TROPICAL COMPOUND MATRIX IDENTITIES

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Abstract. We prove identities on compound matrices in extended tropical semirings. Such identities include analogues to properties of conjugate matrices, powers of matrices and $\text{adj}(A) \text{det}(A)^{-1}$, all of which have implications on the eigenvalues of the corresponding matrices. A tropical Sylvester-Franke identity is provided as well.

Keywords: Tropical linear algebra; characteristic polynomial; compound matrix; eigenvalues; permanent; definite matrices; pseudo-inverse.

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1. Introduction

The max-plus or tropical semiring $\mathbb{R}_{\max}$ is the set of real numbers $\mathbb{R}$, completed with the element $-\infty$, and equipped with the operations $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$ (also denoted as $ab$). The zero-element of this structure is therefore $-\infty$, which is its minimal element. See for instance [BCOQ92, But10, MS15, ABG07] and the references therein for more background on linear algebra over the max-plus semiring.

The lack of additive inverses is a source of difficulties in the study of tropical structures. In particular, the notion of “vanishing” has to be adapted: a tropical polynomial vanishes at a point if the maximum of the values of its monomials, evaluated at this point, is achieved twice at least. In applications coming from real geometry [Vir01], one considers tropical polynomials enriched with a sign information. Then, vanishing tropically means that the maximum of the value of the monomials with a positive tropical sign coincides with the maximum of the value of the monomials with a negative tropical sign. In this way, one can define the notion of polynomial identity over the tropical semiring. Such polynomial identities can often be proved by direct combinatorial methods, i.e., by “bijective proofs”, along the lines of [Str83, Zei85] or of [Gon83] (see also [GM84]). It was observed in [RS84] that certain determinantal identities over semirings can be derived from their classical analogues, avoiding the recourse to bijective proofs. This idea led to a transfer principle in [Gau92], later refined in [AGG09]. As an application of the transfer principle, a number of determinantal identities (Laplace type expansions [RS84, Plu90, Gau92, AGG09], Binet-Cauchy theorem [Gau92, GBCG98, AGG09]) or more advanced polynomial identities (Amitsur-Levitzki [Gau96, AGG09]), were obtained, with several applications. Polynomial identities have also appeared more recently in works on the “supertropical” extension of the tropical semiring [IR11a, IR11b].

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In the present paper, we establish tropical analogues of several classical identities in the theory of determinants [Pri47]. Unlike the previously mentioned tropical determinantal identities, the identities that we establish have the remarkable feature that they do not follow from the transfer principle: more precisely, an application of the transfer principle would lead to weaker identities.

In order to formulate these identities, it is convenient to use the setting of extensions of semirings. This has been developed in a number of works [Plu90, Gau92, Izh09, AGG09, IR11a, IR11b, AGG14]. Two basic extensions have been considered so far. The supertropical semiring [Izh09, IR11a, IR11b] is the union of two copies of the set of tropical numbers, one copy represent the ordinary numbers, whereas the other copy represent “ghost” numbers, encoding the fact that a maximum is achieved twice. The symmetrized max-plus semiring [Plu90, Gau92, AGG09, AGG14] is the union of three copies of the set of tropical numbers, representing respectively tropically positive elements, tropically negative elements, and “balanced” or singular elements, of the form $a \oplus (\ominus a)$. These two extensions may be thought of as special hyperfields [Kra83, CCI1, Vir10, BB16] (a hyperfield is a structure with a multivalued addition on a base set, the supertropical and symmetrized semirings can be identified to the powerset semirings of two hyperfields). The supertropical numbers arise when considering images of complex Puiseux series by the nonarchimedean valuation, whereas symmetrized tropical numbers arise as images of real Puiseux series.

We further elaborate on these different structures in Section 2. In particular, the obvious resemblance between them can be formalized thanks to the notion of semiring with symmetry [AGG09, AGG14]. The latter are semirings equipped with an operation $a \mapsto \ominus a$, in which singular elements (playing the role of the zero element) are of the form $a \ominus a$. In the supertropical case, the symmetry is just the identity map, whereas the symmetry operation behaves formally as an “opposite sign” in the case of the symmetrized tropical semiring. Then, the notion of polynomial identity has to be revised. Instead of looking for identities of the form $a = b$, where $a, b$ can be polynomial expressions, we shall look for identities of the form $a \succ b$, to be read “$a$ surpasses $b$”. The latter relation is defined by

$$a \succ b \iff a = b \oplus (c \ominus c),$$

for some $c$.

This way of writing identities may surprise at the first sight. However, some of the most handy tropical polynomial identities are expressed in this way. The reader will not be wrong in imagining that the presence of the singular term $c \ominus c$ accounts for the irreversibility of algebraic computations in the tropical setting: when doing such computations, some terms of the form $c \ominus c$, which vanish in the usual algebra, remain in the tropical algebra.

Let us now come to our main topic. We shall denote by $A^{\wedge k}$ the $k$th compound matrix or Grassman power of $A$, obtained by taking the $k \times k$ minors of $A$ (see section 2). The compound matrix has been widely studied. One can find definitions, identities and basic algebraic properties in [HJ13]. In contrast with the situation over rings, the invertibility of the determinant does not imply that a matrix is invertible. Nevertheless, as shown in several works, especially [Plu90, Niv15], the familiar expression

$$A^\nabla = \text{adj}(A) \det(A)^{-1},$$

is
which provides in classical algebra the inverse of an invertible matrix, can be defined as soon as the determinant of \( A \) is invertible. It does inherit classical properties, which justify the name of quasi-inverse. Among others, \( A^\nabla \) can be factored as a product of elementary matrices \([Niv14b]\), a property which is equivalent to nonsingularity over fields, but not over semifields such as the tropical one.

In Sections 3, 4 and 5 we use graph theory to provide tropical analogues for identities concerning the compounds, quasi-inverse, powers, and so-called conjugations of matrices. These analogues can be interpreted in terms perfect matchings: they are concerned with the existence of different permutations, described by the same subset of arcs.

In Section 3, Theorem 3.2, we prove the analogue of Jacobi’s identity

\[
\det(A) \left( D A^\nabla D \right)^{\nabla n - k} \succ^o A^\nabla k, \quad \text{where } D \text{ is diagonal with } D_{i,i} = (\ominus 1)^i,
\]

which implies (Corollary 3.3)

\[
\det(A) \tr \left( (A^\nabla)^{\nabla n - k} \right) \succ^o \tr \left( A^\nabla k \right).
\]

This establishes in particular \([Niv15, Conjecture 6.2]\).

Recall that \( \models \) denotes the specialization of the relation \( \succ^o \) to the supertropical case. In Section 4.2 we use \( (A^m)^{\nabla k} \models^o (A^\nabla k)^m \) in order to prove the supertropical identity

\[
\tr \left( (A^m)^{\nabla k} \right) \models \left( \tr(A^\nabla k) \right)^m, \quad k = 1, \ldots n
\]

stated in Corollary 4.5. The identity

\[
E^\nabla k (E^\nabla)^{\nabla k} \succ^o \mathcal{I} \quad \text{ (Proposition 4.2)},
\]

leads to Theorem 4.9 concerning a so-called tropical conjugation

\[
\tr \left( (E^\nabla AE)^{\nabla k} \right) \succ^o \tr(A^\nabla k).
\]

Determinants have been studied in association to linear algebra, graph theory and algebraic geometry. One can see \([Pri47]\) for a survey on identities of determinantal identities, including the Sylvester–Franke identity. In Section 5 we provide a tropical Sylvester-Franke identity, which holds in particular over \( \mathbb{R}_{\text{max}} \)

\[
\per(A^\nabla k) = \per(A)\left(\frac{n-1}{k-1}\right) \quad \text{ (Theorem 5.1)}.
\]

The characteristic polynomial of a square matrix is known for its applications in linear algebra. Yet, due to its connection with compound matrices, the tropical characteristic polynomial (see \([AGM14]\) and \([BM00]\)) has applications in graph theory. We conclude this paper in Section 6 by applying the identities of Section 3 and 4 to the tropical characteristic polynomials of the corresponding matrices. Theorem 6.4 states

\[
f_M(X) \models^o \begin{cases} 
\det(A)^{-1} X^n f_A(X^{-1}) , & M = A^\nabla , \\
 f_A(x) , & M = E^\nabla AE , 
\end{cases}
\]

where \( f_A \) denotes the characteristic polynomial of the matrix \( A \), and it also relates the coefficients of the characteristic polynomials of \( A \) and its powers. We then establish the connection to their corresponding eigenvalues, and deal with the special case of equality in Corollaries 6.6 and 6.5 respectively.

Note that the supertropical version of the identities in \((1.1)\) were proved in \([Niv14a, Niv15, Shi16]\).
2. Preliminaries

Let \( \mathcal{P}(S) \) be the power set of a totally ordered set \( S \), and \( \mathcal{P}_k(S) = \{ I \in \mathcal{P}(S) : |I| = k \} \). We order \( \mathcal{P}_k(S) \) by the (total) lexicographic order, with respect to the order in \( S \).

We denote by \( \mathcal{G}_{I,J} \) the set of bijections between the two totally ordered sets \( I \) to \( J \). A bijection from a set to itself is called a permutation, and \( \mathcal{G}_I \) denotes the set of permutations on \( I \). The permutation of \( S \) such that \( \pi(i) = i \), for all \( i \in S \), is called the identity permutation, and denoted by \( \text{Id} \). Note that the restriction of a permutation is a bijection, that is \( \pi|_J \in \mathcal{G}_{J,\pi[J]} \), \( \forall J \in \mathcal{P}(S) \) and \( \pi \in \mathcal{G}_S \).

We denote \( \sigma[I] = \{ \sigma(i) : i \in I \} \), then, for any \( \sigma \in \mathcal{G}_S \) and \( 0 \leq k \leq n \), we may define \( \sigma^{(k)} \in \mathcal{G}_{P_k(S)} \), by \( \sigma^{(k)}(I) = \sigma[I], \forall I \in P_k(S) \).

The remaining part of this section is organized as follows. In Section 2.1 we recall basic notations in graph theory, followed by their matrix interpretation. We then present in Section 2.2 a unified algebraic setting, allowing us to deal with several extensions of the tropical semiring. In Section 2.3 we provide matrix definitions, adjusted to the setting and notation of Sections 2.1 and 2.2.

2.1. Graph theory. We follow the terminology in [Gib85] and [Gon75]. Let \( G \) be a weighted, directed graph (digraph), with \( n \) nodes, possible loops and no multiple arcs. The set of nodes is denoted by \([n] = \{1, 2, ..., n\}\), an arc from \( i \) to \( j \) is denoted by \((i, j)\), and its weight by \( a_{i,j} \). We shall assume here that the weights of the graph \( G \) take their values in a given semiring \((\mathcal{S}, \circ, \odot)\) with \( \odot \) as its zero (in particular \( \odot \) is absorbing for \( \circ \), see Section 2.2). The weight matrix of \( G \) is the \( n \times n \) matrix \( M_G \) over \( \mathcal{S} \), having \( a_{i,j} \) in its \( i, j \) position, when \((i, j)\) is an arc of \( G \), and \( \odot \) otherwise. Conversely, if \( A = (a_{i,j}) \) is an \( n \times n \) matrix with entries in \( \mathcal{S} \), then the graph \( G \) of \( A \) is the weighted, directed graph, with \( n \) nodes, and an arc from \( i \) to \( j \) with weight \( a_{i,j} \) if \( a_{i,j} \neq \odot \).

Definition 2.1. A path of length \( m \) from \( i \) to \( j \) in \( G \) is a set of \( m \) arcs with concatenating nodes. This path is called closed if \( i = j \), open if \( i \neq j \), elementary if its intermediate nodes are distinct, and different from \( i \) and \( j \), and maximal if it includes all the nodes \([n]\). An elementary maximal path is also called Hamiltonian. The in-degree (resp. out-degree) of the node \( i \), denoted by \( d_{in}(i) \) (resp. \( d_{out}(i) \)), is the number of arcs terminating (resp. originating) in \( i \).

A cycle in \( G \) is an elementary closed path. A bijection (and in particular, a permutation) in \( G \) is a bijection \( \pi \in \mathcal{G}_{I,J} \), \( I,J \in \mathcal{P}_k([n]) \), with a graph, \( \mathcal{G}_\pi := \{(i, \pi(i)) : i \in I\} \) composed of arcs in \( G \). In that case the subgraph of \( G \) composed of the arcs in \( \mathcal{G}_\pi \) satisfies
\[
d_{out}(i) = d_{in}(j) = 1, \forall i \in I, j \in J.
\]
In the sequel, we shall identify a bijection with its graph. The bijection \( \pi \in \mathcal{G}_{I,J} \) can be decomposed into disjoint elementary (open or closed) paths. Denote by \( \mathcal{i} \subseteq I \) the subset of nodes of \( I \) which are in the elementary path of \( i \in I \) in \( \pi \), and by \( C_\pi = \{i : i \in I\} \) the quotient set obtained by the partition of \( I \), induced from the elementary path decomposition of \( \pi \). In particular, a permutation \( \pi \in \mathcal{G}_I \) can be decomposed into disjoint cycles, and we shall identify as usual \( \pi \) with the composition of these cycles. A bijection is called elementary if its decomposition has a single open elementary path and trivial cycles (loops).
We abuse these notations by using them for the matrix $M_G = (a_{i,j})$. That is, the product of $m$ entries with concatenating indices describes a path. Notice that over $\mathbb{R}_{\text{max}}$ the weight of a path is the sum of the weights of its arcs (or its entries).

The number of non-0 entries with a right (resp. left) index $i$ is $d_{\text{in}}(i)$ (resp. $d_{\text{out}}(i)$).

The product $\bigodot_{i \in I} a_{i,\pi(i)}$ is the bijection $\pi \in \mathfrak{S}_{I,J}$ of entries of $M_G$, and can be decomposed into disjoint cycles and elementary open paths

\[
\bigodot_{i \in C} a_{j,\pi(j)}, \quad \text{with} \quad \bigodot_{j \in I} a_{j,\pi(j)} = a_{i,\pi(i)}a_{\pi(i),\pi^2(i)} \cdots a_{\pi^{m_i-1}(i),\pi^{m_i}(i)},
\]

where $\pi^{m_i}(i) \begin{cases} \in J \setminus I & \text{if } \bar{i} \not\subseteq J \text{ (that is, } i \in I \setminus J), \\ = i & \text{obtaining an elementary open path,} \\ \bar{i} \subseteq J, & \text{obtaining a cycle.} \end{cases}$

In the special case of $I = J$, we get that $\pi^{m_i}(i) = i$, $\forall \bar{i} \subseteq C_i$.

**Remark 2.2.** If an elementary path has an intermediate index, it can be decomposed into two non-maximal elementary open paths, and a non-maximal elementary open path can be extended at each of its ends into an elementary path.

### 2.2. Semirings with a symmetry

Recall that a **semiring** is a set $S$ with two binary operations, addition, denoted by $+$, and multiplication, denoted by $\cdot$ or by concatenation, such that:

- $S$ is an abelian monoid under addition (with neutral element denoted by $0$ and called zero);
- $S$ is a monoid under multiplication (with neutral element denoted by $1$ and called unit);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in S$.

A semiring is **idempotent** when the addition is idempotent, that is $a + a = a$ for all $a \in S$. It is **commutative** when the multiplication is commutative, that is $ab = ba$ for all $a, b \in S$. The max-plus semiring $\mathbb{R}_{\text{max}}$ described in the introduction is idempotent and commutative. Semimodules over semirings, and morphisms of semirings or semimodules are defined as for modules over rings, and morphisms of rings or modules, respectively.

**Definition 2.3.** A map $\tau : S \to S$ is a **symmetry** of the semiring $S$ if $\tau$ is a left and right $S$-semimodule homomorphism of order 2, from $S$ to itself, which means that it satisfies:

1. $\tau(a + b) = \tau(a) + \tau(b)$,
2. $\tau(0) = 0$,
3. $\tau(ab) = a\tau(b) = \tau(a)b$,
4. $\tau(\tau(a)) = a$.

The typical example of a symmetry is the map $\tau(a) = -a$, where $-a$ is the opposite of $a$ for the addition $+$ on a ring $S$, which satisfies also $a - a = 0$. Another possible symmetry, which works in any semiring is the identity map $\tau(a) = a$.

The concept of semirings with symmetry first appeared in [Gau92], in order to prove some identities on tropical matrices using a transfer principle. It was then further used in [AGG09] in relation with the transfer principle and used in [AGG09, AGG14].
to give a unified view of several extensions of the tropical semiring introduced in the literature, in particular the symmetrized max-plus semiring introduced in [Plu90], see also [Gau92, BCOQ92], and the Izhakian extension of the tropical semiring introduced in [Izh09], see also [IKR13], that we next recall. Due to the lack of inverses in $\mathbb{R}_{\max}$, its 0 often loses its algebraic role in the sense it is usually observed over rings. The following semirings were introduced to give an algebraic framework to the notions of matrix singularity, linear or algebraic equations and algebraic varieties over the tropical semiring.

2.2.1. Symmetrized max-plus semiring. This extension, denoted $\mathbb{S}R_{\max}$, was studied in [Plu90, Gau92, BCOQ92]. Several equivalent constructions were proposed in [AGG09, AGG14]. In particular, it can be seen as the union of three copies of $\mathbb{R}_{\max}$, denoted respectively $\mathbb{S}R_{\max}^{\oplus}$, $\mathbb{S}R_{\max}^{\ominus}$ and $\mathbb{S}R_{\max}^{\circ}$, in which the zero-elements $-\infty$ are identified and denoted 0. The copies of $a \in \mathbb{R}_{\max}$ in $\mathbb{S}R_{\max}^{\oplus}$, $\mathbb{S}R_{\max}^{\ominus}$ and $\mathbb{S}R_{\max}^{\circ}$ are respectively denoted $\oplus a = a$, $\ominus a$ and $a^\circ$. The set $\mathbb{S}R_{\max}$ is endowed with the operations $\oplus$, $\circ$ and $\ominus$, such that $(\mathbb{S}R_{\max}, \oplus, \circ)$ is an idempotent semiring, with the symmetry $a \mapsto \ominus a$, $\mathbb{S}R_{\max} \rightarrow \mathbb{S}R_{\max}$. The symmetry satisfies that $\ominus(\ominus a) = a \in \mathbb{S}R_{\max}^{\oplus}$, $\ominus(\ominus a) = a \in \mathbb{S}R_{\max}^{\ominus}$, $\ominus(\ominus a) = a \in \mathbb{S}R_{\max}^{\circ}$, $\forall a \in \mathbb{R}_{\max}$.

Moreover,

$$a \ominus b := (\ominus a) \oplus (\ominus b) = (\ominus b) \oplus (\ominus a) = \begin{cases} a & \text{if } a > b, \\ \ominus b & \text{if } b > a, \\ a^\circ & \text{if } a = b, \end{cases} \forall a, b \in \mathbb{R}_{\max}.$$ 

With these properties, and denoting $a^\circ := a \ominus a$ for all $a \in \mathbb{S}R_{\max}$, one can show that $\ominus(a^\circ) = a^\circ$, that $\mathbb{S}R_{\max}^{\circ}$ is an ideal of $\mathbb{S}R_{\max}$ and that the map $a \mapsto a^\circ$ is a morphism of $\mathbb{R}_{\max}$-semimodules.

The balance relation $\nabla$ on $\mathbb{S}R_{\max}$ is defined as

$$a \nabla b \iff a \ominus b \in \mathbb{S}R_{\max}^{\circ}.$$ 

As a result $a \nabla 0$ if and only if $a \in \mathbb{S}R_{\max}^{\circ}$. Such an element $a$ is said singular, and equivalently it is non invertible in $\mathbb{S}R_{\max}$. Note that the symmetry $\tau : a \mapsto \ominus a$ in $\mathbb{S}R_{\max}$ is a sort of extension of the usual symmetry of a ring. In general $a \ominus a = a^\circ \neq 0$, but it is singular.

