Classification of the Killing vectors in nonexpanding \( \mathcal{H}\mathcal{H} \)-spaces with \( \Lambda \)

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Received 11 November 2011, in final form 13 April 2012
Published 31 May 2012
Online at stacks.iop.org/CQG/29/135010

Abstract
Conformal Killing equations and their integrability conditions for nonexpanding hyperheavenly spaces with \( \Lambda \) are studied. Classification of homothetic and isometric Killing vectors in nonexpanding hyperheavenly spaces with \( \Lambda \) and homothetic Killing vectors in heavenly spaces is given. Some metrics belonging to the two-sided Walker class are found. An example of the Lorentzian real slice of the type \([N] \otimes [N]\) is explicitly given.

PACS numbers: 04.20.Cv, 04.20.Jb, 04.30.Gz

1. Introduction

The generalization of the heavenly spaces, i.e. hyperheavenly spaces was discovered by Plebański and Robinson in 1976 [1, 2]. In the late 1970s and early 1980s, many papers devoted to these spaces were published. A structure of the heavenly [3–6] and hyperheavenly [7–10] spaces, within the spinorial formalism has been investigated in detail. General properties of the Killing spinors [11] and two-sided algebraically degenerated hyperheavenly spaces [12] have complemented the image of the complex general relativity.

The most important question of how to use the complex solutions of the Einstein field equation to obtain solutions of the real spacetimes with the Lorentzian signature still remains without an answer. The theoretical properties of such slices have been studied in [13]. A different approach [14] employs the connection 1-form to assure the existence of the real Lorentzian slices. The difficulties in obtaining the real solutions with the Lorentzian signature were the main reason why the complex relativity theory seemed to be of little interest in 1990s.

Recently, a compendium of the hyperheavenly spaces has been presented [15]. Then some results from [12] have been applied to obtain the special form of the hyperheavenly equation determining all \([N] \otimes [N]\)-type spaces with nonzero twist [16]. Although searching for real spacetimes with Lorentzian signature is a very difficult task, the real spacetimes with ultrahyperbolic signature \((++--)\) can be found very easily. That is why hyperheavenly
machinery appeared to be such a useful tool in the theory of Walker and Osserman spaces [17, 18].

It is well known that symmetries in spacetime allow us to simplify the field equations. The most important symmetries are Killing symmetries, defined by the set of equations $\nabla_{(a}K_{b)} = \chi g_{ab}$. (In the present paper, we use the following terminology: equation $\nabla_{(a}K_{b)} = \chi g_{ab}$ is called the conformal Killing equation and its solution $K$ the conformal Killing vector. If $\chi = \chi_0 = \text{const}$, then we deal with the homothetic Killing equation and the homothetic Killing vector. Finally, if $\chi = 0$, we have the isometric Killing equation and the isometric Killing vector, respectively).

The problem of isometric and homothetic Killing symmetries in heavenly spaces was introduced in [5]. The isometric case was considered in detail in outstanding work by Finley and Plebański [19]. The nonexpanding hyperheavenly spaces with isometric and homothetic Killing symmetry was presented in [20] and then in our first work [21]. In [21], the nonzero cosmological constant $\Lambda_1$ and the nonconstant conformal factor $\chi$ were introduced. There exist only two papers devoted to the problem of Killing symmetries in expanding hyperheavenly spaces. In [22], Sonnleitner and Finley developed the main ideas, and then in our work [23] nonzero cosmological constant $\Lambda_1$ has been included and some more compact classification of the Killing vectors and the forms of the reduced hyperheavenly equations in all possible subcases has been studied.

The current paper is the third part of our work devoted to the conformal, homothetic and isometric Killing symmetries in hyperheavenly spaces. It can be considered as a continuation of [21]. We collect here all results from [21] in compact and a little more general form. Moreover, the classification of the homothetic and isometric Killing vectors in nonexpanding hyperheavenly spaces is given (such a classification has never been presented before) and we show how to use the general results to classify the homothetic Killing vectors in heavenly spaces. Such a classification together with different types of the reduced heavenly equations (admitting at least one homothetic Killing vector) has not been considered by Plebański and co-workers. Some applications have also been found. Like we mentioned earlier, hyperheavenly spaces are directly related to the Walker spaces, and some examples of the Walker and two-sided Walker spaces with isometric or homothetic symmetry have been presented. More importantly, we have been able to obtain some Lorentzian real slices. This last result follows from the previous considerations in such a natural manner that it seems to be very promising in the solution of a long standing fundamental problem: how to find a Lorentzian spacetime from a given complex spacetime?

Our paper is organized as follows. In section 2, the general structure of nonexpanding hyperheavenly space and general results of [21] are recalled. Section 3 is devoted to the classification of the isometric and homothetic Killing vectors. Some examples of the nonexpanding hyperheavenly spaces admitting isometric, homothetic and null Killing vector field are given in section 4. One of them appears to be a complex extension of the pp-wave. Finally, the Lorentzian real slice of the hyperheavenly space of the type $[N] \otimes [N]$ is found. Concluding remarks end our paper.

Our work gives a new hope for positive answer to the so-called Plebański program, namely: how can one generate real Lorentzian metrics from holomorphic ones? Until now, that step seemed to be very complicated. The simplicity of obtaining the real pp-wave from the complex solution shows that in appropriate coordinate frame Lorentzian real slice could be found very easily. Application of this statement to the expanding hyperheavenly spaces especially of the type $[N] \otimes [N]$ will become one of our main research programs.
2. Preliminary remarks

In this section we present the most important information about the nonexpanding hyperheavenly spaces and Killing equations in the spinorial formalism. For more details see [21] and references therein.

2.1. Nonexpanding hyperheavenly spaces

Nonexpanding \( \mathcal{H} \)-space with a cosmological constant is a four-dimensional complex analytic differential manifold \( \mathcal{M} \) endowed with a holomorphic Riemannian metric \( ds^2 \) satisfying the vacuum Einstein equations with the cosmological constant and such that the self-dual or anti-self-dual part of the Weyl tensor is algebraically degenerate [1, 2, 7]. Here we assume degeneration of the self-dual part. These spaces admit a congruence of totally null, self-dual surfaces, called null strings [4]. ‘Nonexpanding’ means that the expansion form which characterizes the congruence of null strings is zero. Null tetrad can always be chosen in such a way that null string is spanned by \((\partial_2, \partial_4)\). i.e. there exist local coordinates \((q^A, p^B)\) such that

\[
dq_A = \begin{bmatrix} e^A_1 \\ e^A_2 \end{bmatrix}, \quad -dp^A + \Omega^{AB} dq_B = \begin{bmatrix} e^A_1 \\ e^A_2 \end{bmatrix},
\]

where \( \Omega^{AB} \) are the holomorphic functions. Spinorial coordinates \( p^A \) are coordinates on null string given by \( dq_A = \text{const} \). The following operators

\[
\bar{\partial}_A := \frac{\partial}{\partial p^A}, \quad \bar{\partial}^A := \frac{\partial}{\partial q^A}
\]

form the dual basis

\[
- \bar{\partial}_A = \begin{bmatrix} \partial_2 \\ \partial_4 \end{bmatrix}, \quad \bar{\partial}^A = \begin{bmatrix} \partial_1 \\ \partial_3 \end{bmatrix}.
\]

