ON A NONLOCAL PROBLEM INVOLVING THE FRACTIONAL
\( p(x, \cdot) \)-LAPLACIAN SATISFYING CERAMI CONDITION

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ABSTRACT. The present paper deals with the existence and multiplicity of solutions for a class of fractional \( p(x, \cdot) \)-Laplacian problems with the nonlocal Dirichlet boundary data, where the nonlinearity is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz condition. To overcome the difficulty that the Palais-Smale sequences of the Euler-Lagrange functional may be unbounded, we consider the Cerami sequences. The main results are established by means of mountain pass theorem and Fountain theorem with Cerami condition.

1. Introduction. For several years great effort has been devoted to the study of linear and nonlinear equations involving fractional derivatives of functions of one or several variables, and in particular fractional Laplacian \((-\Delta)^s\), \(s \in (0, 1)\), is a more recent phenomenon. This nonlocal operator can be defined using Fourier analysis, functional calculus, singular integrals, or Lévy processes, etc. Its inverse is closely related to the famous potentials introduced by Marcel Riesz in the late 1930s. For a deeper comprehension, we refer the reader to [18] in which the authors established the equivalence between ten different definitions of fractional Laplacian. So this type of operator arises in a quite natural way in many different contexts, such as phase transitions, stratified materials, anomalous diffusion, crystal dislocation, conservation laws, ultrarelativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, geophysical fluid dynamics, and mathematical finance (see for instance [10, 21, 22]).

In the last few years, much research has tended to focus on the nonlocal nonlinear case. More precisely, the problems involving the fractional \( p \)-Laplacian operator \((-\Delta)^p\) have been investigated by many papers, see for example [5, 11, 14] and the references therein. More recently, there are many researchers extended the constant case to include the fractional variable exponent case [2, 3, 4, 6, 9, 16] where the authors introduced some definitions and basic properties of new fractional Sobolev spaces with variable exponents and obtained some existence results for related nonlocal fractional problems with variable exponents by means of different variational methods. For our first goal, we aim to keep on the study of such nonlocal problems.

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with variable exponent by considering the fractional $p(x,\cdot)$-Laplacian problem with Dirichlet boundary data of the following form
\[
(P_s) \begin{cases}
(-\Delta_{p(x,\cdot)})^s u(x) = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where:
- $\Omega$ is a Lipschitz bounded open domain of $\mathbb{R}^N$, $s \in (0,1)$.
- $p : \overline{\Omega} \to (1, +\infty)$ is a bounded continuous function with $Q := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$, $\Omega^c = \mathbb{R}^N \setminus \Omega$, $p^+ = \inf_{(x,y) \in Q^c} p(x,y)$, $p^- = \sup_{(x,y) \in Q^c} p(x,y)$, and $\bar{p}(x) = p(x,x)$ $\forall x \in \overline{\Omega}$.
- The operator $(-\Delta_{p(x,\cdot)})^s$ is the fractional $p(x,\cdot)$-Laplacian defined by
\[
[(\Delta_{p(x,\cdot)})^s u](x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} \, dy \text{ for all } x \in \mathbb{R}^N,
\]
where $p.v.$ is a commonly used abbreviation in the principal value sense. On the one hand, in the constant exponent case, $p(x,\cdot) = p \in (1, +\infty)$, it is reduced to the fractional $p$-Laplacian operator $(-\Delta)^s_{p}$. On the other hand, we remark that it is a fractional version of the well known $p(\cdot)$-Laplacian operator.

In contrast to the classical $p(x)$-Laplacian, which is a local operator, the fractional $p(x,\cdot)$-Laplacian is a paradigm of the vast family of nonlocal nonlinear operators, and this has immediate consequences in the formulation of basic questions such as the Dirichlet problem. For this, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ (which is different from the classical case of the $p(x)$-Laplacian) and not simply on $\partial \Omega$.

Before giving our hypotheses on the nonlinearity $f$, we first define the family of functions
\[
\mathcal{A} = \{ H_\lambda : H_\lambda(x,t) = f(x,t)t - \lambda F(x,t) \text{ with } \lambda \in [p^-, p^+] \},
\]
where $F(x,t) = \int_{0}^{t} f(x,\tau)d\tau$. Note that when $p(x,y) = p$ is constant, the family $\mathcal{A}$ consists of only one element, that is, $\mathcal{A} = \{ H_p(x,t) = f(x,t)t - pF(x,t) \}$.

Now, we make the following assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$:
- $(f_1) : |f(x,t)| < c_1 + c_2|t|^{q(x)-1}$ for all $(x,t) \in \Omega \times \mathbb{R}$, where $q : \overline{\Omega} \to (1, +\infty)$ is a bounded continuous function such that
\[
1 < p^+ < q^- = \inf_{x \in \overline{\Omega}} q(x) < p^*_s(x) = N\bar{p}(x)/[N - sp(x)] \text{ for any } x \in \overline{\Omega}.
\]
- $(f_2) : \lim_{|t| \to \infty} \frac{f(x,t)t}{|t|^{p^+}} = +\infty$ uniformly for a.e. $x \in \Omega$.
- $(f_3) : f(x,t) = o(|t|^{p^*-1})$ as $t \to 0$ uniformly for $x \in \Omega$.
- $(f_4) : \text{There exists a constant } \theta \geq 1 \text{ such that for any } \beta \in [0,1], t \in \mathbb{R}, \text{ and for all } H_\lambda \in \mathcal{A}, \gamma \in [p^-, p^+] \text{, the inequality}
\[
\theta H_\lambda(x,t) \geq H_{\gamma}(x,\beta t)
\]
holds for a.e. $x \in \Omega$.
- $(f_5) : f(x,-t) = -f(x,t) \text{ for all } (x,t) \in \Omega \times \mathbb{R}$. 

