INFLUENCE OF RANDOM FLUCTUATIONS IN THE \(\Lambda\)-EFFECT ON MERIDIONAL FLOW AND DIFFERENTIAL ROTATION

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ABSTRACT

We present a mean field model based on the approach taken by Rempel in order to investigate the influence of stochastic fluctuations in the Reynolds stresses on meridional flow and differential rotation. The stochastic fluctuations found in the meridional flow pattern directly resemble the stochastic fluctuations of the Reynolds stresses, while the stochastic fluctuations in the differential rotation are smaller by almost 2 orders of magnitude. It is further found that the correlation length and timescale of the stochastic fluctuations have only a weak influence on meridional flow, but a significant influence on the magnitude of variations in the differential rotation. We analyze the energy fluxes within the model to estimate timescales for the replenishment of differential rotation and meridional flow. We find that the timescale for the replenishment of differential rotation (~10 years) is nearly 4 orders of magnitude longer than the timescale for the replenishment of meridional flow, which explains the differences in the response to stochastic fluctuations of the Reynolds stress found for both flow fields.

Subject headings: Sun: interior — Sun: rotation

1. INTRODUCTION

Recently, Rempel (2005) presented a mean field model for solar differential rotation and meridional flow, assuming a parameterization of all convective scale processes, most importantly the turbulent angular momentum transport (\(\Lambda\)-effect) introduced by Kitchatinov & Rüdiger (1993). The model presented by Rempel (2005), as well as other mean field models for differential rotation (see, e.g., Kitchatinov & Rüdiger 1995; Rüdiger et al. 1998; Küker & Stix 2001), uses a time-independent parameterization of the turbulent angular momentum transport leading to stationary solutions describing the mean flows. On the other hand, three-dimensional numerical simulations show a significant temporal variation of meridional flow and differential rotation due to the fluctuations in the convective motions leading to the Reynolds stresses (Miesch et al. 2000; Brun & Toomre 2002).

Observations of differential rotation are usually based on 3–4 month sets of data and therefore can only tell us something about variations on timescales longer than this period (Schou et al. 2002). The 1 \(\sigma\) error intervals of inversions within the convection zone are typically around a few nHz or less than 1% of the rotation rate. The differential rotation also shows a systematic cycle variation, the torsional oscillations, which have an amplitude of around 1% of the rotation rate. By contrast, the variability found in observations of the meridional surface flows are much larger compared to the average amplitude. Over timescales of years Zhao & Kosovichev (2004) found variations of the meridional surface flow of the order of 10 ms\(^{-1}\) compared to a mean flow amplitude of 20 ms\(^{-1}\). While these variations are most probably also related to the solar cycle, much less is known about short-term variations, which could be related to the turbulent origin of the flow itself. Comparison of the long-term variations found in differential rotation and meridional flow indicates that the relative amplitude of the meridional flow variability is almost a factor of 100 larger than the variability of the differential rotation.

In this paper we present a mean field model for differential rotation and meridional flow following the approach described in Rempel (2005) to investigate the influence of stochastic fluctuations in the Reynolds stresses on meridional flow and differential rotation. The main intention of this paper is to investigate how much variability can be expected in the meridional flow pattern, given the strong constraints set by helioseismology on the variability of the 3–4 month mean of the differential rotation. The possible influence of the solar cycle on both flow patterns will be addressed later in a separate paper.

2. MODEL

For this investigation we use the model described in detail in Rempel (2005). We add random noise with defined correlation length and timescales to the parameterization of the turbulent Reynolds stress responsible for driving the differential rotation and meridional flow (see Rempel [2005], eqs. [31] and [32]):

\[
\Lambda_{r\phi} = \Lambda_{\phi r} = +L(r, \theta) \cos(\theta + \lambda(r, \theta)) \left[ 1 + \frac{c \zeta_r(r, \theta)}{\sigma_r} \right],
\]

\[
\Lambda_{\theta r} = \Lambda_{r\theta} = -L(r, \theta) \sin(\theta + \lambda(r, \theta)) \left[ 1 + \frac{c \zeta_\theta(r, \theta)}{\sigma_\theta} \right].
\]

\(L(r, \theta)\) denotes the mean amplitude of the turbulent angular momentum flux, whereas \(\lambda(r, \theta)\) describes the mean inclination of the flux vector with respect to the axis of rotation. \(L(r, \theta)\) and \(\lambda(r, \theta)\) need to be antisymmetric across the equator to fulfill the symmetry constraints of the \(\Lambda\)-effect. For further details, see Rempel (2005).

