ON THE ZERO SET OF SEMI-INVARIANTS FOR REGULAR MODULES OVER TAME CANONICAL ALGEBRAS

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Abstract. We investigate sets of the common zeros of non-constant semi-invariants for regular modules over canonical algebras. In particular, we show that if the considered algebra is tame then for big enough vectors these sets are complete intersections.

Throughout the paper $k$ denotes a fixed algebraically closed field of characteristic 0. By $\mathbb{N}$ and $\mathbb{Z}$ we denote the sets of non-negative integers and integers, respectively. Additionally, if $i, j \in \mathbb{Z}$, then $[i, j] = \{l \in \mathbb{Z} \mid i \leq l \leq j\}$.

Introduction and the main result

With a finite dimensional algebra $\Lambda$ and a dimension vector $d$ we may associate the variety of $\Lambda$-modules of dimension vector $d$ (see 2.1). An interesting problem investigated in the representation theory of finite dimensional algebras is the study of geometric properties of these varieties (see for example [8,10,12,17,23,27,28,31,35,38]). In addition to this topic rings of semi-invariants (see 2.2) are also studied (see for example [20,24,30,32,40,46]). Recently, investigations of sets of the common zeros of non-constant semi-invariants were initiated by Chang and Weyman ([15]) and then continued by Riedtmann and Zwara ([36–39]). Their investigations concerned situations of quivers without relations and were based on known results about semi-invariants in these cases (among others Sato–Kimura theorem [42]). An inspiration for their research was an observation that if, for a given dimension vector, the set of the common zeros of non-constant semi-invariants has a “good” codimension then the coordinate ring of the module variety is free as a module over the ring of semi-invariants.

An important class of algebras are the canonical algebras introduced by Ringel [11, 3.7] (see 1.4). These algebras play an important role in representation theory (see for example [22,25,33,44]). Module varieties over canonical algebras were also studied ([5,6]). One may distinguish a special class of modules over canonical algebras, called regular (see 1.6). The rings of semi-invariants for dimension vectors of regular modules

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over canonical algebra were described by Skowroński and Weyman [45] (they were also studied independently by Domokos and Lenzing [20]). This description allows to investigate sets $Z(d)$ of the common zeros of non-constant semi-invariants for the dimension vectors $d$ of regular modules. The first step in this direction was made by the author in [7].

If $d$ is the dimension vector of a regular module, then we have a canonical decomposition $d = ph + d'$ of $d$ (see 1.7), where $h$ is the dimension vector with all coordinates equal to 1 and $d'$ is the dimension vector of a regular module such that $d' - h$ is no longer the dimension vector of a regular module. Recall that an algebra $\Lambda$ is called tame if for each dimension $d$ indecomposable modules of dimension $d$ can be parameterized by a finite number of lines (see for example [16, Definition 6.5] for a precise formulation).

The following theorem is the main result of the paper.

**Main Theorem.** If $\Lambda$ is a tame canonical algebra, then there exists $N$ such that $Z(d)$ is a complete intersection for all dimension vectors $d$ of regular modules such that $p^d \geq N$.

Moreover, we show that also in the case of canonical algebras there is a connection between the codimension of $Z(d)$ and freeness of the coordinate ring over the ring of semi-invariants, for the dimension vector $d$ of a regular module.

The paper is organized as follows. In Section 1 we recall necessary facts about quivers, their representations, and canonical algebras. In Section 2 we present basic properties of module varieties and rings of semi-invariants. In particular, we give a description of the sets of the common zeros of non-constant semi-invariants for the dimension vectors of regular modules over canonical algebras. In Section 3 we use these results to prove Main Theorem, and in final Section 4 we present an interpretation of the main result in terms of freeness of coordinate rings over rings of semi-invariants.

For background on the representation theory of algebras we refer to [34]. Basic algebraic geometry used in the article can be found for example in [31]. Author gratefully acknowledges the support from the Polish Scientific Grant KBN No. 1 P03A 018 27. The result presented in this paper was obtained during the research camp in Szklarska Poręba (June 2006).

**1. Preliminaries on quivers and canonical algebras**

In this section we present basic facts about quivers and their representations. We also define canonical algebras and review their representation theory.

1.1. Recall that by a quiver $\Delta$ we mean a finite set $\Delta_0$ of vertices and a finite set $\Delta_1$ of arrows together with two maps $s, t : \Delta_1 \to \Delta_0$,
which assign to an arrow $\gamma \in \Delta_1$ its starting and terminating vertex, respectively. By a path of length $m \geq 1$ in $\Delta$ we mean a sequence $\sigma = \gamma_1 \cdots \gamma_m$ of arrows such that $s\gamma_i = t\gamma_{i+1}$ for $i \in [1, m - 1]$. We write $s\sigma$ and $t\sigma$ for $s\gamma_m$ and $t\gamma_1$, respectively. For each vertex $x$ of $\Delta$ we introduce a path $x$ of length 0 such that $sx = x = tx$. We only consider quivers without oriented cycles, i.e., we assume that there is no path $\sigma$ of positive length such that $t\sigma = s\sigma$.

With a quiver $\Delta$ we associate its path algebra $k\Delta$, which as a $k$-vector space has a basis formed by all paths in $\Delta$ and whose multiplication is induced by the composition of paths. By a relation $\rho$ in $\Delta$ we mean a linear combination of paths of length at least 2 with common starting and terminating vertices. The common starting vertex is denoted by $s\rho$ and the common terminating vertex by $t\rho$. A set $R$ of relations is called minimal if for every $\rho \in R$, $\rho$ does not belong to the ideal $\langle R \setminus \{\rho\} \rangle$ of $k\Delta$ generated by $R \setminus \{\rho\}$. A pair $(\Delta, R)$ consisting of a quiver $\Delta$ and a minimal set of relations $R$ is called a bound quiver. If $(\Delta, R)$ is a bound quiver, then the algebra $k\Delta/\langle R \rangle$ is called the path algebra of $(\Delta, R)$.

1.2. By a representation of a bound quiver $(\Delta, R)$ we mean a collection $M = (M_x, M_\alpha)_{x \in \Delta_0, \alpha \in \Delta_1}$ of finite dimensional vector spaces $M_x$, $x \in \Delta_0$, and linear maps $M_\alpha : M_{s\alpha} \to M_{t\alpha}$, $\alpha \in \Delta_1$, such that
\[
\sum_{i \in [1, l]} \lambda_i M_{\alpha_{i, 1}} \cdots M_{\alpha_{i, m_i}} = 0
\]
for each relation $\sum_{i \in [1, l]} \lambda_i \alpha_{i, 1} \cdots \alpha_{i, m_i} \in R$. The category of representations of $(\Delta, R)$ is equivalent to the category of $k\Delta/\langle R \rangle$-modules (see for example [3, Theorem III.1.6]), and we identify $k\Delta/\langle R \rangle$-modules and representations of $(\Delta, R)$. For a representation $M$ its dimension vector $\dim M \in \mathbb{Z}^{\Delta_0}$ is defined by $(\dim M)_x = \dim_k M_x$, $x \in \Delta_0$. For a vertex $x \in \Delta_0$ we denote by $e_x$ the corresponding canonical basis vector in $\mathbb{Z}^{\Delta_0}$.

