CESARO-HARDY OPERATORS ON BILATERAL GRAND LEBESGUE SPACES

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Abstract.

We obtain in this short article the non-asymptotic estimations for the norm of (generalized) Cesaro-Hardy integral operators in the so-called Bilateral Grand Lebesgue Spaces. We also give examples to show the sharpness of these inequalities.

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1. Introduction

The linear integral operator $U_{\alpha,\beta,\lambda}[f](x) = U[f](x)$, or, more precisely, the family of operators of a view

$$U(x) = U_{\alpha,\beta,\lambda}[f](x) = x^{-\beta} \int_{0}^{x} \frac{y^{-\alpha} f(y) \, dy}{|x-y|^\lambda}$$

(1.0)

is called generalized Cesaro-Hardy integral operator, or fractional integral.

Here $x, y \in (0, \infty)$, $\alpha, \beta, \lambda = \text{const} \in (0, 1)$, $\alpha + \beta + \lambda < 1$.

We denote as usually the classical $L_p$ Lebesgue - Riesz norm

$$|f|_p = \left( \int_{X} |f(x)|^p \, dx \right)^{1/p}; \quad f \in L_p \iff |f|_p < \infty,$$

(1.1)

and denote $L(a, b) = \cap_{p \in (a, b)} L_p$.

Here $X = R_+$ or $X = (R_+)^d$ or $X = R^d$.

The case of operators of a view

$$I(x) = I_{\alpha,\beta,\lambda}[f](x) = ||x||^{-\beta} \int_{R^d} \frac{||y||^{-\alpha} f(y) \, dy}{||x-y||^\lambda}$$

was considered in [26], chapter 11, see also [27]; it was proved in particular the following estimation. Define
\[ p_{-}^{(d)} = \frac{d}{d - \alpha}, \quad p_{+}^{(d)} = \frac{d}{d - \alpha - \lambda}, \]

\[ p_{-} = p_{-}^{(1)} = \frac{1}{1 - \alpha}, \quad p_{+} = p_{+}^{(1)} = \frac{1}{1 - \alpha - \lambda}, \]

\[ q_{-}^{(d)} = \frac{d}{\beta + \lambda}, \quad q_{+}^{(d)} = \frac{d}{\beta}, \]

\[ q_{-} = q_{-}^{(1)} = \frac{1}{\beta + \lambda}, \quad q_{+} = q_{+}^{(1)} = \frac{1}{\beta}. \]

\[ \kappa^{(d)} = (\alpha + \beta + \lambda)/d; \quad \kappa = \kappa_1 = \alpha + \beta + \lambda. \quad (1.2) \]

Define also for the arbitrary value \( p \) from the set \( p \in (p_{-}^{(d)}, p_{+}^{(d)}) \) the correspondent value \( q = q(p) \), \( q \in (q_{-}^{(d)}, q_{+}^{(d)}) \) as follows:

\[ 1 + \frac{1}{q} = \frac{p}{p} + \frac{\alpha + \beta + \lambda}{d} = \frac{1}{p} + \kappa^{(d)}. \quad (1.3) \]

The identity (1.3) defined uniquely the function \( p = p(q) \) and inversely the function \( q = q(p) \).

It is proved in \[28\] that

\[ |I_{\alpha, \beta, \lambda}[f]|_{q(p)} \leq V(\alpha, \beta, \lambda; p) \cdot |f|_{p}, \quad p \in (p_{-}^{(d)}, p_{+}^{(d)}), \quad (1.4) \]

where for the optimal, i.e. minimal value \( V(\alpha, \beta, \lambda; p) \):

\[ V(\alpha, \beta, \lambda; p) \overset{def}{=} \sup_{f \in L(p), f \neq 0} \left[ \frac{|I_{\alpha, \beta, \lambda}[f]|_{q(p)}}{|f|_{p}} \right] \]

are true the following estimates:

\[ \frac{C_1(\alpha, \beta, \lambda)}{(p - p_{-}^{(d)})(p_{+}^{(d)} - p)} \kappa^{(d)} \leq V(\alpha, \beta, \lambda; p) \leq \frac{C_2(\alpha, \beta, \lambda)}{(p - p_{-}^{(d)})(p_{+}^{(d)} - p)} \kappa^{(d)}, \quad (1.5) \]

\[ C_1(\alpha, \beta, \lambda), \quad C_2(\alpha, \beta, \lambda) \in (0, \infty). \quad (1.5a) \]

