Abstract

We study higher depth algebras. We introduce several examples of such structures starting from the notion of $N$-differential graded algebras and build up to the concept of $A^N_{\infty}$-algebras.

1 Introduction

In this paper we continue our program aim to study algebraic structures of higher depth, i.e., algebraic structures such that the axioms satisfied by the structural operations on it involve triple or higher order compositions. The simplest and most prominent algebraic structure of such type are $N$-complexes. According to Mayer [21] and Kapranov [15] a $N$-complex is a graded vector space $V$ together with a degree one map $d : V \to V$ such that $d^N = 0$. This condition involves $N$ compositions of the structural map $d$, this makes the definition of $N$-complexes quite different from that of a complex which involves only two compositions of the operator $d$, and so it is quadratic, as most algebraic structures we are familiar with. In this paper we introduce other examples of algebraic structures of higher depth. We put special interest in the operadic description of such structures and in the corresponding deformation theory.

Algebraic structures of higher depth arise when one defines an associative product on a $N$-complex. In this situation there are two consistent choices. One choice begins fixing $q$ a primitive $N$-root of unity. Then one defines the notion of $q$-differential graded algebras, which are associative graded algebras provided with a $N$-differential satisfying a twisted graded Leibnitz rule. The condition of associativity and the Leibnitz rule are quadratic, but the condition for the $N$-differential is of higher depth. This sort of algebraic structure was first discussed by Kapranov and Dubois-Violette in [8], [9], [15], and has been further studied in several works, among them [1], [10], [11], [18], [16]. Deformations of
$q$-differential graded algebras are controlled by the $q$-analogue of the Maurer-Cartan equation, which was defined for general $N$ and explicitly computed for small values of $N$ in [5].

The other choice introduced in [3] is that of $N$-differential graded algebras, which are $N$-complexes provided with an associative product such that the $N$-differential satisfies the graded Leibnitz rule. Again this is a sort of hybrid algebraic structure with the higher depth condition coming from the $N$-differential. We remark that by now there are quite a few known examples of algebras of this sort, we summarize these examples in Section 2. In this work we introduce the graded operad $N$-	extit{dga} whose algebras are $N$-differential graded algebras and compute its generating series. We study a couple of related algebraic structures, namely, $N$-differential graded Lie algebras and $N$-codifferential graded coalgebras. We introduce the graded operad $N$-	extit{dgla} whose algebras are $N$-differential graded Lie algebras and compute its generating function. In Section 3 we study deformations of $N$-codifferential graded coalgebras into $M$-codifferential graded coalgebras and show that they are controlled by the $(N,M)$ Maurer-Cartan equation.

All algebraic structures mention so far are of higher depth only because of the condition imposed on a $N$-differential. Our next goal in this paper is to introduce new families of algebras of higher depth, which are homogeneous in the sense that in each axiom present in their definitions, exactly $N$ structural operations are involved. In order to motivate our definitions it is convenient to switch to the world of graded coalgebras and codifferentials. Recall that a differential graded algebra structure on a graded vector space $A$ is the same as a coderivation $\delta$ on the graded coalgebra $T^{\leq 2}(A[1])$ satisfying $\delta^2 = 0$. If we relax the latter condition and demand instead that $\delta^N = 0$, then we arrive to the notion of differential graded algebra of depth $N$. Any such coderivation is determined by a couple of maps $d : A[1] \rightarrow A[1]$ and $m : A[1] \otimes A[1] \rightarrow A[1]$. If the product $m$ vanishes one recovers the definition of a $N$-complex. If instead $d$ is zero then we arrived to the notion of a $N$-associative algebra.

We construct the operad whose algebras are differential graded algebras of depth $N$, and also the operad whose algebras are $N$-associative algebras. We show that infinitesimal deformations of $N$-associative algebras into $M$-associative algebras, $M \geq N$, are controlled by a well-defined cohomology group. We write explicitly the conditions determining if an associative algebra admits a non-trivial deformation into a 3-associative algebra. In the final section we introduce the notion of $A_N^\infty$-algebras which generalizes both $N$-associative algebras and $A_\infty$-algebras. $A_N^\infty$-algebras are defined as nilpotent coderivations on the full cotensor coalgebra $T(A[1])$. We introduce the operad for $A_N^\infty$-algebras, study infinitesimal deformations of $A_N^\infty$-algebras, and show that the moduli space of de-
formations of $A^N_\infty$-algebras into $A^M_\infty$-algebras is controlled by the $(N, M)$ Maurer-Cartan equation mentioned above.

We close the introduction emphasizing that we are only beginning the study higher depth algebras. Much more work, both theoretical and practical, is needed in order to have a better grasp of the meaning and applications of such structures. We believe that new forms of infinitesimal symmetries will be uncover along this line of thought through the notion of Lie $N$-algebroids to be developed in [2].

2 N-differential graded algebras

Throughout this paper we work with the category $gvect$ of graded $k$-vector spaces $V$ over a field $k$ of characteristic zero. The degree of an homogeneous element $v \in V$ is denote by $\bar{v} \in \mathbb{Z}$. $V[k]$ denotes the $\mathbb{Z}$-graded vector space such that $(V[k])^i = V^{i+k}$ for $i \in \mathbb{Z}$. The superdimension of a graded vector space $V$ is given by

$$sdim(V) = \sum_{i \in \mathbb{Z}} dim(V_{2i}) - \sum_{i \in \mathbb{Z}} dim(V_{2i+1}).$$

Definition 1. A $N$-complex is a graded vector space $V$ together with a degree one map $d : V \rightarrow V$ such that $d^N = 0$. An $N$-complex is said to be a proper $N$-complex if $d^{N-1} \neq 0$.

