RELATIVE HYPERBOLICITY AND SIMILAR PROPERTIES 
OF ONE-GENERATOR ONE-RELATOR RELATIVE PRESENTATIONS WITH POWERED UNIMODULAR RELATOR

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A group obtained from a nontrivial group by adding one generator and one relator which is a proper power of a word in which the exponent-sum of the additional generator is one contains the free square of the initial group and almost always (with one obvious exception) contains a non-abelian free subgroup. If the initial group is involution-free or the relator is at least third power, then the obtained group is SQ-universal and relatively hyperbolic with respect to the initial group.

1. Introduction

Let $G$ be a torsion-free group and let a group $\mathring{G}$ be obtained from the group $G$ by adding one generator and one unimodular relator, i.e., a relator in which the exponent sum of the new generator is one:

$$\mathring{G} = \langle G, t \mid w = 1 \rangle \overset{\text{def}}{=} (G * \langle t \rangle)_{\infty} / \langle [w] \rangle,$$

where $w \equiv g_1 t^{\varepsilon_1} \ldots g_n t^{\varepsilon_n}$, $g_i \in G$, $\varepsilon_i \in \mathbb{Z}$, and $\sum \varepsilon_i = 1$.

It is known that a significant part of one-relator group theory extends to such unimodular one-relator relative presentations. In particular:
- $G$ embeds (naturally) into $\mathring{G}$ [Kl93] (see also [FeR96]);
- $\mathring{G}$ is torsion-free [FoR05];
- $\mathring{G}$ is not simple if it does not coincide with $G$ [Kl05];
- $\mathring{G}$ almost always (with some known exceptions) contains a non-abelian free subgroup [Kl07];
- $\mathring{G}$ is SQ-universal if $G$ decomposes nontrivially into a free product [Kl06b];
- the centre of $\mathring{G}$ is almost always (with some known exceptions) trivial [Kl09].

Some generalisations of these results to relative presentations with several additional generators can be found in [Kl09], [Kl07], [Kl06a], and [Kl06b].

It is well known that one-relator groups with powered relator are more similar to free groups than arbitrary one-relator groups. In particular, Newman’s theorem [New68] (see also [LS80]) (reformulated in the modern language) says that one-relator groups are hyperbolic if the relator is a proper power. The following recent result is a partial generalisation of Newman’s theorem.

Le Thi Giang’s theorem [Le09]. If a group $G$ is torsion-free, a word $w \in G * \langle t \rangle_{\infty}$ is unimodular, and $k \geq 2$, then the group

$$\mathring{G} = \langle G, t \mid w^k = 1 \rangle \overset{\text{def}}{=} G * \langle t \rangle_{\infty} / \langle [w^k] \rangle$$

is relatively hyperbolic (in the sense of Osin) with respect to $G$, i.e. presentation $(*)$ satisfies a linear isoperimetric inequality: there exists a constant $C > 0$ such that any word $u$ in the alphabet $G \cup \{t^{\pm 1}\}$ representing the identity element of $\mathring{G}$ decomposes in $G * \langle t \rangle_{\infty}$ into a product of at most $C |u|$ conjugates of $w^{\pm k}$.

Henceforth, the symbol $|u|$ denotes the number of letters $t^{\pm 1}$ in the word $u$.

Relatively hyperbolic groups have many good properties. For example, they are SQ-universal (apart from some obvious exceptions) [AMO07], the word [Far98] and conjugacy [Bum04] problems are solvable in such groups (under some natural restrictions). The same is true for many other algorithmic problems. More details about relatively hyperbolic groups can be found in book [Os06].

It turns out that the torsion-freeness condition in Le Thi Giang’s theorem can be replaced by the absence of only order-two elements. Presently, the following theorem is the unique result about unimodular relative presentations in which torsion-freeness condition is weakened to the absence of small-order elements.

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*) However, the natural mapping $G \to \mathring{G}$ is never surjective, except in the case when $w \equiv gt$ [CR01].
Theorem. If a word \( w \in G * \langle t \rangle_\infty \) is unimodular and \( k \geq 2 \), then the group \( \tilde{G} \) defined by relative presentation (\( * \)) contains \( G \) as a (naturally embedded) subgroup.\(^{**} \) and \( \langle G, G^t \rangle = G * G^t \) in \( \tilde{G} \).

If the group \( G \) is involution-free or \( k \geq 3 \), then \( \tilde{G} \) is relatively hyperbolic with respect to \( G \).

Example 1 [Le09]. The group \( \tilde{G} = \langle g, t \mid g^3 = 1, [g, t]^3 = 1 \rangle \) is not hyperbolic (in particular, it is not relatively hyperbolic with respect to its finite subgroup \( G = \langle g \rangle_3 \)), because the subgroup \( \langle a^2 a, a a^3 \rangle \) is a free abelian group of rank two. This example shows that the unimodularity condition cannot be omitted from the theorem.

Example 2. The Baumslag–Solitar group \( \tilde{G} = \langle g, t \mid t^6 = t^2 \rangle \) is not hyperbolic (in particular it is not relatively hyperbolic with respect to its cyclic subgroup \( G = \langle g \rangle \)), because the centraliser of the element \( t \) is a noncyclic locally cyclic group \( \langle t^{s^{-1}}, t^{s^{-2}}, \ldots \rangle \). This example shows that the condition \( k \geq 2 \) cannot be omitted from the theorem.

Question. Can the involution-freeness condition be omitted from the theorem for \( k = 2 \)?

We conjecture that the answer is no.

Applying the known facts mentioned above about relatively hyperbolic groups, we obtain, e.g., the following corollary.

