A $\mathbb{Z}_3$-graded generalization of supermatrices

Bertrand Le Roy

Laboratoire de Gravitation et Cosmologie Relativistes
Université Pierre et Marie Curie - CNRS URA 769
Tour 22-12, 4$^{e}$ étage, boîte 142

Abstract

We introduce $\mathbb{Z}_3$-graded objects which are the generalization of the more familiar $\mathbb{Z}_2$-graded objects that are used in supersymmetric theories and in many models of non-commutative geometry. First, we introduce the $\mathbb{Z}_3$-graded Grassmann algebra, and we use this object to construct the $\mathbb{Z}_3$-matrices, which are the generalizations of the supermatrices. Then, we generalize the concepts of supertrace and superdeterminant.
I. INTRODUCTION

$
\mathbb{Z}_2$-graded algebras, which are the basic objects of supersymmetry, are well known since the works of F.A. Berezin, G.I. Kac, D.A. Leites, J. Wess and B. Zumino who introduced the concepts of supermatrices, supertrace and superdeterminant. These concepts will be generalized here.

Recently, there have been many attempts to generalize $\mathbb{Z}_2$-graded constructions to the $\mathbb{Z}_3$-graded case. Many such attempts, though, were aimed at the description of exotic statistics. We think that our construction could describe some properties of the quarks, in particular the ternary aspects of their associations.

II. THE $\mathbb{Z}_3$-GRADED GRASSMANN ALGEBRA

The ordinary Grassmann algebra is generated by anticommuting entities:

$$\eta_i\eta_j = -\eta_j\eta_i$$

This can be viewed in the following way. $\mathbb{Z}_2$ is the group of the permutations of two elements. It can be represented by the real numbers $(-1)$ and 1:

$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} \begin{pmatrix} A & B \\ B & A \end{pmatrix} +1 -1$$

When an element of the permutation group is applied to the product of two generators, it is followed by the multiplication by the number representing this permutation.

Similar operations can be performed with $\mathbb{Z}_3$, which is faithfully represented by the complex numbers 1, $j$ and $j^2$, where $j$ is a cubic root of 1, $e^{\frac{2\pi}{3}}$:

$$\begin{pmatrix} A & B & C \\ A & B & C \\ A & B & C \end{pmatrix} \begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix} \begin{pmatrix} A & B & C \\ 1 & j & j^2 \end{pmatrix}$$
When one applies an element of the cyclic permutation group to the product of three generators, it is multiplied by the complex number representing this permutation:

\[ \theta_i \theta_k \theta_l = j \theta_k \theta_l \theta_i \]

The generators defined in this way have the following properties: their cubes vanish, as also does a product of any four generators:

\[ \theta_i^3 = 0 \]
\[ \theta_i \theta_k \theta_l \theta_m = 0 \]

Therefore the dimension of the algebra generated by \( N \) independant elements is

\[ \left( \frac{N}{\theta'} \right) + \left( \frac{N^2}{\theta \theta'} \right) + \left( \frac{N^3 - N}{3 \theta \theta \theta'} \right) \]

To restore the symmetry between grades 1 (the \( \theta' \)'s) and 2 (the \( \theta \theta' \)'s), we can add \( N \) grade 2 generators \( \bar{\theta}_i \) that behave like the products of two \( \theta' \)'s, that is

\[ \bar{\theta}_i \bar{\theta}_k \bar{\theta}_l = j^2 \bar{\theta}_k \bar{\theta}_l \bar{\theta}_i \]
\[ \theta_i \bar{\theta}_k = j \bar{\theta}_k \theta_i \]

