REPRESENTATION THEORY, RADON TRANSFORM AND THE HEAT EQUATION ON A RIEMANNIAN SYMMETRIC SPACE

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Abstract. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. We give a short exposition of the representation theory related to $X$, and discuss its holomorphic extension to the complex crown, a $G$-invariant subdomain in the complexified symmetric space $X_C = G_C/K_C$. Applications to the heat transform and the Radon transform for $X$ are given.

Introduction

In the analysis of Riemannian symmetric spaces one can follow several approaches, with the emphasis for example on differential geometry, partial differential equations, functional analysis, complex analysis, or representation theory. The representation theory associated with a Riemannian symmetric space is of course well known through the work of Harish-Chandra [25, 26], and it is based on important results in the theory of infinite dimensional representations, by Gelfand-Naimark, Mackey, Segal, just to mention a few.

The powerful theory of Harish-Chandra has been widely generalized [2, 6, 28], and successfully applied to many problems related to Riemannian symmetric spaces. The geometric point of view, with emphasis on relations to topics like classical Euclidean analysis and the Radon transform has been represented by Helgason [29, 31, 32]. Both aspects, as well as the connection to the work of the school around Gelfand and Graev are described in a short and clear fashion in the second half of Mackey’s famous Chicago Lectures on unitary group representations [47].

In recent years much research has been directed towards the interplay between the real analysis and the geometry of the symmetric space $G/K$ on the one side, and complex methods in analysis, geometry and representation theory on the other side. This development has a long history tracing back to Cartan’s analysis of bounded symmetric domains, and Harish-Chandra’s construction of the holomorphic discrete series [23]. The so-called Gelfand-Gindikin program [13], suggests to consider functions on $G$ through holomorphic extension to domains in the complexification $G_C$ of $G$, and to study representations of $G$ realized on spaces of such functions, analogous to the classical Hardy spaces on tube domains over $\mathbb{R}^n$. Only partial results have been obtained so far. The program was carried out for the holomorphic discrete series of groups of Hermitian type in [51, 54], the holomorphic discrete series

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for compactly causal symmetric spaces in [38], and finally the holomorphic most continuous series for noncompactly causal symmetric spaces in [17].

Connected to this development is the study of natural domains in $G_C$ to which the spherical function of $G/K$ admit holomorphic extensions, initiated by Akhiezer and Gindikin [1], and continued by several people. Without claiming to be complete we would like to mention [3, 5, 7, 8, 10, 11, 17, 18, 19, 44, 45, 48] representing different aspects of this important development. The most relevant articles related to the present exposition are the papers [15, 45]. One important conclusion is that there exists a maximal $G$-invariant domain, called the complex crown, to which all the spherical functions extend.

A related area of recent research is the study of the heat equation on the symmetric space. There are several generalizations to the well-known Segal-Bargmann transform, which maps an $L^2$-function $f$ on the Euclidean space $\mathbb{R}^n$ to the function $H_t f$ on $\mathbb{C}^n$, which is the holomorphic continuation of the solution at time $t$ to the heat equation with initial Cauchy data $f$. The first work in this direction was by Hall [20], who replaced $\mathbb{R}^n$ by a connected compact semisimple Lie group $U$, and $\mathbb{C}^n$ by the complexification $U_C$. This was put into a general framework using polarization of a restriction map in [50]. The results of Hall were extended to compact symmetric spaces $U/K$ by Stenzel in [56], where the complexification is $U_C/K_C$. It is important to note that in this compact case, all eigenfunctions of the algebra of invariant differential operators on $U/K$, and also the heat kernel itself, extend to holomorphic functions on $U_C/K_C$. This is related to the fact, that each irreducible representation of $U$ extends to a holomorphic representation of $U_C$.

For a symmetric space of the noncompact type $G/K$, the maximal $G$-invariant domain is the complex crown and not the full complexification $X_C = G_C/K_C$. It was shown in [10] that the image of the Segal-Bargmann transform on $G/K$ can be identified as a Hilbert-space of holomorphic functions on the crown. The norm was defined using orbital integrals and the Faraut version of Gutzmer’s formula [8]. Some special cases have also been considered in [21, 22], but without using the Akhiezer-Gindikin domain explicitly. In particular, in [22] Hall and Mitchell give a description of the image of the $K$-invariant functions in $L^2(G/K)$ in case $G$ is complex.

In the present paper we have the following basic aims. Our first aim is to give a short exposition of the basic representation theory related to a Riemannian symmetric space $G/K$, connecting to work of Harish-Chandra and Helgason. The secondary aim is to discuss the complex extension in $X_C$, and to introduce a $G$-invariant Hilbert space of holomorphic functions on the crown, which carries all the representation theory of $G/K$. Thirdly, we combine the Fourier theory and the holomorphic theory in a study of the heat equation on $G/K$. The image of the Segal-Bargmann transform is described in a way similar to that of [49], where we considered the case of $K$-invariant functions on $G/K$ (in a more general setting of root systems), and derived a result containing that of Hall and Mitchell as a special case. The image is characterized in terms of a Fock space on $\mathfrak{a}_C$.

Finally we discuss some aspects of the Radon transform on $G/K$. We introduce a $G$-invariant Hilbert space of CR-functions on a subset of the complexified horocycle space, and we use the Radon transform to construct a unitary $G$-invariant isomorphism between the Hilbert spaces on the crown and on the complex horocycle space, respectively.
We shall now describe the content of the paper in some more detail. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. For the mentioned primary aim, we define a Fourier transform and we state a Plancherel theorem and an inversion formula, following the beautiful formulation of Helgason. The content of the formulation is described by means of representation theory. We also indicate a representation theoretic proof of the reduction to Harish-Chandra’s theorem. At the end of Section 1 we give a short description of the heat transform and its image in $L^2(X)$ as a direct integral over principal series representations.

The space $X$ is naturally contained in the complexified symmetric space $X_C = G_C/K_C$ as a totally real submanifold. Inside the complexification $X_C$ is the $G$-invariant domain $Cr(X) \subset X_C$, the complex crown, which was introduced in [1]. It has the important property, shown in [45], that all joint eigenfunctions for $D(X)$, the algebra of invariant differential operators on $X$, extend to holomorphic functions on it. A second fundamental fact is the convexity theorem of Gindikin, Krötz, and Otto [15, 43], which is recalled in Theorem 2.1. In Section 2 we define a $G$-invariant Hilbert space $\mathcal{H}_X$ of holomorphic functions on $Cr(X)$, such that restriction to $X$ maps continuously into $L^2(X)$, and such that the representation of $G$ on $\mathcal{H}_X$ carries all the irreducible representations found in the decomposition of $L^2(X)$.

The definition, which is essentially representation theoretic, is related to definitions given in [16]. The main results are stated in Theorem 2.3 where a representational description of $\mathcal{H}_X$ is given and the reproducing kernel of the space is determined, see also [18].

The final section deals with the holomorphic extension of $H_t f$ for $f \in L^2(X)$. It was shown in [45] that $H_t f$ extends to a holomorphic function on $Cr(X)$, and a description of the image $H_t(L^2(X)) \subset O(Cr(X))$ was given in [46]. In fact, we show in Theorem 3.1 that $H_t(L^2(X)) \subset \mathcal{H}_X$, and we give an alternative description of the image by means of the Fourier transform.

The Radon transform sets up a relation between functions on $X$ and functions on the space $\Xi = G/MN$ of horocycles on $X$. The Radon transform is used in Theorem 3.3 to give yet another description of the image of $H_t$. We also give an inversion formula for $H_t$.

The space $\Xi$ sits inside a complex space, the space $\Xi_C$ of complex horospheres on $X_C$, and in this complex space one can define a $G$-invariant domain $\Xi(\Omega) \supset \Xi$ analogous to the crown. However, $\Xi(\Omega)$ is not a complex manifold but a CR-submanifold of $\Xi_C$. On this domain, we define a space $\mathcal{H}_\Xi$ similar to $\mathcal{H}_X$, and we show in Theorem 3.5 that the normalized Radon transform can be used to set up an unitary isomorphism $\tilde{\Lambda} : \mathcal{H}_X \rightarrow \mathcal{H}_\Xi$.

1. Representation theory and harmonic analysis

In this section we introduce the standard notation that will be used throughout this article. We will also recall some well known facts about the principal series representations $\pi_\lambda$ and the Fourier transform on $X$.