In [AGG09, AGG14], the relation $\succ^\circ$ is also introduced as follows:

$$a \succ^\circ b \iff \exists c \in \mathbb{S}R_{\max}, \ a = b \oplus c^\circ.$$ 

Let $a \in \mathbb{S}R_{\max}$. We denote by $|a|$ the element $b \in \mathbb{S}R_{\max}^{\oplus}$ such that $a^\circ = b^\circ$.

The symmetrized max-plus semiring is useful to deal with systems of linear equations over $\mathbb{R}_{\max}$.

2.2.2. Supertropical max-plus semiring. In recent years, tropical geometry is in constant development and became the main interest of algebraic geometry groups such as Itenberg, Mikhalkin and Shustin (see for instance [IMS07] and [Mik06]). Izhakian’s extension (see [Izh09] and [IKR13]) was inspired by the observation from these groups that tropical varieties are the corner locus of some set of tropical polynomials.
This extension is obtained by constructing a second copy of \( \mathbb{R} \), called the “ghost” ideal of this structure, denoted \( \mathbb{R}^\nu \). Then, the supertropical extension of the tropical semiring is defined as

\[
\mathbb{E}R_{\max} := \mathbb{R} \cup \mathbb{R}^\nu \cup \{-\infty\},
\]

and endowed with operations \( \oplus, \odot \) such that \((\mathbb{E}R_{\max}, \oplus, \odot)\) is a semiring, and such that the copy of \( a \in \mathbb{R} \) in \( \mathbb{R}^\nu \), denoted \( a^\nu \), satisfies \( a^\nu = a \oplus a \) and is considered to be singular. That is, \( \mathbb{E}R_{\max} \) is not idempotent. However, the \( \oplus \) operation also satisfies \( a \oplus a \oplus a = a \oplus a = a^\nu \) (see below), which means that singularity is obtained when a maximum is attained at least twice.

The above properties imply that the injective map \( a \mapsto a, \mathbb{R}_{\max} \to \mathbb{E}R_{\max} \) is not a morphism of additive monoids, so of semirings. To further define the semiring operations, let us define \( a^\nu := a \) when \( a \in \mathbb{R}^\nu \cup \{-\infty\} \), and apply the usual order of \( \mathbb{R} \cup \{-\infty\} \) in its copy \( \mathbb{R}^\nu \cup \{-\infty\} \). Then, \( \oplus \) and \( \odot \) are such that the map \( a \mapsto a, \mathbb{R}_{\max} \to \mathbb{E}R_{\max} \) is a morphism of multiplicative monoids, the map \( a \mapsto a^\nu, \mathbb{E}R_{\max} \to \mathbb{R}^\nu \cup \{-\infty\} \) is a surjective morphism of semirings, and of \( \mathbb{R}_{\max} \)-semimodules, and the addition is adjusted with respect to the “ghostness” of the elements:

\[
a \oplus b = b \oplus a = \begin{cases} a & \text{if } a^\nu > b^\nu, \\ b & \text{if } b^\nu > a^\nu, \\ a^\nu & \text{if } a^\nu = b^\nu. \end{cases}
\]

On \( \mathbb{E}R_{\max} \), one also defines the relation \( \models \) as

\[a \models b \iff a = b \oplus c^\nu \text{ for some } c \in \mathbb{R} \cup \{-\infty\},\]

and in particular \( a^\nu \models 0, \forall a \in \mathbb{E}R_{\max} \).

Consider on \( \mathbb{E}R_{\max} \) the identity symmetry \( \tau(a) = a \). If the semiring was idempotent, it would satisfy \( a \oplus \tau(a) = a \), which cannot be set as a singular element. However, in \( \mathbb{E}R_{\max} \), we have that \( a \oplus \tau(a) = a^\nu \) is in general non-zero but is singular.

Let \( a \in \mathbb{E}R_{\max} \). Let denote by \( |a| \) the element \( b \in \mathbb{R}_{\max} \) such that \( a^\nu = b^\nu \).

The supertropical max-plus semiring contributes to the understanding of tropical roots over \( \mathbb{R}_{\max} \) in the sense of tropical geometry.

2.2.3. Unified theory. The above semirings are particular examples of semirings with a symmetry. They enjoy some additional properties that are necessary to obtain the identities presented in Sections 3 and 4. We shall present here a unified construction already presented in [AGGL14], which contains more examples.

Let \( \mathcal{S} \) be a commutative semiring with a symmetry. Let us denote by \( \oplus \) its addition, by \( \odot \) its multiplication, by \( \emptyset \) its zero, by \( \mathbb{1} \) its unit, and by \( \ominus \) its symmetry (which means that we write \( \ominus a \) instead of \( \tau(a) \)). Throughout, we use \( a \odot b \) for \( a \odot (\ominus b) \), and set \( a^\ominus := a \ominus a \). This implies that \( \ominus a^\ominus = a^\ominus \). These notations coincide with the ones of Section 2.2.1 when \( \mathcal{S} \) is the symmetrized max-plus semiring \( \mathbb{S}R_{\max} \). When \( \mathcal{S} \) is the supertropical semiring \( \mathbb{E}R_{\max} \), \( \ominus \) has to be understood as \( \oplus \) and \( a^\ominus \) coincides with \( a^\nu \). In the above semirings, an element \( a^\ominus \) is singular and non invertible, but it is not necessarily \( \emptyset \). To generalize this, we consider in \( \mathcal{S} \), the subset

\[
\mathcal{S}^\ominus := \{a^\ominus \mid a \in \mathcal{S}\}.
\]

Let us denote by \( \mathcal{S}^* \) the set of invertible elements of \( \mathcal{S} \). The set \( \mathcal{S}^\ominus \) is an ideal, hence, either \( \mathcal{S}^\ominus = \mathcal{S} \) or \( \mathcal{S}^\ominus \) contains no invertible element of \( \mathcal{S} \) \( (\mathcal{S}^* \subset \mathcal{S} \setminus \mathcal{S}^\ominus) \). If \( \mathcal{S} \) is a totally
ordered idempotent semiring, like \( \mathbb{R}_{\text{max}} \), then the only symmetry on \( S \) is the identity (see \[AGG14\] Prop. 2.11), then \( S^\circ = S \). For any semiring \( S \), we shall also consider the subset
\[
S^\vee := S^* \cup \{0\}.
\]
Note that in \[AGG14\], this notation was used, when \( S^0 \neq S \), to denote any set such that \( S^* \cup \{0\} \subset S^\vee \subset (S \setminus S^0) \cup \{0\} \), the elements of which were called the thin elements. Here we restrict the value of \( S^\vee \) while putting no constraint on \( S \). In the sequel, we shall say that an element of \( S \) is nonsingular if it belongs to \( S^* \), and that it is singular otherwise. Then, if \( S^0 \neq S \), the elements of \( S^0 \) are necessarily singular, and if \( S \) equals \( \mathbb{E}\mathbb{R}_{\text{max}} \) or \( \mathbb{S}\mathbb{R}_{\text{max}} \), these are the only ones. Note however that if \( S = \mathbb{R}_{\text{max}} \), \( 0 \) is the only singular element.

In the sequel, we shall consider particularly semirings \( S \) with a symmetry which are either totally ordered idempotent semifields (that is such that \( S \setminus \{0\} = S^* \) is a totally ordered group and so \( S = S^\vee = S^0 \)) or semirings satisfying \( S^0 \neq S \) (so \( S^\vee \subset (S \setminus S^0) \cup \{0\} \)), and some additional properties described below.

**Definition 2.4.** On any semiring \( S \) with a symmetry, one defines the relations:
\[
a \preceq b \iff b \succeq a \iff b = a \oplus c \text{ for some } c \in S, \\
a \preceq^0 b \iff b \succeq^0 a \iff b = a \oplus c \text{ for some } c \in S^0,
\]
and
\[
a \preceq b \iff b \succeq a \iff b = a \oplus b.
\]

The relations \( \preceq, \preceq^0 \) are preorders (reflexive and transitive), compatible with the laws of \( S \). They may not be antisymmetric. The relation \( \preceq \) is antisymmetric and transitive, compatible with the laws of \( S \) (if \( a \preceq b \), then \( ac \preceq bc \); if \( a \preceq b \) and \( a' \preceq b' \), then \( a \oplus a' \preceq b \oplus b' \)). It is reflexive when \( S \) is idempotent. Obviously, the relation \( \preceq \) implies the relation \( \preceq^0 \).

Note that both \( \preceq \) and \( \preceq^0 \) are such that all elements of \( S \) are nonnegative, that is \( a \preceq 0 \) and \( a \preceq 0 \) for all \( a \in S \).

We apply these relations to matrices (and in particular to vectors) entry-wise, and to polynomials coefficient-wise.

**Definition 2.5.** A semiring \( S \) is said to be naturally ordered when \( \preceq \) (or equivalently \( \succeq \)) is an order relation, and in that case \( \preceq \) is called the natural order on \( S \), and \( \succeq \) is its opposite order.

When \( \preceq \) (or \( \succeq \)) is an order relation, so are \( \preceq^0 \) and \( \succeq^0 \), and their extensions to matrices and polynomials. When \( S \) is an idempotent semiring, \( \preceq \) is equal to \( \preceq^0 \), \( S \) is necessarily naturally ordered, and a naturally ordered semiring is necessarily zero-sum free.

**Definition 2.6.** When \( S \) is naturally ordered and \( (a_n)_{n \geq 0} \) is a nondecreasing (resp. nonincreasing) sequence of scalars, matrices or polynomials over \( S \), we say that the sequence converges towards \( a \), if the supremum (resp. infimum) of the \( a_n \) exists and is equal to \( a \).

**Definition 2.7.** Let \( S \) be a semiring and \( M \) be a totally ordered idempotent semiring. We say that a map \( \mu : S \to M \) is a modulus if it is a surjective morphism of semirings. In this case, we denote \( \mu(a) \) by \( |a| \) for all \( a \in S \), and for \( a, b \in S \), we say that \( b \) dominate \( a \) when \( |a| \preceq |b| \).
The absolute value is applied to matrices (and in particular to vectors) entry-wise, and to polynomials coefficient-wise. A modulus on a semiring $S$ with a symmetry satisfies necessarily $|a \oplus b| = |a|$ and so $|a^o| = |a|$ for all $a \in S$ (see [AGG14, Prop. 2.11]). In particular, $\mu = \mu|_{S^o} \circ c$ where $c : S \to S^o$ is the map $a \mapsto a^o$. If $S^o$ is already idempotent and totally ordered, then the map $c$ is a modulus. This is the case when $S = S \rho_{\max}$ or $S = E \rho_{\max}$, and indeed in these cases, one considered a modulus map with values in $\mathbb{R}_{\max}$, such that $\mu|_{S^o}$ is an isomorphism. This is also the case when $S$ is already an idempotent and totally ordered semiring, in which case $c$ is the identity. On these semirings, we have the following three properties. Moreover, Property 2.8 holds on any semiring of the form $S \times \mathbb{M}$ as in [AGG14].

**Property 2.8.** If $a, b \in S$ such that $|a| < |b| \Rightarrow a \oplus b = b$.

**Property 2.9.** If $a, b \in S$ such that $b \succ a$, $|a| = |b|$ and $b \in S^\nu$, then $a = b$.

**Corollary 2.10.** If $a, b \in S$ such that $a \preceq b$ and $b \in S^\nu$, then $a = \ominus b$ or $a \preceq b$.

**Proof.** From Property 2.8 if $|a| < |a \ominus b| = |b|$, then $a \preceq b$. Since $\ominus \mathbb{1} \in S^*$, $b \in S^\nu$ implies $\ominus b \in S^\nu$. Then, since $a \preceq \ominus b$ implies $a \preceq a$, we get from Property 2.9 that if $|a| = |a \ominus b|$, then $a = \ominus b$.

The following consequence will also be useful.

**Proposition 2.11.** If $S$ satisfies $S^o \neq S$ and Property 2.9, then $a, b \in S$ such that $b \succ^o a$ and $b \in S^\nu$ implies $a = b$.

**Proof.** Assume $b \succ^o a$ and $b \in S^\nu$. Then, $a \preceq b$ and $|a| \preceq |b|$. If $|a| = |b|$, then using Property 2.9 and $b \in S^\nu$, we get $a = b$. Otherwise $|a| < |b|$, so $b \neq 0$ and since $b = a \ominus c^o$, we get $|b| = |c^o|$. Then, due to Property 2.9 we obtain $c^o = b \in S^o \cap S^\nu \setminus \{0\} = \emptyset$, a contradiction.

**Remark 2.12.** In [AGG14], some different properties were considered, which imply the above ones. For instance, $S^\nu$ satisfies Property 5.2 of [AGG14] if

$$\text{(2.2)} \quad (x \in S^\nu \text{ and } x \preceq y) \implies \exists z \in S^\nu \text{ such that } x \preceq z \preceq y, z \triangledown y, 	ext{ and } |z| = |y|, $$

and it satisfies Property 5.4 of [AGG14] if

$$\text{(2.3)} \quad (x, y \in S^\nu, x \preceq y \text{ and } |x| = |y|) \implies x = y.$$  

When $S$ is naturally ordered, these two properties imply Property 2.9. Indeed, if $b \succ a$, $|a| = |b|$, and $b \in S^\nu$, from (2.2), $\exists c \in S^\nu$ s.t. $c \preceq a \preceq b$, $c \triangledown a$, $|a| = |b| = |c|$. Then, since $b, c \in S^\nu$, from (2.3) we obtain $c = b$ and therefore $a = b$.

The following property will also be needed.

**Property 2.13.** $S^o$ is idempotent, that is for all $a \in S$, $a^o \oplus a^o = a^o$.

**Lemma 2.14.** Property 2.13 is equivalent to the condition: $a \oplus a^o = a^o$, for all $a \in S$. It implies $\ominus a \oplus a^o$, for all $a \in S$.

**Proof.** Since $a^o \preceq a \oplus a^o \preceq a^o \oplus a^o$, Property 2.13 implies that, for all $a \in S$, $a \oplus a^o = a^o$. Since $(\ominus a)^o = a^o$, the latter property implies that, for all $a \in S$, $\ominus a \oplus a^o$. Conversely, if, for all $a \in S$, $a \oplus a^o = a^o$, then, for all $a \in S$, $\ominus a \oplus a^o = a^o$, and so $a^o \oplus a^o = a \oplus (\ominus a) \oplus a^o = a \oplus a^o = a^o$. \qed
2.3. Tropical matrix algebra. Throughout the following sections, we shall consider a commutative semiring with a symmetry, denoted by $S$, which may have some additional properties, like having a modulus. To simplify the presentation, we shall say that $S = T$, and denote $S$ by $T$, if $S$ is naturally ordered, has a modulus taking its values in a totally ordered idempotent semifield $M$, and satisfies Properties 2.8, 2.9 and 2.13 (although the latter property is not needed in the present section). Similarly, we shall say that $S = M$, and denote $S$ by $M$, if $S$ is a totally ordered idempotent semifield with its (unique) identity symmetry and the identity modulus. Following the notations of the previous section, we formulate some basic definitions. One may also find in [But03a] further combinatorial motivation for the objects discussed.

Definition 2.15. The trace of $A = (a_{i,j}) \in S^{n \times n}$ is defined as
\[
\text{tr}(A) = \bigoplus_{i \in [n]} a_{i,i},
\]
and if $S$ has a modulus, we refer to any diagonal entry with highest absolute value as a dominant diagonal entry.

Remark 2.16. Denote $B = (b_{i,j})$. As usual, $\text{tr}(AB) = \text{tr}(BA)$, since
\[
\bigoplus_{i \in [n]} \bigoplus_{t \in [n]} a_{i,t}b_{t,i} = \bigoplus_{t \in [n]} \bigoplus_{i \in [n]} b_{t,i}a_{i,t}.
\]

Definition 2.17. Let $S$ be fixed. The sign of a bijection $\sigma \in \mathfrak{S}_{I,J}$ is $\text{sign}(\sigma) = (\ominus 1)^{|\text{inv}(\sigma)|} \in S$, where
\[
\text{inv}(\sigma) = \{(i,j) \in I^2 : i < j \text{ and } \sigma(i) > \sigma(j)\}
\]
is the set of inversions in $\sigma$, taken with respect to the orders in $I$ and $J$.

Similarly to (2.1), we call signed permutation (resp. signed bijection) of a permutation $\pi$ of $I$ (resp. a bijection $\pi : I \to J$) the expression $\text{sign}(\pi) \bigcirc_{i \in I} a_{i,\pi(i)}$. Recall that a cycle is a permutation on the set of its indices (resp. an elementary open path is a bijection from the set of its left indices to the set of its right indices). It is well-known that a signed permutation is the product of its signed cycles. However, a signed bijection is not the product of its signed cycles and signed elementary open paths (e.g. $\text{sign}((1\ 4)(2)) \neq \text{sign}(1\ 4)\text{sign}(2)$). Note that a path can be decomposed into an elementary path of the same source and target and (not necessarily disjoint) cycles, starting and ending at the points of repeating indices.

It is a well known fact that $(\ominus 1)^{|\text{inv}(\sigma)|} = (\ominus 1)^{|\text{tran}(\sigma)|}$, for all $\sigma \in \mathfrak{S}_{I}$, where $\text{tran}(\sigma)$ denotes the number of transpositions which the permutation is factored into.

Property 2.18. Let $\sigma \in \mathfrak{S}_{I,J}$, $I, J \subseteq S$. For $\pi \in \mathfrak{S}_{S}$, the unique bijection $\rho \in \mathfrak{S}_{\pi[I],\sigma[I]}$ such that $\rho \circ \pi(i) = \sigma(i)$, $\forall i \in I$, satisfy $\text{sign}(\sigma) = \text{sign}(\pi|_I)\text{sign}(\rho)$.

This property is a generalization of the multiplicativity of permutation sign, proved analogously by taking the permutations on $|[I]|$ induced from $\sigma$, $\pi|_I$, and $\rho$.

Definition 2.19. We define the determinant of a matrix $A = (a_{i,j}) \in S^{n \times n}$ to be
\[
\text{det}(A) = \bigoplus_{\sigma \in \mathfrak{S}_{[n]}} \text{sign}(\sigma) \bigcirc a_{i,\sigma(i)}.
\]
We define a matrix to be \textit{nonsingular} if \( \det(A) \in \mathcal{S}^* \), and if \( \mathcal{S} \) has a modulus, we refer to any permutation with weight of highest absolute value as a \textit{dominant permutation}.

When \( \mathcal{S} = \mathcal{M} \), a totally ordered idempotent semifield, or \( \mathcal{S} = \mathbb{ER}_{\text{max}} \), we have \( \text{sign}(\sigma) = 1 \), so the determinant is actually the same as the \textit{permanent}

\[
\text{per}(A) = \bigoplus_{\sigma \in \mathfrak{S}[n]} \bigotimes_{i \in [n]} a_{i,\sigma(i)}.
\]

For a general \( \mathcal{S} \) with a modulus, the modulus of the determinant coincides with the permanent of the modulus of \( A \) over \( \mathcal{M} \). Moreover, when \( \mathcal{M} = \mathbb{R}_{\text{max}} \), this is the \textit{max permanent} of the modulus of \( A \) defined in [But03b] and [CG79], and it is the value of an associated optimal assignment problem.

