QUASI–COXETER CATEGORIES AND A RELATIVE
ETINGOF–KAZHDAN QUANTIZATION FUNCTOR

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Abstract. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $U_\hbar \mathfrak{g}$ its quantized enveloping algebra. The quantum Weyl group operators of $U_\hbar \mathfrak{g}$ and the universal $R$–matrices of its Levi subalgebras endow $U_\hbar \mathfrak{g}$ with a natural quasi–Coxeter quasitriangular quasibialgebra structure which underlies the action of the braid group of $\mathfrak{g}$ and Artin’s braid groups on the tensor product of integrable, category $\mathcal{O}$ modules. We show that this structure can be transferred to the universal enveloping algebra $U_\mathfrak{g}[\hbar]$. The proof relies on a modification of the Etingof–Kazhdan quantization functor, and yields an isomorphism between (appropriate completions of) $U_\mathfrak{g}$ and $U_\mathfrak{g}[\hbar]$ preserving a given chain of Levi subalgebras. We carry it out in the more general context of chains of Manin triples, and obtain in particular a relative version of the Etingof–Kazhdan functor with input a split pair of Lie bialgebras. Along the way, we develop the notion of quasi–Coxeter categories, which are to generalized braid groups what braided tensor categories are to Artin’s braid groups. This leads to their succinct description as a 2–functor from De Concini–Procesi associahedra. These results will be used in the sequel to this paper to give a monodromic description of the quantum Weyl group operators of an affine Kac–Moody algebra, extending the one obtained by the second author for a semisimple Lie algebra.

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1. Introduction

1.1. This is the first of a series of three papers the aim of which is to extend the description of the monodromy of the rational Casimir connection in terms of quantum Weyl group operators given in [TL3, TL4, TL5] to the case of an affine Kac–Moody algebra \( \mathfrak{g} \).

The method we follow is close to that of [TL4], and relies on the notion of a quasi–Coxeter quasitriangular quasibialgebra (qCqtqba), which is informally a bialgebra carrying actions of a given generalized braid group and Artin’s braid groups on the tensor products of its modules. A cohomological rigidity result, proved in the second paper of this series [ATL2], shows that there is at most one such structure with prescribed local monodromies on the classical enveloping algebra \( U_\hbar \mathfrak{g} \). It follows that the generalized braid group actions arising from quantum Weyl groups and the monodromy of the Casimir connection [ATL3] are equivalent, provided the quasi–Coxeter quasitriangular quasibialgebra structure responsible for the former can be transferred from \( U_\hbar \mathfrak{g} \) to \( U_\hbar \mathfrak{g}[[\hbar]] \). This result is the purpose of the present article.

1.2. Its proof differs substantially from that given in [TL4]. Indeed, for a semisimple Lie algebra \( \mathfrak{g} \), the transfer of structure ultimately rests on the vanishing of the first and second Hochschild cohomology groups of \( U_\hbar \mathfrak{g}[[\hbar]] \), and in particular on the fact that \( U_\hbar \mathfrak{g} \) and \( U_\mathfrak{g}[[\hbar]] \) are isomorphic as algebras, a fact which does not hold for affine Kac–Moody algebras. Rather than the cohomological methods of [TL4], we use instead the Etingof–Kazhdan (EK) quantization functor [EK1, EK2, EK6], which yields a canonical isomorphism

\[
\Psi_{\text{EK}} : \widehat{U}_\hbar \mathfrak{g} \cong \widehat{U}_\mathfrak{g}[[\hbar]]
\]

between the completions of \( U_\hbar \mathfrak{g} \) and \( U_\mathfrak{g}[[\hbar]] \) with respect to category \( \mathcal{O} \).

Surprisingly perhaps, and despite its functorial construction, the isomorphism \( \Psi_{\text{EK}} \) does not preserve the inclusions of Levi subalgebras

\[
U_\hbar \mathfrak{g}_D \subseteq U_\hbar \mathfrak{g} \quad \text{and} \quad U_\mathfrak{g}_D[[\hbar]] \subseteq U_\mathfrak{g}[[\hbar]]
\]
determined by a subdiagram \( D \) of the Dynkin diagram of \( \mathfrak{g} \), something which is required by the transfer of structure. The bulk of this paper is therefore devoted to modifying \( \Psi_{\text{EK}} \) so as to make it compatible with such inclusions.

1.3. To outline our construction, which works more generally for an inclusion \( (\mathfrak{g}_D, \mathfrak{g}_{D-}, \mathfrak{g}_{D+}) \subset (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \) of Manin triples over a field \( k \) of characteristic zero, recall first that the main steps of the EK construction are as follows.

(i) One considers the Drinfeld category \( \mathcal{D}_{\Phi}(\mathfrak{g}) \) of (deformation) equicontinuous \( \mathfrak{g} \)-modules, with associativity constraints given by a fixed Lie associator \( \Phi \) over \( k \). This category can be thought of as a topological analogue of category \( \mathcal{O} \) when \( \mathfrak{g} \) is the Manin triple associated
to a Kac–Moody algebra. It can equivalently be described as the
category of Drinfeld–Yetter modules over the Lie bialgebra $g_-$.

(ii) One constructs a tensor functor $F$ from $\mathcal{D}_\Phi(g)$ to the category
$\text{Vect}_{k[[\hbar]]}$ of topologically free $k[[\hbar]]$–modules. The algebra of endomorphisms
$H = \text{End}(F)$ is then a topological bialgebra, i.e., it is
endowed with a coproduct $\Delta$ mapping $H$ to a completion of $H \otimes H$.

(iii) Inside $H$, one constructs a subalgebra $U_{\hbar}g_-$ such that $\Delta(U_{\hbar}g_-) \subset
U_{\hbar}g_- \otimes U_{\hbar}g_-$, and which is a quantization of $U_{\hbar}g_-$. The quantum
group $U_{\hbar}g$ is then defined as the quantum double of $U_{\hbar}g_-$.

(iv) By construction, $U_{\hbar}g_-$ acts and coacts on any $F(V), V \in \mathcal{D}_\Phi(g)$,
so that the functor $F$ lifts to $\tilde{F}: \mathcal{D}_\Phi(g) \to \text{Rep}(U_{\hbar}g)$ where, by
definition, the latter is the category of Drinfeld–Yetter modules over
$U_{\hbar}g$.

(v) Finally, one proves that $\tilde{F}$ is an equivalence of categories.

Since $F$ is isomorphic to the forgetful functor $f: \mathcal{D}_\Phi(g) \to \text{Vect}_{k[[\hbar]]}$ as
abelian functors, we obtain the following diagram

\[
\begin{array}{ccc}
\mathcal{D}_\Phi(g) & \xrightarrow{F} & \text{Rep}(U_{\hbar}g) \\
\downarrow f & & \downarrow f_h \\
\text{Vect}_{k[[\hbar]]} & \xrightarrow{F} & \text{Vect}_{k[[\hbar]]}
\end{array}
\]

where $f_h: \text{Rep}(U_{\hbar}g) \to \text{Vect}_{k[[\hbar]]}$ is the forgetful functor. The EK isomorphism
$\Psi_{\text{EK}}$ is then given by the identifications

\[
\hat{U}_g[[\hbar]] := \text{End}(f) \cong \text{End}(F) = \text{End}(f_h \circ \tilde{F}) \cong \text{End}(f_h) =: \hat{U}_{\hbar}g
\]

1.4. Overlaying the above diagrams for an inclusion $i_D : g_D \hookrightarrow g$ of Manin
triples shows that constructing an isomorphism $\hat{U}_{\hbar}g \sim \hat{U}_g[[\hbar]]$ compatible
with $i_D$ may be achieved by filling in the diagram

\[
\begin{array}{ccc}
\mathcal{D}_\Phi(g_D) & \xrightarrow{i_D} & \mathcal{D}_\Phi(g_D) \\
\downarrow f_D & & \downarrow f_D, h \\
\text{Vect}_{k[[\hbar]]} & \xrightarrow{F_D} & \text{Vect}_{k[[\hbar]]}
\end{array}
\]

where $f_D, f_D, h$ are forgetful functors, $F_D$ the EK functor for $g_D$, and $i_{D, h} : U_{\hbar}g_D \to U_{\hbar}g$ is the inclusion derived from the functoriality of the quantization.
To do so, we first construct a relative fiber functor, that is a (tensor) functor $\Gamma$ on $D_\Phi(g)$ whose target category is $D_\Phi(g_D)$ rather than $\text{Vect}_k[[\hbar]]$, and which is isomorphic as abelian functor to the restriction $i^*_D$. We then show the existence of a natural transformation between the composition $\tilde{F}_D \circ \Gamma$ and $i^*_D, \hbar \circ \tilde{F}$. Our constructions do not immediately yield a commutative diagram, i.e., the two factorizations $F \cong F_D \circ \Gamma$ deduced from $f = f_D \circ i^*_D$ and $f_\hbar = f_D, \hbar \circ i^*_D, \hbar$ do not coincide, but this can easily be adjusted by using a different identification $F \cong f$, which amounts to modifying the original EK isomorphism.

1.5. The construction of the functor $\Gamma$ is very much inspired by [EK1]. The principle adopted by Etingof and Kazhdan is the following. In a $k$-linear monoidal category $C$, a coalgebra structure on an object $C \in \text{Obj}(C)$ induces a tensor structure on the Yoneda functor $h_C = \text{Hom}_C(C, -) : C \rightarrow \text{Vect}_k$. If $C$ is braided and $C_1, C_2$ are coalgebra objects in $C$, then so is $C_1 \otimes C_2$, and there is therefore a canonical tensor structure on $h_{C_1 \otimes C_2}$.

If $g$ is finite–dimensional, the polarization $U_g \simeq M_- \otimes M_+$, where $M_\pm$ are the Verma modules $\text{Ind}_g \Phi^\pm \otimes k$, realizes $U_g$ as the tensor product of two coalgebra objects in $D_\Phi(U_g[[\hbar]])$. This yields a tensor structure on the forgetful functor $h_{Ug} : D_\Phi(U_g[[\hbar]]) \rightarrow \text{Vect}_k[[\hbar]]$.

Our starting point is to apply the same principle to the (abelian) restriction functor $i^*_D : D_\Phi(Ug) \rightarrow D_\Phi(Ug_D[[\hbar]])$. We therefore factorize $Ug$ as a tensor product of two coalgebra objects $L_-, N_+$ in the braided monoidal category of $(g, g_D)$–bimodules, with associator $(\Phi \cdot \Phi^{-1})$, where $\Phi^{-1}$ acts on the right. Just as the modules $M_-, M_+$ are related to the decomposition $g = g_\mp \oplus g_\mp$, $L_-$ and $N_+$ are related to the asymmetric decomposition $g = m_- \oplus p_+$, where $m_- = g_- \cap g_D^+$ and $p_+ = g_D \oplus m_+$. This factorization induces a tensor structure on the functor $\Gamma = h_{L_\mp \otimes N_+}$, canonically isomorphic to $i^*_D$ through the right $g_D$–action on $N_+$. As in [EK1, Part II], this tensor structure can also be defined in the infinite–dimensional case.

1.6. To construct a natural transformation making the following diagram commute

$$
\begin{array}{ccc}
D_\Phi(g) & \xrightarrow{\tilde{F}} & \text{Rep}(Ug) \\
\downarrow \Gamma & & \downarrow (i_D \hbar) \\
D_\Phi(g_D) & \xrightarrow{F_D} & \text{Rep}(Ug_D)
\end{array}
$$
we remark, as suggested to us by P. Etingof, that a quantum analogue $\Gamma_h$ of $\Gamma$ can be similarly defined using a quantum version $L_h^-, N_h^+$ of the modules $L_-, N_+$. The functor $\Gamma_h = \text{Hom}_{\mathbf{EK}}^h (L_h^- \otimes N_h^+, -)$ is naturally isomorphic to $(i_D)_h^*$ as tensor functor, since there is no associator involved on this side. Moreover, an identification 
\[ \tilde{F}_D \circ \Gamma \simeq \Gamma \circ \tilde{F} \]
is readily obtained, provided one establishes isomorphisms of $(U_h^\mathbf{EK} g, U_h^\mathbf{EK} g_D)$–bimodules 
\[ \tilde{F}_D \circ \tilde{F}(L_-) \simeq L_h^- \quad \text{and} \quad \tilde{F}_D \circ \tilde{F}(N_+) \simeq N_h^+ \]

1.7. While for $M_{\pm}$ it is easy to construct an isomorphism between $\tilde{F}(M_{\pm})$ and the quantum counterparts of $M_{\pm}$, the proof for $L_-, N_+$ is more involved. It relies on a description of the quantization functor $F^\mathbf{EK}$ in terms of $\text{PROP}$ categories (cf. [EK2, EG]) and the realization of $L_-, N_+$ as universal objects in a suitable colored $\text{PROP}$ describing the inclusion of bialgebras $g_D \subset g$. This yields in particular a relative extension of the EK functor with input a pair of Lie bialgebras $a, b$ which is split, i.e., endowed with maps $a \leftrightarrow_i b$ such that $p \circ i = \text{id}$.

1.8. Given that we work throughout with completions of algebras obtained as endomorphisms of fiber functors, the transfer of structure from $U_h g$ to $U g[[h]]$ is more conveniently phrased in terms of categories. Part of this paper is therefore devoted to rephrasing the definition of quasi–Coxeter quasitriangular quasibialgebra in categorical terms. This yields the notion of a quasi–Coxeter category, which is to a generalized braid group $B$ what a braided tensor category is to Artin’s braid groups, and of a quasi–Coxeter tensor category. Interestingly perhaps, both notions be concisely rephrased in terms of a 2–functor from a combinatorially defined 2–category $\mathbf{qC}(D)$ to the 2–categories $\text{Cat}, \text{Cat}^\otimes$ of categories and tensor categories respectively. The objects of $\mathbf{qC}(D)$ are the subdiagrams of the Dynkin diagram $D$ of $B$ and, for two subdiagrams $D' \subseteq D''$, $\text{Hom}_{\mathbf{qC}(D)}(D'', D')$ is the fundamental 1–groupoid of the De Concini–Procesi associahedron for the quotient diagram $D''/D'$ [DCP2, TL4].

1.9. Outline of the paper. We begin in Section 2 by reviewing a number of combinatorial notions which will be used in later sections. In Section 3 we define quasi–Coxeter (tensor) categories. In Section 4, we review the construction of the Etingof–Kazhdan quantization functor and the isomorphism $\Psi^\mathbf{EK}$ following [EK1, EK6]. In Section 5, we modify this construction by using generalized Verma modules $L_-, N_+$, and obtain a relative fiber functor $\Gamma : D_\emptyset(g) \to D_{\Psi_D}(g_D)$. In Section 6, we define the quantum generalized Verma modules $L_h^-$ and $N_h^+$. Using suitably defined $\text{PROPS}$ we then show,
in Section 7 that these are isomorphic to the EK quantization of their classical counterparts. In Section 8, we use these results to show that, for any given chain of Manin triples ending in a given \( g \), there exists a quantization of \( U_{\mathfrak{g}} \) which is compatible with each inclusion and independent, up to isomorphism, of the choice of the given chain. Finally, in Section 9, we apply these results to the case of a Kac–Moody algebra \( g \) and obtain the desired transport of its quasi–Coxeter quasitriangular quasibialgebra structure to the completion of \( U_{\mathfrak{g}}[[\hbar]] \) with respect to category \( \mathcal{O} \), integrable modules.

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2. Diagrams and nested sets

We review in this section a number of combinatorial notions associated to a diagram \( D \), in particular the definition of nested sets on \( D \) and of the De Concini–Procesi associahedron of \( D \) following [DCP2] and [TL4, Section 2].

2.1. Nested sets on diagrams. By a diagram we shall mean a nonempty undirected graph \( D \) with no multiple edges or loops. We denote the set of vertices of \( D \) by \( \mathcal{V}(D) \) and set \( |D| = |\mathcal{V}(D)| \). A subdiagram \( B \subseteq D \) is a full subgraph of \( D \), that is, a graph consisting of a subset \( \mathcal{V}(B) \) of vertices of \( D \), together with all edges of \( D \) joining any two elements of \( \mathcal{V}(B) \). We will often abusively identify such a \( B \) with its set of vertices and write \( i \in B \) to mean \( i \in \mathcal{V}(B) \). We denote by \( \mathcal{SD}(D) \) the set of subdiagrams of \( D \).

The union \( B_1 \cup B_2 \) of two subdiagrams \( B_1, B_2 \subseteq D \) is the subdiagram having \( \mathcal{V}(B_1) \cup \mathcal{V}(B_2) \) as its set of vertices. Two subdiagrams \( B_1, B_2 \subseteq D \) are orthogonal if \( \mathcal{V}(B_1) \cap \mathcal{V}(B_2) = \emptyset \) and no two vertices \( i \in B_1, j \in B_2 \) are joined by an edge in \( D \). \( B_1 \) and \( B_2 \) are compatible if either one contains the other or they are orthogonal.

Definition. A nested set on a diagram \( D \) is a collection \( \mathcal{H} \) of pairwise compatible, connected subdiagrams of \( D \) which contains the connected components \( D_1, \ldots, D_r \) of \( D \).

2.2. The De Concini–Procesi associahedron. Let \( \mathcal{N}_D \) be the partially ordered set of nested sets on \( D \), ordered by reverse inclusion. \( \mathcal{N}_D \) has a unique maximal element \( 1 = \{D_1\} \) and its minimal elements are the maximal nested sets. We denote the set of maximal nested sets on \( D \) by \( \text{Mns}(D) \). Every nested set \( \mathcal{H} \) on \( D \) is uniquely determined by a collection \( \{\mathcal{H}_i\}_{i=1}^r \) of nested sets on the connected components of \( D \). We therefore obtain canonical identifications

\[
\mathcal{N}_D = \prod_{i=1}^r \mathcal{N}_{D_i} \quad \text{and} \quad \text{Mns}(D) = \prod_{i=1}^r \text{Mns}(D_i)
\]
The De Concini–Procesi associahedron $A_D$ is the regular CW–complex whose poset of (nonempty) faces is $N_D$. It easily follows from the definition that

$$A_D = \prod_{i=1}^{r} A_{D_i}$$

It can be realized as a convex polytope of dimension $|D| - r$. For any $\mathcal{H} \in N_D$, we denote by $\dim(\mathcal{H})$ the dimension of the corresponding face in $A_D$.

2.3. The rank function of $N_D$. For any nested set $\mathcal{H}$ on $D$ and $B \in \mathcal{H}$, we set $i_{\mathcal{H}}(B) = \bigcup_{i=1}^{m} B_i$ where the $B_i$'s are the maximal elements of $\mathcal{H}$ properly contained in $B$.

**Definition.** Set $\alpha_{\mathcal{H}}^B = B \setminus i_{\mathcal{H}}(B)$. We denote by

$$n(B; \mathcal{H}) = |\alpha_{\mathcal{H}}^B| \quad \text{and} \quad n(\mathcal{H}) = \sum_{B \in \mathcal{H}} (n(B; \mathcal{H}) - 1)$$

An element $B \in \mathcal{H}$ is called unsaturated if $n(B; \mathcal{H}) > 1$.

**Proposition.**

(i) For any nested set $\mathcal{H} \in N_D$,

$$n(\mathcal{H}) = |D| - |\mathcal{H}| = \dim(\mathcal{H})$$

(ii) If $\mathcal{H}$ is a maximal nested set if and only if $n(B; \mathcal{H}) = 1$ for any $B \in \mathcal{H}$.

(iii) Any maximal nested set is of cardinality $|D|$.

For any $\mathcal{F} \in \text{Mns}(D)$, $B \in \mathcal{F}$, $i_{\mathcal{F}}(B)$ denotes the maximal element in $\mathcal{F}$ properly contained in $B$ and $\alpha_{\mathcal{F}}^B = B \setminus i_{\mathcal{F}}(B)$ consists of one vertex, denoted $\alpha_{\mathcal{F}}^B$.

For any $\mathcal{F} \in \text{Mns}(D)$, $B \in \mathcal{F}$, we denote by $\mathcal{F}_B \in \text{Mns}(B)$ the maximal nested set induced by $\mathcal{F}$ on $B$.

2.4. Quotient diagrams. Let $B \subseteq D$ a proper subdiagram with connected components $B_1, \ldots, B_m$.

**Definition.** The set of vertices of the quotient diagram $D/B$ is $V(D) \setminus V(B)$. Two vertices $i \neq j$ of $D/B$ are linked by an edge if and only if the following holds in $D$

$$i \not\sim j \quad \text{or} \quad i, j \not\sim B_i \quad \text{for some } i = 1, \ldots, m$$

For any connected subdiagram $C \subseteq D$ not contained in $B$, we denote by $\overline{C} \subseteq D/B$ the connected subdiagram with vertex set $V(C) \setminus V(B)$. 
2.5. Compatible subdiagrams of $D/B$.

**Lemma.** Let $C_1, C_2 \not\subseteq B$ be two connected subdiagrams of $D$ which are compatible. Then

1. $\overline{C}_1, \overline{C}_2$ are compatible unless $C_1 \perp C_2$ and $C_1, C_2 \not\subseteq B_i$ for some $i$.
2. If $C_1$ is compatible with every $B_i$, then $\overline{C}_1$ and $\overline{C}_2$ are compatible.

In particular, if $F$ is a nested set on $D$ containing each $B_i$, then $\overline{F} = \{\overline{C}\}$, where $C$ runs over the elements of $F$ such that $C \not\subseteq B$, is a nested set on $D/B$.

Let now $A$ be a connected subdiagram of $D/B$ and denote by $\tilde{A} \subseteq D$ the connected subdiagram with vertex set

$$V(\tilde{A}) = V(A) \bigcup_{i : B_i \not\subseteq V(A)} V(B_i)$$

Clearly, $A_1 \subseteq A_2$ or $A_1 \perp A_2$ imply $\tilde{A}_1 \subseteq \tilde{A}_2$ and $\tilde{A}_1 \perp \tilde{A}_2$ respectively, so the lifting map $A \rightarrow \tilde{A}$ preserves compatibility.

