Closure of the Laplace-Beltrami Operator on 2D Almost-Riemannian Manifolds and Semi-Fredholm Properties of Differential Operators on Lie Manifolds

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Abstract. The problem of determining the domain of the closure of the Laplace-Beltrami operator on a 2D almost-Riemannian manifold is considered. Using tools from theory of Lie groupoids natural domains of perturbations of the Laplace-Beltrami operator are found. The main novelty is that the presented method allows us to treat geometries with tangency points. This kind of singularity is difficult to treat since those points do not have a tubular neighbourhood compatible with the almost-Riemannian metric.

Mathematics Subject Classification. 58J05, 58J60, 47A53, 47F10.

Keywords. Almost-Riemannian geometry, Laplace-Beltrami operator, Lie groupoids, Semi-Fredholm conditions.

1. Introduction

1.1. Almost-Riemannian Manifolds

Almost-Riemannian manifolds can be seen as singular analogues of Riemannian manifolds. They are closely related to sub-Riemannian manifolds and carry...
some of their characteristic features. This type of structures has been extensively studied both from geometric and analytic perspectives [2,3,10,12–16,38], however, many important and natural questions still remain obscure.

Consider a manifold $M$ of dimension $n$ endowed with a rank varying distribution of planes $\mathcal{D} \subset TM$, such that $\text{rank}\, \mathcal{D}_q = n$. The set of points $q \in M$ for which $\text{rank}\, \mathcal{D}_q < n$ is called the singular set and will be denoted by $\mathcal{Z}$.

We will give a global definition of an almost-Riemannian manifold in Sect. 2. But if we want to capture its essence, it is already possible to do it locally. Let $M$ be a smooth manifold of dimension $n$ and let $X_1, \ldots, X_n \in \Gamma(TM)$ be a set of vector fields. We define $\mathcal{D}$ to be the $C^\infty(M)$-module generated by $X_1, \ldots, X_n$, which can be seen as a possibly rank-varying distribution of subspaces of $TM$

$$\mathcal{D}_q = \text{span}\{X_1(q), \ldots, X_n(q)\}.$$  

The singular set $\mathcal{Z}$ is the set of points $q \in M$ where $\text{rank}\, \mathcal{D}_q < n$ or equivalently, where $X_1(q), \ldots, X_n(q)$ are linearly dependent. In this article it is assumed that $\mathcal{Z}$ is an embedded submanifold of $M$. We can endow $M \setminus \mathcal{Z}$ locally with a metric by declaring $X_1, \ldots, X_n$ to be orthonormal. If the distribution $\mathcal{D}$ is bracket-generating (or is said to satisfy the Hörmander condition, see Sect. 2), then one can use this metric to define the length of curves which cross $\mathcal{Z}$. This gives rise to a distance function on $M$. An almost-Riemannian manifold essentially is a smooth manifold $M$ together with a rank-varying bracket-generating distribution $\mathcal{D}$ of maximal rank equal to $\text{dim}\, M$, and a metric on $\mathcal{D}$, which locally is given by an orthonormal frame on $M \setminus \mathcal{Z}$. They can be seen to be a particular instance of Carnot-Carathéodory metric spaces [1].

Almost-Riemannian manifolds are equipped with an array of canonical Riemannian objects, such as curvature or volume measure. However, all of those quantities tend to infinity as we approach the singular set $\mathcal{Z}$. Thus even though almost-Riemannian manifolds are complete metric spaces on their own, it is often better to consider the associated non-complete open Riemannian manifolds $M \setminus \mathcal{Z}$ obtained by removing the singular set $\mathcal{Z}$.

These two points of view are in sharp contrast with each other. If we look at the corresponding Laplace-Beltrami operator $\Delta$ with domain $D(\Delta)$ being equal to $C^\infty_c(M \setminus \mathcal{Z})$, the space of smooth functions with compact support outside the singular set, then this operator can be essentially self-adjoint[15]. By Stone’s or Hille-Yosida theorems this implies that the associated Schrödinger or the heat equation are well-defined on a subset bounded by $\mathcal{Z}$. And so a quantum particle or heat flow cannot cross the singular set even when geodesics can. This phenomena is now known in the literature as quantum confinement.

If we want to understand this phenomenon better the first step to understand the simplest case of 2D almost-Riemannian manifolds. A local generic classification of such structures was given in [2, Theorem 16]. Here generic means with respect to the $C^2$-Whitney topology of pairs of vector fields (see
Definition 2 in [2]). The authors of that article proved that structurally there are three generic types of local behaviour:

1. There might be no singularity at \( q \in M \), i.e., \( q \notin Z \). We call such points Riemannian points. A good example of a space without any singular points is just the Euclidean plane which can be seen as an almost-Riemannian structure generated by two vector fields

\[
X_1 = \partial_x, \quad X_2 = \partial_y;
\]

2. If \( q \in Z \) and \( \dim D_q = 1 \) with \( D_q \) transversal to the singular set \( Z \) we call \( q \in Z \) a Grushin point. A good example of an almost-Riemannian structure with only Grushin and Riemannian points is given by the Grushin plane generated by two vector fields on \( \mathbb{R}^2 \):

\[
X_1 = \partial_x, \quad X_2 = x \partial_y;
\]

3. If \( q \in Z \) and \( \dim D_q = 1 \) with \( D_q \) tangent to the singular set \( Z \) we call \( q \in Z \) a tangency point. To give an example consider a structure on \( \mathbb{R}^2 \) generated by two vector fields:

\[
X_1 = \partial_x, \quad X_2 = (y - x^2) \partial_y.
\]

For a generic structure the singular set \( Z \) is an embedded manifold and moreover the tangency points are isolated [2, Proposition 14].

The last example contains all three types of points and is depicted in Fig. 1.
Let us write down the Laplace-Beltrami operators for each structure in order to see concretely what kind of singularities occur. We have

\[
\Delta_{\text{Euclidean}} = \partial_x^2 + \partial_y^2, \\
\Delta_{\text{Grushin}} = \partial_x^2 + x^2 \partial_y^2 - \frac{1}{x} \partial_x, \\
\Delta_{\text{Tangency}} = \partial_x^2 + (y - x^2)^2 \partial_y^2 + \frac{2x}{y - x^2} \partial_x + (y - x^2) \partial_y.
\]

One can ask many natural questions about these operators. What are their natural domains? What can we tell about their spectrum? For which classes of functions can we solve the Poisson equation? And many others. However, when we try to answer these questions, we find that techniques that work well for Riemannian structures often do not work in the presence of Grushin points. Similarly the techniques that work for Grushin structures often do not work in the presence of the tangency points. In an excellent series of papers [29,30,54] very general conditions for quantum confinement were derived. However, those conditions rely on an assumption that the distance from the singular set is at least \( C^2 \). While this assumption indeed holds for Grushin structures, it is never true for structures with tangency points as the authors themselves point out in [29].

In this paper we study the closure of the Laplace-Beltrami on a generic compact orientable 2D almost-Riemannian manifold without a boundary. We wish to find a result that would hold for all generic structures including structures with tangency points. We will only consider the singular cases, since the non-singular Riemannian case is covered by the standard elliptic theory. The method that we employ here does not apply to the Laplace-Beltrami operator itself for certain reasons. It works, however, for certain perturbations of the operator and for some non-generic structures. In the non-perturbed generic case it is still possible to extract some useful information about the closure as we will see in a simple model example.

1.2. Lie Groupoids as Desingularisations and the Main Result

In order to deal with singular objects, the first thing one might want to do is to look at how singularities are treated in other branches of mathematics. If we wish to stay in the realm of differential geometry, then Poisson geometry is a good example. Indeed, regular Poisson structures are symplectic manifolds and for them the Poisson tensor is non-degenerate. Singular Poisson structures however arise very naturally in the study of Lie groups as Lie-Poisson structures of duals of Lie algebras. These singularities in some sense are of a similar nature to the almost-Riemannian singularities as in both cases they arise from some rank dropping conditions. In order to study all the various phenomena of singularities, commonly used tools are \textit{Lie groupoids} and \textit{Lie algebroids} which became a standard tool within Poisson geometry [58].
Lie groupoids have many faces and definitions. In Sect. 3 we will give a purely differential geometric definition. They can be seen as objects that interpolate between manifolds and Lie groups or as manifolds with partial symmetries. However, for our purposes they will serve as natural desingularisations of singular spaces. Besides Poisson geometry they are indeed used in this fashion when studying orbifolds [45], foliations [46] or just manifolds with boundary [53]. In the recent years it was understood that Lie groupoids constitute a good class of spaces to do analysis on. One can construct, for example, pseudo-differential calculi [26,53,59] and Fourier integral operators [43] adapted to their algebraic structure, use them to prove index theorems [22] or study the essential spectrum of differential operators [18].

For our problem of finding the closure of the Laplace-Beltrami operator we will only need a special class of Lie groupoids that come from Lie manifolds introduced in [4]. They are also known as manifolds with a Lie structure at infinity, but in this article the shorter name is used. A Lie manifold is a pair \((M, V)\), where \(M\) is a smooth manifold with boundary and \(V\) is a Lie subalgebra of the Lie algebra of vector fields tangent to the boundary \(\partial M\), such that the restriction of \(V\) to the interior coincides with the space of all vector fields and which is in addition a projective \(C^\infty(M)\)-module. To each Lie manifold it is possible to associate compatible structures: metric \(g_V\), volume \(\mu_V\) and Sobolev spaces \(H^k_V(M)\) [4]. We can define the space of differential operators adapted to \(V\) as follows. Let \(X_1, \ldots, X_n\) be a local basis for \(V\) in some coordinate chart. Then we say that \(P \in \text{Diff}^m_V(M)\) if in local coordinates it is a sum of monomials \(aX_{i_1} \ldots X_{i_m}\), where \(i_j \in \{1, \ldots, n\}\) and \(a\) is a smooth function.

Let us discuss the construction of the associated Lie manifold structure for a generic 2D almost-Riemannian manifold. In local coordinates \((x, y)\) close to the singular set any 2D almost-Riemannian manifolds admits an orthonormal frame of the form \(\{\partial_x, f(x, y)\partial_y\}\), where \(f\) is a smooth function (see Sect. 2). For a generic 2D almost-Riemannian structure this function \(f\) is a defining function for \(Z\).

To a generic 2D almost-Riemannian manifold we associate a Lie manifold structure \(V\) by considering the \(C^\infty(M)\)-module generated by vector fields

\[
Y_1 = f(x, y)\partial_x, \quad Y_2 = f^2(x, y)\partial_y.
\]

In particular, for the previously considered Grushin plane we have \(f(x, y) = x\), and for the plane with a tangency point \(f(x, y) = y - x^2\). A compatible metric is the one, for which \(Y_1\) and \(Y_2\) have bounded non-zero length up to the singular set. This way a connected component of \(M \setminus Z\) can be seen as a Lie manifold and as a complete metric space. Note that the Lie manifold structure \(V\) with a compatible metric is different from the original almost-Riemannian structure. We use the former one to study the Laplace-Beltrami operator \(\Delta\) and to formulate the main result concerning the closure of some perturbations of \(\Delta\).
Theorem 1. Consider a compact generic 2D almost-Riemannian manifold \((M, U, f)\), associate to it a Riemannian manifold by removing the singular set \(Z\) and take a connected component, which we denote via \(M\) by abusing the notation. Let \(\mu\) be the Riemannian volume form, \(\Delta\) the associated Laplace-Beltrami operator and \(\mathcal{V}\) the associated Lie manifold structure. Suppose that \(h \in C^\infty(M)\) is a strictly positive on \(Z\) function and let \(s\) be a defining function of the singular set \(Z\). Define

\[ \tilde{\Delta} = \Delta - \frac{h}{s^2} \]

with domain \(D(\tilde{\Delta}) = C^\infty(M \setminus Z)\).

Then the domain of closure of \(\tilde{\Delta}\) in \(L^2(M, \mu)\)

\[ D(\overline{\tilde{\Delta}}) = sH^2_\mathcal{V}(M) \]

If there are no tangency points, then \(h\) can be taken just non-vanishing on \(Z\).

The proof will rely on the following Proposition proved in [35].

Proposition 2. Let \(H_1, H_2\) be Hilbert spaces, and let \(D\) be a Banach subspace of \(H_1\) equipped with a norm \(\| \cdot \|_D\) such that the inclusion map \((D, \| \cdot \|_D) \to (H_1, \| \cdot \|_{H_1})\) is continuous in \(H_1\). Let \(A : D \to H_2\) be a continuous operator and assume that:

1. The range of \(A\) is closed.
2. The kernel \(\ker A \subset H_1\) is closed with respect to \(\| \cdot \|_{H_1}\).

Then the operator \(A\) with domain \(D\) is closed, i.e., \(D\) is complete with respect to the graph norm \(\| u \|_A = \| u \|_{H_1} + \| Au \|_{H_2}\).

Note that in particular \(A\) can be left semi-Fredholm, which means that \(\dim \ker A < +\infty\) and \(\text{Ran} A\) is closed. The strategy of the proof of Theorem 1 will consist of the following steps:

1. Prove that functions \(C^\infty_c(M)\) are dense in \(sH^2_\mathcal{V}(M)\);
2. Prove that \(sH^2_\mathcal{V}(M)\) is continuously embedded into \(L^2(M, \omega)\);
3. Prove that \(\tilde{\Delta} : sH^2_\mathcal{V}(M) \to L^2(M, \omega)\) is left semi-Fredholm.
The first two points will essentially follow from the definitions (see also [5] for the general theory of Sobolev spaces on Lie manifolds). The main difficulty is the third point. A classical way to prove that an operator is left semi-Fredholm is to construct a left parametrix. This is indeed exactly what the authors of [35] did. However, they construct a pseudodifferential calculus adapted to their problem and following this approach each time is quite difficult. In [18] Nistor, Carvalho and Qiao provide an alternative approach for determining whether a pseudodifferential operator (PDO) on an open manifold $M$ is Fredholm. More precisely, they prove the following result.

**Theorem 3** (Carvalho-Nistor-Qiao (CNQ) conditions). Let $P \in \text{Diff}^m(V)(M)$ be an order $m$ differential operator on manifold $M$ compatible with the Lie manifold structure $V$. Then one can associate to $P$ the following data:

1. Smooth manifolds $M_\alpha$, parametrised by a suitable set $I$;
2. Simply connected Lie groups $G_\alpha$ acting freely and properly on $M_\alpha$, $\alpha \in I$;
3. Limit operators $P_\alpha$, which are $G_\alpha$-invariant differential operators on $M_\alpha$;

and the following statement holds:

$$P : H^s_V(M) \rightarrow H^{s-m}_V(M) \text{ is Fredholm } \iff P \text{ is elliptic and } P_\alpha : H^s(M_\alpha) \rightarrow H^{s-m}(M_\alpha) \text{ are invertible for every } \alpha \in I,$$

for some suitable Sobolev spaces $H^k$.

We will see that a similar statement holds for left semi-Fredholm operators. Namely if $P$ is elliptic and all $P_\alpha$ are left invertible, then $P$ is left semi-Fredholm. It should be noted that there exists extensive literature on Fredholm conditions in a great variety of situations (see, for example, [33,34,47–49]).

We will apply the left semi-Fredholm extension of Theorem 3 to the operator $\tilde{\Delta}$ on generic 2D almost-Riemannian manifolds. In this case manifold $I$ is the singular set $Z$ and hence all of the manifolds $M_\alpha$ are parametrised by points $q \in Z$. Manifolds $M_q$ play the role of model spaces similar to the Euclidean space in the scattering calculus and the hyperbolic plane in the 0-calculus. Lie groups $G_q$ are the simply-connected Lie groups which integrate the Lie algebra given by the kernel of the anchor map of the associated Lie algebroid (see Sect. 3.1). For 2D generic almost-Riemannian manifolds it so happens, that for the associated Lie structure $M_q$ coincide with $G_q$. Thus the corresponding limit operators $P_q$ are right-invariant differential operators. At a Grushin point $G_q$ is the connected component of the group of affine transformations of the real line and at a tangency point $G_q$ is the Euclidean space $\mathbb{R}^2$. This means that surprisingly it is much easier to verify the left invertibility of a limit operator at a tangency point than at a Grushin point. The main work will be related to studying left-invertibility of $\tilde{\Delta}$ at the Grushin points.

It is worth emphasising again that using the techniques explained in the article one can also obtain some information about the closure of the non-perturbed operator as well. Theorem 1 is stated mainly to show that various
singularities can be treated in a unified manner. Further generalisations can be proved similarly. Besides this, it seems there are not many results on the Laplace-Beltrami operator in the presence of tangency points. At least the author is not aware of any result in this direction, though there are several works which study structures without tangency points \cite{15,29–32,54} including a work by the author, Ugo Boscain and Eugenio Pozzoli \cite{11}. The main motivation for this work came from the attempt to find self-adjoint extensions of the curvature Laplacian on Grushin manifolds similarly to what was done in \cite{31,32} and explore their unusual quantum mechanical behaviour.

If in the construction of the Laplace-Beltrami operator instead of the Riemannian volume we take any smooth volume, then there is no singularity present (except the degeneracy of the principal symbol) and such operators can be handled using essentially the theory of Hörmander operators even though the singularity still manifests itself in various forms \cite{20,60}. We should also mention separately articles \cite{9,55}, where some results concerning analysis of some structures with similar singularities were obtained. In \cite{55} the authors studied the heat content on domains with characteristic points, while in \cite{9} induced stochastic processes on surfaces in the Heisenberg group are studied.