In the case of the ordinary Grassmann algebra, the products of an odd number of generators are automatically anticommutative, whereas the products of an even number of generators commute with all other elements. In the \( \mathbb{Z}_3 \)-graded case, this is no longer true. For example,

\[ \theta_i \theta_k \bar{\theta}_l = j \theta_i \bar{\theta}_l \theta_k = j^2 \bar{\theta}_l \theta_i \theta_k; \]

but in the same time, \( \theta_k \bar{\theta}_l \) and \( \bar{\theta}_l \theta_i \), as grade 0 elements, should commute with all other elements, leading to the relations \( \theta_i (\theta_k \bar{\theta}_l) = (\theta_k \bar{\theta}_l) \theta_i \) and \( \theta_i \theta_k \bar{\theta}_l = j^2 (\bar{\theta}_l \theta_i) \theta_k = j^2 \theta_k (\bar{\theta}_l \theta_i) \), which are clearly contradictory.
This leads us to impose the following relations on all elements of a definite grade (the grade of the product of two elements being the sum of their grades, modulo 3). Let us denote by $a, b, \ldots$ the elements of grade 0, by $A, B, \ldots$ the elements of grade 1, by $\bar{A}, \bar{B}, \ldots$ the elements of grade 2 and by $\mathcal{X}$ an element of arbitrary grade. Then the rules

\[
\begin{align*}
\{ & a\mathcal{X} = \mathcal{X}a \\
& A\bar{A} = \bar{j}\bar{A}A
\end{align*}
\]

define entirely the $\mathbb{Z}_3$-graded Grassmann algebra. We obtain the ternary rules $\theta_i\theta_k\theta_l = j\theta_k\theta_l\theta_i$ and $\bar{\theta}_i\bar{\theta}_k\bar{\theta}_l = j^2\bar{\theta}_k\bar{\theta}_l\bar{\theta}_i$ directly from the second rule by replacing $A$ and $\bar{A}$ with products of one or two generators $\theta$ or $\bar{\theta}$.

With the unit element $\mathbb{I}$, the algebra contains the following elements:

- Grade 0: $\mathbb{I}, \theta\bar{\theta}, \theta\theta\theta, \bar{\theta}\bar{\theta}\bar{\theta}$
- Grade 1: $\theta, \bar{\theta}\bar{\theta}$
- Grade 2: $\bar{\theta}, \theta\theta$

and its dimension is

\[
D = 1 + 2N + 3N^2 + 2\frac{N^3 - N}{3} = \frac{3 + 4N + 9N^2 + 2N^3}{3}
\]

One can note that the grade 0 elements recall formally the only observable combinations of quark fields in chromodynamics based on the $SU(3)$ symmetry, that is the mesons which are the combinations $q\bar{q}$ of one quark and one antiquark, and the hadrons which are the combinations $qqq$ or $q\bar{q}\bar{q}$ of three quarks or three antiquarks.

From now on, we shall denote the grade of an object $X$ by $\partial X$.

III. $\mathbb{Z}_3$-MATRICES

We define the $\mathbb{Z}_3$-matrices in analogy with the supermatrices, which form a $\mathbb{Z}_2$-graded matrix algebra and whose entries are elements of a Grassmann algebra.
First, we define a $\mathbb{Z}_3$-graded complex matrix algebra by dividing the algebra $\mathcal{A}$ of $3 \times 3$ block matrices into three parts $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2$. A matrix is an element of $\mathcal{A}_0$ (resp. $\mathcal{A}_1$, $\mathcal{A}_2$) and is of grade 0 (resp. 1, 2) if it has the form shown below:

\[
\begin{pmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & C
\end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix}
0 & A & 0 \\
0 & 0 & B \\
C & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & A \\
B & 0 & 0 \\
0 & C & 0
\end{pmatrix}.
\]

This gives a $\mathbb{Z}_3$-graded structure to the algebra of matrices, in the sense that the grade of the product of two matrices is the sum of the grades of these matrices modulo 3.