Note that when \( \mathcal{S} = \mathcal{M} \), and thus when \( \mathcal{S} = \mathbb{R}_{\text{max}} \), a matrix is nonsingular in the above sense if and only if \( \text{per}(A) \neq 0 \) that is if the optimal assignment problem is feasible. So the nonsingularity of a matrix \( A \in \mathbb{R}_{\text{max}}^{n \times n} \) seen as a matrix over \( \mathbb{R}_{\text{max}} \) does not imply its tropical nonsingularity in the sense of [RGST05]. However, using the injection of \( \mathbb{R}_{\text{max}} \) into \( \mathbb{ER}_{\text{max}} \), \( A \) can also be seen as a matrix over \( \mathbb{ER}_{\text{max}} \), that is as an element of \( \mathbb{ER}_{\text{max}}^{n \times n} \), and in that case \( A \) is nonsingular if and only if \( A \) is tropically nonsingular in the sense of [RGST05].

\textbf{Definition 2.20.} Let \( \mathcal{S} \) be fixed. The \( n \times n \) matrix with \( 1 \) on the diagonal and \( 0 \) otherwise is the \textit{identity matrix}, denoted by \( \mathcal{I} \), or by \( \mathcal{I}_n \) if indicating its size is required.

A matrix \( A \) is \textit{invertible} if there exists a matrix \( B \) such that

\[
AB = BA = \mathcal{I}.
\]

Let \( \mathcal{S} = \{ s_i : i \in [n] \} \) be a totally ordered set of cardinality \( n \) with \( s_i < s_j \) \( \forall i < j \). With an abuse of notation, we shall consider matrices indexed by the ordered elements of \( \mathcal{S} \), called \( \mathcal{S} \times \mathcal{S} \) matrices, and identify them to \( n \times n \) matrices. Then, a \( \mathcal{S} \times \mathcal{S} \) matrix \( A \) with entries in \( \mathcal{S} \) is identified to an element of \( \mathcal{S}^{n \times n} \) and its entries will be denoted either by \( A_{s,t} \) with \( s, t \in \mathcal{S} \), or by \( a_{i,j} \) with \( i, j \in [n] \), and \( a_{i,j} = A_{s_i,s_j} \). In particular, when \( \mathcal{S} = [n] \), \( a_{i,j} = A_{i,j} \).

A \( \mathcal{S} \times \mathcal{S} \) matrix \( A = (a_{i,j}) \) is defined to be the \textit{permutation matrix} associated to the permutation \( \pi \in \mathfrak{S}_\mathcal{S} \), and will be denoted \( P_\pi \), if, for all \( i, j \in [n] \),

\[
a_{i,j} = A_{s_i,s_j} = \begin{cases} 1 & \text{if } s_j = \pi(s_i) \\ 0 & \text{otherwise.} \end{cases}
\]

A \( \mathcal{S} \times \mathcal{S} \) matrix \( A = (a_{i,j}) \) is defined to be the \textit{diagonal matrix} with diagonal entries given by the sequence \( q = (q_1, \ldots, q_n) \) of \( \mathcal{S} \) or the map \( q : \mathcal{S} \to \mathcal{S}, \ s_i \mapsto q_i \), and will be denoted \( D_q \), if, for all \( i, j \in [n] \),

\[
a_{i,j} = A_{s_i,s_j} = \begin{cases} q_i & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}
\]

When \( q \) is such that \( q_i \in \mathcal{S}^* \ \forall i \in [n] \), we denote \( q^{-1} \) the sequence or map such that \( (q_i)^{-1} = (q_i)^{-1} \), for all \( i \in [n] \).

\textbf{Remark 2.21.} (see [Rut63]) Let \( \mathcal{S} \) be naturally ordered. It is necessarily zero-sum free, so a matrix \( A \) is invertible in \( \mathcal{S}^{n \times n} \) if and only if it is the product of a permutation matrix \( P_\pi \), and a diagonal matrix \( D_q \) with an invertible determinant. We define \( M = \)
$D_q P_{\pi}$ as a generalized permutation matrix, also known as monomial. In particular $P_{\pi}^{-1} = P_{\pi}^{-1}$, $D_q^{-1} = D_q^{-1}$, $M = P_{\pi} D_{\phi_{\pi}^{-1}}$, and $M^{-1} = D_q^{-1} P_{\pi}^{-1}$.

**Proposition 2.22.** Let $\pi \in \mathfrak{S}_n$, $I \in \mathcal{P}_k([n])$. We define $\tau_I \in \mathfrak{S}_n$ to be the permutation sending $[k]$ to $I$, and such that its restrictions to $[k]$ and $[n] \setminus [k]$ are order-preserving. We define $\tau_{\pi[I]}$ similarly. Then,

1. $\text{sign}(\tau_I) \text{sign}(\tau_{\pi[I]}) = (\otimes 1) \sum_{i \in I} \text{sign}(\tau_{\pi[I]}^{i+\pi(i)})$.
2. $\text{sign}(\pi) = \text{sign}(\pi[I]) \text{sign}(\pi[I]) (\otimes 1) \sum_{i \in I} \text{sign}(\tau_{\pi[I]}^{i+\pi(i)})$.

**Proof.** To show (1), we note that the number of transpositions in $\tau_I$ and $\tau_{\pi[I]}$ are

$$\sum_{j \in [k]} i_j - j \quad \text{and} \quad \sum_{j \in [k]} l_j - j = \sum_{j \in [k]} \pi(i_j) - j,$$

respectively, where $I = \{i_1 < \cdots < i_k\}$, $\pi[I] = \{l_1 < \cdots < l_k\}$. As a result

$$\text{sign}(\tau_I) \text{sign}(\tau_{\pi[I]}) = (\otimes 1) \sum_{i \in I} \text{sign}(\tau_{\pi[I]}^{i+\pi(i)})$$

For (2), The permutation matrix $P_{\tau_{\pi[I]}^{-1}} P_{\pi[I]} = P_{\tau_I} P_{\pi[I]}^{-1}$ has the $I \times \pi[I]$ block of $P_{\pi}$ as its $[k] \times [k]$ block, and the $I^c \times \pi[I^c]$ block of $P_{\pi}$ as its $([n] \setminus [k]) \times ([n] \setminus [k])$ block. The permutation matrix of $\tau_I$ (resp. $\tau_{\pi[I]}$) has $\mathcal{I}_k$ as its $[k] \times I$ (resp. $[k] \times \pi[I]$) block, and has $\mathcal{I}_{n-k}$ as its $([n] \setminus [k]) \times I^c$ (resp. $([n] \setminus [k]) \times \pi[I^c]$) block. Therefore

$$\text{sign}(\tau_{\pi[I]}) \text{sign}(\pi) \text{sign}(\tau_I) = \text{sign}(\tau_{\pi[I]}^{-1} \pi \tau_I) = \left[ \text{sign}(\tau_{\pi[I]}|_{[k]}) \text{sign}(\pi[I]) \text{sign}(\tau_I|_{[k]}) \right] \left[ \text{sign}(\tau_{\pi[I]}|_{[n] \setminus [k]}) \text{sign}(\pi[I^c]) \text{sign}(\tau_I|_{[n] \setminus [k]}) \right].$$

Since $\tau_{\pi[I]}$ and $\tau_I$ have no inversions over $[k]$ and $[n] \setminus [k]$, their signs are 1. As a result

$$\text{sign}(\pi) = \text{sign}(\tau_{\pi[I]}) \text{sign}(\pi[I]) \text{sign}(\pi[I^c]) \text{sign}(\tau_I)^{-1} = \text{sign}(\pi[I]) \text{sign}(\pi[I^c]) (\otimes 1) \sum_{i \in I} \text{sign}(\tau_{\pi[I]}^{i+\pi(i)}).$$

The following corollary is a result of Property 2.18 and Proposition 2.22.

**Corollary 2.23.** For an elementary path $\rho$ from $i$ to $j$, we consider the cycle $\sigma = (i \, \rho(i) \, \rho^2(i) \, \cdots \, j) \in \mathfrak{S}_n$. If $\rho$ is the concatenation $\rho_1 \rho_2$ of elementary paths $\rho_1, \rho_2$ from $i$ to $k$ and from $k$ to $j$, respectively, $\sigma_1 = (i \, \rho_1(i) \, \rho^2_1(i) \, \cdots \, k) \in \mathfrak{S}_n$, $\sigma_2 = (k \, \rho_2(k) \, \rho^2_2(k) \, \cdots \, j) \in \mathfrak{S}_n$ are the corresponding cycles, and $\ell$ denotes the length of $\rho$, then

$$\sigma|_{(j)^c} = \sigma_1|_{(k)^c} \circ \sigma_2|_{(j)^c} \quad \text{and} \quad \text{sign}(\sigma|_{(j)^c}) = (\otimes 1)^\ell (\otimes 1)^{i+j} = \text{sign}(\sigma_1|_{(k)^c}) \text{sign}(\sigma_2|_{(j)^c}).$$

**Proof.** We get $\sigma|_{(j)^c} = \sigma_1|_{(k)^c} \circ \sigma_2|_{(j)^c}$, since $\rho_1 \rho_2$ is an elementary path. Then, from Proposition 2.22 we get

$$\text{sign}(\sigma|_{(j)^c}) = \text{sign}(\sigma) \text{sign}(\sigma|_{(j)^c}) (\otimes 1)^{i+j} = (\otimes 1)^\ell (\otimes 1)^{i+j} = \text{sign}(\sigma_1|_{(k)^c}) \text{sign}(\sigma_2|_{(j)^c}).$$

**Theorem 2.24.** For $A, B \in S^{n \times n}$, we have that

$$(2.4) \quad \text{det}(AB) \geq \text{det}(A) \text{det}(B),$$

with equality when
(1) \( \mathcal{S} \) is naturally ordered and \( A \) or \( B \) are invertible,
(2) \( \mathcal{S} = \mathcal{T} \neq \mathcal{T}^o \) and \( \det(AB) \in \mathcal{T}' \).

Identity \([2.4]\) was proved in \([Gan92, \text{Proposition 2.1.7}]\). It subsequently appeared in \([GBCG98, \text{Lemma 3.2}]\), with an application to minimal realization of linear recurrent sequences. It also appeared in \([IR11a, \text{Theorem 3.5}]\) over \( \mathbb{E}_{\text{max}} \). Statement \([1]\) follows from \([AGG14, \text{Proposition 3.4}]\) and the property that an invertible matrix is necessarily a monomial matrix. \([2]\) is an immediate consequence of Proposition \(2.11\).

**Definition 2.25.** A matrix \( A = (a_{i,j}) \in S_{n \times n} \) is **definite** if \( \det(A) = a_{i,i} = 1 \ \forall \ i \in [n] \).

Obviously, a definite matrix is nonsingular (over \( S \)). When \( \mathcal{S} = \mathbb{R}_{\text{max}} \), the above definition is equivalent to the one of \([B03a]\) of definite matrices. However, when \( A \in \mathbb{R}_{\text{max}}^{n \times n} \) is seen as a matrix over \( \mathbb{E}_{\text{max}} \), \( \det(A) = 1 \) implies that the optimal assignment problem has a unique solution, hence \( A \) is definite over \( \mathbb{E}_{\text{max}} \) if and only if \( A \) is **strictly definite** in the sense of \([B03a]\). Note also that definite matrices have to be distinguished from **normal matrices**, defined in \([B03a]\) to have non-positive (that is \( \preceq 1 \)) non-diagonal entries, that will not be in use in the present paper.

When \( \mathcal{S} = \mathcal{T} \) and \( A \) is nonsingular, then a dominant permutation of \( A \) has a weight equal to the determinant of \( A \) (due to Property \(2.9\)). Then, this dominant permutation may be normalized and relocated to the diagonal, using an invertible matrix, obtaining a definite matrix with a dominant normalized Id-permutation. That is, \( A = P \bar{A} \) or \( A = AP \), where \( P \) is an invertible matrix such that \( \det(A) = \det(P) \), and the matrix \( \bar{A} \) is definite. We then use the following definition.

**Definition 2.26.** A definite matrix \( \bar{A} \) is a left (resp. right) **definite form** of a nonsingular matrix \( A \in T_{n \times n} \), if \( P \) normalizes it by acting on its rows (resp. columns): \( A = P \bar{A} \) (resp. \( A = \bar{A}P \)). The invertible matrix \( P \) is the left (resp. right) **normalizer** of \( A \), corresponding to its left (resp. right) **definite form**.

We introduce a standard combinatorial property of definite matrices.

**Lemma 2.27.** Let \( A \) be an \( n \times n \) definite matrix.

(1) For every signed permutation \( \sigma \in \mathfrak{S}_I \setminus \{\text{Id}\} \), \( I \in \mathcal{P}_k([n]) \), in \( A \) we have
\[
\operatorname{sign}(\sigma) \bigcirc \ A_{i,\sigma(i)} \preceq 1.
\]

(2) The signed cycle of every \( \bar{i} \in C_{\sigma} \) s.t. \( \bar{i} \neq \{i\} \) satisfies
\[
(\ominus 1)[\bar{i}]^{-1} \bigcirc \ A_{j,\sigma(j)} \preceq 1.
\]

(3) Consider a signed bijection \( \sigma \in \mathfrak{S}_{(t)^c,s}\{s\}\) in \( A \), with \( s,t \in [n] : \ s \neq t \). Let \( \bar{s} \in C_{\sigma} \) and let \( \tau \in \mathfrak{S}_{(t)^c,s\{t\}} \) be such that \( \tau|_{\bar{s}} = \sigma|_{\bar{s}} \) and \( \tau|_{\{s,t\}} = \text{Id} \). We have
\[
\operatorname{sign}(\sigma) \bigcirc \bigcirc \ A_{j,\sigma(j)} \preceq \operatorname{sign}(\tau) \bigcirc \ A_{j,\tau(j)}.
\]

**Proof.** Let \( \sigma \neq \text{Id} \) be a permutation. Using Definition \(2.4\)
\[
1 \preceq 1 \oplus \operatorname{sign}(\sigma) \bigcirc \ A_{i,\sigma(i)} \preceq 1 \oplus \bigoplus_{\sigma \neq \text{Id}} \operatorname{sign}(\sigma) \bigcirc \ A_{i,\sigma(i)} = 1,
\]
and therefore $1 \oplus \text{sign}(\sigma) \bigotimes_{i \in [n]} A_{i, \sigma(i)} = 1$, $\forall \sigma \in \mathcal{S}_{[n]} \setminus \{\text{Id}\}$. As a result

$$1 \oplus \text{sign}(\sigma)|_i \bigotimes_{j \in \mathcal{I}} A_{j, \sigma(j)} \bigotimes_{j \in \mathcal{I}_i} A_{j, j} = 1 \oplus (\ominus 1)^{|\mathcal{I}_i|-1} \bigotimes_{j \in \mathcal{I}_i} A_{j, \sigma(j)} = 1, \forall i \neq \{i\},$$

and

$$1 \oplus \text{sign}(\sigma) \bigotimes_{i \in \mathcal{I}} A_{i, \sigma(i)} \bigotimes_{i \in \mathcal{I}} A_{i, i} = 1 \oplus \text{sign}(\sigma) \bigotimes_{i \in \mathcal{I}} A_{i, \sigma(i)} = 1, \forall \sigma \in \mathcal{S}_I \setminus \{\text{Id}\}$$

which shows Points (1) and (2).

Finally, consider $\sigma$ and $\tau$ as in Point (3). Let $\pi \in \mathcal{S}_{[n]}$ s.t. $\pi|_{\{t\}}^c = \sigma \in \mathcal{S}_{[t]}c_{\{s\}}^{c}$ and $\pi(t) = s$, and denote by $\bar{s}$ the element of $C_\pi$ containing $s$. From Proposition 2.22 we have that $\text{sign}(\sigma)(\ominus 1)^{t+s} = \text{sign}(\pi) = \text{sign}(\pi|_{\bar{s}})\text{sign}(\pi|_{S}) = (\ominus 1)^{|\bar{s}|-1}\text{sign}(\pi|_{S})$. Notice that $|\bar{s}| - 1 = |\bar{s}|$ and that the path $\bigcirc_{j \in \mathcal{S}} A_{j, \sigma(j)}$ from $s$ to $t$ is the only elementary open path in $\sigma$. Using Point (1), we get

$$\text{sign}(\sigma) \bigotimes_{i \in C_\sigma} \bigotimes_{j \in \mathcal{I}_i} A_{j, \sigma(j)} = (\ominus 1)^{|\bar{s}|-1}(\ominus 1)^{s+t} \text{sign}(\pi|_{\bar{s}}) \left( \bigotimes_{i \in C_{\bar{s}}: \ j \in \mathcal{I}} A_{j, \pi(j)} \right) \left( \bigotimes_{j \in \mathcal{S}} A_{j, \sigma(j)} \right)$$

where $(\ominus 1)^{|\bar{s}|}(\ominus 1)^{s+t}$ is the sign of the elementary bijection $\tau \in \mathcal{S}_{[t]}c_{\{s\}}^{c}$, corresponding to the elementary path $\bigcirc_{j \in \mathcal{S}} A_{j, \sigma(j)} = \bigcirc_{j \in \mathcal{S}} A_{j, \tau(j)}$ of $\sigma$ (Corollary 2.23).

**Definition 2.28.** A quasi-identity (or pseudo-identity) matrix over $\mathcal{S}$ is a nonsingular, multiplicatively idempotent matrix, with 1 on the diagonal, and off-diagonal entries in $\mathcal{S}^\circ$.

**Definition 2.29.** The $(r, c)$-submatrix $A_{(r, c)}$ of a matrix $A = (a_{i,j}) \in \mathcal{S}_{n \times n}$ is obtained by deleting row $r$ and column $c$ of $A$, and its determinant is called the $(r, c)$-minor of $A$. The adjoint matrix of $A$ is defined as $\text{adj}(A) = (a'_{i,j})$, where

$$a'_{i,j} = (\ominus 1)^{i+j} \text{det}(A_{(j,i)})$$

Notice that $\text{det}(A_{(j,i)})$ is obtained as the sum corresponding to all permutations passing through $(j, i)$, with $a_{j,i}$ removed

$$\text{det}(A_{(j,i)}) = \bigoplus_{\pi \in \mathcal{S}_{[n]}: \ \pi|_{\{j\}}^c \pi(1) \cdots \pi_{j-1} \pi_{j+1} a_{j,1} a_{j,2} \cdots a_{j,n}} \text{sign}(\pi|_{\{j\}}^c) a_{1, \pi(1)} \cdots a_{j-1, \pi_{j-1}} a_{j+1, \pi_{j+1}} \cdots a_{n, \pi(n)}.$$

Writing each permutation as the product of disjoint cycles, and applying Proposition 2.22 for $I = \{j\}$, we get

$$a'_{i,j} = \bigoplus_{\pi \in \mathcal{S}_{[n]}: \ \pi|_{\{j\}}^c \pi(1) \cdots \pi^{i-1} \pi^i \cdots \pi_{j-1}} \text{sign}(\pi) \left( a_{i, \pi(i)^c} a_{\pi(i), \pi^2(i)} \cdots a_{\pi^{i-1}(j), j} \right) \bigotimes_{j \in [k]} a_{j, \pi(j)}$$

If $A \in \mathcal{S}^*$, we denote $A^\nabla = \text{adj}(A) \text{det}(A)^{-1}$, and call it the quasi-inverse of $A$. We also denote $\mathcal{L}_A := AA^\nabla$ and $\mathcal{T}_A := A^\nabla A$. 


Proposition 2.30. Let $A \in S^{n \times n}$ be a nonsingular matrix. We have

$$I_A \succ^o I, T'_A \succ^o I \text{ and } (I_A)_{i,i} = (T'_A)_{i,i} = 1, \ i \in [n].$$

Moreover, if $S$ has a modulus, then $|\det(I_A)| = |\det(T'_A)| = 1$, and if $S = \mathbb{ER}_{\max}$, then $I_A$ and $T'_A$ are quasi-identities.