The metric is given by

\[
ds^2 = 2 e^1 \otimes e^1 + 2 e^3 \otimes e^3 - 2(\Sigma dp^A \otimes dq_A + \Omega^{AB} dq_A \otimes dq_B).
\]

Reduction of Einstein equations [2, 7] gives

\[
Q^{AB} = -\Theta_{pA} p^B + \frac{\lambda}{2} F^{(A} p^{B)},
\]

where \( F^A \) is an arbitrary function of \( q^A \) only, and \( \Theta \) is an arbitrary function of all variables called the key function. Einstein equations can be reduced to one equation called the nonexpanding hyperheavenly equation with \( \Lambda \):

\[
\frac{1}{2} \Theta_{pA} p^A + \frac{1}{2} \frac{\partial F_A}{\partial q^B} p^B + 2 \left( \frac{\partial}{\partial p^A} \Theta_{pA} - \frac{2}{3} p^B \Theta_{pB} + \frac{1}{6} \frac{\partial^2 F}{\partial q^B} p^B + \Lambda \left( p^B \Theta_{pB} - 2 \frac{p^B p^C}{3} \Theta_{pC} \right) = N_A p^A + \gamma, \right.
\]

where \( N_A \) and \( \gamma \) are the arbitrary functions of \( q^C \) only, which appear as the results of integration procedure for Einstein equations and \( \Theta_{pA} = \frac{\partial q^B}{\partial p^A}, \Theta_{qA} = \frac{\partial q^B}{\partial q^A} \), etc. Explicit formulas for the connection 1-forms in the spinorial formalism read

\[
\Gamma_{12} = \frac{1}{2} (F_B + \Lambda p^B) dq_B, \quad \Gamma_{22} = \left( N_B - \frac{1}{2} p^B C^{(2)} \right) dq_B,
\]

\[
\Gamma_{11} = 0, \quad \Gamma_{AB} = \left( -\Theta_{pA} p^B - \frac{1}{2} (F_{iA} + \Lambda p_{iA}) B_{ja} \right) dq^C.
\]
The conformal curvature is given by

\begin{equation}
C^{(5)} = 0 \quad C^{(4)} = 0 \quad C^{(3)} = -\frac{2}{3} \Lambda \quad C^{(2)} = -\frac{\partial F^A}{\partial q^A}
\end{equation}

(2.8)

where

\begin{equation}
Z_A := F_A C^{(2)} + \frac{\partial C^{(2)}}{\partial q^A} + 2 \Lambda N_A.
\end{equation}

(2.9)

The nonexpanding hyperheavenly space admits coordinate gauge freedom. (This problem was considered in detail in [21, 2, 7, 18].) The form of metric (2.4) remains invariant under the transformation

\begin{equation}
q_A' = q_A(q_B) \quad D_A^B := \frac{\partial q_A'}{\partial q_B} = \Delta \frac{\partial q^B}{\partial q^A} \quad \Delta := \det \left( \frac{\partial q_A'}{\partial q_B} \right)
\end{equation}

(2.10)

where $\sigma^A$ are the arbitrary functions of $q^A$ only. Straightforward but somewhat tedious calculations lead to

\begin{equation}
F^{\alpha} = D^{-1, \alpha} F^B - \frac{\partial \ln \Delta}{\partial q_A'} - \Lambda \sigma^A
\end{equation}

(2.11a)

\begin{equation}
N^A = \Delta^{-1} D^{-1, \alpha} N^B + \frac{\partial h}{\partial q_A'} - h F^A - \frac{1}{2} \frac{\partial F^{\alpha}}{\partial q^A} \sigma^A - \Lambda h \sigma^A
\end{equation}

(2.11b)

\begin{equation}
\Delta^2 \gamma' = \gamma + \frac{\partial H^A}{\partial q^A} + F_A H^A + \frac{1}{6} \Delta^2 \sigma^A \sigma^B \frac{\partial F^A}{\partial q^B} + \frac{1}{18} \Delta^2 (F^A \sigma_A)^2
\end{equation}

(2.11c)

\begin{equation}
\Delta^2 \Theta' = \Theta + \frac{1}{6} L^{ABC} p_A p_B p_C + \frac{1}{2} N^{AB} p_A p_B + H^A p_A + M,
\end{equation}

(2.11d)

where

\begin{equation}
L^{ABC} := D_{\chi}^{(A} \frac{\partial}{\partial q_{B)}} (D^{-1, \chi} C) + \frac{2}{3} \delta^{(A}_C \frac{\partial \ln \Delta}{\partial q_{B)}},
\end{equation}

\begin{equation}
N^{AB} := D_{\chi}^{(A} D_{\gamma}^{B)} \left( \frac{2}{3} F^{(\alpha} \sigma^\gamma) - \frac{\partial \sigma^{(\alpha}}{\partial q_{B)} + \frac{1}{3} \Lambda \sigma^{\chi} \sigma^\gamma \right).
\end{equation}

(2.12)

$H^A = H^A(q_B)$ and $M = M(q_B)$ are the arbitrary functions of $q_B$ only and $2h := \frac{\partial \sigma^A}{\partial q^A}$.

2.2. Conformal Killing symmetries

Conformal Killing equations with $\nabla(q_bK_b) = \chi g_{ab}$ in the spinorial form read

\begin{equation}
\nabla^b K_c^D = l_{AC} \epsilon^{BD} + \bar{I}^{BD} \epsilon_{AC} - 2 \chi \epsilon_{AC} \epsilon^{BD},
\end{equation}

(2.13)

where $l_{AC}$ and $\bar{I}^{BD}$ are symmetric spinors and $K_{AB} = -\sqrt{2} (\delta^A_1 k_B + \delta^B_1 h_B)$ are spinorial components of the Killing vector, connected with the null tetrad components by the relation

\begin{equation}
K = K^a \partial_a = -\frac{1}{2} K_{AB} \sigma^A \sigma^B = k_B \sigma^B = h_B \sigma^B
\end{equation}

(2.14)

\begin{equation}
\sigma^A \sigma^B = \sqrt{2} (\delta^A_1 \sigma^B - \delta^B_2 \sigma^A).
\end{equation}
In [11] the integrability conditions of (2.13) have been found. For the Einstein space \( (C_{ABCD} = 0, R = -4\Lambda) \), they have been considered in detail in [20]. In [21] the generalization for the nonconstant conformal factor \( \chi \) and nonzero cosmological constant \( \Lambda \) has been done. We do not repeat here detailed considerations and present only the final results which are slightly more general than formulas presented in [21] where algebraic types of the Weyl curvature spinor were considered individually. Results presented here are valid for all nonexpanding hyperheavenly spaces of the types [II, D, III, N]⊗[any] and the only assumption is that \( C_{ABCD} \neq 0 \), i.e. the heavenly spaces are excluded from the considerations. (To be more precise, the results below can be used in heavenly spaces, but only for the \( \chi = \chi(q^M) \). However, in [21] it has been shown, that in heavenly spaces the conformal factor \( \chi \) can have a little more general form.) Finally, we obtain

\[
\begin{align*}
l_{11} &= 0 \quad l_{12} = \delta_N(F^N + \Lambda p^N) - 2\chi + \frac{\partial\delta^N}{\partial q^N} \\
l_{22} &= -C(2)\delta_Np^N - 2\frac{\partial\chi}{\partial q^N}p^N + 2\delta_{NN} - \frac{\partial\chi}{\partial q^N}.
\end{align*}
\tag{2.15}
\]

