We explore the possibility to overcome the standard quantum limit (SQL) in a free-fall atom interferometer using a Bose-Einstein condensate (BEC) in either of the two relevant cases of Bragg or Raman scattering light pulses. The generation of entanglement in the BEC is dramatically enhanced by amplifying the atom-atom interactions via the rapid action of an external trap focusing the matter-waves to significantly increase the atomic densities during a preparation stage – a technique we refer to as delta-kick squeezing (DKS). The action of a second DKS operation at the end of the interferometry sequence allows to implement a non-linear readout scheme making the sub-SQL sensitivity highly robust against imperfect atom counting detection. We predict more than 30 dB of sensitivity gain beyond the SQL for the variance, assuming realistic parameters and 10^6 atoms.

Free-fall atom interferometers [1–3] are extraordinarily sensitive to external forces and find key applications as gravimeters, gradiometers and gyroscopes in applied physics as well as in fundamental science [4–6]. State-of-the-art devices use N uncorrelated atoms and their phase estimation uncertainty is lower bounded by the standard quantum limit (SQL), \( \Delta \eta_{\text{SQL}} = 1/\sqrt{N} \). Since N is generally constrained by the experimental apparatus or by the onset of unwanted systematic effects due to the high density, the possibility to overcome the SQL by engineering specific quantum correlations [7] between the atoms is attracting increasing interest [8].

While proof-of-principle entanglement-enhanced atom interferometry [8–13] have been largely investigated both theoretically and experimentally in the context of atomic clocks [14–20] and magnetometry [21–26], free-fall atom interferometers have received surprisingly less attention [27–32]. The main reason is that these measurement devices have stringent practical requirements: in particular, the generation of atomic entanglement must be compatible with the splitting of the atomic wave-packets in momentum modes. Bose-Einstein condensates (BECs) have been pinpointed as optimal candidates for the realization of entanglement-enhanced free-fall atom interferometers [31]. Indeed, the narrow momentum dispersion guarantees ideal splitting [33] and entanglement can be generated via particle-particle interactions [8–10, 31, 34, 35]. However, since the interaction vanishes due to free-fall expansion after a short transient time prior to the interferometer operations, current theoretical studies predict only a modest sub-SQL sensitivity gain [31].

In this paper, we overcome these limitations by proposing a novel method to enhance the generation of entanglement in free-fall atom-interferometers using BECs. The key idea consists in focusing the matter-waves through the rapid application of an external trapping potential in analogy to optics, where the trap plays the role of a converging lens. Going through the focal point increases the matter-wave density and thus the effective strength of the particle-particle interactions preparing the atoms in a highly-entangled spin-squeezed state. Considering previous works on delta-kick collimation [36–40], we designate our technique delta-kick collimation (DKS).

The method is explored for Raman and Bragg scattering and is made fully compatible with the requirement of linear atom interferometer operations. The DKS technique leads to a substantial phase sensitivity gain beyond the SQL, e.g. more than 30 dB in a realistic experimental configuration with 10^6 atoms. For Bragg diffraction, in particular, using a second DKS pulse at the end of the interferometer sequence allows the realization of a nonlinear readout protocol [41–48]. In this case, the twisting dynamics generating the spin-squeezed state is inverted before the final measurement of atom numbers in the two interferometer output ports. This operation makes the interferometer exceptionally robust against detection noise.

Entanglement enhancement by the DKS state preparation protocol. The preparation step illustrated in Fig. 1(a) starts with a BEC suddenly released from an external trap. A short free expansion time \( T_0 \) dilutes the BEC and guarantees, by applying a first beam splitter pulse (BS1), the preparation of the quantum superposition \( |\psi_0\rangle = (|g\rangle + |e\rangle)\otimes N/2^{N/2} \), \( N \) being the number of atoms, and \(|g\rangle \) and \(|e\rangle \) two momentum states [49]. Entanglement is then generated via particle-particle interactions in the BEC, such that \( |\psi_0\rangle \) evolves according to the one-axis twisting dynamics \(|\psi_{tw}(t)\rangle = e^{-i\tau(t)g} |\psi_0\rangle \) [34, 35, 50], where \( \tau(t) = \int_0^t \chi(t') dt' \). Notably, the time dependent non-linear coefficient \( \chi(t) \) is given by [51]

\[
\chi^R(t) = \chi_S(t) - \chi_C(t),
\]

(1a)

\[
\chi^B(t) = \chi_S(t) - 2\chi_C(t),
\]

(1b)

for the Raman (R) or the Bragg (B) scattering, respectively. Here, \( \chi_S(t) = g \int_{-\infty}^{\infty} d\tau |\Phi_{g\langle\tau\rangle}(r, t)|^2 / \hbar \) and \( \chi_C(t) = g \int_{-\infty}^{\infty} d\tau |\Phi_{e\langle\tau\rangle}(r, t)|^2 / \hbar \) denote the self-phase and cross-phase modulation terms, respectively, where, \( \Phi_{g\langle\tau\rangle}(r, t) \) carries the spatial evolution of the state \(|g\rangle \) \((|e\rangle)\). For simplicity, in the following we assume the same intra- and interspecies scattering coefficient \( g = g_{11} = g_{21} = g_{22} > 0 \) [52]. The factor 2 in front of the cross-phase modulation terms in Eq. (1b) [53, 54] is due to the interference of the two modes (see [51]) and is rich of consequences. In particular, when the wave functions overlap \( \chi_S \) and \( \chi_C \) are equal and thus \( \chi^B \approx 0 \) in the Raman case [9, 10]. In contrast, during overlap, \( \chi^B \approx -\chi_S \neq 0 \), making the one-axis-twisting evolution active...
in the Bragg case. Furthermore, $\chi^B$ can assume either positive or negative values (see below), while $\chi^R \geq 0$.

