UNIFORM APPROXIMATION OF EXTREMAL FUNCTIONS IN WEIGHTED BERGMAN SPACES

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Abstract. We discuss approximation of extremal functions by polynomials in the weighted Bergman spaces $A^p_\alpha$ where $-1 < \alpha < 0$ and $-1 < \alpha < p - 2$. We obtain bounds on how close the approximation is to the true extremal function in the $A^p_\alpha$ and uniform norms. We also discuss several results on the relation between the Bergman modulus of continuity of a function and how quickly its best polynomial approximants converge to it.

1. Introduction

In this article we discuss uniform approximation of extremal functions in weighted Bergman spaces. In general, we approximate these functions by solutions to extremal problems restricted to spaces of polynomials.

Definition 1.1. For $1 < p < \infty$ and $-1 < \alpha < \infty$ we define the weighted Bergman space $A^p_\alpha$ to be the space of all analytic functions in $\mathbb{D}$ such that

$$
\|f\|_{p,\alpha} = \left( \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty,
$$

where $dA_\alpha = (\alpha + 1)^{-1}(1 - |z|^2)^\alpha dA(z)$ and $dA$ is Lebesgue measure.

For $1 < p < \infty$, it is known that the dual of $A^p_\alpha$ is isomorphic to $A^q_{\alpha^*}$, where $1/p + 1/q = 1$. Also, if $\phi \in (A^p_\alpha)^*$ and $k \in A^q_{\alpha^*}$ correspond to each other, then $\|\phi\|_{(A^p_\alpha)^*} \leq \|k\|_{A^q_{\alpha^*}} \leq C\|\phi\|_{(A^p_\alpha)^*}$, where $C$ is some constant depending of $p$ and $\alpha$.

Definition 1.2. Let $k \in A^q_{\alpha^*}$ be given, where $1 < q < \infty$ and $k$ is not identically 0. Let $F \in A^p_\alpha$ be such that $\|F\| = 1$ and $\text{Re} \int_{\mathbb{D}} F \overline{k} dA_\alpha$ is as large as possible, where $1/p + 1/q = 1$. There is always a unique function $F$ with this property. We say that $F$ is the extremal function for the integral kernel $k$, and also that $F$ is the extremal function for the functional $\phi$ defined by $\phi(f) = \int_{\mathbb{D}} f \overline{k} dA_\alpha$.

We do not usually discuss the case $p = 2$ because in this case $F$ is a scalar multiple of $k$.

It is known (see [4]) that the spaces $A^p_\alpha$, since they are subspaces of $L^p$ spaces, are uniformly convex. In [7], general results are proven about approximating extremal functions in uniformly convex spaces, and a proof is given there of the well known fact that extremal functions are unique in uniformly convex spaces. See [2,3] for more information on extremal problems in spaces of analytic functions. See also [8,10,12,14] for more information on regularity questions related the extremal problems we discuss.

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Definition 1.3. Let $f \in A^p_\alpha$. Suppose
\[ \|f(e^{it}) + f(e^{-it}) - 2f(\cdot)\|_{p,\alpha} \leq C|t|^\beta \]
for some constant $C$. We then say that $f \in \Lambda^{\ast,\beta,A^p_\alpha}$. Furthermore, we define $\|f\|_{\Lambda^{\ast,\beta,A^p_\alpha}}$ to be the infimum of the constants $C$ such that the above inequality holds.

We refer to functions in the $\Lambda^{\ast}$ classes as being (mean) Bergman-Hölder continuous (see [8]). We discuss several estimates that relate the mean Bergman-Hölder continuity of $A^p_\alpha$ functions to the minimum error in approximating these functions with polynomials of fixed degree. We apply these results to obtain estimates for how close the solution of an extremal problem is to the solution to the problem with the same linear functional posed over the space of polynomials of degree at most $n$. By using inequalities related to uniform convexity due to Clarkson [4] and Ball, Carlen and Lieb [1], we are able to obtain quantitative estimates for distance from approximate extremal functions to the true extremal functions.

The estimates just mentioned are all in the $A^p_\alpha$ norm. However, our goal is to approximate (in certain cases) extremal functions in the uniform norm (i.e. the $L^\infty$ norm). To do so, we use results from [8] to obtain bounds on the $C^{\beta}$ norm of the extremal functions and the functions approximating them for certain $\beta$, as long as the integral kernels are sufficiently regular. We also use Theorem 4.2, which allows us to conclude that two functions that are each not too large in the $C^{\beta}$ norm and that are close in the $A^p_\alpha$ norm must actually be close in the uniform norm. In stating the theorems, we do not aim for the most general estimates possible; however, the estimates we state do apply to the case where $k$ is a polynomial, or even in $C^2(\mathbb{D})$.

We note that in [11], Khavinson and Stessin derive Hölder regularity results for extremal problems in unweighted Bergman spaces, However, they do not state explicit bounds on the exponent $\beta$ or on the $C^{\beta}$ norm of the extremal function, so we cannot use their result to get explicit bounds on extremal functions.

The following lemma about the uniform convexity of $L^p$ will be needed. The inequality for $1 < p \leq 2$ can be proved from Theorem 1 in [4]. The other inequality follows from equation (3) in Theorem 2 in [4].

Lemma 1.1. Let $\|f\|_p = \|g\|_p = 1$ and $(1/2)\|f + g\|_p > 1 - \delta$. Let $\|f - g\|_p = \epsilon$.
If $1 < p \leq 2$ then $\epsilon < \sqrt{p-1}\delta^{1/2}$. If $p \geq 2$ then $\epsilon < 2p^{1/p}\delta^{1/p}$.