1.3. Let $\Lambda$ be the path algebra of a bound quiver $(\Delta, R)$. Assume in addition that $\text{gl. dim } \Lambda \leq 2$. We have the bilinear form $\langle - , - \rangle = \langle - , - \rangle_\Lambda : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ given by
\[
\langle \mathbf{d}', \mathbf{d}'' \rangle = \sum_{x \in \Delta_0} d'_x d''_x - \sum_{\alpha \in \Delta_1} d'_{s\alpha} d''_{t\alpha} + \sum_{\rho \in R} d'_{s\rho} d''_{t\rho}.
\]
It is known (see [11, 2.2]), that if $M$ and $N$ are $\Lambda$-modules, then
\[
\langle \dim M, \dim N \rangle = [M, N] - [M, N]^1 + [M, N]^2,
\]
where following Bongartz [13] we write
\[
[M, N] = [M, N]_\Lambda = \dim_k \text{Hom}_\Lambda(M, N), \quad [M, N]^1 = [M, N]^1_\Lambda = \dim_k \text{Ext}_\Lambda^1(M, N),
\]
and

$$[M, N]^2 = [M, N]^2_\Lambda = \dim_k \operatorname{Ext}^2_\Lambda (M, N).$$

1.4. Let \( m = (m_1, \ldots, m_n) \), \( n \geq 3 \), be a sequence of integers greater than 1 and let \( \lambda = (\lambda_3, \ldots, \lambda_n) \) be a sequence of pairwise distinct non-zero elements of \( k \) with \( \lambda_3 = 1 \). By definition \( \Lambda(m, \lambda) \) is the path algebra of the bound quiver \((\Delta(m), R(m, \lambda))\), where \( \Delta(m) \) is the quiver

![Diagram](image)

and \( R(m, \lambda) \) is the set of the following relations:

$$\alpha_{1,1} \cdot \cdot \cdot \alpha_{1,m_1} + \lambda_0 \alpha_{2,1} \cdot \cdot \cdot \alpha_{2,m_2} - \alpha_{i,1} \cdot \cdot \cdot \alpha_{i,m_i}, \quad i \in [3, n].$$

The algebras of the above form are called canonical. In particular, we say that \( \Lambda(m, \lambda) \) is a canonical algebra of type \( m \). If we fix \( m \) and \( \lambda \), then we usually write \( \Lambda, \Delta, \) and \( R \), instead of \( \Lambda(m, \lambda), \Delta(m), \) and \( R(m, \lambda) \), respectively. From now till the end of the section we assume that \( \Lambda = \Lambda(m, \lambda) \) is a fixed canonical algebra. The following invariant

$$\delta = \delta_\Lambda = \frac{1}{2} (n - 2 - \frac{1}{m_1} - \cdots - \frac{1}{m_n})$$

controls the representation type of \( \Lambda \). Namely, \( \Lambda \) is tame if and only if \( \delta \leq 0 \). Moreover, it is known that \( \operatorname{gl.dim} \Lambda = 2 \).

1.5. We abbreviate \( e_{i,j} \) by \( e_{i,j} \) for \( i \in [1, n] \) and \( j \in [1, m_i - 1] \). We put

$$h = \sum_{x \in \Delta_0} e_x \quad \text{and} \quad e_{i,0} = h - (e_{i,1} + \cdots + e_{i,m_i - 1}).$$

We extend the above definitions by \( e_{i,lm_i+j} = e_{i,j} \) for \( i \in [1, n], \ j \in [0, m_i - 1], \) and \( l \in \mathbb{Z} \).

For \( d \in \mathbb{Z}^{\Delta_0} \) let \( \delta_{i,j}(d) = d_{i,j} - d_{i,j} \) for \( i \in [1, n] \) and \( j \in [1, m_i] \). In the paper we use convention that \( d_{i,0} = d_0 \) and \( d_{i,m_i} = d_\infty \) for \( d \in \mathbb{Z}^{\Delta_0} \) and \( i \in [1, n] \), and \( d_{i,j} = d_{i,j} \) for \( i \in [1, n] \) and \( j \in [1, m_i - 1] \). Similarly as above we extend this definition by \( \delta_{i,lm_i+j}(d) = \delta_{i,j}(d) \) for \( i \in [1, n], \ j \in [1, m_i], \) and \( l \in \mathbb{Z} \). We also put \( \delta_{i,j}^{[j_1,j_2]}(d) = \sum_{j \in [j_1,j_2]} \delta_{i,j}(d) \) for \( i \in [1, n] \) and \( j_1 \leq j_2 \). Observe that

$$\langle e_{i,j}, d \rangle = -\delta_{i,j}(d) \quad \text{and} \quad \langle d, e_{i,j} \rangle = \delta_{i,j+1}(d)$$
for \( i \in [1, n] \) and \( j \in \mathbb{Z} \), and consequently

\[
\langle e_i^{[j_1,j_2]}, d \rangle = -\delta_i^{[j_1,j_2]}(d) \quad \text{and} \quad \langle d, e_i^{[j_1,j_2]} \rangle = \delta_i^{[j_1+1,j_2+1]}(d)
\]

for \( i \in [1, n] \) and \( j_1 \leq j_2 \), where as above \( e_i^{[j_1,j_2]} = \sum_{j \in [j_1,j_2]} e_{i,j} \) for \( i \in [1, n] \) and \( j_1 \leq j_2 \). Finally

\[
\langle d, h \rangle = d_0 - d_\infty = -\langle h, d \rangle.
\]

1.6. Let \( \mathcal{P} (\mathcal{R}, \mathcal{Q}, \text{respectively}) \) be the subcategory of all \( \Lambda \)-modules which are direct sums of indecomposable \( \Lambda \)-modules \( X \) such that

\[
\langle \dim X, h \rangle > 0 \quad (\langle \dim X, h \rangle = 0, \langle \dim X, h \rangle < 0, \text{respectively}).
\]

The modules from the category \( \mathcal{R} \) are called regular. We have the following properties of the above decomposition of the category of \( \Lambda \)-modules (see [11, 3.7]).

First, \([N, M] = 0 \) and \([M, N] \leq 0 \) if either \( N \in \mathcal{R} \vee \mathcal{Q} \) and \( M \in \mathcal{P} \), or \( N \in \mathcal{Q} \) and \( M \in \mathcal{P} \vee \mathcal{R} \). Here, for two subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of the category of \( \Lambda \)-modules we denote by \( \mathcal{X} \vee \mathcal{Y} \) the additive closure of their union. Moreover, one knows that \( \text{pd}_\Lambda M \leq 1 \) for \( M \in \mathcal{P} \vee \mathcal{R} \) and \( \text{id}_\Lambda N \leq 1 \) for \( N \in \mathcal{R} \vee \mathcal{Q} \). Secondly, \( \mathcal{R} \) decomposes into a \( \mathbb{P}^1(k) \)-family \( \coprod_{\lambda \in \mathbb{P}^1(k)} \mathcal{R}_\lambda \) of uniserial categories. In particular, \([M, N] = 0 \) and \([M, N] \leq 0 \) if \( M \in \mathcal{R}_\lambda \) and \( N \in \mathcal{R}_\mu \) for \( \lambda \neq \mu \). If \( \lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1, \ldots, \lambda_n\} \), where \( \lambda_1 = 1 \) and \( \lambda_2 = \infty \), then there is a unique (up to isomorphism) simple object \( R_\lambda \) in \( \mathcal{R}_\lambda \) and its dimension vector is \( h \). On the other hand, if \( \lambda = \lambda_i \) for \( i \in [1, n] \), then there are \( m_i \) pairwise non-isomorphic simple objects \( R_{i,0}, \ldots, R_{i,m_i-1} \) in \( \mathcal{R}_{\lambda_i} \) and their dimension vectors are \( e_{i,0}, \ldots, e_{i,m_i-1} \), respectively.

For \( i \in [1, n] \) and \( j_1 \leq j_2 \) there is a unique (up to isomorphism) indecomposable module \( R_i^{[j_1,j_2]} \) in \( \mathcal{R}_{\lambda_i} \) with regular socle (i.e., the socle in the category \( \mathcal{R} \)) \( R_{i,j_1} \) and of dimension vector \( e_i^{[j_1,j_2]} \), where similarly as usual \( R_{i,m_i,j} = R_{i,j} \) for \( i \in [1, n] \), \( j \in [0, m_i - 1] \), and \( l \in \mathbb{Z} \). Every indecomposable module from \( \mathcal{R}' = \coprod_{i \in [1, n]} \mathcal{R}_{\lambda_i} \) is isomorphic to \( R_i^{[j_1,j_2]} \) for some \( i \in [1, n] \) and \( j_1 \leq j_2 \). Moreover, \( R_i^{[j_1,j_2]} \cong R_i^{[j_1+1,j_2+1]} \) if and only if \( j_2 - j_1 = l_2 - l_1 \), and \( j_1 \) and \( l_1 \) have the same remainder of division by \( m_i \). The regular length (i.e. the length in the category \( \mathcal{R} \)) of \( R_i^{[j_1,j_2]} \) is \( j_2 - j_1 + 1 \) and \( \tau R_i^{[j_1,j_2]} = R_{i,j_2-j_1-1}^{[j_1+1,j_2+1]} \), where \( \tau \) denotes the Auslander–Reiten translation. We have the following rule of calculating dimensions of homomorphism spaces between modules in \( \mathcal{R}' \):

\[
(1.6.1) \quad [R_i^{[j_1,j_2]}, R_i^{[l_1,l_2]}] = \# \{ u \in \mathbb{Z} \mid j_1 \leq l_1 + um_i \leq j_2 \leq l_2 + um_i \}.
\]

We also put \( \mathcal{R}' = \coprod_{\lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1, \ldots, \lambda_n\}} \mathcal{R}_\lambda \).