Finally, we note that Lie groupoids were used in sub-Riemannian geometry. Indeed, they were used in articles \cite{21,24,59}, where the authors mainly focus on equiregular structures and are aimed towards index theory on filtered manifolds. This theory was later extended to general Kolmogorov type operators in \cite{6}.

1.3. Structure of the Paper

We end this rather lengthy introduction by explaining the structure of the paper. It is written for two separate communities. On one hand for the sub-Riemannian community, on the other for people working on analysis and $C^*$-algebras on Lie groupoids. For this reason considerable part of the article is an explanation of basic notions both from almost-Riemannian geometry and Lie theory with many pictures, so that both communities can understand the idea behind definitions and methods used in the paper. In Sect. 2 basic notions from sub-Riemannian and almost-Riemannian geometry are given. In Sect. 3 the basics of Lie groupoids, Lie algebroids and Lie manifolds with relevant examples are discussed. In Sect. 4 the compatible PDO calculus and Sobolev spaces are defined. In Sect. 5 generalised Carvalho-Nistor-Qiao conditions are stated and an idea of the proof is given. In Sect. 6 we give a simple application of the Carvalho-Nistor-Qiao conditions to the study of a model 1D example that will be useful in the study of limit operators at Grushin points. Section 7 is entirely dedicated to the proof of Theorem 1. The goal of this article is to show that it is possible to study different singularities in sub-Riemannian geometry via a unique single theory. For this reason we do not strive for the most general results, nevertheless in the final Sect. 8 we discuss the applicability of this method to other structures. The reader familiar with Lie groupoids...
and their representations can simply skim Sects. 3-5 to become familiar with the notations of the article and go directly to the proof of Theorem 1. For people familiar with sub-Riemannian geometry a short summary on the limit operators and how to write them explicitly is given in the beginning of Sect. 6, where we consider a relevant 1D example. Sections 3–4 essentially collect all the definitions and necessary for this paper results which are scattered over the literature. The author hope that both communities will find the results interesting either in terms of techniques, or examples and possible applications.

2. Almost-Riemannian Geometry

Let us recall the definition of an almost-Riemannian manifold from [1]. Let \( F \) be a set of vector fields on a manifold \( M \). The Lie algebra generated by \( F \) is defined as the smallest Lie subalgebra of the Lie algebra of vector fields on \( M \) containing all the commutators of \( F \):

\[
\text{Lie } F = \text{span}\{[X_1, \ldots, [X_{j-1}, X_j]], X_i \in \Gamma(F), j \in \mathbb{N}\}.
\]

We denote by

\[
\text{Lie}_q F = \text{span}\{X(q) : X \in F\}
\]

and say that \( F \) satisfies the Hörmander condition or is bracket-generating if

\[
\text{Lie}_q F = \{X(q) : X \in \text{Lie } F\} = T_q M, \quad \forall q \in M.
\]

**Definition 4.** Let \( M \) be a connected smooth manifold. An almost-Riemannian structure on \( M \) is a pair \((U, f)\), where

1. \( \pi_U : U \to M \) is a Euclidean bundle;
2. \( f : U \to TM \) is a fiber-wise linear smooth morphism of bundles surjective on an open dense subset of \( M \). In particular the following diagram is commutative

\[
\begin{array}{ccc}
U & \xrightarrow{f} & TM \\
\downarrow{\pi_U} & & \downarrow{\pi} \\
M & & \\
\end{array}
\]

3. The family of vector fields \( \mathcal{D} = f(\Gamma(U)) \) satisfies the Hörmander condition.

The family \( \mathcal{D} \) is often called the distribution of the almost-Riemannian structure \((U, f)\). Denote \( \mathcal{D}_q = f(U_q) \). A Lipschitz curve \( \gamma : [0,1] \to M \) is called admissible if for almost every \( t \in [0,1] \) the velocity vector \( \dot{\gamma}(t) \) belongs to \( D_{\gamma(t)} \).

To explain the Hörmander condition define the flag \( \mathcal{D}^1_q \subset \mathcal{D}^2_q \subset \cdots \subset T_q M \) of an almost-Riemannian manifold at a point \( q \) recursively as

\[
\mathcal{D}^1_q = \mathcal{D}_q, \quad \mathcal{D}^{k+1}_q = \mathcal{D}^k_q + [\mathcal{D}^k_q, \mathcal{D}]_q.
\]
We say that $\mathcal{D}$ satisfies the Hörmander condition if there exists a number $k \in \mathbb{N}$ such that $\mathcal{D}_q^k = T_q M$ for every point $q \in M$. The importance of the Hörmander condition comes from the following result.

**Theorem 5** (Rashevsky-Chow). Let $M$ be a smooth manifold $\mathcal{D} \subset TM$ a possible rank-varying distribution of planes, which satisfies the Hörmander condition. Then any two points of $M$ can be connected by an admissible curve.

We can measure the lengths of admissible curves as follows. The almost-Riemannian norm of a vector $v \in \mathcal{D}_q$ is defined as

$$\|v\| = \min\{|u| : u \in \mathcal{U}_q, v = f(q, u)\}.$$ 

Then the length of an admissible curve $\gamma : [0, 1] \to M$ is defined as

$$l(\gamma) = \int_0^1 \|\dot{\gamma}\| dt.$$ 

Hörmander condition under some additional completeness assumptions guarantees that any two points can be connected by a minimiser. In this case an almost-Riemannian manifold becomes a well-defined metric space.

**Definition 6.** Points where $\dim \mathcal{D}_q < \dim M$ is called the singular set and we denote it by $\mathcal{Z}$.

**Example 7.** The trivial example is when $M$ is a Riemannian manifold and the singular set is empty. In this case we can take $\mathcal{U}$ to be its tangent bundle and $f$ just to be the identity map. We can also give an equivalent, but different description of the same Riemannian structure. Nash embedding theorem states that any Riemannian manifold $M$ can be isometrically embedded into a Euclidean space $\mathbb{R}^k$ of sufficiently big dimension. We can consider the trivial bundle $M \times \mathbb{R}^k$ and a fibre-linear smooth morphism $f : M \times \mathbb{R}^k \to TM$ defined as follows. We take the normal projection of $\mathbb{R}^k$ centred at a point $q \in M \subset \mathbb{R}^k$ to $T_q M \subset \mathbb{R}^k$. It follows from our definitions that the length of a vector $v \in T_q M$ in both constructions is the same.

More generally, if $f$ in the definition of almost-Riemannian manifolds is surjective everywhere, i.e., $\dim \mathcal{D}_q = \dim M$ for all points $q \in M$, then the almost-Riemannian structure is just Riemannian. We can reduce the rank of the bundle $\mathcal{U}$ by taking the sub-bundle $\ker df$ with a metric given by the restriction.

**Example 8.** Let us now consider the basic singular example given by the Grushin plane. In this case we have a map $f : \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathcal{U} \to T\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ defined as follows

$$f : \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ u \\ xv \end{pmatrix}.$$
We denote as before \( q = (x, y) \in \mathbb{R}^2 \) and assume that the metric on \( U_q \) is just the standard Euclidean metric, i.e. \( \|(u, v)\|_{U_q} = \sqrt{u^2 + v^2} \). Note that outside the set \( \{x = 0\} \) the morphism \( f \) is a fibre-wise bijection and allows us to recover the metric by inverting the map \( f \)

\[
\|(u, v)\|_{T_q\mathbb{R}^2} = \|f^{-1}(u, v)\|_{U_q} = \sqrt{u^2 + \frac{v^2}{x^2}}.
\]

Using the polarisation identity we then find the metric tensor

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix},\]

which is well-defined outside the singular set \( \{x = 0\} \). At the singular set the metric tensor explodes, but the length of vectors in \( D_q \) is well-defined regardless. Indeed, if \( q \in Z \) and \( (a, 0) \in D_q \), then \( f^{-1}(a, 0) = \{(a, v) \in \mathbb{R}^2 : v \in \mathbb{R}\} \) and as the result \( \|(a, 0)\|_{T_q\mathbb{R}^2} = |a| \).

**Remark 9.** Note that in the Grushin example we could have taken a Euclidean bundle \( U \) of a bigger rank, where the extra directions would be in the kernel of the morphism \( f \). Increasing the rank of the bundle \( U \) can be useful in general. For example, one can prove that any sub-Riemannian structure can be defined via a trivial bundle \( U \) of a sufficiently high rank \([1, Corollary 3.27]\).

From here on we will consider 2D compact almost-Riemannian manifolds without boundary satisfying the following genericity assumption (H0) from \([2, Proposition 2]\):

1. \( Z \) is an embedded one-dimensional smooth submanifold of \( M \);
2. The points \( q \in M \) at which \( D_q^2 \) is one-dimensional are isolated;
3. \( D_q^3 = T_qM \) for all \( q \in M \).

Locally we can take two orthonormal sections \( \sigma_1, \sigma_2 \in \Gamma(U) \) and construct two vector fields \( X_i = f(\sigma_i), \ i = 1, 2 \), which generate our distribution in a neighbourhood of a given point \( q \in M \). Note that \( X_1 \) and \( X_2 \) cannot vanish at the same point as it would violate the Hörmander condition. Thus for every point \( q \in M \) there exists an open neighbourhood \( O_q \) and a non-vanishing vector field in \( O_q \) which we call again \( X_1 \). By rectifying this vector field we can find local coordinates \((x, y)\) on \( O_q \) centred at \( q \) such that

\[
X_1 = \partial_x, \quad X_2 = f(x, y)\partial_y, \quad (1)
\]

for some smooth function \( f \). In particular \( Z \cap O_q \) coincides with the zero locus of \( f \). Under the genericity assumption \( df|_Z \neq 0 \). In this local frame we can write down

1. an expression for the metric

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{f(x,y)^2} \end{pmatrix};
\]
2. an expression for the associated canonical volume form:

\[ \omega = \frac{dx \wedge dy}{|f|}; \]

3. an expression for the Laplace-Beltrami operator:

\[ \Delta = \partial_x^2 + f^2 \partial_y^2 - \frac{\partial_x f}{f} \partial_x + f(\partial_y f) \partial_y. \]  

(2)

Note that all of those quantities tend to infinity as we get closer to the zero locus of \( f \).

We have already seen that under the genericity assumption there are three types of points that can occur. One can construct normal forms for each type which improve (1).

**Theorem 10** ([3]). Let \( q \) be a point of a generic 2D almost-Riemannian manifold. Then there exists local coordinates \((x, y)\) such that \( D \) is locally generated by two vector fields \( X_1, X_2 \) of one of the following normal forms:

1. **Riemannian points:**

\[ X_1 = \partial_x, \quad X_2 = e^{\phi(x,y)} \partial_y; \]  

(3)

2. **Grushin points:**

\[ X_1 = \partial_x, \quad X_2 = xe^{\Phi(x,y)} \partial_y; \]  

(4)

3. **Tangency points:**

\[ X_1 = \partial_x, \quad X_2 = (y - x^2 \psi(x))e^{\Psi(x,y)} \partial_y; \]  

(5)

where \( \phi, \psi, \Psi \) are smooth functions, \( \Phi(0, y) = \Psi(0, y) = 0, \psi(0) \neq 0 \).

We will mostly work with the general form (1) and use Theorem 10 only to compute the limit operators. Next we recall all the necessary theory that would allows us to prove semi-Fredholm properties of the Laplace-Beltrami operator \( \Delta \) and compute explicitly the closure of the perturbed operator.

### 3. Lie Groupoids, Lie Algebroids and Lie Manifolds

#### 3.1. Definitions and Examples

Lie groupoids were introduced by Charles Ehresmann and nowadays they represent an indispensable tool in many fields of mathematics like Poisson geometry [61] and foliation theory [46]. In analysis Lie groupoids are often used in global geometry in studying various geometric objects via differential operators naturally associated to them. Probably the most famous application of this kind is given by the Connes’ proof of the index theorem [22, Chapter 2, Section 10]. As many geometric objects are singular, Lie groupoids provide a language that allows to treat regular and singular objects in a unified manner. In the following exposition we follow closely papers [18, 42, 52].
A particular type of singularity that we will be interested in are corners. A manifold $M$ with corners is second countable, Hausdorff topological space locally modelled by open subsets of $[-1,1]^n$. We will abbreviate their names just to manifolds. If a manifold actually does not have a boundary, we will call it a smooth manifold. Every point $q \in M$ has an inward pointing tangent cone $T_q^+M$. In the interior of $M$ we have $T_q^+M = T_qM$. A tame submersion $h : M_1 \rightarrow M$ is a smooth map such that its differential is surjective and $dh(v)$ is inward pointing if and only if $v \in T_q^+M_1$. Given a tame submersion each preimage $h^{-1}(q), q \in M$ is actually a smooth manifold. We will use $M_0$ to denote the interior of $M$.

We can now define Lie groupoids. Intuitively one can think of them in many ways, but probably the most useful is to look at them as manifolds with a compatible “partial multiplication” which means that we cannot multiply every element by every other element. This multiplication function must satisfy certain compatibility conditions that are listed in the following formal definition.

**Definition 11.** A Lie groupoid $\mathcal{G} \rightrightarrows M$ is a pair of manifolds $(\mathcal{G}, M)$ with the following structure maps:

- an injection $u : M \rightarrow \mathcal{G}$ called the unit map. The space $u(M) \subset \mathcal{G}$ which we identify with $\mathcal{G}$ is called the space of units;
- a pair of tame submersions $d, r : \mathcal{G} \rightarrow M$ called the domain (or source) and range (or target) maps such that $d \circ u = id, r \circ u = id$;
- a multiplication map $m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ from the set of composable pairs: $\mathcal{G}^{(2)} = \{(g', g) \in \mathcal{G} \times \mathcal{G} : r(g) = d(g')\}$, and analogously to the theory of Lie groups we shorten $m(g, g')$ to $gg'$.

The multiplication map satisfies:

\[
d(g'g) = d(g), \quad r(g'g) = r(g'), \quad (g''g')g = g''(g'g), \quad gd(g) = g, \quad r(g)g = g;
\]

- an inversion map $i : \mathcal{G} \rightarrow \mathcal{G}, i : g \mapsto g^{-1}$ such that

\[
gg^{-1} = r(g), \quad g^{-1}g = d(g).
\]

**Remark 12.** Some modifications are possible. For example, $\mathcal{G}$ often is not assumed to be Hausdorff even though $M$ is always assumed to be Hausdorff. In this paper we will only work with Hausdorff Lie groupoids.

**Example 13.** Let $G$ be a Lie group. We can consider it as a Lie groupoid $G \rightrightarrows \{id_G\}$ with multiplication given by the usual Lie group multiplication. The groupoid definition of a Lie group emphasises more its algebraic structure than its geometric structures.

**Example 14.** Let $M$ be a manifold. Then we can consider a Lie groupoid $M \rightrightarrows M$, where $r(q) = d(q) = q$ for all $q \in M$ and hence all multiplications are just multiplications by units.
For this reason one sometimes says that Lie groupoids interpolate between manifolds and Lie groups. Let us see some other examples which lie in between.

**Example 15.** Let $G$ be a Lie group acting on a manifold $M$. It is known that the quotient $M \setminus G$ is not always a manifold. For this reason it is often more natural to consider the action groupoid $G \ltimes M \to M$ which topologically is just $M \times G$. The domain and range maps are given by $d(g, q) = q$, $r(g, q) = gq$ and the multiplication is $(g', gq)(g, q) = (g'g, q)$.

**Example 16.** Another important example is the pair groupoid $M \times M \to M$. The domain and the range are given by $d(q', q) = q$, $r(q', q) = q'$ and the multiplication by $(q'', q'), (q', q) = (q'', q)$. Under a suitable natural notion of isomorphisms of groupoids [44] one can show that for a Lie group $G$ the pair groupoid $G \times G$ is isomorphic to the action groupoid $G \ltimes G$.

The pair groupoid $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a great visual aid for understanding groupoids and their multiplication. The space of units for this groupoid is given by the diagonal, range and domain maps are projections to the diagonal parallel to one of the axis. The multiplication of two elements $g'g = g''$ is depicted in Fig. 3.

Given two subsets $U, V \subset M$ we define the following subsets of $\mathcal{G}$: $\mathcal{G}_U = d^{-1}(U)$, $\mathcal{G}_V = r^{-1}(V)$, $\mathcal{G}_U^V = d^{-1}(U) \cap r^{-1}(V)$. We call $\mathcal{G}_U^V$ the reduction of $\mathcal{G}$ to $U$. If both $U$ and $\mathcal{G}_U$ are manifolds, then $\mathcal{G}_U^V$ is called the reduced groupoid or reduction to $\mathcal{G}_U$. In order to simplify notations we will denote the reduced groupoid $\mathcal{G}_U^V$ by $\mathcal{G}|_U$. As a special case to every point $q \in M$ we can associate a Lie group $G_q = \mathcal{G}|_q = d^{-1}(q) \cap r^{-1}(q)$ known as the isotropy group of $q$. If $U$ is $\mathcal{G}$-invariant, meaning that $\mathcal{G}_U^U = \mathcal{G}^U = \mathcal{G}_U$, then the reduction $\mathcal{G}_U$ is a Lie groupoid called the restriction of $\mathcal{G}$ to $U$. Reduced groupoids will play an important role in the construction of Lie groupoids, while restrictions and isotropy groups will be used for defining the limit operators.

