Decremental APSP in Directed Graphs Versus an Adaptive Adversary

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Abstract

Given a directed graph $G = (V, E)$, undergoing an online sequence of edge deletions with $m$ edges in the initial version of $G$ and $n = |V|$, we consider the problem of maintaining all-pairs shortest paths (APSP) in $G$.

Whilst this problem has been studied in a long line of research [ACM’81, FOCS’99, FOCS’01, STOC’02, STOC’03, SWAT’04, STOC’13] and the problem of $(1+\epsilon)$-approximate, weighted APSP was solved to near-optimal update time $O(mn)$ by Bernstein [STOC’13], the problem has mainly been studied in the context of oblivious adversaries, which assumes that the adversary fixes the update sequence before the algorithm is started.

In this paper, we make significant progress on the problem in the setting were the adversary is adaptive, i.e. can base the update sequence on the output of the data structure queries. We present three new data structures that fit different settings:

1. We first present a deterministic data structure that maintains the exact distances with total update time $\tilde{O}(n^3)$.
2. We also present a deterministic data structure that maintains $(1+\epsilon)$-approximate distance estimates with total update time $\tilde{O}(\sqrt{mn^2}/\epsilon)$ which for sparse graphs is $\tilde{O}(n^{2+1/2}/\epsilon)$.
3. Finally, we present a randomized $(1+\epsilon)$-approximate data structure which works against an adaptive adversary; its total update time is $\tilde{O}(m^{2/3}n^{5/3}+n^{8/3}/(m^{1/3}\epsilon^2))$ which for sparse graphs is $\tilde{O}(n^{2+1/3})$.

Our exact data structure matches the total update time of the best randomized data structure by Baswana et al. [STOC’02] and maintains the distance matrix in near-optimal time. Our approximate data structures improve upon the best data structures against an adaptive adversary which have $\tilde{O}(mn^2)$ total update time [JACM’81, STOC’03].

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We use $\tilde{O}$-notation to hide logarithmic factors.
1 Introduction

Shortest paths is a classical algorithmic problem dating back to the 1950s. The two main variants are the all-pairs shortest paths (APSP) problem and the single-source shortest paths (SSSP) problem, both of which have been extensively studied in various models, including the partially and fully dynamic setting.

A dynamic graph algorithm is an algorithm that maintains information about a graph that is subject to updates such as insertions and deletions of edges or vertices. Such a graph can model real-world networks that change over time, such as road networks where traffic changes and roads are blocked from time to time. We say that a dynamic graph problem is decremental if it only allows deletions, incremental if it only allows insertions and fully-dynamic if it allows both. Incremental and decremental graphs are referred to as being partially-dynamic. A dynamic graph algorithm aims to efficiently process a sequence of online updates interspersed with queries about some property of the underlying dynamic graph.

1.1 Problem Definition

In this paper, we consider the decremental all-pairs shortest-paths problem where the goal is to efficiently maintain shortest path distances between all pairs of vertices in a decremental directed graph $G = (V, E)$. We shall restrict our attention to the case where $G$ is unweighted. Letting $m$ denote the initial number of edges and $n = |V|$, we want a data-structure which for any $u, v \in V$ supports the following operations:

- $\text{DIST}(u, v)$: reports the shortest path distance $d_G(u, v)$ from $u$ to $v$ in the current version of $G$,
- $\text{DELETE}(u, v)$: deletes an edge $(u, v)$ from $E$.

We furthermore consider the problem also in its relaxed version where we only aim to maintain approximate distance estimates which can then be queried. We denote by $\tilde{d}_G(u, v)$ a distance estimate for the distance from $u$ to $v$ and we say that an APSP algorithm has an approximation ratio (or stretch) of $t > 1$ if for any $u, v \in V$, we have that $d_G(u, v) \leq \tilde{d}_G(u, v) \leq t \cdot d_G(u, v)$. This paper will be concerned with both the exact and the $(1 + \epsilon)$-approximate version of the problem.

Another focus of this article is the adversarial model; the adversarial model defines the model under which the sequence of updates and queries are assumed to be made by an adversary. We say that a performance guarantee of an algorithm works against an oblivious adversary if the adversary must define the sequence of updates before the algorithm starts for the guarantee to hold. Thus the sequence of updates is independent of any random bits used by the algorithm. This is opposed to algorithms that work against an adaptive adversary, where the adversary is allowed to create the update sequence “on the go”, e.g. based on answers to previous queries made to the data structure. Depending on the data structure, these choices may not be independent on the random choices made, which may result in the data structure performing poorly. One key advantage of a data structure that works against an adaptive adversary is that it can be used inside an algorithm as a black box, regardless of whether that algorithm adapts its updates to answers to queries. We point out that deterministic data structures always work against an adaptive adversary.
The performance of a partially-dynamic algorithm is usually measured in terms of the total update time. That is, the accumulated time it takes to process all updates (edge deletions). The query time, on the other hand, is the time to answer a single distance query. A natural goal is to minimize the total update time while keeping the stretch and query time small. Since all the structures presented in this paper explicitly maintain a distance matrix, the query time is constant.

1.2 Prior Work

The naive approach to dynamic APSP is to recompute the shortest path distances after each update using the best static algorithm. The query time is then constant and the time for a single update is $O(mn)$ for APSP and $O(m)$ for SSSP. At the other end of the spectrum one could achieve optimal update time by simply updating the input graph and only running an SSSP algorithm whenever a query is processed. Running a static algorithm each time, however, fails to reuse any information between updates whatsoever and gives a high query time, motivating more efficient dynamic approaches that do this.

In 1981, Even and Shiloach [ES81] gave a deterministic data-structure for maintaining a shortest path tree to given depth $d$ in an undirected, unweighted decremental graph in total time $O(md)$. Henzinger and King [HK95] and King [Kin99] later adapted this to directed graphs with integer weights. Running their structure for each vertex solves the decremental all-pairs shortest paths problem in $O(mn^2 W)$ time, where edge weights are integers in $[1, W]$. Henzinger and King were the first to improve upon this bound, giving an algorithm with total update time $\tilde{O}(mn^{2.5} \sqrt{W})$ [Kin99] which is an improvement for $W = \omega(n)$. Demetrescu and Italiano [DI06] improved this data structure slightly and showed that the restriction to integral edge weights can be removed. Finally, the same authors [DI04] presented a data structure with total update time $\tilde{O}(mn^2)$ which is the state of the art for any data structure against an adaptive adversary up to today. In fact, their algorithm can be extended to a fully-dynamic algorithm with $\tilde{O}(n^2)$ amortized update time and which can handle vertex updates\(^2\). We also point out that this data structure was later simplified and generalized by Thorup [Tho04].

Around the same time Baswana, Hariharan, and Sen [BHS02] gave an oblivious Monte-Carlo construction with total update time $\tilde{O}(n^3)$ for unweighted graphs. Further, they showed that their data structure could be adapted to give an $(1 + \epsilon)$-approximate APSP algorithm for weighted graphs with total update time of $\tilde{O}(\sqrt{mn^2}/\epsilon)$. Finally, Bernstein presented a $(1 + \epsilon)$-approximate algorithm with total running time $\tilde{O}(mn \log W/\epsilon)$ by using a clever approach of shortcutting paths [Ber16]. Whilst his algorithm achieves near-optimal running time, again, the algorithm has to assume an oblivious adversary.

More recently, Karczmarz and Łącki [KL20] gave a deterministic $(1 + \epsilon)$-approximate APSP algorithm for decremental graphs that runs in total time $\tilde{O}(n^3 \log W/\epsilon)$. They also presented the first non-trivial algorithm for incremental graphs [KL19] achieving total update time $\tilde{O}(mn^{4/3} \log W/\epsilon)$.

We refer the reader to Appendix A for a more comprehensive treatment of related work which also includes algorithms for undirected graphs and algorithms with larger stretch.

\(^2\)In this case, vertex updates refers to insertions or deletions of vertices with up to $n - 1$ incident edges.
1.3 Our Contributions

In this paper, we present three new data structures for the all-pairs shortest paths problem. Our first theorem gives a deterministic data structure for the exact variant of the problem with near-optimal $\tilde{O}(n^3)$ total update time. It also matches the best randomized algorithm by Baswana et al. [BHS02] and constitutes a significant improvement over the previous best bound of $\tilde{O}(mn^2)$ which is obtained by running an ES-tree [ES81] from every source or by the data structure Italiano et al. [DI04] and improves over all but the sparsest graph densities. Our data structure is near-optimal as we will show an $\Omega(n^3)$ lower bound on the total update time of any decremental data structure that explicitly maintains the distance matrix.

Theorem 1. Let $G$ be an unweighted directed graph with $n$ vertices and initially $m$ edges. Then there exists a deterministic data structure which maintains all-pairs shortest path distances in $G$ undergoing an online sequence of edge deletions using a total time of $O(n^3 \log^3 n)$. The $n \times n$ distance matrix is explicitly maintained so that at any point, a shortest path distance query can be answered in constant time. The data structure can report a shortest path between any query pair in time proportional to the length of the path.

Our second result is concerned with maintaining $(1 + \epsilon)$-approximate all-pairs shortest path distances. This constitutes the first deterministic data structure that solves the problem in subcubic time with small approximation error (except for graphs that are not extremely dense). In fact, for very sparse graphs with $m = \tilde{O}(n)$, our update time even matches the near-optimal result by Bernstein [Ber16] with total update time $\tilde{O}(mn)$.

Theorem 2. Let $G$ be an unweighted directed graph with $n$ vertices and initially $m$ edges. Then given $\epsilon > 0$, there exists a deterministic data structure that maintains all-pairs $(1 + \epsilon)$-approximate shortest path distances in $G$ undergoing an online sequence of edge deletions using a total time of $O(\sqrt{mn^2 \log^2 n/\epsilon})$. At any point, a $(1 + \epsilon)$-approximate shortest path distance query can be answered in constant time and a $(1 + \epsilon)$-approximate shortest path between the query pair can be reported in time proportional to the length of the path.

Our third result gives a data structure achieving a better time bound. While we use randomization to achieve the improved time bound, our algorithm again works against an adaptive adversary.

Theorem 3. Let $G$ be an unweighted directed graph with $n$ vertices and initially $m$ edges. Then given any $\epsilon > 0$, there exists a Las Vegas data structure that maintains all-pairs $(1 + \epsilon)$-approximate shortest path distances in $G$ under an online sequence of edge deletions using a total expected time of $\tilde{O}(m^{2/3}n^{5/3}/\epsilon + n^{8/3}/(m^{1/3}\epsilon^2))$. This bound holds w.h.p. and the data structure works against an adaptive adversary. At any point, a $(1 + \epsilon)$-approximate shortest path distance query can be answered in constant time.

We summarize our results as well as previous state-of-the-art results in Table 1.

