A MULTISCALE ADVECTION SCHEME FOR DENSITIES ON MANIFOLDS

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Abstract. The task of computing a probability density advected by an dynamical system may be viewed as an infinite dimensional problem on the positive cone of the unsigned densities. Existing schemes exhibit numerical artifacts such as “negative probabilities”. In this article we present a method which preserves the positivity of probability densities at arbitrarily low resolutions. Moreover, we can use a single dense chart to transport the machinery of wavelet analysis to a manifolds. This allows us to avoid the use of transition maps between multiple charts, and to implement our method on a variety of non-Euclidean spaces at multiple length scales with existing wavelet algorithms designed on $\mathbb{R}^n$.

1. Introduction

The task of advecting a probability density presents itself in a variety of scenarios. Engineers are often presented with dynamical systems and incomplete knowledge of the initial conditions. If there is some region, $S$, of state-space which is “dangerous” he or she may wish to compute the probability of landing in this dangerous region at time $T$. If the initial condition is given in the form of a probability density $p(\cdot \mid t = 0)$ the probability of landing in $S$ at time $T$ is

$$P(x \in S \mid t = T) = \int_S p(dx \mid t = T)$$

where $p(dx \mid t = T)$ is the advected probability density at time $T$. In order to compute $p(dx \mid t = T)$ one must advect $p(dx \mid t = 0)$ under the flow of the dynamics.

Computing $p(dx \mid t = T)$ is expensive, as in the solution of any functional evolution equation. This cost is exponentially exacerbated by the curse of
we can define the scalars \( \rho \) condition \( p \) to numerically solve (2) (a first order linear evolution PDE) with the initial \( f \). The simplest type of spectral discretization of (2) is given by (2). This will be addressed in \( \rho \). Using (1) to compute \( \rho \) is difficult because \( \Phi_X \) is typically a non-linear map with no closed form expression. To compute \( \rho(x; t) \), it is often easier to numerically solve (2) (a first order linear evolution PDE) with the initial condition \( p_0 \). On a manifold \( M \), the advection equation is a linear evolution PDE involving the Lie-derivative where the description in a local coordinate chart is given by (2). This will be addressed in \( \rho \) (see (3)).

1.1. Background material. Let \( p_0 \) be a probability density on \( \mathbb{R}^n \) and let \( X \) be a vector-field on \( \mathbb{R}^n \). We denote the flow of \( X \) by \( \Phi_X \). We can define the time-dependent probability density \( p \) by

\[
(1) \quad p(x; t) = \det \left[ (D\Phi_X^t)^{-1}(x) \right] p_0 \left( \left[ \Phi_X^t \right]^{-1}(x) \right),
\]

where \( D\Phi_X^t(x) \) is the Jacobian matrix of \( \Phi_X^t \) at the point \( x \in \mathbb{R}^n \). The density \( p(x; t) \) represents how the density \( p_0(x) \) transforms under the flow of \( X \). We observe that \( p(x; 0) = p_0(x) \) and, upon taking a time derivative,

\[
(2) \quad \partial_t p(x; t) + \partial_i(X^i p)(x) = 0
\]

for all \( x \in \mathbb{R}^n \). Equation (2) is known as the Louiville equation (see [1]). Using (1) to compute \( p \) is difficult because \( \Phi_X \) is typically a non-linear map with no closed form expression. To compute \( p(x; t) \), it is often easier to numerically solve (2) (a first order linear evolution PDE) with the initial condition \( p_0 \). On a manifold \( M \), the advection equation is a linear evolution PDE involving the Lie-derivative where the description in a local coordinate chart is given by (2). This will be addressed in \( \rho \) (see (3)).