1.7. Let \( \mathcal{P}, \mathcal{R} \) and \( \mathcal{Q} \) denote the sets of the dimension vectors of modules from \( \mathcal{P}, \mathcal{R} \) and \( \mathcal{Q} \), respectively. We know from [6, 2.6] that \( d \in \mathcal{P} \) (\( d \in \mathcal{Q} \)) if and only if either \( d = 0 \) or \( d_0 > d_\infty \geq 0 \) (\( 0 \leq d_0 < d_\infty \),
respectively) and $\delta_{i,j}(d) \geq 0$ ($\delta_{i,j}(d) \leq 0$, respectively) for all $i \in [1, n]$ and $j \in [1, m_i]$.

With a dimension vector $d \in \mathbb{R}$ we may associate its canonical decomposition (compare [40, Section 1])

$$d = p^d h + \sum_{i \in [1, n]} \sum_{j \in [0, m_i - 1]} p^d_{i,j} e_{i,j}$$

in the following way. First, for each $i \in [1, n]$ fix $j_i \in [0, m_i - 1]$ such that $d_{i,j_i} = \min\{d_{i,j} \mid j \in [0, m_i - 1]\}$. Then we put

$$p^d_{i,j} = d_{i,j} - d_{i,j_i}, \quad i \in [1, n], \quad j \in [0, m_i - 1],$$

and

$$p^d = (d_{1,j_1} + \cdots + d_{n,j_n}) - (n - 1)d_0.$$ 

The condition $d \in \mathbb{R}$ implies that $p^d \geq 0$. We also put $p^d_{i,l,m_i+j} = p^d_{i,j}$ for $i \in [1, n]$, $j \in [0, m_i - 1]$, and $l \in \mathbb{Z}$. The canonical decomposition of $d$ is the unique presentation

$$d = ph + \sum_{i \in [1, n]} \sum_{j \in [0, m_i - 1]} p_{i,j} e_{i,j}$$

such that $p \geq 0$, $p_{i,j} \geq 0$ for $i \in [1, n]$ and $j \in [0, m_i - 1]$, and for each $i \in [1, n]$ there exists $j \in [0, m_i - 1]$ such that $p_{i,j} = 0$.

2. PRELIMINARIES ON MODULE VARIETIES AND SEMI-INvariANTS

Throughout this section $\Lambda$ is the path algebra of a bound quiver $(\Delta, R)$.

2.1. For $d \in \mathbb{N}^{\Delta_0}$ let $A(d) = \prod_{\alpha \in \Delta_1} \mathbb{M}(d_{t\alpha}, d_{s\alpha})$. The variety $\text{mod}_\Lambda(d)$ of $\Lambda$-modules of dimension vector $d$ is by definition the subset of $A(d)$ formed by all tuples $(M_\alpha)_{\alpha \in \Delta_1}$ such that

$$\sum_{i \in [1, l]} \lambda_i M_{\alpha_1,1} \cdots M_{\alpha_i,m_i} = 0$$

for each relation $\sum_{i \in [1, l]} \lambda_i \alpha_{1,1} \cdots \alpha_{i,m_i} \in R$. We identify the points $M$ of $\text{mod}_\Lambda(d)$ with $\Lambda$-modules of dimension vector $d$ by taking $M_x = k^{d_x}$ for $x \in \Delta_0$. The product $\text{GL}(d) = \prod_{x \in \Delta_0} \text{GL}(d_x)$ of general linear groups acts on $\text{mod}_\Lambda(d)$ by conjugations:

$$(g \cdot M)_\alpha = g_{t\alpha} M_\alpha g^{-1}_{s\alpha}, \quad \alpha \in \Delta_1,$$

for $g \in \text{GL}(d)$ and $M \in \text{mod}_\Lambda(d)$. The orbits with respect to this action correspond bijectively to the isomorphism classes of $\Lambda$-modules of dimension vector $d$. For $M \in \text{mod}_\Lambda(d)$ we denote by $\mathcal{O}(M)$ the $\text{GL}(d)$-orbit of $M$. It is known (see for example [23, 2.2]) that

$$\dim \mathcal{O}(M) = \dim \text{GL}(d) - [M, M].$$
We put
\[ a(d) = a_\Lambda(d) = \dim \Lambda(d) - \sum_{pR} d_{sp}d_{tp}. \]
Note that \( a(d) = \dim GL(d) - (d, d) \) for \( d \in \mathbb{N}^{\Delta_0} \).

2.2. The action of \( GL(d) \) on \( \text{mod}_d(\Lambda) \) induces an action of \( GL(d) \) on the coordinate ring \( k[\text{mod}_d(\Lambda)] \) of \( \text{mod}_d(\Lambda) \) in the usual way:
\[ (g \cdot f)(M) = f(g^{-1} \cdot M) \]
for \( g \in GL(d), f \in k[\text{mod}_d(\Lambda)], \) and \( M \in \text{mod}_d(\Lambda) \). If \( \sigma \in \mathbb{Z}^{\Delta_0} \) is a weight, then we define the weight space
\[ \text{SI}(\Lambda, d)_\sigma = \left\{ f \in k[\text{mod}_d(\Lambda)] \mid g \cdot f = \left( \prod_{x \in \Delta_0} \det^\sigma(x)(g) \right) f \right\}. \]
The elements of \( \text{SI}(\Lambda, d)_\sigma \) are called the semi-invariants of weight \( \sigma \).

By the ring of semi-invariants we mean
\[ \text{SI}(\Lambda, d) = \bigoplus_{\sigma \in \mathbb{Z}^{\Delta_0}, \sigma \neq 0} \text{SI}(\Lambda, d)_\sigma. \]
One knows that \( \text{SI}(\Lambda, d)_0 = k \) (since lack of cycles in \( \Delta \) implies that there is a unique closed orbit in \( \text{mod}_d(\Lambda) \)). By \( Z(d) = Z_\Lambda(d) \) we denote the set of the common zeros of semi-invariants with non-zero weight for \( d \in \mathbb{N} \).

2.3. We present now necessary facts about the rings of semi-invariants for canonical algebras. For the rest of the section we assume that \( \Lambda = \Lambda(m, \lambda) \) is a canonical algebra and \( \Delta = \Delta(m) \).

Fix \( i \in [1, n] \). An interval \( [j_1, j_2] \) with \( j_1 < j_2 \) is called \( i \)-admissible for \( d \in \mathbb{N} \) if \( p_{i, j_1}^d = p_{i, j_2}^d \) and \( p_{i, j_2}^d > p_{i, j_1}^d \) for all \( j \in [j_1 + 1, j_2 - 1] \). Note that \( j_2 \) is uniquely determined by \( j_1 \) and \( j_2 \leq j_1 + m_i \). We say that two \( i \)-admissible intervals \( [j_1, j_2] \) and \( [l_1, l_2] \) are equivalent if \( j_1 \) and \( l_1 \) have the same reminder of the division by \( m_i \) (consequently, \( j_2 \) and \( l_2 \) have the same reminder of the division by \( m_i \)) — in other words there exists \( u \in \mathbb{Z} \) such that \( l_1 = j_1 + um_i \) and \( l_2 = j_2 + um_i \). We will usually identify equivalent intervals. Let \( \mathcal{A}_i(d) \) be the set of equivalence classes of \( i \)-admissible intervals for \( d \) and
\[ \text{ad}(d) = \# \mathcal{A}_1(d) + \cdots + \# \mathcal{A}_n(d). \]
We will use the following consequence of [45, Theorem 1.1].

