ALEXANDROV IMMERSED MINIMAL TORI IN $S^3$

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Abstract. We show that any minimal torus in $S^3$ which is Alexandrov immersed must be rotationally symmetric. An analogous result holds for surfaces of constant mean curvature.

1. Introduction

In a recent paper [2], we showed the Clifford torus is the only embedded minimal surface in $S^3$ of genus 1, thereby confirming a conjecture of Lawson. In this note, we classify minimal tori in $S^3$ that are immersed in the sense of Alexandrov.

Theorem 1. Let $F : \Sigma \to S^3$ be an immersed minimal surface in $S^3$ of genus 1. Moreover, we assume that $F$ is an Alexandrov immersion; this means that there exists a compact manifold $N$ and an immersion $\bar{F} : N \to S^3$ such that $\partial N = \Sigma$ and $\bar{F}|_{\Sigma} = F$. Then $\Sigma$ is rotationally symmetric. That is, we can find an anti-symmetric matrix $Q \in \mathfrak{so}(4)$ of rank 2 such that $Q F(x) \in \text{span}\{\frac{\partial F}{\partial x_1}(x), \frac{\partial F}{\partial x_2}(x)\}$ for all $x \in \Sigma$.

There is a complete classification of all rotationally symmetric minimal tori in $S^3$; for details, we refer to [8] or [4], Theorem 1.4. Besides the Clifford torus, there is a large class of additional examples which are Alexandrov immersed but fail to be embedded.

We will present the proof of Theorem 1 in Section 2. The argument is similar in spirit to the case of embedded surfaces studied in [2], and we will only indicate the necessary modifications.

After the paper [2] was published, Andrews and Li [1] observed that the arguments in [2] can be extended to the setting of constant mean curvature surfaces. As a result, they showed that every embedded constant mean curvature surface in $S^3$ is rotationally symmetric. More recently, we showed that the arguments in [2] can be generalized to a class of Weingarten surfaces (see [3]).

Our proof of Theorem 1 also extends to the setting of constant mean curvature surfaces. This yields the following result:

Theorem 2. Let $F : \Sigma \to S^3$ be an immersed constant mean curvature surface in $S^3$ of genus 1. Suppose that $F$ extends to an immersion $\bar{F} : N \to S^3$ such that $\partial N = \Sigma$ and $\bar{F}|_{\Sigma} = F$. Then $\Sigma$ is rotationally symmetric. That is, we can find an anti-symmetric matrix $Q \in \mathfrak{so}(4)$ of rank 2 such that $Q F(x) \in \text{span}\{\frac{\partial F}{\partial x_1}(x), \frac{\partial F}{\partial x_2}(x)\}$ for all $x \in \Sigma$.

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where \( \partial N = \Sigma \) and that \( \partial N \) is mean convex with respect to the pull-back of the standard metric on \( S^3 \) under \( \tilde{F} \). Then \( \Sigma \) is rotationally symmetric.

The proof of Theorem \(^2\) is similar to Theorem \(^1\). The condition that the surface is Alexandrov immersed is quite natural in light of the work of Korevaar, Kusner, and Ratzkin \(^5\), Korevaar, Kusner, and Solomon \(^6\), and Kusner, Mazzeo, and Pollack \(^7\), where Alexandrov immersed constant mean curvature surfaces in \( \mathbb{R}^3 \) have been studied.

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\section{2. Proof of Theorem \(^1\)}

For convenience, we put a Riemannian metric on \( N \) so that \( \tilde{F} \) is a local isometry. In particular, there exists a real number \( \delta > 0 \) so that \( \tilde{F}(x) \neq \tilde{F}(y) \) for all points \( x, y \in N \) satisfying \( d_N(x, y) \in (0, \delta) \).

For each point \( x \in \Sigma \) and any number \( \alpha \geq 1 \), we define

\[
D_\alpha(x) = \left\{ p \in S^3 : \alpha \sqrt{2} |A(x)| (1 - \langle F(x), p \rangle) + \langle \nu(x), p \rangle \leq 0 \right\}.
\]

Note that \( D_\alpha(x) \) is a geodesic ball in \( S^3 \) whose boundary passes through the point \( F(x) \). Moreover, the outward-pointing unit normal vector to \( \partial D_\alpha(x) \) at the point \( F(x) \) is given by \( \nu(x) \).

Let \( I \) denote the set of all points \( (x, \alpha) \in \Sigma \times [1, \infty) \) with the property that there exists a smooth map \( G : D_\alpha(x) \to N \) such that \( \tilde{F} \circ G = \text{id}_{D_\alpha(x)} \) and \( G(F(x)) = x \).