In the literature, when $s = 1$ and $p = 2$ in $(P_s)$, many researchers studied such type of problem. For example in [19] the authors considered the following problem

$$
\begin{align*}
-\Delta u(x) &= f(x,u), \\
u &\in W^{1,2}_0(\Omega) \setminus \{0\},
\end{align*}
$$

and they establish the existence of nontrivial nonnegative solutions to the above equation in $\mathbb{R}^2$ when the nonlinear term $f$ has subcritical or critical exponential growth and does not satisfy the well-known (AR) condition introduced by Ambrosetti and Rabinowitz in their celebrated paper [1]. In that context, for the nonlinear case, many paper investigated the problem of the following form

$$
\begin{align*}
-\Delta_p u &= f(x,u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{align*}
$$

By means of Morse theory, in [13] Fang et al. established the existence of nontrivial solutions for the above problem, where the nonlinearity is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz condition near infinity, or its dual version near zero. On the other hand, the (AR) condition has appeared in most of the studies for superlinear problems

(AR): There exist $\gamma > p$ and $M > 0$ such that

$$
0 < \gamma \int_0^t f(x,\tau)d\tau = \gamma F(x,t) \leq f(x,t)t \quad \text{for any } x \in \Omega, \text{ and all } |t| \geq M.
$$

The role of the (AR) condition is to ensure the boundedness of the Palais-Smale sequences of the Euler-Lagrange functional. This is very much crucial in the applications of critical point theory. However, although (AR) is a quite natural condition, it is somewhat restrictive and eliminates many nonlinearities. For example the function

$$
f(x,t) = |t|^{p-2}t \log(1 + |t|)
$$

does not satisfy the (AR) condition, but it satisfies our conditions $(f_1)$-$(f_5)$.

Next, for the variable exponent case, the equation $-\Delta_{p(x)} u = f(x,u)$ with Dirichlet boundary conditions have been studied by several authors, see for instance [23] in which the author used critical point theorems with Cerami condition to prove the existence and multiplicity of solutions for the aforementioned equation without the well-known (AR) type growth condition, see also [24] and the references given there.

As far as we know, no previous research has investigated the problem $(P_s)$ with Cerami condition and without the (AR) condition. Hence, Motivated by the aforementioned papers, our main work is to obtain the existence and multiplicity results of solutions to the problem $(P_s)$ via critical point theory.

The rest of this paper is organized as follows. In section 2, we give some definitions and fundamental properties of generalized Lebesgue spaces and fractional Sobolev spaces with variable exponent. In section 3, using the mountain pass theorem with Cerami condition, we discuss the existence of nontrivial weak solutions of problem $(P_s)$. Moreover, we show that problem $(P_s)$ has infinitely many (pairs) of solutions with unbounded energy by means of Fountain theorem with Cerami condition. Finally, in section 4, we give some examples and particular cases of the main results.
2. Preliminaries on variable exponent fractional Sobolev spaces. In order to study problem \((P_q)\), some properties on variable exponent Lebesgue spaces and Sobolev spaces, \(L^{q(x)}(\Omega)\) and \(W^{s,p(x)}(\Omega)\) respectively, are required and listed below. We refer the reader to [12, 17] and [2, 3, 6, 16] for exhaustive details on properties of those spaces. Moreover, we give several important properties of \(\rho(x, \cdot)\)-Laplacian operator.

Suppose that \(\Omega\) be a Lipschitz bounded open set in \(\mathbb{R}^N\). Let us denote

\[ C_+(\overline{\Omega}) = \{ q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega} \} . \]

For all \(q \in C_+(\overline{\Omega})\), we set

\[ q^+ = \sup_{x \in \Omega} q(x) \quad \text{and} \quad q^- = \inf_{x \in \Omega} q(x) \text{ such that } 1 < q^- \leq q(x) \leq q^+ < +\infty. \]

For any \(q \in C_+(\overline{\Omega})\), the variable exponent Lebesgue space is defined by

\[ L^{q(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{q(x)} dx < +\infty \right\}. \]

For any \(u \in L^{q(x)}(\Omega)\) we define the so-called Luxemburg norm on this space by

\[ \|u\|_{L^{q(x)}(\Omega)} = \|u\|_{q(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\} . \]

The variable exponent Lebesgue spaces have many properties similar to those of classical Lebesgue spaces, namely they are separable Banach spaces and they are reflexive if and only if \(1 < q^- \leq q^+ < +\infty\), the inclusions between Lebesgue spaces are also naturally generalized, that is, if \(0 < |\Omega| < \infty\) and \(q_1, q_2\) are two variable exponents such that \(q_1(x) < q_2(x)\) a.e. in \(\Omega\) then there exists a continuous embedding \(L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)\). Moreover, The Hölder inequality holds, that is, for \(q, \hat{q} \in C_+(\overline{\Omega})\) such that \(\frac{1}{q} + \frac{1}{\hat{q}} = 1\), if \(u \in L^{q(x)}(\Omega)\) and \(v \in L^{\hat{q}(x)}(\Omega)\), then

\[ \left| \int_\Omega uv dx \right| \leq \left( \frac{1}{q^-} + \frac{1}{\hat{q}^-} \right) \|u\|_{q(x)} \|v\|_{\hat{q}(x)} \leq 2\|u\|_{q(x)} \|v\|_{\hat{q}(x)}, \]

(1)

The modular, which is the mapping \(\rho_{q(x)} : L^{q(x)}(\Omega) \rightarrow \mathbb{R}\) given by

\[ \rho_{q(x)}(u) = \int_\Omega |u(x)|^{q(x)} dx, \]

is at many aspects an important tool in studying generalized Lebesgue spaces. It is worth noticing that the relation between the norm and the modular shows an equivalence between the topology defined by the norm and that defined by the modular.

**Proposition 1.** Let \(u \in L^{q(x)}(\Omega)\), \(\{u_k\} \subset L^{q(x)}(\Omega)\), \(k \in \mathbb{N}\), then we have

(i) \(\|u\|_{L^{q(x)}(\Omega)} < 1 \) (resp. \(= 1, > 1\)) \(\Leftrightarrow\) \(\rho_{q(x)}(u) < 1 \) (resp. \(= 1, > 1\)),

(ii) for \(u \in L^{q(x)}(\Omega) \setminus \{0\}\), \(\|u\|_{q(x)} = \lambda \Leftrightarrow \rho_{q(x)} \left( \frac{u}{\lambda} \right) = 1\),

(iii) \(\|u\|_{q(x)} < 1 \Rightarrow \|u\|_{q(x)}^{q^+} \leq \rho_{q(x)}(u) \leq \|u\|_{q(x)}^{q^-}\),

(iv) \(\|u\|_{q(x)} > 1 \Rightarrow \|u\|_{q(x)}^{q^-} \leq \rho_{q(x)}(u) \leq \|u\|_{q(x)}^{q^+}\),

