Research Article

The Chebyshev Set Problem in Riesz Space

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1. Introduction

The order structures occurred in a natural manner in many fields, such as spaces of real continuous functions, spaces of measurable functions and so on. In 1928, Riesz [1] first proposed the concept of Riesz space and emphasized that we must consider the order structure, while studying the algebraic and topological properties of normed spaces. Therefore, the research on partially ordered vector space has caused great interest of researchers and developed greatly in recent years. In 1936, Kantorovich [2] established the axiomatic basis of lattice structure in Riesz space and studied some properties of lattice structure in Riesz space. In the mid-1930s, Birkhoff’s work [3–7] started the general development of lattice theory. In a series of wonderful articles [8–11], they presented the importance of lattice theory which can provide a unified framework for the hitherto unrelated development of many mathematical disciplines. In particular, numerous scholars combined functional analysis with lattice theory to transform specific analysis problems into pure algebra, topology, and order structure problems, which made the application of concepts, theorems, and methods more extensive and in-depth. For example, Schaefer [12] studied the basic theories of vector lattices and Banach lattices, which laid the foundation for operator theory. After that, Aliprantis and Burkinshaw [13] had further exploration and promotion of operator theory in Riesz space. Luxemburg [14] pointed out that the order structure was another crucial structure in real analysis in addition to the basic arithmetic structure which allowed the comparison of the quantities represented by inequalities and studies many other characteristic properties. Luxemburg and Zaanen [15] and Zaanen [16] systematically summarized some basic theories of Riesz Space, making the research of Riesz Space more systematic and efficient. Many analytical properties such as various kinds of convergences in terms of the partial ordering were studied in [17–21].

The idea of approximation has always been an important idea in various fields of mathematics. Based on algebraic structures, the classical approximation theory studies the approximation characteristics of various function classes in normed linear space [21–23]. More and more scholars also try to combine the order structures with the approximation theory and obtain some approximation characteristics in Banach Lattices. In 1992, Kurc [24] discussed some monotonic properties of Luxemburg norm in Musielak-Orlicz space and gave some applications to the best approximation. Then, Hudzik and Kurc [25] and Chen et al. [26] continued Kurc’s work and discussed the more general best approximation problems in Banach Lattices by using some monotonicities. Therefore, the purpose of this paper is to continue the above research by combining the order structure with the best approximation problems to study the best approximation theory in Riesz space.

This paper is organized as follows. In Section 2, we recall some notations of the best approximation in normed linear...
spaces and Riesz space. In Section 3, we consider the problem of the best approximation in Riesz space which is the main content and innovation of this paper. In Section 4, we further study the order best approximation projection and some related properties such as the boundedness and continuity.

2. Preliminary

First, we introduce the definition of the classical best approximation in normed linear spaces.

**Definition 1.** See [27]. Let \( E \) be a normed linear space and \( S \) be a nonempty subset of \( E \). Given \( f \in E \), the best approximation of \( S \) to \( f \) is defined as

\[
e(f, S) = \inf_{u \in S} \| f - u \|.
\]

The definition of the classical best approximation in normed linear spaces is constructed by the norm. In this paper, we would study the best approximation theory which is not constructed by the norm, but only rely on the order structure. We consider our problem by recalling some definitions in Riesz space. For details of Riesz Space, refer to [28].

**Definition 2.** See [28]. The relation “\( \leq \)” in \( E \) is called a partial ordering if

1. \( a \leq a \) for every \( a \in E \)
2. \( \forall a, b, c \in E, a \leq b \) and \( b \leq c \) imply \( a \leq c \)
3. \( \forall a, b \in E, a \leq b, b \leq a \) imply \( a = b \)

The set \( E \) with a partial ordering is called a partially ordered set, denoted as \( (E, \leq) \). With out of generality, we denote \( (E, \leq) \) as \( E \).

Suppose that \( E \) is a partially ordered set, and \( Y \) is a nonempty subset of \( E \). If \( x \) and \( y \) are points in \( E \) such that \( x \leq y \) or \( y \leq x \), we say that \( x \) and \( y \) are comparable. The point \( x_0 \in E \) satisfies \( y \leq x_0 \) for all \( y \in Y \), and then \( x_0 \) is called an upper bound of \( Y \). The subset \( Y \) is now said to be bounded above. If \( x_0 \) is an upper bound of \( Y \) such that \( x_0 \leq x' \) for any other upper bound \( x' \) of \( Y \), then \( x_0 \) is called a supremum of \( Y \), denoted by sup \( Y \). Similarly, the definitions of lower bound, bounded below, and infimum are analogous, denoted the infimum of \( Y \) as inf \( Y \).