\textbf{Lemma 3.} Let us fix a pair \( (x, \alpha) \in I \). Then there is a unique map \( G : D_\alpha(x) \to N \) such that \( \tilde{F} \circ G = \text{id}_{D_\alpha(x)} \) and \( G(F(x)) = x \).

\textbf{Proof.} Suppose that there are two maps \( G, \tilde{G} : D_\alpha(x) \to N \) such that \( \tilde{F} \circ G = \tilde{F} \circ \tilde{G} = \text{id}_{D_\alpha(x)} \) and \( G(F(x)) = \tilde{G}(F(x)) = x \). For each point \( p \in D_\alpha(x) \), we have \( \tilde{F}(G(p)) = \tilde{F} (\tilde{G}(p)) \), hence \( d_N(G(p), \tilde{G}(p)) \notin (0, \delta) \). By continuity, we either have \( G(p) = \tilde{G}(p) \) for all \( p \in D_\alpha(x) \) or we have \( G(p) \neq \tilde{G}(p) \) for all \( p \in D_\alpha(x) \). The second case can be ruled out, as \( G(F(x)) = \tilde{G}(F(x)) \). Thus, we conclude that \( G(p) = \tilde{G}(p) \) for all \( p \in D_\alpha(x) \).

This shows that \( G \) is unique.

\textbf{Lemma 4.} The set \( I \) is closed. Furthermore, the map \( G \) depends continuously on the pair \( (x, \alpha) \).

\textbf{Proof.} Consider a sequence of pairs \( (x^{(m)}, \alpha^{(m)}) \in I \) such that \( \lim_{m \to \infty}(x^{(m)}, \alpha^{(m)}) = (\hat{x}, \hat{\alpha}) \). By definition of \( I \), we can find a sequence of smooth maps \( G^{(m)} : D_{\alpha^{(m)}}(x^{(m)}) \to N \) such that \( \tilde{F} \circ G^{(m)} = \text{id}_{D_{\alpha^{(m)}}(x^{(m)})} \) and \( G^{(m)}(F(x^{(m)})) = x^{(m)} \). Since \( \tilde{F} \) is a smooth immersion, the maps \( G^{(m)} \) are uniformly bounded.
in $C^2$ norm. Hence, after passing to a subsequence, the maps $G^{(m)}$ converge in $C^1$ to a map $G : D_\alpha(x) \to N$. The map $G$ satisfies $\hat{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(\hat{x})) = \hat{x}$. From this, we deduce that $G$ is smooth. Thus, $(\hat{x}, \hat{\alpha}) \in I$, and the assertion follows.

**Lemma 5.** We have $(x, \alpha) \in I$ if $\alpha$ is sufficiently large.

**Proof.** By a result of Lawson, we have $|A(x)| > 0$ for all $x \in \Sigma$. Hence, if we choose $\alpha$ sufficiently large, the radius of the geodesic ball $D_\alpha(x) \subset S^3$ can be made arbitrarily small. Therefore, if $\alpha$ is sufficiently large, the implicit function theorem guarantees the existence of a smooth map $G : D_\alpha(x) \to N$ such that $\hat{F} \circ G = \text{id}_{D_\alpha(x)}$ and $G(F(\hat{x})) = x$.

After these preparations, we now describe the proof of Theorem 1. Let us define

$$\kappa = \inf \{ \alpha : (x, \alpha) \in I \text{ for all } x \in \Sigma \}.$$  

Clearly, $\kappa \in [1, \infty)$. For each point $x \in \Sigma$, there is a unique map $G_x : D_\kappa(x) \to N$ such that $\hat{F} \circ G_x = \text{id}_{D_\kappa(x)}$ and $G_x(F(x)) = x$. For each point $x \in \Sigma$, the map $G_x$ and the map $\hat{F} \circ G_{x(D_{\alpha}(x))}$ are one-to-one.

We next define a smooth function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) = \frac{\kappa}{\sqrt{2}} |A(x)| (1 - \langle F(x), F(y) \rangle) + \langle \nu(x), F(y) \rangle$$

for $x, y \in \Sigma$. In contrast to [2], the function $Z(x, y)$ might be negative somewhere.