(v) \(\lim_{k \to +\infty} \|u_k - u\|_{q(x)} = 0 \Leftrightarrow \lim_{k \to +\infty} \rho_{q(x)}(u_k - u) = 0\).
Now, let denote by Q the set
\[ Q := \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c), \quad \text{where} \quad \Omega^c = \mathbb{R}^N \setminus \Omega. \]
Let \( p : \overline{Q} \to (1, +\infty) \) be a continuous bounded function such that
\[ 1 < p^- = \inf_{(x,y) \in Q} p(x,y) \leq p(x,y) \leq p^+ = \sup_{(x,y) \in \overline{Q}} p(x,y) < +\infty \quad (2) \]
and
\[ p \text{ is symmetric, that is, } p(x,y) = p(y,x) \text{ for all } (x,y) \in \overline{Q}. \quad (3) \]
We set
\[ \bar{p}(x) = p(x,x) \text{ for any } x \in \overline{\Omega}. \]
For \( s \in (0, 1) \), and due to the nonlocality of the operator \((-\Delta_{p(x,.)})^s\), we introduce the general fractional Sobolev space with variable exponent as in \([2]\) as follows
\[ X = W^{s,p(x,y)}(Q) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable such that } u_{|\Omega} \in L^{\bar{p}(x)}(\Omega) \text{ with} \right\} \]
\[ \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy < +\infty, \quad \text{for some } \lambda > 0 \]
with the norm
\[ \|u\|_X = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_X, \]
where \([.]_X\) is a Gagliardo seminorm with variable exponent defined by
\[ [u]_X = [u]_{s,p(x,y)}(Q) = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy \leq 1 \right\}. \]
The space \((X, \|.,\|_X)\) is a separable reflexive Banach space. The modular on X is the mapping \( \rho_{p(x,y)} : X \to \mathbb{R} \) defined as follows
\[ \rho_{p(x,y)}(u) = \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \int_{\Omega} |u(x)|^{\bar{p}(x)} \, dx. \]
For any \( u \in X \), the modular norm is given by
\[ \|u\|_{\rho_{p(x,y)}} = \inf \left\{ \lambda > 0 : \rho_{p(x,y)} \left( \frac{u}{\lambda} \right) \leq 1 \right\}. \]
Next, let us denote by \( X_0 \) the following linear subspace of X
\[ X_0 = \{ u \in X : u = 0 \quad \text{a.e. in } \mathbb{R}^N \setminus \Omega \}, \]
with the norm
\[ \|u\|_{X_0} := [u]_X = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy \leq 1 \right\}. \]
The space \((X_0, \|.,\|_{X_0})\) is a separable reflexive Banach space (see \([2, \text{Lemma 2.3}]\)). We define the modular \( \rho_{p(x,y)}^0 : X_0 \to \mathbb{R} \) by
\[ \rho_{p(x,y)}^0(u) = \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy. \]
Consequently,
\[ \|u\|_{\rho_{p(x,y)}^0} = \inf \left\{ \lambda > 0 : \rho_{p(x,y)}^0 \left( \frac{u}{\lambda} \right) \leq 1 \right\} = [u]_X. \]
Similar to Propositions 1, \( \rho_{p(x,y)}^0 \) and \( \|u\|_{X_0} \) satisfy the following assertions.

**Proposition 2.** For any \( u \in X_0 \) and \( \{u_k\} \subset X_0 \), we have

(i) \( \|u\|_{X_0} < 1 \) (resp. \( 1, 1 \)) \( \Leftrightarrow \rho_{p(x,y)}^0(u) < 1 \) (resp. \( 1, 1 \)).
Lemma 2.2.\textit{ Let }$\Omega$\textit{ be a Lipschitz bounded domain in }$\mathbb{R}^N$\textit{ and let }$s \in (0, 1)$.\textit{ Let }$p : \overline{\Omega} \rightarrow (1, +\infty)$\textit{ be a continuous function satisfying (2) and (3) on }$\overline{\Omega}$\textit{ with }$sp^+ < N$. \textit{If }$r : \Omega \rightarrow (1, +\infty)$\textit{ be a continuous variable exponent such that }$1 < r^- \leq r(x) < p^*_s(x)$\textit{ for all }$x \in \Omega$.\textit{ Then, there exists a constant }$C = C(N, s, p, r, \Omega) > 0$\textit{ such that for any }$u \in X$,\textit{ \begin{equation*}
abla\nabla \left\| u \right\|_{L^{r(x)}(\Omega)} \leq C\left\| u \right\|_X. \end{equation*}$\textit{That is, the space }$X$\textit{ is continuously embedded in }$L^{r(x)}(\Omega)$.\textit{ Moreover, this embedding is compact.}\textit{ Remark 1. (1)– Theorem 2.1 remains true if we replace }$X$\textit{ by }$X_0$.\textit{ (2)– Since }$1 < p^- \leq \tilde{p}(x) < p^*_s(x)$\textit{ for all }$x \in \Omega$, \textit{then by Theorem 2.1, we have that, }$\| . \|_{X_0} = \| . \|_X$\textit{ and }$\| . \|_X$\textit{ are equivalent on }$X_0$.\textit{ Let us consider the functional }$L : X_0 \rightarrow \mathbb{R}$\textit{ defined by}\textit{ \begin{equation*}
abla\nabla L(u) = \int_Q \frac{1}{p(x, y)} \left| u(x) - u(y) \right|^{p(x, y)} dx dy. \end{equation*}$\textit{Then, using the same argument as in [3, Lemma 3.1], we have}\textit{ \begin{equation*}
abla\nabla \langle L'(u), \varphi \rangle = \int_Q \frac{|u(x) - u(y)|^{p(x, y) - 2}(u(x) - u(y)) \langle \varphi(x) - \varphi(y) \rangle}{|x - y|^{N + sp(x, y)}} dx dy, \end{equation*}$\textit{for all }$\varphi \in X_0$, \textit{where }$\langle . , . \rangle$\textit{ denotes the usual duality between }$X_0$\textit{ and its dual space }$X_0'$.\textit{ Lemma 2.2. ([6]). Assume that assumptions (2) and (3) are satisfied and that }$0 < s < 1$. \textit{Then, the following assertions hold: a) }$L' : X_0 \rightarrow X_0'$\textit{ is a bounded and strictly monotone operator. b) }$L'$\textit{ is a mapping of type }$(S_+)$\textit{, that is, if }$u_k \rightharpoonup u$\textit{ in }$X_0$\textit{ and }$
abla\nabla \limsup_{k \rightharpoonup +\infty} \langle L'(u_k) - L'(u), u_k - u \rangle \leq 0$, \textit{then }$u_k \rightarrow u$\textit{ in }$X_0$. c) }$L'$\textit{ is a homeomorphism.}\textit{ Next, we recall the definition of the Cerami condition (C) which is introduced by G. Cerami in [8].}\textit{ Definition 2.3. Let }$(\mathcal{E}, \| . \|)$\textit{ be a Banach space and }$\Phi \in C^1(\mathcal{E}, \mathbb{R})$. \textit{Given }$c \in \mathbb{R}$, \textit{we say that }$\Phi$\textit{ satisfies the Cerami }$c$\textit{ condition (we denote condition (C$_c$)), if (C$_1$): any bounded sequence }$\{u_n\} \subset \mathcal{E}$\textit{ such that }$\Phi(u_n) \rightarrow c$\textit{ and }$\Phi'(u_n) \rightarrow 0$\textit{ has a convergent subsequence, (C$_2$): there exist constants }$\delta, R, \beta > 0$\textit{ such that}\textit{ \begin{equation*}
abla\nabla \| \Phi'(u) \|_{\mathcal{E}^*}, \| u \| \geq \beta \quad \text{for all } u \in \Phi^{-1}([c - \delta, c + \delta]) \quad \text{with} \quad \| u \| \geq R. \end{equation*}$
If \( \Phi \in C^1(X, \mathbb{R}) \) satisfies condition \((C_c)\) for every \( c \in \mathbb{R} \), we say that \( \Phi \) satisfies condition \((C)\).