**Proposition.** If \( d \in \mathbb{N} \), \( p^d \geq n - 1 \), and \( \text{mod}_d(\Lambda) \) is irreducible, then \( \text{SI}(\Lambda, d) \) is a polynomial ring generated by \( p^d + 1 + \text{ad}(d) - n \) elements.

If \( i \in [1, n], [j_1, j_2] \in \mathcal{A}_i(d), \) and \( j \in [0, m_i - 1] \), then we say that \( j \) lies inside \([j_1, j_2]\) if \( j_1 + um_i \leq j < j_2 + um_i \) for some \( u \in \mathbb{Z} \). We will need the following.
Observation. Let \( \mathbf{d} \in \mathbb{R}, \ i \in [1,n], \text{ and } j \in [0, m_i - 1]. \) The number of \( [j_1, j_2] \in \mathcal{A}_i(\mathbf{d}) \) such that \( j \) lies inside \( [j_1, j_2] \) is bounded above by \( p_{i,j+1}^d \).

Proof. Let \( [j_{1,1}, j_{1,2}], \ldots, [j_{s,1}, j_{s,2}] \) be the \( i \)-admissible intervals for \( \mathbf{d} \) with the above property. Without loss of generality we may assume that

\[
 j_{1,1} < \cdots < j_{s,1} \leq j < j + 1 \leq j_{s,2} < \cdots < j_{1,2}.
\]

Then \( p_{i,j_{s,2}}^d > \cdots > p_{i,j_{1,2}}^d \) is a decreasing sequence of \( s \) non-negative integers, hence \( p_{i,j+1}^d \geq p_{i,j_{s,2}}^d \geq s - 1. \)

2.4. Now we derive consequences of the connection of semi-invariants with modules given in [19] (see also [18]). Namely, we have the following description of \( \mathcal{Z}(\mathbf{d}) \) for \( \mathbf{d} \in \mathbb{R} \) with \( p^d > 0. \)

Proposition. Let \( \mathbf{d} \in \mathbb{R} \) and \( p^d > 0. \) If \( M \in \text{mod}_{\Lambda}(\mathbf{d}), \) then \( M \in \mathcal{Z}(\mathbf{d}) \) if and only if the following conditions are satisfied:

1. \( [R_{\lambda}, M] \neq 0 \) for all \( \lambda \neq \lambda_1, \ldots, \lambda_n. \)
2. \( [R_{i}^{[j_1,j_2]}, M] \neq 0 \) for all \( i \in [1,n] \) and \( [j_1, j_2] \in \mathcal{A}_i(\mathbf{d}). \)

By an easy application of the Auslander–Reiten formula ([3, Theorem IV.2.13]) we get the following dual version of the above conditions (see also [19, Section 4]).

Observation. Let \( \mathbf{d} \in \mathbb{R} \) and \( M \in \text{mod}_{\Lambda}(\mathbf{d}). \)

1. If \( \lambda \neq \lambda_1, \ldots, \lambda_n, \) then
   \[
   [R_{\lambda}, M] \neq 0 \iff [M, R_{\lambda}] \neq 0.
   \]
2. If \( i \in [1,n], j_1 < j_2, \) and \( \delta_{i}^{[j_1,j_2]}(\mathbf{d}) = 0, \) then
   \[
   [R_{i}^{[j_1,j_2]}, M] \neq 0 \iff [M, R_{i}^{[j_1,j_2-1]}] \neq 0.
   \]

Note that for \( i \in [1,n] \) and \( j_1 < j_2 \) the condition \( \delta_{i}^{[j_1,j_2]}(\mathbf{d}) = 0 \) is equivalent to \( p_{i,j_{1}}^d = p_{i,j_{2}}^d. \)

2.5. For a subcategory \( \mathcal{X} \) of the category of \( \Lambda \)-modules and a dimension vector \( \mathbf{d} \) denote by \( \mathcal{X}(\mathbf{d}) \) the set of \( M \in \text{mod}_{\Lambda}(\mathbf{d}) \) such that \( M \in \mathcal{X} \).

For \( \mathbf{d} \in \mathbb{R} \) let \( \mathcal{C} = \mathcal{C}(\mathbf{d}) \) be the set of quadruples \( (\mathbf{d}', \mathbf{d}'', [X], q) \) such that \( \mathbf{d}' \in \mathcal{P}, \mathbf{d}'' \in \mathcal{Q}, X \in \mathcal{R}', q \in \mathbb{N}, \) and \( \mathbf{d}' + \mathbf{d}'' + \dim X + qd = \mathbf{d}. \) Observe that \( \mathcal{C} \) is a finite set. For \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathcal{C} \) let \( \mathcal{C}(\mathbf{d}', \mathbf{d}'', [X], q) \) be the set of \( M \in \text{mod}_{\Lambda}(\mathbf{d}) \) which are isomorphic to modules of the form \( M' \oplus M'' \oplus X \oplus Y \) with \( M' \in \mathcal{P}, \dim M' = \mathbf{d}', \ M'' \in \mathcal{Q}, \dim M'' = \mathbf{d}'', \) and \( Y \in \mathcal{R}'', \dim Y = qh. \) Obviously \( \text{mod}_{\Lambda}(\mathbf{d}) \) is a finite disjoint union of the sets \( \mathcal{C}(\mathbf{d}', \mathbf{d}'', [X], q), (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathcal{C}. \) We will need the following properties of these sets.
Lemma. If \( \mathbf{d} \in \mathbb{R} \) and \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C} \), then \( \mathcal{C}(\mathbf{d}', \mathbf{d}'', [X], q) \) is an irreducible constructible set of dimension\( a(\mathbf{d}) + \langle \mathbf{d} - \mathbf{d}', \mathbf{d} - \mathbf{d}'', [X], X \rangle. \)

Proof. Compare the proof of [12, Lemma 3.5]. \( \square \)

Let \( \mathfrak{C}' = \mathfrak{C}'(\mathbf{d}) \) be the set of all \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C} \) such that the following conditions are satisfied:

1. \( \mathbf{d}' \neq 0 \) (equivalently, \( \mathbf{d}'' \neq 0 \)),
2. for each \( i \in [1, n] \) and each \( i \)-admissible interval \( [\delta_i, \delta_i + 1] \) either \( \delta_i \mathbf{d}' = 0 \) or \( [X, R_i] \neq 0 \) (equivalently, either \( \delta_i \mathbf{d}'' = 0 \) or \( [X, q] \neq 0 \)).

Observe that for \( M' \in \mathcal{P} \) the condition \( \delta_i (\mathbf{d}) > 0 \) is equivalent to \( [M', R_i] \neq 0 \). Similarly, for \( M'' \in \mathcal{Q} \) the condition \( \delta_i (\mathbf{d}) < 0 \) is equivalent to \( [R_i, M''] \neq 0 \).

Another important property, which follows easily from (2.4) (compare [12, Lemma 3.6]) is the following.

Observation. Let \( \mathbf{d} \in \mathbb{R} \) and \( p^d > 0 \). If \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C} \), then

\[
\mathcal{C}(\mathbf{d}', \mathbf{d}'', [X], q) \cap \mathcal{Z}(\mathbf{d}) \neq \emptyset \quad \iff \quad (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C}'
\]

Recall that if \( \mod_{\Lambda}(\mathbf{d}) \) is irreducible, then it is a complete intersection of dimension \( a(\mathbf{d}) \) (see for example [6]). Hence we get the following corollary, which determines our strategy of the proof.

Corollary. Let \( \mathbf{d} \in \mathbb{R} \), \( p^d > n - 1 \), and assume that \( \mod_{\Lambda}(\mathbf{d}) \) is irreducible. Then \( \mathcal{Z}(\mathbf{d}) \) is a complete intersection provided

\[
[X, X] - \langle \mathbf{d} - \mathbf{d}', \mathbf{d} - \mathbf{d}'', [X], X \rangle \geq p^d + 1 + \text{ad}(\mathbf{d}) - n
\]

for all \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C}' \).

3. Proof of the main result

Throughout this section \( \Lambda = \Lambda(\mathbf{m}, \lambda) \) is a fixed canonical algebra and \( \Delta \) is its quiver. Our aim in this section is to prove Main Theorem.