We can then tensorize this graded algebra with our $\mathbb{Z}_3$-graded Grassmann algebra, the grade of a $\mathbb{Z}_3$-matrix being the sum modulo 3 of the grade of the matrix and of the grade of the Grassmann element. So a $\mathbb{Z}_3$-matrix is of grade 0 (resp. 1, 2) if its blocks contain only Grassmann elements with the respective grades:

\[
\begin{pmatrix}
0 & 2 & 1 \\
1 & 0 & 2 \\
2 & 1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 2 \\
2 & 1 & 0 \\
0 & 2 & 1
\end{pmatrix}, \quad \begin{pmatrix}
2 & 1 & 0 \\
0 & 2 & 1 \\
1 & 0 & 2
\end{pmatrix}
\]

grade 0 \quad \text{grade 1} \quad \text{grade 2}

It is easy to verify that the grade of the product of two $\mathbb{Z}_3$-matrices is the sum of their grades modulo 3.

The algebra of $\mathbb{Z}_3$-matrices contains a neutral element, $\mathbb{I}$, the $\mathbb{Z}_3$-matrix whose only non-vanishing elements are the unit of the Grassmann algebra occupying the main diagonal. The existence of this element enables us to define the invertibility of a $\mathbb{Z}_3$-matrix.

**Theorem 1** (a) A block of a $\mathbb{Z}_3$-matrix is invertible if and only if the block matrix formed with the coefficients of $\mathbb{I}$ in the development of the elements of the block over the grade 0 elements of the Grassmann algebra ($\mathbb{I}, \theta \bar{\theta}, \theta \theta \theta, \bar{\theta} \bar{\theta} \bar{\theta}$) is invertible.

(b) A $\mathbb{Z}_3$-matrix is invertible if and only if its grade 0 blocks are invertible. The inverse of a grade 1 (resp. 2, 0) $\mathbb{Z}_3$-matrix is a grade 2 (resp. 1, 0) $\mathbb{Z}_3$-matrix.
The product of two invertible $\mathbb{Z}_3$-matrices is an invertible $\mathbb{Z}_3$-matrix.

Proof.

(a) Let us decompose a $\mathbb{Z}_3$-matrix $M$ into its complex component matrices along the elements of the Grassmann algebra: $M = M_\emptyset I + M_\mu \theta_\mu + \bar{M}_\mu \bar{\theta}_\mu + M_{\mu\nu} \theta_\mu \theta_\nu + \bar{M}_{\mu\nu} \bar{\theta}_\mu \bar{\theta}_\nu + \tilde{M}_{\mu\nu\eta} \theta_\mu \theta_\nu \theta_\eta + \tilde{M}_{\mu\nu\eta} \bar{\theta}_\mu \bar{\theta}_\nu \bar{\theta}_\eta$. We also consider another $\mathbb{Z}_3$-matrix $N$ that we decompose in the same way. Then it is easy to see that the component of $MN$ along $I$ is $M_\emptyset N_\emptyset$ (no product of two generators different from $I$ can give something proportional to $I$) so that for $M$ to be invertible, its component $M_\emptyset$ must be invertible. Conversely, if $M_\emptyset$ is invertible, let us put $N_\emptyset = (M_\emptyset)^{-1}$. Then the product $MN$ differs from $I$ only in terms of degree 2 and higher. However, we can choose $N_\mu = -(M_\emptyset)^{-1}M_\mu (M_\emptyset)^{-1}$ and $\bar{N}_\mu = -(M_\emptyset)^{-1}\bar{M}_\mu (M_\emptyset)^{-1}$. This way, $MN$ differs from $I$ only in terms of degree 3 and higher. Choosing the higher-level components of $N$ in the same way, it is easy to expel these terms to higher degrees, until we have actually constructed the inverse of $M$, because of the finite number of terms in the development and of their cubic nilpotence.

(b) For this part of the theorem, we can use the proof of part (a) by noting that the component $M_\emptyset$ of a matrix $M$ has non-zero coefficients only in positions corresponding to grade 0 blocks. $I$ being of grade 0, it is obvious that the sum of the grades of a $\mathbb{Z}_3$-matrix and of its inverse should be 0 modulo 3.