Components of the Killing vector read

\[
k^A = \delta^A \quad h^A = \delta_A Q^A - 2\chi p^A - \frac{\partial\delta^A}{\partial q_A}p_N - \epsilon^A
\tag{2.16}
\]

what gives the form of the Killing vector

\[
K = \delta^B \frac{\partial}{\partial q^B} + \left( 2\chi p^B + \frac{\partial\delta^B}{\partial q_B} p_M + \epsilon^B \right) \frac{\partial}{\partial p^B}.
\tag{2.17}
\]

Ten conformal Killing equations can be reduced to one master equation

\[
\mathcal{L}_K \Theta = 2\Theta \left( \dot{3}\chi - \frac{\partial\delta^N}{\partial q^N} \right) + \frac{1}{6} \frac{\partial^2\delta}{\partial q^A \partial q^C} p^A p^B p^C + \left( \frac{1}{3} F_A \epsilon_B + \frac{1}{2} \frac{\partial\epsilon_A}{\partial q^B} \right) p^A p^B + \zeta_A p^A + \xi.
\tag{2.18}
\]

The functions \( \delta^A, \epsilon^A, \zeta^A \) and \( \xi \) are the functions of \( q^M \) only, which have to satisfy integrability conditions of the Killing equations:

\[
\begin{align*}
\Lambda \chi &= 0, \quad \chi = \chi(q^M) \quad \tag{2.19a}
\quad \frac{C(2)}{2} \frac{\partial\chi}{\partial q^A} &= 0 \quad \tag{2.19b}
\quad \frac{\partial}{\partial q^A} \left( \delta_A Z^N + 2\delta^A \frac{\partial Z^N}{\partial q^N} + \delta^A \frac{\partial Z^N}{\partial q^M} \right) &= 0 \quad \tag{2.19c}
\quad \frac{\partial G_A}{\partial q^A} + F_A G^A - 2\Lambda \frac{\partial\chi}{\partial q^A} &= 0 \quad \tag{2.19d}
\quad \frac{\partial\chi}{\partial q^N} \frac{\partial\Theta}{\partial p_N} &= \left( \frac{1}{2} \frac{\partial^2\chi}{\partial q^A \partial q^B} + \frac{1}{3} F_A \frac{\partial\chi}{\partial q^B} \right) p^A p^B + \frac{1}{2} G_A p^A + G. \quad \tag{2.19e}
\end{align*}
\]

where

\[
G_A := \frac{\partial}{\partial q_A} \left( 2\delta_A N^N - \frac{\partial\epsilon^N}{\partial q^N} \right) + F_A \epsilon^N + \frac{\partial\delta^N}{\partial q^A} + 2\delta^A \frac{\partial\epsilon^N}{\partial q^N} + 2\delta^A \frac{\partial\epsilon^N}{\partial q^M} - 4\chi N^A
\tag{2.20}
\]
G := \frac{\partial \xi^A}{\partial q^1} + F_A \xi^A + N_A \xi^A + \Lambda \xi + \delta^A \frac{\partial \chi}{\partial q^A} + 2\chi \frac{\partial \delta^N}{\partial q^N} - 4\chi \chi \tag{2.21}

and Z^A is given by (2.9).

Under the gauge given by (2.10), (2.11a)–(2.11d) and (2.12), \delta^M, \epsilon^M, \xi^M and \xi transform according to

$$\delta^M = \Delta D^{-1} \frac{\partial}{\partial q^M} \delta^h \tag{2.22a}$$

$$\epsilon^M = D^{-1} \frac{\partial}{\partial q^M} \epsilon^b = \sigma_{gb} \frac{\partial \delta^g}{\partial q^M} + \delta^h \frac{\partial \sigma^M}{\partial q^h} - 2\chi \sigma^M \tag{2.22b}$$

$$\Delta^2 \xi^M = D^R \left[ \xi^M - 2\delta^M \frac{\partial H_N}{\partial q^N} + H_N \frac{\partial \delta^N}{\partial q^M} + \epsilon^M N_{MN} + 2H_N \left( 2\chi - \frac{\partial \delta^M}{\partial q^N} \right) \right]$$

$$= \Delta^2 \frac{\partial^2 \delta^M}{\partial q^1 \partial q^M} \sigma^1 \sigma^M - \Delta^2 \sigma^M \left( \frac{\partial \epsilon^M}{\partial q^M} + \frac{2}{3} F^M \epsilon^R \right) \tag{2.22c}$$

$$\Delta^2 \xi^M = \xi + 2M \left( \frac{\partial \delta^N}{\partial q^N} - 3\chi \right) + \delta^N \frac{\partial M}{\partial q^N} - \epsilon ^N H_N$$

$$- \Delta^2 \left[ \frac{1}{6} \frac{\partial^2 \delta^M}{\partial q^1 \partial q^M} \sigma^1 \sigma^1 \delta^M + \delta^N \frac{\partial \sigma^M}{\partial q^N} + \frac{1}{3} F^M \epsilon^R \right] \sigma^R \sigma^R + \epsilon^R \sigma^R \right]. \tag{2.22d}$$

The following transformation formulas for spinorial divergences \( \frac{\delta^M}{\partial q^M} \) and \( \frac{\delta^M}{\partial q^M} \) are useful

$$\frac{\partial \delta^M}{\partial q^M} = \frac{\partial \delta^M}{\partial q^M} + \delta^M \frac{\partial \ln \Delta}{\partial q^M} \tag{2.23a}$$

$$\frac{\partial \epsilon^M}{\partial q^M} = D^L \frac{\partial \epsilon^M}{\partial q^M} + 2\delta^M \frac{\partial h}{\partial q^M} + 2h \left( \frac{\partial \delta^M}{\partial q^M} - 2\chi \right) - 2\sigma^M \frac{\partial \chi}{\partial q^M} \tag{2.23b}$$

3. Classification of the isometric and homothetic Killing vectors in nonexpanding hyperheavenly spaces

3.1. Criterion used in classification

The classification of the Killing symmetries that we propose is based on the algebraic properties of the components of the Killing vector, \( \delta^M \) and \( \epsilon^M \). In particular, properties of \( \delta^A \) are closely connected with the invariant properties of the Killing vector (compare subsection 4.3). If the \( \delta^M \neq 0 \) (nonnull Killing vector), then from (2.23a) it follows that \( \frac{\delta^M}{\partial q^M} \) can be always gauged away, bringing \( \delta^M \) to the gradient. The appropriate gauge transformation allows then to bring \( \delta^M \) to the form \( \delta^M = \frac{\delta^M}{Z} \), with \( Z \) being some fixed index, 1 or 2. We always choose \( Z = 1 \), namely \( \delta^M = \delta^M \). The second choice is completely analogous and does not lead to different results (since the structure of the nonexpanding hyperheavenly space is invariant under the interchanging coordinates \( q^1 \) and \( q^2 \)). This case we call type HK1 or IK1 (homothetic Killing vector of the type 1 or isometric Killing vector of the type 1, respectively). After setting
\( \delta^A = \delta^A_1 \), we still have the coordinate gauge transformations: 
\( q^i = q^1 + f(q^2) \), \( q^2 = q^2(q^2) \) with \( \Delta = \Delta(q^2) = \frac{\partial q^2}{\partial q^2} \) and \( f \) being the arbitrary function.