Next we pass to the definition of a Lie algebroid.
Definition 17. A Lie algebroid is a triple \((A, \cdot, \rho)\) consisting of a vector bundle \(A\) endowed with a Lie bracket \(\cdot, \cdot\) and a morphism of vector bundles \(\rho : A \to TM\) called the anchor map which satisfies the following identities:
- The Leibniz rule: \([X, fY] = (\rho(X)f)Y + f[X,Y]\), where \(X, Y \in \Gamma(A)\) and \(f \in C^\infty(M)\).
- Lie algebra homorphism: \(\rho([X,Y]) = [\rho(X), \rho(Y)]\).

Similarly to Lie groups we can associate the right multiplication map \(R_g : \mathcal{G}_{r(g)} \to \mathcal{G}_{d(g)}\) as
\[
R_g : h \mapsto hg.
\]
and it establishes a diffeomorphism between \(\mathcal{G}_{r(g)}\) and \(\mathcal{G}_{d(g)}\). This allows us to push vectors between the corresponding tangent bundles and define the notion of right invariant vector fields as sections of \(\bigcup_{q \in M} TV_q\) which are invariant under right multiplication. Define the vector bundle \(A(\mathcal{G}) = \bigcup_{q \in M} TV_q G = \bigcup_{q \in M} TV_q (d^{-1}(q))\). Note that there is one-to-one correspondence between right invariant vector fields and section of \(A(\mathcal{G})\) exactly in the same manner as there is a one-to-one correspondence between right invariant vector fields on a Lie group and vectors from a Lie algebra. Space \(A(\mathcal{G})\) is called the Lie algebroid of the Lie groupoid \(\mathcal{G} \rightrightarrows M\). The Lie bracket on \(A(\mathcal{G})\) is the restriction of the Lie bracket between right invariant vector fields to the space of units. The anchor map is given by \(\rho = r_*|_{A(\mathcal{G})}\). The image of \(\Gamma(A(\mathcal{G}))\) under the anchor map we denote by \(Lie(\mathcal{G})\). Finally note that equivalently \(A(\mathcal{G})\) is the restriction to the space of units of the vector bundle \(ker d_*\), where \(d_* : TV \to TM\) is the differential of \(d\).

Example 18. Let \(G \rightrightarrows \{\text{id}_G\}\) be a Lie group viewed as a Lie groupoid. Then \(d^{-1}(\text{id}_G) = G\), \(A(\mathcal{G}) = T\text{id}_G G = g\). The Lie bracket is the usual Lie algebra bracket. The anchor map then maps \(T\text{id}_G G\) to 0 as it should, since \(T(\text{id}_G) = \{0\}\). Hence \(Lie(\mathcal{G}) = \{0\}\).

Example 19. Let \(M \rightrightarrows M\) be a manifold viewed as a Lie groupoid. In this case \(d^{-1}(q) = q\). Hence \(A(\mathcal{G}) = M\). The anchor map \(\rho : M \to TM\) is the embedding of \(M\) as the zero section.

Example 20. Let \(G \ltimes M \rightrightarrows M\) be an action groupoid. We have that \(d^{-1}(q) \simeq G\). And the set of right invariant vector fields coincides with vector fields \(\tilde{X}(q, q) = X(q) \oplus 0\), where \(X(q)\) is a right invariant vector field on \(G\). Hence Lie bracket on \(A(\mathcal{G})\) coincides with the point-wise Lie bracket. Elements of the Lie algebra \(g\) can be identified with the constant sections of \(A(\mathcal{G})\). Under the anchor map they are mapped to infinitesimal actions of the Lie algebra \(g\) on \(M\). The Lie bracket for general sections of \(A(\mathcal{G})\) can be found in [44, Example 3.5.14].

Example 21. Consider the pair groupoid \(M \times M \rightrightarrows M\). We have \(d^{-1}(q) = M \times \{q\}\). Hence \(A(\mathcal{G}) = \bigcup_{q \in M} (T_q M \times \{q\}) = TM\). One can identify the
Figure 4. Translation of vectors via right multiplication in $\mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$

section of $A(G)$, which are vector fields $X \in \Gamma(TM)$, with right invariant vector fields $\tilde{X}(q', q) = 0 \oplus X(q)$ and inherit a Lie bracket that is just given by the usual Lie bracket of vector fields on $TM$. Hence the anchor map is just the identity.

Once again the pair groupoid $\mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$ provides a great visual aid for understanding $A(G)$ and right invariant vector fields. In Fig. 4 the process of translating a vector from $A(G)$ inside $TG$ is considered. In this particular case it is simply given by parallel transport of the vector along the range fibers. Thus all right invariant vector fields are vector fields constant on the vertical line.

Similarly we can consider the dual Lie algebroid $p : A^*(G) \to M$, which is just the dual bundle to $A(G) \to M$. It has a canonical Poisson bracket defined as follows. Let $X \in \Gamma(A(G))$ and $\lambda \in A^*(G)$. Define a linear on fibres Hamiltonian as

$$h_X = \langle \lambda, X \rangle.$$

Then the Possion bracket satisfies the following conditions

$$\{h_X, h_Y\} = -h_{[X,Y]}, \quad \{f \circ \pi, h_X\} = \rho(X)(f) \circ \pi$$

for all $X, Y \in \Gamma(A(G))$ and $f \in C^\infty(M)$. (Note that the minus sign comes from the right invariance).
We will deal with a special case of Lie algebroids, which can be integrated to Lie groupoids $\mathcal{G} \rightrightarrows M \times M$, such that the interior $M_0$ is $\mathcal{G}$-invariant.

**Definition 22.** A Lie manifold is a pair $(M, \mathcal{V})$ consisting of a compact manifold $M$ and $\mathcal{V} \subset \Gamma(TM)$ of vector fields tangent to $\partial M$ satisfying the following properties:
- $\mathcal{V}$ is closed under the Lie bracket $[\cdot, \cdot]$ on $TM$;
- $\mathcal{V}$ is a $C^\infty(M)$-module that is generated in a neighbourhood of $q \in M_0$ by a set of linearly independent vector fields $X_1, \ldots, X_n$;
- in an open neighbourhood $U \subset M_0$ of $q \in M_0$ inside the interior $M_0 \subset M$ we have isomorphism of $\mathcal{V}|_U$ and $TU$ as Lie algebras.

Due to Serre-Swan theorem [39, Theorem 6.18] there exists a Lie algebroid $A\mathcal{V}$ such that $\mathcal{V} = \text{Lie}(A\mathcal{V})$ and $\rho : A\mathcal{V} \to TM$ is an isomorphism over $M_0$. It is well known that not any Lie algebroid comes from a Lie groupoid and a complete set of obstructions were found by Crainic and Fernandes [23]. Lie manifolds and the associated Lie algebroids can be always integrated by results of [25,50]. The only problem is that this groupoid may fail to be Hausdorff. Similar to the theory of Lie groups, if a Lie algebroid is integrable, it has a unique $d$-simply connected Lie groupoid integrating it, meaning that $\mathcal{G}_q$ is simply connected for each $q \in M$. Such integrations are often called maximal and any other integration would be a quotient by a discrete, totally disconnected normal Lie subgroupoid (see, for example, [37, Theorem 1.20]). Lie manifolds are often used in the study of open and singular manifolds [4].

Let us see a couple of examples that do integrate to Hausdorff Lie groupoids, namely to action groupoids. These two examples will be relevant to the 1D model studied in Sect. 6 and in the construction of a Lie groupoid associated to a 2D AR manifold in Sect. 7.1.

**Example 23.** Consider the half-line $\mathbb{R}_+$ given by $\{x \in \mathbb{R} : x \geq 0\}$ and the Lie manifold structure $\mathcal{V}$ given by a $C^\infty$-module generated by $x \partial_x$. The Lie algebroid structure is particularly simple since it is given by the trivial bundle $A\mathcal{V} = \mathbb{R} \times \mathbb{R}_+$ and the anchor map $\rho$ maps some section $\sigma \in \Gamma(A\mathcal{V})$ to $x \partial_x$. This Lie algebroid can be integrated to an action groupoid, where the action of $\mathbb{R}$ on $\mathbb{R}_+$ is given by

$$ x \mapsto e^t x, \quad x \in \mathbb{R}_+, \quad t \in \mathbb{R}. $$

Thus the Lie groupoid which integrates $A\mathcal{V}$ is given by

$$ (\mathbb{R} \times \mathbb{R}_+) \rightrightarrows \mathbb{R}_+, $$

with domain, target and multiplication maps

$$ d(t, x) = x, \quad r(t, x) = e^t x, \quad (t_2, e^{t_1} x)(t_1, x) = (t_2 + t_1, x). $$

Figure 5 gives a graphical interpretation of the leaves $\mathcal{G}_x = d^{-1}(x)$, $\mathcal{G}^x = r^{-1}(x)$. 
Note that $x = 0$ is $\mathcal{G}$-invariant subset and $\mathcal{G}_0 = \mathcal{G}^0$ is the isotropy group given by $\mathbb{R}$ with the standard addition operation. The set $x > 0$ is also a $\mathcal{G}$-invariant subset of $\mathbb{R}_+$. The restriction groupoid $\mathcal{G}_{x>0}$ is equivalent to the pair groupoid. Indeed, to see this it is enough to make a change of variables

$$(t, x) \mapsto (e^t x, x), \quad x > 0.$$

Example 24. We can consider a generalisation of the last example relevant to the study of Grushin points. Let $\mathbb{R}^2_+$ be the half-space given by $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ and the Lie manifold structure $\mathcal{V}$ which is a $C^\infty$-module over $\{x \partial_x, x^2 \partial_y\}$. Consider a solvable Lie algebra $\mathfrak{g}$ generated by two vector fields $X_1, X_2$ which satisfy $[X_1, X_2] = 2X_2$. The Lie algebroid $A_{\mathcal{V}}$ is a trivial bundle $\mathbb{R}^2_+ \times \mathfrak{g}$ with the anchor map given by $\rho(X_1) = x \partial_x, \rho(X_2) = x^2 \partial_y$ and extended by linearity to $\Gamma(A)$. The Lie groupoid structure is given by the action groupoid $G \ltimes \mathbb{R}^2_+$, where $G$ is a group isomorphic to the affine group of the real line:

$$G = \left\{ \begin{pmatrix} a^2 & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}. \tag{6}$$

The action of $G$ given by

$$(x, y) \mapsto (ax, bx^2 + y).$$
We can identify \( \mathbb{R}_+^2 \setminus \{x = 0\} \) with \( G \) via a map
\[
(x, y) \mapsto \begin{pmatrix} x^2 y \\ 0 & 1 \end{pmatrix}.
\]
On the boundary \( \mathbb{R} \) the action is trivial, while in the interior it coincides with the right action of \( G \) on itself.

As we have discussed in the Example 16, \( G \times G \cong G \times G \). Therefore the \( d \)-simply connected Lie groupoid which integrates \( A_V \) topologically is given by
\[
(G \times \mathbb{R}) \sqcup (G \times G) : \mathbb{R}_+^2 \cong \mathbb{R}_+^2.
\]

### 3.2. Integrating Lie Algebroids and Glueing Lie Groupoids

As mentioned before, in contrast to Lie algebras not any Lie algebroid can be realised as a Lie algebroid of some Lie groupoid. However, this is always true for Lie algebroids coming from Lie manifolds. For our problem of determining the closure of differential operators we will require additional properties, in particular the integrating Lie groupoid should be Hausdorff.

One of the techniques of constructing new groupoids from the old ones is the **gluing construction** or the **fibered coproduct**, which was used successfully in [37]. Let \( M_1, M_2 \) be two manifolds and consider two open immersions \( i_j : U \hookrightarrow M_j, \ j = 1, 2 \) of a manifold \( U \). We can declare two points \( q_j' \in M_j, \ j = 1, 2 \) to be equivalent if there exist a point \( q \in U \) such that \( q_j' = i_j(q) \). Thus the fibre coproduct is defined as the quotient
\[
\frac{M_1 \sqcup M_2}{i_1(q) \sim i_2(q), \forall q \in U}
\]
and will be denoted as
\[
M_1 \sqcup M_2 / \sim.
\]

Let \( A \to M \) be a Lie algebroid. Assume that \( M \) has an open cover \( (U_i)_{i \in I} \) and that we can construct Lie groupoids \( G_i \) which integrate the restrictions \( A|_{U_i} \). A natural idea would be to glue \( G_i \) along the reductions \( (G_i)|_{U_i \cap U_j} \). Unfortunately the result may fail to be a Lie groupoid. For example, consider the pair groupoid \( G = \mathbb{R} \times \mathbb{R} \) and assume that we want to glue together its reductions \( G|_{(-\infty, \varepsilon)}, \ G'|_{(-\varepsilon, +\infty)} \). The result is not a Lie groupoid since the multiplication is not well-defined for all elements as can be easily seen from Fig. 6.

We see now that extra conditions must be imposed on the glued Lie groupoids in order to ensure that the multiplication is well defined after taking the fibered coproduct. In [17,37] the following natural gluing condition was proposed.

**Definition 25** ([17]). Let \( (U_i)_{i \in I} \) be a locally finite open cover of \( M \), \( (G_i \supseteq U_i)_G \) be Lie groupoids and
\[
\varphi_{ij} : G_i|_{U_i \cap U_j} \to G_j|_{U_i \cap U_j},
\]

\[ \]
be isomorphisms which satisfy $\varphi_{ij} = \varphi_{ji}$ and $\varphi_{ij} \varphi_{jk} = \varphi_{ik}$ for all $i, j, k \in I$ on common domains. We say that $(G_i)_{i \in I}$ satisfies the weak gluing condition if for any two composable arrows $(g, g')$ of the fibered co-product

$$G = \bigsqcup_{i \in I} G_i$$

there exists $i \in I$ such that both $g, g'$ have a representative in $I$.

For instance, consider the Lie groupoid $G$ from Example 23 and consider the cover $U_1 = [0, \varepsilon), U_2 = (0, +\infty)$. If $g, g'$ belong to $G|_{U_2}$, then we just use the multiplication in $G|_{U_2}$. If instead $g \in G \setminus G|_{U_2} = G_0$, then it can only be composed with $g' \in G_0$. But $G_0$ is a Lie subgroupoid of $G_1$ and hence the weak gluing conditions is satisfied.

**Proposition 26.** Consider $M, (U_i)_{i \in I}, (G_i)_{i \in I}$ and $G$ to be as in Definition 25. Then $G$ is a Lie groupoid. Moreover, if all $(G_i)_{i \in I}$ are Hausdorff, then $G$ is Hausdorff as well.
Consider again the Example 23. Suppose that we are only given the Lie algebroid $\mathcal{A}(\mathcal{G})$ and we do not know yet whether it is integrable or not. There are two $\mathcal{A}(\mathcal{G})$-invariant subsets of $[0, +\infty)$ given by $\{0\}$ and $(0, +\infty)$. We could separately integrate the restrictions of $\mathcal{A}(\mathcal{G})$ to the two invariant subsets and take

$$\mathcal{G} = \mathcal{G}_0 \sqcup \mathcal{G}_{(0, +\infty)},$$

For now we can only deduce that $\mathcal{G}$ is a groupoid, because there is yet no topology on $\mathcal{G}_0 \sqcup \mathcal{G}_{(0, +\infty)}$. In [50] the author describes a way of topologising this groupoid and many others.

**Definition 27 ([17]).** By a *stratified manifold* we mean a manifold together with a disjoint decomposition

$$M = \bigsqcup_{i \in I} S_i,$$

by a local family of smooth manifolds such that the closure is a submanifold and each $S$ is contained in a unique open face of $M$. A stratification $\sqcup_{i \in I} S_i = M$ is called $\mathcal{A}$-invariant if sections $\Gamma(\rho(A))$ preserve the stratification.

The following theorem is the main result of [50].

**Theorem 28 ([50]).** Let $A \to M$ be a Lie-algebroid and $\sqcup_{i \in I} S_i$ an $\mathcal{A}$-invariant stratification. If there exist $d$-simply connected Lie groupoids $\mathcal{G}_S_i$ integrating $A|_{S_i}$, then

$$\mathcal{G} = \bigsqcup_{i \in I} \mathcal{G}_S_i$$

is a Lie groupoid.

The idea of the proof goes as follows. The multiplication, domain and range maps are well defined on $\mathcal{G}$. So in order to transform $\mathcal{G}$ into a Lie groupoid we only need to define topology in a neighbourhood of the units and then use the multiplication map in order to extend the topology to the rest of $\mathcal{G}$.

More precisely, let $X_1, \ldots, X_m$ be complete right invariant vector fields, and $Y_1, \ldots, Y_n$ be complete right invariant vector fields which form a basis of $A_q$ for a fixed $q \in M$, such that $\rho(X_i)$ and $\rho(Y_j)$ are smooth sections of $TM$. If $U \subset M$ is a neighbourhood of $q$ and $B_\varepsilon$ is a sufficiently small neighbourhood of $0$ in $\mathbb{R}^n$, we can define maps $\varphi_Y^X : B_\varepsilon \times U \to \mathcal{G}$

$$\varphi_Y^X : (t_n, \ldots, t_1, q) \mapsto e^{t_m X_m} \circ \cdots \circ e^{t_1 X_1} \circ e^{t_n Y_n} \circ \cdots \circ e^{t_1 Y_1} q,$$  \(7\)

where $e^{tY}$ is the flow of a vector field $Y$ at a moment of time $t$. It should be noted that if we choose $X_1, \ldots, X_m$ to be zero, we essentially recover the local Lie groupoid structure constructed in [25]. Denote by $\Phi$ all collections of maps $\varphi_Y^X$ over different choices of $X, Y$ and open neighbourhoods $U$. Then the following theorem holds.
Theorem 29 ([50]). Let $A \to M$ be a Lie algebroid and $M = \bigsqcup_{i \in I} S_i$ be an $A$-invariant stratification of $M$. Assume that each $A|_{S_i}$ is integrable and let $G_{S_i}$ be the corresponding $d$-simply connected Lie groupoids. Then
\[ G = \bigsqcup_{i \in I} G_{S_i} \]
has a differentiable structure if and only if the family $\Phi$ consisting of maps (7) is a differentiable atlas.