(c) The component along $I$ of the product of two invertible matrices is the product of their components along $I$, which are invertible matrices by virtue of (b) and (a). These components being ordinary matrices, their product is an invertible matrix. We use (b) once more to conclude.
Generalizing this idea, we define the left (resp. right) product of a $\mathbb{Z}_3$-matrix by an element $\lambda$ of the Grassmann algebra as its left (resp. right) multiplication by the following diagonal $\mathbb{Z}_3$-matrix:

\[
\begin{pmatrix}
\lambda \\
(-1)^{\partial \lambda} \lambda
\end{pmatrix}
\]

Note that in general, $\lambda M \neq M \lambda$, but we have $\lambda(MN) = (\lambda M)N$, $(M\lambda)N = M(\lambda N)$, $M(N\lambda) = (MN)\lambda$ and $\lambda(M\mu) = (\lambda M)\mu$ which give our algebra the structure of a bimodule with respect to the Grassmann algebra.

**Definition 2** Let us define a $\mathbb{Z}_3$-matrix $M$ with the following block structure:

\[
M = \begin{pmatrix}
A & B & C \\
D & E & F \\
G & H & I
\end{pmatrix}
\]

Its $\mathbb{Z}_3$-trace is defined by: $\text{tr}_{\mathbb{Z}_3}(M) = \text{tr}(A) + j^{2(1-\partial M)} \text{tr}(E) + j^{(1-\partial M)} \text{tr}(I)$

that is:

- If $M$ is of grade 0, then $\text{tr}_{\mathbb{Z}_3}(M) = \text{tr}(A) + j^2 \text{tr}(E) + j \text{tr}(I)$
- If $M$ is of grade 1, then $\text{tr}_{\mathbb{Z}_3}(M) = \text{tr}(A) + \text{tr}(E) + \text{tr}(I)$
- If $M$ is of grade 2, then $\text{tr}_{\mathbb{Z}_3}(M) = \text{tr}(A) + j \text{tr}(E) + j^2 \text{tr}(I)$

The supertrace of a supermatrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ was defined by $\text{str} M = \text{tr}(A) + (-1)^{1-\partial M} \text{tr}(D)$.

The following theorem can be easily proved:
Theorem 3 (a) If $M$ and $N$ are of the same grade,

$$\text{tr}_{Z_3}(M + N) = \text{tr}_{Z_3}(M) + \text{tr}_{Z_3}(N)$$

(b) $\text{tr}_{Z_3}(\lambda M) = \lambda \text{tr}_{Z_3}(M)$ and $\text{tr}_{Z_3}(M\lambda) = \text{tr}_{Z_3}(M)\lambda$

(c) • If $M$ and $N$ are of grade 0 then $\text{tr}_{Z_3}(MN) = \text{tr}_{Z_3}(NM)$

• If $M$ is of grade 1 and $N$ of grade 2, then $\text{tr}_{Z_3}(MN) = j \text{tr}_{Z_3}(NM)$

Corollary 4 • If $M, N$ and $P$ are of grade 1 then $\text{tr}_{Z_3}(MNP) = j \text{tr}_{Z_3}(NPM)$

• If $M, N$ and $P$ are of grade 2 then $\text{tr}_{Z_3}(MNP) = j^2 \text{tr}_{Z_3}(NPM)$

The proofs are straightforward, and the only non trivial observation concerns the fact that $\text{tr}(MN) = \text{tr}(NM)$ is not always true if $M$ and $N$’s coefficients are not numbers.