Isometric symmetries with \( \delta^M = 0 \) correspond to the null Killing vectors and have two subcases. Isometric-type IK2a is characterized by \( \frac{\partial q^2}{\partial q^2} = 1 \), (compare (2.23b)). Since in two dimensions each vector is proportional to a gradient, one can finally bring \( \epsilon^M \) to the form \( \epsilon^1 = q^1, \epsilon^2 = 0 \). In that case admissible coordinate gauge transformations are \( q^1 = q^1 f(q^2), q^2 = q^2(q^2) \) with \( \Delta = 1 \) and \( \frac{\partial q^2}{\partial q^2} = f^{-1} \). But if from the very beginning \( \frac{\partial q^2}{\partial q^2} = 0 \), without any loss of generality one can put \( \epsilon^M = \delta^M_1 \). This case we call the type IK2b. After that choice we are left with \( q^1 = q^1(q^1) \), \( q^2 = q^2 + \text{const} \), \( \Delta = \Delta(q^1) = \frac{\partial q^1}{\partial q^1} \).

Homothetic symmetries with \( \delta^M = 0 \) do not involve two different types. Nonzero \( \chi_0 \) allows us to gauge away \( \epsilon^M \). That is why this case we simply call the type HK2.

Gathering, we obtain the following types of the isometric and homothetic Killing vectors in nonexpanding hyperheavenly spaces:

| Isometric Killing vectors | Homothetic Killing vectors |
|--------------------------|---------------------------|
| IK1                      | HK1                       |
| IK2a                     | HK2                       |
| \( \delta^1 = \delta^A \) | \( \delta^A_1 = \delta^A_1 \) |
| \( \epsilon^1 = 0 \)     | \( \epsilon^1 = q^1 \)    |

We do not use this way to classify the conformal symmetries. It seems that the way chosen in [21] is the best description of the conformal symmetries. Moreover, the only nonconformally flat space which admits conformal symmetries is a space of the type \([N] \otimes [N]\). There is no need to classify one algebraic type.

The main aim of our considerations is to use coordinate gauge freedom and bring the master equation to the simplest possible form and then to solve it in order to obtain the form of the key function. The remaining gauge freedom was used to simplify maximally the arbitrary structural functions \( F^A, N^A \) and \( \gamma \) appearing in the hyperheavenly equation. It is the main reason why the structural functions in the algebraic types \([III,N] \otimes [any]\) sometimes have a different form of the standard one obtained in many other works devoted to the problem of the nonexpanding hyperheavenly spaces [7, 9, 20].

3.2. Conformal Killing symmetries

Conformal Killing symmetries are allowed only when \( \Lambda = 0 = C^{(2)} \), and \( C^{ABC}_{\delta} \frac{\delta^A}{\delta q^0} = 0 \), i.e. for the types \([N, -] \otimes [N, -]\). The \([N] \otimes [N]\) case was completely solved in [21].

3.3. Homothetic Killing symmetries

We have here \( \chi = \chi_0 = \text{const} \) and we assume that \( \chi_0 \neq 0 \). From (2.19a), we obtain immediately \( \Lambda = 0 \). From (2.19f), it follows that \( G^A = 0 = G \). The integrability conditions (2.19b) and (2.19e) are automatically satisfied. Thus, one obtains the following cases.
3.3.1. Type HK1 ($K = \partial_{\phi^i} + 2\chi_0 p^R \partial_{\rho^R}$). The forms of Killing vector and the key function are given by

| Functions | Killing vector | Master equation | The key function |
|-----------|---------------|-----------------|-----------------|
| $\delta^A = \delta^A_t, e^A = \xi^A = \xi = 0$ | $K = \partial_{\phi^i} + 2\chi_0 p^R \partial_{\rho^R}$ | $\xi_\rho \Theta = 6\chi_0 \Theta$ | $\Theta = e^{\delta^A_\rho^R} T(x, y, q^2)$ |

where $x := p^i e^{-2q^2}$ and $y := p^3 e^{-2q^2}$ and $T$ is an arbitrary function of its variables. With such a Killing vector the integrability conditions of the master equation immediately give

$$F^A = F^A(q^2), \quad N^A = n^A(q^2) e^{2q_i q^i}, \quad \gamma = g(q^2) e^{4q_i q^i},$$  \hspace{1cm} (3.1)

where $n^A$ and $g$ are the arbitrary functions of their variable. The hyperheavenly equation takes the form

$$T_{xy} - T_{xy}^2 + 2\chi_0(2T_y - x T_{xy} - y T_{xy}) - T_{xy}^2$$

$$+ F^1 \left( T_x + \frac{2}{3} x T_{xx} - \frac{2}{3} y T_{xy} \right) + F^2 \left( T_y - \frac{2}{3} x T_{xy} - \frac{2}{3} y T_{xy} \right)$$

$$+ \frac{1}{18} \left( F^3 - F^2 x \right)^2 + \frac{1}{6} \left( \frac{\partial F^2}{\partial q^2} x y - \frac{\partial F^1}{\partial q^2} \right)^2 = n^2 x - n^1 y + g.$$  \hspace{1cm} (3.2)

The remaining gauge freedom can be employed in particular algebraic types:

| [III] $\otimes$ [any] | [N] $\otimes$ [any] |
|-----------------------|-----------------------|
| $F^1 = 0, F^2 = F_0 q^2, F_0 = \text{const} \neq 0, F_0 = \Delta_0^{-1} F_0$ | $F^1 = 0, F^2 = F_0 = \text{const}$ |
| $n^1 = 0, g = 0$ | $n^1 = 0, n^2 = n(q^2), g = 0$ |
| $C^{(2)} = -F_0, C^{(1)} = -F_0 q^2 p^i$ | $C^{(2)} = 0, C^{(1)} = -2 \frac{\Delta_0}{m^2}$ |
| $l_{12} = -F_0 q^2 - 2\chi_0, l_{22} = -F_0 p^2$ | $l_{12} = -F_0 - 2\chi_0, l_{22} = -2n^2$ |

(where $\Delta_0$ is a constant determinant of the gauge transformation (2.10) and $n = n(q^2)$ is an arbitrary function of $q^2$.)