**Definition 3.** See [28]. Let \( E \) be a linear space over a field \( F \) (\( R \) or \( C \)) with a partially order “\( \leq \)” Then, \( E \) is called the Riesz space if for all \( f, g, h \in E \) and \( a \in \mathbb{R}^+ \),

\[
f \leq g \Rightarrow f + h \leq g + h,
\]

\[
\theta \leq f \Rightarrow \theta \leq af.
\]

(1) \( E \) is a lattice
where \( \theta \) is the zero element in \( E \). Unless otherwise specified, \( \theta \) is used to represent the zero element in the Riesz space.

Assume that \( E \) is a Riesz space, and \( f, g \in E \), we denote the following notations as

\[
f^* := f \lor \theta, f^- := (-f) \lor \theta, |f| := f \lor (-f), \text{ and } E^+ := \{ f : \theta \leq f \in E \}.
\]

(3)

If \( |f| \land |g| = \theta \), then \( f \) is called disjoint with \( g \) and denoted as \( f \perp g \). \( D \) is a nonempty subset of \( E \), and then we can define the disjoint complement of \( D \) by

\[
D^d = \{ f \in E : f \perp g, \forall g \in D \}.
\]

**Definition 4.** See [28]. Let \( E \) be a Riesz space, \( a, b \in E \), and \( a \leq b \). Then, the order interval is defined as

\[
[a, b] := \{ f \in E : a \leq f \leq b \}.
\]

**Definition 5.** See [28]. Let \( E \) be a Riesz space. The subset \( S \) is called order bounded if there exist an order interval \( [a, b] \) such that \( x \in [a, b] \), for all \( x \in S \).

**Definition 6.** Let \( E \) be a Riesz space. Then, \( E \) is called a linearly ordered space if for any \( x \) and \( y \) in \( E \), and then \( x \) and \( y \) are comparable.

**Definition 7.** See [28]. Let \( E \) be a Riesz space.

(1) The linear subspace \( V \) of \( E \) is called a Riesz subspace of \( E \) if for all members \( f \) and \( g \) of \( V \), and the elements \( f \lor g \in V \) and \( f \land g \in V \) hold

(2) The Riesz subspace \( S \) of \( E \) is called an ideal in \( E \) if for any \( f \in S \) and \( |g| \leq |f| \), and then \( g \in S \)

(3) The ideal \( B \) in \( E \) is called a band if, whenever a subset of \( B \) possesses a supremum in \( E \), this supremum is a member of \( B \).

**Definition 8.** See [28]. Let \( E \) be a Riesz space, and \( \{ f_n \} \) be a sequence in \( E \). Then,

(1) \( \{ f_n \} \) is said to be increasing (decreasing), if \( f_1 \leq f_2 \leq \cdots \leq f_j \geq f_{j+1} \geq \cdots \). This will be denoted by \( f_n \uparrow (f_n \downarrow) \). If \( f_n \uparrow \) and \( \sup f_n = f \), we say \( f_n \uparrow f \). Similarly, if \( f_n \downarrow \) and \( \inf f_n = f \), we say \( f_n \downarrow f \).
(2) The \( \{ f_n \} \) is said to converge in order to \( f \) if there exists a sequence \( \{ p_n \} \) and \( p_n \{ \theta \} \) such that \( |f - f_n| \leq p_n \) for all \( n \), we call \( f \) the order limit of \( f_n \). We shall denote this by \( f_n \longrightarrow^o f \) or \( \lim_{n \rightarrow \infty} f_n = f \).

(3) A subset \( V \) of \( E \) is said to be an order closed set if for any sequence \( \{ f_n \} \) in \( V \) and \( f_n \longrightarrow^o f \), then \( f \in V \).

Some properties and conclusions of the order convergence have been introduced in the following lemmas, which are crucial to our subsequent research on the order best approximation.

**Lemma 9.** See [28].

1. If \( f_n \longrightarrow^o f \), then \( f = g \).
2. If \( f_n \longrightarrow^o f \), then for any \( \alpha, \beta \in E \), we have \( \alpha f_n + \beta g_n \longrightarrow^o \alpha f + \beta g \). Therefore, \( f_n \wedge g_n \longrightarrow^o f \wedge g \), and \( f_n \longrightarrow^o f \).
3. If \( f_n \longrightarrow^o f \) and \( f_n \geq g \), then \( f \geq g \).
4. If \( f_n \longrightarrow^o f \), then for any subsequence \( \{ f_{n_k} \} \), we have \( f_{n_k} \longrightarrow^o f \).

**3. Order Best Approximation in Riesz Space**

In this section, we consider the problem of the best approximation in Riesz space which is the main content and innovation of this paper. Based on the concept of the best approximation in normed linear spaces and the order structure characteristics in Riesz spaces, we propose the concept of the order best approximation in Riesz spaces and further obtain some related properties.

**Definition 10.** Let \( E \) be a Dedekind complete Riesz space, \( \emptyset \neq S \subset E \), \( f \in E \). The quantity

\[
é(f, S) := \inf_{u \in S} |f - u|
\]

is called the order best approximation of \( f \) by elements in \( S \). If \( e(f, S) = |f - u_0| \) for some \( u_0 \in S \), then \( u_0 \) is said to be the order best approximation element of \( f \) by elements in \( S \). Denoted by \( \mathcal{L}_S(f) \), the order best approximation element set of \( f \), which contains all the order best approximation elements of \( f \) in \( S \).

