Quantum geometric tensor away from Equilibrium

Davide Rattacaso,1 Patrizia Vitale,1,2 and Alioscia Hamma3

1Dipartimento di Fisica Ettore Pancini, Università di Napoli Federico II
2 INFN-Sezione di Napoli
3Physics Department, University of Massachusetts Boston, 02125, USA

The manifold of ground states of a family of quantum Hamiltonians can be endowed with a quantum geometric tensor whose singularities signal quantum phase transitions and give a general way to define quantum phases. In this paper, we show that the same information-theoretic and geometrical approach can be used to describe the geometry of quantum states away from equilibrium. We construct the quantum geometric tensor \( Q_{\mu\nu} \) for ensembles of states that evolve in time and study its phase diagram and equilibration properties. If the initial ensemble is the manifold of ground states, we show that the phase diagram is conserved, that the geometric tensor equilibrates after a quantum quench, and that its time behavior is governed by out-of-time-order commutators (OTOCs). We finally demonstrate our results in the exactly solvable Cluster-XY model.

Introduction.— The notion of a quantum phase is that of an equivalence class of quantum states, in particular, of ground states of a family of Hamiltonians. The states in the same phase are states that look alike according to some salient characteristics that allows us to classify the states in different classes. For instance, all the states that break the symmetry of a parent Hamiltonian in a certain way are labeled by the correspondent local order parameter. A quantum phase transition[1] is the transition between different quantum phases and is usually signaled by the shrinking of the gap between the ground state and the first excited state. In order to go beyond symmetry breaking phases, it has proven very fruitful to use tools from quantum information theory, in particular measures of distinguishability of quantum states. The main idea is that from a state in a quantum phase one can find other states that are infinitesimally close and in this way can connect any two states in the same phase. On the other hand, crossing a quantum phase transition means that at some point the distance between two quantum states is not analytic.

A metric on the space of quantum states can be easily obtained by the so called fidelity \( F = |\langle \phi | \psi \rangle| \) between the pure quantum states \( |\phi\rangle \), \( |\psi\rangle \). As Wooters showed in[2],

\[
d_{FS}(\phi, \psi) = \cos^{-1} F \text{ is the maximum over all the possible projective measurements of the Fisher-Rao statistical distance between the probability distributions obtained from } |\phi\rangle, |\psi\rangle.
\]

The infinitesimal version of this metric distance gives rise to the Fubini-Study metric \( d_{FS}^2(\psi, \psi + \delta \psi) \approx 2(1 - F) \). This information-theoretic distance is closely related to the quantum geometric tensor \( Q_{\mu\nu} \), namely the natural metric structure on the projective Hilbert space[3]. Quantum phase transitions can then be studied in a more elegant and general way by looking at the scaling of the norm of such tensor[4, 5]. It turns out that quantum critical points are marked by divergences of the norm of the real part of \( Q_{\mu\nu} \). This approach has proven useful to study topological quantum phase transitions[6] and it has been generalized to mixed (e.g., thermal) states[7].

An important question is that of clarifying the notion of quantum phase for states away from equilibrium. This would prove very useful to understand dynamical phase transitions like the transition between thermal and many-body localized states[2] or the transition between scrambling and unscrambling behavior. To this end, in this paper we ask the following question: can we endow a manifold of quantum states of quantum states away from equilibrium with a similar geometric structure?

We start from a manifold of ground states and throw them away from equilibrium by means of a quantum quench[10, 11]. In this paper, we exploit the formalism of quantum quenches to construct foliated manifolds of quantum states \( M_t \) and show it can be given a metric structure \( Q_{\mu\nu}(t) \). That is to say that the phase diagram on \( M_t \) is conserved, find conditions for the equilibration of the geometric tensor and find that the time evolution of the tensor is expressed in terms of out-of-time-order commutators (OTOCs). We finally apply these results to an integrable model, the Cluster-XY model[12].

Setup.— Consider a Hamiltonian \( H(\lambda) \) smooth in the parameters \( \lambda = (\lambda^1, ..., \lambda^n) \) and consider the mapping \( \lambda \mapsto |\psi(\lambda)\rangle \) to the (unique) ground state of \( H(\lambda) \). The projective Hilbert space \( \mathcal{P}H \) of the rays is the base manifold of a \( U(1) \) fiber bundle naturally endowed with a complex metrics \( G(u, v) = \langle u \mid (1 - |\psi(\lambda)\rangle \langle \psi(\lambda)|) v \rangle \). The pull back of this metrics to \( M \) gives the Hermitean quantum geometric tensor \( Q_{\mu\nu} := \langle \partial_\mu \psi \mid (1 - |\psi\rangle \langle \psi|) \partial_\nu \psi \rangle \). The real part of this tensor \( g_{\mu\nu} = \Re Q_{\mu\nu} \) is a Riemanian real geometric tensor on \( M \) while its imaginary part is the Berry adiabatic curvature. For real Hamiltonians the ground state manifold is real and one has \( Q_{\mu\nu} = g_{\mu\nu} \). Now let us introduce the unitary operator \( U_t(\lambda) \) that evolves the states in \( M \) in time. This operation defines a new family of states \( M_t = \{ |\psi(\lambda)\rangle \mapsto U_t(\lambda)|\psi(\lambda)\rangle \} \). We can pull back the complex metric \( G \) to the manifold \( M_t \) and similarly obtain the time dependent quantum geometric tensor \( Q_{\mu\nu}(t) := \langle \partial_\mu \psi(t) \mid (1 - |\psi(t)\rangle \langle \psi(t)|) \partial_\nu \psi(t) \rangle \). One can easily show that \( Q_{\mu\nu}(t) = \sum_{n\neq 0} \langle \psi(\lambda) | \partial_\nu \phi_n(t) \psi(\lambda) | \partial_\mu \phi_n(t) \psi(\lambda) \rangle (E_0 - E_n)^{-2} \) with \( H_{\mu} = U H U^\dagger \). Let \( q_{\mu}(t) \) be the eigenvalues of eigenvector \( |\nu(\lambda)\rangle \) for the geometric tensor \( Q_{\mu\nu}(t) \) in the \( \mu \nu \) space.

