A New Anomaly Matching Condition?

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Abstract

We formulate “Witten” matching conditions for confining gauge theories. The conditions are analogous to ’t Hooft’s, but involve Witten’s global SU(2) anomaly. Using a group theoretic result of Geng, Marshak, Zhao and Okubo, we show that if the fourth homotopy group of the flavor group $H$ is trivial ($\Pi_4(H) = 0$) then realizations of massless composite fermions that satisfy the ’t Hooft conditions also satisfy the Witten conditions. If $\Pi_4(H)$ is nontrivial, the new matching conditions can yield additional information about the low energy spectrum of the theory. We give a simple physical proof of Geng, et. al.’s result.

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1 The Global SU(2) Anomaly

Witten [1] has shown that SU(2) gauge theories are subject to a nonperturbative anomaly which can render them mathematically inconsistent. For example, SU(2) theories with an odd number of chiral doublets (and no other fermions in higher representations) are inconsistent. The reason behind the inconsistency is as follows: The fourth homotopy group of SU(2) is nontrivial, \( \Pi_4(SU(2)) = \mathbb{Z}_2 \). This means that there is a homotopically nontrivial class of four dimensional SU(2) gauge configurations that cannot be continuously deformed to the identity. Now consider the fermion integration in the Euclidean path integral for an odd number \( N \) of Weyl fermions:

\[
\int D\psi D\bar{\psi} \exp\left( \int d^4x \sum_j \bar{\psi}_j iD/\psi_j \right) = \det^{N/2}[iD(A)], \tag{1}
\]

where \( A \) is the background gauge field. It can be shown using the Atiyah-Singer index theorem [2] that the fermion determinant, defined as the product of either the positive or negative eigenvalues of \( iD(A) \), changes sign under a topologically nontrivial gauge transformation \( U \):

\[
\det^{N/2}[iD(A)] = (-)^N \det^{N/2}[iD(U^{-1}AU - iU^{-1}\partial U)], \tag{2}
\]

where \( A^{\mu} = U^{-1}A\mu U - iU^{-1}\partial U \). Hence for \( N \) odd, when the integration over gauge configurations \( A_\mu \) is performed in the partition function \( Z \), the homotopically trivial and nontrivial gauge sectors cancel exactly, yielding zero. Similarly, the path integral \( Z_X \) with insertion of any gauge invariant operator \( X \) is identically zero. Therefore any expectation value \( \langle X \rangle = Z_X/Z \) is ill defined.

For the more general case where there exist not only fermion doublets but also other representations, \( N \) in the above equations gets replaced by \( N_0 \), which is the number of zero modes in an instanton background gauge field and equals \( N_0 = \sum_{R_i} 2T(R_i) \) where \( R_i \) is an \( SU(2) \) representation and \( T(R_i) \) is the index defined by \( \text{Tr} T_a T_b |_{R_i} = T(R_i)\delta_{ab} \).

The Witten anomaly, unlike the familiar triangle anomaly, is intrinsically nonperturbative and is often referred to as a global anomaly (this is not to be confused with anomalies in global currents, which we will not discuss here). While the triangle anomaly, due to its non-renormalization properties, can be computed at one loop order, the Witten anomaly does not appear at any order in perturbation theory.

In this letter we will discuss some issues related to the Witten anomaly. The first is its use in deriving “Witten” matching conditions, similar to those of ’t Hooft [3], which constrain the possible realizations of massless composite fermions in confining gauge theories. Such matching conditions are remarkable as they yield nonperturbative, dynamical information in
terms of simple algebraic relations. We will find, perhaps suprisingly since the two types of anomaly are seemingly very different, that if the full flavor group $H$ satisfies $\Pi_4(H) = 0$ then a realization of massless composite fermions which satisfies the ’t Hooft matching conditions automatically satisfies the Witten matching conditions. This result follows from the group theoretic results of Geng, Marshak, Zhao and Okubo regarding global and local anomaly cancellation. We will find that their results can be neatly understood in a physical way in terms of the gauge invariance properties of gauge field effective actions. Alternatively, if $\Pi_4(H)$ is nontrivial, the Witten matching conditions can yield new information about the massless bound states in the theory.