The quantities \(\zeta_r\) and \(\zeta_\theta\) denote random functions, which are described below in more detail. The parameter \(c\) determines the amplitude of the random noise in units of the standard deviations \(\sigma_r\) and \(\sigma_\theta\). By perturbing both components of the turbulent angular momentum flux with uncorrelated random functions, we allow for a change of amplitude and direction of the angular momentum flux.

In order to generate a random function with a defined correlation length scale, we construct a two-dimensional random field in
the \( r-\theta \) plane by superposing Gaussian functions with a fixed width that determines the length scale. The position in the \( r-\theta \) plane of individual Gaussians, as well as the amplitudes, are random. We introduce a correlation timescale into the problem by performing a temporal average over the two-dimensional random fields. In principle this averaging could be done by creating a long time series of a two-dimensional random field and using then a running mean. However, this approach would require a fairly large amount of memory. Instead we produce a two-dimensional random field \( g \) each time step and solve the following equation to introduce a correlation timescale:

\[
\frac{\partial \zeta}{\partial t} = -\frac{\zeta}{\tau_c} + \frac{g}{\tau_c},
\]

where \( \zeta \) denotes the random field with a correlation timescale \( \tau_c \). The main disadvantage of this approach is that the temporal average is always dominated by the most recent realizations of \( g \). This effect can be reduced by applying this method recursively as follows:

\[
\frac{\partial \zeta_1}{\partial t} = -\frac{\zeta_1}{\tau_c/2} + \frac{\zeta_2}{\tau_c/2},
\]

\[
\frac{\partial \zeta_2}{\partial t} = -\frac{\zeta_2}{\tau_c/2} + \frac{\zeta_3}{\tau_c/2},
\]

\[
\vdots
\]

\[
\frac{\partial \zeta_n}{\partial t} = -\frac{\zeta_n}{\tau_c/2} + \frac{\zeta_{n+1}}{\tau_c/2},
\]

where \( \zeta_1 \) is the random field used for the simulation. The functions \( \zeta_2 \) to \( \zeta_n \) are auxiliary functions required for the averaging process. The results presented in this paper are obtained using a value of \( n = 5 \) for the temporal averaging. The free parameters in this approach for generating a two-dimensional random field are the correlation length scale \( \Delta \) and the correlation timescale \( \tau_c \). The quantity \( \Delta \) measures the half-width of the Gaussians at half-maximum in units of the domain size in the radial and latitudinal direction (therefore the radial and latitudinal widths are not the same). For the results presented here we use an amplitude (defined by the parameter \( c \) in eqs. [1] and [2]) of 0.33, meaning that the stochastic fluctuation imposed on the Reynolds stress has a 1 \( \sigma \) fluctuation corresponding to 33\% of the mean.

In order to allow for a detailed statistical analysis of meridional flow and differential rotation, we compute during the simulation averages of all variables and the square of all variables for each time step and solve the following equation to introduce a correlation timescale:

\[
\frac{\partial \zeta}{\partial t} = -\frac{\zeta}{\tau_c} + \frac{g}{\tau_c},
\]

where \( \zeta \) denotes the random field with a correlation timescale \( \tau_c \). The main disadvantage of this approach is that the temporal average is always dominated by the most recent realizations of \( g \). This effect can be reduced by applying this method recursively as follows:

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\]

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\frac{\partial \zeta_2}{\partial t} = -\frac{\zeta_2}{\tau_c/2} + \frac{\zeta_3}{\tau_c/2},
\]

\[
\vdots
\]

\[
\frac{\partial \zeta_n}{\partial t} = -\frac{\zeta_n}{\tau_c/2} + \frac{\zeta_{n+1}}{\tau_c/2},
\]

where \( \zeta_1 \) is the random field used for the simulation. The functions \( \zeta_2 \) to \( \zeta_n \) are auxiliary functions required for the averaging process. The results presented in this paper are obtained using a value of \( n = 5 \) for the temporal averaging. The free parameters in this approach for generating a two-dimensional random field are the correlation length scale \( \Delta \) and the correlation timescale \( \tau_c \). The quantity \( \Delta \) measures the half-width of the Gaussians at half-maximum in units of the domain size in the radial and latitudinal direction (therefore the radial and latitudinal widths are not the same). For the results presented here we use an amplitude (defined by the parameter \( c \) in eqs. [1] and [2]) of 0.33, meaning that the stochastic fluctuation imposed on the Reynolds stress has a 1 \( \sigma \) fluctuation corresponding to 33\% of the mean.