Note that condition \((C)\) is weaker than the Palais-Smale \((PS)\) condition. However, it was shown in [7] that from condition \((C)\) it is possible to obtain a deformation lemma, which is fundamental in order to get some min-max theorems. More precisely, let us recall the following version of the mountain pass lemma with Cerami condition lemma, which is fundamental in order to get some min-max theorems. More precisely, let us recall the following version of the mountain pass lemma with Cerami condition which will be used in the sequel.

**Proposition 3.** Let \((E, ||.||)\) a Banach space, \( \Phi \in C^1(X, \mathbb{R}) \), \( u_0 \in E \) and \( r > 0 \), be such that \( \|u_0\| > r \) and

\[
  b := \inf_{\|u\|=r} \Phi(u) > \Phi(0) \geq \Phi(u_0).
\]

If \( \Phi \) satisfies the condition \((C_c)\) with

\[
  c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)),
\]

\[
  \Gamma := \{ \gamma \in C([0,1], X) \mid \gamma(0) = 0, \gamma(1) = u_0 \}.
\]

Then \( c \) is a critical value of \( \Phi \).

**Remark 2.** Since \( X_0 \) is a separable and reflexive Banach space, there exist \( \{e_n\}_{n=1}^\infty \subset X_0 \) and \( \{e_n^*\}_{n=1}^\infty \subset X^* \) such that

\[
  \langle e_n^*, e_m \rangle = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}
\]

Hence, \( X_0 = \text{span}\{e_n, \ n = 1, 2, \ldots \} \) and \( X_0^* = \text{span}\{e_n^*, \ n = 1, 2, \ldots \} \). For \( k = 1, 2, \ldots, \) denote

\[
  X_k^0 = \text{span}\{e_k\}, \quad Y_k = \oplus_{i=0}^k X_0^k, \quad Z_k = \overline{\oplus_{i=k}^\infty Y_0^k}.
\]

Next, we introduce the Fountain theorem with the condition \((C)\) as in [25].

**Proposition 4.** Assume that \( X \) is a separable Banach space, \( \Phi \in C^1(X, \mathbb{R}) \) is an even functional satisfying the Cerami condition \((C)\). Moreover, for each \( k = 1, 2, \ldots, \) there exist \( R_k > r_k > 0 \) such that

- (A1) : \( \inf_{\{u \in Z_k : ||u||=r_k\}} \Phi(u) \to +\infty \) as \( k \to \infty \),

- (A2) : \( \max_{\{u \in Y_k : ||u||=R_k\}} \Phi(u) \leq 0 \).

Then, \( \Phi \) has a sequence of critical values which tends to +\( \infty \).

Given two Banach spaces \( E \) and \( F \), the symbol \( E \hookrightarrow F \) means that \( E \) continuously embedded in \( F \) and also the symbol \( E \hookrightarrow \hookrightarrow F \) means that there is a compact embedding of \( E \) in \( F \). Moreover, we will use the symbols \( \to \) to denote the strong convergence and \( \rightharpoonup \) to denote the weak convergence. For simplicity, we use \( c_i \), to denote the general non-negative or positive constant.

3. Existence and multiplicity results. In this section, we state our main results and we establish their proofs.

**Definition 3.1.** we say that \( u \in X_0 \) is a weak solution of problem \((P_s)\), if

\[
  \int_Q \frac{|u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y))}{|x - y|^{N + sp(x,y)}} \, dxdy - \int_{\Omega} f(x, u) \phi \, dx = 0,
\]

for all \( \phi \in X_0 \).
Let us consider the energy functional $\Psi : X \rightarrow \mathbb{R}$ associated to problem $(P_s)$ defined by

$$\Psi(u) = \int_Q \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy - \int_{\Omega} F(x,u) \, dx.$$ 

By conditions $(f_1)$ and Theorem 2.1 the functional $\Psi \in C^1(X,\mathbb{R})$ is well defined. Moreover, using the same argument as in [3, Lemma 3.1], we find that its Gâteaux derivative is given by

$$\langle \Psi'(u), \phi \rangle = \int_Q \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+sp(x,y)}} \, dx \, dy - \int_{\Omega} f(x,u) \phi \, dx,$$

for all $u, \phi \in X_0$. Thus, the weak solutions of $(P_s)$ corresponds to the critical points of $\Psi$.

Next, the functional $\Psi$ satisfies the following compactness result which plays the most important role in this paper.

**Lemma 3.2.** Assume that the assumptions $(f_1), (f_2)$ and $(f_4)$ are verified. Then, the functional $\Psi$ satisfies the Cerami condition $(C)$.