We state this theorem in order to emphasize that the maps (7) are the charts of the Lie groupoid $G$ and hence can be used to prove Hausdorff properties of $G$.

4. Analysis on Lie Manifolds and Lie Groupoids

4.1. Compatible Structures

Theorem 28 and Serre-Swan theorem allow us to obtain a groupoid structure starting from a Lie manifold $(M, V)$. The biggest strata in this case is the interior $M_0$ and restriction $A_V|_{M_0}$ will be integrated to the pair groupoid $G_{M_0} \simeq M_0 \times M_0$. This construction and right invariance allows us to pull structures from the Lie manifold $(M, V)$ to the corresponding Lie groupoid.

Consider for example a metric $g_0$ defined on $M_0$. We can consider the pull-back of this metric to the Lie algebroid $\rho^*g_0$ via the anchor map. This allows to define the scalar product on the sublagebra of right invariant vector fields or one can consider it as a family of metrics on $G_q$ for $q \in M$. The problem is that this metric may not be well defined on all of $G$ because the anchor map fails to be a bijection on the boundary. In order to remedy this we need to consider a class of metric that we call compatible, which are restrictions of a smooth metric $g$ on $A(G)$ to $M_0$ under the anchor map. In the language of Lie manifolds we can define them as follows.

**Definition 30.** Let $(M, V)$ be a Lie manifold. A compatible metric on $M_0$ is a metric $g_0$ such that, for any $q \in M$, we can choose the basis $X_1, \ldots, X_n$ from Definition 22 to be orthonormal with respect to this metric on a neighbourhood $O_q \cap M_0$ of $q$.

Such metrics have very nice properties. With a compatible metric $(M_0, g_0)$ is a complete Riemannian manifold of infinite volume with bounded covariant derivatives of curvature. Under some additional assumptions one can also guarantee that the injectivity radius is positive [4]. Manifolds of bounded geometry have many useful analytic properties. For example, the three definitions of Sobolev spaces (via connections, via orthonormal frames and via the Laplace-Beltrami operator) coincide [36, 56].

In a very similar way we can consider constructions of other natural geometric objects on $G$. For example, since we have a metric on $A(G)$, we
can construct a volume form as an element of $\Lambda^n A^* (\mathcal{G})$. Extending by right invariance gives a system of volume forms on $\mathcal{G}_q$ which is an example of right Haar system.

**Definition 31.** A right Haar system of measures for a locally compact groupoid $\mathcal{G}$ is a family $(\lambda_q)_{q \in M}$, where $\lambda_q$ are Borel regular measures on $\mathcal{G}$ with support on $\mathcal{G}_q$ for every $q \in M$ and satisfying:

1. The continuity condition:

   $$M \ni q \mapsto \lambda_q (\varphi) := \int_{\mathcal{G}_q} \varphi (g) d\lambda_q (g)$$

   is continuous for every $\varphi \in C_c(\mathcal{G})$.

2. The invariance condition:

   $$\int_{\mathcal{G}_{r(g)}} \varphi (hg) d\lambda_{r(g)} (h) = \int_{\mathcal{G}_{d(g)}} \varphi (h) d\lambda_{d(g)} (h).$$

   In the very same spirit we can consider differential operators on $M$ whose lifts to $\mathcal{G}$ are right invariant differential operators.

**Definition 32.** Let $(M, V)$ be a Lie manifold. The set of $V$-differential operators $\text{Diff}_V (M)$ is the algebra of differential operators generated by compositions of vector fields from $V$ and multiplication by $C^\infty (M)$. In the frame described in the Definition 22 elements of $\text{Diff}_V (M)$ are locally generated by differential operators of the form

$$P = a X_{i_1} \ldots X_{i_m}, \quad a \in C^\infty (M), \quad i_j \in \{1, \ldots, n\}. \quad (8)$$

The subset of $V$-differential operators of order $m$ is denoted as $\text{Diff}^m_V (M)$.

To each element of $V$ we can associate a section $\sigma \in A (\mathcal{G})$, and to each section $\sigma$ we can associate a right invariant vector field $Z$ on $\mathcal{G}$. Thus an element of the form $(8)$ will be mapped to an element

$$\tilde{P} = (a \circ r) Z_{i_1} \ldots Z_{i_m}.$$ 

One can check that

$$\tilde{P} (f \circ r)(g) = P f (r(g)), \quad \forall f \in C^\infty (M), \quad \forall g \in \mathcal{G}.$$

**Example 33.** If $G \Rightarrow \text{id}$ is a Lie group, then instead of a Haar system we have a single Haar measure, right invariant vector fields are the usual right invariant vectors fields on $G$. Even though there are no non-trivial operators on the base space $\text{id}$, the Lie algebra $A (G)$ indeed contains interesting non-trivial objects. For example, right invariant differential operators restricted to $A (G) = T_{\text{id}} G = \mathfrak{g}$ can be identified with the elements of the universal enveloping algebra $U (\mathfrak{g})$. 

Example 34. If we consider the pair groupoid $G = M \times M \rightrightarrows M$, where $M$ is a smooth manifold, then every smooth metric on $g$ is a compatible metric. Each $G_q$ is a copy of $M$ and the lift of the metric induces the same metric $g$ on each $G_q$. The same happens with other objects: right invariant vector fields are just copies of the same vector field on each $G_q$ and differential operators are lifted to the same differential operator on every $G_q$.

More generally it is possible to define a pseudo-differential operators and pseudo-differential calculus adapted to the Lie groupoid structure. A pseudo-differential operator (PDO) $P$ on a Lie groupoid is a family of classical PDOs $P_q : C_c^\infty(G_q) \to C_c^\infty(G_q)$ parametrised by points $q \in M$ satisfying certain conditions (see Definition 3.2 in [42]). First of all it must be a differentiable right invariant family. Secondly, if $P$ is a PDO, then $P_q$ have distribution kernels $k_q$ and support of $P$ defined as

$$ \text{supp}(P) = \bigcup_{q \in M} \text{supp}(k_q) $$

must be a closed subset of $\{(g, g'), d(g) = d(g')\}$. It will be assumed that the reduced support given by

$$ \{g'g^{-1} : (g, g') \in \text{supp}(P)\} $$

is a compact subset of $G$.

We will denote pseudo-differential operators of order $m$ as $\Psi^m(G)$ and as usual

$$ \Psi^\infty(G) = \bigcup_{m \in \mathbb{Z}} \Psi^m(G), \quad \Psi^{-\infty} = \bigcap_{m \in \mathbb{Z}} \Psi^m(G). $$

We can also define the principal symbol of a pseudodifferential $P \in \Psi^m(G)$ as an order $m$ homogeneous function $\sigma_m$ on $A^*_q(G) \setminus \{0\}$:

$$ \sigma_m(P)(\xi) = \sigma_m(P_q)(\xi), \quad \forall \xi \in A^*_q(G) = T^*_q G. $$

The following result states that the symbol has the usual properties which allow to construct parametrixes.

**Theorem 35.** ([42]) Let $G \rightrightarrows M$ be a Lie groupoid. Then $\Psi^\infty(G)$ is an algebra with the following properties:

1. The principal symbol map

$$ \sigma_m : \Psi^m(G) \to S^m_c(A_q^*(G))/S^{m-1}_c(A_q^*(G)). $$

is surjective with kernel $\Psi^{m-1}(G)$.

2. If $P \in \Psi^m(G)$ and $Q \in \Psi^{m'}(G)$, then $PQ \in \Psi^{m+m'}(G)$ and satisfies

$$ \sigma_{m+m'}(PQ) = \sigma_m(P)\sigma_{m'}(Q). $$

Consequently $[P, Q] \in \Psi^{m+m'-1}(G)$ and

$$ \sigma_{m+m'-1}([P, Q]) = \frac{1}{i}\{\sigma_m(P), \sigma_{m'}(Q)\}, $$

where the Poisson bracket is the one that comes from the Lie algebroid structure.
A PDO $P \in \Psi^m(\mathcal{G})$ is called \textit{elliptic} if $\sigma_m(P)$ does not vanish on $A^*(\mathcal{G}) \setminus M$. Similarly to the classical pseudo-differential calculus it is possible to construct a parametrix for an elliptic operator of order $m$ and to invert it modulo elements of $\Psi^{-\infty}(\mathcal{G})$ (see [62, Section 3] for a conceptual introduction). For future reference it should be also emphasised that right invariant differential operators belong to $\Psi^\bullet(\mathcal{G})$.

We now see that in the case of Lie manifolds we can lift differential operators from $(M, \mathcal{V})$ to invariant differential operators on $\mathcal{G}$. More generally we can start with any Lie groupoid $\mathcal{G}$ and use its structure to study operators generated by vector fields from $\text{Lie}(\mathcal{G})$. We can view the lift $\tilde{P}$ of an operator $P$ as a family of (pseudo-)differential operators $\tilde{P}_q$ acting on $C^\infty(\mathcal{G}_q)$. We thus can consider the restrictions of those operators to various subgroupoids. In particular, in the context of stratified Lie groupoids the following definition will be of great importance.

\textbf{Definition 36.} Let $(M, \mathcal{V})$ be a stratified Lie manifold and $\mathcal{G} \rightrightarrows M$ be a Lie groupoid that integrates $A_\mathcal{V}$. If $P \in \Psi^m(M_0)$ and $\tilde{P}$ is its lift to $\Psi^m(\mathcal{G})$, then the operators $\tilde{P}_q$, where $q \in \partial M$ are called \textit{limit operators}.

Limit operators play a central role in the study of differential operators on Lie manifolds. For example, as we have already mentioned in the introduction, one can formulate necessary and sufficient conditions for Fredholmness in terms of the invertibility of limit operators. As consequence one can also deduce information about the essential spectrum of an operator $P$ from the spectrum of its limit operators [18].

\subsection*{4.2. Sobolev Spaces}

Given a compatible metric on a Lie manifold $(M, \mathcal{V})$, we denote the corresponding volume form as $\mu_\mathcal{V}$ and the corresponding Laplace operator as $\Delta$. The $L^2$-norms in this subsection are taken with respect to $\mu_\mathcal{V}$. Recall that $M_0$ is complete when endowed with a compatible metric and hence $\Delta$ is essentially self-adjoint. We can give three different definitions of Sobolev spaces via three different norms.

First let $\nabla$ be the covariant derivative of the associated Levi-Civita connection and $k \in \mathbb{N}$. We can define the norm

$$\|u\|_{2, \nabla, H^k_\mathcal{V}(M)}^2 = \sum_{i=1}^{k} \|\nabla^i u\|_{L^2_\mathcal{V}(M)}^2.$$ 

If $\mathcal{X} \subset \mathcal{V}$ is a finite set of vector fields such that $C^\infty(M)\mathcal{X} = \mathcal{V}$, then we can define

$$\|u\|_{2, \mathcal{X}, H^k_\mathcal{V}(M)}^2 = \sum \|X_1 X_2 \ldots X_i u\|_{L^2_\mathcal{V}(M)}^2,$$

where the sum is taken over $i \in \{1, \ldots, k\}$ and all possible $X_1, X_2, \ldots, X_k \in \mathcal{X}$. Finally we can use functional calculus to define the norms

$$\|u\|_{2, \Delta, H^k_\mathcal{V}(M)}^2 = \sum \|(1 + \Delta)^{\frac{k}{2}} u\|_{L^2_\mathcal{V}(M)}^2.$$
Definition 37. Sobolev spaces $H^k_V(M)$ are defined to be completions of $C^\infty_c(M_0)$ in one of the norms above.

Theorem 38 ([5]). All three Sobolev norms are equivalent. Moreover different Sobolev spaces constructed using different compatible metrics or different choices of $\mathcal{X} \subset \mathcal{V}$ are equivalent as well.

In view of this theorem we will denote all three norms by $\| \cdot \|_{H^k_V(M)}$.

We should note that one can define Sobolev spaces for every $s \in \mathbb{R}$ using interpolation.

In order to deal with the Laplace-Beltrami operator we need weighted Sobolev spaces.

Definition 39. Fix a boundary hyperface $H$ of $M_0$ and a defining function $s_H$. Define

$$s = \prod a_H^{s_H},$$

where $a_H \in \mathbb{R}$ and the product is taken over all hyperfaces of $M_0$. Then the weighted Sobolev spaces are defined as

$$sH^k_V(M) = \{su : u \in H^k_V(M)\}$$

with the norm $\|su\|_{sH^k_V(M)} = \|u\|_{H^k_V(M)}$.

Lemma 40. If $a_H > 0$ for each hypersurface $H$ in the definition of $s$, then for any $\alpha > \alpha'$ and $0 \leq m \leq k$ Sobolev space $s^\alpha H^k_V(M)$ is continuously embedded into $s^{\alpha'} H^{k-m}_V(M)$.

The proof is a direct consequence of definitions and the fact that under the assumptions $s^{\alpha-\alpha'}$ is a smooth bounded function.

It should be noted that although for the applications that we have in mind this is all the information we need, Sobolev spaces on Lie manifolds were extensively studied with many of the classical results extended to this setting. In particular, Rellich-Kondrachov theorems for general $s^\alpha W^{k,p}$ spaces hold [5, Theorem 4.6] as well as Gagliardo-Nirenberg-Sobolev inequalities [5, Proposition 3.14]. One can also define the traces [5, Theorem 4.3] and use them for solving boundary value problems in this setting (see [5] for applications to solutions of boundary value problems of elliptic operators on polyhedral domains).

Given a Lie manifold we can consider the associated stratified Lie groupoid $G \Rightarrow M$. Assume that $G_{M_0}$ can be identified with the pair groupoid $M_0 \times M_0 \Rightarrow M_0$. As discussed in Example 34, each $G_q$ for $q \in M_0$ can be identified with a copy of $M_0$, and given an operator $P$ on $M_0$, its lift to the pair groupoid will consist of copies of operators $\tilde{P}_q = \tilde{P}$ on each $G_q$. From this point of view Sobolev spaces $H^k_V(M)$ can be seen as natural functional spaces on $G_q$ for $q \in M_0$.

By analogy we can define associated Sobolev spaces on any $G_q$ for a stratified Lie groupoid. We start with a compatible metric and construct a
Laplace-Beltrami operator $\Delta$ on $M_0$. Then we lift it to the Lie groupoid $\mathcal{G}$ and consider it as a family of operators $\tilde{\Delta} = \tilde{\Delta}_q$ of $\mathcal{G}_q$, $q \in M$.

**Definition 41.** Sobolev space $H^k_V(\mathcal{G}_q)$, $q \in M$ are completions of $C_c(\mathcal{G}_q)$ in the norm

$$
\|u\|^2_{H^k_V(\mathcal{G}_q)} = \sum \|(1 + \tilde{\Delta}_q)^{\frac{k}{2}} u\|^2_{L^2_v(M)}
$$

Similarly to the $H^k_V(M)$ spaces defined above we have the following result [51, Corollary 7.10].

**Proposition 42.** Spaces $H^k_V(\mathcal{G}_q)$ do not depend on the choice of a compatible metric on $M_0$. If $P \in \Psi^m(\mathcal{G})$, then $P_q : H^\alpha_V(\mathcal{G}_q) \to H^{\alpha - m}_V(\mathcal{G}_q)$ are continuous operators for any $\alpha \in \mathbb{R}$.

5. Groupoid Representations and CNQ Conditions

5.1. Groupoid $\mathcal{C}^*$-Algebras and Their Representations

Our goal for this section is to show how CNQ conditions appear from the representation theory of groupoid $\mathcal{C}^*$-algebras. A good reference on $\mathcal{C}^*$-algebras is the book [28].

Recall that a $\mathcal{C}^*$-algebra $A$ is a complex algebra endowed with an involution $\ast : A \to A$ and a complete norm satisfying for all $a, b \in A$

- $(ab)^\ast = b^\ast a^\ast$;
- $\|ab\| \leq \|a\|\|b\|$;
- $\|a^\ast a\| = \|a\|^2$.

Given a complex Hilbert space $\mathcal{H}$, the space of bounded linear operators $L(\mathcal{H})$ with involution given by taking the adjoint is the main example of a $\mathcal{C}^*$-algebra. The celebrated Gelfand-Naimark theorem states that every $\mathcal{C}^*$-algebra is $\ast$-isometric to such an algebra. This also motivates the following definition

**Definition 43.** A representation of a $\mathcal{C}^*$-algebra $A$ is a $\ast$-morphism $\pi : A \to L(\mathcal{H}_\pi)$ to the space of linear operators on a Hilbert space $\mathcal{H}_\pi$. We say that a representation $\pi$ of $A$ is contractive, if $\|\pi(a)\| \leq \|a\|$.

If a $\mathcal{C}^*$-algebra $A$ is not unital, we can always use it to construct a unital $\mathcal{C}^*$-algebra $A'$ as follows. We can adjoin to $A$ the identity element to obtain $A'$ and define $(\lambda, a)^\ast = (\overline{\lambda}, a^\ast)$. Then the norm on $A$ extends in unique way to $A'$. Thus for all invertibility results below if $A$ is not unital, one has to replace $A$ by $A'$.