Finally, we can define the generalization of the superdeterminant this way:

Definition 5 $Z_3$-determinant: If $M$ is a grade 0 invertible $Z_3$-matrix, then its $Z_3$-determinant is

$$\text{det}_{Z_3}(M) = \text{det}(A - CI^{-1}G - (B - CI^{-1}H)(E - FI^{-1}H)^{-1}(D - FI^{-1}G)) \times \times (\text{det}(E - FI^{-1}H))^{j^2}(\text{det} I)^j$$

Theorem 6 If $M$ and $N$ are two invertible grade 0 $Z_3$-matrices,

$$\text{det}_{Z_3}(MN) = (\text{det}_{Z_3} M)(\text{det}_{Z_3} N)$$

Proof. Any grade 0 invertible $Z_3$-matrix can be decomposed into the product of three $Z_3$-matrices: $M = M_1M_0M_2$ with
It is very easy to show that the theorem is true for any $M$ if $N$ is block-diagonal or inferior-block-triangular with the identity in the diagonal blocks. It is also easy to generalize to an arbitrary $N$ in the case where $M$ is block-diagonal or superior-block-triangular with the identity in the diagonal blocks. Therefore, we have:

$$\det_{Z_3}(MN) = \det_{Z_3}(M_1) \det_{Z_3}(M_0) \det_{Z_3}(M_2 N_1) \det_{Z_3}(N_0) \det_{Z_3}(N_2) =$$

$$= \det_{Z_3}(M) \det_{Z_3}(M_2 N_1) \det_{Z_3}(N)$$

and the theorem remains to be proved only for $M$ of the form \( \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ B & C & 1 \end{pmatrix} \) and $N$ of the form \( \begin{pmatrix} 1 & D & E \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix} \). One can note that \( \begin{pmatrix} 1 & 0 & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 & 0 \\ A & 0 & 0 \\ 0 & 1 \end{pmatrix} \) so that we can construct the matrix $N_1$, element by element, starting from the matrix that contains only its diagonal blocks, using left multiplication by a series of matrices that contain only one non-zero element out of the diagonal blocks.

We show the theorem for $M$ of the form of $M_2$ and $N$ containing only one element out of the diagonal blocks, using the fact that if $\partial X \neq 0$, then $(1 + X)^\alpha = 1 + \alpha X + \frac{\alpha(\alpha-1)}{2}X^2$. If $N_1 = N_1^0N_1^1$ we can decompose $M_2N_1^0$ in the form \( \begin{pmatrix} 1 & D & E \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix} \). Then,

$$\det_{Z_3}(M_2N_1) = \det_{Z_3}((M_2N_1^0)_1(M_2N_1^0)_0(M_2N_1^0)_2N_1^1) =$$

$$= \det_{Z_3}((M_2N_1^0)_0) \det_{Z_3}((M_2N_1^0)_2N_1^1) = \det_{Z_3}((M_2N_1^0)_0)$$

if $N_1^1$ contains only one element out of its diagonal blocks. But $\det_{Z_3}((M_2N_1^0)_0) = \det_{Z_3}(M_2N_1^0)$. We can perform the same operation, until only the product of $M_2$ and the matrix formed with the diagonal blocks of $N_1$ (which are the identity) remains. Thus $\det_{Z_3}(M_2N_1) = 1$. \fbox{\textbullet}
Theorem 7 If $M$ is a grade 0 invertible $\mathbb{Z}_3$-matrix, its $\mathbb{Z}_3$-determinant can also be expressed in the following five alternative ways:

$$\det_{\mathbb{Z}_3}(M) = \det(A - CI^{-1}G) \times$$

$$\times (\det(E - FI^{-1}H - (D - FI^{-1}G)(A - CI^{-1}G)^{-1}(B - CI^{-1}H)))^j \det(I)^j$$

$$\det_{\mathbb{Z}_3}(M) = (\det A)(\det(E - DA^{-1}B - (F - DA^{-1}C)(I - GA^{-1}C)^{-1}(H - GA^{-1}B)))^j \times$$