2. Mean Hölder Continuity and Best Polynomial Approximation

In this section we discuss several results relating mean Hölder continuity of functions to their distance from the space of polynomials of degree at most $n$. Some of these results are used in the rest of the paper. The proofs of these results are similar to the proofs for similar results about classical Hölder continuity that can be found in [15], Volume 1, starting on p. 115.

Definition 2.1. Let $f \in A^p_\alpha$. We define
\[ E^p_\alpha(f) = \min\{\|f - P\|_{p,\alpha} : P \text{ is a polynomial of degree at most } n\}. \]

Theorem 2.1. Let $0 < \beta < 1$. Suppose that $\|f\|_{\Lambda^{\ast,\beta,A^p_\alpha}} = M$. Let
\[ A_\beta = \frac{2^{1+\beta}}{\pi} \int_0^\infty |\cos(t) - \cos(2t)|t^{\beta-2}dt. \]
Then
\[ E_{n}^{p,\alpha} \leq A_{\beta}n^{-\beta}\|f\|_{\Lambda^{\ast},\beta,A_{\beta}^{\alpha}}. \]

**Proof.** Let \( M = \|f\|_{\Lambda^{\ast},\beta,A_{\beta}^{\alpha}}. \) Let \( f_{r} \) represent the function \( f \) restricted to the circle of radius \( r \). Let \( T_{n} \) be the best polynomial approximant of \( f \), let \( R_{n} = f - T_{n} \) be the remainder and let \( \rho_{k} \) be the \( k \)th Cesàro sum of the remainder. Let \( K_{m} \) be the the Fejér kernel for the \( m \)th Cesàro sum. Then \( K_{m} \) has \( L^{1} \) norm of 1, and Young’s inequality for convolutions shows that \( M_{p}(r,\rho_{k}) = \|R_{n}|r * K_{m}\|_{p} \leq M_{p}(r,R_{n})\|K_{m}\|_{1} = M_{p}(r,R_{n}) \). Let \( \sigma_{k} \) be the \( k \)th Cesàro sum of \( f \). From [13] eq. (13.4), p. 115, Volume 1 we see that
\[ \left(1 + \frac{n}{h}\right)\rho_{n+h-1} - \frac{n}{h}\rho_{n-1} = T_{n} + \left(1 + \frac{n}{h}\right)\rho_{n+h-1} - \frac{n}{h}\rho_{n-1}. \]
Using this equation with \( h = n \), subtracting \( f \) from both sides and using the fact that \( M_{p}(r,\rho_{k}) \leq M_{p}(r,R_{n}) \) shows that \( M_{p}(r,2\sigma_{2n-1} - \sigma_{n-1} - f) \leq 4M_{p}(r,R_{n}) \). Multiply by \((\alpha + 1)2r(1 - r^{2})^{\alpha}\) and integrate \( r \) from 0 to 1 to see that
\[ \|2\sigma_{2n-1} - \sigma_{n-1} - f\|_{A_{\beta}^{\alpha}} \leq 4\|R_{n}\|_{A_{\beta}^{\alpha}}. \]

Let \( \tau_{n} = 2\sigma_{2n-1} - \sigma_{n-1} \). Now (2.1)
\[ \tau_{n}(re^{ix}) - f(re^{ix}) = \frac{2}{\pi} \int_{0}^{\infty} \left[ f(re^{i(x+t/m)}) + f(re^{i(x-t/m)}) - 2f(re^{ix}) \right] \frac{h(t)}{t^{2}} dt \]
where \( h(t) = (\cos(t) - \cos(2t))/2 \). Apply Minkowski’s inequality to see that
\[ E_{2m-1}^{p,\alpha} \leq \|\tau_{n}(re^{ix}) - f(re^{ix})\|_{p,\alpha} \leq \frac{2}{\pi} \int_{0}^{\infty} t^{\beta}Mm^{\alpha-\beta}\left|\frac{h(t)}{t^{2}}\right| dt \]
\[ = A_{\beta}M\left(2m\right)^{-\beta}. \]
Since \( E_{2m}^{p,\alpha} \leq E_{2m-1}^{p,\alpha} \), the theorem follows. \( \square \)

We can also prove the following theorem.

**Theorem 2.2.** Let \( K \geq 0 \) be an integer. Suppose that \( \|D_{\beta}^{K}f(re^{i\theta})\|_{p,\alpha} \leq M \). Let
\[ C_{K} = \frac{4}{\pi} \int_{0}^{\infty} |H_{k}(t)| dt \]
where
\[ H_{0}(t) = h(t)/t^{2}, \quad H_{k}(t) = \int_{t}^{\infty} H_{k-1}(x) dx. \]
Then \( E_{n}^{p,\alpha} \leq 2^{K}C_{K}Mn^{-K} \).

**Proof.** Let \( f^{(n,\theta)}(re^{i\theta}) = \frac{\partial^{n}}{\partial\theta^{n}}f(re^{i\theta}) \). Then integrating by parts in equation (2.1) shows that
\[ \tau(re^{ix}) - f(re^{ix}) = \frac{2}{\pi m^{K}} \int_{0}^{\infty} \left[ f^{(k,\theta)}(re^{i(x+t/m)}) + (-1)^{K}f^{(K,\theta)}(re^{i(x-t/m)}) \right] H_{K}(t) dt \]
Applying Minkowski’s inequality shows that
\[ \|\tau_{n}(x) - f(x)\|_{p,\alpha} \leq C_{K}m^{-K}\|D_{\beta}^{n}f\|_{p,\alpha}. \]
As above, this implies that \( E_{n}^{p,\alpha} \leq C_{K}2^{K}Mn^{K} \). \( \square \)

Define the \( A_{\gamma}^{\alpha} \) modulus of continuity for \( f \) by
\[ \omega_{p,\alpha}(\delta,f) = \sup_{|z| \leq \delta} \|f(e^{it}z) - f(z)\|_{p,\alpha}. \]
Theorem 2.3. Let $k \geq 0$ be an integer. Suppose $D_{\theta}^k f$ has modulus of continuity $\omega_{p,\alpha}(\delta)$. Then

$$E_{n}^{p,\alpha}(f) \leq B_{k}\omega_{p,\alpha}\left(\frac{2\pi}{n}\right)n^{-K}$$

where $B_{k} = 2K C_{K+1}/\pi + 2K C_{K}$.