1.4 Overview

Our overall approach for the deterministic data structures is similar to that of Baswana et al. [BHS02] but with a key difference that allows us to avoid using a randomized hitting set.
Table 1: Our results and previous state-of-the-art results for decremental APSP in the exact setting.

| Time                  | Approximation | Adversary/ Deterministic | Reference |
|-----------------------|---------------|---------------------------|-----------|
| $O(mn^2)$             | exact         | deterministic             | [ES81; DI04] |
| $O(n^3)$              | exact         | deterministic             | New Result |
| $O(n^3)$              | exact         | adaptive                  | [BHS02]   |

Table 2: Our results and previous state-of-the-art results for decremental APSP in the approximate setting.

| Time                  | Approximation | Adversary/ Deterministic | Reference |
|-----------------------|---------------|---------------------------|-----------|
| $\tilde{O}(\sqrt{mn^2}/\epsilon)$ | $(1 + \epsilon)$ | deterministic             | New Result |
| $\tilde{O}(m^{2/3}n^{5/3}/\epsilon + n^{8/3}/(m^{1/3}2))$ | $(1 + \epsilon)$ | adaptive                  | New Result |
| $\tilde{O}(\sqrt{mn^2}/\epsilon)$ | $(1 + \epsilon)$ | oblivious                  | [BHS02]   |
| $\tilde{O}(mn)$       | $(1 + \epsilon)$ | oblivious                  | [Ber16]   |

and instead rely on deterministic separators. The idea of the construction by Baswana et al. relies on a well-known result which says that if we sample a subset $H_{\rho}^{i}$ of the vertices of size $\tilde{O}(n/\rho^{i})$ (where $\rho$ is some constant strictly larger than 1), each with uniform probability, then, w.h.p. we "hit" each shortest-path of length $[\rho^{i}, \rho^{i+1}]$ between any pair of vertices in any version of the graph $G$.

Phrased differently, given vertices $u, v \in V$, we have that if the the shortest path from $u$ to $v$ is of length $\ell \in [\rho^{i}, \rho^{i+1})$, then there is some vertex $w \in H_{\rho}^{i}$, such that the concatenation of the shortest path from $u$ to $w$ and the shortest path from $w$ to $v$ is of length $\ell$. For each such $w$, we say $w$ is a witness for the tuple $(u, v)$ for distance $\ell$.

Now for each $u, v \in V$, if the initial distance from $u, v$ was $\ell \in [\rho^{i}, \rho^{i+1})$, we can check $H_{\rho}^{i}$ to find a witness $w$. If the length of the path from $u$ to $v$ is increased, we can continue our scanning of $H_{\rho}^{i}$ to see whether another witness exists. If there is no witness $w \in H_{\rho}^{i}$ left at some stage, we know that there is no path of length $\ell$ left in $G$ w.h.p. and increase our guess by setting $\ell \mapsto \ell + 1$.

Sampling initially a hitting set $H_{\rho}^{i}$ for every $i \in [0, \log_{\rho} n]$, we can find the 'right' hitting set for each distance $\ell$. Observe now that for each tuple $(u, v) \in V^{2}$, we have to scan a hitting set of size $\tilde{O}(n/\rho^{i})$ for $\rho^{i+1} - \rho^{i} \sim \rho^{i+1}$ levels before the hitting set index $i$ is increased which only occurs $O(\log n)$ times, thus we only spend time $\tilde{O}(n)$ for each vertex tuple $(u, v)$. Thus, the total running time of the searches for witnesses can be bound by $\tilde{O}(n^{3})$.

The Deterministic Exact Data Structure Our construction is similar in the sense that we maintain witnesses for each distance scale $[\rho^{i}, \rho^{i+1})$ for each $i \in [0, \log_{\rho} n]$ such that each distance $\ell$ is in one such distance scale. The key difference is that instead of using a randomized global hitting set $H_{\rho}^{i}$ for a distance scale $[\rho^{i}, \rho^{i+1})$, our construction relies on deterministically maintaining a small local vertex separator $S_{i}(u)$ for every vertex $u \in V$ of
size $\tilde{O}(n/\rho)$ separating all shortest paths starting in $u$ with a distance in $[\rho^i, \rho^{i+1})$.

More precisely, for each distance scale $[\rho^i, \rho^{i+1})$ and vertex $u \in V$, we maintain a separator $S_i(u)$ that satisfies the invariant that every shortest path from $u$ to a vertex $v$ at distance at least $\rho^i$ is intersected by a vertex in $S_i(u)$. If this invariant is violated after an adversarial update, then we find such a vertex $v$ and need to add additional vertices to $S_i(u)$ during the time step. The challenge is to take these additional separator vertices such that the total size of $S_i(u)$ is not increased beyond $\tilde{O}(n/\rho^i)$. We defer the details of the separator procedure to a later section and continue our discussion of the APSP data structure.

Since we need to detect whether vertices have distance less than $\rho^i$ from $u$ or not in $G$, we further have to use a bottom-up approach to compute distances, i.e., we start with the smallest possible distance range and find all small distances and then find larger distances using the information already computed. This issue did not arise in Baswana et al. [BHS02] but can be handled by a careful approach.

It is now easy to see that the scanning for witnesses can be implemented in the same time as in the analysis sketched above by scanning the list of local separator vertices which serve as witnesses instead of the hitting set. Further, we can maintain local vertex separators using careful arguments in total time $\tilde{O}(mn)$ giving our result in Theorem 1.

![Diagram of separators and path "hierarchy". Here $u \rightarrow v$ goes through a witness $w$, and $w \rightarrow v$ goes through $w'$. If the length of the path $w' \rightarrow v$ is increased by $\Delta$, the distance estimates of all 2-hop-paths that use $w' \rightarrow v$ as a sub-path are increased by that amount. In this case, the estimate for $w \rightarrow w' \rightarrow v$ is increased and is propagated to the next level where subsequently the estimate for $s \rightarrow w \rightarrow v$ is increased.](image)

**The Deterministic Approximate Data Structure**  
In order to improve the running time for sparse graphs, we can further focus on only considering distances that are roughly at a $(1 + \epsilon)$-multiplicative factor from each other. More concretely, instead of increasing the expected distance from $\ell$ to $\ell + 1$ when we cannot find a witness for some path from $u$ to $v$ for distance $\ell$, we can increase the next expected distance level $\ell'$ to $\sim (1 + \epsilon)\ell$ and consider every vertex $w$ a witness if there is a path $u \rightarrow w \rightarrow v$ of length at most $\ell'$. Thus, we search less distances and can thereby increase the time to maintain distances that are at least $d$ in total time $\tilde{O}(n^3/d + mn)$. Again, a careful approach is necessary to ensure that approximations do not add up over distance scales.

This is no faster than the data structure for exact distances when $d$ is small so in order to get Theorem 2, we use the $O(mnd)$ data structure of Even and Shiloach [ES81] to maintain
distances up to $d$. Picking $d$ such that $mnd = n^3/d$ gives the result of Theorem 2 (the term $\tilde{O}(mn)$ vanishes since it is subsumed by the two other terms, also we assumed $\epsilon > 0$ to be a constant to simplify the presentation).

**Maintaining Separators** We now describe how to deterministically maintain the “small” local separator for a vertex $s \in V$ with some useful invariants.

Let $S$ be the local separator for $s$. The first invariant that will be useful is that any vertex $t \in V$ that is reachable from $s$ in $G \setminus S$, is “close” to $s$ or roughly within distance $d$. As edges are deleted from $G$, the distances from $s$ to vertices in $S$ may increase, meaning that separator vertices may move away from $s$ as edges are deleted. When some separator vertex $w \in S$ moves too far away, the invariant is re-established by growing BFS trees in parallel, one layer at a time, from $s$ in $G \setminus S$ and from $w$ in $G \setminus S$ with the edges reversed. The search halts when a layer (corresponding to the leaves of the BFS tree at the current iteration) that is “thin” is found, and its vertices are added to $S$; vertices that are on the opposite side of the separator than $s$ are cut off as they must all be too far away from $s$. Here, "thin" refers to a BFS layer such that the number of vertices added to the separator is only a factor $\tilde{O}(1/d)$ times the number of vertices cut off. It is well known that such a layer exists (cfr. Lemma 2 for the details). Summing up, it follows that $|S| = \tilde{O}(n/d)$ at all times. By marking vertices as they are searched (according to the side of the BFS layer on which they are found), the vertices that are “cut off” from $s$ by the augmented separator will never be searched again, and the cost of searching the edges of either side of the search can be charged to sum of the degree of these vertices, for a total update time of $O(m)$.

For our randomized data structure, we need an additional property that essentially allows us to take a snapshot of the current separator and use it in later updates rather than having to repeatedly update the separator. This will be key to getting an improved randomized time bound. Details can be found in Lemma 3 which states our separator result.

**The Randomized Approximate Data Structure** The randomized approximate data structure of Theorem 3 follows the same overall approach but is technically more involved. Instead of keeping track of all 2-hop paths $u \leadsto s \leadsto v$ for every $s \in S_i(u)$, the randomized data structure samples a subset of these by picking each vertex of $S_i(u)$ independently with some probability $p$. It only keeps track of approximate shortest path distances through this subset rather than the full set $S_i(u)$. This will speed up the above since priority queue sizes are reduced in expectation by a factor $p$. However, this approach fails once no short 2-hop path intersects the sampled subset. At this point, w.h.p. there should only be short 2-hop paths through $O(\log n/p)$ vertices of $S_i(u)$ so also in this case, the priority queue sizes can be kept small. However, scanning linearly through $S_i(u)$ to find this small subset will take $\tilde{O}(n/d)$ time and over all pairs $(u,v)$.

Our solution is roughly the following. Suppose no sampled vertices certify an approximate short path from $u$ to $v$. Then $v$ scans linearly through $S_i(u)$ to find the small $O(\log n/p)$ size subset $S'_i(u)$. Consider the set $W$ of vertices $w$ such that $d_G(w, v)$ is small compared to $d$, i.e., $d_G(w, v) \leq ed$ for some small constant $\epsilon > 0$. Then we show that the small subset $S'_i(u)$ found for $v$ can also be used for each vertex $w \in W$. The intuition is that for any vertex $s \in S_i(u) \setminus S'_i(u)$, the approximate shortest path distance from $u$ to $w$ through $s$ must be large since otherwise we get a short path $u \leadsto s \leadsto w \leadsto v$ from $u$ to $v$ through $s$,
contradicting that $s \notin S'_i(u)$.

It follows that if $|W|$ is large, the $\tilde{O}(n/d)$ cost of scanning $S_j(u)$ can be distributed among a large number of vertices of $W$. Dealing with the case where $|W|$ is small is more technical so we omit it here.

The way we deal with an adaptive adversary is roughly as follows. Consider a deterministic data structure that behaves like the randomized data structure above, except that it maintains 2-hop paths $u \leadsto s \leadsto v$ for all $S_i(u)$ rather than only through a sampled subset. The slack from the approximation allows us to round up all “short” approximate distances to the same value. Hence, as long as the randomized data structure has short 2-hop paths, it maintains exactly the same approximate distances as the deterministic structure and hence the approximate distances output to the adversary is independent of the random bits used.

2 Definitions and Notation

In the following, let $G = (V, E)$ be a directed unweighted graph. The graph $G_{\text{rev}}$ is obtained from $G$ by reversing the orientation of each edge. For any two vertices $u, v \in V$, we denote by $u \leadsto v$ a shortest path from $u$ to $v$ in $G$ and let $d_G(u, v)$ denote the distance of such a path. We extend this notation to sets so that, e.g., $d_G(u, V') = \min\{d_G(u, v) | v \in V'\}$ for $V' \subseteq V$.

