Communication Compression for Decentralized Learning With Operator Splitting Methods

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Abstract—In decentralized learning, operator splitting methods using a primal-dual formulation (e.g., Edge-Consensus Learning (ECL)) have been shown to be robust to heterogeneous data and have attracted significant attention in recent years. However, in the ECL, a node needs to exchange dual variables with its neighbors. These exchanges incur significant communication costs. For the Gossip-based algorithms, many compression methods have been proposed, but these Gossip-based algorithms do not perform well when the data distribution held by each node is statistically heterogeneous. In this work, we propose a novel framework of the compression methods for the ECL, called the Communication Compressed ECL (C-ECL). Specifically, we reformulate the update formulas of the ECL and propose to compress the update values of the dual variables. We demonstrate experimentally that the C-ECL can achieve a nearly equivalent performance with fewer parameter exchanges than the ECL. Moreover, we demonstrate that the C-ECL is more robust to heterogeneous data than the Gossip-based algorithms.

Index Terms—Douglas-Rachford splitting, operator splitting, decentralized learning.

I. INTRODUCTION

In recent years, neural networks have shown promising results in various fields, including image processing [4], [8] and natural language processing [7], [40], and have thus attracted considerable attention. To train a neural network, we generally need to collect a large number of training data. Owning to the use of crowdsourcing services, it is now easy to collect a large number of annotated images and texts. However, because of privacy concerns, it is difficult to collect a large number of personal data, such as medical data including gene expression and medical images, on a single server. One of the powerful tools to address this issue is federated learning [14], [26], which trains neural networks without sharing the training data among servers. However, because federated learning requires nodes to communicate with the central server, it suffers from large communication costs. To reduce these communication costs, decentralized learning [21] has emerged, which allows nodes to communicate with only their neighbors and train neural networks without the central server.

One of the most widely used algorithms for decentralized learning is the Gossip-based algorithm [2], [21]. In the Gossip-based algorithm, each node (i.e., server) updates the model parameters using its own gradient, exchanges model parameters with its neighbors, and then takes the average value to reach a consensus. The Gossip-based algorithm is a simple yet effective approach. When the distribution of the data subset held by each node is statistically homogeneous, the Gossip-based algorithm can perform as well as the vanilla SGD, which trains the model on a single node using all of the training data [21]. However, when the data distribution of each node is statistically heterogeneous (e.g., each node has only images of some classes), parameters of each node drift away, and the Gossip-based algorithms do not perform well [39], [41]. To address this issue, the various methods, including $D^2$ [39], Gradient Tracking [24], [27], etc [20], [47], have been proposed.

Recently, the operator splitting method using the primal-dual formulations, called the Edge-Consensus Learning (ECL) [28], has been proposed. The primal-dual algorithm, including the Alternating Direction Method of Multipliers (ADMM) [3] and the Primal-Dual Method of Multipliers (PDMM) [48], can be applied in decentralized learning by representing the model consensus as the linear constraints. It has recently been shown that the ADMM and PDMM for decentralized learning can be derived by solving the dual problem using operator splitting methods [33] (e.g., the Douglas-Rachford splitting [9] and the Peaceman-Rachford splitting [30]). In addition, Niwa et al. [28], [29] applied these operator splitting methods to the neural networks, named them the Edge-Consensus Learning (ECL). The ECL can be regarded as a variant of the Gradient Tracking that halves the communication costs of the Gradient Tracking [37]. Thus, the ECL is more robust to data heterogeneity than the Gossip-based algorithm, although the communication costs of the ECL are the same as that of the Gossip-based algorithm [28], [29].

However, for both the Gossip-based algorithm and the ECL, each node needs to exchange the model parameters and/or dual variables with its neighbors, and such exchanges incur significant communication costs. Recently, in the Gossip-based algorithm, many studies have proposed methods for compressing the
exchange of parameters [16], [25], [38], [42]. Then, they showed that these compression methods can train a model with fewer parameter exchanges than the uncompressed Gossip-based algorithm. However, these compression methods are based on the Gossip algorithms and do not perform well when the data distribution of each node is statistically heterogeneous.

In this work, we propose a novel framework for the compression methods applied to the ECL, which we refer to as the Communication Compressed ECL (C-ECL). Specifically, we reformulate the update formulas of the ECL, and propose to compress the update values of the dual variables. Theoretically, we analyze how our proposed compression affects the convergence rate of the ECL and show that the C-ECL converges linearly to the optimal solution as well as the ECL (i.e., without compression). Experimentally, we show that the C-ECL can achieve almost the same accuracy as the ECL with fewer parameter exchanges. Furthermore, the experimental results show that the C-ECL is more robust to heterogeneous data than the Gossip-based algorithm, and when the data distribution of each node is statistically heterogeneous, the C-ECL can outperform the uncompressed Gossip-based algorithm, in terms of both accuracy and communication costs.

Notation: In this work, \(|\cdot|\) denotes the L2 norm, 0 denotes a vector with all zeros, and \(I\) denotes an identity matrix.

II. RELATED WORK

In this section, we briefly introduce the problem setting of decentralized learning, and then introduce the Gossip-based algorithm and the ECL.

A. Decentralized Learning

Let \(G = (\mathcal{V}, \mathcal{E})\) be an undirected connected graph that represents the network topology where \(\mathcal{V}\) denotes the set of nodes and \(\mathcal{E}\) denotes the set of edges. For simplicity, we denote \(V\) as the set of integers \(\{1, 2, \ldots, |\mathcal{V}|\}\). We denote the set of neighbors of the node \(i\) as \(\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}\). The goal of decentralized learning is formulated as follows:

\[
\inf_{\mathbf{w}} \sum_{i \in \mathcal{V}} f_i(\mathbf{w}), \quad f_i(\mathbf{w}) := \mathbb{E}_{\zeta_i \sim \mathcal{D}_i} [F(\mathbf{w}; \zeta_i)],
\]

(1)

where \(\mathbf{w} \in \mathbb{R}^d\) is the model parameter, \(F\) is the loss function, \(\mathcal{D}_i\) represents the data held by the node \(i\), \(\zeta_i\) is the data sample from \(\mathcal{D}_i\), and \(f_i\) is the loss function of the node \(i\).

B. Gossip-Based Algorithm

A widely used approach for decentralized learning is the Gossip-based algorithm (e.g. D-PSGD [21]). In the Gossip-based algorithm, each node \(i\) computes its own gradient \(\nabla f_i\), exchanges parameters with its neighbors, and then takes their average. Although, the Gossip-based algorithm is a simple and effective approach, there are two main issues: high communication costs and sensitivity to the heterogeneity of the data distribution.

In the Gossip-based algorithm, the node needs to receive the model parameter from its neighbors. Because the number of parameters in a neural network is large, the exchanges of the model parameters incur huge communication costs. To reduce these communication costs of the Gossip-based algorithm, many methods that compress the exchanging parameters by using sparsification, quantization, and low-rank approximation have been proposed [11], [15], [16], [22], [25], [34], [38], [42]. Then, it was shown that these compression methods can achieve almost the same accuracy as the uncompressed Gossip-based algorithm with fewer parameter exchanges.

The second issue is that the Gossip-based algorithm is sensitive to the heterogeneity of the data distribution. When the data distribution of each node is statistically heterogeneous, because the optimal solution of the loss function \(\sum_i f_i\) and the optimal solution of the loss function of each node \(f_i\) are far from each other, the Gossip-based algorithms do not perform well [28], [41]. To address this issue of the Gossip-based algorithm, Tang et al. [39] and Xin et al. [46] applied variance reduction methods [6], [13] to the Gossip-based algorithm. Lorenzo and Scutari [24] proposed Gradient Tracking, and Takezawa et al. [36] extended Gradient Tracking and proposed Momentum Tracking. Recently, Vogels et al. [41] proposed a novel averaging method, which is called the RelaySum.