$$\times (\det(I - GA^{-1}C))^j$$

$$\det_{\mathbb{Z}_3}(M) = (\det A)(\det(E - DA^{-1}B))^j \times$$

$$\times (\det(I - GA^{-1}C - (H - GA^{-1}B)(E - DA^{-1}B)^{-1}(F - DA^{-1}C)))^j$$

$$\det_{\mathbb{Z}_3}(M) = (\det(A - BE^{-1}D - (C - BE^{-1}F)(I - HE^{-1}F)^{-1}(G - HE^{-1}D)))^j \times$$

$$\times (\det E)^j (\det(I - HE^{-1}F))^j$$

$$\det_{\mathbb{Z}_3}(M) = (\det(A - BE^{-1}D)(\det E))^j \times$$

$$\times (\det(I - HE^{-1}F - (G - HE^{-1}D)(A - BE^{-1}D)^{-1}(C - BE^{-1}F)))^j$$

Proof. We use the following decompositions of $M$:

$$\begin{pmatrix}
A - CI^{-1}G & 0 \\
D - FI^{-1}G & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
C \\
F \\
I
\end{pmatrix}
\begin{pmatrix}
1 & (A - CI^{-1}G)^{-1}(B - CI^{-1}H) & 0 \\
0 & 1 & 0 \\
I^{-1}G & I^{-1}H & 1
\end{pmatrix}$$

$$\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
A B & 0 & 0 & 0 \\
0 & E - DA^{-1}B - (F - DA^{-1}C)(I - GA^{-1}C)^{-1}(H - GA^{-1}B) & 0 & 0 \\
0 & H - GA^{-1}B & 0 & 0 \\
I & I^{-1}G & I & I^{-1}H
\end{pmatrix}$$

$$\begin{pmatrix}
A B & 0 & 0 & 0 \\
0 & E - DA^{-1}B & 0 & 0 \\
0 & H - GA^{-1}B & 0 & 0 \\
I & I^{-1}G & I & I^{-1}H
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
A - BE^{-1}D - (C - BE^{-1}F)(I - HE^{-1}F)^{-1}(G - HE^{-1}D) & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
0 & 0 & 1 & 0 \\
E & I & I^{-1}G & I^{-1}H
\end{pmatrix}$$

$$\begin{pmatrix}
A - BE^{-1}D & B & 0 & 0 \\
0 & E & 0 & 0 \\
G & H & I & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A - BE^{-1}D & 0 & 0 & 0 \\
0 & E & 0 & 0 \\
G & H & I & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$
and make use of the theorem \( \Box \). 

In the case of the superdeterminant, the following two expressions are equivalent:

\[
\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C). (\det D)^{-1} = \det A.(\det(D - CA^{-1}B))^{-1}
\]

This restores the symmetry between the two diagonal elements on the one hand, and between the two off diagonal elements on the other hand. Here we have six expressions, restoring the \( S_3 \) symmetry between the three grade 0, grade 1 and grade 2 elements of the \( \mathbb{Z}_3 \)-matrix which is broken in any expression separately chosen.

**Theorem 8** If \( M \) is a grade 0 invertible \( 3 \times 3 \) \( \mathbb{Z}_3 \)-matrix, then

\[
\det_{\mathbb{Z}_3}(\exp M) = \exp(\text{tr}_{\mathbb{Z}_3} M)
\]

Proof. Let \( M \) be the matrix:

\[
M = \begin{pmatrix} a & \bar{A} & A \\ B & b & \bar{B} \\ \bar{C} & C & c \end{pmatrix}
\]

where \( a, b, c \) are three grade 0 invertible elements of the Grassmann algebra (they must be invertible in order for \( M \) to be invertible, see theorem \( \Box \)), \( A, B, C \) are three grade 1 elements of the Grassmann algebra, and \( \bar{A}, \bar{B}, \bar{C} \) are three grade 2 elements of the Grassmann algebra. Assuming that \( a, b, c \) are distinct, straightforward calculus give the following expression for the exponential of the matrix \( M \):

