Abstract

Pontryagin’s Maximum Principle is an outstanding result for solving optimal control problems by means of optimizing a specific function on some particular variables, the so-called controls. However, this is not always enough for solving all these problems. A high order maximum principle (Krener, 1977) must be used in order to obtain more necessary conditions for optimality. These new conditions determine candidates to be optimal controls for a wider range of optimal control problems.

Here, we focus on control-affine systems. Krener’s high order perturbations are redefined following the notions introduced in Aguilar–Lewis (2008). A weaker version of Krener’s high order maximum principle is stated in the framework of presymplectic geometry. As a result, the presymplectic constraint algorithm in the sense of Gotay–Nester–Hinds (1979) can be used. We establish the connections between the presymplectic constraint algorithm and the candidates to be optimal curves obtained from the necessary conditions in Krener’s high order maximum principle. In this paper we obtain weaker geometric necessary conditions for optimality of abnormal solutions than the ones in Krener (1977) and the ones in the weak high order maximum principle. These new necessary conditions are more useful, computationally speaking, for finding curves candidate to be optimal. The theory is supported by describing specifically some of the above-mentioned conditions for some mechanical control systems.

Keywords: Optimal control problem, high order perturbations, presymplectic constraint algorithm.

Mathematics Subject Classification (2000): 49E25, 49J15, 70H05, 70H50, 93.40.

1 Introduction

Pontryagin’s Maximum Principle (PMP) is a useful result to try to solve optimal control problems since it provides necessary conditions for optimality. This Principle is understood as a first order test of optimality for two reasons. If the controls take values in the interior of the control set, the condition of maximization of the Hamiltonian can be replaced by the fact that the partial derivative of the Hamiltonian with respect to the controls vanishes. On the other hand, the perturbations of the optimal curves at a point are linearly approximated by the so-called perturbation vectors in a neighbourhood of the point.

Despite the utility of Pontryagin’s Maximum Principle it is not always possible to determine the optimal controls from its necessary conditions. For instance, the singular optimal curves are those curves whose controls are not uniquely determined from the condition of maximization of the Hamiltonian. There are also the so-called abnormal extremals that cannot be determined in general by means of Pontryagin’s Maximum Principle. That is why research toward a high
order maximum principle started to be developed \[1, 13, 14, 15\]. Hermes \[12\] was one of the first to introduce high order perturbations to study controllability along a fixed trajectory. Later these perturbations were adapted to provide necessary conditions for optimality \[14, 15\].

Geometrically speaking, optimal control problems can be studied from the viewpoint of symplectic geometry or of presymplectic one. The former consists of fixing the controls in such a way that the natural symplectic structure of the cotangent bundle of the state space is considered \[1, 3, 21\]. The latter consists of using the presymplectic structure \[4\] of the above cotangent bundle times the control set. In the presymplectic formalism, we are able to state a weaker version of Krener’s high order maximum principle in the same way as stated the weaker version of Pontryagin’s Maximum Principle in \[4\]. Some of the necessary conditions for optimality coming from the weak high order maximum principle give a geometric meaning to previous techniques used in the literature \[13, 14, 15\].

The paper is organized as follows. In Section 2 we review the notion of an optimal control problem using two equivalent statements. Section 3 contains all the necessary elements to state Krener’s high order Maximum Principle \[15\] for control-affine systems by using the constructions in \[2\]. In Section 3.1 a weak high order maximum principle is stated. The main result of this paper consists of establishing a connection between the above-mentioned weak principle and the classical high order maximum principle \[15\] as described in Sections 3.2 and 3.3 under the assumption of having an abnormal optimal solution. In order to clarify ideas, we conclude with a detailed description of some elements that appear in Section 3.2 for some mechanical control systems in Section 3.3.

Before concluding this introduction we want to point out that most of the theory in the first part of Section 3 and Section 3.1 can be rewritten for any control system. We restrict ourselves to control-affine systems because the aim of this paper is to establish a relationship between the above-mentioned presymplectic constraint algorithm and the high order maximum principle as explained in Section 3.2 for abnormal optimal solutions. This relationship is obtained by identifying brackets of vector fields in the control system with high order perturbation vectors. To our best knowledge, it is not known how to deal with that when the control systems are not control-affine.

## 2 Background in optimal control theory

First, let us define geometrically the notion of control system.

**Definition 1.** Let \( M \) be an \( m \)-dimensional manifold and \( U \) be a subset of \( \mathbb{R}^k \). A control system on \( M \) is a smooth vector field \( X \) defined along the projection \( \pi: M \times U \to M \). A trajectory or an integral curve of the control system \( X \) is a curve \((\gamma, u): I \subset \mathbb{R} \to M \times U \) such that \( \gamma \) is absolutely continuous, \( u \) is measurable and bounded, and \( \dot{\gamma}(t) = X(\gamma(t), u(t)) \) a.e. for \( t \in I \).

The set of control systems is denoted by \( \mathcal{X}(\pi) \); that is, the set of smooth vector fields defined along the projection \( \pi \). The curve \( u: I \to U \) is called the control.

Given a control system \( X \in \mathcal{X}(\pi) \), we are usually interested in the set of points that can be reached from an initial point through trajectories \((\gamma, u): I \subset \mathbb{R} \to M \times U \) of \( X \), where \( I = [a, b] \). In this regard, the following definition is essential.