In order to allow for a detailed statistical analysis of meridional flow and differential rotation, we compute during the simulation averages of all variables and the square of all variables over the output sampling period, which allows the computing of the mean flows and the standard deviations including all simulation time steps and therefore avoiding the influence of the output sampling.

3. RESULTS

For evaluating the effect of random noise in the turbulent angular momentum transport we use the model described in case 1 of Rempel (2005) as reference. Table 1 summarizes the parameters we use for the generation of the random noise fields used to randomize the Reynolds stress according to equations (1) and (2). All other model parameters are given in Rempel (2005). Cases 1–4 cover a range in the correlation timescale of a factor of 64 (from roughly one day to two months), while the series of cases 5, 3, and 6 encompasses a variation in the correlation length scale of a factor of 4 (from a radial half-width of about 10 to 40 Mm).

In the following discussion we focus on the variance of the meridional flow \( \sigma_v \) and the variance of the differential rotation \( \sigma_{\Omega_\theta} \). Since helioseismic measurements of differential rotation typically involve 4 month averages, we also compute the variance of 4 month averages of our model output.

Figure 1 shows \( \sigma_v \) (left column) and \( \sigma_{\Omega_\theta} \) (right column) computed at the top of the domain \( (r = 0.985 \, R_\odot) \) for models 1–6 listed in Table 1. The top panels show the total variability for cases 1–4 (change of correlation timescale), while the middle panels show cases 5, 3, and 6 (variation of correlation length scale). The bottom panels show the same properties as the top panels, but computed from a 4 month mean.

The most striking feature visible in Figures 1a–1d is that while the fluctuations of the meridional flow show no systematic variation with correlation time and length, the fluctuations of differential rotation show a systematic increase with both. An exception is case 2 in Figures 1a and 1b, since \( \tau_c = \Omega_\odot^{-1} \) is very close to the timescale of inertial oscillations and therefore \( \sigma_v \) is enhanced because of a resonance. The amplitude of \( \sigma_v \) is about 20%–25% of the mean meridional flow (in our model the meridional flow reaches a peak value of around 13.5 m s\(^{-1}\) at 45° latitude), which is close to the noise level added to the Reynolds stress (33\%). On the other hand, except for high latitudes the value of \( \sigma_{\Omega_\theta} \) is less than 1% of the reference rotation rate in all cases.

Figures 1e and 1f show similar quantities as Figures 1a and 1b, but computed from a 4 month mean of the time series, similar to the averaging interval typically used in GONG/MDI inversions of differential rotation profiles. It is not surprising that the amplitude of the fluctuations decreases, however the effect is stronger in case of the meridional flow; e.g., the averaging decreases \( \sigma_v \) for case 3 (dashed line) by a factor of 2, while \( \sigma_{\Omega_\theta} \) is nearly unaffected.

So far we focused on the variability of meridional flow and differential rotation at the top of the domain. Figure 2 shows the depth dependence for the meridional flow at 45° and the differential rotation at the selected latitudes 0°, 15°, 30°, 45°, 60°, and 90°. In both cases solid lines indicate the mean and the gray scale the 1 \( \sigma \) interval. Shown is case 3 with a correlation timescale of 48 \( \Omega_\odot^{-1} \). Both \( \sigma_v \) and \( \sigma_{\Omega_\theta} \) show a maximum at the top of the domain and a nearly monotonic decrease toward the base of the convection zone. We emphasize that this result was obtained by using a constant correlation time and length scale throughout the convection zone. In a more realistic description it can be expected that both correlation time and length scale decrease toward the solar surface, which according to the trends found in Figure 1 would lead to a decrease of \( \sigma_{\Omega_\theta} \) close to the surface, while \( \sigma_v \) remains...
Fig. 1.—Variation of meridional flow (left panels) and differential rotation (right panels) at the top of the domain $r = 0.985 R$. (a, b) Total variance of the time series for cases 1–4 (variation of correlation timescale). The solid line indicates case 1 ($\tau = 0.25 \Omega_0^{-1}$), the dotted line case 2 ($\tau = 1 \Omega_0^{-1}$), the dashed line case 3 ($\tau = 4 \Omega_0^{-1}$), and the dashed-dotted line 4 ($\tau = 16 \Omega_0^{-1}$). (c, d) Cases 5, 3, and 6 (variation of correlation length scale). The solid line indicates case 5 ($\Delta = 0.05$), the dotted line case 3 ($\Delta = 0.1$), and the dashed line case 6 ($\Delta = 0.2$). (e, f) Similar to the top panels; however, the variances are computed from a 4 month mean.

