Inertial Stochastic PALM and its Application for Learning Student-\(t\) Mixture Models

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Abstract

Inertial algorithms for minimizing nonsmooth and nonconvex functions as the inertial proximal alternating linearized minimization algorithm (iPALM) have demonstrated their superiority with respect to computation time over their non inertial variants. In many problems in imaging and machine learning, the objective functions have a special form involving huge data which encourage the application of stochastic algorithms. While the stochastic gradient descent algorithm is still used in the majority of applications, recently also stochastic algorithms for minimizing nonsmooth and nonconvex functions were proposed. In this paper, we derive an inertial variant of the SPRING algorithm, called iSPRING, and prove linear convergence of the algorithm under certain assumptions. Numerical experiments show that our new algorithm performs better than SPRING or its deterministic counterparts, although the improvement for the inertial stochastic approach is not as large as those for the inertial deterministic one.

The second aim of the paper is to demonstrate that (inertial) PALM both in the deterministic and stochastic form can be used for learning the parameters of Student-\(t\) mixture models. We prove that the objective function of such models fulfills all convergence assumptions of the algorithms and demonstrate their performance by numerical examples.

1. Introduction

Recently, duality concepts were successfully applied to minimize nonsmooth and nonconvex functions appearing in certain applications in image and data processing. A frequently

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applied algorithm in this direction is the proximal alternating linearized minimization algorithm (PALM) by Bolte, Teboulle and Sabach [5] based on results in [1, 3]. Pock and Sabach [36] realized that the convergence speed of PALM can be considerably improved by inserting some non-expensive inertial steps and called the accelerated algorithm iPALM. In data driven approaches in imaging and machine learning, parts of the objective function can be often written as sum of a huge number of functions sharing the same structure. In general the computation of the gradient of these parts is too time and storage consuming so that stochastic gradient approximations were applied, see, e.g. [6] and the references therein. A combination of the simple stochastic gradient descent (SGD) estimator with PALM was first discussed by Xu and Yin in [43]. The authors refer to their method as block stochastic gradient iteration and do not mention the connection to PALM. Under rather hard assumptions on the objective function $F$, they proved that the sequence $(x_k)_k$ produced by their algorithm is such that $\mathbb{E}\left(\text{dist}(0, \partial F(x_k))\right)$ converges to zero as $k \to \infty$. Another idea for a stochastic variant of PALM was proposed by Davis et al. [12]. The authors introduce an asynchronous variant of PALM with stochastic noise in the gradient and called it SAPALM. Assuming an explicit bound of the variance of the noise, they proved certain convergence results. Further, we like to mention that a stochastic variant of the primal-dual algorithm of Chambolle and Pock [10] for solving convex problems was developed in [9].

Replacing the simple stochastic gradient descent estimators by more sophisticated so-called variance-reduced gradient estimators, Driggs et al. [15] could weaken the assumptions on the objective function and improve the estimates on the convergence rate of a stochastic PALM algorithm. They called the corresponding algorithm SPRING. Note that the advantages of variance reduction to accelerate stochastic gradient methods were discussed by several authors, see, e.g. [20, 37].

In this paper, we merge the SPRING algorithm with an inertial procedure to obtain a new iSPRING algorithm. We examine its convergence behavior both theoretically and numerically. Under certain assumptions on the parameters of the algorithm which also appear in the iPALM algorithm, we show that iSPRING converges linearly. In particular, we have to adapt the definition of variance-reduced gradient estimators to the sequence produced by iSPRING. In the numerical part, we focus on two examples, namely (i) sparse principal component analysis (PCA), and (ii) parameter learning for Student-t mixture models (MMs).

PCA is a basic tool for data reduction and we refer to [33] as one of the first papers on this topic. There exists a huge amount of work aiming to make the original model more robust against outliers and to enforce a sparse dimension of the affine subspace the data
will be projected to, see, e.g. [24, 25, 30, 40]. For comparison, we restrict our attention in this paper to the sparse PCA model applied in [15].

Learned MMs provide a powerful tool in data and image processing. While Gaussian MMs are mostly used in the field, more robust methods can be achieved by using heavier tailed distributions, as, e.g. the Student-t distribution. In [41], it was shown that Student-t MMs are superior to Gaussian ones for modeling image patches and the authors proposed an application in image compression. Image denoising based on Student-t models was addressed in [23] and image deblurring in [14, 44]. Further applications include robust image segmentation [4, 32, 39] and superresolution [19] as well as registration [16, 45].

For learning MMs a maximizer of the corresponding log-likelihood has to be computed. Usually an expectation maximization (EM) algorithm [22, 28, 34] or certain of its acceleration [7, 29, 42] are applied for this purpose. However, if the MM has many components and we are given large data, a stochastic optimization approach appears to be more efficient. Indeed, recently, also stochastic variants of the EM algorithm were proposed [8, 11], but show various disadvantages and we are not aware of a circumvent convergence result for these algorithms. In particular, one assumption on the stochastic EM algorithm is that the underlying distribution family is an exponential family, which is not the case for MMs. In this paper, we propose for the first time to use the (inertial) PALM algorithms as well as their stochastic variants for maximizing a modified version of the log-likelihood function. So far, the model is smooth so that the algorithms can be considered as block gradient descent algorithms. However, we show that the objective function fulfills all assumptions required for the convergence results of (inertial) PALM, respectively its stochastic variants, so that we have a theoretical convergence guarantee.

This paper is organized as follows: In Section 2, we provide the notation used throughout the paper. To understand the differences of existing algorithms to our novel one, we discuss PALM and iPALM together with convergence results in Section 3. Section 4 contains their stochastic variants, where iSPRING is new. We discuss the convergence behavior of iSPRING in Section 5. In Section 6, we propose a model for learning the parameters of Student-t MMs based on its log-likelihood function. We show how (inertial) PALM and its stochastic variants (inertial) SPRING can be used for optimization. Further, we prove that our model fulfills the assumptions on the convergence of these algorithms. Section 7 compares the performance of the four algorithms for two examples. Finally, conclusions are drawn and directions of further research are addressed in Section 8.
2. Preliminaries

In this section, we introduce the basic notation and results which we will use throughout this paper.

For a proper and lower semi-continuous function \( f : \mathbb{R}^d \to (-\infty, \infty] \) and \( \tau > 0 \) the proximal mapping \( \text{prox}_\tau^f : \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d) \) is defined by

\[
\text{prox}_\tau^f (x) = \arg\min_{y \in \mathbb{R}^d} \left\{ \frac{\tau}{2} \| x - y \|^2 + f(y) \right\},
\]

where \( \mathcal{P}(\mathbb{R}^d) \) denotes the power set of \( \mathbb{R}^d \). The proximal mapping admits the following properties, see e.g. [38].

**Proposition 2.1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be proper and lower semi-continuous with \( \inf_{\mathbb{R}^d} f > -\infty \). Then, the following holds true.

(i) The set \( \text{prox}_\tau^f (x) \) is nonempty and compact for any \( x \in \mathbb{R}^d \) and \( \tau > 0 \).

(ii) If \( f \) is convex, then \( \text{prox}_\tau^f (x) \) contains exactly one value for any \( x \in \mathbb{R}^d \) and \( \tau > 0 \).

To describe critical points, we will need the definition of (general) subgradients.

**Definition 2.2.** Let \( f : \mathbb{R}^d \to (-\infty, \infty] \) be a proper and lower semi-continuous function and \( v \in \mathbb{R}^d \). Then we call

(i) \( v \) a regular subgradient of \( f \) at \( \bar{x} \), written \( v \in \hat{\partial}f(\bar{x}) \), if for all \( x \in \mathbb{R}^d \),

\[
f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|).
\]

(ii) \( v \) a (general) subgradient of \( f \) at \( \bar{x} \), written \( v \in \partial f(\bar{x}) \), if there are sequences \( x_k \to \bar{x} \) and \( v_k \in \partial f(x_k) \) with \( v_k \to v \) as \( k \to \infty \).

The following proposition lists useful properties of subgradients.

**Proposition 2.3** (Properties of Subgradients). Let \( f : \mathbb{R}^d \to (-\infty, \infty] \) and \( g : \mathbb{R}^d \to (-\infty, \infty] \) be proper and lower semicontinuous and let \( h : \mathbb{R}^d \to \mathbb{R} \) be continuously differentiable. Then the following holds true.

(i) For any \( x \in \mathbb{R}^d \), we have \( \hat{\partial}f(x) \subseteq \partial f(x) \). If \( f \) is additionally convex, we have \( \hat{\partial}f(x) = \partial f(x) \).

(ii) For \( x \in \mathbb{R}^d \) with \( f(x) < \infty \), it holds

\[
\hat{\partial}(f + h)(x) = \hat{\partial}f(x) + \nabla h(x) \quad \text{and} \quad \partial(f + h)(x) = \partial f(x) + \nabla h(x).
\]
(iii) If $\sigma(x_1, x_2) = f_1(x_1) + f_2(x_2)$, then
\[
\left( \frac{\partial x_1 f_1(\bar{x}_1)}{\partial x_2 f_2(\bar{x}_2)} \right) \subseteq \partial \sigma(\bar{x}_1, \bar{x}_2) \quad \text{and} \quad \left( \frac{\partial x_1 f_1(\bar{x}_1)}{\partial x_2 f_2(\bar{x}_2)} \right) \subseteq \partial \sigma(\bar{x}_1, \bar{x}_2).
\]

Proof. Part (i) was proved in [38, Theorem 8.6 and Proposition 8.12] and part (ii) in [38, Exercise 8.8]. Concerning part (iii) we have for $v_x \in \partial_x f(\bar{x}_i), i = 1, 2$ that for all $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ it holds
\[
\sigma(x_1, x_2) = f_1(x_1) + f_2(x_2) \geq \sum_{i=1}^{2} f_i(\bar{x}_i) + (v_{x_1}, x_1 - \bar{x}_i) + o(\|x_1 - \bar{x}_i\|).
\]

This proves the claim for regular subgradients.

For general subgradients consider $v_{x_i} \in \partial_x f_i(\bar{x}_i), i = 1, 2$ By definition there exist sequences $x_i^k \rightarrow \bar{x}_i$ and $v_{x_i}^k \rightarrow v_{x_i}$ with $v_{x_i}^k \in \partial_x f_i(x_i^k), i = 1, 2$. By the statement for regular subgradients we know that $(v_{x_1}^k, v_{x_2}^k) \in \partial \sigma(x_1^k, x_2^k)$. Thus, it follows by definition of the general subgradient that $(v_{x_1}, v_{x_2}) \in \partial \sigma(\bar{x}_1, \bar{x}_2)$.

We call $(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ a critical point of $F$ if $0 \in \partial F(x_1, x_2)$. By [38, Theorem 10.1] we have that any local minimizer $\hat{x}$ of a proper and lower semi-continuous function $f : \mathbb{R}^d \rightarrow (-\infty, \infty]$ fulfills
\[
0 \in \partial f(\hat{x}) \subseteq \partial f(\hat{x}).
\]

In particular, it is a critical point of $f$. Further, we have by Proposition 2.3 that $\hat{x} \in \text{prox}^f(x)$ implies
\[
0 \in \tau(\hat{x} - x) + \partial f(y) \subseteq \tau(\hat{x} - x) + \partial f(y).
\]

In this paper, we consider functions $F : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow (-\infty, \infty]$ of the form
\[
F(x_1, x_2) = H(x_1, x_2) + f(x_1) + g(x_2)
\]

with proper, lower semicontinuous functions $f : \mathbb{R}^{d_1} \rightarrow (-\infty, \infty]$ and $g : \mathbb{R}^{d_2} \rightarrow (-\infty, \infty]$ bounded from below and a continuously differentiable function $H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}$. Further, we assume throughout this paper that
\[
F := \inf_{(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} F(x_1, x_2) > -\infty.
\]
By Proposition 2.3 it holds
\[
\begin{pmatrix}
\partial_{x_1} F(x_1, x_2) \\
\partial_{x_2} F(x_1, x_2)
\end{pmatrix} \nabla H(x_1, x_2) + \begin{pmatrix}
\partial_{x_1} f(x_1) \\
\partial_{x_2} g(x_2)
\end{pmatrix}
\subseteq \nabla H(x_1, x_2) + \partial (f + g)(x_1, x_2) = \partial F(x_1, x_2).
\] (3)

The generalized gradient of \( F: \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to (-\infty, \infty] \) was defined in [15] as set-valued function
\[
\mathcal{G} F_{\tau_1, \tau_2}(x_1, x_2) := \begin{pmatrix}
\tau_1 (x_1 - \text{prox}^{f}_{\tau_1}(x_1 - \frac{1}{\tau_1} \nabla x_1 H(x_1, x_2))) \\
\tau_2 (x_2 - \text{prox}^{g}_{\tau_2}(x_2 - \frac{1}{\tau_2} \nabla x_2 H(x_1, x_2)))
\end{pmatrix}.
\]

To motivate this definition, note that \( 0 \in \mathcal{G} F_{\tau_1, \tau_2}(x_1, x_2) \) is a sufficient criterion for \((x_1, x_2)\) being a critical point of \( F \). This can be seen as follows: For \((x_1, x_2) \in \mathcal{G} F_{\tau_1, \tau_2}(x_1, x_2)\) we have
\[
x_1 \in \text{prox}^{f}_{\tau_1}(x_1 - \frac{1}{\tau_1} \nabla x_1 H(x_1, x_2)).
\]
Using (1), this implies
\[
0 \in \tau_1 (x_1 - x_1 + \frac{1}{\tau_1} \nabla x_1 H(x_1, x_2)) + \partial f(x_1) = \nabla x_1 H(x_1, x_2) + \partial f(x_1).
\]
Similarly we get \( 0 \in \nabla x_2 H(x_1, x_2) + \partial g(x_2) \). By (3) we conclude that \((x_1, x_2)\) is a critical point of \( F \).

3. PALM and iPALM

In this section, we review PALM [5] and its inertial version iPALM [36].

3.1. PALM

The following Algorithm 3.1 for minimizing (2) was proposed in [5].

To prove convergence of PALM the following additional assumptions on \( H \) are needed:

Assumption 3.1 (Assumptions on \( H \)). (i) For any \( x_1 \in \mathbb{R}^{d_1} \), the function \( \nabla x_2 H(x_1, \cdot) \) is globally Lipschitz continuous with Lipschitz constant \( L_2(x_1) \). Similarly, for any \( x_2 \in \mathbb{R}^{d_2} \), the function \( \nabla x_2 H(\cdot, x_2) \) is globally Lipschitz continuous with Lipschitz constant \( L_1(x_2) \).
Algorithm 3.1 Proximal Alternating Linearized Minimization (PALM)

Input: \((x_0^1, x_0^2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\), parameters \(\tau_k^1, \tau_k^2\) for \(k \in \mathbb{N}_0\).

for \(k = 0, 1, \ldots\) do

Set
\[ x_1^{k+1} \in \text{prox}_{\tau_k^1}^f(x_1^k - \frac{1}{\tau_k^1} \nabla x_1 H(x_1^k, x_2^k)) \]

Set
\[ x_2^{k+1} \in \text{prox}_{\tau_k^2}^g(x_2^k - \frac{1}{\tau_k^2} \nabla x_2 H(x_1^{k+1}, x_2^k)) \]

(ii) There exist \(\lambda^-_1, \lambda^-_2, \lambda^+_1, \lambda^+_2 > 0\) such that
\[
\inf\{L_1(x_2^k) : k \in \mathbb{N}\} \geq \lambda^-_1 \quad \text{and} \quad \inf\{L_2(x_1^k) : k \in \mathbb{N}\} \geq \lambda^-_2 \\
\sup\{L_1(x_2^k) : k \in \mathbb{N}\} \leq \lambda^+_1 \quad \text{and} \quad \sup\{L_2(x_1^k) : k \in \mathbb{N}\} \leq \lambda^+_2.
\]

Remark 3.2. Assume that \(H \in \mathcal{C}^2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\) fulfills assumption 3.1(i). Then, the authors of [5] showed, that there are partial Lipschitz constants \(L_1(x_2)\) and \(L_2(x_1)\), such that Assumption 3.1(ii) is satisfied.

