Off-Diagonal Long Range Order and Scaling in a Disordered Quantum Hall System

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Abstract

We have numerically studied the bosonic off-diagonal long range order, introduced by Read to describe the ordering in ideal quantum Hall states, for noninteracting electrons in random potentials confined to the lowest Landau level. We find that it also describes the ordering in disordered quantum Hall states: the proposed order parameter vanishes in the disordered ($\sigma_{xy} = 0$) phase and increases continuously from zero in the ordered ($\sigma_{xy} = e^2/h$) phase. We study the scaling of the order parameter and find that it is consistent with that of the one-electron Green’s function.
The quantum Hall effect (QHE) is a consequence of novel correlated states that arise in a two dimensional electron gas placed in a transverse magnetic field [1]. In ideal, i.e. translationally invariant, systems these states exist at isolated filling factors at which the system is incompressible. In systems with impurities these states broaden into phases — ranges of filling factor ($\nu$) which exhibit the same transport properties as the parent ideal states — thus giving rise to the characteristic plateaux structure of the QHE. The nature of the ordering in the ideal states was elucidated by Girvin and MacDonald (GM) [2] and later by Zhang, Hansson and Kivelson (ZHK) [3] and by Read [4], who showed that the Laughlin states could be viewed as condensates of composite bosons consisting of electrons that “carry” flux. The GM/ZHK formulation is distinct from that of Read and there is no proof that they are equivalent. In this paper we will be concerned, for reasons of computational convenience, solely with Read’s formulation; we comment briefly on the GM case at the end.

In this work we address the following questions: Is Read’s bosonic off-diagonal long range order (ODLRO) a property of real, dirty quantum Hall systems, i.e. is it non-zero in the entire phase descended from an ideal state and does it vanish outside its boundaries? (That the ODLRO vanishes for a clean system when the Laughlin states are destabilized by varying the electron-electron interaction was shown already by Rezayi and Haldane [5].) We emphasize that at issue here is whether the ODLRO can serve as a sufficient characterization of the real systems that exhibit the QHE; note that incompressibility is lost when disorder is introduced into the system. Anticipating that the ODLRO does survive the introduction of disorder, we are led to the derivative questions of the critical behavior of the Read bosons near the transition between neighboring Hall plateaux, “seen” by them as a superfluid-insulator transition [6], and its relationship to the conventional measures of delocalization.

We will present likely answers to these questions based on numerical studies of non-interacting electrons subject to random potentials which are confined to the lowest Landau level (LLL) — a problem for which the localization properties of single-particle states have been investigated extensively [7]. This choice may seem perplexing on account of the historical association of the ODLRO with the fractional states, but we remind the reader that
the integer Hall state at $\nu = 1$, the filled LLL, is just the first state in the Laughlin sequence $1, 1/3, 1/5 \ldots$ and does exhibit ODLRO; more generally, there is no distinction between the integer QHE and fractional QHE in this regard and we expect our qualitative results to hold quite generally. We note that even for non-interacting electrons the ODLRO is a property of the many-body state, and is not a one-electron quantity averaged over the occupied single-particle states.

Most of the calculations are done on the sphere $\mathbb{S}$ using a density per flux quantum, $\rho_i$, of delta-function scatterers, and a modified version of Read’s operator introduced by Rezayi and Haldane $\mathbb{R}$. For a sphere containing $N_\phi = 2S + 1$ flux quanta the single-particle states can be labeled by the eigenvalues, $m$, of $L_z$ which run from $-S$ to $S$. If $L_z$ generates rotations about the point $x$, then Read’s operator for the $\nu = 1$ state $\mathbb{R}$ takes the form

$$\phi_R^\dagger(x; S) = c_{m=S+1/2}^\dagger F(x; S).$$

(1)