As in [2], we distinguish two cases:

**Case 1:** Suppose first that $\kappa = 1$.

**Lemma 6.** We have $Z(x, y) \geq 0$ if $x$ and $y$ are sufficiently close.

**Proof.** The proof is by contradiction. Suppose that there exist two sequences of points $x^{(m)}, y^{(m)} \in \Sigma$ such that $\lim_{m \to \infty} x^{(m)} = \lim_{m \to \infty} y^{(m)}$ and $Z(x^{(m)}, y^{(m)}) < 0$ for all $m$. Since $Z(x^{(m)}, y^{(m)}) < 0$, the point $F(y^{(m)})$ lies in the interior of the geodesic ball $D_\kappa(x^{(m)})$. Since $G_{x^{(m)}}$ is an immersion, the point $\tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)}))$ must lie in the interior of $N$. Since $y^{(m)}$ lies on the boundary $\Sigma$, it follows that $\tilde{y}^{(m)} \neq y^{(m)}$.

On the other hand, we have

$$\hat{F}(\tilde{y}^{(m)}) = F(y^{(m)})$$

and

$$\lim_{m \to \infty} \hat{y}^{(m)} = \lim_{m \to \infty} G_{x^{(m)}}(F(y^{(m)})) = \lim_{m \to \infty} G_{x^{(m)}}(F(x^{(m)})) = \lim_{m \to \infty} x^{(m)} = \lim_{m \to \infty} y^{(m)}.$$  

This is impossible since $\hat{F}$ is an immersion.
Lemma 7. Fix a point \( x \in \Sigma \), and let \( \{e_1, e_2\} \) be an orthonormal basis of \( T_x \Sigma \) such that \( h(e_1, e_1) > 0 \), \( h(e_1, e_2) = 0 \), and \( h(e_2, e_2) < 0 \). Then \( e_1(|A|) = 0 \).

Proof. The proof is analogous to arguments in [2]. Let \( \gamma : \mathbb{R} \to \Sigma \) be a geodesic such that \( \gamma(0) = x \) and \( \gamma'(0) = e_1 \). Since the function \( Z \) is nonnegative when \( x \) and \( y \) are sufficiently close, the function \( f(t) = Z(x, \gamma(t)) \) is nonnegative when \( t \) is sufficiently small. Since \( \kappa = 1 \), we have \( f(0) = f'(0) = f''(0) = 0 \). Since the function \( f(t) \) is nonnegative in a neighborhood of 0, we conclude that \( f'''(0) = 0 \). From this, we deduce that \( (D_{e_1}^\Sigma h)(e_1, e_1) = 0 \). From this, the assertion follows.

Lemma 7 implies that the function \(|A|\) is constant along one family of curvature lines on \( \Sigma \). We claim that \( F \) is rotationally symmetric. To see this, we define a vector field \( V \) on \( \Sigma \) by \( V = |A|^{-\frac{1}{2}} e_1 \). The identity \( e_1(|A|) = 0 \) implies that \([V, e_1] = 0 \). Moreover, we compute

\[
D_{e_1}^\Sigma e_2 = -\frac{1}{2 |A|} e_2(|A|) e_1
\]

and

\[
D_{e_2}^\Sigma e_1 = 0.
\]

This implies

\[
[e_1, e_2] = D_{e_1}^\Sigma e_2 - D_{e_2}^\Sigma e_1 = -\frac{1}{2 |A|} e_2(|A|) e_1,
\]

hence

\[
[V, e_2] = |A|^{-\frac{1}{2}} [e_1, e_2] - e_2(|A|^{-\frac{1}{2}}) e_1 = 0.
\]

Using the identities \([V, e_1] = [V, e_2] = 0 \), it is straightforward to check that \( \mathcal{L}_V g = \mathcal{L}_V h = 0 \). Therefore, \( V \) is the restriction of an ambient Killing vector field on \( S^3 \). Hence, we can find a constant matrix \( Q \in \mathfrak{so}(4) \) such that \( V(x) = Q F(x) \) for all \( x \in \Sigma \). We now differentiate this relation along \( e_2 \). Since \( g(e_2, V) = h(e_2, V) = 0 \), we obtain the relation \( D_{e_2}^\Sigma V = Q e_2 \).

Since \( D_{e_2}^\Sigma V = e_2(|A|^{-\frac{1}{2}}) e_1 \), we conclude that

\[
Q \left( e_2 + \frac{1}{2 |A|} e_2(|A|) F(x) \right) = Q e_2 + \frac{1}{2 |A|^\frac{1}{2}} e_2(|A|) e_1 = 0.
\]

Therefore, the matrix \( Q \) has non-trivial nullspace. Thus, \( Q \) has rank 2 and \( F \) is rotationally symmetric.

Case 2: Suppose next that \( \kappa > 1 \).