The atomic interactions, proportional to the density of the freely-expanding cloud, are vanishing a few milliseconds after release in a free-fall interferometer and therefore prohibit the generation of highly entangled states [31]. This problem is overcome here by switching on, after a pre-expansion time $t_{\text{exp}}$, an external harmonic trap for a time $\Delta t$, to induce a size focusing of the atomic cloud (Fig. 2a) [55]. This re-focusing increases the density of the cloud and thus the effective interaction coefficient $\tau$. Right after this delta-kick pulse, at time $T_\tau = t_{\text{exp}} + \Delta t_1$, a first mirror pulse (M) is applied. Since the DKS also imprints a phase equivalent to a classical center-of-mass motion, the detuning of the mirror pulse has to be adjusted to absorb the change of trajectories [56–58]. Finally, the state preparation stage ends after an additional time $2T_\tau'$ (notice that a second mirror pulse is applied after a time $T_{\tau}'$) necessary to dilute the cloud density and to guarantee that the interferometer sequence is implemented with non-interacting atoms. The time $T_\tau'$ can be tuned depending on the specific DKS parameters. In Fig. 2(a-c) we present a realistic example of state preparation using the DKS [59]: we show the size of the BEC cloud in Fig. 2(a) and the non-linear coefficient $\chi$ as a function of time in Fig. 2(b). The latter clearly shows how $\chi(t)$ is enhanced by the DKS and is different for Raman and Bragg pulses. Figure 2(c) shows $\tau$ as a function of the duration of the DKS, $\Delta t$. In particular, for Bragg scattering, $\tau$ can have either positive or negative values, depending on $\Delta t$.  

**Atom interferometry with linear detection.** The Mach-Zehnder interferometer sequence illustrated in Fig. 1(b) consists of two beam-splitters (BS2 and BS3) and a mirror pulse (M) equally spaced in time by $T_\theta$. We define the sensitivity gain over the standard quantum limit, $G = \Delta S_{\text{BSN}}/\Delta \theta$ calculated at $\theta = 0$ [60], where $\theta$ denotes the phase accumulated during the interferometer and $(\Delta \theta)^2 = (\Delta S_2)\langle (d_2 S_2)/d(\theta_\text{ BS N}) \rangle^2$ is obtained by error propagation. As discussed previously, the linear interferometer condition $\tau_I = \int_0^{T_0 + 2T_\tau} \chi(t')dt' = 0$, with $T_\tau = T_0 + 2T_\tau + 2T_\tau'$, can be realized independently of the DKS parameters, $t_{\text{exp}}$ and $\Delta t$, by adjusting the time $T_\tau'$. In this case, the ideal interferometer sequence can be described by the linear transformation $\hat{U}_{\text{AI}} = e^{i\theta_\text{BS}}$ such that the output state is $|\psi_{\text{out}}\rangle = \hat{U}_{\text{AI}}|\psi_{\text{in}}\rangle$ [61]. The sensitivity gain hence reads

$$G = \frac{2 \cos(\tau)^{N-1}}{[4 + (N - 1)(A - \sqrt{A^2 + B^2})]^{1/2}},$$  

with $A = 1 - \cos(2\tau)^{N-2}$ and $B = 4 \sin(\tau)\cos(\tau)^{N-2}$, upon an opportune rotation [35] of the squeezed state $|\psi_{\text{in}}\rangle$ at BS2. The maximum value of $G$ is reached for $\tau_{\text{opt}} \approx 1.2N^{-1/3}$ [7, 35]. In Fig. 2(d,e) we plot the gain $G$ as a function of the DKS parameters $t_{\text{exp}}$ and $\Delta t$, for Bragg and Raman scattering, respectively. While for $\Delta t = 0$, no significant gain can be obtained ($G \approx 1$), the DKS enables the creation of highly-entangled input states in these freely expanding configurations: a large gain is possible for a large parameters range of pre-expansion, $t_{\text{exp}}$, and DKS duration, $\Delta t$.