**Definition 2.** Let \( M \) be a manifold, \( U \) be a set in \( \mathbb{R}^k \) and \( X \) be a smooth vector field along the projection \( \pi: M \times U \to M \). The reachable set from \( x_0 \in M \) at time \( T \in I \) is the set of points described by

\[
\mathcal{R}(x_0, T) = \{ x \in M \mid \text{there exists } (\gamma, u): [a, b] \to M \times U \text{ such that } \dot{\gamma}(t) = X(\gamma(t), u(t)) \text{ a.e., } \gamma(a) = x_0, \gamma(T) = x \}.
\]
The reachable set from a point $x_0 \in M$ up to time $T$ is

$$\mathcal{R}(x_0, \leq T) = \bigcup_{a \leq t \leq T} \mathcal{R}(x_0, t).$$

The topological properties of the reachable set are closely related to different properties of the control system such as reachability, controllability, see [7] for more details. Moreover, in the sequel we briefly describe why the linear approximation of the reachable sets is the key point to prove Pontryagin’s Maximum Principle [18] and also Krener’s high order maximum principle [15].

2.1 Optimal control problem

Let us define an optimal control problem associated to the control system in Definition 1. Given $F : M \times U \rightarrow \mathbb{R}$, consider the functional

$$\mathcal{S}[\gamma, u] = \int_I F(\gamma(t), u(t)) \, dt$$

defined on curves $(\gamma, u)$ with a compact interval as domain. The function $F : M \times U \rightarrow \mathbb{R}$ is continuous on $M \times U$ and continuously differentiable with respect to $M$ on $M \times U$.

If we fix the control $u$, the vector field $X \in \mathfrak{X}(\pi)$ can be rewritten as a time-dependent vector field $X^{(u)} : I \times M \rightarrow TM$, $X^{(u)}(t, x) = X(x, u(t))$.

**Statement 3.** (Optimal Control Problem, OCP) Given the elements $M$, $U$, $X$, $F$, $I = [a, b]$ and the endpoint conditions $x_a$, $x_b \in M$, consider the following problem.

Find $(\gamma, u)$ such that

1. $\gamma(a) = x_a$, $\gamma(b) = x_b$ (endpoint conditions),
2. $\gamma$ is an integral curve of $X^{(u)}$, i.e. $\dot{\gamma}(t) = X^{(u)}(t, \gamma(t))$, for a.e. $t \in I$, and
3. $\mathcal{S}[\gamma, u]$ is minimum over all curves satisfying (1) and (2) (minimal condition).

The function $F$ is called the cost function of the problem. A solution to the OCP is a curve $(\gamma, u)$ satisfying conditions (1)-(3) in Problem 3 such that $\gamma$ is absolutely continuous and $u$ is measurable and bounded. These problems are also known in the literature as Lagrange problems.

2.2 Extended optimal control problem

A usual technique in optimal control theory is to consider an equivalent problem to Problem 3 but defined on the extended manifold $\tilde{M} = \mathbb{R} \times M$. Let $\bar{X}$ be the following vector field defined along the projection $\tilde{\pi} : \tilde{M} \times U \rightarrow \tilde{M}$,

$$\bar{X}(x_0, x, u) = F(x, u)\partial / \partial x_0\big|_0 + X(x, u),$$

where $x_0$ is the natural coordinate on $\mathbb{R}$. In this new manifold it is easier to identify the direction of decreasing of the cost function. Now the integral curves of the control system $\bar{X}$ are curves $(\tilde{\gamma}, u) = ((x_0 \circ \tilde{\gamma}, \gamma), u) : I \rightarrow \tilde{M} \times U$ such that $\tilde{\gamma}$ is absolutely continuous and $u$ is measurable and bounded.

**Statement 4.** (Extended Optimal Control Problem, $\tilde{OCP}$) Given the elements in Problem 3 $\tilde{M}$ and $\bar{X}$, consider the following problem.

Find $(\tilde{\gamma}, u) = (\gamma^0, \gamma, u)$ such that
(1) $\hat{\gamma}(a) = (0, x_a)$, $\gamma(b) = x_b$ (endpoint conditions),

(2) $\hat{\gamma}$ is an integral curve of $\hat{X}^{(u)}$, i.e. $\hat{\gamma}(t) = \hat{X}^{(u)}(t, \hat{\gamma}(t))$, for a.e. $t \in I$, and

(3) $\gamma^0(b)$ is minimum over all curves satisfying (1) and (2) (minimal condition).

A solution to the OCP is a curve $(\hat{\gamma}, u)$ satisfying conditions (1)-(3) in Problem $\mathbb{P}$ such that $\hat{\gamma}$ is absolutely continuous and $u$ is measurable and bounded. In the literature this problem is also known as Mayer problem.

3 High order Maximum Principle for control-affine systems

First order necessary conditions for optimality do not always provide enough information for finding the optimal solution. High order variations of a curve depending on parameters must be studied, as is done for instance in [5, 6, 13, 14, 15], in order to obtain high order necessary conditions for the optimal solution. High order variations of a curve depending on parameters must be studied, as First order necessary conditions for optimality do not always provide enough information for finding

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Given a fixed reference trajectory $(\gamma, u)$ defined in $I \subset \mathbb{R}$, we use the description of high order variations for control-affine systems considered in [2] to study controllability at a point $x_0 \in M$ in order to obtain the variations along a reference trajectory defined in [15].

Let $\Sigma = (M, \{X_0, X_1, \ldots, X_k\}, U)$ be a control-affine system whose trajectories $(\gamma, u)$ on $M \times U$ satisfy

$$\dot{\gamma}(t) = X_0(\gamma(t)) + \sum_{i=1}^k u^c X_c(\gamma(t)).$$

The vector field $X_0$ is usually called the drift vector field and $X_1, \ldots, X_k$ are the control or input vector fields. For the sake of simplicity all the vector fields are assumed to be complete from now on.