Fig. 2.—Depth variation of the standard variations of meridional flow and differential rotation for case 3. (a) Meridional flow profile at 45° latitude and the 1 $\sigma$ intervals as gray scale. (b) Differential rotation at 0°, 15°, 30°, 45°, 60°, and 90° latitude with 1 $\sigma$ intervals as gray scale.
more or less unchanged (except for the enhancement of the signal close to the resonance). In our investigation we also use a constant amplitude of the stochastic fluctuations of the Reynolds stress throughout the convection zone. A depth dependence of this amplitude would influence both $\sigma_c$ and $\sigma_{\Omega}$ in a similar way.

We computed all models shown in this paper with an amplitude of $c = 0.33$ for the random noise in the $\Lambda$-effect. Note that $\sigma_c$ and $\sigma_{\Omega}$ scale linearly with $c$, meaning that the results given here can be rescaled to apply to a different amplitude of the random noise.

We already indicated above that the variance of the meridional flow and differential rotation is enhanced around $\tau_c \sim \Omega_0^{-1}$ because of the excitation of inertia oscillations. Figure 3 shows the variances of meridional flow and differential rotation at the top of the domain ($r = 0.985$ $R_\odot$) for the latitudes 22.5° (solid line), 45° (dotted line), and 67.5° (dashed line) as function of $\tau_c$. Figure 3a shows the total variance of the meridional flow, clearly indicating a resonance around $\tau_c \approx (1-2)\Omega_0^{-1}$. Since our timescale $\tau_c$ is defined as an $e$-folding timescale for the random perturbations, we expect this resonance around $\tau_c \approx \tau/2 = \pi/\omega_i$, with the frequency of inertia oscillations given by $\omega_i = 2\Omega_0$. This leads to a location of the resonance around

$$\tau_c \approx \frac{\pi}{2\Omega_0}. \quad (5)$$

Figure 3a shows a weak tendency of increasing $\tau_c$ with decreasing latitude, which indicates that the geometry of the problem and the stratification leads to some confinement of motions in latitude (for motions in latitude only we would have $\omega_i = 2\Omega_0 \cos \theta$).

The resonance also affects the variance of the differential rotation; however, this influence is much weaker and disappears at low latitudes (variations in the meridional flow have less influence on $\Omega$ close to the equator). The resonance is not visible in the variance of the 4 month mean flows (Figs. 3c–3d), which shows the monotonic increase of the $\sigma_c$ and $\sigma_{\Omega}$ with $\tau_c$ at all latitudes as already indicated in Figure 1.

4. INTERPRETATION

Our main results presented can be understood in terms of different response timescales of meridional flow and differential rotation with respect to changes in the Reynolds stress. The difference in the response timescale follows from the different size of the kinetic energy reservoirs associated with meridional flow and differential rotation. Here we make a detailed analysis of the energy fluxes within the differential rotation model in order to estimate typical timescales related to the meridional flow and differential rotation. In the following derivation we use the assumptions $\vartheta_1 \ll \vartheta_0, p_1 \ll p_0$, and $|\delta| = |\nabla - \nabla_d| \ll 1$. We also omit terms proportional to $\nabla \cdot \langle \theta \varrho m \rangle$, which are very small for the low Mach number meridional flow and vanish for the stationary solution finally considered. We start from equations (2)–(4) of Rempel (2005) to derive two separate energy equations for the meridional flow and differential rotation. To this end we multiply their equation (3) by $\vartheta_0 v_r$ and add both equations. Rearrangement of the different terms leads to