Corollary 1. Suppose that a word \( w \) is unimodular and either \( k \geq 3 \) or \( k \geq 2 \) and \( G \) is involution-free. Then
1) if \( G \) is nontrivial, then \( \tilde{G} \) is SQ-universal, i.e. any countable group embeds into a quotient of \( \tilde{G} \);
2) the word and conjugacy problems are solvable in \( \tilde{G} \) if the corresponding problems are solvable in \( G \) and it is finitely generated.

Proof. The second assertion follows immediately from the theorem and the results of Farb [Far98] and Bumagina [Bum04] mentioned above.

To prove the first assertion, it suffices to apply the Arzhantseva–Minasyan–Osin theorem [AMO07] mentioned above, which says that a group relatively hyperbolic with respect to its proper subgroup is either SQ-universal or virtually cyclic.

The group \( \tilde{G} \) is relatively hyperbolic with respect to \( G \) by the theorem. The subgroup \( G \subseteq \tilde{G} \) is proper, because \( \tilde{G}/\langle \langle G \rangle \rangle = \langle t \mid t^k = 1 \rangle \) is the cyclic group of order \( k \geq 2 \). Finally, \( \tilde{G} \) is not virtually cyclic, since according to the theorem \( \tilde{G} \) contains the free square of \( G \) and it is well-known that the free square of a group of order larger than two (in particular, any nontrivial group without involutions) is not virtually cyclic. The remaining case \( G \simeq \mathbb{Z}_2 \) and \( k \geq 3 \) is covered by the Baumslag–Morgan–Shalen theorem [BMS87] implying that, in this case, \( \tilde{G} \) contains a non-abelian free subgroup and, therefore, is not virtually cyclic.

Corollary 2. If a word \( w \) is unimodular, \( G \neq \{1\} \), and \( k \geq 2 \), then \( \tilde{G} \) contains a non-abelian free subgroup, except in the case where \( G \) consists of two elements, \( k = 2 \), and \( w \) is conjugate in \( G * \langle t \rangle_\infty \) to a word of the form \( g t \), where \( g \in G \) (in this case, \( \tilde{G} \) is infinite dihedral).

Proof. According to the theorem, \( \tilde{G} \) contains the free square of \( G \), which contains a non-abelian free group, except in the case where \( G \simeq \mathbb{Z}_2 \). In this exceptional case, if \( k \geq 3 \), then the presence of a non-abelian free subgroup follows from Corollary 1. If \( k = 2 \), then the generalised triangle group \( \tilde{G} = \langle g, t \mid g^2 = w^2 = 1 \rangle \) satisfies the conditions of a theorem of Howie [How98], which (in particular) describes generalised triangle groups of such form without free subgroups.

Remark. Our proof shows also that relative presentation (\( * \)) is aspherical (if \( w \) is unimodular and \( k \geq 2 \)). In particular, this means (see [FoR05]) that each finite subgroup of \( \tilde{G} \) is conjugate to either a subgroup of \( G \) or a subgroup of the cyclic group \( \langle w \rangle \).

If we do not assume unimodularity condition in presentation (\( * \)) and suppose only that \( w \) is not conjugate to elements of \( G \) in \( G * \langle t \rangle_\infty \), then, as is known, we have, e.g., the following:

- the group \( G \) embeds naturally into \( \tilde{G} \) if either \( G \) is locally indicable [B84], or \( G \) is cyclic and \( k \geq 2 \) [BMS87], [Boy88], or \( k \geq 4 \) [How90], or \( k \geq 3 \) and \( G \) is involution-free [DuH92];
- \( \tilde{G} \) is relatively hyperbolic with respect to \( G \) if either \( G \) is locally indicable and \( k \geq 2 \) [DuH91] or \( G \) is involution-free and \( k \geq 4 \) [DuH93].

A survey of results on one-relator relative presentations with a powered relator can be found in [DuH93] and [FiR99].

Our approach to the proof of the theorem, as well as Le Thi Giang’s approach, is based on the use of a standard algebraic trick (Section 2) and geometric technique: Howie’s diagrams (Section 4) and car crashes (Sections 6 and 7). The difference is that we use the crashes in combination with the weight test, i.e. the combinatorial Gauss–Bonnet formula (Section 3). Actually, the major part of the theorem is proven (in Section 5) without any “automobile technique”. The cars are needed only to prove the relative hyperbolicity when \( k = 2 \) and \( G \) is involution-free (Section 8).

\( ** \) In other words, an equation of the form \( (w(t))^k = 1 \), where \( k \geq 2 \) and the word \( w(t) \in G * \langle t \rangle_\infty \) is unimodular is solvable over any group \( \tilde{G} \), i.e., there exists a group \( H \) containing \( G \) as a subgroup and an element \( h \in H \) such that \( w(h) = 1 \) in \( H \).
Notation which we use is mainly standard. Note only that if \( k \in \mathbb{Z}, \) \( x \) and \( y \) are elements of a group, and \( \varphi \) is a homomorphism from this group into another, then \( x^y, x^{k y}, x^{-y}, x^k, \) and \( x^{-\varphi} \) denote \( y^{-1} x y, y^{-1} x^k y, y^{-1} x^{-1} y, \) \( \varphi(x), \varphi(x^{-1}), \) respectively. If \( X \) is a subset of a group, then \( \langle X \rangle \) and \( \langle X \rangle' \) are the subgroup generated by \( X \) and the normal subgroup generated by \( X, \) respectively. The letters \( \mathbb{Z}, \mathbb{N}, \) and \( \mathbb{R} \) denote the set of integers, positive integers, and real numbers, respectively. The symbol \( \bar{G} \) always denotes the group defined by presentation \((*)\).

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2. An algebraic lemma

The following lemma is an easy generalisation of Lemma 2.1 from [Le09]; a similar trick with the change of presentation was used in [Kl05] and later in many other works (see, e.g., [KP05], [CG59], [CG00], [CR01], [FrR69], [FrR98], [FoR05], [Kl05], [Kl06b], [Kl07], and [Kl09]). A geometric interpretation of this trick can be found in [FoR05].

