A simplification of the vorticity equation and an extension of the vorticity persistence theorem to three dimensions

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A simplified form of the vorticity equation is derived for arbitrary coordinate systems. The present work unifies and extends the previous findings that vorticity is conserved in planar Euler flow, while in axisymmetric Euler rings it is the ratio of the vorticity to the distance from the symmetry axis that is conserved. The unifying statement is that in any Euler flow, all components of the vorticity tensor of a streamline coordinate system that are normal to the streamline direction are conserved along streamlines. This is true for both two- and three-dimensional flows, whether the flow is axisymmetric or not, with or without swirl. What remains of the nonlinear convective terms in the vorticity equation, after the mathematical simplification, is the Lie derivative of the vorticity tensor with respect to fluid velocity. A temporal derivative is defined which, when set to zero, expresses either the continuity or vorticity equation (excluding the viscous term), depending upon the argument supplied to it.

1. Introduction

The Navier-Stokes equation presents various difficulties to those seeking to solve it. Because of the presence of partial derivatives in all three spatial variables and the vectorial nature of the equation, the most promising solution methods involve the use of streamlined coordinate systems, which can consolidate the spatial dependence into a single variable. The solution can then be found by integrating along streamlines. An example of this is the following two-dimensional solution of Oseen (1910), given by:

\[ \bar{\omega}^3 = \frac{\Gamma_O}{4\pi \nu t} e^{-(\bar{x}^2)/(4\nu t)}, \quad \bar{\omega}^1, \bar{\omega}^2 = 0, \quad (1) \]

\[ \bar{v}^2 = \frac{\Gamma_O}{2\pi (\bar{x}^2)^{3/2}} \left( 1 - e^{-(\bar{x}^2)/(4\nu t)} \right), \quad \bar{v}^1, \bar{v}^3 = 0. \quad (2) \]

Here, variables in the streamline coordinate system are denoted with overbars, \( \Gamma_O \) is the total circulation, and \( \bar{x}^k \) are components of the vorticity tensor in the streamline coordinate system. In this case the streamline coordinate system happens to be a cylindrical coordinate system. Therefore, \( \bar{x}^2 \) is the spatial variable \( \theta \) in the streamwise direction, and \( \bar{v}^k = d\bar{x}^k/dt \) is the velocity tensor. The variable \( \bar{x}^1 \) in this case is the radius \( R \) from the vortex center.

Some numerical solution schemes are limited to two dimensions due, in part, to the computational load and large storage requirements associated with the connection coefficients (Christoffel symbols) inherent in general coordinate systems (Lee & Soni 1997; Wesseling et al. 1998). It is has been known for more than a century that in 2-D (planar) Euler flow, the vorticity, \( \omega \), is conserved along streamlines (Majda & Bertozzi 2002). Equally well known, however, is the fact that such a result has not been established for 3D Euler flows (see, e.g., Majda & Bertozzi 2002; Shariff & Leonard 1992; Friedlander 2002). In fact, research in the
last few decades suggests that in axisymmetric Euler rings, it is the quantity \( \omega/R \) (where \( R \) is the distance from the symmetry axis), not \( \omega \) itself, which seems to persist (Fraenkel 1970, Norbury 1972, Hunt and Eames 2002). It will be shown in §5 that these two phenomena may be unified by stating that all off-trajectory components of the vorticity tensor persist along streamlines, not only in Euler rings and not only in axisymmetric flows, but in any three-dimensional flow for which \( \nu \nabla^2 \omega = 0 \).

The restriction of the vorticity persistence theorem to planar flow obviously limits the assumptions that can be made when attempting to solve for general three-dimensional flows analytically. The purpose of this paper is to present a mathematical simplification of the vorticity equation valid in arbitrary coordinate systems and to extend the vorticity persistence theorem to three dimensions. It will be shown that in three-dimensional Euler flow, as in the two-dimensional case, any component of the vorticity tensor not directed along fluid particle trajectories must remain constant along particle paths. The key to the proof is a mathematical simplification of the nonlinear convective terms in the vorticity equation. It turns out that vortex stretching is closely related to the Christoffel symbols of the streamline coordinate system.