Definition 5. Let $\xi = (\xi_1, \ldots, \xi_r)$ be a finite sequence of vector fields on $M$ where $\xi_i(x) \in \{X_0(x) + \sum_{c=1}^k u^c X_c(x) | u \in U\}$ for each $i \in \{1, \ldots, r\}$ and $x \in M$.

(i) The flow associated with $\xi$ called $\xi$-flow is the map $\Phi^\xi : \mathbb{R}^r \times M \to M$ given by

$$\Phi^\xi(t, x) = \left(\Phi^\xi_{t_1} \circ \cdots \circ \Phi^\xi_{t_r}\right)(x),$$

where $t = (t_1, \ldots, t_r)$. For a fixed $x \in M$ define $\Phi^\xi : \mathbb{R}^r \to M$ by $\Phi^\xi(t) = \Phi^\xi(t, x)$, and for a fixed $t \in \mathbb{R}^r$ define $\Phi^\xi_t : M \to M$ by $\Phi^\xi_t(x) = \Phi^\xi(t, x)$.

(ii) A positive $r$-end-time variation is a smooth map $\tau : [0, \infty) \to \mathbb{R}^r_{\geq 0}$ with the property $\tau(0) = 0$. The set of all the positive $r$-end-time variation is denoted by $\mathcal{ET}^r_\tau$.

As mentioned, the variations in Definition 5 are useful for studying particular notions of controllability as shown in [2]. However, to study optimality the variations described in [15] are necessary. We describe them similarly to the ones in Definition 5 using some usual notation from the theory of jet bundles [19].

Definition 6. Let $(\gamma, u)$ be a reference trajectory associated with the vector field $\xi_0$ and $\xi = (\xi_1, \ldots, \xi_r)$ be a finite sequence of vector fields on $M$ where $\xi_i(x) \in \{X_0(x) + \sum_{c=1}^k u^c X_c(x) | u \in U\}$ for each $i \in \{0, \ldots, r\}$, $x \in M$, $\tau \in \mathcal{ET}^r_\tau$.

(i) A $(r + 2)$-end-time variation is a smooth map $\tau_2 : [0, \infty) \to \mathbb{R}^2 \times \mathbb{R}^r_{\geq 0}$ such that $\tau_2(s) = (q_1(s), q_2(s), \tau(s))$ and $\tau_2(0) = 0$. 

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(ii) The \((\xi, \tau_2)\)-variation along \(\gamma\) at time \(t_0\) is the curve \(\nu_{\xi, \tau_2} : [0, \infty) \rightarrow M\) given by
\[
\nu_{\xi, \tau_2}(s) = (\Phi_{q_2(s)}^{\xi_0} \circ \Phi_{\tau(s)}^{\xi_1} \circ \Phi_{q_1(s)}^{\xi_0})(\gamma(t_0)).
\] (2)

(iii) If \(\tau_2\) is \((r + 2)\)-end-time variation, the order of the pair \((\xi, \tau_2)\) at \(\gamma(t_0) \in M\), denoted \(\text{ord}_{\gamma(t_0)}(\xi, \tau_2)\), is the smallest positive integer \(l\) such that there exists \(\epsilon > 0\) satisfying
\[
j^l(\nu_{\xi, \tau_2})(s) \neq 0_{\nu_{\xi, \tau_2}(s)}, \quad |s| < \epsilon.
\] (3)

If no such \(s\) exists then the order is set to \(\infty\).

(iv) If \(\text{ord}_{\gamma(t_0)}(\xi, \tau_2) = l\), then the \((\xi, \tau_2)\)-infinitesimal variation at \(\gamma(t_0)\) is the tangent vector
\[
V_{\xi, \tau_2} = j^l(\nu_{\xi, \tau_2})(0).
\]

A \((\xi, \tau_2)\)-infinitesimal variation of order \(l\) is also called a high order elementary perturbation vector at \(\gamma(t_0)\) of order \(l\).

Note that the notion of order is defined by a condition that must be satisfied in a neighborhood of 0, see [3]. As the controls are assumed to be measurable, this condition is necessary to guarantee that perturbations at different times can be summed as explained in [15].

These infinitesimal variations include the elementary perturbation vectors in the classical Pontryagin’s Maximum Principle [18]. They are defined as follows. Take \(\xi = (\xi_1)\) such that \(\xi_1(x) = X_0(x) + \sum_{i=1}^k u^i X_c(x)\) being \(u_1 \in U\), \(\tau(s) = l_1 s\) for \(l_1 \in \mathbb{R}^+\), \(q_1(s) = -l_1 s\), \(q_2(s) = 0\). Then the \((\xi, \tau_2)\)-variation along \(\gamma\) corresponding with \(\xi_0(x) = X_0(x) + \sum_{i=1}^k u^i X_c(x)\) at time \(t_0\) is
\[
\nu_{\xi, \tau_2}(s) = (\Phi_{l_1 s}^{\xi_1} \circ \Phi_{-l_1 s}^{\xi_0})(\gamma(t_0)).
\]
The \((\xi, \tau_2)\)-infinitesimal variation is given by
\[
\frac{d}{ds}\bigg|_{s=0} \nu_{\xi, \tau_2}(s) = \frac{d}{ds}\bigg|_{s=0} (\Phi_{t_1(s)}^{\xi_1}(\tau(s), \Phi_{t_2(s)}^{\xi_0}(q_1(s), \gamma(t_0))))
\] + \left. \frac{\partial \Phi_{\xi_1}}{\partial x^1} \bigg|_{s=0} \right) (\tau(s), \Phi_{t_2(s)}^{\xi_0}(q_1(s), \gamma(t_0))) \frac{d}{ds}\bigg|_{s=0} (\Phi_{t_2(s)}^{\xi_0}(q_1(s), \gamma(t_0)))^i (q_1(s), \gamma(t_0))
\] = \(l_1 \xi_1(\gamma(t_0)) + \frac{\partial \Phi_{\xi_1}}{\partial x^1} \bigg|_{s=0} \right) (0, \Phi_{t_2(s)}^{\xi_0}(q_1(s), \gamma(t_0)))\xi_0^i(\gamma(t_0)) (-l_1)
\] = \(l_1(\xi_1(\gamma(t_0)) - \xi_0(\gamma(t_0))) = \sum_{c=1}^k l_1(u^c - u^{c-1}) X_c(\gamma(t_0))\),