Our purpose is the extension of inequality (1.5) into the generalized operator of Cesaro - Hardy view:

\[ |U_{\alpha, \beta, \lambda}[f]|_{q(p)} \leq K(\alpha, \beta, \lambda; p) \cdot |f|_{p}, \quad p \in (p_{-}^{(d)}, p_{+}^{(d)}), \quad (1.6) \]

with exact “constant” \( K(\alpha, \beta, \lambda; p) \) estimation, alike (1.5) - (1.5a).

As before, we will understood in the capacity of the coefficient \( K = K(\alpha, \beta, \lambda; p) \) its minimal value:

\[ K(\alpha, \beta, \lambda; p) \overset{def}{=} \sup_{f \in L(p), f \neq 0} \left[ \frac{|U_{\alpha, \beta, \lambda}[f]|_{q(p)}}{|f|_{p}} \right]. \quad (1.7) \]
Notice that the case $\alpha + \beta + \lambda = 1$ was investigated in the classical book [23], with exact constant computation. Therefore, we do not impose this condition.

We will obtain also the generalization of these estimations on the so-called Grand Lebesgue Spaces (GLS). Note that the Sobolev’s weight space estimates for these operators are obtained in a recent article [25], without constants evaluating.

These operators are used in the theory of Fourier transform, probability theory, theory of PDE, in the functional analysis, in particular, in the theory of interpolation of operators etc., see for instance [1], [3], [11], [19], [20].

One of absolutely unexpected application of these estimations are in the theory of Navier-Stokes equations, see e.g. [21], [22], [24], [29]. Authors hope to use further the results of this report in the theory of Navier-Stokes equation.

We use symbols \( C(X,Y), C(p,q;\psi) \), etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like \( C_1(X,Y) \) and \( C_2(X,Y) \).

The relation \( g(\cdot) \approx h(\cdot), p \in (A,B) \), where \( g = g(p), h = h(p), g,h : (A,B) \to R_+ \), denotes as usually

\[
0 < \inf_{p \in (A,B)} h(p)/g(p) \leq \sup_{p \in (A,B)} h(p)/g(p) < \infty.
\]

The symbol \(~\) will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

\[
I(x \in A) = 1, x \in A, I(x \in A) = 0, x / \in A;
\]

here \( A \) is a measurable set.

All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

2. Main result: upper and lower estimations for Cesaro-Hardy operator

We consider in this section only the one-dimensional case for these operators: \( d = 1 \).

**Lemma 2.1.** If the inequality (1.6) there holds for every function \( f \) from the Schwartz space \( S(R_+) \), then

\[
1 + \frac{1}{q} = \frac{1}{p} + \alpha + \beta + \lambda = \frac{1}{p} + \kappa.
\]

**Proof.** We will use the well-known scaling, or equally, dilation method, see [20], [30]. Indeed, let the inequality (1.6) be satisfied for some function \( f(\cdot) \neq 0 \) from the set \( S(R_+) \).

Let \( \gamma = \text{const} \in (0,\infty) \); consider the dilation function

\[
f_\gamma(x) = T_\gamma[f](x) = f(\gamma x).
\]

Evidently, \( f_\gamma(\cdot) \in S(R_+) \). Therefore

\[
|U_{\alpha,\beta,\lambda}[T_\gamma[f]]_q | \leq K(\alpha, \beta, \lambda; p) \cdot |T_\gamma[f]|_p.
\]