$$\frac{\partial}{\partial t} \left( \frac{v_r^2}{2} \right) + \nabla \cdot \mathbf{F}_m = - (r \sin \theta)^2 \vartheta_0 v_m \cdot \nabla \frac{\Omega^2}{2}$$

$$- \frac{1}{2} \left( R_{\vartheta r} E_{\vartheta r} + 2R_{\vartheta \vartheta} E_{\vartheta \vartheta} + R_{\vartheta \vartheta} E_{\vartheta \vartheta} + R_{\vartheta \vartheta} E_{\vartheta \vartheta} \right)$$

$$+ v_\vartheta \varrho_0 \frac{s}{\tau} \quad (6)$$

with the energy flux

$$\mathbf{F}_m = \vartheta_0 v_m \left( \frac{v_r^2}{2} + \frac{p_1}{\vartheta_0} \right) - \vartheta_0 v_m (r \sin \theta)^2 \frac{\Omega^2}{2} - \frac{\Omega^2}{2} + \mathbf{R} \cdot \mathbf{v}_m.$$  

(7)
Here $\mathbf{R}$ denotes the Reynolds stress tensor, $\mathbf{E}$ the deformations tensor. For the definition we refer to Rempel (2005). Multiplying equation (4) of Rempel (2005) by $\partial \Omega (r \sin \theta)^2 \Omega$ leads to
\[
\frac{\partial}{\partial t} \left[ \frac{\partial \Omega (r \sin \theta)^2 \Omega}{2} \right] + \nabla \cdot \mathbf{F}_\Omega =
\]
\[\quad - \nu \partial \Omega (r \sin \theta)^2 \left[ \left( \frac{\partial \Omega}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \right)^2 \right]\]
\[\quad - \nu \partial \Omega^2 (r \sin \theta) \partial \Omega \Lambda_{r \phi} + \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \Lambda_{\theta \phi} \]
\[\quad + (r \sin \theta)^2 \partial \Omega v_m \cdot \nabla \frac{\Omega^2}{2} \quad (8)
\]
with the energy flux
\[
\mathbf{F}_\Omega = \partial \Omega v_m (r \sin \theta)^2 - r \sin \theta \partial \Omega \mathbf{R}_{\phi} . \quad (9)
\]
Here $\mathbf{R}_{\phi}$ denotes a vector with components $R_{\rho \rho}$ and $R_{\phi \theta}$. Since for the boundary conditions we use, the energy fluxes (eqs. [7] and [9]) vanish at the boundaries, the divergence terms in equations (6) and (8) do not contribute after integration over the entire domain, leading to the energy balances
\[
\frac{\partial E_m}{\partial t} = Q_C + Q_{m}^\text{visc} + Q_{w}^m , \quad (10)
\]
\[
\frac{\partial E_\Omega}{\partial t} = -Q_C + Q_{m}^\text{visc} + Q_{w}^\text{differential} , \quad (11)
\]
with the terms
\[
E_m = \int dV \partial \Omega v_m^2 ,
\]
\[
Q_C = -\int dV (r \sin \theta)^2 \partial \Omega v_m \cdot \nabla \frac{\Omega^2}{2} ,
\]
\[
Q_{w}^m = -\int dV \frac{1}{2} \left( R_{\rho \rho} E_{rr} + 2 R_{\rho \theta} E_{r \theta} + R_{\theta \theta} E_{\theta \theta} + R_{\phi \phi} E_{\phi \phi} \right) ,
\]
\[
Q_{m}^\text{visc} = \int dV \nu \partial \Omega \gamma (r \sin \theta)^2 (\Omega^2 - \Omega_0^2) ,
\]
\[
E_\Omega = \int dV \frac{1}{2} \partial \Omega (r \sin \theta)^2 (\Omega^2 - \Omega_0^2) ,
\]
\[
Q_{m}^\text{visc} = -\int dV \nu \partial \Omega (r \sin \theta)^2 \left[ \left( \frac{\partial \Omega}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \right)^2 \right] ,
\]
\[
Q_{w}^\text{differential} = -\int dV \nu \partial \Omega (r \sin \theta) \partial \Omega \Lambda_{r \phi} + \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \Lambda_{\theta \phi} . \quad (12)
\]
Here
\[
\int dV = 4\pi \int_{r_{\min}}^{r_{\max}} dr \int_0^{\pi/2} d\theta r^2 \sin \theta \quad (13)
\]
denotes the integral over the entire volume of the sphere from $r = r_{\min}$ to $r = r_{\max}$. We emphasize that we solve our model only for the northern hemisphere but we compute from that the energy conversion for the entire sphere. $E_m$ and $E_\Omega$ denote energy available in the reservoir of meridional flow and differential rotation, respectively. For the latter we subtracted the core rotation rate to obtain a value representative of the differential rotation rather than total rotation. $Q_C$ denotes the amount of energy that is converted by means of the Coriolis force (which is a sink for differential rotation and a source for meridional flow). The terms $Q_{w}^m$ and $Q_{w}^\text{differential}$ represent the losses through viscous dissipation for meridional flow and differential rotation, respectively. $Q_{m}^\text{visc}$ is the energy that is converted through the $\Lambda$-effect from internal energy to energy of differential rotation, and $Q_{m}^\text{visc}$ denotes the work of the meridional flow against the buoyancy force arising from the entropy perturbation within the convection zone.