### 2. Simplification of the Vorticity Equation

The steady vorticity equation, obtained by taking the curl of the steady Navier-Stokes equation, can be written in contravariant form for an arbitrary inertial coordinate system, as follows:

\[
v^k \omega^i_{,k} - \omega^i v^i_{,j} = \nu \left( g^{pk} \omega^i_{,k} \right)_{,p},
\]

where \( g^{pk} \) are components of the inverse of the metric tensor of the arbitrary coordinate system, the comma before an index represents covariant differentiation, and body forces are zero. What has limited three-dimensional approaches until now is the fact that in 3D flows, it can happen that \( \omega \cdot \nabla \mathbf{v} \neq 0 \) in (3); in other words, vortex stretching is possible in three-dimensional flows. Therefore, a vast arsenal of methods available for two-dimensional flows is not available in three-dimensions (Majda & Bertozzi 2002). The added complexity of three-dimensional flow can be reduced, however, by taking a more general view of (3). Expanding the left side of (3) gives:

\[
v^1 \left( \frac{\partial \omega^i}{\partial x^1} + \Gamma^i_{11} \omega^1 + \Gamma^i_{12} \omega^2 + \Gamma^i_{13} \omega^3 \right) + v^2 \left( \frac{\partial \omega^i}{\partial x^2} + \Gamma^i_{21} \omega^1 + \Gamma^i_{22} \omega^2 + \Gamma^i_{23} \omega^3 \right) + v^3 \left( \frac{\partial \omega^i}{\partial x^3} + \Gamma^i_{31} \omega^1 + \Gamma^i_{32} \omega^2 + \Gamma^i_{33} \omega^3 \right) - \omega^1 \left( \frac{\partial v^i}{\partial x^1} + \Gamma^i_{11} v^1 + \Gamma^i_{12} v^2 + \Gamma^i_{13} v^3 \right) - \omega^2 \left( \frac{\partial v^i}{\partial x^2} + \Gamma^i_{21} v^1 + \Gamma^i_{22} v^2 + \Gamma^i_{23} v^3 \right) - \omega^3 \left( \frac{\partial v^i}{\partial x^3} + \Gamma^i_{31} v^1 + \Gamma^i_{32} v^2 + \Gamma^i_{33} v^3 \right)
\]

where \( \Gamma^i_{jk} \) are the connection coefficients, or Christoffel symbols of the second kind, given by:

\[
\Gamma^i_{jk} = \frac{g^{pq}}{2} \left( \frac{\partial g_{jq}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^q} - \frac{\partial g_{jk}}{\partial x^q} \right).
\]

Since \( \Gamma^i_{jk} \) are not tensors, they do not vanish in some coordinate systems, such as axisymmetric coordinate systems, even though they vanish in a rectangular system. By the symmetry in the two lower indices of the Christoffel symbols however, 18 terms in (4) cancel each other, so that (3) becomes:
\[ v^k \frac{\partial \omega^i}{\partial x^k} - \omega^k \frac{\partial v^i}{\partial x^k} = \nu \left( g^{pk} \omega^i \right)_p \]  

(5)

for any coordinate system with a symmetric connection. This is, of course, not limited to rectangular, cylindrical, and spherical coordinate systems but also includes non-orthogonal systems as well as arbitrary streamlined coordinate systems. Replacing (3) with (5) can dramatically simplify computations in some numerical solution schemes, particularly when complicated coordinate systems are used. For Euler flows in particular, it may allow the use of more complex three-dimensional coordinate systems without additional computational overhead. Therefore, the statement made previously by the author (Morton 2004) that the vortex stretching term in (3) is zero if \( v^2 \) and \( \omega^3 \) are the only non-zero components of their respective tensors [setting \( i = 2 \) and \( j = 3 \) in (3)] and \( v^2 \) is independent of \( x^3 \), should be qualified as follows: What remains of the vortex stretching term in (3), after cancellation of terms containing Christoffel symbols, is zero if \( v^2 \) and \( \omega^3 \) are the only non-zero components of their respective tensors and \( v^2 \) is independent of \( x^3 \) [set \( i = 2 \) and \( k = 3 \) in (5)].

The mathematical simplification above may also be useful in studies of electricity and magnetism, since the same nonlinear convective term arises there, but with the vorticity field replaced by the magnetic field strength (see, e.g., Hughes & Young 1989, chapter 6).