Definition 2. A $N$-differential graded algebra (N-dga) is a triple $(A, m, d)$, where $A$ is a graded vector space, $m : A \otimes A \rightarrow A$ and $d : A \rightarrow A$ are maps of degree zero and one, respectively, such that

1. $(A, m)$ is a graded associative algebra.

2. $d$ satisfies the graded Leibnitz rule $d(ab) = d(a)b + (-1)^{\bar{a}}a d(b)$ for $a, b \in A$.

3. $d^N = 0$.

We said that a $A$ is a proper $N$-dga if $d^{N-1} \neq 0$.

Definition 3. 1. Let $N^{ul}$-dvect be the category whose objects are $N$-differential graded vector spaces for some $N$. Morphisms in $N^{ul}$-dvect are maps $T : V \rightarrow W$ such that $dT = Td$.

2. Let $N^{ul}$-dga be the category whose objects are $N$-differential graded algebras for some $N$. Morphisms in $N^{ul}$-dga are maps $T : V \rightarrow W$ such that $dT = Td$ and $m(T \otimes T) = Tm$. 3
The following result justifies from the point of view of category theory our definition of $N$-differential graded algebras.

Lemma 4. Tensor product gives $N^{il}$-$dgvect$ the structure of a monoidal category. $N^{il}$-$dga$ is the category of monoids in $N^{il}$-$dgvect$.

We remark that by now there are several known examples of nil-differential graded algebras, see [2], [3], [4] and [5], which may be classified as follows:

1. Differential graded algebras are the same as 2-differential graded algebras.

2. If $C$ is a $N$-complex, then $End(C)$ is a $(2n + 1)$-differential graded algebra. There are plenty of examples of $N$-complexes [1], [10], [11], [15], [16], [18].

3. $N$-flat connections and $N$-flat Riemannian metrics gives rise to nil-differential graded algebras. A connection $A$ on a vector bundle is called $N$-flat if and only if its curvature $F_A$ satisfies $(F_A)^N = 0$. A Riemanniann metric $g$ on a manifold is $N$-flat if the Levi-Civita connection on the tangent bundle is $N$-flat.

4. Differential forms of depth $N$ on affine manifolds of dimension $m$ are $m(N-1)+1$ differential graded algebras.

5. One can generalize Sullivan’s notion of algebraic differential forms on a simplicial sets, to the notion of differential forms of depth $N$. For finitely generated simplicial sets these algebras are examples of Nil-differential graded algebras.

6. Zeilberger in [24] defines the notion of difference forms on the integral lattice in Euclidean space. His construction can be generalized to define the algebra of difference forms of depth $N$ on simplicial sets. Difference forms of depth $N$ on finitely generated simplicial sets are twisted Nil-differential graded algebras.

7. Lie $N$-algebroids are examples of $N$-differential graded algebras.

We are going to use the language of operads, the reader may consult [20] for more on operads. The generating series of an operad $O$ in $vect$ is $\sum_{n=0}^{\infty} \dim(O(n)) \frac{N^n}{n!}$, the generating series of an operad $O$ in $gvect$ is $\sum_{n=0}^{\infty} sdim(O(n)) \frac{N^n}{n!}$.

Definition 5. Let $N$-$dga$ be the free graded operad generated by a degree one element $d \in N$-$dga(1)$, and a degree zero element $m \in N$-$dga(2)$, subject to the relations $m^2 = 0$ and $dm = dm$, $d^N = 0$.

Graphically $N$-$dga$ is the free operad generated by the tree
representing the $N$-differential and the tree

representing the product, subject to the relations

The following result is proved using the graphical description above.

**Proposition 6.**

1. Algebras in $gvect$ over $N$-dga are $N$-differential graded algebras.

2. $\dim(N$-dga$(n)) = n! N^n$. The generating series of $N$-dga is $\frac{N x}{1 - N x}$.

3. If $N$ is even then $s\dim(N$-dga$(n)) = 0$, for $n \geq 2$. The generating series of $N$-dga as a graded linear operad is $x$.

4. If $N$ is odd then $s\dim(N$-dga$(n)) = n!$, for $n \geq 1$. The generating series of $N$-dga as a graded operad is $\frac{x}{1 - x}$.
Let us introduce the corresponding notion in the context of Lie algebras.

**Definition 7.** A $N$-differential graded Lie algebra ($N$-dgla) is a $\mathbb{Z}$-graded vector space $L$ together with a degree zero map $[\ ,
\ ] : L \otimes L \to L$ and a degree one map $d : L \to L$ such that

1. $(L, [\ ,
\ ])$ is a graded Lie algebra.
2. $d[a, b] = [d(a), b] + (-1)^a[a, d(b)]$.
3. $d^N = 0$.

Let $\tau : \{1, 2\} \to \{1, 2\}$ be the non-trivial permutation on $\{1, 2\}$.

**Definition 8.** Let $N$-dgla be the free graded operad generated by $d \in N$-dgla(1) of degree one and $m \in N$-dgla(2) of degree zero, subject to the relations $m\tau = -m$, $m^2 = 0$, $dm = dm$ and $d^N = 0$.

**Proposition 9.** 1. Algebras over $N$-dgla are $N$-differential graded Lie algebras.
2. $\dim(N$-dgla(\(n\))) = (n - 1)!N^n$. The generating series of $N$-dgla as a linear operad is $\ln(\frac{1}{1-Nx})$.
3. If $N$ is even the superdim($N$-dgla(\(n\))) = 0, for $n \geq 2$. The generating series of $N$-dgla as a graded operad is $x$.
4. If $N$ is odd the superdim($N$-dgla(\(n\))) = $(n - 1)!$, for $n \geq 2$. The generating series of $N$-dgla as a graded operad is $\ln(\frac{1}{1-x})$.