For the stationary reference model we can write the energy balance as
\[
Q_C = -Q_{m}^\text{visc} - Q_{w}^m ,
\]
\[
Q_{m}^\text{visc} = -Q_{m}^\text{visc} + Q_C , \quad (15)
\]
where the left-hand side terms are the sources and the right-hand terms are the sinks for the reservoirs of meridional flow and differential rotation, respectively. Figure 4 shows a qualitative diagram indicating the energy fluxes between the different energy reservoirs. A quantitative analysis of model 1 of Rempel (2005), which we used in this paper as reference model, leads to
\[
E_\Omega = 1.55 \times 10^{33} \text{ J} ,
\]
\[
Q_{m}^\text{visc} = 6.6 \times 10^{24} \text{ W} \quad 0.017 L_\odot ,
\]
\[
\tau_\Omega = \frac{E_\Omega}{Q_{m}^\text{visc}} = 7.5 \text{ yr} ,
\]
\[
Q_{w}^m = -0.54 Q_{m}^\text{visc} \quad Q_C = 0.46 Q_{m}^\text{visc} , \quad (16)
\]
for the differential rotation and to
\[
E_m = 10^{29} \text{ J} ,
\]
\[
Q_C = 3 \times 10^{24} \text{ J} \quad 0.008 L_\odot ,
\]
\[
\tau_m = \frac{E_m}{Q_C} = 9.4 \text{ hr} ,
\]
\[
Q_{w}^m = -0.02 Q_C , \quad Q_{w}^\text{differential} = -0.98 Q_C , \quad (17)
\]
for the meridional flow.
In order to maintain the differential rotation in this model, the $\Lambda$-effect has to convert an amount of energy equivalent to around 1.7% of the solar luminosity. Comparing this energy flux to the energy stored in the reservoir of differential rotation leads to a timescale of around 8 years for the replenishment of energy. Around 54% of the energy converted by the $\Lambda$-effect returns directly to the reservoir of internal energy through viscous dissipation, while 46% flows through the action of the Coriolis force into the reservoir of meridional flow. Dividing the energy in this reservoir by the energy flux related to the Coriolis force leads to a timescale for the replenishment of energy in the meridional flow of only 10 hours, which is around a factor of 6500 shorter than the corresponding timescale for the differential rotation. The direct viscous loss does not play an important role for the meridional flow (only around 2% of the energy is dissipated); most of the energy returns to the reservoir of internal energy through work against the buoyancy force.

The numbers presented here vary with different model assumptions made but the fact that $\tau_{\Omega}$ is around 4 orders of magnitude longer than $\tau_m$ is a very robust result, since it follows mainly from the different sizes of the energy reservoirs of differential rotation and meridional flow. Another very robust result is that the viscous dissipation does not play an important role for the meridional flow (unless the assumed turbulent viscosity would be more than 1 order of magnitude larger than the value of $5 \times 10^8$ m$^2$ s$^{-1}$ used here). This emphasizes the importance of the pole-equator entropy variation for avoiding the Taylor-Proudman state, as discussed in Rempel (2005), since the work of the meridional flow against the buoyant force is the primary sink of energy.