3. Vorticity Persistence in 3D Flows with Fixed Streamlines in Steady Translation

For steady flow in which the right side of (5) is zero, we have:

\[ v^k \frac{\partial \omega^i}{\partial x^k} - \omega^k \frac{\partial v^i}{\partial x^k} = 0 . \]  

(6)

Since (5) is valid in any inertial coordinate system, it is valid in any streamlined coordinate system in which the streamlines are fixed or translating with constant velocity. The velocity tensor in such a streamlined coordinate system will be denoted with overbars, and the nonzero component will be, say, \( \bar{v}^2 \). Then when \( i = 1 \) or \( 3 \), (5) simplifies to:

\[ \bar{v}^2 \frac{\partial \bar{\omega}^j}{\partial \bar{x}^2} = \nu \left( \bar{g}^{pk} \bar{\omega}^i \right)_p \]  

(7)  \( (i \neq 2) \)

Correspondingly, (6) becomes:

\[ \frac{\partial \bar{\omega}^j}{\partial \bar{x}^2} = 0 . \]  

(8)  \( (i \neq 2) \)

Since the directions \( \bar{x}^1 \) and \( \bar{x}^3 \) were arbitrary, integrating (8) shows that any component of the vorticity tensor not directed along fluid particle trajectories in a flow described by (6), which includes 3D steady Euler flow, must remain constant throughout the fluid particle motion. This result is valid with or without swirl, and whether the flow is axisymmetric or not. When written as in (8), the vorticity equation can be integrated; however, when written as an inviscid form of (3), it cannot, generally, due to the presence of the Christoffel symbols.

The applicability of (6), and consequently (8), is not limited to inviscid flow, since it is satisfied when either \( \nu = 0 \) or \( \nabla^2 \omega = 0 \), examples of the latter case being the viscous core of Hill’s spherical vortex and a stationary elliptical patch of uniform vorticity such as that used by Morton (2007). It can happen that \( \nabla^2 v \neq 0 \) while \( \nabla^2 \omega = 0 \), as in the case of Hill’s spherical vortex. Equation (6) is also not necessarily limited to incompressible flows. Finally, (6) can be valid even when both \( \nu = 0 \) and \( \nabla^2 \omega \neq 0 \). The unsteady Oseen solution, given by (1) and (2), is such a flow. It satisfies (6), which, in the streamlined (cylindrical)
coordinates \( \pi^k = (R, \theta, z) \), simplifies to (8), and also satisfies the remaining terms of the unsteady vorticity equation, namely

\[
\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega.
\]  

(9)

Oseen’s solution satisfies (8) because, as seen from (1), \( \omega^3 \) is not a function of the streamwise direction, \( x^3 \).

The left side of (5) is the Lie derivative, \( L_v \), of the vorticity tensor with respect to the fluid velocity tensor, \( v \). Therefore, combining (6) and (9) gives the following statement:

\[
L_v \omega = 0, \quad \frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega,
\]  

(10)

whose applicability is nearly as broad as the vorticity equation. Describing (10) as being valid for Euler flow can be too restrictive because flows obeying (10) may be viscous, as in the core of Hill’s spherical vortex. Likewise, referring to (10) as valid for “flows with non-diffusive vorticity,” wherein \( \nu \nabla^2 \omega = 0 \), either because \( \nu = 0 \) or \( \nabla^2 \omega = 0 \), is also too restrictive because, as in the case of Oseen’s unsteady vortex, the vorticity may still be diffusive and yet (10) remain satisfied. Therefore, an appropriate label for flows satisfying (10) might be: “flows with streamlines that are fixed in some steadily translating coordinate system.”

The unsteady continuity equation, written for a general coordinate system, can be derived from the Reynolds transport theorem, as follows:

\[
\frac{d}{dt} \int_{\text{fluid particle}} \rho \sqrt{g} \, d\mathcal{V} = \int_{\text{alternate frame}} \left( \frac{\partial}{\partial t} \left( \rho \sqrt{g} \right) + \left( \rho \sqrt{g} \, v^k \right)_k \right) d\mathcal{V} = 0.
\]

Since the integrands must be zero, the middle term above gives the general continuity equation in an arbitrary coordinate frame:

\[
\frac{\partial}{\partial t} \left( \rho \sqrt{g} \right) + L_v \left( \rho \sqrt{g} \right) = 0.
\]  

(11)

For the case of a streamline coordinate system “frozen” in a steadily translating frame, the unsteady term above simplifies to \( \partial \rho/\partial t \).