The following theorem was proven in [5, Lemma 3, Theorem 1]. For the definition of KL functions see Appendix A. Here we just mention that semi-algebraic functions are KL functions, see, e.g. [5].

Theorem 3.3 (Convergence of PALM). Let \(F : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to (-\infty, \infty]\) be given by (2), fulfills the Assumptions 3.1 and that \(\nabla H\) is Lipschitz continuous on bounded subsets of \(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\). Let \((x_1^k, x_2^k)_k\) be the sequence generated by PALM, where the step size parameters fulfill
\[ \tau_k^1 \geq \gamma_1 L_1(x_2^k), \quad \tau_k^2 \geq \gamma_2 L_2(x_1^{k+1}) \]
for some \(\gamma_1, \gamma_2 > 1\). Then, for \(\eta := \min\{(\gamma_1-1)\lambda^-_1, (\gamma_2-1)\lambda^-_2\}\), the sequence \((F(x_1^k, x_2^k))_k\) is nonincreasing and
\[
\frac{\eta}{2} \|(x_1^{k+1}, x_2^{k+1}) - (x_1^k, x_2^k)\|_2^2 \leq F(x_1^k, x_2^k) - F(x_1^{k+1}, x_2^{k+1}).
\]
If \(F\) is in addition a KL function and the sequence \((x_1^k, x_2^k)_k\) is bounded, then it converges to a critical point of \(F\).
3.2. iPALM

To speed up the performance of PALM the inertial variant iPALM in Algorithm 3.2 was suggested in [36].

**Algorithm 3.2** Inertial Proximal Alternating Linearized Minimization (iPALM)

Input: \((x_1^{-1}, x_2^{-1}) = (x_1^0, x_2^0) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\), parameters \(\alpha_1^k, \alpha_2^k, \beta_1^k, \beta_2^k, \tau_1^k, \tau_2^k\) for \(k \in \mathbb{N}_0\).

for \(k = 0, 1, \ldots\) do

Set

\[
y_1^k = x_1^k + \alpha_1^k(x_1^k - x_1^{k-1}) \\
z_1^k = x_1^k + \beta_1^k(x_1^k - x_1^{k-1}) \\
x_1^{k+1} \in \text{prox}_{\tau_1^k}(y_1^k - \frac{1}{\tau_1^k} \nabla_{x_1} H(z_1^k, x_2^k)).
\]

Set

\[
y_2^k = x_2^k + \alpha_2^k(x_2^k - x_2^{k-1}) \\
z_2^k = x_2^k + \beta_2^k(x_2^k - x_2^{k-1}) \\
x_2^{k+1} \in \text{prox}_{\tau_2^k}(y_2^k - \frac{1}{\tau_2^k} \nabla_{x_2} H(x_1^{k+1}, z_2^k)).
\]

To prove the convergence of iPALM the parameters of the algorithm must be carefully chosen.

**Assumption 3.4** (Conditions on the Parameters of iPALM). Let \(\lambda^+_i\), \(i = 1, 2\) and \(L_1(x_2^k), L_2(x_1^k)\) be defined by Assumption 3.1. There exists some \(\epsilon > 0\) such that for all \(k \in \mathbb{N}\) and \(i = 1, 2\) the following holds true:

(i) There exist \(0 < \bar{\alpha}_i < \frac{1-\epsilon}{2}\) such that \(0 \leq \alpha_i^k \leq \bar{\alpha}_i\) and \(0 < \beta_i \leq 1\) such that \(0 \leq \beta_i^k \leq \bar{\beta}_i\).

(ii) The parameters \(\tau_1^k\) and \(\tau_2^k\) are given by

\[
\tau_1^k := \frac{(1+\epsilon)\delta_1 + (1+\bar{\beta}_1)L_1(x_1^k)}{1-\alpha_1^k} \quad \text{and} \quad \tau_2^k := \frac{(1+\epsilon)\delta_2 + (1+\bar{\beta}_2)L_2(x_1^{k+1})}{1-\alpha_2^k},
\]

and for \(i = 1, 2\),

\[
\delta_i := \frac{\bar{\alpha}_i + \bar{\beta}_i}{1-\epsilon-2\bar{\alpha}_i}\lambda^+_i.
\]

The following theorem was proven in [36, Theorem 4.1].
Theorem 3.5 (Convergence of iPALM). Let $F : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to (-\infty, \infty]$ given by (2) be a KL function. Suppose that $H$ fulfills the Assumptions 3.1 and that $\nabla H$ is Lipschitz continuous on bounded subsets of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Further, let the parameters of iPALM fulfill the parameter conditions 3.4. If the sequence $(x_1^k, x_2^k)_k$ generated by iPALM is bounded, then it converges to a critical point of $F$.

Remark 3.6. Even though we cited PALM and iPALM just for two blocks $(x_1, x_2)$ of variables, the convergence proofs from [5] and [36] even work with more than two blocks.

4. Stochastic Variants of PALM and iPALM

For many problems in imaging and machine learning the function $H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ in (2) is of the form

$$H(x_1, x_2) = \frac{1}{n} \sum_{i=1}^{n} h_i(x_1, x_2),$$

where $n$ is large. Then the computation of the gradients in PALM and iPALM is very time consuming. Therefore stochastic approximations of the gradients were considered in the literature.

4.1. Stochastic PALM and SPRING

The idea to combine stochastic gradient estimators with a PALM scheme was first discussed by Xu and Yin in [43]. The authors replaced the gradient in Algorithm 3.1 by the stochastic gradient descent (SGD) estimator

$$\hat{\nabla}_x H(x_1, x_2) := \frac{1}{b} \sum_{j \in B} \nabla_x h_j(x_1, x_2),$$

where $B \subset \{1, \ldots, n\}$ is a random subset (mini-batch) of fixed batch size $b = |I|$. This gives Algorithm 4.1 which we call SPALM.

Xu and Yin showed under rather strong assumptions, in particular $f, g$ have to be Lipschitz continuous and the variance of the SGD estimator has to be bounded, that there exists a subsequence $(x_1^k, x_2^k)_k$ of iterates generated by Algorithm 4.1 such that the sequence $\mathbb{E} \left( \text{dist}(0, \partial F(x_1^k, x_2^k)) \right)$ converges to zero as $k \to \infty$. If $F$, $f$ and $g$ are strongly convex, the authors proved also convergence of the function values to the infimum of $F$. Driggs et al. [15] could weaken the assumptions and improve the convergence rate by replacing the SGD estimator by so-called variance-reduced gradient estimators. Let $\mathbb{E}_k =$
Algorithm 4.1 Stochastic Proximal Alternating Linearized Minimization (SPALM/SPRING)

Input: \((x_1^0, x_2^0) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\), parameters \(\tau_1^k, \tau_2^k\) for \(k \in \mathbb{N}_0\).

for \(k = 0, 1, \ldots\) do

\[x_1^{k+1} \in \text{prox}_{\tau_1^k} \left( x_1^k - \frac{1}{\tau_1^k} \nabla_{x_1} H(x_1^k, x_2^k) \right).\]

Set

\[x_2^{k+1} \in \text{prox}_{\tau_2^k} \left( x_2^k - \frac{1}{\tau_2^k} \nabla_{x_2} H(x_1^{k+1}, x_2^k) \right).\]

\(\mathbb{E}(|(x_1^1, x_1^2), (x_1^2, x_1^3), \ldots, (x_1^s, x_1^s)|)\) be the conditional expectation on the first \(k\) sequence elements. Then these estimators have to fulfill the following properties.

**Definition 4.1** (Variance-Reduced Gradient Estimator). A gradient estimator \(\tilde{\nabla}\) is called variance-reduced for a differentiable function \(H: \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}\) with constants \(V_1, V_2, V_T \geq 0\) and \(\rho \in (0, 1]\), if for any sequence \((x^k)_k = (x_1^k, x_2^k)_k\) the following holds true:

(i) There exist random vectors \(v_i^k, k \in \mathbb{N}_0, i = 1, \ldots, s\) such that for \(\Upsilon_k = \sum_{i=1}^s \|v_i^k\|^2\),

\[
\mathbb{E}_k(\|\tilde{\nabla}_{x_1} H(x_1^k, x_2^k) - \nabla_{x_1} H(x_1^k, x_2^k)\|^2 + \|\tilde{\nabla}_{x_2} H(x_1^{k+1}, x_2^k) - \nabla_{x_2} H(x_1^{k+1}, x_2^k)\|^2) \\
\leq \Upsilon_k + V_1(\mathbb{E}_k(\|x^{k+1} - x^k\|^2) + \|x^k - x^{k-1}\|^2) \\
\text{and for } \Gamma_k = \sum_{i=1}^s \|v_i^k\|,
\]

\[
\mathbb{E}_k(\|\tilde{\nabla}_{x_1} H(x_1^k, x_2^k) - \nabla_{x_1} H(x_1^k, x_2^k)\| + \|\tilde{\nabla}_{x_2} H(x_1^{k+1}, x_2^k) - \nabla_{x_2} H(x_1^{k+1}, x_2^k)\|) \\
\leq \Gamma_k + V_2(\mathbb{E}_k(\|x^{k+1} - x^k\|) + \|x^k - x^{k-1}\|). \\
\]

(ii) The sequence \((\Upsilon_k)_k\) decays geometrically, that is

\[
\mathbb{E}_k(\Upsilon_{k+1}) \leq (1 - \rho) \Upsilon_k + V_T(\mathbb{E}_k(\|x^{k+1} - x^k\|^2) + \|x^k - x^{k-1}\|).
\]

(iii) If \(\lim_{k \to \infty} \mathbb{E}(\|x^k - x^{k-1}\|^2) = 0\), then it holds \(\mathbb{E}(\Upsilon_k) \to 0\) and \(\mathbb{E}(\Gamma_k) \to 0\) as \(k \to \infty\).

While the SGD estimator is not variance-reduced, many popular gradient estimators as the SAGA [13] and SARAH [31] estimators have this property. Since for many problems in image processing and machine learning the SAGA estimator is not applicable due to of its high memory requirements, we will focus on the SARAH estimator in this paper.
**Definition 4.2** (SARAH Estimator). The SARAH estimator reads for $k = 0$ as

$$\tilde{\nabla}_{x_i} H(x_1^0, x_2^0) = \nabla_{x_i} H(x_1^0, x_2^0).$$

For $k = 1, 2, \ldots$ we define random variables $p_i^k \in \{0, 1\}$ with $P(p_i^k = 0) = \frac{1}{p}$ and $P(p_i^k = 1) = 1 - \frac{1}{p}$, where $p \in (1, \infty)$ is a fixed chosen parameter. Further, we define $B_i^k$ to be random subsets uniformly drawn from $\{1, \ldots, n\}$ of fixed batch size $b$. Then for $k = 1, 2, \ldots$ the SARAH estimator reads as

$$\tilde{\nabla}_{x_i} H(x_1^k, x_2^k) = \begin{cases} \nabla_{x_i} H(x_1^k, x_2^k), & \text{if } p_i^k = 0, \\ \frac{1}{b} \sum_{i \in B_i^k} \nabla_{x_i} h_i(x_1^k, x_2^k) - \nabla_{x_i} h_i(x_1^{k-1}, x_2^{k-1}) + \tilde{\nabla}_{x_i} H(x_1^{k-1}, x_2^{k-1}), & \text{if } p_i^k = 1, \end{cases}$$

and analogously for $\tilde{\nabla}_{x_2} H$. In the sequel, we assume that the family of the random elements $p_i^k, B_i^k$ for $i = 1, 2$ and $k = 1, 2, \ldots$ is independent.

The following proposition was shown in [15].

**Proposition 4.3.** Let $H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be given by (4) with functions $h_i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ having a globally $M$-Lipschitz continuous gradient. Then the SARAH gradient estimator is variance-reduced with parameters $\rho = \frac{1}{p}$ and $V_1 = V_T = 2M^2$, $V_2 = 2M$.

The convergence results in the next theorem were proven in [15]. We refer to the type of convergence in (6) as linear convergence. Note that the parameter $V_2$ from the definition of variance reductions does not appear in the theorem. Actually, Driggs et al. [15] need the assumption containing $V_2$ to prove tighter convergence rates for semi-algebraic functions $F$.

**Theorem 4.4** (Convergence of SPRING). Let $F$ by given as in (2), where $H$ fulfills the Assumptions 3.1. Let $\tilde{\nabla}$ be a variance-reduced estimator for $H$ with parameters $V_1, V_T \geq 0$ and $\rho \in (0, 1]$. Assume that $\bar{\gamma}_k := \max(\frac{1}{\tau_1}, \frac{1}{\tau_2})$ is non-increasing and that $0 < \beta := \inf_k \min(\frac{1}{\tau_1}, \frac{1}{\tau_2})$. Further suppose that for all $k \in \mathbb{N},$

$$\tilde{\gamma}_k \leq \frac{1}{16} \sqrt{\frac{M^2}{V_1 + \frac{V_T}{\rho}}} + \frac{16}{V_1 + \frac{V_T}{\rho}} - \frac{M}{16(V_1 + \frac{V_T}{\rho})}.$$

Let the stepsize in SPRING fulfill $\tau_1^k > 4\lambda_1^+$, $\tau_2^k > 4\lambda_2^+$ and set $\eta := \max(\frac{\tau_1^k}{4} - \lambda_1^+, \frac{\tau_2^k}{4} - \lambda_2^+)$. Then, with $t$ drawn uniformly from $\{0, \ldots, T - 1\}$, the generalized gradient at $(x_1^t, x_2^t)$ after
\( T \) iterations of SPRING satisfies
\[
\mathbb{E}(\|G_{F_1, F_2}(x_1^T, x_2^T)\|^2) \leq \frac{4(F(x_1^0, x_2^0) + \frac{2\gamma_0}{\rho} \Upsilon_0)}{T \eta \beta^2}.
\]

Furthermore, if for some \( \gamma > 0 \), the function \( F \) fulfills the error bound
\[
F(x_1, x_2) - F \leq \gamma \|G_{F_1, F_2}(x_1, x_2)\|^2
\]
for all \((x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\), then it holds after \( T \) iterations of SPRING that
\[
\mathbb{E}(F(x_1^T, x_2^T) - F) \leq (1 - \Theta)^T(F(x_1^0, x_2^0) - F + \frac{4\gamma_0}{\rho} \Upsilon_0),
\]
where \( \Theta = \min(\frac{4\eta \beta^2}{\delta^2}, \frac{\gamma}{2}) \). In particular, the sequence \( \mathbb{E}(F(x_1^T, x_2^T)) \) converges linearly to \( F \) as \( T \to \infty \).

Finally, we like to mention that Davis et al. [12] considered an asynchronous variant of PALM with stochastic noise in the gradient. Their approach requires an explicit bound on the noise, which is not fulfilled for the above gradient estimators. Thus, focus and setting in [12] differ from those of SPRING.

4.2. iSPRING

Inspired by the inertial PALM, we propose the inertial SPRING (iSPRING) algorithm outlined in Algorithm 4.2.

