1$ nodes of the same rank | $\leq +2$ | $\leq +5$ | $\leq +6$ | $\leq 5$ |

Here $p$ again denotes number of pointer changes not reflected in heap trees.

Link($x,y$): There could be assert that violation subtype of $x$ and $y$ should be the same and from $\{A, L_1\}$, ranks of $x$ and $y$ should be equal in the case $L_1$. The keys of $x$ and $y$ are compared, let node $s$ be the one with smaller key while $h$ the other. CutFromParent() for $h$ is called and $h$ is added as leftmost child of $s$, parent pointer of $h$ is set to $s$. If ranks of $s$ and $h$ are equal, let $H = N$, otherwise let $H = A$. SetViolationSubtype($H$) for $h$ is called unless violation subtype of $h$ is $H$. If $H = N$ let $S$ is obtained from subtype of $s$ by tabulated conversion by description ($A \rightarrow A$), ($L_1 \rightarrow N$), SetViolationSubtype($S$) is called for $s$ and rank of $s$ is incremented.

Pushing and Popping to the stacks $C[X]$ using stack pointer indices $P[X]$ is standard (left as an excersise).

Let us repeat nentivial cases of stack size reductions using private methods. $|LC|$ reduction for $x$ of subtype $L_2$ just calls SetViolationSubtype($A$) for $x$ and DecrementRank() for its parent. $|LC|$ reduction for $x$, $y$ of subtype $L_1$ and the same rank just calls Link($x,y$). $|AC|$ reduction for $x$, $y$ of subtype $A$ and the same rank just calls Link($x,y$).
Public methods

Insert\( (k) \) creates new node \( x \) with violation subtype \( N \), key \( k \), rank 0, \{no parent\} and no child. It adds \( x \) as a new root to the list of heap tree roots and it invokes FindMin. Insert returns \( x \) for future references.

FindMin in the phase 0 traverses list of roots and calls SetViolationSubtype(\( A \)) for each of them which does not have subtype \( A \) and it sets their parent pointers to null.

In the phase 1 are made stack sizes reductions. In the amortized variant they are performed until stacks are empty. In the worst case variant the reductions are applied according to the strategy (1st one makes \( \Delta \Phi_X \) nonpositive or stack \( C[X] \) empty, 2nd makes enough reductions to be sure the same holds).

Then in phase 2 FindMin traverses the heap tree roots leftwise linking two neighbouring roots interlaced with steps to left in the circular list (to link the roots as even as possible). We finish when only one tree remains. It’s root points to minimum and it will be returned.

In the phase 3, which is last are made stack sizes reductions again. Amortized variant until stacks are empty while worst case variants according to the strategy to reflect real \( \Delta \Phi_A \) change during second phase (or maximal possible \( \Delta \Phi_A \) change).

DeleteMin implements only amortized variant, which has guaranteed worst case time \( O(\log n) \), so no maintainance of \( \Phi \) coordinates is needed. Let \( \rho \) be the only tree root. It updates pointer to the list of roots to point to the leftmost child of \( \rho \). SetViolationSubtype(\( \emptyset \)) is called for \( \rho \). At the end FindMin is called and \( \rho \) discarded.

DecreaseKey\( (x, k) \) calls CutFromParent() for \( x \) and \( x \) is added as a new root to the list of heap tree roots. Than in all cases it updates key at node \( x \) to \( k \). It invokes FindMin at the end.

Second worst case strategy

We can see the effect of public methods on \( \Phi \) coordinates and pointer overhead in the table.

The DeleteMin analysis is part of the table.

For amortized version of DecreaseKey we got stack size reduction requirements by at most 8 and 6 in additional pointer overhead, and for amortized version of Insert we got stack size reduction requirements by at most 3 and 3 in additional pointer overhead. Including stack size reductions this makes amortized pointer overhead at most 30 per DecreaseKey, and at most 12 per Insert.

Only DecreaseKey can increase \( \Phi_L \). It is increased by at most 5. Each \( |C_L| \) reduction reduces \( \Phi_L \) by at least 2, so 3 \( |C_L| \) reduction steps when DecreaseKey calls FindMin are sufficient to maintain \( \Phi_L \) in bounds. Largest increase of \( \Phi_A \) per \( |C_L| \) reduction is by 3, so 3 reduction steps by second strategy increase \( \Phi_A \) by at most 9. So \( \Phi_A \) could increase by 12 before \( |C_A| \) reductions start at FindMin phase 1. Each reduction decreases \( \Phi_A \) by at least 1, so 12 \( |C_A| \) reductions in phase 1 are sufficient. At most 1 additional \( |C_A| \) reduction should be performed at phase 3. The stack size reductions generate pointer maintainance overhead at most 35 so pointer maintainance overhead for DecreaseKey according the second worst case strategy is at most 41.