2.6. Nested sets on quotients. For any connected subdiagrams $A \subseteq D/B$ and $C \subseteq D$, we have

$$\overline{A} = A \quad \text{and} \quad \overline{C} = C \bigcup_{i : B_i \not\subseteq C} B_i$$

In particular, $\overline{C} = C$ if, and only if, $C$ is compatible with $B_1, \ldots, B_m$ and not contained in $B$. The applications $C \rightarrow \overline{C}$ and $A \rightarrow \tilde{A}$ therefore yield a bijection between the connected subdiagrams of $D$ which are either orthogonal to or strictly contain each $B_i$ and the connected subdiagrams of $D/B$. This bijection preserves compatibility and therefore induces an embedding $N_{D/B} \hookrightarrow N_D$. This yields an embedding

$$N_{D/B} \times N_B = N_{D/B} \times (N_{B_1} \times \cdots \times N_{B_m}) \hookrightarrow N_D$$

with image the poset of nested sets on $D$ containing each $B_i$. Similarly, for any $B \subseteq B' \subseteq B''$, we obtain a map

$$\bigcup : N_{B''/B'} \times N_{B'/B} \hookrightarrow N_{B''/B}$$

The map $\bigcup$ restricts to maximal nested sets. For any $B \subseteq B'$, we denote by $\text{Mns}(B', B)$ the collection of maximal nested sets on $B'/B$. Therefore, for any $B \subseteq B' \subseteq B''$, we obtain an embedding

$$\bigcup : \text{Mns}(B'', B') \times \text{Mns}(B', B) \rightarrow \text{Mns}(B'', B)$$

such that, for any $\mathcal{F} \in \text{Mns}(B'', B'), \mathcal{G} \in \text{Mns}(B', B)$,

$$(\mathcal{F} \cup \mathcal{G})_{B'/B} = \mathcal{G}$$
2.7. Elementary and equivalent pairs.

**Definition.** An ordered pair \((G, F)\) in \(\text{Mns}(D)\) is called elementary if \(G\) and \(F\) differ by one element. A sequence \(H_1, \ldots, H_m\) in \(\text{Mns}(D)\) is called elementary if \(|H_{i+1} \setminus H_i| = 1\) for any \(i = 1, 2, \ldots, m - 1\).

**Definition.** The support \(\text{supp}(F, G)\) of an elementary pair in \(\text{Mns}(D)\) is the unique unsaturated element of \(F \cap G\). The central support \(\text{z supp}(F, G)\) is the union of the maximal elements of \(F \cap G\) properly contained in \(\text{supp}(F, G)\). Thus
\[
\text{z supp}(F, G) = \text{supp}(F, G) \setminus \alpha_{\text{supp}(F, G)} \text{supp}(F, G)
\]

**Definition.** Two elementary pairs \((F, G), (F', G')\) in \(\text{Mns}(D)\) are equivalent if
\[
\text{supp}(F, G) = \text{supp}(F', G') \quad \alpha_F^{\text{supp}(F, G)} = \alpha_F^{\text{supp}(F', G')} \quad \alpha_G^{\text{supp}(F, G)} = \alpha_G^{\text{supp}(F', G')}
\]

3. Quasi–Coxeter categories

The goal of this section is to rephrase the notion of quasi–Coxeter quasi-triangular quasibialgebra defined in [TL4] in terms of terms of categories of representations.

3.1. Algebras arising from fiber functors. We shall repeatedly need the following elementary

**Lemma.** Consider the following situation

\[
\begin{array}{ccc}
C & \xrightarrow{H} & D \\
\downarrow{F} & \alpha & \downarrow{G} \\
A & \xleftarrow{} & \\
\end{array}
\]

where \(A, C, D\) are additive \(k\)--linear categories, \(F, G, H\) functors, and \(\alpha\) is an invertible transformation. If \(H\) is an equivalence of categories, the map \(\text{End}(G) \rightarrow \text{End}(F)\) given by
\[
\{g_W\} \mapsto \{\text{Ad}(\alpha_V^{-1})(g_{H(V)})\}
\]

is an algebra isomorphism.

3.2. \(D\)--categories. Recall [TL4, Section 3] that, given a diagram \(D\), a \(D\)--algebra is a pair \((A, \{A_B\}_{B \in \text{SD}(D)})\), where \(A\) is an associative algebra and \(\{A_B\}_{B \in \text{SD}(D)}\) is a collection of subalgebras indexed by \(\text{SD}(D)\) and satisfying

\[
A_B \subseteq A_{B'} \quad \text{if} \ B \subseteq B' \quad \text{and} \quad [A_B, A_{B'}] = 0 \quad \text{if} \ B \perp B'
\]

The following rephrases the notion of \(D\)--algebras in terms of their category of representations.
**Definition.** A $D$–category

$$C = \{\{C_B\}, \{F_{BB'}\}\}$$

is the datum of

- a collection of $k$–linear additive categories $\{C_B\}_{B \subseteq D}$
- for any pair of subdiagrams $B \subseteq B'$, an additive $k$–linear functor $F_{BB'} : C_B \to C_{B'}$
- for any $B \subset B'$, $B' \perp B''$, $B', B'' \subset B'''$, a homomorphism of $k$–algebras

$$\eta : \text{End}(F_{BB'}) \to \text{End}(F_{(B \cup B' \cup B'')}B'')$$

satisfying the following properties

- For any $B \subseteq D$, $F_{BB} = \text{id}_{C_B}$.
- For any $B \subseteq B' \subseteq B''$, $F_{BB'} \circ F_{B'B''} = F_{BB''}$.
- For any $B \subset B'$, $B' \perp B''$, $B', B'' \subset B'''$, the following diagram of algebra homomorphisms commutes:

$$
\begin{array}{ccc}
\text{End}(F_{BB'}) \otimes \text{End}(F_{B'B''}) & \xrightarrow{\text{id} \otimes \eta} & \text{End}(F_{BB''}) \\
\circlearrowright & & \circlearrowright \\
\text{End}(F_{B(B \cup B' \cup B'')}B'') \otimes \text{End}(F_{(B \cup B' \cup B'')}B'') & \xrightarrow{\circlearrowright} & \text{End}(F_{BB''})
\end{array}
$$

**Remark.** It may seem more natural to replace the equality of functors $F_{BB'} \circ F_{B'B''} = F_{BB''}$ by the existence of invertible natural transformations $\alpha_{BB''}^{B'B''} : F_{BB'} \circ F_{B'B''} \Rightarrow F_{BB''}$ for any $B \subseteq B'$ satisfying the associativity constraints $\alpha_{BB''}^{B'B''} \circ F_{BB''}(\alpha_{BB''}^{B'B''}) = \alpha_{BB''}^{B'B''} \circ (\alpha_{BB''}^{B'B''})F_{B'B''}$, for any $B \subseteq B' \subseteq B'' \subseteq B'''$. A simple coherence argument shows however that this leads to a notion of $D$–category which is equivalent to the one given above.

**Remark.** We will usually think of $C_\emptyset$ as a base category and at the functors $F$ as forgetful functors. Then the family of algebras $\text{End}(F_B)$ defines, through the morphisms $\alpha$, a structure of $D$–algebra on $\text{End}(F_D)$. Conversely, every $D$–algebra $A$ admits such a description setting $C_B = \text{Rep} A_B$ for $B \neq \emptyset$ and $C_\emptyset = \text{Vect}_k$, $F_{BB'} = i_{B'B''}^* B''$, where $i_{B'B''} : A_B \subset A_{B''}$ is the inclusion.

**Remark.** The conditions satisfied by the maps $\eta$ imply that, given $B = \bigsqcup_{j=1}^r B_j$, with $B_j \in \text{SD}(D)$ pairwise orthogonal, the images in $\text{End}(F_B)$ of the maps

$$\text{End}(F_{B_j}) \to \text{End}(F_{B_j}F_{B_j}B) = \text{End}(F_B)$$

pairwise commute. This condition rephrases for the endomorphism algebras the $D$–algebra axiom

$$[A_{B'}, A_{B''}] = 0 \quad \forall \ B' \perp B''$$

$^1$When $B = \emptyset$ we will omit the index $B$. 

\[\text{(Definition)}\]
that is equivalent to the condition, for any $B \supset B', B''$,

$$A_{B'} \subset A_{B''}$$

**Remark.** The above definition of $D$–category may be rephrased as follows. Let $I(D)$ be the category whose objects are subdiagrams $B \subseteq D$ and morphisms $B' \to B$ the inclusions $B \subset B'$. Then a $D$–category is a functor

$$C : I(D) \to \text{Cat}$$

3.3. **Strict morphisms of** $D$–**categories.** The interpretation of $D$–categories in terms of $I(D)$ suggests that a morphism of $D$–categories $C, C'$ is one of the corresponding functors

![Diagram](image)

This yields the following definition. For simplicity, we assume that $C_{\emptyset} = C'_{\emptyset}$.

**Definition.** A *strict morphism* of $D$–categories $C, C'$ is the datum of

- for any $B \subseteq D$, a functor $H_B : C_B \to C'_B$
- for any $B \subseteq B'$, a natural transformation $\gamma_{BB'} : H_B \to H_{B'}$

such that

- $H_{\emptyset} = \text{id}$
- $\gamma_{BB} = \text{id}_{H_B}$
- For any $B \subseteq B' \subseteq B''$,

$$\gamma_{BB''} = \gamma_{BB'} \circ \gamma_{B'B''}$$

where $\circ$ is the composition of natural transformations defined by

![Diagram](image)
The diagram (3.1), with $B = \emptyset$, induces an algebra homomorphism $\text{End}(F'_B) \to \text{End}(F'_{B'})$ which, by (3.2) is compatible with the maps $\text{End}(F_B) \to \text{End}(F'_B)$ and $\text{End}(F'_B) \to \text{End}(F'_{B'})$ for any $B \subseteq B'$. As pointed out in [TL4, 3.3], this condition is too restrictive and will be weakened in the next paragraph.

3.4. Morphisms of $D$–categories.

**Definition.** A morphism of $D$–categories $C, C'$, with $C_\emptyset = C'_\emptyset$, is the datum of

- for any $B \subseteq D$ a functor $H_B : C_B \to C'_B$
- for any $B \subseteq B'$ and $F \in \text{Mns}(B, B')$, a natural transformation

\[
\begin{array}{c}
C_B \\
\downarrow \gamma^F_{BB'}
\end{array}
\begin{array}{c}
C'_B \\
\downarrow \gamma^F_{BB'}
\end{array}
\]

such that

- $H_\emptyset = \text{id}$
- $\gamma^F_{BB} = \gamma^F_{BB'}$
- for any $B \subseteq B' \subseteq B''$, $F \in \text{Mns}(B, B')$, $G \in \text{Mns}(B', B'')$,

\[\gamma^F_{BB'} \circ \gamma^G_{B'B''} = \gamma^{F \circ G}_{BB''}\]

**Remark.** For any $F \in \text{Mns}(B')$, the natural transformation $\gamma^F_{BB'}$ induces an algebra homomorphism $\Psi^F_{BB'} : \text{End}(F'_{B'}) \to \text{End}(F_{B'})$ such that the following diagram commutes for any $B \in F$

\[
\begin{array}{c}
\text{End}(F'_{B'}) \\
\downarrow \Psi^F_{BB'}
\end{array}
\begin{array}{c}
\text{End}(F_{B'}) \\
\downarrow \Psi^F_{BB'}
\end{array}
\]

In particular, the collection of homomorphisms $\{\Psi^F_{BB'}\}$ defines a morphism of $D$–algebras $\text{End}(F'_{B'}) \to \text{End}(F_{B'})$ in the sense of [TL4, 3.4].

**Remark.** The above definition may be rephrased as follows. Let $M(D)$ be the category with objects the subdiagrams $B \subseteq D$ and morphisms $\text{Hom}(B', B) = \text{Mns}(B', B)$, with composition given by union. There is a forgetful functor $M(D) \to I(D)$ which is the identity on objects and maps $F \in \text{Mns}(B', B)$ to the inclusion $B \subseteq B'$. Given two $D$–categories $C, C' : I(D) \to \text{Cat}$ a morphism $C \to C'$ as defined above coincides with a morphism of the functors $M(D) \to \text{Cat}$ given by the composition

\[
M(D) \xrightarrow{C} I(D) \xrightarrow{C'} \text{Cat}
\]
3.5. Quasi–Coxeter categories.

**Definition.** A *labelling* of the diagram $D$ is the assignment of an integer $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair $i, j$ of distinct vertices of $D$ such that

$$m_{ij} = m_{ji}$$

if and only if $i \perp j$.

Let $D$ be a labeled diagram.

**Definition.** The *Artin braid group* $B_D$ is the group generated by elements $S_i$ labeled by the vertices $i \in D$ with relations

$$S_i S_j \cdots = S_j S_i \cdots$$

for any $i \neq j$ such that $m_{ij} < \infty$. We shall also refer to $B_D$ as the braid group corresponding to $D$.

**Definition.** A *quasi–Coxeter category of type $D$* $\mathcal{C} = (\{C_B\}, \{F_{BB'}\}, \{\Phi_{FG}\}, \{S_i\})$ is the datum of

- a $D$–category $\mathcal{C} = (\{C_B\}, \{F_{BB'}\})$
- for any elementary pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mns}(B, B')$, a natural transformation $\Phi_{\mathcal{F}\mathcal{G}} \in \text{Aut}(F_{BB'})$
- for any vertex $i \in V(D)$, an element $S_i \in \text{Aut}(F_i)$

satisfying the following conditions

- **Orientation.** For any elementary pair $(\mathcal{F}, \mathcal{G})$,
  $$\Phi_{\mathcal{G}\mathcal{F}} = \Phi_{\mathcal{F}\mathcal{G}}^{-1}$$

- **Coherence.** For any elementary sequences $\mathcal{H}_1, \ldots, \mathcal{H}_m$ and $\mathcal{K}_1, \ldots, \mathcal{K}_l$ in $\text{Mns}(B, B')$ such that $\mathcal{H}_1 = \mathcal{K}_1$ and $\mathcal{H}_m = \mathcal{K}_l$,
  $$\Phi_{\mathcal{H}_m \cdots \mathcal{H}_1} \Phi_{\mathcal{H}_1 \mathcal{H}_2} = \Phi_{\mathcal{K}_l \cdots \mathcal{K}_1} \Phi_{\mathcal{K}_1 \mathcal{K}_2}$$

- **Factorization.** The assignment
  $$\Phi : \text{Mns}(B, B')^2 \to \text{Aut}(F_{BB'})$$

is compatible with the embedding

$$\cup : \text{Mns}(B, B') \times \text{Mns}(B', B'') \to \text{Mns}(B, B'')$$

for any $B'' \subset B' \subset B$, i.e., the diagram

$$\begin{array}{ccc}
\text{Mns}(B, B')^2 \times \text{Mns}(B', B'')^2 & \overset{\Phi \times \Phi}{\longrightarrow} & \text{Aut}(F_{BB'}) \times \text{Aut}(F_{B'B}) \\
\downarrow & & \downarrow \circ \\
\text{Mns}(B, B'')^2 & \overset{\Phi}{\longrightarrow} & \text{Aut}(F_{B''B})
\end{array}$$
is commutative.

- **Braid relations.** For any pairs $i, j$ of distinct vertices of $B$, such that $2 < m_{ij} < \infty$, and elementary pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mns}(B)$ such that $i \in \mathcal{F}, j \in \mathcal{G}$, the following relations hold in $\text{End}(F_B)$

$$\text{Ad}(\Phi_{\mathcal{G,F}})(S_i) \cdot S_j \cdot \ldots = S_j \cdot \text{Ad}(\Phi_{\mathcal{G,F}})(S_i) \cdot \ldots$$

where, by abuse of notation, we denote by $S_i$ its image in $\text{End}(F_B)$ and the number of factors in each side equals $m_{ij}$.

The elements $S_i$ will be commonly referred to as *local monodromies*.

**Remark.** It is clear that the factorization property implies the support and forgetful properties as stated in [TL4, Def. 3.12].

- **Support.** For any elementary pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mns}(B, B')$, let $S = \text{supp}(\mathcal{F}, \mathcal{G})$, $Z = \text{supp}(\mathcal{F}, \mathcal{G}) \subseteq D$ and

$$\tilde{\mathcal{F}} = \mathcal{F}|^{\text{supp}(\mathcal{F}, \mathcal{G})} \quad \tilde{\mathcal{G}} = \mathcal{G}|_{\text{supp}(\mathcal{F}, \mathcal{G})}$$

Then

$$\Phi_{\tilde{\mathcal{F}}\tilde{\mathcal{G}}} = \text{id}_B \circ \Phi_{\tilde{\mathcal{F}}\tilde{\mathcal{G}}} \circ \text{id}_{B'}$$

where the expression above denotes the composition of natural transformations

- **Forgetfulness.** For any equivalent elementary pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ in $\text{Mns}(B, B')$

$$\Phi_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{F}'\mathcal{G}'}$$

**Remark.** To rephrase the above definition, consider the 2–category $\mathbf{qC}(D)$ obtained by adding to $M(D)$ a unique 2–isomorphism $\varphi^{B'B'}_{\mathcal{F}\mathcal{G}} : \mathcal{F} \to \mathcal{G}$ for any pair of 1–morphisms $\mathcal{F}, \mathcal{G} \in \text{Mns}(B', B)$, with the compositions

$$\varphi^{B'B'}_{\mathcal{H}\mathcal{G}} \circ \varphi^{B'B'}_{\mathcal{F}\mathcal{G}} = \varphi^{B'B'}_{\mathcal{H}\mathcal{G}} \quad \text{and} \quad \varphi^{B'B''}_{\mathcal{F}\mathcal{G}_2} \circ \varphi^{B'B''}_{\mathcal{F}_1\mathcal{G}_1} = \varphi^{B'B''}_{\mathcal{F}_2\mathcal{G}_2 \cup \mathcal{F}_1 \mathcal{G}_1}$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B', B), \ B \subset B' \subseteq B''$ and $\mathcal{F}_1, \mathcal{G}_1 \in \text{Mns}(B'', B'), \mathcal{F}_2, \mathcal{G}_2 \in \text{Mns}(B', B)$. There is a unique functor $\mathbf{qC}(D) \to I(D)$ extending
$M(D) \to I(D)$, and a quasi–Coxeter category is the same as a 2–functor $qC(D) \to \text{Cat}$ fitting in a diagram

$$
\begin{array}{ccc}
qC(D) & \longrightarrow & \text{Cat} \\
\downarrow & & \downarrow \\
I(D) & \longrightarrow & \text{Cat}
\end{array}
$$

Note that, for any $B \subset B'$, the category $\text{Hom}_{qC(D)}(B', B)$ is the 1–groupoid of the De Concini–Procesi associahedron on $B'/B$ [TL4].

### 3.6. Morphisms of quasi–Coxeter categories.

**Definition.** A morphism of quasi–Coxeter categories $C, C'$ of type $D$ is a morphism $(H, \gamma)$ of the underlying $D$–categories such that

- For any $i \in B$, the corresponding morphism $\Psi_i : \text{End}(F'_i) \to \text{End}(F_i)$ satisfies
  $$
  \Psi_i(S'_i) = S_i
  $$
- For any elementary pair $(F, G)$ in $\text{Mns}(B, B')$, $H_B(\Phi_{FG}) \circ \gamma_{BB'} \circ (\Psi'_G\gamma)_{H_{B'}} = \gamma_{BB'}$

in $\text{Nat}(F_{BB'} \circ H_{B'}, H_B \circ F_{BB'})$, as in the diagram

**Remark.** Note that the above condition can be alternatively stated in terms of morphisms $\Psi_F$ as the identity

$$
\Psi_G \circ \text{Ad}(\Phi_{GF}) = \text{Ad}(\Phi'_{GF}) \circ \Psi_F
$$

### 3.7. Strict $D$–monoidal categories.

**Definition.** A strict $D$–monoidal category $C = (\{C_B\}, \{F_{BB'}\}, \{J_{BB'}\})$ is a $D$–category $C = (\{C_B\}, \{F_{BB'}\})$ where

- for any $B \subseteq D$, $(C_B, \otimes_B)$ is a strict monoidal category
- for any $B \subseteq B'$, the functor $F_{BB'}$ is endowed with a tensor structure $J_{BB'}$

with the additional condition that, for every $B \subseteq B' \subseteq B''$, $J_{BB'} \circ J_{B'BB''} = J_{BB''}$.
Remark. The tensor structure $J_B$ induces on $\text{End}(F_B)$ a coproduct $\Delta_B : \text{End}(F_B) \to \text{End}(F_B^2)$, where $F_B^2 := \otimes \circ (F_B \boxtimes F_B)$, given by

$$\{g_V\}_{V \in C_B} \mapsto \{\Delta_B(g)_{V W} := \text{Ad}(J_B^{V W})(g_V \otimes W)\}_{V, W \in C_B}$$

Moreover, for any $B \subseteq B'$, $\text{End}(F_B)$ is a subbialgebra of $\text{End}(F_{B'})$, i.e., the following diagram is commutative

$$\begin{array}{ccc}
\text{End}(F_B) & \xrightarrow{\Delta_B} & \text{End}(F_B^2) \\
\downarrow & & \downarrow \\
\text{End}(F_{B'}) & \xrightarrow{\Delta_{B'}} & \text{End}(F_{B'}^2)
\end{array}$$

Remark. Note that a strict $D$–monoidal category can be thought of as a functor

$$C : I(D) \to \text{Cat}_0^\otimes$$

where $\text{Cat}_0^\otimes$ denotes the 2–category of strict monoidal category, with monoidal functors and gauge transformations.

Definition. A morphism of strict $D$–monoidal categories is a natural transformation of the corresponding 2–functors $M(D) \to \text{Cat}_0^\otimes$, obtained by composition with $M(D) \to I(D)$.

3.8. $D$–monoidal categories.

Definition. A $D$–monoidal category

$$C = (\{(C_B, \otimes_B, \Phi_B)\}, \{F_{BB'}\}, \{J_{BB'}^F\})$$

is the datum of

- A $D$–category $\{(C_B), \{F_{BB'}\}\}$ such that each $(C_B, \otimes_B, \Phi_B)$ is a tensor category, with $C_\emptyset$ a strict tensor category, i.e., $\Phi_\emptyset = \text{id}$.
- for any pair $B \subseteq B'$ and $F \in \text{Mns}(B, B')$, a tensor structure $J_{BB'}^F$ on the functor $F_{BB'} : C_{B'} \to C_B$

with the additional condition that, for any $B \subseteq B' \subseteq B''$, $F \in \text{Mns}(B'', B')$, $G \in \text{Mns}(B', B)$,

$$J_{BB''}^G \circ J_{BB'}^F = J_{BB''}^{G F}$$

Remark. The usual comparison with the algebra of endomorphisms leads to a collection of bialgebras $(\text{End}(F_B), \Delta_F, \varepsilon)$ endowed with multiple coproducts, indexed by $\text{Mns}(B)$.