We can associate two main $\mathcal{C}^*$-algebras to a Lie groupoid called the *full* and *reduced* Lie groupoid $\mathcal{C}^*$-algebras. Let $C_c(\mathcal{G})$ be the space of continuous complex-valued compactly supported functions on $\mathcal{G}$. If $\mathcal{G}$ is endowed with a
right Haar system, then this space has a convolution product defined as
\[(\varphi_1 * \varphi_2)(g) = \int_{G_{d(g)}} \varphi_1(gh^{-1})\varphi_2(h)d\lambda_{d(g)}(h)\]

**Example 44.** For a Lie group \(G \Rightarrow \text{id}\) (Example 13) we have \(G_{d(g)} = G\) and \(\lambda_{d(g)} = \lambda\) is just the right invariant Haar measure. The convolution product is the standard convolution product on \(G\).

**Example 45.** For the trivial groupoid structure (Example 14) the right Haar system is given by the singletons \(\lambda_q = \delta_q\), i.e.,
\[\lambda_q(\varphi) = \varphi(q)\]

Then the convolution product is nothing but the pointwise multiplication in \(C_c(M)\).

**Example 46.** Consider the pair groupoid \(M \times M \Rightarrow M\) with \(M\) smooth (Example 16). Since each \(G_q\) is diffeomorphic to \(M\) under the range map, we can take as a Haar system copies of the same smooth volume form \(\omega\) on \(M\). Then the convolution product transforms to
\[(\varphi_1 * \varphi_2)(q', q) = \int_M \varphi_1(q', q'')\varphi_2(q'', q)d\omega(q''),\]
which is a formula for the integral kernel of composition of two linear operators with kernels \(\varphi_1, \varphi_2\).

The space \(C_c(G)\) with convolution product and involution
\[\varphi^*(g) = \varphi(g^{-1})\]
becomes an associative \(*\)-algebra. One can also endow it with a natural norm
\[\|\varphi\|_I = \max \left\{ \sup_{q \in M} \int_{G_q} |\varphi|d\lambda_q, \sup_{q \in M} \int_{G_q} |\varphi^*|d\lambda_q \right\} .\]

Completion of \(C_c(G)\) with respect to \(\|\cdot\|_I\) is denoted by \(L^1(G)\)

**Definition 47.** The **full \(C^*\)-algebra** associated to \(G \Rightarrow M\), denoted as \(C^*(G)\) is the completion of \(C_c(G)\) in the norm
\[\|\varphi\| = \sup_{\pi} \|\pi(\varphi)\|,\] (9)
where \(\pi\) ranges over all contractive representations of \(C_c(G)\), i.e., for which
\[\|\pi(\varphi)\| \leq \|\varphi\|_I\]

This \(C^*\)-algebra is difficult to handle. For this reason the reduced \(C^*\)-algebra is often considered instead.
Definition 48. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, a right Haar system $\lambda$ and a point $q \in M$, the regular representation $\pi_q : C^\infty(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_q, \lambda_q))$ is defined by the formula

$$(\pi_q(\varphi)\psi)(g) = \varphi * \psi(g), \quad \forall \varphi \in C_c(\mathcal{G}).$$

Definition 49. The reduced $C^*$-algebra $C^*_r(\mathcal{G})$ is defined as the completion of $C_c(\mathcal{G})$ in the norm

$$\|\varphi\|_r = \sup_{q \in M} \|\pi_q(\varphi)\|.$$

Since regular representations are contractive, there is a natural $\ast$-homomorphism $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$. The groupoid $\mathcal{G}$ is called metrically amenable if this homomorphism is a bijection.

We can also construct other subalgebras of $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$ by taking restrictions to sub-groupoids. If $A$ is a $\mathcal{G}$-invariant locally closed subset of $M$, then we can define $C^*(\mathcal{G}_A)$ and $C^*_r(\mathcal{G}_A)$ together with $\ast$-homomorphisms $\rho_A : C^*(\mathcal{G}) \to C^*(\mathcal{G}_A)$ and $\rho_{A,r} : C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}_A)$.

Consider now an operator $P \in \Psi^\infty(\mathcal{G})$ and its distributional kernel $k_q$. The action of $P$ is then given by

$$(P\varphi)(g) = \int_{\mathcal{G}_d(g)} k_{d(g)}(g, g')\varphi(g')d\lambda_{d(g)}(g') \quad (10)$$

Since $P$ is right invariant we have

$$P \circ R_h = R_h \circ P,$$

where $R_h$ is the multiplication on $h$ from right. Using the invariant property of Haar systems and (10) this invariance conditions in terms of kernels reads as

$$k_{d(h)}(gh, g'h) = k_{r(h)}(g, g').$$

For this reason it makes sense to define the reduced kernel

$$\kappa_P(g) = k_{d(g)}(g, d(g))$$

which encodes all of the information about the corresponding operator. If we consider the convolution with the reduced kernel, we obtain using the invariance property

$$(\kappa_P * \varphi)(g) = \int_{\mathcal{G}_d(g)} k_{r(h)}(gh^{-1}, r(h))\varphi(h)d\lambda_{d(g)}(h) =$$

$$= \int_{\mathcal{G}_d(g)} k_{d(h)}(g, h)\varphi(h)d\lambda_{d(g)}(h) = (P\varphi)(g).$$

Similarly, if $P, Q \in \Psi^\infty(\mathcal{G})$ are operators with reduced kernels $\kappa_P, \kappa_Q$, in the same fashion one sees that

$$\kappa_{PQ}(g) = (\kappa_P * \kappa_Q)(g)$$
Thus \( \Psi^m(\mathcal{G}) \), where \( m = \pm \infty, 0 \) form an involutive \( \ast \)-algebra similar to compactly supported functions. For explaining CNQ conditions we will need to transform \( \Psi^0(\mathcal{G}) \) into a \( C^\ast \)-algebra. In order to do this we note that \( \Psi^{-\infty}(\mathcal{G}) \) by construction consists of smooth functions (like in the classical pseudodifferential calculus) with compact support (because of the compactness of the reduced support). Hence \( \Psi^{-\infty}(\mathcal{G}) \) is exactly the involutive \( \ast \)-algebra \( C^\infty_c(\mathcal{G}) \) and the full and reduced \( C^\ast \)-algebras are its completions.

The following theorem was proven in [42, Theorem 4.2].

**Theorem 50.** Let \((\pi, \mathcal{H})\) be a bounded representation of \( \Psi^{-\infty}(\mathcal{G}) \). Then it can be extended to a bounded representation of \( \Psi^0(\mathcal{G}) \).

**Definition 51.** We denote by \( \overline{\Psi}(\mathcal{G}) \) the completion of \( \Psi^0(\mathcal{G}) \) in the full norm (9), where \( \pi \) ranges over all extensions of bounded representations of \( \Psi^{-\infty}(\mathcal{G}) \).

The importance of \( C^\ast \)-algebra \( \overline{\Psi}(\mathcal{G}) \) comes from the fact that it contains resolvents of differential operators.

Many algebraic properties of \( C^\ast \)-algebras can be studied via their ideals. Two sided ideals are \( C^\ast \)-subalgebras. A two sided ideal \( I \) of a \( C^\ast \)-algebra is said to be primitive, if it is a kernel of an irreducible representation of \( A \). We denote the set of primitive ideals by \( \text{Prim}(A) \). Given a representation \( \pi \) of \( A \) its support is defined as

\[
\text{supp} \, \pi = \{ J \in \text{Prim}(A) : J \neq A, \ker \pi \subset J \}.
\]

**Definition 52.** Let \( \mathcal{F} \) be a set of representations of a \( C^\ast \)-algebra \( A \). \( \mathcal{F} \) is called exhaustive if

\[
\text{Prim}(A) = \bigcup_{\pi \in \mathcal{F}} \text{supp} \, \pi
\]

**Definition 53.** We say that a groupoid \( \mathcal{G} \) has Exel’s property if the set of regular representations is exhaustive in \( C^\ast_r(\mathcal{G}) \). A groupoid \( \mathcal{G} \) is said to have the strong Exel’s property if the set of regular representations is exhaustive in \( C^\ast(\mathcal{G}) \).

Exel’s property is one of the main ingredients in the Carvalho-Nistor-Qiao Fredholm conditions. All the groupoids that we consider in this article have Exel’s property as follows from Proposition 3.10 of [18].

**5.2. Semi-Fredholm Conditions**

In this subsection we consider Lie groupoids \( \mathcal{G} \rightrightarrows M \) for which there exists an open dense set \( U_0 \subset M \) such that \( \mathcal{G}_{U_0} \simeq U_0 \times U_0 \). All regular representations \( \pi_q \) for \( q \in U_0 \) are thus isomorphic and we denote them by \( \pi_0 \). If the Lie groupoid \( \mathcal{G} \) is Hausdorff, then the regular representation \( \pi_0 \) is non-degenerate [40], and we have an isomorphism \( C^\ast_r(\mathcal{G}_{U_0}) \simeq \pi_0(C^\ast_r(\mathcal{G}_{U_0})) \simeq \mathcal{K} \), where \( \mathcal{K} \) are compact operators on \( L^2(U_0) \). Indeed, elements of \( \pi_0(C^\ast_r(\mathcal{G}_{U_0})) \) can be identified with completion in the \( L^2 \)-norm of integral operators with compact continuous kernels which are themselves compact.
Theorem 54 ([18]). Let $\mathcal{G} \Rightarrow M$ be a Lie groupoid. Assume that
1. There exists and open dense $\mathcal{G}$-invariant set $U_0 \subset M$;
2. The canonical projection $C_r^\ast(\mathcal{G}) \to C_r^\ast(\mathcal{G}_{M\setminus U_0})$ induces an isomorphism
   $$C_r^\ast(\mathcal{G})/C_r^\ast(\mathcal{G}_{U_0}) \simeq C_r^\ast(\mathcal{G}_{M\setminus U_0});$$
3. $\mathcal{G}_{M\setminus U_0}$ has Exel’s property.

Then for any unital $C^\ast$-algebra $A$ containing $C_r^\ast(\mathcal{G})$ as an essential ideal and for any $a \in A$ we have that $\pi_0(a)$ is Fredholm if and only if the image of $a$ in $A/C_r^\ast(\mathcal{G})$ is invertible and all $\pi_q(a)$, $q \in M\setminus U_0$, are invertible.

For the proof of this Theorem see [18, Theorem 4.6]. We emphasize again that the proof uses the fact that $\pi_0$ is injective.

We will need a slightly weaker result since the operators we are interested in general are not Fredholm. We will see that they are however left semi-Fredholm. We have the following consequence of Theorem 54.

Corollary 55. The statement of Theorem 54 remains true if we replace simultaneously “Fredholm operators” with “left (or right) semi-Fredholm” and “invertible” with “left (or right) invertible”.

Indeed, this is a consequence of the following lemmas which are folklore and very likely to be written somewhere. We prove some of them for the sake of completeness.

Lemma 56. Let $T : H_1 \to H_2$ is a bounded linear operator, where $H_1, H_2$ are Hilbert spaces. Then the following statements are equivalent:
1. $T$ is left invertible;
2. $\ker T = \{0\}$, $\operatorname{Ran} T$ is closed;
3. There exists a constant $c > 0$ such that
   $$\|Tu\|_{H_2} \geq c\|u\|_{H_1}, \quad \forall u \in H_1.$$ (11)

The proof can be found in Section 4.5 of [8].

Lemma 57. An element $a$ of a $C^\ast$-algebra $A$ is left invertible if and only if $a^*a$ is invertible. Similarly $a \in A$ is right invertible if and only if $aa^*$ is invertible.

Proof. It is obvious, that if $a \in A$ is right invertible, then $a^*$ is left invertible. So it is sufficient to prove the condition for left invertibility. If $a^*a$ is invertible, then we can simply take $(a^*a)^{-1}a^*$ as the left inverse of $a$.

To prove the other direction we use the Gelfand-Naimark theorem and assume that $a \in L(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and that $a$ is left invertible. From the previous Lemma it follows that $\ker a = \{0\}$. Clearly $\ker a \subset \ker(a^*a)$.

We claim that this inclusion is actually an equality. Indeed, if $v \in \ker(a^*a)$, then for any $w \in \mathcal{H}$
$$0 = \langle w, a^*av \rangle_{\mathcal{H}} = \langle aw, av \rangle_{\mathcal{H}} \iff av \in \operatorname{Ran}(a)^\perp,$$
which is possible only if \( av = 0 \). Since \( a^*a \in L(H) \) is a self-adjoint operator, \( \text{Ran}(a^*a)^\perp = \ker a^*a = \{0\} \). Hence \( a^*a \) has zero kernel and cokernel and therefore it is invertible by the bounded inverse theorem.

**Lemma 58.** A continuous linear operator \( T : H_1 \to H_2 \) between two Hilbert spaces is left (right) semi-Fredholm if and only if \( T^*T \) (correspondingly \( TT^* \)) is Fredholm.

**Proof.** By Atkinson’s theorem an operator is left (right) semi-Fredholm if it is left (right) invertible modulo compact operators. Compact operators \( K(H_1,H_2) \) form a two-sided ideal in \( L(H_1,H_2) \) and hence the quotient of spaces \( L(H_1,H_2)/K(H_1,H_2) \) is a \( C^* \)-algebra known as the Calkin algebra. Since the projection operator from \( L(H_1,H_2) \) to the Calkin algebra is a \( \ast \)-homomorphism, the result is a direct consequence of Lemma 57. □

Corollary 55 now follows directly from Lemmas 57 and 58.

We are now ready to explain how the CNQ conditions are derived. The following Theorem is an analogue of Theorem 4.12 in [18] and a direct consequence of Theorem 54 and Corollary 55.

**Theorem 59.** (Modified Carvalho-Nistor-Qiao conditions). Let \((M,V)\) be a compact Lie manifold and \( \mathcal{G} \to M \) be the associated Hausdorff Lie groupoid which satisfies conditions of Theorem 54 with \( U_0 = M_0 \). Let \( \alpha \in \mathbb{R} \) and assume that \( P_0 \in \Psi^m(M_0) \) is such that \( P_0 = \pi_0(P) \) for some \( P \in \Psi^m(\mathcal{G}) \). Then

\[
  P_0 : H_0^\alpha(M) \to H_0^{\alpha-m}(M)
\]

is left semi-Fredholm if \( P_0 \) is elliptic and

\[
  P_q : H^\alpha(\mathcal{G} q) \to H^{\alpha-m}(\mathcal{G} q)
\]

are left invertible for every \( q \in M \setminus M_0 \).

**Proof.** The idea of proof as one can see is to consider a PDO \( P_0 \in \Psi^m(M_0) \) as an operator \( \pi_0(P) \), where \( P \in \Psi^m(\mathcal{G}) \). Then we can apply Theorem 54 to \( P \). There are however some technical issues. First of all we need to reformulate the question of determining whether \( \pi_0(P) \) is left semi-Fredholm as a question about elements of some \( C^* \)-algebra. For this reason we use a smooth metric on \( A(\mathcal{G}) \) and construct the corresponding right invariant Laplace operator \( \Delta \). One then can replace \( P \) with

\[
  a = (1 + \Delta)^{(s-m)/2} P (1 + \Delta)^{-s/2}.
\]

Operator \( a \) now belongs to \( \Psi^0(\mathcal{G}) \). Note that \( (1 + \Delta_q)^{1/2} \) by the definitions is an isometry between \( H^k(\mathcal{G} q) \) and \( H^{k-1}(\mathcal{G} q) \). Hence everything we say about \( \pi_q(a) \) can be transformed to statements about \( \pi_q(P) \) after a suitable change of functional spaces. We view \( a \) as an element of the completion \( \overline{\Psi}(\mathcal{G}) \) of \( \Psi^0(\mathcal{G}) \) from Definition 51. We can now apply Theorem 54 and Corollary 55 with \( A = \overline{\Psi}(\mathcal{G}) \) (see Definition 51). We obtain that \( \pi_0(a) \) is left semi-Fredholm if and only if \( \pi_q(a) : L^2(\mathcal{G} q) \to L^2(\mathcal{G} q) \) for \( q \in M \setminus M_0 \) are left invertible and the
image of $a$ is left invertible in $\overline{\Psi(G)}/C^*_r(G)$. Left invertibility of $\pi_q(a)$ is a part of the statement. So it only remain to prove left invertibility of $a$ in $\overline{\Psi(G)}/C^*_r(G)$. The fact that $P$ is elliptic implies that $a$ is elliptic. But then we can use the symbolic calculus for constructing a parametrix of an elliptic operator and invert it modulo $\Psi^{-\infty}(G) \subset C^*_r(G)$. Hence $a$ is invertible in $\overline{\Psi(G)}/C^*_r(G)$ and in particular left invertible.

\[\square\]

Remark 60. In [18] the authors introduce a special class of groupoids for which the conditions of Theorem 54 are satisfied which they call \textit{(amenable) stratified submersion groupoids}. Those are Lie groupoids $G \rightrightarrows M$ for which $M$ has a $G$-invariant stratification

\[\emptyset \subset U_0 \subset U_{N-1} \subset \cdots \subset U_0 \subset M\]

with $U_0$ open dense and for which moreover the restrictions of $G$ to each strata in $M \setminus U_0$ are certain fibered pull-back groupoids of amenable Lie group bundles. All the examples in this article are of this form since on the boundary we will always have Lie group bundles of solvable Lie groups.