because \(\Phi_{t_1(s)}^{\xi_1}\) is the identity map. The last equality in (4) is exactly Pontryagin’s elementary perturbation vectors for control-affine systems.

**Remark 7.** All the theory developed in this section so far is true for general control systems apart from the last equality in (4), only true for control-affine systems.

**Definition 8.** Let \((\gamma, u)\) be a reference trajectory associated with \(\xi_0, t_0 \in I\). The set of all the high order elementary perturbation vectors at \(\gamma(t_0)\) of order \(l\) is
\[
\mathcal{V}_{\Sigma_2}^l(\gamma(t_0)) = \{ V_{\xi, \tau_2} \in T_{\gamma(t_0)}M \mid \tau_2 \in C^\infty(\mathbb{R}, \mathbb{R}^2) \times ET^+_r, \text{ord}_{\gamma(t_0)}(\xi, \tau_2) = l, \xi = (\xi_1, \ldots, \xi_r), \xi_i \in \{X_0 + \sum_{c=1}^k u^c X_c \mid u \in U\} \text{ for each } i = 1, \ldots, r, r \in \mathbb{Z}_{\geq 1} \}.
\]

The set of all the high order elementary perturbation vectors at \(\gamma(t_0)\) is
\[
\mathcal{V}_{\Sigma_2}(\gamma(t_0)) = \bigcup_{l \geq 1} \mathcal{V}_{\Sigma_2}^l(\gamma(t_0)).
\]
Remark 9. If the control system is control-affine and $0 \in \text{int} \, U$ then (4) guarantees that
\[
\text{cone}(\text{conv}\{X_1(\gamma(t)), \ldots, X_k(\gamma(t))\}) \subseteq \mathcal{Y}_{\Sigma_2}^1(\gamma(t)),
\]
where conv denotes convex hull.

Proposition 10. The set $\mathcal{Y}_{\Sigma_2}^1(\gamma(t))$ is a convex cone.

Proof. The set $\mathcal{Y}_{\Sigma_2}^1(\gamma(t))$ is a convex cone if it is closed under addition and closed under $\mathbb{R}_{>0}$-multiplication.

Then [15, Lemma 3.1] guarantees that the order of any variation can be shifted upward and [15, Lemma 3.4] assures that the sum of variations of different order is a variation of higher order. Thus $\mathcal{Y}_{\Sigma_2}^1(\gamma(t))$ is closed under addition. To prove that $\mathcal{Y}_{\Sigma_2}^1(\gamma(t))$ is closed under $\mathbb{R}_{>0}$-multiplication, the time must be re-scaled by using a different $(r + 2)$-end-time variation.

Definition 11. Let $(\gamma, u)$ be a reference trajectory associated with $\xi_0$, $t \in I$. The high order tangent perturbation cone $K_t$ at $\gamma(t)$ is the smallest closed convex cone in $T_{\gamma(t)}M$ that contains all the displacements by the flow of $\xi_0$ of all the $(\xi, \tau_2)$-infinitesimal variations along $\gamma$ at all Lebesgue times smaller than $t$, i.e.,
\[
K_t = \text{conv} \left( \bigcup_{a<t_0 \leq t} (\Phi_{\xi_0}^{t-t_0}(\mathcal{Y}_{\Sigma_2}^1(\gamma(t_0)))) \right).
\]

(5)

Remark 12. The notion of Lebesgue time comes from considering Carathéodory vector fields, see for instance [9] for more details.

This cone generalizes the notion of tangent perturbation cone introduced in Pontryagin’s Maximum Principle [1, 3, 18] so as to state and prove the high order maximum principle, see Theorem 15.

The high order tangent perturbation cone may be understood as an approximation of the reachable set along a reference trajectory that is perturbed as shown in the following result.

Proposition 13. Let $(\gamma, u)$ be a reference trajectory and $t \in (a, b]$. If $v$ is a nonzero vector in the interior of the high order tangent perturbation cone $K_t$, then there exist $\epsilon > 0$ and a $(\xi, \tau_2)$-variation of order $l$ such that
\[
\nu_{\xi, \tau_2}(s) = \gamma(t) + v \frac{s^l}{l!} + o(s^l) \quad \text{for} \quad 0 < s < \epsilon.
\]

Proof. It is part of the proof of Theorem 3.6 in [15, page 266].

We have constructed the above perturbations on the manifold $M$ for the sake of simplicity, but in order to state the high order maximum principle these perturbations must be considered for the extended control system defined on $\tilde{M} = \mathbb{R} \times M$. On $\tilde{M}$ the perturbations are defined analogously as the ones in Definition 6. Note that in general a control-affine system on $M$ is not control-affine on $\tilde{M}$. However, as mentioned in Remark 7 all the above constructions and results are true for any control system apart from Remark 9.