3.3.2. Type HK2 ($K = 2\chi_0 p^R \partial_{\rho^R}$). The forms of Killing vector and the key function are given by

| Functions | Killing vector | Master equation | The key function |
|-----------|---------------|-----------------|-----------------|
| $\delta^A = e^A = \xi^A = \xi = 0$ | $K = 2\chi_0 p^R \partial_{\rho^R}$ | $\xi_\rho \Theta = 6\chi_0 \Theta$ | $\Theta = (p^2)^3 T(x, q^4)$ |

$T$ is an arbitrary function of its variables and $x := l_1/l_2^2$. Integrability conditions of the master equation give $N^A = \gamma = 0$. Moreover, $C^{(2)}$ must be nonzero; in the other case we obtain automatically a heavenly space. All components of the anti-self-dual part of the curvature Weyl tensor are proportional to $T_{xxx}$, so the only allowed algebraic type is [III] $\otimes$ [N] in which the structural functions can be brought to the form.
This type is not allowed in this case, since from (2.19c) with $\delta^i = 0$ it immediately follows that $\epsilon^i = 0$. Therefore, when $\Lambda \neq 0$, there is no Killing vector of the type IK2.
3.5. Isometric Killing symmetries with $\Lambda = 0$

As in the previous case, $\chi = 0$, $G^0 = 0 = G$, (2.19b) and (2.19e) reduce to the identities $0 = 0$. However, $\Lambda = 0$ implies the types $\text{III}[N] \otimes \text{[any]}$.

3.5.1. Type I$K1$ ($K = \partial_{q^2}$). The forms of the Killing vector and the key function are given by

| Functions | Killing vector | Master equation | The key function |
|-----------|----------------|-----------------|-----------------|
| $\delta^4 = \delta^4_1$, $\epsilon^4 = \zeta^4 = \xi = 0$ | $K = \partial_{q^2}$ | $\mathbf{E}_k \Theta = 0$ | $\Theta = \Theta(q^2, p^M)$ |

The integrability conditions of the master equation give $F^A = F^A(q^2)$, $N^A = N^A(q^2)$, $\gamma = \gamma(q^2)$. Concrete algebraic types are characterized by

| [III] $\otimes$ [any] | [N] $\otimes$ [any] |
|------------------------|------------------------|
| $F^1 = 0, F^2 = F_0 q^2$, $F_0 = \text{const} \neq 0, F_0' = \Delta_0^{-1} F_0$ | $F^1 = 0, F^2 = F_0 = \text{const}$ |
| $N^1 = 0, N^2 = N^2(q^2), \gamma = 0$ | $N^1 = 0, N^2 = N^2(q^2), \gamma = 0$ |
| $C^{(1)} = -F_0$, $C^{(3)} = -F_0 q^2 p^1$ | $C^{(2)} = 0$, $C^{(1)} = -2 \frac{\Delta_0}{2q^2}$ |
| $l_{12} = -F_0 q^2$, $l_{22} = -F_0 p^2$ | $l_{12} = -F_0$, $l_{22} = -2N^2$ |

3.5.2. Type I$K2a$ ($K = q^2 \partial_{q^2}$). The forms of the Killing vector and the key function read

| Functions | Killing vector | Master equation | The key function |
|-----------|----------------|-----------------|-----------------|
| $\delta^4 = \zeta^4 = \xi = 0$ | $K = q^2 \partial_{q^2}$ | $\mathbf{E}_k \Theta = 1$ | $\Theta = T(q^2, p^2)$ |
| $\epsilon^4 = q^1$, $\epsilon^3 = 0$ | $\frac{1}{2} F_0 p^2 p^2 - \frac{1}{2} p^1 p^2 + \frac{1}{2} \left(\frac{1}{2} F_0 p^1 p^2 p^2 - \frac{1}{2} p^1 p^1 p^2\right)$ |

The only nonzero $C_{ABCD}$ is $C^{(1)}$, so the space considered must be of the type $\text{III}[N] \otimes [N]$. The integrability conditions of the master equation allow us to find the forms of the $F^A$ and $N^A$. Concrete algebraic types are characterized by

| [III] $\otimes$ [N] | [N] $\otimes$ [N] |
|------------------------|------------------------|
| $F^1 = F_0 \frac{1}{\sqrt{q^2}}$, $F^2 = 0$, $F_0 = \text{const} \neq 0, F_0' = f_0 F_0$ | $F^A = 0 \rightarrow F_0 = 0$ |
| $N^2 = 0$, $N^2 = N^2(q^2)$, $\gamma = 0$ | $N^2 = 0$, $N^2 = N^2(q^2)$, $\gamma = 0$ |
| $C^{(2)} = 0$, $C^{(3)} = 0$ | $C^{(1)} = -2 \frac{\Delta_0}{2q^2}$ |
| $l_{12} = 0$, $l_{22} = -1$ | $l_{12} = 0$, $l_{22} = -1$ |

$f_0$ is an arbitrary complex, gauge constant, which can be used in order to bring $F_0$ to 1 if desired. The hyperheavenly equation takes the form

$$
- \frac{1}{2} \frac{p^2}{q^1} T(p^2 p^2) + T(p^2 q^1) - \frac{1}{2} F_0^2 \left(\frac{p^2}{q^1}\right)^2 + N^1 p^2 = 0.
$$

(3.5)
3.5.3. Type IK2b \((K = \partial_{\rho \lambda})\). The forms of the Killing vector and the key function are

| Functions | Killing vector | Master equation | The key function |
|-----------|----------------|-----------------|-----------------|
| \(\epsilon^A = \delta^A, \delta^A = \zeta^A = \xi = 0\) | \(K = \partial_{\rho \lambda}\) | \(\xi_k \Theta = 0\) | \(\Theta = \Theta(p^2, q^M)\) |

Integrability conditions of the master equation give \(C^{(2)} = 0\), and the only nonzero anti-self-dual curvature coefficient is \(C^{(1)}\), so the only allowed algebraic type is \([N] \otimes [N]\).
Coordinate gauge freedom can be used to set

\[
[N] \otimes [N] \\
F^A = 0, N^2 = 0, N^4 = N^4(q^M), \gamma = 0 \\
C^{(2)} = 0, C^{(1)} = -2 \frac{\delta^A}{\omega^A} \\
l_{12} = 0, l_{23} = 0
\]

The hyperheavenly equation takes a very simple form

\[
\Theta_{p^A q^j} = -N^j p^2. \tag{3.6}
\]

3.6. Classification of the homothetic Killing vectors in heavenly spaces

Isometric Killing vectors in heavenly spaces were considered and classified in detail in [19]. Here we present the classification of the homothetic Killing vectors.