**Lemma 1.** If a word \( w = g_1 t^{e_1} \ldots g_n t^{e_n} \) is unimodular and cyclically reduced and \( n > 1, \) then the group \( \bar{G} \) has a relative presentation of the form

\[
\bar{G} = \left\langle H, t \right| \{p^i = p^\varphi, p \in P \setminus \{1\}, \left( ct^{\sum_{i=0}^{m} (b_ia_i^e)\right)^k = 1\}, \right. \tag{1}
\]

where \( a_i, b_i, c \in H, \) \( P \) and \( P^\varphi \) are isomorphic subgroups of the group \( H, \) and \( \varphi : P \to P^\varphi \) is an isomorphism between them. In addition,

1) \( m \geq 0 \) (i.e. the product in formula (1) is nonempty);
2) \( a_i \notin P \) and \( b_i \notin P^\varphi; \)
3) \( \langle P, a_i \rangle = P \ast \langle a_i^\varphi \rangle \) and \( \langle P^\varphi, b_i \rangle = P^\varphi \ast \langle b_i^\varphi \rangle \) in \( H, \) where \( a_i' \in Pa_i, b_i' \in P^\varphi b_i; \)
4) the groups \( H, P, \) and \( P^\varphi \) are free products of finitely many isomorphic copies of \( G; \) \( H = G^{(0)} \ast \ldots \ast G^{(s)}, \)
\( P = G^{(0)} \ast \ldots \ast G^{(s-1)}; \) and \( P^\varphi = G^{(1)} \ast \ldots \ast G^{(s)}, \) where \( s \geq 0 \) (if \( s = 0, \) the groups \( P \) and \( P^\varphi \) are trivial) and the isomorphism \( \varphi \) is the shift: \( \langle G^{(i)} \rangle^\varphi = G^{(i+1)} \).

**Proof.** First, we show that \( \bar{G} \) has at least one presentation of the form (1) satisfying condition 4). Since \( \sum \varepsilon_i = 1, \) the word \( w \) can be written in the form

\[
w = \left( \prod g_i^{k_i} \right) t.
\]

Conjugating, if necessary, \( w \) by \( t, \) we can assume that \( k_i \geq 0. \) Setting \( g_1 = g_i^t \) for \( g \in G, \) \( G^{(i)} = G_t, s = \max k_i, \) and \( c = \prod g_i^{k_i}, \) we see that \( \bar{G} \) has presentation

\[
\bar{G} \simeq \left\langle G^{(0)} \ast \ldots \ast G^{(s)}, t \right| \left\{ \left(g_i^{(i)} \right)^t = g_i^{(i+1)}, i = 0, \ldots, s - 1, g \in G \right\}, \right. \tag{2}
\]

i.e., a presentation of the form (1) (with \( m = -1 \) satisfying condition 4).

Now, from all presentations of the form (1) satisfying condition 4) we choose presentations with minimal \( s, \) and from all these presentations with minimal \( s \) we choose one with minimal \( m. \) The obtained presentation (1) is as required.

Indeed, if \( m < 0 \) (i.e., \( w = ct, \) where \( c \in H, \) then \( s = 0, \) because otherwise we might decrease \( s \) replacing all fragments \( g^{(i)} \) in the word \( c \) by \( (g^{(s-1)})^t. \) But the conditions \( m < 0 \) and \( s = 0 \) mean that the initial word \( w \) has the form \( w = ct, \) where \( c \in G, \) which contradicts the assumption \( n > 1. \) Thus, condition 1) holds.

Condition 2) holds because otherwise in presentation (1) we might replace a fragment \( t^{-1} a_i t \) with \( a_i \in P \) (or a fragment \( t h_i^{-1} \) with \( b_i \in P^\varphi \)) by \( a_i^\varphi \) (or by \( b_i^\varphi^{-1} \), respectively), thereby decreasing \( m \) (and not increasing \( s \)).

Condition 3) follows from conditions 2) and 4) by virtue of the following simple fact, whose proof we leave to the reader as an exercise.

If \( u \in A \ast B, \) then \( \langle A, u \rangle = A \ast \langle u' \rangle \) for some \( u' \in Au. \)

Lemma 1 is proven.

**Corollary.** If for some \( i \) an equality of the form \( a_i^{n_1} p_1 \ldots a_i^{n_r} p_s = 1 \) or \( b_i^{n_1} p_1^\varphi \ldots b_i^{n_r} p_s^\varphi = 1, \) where \( s \geq 1, n_j \in \mathbb{Z} \setminus \{0\}, \)
\( p_j \in P, \) and \( p_j \neq 1 \) for \( j \neq s, \) holds in \( H, \) then the minimal order of a nonidentity element of \( G \) is at most \( \max_{k < t} \left| \sum_{j=k}^{t} n_j \right|.
\)

**Proof.** This follows immediately from assertions 4), 3), and 2) of Lemma 1.
3. Maps and weight test

Throughout this paper, the term “surface” means a closed two-dimensional oriented surface.

A map $M$ on a surface $S$ is a finite set of continuous mappings $\{\mu_i : D_i \rightarrow S\}$, where $D_i$ is a compact oriented two-dimensional disk, called the $i$th face or cell of the map; the boundary of each face $D_i$ is partitioned into finitely many intervals $e_{ij} \subset \partial D_i$, called the pre-edges of the map, by a nonempty set of points $c_{ij} \in \partial D_i$, called the corners of the map. The images of the corners $\mu_i(c_{ij})$ and the pre-edges $\mu_i(e_{ij})$ are called the vertices and the edges of the map, respectively. It is assumed that

1) the restriction of $\mu_i$ to the interior of each face $D_i$ is a homeomorphic embedding preserving orientation; the restriction of $\mu_i$ to each pre-edge is a homeomorphic embedding;
2) different edges do not intersect;
3) the images of the interiors of different faces do not intersect;
4) $\bigcup \mu_i(D_i) = S$.