Remark. A $D$–monoidal category can be thought of as a functor $M(D) \to \text{Cat}_0^\otimes$ fitting in a diagram

$$\begin{array}{ccc}
M(D) & \longrightarrow & \text{Cat}_0^\otimes \\
\downarrow & & \downarrow \\
I(D) & \longrightarrow & \text{Cat}
\end{array}$$
Accordingly, a morphism of monoidal $D$–categories is one of the corresponding functors.

![Diagram]

3.9. Fibered $D$–monoidal categories. We shall often be concerned with $D$–monoidal categories such that the underlying categories $(C'_B, \otimes_B)$ are strict, and the functors $F_{BB'} : (C'_B, \otimes_B) \to (C'_B, \otimes_B)$ are tensor functors. This may be described in terms of the category $M(D)$ as follows. Let $\text{DCat}^\otimes$ be the 2–category of Drinfeld categories, that is strict tensor categories $(C, \otimes)$ endowed with an additional associativity constraint $\Phi$ making $(C, \otimes, \Phi)$ a monoidal category. There is a canonical forgetful 2–functor $\text{DCat}^\otimes \to \text{Cat}^\otimes$.

We shall say that a $D$–monoidal category fibers over a strict $D$–monoidal category if the corresponding functor $M(D) \to \text{Cat}^\otimes$ maps into $\text{DCat}^\otimes$ and fits in a commutative diagram

![Diagram]

In this case, the coproduct $\Delta_F$ on a bialgebra $\text{End}(F_B)$ is the twist of a reference coassociative coproduct $\Delta_0$ on $\text{End}(F_D)$ such that $\Delta_0 : \text{End}(F_B) \to \text{End}(F_D)$.

3.10. Braided $D$–monoidal categories.

**Definition.** A braided $D$–monoidal category

$$C = \{(C_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'})\}$$

is the datum of

- a $D$–monoidal category $\{(C_B, \otimes_B, \Phi_B)\}, \{(F_{BB'}, J_{BB'})\}$
- for every $B \subseteq D$, a commutativity constraint $\beta_B$ in $C_B$, defining a braiding in $(C_B, \otimes_B, \Phi_B)$.

**Remark.** Note that the tensor functors $F_{BB'} : C'_{B'} \to C_B$ are not assumed to map the commutativity constraint $\beta_{B'}$ to $\beta_B$.

**Definition.** A morphism of braided $D$–monoidal categories from $C$ to $C'$ is a morphism of the underlying $D$–monoidal categories such that the functors $H_B : C_B \to C'_B$ are braided tensor functors.

**Remark.** The fact that $H_B$ are braided tensor functors automatically implies that

$$\Psi^2_F((R_B)_{J_F}) = (R'_B)_{J'_F}$$
in analogy with [TL4], where \( R_B = (12) \circ \beta_B \), and we are assuming that \( \mathcal{C}_\emptyset = \mathcal{C}_\emptyset' \) is a symmetric strict tensor category.

### 3.11. Quasi–Coxeter braided monoidal categories.

**Definition.** A quasi–Coxeter braided monoidal category of type \( D \)

\[
\mathcal{C} = (\{ (\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B) \}, \{ (F_{BB'}, J_{BB'}^F) \}, \{ \Phi_{FG} \}, \{ S_i \})
\]

is the datum of

- a quasi–Coxeter category of type \( D \),
  \[
  \mathcal{C} = (\{ (\mathcal{C}_B) \}, \{ (F_{BB'}) \}, \{ (\Phi_{FG}) \}, \{ S_i \})
  \]
- a braided \( D \)-monoidal category
  \[
  \mathcal{C} = (\{ (\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B) \}, \{ (F_{BB'}, J_{BB'}^F) \})
  \]

satisfying the following conditions

- for any \( B \subseteq B' \), and \( G, F \in \text{Mns}(B, B') \), the natural transformation \( \Phi_{FG} \in \text{Aut}(F_{BB'}) \) determines an isomorphism of tensor functors
  \[
  (F_{BB'}, J_{BB'}^G) \rightarrow (F_{BB'}, J_{BB'}^F), \text{ that is for any } V, W \in \mathcal{C}_{B'},
  \]
  \[
  (\Phi_{GF})_{V \otimes W} \circ (J_{BB'}^G)_{V,W} = (J_{BB'}^F)_{V,W} \circ ((\Phi_{GF})_V \otimes (\Phi_{GF})_W)
  \]
- for any \( i \in D \), the following holds:
  \[
  \Delta_i (S_i) = (R_i)_{J_i}^{21} \cdot (S_i \otimes S_i)
  \]

A morphism of quasi–Coxeter braided monoidal categories of type \( D \) is a morphism of the underlying quasi–Coxeter categories and braided \( D \)-monoidal categories.

**Remark.** A quasi–Coxeter braided monoidal category of type \( D \) determines a 2–functor \( \mathfrak{qC}(D) \rightarrow \text{Cat}^{\otimes} \) fitting in a diagram

\[
\begin{array}{ccc}
\mathfrak{qC}(D) & \longrightarrow & \text{Cat}^{\otimes} \\
\downarrow & & \downarrow \\
I(D) & \longrightarrow & \text{Cat}
\end{array}
\]

Note however that this functor does not entirely capture the quasi–Coxeter braided monoidal category since it does not encode the commutativity constraints \( \beta_B \) and automorphisms \( S_i \).

### 4. Etingof-Kazhdan quantization

We review in this section the results obtained in [EK1, EK6]. More specifically, we follow the quantization of Lie bialgebras given in [EK1, Part II] and the case of generalized Kac–Moody algebras from [EK6].
4.1. **Topological vector spaces.** The use of topological vector spaces is needed in order to deal with convergence issues related to duals of infinite dimensional vector spaces and tensor product of such spaces.

Let $k$ be a field of characteristic zero with the discrete topology and $V$ a topological vector space over $k$. The topology on $V$ is **linear** if open subspaces in $V$ form a basis of neighborhoods of zero. Let $V$ be endowed with a linear topology and $p_V$ the natural map

$$p_V : V \to \lim(V/U)$$

where the limit is taken over the open subspaces $U \subseteq V$. Then $V$ is called **separated** if $p_V$ is injective and **complete** if $p_V$ is surjective. Throughout this section, we shall call topological vector space a linear, complete, separated topological space.

If $U$ is an open subspace of a topological vector space $V$, then the quotient $V/U$ is discrete. It is then possible, given two topological vector spaces $V$ and $W$, to define the topological tensor product as

$$V \hat{} \otimes W := \lim V/V' \otimes W/W'$$

where the limit is take over open subspaces of $V$ and $W$. We then denote by $\text{Hom}_k(V,W)$ the topological vector space of continuous linear operators from $V$ to $W$ equipped with the weak topology. Namely, a basis of neighborhoods of zero in $\text{Hom}_k(V,W)$ is given by the collection of sets

$$Y(v_1,\ldots,v_n,W_1,\ldots,W_n) := \{f \in \text{Hom}_k(V,W) \mid f(v_i) \in W_i, i = 1,\ldots,n\}$$

for any $n \in \mathbb{N}, v_i \in V$ and $W_i$ open subspace in $W$ for all $i = 1,\ldots,n$. In particular, if $W = k$ with the discrete topology, the space $V^* = \text{Hom}_k(V,k)$ has a basis of neighborhoods of zero given by orthogonal complements of finite-dimensional subspaces in $V$. When $V$ is finite-dimensional, $V^*$ coincides with the linear dual and the weak topology coincides with the discrete topology. The canonical map $V \to V^{**}$ is a linear isomorphism, when $V$ is discrete, and it is not topological in general.

The space of formal power series in $h$ with coefficients in a topological vector space $V$, $V[[h]] = V \hat{} k[[h]]$, is also a complete topological space with a natural structure of a topological $k[[h]]$-module. A topological $k[[h]]$-module is complete if it is isomorphic to $V[[h]]$ for some complete $V$. The additive category of complete $k[[h]]$-module, denoted $\mathcal{A}$, where morphisms are continuous $k[[h]]$-linear maps, has a natural symmetric monoidal structure. Namely, the tensor product on $\mathcal{A}$ is defined to be the quotient of the tensor product $V \hat{} W$ by the image of the operator $h \otimes 1 - 1 \otimes h$. This tensor product will be still denoted by $\hat{}$. There is an extension of scalar functor from the category of topological spaces to $\mathcal{A}$, mapping $V$ to $V[[h]]$. This functor respects the tensor product, i.e., $(V \hat{} W)[[h]]$ is naturally isomorphic to $V[[h]] \hat{} W[[h]]$. 
4.2. **Equicontinuous modules.** Fix a topological Lie algebra $\mathfrak{g}$.

**Definition.** Let $V$ be a topological vector space. We say that $V$ is an *equicontinuous* $\mathfrak{g}$-module if:

- the map $\pi_V : \mathfrak{g} \to \text{End}_k V$ is a continuous homomorphism of topological Lie algebras;
- $\{\pi_V(g)\}_{g \in \mathfrak{g}}$ is an equicontinuous family of linear operators, *i.e.*, for any open subspace $U \subseteq V$, there exists $U'$ such that $\pi_V(g)U' \subseteq U$ for all $g \in \mathfrak{g}$.

Clearly, a topological vector space with a trivial $\mathfrak{g}$-module structure is an equicontinuous $\mathfrak{g}$-module. Moreover, given equicontinuous $\mathfrak{g}$-modules $V,W,U$, the tensor product $V \hat{\otimes} W$ has a natural structure of equicontinuous $\mathfrak{g}$-module and $(V \hat{\otimes} W) \hat{\otimes} U$ is naturally identified with $V \hat{\otimes} (W \hat{\otimes} U)$. The category of equicontinuous $\mathfrak{g}$-modules is then a symmetric monoidal category, with braiding defined by permutation of components. We denote this category by $\text{Rep}^{eq} \mathfrak{g}$.

4.3. **Lie bialgebras and Manin triples.** A Manin triple is the data of a Lie algebra $\mathfrak{g}$ with

- a nondegenerate invariant inner product $\langle , \rangle$;
- isotropic Lie subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$;

such that

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector space;
- the inner product defines an isomorphism $\mathfrak{g}_+ \cong \mathfrak{g}_-^*$;
- the commutator of $\mathfrak{g}$ is continuous with respect to the topology obtained by putting the discrete and the weak topology on $\mathfrak{g}_-, \mathfrak{g}_+$ respectively.

Under these assumptions, the commutator on $\mathfrak{g}_+ \cong \mathfrak{g}_-^*$ induces a co-bracket on $\mathfrak{g}_-$, satisfying the cocycle condition [D1]. Therefore, $\mathfrak{g}_-$ is canonically endowed with a Lie bialgebra structure. Notice that, in absolute generality, $\mathfrak{g}_+$ is only a topological Lie bialgebra, *i.e.*, $\delta(\mathfrak{g}_+) \subset \mathfrak{g}_+ \hat{\otimes} \mathfrak{g}_+$. The inner product also gives rise to an isomorphism of vector spaces $\mathfrak{g}_- \cong \mathfrak{g}_+^{**} \cong \mathfrak{g}_+^*$, where the latter is the continuous dual, though this isomorphism does not respect the topology. Conversely, every Lie bialgebra $\mathfrak{a}$ defines a Manin triple $(\mathfrak{a} \oplus \mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$.

4.4. **Verma modules.** In [EK1], Etingof and Kazhdan constructed two main examples of equicontinuous $\mathfrak{g}$-modules in the case when $\mathfrak{g}$ belongs to a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. The modules $M_{\pm}$, defined as

$$M_+ = \text{Ind}_{\mathfrak{g}_-}^\mathfrak{g} k \quad M_- = \text{Ind}_{\mathfrak{g}_+}^\mathfrak{g} k$$

are freely generated over $U(\mathfrak{g}_\pm)$ by a vector $1_{\pm}$ such that $\mathfrak{g}_1 1_{\pm}$. Therefore, they are naturally identified, as vector spaces, to $U(\mathfrak{g}_\pm)$ via $x 1_{\pm} \to x$. The
modules $M_-$ and $M_+^*$, with appropriate topologies, are equicontinuous $\mathfrak{g}$-modules.

The module $M_-$ is an equicontinuous $\mathfrak{g}$-module with respect to the discrete topology. The topology on $M_+$ comes, instead, from the identification of vector spaces

$$M_+ \simeq U(\mathfrak{g}_+) = \bigcup_{n \geq 0} U(\mathfrak{g}_+)n$$

where $U(\mathfrak{g}_+)n$ is the set of elements of degree at most $n$. The topology on $U(\mathfrak{g}_+)n$ is defined through the linear isomorphism

$$\xi_n : \bigoplus_{j=0}^n S^j \mathfrak{g}_+ \to U(\mathfrak{g}_+)n$$

where $S^j \mathfrak{g}_+$ is considered as a topological subspace of $(\mathfrak{g}_-)^j$, embedded with the weak topology. Finally, $U(\mathfrak{g}_+)n$ is equipped with the topology of the colimit. Namely, a set $U \subseteq U(\mathfrak{g}_+)n$ is open if and only if $U \cap U(\mathfrak{g}_+)m$ is open for all $m$. With respect to the topology just described, the action of $\mathfrak{g}$ on $M_+$ is continuous.

Consider now the vector space of continuous linear functionals on $M_+$

$$M_+^* = \text{Hom}_k(M_+, k) \simeq \text{colim} \text{Hom}_k(U(\mathfrak{g}_+)n, k)$$

It is natural to put the discrete topology on $U(\mathfrak{g}_+)n^*$, since, as a vector space,

$$U(\mathfrak{g}_+)n^* \simeq \bigoplus_{j=0}^n S^j \mathfrak{g}_+^* \simeq \bigoplus_{j=0}^n S^j \mathfrak{g}_-^* \simeq U(\mathfrak{g}_-)n$$

We then consider on $M_+^*$ the topology of the limit. This defines, in particular, a filtration by subspaces $(M_+^*)n$ satisfying

$$0 \to (M_+^*)n \to M_+^* \to (U(\mathfrak{g}_+)n)^* \to 0$$

and such that $M_+^* = \text{lim} M_+^*/(M_+^*)n$. The topology of the limit on $M_+^*$ is, in general, stronger than the weak topology of the dual. Since the action of $\mathfrak{g}$ on $M_+$ is continuous, $M_+^*$ has a natural structure of $\mathfrak{g}$-module. In particular, this is an equicontinuous $\mathfrak{g}$-action.

4.5. Drinfeld category. The natural embedding

$$\mathfrak{g}_- \otimes \mathfrak{g}_+^* \subset \text{End}_k(\mathfrak{g}_-)$$

induces a topology on $\mathfrak{g}_- \otimes \mathfrak{g}_+^*$ by restriction of the weak topology in $\text{End}_k(\mathfrak{g}_-)$. With respect to this topology, the image of $\mathfrak{g}_- \otimes \mathfrak{g}_+^*$ is dense in $\text{End}_k(\mathfrak{g}_-)$ and the topological completion $\mathfrak{g}_- \otimes \mathfrak{g}_+^*$ is identified with $\text{End}_k(\mathfrak{g}_-)$. Under this identification, the identity operator defines an element $r \in \mathfrak{g}_- \otimes \mathfrak{g}_+^*$.

Given two equicontinuous $\mathfrak{g}$-modules $V, W$, the map

$$\pi_V \otimes \pi_W : \mathfrak{g}_- \otimes \mathfrak{g}_+^* \to \text{End}_k(V \otimes W)$$
naturally extends to a continuous map \( g_\hat{\otimes}g^\ast \to \text{End}_k(V \hat{\otimes} W) \). Therefore, the Casimir operator

\[
\Omega = r + r^{\text{op}} \in g_\hat{\otimes}g^\ast \otimes g^\ast_\hat{\otimes}g
\]
defines a continuous endomorphism of \( V \hat{\otimes} W \), \( \Omega_{VW} = (\pi_V \otimes \pi_W)(\Omega) \), commuting with the action of \( g \).

Following [D2], it is possible to define a structure of braided monoidal category on the category of deformed equicontinuous \( g \)-module, depending on the choice of a Lie associator \( \Phi \), the bifunctor \( \hat{\otimes} \) and the Casimir operator \( \Omega \). The commutativity constraint is explicitly defined by the formula

\[
\beta_{VW} = (12) \circ e^{\frac{\Omega_{VW}}{2}} \in \text{Hom}_g(V \hat{\otimes} W, W \hat{\otimes} V)[[h]]
\]
We denote this braided tensor category braided tensor category \( \mathcal{D}_\Phi(Ug) \). The category of equicontinuous \( g \)-modules is equivalent to the category of Yetter-Drinfeld module over \( g_\mathcal{D}(g_-) \). The equivalence holds at the level of tensor structure induced by the choice of an associator \( \Phi \),

\[
\mathcal{D}_\Phi(Ug) \simeq \mathcal{YD}_\Phi(Ug_-[[h]])
\]

4.6. Verma modules. The modules \( M_\pm \) are identified, as vector spaces, with the enveloping universal algebras \( Ug_\pm \). Their comultiplications induce the \( Ug \)-intertwiners \( i_\pm : M_\pm \to M_\pm \hat{\otimes} M_\pm \), mapping the vectors \( 1_\pm \) to the \( g_\mp \)-invariant vectors \( 1_\pm \otimes 1_\pm \).

For any \( f, g \in M_\ast \), consider the linear functional \( v \mapsto (f \otimes g)(i_+(v)) \). This functional defines a map \( i_+^* : M_\ast \otimes M_\ast^+ \to M_\ast^+ \), that is continuous and extends to a morphism in \( \text{Rep}\ g[[h]] \), \( i_+^* : M_\ast^+ \otimes M_\ast^+ \to M_\ast^+ \). The pairs \( (M_-, i_-) \) and \( (M_+, i_+) \) form, respectively, a coalgebra and an algebra object in \( \mathcal{D}_\Phi(Ug) \).

For any \( V \in \mathcal{D}_\Phi(Ug) \), the vector space \( \text{Hom}_g(M_-, M_\ast^+ \hat{\otimes} V) \) is naturally isomorphic to \( V \), as topological vector space, through the isomorphism \( f \mapsto (1_+ \otimes 1)f(1_-) \).

4.7. The fiber functor and the EK quantization. We will now recall the main results from [EK1, EK2]. Where no confusion is possible, we will abusively denote \( \hat{\otimes} \) by \( \otimes \). Let then \( F \) be the functor

\[
F : \mathcal{D}_\Phi(Ug) \to \mathcal{A} \quad F(V) = \text{Hom}_{\mathcal{D}_\Phi(Ug)}(M_-, M_\ast^+ \otimes V)
\]
There is a natural transformation

\[
J \in \text{Nat}(\otimes \circ (F \otimes F), F \circ \otimes)
\]
defined, for any \( v \in F(V), w \in F(W) \), by

\[
J_{VW}(v \otimes w) = (i_+^* \otimes 1 \otimes 1)A^{-1}\beta_{23}^{-1}A(v \otimes w)i_-
\]
where $A$ is defined as a morphism
\[(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \to V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)\]
by the action of $(1 \otimes \Phi_{2,34})\Phi_{1,2,34}$.

**Theorem.** The natural transformation $J$ is invertible and defines a tensor structure on the functor $F$.

The tensor functor $(F, J)$ is called fiber functor. The algebra of endomorphisms of $F$ is therefore naturally endowed with a topological bialgebra structure, as described in the previous section.\(^1\)

The object $F(M_-) \in A$ has a natural structure of Hopf algebra, defined by the multiplication
\[m : F(M_-) \otimes F(M_-) \to F(M_-) \quad m(x, y) = (i^+ \otimes 1)\Phi^{-1}(1 \otimes y)x\]
and the comultiplication
\[\Delta : F(M_-) \to F(M_-) \otimes F(M_-) \quad \Delta(x) = J^{-1}(F(\iota_-(x)))\]
The algebra $F(M_-)$ is naturally isomorphic as a vector space with $M_-[[\hbar]] \simeq U\mathfrak{g}_-[[\hbar]]$ and

**Theorem.** The algebra $U^\text{EK}_h \mathfrak{g}_- = F(M_-)$ is a quantization of the algebra $U\mathfrak{g}_-.$

In [EK2], it is shown that this construction defines a functor
\[Q^\text{EK} : \text{LBA}(k) \to \text{QUE}(K)\]
where $\text{LBA}(k)$ denotes the category of Lie bialgebras over $k$ and $\text{QUE}(K)$ denotes the category of quantum universal enveloping algebras over $K = k[[\hbar]]$. Another important result in [EK2] states the invertibility of the functor $Q^\text{EK}$.

The map
\[m_- : U^\text{EK}_h \mathfrak{g}_- \to \text{End}(F) \quad m_-(x)V(v) = (i^+ \otimes 1)\Phi^{-1}(1 \otimes v)x\]
where $V \in \mathcal{YD}_\Phi(U\mathfrak{g}_-[[\hbar]])$ and $v \in F(V)$, is, indeed, an inclusion of Hopf algebras. The map $m_-$ defines an action of $U^\text{EK}_h \mathfrak{g}_-$ on $F(V)$. Moreover, the map
\[F(V) \to F(M_-) \otimes F(V) \quad v \mapsto R_J(1 \otimes v)\]
where $R_J$ denotes the twisted $R$–matrix, defines a coaction of $U^\text{EK}_h \mathfrak{g}_-$ on $F(V)$ compatible with the action, therefore

**Theorem.** The fiber functor $F : \mathcal{YD}_\Phi(U\mathfrak{g}_-[[\hbar]]) \to A$ lifts to an equivalence of braided tensor categories
\[\tilde{F} : \mathcal{YD}_\Phi(U\mathfrak{g}_-[[\hbar]]) \to \mathcal{YD}(U^\text{EK}_h \mathfrak{g}_-)\]

\(^1\)By topological bialgebra we do not mean topological over $k[[\hbar]]$. We are instead referring to the fact that the algebra $\text{End}(F)$ has a natural comultiplication $\Delta : \text{End}(F) \to \text{End}(F^2)$, where $\text{End}(F^2)$ can be interpreted as an appropriate completion of $\text{End}(F)^{\otimes 2}$.
4.8. Generalized Kac-Moody algebras. Denote by $k$ a field of characteristic zero. We recall definitions from [Ka] and [EK6]. Let $A = (a_{ij})_{i,j \in I}$ be an $n \times n$ symmetrizable matrix with entries in $k$, i.e., there exists a (fixed) collection of nonzero numbers $\{d_i\}_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. Let $(h, \Pi, \Pi^\vee)$ be a realization of $A$. It means that $h$ is a vector space of dimension $2n - \text{rank}(A)$, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset h^*$ and $\Pi^\vee = \{h_1, \ldots, h_n\} \subset h$ are linearly independent, and $(\alpha_i, h_j) = a_{ij}$.