**Proof.** For any $c \in \mathbb{R}$, we first show that $\Psi$ satisfies the assertion $(C_1)$ of Cerami condition $(C)$ (see Definition 2.3). In fact, let $\{u_n\} \subset X_0$ be a bounded sequence such that

$$\Psi(u_n) \xrightarrow{n \to \infty} c \ \text{and} \ \Psi'(u_n) \xrightarrow{n \to \infty} 0. \quad (4)$$

Since $X_0$ is a reflexive space, then $\{u_n\}$ has a weakly convergent subsequence in $X_0$. Without loss of generality, we assume that $u_n \xrightarrow{n \to \infty} u$. By (4), we have

$$\langle \Psi'(u_n), u_n - u \rangle \xrightarrow{n \to \infty} 0,$$

that is,

$$\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))((u_n(x) - u_n(y)) - (u(x) - u(y)))}{|x-y|^{N+sp(x,y)}} \, dx \, dy$$

$$- \int_{\Omega} f(x,u_n)(u_n - u) \, dx \xrightarrow{n \to \infty} 0. \quad (5)$$

On the other hand, using $(f_1)$ and the Hölder inequality in (1), we obtain

$$\left| \int_{\Omega} f(x,u)(u_n - u) \, dx \right| \leq 2c_1 \|u_n\|_{q(x)} \|u_n - u\|_{q(x)} + 2c_1 \|u_n\|_{q(x)} \|u_n - u\|_{q(x)},$$

where $1/q(x) + 1/q(x) = 1$. As $1 < q^- \leq q(x) < p^*_s(x)$ for all $x \in \Omega$, by Remark 1-(1), $X_0$ is compactly embedded in $L^{q(x)}(\Omega)$. It follows that

$$\int_{\Omega} f(x,u_n)(u_n - u) \, dx \xrightarrow{n \to \infty} 0. \quad (6)$$

Combining (5) and (6), we get

$$\int_Q \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))((u_n(x) - u_n(y)) - (u(x) - u(y)))}{|x-y|^{N+sp(x,y)}} \, dx \, dy \xrightarrow{n \to \infty} 0,$$

that is,

$$\langle L'(u_n), u_n - u \rangle \xrightarrow{n \to \infty} 0. \quad (7)$$
Next, since \( u_n \xrightarrow{n \to \infty} u \), from (4), we have
\[ \langle \Psi'(u), u_n - u \rangle \xrightarrow{n \to \infty} 0. \]
Using the same argument as before, we deduce that
\[ \langle \mathcal{L}'(u), u_n - u \rangle \xrightarrow{n \to \infty} 0. \tag{8} \]
Hence, by (7) and (8), we infer
\[ \limsup_{k \to +\infty} (\mathcal{L}'(u_k) - \mathcal{L}'(u), u_n - u) \leq 0. \]
Then, as \( u_n \xrightarrow{n \to \infty} u \) in \( X_0 \) and since \( \mathcal{L}' \) is a mapping of type \((S_+)\) (see Lemma 2.2-(b)), it follows that \( u_n \xrightarrow{n \to \infty} u \) in \( X_0 \).

Now, we check that \( \Psi \) satisfies the assertion \((C_2)\) of Cerami condition \((C)\). We argue by contradiction. Indeed, we assume that there exists \( c \in \mathbb{R} \) and \( \{u_n\} \subset X_0 \) satisfying
\[ \Psi(u_n) \xrightarrow{n \to +\infty} c, \quad \|u_n\|_{X_0} \xrightarrow{n \to +\infty} \infty, \quad \text{and} \quad \|\Psi'(u_n)\|_{X_0^*} \|u_n\|_{X_0} \xrightarrow{n \to +\infty} 0. \tag{9} \]
Let
\[ \gamma_n = \frac{\rho_{p(x,y)}(u_n)}{\mathcal{L}(u_n)}. \]
We can assume that \( \|u_n\|_{X_0} > 1 \), for \( n \in \mathbb{N} \), then we have
\[ c = \lim_{n \to +\infty} \left\{ \Psi(u_n) - \frac{1}{\gamma_n} \langle \Psi'(u_n), u_n \rangle \right\} \\
= \lim_{n \to +\infty} \left\{ \frac{1}{\gamma_n} \int_{\Omega} f(x, u_n) u_n \, dx - \int_{\Omega} F(x, u_n) \, dx \right\}. \tag{10} \]
Denote \( \phi_n = \frac{u_n}{\|u_n\|_{X_0}} \), so \( \|\phi_n\|_{X_0} = 1 \), which implies that \( \{\phi_n\} \) is bounded in \( X_0 \).
Hence for a subsequence of \( \{\phi_n\} \) still denoted by \( \{\phi_n\} \), and \( \phi \in X_0 \), we obtain
\[ \phi_n \rightharpoonup \phi \text{ in } X_0, \tag{11} \]
\[ \phi_n \rightarrow \phi \text{ in } L^{q(x)}(\Omega), \tag{12} \]
\[ \phi_n(x) \rightarrow \phi(x) \text{ a.e. in } \Omega. \tag{13} \]
where \( q \) is given in assumption \((f_1)\).

- If \( \phi \equiv 0 \), then from the proof of Lemma 3.6 in [15], we can define a sequence \( \{t_n\} \subset [0, 1] \) such that
\[ \Psi(t_n u_n) = \max_{t \in [0, 1]} \Psi(t u_n). \tag{14} \]
If for \( n \in \mathbb{N} \), \( t_n \) satisfying (14) is not unique, then we choose the smaller positive value. Fix \( K > \frac{1}{2p^+} \), let \( \tilde{\phi}_n = (2p^+K)^{\frac{1}{p^+}} \phi_n \). So by (12), we have
\[ \tilde{\phi}_n \rightarrow 0 \text{ in } L^{q(x)}(\Omega). \]
From \((f_1)\), we have
\[ |F(x, t)| \leq c_1 |t| + c_2 |t|^{q(x)}. \]
Hence, from the continuity of \( t \mapsto F(., t) \), we obtain
\[ F(., \tilde{\phi}_n) \xrightarrow{n \to +\infty} 0 \text{ in } L^1(\Omega). \]
Thus,

$$\lim_{n \to +\infty} \int_{\Omega} F(x, \bar{\phi}_n) dx = 0. \quad (15)$$

Then, for $n$ large enough,

$$\left(2Kp_{\max}^{+}\right)^{\frac{1}{p^+}} \in (0, 1),$$

and

$$\Psi(t_n u_n) \geq \Psi(\bar{\phi}_n)$$

$$= \int_{Q} \frac{1}{p(x, y)} \left| (2p^+ K)^{\frac{1}{p^+}} \left( \phi_n(x) - \phi_n(y) \right) \right|^{p(x, y)} \frac{dx dy}{|x - y|^{N + sp(x, y)}} - \int_{\Omega} F(x, \bar{\phi}_n) dx$$