### 3 Deformation of coalgebras

In this section we introduce the notion of $N$-codifferential graded coalgebras, and study the deformation theory of $N$-codifferentials.

**Definition 10.** A graded coalgebra is a $\mathbb{Z}$-graded vector space $C$ together with a degree zero map $\Delta : C \to C \otimes C$ such that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow \Delta & & \downarrow \Delta \otimes 1 \\
C \otimes C & \xrightarrow{1 \otimes \Delta} & C \otimes C \otimes C
\end{array}
$$
commutes. If in addition we have a degree zero map $\epsilon : C \to k$ such that the diagram
\[
\begin{array}{c}
\Delta \\
\downarrow \\
C \otimes C \\
\downarrow \\
C
\end{array}
\]
commutes, then we say that $C$ is a graded coalgebra with counit.

**Definition 11.** A coderivation $\delta : C \to C$ on a graded coalgebra $(C, \Delta)$ is a linear map such that the diagram
\[
\begin{array}{c}
\Delta \\
\downarrow \\
C \otimes C \\
\downarrow \\
C \otimes C
\end{array}
\]
commutes. We denote by $\text{Coder}(C)$ the space of coderivations on $C$.

**Definition 12.** A $N$-codifferential graded coalgebra ($N$-cgc) is a pair $(C, \delta)$ where $C$ is a $\mathbb{Z}$-graded coalgebra and $\delta : C \to C$ is a degree one coderivation such that $\delta^N = 0$.

A 1-cgc is a graded coassociative coalgebra. A 2-cgc is a codifferential graded coalgebra. Below we use the fact that a $N$-differential graded algebra $A$ may be regarded as a $N$ differential graded Lie algebra with bracket $[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$, for $a, b \in A$.

**Proposition 13.** Let $(C, \delta)$ be a $N$-cgc.

1. $\text{End}(C)$ is a $(2N - 1)$-dga.
2. $\text{Coder}(C)$ is a $(2N - 1)$-differential graded Lie algebra.

**Proof.**
1. Composition gives $\text{End}(C)$ the structure of an associative algebra. Differential $d$ on $\text{End}(C)$ is defined by $d(f) = \delta \circ f - (-1)^{\bar{f}} f \circ \delta$, for $f \in \text{End}(C)$. $d$ and $m$ satisfy the graded Leibniz rule. The identity
\[
d^n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \delta^k \circ f \circ \delta^{n-k}
\]
implies that $(\text{End}(C), m, d)$ is a $(2N - 1)$-dga.

2. By the previous remark $\text{End}(C)$ is a $(2N - 1)$-dgla. It is easy to show that $\text{Coder}(C) \subseteq \text{End}(C)$ is closed under Lie brackets, thus it is a $(2N - 1)$-dgla. \qed
We consider deformations of \( N \)-codifferentials and show that such deformations are controlled by the \((N,M)\) Maurer-Cartan equation. Let Artin be the category whose objects are local \( k \)-algebras.

**Definition 14.** Let \( C \) be a \( N \)-cgc and \( a \) and object in Artin with maximal ideal \( a_+ \). A \( M \)-deformation of \( C \) over \( a \) is a \( M \)-cgc \( C_a \) over \( a \), with \( M \geq N \), such that \( C_a/a_+C_a \) is isomorphic to \( C \) as \( N \)-cgc.

As usual in deformation theory the core of Definition 14 is that \( \delta C_a \) reduces to \( \delta C \) and \( \Delta C_a \) reduces to \( \Delta C \) under the natural projection \( \pi : C_a \to C_a/a_+C_a \sim C \). Assume that \( C_a = C \otimes a \) as graded coalgebras over \( a \). We have a vector spaces decomposition

\[
C_a = C \otimes a = C \otimes (k \oplus a_+) = (C \otimes k) \oplus (C \otimes a_+) = C \oplus (C \otimes a_+).
\]

Since \( \delta C_a \) reduces to \( \delta C \) under projection \( \pi \) we must have that \( \delta C_a = \delta C + e \), where \( e \in \text{Coder}(C \otimes a_+) \) is a degree one coderivation. The fact that \( \delta C_a = 0 \) implies that \( e \) should satisfy certain identities which we call the \((N,M)\) Maurer-Cartan equation.

Let us review the construction of \((N,M)\) Maurer-Cartan equation [3]. For \( s = (s_1, \ldots, s_n) \in \mathbb{N}^n \) we set \( l(s) = n \) and \( |s| = \sum_i s_i \). For \( 1 \leq i < n \), let \( s_{\geq i} = (s_{i+1}, \ldots, s_n) \), for \( 1 < i \leq n \), let \( s_{\leq i} = (s_1, \ldots, s_{i-1}) \), also set \( s_{>n} = s_{<1} = \emptyset \). \( \mathbb{N}^{(\infty)} \) denotes the set \( \bigsqcup_{n=0}^{\infty} \mathbb{N}^n \), where by convention \( \mathbb{N}^0 = \{\emptyset\} \). For \( e \in \text{Coder}(C) \) we set \( e^{(s)} = e^{(s_1)} \cdots e^{(s_n)} \), where \( e^{(a)} = d^a(e) \) if \( a \geq 1 \), \( e^{(0)} = e \) and \( e^{(\emptyset)} = 1 \). For \( M \in \mathbb{N} \) we let

\[
E_M = \{ s \in \mathbb{N}^{(\infty)} : |s| + l(s) \leq M \}
\]

and for \( s \in E_M \) we define integer \( M(s) \) by \( M(s) = M - |s| - l(s) \).