To prove that the generalized gradients on the sequence of iterates produced by iSRING converge to zero, some properties of the gradient estimator are required. The authors of [15] assumed that the estimators are evaluated at \((x_1^k, x_2^k)\) and \((x_1^{k+1}, x_2^{k})\), \(k \in \mathbb{N}_0\). In contrast, we require that the gradient estimators are evaluated at \((z_1^k, x_2^k)\) and \((z_1^{k+1}, z_2^k)\) for \(k \in \mathbb{N}_0\). To prove a counterpart of Theorem 4.4, we modify Definition 4.1. In particular, we need only the first part in (i).

**Definition 4.5** (Inertial Variance-Reduced Gradient Estimator). A gradient estimator \( \tilde{\nabla} \) is called inertial variance-reduced for a differentiable function \( H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R} \) with constants \( V_1, V_2 \geq 0 \) and \( \rho \in (0, 1] \), if for any sequence \( (x_1^k)_{k \in \mathbb{N}_0}, x^{-1} := x^0 \) and any \( 0 \leq \bar{\beta}^i < \bar{\beta}_i, i = 1, 2 \) there exists a sequence of random variables \( (\Upsilon_k)_{k \in \mathbb{N}} \) with \( \mathbb{E}(\Upsilon_1) < \infty \) such that following holds true:
Algorithm 4.2 Inertial Stochastic Proximal Alternating Linearized Minimization (iSPRING)

Input: \((x_1^{-1}, x_2^{-1}) = (x_0^1, x_0^2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\), parameters \(\alpha_i^k, \alpha_2^k, \beta_i^k, \beta_2^k, \tau_i^k, \tau_2^k\) for \(k \in \mathbb{N}_0\).

for \(k = 0, 1, \ldots\) do

Set

\[
y_1^k = x_1^k + \alpha_1^k(x_1^k - x_1^{k-1}) \\
z_1^k = x_1^k + \beta_1^k(x_1^k - x_1^{k-1}) \\
x_1^{k+1} \in \text{prox}^i_{\tau_1^k}(y_1^k - \frac{1}{\tau_1^k} \nabla x_1 H(z_1^k, x_2^k)).
\]

Set

\[
y_2^k = x_2^k + \alpha_2^k(x_2^k - x_2^{k-1}) \\
z_2^k = x_2^k + \beta_2^k(x_2^k - x_2^{k-1}) \\
x_2^{k+1} \in \text{prox}^i_{\tau_2^k}(y_2^k - \frac{1}{\tau_2^k} \nabla x_2 H(x_1^{k+1}, z_2^k)).
\]

(i) For \(z_i^k := x_i^k + \beta_i^k(x_i^k - x_i^{k-1}), i = 1, 2\), we have

\[
\mathbb{E}_k(||\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)||^2 + ||\nabla x_2 H(x_1^{k+1}, z_2^k) - \nabla x_2 H(x_1^{k+1}, z_2^k)||^2) \\
\leq \Upsilon_k + V_1 \left(\mathbb{E}_k(||x^{k+1} - x^k||^2) + ||x^k - x^{k-1}||^2 + ||x^{k-1} - x^{k-2}||^2\right).
\]

(ii) The sequence \((\Upsilon_k)_k\) decays geometrically, that is

\[
\mathbb{E}_k(\Upsilon_{k+1}) \leq (1 - \rho) \Upsilon_k + V_1(\mathbb{E}_k(||x^{k+1} - x^k||^2) + ||x^k - x^{k-1}||^2 + ||x^{k-1} - x^{k-2}||^2).
\]

(iii) If \(\lim_{k \to \infty} \mathbb{E}(||x^k - x^{k-1}||^2) = 0\), then \(\mathbb{E}(\Upsilon_k) \to 0\) as \(k \to \infty\).

To prove that the SARAH gradient estimator is inertial variance-reduced and that iSPRING converges, we need the following auxiliary lemma, which can be proved analogously to [36, Proposition 4.1].

Lemma 4.6. Let \((x_i^k, x_2^k)_k\) be an arbitrary sequence and \(\alpha_i^k, \beta_i^k \in \mathbb{R}, i = 1, 2\). Further define

\[
y_i^k := x_i^k + \alpha_i^k(x_i^k - x_i^{k-1}), \quad z_i^k := x_i^k + \beta_i^k(x_i^k - x_i^{k-1}), \quad i = 1, 2,
\]

and

\[
\Delta_i^k := \frac{1}{2}||x_i^k - x_i^{k-1}||^2, \quad i = 1, 2.
\]
Then, for any $k \in \mathbb{N}$ and $i = 1, 2$, we have

(i) $\|x_i^k - y_i^k\|^2 = 2(\alpha_i^k)^2 \Delta_i^k$,

(ii) $\|x_i^k - z_i^k\|^2 = 2(\beta_i^k)^2 \Delta_i^k$,

(iii) $\|x_i^{k+1} - y_i^k\|^2 \geq 2(1 - \alpha_i^k \Delta_i^{k+1} + 2\alpha_i^k)(\alpha_i^k - 1)\Delta_i^k$.

Now we can show the desired property of the SARAH gradient estimator.

**Proposition 4.7.** Let $H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be given by (4) with functions $h_i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ having a globally $M$-Lipschitz continuous gradient. Then the SARAH estimator $\tilde{\nabla}$ is inertial variance-reduced with parameters $\rho = \frac{1}{p}$ and

$$V_\Upsilon = 3(1 - \frac{1}{p})M^2 \left(1 + \max \left(\bar{\beta}_1^2, \bar{\beta}_2^2\right)\right).$$

Furthermore, we can choose

$$\Upsilon_{k+1} = \|\tilde{\nabla}_{x_1} H(z_{1}^k, x_{2}^k) - \nabla_{x_1} H(z_{1}^k, x_{2}^k)\|^2 + \|\tilde{\nabla}_{x_2} H(x_{1}^{k+1}, z_{2}^k) - \nabla_{x_2} H(x_{1}^{k+1}, z_{2}^k)\|^2.$$

The proof is given in Appendix B.

5. Convergence Analysis of iSPRING

We assume that the parameters of iSPRING fulfill the following conditions.

**Assumption 5.1** (Conditions on the Parameters of iSPRING). Let $\lambda_i^k$, $i = 1, 2$ and $L_1(x_2^k)$, $L_2(x_1^k)$ be defined by Assumption 3.1 and $\rho, V_1, V_\Upsilon$ by Definition 4.5. Further, let $H : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be given by (4) with functions $h_i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ having a globally $M$-Lipschitz continuous gradient. There exist $\epsilon, \varepsilon > 0$ such that for all $k \in \mathbb{N}$ and $i = 1, 2$ the following holds true:

(i) There exist $0 < \bar{\alpha}_i < \frac{1 - \epsilon}{2}$ such that $0 \leq \alpha_i^k \leq \bar{\alpha}_i$ and $0 < \bar{\beta}_i \leq 1$ such that $0 \leq \beta_i^k \leq \bar{\beta}_i$.

(ii) The parameters $\tau_i^k$, $i = 1, 2$ are given by

$$\tau_1^k := \frac{(1 + \epsilon)\delta_1 + M + L_1(x_2^k) + S}{1 - \alpha_1^k}, \quad \text{and} \quad \tau_2^k := \frac{(1 + \epsilon)\delta_2 + M + L_2(x_1^{k+1}) + S}{1 - \alpha_2^k}.$$
where \( S := \frac{4^2 \lambda_1^\dagger \lambda_1^\dagger + \varepsilon}{M} \) and for \( i = 1, 2, \)

\[
\delta_i := \frac{(M + \lambda_1^\dagger)\overline{\alpha}_i + 2\lambda_1^\dagger \overline{\beta}_i^2 + S}{1 - 2\overline{\alpha}_i - \varepsilon}.
\]

To analyze the convergence behavior of iSPRING, we start with an auxiliary lemma which can be proven analogously to [36, Lemma 3.2].

**Lemma 5.2.** Let \( \psi = \sigma + h \), where \( h: \mathbb{R}^d \to \mathbb{R} \) is a continuously differentiable function with \( L_h \)-Lipschitz continuous gradient, and \( \sigma: \mathbb{R}^d \to (-\infty, \infty] \) is proper and lower semicontinuous with \( \inf_{\mathbb{R}^d} \sigma > -\infty \). Then it holds for any \( u, v, w \in \mathbb{R}^d \) and any \( u^+ \in \mathbb{R}^d \) defined by

\[
u^+ \in \text{prox}_\theta(v - \frac{1}{t}\nabla h(w)), \quad t > 0
\]

that

\[
\psi(u^+) \leq \psi(u) + \langle u^+ - u, \nabla h(u) - \nabla h(w) \rangle + \frac{L_h^2}{2} \|u - u^+\|^2 + \frac{t}{2} \|u - v\|^2 - t \frac{1}{2} \|u^+ - v\|^2.
\]

Now we can establish a result on the expectation of squared subsequent iterates.

**Theorem 5.3.** Let \( F: \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to (-\infty, \infty] \) be given by (2) and fulfill Assumption 3.1. Let \( (x^k_1, x^k_2) \) be generated by iSPRING with parameters fulfilling Assumption 5.1, where we use an inertial variance-reduced gradient estimator \( \tilde{\nabla} \). Then it holds for \( \Psi: (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^3 \to \mathbb{R} \) defined for \( u = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}) \in (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^3 \) by

\[
\Psi(u) := F(u_{11}, u_{12}) + \frac{\delta_1}{T} \|u_{11} - u_{21}\|^2 + \frac{\delta_2}{T} \|u_{12} - u_{22}\|^2 + S \left( \|u_{21} - u_{31}\|^2 + \|u_{22} - u_{32}\|^2 \right)
\]

that there exists \( \gamma > 0 \) such that

\[
\Psi(u^1) - \inf_{u \in (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})^2} \Psi(u) + \frac{1}{M \rho} E(\Upsilon_1) \geq \gamma \sum_{k=0}^{T} E(\|u^{k+1} - u^k\|^2),
\]

where \( u^k := (x^k_1, x^k_2, x^{k-1}_1, x^{k-1}_2, x^{k-2}_1, x^{k-2}_2) \). In particular, we have

\[
\sum_{k=0}^{\infty} E(\|u^{k+1} - u^k\|^2) < \infty.
\]

**Proof.** By Lemma 5.2 with \( \psi := H(\cdot, x_2) + f \), we obtain

\[
H(x_1^{k+1}, x_2^k) + f(x_1^{k+1}) \leq H(x_1^k, x_2^k) + f(x_1^k)
\]
\[+ \langle x_1^{k+1} - x_1^k, \nabla x_1 H(x_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k) \rangle + \frac{L_1(x_2^k)}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{\tau}{2} \|x_1^k - y_1^k\|^2 - \frac{\tau^2}{2} \|x_1^{k+1} - y_1^k\|^2 \tag{7}\]

Using \(ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2\) for \(s > 0\) and \(\|a-c\|^2 \leq 2\|a-b\|^2 + 2\|b-c\|^2\) the inner product is smaller or equal than

\[
\frac{s^k}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{1}{s^k} \|\nabla x_1 H(x_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 \\
\leq \frac{s^k}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{1}{s^k} \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 \\
+ \frac{1}{s^k} \|\nabla x_1 H(x_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 \\
= \frac{s^k}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{1}{s^k} \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 + \frac{L_1(x_2^k)^2}{s^k} \|x_1^k - z_1^k\|^2.
\]

Combined with (7) this becomes

\[
H(x_1^{k+1}, x_2^k) + f(x_1^{k+1}) \\
\leq H(x_1^k, x_2^k) + f(x_1^k) + \frac{L_1(x_2^k)}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{\tau}{2} \|x_1^k - y_1^k\|^2 - \frac{\tau^2}{2} \|x_1^{k+1} - y_1^k\|^2 \\
+ \frac{s^k}{2} \|x_1^{k+1} - x_1^k\|^2 + \frac{1}{s^k} \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 + \frac{L_1(x_2^k)^2}{s^k} \|x_1^k - z_1^k\|^2.
\]

Using Lemma 4.6 we get

\[
H(x_1^{k+1}, x_2^k) + f(x_1^{k+1}) \\
\leq H(x_1^k, x_2^k) + f(x_1^k) + \left(L_1(x_2^k) + s^k - \tau_1^k(1 - \alpha_1^k)\right) \Delta_1^{k+1} \\
+ \frac{1}{s^k} \left(2L_2(x_2^k)^2(\beta_2^k)^2 + s^k \tau_1^k \alpha_1^k\right) \Delta_1^k + \frac{1}{s^k} \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2.
\]

Analogously we conclude for \(\psi := H(x_1, \cdot) + g\) that

\[
H(x_1^{k+1}, x_2^{k+1}) + g(x_2^{k+1}) \\
\leq H(x_1^{k+1}, x_2^k) + \left(L_2(x_2^{k+1}) + s^k - \tau_2^k(1 - \alpha_2^k)\right) \Delta_2^{k+1} \\
+ \frac{1}{s^k} \left(2L_2(x_2^{k+1})^2(\beta_2^k)^2 + s^k \tau_2^k \alpha_2^k\right) \Delta_2^k + \frac{1}{s^k} \|\nabla x_2 H(x_1^{k+1}, z_2^k) - \nabla x_2 H(x_1^{k+1}, z_2^k)\|^2.
\]

Adding the last two inequalities and using the abbreviation \(L_1^k := L_1(x_2^k)\) and \(L_2^k := L_2(x_2^{k+1})\), we obtain

\[
F(x_1^{k+1}, x_2^{k+1}) \leq F(x_1^k, x_2^k)
\]

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Reformulating (8) in terms of $s$

Now, we set $\Psi(\tau^k_s(L^k) - s^k - L^k - \delta_i) \Delta^{k+1} + \frac{1}{s^k_t} \left(2(L^k)^2(\beta^k)^2 + s^k_t \tau^k_s \alpha^k_t \Delta^k\right)$

\[+ \frac{1}{s^k_t} \left\|\nabla x, H(z^k, x^k_2) - \nabla x_1 H(z^k_1, x^k_2)\right\|^2 + \frac{1}{s^k_t} \left\|\nabla x_2 H(x^k_1 + 1, z^k_2) - \nabla x_2 H(x^k_1, z^k_2)\right\|^2. \tag{8}\]

Reformulating (8) in terms of $\Psi(u^k) = F(x^k_1, x^k_2) + \delta_1 \Delta^k_1 + \delta_2 \Delta^k_2 + \frac{S}{2} (\delta_1^{k-1} + \delta_2^{k-1}) \tag{9}$

leads to

\[\Psi(u^k) - \Psi(u^{k+1}) = F(x^k_1, x^k_2) - F(x^k_1, x^{k+1}_2) + \delta_1 \Delta^k_1 + \delta_2 \Delta^k_2 + \frac{S}{2} (\delta_1^{k-1} + \delta_2^{k-1}) \]

\[+ \frac{S}{2} \left(\Delta^{k-1}_1 + \Delta^{k-1}_2 - \Delta^k_1 - \Delta^k_2\right) \]

\[\geq \sum_{i=1}^{2} \left(\left(\tau^k_s(1 - \alpha^k_t) - s^k - L^k - \delta_i\right) \Delta^{k+1}_i\right) + \frac{2}{s^k_t} \left(\frac{\delta_i}{s^k_t} (L^k)^2(\beta^k)^2 - \tau^k_s \alpha^k_t \right) \Delta^k_i \]

\[+ \frac{1}{s^k_t} \left\|\nabla x, H(z^k, x^k_2) - \nabla x_1 H(z^k_1, x^k_2)\right\|^2 + \frac{1}{s^k_t} \left\|\nabla x_2 H(x^k_1 + 1, z^k_2) - \nabla x_2 H(x^k_1, z^k_2)\right\|^2 \]

\[+ \frac{S}{2} \left(\Delta^{k-1}_1 + \Delta^{k-1}_2 - \Delta^k_1 - \Delta^k_2\right). \tag{10}\]

