A REMARK ON GEODESICS IN THE BANACH MAZUR DISTANCE

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Abstract. We show that there are uncountably many geodesics between any two non-isometric \( n \)-dimensional normed spaces. We construct two explicit geodesics that can be used to describe all the points of the other geodesics.

1. Introduction

The (multiplicative) Banach-Mazur distance between the \( n \)-dimensional normed spaces \( E \) and \( F \) is given by

\[
d(E, F) = \inf \{ \|T\|\|T^{-1}\| : T : E \to F \text{ is an isomorphism} \}.
\]

This quantity was introduced by Pełczyński and it measures how close \( E \) and \( F \) are isomorphic to each other. One can easily check that the infimum is attained, that \( d(E, F) = 1 \) iff \( E \) and \( F \) are isometric, and that \( d(E, F) \leq d(E, H)d(H, F) \). It follows that \( \log d(E, F) \) is a distance in the Banach Mazur compactum \( BM_n \), the set of isometry classes of \( n \)-dimensional normed spaces, that turns out to be compact with this metric.

The Banach Mazur distance has a geometric interpretation. Suppose that \( E = (\mathbb{R}^n, B_E) \) and \( F = (\mathbb{R}^n, B_F) \), where \( B_E \) and \( B_F \) are the unit balls of \( E \) and \( F \). Then one can easily check that \( d(E, F) \) is the smallest number \( L \geq 1 \) such that there exists an invertible map \( T : F \to E \) such that \( B_E \subset T(B_F) \subset LB_E \). If \( T : F \to E \) attains the distance and if we replace \( F \) by its isometric version \( (\mathbb{R}^n, T(B_F)) \), we assume without loss of generality that

\[
(1) \quad B_E \subset B_F \subset d(E, F)B_E.
\]

This means that when the normed spaces are put in this canonical position, the distance is attained by the identity map.

Suppose that \((M, \rho)\) is a metric space. The length of a path \( \gamma : [a, b] \to (M, \rho) \) is defined by

\[
L(\gamma) = \sup_{a=t_0 \leq t_1 \leq \cdots \leq t_n = b} \sum_{i=1}^{n} \rho(\gamma(t_i), \gamma(t_{i-1})),
\]

where the supremum runs over all possible partitions. By the triangle inequality, we have that \( \rho(\gamma(a), \gamma(b)) \leq L(\gamma) \). A geodesic between \( x \in M \) and \( y \in M \) is a path \( \gamma \) that starts at \( x \), ends at \( y \), and that has length equal to \( \rho(x, y) \).

Definition 1. A metric space \((M, \rho)\) is a geodesic space if every two points in \( M \) are joined by a geodesic.

For the (multiplicative) distance in the Banach Mazur compactum \( BM_n \), the relevant concept is the following:

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Definition 2. The \( n \)-dimensional normed space \( H \) is an intermediate space between \( E \) and \( F \) if \( d(E, F) = d(E, H) d(H, F) \).

We easily see that

Lemma 1. A path \( \gamma : [a, b] \to BM_n \) is a geodesic from \( E \) to \( F \) iff \( \gamma(a) = E, \gamma(b) = F \), and for every \( a = t_0 < t_1 < \cdots < t_n = b \), \( d(E, F) = \prod_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) \).

Classical spaces provide examples of geodesics. It follows from Holder’s inequality that the paths \( \{ \ell_p^n : 1 \leq p \leq 2 \} \) and \( \{ \ell_p^n : 2 \leq q \leq \infty \} \) are geodesics in \( BM_n \). Indeed, if \( 1 \leq r < s \leq \infty \), \( B_{\ell_p^r} \subset B_{\ell_p^s} \subset n^{\frac{1}{r} - \frac{1}{s}} B_{\ell_p^2} \). When \( 2 \leq q < \infty \) we have

\[
B_{\ell_2^q} \subset B_{\ell_\infty^q} \subset n^{\frac{1}{q}} B_{\ell_2^q} \quad \text{and} \quad B_{\ell_2^q} \subset B_{\ell_\infty^q} \subset n^{\frac{1}{q} - \frac{1}{\infty}} B_{\ell_2^q}.
\]