C. Edge-Consensus Learning

In this section, we briefly introduce the Edge-Consensus Learning (ECL) [28]. By reformulating (1), the primal problem can be defined as follows:

\[
\inf_{\{\mathbf{w}_i\}} \sum_{i \in \mathcal{V}} f_i(\mathbf{w}_i)
\]

s.t. \(\mathbf{A}_{ij}\mathbf{w}_i + \mathbf{A}_{ji}\mathbf{w}_j = 0, \ (\forall (i, j) \in \mathcal{E})\),

(2)

where \(\mathbf{A}_{ij} = \mathbf{I}\) when \(j \in \mathcal{N}_i\) and \(i < j\), and \(\mathbf{A}_{ij} = -\mathbf{I}\) when \(j \in \mathcal{N}_i\) and \(i > j\). By contrast, the Gossip-based algorithm explicitly computes the average at each round, whereas the primal problem of (2) represents the consensus based on the linear constraints. Subsequently, by solving the dual problem of (2) using the Douglas-Rachford splitting [9], the update formulas can be derived as follows [33]:

\[
\mathbf{w}_i^{(r+1)} = \arg\min_{\mathbf{w}_i} \left\{ f_i(\mathbf{w}_i) + \frac{\alpha}{2} \sum_{j \in \mathcal{N}_i} \left\| \mathbf{A}_{ij}\mathbf{w}_i - \frac{1}{\alpha} \mathbf{z}_{ij}^{(r)} \right\|^2 \right\},
\]

(3)

\[
\mathbf{y}_{ij}^{(r+1)} = \mathbf{z}_{ij}^{(r)} - 2\alpha \mathbf{A}_{ij}\mathbf{w}_i^{(r+1)},
\]

(4)

\[
\mathbf{z}_{ij}^{(r+1)} = (1 - \theta) \mathbf{z}_{ij}^{(r)} + \theta \mathbf{y}_{ij}^{(r+1)},
\]

(5)

where \(\theta \in (0, 1]\) and \(\alpha > 0\) are hyperparameters, and \(\mathbf{y}_{ij}, \mathbf{z}_{ij} \in \mathbb{R}^d\) are dual variables. We present detailed derivation of these update formulas in Section B-B in Appendix. When \(\theta = 1\), the Douglas-Rachford splitting is specifically called the Peaceman-Rachford splitting [30]. When \(f_i\) is non-convex (e.g., a loss function of a neural network), (3) can not be generally solved.
Then, Niwa et al. [28] proposed the Edge-Consensus Learning (ECL), which approximately solves (3) as follows:

$$w_i^{(r+1)} = \arg\min_{w_i} \left\{ \langle w_i, \nabla f_j(w_i^{(r)}) \rangle + \frac{1}{2\eta} \|w_i - w_i^{(r)}\|^2 \right\} + \frac{\alpha}{2} \sum_{j \in \mathcal{N}_i} \left\| A_{ij} w_i - \frac{1}{\alpha} z_{ij}^{(r)} \right\|^2,$$

(6)

where $\eta > 0$ corresponds to the learning rate. Then, Niwa et al. [29] showed that the ECL can be interpreted as a stochastic variance reduction method and demonstrated that the ECL is more robust to heterogeneous data than the Gossip-based algorithms. However, as shown in (5), the node $i$ must receive dual variables $y_{ji}$ from its neighbor $j$ in the ECL. Therefore, in the ECL as well as in the Gossip-based algorithm, large communication costs occur during training.

In addition to the ECL, other methods using primal-dual formulations have been proposed [20], and recently, the compression methods for these primal-dual algorithms have been studied [17], [23]. However, the compression methods for the ECL have not been studied. In this work, we propose the compression method for the ECL, called the C-ECL, that can train a model with fewer parameter exchanges and is robust to heterogeneous data.

### III. PROPOSED METHOD

#### A. Compression Operator

Before proposing the C-ECL, we first introduce the compression operator used in this work.

**Assumption 1 (Compression Operator):** For some $\tau \in (0, 1)$, we assume that the compression operator $\text{comp} : \mathbb{R}^d \to \mathbb{R}^d$ satisfies for all $x, y \in \mathbb{R}^d$ and $\omega$:

$$E_\omega \|\text{comp}(x; \omega) - x\|^2 \leq (1 - \tau) \|x\|^2, \quad (7)$$

$$\text{comp}(x + y; \omega) = \text{comp}(x; \omega) + \text{comp}(y; \omega), \quad (8)$$

$$\text{comp}(-x; \omega) = -\text{comp}(x; \omega), \quad (9)$$

where $\omega$ represents the parameter of the compression operator. In the following, we abbreviate $\omega$ and write $\text{comp}(x)$ as the operator containing the randomness.

The assumption of (7) is commonly used for the compression methods [5], [16], [25], [42]. In addition, we assume that the compression operator satisfies Eqs. (8-9), and the low-rank approximation [42] and the following sparsification used in the compression methods for the Gossip-based algorithm satisfies Eqs. (8-9).

**Example 1:** For some $k \in (0, 100]$, we define the operator $\text{rand}_{k\%} : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$\text{rand}_{k\%}(x) := s \odot x, \quad (10)$$

where $s$ is the Hadamard product and $s \in \{0, 1\}^d$ is a uniformly sampled sparse vector whose element is one with probability $k\%$. Here, the parameter $\omega$ of the compression operator corresponds to the randomly sampled vector $s$. Then, $\text{rand}_{k\%}$ satisfies Assumption 1 [35].

#### B. Communication Compressed Edge-Consensus Learning

In this section, we propose the Communication Compressed ECL (C-ECL), the method for compressing the dual variables exchanged in the ECL using the compression operator introduced in the previous section.

In the ECL, to update $z_{ij}$ in (5), the node $i$ needs to receive $y_{ji}$ from the node $j$. Because the number of elements in $y_{ji}$ is the same as that of the model parameter $w_i$, this exchange incurs significant communication costs. A straightforward approach to reduce this communication cost is compressing $y_{ji}$ in (5) as follows:

$$z_{ij}^{(r+1)} = (1 - \theta)z_{ij}^{(r)} + \theta \text{comp}(y_{ji}^{(r+1)}). \quad (11)$$

However, we found experimentally that compressing $y_{ji}$ does not work. In the compression methods for the Gossip-based algorithm, Lu and De Sa [25] showed that the model parameters are not robust to the compression. This is because the optimal solution of the model parameter is generally not zero, and thus the error caused by the compression does not approach zero even if the model parameters are near the optimal solution. Therefore, the successful compression methods for the Gossip-based algorithms compress the gradient $\nabla f_i(w_i)$ or the model difference $(w_j - w_i)$, which approach zero when the model parameters are near the optimal solution [16], [25], [42].

Inspired by these compression methods for the Gossip-based algorithms, we reformulate (5) into (12) so that we can compress the parameters which approach zero when the model parameters are near the optimal solution.

$$z_{ij}^{(r+1)} = z_{ij}^{(r)} + \theta(y_{ji}^{(r+1)} - z_{ij}^{(r)}). \quad (12)$$

In the Douglas-Rachford splitting, $z_{ij}$ approaches the fixed point (i.e., $z_{ij}^{(r)} = z_{ij}^{(r+1)}$) when the model parameters approach the optimal solution. (See Appendix for details of the Douglas-Rachford splitting and the definition of the fixed point). Then, from (12), when the model parameters approach the optimal solution, $(y_{ji} - z_{ij})$ in (12) approaches zero. Then, instead of compressing $y_{ji}$ as in (11), we propose compressing $(y_{ji} - z_{ij})$ as follows:

$$z_{ij}^{(r+1)} = z_{ij}^{(r)} + \theta \text{comp}(y_{ji}^{(r+1)} - z_{ij}^{(r)})$$

$$= z_{ij}^{(r)} + \theta (\text{comp}(y_{ji}^{(r+1)}) - \text{comp}(z_{ij}^{(r)})), \quad (13)$$

where we use Assumption 1 in the last equation. When rand_{k\%} is used as the compression operator, $y_{ji}$ and $z_{ij}$ must be compressed using the same sparse vector $s$ in (10) to use Assumption 1. In Algorithm 1, we provide the pseudo-code of the C-ECL. For simplicity, the node $i$ and the node $j$ exchange $\omega_{ij}$ and $\omega_{ji}$ at Lines 5-6 in Algorithm 1. However, by sharing the same seed value to generate $\omega_{ij}$ and $\omega_{ji}$ before starting the training, the node $i$ and the node $j$ can obtain the same values $\omega_{ij}$ and $\omega_{ji}$ without any exchanges. Moreover, when rand_{k\%} is used as the compression operator, the node $i$ can obtain $\omega_{ij}$ from the received value $\text{comp}(y_{ji}; \omega_{ij})$ because $\text{comp}(y_{ji}; \omega_{ij})$ is stored in a sparse matrix format (e.g., COO format). Thus, these exchanges of $\omega_{ij}$ and $\omega_{ji}$ can be omitted in practice. Therefore, in the C-ECL, each node only needs to exchange the
Algorithm 1: Update procedure at the node $i$ of the C-ECL.