Proof. The first assertion can be deduced from [Gau92 Proposition 2.1.2]. See also [RS84]. The property that $I_A$ and $T'_A$ are quasi-identities when $S = \mathbb{ER}_{\max}$ is shown in [IR11b Theorem 2.8]. When $S$ has a modulus, applying the modulus to the expression of $I_A$, we obtain that $|I_A| = |I_A|$, hence applying the previous property to $|A|$, we get the second assertion of the proposition, that is $|\det(I_A)| = |\det(T'_A)| = 1$. □

Definition 2.31. Let $S = \{s_i : i \in [n]\}$ be a totally ordered set of cardinality $n$, with $s_i < s_j \ \forall i < j$, and let $k \in \{0, \ldots, n\}$. The $k$th compound matrix of a $S \times S$ matrix $A = (a_{i,j})$ with entries in $S$ is the $P_k(S) \times P_k(S)$ matrix with entries in $S$, denoted $A^{\wedge k}$, defined by

$$A^{\wedge k}_{I,J} = \bigoplus_{\sigma \in \mathfrak{S}_I} \operatorname{sign}(\sigma) \cdot a_{s_i,s_{\sigma(i)}}, \ I, J \in P_k(S),$$

and identified as a $\binom{n}{k} \times \binom{n}{k}$ matrix. Identification, signs of bijections in $A^{\wedge k}$, and signs of entries in $\operatorname{adj}(A^{\wedge k})$, are taken with respect to the lexicographic order in $P_k(S)$.

Remark 2.32. (1) In particular if $S = [n]$, then

$$A^{\wedge k}_{I,J} = \bigoplus_{\sigma \in \mathfrak{S}_I} \operatorname{sign}(\sigma) \cdot a_{i_i,\sigma(i)}, \ I, J \in P_k([n]), \ k \in \{0, \ldots, n\}.$$ 

So

$$A^{\wedge k} = \begin{cases} \mathbb{1}, & k = 0 \\ A, & k = 1 \end{cases}, \ \text{and } A^{\wedge n-1} = Q^{-1} \operatorname{adj}(A)^T Q \in S^{n \times n},$$

where $A^T$ denotes the transpose of matrix $A$, and $Q$ is the monomial matrix defined by $Q_{i_n + i - 1} = (\oplus 1)^i$, and 0 otherwise, which is orthogonal and satisfies $Q^T = (\oplus 1)^{n+1} Q$. That is $\operatorname{adj}(A) = Q(A^{\wedge n-1})^T Q$, or equivalently, $\operatorname{adj}(A)_{i,j} = (\oplus 1)^{i+j} A_{\{j\}^c \cup \{i\}}^{\wedge n-1}$. If $A$ is diagonal, then the transpose and the entry-signs vanish so $\operatorname{adj}(A) = P_\pi A^{\wedge n-1} P_\pi$, where $\pi(i) = n + 1 - i, \ i \in [n]$.

(2) For $k \in \{0, \ldots, n\}$ and $\pi \in \mathfrak{S}_S$, define the maps $g^{k,\pi}, f^{k,\pi} : P_k(S) \to S$ by

$$g^{k,\pi}(I) = \operatorname{sign}(\pi[I]) \text{ and } f^{k,\pi}(I) = \operatorname{sign}(\pi_{[\pi^{-1}(\pi[I])]}), \ I \in P_k(S).$$

Note that $g^{k,\pi}(I) = \operatorname{sign}(\pi[I]) = \operatorname{sign}(\pi^{-1}[\pi[I]]) = f^{k,\pi^{-1}}(I)$, for all $I \in P_k(S)$.

Recall that for $\pi \in \mathfrak{S}_S$, $\pi^{(k)} \in \mathfrak{S}_{P_k(S)}$ is s.t. $\pi^{(k)}(I) = \pi[I]$ for all $I \in P_k(S)$.

If $A$ is the permutation matrix $P_\pi$ associated to $\pi \in \mathfrak{S}_S$, then

$$A^{\wedge k}_{I,J} = \begin{cases} \operatorname{sign}(\pi[I]) & J = \pi[I] \\ 0 & \text{otherwise} \end{cases},$$

so that it can be written as $A^{\wedge k} = D_{g^{k,\pi}} P_{\pi^{(k)}} = P_{\pi^{(k)}} D_{f^{k,\pi}}$.

If $A$ is the $S \times S$ diagonal matrix $D_q$ with diagonal entries given by the map $q : S \to S$, $i \mapsto q_i$, then $A^{\wedge k}$ is the diagonal matrix $D_h$ with diagonal entries given by the map $h : P_k(S) \to S$, $I \mapsto h_I = \bigoplus_{i \in I} q_i$. 

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As a result of the (classical) Cauchy-Binet formula, if \( A = D_q P_{\pi} \) (resp. \( P_{\pi} D_q \)), then
\[
A^{\wedge k} = D_h D_{g^{k,\pi}} P_{\pi(k)} \quad \text{(resp. } P_{\pi(k)} D_{f^{k,\pi}} D_h)\).
\]

**Proposition 2.33.** (Tropical Cauchy-Binet, [Gau92, Proposition 2.18]) If \( A, B \in S^{n \times n} \), then
\[
(AB)^{\wedge k} \succ^\circ A^{\wedge k} B^{\wedge k}.
\]

**Theorem 2.34.** For \( A, B \in S^{n \times n} \), we have
\[
\text{adj}(AB) \succ^\circ \text{adj}(B)\text{adj}(A).
\]

**Proof.** Using the properties in (1) of Remark 2.32 this theorem follows from Proposition 2.33
\[
\text{adj}(AB) = Q((AB)^{\wedge n-1})^T Q^{-1} \succ^\circ Q(A^{\wedge n-1} B^{\wedge n-1})^T Q^{-1} =
Q(B^{\wedge n-1})^T (A^{\wedge n-1})^T Q^{-1} = Q(A^{\wedge n-1})^T Q^{-1} = \text{adj}(B)\text{adj}(A) \ .\]

**Corollary 2.35.** Let \( S \) be naturally ordered, and let \( A, B \in S^{n \times n} \). If \( A \) is invertible, then
1. Equality holds in Theorem 2.34, and \( A^{\wedge} = A^{-1} \).
2. Equality holds in Proposition 2.33 and \( (A^{-1})^{\wedge k} = (A^{\wedge k})^{-1} \).

Moreover, similar assertions hold if \( B \) is invertible.

**Proof.** Let \( A \) be invertible. Since \( S \) is naturally ordered, it can be written \( A = D_q P_{\pi} \). So \( A^{-1} = P_{\pi^{-1}} D_{q^{-1}} \).
1. From [AGG14, Lemma 3.6], equality holds in Theorem 2.34 when \( A \) or \( B \) is invertible. Therefore, using Theorem 2.24 and Definition 2.29, we get
\[
A^{\wedge} = \det(P_{\pi})^{-1}\text{adj}(P_{\pi}) \det(D_q)^{-1}\text{adj}(D_q) = P_{\pi^{-1}} D_{q^{-1}} = A^{-1}.
\]
2. From (1) of Remark 2.32 applied to \( A \) and \( A^{-1} \), \( k \in \{0, \ldots, n\} \) being fixed, we have \( A^{\wedge k} = D_h D_{g^{k,\pi}} P_{\pi(k)} \) and
\[
(A^{-1})^{\wedge k} = P_{\pi(k)^{-1}} D_{f^{k,\pi^{-1}} D_{h^{-1}}} \text{ with } h_I = \bigoplus_{i \in I} q_i \text{ for } I \in P_k(S) \ .
\]

Since \( g^{k,\pi} = f^{k,\pi^{-1}} \) and \( D_{g^{k,\pi}} \) is involutory, we get \( (A^{\wedge k})^{-1} = (A^{-1})^{\wedge k} \).

Now, applying Proposition 2.33 to \( A \) and \( B \) and then to \( A^{-1} \) and \( AB \), we get that
\[
(AB)^{\wedge k} \succ^\circ A^{\wedge k} B^{\wedge k} \succ^\circ A^{\wedge k} (A^{-1})^{\wedge k} (AB)^{\wedge k} \text{ and } A^{\wedge k} (A^{-1})^{\wedge k} (AB)^{\wedge k} = (AB)^{\wedge k}.
\]
Since \( \succ^\circ \) is an order, we deduce the equality. \qed

Note that when \( k = n \), Proposition 2.33 and its equality were already stated in Theorem 2.24.

**Lemma 2.36.** If \( A \in \mathcal{T}^{n \times n} \) is definite, then \( A^{\wedge k} = 1, \forall I \in P_k([n]), \forall k \in [n] \).

**Proof.** From Point 1 of Lemma 2.27
\[
A_{i,j}^{\wedge k} = 1 \oplus \bigoplus_{\sigma \in S_I \setminus \{\text{Id}\}} \text{sign}(\sigma) \bigoplus_{i \in I} A_{i,\sigma(i)} = 1 \ . \]
Remark 2.37. Notice that $A^\wedge k$ is not necessarily definite when $A$ is definite.

For example, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{\wedge 2} = \begin{pmatrix} 1 & 1 & 1 \\ \ominus 1 & 1 & 1 \\ \ominus 1 & \ominus 1 & 1 \end{pmatrix} \text{ over } \mathbb{R}_{\text{max}}, \text{ which is singular.}

Definition 2.38. Let $A$ be a matrix over $\mathcal{T}$ and let $A^0 = \mathcal{T}$. If $\oplus_{k \geq 0} A^k$ converges to a matrix over $\mathcal{T}$, then this matrix is defined as the Kleene star of $A$, denoted by $A^*$.

Theorem 2.39. Let $A = \mathcal{T} \oplus B$ over $\mathcal{T}$, with $B_{i,i} = 0, i \in [n]$. Assume that $A$ is definite. Then $|A|^*$ exists and is definite, the weight of every cycle in $B$ is $\nless \ominus 1$, and

$$|B|^* = |A|^* = |A^k| = |\text{adj}(A)| = |A^\nabla| = |A^{\nabla \nabla}|, \forall k \geq n - 1.$$ If $A^*$ exists, then $|A|^* = |A|^*$. Moreover, if $\mathcal{T}$ is idempotent, then

$$A^{**} = A^* = (\ominus B)^*.$$ If the weight of every cycle in $B$ is $\nless 1$ (in particular if $\ominus 1 = 1$ or if the modulus of the weight of every cycle in $B$ is strictly dominated by $1$), $\mathcal{T}$ being not necessarily idempotent, then $B^*$ exists and

$$A^\nabla = B^*.$$ Proof. Since $A$ is definite and $A_{i,j} = \ominus B_{i,j}$ when $i \neq j$, Point (2) of Lemma 2.27 implies that the weight of every cycle in $B$ is $\nless 1$. Since the modulus is a morphism, we have $\text{det}(|A|) = |\text{det}(A)|$, $|A|$ is definite when $A$ is definite, $|A^k| = |A|^k$, $|\text{adj}(A)| = \text{adj}(|A|)$, $|A^\nabla| = (|A|^\nabla)$, $|A^{\nabla \nabla}| = (|A|)^{\nabla \nabla}$, and $|A|^* = |A|^*$ when both matrices exist. So to show the two first assertions, it is sufficient to show that $A^*$ exists and that (2.6) holds if $A$ is a definite matrix over $\mathcal{M}$. In this case, the existence of $A^*$ and the first and second equalities of (2.6) date back to the 60’s (see [Yoe61]), and have received several proofs, using various techniques (see for instance [Dua04, Theorems 2 and 6] and [AGG14, Theorem 3.9]). The third equality in (2.6) is true by definition, since $A$ is definite, and the last equality has been proved in [Niv14b, Lemma 6.7 and Claim 6.8]. The main argument is made due to Lemma 2.27 by factoring into cycles. As a result, the diagonal entries of the $A^{\nabla \nabla}$ are $1$, and the $i, j$ off-diagonal entry is the sum of dominant elementary paths from $i$ to $j$.

When $\mathcal{T}$ is idempotent, then the equalities in (2.7) hold since

$$A^k = (\mathcal{T} \oplus B)^k = \bigoplus_{m=0}^k (\ominus B)^m \text{ implies } A^* = \bigoplus_{k \geq 0} A^k = \bigoplus_{k \geq 0} (\mathcal{T} \oplus B)^k = (\ominus B)^*.$$ Assume now that the weight of every cycle in $B$ is $\nless 1$. Because it is already $\nless 1$, this holds when $\ominus 1 = 1$. Also, from Property 2.28 this also holds when the modulus of the weight of every cycle in $B$ is strictly dominated by $1$. By Point (3) of Lemma 2.27 and (2.5), we get that the $i, j$ entry of $A^\nabla$ is the sum $\bigoplus \text{sign}(\pi) A_{i,\pi(i)} \cdots A_{\pi^{-1}(j), j}$, where $\pi$ is a permutation s.t. $\pi(j) = i$ with at most one nontrivial cycle. Denote $k_\pi$ the length of the nontrivial cycle of $\pi$, with $k_\pi = 1$ if there is no such cycle, then $\text{sign}(\pi) = (\ominus 1)^{k_\pi - 1}$. Since $A_{i,j} = \ominus B_{i,j}$ when $i \neq j$, this implies that the $i, j$ entry of $A^\nabla$ is the sum of the weights of all elementary paths from $i$ to $j$ with respect to the weight matrix $B$, when $i \neq j$, and that it is equal to $1$ if $i = j$.

By definition, the $i, j$ entry of $B^*$ is the sum of the weights of all paths from $i$ to $j$ with respect to the weight matrix $B$ (when this sum converges). Since any path from
i to j is the product of an elementary path from i to j and of not necessarily disjoint cycles, and since the weight of every cycle in B is ⪯ 1, we deduce that the i, j entry of \(B^*\) is equal to the sum of the weights of elementary paths from i to j when \(i \neq j\), or to 1 when \(i = j\), and that \(B^*\) exists. \(\square\)

**Theorem 2.40** (Frobenius property. See [IR11a Remark 1.3]). If \(S = \mathcal{M}\) or \(S = \mathbb{ER}_{\max}\), then

\[(a \oplus b)^n = a^n \oplus b^n, \ \forall a, b \in S.\]

3. JACOBI’S IDENTITY

Over a ring, one can easily obtain the identity of Jacobi (see for instance [FJ11, Section 1.2])

\[\det(A)\left(DA^{-1}\right)_{Jc,Ic}^{\land n-k} = A_{i,j}^\land k, \ \text{where} \ D_{i,i} = (-1)^i, \ \text{and} \ 0 \ \text{otherwise},\]

from the multiplicativity of the compound matrix and the multiplicativity of the determinant function. The following result shows that the same holds in rather general semirings if A is invertible.

**Lemma 3.1.** Let \(S\) be naturally ordered and \(A \in S^{n \times n}\) be invertible. Consider the diagonal matrix \(D\) over \(S\) such that \(D_{i,i} = (\oplus 1)^i, \ i \in [n]\). Then for every \(I, J \in \mathcal{P}_k([n])\)

\[\det(A)\left(DA^{-1}\right)_{Jc,Ic}^{\land n-k} = (A)_{i,j}^\land k, \ \forall k \in \{0, \ldots, n\}.\]

**Proof.** Since \(S\) is naturally ordered, an invertible matrix \(A\) is necessarily of the form \(A = DgP_\pi\) with \(g_i \in S^*, \ i \in [n]\), and \(\pi \in \mathfrak{S}_n\). Moreover, using Point (2) of Corollary 2.35 (with \(k\) and \(n\)) and \(D^{-1} = D\), we only need to prove the equality in the lemma when \(A\) is the diagonal matrix \(A = Dg\) and when \(A\) is the permutation matrix \(A = P_\pi\).

The lemma holds for an invertible diagonal matrix \(A = Dg\), since, when \(I = J\),

\[\det(A)\left(DA^{-1}\right)_{Jc,Ic}^{\land n-k} = \left(\bigoplus_{i \in [n]} g_i\right)\left(Dg^{-1}\right)_{Jc,Ic}^{\land n-k} = \left(\bigoplus_{i \in [n]} g_i\right)\left(\bigoplus_{i \in Ic} g_i^{-1}\right) = \bigoplus_{i \in I} g_i = A_{i,j}^\land k,\]

and both sides of the equality are equal to \(0\) otherwise.

When \(A = P_\pi\), then \(A^{-1} = P_{\pi^{-1}}\). Using Point (2) of Remark 2.32 and then Proposition 2.22, we get

\[\det(A)\left(DA^{-1}\right)_{Jc,Ic}^{\land n-k} = \begin{cases} \text{sign}(\pi) \bigcirc_{j \in Jc} (\oplus 1)^j \text{sign}(\pi^{-1}|_{Jc}) \bigcirc_{i \in Ic} (\oplus 1)^i & \text{if } \pi^{-1}[Jc] = Ic \\ 0 & \text{otherwise} \end{cases}\]

\[= \begin{cases} \text{sign}(\pi|_I) & \text{if } \pi[I] = Jc \\ 0 & \text{otherwise} \end{cases} = A_{i,j}^\land k. \ \square\]

The following theorem uses Corollary 2.35 to generalize Lemma 3.1 to any nonsingular matrix, using its definite form.

**Theorem 3.2** (Tropical Jacobi). If \(A \in \mathcal{T}^{n \times n}\) is nonsingular, then \(\forall I, J \in \mathcal{P}_k([n])\)

\[\det(A)\left(DA^{-1}\right)_{Jc,Ic}^{\land n-k} = \det(A)\left(\det(A)^{-1}A^{\land n-k}\right)_{\pi(I),\pi(J)} \cong 0 \ A_{i,j}^\land k, \ \forall k \in \{0, \ldots, n\},\]

where \(\pi: \mathcal{P}_k([n]) \to \mathcal{P}_n-k(\mathcal{P}_{n-1}([n]))\) is defined by \(I \mapsto \{c\ i: i \in I^c\}.\)
Proof. From Point (1) of Remark 2.32 we have for all nonsingular matrices $A$, $DA^\nabla D = \det(A)^{-1}DQ(A^{n-1})^\nabla Q^{-1}D$, with $Q$ as in Remark 2.32. Then $DQ = P_{\sigma}$ and $Q^{-1}D = P_{\sigma^{-1}}$ where $\sigma(i) = n + 1 - i$ or simply $\sigma(i) = \{i\}^c$ when $A^{n-1}$ is indexed by the sets $\{i\}^c$. Since the bijection $\pi$ of the theorem satisfies $\pi(I) = (n-k)(I)$, we get the (first) equality of the theorem.

From Point (1) of Corollary 2.35 and Lemma 3.1 and the first equality of the theorem, the (second) inequality of the theorem is true (and is an equality) for all invertible matrices. Then, using any definite form of a nonsingular matrix, $A = PA$, and using Corollary 2.35 and $D = D^{-1}$, we see that it is sufficient to prove the inequality when $A$ is definite, in which case it reduces to

$$\left(A^{n-1}\right)^{\pi(I),\pi(J)}_{\pi(I),\pi(J)} \geq A^k_{\pi(I),\pi(J)}, \forall k \in \{0, \ldots, n\}.$$  

We thus assume now that $A = (a_{i,j})$ is definite. Then, from Point (3) of Lemma 2.27 we get that $A^{n-1}_{\{i\}^c,\{j\}^c}$ is the sum of signed elementary bijections, with the elementary path being from $j$ to $i$. In particular, it is equal to $\mathbb{1}$ when $i = j$. Then, $\left(A^{n-1}\right)^{\pi(I),\pi(J)}$ is the sum of signed products of signed elementary bijections.