This implies that \( d(\ell_p^n, \ell_\infty^q) \leq n^{\frac{1}{q} - \frac{1}{\infty}} \) and that \( d(\ell_p^n, \ell_\infty^q) \leq n^{\frac{1}{q}} \). Since

\[
\sqrt{n} = d(\ell_2^n, \ell_\infty^n) \leq d(\ell_2^n, \ell_q^n) d(\ell_q^n, \ell_\infty^n) \leq n^{\frac{1}{q} - \frac{1}{\infty}} n^{\frac{1}{q}} = \sqrt{n},
\]

we conclude that \( d(\ell_2^n, \ell_\infty^n) = n^{\frac{1}{q} - \frac{1}{\infty}} \) and \( d(\ell_2^n, \ell_\infty^n) = n^{\frac{1}{q}} \). If \( 2 < q_1 < q_2 < \infty \), then \( B_{\ell_2^{q_1}} \subset B_{\ell_2^{q_2}} \subset n^{\frac{1}{q_1} - \frac{1}{q_2}} B_{\ell_2^2} \), which implies that \( d(\ell_2^{q_1}, \ell_2^{q_2}) \leq n^{\frac{1}{q_1} - \frac{1}{q_2}} \). Since

\[
n^{\frac{1}{q_1} - \frac{1}{q_2}} = d(\ell_2^{q_1}, \ell_2^{q_2}) \leq d(\ell_2^{q_1}, \ell_\infty^{q_2}) d(\ell_\infty^{q_1}, \ell_\infty^{q_2}) \leq n^{\frac{1}{q_2} - \frac{1}{\infty}} n^{\frac{1}{q_1} - \frac{1}{q_2}} = n^{\frac{1}{q_1} - \frac{1}{q_2}},
\]

we conclude that \( d(\ell_\infty^{q_1}, \ell_\infty^{q_2}) = n^{\frac{1}{q_1} - \frac{1}{q_2}} \). And from this, it follows that \( \{ \ell_q^n : 2 \leq q < \infty \} \) is a geodesic. The case \( \{ \ell_p^n : 1 \leq p \leq 2 \} \) is similar.

Notice that the path \( \{ \ell_p^n : 1 \leq p \leq \infty \} \) is not a geodesic because \( d(\ell_\infty^n, \ell_\infty^n) < d(\ell_2^n, \ell_2^n) d(\ell_2^n, \ell_\infty^n) = \sqrt{n} = n \) (see [4], for the case of \( n = 3 \) see [5]).

Interpolation spaces can be used to obtain geodesics and the previous examples illustrate this for \( \ell_p^n \) spaces. The technique works in more general settings, including in operator spaces (see [1]). Our results can be interpreted using the real interpolation \( K \)-method and \( J \)-method. However, we prefer to use geometric language.

2. Characterization of Intermediate Spaces

In this section we show that there are many intermediate spaces between any two non-isomorphic \( n \)-dimensional normed spaces. We first identify the “extreme” intermediate spaces and then use them to determine all of them and to describe geodesics.

We fix some notation. \( E \) and \( F \) are two non-isomorphic \( n \)-dimensional spaces that satisfy \( B_E \subset B_F \subset d(E, F) B_E \). For \( 0 < \lambda < 1 \), let \( E_\lambda = (\mathbb{R}^n, B_\lambda) \) and \( F_\lambda = (\mathbb{R}^n, C_\lambda) \) be the normed spaces with unit balls defined by

\[
B_\lambda = (d(E, F)^\lambda B_E) \cap B_F \quad \text{and} \quad C_\lambda = \text{Conv} \left[ B_E \bigcup \frac{1}{d(E, F)^{1-\lambda}} B_F \right],
\]

where \( \text{Conv}(S) \) is the convex hull of the set \( S \).

Lemma 2. For \( 0 < \lambda < 1 \), \( B_E \subset C_\lambda \subset B_\lambda \subset B_F \)

Proof. Let \( 0 < \lambda < 1 \) and let \( d = d(E, F) \). \( B_E \subset C_\lambda \) and \( B_\lambda \subset B_F \) follow from the definition of \( C_\lambda \) and \( B_\lambda \). We need to show that \( C_\lambda \subset B_\lambda \). By convexity of \( B_\lambda \), we only need to check that \( B_E \subset B_\lambda \) and that \( B_F \subset d(E, F)^{1-\lambda} B_\lambda \). Since \( d^\lambda \geq 1 \), \( B_E \subset d^\lambda B_E \); and since \( B_E \subset B_F \) (from (1)), we conclude that \( B_E \subset B_\lambda \). To check
the other inclusion, notice that \( d^{1-\lambda}B_\lambda = (dB_E) \cap (d^{1-\lambda}B_F) \). Then \( B_F \subset dB_E \) from (1) and since \( d^{1-\lambda} \geq 1 \), \( B_F \subset d^{1-\lambda}B_F \).