1: for $r = 0$ to $R$ do
2: \begin{align*}
&\text{argmin}_{w_i} \left\{ f_i(w_i) + \frac{\nu}{2} \sum_{j \in N_i} \| A_{ij} w_i - \frac{1}{\alpha_i} y_{ij}^{(r)} \|^2 \right\}.
&\text{for } j \in N_i \text{ do}
&y_{ij}^{(r+1)} \leftarrow y_{ij}^{(r)} - 2\alpha_i A_{ij} w_i^{(r)}.
&\text{Receive}_{i-j}(w_i^{(r+1)}).
&\text{Transmit}_{i-j}(\omega_{ij}^{(r+1)}).
&\text{Receive}_{i-j}(\text{comp}(y_{ij}^{(r+1)}; \omega_{ij}^{(r+1)})).
&\text{Transmit}_{i-j}(\text{comp}(y_{ij}^{(r+1)}; \omega_{ij}^{(r+1)})).
&z_{ij}^{(r+1)} \leftarrow z_{ij}^{(r)} + \theta \text{comp}(y_{ij}^{(r+1)} - z_{ij}^{(r)}; \omega_{ij}^{(r+1)}).
&\text{end for}
&\text{end for}
\end{align*}

The compressed value of $y_{ij}$, and the C-ECL can train a model with fewer parameter exchanges than the ECL.

IV. CONVERGENCE ANALYSIS

In this section, we analyze how compression in the C-ECL affects the convergence rate of the ECL. Our convergence analysis is based on the analysis of the Douglas-Rachford splitting [10], and the proofs are presented in Section C.

A. Assumptions

In this section, we introduce additional notations and assumptions used in our convergence analysis. We define $N := |V|$, $N_{\min} := \min_i \{|N_i|\}$, $N_{\max} := \max_i \{|N_i|\}$ where $N_i$ is the set of neighbors of the node $i$. Let $N_i(j)$ be the $j$-th smallest index of the node in $N_i$, we define $w \in \mathbb{R}^{dN}$, $z_i \in \mathbb{R}^{|N_i|}$, and $z \in \mathbb{R}^{2d|E|}$ as follows:

\begin{align*}
w &:= (w_i^1, \ldots, w_i^N)^T, \tag{14} \\
z_i &:= (z_i|_{N_i(1)}, \ldots, z_i|_{N_i(|N_i|)})^T, \tag{15} \\
z &:= (z_1^T, \ldots, z_N^T)^T. \tag{16}
\end{align*}

For simplicity, we drop the superscripts of the number of round $r$. Let $\{w_i^r\}_r$ be the optimal solution of (2), we define $w^* \in \mathbb{R}^{dN}$ in the same manner as the definition of $w$ in (14). We define the loss function as $f(w) := \sum_{i \in V} f_i(w_i)$. Next, we introduce the assumptions used in the convergence analysis.

**Assumption 2:** We assume that $f$ is proper, closed and convex.

**Assumption 3:** We assume that $f$ is $L$-smooth and $\mu$-strongly convex with $L > 0$ and $\mu > 0$.

**Assumption 4:** We assume that the graph $G$ has no isolated nodes (i.e., $N_{\min} > 0$).

Assumptions 2 and 3 are the standard assumptions used for the convergence analysis of the operator splitting methods [10], [32]. Assumption 3 is weaker than the assumptions of the smoothness and the strong convexity of $f_i$ for all $i \in V$, which are commonly used in decentralized learning. Assumption 4 holds in general because decentralized learning assumes that the graph $G$ is connected. In addition, we define $\delta \in \mathbb{R}$ as follows:

$$
\delta := \max \left( \frac{\alpha N_{\max} - \mu}{\alpha N_{\min}}, \frac{L - \alpha N_{\min}}{L + \alpha N_{\min}} \right) > 0.
$$

Suppose that Assumptions 2, 3, and 4 hold and $\alpha \in (0, \infty)$ holds, then $\delta \in (0, 1)$ holds because $L \geq \mu > 0$ and $N_{\max} \geq N_{\min} > 0$.

B. Convergence Rates

**Theorem 1:** Let $\bar{y} \in \mathbb{R}^{2d|E|}$ be the fixed point of the Douglas-Rachford splitting.\(^1\) Suppose that Assumptions 1, 2, 3, and 4 hold. If $\tau \geq 1 - \left(\frac{1}{1+\delta}\right)^2$ and $\theta$ satisfies

$$
\theta \in \left(\frac{2\delta}{(1-\delta)(1-\sqrt{1-\tau})}, \frac{2}{(1+\delta)(1+\sqrt{1-\tau})}\right) \tag{17}
$$

then $w^{(r+1)}$ generated by Algorithm 1 linearly converges to the optimal solution $w^*$ of (2) as follows:

$$
\mathbb{E}\|w^{(r+1)} - w^*\|^2 \leq \frac{\sqrt{N_{\max}}}{\mu + \alpha N_{\min}} \times \left\{ |1 - \theta| + \theta \delta + \sqrt{1 - \tau} (\theta + |1 - \theta| \delta + \delta) \right\}^r \|z^{(0)} - \bar{y}\|^2. \tag{18}
$$

**Corollary 1:** Let $\bar{z} \in \mathbb{R}^{2d|E|}$ be the fixed point of the Douglas-Rachford splitting. Under Assumptions 1, 2, 3, and 4, when $\tau = 1$ and $\theta \in (0, \frac{2}{1+\delta})$, $w^{(r+1)}$ generated by Algorithm 1 linearly converges to the optimal solution $w^*$ of (2) as follows:

$$
\mathbb{E}\|w^{(r+1)} - w^*\|^2 \leq \frac{\sqrt{N_{\max}}}{\mu + \alpha N_{\min}} \left\{ |1 - \theta| + \theta \delta \right\}^r \|z^{(0)} - \bar{y}\|^2. \tag{19}
$$

Because $\tau = 1$ implies that $\text{comp}(x) = x$ in the C-ECL, Corollary 1 shows the convergence rate of the ECL under Assumptions 1, 2, 3, and 4, which is almost the same as that shown in the previous work [10] that analyzed the convergence rate of the Douglas-Rachford splitting. Comparing the domain of $\theta$ for the ECL and the C-ECL to converge, as $\tau$ decreases, the domain in (17) becomes smaller. Subsequently, in order for the domain in (17) to be non-empty, $\tau$ must be greater than or equal to $1 - (1 - \delta)^2 / (1 + \delta)^2$. Next, comparing the convergence rate of the ECL and the C-ECL, the compression in the C-ECL slows down the convergence rate of the ECL by the term $(\sqrt{1 - \tau} (\theta + |1 - \theta| \delta + \delta))$. Moreover, similar to the convergence analysis of the Douglas-Rachford splitting [10], Theorem 1 and Corollary 1 imply that the optimal parameter of $\theta$ can be determined as follows:

**Corollary 2:** Suppose that Assumptions 1, 2, 3, and 4 hold, and $\tau \geq 1 - \left(\frac{1}{1+\delta}\right)^2$, the optimal convergence rate of (18) in the C-ECL is achieved when $\theta = 1$.

**Corollary 3:** Suppose that Assumptions 1, 2, 3, and 4 hold, and $\tau = 1$, the optimal convergence rate of (19) is achieved when $\theta = 1$.

Note that $\theta$ is generally set as $\theta \in (0, 1]$, but Theorem 1 and Corollary 1 indicate that for some $\delta$, ECL and C-ECL can

\(^1\) A more detailed definition is shown in Section A-B in Appendix.
converge even when $\theta$ is greater than 1. The same phenomenon has been reported by the previous work [10] that analyzed the convergence rate of the Douglas-Rachford splitting.

