Integral manifolds of the reduced system in the problem of inertial motion of a rigid body about a fixed point

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Let us point out one more property of manifolds \( J_h \). Let \( Q \) be the closed ball in \( \mathbb{R}^3 \) of radius \( \pi \) with the center at the coordinates origin. Declare the diametrically opposite points of the ball boundary equivalent and denote by \( P \) the quotient space of the topological space \( Q \) with respect to this equivalence. For each \( \nu \in P \), we denote by \( \nu_\nu \in SO(3) \) the element for which \( \nu \) is the defining vector (see [1]). Let \( \omega_0 \in T_1^*S^2 \) have the coordinates \( \xi = 1, \eta = \zeta = 0, p_\eta = 1, p_\xi = p_\zeta = 0 \). The map \( \beta : P \to J_h \) defined as \( \beta(\nu) = (\nu_\nu \circ \alpha)^{-1}(\omega_0) \) is a homeomorphism. We use the map \( \beta \) for a geometric interpretation.

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Let $\lambda, \mu$ be the elliptic coordinates on $E^2$
\[ x^2 = a \frac{(a - \lambda)(a - \mu)}{(a - b)(a - c)}, \quad y^2 = b \frac{(\lambda - b)(b - \mu)}{(a - b)(b - c)}, \quad z^2 = c \frac{(\lambda - c)(\mu - c)}{(a - c)(b - c)}, \]
where $a = 1/A, b = 1/B, c = 1/C$. The elliptic coordinates change in the regions $a \geq \lambda \geq b \geq \mu \geq c$. Denote $F(t) = (a - t)(b - t)(c - t)/t$. The Hamilton function (1) takes the form
\[ H = 2 \frac{\lambda \mu}{\lambda - \mu} |F(\lambda)p_\lambda^2 - F(\mu)p_\mu^2|. \]
Let us introduce on $E^2$ the Liouville coordinates by the formulas
\[ u = \int_b^\lambda \frac{dt}{\sqrt{F(t)}}, \quad v = \int_c^\mu \frac{dt}{\sqrt{F(t)}}. \]
For them, the regions are
\[ 0 \leq u \leq m = \int_b^a \frac{dt}{\sqrt{F(t)}}, \quad 0 \leq v \leq n = \int_c^b \frac{dt}{\sqrt{F(t)}}. \]
In Fig. 1 we show parametric curves of $u$ and $v$ on the ellipsoid. In the coordinates $(u, v)$,
\[ H = 2[V(v) - U(u))]^{-1}(p_u^2 + p_v^2), \]
where $U(u) = 1/\lambda(u), V(v) = 1/\mu(v)$. Note that $dU/du = -\lambda^{-2} \sqrt{F(\lambda)}$, i.e., $dU/du = 0$ at $u = 0$, $u = m$ and $dU/du < 0$ at $0 < u < m$. Similarly, $dV/dv = 0$ at $v = 0$, $v = n$ and $dV/dv < 0$ at $0 < v < n$.

**Figure 1: Coordinates on the ellipsoid.**

In the domain where $u$ and $v$ are local coordinates the restriction of the initial system to the manifold $J_h$ admits the integrals
\[ p_u^2 + hU(u) = h\kappa, \quad p_v^2 - hV(v) = -h\kappa. \]
Denote by $J_{h, \kappa}$ the subset of $J_h$ defined by equations (2). The admissible values of $\kappa$ are $A \leq \kappa \leq C$. Let us find out the topological type of the integral manifolds $J_{h, \kappa}$ in the following cases: 1) $A \leq \kappa \leq B$; 2) $B \leq \kappa \leq C$; 3) $\kappa = B$.

Let $\mathcal{W} = E^2 \setminus \{u = 0\}$ and $\mathcal{S} = E^2 \setminus \{v = n\}$ be the regions on the ellipsoid surface. In them, we introduce the local coordinates $\mathcal{W} = \{(w, \varphi \bmod 4n\}$, $\mathcal{S} = \{(s, \theta \bmod 4m\}$ similar to cylindrical ones putting

$w = \begin{cases} u & \text{при } x \leq 0; \\ 2m - u & \text{при } x \geq 0. \end{cases}$ \hspace{1cm} $s = \begin{cases} v & \text{при } z \leq 0; \\ -v & \text{при } z \geq 0. \end{cases}$

$\varphi = \begin{cases} v & \text{при } y \geq 0, z \geq 0; \\ 2n - v & \text{при } y \leq 0, z \geq 0; \\ 2n + v & \text{при } y \leq 0, z \leq 0; \\ 4n - v & \text{при } y \geq 0, z \leq 0, \end{cases}$ \hspace{1cm} $\theta = \begin{cases} u & \text{при } x \geq 0, y \geq 0; \\ 2m - u & \text{при } x \leq 0, y \geq 0; \\ 2m + u & \text{при } x \leq 0, y \leq 0; \\ 4m - u & \text{при } x \geq 0, y \leq 0. \end{cases}$
It is easily shown that these coordinates are compatible with the smooth structure of the ellipsoid.