Let $f_{\delta}(z) = \frac{1}{2\pi} \int_{\delta}^{\delta} f(e^{it}z) dt$. Note that $D_{\theta} f_{\delta} = (D_{\theta} f)_{\delta}$. Minkowski’s inequality shows that $\|f_{\delta} - f\|_{p,\alpha} \leq \omega_{p,\alpha}(\delta, f)$. Let $f = f_{\delta} + g$. Then using the fundamental theorem of calculus, we see that

$$\|D_{\theta}^{K+1} f_{\delta}\|_{p,\alpha} = \|D_{\theta}^{K} f(ze^{i\delta}) - D_{\theta}^{K} f(ze^{-i\delta})\|_{p,\alpha} \leq 2\delta^{-1}\omega_{p,\alpha}(2\delta; D_{\theta}^{K} f)$$

Also $\|D_{\theta}^{K} g\|_{p,\alpha} \leq \omega_{p,\alpha}(\delta, D_{\theta}^{K} f)$.

Thus by Theorem 2.2

$$E_{n}^{p,\alpha}(f) \leq 2K^{K+1} C_{K+1} n^{-(K+1)}(2\delta)^{-1}\omega_{p,\alpha}(2\delta, D_{\theta}^{K} f) + 2K C_{K} n^{-K}\omega(\delta; D_{\theta}^{K} f).$$

Taking the supremum over $|t| < \delta$ in the inequality

$$\|f(\cdot) - f(e^{-2it} \cdot)\|_{p,\alpha} \leq \|f(\cdot) - f(e^{-it} \cdot)\|_{p,\alpha} + \|f(e^{-it} \cdot) - f(e^{-2it} \cdot)\|_{p,\alpha}$$

shows that $\omega_{p,\alpha}(2\delta, f) \leq 2\omega_{p,\alpha}(\delta, f)$. Thus

$$E_{n}^{p,\alpha}(f) \leq 2K^{K+1} C_{K+1} n^{-(K+1)}\delta^{-1}\omega_{p,\alpha}(\delta, D_{\theta}^{K} f) + 2K C_{K} n^{-K}\omega(\delta; D_{\theta}^{K} f).$$

Now choose $\delta = 2\pi/n$ to see that

$$E_{n}^{p,\alpha}(f) \leq B_{k}\omega_{p,\alpha}\left(\frac{2\pi}{n}\right)n^{-K}$$

where $B_{k} = 2K C_{K+1}/\pi + 2K C_{K}$.

From this it follows that if $f \in \Lambda_{\beta,\alpha}^n$ for $0 < \beta < 1$ then $E_{n}^{p,\alpha}(f) \leq (2\pi)^{\beta}B_{0}\|f\|_{\Lambda_{\beta,\alpha}^n} n^{-\beta}$.

Theorem 2.4. Suppose that $f_{\theta,K} \in \Lambda_{1,\alpha}^n$ and that $\|f_{\theta,K}\|_{\Lambda_{1,\alpha}^n} = M$. Then

$$E_{n}^{p,\alpha}(f) \leq \widetilde{B}_{k} M n^{-K-1}$$

where

$$\widetilde{B}_{k} = 2K(C_{K+2}/\pi + \pi C_{k}).$$

Proof. Write $f = f_{\delta\delta} + g$ where $f_{\delta\delta} = (f_{\delta})_{\delta}$. Then

$$\partial_{t}^{K+2} f_{\delta\delta}(re^{it}) = \frac{f_{t}(\theta,K)(r e^{i(t+2\delta)}) + f_{t}(\theta,K)(r e^{i(t-2\delta)}) - 2f_{t}(\theta,K)(r e^{it})}{4\delta^2}$$

as in the last equation on [15] Volume 1, p. 117]. Thus

$$\|\partial_{t}^{K+2} f_{\delta\delta}\|_{p,\alpha} \leq \frac{M}{2\delta}.$$ 

Following the first and second equations on [15] Volume 1, p. 118] shows that

$$\|g_{t}(\theta,K)(z)\|_{p,\alpha} = \frac{1}{4\delta^2} \left\| \int_{0}^{2\delta} f_{t}(\theta,K)(z e^{it}) + f_{t}(\theta,K)(z e^{-it}) - 2f_{t}(\theta,K)(z)(2\delta - t) dt \right\|_{p,\alpha}$$

which shows that $\|g_{t}(\theta,K)(z)\|_{p,\alpha} \leq (1/2)M\delta$. Applying Theorem 2.2 to $g$ and $f_{\delta\delta}$ and setting $\delta = 2\pi/n$ now yields the result. \qed
3. Approximation of Extremal Functions by Polynomials in the Bergman Norm

We now discuss extremal problems restricted to the space of polynomials of degree \( n \). Let \( F_n \) denote the extremal polynomial of degree \( n \), for the extremal problem of maximizing \( \text{Re} \phi(f) \) where \( f \) ranges over all polynomials of degree at most \( n \) with norm 1. We will need the following theorem from [3].