Using the notation of (2) we have

**Proposition 1.** For \( 0 < \lambda < 1 \), the spaces \( E_\lambda \) and \( F_\lambda \) are intermediate spaces between \( E \) and \( F \). More precisely, \( d(E, E_\lambda) = d(E, F_\lambda) = d(E, F)^\lambda \) and \( d(E_\lambda, F) = d(F_\lambda, F) = d(E, F)^{1-\lambda} \).

**Proof.** Let \( 0 < \lambda < 1 \) and \( d = d(E, F) \). We start with \( E_\lambda \). We claim that \( B_E \subset B_\lambda \subset dB_E \). The first inclusion follows from Lemma 2 and the second follows from the definition of \( B_\lambda \). Notice that this implies that \( d(E, E_\lambda) \leq d^\lambda \). Now we claim that \( B_\lambda \subset B_F \subset d^{1-\lambda}B_\lambda \). The first inclusion follows from Lemma 2. The definition of \( C_\lambda \) implies that \( B_F \subset d^{1-\lambda}C_\lambda \). Then by Lemma 2 again, \( B_F \subset d^{1-\lambda}C_\lambda \subset d^{1-\lambda}B_\lambda \). Notice that this implies that \( d(E_\lambda, F) \leq d^{1-\lambda} \). Since \( d = d(E, F) \leq d(E, E_\lambda) d(E_\lambda, F) \leq d^\lambda d^{1-\lambda} = d \), we conclude that \( d(E, E_\lambda) = d^\lambda \) and that \( d(E_\lambda, F) = d^{1-\lambda} \).

The proof of \( F_\lambda \) is similar. We claim that \( B_E \subset C_\lambda \subset dB_E \). The first inclusion follows from Lemma 2 and the second one follows from \( C_\lambda \subset B_\lambda \) and \( B_\lambda \subset dB_E \), which we proved in the previous paragraph. We also claim that \( C_\lambda \subset B_F \subset d^{1-\lambda}C_\lambda \). The first inclusion follows from Lemma 2 and the second one follows from the definition of \( C_\lambda \). Following the argument of the previous paragraph, we conclude that \( d(E, F_\lambda) = d^\lambda \) and \( d(F_\lambda, F) = d^{1-\lambda} \).

**Corollary 1.** The sets \( \{B_\lambda : 0 \leq \lambda \leq 1\} \) and \( \{C_\lambda : 0 \leq \lambda \leq 1\} \) are geodesics from \( F \) to \( E \).

**Proof.** Let \( d = d(E, F) \) and \( 0 < \lambda_1 < \lambda_2 < 1 \). We claim that \( B_{\lambda_1} \subset B_{\lambda_2} \subset d^{\lambda_2-\lambda_1}B_{\lambda_1} \). The first inclusion follows from the definition of \( B_{\lambda_1} \). To check the second inclusion, notice that \( d^{\lambda_2-\lambda_1}B_{\lambda_1} = (dB_E) \cap (d^{\lambda_2-\lambda_1}B_F) \) and \( B_{\lambda_2} = (dB_E) \cap B_F \). Since \( d^{\lambda_2-\lambda_1} \geq 1 \), \( B_F \subset d^{\lambda_2-\lambda_1}B_{\lambda_1} \) and this implies that \( B_{\lambda_2} \subset d^{\lambda_2-\lambda_1}B_{\lambda_1} \).

From \( B_{\lambda_1} \subset B_{\lambda_2} \subset d^{\lambda_2-\lambda_1}B_{\lambda_1} \) it follows that \( d(E_{\lambda_1}, E_{\lambda_2}) \leq d^{\lambda_2-\lambda_1} \). Since \( d(E, E_{\lambda_1}) = d^{\lambda_1} \) and \( d^{\lambda_2} = d(E, E_{\lambda_2}) \leq d(E, E_{\lambda_1})d(E_{\lambda_1}, E_{\lambda_2}) \leq d^{\lambda_1}d^{\lambda_2-\lambda_1} = d^{\lambda_2} \), we conclude that \( d(E_{\lambda_1}, E_{\lambda_2}) = d^{\lambda_2-\lambda_1} \).