We define a BFS-layer to mean the set of nodes at some fixed distance from some $v$ in $G$. An in-tree in $G$ is a BFS tree in $G_{\text{rev}}$.

We will need notation to refer to dynamically changing data at specific points in time. Consider a sequence of updates to some object $X$ where each update takes place at a time step $t \in \mathbb{N}$. We denote by $X(t)$ the object just after update $t$. Here, $X$ could be a graph, a shortest path distance, etc.

For handling small distances, we rely on the data-structure of Even and Shiloach [ES81], the properties of which we will state in the following lemma:

**Lemma 1 ([ES81]).** Given a directed unweighted graph $G$ undergoing a sequence of edge deletions, a source vertex $s \in V$, and $d > 0$, a shortest path tree in $G$ rooted at $s$ can be maintained up to distance $d$ in total time $O(md)$. The structure requires $O(m)$ space and can be constructed in time $O(m + n)$.

3 Maintaining Separators

Lemma 3 below provides a key tool used in all of our data structures. It gives an efficient data structure that maintains a growing separator set $S$ of small size in a decremental graph $G$ with the following guarantees. Let $s$ be a fixed vertex and let $d$ be some given threshold distance. Then at every time step, vertices reachable from $s$ in $G \setminus S$ are of distance slightly less than $d$ from $s$ in $G$. Conversely, for vertices $v$ not reachable from $s$ in $G \setminus S$, we have $d_G(s, v) = \Omega(d)$; furthermore, if $d_G(s, v)$ is larger than $d$ by some small constant factor then any shortest path $s \leadsto v$ in $G$ can be decomposed into $s \leadsto w \leadsto v$ such that $s \in S$, $d_G(s, w) \leq d$, and $d_G(w, v) \leq d$. In fact, the lemma states that $w$ can be chosen in $S_{t_0}^t$ where $t_0$ is the first time step in which $d_G(s, v)$ became (slightly) larger than $d$; note that this is a stronger statement since $S$ is growing over time. Before proving Lemma 3, we need the following well-known result.
Lemma 2. Given a directed unweighted $n$-vertex graph $G = (V, E)$, given $d_1, d_2 \in \mathbb{N}$ with $d_2 - d_1 + 1 \geq \log n$, and given vertices $u, v \in V$ with $d_G(u, v) \geq d_2$, a BFS tree in $G$ with root $u$ contains a layer $L \subseteq V$ with $d_1 \leq d_G(u, L) \leq d_2$ and $|L| \leq |L_\leq| \log n/(d_2 - d_1 + 1)$ where $L_\leq = \{w \in V \mid d_G(u, w) < d_G(u, L)\}$ is the union of layers closer to $u$ than $L$.

Proof. Denote by $L_i$ the $i$th layer of the BFS tree from $u$. For each $i$, let $L_{<i} = \cup_{j<i} L_j$. Let $q = (d_2 - d_1 + 1)/\log n$. Assume for contradiction that $L$ does not exist. Then for $i = d_1, \ldots, d_2$, $|L_i| > |L_{<i}|/q$ so $|L_{<i+1}| = |L_i| + |L_{<i}| > (1 + 1/q)|L_{<i}|$. Since $q \geq 1$, we have $(1 + 1/q)^q \geq 2$ so

$$|L_{<i+1}| > (1 + 1/q)^{d_2-d_1+1}|L_{<i}| \geq 2^{d_2-d_1+1}/q = n,$$

contradicting that there are only $n$ vertices in $G$. \hfill \Box

Lemma 3. Given a directed unweighted $n$-vertex graph $G = (V, E)$ undergoing a sequence of edge deletions, a source $s \in V$, and a value $d \in \mathbb{N}$ with $d > 33 \log n$. Let $O$ be a data structure that maintains for each $v \in V$ a distance estimate $\hat{d}(s, v) \geq d_G(s, v)$ such that if $d_G(s, v) \leq d$ then $\hat{d}(s, v) \leq \frac{1}{2}d_G(s, v)$. Whenever an estimate $\hat{d}(s, v)$ grows to a value of at least $\frac{34}{33}d$, $O$ outputs $v$. Then there is a data structure $D$ with access to $O$ which maintains a growing set $S \subseteq V$ such that for each $v \in V$,

1. at the end of each update, if $v$ is reachable from $s$ in $G \setminus S$ then $d_G(s, v) < \frac{34}{33}d$ and otherwise $d_G(s, v) > \frac{2}{3}d$,

2. if $t_0$ is a time step in which $d < d^{(t_0)}_G(s, v) \leq \frac{34}{33}d$ then for every time step $t_1 \geq t_0$ in which $d^{(t_1)}(s, v) \leq \frac{34}{33}d$, any shortest $s$-to-$v$ path $P$ in $G^{(t_1)}$ intersects $S^{(t_0)}$ and for the first such intersection vertex $w$ along $P$, $d^{(t_1)}_G(s, w) \leq d$, and $d^{(t_1)}(w, v) \leq d$.

At any time, $|S| = O(n \log n/d)$ and $D$ has total update time $O(m)$, excluding the time spent by $O$.

Proof. Let $\epsilon = \frac{1}{m}$. For each $v \in V$, let $\hat{d}(v)$ be obtained from the degree of $v$ in the initial graph $G$ by rounding up to the nearest multiple of $\Delta = \lfloor m/n \rfloor$. In the description of $D$ below, processing one edge takes at most one unit of time.

Data structure $D$ initializes $S = \emptyset$ and unmarks all vertices of $V$. Whenever $O$ outputs an unmarked vertex $v$, $D$ runs a modified BFS from $s$ in $G_S = G \setminus S$ which for each vertex $w$ spends $\hat{d}(w)$ time to process its outgoing edges; this can always be achieved by busy-waiting at $w$ if needed. In parallel, $D$ runs a similar modified BFS from $v$ in $G'_S = (G \setminus S)_{\text{rev}}$. The search from $s$ halts if a layer $L_s$ is found such that $\frac{2}{3}d < d_{G_S}(s, L_s) \leq (\frac{2}{3} + \epsilon)d$ and $|L_s| = O((x \log n)/d)$ where $x$ is the number of vertices visited by the search, excluding $L_s$. Similarly, the search from $v$ halts if a layer $L_v$ is found such that $d_{G_S}(v, L_v) < \epsilon d$ and $|L_v| = O((y \log n)/d)$ where $y$ is the number of vertices visited by the search excluding $L_v$. Let $L$ be the first of the two layers found. $D$ halts both searches when $L$ is found. Then $L$ is added to $S$. The existence of $L$ follows from Lemma 2 which applies since by assumption, $\epsilon d > \log n$.

Observe that when $O$ outputs $v$, we have $d_G(s, v) \geq (1 - \epsilon)d/(4/3) = (\frac{2}{3} + 2\epsilon)d$ as otherwise, $\hat{d}(s, v) < 1 - \epsilon = \frac{34}{33}d$. This shows the existence of $L_s$ and $L_v$ and that no edge
is visited by both searches. We have $d_{G_S}(s,v) \geq d_G(s,v) \geq (\frac{2}{3} + 2\epsilon)d$ and $d_{G_S}(s, L_s) > \frac{2}{3}d$ and for every $w \in L_v$, \[
d_{G_S}(s, w) \geq d_{G_S}(s, v) - d_{G_S}(w, v) \\
\geq \left(\frac{2}{3} + 2\epsilon\right)d - d_{G'_S}(v, w) \\
= \left(\frac{2}{3} + 2\epsilon\right)d - d_{G'_S}(v, L_v) \\
\geq \left(\frac{2}{3} + \epsilon\right)d,
\]
implying that $d_{G_S}(s, L_v) > (\frac{2}{3} + \epsilon)d$. It follows that $d_{G_S}(s, L) = \min\{d_{G_S}(s, L_s), d_{G_S}(s, L_v)\} > \frac{2}{3}d$.

**Showing part 1:** Let $v \in V$ and consider any point during the sequence of updates. Assume first that $v$ is reachable from $s$ in $G_S$. Then $O$ has not yet output $v$ (otherwise, the above procedure separates $v$ from $s$ with $S$) so $d_{G}(s, v) \leq d(s, v) < \frac{2}{3}d$, as desired.

Now, assume that $v$ is not reachable from $s$ in $G_S$. We may assume that there is a shortest path $P$ from $s$ to $v$ in $G$ since otherwise $d_{G}(s, v) = \infty > \frac{2}{3}d$. Let $w$ be the first vertex of $S$ along $P$. It suffices to show that $|P| > \frac{2}{3}d$. At some earlier point in time, the procedure added $w$ to $S$; just prior to this, $P$ was contained in $G_S$ so from the above $|P| > \frac{2}{3}d$, as desired.

**Showing part 2:** Let $t_0 \leq t_1$ satisfy the second part of the lemma. Since $d_{G^{(t_0)}}^{(t_0)}(s, v) > d$ by assumption, the first part of the lemma implies that $v$ is not reachable from $s$ in $G^{(t_0)}_S$ and hence $v$ is also not reachable from $s$ in $G^{(t_1)}_S$.

Let $P$ be a shortest path from $s$ to $v$ in $G^{(t_1)}$. From what we have just shown, $P$ must intersect $S^{(t_0)}$ along $P$. Then clearly, $d_{G^{(t_1)}}^{(t_1)}(s, v) = d_{G^{(t_1)}}^{(t_1)}(s, w) + d_{G^{(t_1)}}^{(t_1)}(w, v)$. Since the vertex $w'$ preceding $w$ on $P$ is reachable from $s$ in $G^{(t_0)}_S$, the first part of the lemma implies that $d_{G^{(t_0)}}^{(t_0)}(s, w') \leq d_{G^{(t_0)}}^{(t_0)}(s, w') + 1 < \frac{2}{3}d + 1$ and $d_{G^{(t_0)}}^{(t_0)}(s, w) > \frac{2}{3}d$.

The latter implies that $d_{G^{(t_0)}}^{(t_1)}(w, v) = d_{G^{(t_0)}}^{(t_1)}(s, v) - d_{G^{(t_0)}}^{(t_1)}(s, w) \leq \frac{34}{33}d - d_{G^{(t_0)}}^{(t_0)}(s, w) < \frac{34}{33}d - \frac{2}{3}d < d$, showing one of the two inequalities in the second part of the lemma.