**Theorem 3.1.** Suppose that \( k \in \Lambda^*_\beta,\Lambda^*_n \), and let \( F \) be the extremal function for \( k \). Then if \( 2 \leq p < \infty \) we have \( F \in \Lambda^*_\beta/p,\Lambda^*_n \) while if \( 1 < p \leq 2 \) we have \( F \in \Lambda^*_\beta/2,\Lambda^*_n \).

Furthermore, suppose that \( \int_0^\infty F(t)k(t)\alpha(t)\,d\alpha \leq 1 \) and \( \|k(e^{it} \cdot) + k(e^{-it} \cdot) - 2k(\cdot)\|_{q,\alpha} \leq B|t|^\beta \). If \( p \geq 2 \) then \( \|F\|_{\Lambda^*_\beta,\beta/p,\Lambda^*_n} \leq 2p^{1/p}(B/2)^{1/p} \leq 2e^{1/\epsilon}(B/2)^{1/p} \) whereas if \( 1 < p < 2 \) then \( \|F\|_{\Lambda^*_\beta/2,\Lambda^*_n} \leq 2(p-1)^{-1/2}B^{1/2} \).

The space of polynomials of degree \( n \) is isomorphic with \( \mathbb{R}^{2n+2} \). The set of all \( x \in \mathbb{R}^{2n+2} \) for which the corresponding polynomial has norm of at most 1 is a convex set. Thus, the extremal problem for finding \( F_n \) can be thought of as a problem of maximizing a (real) linear functional over a convex set in \( \mathbb{R}^{2n+2} \). This is a convex optimization problem, and many algorithms for approximating the solution are known.

We first discuss a worst case rate of convergence of \( F_n \) to \( F \) in the Bergman space norm.

**Theorem 3.2.** Let \( F \) be the extremal function for \( \phi \) and let \( F_n \) be the extremal polynomial of degree \( n \), when the problem is posed over polynomials of degree \( n \). Suppose \( k \in \Lambda^*_\beta,\Lambda^*_n \). Then for \( p < 2 \) we have \( \|F - F_n\|_{p,\alpha} = O(n^{-\beta/4}) \). Similarly if \( p > 2 \) we have \( \|F - F_n\|_{p,\alpha} = O(n^{-\beta/p^2}) \).

More precisely, for \( p < 2 \) and \( 0 < \beta < 2 \),
\[
\|F - F_n\|_{p,\alpha} \leq 4(p-1)^{-3/4}A_{\beta/p}^{1/2}\|k\|_{\Lambda^*_\beta,\Lambda^*_n}^{1/4}n^{-\beta/4};
\]
for \( p < 2 \) and \( \beta = 2 \),
\[
\|F - F_n\|_{p,\alpha} \leq 4(p-1)^{-3/4}\tilde{B}_0^{1/2}\|k\|_{\Lambda^*_\beta,\Lambda^*_n}^{1/4}n^{-\beta/4};
\]
for \( p > 2 \) and \( 0 < \beta < 2 \),
\[
\|F - F_n\|_{p,\alpha} \leq 2^{1+1/p-1/p^2}p^{1/p+1/p^2}A_{\beta/p}^{1/p}\|k\|_{\Lambda^*_\beta,\Lambda^*_n}^{1/p}n^{-\beta/p};
\]
and for \( p > 2 \) and \( \beta = 2 \),
\[
\|F - F_n\|_{p,\alpha} \leq 2^{1+1/p-1/p^2}p^{1/p+1/p^2}\tilde{B}_0^{1/p}\|k\|_{\Lambda^*_\beta,\Lambda^*_n}^{1/p}n^{-1/p}.
\]

**Proof.** Let \( \|\phi\| \) denote \( \|\phi\|_{(\Lambda_2^*,\alpha)} \). The argument in [7, Theorem 4.1] shows that, if \( T_n \) is the best approximate of \( F \) of degree \( n \) and \( E_n^\alpha < \delta \) and \( \bar{T}_n = T_n/\|T_n\|_{p,\alpha}, \) then \( \text{Re} \phi(T_n) \geq (1/\delta^2)\|\phi\| \). This also shows that \( \text{Re} \phi(F_n) \geq (1/\delta^2)\|\phi\| \). Thus
\[
\phi((F_n + F)/2) \geq \|\phi\| \left( \frac{1}{2} + \frac{1-\delta}{2(1+\delta)} \right).
\]
Therefore \((1/2)\|F_n + F\| \geq \frac{1}{2} + \frac{1-\delta}{2(1+\delta)} \geq 1 - \delta \). This shows that \( \|F_n - F\| \leq \sqrt{\frac{8}{p-1}}\delta^{1/2} \) for \( p < 2 \) and \( \|F_n - F\| \leq 2p^{1/p}\delta^{1/p} \) for \( p > 2 \).

\[ \square \]

The convergence rate in the previous theorem may be slow, especially for large \( p \). However, this is a worst case scenario and a given \( F_n \) may be more accurate than this predicts. The following theorem give a way to bound the distance of a
any positive scalar multiple of $k$. An advantage of the theorem is that it applies to any $A^p_\alpha$ function $g$, so we can directly apply it to an approximation of $F_n$, and not just $F_n$ itself. In the theorem statement, $\mathcal{P}_\alpha$ denotes the Bergman projection for $A^p_\alpha$, which is the orthogonal projection from $L^2_\alpha$ onto $A^2_\alpha$. Also $|F|^p/|F| = F^{p/2}F^{-1} = F^{p-1}$ sgn $F$ should be interpreted to equal 0 when $F$ has a zero. It is known that $\mathcal{P}_\alpha$ is bounded from $L^p_\alpha$ to $A^p_\alpha$ for $1 < p < \infty$ (see [9]).