Let us consider the cases 1 – 3.

If $A \leq \kappa < B$, then the motion takes place in the region $\mathfrak{M}$ and the equations admit the first integrals
\[ H_w = p_w^2 + hW(w) = h\kappa, \quad H_\varphi = p_\varphi^2 - h\Phi(\varphi) = -h\kappa, \]
where $W(w) = U(u(w))$, $\Phi(\varphi) = V(v(\varphi))$. The qualitative picture of the functions $W$ and $\Phi$ is shown in Fig. 2.

In Fig. 3 we show the phase portraits of one-dimensional systems corresponding to the Hamilton functions $H_w$ and $H_\varphi$. Each manifold $J_{h,\kappa}$ is the product of level lines of the functions $H_w$ and $H_\varphi$ defined by (3). Thus, $J_{h,A}$ is two non-intersecting circles (they correspond to the cross section of the ellipsoid by the plane $x = 0$ with two different directions of motion). If $A < \kappa < B$, then $J_{h,\kappa}$ consists of two two-dimensional tori each of which concentrically envelopes one of the circles out of $J_{h,A}$.

In Fig. 4, where the diametrically opposite points of the ball boundary are identified, we show the sets corresponding to the manifolds $J_{h,A}$ and $J_{h,C}$ under the homeomorphism $\beta : P \to J_h$. The union of the circles 1 and 2 is the set $\beta^{-1}(J_{h,C})$. The set $\beta^{-1}(J_{h,A})$ consists of the circles 3 and 4.

Now let us consider the case $\kappa = B$. We denote by $K_1$, $K_2$, $K_3$, and $K_4$ the umbilical points $(u = 0, v = n)$ on the ellipsoid surface lying respectively in the regions $\{x > 0, z > 0\}$, $\{x < 0, z > 0\}$, $\{x < 0, z < 0\}$, and $\{x > 0, z < 0\}$. 
Proposition 2. The cross section of the ellipsoid by the plane $y = 0$ is a closed geodesic of the metric $d\Sigma$. All geodesics starting from an umbilical point at $t = 0$ meet simultaneously at the opposite umbilical point.

Proof. Let us use the coordinates $(w, \varphi)$. Introducing the “reduced time” $\tau$ by the formula $d\tau = [\Phi(\varphi) - W(w)]^{-1}dt$ and using equations (3) with $\kappa = B$, we get the equations of geodesics in the form

$$\frac{dw}{d\tau} = \pm \sqrt{h(B - W(w))}, \quad \frac{d\varphi}{d\tau} = \pm \sqrt{h(\Phi(\varphi) - B)}.$$

(4)

Denote

$$F(w, w_0) = \int_{w_0}^{w} \frac{dw}{\sqrt{h(B - W(w))}}, \quad G(\varphi, \varphi_0) = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{h(\Phi(\varphi) - B)}}.$$

Let $w = f(\tau, w_0)$ and $\varphi = g(\tau, w_0)$ be the inverse for the dependencies $\tau = F(w, w_0)$ and $\tau = G(\varphi, \varphi_0)$ respectively. Equations (4) admit the solutions

$$(w \equiv 0, \varphi = g(\pm \tau, \varphi_0)), \quad (w \equiv 2m, \varphi = g(\pm \tau, \varphi_0)),$$

$$(w = f(\pm \tau, w_0), \varphi \equiv n), \quad (w = f(\pm \tau, w_0), \varphi \equiv 3n).$$

This proves the first statement.

Consider an arbitrary trajectory of equations (4) starting at a point $\{w_0, \varphi_0\}$ not belonging to the cross section $y = 0$. Let, for definition, this point lie in the first octant, i.e., $m < w_0 < 2m$, $0 < \varphi_0 < n$. The initial velocity may have four directions according to the choice of the signs in (4). Suppose, for example, that $dw/d\tau|_{\tau=0} > 0$, $d\varphi/d\tau|_{\tau=0} > 0$. Then (see Fig. 3) as $\tau \to +\infty$, the coordinates $w$ and $\varphi$ monotonously increase and $w \to 2m$, $\varphi \to n(\mod 4n)$. As $\tau \to -\infty$ we have monotonous decreasing $w \to 0$ and $\varphi \to -n(\mod 4n)$. 