Lemma 8. Fix a point \( x \in \Sigma \). Then there exists a constant \( \beta > 0 \) such that

\[
d_N(G_x(p), \Sigma) \geq \beta |p - F(x)|^2
\]

for all points \( p \in \partial D_n(x) \) that are sufficiently close to \( F(x) \).
Proof. Let us fix a point \( x \in \Sigma \). Moreover, we consider the function
\[
\varphi_x : \partial D_\kappa(x) \to \mathbb{R}, \quad p \mapsto d_N(G_x(p), \Sigma).
\]
Clearly, \( \varphi_x(F(x)) = 0 \), and the gradient of the function \( \varphi_x \) at the point \( F(x) \) vanishes. Moreover, since \( \kappa > 1 \), the Hessian of the function \( \varphi_x \) at the point \( F(x) \) is positive definite. Hence, we can find a positive constant \( \beta > 0 \) such that \( \varphi_x(p) \geq \beta |p - \varphi_x| \) for all points \( p \in \partial D_\kappa(x) \) that are sufficiently close to \( F(x) \).

Lemma 9. There exists a point \( \hat{x} \in \Sigma \) such that \( \Sigma \cap G_{\kappa}(D_\kappa(\hat{x})) \neq \{ \hat{x} \} \).

Proof. Suppose this is false. Then \( \Sigma \cap G_{\kappa}(D_\kappa(x)) = \{ x \} \) for all \( x \in \Sigma \). This implies that \( d_N(G_x(p), \Sigma) > 0 \) for all \( x \in \Sigma \) and all points \( p \in \partial D_\kappa(x) \setminus \{ F(x) \} \). Using Lemma 8 we conclude that there exists a positive constant \( \gamma > 0 \) such that \( d_N(G_x(p), \Sigma) \geq \gamma |p - \varphi_x|^2 \) for all \( x \in \Sigma \) and all \( p \in \partial D_\kappa(x) \). By the implicit function theorem, there exists a small number \( \varepsilon > 0 \) such that \( (x, \kappa - \varepsilon) \in I \) for all \( x \in \Sigma \). This contradicts the definition of \( \kappa \).

Let \( \hat{x} \) be chosen as in Lemma 9. Moreover, let us pick a point \( \hat{y} \in \Sigma \cap G_{\kappa}(D_\kappa(\hat{x})) \) such that \( \hat{x} \neq \hat{y} \). Since \( \hat{y} \in G_{\kappa}(D_\kappa(\hat{x})) \), we conclude that \( F(\hat{y}) \in D_\kappa(\hat{x}) \) and \( G_{\kappa}(F(\hat{y})) = \hat{y} \). Moreover, we claim that \( F(\hat{x}) \neq F(\hat{y}) \); indeed, if \( F(\hat{x}) = F(\hat{y}) \), then \( \hat{x} = G_{\kappa}(F(\hat{x})) = G_{\kappa}(F(\hat{y})) = \hat{y} \), which contradicts our choice of \( \hat{y} \).

We next consider the function \( Z \) defined above. We claim that the function \( Z \) is nonnegative in a neighborhood of the point \( (\hat{x}, \hat{y}) \).

Lemma 10. We have \( Z(x, y) \geq 0 \) if \( (x, y) \) is sufficiently close to \( (\hat{x}, \hat{y}) \).