6. Outline of the Method and a 1D Model Example

Let us sum up all of the results and explain the rough informal algorithm for finding closure of singular elliptic operators using CNQ-conditions. Assume that we are given a differential operator $P$ of order $k$ on a manifold $M$ with boundary or some other singularity that we denote by $Z$. Let $\mu$ be a volume form on $M$ smooth outside $Z$. Let $M_0 = M \setminus Z$. In order to find the closure of $P$ defined on $C^\infty(M_0)$ in $L^2(M, \mu)$ topology one should follow the following steps:

1. Represent operator $P$ as an operator $s_Z^{-1}\text{Diff}^k_\nu(M)$, where $(M, \nu)$ is a Lie manifold that extends $M_0$ and $s_Z$ is some function of the defining function of the singular set $Z$;
2. Find a compatible volume $\mu_\nu$ and represent $L^2(M_0, \mu)$ as $wL^2_\nu(M)$, where $w$ is a weight function;
3. Write down the continuous operator $\tilde{P} = s_Zw^{-1}Pw : H^k_\nu(M) \to L^2_\nu(M)$;
4. Check first that $w^{-1}H^k_\nu(M)$ is continuously embedded in $L^2(M_0, \mu)$ as required by Proposition 2;
5. Check that there is a Hausdorff Lie groupoid integrating $A_\nu$ satisfying conditions of Theorem 54;
6. Write down the limit operators $\tilde{P}_q$ for $q \in Z$ and prove that they are invertible. Then from the previous point, Theorem 59 and Proposition 2 it will follow that $D(\overline{P}) = w^{-1}H^k_\nu(M)$.

Remark 61. This is just the outline of the method although in this paper we follow it word-by-word. In practice one often has to perform other actions, like doubling domains, performing blow-ups and others.
Let us illustrate the method on a relevant 1D example that will be useful for us in the study of Grushin points. Consider

$$\Delta = \partial_x^2 - \left(\frac{3}{4} + \alpha\right) \frac{1}{x^2(1-x)^2}$$

defined on $(0, 1)$. Here $\alpha \in \mathbb{R}$ is an arbitrary fixed parameter. Operator $\Delta$ is a compact version of the inverse-square potential. Note that there is a symmetry $x \mapsto 1-x$, which allows to concentrate our discussion only on one of the ends.

Suppose that we are interested in the closure of this operator defined on $D(\Delta) = C^\infty_c(0, 1)$ in the standard $L^2$-topology. Function $x$ is a defining function for the left boundary and function $(1-x)$ is a defining function for the right boundary. Thus we define $s = x(1-x)$. We have

$$s^2 \Delta = x^2(1-x)^2 \partial_x^2 - \left(\frac{3}{4} + \alpha\right).$$

The Lie manifold structure $\mathcal{V}$ is given by a single vector field

$$X = x(1-x) \partial_x.$$

Note that locally at the boundary $X$ is diffeomorphic to $x \partial_x$. Hence we see that $\Delta \in s^{-2} \text{Diff}(M)$.

A compatible metric can be chosen to be

$$g = \frac{1}{x^2(1-x)^2} dx^2$$

and the corresponding volume form as

$$\omega = \frac{dx}{s}.$$

If we want the operator $\tilde{P}$ to have range in the standard space $L^2(0, 1)$, then $s^2 \Delta$ should have range in $s^2 L^2(0, 1)$ or equivalently in $s^{\frac{3}{2}} L^2_\gamma(0, 1)$. Now we need to get rid of the weight via conjugation by the $s^{\frac{3}{2}}$ factor. We find a slightly more general formula

$$\tilde{P} = s^{2-\gamma} \Delta s^\gamma$$

$$= x^2(1-x)^2 \partial_x^2 + 2\gamma(1-2x)x(1-x) \partial_x$$

$$+ \gamma(\gamma - 1 + 2(x-1)x(2\gamma - 1)) - \left(\frac{3}{4} + \alpha\right)$$

$$= X^2 + (2\gamma - 1)(1-2x)X + \gamma(\gamma - 1 + 2(x-1)x(2\gamma - 1)) - \left(\frac{3}{4} + \alpha\right),$$

where $\gamma \in \mathbb{R}$.

Operator $\tilde{P} : H^2_\gamma(0, 1) \to L^2_\gamma(0, 1)$ is a continuous operator and hence $\Delta$ is a continuous operator from $s^{\frac{3}{2}} H^2_\gamma(0, 1)$ to $s^{-\frac{3}{2}} L^2_\gamma(0, 1) \simeq L^2(0, 1)$ and the former is continuously embedded into the latter by Lemma 40. So by
Proposition 2 we only need to check that \( \tilde{P} : H^2_V(0,1) \to L^2_V(0,1) \) is left semi-Fredholm.

We will do this using CNQ conditions. The first step is to verify that there exists a Hausdorff Lie groupoid \( \mathcal{G} \rightrightarrows [0,1] \) integrating \( A_V \) and such that \( \mathcal{G}_{(0,1)} \simeq (0,1) \times (0,1) \). There are several ways to do this. We will use Example 23 and the gluing Proposition 26. The restriction \( A_V|_{[0,\varepsilon)} \) coincides with the restriction of the Lie algebroid from Example 23 to the same semi-interval. Hence we can take the reduction of the action groupoid from Example 23 to the interval \([0,\varepsilon)\) as integration of \( A_V|_{[0,\varepsilon)} \). The symmetry argument allows us to use the same action groupoid for integrating \( A_V|_{(1-\varepsilon,1]} \). By Proposition 26 we can glue those groupoid to the pair groupoid \((0,1)^2 \rightrightarrows (0,1)\) to obtain a Hausdorff Lie groupoid \( \mathcal{G} \) integrating \( A_V \). Moreover this Lie groupoid is a stratified submersion groupoid as stated in Remark 60. Restrictions \( \mathcal{G}_0, \mathcal{G}_1 \) are just \( \mathbb{R} \), which are amenable. Thus all the conditions of Theorem 59 are verified and we can use the CNQ conditions in order to prove that \( \tilde{P} \) is left semi-Fredholm.

Since \( \tilde{P} \) is elliptic in \((0,1)\), by Theorem 59 it will be left semi-Fredholm if and only if both limit operators \( \tilde{P}_0 \) and \( \tilde{P}_1 \) are left invertible. The limit operators in the case when the restriction \( \mathcal{G}|_{\partial M} \) is a bundle of Lie groups can be computed by replacing the generators of \( \mathcal{V} \) by the corresponding generators of the Lie algebra. For example, in our case for the left boundary point it means replacing \( X = x\partial_x \) with \( Z = \partial_y, y \in \mathbb{R} \) and evaluating the rest of functions at \( x = 0 \). We only concentrate on the left boundary point, since the right boundary points is handled exactly in the same way via symmetry. We find for \( \gamma = 3/2 \):

\[
\tilde{P}_0 = Z^2 + 2Z - \alpha = \partial^2_y + 2\partial_y - \alpha.
\]

We only need to check that \( \tilde{P}_0 : H^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is left invertible. In order to do this we use a slight generalisation of Lemma 56 that will be also useful in the general case.

**Lemma 62.** Let \( H_1, H_2 \) be Hilbert spaces and \( T : D(T) \subset H_1 \to H_2 \) a closed operator. \( T \) is injective with closed range if and only if there exists a constant \( c > 0 \) such that

\[
\|Tu\|_{H_2} \geq c\|u\|_{H_1}, \quad \forall u \in D(T).
\]

This is a well known result and a proof can be found in [19, Proposition 2.14].

We apply this lemma to \( \tilde{P}_0 \). Using the fact that the Fourier transform is an isometry between \( L^2 \) spaces we find in the frequency domain

\[
\|\tilde{P}_0u\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\xi^2 - 2i\xi + \alpha|^2|\hat{u}(\xi)|^2d\xi = \int_{\mathbb{R}} ((\xi^2 + \alpha^2 + 4\xi^2)|\hat{u}(\xi)|^2d\xi
\]

Since expression in the brackets is a sum of two non-negative functions, it will be bounded by a constant if and only if the whole polynomial has no zeros.
It is easy to see, that this is indeed the case if and only if $\alpha \neq 0$. Thus if $\alpha \neq 0$, by Lemma 62, we have that $\tilde{P}_0 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ has closed range and is injective and therefore by Lemma 56 $\tilde{P}_0$ is left invertible. This proves the following proposition.

**Proposition 63.** Let $\Delta$ be the operator

$$\Delta = \partial_x^2 - \left(\frac{3}{4} + \alpha\right) \frac{1}{x^2(1-x)^2}$$

defined on $C_c^\infty(0,1)$. If $\alpha \neq 0$, then $D(\Delta)$ as an operator from $L^2(0,1)$ to itself is given by $s^{3/2}H^2_V(0,1)$, where $s = x(1-x)$ and $V$ is the $C^\infty$-module generated by $X = s\partial_x$.

Can we say something about $D(\Delta)$, when $\alpha = 0$? As we have seen, the operator is not left semi-Fredholm in this case due to the problems in the range. What one can do is to consider a smaller Hilbert space $(A, \| \cdot \|_A)$ which is continuously embedded in $L^2(0,1)$ and contains smooth functions as a dense subset. Then the closure $D_A(\Delta)$ in this smaller space is contained in $D(\Delta)$ providing us with some useful information on the domain of the closure. Let us apply this idea to our example.

Suppose that $\alpha = 0$. Let $\varepsilon > 0$, then $s^\varepsilon L^2(0,1) \hookrightarrow L^2(0,1)$ is a continuous map. We repeat the whole procedure one more time for this operator, but with $\gamma = \frac{3}{2} + \varepsilon$. The limit operator is then given by

$$\tilde{P}_0 = \partial_y^2 + 2(1+\varepsilon)\partial_y + \varepsilon(2+\varepsilon),$$

which is left invertible. Thus we obtain that

$$\bigcup_{\varepsilon > 0} s^{3/2+\varepsilon} H^2_V(0,1) \subset D(\Delta)$$

In a completely similar fashion we can now assume $-2 < \varepsilon < 0$. Then $L^2(0,1) \subset s^\varepsilon L^2(0,1)$ and by the natural inclusions of Sobolev space with different weights we find that

$$D(\Delta) \subset \bigcap_{\varepsilon < 0} s^{3/2+\varepsilon} H^2_V(0,1).$$

Thus we have proven

**Proposition 64.** In notations of Proposition 63 for $\alpha = 0$ one has

$$\bigcup_{\varepsilon > 0} s^{3/2+\varepsilon} H^2_V(0,1) \subset D(\Delta) \subset \bigcap_{\varepsilon > 0} s^{3/2-\varepsilon} H^2_V(0,1).$$

Operators similar to (12) are often encountered in practice and were extensively studied in the past. Let us compare Propositions 63 and 64 with results which exist in the literature, more precisely with [27] which has a
literature overview and the most up-to-date results. In [27] the authors study the operator of the form

\[ L_\beta = -\partial_x^2 + \left( \beta - \frac{1}{4} \right) \frac{1}{x^2} \]  \hspace{1cm} (14)

even more generally with complex \( \beta \). They state at the end of Section 1.2 and prove later that

1. if \( \beta < 1 \), then \( L_\beta \) is Hermitian (symmetric) but not self-adjoint and its domain is given by \( H^2_0(\mathbb{R}_+) \);
2. if \( \beta = 1 \), then \( L_\beta \) is self-adjoint and \( H^2_0(\mathbb{R}_+) \) is dense in its domain;
3. if \( \beta > 1 \), then \( L_\beta \) is self-adjoint and its domain is given by \( H^2_0(\mathbb{R}_+) \)

where

\[ H^2_0(\mathbb{R}_+) = \{ u \in H^2(\mathbb{R}_+) : u(0) = \partial_x u(0) = 0 \}. \]

Note that if we take \( \beta = \alpha + 1 \) in (14), then we obtain an operator similar to (12) and we expect that close to zero the behaviour of functions in \( \overline{L}_{\alpha+1} \) and \( \Delta \) should be the same.

Indeed, if \( u \in H^2_0(\mathbb{R}_+) \), then \( u = o(x^{3/2}) \) as \( x \to 0^+ \). On the other side close to zero \( s \sim x \). Thus

\[ \omega \sim \frac{dx}{x}, \quad X \sim x\partial_x, \quad x \to 0^+ \]

and if \( u \in s^{3/2}H^2_0(0,1) \), then for \( 1 > \varepsilon > 0 \) we have that

\[ \int_0^\varepsilon \left| ux^{-\frac{3}{2}} \right|^2 \frac{dx}{x} + \int_0^\varepsilon \left| x\partial_x(ux^{-\frac{3}{2}}) \right|^2 \frac{dx}{x} + \int_0^\varepsilon \left| (x\partial_x)^2(ux^{-\frac{3}{2}}) \right|^2 \frac{dx}{x} < +\infty. \]

Each of those integrals is finite if and only if \( u(x) = o(x^{3/2}) \) as \( x \to 0^+ \) recovering the required asymptotics.

If \( \alpha = 0 \), then both \( \Delta \) and \( L_1 \) are self-adjoint real symmetric operators. One can check that functions that go to zero as \( O(x^{3/2}) \) when \( x \to 0^+ \) lie in the domain of the adjoint (those are \( L^2 \) functions mapped to \( L^2 \) functions) and hence lie in the domain of the closure as well by the results of [27]. This asymptotics is indeed consistent with Proposition 64 and is also in accordance with the results of [35], which can be seen as the generalisation of operators (12) and (14).
7. Closure of the Laplace-Beltrami Operator on Generic 2D AR Manifolds

7.1. Almost-Riemannian Manifolds as Lie Manifolds and Associated Lie Groupoids

Let us go back to the study of the Laplace operator on a generic 2D ARS structure. In this section we prove Theorem 1. We start by considering a connected component of \( M \setminus \mathcal{Z} \), which we denote tautologically again by \( M \) and whose boundary is \( \partial M \subset \mathcal{Z} \times \mathcal{Z} \) (see Fig. 2).

If we are given a local frame of orthonormal vector fields \( X_1, X_2 \), we can write the Laplace operator as

\[
\Delta = X_1^2 + X_2^2 + \text{div}_\omega X_1 + \text{div}_\omega X_2
\]  

(15)

or if have chosen local coordinates such that \( X_1, X_2 \) are of the form

\[
X_1 = \partial_x, \quad X_2 = f(x, y)\partial_y,
\]

the Laplace operator is given by (2).

Let \( s \) be a defining function of \( \partial M \). From the normal forms (4), (5) we see that for a generic structure \( \pm f \) have non-zero differential and hence satisfy the definition of a defining function. Function \( f \) is not global, but any defining functions locally can be written as

\[
s = f e^g
\]

for some smooth function \( g \). A simple way to define invariantly a global defining function for a generic 2D ARS is to take a smooth volume form \( \nu \) and take the Radon-Nykodim derivative with respect to the Riemannian volume \( \omega \). Nevertheless remember that the definition of Sobolev spaces and Fredholm properties do not depend on the particular choice of the defining function. For this reason we will assume in concrete calculations that above \( g = 1 \), since it does not influence the results, but greatly shortens formulas under consecutive differentiations.

The Lie manifold structure \( V \) is the \( C^\infty(M) \)-module over vector fields which close to the singular set looks like the span of

\[
Y_1 = sX_1, \quad Y_2 = sX_2.
\]

(16)

From formula (2) it then follows that \( \Delta \in s^{-2} \text{Diff}^2 V(M) \).

**Theorem 65.** The Lie algebroid \( A_V \to M \) can be integrated to a Lie groupoid \( G \), such that

1. \( G \) is Hausdorff;
2. \( G|_{M_0} \) is equivalent to the pair groupoid \( M_0 \times M_0 \to M_0 \);
3. If \( q \in Z \) is a Grushin point, then \( G_q \) is isomorphic to the isotopic to the identity component of the affine group of transformations of the real line;
4. If \( q \in Z \) is tangency point, then \( G_q \) is isomorphic to the abelian group \( \mathbb{R}^2 \).
Proof. Assume first that there are no tangency points. Since both $Y_1, Y_2$ vanish identically on $Z$, we have that the restriction $G_q$ for $q \in Z$ are Lie groups. If we at a Grushin point, then in a local neighbourhood by \eqref{eq:4} we have $f(x, y) = xe^{\Phi(x, y)}$. The $C^\infty(M)$-module locally generated by

$$
Y_1 = xe^\Phi \partial_x, \quad Y_2 = x^2 e^{2\Phi} \partial_y
$$

coincides with $C^\infty(M)$-module locally generated by

$$
\tilde{Y}_1 = x \partial_x, \quad \tilde{Y}_2 = x^2 \partial_y.
$$

But those are the same generators as in the Example 24. This allows us immediately to construct an integrating Lie groupoid using glueing Proposition 26. To see this consider the Lie algebroid $A_V$ coming from $V$. Cover the singular set by open sets $U_i, i = 1, \ldots, N$, such that on each $U_i$ vector fields $X_1, X_2$ are given by \eqref{eq:4}. Then we can view $A_V|_{U_i}$ as the restriction of the Lie algebroid from the Example 24. But we have already integrated this Lie algebroid. So we can use reductions of the integrating Lie groupoid to $U_i$, that we call $G_i$ as Lie groupoids which integrate $A_V|_{U_i}$. Then using Proposition 26 we find that

$$
G = M_0 \times M_0 \sqcup \left( \bigsqcup_{i=1}^N G_i \right) / \sim
$$

integrates $A_V$, is Hausdorff and the isotropy groups $G_q$ for $q \in Z$ coincide with the affine group of transformation of the real line.