We get consequently after simple calculations:
\[ |T_\gamma[f]|_p = \gamma^{-1/p} |f|_p, \]
\[ |U_{\alpha,\beta,\lambda}[T_\gamma[f]]|_q = \gamma^{\alpha+\beta+\lambda-1-1/q} |U_{\alpha,\beta,\lambda}[f]|_q. \]

We conclude substituting into (1.6)
\[ \gamma^{\alpha+\beta+\lambda-1-1/q} \cdot |U_{\alpha,\beta,\lambda}[f]|_q \leq K(\alpha, \beta, \lambda; p) \cdot \gamma^{-1/p} \cdot |f|_p. \]

Since \( \gamma \) is arbitrary positive number, we see
\[ \alpha + \beta + \lambda - 1 - 1/q = -1/p, \]
which is equivalent to the assertion of Lemma 1.

**Lemma 2.2.** If the inequality (1.6) there holds with finite value of \( K(\alpha, \beta, \lambda; p) \) for every function \( f \) from the space \( L^p(R_+) \), then
\[ p_- < p \leq p_+. \]

Correspondingly, \( q_- < q \leq q_+ \).

**Proof. A. Case** \( p = p_- = 1/(1 - \alpha) \).
Let us consider the function
\[ f_{\Delta, \theta}(x) = x^{-\Delta} \log x|^\theta I(x \in (0, 1)), \quad \Delta = 1 - \alpha, \quad \theta = \text{const} > -(1 - \alpha). \]
Then \( |f_{\Delta, \theta}|_{p_-} < \infty \), but it is easy to verify that \( U_{\alpha,\beta,\lambda}[f_{\Delta, \theta}] \notin L_{q_-} \).

**Proof. B. Case** \( p = p_+ = 1/(1 - \alpha - \lambda) \).
Define a function
\[ g(x) = x^{-(1-\alpha-\lambda)} I(x \in (1, \infty)); \]
then
\[ \forall p > p_+ \Rightarrow g(\cdot) \in L_p, \]
but
\[ U_{\alpha,\beta,\lambda}[g] \notin L_{q_+}, \quad q_+ = q(p_+). \]

Now we formulate and prove the main result of this article.

**Theorem 2.1.**
\[ \frac{C_3(\alpha, \beta, \lambda)}{|p - p_-|^\kappa} \leq K(\alpha, \beta, \lambda; p) \leq \frac{C_4(\alpha, \beta, \lambda)}{|p - p_-|^\kappa}, \quad p \in (p_-, p_+], \]
\[ C_3(\alpha, \beta, \lambda), \quad C_4(\alpha, \beta, \lambda) \in (0, \infty). \]

**Proof. Upper bound.**
The upper bound follows immediately from the inequality
\[ K(\alpha, \beta, \lambda; p) \leq \left[ \frac{\Gamma((1 - 1/p - \alpha)/\kappa) \Gamma((\alpha + \beta)/\kappa)}{\Gamma((1 - 1/p + b)/\kappa)} \right]^{\kappa}, \]

see [26], p. 213-215.

**Proof. Lower bound.**

Let us consider the following example.

\[ f_0(x) := x^{-(1-\alpha)} I(x \geq 1); \]

then

\[ |f_0|_p \asymp c (p - p_-)^{-1/p}, \quad p \in (p_-, p_+]. \]

Further, we have denoting for the values \( x > 1, \ x \to \infty \) and \( q = q(p) \):

\[ u_0(x) = U_{\alpha, \beta, \lambda}[f_0](x) : \]

\[ u_0(x) = x^{-\beta} \int_0^x \frac{y^{-\alpha} y^{-1+\alpha} dy}{|x - y|^\lambda} = x^{-(\beta+\lambda)} \int_{1/x}^1 \frac{z^{-1} dz}{(1 - z)^\lambda} \sim \]

\[ x^{-(\beta+\lambda)} \int_{1/x}^1 z^{-1} dz = x^{-(\beta+\lambda)} \log x; \]

\[ |u_0|_q \asymp \int_1^\infty x^{-q(\beta+\lambda)} (\log x)^q dx = \frac{\Gamma(q+1)}{[q(\beta + \lambda) - 1]^q}; \]

\[ |u_0|_q \asymp \frac{1}{(q - q_-)^{1+1/q}} \asymp \frac{1}{(p - p_-)^{1+1/q}}; \]

\[ \frac{|u_0|_q}{|f|_p} \asymp C(\alpha, \beta, \lambda). \]

This completes the proof of theorem 2.1.