**Lemma 3.3.** Suppose that $F_1$ and $F_2$ are the $A^p_\alpha$ extremal functions for $\phi_1$ and $\phi_2$ respectively. Suppose that $\|\phi_1\| = \|\phi_2\| = 1$ and $\|\phi_1 - \phi_2\| < \delta$. Then for $p > 2$

$$\|F_1 - F_2\| < 2^{1-1/p}p^{1/2}\delta^{1/p};$$

for $p < 2$

$$\|F_1 - F_2\| < 2(p - 1)^{-1/2}\delta^{1/2}.$$  

**Proof.** Note that 

$$|\phi_1(F_1) + \phi_1(F_2)| \geq |\phi_1(F_1) + \phi_2(F_2)| + |(\phi_1 - \phi_2)(F_2)| = \|\phi_1\| + \|\phi_2\| - \delta \geq 2 - \delta.$$  

This implies that 

$$\left\|\frac{F_1 + F_2}{2}\right\| > 1 - \frac{\delta}{2}.$$  

The result now follows by Lemma [11].

It is known that if $k$ is a positive scalar multiple of $\mathcal{P}_\alpha(|F|^p/|F|)$, where $F$ has unit norm, then $F$ is the extremal function for $k$. Since $\int_D F\overline{\mathcal{P}_\alpha(|F|^p/|F|)} dA = \int_D F\overline{\mathcal{P}_\alpha(|F|^p/|F|)} dA = 1$, we see that if $k$ is scaled so that $\int_D F\overline{k}dA = 1$, then $k = \mathcal{P}_\alpha(|F|^p/|F|)$.

**Theorem 3.4.** Let $k \in A^p_\alpha$, and let $F$ be the extremal function for $k$. Let $\overline{k}$ be any positive scalar multiple of $k$ (so that $\overline{k}$ also has $F$ as extremal function.) Let $G \in A^p_\alpha$ and suppose that for some $\delta$ such that $0 < \delta < 1$ the inequality

$$\|\mathcal{P}_\alpha(|G|^p/|G|) - \overline{k}\|_{A^p_\alpha} < \delta$$

is satisfied. Then for $p > 2$,

$$\|F - G\| < 2p^{1/p}\delta^{1/p}$$

and for $p < 2$

$$\|F_1 - F_2\| < 2\sqrt{2}(p - 1)^{-1/2}\delta^{1/2}.$$  

**Proof.** Let $\psi$ be the functional of unit norm for which $G$ is the extremal function. Then $\psi$ has kernel $\mathcal{P}_\alpha(|G|^p/|G|)$ and $\|\psi\| = 1$. Let $\phi$ be the functional with kernel $\overline{k}$. We then have 

$$\|\phi - \psi\|_{(A^p_\alpha)^*} \leq \|\mathcal{P}_\alpha(|G|^p/|G|) - \overline{k}\|_{A^p_\alpha} < \delta.$$  

This implies that $1 - \delta < \|\phi\| < 1 + \delta$. Let $\tilde{\phi} = \phi/\|\phi\|$. Then $\|\phi - \tilde{\phi}\| < \delta$ and thus $\|\tilde{\phi} - \psi\| < 2\delta$. The conclusion now follows from the previous lemma. □
4. Approximation of Extremal Functions by Polynomials in the Supremum Norm

We now discuss how to use the results in the previous section to bound the distance from a given function to \( F \) in the supremum norm. We will use the following theorem found in [8, Corollary 4.3]. The proof of this theorem shows that the same results hold if \( F \) is replaced by \( F_n \). However, we may need to multiply \( k \) by a positive scalar constant greater than 1 so that the condition \( \int_{\mathbb{D}} F_n k \, dA_\alpha = 1 \) holds.

**Theorem 4.1.** Let \( 1 < p < \infty \) and let \( p \) and \( q \) be conjugate exponents. Suppose \( k \in \Lambda_{r,2,A_r}^p \) and that \( \int_{\mathbb{D}} F_n k \, dA_\alpha \geq 1 \). If \( 2 \leq p < \infty \) and \(-1 < \alpha < 0\), then \( f \) has Hölder continuous boundary values. If \( 1 < p < 2 \) and \(-1 < \alpha < p - 2\), the same conclusion holds.

Let \( B = \|k\|_{\Lambda_{r,2,A_r}^p} \). For \( p > 2 \), The Hölder exponent is \(-\alpha/p\). The Hölder constant is bounded above by

\[
2^{p/\alpha}(B/2)^{1/p} \cdot 383\left(1 - \frac{2}{p}\right)^{-1/\alpha} \cdot 2 \left(\frac{\Gamma(q-1)}{\Gamma(q/2)^{2}}\right)^{1/q} \cdot \left(1 - \frac{2p}{\alpha}\right)
\]

For \( p < 2 \), if we let \( \eta \) be any number greater than 0, then the Hölder exponent can be taken to be \( 1 - 2/p - \alpha/p - \eta \) (if the indicated exponent is positive). The Hölder constant is bounded above by

\[
2(p-1)^{-1/2} B^{1/2} \cdot 192\left(1 - \frac{2}{p}\right)^{-1/\alpha} \cdot 2 \left(\frac{\Gamma(q-1)}{\Gamma(q/2)^2}\right)^{1/q} \cdot \left(1 - \frac{2}{1 - 2/p - \alpha/p - \eta}\right)
\]

For ease of notation, we will call the Hölder exponent \( \beta(p, \alpha) \) for \( p > 2 \) and \( \beta(p, \alpha, \eta) \) for \( p < 2 \). We will denote the constant by \( C(B, p, \alpha) \) and \( C(B, p, \alpha, \eta) \) respectively. For \( p > 2 \) if we refer to \( \beta(p, \alpha, \eta) \) and \( C(p, \alpha, \eta) \), we mean \( \beta(p, \alpha) \) and \( C(p, \alpha) \) respectively.