2 Anomaly Matching Conditions

Consider a confining gauge theory with color group $G$ and fundamental chiral fermion fields $\psi$. Let the group of global flavor symmetries for this theory be $H$. If the global chiral symmetries are not broken in the confined phase of this theory, the physical gauge singlet states will include massless composite fermions (“baryons” - denoted here by $B$). The $B$’s will form complicated representations of $H$, as typically many $\psi$’s will have to be combined to yield gauge singlet state. ’t Hooft’s matching conditions provide very nontrivial constraints on the possible representations $B$. To arrive at his conditions ’t Hooft considered gauging the flavor group with an extremely small gauge coupling (so as not to affect the strong color dynamics). In order to have a consistent theory, ’t Hooft introduced massless spectator fermions $S$ which carry flavor charges, but are color singlets. The spectators are chosen to exactly cancel any triangle anomalies due to the fundamental fermions $\psi$. Symbolically,

$$A_\psi^\triangle = -A_S^\triangle. \quad (3)$$

Now consider the low energy, confined limit of the theory. One expects the effective theory here to be described by the massless composites, which now carry only the flavor charge. However, a consistent effective theory requires that the triangle anomalies of the composite fermions cancel those of the massless spectators. Therefore, one has the relation

$$A_B^\triangle = -A_S^\triangle = A_\psi^\triangle. \quad (4)$$

In other words, the perturbative flavor anomalies of the composite fermions must exactly equal those of the fundamental fermions. Since the possible representations $B$ are limited by the requirement of color neutrality, these matching conditions are in many cases nontrivial to satisfy. If they cannot be satisfied one of the assumptions of the construction must be
abandoned - for example, conservation of chiral symmetries or confinement. If one assumes confinement, then this line of reasoning, combined with the decoupling or “persistence of mass” condition, shows that QCD must break its chiral symmetries $SU(n_f)_L \otimes SU(n_f)_R \otimes U(1)_V$ down to $SU(n_f)_V \otimes U(1)_V$, if $n_f > 2$. Persistence of mass, which requires that a \textit{massless} composite not contain a \textit{massive} constituent, has been demonstrated rigorously for vectorlike theories by Vafa and Witten \cite{8}. (See \cite{9} for a generalization of this result when fundamental scalars are present.)

More detailed and rigorous arguments for ‘t Hooft’s conditions have been given by Frischman, Schwimmer, Banks and Yankielowicz \cite{4} and by Coleman and Grossman \cite{5}. Their strategy is to compare the anomalies of the fundamental and bound state spectra by comparing the singularities and discontinuities of the anomaly equation implied by the triangle diagram in the confined and deconfined phases of the theory.

We can now generalize ‘t Hooft’s construction by considering SU(2) subgroups of $H$, and adding spectator fermions $S'$ (note that $S'$ and $S$ are not \textit{a priori} the same representations) to cancel the global anomaly of the fundamental fermions, $\psi$. Reasoning similar to that of the above analysis yields the condition:

$$A_W^B = A_W^\psi.$$  

(5)

In other words, the composite fermions must exactly reproduce the Witten anomaly of the fundamental fermions. Of course, the global matching conditions cannot be justified by examining the structure of specific perturbative graphs such as the triangle, and so the above result cannot be justified at the same level of rigor as the ‘t Hooft conditions. However, if correct, as the spectator argument suggests, the above conditions \textit{seem} to again yield, at first sight, nontrivial constraints on the representations $B$, and can provide new information about the bound state spectrum of chiral gauge models.