1.1. Notation. Let $G$ be a connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g}$, let $\theta : G \to G$ be a Cartan involution and set

$$K = G^\theta = \{ x \in G \mid \theta(x) = x \}.$$  

The space $X = G/K$ is Riemannian symmetric space of noncompact type. It does not depend on which one of the locally isomorphic groups with Lie algebra $\mathfrak{g}$ is
used, as the center of $G$ is always contained in $K$. We will therefore assume that $G$ is contained in a simply connected Lie group $G_C$ with Lie algebra $g_C = g \otimes_R \mathbb{C}$. In particular, $G$ has finite center and $K$ is a maximal compact subgroup of $G$.

We will denote by the same symbol $\theta$ the holomorphic extension of $\theta$ to $G_C$, as well as the derived Lie algebra homomorphisms $\theta : g \to g$ and its complex linear extension to $g_C$. Denote by $\sigma : g_C \to g_C$ the conjugation on $g_C$ with respect to $g$. As $G_C$ is simply connected, $\sigma$ integrates to an involution on $G_C$ with $G = G_C^\theta$. Denote by $K_C \subset G_C$ the complexification of $K$ in $G_C$ and $X_C = G_C/K_C$. Then $X_C$ has a complex structure, with respect to which $X \simeq G \cdot x_0 \subset X_C$ is a totally real submanifold. It can be realized as the connected component containing $x_0 = eK_C \in X_C$ of the fixed point set of the conjugation $g \cdot x_0 \mapsto \sigma(g \cdot x_0) := \sigma(g) \cdot x_0$.

Let $g = t \oplus p$ be the Cartan decomposition defined by $\theta$, and let $a$ be a maximal abelian subspace of $p$, $\Delta$ the set of roots of $a$ in $g$, and $\Delta^+$ a fixed set of positive roots. For $\alpha \in \Delta$ let

$$g^\alpha = \{ X \in g \mid (\forall H \in a) [H, X] = \alpha(H)X \}$$

be the joint $\alpha$-eigenspace. Let

$$n = \bigoplus_{\alpha \in \Delta^+} g^\alpha,$$

then $n$ is a nilpotent subalgebra of $g$. Let $m = z_t(a)$, then $m \oplus a \oplus n$ is a minimal parabolic subalgebra and

$$g = t \oplus a \oplus n.$$

Let $A = \exp a$, $N = \exp n$, $M = Z_K(A)$ and $P = MAN$. We have the Iwasawa decompositions

$$G = KAN \subset K_C A_C N_C \subset G_C$$

where the subscript $C$ on a subgroup of $G$ stands for its complexification in $G_C$. The map $K \times A \times N \ni (k, a, n) \mapsto k a n \in G$ is an analytic diffeomorphism (the analogous statement fails for the complexified Iwasawa decomposition). We shall denote its inverse $x \mapsto (k(x), a(x), n(x))$.

Let $B = K/M$, this is the so-called Furstenberg boundary of $X$. Note that since $B \simeq G/P$, it carries the action of $G$ given by $g \cdot (kM) = k(g)M$.

1.2. Integration. If $C$ is a Lie group or a homogeneous space of a Lie group, then we denote by $dc$ a left invariant measure on $C$. We normalize the invariant measure on compact groups and compact homogeneous spaces such that the total measure is one. We require that the Haar measures on $A$ and $\text{ia}^*$ are normalized such that if

$$\mathcal{F}_A(f)(\lambda) = \int_A f(a)a^{-\lambda} da$$

is the Fourier transform on the Abelian group $A$, then

$$f(x) = \int_{\text{ia}^*} \mathcal{F}_A(f)(\lambda) a^\lambda d\lambda.$$

Finally, we normalize the Haar measure on $N$ as usual by $\int_N a(\theta(n))^{-2\rho} dn = 1$, where $2\rho = \sum_{\alpha \in \Delta^+} m_\alpha \alpha$ and $m_\alpha = \dim g^\alpha$. Then we can normalize the Haar measure on $G$ such that for all $f \in C_c(G)$ we have

$$\int_G f(x) dx = \int_N \int_A \int_K f(ank) dk d\alpha = \int_K \int_A \int_N f(kan) a^{2\rho} d\alpha dk d\alpha.$$
In this normalization the invariant measure on \( X \) is given by
\[
\int_X f(x) \, dx = \int_G f(g \cdot x_o) \, dg = \int_{A} \int_N f(an \cdot x_o) \, dadn.
\]
Finally, it follows from (1.2) that the \( K \)-invariant measure on \( B \) transforms under the \( G \)-action according to
\[
\int_B f(g \cdot b) \, db = \int_B f(b) a(g^{-1}b)^{-2\rho} \, db
\]
for all \( f \in L^1(B) \) and \( g \in G \).

1.3. **Spherical principal series and spherical functions.** Denote by \( L \) the left regular representation \( L_a f(x) = f(a^{-1}x) \) and by \( R \) the right regular representation \( R_a f(x) = f(xa) \). If \( C/D \) is a homogeneous space, then we identify functions on \( C/D \) with right \( D \)-invariant functions on \( C \).

The spherical principal series of representations are defined as follows. For \( \lambda \in \mathfrak{a}_C^* \), denote by \( H_\lambda \) the Hilbert space of measurable functions \( f : G \to \mathbb{C} \) such that for all \( man \in P \) (with the obvious notation)
\[
R_{man} f = a^{-\lambda} f \quad \text{and} \quad \| f \|^2 = \int_K |f(k)|^2 \, dk < \infty.
\]
Define the representation \( \pi_\lambda \) of \( G \) on \( H_\lambda \) by
\[
\pi_\lambda(x)f(y) = L_x f(y) = a(x^{-1}y)^{-\lambda} f(k(x^{-1}y)).
\]
A different picture (called the compact picture) of the representations \( \pi_\lambda \) is obtained by noting that the Iwasawa decomposition implies that the restriction map \( f \mapsto f|_K \), is a unitary \( K \)-isomorphism \( H_\lambda \to L^2(B) \). Thus we may view \( \pi_\lambda \) as a representation on the latter space. By (1.5) the representation is then given by
\[
\pi_\lambda(x)f(b) = a(x^{-1}k)^{-\lambda} f(x^{-1} \cdot b).
\]
The advantage of the compact picture is that the Hilbert space is independent of \( \lambda \).

The representations \( \pi_\lambda \) are known to be unitary when \( \lambda \) is purely imaginary on \( a \). Furthermore, it is known that \( \pi_\lambda \) is irreducible for almost all \( \lambda \in \mathfrak{a}_C^* \), and that \( \pi_{ua\lambda} \) and \( \pi_\lambda \) are equivalent representations, also for almost all \( \lambda \in \mathfrak{a}_C^* \), see \( [4] [12] \). Here \( W \) denotes the Weyl group of the root system \( \Delta \).

The representations \( \pi_\lambda \) all restrict to the same representation of \( K \), the left regular representation on \( L^2(B) \). In particular, the trivial representation of \( K \) has multiplicity one and is realized on the space of constant functions on \( B \). We fix \( p_\lambda \in H^K_\lambda \) as the constant function 1 on \( B \), that is, on \( G \) it is
\[
p_\lambda(g) = a(g)^{-\lambda}.
\]
We define the following function \( e_{\lambda,b} \) on \( X \) for \( (\lambda, b) \in \mathfrak{a}_C^* \times B \),
\[
e_{\lambda,b}(x) = p_\lambda(g^{-1}k), \quad k \in K, g \in G.
\]
then \( (x, b) \mapsto e_{\lambda,b}(x) \) is the generalized Poisson kernel on \( X \times B \). The spherical functions on \( X \) are the \( K \)-biinvariant matrix coefficients of \( \pi_\lambda \) defined by
\[
\varphi_\lambda(x) = (\pi_\lambda(g)p_\lambda, p_\lambda) = \int_B e_{\lambda,b}(x) \, db
\]
where \( x = gK \) and \( \lambda \in \mathfrak{a}_C^* \). The latter integral is exactly Harish-Chandra’s formula for the spherical functions.
1.4. The standard intertwining operators. As mentioned, the representation \( \pi_{w\lambda} \) is known to be equivalent with \( \pi_\lambda \) for almost all \( \lambda \in i\mathfrak{a}^* \). Hence for such \( \lambda \) there exists a unitary intertwining operator

\[
A(w, \lambda) : H_\lambda \rightarrow H_{w\lambda}.
\]

The operator is unique, up to scalar multiples, by Schur’s lemma. The trivial \( K \)-type has multiplicity one and is generated by the function \( p_\lambda \in H_\lambda \), which has norm 1 in \( L^2(B) \). It follows that \( A(w, \lambda)p_\lambda \) is a unitary multiple of \( p_{w\lambda} \). We normalize the intertwining operator so that

\[
A(w, \lambda)p_\lambda = p_{w\lambda}.
\]

The intertwining operator so defined is called the normalized standard intertwining operator. It is known that the map \( \lambda \mapsto A(w, \lambda) \) extends to a rational map (which we denote by the same symbol) from \( i\mathfrak{a}^* \) into the bounded operators on \( L^2(B) \).