From the definition of the order best approximation set, the following proposition can be obtained.

**Proposition 11.** If \( S \) is a nonempty subset of Riesz space \( E \), \( x \in E \), and \( z \in \mathcal{L}_S(x) \), then \( z \in \mathcal{L}_S(y) \) for all \( y \in [x \wedge z, x \vee z] \).

**Proof.** We first prove \( |y - z| = |x - z| - |x - y| \), for any \( y \in [x \wedge z, x \vee z] \).

In fact, we have

\[
|y - z| = |x - y| - |x - z| + |x \wedge z - z| - |x \wedge z - y| = |x \wedge z - z| - |x \wedge z - y|.
\]

Then for any \( k \in S \), by Definition 10, there is

\[
|y - z| = |x - z| - |x - y| \leq |x - y| - |y - k| = \inf_{z \in \mathcal{L}_S(y)} |y - k|.
\]

So, \( |y - z| \leq \inf_{\mathcal{L}_S(z)} |y - k| \). Because \( z \in \mathcal{L}_S(x) \), then \( |y - z| = \inf_{\mathcal{L}_S(z)} |y - k| \).

Obviously, according to the definition of the order best approximation set \( \mathcal{L}_S(f) \), there are three cases:

1. \( \mathcal{L}_S(f) = \emptyset \)
2. \( \mathcal{L}_S(f) \neq \emptyset \), and \( |\mathcal{L}_S(f)| = 1 \) (where \( |\mathcal{L}_S(f)| \) is the cardinality of the set \( \mathcal{L}_S(f) \)). Based on above cases of the order best approximation set \( \mathcal{L}_S(f) \), we introduce the following definitions.

**Definition 12.** Let \( S \) be a nonempty subset of Riesz space \( E \).

1. \( S \) is said to be an order proximinal set if \( \mathcal{L}_S(f) \neq \emptyset \) for every \( f \in E \).
2. \( S \) is said to be an order uniqueness set if \( \mathcal{L}_S(f) \neq \emptyset \) or \( |\mathcal{L}_S(f)| = 1 \) for every \( f \in E \).
3. \( S \) is said to be an order Chebyshev set if \( |\mathcal{L}_S(f)| = 1 \) for every \( f \in E \).

In [27], according to the classical definition of the best approximation set, a linear subspace must be an order proximinal set. However, in this paper, a linear subspace in Riesz space is not an order proximinal set, and the following example illustrates this result.

**Example 13.** Let \( X \neq \emptyset \) and \( \mu \) be the nonnegative countable addable measure. \( \mathfrak{F} \) is the finite or countable union of the set \( X \) with finite measure, and \( M(\mathfrak{F}, \mu) \) is the real vector space formed by all the measurable functions defined on \( \mathfrak{F} \), \( \sigma \) is a set with a measure zero on \( \mathfrak{F} \). For all \( f, g \in M(\mathfrak{F}, \mu) \), \( f \approx g \) and \( g \) are not equal only on the set of measure zero. Obviously, "\( \approx \)" is an equivalence relationship. Denote by \( L_0(\mathfrak{F}, \mu) \), the space formed by equivalence classes in \( M(\mathfrak{F}, \mu) \). Define the partial ordering "\( \leq \)" on \( L_0(\mathfrak{F}, \mu) \) with \( |f| \leq |g| \), if \( f(x) \leq g(x) \) for all \( x \in \{ \mathfrak{F} \setminus \sigma \} \). Then, \( L_0(\mathfrak{F}, \mu) \) is Dedekind complete (ref. [28]).

Now, we start to illustrate that the linear space of \( L_0(\mathfrak{F}, \mu) \) is not an order proximinal set. For all \( f \in L_0(\mathfrak{F}, \mu) \), we consider the linear space \( L^o_0(\mathfrak{F}, \mu) = \text{span} \{ f \} \). Define the function \( g \) as follows:

\[
g = \begin{cases} 
3f, & x \in \Omega, \\
f, & x \in \mathfrak{F} \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is a set with a measure of nonzero on \( X \). The order best approximation of \( g \) in linear space \( L^o_0(X, \mu) \) can be
estimated as
\[ 0 \leq c(g, L^1_0(E, \mu)) \leq (g - 3f) \land (g - f) = 0. \tag{10} \]

Here, 0 represents zero transformation. Thus, we have
\[ c(g, L^1_0(E, \mu)) = 0 \Rightarrow |g - kf| = 0 \Rightarrow g = kf \in L^1_0(E, \mu). \tag{11} \]

This contradicts the construction of \( g \). Therefore, the linear space in \( L^1_0(E, \mu) \) is not necessarily an order proximinal set.