**Atom interferometry with non-linear readout.** Using the DSK to tune the sign of the effective interaction for Bragg scattering can be exploited to realize a non-linear readout scheme. After the interferometer sequence, a second DKS is applied, see Fig. 1(c), such that the output state is now given by

$$|\psi_{\text{out}}\rangle = e^{-i\theta/2\hat{S}_\tau}e^{-i\hat{S}_z/2}\hat{U}_{\text{AI}}e^{-i\theta/2\hat{S}_\tau}|\psi_0\rangle.$$  

Notice that, for simplicity, the final rotation $e^{-i\theta/2\hat{S}_\tau}$ is not included in Fig. 1. For the sake of readability, we distinguish the sensitivity gain of the linear detection, $G$, to the non-linear readout, $\Omega$. In the case where

$$\tau_1 = -\tau_2 \equiv \tau$$  

and $\theta = 0$, it has already been shown that non-linear readout provides (i) the possibility to reach a higher sensitivity gain, $\max(\Omega) > \max(G)$, with respect to the linear detection case for sufficiently large values of $\tau$ (see Fig. 3a) [42]; and

![Diagram](image-url)
(ii) a phase magnification robust against atom number detection imperfection (see Fig. 3b) [42, 45]. The later being one of the most critical limitations in quantum-enhanced atom interferometers [8]. Throughout this paper, the imperfect detection resolution is modeled by a Gaussian noise of variance \((\Delta n)^2\) [8, 31].

It should be noticed that satisfying the conditions (3) and (4) is not straightforward in quantum systems as it requires inverting the twisting evolution that generated entanglement in the probe state. Such a possibility has been predicted for Rydberg atoms [42, 44] and has been experimentally realized for cold atoms in a cavity [45] and trapped ions [41, 63]. Here the condition (4) can be naturally satisfied by using Bragg scattering and tuning the DKS parameters. This is shown in Fig. 3(c-d) where we plot \(\tau_2\) as a function of the DKS durations \(\Delta t_2\) and compare it to \(\tau_1\). The condition (4) can be satisfied in realistic experimental conditions. In particular, panel (c) shows the case where the sizes of the BECs are kept constant after BS2 until the second DKS during the non-linear readout. This configuration can be engineered through the action of a delta-kick collimation pulse [36–40] before BS2 when the two clouds are dilute enough. This manipulation leads to record low-expansion rates as low as 50 pK [39] enabling interferometer sequences longer than \(T_B = 1\) s. In addition collimated atomic ensembles brings the possibility to perfectly "un-twist" the quantum states with a second DKS action of a delta-kick collimation pulse [36–40] before BS2.

In Fig. 3a we give a more complete overview of the nonlinear readout parameters beyond the condition (4). There, we plot

FIG. 2. DKS engineering. (a) Thomas-Fermi Radii along the \(z\) \((R_z)\) and transverse directions \((R_\perp)\) as a function of time during state preparation (see [59] for the parameters used). The different laser pulses are highlighted by the vertical black lines with \(T = T'_* = 5.25\) ms and the DKS time is fixed to \(\Delta t = 0.25\) ms (vertical blue lines). (b) Corresponding non-linear coefficient \(\chi(t)\) for Raman and Bragg scattering. (c) Effective non-linear coefficient, \(\tau\), as a function of the DKS duration, \(\Delta t\), during the state preparation (solid lines) and during the interferometer sequence (dashed). The vertical line denotes the case shown in panels (a) and (b). Panels (d) and (e) show the sensitivity gain (color scale) for the Bragg (d) and Raman (e) configurations as a function of the pre-expansion time and DKS duration. The white lines denotes \(\tau^R = 0\) and distinguished the regions \(\tau^D > 0\) from \(\tau^D < 0\). Here \(G = 40\) corresponds to a variance of 32 dB below the SQL.

FIG. 3. Non-linear readout. (a) Sensitivity gain for linear, \(G(\tau/\tau_{opt})\), and non-linear, \(Q(\tau/\tau_{opt})\), readout as a function of the detection noise \(\Delta n\) for linear and non-linear readout, \(\tau = \tau_{opt}\) (solid) and \(\tau = 0.1\tau_{opt}\) (dashed). The number of atoms is \(N = 10^4\). Panels (c) and (d) show the non-linear coefficient as a function of the second DKS duration, \(\Delta t_2\) (solid blue line). In panel (c) the size of the BECs are assumed to be constant after the BS2 while in panel (d) the BECs continue to expand after BS2. For specific values of \(\Delta t_2\), the non-linear coefficient matches \(-\tau_1 \approx -0.1\tau_{opt}\) [62].
In Fig. 4(e) we plot the corresponding robustness to atom number resolution of the non-linear readout for $\tau = 0.1 \tau_{\text{opt}}$ and verify that there is no significant gain difference between the different configurations. While the presence of residual interactions and imperfect detection prohibit sub-SQL sensitivity with linear detection schemes, non-linear readout enables the creation of a quantum-enhanced interferometer sequence, in the regime of low interactions, robust against imperfect detection.

**Conclusion.** In this manuscript we have proposed and exploited a phase-space engineering technique to focus matter waves that (i) substantially enhances the amount of entanglement generated in the probe state of a free-fall atom interferometer and (ii) realizes, in the case of Bragg diffraction, a non-linear readout protocol making the interferometer sensitivity extraordinarily robust against detection noise. This robustness is crucial atom interferometers with non-classical states to avoid dramatic effects of imperfect atom number detection on sub-SQL sensitivities. Our predictions have been obtained for realistic experimental parameters and assuming ideal beamsplitter and mirror pulses. Different effects may degrade the sensitivity of the interferometer as atomic losses [64, 65], mode mismatch [66] and shape deformation [67] due to BEC crossing, and residual interaction during splitting pulses [68]. A careful analysis of these effects depends on the specific interferometer configuration and is beyond the scope of this work. The assumption of harmonic traps to realize the DKS is justified for BAC sizes and the proposed spatial separations. Within these assumptions, a sensitivity gain of more than 30 dB beyond the SQL with $10^6$ atoms is predicted for both Bragg and Raman diffraction. Larger harmonic traps [69, 70] could enable even higher sensitivity gains.