Recall that a discrete quantum mechanical system is given by a directed graph together with a weight attached to each of its edges. Let us introduce a discrete quantum mechanical system given by the following data

1. The set of vertices is \( \mathbb{N}^{(\infty)} \).
2. There is a unique directed edge from vertex \( s \) to vertex \( t \) if and only if

\[
t \in \{(0,s)\} \cup \{s\} \cup \{s + e_i \mid 1 \leq i \leq l(s)\}
\]

where \( e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^{l(s)} \).
3. An edge \( e \) with source \( s(e) \) and target \( t(e) \) is weighted according to the table
The set $P_M(\emptyset, s)$ consists of all paths $\gamma = (e_1, \ldots, e_M)$ such that $s(e_1) = \emptyset$ and $t(e_M) = s$. The weight $v(\gamma)$ of $\gamma \in P_M(\emptyset, s)$ is $v(\gamma) = \prod_{i=1}^M v(e_i)$.

The $(N, M)$ Maurer-Cartan equation is
$$\sum_{k=1}^{M-1} c_k \delta^k = 0,$$
where
$$c_k = \sum_{\begin{subarray}{c} s \in E_M \\ M(s) = k \\ s_i < N \end{subarray}} c(s, M) e^{(s)} \quad \text{and} \quad c(s, M) = \sum_{\gamma \in P_M(\emptyset, s)} v(\gamma).$$

In Section 6 we shall encounter the problem of understanding the deformations of a codifferential.

4 Differential graded algebras of depth $N$

In this section we introduce the notion of differential graded algebras of depth $N$ and construct explicitly the operad controlling such algebraic structures.

Let $A$ be a $\mathbb{Z}$-graded vector space. Let $T(A[1])$ be the cofree coalgebra cogenerated by $A[1]$, that is,
$$T(A[1]) = \bigoplus_{n \geq 1} A[1]^\otimes n$$
with $\Delta : T(A[1]) \to T(A[1])$ given for $n \geq 2$ by
$$\Delta(a_1, \ldots, a_n) = \sum (a_1, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_n).$$

We also consider sub-coalgebras
$$T^{\leq k}(A[1]) = \bigoplus_{1 \leq n \leq k} A[1]^\otimes n.$$

Recall that a sequence of homomorphisms $m_k : A[1]^\otimes k \to A[1]$ of degree one, for $k \in \mathbb{N}_+$, defines a unique coderivation $\delta : T(A[1]) \to T(A[1])$ on $A[1]^\otimes n$ given by
$$\delta(a_1, \ldots, a_n) = \sum_{k=1}^n \sum_{i=1}^{n-k+1} (-1)^{\sum_{j=1}^{i-1} n_k} (a_1, \ldots, a_{i-1}, m_k(a_i, \ldots, a_{i+k-1}), a_{i+k+1}, \ldots, a_n).$$
One may think of a dga-structure on a graded vector space $A$ as being a codifferential $\delta \in \text{Coder}(T^{\leq 2}A[1])$ of degree one given by $\delta = d + m$. The condition $\delta^2 = 0$ is equivalent to the properties defining a differential graded algebra. Explicitly,

$$\delta^2 = (d + m)(d + m) = d^2 + dm + md + m^2.$$ 

We see that $\delta^2 = 0$ if and only if $d^2 = 0$, $dm + md = 0$ and $m^2 = 0$, i.e., $d$ is a square free map, satisfying the graded Leibnitz rule with respect to the associative product $m$.

We come to the main idea of this paper. Instead of looking at codifferentials on $T^{\leq 2}(A[1])$, we may as well consider 3-codifferentials on $T^{\leq 2}(A[1])$, or more generally a $N$-codifferentials for $N \geq 2$. This idea leads to a new mathematical entity which we define below.

**Definition 15.** Let $A$ be a $\mathbb{Z}$-graded vector space. A structure of graded algebra of depth $N$ on $A$ is given by a degree one coderivation $\delta \in \text{Coder}(T^{\leq 2}(A[1]))$ such that $\delta^N = 0$.

By the previous remarks a graded algebra of depth two is the same as a differential graded algebra. From the decomposition of $\delta = d + m$ in homogeneous components with $d : A[1] \to A[1]$ and $m : A[1]^{\otimes 2} \to A[1]$, we can see that for a differential graded algebra of depth 3 we have

$$\delta^3 = (d + m)(d + m)(d + m) = (d + m)(d^2 + dm + md + mm) = d^3 + d^2m + dmd + dmm + md^2 + mdm + mmd + mm^2 = 0.$$ 

Looking at the homogeneous components of the identity above, we conclude that $d$ and $m$ should satisfy:

1. $m^3 = 0$. It is natural to call this property the condition of 3-associativity for $m$.

2. $d$ satisfies two generalized graded Leibnitz rules

   $$d^2m + dmd + md^2 = 0,$$

   $$dmm + mdm + mmd = 0.$$ 

3. $d^3 = 0$, i.e., $d$ is a 3-differential on $A$. 

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Before we continue with our study of algebras of depth $N$ we introduce a few combinatorial tools.

**Definition 16.** A finite planar rooted tree $\Gamma$ consists of a pair of finite sets $V_{\Gamma}$ and $E_{\Gamma}$, called the vertices and edges of $\Gamma$, and maps $s, t : E_{\Gamma} \to V_{\Gamma}$ satisfying:

1. There exists a unique distinguished vertex $r \in V_{\Gamma}$ called the root of $\Gamma$, such that $|t^{-1}(v_r)| = 1$ and $|s^{-1}(v_r)| = 0$.