Here $F(x; S)$ is the flux insertion operator at $x$, which replaces a state of the system with $N_\phi$ flux quanta and occupations $n_{N_\phi}(m)$ by a state of the system with $N_\phi + 1$ flux quanta and occupations $n_{N_\phi+1}(m - 1/2) = n_{N_\phi}(m)$. Note that the electron creation operator $c^\dagger$ acts on the states of the $N_\phi + 1$ flux system. The intuitive content of this definition is more easily understood in the planar disk geometry, where the action of $\phi_R^\dagger(0)$ is the insertion of one flux quantum at the origin causing each of the single-particle states to move outwards into its neighbor and the subsequent injection of an electron into the central Gaussian orbital. At $\nu = 1$ it is clear that this operation takes the $N$ particle ground state to the $N + 1$ particle ground state, much as the operation of the field operator on the $N$ particle ground state of a superfluid produces a state with an $O(1)$ overlap with the $N + 1$ particle ground state. For dirty systems, it is not immediately evident that (in obvious notation), $\langle N + 1|\phi_R^\dagger(x)|N \rangle$ continues to be $O(1)$ for all $\nu$ at which $\sigma_{xy} = e^2/h$ or that it vanishes when $\sigma_{xy} = 0$. There is, however, a suggestive connection of $\phi_R^\dagger$ with the flux insertion in Laughlin’s gauge argument $\mathbb{L}$. The latter can be interpreted as the statement that flux insertion followed by transferring an electron between edges has no effect on the ground state in the $\nu = 1$ phase.
but it is important to note that in the gauge argument the electrons in the localized states are unaffected by adiabatic flux insertion while in the action of $\phi_R^\dagger$ all the electrons are affected. We note that, in the gauge argument, the inertness of the localized electrons is essential in order to get a quantized $\sigma_{xy}$. Consequently, and in contrast, the expectation value of $\phi_R^\dagger$ is not quantized.

The principal numerical results concern the disorder-averaged absolute value of the antipodal correlator:

$$G_R(\nu, S) = \left| \langle \phi_R(\theta = 0) \phi_R^\dagger(\theta = \pi) \rangle \right|_{\text{dis}},$$

where $\theta = 0, \pi$ denote the north and south poles, respectively, and the expectation value $\langle \rangle$ is evaluated in the $N_e$-electron ground state. (Note that the filling factor $\nu = N_e/N_\phi$.) We find:

1. In the thermodynamic limit ($S \to \infty$), for $\nu \leq 1/2$, $G_R(\nu, S) \to 0$, while for $\nu > 1/2$, $G_R(\nu, S) \to m_R^2(\nu) \neq 0$, so that $\phi_R^\dagger$ defines a legitimate order parameter for the dirty system.

As seen from Fig. 1, $m_R^2$ vanishes continuously on approaching $\nu = 1/2$ from above. The critical point is at $\nu = 1/2$ because the random potentials employed were strictly particle-hole symmetric. We have also calculated the disorder-averaged absolute value of the order parameter matrix element, $M_R = \left| \langle N_e + 1 | \phi_R^\dagger(\theta = 0) | N_e \rangle \right|_{\text{dis}}$, and checked that it behaves as expected. The antipodal correlator is, however, easier to calculate and vanishes more rapidly (exponentially with $S$ rather than as $1/S$) in the low-$\nu$ phase, so it was investigated more thoroughly. For other practical reasons, we have focused on the sphere, rather than the disk or torus geometries.

Previous work on destruction of the ODLRO in the ideal states [2,4,5] identified the unbinding of zeroes of the wavefunction from the particles as the relevant mechanism. In our problem there is always one zero per particle due to fermi statistics; the ODLRO is destroyed instead by the delocalization of extra zeroes (quasiholes) introduced by varying $\nu$. Heuristically, the variation of $m_R$ with $\nu$ appears to reflect a geometrical property of the states, namely the extent of a “percolating” cluster of $\nu = 1$ liquid. The quantized Hall
conductance, however, is sensitive only to the existence of the cluster.

(2) For $1/2 < \nu < 1$ one expects a state in which randomly localized quasiholes are interspersed with a $\nu = 1$ condensate. In the bosonic description, this state resembles a vortex glass \cite{12} where the phase of the order parameter at a given point in space fluctuates randomly between disorder configurations on account of fluctuations in the locations of the vortices (quasiholes). We have confirmed that not taking the absolute value in Eq. (2) causes the disorder average to vanish at all filling factors $0 < \nu < 1$.

(3) The data in the critical region admit a finite-size scaling analysis consistent with the critical behavior found in previous studies of one-electron measures of localization. The appropriate scaling ansatz is

$$G_R \sim S^{-n/2} g[(\nu - \frac{1}{2})S^x],$$

where $x = 1/(2\nu_\xi)$ and $\nu_\xi$ is the correlation length exponent. Note that the antipodal distance varies as the square root of $S$. We estimate $\nu_\xi > 2$ and $\eta \simeq 1.6$; consequently $m^2_R(\nu) \sim (\nu - 1/2)^{2\beta}$ where $2\beta = \nu_\xi \eta > 3.2$. However, the analysis is complicated by a slow crossover, apparently reflecting a weakly irrelevant operator. The analysis is described below.