V. EXPERIMENTS

In this section, we demonstrate that the C-ECL can achieve almost the same performance with fewer parameter exchanges than the ECL. Furthermore, we show that the C-ECL is more robust to heterogeneous data than the Gossip-based algorithm. Our implementation is available at https://github.com/yukiTakezawa/C-ECL.

A. Experimental Setting

**Dataset and Model:** We evaluate the C-ECL using Fashion-MNIST [44] and CIFAR10 [18], which are datasets of 10-class image-classification tasks. As the models for both datasets, we use 5-layer convolutional neural networks [19] with group normalization [43]. Following the previous work [28], we distribute data to the nodes in two settings: the homogeneous and heterogeneous settings. In the homogeneous setting, the data are distributed such that each node has the data of all 10 classes and has approximately the same number of data of each class. In the heterogeneous setting, the data are distributed such that each node has the data of randomly selected 8 classes. In both settings, the data are distributed to the nodes such that each node has the same number of data.

**Network:** In Section V-B, we evaluate all comparison methods on a network of a ring consisting of eight nodes. In addition, in Section V-C, we evaluate all comparison methods in four settings of the network topology: chain, ring, multiplex ring, and fully connected graph, where each setting consists of eight nodes. In Appendix, we show a visualization of the network topology. Each node exchanges parameters with its neighbors per five local updates. We implement with PyTorch using gloo as the backend and run all comparison methods on eight GPUs (NVIDIA RTX 3090).

**Comparison Methods:** (1) D-PSGD [21]: The uncompressed Gossip-based algorithm. (2) PowerGossip [42]: The Gossip-based algorithm that compresses the exchanging parameters by using a low-rank approximation. We use the PowerGossip as the compression method for the Gossip-based algorithm because the PowerGossip has been shown to achieve almost the same performance as other existing compression methods without additional hyperparameter tuning. (3) ECL [28], [29]: The primal-dual algorithm described in Section II-C. Because Niwa et al. [28] showed that the ECL converges faster when $\theta = 1$ than when $\theta = 0.5$, we set $\theta = 1$. (4) C-ECL: Our proposed method

![Graphs showing test accuracy on FashionMNIST when varying the network topology.](image)
described in Section III. We use $\text{rand}_k$ as the compression operator. Following Corollary 2, we set $\theta = 1$. We initialize $z_{i|j}$ and $y_{i|j}$ to zeros. However, we found that when we compress the update values of $z_{i|j}$ by using $\text{rand}_k$, the convergence becomes slower because $z_{i|j}$ remains to be sparse in the early training stage. Then, we set $k_%$ of $\text{rand}_k$ to 100% only during the first epoch.

In addition, for reference, we show the results of the Stochastic Gradient Descent (SGD), in which the model is trained on a single node containing all training data. In our experiments, we set the learning rate, number of epochs, and batch size to the same values for all comparison methods. In Section D-A in Appendix, we show the detailed hyperparameters used for all comparison methods.

### B. Experimental Results

In this section, we evaluate the accuracy and the communication costs when setting the network topology to be a ring.

**Homogeneous Setting:** First, we discuss the results on the homogeneous setting. Table I shows the accuracy and the communication costs on the homogeneous setting. The results show that the D-PSGD and the ECL achieve almost the same accuracy on both datasets. Then, the C-ECL and the PowerGossip are comparable and achieve almost the same accuracy as the ECL and the D-PSGD even when we set $k_%$ of $\text{rand}_k$ to 1% and the number of the power iteration steps to 1 respectively. Therefore, the C-ECL can achieve comparable accuracy with approximately 50-times fewer parameter exchanges than the ECL and the D-PSGD on the homogeneous setting.

**Heterogeneous Setting:** Next, we discuss the results of the heterogeneous setting. Table II shows the accuracy and the communication costs in the heterogeneous setting. In the D-PSGD, the accuracy on the heterogeneous setting decreases by approximately 3% compared to that on the homogeneous setting. In the PowerGossip, even if the number of power iteration steps is increased, the accuracy does not approach that of the D-PSGD and the ECL. On the other hand, the accuracy of the ECL is almost the same on both the homogeneous and heterogeneous settings, and the results indicate that the ECL is more robust to heterogeneous data than the D-PSGD. In the C-ECL, when we set $k_%$ of $\text{rand}_k$ to 1%, the accuracy on the heterogeneous setting decreases by approximately 10% compared to that of the ECL. However, when we increase $k_%$ of $\text{rand}_k$, the C-ECL is competitive with the ECL and outperforms the D-PSGD and the PowerGossip. Specifically, on FashionMNIST, when we set $k_%$ to 10%, the C-ECL is competitive with the ECL and outperforms the D-PSGD and the PowerGossip. On CIFAR10, when we set $k_%$ to 20%, the C-ECL is competitive with the ECL and outperforms the D-PSGD and the PowerGossip.

In summary, on the homogeneous setting, the C-ECL and the PowerGossip can achieve almost the same accuracy as

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**TABLE I**

**Test Accuracy and Communication Costs on the Homogeneous Setting**

|                | FashionMNIST | CIFAR10 |
|----------------|--------------|---------|
|                | Accuracy     | Send/Epoch | Accuracy     | Send/Epoch |
| SGD            | 88.7         | -         | 75.7         | -         |
| D-PSGD         | 84.1         | 5336 KB ($\times 1.0$) | 72.8         | 6255 KB ($\times 1.0$) |
| ECL            | 84.4         | 5336 KB ($\times 1.0$) | 72.6         | 6255 KB ($\times 1.0$) |
| PowerGossip (1) | 84.0         | 138 KB ($\times 38.7$) | 72.0         | 135 KB ($\times 46.3$) |
| PowerGossip (10)| 84.3         | 1079 KB ($\times 5.0$) | 72.3         | 1102 KB ($\times 5.7$) |
| PowerGossip (20)| 84.2         | 2124 KB ($\times 2.5$) | 72.2         | 2175 KB ($\times 2.9$) |
| C-ECL (1%)     | 84.0         | 115 KB ($\times 48.1$) | 71.5         | 132 KB ($\times 47.4$) |
| C-ECL (10%)    | 84.0         | 1075 KB ($\times 5.1$) | 71.4         | 1257 KB ($\times 5.0$) |
| C-ECL (20%)    | 84.0         | 2142 KB ($\times 2.5$) | 71.1         | 2507 KB ($\times 2.5$) |

For the C-ECL, the number in the bracket is $k%$ of $\text{rand}_k$ for the power gossip, the number in the bracket is the number of the power iteration steps. As the communication costs, the average amount of parameters sent per epoch is shown.

---

**TABLE II**

**Test Accuracy and Communication Costs on the Heterogeneous Setting**

|                | FashionMNIST | CIFAR10 |
|----------------|--------------|---------|
|                | Accuracy     | Send/Epoch | Accuracy     | Send/Epoch |
| SGD            | 88.7         | -         | 75.7         | -         |
| D-PSGD         | 79.4         | 5336 KB ($\times 1.0$) | 70.8         | 6155 KB ($\times 1.0$) |
| ECL            | 84.5         | 5336 KB ($\times 1.0$) | 72.7         | 6155 KB ($\times 1.0$) |
| PowerGossip (1) | 77.5         | 138 KB ($\times 38.7$) | 64.3         | 133 KB ($\times 46.3$) |
| PowerGossip (10)| 77.7         | 1079 KB ($\times 5.0$) | 67.2         | 1084 KB ($\times 5.7$) |
| PowerGossip (20)| 77.4         | 2124 KB ($\times 2.5$) | 67.9         | 2141 KB ($\times 2.9$) |
| C-ECL (1%)     | 77.7         | 115 KB ($\times 48.1$) | 60.2         | 129 KB ($\times 47.7$) |
| C-ECL (10%)    | 83.4         | 1075 KB ($\times 5.1$) | 61.8         | 1237 KB ($\times 5.0$) |
| C-ECL (20%)    | 83.6         | 2142 KB ($\times 2.5$) | 72.3         | 2467 KB ($\times 2.5$) |

For the C-ECL, the number in the bracket is $k%$ of $\text{rand}_k$ for the power gossip, the number in the bracket is the number of the power iteration steps. As the communication costs, the average amount of parameters sent per epoch is shown.
the ECL and the D-PSGD with approximately 50-times fewer parameter exchanges. On the heterogeneous setting, the C-ECL can achieve almost the same accuracy as the ECL with approximately 4-times fewer parameter exchanges and can outperform the PowerGossip. Furthermore, the results show that the C-ECL can outperform the D-PSGD, the uncompressed Gossip-based algorithm, in terms of both the accuracy and the communication costs.