**Definition.** The Lie algebra $\tilde{g} = \tilde{g}(A)$ is generated by $h, \{e_i, f_i\}_{i \in I}$ with defining relations

$$[h, h'] = 0 \quad h, h' \in h; \quad [h, e_i] = (\alpha_i, h) e_i$$

$$[h, f_i] = -(\alpha_i, h) f_i; \quad [e_i, f_j] = \delta_{ij} h_i$$

There exists a unique maximal ideal $r$ in $\tilde{g}$ that intersect $h$ trivially. Let $g := \tilde{g}/r$. The algebra $g$ is called *generalized Kac-Moody algebra*. The Lie algebra $g$ is graded by principal gradation $\text{deg}(e_i) = 1, \text{deg}(f_i) = -1, \text{deg}(h) = 0$, and the homogenous component are all finite-dimensional.

Let us now choose a non-degenerate bilinear symmetric form on $h$ such that $\langle h, h_i \rangle = d_i^{-1}(\alpha_i, h)$. Following [Ka], there exists a unique extension of the form $\langle \cdot, \cdot \rangle$ to an invariant symmetric bilinear form on $\tilde{g}$. For this extension, one gets $\langle e_i, f_j \rangle = \delta_{ij} d_i^{-1}$. The kernel of this form on $\tilde{g}$ is $r$, therefore it descends to a non-degenerate bilinear form on $g$.

Let $n_{\pm}, b_{\pm}$ be the nilpotent and the Borel subalgebras of $g$, i.e., $n_\pm$ are generated by $\{e_i\}, \{f_i\}$, respectively, and $b_\pm := n_\pm \oplus h$. Since $[n_\pm, h] \subset n_\pm$, we get Lie algebra maps $\eta : b_\pm \to h$ and we can consider the embeddings of Lie subalgebras $\eta_{\pm} : b_\pm \to g \oplus h$ given by

$$\eta_{\pm}(x) = (x, \pm \bar{x})$$

Define the inner product on $g \oplus h$ by $\langle \cdot, \cdot \rangle_{g \oplus h} = \langle \cdot \rangle_g - \langle \cdot \rangle_h$.

**Proposition.** The triple $(g \oplus h, b_+, b_-)$ with inner product $\langle \cdot, \cdot \rangle_{g \oplus h}$ and embeddings $\eta_{\pm}$ is a graded Manin triple.

Under the embeddings $\eta_{\pm}$, the Lie subalgebras $b_{\pm}$ are isotropic with respect to $\langle \cdot, \cdot \rangle_{g \oplus h}$. Since $\langle \cdot \rangle_g$ and $\langle \cdot \rangle_h$ are invariant symmetric non-degenerate bilinear form, so is $\langle \cdot, \cdot \rangle_{g \oplus h}$.

The proposition implies that $g \oplus h, b_+, b_-$ are Lie bialgebras. Moreover, $b_+^* \simeq b_-^{\text{cop}}$ as Lie bialgebras (where $b_+^* := \bigoplus (b_+)^*_n$ denotes the restricted dual space and by $\text{cop}$ we mean the opposite cocommutator). The Lie bialgebra structures on $b_\pm$ are then described by the following formulas:

$$\delta(h) = 0, \quad h \in h \subset b_\pm;$$

$$\delta(e_i) = \frac{d_i}{2}(e_i \otimes h_i - h_i \otimes e_i) = \frac{d_i}{2} e_i \wedge h_i; \quad \delta(f_i) = \frac{d_i}{2} f_i \wedge h_i$$
The Lie subalgebra \( \{(0, h) \mid h \in \mathfrak{h} \} \) is therefore an ideal and a coideal in \( \mathfrak{g} \oplus \mathfrak{h} \). Thus, the quotient \( \mathfrak{g} = (\mathfrak{g} \oplus \mathfrak{h})/\mathfrak{h} \) is also a Lie bialgebra with Lie subbialgebras \( \mathfrak{b}_\pm \) and the same cocommutator formulas.

4.9. **Quantization of Kac–Moody algebras and category \( \mathcal{O} \).** In [EK6], Etingof and Kazhdan proved that, for any symmetrizable irreducible Kac–Moody algebra \( \mathfrak{g} \), the quantization \( U^\mathrm{EK}_h \mathfrak{g} \) is isomorphic with the Drinfeld–Jimbo quantum group \( U_h \mathfrak{g} \).

In particular, they construct an isomorphism of Hopf algebras \( U_h \mathfrak{b}_+ \simeq U^\mathrm{EK}_h \mathfrak{b}_+ \), inducing the identity on \( U_h[[\mathfrak{h}]] \), where \( \mathfrak{b}_+ \) is the Borel subalgebra and \( \mathfrak{h} \) is the Cartan subalgebra of \( \mathfrak{g} \). Thanks to the compatibility with the doubling operations

\[
\mathcal{D} U^\mathrm{EK}_h \mathfrak{b}_+ \simeq U^\mathrm{EK}_h \mathfrak{b}_+
\]

proved by Enriquez and Geer in [EG], the isomorphism for the Borel subalgebra induces an isomorphism \( U_h \mathfrak{g} \simeq U^\mathrm{EK}_h \mathfrak{g} \).

Recall that the category \( \mathcal{O} \) for \( \mathfrak{g} \), denoted \( \mathcal{O}_\mathfrak{g} \) is defined to be the category of all \( \mathfrak{h} \)-diagonalizable \( \mathfrak{g} \)-modules \( V \), whose set of weights \( \mathcal{P}(V) \) belong to a union of finitely many cones

\[
\mathcal{D}(\lambda_s) = \lambda_s + \sum_i \mathbb{Z}_{\geq 0} \alpha_i, \quad \lambda_s \in \mathfrak{h}^*, s = 1, ..., r
\]

and the weight subspaces are finite-dimensional. We denote by \( \mathcal{O}_\mathfrak{g}[[\mathfrak{h}]] \) the category of deformation \( \mathfrak{g} \)-representations, i.e., representations of \( \mathfrak{g} \) on topologically free \( k[[\mathfrak{h}]] \)-modules with the above properties (with weights in \( \mathfrak{h}^*[[\mathfrak{h}]] \)).

In a similar way, one defines the category \( \mathcal{O}_{U_h \mathfrak{g}} \): it is the category of \( U_h \mathfrak{g} \)-modules which are topologically free over \( k[[\mathfrak{h}]] \) and satisfy the same conditions as in the classical case.

The morphism of Lie bialgebras

\[
\mathfrak{D} \mathfrak{b}_+ \rightarrow \mathfrak{g} \simeq \mathfrak{D} \mathfrak{b}_+/(\mathfrak{h} \simeq \mathfrak{h}^*)
\]

gives rise to a pullback functors

\[
\mathcal{O}_\mathfrak{g} \rightarrow \mathcal{YD}(U\mathfrak{b}_+) \quad \mathcal{O}_{\mathfrak{g}_\Phi}[[\mathfrak{h}]] \rightarrow \mathcal{YD}_\Phi(U\mathfrak{b}_+[[\mathfrak{h}]])
\]

where \( \mathcal{O}_{\mathfrak{g}_\Phi} \) denotes the category \( \mathcal{O}_\mathfrak{g} \) with the tensor structure of the Drinfeld category. Similarly, the morphism of Hopf algebras

\[
DU^\mathrm{EK}_h \mathfrak{b}_+ \rightarrow U^\mathrm{EK}_h \mathfrak{g} \simeq U_h \mathfrak{g}
\]

gives rise to a pullback functor

\[
\mathcal{O}_{U_h \mathfrak{g}} \rightarrow \mathcal{YD}(U^\mathrm{EK}_h \mathfrak{b}_+)
\]
Theorem. The equivalence $\tilde{F}$ reduces to an equivalence of braided tensor categories

$$\tilde{F}_\mathcal{O} : \mathcal{O}_\mathfrak{g}[\hbar] \to \mathcal{O}_{U_\mathfrak{g}}$$

which is isomorphic to the identity functor at the level of $\mathfrak{h}$–graded $k[[\hbar]]$–modules.

4.10. The isomorphism $\Psi^{EK}$. In [EK6], Etingof–Kazhdan showed that the equivalence $\tilde{F}$ induces an isomorphism of algebras

$$\Psi^{EK} : \widehat{U}_\mathfrak{g}[[\hbar]] \to \widehat{U}_\hbar \mathfrak{g}$$

where

$$\widehat{U}_\mathfrak{g} = \lim_{\beta} U_\beta \quad U_\beta = U_\mathfrak{g}/I_\beta, \beta \in \mathbb{N}^I$$

$I_\beta$ being the left ideal generated by elements of weight less or equal $\beta$ (analogously for $\widehat{U}_\hbar \mathfrak{g}$, cf. [EK6, Sec. 4]).

Proposition. The isomorphism $\Psi^{EK}$ coincides with the isomorphism induced by the equivalence $F_\mathcal{O}$, as explained in Section 3.1.

Proof. The identification of the two isomorphism is constructed in the following way:

(a) First, we show that there is a canonical map

$$\text{End}(f_\mathcal{O}) \to C_{\text{End}(\mathcal{O})}(\text{End}_\mathfrak{g}(\mathcal{U}))$$

(b) There is a canonical multiplication in $\mathcal{U}$, so that

(i) There is a canonical map

$$C_{\text{End}(\mathcal{U})}(\text{End}_\mathfrak{g}(\mathcal{U})) \to \mathcal{U}$$

(ii) For every $V \in \mathcal{O}$ the action of $U_\mathfrak{g}$ lifts to an action of $\mathcal{U}$

$$U_\mathfrak{g} \quad \rightarrow \quad \text{End}(V) \quad \downarrow \quad \mathcal{U}$$

$$\downarrow \quad \star$$

(c) It defines a map $\mathcal{U} \to \text{End}(f_\mathcal{O})$ and we have an isomorphism of algebras

$$\mathcal{U} \simeq \text{End}(f_\mathcal{O})$$

If $\mathfrak{g}$ is a semisimple Lie algebra, the equivalence of categories $\tilde{F}$ leads to an isomorphism of algebras

$$U(\mathfrak{D}_\mathfrak{b}_+)[[\hbar]] \simeq DU^{EK}_\hbar \mathfrak{b}_+ \Longrightarrow U_\mathfrak{g}[[\hbar]] \simeq U_\mathfrak{h} \mathfrak{g}$$

which is the identity modulo $\hbar$. Toledano Laredo proved in [TL4, Prop. 3.5] that such an isomorphism cannot be compatible with all the isomorphisms

$$U_\mathfrak{sl}_2[[\hbar]] \simeq U_\hbar \mathfrak{sl}_2^{\alpha_i} \quad \forall i$$
where \( \{ \alpha_i \} \) are the simple roots of \( \mathfrak{g} \). This amounts to a simple proof that the isomorphism \( \Psi^{\mathfrak{E}K} \) cannot be, in general, an isomorphism of \( D \)-algebras.

5. A relative Etingof–Kazhdan functor

5.1. In this section, we consider a split inclusion of Manin triples

\[ i_D : (\mathfrak{g}_D, \mathfrak{g}_{D,+}, \mathfrak{g}_{D,-}) \hookrightarrow (\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-) \]

We then define a relative version of the Verma modules \( M_\pm \), and use them to prove the following

**Theorem.** There is a tensor functor

\[ \Gamma : \mathcal{D}_\mathfrak{g}(U \mathfrak{g}) \to \mathcal{D}_{\mathfrak{g}_D}(U \mathfrak{g}_D) \]

canonically isomorphic, as abelian functor, to the restriction functor \( i_D^* \).

5.2. Split inclusions of Manin triples.

**Definition.** An embedding of Manin triples

\[ i : (\mathfrak{g}_D, \mathfrak{g}_{D,-}, \mathfrak{g}_{D,+}) \longrightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \]

is a Lie algebra homomorphism \( i : \mathfrak{g}_D \to \mathfrak{g} \) preserving inner products, and such that \( i(\mathfrak{g}_{D,\pm}) \subseteq \mathfrak{g}_{\pm} \).

Denote the restriction of \( i \) to \( \mathfrak{g}_{D,\pm} \) by \( i_\pm \). \( i_\pm \) give rise to maps \( p_\pm = i_\pm^* : \mathfrak{g}_{\pm} \to \mathfrak{g}_{D,\pm} \), defined via the identifications \( \mathfrak{g}_{\pm} \simeq \mathfrak{g}_{D,\pm}^* \) and \( \mathfrak{g}_{D,\pm} \simeq \mathfrak{g}_{D,\mp}^* \) by

\[ \langle p_\pm(x), y \rangle_D = \langle x, i_\pm(y) \rangle \]

for any \( x \in \mathfrak{g}_{\pm} \) and \( y \in \mathfrak{g}_{D,\mp} \). These map satisfy \( p_\pm \circ i_\pm = \text{id}_{\mathfrak{g}_{D,\pm}} \) since, for any \( x \in \mathfrak{g}_{D,\pm}, y \in \mathfrak{g}_{D,\mp}, \)

\[ \langle p_\pm \circ i_\pm(x) - x, y \rangle_D = \langle i_\pm(x), i_\mp(y) \rangle - \langle x, y \rangle_D = 0 \]

This yields in particular a a direct sum decomposition \( \mathfrak{g}_{\pm} = i(\mathfrak{g}_{D,\pm}) \oplus \mathfrak{m}_{\pm} \), where

\[ \mathfrak{m}_{\pm} = \text{Ker}(p_\pm) = \mathfrak{g}_{\pm} \cap i(\mathfrak{g}_D)^\perp \]

**Definition.** The embedding \( i : (\mathfrak{g}_D, \mathfrak{g}_{D,-}, \mathfrak{g}_{D,+}) \longrightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \) is called *split* if the subspaces \( \mathfrak{m}_{\pm} \subseteq \mathfrak{g}_{\pm} \) are Lie subalgebras.

5.3. Split pairs of Lie bialgebras. For later use, we reformulate the above notion in terms of bialgebras via the double construction.

**Definition.** A *split pair* of Lie bialgebras is the data of

- Lie bialgebras \((\mathfrak{a}, [,], \delta_{\mathfrak{a}})\) and \((\mathfrak{b}, [,], \delta_{\mathfrak{b}})\).
- Lie bialgebra morphisms \( i : \mathfrak{a} \to \mathfrak{b} \) and \( p : \mathfrak{b} \to \mathfrak{a} \) such that \( p \circ i = \text{id}_{\mathfrak{a}} \).

**Proposition.** There is a one–to–one correspondence between split inclusions of Manin triples and split pairs of Lie bialgebras. Specifically,

(i) If \( i : (\mathfrak{g}_D, \mathfrak{g}_{D,-}, \mathfrak{g}_{D,+}) \longrightarrow (\mathfrak{g}, \mathfrak{g}_-, \mathfrak{g}_+) \) is a split inclusion of Manin triples, then \( (\mathfrak{g}_{D,-}, \mathfrak{g}_-, i_-, i_-^*) \) is a split pair of Lie bialgebras.

(ii) Conversely, if \((\mathfrak{a}, \mathfrak{b}, i, p)\) is a split pair of Lie bialgebras, then \( i \oplus p^*: (\mathcal{D}\mathfrak{a}, \mathfrak{a}^*) \longrightarrow (\mathcal{D}\mathfrak{b}, \mathfrak{b}^*) \) is a split inclusion of Manin triples.
5.4. Proof of (i) of Proposition 5.3. Given a split inclusion
\[ i = i_- \oplus i_+ : (\mathfrak{g}_D, \mathfrak{g}_D, \mathfrak{g}_D) \to (\mathfrak{g}, \mathfrak{g}, \mathfrak{g}) \]
we need to show that \( i_- \) and \( i_+^* \) are Lie bialgebra morphisms. By assumption, \( i_- \) is a morphism of Lie algebras, and \( i_+^* \) one of coalgebras. Since \( i_- = (i_+^*)^* \), it suffices to show that \( p_\pm = i_\pm^* \) preserve Lie brackets.

We claim to this end that \( m_\pm \) are ideals in \( \mathfrak{g}_\pm \). Since \( [m_\pm, m_\pm] \subseteq m_\pm \) by assumption, this amounts to showing that \( [i(m_\pm), m_\pm] \subseteq m_\pm \). This follows from the fact that \( [i(D_\pm), m_\pm] \subseteq \mathfrak{g}_\pm \), and from
\[
\langle [i(D_\pm), m_\pm], i(D_\pm) \rangle = \langle m_\pm, i(D_\pm) \rangle \subset \langle m_\pm, i(D_\pm) \rangle + \langle m_\pm, i(D_\pm) \rangle
\]
where the first term is zero since \( \mathfrak{g}_\pm \) is isotropic, and the second one is zero by definition of \( m_\pm \).

Let now \( X_1, X_2 \in \mathfrak{g}_\pm \), and write \( X_j = i_\pm(x_j) + y_j \), where \( x_j \in \mathfrak{g}_{D,\pm} \) and \( y_j \in m_\pm \). Since \( m_\pm = \text{Ker}(p_\pm) \) and \( p_\pm \circ i_\pm = \text{id} \), we have \( [p_\pm(X_1), p_\pm(X_2)] = [x_1, x_2] \), while
\[
p_\pm[X_1, X_2] = p_\pm(i_\pm[x_1, x_2] + [i_\pm x_1, y_2] + [y_1, i_\pm x_2] + [y_1, y_2]) = [x_1, x_2]
\]
where the last equality follows from the fact that \( m_\pm \) is an ideal.

5.5. Proof of (ii) of Proposition 5.3. The bracket on \( \mathfrak{D} \mathfrak{a} \) is defined by
\[
[a, \phi] = \text{ad}^*(a)(\phi) - \text{ad}^*(\phi)(a) = -\langle \phi, [a, -]_\mathfrak{a} \rangle + \langle \phi \otimes \text{id}, \delta_\mathfrak{a}(a) \rangle
\]
for any \( a \in \mathfrak{a}, \phi \in \mathfrak{a}^* \). Analogously for \( \mathfrak{D} \mathfrak{b} \). Therefore, the equalities
\[
\langle p^*(\phi) \otimes \text{id}, \delta_\mathfrak{b}(i(a)) \rangle = \langle \phi \otimes \text{id}, (p \otimes \text{id})(i \otimes i)\delta_\mathfrak{a}(a) \rangle
\]
\[
= \langle \phi \otimes \text{id}, (\text{id} \otimes i)\delta_\mathfrak{a}(a) \rangle = i(\langle \phi \otimes \text{id}, \delta_\mathfrak{a}(a) \rangle)
\]
and
\[
\langle p^*(\phi), [i(a), b]_\mathfrak{b} \rangle = \langle \phi, p([i(a), b]_\mathfrak{b}) \rangle = \langle \phi, [a, p(b)]_\mathfrak{a} \rangle
\]
for all \( a \in \mathfrak{a} \) and \( b \in \mathfrak{b} \), imply that the map \( i \otimes p^* \) is a Lie algebra map. It also respects the inner product, since for any \( a \in \mathfrak{a}, \phi \in \mathfrak{a}^* \),
\[
\langle p^*(\phi), i(a) \rangle = \langle \phi, p \circ i(a) \rangle = \langle \phi, a \rangle
\]
Finally, \( m_- = \text{Ker}(p) \) and \( m_+ = \text{Ker} i^* \) are clearly subalgebras.

5.6. Parabolic Lie subalgebras. Let
\[
i_D = i_- \oplus i_+ : (\mathfrak{g}_D, \mathfrak{g}_D, \mathfrak{g}_D) \to (\mathfrak{g}, \mathfrak{g}, \mathfrak{g})
\]
be a split embedding of Manin triples. We henceforth identify \( \mathfrak{g}_D \) as a Lie subalgebra of \( \mathfrak{g} \) with its induced inner product, and \( \mathfrak{g}_{D,\pm} \) as subalgebras of \( \mathfrak{g}_\pm \) noting that, by Proposition 5.3, \( \mathfrak{g}_{D,-} \) is a sub Lie bialgebra of \( \mathfrak{g}_- \).

The following summarizes the properties of the subspaces \( m_\pm = \mathfrak{g}_\pm \cap \mathfrak{g}_D \) and \( p_\pm = m_\pm \oplus \mathfrak{g}_D \).

Proposition.
(i) $m_\pm$ is an ideal in $g_\pm$, so that $g_\pm = m_\pm \times g_{D,\pm}$.

(ii) $[g_D, m_\pm] \subset m_\pm$, so that $p_\pm = m_\pm \times g_D$ are Lie subalgebras of $g$.

(iii) $\delta(m_-) \subset m_- \otimes g_{D,-} + g_{D,-} \otimes m_-$, so that $m_- \subseteq g_-$ is a coideal.

**Proof.** (i) was proved in 5.4. (ii) Since $\langle [g_D, m_\pm], g_D \rangle = \langle m_\pm, [g_D, g_D] \rangle = 0$ we have $[g_D, m_\pm] \subset g_D^\perp = m_- \oplus m_+$. Moreover, $\langle [g_D, m_\pm], m_\pm \rangle = \langle g_D, [m_\pm, m_\pm] \rangle = \langle g_D, m_\pm \rangle = 0$ since $m_\pm$ is a subalgebra, and it follows that $[g_D, m_\pm] \subset m_\pm$. (iii) is clear since $m_-$ is the kernel of a Lie coalgebra map.

**Remark.** If the inclusion $i_D$ is compatible with a finite type $\mathbb{N}$–grading, then $m_\pm \subset g_\pm$ is a coideal. Moreover, $p_\pm$ are Lie subbialgebras of $g$ such that the projection $p_\pm \rightarrow g_D$ is a morphism of bialgebras. Namely, a finite type $\mathbb{N}$–grading allows to define a Lie bialgebra structure on $g, g_\pm$. We then get a vector space decomposition $g_\pm = m_\pm \oplus g_{D,\pm}$ and a Lie bialgebra map $g_\pm \rightarrow g_{D,\pm}$. It is also possible to define the Lie subalgebras $p_\pm = m_\pm \oplus g_D \subset g$.

