\textit{\(\eta\)-weak-pseudo-Hermiticity generators and exact solvability}

Omar Mustafa\textsuperscript{1} and S.Habib Mazharimousavi\textsuperscript{2}  
Department of Physics, Eastern Mediterranean University,  
G Magusa, North Cyprus, Mersin 10,Turkey  
\textsuperscript{1}\textsuperscript{E-mail: omar.mustafa@emu.edu.tr}  
\textsuperscript{2}\textsuperscript{E-mail: habib.mazhari@emu.edu.tr}  

September 8, 2018

Abstract

Exact solvability of some non-Hermitian \(\eta\)-weak-pseudo-Hermitian Hamiltonians is explored as a byproduct of \(\eta\)-weak-pseudo-Hermiticity generators. A class of \(V_{\text{eff}}(x) = V(x) + iW(x)\) potentials is considered, where the imaginary part \(W(x)\) is used as an \(\eta\)-weak-pseudo-Hermiticity generator to obtain exactly solvable \(\eta\)-weak-pseudo-Hermitian Hamiltonian models.

PACS numbers: 03.65.Ge, 03.65.Fd,03.65.Ca

1 Introduction

Since the early years of quantum mechanics evolution the exact solvability of quantum mechanical models have attracted much attention. Some exactly solvable model have already become typical standard examples in quantum mechanical textbooks. However, it was believed that the reality of the spectra of the Hamiltonians, describing quantum mechanical models, is necessarily attributed to their Hermiticity. It was the non-Hermitian \(\mathcal{PT}\)-symmetric Hamiltonians’ proposal by Bender and Boettcher [1] that relaxed the Hermiticity condition as a necessity for the reality of the spectrum [1-7]. Herein, \(\mathcal{P}\) denotes the parity \((\mathcal{P}x\mathcal{P} = -x)\) and the anti-linear operator \(\mathcal{T}\) mimics the time reflection \((\mathcal{T}i\mathcal{T} = -i)\).

Recently, Mostafazadeh [8] has introduced a broader class of non-Hermitian pseudo-Hermitian Hamiltonians (a generalization of \(\mathcal{PT}\) - symmetry, therefore). In these settings [8-19], a Hamiltonian \(H\) is pseudo-Hermitian if it obeys the similarity transformation:

\[\eta H \eta^{-1} = H^\dagger,\]  

(1)
where \( \eta \) is a Hermitian invertible linear operator and \((\dagger)\) denotes the adjoint. However, if \( H \) is an \( \eta \)-pseudo-Hermitian with respect to the nontrivial “metric” operator

\[
\eta = O^\dagger O,
\]

(2)

for some linear invertible operator \( O : \mathcal{H} \rightarrow \mathcal{H} \) (\( \mathcal{H} \) is the Hilbert space), then its spectrum is real and \( H \) satisfies the intertwining relation

\[
\eta H = H^\dagger \eta,
\]

(3)

where \((\eta H)\) is also Hermitian. Moreover, one can even relax \( H \) to be \( \eta \)-weak-pseudo-Hermitian by not restricting \( \eta \) to be Hermitian (cf., e.g., Bagchi and Quesne [15]), and linear and/or invertible (cf., e.g., Solombrino [15]). Nevertheless, Fityo [13] has implicitly used \( \eta \)-weak-pseudo-Hermiticity without invertibility as a necessary condition on \( O \) and hence on \( \eta \).

Very recently we have followed Fityo’s [13] \( \eta \)-weak-pseudo-Hermiticity condition and introduced a class of spherically symmetric non-Hermitian Hamiltonians and their \( \eta \)-weak-pseudo-Hermiticity generators [13], where a generalization beyond the nodeless 1D state was proposed. The same recipe was extended to deal with a class of \( \eta \)-weak-pseudo-Hermitian \( d \)-dimensional Hamiltonians for quantum particles endowed with position-dependent masses [14].

In this work, we target exact solvability of some non-Hermitian \( \eta \)-weak-pseudo-Hermitian Hamiltonians as a byproduct of our \( \eta \)-weak-pseudo-Hermiticity generators discussed in [13]. In section 2, we present our procedure for a class of non-Hermitian Hamiltonians with 1D effective potentials of the form

\[
V_{\text{eff}}(x) = V(x) + iW(x),
\]

(4)

Then \( H \) has a real spectrum if and only if there is a linear operator \( O : \mathcal{H} \rightarrow \mathcal{H} \) such that \( H \) is \( \eta \)-weak-pseudo-Hermitian with the linear operator

\[
O = \partial_x + F(x) + iG(x) \quad \Rightarrow \quad O^\dagger = -\partial_x + F(x) - iG(x)
\]