Now, we set $s^k_1 = s^k_2 := M$ use that $L^k_i \leq M$, take the conditional expectation $\mathbb{E}_k$ in (10) and use that $\nabla$ is an inertial variance-reduced estimator to get

\[\Psi(u^k) - \mathbb{E}_k(\Psi(u^{k+1})) \]

\[\geq \sum_{i=1}^{2} \left(\left(\tau^k_s(1 - \alpha^k_t) - M - L^k - \delta_i\right) \mathbb{E}_k(\Delta^{k+1}_i)\right) + \left(\delta_i - \frac{2}{M} \left(L^k)^2(\beta^k)^2 - \tau^k_s \alpha^k_t \right)\right) \Delta^k_i \]

\[+ \frac{2}{M} \sum_{i=1}^{2} \left(\mathbb{E}_k(\Delta^{k+1}_i) + \Delta^k_i\right) - \frac{1}{M} \Upsilon_k + \frac{S}{2} \left(\Delta^{k-1}_1 + \Delta^{k-1}_2 - \Delta^k_1 - \Delta^k_2\right) \]

\[\geq \sum_{i=1}^{2} \left(\left(\tau^k_s(1 - \alpha^k_t) - M - L^k - \delta_i - \frac{2}{M} \right) \mathbb{E}_k(\Delta^{k+1}_i)\right) \]

\[+ \sum_{i=1}^{2} \left(\left(\delta_i - 2L^k(\beta^k)^2 - \tau^k_s \alpha^k_t - \frac{2}{M}\right)\right) \Delta^k_i \]

\[- \frac{1}{M} \Upsilon_k + \frac{S}{2} \left(\Delta^{k-1}_1 + \Delta^{k-1}_2 - \Delta^k_1 - \Delta^k_2\right). \tag{11}\]
Applying this in (13), we get

\[ \rho \Upsilon_k \leq \Upsilon_k - \mathbb{E}_k(\Upsilon_{k+1}) + 2V \sum_{i=1}^{2} \left( E_k(\Delta_{i}^{k+1}) + \Delta_{i}^{k} + \Delta_{i}^{k-1} \right). \]  

Inserting this in (11) and using the definition of \( S \) yields

\[
\Psi(u^k) - \mathbb{E}_k \left( \Psi(u^{k+1}) \right) \geq \sum_{i=1}^{2} \left( \left( \tau_{i}^{k}(1 - \alpha_{i}^{k}) - M - L_{i}^{k} \right) - \delta_{i} - \frac{S}{2} \right) \mathbb{E}_k(\Delta_{i}^{k+1}) \\
+ \sum_{i=1}^{2} \left( \left( \delta_{i} - 2L_{i}^{k}(\beta_{i}^{k})^{2} - \tau_{i}^{k}\alpha_{i}^{k} - \frac{S}{2} \right) \Delta_{i}^{k} \right) \\
- \frac{2V}{\rho M}(\Delta_{1}^{k-1} + \Delta_{2}^{k-1}) + \frac{1}{\rho M}(\mathbb{E}_k(\Upsilon_{k+1}) - \Upsilon_k) + \frac{S}{2} \left( \Delta_{1}^{k-1} + \Delta_{2}^{k-1} - \Delta_{i}^{k} - \Delta_{i}^{k} \right) \\
\geq \sum_{i=1}^{2} \left( \left( \tau_{i}^{k}(1 - \alpha_{i}^{k}) - M - L_{i}^{k} - \delta_{i} - S \right) \mathbb{E}_k(\Delta_{i}^{k+1}) \right) \\
+ \sum_{i=1}^{2} \left( \left( \delta_{i} - 2L_{i}^{k}(\beta_{i}^{k})^{2} - \tau_{i}^{k}\alpha_{i}^{k} - S \right) \mathbb{E}_k(\Delta_{i}^{k+1}) \right) \\
+ \frac{1}{\rho M}(\mathbb{E}_k(\Upsilon_{k+1}) - \Upsilon_k) + \left( \frac{S}{2} - \frac{2V}{\rho M} \right) (\Delta_{1}^{k-1} + \Delta_{2}^{k-1}).
\]  

Choosing \( \tau_{i}^{k}, \delta_{i}, i = 1, 2 \) and \( \epsilon \) as in Assumption 5.1(ii), we obtain by straightforward computation for \( i = 1, 2 \) and all \( k \in \mathbb{N} \) that \( a_{i}^{k} = \epsilon \delta_{i} \) and

\[
\begin{align*}
\beta_{i}^{k} &= \frac{1}{1 - \alpha_{i}^{k}} \left( (1 - \epsilon - 2\alpha_{i}^{k})\delta_{i} - \alpha_{i}^{k} M - S - L_{i}^{k} \left( 2(\beta_{i}^{k})^{2}(1 - \alpha_{i}^{k}) + \alpha_{i}^{k} \right) \right) + \epsilon \delta_{i} \\
&\geq \frac{1}{1 - \alpha_{i}^{k}} \left( (1 - \epsilon - 2\tilde{\alpha}_{i})\delta_{i} - \tilde{\alpha}_{i} M - S - L_{i}^{k} \left( 2(\tilde{\beta}_{i})^{2}(1 - \alpha_{i}^{k}) + \tilde{\alpha}_{i} \right) \right) + \epsilon \delta_{i} \\
&= \epsilon \delta_{i} + \frac{2\lambda_{i}^{k} \alpha_{i}^{k} (\tilde{\beta}_{i})^{2}}{1 - \alpha_{i}^{k}} \geq \epsilon \delta_{i}.
\end{align*}
\]

Applying this in (13), we get

\[
\Psi(u^k) - \mathbb{E}_k \left( \Psi(u^{k+1}) \right) \geq \epsilon \min(\delta_{1}, \delta_{2}) \sum_{i=1}^{2} \left( E_k(\Delta_{i}^{k+1}) + \Delta_{i}^{k} \right) \\
+ \frac{1}{\rho M}(\mathbb{E}_k(\Upsilon_{k+1}) - \Upsilon_k) + \left( \frac{S}{2} - \frac{2V}{\rho M} \right) (\Delta_{1}^{k-1} + \Delta_{2}^{k-1}).
\]
By definition of $S$ it holds $(\frac{2V}{pM} - \frac{S}{2}) \geq \varepsilon$. Thus, we get for $\gamma := \frac{1}{2} \min(\epsilon_1, \epsilon_2, \varepsilon)$ that

$$\Psi(u^k) - E_k(\Psi(u^{k+1})) \geq 2\gamma \sum_{i=1}^{2} (E_k(\Delta_i^{k+1}) + \Delta_i^k + \Delta_i^{k-1}) + \frac{1}{M\rho}(E_k(\Upsilon_{k+1}) - \Upsilon_k).$$

Taking the full expectation yields

$$E(\Psi(u^k) - \Psi(u^{k+1})) \geq \gamma E(||u^{k+1} - u^k||^2) + \frac{1}{M\rho}E(\Upsilon_{k+1} - \Upsilon_k), \quad (14)$$

and summing up for $k = 1, ..., T$,

$$E(\Psi(u^1) - \Psi(u^{T+1})) \geq \gamma \sum_{k=0}^{T} E(||u^{k+1} - u^k||^2) + \frac{1}{M\rho}E(\Upsilon_{T+1} - \Upsilon_1).$$

Since $\Upsilon_k \geq 0$, this yields

$$\gamma \sum_{k=0}^{T} E(||u^{k+1} - u^k||^2) \leq \Psi(u^1) - \inf_{u \in (R^d \times R^d)^2 \rightarrow \infty} \Psi(u) + \frac{1}{M\rho}E(\Upsilon_1) < \infty.$$ 

This finishes the proof. \qed

**Theorem 5.4.** Under the assumptions of Theorem 5.3 there exists some $C > 0$ such that

$$E \left( \text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2 \right) \leq C E(||u^{k+1} - u^k||^2) + 3E(\Upsilon_k).$$

In particular, it holds

$$E \left( \text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2 \right) \to 0 \quad \text{as} \ k \to \infty.$$ 

**Proof.** By definition of $x_1^{k+1}$, and (1) as well as Proposition 2.3 it holds

$$0 \in \pi_1^k(x_1^{k+1} - y_1^k) + \nabla x_1^k H(z_1^k, x_2^k) + \partial f(x_1^{k+1}).$$

This is equivalent to

$$\pi_1^k(y_1^k - x_1^{k+1}) + \nabla x_1^k H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1^k H(z_1^k, x_2^k) \in \nabla x_1^k H(x_1^{k+1}, x_2^{k+1}) + \partial f(x_1^{k+1}) \in \partial x_1^k F(x_1^{k+1}, x_2^{k+1}).$$
Analogously we get that
\[ \tau_2^k (y_2^k - x_2^{k+1}) + \nabla x_2 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1 H(x_1^{k+1}, z_2^k) \in \nabla x_2 H(x_1^{k+1}, x_2^{k+1}) + \partial g(x_2^{k+1}) \in \partial x_2 F(x_1^{k+1}, x_2^{k+1}). \]

Then we obtain by Proposition 2.3 that
\[ v := \left( \tau_1 (y_1^k - x_1^{k+1}) + \nabla x_1 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1 H(x_1^{k+1}, z_2^k) \right) \in \partial F(x_1^{k+1}, x_2^{k+1}), \]
and it remains to show that the squared norm of \( v \) is in expectation bounded by \( C \mathbb{E}(\|u^{k+1} - u^k\|^2) + 3 \mathbb{E}(\Upsilon_k) \) for some \( C > 0 \). Using \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) we estimate
\[ \|v\|^2 = \|\tau_1 (y_1^k - x_1^{k+1}) + \nabla x_1 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1 H(x_1^{k+1}, z_2^k)\|^2 \]
\[ + \|\tau_2^k (y_2^k - x_2^{k+1}) + \nabla x_2 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1 H(x_1^{k+1}, z_2^k)\|^2 \]
\[ \leq 3(\tau_1^2 \|y_1^k - x_1^{k+1}\|^2 + 3\|\nabla x_1 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_1 H(x_1^{k+1}, z_2^k)\|^2 \]
\[ + 3\|\nabla x_1 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_2 H(x_1^{k+1}, z_2^k)\|^2 \]
\[ + 3\|\nabla x_2 H(x_1^{k+1}, x_2^{k+1}) - \nabla x_2 H(x_1^{k+1}, z_2^k)\|^2. \]

Since \( \nabla H \) is \( M \)-Lipschitz continuous and \((a + b)^2 \leq 2(a^2 + b^2)\), we get further
\[ \|v\|^2 \leq 12(\tau_1^k)^2 \Delta_1^{k+1} + 6(\tau_1^k)^2 \|y_1^k - x_1^k\|^2 + 12(\tau_2^k)^2 \Delta_2^{k+1} + 6(\tau_2^k)^2 \|y_2^k - x_2^k\|^2 \]
\[ + 3M^2 \|x_1^{k+1} - z_1^k\|^2 + 6M^2 \Delta_2^{k+1} + 3M^2 \|x_2^{k+1} - z_2^k\|^2 \]
\[ + 3\left( \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 + \|\nabla x_2 H(x_1^{k+1}, z_2^k) - \nabla x_2 H(x_1^{k+1}, z_2^k)\|^2 \right). \]

Using Lemma 4.6 and the fact that \( \nabla H \) is inertial variance-reduced, this implies
\[ \|v\|^2 \leq 12(\tau_1^k)^2 \Delta_1^{k+1} + 12(\tau_1^k)^2 (\alpha_1^k)^2 \Delta_1^k + 12(\tau_2^k)^2 \Delta_2^{k+1} + 12(\tau_2^k)^2 (\alpha_2^k)^2 \Delta_2^k \]
\[ + 12M^2 \Delta_1^{k+1} + 6M^2 \|x_1^k - z_1^k\|^2 \]
\[ + 12M^2 \Delta_2^{k+1} + 6M^2 \|x_2^k - z_2^k\|^2 \]
\[ + 3\left( \|\nabla x_1 H(z_1^k, x_2^k) - \nabla x_1 H(z_1^k, x_2^k)\|^2 + \|\nabla x_2 H(x_1^{k+1}, z_2^k) - \nabla x_2 H(x_1^{k+1}, z_2^k)\|^2 \right) \]
\[ \leq 12 \left( (\tau_1^k)^2 + M^2 \right) \Delta_1^{k+1} + 12 \left( (\tau_1^k)^2 (\alpha_1^k)^2 + M^2 (\beta_1^k)^2 \right) \Delta_1^k \]

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+ \left(12(\tau_1^k)^2 + 18M^2\right) \Delta_2^{k+1} + 12\left((\tau_1^k)^2(\alpha_2^k)^2 + M^2(\beta_2^k)^2\right) \Delta_2^k
+ 3\left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \tilde{\nabla}_{x_1} H(z_1^k, x_2^k)\|^2 + \|\nabla_{x_2} H(x_1^{k+1}, z_2^k) - \tilde{\nabla}_{x_2} H(x_1^{k+1}, z_2^k)\|^2\right)
\leq C_0\|u^{k+1} - u^k\|^2
+ 3\left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \tilde{\nabla}_{x_1} H(z_1^k, x_2^k)\|^2\right)
+ 3\left(\|\nabla_{x_2} H(x_1^{k+1}, z_2^k) - \tilde{\nabla}_{x_2} H(x_1^{k+1}, z_2^k)\|^2\right),
\]

where

\[
C_0 = 12 \max \left((\tau_1^k)^2 + M^2, (\tau_1^k)^2(\alpha_2^k)^2 + M^2(\beta_2^k)^2, (\tau_1^k)^2 + \frac{3}{2}M^2, (\tau_1^k)^2(\alpha_2^k)^2 + M^2(\beta_2^k)^2\right).
\]

Noting that \(\text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1})) \leq \|v\|^2\), taking the conditional expectation \(\mathbb{E}_k\) and using that \(\tilde{\nabla}\) is inertial variance-reduced, we conclude

\[
\mathbb{E}_k \left(\text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2\right)
\leq \mathbb{E}_k \left(C_0\|u^{k+1} - u^k\|^2\right)
+ 3\mathbb{E}_k \left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \tilde{\nabla}_{x_1} H(z_1^k, x_2^k)\|^2 + \|\nabla_{x_2} H(x_1^{k+1}, z_2^k) - \tilde{\nabla}_{x_2} H(x_1^{k+1}, z_2^k)\|^2\right)
\leq \mathbb{E}_k \left((C_0 + 3V_1)\|u^{k+1} - u^k\|^2\right) + 3\Upsilon_k.
\]

Taking the full expectation on both sides and setting \(C := C_0 + 3V_1\) proves the claim. \(\square\)

**Theorem 5.5 (Convergence of iSPRING).** Under the assumptions of Theorem 5.3 it holds for \(t\) drawn uniformly from \(\{2, \ldots, T+1\}\) that there exists some \(0 < \sigma < \gamma\) such that

\[
\mathbb{E} \left(\text{dist}(0, \partial F(x_1^t, x_2^t))^2\right) \leq \frac{C}{T(\gamma - \sigma)} \left(\Psi(u^1) - \inf_{u \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \Psi(u) + \frac{3(\gamma - \sigma)}{\rho C} + \frac{1}{M\rho}\right) \mathbb{E}(\Upsilon_1).
\]