Unfortunately, in general this is not the case. It is possible to show that a set of massless composite fermions $B$ which satisfy the ‘t Hooft conditions (4) will also \textit{automatically} satisfy the Witten matching conditions (5) if $\Pi_4(H) = 0$. In order to prove this, we will invoke the result of Geng, Marshak, Zhao and Okubo (GMZO) \cite{6}, which states that a theory with simple gauge group $H$, which satisfies $\Pi_4(H) = 0$ and which has no perturbative anomalies, will have no global anomalies in any of its SU(2) subgroups. The GMZO result is entirely group theoretical, and relates the global SU(2) anomaly to the local anomalies of the larger group $H$. It is important to emphasize that the GMZO result states that the vanishing of the triangle anomaly implies the vanishing of the Witten anomaly, but not vice versa. (A model can still have vanishing Witten anomaly and non-vanishing triangle anomaly.) In the next section we will explain the GMZO result in more detail, from a more physical viewpoint.
Now suppose that the composite fermions $B$ satisfy the 't Hooft conditions (4). Then both the short distance, fundamental theory and the low energy, composite theory are free from perturbative anomalies when the spectators $S$ are included. If $\Pi_4(H) = 0$ and $H$ is simple then the GMZO result applies, and we can conclude that no SU(2) subgroup of $H$ has a Witten anomaly when the spectators are included. But this just implies that

$$A_W^B = -A_W^S = A_W^\psi,$$

which is the Witten matching condition. Here the first equality uses GMZO, and the second equality uses the assumption of unbroken $H$, 't Hooft’s matching condition, and GMZO. The condition (6) also implies that the spectators $S$ from the 't Hooft construction will suffice as the spectators $S'$ for the corresponding Witten construction. (This is always the case for any model, as we can always choose the spectators to be “mirror” fermions, whose addition renders the model vectorlike.) But note that in principle it is possible to have Witten anomaly cancellation without having 't Hooft anomaly cancellation.

Now let us consider the possibility that $\Pi_4(H)$ is nontrivial. In that case the GMZO result cannot be directly applied. For the simplest case, $H = SU(2)$, it is easy to see that the 't Hooft conditions are trivially satisfied due to the fact that representations of SU(2) are pseudo-real and hence cannot contribute to the triangle anomaly. Therefore, if the Witten conditions are not always trivially satisfied, they provide new constraints on the spectrum of massless composites. Here we will consider two specific models: an $SU(N_c) \otimes SU(2_f)_L$, and an $SU(3_c) \otimes SU(2_f)_L \otimes SU(2_f)_R$, where the subscripts $c$ and $f$ denote color and flavor respectively.

1. $SU(N_c) \otimes SU(2_f)_L$: We take $N_c$ colors of fundamental fermions $X$ transforming as doublets (2’s) of $SU(2_f)$. In order to cancel color anomalies, it will be necessary to include additional left and right handed fermions in possibly higher color representations. As a specific example, for $N_c = 3$, consider adding a left handed flavor singlet which transforms as a $15$ of color, $Y$, and 16 right handed flavor singlets which are triplets of color, $Z$. The net color anomaly $\left[\begin{array}{l}10 \\ 0 \end{array}\right]$ (in units of the fundamental 3) is then $\left[\begin{array}{l}A \\ 1 \end{array}\right] = 2 + 14 - 16 = 0$, where the first contribution is from each member of the flavor doublet, the second one from the $15$ of color (a flavor singlet), and the final contribution is from the 16 right handed (hence the relative minus sign) color triplets (which are flavor singlets). The full symmetry of the model is $SU(3_c) \otimes SU(2_f)_L \otimes SU(16_f)_R \otimes U(1)^3$, where the $U(1)$’s are associated with phase rotations of the three types of fermion. One problem with this specific model is that the

\[3\] The interested reader can construct more sophisticated models along these lines by using the toolbox for computing indices given in the appendix.
addition of the extra fermions destroys asymptotic freedom. This may not be a problem for similar constructions with larger $N_c$. In any case, the group theory still provides an example of new information from Witten matching conditions.