Some manipulation (see [28]) leads to the expression

\[
q_{\mu}(t) = \sum_{n\neq 0} \frac{\langle \psi(\lambda) | \partial_\nu H + [H, iD_\nu] | \psi_n(\lambda) \rangle^2}{(E_0 - E_n)^2} \tag{1}
\]

where \( D_\nu := -i\partial_\nu U^\dagger U \).
So far, the evolution operator $U(\lambda, t)$ is completely general. Now we specialize it to the case in which time evolution is obtained by a sudden quantum quench. To this end, we define another family of Hamiltonians $H^q(\lambda)$ on the same manifold $\mathcal{M}$, the so-called *quench* Hamiltonian, and consider the mapping $\lambda \mapsto |\psi_0(\lambda)\rangle = U(\lambda, t)|\psi(\lambda)\rangle$ where $U(\lambda, t) = \exp(-i t H^q(\lambda))$ is the unitary evolution operator associated to the quench Hamiltonian. Typically, a quench can be obtained by $H^q(\lambda) = H(\lambda + \delta \lambda)$, where $\delta \lambda$ is a small variation of the $\lambda$ parameters on $\mathcal{M}$. For a quantum quench, $D_a := -i \partial_a U^{-1} U = \int_0^t dt' U(t')^\dagger \partial_a H U(t')$.

A simplified protocol for the quantum quench consists in preparing the initial state in the ground state $|\psi_0(\lambda_0)\rangle$ of a fixed $H(\lambda_0)$ and then evolving with $H^q = H(\lambda)$. In this case, the term $\partial_a H$ in Eq.\((\ref{eq:partialH})\) vanishes and we have $q_a(t) = \sum_{n \neq 0} |\langle \psi_0| H, iD_a|\psi_n\rangle|^2 (E_0 - E_n)^{-2} \equiv q_{1a}(t)$. Notice that, since $q_a(0) = \sum_{n \neq 0} |\langle \psi_0| H, iD_a|\psi_n\rangle|^2 (E_0 - E_n)^{-2}$, one can bound the absolute value of the $q_a(t)$ as $q_a(t) = q_a(0) + q_a(t) - 2 \sqrt{q(0)} q_1(t) \leq q(t) \leq q(0) + q_1(t) + 2 \sqrt{q(0)} q_1(t)$, where we dropped the subscript $a$ for ease of notation.

**Time evolution of the Phase Diagram.**— The zero-time phase diagram is a consequence of the locality of the Hamiltonian, as divergences of the rescaled quantum geometric tensor $q_a/N$ can only happen when the gap $\Delta = E_1 - E_0$ with the first excited state changes. In order to show the time evolution of the phase diagram on $\mathcal{M}$, we need to exploit locality again. From now on, we will be interested in local Hamiltonians, that is, sum of local operators $H(\lambda) = \sum_i H_i(\lambda)$, and similarly for the quench Hamiltonian $H^q(\lambda) = \sum_i H_i^q(\lambda)$.

Let us start with showing that $q_a(t)$ can be written as a connected correlation function for $D_a$. We use the simplified quench so that we can drop the term $\partial_a H$ in Eq.\((\ref{eq:partialH})\). We also drop the subscript $a$ for the sake of making the notation lighter. A simple calculation\cite{28} then shows that $q_1(t) = \sum_{n \neq 0} |\langle \psi_0| D|\psi_n\rangle|^2 = |\langle \psi_0| D|\psi_0\rangle|^2 = (\langle \psi_0| D^2|\psi_0\rangle - \langle \psi_0| D|\psi_0\rangle^2) := (D^2)_{C}$. Writing the operator $D$ explicitly and exploiting the locality of the Hamiltonian and translational invariance we obtain\cite{28} the rescaled tensor

$$q_1(t)/N = \sum_j \int_0^t dt' \int_0^t dt'' \langle \partial H_0^2(t') \partial H_0^2(t'') \rangle_C.$$  \hfill (2)  

We see that the geometric tensor is sum of unequal-time connected correlation functions of the variation of the quench Hamiltonian. In the general quench scheme instead, we can upper bound $|q_a(t)| \leq \Delta^2 (D^2(H))_C$, where we have defined the covariant derivative $D_a[X] := \partial_a X - [D_a, X]$. The locality of the Hamiltonian and its spectral properties can now be exploited to upper bound the norm of $q_1(t)$. We generalize a result about the Lieb-Robinson bounds\cite{13} using the condition $A_{AB} + B_{AB}$ is bounded by the Lieb-Robinson bound $|\langle A(t) B(0) \rangle| \leq k e^{-d_{AB}/\sqrt{v_{LR}}} (e^{-v_{LR}/\sqrt{4}} + e^{-v_{LR}/\sqrt{4}})^2$, where $v_{LR}$ is the maximum speed of the interactions, the Lieb-Robinson speed, $\chi$ is the correlation length of the state $\psi$, that is, the initial amount of correlations in the state and $a$ is the lattice spacing. In the above $k$ is a constant that does not depend on the size of the system (see\cite{28} for details). We use this result to bound the quantum geometric tensor as this can be written as sum of unequal-time connected correlation functions.

Let $\chi$ be the correlation length of $|\psi_0(\lambda_0)\rangle$. One has $\chi = 2 v_{LR} R \Delta^3$ where $v_{LR}$ is the Lieb-Robinson speed associated to $H(\lambda)$. By exploiting the clustering of correlations into Eq.(2) one obtains $|q_1(t)|/N \leq k ||\partial H^2||_M^2 \sum_j \int_0^t dt' \int_0^t dt'' (e^{-d_0 v_{LR}/\sqrt{4}} + e^{-d_0 v_{LR}/\sqrt{4}})^2$, where $||\partial H^2||_M^2$ is the Euclidean distance of the support of $\partial H^2$ from the origin. A straightforward calculation leads to the following upper bound:

$$|q_1(t)|/N \leq k ||\partial H^2||_M^2 \left[4 e^{2 v_{LR}^2 R^2} \sum_j e^{-d_0 v_{LR}^2 R^2} \right] \hfill (3)$$  

To understand the behavior of the above expression for large $N$, it is crucial the behavior of the initial correlation length $\chi$. For a non critical Hamiltonian $H(\lambda)$, the gap is finite, thus the correlation length is finite and the rescaled geometric tensor does not diverge for any $t$. Notice that this behavior does not depend on the criticality of $H^q$. In view of the inequality $q(t) \leq q(0) + q_1(t) + 2 \sqrt{q(0)} q_1(t)$, we see that the divergences of $q(t)$ can only be those of $q(0)$. We therefore obtain the remarkable result that the phase diagram on $\mathcal{M}$ is preserved in time.

**Equilibration of the geometric tensor.**— In this section, we want to show that the geometric tensor after the simplified quench equilibrates in the sense that its oscillations go to zero in the large $N$ limit. In the previous section, we showed that for the simplified quench the geometric tensor is $q(t) = q_1(t)$ and that this can be written as sum of (connected) correlation functions $C(t', t'')$. We first prove that these correlation functions equilibrate. We will do so by showing that their temporal variance defined as $\sigma^2(C) = \lim_{T \to \infty} T^{-2} \int_0^T dt' \int_0^T dt'' (C(t', t'') - \bar{C}(t', t''))^2$ is small. Indeed, since the probability for $C(t_1, t_2) = x$ is given by $p(x) = \lim_{T \to \infty} T^{-2} \int_0^T dt' \int_0^T dt'' \delta(C(t', t'') - x)$, the expectation value for $C$ over the whole time interval is $E(C) = \lim_{T \to \infty} T^{-2} \int_0^T dt' \int_0^T dt'' C(t', t'') = \bar{C}(t', t'')$ and therefore a small variance $\sigma^2$ means that the probability of observing a correlation function with a value different from its average is small. In other words, $\forall \lambda > 0$, the probability $p(C(t', t'') - \bar{C}(t', t'')) > \lambda \chi^{-2}$.