**Proof.** By (12), Theorem 5.4 and (14) it holds for \(0 < \sigma < \gamma\) that

\[
\mathbb{E} \left(\Psi(u^k) - \Psi(u^{k+1})\right) \geq \gamma \mathbb{E}(\|u^{k+1} - u^k\|^2) + \frac{1}{M\rho} \mathbb{E}(\Upsilon_{k+1} - \Upsilon_k)
\geq \sigma \mathbb{E}(\|u^{k+1} - u^k\|^2) + \frac{2}{C} \mathbb{E} \left(\text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2\right)
- \frac{3(\gamma - \sigma)}{C} \mathbb{E}(\Upsilon_k) + \frac{1}{M\rho} \mathbb{E}(\Upsilon_{k+1} - \Upsilon_k)
\geq \sigma \mathbb{E}(\|u^{k+1} - u^k\|^2) + \frac{2}{C} \mathbb{E} \left(\text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2\right)
+ \left(\frac{3(\gamma - \sigma)}{C} + \frac{1}{M\rho}\right) \mathbb{E}(\Upsilon_{k+1} - \Upsilon_k) - \frac{3(\gamma - \sigma)\rho\tau}{C\rho} \mathbb{E}(\|u^{k+1} - u^k\|^2).
\]
Choosing $\sigma := \frac{3(\gamma - \sigma)V}{\zeta p}$ yields
\[ E \left( \Psi(u^k) - \Psi(u^{k+1}) \right) \geq \frac{\gamma - \sigma}{C} E \left( \text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2 \right) - \left( \frac{3(\gamma - \sigma)}{\zeta p} + \frac{1}{\lambda p} \right) E (Y_{k+1} - Y_k). \]

Adding this up for $k = 1, \ldots, T$ we get
\[ E \left( \Psi(u^1) - \Psi(u^T) \right) \geq \frac{\gamma - \sigma}{C} \sum_{k=1}^{T} E \left( \text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2 \right) + \left( \frac{3(\gamma - \sigma)}{\zeta p} + \frac{1}{\lambda p} \right) E (Y_T - Y_1). \]

Since $Y_T \geq 0$ this yields for $t$ drawn randomly from $\{2, \ldots, T+1\}$ that
\[ E \left( \text{dist}(0, \partial F(x_1^t, x_2^t))^2 \right) = \frac{1}{T} \sum_{k=1}^{T} E \left( \text{dist}(0, \partial F(x_1^{k+1}, x_2^{k+1}))^2 \right) \leq \frac{C}{T(\gamma - \sigma)} \left( \Psi(u^1) - \inf_{u \in (R^d_1 \times R^d_2)^2} \Psi(u) + \left( \frac{3(\gamma - \sigma)}{\zeta p} + \frac{1}{\lambda p} \right) E (Y_1) \right). \]

This finishes the proof. $\square$

In [15] the authors proved global convergence of the objective function evaluated at the iterates of SPRING in expectation if the global error bound
\[ F(x_1, x_2) - F \leq \mu \text{dist}(0, \partial F(x_1, x_2))^2 \quad (15) \]
is fulfilled for some $\mu > 0$. Using this error bound, we can also prove global convergence of iSPRING in expectation with a linear convergence rate.

**Theorem 5.6** (Convergence of iSPRING). *Let the assumptions of Theorem 5.3 hold true. If in addition (15) is fulfilled, then there exists some $\Theta_0 \in (0, 1)$ and $\Theta_1 > 0$ such that
\[ E \left( F(x_1^{T+1}, x_2^{T+1}) - F \right) \leq \left( \Theta_0 \right)^T \left( \Psi(u^1) - F + \Theta_1 E(Y_1) \right). \]

In particular, it holds $\lim_{T \to \infty} E(F(x_1^T, x_2^T) - F) = 0$.\]
Proof. By (14) and Theorem 5.4, we obtain for $0 < d < \min(\gamma, \frac{C_{\mu\nu}}{1 - \rho})$ that

$$\mathbb{E} \left( \Psi(u^{k+1}) - F + \frac{1}{M_P} Y_{k+1} \right) \leq \mathbb{E} \left( \Psi(u^k) - F + \frac{1}{M_P} Y_k \right) - \gamma \mathbb{E}(\|u^{k+1} - u^k\|^2)$$

$$\leq \mathbb{E} \left( \Psi(u^k) - F + \frac{1}{M_P} Y_k \right)$$

$$- \frac{d}{\epsilon} \mathbb{E} \left( \text{dist}(0, \partial F(x_{k+1}, x_k)) \right)^2 + \frac{3d}{\epsilon} \mathbb{E}(Y_k)$$

$$- (\gamma - d) \mathbb{E}(\|u^{k+1} - u^k\|^2).$$

Using (12) in combination with the global error bound (15), we get

$$\mathbb{E} \left( \Psi(u^{k+1}) - F + \left( \frac{3d}{\epsilon C_\mu} + \frac{1}{M_P} \right) Y_{k+1} \right) \leq \mathbb{E} \left( \Psi(u^k) - F + \left( \frac{3d}{\epsilon C_\mu} + \frac{1}{M_P} \right) Y_k \right)$$

$$- \frac{d}{\epsilon M_P} \mathbb{E} \left( F(x_{k+1}, x_k) - F \right) - \left( \gamma - d - \frac{3dV_\gamma}{\epsilon C_\mu} \right) \mathbb{E}(\|u^{k+1} - u^k\|^2).$$

Setting $C_\gamma := \left( \frac{3d}{\epsilon C_\mu} + \frac{1}{M_P} \right)$ and applying the definition (9) of $\Psi$, this implies

$$\left( 1 + \frac{d}{\epsilon M_P} \right) \mathbb{E} \left( \Psi(u^{k+1}) - F \right) - \frac{d}{\epsilon M_P} \mathbb{E}\left( \delta_1 \Delta_{k+1}^1 + \delta_2 \Delta_{k+1}^2 + C_\gamma \mathbb{E}(Y_{k+1}) \right)$$

$$\leq \mathbb{E} \left( \Psi(u^k) - F \right) + C_\gamma \mathbb{E}(Y_k) - \left( \gamma - d - \frac{3dV_\gamma}{\epsilon C_\mu} \right) \mathbb{E}(\|u^{k+1} - u^k\|^2).$$

With $\delta := \max(\delta_1, \delta_2)$ and $\Delta_{k+1}^1 + \Delta_{k+1}^2 \leq \frac{1}{2} \|u^{k+1} - u^k\|^2$ we get

$$\left( 1 + \frac{d}{\epsilon M_P} \right) \mathbb{E} \left( \Psi(u^{k+1}) - F \right) + C_\gamma \mathbb{E}(Y_{k+1})$$

$$\leq \mathbb{E} \left( \Psi(u^k) - F \right) + C_\gamma \mathbb{E}(Y_k) - \left( \gamma - d - \frac{3dV_\gamma}{\epsilon C_\mu} - \frac{\delta}{2C_\mu} \right) \mathbb{E}(\|u^{k+1} - u^k\|^2).$$

Multiplying by $C_d := \frac{1}{1 + \frac{d}{\epsilon M_P}} = \frac{C_\mu}{C_\mu + d}$ this becomes

$$\mathbb{E} \left( \Psi(u^{k+1}) - F \right) + C_d C_\gamma \mathbb{E}(Y_{k+1}) \leq \frac{C_\mu}{C_\mu + d} \mathbb{E} \left( \Psi(u^k) - F \right) + C_\gamma C_d \mathbb{E}(Y_k)$$

$$- \frac{C_d}{C_\mu + d} \left( \gamma - d - \frac{3dV_\gamma}{\epsilon C_\mu} - \frac{\delta}{2C_\mu} \right) \mathbb{E}(\|u^{k+1} - u^k\|^2).$$

(16)

Since $d < \frac{C_{\mu\nu}}{1 - \rho}$ we know that $s := \frac{1 - C_d}{1 - \rho \mu + (\rho - 1)d} > 0$. Thus, adding $sC_d \gamma C_d$ times equation Definition 4.5 (ii) to (16) gives

$$\mathbb{E} \left( \Psi(u^{k+1}) - F \right) + (1 + s)C_d \gamma C_d \mathbb{E}(Y_{k+1}) \leq C_d \mathbb{E} \left( \Psi(u^k) - F \right) + (1 + s)C_d C_\gamma \mathbb{E}(Y_k)$$

$$+ C_d \left( V_\gamma sC_\gamma \left( \gamma - d - \frac{3dV_\gamma}{\epsilon C_\mu} - \frac{\delta}{2C_\mu} \right) \mathbb{E}(\|u^{k+1} - u^k\|^2),

= h(d),

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where we have used that $1 + (1 - \rho)s = C_d(1 + s)$. Since $s$ converges to 0 as $d \to 0$ we have that $\lim_{d \to 0} h(d) = -\gamma$. Thus we can choose $d > 0$ small enough, such that $h(d) < 0$. Then we get
\[ E\left(\Psi(u^{k+1}) - F\right) + (1 + s)C_T C_d E(\Upsilon_{k+1}) \leq C_d E\left(\Psi(u^k) - F + (1 + s)C_T C_d E(\Upsilon_k)\right). \]

Finally, setting $\Theta_0 := C_d$ and $\Theta_1 := (1 + s)C_T C_d$ and applying the last equation iteratively, we obtain
\[ E\left(\Psi(u^{T+1}) - F + \Theta_1 \Upsilon_{T+1}\right) \leq (\Theta_0)^T E\left(\Psi(u^1) - F + \Theta_1 \Upsilon_1\right). \]

Note that $\Psi(u^{T+1}) \geq F(x_1^{T+1}, x_2^{T+1})$ and that $\Upsilon_{T+1} \geq 0$. This yields
\[ E\left(F(x_1^{T+1}, x_2^{T+1}) - F\right) \leq (\Theta_0)^T E\left(\Psi(u^1) - F + \Theta_1 \Upsilon_1\right), \]
and we are done. 

6. Student-t Mixture Models

In this section, we show how PALM and its inertial and stochastic variants can be applied to learn Student-t MMs. To this end, we denote by $\text{Sym}(d)$ the linear space of symmetric $d \times d$ matrices, by $\text{SPD}(d)$ the cone of symmetric, positive definite $d \times d$ matrices and by $\Delta_K := \{\alpha = (\alpha_k)_{k=1}^K : \sum_{k=1}^K \alpha_k = 1, \alpha_k \geq 0\}$ the probability simplex in $\mathbb{R}^K$. The density function of the $d$-dimensional Student-$t$ distribution $T_\nu(\mu, \Sigma)$ with $\nu > 0$ degrees of freedom, location parameter $\mu \in \mathbb{R}^d$ and scatter matrix $\Sigma \in \text{SPD}(d)$ is given by
\[ f(x|\nu, \mu, \Sigma) = \frac{1}{\Gamma\left(\frac{d+\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi^d |\Sigma|^{\frac{d}{2}}} (1 + \frac{1}{\nu}(x - \mu)^T \Sigma^{-1}(x - \mu))^{\frac{d+\nu}{2}}} , \]
with the Gamma function $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t} \, dt$. The expectation of the Student-$t$ distribution is $E(X) = \mu$ for $\nu > 1$ and the covariance matrix is given by $\text{Cov}(X) = \frac{\nu}{\nu-2} \Sigma$ for $\nu > 2$, otherwise these quantities are undefined. The smaller the value of $\nu$, the heavier are the tails of the $T_\nu(\mu, \Sigma)$ distribution. For $\nu \to \infty$, the Student-$t$ distribution $T_\nu(\mu, \Sigma)$ converges to the normal distribution $N(\mu, \Sigma)$ and for $\nu = 0$ it is related to the projected normal distribution on the sphere $S^{d-1} \subset \mathbb{R}^d$. Figure 1 illustrates this behavior for the one-dimensional standard Student-$t$ distribution.

The construction of MMs arises from the following scenario: we have $K$ random number generators sampling from different distributions. Now we first choose one of the random
number generators randomly using the probability weights \( \alpha = (\alpha_1, ..., \alpha_K)^T \in \Delta_K \) and sample from the corresponding distribution. If all random number generators sample from Student-\( t \) distributions we arrive at Student-\( t \) MMs. More precisely, if \( Y \) is a random variable mapping into \( \{1, ..., K\} \) and \( X_1, ..., X_k \) are random variables with \( X_k \sim T_{\nu_k}(\mu_k, \Sigma_k) \), then the random variable \( X_Y \) is a Student-\( t \) MM with probability density function

\[
p(x) = \sum_{k=1}^{K} \alpha_k f(x|\nu_k, \mu_k, \Sigma_k), \quad \alpha \in \Delta_K.
\]

For samples \( \mathcal{X} = (x_1, ..., x_n) \), we aim to find the parameters of the Student-\( t \) MM by minimizing its negative log-likelihood function

\[
\mathcal{L}(\alpha, \nu, \mu, \Sigma|\mathcal{X}) = -\sum_{i=1}^{n} \log \left( \sum_{k=1}^{K} \alpha_k f(x_i|\nu_k, \mu_k, \Sigma_k) \right)
\]

subject to the parameter constraints. A first idea to rewrite this problem in the form (2) looks as

\[
F(\alpha, \nu, \mu, \Sigma) = H(\alpha, \nu, \mu, \Sigma) + f_1(\alpha) + f_2(\nu) + f_3(\mu) + f_4(\Sigma), \quad (17)
\]

where \( H := \mathcal{L} \), \( f_1 := \iota_{\Delta_K} \), \( f_2 := \iota_{\mathbb{S}^+_K} \), \( f_3 := 0 \), \( f_4 := \iota_{\text{PD}(d^K)} \), and \( \iota_{\mathcal{S}} \) denotes the indicator function of the set \( \mathcal{S} \) defined by \( \iota_{\mathcal{S}}(x) := 0 \) if \( x \in \mathcal{S} \) and \( \iota_{\mathcal{S}}(x) := \infty \) otherwise. Indeed one of the authors has applied PALM and iPALM to such a setting without any convergence guarantee in [19]. The problem is that \( \mathcal{L} \) is not defined on the whole Euclidean space and since \( \mathcal{L}(\alpha, \nu, \mu, \Sigma) \to \infty \) if \( \Sigma_k \to 0 \) for some \( k \), the function can
also not continuously extended to the whole \( \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{d \times K} \times \text{Sym}(d)^K \). Furthermore, the functions \( f_2 \) and \( f_4 \) are not lower semi-continuous. Consequently, the function (17) does not fulfill the assumptions required for the convergence of PALM and iPALM as well as their stochastic variants. Therefore we modify the above model as follows: Let \( \text{SPD}_\varepsilon(d) := \{ \Sigma \in \text{SPD}(d) : \Sigma \succeq \varepsilon I_d \} \). Then we use the surjective mappings \( \varphi_1 : \mathbb{R}^K \to \Delta_K \), \( \varphi_2 : \mathbb{R}^K \to \mathbb{R}^{K^2}_\varepsilon \) and \( \varphi_3 : \text{Sym}(d)^K \to \text{SPD}_\varepsilon(d)^K \) defined by

\[
\varphi_1(\alpha) := \frac{\exp(\alpha)}{\sum_{j=1}^K \exp(\alpha_j)}, \quad \varphi_2(\nu) := \nu^2 + \epsilon, \quad \varphi_3(\Sigma) := (\Sigma^T \Sigma_k + \epsilon I_d)^K_{k=1} \tag{18}
\]

to reshape problem (17) as the unconstrained optimization problem

\[
\arg\min_{\alpha \in \mathbb{R}^K, \nu \in \mathbb{R}^K, \mu \in \mathbb{R}^{d \times K}, \Sigma \in \text{Sym}(d)^K} H(\alpha, \nu, \mu, \Sigma) := \mathcal{L}(\varphi_1(\alpha), \varphi_2(\nu), \mu, \varphi_3(\Sigma)|X). \tag{19}
\]

Note that the functions \( f_i \), \( i = 1, \ldots, 4 \) are just zero.