(5)

where \( F(x) \) and \( G(x) \) are real-valued functions and \( \mathbb{R} \ni x \in (-\infty, \infty) \). Equation (2), in turn, implies

\[
\eta = -\partial_x^2 - 2iG(x)\partial_x + F(x)^2 + G(x)^2 - F'(x) - iG'(x),
\]

(6)

\section{\( \eta \)-weak-pseudo-Hermiticity generators}

In this section we consider a class of 1D non-Hermitian Hamiltonians (in \( \hbar = 2m = 1 \) units) of the form

\[
H = -\partial_x^2 + V_{\text{eff}}(x); \quad V_{\text{eff}}(x) = V(x) + iW(x),
\]

(4)

Then \( H \) has a real spectrum if and only if there is a linear operator \( O : \mathcal{H} \rightarrow \mathcal{H} \) such that \( H \) is \( \eta \)-weak-pseudo-Hermitian with the linear operator

\[
O = \partial_x + F(x) + iG(x) \quad \Rightarrow \quad O^\dagger = -\partial_x + F(x) - iG(x)
\]

(5)

where \( F(x) \) and \( G(x) \) are real-valued functions and \( \mathbb{R} \ni x \in (-\infty, \infty) \). Equation (2), in turn, implies

\[
\eta = -\partial_x^2 - 2iG(x)\partial_x + F(x)^2 + G(x)^2 - F'(x) - iG'(x),
\]

(6)
where primes denote derivatives with respect to \( x \). Herein, the operators \( O \) and \( O^\dagger \) are two intertwining operators and the Hermitian operator \( \eta \) leads to the intertwining relation (3) (cf, e.g.,[8,13]). Hence, relation (3) would lead to

\[
W(x) = -2G'(x),
\]

(7)

\[
F(x)^2 - F'(x) = \frac{2G(x) G''(x) - G'(x)^2 + \alpha}{4G(x)^2},
\]

(8)

\[
V(x) = \frac{2G(x) G''(x) - G'(x)^2 + \alpha}{4G(x)^2} - G(x)^2 + \beta
\]

(9)

where \( \alpha, \beta \in \mathbb{R} \) are integration constants.

Now we depart from our formal procedure in [13] and seek the real part, \( V(x) \), of the effective potential, \( V_{\text{eff}}(x) \), using the imaginary part, \( W(x) \), as a generating function. Equation (7) would therefore lead to

\[
G(x) = -\frac{1}{2} \int_x^\infty W(z)dz,
\]

(10)

with an integration constant set equals zero for simplicity/convenience. In a straightforward manner one can show, following (8) and (9) respectively, that

\[
F(x)^2 - F'(x) = W'(x) \left( 2 \int_x^\infty W(z)dz \right)^{-1} - \left( W(x) \left( 2 \int_x^\infty W(z)dz \right)^{-1} \right)^2
\]

\[
+ \alpha \left( \int_x^\infty W(z)dz \right)^{-2}.
\]

(11)

\[
V(x) = W'(x) \left( 2 \int_x^\infty W(z)dz \right)^{-1} - \left( W(x) \left( 2 \int_x^\infty W(z)dz \right)^{-1} \right)^2
\]

\[
+ \alpha \left( \int_x^\infty W(z)dz \right)^{-2} - \left( \frac{1}{2} \int_x^\infty W(z)dz \right)^2 + \beta
\]

(12)

At this point, we should report that \( \alpha, \beta \in \mathbb{R} \) can be used as adjustable real parameters that would serve for the exact solvability of the \( \eta \)-weak-pseudo-Hermitian generators’ productions of the real part \( V(x) \) of the effective potential \( V_{\text{eff}}(x) \).

3 Illustrative examples

In this section, we construct \( \eta \)-weak-pseudo-Hermitian Hamiltonians and exact solvability of some non-Hermitian Hamiltonians using our \( \eta \)-weak-pseudo-Hermiticity generator \( W(x) \) proposed above.
3.1 An \( \eta \)-weak-pseudo-Hermitian \( PT \)-symmetric Scarf II Hamiltonian model

An \( \eta \)-weak-pseudo-Hermitian generator of the form
\[
W(x) = \frac{-A \sinh(x)}{\cosh^2(x)}
\]  \hspace{1cm} (13)

would lead, using (12), to
\[
V(x) = -\frac{3 + A^2}{4 \cosh^2(x)}.
\]  \hspace{1cm} (14)