1.2. A naive pseudo-spectral method. In this section we will present the simplest type of spectral discretization of (2). Let \( f^0, f^1, f^2, \ldots \in L^2(\mathbb{R}^n) \) serve as an orthonormal Hilbert basis (e.g. a Fourier basis). Let \( \rho(x, t) \) be the solution to (2). Assuming \( \rho(\cdot, t) \in L^2 \) and \( \partial_t X^i \in L^\infty \), we can define the scalars \( \rho_j(t), A_j^k \in \mathbb{R} \) by \( \rho_j(t) = \langle \rho(\cdot, t), f^j \rangle_{L^2(\mathbb{R}^n)} \) and \( A_j^k = \langle (\partial_t X^i) f^j, f^k \rangle_{L^2(\mathbb{R}^n)} \). Then \( \rho_j(t) \) satisfies the infinite-dimensional linear ordinary differential equation \( \partial_t \rho_j = -\sum_{k=0}^\infty A_j^k \rho_k \). Moreover, \( \rho(x, t) = \rho_j(t) f_j(x) \). We can truncate this system at some finite \( N \in \mathbb{N} \) to obtain an \( N \)-dimensional linear ordinary differential equation \( \partial_t \rho_{N,j} = -\sum_{k=0}^N A_j^k \rho_{N,k} \) for \( j = 0, \ldots, N \). It is notable that if \( f_0, f_1, \ldots \) is a Haar basis, then at finite resolution, this algorithm is equivalent to partitioning the space into cells and the basis is equivalent to a set of indicator functions on the cells. The matrix \( A_j^k \) is then simply a matrix of fluxes known as a transfer operator. Under the right circumstances, this method converges as the cell width approaches 0 and \( N \to \infty \) [2].

However, for finite \( N \), there is no guarantee that the reconstructed density \( \hat{\rho}(x, t) = \sum_{j=0}^N \rho_{N,j}(t) f_j(x) \) is non-negative. Generically \( \hat{\rho}(x, t) \) will take on both signs, in contrast with the exact solution \( \rho(x, t) \) (see figure [1]).
Figure 1. Red lines indicate numerically computed densities advected by, $\dot{x} = \sin(2x)$, via the transfer operator method with a resolution of $2\pi/2^5$ at time $t = 1$ and $t = 2$. The initial condition is a uniform distribution. Black lines plot the exact solution given by (7).

Moreover, important entities such as the advected moments $\tilde{m}_i^k(t) = \int \tilde{\rho}(x, t)(\Phi_t^X)_*((x_i)^k)dx$ may fluctuate, also in contrast with the exact moments $m_i^k = \int \rho(x, t)(\Phi_t^X)_*((x_i)^k)dx$.

From the standpoint of interpretation, positivity and moments are important aspects of probability densities. There is no agreed upon interpretation of “negative probability”, and aspects such as moments serve as important qualitative characteristics of a given probability density (the zeroth moment is the average, the first moment is the variance, and so on).

When dealing with manifolds of even moderate dimensions (e.g. $d = 3$) it is important for a method to behave well at finite $N$'s because it is infeasible to finely resolve along each dimension. If the qualitative aspects of flows (such as conserved quantities and positivity constraints) can be incorporated into the numerics at arbitrary resolutions it is more likely that the resulting numerical scheme will produce qualitatively accurate results. This allows the analyst to focus on other aspects of the system at hand without worrying about the inconsistency errors which arise in schemes which do not preserve structure.

1.3. Main contributions. In this paper we will present a method for advection of probability densities on manifolds. This method will yield reconstructed probability measures which are non-negative and mass conserving at any finite resolution.

2. Mathematical preliminaries

Throughout this section we will have the following setup. Let $M$ be a smooth manifold. We denote the tangent bundle of $M$ by $TM$. Given any
A $C^1$ function $f : M \to N$, we denote the tangent lift of $f$ by $Tf : TM \to TN$.

2.1. Densities. A smooth density on $M$ is a smooth means of assigning real numbers to measurable sets. Heuristically, it is a map which takes an infinitesimal box (or volume element) on a manifold as input, and outputs the infinitesimal “size” of the box (a real number). Therefore, in order to discuss densities, we must first formalize the notion of an “infinitesimal box.” This motivates our introduction of frames.