For problem (19), PALM and iPALM reduce basically to block gradient descent algorithms as in Algorithm 6.1 and 6.2, respectively. Note that we use \( \beta_i^k = 0 \) for all \( k \) and \( \alpha_i^k = \rho^k \) for \( i = 1, \ldots, 4 \) as iPALM parameters in Algorithm 6.2. For the stochastic variants SPRING and iSPRING, we have just to replace the gradient by a stochastic gradient estimator.

**Algorithm 6.1** Proximal Alternating Linearized Minimization (PALM) for Student-t MMs

**Input:** \( x_1, \ldots, x_n \in \mathbb{R}^d \), \( \alpha^0 \in \mathbb{R}^K \), \( \nu^0 \in \mathbb{R}^K \), \( \mu^0 \in \mathbb{R}^{d \times K} \), \( \Sigma^0 \in \mathbb{R}^{d \times d \times K}, x_1^k, x_2^k, x_3^k, x_4^k \) for \( k \in \mathbb{N} \)

**for** \( k = 1, \ldots \) **do**

\( \alpha \)-Update:

\[
\alpha^{k+1} = \alpha^k - \frac{1}{\tau_1} \nabla_\alpha H(\alpha^k, \nu^k, \mu^k, \Sigma^k)
\]

\( \nu \)-Update:

\[
\nu^{k+1} = \nu^k - \frac{1}{\tau_2} \nabla_\nu H(\alpha^{k+1}, \nu^k, \mu^k, \Sigma^k)
\]

\( \mu \)-Update:

\[
\mu^{k+1} = \mu^k - \frac{1}{\tau_3} \nabla_\mu H(\alpha^{k+1}, \nu^{k+1}, \mu^k, \Sigma^k)
\]

\( \Sigma \)-Update:

\[
\Sigma^{k+1} = \Sigma^k - \frac{1}{\tau_4} \nabla_\Sigma H(\alpha^{k+1}, \nu^{k+1}, \mu^{k+1}, \Sigma^k)
\]

**end for**

Finally, we will show that \( H \) in (19)
Algorithm 6.2 Inertial Proximal Alternating Linearized Minimization (iPALM) for Student-t MMs

**Input:** $x_1, \ldots, x_n \in \mathbb{R}^d$, $\alpha^0 \in \mathbb{R}^K$, $\nu^0 \in \mathbb{R}^{dK}$, $\mu^0 \in \mathbb{R}^{d \times K}$, $\Sigma^0 \in \mathbb{R}^{d \times d \times K}$, $\rho^k \in [0, 1]$, $\tau_1^k, \tau_2^k, \tau_3^k, \tau_4^k$ for $k \in \mathbb{N}$

for $k = 1, \ldots, \tau_n$
do

$\alpha$-Update:

$\alpha^k = \alpha^k + \rho^k (\alpha^k - \alpha^{k-1})$

$\alpha^{k+1} = \alpha^k - \frac{1}{\tau_1} \nabla \alpha H(\alpha^k, \nu^k, \mu^k, \Sigma^k)$

$\nu$-Update:

$\nu^k = \nu^k + \rho^k (\nu^k - \nu^{k-1})$

$\nu^{k+1} = \nu^k - \frac{1}{\tau_2} \nabla \nu H(\alpha^{k+1}, \nu^k, \mu^k, \Sigma^k)$

$\mu$-Update:

$\mu^k = \mu^k + \rho^k (\mu^k - \mu^{k-1})$

$\mu^{k+1} = \mu^k - \frac{1}{\tau_3} \nabla \mu H(\alpha^{k+1}, \nu^{k+1}, \mu^k, \Sigma^k)$

$\Sigma$-Update:

$\Sigma^k = \Sigma^k + \rho^k (\Sigma^k - \Sigma^{k-1})$

$\Sigma^{k+1} = \Sigma^k - \frac{1}{\tau_4} \nabla \Sigma H(\alpha^{k+1}, \nu^{k+1}, \mu^{k+1}, \Sigma^k)$

end

- is a KL function which is bounded from below, and
- satisfies the Assumption 3.1(i).

Since $H \in C^2(\mathbb{R}^K \times \mathbb{R}^{d \times K} \times \mathbb{R}^{d \times K})$ we know by Remark 3.2 that Assumption 3.1(ii) is also fulfilled. Further, $\nabla H$ is continuous on bounded sets. Then, choosing the parameters of PALM, resp. iPALM as required by Theorem 3.3 resp. 3.5, we conclude that the sequences generated by both algorithms converge to a critical point of $H$ supposed that they are bounded. Similarly, if we assume in addition that the stochastic gradient estimators are variance-reduced, resp. inertial variance-reduced, we can conclude that the sequences of SPRING and iSPRING converge as in Theorem 4.4 resp. Theorems 5.5 and 5.6, if the corresponding requirements on the parameters are fulfilled.

We start with the KL property.

**Lemma 6.1.** The function $H : \mathbb{R}^K \times \mathbb{R}^{d \times K} \times \mathbb{R}^{d \times K} \times \mathbb{R}^{d \times K} \rightarrow \mathbb{R}$ defined in (19) is a KL function. Moreover, it is bounded from below.

**Proof.** 1. Since the Gamma function is real analytic, we have that $H$ is a combination of sums, products, quotients and concatenations of real analytic functions. Thus $H$ is real
analytic. This implies that it is a KL function, see [2, Remark 5] and [26, 27].

2. First, we proof that \( f(x|\nu, \mu, \Sigma) \) is bounded from above for \( \nu > \epsilon, \mu \in \mathbb{R}^d \) and \( \Sigma \succeq \epsilon I_d \).

By definition of the Gamma function and since

\[
\frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})} \nu^d \to 1 \quad \text{as} \quad \nu \to \infty
\]  

we have that (20) is bounded from below for \( \nu \in [\epsilon, \infty) \). Further, we see by assumptions on \( \nu \) and \( \Sigma \) that

\[
|\Sigma|^{-\frac{1}{2}} \leq \epsilon^{-\frac{d}{2}} \quad \text{and} \quad (1 + \frac{1}{\nu}(x - \mu)^T\Sigma^{-1}(x - \mu))^{-\frac{d+\nu}{2}} \leq 1.
\]

Thus, \( f(x|\nu, \mu, \Sigma) \) is the product of bounded functions and therefore itself bounded by some \( C > 0 \). This yields for \( \tilde{\alpha} = \varphi_1(\alpha), \tilde{\nu} = \varphi_2(\nu) \) and \( \tilde{\Sigma} = \varphi_3(\Sigma) \) that

\[
-\sum_{i=1}^{n} \log \left( \sum_{k=1}^{K} \tilde{\alpha}_k f(x_i|\tilde{\nu}_k, \tilde{\mu}_k, \tilde{\Sigma}_k) \right) \leq -\sum_{i=1}^{n} \log \left( \sum_{k=1}^{K} \tilde{\alpha}_k C \right) \leq -n \log C,
\]

which finishes the proof. \( \square \)

Here are the Lipschitz properties of \( H \).

**Lemma 6.2.** For \( H : \mathbb{R}^K \times \mathbb{R}^K \times \mathbb{R}^{d \times K} \times \text{Sym}(d)^K \to \mathbb{R} \) defined by (19) and all \( \alpha \in \mathbb{R}^K, \nu \in \mathbb{R}^K, \mu \in \mathbb{R}^{d \times K} \) and \( \Sigma \in \mathbb{R}^{d \times d \times K} \) we have that the gradients \( \nabla_\alpha H(\cdot, \nu, \mu, \Sigma), \nabla_\nu H(\alpha, \cdot, \mu, \Sigma), \nabla_\mu H(\alpha, \nu, \cdot, \Sigma), \) and \( \nabla_\Sigma H(\alpha, \nu, \mu, \cdot) \) are globally Lipschitz continuous.

The technical proof of the lemma is given in Appendix D.

### 7. Numerical Results

In this section, we apply iSPRING with SARAH gradient estimator (iSPRING-SARAH) for two different applications and compare it with PALM, iPALM and SARAH-SPRING. We run all our experiments on a HP Probook with Intel i7-8550U Quad Core processor and 8GB RAM. For the implementation we use Python and Tensorflow.

#### 7.1. Parameter Choice and Implementation Aspects

On the one hand, the algorithms based on PALM have many parameters which enables a high adaptivity of the algorithms to the specific problems. On the other hand, it is often hard to fit these parameters to ensure the optimal performance of the algorithms.

Based on approximations \( \hat{L}_1(x_2^k) \) and \( \hat{L}_2(x_1^{k+1}) \) of the partial Lipschitz constants \( L_1(x_2^k) \) and \( L_2(x_1^{k+1}) \) outlined below, we use the following step size parameters \( \tau_{i}^k, i = 1, 2 \):
For PALM and iPALM, we choose \( \tau_1^k = \tilde{L}_1(x_1^k, x_2^k) \) and \( \tau_2^k = \tilde{L}_2(x_1^{k+1}, x_2^k) \) which was also suggested in [5, 36].

For SPRING-SARAH and iSPRING-SARAH, we choose \( \tau_1^k = s\tilde{L}_1(x_1^k, x_2^k) \) and \( \tau_2^k = s\tilde{L}_2(x_1^{k+1}, x_2^k) \), where the manually chosen scalar \( s > 0 \) depends on the application. Note that the authors in [15] propose to take \( s = 2 \) which was not optimal in our examples.

Computation of Gradients and Approximative Lipschitz Constants

Since the global and partial Lipschitz constants of \( H \) are usually unknown, we estimate them locally using the second order derivative of \( H \) which exists in our examples. If \( H \) acts on a high dimensional space, it is often computationally too costly to compute the full Hessian matrix. Thus we compute a local Lipschitz constant only in the gradient direction, i.e. we compute
\[
\tilde{L}_i(x_1, x_2) := \|\nabla_{x_i}^2 H(x_1, x_2)\|, \quad g := \frac{\nabla_{x_i} H(x_1, x_2)}{\|\nabla_{x_i} H(x_1, x_2)\|} \tag{21}
\]

For the stochastic algorithms we replace \( H \) by the approximated function \( \tilde{H}(x_1, x_2) := \frac{1}{b}\sum_{i \in B_k} h_i(x_1, x_2) \), where \( B_k \) is the current mini-batch. The analytical computation of \( \tilde{L}_i \) in (21) is still hard. Even computing the gradient of a complicated function \( H \) can be error prone and laborious. Therefore, we compute the (partial) gradients of \( H \) or \( \tilde{H} \), respectively, using the reverse mode of algorithmic differentiation (also called backpropagation), see e.g. [17]. To this end, note that the chain rule yields that
\[
\|\nabla_{x_i} (\|\nabla_{x_i} H(x_1, x_2)\|^2)\| = 2\|\nabla_{x_i} H(x_1, x_2)\|\|\nabla_{x_i}^2 H(x_1, x_2)\nabla_{x_i} H(x_1, x_2)\|
\]
\[
= 2\|\nabla_{x_i} H(x_1, x_2)\|^2 \tilde{L}_i(x_1, x_2).
\]

Thus, we can compute \( \tilde{L}_i(x_1, x_2) \) by applying two times the reverse mode. If we neglect the taping, the execution time of this procedure can provably be bounded by a constant times the execution time of \( H \), see [17, Section 5.4]. Therefore, this procedure gives us an accurate and computationally very efficient estimation of the local partial Lipschitz constant.

Inertial Parameters

For the iPALM and iSPRING-SARAH we have to choose the inertial parameters \( \alpha_i^k \geq 0 \) and \( \beta_i^k \geq 0 \). With respect to our convergence results we have to assume that there exist \( \alpha_i^k \leq \bar{\alpha}_i < \frac{1}{2} \) and \( \beta_i^k \leq \bar{\beta}_i < 1, i = 1, 2 \). Note that for convex functions \( f \) and \( g \), the authors in [36] proved that the assumption on the \( \alpha \)'s can be lowered to \( \alpha_i^k \leq \bar{\alpha}_i < 1 \) and suggested to use \( \alpha_i^k = \beta_i^k = \frac{k-1}{k+2} \). Unfortunately,
we cannot show this for iSPRING and indeed we observe instability and divergence in iSPRING-SARAH, if we choose $\alpha_{i}^{k} > \frac{1}{2}$. Therefore, we choose for iSPRING-SARAH the parameters

$$\alpha_{i}^{k} = \beta_{i}^{k} = \frac{k - 1}{2(k + 2)}.$$  

**Initialization** We observed that SPRING-SARAH and iSPRING-SARAH show a slow convergence behavior for a poor initializations. Thus, we use a so-called warm start for the algorithms, similarly as in [21] and [15]. More precisely, we pre-process our initialization by performing two steps of PALM before comparing the algorithms.

### 7.2. Student-$t$ Mixture Models

We estimate the parameters of the Student-$t$ MM (19) with $K$ components and data points $\mathcal{X} = (x_1, \ldots, x_n) \in \mathbb{R}^{d \times n}$. We generate the data by sampling from a Student-$t$ MM as described above. The parameters of the ground truth MM are generate as follows:

- We generate $\alpha = \frac{\bar{\alpha}^2 + 10^{-3}}{\|\bar{\alpha}^2 + 10^{-3}\|_1}$, where the entries of $\bar{\alpha} \in \mathbb{R}^K$ are drawn independently from the standard normal distribution.

- We generate $\nu_i = \min(\bar{\nu}_i^2 + 0.1, 100)$, where $\bar{\nu}_i$, $i = 1, \ldots, n$ is drawn from a normal distribution with mean 0 and standard deviation 10.

- The entries of $\mu \in \mathbb{R}^{d \times K}$ are drawn independently from a normal distribution with mean 0 and standard deviation 2.

- We generate $\Sigma_i = \Sigma_i^T \Sigma_i + I$, where the entries of $\Sigma_i \in \mathbb{R}^{d \times d}$ are drawn independently from the standard normal distribution.

We initialize the algorithms by applying this procedure again. We run the algorithm for $n = 100000$ data points of dimension $d = 5$ and $K = 30$ components. We use a batch size of $b = 10000$. The resulting values of the negative log-likelihood function versus the number of epochs and the execution times, respectively, are given in Figure 2. One epoch contains for SPRING-SARAH and iSPRING-SARAH 10 steps and for PALM and iPALM 1 step. We see that in terms of the number of epochs as well as in terms of the execution time the iSPRING-SARAH is the fastest algorithm.
7.3. Sparse PCA

For comparing the performance of the algorithms, we focus on a special PCA model knowing that there exist more sophisticated approaches in the literature. For a given data matrix $A \in \mathbb{R}^{n \times d}$, we deal with the sparse PCA model considered in [15]:

$$(\hat{X}, \hat{Y}) = \arg\min_{X \in \mathbb{R}^{n \times r}, Y \in \mathbb{R}^{r \times d}} \|A - XY\|_F^2 + \lambda_1 \|\text{vec}(X)\|_1 + \lambda_2 \|\text{vec}(Y)\|_1,$$

where $\text{vec}(X)$ is the columnwise reshaping if of the matrix $X \in \mathbb{R}^{n \times r}$ into a vector of length $nr$. By penalizing the $\ell_1$, this model enforces sparsity in all components of $X$ and $Y$. For applying our algorithms, we set

$$F(X, Y) := H(X, Y) + f(X) + g(Y)$$

with $f(X) := \lambda_1 \|\text{vec}(X)\|_1$ and $g(Y) := \lambda_2 \|\text{vec}(Y)\|_1$ and

$$H(X, Y) := \|A - XY\|_F^2 = \sum_{i=1}^{d} h_i(X, Y), \quad h_i(X, Y) := \|A_i - X_i Y\|^2,$$
where $A_i$ and $X_i$ denote the $i$-th row of $a$ and $X$, respectively. Note that proximal mapping $\text{prox}_{\lambda_1}^f$ is given by the componentwise applied soft shrinkage functions

$$S_{\lambda_1} (x) := \begin{cases} 
  x - \frac{\lambda_1}{\tau_1} & x \geq \frac{\lambda_1}{\tau_1}, \\
  x + \frac{\lambda_1}{\tau_1} & x \leq -\frac{\lambda_1}{\tau_1}, \\
  0 & \text{otherwise.}
\end{cases}$$

For our numerical experiments we use $n = 1000000$, $d = 20$ and $r = 5$. We generate a data matrix $A$ with entries $a_{ij}$ randomly drawn from a uniform distribution on $[0, 50]$ and set $\lambda_1 = \lambda_2 = 0.1$. We initialize $X$ and $Y$ with entries uniformly drawn from the uniform distribution on $[0, 20]$. For the SPRING-SARAH and the iSPRING-SARAH algorithm we use a batch-size of 10000 and perform 10 steps per epoch. For PALM and iPALM we define an epoch to be exactly one step. Then, we apply PALM, iPALM, SPRING and iSPRING on $F$. We see that all of the algorithms converge. We plot the objective value against the number of epochs and the objective value against the execution time in Figure 3. We observe that in both cases the iSPRING-SARAH outperforms the other algorithms. However, the acceleration for the inertial stochastic variant is smaller than those achieved when making the deterministic PALM inertial.