A straightforward implementation of our technique, can boost the sensitivity of a BEC gravimeter [71, 72] of $10^6$ atoms to that of an ensemble with a hundredfold flux. Our DKS technique thus promotes BEC ensembles in free-fall atom interferometers to primary quantum-enhanced sensors to explore timely physics quests such as testing General relativity principles [73, 74] or atom-interferometric gravitational-wave detection [75].

RC thanks Franck Pereira Dos Santos and Peter Wolf, NG thanks Christian Schubert and Clemens Hammerer for fruitful discussions. This work is supported by the European Union’s Horizon 2020 research and innovation programme - Qombs Project, FET Flagship on Quantum Technologies grant no. 820419. NG acknowledges the support of the CRC 1227 (DQmat) within Project A05, the German Space Agency (DLR) for funds provided by the Federal Ministry for Economic Affairs and Energy (BMWi) due to an enactment of the German Bundestag under Grant Nos. 50WM1861 (CAL) and 50WM2060 (CARIOQA) and the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2123 QuantumFrontiers – 390837967.
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[51] See Supplementary Material.

[52] This approximation for instance describes the case of Rubidium-87 species in the internal states: \(|g⟩ \equiv |F = 1, m_F = -1⟩\) and \(|e⟩ \equiv |F = 2, m_F = 1⟩\), where \(a_{11} = 100.4a_0\), \(a_{12} = 97.7a_0\) and \(a_{22} = 95.0a_0\) with \(a_0\) the Bohr radius.

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[62] The calculation have been made for \(T_\pi = 5\text{ ms}, T_r = 5\text{ ms}, \Delta t_i = 0.6\text{ ms}, T_r = 0, T_{\perp} = 1\text{ ms}\) and \(T_{\perp} = 2\text{ ms}\) for (c) and \(T_{\perp} = 10\text{ ms}\) for (d). In this configuration the clouds are diluted at each laser pulse to guarantee ideal beam-splitters or mirrors pulse efficencies.

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Supplemental Materials

RAMAN VS BRAGG SCATTERING

Two-component single external mode Hamiltonian (Raman)

Here we consider the two-component non-linear Hamiltonian,

\[
\hat{H}_\text{int}^R(t) = \frac{g_{11}}{2} \int_{-\infty}^{\infty} \! dr \, \hat{\Psi}_1^\dagger(t) \hat{\Psi}_1 \hat{\Psi}_1(0) + \frac{g_{22}}{2} \int_{-\infty}^{\infty} \! dr \, \hat{\Psi}_2^\dagger(0) \hat{\Psi}_2 \hat{\Psi}_2(0) + g_{12} \int_{-\infty}^{\infty} \! dr \, \hat{\Psi}_1^\dagger(t) \hat{\Psi}_2(0) + g_{12} \int_{-\infty}^{\infty} \! dr \, \hat{\Psi}_1(0) \hat{\Psi}_2^\dagger(t),
\]

(S1)

where the field operators are \(\hat{\Psi}_1 = \phi_1(\mathbf{r}, t) \hat{a}_1\) and \(\hat{\Psi}_2 = \phi_2(\mathbf{r}, t) e^{2\text{ikz}} \hat{a}_2\). Here \(\phi_\ell(\mathbf{r}, t)\) denotes the spatial shape of component \(\hat{\Psi}_\ell\) and \(e^{2\text{ikz}}\) is the phase imprinted by the two-photon transition. \(\hat{a}_i\) (\(\hat{a}_i^\dagger\)) is the bosonic annihilation (creation) operator. It is convenient to introduce the SU(2) pseudo-spin operators of the Lie’s algebra,

\[
\begin{align*}
\hat{S}_x & = (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)/2, \\
\hat{S}_y & = (\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)/2i, \\
\hat{S}_z & = (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)/2, \\
\hat{N} & = (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2),
\end{align*}
\]

(S2)

satisfying the commutation relation \([\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk} \hat{S}_k\) with \(\epsilon_{ijk}\) being the Levi-Civita symbol. \(\hat{H}_\text{int}^R(t)\) read then:

\[
\hat{H}_\text{int}^R(t) = \frac{g_{11}}{2} \int_{-\infty}^{\infty} \! dr \, |\phi_1(\mathbf{r}, t)|^2 \hat{a}_1^\dagger \hat{a}_1 + \frac{g_{22}}{2} \int_{-\infty}^{\infty} \! dr \, |\phi_2(\mathbf{r}, t)|^2 \hat{a}_2^\dagger \hat{a}_2 + g_{12} \int_{-\infty}^{\infty} \! dr \, |\phi_1(\mathbf{r}, t)|^2 |\phi_2(\mathbf{r}, t)|^2 \hat{a}_1^\dagger \hat{a}_2 + \hat{N},
\]