In the following parts of this section, we discuss some main properties of the order proximal set, the order uniqueness set, and the order Chebyshev set. We will henceforth assume, unless otherwise stated, that the Riesz space is Dedekind complete.

**Theorem 14.** The order proximal set is an order closed lattice.

**Proof.** Let \( E \) be a Riesz space and \( S \) is an order proximal set in \( E \). First, we prove \( S \) is an order closed set. Otherwise, according to the definition of order closed set, there exists \( \{ f_n \} \subset S \) such that \( f_n \rightarrow^\sigma f \) and \( f \notin S \). Thus, there is a sequence \( \{ p_n \} \) such that \( |f_n - f| \leq p_n \) for all \( n \).

Then,
\[ \inf_n |f_n - f| = \theta \text{ and } e(f, S) \leq \inf_{m \in S} |f - u| \leq \inf_n |f_n - u| = \theta. \tag{12} \]

\[ \Box \]

Hence, \( e(f, S) = \theta \). Since \( S \) is an order proximal set in \( E \), there exists an order best approximation element \( u_0 \in S \) such that \( e(f, S) = |f - u_0| = \theta \), which implies \( f = u_0 \in S \). In this way, contradictions are arise, and we can obtain that \( S \) is an order closed set.

Now, we prove the order proximal set \( S \) is also a lattice. Otherwise, we assume that \( S \) is not a lattice. By the properties of lattice, we can obtain that there are two elements \( x, y \in S \), but their upper or lower bounds do not exist in \( S \). Without losing generality, we might as well assume that \( x \lor y \notin E \), but \( x \land y \in S \). Since \( S \) is an order proximal set, there exists \( u^* \in S \) which is the best approximation element of \( x \lor y \) with respect to the set \( S \). However, for \( x \lor y \in E \), we have
\[ (x \lor y - x) \land (x \lor y - y) = (\theta \lor (y - x)) \land ((x - y) \lor \theta) = (x - y)^+ \land (x - y)^+ = \theta. \tag{13} \]

Thus, we obtain \( e(x \lor y, S) = \inf_{s \in S} |x \lor y - u| = |x \lor y - u^*| = \theta \), which implies that \( x \lor y \in S \). This creates a contradiction. Then the conclusion can be proved.

**Remark 15.** Theorem 14 shows that the order proximal set is an order closed lattice in Riesz space, but the converse is not all true. The following example will show that a set is an order closed lattice but not an order proximal set.

**Example 16.** Let us consider the space \( \mathbb{R}^2 \) with the coordinate order, i.e., \( x = (x_1, x_2), y = (y_1, y_2) \), and \( x \leq y \) if and only if \( x_1 \leq y_1, x_2 \leq y_2 \). It is obvious that \( \mathbb{R}^2 \) is Dedekind complete. Assume that \( S = \{ (x_1, x_2) | x_1 = x_2, 1 \leq x_2 \leq 2 \} \). Obviously, \( S \) is an order closed lattice. However, we find that \( S \) is not an order proximal set. In fact, for \( x_0 = (1, 2) \in \mathbb{R}^2 \), obviously, \( x_0 \notin S \), there exists \( x' = (1, 1), x'' = (2, 2) \in S \), such that
\[ |x_0 - x'| \land |x_0 - x''| = (0, 0). \tag{14} \]

Therefore, \( E(x, S) = (0, 0) \).

So, if \( S \) is an order proximal set, then \( \exists x^* \in S \), such that \( |x_0 - x^*| = (0, 0) \), then \( x_0 = x^* \in S \) which shows a contradiction. Therefore, it can be concluded that the set may not be an order proximal set although it satisfies the properties of order closed lattice.

What kind of order closed lattice must be an order proximal set? We would discuss it from the following two aspects. On the one hand, the order interval must be an order proximal set (Theorem 18); on the other hand, we would consider it in a linearly ordered space (Theorem 20). In order to prove Theorem 18, we recall some lemmas.

**Lemma 17.** See [28] (minimal decomposition theorem).

Let \( E \) be a Riesz space. If \( f = u - v \) with \( u \) and \( v \) in \( E^+ \), then there exists \( f^+ \leq u \) and \( f^- \leq v \), such that \( f = f^+ - f^- \).

**Theorem 18.** Any order closed interval in Riesz space is an order proximal set.

**Proof.** Let \( E \) be a Dedekind complete Riesz space and \( S = [a, b] \) be an order closed interval of \( E \). Then for each \( x \in E \), the set \( \{ x - g | g \in S \} \) is equivalent to the order interval \( M = [x - b, x - a] \). Thus, we obtain \( \inf_{g \in S} |x - g| = \inf_{m \in M} |m| \). For any \( m \in M \), we have
\[ (x - b)^+ \leq m^+ \leq (x - a)^+, (x - a)^- \leq m^- \leq (x - b)^-. \tag{15} \]