Sometimes, we interpret a map $M$ as a continuous mapping $M : \bigsqcup D_i \rightarrow S$ from a discrete union of disks onto the surface.

The union of all vertices and edges of a map is a graph on the surface, called the 1-skeleton.

We say that a corner $c$ is a corner at a vertex $v$ if $M(c) = v$. There is a natural cyclic order on the set of all corners at a vertex $v$; we call two corners at $v$ adjacent if they are neighboring with respect to this order.

By abuse of language, we say that a point or a subset of the surface is contained in a face $D_i$ if it lies in the image of $\mu_i$. Similarly, we say that a face $D_i$ is contained in some subset $X \subseteq S$ of the surface $S$ if $M(D_i) \subseteq X$.

Figure 1 presents a map on the sphere with 10 faces ($A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $I$, and $K$), 32 corners, 8 vertices, 16 edges, and 32 pre-edges. Note that the number of corners always equals to the number of pre-edges and is twice the number of edges, and the value

$$\chi(S) \overset{\text{def}}{=} (\text{the number of vertices}) - (\text{the number of edges}) + (\text{the number of faces})$$

does not depend on the choice of a map on the surface $S$ and is called the Euler characteristic of this surface. The Euler characteristic of the sphere (the only surface of our real interest in this paper) is two.

We need also the following simple but useful fact, sometimes called the combinatorial Gauss–Bonnet formula.

**Weight test** [Ger87], [Pri88], see also [MCW02]. If each corner $c$ of a map on a surface $S$ is assigned a number $\nu(c)$ (called the weight or the value of the corner $c$), then

$$\sum_v K(v) + \sum_D K(D) + \sum_e K(e) = 2\chi(S).$$

Here the summations are over all vertices $v$ and all cells $D$ of the map and the values $K(v)$, $K(D)$, and $K(e)$, called the curvatures of the corresponding vertex, cell, and edge, are defined by the formulae

$$K(v) \overset{\text{def}}{=} 2 - \sum_c \nu(c), \quad K(D) \overset{\text{def}}{=} 2 - \sum_e (1 - \nu(c)), \quad K(e) \overset{\text{def}}{=} 0.$$
where the first sum is over all corners at the vertex \( v \), and the second sum is over all corners of the cell \( D \).

### 4. Howie diagrams

Suppose that we have a map \( M \) on a surface \( S \), the corners of the map are labeled by elements of a group \( H \), and the edges are oriented (in the figures, we draw arrows on the edges) and labelled by elements of a set \( \{t_1, t_2, \ldots \} \) disjoint from the group \( H \). The label of a corner or an edge \( x \) is denoted by \( \lambda(x) \).

The label of a vertex \( v \) of such a map is defined by the formula

\[
\lambda(v) = \prod_{i=1}^{k} \lambda(c_i),
\]

where \( c_1, \ldots, c_k \) are all corners at \( v \) listed clockwise. The label of a vertex is an element of the group \( H \) determined up to conjugacy. For instance, the label of a vertex in Fig. 1 is \( \lambda(b_2)\lambda(e_1)\lambda(d_1) \).

The label of a face \( D \) is defined by the formula

\[
\lambda(D) = \prod_{i=1}^{k} (\lambda(M(e_i)))^{e_i} \lambda(c_i),
\]

where \( e_1, \ldots, e_k \) and \( c_1, \ldots, c_k \) are all pre-edges and all corners of \( D \) listed anticlockwise, the endpoints of \( e_i \) are \( c_{i-1} \) and \( c_i \) (subscripts are modulo \( k \)), and \( e_i = \pm 1 \) depending on whether the homeomorphism \( e_i \rightarrow M(e_i) \) preserves or reverses orientation. Simply speaking, to obtain the label of a face, we should go around its boundary anticlockwise, writing out the labels of all corners and edges we meet; the label of an edge traversed against the arrow should be raised to the power \(-1\).

The label of a face is an element of the group \( H \ast F(t_1, t_2, \ldots) \) (the free product of \( H \) and the free group with basis \( \{t_1, t_2, \ldots\} \) determined up to a cyclic permutation. More precisely, the right-hand side of our formula for \( \lambda(D) \) is called the label of the face \( D \) written starting with the pre-edge \( e_1 \).

For instance, if the label of each edge in Fig. 1 is \( t \), then the label of the face \( B \) written starting with the pre-edge \( \alpha \) is \( t\lambda(b_1)\lambda(b_2)\lambda(b_3) \).

Such a labelled map is called a Howie diagram (or simply diagram) over a relative presentation

\[ K = \langle H, t_1, t_2, \ldots \mid w_1 = 1, w_2 = 1, \ldots \rangle \] (**)

if

1. some vertices and faces are distinguished and called exterior; the remaining vertices and faces are called interior;
2. the label of each interior face is a cyclic permutation of one of the words \( w_i^{\pm 1} \);
3. the label of each interior vertex is the identity element of \( H \).

Figure 4 presents all possible interior faces of Howie diagrams over presentation (1).

A diagram is said to be reduced if it contains no such edge \( e \) that both faces containing \( e \) are interior, these faces are different and the label of one of these faces written starting with the label of \( e \) is inverse to the label of the other face written ending with the label of \( e \); such a pair of faces with a common edge is called a reducible pair. For example, the faces \( D \) and \( E \) in Fig. 1 form a reducible pair if \( \lambda(d_1) = (\lambda(e_1))^{-1} \) and the labels of all edges are equal.

The following lemma is an analogue of the van Kampen lemma for relative presentations.