Hence, the corresponding \( \eta \)-weak-pseudo-Hermitian Hamiltonian, with \( \alpha = 0 \) and \( \beta = -1/4 \), is given by
\[
H = -\frac{\partial_x^2}{4 \cosh^2(x)} - \frac{3 + A^2}{4 \cosh^2(x)} - i \frac{A \sinh(x)}{\cosh^2(x)}.
\]  \hspace{1cm} (15)

This Hamiltonian model represents an \( \eta \)-weak-pseudo-Hermitian \( PT \)-symmetric Scarf II model which is exactly solvable (cf., e.g., Ahmet [15], Khare [18], and Cooper et al [19]). The eigenvalues and eigenfunctions of which are reported by Ahmet [15] as
\[
\psi_n(x) = C_n \left( \frac{1}{\cosh(x)} \right)^{(s+t-1/2)} \times \exp \left[ \frac{i}{2} (t - s) \tanh^{-1}(\sinh(x)) \right] P_n^{-t-s}(i \sinh(x))
\]  \hspace{1cm} (16)

\[
E_n = \begin{cases} 
- \left( n + \frac{1}{2} \right)^2 & n = 0, 1, \ldots < \frac{A+1}{2}; \quad \text{for} \ A \geq 2, \\
- \frac{1}{4} & \text{for} \ A < 2.
\end{cases}
\]  \hspace{1cm} (17)

where
\[
s = \frac{1}{2} |A - 2| \quad \text{and} \quad t = \frac{1}{2} |A + 2|
\]  \hspace{1cm} (18)

3.2 An \( \eta \)-weak-pseudo-Hermitian periodic-type \( PT \)-symmetric Hamiltonian model

An \( \eta \)-weak-pseudo-Hermiticity generator of the form
\[
W(x) = \frac{4 \sin(2x)}{3(\cos^2(x) - \frac{4}{3})^2}
\]  \hspace{1cm} (19)

would result, with \( \alpha = 0 \) and \( \beta = 1 \), in
\[
V(x) = \frac{\frac{1}{9} \left(-30 \cos^2(x) + 24 \right)}{(\cos^2(x) - \frac{4}{3})^2}
\]  \hspace{1cm} (20)
This in turn yeilds an $\eta$-weak-pseudo-Herrmitian Hamiltonian

$$H = -\partial_x^2 - \frac{6}{(\cos(x) + 2i \sin(x))}. \quad (21)$$

The solution of which is reported by Samsonov [16], in the interval $x \in (-\pi, \pi)$ with the boundary conditions $\psi(-\pi) = \psi(\pi) = 0$, as

$$\psi_n(x) = \left\{ \left[ (16 - n^2) \cos x - 2i \left( n^2 - 4 \right) \sin x \right] \sin \left[ \frac{n}{2} \left( \pi + x \right) \right] \right.$$  
$$-6n \sin x \cos \left[ \frac{n}{2} \left( \pi + x \right) \right] \right\} \times (\cos x + 2i \sin x)^{-1} \quad (22)$$

$$E_n = n^2/4; \; n = 1, 3, 4, 5, ... \quad (23)$$

with a missing state at $n = 2$ (for more details the reader may refer to Samsonov [16]).

### 3.3 An $\eta$-weak-pseudo-Herrmitian non-$PT$-symmetric Morse Hamiltonian

An $\eta$-weak-pseudo-Herrmiticity generator of the form

$$W(x) = -\xi e^{-x} \quad (24)$$

would result, with $\alpha = 0$ and $\beta = -1/4$, in

$$V(x) = -\frac{1}{4} \xi^2 e^{-2x}. \quad (25)$$

This is an $\eta$-weak-pseudo-Herrmitian non-$PT$-symmetric Morse Hamiltonian model

$$H = -\partial_x^2 - \frac{1}{4} \xi^2 e^{-2x} - i\xi e^{-x}, \quad (26)$$

considered by Ahmed [17], who has reported its eigenfunction and eigenvalue, with $A = 0$, $B = \xi$ and $C = 1/2$, as

$$E_0 = -1/4 \quad (27)$$

$$\psi_n(x) = z^{1/2} e^{-z/2} L_0^1(z) \quad (28)$$

where $z = 2i\xi e^{-x}$.

### 4 Conclusion

In this work, we have introduced a byproduct of our $\eta$-weak-pseudo-Herrmiticity generators discussed in [13]. We have shown that the imaginary part, $W(x)$,
of the effective potential, $V_{\text{eff}}(x) = V(x) + iW(x)$, can be used as an $\eta$-weak-pseudo-Hermiticity generator to come out with exactly solvable Hamiltonian models. The utility of the current recipe is demonstrated through a $PT$-symmetric Scarf II, a $PT$-symmetric periodic-type, and a non-$PT$-symmetric Morse models.