3.1. The first step in our proof is the following.

Lemma. If \( \mathbf{d} \in \mathbb{R} \), \( (\mathbf{d}', \mathbf{d}'', [X], q) \in \mathfrak{C}' \), and \( q > 0 \), then there exists \( (\mathbf{x}', \mathbf{x}'', [X'], q') \in \mathfrak{C}' \) such that

\[
[X, X] - \langle \mathbf{d} - \mathbf{d}', \mathbf{d} - \mathbf{d}'', [X], X \rangle \geq [X', X'] - \langle \mathbf{d} - \mathbf{x}', \mathbf{d} - \mathbf{x}'', [X], X \rangle
\]

Proof. Take \( \mathbf{x}' = \mathbf{d}' + q \mathbf{h} \), \( \mathbf{x}'' = \mathbf{d}'' \), \( X' = X \), and \( q' = 0 \). \( \square \)

Let \( \mathfrak{C}'' \) be the set of triples \( (\mathbf{d}', \mathbf{d}'', [X]) \) such that \( (\mathbf{d}', \mathbf{d}'', [X], 0) \in \mathfrak{C}' \). We have the following consequence of the above lemma and Corollary [2.5].
Corollary. Let \( d \in \mathbb{R} \), \( p^d \geq n - 1 \), and assume that \( \text{mod}_A(d) \) is irreducible. Then \( Z(d) \) is a complete intersection provided

\[
[X, X] - \langle d - d', d - d'' \rangle \geq p^d + 1 + \text{ad}(d) - n
\]

for all \((d', d'', [X]) \in \mathcal{C''}\).

3.2. The second, and the most difficult step, is to prove that we may assume that \( p^{\dim X} = 0 \).

Fix \( d \in \mathbb{R} \) and \((d', d'', [X]) \in \mathcal{C''}\) such that \( p^{\dim X} > 0 \). We associate to \((d', d'', [X])\) a new triple \((x', x'', [X'])\) such that \( x' \in P \), \( x'' \in Q \), and \( x' + x'' + \dim X' = d \) in the following way. Write \( X = \bigoplus_{i \in [1, n]} X_i \) with \( X_i \in \mathcal{R}_A \), \( i \in [1, n] \). Since \( p^{\dim X} > 0 \), there exists \( i \in [1, n] \) such that \( p^{\dim X_i} > 0 \). Without loss of generality we may assume that \( p^{\dim X_i} > 0 \). Let \( j_0 \) be the minimal \( j \in [1, m_1] \) such that \( \delta_{1,j}(d') > 0 \), and let \( l_2 \) be the minimal \( l \geq j_0 \) such that \( R_{1[l,j]} \) is a direct summand of \( X \) for some \( j \leq j_0 \) (this definition makes sense since \( p^{\dim X_i} > 0 \)), and let \( l_1 \) be the minimal \( l \) such that \( R_{1[l,1]} \) is a direct summand of \( X \). Note that \( l_2 < j_0 + m_1 \).

Write \( X = Y \oplus R_{1[l,j_2]} \) and put \( x' = d' + e_{1[l,j_2]} \), \( x'' = d'' \), and \( X' = Y \oplus R_{1[l,j_0-1]} \) (where \( R_{1[l,j_0-1]} = 0 \) if \( l_1 = j_0 \)).

In the following lemma and the next subsection we use the above notation.

Lemma. In the above situation

\[
[X, X] - \langle d - d', d - d'' \rangle > [X', X'] - \langle d - x', d - x'' \rangle.
\]

Proof. A crucial role in the proof is played by the following exact sequence

\[
0 \to R_{1[l,j_0-1]} \to R_{1[l,j_2]} \to R_{1[l,j_2]} \to 0.
\]

By applying the functor \( \text{Hom}_A(\cdot, X) \) to this sequence we obtain

\[
[R_{1[l,j_0-1]}, X] \leq [R_{1[l,j_2]}, X] - ([R_{1[j_0,l_2]}, X] - [R_{1[j_0,l_2]}, X])
= [R_{1[l,j_2]}, X] - \langle e_{1[l,j_2]}, \dim X \rangle = [R_{1[l,j_2]}, X] + \delta_{1[l,j_2]}(\dim X).
\]

Moreover, by application of the functor \( \text{Hom}_A(R_{1[l,j_0-1]}, \cdot) \) to this sequence we know that

\[
[R_{1[l,j_0-1]}, R_{1[l,j_0-1]}] \leq [R_{1[l,j_0-1]}, R_{1[l,j_2]}],
\]

and consequently

\[
[R_{1[l,j_0-1]}, X'] = [R_{1[l,j_0-1]}, R_{1[l,j_0-1]}] + [R_{1[l,j_0-1]}, Y]
\leq [R_{1[l,j_0-1]}, R_{1[l,j_2]}] + [R_{1[l,j_0-1]}, Y] = [R_{1[l,j_0-1]}, X].
\]

Finally by applying the functor \( \text{Hom}_A(Y, \cdot) \) to the above sequence we get

\[
[Y, R_{1[l,j_0-1]}] \leq [Y, R_{1[l,j_2]}],
\]
hence
\[ [X', X'] = [R_i^{[j_1, j_0 - 1]}, X'] + [Y, R_i^{[l_1, j_0 - 1]]} + [Y, Y] \]
\[ \leq [R_i^{[j_1, j_0 - 1]}, X] + [Y, R_i^{[l_1, l_2]}] + [Y, Y] \]
\[ \leq [R_i^{[j_1, l_2]}, X] + \delta_i^{[j_0, l_2]}(\dim X) + [Y, R_i^{[l_1, l_2]}] + [Y, Y] \]
\[ = [X, X] + \delta_i^{[j_0, l_2]}(\dim X). \]

On other hand
\[ \langle d - x', d - x'' \rangle = \langle d - d', d - d'' \rangle - \langle e_i^{[j_0, l_2]}, d' + \dim X \rangle \]
\[ = \langle d - d', d - d'' \rangle + \delta_i^{[j_0, l_2]}(d') + \delta_i^{[j_0, l_2]}(\dim X), \]

hence consequently
\[ [X', X'] - \langle d - x', d - x'' \rangle \leq [X, X] - \langle d - d', d - d'' \rangle - \delta_i^{[j_0, l_2]}(d') \]
what finishes the proof. \( \square \)

3.3. Now we check when \((x', x'', [X']) \in \mathcal{C}''\). For \(i \in [1, n]\) and \([j_1, j_2] \in \mathcal{A}_i(d)\) we say that the triple \((x', x'', [X'])\) satisfies the \((i, [j_1, j_2])\)-condition if either \(\delta_i^{[j_1, j_2]}(x') > 0\) or \([X', R_i^{[j_1, j_2 - 1]}] \neq 0\) (equivalently, either \(\delta_i^{[j_1, j_2]}(x'') < 0\) or \([R_i^{[j_1, j_2 - 1]}], X'] \neq 0\). Obviously \((x', x'', [X']) \in \mathcal{C}''\) if and only if \((x', x'', [X'])\) satisfies \((i, [j_1, j_2])\)-condition for all \(i \in [1, n]\) and \([j_1, j_2] \in \mathcal{A}_i(d)\).

We call a pair \((i, [j_1, j_2])\) consisting of \(i \in [1, n]\) and \([j_1, j_2] \in \mathcal{A}_i(d)\) critical for \((d', d'', [X])\) if \(i = 1\), \((d', d'', [X])\) is not a critical pair for \((d', d'', [X])\).

**Lemma.** If \(i \in [1, n]\), \([j_1, j_2] \in \mathcal{A}_i(d)\), and \((i, [j_1, j_2])\) is not a critical pair for \((d', d'', [X])\), then \((x', x'', [X'])\) satisfies the \((i, [j_1, j_2])\)-condition.

**Proof.** If \(i \neq 1\) or \(\delta_i^{[j_1, j_2]}(d'') < 0\), then the claim is obvious. Similarly, the claim follows easily if \(\delta_i^{[j_1, j_2]}(d') > 1\), since \(\delta_i^{[j_1, j_2]}(x') \geq \delta_i^{[j_1, j_2]}(d') - 1\). Hence we may assume that \(i = 1\), \(\delta_i^{[j_1, j_2]}(d') \leq 1\), and \(\delta_i^{[j_1, j_2]}(d'') = 0\).