If we assume the existence of a compatible grading on $g$ and $g_D, i.e.,$ preserved by $i_D$, then the natural maps $p_\pm \subset g$ $p_\pm \rightarrow g_D$ are morphisms of Lie bialgebras.

### 5.7. The relative Verma Modules.

**Definition.** Given a split embedding of Manin triples $g_D \subset g$, and the corresponding decomposition $g = m_\pm \oplus p_\pm$, let $L_-, N_+$ be the relative Verma modules defined by $L_- = \text{Ind}_{p_-}^g k$ and $N_+ = \text{Ind}_{m_-}^g k$

**Proposition.** The $g$–modules $L_-$ and $N_+$ are equicontinuous.

The description of the appropriate topologies on $L_-$ and $N_+$, and the proof of their equicontinuity will be carried out in 5.8–5.11.

### 5.8. Equicontinuity of $L_-$. As vector spaces, $L_- \simeq U m_- \subset U g_-$ so it is natural to equip $L_-$ with the discrete topology. The set of operators $\{\pi_{L_-}(x)\}_{x \in g}$ is then an equicontinuous family, and the continuity of $\pi_{L_-}$ reduces to checking that, for every element $v \in L_-$, the set $Y_v = \{b \in g_+ | b.v = 0\}$
is a neighborhood of zero in \( g_+ \). Since \( U \mathfrak{m}_- \) embeds naturally in \( U g_- \) the proof is identical to [EK1, Lemma 7.2]. We proceed by induction on the length of \( v = a_1 \ldots a_i 1_- \). If \( n = 0 \), then \( v = 1_- \) and \( Y_v = g_+ \). If \( n > 1 \), then assume \( v = a_j w \), with \( w = a_1 \ldots a_{i-1} 1_- \) and \( Y_w \) open in \( g_+ \). For every \( x \in g_+ \),

\[
x.v = x.(a_j w) = [x, a_j]w + (a_j x).w
\]

Call \( Z \) the subset of \( g_+ \)

\[
Z = \{ x \in g_+ \mid [x, a_j] \in Y_w \}
\]

\( Z \) is open in \( g_+ \), by continuity of bracket \([,]\), and clearly \( Z \cap Y_w \subset Y_v \).

5.9. Topology of \( N_+ \). As vector spaces,

\[
N_+ = \text{Ind}_{\mathfrak{m}_-}^{g_+} k \simeq U p_+ \simeq \text{colim} \ U_n p_+
\]

where \( \{ U_n p_+ \} \) denotes the standard filtration of \( U p_+ \), so that

\[
U_n p_+ \simeq \bigoplus_{m=0}^n S^m p_+ = \bigoplus_{i+j \leq n} (S^i g_+ \otimes S^j g_{D,-})
\]

We turn this isomorphism into an isomorphism of topological vector spaces, by taking on \( S^i g_+ \) and \( S^j g_{D,-} \) the topologies induced by the embeddings

\[
S^i g_+ \hookrightarrow (g_+^{[i]})^* \quad \text{and} \quad S^j g_{D,-} \hookrightarrow g_{D,-}^{[j]}
\]

With respect to these topologies, \( U_m p_+ \) is closed inside \( U_n p_+ \) for \( m < n \), and we equip \( N_+ \) with the direct limit topology. We shall need the following

**Lemma.** For any \( x \in g \), the map \( \pi_{N_+}(x) : N_+ \to N_+ \) is continuous.

**Proof.** We need to show that for any neighborhood of the origin \( U \subset N_+ \), there exists a neighborhood of zero \( U' \subset N_+ \) such that \( \pi_{N_+}(x)U' \subset U \). The topology on \( N_+ \) comes from the decomposition \( U p_+ \simeq U g_+ \otimes U g_{D,-} \), so that an open neighborhood of zero in \( N_+ \) has the form \( U \otimes U g_{D,-} + U g_+ \otimes V \), with \( U \) open in \( U g_+ \) and \( V \) open in \( U g_{D,-} \). We apply the same procedure used in [EK1, Lemma 7.3] to construct a set \( U' \otimes U g_{D,-} \), with \( U' \) open in \( U g_+ \), such that

\[
\pi_{N_+}(x)(U' \otimes U g_{D,-}) \subset U \otimes U g_{D,-} \subset U \otimes U g_{D,-} + U g_+ \otimes V
\]

Since the topology on \( U g_{D,-} \) is discrete, the set \( U' \otimes U g_{D,-} \) is open in \( N_+ \) and the lemma is proved. \( \square \)

5.10. Topology of \( N_+^* \). As vector spaces,

\[
N_+^* \simeq (U p_+)^* \simeq \text{lim}(U_n p_+)^*
\]

Define a filtration \( \{ (N_+^*)_n \} \) on \( N_+^* \) by

\[
0 \to (N_+^*)_n \to (U p_+)^* \to (U_n p_+)^* \to 0
\]

so that \( N_+^* \supset (N_+^*)_0 \supset (N_+^*)_1 \supset \cdots \), and we get an isomorphism of vector spaces

\[
N_+^* \simeq \text{lim} N_+^*/(N_+^*)_n
\]
Finally, we use the isomorphism to endow $N_+^*$ with the inverse limit topology.

**Lemma.** $\{\pi_{N_+^*}(x)\}_{x \in \mathfrak{g}}$ is an equicontinuous family of operators.

**Proof.** Since $p_+$ acts on $N_+$ by multiplication,

$$p_+(N_+^*)_n \subset (N_+^*)_{n-1}$$

If $x \in m_-$ and $x_i \in U p_+$ for $i = 1, \ldots, n$, then in $U \mathfrak{g},$

$$xx_1 \cdots x_n = x_1 \cdots x_n x - \sum_{i=0}^{n} x_1 \cdots x_{i-1}[x_i, x]x_{i+1} \cdots x_n$$

where $[x_i, x] \in \mathfrak{g}$. Iterating shows that $(x.f)(x_1 \cdots x_n) = 0$ if $f \in (N_+^*)_n$, so that $x(N_+^*)_n \subset (N_+^*)_{n-1}$. Then, for any neighborhood of zero of the form $U = (N_+^*)_n$, it is enough to take $U' = (N_+^*)_{n+1}$ to get $g(N_+^*)_{n+1} \subset (N_+^*)_n$. \hfill \Box

5.11. Equiscontinuity of $N_+^*$.

**Lemma.** The map $\pi_{N_+^*} : \mathfrak{g} \to \text{End}(N_+^*)$ is a continuous map.

**Proof.** Since $\mathfrak{g}_-$ is discrete, it is enough to check that, for any $f \in N_+^*$ and $n \in \mathbb{N}$, the subset

$$Y(f, n) = \{ b \in \mathfrak{g}_+ | b.f \in (N_+^*)_n \}$$

is open in $\mathfrak{g}_+$, i.e.

$$b^i.f \in (N_+^*)_n \quad \text{for a.a.} \ i \in I$$

Since $f \in N_+^* \simeq \lim N_+^*/(N_+^*)_n$, we have $f = \{f_n\}$ where $f_n$ is the class of $f$ modulo $(N_+^*)_n$. Therefore $b^i.f \in (N_+^*)_n$ iff

$$(b^i.f)_n = b^i.f_{n+1} = 0$$

Now, for any $x_1 \cdots x_n \in U_n p_+$, we have

$$b^i.f_{n+1}(x_1 \cdots x_n) = -f_{n+1}(b^i x_1 \cdots x_n) = 0$$

for a.a. $i \in I$ and the lemma is proved (it is enough to exclude the indices corresponding to the generators involved in the expression of $f_{n+1}$).

As a vector spaces, we can identify

$$p_+^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_{D,-}^* \simeq \mathfrak{g}_- \oplus \mathfrak{g}_{D,+} = p_-$$

We can give as a basis for $p_+$ and $p_-

$$p_+ \supset \{ (b^i)_i, (a_r)_{r \in I(D)} \} \quad p_- \supset \{ (a_i)_i, (b^r)_{r \in I(D)} \}$$

and obvious relations

$$(b^i, a_j) = \delta_{ij} \quad (b^i, b^r) = 0$$

$$(a_r, a_j) = 0 \quad (a_r, b^r) = \delta_{rs}$$

with $i, j \in I$, $r, s \in I(D)$. We can then identify $f_{n+1}$ with an element in $U_{n+1} p_-$. Call $T_{n+1}(f)$ the set of indices of all $a_i$ involved in the expression of $f_{n+1}$. Excluding these finite set of indices we get the result. \hfill \Box
5.12. **Coalgebra structure on** \( L_-, N_+ \). Define \( \mathfrak{g} \)-module maps

\[
i_- : L_- \to L_\ast \otimes L_- \quad \text{and} \quad i_+ : N_+ \to N_\ast \otimes N_+
\]

by mapping \( 1_\pm \) to \( 1_\pm \otimes 1_\pm \). Note that, under the identification \( L_- \simeq U\mathfrak{m}_- \) and \( N_+ \simeq U\mathfrak{p}_+ \), \( i_\pm \) correspond to the coproduct on \( U\mathfrak{m}_- \) and \( U\mathfrak{p}_+ \) respectively.

Following [D3, Prop. 1.2], we consider the invertible element \( T \in (U\mathfrak{g} \otimes U\mathfrak{g})[[\hbar]] \) satisfying relations:

\[
S^{\otimes 3}(\Phi^{321}) \cdot (T \otimes 1) \cdot (\Delta \otimes 1)(T) = (1 \otimes T)(1 \otimes \Delta)(T) \cdot \Phi
\]

\[
T\Delta(S(a)) = (S \otimes S)(\Delta(a))T
\]

Let \( N_\ast^+ \) be as before and \( f, g \in N_\ast^+ \). Consider the linear functional in \( \text{Hom}_k(N_+, k) \) defined by

\[
v \mapsto (f \otimes g)(T \cdot i_+(v))
\]

This functional is continuous, so it belongs to \( N_\ast^+ \) and allow us to define the map

\[
i_+^\vee \in \text{Hom}_k(N_\ast^+ \otimes N_\ast^+ \otimes N_\ast^+)[[\hbar]], \quad i_+^\vee(f \otimes g)(v) = (g \otimes f)(T \cdot i_+(v))
\]

This map is continuous and extends to a map from \( N_\ast^+ \otimes N_\ast^+ \) to \( N_\ast^+ \). For any \( a \in \mathfrak{g} \), we have

\[
i_+^\vee(a(f \otimes g))(v) = (f \otimes g)((S \otimes S)(\Delta(a))T \cdot i_+(v)) = (f \otimes g)(T\Delta(S(a)) \cdot i_+(v)) = i_+^\vee(f \otimes g)(S(a).v) = (a \cdot i_+^\vee(f \otimes g))(v)
\]

and then \( i_+^\vee \in \text{Hom}_\mathfrak{g}(N_\ast^+ \otimes N_\ast^+, N_\ast^+)[[\hbar]] \).

The following shows that \( L_- \) and \( N_+ \) are coalgebra objects in the Drinfeld categories of \( \mathfrak{g} \)-modules and \( (\mathfrak{g}, \mathfrak{g}_D) \)-bimodules respectively.

**Proposition.** The following relations hold

(i) \( \Phi(i_- \otimes 1)i_- = (1 \otimes i_-)i_- \).

(ii) \( i_+^\vee(1 \otimes i_+^\vee)\Phi = i_+^\vee(i_+^\vee \otimes 1)S^{\otimes 3}(\Phi^{-1})^\rho \)

where \((-)^\rho\) denotes the right \( \mathfrak{g}_D \)-action on \( N_\ast^+ \).

**Proof.** We begin by showing that

\[
\Phi(1^{\otimes 3}) = 1_{\otimes 3} \quad \text{and} \quad \Phi(1_+^{\otimes 3}) = \Phi_D(1_+^{\otimes 3})
\]

To prove the first identity, it is enough to notice that, since \( \mathfrak{g}_-1_\ast = 0 \) and

\[
\Omega = \sum (a_i \otimes b^i + b^i \otimes a_i),
\]

\[
\Omega_{ij}(1^{\otimes 3}) = 0
\]

Then \( \Phi(1^{\otimes 3}) = 1_{\otimes 3} \). To prove the second one, we notice that \( \mathfrak{m}_-1_+ = 0 \) and that we can rewrite

\[
\Omega = \sum_{j \in I_D} (a_j \otimes b^j + b^j \otimes a_j) + \sum_{i \in I \setminus I_D} (a_i \otimes b^i + b^i \otimes a_i) = \Omega_D + \sum_{i \in I \setminus I_D} (a_i \otimes b^i + b^i \otimes a_i)
\]
where \( \{a_j\}_{j \in I_D} \) is a basis of \( g_{D,-} \) and \( \{b^j\}_{j \in I_D} \) is the dual basis of \( g_{D,+} \).

Then

\[
\Omega_{ij}(1_+^{\otimes 3}) = \Omega_{D,ij}(1_+^{\otimes 3})
\]

and, since for any element \( x \in g_D \), the right and the left \( g_D \)-action coincide on \( 1_+ \), i.e. \( x.1_+ = 1_+.x \), we have

\[
\Omega_{ij}(1_+^{\otimes 3}) = (1_+^{\otimes 3})\Omega_{D,ij}
\]

and consequently \( \Phi(1_+^{\otimes 3}) = \Phi_D(1_+^{\otimes 3}) \).

To prove (i), note that since the comultiplication in \( Um_- \) is coassociative, we have \((i_- \otimes 1)i_- = (1 \otimes i_-)i_- \). We therefore have to show that \( \Phi(i_- \otimes 1)i_- = (1 \otimes i_-)i_- \). This is an obvious consequence of (5.1) and the fact that \( m_- \) is generated by \( 1_- \).

To prove (ii), consider \( v \in N_+ \),

\[
i_+^\vee(1 \otimes i_+^\vee)(\Phi(f \otimes g \otimes h))(v) =
\]

\[
= (h \otimes g \otimes f)((S^{\otimes 3}(\Phi^3)1) \cdot (T \otimes 1) \cdot (\Delta \otimes 1)(T)) \cdot (i_+ \otimes 1)i_+(v)) =
\]

\[
= (h \otimes g \otimes f)((1 \otimes T)(1 \otimes \Delta)(T) \cdot \Phi(i_+ \otimes 1)i_+(v)) =
\]

\[
= (h \otimes g \otimes f)((1 \otimes T)(1 \otimes \Delta)(T)(1 \otimes i_+)(v)\Phi_D) =
\]

\[
= (S^{\otimes 3}(\Phi_D)^{\rho}(h \otimes g \otimes f))((1 \otimes T)(1 \otimes \Delta)(T)(1 \otimes i_+)(v) =
\]

\[
i_+^\vee(i_+^\vee \otimes 1)(S^{\otimes 3}(\Phi_D^{31})^{\rho}(f \otimes g \otimes h))(v) =
\]

\[
= i_+^\vee(i_+^\vee \otimes 1)S^{\otimes 3}(\Phi_D^{1})^{\rho}(f \otimes g \otimes h)(v)
\]

and (ii) is proved. \( \square \)

5.13. **The fiber functor over** \( g_D \). To any representation \( V[[h]] \in \text{Rep}_{Ug}[[h]] \), we can associate the \( k[[h]] \)-module

\[
\Gamma(V) = \text{Hom}_g(L_-, N_+^* \hat{\otimes} V)[[h]]
\]

where \( \text{Hom}_g \) is the set of continuous homomorphisms, equipped with the weak topology. The right \( g_D \)-action on \( N_+^* \) endows \( \Gamma(V) \) with the structure of a left \( g_D \)-module.

**Proposition.** The complete vector space \( \text{Hom}_g(L_-, N_+^* \hat{\otimes} V) \) is isomorphic to \( V \) as equicontinuous \( g_D \)-module. The isomorphism is given by

\[
\alpha_V : f \mapsto (1_+ \otimes 1)f(1_-)
\]

for any \( f \in \text{Hom}_g(L_-, N_+^* \hat{\otimes} V) \).

**Proof.** By Frobenius reciprocity, we get an isomorphism

\[
\text{Hom}_g(L_-, N_+^* \hat{\otimes} V) \simeq \text{Hom}_{p^+}(k, N_+^* \hat{\otimes} V) \simeq \text{Hom}_k(k, V) \simeq V
\]

given by the map

\[
f \mapsto (1_+ \otimes 1)f(1_-)
\]

For \( f \in \Gamma(V) \) and \( x \in U_{g_D} \), \( x.f \in \Gamma(V) \) is defined by

\[
x.f = (S(x)^{\rho} \otimes \text{id}) \circ f
\]
For any \( x \in U\mathfrak{g}_D \), we have
\[
\sum_{i,j} x^{(1)}_i f_j \otimes x^{(2)}_i v_j = \varepsilon(x) f(1_-)
\]
where \( \Delta(x) = \sum_i x^{(1)}_i \otimes x^{(2)}_i \) and \( f(1_-) = \sum_j f_j \otimes v_j \). Using the identity
\[
1 \otimes x = \sum_i (S(x^{(1)}_i) \otimes 1) \cdot \Delta(x^{(2)}_i)
\]
holding in any Hopf algebra, we obtain
\[
(1 \otimes x) f(1_-) = \sum_i (S(x^{(1)}_i) \varepsilon(x^{(2)}_i)) \otimes 1) f(1_-) = (S(x) \otimes 1) f(1_-)
\]
Finally, we have
\[
x.\alpha_V(f) = \langle 1_+ \otimes \text{id}, (1 \otimes x) f(1_-) \rangle = \langle 1_+ \otimes \text{id}, (S(x) \otimes 1) f(1_-) \rangle = \langle 1_+ \otimes \text{id}, (S(x)^\rho \otimes 1) f(1_-) \rangle = \alpha_V(x.f)
\]
Therefore, \( \Gamma(V) \) is isomorphic to \( V[[h]] \) as equicontinuous \( g_D \)-module. \( \square \)

5.14. For any continuous \( \varphi \in \text{Hom}_\mathfrak{g}(V, V') \), define a map \( \Gamma(\varphi) : \Gamma(V) \to \Gamma(V') \) by
\[
\Gamma(\varphi) : f \mapsto (\text{id} \otimes \varphi) \circ f
\]
This map is clearly continuous and for all \( x \in \mathfrak{g}_D \)
\[
\Gamma(\varphi)(x.f) = (S(x)^\rho \otimes \varphi) \circ f = x.\Gamma(\varphi)(f)
\]
then \( \Gamma(\varphi) \in \text{Hom}_{\mathfrak{g}_D}(\Gamma(V), \Gamma(V')) \).
Since the diagram
\[
\begin{array}{ccc}
\Gamma(V) & \xrightarrow{\Gamma(\varphi)} & \Gamma(V') \\
\alpha_V \downarrow & & \downarrow \alpha_{V'} \\
V[[h]] & \xrightarrow{\varphi} & V'[[h]]
\end{array}
\]
is commutative for all \( \varphi \in \text{Hom}_\mathfrak{g}(V, V') \), we have a well-defined functor
\[
\Gamma : \text{Rep}^{eq} U\mathfrak{g}[[h]] \to \text{Rep}^{eq} U\mathfrak{g}_D[[h]]
\]
which is naturally isomorphic to the pullback functor induced by the inclusion \( i_D : \mathfrak{g}_D \hookrightarrow \mathfrak{g} \) via the natural transformation
\[
\alpha_V : \Gamma(V) \simeq i_D^*V[[h]]
\]
5.15. **Tensor structure on $\Gamma$.** Denote the tensor product in the categories $\mathcal{D}_q(Ug)$, $\mathcal{D}_q(U\mathfrak{g}_D)$ by $\otimes$, and let $B_{1234}$ and $B'_{1234}$ be the associativity constraints

$$B_{1234} : (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \rightarrow V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)$$

and

$$B'_{1234} : (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \rightarrow (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4$$

For any $v \in \Gamma(V), w \in \Gamma(W)$, define $J_{VW}(v \otimes w)$ to be the composition

$$L_- \xrightarrow{i} L_- \otimes L_- \xrightarrow{\nu \otimes w} (N^*_+ \otimes V) \otimes (N^*_+ \otimes W) \xrightarrow{A} N^*_+ \otimes ((V \otimes N^*_+) \otimes W)$$

$$\xrightarrow{\beta^{-1}} \otimes (N^*_+ \otimes V \otimes W) \xrightarrow{A'} (N^*_+ \otimes N^*_+) \otimes (V \otimes W) \xrightarrow{i'_+ \otimes 1} N^*_+ \otimes (V \otimes W)$$

where the pair $(A, A')$ can be chosen to be $(B_{N^*_+,V,N^*_+,W}, B_{N^*_+,V,N^*_+,W}^{-1})$ or $(B'_{N^*_+,V,N^*_+,W}, B'_{N^*_+,V,N^*_+,W}^{-1})$. The map $J_{VW}(v \otimes w)$ is clearly a continuous $\mathfrak{g}$-morphism from $L$ to $N^*_+ \otimes (V \otimes W)$, so we have a well-defined map

$$J_{VW} : \Gamma(V) \otimes \Gamma(W) \rightarrow \Gamma(V \otimes W)$$

**Proposition.** The maps $J_{VW}$ are isomorphisms of $\mathfrak{g}_D$–modules, and define a tensor structure on the functor $\Gamma$.

The proof of Proposition 5.15 is given in 5.16–5.19.

5.16. The map $J_{VW}$ is compatible with the $\mathfrak{g}_D$–action. Indeed, $i'_+ \otimes 1$ is a morphism of right $\mathfrak{g}_D$–modules and, for any $x \in \mathfrak{g}_D,

$$x.J_{VW}(v \otimes w) = (S^\rho(x) \otimes \text{id})((i'_+ \otimes \text{id} \otimes \text{id})\tilde{A}(v \otimes w)i_- =$$

$$= (i'_+ \otimes \text{id} \otimes \text{id})(\Delta(S(x))^{\rho})_{12} \tilde{A}(v \otimes w)i_- =$$

$$= (i'_+ \otimes \text{id} \otimes \text{id})\tilde{A}((S \otimes S)(\Delta(x)))^{\rho}_{13}(v \otimes w)i_- = J_{VW}(x.(v \otimes w))$$

where $\tilde{A} = A'\beta^{-1}_2 A$.