Finally, we may report that although the choice of $W(x) = W_\circ \in \mathbb{R}$, where $W_\circ$ is constant, is feasible for our $\eta$-weak-pseudo-Hermiticity generators, one should avoid such setting in $W(x)$. This choice would result in

$$V_{\text{eff}}(x) = \frac{\alpha - W_\circ^2 / 4}{(W_\circ x + C_\circ)^2} - \frac{1}{4} (W_\circ x + C_\circ)^2 + iW_\circ + \beta$$

where $C_\circ \in \mathbb{R}$ is an integration constant. It is obvious that such an effective potential does not support bound states (i.e., the spectrum discreteness of the $\eta$-pseudo-Hermiticity required by Mostafazadeh’s theorem in [8] is violated). Of course, this is only valid for our settings of the $\eta$-weak-pseudo-Hermiticity generators. That is, only for properly chosen $W(x)$ does the effective potential become exactly solvable and there is no known systematic way of making such choices. It seems that this is the only sacrifice we have to make for the sake of exactly solvable $\eta$-weak-pseudo-Hermitian Hamiltonian models.
References

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 24 (1998) 5243
   C. M. Bender, S. Boettcher and P. N. Meisinger: J. Math. Phys. 40 (1999) 2201
   B. Bagchi, F. Cannata and C. Quesne, Phys. Lett. A 269 (2000) 79

[2] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541

[3] M. Znojil and G. Lévai, Phys. Lett. A 271 (2000) 327

[4] Z. Ahmed: Phys. Lett. A 286 (2001) 231
   B. Bagchi, S. Mallik, C. Quesne and R. Roychoudhury, Phys. Lett. A 289 (2001) 34

[5] P. Dorey, C. Dunning and R. Tateo, J. Phys. A: Math. Gen. 4 (2001) 5679
   R. Kretschmer and L. Szymanowski, Czech. J. Phys. 54 (2004) 71

[6] M. Znojil, F. Gemperle and O. Mustafa, J. Phys. A: Math. Gen. 35 (2002) 5781
   O. Mustafa and M. Znojil, J. Phys. A: Math. Gen. 35 (2002) 8929

[7] O. Mustafa, J. Phys. A: Math. Gen. 36 (2003) 5067
   F. Fernandez, R. Guardiola, J. Ros and M. Znojil: J. Phys. A: Math. Gen. 31 (1998) 10105

[8] A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814
   A. Mostafazadeh, Nucl. Phys. B640 (2002) 419

[9] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205
   M. Znojil, "Pseudo-Hermiian version of the charged harmonic oscillator and its "forgotten" exact solution" (2002) (arXiv: quant-ph/0206085)
   A. Sinha and P. Roy, Czech. J. Phys. 54 (2004) 129

[10] A. Mostafazadeh, J. Math. Phys. 43 (2002) 3944
    A. Mostafazadeh, J. Math. Phys. 44 (2003) 974

[11] L. Jiang, L. Z. Yi and C. S. Jia, Phys Lett A 345 (2005) 279
    B. P. Mandal, Mod. Phys. Lett. A 20 (2005) 655
    M. Znojil, H. Bila and V. Jakubsky, Czech. J. Phys. 54 (2004) 1143
    A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37 (2004) 11645

[12] A. Mostafazadeh, J. Phys. A: Math. Gen. 38 (2005) 3213
[13] T. V. Ftiyo, J. Phys. A: Math. Gen. 35 (2002) 5893
O. Mustafa and S. H. Mazharimousavi: "Generalized $\eta$-pseudo-Hermiticity generators; radially symmetric Hamiltonians" (2006) (arXiv: hep-th/0601017)

[14] O. Mustafa and S. H. Mazharimousavi, Czech. J. phys. (2006), in press (arXiv: quant-ph/0603237).

[15] Z. Ahmed, Phys. Lett. A 282 (2001) 343
L. Solombrino, J. Math. Phys. 43 (2002) 5439
B. Bagchi and C. Quesne, Phys. Lett. A 301 (2002) 173

[16] B. F. Samsonov and P Roy, J. Phys. A; Math. Gen. 38 (2005) L249

[17] Z. Ahmed, Phys. Lett. A 290 (2001) 19
B. Bagchi and C. Quesne, Phys. Lett. A 273 (2000) 285

[18] A. Khare, Phys. Lett. A 288 (2001) 69

[19] F. Cooper, A. Khare, and U. P. Sukhatme, Phys Rep 251 (1995) 267