(S3)

Using the two relations,

\[
\begin{align*}
\hat{N}^2 & = \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_2 + 2\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2 \hat{a}_2 + \hat{N}, \\
(2\hat{S}_z)^2 & = \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_2 - 2\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2 \hat{a}_2 + \hat{N},
\end{align*}
\]

(S4)

we find

\[
\begin{align*}
\hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_2 & = \frac{\hat{N}^2 + (2\hat{S}_z)^2}{2} - \hat{N} \equiv 2(\hat{S}_z)^2, \\
\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2 \hat{a}_2 & = \frac{\hat{N}^2 - (2\hat{S}_z)^2}{4} \equiv -(\hat{S}_z)^2.
\end{align*}
\]

(S5)

Typically one find,

\[
\hat{H}_\text{int}^R(t) = \hbar [\chi_{11}(t) + \chi_{22}(t) - 2\chi_{12}(t)] (\hat{S}_z)^2,
\]

(S6)

where \(\chi_{ij}(t) = g_{ij} \int \! dr \, |\Phi_\ell(\mathbf{r}, t)|^2 |\Phi_\ell(\mathbf{r}, t)|^2/2\hbar\). The pre-factor \(g_{ij}/2\hbar = \pi \hbar a_{ij}/m_{ij}\) is associated to the s-wave scattering length \(a_{ij}\) with \(m_{ij}\) being the reduced mass \(m_{ij} = m_{i} m_{j}/(m_{i} + m_{j})\). In the case where \(g_{11} \approx g_{22} \approx g_{12} = g\) and \(|\phi_1(\mathbf{r}, t)|^2 = |\phi_2(\mathbf{r}, t)|^2\), \(\hat{H}_\text{int}^R(t)\) simplified to

\[
\hat{H}_\text{int}^R(t) = \hbar [\chi_S(t) - \chi_C(t)] \hat{S}_z^2,
\]

(S7)

where \(\chi_S(t) = g \int_{-\infty}^{\infty} \! dr \, |\Phi(\mathbf{r}, t)|^4/\hbar\) denotes the self-phase modulation non-linear coefficient \(|\Phi_1(\mathbf{r}, t)|^2 = |\Phi_2(\mathbf{r}, t)|^2 \equiv |\Phi(\mathbf{r}, t)|^2\) and \(\chi_C(t) = g \int_{-\infty}^{\infty} \! dr \, |\Phi_1(\mathbf{r}, t)|^2 |\Phi_2(\mathbf{r}, t)|^2 /\hbar\) denotes the cross-phase modulation non-linear coefficient.
where the state at the end of the interferometer sequence is given by:

\[ \hat{H}_{\text{int}}(t) = \frac{g}{2} \int_{-\infty}^{\infty} \, dr \, \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \hat{\Psi}, \]  

(S8)

where the field operators, \( \hat{\Psi} = \phi_1(r, t) \hat{a}_1 + \phi_2(r, t) e^{i k z^2} \hat{a}_2 \). As before \( \phi_i(r, t) \) denotes the spatial shape of component \( \hat{\Psi}_i \) and \( e^{i k z^2} \) is the phase imprinted by the two-photon transition. The relations given in Eqs.\((S2), (S4)\) and \((S5)\) still hold. In this case the term, \( \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \hat{\Psi} \hat{\Psi} \), read as:

\[ \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} = |\phi_1(r, t)|^2 \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + |\phi_2(r, t)|^2 \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + 4 |\phi_1(r, t)|^2 |\phi_2(r, t)|^2 \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_1 \]

\[ + \phi_1(r, t)^2 \phi_2(r, t)^2 e^{i k z^2} \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_1 + 2 |\phi_1(r, t)|^2 |\phi_2(r, t)|^2 e^{i k z^2} \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \hat{a}_2 \]

\[ + 2 \phi_1(r, t) |\phi_1(r, t)|^2 \phi_2(r, t) e^{i k z^2} \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2 \hat{a}_1 + 2 \phi_2(r, t) |\phi_2(r, t)|^2 e^{i k z^2} \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_1, \]  

(S9)

and after simplification, where we neglect the terms \( \int_{-\infty}^{\infty} \, dr \, \phi_i(r, t) \phi_j(r, t) \phi_k(r, t) \phi_l(r, t) e^{i k z^2} \approx 0 \), we find:

\[ \hat{H}_{\text{int}}(t) = \frac{g}{2} \int_{-\infty}^{\infty} \, dr \left( |\phi_1(r, t)|^2 \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + |\phi_2(r, t)|^2 \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 + 4 |\phi_1(r, t)|^2 |\phi_2(r, t)|^2 \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2 \hat{a}_1 \right). \]  

(S10)

In the case where \(|\phi_1(r, t)|^2 = |\phi_2(r, t)|^2\), \( \hat{H}_{\text{int}}(t) \) simplified to

\[ \hat{H}_{\text{int}}(t) = \hbar \left[ \chi_S(t) - 2 \chi_C(t) \right] \hat{S}_z. \]  

(S11)

**NON-LINEAR READOUT**

**Sensitivity Gain**

We consider the initial coherent state

\[ |\psi_0 \rangle = 2^{-S} \sum_{n=0}^{2S} \binom{2S}{n}^{1/2} |S, S - n \rangle, \]  

(S12)

where \( S = N/2 \).