It is known that the temporal variance for the expectation value of observables evolving under the non-resonance condition of the Hamiltonian\cite{17} is bounded as $\sigma^2_{\chi} \leq ||A||^2 T \varphi(\vec{p}^2)$ where $\varphi(\vec{p}^2)$ is the purity of the completely dephased initial state in the basis of the evolving Hamiltonian, here $H^q$. This result can be extended to unequal-time correlation functions (see\cite{28} for the proof):  

**Theorem.** Consider a Hamiltonian $H = \sum_n E_n P_n$ satisfying the non-resonance condition, that is, being non degen-
ate and also having non-degenerate gaps: $E_n - E_m = E_k - E_l \Rightarrow n = k \land m = l$. Then, the temporal variance $\sigma^2(t)$ of unequal-time correlation functions $C(t', t'') = \langle A(t')A(t'') \rangle$ is upper bounded as $\sigma^2(t) \leq \|A\|^2 \text{Tr} \rho^2$. At this point, it is a corollary that the same bound holds also for connected correlation functions.

Let us now apply these results to the temporal variance for the geometric tensor. From Eq. (2), we can write $q_1(t)$ in the form $q(t) = \int_0^t q(t')dt'$. We can then write $q(t) = t^2 f(t', t'') + X(t)$ where $X(t) = \int_0^t \int_0^t \int_0^t \int_0^t (f(t', t') - f(t', t'')) dt'' dt'''. \quad \text{Obviously,}$

\[ |X(t)|/t^2 \leq \sqrt{t^2 f(t', t''') dt'' dt'''} f(t', t') - f(t', t'')|^2 \leq \sqrt{t^2 f(t', t''') dt'' dt'''} f(t', t') - f(t', t'')|^2 |X(t)| \leq 2t^2 \sigma. \quad \text{The long time behavior of the geometric tensor is thus, applying the result of the above theorem, given by}$

\[ q_1(t) = t^2 \partial H q(t') \partial H q(t''') C + X |X| \leq t^2 \partial H q|^2 \text{Tr}(\rho_0)^2 \quad (4) \]

We can now state one of the main results of the paper. Let us consider a Hilbert space with $N$ bodies, such that its dimensions go like $d = q^N$. If the initial state $\rho_0$ is sufficiently spread in the eigenbasis of $H^q$, that is, its purity goes like $1/d$, then the time fluctuation $X$ is upper bounded as $|X| \leq q^{-N} \partial H q|^4 \leq q^{-N} O(N^4)$ so that they vanish very fast for a large system.

In the case of the general quench, one has to make use of the inequality $q(t) \leq q(0) + t^2 (\partial H q(t') \partial H q(t'')) C + X$. Obviously, therefore we can state that as for large $N$ the purity suppresses $|X|$, the QGT $q(t)$ oscillates around $q(0) + t^2 (\partial H q(t') \partial H q(t'')) C$ at most linearly in time.

We want to conclude this section with a remark. In the simplified quench protocol, the geometric tensor $q(t)$ can be expressed as in Eq. (3) with $\partial H = 0$. We can then also see that $|q(t)| \leq \Delta^2 \sum_{n \neq 0} (\psi_0 [H, D] \psi_n)^2 = \Delta^2 (H, D)[H, D]$ and thus

\[ |q(t)| \leq \Delta^2 \sum_{i,j,k,l} \int_0^t dt' \int_0^t dt'' \langle [H_i, \partial H q(t')][H_k, \partial H q(t'')] \rangle \quad (6) \]

The eigenvalues of the geometric tensor are thus upper bounded by the sum of OTOCs. This means that regions of higher distinguishability correspond to the large operator spreading of the local terms in the Hamiltonian. Moreover, one can see that the time fluctuations of the quantum geometric tensor are directly connected to the time fluctuations of the OTOCs, and this, in turn, provides a framework to unify different aspects of quantum dynamics like quantum chaos and scrambling.\[20,22\]. Recently, it has been shown that in the quantum Dicke model a fidelity out-of-time-order notion is connected to both entanglement spreading and chaos.\[24\]. In our approach, the appearance of OTOCs is a generic feature of the space-time description for the geometry of quantum states.

QGT in the Cluster-XY model.— We now apply these findings in an exactly solvable spin chain. In order to demonstrate meaningfully the time evolution of the geometric tensor after the orthogonal quench, we need a model with at least three parameters. We consider the Cluster-XY model.\[12\]. The model interpolates between a stabilizer Hamiltonian and the quantum XY model. The stabilizer Hamiltonian is the sum of terms of the form $H = \sum_{\mu \neq \mu} \sigma_\mu^z \sigma_\nu^z$ where $\mu, \nu$ label the sites of a lattice and $\sigma^z \sim \mu$ denotes that $\sigma$ is connected to $\mu$. The ground state for this Hamiltonian is important as it is a universal resource for measurement-based quantum computation.\[13,19\].

The Hamiltonian reads

\[ H = -\sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{N} \sigma_i^z \lambda_j \sum_{i=1}^{N} \sigma_i^y \sigma_{i+1}^y + \lambda_x \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x, \quad (7) \]

A somehow canonical way to prepare the quench is the following. Write the manifold parameters as $\lambda = (\lambda', \lambda)$ and consider the submanifold $M_{\lambda}$ where the parameters $\lambda$ are fixed. The orthogonal quench is given by $H(\lambda', \lambda) \mapsto H(\lambda', \lambda + \delta \lambda)$. This sudden quench produces a time evolution on the quantum geometric tensor on $M_{\lambda}$.