C. Results With Various Network Topologies

In this section, we show the accuracy and the communication costs when the network topology is varied. Table III and Fig. 1 show the communication costs and the accuracy on Fashion-MNIST when the network topology is varied as a chain, ring, multiplex ring, or fully connected graph.

On the homogeneous setting, Fig. 1 shows that the accuracy of all comparison methods are almost the same and reach that of the SGD on all network topologies. On the heterogeneous setting, Fig. 1 shows that the accuracy of the D-PSGD and the PowerGossip are decreased compared to that on the homogeneous setting. On the other hand, the accuracy of the ECL is almost the same as on the homogeneous setting on all network topologies. Then, on all network topologies, the C-ECL achieves almost the same accuracy as the ECL with fewer parameter exchanges and consistently outperforms the PowerGossip. Moreover, the results show that on the heterogeneous setting, the C-ECL can outperform the D-PSGD, the uncompressed Gossip-based algorithm, on all network topologies in terms of both accuracy and communication costs.

VI. CONCLUSION

In this work, we propose the Communication Compressed ECL (C-ECL), the novel framework for the compression methods of the ECL. Specifically, we reformulate the update formula of the ECL and propose compressing the update values of the dual variables. Theoretically, we analyze the convergence rate of the C-ECL and show that the C-ECL converges linearly to the optimal solution as well as the ECL. Experimentally, we demonstrate that, even if the data distribution of each node is statistically heterogeneous, the C-ECL can achieve almost the same accuracy as the ECL with fewer parameter exchanges. Moreover, we show that, when the data distribution of each node is statistically heterogeneous, the C-ECL outperforms the uncompressed Gossip-based algorithm in terms of both accuracy and communication costs.

APPENDIX A

Preliminary

In this section, we introduce the definitions used in the following section and briefly introduce the Douglas-Rachford splitting. (See [1], [32] for more details.)

A. Definition

Definition 1 (Smooth Function): Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a closed, proper, and convex function. $f$ is $L$-smooth if $f$ satisfies for all $x, y \in \mathcal{H}$,

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \|x - y\|^2.$$

Definition 2 (Strongly Convex Function): Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ be a closed, proper, and convex function. $f$ is $\mu$-strongly convex if $f$ satisfies for all $x, y \in \mathcal{H}$,

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \|x - y\|^2.$$

Definition 3 (Conjugate Function): Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. The conjugate function of $f$ is defined as follows:

$$f^*(y) = \sup_{x \in \mathcal{H}} \{\langle y, x \rangle - f(x)\}.$$

Definition 4 (Nonexpansive Operator): Let $D$ be a non-empty subset of $\mathcal{H}$. An operator $T : D \to \mathcal{H}$ is nonexpansive if $T$ is 1-Lipschitz continuous.

Definition 5 (Contractive Operator): Let $D$ be a non-empty subset of $\mathcal{H}$. An operator $T : D \to \mathcal{H}$ is $\beta$-contractive if $T$ is Lipschitz continuous with constant $\beta \in [0, 1)$.

Definition 6 (Proximal Operator): Let $f : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$. $f$ is a closed, proper, and convex function. The proximal operator of $f$ is defined as follows:

$$\text{prox}_f(x) = \arg\min_y \left\{ f(y) + \frac{1}{2} \|y - x\|^2 \right\}.$$

Definition 7 (Resolvent): Let $A : \mathcal{H} \to 2^\mathcal{H}$, the resolvent of $A$ is defined as follows:

$$J_A = (\text{Id} + A)^{-1}.$$

Definition 8 (Reflected Resolvent): Let $A : \mathcal{H} \to 2^\mathcal{H}$, the reflected resolvent of $A$ is defined as follows:

$$R_A = 2J_A - \text{Id}.$$

| TABLE III |
|---|
| Communication Costs on Fashion-MNIST When Varying the Network Topology |
| Chain | Ring | Multiplex Ring | Fully Connected Graph |
| D-PSGD | 4670 KB | 5336 KB | 10673 KB | 18677 KB |
| ECL | 4670 KB | 5336 KB | 10673 KB | 18677 KB |
| PowerGossip (10) | 944 KB | 1075 KB | 2158 KB | 3776 KB |
| C-ECL (10%) | 941 KB | 1075 KB | 2151 KB | 3784 KB |

As the communication costs, the average amount of parameters sent per epoch on homogeneous and heterogeneous settings is shown.

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B. Douglas-Rachford Splitting

In this section, we briefly introduce the Douglas-Rachford splitting [9].

The Douglas-Rachford splitting can be applied to the following problem:

$$\inf \limits_{x} f(x) + g(x),$$

where $f$ and $g$ are closed, proper, and convex functions. Let $x^*$ be the optimal solution of the above problem, the optimality condition can be written as follows:

$$0 \in \partial f(x^*) + \partial g(x^*).$$

From [1, Proposition 26.1], the optimality condition above is equivalent to the following:

$$x^* = J_{\alpha f}(\text{Fix}(R_{\alpha g}R_{\alpha f})), $$

where $\text{Fix}(R_{\alpha g}R_{\alpha f}) = \{z | R_{\alpha g}R_{\alpha f}z = z\}$ and $\alpha > 0$.

The Douglas-Rachford splitting then computes the fixed point $z \in \text{Fix}(R_{\alpha g}R_{\alpha f})$ as follows:

$$z^{(r+1)} = ((1-\theta)I + \theta R_{\alpha g}R_{\alpha f})z^{(r)},$$

where $\theta \in (0,1]$ is the hyperparameter of the Douglas-Rachford splitting. Under certain assumptions, $((1-\theta)I + \theta R_{\alpha g}R_{\alpha f})$ is contractive [10], and the update formula above is guaranteed to converge at the fixed point $z$. Then, after converging to the fixed point $z$, the Douglas-Rachford splitting obtains the optimal solution $x^*$ as follows:

$$x^* = J_{\alpha f}z.$$

Moreover, by the definition of the reflected resolvent, the Douglas-Rachford splitting of (21) can be rewritten as follows:

$$x^{(r+1)} = J_{\alpha f}z^{(r)},$$

$$y^{(r+1)} = 2x^{(r+1)} - z^{(r)},$$

$$z^{(r+1)} = (1-\theta)z^{(r)} + \theta R_{\alpha g}y^{(r+1)}.$$  

APPENDIX B

DERIVATION OF UPDATE FORMULAS OF ECL

In this section, we briefly describe the derivation of the update formulas of the ECL. (See [28], [29], [33] for more details.)

A. Derivation of Dual Problem

First, we derive the Lagrangian function. The Lagrangian function of (2) can be derived as follows:

$$\sum_{i \in V} f_i(w_i) - \sum_{i \in V} \sum_{j \in N_i} (\lambda_{ij} + \hat{\lambda}_{ij})w_i + \hat{\lambda}_{ij}w_j$$

$$= \sum_{i \in V} f_i(w_i) - \sum_{i \in V} \sum_{j \in N_i} (\lambda_{ij}, \hat{\lambda}_{ij}w_i)$$

$$- \sum_{i \in V} \sum_{j \in N_i} (\lambda_{ij} + \hat{\lambda}_{ij})w_j$$

$$= \sum_{i \in V} f_i(w_i) - \sum_{i \in V} \sum_{j \in N_i} (\lambda_{ij} + \lambda_{ij}, \hat{\lambda}_{ij}w_i),$$

where $\lambda_{ij} \in \mathbb{R}^d$ is the dual variable. Defining $\lambda_{ij} = \lambda_{ij} + \lambda_{ij}$, the Lagrangian function can be rewritten as follows:

$$\sum_{i \in V} f_i(w_i) - \sum_{i \in V} \sum_{j \in N_i} (\lambda_{ij}, A_{ij}w_i).$$