**Definition 2.1.** Given a manifold $M$, and a point $x \in M$, a frame at $x$ is a basis on the tangent space $T_x M$. We denote the set of frames at $x$ by $Fr_x M$. The frame bundle is $Fr M = \cup_{x \in M} Fr_x M$.

**Proposition 2.2.** There is a transitive action of $GL(n)$ on each fiber of $Fr M$.

**Proof.** Let $e = \{e_1, \ldots, e_n\} \in Fr_x M$ for some $x \in M$. For each $A \in GL(n)$, define the left action

$$A \cdot (e_1, \ldots, e_n) := (A^1_1 e_j, \ldots, A^1_n e_j).$$

By inspection this actions is free and transitive. □

Now that we understand frames (i.e. infinitesimal boxes) sufficiently well, we may introduce the notion of densities.

**Definition 2.3** (Appendix A [3]). Let $\alpha > 0$. An $\alpha$-density on a manifold, $M$, is a map $\rho : Fr M \to \mathbb{R}$ such that for any frame $e \in Fr M$ and $A \in GL(n)$, $\rho(A \cdot e) = |\det(A)|^\alpha \rho(e)$. We denote the space of densities by $Dens^\alpha(M)$. A 1-density is simply called a density, and so we denote $Dens^1(M)$ by $Dens(M)$. The integral of a density is defined via the same construction as that for $n$-forms [4, Ch. 14]. A mass density is a density which is non-negative, and it is called a probability density if its integral is unity.

One can observe immediately that $Dens^\alpha(M)$ is a vector-space. Despite this commonality with tensors, 1-densities are not tensors. Densities are very close in spirit to $n$-forms, but unlike $n$-forms, densities are non-oriented due to the use of “$|\det(A)|$” rather than “$\det(A)$” in the definition. Therefore, a density will not flip signs under a change of basis. This allows for the integral of a density to be well defined on non-orientable manifolds as well [4, Ch. 14].

**Proposition 2.4** (Appendix A [3]). Let $\psi_1, \psi_2 \in Dens^{1/2}(M)$. The function, $\psi_1 \psi_2$, obtained by scalar multiplication is a 1-density. The pairing $\langle \psi_1, \psi_2 \rangle := \int_M \psi_1 \psi_2$ is a real inner-product. Finally, for any density $\rho$, the functions $\pm \sqrt{\rho}$ are $\frac{1}{2}$-densities.

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1This makes $Fr M$ a $GL(n)$ principal bundle over $M$. 
\textbf{Proof.} Let \( e \in \text{Fr} \, M \) and \( A \in \text{GL}(n) \). We observe \( \psi_1(A \cdot e)\psi_2(A \cdot e) = |\det(A)|\psi_1(e)\psi_2(e) \). Thus \( \psi_1\psi_2 \in \text{Dens}(M) \). Conversely

\[ \pm \sqrt{\rho(A \cdot e)} = \pm |\det(A)|^{1/2} \sqrt{\rho(e)}. \]

So \( \pm \sqrt{\rho} \in \text{Dens}^{1/2}(M) \). Finally, if \( \psi \neq 0 \) we see that \( \|\psi\| := \langle \psi, \psi \rangle \neq 0 \). Thus \( \langle \cdot, \cdot \rangle \) is weakly non-degenerate and defines an inner-product on \( \text{Dens}^{1/2}(M) \).

Note that for any \( C^1 \) diffeomorphism \( \Phi : M \to M \), there is a map \( \text{Fr}(\Phi) : \text{Fr} \, M \to \text{Fr} \, M \) given by \( \left(e_1, \ldots, e_n\right) \mapsto \left(\Phi \cdot e_1, \ldots, \Phi \cdot e_n\right) \). This defines the \textit{pull-back} of an \( \alpha \)-density \( \nu \in \text{Dens}^\alpha(N) \) by \( \Phi^* \nu := \nu \circ \text{Fr}(\Phi) \in \text{Dens}^\alpha(M) \).