The Hamiltonian Eq. (7) can be diagonalized by the standard technique of Jordan-Wigner transformation $c_i = (\prod_{m=1}^{i-1} \sigma_m^z) \sigma_i^x$ that maps the model into a quadratic Hamiltonian of spinless fermions $\{c_m, c_n\} = 0, \{c_m, c_n^\dagger\} = \delta_{nm}$, followed by Fourier transform $c_k = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{ikm} c_m$, $k = \frac{x}{N}(2m + 1)$ after

\[ \text{FIG. 1. Time evolution of the logarithm of the norm of the rescaled metric } g_{\mu\nu}(t)/N \text{ after the orthogonal quench } (\lambda_x, \lambda_y, h = 0) \mapsto (\lambda_x, \lambda_y, h = 0.2) \text{ for } N = 500 \text{ spin. The red dashed lines represent the critical lines at } t = 0. \text{ The white dashed lines represent the states under a critical quench. We see that the singularities of the metric tensor only depend on those at } t = 0. \]
which the Hamiltonian reads

\[ H = 2 \sum_{0 \leq k \leq N} \left[ \epsilon_k (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + i \delta_k (c_k^\dagger c_{-k}^\dagger + c_k c_{-k}) \right], \]

Finally, a Bogoliubov transformation diagonalizes the above Hamiltonian in each \( k \) block by \( \gamma_k = \cos(\theta_k) c_k - i \sin(\theta_k) c_k^\dagger \), where \( \theta_k = -1/2 \arctan \delta_k/\epsilon_k \) with \( \delta_k = \sin(2k) - (\lambda_x - \lambda_y) \sin(k) \) and \( \epsilon_k = \cos(2k) - (\lambda_x + \lambda_y) \cos(k) - h \).

This allows to write the time evolution after a quantum quench in an exact way. Let \( \Omega(\lambda) \) be the ground state of \( H(\lambda) \), its time evolution by the quench Hamiltonian \( H(\lambda') \) is then given by\[28]:

\[ |\Omega(\lambda, t)\rangle = \prod_{0 \leq k \leq N} \left\{ \cos(\chi_k) (\cos(\theta_k) + i \sin(\theta_k) c_k^\dagger c_{-k}^\dagger) + i e^{-4it\Delta_k} \sin(\chi_k) (\cos(\theta_k) c_k^\dagger c_{-k}^\dagger + i \sin(\theta_k)) \right\} |0\rangle \]

where the \( c_k^\dagger \) are the creation operators of the Bogoliubov particles on the vacuum \( |0\rangle \). The energy of the particle of momentum \( k \) is given by \( \Delta_k = \sqrt{\epsilon_k^2 + \chi_k^2} \). At this point one can compute directly the quantum geometric tensor from \( Q_{\mu\nu} = \langle 0 | \partial_\mu | (1 - |\psi_0\rangle \langle \psi_0|) | \partial_\nu | 0 \rangle \) (or from the fidelity \( F \)). In the following, we use the quantum quench defined by \( \lambda \rightarrow \lambda' = \lambda + \delta \lambda \). The angle \( \chi_k \) then contains the quench information: \( \chi_k = \theta_k(\lambda) - \theta_k(\lambda + \delta \lambda) \). Details of the calculation are given in\[28].

Since the Hamiltonian is real, the quantum geometric tensor is real and it is thus a Riemannian metric reading

\[ g_{\mu\nu}(t) = g_{\mu\nu}(0) + \Delta g_{\mu\nu}(t) \] (8)

where \( g_{\mu\nu}(0) = N^{-1} \sum_{0 \leq k \leq \pi} \partial_\mu \theta_k \partial_\nu \theta_k \) and the time dependent part is given by

\[ \Delta g_{\mu\nu}(t) = \frac{1}{N} \sum_{0 \leq k \leq \pi} \left\{ \partial_\mu \theta_k^\prime \partial_\nu \theta_k^\prime (2 - 2 \cos(4t\Delta_k^\prime)) - 4 \sin^2(4t\Delta_k^\prime) \cos^2 \chi_k \sin^2 \chi_k + \left[ \partial_\mu \theta_k^\prime \partial_\nu \theta_k^\prime + \partial_\nu \theta_k^\prime \partial_\mu \theta_k^\prime \right] (\cos(4t\Delta_k^\prime) - 1) - 4t \sin(4t\Delta_k^\prime) \left[ \partial_\mu \theta_k^\prime \partial_\nu \theta_k^\prime + \partial_\nu \theta_k^\prime \partial_\mu \theta_k^\prime \right] \right\} \]

where the primed quantities are those pertaining the quench Hamiltonian, namely \( \theta_k^\prime = \theta_k(\lambda + \delta \lambda) \), \( \Delta_k^\prime = \Delta_k(\lambda + \delta \lambda) \). The study of Eq.\[9\] can give us all the information regarding the time evolution the metric tensor, including the time evolution of the phase diagram and its equilibration. Let us first show that the phase diagram is conserved, that is, no new critical lines are added on top of the ones at \( t = 0 \), nor the original ones are deformed. Notice that the first term in Eq.\[8\] is the geometric tensor at the time zero, \( g_{\mu\nu}(0) \). Divergences may only happen when \( \Delta_k \) or \( \Delta_k^\prime \) becomes null for some \( k \in [-\pi, \pi] \) in the thermodynamic limit. Since these gaps appear in different terms, the phase diagram is not deformed. It must be the one at time zero, plus possibly the phase diagram of the quench Hamiltonian. However, one has to take a limiting procedure because as \( \Delta_k \) shrinks to zero there are competing terms. The details of the limit are given in\[28\]. The result is that as \( \Delta_k^\prime \rightarrow 0 \) no new divergence in \( g_{\mu\nu}(t) \) is introduced: the phase diagram is conserved by the temporal evolution.

In Fig[1] we plot the time evolution of \( \log|g_{\mu\nu}(t)| \) in the plane \( \lambda_x, \lambda_y \) after the orthogonal quench \( h = 0 \rightarrow h = 0.2 \). The initial phase diagram is clearly visible. We have superimposed the lines corresponding to the criticality of the quenching Hamiltonian. As the time evolution proceeds, the initial divergences stay constant but they are thickened as regions of higher distinguishability (which is also, larger curvature). However, this higher curvature never diverges. In other words, the phase diagram is constant. In Fig[2], we plot the log of the norm of the rescaled \( \Delta g \) that is, the time dependent part.

In order to show the equilibrating properties, we first analytically compute the purity as \( Tr \tilde{\rho}^2 = \sum_k \cos(\chi_k)^4 + \sin(\chi_k)^4 \). This shows that for a quench \( \chi_k = \theta_k - \theta_k' \) the purity of the dephased state \( \tilde{\rho} \) is exponentially small in \( N \) and therefore, in view of the theorem, equilibration should ensue.

Conclusions and Outlook.— In this Letter, we have shown that the metric structure that describes the geometry of the manifold of ground states of a family of Hamiltonians can be extended to non-equilibrium states, for example states that evolve unitarily under a quantum quench. We have shown that the initial phase diagram is conserved and that the geometric tensor equilibrates. One of the interesting aspects of this formulation is that the geometric tensor can be written in terms of out-of-time-order commutators. This suggests that the study
of the fluctuations of the geometric tensor can be useful to investigate questions of quantum chaos and the transition to non-integrability. Moreover, the connection of a space-time metrics with OTOCs can be important in order to understand scrambling in black holes\cite{23}. It would be interesting to show how the fluctuations of the QGT are connected to spreading and complexity of entanglement\cite{25} - and their relation to the many-body localization transition - and the emergence of irreversibility in quantum mechanics\cite{26}. In another setting, it would be important to use these methods to address questions in Adiabatic Quantum computation as its performance depends on the curvature along adiabatic evolution\cite{27}. In perspective, we think that the study of the space-time behavior of the quantum geometric tensor may provide an unifying framework to the study of quantum dynamics.