Presence of tangency points introduces certain difficulties into the integration procedure. Theorem 28 guarantees that there is an integrating Lie groupoid, however, it may fail to be Hausdorff due to the restrictive condition of being $d$-simply connected. In order to construct a Hausdorff integrating Lie groupoid also in the presence of tangency points we repeat the first step of the previous argument. We take an open cover $U_i \subset M, i = 1, \ldots, N$ of $Z$. Assume that $q$ is a tangency point and that it is contained in a unique $U_j$, which can be always achieved since tangency points cannot cluster. Our goal is to construct a Hausdorff Lie groupoid which would integrate $A_V|_{U_j}$.

Similarly to Grushin points we will construct a Lie groupoid whose reduction integrates $A_V|_{U_j}$ and whose restriction to the interior $U_j \cap M_0$ is the pair groupoid. To do this we consider a slightly bigger neighbourhood $\tilde{U}_j \supset U_j$, which satisfies the following extra assumption. The boundary of the closure of $\tilde{U}_j$ in $M$ has a one-dimensional face $S_1 = \tilde{U}_j \cap \partial M$ and a one-dimensional face $S_2 \subset M_0$, which intersects transversally $\partial M$ (see Fig. 7).

Let $s_2$ be the defining function of $S_2$, such that $s_2 \equiv 1$ on $U_j$. We define a new Lie manifold structure $\tilde{V}$ as a $C^\infty(M)$-module generated by

$$
\tilde{Y}_1 = s_2^2 Y_1, \quad \tilde{Y}_2 = s_2^2 Y_2.
$$

Note that by construction $A_{\tilde{V}}|_{U_j} = A_V$. This is just a particular choice for the extension of $A_V$ in order to guarantee completeness of vector fields $\tilde{Y}_1, \tilde{Y}_2$.
and other choices are, of course, are possible. We wish to apply Theorem 28 to show that there exists a Lie groupoid integrating $A\bar{V}$. Indeed, on $S_2$ we have

$$[\bar{Y}_1, \bar{Y}_2]|_{S_2} = 0.$$ 

and hence a trivial bundle of abelian groups would integrate $A\bar{V}|_{S_2}$. For the face $S_1$ we have

$$[\bar{Y}_1, \bar{Y}_2]|_{S_1} = (X_1(s_2^2 s)\bar{Y}_2 - X_2(s_2^2 s)\bar{Y}_1)|_{S_1}.$$ 

We can without loss of generality as discussed previously take $s = f$ and then from (1) we find that $X_2(s_2^2 s)|_{S_1} = 0$. Let us shorten $\alpha(q) = X_1(s_2^2 s)(q)$. Then we can integrate $A\bar{V}|_{S_1}$ to a trivial bundle $S_1 \times \mathbb{R}^2$, such that

$$d(q, t, \tau) = r(q, t, \tau) = q, \quad u(q) = (q, 0, 0)$$

and the multiplication is given by

$$(q, t_2, \tau_2)(q, t_1, \tau_1) = (q, t_1 + t_2, e^{\alpha(q)t_2 \tau_1 + \tau_2}).$$

Hence there exists a $d$-simply connected Lie groupoid $G$ that integrates $A\bar{V}$ and such that its restriction to the interior of $\bar{U}_j$ is the pair groupoid. It remains to show that $G$ is Hausdorff.
If two points belong to the restriction of $G$ to the interior of $\tilde{U}_j$, then clearly there exists two neighbourhoods separating them. The same is true if only one of the points belongs to the restriction to the interior. The only problem that may arise is that two points in $G_{\partial\tilde{U}_j}$ may not be separable inside $G$. To see that this is not the case we use Theorem 29, which states that the maps (7) form charts. Let $g, g'$ be points in $G_q, G_{q'}$ and $q, q' \in \partial\tilde{U}_j$. Then we can consider two charts of the form

$$\begin{align*}
(y', t'_1, t'_2) &\mapsto e^{t'_1 Y_1} \circ e^{t'_2 Y_2}(y'), \\
(y, t_1, t_2) &\mapsto e^{t_1 Y_1} \circ e^{t_2 Y_2}(y),
\end{align*}$$

(17)

where $y, y'$ belong to small disjoint neighbourhoods $U, U' \subset \tilde{U}_j$ of $q, q'$. We have also slightly abused notations and identified $\tilde{Y}_i$ with right invariant vector fields using the range map. We do not need vector fields $X_i$ like in (7), because each of those charts already contains $G_q$ and $G_{q'}$ entirely. Indeed, if a right invariant vector field $Y$ is mapped to a complete vector field under the range map, then $Y$ is complete in $G$ [41, Appendix, Section 33]. Hence by our construction maps (17) are defined for all $t_i, t'_i, i = 1, 2$ and for $y = q$ and $y' = q'$ they represent coordinates of the second kind for the isotropy groups $G_q, G_{q'}$, which are global, since $G_q$ and $G_{q'}$ are solvable.

If $q = q'$, then $g, g' \in G_q$ are contained in a single coordinate chart and hence can be separated by taking smaller neighbourhoods in this chart. If $q \neq q'$, we can take two disjoint neighbourhoods $U \ni q, U' \ni q'$ and consider charts (17) with $y \in U$ and $y' \in U'$. We claim that we can make $U, U'$ so small that those charts do not overlap. Indeed, let us consider the images of $U \times B_{\varepsilon}, U' \times B_{\varepsilon}$ under (17) projected to $\tilde{U}_j$ via the range map which we denote by $V, V'$. They will be given by the orbits of $U, U'$ under the flows of the corresponding vector fields from $\tilde{V}$. We assume that $\varepsilon > 0$ is big enough to ensure that $g, g'$ lie in these two charts. If $U, U'$ would have been only subsets of $\partial\tilde{U}_j$, they would have stayed invariant no matter how big $\varepsilon$ is chosen. Thus by smooth dependence of solutions of ODEs on the initial value we can find $U, U'$ so small that $V \cap V' = \emptyset$. Hence the charts do not overlap as well and $g, g'$ are separated.

Thus we have proven that $A_\tilde{V}$ can be integrated to a Hausdorff Lie groupoid. Now it is enough to take its restriction to $U_j$ and glue to all of the other Lie groupoids obtained in a similar fashion via Proposition 26. □

Theorem 65 allows us to apply the machinery of Sects. 4 and 5 in order to determine the closure of the Laplace operator $\Delta$. However, $\Delta$ in many aspects is similar to the critical case $\alpha = 0$ in the 1D example (12). CNQ conditions are not directly applicable to $\Delta$, but they are applicable to certain perturbations of $\Delta$. For this reason in Theorem 1 we considered instead

$$\tilde{\Delta} = \Delta - \frac{h}{s^2},$$

(18)

where $h \in C^\infty(M)$ such that $h|_Z$ is a strictly positive function. If there are no tangency points, a nice geometric perturbation of this kind exists, namely
one can consider the operator $\Delta + cK$, where $c \in \mathbb{R}$ is a constant and $K$ is the Gaussian curvature. This operator can be considered as a possible covariant quantization of the classical energy Hamiltonian on a Grushin manifold (see [11] for further explanation).

We can now follow the algorithm outlined in Sect. 6. Exactly as $\Delta$, the operator $\tilde{\Delta}$ belongs to $s^{-2} \text{Diff}^2_V(M)$. If we want its image to lie in $L^2(M, \omega)$, then $\text{Ran}(s^2\tilde{\Delta})$ must be contained in $s^2L^2(M, \omega) = sL^2(M, \omega/s^2)$. We let $\mu_V = \omega/s^2$ and note that it is a volume coming from a compatible metric. In order to mitigate the weight of the $L^2$ space we conjugate by $s$ obtaining an operator

$$\tilde{P} = s\tilde{\Delta}s.$$

Note that $\tilde{P} \in \text{Diff}^2_V(M)$ and hence it defines a bounded operator $\tilde{P} : H^2_V(M) \to L^2_V(M)$. If $\tilde{P}$ is left semi-Fredholm, then it would imply that

$$\tilde{\Delta} : sH^2_V(M) \to s^{-1}L^2_V(M) \simeq L^2(M, \omega)$$

is left semi-Fredholm as well. From Lemma 40 it follows that there exists a continuous inclusion between $sH^2_V(M)$ and $L^2(M, \omega)$ and hence as consequence of Proposition 2 we would find that

$$D(\tilde{\Delta}) = sH^2_V(M)$$

thus proving Theorem 1.

So it only remains to prove that $\tilde{P}$ is a left semi-Fredholm. The proof will be a consequence of the modified Carvalho-Nistor-Qiao conditions stated in Theorem 59. Exactly as in the 1D model example we only need to check the left invertibility of limit operators $\tilde{P}_q$, which we study in the next subsection.

7.2. Limit Operators and Their Invertibility

In a local trivialisation $X_1, X_2$, from (15) and (16) we find that

$$s\Delta s = \sum_{i=1}^2 Y_i^2 + (X_i(s) + s \text{ div } X_i)Y_i + (sX_i^2(s) + sX_i(s) \text{ div } X_i).$$

Note that now all of the coefficients are smooth on the singular set $Z$ as can be easily seen from the local coordinate expressions and moreover the last term is exactly $s\Delta(s)$, which means that the free term is a globally defined smooth function.

Let us now write down the boundary operators. Let $q \in \partial M$, $G_q$ be the restriction of $\mathcal{G}$ to $q$, which coincides with the isotropy group $G_q$ and let $\mathfrak{g}_q$ be its Lie algebra. Suppose that $Z_1, Z_2$ is a basis of left invariant vector fields on $G_q$ at a point $q \in \partial M$, which correspond to the vector fields $Y_1, Y_2$. Then a monomial $aY_{i_1}Y_{i_2} \ldots Y_{i_n}$ will correspond to $a(q)Z_{i_1}Z_{i_2} \ldots Z_{i_n}$, which is an element of the universal enveloping algebra $U(\mathfrak{g}_q)$. Since as discussed earlier the Sobolev spaces $H^k_V(M)$ do not depend on the choice of the compatible metric or the defining function the semi-Fredholm property of the operator $\tilde{P}$.
does not depend as well on those things. Thus we can assume that around \( q \) the defining function \( s \) coincides with \( f \). Recall that in coordinates around \( q \in \partial M \) centered at zero we have \( \text{div} X_1 = -\partial_x f/f \), \( \text{div} X_2 = f \partial_y f \) and \( f(0,0) = 0 \). So for the operator \( \tilde{P} \) we obtain

\[
\tilde{P}_q = Z_1^2 + Z_2^2 - \partial_x f(0,0)^2 - h(0,0).
\]

Assume first that \( q \) is a tangency point. From the normal form (5) it follows that \( f(0,0) = 0 \). Since \( G_q = \mathbb{R}^2 \) and \( [Y_1, Y_2](0,0) = 0 \), we have

\[
\tilde{P}_q = \Delta_{\mathbb{R}^2} - h(0,0).
\]

Thus a Fourier transform arguments proves the following proposition.

**Proposition 66.** If \( q \) is a tangency point, then \( G_q = \mathbb{R}^2 \) and \( \tilde{P}_q : H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) is left invertible if and only if \( h(0,0) > 0 \).

The rest of this section is dedicated to the proof of an analogous result when \( q \) is a Grushin point. More precisely

**Proposition 67.** If \( q \) is a Grushin point, then \( G_q \) is the isotopic to the identity component of the affine group of transformations of the real line and \( \tilde{P}_q : H^2(G_q) \to L^2(G_q) \) is left invertible if and only if \( h(0,0) \neq 0 \).

For the rest of this subsection we will write \( G \) instead of \( G_q \) and \( g \) instead of \( g_q \). Spaces \( H^k(G) \) are the corresponding Sobolev spaces with respect to right invariant Haar measure \( \mu_G \). Recall that we can define these Sobolev spaces as the completion of compactly supported smooth functions in the Sobolev norm

\[
\|u\|^2_{H^k(G)} = \|(1 + \Delta_G)^{\frac{k}{2}} u\|^2_{L^2(G)}
\]

where \( \Delta \) is the invariant Laplace operator, or

\[
\|u\|^2_{H^k(G)} = \sum \|Z_{i_1} Z_{i_2} \ldots Z_{i_m} u\|^2_{L^2(G)},
\]

where \( m \in \{1, \ldots, k\} \) and \( Z_{i_j}, i_j \in \{1, \ldots, \text{dim} g\} \) is a basis of right invariant vector fields on \( G \). Since Lie groups with left invariant metrics are geodesically complete and have constant curvature, they are manifolds of bounded geometry and the two norms are equivalent.

From the normal form (4) it follows that in a chart centred at a Grushin point

\[
[Y_1, Y_2](0,0) = 2Y_2(0,0).
\]

Thus

\[
\tilde{P}_q = Z_1^2 + Z_2^2 - 1 - h_0,
\]

where \( Z_1, Z_2 \) are two right invariant vector fields on \( G \) which satisfy

\[
[Z_1, Z_2] = 2Z_2.
\]
It turns out that the right invariant definitions in the particular instance of $G$ result in more cumbersome calculations compared to the left invariant ones. We can pass from right invariant objects to left invariant ones by using the usual involution $i: g \mapsto g^{-1}$. In particular, right invariant vector fields are mapped to minus left invariant vector fields and a right invariant volume to a left invariant one. In the matrix representation (6) we have a basis of left invariant vector fields $a\partial_a$, $a^2\partial_b$ which satisfy

$$[a\partial_a, a^2\partial_b] = 2a^2\partial_b.$$ 

Thus we can take $Z_1 = -a\partial_a$, $Z_2 = -a^2\partial_b$ (here minus sign is a direct consequence of our a priori right invariant construction). The left invariant volume form is given by

$$\mu_G = \frac{dadb}{a^3}.$$ 

This way we arrive at the following coordinate representation of $\tilde{P}_q$:

$$\tilde{P}_q = (a\partial_a)^2 + (a^2\partial_b)^2 - 1 - h_0. \quad (19)$$

We need to prove that $\tilde{P}_q$ is left invertible. To do this we first transform $\tilde{P}_q$ to a simpler form via some changes of variables. First we apply a partial Fourier transform with respect to the $b$ variable.

$$\mathcal{F}: u(a, b) \mapsto \hat{u}(a, \xi) = \int_{\mathbb{R}} u(a, b) e^{-i\xi b} db.$$ 

This partial Fourier transform is a $L^2$-isometry

$$\mathcal{F}: L^2 \left(G, \frac{dadb}{a^3}\right) \to L^2 \left(G, \frac{dadx}{a^3}\right)$$

and gives us

$$\hat{\tilde{P}}_q = (a\partial_a)^2 - a^4\xi^2 - 1 - h_0.$$ 

We can make a change of variables $x = a\sqrt{|\xi|}$ and obtain

$$\hat{\tilde{P}}_q = (x\partial_x)^2 - x^4 - 1 - h_0. \quad (20)$$

Note that this is a singular change variables, however, the volume form is well defined because the singularity has measure zero. In this new coordinates the dual volume is equal to

$$\hat{\mu}_G = \frac{|\xi|dxd\xi}{x^3}.$$ 

Let us consider an operator of the same form as $\tilde{P}_q$ that we will denote as

$$T = (x\partial_x)^2 - x^4 - 1 - h_0$$

acting on $C^\infty_0(\mathbb{R}_+)$. We want to extend its domain to a subspace $H \subset L^2(\mathbb{R}_+, dx/x^3)$ such that the operator $T: H \to L^2(\mathbb{R}_+, dx/x^3)$ would be left invertible.
and $H$ would contain smooth compactly supported functions as a dense subset. After that we will use $H$ to prove left-invertibility of the operator $\tilde{P}_q$.

To prove left invertibility $T$ we will use once again the CNQ conditions, by proving that $T$ is injective and left semi-Fredholm. We want to first represent $T$ as an operator coming from a differential operator compatible with a structure of a Lie manifold. We need to compactify $\mathbb{R}^+$. This can be done easily by local consideration around zero and infinity. The Lie manifold structure, which we denote by $\mathcal{V}_X$, will be a $C^\infty$-module generated by a single vector field $X$ non-zero in $(0, +\infty)$ with certain asymptotics when $x \to 0$ and $x \to +\infty$.

From explicit form of $T$ we can see that it is compatible with the Lie manifold structure that is locally generated by $x\partial_x$ around zero. In order to consider what happens at infinity, we make a change of variables $x \to y = 1/x$ and find that

$$T = (y\partial_y)^2 - \frac{1}{y^4} - 1 - h_0.$$ 

After multiplying by $y^4$ we find an operator which is compatible with the Lie manifold structure that is locally generated by $y^3\partial_y$. Thus let

$$h : x \mapsto y = \frac{1}{x}$$ 

and $X$ to be any vector field, such that

$$X = x\partial_x, \quad 0 < x < \varepsilon;$$ 

$$h_*X = y^3\partial_y, \quad 0 < y < \varepsilon.$$ 

Then we choose $\mathcal{V}_X$ to be the Lie manifold structure on $\mathbb{R}^+$ defined as a $C^\infty_b(\mathbb{R}^+)$-module generated by $X$, where $C^\infty_b(\mathbb{R}^+)$ are smooth bounded functions. This way we perform a two-point compactification of $\mathbb{R}^+$ by adding zero and infinity as boundaries. We denote this compactification by $\overline{\mathbb{R}^+}$.

**Proposition 68.** Let $r : [0, +\infty)$ be a bounded smooth function such that $r(x) = x$ for $x \leq \varepsilon$ and $r(x) = 1$ for $x \geq 2\varepsilon$ and let $\tilde{r}(x) = r(1/x)$. Then $T : (r\tilde{r}^2)H^2_{\mathcal{V}_x}(\mathbb{R}^+) \to L^2(\mathbb{R}^+, dx/x^3)$ is left invertible.