In order to prove Inequality (3.1), we shall show the following properties:

1. every signed bijection in $A^k_{i,j}$ is a signed product of signed elementary bijections in $\left(A^{n-1}\right)^{\pi(I),\pi(J)}$;
2. every other signed product of signed elementary bijections in $\left(A^{n-1}\right)^{\pi(I),\pi(J)}$ reappears, with an opposite sign, creating an element of $\mathcal{T}^\circ$ which is added to $A^k_{i,j}$.

for every $I, J \in \mathcal{P}_k([n])$, up to terms that are $\leq$ to other terms in the same sum (and then can be omitted). Property (2) means that an element in $\mathcal{T}^\circ$ is added to $A^k_{i,j}$. Moreover, due to Property 2.13 and Lemma 2.14 we do not need to count the number of times a term appear in (2), so that (1) and (2) are sufficient to prove (3.1).

Proof of (1): If $I = J$, from Lemma 2.30 $A^k_{I,I} = \mathbb{1}$, $\forall I \in \mathcal{P}_k([n]), \forall k \in [n]$, which corresponds to the identity permutation in $\left(A^{n-1}\right)^{\pi(I),\pi(I)}$.

Assume now that $I \neq J$. Using (2.1), we factor every bijection in $A^k_{i,j}$ into disjoint elementary open paths and cycles

$$\left(A^{n-1}\right)^{\pi(I),\pi(J)} = \text{sign}(\tau) \bigotimes_{\bar{i} \in C_{\tau}} \bigotimes_{j \in i} a_{j,\tau(j)} = \text{sign}(\tau) \bigotimes_{\bar{i} \in C_{\tau}: j \in i} a_{j,\tau(j)} \bigotimes_{\bar{i} \in C_{\tau}, j \in J} a_{j,\tau(j)} \ldots \cdot a_{\tau^{-1}m_j(j),\tau^{-1}(m_j(j))},$$

where $\tau \in \mathfrak{S}_{I,J}$, and $\tau^{-1}(m_j(j)) \in J \setminus I$ for all $j \in I \setminus J$. Define $\sigma \in \mathfrak{S}_{[n]}$ s.t. $\sigma|_I = \tau$, $\sigma|_{J \setminus I} = \mathbb{1}$ and let $K = \bigcup_{i \in C_{\tau}, \bar{i} \in J} \bar{i} \subseteq I \cap J$. Using
Proposition 2.22, we obtain
\[
\text{sign}(\sigma|_K) \cdot \text{sign}(\sigma|_{K^c}) = \text{sign}(\sigma)
\]
where the underbraced expression is a signed elementary bijection in \(A_{\{\sigma^m_j\}_{j \in I}}^{n-k}\) corresponding to the permutation \(\sigma^j\).

Since \(\{j\}^c \in \pi(J) \setminus \pi(I) \Leftrightarrow j \in I \setminus J \Leftrightarrow \tau^m_j(j) \in J \setminus I \Leftrightarrow \{\tau^m_j(j)\}^c \in \pi(I) \setminus \pi(J),\) the map \(\pi(I) \setminus \pi(J) \to \pi(J) \setminus \pi(I),\) defined by \(\{\sigma^m_j\}_{j \in I} \mapsto \{j\}^c,\) is a bijection, which is conjugate to the bijection \(\sigma|_{J^c}.\) Moreover, by taking \(\rho|_{\pi(I) \cap \pi(J)} = \text{Id},\) it can be extended to a bijection \(\rho : \pi(I) \to \pi(J).\) Then, \(\text{sign}(\rho|_{\pi(I) \cap \pi(J)}) = \text{sign}(\rho|_{J^c})\) and since \(\sigma\) is the identity on \((I \cup J)^c,\) we get that \(\text{sign}(\rho) = \text{sign}(\sigma|_{J^c}).\) Therefore, recalling that \(A_{I^c}(\pi(I)\cdot \pi(J)) = 1\) for all \(i \in [n],\) we get that (3.4) is a signed product of signed elementary bijections in \((A^{n-1})_{\pi(I)\cdot \pi(J)},\) as desired.

Proof of (2): Consider the expression
\[
\text{sign}(\rho) \cdot A_{\{i\}^c \cdot \{q(i)\}^c},
\]
where \(\rho \in \mathfrak{S}_{\pi(I), \pi(J)}\) and \(q : I^c \to J^c\) is s.t. \(\{q(i)\}^c = \rho(\{i\}^c).\) Up to terms that are \(\leq\) to other ones, a signed product of signed elementary bijections in \((A^{n-1})_{\pi(I)\cdot \pi(J)}^{n-k}\) is obtained by replacing \(A_{\{\tau^m_j\}_{j \in I}}^{n-1}\) in (3.5), by a signed elementary bijection, with the single elementary path being from \(q(i)\) to \(i.\) In particular, when \(q(i) = i,\) the latter is replaced by \(1.\) In view of the arguments above, such a signed product of signed elementary bijections in \((A^{n-1})_{\pi(I)\cdot \pi(J)}^{n-k}\) is also a bijection of \(A_{I^c, I}^{n-k}\) if \(\rho\) and the signed elementary bijections satisfy all of the following conditions:
(a) for every \( i \in I^c, \ i \neq q(i) \), we have \( \{i\}^c \in \pi(I) \setminus \pi(J) \) and \( \{q(i)\}^c \in \pi(J) \setminus \pi(I) \) (which means \( i \in J \setminus I \) and \( q(i) \in I \setminus J \)),

(b) for every \( i \in I^c, \ i \neq q(i) \), the intermediate indices of the signed elementary path of \( A \) from \( q(i) \) to \( i \) (as a bijection in \( A^{\wedge n^{-1}}_{(i)^c,\{q(i)\}^c} \)) are in \( I \cap J \),

(c) the sets of intermediate indices of the elementary paths in the nontrivial signed elementary bijections of \( A \) in \( (3.5) \) are disjoint.

Note that these conditions are satisfied in particular when \( \rho \) is the identity permutation and thus \( I = J \). We need to show that if a signed product of signed elementary bijections in \( (A^{\wedge n^{-1}})_{\pi(I),\pi(J)} \) does not satisfy all of the above properties, then it reappears with an opposite sign.

**Condition \([a]\) fails.** Then, there exists \( i \in I^c, \ i \neq q(i) \), s.t. \( \{i\}^c \in \pi(I) \cap \pi(J) \) (resp. \( \{q(i)\}^c \in \pi(I) \cap \pi(J) \)), which means that \( i \in I^c \cap J^c \) and \( \exists j \neq q(j) = i \) s.t. \( \{q(j)\}^c \in \pi(I) \cap \pi(J) \) (resp. \( \{q(i)\}^c \in \pi(I) \cap \pi(J) \)). W.l.o.g., we consider the first situation. Let \( b \) be the product of the signed elementary bijections in \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \) and \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \) respectively, and let \( p \) be the corresponding path, that is \( p \) is the concatenation of the elementary paths from \( q(i) \) to \( i \) and from \( q(j) = i \) to \( j \) corresponding to the former signed elementary bijections. If the path \( p \) is elementary, then Corollary \([2.23]\) implies that \( b \) is a signed elementary bijection (or a cycle if \( q(i) = j \)) with path \( p \). Then, \( b \) appears in the factor \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c}A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \) of the opposite sign bijection \( q \circ (i \ j) \), where \( (i \ j) \) denotes the transposition of \( i \) and \( j \). Then, the signed product of signed elementary bijections considered initially reappears in \( (A^{\wedge n^{-1}})_{\pi(I),\pi(J)} \) with an opposite sign.

If the path \( p \) is not elementary, it may be decomposed into an elementary path \( p' \) from \( q(i) \) to \( j \) (or a loop if \( q(i) = j \)), and the union of not necessarily disjoint cycles. Let \( b' \) be the signed elementary bijection of \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \) corresponding to \( p' \) and \( a \) be the product of the signed cycles corresponding to the cycles in \( p \). Then, \( b = ab' \) or \( b = \ominus ab' \). Again, the signed elementary bijection \( b' \) appears in the factor \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c}A^{\wedge n^{-1}}_{\{i\}^c,\{i\}^c} \) of the opposite sign bijection \( q \circ (i \ j) \). So, the signed product of signed elementary bijections of \( (A^{\wedge n^{-1}})_{\pi(I),\pi(J)} \) considered initially is equal to the product of a term in \( (A^{\wedge n^{-1}})_{\pi(I),\pi(J)} \) corresponding to the bijection \( q \circ (i \ j) \), with a factor \( a' \), equal either to \( a \) or to \( \ominus a \). From Lemma \([2.27]\) we have \( a \preceq 1 \). This implies that \( a' \preceq 1 \) or \( a' = \ominus 1 \). Indeed, if \( a' = a \), then \( a' \preceq 1 \), whereas if \( a' = \ominus a \), we get that \( a' \preceq 1 \), and Corollary \([2.10]\) shows that \( a' \preceq 1 \) or \( a' = \ominus 1 \). Then, if \( a' \preceq 1 \), the signed product of signed elementary bijections considered initially is \( \equiv \) to another one in \( (A^{\wedge n^{-1}})_{\pi(I),\pi(J)} \), whereas if \( a' = \ominus 1 \), it is equal to the opposite of another term.

**Condition \([a]\) holds, but condition \([b]\) fails.** We consider an intermediate index \( i \in I^c \) (resp. \( i \in J^c \)) in the elementary path of the signed elementary bijection \( \tau \) in \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \). Since \( \{i\}^c \in \pi(I) \) (resp. \( \{i\}^c \in \pi(J) \)) and \( \rho \in \mathfrak{S}_{\pi(I),\pi(J)} \), this index also appears as the last (resp. first) index of the elementary paths of the signed elementary bijections in \( A^{\wedge n^{-1}}_{\{i\}^c,\{q(i)\}^c} \) (resp. \( A^{\wedge n^{-1}}_{\{q^{-1}(i)\}^c,\{i\}^c} \)). We shall only consider the first situation, since the second one can be handled similarly. From Remark \([2.2]\) an elementary path

can be factored into two non-maximal elementary paths, each of which can be extended into an elementary path at its ends, and from Corollary 2.23 the composition of the corresponding signed elementary bijections is a signed elementary bijection. Let $\sigma, \sigma_1, \sigma_2 \in \mathcal{S}_{[n]}$ be such that $\sigma|_{\{j\}^c} = \tau$, $\sigma(j) = q(j)$, $\sigma_1 = (i \tau(i) \cdots \tau^{-1}(j) j)$ and $\sigma_2 = (q(j) \tau(q(j)) \cdots \tau^{-1}(i) i)$. From Corollary 2.23 the product of $A_{\{i\}^c, \{q(i)\}^c}$ with the signed elementary bijection $\tau$ in $A_{\{j\}^c, \{q(j)\}^c}$ satisfies

$$A_{\{i\}^c, \{q(i)\}^c} \cdot \text{sign}(\tau) \cdot (a_{q(j), \tau(q(j))} \cdots a_{\tau^{-1}(i), i} \cdot a_{i, \tau(i)} \cdots a_{\tau^{-1}(j), j}) =$$

$$A_{\{i\}^c, \{q(i)\}^c} \cdot \text{sign}(\sigma_1|_{\{j\}^c}) \cdot (a_{i, \sigma(i)} \cdots a_{\sigma^{-1}(j), j}) \cdot \text{sign}(\sigma_2|_{\{i\}^c}) \cdot (a_{q(j), \sigma(q(j))} \cdots a_{\sigma^{-1}(i), i}).$$

Let $b$ be obtained by replacing $A_{\{i\}^c, \{q(i)\}^c}$ by a signed elementary bijection in $\tau$, and let $p$ be the corresponding path, that is $p$ is the concatenation of the path from $q(i)$ to $i$, corresponding to the signed elementary bijection in $A_{\{i\}^c, \{q(i)\}^c}$, and the path from $i$ to $j$ corresponding to $\sigma_1$. If $p$ is elementary, then, from Corollary 2.23 again, $b$ is a signed elementary bijection in $A_{\{j\}^c, \{q(j)\}^c}$, then the product of the signed elementary bijections in the factor $A_{\{i\}^c, \{q(i)\}^c} \cdot A_{\{j\}^c, \{q(j)\}^c}$ of $q$ reappears in the factor $A_{\{i\}^c, \{q(i)\}^c} \cdot A_{\{j\}^c, \{q(j)\}^c}$ of the opposite sign bijection $q \circ (i j)$. So as above, the signed product of signed elementary bijections considered initially reappears in $(A_{\{i\}^c, \{q(i)\}^c})\pi_{(I), \pi(J)}$ with an opposite sign.

If $p$ is not elementary, then it may be decomposed into an elementary path $p'$ from $q(i)$ to $j$ (note that we cannot have a loop due to Condition (a)], and the union of not necessarily disjoint cycles. Let $b'$ be the signed elementary bijection of $A_{\{j\}^c, \{q(j)\}^c}$ corresponding to $p'$ and $a$ be the product of the signed cycles corresponding to the cycles in $p$. Then, $b = ab'$ or $b = \emptyset ab'$, so the product of the signed elementary bijections in the factor $A_{\{i\}^c, \{q(i)\}^c} \cdot A_{\{j\}^c, \{q(j)\}^c}$ of $q$ is equal to the product of signed elementary bijections in the factor $A_{\{i\}^c, \{q(i)\}^c} \cdot A_{\{j\}^c, \{q(j)\}^c}$ of the opposite sign bijection $q \circ (i j)$, times a factor equal to $a$ or $\emptyset a$. With the same arguments as in the first case (in which Condition (a) fails), we obtain that the signed product of signed elementary bijections considered initially is either $\leq$ to another one in $(A_{\{i\}^c, \{q(i)\}^c})\pi_{(I), \pi(J)}$, or equal to the opposite of such a term.

Conditions (a) hold, but condition (b) fails. We consider $i \in I \cap J$, such that $i$ is an intermediate index in the signed elementary bijections $\tau$ in $A_{\{i\}^c, \{q(i)\}^c}$ and $\psi$ in $A_{\{t\}^c, \{q(t)\}^c}$. Similarly to the previous case, elementary paths can be factored into non-maximal elementary paths, which can be extended into elementary paths, where the composition of the corresponding signed elementary bijections is a signed elementary bijection. Let $\sigma, \phi \in \mathcal{S}_{[n]}$ be such that $\sigma|_{\{j\}^c} = \tau$, $\sigma(j) = q(j)$, $\phi|_{\{t\}^c} = \psi$ and $\phi(t) = q(t)$. From Corollary 2.23 we have $\tau = \sigma|_{\{j\}^c} = \sigma_2|_{\{i\}^c} \circ \sigma_1|_{\{j\}^c}$, and $\psi = \phi|_{\{t\}^c} = \phi_2|_{\{i\}^c} \circ \phi_1|_{\{t\}^c}$, where

$$\sigma_1 = (i \tau(i) \cdots \tau^{-1}(j) j) \in \mathcal{S}_{[n]} \ , \ \sigma_2 = (q(j) \tau(q(j)) \cdots \tau^{-1}(i) i) \in \mathcal{S}_{[n]},$$

$$\phi_1 = (i \psi(i) \cdots \psi^{-1}(t) t) \in \mathcal{S}_{[n]} \ , \ \phi_2 = (q(t) \psi(q(t)) \cdots \psi^{-1}(i) i) \in \mathcal{S}_{[n]}.$$
As a result, the product $b$ of the corresponding signed elementary bijections can be factored as follows:

\begin{equation}
(3.7) \quad b = \text{sign}(\tau)(a_{q(j),\tau(q(j))} \cdots a_{t^{-1}(i),i} a_{i,\tau(t)} \cdots a_{t^{-1}(j),j})
\end{equation}

\begin{equation}
\circ \text{sign}(\psi)(a_{q(t),\psi(q(t))} \cdots a_{\psi^{-1}(i),i} a_{i,\psi(i)} \cdots a_{\psi^{-1}(t),t})
\end{equation}

\begin{equation}
(3.8) \quad = \text{sign}(\sigma_1|1^n\rangle)(a_{q(j),\sigma(q(j))} \cdots \text{sign}(\sigma_1|t^n\rangle) a_{\sigma^{-1}(i),i})(a_{i,\phi(i)} \cdots a_{\phi^{-1}(t),t})
\end{equation}

\begin{equation}
(3.9) \quad \circ \text{sign}(\phi_2|1^n\rangle)(a_{q(t),\phi(q(t))} \cdots a_{\phi^{-1}(i),i} \text{sign}(\sigma_1|t^n\rangle)(a_{i,\phi(i)} \cdots a_{\phi^{-1}(j),j})).
\end{equation}

Let $b_1$ and $b_2$ be equal to (3.8) and (3.9) respectively, and let $p_1$ and $p_2$ be the corresponding paths. If $p_1$ and $p_2$ are elementary paths, then $b_1$ and $b_2$ are signed elementary bijections in $A_{(t^n),(q(t))}^{n,n-k}$ and $A_{(t^n),(q(t))}^{n,n-k}$ respectively. So their product $b$ appears in the factor $A_{(t^n),(q(t))}^{n,n-k}$, of the opposite sign bijection $\rho \circ (J t)$.

If $p_1$ or $p_2$ (or both) is not elementary, then they may be decomposed into elementary paths $p'_1$ and $p'_2$ from $q(j)$ to $t$, and from $q(t)$ to $j$ respectively, and the union of not necessarily disjoint cycles. Let $b'_1$ and $b'_2$ be signed elementary bijections in $A_{(t^n),(q(j))}^{n,n-k}$ and $A_{(j^n),(q(t))}^{n,n-k}$, corresponding to the paths $p'_1$ and $p'_2$, respectively, and $a$ be the product of all the signed cycles of $p_1$ and $p_2$. Then, $b = b'_1 b'_2 a$ or $b = \ominus b'_1 b'_2 a$. We conclude as for the previous cases.

\[\square\]

**Corollary 3.3.** Let $A \in \mathcal{T}^{n \times n}$ be nonsingular, then

$$\det(A) \text{tr} \left( \left( A^\top \right)^{n-k} \right) \succ^o \text{tr} \left( A^k \right).$$

If $\ominus 1 = 1$, we have

$$\det(A) \left( A^\top \right)^{n-k} \succ^o A^k_{i,j},$$

in particular over $\mathbb{E}^{\max}$ we have

$$\det(A) \left( A^\top \right)^{n-k} \models A^k_{i,j}.$$

**Proof.** By Corollary 2.35, we have $(DA^\top V D)^{n-k} = D^{n-k}(A^\top)^{n-k} D^{n-k}$, and since $D^{n-k}$ is diagonal and equal to its inverse, we get that $(DA^\top V D)^{n-k} I_{i,j} = (A^\top)^{n-k} I_{i,j}$ for all $I \in \mathcal{P}_{n-k}$. Using Theorem 3.2 we deduce the first assertion of the corollary. The second one follows from the property that $DA^\top V D = A^\top V$ when $\ominus 1 = 1$.

In Section 6 we shall apply Corollary 3.3 to relate the characteristic polynomials of a matrix and its quasi-inverse, and deduce an analogue relation between their eigenvalues.

**Corollary 3.4.** Let $A \in \mathcal{T}^{n \times n}$ be nonsingular and assume that $\mathcal{T} \neq \mathcal{T}^o$ and that all entries of $(A^{\lambda_{n-1}})^{n-k}$ belong to $\mathcal{T}^\prime$. Then, the inequalities of Theorem 3.2 and Corollary 3.3 become equalities.