Taking \( u = (x - b)^+ - (x - a)^- \), it is truly that \( u \in [x - b, x - a] \). From Lemma 17, there is a minimum positive decomposition
\[ u^+ \leq (x - b)^+, u^- \leq (x - a)^-, \tag{16} \]

such that \( u = u^+ - u^- \) and \( u^+ \land u^- = \theta \). Thus,
\[ |u| \leq (x - b)^+ + (x - a)^- \leq m^+ + m^- = |m|, \tag{17} \]

which shows that \( |u| \) is a lower bound of the set \( \{ |m| | \forall m \in M \} \). Taking into account \( u \in [x - b, x - a] \), we obtain \( |u| = \inf_{m \in M} |m| \). So, for each \( x \in E \), there exists \( u \in [x - b, x - a] \) such that
\[ \inf_{g \in S} |x - g| = \inf_{m \in M} |m| = |u|. \tag{18} \]

And for \( u \in [x - b, x - a] \), there exists a \( g_0 \in S \) such that
\[ u = x - g_0. \] Then,
\[
\inf_{g \in S} |x - g| = |x - g_0|, \tag{19}
\]
which shows \( S \) is an order proximinal set by definition.

In real number space, we have the Bolzano-Weierstrass theorem. In linearly ordered spaces, we find that there is a similar conclusion as follows.

**Lemma 19.** For every order bounded sequence in linearly ordered spaces, there exists an order convergent subsequence.

**Proof.** Assuming that \( \{x_k\} \) is an order bounded sequence in the strictly ordered space \( E \), then there exists an order interval \( [a, b] \) such that \( \{x_k\} \subseteq [a, b] \). Taking \( c = 1/2(a + b) \in [a, b] \), we can find that all points of \( \{x_k\} \) are included in the order interval \([a, c]\) and \([c, b]\). Because any two elements in \( E \) can be compared, there is at least one interval of \([a, c]\) and \([c, b]\) which contains infinite points of \( \{x_k\} \) and denoted as \([a_1, b_1]\). Obviously, \( b_1 - a_1 = 1/2(b - a) \) holds. Continuing the above process, we can get an ordered interval sequence \( \{a_n, b_n\} \) satisfying
\[
[a, b] > [a_1, b_1] > [a_2, b_2] > \cdots > [a_n, b_n] > \cdots,
\]
\[
b_n - a_n = \frac{1}{2^n} (b - a). \tag{20}
\]

(1) \( [a_n, b_n] \) contains infinite number of points of \( \{x_k\} \)

Since \( E \) is Dedekind complete, we have \( a_n \leq b_n \), \( n = 1, 2, \ldots \), which means \( \{a_n\} \) has a supremum denoted as \( \xi \). Choosing a subsequence \( \{k_n\} \) of \( \{k\} \) and \( x_{k_n} \in [a_n, b_n] \), then
\[
|x_{k_n} - \xi| \leq b_n - a_n = \frac{1}{2^n} (b - a). \tag{21}
\]

With the Dedekind completeness of \( E \), we obtain \( 1/2^n \longrightarrow 0 \). Therefore, \( \{x_{k_n}\} \) is order convergent to \( \xi \).

**Theorem 20.** Every order closed set in a linearly ordered space is an order proximinal set.

**Proof.** Let \( E \) be a linearly ordered space and \( S \) be an order closed set in \( E \). For all \( x \in E \), there exists a sequence \( \{p_n\} \subseteq E \), and \( p_n \downarrow \theta \), such that \( e(x, S) \leq e(x, S) + p_n \). Since \( e(x, S) = \inf_{u \in S} |x - u| \), there exists a sequence \( \{u_n\} \subseteq S \) such that \( |x - u_n| \leq e(x, S) + p_n \). In fact, for all \( u \in S \) and \( e(x, S) + p_n \leq |x - u| \) holds. Then, \( e(x, S) + p_n \leq \inf_{u \in S} |x - u| \), which shows the contradiction. Hence, we have
\[
|x - u_n| \leq e(x, S) + p_n \Rightarrow |u_n| \leq |x| + e(x, S) + p_n \leq |x|e(x, S) + p_n \forall n \in \mathbb{N}. \tag{22}
\]

Then, \( \{u_n\} \) is an order bounded sequence. According to Lemma 9 (4), there exists an order convergent subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and denotes the order limit by \( \xi \). Since the sequence \( \{x - u_{n_k}\} \) converges in order to \( e(x, S) \), we obtain that the subsequence \( \{x - u_{n_k}\} \) also order converges to \( e(x, S) \) by Lemma 19. Thus, by Lemma 9 (44), we have
\[
u_{n_k} \xrightarrow{o} \xi \Rightarrow x - u_{n_k} \xrightarrow{o} x - \xi \Rightarrow |x - u_{n_k}| \xrightarrow{o} |x - \xi|. \tag{23}
\]

Hence, \( e(x, S) \leq |x - \xi| = \lim_{n \rightarrow \infty} |x - u_{n_k}| = e(x, S) \) holds. Therefore, we obtain \( e(x, S) = |x - \xi| \), and it follows that \( \xi \in \mathcal{L}_r(x) \).