The heavenly spaces can be easily obtained from the hyperheavenly spaces by setting \(C^{(1)} = C^{(2)} = C^{(3)} = 0\). Using the gauge freedom in \(\Delta, h\) and \(H^A\), the functions \(F^A, N^A\) and \(\gamma\) can be gauged away. After that step, admissible coordinate transformation must have \(\Delta = \Delta_0 = \text{const}\) and \(h = h_0 = \text{const}\), but the hyperheavenly equation reduces to the well-known, second heavenly equation [3]:

\[
\frac{1}{2} \Theta_{p^A p^B} \Theta_{p^c q^d} + \Theta_{p^A q^c} = 0. \tag{3.7}
\]

In [21] the problem of Killing vectors in heavenly spaces was generalized for the case of the most general conformal factor. It has been proved that \(\chi\) can be at most a linear function of the \(p^A\) coordinates and it has the form \(\chi = \frac{\delta \alpha}{\delta q^A} p^A + \chi_1\), with \(\alpha = \alpha(q^M)\) and \(\chi_1 = \chi_1(q^M)\) being the arbitrary functions of their variables. Since we are interested in homothetic symmetries it is enough to set \(\chi_1 = \chi_0 = \text{const}\) and \(\alpha = \alpha_0 = \text{const}\). Moreover, it has been proved in [19] that if \(\epsilon\) is constant, it can be put zero without any loss of generality. Although all results from subsection 3 have been obtained with assumption \(C_{ABCD} \neq 0\) (i.e. for hyperheavenly spaces), detailed considerations show that the choice \(\alpha_0 = 0, \chi = \chi_0 = \text{const}\) allows us to use all that formulas in analysis of the heavenly spaces without any loss of generality. Particularly important for further considerations are integrability conditions of the master equation

\[
\frac{\partial \xi^A}{\partial q^A} = 0, \quad \frac{\partial \delta^A}{\partial q^A} = \delta_0 = \text{const} , \quad \frac{\partial \epsilon^A}{\partial q^A} = -\epsilon_0 = \text{const} \tag{3.8}
\]

with the transformation laws (compare (2.23a) and (2.23b)):

\[
\delta_0 = \delta_0, \quad \epsilon_0 = \Delta^{-1} \epsilon_0 - 2h_0 (\delta_0 - 2\chi_0). \tag{3.9}
\]
The most general type appears when both δ₀ and ε₀ are non-zero. Note that it involves δ₀ = 2χ₀. (In the other way ε₀ could be always gauged to zero.) In that case it is convenient to use Δ₀ and set ε₀ = −4χ₀. This case we call \( \gamma_{HKI} \) (heavenly homothetic Killing vector of the type I).

Second type \( \gamma_{HKII} \) is characterized by δ₀ ≠ 0 and ε₀ = 0. Then δ₀ can be different from 2χ₀ (in that case ε₀, if nonzero, can be always gauged away) or equals 2χ₀ (because there always exists a possibility that ε₀ is equal to zero from the very beginning).

Third type \( \gamma_{HKIII} \) appears when δ₀ = 0 = ε₀. There are two subcases: type \( \gamma_{HKIIIa} \) with δ³ ≠ 0 and type \( \gamma_{HKIIIb} \) for which δ³ = 0.

Remaining gauge freedom allows us to put ζ = \( \frac{\alpha}{\sqrt{q}} \) and \( \xi = 0 \), and brings δ⁴ and ε⁴ to the most plausible form. Gathering, we obtain the following types of the homothetic Killing vectors in heavenly spaces

| \( \gamma_{HKI} \) | \( \gamma_{HKII} \) | \( \gamma_{HKIIIa} \) | \( \gamma_{HKIIIb} \) |
|---|---|---|---|
| δ₁ ≠ 0, δ₀ = 2χ₀ | δ₁ ≠ 0 | δ₁ = 0 | δ₁ = 0 |
| ε₁ ≠ 0, ε₀ = −4χ₀ | ε₁ ≠ 0 | ε₁ = 0 | ε₁ = 0 |
| δ³ = 2χ₀q¹, δ⁴ = 0 | δ³ = δ₀q¹, δ⁴ = 0 | δ³ = δ¹ | δ³ = δ¹ |
| ε⁴ = 2χ₀q¹ | ε⁴ = 0 | ε⁴ = 0 | ε⁴ = 0 |
| ζ⁴ = ζ = 0 | ζ⁴ = ζ = 0 | ζ⁴ = ζ = 0 | ζ⁴ = ζ = 0 |
| \( K = 2\chi₀(q¹ \frac{\alpha}{\sqrt{q}}) \) | \( K = \delta₀q¹ \frac{\alpha}{\sqrt{q}} \) | \( K = \frac{\alpha}{\sqrt{q}} + 2\chi₀p³ \frac{\alpha}{q^p} \) | \( K = 2\chi₀p³ \frac{\alpha}{q^p} \) |
| +p³ \frac{\alpha}{q^p}q⁴ \frac{\alpha}{q^p} \frac{\alpha}{q^p} | −δ₀p³ \frac{\alpha}{q^p}q⁴ \frac{\alpha}{q^p} + 2\chi₀p³ \frac{\alpha}{q^p} \frac{\alpha}{q^p} | | |

\( \xi = 2\chi₀θ \) \( \xi = (6\chi₀ − 2δ₀)θ \) \( \xi = 6\chi₀θ \) \( \xi = 6\chi₀θ \)
\( θ = q¹T(q², x, y) \) \( \Theta = (q¹) \frac{\ln\chi₀}{\sqrt{q}} T(q², x, y) \) \( \Theta = e^{\chi₀q¹} T(q², x, y) \) \( \Theta = (p³)^{3} T(q³, x) \)
\( x := \frac{p³}{q^{p³}} q^{q³} \) \( x := p³ \) \( x := \frac{p³}{q^{p³}} q^{q³} \) \( x := p³ \)
\( y := \frac{q^{q³}}{q^{q³}} \) \( y := p³ \) \( y := \frac{q^{q³}}{q^{q³}} \) \( y := p³ \)

After some calculations we can find the reduced forms of the heavenly equation for the above types to be

Type \( \gamma_{HKI} \):
\[ T_{xx}T_{yy} − T_{xy}^2 + q¹(T_y − T_{yy} − (1 + x − y)T_{xy}) − (q²)^{3}T_{qq²} = 0. \] (3.10)

Type \( \gamma_{HKII} \):
\[ T_{xx}T_{yy} − T_{xy}^2 + 4χ₀ − δ₀ \frac{\delta₀}{\delta₀} T_y − 2\chi₀ \frac{\delta₀}{\delta₀} xT_{xy} − 2\chi₀ \frac{\delta₀}{\delta₀} yT_{yy} − T_{qq²} = 0. \] (3.11)

Type \( \gamma_{HKIIIa} \):
\[ T_{xx}T_{yy} − T_{xy}^2 + 2χ₀(2T_y − xT_{xy} − yT_{yy}) − T_{qq²} = 0. \] (3.12)

Type \( \gamma_{HKIIIb} \):
\[ 6TT_{xx} − 4T_{xx}^2 − T_{qq²} − 3T_{qq²} = 0 \] (3.13)

We are going to investigate equations (3.10)–(3.13) soon.
4. Applications

4.1. Examples of the nonexpanding hyperheavenly spaces admitting isometric Killing vector. Lorentzian real slices

Existence of the Killing vector of the form $K = q^i \partial_{q^i}$ (type IK2a) or $K = \partial_{p^i}$ (type IK2b) assures that nonlinearities in hyperheavenly equation (2.6) disappear. In that cases hyperheavenly equation becomes the linear differential equation and it can be easily solved.