8. Conclusions

We combined a stochastic variant of the PALM algorithm, called SPRING, with the inertial PALM algorithm to a new algorithm, called iSPRING. We analyzed the convergence behavior of iSPRING and proved similar convergence results as for SPRING, if the gradient estimators is inertial variance-reduced. In particular, we showed that the expected distance of the subdifferential to zero converges to zero for the sequence of iterates generated by iSPRING. Additionally the sequence of function values achieves linear convergence for functions satisfying a global error bound. We proved that a modified version of the negative log-likelihood function of Student-$t$ MMs fulfills all necessary convergence assumption of PALM, iPALM. We demonstrated the performance of iSPRING for two quite different applications. In the numerical comparison, it turns out that iSPRING shows the best performance of all four algorithms, although the improvement by using the inertial variant is in the stochastic setting lower than in the deterministic one.

For future work, it would be interesting to compare the performance of the iSPRING algorithm with more classical algorithms for estimating the parameters of Student-$t$ MMs,
in particular with the EM algorithm and some of its accelerations. For first experiments in this direction we refer to our work \cite{18, 19}.

Further, Driggs et al. \cite{15} proved tighter convergence rates for SPRING if the objective function is semi-algebraic. Whether these convergence rates also hold true for iSPRING is still open.

Finally, we intend to apply iSPRING to other practical problems as e.g. in deep learning.

\section{KL Functions}

Finally, let us recall the notation of Kurdyka-Łojasiewicz functions. For $\eta \in (0, \infty]$, we denote by $\Phi_\eta$ the set of all concave continuous functions $\phi : [0, \eta) \to \mathbb{R}_{\geq 0}$ which fulfill the following properties:

\begin{enumerate}[(i)]
  \item $\phi(0) = 0$.
  \item $\phi$ is continuously differentiable on $(0, \eta)$.
  \item For all $s \in (0, \eta)$ it holds $\phi'(s) > 0$.
\end{enumerate}

\begin{definition}[Kurdyka-Łojasiewicz property]
A proper, lower semicontinuous function $\sigma : \mathbb{R}^d \to (-\infty, +\infty]$ has the Kurdyka-Łojasiewicz (KL) property at $\bar{u} \in \text{dom} \partial \sigma = \{ u \in \mathbb{R}^d : \partial \sigma \neq \emptyset \}$ if there exist $\eta \in (0, \infty]$, a neighborhood $U$ of $\bar{u}$ and a function $\phi \in \Phi_\eta$, such that for all

$u \in U \cap \{ v \in \mathbb{R}^d : \sigma(\bar{u}) < \sigma(v) < \sigma(\bar{u}) + \eta \},$
it holds
\[ \phi'(\sigma(u) - \sigma(\bar{u})) \operatorname{dist}(0, \partial \sigma(u)) \geq 1. \]

We say that \( \sigma \) is a KL function, if it satisfies the KL property in each point \( u \in \operatorname{dom} \partial \sigma \).

### B. Proof of Proposition 4.7

The proof follows the path of those in [15, Proposition 2.2]. Let \( \mathbb{E}_{k,p} = \mathbb{E}(\cdot|(x_1^k, x_2^k), \ldots, (x_1^k, x_2^k), p_k^k) \) denote the expectation conditioned on the first \( k \) iterations and the event that we do not compute the full gradient at the \( k \)-th iteration in (5), \( k \geq 1 \). Then we get

\[
\mathbb{E}_{k,p}\left(\nabla_{x_1} H(z_1^k, x_2^k)\right) = \frac{1}{b} \mathbb{E}_{k,p}\left( \sum_{i \in H_k^t} \nabla_{x_1} h_i(z_1^k, x_2^k) - \nabla_{x_1} h_i(z_1^{k-1}, x_2^{k-1}) \right) + \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})
\]

and further

\[
\mathbb{E}_{k,p}\left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \nabla_{x_1} H(z_1^k, x_2^k)\|^2\right)
= \mathbb{E}_{k,p}\left(\|\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) + \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\|^2 + \|\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\|^2\right)
+ \mathbb{E}_{k,p}\left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\|^2\right)
+ 2\left(\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}), \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\right)\text{.}
\]

By (22), we see that

\[
\mathbb{E}_{k,p}\left(\nabla_{x_1} H(z_1^k, x_2^k) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\right) = \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\text{.}
\]

Thus, the first two inner products in (23) sum to zero and the third one is equal to

\[
2\left(\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}), \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\right)\text{.}
\]

It follows that

\[
\mathbb{E}_{k,p}\left(\|\nabla_{x_1} H(z_1^k, x_2^k) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\|^2\right)
= 2\left(\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}), \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\right)
= -2\|\nabla_{x_1} H(z_1^{k-1}, x_2^{k-1}) - \nabla_{x_1} H(z_1^{k-1}, x_2^{k-1})\|^2\text{.}
\]
This yields

\[ E_{k,p} \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) \|^2 \right) \]
\[ \leq \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 - \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) \|^2 \]
\[ + E_{k,p} \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right) \]
\[ \leq \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 + E_{k,p} \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right). \]

Since the function \( x \mapsto \| x \|^2 \) is convex, the second summand fulfills

\[ E_{k,p} \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right) \]
\[ = E_{k,p} \left( \left\| \frac{1}{n} \left( \sum_{j=1}^{n} \nabla x_j H(z^k \mathop{1}, x^k \mathop{2}) - \nabla x_j H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \right) \right\|^2 \right) \]
\[ \leq \frac{1}{n} \sum_{j=1}^{n} \| \nabla x_j H(z^k \mathop{1}, x^k \mathop{2}) - \nabla x_j H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2, \]

so that we obtain

\[ E_{k,p} \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right) \leq \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \]
\[ + \frac{1}{n} \sum_{j=1}^{n} \| \nabla x_j H(z^k \mathop{1}, x^k \mathop{2}) - \nabla x_j H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2. \]

Since the conditional expectation \( E_k \) of \( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \) conditioned on the event that the full gradient is computed in (5) is zero, and taking the \( M \)-Lipschitz continuity of the gradients of the \( h_j \) into account, we get

\[ E_k \left( \| \nabla_x H(z^k \mathop{1}, x^k \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right) \]
\[ \leq (1 - \frac{1}{p}) \left( \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right) \]
\[ + \frac{1}{n} \sum_{j=1}^{n} \| \nabla x_j H(z^k \mathop{1}, x^k \mathop{2}) - \nabla x_j H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \]
\[ \leq (1 - \frac{1}{p}) \left( \| \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) - \nabla_x H(z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 + M^2 \| (z^k \mathop{1}, x^k \mathop{2}) - (z^{k-1} \mathop{1}, x^{k-1} \mathop{2}) \|^2 \right). \]

By symmetric arguments, it holds

\[ E_k \left( \| \nabla_x H^{(k+1)} \mathop{1}, z^k \mathop{2}) - \nabla_x H^{(k+1)} \mathop{1}, z^k \mathop{2}) \|^2 \right) \]
\[ \leq (1 - \frac{1}{p}) \left( \| \nabla_x H^{(k+1)} \mathop{1}, z^k \mathop{2}) - \nabla_x H^{(k+1)} \mathop{1}, z^k \mathop{2}) \|^2 + M^2 \| (z^k \mathop{1}, z^k \mathop{2}) - (z^{k-1} \mathop{1}, z^{k-1} \mathop{2}) \|^2 \right). \]
Using \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) and Lemma 4.6, we can estimate

\[
\| (z_1^k, x_2^k) - (z_1^{k-1}, x_2^{k-1}) \|^2 = \| z_1^k - z_1^{k-1} \|^2 + \| x_2^k - x_2^{k-1} \|^2 \\
\leq 3\| z_1^k - x_1^{k-1} \|^2 + 3\| z_1^k - x_1^{k-1} \|^2 + 3\| x_2^k - x_2^{k-1} \|^2 + \| x_2^k - x_2^{k-1} \|^2 \\
\leq 3(1 + (\beta_1^k)^2)\| x_2^k - x_2^{k-1} \|^2 + 3(\beta_1^{k-1})^2\| x_1^k - x_1^{k-2} \|^2 + \| x_2^k - x_2^{k-1} \|^2.
\]

Further, we have

\[
\mathbb{E}_k(\| (x_1^{k+1}, z_2^k) - (x_1^k, z_2^k) \|^2) = \mathbb{E}_k(\| x_1^{k+1} - x_1^k \|^2) + \| z_2^k - z_2^{k-1} \|^2 \\
\leq \mathbb{E}_k(\| x_1^{k+1} - x_1^k \|^2) + 3\| z_2^k - x_2^k \|^2 + 3\| x_2^k - x_2^{k-1} \|^2 + 3\| x_2^k - z_2^{k-1} \|^2 \\
\leq \mathbb{E}_k(\| x_1^{k+1} - x_1^k \|^2) + 3(1 + (\beta_2^k)^2)\| x_2^k - x_2^{k-1} \|^2 + 3(\beta_2^{k-1})^2\| x_2^k - x_2^{k-2} \|^2.
\]

Altogether we obtain for

\[
\Upsilon_{k+1} := \| \tilde{\nabla}_x H(z_1^k, x_2^k) - \nabla_x H(z_1^k, x_2^k) \|^2 + \| \tilde{\nabla}_x H(x_1^{k+1}, z_2^k) - \nabla_x H(x_1^{k+1}, z_2^k) \|^2.
\]

that

\[
\mathbb{E}_k(\Upsilon_{k+1}) = \mathbb{E}_k \left( \| \tilde{\nabla}_x H(z_1^k, x_2^k) - \nabla_x H(z_1^k, x_2^k) \|^2 + \| \tilde{\nabla}_x H(x_1^{k+1}, z_2^k) - \nabla_x H(x_1^{k+1}, z_2^k) \|^2 \right) \\
\leq (1 - \frac{1}{p})\Upsilon_k + V_T(\mathbb{E}_k(\| x_1^{k+1} - x_1^k \|^2) + \| x_2^k - x_2^{k-1} \|^2 + \| x_2^k - x_2^{k-2} \|^2), \tag{24}
\]

where \(V_T = 3(1 - \frac{1}{p})\)M² \((1 + \max((\beta_1^2), (\beta_2^2)))\). This proves the properties (i) and (ii) of Definition 4.5. Taking the full expectation in \((24)\) and iterating, we

\[
\mathbb{E}(\Upsilon_k) \leq (1 - \frac{1}{p})^{k-l} \mathbb{E}(\Upsilon_1) \\
+ V_T \sum_{l=1}^{k-1} (1 - \frac{1}{p})^{k-l-1} \mathbb{E}(\| x_1^{l+1} - x_1^{l} \|^2 + \| x_2^{l+1} - x_2^{l-1} \|^2 + \| x_2^{l} - x_2^{l-2} \|^2).
\]

We want to show that \(\mathbb{E}(\Upsilon_k) \to 0\) as \(k \to \infty\), if \(\mathbb{E}(\| x_1^k - x_1^{k-1} \|^2) \to 0\) as \(k \to \infty\). Since the first summand converges to zero for \(k\) large enough, it remains to prove that for an arbitrary \(\epsilon > 0\), there exists some \(k_0 \in \mathbb{N}\) such that for all \(k \geq k_0\) the sum becomes not larger than \(\epsilon\) Recall that \(\sum_{l=0}^{\infty} (1 - \frac{1}{p})^l = p\). Now we choose \(k_1 \in \mathbb{N}\) such that for all \(k \geq k_1\) we have \(\mathbb{E}(\| x_1^{k+1} - x_1^k \|^2) < \frac{\epsilon}{2p}\). Further we define \(k_2 \in \mathbb{N}\) such that \((1 - \frac{1}{p})^{k_2} < \frac{\epsilon}{2k_2}\), where \(S := \max_{k \in \mathbb{N}} \mathbb{E}(\| x_1^k - x_1^{k-1} \|^2)\). Then the above sum can be estimated for \(k \geq k_0 := k_1 + k_2\) as

\[
\sum_{l=1}^{k-1} (1 - \frac{1}{p})^{k-l-1} \mathbb{E}(\| x_1^{l+1} - x_1^{l} \|^2 + \| x_2^{l+1} - x_2^{l-1} \|^2 + \| x_2^{l} - x_2^{l-2} \|^2) \\
= \sum_{l=1}^{k_1} (1 - \frac{1}{p})^{k-l-1} \mathbb{E}(\| x_1^{l+1} - x_1^{l} \|^2 + \| x_2^{l+1} - x_2^{l-1} \|^2 + \| x_2^{l} - x_2^{l-2} \|^2) \\
+ \sum_{l=k_1+1}^{k-1} (1 - \frac{1}{p})^{k-l-1} \mathbb{E}(\| x_1^{l+1} - x_1^{l} \|^2 + \| x_2^{l+1} - x_2^{l-1} \|^2 + \| x_2^{l} - x_2^{l-2} \|^2).
\]

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\[ + \sum_{l=k_1+1}^{k-1} (1 - \frac{1}{p})^{k-l-1} \epsilon \left( ||x^{l+1} - x^l||^2 + ||x^l - x^{l-1}||^2 + ||x^{l-1} - x^{l-2}||^2 \right) \]
\[ \leq (1 - \frac{1}{p})^{k_1} \sum_{l=1}^{k_1} (1 - \frac{1}{p})^{k_1-l-1} \epsilon \left( ||x^{l+1} - x^l||^2 + ||x^l - x^{l-1}||^2 + ||x^{l-1} - x^{l-2}||^2 \right) \]
\[ + \sum_{l=k_1+1}^{k} (1 - \frac{1}{p})^{k-l-1} \epsilon \leq \epsilon, \]
and we are done.