**Lemma 2** [How83]. The natural mapping from a group \( H \) to the group with relative presentation (***) is noninjective if and only if there exists a spherical diagram over this presentation with no exterior faces and a single exterior vertex whose label is not 1 in \( G \). A minimal (with respect to the number of faces) such diagram is reduced. If this natural mapping is injective, then we have the equivalence: the image of an element \( u \in H \ast F(t_1, t_2, \ldots) \) \( \{1\} \) is 1 in the group (***) if and only if there exists a spherical diagram over this presentation without exterior vertices and with a single exterior face with label \( u \). A minimal (with respect to the number of faces) such diagram is also reduced.

Diagrams on the sphere with a single exterior face and no exterior vertices are also called disk diagrams, the boundary of the exterior face of such a diagram is called the contour of the diagram.

Let \( \varphi : P \rightarrow P^x \) be an isomorphism between two subgroups of a group \( H \). A relative presentation of the form

\[ \langle H, t \mid \{p^x = p^y; p \in P \ \setminus \ \{1\}\}, w_1 = 1, w_2 = 1, \ldots \rangle \] (***)

is called a \( \varphi \)-presentation. A diagram over a \( \varphi \)-presentation (***) is called \( \varphi \)-reduced if it is reduced and different interior cells with labels of the form \( p^xp^{-y} \), where \( p \in P \), have no common edges.

**Lemma 3** [KI05]. A minimal (with respect to the number of faces) diagram among all spherical diagrams over a given \( \varphi \)-presentation without exterior faces and with a single exterior vertex with nontrivial label is \( \varphi \)-reduced. If no such diagrams exists, then a minimal diagram among all disk diagrams with a given label of contour is \( \varphi \)-reduced. In other words, the complete \( \varphi \)-analogue of Lemma 2 is valid.

The idea of the proof is shown in Fig. 2.
A relative presentation ($\varphi$-presentation) over which there exists no reduced ($\varphi$-reduced) spherical diagrams with no exterior faces and a single exterior vertex are called aspherical (respectively, $\varphi$-aspherical).

Suppose that we have a map on a surface all whose edges are oriented (e.g., a Howie diagram). Such a map has 4 kinds of corners: $(++)$, $(- -)$, $(+ -)$, and $(- +)$ (Fig. 3).

The following lemma is obvious.

**Lemma 4.** In the anticlockwise listing of the corners at a vertex $v$, the corners of type $(++)$ alternate with corners of type $(- -)$. If at a vertex $v$ there are no corners of type $(++)$, or, equivalently, there are no corners of type $(- -)$, then either all corners at $v$ are of type $(+ -)$ (in this case, $v$ is called a sink), or all corners at $v$ are of type $(- +)$ (in this case, $v$ is called a source).

Fig. 2

Fig. 3

Fig. 4a

Fig. 4b

Fig. 4c
5. The proof of a major part of the theorem

In this section, we prove all assertions of the theorem except the relative hyperbolicity for $k = 2$.

If the word $w$ is conjugate to a word $gt$, then the group $G$ is the free product of the group $G$ and a cyclic group of order $k$, and all assertions of the theorem are obvious. If the letters $t^{\pm 1}$ occur more than once in the word $w$, then, by Lemma 1, the group $G$ has presentation (1).

Consider a $\varphi$-reduced spherical Howie diagram over presentation (1) that has either no exterior faces and one exterior vertex or no exterior vertices and one exterior face. Faces with label of the form $p^{-\varphi}p'$ are called digons, the other interior faces are called large faces.

Vertices and edges belonging to the boundary of the exterior face are called boundary. The exterior vertex (if it exists) is also considered as a boundary vertex.

A digon is called special if its both neighboring faces are interior and one of its corners (called positive) is adjacent with corners of types $(++)$ and $(−−)$ (Fig. 5). Note that the other corner of a special digon (called negative) is automatically non-adjacent with corners of type $(++)$ and $(−−)$.

Let us assign a value (weight) $\nu(\gamma)$ to each corner $\gamma$ of the diagram by the following rule:

$$
\nu(\gamma) = \begin{cases} 
0 & \text{if } \gamma \text{ is a corner of a nonspecial digon} \\
-1 & \text{if } \gamma \text{ is a negative corner of a special digon;}
\end{cases}
$$

$$
1, \quad \text{otherwise.}
$$

Let us calculate the curvatures of vertices and faces according to the weight test (see Section 3). For faces, we have

$$
K(\text{digon}) = 0, \quad K(\text{large face}) = 2 - k, \quad K(\text{exterior face}) = 2.
$$

For a vertex $v$, the curvature is

$$
K(v) = 2 + n - l - p - x,
$$

where $l$ is the number of corners of types $(++)$ and $(−−)$ of large faces, $p$ is the number of positive corners of special digons, $n$ is the number of negative corners of special digons, and $x$ is the number of corners of the exterior face (all corners are at the vertex $v$).

Each negative corner of a special digon is adjacent to two corners of type $(++)$ or $(−−)$ of large faces (by the definition of special digons), and no corner of type $(++)$ or $(−−)$ can be adjacent to two negative corners (since otherwise, the corresponding large face would have both a corner of type $(++)$ and a corner of type $(−−)$). Therefore, $l \geq 2n$.