Since \( \|F\| = 1 \), it follows that \( |F(0)| < 1 \). Thus the preceding estimate can be used to bound \( \|F\|_\infty \). However, the estimates do not allow one to conclude directly that \( \|F - F_n\|_\infty \) must be small for large \( n \). The following theorem remedies this situation. It says that if a function is Hölder continuous (with control on the exponent and size of the constant) and the function has small \( L_p^p \) norm, then its uniform norm cannot be too large.

**Theorem 4.2.** Let \( \epsilon > 0 \) and \( 0 < \beta \leq 1 \) be given. Suppose that \( f \in L_p^p(\mathbb{D}) \) and that for some \( C > 0 \) we have \( |f(z) - f(w)| \leq C|z - w|^\beta \) for every \( z, w \in \mathbb{D} \). Then there exists a \( \delta > 0 \) such that if \( \|f\|_{p, \alpha} < \delta \) then \( \|f\|_\infty < \epsilon \). In fact, we may take \( \delta \) to be

\[
(\alpha + 1)\pi \frac{1}{4} B(2/\beta, p + 1)^{1/p} C^{-2/(\beta p)} \epsilon^{1+2/(\beta p)}
\]

as long as \( \epsilon < 2^{\beta/2}C \). Here \( B(x, y) \) is the Beta function.

For ease of notation we will denote the \( \delta \) in the theorem by \( \delta(\epsilon; C, \beta, p, \alpha) \). We let \( \epsilon(\delta; C, \beta, p, \alpha) \) denote the inverse function of \( \delta(\epsilon) = \delta(C, \beta, p, \alpha) \).
where \( B \) and \( P \) are such that \( r_0 = (b/C)^{1/\beta} \).

Now for fixed \( \delta > 0 \), we may estimate \( \| f \|_p \geq \delta(b)^p \). Then if \( |f(z_0)| \geq \delta \) we have \( \| f \|_p \geq \delta(b)^p \). Then if \( \| f \|_p < \delta(e) \) we have \( \| f \|_\infty < \epsilon \).

We may also prove the following theorem. It will not be used in the sequel, but we include it for completeness.

**Theorem 4.3.** Let \( \epsilon > 0 \) and \( 0 < \gamma < \beta \leq 1 \) be given. Suppose that \( f \in L^p(\mathbb{D}) \) and that for some \( C > 0 \) we have \( |f(z) - f(w)| \leq C|z - w|^{\beta \gamma} \) for every \( z, w \in \mathbb{D} \). Then there exists a \( \alpha > 0 \) such that \( \| f \|_p < \delta \) then \( |f(z) - f(w)| < \epsilon|z - w|^{\gamma} \).

**Proof.** Suppose \( |f(z) - f(w)| \geq \epsilon|z - w|^{\gamma} \) for some \( z \) and \( w \). Since \( |f(z) - f(w)| < C|z - w|^{\beta \gamma} \) we have \( |z - w|^{\beta \gamma} > \epsilon/C \). Thus \( |z - w|^{\beta \gamma} > (\epsilon/C)^{\gamma/(\beta \gamma)} \), and so \( |f(z) - f(w)| > \epsilon^{\beta/(\beta \gamma)}C^{-\gamma/(\beta \gamma)} \). But this contradicts the previous theorem if \( \delta \) is small enough.

**Theorem 4.4.** Let \( 1 < p < \infty \) and let \( p \) and \( q \) be conjugate exponents. Suppose \( k \in \Lambda_{2, A^\alpha_q}^\star \). If \( 2 \leq p < \infty \) let \( -1 < \alpha < 0 \). If \( 1 < p < 2 \) let \(-1 < \alpha < p - 2 \). Suppose that \( \int_\mathbb{D} \frac{1}{k} |F| dA_\alpha > 1 \). Then if \( \| F - F_n \|_{A^\alpha_q} < \delta \) then \( \| F - F_n \|_\infty < \epsilon(\delta; C, \beta, p, \alpha) \), where \( \beta = \beta(p, \alpha, \eta) \) and

\[
C = C(||k||_{L^r A^\alpha_q}, p, \alpha, \eta) + C((1 - \delta)^{-1} ||k||_{L^r A^\alpha_q}, p, \alpha, \eta).
\]

**Proof.** This follows from Theorems 4.1 and 4.2. We use the fact that Theorem 4.1 applies to \( F_n \) if \( k \) is first multiplied by \( 1/(1 - \delta) \), which ensures that the condition \( \int_\mathbb{D} \frac{1}{k} |F_n| dA_\alpha \geq 1 \) holds.

5. APPROXIMATION OF EXTREMAL FUNCTIONS FOR EVEN \( p \)

We will give an example of approximating an extremal function. The case where \( p \) is even is in some ways easier than other cases since then we can explicitly compute \( \mathcal{P}(f^p/\mathcal{J}) = \mathcal{P}(f^{p/2}/\mathcal{J}^{p/2 - 1}) \) when \( f \) is a polynomial, due to the fact that \( f^{p/2} \) and \( f^{p/2 - 1} \) are polynomials, so our example will involve this case.