Note that the definition of $\lambda_{ij}$ implies that $\lambda_{ij} = \lambda_{ji}$. Let $N$ be the number of nodes $|V|$. Here, $N_i(j)$ denotes the $j$-th smallest index of the node in $N_i$. We define $w \in \mathbb{R}^{dN}$, $A \in \mathbb{R}^{dN \times 2d|E|}$ and $\lambda \in \mathbb{R}^{2d|E|}$ as follows:

$$w = (w_1^T, \ldots, w_N^T)^T,$$

$$A_i = (A_{i|N_i(1)}, \ldots, A_{i|N_i(|N_i|)}),$$

$$\lambda_i = (\lambda_{i|N_i(1)}, \ldots, \lambda_{i|N_i(|N_i|)})^T,$$

$$\lambda = (\lambda_1^T, \ldots, \lambda_N^T)^T.$$  

We define the function $f$ as follows:

$$f(w) = \sum_{i \in V} f_i(w_i).$$

The Lagrangian function can be rewritten as follows:

$$f(w) - \langle \lambda, A^Tw \rangle.$$

Then, the primal and dual problems can be defined as follows:

$$\inf \limits_{w} \sup \limits_{\lambda} f(w) - \langle \lambda, A^T w \rangle - \iota(\lambda)$$

$$= \inf \limits_{w} f(w) + \iota^*(-A^T w),$$

$$\sup \limits_{\lambda} \inf \limits_{w} f(w) - \langle \lambda, A^T w \rangle - \iota(\lambda)$$

$$= -\inf \limits_{\lambda} f^*(A\lambda) + \iota(\lambda),$$

where $\iota$ is the indicator function defined as follows:

$$\iota(\lambda) = \begin{cases} 0 & \text{if } \lambda_{ij} = \lambda_{ji}, \hspace{0.5cm} (\forall (i,j) \in E) \\ \infty & \text{otherwise} \end{cases}.$$  

B. Derivation of Update Formulas

Next, we derive the update formulas of the ECL. By applying the Douglas-Rachford splitting of Eqs. (23–25) to the dual problem of (27), we obtain the following update formulas:

$$\lambda^{(r+1)} = J_{\alpha A^T}f(A)z^{(r)},$$

$$y^{(r+1)} = 2\lambda^{(r+1)} - z^{(r)},$$

$$z^{(r+1)} = (1-\theta)z^{(r)} + \theta R_{\alpha A^T}y^{(r+1)},$$

where $z \in \mathbb{R}^{2d|E|}$ and $y \in \mathbb{R}^{2d|E|}$ can be decomposed into $z_{ij} \in \mathbb{R}^d$ and $y_{ij} \in \mathbb{R}^d$ as follows:

$$z_i = (z_{i|N_i(1)}, \ldots, z_{i|N_i(|N_i|)})^T,$$

$$z = (z_1^T, \ldots, z_N^T)^T.$$
\[ y_i = (y_{i1}^T, \ldots, y_{iN_i}^T)^T, \]
\[ y = (y_1^T, \ldots, y_N^T)^T. \]

Update formulas of Eqs. (28-29): By the definition of the resolvent \( J_{\alpha A^+}\nabla f_{\alpha}(A) \), we obtain
\[ \lambda^{(r+1)} + \alpha A^+ \nabla f^*(A\lambda^{(r+1)}) = z^{(r)}, \]
\[ \lambda^{(r+1)} = z^{(r)} - a A^+ \nabla f^*(A\lambda^{(r+1)}). \] (31)

We define \( w^{(r+1)} \) as follows:
\[ w^{(r+1)} = \nabla f^*(A\lambda^{(r+1)}). \] (32)

From the property of the convex conjugate function, we obtain
\[ A\lambda^{(r+1)} = \nabla f(w^{(r+1)}). \] (33)

Substituting Eqs. (31-32) into (33), we obtain
\[ 0 = \nabla f(w^{(r+1)} + A(\alpha A^+ w^{(r+1)} - z^{(r)}). \]

We then obtain the update formula of \( w \) as follows:
\[ w^{(r+1)} = \arg\min \left\{ f(w) + \frac{\alpha}{2} \| A^+ w - \frac{1}{\alpha} z^{(r)} \|^2 \right\}. \]

Substituting (32) into (31), we obtain
\[ \lambda^{(r+1)} = z^{(r)} - 2\alpha A^+ w^{(r+1)} \]

Then, the update formula of (29) is written as follows:
\[ y^{(r+1)} = z^{(r)} - 2\alpha A^+ w^{(r+1)}. \]

Update formula of (30): From [33, Lemma VI.2], the update formula of (30) is rewritten as follows:
\[ z^{(r+1)} = (1 - \theta)z^{(r)} + \theta P y^{(r+1)}, \]
where \( P \) denotes the permutation matrix transforming \( y_{ij} \) into \( y_{ji} \) for all \((i, j) \in E\).

In summary, the update formulas of the ECL can be derived as follows:
\[ w^{(r+1)} = \arg\min_w \left\{ f(w) + \frac{\alpha}{2} \| A^+ w - \frac{1}{\alpha} z^{(r)} \|^2 \right\}. \] (34)
\[ y^{(r+1)} = z^{(r)} - 2\alpha A^+ w^{(r+1)}, \] (35)
\[ z^{(r+1)} = (1 - \theta)z^{(r)} + \theta P y^{(r+1)}. \] (36)

Then, by rewriting \( w, z \) and \( y \) with \( w_i, z_{ij}, \) and \( y_{ij} \) respectively, we obtain the update formulas of Eqs. (3-5).

APPENDIX C

CONVERGENCE ANALYSIS OF C-ECL

From the derivation of the ECL shown in Section B-B, the update formulas of (3), (4), and (13) in the C-ECL can be rewritten as follows:
\[ y^{(r+1)} = R_{\alpha A^+ \nabla f_{\alpha}(A)} z^{(r)}, \] (37)
\[ z^{(r+1)} = z^{(r)} + \theta \text{comp}(R_{\alpha A^+ y^{(r+1)}} - z^{(r)}). \] (38)

To simplify the notation, we define \( g := f^*(A\cdot) \). Then, (37) is rewritten as follows:
\[ y^{(r+1)} = R_{\alpha A^+ g} z^{(r)}. \] (39)

In the following, we analyze the convergence rate of the update formulas of (39) and (38).

**Lemma 1:** Under Assumption 4, the maximum singular value and the minimum singular value of \( A \) are \( \sqrt{N_{\max}} \) and \( \sqrt{N_{\min}} \) respectively.

**Proof:** By the definition of \( A \), we have \( AA^+ = \text{diag}(N_1,\ldots,N_N) \). This implies that \( AA^+ \) is not only a block diagonal matrix, but also a diagonal matrix. Then, the eigenvalues of \( AA^+ \) are \( |N_1|,\ldots,|N_N| \). This concludes the proof.

**Remark 1:** Under Assumption 4, the maximum singular value and the minimum singular value of \( A^+ \) are \( \sqrt{N_{\max}} \) and \( \sqrt{N_{\min}} \) respectively.

**Lemma 2:** Under Assumptions 2, 3, and 4, \( f^*(A\cdot) \) is \((N_{\max} - \mu)\)-smooth and \((N_{\min}/L)\)-strongly convex.

**Proof:** From Assumption 3, \( f^* \) is \((1/\mu)\)-smooth and \((1/L)\)-strongly convex. Then, for any \( \lambda \) and \( \lambda' \), by the \((1/\mu)\)-smoothness of \( f^* \) and Lemma 1, we have
\[ f^*(A\lambda) \]
\[ f^*(A\lambda') + \langle \nabla f^*(A\lambda'), A\lambda - A\lambda' \rangle + \frac{1}{2\mu} \| A\lambda - A\lambda' \|^2 \]
\[ f^*(A\lambda') + \langle A^+ \nabla f^*(A\lambda'), \lambda - \lambda' \rangle + \frac{N_{\min}}{2\mu} \| \lambda - \lambda' \|^2. \]

Because \( f^* \) is \((1/L)\)-strongly convex, from Lemma 1, for any \( \lambda \) and \( \lambda' \), we have
\[ f^*(A\lambda) \]
\[ f^*(A\lambda') + \langle \nabla f^*(A\lambda'), A\lambda - A\lambda' \rangle + \frac{1}{2L} \| A\lambda - A\lambda' \|^2 \]
\[ f^*(A\lambda') + \langle A^+ \nabla f^*(A\lambda'), \lambda - \lambda' \rangle + \frac{N_{\min}}{2L} \| \lambda - \lambda' \|^2. \]

This concludes the proof.