2. There is a unique path from each vertex to the root.

A vertex $v \in V_{\Gamma}$ such that $|t^{-1}(v)| = 0$ is called a leave. A vertex $v$ the is neither the root nor a leave is called internal. A vertex $v$ is said to be $n$-ary if $|t^{-1}(v)| = n$. $l(T)$ is the set of leaves in $T$.

We use planar rooted trees to encode in a simpler notation operators on $A$. First we associate rooted trees with the operators $d$ and $m$ as done in Section 2. It should be clear that from these simple rules we can associate with each tree $T$ whose internal vertices are either unary or binary an operator $O_T$ from $A^\otimes |l(T)|$ to $A$. Formally, $O_T$ is defined inductively as follows

1. The operator associated with the unique rooted tree with two vertices is the identity.

2. For any other tree $T$ let $v$ be the unique vertex connected with one edge to the root. Assume that $v$ is unary. Then a tree $S$ is attached to $v$ and $O_T = d \circ O_S$.

3. Assume now that $v$ is binary. Then trees $T_1$ and $T_2$ are attached to $v$ and $O_T = m \circ (O_{T_1} \otimes O_{T_2})$.

The condition for $A$ to be a differential graded algebra of depth 3 may be described graphically as follows. The condition $d^3 = 0$ corresponds with

The generalized Leibniz rules for $d$ and $m$ are

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$^1 |X|$ denotes the cardinality of a finite set $X$. 

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The condition for $m$ to be 3-associative is depicted as

$$\sum \sum \sum \sum = 0$$

Let us consider the graphical description of the axioms satisfied by the operators defining a differential graded algebra of depth $N$.

**Definition 17.** For $u + b = N$, Let $RT_{l}^{u,b}$ be the set of isomorphisms classes of planar rooted tress with $l$ leaves, $u$ unary internal vertices and $b$ binary internal vertices.

We are ready to give a graphical description of the equations defining a differential graded algebra of depth $N$.

**Theorem 18.** Maps $d : A[1] \rightarrow A[1]$ and $m_{A} : A[1] \otimes 2 A[1] \rightarrow A[1]$ of degree one define a differential graded algebra of depth $N$ structure on $A$ if and only if for $l = 1, \ldots, N + 1$ and $u + b = N$ the following identities hold

$$\sum \sum \sum = 0.$$ 

**Proof.** Follows from the fact that if $T \in RT_{l}^{u,b}$ then $O_{T}$ is a degree $u$ operator, and from the identity

$$(d + m)^{N} = \sum_{l=1}^{N+1} \sum_{u+b=N} \left( \sum_{T \in RT_{l}^{u,b}} O_{T}. \right).$$

Theorem 18 has several interesting consequences. First, it allows us to define an operad over $gvect$ whose algebras in $gvect$ are differential graded algebras of depth $N$. Second, the existence of such an operad implies the existence of the free differential graded algebra of depth $N$ generated by a graded vector space.
Definition 19. The operad $\text{dgass}^N$ is given by $\text{dgass}^N = \frac{\text{dgass}^N}{\mathcal{I}}$, where
\[
\text{dgass}^N(n) = \langle (T, f) \mid T \in \mathcal{RUBT}_n \text{ and } f : l(T) \to [n] \text{ bijection} \rangle,
\]
and $\mathcal{RUBT}_n = \bigsqcup_{u=0, b=0}^{\infty} \mathcal{RT}_n^{u:b}$. An element $(T, f)$ with $T \in \mathcal{RT}_n^{u:b}$ is placed in degree $u$. Compositions are given by grafting of trees. $\mathcal{I}$ is the operadic ideal
\[
\mathcal{I} = \langle \sum_{T \in \mathcal{RT}_n^{u:b}} (T, f) \mid u + b = N \rangle.
\]

Proposition 20. 1. A $\text{dgass}^N$-algebra is a differential graded algebra of depth $N$.

2. $\bigoplus_{n=0}^\infty \text{dgass}^N(n) \otimes \mathcal{S}_n V^\otimes n$ is the free differential graded algebra of depth $N$ generated by the graded vector $V$.

5 Deformations of N-associative algebras

In this section we study $N$-associative algebras. We define the operad whose algebras are $N$-associative algebras and study the infinitesimal deformations of such structures.

Definition 21. A $N$-associative algebra is a vector space $A$ together with a linear map $m : A \otimes A \to A$ such that $m^3 = 0$.

Explicitly a product on a vector space $A$ is 3-associative if for $a, b, c, d \in A$ we have that
\[
(ab)(cd) + (a(bc))d + ((ab)c)d = a((bc)d) + a(b(cd)).
\]

Let us provide an elementary example of a 3-associative algebra.

Example 22. Let $A$ be the free non-associative algebra generated by $a, b, c, d$ subject to the relations: $a^2 = b, ab = d, ba = c$, the product of two other letters is zero. $A$ is 3-associative algebra.

Let us now define explicitly the operad on vect whose algebras in vect are $N$ associative algebras. Let $\mathcal{RBT}_n$ be the set of isomorphisms classes rooted trees with binary internal vertices and $n$ leaves.

Definition 23. The operad $\text{ass}^N$ is given by $\text{ass}^N = \frac{\text{ass}^N}{\mathcal{I}}$, where
\[
\text{ass}^N(n) = \langle (T, f) \mid T \in \mathcal{RBT}_n \text{ and } f : l(T) \to [n] \text{ bijection} \rangle.
\]
Compositions are given by grafting of trees. $\mathcal{I}$ is the operadic ideal
\[
\mathcal{I} = \langle \sum_{T \in \mathcal{RBT}_{n+1}} (T, f) \rangle.
\]
Proposition 24. 1. An ass$^N$-algebra is the same as a $N$-associative algebra.