The sensitivity of the interferometer, \( \Delta \theta \), is evaluated via the error propagation formula,

\[ (\Delta \theta)^2 = \frac{(\Delta S_y)^2}{(d(S_y)/d\theta)^2}_{|_{\theta=0}}, \]  

(S13)

where the state at the end of the interferometer sequence is given by:

\[ |\Psi_{\text{out}} \rangle = \hat{U}^\dagger e^{i (\delta \hat{S}_y + \eta \hat{S}_z^2)} \hat{U} |\Psi_0 \rangle, \]  

(S14)

where \( \eta = \tau_{\text{AI}} \).

We define the sensitivity gain in the case of a non-linear readout by \( \mathcal{Q} \) such as

\[ \mathcal{Q}^2 = \frac{1}{N(\Delta \theta)^2} \]  

(S15)

**Case where the phase and non-linear term during the interferometer are considered small**

In the situation where \( \theta \) and \( \eta \) are small enough to approximate \( e^{i (\delta \hat{S}_y + \eta \hat{S}_z^2)} \approx 1 + i (\theta \hat{S}_y + \eta \hat{S}_z^2) \) we have,
After simplification the di 

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Using the results derived in Appendix we find,

\[ \langle \Psi_{\text{out}} | \hat{S}_y, \hat{O}^\dagger \hat{S}_y, \hat{O} | \Psi_{\text{out}} \rangle = iS (2S - 1) \sin(\tau) \cos(\tau)^{2S-2} \] (S19a)
\[ \langle \Psi_{\text{out}} | \hat{S}_y^2, \hat{O}^\dagger \hat{S}_y, \hat{O} | \Psi_{\text{out}} \rangle = -2iS (2S - 1) \sin(\tau)^2 \cos(\tau)^{2S-1} \] (S19b)
\[ \langle \Psi_{\text{out}} | \hat{S}_y, \hat{O}^\dagger \hat{S}_y^2, \hat{O} | \Psi_{\text{out}} \rangle = 0 \] (S19c)
\[ \langle \Psi_{\text{out}} | \hat{S}_y^2, \hat{O}^\dagger \hat{S}_y^2, \hat{O} | \Psi_{\text{out}} \rangle = \frac{i}{4} S \left[ (3 + 4S (S - 1) + 2(S + 1) \cos(4\tau) + \cos(8\tau)) \sin(2\tau) \cos(2\tau)^{2S-3} \right. \]
\[ - (2S + 1) \sin(4\tau) \biggr] \] (S19d)
\[ \langle \Psi_{\text{out}} | \hat{O}^\dagger \hat{S}_y, \hat{S}_y, \hat{O}^\dagger \hat{S}_y, \hat{O}^\dagger \hat{S}_y^2, \hat{O}^\dagger \hat{S}_y^2, \hat{O}^\dagger \hat{S}_y, \hat{O} | \Psi_{\text{out}} \rangle = \frac{S}{8} (2S - 1) \cos(3\tau)^{2S-3} \left[ (2S - 1) \cos(2\tau) + S \cos(4\tau) \right] \]
\[ + \frac{S}{8} \cos(\tau)^{2S-3} \left[ 1 - 4S + 8S^2 + (-1 + 2S + 4S^2) \cos(2\tau) - 3(S - 1)(2S - 1) \cos(4\tau) \right] \] (S19e)

**Sensitivity Gain in the perturbative regime**

The sensitivity gain read,

\[ Q^2 = (2S - 1)^2 \sin(\tau)^2 \cos(\tau)^{4S-4} + \mathcal{A} \eta + \mathcal{O}(\eta^3) \] (S20)

with

\[ \mathcal{A} = \frac{1}{4} (2S - 1)^2 \sin(\tau) \cos(\tau)^{2S-2} \cos(3\tau)^{2S-3} \left[ (2S - 1) \cos(2\tau) + S \cos(4\tau) \right] \]
\[ + 2(2S - 1)^2 \sin(\tau)^3 \cos(\tau)^{4S-3} \left[ -2(2S + 1) \cos(2\tau) + \cos(2\tau)^{2S-3} (3 + 4S (S - 1) + 2(S + 1) \cos(4\tau) + \cos(8\tau)) \right] \]
\[ + \frac{1}{4} (2S - 1) \sin(\tau) \cos(\tau)^{4S-5} \left[ -1 + 4S - 8S^2 + (1 - 2S (2S + 1)) \cos(2\tau) + (3 - 9S + 6S^2) \cos(4\tau) \right] \] (S21b)

To first order in \( \tau \) and \( \eta \) one finds,

\[ Q^2(\tau, \eta) = \frac{1}{4} [(4\tau - 3\eta) S - (2\tau - \eta)]^2 + \mathcal{O}(\tau^3) + \mathcal{O}(\eta^3). \] (S22)