In fact, one can give a formula for the operator \( A(w, \lambda) \) as follows. The (unnormalized) standard intertwining operator \( A(w, \lambda) : H_\lambda \rightarrow H_{w\lambda} \) is defined by the formula

\[
A(w, \lambda)f(g) = \int_{\tilde{N}_w} f(gw\bar{n}) \, d\bar{n}
\]

where \( \tilde{N}_w = \theta(N) \cap w^{-1}Nw \), see [41]. The integral converges when \( f \) is continuous and \( \lambda \in \mathfrak{a}_\mathbb{C}^* \) satisfies that \( \text{Re}(\lambda, \alpha) > 0 \) for all \( \alpha \in \Delta^+ \). It is defined by meromorphic continuation for other values of \( \lambda \), and by continuous extension for \( f \in L^2(B) \) (see [40], Ch. 7). From the definition of \( A(w, \lambda) \) we see

\[
A(w, \lambda)p_\lambda = c_w(\lambda)p_{w\lambda}
\]

where \( c_w(\lambda) = \int_{\tilde{N}_w} p_\lambda(\bar{n}) \, d\bar{n} \) (see [39] page 446). Hence

\[
A(w, \lambda) = c_w(\lambda)^{-1}A(w, \lambda).
\]

1.5. The Fourier transform. In this section we introduce the Fourier transform on \( X \), following Helgason [31, 32]. While Helgason introduced it from a more geometric point of view, we shall show here that it can also be done from a representation theory point of view, resulting in alternative proofs of the inversion formula and the Plancherel theorem.

From the point of view of representation theory, the Fourier transform of an integrable function on \( G \) is the operator \( \pi(f) \) on \( \mathcal{H} \) defined by

\[
\pi(f)v = \int_G f(g)\pi(g)v \, dg, \quad v \in \mathcal{H},
\]

for each unitary irreducible representation \((\pi, \mathcal{H})\).

If \( f \) is a function on \( X \), this operator will be trivial on the orthocomplement of the space of \( K \)-fixed vectors. This space is always one dimensional in an irreducible representation, and hence it becomes natural to define the Fourier transform of \( f \) as the vector \( \pi(f)v \) in the representation space of \( \pi \), where \( v \) is a specified \( K \)-fixed vector. For the spherical principal series, we thus arrive at the following definition of the Fourier transform:

\[
\hat{f}_\lambda := \pi_{-\lambda}(f)\in H_{-\lambda}
\]

for \( \lambda \in i\mathfrak{a}^* \) (the reason for the minus is just historical).
In the notation of the compact picture, it is

\[
(1.9) \quad \hat{f}_\lambda(b) = (\pi_\lambda(f)p_\lambda)(b) = \int_G f(x)p_\lambda(x^{-1}b)\,dx = \int_G f(x)e_{\lambda,b}(x)\,dx.
\]

where \( b \in B \). Thus the Fourier transform of \( f \) may be viewed as a map

\[
\mathfrak{a}_c^+ \times B \ni (\lambda, b) \mapsto \hat{f}(\lambda, b) := \hat{f}_\lambda(b) \in \mathbb{C}
\]

Apart from the replacement of \( \lambda \) by \( i\lambda \), this is the Fourier transform as it was introduced by Helgason in [31].

Let

\[
a^+ := \{ H \in \mathfrak{a} \mid (\forall \alpha \in \Delta^+) \alpha(X) > 0 \}
\]

be the positive Weyl chamber corresponding to \( \Delta^+ \), and let \( \mathfrak{a}^*_c \) denote the corresponding open chamber in \( \mathfrak{a}^* \). Let \( c(\lambda) \) be the Harish-Chandra \( c \)-function, which for \( \text{Re} \lambda \in \mathfrak{a}^*_c \) is given by (see [36] p. 447)

\[
c(\lambda) = c_{w^*}(\lambda) = \int_N p_\lambda(n)d\bar{n},
\]

where \( w^* \in W \) is the long element and \( \bar{N} = \theta N \). We recall that an explicit formula for \( c(\lambda) \) was determined by Gindikin and Karpelevich, see [14] or [36], p. 447. Furthermore, we define a measure \( \mu \) on \( \mathfrak{a}_c^+ \times B \) by

\[
d\mu(\lambda, b) = |c(\lambda)|^{-2} \, d\lambda \, db.
\]

We will also denote by \( d\mu \) the measure \( |c(\lambda)|^{-2} d\lambda \) on \( \mathfrak{a}_c^* \).

Let \( L^2_W(\mathfrak{a}^* \times B, d\mu_{W}) \) be the space of all \( F \in L^2(\mathfrak{a}^* \times B, d\mu_{W}) \) such that for all \( w \in W \) we have

\[
F(w\lambda, \cdot) = \mathcal{A}(w, -\lambda) F(\lambda, \cdot)
\]

in \( L^2(B) \), for almost all \( \lambda \in \mathfrak{a}^* \). Notice that, since \( \mathcal{A}(w, -\lambda) \) is an intertwining operator for each \( \lambda \in \mathfrak{a}^* \), this is an invariant subspace for the unitary action of \( G \) on \( L^2(\mathfrak{a}^* \times B, d\mu_{W}) \), defined by

\[
(g \cdot F)(\lambda, \cdot) = \pi_{-\lambda}(g) F(\lambda, \cdot).
\]

We recall the following theorem of [32] [33]:

**Theorem 1.1.** The Fourier transform is an intertwining unitary isomorphism

\[
L^2(X) \simeq L^2_W(\mathfrak{a}^* \times B, d\mu_{W}).
\]

Furthermore, if \( f \in C_c^\infty(X) \), then

\[
f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \int_B \hat{f}(\lambda, b)e_{\lambda,b}(x)\,d\mu(\lambda, b).
\]

For left \( K \)-invariant functions on \( X \), this is Harish-Chandra’s Plancherel theorem for \( X \). The Fourier transform \( \hat{f}_\lambda \) is then a constant function on \( B \), and the constant is

\[
\hat{f}_\lambda(b) = \int_K \hat{f}(\lambda, kM)\,dk = \int_X f(x)\varphi_{-\lambda}(x)\,dx
\]

(see (1.8)), the spherical Fourier transform of \( f \). By definition, the normalized intertwining operator maps \( p_{-\lambda} \) to \( p_{-w\lambda} \), and hence in this case the intertwining relation (1.11) is just \( F(w\lambda) = F(\lambda) \).
The proof of the spherical Plancherel theorem is given in [26] (see also [24]), but it depends on two conjectures, see p. 611-612. One conjecture was affirmed with [14] by the mentioned formula for $c(\lambda)$. The second conjecture is affirmed in [27], see p. 4. A simpler proof has later been given in [52], see also [36] p. 545.

As explained in [30] p. 50, Theorem 1.1 is proved by reduction to the spherical case. Because of the modified point of view invoking the representation theory, and since we have stated the relations (1.11) differently, we discuss some aspects of the proof. For more details, we refer to [32, 33, 37].

1.6. The intertwining relation. For the Fourier transformed function $F = \hat{f}$, the intertwining relation (1.11) is a direct consequence of the definition of $\hat{f}$:

$$A(w,-\lambda)f_\lambda = A(w,-\lambda)\pi_{-\lambda}(f)p_{-\lambda}$$
$$= \pi_{-w\lambda}(f)A(w,-\lambda)p_{-\lambda}$$
$$= \pi_{-w\lambda}(f)p_{-w\lambda}$$
$$= \hat{f}_{w\lambda}. \quad (1.14)$$

The equations (1.11) allow the following important reformulation (which is the original formulation of Helgason, see [33], p. 132)

$$\int_B e_{w\lambda,b}(x)F(w\lambda,b)\,db = \int_B e_{\lambda,b}(x)F(\lambda)\,db \quad (\forall x \in X). \quad (1.15)$$

It follows that the integral over $i\mathfrak{a}^*$ in (1.13) is $W$-invariant, and thus can be written as an integral over the Weyl chamber $a^+_\mathfrak{c}$.

The equivalence of (1.11) and (1.15) is an immediate consequence of the following lemma.