We define \( \delta \) as follows:
\[ \delta := \max \left\{ \frac{\alpha N_{\max} - 1}{\alpha N_{\min} + 1}, \frac{1}{\alpha N_{\max} + 1} \right\}. \]

Note that when Assumptions 2, 3, and 4 are satisfied, and when \( \alpha > 0, 0 \leq \delta < 1 \) is satisfied.

**Lemma 3:** Under Assumptions 2, 3, and 4, \( R_{\alpha A^+ g} \) is \( \delta \)-contractive for any \( \alpha \in (0, \infty) \).

**Proof:** The statement follows from [10, Theorem 1] and Lemma 2.

**Lemma 4:** Under Assumptions 2, 3, and 4, \( R_{\alpha A^+ A^+ g} \) is \( \delta \)-contractive for any \( \alpha \in (0, \infty) \).

**Proof:** From [1, Corollary 23.9] and [1, Theorem 20.25], \( R_{\alpha A^+ A^+} \) is nonexpansive for any \( \alpha \in (0, \infty) \). Then, from Lemma 3, for any \( z \) and \( z' \), we have
\[ \| R_{\alpha A^+ A^+ g} z - R_{\alpha A^+ A^+} R_{\alpha A^+ A^+ g} z' \| \leq \| R_{\alpha A^+ g} z - R_{\alpha A^+ g} z' \| \]
\[ \leq \delta \| z - z' \|. \]
This concludes the proof.

In the following, we use the notation $E_r[\cdot]$ as the expectation over the randomness in the round $r$.

**Lemma 5:** Let $\bar{z} \in \text{Fix}(R_{\alpha \partial_z}R_{\alpha \nabla g})$. Under Assumptions 1, 2, 3, and 4, $z^{(r+1)}$ and $z^{(r)}$ generated by Eqs. (37-38) satisfy the following:

$$E_r\|z^{(r+1)} - \bar{z}\| \\ \leq \{1 - \theta + \theta \delta + \sqrt{1 - \tau(\theta + |1 - \theta|\delta + \delta)}\} \|z^{(r)} - \bar{z}\|,$$

(40)

**Proof:** Combining (38) and (39), the update formula of $z$ can be written as follows:

$$z^{(r+1)} = z^{(r)} + \theta \text{comp}((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z^{(r)}).$$

Let $\omega^{(r)}$ be the parameter used to compress $((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z^{(r)})$. In the following, to use (8) and (9) in Assumption 1, we rewrite the update formula of $z$ as follows:

$$z^{(r+1)} = z^{(r)} + \theta \text{comp}((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z^{(r)}; \omega^{(r)}).$$

Because $\bar{z} \in \text{Fix}(R_{\alpha \partial_z}R_{\alpha \nabla g})$, we have $(R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})\bar{z} = 0$. Under Assumption 1, because $\text{comp}(0; \omega) = 0$ for any $\omega$, we have

$$\bar{z} = \bar{z} + \theta \text{comp}((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z; \omega^{(r)}).$$

We then have

$$E_r\|z^{(r+1)} - \bar{z}\| = E_r\|z^{(r)} - \bar{z} + \theta \text{comp}((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z^{(r)}; \omega^{(r)})\|$$

$$- \theta \text{comp}((R_{\alpha \partial_z}R_{\alpha \nabla g} - \text{Id})z; \omega^{(r)})\|$$

$$\leq \|(1 - \theta)(z^{(r)} - \bar{z}) + \theta(R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$+ E_r\|\theta(z^{(r)} - \bar{z} - R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} + R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$+ \theta\{\text{comp}(z^{(r)} - \bar{z} - R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} + R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z}; \omega^{(r)})\|$$

$$\leq \|(1 - \theta)(z^{(r)} - \bar{z}) + \theta(R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$+ \sqrt{1 - \tau}\|\theta(z^{(r)} - \bar{z}) - \theta(R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$\leq \|(1 - \theta)(z^{(r)} - \bar{z}) + \theta(R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$+ \sqrt{1 - \tau}\|R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z}\|,$$

where we use Assumption 1 for (a) and (b). From Lemma 4, $T_1$, $T_2$, and $T_3$ are upper bounded as follows:

$$T_1 \leq |1 - \theta|\|z^{(r)} - \bar{z}\| + \theta\|R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z}\|$$

$$\leq \|(1 - \theta + \theta\delta)\|z^{(r)} - \bar{z}\|,$$

$$T_2 \leq \sqrt{1 - \tau}$$

$$T_3 \leq \sqrt{1 - \tau}\|R_{\alpha \partial_z}R_{\alpha \nabla g}z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z}\|,$$

$$\leq \sqrt{1 - \tau}\|z^{(r)} - \bar{z}\| + |1 - \theta|\|R_{\alpha \partial_z}R_{\alpha \nabla g}(z^{(r)} - R_{\alpha \partial_z}R_{\alpha \nabla g}\bar{z})\|$$

$$\leq \sqrt{1 - \tau}\|z^{(r)} - \bar{z}\|.$$
\[
\arg\min_w \left\{ f(w) + \frac{\alpha}{2} \| A^T w \|^2 - \frac{1}{2} \| w \|^2 \right\} \\
+ \frac{1}{2} \| w - A z^{(r)} \|^2
\]

From [1, Example 23.3], the proximal operator of \( f + \frac{\alpha}{2} \| A^T \cdot \| \|^2 \) is the resolvent of \( \nabla (f + \frac{\alpha}{2} \| A^T \cdot \| \)^{-1}, and the resolvent can be rewritten as follows:

\[
J_{\nu(f + \frac{\alpha}{2} \| A^T \cdot \| \)^{-1}}(z) = \left( \text{Id} + \nabla \left( f + \frac{\alpha}{2} \| A^T \cdot \| \right) \right)^{-1} z
\]

From Assumption 3 and Remark 1, \( f + \frac{\alpha}{2} \| A^T \cdot \| \) is strongly convex. Then, \( f + \frac{\alpha}{2} \| A^T \cdot \| \) is Lipschitz continuous. Then, we have

\[
\left\| \text{prox}_{f + \frac{\alpha}{2} \| A^T \cdot \| \}(z) - \text{prox}_{f + \frac{\alpha}{2} \| A^T \cdot \| \}(z') \right\| \\
\leq \frac{1}{\mu + \alpha N_{\text{min}}} \| A(z - z') \|
\]

Lemma 8: Suppose that Assumptions 2, 3, and 4 hold. Let \( \lambda^* \) be the optimal solution of the dual problem of (27). Then, \( \nabla f^*(\lambda^*) \) is the optimal solution of the primal problem of (26).

Proof: By using the optimality condition [31, Theorem 27.1] of \( \lambda^* \), we have

\[
-A^T \nabla f^*(\lambda^*) \in \partial \varphi(\lambda^*)
\]

From the property of the convex conjugate function of \( \varphi \), we get

\[
\lambda^* \in \partial \varphi(-A^T \nabla f^*(\lambda^*))
\]

Then, by multiplying \( A \) and using the property of the convex conjugate function of \( f \), we have

\[
\nabla f(\nabla f^*(\lambda^*)) \in A \partial \varphi(-A^T \nabla f^*(\lambda^*))
\]

From the above equation, \( \nabla f^*(\lambda^*) \) satisfies the optimality condition of (26). This concludes the proof.

Lemma 9: Let \( \bar{z} \in \text{Fix}(R_{\alpha N_{\text{min}}}) \), and let \( w^* \) be the optimal solution of (2). Under Assumptions 1, 2, 3, and 4, \( w \) generated by Algorithm 1 satisfies the following:

\[
\max_{1 \leq i \leq N} \left\{ \frac{\sqrt{\lambda_{\text{max}}}}{\mu + \alpha N_{\text{min}}} \| z - z^{(r)} \| \right\}
\]

Lemma 10: Let \( \bar{z} \in \text{Fix}(R_{\alpha N_{\text{min}}}) \), and let \( w^* \) be the optimal solution of (2). Under Assumptions 1, 2, 3, and 4, \( w \) generated by Algorithm 1 linearly converges to the optimal solution \( w^* \) as follows:

\[
\| w^{(r+1)} - w^* \| \leq \frac{\sqrt{\lambda_{\text{max}}}}{\mu + \alpha N_{\text{min}}} \| z^{(r)} - \bar{z} \|
\]

Proof: The statement follows from Lemma 6 and Lemma 9.