4. Vorticity Persistence in 3D Flows with General, Time-dependent Streamlines

The unsteady version of (6) is:

\[
\frac{\partial \omega^i}{\partial t} + v^k \frac{\partial \omega^i}{\partial x^k} - \omega^k \frac{\partial v^i}{\partial x^k} = 0.
\]  

(12)

However, (12) is only applicable to an inertial coordinate system, which poses a problem if the coordinate system is to coincide with general unsteady streamlines. In such a coordinate system, the metric tensor will be time-dependent; therefore, the Christoffel symbols will be also. At every instant of time, however, the cancellation in §2 still occurs, since the Christoffel symbols only involve spatial relationships. In a manner analogous to the coordinate-independent definition of differentiation afforded by the covariant derivative, a general definition of temporal differentiation will be constructed so that (12) may be written in terms of non-inertial coordinates. Temporal differentiation of the contravariant first-order
tensor $\omega$ can be expressed in terms of components $\bar{\omega}^k$ of the vorticity tensor referenced to a non-inertial coordinate system, as follows:

$$\frac{d\omega^i}{dt} = \frac{d}{dt} \left( \frac{\partial x^i}{\partial \bar{x}^k} \bar{\omega}^k \right).$$  \hspace{1cm} (13)

Performing the indicated differentiation with respect to time gives:

$$\frac{d\omega^i}{dt} = \frac{d}{dt} \left( \frac{\partial x^i}{\partial \bar{x}^k} \right) \bar{\omega}^k + \frac{\partial x^i}{\partial \bar{x}^k} \frac{d\bar{\omega}^k}{dt}.$$  \hspace{1cm} (14)

Replacing $\bar{\omega}^k$ in the middle term with $\omega^j \frac{\partial \bar{x}^k}{\partial x^j}$ gives:

$$\frac{d\omega^i}{dt} = \frac{d}{dt} \left( \frac{\partial x^i}{\partial \bar{x}^k} \right) \left( \frac{\partial \bar{x}^k}{\partial x^j} \omega^j \right) = \frac{\partial x^i}{\partial \bar{x}^k} \frac{d\omega^k}{dt}.$$  \hspace{1cm} (15)

Therefore, if a temporal differentiation is defined as the left side of (14), that is:

$$\omega^j,_{T} \equiv \frac{d\omega^j}{dt} = \frac{d}{dt} \left( \frac{\partial x^i}{\partial \bar{x}^k} \right) \frac{\partial \bar{x}^k}{\partial x^j} \omega^j,$$  \hspace{1cm} (16)

where the subscript “,$T$” denotes differentiation with respect to time, then (14) may be abbreviated as:

$$\omega^j,_{T} = \frac{\partial x^i}{\partial \bar{x}^k} \frac{d\omega^k}{dt}. \hspace{1cm} (17)$$

Obviously, the middle term of (14) treats the temporal characteristics in the same manner that the Christoffel symbols treat spatial characteristics.

Reversing the order of differentiation in (15) gives:

$$\omega^j,_{T} = \frac{d\omega^j}{dt} - \frac{\partial}{\partial \bar{x}^j} \left( \frac{dx^j}{dt} \right) \frac{\partial \bar{x}^k}{\partial x^j} \omega^j.$$  \hspace{1cm} (18)

The derivative in the parentheses above is $v^i$; therefore, (17) simplifies to

$$\omega^j,_{T} = \frac{d\omega^j}{dt} - \frac{\partial v^j}{\partial x^j} \omega^j.$$  \hspace{1cm} (19)

Therefore, the vorticity equation in (12) may be written simply as

$$\omega^j,_{T} = 0.$$  \hspace{1cm} (20)

Now suppose that the non-inertial (barred) system coincides with an unsteady streamline coordinate system. Writing the definition (18) for this new coordinate system, we have:

$$\bar{\omega}^j,_{T} = \frac{d\bar{\omega}^j}{dt} - \frac{\partial \bar{\omega}^j}{\partial \bar{x}^j} \bar{\omega}^j.$$  \hspace{1cm} (21)

If $\bar{x}^2$ is the streamline direction, we may set $\bar{\omega}^i = 0$ when $i \neq 2$, so that (20), written for directions normal to the velocity vector, becomes:

$$\bar{\omega}^j,_{T} = \frac{d\bar{\omega}^j}{dt}.$$  \hspace{1cm} (22)

and the coordinate transformation in (16) then becomes
\[ \omega^i_{,T} = \frac{\partial x^i}{\partial T} \varpi^j_{,T}, \quad (i \neq 2) \quad (22) \]