After an appropriate choice of a representative we may assume that \(j_0 \leq j_2 < j_0 + m_1\). Consider first the case \(j_1 \geq j_0\). If \(j_1 \leq l_2 < j_2\), then \(\delta_i^{[j_1, j_2]}(x') > 0\), hence \((x', x'', [X'])\) satisfies \((1, [j_1, j_2])\)-condition in this case. On the other hand, if either \(j_1, j_2 > l_2\) or \(j_1, j_2 \leq l_2\), then \(\delta_i^{[j_1, j_2]}(x') = \delta_i^{[j_1, j_2]}(d')\), thus the claim will follow if we show that \([X', R_1^{[j_1, j_2 - 1]}] = [X, R_1^{[j_1, j_2 - 1]}]\) in this case. In order to prove this equality it is enough to show that \([R_1^{[l_2, j_1]}], R_1^{[j_1, j_2 - 1]}] = [R_1^{[l_2, j_1 - 1]}], R_1^{[j_1, j_2 - 1]}\].
By applying the functor $\text{Hom}_A(-, R_i^{[j_1, j_2-1]})$ to the short exact sequence

$$0 \to R_i^{[j_1, j_0-1]} \to R_i^{[l_1, l_2]} \to R_i^{[j_0, j_2]} \to 0,$$

we get a sequence

$$0 \to \text{Hom}_A(R_i^{[l_1, l_2]}, R_i^{[j_1, j_2-1]}) \to \text{Hom}_A(R_i^{[l_1, l_2]}, R_i^{[j_1, j_2-1]}) \to \text{Hom}_A(R_i^{[l_1, j_0-1]}, R_i^{[j_1, j_2-1]}) \to \text{Ext}^1_A(R_i^{[j_0, j_2]}, R_i^{[j_1, j_2-1]}).$$

By using (1.6.1) and the Auslander–Reiten formula we obtain that

$$\text{Ext}^1_A(R_i^{[j_0, j_2]}, R_i^{[j_1, j_2-1]}) = \text{Hom}_A(R_i^{[j_1, j_2-1]}, R_i^{[j_0-1, j_2-1]}) = 0,$$

hence we get the required equality and finish the proof in this case.

In the second case, i.e. when $l_1 \leq j_1 < j_0$, $[R_i^{[l_1, j_0-1]}, R_i^{[j_1, j_2-1]}] \neq 0$ and the claim follows again.

Finally, assume $j_1 < l_1$. If $j_2 > l_2$, then $\delta_i^{[j_1+1, j_2]}(\lambda') = \delta_i^{[j_1+1, j_2]}(\lambda') > 0$. On the other hand, if $j_2 = j_0$ then $[X, R_i^{[j_1, j_2-1]}] \neq 0$, since the pair $(1, [j_1, j_2])$ is not critical. If $j_1 + m_1 > l_2$ then $[R_i^{[l_1, l_2]}, R_i^{[j_1, j_2-1]}] = 0$ and we get $[X, R_i^{[j_1, j_2-1]}] = [X, R_i^{[j_1, j_2-1]}] = 0$, which imply $\delta_i^{[j_1+1, j_2]}(\lambda') = 0$, if $j_1 + m_1 \leq l_2$ then $\delta_i^{[j_1+1, j_2]}(\lambda') = 0$, together with the inequality $d_{j_1, j_0} > d_{l_1, l_2}$, would mean that $p_{\dim X} > p_{\dim X}$ if $j_0 < j_2 \leq l_2$. As a consequence, there would exist a direct summand of $X$ of the form $R_i^{[j_1]}$ for $j \leq j_0 \leq l < j_2 \leq l_2$ in this case — a contradiction to the definition of $l_2$. \square

3.4. We use now the results of the two previous subsections to make the next step in the proof.

**Lemma.** If $d \in R$, $(d', d'', [X]) \in \mathcal{C}''$, and $p_{\dim X} > 0$, then there exists $(\lambda', \lambda'', [X']) \in \mathcal{C}''$ such that $\dim_k X' < \dim_k X$ and

$$[X, X] - \langle d - d', d - d'' \rangle > [X', X'] - \langle d - \lambda', d - \lambda'' \rangle.$$

Moreover, the inequality is strict if $p_{\dim X'} = 0$.

**Proof.** Without loss of generality we may assume that $p_{\dim X_1} > 0$, where $X = \bigoplus_{i \in [1, n]} X_i$ for $X_i \in R_{\lambda_1}$, $i \in [1, n]$. Suppose first there are no critical pairs for $(d', d'', [X])$. Then it follows from Lemmas 3.2 and 3.3 that the triple $(\lambda', \lambda'', [X'])$ obtained from $(d', d'', [X])$ by applying the construction described in (3.2) belongs to $\mathcal{C}''$, $\dim_k X' < \dim_k X$, and

$$[X, X] - \langle d - d', d - d'' \rangle > [X', X'] - \langle d - \lambda', d - \lambda'' \rangle.$$
Assume now that there exists a critical pair for \((\mathbf{d}', \mathbf{d}'', [X])\). Without loss of generality we may assume this pair is of the form \((1, [j_1, j_0])\) for \(j_1 < l_1\), where \(j_0\) and \(l_1\) (and also \(l_2\)) have the same meaning as in (3.2).

If \(R\) is an indecomposable direct summand of \(X\) of the form \(R^{[u_1, u_2]}\) for \(u_1 \leq j_1 \leq u_2\), then it follows that \(u_2 \geq j_0\) (since \([X, R^{[j_1, j_0 - 1]}] = 0\)), and consequently \(u_2 > l_2\) by the definition of \(l_1\) and \(l_2\). In particular this means that \(p^{\dim R}_{1, j_0} \leq p^{\dim R}_{1, j_1}\) for each direct summand \(R\) of \(X\). Since \(p^{\dim X}_{1, j_1} = p^{\dim X}_{1, j_0} - 1\), this implies that if \(R\) is a direct summand of \(X\) of the form \(R^{[u_1, u_2]}\) for \(u_1 \leq j_0 \leq u_2 < j_0 + m_1\) different from \(R^{[l_1, l_2]}\), then \(u_1 \leq j_1\) and \(u_2 > l_2\).

Let \(v_2\) be the minimal \(u_2\) such that \(R^{[u_1, u_2]}\) is a direct summand of \(X\) for \(u_1 \leq j_1 \leq u_2\) and let \(v_1\) be the maximal \(u_1\) such that \(R^{[v_1, v_2]}\) is a direct summand of \(X\). Recall that \(v_1 < l_1 \leq l_2 < v_2\). Moreover, the minimality of \(v_2\) implies that \(v_2 < l_2 + m_1\). Write \(X = Y \oplus R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}\), and let \(X' = Y \oplus R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}\). Our definitions imply that \(Y\) has no direct summands of the form \(R^{[u_1, u_2]}\) with either \(v_1 < u_1 \leq l_1\) and \(l_2 < u_2 \leq v_2\), or \(v_1 \leq u_1 \leq l_1\) and \(l_2 \leq u_2 < v_2\), hence

\[ [Y, R^{[v_1, l_2]} \oplus R^{[l_1, l_2]}] = [Y, R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}] \]

and

\[ [R^{[v_1, l_2]} \oplus R^{[l_1, l_2]}, Y] = [R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}, Y] \]

In addition, tedious analysis shows that

\[ [R^{[v_1, l_2]} \oplus R^{[l_1, l_2]}, R^{[v_1, v_2]}] =
\[ [R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}, R^{[l_1, l_2]} \oplus R^{[v_1, v_2]}] + 1 \]