\textbf{Proposition 2.5.} Let \( \Phi \in \text{Diff}(M) \). The transformation \( \psi \mapsto \Phi^*\psi \) for \( \psi \in \text{Dens}^{1/2}(M) \) is an isometry with respect to the inner product on \( \text{Dens}^{1/2}(M) \).

\textbf{Proof.} Let \( \psi_1, \psi_2 \in \text{Dens}^{1/2}(M) \) and observe

\[ \langle \Phi^*\psi_1, \Phi^*\psi_2 \rangle = \int \Phi^*(\psi_1\psi_2) = \int \psi_1\psi_2 = \langle \psi_1, \psi_2 \rangle, \]

where the equivalence of the integrals follows from [3] Proposition 14.32(c). \( \square \)

Given a vector field \( X \in \mathfrak{X}(M) \) we can denote the flow by \( \Phi^t_X \in \text{Diff}(M) \) and define the Lie-derivative of an alpha density by \( \mathcal{L}_X[\nu] := \frac{d}{dt} |_{t=0} (\Phi^t_X)^* \nu \). This yields the following corollary to proposition 2.5.

\textbf{Corollary 2.6.} For any \( X \in \mathfrak{X}(M) \), \( \mathcal{L}_X[\cdot] \) is an anti-symmetric linear operator on \( \text{Dens}^{1/2}(M) \).

With the Lie-derivative defined we can write the advection PDE for \( \alpha \)-densities as

\begin{equation}
\partial_t \nu + \mathcal{L}_X[\nu] = 0 \quad , \quad \nu(t) \in \text{Dens}^\alpha(M).
\end{equation}

This is the equation for a time-dependent \( \alpha \)-density which is advected by the vector field \( X \in \mathfrak{X}(M) \). If \( \alpha = 1 \), [3] is written in local coordinates as [2]. If \( \nu(t) = \psi(t) \in \text{Dens}^{1/2}(M) \) is a half-density, then [3] is written in local coordinates as

\begin{equation}
\partial_t \psi(x) + \frac{1}{2} X^i(x) \partial_i \psi(x) + \frac{1}{2} \partial_i(\psi \cdot X^i)(x) \psi(x) = 0.
\end{equation}

Note that this equation appears to be the average of the advection equation for densities (\( \text{Dens}^1(M) \)) and the advection equation for functions (\( \text{Dens}^0(M) \)).

\textbf{Theorem 2.7.} Let \( \rho(t) \in \text{Dens}(M) \) be a time-dependent probability density and let \( \psi \in \text{Dens}^{1/2}(M) \) be such that \( \rho = \psi^2 \). Assume that \( \rho \) is \( C^1 \). Let \( X \in \mathfrak{X}(M) \). The following are equivalent:
(1) \( \rho \) satisfies the advection equation \([3]\) with \( \alpha = 1 \) (locally given by \([2]\)).
(2) \( \psi \) satisfies the advection equation \([3]\) with \( \alpha = 1/2 \) (locally given by \([4]\)).

Proof. Let \( \psi \) satisfy \([3]\). We find
\[
\partial_t(\psi^2) = 2\partial_t\psi \cdot \psi = -2\mathcal{L}_X[\psi] \psi = -2 \left. \frac{d}{dt} \right|_{t=0} (\Phi_X^t)^* \psi \cdot \psi \\
= - \left. \frac{d}{dt} \right|_{t=0} [(\Phi_X^t)^* \psi \cdot (\Phi_X^t)^* \psi] = - \left. \frac{d}{dt} \right|_{t=0} [(\Phi_X^t)^*(\psi^2)] \\
= -\mathcal{L}_X[\psi^2].
\]
Therefore \( \rho = \psi^2 \) satisfies \([3]\). Conversely, if \( \rho \) satisfies \([3]\) and \( \psi^2 = \rho \) then
\[
\partial_t(\psi^2) = -\mathcal{L}_X[\psi^2] = -\mathcal{L}_X[\psi] \psi.
\]
Moreover, the right hand side is \( 2(\partial_t \psi) \psi \). We can divide both side by \( \psi \) at any point where \( \psi(x) \neq 0 \). By continuity of \( \partial_t \rho \) and \( \partial_t \psi \) we can verify \([3]\) on the entire support of \( \psi \) (which is also the support of \( \rho \)). Outside the support it is necessarily the case that \( \rho = 0 \) and \( \partial_t \rho = 0 \). We observe that \( \mathcal{L}_X[\rho] = 0 \) as well. In this case \( \mathcal{L}_X[\psi] = 0 \) by the same argument. So we’ve verified \([3]\) on the entire domain.