\textbf{SUPPLEMENTAL MATERIAL}

\textbf{QGT after the quench}

Let $H(\lambda)$ be a smooth family of Hamiltonians and let $\Sigma$ the manifold of (unique) ground states $|\psi_0(\lambda)\rangle$ of this family of Hamiltonians. In \cite{5} the QGT on $\Sigma$ has been calculated:

$$q_{\mu\nu}(\lambda) = \sum_{n\neq 0} \frac{\langle \psi_0(\lambda)|\partial_\mu H(\lambda)|\psi_n(\lambda)\rangle \langle \psi_n(\lambda)|\partial_\nu H(\lambda)|\psi_0(\lambda)\rangle}{(E_0(\lambda) - E_n(\lambda))^2}$$

Let us now consider the time evolved manifold $\Sigma_t = \{|\psi_0(\lambda)\rangle\} = \{U(\lambda,t)|\psi_0(\lambda)\rangle\}$, where $U(\lambda,t)$ is a smooth family of unitary evolutions. Since $(UHU^\dagger)|\psi_0\rangle = UH|\psi_0\rangle = E_0(U)|\psi_0\rangle$, $\Sigma_t$ is the manifold of (unique) ground states of the smooth family of Hamiltonians $H(\lambda)_{\rightarrow t} = U(\lambda,t)H(\lambda)U(\lambda,t)^\dagger$. Thus the Eq.[10] on the manifold $\Sigma_t$ becomes

$$q_{\mu\nu}(t) = \sum_{n\neq 0} \frac{\langle \psi_0|\partial_\mu H_{\rightarrow t}|\psi_n\rangle \langle \psi_n|\partial_\nu H_{\rightarrow t}|\psi_0\rangle}{(E_0 - E_n)^2}$$

where we have omitted $\lambda$ and $t$ for the sake of simplicity.

If we choose a time dependent coordinate map that diagonalizes the QGT, we obtain a diagonal tensor with eigenvalues $q_\alpha(t) = \sum_{n\neq 0} \frac{\langle \psi_0|\partial_\alpha H_{\rightarrow t}|\psi_n\rangle \langle \psi_n|\partial_\alpha H_{\rightarrow t}|\psi_0\rangle}{(E_0 - E_n)^2}$

\begin{align*}
q_\alpha(t) &= \sum_{n\neq 0} \frac{\langle \psi_0|\partial_\alpha H_{\rightarrow t} + [H, iD_\alpha]|\psi_n\rangle \langle \psi_n|\partial_\alpha H_{\rightarrow t} + [H, iD_\alpha]|\psi_0\rangle}{(E_0 - E_n)^2} \\
&= \sum_{n\neq 0} \frac{|\langle \psi_0|U\partial_\alpha (UHU^\dagger)U|\psi_n\rangle|^2}{(E_0 - E_n)^2}
\end{align*}

where we used $\partial U^\dagger U = -U^\dagger \partial U$. Let us define $iD_\alpha := \partial_\alpha U^\dagger U$. Notice $D_\alpha$ is a hermitian operator. We have

$$q_\alpha(t) = \sum_{n\neq 0} \frac{|\langle \psi_0|\partial_\alpha H + [H, iD_\alpha]|\psi_n\rangle|^2}{(E_0 - E_n)^2}$$

For the sake of not burdening the notation, we can drop the index $\alpha$

$$q(t) = \sum_{n\neq 0} \frac{|\langle \psi_0|\partial H + [H, iD]|\psi_n\rangle|^2}{(E_0 - E_n)^2}$$

We are going to show that if the time evolution is generated by a quantum quench, that is $U(\lambda, t) = e^{-itH_0(\lambda)}$, $D = \int_0^t dt'U(t')^\dagger \partial H U(t')$.

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Since the time evolution $U$ is generated by the quench Hamiltonian $H^q$, that is a smooth function of the coordinates, the following equation holds:

$$
\partial U^\dagger U = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [e^{itH^q(\lambda + \Delta t)} e^{-itH^q(\lambda)} - e^{itH^q(\lambda)} e^{-itH^q(\lambda)}] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [e^{itH^q + i\partial H^q} e^{-itH^q} - 1]
$$

(15)

Deriving in time one obtains:

$$
\frac{d}{dt} [\partial U^\dagger U] = \frac{d}{dt} \lim_{\Delta t \to 0} \frac{1}{\Delta t} [e^{itH^q + i\partial H^q} e^{-itH^q} - 1] = \frac{1}{i} \lim_{\Delta t \to 0} \frac{1}{d\Delta t} [e^{itH^q + i\partial H^q} (H^q + i\partial H^q) e^{-itH^q} - e^{itH^q + i\partial H^q} H^q e^{-itH^q}]
$$

$$
= i[e^{itH^q + \partial H^q} e^{-itH^q}] = iU^\dagger \partial H^q U
$$

(16)

Finally we integrate both the RHS and the LHS of the last equation. Considering that for $t = 0$, $U^\dagger U = 0$ (indeed $U(\lambda, t = 0) = 1$) the integration leads to:

$$
D = -i\partial U^\dagger U = \int_0^t dt' U(t') \partial H^q U(t')
$$

(17)

The simplified protocol and its relation to the general protocol

We have defined an evolved quantum geometric tensor which eigenvalues can be written as

$$
q(t) = \sum_{n \neq 0} \frac{|\langle \psi_0 | \partial H + [H, iD] | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

(18)

If we start with a family of identical Hamiltonians $H(\lambda) = H_0$ this function assumes a simplified form

$$
q_1(t) = \sum_{n \neq 0} \frac{|\langle \psi_0 | [H, iD] | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

(19)

that we call eigenvalues of the simplified protocol, in opposition to the general protocol that defines $q(t)$. Making statements about $q_1(t)$ is obviously easier that making statements about $q(t)$, thus all the general statement in this paper are proven on $q_1(t)$ and then extended to $q(t)$. In order to extend these statements we need to clarify the relationship between $q(t)$ and $q_1(t)$.