(4) In order to further test the basic question of the order parameters for dirty quantum Hall systems, we calculated the antipodal correlations for the $\nu = 1/3$ order parameter (i.e., insertion of three flux quanta at a point followed creation of an electron there) in the noninteracting system. For finite $S$ (and at $\rho_\xi = 4$), the correlations are nonzero, with a maximum at a filling somewhat less than 1/3. The maximum value of the correlation function decays rapidly as the system size increases, perhaps as $1/S^3$. These results support the notion that there is a unique bosonic order parameter associated with each quantum Hall phase.

We now turn to the details of the scaling analysis for $G_R$. We have data for $G_R$, as well as the equal-time one-electron Green’s function, at $\rho_\xi = 2, 4, 8, 16$ and for system sizes up to $2S = 320$. At any given $\rho_\xi$, Eq. (3) describes the data well but the estimated $\nu_\xi$ (but not
\( \eta \) drifts substantially with the impurity concentration.

Let us focus on the \( \rho_i = 4 \) data for the moment. The band-center data exhibit systematic deviations from scaling at the smaller system sizes, see Fig. 2; we estimate \( \eta/2 = 0.78(2) \) based on the data for \( 2S \geq 120 \), but that value may be too low \[13\]. Likewise, a log-log plot of \( dG_R/d\nu \) at \( \nu = 1/2 \) (determined by fitting, as were the band-center values themselves) versus \( 2S \) exhibits curvature. However, the ratio of the slope to the value scales nicely, as shown in Fig. 2, leading to an estimate of \( x = 0.344(10) \), corresponding to \( \nu_\xi = 1.45(4) \). The data collapse in the corresponding scaling plot, Fig. 3, is evident. This estimate of \( \nu_\xi \) above is significantly different from the value 2.34(4) determined by Huckestein and Kramer \[7\] from the spatial decay of the Schrödinger Green’s function. A similar analysis for the other impurity densities yields \( \nu_\xi = 1.00(2), 1.64(5), 1.67(5) \) for \( \rho_i = 2, 8, 16 \) respectively.

These estimates suggest that \( \rho_i \) is associated with a weakly irrelevant operator responsible for a slow crossover to the “quantum percolation” fixed point \[14\], perhaps flowing from the classical percolation fixed point. In an attempt to investigate the issue of the correlation length exponent more closely we were led to calculations of the disorder-averaged absolute value of the antipodal Green’s function, \( G_e(\nu, S) \), for the same systems. Here we can only summarize the findings, which will be reported in detail elsewhere. At \( \rho_i = 4 \) and for system sizes up to \( 2S = 241 \), we find that there is excellent data collapse (see Fig. 4) with \( \nu_\xi = 1.64(3) \). For the remaining impurity densities we estimate \( \nu_\xi = 1.38(3), 2.02(7), 2.16(8) \) for \( \rho_i = 2, 8, 16 \) respectively. As the two sets of estimates for \( \nu_\xi \), from \( G_R \) and \( G_e \), show the same systematics we do not view the difference between them as significant \[15\].

A comment on the stability of our estimate for \( \eta \) is in order. This stability is somewhat surprising given the crossover effects in our estimates of \( \nu_\xi \). One possible explanation relies upon the numerically plausible identification of \( \eta \) with the fractal dimension of the critical eigenstates which has been estimated to be approximately 1.6 \[13\] and is not greatly different from the value 1.75 for classical percolation. If the latter is indeed the point of departure for the crossover the robustness of \( \eta \) becomes plausible.

To recapitulate, we have established that Read’s operator does define a legitimate order
parameter for the $\sigma_{xy} = e^2/h$ disordered quantum Hall system and by analogy, for other quantum Hall systems. At the transition between the ordered ($\sigma_{xy} = e^2/h$) and disordered ($\sigma_{xy} = 0$) phases, we find that the ODLRO of the Read bosons exhibits a scaling behavior consistent with that of the one-electron properties. It is worth noting that the ordered phase is, in terms of the bosonic description, a superfluid state that resembles a vortex glass while the disordered phase resembles a bose glass; hence, there is a very suggestive analogy to the field tuned transition in dirty superconductors as discussed by Fisher [17] in his treatment of the latter [18].