\[
\exp M = \begin{cases} 
    e^a + f(a, b)\bar{A}B + f(a, c)A\bar{C} + g(a, b, c)(\bar{A}\bar{B}\bar{C} + j.CBA) \\
    h(a, b)B + l(a, b, c)\bar{B}\bar{C} \\
    h(a, c)\bar{C} + l(a, b, c)CB \\
    h(a, b)\bar{A} + l(a, b, c)AC \\
    e^b + j^2 f(b, c)CB + j.f(b, a)\bar{A}B + j.g(b, c, a)(\bar{A}\bar{B}\bar{C} + j.CBA) \\
    h(b, c)C + l(a, b, c)\bar{C}\bar{A}
\end{cases}
\]
where

\[ f(x, y) = \frac{(x - y - 1)e^x + e^y}{(x - y)^2} \]

\[ g(x, y, z) = \left[ \frac{1}{(x-y)(y-z)} - \frac{1}{(x-z)(y-z)} - \frac{1}{(x-z)(x-y)} \right] e^x + n \]

\[ + \frac{e^y}{(y-z)(y-x)} + \frac{e^z}{(z-y)(z-x)} \]

\[ h(x, y) = \frac{e^x - e^y}{x - y} \]

\[ l(x, y, z) = \frac{e^x}{(x-y)(x-z)} + \frac{e^y}{(y-z)(y-x)} + \frac{e^z}{(z-x)(z-y)} \]

From this we compute

\[
\det_{Z_3}(\exp M) = e^{a+j^2b+je^{2k}} \left[ I + (e^{-a}f(a, b) + e^{-b}f(b, a) - e^{-(a+b)}(h(a, b))^2)\bar{A}\bar{B} + (e^{-a}f(a, c) + e^{-c}f(c, a) - e^{-(a+c)}(h(a, c))^2)\bar{A}\bar{C} + (e^{-b}g(b, c) + e^{-c}g(c, b) - e^{-(b+c)}(h(b, c))^2)\bar{B}\bar{C} + (e^{-a}g(a, b, c) + e^{-b}g(b, c, a) + e^{-c}g(c, a, b) - e^{-(a+b+c)}h(a, b)h(b, c)h(c, a) - \bar{A]\bar{B}\bar{C} + j.CBA) \right]
\]

and it is then easy to verify that all terms inside the brackets vanish, except for \( I \). The cases where \( a = b, b = c \) or \( c = a \) are just limits of the previous case. \( \Box \)

These constructions will be used in a forthcoming paper concerning the construction of a gauge theory based on a \( Z_3 \)-graded non-commutative geometry model similar to the one used by Coquereaux et al., using instead Kerner’s differential whose cube is zero, whereas its square is not.\[\]
IV. ACKNOWLEDGMENTS

I wish to thank Profs. Kerner and Abramov for fruitful discussions. This work has been financed by a grant from the French Ministère de l’Enseignement Supérieur et de la Recherche.
REFERENCES

1 F.A. Berezin, G.I. Kac, Math. USSR Sbornik, 11, 311 (1970)

2 F.A. Berezin, D.A. Leites, Soviet Math. Dokl. 16, 1218 (1975)

3 J. Wess, B. Zumino, Nucl.Phys. B70, 39 (1974)

4 J. Wess, B. Zumino, Phys.Lett. 66B, 361 (1977)

5 J. Wess, B. Zumino, Phys.Lett. 74B, 51 (1978)

6 R. Kerner, J. Math. Phys. 33, 403 (1991)

7 R. Kerner, in Symmetries in Science VI Plenum Press, 373 (1993)

8 R. Kerner, to appear in Lett.Math.Phys. (1995)

9 W.S. Chung, J. Math. Phys. 35, 2497 (1993)

10 N. Mohammedi, hep-th/9412133 (1994)

11 R. Coquereaux, G. Esposito-Farese, G. Vaillant, Nuclear Physics B353, 689 (1991)