2. $\bigoplus_{n=0}^{\infty} \text{ass}^N(n) \otimes S_n V^{\otimes n}$ is the free $N$-associative algebra generated by the vector $V$.

It would be interesting to compute the generating series of the operad ass$^N$. For $2 \leq n \leq N$ the dimension of ass$^N(n)$ is $n!C_{n-1}$, where $C_{n-1} = \frac{1}{n}(2n-2)^{n-1}$ is the Catalan number. The dimension of ass$^N(N + 1)$ is $\frac{1}{N!(2N)} - (N + 1)!$.

Notice that ass$^N$ is an $(N+1)$ homogeneous operad in the sense that it is the quotient of a free operad ass$^N$ generated by elements in ass$^N(2)$ by an ideal generated by elements in ass$^N(N + 1)$. Koszul duality was originally introduced by Priddy [22] in the context of quadratic algebras. Ginzburg and Kapranov in their seminal paper [13] defined Koszul duality for quadratic operads. Berger in [6] and Berger, Dubois-Violette and Wambst in [7], developed a theory of Koszul duality for $N$-homogeneous algebras. It is natural to wonder if there exists a notion of Koszul duality for $N$-homogeneous operads, and in particular what the Koszul dual of ass$^N$ might be.

Our next goal is to define the analogue for $N$-associative algebras of the fact that infinitesimal deformations of an associative algebra are controlled by its second Hochschild cohomology group.

Definition 25. 1. For a graded vector space $V$, let $C(V,V)$ be the graded vector space given by $C(V,V) = \bigoplus_{n=1}^{\infty} \text{Hom}(V^{\otimes n}, V)[1-n]$.

2. For $(A,m)$ a differential graded algebra of depth $N$ and $k \geq N$, let

$$t_k : C^2(A,A) \longrightarrow C(A,A)$$

be given by $t_k(f) = \sum_{i=0}^{k-1} m^i f m^{k-i}$.

3. The cohomology group $H^2_{N,M}(A,A)$ is given by

$$H^2_{N,M}(A,A) = \text{Ker}(t_M : C^2(A,A) \longrightarrow C(A,A))/\text{Im}(t_1 : C^1(A,A) \longrightarrow C^2(A,A)),$$

where $t_1 : C^1(A,A) \longrightarrow C^2(A,A)$ is given by $t_1(g) = mg - gm$.

The definition above is consistent by the following result.

Lemma 26. $t_k \circ t_1 = 0$ for $k \geq N$. 

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Proof. For $g \in C(A,A)$ we have

$$t_k \circ t_1(g) = t_k(t_1(g))$$

$$= \sum_{i=0}^{k-1} m^it_1(g)m^{k-1-i}$$

$$= \sum_{i=0}^{k-1} m^i(mg - gm)m^{k-1-i}$$

$$= \sum_{i=1}^{k} m^i gm^{k-i} - \sum_{i=0}^{k-1} m^i gm^{k-i}$$

$$= 0.$$

We are ready to state the main result of this section.

**Theorem 27.**

1. Infinitesimal deformations of a $N$-associative algebra $(A, m)$ into a $M$-associative algebra are determined by the cohomology group $H^2_{N,M}(A, A)$.

2. There are inclusion maps $H^2_{N,M}(A, A) \rightarrow H^2_{N,M+1}(A, A)$.

3. Infinitesimal deformations of a $N$-associative algebra into a proper $M$-associative algebra, $M > N$, are determined by the quotient space $H^2_{N,M}(A, A)/H^2_{N,M-1}(A, A)$.

Proof. Let $m_h(a,b) = m(a,b) + hf(a,b)$ be an infinitesimal deformation of $A$ into a $M$-associative algebra with $f : A \otimes A \rightarrow A$ and $h$ a formal variable such that $h^2 = 0$. The condition for $m_h$ to be $M$-associative is that $m_h^M = 0$. We have

$$m_h^M = (m + hf)^M = h \sum_{i=0}^{M-1} m^i ft_m^{M-1-i} = ht_M(f).$$

Thus $m_h$ is an infinitesimal deformation of $m$ into a $M$-associative product if and only if $f$ is a $t_M$ closed element in $C^2(A, A)$, i.e., $f$ belongs to $Ker(t_M)$.

Any $g \in C^1(A,A)$ is a linear map $g : A \rightarrow A$ and gives rise to a formal isomorphisms connected to the identity $\rho = I + hg$ with inverse $\rho^{-1} = I - hg$. Thus any $g \in C^1(A, A)$
defines a infinitesimal deformation of $m$ given by $m_g = \rho \circ m \circ (\rho^{-1} \otimes \rho^{-1})$. We have that
\[ m_g(a, b) = \rho(m(\rho^{-1}(a), \rho^{-1}(b))) = \rho(m(a - hg(a), b - hg(b))) = \rho(m(a, b) - h[\rho(m(g(a), b)) + \rho(m(a, g(b)))] = m(a, b) + h[g(m(a, b)) - m(g(a), b) - (m(a, g(b)))] = m(a, b) + h[t_1(g)(a, b)] \]

We see that $m_g$ differs from $m$ by an element $Im(t_1)$. We have shown that $H^2_{N,M}(A, A)$ controls the infinitesimal deformations of $m$ into a $M$-associative product. Properties 2 and 3 follow from property 1.

Let us apply this result to the case of associative algebras.