A similar argument shows that \( C_{\lambda_1} \subset C_{\lambda_2} \subset d^{\lambda_2-\lambda_1}C_{\lambda_1} \) and this implies that \( d(F_{\lambda_1}, F_{\lambda_2}) = d^{\lambda_2-\lambda_1} \).

**Remark.** One can check that the norms of \( E_\lambda \) and \( F_\lambda \) are given the real K-method and J-method ([3], pp. 96 - 105).

\[ \|x\|_{E_\lambda} = K(x, d(E, F)^\lambda, E, F) \quad \text{and} \quad \|x\|_{F_\lambda} = J \left( x, \frac{1}{d(E, F)^{1-\lambda}}, E, F \right) \]

We now use the intermediate spaces of Proposition 1 to describe all other intermediate spaces. If \( X \) is an intermediate space between \( E \) and \( F \), then \( d(E, X) = d(E, F)^\lambda \) for some \( \lambda \in [0, 1] \). This number and the unit balls of the previous Proposition determine the intermediate spaces.

**Theorem 1.** Suppose that \( E, F, X \) are n-dimensional normed spaces. Then \( X \) is an intermediate space between \( E \) and \( F \) iff there exist \( \lambda \in [0, 1] \) and isometric copies of \( E, F, X \) in \( \mathbb{R}^n \) such that \( d(E, X) = d(E, F)^\lambda \), \( B_E \subset B_F \subset d(E, F)B_F \) and \( C_\lambda \subset B_X \subset B_\lambda \).
Proof. Suppose that $X$ is an intermediate space between $E$ and $F$. Then there exists $\lambda \in [0,1]$ such that $d(E, X) = d(E, F)^\lambda$ and $d(X, F) = d(E, F)^{1-\lambda}$. Find $T : X \to F$ and $S : E \to X$ such that $\|T\| = 1$, $\|T^{-1}\| = d(E, F)^{1-\lambda}$, $\|S\| = 1$ and $\|S^{-1}\| = d(E, F)^\lambda$. Then $T(B_X) \subset B_F \subset d(E, F)^{1-\lambda}T(B_X)$ and $S(B_E) \subset B_X \subset d(E, F)^\lambda S(B_E)$. 

Replacing $E$ and $X$ by their isometries $(\mathbb{R}^n, T(S(B_E)))$ and $(\mathbb{R}^n, T(B_X))$ we get

$$B_E \subset B_X \subset d(E, F)^\lambda B_E \quad \text{and} \quad B_X \subset B_F \subset d(E, F)^{1-\lambda} B_X.$$ 

Combining these inclusions we get that $B_E \subset B_F \subset d(E, F)B_E$. Moreover, we clearly have $B_X \subset (d(E, F)^\lambda B_E) \cap B_F$ and $\text{Conv} \left[ B_E \bigcup \frac{1}{d(E, F)\lambda} B_F \right] \subset B_X$. 

On the other hand, suppose that there are isometric versions of $E$, $F$, and $X$ that satisfy $B_E \subset B_F \subset d(E, F)B_E$ and $C \subset B_X \subset B_\lambda$. From the proof of Proposition [1] we have that $B_E \subset C_B \subset d(E, F)^\lambda B_E$ and $B_E \subset B_\lambda \subset d(E, F)^\lambda B_E$. Therefore we have that $B_E \subset C_B \subset B_X \subset B_\lambda \subset d(E, F)^\lambda B_E$, that implies that $d(E, X) \leq d(E, F)^\lambda$. 

Similarly, $C_B \subset B_F \subset d(E, F)^{1-\lambda} C_B$ and $C_B \subset B_F \subset d(E, F)^{1-\lambda} B_F$. Then

$$B_X \subset B_\lambda \subset B_F \subset d(E, F)^{1-\lambda} C_B \subset d(E, F)^{1-\lambda} B_X,$$

and this implies that $d(X, F) \leq d(E, F)^{1-\lambda}$. Since $d(E, F) \leq d(E, X)d(X, F) \leq d(E, F)^\lambda d(E, F)^{1-\lambda} = d(E, F)$, we conclude that $X$ is an intermediate space between $E$ and $F$. 

We now refine Lemma [2]

Lemma 3. Suppose $E$ and $F$ are non-isometric $n$-dimensional normed spaces satisfying $B_E \subset B_F \subset d(E, F)B_E$. Then $C_\lambda \not\subset B_\lambda$ for all $\lambda \in (0,1)$.