**Lemma 1.2.** Let $f, g \in L^2(B)$ and $w \in W$ be given. Then the following holds for every $\lambda \in a^+_\mathfrak{c}$ outside a locally finite set of complex hyperplanes. The relation

$$\int_B e_{w\lambda,b}(x)g(b)\,db = \int_B e_{\lambda,b}(x)f(b)\,db. \quad (1.16)$$

holds for all $x \in X$, if and only if

$$g = A(w,-\lambda)f. \quad (1.17)$$

**Proof.** The relation (1.16) can be written in terms of the Poisson transformation for $X$. Recall (see [32, 53]) that the Poisson transform is the operator $P_\lambda: H_{-\lambda} \to C^\infty(X)$ which is defined by

$$P_\lambda f(x) = \int_B f(b)e_{\lambda,b}(x)\,db$$

or equivalently (see [53] page 80-81),

$$P_\lambda f(gk) = \int_K f(gk)\,dk.$$

The latter formula shows that $P_\lambda$ is a $G$-equivariant operator for the left action. Notice that $P_{\lambda p_{-\lambda}}$ is exactly the spherical function $\varphi_\lambda$. It is now seen that (1.16) holds for all $x$ if and only if

$$P_{w\lambda}g = P_\lambda f. \quad (1.18)$$
On the other hand, the following can be seen to hold for all \( \lambda \), for which the normalized standard intertwining operator is non-singular,
\[
\mathcal{P}_{w, \lambda} \circ \mathcal{A}(w, -\lambda) = \mathcal{P}_{\lambda}.
\]
Indeed, it suffices to verify the identity for almost all \( \lambda \), so we can assume that \( \pi_{\lambda} \) is irreducible. It follows from the identity \( \varphi_{w, \lambda} = \varphi_{\lambda} \) that the two operators agree when applied to the element \( p_{-\lambda} \in H_{-\lambda} \). Since the operators are \( G \)-equivariant, they must then agree everywhere. This proves (1.19).

The equivalence of (1.18) and (1.17) follows immediately, for all \( \lambda \) such that \( \mathcal{P}_{w, \lambda} \) is injective. The latter is obviously true if \( \pi_{-w, \lambda} \) is irreducible. In fact it is known that the Poisson transformation is injective for all \( \lambda \), except on a singular set of hyperplanes (see [34] and [53] Thm 5.4.3).

\[\Box\]

1.7. The inversion formula. Let \( f \in C^\infty_c(X) \). Then, at the origin of \( X \), \( f(eK) = f^K(eK) \) where \( f^K(x) = \int_K f(kx) \, dk \). By the inversion formula of Harish-Chandra one can determine \( f(eK) \) through the expression
\[
f(eK) = \int_{\mathfrak{a}_+^*} \text{Tr}(\pi_{-\lambda}(f)) \, d\mu(\lambda).
\]

Applied to the function \( L_{g^{-1}}f \) it gives
\[
f(gK) = \int_{\mathfrak{a}_+^*} \text{Tr}(\pi_{-\lambda}(g^{-1})\pi_{-\lambda}(f)) \, d\mu(\lambda).
\]

Since \( \pi_{-\lambda}(f) \) annihilates the orthocomplemet of \( H^K_\lambda \),
\[
\text{Tr}(\pi_{-\lambda}(g^{-1})\pi_{-\lambda}(f)) = (\pi_{-\lambda}(g^{-1})\pi_{-\lambda}(f))p_{-\lambda, p_{-\lambda}}L^2(B),
\]
and as the representations are unitary this equals
\[
(\pi_{-\lambda}(f)p_{-\lambda, \pi_{-\lambda}(g)p_{-\lambda}}L^2(B) = \int_B \hat{f}(\lambda, b)e_{\lambda, b}(gK) \, db.
\]

It follows that
\[
f(x) = \int_{\mathfrak{a}_+^* \times B} \hat{f}(\lambda, b)e_{\lambda, b}(x) \, d\mu(\lambda, b).
\]

1.8. The \( L^2 \)-isomorphism. For \( f : G \to \mathbb{C} \) let \( f^\ast(g) = \overline{f(g^{-1})} \). Then, if \( \pi \) is an unitary representation, we have \( \pi(f^\ast) = \pi(f)^\ast \) and \( \pi(f \ast g) = \pi(f)\pi(g) \) for \( f, g \in L^1(G) \). Thus, for \( \lambda \in i\mathfrak{a}^* \) (so \( \pi_{\lambda} \) is unitary), we have
\[
\text{Tr}(\pi_{-\lambda}(f^\ast \ast f)) = (\pi_{-\lambda}(f)p_{-\lambda, \pi_{-\lambda}(f)p_{-\lambda}}) = \int_B |\hat{f}(\lambda, b)|^2 \, db.
\]

As \( ||f||^2 = f^\ast \ast f(eK) \) the equations (1.20) and (1.21) imply that
\[
||f||^2_{L^2(X)} = \int_{\mathfrak{a}_+^* \times B} |\hat{f}(\lambda, b)|^2 \, d\mu(\lambda, b) = \int_{\mathfrak{a}_+^*} ||\hat{f}_{\lambda}||^2 \, d\mu(\lambda).
\]

Thus \( L^2(X) \ni f \mapsto \hat{f}_{\lambda} \in L^2(B) \) sets up a unitary map into \( f_{\lambda_{\mathfrak{a}_+}^*}^{\oplus} H_{-\lambda, \pi_{-\lambda}} \, d\mu(\lambda) \). By Harish-Chandra’s Plancherel formula this is an unitary isomorphism on the level of \( K \)-invariant elements. We want to show that this implies it is onto. Let \( K \) be the orthogonal complement of the image in \( f_{\lambda_{\mathfrak{a}_+}^*}^{\oplus} H_{-\lambda} \, d\mu(\lambda) \). Then \( K \) is \( G \)-invariant and \( K^K = \{0\} \). By [37], Thm. 2.15, there exists measurable subset \( \Lambda \subset i\mathfrak{a}_+^* \) such that, as a representation of \( G \), we have \( K \simeq \int_{\Lambda}^{\oplus} H_{-\lambda} \, d\mu(\lambda) \). In particular,
Observation 1.3. Let \( K^K \simeq \int_{\Lambda} H^K d\mu(\lambda) \) and hence it follows from \( K^K = \{0\} \) that \( \mu(\Lambda) = 0 \). Hence \( K = \{0\} \) and

\[
(1.22) \quad (L^2(X), L) \simeq \int_{\mathbb{R}^+} (H_{-\lambda}, \pi_{-\lambda}) d\mu(\lambda)
\]

The isomorphism statement in \((1.12)\) follows easily. This completes the discussion of Theorem 1.1.

1.9. The heat equation. We illustrate the use of the Fourier theory by applying it to the heat equation on \( X \). Denote by \( L_X \) the Laplace operator on \( X \). It is known that \( e_{\lambda, b} \) is an eigenfunction for \( L_X \), for all \( \lambda, b \), with the eigenvalues \( -(|\lambda|^2 + |b|^2) \) for \( \lambda \in i\mathbb{R} \).

The heat equation on \( X \) refers to the Cauchy problem

\[
(1.23) \quad L_X u(x, t) = \partial_t u(x, t) \quad \text{and} \quad u(x, 0) = f(x)
\]

where \( f \in L^2(X) \). The problem has a unique solution, which is easily found by using the Fourier transform in Theorem 1.1. It is given by

\[
(1.24) \quad u(x, t) = \frac{1}{|W|} \int_{ia^* \times B} e^{-t(|\lambda|^2 + |b|^2)} \hat{f}(b, \lambda) e_{\lambda, b}(x) d\mu(\lambda, b).
\]

We denote by \( h_t \) the heat kernel on \( X \) defined by \( \hat{h}_t(\lambda, b) = e^{-t(|\lambda|^2 + |b|^2)} \) for all \( \lambda, b \), or equivalently,

\[
(1.25) \quad h_t(x) = \frac{1}{|W|} \int_{ia^*} e^{-t(|\lambda|^2 + |b|^2)} \varphi_{\lambda}(x) d\mu(\lambda)
\]

(see \cite{12}).