### 3. Multidimensional case

We recall here the definition of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces. More detail information about this spaces see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [2], chapter 16,17; Leoni G. [13], chapter 11; using for us theory of operators interpolation in this spaces see in [2], [1].

Let \( (X_j, A_j, j = 1, 2, \ldots, d) \) be measurable spaces with sigma-finite non-trivial measures \( j \). (It is clear that in this article \( X_j = R_+ \) and \( \mu_j \) is ordinary Lebesgue measure.)

Let \( p = \vec{p} = (p_1, p_2, \ldots, p_d) \) be \( d \) dimensional vector such that \( 1 \leq p_j \leq \infty \). Recall that the **anisotropic** Lebesgue space \( L(\vec{p}) \) consists on all the total measurable real valued function \( f = f(x_1, x_2, \ldots, x_d) = f(x) = f(\vec{x}), \ x_j \in X_j \) with finite norm \( |f|_{\vec{p}} \defeq \]

\[ \left( \int_{X_d} \mu_d(dx_d) \left( \int_{X_{d-1}} \mu_{d-1}(dx_{d-1}) \ldots \left( \int_{X_1} \mu_1(dx_1) |f(x_1, x_2, \ldots, x_d)|^{p_1} \right)^{p_2/p_1} \right)^{p_3/p_2} \right)^{1/p_d}. \]
Note that in general case \( |f|_{p_1, p_2} \neq |f|_{p_2, p_1} \) but \( |f|_{p, p} = |f|_p \). Observe also that if \( f(x_1, x_2) = g_1(x_1)g_2(x_2) \), (condition of factorization), then \( |f|_{p_1, p_2} = |g_1|_{p_1} |g_2|_{p_2} \), (formula of factorization).

Let also

\[
\tilde{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}, \quad \tilde{\beta} = \{\beta_1, \beta_2, \ldots, \beta_d\},
\]

\[
\tilde{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}
\]

be three numerical \( d \) - dimensional vectors such that

\[
0 < \alpha_i, \beta_i, \lambda_i; \quad \alpha_i + \beta_i + \lambda_i < 1, \quad i = 1, 2, \ldots, d.
\]

We define the multidimensional (generalized) Cesaro - Hardy operator \( U_{\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}}[f](\vec{x}), \quad \vec{x} \in (R_+)^d \) as follows: \( U_{\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}}[f](\vec{x}) \overset{\text{def}}{=} \)

\[
\int_0^{x_1} \frac{x_1^{-\beta_1} y_1^{-\alpha_1}}{|x_1 - y_1|^\lambda_1} dy_1 \left[ \int_0^{x_2} \frac{x_2^{-\beta_2} y_2^{-\alpha_2}}{|x_2 - y_2|^\lambda_2} dy_2 \left[ \cdots \left[ \int_0^{x_d} \frac{x_d^{-\beta_d} y_d^{-\alpha_d}}{|x_d - y_d|^\lambda_d} f(y) dy_d \right] \right] \right].
\]

Let \( p = \vec{p} = (p_1, p_2, \ldots, p_d) \) and \( q = \vec{q} = (q_1, q_2, \ldots, q_d) \) be two \( d \) - dimensional vectors such that \( 1 < p_j, q_j < \infty \).