Note also that corners of types $(++)$ and $(−−)$ at a non-boundary vertex alternate (Lemma 4) and cannot be adjacent (since the diagram is reduced): between two such corners there must be a corner of weight 1 (either a corner of type $(++)$ or $(−−)$ of a large face or a positive corner of a special digon). Taking into account the preceding remark about negative corners, we conclude that the sum of weights of corners lying between corners of type $(++)$ and $(−−)$ (if we list them clockwise around the vertex $v$) is at least one (Fig. 6, left). Therefore, a non-boundary vertex with positive curvature must be either a source or a sink and, for such vertex, $p = 0$, and either $n = 1$ and $l = 2$ or $n = 0$ and $l = 1$ or $n = 0$ and $l = 0$ ($n < 2$, since otherwise, formula (2) and the inequality $l \geq 2n$ mentioned above would give a nonpositive curvature). See Fig. 6, the boldface digits denote the values of corners.
The first case \((n = 1 \text{ and } l = 2)\) for a non-boundary vertex is impossible, because the label of such a vertex, i.e., the product of labels of corners, is \(a_{\nu}^{-1}p_1a_{\mu}p_2\) (if the vertex is a source) or \(b_0^{-1}p_1b_0p_2^2\) (if the vertex is a sink), where \(p_1\) and \(p_2\) lie in \(P\) and are not 1 (since the diagram is reduced) and, therefore, the label of the vertex is not 1 by Corollary of Lemma 1; thus this vertex cannot be interior. The second and third cases \((n = 0 \text{ and } l \in \{0, 1\})\) for a non-boundary vertex are impossible by nearly the same reason: they would imply an equality of the form \(a_i^{\pm 1}p_1 = 1\), \(b_i^{\pm 2}p_i = 1\), \(p_2 = 1\), or \(p_2^2 = 1\), where \(p_1 \in P \neq p_2 \neq 1\).

Thus, the curvature of any non-boundary vertex \(v\) is nonpositive. The curvatures of interior faces are also nonpositive (for \(k \geq 2\)), the curvature of a boundary vertex is at most two (this follows from formula (2) and the inequality \(l \geq 2n\)), while the total curvature must be four according to the weight test.

This means that, first, there exist no diagrams without exterior faces and with single exterior vertex, i.e. the natural mapping \(H \to G\) (and, hence, the natural mapping \(G \to \widetilde{G}\)) is injective by Lemma 2; and secondly, if there is one exterior face and no exterior vertices and \(k \geq 3\), then the number of interior large faces is bounded by a linear function of the perimeter of the exterior face:

\[
2 \cdot (\text{the perimeter of the exterior face}) - (k - 2) \cdot (\text{the number of large interior faces}) + 2 \geq 4.
\]

It is easy to see that such an isoperimetric inequality for presentation (1) implies the usual linear isoperimetric inequality for presentation \((\ast)\) (see [Le09]), i.e., the relative hyperbolicity of \(G\) for \(k \geq 3\). For the sake of completeness, we prove this fact here.

**Proposition 1.** Suppose that some word \(u \in G * (t)_\infty\) represents the identity element of the group \(\widetilde{G}\), i.e. \(u\) can be represented as a product of the form

\[
u = v_1 \ldots v_sp_1v_1 \ldots v_s,
\]

where each \(v_i\) is conjugate to a word of the form \(p^{-i}p^r\) \((p \in P)\) in the group \(H * (t)_\infty\) and each \(w_i\) is conjugate to the word \(\left(c t \prod_{i=0}^{m} (b_0a_i^r) \right)^{\pm k}\) in \(H * (t)_\infty\) (in the notation of Lemma 1, where \(G\) is the same as \(G^{(0)}\)). Then \(u\) can be represented as a product of \(s\) words conjugate to \(w^{\pm k}\) in \(G * (t)_\infty\).

Informally, any isoperimetric inequality for presentation (1) counting only long relators (only large faces) implies the same isoperimetric inequality for presentation \((\ast)\).

**Proof.** In the group \(\langle H, t \mid \{p^l \neq p^r \mid p \in P\}\rangle\) (isomorphic to \(G * (t)_\infty\)), the words \(v_i\) represent the identity element, the words \(w_i\) are conjugate to \(w^{\pm k}\) (because \(ct \prod_{i=0}^{m} (b_0a_i^r)\) is equal to a cyclic shift of \(w\) by the construction). This implies the assertion of Proposition 1.

Resuming the proof of the theorem, let us show that \(\langle G, G^t \rangle = G * G^t\) in the group \(\widetilde{G}\). If \(H \neq G\), i.e., if \(P \neq \{1\}\), i.e., if \(s > 0\) in Lemma 1, then we have nothing to prove, because it is already proven that the natural mapping \(H = G * G^t \ast \ast \to G\) is injective.

It remains to consider the case \(H = G\) (i.e., \(P = \{1\}\)). Suppose that \(u \in G * G^t\) is a reduced nonempty word representing the identity element of \(\widetilde{G}\). By Lemma 2, \(u\) is the label of the exterior face of some \(\varphi\)-reduced spherical diagram over presentation \((\ast)\) (which coincides with presentation (1) in the case under consideration) without exterior vertices and with a unique exterior face. Since digons are absent and the exterior face has no corners of types \((++)\) and \((-\cdot-\cdot-\cdot)-\), the curvature of each boundary vertex is nonpositive. The sum of curvatures of all faces and vertices must be four, but the unique positive term in this sum is two (the curvature of the exterior face). This contradiction with the weight test completes the proof of the theorem, except the assertion about relative hyperbolicity for \(k = 2\).

**Remark.** This argument proves also the \(\varphi\)-asphericity of presentation (1) (for \(k \geq 2\)), which implies (see [FoR05]) the asphericity of presentation \((\ast)\).

In the remaining part of this paper, we prove relative hyperbolicity for \(k = 2\).

6. Motions

All definitions and facts of this section are taken from paper [Ki05].

Consider a map \(M\) on a closed oriented surface \(S\). Some corners of this map are distinguished and called stop corners.

A \emph{car} moving around a face \(D\) of this map is a continuous locally nondecreasing* mapping from an oriented circle \(R\) \emph{(the circle of time)} to the boundary \(\partial D\) of the face \(D\) such that the preimage of each point, except possibly stop corners, is discrete.