The convolution product of a function \( f \) on \( X \) with a \( K \)-invariant function \( h \) on \( X \) (both viewed as functions on \( G \)), is again a function on \( X \). Moreover, since \( \pi(f * h) = \pi(f) \pi(h) \), it follows that

\[
(f * h)^\wedge(\lambda, b) = \hat{f}(\lambda, b) \hat{h}(\lambda)
\]

It follows that we can write, for all \( f \in L^2(X) \),

\[
(1.26) \quad u(\cdot, t) = f * h_t.
\]

We define the heat transform as the map \( f \mapsto H_t f = f * h_t \) that associates the solution at time \( t > 0 \) to the initial function \( f \). It follows from the Fourier analysis that the heat transform is injective. Notice that the map \( H_t \) is a \( G \)-equivariant bounded operator from the space \( L^2(X) \) to itself. We equip the image \( \text{Im}(H_t) \), not with the norm of \( L^2(X) \), but with the norm that makes the heat transform a unitary isomorphism. As a consequence we obtain the following

Observation 1.3. Let \( d\mu_t(\lambda) = e^{2t(|\lambda|^2 + |b|^2)} d\mu(\lambda) \). Then, as a representation of \( G \), the image of the heat transform decomposes as

\[
(1.27) \quad (\text{Im}(H_t), L) \simeq \int_{\mathbb{R}^+} (H_{-\lambda}, \pi_{-\lambda}) d\mu_t(\lambda).
\]

Notice also the semigroup property \( H_t(H_s f) = H_{s+t} f \) which follows from \((1.26)\).
2. The complex crown of $X$ and the space $\mathcal{H}_X$

In this section we discuss some aspects of the interplay between complex geometry and the harmonic analysis on $X$. We introduce the crown, and we construct a $G$-invariant Hilbert space of holomorphic functions on it. The construction is motivated by the analysis in [17] where a similar construction was carried out on a subdomain $\text{Cr}(X)_j \subseteq \text{Cr}(X)$. There the purpose was to obtain a Hardy space realization of a part of the most continuous spectrum of a pseudo-Riemannian symmetric space $G/H_j$, which is embedded in the boundary of $\text{Cr}(X)_j$. In the present paper, our purpose is obtain a holomorphic model that carries all the representations in the Plancherel decomposition of $L^2(X)$.

The main references for Subsection 2.1 are [15, 45, 17]. For a nice overview article see [7].

2.1. The complex convexity theorem. Let

$$\Omega = \{ X \in \mathfrak{a} \mid (\forall \alpha \in \Delta) |\alpha(X)| < \pi/2 \}.$$  

For $\emptyset \neq \omega \subset \mathfrak{a}$ set $T(\omega) = \mathfrak{a} \oplus i\omega$, $A(\omega) = \exp T(\omega)$ and $X(\omega) = G \exp i\omega \cdot x_o = G \exp \omega \cdot x_o$. Then $X(\omega)$ is $G$-invariant by construction. The set $\text{Cr}(X) := X(\Omega)$, introduced in [1], is called the complex crown of $X$ or the Akhiezer-Gindikin domain. Note, that by Theorem 11.2 in [35] it follows that $\exp : T(\Omega) \rightarrow A(\Omega)$ is a diffeomorphism.

The crown is an open $G$-invariant complex submanifold of $X$ containing $X$ as a totally real submanifold. Furthermore, the $G$-action on $\text{Cr}(X)$ is proper, $\text{Cr}(X) \subset N_{\mathfrak{a}C} \cdot x_o$ and $\text{Cr}(X)$ is Stein. The importance of $\text{Cr}(X)$ for harmonic analysis comes from the fact, that it allows a holomorphic extension of every eigenfunction of the algebra of invariant differential operators. The main step in proving this, see [45] Proposition 1.3, is to show that each function $e_{\lambda, b}$ extends to a holomorphic function on $\text{Cr}(X)$ and then to use the affirmative solution to the Helgason conjecture [39].

The important complex convexity theorem of Gindikin-Krötz [15] is the inclusion $\subseteq$ of the following theorem. The equality was recently established by Krötz-Otto in [43].

**Theorem 2.1.** Let $Y \in \Omega$, then

$$a(\exp(iY)G) = A \exp(i \text{conv}(W \cdot Y)).$$  

Here $\text{conv}$ stands for convex hull. Note that (2.1) follows from Theorem 5.1 in [43] by $a(\exp iYg) = a(\exp iYk(g)) a(g)$.

2.2. The space $\mathcal{H}_X$. We define the following function $\omega : \mathfrak{a}^* \times \mathfrak{a} \rightarrow \mathbb{R}^+.$

$$\omega(i\nu, Y) := \frac{1}{|W|} \sum_{w \in W} e^{2\nu(wY)} \in \mathbb{R}^+.$$  

Furthermore, we define

$$\omega(\lambda) = \sup_{Y \in \Omega} \omega(\lambda, Y)$$

for $\lambda \in \mathfrak{a}^*$, and we define a measure $\mu_\omega$ on $\mathfrak{a}^* \times B$ by $d\mu_\omega(\lambda, b) = \omega(\lambda) d\mu(\lambda, b)$.

**Lemma 2.2.** Let $Y \in \Omega$. There exists $T \in \mathfrak{a}^+$, and for each $g \in G$ a constant $C > 0$, such that

$$|e_{i\nu, b}(g \exp iY \cdot x_o)\sqrt{\omega(i\nu)}| \leq Ce^{-\nu(T)}$$
for all $\nu \in \mathfrak{a}_+^*$ and $b \in B$. The element $T$ can be chosen locally uniformly with respect to $Y$, and the constant $C$ can be chosen locally uniformly with respect to $Y$ and $g$. In particular $(\lambda, b) \mapsto e_{\lambda, b}(z)/\omega(\lambda)$ is in $L^2(\mathfrak{a}_+^* \times B, d\mu_\omega)$ for all $z \in \Cr(X)$.

Proof. Let $b = kM \in B$. Using Theorem 2.1 we can write

$$a(\exp(-iY)g^{-1}k) = a \exp iZ$$

where $Z \in -\conv(W \cdot Y) \subset \Omega$ and $a \in A$ are unique and depend continuously on $Y$, $g$ and $b$. In particular, if $Q \subset G \times \Omega$ is compact, then there exists a constant $C = C_Q > 0$ such that $a^{-\rho} \leq C$ for all $(g, Y) \in Q$ and $b \in B$. Hence, for all $\lambda = i\nu \in \mathfrak{a}_+^*$:

$$|e_{\lambda, b}(g \exp iY \cdot x_o)| = |(a \exp(iZ))^{-i\nu - \rho}| \leq Ce^{\nu(Z)}.$$ 

Let $U \subset \Omega$ be a compact neighborhood of $Y$. Since $-\conv(W \cdot U)$ is compact and contained in $\Omega$, we can find an element $T \in \mathfrak{a}^+$ such that, $-\conv(W \cdot U) + T \subset \Omega$. Thus by (2.2)

$$\frac{1}{|W|}e^{2\nu(Z + T)} \leq \frac{1}{|W|} \sum_{w \in W} e^{2\nu(w(Z + T))} = \omega(i\nu, Z + T) \leq \omega(\lambda)$$

for all $Z \in -\conv(W \cdot U)$. Notice that $T$ was chosen independently of $g$ and $b$. Hence

$$\frac{1}{\omega(\lambda)} \leq |W|e^{-2\nu(Z)}e^{-2\nu(T)}.$$ 

It follows that there exists a constant $C > 0$ as claimed such that (2.3) holds.

As $|c(i\nu)|^{-2}$ has a polynomial growth on $\mathfrak{a}^*$ it follows that $|c(i\nu)|^{-2}e^{-\nu(T)}$ is bounded. The last statement follows as $\mathfrak{a}_+^* \ni \nu \mapsto e^{-\nu(T)}$ is integrable for all $T \in \mathfrak{a}^+$, see [9], p. 10.

Denote by $\mathcal{H}_X$ the space of holomorphic functions $F : \Cr(X) \to \mathbb{C}$ such that $F|_X \in L^2(X)$ and

$$\|F\|_{\mathcal{H}_X}^2 := \frac{1}{|W|} \int_{\mathfrak{a}_+^* \times B} |F|_X(\lambda, b)|^2 d\mu_\omega(\lambda, b) < \infty.$$ 

**Theorem 2.3.** The space $\mathcal{H}_X$ is a $G$-invariant Hilbert space. The action of $G$ is unitary and

$$\langle \mathcal{H}_X, L_\lambda \rangle \simeq \int_{\mathfrak{a}_+^*} (H_{-\lambda}, \pi_{-\lambda}) d\mu_\omega(\lambda).$$

Furthermore, the following holds:

1. Let $F \in \mathcal{H}_X$ and $f = F|_X$. Then

$$F(z) = \int \hat{f}(\lambda, b)e_{\lambda, b}(z) d\mu(\lambda, b)$$

for all $z \in \Cr(X)$.