Finally, let us note some speculative implications for a couple of related issues. First, the consistent scaling of the ODLRO and Green’s function implies, roughly speaking, that multiplication of the wavefunctions of the disordered Bose problem by the Jastrow factor $\Pi_{ij}(z_i - z_j)$ is innocuous (at least as far as the correlation length exponent is concerned). This appears to give support to the claim of Jain, Kivelson and Trivedi [19] that their “composite fermion” wavefunctions for the fractional QHE yield transitions in the same universality class as those for the integer QHE. Second, the GM/ZHK bosonization differs from Read’s in that the former involves only the phase of the above Jastrow factor and that the resulting ODLRO is algebraic, even in the ideal states. Our cluster interpretation of the Read correlator results suggests that the GM correlator in the disordered phase will be characterized by the same $\nu_\xi$. However, it is less clear whether the GM correlator exhibits the same algebraic decay throughout the ordered phase, and we believe that this remains an interesting topic for future work.

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A somewhat puzzling point is that for different \( \rho_i \) the curves of \( G_R \) versus \( \nu \) at the same \( S \) exhibit crossings very close to half filling, even for \( 2S \) as small as 20. Indeed, the entire distribution function for the absolute-value of the antipodal correlations appears to be universal close to \( \nu = 1/2 \). This is puzzling only because we are manifestly not in the scaling regime.

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Two remarks are in order. At a fixed impurity density a generic renormalization group scenario would require that different operators define correlation lengths that diverge with the same exponent which strengthens our belief that the observed discrepancy is a crossover effect. However, we have not observed the crossover directly as a function of system size. We are currently extending our studies to larger system sizes in an attempt to verify this directly.

The critical eigenstates are multifractal; hence the relevant quantity is the generalized dimension \( D(2) \), see W. Pook and M. Janssen, Z. Phys. B 82, 295 (1991) and H. Aoki, Phys. Rev. B 33, 7310 (1986).

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One can also define a Read operator for the holes (vortices), \( \phi_H^\dagger(x; S) = F(x; S) \), which condenses for \( \nu < 1/2 \) and vanishes for \( \nu > 1/2 \). This dual description of the transition is a feature of any QH transition and also of the purely bosonic description of the vortex glass/insulator transition (previous reference).

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FIGURES

FIG. 1. The antipodal Read correlator, $G_R(\nu, S)$, plotted versus $\nu$ for $\rho_i = 4$ at $2S = 20, 60,$ and 180 (from top to bottom). The data are averages over 3600 samples; the statistical errors are less than 1%. Inset: $G_R(\nu, S)$ plotted versus $2S$ for $\nu = 0.9, 0.5,$ and 0.1, from top to bottom. Note that the $\nu = 0.9$ data saturates at large $S$, while at $\nu = 0.1$ it vanishes exponentially with $S^{1/2}$.

FIG. 2. Double logarithmic plots of $G_R(1/2, S)$ ($\times$), $dG_R/d\nu(1/2, S)$ ($\diamond$), and their ratio (+), versus $2S$. The line is a least-squares fit to the data for the ratio ($2S = 40$ through $320$); the latter have been shifted vertically to accommodate them on the plot.

FIG. 3. Scaling plot, of $G_R(\nu, S)/G_R(1/2, S)$ versus $S^{x}(\nu - 1/2)$ at $\rho_i = 4$, for $2S = 20, 40,$ $80, 100, 140, 180, 240, 320$ and $|\nu - 1/2| < 0.1$. The continuous curve is the data at $2S = 320$. The value of $x$ used is 0.342, corresponding to $\nu_\xi = 1.46$.

FIG. 4. Scaling plot of the Green’s function, $G_e(\nu, S)$, data at $\rho_i = 4$ for all $\nu$ and $2S = 41, 61,$ $81, 101, 141, 181, 241$. The ordinate is $(\ln[G_e(1/2, S)/G_e(\nu, S)])^{1/2}$ and the abscissa is $S^{x}(\nu - 1/2)$ with $x = 0.31$ corresponding to $\nu_\xi = 1.6$. Note that $G_e(\nu, S) = G_e(1 - \nu, S)$. The different curves correspond to different $S$ values. The data are plotted as curves rather than as points for clarity; the error bars are smaller than the width of the curves except for points near the origin.