Proof. Let $C := \partial B_E \cap \partial B_F$. Then $C$ is closed, non-empty (we need to have contact points between the spheres to attain the distance) and $C \neq \partial B_F$ (or $d(E, F) = 1$). Then $\partial B_F \setminus C$ is open in the relative topology of $\partial B_F$ and $\partial B_F \setminus C$ is not closed (because $\partial B_F$ is connected). Find $x \in C$ and $x_n \in \partial B_F \setminus C$ such that $x_n \to x$.

Since $x \in \partial B_E$, $x \in d(E, F)^\lambda B_E^\circ$ (the interior) and we can find $n$ large enough so that $x_n \in d(E, F)^\lambda B_E^\circ$. It is clear that this $x_n$ belongs to $B_\lambda$ and we will show that it does not belong to $C_\lambda$. Find $f : \mathbb{R}^n \to \mathbb{R}$ linear, separating $x_n$ from $B_F^\circ$. Then we can assume that $f(x_n) = 1$ and that for all $y \in B_F$, $f(y) \leq 1$. If $x_n$ belonged to $C_\lambda$ we would write it as $x_n = \alpha y_1 + (1-\alpha)y_2$ for some $\alpha \in [0,1]$, $y_1 \in B_E$, $y_2 \in \frac{1}{d(E, F)\lambda} B_F$. Note that $f(y_1) \leq 1$ and $f(y_2) < 1$. Then $1 = f(x_n) = \alpha f(y_1) + (1-\alpha)f(y_2)$ implies that $\alpha = 1$ and $x_n = y_1$ which is not possible. Therefore, $x_n \not\in C_\lambda$. 

3. Main Result

In this section we show there are uncountably many different geodesics between two non-isometric $n$-dimensional normed spaces. We will start recalling some standard definitions.

Definition 3. Set $(x, y) := \{\alpha x + (1-\alpha)y : \alpha \in (0,1)\}$. A face $F$ of a convex set $K$ is a subset of $K$ satisfying the following: if $z \in F$, $x, y \in K$ and $z \in (x, y)$ then $x, y \in F$. A face $F$ is exposed if there exists a separating hyperplane $H$ such that $F = H \cap K$. If $F$ has a non-empty relative interior in $F \cap H$, $F$ is an
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(n−1)-dimensional face of K. The set of (n−1)-dimensional faces is denoted by \( F_{n-1}(K) \).

Lemma 4. The unit ball of a finite dimensional normed space \( E \) has at most countably many \((n−1)\)-dimensional faces.

Proof. Let \( \{ F_\alpha : \alpha \in I \} \) be the set of \((n-1)\)-dimensional faces of \( B_E \subset \mathbb{R}^n \), the unit ball of \( E \). Their interiors with respect to the relative topology of the sphere \( S_E \), the boundary of \( B_E \), form a disjoint family of non-empty open sets in \( S_E \). Since \( S_E \) is second countable, \( I \) is at most countable. □

The next proposition states if \( E \) and \( F \) are isometric, then \( F_{n-1}(B_E) \) and \( F_{n-1}(B_F) \) are equal up to affine maps. This provides a criterion to show that two normed spaces are not isometric. We are to exhibit an \( n−1 \) dimensional face that is not in the other. Notice that there are uncountably many non-isometric convex bodies in \( \mathbb{R}^{n-1} \), if \( n \geq 3 \). Since (after an affine map) any convex body \( K \) in \( \mathbb{R}^{n-1} \) can be a face of an \( n \)-dimensional normed space, there are a lot of options.

We need the following result, that is easy to prove:

Lemma 5. Let \( E \) and \( F \) be isometric \( n \)-dimensional normed spaces and let \( Q \in F_{n-1}(B_E) \). Then there exists \( Q' \in F_{n-1}(B_E) \) that is an affine copy of \( F \).

We state and show our main result in two parts, first one deals with \( BM_n \) for \( n \geq 3 \), the other with \( BM_2 \).

Theorem 2. Let \( n \geq 3 \) and let \( E \) and \( F \) be two non-isometric \( n \)-dimensional normed spaces satisfying \( B_E \subset B_F \subset d(E,F)B_E \). Then for each \( \lambda \in (0,1) \) the set \( \{ X \in BM_n : C_\lambda \subset B_X \subset B_\lambda \} \) contains uncountably many non-isometric spaces.