**Help for the calculation**

\[ \langle \Psi_0 | \hat{S}_y \hat{W} | \Psi_0 \rangle = S \cos(\tau)^{2S-1} \]
\[ \langle \Psi_0 | \hat{S}_y^2 \hat{W} | \Psi_0 \rangle = \frac{S}{2} (2S - 1) \cos(\tau)^{2S-2} \exp(-i\tau) \]
\[ \langle \Psi_0 | \hat{S}_y^3 \hat{W} | \Psi_0 \rangle = \frac{S}{2} (2S - 1)(S - 1) \cos(\tau)^{2S-3} \exp(-2i\tau) \]
\[ \langle \Psi_0 | \hat{S}_y^4 \hat{W} | \Psi_0 \rangle = \langle \Psi_0 | \hat{S}_y^4 \hat{W} | \Psi_0 \rangle \]
\[ \langle \Psi_0 | \hat{S}_y^2 \hat{W} \hat{S}_y | \Psi_0 \rangle = -\frac{S}{2} \cos(\tau)^{2S-1} + i\frac{S}{2} (2S - 1) \sin(\tau) \cos(\tau)^{2S-2} \]
\[ \langle \Psi_0 | \hat{S}_y^2 \hat{W}^\dagger \hat{S}_y | \Psi_0 \rangle = -\frac{S}{2} \cos(\tau)^{2S-1} - i\frac{S}{2} (2S - 1) \sin(\tau) \cos(\tau)^{2S-2} \]
\[ \langle \Psi_0 | \hat{S}_y \hat{W} | \Psi_0 \rangle = iS \sin(\tau) \cos(\tau)^{2S-1} \exp(i\tau) \]
\[ \langle \Psi_0 | \hat{S}_y \hat{W} \hat{S}_y | \Psi_0 \rangle = S \cos(\tau)^{2S-1} + \frac{S}{2} (2S - 1) \cos(\tau)^{2S-2} \exp(-i\tau) \]
\[ \langle \Psi_0 | \hat{S}_y^2 \hat{W}^\dagger \hat{S}_y | \Psi_0 \rangle = S (2S - 1) \cos(\tau)^{2S-2} \exp(i\tau) + \frac{S}{2} (2S - 1)(S - 1) \cos(\tau)^{2S-3} \exp(2i\tau) \]
\begin{align*}
\langle \Psi_0 | \hat{S}_z^2 \hat{W} | \Psi_0 \rangle &= \frac{S}{2} \cos(\tau)^{2S-2} (1 - S + S \cos(2\tau)) \exp(i\tau) \\
\langle \Psi_0 | \hat{S}_z^2 \hat{S}_z \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1) \cos(\tau)^{2S-3} \left( (S - 2) - S \exp(-2i\tau) \right) \\
\langle \Psi_0 | \hat{S}_x \hat{S}_y \hat{W} | \Psi_0 \rangle &= \frac{S}{4} \left[ 1 + (S - 1) \exp(2i\tau) - 2S^2 \exp(-2i\tau) + S(2S - 5) \right] \cos(\tau)^{2S-3} \\
\langle \Psi_0 | \hat{S}_x^2 \hat{S}_z \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1) \left( (S - 2) \exp(-2i\tau) - S \exp(-4i\tau) \right) \cos(3\tau)^{2S-3} \\
\langle \Psi_0 | \hat{S}_y \hat{S}_z \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 1)(2S - 3) \cos(3\tau)^{2S-4} \exp(-7i\tau) \\
\langle \Psi_0 | \hat{S}_z^2 \hat{W}^2 \hat{S}_z | \Psi_0 \rangle &= \frac{S}{4} (2S - 1) \left[ \exp(3i\tau) + S(2S - 1) \exp(-i\tau) + (4S - 2) \exp(i\tau) \right] \cos(\tau)^{2S-4} \\
\langle \Psi_0 | \hat{S}_x^3 \hat{V} \hat{S}_z \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 1) \left[ 3 \exp(i\tau) + 2S \exp(-7i\tau) \right] \cos(3\tau)^{2S-4} \\
\langle \Psi_0 | \hat{S}_y \hat{S}_z \hat{V} \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 1) \left[ 3 \exp(i\tau) + 2S \exp(-i\tau) \right] \cos(\tau)^{2S-4} \\
\langle \Psi_0 | \hat{S}_z^2 \hat{V} \hat{W}^2 \hat{S}_z | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 2) \exp(2i\tau) - S \cos(\tau)^{2S-3} \\
\langle \Psi_0 | \hat{S}_x^2 \hat{V} \hat{W}^2 | \Psi_0 \rangle &= \frac{S}{2} (2S - 1) \cos(\tau)^{2S-2} \exp(i\tau) \\
\langle \Psi_0 | \hat{S}_y \hat{S}_z \hat{W} \hat{V}^2 | \Psi_0 \rangle &= \frac{S}{4} [-1 + S + \exp(2i\tau) (1 - S (S + 2S \exp(2i\tau))))] \cos(\tau)^{2S-3} \exp(-4i\tau) \\
\langle \Psi_0 | \hat{S}_x \hat{V} \hat{W} \hat{S}_z | \Psi_0 \rangle &= \frac{S}{2} [2S \exp(-i\tau) + \exp(-3i\tau)] \cos(\tau)^{2S-2} \\
\langle \Psi_0 | \hat{S}_z^2 \hat{V} \hat{W} | \Psi_0 \rangle &= \frac{S}{2} [1 - S + S \cos(2\tau)] \cos(\tau)^{2S-2} \exp(-3i\tau) \\
\langle \Psi_0 | \hat{S}_x^3 \hat{V} | \Psi_0 \rangle &= -iS \sin(\tau) \cos(\tau)^{2S-1} \exp(-3i\tau) \\
\langle \Psi_0 | \hat{S}_y \hat{W} | \Psi_0 \rangle &= \frac{S}{2} (2S - 1) \cos(3\tau)^{2S-2} \exp(-i\tau) \\
\langle \Psi_0 | \hat{S}_z^2 \hat{W} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 1) \left[ 2S \exp(-3i\tau) + 3 \exp(-i\tau) \right] \cos(\tau)^{2S-4} \\
\langle \Psi_0 | \hat{S}_x^2 \hat{W}^2 | \Psi_0 \rangle &= \frac{S}{4} (2S - 1) \left[ \exp(i\tau) + S(2S - 1) \exp(-3i\tau) + (4S - 2) \exp(-i\tau) \right] \cos(\tau)^{2S-4} \\
\langle \Psi_0 | \hat{S}_x | \Psi_0 \rangle &= S \\
\langle \Psi_0 | \hat{S}_x^2 | \Psi_0 \rangle &= \frac{S}{2} (2S - 1) \\
\langle \Psi_0 | \hat{S}_y | \Psi_0 \rangle &= S \cos(2\tau)^{2S-1} \exp(2i\tau) \\
\langle \Psi_0 | \hat{S}_z \hat{V} | \Psi_0 \rangle &= \frac{S}{2} (2S - 1) \cos(2\tau)^{2S-2} \\
\langle \Psi_0 | \hat{S}_z^2 \hat{V} | \Psi_0 \rangle &= \frac{S}{2} (2S - 1)(S - 1) \cos(2\tau)^{2S-3} \exp(-2i\tau) \\
\langle \Psi_0 | \hat{S}_x^2 \hat{V} | \Psi_0 \rangle &= \frac{S}{4} (2S - 1)(S - 1)(2S - 3) \cos(2\tau)^{2S-4} \exp(-4i\tau) \\
\langle \Psi_0 | \hat{S}_x \hat{S}_z | \Psi_0 \rangle &= -\frac{S}{2} \\
\langle \Psi_0 | \hat{S}_y \hat{S}_z | \Psi_0 \rangle &= -S \left( \frac{S}{2} (2S - 1) \right) \\
\langle \Psi_0 | \hat{S}_z \hat{V} | \Psi_0 \rangle &= iS \sin(2\tau) \cos(2\tau)^{2S-1} \exp(4i\tau) 
\end{align*}
\[ \langle \Psi_0 | \hat{S}_z^2 \hat{V} | \Psi_0 \rangle = \frac{S}{2} [S \cos(4\tau) - (S - 1)] \exp(4i\tau) \cos(2\tau)^{2S-2} \]