C. Derivatives of the Likelihood of Student-\(t\) MMs

In this section, we compute the outer derivatives of the objective function in (19) which we need in the numerical computations and in the proof of Lemma 6.2. In [18], the derivatives of \(g(\nu, \mu, \Sigma) := \log(f(x|\nu, \mu, \Sigma))\) were computed as follows:

\[
\frac{\partial g}{\partial \mu}(\nu, \mu, \Sigma) = \frac{d + \nu}{\nu + s} \Sigma^{-1} (x - \mu),
\]

(25)
\[
\frac{\partial g}{\partial \Sigma}(\nu, \mu, \Sigma) = \frac{1}{2} \left( \frac{d + \nu}{\nu + s} \Sigma^{-1} (x - \mu)^T \Sigma^{-1} - \Sigma^{-1} \right),
\]

(26)
\[
\frac{\partial g}{\partial \nu}(\nu, \mu, \Sigma) = \frac{1}{2} \left( \psi \left( \frac{\nu + d}{2} \right) - \psi \left( \frac{\nu}{2} \right) - \frac{d - s}{\nu + s} \log \left( 1 + \frac{s}{\nu} \right) \right),
\]

(27)
where \(s := (x - \mu)^T \Sigma^{-1} (x - \mu)\) and \(\psi\) is the digamma function defined by

\[
\psi(x) := \frac{d}{dx} \log (\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]

We use the abbreviations

\[
f_{i,k} := f(x_i|\nu_k, \mu_k, \Sigma_k), \quad \gamma_i := \left( \sum_{k=1}^{K} \alpha_k f_{i,k} \right)^{-1}, \quad s_{i,k} := (x_i - \mu_k)^T \Sigma_k^{-1} (x_i - \mu_k).
\]

Then we obtain for

\[
\mathcal{L}(\alpha, \nu, \mu | \mathcal{X}) = -\sum_{i=1}^{n} \log \left( \sum_{k=1}^{K} \alpha_k f_{i,k} \right)
\]

that the derivative with respect to \(\alpha\) is given by

\[
\frac{\partial \mathcal{L}(\alpha, \nu, \mu | \mathcal{X})}{\partial \alpha_l} = -\sum_{i=1}^{n} \gamma_i f_{i,k}
\]

(28)
Using that for \( g = \log f \), the relation \( g'f = f' \) holds true, the derivatives with respect to \( \mu_l, \Sigma_l \) and \( \nu_l \) have the form
\[
\frac{\partial L(\alpha, \nu, \mu, \Sigma | X)}{\partial \mu_l} = -\sum_{i=1}^{n} \gamma_i \alpha_i f_i \frac{\partial f(\alpha, \nu, \mu, \Sigma | X)}{\partial \mu_l}.
\]
Together with (25) - (27), we obtain
\[
\nabla_{\alpha_l} L(\alpha, \nu, \mu, \Sigma | X) = \sum_{i=1}^{n} \gamma_i \alpha_i f_i \left( \frac{d + \nu_l}{\nu_l + s_{i,l}} \right) \Sigma_{i,l}^{-1} (\mu_l - x_i),
\]
\[
\nabla_{\Sigma_l} L(\alpha, \nu, \mu, \Sigma | X) = \frac{1}{2} \sum_{i=1}^{n} \gamma_i \alpha_i f_i \left( \Sigma_{i,l}^{-1} - \frac{d + \nu_l}{\nu_l + s_{i,l}} \Sigma_{i,l}^{-1} (x_i - \mu_l)(x_i - \mu_l)^T \right),
\]
\[
\frac{\partial L(\alpha, \nu, \mu, \Sigma | X)}{\partial \nu_l} = \frac{1}{2} \sum_{i=1}^{n} \gamma_i \alpha_i f_i \left( \psi \left( \frac{\nu_l}{2} \right) - \psi \left( \frac{\nu_l + d}{2} \right) + \frac{d - s_{i,l}}{\nu_l + s_{i,l}} + \log \left( 1 + \frac{s_{i,l}}{\nu_l} \right) \right).
\]

**D. Proof of Lemma 6.2**

Since the sum of Lipschitz continuous functions is Lipschitz continuous, it is sufficient to show the claim for the summands of \( H \) in (19). Hence we consider only
\[
h(\alpha, \nu, \mu, \Sigma) := \log \left( \sum_{k=1}^{K} \tilde{\alpha}_k f_k \right), \quad f_k := f(x | \tilde{\nu}_k, \tilde{\mu}_k, \tilde{\Sigma}_k),
\]
where \( \tilde{\alpha} = \varphi_1(\alpha), \tilde{\nu} = \varphi_2(\nu), \tilde{\Sigma} = \varphi_3(\Sigma) \) are given by (18). Set
\[
\gamma := \left( \sum_{k=1}^{K} \tilde{\alpha}_k f_k \right)^{-1}, \quad s_k := (x - \tilde{\mu}_k)^T \tilde{\Sigma}_k^{-1} (x - \tilde{\mu}_k).
\]

1. By (28) we obtain
\[
\frac{\partial h(\alpha, \nu, \mu, \Sigma)}{\partial \alpha_l} = \frac{\exp(\alpha_l) f_l}{\sum_{k=1}^{K} \exp(\alpha_k) f_k} - \frac{\exp(\alpha_l)}{\sum_{k=1}^{K} \exp(\alpha_k)}
\]
and further for the Hessian of \( h \) with respect to \( \alpha \),
\[
\frac{\partial h(\alpha, \nu, \mu, \Sigma)}{\partial \alpha_l \partial \alpha_j} = \delta_{l,j} \left( \frac{\exp(\alpha_l) f_l}{\sum_{k=1}^{K} \exp(\alpha_k) f_k} - \frac{\exp(\alpha_l)}{\sum_{k=1}^{K} \exp(\alpha_k)} \right) - \frac{\exp(\alpha_l) \exp(\alpha_j) f_l f_j}{\left( \sum_{k=1}^{K} \exp(\alpha_k) f_k \right)^2} + \frac{\exp(\alpha_l) \exp(\alpha_j)}{\left( \sum_{k=1}^{K} \exp(\alpha_k) \right)^2}.
\]
This is bounded so that \( \nabla_{\alpha} h(\cdot, \nu, \mu, \Sigma) \) is globally Lipschitz continuous.
2. Using (27), we get
\[
\frac{\partial}{\partial \nu_i} h(\alpha, \nu, \mu, \Sigma) = \tilde{\alpha}_i \frac{\gamma_2}{g_1} \left( \psi \left( \psi \left( \frac{d + \tilde{\nu}_i}{2} \right) - \frac{d - s_i}{\tilde{\nu}_i + s_i} \right) - \log \left( \frac{1 + s_i}{\tilde{\nu}_i} \right) \right) \nu_i.
\]

We show that the functions \( g_i, i = 1, 2 \) are Lipschitz continuous and bounded. This implies that \( \frac{\partial}{\partial \nu_i} h(\alpha, \cdot, \mu, \Sigma) \) is Lipschitz continuous. It holds
\[
|g_2(\nu)| \leq |\nu| \left| \psi \left( \frac{d + \tilde{\nu}_i}{2} \right) - \psi \left( \frac{\tilde{\nu}_i}{2} \right) \right| + |\nu| \left| \frac{d - s_i}{\tilde{\nu}_i + s_i} \right| + \log \left( 1 + \frac{s_i}{\tilde{\nu}_i} \right)^{|\nu|}.
\]

Using the summation formula
\[
\psi(x + 1) = \psi(x) + \frac{1}{x}
\]
and the fact that the digamma function is monotone increasing we conclude
\[
|g_2(\nu)| \leq \sum_{r=1}^{\left\lfloor \frac{d}{2} \right\rfloor} \frac{|\nu|}{d + \nu_i + r} + |\nu| \left| \frac{d - s_i}{\tilde{\nu}_i + s_i} \right| + \log \left( 1 + \frac{s_i}{\tilde{\nu}_i} \right)^{|\nu|}.
\]

Since
\[
\lim_{\nu_i \to \pm \infty} \log(1 + \frac{1}{\nu_i} s_i)^{|\nu|} \leq \lim_{\nu_i \to \pm \infty} \log \left( 1 + \frac{1}{\tilde{\nu}_i} s_i \right)^{\nu^2} = s_i
\]
and \( g_2 \) is continuous we conclude that \( g_2 \) is bounded. Further, it holds
\[
g'_2(\nu) = \psi \left( \frac{d + \tilde{\nu}_i}{2} \right) - \psi \left( \frac{\tilde{\nu}_i}{2} \right) - \frac{d - s_i}{\tilde{\nu}_i + s_i} - \log \left( 1 + \frac{s_i}{\tilde{\nu}_i} \right) + \nu_i^2 \left( \psi' \left( \frac{d + \tilde{\nu}_i}{2} \right) - \psi' \left( \frac{\tilde{\nu}_i}{2} \right) \right) + \frac{2(d - s_i)\nu_i}{(\tilde{\nu}_i + s_i)^2} + \frac{2\nu_i \nu_i}{(\tilde{\nu}_i + s_i)^2 + s_i(\tilde{\nu}_i^2 + s_i)}.
\]

Using again (30) and \( \psi'(x + 1) = \psi'(x) + \frac{1}{x^2} \) as well as the fact that the digamma function and its derivatives are monotone increasing we get,
\[
\lim_{\nu_i \to \pm \infty} \left| g'_2(\nu) \right| \leq \lim_{\nu_i \to \pm \infty} \left( \sum_{r=0}^{\left\lfloor \frac{d}{2} \right\rfloor} \frac{1}{r^2 + r} + \sum_{r=0}^{\left\lfloor \frac{d}{2} \right\rfloor} \frac{|\nu|^2}{(\nu_i^2 + r)^2} \right) = 0.
\]

Since \( g'_2(\nu) \) is continuous, this yields that it is bounded which implies that \( g_2 \) is Lipschitz continuous.
The function $g_1$ is obviously bounded. Further we obtain for $j \neq l$ that
\[
\nabla_{\nu_j} g_1(\nu) = -2\tilde{\alpha}_j f_1 \gamma^2 \frac{\partial f_j}{\partial \nu_j} = -2g_2(\nu_j)\tilde{\alpha}_j f_j \gamma^2
\]
so that
\[
|\nabla_{\nu_j} g_1(\nu_l)| = 2|g_2(\nu_j)||\tilde{\alpha}_j f_j \gamma^2|.
\]
This expression is bounded. Similarly, we get for $j = l$ that
\[
|\nabla_{\nu_l} g_1(\nu_l)| = 2|g_2(\nu_l)||\tilde{\alpha}_l f_l \gamma^2 + f_l|.
\]
Thus, $g_1$ is Lipschitz continuous.

3. By (29) we obtain
\[
\nabla_{\mu_l} h(\alpha, \nu, \mu, \Sigma) = \tilde{\alpha}_l f_l \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + s_l} \tilde{\Sigma}_l^{-1}(x - \mu) \cdot \frac{\nabla_{\mu_l} g_2(\mu_l)}{g_2} + \tilde{\nu}_l \tilde{\Sigma}_l^{-1} - 1 \frac{\mu_l - x}{\tilde{\nu}_l + s_l} \tilde{\Sigma}_l^{-1} + \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + s_l} \tilde{\Sigma}_l^{-1}
\]
and taking the Frobenius norm $\| \cdot \|_F$, we obtain
\[
\| \nabla_{\mu_l} g_2(\mu_l) \|_F \leq \frac{d + \tilde{\nu}_l}{(\tilde{\nu}_l + s_l)^2} \tilde{\Sigma}_l^{-1}(x - \mu) \|_F + \frac{d + \tilde{\nu}_l}{(\tilde{\nu}_l + s_l)^2} \tilde{\Sigma}_l^{-1} \|_F + \text{const.}
\]
Thus $g_2$ is Lipschitz continuous. Since
\[
g_2(\mu_l) \leq \frac{d + \tilde{\nu}_l}{(\tilde{\nu}_l + s_l)^2} \tilde{\Sigma}_l^{-1} \|_F + \frac{d + \tilde{\nu}_l}{(\tilde{\nu}_l + s_l)^2} \tilde{\Sigma}_l^{-1} \|_F \leq \frac{d + \tilde{\nu}_l}{2 \min(1, \tilde{\nu}_l)} \tilde{\Sigma}_l^{-1} \|_F,
\]
g_2 is bounded.
The function $g_1$ is obviously bounded. Further, it holds
\[
\|\nabla_{\mu} g_1(\mu)\| = \begin{cases} 
\|g_2(\mu_l)\| \|\gamma^2 f_l f_j \tilde{\alpha}_j\| & \text{if } j \neq l, \\
\|g_2(\mu_l)\| \|\gamma^2 f_l^2 \tilde{\alpha}_l + \gamma f_l\| & \text{if } j = l,
\end{cases}
\]
so that $g_1$ is Lipschitz continuous.

4. At the end, we use (26) to compute
\[
\nabla_{\Sigma_l} h(\alpha, \nu, \mu, \Sigma) = \tilde{\alpha}_l \gamma f_l \left( \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + s_l} \tilde{\Sigma}_l^{-1} (x - \mu_l)(x - \mu_l)^T \tilde{\Sigma}_l^{-1} - \tilde{\Sigma}_l^{-1} \right) \Sigma_l.
\]

We show that $g_i$, $i = 1, 2$ are bounded and Lipschitz continuous. We have
\[
g_2(\Sigma_l) = \left( \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + s_l} \tilde{\Sigma}_l^{-1} (x - \mu_l)(x - \mu_l)^T - I_d \right) \tilde{\Sigma}_l^{-1} \Sigma_l h_2(\Sigma_l)
\]
Obviously, $h_1$ is bounded. The second factor $h_2$ is bounded, since with the spectral decomposition $\Sigma_l = PDP^T$ it holds
\[
\tilde{\Sigma}_l^{-1} \Sigma_l = (P(D^2 + \epsilon I)P^T)^{-1} PDP^T = P(D^2 + \epsilon I)^{-1} PDP^T,
\]
so that the absolute value of the largest eigenvalue of $h_2(\Sigma_l)$ is smaller than 1.

To prove the Lipschitz continuity of $g_2$ we compute the directional derivative using the computation rules from [35]:
\[
D_{\Sigma_l}(\tilde{\Sigma}_l^{-1})[H] = D_{\Sigma_l}(\tilde{\Sigma}_l^{-1} + \epsilon I_d)^{-1}[H] = \tilde{\Sigma}_l^{-1} D_{\Sigma_l}(\tilde{\Sigma}_l^{-1} + \epsilon I_d) \tilde{\Sigma}_l^{-1} = 2\tilde{\Sigma}_l^{-1} \Sigma_l \tilde{\Sigma}_l^{-1}.
\]
Then we obtain
\[
\|D_{\Sigma_l} h_2[H]\|_F = \|2\tilde{\Sigma}_l^{-1} \Sigma_l \tilde{\Sigma}_l^{-1} \Sigma_l + \tilde{\Sigma}_l^{-1}\|_F \leq 4\|h_2(\Sigma_l)\|_F + \|\tilde{\Sigma}_l^{-1}\|_F
\]
which is bounded. Thus, $h_2$ is Lipschitz continuous.

To show, that $h_1$ is Lipschitz continuous, note that the mapping $x \mapsto \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + (x - \mu_l)\Sigma_l^{-1} (x - \mu_l)}$ has a bounded derivative and is Lipschitz continuous. Further, the mapping $A \mapsto (x - \mu)A(x - \mu)$ has a bounded derivative, if $A$ is bounded. Together with the fact that $\Sigma_l \mapsto \tilde{\Sigma}_l^{-1}$ has a bounded derivative by (31) and is bounded, this yields that the mapping
\[
\Sigma_l \mapsto \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l + (x - \mu_l)\tilde{\Sigma}_l^{-1} (x - \mu_l)} \leq \frac{d + \tilde{\nu}_l}{\tilde{\nu}_l}
\]
is Lipschitz continuous as a concatenation of Lipschitz continuous functions. Since also $\Sigma_l \mapsto \tilde{\Sigma}_l^{-1}$ is Lipschitz continuous by (31) and bounded, we get that $h_1$ is Lipschitz continuous. Now, $h_1$
and $h_2$ are Lipschitz continuous and bounded. Thus also $g_2 = h_1 h_2$ is Lipschitz continuous and bounded.

Finally, the function $g_1$ maps into the interval $[0, 1]$ and is Lipschitz continuous by the same arguments as in the second part of the proof. □

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