**Corollary 28.** An associative algebra admits an infinitesimal deformation into a proper 3-associative algebra if and only if there exists a map $f : A \otimes A \to A$ such that

1. For all $a, b, c, d \in A$ the following identity holds
   \[ f(ab, c)d + af(bc, d) + f(a, b)cd = abf(c, d) + f(a, bc)d + af(b, cd). \]

2. The following identity does not hold for all $a, b, c \in A$
   \[ f(a, b)c + f(ab, c) = af(b, c) + f(a, bc). \]

**Proof.** $f$ defines a infinitesimal deformation of $m$ into a 3-associative product if $mfm = 0$. It is also a infinitesimal deformation into an associative product if $mf + fm = 0$. □

### 6 Introduction to $A^N_\infty$-algebras

In this section we introduce the notion of $A^N_\infty$-algebras which generalizes both $N$-associative algebras and $A_\infty$ algebras. The interested reader may consult [12], [14], [17] and [19] for more on $A_\infty$-algebras.

**Definition 29.** A structure of $A^N_\infty$-algebra on a graded vector space $A$ is given by a sequence of degree one maps $m_k : A[1] \otimes^k \to A[1]$, for $k \in \mathbb{N}_+$, such that the associated coderivation $\delta = \sum m_k$ on $T(A[1])$ satisfies $\delta^N = 0$.
The condition $\delta^2 = 0$ defining a $A_\infty^2$-algebra is usual condition for an $A_\infty$-algebra

$$\sum_{r+s+t=n} m_{r+1+t} \circ (1^\otimes r \otimes m_s \otimes 1^\otimes t) = 0.$$ 

The condition $\delta^3 = 0$ defining an $A_\infty^3$-algebra is

$$\sum_{a+b+c+d+e=n} m_{a+e+1} \circ (1^\otimes a \otimes m_{b+d+1} \otimes 1^\otimes e) \circ (1^\otimes a+b \otimes m_c \otimes 1^\otimes d+e) +$$

$$m_{a+b+c+e+1} \circ (1^\otimes a \otimes m_{b} \otimes 1^\otimes c+e+1) \circ (1^\otimes a+b+c \otimes m_d \otimes 1^\otimes e) +$$

$$m_{a+c+d+e+1} \circ (1^\otimes a+c+1 \otimes m_d \otimes 1^\otimes e) \circ (1^\otimes a \otimes m_{b} \otimes 1^\otimes c+d+e) = 0.$$ 

It becomes difficult to write explicitly the condition $\delta^N = 0$ for $N \geq 4$, so we shall write it in terms of trees. Denote by $RT^n_l$ the set of isomorphisms classes of rooted planar trees with $l$ leaves and $n$ internal vertices. For example the following trees are in $RT^3_{16}$.

The condition $\delta^N = 0$ holds if and only if for each $l \in \mathbb{N}_+$

$$\sum_{\Gamma \in RT^N_l} m_{\Gamma} = 0,$$

where $m_{\Gamma}$ is defined by a procedure similar to that explained in the previous section, i.e., putting the $m_s$ operator on each vertex with $s$ incoming edges attached to it.

**Definition 30.**

1. An $A_\infty^N$-morphism between $A_\infty^N$-algebras $(A, \delta_A)$ and $(B, \delta_B)$ is a degree zero coalgebra morphism $f : T(A[1]) \to T(B[1])$ such that $f \circ \delta_A = \delta_B \circ f$.

2. Let $(A, \delta)$ be $A_\infty^N$-algebra. Let $m_1$ is the Hom$(A[1], A[1])$ component of $\delta$. It is not hard to check that $m_1^N = 0$. The cohomology of $A$ is the cohomology of the $N$-complex $(A, m_1)$.

3. $A_\infty^N$-algebras $A$ and $B$ are quasi-isomorphic if there exist an $A_\infty^N$-morphism between them inducing an isomorphism in cohomology.

The following result gives examples of $A_\infty^N$-algebras.
Lemma 31. Let \((A, m, d)\) be a graded algebra of depth \(N\). Setting \(m_1(a) = d(a)\), \(m_2(a, b) = ab\) and \(m_k = 0\) for \(k \geq 3\) defines an \(A^N\)-structure on \(A\).

One can define the category of \(A^{\mathbb{N}}\)-algebras whose objects are \(A^{\mathbb{N}}\)-algebras for some \(N\). Next result shows that the category of \(A^{\mathbb{N}}\)-algebras is monoidal.

Theorem 32. Let \((A, \delta_A)\) be an \(A^{\mathbb{N}}\)-algebra and \((B, \delta_B)\) be an \(A^{\mathbb{M}}\)-algebras. Setting \(\delta_{A \otimes B} = \delta_A \otimes \text{Id} + \text{Id} \otimes \delta_B\), we get that \((A \otimes B, \delta_{A \otimes B})\) is a \(A^{(N+M-1)}\)-algebra.

Let us define the graded operad \(A^{\mathbb{N}}\) whose algebras in \(gvect\) are \(A^{\mathbb{N}}\)-algebras.

Definition 33. The operad \(A^{\mathbb{N}}\) is given by \(\overline{A}^N = A^N/\mathbb{I}\), where for \(n \in \mathbb{N}_+\)

\[
\overline{A}^N(n) = \langle (T, f) \mid T \in RT_n \text{ and } f : l(T) \to [n] \text{ bijection} \rangle.
\]

If \(T\) has \(v_k\) vertices with valence \(k\), then \((T, f)\) is placed in degree \(\sum_k v_k(2-k)\). Compositions are given by grafting of trees. \(\mathbb{I}\) is the operadic ideal

\[
\mathbb{I} = \langle \sum_{T \in RT^N} (T, f) \rangle.
\]

Proposition 34. 1. Algebras over \(A^{\mathbb{N}}\) in \(gvect\) are \(A^{\mathbb{N}}\)-algebras. 2. \(\bigoplus_{n=1}^{\infty} \text{ass}^N(n) \otimes_{\mathbb{S}_n} V^\otimes n\) is the free \(A^{\mathbb{N}}\)-algebra generated by the graded space \(V\).