To solve optimal control problems is difficult. That is why Pontryagin’s Maximum Principle [18] transforms them into Hamiltonian problems that provide more conditions to characterize optimality. To state Krener’s maximum principle [15] we need to define a Hamiltonian system associated with the control system $\tilde{X}$. First, consider the cotangent bundle $T^*\tilde{M}$ with its natural symplectic structure that will be denoted by $\Omega$. If $(\tilde{x}, \tilde{p}) = (x^0, x, p_0, p) = (x^0, x^1, \ldots, x^m, p_0, p_1, \ldots, p_m)$ are local natural
coordinates on $T^*\hat{M}$, the form $\Omega$ has as local expression $\Omega = dx^0 \wedge dp_0 + dx^i \wedge dp_i$. For each $u \in U$, $H^u: T^*\hat{M} \to \mathbb{R}$ is the Hamiltonian function defined by

$$H^u(\hat{p}) = H(\hat{p}, u) = \langle \hat{p}, \hat{X}(\hat{x}, u) \rangle = p_0 f(x, u) + \sum_{i=1}^{m} p_i f_i(x, u),$$

where $\hat{p} \in T^*_x \hat{M}$ and $\hat{X} = \mathcal{F} \partial / \partial x^0 + f \partial / \partial x^i$. The tuple $(T^*\hat{M}, \Omega, H^u)$ is a Hamiltonian system. The associated Hamiltonian vector field $\hat{X}_H^u$ on $T^*\hat{M}$ defined by $\hat{X}_H^u(\hat{p}) = \hat{X}(\hat{p}, u)$ satisfies the equation

$$i_{\hat{X}_H^u} \Omega = \text{d}H^u.$$

It should be noted that $\hat{X}_H^u = \left(\hat{X}^u\right)^T$; that is, $\hat{X}_H^u$ is the cotangent lift of $\hat{X}^u$. Hence we get a family of Hamiltonian systems parameterized by $u$ and given by $H^u$.

With the above constructions, the Hamiltonian system in mind and the following definition, we are able to state the high order maximum principle [15].

**Definition 14.** Let $C$ be a cone with vertex at $0 \in T_x \hat{M}$. A **supporting hyperplane to $C$ at 0** is a hyperplane such that $C$ is contained in one of the half-spaces defined by the hyperplane.

**Theorem 15.** (High order maximum principle, HOMP) Let $(\hat{\gamma}, u): I \to \hat{M} \times U$ be a solution to $\hat{OCP}$, Problem 4, and let $H$ be the Hamiltonian function defined by

$$H: T^*\hat{M} \times U \rightarrow \mathbb{R}$$

$$(\hat{p}, u) \mapsto H(\hat{p}, u) = \langle \hat{p}, \hat{X}(\hat{x}, u) \rangle$$

where $\hat{p} \in T^*_x \hat{M}$. Then there exists $\hat{\lambda}: I \to T^*\hat{M}$ along $\hat{\gamma}$ such that:

1. $\hat{\lambda}$ is an integral curve of a Hamiltonian vector field $\hat{X}_H^u$ on $T^*\hat{M}$, i.e. solution to Hamilton’s equations

$$i_{\hat{X}_H^u} \Omega = \text{d}H^u;$$

2. (a) $H(\hat{\lambda}(t), u(t)) = \sup_{w \in U} H(\hat{\lambda}(t), w)$ almost everywhere;
   (b) $\sup_{w \in U} H(\hat{\lambda}(t), w)$ is constant everywhere;
   (c) for every $t \in I$, ker $\lambda(t)$ is a supporting hyperplane to $\hat{K}_t$ at 0;
   (d) $\hat{\lambda}(t) \neq 0 \in T^*_x \hat{M}$ for each $t \in [a, b]$;
   (e) $\lambda_0$ is constant and $\lambda_0 \leq 0$.

**Proof.** It follows similarly to the proof of Pontryagin’s Maximum Principle [11, 3, 18] having in mind Proposition [15]. For more details see proof in [15, Section 3].

The condition that does not appear in the classical Pontryagin’s Maximum Principle (PMP) is item (2c). The existence of this supporting hyperplane is not explicitly stated in PMP, though it is also fulfilled. Thus all the constructions related with the cones are not necessary to state the classical PMP, but they are completely necessary to prove it. After Definition 6 we have already proved how to relate $(\xi, \tau)$-variations of order 1 with the elementary perturbation vectors used in the classical PMP, see [4]. For these vectors the existence of a supporting hyperplane is equivalent to

$$\langle \hat{p}, \hat{X}(\hat{\gamma}(t), w(t)) - \hat{X}(\hat{\gamma}(t), u(t)) \rangle \leq 0 \text{ \forall w s.t. } w(t) \in U, \forall t \in I,$$

that is,

$$\langle \hat{p}, \hat{X}(\hat{\gamma}(t), w(t)) \rangle \leq \langle \hat{p}, \hat{X}(\hat{\gamma}(t), u(t)) \rangle \text{ \forall w s.t. } w(t) \in U, \forall t \in I.$$
In other words, the existence of a supporting hyperplane to the first order tangent perturbation cone is equivalent to the classical condition of maximization in PMP corresponding with (2a) in Theorem 15. That is why in PMP the supporting hyperplane is not mentioned in the statement.

On the contrary, in the high order version of this Principle more perturbations are considered. The usual necessary conditions in PMP appear in Theorem 15, but also some extra conditions as given in (2c). These extra conditions consist of determining a hyperplane that supports a greater cone than the first order tangent perturbation cone. Remember that the need of a high order maximum principle comes from the need to obtain more conditions to determine the optimal controls.