$J_{VW}$ is an isomorphism, since it is an isomorphism modulo $h$. Indeed,

$$J_{VW}(v \otimes w) \equiv (i'_+ \otimes 1)(1 \otimes s \otimes 1)(v \otimes w)i_- \mod h$$

To prove that $J_{VW}$ define a tensor structure on $\Gamma$, we need to show that, for any $V_1, V_2, V_3 \in \mathcal{D}_q(Ug)$ the following diagram is commutative

$$\begin{array}{c}
\xymatrix{
\Gamma(V_1) \otimes (\Gamma(V_2) \otimes \Gamma(V_3)) \ar[r]^{J_{123}} & \Gamma((V_1 \otimes V_2) \otimes V_3) \\
\Gamma(V_1) \otimes (\Gamma(V_2) \otimes \Gamma(V_3)) \ar[r]^{1 \otimes J_{23}} & \Gamma(V_1) \otimes (V_2 \otimes V_3) & \Gamma(V_1 \otimes (V_2 \otimes V_3)) \\
\xymatrix{
\Phi_D & \ar[l]_{J_{12} \otimes 1} (\Gamma(V_1) \otimes (\Gamma(V_2) \otimes \Gamma(V_3))) & \ar[l]_{J_{123}} \Gamma((V_1 \otimes V_2) \otimes V_3) \\
\Gamma(V_1) \otimes (\Gamma(V_2) \otimes \Gamma(V_3)) & \Gamma(V_1) \otimes (V_2 \otimes V_3) & \Gamma(V_1 \otimes (V_2 \otimes V_3)) \\
\end{array}
$$

where $J_{ij}$ denotes the map $J_{V_i,V_j}$ and $J_{ij,k}$ the map $J_{V_i \otimes V_j \otimes V_k}$.
5.17. For any \( v_i \in \Gamma(V_i), i = 1, 2, 3 \), the map \( \Gamma(\Phi)J_{12,3}J_{12} \otimes 1(v_1 \otimes v_2 \otimes v_3) \) is given by the composition

\[
(1 \otimes \Phi)(i^*_+ \otimes 1)^{\otimes 3})A_4(1 \otimes \beta_{12, N^*_+} \otimes 1)A_3((i^*_+ \otimes 1) \otimes 1)^{\otimes 3})(A_2 \otimes 1 \otimes 1)
\]

\[
\cdot (1 \otimes \beta_{N^*_+,1} \otimes 1)^{\otimes 3})(A_1 \otimes 1 \otimes 1)(v_1 \otimes v_2 \otimes v_3)(i_- \otimes 1)i_-
\]

where

\[
A_1 = B_{N^*_+,1 \otimes 2, N^*_+,2} \quad A_3 = B_{N^*_+,1 \otimes 2, N^*_+,3} \quad A_4 = B_{N^*_+,1 \otimes 2, 2, 3}
\]

illustrated by the diagram

\[
\begin{array}{cccc}
L_- & \overset{i_-}{\longrightarrow} & L_- \otimes L_- & \overset{i_- \otimes 1}{\longrightarrow} (L_- \otimes L_-) \otimes L_-
\end{array}
\]

\[
\begin{array}{c}
\overset{1 \otimes \beta_{N^*_+,1} \otimes 1}{\longrightarrow}
\overset{1 \otimes \beta_{N^*_+,1} \otimes 1}{\longrightarrow}
\overset{1 \otimes \beta_{N^*_+,1} \otimes 1}{\longrightarrow}
\overset{1 \otimes \beta_{N^*_+,1} \otimes 1}{\longrightarrow}
\end{array}
\]

By functoriality of associativity and commutativity isomorphisms, we have

\[
A_3(i^*_+ \otimes 1)^{\otimes 4}) = (i^*_+ \otimes 1)^{\otimes 4})A_5
\]

where \( A_5 = B_{N^*_+,1 \otimes 2, 12, N^*_+,3} \)

\[
(1 \otimes \beta_{12, N^*_+} \otimes 1)(i^*_+ \otimes 1)^{\otimes 4}) = (i^*_+ \otimes 1)^{\otimes 4})(1^{\otimes 2} \otimes \beta_{12, N^*_+} \otimes 1^{\otimes 2})
\]

and

\[
A_4(i^*_+ \otimes 1)^{\otimes 4}) = (i^*_+ \otimes 1)^{\otimes 4})A_6
\]

where \( A_6 = B_{N^*_+,1 \otimes 2, 12, N^*_+,3} \)

Finally, we have

\[
\Gamma(\Phi)J_{12,3}(J_{12} \otimes 1)(v_1 \otimes v_2 \otimes v_3)
\]

\[
= (1^{\otimes 3} \otimes \Phi_{123})((i^*_+ \otimes 1)^{\otimes 3})A(v_1 \otimes v_2 \otimes v_3)(i_- \otimes 1)i_- \quad (5.2)
\]

where

\[
A = A_6(1^{\otimes 2} \otimes \beta_{1 \otimes 2, N^*_+,1} \otimes 1^{\otimes 2})A(1 \otimes \beta_{N^*_+,1} \otimes 1^{\otimes 3})(A_1 \otimes 1 \otimes 1)
\]
5.18. On the other hand, \( J_{1,23}(1 \otimes J_{23}) \Phi_D(v_1 \otimes v_2 \otimes v_3) \) corresponds to the composition

\[
(i_+^* \otimes 1^{\otimes 3}) A'_1 (1 \otimes \beta_{N^*_+ \otimes 1} \otimes 1^{\otimes 2}) A'_2 (1^{\otimes 3} \otimes i_+^* \otimes 1^{\otimes 2}) (1 \otimes 1 \otimes A'_3) \]

where

\[
A'_1 = B_{N^*_+ \otimes 2, 2, 3} \quad A'_3 = B_{N^*_+ \otimes 1, N^*_+ \otimes 2, 2^{\otimes 3}}
\]

\[
A'_2 = B_{N^*_+ \otimes 1, 2, 3} \quad A'_4 = B_{N^*_+ \otimes 1, 1, 2^{\otimes 3}}
\]

illustrated by the diagram

\[
\begin{align*}
\Phi_D(v_1 \otimes v_2 \otimes v_3) &\quad (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes V_2) \otimes (N^*_+ \otimes V_3)) \quad 1^{\otimes 1} A'_1 \quad (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes (V_2 \otimes N^*_+) \otimes V_3)) \\
1^{\otimes 2} \otimes B_{1, N^*_+ \otimes 1} &\quad (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes V_2) \otimes V_3) \quad 1^{\otimes 1} A'_2 \quad (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes N^*_+) \otimes (V_2 \otimes V_3)) \\
1^{\otimes 2} \otimes i_+^* &\quad (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes N^*_+) \otimes (V_2 \otimes V_3)) \quad A'_3 \quad (N^*_+ \otimes V_1) \otimes (V_2 \otimes V_3) \\
1^{\otimes 1} \otimes i_+^* &\quad N^*_+ \otimes ((N^*_+ \otimes V_1) \otimes (V_2 \otimes V_3)) \quad A'_4 \quad (N^*_+ \otimes N^*_+) \otimes (V_2 \otimes V_3) \\
&\quad N^*_+ \otimes (V_2 \otimes V_3)
\end{align*}
\]

By functoriality of associativity and commutativity isomorphisms, we have

\[
A'_3 (1^{\otimes 2} \otimes i_+^* \otimes 1^{\otimes 2}) = (1^{\otimes 2} \otimes i_+^* \otimes 1^{\otimes 2}) A'_5
\]

where \( A'_5 = B_{N^*_+ \otimes 1, N^*_+ \otimes 2^{\otimes 3}} \).

\[
(1 \otimes \beta_{1, N^*_+ \otimes 1} \otimes 1^{\otimes 2}) (1^{\otimes 2} \otimes i_+^* \otimes 1^{\otimes 2}) = (1 \otimes i_+^* \otimes 1^{\otimes 3}) (1 \otimes \beta_{1, N^*_+ \otimes N^*_+} \otimes 1^{\otimes 2})
\]

and

\[
A'_4 (1 \otimes i_+^* \otimes 1^{\otimes 3}) = (1 \otimes i_+^* \otimes 1^{\otimes 3}) A'_6
\]

where \( A'_6 = B_{N^*_+ \otimes 1, 1, 2^{\otimes 3}} \). Thus,

\[
J_{1,23}(1 \otimes J_{23}) \Phi_D(v_1 \otimes v_2 \otimes v_3) = (i_+^* \otimes 1^{\otimes 3}) (1 \otimes i_+^* \otimes 1^{\otimes 3}) B \Phi_D(v_1 \otimes v_2 \otimes v_3) (1 \otimes i_-) i_-
\]

where

\[
B = A'_6 (1 \otimes \beta_{1, N^*_+ \otimes N^*_+} \otimes 1^{\otimes 2}) A'_0 (1^{\otimes 2} \otimes A'_2) (1^{\otimes 3} \otimes \beta_{2, N^*_+} \otimes 1) (1 \otimes 1 \otimes A'_4)
\]
5.19. Comparing (5.2) and (5.3), we see that it suffices to show that the outer arrows of the following form a commutative diagram.

\[
\begin{array}{ccc}
(L_- \otimes L_-) \circ L_- & \overset{(i_- \otimes \varphi)_{i_-}}{\longrightarrow} & L_- \otimes (L_- \otimes L_-) \\
((N^*_+ \otimes V_1) \otimes (N^*_+ \otimes V_2)) \otimes (N^*_+ \otimes V_3) & \overset{\Phi}{\longrightarrow} & (N^*_+ \otimes V_1) \otimes ((N^*_+ \otimes V_2) \otimes (N^*_+ \otimes V_3)) \\
((N^*_+ \otimes N^*_+) \otimes N^*_+) \otimes ((V_1 \otimes V_2) \otimes V_3) & \overset{\Phi \otimes \Phi}{\longrightarrow} & (N^*_+ \otimes (N^*_+ \otimes N^*_+)) \otimes (V_1 \otimes (V_2 \otimes V_3)) \\
(N^*_+ \otimes ((V_1 \otimes V_2) \otimes V_3) & \overset{1 \otimes \Phi}{\longrightarrow} & N^*_+ \otimes (V_1 \otimes (V_2 \otimes V_3)) \\
\end{array}
\]

Using the pentagon and the hexagon axiom, we can show that

\[(\Phi \otimes \Phi)A = B\Phi\]

We have to show that

\[\Gamma(\Phi)J_{12,3}(J_{12} \otimes 1)(v_1 \otimes v_2 \otimes v_3) = J_{1,23}(1 \otimes J_{23})\Phi_D(v_1 \otimes v_2 \otimes v_3)\]

in \(\text{Hom}_q(L_-, N^*_+ \otimes (V_1 \otimes (V_2 \otimes V_3)))\):

\[
J_{1,23}(\text{id} \otimes J_{23})\Phi_D(v_1 \otimes v_2 \otimes v_3) =
\]

\[
= (i_+^\vee (\text{id} \otimes i_+^\vee) \otimes \text{id}^\otimes 3) \Phi_D(v_1 \otimes v_2 \otimes v_3)(\text{id} \otimes i_-)_-i_-
\]

\[
= (i_+^\vee (\text{id} \otimes i_+^\vee) \otimes \text{id}^\otimes 3) \Phi_D(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \text{id})i_-
\]

\[
= (i_+^\vee (\text{id} \otimes i_+^\vee)(\Phi \otimes \Phi)A\Phi_D(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \text{id})i_-
\]

\[
= (i_+^\vee (\text{id} \otimes i_+^\vee)(\Phi \otimes \Phi)(S^\otimes 3(\Phi_D)^{\rho} \otimes \text{id}^\otimes 3)A(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \text{id})i_-
\]

\[
= (i_+^\vee (\text{id} \otimes i_+^\vee)(\Phi \otimes \Phi)(S^\otimes 3(\Phi_D)^{\rho} \otimes \Phi)A(v_1 \otimes v_2 \otimes v_3)(i_- \otimes \text{id})i_-
\]

\[
= \Gamma(\Phi)J_{12,3}(J_{12} \otimes \text{id})(v_1 \otimes v_2 \otimes v_3)
\]

where the second and seventh equalities follow from Proposition 5.12, the fifth one from the definition of the \(\Phi_D\)-action on the modules \(\Gamma(V^\vee_j)\) and the others from functoriality of the associator \(\Phi\). This complete the proof of Theorem 5.1.
5.20. **1–Jets of relative twists.** The following is a straightforward extension of the computation of the 1–jet of the Etingof–Kazhdan twist given in [EK1].

**Proposition.** Under the natural identification

$$\alpha_V : \Gamma(V) \to V[[\hbar]]$$

the relative twist \( J_\Gamma \) satisfies

$$\alpha_V \otimes W \circ J_\Gamma \circ (\alpha_V^{-1} \otimes \alpha_W^{-1}) \equiv 1 + \hbar \frac{1}{2} (r + r_D^{21}) \mod \hbar^2$$

in \( \text{End}(V \otimes W)[[\hbar]] \).

**Proof.** For \( v \in V, w \in W \), let

$$\alpha_V^{-1}(v)(1_{-}) = \sum f_i \otimes v_i \quad \alpha_W^{-1}(w)(1_{-}) = \sum g_j \otimes w_j$$

in \((N^*_+ \otimes V)^{p^+}\) and \((N^*_+ \otimes W)^{p^+}\) respectively. Then using

$$\langle (1_+ \otimes 1)^{\otimes 2}, \Omega_{23} \sum_{i,j} f_i \otimes v_i \otimes g_j \otimes w_j \rangle = -\Omega(v \otimes w)$$

and

$$\langle (1_+ \otimes 1)^{\otimes 2}, \Omega_{D,23} \sum_{i,j} f_i \otimes v_i \otimes g_j \otimes w_j \rangle = -\Omega_D(v \otimes w)$$

where \( \Omega = \Omega + \Omega_D \), we get

$$\alpha_V \otimes W \circ J_\Gamma \circ (\alpha_V^{-1} \otimes \alpha_W^{-1})(v \otimes w) \equiv v \otimes w + \hbar \frac{1}{2} (r + r_D^{21})(v \otimes w) \mod \hbar^2$$

because the definition of \( J_\Gamma \) involves the braiding \( \beta_{XY}' = \beta_{YX}^{-1} \). \( \square \)

**Corollary.** The relative twist \( J_\Gamma \) satisfies

$$\text{Alt}_2 J_\Gamma \equiv \frac{\hbar}{2} \left( \frac{r - r_D^{21}}{2} - \frac{r_D - r_D^{21}}{2} \right) \mod \hbar^2$$

6. **Quantization of Verma modules**

This section and the next contain results about the quantization of classical Verma modules, which are required to construct the morphism of \( D \)-categories between the representation theory of \( U^\hbar \mathfrak{g}[[\hbar]] \) and that of \( U_\hbar \mathfrak{g} \). In particular, from now on, we will assume the existence of a finite \( \mathbb{N} \)–grading on \( \mathfrak{g} \), which induces on \( \mathfrak{g} \) a Lie bialgebra structure and allows us to consider the quantization of \( \mathfrak{g} \) through the Etingof–Kazhdan functor, \( U^\hbar_{\text{EK}} \mathfrak{g} \).
6.1. Quantum Verma Modules. Because of the functoriality of the quantization defined by Etingof and Kazhdan in [EK2], in the category of Drinfeld-Yetter modules over $U^\mathcal{E}_h g_-$ we can similarly define quantum Verma modules.

The standard inclusions of Lie bialgebras $g_\pm \subset g \simeq Dg_-$ lift to $U^\mathcal{E}_h g_\pm \subset U^\mathcal{E}_h h g_-$, and we can define the induced modules of the trivial representation over $U^\mathcal{E}_h h g_-$

$$M^h_\pm = \text{Ind}_{U^\mathcal{E}_h h g_\pm}^{U^\mathcal{E}_h h g_-} k[[h]]$$

Similarly, we have Hopf algebra maps $U^\mathcal{E}_h p_\pm \subset U^\mathcal{E}_h h g$ and $U^\mathcal{E}_h p_\pm \to U^\mathcal{E}_h h g_D$, and we can define induced modules

$$L^h_- = \text{Ind}_{U^\mathcal{E}_h p_-}^{U^\mathcal{E}_h h g_-} k[[h]] \quad N^h_+ = \text{Ind}_{U^\mathcal{E}_h p_+}^{U^\mathcal{E}_h h g_-} U^\mathcal{E}_h h g_D$$

We want to show that the equivalence $\tilde{F} : \mathcal{YD}_U(Ug_-)[[h]] \to \mathcal{YD}_{U^\mathcal{E}_h h g_-}$ matches these modules. We start proving the statement for $M_-, M^*_+.$

6.2. Quantization of $M_\pm$. We denote by $(M^*_+)\ast$ the $U^\mathcal{E}_h h g_-$-module

$$\text{Hom}_k(\text{Ind}_{U^\mathcal{E}_h h g_-}^{U^\mathcal{E}_h h g_-} k[[h]], k[[h]])$$

**Theorem.** In the category of left $U^\mathcal{E}_h h g_-$-modules,

(a) $F(M_-) \simeq M^h_-$

(b) $F(M^*_+) \simeq (M^h_+)^\ast$

**Proof.** The Hopf algebra $U^\mathcal{E}_h h g_-$ is constructed on the space $F(M_-)$ with unit element $u \in F(M_-)$ defined by $u(1_-) = \epsilon_+ \otimes 1_-$, where $\epsilon_+ \in M^*_+$ is defined as $\epsilon_+(x1_+) = \epsilon(x)$ for any $x \in U_\pm g$. Consequently, the action of $U^\mathcal{E}_h h g_-$ on $u \in F(M_-)$ is free, as multiplication with the unit element. The coaction of $U^\mathcal{E}_h h g_-$ on $F(M_-)$ is defined using the $\mathcal{R}$-matrix associated to the braided tensor functor $F$, i.e.,

$$\pi^*_M : F(M_-) \to F(M_-) \otimes F(M_-), \quad \pi^*_M(x) = \mathcal{R}(u \otimes x)$$

where $x \in F(M_-)$ and $\mathcal{R}_{VW} \in \text{End}_{U^\mathcal{E}_h h g_-} (F(V) \otimes F(W))$ is given by $\mathcal{R}_{VW} = \sigma \mathcal{J}_{WV} \mathcal{J}_{VW}$, $\mathcal{J}_{VW}$ being the tensor structure on $F$.

It is easy to show that $\mathcal{J}(u \otimes u)|_{1_-} = \epsilon_+ \otimes 1_- \otimes 1_-$, and, since $\Omega(1_- \otimes 1_-) = 0$, we have

$$\mathcal{R}(u \otimes u) = u \otimes u$$

For a generic $V \in \mathcal{YD}_{Ug_-}[[h]]$, the action of $U^\mathcal{E}_h h g^\ast$ is defined as

$$F(M_-) \ast \otimes F(V) \to F(M_-) \ast \otimes F(M_-) \otimes F(V) \to F(V)$$
This means, in particular that, for every \( \phi \in I \subset U_h^{EK} g^\perp \), where \( I \) is the maximal ideal corresponding to \( u^\perp \), we have \( \phi.u = 0 \). This proves (a).

The module \( M^*_+ \) satisfies the following universal property: for any \( V \) in the Drinfeld category of equicontinuous \( U g \)-modules, we have

\[
\text{Hom}_{U g}(V, M^*_+) \simeq \text{Hom}_{U g_-(V, k)}
\]

Indeed, to any map of \( U g \)-modules \( f : V \to M^*_+ \), we can associate \( \hat{f} : V \to k \), \( \hat{f}(v) = \langle f(v), 1_+ \rangle \). It is clear that \( \hat{f} \) factors through \( V / g_- \cdot V \). The equicontinuity property is necessary to show the continuity of \( \hat{f} \) with respect to the topology on \( V \).

Since \( F \) defines an equivalence of categories, we have

\[
\text{Hom}_{U_h^{EK} g_-}(F(V), F(M^*_+)) \simeq \text{Hom}_{U g}(V, M^*_+)[[h]] \simeq \text{Hom}_{U g_-(V, k)[[h]]}
\]

Using the natural isomorphism \( \alpha_V : F(V) \to V[[h]] \), defined by

\[
\alpha_V(f) = \langle f(1_\perp), 1_+ \otimes \text{id} \rangle
\]

we obtain a map \( \text{Hom}_{U g_-(V, k)[[h]]} \to \text{Hom}_k(F(V), k[[h]]) \). Consider now the linear isomorphism \( \alpha : U_h^{EK} g_- \to U g_-[h] \) and for any \( x \in U g_- \) consider the \( g \)-intertwiner \( \psi_x : M_- \to M^*_+ \otimes M_- \) defined by \( \psi_x(1_-) = \epsilon_+ \otimes x \cdot 1_- \). It is clear that, if \( f(1_\perp) = f(1_\perp) \otimes f(2) \) in Swedler’s notation,

\[
\alpha_V(\psi_x.f) = \langle (i_+^\vee \otimes \text{id}) \Phi^{-1}(\text{id} \otimes f)(\epsilon_+ \otimes x \cdot 1_-), 1_+ \otimes \text{id} \rangle
\]

\[
= \langle \Phi^{-1}(\epsilon_+ \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta(x))(\text{id} \otimes f(1) \otimes f(2)), (\text{id} \otimes f(1) \otimes f(2))(1_+ \otimes f(2)) \rangle
\]

\[
= \langle f(1), 1_+ \rangle x \cdot f(2)
\]

\[
= x. \alpha_V(f)
\]

using the fact that \( (\epsilon \otimes 1 \otimes 1)(\Phi) = 1^ {\otimes 2} \) and \( (\epsilon \otimes 1)(T) = 1 \). So, clearly, if \( \phi \in \text{Hom}_{U g_-(V, k)} \), then \( \phi \circ \alpha_V \in \text{Hom}_{U_h^{EK} g_-}(F(V), k[[h]]) \). Then \( F(M^*_+) \)

satisfies the universal property of \( \text{Hom}_k(\text{Ind}_{U_h^{EK} g_-}^{U_h^{EK} g_-} k[[h]], k[[h]]) \) and (b) is proved. \( \square \)

6.3. Quantization of relative Verma modules. The proof of (b) shows that the linear functional \( F(M^*_+) \to k[[h]] \) is, in fact, the trivial deformation of the functional \( M^*_+ \to k \). These results extend to the relative case and hold for the right \( \mathfrak{g} \mathfrak{D} \)-action on \( L_-, N^*_+ \).