We show the other inequality by contradiction so assume that $d_{G^{(t_1)}}^{(t_0)}(s, w) > d$. Then $d_{G^{(t_1)}}^{(t_0)}(s, w) \geq d + 1$ so by the above $d_G(s, w)$ would have increased by more than $d + 1 - (\frac{34}{33}d + 1) = \frac{1}{33}d$ from time step $t_0$ to $t_1$. Combining this with $d_{G^{(t_1)}}^{(t_1)}(s, v) = d_{G^{(t_0)}}^{(t_1)}(s, w) + d_{G^{(t_0)}}^{(t_1)}(w, v)$, $d_{G^{(t_0)}}^{(t_1)}(w, v) \leq d_{G^{(t_1)}}^{(t_1)}(w, v)$, and the triangle inequality, we get

\[
d_{G^{(t_1)}}^{(t_1)}(s, v) - d_{G^{(t_0)}}^{(t_0)}(s, v) \geq d_{G^{(t_1)}}^{(t_1)}(s, w) + d_{G^{(t_0)}}^{(t_1)}(w, v) - (d_{G^{(t_0)}}^{(t_0)}(s, w) + d_{G^{(t_0)}}^{(t_1)}(w, v)) > \frac{1}{33}d
\]

This contradicts the assumption $d < d_{G^{(t_0)}}^{(t_0)}(s, v) \leq d_{G^{(t_1)}}^{(t_1)}(s, v) \leq \frac{34}{33}d$. We conclude that $d_{G^{(t_1)}}^{(t_1)}(s, w) \leq d$ and $d_{G^{(t_1)}}^{(t_1)}(w, v) \leq d$ which shows the second part of the lemma.
Bounding $|S|$ and running time: To bound $|S|$, consider the two parallel searches from $s$ and from $v$, respectively, in some update. As argued earlier, there cannot be an edge visited by both searches. Let $X$ resp. $Y$ be the set of vertices visited by the BFS from $s$ resp. $v$, excluding $L_s$ resp. $L_v$ and let $x = |X|$ and $y = |Y|$.

Assume first that $L = L_s$. Then all vertices in $Y \cup L_v$ become unreachable in $G_S$ once $L$ has been added to $S$. Since $\hat{d}(w)/\Delta \geq 1$ for each $w \in V$, since each BFS spends $\hat{d}(w)$ time to process edges incident to each vertex $w$, and since the two searches run in parallel, we have

$$|L| = O((x \log n)/d) = O\left(\frac{\log n}{d} \sum_{w \in X} \frac{\hat{d}(w)}{\Delta}\right) = O\left(\frac{\log n}{d} \sum_{w \in Y \cup L_v} \frac{\hat{d}(w)}{\Delta}\right)$$

Now, assume that $L = L_v$. Then all vertices of $Y \cup L_v$ become unreachable in $G_S$ once $L$ has been added to $S$ so again,

$$|L| = O((y \log n)/d) = O\left(\frac{\log n}{d} \sum_{w \in Y \cup L_v} \frac{\hat{d}(w)}{\Delta}\right)$$

In both cases, $|L|$ can be paid for by charging each vertex $w$ no longer reachable from $s$ in $G_S$ a cost of $O((\log n \cdot \hat{d}(w))/\Delta)$. Since a vertex is only charged once during the course of the algorithm, we get that for the final separator $S$ (and hence for each intermediate separator)

$$|S| = O\left(\frac{\log n}{d} \sum_{w \in V} \frac{\hat{d}(w)}{\Delta}\right)$$

$$= O\left(\frac{\log n}{d} \sum_{w \in V} \frac{d(w) + \Delta}{\Delta}\right)$$

$$= O\left(\frac{\log n(m + n \lceil m/n \rceil)}{d \lceil m/n \rceil}\right)$$

$$= O\left(\frac{n \log n}{d}\right)$$

where the last bound follows since we may assume that all vertices are initially reachable from $s$ in $G$, implying $m \geq n - 1$ and hence $\lceil m/n \rceil = \Theta(m/n)$. This shows the desired bound on $|S|$.

The running time cost of any two parallel searches can be charged to the total degree of the vertices that become unreachable from $s$ in $G_S$ after extending $S$ with $L$. This shows that the total running time of parallel searches over all updates is $O(m)$, as desired.

4 Deterministic Decremental APSP

In this section, we present our deterministic data structures for the exact resp. $(1 + \epsilon)$-approximate decremental APSP problem and show Theorems 1 and 2. In the following, let $G = (V, E)$ denote the decremental graph.
4.1 Exact distances

Let \( \rho = \frac{34}{33} \) and \( D_i = \rho^i \) for \( i = 0, \ldots, \lfloor \log_\rho n \rfloor \). For each \( i \) and each \( u \in V \), we give a data structure \( D_i(u) \) which for any query vertex \( v \) maintains a value \( d_i(u, v) \geq d_G(u, v) \) with equality if \( d_G(u, v) \in (D_i, D_{i+1}] \). In each update, these data structures will be updated in order of increasing \( i \).

Handling all-pairs shortest path distances up to at most \( 33 \log n \) can be done in \( O(mn \log n) \) using the data structure of Even and Shiloach so we only consider \( i \) such that \( D_i \geq 33 \log n \). This allows us to apply Lemma 3. Consider such an \( i \) and assume that we already have data structures for all values smaller than \( i \).

Data structure \( D_i(u) \) maintains a separator set \( S_i(u) \) using an instance \( S_i(u) \) of the data structure of Lemma 3 with \( s = u \), \( d = D_i \), and with \( D_{i-1}(u) \) playing the role of \( O \). At the beginning of each update, \( S_i(u) \) updates \( S_i(u) \). Then for each \( v \), if \( O \) reports that \( d(u, v) \) has increased from a value of at most \( D_i \) to a value strictly greater than \( D_i \), \( D_i(u) \) sets \( S_i(u, v) \) equal to the current separator set \( S_i(u) \); \( D_i(u) \) then sets up a priority queue \( Q_i(u, v) \) where elements are all \( S_i(u, v) \) with corresponding keys \( d_{i-1}(u, s) + d_{i-1}(s, v) \). During updates, whenever \( D_{i-1}(u) \) resp. \( D_{i-1}(s) \) reports that \( d_{i-1}(u, s) \) resp. \( d_{i-1}(s, v) \) increases, the key value of \( s \) in \( Q_i(u, v) \) increases by the same amount.

For each vertex \( v \), \( D_i(u) \) maintains \( d_i(u, v) \) as the min key value in \( Q_i(u, v) \). This completes the description of each structure \( D_i(u) \).

The overall data structure \( D \) maintains a priority queue \( Q(u, v) \) for each vertex pair \( (u, v) \) with an element for each \( i \) with key value \( d_i(u, v) \). For \( i \) in increasing order, \( D \) updates \( D_i(u) \) for each \( u \). Whenever a data structure \( D_i(u) \) increases a value \( d_i(u, v) \), the corresponding key in \( Q(u, v) \) is increased accordingly. On a query \( (u, v) \), \( D \) reports the min key value in \( Q(u, v) \).

**Correctness:** Consider a vertex pair \( (u, v) \) at any time step \( t_1 \) in the sequence of edge deletions. If \( d_G^{(t_1)}(u, v) = \infty \) then correctness is clear so assume otherwise and pick \( i \) such that \( d_G^{(t_1)}(u, v) \in (D_i, D_{i+1}] \) where \( D_i \geq 33 \log n \). Let \( t_0 \leq t_1 \) be the first time step such that \( d_G^{(t_0)}(u, v) \in (D_i, D_{i+1}] \). Note that \( S_i(u, v) = S_i(u)^{(t_0)} \). By the second part of Lemma 3 combined with the observation that no key value in \( Q_i(u, v) \) is below \( d_G(u, v) \), it follows that the min key value in \( Q_i(u, v) \) equals \( d_G^{(t_1)}(u, v) \). This shows correctness.

**Running time:** Consider an \( i \) \( \in \{0, \ldots, \lfloor \log_\rho n \rfloor \} \) with \( D_i \geq 33 \log n \) and a vertex \( u \in V \). We will show that maintaining \( D_i(u) \) takes \( O(n^2 \log^2 n) \) time using a standard binary heap. Total time over all \( i \) and \( u \) will thus be \( O(n^3 \log^3 n) \). This dominates the \( O(n^3 \log^2 n) \) time to maintain priority queues \( Q(u, v) \) and the \( O(mn \log n) \) time for the data structure of Even and Shiloach for small values of \( i \).

Maintaining \( S_i(u) \) takes a total of \( O(m) \) time by Lemma 3. The total number of elements in priority queues \( Q_i(u, v) \) over all \( v \in S_i(u, v) \) is \( O(n^2 \log n / D_i) \), again by Lemma 3. The number of increase-key operations for a single priority queue element \( s \) of \( Q_i(u, v) \) is \( O(D_i) \) which takes a total of \( O(D_i \log n) \) time. Over all elements of priority queues \( Q_i(u, v) \), this is \( O(n^2 \log^2 n) \).
Lower bound: We show that any data structure that explicitly maintains the distance matrix of \( G \) during the sequence of deletions must use \( \Omega(n^3) \) time.

Let the initial graph \( G \) consist of a simple path \( v_1 \to v_2 \to \cdots \to v_n \) augmented with edges \( e_i = (v_i, v_{i+2}) \) for \( i = 1, 3, 5, \ldots, n \) (assuming \( n \) is odd; otherwise, \( i = 1, 3, 5, \ldots, n-1 \)). Deleting the edges not on the simple path in any order, say, by increasing index, results in \( \Theta(n^2) \) vertex pairs each increasing their pairwise distance \( \Theta(n) \) times. Hence, there are \( \Omega(n^3) \) changes to the distance matrix, showing the lower bound.

Note that our choice of \( G \) for the lower bound is sparse; it is straightforward to extend the above to any edge density: simply take the above graph and arbitrarily insert additional edges to reach the desired density. Then consider a sequence starting with the deletion of these additional edges followed by the sequence above.

Reporting paths: It is easy to extend our data structure to efficiently answer queries for shortest paths (rather than shortest path distances) between any vertex pair \((u, v)\). Associated with the min element of \( Q(u, v) \) is a vertex \( s \) such that for the associated index \( i, d_i(u, v) = d_G(u, v) = d_G(u, s) + d_G(s, v), \) \( d_{i-1}(u, s) = d_G(u, s) \), and \( d_{i-1}(s, v) = d_G(s, v). \) Hence, by recursively querying for pairs \((u, s)\) and \((s, v)\), we get a shortest \( u \)-to-\( v \) path in \( G \) in time proportional to its length.

We have shown our first main result, Theorem 1.

4.2 Approximate distances

Let \( \epsilon > 0 \) be given. We now present our deterministic data structure for the \( (1 + \epsilon) \)-approximate variant of the problem.

The data structure is quite similar to the one for the exact variant so we only describe the changes needed. For \( i > 0 \) and \( u \in V \), we describe data structure \( D_i(u) \) and assume that we have data structures for values less than \( i \). As before, we only consider \( i \) with \( D_i \geq 33 \log n \).

Let \( \epsilon' > 0 \) be a value depending on \( \epsilon \) such that \((1 + \epsilon')^c = \rho \) for some \( c \in \mathbb{N} \); we will specify \( \epsilon' \) later. For \( j = 0, \ldots, c = \log_{1+\epsilon'} \rho \), let \( d_{i,j} = D_i(1 + \epsilon')^j \). This partitions each interval \((D_i, D_{i+1}]\) into \( c \) sub-intervals \((D_i(1 + \epsilon')^j, D_i(1 + \epsilon')^{j+1}]\) for \( j = 0, \ldots, c - 1 \).