Observe that HOMP guarantees the existence of a covector along the optimal curve, but it does not say anything about the uniqueness of the covector, see Definition 17 for the different kind of curves. Indeed, the key point in the proof of HOMP consists of choosing an initial condition for \( \hat{\lambda} \) to integrate (7) in such a way that the resulting momentum satisfies all the necessary conditions in Theorem 15. Moreover, as \( \hat{\gamma} \) is an optimal trajectory, the momentum must separate the cone \( \hat{K}_t \) and the direction of decreasing of the cost function. Note that not all the supporting hyperplanes fulfill necessarily this separating condition. The non existence of a supporting hyperplane being also separating would immediately contradict the hypothesis of optimality, see proof in [1, 3, 15, 18] for more details.

Remark 16. We always refer to the maximum instead of supremum because the set of necessary conditions in HOMP guarantee that the supremum is achieved, so it can be called maximum.

Definition 17. A curve \( (\hat{\gamma}, u): [a, b] \rightarrow \hat{M} \times U \) for \( \hat{\text{OCP}} \), Problem 4, is

(i) an extremal if there exists \( \hat{\lambda}: [a, b] \rightarrow T^*\hat{M} \) along \( \hat{\gamma} = \pi_{\hat{M}} \circ \hat{\lambda} \) being \( \pi_{\hat{M}}: T^*\hat{M} \rightarrow \hat{M} \) and \( (\hat{\lambda}, u): [a, b] \rightarrow T^*\hat{M} \times U \) satisfies the necessary conditions of HOMP. The pair \( (\hat{\lambda}, u) \) is called biextremal for \( \hat{\text{OCP}} \);

(ii) a normal extremal if it is an extremal with \( \lambda_0 = -1 \);

(iii) an abnormal extremal if it is an extremal with \( \lambda_0 = 0 \);

(iv) a strictly abnormal extremal if it is not a normal extremal, but it is abnormal.

3.1 Weak high order maximum principle

From now on the control set \( U \) is assumed to be an open set. In the sequel, we use a presymplectic framework to state a weaker high order Maximum Principle, as was already considered for PMP in [4]. We will show that the presymplectic framework provides some useful techniques to characterize optimality in a geometric way. Some of the new conditions obtained are easier to deal with than to compute the \( (\xi, \tau_2) \)-infinitesimal variations in order to find a supporting hyperplane to all of them.

Observe that \( (T^*\hat{M} \times U, \Omega, H) \) is a presymplectic Hamiltonian system with \( \Omega \) being the pullback of the canonical 2–form on \( T^*\hat{M} \) through \( \pi_1: T^*\hat{M} \times U \rightarrow T^*\hat{M} \) and \( H \) the function in (6).

Theorem 18. (Weak high order maximum principle, WHOMP) Let \( (\hat{\gamma}, u): I \rightarrow \hat{M} \times U \) be a solution to \( \hat{\text{OCP}} \), Problem 4, and let \( H \) be the Hamiltonian function defined in (6). Then there exists \( \hat{\lambda}: I \rightarrow T^*\hat{M} \) along \( \hat{\gamma} \) such that:

(i) \( (\hat{\lambda}, u) \) is an integral curve of a Hamiltonian vector field \( \hat{X}_H \) such that

\[
i_{\hat{X}_H} \Omega = dH, \text{ i.e. } i_{(\hat{\lambda}(t), u(t))} \Omega = dH(\hat{\lambda}(t), u(t)); \quad (8)
\]

(ii) \( H(\hat{\lambda}(t), u(t)) \) is constant almost everywhere;
(b) for every $t \in I$, $\ker \hat{\lambda}(t)$ is a supporting hyperplane to $\hat{K}_t$ at 0;
(c) $\hat{\lambda}(t) \neq 0 \in T^*_{\gamma(t)}\hat{M}$ for each $t \in [a,b]$;
(d) $\lambda_0$ is constant and $\lambda_0 \leq 0$.

As $\Omega$ is degenerate, \cite{9} does not necessarily have solution on the entire manifold $T^*\hat{M} \times U$. It may have a solution, see \cite{11}, if we restrict the equation to the primary constraint submanifold defined by
\[ N_0 = \left\{ (\hat{\lambda}, u) \in T^*\hat{M} \times U \mid i_v \, dH = 0, \quad \text{for } v \in \ker \Omega(\hat{\lambda}, u) \right\}. \]
Locally, $N_0 = \left\{ (\hat{\lambda}, u) \in T^*\hat{M} \times U \mid \frac{\partial H}{\partial u^c}(\hat{\lambda}, u) = 0, \quad c = 1, \ldots, k \right\}$. If $X_0$ is a solution to the presymplectic equation, then $X^N_0 = X_0 + \ker \Omega$. At this point, a presymplectic constraint algorithm in the sense given in \cite{11} starts. The stabilization process gives inductively $N_1 = \{(\hat{\lambda}, u) \in N_0 \mid \exists X \in X^N_0, \, X(\hat{\lambda}, u) \in T(\hat{\lambda}, u)N_0 \}, \, N_2, \ldots, N_r$ such that $N_r \subseteq N_{r-1}$ and finishes when $N_i = N_{i-1}$ for some $i \in \mathbb{N}$. The algorithm just imposes tangency conditions in order to find a submanifold where the optimal trajectories live and stay. This algorithm has been adapted when the sets are not submanifolds \cite{16}, thus this method is not that restrictive.

Observe that $N_0$ is defined implicitly by a necessary condition for the Hamiltonian to have an extremum over the controls as long as $U$ is an open set. In Theorem \cite{15} the Hamiltonian is equal to the maximum of the Hamiltonian over the controls. Therefore, Theorem \cite{18} is just a weaker version of Theorem \cite{15} and the proof is straightforward from Theorem \cite{15} and the explanation given about the presymplectic constraint algorithm.