**Proof.** It is clear that $r$ and $\tilde{r}$ are defining functions for the zero boundary and boundary at infinity. We may choose a compatible volume form as

$$\mu_{\mathcal{V}_X} = \frac{r^2(x)dx}{x^3\tilde{r}^4(x)}$$

and consider the corresponding Sobolev spaces $H^k_{\mathcal{V}_X}(\overline{\mathbb{R}^+})$.

First we prove that $T$ is injective. For this we need to solve $Tu = 0$ and show that no solution lies in $(r\tilde{r}^2)H^2_{\mathcal{V}_x}(\mathbb{R}^+)$. Even more we will see that none of the solutions lies in $(r\tilde{r}^2)L^2_{\mathcal{V}_x}(\mathbb{R}^+)$. Indeed, there are two independent solutions to $Tu = 0$ described via Bessel functions:

$$u_-(x) = K_\sqrt{\frac{1}{1+h_0}} \left( \frac{x^2}{2} \right), \quad u_+(x) = I_\sqrt{\frac{1}{1+h_0}} \left( \frac{x^2}{2} \right),$$

where $K_{\alpha}$ and $I_{\alpha}$ are modified Bessel functions of the first kind and the second kind, respectively.
and any other solution is a linear combination:

$$u(x) = c_- u_-(x) + c_+ u_+(x).$$

However, neither $u_-$ nor $u_+$ are in $L^2(dx/x^3)$. This follows from the asymptotics of Bessel functions of the second kind. For $\nu > 0$ the following asymptotic relations are valid $\nu > 0$:

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu,$$

$$K_\nu(x) \sim \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu$$

for $x \to 0+$ and

$$I_\nu(x) \sim \frac{e^x x^{-1/2}}{\sqrt{2\pi}},$$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2}} e^{-x} x^{-1/2}$$

for $x \to \infty$.

From here we see that $u_-$ and $u_+$ have different asymptotics close to zero and different asymptotics close to infinity. Close to zero functions from $r\tilde{r}^2 L^2_{V_X}(\mathbb{R}_+)$ behave exactly like functions in $L^2(\mathbb{R}_+,dx/x^3)$. Function $u_+$ is never $dx/x^3$ square integrable for $x < \varepsilon$ small. On the other hand for $x > \varepsilon$ big $u_-$ is never locally square integrable because of the exponential growth. Hence a solution of $Tu = 0$ cannot be in $r\tilde{r}^2 L^2_{V_X}(\mathbb{R}_+)$. 

Now we need to prove that $T$ is left semi-Fredholm. This is equivalent to proving that the operator

$$\tilde{T} = r^{-1} \tilde{r}^2 T(\tilde{r}^2) : H^2_{V_X}(\mathbb{R}_+) \to L^2_{V_X}(\mathbb{R}_+)$$

is left semi-Fredholm. First note that $A_{V_X}$ is integrable to a Hausdorff simply connected Lie groupoid whose restriction to $(0,\infty)$ is the pair groupoid. Indeed, $X$ is a complete vector field on $\mathbb{R}_+$ and its flow defines the action of $\mathbb{R}$ on $\mathbb{R}_+$. Hence we can take the integrating Lie groupoid to be the corresponding action groupoid and CNQ conditions can be applied.

We have to study invertibility of limit operators $\tilde{T}_0$ and $\tilde{T}_\infty$ which will be constant coefficient differential operators on $\mathbb{R}$. Let $z \in \mathbb{R}$ be the variable on $\mathbb{R}$. Recall that for $x = 0$ in order to compute limit operators we have to replace $x \partial_x$ with $\partial_z$. After a lengthy computation we find that

$$\tilde{T}_0 = \partial_z^2 + 2\partial_z - h_0.$$

But this is exactly the operator (13) as in the 1D example from Sect. 6. So we already know that this operator is left invertible.

Let us now compute the limit operator $\tilde{T}_\infty$. In this case we have to replace $y^3 \partial_y$ with $\partial_z$. After a change of variables and another lengthy computation we
obtain
$$\tilde{T}_\infty = \partial_z^2 - 1.$$ This operator is also invertible on $\mathbb{R}$.

The left semi-Fredholm property now follows from Theorem 59 and this finishes the proof that the operator $T$ is left invertible. □

Next step is to extend the action of $T$ to the Fourier dual of $G$ such that it would remain to be left invertible. For this we will need some corollaries of left invertibility conditions of Lemmas 56 and 62.

**Corollary 69.** Let $A_i : H_i \to H_3$, $i = 1, 2$ be closed operators between the corresponding Hilbert spaces. Assume that there exist continuous inclusions $H_1 \hookrightarrow H_2 \hookrightarrow H_3$ such that $H_i$ is dense in $H_{i+1}$ and that $A_2$ extends $A_1$. Then $A_1$ is left invertible if and only if $A_2$ is left invertible.

**Proof.** We can treat $H_i$ as subspaces of $H_3$. In order to prove the result by Lemma 62 it is enough to show that there exists a constant $c > 0$, such that
$$\|A_1 u\|_{H_3} \geq c \|u\|_{H_3}, \quad \forall u \in H_1 \iff \|A_2 v\|_{H_3} \geq c \|v\|_{H_3}, \quad \forall v \in H_2.$$ Since $H_1 \subset H_2$ the arrow in the left direction is true by restriction. The right arrow is true because the inclusions $H_1 \subset H_2 \subset H_3$ are continuous. Thus $A_2$ is the closure of $A_1$ in the $H_2$-norm and for any sequence $u_n \subset H_1$ converging to $u \in H_2$ in $H_2$, we have that $u_n \to u$ and $A u_n \to A u$ in $H_3$. □

**Corollary 70.** Let $A : D(A) \subset H_1 \to H_1$ be a left invertible operator in a Hilbert space $H_1$. Let $H_2$ be another Hilbert space. Then the operator $A \otimes \text{id} : D(A) \otimes H_2 \subset H_1 \otimes H_2 \to H_1 \otimes H_2$. Is left invertible.

**Proof.** From Lemma 62 and definition of the tensor product it readily follows that
$$\|(A \otimes \text{id})(u_1 \otimes u_2)\|_{H_1 \otimes H_2} \geq c \|u_1 \otimes u_2\|_{H_1 \otimes H_2}, \quad \forall u_1 \in D(A), u_2 \in H_2$$ for some $c > 0$. The rest follows by taking the closure and Lemma 56. □

We apply now Corollary 70 to the operator $A = T \otimes \text{id}$ from $L^2(\mathbb{R}_+, dx/x^3) \otimes L^2(\mathbb{R}, |\xi|d\xi) \approx L^2(G, \hat{\mu}_G)$ to itself with domain $r \sqrt{2} H^2_{x\partial_x}(\mathbb{R}_+) \otimes L^2(\mathbb{R}, |\xi|d\xi)$ and find that this operator is left invertible. We can also consider $A$ as an operator
$$A : H^2_{x\partial_x}(\mathbb{R}_+, \frac{dx}{x^3}) \otimes L^2(\mathbb{R}_+, |\xi|d\xi) \to L^2(\mathbb{R}_+, \frac{dx}{x^3}) \otimes L^2(\mathbb{R}_+, |\xi|d\xi),$$
where, as the notation suggests, the Sobolev space on the left is the closure of $C_c^\infty(\mathbb{R}_+)$ in the norm
$$\|u\|^2_{H^2_{x\partial_x}} = \int_{\mathbb{R}_+} \left( |u|^2 + |x\partial_x u|^2 + |(x\partial_x)^2 u|^2 \right) \frac{dx}{x^3}.$$
If we can prove that $A$ is left invertible on this new domain, then we can apply Corollary 69 to prove that $A$ is left invertible on $H^2(G)$. To see this let us write down the norm for the Hilbert space $H^2(G)$ in the $(x, \xi)$ coordinates. Using the partial Fourier transform $\mathcal{F}$ and change of variables $(a, \xi) \mapsto (x, \xi)$ we can write it as

$$
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^+} \left( |u|^2 + |x \partial_x u|^2 + |(x \partial_x)^2 u|^2 + |x^2 u|^2 + |x^3 \partial_x u|^2 + |x \partial_x (x^2 u)|^2 + |x^4 u|^2 \right) \frac{dx}{x^3} \right) |\xi| d\xi
$$

We can now see from the explicit forms of the norms that $H^2_{x\partial_x} (\mathbb{R}^+, dx/3) \otimes L^2 (\mathbb{R}, |\xi| d\xi)$ continuously embeds in $H^2(G, \hat{\mu})$. Thus left invertibility of $A$ on $H^2(G)$ will follow from Corollary 69.

This means that in order to finish the proof of Proposition 67 and the main Theorem 1, we only have to prove that the operator $A$ defined in (21) is left invertible.

**Lemma 71.** Operator $A$ defined in (21) is left invertible.

**Proof.** We have seen that $A$ as an operator

$$
A : \tilde{r}^2 H^2_{\hat{V}_x} (\mathbb{R}^+) \otimes L^2 (\mathbb{R}, |\xi| d\xi) \to L^2(G, \hat{\mu})
$$

is left invertible. Note also that \( \tilde{r}^2 H^2_{\hat{V}_x} (\mathbb{R}^+) \otimes L^2 (\mathbb{R}, |\xi| d\xi) \) embeds continuously into $L^2(G, \hat{\mu})$ by Lemma 40 and definition of the tensor product. So it only remains to prove that $\tilde{r}^2 H^2_{\hat{V}_x} (\mathbb{R}^+)$ embeds continuously into $H^2_{x\partial_x} (\mathbb{R}^+, dx/3)$, i.e., that there exists a constant $c > 0$ such that

$$
\| u \|_{H^2_{x\partial_x}}^2 \leq c \| u \|_{\tilde{r}^2 H^2_{\hat{V}_x}}^2, \quad \forall u \in \tilde{r}^2 H^2_{\hat{V}_x} (\mathbb{R}^+).
$$

We can write the $\tilde{r}^2 H^2_{\hat{V}_x}$ norm as

$$
\| u \|_{\tilde{r}^2 H^2_{\hat{V}_x}}^2 = \int_{\mathbb{R}^+} \left( \left( \frac{u}{\tilde{r}^2} \right)^2 + \left| x \partial_x \left( \frac{u}{\tilde{r}^2} \right) \right|^2 + \left| (x \partial_x)^2 \left( \frac{u}{\tilde{r}^2} \right) \right|^2 \right) \frac{r^2 dx}{x^3}.
$$

We will prove (22) for smooth function with compact support and the rest will follow by completion. The proof is straightforward and is given mainly for completeness.

Let $u = \tilde{r}^2 v$. Then we have

$$
\| \tilde{r}^2 v \|_{H^2_{x\partial_x}}^2 = \int_{\mathbb{R}^+} |\tilde{r}^2 v|^2 \frac{dx}{x^3} + \int_{\mathbb{R}^+} |x \partial_x (\tilde{r}^2 v)|^2 \frac{dx}{x^3} + \int_{\mathbb{R}^+} |(x \partial_x)^2 (\tilde{r}^2 v)|^2 \frac{dx}{x^3} = I_0 + I_1 + I_2.
$$

Recalling that $\tilde{r}(x) \in [0, 1]$ we obtain

$$
I_0 = \int_{\mathbb{R}^+} \tilde{r}^2 |v|^2 \mu x \leq \int_{\mathbb{R}^+} |v|^2 \mu x.
$$
Terms $I_1$ and $I_2$ are proved similarly. We only show the argument for $I_1$ since for $I_2$ it is almost identical but with more steps. We have by the chain rule and Young inequality

\[ I_1 = \|x \partial_x (r \tilde{r}^2) v\|_{L^2(\mathbb{R}_+, dx/x^3)}^2 + \langle x \partial_x (r \tilde{r}^2) v, r \tilde{r}^2 x \partial_x u \rangle_{L^2(\mathbb{R}_+, dx/x^3)} + \|r \tilde{r}^2 x \partial_x u\|_{L^2(\mathbb{R}_+, dx/x^3)}^2 \leq 2 \|x \partial_x (r \tilde{r}^2) v\|_{L^2(\mathbb{R}_+, dx/x^3)}^2 + 2 \|r \tilde{r}^2 x \partial_x u\|_{L^2(\mathbb{R}_+, dx/x^3)}^2, \]

Now for the first term we find

\[ \|x \partial_x (r \tilde{r}^2) v\|_{L^2(\mathbb{R}_+, dx/x^3)}^2 = \int_{R^+} (x \partial_x (r \tilde{r}^2))^2 |v|^2 \tilde{r}^4 \frac{r^2}{r^2} \mu_X. \]

Function

\[ f(x) = \frac{x \partial_x (r(x) \tilde{r}^2(x))^2 \tilde{r}^4(x)}{r^2(x)} \]

is smooth and bounded on $\mathbb{R}_+$. Indeed, $r, \tilde{r}$ are smooth and bounded. So the only problem can occur close to $x = 0$. Recall that close to zero $r = x$ and $\tilde{r} = 1$ by construction. So $f(x) \equiv 1$ close to $x = 0$. Hence

\[ \|x \partial_x (r \tilde{r}^2) v\|_{L^2(\mathbb{R}_+, dx/x^3)}^2 \leq C \int_{R^+} |v|^2 \mu_X. \]

The other term is handled exactly as $I_0$ and $I_2$ is bounded by a similar argument. The lemma now follows by completion. □

This proves Proposition 67 and finishes the proof of Theorem 1.

8. Conclusions, Final Remarks, Extensions

Theorem 1 illustrates that it is possible to treat various singularities in almost-Riemannian geometry from a unified perspective. This raises the question of what is the domain of applicability and where are the practical limits of this method for obtaining information about geometric operators on sub-Riemannian manifolds and more generally about geometric differential operators on singular spaces.

First of all we note that it is possible to prove an analogue of Proposition 64 in the case when there are no tangency points for the Laplace operator $\Delta$ itself. For this it is enough to consider instead of $\tilde{P}^\gamma$ operators $\tilde{\Delta}^\gamma$ defined as

\[ \tilde{P}^\gamma = s^{2-\gamma} \Delta s^\gamma, \]

for $\gamma = 1 \pm \varepsilon$ and repeat the proof. But it would have resulted in much longer formulas and a more convoluted analysis. Theorem 1 is already enough to explain the concept. In the case of the tangency points weights $s^\gamma$ are not enough. They behave similarly to cusp points and irregular singular points which require non-polynomial weights.
Secondly we should note that the method applies to non-generic structures as well, for examples manifold analogues of \( \alpha \)-Grushin planes locally modelled by vector fields \( \partial_x, |x|^\alpha e^{\phi(x,y)} \partial_y \). Indeed, the construction of the integrating Lie groupoid would be very similar and the analysis as well. Recall that in the dimension two there are only two simply connected Lie groups: the Euclidean space and the group of affine transformations of the real line. In both cases their harmonic analysis is relatively simple. For this reason we did not need to use the machinery of non-commutative harmonic analysis in this article. Partial Euclidean Fourier transform was enough.

One can then ask how the method would work in higher dimensions. Two problems are encountered in this case. The first obstacle is that we need to prove the existence of a Hausdorff integrating Lie groupoid. Even though there are no obstructions to Hausdorffness in dimension two, one can imagine that there might be some in higher dimensions. The second obstacle is that the left invertibility of limit operators might be difficult to prove in practice. From this point of view the best structures would be those that have a relatively simple harmonic analysis associated to the isotropy groups at the singular set. Having small dimensional coadjoint orbits would certainly help. In particular, if orbits are two-dimensional, then the non-commutative Fourier transform would transform limit operators to one-dimensional operators. All Lie groups of dimensions three and smaller have this property. Lie groups with coadjoint orbits of dimensions less or equal than two were classified in [7]. One of the consequences of this classification is that in dimensions greater than six all Lie groups having two-dimensional coadjoint orbits are semidirect products of abelian groups. This partially explains the prevalence in the literature of asymptotically Euclidean and asymptotically Hyperbolic manifolds which have isotropy groups of this type.

One can also use the algorithm outlined in the beginning of the Sect. 6 to already known results. For example, it is possible to recover the closure results for wedge operators from [35]. The advantage of the method presented in the article is that CNQ Fredholm conditions can be used as a black box. There is no need for constructing an associated pseudo-differential calculus by hand. However, going into the PDO structure of the problem allows to prove deeper results. For example, in [35] as a by-product of constructing the parametrix using a hand-made PDO calculus the authors also proved asymptotics of the trace of the resolvent of elliptic wedge operators.

Finally it would be interesting to generalise this kind of results to other classes of singular sub-Riemannian manifolds. For example, an analogue of the main Theorem 1 for generic Martinet manifolds [57] might be proven using a similar strategy modulo some modifications required due to hypoellipticity. But of course, it would be good to have a general theoretical basis for any sub-Riemannian structure regardless of dimensions or singularities.
Acknowledgements
The author would like to thank Victor Nistor for his patient explanations regarding Lie groupoids and results of [18], Eugenio Pozzoli for many fruitful discussions and the anonymous referees for useful remarks.

Funding Open access funding provided by FCT—FCCN (b-on). French ANR project Quaco ANR-17-CE40-0007-01 and Portuguese FCT Project UIDB/04106/2020.

Declarations
Conflict of interest The authors declare that they have no conflict of interest.

Data Availability Statement No data was gathered for this article.

Code availability No code was written for this article.

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Received: April 16, 2021.
Accepted: December 29, 2022.

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