Let us define the following vectors in the Hilbert space $l_2$ endowed with the norm $| \cdot |$

$$
a_n : = \langle \psi_0 | \partial H | \psi_n \rangle / (E_0 - E_n)
$$

$$
b(t)_n : = \langle \psi_0 | [H, iD] | \psi_n \rangle / (E_0 - E_n) = i \langle \psi_0 | D | \psi_n \rangle
$$

(20)

where $n \in \{1; 2; \ldots; N\}$. In this way the following equations hold

$$
q(0) = \sum_{n \neq 0} \frac{|\langle \psi_0 | \partial H | \psi_n \rangle| \langle \psi_0 | D | \psi_n \rangle}{(E_0 - E_n)^2}
$$

$$
= \sum_{n \neq 0} |a_n|^2 = |a|^2
$$

$$
q_1(t) = \sum_{n \neq 0} \frac{|\langle \psi_0 | [H, iD] | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

$$
= \sum_{n \neq 0} |b(t)_n|^2 = |b(t)|^2
$$

$$
q(t) = \sum_{n \neq 0} \frac{|\langle \psi_0 | \partial H + [H, iD] | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

$$
= \sum_{n \neq 0} |(a + b(t))_n|^2 = |a + b(t)|^2
$$

(21)

At this point we can write the triangular inequality and the reverse triangular inequality

$$
|a + b(t)| \leq |a| + |b(t)|
$$

$$
|a + b(t)| \geq ||a| - |b(t)||
$$

(22)

as

$$
q(t) \leq q(0) + q_1(t) + 2 \sqrt{q(0)q_1(t)}
$$

$$
q(t) \geq q(0) + q_1(t) - 2 \sqrt{q(0)q_1(t)}
$$

(23)

We will exploit these bounds to extend proofs for the simplified QGT to the general QGT.

QGT and correlation functions

We are going to show the relationship between the simplified time dependent QGT and the connected correlation functions. In the simplified protocol we can write:

$$
q_1(t) = \sum_{n \neq 0} \frac{|\langle \psi_0 | [H, iD] | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

$$
= \sum_{n \neq 0} \frac{|\langle \psi_0 | E_0 D - E_n D | \psi_n \rangle|^2}{(E_0 - E_n)^2}
$$

$$
= \sum_{n \neq 0} |\langle \psi_0 | D | \psi_n \rangle|^2
$$

$$
= \langle \psi_0 | D^2 | \psi_0 \rangle - \langle \psi_0 | D | \psi_0 \rangle = \langle D^2 \rangle
$$

(24)

Now we make explicit the the role of the involved points of the space-time, and thus we write the operator $D$ in the explicit form. Taking into account the translational invariance of the system we have:

$$
q_1(t) = \langle D^2 \rangle - \langle D \rangle^2
$$

$$
= \int_0^t dt' \int_0^t dt'' \left[ \langle \partial H^q(t') \partial H^q(t'') \rangle - \langle \partial H^q(t) \rangle \langle \partial H^q(t'') \rangle \right]
$$

$$
= N \sum_j \int_0^t dt' \int_0^t dt'' \left[ \langle \partial H^q_j(t') \partial H^q_j(t'') \rangle \right]
$$

(25)
Spreading of unequal times correlation functions

Here we extend the results of \cite{13} about the equal-times correlation functions in local quantum systems to the different-times correlation functions. As a consequence of Lieb-Robinson bound, given a local normalized operator $O_A$ with support on $A$, the following inequality holds \cite{13}:

$$\|O_A(t) - O_A'(t)\| \leq c|A|\exp\left(-\frac{l - v_{LR}|t|}{a^2/2}\right) \tag{26}$$

where $O^I_A(t)$ is the restriction of $O_A$ to the sites of the lattice that are less than $l$ away from $A$, $v_{LR}$ is the Lieb-Robinson velocity, $|A|$ is the cardinality of $A$, $a$ is the lattice spacing and $c$ is a constant that depends only on the maximum norm of the local interactions and on the maximum degree of the interaction vertices. On the other hand, for the exponential clustering theorem \cite{14} the following upper bound on the ground states of a gapped system holds:

$$\|O_A O_B\| \leq \alpha \exp\left(-\frac{d_{AB}}{\chi}\right) \tag{27}$$

where $O_A$ and $O_B$ are normalized local operators, $\chi$ is the correlation length, $d_{AB}$ is the distance between the supports $A$ and $B$ and $\alpha$ does not depend on the size of the system. Once $\Delta_A(t', t'') := O_A(t') - O_A'(t'')$ is defined we can write

$$\|\langle O_A(t') O_B(t'')\rangle\|_C \leq |\langle [\Delta_A(t', t'') O_B(t'')]\rangle| + |\langle O_A'(t') O_B'(t'')\rangle| + |\langle O_A(t') O_B'(t'')\rangle| \tag{28}$$

Since $\langle O_A O_B\rangle \leq \langle O_A O_B\rangle \leq \|O_A\|\|O_B\|$ and $\|O_A'\| \leq \|O_A\|$ the following inequality holds:

$$\|\langle O_A(t') O_B(t'')\rangle\|_C \leq \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \tag{29}$$

at this point the triangular inequality $\|\Delta_A(t', t'')\| \leq \|\Delta_A(t', t'')\| + \|\Delta_A(t', t'')\| \leq 2$ leads to

$$\|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \leq \frac{1}{2} \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \leq \frac{1}{2} \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \tag{30}$$

and thus

$$\|\langle O_A(t') O_B(t'')\rangle\| \leq \frac{1}{2}\|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \leq \frac{1}{2}\|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \tag{31}$$

Now we can replace Eq.\,(26) and Eq.\,(27) in the inequality above and we obtain the following:

$$\|\langle O_A(t') O_B(t'')\rangle\| \leq 2\|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B + \|\Delta_A(t', t'')\|_B \tag{32}$$

Finally we replace $t'$ and $t''$ with the optimal values $t' = \sum_{l=0}^{\infty} e^{-\lambda l} \rho(l)$ and $t'' = \sum_{l=0}^{\infty} e^{-\lambda l} \rho(l)$ and obtain the following upperbound on different-times connected correlation functions:

$$\|\langle O_A(t') O_B(t'')\rangle\| \leq e^{-\frac{d_{AB}}{\chi}} \left[2|\langle O_{A'}(t'')\rangle| + 2|\langle O_{B'}(t''\rangle| + \alpha e^{-\frac{d_{AB}}{\chi + \delta}} \right] \tag{33}$$

where $k = \max\{2|A|, 2|B|; \frac{\alpha}{\chi}\}$.

Equilibration of correlation functions

**Theorem.** Consider a Hamiltonian $H = \sum_n E_n P_n$, satisfying the non-resonance condition, that is, being non degenerate and also having no degenerate gaps: $E_n - E_m = E_k - E_l \Rightarrow n = k \land m = l$. Then, the temporal variance $\sigma^2(C)$ of unequal-time correlation functions $C(t', t'') = \langle A(t') A(t'')\rangle$ is upper bounded as $\sigma^2(C) \leq \|A\|^4 Tr \tilde{P}^2$.