Define

\[
\gamma(n, \alpha) = \|z^n\|_{A^\alpha_q}^2 = (\alpha + 1)B(n + 1, \alpha + 1) = \frac{\Gamma(\alpha + 2)\Gamma(n + 1)}{\Gamma(n + \alpha + 2)}.
\]

Then

\[
\mathcal{P}_\alpha(z^m) = \begin{cases} 
\frac{\gamma(m, \alpha)}{\gamma(m - n, \alpha)}z^{m - n} & \text{if } m \geq n \\
0 & \text{if } m < n
\end{cases}
\]

(5.1)
Example 5.1. Let us approximate the solution to the problem of maximizing the (real part of) the functional $f \mapsto a_0 + a_1 + a_2$, where the $a_n$ are the Taylor series coefficients of $f$ about 0, and where $p = 4$ and $\alpha = -1/2$ (and where $f$ has unit norm). Then $k = a_0 + a_1/\gamma(1, -1/2) + a_2/\gamma(2, -1/2) = 1 + (3/2)z + (15/8)z^2$.

This problem is made simpler because the uniqueness of $F$ implies that it must have real coefficients. Let us take the approximation of degree $N = 20$. We thus seek to maximize $a_0 + a_1 + a_2$ subject to the constraint $\|f\|_{4, -1/2} = \|f^2\|_{2, -1/2} \leq 1$, i.e.

$$\sum_{n=0}^{N} \left( \sum_{m=0}^{n} a_m a_{n-m} \right)^2 \gamma(n, -1/2) \leq 1.$$ 

This is a convex optimization problem, and we are aided by the fact that any local maximum must be a global maximum, since if $F$ is any local maximum (necessarily of norm 1) then a variational argument similar to the one in the proof of [6, Chapter 5, Lemma 2] shows that the $P_{\alpha, 20}(|F^p/F|)$ is a scalar multiple of $k$, and thus $F$ is the extremal function (see [13, p. 55]). Here we let $P_{\alpha, 20}$ denote the orthogonal projection from $L^p_n(\mathbb{D})$ onto the subspace of $A^p_n$ of consisting of polynomials of degree at most 20.

Using Mathematica (for example) to approximate a solution yields a maximum functional value of 1.78785 and

$$F_{20} = 0.431458 + 0.496144x + 0.860246x^2 - 0.341597x^3 - 0.0225992x^4 + 0.110915x^5$$
\[-0.0520239x^6 - 0.00952809x^7 + 0.0235908x^8 + \cdots + -0.00599527x^{15}.\]

All of the omitted terms have coefficients of less than 1/100. If we compute $\tilde{k} = P_{\alpha, 20}(F_{20}^4/F_{20})$, we find that it is $$.559332 + 0.838998z + 1.04875z^2 + \cdots + 1.55098 \cdot 10^{-9}z^{40}.\]

All of the omitted terms have coefficients of at most .0001. We must now find a multiple of $k$ close to $\tilde{k}$. We could find the closest one as an optimization problem, but we will choose $\tilde{k} = .559332k$ in order to make the first coefficients of $\tilde{k}$ and $\tilde{k}$ match, since this is simpler and yields a result close to $\tilde{k}$. If we now compute $\|\tilde{k} - k\|_{A^{4/3}}$, we see that it is about .000018. Theorem [13] shows that $\|F - F_{20}\|_{4, -1/2}$ is less than .185. In fact, I suspect that the true error is much smaller. For example, $\|F_{25} - F_{20}\|_{4, -1/2} < 1.85 \cdot 10^{-5}$, so the true error may be closer to this order of magnitude.

We find that .559332 times the sum of the first three coefficients of $F$ is bigger than 1 by about $2.5 \cdot 10^{-7}$. The second $\theta$ derivative of $\tilde{k}$ is most $5.034$, so $\|\tilde{k}\|_{A^{*, 2}, A^{4/3}_{-1/2}}$ is at most $5.034$. Thus Theorem [13] shows that $\|F - F_{20}\|_{\infty} < 5181$. Again, I suspect the true error is much smaller. For example, $\|F_{25} - F_{20}\|_{\infty} < .00006$, and the true error may be this order of magnitude.

It would be interesting to see if the estimates in this paper can be substantially improved in order to yield better estimates on the approximation of extremal functions in the uniform norm. The example above shows that the estimates in the paper are likely too large by a substantial margin. However, the estimates in this paper are the only ones known (as far as I know) that allow approximation of these
extremal functions in the uniform norm, and they have the advantage of being explicitly computable without great difficulty.

6. Non-zero Extremal Functions

The proceeding results can be used to find explicit conditions on $k$ that guarantee that $F$ is non-zero. In Theorem 6.2 we give one such result.

**Theorem 6.1.** Let $0 < \theta < 2\pi$ and $\theta < 2\pi(p-1)$. Suppose that $k \in A^p_\alpha$ has range that is a subset of the sector $-\theta/2 < \arg z < \theta/2$, and that $\|k\|_{q,\alpha} = 1$. Let $F$ be the extremal function for $k$ and let

$$C_\theta = 2C_{p,\alpha} \sin \left( \frac{(p-2)\theta}{4(p-1)} \right),$$

where $C_{p,\alpha}$ is the bound for the Bergman projection from $L^p_\alpha$ onto $A^p_\alpha$. Then if $p > 2$ we have $\|F - k^{1/(p-1)}\|_{p,\alpha} \leq 2p^{1/p}C_{1/p}$ and if $p < 2$ we have $\|F - k^{1/(p-1)}\|_{p,\alpha} \leq 2\sqrt{2}(p-1)^{-1/2}C_1^{1/2}.$