Let us study infinitesimal deformations of \(A^{\mathbb{N}}\)-algebras.

Definition 35. 1. For \((A, \delta)\) an \(A^{\mathbb{N}}\)-algebra and \(k \geq N\), let

\[
t_k : C(A, A) \longrightarrow C(A, A) \text{ be given by } t_k(f) = \sum_{i=0}^{k-1} m^i f m^{k-i}.
\]

2. The cohomology group \(H_{N,M}(A, A)\) is given by

\[
H_{N,M}(A, A) = \text{Ker}(t_M : C(A, A) \longrightarrow C(A, A))/\text{Im}(t_1 : C^1(A, A) \longrightarrow C(A, A)),
\]

where \(t_1 : C^1(A, A) \longrightarrow C(A, A)\) is given by \(t_1(g) = mg - gm\).

Next couple of results are proved as Lemma 26 and Theorem 27, respectively.

Lemma 36. \(t_k \circ t_1 = 0\) for \(k \geq N\).

Theorem 37. 1. Infinitesimal deformations of an \(A^{\mathbb{N}}\)-algebra \(A\) into an \(A^{\mathbb{M}}\)-algebra are determined by the cohomology group \(H_{N,M}(A, A)\).
2. There are inclusion maps $H_{N,M}(A,A) \rightarrow H_{N,M+1}(A,A)$.

3. Infinitesimal deformations of an $A^N_{\infty}$-algebra into a proper $A^M_{\infty}$-algebra, $M > N$, are determined by the quotient space $H_{N,M}(A,A)/H_{N,M-1}(A,A)$.

Let us apply this result to $A_{\infty}$-algebras.

**Corollary 38.** An $A_{\infty}$-algebra $A$ admits an infinitesimal deformation into a proper $A^M_{\infty}$-algebra if and only if there exists $f \in C(A,A)$ such that the following identity holds

$$
\sum_{a+b+c+d+e=n} m_{a+e+1} \circ (1^{\otimes a} \otimes f_{b+d+1} \otimes 1^{\otimes e}) \circ (1^{\otimes a+b} \otimes m_e \otimes 1^{\otimes d+e}) + \\
m_{a+b+c+e+1} \circ (1^{\otimes a} \otimes f_b \otimes 1^{\otimes c+e+1}) \circ (1^{\otimes a+b+c} \otimes m_d \otimes 1^{\otimes e}) + \\
m_{a+c+d+e+1} \circ (1^{\otimes a+c+1} \otimes m_d \otimes 1^{\otimes e}) \circ (1^{\otimes a} \otimes f_b \otimes 1^{\otimes c+d+e}) = 0.
$$

but the following condition fails

$$
\sum_{a+b+c=n} m_{a+1+b} \circ (1^{\otimes a} \otimes f_b \otimes 1^{\otimes c}) + \sum_{a+b+c=n} f_{a+1+b} \circ (1^{\otimes a} \otimes m_b \otimes 1^{\otimes c}) = 0.
$$

We consider full deformations of $A^N_{\infty}$-algebras. Let $k$ be a field and consider a local $k$-algebra $a$ such that $k \cong a/a_+$ where $a_+$ is the unique maximal ideal in $a$. Then $a \cong a \oplus a_+$ as vector spaces.

**Definition 39.** Let $(A,\delta)$ be an $A^N_{\infty}$-algebra. A deformation of $(A,\delta)$ over $a$ is a $A^N_{\infty}$-algebra $(A_a,\delta_a)$ over $a$ such that $A_a/a_+A_a$ is isomorphic to $A$ as $A^N_{\infty}$-algebra. Equivalently, $\delta_a$ maps to $\delta$ under the natural projection $\pi : A_a \rightarrow A_a/a_+A_a \cong A$.

Suppose that $A_a = A \otimes a$ as $a$-modules. Then $\delta_a = \delta + e$ where $e \in Coder(T(A[A]) \otimes a_+)$ is a coderivation of degree one. The next result follows from the discussion on the deformations of $N$-codifferentials at the end of Section 3.

**Theorem 40.** Let $e \in Coder(T(A[A]) \otimes a_+)$. $\delta + e$ defines an $A^M_{\infty}$-algebra structure on $A_a = A \otimes a$ if and only if the $(N,M)$ Maurer-Cartan equation $\sum_{k=0}^{M-1} c_k \delta^k = 0$ holds, where

$$
c_k = \sum_{s \in E_M N(s) = k \atop s_i < N} c(s,M)e(s) \quad \text{and} \quad c(s,M) = \sum_{\gamma \in \Gamma_M(0,s)} v(\gamma).
$$

We close this paper with a discussion of some open problems. The operadic description allows us to construct free $A^N_{\infty}$-algebras, however further examples are needed. One can define a notion of $L^N_{\infty}$-algebras along the lines of our definition of $A^N_{\infty}$-algebras. It would be interesting to describe explicitly $L^N_{\infty}$-algebras and find examples of such structures. There
are of course many open questions regarding $A^\infty_N$-algebras and $L^\infty_N$-algebras. What are the generating series of $A^\infty_N$ and $L^\infty_N$? Is any $A^\infty_N$-algebra quasi-isomorphic to a differential graded algebra of depth $N$? $A_\infty$-algebras can be defined geometrically via Stasheff’s associahedra [23]. Is there an analogous geometrical description for $A^\infty_N$-algebras? Notice that the equation defining $A^N_\infty$-algebras describes the $N - 1$ differential of the operation $m_k$ with $k \geq 2$ as linear sum of operators constructed from the composition of $N$ operators $m_\alpha$, each differentiated at most $N - 2$ times.

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