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Therefore the chosen trajectory of \( \Phi \) asymptotically approaches \( K_1 \) as \( \tau \to +\infty \) and \( K_3 \) as \( \tau \to -\infty \). Another possible cases of the inial directions are considered analogously.

So, since the geodesics starting at an umbilical point can correspond only to the value \( \kappa = B \), each such geodesic meets the cross section \( y = 0 \) for the first time at the opposite umbilical point.

Let \( \gamma_1(t) \) and \( \gamma_2(t) \) be two geodesics such that \( \gamma_1(0) = \gamma_2(0) = K_3 \). Suppose that some time value \( t = t_0 > 0 \) corresponds to the value \( \tau = 0 \) of the “reduced time”. Let \( \gamma_1(t_0) = (w_1, \varphi_1) \), \( \gamma_2(t_0) = (w_2, \varphi_2) \). Then the dependency of \( \gamma_1 \) on the “reduced time” is \( w = f(\tau, w_1) \), \( \varphi = g(\tau, \varphi_1) \), and the equations of \( \gamma_2 \) are \( w = f(\tau, w_2) \), \( \varphi = g(\tau, \varphi_2) \). Denote by \( t_1 \) and \( t_2 \) the minimal positive values of \( t \) for which \( \gamma_1(t_1) = \gamma_2(t_2) = K_1 \). Then

\[
\begin{align*}
  t_1 &= \int_{-\infty}^{+\infty} \left[ \Phi(g(\tau, \varphi_1)) - W(f(\tau, \varphi_1)) \right] d\tau, \\
  t_2 &= \int_{-\infty}^{+\infty} \left[ \Phi(g(\tau, \varphi_2)) - W(f(\tau, \varphi_2)) \right] d\tau.
\end{align*}
\]

The integrals in (5) and (6) converge since the metric \( d\Sigma \) does not have singularities.

Let us show that \( t_1 = t_2 \). For this purpose we use the obvious relations

\[
f(\tau, w_1) = f(\tau - F(w_1, w_2), w_2), \quad g(\tau, w_1) = g(\tau - G(w_1, w_2), w_2)
\]

and the following almost obvious statement. Suppose that for a function \( \psi(\tau) (-\infty < \tau < +\infty) \) there exists such a point \( \tau_0 \) that \( \chi(\tau) = \psi(\tau + \tau_0) \) is an even function. If the integral

\[
\int_{-\infty}^{+\infty} [\psi(\tau) - \psi(\tau + k)] d\tau,
\]

with some constant \( k \) converges, then it equals zero. Using (7), we transform (6) as follows

\[
t_1 = \int_{-\infty}^{+\infty} \left[ \Phi(f(\tau, \varphi_2)) - W(f(\tau + G(\varphi_2, \varphi_1)) - F(w_2, w_1)) \right] d\tau.
\]

Then we subtract the equality (6):

\[
t_1 - t_2 = \int_{-\infty}^{+\infty} [W(f(\tau, w_2)) - W(f(\tau + k, w_2))] d\tau.
\]

Here

\[ k = G(\varphi_2, \varphi_1) - F(w_2, w_1) \]

does not depend on \( \tau \).

It is easy to check that \( W(f(\tau, w_2)) \) as a function of \( \tau \) satisfies the condition of the just formulated statement. For this, it is sufficient to choose \( \tau_0 \) in such a way that \( f(\tau_0, w_2) = m \). Consequently, \( t_1 = t_2 \). The proposition is proved.

Let us now describe the type of the set \( J_{h,b} \). The curves \( O_i = J_h \cap T_h^* E^2 \) \((i = 1, 2, 3, 4)\) are topological circles. According to Proposition 2, all trajectories starting at \( O_1 \) simultaneously cross \( O_3 \) and simultaneously return to \( O_1 \). Therefore this family of trajectories fills a closed flow tube, i.e., they fill a two-dimensional torus \( T_1 \) in \( J_h \). In the same way the family of geodesics crossing \( K_2 \) and \( K_4 \) fills a two-dimensional torus \( T_2 \) in \( J_h \). The tori \( T_1 \) and \( T_2 \) intersect by two circles corresponding to the cross section of the ellipsoid by the plane \( y = 0 \) with two different directions of motion.

In Fig. 3 we show how the set \( \beta^{-1}(J_{h,b}) \) is embedded in \( P \) (the diametrically opposite points of the ball boundary are identified). The regions \( I - IV \) are filled with the one-parameter families of the integral tori enveloping concentrically the circles \( I - 4 \) respectively (see Fig. 4).

References

[1] Kharlamov M.P. Reduction in mechanical systems with symmetry // Mekh. Tverd. Tela. – 1976. – N 8. – P. 4–18. \texttt{arXiv:1401.4393}