For normal extremals, $p_0 = -1$, and for cost functions quadratic on the controls, the presymplectic algorithm stops at $N_0$. There the controls can already be determined. More discussion about how the algorithm behaves for abnormality is developed in the following section.

### 3.2 Tangent perturbation cone versus presymplectic constraint algorithm

The high order tangent perturbation cone in Definition \cite{11} gives directions that approximate the perturbations of the optimal trajectory in the sense given in Proposition \cite{13}. On the other hand, the submanifolds that appear in the presymplectic constraint algorithm live in $T^*\hat{M} \times U$ and they are the zero sets of a family of vector fields $Z_j : \hat{M} \times U \to T\hat{M}$, $j \in \mathbb{N}$ as is proved in the sequel.

Our goal now is to establish a relationship between $\hat{K}_t$ and the sequence of constraint submanifolds $N_i$. From now on, we focus on abnormal optimal solutions because in general the normal optimal solutions admit a lift to the cotangent bundle $T^*\hat{M}$ that lies in the primary constraint submanifold and so the algorithm stops in the first step. For abnormal extremals, the existence of a supporting hyperplane $\ker \hat{\lambda}$ to $\hat{K}_t$ is equivalent to the existence of a supporting hyperplane given by $\ker \lambda$ to $K_t$ without considering the extended manifold.

**Theorem 19.** Let $U$ be an open set in $\mathbb{R}^k$, $\Sigma = (M, \{X_0, \ldots, X_k\}, U)$ be a control-affine system such that $0 \in \text{int} \, U$ and $(\gamma, u)$ be an abnormal optimal solution associated with $\xi_0$. Then there exists $\lambda : I \to T^*M$ along $\gamma$ and $l \in \mathbb{N}$ satisfying $N_l = N_{l+1}$ such that for every $i = 0, \ldots, l$ there exists a family of vector fields
\[ Z_i(\gamma(t), u) = \{Z_1(\gamma(t), u), \ldots, Z_{n_i}(\gamma(t), u)\} \subseteq V^l_{\Sigma_2}(\gamma(t)) \]
fulfilling that

(i) the sequence of constraint submanifolds $N_i$ obtained from the presymplectic constraint algorithm is given by
\[ N_i = \{ (\beta, u) \in N_{i-1} \mid \langle \beta, Z_j(x, u) \rangle = 0, \quad j = 1, \ldots, n_i \}, \quad \beta \in T^*_xM, \]


for $i = 1, \ldots, l$;

(ii) and $(\lambda, u) \in N_l$.

Proof. Since $(\gamma, u)$ is optimal, Theorem 18 can be applied and the presymplectic constraint algorithm can be used. For $i = 0$, $N_0 = \{(\beta, u) \mid \langle \beta, X_c(x) \rangle = 0, c = 1, \ldots, k \}$. For control-affine systems the assumption of abnormality assures that Remark 9 is true provided that 0 is in the interior of the control set, i.e.

$$\text{cone}(\text{conv}(\sum_{c=1}^{k} u^c X_c(\gamma(t)) \mid u \in U)) \subseteq \mathcal{I}_{S_2}^{1}(\gamma(t)).$$

Thus the primary constraint submanifold is defined as in (10). Hence

$$\mathcal{Z}_0(\gamma(t), u) = \{X_1(\gamma(t)), \ldots, X_k(\gamma(t))\}.$$

Inductively, assume the result holds for $i - 1$ and prove it for $i$. If $(\lambda, u) \in N_{i-1}$, then

$$\langle \lambda, Z_j(x, u) \rangle = 0, \quad j = 1, \ldots, n_{i-1},$$

where $Z_j(\gamma(t), u) \in \mathcal{Z}_{i-1}(\gamma(t), u) \subseteq \mathcal{I}_2(\gamma(t))$. In order to obtain the following constraint submanifold we compute

$$\frac{d}{dt} \langle \lambda, Z_j(x, u) \rangle = \langle \lambda, [\xi_0, Z_j](x, u) \rangle = 0.$$

In [15, Section 4] it is proved that $[\xi_0, Z_j]$ is a variation of order $i + 1$. Thus it is true that the abnormal lift lies in a constraint submanifold $N_i$ defined as in [10].

Observe that for driftless control systems this result is proved straightforward because the control system is control-linear. Then,

$$[\xi_0, Z_j] = j^2(\nu_{\xi, x_2})(0) = j^2 \left( \Phi_s^{-\xi_0} \Phi_s^{-Z_j} \Phi_s^{\xi_0} \Phi_s^{Z_j} \right)(\gamma(t))(0),$$

where the juxtaposition of maps denotes composition. Thus the vector field $[\xi_0, Z_j]$ is obtained as a high order elementary perturbation vector at $\gamma(t)$ of order 2.

As in PMP, the above proposition does not guarantee the uniqueness of the momentum, there might be more than one lift to the cotangent bundle of the optimal curve satisfying the necessary conditions.

If the controls do not appear explicitly in the constraints defining the submanifolds in [10], then the supporting hyperplane contains subspaces generated by the family of vectors [9] in the high order tangent perturbation cone. These conditions impose restrictions on the momentum. If the controls appear in the constraints, then we might be able to determine the vector field whose integral curves are the biextremals of the OCP.

Remember that to solve uniquely Hamilton’s equations “only” the controls and the initial condition for the momentum must be determined. After applying the presymplectic constraint algorithm it could happen that the initial condition for the momentum is not completely determined. If we prefer to restrict more the candidates to be optimal, then it must be imposed that the momentum defines a supporting hyperplane to all the infinitesimal variations and not only to the ones appearing in [10].