With the arbitrariness of \( x \), we could easily conclude that \( S \) is an order proximinal set.

In this position, we discuss the properties of the order uniqueness sets.

**Lemma 21.** See [28]. If \( f \perp g \), then it has the following properties:

1. \( f \perp g \), for any \( a \in \mathbb{R} \), we have \( af \perp g \)
2. \( f \perp g \), and then \( |f + g| = |f - g| = |f| + |g| = ||f| - |g|| \)

**Theorem 22.** A convex set in Riesz space is an order uniqueness set.

**Proof.** Let \( E \) be a Dedekind complete Riesz space, and \( S \) be a convex set in \( E \). Assume that the conclusion is not true, then there exists a \( f \in E \), such that \( f_1, f_2 \in \mathcal{L}_r(f) \), \( f_1 \neq f_2 \), i.e., \( e(f, S) = |f - f_1| = |f - f_2| \). Since \( S \) is a convex set in \( E \), then \( f_1 + f_2 / 2 \in S \). From the definition of the order best approximation, we obtain
\[
|f - f_1| \leq \left| f - \frac{f_1 + f_2}{2} \right| \leq \left| f - f_1 \right| + \left| f - f_2 \right| = e(f, S). \tag{24}
\]

Then, \( |f - f_1 + f_2/2| = e(f, S) \). Therefore, we have
\[
|2f - f_1 - f_2 / 2| = ||f - f_1| - |f - f_2|| = \theta. \tag{25}
\]

So, \( 2|f - f_1| \leq |f - f_2 - f - f_1| = ||f - f_1| - |f - f_2|| = \theta \).

Therefore, we obtain \( |f_2 - f_1| = \theta \Rightarrow f_1 = f_2 \), and the conclusion is proved.

**Remark 23.** Theorem 22 also shows that for any point of Riesz space, if there exists an order best approximation element in some convex set, then the order best approximation element must be unique.

Considering that the order interval and the linear subspace of Riesz space are convex sets, we can obtain some corollaries.
Corollary 24. Every order interval of Riesz space is an order uniqueness set.

Corollary 25. Every linear subspace of Riesz space is an order uniqueness set.

Theorem 26. If $S$ is an order uniqueness set and the elements of $S$ are disjoint with each other, then $S$ must be a single point set.

Proof. Let $E$ be a Riesz space and $S$ be an order uniqueness set with the elements being disjoint with each other in $E$. Assume that $S$ is not a single point set, then there exists $a, b \in S$ and $a \neq b$, such that $a \bot b$, and at least one of $a, b$ is not equal to $\theta$. Let us set $a \neq \theta$, and we have $a + b/2 \notin S$ (in fact, if $a + b/2 \in S$, then by Lemma 21, there is $|a + b/2| = (|a|/2 + |b|/2)|a| = |a|/2 \neq \theta$, i.e., $a + b/2$ and $a$ are not disjoint). Thus, we could discuss it from the following two cases.

(1) If $S = \{a, b\}$, then $\mathcal{L}_S(a + b/2) = \{a, b\}$ is not unique

(2) If there exists $c \in S$ such that $c \bot a, c \bot b$, then

\[
\left| c - \frac{a + b}{2} \right| = |c| + \frac{|a|}{2} + \frac{|b|}{2} \geq \frac{|a|}{2} + \frac{|b|}{2}.
\]  

(27)

Meanwhile, since $|a - a + b/2| = |b - a + b/2| = |a| + |b|/2$, we have

\[
e\left(\frac{a + b}{2}, S\right) = \frac{|a| + |b|}{2}.
\]

(28)

Therefore, $S$ cannot be an order uniqueness set, which contradicts the suppose of the theorem. \hfill \Box

Lemma 27. See [28]. Let $E$ be a Dedekind complete Riesz space. Then, $E = B \oplus B'$ holds for every band $B$ in $E$, i.e., for any $u \in E$, there exists a unique decomposition.

\[ u = u_1 + u_2, \quad \text{where} \quad u_1 \in B, u_2 \in B'. \]

Theorem 28. The band in Riesz space is order Chebyhsev set.

Proof. Let $E$ be a Riesz space and $B$ be a band in $E$. For any $x \in E$, considering the Lemma 27, we have $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B'$. Then,

\[
\inf_{g \in B'} |x - g| = \inf_{g \in B'} |x_1 + x_2 - g| = \inf_{g \in B'} |x_1 - g + x_2|.
\]

(29)

holds. Since $x_1 - g \in B$, we have $(x_1 - g) \bot x_2$. So,

\[
\inf_{g \in B'} |x_1 - g + x_2| = \inf_{g \in B'} (|x_1 - g| + |x_2|) \geq |x_2|.
\]

(30)

Therefore, there exists $g_0 = x_1 \in B$, such that

\[
e(x, B) = \inf_{g \in B} |x - g| = |x - x_1| = |x_2|.
\]

(31)

Hence, $x_1 \in \mathcal{L}_B(x)$.