\( D_i(u) \) maintains \( S_i(u) \) as in the exact version. For each \( v \in V, \) \( D_i(u) \) maintains an initially empty set \( S_i(u, v) \). Once \( D_{i-1}(u) \) reports that \( d_{i-1}(u, v) \) increased from a value of at most \( D_i(1 + \epsilon')^j \) to a value strictly greater than \( D_i(1 + \epsilon')^j \), \( D_i(u) \) sets \( S_i(u, v) \) equal to the current set \( S_i(u) \).

For each \( j = 0, \ldots, c - 1 \), a data structure \( D_{i,j}(u) \) maintains approximate distances \( d_{i,j}(u, v) \) for each \( v \) as follows. The following set is maintained:

\[
Q_{i,j}(u, v) = \left\{ s \in S_i(u, v) \mid \tilde{d}_{i-1}(u, s) + \tilde{d}_{i-1}(s, v) \leq (1 + \epsilon')^j d_{i,j} \right\}
\]

For ease of analysis, \( Q_{i,j}(u, v) \) is maintained as a queue in which every \( s \in Q_{i,j}(u, v) \) has key \( \tilde{d}_{i-1}(u, s) + \tilde{d}_{i-1}(s, v) \) and is removed from \( Q_{i,j}(u, v) \) (or increased to \( \infty \)) when this value exceeds \((1 + \epsilon')^j d_{i,j} \).

For each vertex \( v, \) \( \tilde{d}_{i,j}(u, v) = (1 + \epsilon')^j d_{i,j} \) if \( Q_{i,j}(u, v) \) contains at least one element and otherwise \( \tilde{d}_{i,j}(u, v) = \infty \).
Data structure $D_i(u)$ maintains for each $v$ a min-priority queue $Q_i(u)$ with an element of key value $\tilde{d}_{i,j}(u,v)$ for each $j$. On query $v$, it outputs $\tilde{d}_i(u,v) = \min \{ k, \tilde{d}_{i-1}(u,v) \}$ where $k$ is the min-key of this queue, i.e., $\tilde{d}_i(u,v) = \min \{ \tilde{d}_{i-1}(u,v), \min_j \tilde{d}_{i,j}(u,v) \}$.

The overall data structure $D$ works in the same manner as for the exact data structure.

**Correctness:** Consider any point during the sequence of edge deletions. We will show that for suitable choice of $\epsilon'$, the estimate $\tilde{d}(u,v)$ that $D$ outputs satisfies $d_G(u,v) \leq \tilde{d}(u,v) \leq (1 + \epsilon)d_G(u,v)$ for every vertex pair $(u,v)$.

We first show that $d_G(i,u,v) \leq \tilde{d}(u,v)$. It suffices to prove by induction on $i \geq 0$ that $d_G(u,v) \leq \tilde{d}_i(u,v)$. The proof holds for small $i$ such that $D_i < 33 \lfloor \log n \rfloor$ since then we use the data structure of Even and Shiloach, implying $\tilde{d}_i(u,v) = d_G(u,v)$. Now, consider an $i$ such that $D_i \geq 33 \lfloor \log n \rfloor$ and assume that the claim holds for smaller values than $i$. Since $\tilde{d}_i(u,v) = \min \{ \tilde{d}_{i-1}(u,v), \min_j \tilde{d}_{i,j}(u,v) \}$ and $\tilde{d}_{i,j}(u,v) \geq (1 + \epsilon')^{i} \tilde{d}_{i,j} \geq \tilde{d}_{i-1}(u,s) + \tilde{d}_{i-1}(s,v)$, the induction hypothesis implies $\tilde{d}_i(u,v) \geq d_G(u,v)$, showing the induction step. Thus, $d_G(u,v) \leq \tilde{d}(u,v)$.

To show that $\tilde{d}(u,v) \leq (1 + \epsilon)d_G(u,v)$, we prove by induction on $i \geq 0$ that during all updates and for all vertex pairs $(u,v)$, if $d_G(u,v) \in (0, D_{i+1}]$ then $\tilde{d}_i(u,v) \leq (1 + \epsilon')\tilde{d}_i(u,v)$. If we can show this then picking $\epsilon' \leq \ln(1 + \epsilon)/(|\log n|)$ gives $\tilde{d}(u,v) \leq (1 + \epsilon')^{\lfloor \log n \rfloor}d_G(u,v) \leq e^{\epsilon' \lfloor \log n \rfloor}d_G(u,v) \leq (1 + \epsilon)d_G(u,v)$ for every vertex pair $(u,v)$.

We only need to consider $i$ with $D_i \geq 33 \lfloor \log n \rfloor$ since otherwise, we use the data structure of Even and Shiloach. Assume inductively that the claim holds for values less than $i$.

Let $t_1$ be the current time step and consider a vertex pair $(u,v)$ with $d_G^{(t_1)}(u,v) \in (0, D_{i+1}]$. By the induction hypothesis, we may assume that $d_G^{(t_1)}(u,v) \in (D_i, D_{i+1})$. We may further assume that $d_G^{(t_1)}(u,v) > D_i(1 + \epsilon')i$ since otherwise,

$$d_G^{(t_1)}(u,v) \leq d_G^{(t_1)}(u,v) \leq D_i(1 + \epsilon')i < (1 + \epsilon')^{i}d_G^{(t_1)}(u,v).$$

Let $t_0 \leq t_1$ be the first time step where $d_G^{(t_0)}(u,v) > D_i(1 + \epsilon')i$. We must have $d_G^{(t_0)}(u,v) > D_i$ since otherwise, the induction hypothesis would imply $d_G^{(t_0)}(u,v) \leq d_G^{(t_0)}(u,v)(1 + \epsilon')^{i-1} \leq D_i(1 + \epsilon')^{i-1}$, contradicting the choice of $t_0$. Since also $d_G^{(t_0)}(u,v) \leq d_G^{(t_1)}(u,v) \leq D_i$, Lemma 3 implies that there is a vertex $s \in S_i^{(t_0)}(u) = S_i^{(t_1)}(u,v) = S_i^{(t_1)}(u,v)$ such that $d_G^{(t_1)}(u,v) = d_G^{(t_1)}(u,s) + d_G^{(t_1)}(s,v)$, $d_G^{(t_1)}(u,s) \leq D_i$, and $d_G^{(t_1)}(s,v) \leq D_i$.

Pick $j$ such that $d_G^{(t_1)}(u,v) \in (d_{i,j}, d_{i,j+1})$. By the induction hypothesis,

$$\tilde{d}_{i,j}(u,s) + \tilde{d}_{i,j}(s,v) \leq (1 + \epsilon')^{i-1}d_G^{(t_1)}(u,v) \leq (1 + \epsilon')^{i-1}d_{i,j+1} = (1 + \epsilon')^{i}d_{i,j}.$$

Hence, $Q_{i,j}(u,v)$ is non-empty at time step $t_1$ so

$$d_G^{(t_1)}(u,v) \leq d_G^{(t_1)}(u,v) = (1 + \epsilon')^{i}d_{i,j} \leq (1 + \epsilon')^{i}d_G^{(t_1)}(u,v).$$

This shows the induction step.

**Running time:** The analysis is similar to the one for exact distances. Pick an $i \in \{0, \ldots, \lceil \log \rho \rceil \}$ with $D_i \geq 33 \lfloor \log n \rfloor$. The total time to maintain $S_i(u)$ over all $u$ is $O(mn)$.  

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We start by giving a high-level description of our data structure and sketch its analysis.

Theorem. We obtain a speed-up by only maintaining a sampled subset of $S$ by Lemma Q number of elements in $\epsilon$ that $(1 + \epsilon')^d i,j$ if and only if the counter is strictly greater than 0. Every queue operation for $Q_{i,j}(u,v)$ can then be supported in $O(1)$ time. The number of elements in $Q_{i,j}(u,v)$ over all $u,v$, and $j$ is $O(cn^3 \log n/D_i) = O(n^3 \log n/(D_i/\epsilon'))$ by Lemma 3. This gives a total time bound of $O(mn + n^3 \log^2 n/(D_i(\epsilon')^2))$. This dominates the time spent on maintaining priority queues $Q_i(u)$.

Recall from above that $\epsilon' \leq \ln(1+\epsilon)/([\log \rho n])$. The only additional constraint on $\epsilon'$ is that $(1 + \epsilon')^c = \rho$ for some $c \in \mathbb{N}$. This can be achieved with $\epsilon' = \Theta(\ln(1+\epsilon)/([\log \rho n]))$. Hence, we get a time bound of $O(mn + n^3 \log^4 n/(D_i\epsilon'^2))$.

Note that this bound is no better than the exact data structure for small $D_i$. We thus consider a hybrid data structure that only applies our data structure when $D_i$ is above some distance threshold $d$ and otherwise applies the data structure of Even and Shiloach which takes a total of $O(mnd)$ time. Summing over all $D_i > d$ and applying a geometric sums argument, the total time for our hybrid data structure is

$$O(mn + \sum_{i:D_i>d} n^3 \log^4 n/(D_i\epsilon'^2)) = O(mn + n^3 \log^4 n/(d\epsilon'^2)))$$

Setting $d = n \log^2 n/\sqrt{m}$ gives Theorem 2. Showing the bound for reporting approximate shortest paths in the theorem is done in the same way as in Section 4.1.

5 Randomized Incremental APSP

In this section, we provide a randomized $(1 + \epsilon)$-approximate data structure and show Theorem 3. The data structure is Las Vegas and works against an adaptive adversary. In contrast, the data structures of [BHS02] and [Ber16] are both Monte Carlo and can only handle an oblivious adversary.

5.1 High-level description

We start by giving a high-level description of our data structure and sketch its analysis. We focus our attention on maintaining approximate distances close to the value $D_i$ from a single vertex $u$ and for now we assume an oblivious adversary.

Maintaining a sampled separator subset: Instead of maintaining each separator $S_{i,j}(u,v)$ (with associated with priority queue $Q_{i,j}(u,v)$) as the full vertex separator $S_i(u)$, we obtain a speed-up by only maintaining a sampled subset of $S_i(u)$. As long as this sampled subset certifies that there is a short two-hop path from $u$ to $v$, the data structure proceeds as in the previous section. When this is no longer the case, there might still be a short two-hop path from $u$ to $v$ through a non-sampled vertex $s$ in the full separator set $S_i(u)$. However, since there are no more sampled candidates, the expected number of vertices of
that provide a short two-hop path is small and we can update $S_{i,j}(u,v)$ to be this small subset.

It follows that $S_{i,j}(u,v)$ can be kept small at all times, which is needed to give a speed-up.

A speed-up using shallow in-trees: The problem with the data structure sketched above is that the entire set $S_i(u)$ had to be scanned in order to update $S_{i,j}(u,v)$ which means that the data structure will not be faster than our deterministic structure from the previous section. To deal with this, consider the following modification. The set $S_{i,j}(u,v)$ is updated as before by scanning over the entire set $S_i(u)$. Now, an in-tree $T(v)$ is grown from $v$ of radius at most $\epsilon'D_i$. Each vertex $v'$ in $T(v)$ then inherits the set of $v$, i.e., $S_{i,j}(u,v')$ is updated to the set $S_{i,j}(u,v)$ and this update is fast since $S_{i,j}(u,v)$ is small in expectation. This works since $v$ is a proxy for $v'$ in the sense that a short two-hop path from $u$ to $v'$ is also a short two-hop path from $u$ to $v$ (as $T(v)$ is an in-tree of small radius). Now, the time spent on the single scan of $S_i(u)$ can be distributed among all vertices of $T(v)$ and the number of such vertices must be at least $\epsilon'D_i + 1$ (if not, $v$ would be within distance $\epsilon'D_i$ from $u$).