We impose on the parameters \( \{p_j, q_j\} \) in this section the following condition:

\[
1 + \frac{1}{q_j} = \frac{1}{p_j} + \alpha_j + \beta_j + \lambda_j, \quad j = 1, 2, \ldots, d.
\]

Denote

\[
p_-^{(j)} = \frac{1}{1 - \alpha_j}, \quad p_+^{(j)} = \frac{1}{1 - \alpha_j - \lambda_j}, \tag{3.5a}
\]

\[
q_-^{(j)} = \frac{1}{\beta_j + \lambda_j}, \quad q_+^{(j)} = \frac{1}{\beta_j}, \tag{3.5b}
\]

\[
\kappa_j = \alpha_j + \beta_j + \lambda_j. \tag{3.6}
\]

The equations (3.5a) and (3.5b) uniquely define the functions \( q_j = q_j(p_j) \) and conversely the functions \( p_j = p_j(q_j) \); wherein \( p_j \in (p_-^{(j)}, p_+^{(j)}) \) and correspondingly \( q_j \in (q_-^{(j)}, q_+^{(j)}) \).

Introduce as before the following function:

\[
K_{\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}}(\vec{p}) = \sup_{f \in L(\vec{p}), f \neq 0} \left[ \frac{|U_{\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}}[f]|_{\vec{q}}}{|f|_{\vec{p}}} \right], \quad \vec{q} = q(\vec{p}). \tag{3.7}
\]

**Theorem 3.1. A.** The "constant" \( K_{\tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}}(\vec{p}) \) is finite iff

\[
\forall j = 1, 2, \ldots, d \Rightarrow p_-^{(j)} < p_j \leq p_+^{(j)} \tag{3.8}
\]

and equation (3.4) is satisfied.

**B.** If both these conditions are satisfied, then
\[
\frac{C_6(\bar{\alpha}, \bar{\beta}, \bar{\lambda})}{\prod_{j=1}^{d}(p_j - p_{-j}^{(j)})^{\kappa_j}} \leq K_{\bar{\alpha}, \bar{\beta}, \bar{\lambda}}(\bar{p}) \leq \frac{C_6(\bar{\alpha}, \bar{\beta}, \bar{\lambda})}{\prod_{j=1}^{d}(p_j - p_{-j}^{(j)})^{\kappa_j}}.
\]

(3.9)

**Proof** is quite similar to the analogous proof for the weight Riesz potential, see [18] and may be omitted.

In particular, the example for lower estimate may be constructed as a *factorable* function of a view

\[
f_0(\vec{x}) = \prod_{j=1}^{d} g_j(x_j).
\]

4. **Generalization on the Grand Lebesgue Spaces (GLS).**

We recall first of all here for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [4], [5], [6], [7], [8], [9], [10], [12], [15], [16] etc. appear the so-called Grand Lebesgue Spaces GLS

\[
G(\psi) = G = G(\psi; A; B); \quad A; B = \text{const; } A \geq 1, \ B \leq \infty
\]

spaces consisting on all the measurable functions \( f : X \to R \) with finite norms

\[
||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (A; B)} \left[ \frac{|f|_p}{\psi(p)} \right].
\]

(4.1)

Here \( \psi = \psi(p), \ p \in (A, B) \) is some continuous positive on the open interval \((A; B)\) function such that

\[
\inf_{p \in (A; B)} \psi(p) > 0.
\]

(4.2)

We will denote

\[
\text{supp}(\psi) \overset{def}{=} (A; B).
\]

The set of all such a functions with support \( \text{supp}(\psi) = (A; B) \) will be denoted by \( \Psi(A; B) \).

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue - Riesz spaces \( L_p \) are extremal case of Grand Lebesgue Spaces, see [16], [17].

Let a function \( f : R_+ \to R \) be such that

\[
\exists (A, B) : 1 \leq A < B \leq \infty \Rightarrow \forall p \in (A, B) \ |f|_p < \infty.
\]

Then the function \( \psi = \psi(p) \) may be naturally defined by the following way:

\[
\psi_f(p) := |f|_p, \ p \in (A, B).
\]

(4.3)
Let now the (measurable) function $f : \mathbb{R}^+ \to \mathbb{R}$, $f \in G_\psi$ for some $\psi(\cdot)$ with support $\text{supp} \, \psi = (A, B)$ for which

$$ (a, b) := (A, B) \cap (p_-, p_+) \neq \emptyset. \quad (4.4) $$

We define a new $\psi$ - function, say $\psi_K = \psi_K(q)$ as follows.