Proof. By Lemma 3 there exists \( x \in B_1 \setminus C_\lambda \) and since \( C_\lambda \) is closed, we can assume that \( x \in B_\lambda \setminus C_\lambda \). Find \( \epsilon > 0 \) such that \( B_\epsilon(x) \subset B_\lambda \setminus C_\lambda \). Then find a linear function \( f : \mathbb{R}^n \to \mathbb{R} \) separating \( \{ x \} \) from \( C_\lambda \). Assume that \( f(x) = 1 \) and that for all \( y \in C_\lambda, f(y) < 1 \). Let \( H = \{ z \in \mathbb{R}^n : f(z) = 1 \} \) be the hyperplane induced by \( f \) at \( x \). By Lemma 4 the collections \( F_{n-1}B_\lambda, F_{n-1}C_\lambda \) are countable. Since there are uncountably many non-isometric \( n−1 \)-dimensional Banach spaces, choose a unit ball \( K \in \mathbb{R}^{n-1} \) that is not affinely isometric to a ball in either collection. Find an affine copy of \( K \) inside \( H \cap B_\lambda(x) \) and call it \( K' \). Define the \( n \)-dimensional normed space \( X_K \) to have unit ball equal \( B_X := B_{X_K} := \text{Conv}(C_\lambda \cup K' \cup -K') \). Then \( C_\lambda \subset B_{X_K} \subset B_\lambda \) and \( B_{X_K} \) has an \( n−1 \) dimensional face isometric to \( K \). So by Lemma 5 it is not isometric to \( C_\lambda \) nor \( B_\lambda \). Since the construction produces uncountably many non-isometric spaces the claim follows. □

The above construction fails for \( n = 2 \) since all \( 1 \)-dimensional normed spaces are isometric. We tackle the \( n = 2 \) case in the next section. We need some elementary results to show that there are uncountably many geodesics between \( E \) and \( F \).

Lemma 6. Suppose that \( X \) is an intermediate space between \( E \) and \( F \), that \( Y \) is an intermediate space between \( E \) and \( X \) and that \( Z \) is an intermediate space between \( X \) and \( F \). Then \( X \) is an intermediate space between \( Y \) and \( Z \).

Proof. This follows from the triangle inequality: \( d(E,F) \leq d(E,Y)d(Y,Z)d(Z,F) \leq d(E,Y)d(Y,X)d(X,Z)d(Z,F) = d(E,X)d(X,F) = d(E,F) \). Then the inequalities are equalities and \( d(Y,Z) = d(Y,X)d(X,Z) \). □
Corollary 2. If \( X \) is an intermediate space between \( E \) and \( F \), then there exists a geodesic from \( E \) to \( F \) containing \( X \).

Proof. Use Corollary 1 to construct geodesics from \( E \) to \( X \), from \( X \) and \( F \) and put them together. To see that the joined path is a geodesic from \( F \) to \( E \) we use Lemma 3 to show it satisfies the condition of Lemma 1. If the partition contains \( X \), it follows from the fact that \( X \) is an intermediate space and that both pieces are geodesics. And if the partition does not include \( X \), we use Lemma 3 adding \( X \) to the partition. \( \square \)

Combining Theorem 2 with the previous Corollary we get:

Corollary 3. There are uncountably many geodesics between any two non-isomorphic \( n \)-dimensional normed spaces for \( n \geq 3 \).

4. Dimension 2

In what follows \( K \) is a compact, convex and symmetric body of \( \mathbb{R}^2 \). By Lemma 4 there are at most countably many perfect 1-faces i.e. line segments \( F^1K \). Consider the set of triangles formed by joining the endpoints of line segments \([p, q]\) in \( F^1K \) to the origin, \( \Delta 0pq \). Call this set \( S_K := \{ \Delta 0pq : [p, q] \in F^1K \} \) and consider all possible ratios of areas of these triangles \( A_K := \{ \mu(\Delta 0pq) : T_1, T_2 \in S_K \} \), where \( \mu \) is the usual Lebesgue area measure. Since \( \mu(\phi(T)) = \det(\phi)(\mu(T)) \) we arrive at the following countable isometric invariant:

Lemma 7. Let \( \phi : (\mathbb{R}^2, B_E) \to (\mathbb{R}^2, B_F) \) be an invertible linear map with \( \phi(B_E) = B_F \). Then \( A_{B_E} = A_{B_F} \).