Unfortunately, the time analysis for the above procedure breaks down if the in-trees grown during the sequence of updates overlap too much. We now sketch how to deal with this. Mark vertices of each in-tree grown so far. When the BFS procedure grows a new in-tree $T(v)$, this procedure is modified by having it backtrack at previously marked vertices which thus become leaves of $T(v)$; this set of marked leaves will be referred to as $L$ in the detailed description below.

Case 1, dealing with a large in-tree: If the number of unmarked vertices visited in $T(v)$ is greater than $\epsilon'D_i$, the above procedure and analysis can be applied; this is referred to as Case 1 in the detailed description below.

Case 2, dealing with a small in-tree: Otherwise, we are in Case 2; here we recall that $T(v)$ has small radius and observe that the only way to enter $T(v)$ from $G \setminus T(v)$ is through $L$. Hence, for every vertex $s$ in the union $\cup_{v' \in L} S_{i,j}(u,v')$, there is a good two-hop path from $u$ to $v$ through $s$. But since we know that there is only a small number of such vertices left (in expectation), this union must be small. Furthermore, the union must contain a good separator for every vertex in $T(v)$ (again because $T(v)$ has small radius and because $T(v)$ must be entered through $L$) and we thus have an efficient way to update $S_{i,j}(u,w)$ for all $w \in T(v)$.

Handling an adaptive adversary: Above we assumed an oblivious adversary. When the adversary is adaptive, we need to be more careful since the approximate distances reported might reveal information about which vertices have been sampled. To deal with this, we round up every two-hop distance on a given distance scale to the same upper bound value (this will only increase the weight of each two-hop path by a small factor so that the output to a query will still be $(1 + \epsilon)$-approximate). Hence, the rounded up approximate weight of a two-hop path $u \leadsto s \leadsto v$ is the same for every choice of "good" separator vertex $s$ regardless of whether it was sampled or not. It follows that our randomized structure outputs the same distance estimates as a slower deterministic algorithm that maintains the
full separator sets. Hence, the updates done by the adversary does depend on answers to previous approximate distance queries, as desired.

This completes the high-level description of our data structure.

5.2 The data structure

We now make the above formal. First, redefine \( \rho = \frac{34-\frac{1}{c}}{34} = \frac{67}{60} \) and pick \( \epsilon' \) such that \((1 + \epsilon')^c = \rho \) for some \( c \in \mathbb{N} \) and such that \( \rho(1 + \epsilon') \leq \frac{34}{33} \). For each \( u \) and \( i \) such that \( D_i \geq \frac{33}{34} \) lg \( n \), a separator \( S_i(u) \) is maintained with a data structure \( S_i(u) \) as in Section 4.

We extend the range of index \( j \) by 1 so that \( j \in \{0, \ldots, c + 1\} \). Each structure \( D_{i,j}(u) \) maintains a growing set \( M_{i,j}(u) \) of marked vertices; this set is initially empty. In the following, let \( U_{i,j}(u) = V \setminus M_{i,j}(u) \) denote the set of unmarked vertices and let \( G_{U_{i,j}(u)} \) denote the graph with vertex set \( V \) and containing the edges of \( G \) having at least one unmarked endpoint.

In each update, \( D_{i,j}(u) \) maintains \( S_{i,j}(u,v) \) and \( Q_{i,j}(u,v) \) for \( v \in V \) in the following way. For each \( v \in V \) and every vertex \( s \) added to \( S_i(u) \) in the current update, \( s \) is added to \( S_{i,j}(u,v) \) with some probability \( p \) to be fixed later. Note that only vertices \( v \) for which \( s \) is actually added to \( S_{i,j}(u,v) \) need to be processed. In Appendix B, we employ a different sampling scheme that avoids having to flip a coin for every vertex \( v \in V \) in every update.

For vertices \( v \) such that \( v \in M_{i,j}(u) \) or \( v \in U_{i,j}(u) \) and \( \tilde{d}_{i-1}(u,v) \leq D_i(1 + \epsilon')^{2i} \), no further processing is done.

Now, assume that \( v \in U_{i,j}(u) \) and that \( \tilde{d}_{i-1}(u,v) > D_i(1 + \epsilon')^{2i} \). If this inequality did not hold in the previous update, each vertex of \( S_{i,j}(u,v) \) is added to a new min-queue \( Q_{i,j}(u,v) \) with key values as in the previous section. Conversely, if the inequality did hold in the previous update, each new vertex added to \( S_{i,j}(u,v) \) in the current update is added to \( Q_{i,j}(u,v) \).

If the min key value of \( Q_{i,j}(u,v) \) is greater than \( d_{i,j}(1 + \epsilon')^{2i} \), \( D_{i,j}(u) \) grows an in-tree \( T(v) \) from \( v \) in \( G_{U_{i,j}(u)} \) up to radius \( \epsilon' D_i \).

There are now two cases: \( |V(T(v)) \setminus M_{i,j}(u)| > \epsilon' D_i \) and \( |V(T(v)) \setminus M_{i,j}(u)| \leq \epsilon' D_i \).

**Case 1:** If \( |V(T(v)) \setminus M_{i,j}(u)| > \epsilon' D_i \) then \( D_{i,j}(u) \) scans once over \( S_i(u) \) to find the subset of vertices \( s \in S_i(u) \) for which \( \tilde{d}_{i-1}(u,s) + \tilde{d}_{i-1}(s,v) \leq d_{i,j}(1 + \epsilon')^{2i} \). For each \( v' \in V(T(v)) \), \( Q_{i,j}(u,v') \) is set to contain exactly this subset of vertices \( s \) but with key value \( \tilde{d}_{i-1}(u,s) + \tilde{d}_{i-1}(s,v') \).

**Case 2:** If \( |V(T(v)) \setminus M_{i,j}(u)| \leq \epsilon' D_i \) then let \( L = V(T(v)) \cap M_{i,j}(u) \) and let \( Q = \cup_{v' \in L} Q_{i,j}(u,v') \). For each \( v' \in V(T(v)) \setminus L \), \( D_{i,j}(u,v) \) sets \( Q_{i,j}(u,v') \) to contain the subsets of elements \( s \in Q \) with \( \tilde{d}_{i-1}(u,s) + \tilde{d}_{i-1}(s,v) \leq d_{i,j}(1 + \epsilon')^{2i} \); their key values are \( \tilde{d}_{i-1}(u,s) + \tilde{d}_{i-1}(s,v') \).

In both cases, \( D_{i,j}(u) \) then marks all vertices of \( T(v) \), i.e., \( M_{i,j}(u) \leftarrow M_{i,j}(u) \cup V(T(v)) \).

Approximate distances \( \tilde{d}_{i,j}(u,v) \) are maintained by \( D_{i,j}(u) \) in a way similar to that in Section 4.2: \( \tilde{d}_{i,j}(u,v) = (1 + \epsilon')^{2i} d_{i,j} \) if the min key value of \( Q_{i,j}(u,v) \) is at most \( (1 + \epsilon')^{2i} d_{i,j} \) and otherwise \( \tilde{d}_{i,j}(u,v) = \infty \).

Data structures \( D_{i}(u) \) as well as the overall data structure \( D \) work exactly as in Section 4.2.
5.3 Correctness

Consider any point during the sequence of edge deletions. We will show that for suitable choice of \( \epsilon' \), we have \( d_G(u, v) \leq \tilde{d}(u, v) \leq (1 + \epsilon)d_G(u, v) \).

We will show that during all updates and for all vertex pairs \((u, v)\), if \( d_G(u, v) \in (0, D_{i+1}(1+\epsilon')) \) then \( \tilde{d}(u, v) \leq (1+\epsilon')^2d_G(u, v) \). By picking \( \epsilon' = \ln(1+\epsilon)/(2[\log_p n]) \), we will then get \( d_G(u, v) \leq \tilde{d}(u, v) \leq (1+\epsilon')^2[\log_p n]d_G(u, v) \leq e^{2[\log_p n]\epsilon'}d_G(u, v) \leq (1+\epsilon)d_G(u, v) \), as desired.

The proof is by induction on \( i \). The claim is clear for \( i \) with \( D_i < 33 \log n \) since then we use the data structure of Even and Shiloach. Now, consider an \( i \) with \( D_i \geq 33 \log n \) and assume that the claim holds for values less than \( i \). By the induction hypothesis, we only need to consider pairs \((u, v)\) with \( d_G(u, v) \in (D_i(1+\epsilon'), D_{i+1}(1+\epsilon'))\), i.e., \( d_G(u, v) \in (d_{i,j}, d_{i,j+1}) \) with \( j > 0 \).

We first show the following invariant for marked vertices that holds prior to each update over the entire sequence of updates:

**Invariant 1.** At the end of each update, for every \( w \in M_{i,j}(u) \) with \( d_G(u, w) \in (d_{i,j}, d_{i,j+1}) \), each shortest \( u\)-to-\( w \) path in \( G \) intersects a vertex \( s \in Q_{i,j}(u, w) \) such that \( d_G(u, s) \leq D_i \) and \( d_G(s, w) \leq D_i \).

**Proof.** The invariant is shown by induction on the rank of \( w \) in the order in which vertices are marked. Note that this is a proof by induction inside a step of the main proof by induction on \( i \); in addition to the induction hypothesis stated above, we may thus assume that the invariant holds for values less than \( i \). Additionally, for the current value of \( i \), we may assume by induction that the invariant holds for vertices of lower rank than \( w \).

Let \( t_1 \) be a time step with \( w \in M_{i,j}(u)(t_1) \) and \( d_G^{(t_1)}(u, w) \in (d_{i,j}, d_{i,j+1}) \), let \( t_0 \leq t_1 \) be the time step in which \( w \) was marked, and let \( r \) be the vertex from which an in-tree \( T(r) \ni w \) was grown in time step \( t_0 \). Let \( P \) be a shortest \( u\)-to-\( w \) path in \( G^{(t_1)} \).

We must have \( d_{i-1}^{(t_0)}(u, r) > D_i(1+\epsilon')^2 \) since otherwise, no processing would be done for \( r \) in time step \( t_0 \), contradicting that \( T(r) \) is grown in that time step. We also have \( d_G^{(t_0)}(u, r) > D_i(1+\epsilon') \) since otherwise the induction hypothesis would give the contradiction \( D_i(1+\epsilon') \geq d_G^{(t_0)}(u, r) \geq d_{i-1}^{(t_0)}(u, r)/(1+\epsilon')^{2(i-1)} > D_i(1+\epsilon')^{2i-2}(i-1) = D_i(1+\epsilon')^2 \).