4.1.1. Type IK2a. The hyperheavenly equation reduces to the form (3.5) (if $F_0 = 0$ the algebraic type is [N] ⊗ [N]). Writing $N^1$ in the form

$$N^1 = \frac{1}{2q^1} N - \frac{\partial N}{\partial q^1},$$

(4.1)

with $N$ being the arbitrary function of the $q^M$, one can find the solution for $T$ and finally, for $\Theta$:

$$\Theta = p^2 S(q^2, q^1 p^3 p^2) - \frac{1}{12q^1} (F_0 p^2 - p^1)(F_0 p^2 - 3p^1) + \frac{1}{2} N (p^3)^2 + r(q^M),$$

(4.2)

where $S = S(q^2, q^1 p^3 p^2)$ is an arbitrary function of its variables. The arbitrary function $r = r(q^M)$ can be gauged to 0 by using the gauge function $M$—see the transformation formula (2.11d)). The only nonzero curvature coefficients read

$$C^{(2)} = F_0 \frac{1}{q^1 q^1},$$

$$C^{(1)} = -2 \frac{\partial}{\partial q^1} \left( \frac{1}{2q^1} N - \frac{\partial N}{\partial q^1} \right) - 2F_0 \frac{p^1}{(q^1)^3} - F_0 \frac{p^2}{(q^1)^3}$$

(4.3)

$$\dot{C}^{(1)} = 2 (p^2 S(q^2, q^1 p^3 p^2)) \frac{p^1 p^2 p^3 p^4}{p^1 p^2 p^3}.$$  

From (4.3), it follows that the only allowed algebraic types are [III,N] ⊗ [N]:

$$ds^2 = -2 dp^1 \otimes dq_1 - 2 \left( (p^2 S) \frac{p^1 p^2 p^3 p^4}{q^1} + N \right) dq_1 \otimes dq_1$$

$$+ \frac{p^2}{q^1} dq_2 \otimes dq_2 + 2 \left( 2F_0 \frac{p^2}{q^1} - \frac{p^1}{q^1} \right) dq_1 \otimes dq_2.$$

(4.4)

4.1.2. Type IK2b. Writing $N^1$ in the form $N^1 = -\frac{\partial N}{\partial q^1}$ with $N = N(q^M)$ we obtain the solution of equation (3.6):

$$\Theta = \frac{1}{2} N p^2 p^2 + A(p^3, q^2) + r(q^M),$$

(4.5)

where $A = A(p^3, q^2)$ is an arbitrary function of its variables. (As before, the arbitrary function $r(q^M)$ can be gauged away.) The metric has the form

$$ds^2 = -2 dp^1 \otimes dq_A - 2 (N + A_{p^3 p^2}) dq_1 \otimes dq_1.$$

(4.6)

Since

$$C^{(1)} = 2 \frac{\partial^2 N}{\partial q^1 \partial q^1}, \quad \dot{C}^{(1)} = 2A_{p^3 p^2 p^2},$$

(4.7)

the space considered can be of types [N, -] ⊗ [N, -].
The Lorentzian real slices of the complex spacetime can exist only if there exists a coordinate frame such that \( \tilde{C}_{ABCD} = C_{ABCD} \), where the overbar denotes complex conjugation (see [13]). Here we have to assure that \( C^{(1)} = \tilde{C}^{(1)} \) and surprisingly this condition can be easily investigated. Denoting

\[
\begin{align*}
    p^1 &\equiv v, \quad p^2 \equiv \tilde{\zeta}, \quad q^1 \equiv \zeta, \quad q^2 \equiv u \\
    N(q^1, q^2) &\equiv f(\zeta, u), \quad A_{\rho\mu}(p^2, q^2) \equiv \tilde{f}(\tilde{\zeta}, u),
\end{align*}
\]

we automatically obtain \( \tilde{C}^{(1)} = C^{(1)} \). Of course, \( \tilde{C}_{ABCD} = C_{ABCD} \) is only the necessary condition of existing the real spacetime with the Lorentzian signature. However, in this particular case the metric (4.6) becomes

\[
dx^2 = 2 (d\zeta \otimes d\tilde{\zeta} - du \otimes dv) - 2 H(\zeta, \tilde{\zeta}, u) du \otimes du,
\]

with \( H(\zeta, \tilde{\zeta}, u) = f(\zeta, u) + \tilde{f}(\zeta, u) \), so (4.9) is the metric for the pp-wave solution with the Einstein field equation being assumed [24]. With (4.8) being assumed the key function takes the form

\[
\Theta = \frac{1}{2} f(\zeta, u) \tilde{\zeta}^2 + \int \tilde{f}(\tilde{\zeta}, u) d\tilde{\zeta} d\zeta.
\]

In the hyperheavenly language, (4.10) is the key function for the pp-wave solution.

It is worthwhile to note that the type IK2a does not have any Lorentzian real slice (compare with [24]).

4.2. Examples of Walker spaces with Killing symmetry

Two-sided Walker space is a real space of signature \((++--)\), which admits self-dual and anti-self-dual congruences of nonexpanding null strings [17]. General results obtained in subsection 3 allow us to find explicit examples of such spaces admitting homothetic or isometric symmetry.

4.2.1. Two-sided Walker space admitting homothetic symmetry. Homothetic symmetry of the type HK2 is allowed only for the type \([\text{III}] \otimes [\text{N}]\) (so it does not admit any Lorentzian real slices). It has self-dual, nonexpanding null string (defined by the 2-form \( e^1 \wedge e^3 \)) (like all nonexpanding hyperheavenly spaces). If \( \Gamma_{11} = 0 \), then there exists anti-self-dual null string, defined by \( e^2 \wedge e^3 \). Condition \( \Gamma_{11} = 0 \) is equivalent to \( T_{\alpha\alpha} = 0 \), so the space is automatically reduced to the heavenly space of the type \([\text{III}] \otimes [\text{I}]\). The solution of the hyperheavenly equation (3.3) reads

\[
T(x, q^\alpha) = \left( -\frac{F_0}{24} q^4 + f_1 \right) x + \frac{F_0^2}{144} (q^1)^3 - \frac{1}{3} \left( \frac{\partial f_1}{\partial q^2} + 4 f_2^2 \right) q^3 + f_2
\]

where \( f_1 \) and \( f_2 \) are the arbitrary functions of the variable \( q^2 \). The remaining gauge freedom enable us to gauge away \( f_1 \). Finally, the metric reads

\[
\frac{1}{2} ds^2 = -d\rho^4 \otimes dq_4 + F_0 \rho^4 p^2 dq_1 \otimes dq_2 + \left( \frac{3}{4} F_0 q^4 p^2 - \frac{F_0^2}{24} (q^1)^3 p^2 - 6 f_2 p^2 \right) dq_1 \otimes dq_1.
\]

Of course, taking all coordinates and functions real we obtain a two-sided Walker metric of signature \((++--)\).
4.2.2. Two-sided Walker space admitting isometric symmetry. Let us consider the space of the type \([N] \otimes [N]\) admitting the isometric Killing vector of the type IK1. Type \([N]\) on the anti-self-dual side we obtain by demanding \(C^{(4)} = C^{(3)} = C^{(2)} = 0, C^{(1)} \neq 0\). In that way we obtain the key function which is the third-order polynomial in \(p^1\). (Remember that this is not the most general form of the key function determining the \([N] \otimes [N]\)-type spaces.) Taking for simplicity \(F_0 = 0\), we find the solution of the hyperheavenly equation. Using the remaining coordinate gauge freedom we finally obtain

\[
\Theta(x, y, q^2) = f_1 x y^2 + f_2 x^2 + \frac{1}{28} \hat{C}^{(1)} y^4,
\]

where \(f_1 \neq 0, f_2 \neq 0\) are the arbitrary functions of \(q^2\) only, and we denote \(x := p^1, y := p^2\).