$$\geq \int_{Q} \frac{2p^+ K}{p^+} \left| (\phi_n(x) - \phi_n(y)) \right|^{p(x, y)} \frac{dx dy}{|x - y|^{N + sp(x, y)}} - \int_{\Omega} F(x, \bar{\phi}_n) dx$$

$$= 2K \rho_0^{p(x, y)}(\phi_n) - \int_{\Omega} F(x, \bar{\phi}_n) dx$$

$$\geq 2K - \int_{\Omega} F(x, \bar{\phi}_n) dx \geq K,$$

that is,

$$\lim_{n \to +\infty} \Psi(t_n u_n) = +\infty. \quad (16)$$

As $\Psi(0) = 0$ and $\Psi(u_n) \to c$, then for $n$ large enough, $t_n \in (0, 1)$ and

$$\rho_0^{p(x, y)}(t_n u_n) - \int_{\Omega} f(x, t_n u_n) t_n u_n dx = \Psi(t_n u_n), \quad t_n \to +\infty,$$

Combining (16) and (17), we find

$$\int_{\Omega} \left( \frac{1}{\gamma_n} f(x, t_n u_n) - F(x, t_n u_n) \right) dx = \frac{1}{\gamma_n} \rho_0^{p(x, y)}(t_n u_n) - \int_{\Omega} F(x, t_n u_n) dx$$

$$= \Psi(t_n u_n) \to +\infty,$$

where

$$\gamma_n = \frac{\rho_0^{p(x, y)}(t_n u_n)}{\mathcal{L}(t_n u_n)}.$$

From the definition of $\gamma_n$ and $\gamma_{t_n}$, we have $\gamma_n, \gamma_{t_n} \in [p^-, p^+]$, it follows that $H_{\gamma_n}, H_{\gamma_{t_n}} \in \mathcal{A}$. Then, by (14), we obtain

$$\int_{\Omega} \left( \frac{1}{\gamma_n} f(x, u_n) u_n - F(x, u_n) \right) dx = \frac{1}{\gamma_n} \int_{\Omega} H_{\gamma_n}(x, u_n) dx$$

$$\geq \frac{1}{\theta_{\gamma_n}} \int_{\Omega} H_{\gamma_{t_n}}(x, t_n u_n) dx$$

$$= \frac{\gamma_{t_n}}{\theta_{\gamma_n}} \int_{\Omega} \left( \frac{1}{\gamma_{t_n}} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) \right) dx.$$

Since $\frac{\gamma_{t_n}}{\theta_{\gamma_n}} > 0$, so from (18), we deduce that

$$\frac{1}{\gamma_n} \int_{\Omega} f(x, u_n) u_n dx - \int_{\Omega} F(x, u_n) dx \to +\infty, \quad n \to +\infty,$$
which contradicts (10).
• If $\phi \neq 0$, from (9), we write
  
  \[\rho_{p(x,y)}^0(u_n) - \int_\Omega f(x,u_n)u_n\,dx = \langle \Psi'(u_n), u_n \rangle = o(1)\|u_n\|_{X_0},\]
  
  namely
  
  \[1 - o(1) = \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|_{X_0}^{p^+}}\|\phi_n\|^{p^+} dx.\]  

Next, we define the set $\Omega_0 = \{x \in \Omega : \phi(x) = 0\}$.
So for any $x \in \Omega \setminus \Omega_0 = \{x \in \Omega : \phi(x) \neq 0\}$, we have
  
  \[|u_n| \xrightarrow{n \to +\infty} +\infty.\]

Then by (f2), we infer
  
  \[\frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\phi_n|^{p^+} \xrightarrow{n \to +\infty} +\infty.\]

Since $|\Omega \setminus \Omega_0| > 0$, we deduce via the Fatou lemma that
  
  \[\int_\Omega \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\phi_n|^{p^+} dx \xrightarrow{n \to +\infty} +\infty.\]  

By (f1) and (f2) there exists $L > 0$ such that \(\frac{f(x,t)t}{|t|^{p^+}} > L\) for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Moreover, it is easy to see that \(\int_{\Omega_0} |\phi_n|^{p^+} dx \xrightarrow{n \to +\infty} 0\). Hence, there exists $d > -\infty$ such that
  
  \[\int_{\Omega_0} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|\phi_n|^{p^+} dx \geq L \int_{\Omega_0} |\phi_n|^{p^+} dx \geq d > -\infty.\]  

Combining (19)-(21), we get a contradiction. Thus, the functional $\Psi$ satisfies the assertion (C2) of Cerami condition (C). This completes the proof of Lemma 3.2. \qed

Now, we are in position to state and prove our first existence result.

**Theorem 3.3.** Suppose that (f1)-(f4) are verified. If $p^+ < q^-$, then problem (Ps) has at least one nontrivial weak solution.

**Proof.** From Lemma 3.2, we infer that $\Psi$ satisfies the Cerami condition (C) on $X_0$. In order to apply Proposition 3, we need to check that $\Psi$ possesses the mountain pass geometry.

• Firstly, we show that there exists $R, a > 0$ such that
  
  \[\Psi(u) \geq a \quad \text{for any } u \in X_0 \quad \text{with } \|u\|_{X_0} = R.\]  

Indeed, Let $\|u\|_{X_0} < 1$. Then by Proposition 2, we get
  
  \[\Phi(u) \geq \frac{1}{p^+} \rho_{p(x,y)}^0(u) - \int_\Omega F(x,u)\,dx \geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} - \int_\Omega F(x,u)\,dx.\]  

Since, $p^+ < q^- \leq q(x) < p_s(x)$ for all $x \in \overline{\Omega}$, then by Remark 1-(1), we have that $X_0 \hookrightarrow L^{p^+}(\Omega)$ and $X_0 \hookrightarrow L^{q(x)}(\Omega)$, that is, there exist two positive constants $c_3, c_4 > 0$ such that

$$\|u\|_{p^+} \leq c_3 \|u\|_{X_0} \quad \text{and} \quad \|u\|_{q(x)} \leq c_4 \|u\|_{X_0} \quad \text{for all} \quad u \in X_0. \quad (24)$$