In this model there are Lorentz invariant color singlet operators which transform under $SU(16f)_R \otimes U(1)^3$, but are singlets under $SU(2f)_L$. For instance,
\[ (e^{ij} X^a_i X^{\beta j})^3 \]
(7)
carries the first $U(1)$ charge, but is an $SU(2f)_L$ singlet ($i, j$ are flavor indices and $\alpha, \beta$ are color indices). We can also form operators out of $Y$ and $Z$ fields that are spin zero, color and $SU(2f)_L$ singlet, but transform under $SU(16)$ and the remaining two $U(1)$’s. We can consider the case where condensates of the above operators form, breaking all the chiral symmetries except $SU(2f)_L$. Then the ’t Hooft anomaly matching condition is trivial, since there is no triangle anomaly for the $SU(2)$ group because all representations are pseudoreal. Similar constructions are possible for larger values of $N_c$.

In models of this sort it is possible to obtain nontrivial constraints by considering the Witten anomaly matching condition presented above. To classify the color singlet baryons in the confined theory, we consider the tensor product $\underbrace{\mathbf{2} \otimes \mathbf{2} \otimes \ldots \otimes \mathbf{2}}_{N_c \text{ times}}$, and decompose this into representations of the flavor group $SU(2f)_L$. GMZO point out that there are an even number of fermion zero modes in any representation of the $SU(2)$ gauge group except those of dimension
\[ n = 2(2m + 1), \quad m = 0, 1, 2, \ldots \] 
(8)
The number of zero modes is given by $N_0 = \frac{1}{6} n(n^2 - 1)$, $n = 1, 2, 3, \ldots$ which is odd only for $n$ given in (8). Therefore it is only representations of dimensionality $n$ that contribute to the Witten anomaly. These ‘anomalous’ $n$ are all even, i.e. $n = 2, 6, 10, 14, 18, \ldots$

It is easy to see from Young tableaux that only two kinds of irreps may appear in the decomposition of the product of $N_c$ 2’s, all even-dimensional or all odd dimensional. When $N_c$ is even, the only irreps are of dimensions $N_c + 1, N_c - 1, N_c - 3, \ldots, 1$, i.e. all odd dimensional. Since (8) requires even dimensional irreps. for a Witten anomaly, it is clear that when $N_c$ is even, it is impossible to get a Witten anomaly in the confining theory. This is consistent, since for even $N_c$, the fundamental theory does not have a Witten anomaly either. In the case $N_c$ even, the Witten anomaly matching thus gives no new constraints on the representations of the flavor group in the confining phase.

For $N_c$ odd, we know that the fundamental theory has a Witten anomaly which must be reproduced in the low energy confining theory. For an $N_c$-fold tensor product of the 2’s
transforming under $SU(2_f)$, ($N_c$ odd) it is again easy to see that the only irreps appearing are of dimension $N_c + 1, N_c - 1, N_c - 3, \ldots, 0$, i.e. all even dimensional. Thus for odd $N_c$, there must be at least one massless ‘baryon’ that transforms under $SU(2_f)$ as one of the anomalous representations given by (8). The Witten matching condition requires irreps that replicate the Witten anomaly in the low energy theory, i.e. the ones of dimension 2, 6, 10, … Massless composites must appear in at least one of those irreps, a fact which could not have been deduced from the ’t Hooft conditions.

2. $SU(3_c) \otimes SU(2_f)_L \otimes SU(2_f)_R \otimes U(1)_V$: Since this is a vector-like theory, the perturbative anomaly constraints can be combined with the persistence of mass condition \cite{7, 8}. However, there are many solutions to these combined conditions and it is therefore not possible to prove that chiral symmetry breaking occurs for two flavors. (It may be that a light strange quark is necessary for chiral symmetry breaking in QCD!)