**Proof.** Note that $G = k^{1/(p-1)}$ is well defined, where we take the branch with $1^{1/(p-1)} = 1$. Notice that $|G|^{p-1} \text{sgn } G = |k|e^{i\arg (k)/(p-1)}$. Thus

$$|k - |G|^{p-1} \text{sgn } G| = |k| \left| e^{i\arg (k)} - e^{i\arg (k)/(p-1)} \right| \leq 2|k| \sin \left( \frac{(p-2)\theta}{2(p-1)} \right)$$

and therefore

$$\|k - |G|^{p-1} \text{sgn } G\|_{p,\alpha} \leq 2\|k\|_{p,\alpha} \sin \left( \frac{(p-2)\theta}{2(p-1)} \right).$$

Let $C_{p,\alpha}$ be the bound for the Bergman projection from $L^p_\alpha$ onto $A^p_\alpha$. Then

$$\|k - P_{\alpha}(|G|^{p-1} \text{sgn } G)\|_{q,\alpha} \leq 2C_{p,\alpha} \sin \left( \frac{(p-2)\theta}{4(p-1)} \right) \|k\|_{q,\alpha},$$

since $P_{\alpha}(k) = k$. The result now follows from Theorem 3.4. \hfill \Box

**Theorem 6.2.** Let $0 < d < 1$ and $2 \leq p < \infty$ and $-1 < \alpha < 0$. If $1 < p < 2$ also suppose $-1 < \alpha < p - 2$. Let $\|k\|_{q,\alpha} = 1$ and suppose that $\|k\|_{\Lambda^{2,2},A^p_\alpha} < B$. Then there exists a $\theta > 0$ depending only on $d$, $B$, $p$, and $\alpha$ such that if the range of $k$ is a subset of $\{z : -\theta/2 < \arg z < \theta/2$ and $|z| > d\}$ then $F$ is non-zero.

**Proof.** Let $\theta > 0$ be given. This $\theta$ will make the conclusion of the theorem true if the assumptions show that $\|F - k^{1/(p-1)}\|_{\infty} < d^{1/(p-1)}$. Let $\lambda = d^{1/(p-1)}$. For $p < 2$ choose $0 < \eta < 1 - 2/p - \alpha/p$ and let $\beta = \beta(p, \alpha, \eta)$; otherwise let $\beta = \beta(p, \alpha)$.

Note that by [3] Theorems 3.1 and 1.2] and [5] Theorems 5.9 and 5.1], we have $k \in C^{2-2/p-\alpha/p} \subset C^\beta$ with Hölder constant depending only on $B$, $p$, and $\alpha$. Since $k$ is bounded away from 0, we also have that $k^{1/(p-1)} \subset C^\beta$ with constant depending only on $B$, $d$, $p$, and $\alpha$. Let $D$ be the smallest constant such that $|k(z)^{1/(p-1)} - k(w)^{1/(p-1)}| \leq D|z - w|^{\beta}$.

By Theorem 4.2 we will be done if we can show that

$$\|F - k^{1/(p-1)}\|_{p,\alpha} < \delta$$

where $\delta = \delta(\lambda, C + D, \beta, p, \alpha)$, where $C = C(B, p, \alpha, \eta)$. But by the previous theorem, this is true if $\theta$ is small enough. \hfill \Box

Notice that, given $B$, $d$, $\epsilon$, $p$ and $\alpha$, we could if we wish calculate an explicit value for $\theta$. 
References

[1] Keith Ball, Eric A. Carlen, and Elliott H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, Invent. Math. 115 (1994), no. 3, 463–482. MR 1262940

[2] Catherine Bénétéeau and Dmitry Khavinson, A survey of linear extremal problems in analytic function spaces, Complex analysis and potential theory, CRM Proc. Lecture Notes, vol. 55, Amer. Math. Soc., Providence, RI, 2012, pp. 33–46. MR 2986891

[3] Catherine Bénétéeau and Dmitry Khavinson, Selected problems in classical function theory, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 255–265. MR 3309357

[4] James A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936), no. 3, 396–414. MR MR1501880

[5] Peter Duren, Theory of $H^p$ spaces, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970. MR MR0268655 (42 #3552)

[6] Peter Duren and Alexander Schuster, Bergman spaces, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004. MR MR2033702 (2005c:30053)

[7] Timothy Ferguson, Continuity of extremal elements in uniformly convex spaces, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2645–2653.

[8] Timothy Ferguson, Bergman– Hölder functions, area integral means and extremal problems, Integral Equations and Operator Theory 87 (2017), no. 4, 545–563.

[9] Håkan Hedenmalm, Boris Korenblum, and Kehe Zhu, Theory of Bergman spaces, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000. MR 1758653 (2001c:46043)

[10] Dmitry Khavinson, John E. McCarthy, and Harold S. Shapiro, Best approximation in the mean by analytic and harmonic functions, Indiana Univ. Math. J. 49 (2000), no. 4, 1481–1513. MR 1836538 (2002b:41023)

[11] Dmitry Khavinson and Michael Stessin, Certain linear extremal problems in Bergman spaces of analytic functions, Indiana Univ. Math. J. 46 (1997), no. 3, 933–974. MR 1488342 (99k:30080)

[12] V. G. Ryabykh, Extremal problems for summable analytic functions, Sibirsk. Mat. Zh. 27 (1986), no. 3, 212–217, 226 ((in Russian)). MR 853902 (87j:30058)

[13] Harold S. Shapiro, Topics in approximation theory, Springer-Verlag, Berlin, 1971, With appendices by Jan Boman and Torbjörn Hedberg, Lecture Notes in Math., Vol. 187. MR 0437981 (55 #10902)

[14] ________, Regularity properties of the element of closest approximation, Trans. Amer. Math. Soc. 181 (1973), 127–142. MR 0326066

[15] A. Zygmund, Trigonometric series: Vols. I, II, Second edition, reprinted with corrections and some additions, Cambridge University Press, London, 1968. MR 0236587 (38 #4882)

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