The only nonzero curvature coefficients are

\[
\frac{1}{2} C^{(1)} = \frac{\partial}{\partial q^2} \left( 2 \frac{\partial f_2}{\partial q^2} - 4 f_1 f_2 \right), \quad \frac{1}{2} \hat{C}^{(1)} = \frac{4 f_1^2}{f_2} + \frac{1}{8} \frac{\partial f_1}{f_2} \hat{q}^2.
\]

Here we have \(\Gamma_{11} = \Gamma_{1i} = 0\), so there exist two independent self-dual and anti-self-dual, nonexpanding null strings. This space belongs to the class of two-sided Walker–Einstein spaces if one takes all coordinates and functions real. The metric reads

\[
\frac{1}{2} ds^2 = dy \otimes dq^1 - dx \otimes dq^2 - 2 f_2 dq^1 \otimes dq^1 - 4 f_1 y dq^1 \otimes dq^2 - \left( 2 f_1 x + \frac{1}{4} \hat{C}^{(1)} y^2 \right) dq^2 \otimes dq^2.
\]

4.2.3. Walker space with nonzero \(\Lambda\) admitting isometric symmetry. In the last example of the isometric Killing symmetry in the Walker space cosmological constant \(\Lambda \neq 0\). We choose the case with \(N^2 = 0\) and use the arbitrary gauge constant \(\Delta_0\) to set \(N^2 = -\frac{2}{3} \Lambda\). Additionally, we assume \(\Theta_{0\theta} = 0\), what immediately gives us the space of the type \([II] \otimes [N]\) (which is rather rare). The solution of equation (3.4) gives

\[
\Theta(x, q^2) = x \ln x + \frac{1}{2} f_1 (q^2)^2 x^3 + f_2 (q^2) x,
\]

where we denoted \(x := p^1\). \(f_1\) and \(f_2\) are the arbitrary functions of \(q^2\) only, but they can be gauged to zero by the remaining gauge freedom. The conformal curvature reads

\[
C^{(3)} = -\frac{2}{3} \Lambda, \quad \hat{C}^{(1)} = \frac{4}{3} \Lambda^2 x, \quad C_{2222} = -\frac{2}{x}.
\]

\(\Lambda \neq 0\) causes \(\Gamma_{\delta \delta} \neq 0\), so any anti-self-dual null string is necessarily expanding. This space does not belong to the class of two-sided Walker space. The metric reads

\[
\frac{1}{2} ds^2 = -dp^1 \otimes dq_A + \left( \frac{\Lambda}{3} p^1 p^3 - \frac{\delta_A^1}{\delta_1} - \frac{1}{x} \right) dq_A \otimes dq_\theta.
\]

4.3. Null Killing vector fields

The Killing vector field is null when \(0 = K^a K_a = 2 \delta_\theta h^\theta\). There are two possibilities: \(\delta_{\Lambda} = 0\) and \(\delta_{\Lambda} \neq 0\); additionally, we assume that \(x = \chi_0 = \text{const}\).

4.3.1. \(\delta_{\Lambda} \neq 0\). Here we deal with the types IK1 and HK1. After using the gauge freedom to set \(\delta_{A} = \delta_{1}^A, e^A = \zeta^A = \xi = \gamma = 0\), the condition \(\delta_{\theta} h^\theta = 0\) gives \(0 = Q^{22} + 2 x_0 p^2\) and, finally, one obtains the solution for the key function which becomes the second-order polynomial in \(p^1\). Inserting this key function into the master equation and its integrability conditions we find that the cosmological constant \(\Lambda\) must vanish. Using the remaining gauge
freedom to set $F^1 = 0 = N^1$ and then inserting the key function into the hyperheavenly equation we find that its solution leads to the space of the type $[-] \otimes [\text{any}]$, i.e. the heavenly space. Summing up: null Killing vector fields with nonzero $\delta_A$ in nonexpanding hyperheavenly spaces of the type $[\text{II,D,III,N}] \otimes [\text{any}]$ do not exist.

4.3.2. $\delta_A = 0$. With $\delta^A = 0$, the condition $\delta_B h^B = 0$ is automatically satisfied. Killing vector lies on the null string and is null. In that case we deal with the types HK2, IK2a and IK2b. The cases IK2a and IK2b have been solved completely in the previous subsection. The hyperheavenly equation for the type HK2 reduces to equation (3.3), and we are going to investigate this equation in a future work.

5. Concluding remarks

In this paper we end the theoretical considerations on the Killing symmetries in nonexpanding $\mathcal{H}^\Lambda$-spaces with $\Lambda$. Results from [21] and [20] have been gathered in a compact form. The classification of these symmetries is the original part of the theory of the nonexpanding hyperheavenly spaces. The reduction of the ten Killing equations to one master equation has been done under the assumption that $C_{ABCD} \neq 0$. However, the final results can be quickly carried over to the case of the isometric and homothetic symmetries in heavenly spaces ($C_{ABCD} = 0$). This allows us to classify homothetic Killing symmetries in heavenly spaces. Only the conformal Killing symmetries in heavenly spaces are outside the scope of this paper, but they have been considered in [21].

An interesting part of our work consists of presenting the solution of the hyperheavenly space admitting isometric Killing vector of the type IK2b. It appeared to be a complex pp-wave. In this case we have been able to find the Lorentzian real slice in a very easy way. Using the results of [13], especially condition $\tilde{C}_{ABCD} = C_{ABCD}$, and substituting (4.8) we immediately find the real metric with Lorentzian signature describing the real pp-wave (4.9). An important question arises: can the condition $\tilde{C}_{ABCD} = C_{ABCD}$ be used in a simple way to the more complicated problems, for example, complex metrics of the expanding type $[\text{N}] \otimes [\text{N}]$ found in [23]?

Finally, the results presented in this paper can be used to describe the symmetries of Einstein–Walker spaces. All Einstein–Walker spaces have been found in [17]. It is enough to take the metrics from [17] and insert them into the master equation and its integrability conditions. Solutions of these equations give the Einstein–Walker space with additional Killing symmetry. Can the Einstein–Walker spaces with Killing symmetry be classified according to the types of Killing vectors? We are going to answer this question soon.

This paper is the third part of the work devoted to the Killing symmetries in $\mathcal{H}^\Lambda$-spaces. In the last part we will deal with the symmetries of heavenly spaces with $\Lambda$. Nonzero $\Lambda$ changes dramatically the structure of the self-dual (anti-self-dual) heavenly spaces. There are no nonexpanding congruences of anti-self-dual (self-dual) null strings. One must use the expanding hyperheavenly equation with the condition $C_{ABCD} = 0$. Can the (conformal) Killing equations be reduced to one master equation in that case, as has been done in the case of the Euclidian signature [25, 26]?

Acknowledgments

I am indebted to Professor M Przanowski for many enlightening discussions and help in many crucial matters.
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