Remark 20. The constructions in Definitions [6] and [8] have been successfully related with the presymplectic constraint algorithm. However, for each control system, only some infinitesimal variations appear in the constraint algorithm as the inclusion in [9] indicates. More details are given in the following section.
3.3 Example: Affine connection control systems

The affine connection control systems are a class of control mechanical systems widely-studied in the literature [7]. If $Q$ is the configuration manifold and $\nabla$ is an affine connection on $Q$, the control system on $TQ$ is the following control-affine system

$$\dot{q}(t) = (Z + \sum_{c=1}^{k} u^c Y^V_c)(q(t)),$$

where $\gamma: I \to TQ$, $Z$ is the geodesic spray associated with $\nabla$ and $Y^V_c$ is the vertical lift of $Y_c$.

Let $\zeta_0$ be the reference trajectory with control $u_0$. The Campbell–Baker–Hausdorff formula [20] is useful to describe specifically the sets $\mathcal{Z}_j$ whose existence has been proved in Theorem [19]

$$\mathcal{Z}_0(\gamma(t), u) = \left\{ \left\{ j^1 \left( (\Phi^\xi_{s-1} \Phi^\xi_{-s})(\gamma(t)) \right) (0) \mid \xi_i = Z + \sum_{k=1}^{c} u^c_0 Y^V_c + Y_i^V \right\} \right\} = \{ Y^V_1(\gamma(t)), \ldots, Y^V_k(\gamma(t)) \}.$$

Moreover, it can be also proved that

$$\mathcal{Z}_1(\gamma(t), u) = \left\{ \left\{ j^2 \left( (\Phi^\xi_{s-1} \Phi^\xi_{s})(\gamma(t)) \right) (0) \mid \xi_{\pm i} = Z + \sum_{k=1}^{c} u^c_0 Y^V_c \pm Y_i^V \right\} \right\} = \{ [\xi_0, Y^V_1(\gamma(t))], \ldots, [\xi_0, Y^V_k(\gamma(t))] \}.$$

Note that if $(\lambda, u) \in N_0$, then $\lambda$ annihilates $\mathcal{Z}_0$ and

$$\langle \lambda, [\xi_0, Y^V_1] \rangle = \langle \lambda, [Z, Y^V_1] - \sum_{c=1}^{k} Y^V_1(u^c_0)Y^V_c \rangle = \langle \lambda, [Z, Y^V_1] \rangle = 0.$$

Thus the controls are not determined in these first two primary constraint submanifolds. Only the momenta has been restricted because the high order tangent perturbation cone contains the subspaces generated by $\mathcal{Z}_0(\gamma(t), u)$ and $\mathcal{Z}_0(\gamma(t), u)$.

Note that

$$j^1 \left( (\Phi^\xi_{s-1} \Phi^\xi_{-s})(\gamma(t)) \right) (0) = -Y^V_1(\gamma(t)),$$

$$\xi_{-i} = Z + \sum_{k=1}^{c} u^c_0 Y^V_c - Y^V_i,$$

$$j^2 \left( (\Phi^\xi_{s-1} \Phi^\xi_{s})(\gamma(t)) \right) (0) = -[\xi_0, Y^V_1(\gamma(t))],$$

$$\xi_{\pm i} = Z + \sum_{k=1}^{c} u^c_0 Y^V_c \pm Y^V_i.$$

4 Conclusions and future work

To sum up, HOMP provides extra conditions for optimality related with the existence of a supporting hyperplane. Computationally speaking, it is difficult to construct explicitly all the infinitesimal
variations. In the same way, it was difficult to deal with the condition of maximization of the Hamiltonian over the controls and that is why it is usually replaced by the necessary condition \( \partial H/\partial u = 0 \). From this weak first order necessary condition, the presymplectic constraint algorithm gives an idea about how to replace the existence of the supporting hyperplane in HOMP by the conditions in Theorem 19. These last ones provide less information, but they establish a connection between the weak high order maximum principle and the application of the presymplectic constraint algorithm to optimal control theory. The conditions obtained by this algorithm are more suitable to work with than to try to write down all the \((\xi, \tau_2)\)-infinitesimal variations as pointed out in the following section.

This paper sets up the foundations to characterize abnormality by means of geometric high order necessary conditions for optimality. We have weakened Krener’s high order maximum principle to Theorem 18, which in turn has been weakened to Theorem 19 to obtain easier necessary conditions for optimality to compute. Next step is to describe when these weaker necessary conditions can also satisfy the above-mentioned stronger necessary conditions for optimality. In particular, how it can be detected by means of the presymplectic algorithm if the momentum, apart from defining a supporting hyperplane to some \((\xi, \tau_2)\)-infinitesimal variations, also determines the supporting hyperplane in Theorems 15 and 18. As mentioned earlier, this supporting hyperplane must be indeed a separating hyperplane of the cone and the decreasing direction of the cost function in order not to contradict the optimality condition. Thus, it is also necessary to study how to detect when the supporting hyperplane is the separating hyperplane used in the proof of Krener’s Maximum Principle.

In this regard, it could be useful to be able to define some vector-valued quadratic forms generalizing the ones considered in [7] to study second order conditions for controllability. It should be analyzed how those quadratic forms take part in the presymplectic constraint algorithm.

In [2] some notions of controllability for control-affine systems have been characterized for first time in the literature in a coordinate-free way. Thus a future research line consists of describing necessary conditions for optimality using the notion of affine bundles and the jet bundle theory.

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