By the triangle inequality and the fact that \( w \in T(r) \) and \( T(r) \) has radius at most \( \epsilon'D_i \), we get \( d_G^{(t_0)}(u, w) > d_G^{(t_0)}(u, r) - d_G^{(t_0)}(w, r) > D_i(1+\epsilon') - \epsilon'D_i = D_i \). Hence, \( D_i < d_G^{(t_0)}(u, w) \leq d_G^{(t_1)}(u, w) \leq d_{i,j+1} \leq 3\frac{4}{33}D_i \) so by Lemma 3, \( P \) intersects \( S^{(t_0)}(u) \) and for the first such intersection vertex \( s \) along \( P \), \( d_G^{(t_0)}(u, s) \leq d_G^{(t_1)}(u, s) \leq D_i \) and \( d_G^{(t_0)}(s, w) \leq d_G^{(t_1)}(s, w) \leq D_i \). We consider the two cases in the description of \( D_{i,j}(u) \): 

**Case 1:** It suffices to show that \( s \in Q_{i,j}^{(t_1)}(u, w) \).
We have $d_G^{(t_0)}(s, r) \leq d_G^{(t_0)}(s, w) + d_G^{(t_0)}(w, r) \leq (1 + \epsilon')D_i$. By the induction hypothesis,

$$d_{i-1}^{(t_0)}(u, s) + d_{i-1}^{(t_0)}(s, r) \leq (1 + \epsilon')^{2(i-1)}d_G^{(t_0)}(u, r)$$

$$\leq (1 + \epsilon')^{2i-2}(d_G^{(t_0)}(u, w) + \epsilon'D_i)$$

$$\leq (1 + \epsilon')^{2i-2}(d_G^{(t_0)}(u, w) + \epsilon'D_i + d_{i-1}(s, w))$$

$$= (1 + \epsilon')^{2i}d_{i,j},$$

so $s \in Q_{i,j}^{(t_0)}(u, w) = Q_{i,j}^{(t_1)}(u, w)$, showing maintenance of the invariant.

**Case 2:** We first show that $P$ must intersect the set $L$ formed when growing $T(r)$ in time step $t_0$. Since we are in Case 2, every leaf of $T(r)$ either belongs to $L$ or has no ingoing edges from vertices not in $T(r)$; otherwise, $T(r)$ would contain more than $\epsilon'D_i$ vertices since it is grown up to radius $\epsilon'D_i$. Hence, the only way that $P$ could not intersect $L$ would be if $P$ were fully contained in $T(r)$. But this is not possible since then $T(r)$ would contain at least $|P| + 1 \geq d_{i,j} + 1 > D_i \geq \epsilon'D_i$ unmarked vertices at the beginning of time step $t_0$, contradicting that we are in Case 2.

Thus, $P$ intersects $L$ and we have $w \notin L$ since $w$ was an unmarked vertex of $T(r)$ when growing this tree. Let $x$ be the last vertex of $P$ belonging to $L$. Since $x$ was marked earlier than $w$, the induction hypothesis implies that the subpath of $P$ from $u \sim x$ to $x \sim w$ is $\epsilon^D_i$. By the induction hypothesis, $d_G^{(t_0)}(s_x, x) \leq d_G^{(t_1)}(s_x, x) \leq D_i$. The latter implies $d_G^{(t_0)}(s_x, r) \leq (1 + \epsilon')D_i$. By the induction hypothesis, $d_{i-1}^{(t_0)}(u, s_x) + d_{i-1}^{(t_0)}(s_x, r) \leq (1 + \epsilon')^{2(i-1)}(d_G^{(t_0)}(u, s_x) + d_G^{(t_0)}(s_x, r)) = (1 + \epsilon')^{2i}d_G^{(t_0)}(u, r)$ which by the same calculations as in Case 1 is at most $(1 + \epsilon')^{2i}d_{i,j}$. Inspecting the execution of $D_{i,j}(u)$ in Case 2, it follows that $s_x \in Q_{i,j}^{(t_0)}(u, w) = Q_{i,j}^{(t_1)}(u, w)$. We have $s_x \in Q_{i,j}^{(t_0)}(u, x) \subseteq S_{i,j}^{(t_0)}(u)$. Since $s$ is the first vertex of $S_{i,j}^{(t_0)}(u)$ along $P$, $P$ can thus be decomposed into $u \sim s \sim s_x \sim x \sim w$ and we get $d_G^{(t_1)}(u, s_x) \leq D_i$ (as shown above) and $d_G^{(t_1)}(s_x, w) \leq d_G^{(t_1)}(s, w) \leq D_i$. This shows maintenance of the invariant with $s_x$ in place of $s$. 

Now, we continue with our proof by induction on $i$. Consider any vertex pair $(u, v)$ at the end of an update with $d_G(u, v) \in (d_{i,j}, d_{i,j+1}]$ and $j > 0$.

If $v \notin M_{i,j}(u)$ and $d_{i-1}(u, v) \leq (1 + \epsilon')^{2i}D_i$ then $d_G(u, v) \leq d_{i,j}(u, v) \leq (1 + \epsilon')^{2i}D_i < (1 + \epsilon')^{2i}d_G(u, v)$, as desired.

Now assume that $v \notin M_{i,j}(u)$ and $d_{i-1}(u, v) > (1 + \epsilon')^{2i}D_i$. Since $v$ was not marked in the current update, the min key value of $Q_{i,j}(u, v)$ at the end of the update is at most $d_{i,j}(1 + \epsilon')^{2i}$ so $d_G(u, v) \leq d_{i,j}(u, v) \leq (1 + \epsilon')^{2i}d_{i,j} < (1 + \epsilon')^{2i}d_G(u, v)$, as desired.

Finally assume that $v \in M_{i,j}(u)$. By Invariant 1, there is an $s \in Q_{i,j}(u, v)$ such that $d_G(u, v) = d_G(u, s) + d_G(s, v)$, $d_G(u, s) \leq D_i$, and $d_G(s, v) \leq D_i$. By the induction hypothesis, $d_G(u, v) \leq d_{i,j}(u, v) \leq d_{i,j}^{-1}(u, s) + d_{i,j}(s, v) \leq (1 + \epsilon')^{2(i-1)}d_G(u, v)$, as desired. This completes the inductive proof and correctness follows.
5.4 Running time

Maintaining separators $S_i(u)$ over all $u$ and $i$ takes $O(mn \log \rho n) = O(mn \log n)$ time by Lemma 3. For the remaining time analysis, we focus on a single data structure $D_{i,j}(u)$. It is useful in the following to regard this structure as handling an adversarial sequence of updates consisting of changes to approximate distances maintained by structures $D_i(v)$ for $i' < i$ and $v \in V$. We will give an expected time bound for $D_{i,j}(u)$ and we shall rely on the following key lemma.

**Lemma 4.** Let $r \in V$. If at some point in the sequence of updates, $D_{i,j}(u)$ grows an in-tree from $r$ then at the end of that update, the expected number of vertices $s \in S_i(u)$ satisfying $d_{i-1}(u,s) + \tilde{d}_{i-1}(s,r) \leq D_i(1 + \epsilon')^2$ is $O(\ln n/p)$. This bound holds against an adaptive adversary.

**Proof.** Assume that an in-tree is grown from $r$ at some time step $t_1$. Then we must have $\tilde{d}^{(t_1)}_{i-1}(u,r) > D_i(1 + \epsilon')^2$. Let $t_0 \leq t_1$ be the earliest time step where $\tilde{d}^{(t_0)}_{i-1}(u,r) > D_i(1 + \epsilon')^2$. For the analysis, consider a modification $D'_{i,j}(u)$ of $D_{i,j}(u)$ when which processing $r$ in each update $t \in \{t_0, \ldots, t_2\}$ applies a deterministic algorithm that maintains $Q_{i,j}(u,r)$ as the set of all vertices $s \in S_{i,j}(u)$ with corresponding key values $\tilde{d}^{(t_1)}_{i-1}(u,s) + \tilde{d}^{(t_1)}_{i-1}(s,r)$; here $t_2$ is the time step of $D'_{i,j}(u)$ in which $r$ is marked. Note that from time step $t_0$ to $t_2 - 1$, the min key value of $Q_{i,j}(u,r)$ is at most $d_{i,j}(1 + \epsilon')^2$. Thus, $t_1 \leq t_2$ and $D'_{i,j}(u)$ and $D_{i,j}(u)$ maintain exactly the same approximate distances for each time step $t \in \{t_0, \ldots, t_1 - 1\}$, namely $\tilde{d}^{(t)}_{i}(u,r) = d_{i,j}(1 + \epsilon')^2$. Hence, the output revealed to the adversary during these updates is the same when using $D'_{i,j}(u)$ as when using $D_{i,j}(u)$. From this and from the observation that the update done by the adversary in time step $t_1$ only depends on outputs from earlier updates, the sequence of updates from time step $t_0$ to $t_1$ is independent of which elements of $S_i(u)$ are sampled.

The above relates to the following experiment. We have a dynamic set $X$ undergoing a sequence of $k$ updates consisting of insertions and deletions of single elements. For $t = 0, \ldots, k$, let $n_t$ denote the number of elements in $X$ after the $t$’th update. Associated with $X$ we have a dynamic subset $Y$. The initial set $Y$ is obtained by sampling each element of the initial set $X$ independently with some probability $q$. Whenever a new element is inserted into $X$, it is added to $Y$ with probability $q$. We assume that the sequence of updates to $X$ is independent of which elements are sampled. Then for $t = 0, \ldots, k$, $\Pr[Y^{(t)} = \emptyset] = (1 - q)^{n_t}$.

Returning to the analysis of our algorithm, let $q = p$ and let $X$ denote the dynamic subset of $S_i(u)$ consisting of elements $s$ with $d_{i-1}(u,s) + \tilde{d}_{i-1}(s,r) \leq D_i(1 + \epsilon')^2$ during time steps $t_0, \ldots, t_2$. Note that $Y^{(t_1)} = \emptyset$. Let $T = \{t_0, \ldots, t_2\}$. For each $t \in T$,

$$\Pr[t_1 = t] \leq \Pr[Y^{(t)} = \emptyset] = (1 - p)^{|X^{(t)}|} \leq e^{-p|X^{(t)}|}.$$  

Let $T_1$ be the set of elements $t \in T$ where $|X^{(t)}| \leq \ln((m + 1)n)/p$ and let $T_2 = T \setminus T_1$. Since
\(|T_2| \leq |T| \leq m + 1\) and since \(|X(t)| \leq n\) for each \(t \in T\),
\[
E[|X(t_i)|] = \sum_{t \in T_1} |X(t)| \Pr[t_1 = t] + \sum_{t \in T_2} |X(t)| \Pr[t_1 = t]
\]
\[
\leq \ln((m + 1)n)/p \sum_{t \in T_1} \Pr[t_1 = t] + \sum_{t \in T_2} |X(t)| e^{-p|X(t)|}
\]
\[
\leq 3\ln \frac{n}{p} + \sum_{t \in T_2} |X(t)|/((m + 1)n)
\]
\[
\leq 3\ln \frac{n}{p} + 1
\]
\[= O(\ln n/p),\]
as desired. \(\square\)

**Corollary 1.** When a vertex \(v\) is marked, \(E[|Q_{i,j}(u, v)|] = O(\ln n/p)\) and this bound holds against an adaptive adversary.