Perhaps the Witten conditions can eliminate all or some of the solutions: consider gauging the $SU(2_f)_L$ gauge group. For $N_c = 3$, the ’t Hooft matching equation obtained from the $[SU(2_f)_L]^2 \otimes U(1)_V$ matching condition is then \cite{3}

$$10a - 5b + c = 1 \tag{9}$$

where $a, b$ and $c$ are non-negative integers denoting respectively the number of fermions transforming under $SU(2_f)_L \otimes SU(2_f)_R$ as $(4, 1), (2, 3)$ and $(2, 1)$. The ’t Hooft anomaly condition is satisfied by any set of $a, b, c$ which satisfies (8). There is also an $SU(2)_L$ Witten anomaly in the fundamental theory with three colors, which needs to be matched in the confining theory.

As a special case, choose $a = 0, b = 0, c = 1$ in the ’t Hooft matching equation which gives the low energy spectrum containing simply the representation $(2, 1)$ and its parity double $(1, 2)$, which are the nucleons of the $\sigma$ model. The representation $(2, 1)$ is also one of those in (8) that can give a Witten anomaly, so the Witten anomaly is also satisfied.

Another solution of the ’t Hooft matching equation is afforded by $a = 1, b = 2, c = 1$, which corresponds to a low energy spectrum of $(4, 1) \oplus 2(2, 3) \oplus (2, 1)$ plus their parity doubles. The ’t Hooft conditions cannot distinguish this realization from the previous one. Since by (8) the $(4, 1)$ representation gives no contribution to the Witten anomaly, we need only consider the contribution of the irreps $(2, 3)$ and $(2, 1)$. In toto, these two contribute seven $SU(2_f)_L$ doublets, so the Witten matching condition is again satisfied, yielding no new information.

In general, only the irreps $(2, 3)$ or $(2, 1)$ can contribute to the Witten anomaly in the confined phase, so we can ignore the index $a$. If the Witten matching conditions are to
eliminate representations allowed by the 't Hooft matching conditions, there must exist \(b\) and \(c\) such that \(3b + c\) is even. Setting \(3b + c = 2m, m = 0, 1, 2, \ldots\), and using the 't Hooft equation, we obtain \(10a - 8b + 2m = 1\), which can never be satisfied by integral \(a, b\) and \(m\). In this case we conclude that the Witten anomaly matching conditions are subsumed by the 't Hooft anomaly matching conditions. We have checked that this is the also the case in five color QCD with two flavors, with or without parity doubling of nucleons.

It is possible to give a general proof of this result for any \(SU(N_c) \otimes SU(2f)_L \otimes SU(2f)_R \otimes U(1)_V\) theory. \((N_c\) must be odd in order that the confined phase have chiral baryons with spin \(1/2\)). Consider the 't Hooft matching condition resulting from the \([SU(2f)_L]^2U(1)_V\] anomaly. In the fundamental theory we have \(N_c\) left handed doublets of \(SU(2f)_L\), which contribute \(N_c/2\) to the mixed anomaly (we define \(T(R) = 1/2\) in the fundamental representation, where \(T(R)\) satisfies \(TrT^aT^b|_R = T(R)\delta^{ab}\)). In the confined phase of the theory we can have any number of baryons transforming in higher representations of \(SU(2f)_L\). However, in the simplest case these baryons will consist of \(N_c\) fundamental fermions, and therefore have \(U(1)_V\) charge \(N_c\). The anomaly matching equation is then

\[
\sum_i 2l_i T(R_i) = 1, \tag{10}
\]

where the sum is over all baryon states, with multiplicity \(l_i\). Now consider Witten matching conditions, which require an odd number \(N_0\) of zero modes in an \(SU(2f)_L\) instanton background (recall \(N_c\) is odd, so we have an odd number of doublets in the fundamental theory). Since \(N_0(R_i) = 2T(R_i)\) \(\Box\), the Witten condition becomes

\[
\sum_i 2l_i T(R_i) = \text{odd}. \tag{11}
\]

It is clear that (11) is implied by (10). Note that if the \(U(1)\) symmetry is spontaneously broken in the low energy theory (as was assumed in the first example), (11) no longer applies and the Witten conditions may contain new information.