Now we prove Theorem 2 for \( n = 2 \). The idea is to find balls \( B_q \) satisfying \( C_\lambda \subset B_q \subset B_\lambda \), with two contiguous faces \([p_1, q]\) and \([q, p_2]\) such that \( \frac{\mu(\Delta 0p_1q)}{\mu(\Delta 0p_2q)} \) does not belong to \( A_{B_E} \) or \( A_{B_F} \).

Theorem 3. Suppose that \( n = 2 \) and that \( E \) and \( F \) are two non-isometric normed spaces satisfying \( B_E \subset B_F \subset \partial(E, F)B_E \). Then for each \( \lambda \in (0, 1) \) the set \( \{ X \in BM_n : C_\lambda \subset B_X \subset B_\lambda \} \) contains uncountably many non-isometric spaces.

Proof. By Lemma 3 there exists \( x \in B_\lambda \setminus C_\lambda \) and since \( C_\lambda \) is closed, we can assume that \( x \in B_\lambda^\circ \setminus C_\lambda \). Find \( \epsilon > 0 \) so that \( B_\epsilon(x) \subset B_\lambda^\circ \setminus C_\lambda \) and find a linear function \( f : \mathbb{R}^n \to \mathbb{R} \) separating \( B(x, \epsilon) \) from \( C_\lambda \). Then \( H = \{ z \in \mathbb{R}^2 : f(z) = f(x) \} \) is the hyperplane (line in this case) induced by \( f \) at \( x \); and \( H : B(x, \epsilon) \) is a segment that we denote \([p_1, p_2]\).

As a first step, let \( B = \text{conv}(C_\lambda \cup [p_1, p_2] \cup [-p_1, -p_2]) \). Notice that \( C_\lambda \subset B \subset B_\lambda \) and that the segment \([p_1, p_2]\) is a face of \( B \). For each \( q \in B(x, \epsilon) \) with \( f(q) > f(x) \) (i.e., \( q \) is inside \( B(x, \epsilon) \)) but outside the triangle \( \Delta 0p_1p_2 \) define

\[ B_q = \text{conv}(C_\lambda \cup [p_1, p_2] \cup [-p_1, -p_2] \cup \{q, -q\}) \]

We still have that \( C_\lambda \subset B_q \subset B_\lambda \) and it is easy to check that, for \( i = 1, 2 \), the segment \([p_i, q]\) is a face of \( B_q \) if the line going through \( p_i \) and \( q \) does not intersect \( C_\lambda \). Since \( f \) separates \( B(x, \epsilon) \) and \( C_\lambda \) there are many points \( q \) with this property. In fact, it is easy to see that there exists \( 0 < \delta < \epsilon \) small enough so that whenever \( q \in B(x, \delta) \) and \( f(q) > f(x) \), then the segments \([p_1, q]\) and \([q, p_2]\) are faces of \( B_q \).
Find two such points $q_1$ and $q_2$ that satisfy $f(q_1) = f(q_2) > f(x)$. Notice that the segment $[q_1, q_2]$ is parallel to the segment $[p_1, p_2]$ and that for any $q \in [q_1, q_2]$, the segments $[p_1, q]$ and $[q, p_2]$ are faces of $B_q$. Since $[q_1, q_2]$ is connected and since the values of $\frac{\mu(\triangle 0 p_1 q)}{\mu(\triangle 0 p_2 q)}$ when $q = q_1$ and $q = q_2$ are clearly different, the set of points $\left\{ \frac{\mu(\triangle 0 p_1 q)}{\mu(\triangle 0 p_2 q)} : q \in [q_1, q_2] \right\}$ is uncountable. This allows us to choose $q$’s such that $(\mathbb{R}^2, B_q)$ is not isometric to $(\mathbb{R}^2, C_\lambda)$ or to $(\mathbb{R}^2, B_\lambda)$. Moreover, we can easily choose uncountably many non-isometric $(\mathbb{R}^2, B_q)$’s and the result follows. 

This Theorem combined with Corollary\textsuperscript{2}\textsuperscript{3} imply that there are uncountably many geodesics between $E$ and $F$, which completes the case $n \geq 2$. The proof of Theorem\textsuperscript{2} can be adapted to prove Theorem\textsuperscript{2} for $n \geq 3$.

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