**Proof.** Let $\langle A(t') A(t'')\rangle$ be a unequal-times correlation function. First of all the average of the function respect to the two involved times is:

$$\langle Tr(A(t) A(t'))\rangle = \sum_{mn} A_{mn} A_{n} P_{m} e^{i(t (E_n - E_m))} e^{it' (E_n - E_l)}$$

$$= \sum_n A_{nn}^2 P_{nn} \tag{34}$$

and thus the oscillations around the average:

$$\langle Tr(A(t) A(t'))\rangle - \langle Tr(A(t) A(t'))\rangle \leq \sum_{mn} A_{mn} A_{n} P_{m} e^{i(t (E_n - E_m))} e^{it' (E_n - E_l)} (1 - \delta_{mn} \delta_{nl})$$

In order to find the variance we need the square of the oscillations above, that is

$$\langle (Tr(A(t) A(t'))\rangle - \langle Tr(A(t) A(t'))\rangle)^2 \leq \langle \sum_{mn} A_{mn}^2 A_{n} P_{m} e^{i(t (E_n - E_m))} e^{it' (E_n - E_l)} (1 - \delta_{mn} \delta_{nl})^2 \rangle$$

$$= \langle \sum_{mn} \sum_{rs} A_{mn} A_{n} P_{m} e^{i(t (E_n - E_m))} e^{it' (E_n - E_l)} \times (1 - \delta_{mn} \delta_{nl}) A_{rs} A_{s} P_{r} e^{i(t (E_n - E_m))} e^{it' (E_n - E_l)} (1 - \delta_{rs} \delta_{st}) \rangle$$
Finally we consider the double-times average of the above equation, that is the variance:

\[
\sigma^2 = (\text{Tr}(A(t)A(t')\rho) - \text{Tr}(A(t)A(t')\rho))^2
\]

\[
= \sum_{mnl} \sum_{rst} A_{mn} A_{nl} A_{rs} A_{st} \times \rho_{tm}\rho_{tr}\delta_{ms}\delta_{sr}\delta_{nt}(1 - \delta_{mn}\delta_{nl})(1 - \delta_{rs}\delta_{st})
\]

\[
= \sum_{mn} A_{mn} A_{nm} A_{mn} \rho_{nm} \rho_{nn} (1 - \delta_{mn})
\]

\[
= \sum_{m \neq n} |A_{mn}|^2 \rho_{mn} \rho_{nn}
\]

\[
\leq \sum_{mn} |A_{mn}|^2 A_{nm} \rho_{mm} A_{nn} \rho_{nn} \leq \max_{mn} |A_{mn}|^2 \text{Tr}(\overline{A}p)^2
\]

At this point is useful to remark the following inequality:

\[
\max_{mn} |A_{mn}| = \max_{mn} |\langle m|A|n\rangle|^2 \leq \max_k ||A||^2 \langle k|A|k\rangle \leq ||A||^2
\]

where in the last line we have used \(\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}\).

Finally we can state that for sufficiently large \(N\) and sufficiently large \(t\) the QGT \(q(t)\) oscillates around \(q(0) + t^2 (\partial H^q(t')\partial H^q(t''))_C\) at most linearly in time.

**QGT and OTOCs**

The metric tensor of the simplified protocol can be upper-bounded with an OTOC-related function as follows:

\[
q_1(t) = \sum_{n \neq 0} \left[ \frac{||\langle \psi_0 | [H,D] | \psi_0 \rangle ||^2}{(E_0 - E_n)^2} \right]
\]

\[
\leq \frac{1}{\Delta^2} \sum_{n \neq 0} ||\langle \psi_0 | [H,D] | \psi_n \rangle ||^2
\]

\[
= \frac{1}{\Delta^2} \sum_n ||\langle \psi_0 | [H,D] | \psi_n \rangle ||^2
\]

\[
= \frac{1}{\Delta^2} \sum_{ijkl} \int_0^t \int_0^t dt' dt'' \langle [H_i, \partial H^q_j(t')] [H_k, \partial H^q_l(t'')] \rangle^1
\]

where we have called \(\Delta\) the first energy gap of the Hamiltonian \(H\) of which the \(|\psi_n\rangle\) are eigenstates.

Since \(q(t) \leq q_1(t) + q(0) + 2\sqrt{q_1(t)q(0)}\) the upperbound above is equally significant for the general time dependent QGT:

\[
q(t) \leq q(0) + \frac{1}{\Delta^2} \sum_{ijkl} \int_0^t \int_0^t dt' dt'' \langle [H_i, \partial H^q_j(t')] [H_k, \partial H^q_l(t'')] \rangle^1
\]

\[
+ 2\sqrt{q(0) \frac{1}{\Delta^2} \sum_{ijkl} \int_0^t \int_0^t dt' dt'' \langle [H_i, \partial H^q_j(t')] [H_k, \partial H^q_l(t'')] \rangle^1}
\]

\[
(40)
\]

**Evolution of the metric in Cluster XY model and its criticalities**

Given the ground state \(|\Omega(\lambda)\rangle\) for the Cluster XY model with control parameters \(\lambda = (\lambda_x, \lambda_y, h)\), the evolved state after the application of a quench Hamiltonian \(H(\lambda + q)\) has been calculated in [12], where it is expressed through the fermionic creation and annihilation operators \(\gamma^+_k(\lambda)\) and \(\gamma_k(\lambda)\):

\[
|\Omega(\lambda, t)\rangle := e^{-iH(\lambda + \delta \lambda) t} |\Omega(\lambda)\rangle
\]

\[
= \prod_{0 \leq k < \pi} \left[ \cos(\chi_k(\lambda, \lambda^g)) + ie^{-4it\Delta_k(\lambda^g)} \times \sin(\chi_k(\lambda, \lambda^g)) \gamma^+_k(\lambda^g) \gamma^+_k(\lambda^g) \right]|\Omega(\lambda^g)\rangle
\]

\[
(41)
\]
where
\[ \theta_k(\lambda) = -\frac{1}{2} \arctan\left( \frac{\delta_k}{\epsilon_k} \right) \]
\[ \Delta_k(\lambda) = \sqrt{\delta_k^2 + \epsilon_k^2} \]
\[ \delta_k = \sin(2k) - (\lambda_x - \lambda_y) \sin(k) \]
\[ \epsilon_k = \cos(2k) - (\lambda_x + \lambda_y) \cos(k) - \hbar \]
\[ \chi_k(\lambda, \lambda^*) = \theta_k(\lambda) - \theta_k(\lambda^*) \]

We are going to write \( |\Omega(\lambda, t)\rangle \) in terms of the fermionic operators \( c^\dagger_k \) and \( c_k \) that are independent on \( \lambda \). They are related to the \( \gamma_k(\lambda) \) by
\[ \gamma_k = \cos(\theta_k(\lambda))c_k - i \sin(\theta_k(\lambda))c_k^\dagger \]