### 3 Perturbative and Global Anomalies

In this section we address the relation between perturbative and global anomalies. This relation will yield a particularly simple understanding of the GMZO result. The point of view taken here follows that of Alvarez-Gaume and Witten \(\square\).

Consider integrating out chiral fermions in the background of an arbitrary gauge field \(A_\mu\) (see equation (1)). The result is an effective action

\[
\Gamma[A_\mu] = -\ln \det^{N/2}[iD_\mu(A_\mu)] \tag{12}.
\]
If we wish to quantize the gauge fields (i.e. to treat \( A_\mu \) as dynamical quantum fields), we require that the effective action \( \Gamma[A_\mu] \) be invariant under all gauge transformations. Gauge invariance is a classical symmetry of the theory, but may be violated at the quantum level by the introduction of chiral fermions. When \( \Gamma[A_\mu] \) is not gauge invariant, we say that the theory has a gauge anomaly.

Perturbative, or triangle, anomalies correspond to noninvariance of \( \Gamma[A_\mu] \) under gauge transformations \( U \) which can be smoothly deformed to the identity. The explicit form of the effective action was computed by Wess and Zumino \[12\]. Global anomalies correspond to noninvariance of \( \Gamma[A_\mu] \) under \( U \)'s which are nontrivial mappings of \( S^4 \) into the gauge group. In a theory with a global anomaly the fermion determinant changes sign under such a nontrivial gauge transformation \( U_\star \). But this tells us that in such a theory

\[
\Gamma[A_\mu^{U_\star}] = \Gamma[A_\mu] + im\pi,
\] (13)

where \( m \) is odd. This was originally noted in \[13\].

Now we turn to the GMZO result, which can be easily formulated in this language. Consider a theory with gauge group \( H \) and chiral fermions \( \psi \). Suppose that this theory exhibits neither a Witten anomaly (for example, \( \Pi_4(H) = 0 \)) nor a triangle anomaly. Then \( \Gamma[A_\mu] \) is completely gauge invariant and we can define a consistent quantum gauge theory based on \( H \) with fermions \( \psi \). Now consider any \( SU(2) \) subgroup of \( H \) (or in general any subgroup \( H' \) with nontrivial \( \Pi_4 ) \). It is clear that \( H' \) cannot have either a global or local anomaly. Either type of anomaly would require noninvariance of \( \Gamma[A_\mu^H/H' = 0, A_\mu'^H] \), where \( A^H \) are gauge fields in \( H' \) and \( A^{H/H'} \) are gauge fields in \( H/H' \). \( \Gamma[A_\mu^{H/H'} = 0, A_\mu'^H] \) is simply the effective action for gauge fields of \( H' \) which arises from integrating out the fermions \( \psi \). However, since by assumption \( \Gamma[A_\mu^{H/H'}, A_\mu'^H] \) is completely gauge invariant, \( H' \) cannot have either a global or local anomaly.

In other words, the nontrivial gauge transforms \( U_\star \subset H' \) are a subset of the gauge transformations of the full theory, \( U \subset H \). Indeed, if \( \Pi_4(H) = 0 \) then the \( U_\star \) maps can be continuously deformed to the identity in \( H \). Therefore, an anomalous global transformation in \( H' \) must correspond to an anomalous local transformation in \( H \). If these are absent from the full theory (cancelling triangle anomalies for \( H \)), then they must be absent from \( H' \) (no global anomaly in \( H' \)).

Note that these results are somewhat more general those of GMZO. They apply to groups \( H \) which are not simple, and also to groups \( H \) with \( \Pi_4(H) \neq 0 \) but in which the Witten anomaly is cancelled.
Finally, we mention an even more physical argument for the GMZO result. Suppose, contrary to GMZO, that one could formulate a theory with no triangle or Witten anomalies, but which has a Witten anomaly in a subgroup $H'$. Then, one could consider adding scalars whose vacuum expectations break the full group to $H'$ while not affecting the fermions. As the vacuum expectation values $v$ are taken to infinity the gauge fields corresponding to broken generators become arbitrarily massive. We are then left with a trivial effective low energy theory - any correlator computed to lowest order in $1/v$ is exactly zero! This is extremely pathological behavior for a theory which is perfectly well defined at short (less than $v^{-1}$) distances. Surely nature cannot allow the realization of such a model.