\[ \langle \Psi_0 | \hat{S}_+ \hat{S}_z \hat{V} | \Psi_0 \rangle = -\frac{S}{2} \cos(2\tau)^{2S-2} \exp(4i\tau) + iS^2 \sin(2\tau) \cos(2\tau)^{2S-2} \exp(2i\tau) \]

\[ \langle \Psi_0 | \hat{S}_+^2 \hat{S}_z \hat{V} | \Psi_0 \rangle = -\frac{S}{2} (2S - 1) \cos(2\tau)^{2S-2} + i\frac{S}{2} (2S - 1)(S - 1) \sin(2\tau) \cos(2\tau)^{2S-3} \]

\[ \langle \Psi_0 | \hat{S}_+^2 \hat{S}_- | \Psi_0 \rangle = \frac{S}{2} (S + \exp(4i\tau))(2S - 1) \cos(2\tau)^{2S-3} \exp(-2i\tau) \]

\[ \langle \Psi_0 | \hat{S}_+^2 \hat{S}_-^2 | \Psi_0 \rangle = \frac{S}{4} (2S - 1)[\exp(4i\tau) + 2(2S - 1) + S (2S - 1) \exp(-4i\tau)] \cos(2\tau)^{2S-4} \]

\[ \langle \Psi_0 | \hat{S}_-^2 \hat{S}_+ | \Psi_0 \rangle = -S^2 \]

\[ \langle \Psi_0 | \hat{S}_- \hat{S}_z \hat{V} \hat{S}_+ | \Psi_0 \rangle = \frac{S}{4} [(S - 1) \exp(6i\tau) + (1 - 5S) \exp(2i\tau) + 4iS^2 \sin(2\tau)] \cos(2\tau)^{2S-3} \]

\[ \langle \Psi_0 | \hat{S}_- \hat{V} \hat{S}_+ | \Psi_0 \rangle = \frac{S}{2} (2S - 1) \cos(2\tau)^{2S-2} + S \cos(2\tau)^{2S-1} \exp(2i\tau) \]