The analysis would give the same asymptotic complexities if phase 1 is omitted, but \( |C_A| \) reductions early attempts to create rank edges directly rather to converting created nonrank edges by \( |C_A| \) reductions later. In only Insert case it reduces number of comparisons by factor about 3/4.

Skipping phase 3 would complicate the argument, but doing all reductions in phase 1 is an alternative.
Table 3 Effect of public methods to violations

| Method                  | $\Phi_A$  | $\Phi_L$ | $\Phi$ | $p$  |
|-------------------------|-----------|----------|--------|------|
| Insert                  | $\leq +3$ | 0        | $\leq +3$ | $\leq 3$ |
| + FindMin phase 0       | +2        | 0        | $+2$   | 1    |
| + FindMin phase 2       | $\leq +1$ | 0        | $\leq +1$ | $\leq 2$ |
| FindMin                 | 0         | 0        | 0      | 0    |
| DeleteMin               |           |          |        |      |
| + SetViolationSubtype($\wp$) of $\rho$ | $\leq 0$ | $\leq 0$ | $\leq 0$ | $\leq 1$ |
| + FindMin phase 0       | $\leq 2R(n) + 2$ | 0   | $\leq 2R(n) + 2$ | $\leq R(n) + 1$ |
| + FindMin phase 2       | 0         | 0        | 0      | 0    |
| DecreaseKey             | $\leq +4$ | $\leq +5$ | $\leq +8$ | $\leq 6$ |
| + CutFromParent() for $x$ | $\leq +1$ | $\leq +5$ | $\leq +5$ | $\leq 2$ |
| + FindMin phase 0       | $\leq +2$ | 0        | $\leq +2$ | $\leq 2$ |
| + FindMin phase 2       | $\leq +1$ | 0        | $\leq +1$ | $\leq 2$ |

Here $p$ again denotes number of pointer changes not reflected in heap trees. **DeleteMin** requires at most $2R(n) + 2 \leq 2.4 \log_2(n) + 10$ stack size reductions in amortized sense. There is at most 3 pointer overhead per stack size reduction so altogether it would generate at most $7R(n) + 8 \leq 8.4 \log_2 n + 36$ pointer overhead in amortized sense. If $\Phi^0$ be potential before and $\Phi^E$ after **DeleteMin**, we should include the difference into account as well. But $\Phi^0 \leq 4R(n) + 4$ and $\Phi^E \geq 0$. So we have worst case bound $6R(n) + 6 \leq 7.2 \log_2(n) + 30$ stack reductions. At the worst case there will be at most $19R(n) + 20 \leq 22.8 \log_2 n + 96$ pointer overhead.

Carefull look for first strategy will show maximal increase of $\Phi_A$ would be 5 obtained by first reduction increasing $\Phi_A$ by 2 and second by 3, while $\Phi_L$ reaches smaller value than before the **DecreaseKey** started. **DecreaseKey** increases $\Phi_A$ by at most 7 before $|C_A|$ reductions start at **FindMin** phase 1 (CutFromParent does not increase $\Phi_A$ and $\Phi_L$ simultaneously) so at most 7 $|C_A|$ reductions are performed in phase 1 of **FindMin** according to the first strategy, and at most 1 $|C_A|$ reduction is performed at phase 3. The stack size reductions generate pointer maintainance overhead at most 22 so pointer maintainance overhead for **DecreaseKey** according the first worst case strategy is at most 28.

The situation with stack size reductions is much easier for **Insert**. There are no $|C_L|$ reductions at all and at most 2 $|C_A|$ reductions in phase 1 of **FindMin**, and at most 1 $|C_A|$ reduction in phase 3. This generates (in both worst case strategies) at most 6 pointer maintainance overhead so worst case **Insert** has at most 9 pointer maintainance overhead in total.

Let us repeat at the end that second worst case strategy plans for **FindMin** called from **DecreaseKey** 3 $|C_L|$ and 12 $|C_A|$ reductions to the phase 1, and 1 $|C_A|$ reduction to the phase 3. It plans for **FindMin** called from **Insert** 2 $|C_A|$ reductions to the phase 1, and 1 $|C_A|$ reduction to the phase 3.

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**References**

1. Vladan Majerech. Fast fibonacci heaps with worst case extensions, 2019. [arXiv:191111637](https://arxiv.org/abs/1911.11637)