Because of the uniqueness of the order direct sum decomposition of $E$, the conclusion is proved.

From Theorem 18 and Corollary 24, we can further obtain the following corollary.

Corollary 29. Every order closed interval in Riesz space is an order Chebyhsev set.

Theorem 30. The order Chebyhsev set in linearly ordered space is a convex set.

Proof. Let $E$ be the linearly ordered space, and $B$ be an order Chebyhsev set in $E$. According to Theorem 14, the order proximinal set must be an order closed lattice. Therefore, we can infer that $S$ is a convex set. In fact, if $S$ is a nonconvex set, then there exists $x, y \in S$, and $a_0 \in (0, 1)$, such that $a_0 x + (1 - a_0) y \notin S$. Thus, there exists $m \in S$ by the definition of the order Chebyhsev set, such that

\[
e(a_0 x + (1 - a_0) y, S) = |m - a_0 x + (1 - a_0) y| := q.
\]

(32)

$E$ is a linearly ordered space; so, $m$ and $a_0 x + (1 - a_0) y$ are comparable. Without loss of generality, we assume that $m \leq a_0 x + (1 - a_0) y$. Taking $m_1 = a_0 x + (1 - a_0) y + q$, by the definition of order Chebyhsev set, then $m \neq m_1$ (if $m \geq a_0 x + (1 - a_0) y$, take $m_1 = a_0 x + (1 - a_0) y - q$). Therefore, the order best approximation element of $m_1$ in $S$ exists (notation $m_1$), and then we have $\{m, m_1\} \cap S = \emptyset$. Taking $m_2 = m + m_1/2$, we get

\[
e(m_2, S) = |m - m_2| = |m_1 - m_2|.
\]

(33)

There is a contradiction with the definition of the order Chebysev set; so, the conclusion is proved. \hfill \Box

4. Order Best Approximation Projection

In this section, we would further study the properties of the order best approximation from the perspective of operator by constructing a projection operator from the elements to the order proximinal set and study the related properties such as the local order boundedness and order continuity.

Definition 31. Let $E$ be a Riesz space and $S$ be an order proximinal set of $E$. Define the mapping $\mathcal{L}_S : x \longrightarrow \mathcal{L}_S(x)$ as

\[
\mathcal{L}_S(x) := \{y \in S : |x - y| = e(x, S)\},
\]

(34)

and then $\mathcal{L}_S$ is called the order best approximation projection.

Definition 32. Let $X$ and $Y$ be two Riesz spaces and $f$ be a mapping from $X$ to $Y$ (the mapping here could be...
multivalued). Then, \( f \) is said to be locally order bounded, and if for all \( x_0 \in X \), there exists \( p_0 \in X \) and \( M \in \mathbb{Y}^* \), such that \( y = f(x) \in [-M, M] \) for all \( x \in [x_0 - p_0, x_0 + p_0] \), i.e., \( f([x_0 - p_0, x_0 + p_0]) \) is order bounded in \( Y \).

**Definition 33.** Let \( X \) and \( Y \) be two Riesz spaces and \( f \) be the mapping from \( X \) to \( Y \). Given \( x \in X \), then \( f \) is said to be order continuous at \( x \), if \( f \) is single-valued, and for all \( \{x_n\} \) of \( X \), \( x_n \xrightarrow{\text{ol}} x \), then \( f(x_n) \xrightarrow{\text{ol}} f(x) \). If \( f \) is order continuous at every point of \( X \), then \( f \) is said to be order continuous at \( X \).

The main purpose of this section is to discuss the properties of the order best approximation projection from two aspects: local order boundedness and order continuity. In order to discuss local order boundedness of the order best approximation projection, we introduce the following lemma.

**Lemma 34.** Let \( E \) be a Riesz space and \( S \) be a nonempty subset of \( E \); then for any \( x, y \in E \), we have

\[
|e(x, S) - e(y, S)| \leq |x - y|. \tag{35}
\]

**Proof.** For any \( x, y \in E \), \( z \in S \), we have

\[
e(x, S) \leq |x - z| \leq |x - y| + |y - z| \Rightarrow e(x, S) - |x - y| \leq |y - z|. \tag{36}
\]

Hence, from the definition of \( e(y, S) \),

\[
e(x, S) - e(y, S) \leq |x - y|. \tag{37}
\]

Similarly, \( e(y, S) - e(x, S) \leq |x - y| \). Therefore, \( |e(x, S) - e(y, S)| \leq |x - y| \).

**Theorem 35.** Let \( E \) be a Riesz space and \( S \) be an order proximinal set in \( E \), and then the order best approximation projection \( L^*_S : x \mapsto L^*_S(x) \) is locally order bounded.