**Proof.** This follows from Proposition 2.11. \[\square\]

**Corollary 3.5.** Let $A \in \mathcal{T}^{n \times n}$ be nonsingular and assume that $A^*$ exists and $A = A^*$. Then, $|A^{n-k}|_{i,j} = |A^k|_{i,j}$, for all $k \in \{0, \ldots, n\}$, and $I, J \in \mathcal{P}_k([n])$.

**Proof.** Since $A$ is nonsingular and $A = A^*$, we deduce that $A$ is definite. From Theorem 2.39 we have $|A^\top| = |A^*|$ so $|A^\top| = |A|$. Since the modulus is a morphism, we get, for $k \in [n]$,

$$|A^{n-k}| = (|A|)^{n-k} = (|A^\top|)^{n-k} = (|DA^\top D|)^{n-k} = |(DA^\top D)^{n-k}|.$$
where all operations on moduli of matrices are done with respect to the semiring $\mathcal{M}$. Hence, applying the modulus to the inequality of Theorem 3.2, we get $|A_{I,J}^{n-k}| \succ |A_{I,J}^{n-k}|$, for all $I,J \in \mathcal{P}_n([n])$. Applying the same inequality to $n-k$ instead of $k$, we get $|A_{I,J}^{n-k}| \succ |A_{I,J}^{n-k}|$, thus the equality. 

\section{4. Other identities on compound matrices}

\subsection{4.1. The quasi-inverse matrix}

We define the relation $\succ^0$ by: $a \succ^0 b \iff a \succ b$ and $|a| = |b|$. With the same arguments as for Theorem 3.2 we obtain the following identities.

**Proposition 4.1.** Let $A \in \mathcal{T}^{n \times n}$. Recall that $\mathcal{I}_A = AA^\nabla$.

1. If $A$ is definite, then $A_{i,i}^{\nabla} = (\mathcal{I}_A)_{i,i} = 1$, $\forall i$, and

$$\det(\mathcal{I}_A) \succ^0 \det(A^\nabla) = \det(A^{n-1}) \succ^0 1.$$  

2. If $A$ is nonsingular, then

$$\det(B) \succ^0 \begin{cases} \det(A)^{n-1} & B = \adj(A) \\ \det(A)^n & B = A\adj(A) \\ \det(A)^{-1} & B = A^\nabla. \end{cases}$$

**Proof.** (1) Let $A$ be definite. From Lemma 2.36 we get $A_{i,i}^{\nabla} = A_{\{i\}^c,i}^{n-1} = 1$, and from Proposition 2.30 we have $(\mathcal{I}_A)_{i,i} = 1$, for every $i \in [n]$. From Remark 2.32, $\det(A^\nabla) = \det(Q(A^{n-1})^TQ^{-1}) = \det(A^{n-1})$.

Applying Theorem 3.2 for $k = 0$, we get $\det(A^{n-1}) \succ^0 1$. Now by the same arguments as in the proof of Theorem 3.2, $\det(A^{n-1})$ is the sum of signed products of signed elementary bijections, each of them being equal to the product of (not necessarily disjoint) signed cycles possibly times $\ominus 1$. Then, each term is equal to $\ominus 1$ or $\leq 1$. Since the term corresponding to the identity is equal to $1$, we obtain that $\det(A^{n-1})$ equals $1$ or $1^\circ$, which implies $\det(A^{n-1}) = 1$ and so $\det(A^{n-1}) \succ^0 1$.

Straightforward, $\det(AA^\nabla) \succ^0 \det(A) \det(A^\nabla) = \det(A^\nabla) = \det(A^{n-1}) \succ^0 1$. As above, one can write $\det(AA^\nabla)$ as the sum of signed products of signed elementary bijections, each of them being equal to the product of (not necessarily disjoint) signed cycles possibly times $\ominus 1$. So we obtain again that $\det(AA^\nabla)$ equals $1$ or $1^\circ$, which implies $\det(AA^\nabla) \succ^0 \det(A^\nabla)$.

(2) Denote the right normalization of $A$ by $A = \bar{A}P$. From Corollary 2.35, $\det(\adj(A)) = \det(\adj(P)\adj(\bar{A})) = \det(\adj(P))\det(\adj(\bar{A})) = (\det(P))^n \det(P^{-1})\det(\adj(\bar{A})) \succ^0 \det(A)^{n-1}$,

$$\det(A\adj(A)) = \det(A)\adj(A)^T \adj(\bar{A})^{-1} = \det(A)^n \det(\mathcal{I}_A) \succ^0 \det(A)^n,$$

$$\det(A^\nabla) = \det(A)^{-n} \det(\adj(A)) \succ^0 \det(A)^{-1}.$$ 

Recalling Proposition 2.30 and Theorem 2.24, the matrices $\mathcal{I}_A$ and $A^\nabla$ in (1) are definite over $\mathbb{E}_{\max}$, as proved in [Niv15, Remark 2.18], and the $\succ^0$ relations in (1) and (2) become equalities over $\mathbb{E}_{\max}$, as proved in [IR11a, Theorem 4.9]. Note that Point (2) of Proposition 4.1 is an equality when $\mathcal{T} = \mathcal{M}$ and in particular when $\mathcal{T} =$
The right hand side of (4.1) is the sum of all expressions of the form
\((4.1)\) property 2.18, sign(\(S\)) for any matrix \(D\) that
then it is sufficient to prove the identity for \(A\) definite. Using Theorem 3.2 and the fact
that \(D^{\wedge k}\) is a diagonal matrix equal to its inverse, we get
\[
(A^{\wedge k}(A^{\wedge k})^\wedge k)_{I,J} = \bigoplus_{L \in P_k([n])} A_{I,L}^{\wedge k}(A^{\wedge k})_{L,J}
\]
\[
= \bigoplus_{L \in P_k([n])} A_{I,L}^{\wedge k}D_{L,L}^{\wedge k}(DA^\wedge D)^{\wedge k}_{L,J}D_{L,J}^{\wedge k}
\]
\[
(4.1)
\]
(4.2) \(\text{sign}(\sigma) \bigcirc a_{i,\sigma(i)}(\ominus 1)\sum_{i \in I} j + \sum_{j \in J} j \text{sign}(\tau) \bigcirc a_{j,\tau(j)}\),
where \(\sigma \in \mathcal{S}_{I,L}\), \(\tau \in \mathcal{S}_{J^c,L^c}\).
If \(I \cap J^c = \emptyset\), then \(I = J\) and one can extend the bijection \(\sigma\) into a permutation \(\rho \in \mathcal{S}_{[n]}\) such that \(|\rho|_I = \sigma\) and \(|\rho|_{J^c} = \tau\). Using Proposition 2.22, we obtain that (4.2) is
equal to sign(\(\rho\)) \(\bigcirc\) \(a_{i,\rho(i)}\), so the right hand side of (4.1) is equal to det(\(A\)).
If \(I \cap J^c \neq \emptyset\), denote \(I = \{i_1 < \cdots < i_k\}\) and \(J = \{j_1 < \cdots < j_k\}\) and define \(\delta \in \mathcal{S}_{I,L}\) by \(\delta(j_i) = j_t\), and let \(\pi = \sigma \circ \delta \in \mathcal{S}_{J,L}\). Then, sign(\(\delta\)) = 1 and, by Property 2.18 sign(\(\pi\)) = sign(\(\sigma\)). Let \(\rho \in \mathcal{S}_{[n]}\) be such that \(|\rho|_I = \pi\) and \(|\rho|_{J^c} = \tau\). From Proposition 2.22 again, we have that
\[
(4.3)
\]
Let \(m \in I \cap J^c\). Define \(L' = \{\tau(m)\} \cup L \setminus \{\sigma(m)\} \in P_k([n])\) and \(\sigma' \in \mathcal{S}_{I,L'}, \tau' \in \mathcal{S}_{J^c,L^c}\by \sigma'|_{\{m\}} = \sigma|_{\{m\}}\), \(\sigma'(m) = \tau'(m)\).
Next, consider \(\rho' \in \mathcal{S}_{[n]}\) such that \(|\rho'|_I = \sigma' \circ \delta\) and \(|\rho'|_{J^c} = \tau'\). Since \(\rho' = (\sigma(m) \tau(m)) \circ \rho\), we have that
\(\ominus\text{sign}(\rho) = \text{sign}(\rho') = \text{sign}(\sigma')(\ominus 1)\sum_{j \in J} j + \sum_{i \in I} i\).
Moreover,
\[
\bigcirc a_{i,\sigma(i)} \bigcirc a_{j,\tau(j)} = \bigcirc a_{i,\sigma'(i)} \bigcirc a_{j,\tau'(j)}\.
\]
Therefore, (4.2) reappears in the right hand side of (4.1), with \(\sigma, \tau\) and \(L\) replaced by \(\sigma', \tau'\) and \(L'\) respectively, and with an opposite sign.

Proposition 4.2. If \(A\) is nonsingular, then \(A^{\wedge k}(A^{\wedge k})^\wedge k \succ^o I\).

(Not to be confused with \(A^{\wedge k}(A^{\wedge k})^\wedge k\) and \((AA^{\wedge k})^\wedge k\) which are obviously \(\succ^o I\).)

Proof. Let \(\bar{A}P\) be a right normalization of \(A\). By Corollary 2.35, we have
\[
A^{\wedge k}(A^{\wedge k})^\wedge k = \bar{A}^{\wedge k}P^{\wedge k}(P^{-1})^{\wedge k}(\bar{A}^{\wedge k})^\wedge k = \bar{A}^{\wedge k}P^{\wedge k}(P^{\wedge k})^{-1}(\bar{A}^{\wedge k})^\wedge k = \bar{A}^{\wedge k}(\bar{A}^{\wedge k})^\wedge k,
\]
then it is sufficient to prove the identity for \(A\) definite. Using Theorem 3.2
4.2. Powers of matrices. Let \( M \) be the weight matrix of a weighted directed graph \( G \). Considering a permutation of \([n]\) in \( G \), we analyze the corresponding permutation in the graph having \( M^m \) as its weight matrix. As one can see in Theorem 2.39, the elementary paths in graphs having powers of matrices as a weight matrix, satisfy rather unusual characteristics. In this section we provide two additional properties. The first is an analogue to a classical property, and the second holds over \( \mathbb{ER}_{\max} \), but neither classically, over \( \mathbb{SR}_{\max} \) or over \( \mathbb{R}_{\max} \), as shown in the following example. Nevertheless, the second property leads to an analogue of a classical result, stated in Corollary 6.6.

Example 4.3.

If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then \( A^2 = \begin{pmatrix} a^2 \oplus bc & b(a \oplus d) \\ c(a \oplus d) & d^2 \oplus bc \end{pmatrix} \), and we get

\[
\text{tr}(A^2) = \begin{cases} 
2^2 \oplus 2^2 \oplus bc \oplus bc & \text{over a field of characteristic 2} \\
2^2 \oplus 2^2 \oplus bc \gg 2^2 \oplus 2^2 & \text{over } \mathbb{SR}_{\max} \text{ or } \mathbb{R}_{\max} \\
2^2 \oplus 2^2 \oplus bc \mid 2^2 \oplus 2^2 & \text{over } \mathbb{ER}_{\max}
\end{cases}
\]

The first identity is the elementary consequence \( \text{tr}(A^p) = \text{tr}(A)^p \) of Frobenius property over a field of characteristic \( p \). In the following theorem and corollary, we extend an identity proved in \([Niv14a, \text{Theorem 3.6}]\), and provide the supertropical property motivated by the third identity in (4.3).

Theorem 4.4. Let \( A \in \mathcal{T}^{n \times n} \), \( m \in \mathbb{N} \) and \( k \in \{0, \ldots, n\} \).

1. We have \((A^m)^k \gg^o (A^k)^m\).
2. If \( \mathcal{T} = \mathcal{M} \) or \( \mathcal{T} = \mathbb{ER}_{\max} \), we have \( \text{tr}((A^k)^m) \gg^o (\text{tr}(A^k))^m \).

Proof. (1) is obtained by induction on \( m \), using Proposition 2.33 applied with \( B = A^{m-1} \) and using the compatibility of \( \gg^o \) with the laws of \( \mathcal{T} \).

For (2), we show more generally that \( \text{tr}(B^m) \gg^o (\text{tr}(B))^m \) for any \( B \in \mathcal{T}^{n \times n} \) when \( \mathcal{T} = \mathcal{M} \) or \( \mathcal{T} = \mathbb{ER}_{\max} \). We have

\[
\text{tr}(B^m) = \bigoplus_{i \in [n]} (B^m)_{i,i} = \bigoplus_{i \in [n]} \bigoplus_{t_i, l \in [n]} B_{t_{i,1}, t_{i,1}, t_{i,2}, \ldots, t_{i,m-1}, i},
\]

and using Theorem 2.40, we have

\[
(\text{tr}(B))^m = \bigoplus_{i \in [n]} B_{i,i}^m.
\]

The summing terms in the right hand side of (4.6) are also terms in the right hand side of (4.5) with \( t_{i,\ell} = i \), \( \forall \ell \in [m-1] \), \( \forall i \in [n] \). For every other term of (4.5), there exists \( \ell \) s.t. \( t_{i,\ell} \neq i \), and therefore this term reappears as a term of \( (B^m)_{t_{i,\ell}, t_{i,\ell}} \):

\[
B_{t_{i,1}, \ldots, t_{i,\ell}, t_{i,\ell+1}, \ldots, t_{i,m-1}, i} = B_{t_{i,\ell}, \ldots, t_{i,m-1}, i} B_{t_{i,1}, \ldots, t_{i,\ell-1}, i}.
\]

Since \( \oplus \mathbb{1} = \mathbb{1} \) in \( \mathcal{T} \), we deduce \( \text{tr}(B^m) \gg^o (\text{tr}(B))^m \).

Using the compatibility of \( \gg^o \) with the laws of \( \mathcal{T} \), we deduce the following result.
Corollary 4.5. Let $\mathcal{T} = \mathcal{M}$ or $\mathcal{T} = \mathbb{ER}_{\max}$, $A \in \mathcal{T}^{n \times n}$, $m \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$. We have:

$$\text{tr} \left( (A^m)^\wedge k \right) \preceq \left( \text{tr}(A^k) \right)^m.$$ 

Note that the previous inequality can be written as:

$$\text{tr} \left( (A^m)^\wedge k \right) \models \left( \text{tr}(A^k) \right)^m \text{ over } \mathbb{ER}_{\max},$$

$$\text{tr} \left( (A^m)^\wedge k \right) \succeq \left( \text{tr}(A^k) \right)^m \text{ over } \mathcal{M}.$$ 

In Section 3 we shall apply Corollary 4.5 to the characteristic polynomials of $A$ and its powers, and provide over $\mathbb{ER}_{\max}$ an analogue to the property: if $\lambda$ is an eigenvalue of $A$, then $\lambda^m$ is an eigenvalue of $A^m$, which holds over rings.

4.3. Conjugate matrices.

Definition 4.6. We say that a matrix $A$ is tropically conjugate to $A'$ if there exists a nonsingular matrix $E$ such that $A' = E^\nabla AE$.

Tropical conjugation is not a symmetric relation, as shown in the following example.

Example 4.7. We take $A = \mathcal{I}$, $E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A' = \mathcal{I}_E$ in $\mathbb{ER}_{\max}^{2 \times 2}$.

Obviously $E^\nabla AE = A'$. Exchanging the roles of $A$ and $A'$, we look for a nonsingular matrix $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $F^\nabla A'F = A$, which means that

$$\text{adj}(F)A'F = \begin{pmatrix} d & b \\ c & a \end{pmatrix}_E \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad \oplus bc \oplus c^2d & (bd \oplus d^2)^\nu \\ (ca \oplus c^2)^\nu & ad \oplus bc \oplus d^2c \end{pmatrix} = \det(F)\mathcal{I}.$$ 

In order for the $(1, 2)$ and $(2, 1)$ positions to be $0$, we must require $c = d = 0$, which means $\det(F) = 0$, and therefore such a nonsingular matrix does not exist.

Conjugate matrices have algebraic value due to the various properties being preserved. In particular, it is a key tool in representation theory for describing equivalent representations (see [FH91]), and in linear algebra for diagonalizing, triangularizing or Jordanizing matrices (see [Str09]). In the present section we are interested in a well known identity on the compound matrix of a conjugation. Whereas over fields the proof is straightforward due to the multiplicativity of the compound operation, in the tropical case this property becomes an inequality, unless the conjugating matrix is invertible. Nevertheless, due to the invertible case, we can reduce the desired property to definite matrices.

Lemma 4.8. Let $E, A \in \mathcal{T}^{n \times n}$. If $E$ is nonsingular, with right definite form $\tilde{E}$, then

$$\text{tr} \left( (E^\nabla AE)^\wedge k \right) = \text{tr} \left( (E^\nabla AE)^\wedge k \right).$$

Proof. We recall that the trace function satisfies $\text{tr}(MN) = \text{tr}(NM)$ for any two square matrices $M, N$. Let $P$ be the right normalizer of $E$, that is $E = \tilde{E}P$. Using Corollary 2.33 we get

$$\text{tr} \left( (E^\nabla AE)^\wedge k \right) = \text{tr} \left( (P^{-1}\tilde{E}^\nabla A\tilde{E}P)^\wedge k \right) = \text{tr} \left( (P^{-1})^\wedge k (\tilde{E}^\nabla A\tilde{E})^\wedge k (P)^\wedge k \right) = \text{tr} \left( (P)^\wedge k (P^{-1})^\wedge k (\tilde{E}^\nabla A\tilde{E})^\wedge k \right) = \text{tr} \left( (\tilde{E}^\nabla A\tilde{E})^\wedge k \right). \quad \square$$
Theorem 4.9. Let $E, A \in T^{n \times n}$. If $E$ is nonsingular, then
\[
\text{tr}\left((E^\nabla AE)^\wedge k\right) \geq \circ \text{tr}\left(A^\wedge k\right), \forall k \in [n].
\]

Proof. Using Proposition 2.33 the compatibility of $\geq \circ$ with the laws of $\mathcal{T}$, the commutativity of the trace, and Proposition 4.2, we obtain
\[
\text{tr}\left((E^\nabla AE)^\wedge k\right) \geq \circ \text{tr}\left(A^\wedge k\right) \geq \circ \text{tr}(A^\wedge k).
\]

In Section 6 we apply Theorem 4.9 to the characteristic polynomials of $A$ and its conjugates, which provide the desired connection between their eigenvalues.

5. The Sylvester–Franke identity

The Sylvester–Franke identity has received a few proofs over the years, including diagonalization (see [Con17]) and factorization to elementary matrices (see [Tor52]). The tropical version for this identity is provided in this section combinatorially, proving equality over $\mathbb{R}_{\max}$ and $\mathbb{ER}_{\max}$ in particular.

Theorem 5.1 (Tropical Sylvester–Franke theorem). Let $A \in T^{n \times n}$, and $k \in \{0, \ldots, n\}$. The identity
\[
\det(A^\wedge k) = \det(A)^{(n-1)}_{(k-1)},
\]
holds under one of the following conditions:

(1) $\mathcal{T} = \mathcal{M}$, in which case $\det$ coincides with $\text{per}$;
(2) $A$ is invertible in $\mathcal{T}$;
(3) $A = I \ominus B$ is definite with $B_{ii} = 0$ for all $i \in [n]$ and the weight of every cycle in $B$ is $\preceq 1$ (which holds in particular when the modulus of the weight of every cycle in $B$ is strictly dominated by 1).

Proof. Let $A = (a_{i,j}) \in T^{n \times n}$, and let $A^\wedge k = (A^\wedge k_{i,j})$ be its $k$th compound matrix.