Corollary 1: Let \( \bar{z} \in \text{Fix}(R_{\alpha N_{\text{min}}}) \), and let \( w^* \) be the optimal solution of (2). Under Assumptions 1, 2, 3, and 4, \( w \) generated by Algorithm 1 linearly converges to the optimal solution \( w^* \) as follows:

\[
\| w^{(r+1)} - w^* \| \leq \frac{\sqrt{\lambda_{\text{max}}}}{\mu + \alpha N_{\text{min}}} \| z^{(r)} - \bar{z} \|
\]
The statement follows from Theorem 1.

Corollary 2: Under Assumptions 1, 2, 3, and 4, when $\tau \geq 1 - \left(1 + \frac{1}{T}\right)^2$, the optimal convergence rate of (18) in the C-ECL is achieved when $\theta = 1$.

Proof: By Theorem 1, when $\theta \leq 1$, we have

$$E\|w^{(r+1)} - w^*\| \leq \frac{\sqrt{N_{max}}}{\mu + \alpha N_{min}} \times \left\{ 1 + 2\delta \sqrt{1 - \tau} - \theta(1 - \sqrt{1 - \tau})(1 - \delta) \right\}^r \|z^{(0)} - \bar{z}\|.$$  

Because $(1 - \sqrt{1 - \tau})(1 - \delta) > 0$, $\left\{ 1 + 2\delta \sqrt{1 - \tau} - \theta(1 - \sqrt{1 - \tau})(1 - \delta) \right\}$ decreases when $\theta$ approaches 1. When $\theta \geq 1$, we have

$$E\|w^{(r+1)} - w^*\| \leq \frac{\sqrt{N_{max}}}{\mu + \alpha N_{min}} \times \left\{ -1 + \theta(1 + \delta)(1 + \sqrt{1 - \tau}) \right\}^r \|z^{(0)} - \bar{z}\|.$$  

Because $(1 + \delta)(1 + \sqrt{1 - \tau}) > 0$, $\left\{ -1 + \theta(1 + \delta)(1 + \sqrt{1 - \tau}) \right\}$ decreases when $\theta$ approaches 1. Therefore, the optimal convergence rate is achieved when $\theta = 1$. □

Corollary 3: Under Assumptions 1, 2, 3, and 4, when $\tau = 1$, the optimal convergence rate of (19) is achieved when $\theta = 1$.

Proof: The statement follows from Corollary 2. □

APPENDIX D
EXPERIMENTAL SETTING

A. Hyperparameter

In this section, we describe the detailed hyperparameters used in our experiments.

FashionMNIST: We set the learning rate to 0.001, batch size to 100, and the number of epochs to 1500. To avoid overfitting, we use RandomCrop of PyTorch for the data augmentation. In the ECL, following the previous work [29], we set $\alpha$ as follows:

$$\alpha = \frac{1}{\eta |N_i| (K - 1)}, \quad (47)$$

where $K$ is the number of local steps. In the C-ECL, when we use rand$_{kK}$ as the compression operator, the number of local steps can be regarded to be $\frac{100K}{k}$. Then, in the C-ECL, we set $\alpha$ as follows:

$$\alpha = \frac{1}{\eta |N_i| (\frac{100K}{k} - 1)}. \quad (48)$$

Note that $\alpha$ is set to the different values between the nodes by the definition of $\alpha$ in Eqs. (47-48). For the D-PSGD and the PowerGossip, we use Metropolis-Hastings weights [45] as the weights of the edges.

CIFAR10: We set the learning rate to 0.005, batch size to 100, and the number of epochs to 2500. To avoid overfitting, we use RandomCrop and RandomHorizontalFlip of PyTorch for the data augmentation. For the ECL and the C-ECL, we set $\alpha$ in the same way as the FashionMNIST. For the D-PSGD and the PowerGossip, we use Metropolis–Hastings weights as the weights of the edges.

B. Network Topology

Fig. 2 shows the visualization of the network topology used in our experiments.

APPENDIX E
ADDITIONAL EXPERIMENT

A. Results on Synthetic Dataset

In this section, we demonstrate the effectiveness of the C-ECL using the synthetic dataset.

Synthetic Dataset: We set the number of nodes $n = 25$ and use the ring as an underlying network topology. We set the loss function of node $i$ to be $f_i(\mathbf{X}) = \frac{1}{2} \| \mathbf{X} - \mathbf{B}_i \|_{F}^2$, where $\mathbf{X} \in \mathbb{R}^{10 \times 10}$, $\mathbf{B}_i \in \mathbb{R}^{10 \times 10}$, and each element of $\mathbf{B}_i$ is drawn from $\mathcal{N}(0, \frac{\zeta^2}{100})$. As $\zeta^2$ becomes larger, the data distributions held by each node can be considered to be more heterogeneous.

Results: Fig. 3 shows how the parameters reach the optimal solution $\mathbf{X}^*$. The results indicate that in the D-PSGD and PowerGossip, the error only goes down to $10^{-2}$ when $\zeta^2 > 0$. On the other hand, in the ECL and C-ECL, the error can go down to $10^{-8}$ even when $\zeta^2 > 0$. Thus, the C-ECL is more robust to data heterogeneity than the D-PSGD and PowerGossip.

B. Comparision With ChocoSGD

In this section, we also compared the C-ECL with the ChocoSGD [15], showing the results in Table IV. In the homogeneous setting, all methods achieve comparable accuracy, while in the heterogeneous setting, the C-ECL can consistently outperform the PowerGossip and ChocoSGD. Thus, the C-ECL is more robust to data heterogeneity than the PowerGossip and ChocoSGD.
Fig. 3. Comparison of error $\frac{1}{n} \sum_i \|X_i - X^*\|^2$ when varying data heterogeneity. We omit the results of the PowerGossip when $\zeta^2 = 0$ because the power iteration for a zero matrix numerically collapses.

TABLE IV
TEST ACCURACY AND COMMUNICATION COSTS ON FASHIONMNIST

|                  | Homogeneous Setting | Heterogeneous Setting |
|------------------|---------------------|-----------------------|
|                  | Accuracy | Send/Epoch | Accuracy | Send/Epoch |
| C-ECL (1%)        | 84.4     | 577 KB     | 79.1     | 577 KB     |
| C-ECL (10%)       | 84.4     | 5377 KB    | 84.1     | 5377 KB    |
| C-ECL (20%)       | 84.2     | 10709 KB   | 84.5     | 10709 KB   |
| BEER (0.5%)       | nan      | 550 KB     | nan      | 550 KB     |
| BEER (5%)         | 84.6     | 5323 KB    | 82.2     | 5322 KB    |
| BEER (10%)        | 84.8     | 10619 KB   | 84.0     | 10618 KB   |

We use the ring as the underlying network topology and set the number of local steps to one. The number in the bracket is $k$ of rand$_{init}$. For the PowerGossip, the number in the bracket is the number of the power iteration steps.

C. Comparison With BEER

We compared C-ECL with BEER [49], which is the communication compression method for Gradient Tracking [27].

In the original BEER, the number of local steps was set to one, and we found that BEER collapses when the number of local steps is larger than one. Thus, we compared C-ECL with BEER by setting the number of local steps to one. Table V lists accuracy and communication costs. The results indicate that C-ECL achieves comparable accuracy to BEER in the homogeneous setting and can outperform BEER in the heterogeneous setting. BEER requires nodes to exchange two variables with their neighbors, while C-ECL requires nodes to exchange one variable. Therefore, C-ECL can outperform BEER when the communication costs are the same.

D. Results With Other Heterogeneous Setting.

In this section, we show the results when varying the data heterogeneity by using Dirichlet distribution with hyperparameter $\alpha$ [12]. As $\alpha$ decreases, the data distributions held by each node become more heterogeneous. The results are listed in Table VI and are consistent with the results shown in Section V-B. Thus, the C-ECL is more robust to data heterogeneity than the D-PSGD and PowerGossip and can achieve almost the same accuracy with approximately 4-time fewer parameter exchanges than the ECL.

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