The ground state then reads \( |\Omega(\lambda)\rangle \):
\[ |\Omega(\lambda)\rangle = \prod_{0 \leq k \leq \pi} \left( \cos(\theta_k(\lambda)) + i \sin(\theta_k(\lambda))c_k^\dagger c_{-k} \right) |0\rangle_c \]

where \( |0\rangle_c \) is the vacuum of the \( c_k \) operators. Substituting in \( |\Omega(\lambda, t)\rangle \) and exploiting the algebra of the \( c_k \) we obtain:
\[ |\Omega(\lambda, t)\rangle = \prod_{0 \leq k \leq \pi} \left\{ \cos(\chi_k(\lambda, \lambda^*))\cos(\theta_k(\lambda^*)) + i \sin(\theta_k(\lambda^*))c_k^\dagger c_{-k} + ie^{-4it\Delta_k} \sin(\chi_k(\lambda, \lambda^*))\cos(\theta_k(\lambda^*))c_k^\dagger c_{-k} + i \sin(\theta_k(\lambda^*)) \right\} |0\rangle \]

From direct calculation of the fidelity we then obtain
\[ F^2 = |\langle \Omega(\lambda', t) | \Omega(\lambda, t) \rangle|^2 \]
\[ = \prod_{0 \leq k \leq \pi} \left| \cos(\theta'_k - \theta_k) \cos(\chi'_k) - \cos(\theta'_k - \theta_k) \sin(\chi'_k) \sin(\chi_k) + \sin(\theta'_k - \theta_k) e^{4it\Delta_k} \cos(\chi'_k) \sin(\chi_k) - \sin(\theta'_k - \theta_k) e^{4it\Delta_k} \sin(\chi'_k) \cos(\chi_k) \right|^2 \]

with
\[ \theta_k = \theta_k(\lambda + \delta \lambda) \]
\[ \Delta_k = \Delta_k(\lambda + \delta \lambda) \]
\[ \chi_k = \theta_k(\lambda' + q) - \theta_k(\lambda' + q) \]

To get the metric, we look at the fidelity for an infinitesimal shift \( \lambda' = \lambda + \delta \lambda \) and develop the above expression to the second order in \( \delta \lambda \). We finally obtain the rescaled time-dependent metric tensor
\[ g_{\mu\nu} = \frac{1}{N} \sum_{0 \leq k \leq \pi} \left[ \left( \partial_{\mu} \theta_k \right) \left( \partial_{\nu} \theta_k \right) + \left( \partial_{\mu} \theta'_k \right) \left( \partial_{\nu} \theta'_k \right) \times \left( 2 - 2 \cos(4t\Delta_k') - 4 \sin^2(4t\Delta_k') \cos^2(\chi_k) \sin^2(\chi_k) \right) \right. \]
\[ + \left[ \left( \partial_{\mu} \theta_k \right) \left( \partial_{\nu} \theta_k \right) + \left( \partial_{\mu} \theta'_k \right) \left( \partial_{\nu} \theta'_k \right) \right] \left( \cos(4t\Delta_k') - 1 \right) \]
\[ - 4t \sin(4t\Delta_k') \left[ \left( \partial_{\mu} \theta_k \right) \left( \partial_{\nu} \Delta_k' \right) + \left( \partial_{\mu} \theta'_k \right) \left( \partial_{\nu} \Delta_k' \right) \right] \]
\[ \times \cos(\chi_k) \sin(\chi_k) [1 - 2 \sin^2(\chi_k)] \]
\[ + 16t^2 \sin^2(\chi_k) (1 - \sin^2(\chi_k)) \left( \partial_{\mu} \Delta_k' \right) \left( \partial_{\nu} \Delta_k' \right) \]
Consider now
\[ \partial_{\mu} \theta'_k = \frac{-1}{2} \epsilon' k \partial_{\mu} \delta'_k - \delta'_k \partial_{\mu} \epsilon'_k \]
\[ \partial_{\mu} \Delta'_k = \frac{\epsilon' k \partial_{\mu} \delta'_k + \delta'_k \partial_{\mu} \epsilon'_k}{\Delta'_k} \]  
(52)

and observe that both \( \partial_{\mu} \delta'_k \) and \( \partial_{\mu} \epsilon'_k \) are bounded, therefore
\[ |\partial_{\mu} \theta'_k| \leq \frac{1}{2} \left( \left| \epsilon' k \partial_{\mu} \delta'_k \right| + \left| \delta'_k \partial_{\mu} \epsilon'_k \right| \right) \frac{1}{\Delta'_k} \leq F_{\mu} \frac{1}{\Delta'_k} \]
\[ |\partial_{\mu} \Delta'_k| \leq \frac{\epsilon' k \partial_{\mu} \delta'_k}{\sqrt{\delta'^2_k + \delta'^2_k}} + \frac{\delta'_k \partial_{\mu} \epsilon'_k}{\sqrt{\epsilon'^2_k + \epsilon'^2_k}} \leq G_{\mu} \]  
(53)

where \( F_{\mu} \) and \( G_{\mu} \) are not diverging. Then only \( \partial_{\mu} \theta'_k \) may diverge at most as \( \Delta'_k^{-1} \). So the term \( D \) cannot diverge, while the terms \( A, B \) and \( C \) cannot diverge because the divergences of \( \partial_{\mu} \theta'_k \) are cancelled by multiplication with terms that go to zero at the first order in \( \Delta'_k \).

Dephased state purity in the Cluster XY model

Here we calculate the purity of the ground state \( |\Omega(\lambda)\rangle \) of the Cluster XY Hamiltonian \( H(\lambda) \) dephased in the eigenbasis of the quench Hamiltonian \( H(\lambda + q) \). The purity is
\[ \text{Tr}[|\Phi\rangle\langle\Phi|] = \sum_n |c_n|^4 \]  
where \( |c_n|^2 \) are the populations in the eigenbasis of \( H(\lambda + q) \). The expansion of the ground state \( |\Omega(\lambda)\rangle \) of \( H(\lambda) \) in the eigenbasis of \( H(\lambda + q) \) reads\[12\]
\[ |\Omega(\lambda)\rangle = \prod_{0 \leq k \leq \pi} \left( \cos(\chi_k) + i \sin(\chi_k) \gamma^+_{+k}(\lambda^q) \gamma^+_{-k}(\lambda^q) \right) |\Omega(\lambda + q)\rangle \]  
(54)

where \( \chi_k \) is the same of Eq.[42], from which one easily obtains
\[ \text{Tr}\left(|\Phi\rangle\langle\Phi|\right) = \prod_{0 \leq k \leq \pi} \left( \cos(\chi_k)^4 + \sin(\chi_k)^4 \right) \]  
(55)