2. For each $\varphi \in L^2(\mathfrak{a}_+^* \times B, d\mu_\omega)$ the function defined by

$$F(z) = \int \varphi(\lambda, b)e_{\lambda, b}(z) d\mu(\lambda, b)$$

belongs to $\mathcal{H}_X$ and has $F|_X = \varphi$. 


(3) The point evaluation maps
\[ \mathcal{H}_X \ni F \mapsto F(z) \in \mathbb{C} \]
are continuous for all \( z \in \text{Cr}(X) \).

(4) The reproducing kernel of \( \mathcal{H}(X) \) is given by
\[ K(z, w) = \int_{i\mathbb{A}_+ \times B} e_{\lambda, b}(z) e_{-\lambda, b}(\sigma(w)) d\mu(\lambda, b) \]
where \( \sigma \) is the conjugation introduced in Subsection 1.1.

Proof. We first establish the property (2). For \( z \in \text{Cr}(X) \) let \( f_z(\lambda, b) = e_{\lambda, b}(z) \omega(\lambda)^{-1} \). Assume that \( \varphi \in L^2(i\mathbb{A}_+ \times B, d\mu_\omega) \). Then by Cauchy-Schwartz and the last part of Lemma 2.2 we have
\[ \int_{i\mathbb{A}_+ \times B} |\varphi(\lambda, b)e_{\lambda, b}(z)| d\mu(\lambda, b) = \int_{i\mathbb{A}_+ \times B} |\varphi(\lambda, b) f_z(\lambda, b)(z) \omega(\lambda)| d\mu_\omega(\lambda, b) \]
\[ \leq \| \varphi \|_{L^2(d\mu_\omega)} \| f_z \|_{L^2(d\mu_\omega)} < \infty \]
Recall that there is a compact neighborhood \( U \) of \( z \) such that \( \| f_z \| \) is uniformly bounded on \( U \). It follows that the function
\[ \text{Cr}(X) \ni z \mapsto G_{\varphi}(z) := \int \varphi(\lambda, b)e_{\lambda, b}(z) d\mu(\lambda, b) \in \mathbb{C} \]
exists and is holomorphic. Furthermore,
\[ |G_{\varphi}(z)| \leq C \| \varphi \|_{L^2(d\mu_\omega)} \]
for all \( \varphi \in L^2(i\mathbb{A}_+ \times B, d\mu_\omega) \) with \( C \) depending locally uniformly on \( z \). Now part (2) follows.

Let \( F \in \mathcal{H}_X \) and \( f = F|_X \). Then it follows that
\[ G_f(z) = \int \hat{f}(\lambda, b)e_{\lambda, b}(z) d\mu(\lambda, b) \]
is holomorphic and satisfies \( G|_X = F|_X \). Hence \( G = F \). This proves part (1), and the part (3) now follows from (2.4).

It follows from part (3) that there exists a reproducing kernel \( K(z, w) = K_w(z) \) for \( \mathcal{H}_X \). Let \( F \in \mathcal{H}_X \). Then on the one hand
\[ F(w) = (F, K_w)_{\mathcal{H}_X} = \int F|_X(\lambda, b)\overline{K_w|_X(\lambda, b)} \omega(\lambda) d\mu(\lambda, b), \]
and on the other, using part (1)
\[ F(w) = \int \overline{F|_X(\lambda, b)e_{\lambda, b}(w)} d\mu(\lambda, b). \]
It follows from part (2) that \( \overline{F|_X(\lambda, b)} \) can be any function in \( L^2(i\mathbb{A}_+ \times B, d\mu_\omega) \).
Since \( e_{\lambda, b}/\omega \) also belongs to this space by Lemma 2.2, it follows that
\[ K_w|_X(\lambda, b) = e_{\lambda, b}(w)/\omega(\lambda) = e_{-\lambda, b}(\sigma(w))/\omega(\lambda). \]
Thus
\[K(z, w) = (K_w, K_z)_{\mathcal{H}_X} = \int \frac{e^{-\lambda b(\sigma(w))}e_{\lambda b}(z)}{\omega(\lambda)^2} \omega(\lambda) d\mu = \int \frac{e_{\lambda b}(z)e^{-\lambda b(\sigma(w))}}{\omega(\lambda)} d\mu .\]

It suffices to establish the final formula in (4) for \(z, w \in X\). Moreover, by \(G\)-invariance of the kernel we may assume \(w = e\). Then the formula follows immediately from \(1.8\) (see also Theorem 1.1 in \[37\], p. 224). □

Remark 2.4. It is clear that the space \(\mathcal{H}_X^K\) also is a reproducing kernel Hilbert space. The reproducing kernel is obtained by averaging over \(K\),
\[K_{\mathcal{H}_X^K}(z, w) = \int_{a^+} \frac{\varphi_{\lambda}(z)\varphi_{-\lambda}(\sigma(w))}{\omega(\lambda)} d\mu(\lambda),\]
as stated in \[18\], Proposition 8.7.

3. The Image of the Segal-Bargmann Transform

In this section we introduce the Segal-Bargmann transform on \(X\) and give two characterizations of its image as a \(G\)-invariant Hilbert space of holomorphic functions on \(\text{Cr}(X)\), both different from the one given in \[46\]. The first characterization is a natural extension of Observation 1.27. The second characterization uses the normalized Radon transform \(\Lambda\) from \[32\]. A similar result was proved in \[49\] for the Segal-Bargmann transform related to positive multiplicity functions, but only for the \(K\)-spherical case.

In the last part of the section we show that the normalized Radon transform allows an analytic extension as a unitary isomorphism of \(\mathcal{H}_X^K\) into a function space over a domain in the complexified horocycle space \(\Xi_C\). A similar construction for the Hardy space on a subdomain \(\text{Cr}(X)_j\) was given in \[18\].

3.1. The Segal-Bargmann transform on \(X\). The description of the heat transform in Subsection 1.9 implies that the heat kernel \(h_t\) as well as every function \(H_t f, f \in L^2(X)\), extends to a holomorphic function on \(\text{Cr}(X)\) (see also \[45\], Prop. 6.1). In fact, it follows from Theorem \[23\] (2), that these extensions belong to \(\mathcal{H}_X^K\). We shall denote the holomorphic extensions by the same symbols. The map \(H_t : L^2(X) \rightarrow \mathcal{H}_X \subset \mathcal{O}(\text{Cr}(X))\) is the Segal-Bargmann transform. Note, \(\|H_t f\|_{\mathcal{H}_X} \leq \|f\|_{L^2}\).

Our first description of the image of \(H_t\) in \(\mathcal{O}(\text{Cr}(X))\) is given by the following.

**Theorem 3.1.** The image \(H_t(L^2(X))\) of the Segal-Bargmann transform is the space \(\mathcal{O}_t(\text{Cr}(X))\) of holomorphic functions \(F\) on \(\text{Cr}(X)\) such that \(F|_X \in L^2(X)\) and
\[\|F\|_t^2 := \int_{a^+ \times B} |\widehat{F}|^2 \mu_t(\lambda, b) < \infty .\]

Here \(\mu_t\) is the measure defined in Observation 1.3

Furthermore, point evaluation is continuous on \(\mathcal{O}_t(\text{Cr}(X))\), and the reproducing kernel is given by
\[K_t(z, w) = h_{2t}(\sigma(w)^{-1}z) .\]
Proof. The first statement follows from Observation 1.3 and Theorem 2.3. The reproducing kernel is obtained by a standard argument, using the semigroup property of the convolution with $h_t$. Let $F = f * h_t$ and $z \in \text{Cr}(X)$. Note that $L_2 h_t$ is well defined for $z \in \text{Cr}(X)$ as $h_t$ is $K$-bi-invariant. We have
\[
F(w) = f * h_t(w) = (f, L_{\sigma(w)} h_t)_{L^2(X)} = (H_t f, H_t (L_{\sigma(w)} h_t))_{\mathcal{O}_z}.
\]

Hence point evaluation is continuous and
\[
K_w(z) = H_t(L_{\sigma(w)} h_t)(z) = (L_{\sigma(w)} h_t) * h_t(z) = h_{2t}(\sigma(w)^{-1}z).
\]

3.2. The Radon transform on $X$ and the Segal-Bargmann transform. Let $\Xi = G/MN$ and put $\xi_o = eMN \in \Xi$. By the Iwasawa decomposition it follows that
\[
B \times A \simeq \Xi \quad (kM,a) \mapsto ka \cdot \xi_o
\]
is a diffeomorphism. A subset $\xi \subset X$ is said to be a horocycle if there exists $x \in X$ and $g \in G$ such that $\xi = gNx$. It is well known, see [29, 32], that the map $g\xi_0 \mapsto gN\xi_0$ is a bijection of $\Xi$ onto the set of horocycles. Using this identification the space of horocycles becomes an analytic manifold with a transitive $G$ action.