*) We call a continuous mapping \(\alpha: X \to Y\) from an oriented circle \(X\) to an oriented circle \(Y\) \emph{(locally) nondecreasing} if the preimage of any interval \(U \subset Y\) is a union of intervals such that the restriction of \(\alpha\) to each of these intervals is a nondecreasing function (in the usual sense, as a function from one oriented interval to another).
Simply speaking, each car moves without U-turns and infinite decelerations and accelerations along the boundary of its face anticlockwise, possibly stopping for a finite time at some corners. And this motion is periodic.

We say that a car \( \alpha_i \) is at a corner \( c \in \partial D_i \) at a moment of time \( t \in R \) if \( \alpha_i(t) = c \); we also say that a car \( \alpha_i \) is at a point \( p \in S \) at a moment \( t \in R \) if \( \mu_i(\alpha_i(t)) = p \). If the number of cars being at a moment \( t \in R \) at a point \( p \) of the 1-skeleton of \( S \) equals the multiplicity of this point (in other words, \( \bigcup \alpha_i(t) \supseteq M^{-1}(p) \)), then we say that at the point \( p \) at the moment \( t \) a complete collision occurs; the point \( p \) is called a point of complete collision. Points of complete collision lying on edges are called simply points of collision.

A multiple motion of period \( T \) with separated stops on a map \( M \) is a set of cars \( \alpha_{D,j} : R \rightarrow \partial D \), where \( j = 1, \ldots, d_D \), such that

1) \( d_D \geq 1 \) (i.e. each face is moved around by at least one car);
2) at each vertex \( v \) at which there are stop corners, the stops are separated in the following sense: let \( c_1, \ldots, c_k \) be all stop corners at \( v \) enumerated anticlockwise; it is required that, for each \( i \), at corners \( c_i \) and \( c_{i+1} \) (subscripts are modulo \( k \)), cars are never located simultaneously. (In particular, this implies that \( k \geq 2 \).)
3) \( \alpha_{D,j}(t + T) = \alpha_{D,j+1}(t) \) for any \( t \in R \) and \( j = \{1, \ldots, d_D\} \) (subscripts are modulo \( d_D \), and the addition of points of the circle \( R \) is defined naturally: \( R = \mathbb{R}/\mathbb{Z} \));
4) there exists a partition of each circle \( \partial D \) into \( d_D \) arcs (with disjoint interiors) such that during the interval of time \( [0, T) \) each car \( \alpha_{D,j} \) moves along the \( j \)th arc.

**Car-crash test** [Kl05], [Kl97]. For any multiple motion with separated stops on a map \( M \) on a surface \( S \), we have

\[
\sum_v K'(v) + \sum_e K'(e) + \sum_D K'(D) = \chi(S),
\]

where the sums are over all vertices \( v \), edges \( e \), and faces \( D \) of the map \( M \).

Here \( K'(D) = 1 - d_D \), the value \( K'(e) \) is the number of collision points on an edge \( e \) (not counting the end-points), and \( K'(v) = 1 \) if at the vertex \( v \) a complete collision occurs; otherwise \( K'(v) \) is an integer nonpositive number (whose exact definition can be found in [Kl05]).

Throughout this paper, the surface is always the sphere, its Euler characteristic is 2.

### 7. Standard multiple motion

In this section, we define some particular multiple motion on Howie diagrams over presentation (1). Our definition almost literally repeats a definition from [Le09]. A similar motion was considered in [Kl05].

The following motion on a Howie diagram over presentation (1) is called standard:

a) the car going around an interior face with label \( p^{-e}p^t \) moves anticlockwise uniformly with unit speed (one edge per a unit time) visiting the corner of type \((+-)\) at the even moments of time (Fig. 4a);

b) An interior face with label \( \left( ct \prod_{i=0}^{m} b_i a_i^t \right)^k \) are moved around by \( k \) cars; for \( m > 0 \), they stay at the corners of type \((++)\) during the time intervals \([2m + 2, 4m + 1 + (4m + 2)\mathbb{Z}]\), and moves anticlockwise uniformly with unit speed all the remaining time; for \( m = 0 \), each car moves without stops with speed 2 when it moves in the direction of an edge, and with speed 1 when it moves against the direction of an edge; at time zero the car is at a corner of type \((+-)\) (Fig. 4b);

c) An interior face with label \( \left( ct \prod_{i=0}^{m} b_i a_i^t \right)^{-k} \) are moved around by \( k \) cars; for \( m > 0 \), they stay at the corners of type \((-+)\) during the time intervals \([1, 2m]\cup(4m + 2)\mathbb{Z}]\), and moves anticlockwise uniformly with unit speed all the remaining time; for \( m = 0 \), each car moves without stops with speed 2 when it moves against the direction of an edge, and with speed 1 when it moves in the direction of an edge; at time zero the car is at a corner of type \((+-)\) (Fig. 4c);

d) An exterior face is moved around by one car; it moves with period \( 4m + 2 \); at time zero, it is at some vertex; during the interval \([0, \frac{1}{2}]\), it (rapidly) moves counterclockwise along the entire boundary of the face, except the last edge; and at the remaining time it (slowly) goes along this edge.

The standard motion is periodic with period \( 4m + 2 \) (on faces with label \( p^{-e}p^t \) minimal period is two). Figure 4 shows the detailed schedule of the motion of cars moving around interior cells during the interval \([0, 4m + 2]\); the framed numbers near edges denote the speed of the cars on these edges (the default speed is unit).

**Lemma 5** (cf. [Le09], [Kl05]). Suppose that a Howie diagram over presentation (1) has at most one exterior face. Then the standard motion is a motion with separated stops. Complete collisions which occur not on the boundary of the exterior face can occur only at vertices being sinks or sources and only at integer moments of time. On each edge of the boundary of the exterior face there are at most \( k(2m + 1) \) points of complete collision.