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A A Group Theory Toolbox

Here we collect some important group theoretical results that are utilized in this paper. For any SU(N), and indeed for any classical Lie algebra $H$, a famous theorem due to Dynkin states that a sequence of rank($H$) non-negative integers is a highest weight vector for some unique irreducible representation. The converse is also true, that any irreducible representation can be constructed by specifying a suitable sequence of non-negative integers from which the whole irrep. can be constructed by the action of the lowering operators.

Specify a highest weight for SU(N) (rank N-1) by N-1 non-negative integers

$$\Lambda \equiv (a_1, a_2, \ldots, a_{N-1}).$$  \hspace{1cm} (14)

In terms of Young tableaux, this is just the tableaux which has $a_i$ boxes over-hanging between row $i$ and row $i+1$. Every Young tableaux constructed by this prescription is admissible and corresponds to a valid SU(N) tensor. The dimension of this irrep. can then be computed using the factors over hooks rule. For example, in SU(3), the Dynkin labels $(1, 0), (0, 1)$ and $(1, 1)$ correspond to the 3, $\bar{3}$ and 8 respectively.

SDH thanks Richard Holman for discussions on this subject.
The advantage of using the Dynkin language is that not only can all irreps. be constructed by subtracting the rows of the Cartan matrix from the highest weight (the number of times indicated by the Dynkin indices), but also that it is rather straightforward to compute the eigenvalues of the invariant Casimir operators of both second and third order, which have appeared in this paper a number of times: the quadratic index has appeared in the computation of the number of zero modes and in the computation of the mixed $SU(2)^2 \otimes U(1)$ anomaly, whereas the cubic index has appeared in the computation of the pure $SU(2)^3$ anomaly contribution.

For $SU(3)$, we specify the highest weight of an irrep. by $(a_1, a_2)$. Then the dimension is $N(a_1, a_2) = (a_1 + 1)(a_2 + 1)(a_1+a_2+1)$, the quadratic index is $Q_2(a_1, a_2) = \frac{N(a_1, a_2)}{12}(a_1^2 + 3a_1 + a_1a_2 + 3a_2 + a_2^2)$, and the anomaly index is $Q_3(a_1, a_2) = \frac{N(a_1, a_2)}{60}(a_1 - a_2)(a_1 + 2a_2 + 3)(2a_1 + a_2 + 3) [10]$. A real representation of any $SU(N)$ is obtained if reversing the order of the Dynkin indices leaves the irrep. unchanged. For $SU(3)$ irreps., if $a_1 = a_2$ we hence obtain a real irrep., and since $Q_3$ is then zero, we find that the real representations (e.g. the adjoint), do not contribute to the anomaly. It is not true though that only real irreps. are anomaly free for higher $SU(N)$ groups. For instance, the 3048474 dimensional Dynkin irrep. $(5, 1, 8, 1)$ is complex but anomaly free in $SU(5)$. Note also that on complex conjugation of irreps., i.e. $\Lambda \leftrightarrow \bar{\Lambda}$, $Q_2(\Lambda) = Q_2(\bar{\Lambda})$, but $Q_3(\Lambda) = -Q_3(\bar{\Lambda})$, which is relevant when the contribution of opposite handedness fermions is included.

The total number of zero modes in an instanton background field is given by the $SU(2)$ quadratic index. In the normalization of this paper, for an isospin $I$ representation, it is given by $N_0 = \frac{2}{3}I(I + 1)(2I + 1)$. This same index, under a different guise, also appears in the computation of the mixed anomaly matching condition of the examples given in this paper. For tables of the quadratic indices for most Lie algebras of interest, and for other group theoretical data, please refer to [14].
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