The Radon transform $R(f)(g \cdot \xi_o) = \int_X f(gn \cdot \xi_o) \, dn$ is a $G$-intertwining operator $C^\infty_c(X) \to C^\infty(\Xi)$. The importance of this observation comes from the fact that the regular representation of $G$ on $L^2(\Xi)$ is much easier to decompose than that on $L^2(X)$. As induction commutes with direct integral, induction by stages shows that
\[
(L^2(\Xi), L) \simeq \int_{\mathfrak{a}^*} (H_\lambda, \pi_\lambda) \, d\lambda,
\]
see [17], p. 284 and 287. In fact, let $\chi_\lambda(man) = a^\lambda$. Denote by $\epsilon$ the trivial representation of $MN$ and by $L_\Xi$ the regular representation of $G$ on $L^2(\Xi)$. As $MAN/MN \simeq A$ and $MN$ acts trivially on $L^2(MAN/MN) \simeq L^2(A)$, it follows that
\[
L_\Xi \simeq \text{ ind}^G_{MAN} \epsilon \\
\simeq \text{ ind}^G_{MAN} \text{ ind}^{MAN} \epsilon \\
\simeq \text{ ind}^G_{MAN} \int_{\mathfrak{a}^*} \chi_\lambda \, d\lambda \\
\simeq \int_{\mathfrak{a}^*} \pi_\lambda \, d\lambda.
\]

Equation (1.2) implies that (up to a constant) the $G$-invariant measure on $\Xi$ is given by
\[
\int_{\Xi} f(\xi) \, d\xi = \int_B \int_A f(ka \cdot \xi_o) a^{2\rho} \, dak.
\]

It follows that
\[
L^2(\Xi) \ni f \mapsto [(kM,a) \mapsto a^\rho f(ka \cdot \xi_o)] \in L^2(B \times A)
\]
is a unitary isomorphism. We also note that by (1.2),
\[
\hat{f}(\lambda, kM) = \int_{AN} f(kan \cdot x_o) a^{-\lambda + \rho} \, dnda = \int_A a^\rho R(f)(kM,a) a^{-\lambda} \, da
\]
for \( f \in C_c(X) \). It is therefore natural to define a \( \rho \)-twisted Radon transform by
\[
(3.4) \quad \mathcal{R}_\rho(f)(b, a) = a^\rho \mathcal{R}(f)(b, a).
\]
Notice that this is then an intertwining operator for the regular action of \( G \) on functions over \( X \) and \( \Xi \), respectively, when the action on functions over \( \Xi \) is transferred to a \( \rho \)-twisted action on functions over \( B \times A \) through (3.2), that is,
\[
(3.5) \quad (g \cdot \phi)(b, a) := a(g^{-1}b)^{-\rho} \phi(k(g^{-1}b), a(g^{-1}b)a).
\]

Identifying \( L^2(B \times A) \) with \( L^2(A, L^2(B)) \) in a natural way, equation (3.3) now reads
\[
(3.6) \quad \hat{\mathcal{F}}^A = \mathcal{F}_A(\mathcal{R}_\rho(f))\lambda).
\]
Note, if \( f \) is \( K \)-invariant the \( \rho \)-twisted Radon transform reduces to the Abel transform \( f \mapsto F_f \) introduced in [26], p. 261, and conjectured to be injective. The proof of that conjecture was the final step towards the Plancherel formula, see [27], p. 4.

By Theorem 1.1 and (3.6) we have the following commutative diagram
\[
\begin{array}{ccc}
L^2(X) & \xrightarrow{\mathcal{F}_A} & L^2_{\mathcal{H}_d}(B \times A, d\tau) \\
\mathcal{R}_\rho \downarrow & & \mathcal{R}_\rho \\
L^2_{\mathcal{H}_d}(B \times A^*, |W|^{-1}d\mu) & \xrightarrow{\mathcal{F}_A \times \mathcal{F}_A} & L^2_{\mathcal{H}_d}(B \times A^*, |W|^{-1}d\lambda)
\end{array}
\]

The vertical maps and the lower horizontal map are unitary isomorphisms. It follows that the linear operator
\[
(3.9) \quad \Lambda := (\mathcal{F}_A \times \mathcal{F}_A) \circ \mathcal{R}_\rho : L^2(X) \to L^2_{\mathcal{H}_d}(B \times A, d\tau)
\]
is an intertwining operator. By Lemma 3.3 in [32], p. 42, we know that \( \Lambda \) is an intertwining operator. Here the action of \( G \) on \( L^2(B \times A, d\tau) \) is the \( \rho \)-twisted action defined through the identification of \( B \times A \) with \( \Xi \) (see (3.3)).

By Theorem 5.3 in [29] it follows that for each invariant differential operator \( D \) on \( X \), there exists a differential operator \( \tilde{D} \) on \( A \) such that \( \Lambda(Df)(b, a) = \tilde{D}_a \Lambda(f)(b, a) \).
Here the subscript indicates that $\hat{D}$ acts on the $a$ variable only. In particular, this applies with $D$ equal to the Laplace operator $L_X$. Tracing the commutative diagram \[\text{(3.10)}\] and using that $L_X \tilde{f}(\cdot, \lambda) = -(|\lambda|^2 + |\rho|^2)\tilde{f}(\cdot, \lambda)$, $\lambda \in \mathfrak{a}^*$, it is easily seen that $L_X = L_A - |\rho|^2$, i.e., for $f$ sufficiently smooth
\[
\Lambda(L_X f) = (L_A - |\rho|^2)\Lambda(f).
\]

Let $r = \dim A$.

**Lemma 3.2.** Let $f \in L^2(X)$. Then $e^{t|\rho|^2}\Lambda(H_t f)$ solves the heat equation on $A$ with initial value $\Lambda(f) \in L^2(B)$. In particular, the map $a \ni X \mapsto \Lambda(H_t f)(\cdot, \exp X) \in L^2(B)$ extends to a holomorphic function on $a_C$, again denoted by $\Lambda(H_t f)$ such that
\[
\int_{B \times a_C} |e^{t|\rho|^2}\Lambda(H_t f)(b, \exp(X + iY))|^2 e^{-|Y|^2/2t} \, db dY < \infty.
\]

**Proof.** See the proof of Lemma 2.5 in \cite{49}. \[\square\]

Let $\mathcal{F}_{W,t}(B \times a_C)$ denote the space of holomorphic functions on $B \times a_C$ such that
\[
a_C \ni Z \mapsto F(\cdot, Z) \in L^2(B)
\]

is holomorphic with $F \circ (\cdot, \log) \in L^2_W(B \times A)$, and satisfies
\[
\|F\|_t^2 = \int_{a_C} \int_{B} |F(b, X + iY)|^2 e^{-|Y|^2/2t} \, dX dY
\]

Thus, $\mathcal{F}_{W,t}(B \times a_C)$ is analog to a $L^2(B)$ valued Fock space describing the image of the Segal-Bargmann transform on the Euclidean space $a$ with the addition of the Weyl group relations derived from \[\text{(3.4)}\]. For $t > 0$ define $\Lambda_t : \mathcal{O}_t(\text{Cr}(X)) \to \mathcal{F}_{W,t}(B \times a_C)$ in the following way. Let $F \in \mathcal{O}_t(\text{Cr}(X))$. By Theorem 3.1 there exists a unique $f \in L^2(X)$ such that $F|X = H_t f$. Let $\Lambda_t(F)$ be the holomorphic extension of $e^{t|\rho|^2}\Lambda(H_t f)$ given by Lemma 3.2. By the same lemma the Weyl group relations are satisfied and $\|\Lambda_t(F)\|_t < \infty$. Thus $\Lambda_t(F) \in \mathcal{F}_{W,t}(B \times a_C)$. The following theorem gives an alternative description of the Hilbert space $\mathcal{O}_t(\text{Cr}(X))$.