(here it is important that \(v_2 - l_2 < m_1\)), hence it follows that \([X', X'] = [X, X] + 1\). Observe that \((\mathbf{d}', \mathbf{d}'', [X'])\) \(\in\) \(\mathfrak{C}''\) (since \([X', R^{[u_1, u_2]}] \geq [X, R^{[u_1, u_2]}]\) for all \(i \in [1, n]\) and \(u_1 \leq u_2\)). Moreover, there are no critical pairs for \((\mathbf{d}', \mathbf{d}'', [X'])\), thus it follows from Lemmas 3.2 and 3.3 that for the triple \((\mathbf{x}', \mathbf{x}'', [X''])\) obtained from \((\mathbf{d}', \mathbf{d}'', [X'])\) by applying the construction of (3.2) we have: \((\mathbf{d}', \mathbf{d}'', [X'']) \in \mathfrak{C}''\), \(\dim_k X'' < \dim_k X' = \dim_k X\), and

\[ [X'', X''] - \langle \mathbf{d} - \mathbf{x}', \mathbf{d} - \mathbf{x}'' \rangle \leq [X', X'] - \langle \mathbf{d} - \mathbf{d}', \mathbf{d} - \mathbf{d}'' \rangle - 1 = [X, X] - \langle \mathbf{d} - \mathbf{d}', \mathbf{d} - \mathbf{d}'' \rangle. \]

Since \(p^{\dim X''} > 0\), this finishes the proof. \(\square\)

Let \(\mathfrak{C}''\) be the set of triples \((\mathbf{d}', \mathbf{d}'', [X])\) \(\in\) \(\mathfrak{C}''\) such that \(p^{\dim X} = 0\). We have the following consequence of the above lemma and Corollary 3.1.
**Corollary.** Let \( d \in \mathbb{R}, \) \( p^d \geq n - 1, \) and assume that \( \text{mod}_A(d) \) is irreducible. Then \( Z(d) \) is a complete intersection provided
\[
[X, X] - \langle d - d', d - d'' \rangle \geq p^d + 1 + \text{ad}(d) - n
\]
for all \( (d', d'', [X]) \in \mathcal{C}^m. \)

3.5. For \( d \in \mathbb{R} \) and \( (d', d'', [X]) \in \mathcal{C}^m, \) let
\[
\text{ad}^{(1)} = \# \{(i, [j_1, j_2]) \in [1, n] \times \mathcal{A}_i(d) \mid d_{i[j_1+1,j_2]} > 0\},
\]
\[
\text{ad}^{(2)} = \# \{(i, [j_1, j_2]) \in [1, n] \times \mathcal{A}_i(d) \mid d_{i[j_1+1,j_2]}(d') = 0, d_{i[j_1+1,j_2]}(d'') < 0\},
\]
and
\[
\text{ad}^{(3)} = \# \{(i, [j_1, j_2]) \in [1, n] \times \mathcal{A}_i(d) \mid d_{i[j_1+1,j_2]}(d') = 0 = d_{i[j_1+1,j_2]}(d'')\}.
\]

Obviously
\[
\text{ad}(d) = \text{ad}^{(1)} + \text{ad}^{(2)} + \text{ad}^{(3)}.
\]
The final auxiliary step in the proof is as follows.

**Lemma.** Let \( d \in \mathbb{R} \) and \( (d', d'', [X]) \in \mathcal{C}^m. \)

1. \( -\langle d, d' \rangle \geq (p^d - n)(d'_0 - d''_0) + \text{ad}^{(1)}, \)
2. \( -\langle d'', \dim X \rangle \geq \text{ad}^{(2)}, \)
3. \( [X, X] \geq \langle \dim X, \dim X \rangle + \text{ad}^{(3)}. \)

Before we present the proof of the above lemma, we show how it implies Main Theorem. Note that
\[
\langle d - d', d - d'' \rangle = -\langle d', d' \rangle + \langle d, d' \rangle + \langle d'', \dim X \rangle + \langle \dim X, \dim X \rangle.
\]

Consequently, we have the following corollary being a consequence of the above lemma and Corollary 3.4.

**Corollary.** Let \( d \in \mathbb{R}, \) \( p^d \geq n - 1, \) and assume that \( \text{mod}_A(d) \) is irreducible. Then \( Z(d) \) is a complete intersection provided
\[
\langle \langle d', d' \rangle - 1 \rangle + (p^d - n)(\langle d', h \rangle - 1) \geq 0
\]
for all \( (d', d'', [X]) \in \mathcal{C}^m. \)

**Proof of Main Theorem.** By repeating arguments used in \[7, \] Proofs of Propositions 4.1 and 4.2] we get that
\[
\langle \langle d', d' \rangle - 1 \rangle + (p^d - n)(\langle d', h \rangle - 1) - 1 \geq 0
\]
for \( d' \in \mathbb{P}, d' \neq 0, \) if \( \delta \leq 0 \) and \( p^d \geq N, \) where \( N = n \) if \( \delta < 0, N = n + 1 \) if \( \delta = 0. \) Recall from \[8, \] Theorem 1 \] (compare also \[6, \] Theorems 1.1 and 1.3 (2))] that \( \text{mod}_A(d) \) is irreducible, if \( \Lambda \) is a tame canonical algebra, hence the claim follows from the previous corollary.
\[\square\]
3.6. We prove now points (1) and (2) of Lemma 3.5.

**Proof of Lemma 3.5 (1).** Let \( s = d'_0 - d'_\infty \) and \( t = d'_\infty \). For each \( i \in [1, n] \) there exists a sequence \( 0 \leq l_{i,1} \leq \cdots \leq l_{i,s} < m_i \) such that

\[
d' = th + \sum_{j \in [1,s]} e(l_{1,j}, \ldots, l_{n,j})
\]

where for a sequence \((l_1, \ldots, l_n)\) such that \( l_i \in [0, m_i - 1] \) for \( i \in [1, n] \) we put

\[
e(l_1, \ldots, l_n) = e_0 + \sum_{i \in [1,n]} \sum_{j \in [1,l_i]} e_{i,j}.
\]

Note that for \((l_1, \ldots, l_n)\) as above

\[
\langle d, e(l_1, \ldots, l_n) \rangle = -p^d - \sum_{i \in [1,n]} p_i^{d_{l_i+1}},
\]

and for \( i \in [1, n] \) and \( j_1 < j_2 \)

\[
\delta_i^{[j_1+1,j_2]}(e(l_1, \ldots, l_n)) = \begin{cases} 1 & \text{ if } l_i \text{ lies inside } [j_1, j_2], \\ 0 & \text{ otherwise.} \end{cases}
\]

Since for \( i \in [1, n] \) and \( j_1 < j_2 \),

\[
\delta_i^{[j_1+1,j_2]}(d) > 0 \iff \exists j \in [1,s] \quad \delta_i^{[j_1+1,j_2]}(e(l_1, \ldots, l_n)) > 0,
\]

the claim follows from Observation 2.3. \( \square \)

**Proof of Lemma 3.5 (2).** Similarly as above for each \( i \in [1, n] \) there exists \( 0 < l_{i,1} \leq \cdots \leq l_{i,s} \leq m_i \) such that

\[
d'' = th + \sum_{j \in [1,s]} e'(l_{1,j}, \ldots, l_{n,j})
\]

for \( s = d''_\infty - d''_0 \) and \( t = d''_0 \), where for a sequence \((l_1, \ldots, l_n)\) such that \( l_i \in [1, m_i) \) for \( i \in [1, n] \) we put

\[
e'(l_1, \ldots, l_n) = e_\infty + \sum_{i \in [1,n]} \sum_{j \in [l_i, m_i-1]} e_{i,j}.
\]

We also have

\[
\langle e'(l_1, \ldots, l_n), \dim X \rangle = -\sum_{i \in [1,n]} p_{i,l_i-1}^{\dim X}
\]

for \((l_1, \ldots, l_n)\) as above.