From the very short response timescale of $\tau_m \sim 10$ hours it can be expected that the meridional flow is affected nearly instantaneously by fluctuations in the Reynolds stress, while the differential rotation only responds to a long-term average over several years. As a consequence, the relative amplitudes of fluctuations in meridional flow and differential rotation are expected to scale with a factor of $(\tau_{\Omega}/\tau_m)^{1/2} \sim 80$, which is seen in the results presented before.

An order-of-magnitude estimate based on the energy yields

$$\Delta E_m \sim v_{rms}^m \Delta \tau_m$$

$$\Delta E_{\Omega} \sim v_{rms}^\Omega \Delta \tau_m$$

(18)

(19)

Given the fact that the amount of energy transferred into the meridional flow is roughly half the energy transferred into the differential rotation, the expected change in the flow amplitudes scales like

$$\frac{\Delta v_{rms}^m}{\Delta v_{rms}^\Omega} \sim \frac{1}{2} \frac{v_{rms}^\Omega}{v_{rms}^m} \sim 100,$$

(20)

with $v_{rms}^\Omega \sim 1$ km s$^{-1}$ and $v_{rms}^m \sim 5$ m s$^{-1}$. This is close to the ratio we obtained in the more detailed analysis of the energy fluxes within the model.

Since the large kinetic energy reservoir of the differential rotation tends to average fluctuations, a systematic variation of $\sigma_{\Omega}$ with correlation timescale and length scale is seen in the data (longer lasting, large-scale perturbations manifest more easily in the differential rotation). On the other hand, the meridional flow shows a nearly immediate response to changes in the Reynolds stress. Therefore, $\sigma_\Omega$ is mainly determined by the amplitude of the random fluctuations in the Reynolds stress assumed and not by the correlation time and length scale (unless the timescale for fluctuations in the Reynolds stress is much shorter than $\tau_m$, which is not reasonable in the bulk of the convection zone). The only exception is the resonance found in case 2, which enhances the amplitude by about 50%.

5. CONCLUSIONS

We presented a mean field model for differential rotation and meridional flow including stochastic fluctuations in the Reynolds stress driving the differential rotation. We found that the variations in the differential rotation are around 2 orders of magnitude less than the variability observed in the meridional flow pattern. For example, case 3 discussed above shows a 1 $\sigma$ variation of the 4 month mean of less than 0.3% of the rotation rate for the differential rotation (in latitudes lower than 50°), while the 1 $\sigma$ variation for the meridional flow is around 8% of the mean flow value (4 month mean) or even 20% of the mean flow value if also the short-term variation is considered. The maximum amplitude of the short-term meridional flow variability in this models is around 10 ms$^{-1}$ and therefore close to 100% of the mean flow.

This result is a consequence of different response timescales of the differential rotation and meridional flow to changes in the Reynolds stress. Our model shows that the energy flows through the reservoir of differential rotation and meridional flow are comparable, while the kinetic energy of differential rotation exceeds that of meridional flow by about 4 orders of magnitude. As a consequence the meridional flow shows a nearly instantaneous response (on a timescale of less than a day) to changes in the Reynolds stress, while the differential rotation is affected only by changes that have a long-term average of at least 10 years.

Our result is in agreement with observations, which indicate that the 1 $\sigma$ variation of the 4 month differential rotation inversions is a few nHz, but also show that the variations in the meridional flow can be of up to 10 ms$^{-1}$. However, most of the meridional flow measurements so far only address long-term variability (timescale of years) rather than short-term variability, which is the main focus of the model presented here.

We conclude that a fairly significant amount of random noise in the Reynolds stresses driving differential rotation can be tolerated without leading to variations in the differential rotation contradicting helioseismic inversions. However, the meridional flow will show significant variability on the same order of magnitude as the fluctuations of the Reynolds stress.

We found that the random forcing through the Reynolds stress excites inertia oscillations for timescales $\tau_c \sim (1-2)\Omega^{-1} \sim 4-8$ days. This resonance enhances the total variance $\sigma_\Omega$ by about 50% and has a visible but much weaker effect on $\sigma_{\Omega}$. However, this effect is not visible if the variance of the 4 month mean is considered. Therefore, this effect could be observable in the variability of daily flow maps of the surface meridional flow but not in helioseismic inversions normally considering longer averages.

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