We will use theorem \([2.7]\) later to justify building a numerical scheme to solve \([3]\) with \( \alpha = 1/2 \) in lieu of solving \([3]\) with \( \alpha = 1 \).

2.2. Euclidean realizations. We would like to apply wavelet theory later for the analytical tools and sparsity structure they carry. However, the notion of wavelets on manifolds is still young, and virtually all of the available wavelet analysis machinery is developed on Euclidean spaces and tori. In this section we will present theorems which allow us to transform analysis on manifolds into problems of analysis on subspaces of functions on \( \mathbb{R}^n \).

**Theorem 2.8.** Let \( \varphi : U \subset M \to V \subset \mathbb{R}^n \) be a chart. As \( \varphi \) is injective, we can invert it on the range \( V \). If \( U \) is dense in \( M \) then the maps
\[
f \in C^k(M) \mapsto \varphi_* f = f \circ \varphi^{-1} \in C^k(V) \\
\nu \in \text{Dens}^\alpha(M) \mapsto \varphi_* \nu = \nu \circ \text{Fr}(\varphi^{-1}) \in \text{Dens}^\alpha(V) \\
X \in \mathfrak{X}(M) \mapsto \varphi_* X = T\varphi \cdot X \circ \varphi^{-1} \in \mathfrak{X}(V)
\]
are injective ring/vector-space/Lie-algebra morphisms respectively.

Proof. Let \( f, g \in C^k(M) \) be such that \( \varphi_* f = \varphi_* g \). Assume \( f \neq g \). Since the range of \( \varphi^{-1} \) is \( U \), it must be the case that \( f(x) \neq g(x) \) for some \( x \notin U \). As \( U \) is dense in \( M \) there is a sequence \( x_0, x_1, \ldots \) in \( U \) which converges to \( x \). As \( f \) and \( g \) are continuous \( g(x_i) = f(x_i) \) must converge to a unique limit. Thus \( g(x) = f(x) \), contradicting the assumption that \( f \neq g \). Therefore the map \( f \in C^k(M) \to \varphi_* f \in C^k(V) \) is injective. That it is a ring morphism can then be viewed directly, \( \varphi_*(f \cdot g) = \varphi_*(f) \cdot \varphi_*(g) \) for any \( f, g \in C^k(M) \).
The same argument applies to vector-fields upon noting \( \varphi_*(X + cY) = \varphi_*X + c \varphi_*Y \) for any \( X, Y \in \mathfrak{X}(M) \) and \( c \in \mathbb{R} \), and \( \varphi_*([X, Y]) = [\varphi_*X, \varphi_*Y] \).

Finally, the same argument applies to \( \alpha \)-densities upon noting \( \varphi_*(\nu + c\mu) = \varphi_*\nu + c\varphi_*\mu \) for any \( \nu, \mu \in \text{Dens}^\alpha(M) \).  

\[ \square \]

**Corollary 2.9.** Assume the setup of theorem 2.8. The map

\[ \psi \in \text{Dens}^{1/2}(M) \mapsto \varphi_*\psi \in \text{Dens}^{1/2}(V) \]

is an isometry.