This is the transformation required of a contravariant first-order tensor. Provided it is non-singular, the transformation in (22) shows also that, according to (19), the vorticity equation in the unsteady streamline coordinate system may be written

\[ \varpi^i_{,T} = 0 \quad (i \neq 2) \quad (23) \]

for directions normal to streamlines. According to (21), this may also be written:

\[ \frac{d \varpi^i}{dt} = 0 \quad (i \neq 2) \quad (24) \]

Therefore, as in the 2-D Euler case, in any 3-D flow satisfying (12), all components of the vorticity tensor that are orthogonal to the streamline direction are conserved along streamlines. This result is valid in any coordinate system, both inertial and non-inertial, provided its transformation to inertial coordinates is non-singular.

If the temporal derivative defined by (15), or equivalently by (18), is written as:

\[ \frac{\partial}{\partial t} + L, \quad (3) \]

the continuity equation (11) and the vorticity equation (12) may both be written simply as \( (\_)_T = 0 \), with \( \rho \sqrt{g} \) and \( \omega \) as their respective arguments.

5. **A Unifying Statement about Vorticity Persistence**

In axisymmetric flows, a spherical coordinate system is almost as convenient as a streamline coordinate system because the azimuthal variables of the two coincide. Let \( \tilde{x}^k = (r, \theta, \phi) \) represent components of a spherical coordinate system, \( \tilde{\omega}(3) \) the azimuthal component of the physical vorticity in an axisymmetric flow, and \( R = r \sin \theta \) the distance to the symmetry axis. The quantity \( \tilde{\omega}(3)/R \) is one that has been employed in several past studies. For vortex rings of small cross-section, Fraenkel (1970) proved that if the ratio \( \tilde{\omega}(3)/R \) is constant along streamlines, then steady solutions exist. Fraenkel's method of proof relied upon the nearly “two-dimensional nature of the flow in the neighborhood of a small cross-section.” Norbury (1972) remarked that it had long been thought that certain vortex rings of small cross-section are steady if the above ratio is constant throughout their cores. Norbury (1973) assumed this ratio to be constant in obtaining his well-known family of solutions. The ratio \( \tilde{\omega}(3)/R \) has been referred to as a “vorticity constant” (Norbury 1973) as well as a “vorticity density” (Mohseni 2001; Mohseni & Gharib 1998). Hunt and Eames (2002) noted that during axisymmetric vortex stretching in a straining flow, this ratio is conserved.

The ratio referred to above is actually the azimuthal component of the vorticity tensor of the flows referenced in the above studies, since the physical component of vorticity is \( \tilde{\omega}(3) = \sqrt{g_{33}} \tilde{\omega}^3 = (r \sin \theta) \tilde{\omega}^3 = R \tilde{\omega}^3 \). Here, \( \omega^3 \) is the azimuthal component of the vorticity tensor in a spherical coordinate system (and, for the flows mentioned in the above paragraph, a streamline coordinate system as well), and \( r \) is the radial coordinate in a spherical coordinate system. However, the result in (8), and more generally (21), indicates that all off-trajectory components of the vorticity tensor (not only the azimuthal component) are conserved in any flow satisfying \( \nu \nabla^2 \omega = 0 \) (not only in Euler rings of small cross-section and not only in planar or axisymmetric flows).
6. Conclusion

The vorticity equation for arbitrary coordinate systems is simplified by a complete cancellation of all Christoffel symbols from the nonlinear convective terms. Whereas vorticity has been known for more than a century to persist in 2-D Euler flow, in the last few decades researchers have established that for inviscid axisymmetric Euler rings, it is the quantity $\omega / R$, rather than $\omega$, which seems to persist. It was shown herein that a unifying statement is that all off-trajectory components of the vorticity tensor of a streamline coordinate system persist along streamlines if $\nu \nabla^2 \omega = 0$. Therefore, the known persistence of vorticity in two-dimensional Euler flow has been extended to the more general three-dimensional case.

Another consequence of the simplification of the vorticity equation is the finding that what remains of the non-linear convective terms, after the simplification, constitutes the Lie derivative of the vorticity tensor with respect to the fluid velocity.

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