**Theorem.** In the category \( \mathcal{YD}_{U_h^{EK} g_-} \)

(a) \( F(L_-) \simeq L_-^h \)

(b) \( F(N^*_+) \simeq (N^*_+)^h \)

Moreover, as right \( U_h^{EK} \mathfrak{g} \mathfrak{D} \)-module
(c) $F_D(L_-) \simeq L_\hbar$

(d) $F_D(N^*_+) \simeq (N^\hbar_+)^*$

The proof of (a) and (b) amounts to constructing the morphisms

$$k[[\hbar]] \to F(L_-) \quad F(N^*_+) \to U_h^{\text{EK}} D$$

equivariant under the action of $U_h^{\text{EK}} p_+$ and $U_h^{\text{EK}} p_-$ respectively.

A direct construction along the lines of the proof of Theorem 6.2 is however not straightforward. We prove this theorem in the next section by using a description of the modules $L_-, N^*_+$ and their images through $\bar{F}$ via PROP categories. These descriptions show that the classical intertwiners

$$k \to L_- \quad N^*_+ \to U_D \hbar$$

satisfy the required properties and yield canonical identifications

$$\bar{F}(L_-) \simeq L_\hbar \quad \bar{F}(N^*_+) \simeq (N^\hbar_+)^*$$

7. Universal relative Verma modules

In this section, we prove Theorem 6.3 by using suitable PROP (product-permutation) categories compatible with the EK universal quantization functor [EK2, EG].

7.1. PROP description of the EK quantization functor. We will briefly review the construction of Etingof–Kazhdan in the setting of PROP categories [EK2].

A PROP is a symmetric tensor category generated by one object. More precisely, a cyclic category over $S$ is the datum of

- a symmetric monoidal $k$–linear category $(\mathcal{C}, \otimes)$ whose objects are non–negative integers, such that $[n] = [1]^{\otimes n}$ and the unit object is $[0]$
- a bigraded set $S = \bigcup_{m,n \in \mathbb{Z}_{\geq 0}} S_{nm}$ of morphism of $\mathcal{C}$, with $S_{nm} \subset \text{Hom}_\mathcal{C}([m], [n])$

such that any morphism of $\mathcal{C}$ can be obtained from the morphisms in $S$ and permutation maps in $\text{Hom}_\mathcal{C}([m], [m])$ by compositions, tensor products or linear combinations over $k$. We denote by $\mathcal{F}_S$ the free cyclic category over $S$. Then there exists a unique symmetric tensor functor $\mathcal{F}_S \to \mathcal{C}$, and the following holds (cf. [EK2])

**Proposition.** Let $\mathcal{C}$ be any cyclic category generated by a set $S$ of morphisms. Then $\mathcal{C}$ has the form $\mathcal{F}_S/I$, where $I$ is a tensor ideal in $\mathcal{F}_S$.

Let $\mathcal{N}$ be a symmetric monoidal $k$–linear category, and $X$ an object in $\mathcal{N}$. A linear algebraic structure of type $\mathcal{C}$ on $X$ is a symmetric tensor functor $G_X : \mathcal{C} \to \mathcal{N}$ such that $G_X([1]) = X$. A linear algebraic structure of type
\( \mathcal{C} \) on \( X \) is a collection of morphisms between tensor powers of \( X \) satisfying certain consistency relations.

We mainly consider the case of non-degenerate cyclic categories, i.e., symmetric tensor categories with injective maps \( k[\mathfrak{S}_n] \to \text{Hom}_\mathcal{C}([n],[n]) \). We first consider the Karoubian envelope of \( \mathcal{C} \) obtained by formal addition to \( \mathcal{C} \) of the kernel of the idempotents in \( k[\mathfrak{S}_n] \) acting on \([n] \). Furthermore, we consider the closure under inductive limits. In this category, denoted \( S(\mathcal{C}) \), every object is isomorphic to a direct sum of indecomposables, corresponding to irreducible representations of \( \mathfrak{S}_n \) (cf. [EK2, EG]). In particular, in \( S(\mathcal{C}) \), we can consider the symmetric algebra

\[
S[1] = \bigoplus_{n \geq 0} S^n[1]
\]

If \( \mathcal{N} \) is closed under inductive limits, then any linear algebraic structure of type \( \mathcal{C} \) extends to an additive symmetric tensor functor \( G_X : S(\mathcal{C}) \to \mathcal{N} \).

We introduce the following fundamental props.

- **Lie bialgebras.** In this case the set \( S \) consists of two elements of bidegrees \((2,1)\), \((1,2)\), the universal commutator and cocommutator. The category \( \mathcal{C} = \mathcal{LBA} \) is \( F_S/\mathcal{I} \), where \( \mathcal{I} \) is generated by the classical five relations.

- **Hopf algebras.** In this case, the set \( S \) consists of six elements of bidegrees \((2,1)\), \((1,2)\), \((0,1)\), \((1,0)\), \((1,1)\), \((1,1)\), the universal product, coproduct, unit, count, antipode, inverse antipode. The category \( \mathcal{C} = \mathcal{HA} \) is \( F_S/\mathcal{I} \), where \( \mathcal{I} \) is generated by the classical four relations.

The quantization functor described in Section 4 can be described in this generality, as stated by the following (cf. [EK2, Thm.1.2])

**Theorem.** There exists a universal quantization functor \( Q : \mathcal{HA} \to S(\mathcal{LBA}) \).

Let \( \mathfrak{g}_- \) be the canonical Lie dialgebra \([1]\) in \( \mathcal{LBA} \) with commutator \( \mu \) and cocommutator \( \delta \). Let \( U\mathfrak{g}_- := S\mathfrak{g}_- \in S(\mathcal{LBA}) \) be the universal enveloping algebra of \( \mathfrak{g}_- \). The construction of the Etingof–Kazhdan quantization functor amounts to the introduction of a Hopf algebra structure on \( U\mathfrak{g}_- \), which coincides with the standard one modulo \( \langle \delta \rangle \), and yields the Lie bialgebra structure on \( \mathfrak{g}_- \) when considered modulo \( \langle \delta^2 \rangle \). This Hopf algebra defines the object \( Q[1] \), where \([1]\) is the generating object in \( \mathcal{HA} \). The formulae used to defined the Hopf structure coincide with those defined in [EK1, Part II] and described in Section 4. In particular, they rely on the construction of the Verma modules

\[
M_- := S\mathfrak{g}_- M_+^* = \hat{S}\mathfrak{g}_-
\]

realized in the category of Drinfeld–Yetter modules over \( \mathfrak{g}_- \) as object of \( \mathcal{LBA} \).
7.2. **Props for split pairs of Lie bialgebras.** Let \((\mathfrak{g}_-, \mathfrak{g}_{D,-})\) be a split pair of Lie bialgebras, i.e., there are Lie bialgebra maps

\[
\mathfrak{g}_{D,-} \xrightarrow{i} \mathfrak{g}_- \xrightarrow{p} \mathfrak{g}_{D,-}
\]

such that \(p \circ i = \text{id}\). These maps induce an inclusion \(\mathfrak{g}_{D,-} \subset \mathfrak{g}_-\) and consequently an inclusion of Manin triple \((\mathfrak{g}_D, \mathfrak{g}_{D,-}, \mathfrak{g}_{D,+}) \subset (\mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g}_+),\) as described in Section 5.6.

**Definition.** We denote by \(\text{PLBA}\) the Karoubian envelope of the multicolored \(\text{Prop}\), whose class of objects is generated by the Lie bialgebra objects \([\mathfrak{g}_-], [\mathfrak{g}_{D,-}],\) related by the maps \(i : [\mathfrak{g}_{D,-}] \to [\mathfrak{g}_-],\ p : [\mathfrak{g}_-] \to [\mathfrak{g}_{D,-}],\) such that \(p \circ i = \text{id}_{[\mathfrak{g}_{D,-}]}\).

The Karoubian envelope implies that \([m_-] := \ker(p) \in \text{PLBA}\).

**Proposition.** The multicolored \(\text{Prop} \ \text{PLBA}\) is endowed with a pair of functors \(U, L\)

\[
U, L : \text{LBA} \to \text{PLBA} \quad U[1] := [\mathfrak{g}_-], \ L[1] := [\mathfrak{g}_{D,-}]
\]

and natural transformations \(i, p,\) induced by the maps \(i, p\) in \(\text{PLBA},\)

\[
\begin{array}{ccc}
\text{LBA} & \xrightarrow{i} & \text{PLBA} \\
\downarrow{p} & & \downarrow{p} \\
\text{PLBA} & \xrightarrow{i} & \text{C}
\end{array}
\]

such that \(p \circ i = \text{id}\). Moreover, it satisfies the following universal property: for any tensor category \(\mathcal{C},\) closed under kernels of projections, with the same property as \(\text{PLBA}\), there exists a unique tensor functor \(\text{PLBA} \to \mathcal{C}\) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{LBA} & \xrightarrow{i} & \text{PLBA} \\
\downarrow{p} & & \downarrow{p} \\
\text{PLBA} & \xrightarrow{i} & \mathcal{C}
\end{array}
\]

7.3. **Props for split pairs of Hopf algebras.** We can analogously define suitable \(\text{Prop}\) categories corresponding to split pairs of Hopf algebras. In particular, we consider the \(\text{Prop} \ \text{PHA}\) characterized by functors \(U_h, L_h\) and natural transformations \(p_h, i_h\) satisfying

\[
\begin{array}{ccc}
\text{HA} & \xrightarrow{i_h} & \text{PHA} \\
\downarrow{p_h} & & \downarrow{p_h} \\
\text{PHA} & \xrightarrow{i} & \mathcal{C}
\end{array}
\]
where $\text{HA}$ denotes the $\text{Prop}$ category of Hopf algebras. These also satisfy

\[
\begin{array}{ccc}
\text{HA} & \xrightarrow{Q_{\text{EK}}} & S(\text{LBA}) \\
\downarrow & & \downarrow \\
\text{PHA} & \xrightarrow{Q_{\text{PLBA}}} & S(\text{PLBA})
\end{array}
\]

where $Q_{\text{PLBA}}$ is the extension of the Etingof–Kazhdan quantization functor to $\text{PLBA}$, obtained by the universal property described above with $\mathcal{C} = S(\text{PLBA})$.

7.4. Props for parabolic Lie subalgebras. In order to describe the module $N^+_\ast$ it is necessary to deal with the Lie bialgebra object $p_-$ or, in other words to introduce the double of $\mathfrak{g}_{D,-}$ and the $\text{Prop}$ $D_{\otimes}(\text{LBA})$ [EG]. We then introduce the multicolored $\text{Prop}$ as a cofiber product of $\text{PLBA}$ and $D_{\otimes}(\text{LBA})$ over $\text{LBA}$.

**Proposition.** The multicolored $\text{Prop}$ $\text{PLBAD}$ is endowed with canonical functors

\[
\begin{array}{ccc}
D_{\otimes}(\text{LBA}) & \rightarrow & \text{PLBAD} \\
\downarrow & & \downarrow \\
\text{PHA} & \rightarrow & \text{PLBA}
\end{array}
\]

and satisfies the following universal property:

\[
\begin{array}{ccc}
\text{LBA} & \xrightarrow{\text{double}} & D_{\otimes}(\text{LBA}) \\
\downarrow & & \downarrow \\
\text{PLBA} & \rightarrow & \text{PLBAD}
\end{array}
\]

where double is the $\text{Prop}$ map introduced in [EG].

In $\text{PLBAD}$ we can consider the Lie bialgebra object $[p_-]$.

7.5. Props for parabolic Hopf subalgebras. Similarly, we introduce the multicolored $\text{Prop}$ $\text{PHAD}$, endowed with canonical functors (cf. [EG])

\[
\begin{array}{ccc}
D_{\otimes}(\text{HA}) & \rightarrow & \text{PHAD} \\
\downarrow & & \downarrow \\
\text{PHA} & \rightarrow & \text{PHAD}
\end{array}
\]

and satisfying an analogous universal property:

\[
\begin{array}{ccc}
\text{HA} & \xrightarrow{\text{double}} & D_{\otimes}(\text{HA}) \\
\downarrow & & \downarrow \\
\text{PHA} & \rightarrow & \text{PHAD}
\end{array}
\]

where $C$ is a universal property.

\[\begin{array}{ccc}
\text{HA} & \xrightarrow{Q_{\text{EK}}} & S(\text{LBA}) \\
\downarrow & & \downarrow \\
\text{PHA} & \xrightarrow{Q_{\text{PLBA}}} & S(\text{PLBA})
\end{array}\]
Moreover, we then have a canonical functor

\[ Q_{PLBAD} : PHAD \rightarrow S(PLBAD) \]

obtained applying such universal property with \( C = S(PLBAD) \) and satisfying

\[
\begin{array}{c}
\text{HA} \\
\downarrow \text{double} \\
\text{PHA}
\end{array}
\quad \begin{array}{c}
\text{L}_{HA} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{D}_{S}(HA) \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{Q}^{EK} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{PHAD} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{Q}^{PLBA} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{S(LBA)} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{S(double)} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{S(D} \oplus \text{LBA)} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{S(PLBA)} \\
\rightarrow \\
\leftarrow
\end{array}
\quad \begin{array}{c}
\text{S(PLBAD)} \\
\rightarrow \\
\leftarrow
\end{array}
\]

The commutativity of the square on the back is given by the compatibility of the quantization functor with the doubling operations, proved in [EG].

7.6. Prop description of \( L_{-}, N_{+}^{*} \). The modules \( L_{-}, N_{+}^{*} \) can be realized in \( S(PLBAD) \). The module \( L_{-} \) is constructed over the object \( \text{Sm}_{-} \in S(PLBA) \). The structure of Drinfeld-Yetter module over \( g_{-} \) is determined in the following way:

- the free action of the Lie algebra object \( m_{-} \) is defined by the map
  
  \[ \text{Sm}_{-} \otimes \text{Sm}_{-} \rightarrow \text{Sm}_{-} \]

  given by Campbell-Hausdorff series, describing on \( \text{Sm}_{-} \) the multiplication in \( U\text{m}_{-} \).

- we define the action of \( g_{D,-} \) to be trivial on \( 1 \rightarrow \text{Sm}_{-} \).

- The actions of \( m_{-}, g_{D,-} \), the relation
  
  \[ \pi \circ ([,] \otimes 1) = \pi \circ (1 \otimes \pi) - \pi \circ (1 \otimes \pi) \circ \sigma_{12} \]

  and the map \([,] : g_{D,-} \otimes m_{-} \rightarrow m_{-} \) define the action of \( g_{-} \).

- We then impose the trivial coaction on \( 1 \rightarrow \text{Sm}_{-} \) and the compatibility condition between action and coaction
  
  \[ \pi^{\ast} \circ \pi = (1 \otimes \pi)\sigma_{12}(1 \otimes \pi^{*}) - (1 \otimes \pi)(\delta \otimes 1) + (\mu \otimes 1)(1 \otimes \pi^{*}) \]

  determines the coaction for \( \text{Sm}_{-} \). The action defined is compatible with \([,] : g_{D,-} \otimes m_{-} \rightarrow m_{-} \).
Similarly, the module $N^*_+$ can be realized on the object $\widehat{Sp}_-$, formally added to $S(PLBAD)$.

We determine the formulae for the action and the coaction of $g_-$ by direct inspection of the action of $g = g_- \oplus g_+$ on $N_+$ in the category Vect. Namely, the identification $N^*_+ = \widehat{Sp}_-$ is clearly obtained through the invariant bilinear form $\langle -, - \rangle$ and there are topological formulae expressing the action of $g$ on $N_+$. Therefore we determine action and coaction on $N^*_+$ in the following way:

- the $g_+$-action on $N_+ = Sp_+ = Sg_+ \otimes Sg_{D,-}$ is given by the free action on the first factor $Sg_+$ expressed by Campbell–Hausdorff series.

- the action of $g_- = m_- \oplus g_{D,-}$ on the subspace $Sg_{D,-} \subset Sp_+$ is given by the trivial action of $m_-$ and the usual free action of $g_{D,-}$ by multiplication.

- The action of $g_-$ is then interpreted as a topological coaction of $g_+$ and the aforementioned compatibility condition between action and coaction allows to extend the formula for the topological $g_+$ coaction on the entire space $Sp_+$.

- Through the invariant bilinear form $\langle -, - \rangle$, these formulae are carried over $N^*_+ = \widehat{Sp}_-$, by switching, in particular, the bracket and the topological cobracket on $g_+$ with the cobracket and the bracket in $g_-$, respectively.

- The obtained formulae, describing the action and the coaction of $g_-$ on $N^*_+$, are well–defined in the category $PLBAD$ and define the requested structure of Drinfeld-Yetter module over $g_-$.

7.7. **Proof of Theorem 6.3.** The relative Verma module

$$N_+ = \text{Ind}_{m_-}^{g_-} k \simeq \text{Ind}_{p_-}^{g_-} U_{g_D}$$

satisfies

$$\text{Hom}_{U_{\mathfrak{g}}}(N_+, V) \simeq \text{Hom}_{U_{p_-}}(U_{g_D}, V)$$

for every $U_{\mathfrak{g}}$-module $V$. We have a canonical map of $p_-$-modules $\rho_D : U_{g_D} \to N_+$ corresponding to the identity in the case $V = N_+$. We get a map of $p_-$-modules $\rho^*_D : N^*_+ \to U_{g_D}^*$ inducing an isomorphism

$$\text{Hom}_{U_{\mathfrak{g}}}(V, N^*_+) \simeq \text{Hom}_{U_{p_-}}(V, U_{g_D}^*)$$
The morphism $\rho_D^*$ can indeed be thought as

\[ \begin{array}{ccc}
U_\mathfrak{p}^- \otimes N_+^* & \longrightarrow & N_+^* \\
\downarrow & & \downarrow \\
U\mathfrak{g}_D \otimes U\mathfrak{g}_D^* & \longrightarrow & U\mathfrak{g}_D^*
\end{array} \]

Assuming the existence of a suitable finite $\mathbb{N}$–grading, a split pair of Lie bialgebras $(\mathfrak{g}_-, \mathfrak{g}_D, -)$, gives rise to a functor

$\text{PLBAD} \rightarrow \text{Vect}$

Consider now the trivial split pair given by $(\mathfrak{g}_D, -, \mathfrak{g}_D, -)$. We have a natural transformation

\[ \begin{array}{ccc}
\text{PLBAD} & \xrightarrow{\rho} & \text{Vect} \\
\downarrow & & \downarrow \\
(\mathfrak{g}, \mathfrak{g}_D, -) & \longrightarrow & (\mathfrak{g}_D, -, \mathfrak{g}_D, -)
\end{array} \]

where $p$ naturally extends to the projection $p_- \rightarrow \mathfrak{g}_D$.

The module $U(\mathfrak{g}_D)^*$ is indeed the module $N_+^*$ with respect to the trivial pair $(\mathfrak{g}_D, -, \mathfrak{g}_D, -)$. Consequently, the existence of the $p_-$–intertwiner $\rho_D^*$ can be interpreted as a simple consequence of the existence of natural transformation $p$.

The quantization functor $Q_{\text{PLBAD}}$ extends the natural transformation $p$ to

\[ \begin{array}{ccc}
\text{PHAD} & \xrightarrow{S} & S(\text{PLBAD}) \\
\downarrow & & \downarrow S(p) \\
(\mathfrak{g}, \mathfrak{g}_D, -) & \longrightarrow & (\mathfrak{g}_D, -, \mathfrak{g}_D, -)
\end{array} \]

and shows that

$F(N_+^*) \simeq (N_+^h)^*$

Similarly, we can consider the natural transformation $S(i)$ and the diagram

\[ \begin{array}{ccc}
\text{PHAD} & \xrightarrow{S} & S(\text{PLBAD}) \\
\downarrow & & \downarrow S(i) \\
(\mathfrak{g}, \mathfrak{g}_D, -) & \longrightarrow & (\mathfrak{g}_D, -, \mathfrak{g}_D, -)
\end{array} \]

implying

$F(L_-) \simeq L_-^h$
We can make analogous consideration for the right \( g_D \)-action on \( L_-, N_+^* \). This leads to isomorphisms of right \( U_h \) \( g_D \)-modules

\[
\tilde{F}_D(N_+^*) \simeq (N_+^h)^* \quad F^E_D(L_-) \simeq L_-^h
\]

8. Chains of Manin triples

8.1. Chains of length 2. In Section 6, given an inclusion of Manin triples \( i_D : g_D \subseteq g \), we introduced the relative quantum Verma modules

\[
L^h_- = \text{Ind}_{U_{h}^{E} \text{ } g}^{U_{h}^{E} \text{ } g_D} k[[h]] \quad N^h_+ = \text{Ind}_{U_{h}^{E} \text{ } p_-}^{U_{h}^{E} \text{ } g_D} U_{h}^{E} \text{ } g
\]

These modules allow to define the functor \( \Gamma_h : \mathcal{D}(U_{h}^{E} \text{ } g) \rightarrow \mathcal{D}(U_{h}^{E} \text{ } g_D) \)

by

\[
\Gamma_h(V) = \text{Hom}_{U_{h}^{E} \text{ } g}(L^h_-, (N^h_+)^* \otimes V)
\]

Lemma. The functor \( \Gamma_h \) is naturally tensor isomorphic to the restriction functor (\( U_{h}^{E} (i_D^h) \))^*.

Proof. The proof of the existence of the natural isomorphism as \( U_{h}^{E} \) \( g_D \)-module is identical to that of Proposition 5.13. The isomorphism respects the tensor structures, because there are only trivial associators involved. \( \square \)

8.2. We now prove the following

Theorem. Let \( g, g_D \) be Manin triples with a finite \( \mathbb{Z} \)-grading and \( i_D : g_D \subseteq g \) an inclusion of Manin triples compatible with the grading. Then, there exists an algebra isomorphism

\[
\Psi : \hat{U}_{h}^{E} \text{ } g \rightarrow \hat{U}_{h}^{E} \text{ } g[[h]]
\]

restricting to \( \Psi^E_D \) on \( U_{h}^{E} \) \( g_D \), where the completion is given with respect to Drinfeld–Yetter modules.