**Proof.** Simply observe

\[ \langle \varphi_*\psi_1, \varphi_*\psi_2 \rangle = \int_V \varphi_*(\psi_1) \varphi_*(\psi_2) = \int_V \varphi_*(\psi_1 \cdot \psi_2) = \int_U \psi_1 \cdot \psi_2. \]

As \( U \) is dense in \( M \), the above integral is unchanged by integration over \( M \). Thus we’ve verified \( \langle \varphi_*\psi_1, \varphi_*\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle \).  

\[ \square \]

The upshot of theorem 2.8 is that we may represent PDEs on manifolds as PDE’s on Euclidean domains. In particular, theorem 2.8 equates the function space \( C^k(M) \) with the subring

\[ \varphi_*C^k(M) := \{ g \in C^k(V) \mid g = f \circ \varphi^{-1}, f \in C^k(M) \} \subset C^k(V). \]

Thus any, PDE on \( M \) can be fully represented as a PDE on a subring of functions on \( V \).

This is useful for solving the PDE under concern because it allows us to implement wavelet analysis on manifolds. Specifically, given our PDE on \( \text{Dens}^{1/2}(M) \), we may invoke the above isometry to get a PDE on the space \( \text{Dens}^{1/2}(V) \equiv L^2(V) \). We can then solve this PDE on \( L^2(V) \) in a wavelet basis. This will give us time dependent function which we can pull-back to \( M \).

2.3. **Example: A Euclidean realization of \( S^2 \).** Consider the 2-sphere, \( S^2 \subset \mathbb{R}^3 \). If we use spherical coordinates we obtain a map \( \varphi : (-\pi, \pi) \times (0, \pi) \to U \subset S^2 \) given by

\[ \chi(\phi, \theta) = \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}. \]

Then we find

\[ \varphi_*C^0(S^2) = \{ f \in C^0((-\pi, \pi) \times (0, \pi) \mid \lim_{\theta \to 0} f(\phi, \theta) = f_{\text{north}} \text{ for some } f_{\text{north}} \in \mathbb{R} \}
\lim_{\theta \to \pi} f(\phi, \theta) = f_{\text{south}} \text{ for some } f_{\text{south}} \in \mathbb{R} \} \]

The canonical volume form on \( S^2 \) viewed as a subset of \( \mathbb{R}^3 \) is given by

\[ \mu_{S^2} = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy. \]

In spherical coordinates, this volume form is given by

\[ \varphi_*(\mu_{S^2}) = \sin(\theta)d\theta \wedge d\phi. \]
It is easy to observe that $|\mu|_2^\alpha \in \text{Dens}^\alpha(M)$ and arbitrary $\nu \in \text{Dens}^\alpha(M)$ can always be written as $\nu = f \otimes |\mu|_2^\alpha$ for some function $f \in C^0(M)$. The push-forward of $\nu$ by $\varphi$ is given by $\varphi_*\nu = (\varphi_* f) \otimes (\varphi_* |\mu|_2^\alpha)$. Therefore, by theorem 2.8 the inner-product space of half densities $\text{Dens}_{1/2}(S^2)$ is isometric to the subspace

$$\varphi_* \text{Dens}_{1/2}(S^2) = \left\{ \psi \in \text{Dens}_{1/2}(V) \mid \psi = f \otimes (\sin(\theta)d\theta \wedge d\phi)^{1/2}, \right\}.$$

3. An advection scheme

In this section we will use Euclidean realizations of wavelet space to implement an advection scheme for densities on manifolds. More specifically, we want to solve the evolution equation

$$\partial_t \rho + \mathcal{L}_X[\rho] = 0, \quad \rho(0) = \rho_0 \in \text{Dens}^1(M).$$

We are going to do this by first solving the half-density advection equation

$$\partial_t \psi + \mathcal{L}_X[\psi] = 0, \quad \psi(0) = \psi_0 := \rho^{1/2} \in \text{Dens}^{1/2}(M).$$

If $\psi(t)$ is a solution to (6), then theorem 2.7 tells us that $\rho(t) := |\psi(t)|^2$ is a solution to our original problem (5). Therefore we seek to solve (6) in lieu of solving (5). There are two reasons for doing this. Firstly, since $\rho(t)$ will be obtained by squaring something, it will necessarily be positive. Secondly, we may invoke the natural inner-product on $\text{Dens}^{1/2}(M)$ to construct our advection scheme. There is no need to introduce a non-canonical innerproduct, as is the case in many spectral methods.