Proof. We argue by contradiction. Suppose that there exist sequences of points \( x^{(m)}, y^{(m)} \in \Sigma \) such that \( \lim_{m \to \infty} x^{(m)} = \hat{x}, \lim_{m \to \infty} y^{(m)} = \hat{y} \), and \( Z(x^{(m)}, y^{(m)}) < 0 \). Since \( Z(x^{(m)}, y^{(m)}) < 0 \), the point \( F(y^{(m)}) \) lies in the interior of the ball \( D_\kappa(x^{(m)}) \). Since \( G_{x^{(m)}} \) is an immersion, we conclude that the point \( \tilde{y}^{(m)} := G_{x^{(m)}}(F(y^{(m)})) \) lies in the interior of \( N \). In particular, we have
\[
\tilde{y}^{(m)} \neq y^{(m)}.
\]
On the other hand, we have
\[
\tilde{F}(\tilde{y}^{(m)}) = F(\tilde{y}^{(m)})
\]
and
\[
\lim_{m \to \infty} \tilde{y}^{(m)} = \lim_{m \to \infty} G_{x^{(m)}}(F(y^{(m)})) = G_{\hat{x}}(F(\hat{y})) = \hat{y} = \lim_{m \to \infty} y^{(m)}.
\]
This contradicts the fact that \( \tilde{F} \) is an immersion. Thus, \( Z(x, y) \geq 0 \) for \( (x, y) \) close to \( (\hat{x}, \hat{y}) \).
Therefore, we can find disjoint open sets \( U, V \subset \Sigma \) such that \( \hat{x} \in U \), \( \hat{y} \in V \), and \( Z(x, y) \geq 0 \) for all points \( (x, y) \in U \times V \). As in [2], we define
\[
\Omega = \{ x \in U : \text{there exists a point } y \in V \text{ such that } Z(x, y) = 0 \}.
\]
Since \( F(\hat{y}) \in D_\kappa(\hat{x}) \), we have \( Z(\hat{x}, \hat{y}) \leq 0 \). This implies that \( Z(\hat{x}, \hat{y}) = 0 \), hence \( \hat{x} \in \Omega \). We can now use the calculation in [2] to conclude that \( Z \) is a supersolution of a degenerate elliptic equation. More precisely, suppose that \( (\hat{x}, \hat{y}) \) is an arbitrary point in \( U \times V \). Then we can find a system of geodesic normal coordinates \( (x_1, x_2) \) around \( \hat{x} \) and a system of geodesic normal coordinates \( (y_1, y_2) \) around \( \hat{y} \) such that
\[
\sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i^2}(\hat{x}, \hat{y}) + 2 \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial x_i \partial y_i}(\hat{x}, \hat{y}) + \sum_{i=1}^{2} \frac{\partial^2 Z}{\partial y_i^2}(\hat{x}, \hat{y}) \leq -\frac{\kappa^2 - 1}{\sqrt{2} \kappa} \left( 1 - \langle F(\hat{x}), F(\hat{y}) \rangle \right) \sum_{i=1}^{2} \left( \frac{\partial F}{\partial x_i}(\hat{x}), F(\hat{y}) \right) \right)^2
\]
\[
+ \Lambda \left( Z(\hat{x}, \hat{y}) + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial x_i}(\hat{x}, \hat{y}) \right| + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial y_i}(\hat{x}, \hat{y}) \right| \right),
\]
where \( \Lambda \) is a positive constant. Using Bony’s version of the strict maximum principle, we conclude that the set \( \Omega \) contains an open neighborhood of \( \hat{x} \). Moreover, the gradient of \( |A| \) vanishes on the set \( \Omega \). By analytic continuation, \( |A| \) is a constant function on \( \Sigma \). This implies that \( F \) is congruent to the Clifford torus. This completes the proof of Theorem [1].

3. Sketch of the proof of Theorem [2]

Finally, let us sketch the proof of Theorem [2]. Let \( F : \Sigma \to S^3 \) be an immersed constant mean curvature surface in \( S^3 \) of genus 1. We assume that \( F \) extends to an immersion \( \hat{F} : N \to S^3 \) where \( \partial N = \Sigma \) and that \( \partial N \) is mean convex with respect to the pull-back of the standard metric on \( S^3 \) under \( \hat{F} \). Given a point \( x \in \Sigma \) and a real number \( \alpha \geq 1 \), one defines
\[
D_\alpha(x) = \left\{ p \in S^3 : \left( \frac{H}{2} + \frac{\alpha}{\sqrt{2}} |\hat{A}(x)| \right) (1 - \langle F(x), p \rangle) + \langle \nu(x), p \rangle \leq 0 \right\}
\]
(cf. [1]). Here, \( H \) is the mean curvature (i.e. the sum of the principal curvatures) and \( \hat{A} \) is the trace-free part of the second fundamental form. As above, let \( I \) denote the set of all points \( (x, \alpha) \in \Sigma \times [1, \infty) \) with the property that there exists a smooth map \( G : D_\alpha(x) \to N \) such that \( \hat{F} \circ G = \text{id}_{D_\alpha(x)} \) and \( G(F(x)) = x \). It is well known that a constant mean curvature torus in \( S^3 \) has no umbilic points, so \( |\hat{A}(x)| > 0 \) for all points \( x \in I \). Hence, if \( \alpha \) is sufficiently large, then the radius of the geodesic ball \( D_\alpha(x) \) will be very small. From this, we deduce that \( (x, \alpha) \in I \) if \( \alpha \) is sufficiently large. We then define
\[
\kappa = \inf \{ \alpha : (x, \alpha) \in I \text{ for all } x \in \Sigma \}.
\]
If \( \kappa = 1 \), we can argue as above to conclude that \( e_1(|A|) = 0 \), where \( e_1 \) is one of the eigenvectors of the second fundamental form. This implies that \( F \) is rotationally symmetric. On the other hand, if \( \kappa > 1 \), we can combine the arguments above with the calculations in \([1]\) and \([2]\) to arrive at a contradiction.

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