Let $\varepsilon > 0$ be small enough such that $\varepsilon < \frac{1}{p^+ c_3^3}$. Combining $(f_1)$ and $(f_2)$, we have

$$|F(x, t)| \leq \varepsilon |t|^{p^+} + c_\varepsilon |t|^{q(x)} \quad \text{for all} \quad (x, t) \in \Omega \times \mathbb{R}. \quad (25)$$

Then, using (23)-(25), Proposition 2, and Remark 1-(1), for all $\|u\|_{X_0}$ sufficiently small, we get

$$\Phi(u) \geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} - \varepsilon \int_{\Omega} |u|^{p^+} dx - c_\varepsilon \int_{\Omega} |u|^{q(x)} dx$$

$$\geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} - \varepsilon \|u\|_{p^+}^{p^+} - c_\varepsilon \|u\|_{q(x)}^{q^-}$$

$$\geq \frac{1}{p^+} \|u\|_{X_0}^{p^+} - \varepsilon c_3^{p^+} \|u\|_{X_0}^{p^+} - c_\varepsilon c_4^{q^-} \|u\|_{X_0}^{q^-}$$

$$= \left( \frac{1}{p^+} - \varepsilon c_3^{p^+} \right) \|u\|_{X_0}^{p^+} - c_\varepsilon c_4^{q^-} \|u\|_{X_0}^{q^-}.$$

As $p^+ < q^-$, then there exist two positive real number $R$ and $a$ such that (22) holds true.

- Secondly, we claim that there exists $u_0 \in X_0 \setminus B_R(0)$ such that

$$\Psi(u_0) < 0. \quad (26)$$

In fact, let $\phi_0 \in X_0 \setminus \{0\}$, by $(f_2)$, we can choose a constant $\mu > \frac{1}{p^+} \frac{\rho_0}{\int_{\Omega} |\phi_0|^{p^+} dx}$ and a constant $c_\mu > 0$ depending on $\mu$ such that

$$F(x, t) \geq \mu |t|^{p^+} \quad \text{for all} \quad |t| > c_\mu \quad \text{and uniformly in} \quad \Omega.$$
Let \( l > 1 \) be large enough, using the above inequality, we obtain
\[
\Psi(l\phi_0) = \int_{Q} \frac{1}{p(x, y)} \frac{|l(\phi_0(x) - \phi_0(y))|^{p(x, y)}}{|x - y|^{N + sp(x, y)}} dxdy - \int_{\Omega} F(x, l\phi_0)dx
\]
\[
\leq \frac{l^{p^+}}{p} \int_{Q} \frac{|\phi_0(x) - \phi_0(y)|^{p(x, y)}}{|x - y|^{N + sp(x, y)}} dxdy - \int_{\{l|\phi_0| > c_\mu\}} F(x, l\phi_0)dx
\]
\[
- \int_{\{l|\phi_0| \leq c_\mu\}} F(x, l\phi_0)dx
\]
\[
\leq \frac{l^{p^+}}{p} \rho_0^{p(x, y)}(\phi_0) - \int_{\{l|\phi_0| \leq c_\mu\}} F(x, l\phi_0)dx - \mu l^{p^+} \int_{\Omega} |\phi_0|^{p^+} dx
\]
\[
+ \mu l^{p^+} \int_{\{l|\phi_0| \leq c_\mu\}} |\phi_0|^{p^+} dx
\]
\[
= \frac{l^{p^+}}{p} \left( \frac{\rho_0^{p(x, y)}(\phi_0)}{p^+} \right) - \mu l^{p^+} \int_{\Omega} |\phi_0|^{p^+} dx + c_5
\]
which implies that \( \Psi(l\phi_0) \xrightarrow{l \to +\infty} -\infty \). Then there exists \( l_0 > 0 \) and \( u_0 = l_0\phi_0 \in X_0 \setminus \overline{B_R(0)} \) such that (26) hold true. Consequently, Proposition 3 guarantees that problem \((\mathcal{P}_s)\) has at least a nontrivial weak solution. This complete the proof. \( \square \)

Now, by means of the Fountain theorem (see Proposition 4), we can show the existence of infinitely many (pairs) of solutions with unbounded energy for problem \((\mathcal{P}_s)\).

**Theorem 3.4.** Assume that the conditions \((f_1), (f_2), (f_4)\) and \((f_5)\) hold. If \( p^+ < q^- \), then problem \((\mathcal{P}_s)\) possesses a sequence of weak solutions \( \{\pm u_n\}_{n=1}^\infty \) such that
\[
\Psi(\pm u_n) \to +\infty \quad \text{as} \quad n \to +\infty.
\]

In order to establish the proof of Theorem 3.4, we will use the mean value theorem in the following form:
For any \( \alpha \in C_+(\overline{\Omega}) \) and \( u \in L^{\alpha}(\Omega) \), there exists \( z \in \Omega \) such that
\[
\int_{\Omega} |u|^{\alpha}(x) dx = \|u\|_{\alpha(x)}^{\alpha(z)}.
\]
Indeed, on the one hand, using Proposition 1-\((ii)\), we have
\[
\rho(x) \left( \frac{u}{\|u\|_{\alpha(x)}} \right) = \int_{\Omega} \left( \frac{|u|}{\|u\|_{\alpha(x)}} \right)^{\alpha(x)} dx = 1.
\]
On the other hand, by the mean value theorem for integrals, there exists a positive constant \( \tilde{\alpha} \in [\alpha^-, \alpha^+] \) depending on \( \alpha \) such that
\[
\int_{\Omega} \left( \frac{|u|}{\|u\|_{\alpha(x)}} \right)^{\alpha(x)} dx = \left( \frac{1}{\|u\|_{\alpha(x)}} \right)^{\tilde{\alpha}} \int_{\Omega} |u|^{\alpha(x)} dx.
\]
Moreover, from the continuity of $\alpha$, there exists $z \in \Omega$ such $\alpha(z) = \bar{\alpha}$. Combining this fact with the above equalities, we deduce the relation (27).

Respecting the same notations in Remark 2, and in the similar manner in [24, Lemma 3.4], we can obtain the following auxiliary lemma.

**Lemma 3.5.** Let $r \in C_+(\overline{\Omega})$ such that $r(x) < p^*_s(x)$ for any $x \in \overline{\Omega}$, define

$$\theta_k = \sup \left\{ \|u\|_{L^{r^*}(\Omega)} : \|u\|_{X_0} = 1, u \in Z_k \right\}.$$

Then $\lim_{k \to +\infty} \theta_k = 0$.