(1) Assume $\mathcal{T} = \mathcal{M}$. Then, $\det$ coincides with $\text{per}$. We have
\[
\text{per}(A^\wedge k) = \bigoplus_{\pi \in \mathcal{P}_k([n])} \bigotimes_{I \in \mathcal{P}_k([n])} A^\wedge k_{I, \pi(I)} = \bigoplus_{\pi \in \mathcal{P}_k([n])} \bigotimes_{I \in \mathcal{P}_k([n])} \left(\bigoplus_{\sigma \in \mathcal{I}_I} \bigotimes_{i \in I} a_{i, \sigma(i)}\right),
\]
and
\[
\text{per}(A)^{(n-1)}_{(k-1)} = \left(\bigoplus_{\rho \in \mathcal{E}_n} \bigotimes_{i \in [n]} a_{i, \rho(i)}\right)^{(n-1)}_{(k-1)}.
\]
Moreover, using Theorem 2.40, we also have
\[
\text{per}(A)^{(n-1)}_{(k-1)} = \bigoplus_{\rho \in \mathcal{E}_n} \left(\bigotimes_{i \in [n]} a_{i, \rho(i)}\right)^{(n-1)}_{(k-1)}.
\]
Developing (5.2), (5.3) and (5.4) using the distributivity of the multiplication with respect to addition, we arrive at a sum such that each summand is a product of $\binom{n}{k} \cdot k =$
\[
\binom{n-1}{k-1} \cdot n \quad \text{entries of } A. \quad \text{We shall show that each summand in (5.2) is a summand in (5.3) and (5.4), and that each summand in (5.4) is a summand in (5.2).}
\]

For each summand in (5.2), (5.3) or (5.4), we shall consider the \( n \times n \) integer matrix \( B = (b_{i,j}) \) such that \( b_{i,j} \) is the number of occurrences of the factor \( a_{i,j} \) in the summand, which means that the summand is equal to \( \bigcirc_{i,j \in [n]} a_{i,j} \). Equivalently, one may consider the multigraph with set of nodes \([n]\), and \( b_{i,j} \) arcs between \( i \) and \( j \), for each \( i, j \in [n] \). From the above remark, \( B \) satisfies necessarily \( \sum_{i,j \in [n]} b_{i,j} = \binom{n}{k} \cdot k \).

Summands in (5.2) are summands in (5.3): Let \( B \) be an integer matrix associated to a summand in (5.2). For every \( m \in [n] \), the number of sets \( I \in \mathcal{P}_k([n]) \) such that \( m \in I \), is \( \binom{n-1}{k-1} \). Thus, \( \sum_{j \in [n]} b_{i,j} = \binom{n-1}{k-1} \). By symmetry, that is indexing the product in (5.2) by the image \( J \) of \( \pi \), we get that \( \sum_{i \in [n]} b_{i,m} = \binom{n-1}{k-1} \) for all \( m \in [n] \). Using the Birkhoff-von Neumann theorem (or Hall’s Theorem, see e.g. Hall, Hall98 Theorem 5.1.9), we get that \( B \) can be written as the sum of \( \binom{n-1}{k-1} \) permutation matrices (in the usual sense). Since the matrix associated to the permutation \( \rho \in \mathcal{S}_{[n]} \) corresponds to \( \bigcirc_{i \in [n]} a_{i,\rho(i)} \), we obtain that \( B \) is the integer matrix associated to the product of \( \binom{n-1}{k-1} \) such products. This means that the summand of (5.2) considered initially is also a summand in (5.3).

Summands in (5.4) are summands in (5.2): Let \( B \) be the integer matrix associated to a summand in (5.4). We have \( B = \binom{n-1}{k-1} P \), where \( P \) is the matrix of some permutation \( \rho \in \mathcal{S}_{[n]} \). Consider \( \pi = \rho^k \in \mathcal{S}_{\mathcal{P}_k([n])} \), and, for all \( I \in \mathcal{P}_k([n]) \), take \( \sigma = \rho^k \) in (5.2). We get that \( \sigma \in \mathcal{S}_{I,\pi(I)} \) and \( \sigma(i) = \rho^k(i) \) for all \( i \in I \). Since the number of sets \( I \in \mathcal{P}_k([n]) \) such that \( i \in I \) is equal to \( \binom{n-1}{k-1} \), we obtain that \( \bigcirc_{i \in [n]} a_{i,\rho^k(i)} = \binom{n-1}{k-1} \) = \( \bigcirc_{I \in \mathcal{P}_k([n])} \left( \bigcirc_{i \in [n]} a_{i,\rho^k(i)}(I) \right) \), so that the summand of (5.4) considered initially is also a summand in (5.2).

Now since \( T = \mathcal{M} \), \( \mathcal{T} \) is totally ordered, thus idempotent. Hence, \( \text{per}(A^\wedge k) \cong \text{per}(A^\wedge \binom{n-1}{k-1}) \) because summands in (5.2) are summands in (5.3), and \( \text{per}(A^\wedge \binom{n-1}{k-1}) \cong \text{per}(A^\wedge k) \) because summands in (5.4) are summands in (5.2). This shows (5.1).

2 Assume that \( A \) is invertible in \( \mathcal{T} \). Then, \( A \) is a monomial matrix. From Remark 2.32 for a monomial matrix \( A \) on any semiring \( \mathcal{S} \), \( A^k \) is a monomial matrix. This implies that in that case, both sides of (5.1) are monomials in the non-zero entries of \( A \) with a coefficient equal to \( 1 \) or \( \ominus 1 \), depending on the sign of the permutation associated to \( A \), on \( k \) and \( n \). Since the equality in (5.1) holds on any commutative ring, it also holds on any semiring \( \mathcal{T} \) for a monomial matrix (see for instance [RS84] or [AGG09]).

3 Assume that \( A = I \ominus B \) is definite with \( B_{ii} = 0 \) for all \( i \in [n] \) and that the weight of every cycle in \( B \) is \( \leq 1 \). Recall that this condition holds in particular when the modulus of the weight of every cycle in \( B \) is strictly dominated by \( 1 \), see Theorem 2.39. Also by Theorem 2.39 under the above conditions, we also have that the weight of every cycle in \( B \) is \( \leq \ominus 1 \). Since \( \det(A) = 1 \), we only need to show that \( \det(A^k) = 1 \).
We have
\[
\det(A^k) = \bigoplus_{\pi \in \mathcal{P}_k([n])} \text{sign}(\pi) \cdot \bigoplus_{I \in \mathcal{P}_k([n])} A^k_{I,\pi(I)}
\]
(5.5)
\[
= \bigoplus_{\pi \in \mathcal{P}_k([n])} \text{sign}(\pi) \cdot \bigoplus_{I \in \mathcal{P}_k([n])} \left( \bigoplus_{\sigma \in \mathcal{S}_I,\pi(I)} \text{sign}(\sigma) \cdot a_{I,\sigma(I)} \right).
\]
By the arguments above, each summand in (5.5) is equal to a summand in (5.3) times 1 or \(\ominus 3\) or \(\ominus 1\), thus it is equal to the product of \(\binom{n-1}{k-1}\) signed permutations of \(A\) times 1 or \(\ominus 1\). Decomposing permutations into cycles, and using that \(A = I \ominus B\) and \(B_{i,i} = 0\), we get that if one of these permutations is not equal to the identity, the summand is equal to a (nonempty) product of weights of nontrivial cycles in \(B\) times 1 or \(\ominus 1\). In that case, using that the weight of every cycle in \(B\) is \(\leq 1\), and also \(\leq \ominus 1\), we deduce that the summand is \(\leq 1\). Otherwise, if all the permutations are equal to the identity, the summand corresponds to the permutation \(\pi\) equal to identity, and all bijections \(\sigma\) equal to the identity, in which case the summand is equal to 1, since all diagonal entries of \(A\) are equal to 1. In all, this imply that (5.5) is equal to 1 and so (5.1) holds. \(\square\)

**Corollary 5.2.** Let \(A \in \mathcal{T}^{n \times n}\), and \(k \in \{0, \ldots, n\}\). We have

(5.6)
\[|\det(A^k)| = |\det(A)|^{\binom{n}{k-1}}.\]

Moreover, (5.1) holds under one of the following conditions:

1. \(\ominus 1 = 1\) and \(A\) is nonsingular;
2. \(T = \mathbb{ER}_{\max}\), \(A\) being not necessarily nonsingular.

**Proof.** The first assertion follows from Point (1) of Theorem 5.1 applied to \(|A|\), and the property that the modulus is a morphism.

(1) Let \(A\) be nonsingular, and let \(A = PA\bar{A}\) be any definite form. Using Corollary 2.35 and Point (2) of Theorem 5.1 we obtain that (5.1) holds for \(A\) as soon as it holds for the definite matrix \(\bar{A}\). When \(A\) is definite, writing it as \(A = I \ominus B\) as in Theorem 2.39 and using Theorem 2.39 and \(I = \ominus 1\), we get that the weight of every cycle in \(B\) is \(\leq 1\). Then, by Point (3) of Theorem 5.1 we deduce that (5.1) holds for \(A\).

(2) Assume \(T = \mathbb{ER}_{\max}\). If \(A\) is nonsingular, then (5.1) holds by the previous point since \(\ominus 1 = 1\). Assume now that \(A\) is singular. Let \(a = \det(A^k)\) and \(b = \det(A)^{\binom{n}{k-1}}\) be the left and right hand sides of (5.1), respectively. From the first assertion of the present corollary, we have \(|a| = |b|\). Since \(A\) is singular, we get that \(b \in T^o\) (that is \(b \in \mathbb{R}^+ \cup \{-\infty\}\)), thus \(b\) is the maximal element of \(T\) with modulus equal to \(|b|\), which implies that \(a \leq b\).

Now, since \(\det\) coincides with \(\text{per}\), and Theorem 2.40 holds for \(T = \mathbb{ER}_{\max}\), Equalities (5.2), (5.3) and (5.4) hold true. Also, the implications between summands shown in the proof of Point (1) of Theorem 5.1 are also valid. In addition, the correspondence between summands in (5.4) and summands in (5.2) constructed in that proof is one to one. Indeed, if \(\rho \neq \rho'\), then \(\rho^{(k)} \neq (\rho')^{(k)}\). This implies that the sum in (5.4) is \(\leq\) to the sum in (5.2), that is \(b \leq a\), so \(b = a\), which finishes the proof of (5.1). \(\square\)
Example 5.3. Note that the condition $\ominus 1 = 1$ in Point (1) of Corollary 5.2 is necessary. Indeed, let $\mathcal{T} = \mathbb{R}_{\max}$ and
\[
A = \begin{pmatrix}
1 & \ominus 1 & 0 \\
1 & 1 & \ominus 1 \\
1 & 1 & 1
\end{pmatrix}.
\]
Then $\det(A) = 1$ so that $A$ is nonsingular. However,
\[
A^{\wedge 2} = \begin{pmatrix}
1 & \ominus 1 & 1 \\
1 & 1 & \ominus 1 \\
1^\circ & 1 & 1
\end{pmatrix},
\]
and $\det(A^{\wedge 2}) = 1^\circ \neq \det(A)^2$.

6. The tropical characteristic polynomial

The characteristic polynomial over the symmetrized and supertropical semirings are studied in [AGG09] and [IKR13], respectively, whose terminology we follow. In this section, we investigate tropical characteristic polynomials of a matrix, its powers, its quasi-inverse and its conjugates, using the results of Sections 3 and 4. We obtain an analogue to properties connecting the eigenvalues of these matrices, which are wider in the extended tropical semiring, and we address the special case where the coefficients of the characteristic polynomial are invertible.

Let us first give some definitions. From Properties 2.8 and 2.9, a polynomial over $\mathcal{T}$ takes generally the value of monomials of highest absolute value. Yet some monomials do not dominate for any $x \in \mathcal{T}$.

Definition 6.1. Let $f = \oplus_{k=0}^n a_k x^k \in \mathcal{T}[X]$ be a formal polynomial over $\mathcal{T}$ and let $f(x)$ be its evaluation in $x \in \mathcal{T}$. We call monomials $a_k x^k$ that dominate $f(x)$ at some $x \in \mathcal{T}$ (that is such that $|a_k x^k| = |f(x)|$) essential at $x$, and monomials that do not dominate $f(x)$ for any $x \in \mathcal{T}$ inessential.

We call an element $r \in \mathcal{T}^\circ$ a root of $f$, if $f(r) \in \mathcal{T}^\circ$. If $r$ is a root such that either $r \in \mathcal{T}^*$ and $f(r)$ is the sum of at least two essential monomials at $r$ that have nonsingular coefficients, that is, there exists a set $S \subset \{0 \leq k \leq n \mid |a_k r^k| = |f(r)|\}$ and $a_k \in \mathcal{T}^*$ with at least two elements, such that $f(r) = \oplus_{k \in S} a_k r^k$, or $r = 0$ and $f(r) = 0$, then $r$ will be called a corner root. Roots that are not corner roots will be called non-corner roots.

One can also give the following different definitions of a corner root.

Lemma 6.2. Let $r \in \mathcal{T}^*$ be a root of the polynomial $f = \oplus_{k=0}^n a_k x^k \in \mathcal{T}[X]$. Then, the following are equivalent:

1. $r$ is a corner root of $f$;
2. $S = \{0 \leq k \leq n \mid |a_k r^k| = |f(r)|\}$ has at least two elements and $f(r) = \oplus_{k \in S} a_k r^k$.

If $\mathcal{T} \neq \mathcal{T}^\circ$, they are also equivalent to:

3. $f(r) = \oplus_{k \in S} a_k r^k$ with $S = \{0 \leq k \leq n \mid |a_k r^k| = |f(r)|\}$ and $a_k \in \mathcal{T}^*$;
4. $f(r) = \oplus_{k \in S} a_k r^k$ with $S = \{0 \leq k \leq n \mid a_k \in \mathcal{T}^*\}$;
5. $f(r) = \oplus_{k \in S} a_k r^k$ for some set $S \subset \{0 \leq k \leq n \mid a_k \in \mathcal{T}^*\}$.
Moreover, an invertible root such that all essential monomials have nonsingular coefficients is necessarily a corner root.

Proof. The implications (2) ⇒ (1) and (1) ⇒ (3) are trivial.

(1) ⇒ (2): let \( S \subseteq S' := \{ 0 \leq k \leq n \mid |a_k r^k| = |f(r)| \text{ and } a_k \in T^* \} \) such that \( f(r) = \oplus_{k \in S} a_k r^k \) and \( S' \) has at least two elements. Then, \( S' \) has at least two elements and since \( f(r) = \oplus_{k \in S} a_k r^k \prec \oplus_{k \in S'} a_k r^k \prec f(r) \), we deduce that \( f(r) = \oplus_{k \in S} a_k r^k \), which implies (2).

(5) ⇒ (4) follows from the same arguments as for the previous implication.

(4) ⇒ (3) follows from Property 2.8.

Assume now that \( T \neq T^0 \).

(3) ⇒ (1) Assume that \( r \in T^* \) is a root of \( f \), and that \( f(r) = \oplus_{k \in S} a_k r^k \) with \( S = \{ 0 \leq k \leq n \mid |a_k r^k| = |f(r)| \text{ and } a_k \in T^* \} \). Since \( r \in T^* \), we have \( a_k r^k \in T^* \) for all \( k \in S \), and since \( r \) is a root of \( f \), so \( f(r) \in T^0 \), and \( T \neq T^0 \), so \( T^* \) and \( T^0 \) are disjoint, we get that \( f(r) \neq a_k r^k \) for all \( k \in S \). Therefore, \( S \) has at least two elements, and \( r \) is a corner root.

By Property 2.8, \( f(r) \) is the sum of all essential monomials of \( f \) at \( r \), so if all essential monomials have nonsingular coefficients, then \( r \) satisfies (3), which implies the last assertion of the lemma.

When \( T = T^0 \), a root is simply any element of \( T^\vee \). However, for \( T = \mathbb{R}_{\text{max}} \) for instance, \( r \neq 0 \) is a corner root if and only if the maximum in \( f(r) = \oplus_{k \in n} a_k r^k \) is attained at least twice. Then, corner roots over \( \mathbb{R}_{\text{max}} \) coincide with roots of tropical polynomials in the sense of tropical geometry. If now \( M = \mathbb{R}_{\text{max}} \), in particular for \( T = \mathbb{S}\mathbb{R}_{\text{max}} \) or \( T = \mathbb{E}\mathbb{R}_{\text{max}} \), a corner root \( r \) of \( f \) is such that \( |r| \) is a corner root (or a root in the sense of tropical geometry) of \( |f| \), that is either \( |r| = 0 \) and \( |a_0| = 0 \), or \( |r| \neq 0 \) and the maximum in the expression \( \max(|a_k||r|^k, k = 0, \ldots, n) \) is attained at least twice. The converse is not true in general. For instance over \( T = \mathbb{S}\mathbb{R}_{\text{max}} \), the polynomial \( f = 1 \oplus X^2 \) is such that \( |f| = 1 \oplus X^2 \) over \( \mathbb{R}_{\text{max}} \), so \( 1 \) is a corner root of \( |f| \) (with multiplicity 2), but the only elements \( r \in \mathbb{S}\mathbb{R}_{\text{max}}^\vee \) such that \( |r| = 1 \) are \( \oplus 1 \) and \( \ominus 1 \) and both satisfy \( f(r) = 1 \notin T^0 \), so there exist no (corner) roots of \( f \) such that \( |r| \) is a corner root of \( |f| \). Also on \( T = \mathbb{E}\mathbb{R}_{\text{max}} \), \( f = 1^\vee \ominus X \) is such that \( |f| = 1 \ominus X \) over \( \mathbb{R}_{\text{max}} \), so \( 1 \) is a corner root of \( |f| \), but the only element \( r \in \mathbb{E}\mathbb{R}_{\text{max}}^\vee \) such that \( |r| = 1 \) is \( 1 \), which is a non-corner root of \( f \).

For the next definition, we follow the terminology in \cite{AGM14, ABG16} and \cite{IKR13}.

**Definition 6.3.** The (formal) characteristic polynomial of \( A \in T^{n \times n} \) is defined to be \( f_A = \det(XI \ominus A) \in T[X] \), and its characteristic polynomial function is \( f_A(x) = \det(xI \ominus A) \). The eigenvalues of \( A \) are defined as the corner roots of \( f_A \).

Recall that over \( \mathbb{R}_{\text{max}} \) and \( \mathbb{E}\mathbb{R}_{\text{max}} \), \( \ominus \) means \( \oplus \), and \( f_A \) is called the maxpolynomial. Also, over \( \mathbb{R}_{\text{max}} \), the eigenvalues of \( A \) are the roots of \( f_A \) in the sense of tropical geometry, so they coincide with the algebraic tropical eigenvalues in \cite{AGM14, ABG16}.

The coefficient of \( X^k \) in the formal characteristic polynomial of \( A \) times \((\ominus 1)^{n-k}\) is the sum of the determinants of its \( n-k \times n-k \) principal submatrices (that is, obtained by deleting \( k \) chosen rows, and their corresponding columns). Thus, this is the trace of...
the \((n-k)\)th compound matrix of \(A\):

\[
(6.1) \quad f_A = \bigoplus_{k=0}^{n} (\ominus 1)^{n-k} \text{tr}(A^{\wedge n-k}) X^k.
\]

The combinatorial motivation for the tropical characteristic polynomial is the Best Principal Submatrix problem, and has been studied by Butkovic in [But03b] and [BM00].

Recall that orders over \(\mathcal{T}\) are applied to polynomials coefficient-wise. Moreover, polynomials with possibly negative exponents can be composed formally.