The following is the algorithm we wish to consider in this paper. Let $E_n = \{f_0, \ldots, f_n\}$ be set of half-densities on $M$ and let $\text{pr}_n : \text{Dens}^{1/2}(M) \rightarrow E_n$ be the orthogonal projection with respect to the inner-product on half-densities. For $X \in \mathfrak{X}(M)$, if we let $\mathcal{L}_X$ denote the Lie derivative on $\text{Dens}^{1/2}(M)$ then we can consider the operator

$$[\mathcal{L}_X]_n = \text{pr}_n \circ \mathcal{L}_X|_{E_n} : E_n \rightarrow E_n$$

as a finite-dimensional approximation of $\mathcal{L}_X$. If this approximation is any good (i.e. convergent as $n \rightarrow \infty$ a dense subspace of $\text{Dens}^{1/2}(M)$) , we may solve the finite dimensional linear ODE

$$\dot{\psi}_n = [\mathcal{L}_X]_n \cdot \psi_n, \quad \psi_n(0) = \text{pr}_n(\psi_0) \in E_n.$$

Then $\psi(t) \approx \psi_n(t)$ for sufficiently large $n$, and we have an approximate solution to (6), and therefore an approximation to (5).

4. A Benchmark computation

Consider the ODE $X(x) = \sin(2x)\partial_x$ on the unit circle, $S^1$. We can solve for the flow in closed from. We find $\Phi^t_X(x) = \arccot(e^{-2t}\cot(x))$, and the inverse of $\Phi^t_X$ is the map $[\Phi^t_X]^{-1}(y) = \arccot(e^{2t}\cot(y))$. Using (1) we
obtain the solution to (5) with initial condition $p_0(x)$ as the time-dependent density

\begin{equation}
  p(x, t) = p_0 \left[ \arccot \left( \cot(x)e^{2t} \right) \right] \left( e^{2t} \cos^2(x) + e^{-2t} \sin^2(x) \right)^{-1}.
\end{equation}

Similarly, the solution to (6) with the initial condition $\psi_0(x)$ is the time-dependent half-density

\[
  \psi(x, t) = \psi_0 \left[ \arccot \left( \cot(x)e^{2t} \right) \right] \left( e^{2t} \cos^2(x) + e^{-2t} \sin^2(x) \right)^{-1/2}.
\]

We can use the Haar-wavelet to implement the method mentioned in section 1.2. This equivalent to the transfer operator approach wherein one partitions the space into cells and computes fluxes [2]. We do this for a uniform distribution on the circle, and depict the numerically computed solutions at time $t = 1, 2$ in figure 1 (page 3). This scheme appears to be at least qualitatively accurate at time $t = 1$ and earlier. However, the transfer operator method exhibits spurious spatial oscillations and negative probability densities at $t = 2$ and beyond.

**Figure 2.** Snapshots of numerically computed densities at $t = 1, 2$ with respect to the ODE “$\dot{x} = \sin(2x)$”. Red indicates densities computed via our numerical scheme (3) with a resolution of $2\pi/2^5$ using the Haar wavelet. Black indicates the exact solution, (7). The initial condition is a uniform density.

For the purpose of comparison, we can also use the Haar wavelet to implement the half-density based method described in section 3. Numerically computed solutions are with respect to our new method are depicted in figure 2 (page 9). We observe that the half-density advection scheme maintains positivity and the regularity of the exact solution.

Both schemes begin to fail once the exact solution becomes concentrated in a space below the chosen resolution of $2\pi \times 2^{-6}$.