**Proof.** Let us declare all corners of types \((++)\) and \((-+)\) to be stop corners. The schedule of the standard motion is such that cars are never located simultaneously at corners of types \((++)\) and \((-+)\); the corners of type \((-+)\) are
visited only during the first half of the period, while the corners of type (++) are visited during the second half of the period. The car moving around the exterior face is not at corners at all at such moments. This and Lemma 4 imply that the standard motion is a motion with separated stops. A collision on an edge separating two interior faces at a moment $t$ means that at this moment the direction of the motion of one of the cars coincides with the direction of the edge, while the direction of the motion of the other colliding car is opposite to the direction of the edge. But the schedule of the standard motion is such that, at each moment $t$, either all cars moving around interior faces and being on edges move in the direction of the edge (this is so when the integer part of $t$ is even), or all cars being on edges move in the direction opposite to the direction of the edge (this is so when the integer part of $t$ is odd). Note also that the definition of multiple motion implies that there are no overtakings. Therefore, collisions can occur only at vertices; the separatedness of stops implies that a vertex of complete collision can not have stop corners and, therefore, is a source or a sink. The cars visit such vertices only at integer moments of time (even for sinks and odd for sources).

The car $\beta$ moving around the exterior face can collide with at most $k$ cars on each edge $e$. During the period $[0; 4m + 2)$ the car $\beta$ occurs on each edge only once, while each car moving along this edge in the opposite direction occurs on $e$ at most $2m + 1$ times (this value is attained on digons). Therefore, during the period, on each edge of the boundary of the exterior face at most $k(2m + 1)$ collisions occur. This very rough estimate completes the proof.

8. Completion of the proof of the theorem

In this section, we complete the proof of the theorem, i.e., we prove that $\tilde{G}$ is relatively hyperbolic with respect to $G$ if $G$ contains no involutions and $k = 2$ (however, the proof below is suitable for any $k \geq 2$).

If the word $w$ is conjugate to $gt$, then $\tilde{G}$ is the free product of $G$ and the cyclic group of order $k$, and we have nothing to prove. If letters $t^\pm 1$ occur more then twice in $w$, then by Lemma 1 $\tilde{G}$ has presentation (1).

Consider a $\varphi$-reduced spherical Howie diagram over presentation (1) without exterior vertices and with one exterior face. As in Section 5, it suffices to show that the diagram satisfies a linear isoperimetric inequality, i.e., the number of large interior faces is bounded by a linear function of the perimeter of the exterior face.

Let us assign a value (weight) to each corner of the diagram as in Section 5. Recall that, for such weights, the curvatures of interior vertices are nonpositive. Moreover, according to formula (2), the curvature of an interior vertex can be zero only in the following cases:

a) $p > 0$ (and, therefore, the vertex is neither a source nor a sink);

b) $p = 0$, $n = 0$, $l = 2$;

c) $p = 0$, $n = 1$, $l = 3$;

d) $p = 0$, $n = 2$, $l = 4$ (Fig. 7).

Note that in cases a), b), c), d) a complete collision cannot occur at the vertex $v$ under the standard motion (Section 7). Indeed, by virtue of Lemma 5, a vertex of complete collision must be either a source or a sink; therefore, in case a) we have no complete collision. A complete collision in case b), when the vertex $v$ is, e.g., a source, would imply, according to the schedule of the motion, that both corners of large faces at this vertex have labels $a_i^\pm 1$ with the same subscript $i$, and the product of all these labels is 1 in the group $G$; this is impossible by virtue of the reducedness of the diagram, the absence of involutions, and the corollary of Lemma 1. For the same reason, complete collisions cannot occur in cases c) and d): in these cases, all corners of large faces must have labels $a_i^\pm 1$ if the vertex is a source, or $b_0^\pm 1$ if the vertex is a sink.

Note also that, for the standard motion (Section 7), we have

$$K'(\text{digon}) = 0, \quad K'(\text{large face}) = 1 - k, \quad K'(\text{non-boundary edge}) = 0, \quad K'(\text{boundary edge}) \leq k(2m + 1),$$

where the value $K'$ is defined in Section 4 (car-crash test). The last two inequalities follow from Lemma 5.

Now, we define the combined curvature of vertices, faces, and edges by the formula

$$K_\Sigma(\cdot) \overset{\text{def}}{=} K(\cdot) + K'(\cdot).$$

Clearly, $K_\Sigma(v) \leq 0$ for any interior vertex $v$, because $K(v)$ is either a negative integer or zero, but in the latter case, as we have seen, there are no complete collision at the vertex $v$ and, therefore, $K'(v) \leq 0$.

It remains to note that, for any non-boundary edge $e$ and any interior large face $\Gamma$,

$$K_\Sigma(e) = K'(e) = 0 \quad \text{and} \quad K_\Sigma(\Gamma) = K(\Gamma) + K'(\Gamma) = 2 - k + (1 - k) \leq -1 \text{ for } k \geq 2.$$
The combined curvature of a boundary edge is bounded by some constant (depending only on $k$ and $m$) by Lemma 5. The combined curvature of a boundary vertex is at most three (since $K(v) \leq 2$, as mentioned in Section 5). The combined curvature of the exterior face is two. On the other hand, the sum of the combined curvatures of all vertices, edges, and faces must be $4 + 2$, according to the weight test and the car-crash test.

This means that the number of interior large faces is bounded by a linear function of the perimeter of the exterior face:

$$(D + 3) \cdot \text{(perimeter of the exterior face)} - \text{(number of large interior faces)} + 2 \geq 4 + 2,$$

where $D = k(2m + 1)$ is the constant from Lemma 5 (this is a very rough estimate). This isoperimetric inequality completes the proof (by virtue of Proposition 1).

Other applications of the combined test and a description of all possible tests (in some exact sense) can be found in [KI97].

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