**Proof.** of Theorem 3.4. We check that $\Psi$ verify the assumptions of Fountain theorem (see Proposition 4). Indeed, from (f3), $\Psi$ is an even functional. According to Lemma 3.2, $\Psi$ satisfies the the Cerami condition (C). Next, we need to verify that the assertions (A1) and (A2) are satisfied.

(A1): For all $u \in Z_k$ such that $r_k = \|u\|_{X_0} > 1$ ($r_k$ will be given below), using (f1), Proposition 2, Remark 1, and (27), we have

$$\Psi(u) = \int_Q \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} |x - y|^{N + sp(x,y)} \, dx \, dy - \int_\Omega F(x,u) \, dx \geq \frac{1}{p^+} \rho_p(x,y)(u) - c_5 \int_\Omega |u|^{q(x)} \, dx - c_6 \int_\Omega |u| \, dx \geq \frac{1}{p^+} \|u\|_{p^+}^{p^-} - c_5 \|u\|_{q(x)} - c_7 \|u\|_{X_0},$$

where $z \in \Omega$

$$\geq \begin{cases} \frac{1}{p^+} \|u\|_{p^+}^{p^-} - c_5 - c_7 \|u\|_{X_0} & \text{if } \|u\|_{q(x)} \leq 1 \\ \frac{1}{p^+} \|u\|_{p^+}^{p^-} - c_5 (\theta_k \|u\|_{X_0})^{q^+} - c_7 \|u\|_{X_0} & \text{if } \|u\|_{q(x)} > 1 \end{cases}$$

$$\geq \frac{1}{p^+} \|u\|_{p^+}^{p^-} - c_7 (\theta_k \|u\|_{X_0})^{q^+} - c_7 \|u\|_{X_0} - c_5$$

$$\geq r_k^{p^-} \left( \frac{1}{p^+} - c_5 \theta_k^{q^+} r_k^{q^+ - p^-} \right) - c_7 \|u\|_{X_0} - c_5.$$

We fix $r_k$ as follows

$$r_k = \left( q^+ c_5 \theta_k^{q^+} \right)^{\frac{1}{q^+ - p^-}}.$$

Then

$$\Psi(u) \geq r_k^{p^-} \left( \frac{1}{p^+} - \frac{1}{q^+} \right) - c_7 \|u\|_{X_0} - c_5.$$

According to Lemma 3.5, we know that $\lim_{k \to +\infty} \theta_k = 0$. Then, since $1 < p^- < q^+$, it follows that $r_k \to +\infty$ as $k \to +\infty$. Consequently,

$$\Psi(u) \to +\infty \text{ as } k \to +\infty.$$

Hence, the assertion (A1) is verified.

(A2): As $Y_k = \bigoplus_{i=0}^k X^k_0$, then $\dim Y_k < +\infty$, and since all norms are equivalent in
finite dimensional space, then there exists $b_k > 0$, for any $u \in Y_k$ with $\|u\|_{X_0}$ is big enough, we get
\[
\mathcal{L}(u) \leq \frac{1}{p^*}p_0^{p(x,y)}(u) \leq \frac{1}{p^*}\|u\|_{X_0}^{p^*} \leq b_k\|u\|_{P^+}^{p^*}.
\] (28)

Next, from $(f_2)$, there exists $L_k > 0$ such that for all $|t| \geq L_k$, we obtain
\[
F(x,t) \geq b_k|t|^{p^*} \text{ for any } x \in \Omega.
\]
Furthermore, by $(f_1)$, there exist a positive $M_k$ such that
\[
F(x,t) \leq M_k \text{ for all } (x,t) \in \Omega \times [-L_k, L_k].
\]
Thus, for all $(x,t) \in \Omega \times \mathbb{R}$, we deduce
\[
F(x,t) \leq 2b_k|t|^{p^*} - M_k.
\] (29)

Combining (28) and (29), for any $u \in Y_k$ such that $\|u\|_{X_0} = R_k > r_k$, we infer
\[
\Psi(u) = \mathcal{L}(u) - \int_{\Omega} F(x,u)dx
\]
\[
\leq b_k\|u\|_{P^+}^{p^*} - 2b_k\|u\|_{P^+}^{p^*} + M_k|\Omega|
\leq -b_k\|u\|_{P^+}^{p^*} + M_k|\Omega|
\leq -b_kc_k\|u\|_{X_0}^{P^*} + M_k|\Omega|
\]
Hence, from the above inequalities, for $R_k$ large enough ($R_k > r_k$), we find
\[
\max_{\{u \in Y_k : \|u\|_{X_0} = R_k\}} \Psi(u) \leq 0,
\]
which implies that the assertion $(A_2)$ is verified. Finally, in the light of Fountain theorem, we achieve the proof of Theorem 3.4. \qed

4. Examples. In this section, we give some particular cases of the nonlinearity $f$.

To this end, we start by the constant case:

- if $p = 2$, then, we can take $f(x,t) = 2t \log(1 + |t|)$,
- if $p \in (1, +\infty)$, then, we can take $f(x,t) = |t|^{p-2}t \log(1 + |t|)$. In this case $\mathcal{A} = \{H_p(x,t) = f(x,t)\}$. Moreover, $f$ does not satisfy the $(AR)$ condition, but it satisfies the conditions $(f_1)$-$(f_4)$.

Next, for the variable exponent case, we give the following example:
\[
f(x,t) = |t|^\alpha(x)-2t \log(1 + |t|), \text{ where } p^+ \leq \alpha(x) < q(x) \text{ for all } x \in \overline{\Omega}.
\]

Hence, problem $(P_\alpha)$ becomes
\[
(P) \left\{ \begin{array}{ll}
(-\Delta_{p(x,\cdot)})^s u(x) = |t|^\alpha(x)-2t \log(1 + |t|) \quad & \text{in } \Omega, \\
u = 0 \quad & \text{in } \mathbb{R}^N \setminus \Omega.
\end{array} \right.
\]
It is easy to see that $f$ satisfies the assumptions $(f_1)$-$(f_5)$, but it does not satisfy the $(AR)$ condition. Therefore, The results corresponding to Theorem 3.4. and Theorem 3.3 can be obtained and stay true for problem $(P)$. The problem and results are all new.
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