$$ \psi_K(q) = K(\alpha, \beta, \lambda; \psi(q)) \cdot \psi(q), \ p \in (a, b). \quad (4.5) $$

**Theorem 4.1.** Denote

$$ \psi_{a,b}(p) = \psi(p) \cdot I(p \in (a, b)). $$

We assert under condition (4.4):

$$ ||U_{\alpha,\beta,\lambda}[f]||_{G_\psi K} \leq 1 \cdot ||f||_{G_\psi a,b}, \quad (5.6) $$

where the constant "1" is the best possible.

**Proof. Upper bound.**

Let further in this section $p \in (a, b)$. We can and will suppose without loss of generality $||f||_{G_\psi a,b} = 1$. Then

$$ |f|_p \leq \psi_{a,b}(p), \ p \in (a, b). \quad (4.7) $$

We conclude after substituting into the inequality (1.6)

$$ |U_{\alpha,\beta,\lambda}[f]|_{q(p)} \leq K(\alpha, \beta, \lambda; \psi_{a,b}(p)) = \psi_K(p), \ p \in (a, b). \quad (4.8) $$

The inequality (3.6) follows from (3.8) after substitution $p = p(q)$.

**Proof. Exactness.**

The exactness of the constant "1" in the proposition (3.8) follows from the theorem 2.1 in the article [17].

\[ \square \]

5. **Concluding remarks**

1. Analogously may be investigated the "conjugate" operator of a view

$$ W_{\alpha,\beta,\lambda}[f](x) = \frac{x^{-\beta}}{\Gamma(\alpha)} \int_x^\infty \frac{y^{-\alpha} f(y)dy}{(y-x)^\lambda}. \quad (5.1) $$

see [26], p. 213, [14], p. 173-176.

We retain at the same notations and the restrictions on the parameters $(\alpha, \beta, \lambda; p, q; p_-, p_+)$ as in the second section; in opposite case $K^{(W)}(\alpha, \beta, \lambda; p) = \infty$.

In particular, again

$$ 1 + \frac{1}{q} = \frac{1}{p} + \kappa, \ q = q(p). $$

We denote as before
\[ K^{(W)}(\alpha, \beta, \lambda; p) = \sup_{f \in L(p), f \neq 0} \left[ \frac{|W_{\alpha,\beta,\lambda}[f]|_{q(p)}}{|f|_p} \right]. \] (5.2)

**Proposition 5.1.**

\[ K^{(W)}(\alpha, \beta, \lambda; p) \leq C^*(\alpha, \beta, \lambda) \frac{(p - p_-)^\kappa}{(p_+ - p_-)}, \quad p \in (p_-, p_+). \] (5.3)

2. For the weighted convolution operator

\[ V_S[f](x) = \frac{1}{S(x)} \int_0^x s(t - x) f(t) \, dt, \]

where

\[ s(t) > 0, \quad S(t) = \int_0^t s(x) \, dx, \quad L := \sup_{x, y, x > y} \frac{s(x)}{s(y)} < \infty, \]

it is known [13], p. 173-176 that

\[ |V_S[f]|_p \leq L \frac{p^2}{p - 1} |f|_p. \] (5.4)

Note that this operator contains as a particular case the well-known Riemann-Liouville fractional derivative operator, in which \( s(x) = x^{\beta - 1}, \beta = \text{const} \in (0, 1) \).

3. It may be investigated analogously the discrete version of considered inequalities, i.e. when

\[ M[a](n) = \sum_{m=0}^{\infty} M(m, n) a(m), \quad m, n = 0, 1, 2, \ldots \] (5.5)

at least in the case when the function \( M(x, y), \ x, y > 0 \) is homogeneous of degree \(-1\); see [23].

Here as usually

\[ |\vec{a}|_p = |\{a(n)\}|_p = \left[ \sum_{n=0}^{\infty} |a(n)|^p \right]^{1/p}. \]

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