**Theorem 6.4.** Let \(A, E \in \mathcal{T}^{n \times n}\) and \(m \in \mathbb{N}\). We have

\[
(6.2a) \quad f_{E \land AE} \succeq f_A, \quad \text{when } E \text{ is nonsingular};
\]

\[
(6.2b) \quad f_{A \lor} \succeq \det(A)^{-1} X^n f_A(X^{-1}), \quad \text{when } A \text{ is nonsingular};
\]

\[
(6.2c) \quad f_{A^m} \succeq \bigoplus_{k=0}^{n} (f_A)_k X^k, \quad \text{when } \mathcal{T} = \mathcal{M} \text{ or } \mathbb{E}\mathbb{R}_{\max}.
\]

Moreover, \((6.2c)\) implies

\[
(6.3) \quad f_{A^m}(x^m) \succeq (f_A(x))^m \quad \forall x \in \mathcal{T}.
\]

**Proof.** The three first inequalities follow from \((6.1)\), together with Theorem 4.9, Corollary 3.3 and Corollary 4.5, respectively. The last one follows from Theorem 2.40. □

Note that \((6.3)\) concerns the polynomial functions \(f_{A^m}\) and \(f_A\), and that the inequality is false for the corresponding formal polynomials.

**Corollary 6.5.** Assume that \(\mathcal{T} \neq \mathcal{T}^o\). Equality holds for those coefficients in \((6.2)\) such that the coefficient in the left hand side is in \(\mathcal{T}^\lor\). In particular, if \(f_{M} \in \mathcal{T}^\lor[X]\), then

\[
\begin{align*}
f_M &= f_A, \quad \text{for } M = E \land AE \text{ and } E \text{ nonsingular};
\end{align*}
\]

\[
\begin{align*}
f_M &= \det(A)^{-1} X^n f_A(X^{-1}), \quad \text{for } M = A \lor \text{ and } A \text{ nonsingular};
\end{align*}
\]

\[
\begin{align*}
f_M &= \bigoplus_{k=0}^{n} (f_A)_k X^k, \quad \text{for } M = A^m \text{ and } \mathcal{T} = \mathbb{E}\mathbb{R}_{\max}.
\end{align*}
\]

Moreover, if \(A \lor\) is nonsingular and \(f_{A \lor} \in \mathcal{T}^\lor[X]\), then \(f_{A \lor} = f_A\).

**Proof.** This is straightforward from Theorem 6.4 and Proposition 2.11 □

**Corollary 6.6.** Assume that \(\mathcal{T} \neq \mathcal{T}^o\), \(A, E, M \in \mathcal{T}^{n \times n}\), \(m \in \mathbb{N}\) and \(g\) is an invertible map satisfying one of the following conditions:

\[
(6.4a) \quad M = E \land AE, \quad g : \mathcal{T} \to \mathcal{T}, x \mapsto x, \quad \text{and } E \text{ nonsingular};
\]

\[
(6.4b) \quad M = A \lor, \quad g : \mathcal{T}^* \to \mathcal{T}^*, x \mapsto x^{-1}, \quad \text{and } A \text{ nonsingular};
\]

\[
(6.4c) \quad M = A^m, \quad g : \mathcal{T} \to \mathcal{T}, x \mapsto x^m, \quad \text{and } \mathcal{T} = \mathbb{E}\mathbb{R}_{\max}.
\]

We have

- (1) if \(\gamma\) is a root of \(f_A\), then \(g(\gamma)\) is a root of \(f_M\);
- (2) if \(\lambda\) is an eigenvalue of \(M\), then \(g^{-1}(\lambda)\) is an eigenvalue of \(A\).

The proof of Corollary 6.6 uses the following general lemmas.
Lemma 6.7. Let $P, Q \in \mathcal{T}[X]$ with $\mathcal{T} \neq \mathcal{T}^\circ$ and assume that $P \succcurlyeq^o Q$. Then the degree of $P$ is greater or equal to the degree of $Q$ and we have:

(1) If $r$ is a root of $Q$, then $r$ is a root of $P$.

(2) If $r$ is a corner root of $P$, then $r$ is a corner root of $Q$.

Proof. Let $P, Q \in \mathcal{T}[X]$ be given by $P = \bigoplus_{k=0}^n p_k X^k$ and $Q = \bigoplus_{k=0}^n q_k X^k$ and such that $p_k \succcurlyeq^o q_k$ for all $k = 0, \ldots, n$, with $p_n$ or $q_n$ possibly equal to zero. If $p_n = 0$ (that is if the degree of $P$ is less than $n$) then $q_n \preceq^o 0$ and since $0 \in \mathcal{T}^\vee$, we get that $q_n = 0$ by Proposition 2.11 so the degree of $Q$ is also less than $n$. This shows that the degree of $P$ is greater or equal to the degree of $Q$.

1. If $r \in \mathcal{T}^\vee$ is a root of $Q$, then $Q(r) \in \mathcal{T}^\circ$. Since an inequality between formal polynomials implies the same for the corresponding polynomial functions, we have $Q(r) \preceq^o P(r)$. This implies that $P(r) \in \mathcal{T}^\circ$ and so $r$ is a root of $P$.

2. Let $r \in \mathcal{T}^\vee$ be a corner root of $P$. If $r = 0$ this means that $P(0) = 0$. Then $Q(0) \preceq^o 0$, which implies that $Q(0) = 0$ by Proposition 2.11. So $r$ is a corner root of $Q$. If $r \in \mathcal{T}^*$, then by definition $P(r) \in \mathcal{T}^\circ$ and by Point (5) of Lemma 6.2 $P(r) = \bigoplus_{k \in S} p_k r^k$ for some set $S \subset \{0 \leq k \leq n \mid p_k \in \mathcal{T}^*\}$. Let $k \in S$. Since $p_k \succcurlyeq^o q_k$, and $p_k \in \mathcal{T}^* \subset \mathcal{T}^\circ$, Proposition 2.11 implies that $p_k = q_k$. So $P(r) = \bigoplus_{k \in S} p_k r^k = \bigoplus_{k \in S} q_k r^k \preceq Q(r)$ and since we also have $Q(r) \preceq^o P(r)$, so $Q(r) \preceq P(r)$, we deduce that $P(r) = Q(r) = \bigoplus_{k \in S} q_k r^k$. In particular $Q(r) \in \mathcal{T}^\circ$, therefore $r$ is a root of $Q$. Moreover, since $q_k = p_k \in \mathcal{T}^*$ for all $k \in S$ and $Q(r) = \bigoplus_{k \in S} q_k r^k$, we obtain by Point (5) of Lemma 6.2 that $r$ is a corner root of $Q$. \hfill \Box

Lemma 6.8. Assume that $\mathcal{T} \neq \mathcal{T}^\circ$, $m \in \mathbb{N}$, $f = \bigoplus_{k=0}^n f_k X^k$, $Q \in \mathcal{T}[X]$, and $g$ is an invertible map satisfying one of the following conditions:

\begin{align*}
(6.5a) & \hspace{1cm} Q = (f_0)^{-1} X^n f(X^{-1}), \quad g : \mathcal{T}^* \to \mathcal{T}^*, x \mapsto x^{-1}, \quad f_0 \in \mathcal{T}^* \text{ and } f_n = 1; \\
(6.5b) & \hspace{1cm} Q = \bigoplus_{k=0}^n (f_k)^m X^k, \quad g : \mathcal{T} \to \mathcal{T}, x \mapsto x^m, \quad \text{and } \mathcal{T} = \mathbb{ER}_{\max}.
\end{align*}

We have

(1) $\gamma$ is a root of $f$ if, and only if, $g(\gamma)$ is a root of $Q$;

(2) $\gamma$ is a corner root of $f$ if, and only if, $g(\gamma)$ is a corner root of $Q$.

Proof. In Case (6.5a), $f_0 \in \mathcal{T}^*$, $f_n = 1$, and $Q = \bigoplus_{k=0}^n q_k X^k$ with $q_k = (f_0)^{-1} f_{n-k}$. If $\gamma \in \mathcal{T}^\vee$ is a root of $f$, then $f(\gamma) \in \mathcal{T}^\circ$. Since $f(0) = f_0 \in \mathcal{T}^*$, and $\mathcal{T}^*$ and $\mathcal{T}^\circ$ are disjoint, $0$ is not a root of $f$, so $\gamma \in \mathcal{T}^*$ and $g(\gamma)$ exists and is in $\mathcal{T}^\vee$. Then $Q(g(\gamma)) = (f_0)^{-1} \gamma^n f(\gamma) \in \mathcal{T}^\circ$, so $g(\gamma)$ is a root of $Q$. If now $\gamma \in \mathcal{T}^\vee$ is a corner root of $f$, then $g(\gamma)$ is a root of $Q$. Moreover, $f(\gamma) = \bigoplus_{k \in S} f_k \gamma^k$, where $S \subset \{k \geq 0 \mid f_k \in \mathcal{T}^*\}$. This implies that $Q(g(\gamma)) = (f_0)^{-1} \gamma^n f(\gamma) = \bigoplus_{k \in S} q_{n-k} g(\gamma)^{n-k}$, with $q_{n-k} \in \mathcal{T}^*$ for all $k \in S$. Then, by Point (5) of Lemma 6.2, $g(\gamma)$ is a corner root of $Q$. Using that $f = (f_0)^{-1} X^n Q(X^{-1})$ and $q_0 = (f_0)^{-1} \in \mathcal{T}^*$ and $q_n = 1$, we obtain also the reverse implications, which shows (1) and (2) for Case (6.5a).

Let us now consider the case (6.5b) in which $\mathcal{T} = \mathbb{ER}_{\max}$. Let us first remark that $g$ is a bijection from $\mathcal{T}$ (resp. $\mathcal{T}^*$, $\mathcal{T}^\vee$, $\mathcal{T}^\circ$) to itself. If $\gamma \in \mathcal{T}^\vee$ is a root of $f$, then $f(\gamma) \in \mathcal{T}^*$. By Theorem 2.40 we have $Q(g(\gamma)) = \bigoplus_{k=0}^n (f_k)^m \gamma^{km} = \bigoplus_{k=0}^n (f_k \gamma^k)^m = (\bigoplus_{k=0}^n f_k \gamma^k)^m = g(f(\gamma))$. Since $g$ is a bijection from $\mathcal{T}^\vee$ (resp. $\mathcal{T}^\circ$) to itself, we get that $g(\gamma) \in \mathcal{T}^\vee$ and $Q(g(\gamma)) \in \mathcal{T}^\circ$, so $g(\gamma)$ is a root of $Q$. Conversely, if $g(\gamma) \in \mathcal{T}^\vee$ is a root
of $Q$, then $g(f(\gamma)) \in T^\circ$, and since $g$ is a bijection from $T^\circ$ (resp. $T^\circ$) to itself, $\gamma \in T^\circ$ and $f(\gamma) \in T^\circ$ and so $\gamma$ is a root of $f$. This shows Point (1).

Now, let $\gamma$ be a corner root of $f$, then $g(\gamma)$ is a root of $Q$. Moreover, $f(\gamma) = \oplus_{k \in S} f_k \gamma^k$, where $S = \{k \geq 0 \mid f_k \in T^*\}$. By Theorem 2.40 we have $Q(g(\gamma)) = (f(\gamma))^m = (\oplus_{k \in S} f_k \gamma^k)^m = \oplus_{k \in S} (f_k)^m \gamma^{km} = \oplus_{k \in S} q_k (g(\gamma))^k$, where $q_k = (f_k)^m$ is the $k$th coefficient of $Q$ and $q_k \in T^*$ for all $k \in S$. Then, by Point (5) of Lemma 6.2, $g(\gamma)$ is a corner root of $Q$. Conversely, if $g(\gamma) \in T^\circ$ is a corner root of $Q$, then $\gamma$ is a root of $f$ and $Q(g(\gamma)) = \oplus_{k \in S} q_k (g(\gamma))^k$, where $S = \{k \geq 0 \mid q_k \in T^*\}$. This implies that $(f(\gamma))^m = Q(g(\gamma)) = \oplus_{k \in S} q_k (g(\gamma))^k = (\oplus_{k \in S} f_k \gamma^k)^m$ and since $g$ is a bijection, we deduce that $f(\gamma) = \oplus_{k \in S} f_k \gamma^k$. Moreover, $f_k = g^{-1}(q_k) \in T^*$ for all $k \in S$. Then, by Point (5) of Lemma 6.2, $\gamma$ is a corner root of $f$. □

**Proof of Corollary 6.6.** Consider the polynomials $P, Q, f \in T[X]$ of degree $n$ with $P = f_M$, $f = f_A$ and $Q = f_A$ in case (6.4a) and $Q$ as in (6.5a) and (6.5b) in cases (6.4b) and (6.4c) respectively. By Theorem 6.4, we have $P \succ \succ Q$. By definition, an eigenvalue of a matrix $M$ is a corner root of $f_M$, and since $P = f_M$, Lemma 6.7 shows that if $\lambda$ is a root of $Q$ then $\lambda$ is a root of $f_M$ and if $\lambda$ is an eigenvalue of $M$, then $\lambda$ is a corner root of $Q$. The assertions of the corollary follow in Case (6.4a) since $Q = f_A$ and $g$ is the identity map. They also follow in Cases (6.4b) and (6.4c), using Lemma 6.8. □

**Example 6.9.** Let $A = \begin{pmatrix} 3 & 2^\circ \\ 1 & 1 \end{pmatrix}$. Then

$$A^\circ = \begin{pmatrix} -3 & (2^\circ)^\circ \\ \oplus(-3) & -1 \end{pmatrix} \text{ over } \mathbb{S}\mathbb{R}_{\text{max}}, \text{ and } A^2 = \begin{pmatrix} 6 & 5^\circ \\ 4 & 3^\circ \end{pmatrix} \text{ over } \mathbb{E}\mathbb{R}_{\text{max}}.$$  

The corresponding characteristic polynomials are

$$f_A = X^2 \oplus 3X \oplus 4 \text{ over } \mathbb{S}\mathbb{R}_{\text{max}}, \text{ and } f_A = X^2 \oplus 3X \oplus 4 \text{ over } \mathbb{E}\mathbb{R}_{\text{max}},$$

$$f_{A^\circ} = X^2 \oplus (-1)X \oplus (-4) = (-4)X^2(4 \oplus 3X^{-1} \oplus X^{-2}) = \det(A)^{-1}X^n f_A(X^{-1}),$$

$$f_{A^2} = X^2 \oplus 6X \oplus 9^\circ = X^2 \oplus 6X \oplus 8 = X^2 \oplus (f_A)^2 X \oplus (f_A)^2_0.$$ 

The polynomials $f_A$ and $f_{A^\circ}$ have only two roots: 3 and 1 for $f_A$ (either in $\mathbb{E}\mathbb{R}_{\text{max}}$ or $\mathbb{S}\mathbb{R}_{\text{max}}$) and $(-3) = 3^{-1}$ and $(-1) = 1^{-1}$ for $f_{A^\circ}$. These roots are also corner roots so eigenvalues of $A$ and $A^\circ$ respectively. The polynomial $f_{A^2}$ has a unique corner root $3^2$, that is $A^2$ has a unique eigenvalue, whereas all the $x \in \mathbb{E}\mathbb{R}_{\text{max}}^\circ$ such that $x \leq 1.5^2$ are roots of $A^2$.

In Corollary 6.6 we related the eigenvalues of the matrices of Theorem 6.4 under the assumption that $T \neq T^\circ$. The typical example where $T = T^\circ$ is when $T = M = \mathbb{R}_{\text{max}}$. In that case, roots are any elements, so Point (1) of Corollary 6.6 is true but has no interest. Moreover, $\succ \succ$ is simply the order $\succ$, so one cannot expect an exact correspondence between eigenvalues of $A$ and $M$ as in Corollary 6.6. Nevertheless, one can apply [ABG16] Lemma 4.2 to obtain the following majorization inequality. Recall that a matrix $A$ over $\mathbb{R}_{\text{max}}$ is nonsingular if and only if per $A \neq 0$. For a polynomial $P$ over $\mathbb{R}_{\text{max}}$, we define as in [ABG16], the multiplicity of a corner root $r$ as the difference between right and left slopes of the polynomial function $P$ at point $r$. 


Corollary 6.10. Assume that $\mathcal{T} = \mathbb{R}_{\text{max}}$, $A, E, M \in \mathcal{T}^{n \times m}$, $m \in \mathbb{N}$ and $g$ is an invertible map satisfying one of the following conditions:

(6.6a) \[ M = E^V A E, \quad g : \mathcal{T} \to \mathcal{T}, x \mapsto x, \] and $E$ is nonsingular;

(6.6b) \[ M = A^V, \quad g : \mathcal{T}^* \to \mathcal{T}^*, x \mapsto x^{-1}, \] and $A$ is nonsingular;

(6.6c) \[ M = A^m, \quad g : \mathcal{T} \to \mathcal{T}, x \mapsto x^m. \]

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ denote the eigenvalues of $M$ and $A$, respectively, counted with multiplicities and ordered so that $\lambda_1 \succeq \lambda_2 \succeq \ldots \succeq \lambda_n$ and $g(\gamma_1) \succeq g(\gamma_2) \succeq \ldots \succeq g(\gamma_n)$. Then

$$\lambda_1 \cdots \lambda_k \succeq g(\gamma_1) \cdots g(\gamma_k) \quad \forall k \in [n].$$

Moreover, in Case (6.6b), the latter inequality becomes an equality for $k = n$.

Proof. \cite[Lemma 4.2]{ABG16} states that if $P = \bigoplus_{k=0}^{n} p_k X^k$, $Q = \bigoplus_{k=0}^{n} q_k X^k \in \mathbb{R}_{\text{max}}[X]$ are such that $P \succ Q$ with $p_n = q_n$, and the corner roots of $P$ and $Q$ are respectively $\lambda_1 \succeq \lambda_2 \succeq \ldots \succeq \lambda_n$ and $\delta_1 \succeq \delta_2 \succeq \ldots \succeq \delta_n$, counted with multiplicities, then

$$\lambda_1 \cdots \lambda_k \succeq \delta_1 \cdots \delta_k \quad \forall k \in [n].$$

If in addition $p_0 = q_0$, then (6.7) becomes an equality for $k = n$.

Consider, as in the proof of Corollary 6.8, the polynomials $P, Q, f \in \mathbb{R}_{\text{max}}[X]$ of degree $n$ with $P = f_M$, $f = f_A$ and $Q = f_A$ in case (6.6a) and $Q$ as in (6.5a) and (6.5b) in cases (6.6b) and (6.6c) respectively. By Theorem 6.4, we have $P \succ Q$. Moreover $p_n = q_n = 1$ in all cases. So (6.7) holds in all cases for the corner roots $\lambda_1 \succeq \ldots \succeq \lambda_n$ of $P$ and $\delta_1 \succeq \ldots \succeq \delta_n$ of $Q$. The corner roots of $P$ are the eigenvalues of $M$ by definition. Moreover, the corner roots of $Q$ are the eigenvalues of $A$ in Case (6.6a) since $Q = f_A$.

It is easy to see from the definition of corner roots and multiplicities, that Point (2) of Lemma 6.3 holds true for $\mathcal{T} = \mathbb{R}_{\text{max}}$, that is the corner roots of $Q$ are the images by $g$ of the corner roots of $f$ and that in addition the multiplicities coincide. So in Cases (6.4b) and (6.4c), the corner roots of $Q$ are the images by $g$ of the eigenvalues of $A$, so $\delta_i = g(\gamma_i)$. This shows the first assertion of the corollary.

Now, in Case (6.4b), we have $p_0 = \det(A^V)$ and $q_0 = \det(A)^{-1}$. By Point (2) of Proposition 4.1 applied to $\mathcal{T} = \mathbb{R}_{\text{max}}$, we obtain that $p_0 = q_0$, hence (6.7) becomes an equality for $k = n$, which shows the last assertion of the corollary. \qed

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