Finally, we can also implement our own method using any wavelet. Snapshots of our method using two different Daubechies (DB) wavelets (the DB 4 and 6 wavelets) each at two different scales are illustrated in figure 3 (page 10). Notice that although four alternatives appear accurate at the two time instances depicted, the DB 6 wavelet appears to perform at both scales.
Figure 3. Ground truth densities (black) and densities computed using our method (red) with different DB wavelets (rows 1 and 2 use the DB 4 wavelet and rows 3 and 4 use the DB 6 wavelet) at different scales (rows 1 and 3 are computed via bases functions with a resolution of $2^{-1}$ and rows 2 and 4 are computed via bases functions with a maximum resolution of $2^{-2}$) for the vector field $\dot{x} = \sin(2x)$ at $t = 1$ (left column) and 2 (right column) and with a uniform density at $t = 0$. Beginning from the top row the number of bases functions used to compute the densities in each row are 88, 160, 100, and 176.
5. Example: the rigid body equations

In the absence of external forces the equations of motion for the angular momentum, $\Pi \in \mathbb{R}^3$, of a rigid body are

$$\dot{\Pi} = \Pi \times (I^{-1} \cdot \Pi),$$

where $I = \text{diag}(I_1, I_2, I_3)$, and $I_1, I_2, I_3 > 0$ are rotational inertias along the principal axes of rotation. As $\Pi$ is orthogonal to $\Pi$, we observe that $\|\Pi\|$ is constant. Thus the dynamics are constrained to spheres. Moreover, the dynamics on a sphere of radius $r > 0$ are identical to the dynamics on a sphere of unit radius upon rescaling time by $r^{-2}$. Therefore we may (literally) restrict our analysis of this system to an ode on the unit sphere $S^2 \subset \mathbb{R}^3$. In spherical coordinates, the dynamics are given by

$$\dot{\phi} = -\sin^2(\phi) \cos(\theta) \frac{I_3 - I_2}{I_3 I_2} + \cos^2(\phi) \cos(\theta) \frac{I_1 - I_3}{I_1 I_3}$$

$$\dot{\theta} = -\sin(\theta) \sin(\phi) \cos(\phi) \frac{I_2 - I_1}{I_2 I_1}$$

$$\dot{r} = 0$$

A plot of various trajectories on $S^2$ is depicted in figure 4.

![Figure 4. trajectories of (8) on the unit sphere.](image.png)

We can consider the global chart induced by spherical coordinates. Specifically, this is the chart with

$$U = S^2 \setminus (0, 0, -1) \quad , \quad V = (0, \pi) \times (-\pi, \pi) \quad , \quad \varphi(\theta, \phi) = \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$
We may approximate $C^0(V)$ using the Haar wavelet, and then push-forward these approximations to obtain an advection scheme on $S^2$.

![Figure 5. Snapshots at time $t = 0.0, 2.5, 5.0, 7.5$ of evolution on the 2-sphere, according to the rigid body equations.](image)

We simulate an initial density which consists of a smooth bump function around the point $(-1,0,0)$. There is a saddle point at $(-1,0,0)$, and so we should expect the distribution to tend towards a singular distribution concentrated along the unstable submanifold associated to the saddle point. The result is depicted in figure 5.

6. Conclusion and future work

In this article we have presented a spectral advection scheme which preserves the positivity of probability densities at arbitrarily low resolutions. However, there are two major items which will constitute future work for clarifying the range of applicability of this scheme. Firstly, we have not shown under which circumstance the scheme is convergent. It may be intuitively clear that this scheme will converge under a “reasonable” choice of basis, but this is not precise. Future work entails clarifying what “reasonable” means. Secondly, the scheme appears to conserve a number of other properties at arbitrarily low resolutions. For example, the space of functions is discretized as a ring of Hermitian operators. This ring structure is preserved under our scheme. Enumerating the many other structures which are preserved by the scheme will be elucidated in future work as well.

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