**Proof.** Taking \( x_0 \in E \), \( p_0 \in E^* \). For all \( x \in [x_0 - p_0, x_0 + p_0] \), \( y \in L^*_S(x) \), we have \( y = f(x) \in [y_0 + |y - x_0| + |x - x_0|, y_0 + |y - x_0| + |x - x_0| + |x_0|] \). From Lemma 34, we obtain \( |y - y_0| \leq e(x_0, S) + 2|x - x_0| + |x_0| \). Hence, \( L^*_S([x_0 - p_0, x_0 + p_0]) \) is order bounded.

**Remark 36.** As previously studied, if \( S \) is an order proximinal set in \( E \), then the order best approximation element set \( L^*_S(x) \) can be multi-point set. Therefore, we would discuss the continuity of order best approximation projection defined on the set that is the order Chebyshev set.

**Proposition 37.** Let \( E \) be a Riesz space, \( S \) be an order proximinal set in \( E \) and \( \{x_n\}, \{y_n\} \) be two sequences of \( E \). If \( y_n \in L^*_S(x_n) \) for all \( n \in \mathbb{N} \), \( x_n \xrightarrow{\text{ol}} x \) and \( y_n \xrightarrow{\text{ol}} y \), then \( y \in L^*_S(x) \).

**Proof.** From Theorem 14, the order proximinal set \( S \) is order closed in \( E \). So, for the sequence \( \{y_n\} \subset S \), \( y_n \xrightarrow{\text{ol}} y \Rightarrow y \in S \), according to Lemma 34, we have

\[
||x_n - y_n| - e(x, S)| = |e(x_n, S) - e(x, S)| \leq |x_n - x|. \tag{38}
\]

Taking the limits of both sides of above inequality, we obtain

\[
||x - y| - e(x, S)| \leq \theta. \tag{39}
\]

Thus, \( |x - y| = e(x, S) \Rightarrow y \in L^*_S(x) \).

**Theorem 38.** Let \( E \) be a Riesz space and \( S \) be a convex order Chebyshev set in \( E \), and then the order best approximation projection \( L^*_S : x \mapsto L^*_S(x) \) is order continuous.

**Proof.** For \( \forall x \in E \), there exists a sequence \( \{x_n\} \) of \( E \) converges in order to \( x \), i.e., there exists \( \{p_n\} \) and \( p \) such that \( |x_n - x| \leq \theta \). By the definition of order Chebyshev set, there is a unique \( y_n \in L^*_S(x_n) \) for all \( n \), and \( y \in L^*_S(x) \). Therefore,

\[
|y_n - y| - |x_n - y| \leq |x_n - y| - |x_n - x| \leq p_n,
\]

\[
|x_n - y_n| - |x_n - y| \leq |x_n - y| - |x_n - x| \leq |x_n - x| \leq p_n.
\]

So,

\[
\lim_{n \to \infty} |y_n - x_n| = \lim_{n \to \infty} |y - x_n| = |y - x|. \tag{41}
\]

By the definition of the order best approximation \( |y_n - x_n| \),

\[
|y_n - x_n| |y - x_n| - |y - x_n| \]

\[
\leq |(y_n - x_n) - (y_n - x_n)| \]

\[
\leq |y - x_n| - |y_n - x_n| \]

holds. And we know that \( S \) is convex; so, \( |y_n - x_n| \leq |y_n + y| / 2 - x_n| \). Hence,

\[
\lim_{n \to \infty} |y_n - x_n| \leq \lim_{n \to \infty} \frac{|y_n + y - x_n|}{2} \]

\[
\leq \lim_{n \to \infty} \left( \frac{|y_n - x_n|}{2} + \frac{|y - x_n|}{2} \right) = |y - x|. \tag{43}
\]

Therefore,

\[
\lim_{n \to \infty} \left| \frac{y_n + y}{2} - x_n \right| - |y_n - x_n| = \theta. \tag{44}
\]

This means that there is \( q_n \) and \( q_n / \theta \) such that

\[
\left| \frac{y_n + y}{2} - x_n \right| - |y_n - x_n| \leq q_n. \tag{45}
\]
So,
\[
|y_n - y| = |y_n - x_n + x_n - y| = |y_n - x_n - (y - x_n)| \\
= 2(|y_n - x_n||y-x_n|) - |y_n + y - 2x_n| \\
= 2(|y_n - x_n||y-x_n|) - |y_n + y - 2x_n| \\
\leq 2\left[\left(|y-x_n| + |y_n - x_n|\right) + |y_n - x_n| - \frac{|y_n + y - 2x_n|}{2}\right].
\]
(46)

By the equations (41), (44), and definition of order convergence, there is \(|y_n - y| \leq 2(p_n + q_n)\). From the Lemma 9, we can conclude that \(\{y_n\}\) order converges to \(y\). So, \(\mathcal{L}_S : x \rightarrow \mathcal{L}_S(x)\) is continuous.

Data Availability

This paper belongs to basic theoretical research without any data. If you need a detailed proof of theorems, you can ask the corresponding author for them.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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