**Theorem 3.3.** The map $\Lambda_t : \mathcal{O}_t(\text{Cr}(X)) \to \mathcal{F}_{W,t}(B \times a_C)$ is an unitary isomorphism. Furthermore, let $F \in \mathcal{O}_t(\text{Cr}(X))$. Define $f \in L^2(X)$ by applying $\Lambda^*$ to the function on $B \times A$ given by
\[
(b, a) \mapsto (4\pi t)^{-r/2} \lim_{R \to \infty} \int_{|Y| \leq R} \Lambda_t(F)(b, \log a + iY)e^{-|Y|^2/4t} \, dY.
\]

Then $H_t(f) = F$.

**Proof.** The proof is a simple adaption of the standard argument for $\mathbb{R}^r$, as described in \cite{21}, to the $L^2(B)$-valued case. \[\square\]

3.3. Holomorphic properties of the normalized Radon transform. Consider as before $X$ as a subset of $X_C = G_C/K_C$. A complex horocycle in $X_C$ is a set of the form $gN_C x_o \subset X_C$ for some $g \in G_C$, see \cite{18}. Let $\xi^C_o = N_C \cdot x_o$, then
\[
\Xi_C = \{g \cdot \xi^C_o \subset G_C/K_C \mid g \in G_C\} \simeq G_C/M_C N_C.
\]
is the set of complex horocycles. The map
\[ \Xi \ni g \cdot \xi_o \mapsto g \cdot \xi_o^C \in \Xi_C, \quad g \in G \]
is well defined and injective.

Define
\[ \Xi(\Omega) = G \exp i\Omega \cdot \xi_o = K A(\Omega) \cdot \xi_o \subset \Xi_C. \]

Then \( \Xi(\Omega) \simeq B \times A(\Omega) \) is a \( G \)-invariant CR-submanifold of \( \Xi_C \). Let \( \mathcal{H}_\Xi \) be the space of function \( F : \Xi(\Omega) \to \mathbb{C} \) such that the map \( A(\Omega) \ni z \mapsto F(\cdot, z) = F_z \in L^2(B) \) is holomorphic and for each fixed \( Y \in \Omega \) the function \( (b, a) \mapsto F(b, a \exp iY) \) is in \( L^2(B \times A, d\tau) \) (Lemma 3.4). By the explicit formula for \( \Lambda(\phi) \) and \( \Lambda(F|_X)(\cdot, a) \) extends to a holomorphic \( L^2(B) \)-valued function on \( A(\Omega) \), also denoted by \( \Lambda(F|_X) \), which belongs to \( \mathcal{H}_\Xi \) and satisfies
\[ \| \Lambda(F|_X) \|_{\mathcal{H}_\Xi} = \| F \|_{\mathcal{H}_X}. \]

Moreover, the map \( f \mapsto \Lambda(F|_X) \) is intertwining for the actions of \( G \).

**Proof.** Let \( \phi = F|_X \). It follows by Lemma 3.2 that \( a \mapsto \Lambda(\phi)(\cdot, a) \) extends to a holomorphic \( L^2(B) \)-valued function on \( A(\Omega) \). In fact
\[ \Lambda(\phi)(\cdot, \exp(X+iY)) = e^{-t|\rho|^2} \Lambda(f) * A h^A(\cdot, \exp(X+iY)) \]
where \( h^A_t(\exp X) = (4\pi t)^{-\tau/2} e^{-|X|^2/4t} \) is the heat kernel on \( A \) and the convolution is on the abelian group \( A \). For \( Y \in \Omega \) the function \( g_Y : (b, a) \mapsto \Lambda(\phi)(b, a \exp iY) \) is in \( L^2(B \times A, d\tau) \). By the explicit formula for \( h^A_t \) there exists a positive constant \( C > 0 \) such that for \( a \in A \) and \( Y \in \Omega \)
\[ \| g_Y(\cdot, \exp X) \|_{L^2(B)} \leq C e^{-(|X|-1)^2/4t} e^{Y^2/4t} \leq C_1 e^{-(|X|-1)^2/4t} \]
where
\[ C_1 = C \sup_{Y \in \Omega} e^{Y^2/4t}. \]

Let \( g_{b,Y}(a) = g_Y(b, a) \). The estimate (3.13) allows us to change the path of integration to derive
\[ \mathcal{F}_A(g_{b,Y})(\lambda) = \mathcal{F}_A(g_{b,0})(\lambda) e^{i\lambda Y}. \]
Thus the integral over $B \times A$ in (3.11) is

$$
\int_B \int_A |g_Y(b, a)|^2 \, dadb = \int_B \int_{a^*} |\mathcal{F}_A(g_b, \lambda)(\lambda)|^2 \, d\lambda \, db
$$

$$
= \int_{a^*} \int_B |\mathcal{F}_A(g_b, 0)(\lambda)|^2 e^{2i\lambda(Y)} \, db \, d\lambda
$$

$$
= \int_{a^*} \int_B |\mathcal{F}_A(\Lambda(\varphi))(b, \lambda)|^2 \, db \, e^{2i\lambda(X)} \, d\lambda
$$

(3.14)

$$
= \int_{B \times a^*} |\hat{\varphi}(b, \lambda)|^2 e^{2i\lambda(Y)} \frac{db \, d\lambda}{|c(\lambda)|^2}
$$

where (3.14) follows from the definition of $\Lambda$ in (3.8). According to (1.14) we have $\mathcal{A}(w, -\lambda)\hat{\varphi}(\cdot, \lambda) = \hat{\varphi}(\cdot, w\lambda)$. Hence $\int_B |\hat{\varphi}(b, \lambda)|^2 \, db$ is $W$-invariant as the intertwining operator $\mathcal{A}(w, -\lambda)$ is unitary. Summing over the Weyl group and using that $|c(\lambda)|^{-2}$ is $W$-invariant, we obtain

$$
\int_{B \times a^*} |\hat{\varphi}(b, \lambda)|^2 e^{2i\lambda(Y)} \frac{db \, d\lambda}{|c(\lambda)|^2} = \int_{B \times a^*} |\hat{\varphi}(b, \lambda)|^2 \omega(\lambda, -Y) \, d\mu(b, \lambda).
$$

Divide by $|W|$ and take the supremum over $Y \in \Omega = -\Omega$ to get that the norms are equal.

The intertwining property of the map follows from the corresponding properties for $H_t$ and for $\Lambda$ on $B \times A$. The latter property was remarked below (3.9). □

Let $F \in \mathcal{H}_X$ and $\varphi = F|_X$. Let $t_n \to 0$, $t_n > 0$, and view $\varphi_n := H_{t_n} \varphi$ as an element of $\mathcal{H}_X$. Then

$$
\|\varphi_n - F\|_{\mathcal{H}_X}^2 = \int_{B \times a^*} |e^{-t_n(\ell^2 + |\varphi|^2)} - 1|^2 |\hat{\varphi}(b, \lambda)|^2 \omega(\lambda) \, d\mu(b, \lambda).
$$

As $(b, \lambda) \mapsto |\hat{\varphi}(b, \lambda)|^2 \omega(\lambda)$ is integrable with respect to $d\mu$ it follows by the Lebesque dominant convergence theorem that $\lim_n \varphi_n \to F$ in $\mathcal{H}_X$. By Lemma (3.4) it follows that $\lim_n \Lambda(H_{t_n} \varphi)$ exists in $\mathcal{H}_\Xi$ and is independent of the sequence $t_n$. Define $\hat{\Lambda} : \mathcal{H}_X \to \mathcal{H}_\Xi$ by

$$
\hat{\Lambda}(F) = \lim_{n \to \infty} \Lambda(H_{t_n} \varphi).
$$

**Theorem 3.5.** The map $\hat{\Lambda} : \mathcal{H}_X \to \mathcal{H}_\Xi$ is an unitary intertwining isomorphism.

**Proof.** We only have to show that $\hat{\Lambda}$ is surjective. Let $F \in \mathcal{H}_\Xi$. Then $F|_{B \times A} \in L^2_W(B \times A, d\tau)$. Define $f = \Lambda^*(F|_{B \times A})$. Then the argument in the proof of Lemma 3.4 shows that

$$
|W|^{-1} \int_{B \times a^*} |\hat{f}(b, \lambda)|^2 \omega(\lambda) \, d\mu(b, \lambda) = \|F\|_{\mathcal{H}_\Xi}^2 < \infty.
$$

In particular it follows that $f$ extends to a holomorphic function on $\text{Cr}(X)$, denoted by $\varphi$, and $\varphi \in \mathcal{H}_X$. By construction we have $\Lambda(\varphi|_X) = F|_{B \times a^*}$. By construction it follows now easily that $\hat{\Lambda}(\varphi) = F$. □

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