**Proof.** Consider the update in which \(v\) is marked and let \(r\) be the root of the in-tree \(T(r)\) containing \(v\). If \(|T(r)| \geq \epsilon'D_i\) then \(Q_{i,j}(u, v) = Q_{i,j}(u, r) \subseteq S_i(u)\) and all \(s \in Q_{i,j}(u, r)\) satisfy the inequality of Lemma 4. In the case where \(|T(r)| < \epsilon'D_i\), let \(Q\) be as defined in the description of the data structure. Then vertices \(s \in Q \subseteq S_i(u)\) are only added to \(Q_{i,j}(u, v)\) if they satisfy the inequality of Lemma 4. The corollary now follows. \(\square\)

Now, we can bound the time spent by \(D_{i,j}(u)\). The total time spent on growing in-trees is \(O(m)\) since every edge \((w_1, w_2)\) visited must have \(w_1 \notin M_{i,j}(u)\) at the beginning of the BFS search and \(w_1 \in M_{i,j}(u)\) immediately afterwards and a vertex can never be unmarked. This also bounds the time spent on marking vertices.

The total expected number of sampled vertices added to \(Q_{i,j}(u, v)\) prior to \(v\) being marked is at most \(xp\) where \(x\) is the size of the set \(S_i(u)\) after the final update. By Lemma 3, \(x = O(n \log n/D_i)\). By Corollary 1, the expected size of \(Q_{i,j}(u, v)\) after \(v\) is marked is \(O(\ln n/p)\). Using the same argument as in the running time analysis of Section 4.2, the number of increase-key operations applied to a single element of \(Q_{i,j}(u, v)\) is \(O(\log n/\epsilon')\). Hence, the total expected time spent on operations on \(Q_{i,j}(u, v)\) is \(O((\log n \cdot p/D_i + \log n/p)^2 n/\epsilon)\).

Whenever \(D_{i,j}(u)\) grows an in-tree \(T(r)\) with \(|V(T(r)) \setminus M_{i,j}(u, v)| > \epsilon'D_i\), scanning \(S_i(u)\) takes \(O(n \log n/D_i)\) time by Lemma 3. Since all vertices of \(V(T(r)) \setminus M_{i,j}(u, v)\) are marked just after \(T(r)\) is grown and since vertices are never unmarked, the number of such trees over the course of the updates is at most \(n/(\epsilon'D_i)\) so the total time for all these scans is \(O(n^2 \log^2 n/(\epsilon D_i^2))\).

Whenever \(D_{i,j}(u)\) grows an in-tree \(T(r)\) with \(|V(T(r)) \setminus M_{i,j}(u, v)| \leq \epsilon'D_i\), the set \(\cup_{x \in L} Q_{i,j}(u, x)\) needs to be computed. Note that for each \(x \in L, E[|Q_{i,j}(u, v)|] = O(\log n/p)\) by Corollary 1. At least one edge \((y, x)\) ingoing to \(x\) belongs to \(T(r)\) and this edge is not part of any later grown in-tree since \(x\) is marked immediately after \(T(r)\) is grown. We charge a cost of \(O(\log n/p)\) to \((y, x)\) for computing \(Q_{i,j}(u, x)\). Over all \(x \in L\), this pays for computing \(\cup_{x \in L} Q_{i,j}(u, x)\) and we get a total expected time bound for this part of \(O(m \log n/p)\).

Summing the above over all \(u, v, i, j\), we get a total expected time bound for our data structure of
\[
O(mn/\epsilon + \sum_{i} \sum_{j} (n^3 \cdot p/(D_i \epsilon) + n^2/(pe) + n^3/(\epsilon D_i^2)) + mn/p).
\]
Since this bound is only fast for sufficiently large $i$, we pick a distance threshold $d$ and apply the algorithm of Even and Shiloach for distances of at most $d$ and our data structure for distances above $d$. By a geometric sums argument, our hybrid algorithm has an expected total time bound of
\[
\tilde{O}(mnd + mn/\epsilon + n^3 \cdot p/(d\epsilon^2) + n^2/(p\epsilon^2) + n^3/(d^2\epsilon^2) + mn/(p\epsilon))
\]
Setting the second and fifth terms equal to each other, we get $p = \tilde{\Theta}(\sqrt{md}/n)$ and the time bound simplifies to
\[
\tilde{O}(mnd + \sqrt{mn^2}/(\sqrt{d}\epsilon^{3/2}) + n^3/(\sqrt{md}\epsilon^{5/2}) + n^3/(d^2\epsilon^2)).
\]
We balance the first two terms by setting $d = \tilde{\Theta}(n^{2/3}/(m^{1/3}\epsilon))$ and we get a time bound of
\[
\tilde{O}(m^{2/3}n^{5/3}/\epsilon + n^{8/3}/(m^{1/3}\epsilon^2)),
\]
which shows the time bound of Theorem 3.

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A Related Work

Undirected APSP. In undirected graphs, maintaining all-pairs shortest paths with stretch \((1 + \epsilon)\) was first studied by Roditty and Zwick [RZ04] who achieved total running time \(O(mn)\) for unweighted graphs. Their data structure was then derandomized by Henzinger et al. [HKN16] which was in turn simplified in [GW20b]. In [HKN16], the authors also give a \((2 + \epsilon)\)-approximate all-pairs shortest path for undirected, unweighted graphs with total update time \(O(n^{2.5}/\epsilon)\). For high stretch, Henzinger et al. [HKN14b] gave an oblivious data structure with total update time \(O(mn^{1/k+o(1)\text{polylog } W})\) with stretch \(O(k^b)\) for any integer \(k \geq 1\). Chechik [Che18] obtained the same running time but obtained near-optimal stretch \(2(1 + \epsilon)k - 1\). Very recently, Chuzhoy and Saranurak [CS20] gave the first deterministic
decremental all pair shortest paths algorithm that achieves subcubic running time for any graph density. However, their algorithm only achieves a large constant approximation factor.

Finally, we also point out that there is a $(2 + \epsilon)$-approximate all-pairs shortest paths by Bernstein [Ber09] with $\tilde{O}(m)$ amortized update time for fully-dynamic graphs and that Abraham et al. [AC13] showed that sublinear amortized update time is possible for constant stretch.

APSP with Worst-case Update Time. For the fully-dynamic setting, Thorup also introduced the problem of maintaining APSP with worst-case update time. In [Tho05], he presents a deterministic data structure that achieves $\tilde{O}(n^{2.75})$ worst-case time. This bound was recently improved to $\tilde{O}(n^{2.66})$ by Abraham et al. [ACK17] who presented an adaptive randomized algorithm. Recently, Probst Gutenberg and Wulff-Nilsen [GW20c] gave a deterministic algorithm that breaks the $\tilde{O}(n^{2.75})$ update time bound by Thorup.

Single-Source Shortest Path. For the single-source shortest-path problem, a recent line of research [BR11; HKN14a] has culminated in a $(1 + \epsilon)$-approximate data structure by Henzinger et al. [HKN14b] for partially-dynamic undirected graphs. They achieve total update time $m^{1+o(1)}$, however, they need to assume an oblivious adversary. To overcome this restriction, Bernstein and Chechik, recently introduced a framework [BC16; BC17; Ber17] to maintain $(1 + \epsilon)$-approximate shortest paths against an adaptive adversary in time $\tilde{O}(\min\{n^2, mn^{3/4}\})$. Even more recently, Chuzoy and Khanna [CK19] extended their framework and showed that it can be used to improve the static problems of vertex-capacitated max-flow and sparsest vertex cut. Probst Gutenberg and Wulff-Nilsen [GW20b] recently presented a deterministic algorithm that improves on the former bounds for sparse graphs with total update time $mn^{0.5+o(1)}$. The existing data structures where further extended in [Ber+20] to be path-reporting.

In the directed, weighted setting, Henzinger et al. [HKN14c] presented a $(1 + \epsilon)$-approximate data structure with total update time $\tilde{O}(mn^{0.9+o(1)})$ for the decremental setting. A new approach by Bernstein et al. [GW20a; BGW20] has recently obtained running time $\tilde{O}(n^2, mn^{2/3})$ for decremental weighted digraphs, which is near-optimal when the graph is dense e.g. $m = \Theta(n^2)$. The simpler problem of maintain Single-Source Reachability in a decremental digraph was further solved to near-optimality [BPW19]. For Decremental Single-Source Reachability and SSSP, deterministic algorithms that improve over the classic ES-tree were given by Bernstein et al. [BPS20]. For the incremental setting, Probst Gutenberg et al. [PVW20] recently obtained a deterministic $(1 + \epsilon)$-approximate algorithm with total update time $\tilde{O}(n^2)$.

In both settings, the exact partially-dynamic SSSP problem was proven to require $\Omega(mn^{1-o(1)})$ total update time [RZ04; AW14; Hen+15; PVW20], assuming in various popular conjectures.

B Sampling technique

Here we describe how we use the sampling technique used by Wulff-Nilsen [Wul17] lemma 28 when $D_{i,j}(u)$ adds vertices to $S_{i,j}$. Remember that when a vertex $s$ is added to $S_i(u)$ it
is added to $S_{i,j}(u,v)$ with probability $p$ for every $v \in V$, but since flipping a coin for every vertex is too slow, we do something different to simulate that process:

Enumerate the vertices arbitrarily $v_1 \ldots v_n$ and let $\mathcal{E}_{k,s}$ denote the event that $v_k$ is the first to sample a vertex $s \in S_i(u)$. Clearly $p_{k,s} = \Pr[\mathcal{E}_{k,s}] = (1 - p)^{k-1}p$ for every $k \leq n$, and $p_{n+1,s} = (1 - p)^n$ i.e. the vertex wasn’t added to any $S_{i,j}(u,v)$. The algorithm can then sample from this distribution to pick $k$ and then recurse on the remaining $n - k$ vertices.

It remains to show how to sample from this distribution. Let $p_{i_1,i_2,s} = \sum_{i=i_1}^{i_2} p_{i,s}$ i.e. the probability that the we want $k$ is between $i_1$ and $i_2$. We precompute these values for $1 \leq i_1 \leq i_2 \leq n + 1$ in time $O(n^2)$ using dynamic programming, note that these numbers are the same for every $D_{i,j}$ so we only need to compute them once.

Using these numbers we sample $k$ as follows: Start with $i_1 = 1$, $i_2 = n + 1$ and $j = \lceil i_1 + i_2/2 \rceil$ With probability $\frac{p_{i_1,i_2}}{p_{i_1,i_2,s}}$ set $i_2 = j$ and $i_1 = j + 1$ otherwise, and recurse like binary search would until $i_1 = i_2 = k$. Clearly we can find $k$ in logarithmic time using the precomputed probabilities, so for every sampled vertex we can charge $O(\log n)$ running time to every $D_{i,j}(u)$ for which the vertex was added to $S_{i,j}(u,v)$, in total $O(|S_i(u)|p \log n)$ for every $D_{i,j}(u)$.