Fix \( i \in [1, n] \). Let \( A_i^{(2)}(d) \) be the set of \([j_1, j_2] \in A_i(d)\) such that \( \delta_i^{[j_1+1,j_2]}(d') = 0 \) and \( \delta_i^{[j_1+1,j_2]}(d'') < 0 \). We define a function \( f : A_i^{(2)}(d) \to [1, s] \) given by

\[
f([j_1, j_2]) = \min\{j \in [1, s] \mid l_i,j - 1 \text{ lies inside } [j_1, j_2]\}.
\]
Our claim will follow if we show that the inverse image of \( j \in [1, s] \) has at most \( p_{d_{i,j,1}}^{\text{dim}X} \) elements. Fix \( j \in [1, s] \) and let \([j_1, j_1, j_2], \ldots, [j_s, j_s, j_s]\) be the intervals in \( \mathcal{A}_i^{(2)} \) whose image under \( f \) is \( j \). We may assume that

\[
j_{1,1} < \cdots < j_{s,1} \leq l_{i,j} - 1 < j_{i,j} \leq j_{s,2} < \cdots < j_{1,2}.
\]

Then \( p_{i,j,1}^d < \cdots < p_{d_{i,j,1}} \), hence

\[
s \leq p_{d_{i,j,1}} - p_{d_{i,j,1}}^d + 1 \leq p_{d_{i,j,1}}^d - p_{d_{i,j,1}}^d + 1.
\]

The definitions of \( \mathcal{A}_i^{(2)}(d) \) and \( f \) imply that \( d_{i,j,1}^s = d_{i,j,1}^d \) and \( d_{i,j,1}^u = d_{i,j,1}^d \), hence

\[
p_{i,j,1}^d - p_{d_{i,j,1}}^d = p_{d_{i,j,1}}^d - d_{i,j,1} = x_{i,j,1} = p_{d_{i,j,1}}^d - p_{d_{i,j,1}}^d,
\]

where \( X = \text{dim} \), thus in order to finish the proof it remains to show that \( p_{d_{i,j,1}}^d > 0 \). This follows since the conditions \( p_{d_{i,j,1}}^d = p_{d_{i,j,1}}^d, \delta_{d_{i,j,1},j_2}^d(d') = 0, \) and \( \delta_{d_{i,j,1}+j_2}^d(d'') < 0, \) imply that \( p_{d_{i,j,1}}^d > p_{d_{i,j,1}}^d \) at

3.7. Before we give the proof of the last point of Lemma 3.5 we present some auxiliary facts. For \( m \geq 1 \) let \( A_m \) be the path algebra of the quiver

\[
\sum_{m} = \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet
\]

For an interval \([j_1, j_2]\) with \( 1 \leq j_1 < j_2 \leq m \) let \( X_{j_1:j_2} \) be the unique indecomposable \( A_m \)-module of dimension vector \( \sum_{j \in [j_1:j_2]} e_j \). An interval \([j_1, j_2]\) with \( 1 \leq j_1 < j_2 \leq m \) is called admissible for \( d \in \mathbb{N}^{(\sum_m)} \) if \( d_{j_1} = d_{j_2} > 0 \) and \( d_{j} > d_{j_2} \) for all \( j \in [j_1 + 1, j_2 - 1] \). Let \( \mathcal{A}(d) \) be the set of admissible intervals for \( d \). The following is a consequence of [15] Theorem 5.7 and the description of semi-invariants for \( \sum_m \) obtained in [2] (see also [11]).

**Proposition.** Let \( d \in \mathbb{N}^{(\sum_m)} \), \( \mathcal{A}' \) be a subset of \( \mathcal{A}(d) \), and \( M \in \text{mod}_{A_m}(d) \). If \([X_{j_1+j_2}^s, M]_{A_m} \neq 0 \) for all \([j_1, j_2] \in \mathcal{A}' \), then

\[
[M, M]_{A_m} \geq \langle d, d \rangle_{A_m} + \# \mathcal{A}'.
\]

**Proof of Lemma 3.5**. For each \( i \in [1, n] \) fix \( l_i \in [0, m_i - 1] \) such that \( p_{i,l_i}^{\text{dim}X} = 0 \). For \( i \in [1, n] \) let \( \mathcal{S}_i \) be the full subcategory of \( \mathcal{R}_s \) formed by the objects \( F_i(R_i^{j_1:j_2}) \) such that \( l_i < j_1 \leq j_2 < l_i + m_i \). It is known that there exists an equivalence \( F_i \) between \( \mathcal{S}_i \) and the category \( \mathcal{A}_{m_i-1} \)-modules such that \( F_i(R_i^{j_1:j_2}) = X_{j_i-j_1,j_2-l_i} \) for \( l_i < j_1 \leq j_2 < l_i + m_i \) (in particular, \( \text{dim}(F_iR)_{j} = p_{i,l_i+j}^{\text{dim}R} \) for \( j \in [1, m_i - 1] \)). Write

\[
[X, X]_{\Lambda} = [X_1, X_1]_{\Lambda} + \cdots + [X_n, X_n]_{\Lambda} = [F_1X_1, F_1X_1]_{A_{m_1-1}} + \cdots + [F_nX_n, F_nX_n]_{A_{m_n-1}}.
\]
Fix $i \in [1, n]$. Note that for each $[j_1, j_2] \in \mathcal{A}_i(d)$ with $\delta_{i}^{[j_1+1,j_2]}(d') = 0 = \delta_{i}^{[j_1,j_2]}(d'')$ we have $P_{i,j_1}^{\dim X_i} = P_{i,j_2}^{\dim X_i}$ and $P_{i,j}^{\dim X_i} > P_{i,j_1}^{\dim X_i}$ for $j \in [j_1+1, j_2-1]$. Moreover, $[P_{i,j_1+1,j_2}, X_i]_{\Lambda} \neq 0$. This implies in particular that $P_{i,j_2}^{\dim X_i} > 0$. Consequently, $[j_1 - l_i, j_2 - l_i] \in \mathcal{A}(\dim F_i X_i)$ and $[X^{[j_1-l_i,j_2-l_i]}, F_i X_i]_{\Lambda} \neq 0$ (here we assume that $[j_1, j_2]$ is chosen in such a way that $l_i < j_1 < j_2 < l_1 + m_i$). Thus it follows from the above proposition that

$$[F_i X_i, F_i X_i]_{\Lambda} \geq \langle \dim F_i X_i, \dim F_i X_i \rangle_{\Lambda} + \text{ad}_i^{(3)},$$

where

$$\text{ad}_i^{(3)} = \#\{(j_1, j_2) \in \mathcal{A}_i(d) \mid \delta_{i}^{[j_1+1,j_2]}(d') = 0 = \delta_{i}^{[j_1,j_2]}(d'')\}.$$

Since $\langle \dim F_i X_i, \dim F_i X_i \rangle_{\Lambda} = \langle \dim X_i, \dim X_i \rangle_{\Lambda}$, we have

$$\langle \dim X_i, \dim X_i \rangle_{\Lambda} = \langle \dim X_1, \dim X_1 \rangle_{\Lambda} + \cdots + \langle \dim X_n, \dim X_n \rangle_{\Lambda},$$

and $\text{ad}^{(3)} = \text{ad}_1^{(3)} + \cdots + \text{ad}_n^{(3)}$, the claim follows. □

4. Application to modules of covariants

Let $\Lambda = \Lambda(m, \lambda)$ be a canonical algebra and $d \in \mathbb{R}$ with $p^d > 0$. The aim of this section is to prove the following.

**Theorem.** If $\text{mod}_A(d)$ is irreducible, $\text{SI}(\Lambda, d)$ is a polynomial ring in $s$ variables and the codimension of $\mathcal{Z}(d)$ in $\text{mod}_A(d)$ equals $s$, then $k[\text{mod}_A(d)]$ is free as a $\text{SI}(\Lambda, d)$-module.

Note that the conditions of the above theorem are satisfied in the situations covered by Main Theorem and [7, Theorem 3].

The proof of the above theorem basically repeats arguments from [43, Proof of Proposition 17.29].

**Proof.** We introduce a grading in $k[\Lambda(d)]$ in such a way that polynomials defining $\text{mod}_A(d)$ are homogeneous with respect to this grading, and consequently $k[\text{mod}_A(d)]$ is graded (recall that the corresponding scheme is reduced — see [6, (3.3)]). Namely, the degree of $X_{u,i,j}$ is $m_i/m_i$ for $i \in [1, n]$, $j \in [1, m_i]$, $u \in [1, d_{i,j-1}]$ and $v \in [1, d_{i,j}]$, where $m = m_1 \cdots m_n$. Obviously, we may choose generators $f_1, \ldots, f_s$ of $\text{SI}(\Lambda, d)$ which are homogeneous (in fact, one may easily calculate the degrees of the generators from [45, Theorem 1.1]). It follows from the proof of [47, Theorem VII.25, p. 200], that $f_1, \ldots, f_s$ can be extended to a homogeneous system of parameters for $k[\text{mod}_A(d)]$. Since $k[\text{mod}_A(d)]$ is a Cohen–Macaulay ring, the claim is a consequence of arguments given in [26, p. 1036] □
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