Proof. In the previous section, we showed that the quantization of the \( (U_{h}^{E} \text{ } g, U_{h}^{E} \text{ } g_D) \)-modules \( N_+^*, L_- \) gives

\[
\tilde{F}(N_+^*) \xrightarrow{U_{h}^{E} \text{ } g} (N_+^h)^* \xleftarrow{U_{h}^{E} \text{ } g_D} F^E_D(N_+^*)
\]
\[
\tilde{F}(L_-) \xrightarrow{U_{h}^{E} \text{ } g} L_-^h \xleftarrow{U_{h}^{E} \text{ } g_D} F^E_D(L_-)
\]

Recall that the standard natural transformations \( \alpha_V : \tilde{F}(V) \simeq V[[h]] \), \( (\alpha_D)_{\mathcal{V}} : \tilde{F}_D(V) \simeq V[[h]] \) give isomorphisms of right \( U g_D[[h]] \)-modules

\[
\tilde{F}(N_+^*) \simeq N_+^* [[h]] \quad \tilde{F}(L_-) \simeq L_- [[h]]
\]

and isomorphisms of \( U g[[h]] \)-modules

\[
F^E_D(N_+^*) \simeq N_+^* [[h]] \quad F^E_D(L_-) \simeq L_- [[h]]
\]
In particular, we get isomorphisms of right \( U_h^E K \)–modules
\[
F^E K \circ \overline{F}(N^*_{+}) \simeq F^E K(D(N^*_{+})) \simeq (N^h_{+})^* \quad F^E K \circ \overline{F}(L_{-}) \simeq F^E K(D(L_{-})) \simeq L^h_{-}
\]
and isomorphisms of \( U_h^E K \)–modules
\[
F^E K \circ \overline{F}(N^*_{+}) \simeq \overline{F}(N^*_{+}) \simeq (N^h_{+})^* \quad F^E K \circ \overline{F}(L_{-}) \simeq \overline{F}(L_{-}) \simeq L^h_{-}
\]
We have a natural isomorphism through \( J \):
\[
\text{Hom}_{U_h^E K}(F(L_{-}), F(N^*_{+}) \otimes F(V)) \simeq \text{Hom}_g(L_{-}, N^*_{+} \otimes V)[[h]]
\]
This is indeed an isomorphism of \( U g_D[[h]] \)–modules, since, for \( x \in U g_D \),
\[
\phi \in \text{Hom}_{U_h^E K}(F(L_{-}), F(N^*_{+}) \otimes F(V)), \text{ we have}
\]
\[
x. \phi := (F(x) \otimes \text{id}) \circ \phi \quad J \circ (F(x) \otimes \text{id}) = F(x \otimes \text{id}) \circ J
\]
Quantizing both sides and using the isomorphism \( F^E K \circ F(N^*_{+}) \simeq (N^h_{+})^* \), we obtain a natural transformation
\[
\gamma_D : \Gamma_h \circ \overline{F} \simeq \overline{F}_D \circ \Gamma
\]
making the following diagram commutative
\[
\begin{array}{ccc}
\mathcal{D}_\Phi(U g) & \xrightarrow{\overline{F}} & \mathcal{D}(U_h g) \\
\gamma_D & \downarrow & \downarrow \Gamma^h \\
\mathcal{D}_\Phi(U g_D[[h]]) & \xrightarrow{\overline{F}_D} & \mathcal{D}(U_h g_D)
\end{array}
\]
Applying the construction above to the algebra of endomorphisms of the fiber functor, we get the result.

\[\square\]

8.3. Chains of arbitrary length. For any chain
\[
\mathcal{C} : 0 = g_0 \subseteq g_1 \subseteq \cdots \subseteq g_{n-1} \subseteq g_n = g
\]
of inclusions of Manin triples, the natural transformations
\[
\gamma_{i,i+1} \in \text{Nat}_\otimes (\Gamma^h_{i,i+1} \circ \overline{F}_{i+1}, \overline{F}_i \circ \Gamma_{i,i+1})
\]
where \( 0 \leq i \leq n - 1 \), \( \Gamma_{0,1} : \mathcal{D}_\Phi(g_1) \to \text{Vect}_k[[h]] \) is the EK fiber functor, and \( F^E K_0 = \text{id} \), yield a natural transformation
\[
\gamma \mathcal{C} = \gamma_{0,1} \circ \cdots \circ \gamma_{n-1,n}
\]
\[
\in \text{Nat}_\otimes (\Gamma_{0,1} \circ \cdots \circ \Gamma^h_{n-1,n} \circ \overline{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]
\[
\simeq \text{Nat}_\otimes ((i^*_n)^h \circ \overline{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]
\[
= \text{Nat}_\otimes (\overline{F}_n, \Gamma_{0,1} \circ \cdots \circ \Gamma_{n-1,n})
\]
where we used \( \Gamma^h_{i,i+1} \cong (i^*_i)^h \), and the fact that the composition \( (i^*_0)^h \circ \overline{F}_n \)
is the EK fiber functor for \( g_n \), which we denote by the same symbol as \( \overline{F}_n \).
This proves the following

Theorem.
(i) For any chain of Manin triples
\[ C \colon g_0 \subseteq g_1 \subseteq \cdots \subseteq g_n \subseteq g \]
there exists an isomorphism of algebras
\[ \Psi_C : \hat{U}_h g \to \hat{U} g[[h]] \]
such that \( \Psi_C(U^{EK}_h g_i) = \hat{U} g_i[[h]] \) for any \( g_i \in C \).

(ii) Given two chains \( C, C' \), the natural transformation
\[ \Phi_{CC'} := \gamma^{-1}_C \circ \gamma_{C'} \in \text{Aut}(F^{EK}) \]
satisfies
\[ \text{Ad}(\Phi_{CC'}) \Psi_{C'} = \Psi_C \]

Proposition. The natural transformations \( \{ \Phi_{CC'} \}_{C,C'} \) satisfy the following properties

(i) **Orientation.** Given two chains \( C, C' \)
\[ \Phi_{CC'} = \Phi^{-1}_{C'C} \]

(ii) **Transitivity.** Given the chains \( C, C', C'' \)
\[ \Phi_{CC'} \circ \Phi_{C'C''} = \Phi_{CC''} \]

(iii) **Factorization.** Given the chains
\[ C, C' : g_0 \subseteq g_0 \subseteq \cdots \subseteq g_n \quad D, D' : g_n \subseteq \cdots \subseteq g_n+n' \]
\[ \Phi_{(C \cup D)(C' \cup D')} = \Phi_{CC'} \circ \Phi_{DD'} \]

8.4. **Abelian Manin triples and central extensions.** We will now consider the following special case, that generalizes the role of Levi subalgebras for Kac–Moody algebras.

**Proposition.** If \( g \) admits a Manin subtriple \( l_D \), obtained by a central extension of \( g_D \), then the relative twists and the gauge transformations are invariant under \( l_D \). In particular, the Etingof–Kazhdan constructions are invariant under abelian Manin subtriples.

**Proof.** For \( g_D = \{0\} \), the statement reduces to prove that the Etingof–Kazhdan functor preserves the action of an abelian Manin subalgebra \( a \subset g \) (cf. [EK6, Thm. 4.3], with \( a = h \)). Under this assumption, the natural map
\[ U a_\ldots \longrightarrow U^{EK}_h g_\ldots := \overline{F}(M_\ldots) \]
\[ a \longrightarrow \{ \psi_a : 1_\ldots \mapsto \varepsilon \otimes a 1_\ldots \} \]
defines an inclusion of bialgebras. For any \( V[[h]] \in D_\Phi(U g) \), the natural identification
\[ \alpha_V : F(V) \to V[[h]] \]
is then an isomorphism of $U\mathfrak{a}$–modules. This gives the following commutative diagram

$$
\begin{array}{c}
\mathcal{D}_\Phi(U\mathfrak{g}) & \xrightarrow{\tilde{F}} & \mathcal{D}(U^\text{rk}_h \mathfrak{g}) \\
\downarrow F & & \downarrow \mathcal{D}(U\mathfrak{a}[[h]]) \\
\mathcal{D}(U\mathfrak{a}[[h]]) & & \\
\end{array}
$$

We can observe that the tensor restriction functor fits in an analogous diagram. It is easy to show that the object $\Gamma_D(L_-)$ is naturally a pointed Hopf algebra in the category $\mathcal{D}_\Phi(D(U\mathfrak{g}))$. We denote by $\mathcal{D}_g(\Gamma_D(L_-))$ the category of Drinfeld–Yetter modules over $\Gamma_D(L_-)$ in the category $\mathcal{D}_\Phi(D(U\mathfrak{g}))$. This category is naturally equivalent to the category of Drinfeld–Yetter module over the Radford’s product $U\mathfrak{g}_D\cdot[[\hbar]]\#\Gamma_D(L_-)$ and there is a natural identification

$$
\begin{array}{c}
\mathcal{D}_\Phi(U\mathfrak{g}) & \xrightarrow{\Gamma_D} & \mathcal{D}_g(\Gamma_D(L_-)) \\
\downarrow \mathcal{D}_\Phi(D(U\mathfrak{g})) & & \downarrow \mathcal{D}_g(\Gamma_D(L_-)) \\
\end{array}
$$

Moreover, there is a natural inclusion of bialgebras

$$
U\mathfrak{I}_D \subset U\mathfrak{g}_D\cdot[[\hbar]]\#\Gamma_D(L_-)
$$

and a natural $U\mathfrak{I}_D$–module identification $\Gamma_D(V) \to V[[\hbar]]$. This originates natural identifications

$$
\begin{array}{c}
\mathcal{D}_\Phi(U\mathfrak{g}) & \xrightarrow{\Gamma_D} & \mathcal{D}_g(\Gamma_D(L_-)) \\
\downarrow \mathcal{D}(U\mathfrak{I}_D) & & \downarrow \mathcal{D}_\Phi(D(U\mathfrak{g})) \\
\end{array}
$$

This proves that relative twists are invariant under $\mathfrak{I}_D$. It is clear that the Casimir operator $\Omega_D \in (\mathfrak{g}_D \otimes \mathfrak{g}_D)^{\mathfrak{I}_D}$ defines a braided tensor structure on $\mathcal{D}(U\mathfrak{I}_D[[\hbar]])$ that is preserved by the restriction functor induced by the inclusion $j_D : \mathfrak{g}_D \subset \mathfrak{I}_D$. Given the decomposition $\mathfrak{I}_D = \mathfrak{g}_D \rtimes \mathfrak{c}_D$, the natural map $U\mathfrak{c}_D \to \text{End}_{\mathfrak{g}_D}(j_D^*V)$ induces an action of $U\mathfrak{c}_D$ on $\tilde{F}_D(j_D^*V)$, commuting with the action of $U^\text{rk}_h \mathfrak{g}_D$. Therefore, we obtain a naturally commutative
9. An equivalence of quasi–Coxeter categories

The following is the main result of this paper.

**Theorem.** Let \( \mathfrak{g} \) be a symmetrizable Kac–Moody algebra with a fixed \( D_\mathfrak{g} \)-structure. Then the completion \( \hat{U}_h \mathfrak{g} \) is isomorphic to a quasi-Coxeter quasi-triangular quasibialgebra of type \( D_\mathfrak{g} \) on the quasitriangular \( D_\mathfrak{g} \)-quasibialgebra

\[
\left( \hat{U}_h \mathfrak{g}, \{ \hat{U}_h \mathfrak{g}[\hbar] \}, \{ \hat{U}_h \mathfrak{g}_D[\hbar] \}, \Delta_0, \{ \Phi^K_{\mathfrak{g}} \}, \{ R^K_{\mathfrak{g}} \} \right)
\]

where the completion is taken with respect to the integrable modules in category \( \mathcal{O} \).

9.1. \( D \)-structures on Kac–Moody algebras. Let \( A = (a_{ij})_{i,j \in I} \) be a complex \( n \times n \) matrix and \( \mathfrak{g} = \mathfrak{g}(A) \) the corresponding generalized Kac–Moody algebra defined in Section 4. Let \( J \) be a nonempty subset of \( I \). Consider the submatrix of \( A \) defined by

\[
A_J = (a_{ij})_{i,j \in J}
\]

We recall the following proposition from [Ka, Ex.1.2]

**Proposition.** Let

\[
\Pi_J := \{ \alpha_j \mid j \in J \} \quad \Pi_J^\vee := \{ h_j \mid j \in J \}
\]

Let \( \mathfrak{h}_J \) be the subspace of \( \mathfrak{h} \) generated by \( \Pi_J^\vee \) and

\[
t_J = \bigcap_{j \in J} \text{Ker} \alpha_j = \{ h \in \mathfrak{h} \mid \langle \alpha_j, h \rangle = 0 \ \forall j \in J \}
\]
Let $\mathfrak{h}_{\mathcal{J}}''$ be a supplementary subspace of $\mathfrak{h}_{\mathcal{J}}' + t_{\mathcal{J}}$ in $\mathfrak{h}$ and let

$$\mathfrak{h}_{\mathcal{J}} = \mathfrak{h}_{\mathcal{J}}' \oplus \mathfrak{h}_{\mathcal{J}}''$$

Then,

(i) $(\mathfrak{h}_{\mathcal{J}}, \Pi_{\mathcal{J}}, \Pi_{\mathcal{J}}')$ is a realization of the generalized Cartan matrix $A_{\mathcal{J}}$.

(ii) The subalgebra $\mathfrak{g}_{\mathcal{J}} \subset \mathfrak{g}$, generated by $\{e_j, f_j\}_{j \in \mathcal{J}}$ and $\mathfrak{h}_{\mathcal{J}}$, is the Kac–Moody algebra associated to the realization $(\mathfrak{h}_{\mathcal{J}}, \Pi_{\mathcal{J}}, \Pi_{\mathcal{J}}')$ of $A_{\mathcal{J}}$.

Set

$$Q_{\mathcal{J}} = \sum_{j \in \mathcal{J}} \mathbb{Z}\alpha_j \subset Q \quad g = g(A) = \bigoplus_{\alpha \in Q} g_{\alpha}$$

Then,

(iii) $\mathfrak{g}_{\mathcal{J}} = \mathfrak{h}_{\mathcal{J}} \oplus \bigoplus_{\alpha \in Q_{\mathcal{J}} \setminus \{0\}} g_{\alpha}$

Let $A$ be a symmetrizable matrix with a fixed decomposition and $(-|-)$ be the standard normalized non–degenerate bilinear form on $\mathfrak{h}$. Then,

(iv) The restriction of $(-|-)$ to $\mathfrak{h}_{\mathcal{J}}$ is non–degenerate.

**Proof.** Since $\dim(\mathfrak{h}_{\mathcal{J}}' \cap t_{\mathcal{J}}) = \dim(\mathfrak{z}(\mathfrak{g}_{\mathcal{J}})) = n_{\mathcal{J}} - l_{\mathcal{J}}$, where $n_{\mathcal{J}} = |\mathcal{J}|$ and $l_{\mathcal{J}} = \text{rank}(A_{\mathcal{J}})$, it follows that

$$\dim \mathfrak{h}_{\mathcal{J}}' = n_{\mathcal{J}} - l_{\mathcal{J}} \quad \dim \mathfrak{h}_{\mathcal{J}}'' = 2n_{\mathcal{J}} - l_{\mathcal{J}}$$

Moreover, by construction, the restriction of $\{\alpha_j\}_{j \in \mathcal{J}}$ to $\mathfrak{h}_{\mathcal{J}}$ are linearly independent. Indeed, since $\langle \sum c_j \alpha_j, t_{\mathcal{J}} \rangle = 0$ for all $c_j \in \mathbb{C}$,

$$\langle \sum c_j \alpha_j, \mathfrak{h}_{\mathcal{J}} \rangle = 0 \quad \Rightarrow \quad \langle \sum c_j \alpha_j, h \rangle = 0 \quad \Rightarrow \quad c_j = 0$$

This proves (i). The proof of (ii) and (iii) is clear.

Assume now that $A$ is irreducible and symmetrizable and there exists $h \in \mathfrak{h}_{\mathcal{J}}$ such that

$$\langle h|h' \rangle = 0 \quad \forall h' \in \mathfrak{h}_{\mathcal{J}}$$

In particular, $\langle h|\alpha_j^\vee \rangle = 0$ and $h \in \mathfrak{h}_{\mathcal{J}}' \cap t_{\mathcal{J}} \subset \mathfrak{h}_{\mathcal{J}}'$. Therefore, $h = \sum c_j \alpha_j^\vee$ and

$$\langle \sum c_j \alpha_j^\vee |h' \rangle = \sum c_j \langle \alpha_j^\vee |h' \rangle = \langle \sum c_j d_j \alpha_j, h' \rangle = 0$$

Since the operators $\{\alpha_j\}$ are linearly independent over $\mathfrak{h}_{\mathcal{J}}$ and $d_j \neq 0$, we have $c_j = 0$ and $h = 0$. We conclude that $\langle |\rangle$ is non–degenerate on $\mathfrak{h}_{\mathcal{J}}$ and (iv) is proved.

**Remark.** The derived algebra $\mathfrak{g}'_{\mathcal{J}} = [\mathfrak{g}_{\mathcal{J}}, \mathfrak{g}_{\mathcal{J}}]$ is generated by $\{e_j, f_j, h_j\}_{j \in \mathcal{J}}$, where $h_j = [e_j, f_j]$. Therefore, it does not depend on the choice of the subspace $\mathfrak{h}_{\mathcal{J}}''$. The assignment $\mathcal{J} \mapsto \mathfrak{g}'_{\mathcal{J}}$ defines a structure that coincides with the one provided in [TL4, 3.2.2].
Let now $A$ be an irreducible, generalized Cartan matrix. Let $D_\mathfrak{g} = D(A)$ be the Dynkin diagram of $\mathfrak{g}$, that is, the connected graph having $I$ as vertex set and an edge between $i$ and $j$ if $a_{ij} \neq 0$. For any $i \in I$, let $\mathfrak{sl}_2 \subset \mathfrak{g}$ be the three–dimensional subalgebra spanned by $e_i, f_i, h_i$.

Any connected subdiagram $D \subseteq D_\mathfrak{g}$ defines a subset $J_D \subset I$. We would like to use the assignment $J \mapsto \mathfrak{g}_J$ to define a $D_\mathfrak{g}$–algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$.

**Remark.** For any subset $J$ of finite type, $\dim h''_J = n_J - l_J = 0$ and $h_J = h'_J$. Therefore, if $A$ is a generalized Cartan matrix of finite type, $h''_J = \{0\}$ for any subset $J \subset I$. The $D_\mathfrak{g}$–algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$ is then uniquely defined by the subalgebras $\{\mathfrak{sl}_2\}_{i \in I}$ and the Cartan subalgebra is defined for any subdiagram $D \subset D_\mathfrak{g}$ by

$$h_D = \{h_i \mid i \in V(D)\}$$

If $A$ is a generalized Cartan matrix of affine type, we obtain diagrammatic Cartan subalgebras $\mathfrak{h}_D$, where

$$h_D = \begin{cases} \{h_i \mid i \in V(D)\} & \text{if } D \subset D_\mathfrak{g} \\ \mathfrak{h} & \text{if } D = D_\mathfrak{g} \end{cases}$$

If $A$ is an irreducible generalized Cartan matrix of hyperbolic type, i.e., every submatrix is of finite or affine type, it is still possible to define a $D_\mathfrak{g}$–algebra structure, depending upon the choice of the subspaces $h''_J$ for $|I \setminus J| = 1$.

It is not always possible to define a $D_\mathfrak{g}$–algebra structure for a generic matrix of order $\geq 3$. In order to obtain a $D_\mathfrak{g}$–algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$, we have to satisfy the following condition:

$$h_J \subset t_{J^\perp} \cap \bigcap_{J \subset J''} h_{J''}$$

Since $t_{J^\perp} + t_J = \mathfrak{h}$, we can always choose $h_J \subseteq t_{J^\perp}$.

**Lemma.** Assume given a $D_\mathfrak{g}$–algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$. Then for any two subsets $J', J'' \subset I$,

$$\text{corank}(A_{J' \cap J''}) \leq \text{corank}(A_{J'}) + \text{corank}(A_{J''})$$

In particular, if $\text{corank}(A_{J'}) = \text{corank}(A_{J''}) = 0$, then $\text{corank}(A_{J' \cap J''}) = 0$.

**Proof.** The result is an immediate consequence of the estimate, given by the construction,

$$\dim(h_{J'} \cap h_{J''}) \leq |J' \cap J''| + (\text{corank}(A_{J'}) + \text{corank}(A_{J''}))$$

and the constraint

$$h_{J' \cap J''} \subseteq h_{J'} \cap h_{J''}$$

$\square$
Remark. Indeed, it is easy to show that the symmetric irreducible Cartan matrix

\[
A = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

does not admit any \(D_g\)-algebra structure on \(g(A)\), since \(\dim h_{23} = 3\) and \(\dim h_{123} \cap h_{234} = 2\).

The previous condition on the corank is not sufficient to obtain a \(D_g\)-algebra structure on \(g(A)\). Consider the symmetric Cartan matrix

\[
A = \begin{bmatrix}
2 & -2 & 0 & 0 \\
-2 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

A clearly satisfies the above condition. Nonetheless, a suitable \(h'_{12}\), complement in \(h\) of \((h'_{12} + t_{12})\), should satisfy:

\[
\begin{align*}
\langle h'_{12}, h_{123} \rangle &= h'_{123} \\
\langle h'_{12}, t_4 \rangle &= \langle h_{12}, -2\alpha_3^\vee + \alpha_4^\vee \rangle
\end{align*}
\]

that are clearly not compatible conditions. Therefore, there is no suitable structure for \(A\).

In the following, we will consider only symmetrizable Kac–Moody algebras \(g\) that admit such a structure. It automatically defines an analogue structure on \(U_{\hbar}g\).

9.2. \textit{qCqtqba structure on} \(U_{\hbar}g\). Given a fixed \(D_g\)-structure on the Kac–Moody algebra \(g\), the quantum enveloping algebra \(U_{\hbar}g\) is naturally endowed with a quasi–Coxeter quasitriangular quasibialgebra structure of type \(D_g\) defined by

(i) \(D_g\)-algebra: for any \(D \in \text{SD}(D_g)\), let \(g_D \subset g\) be the corresponding Kac–Moody subalgebra. The \(D_g\)-algebra structure is given by the subalgebras \(\{U_{\hbar}g_D\}\).

(ii) Quasitriangular quasibialgebra: the universal \(R\)-matrices \(\{R_{h,D}\}\), with trivial associators \(\Phi_D = 1^\otimes 3\) and structural twists \(F_F = 1^\otimes 2\).

(iii) Quasi–Coxeter: the local monodromies are the quantum Weyl group elements \(\{S_i^h\}_{i \in I}\). The Casimir associators \(\Phi_{g,F} = \text{trivial}\).

We transfer this \(qCqtqba\) structure on \(U_g[[\hbar]]\). More precisely, we define an equivalence of quasi–Coxeter categories between the representation theories of \(U_{\hbar}g\) and \(U_g[[\hbar]]\).

9.3. \textbf{Gauge transformations for} \(g(A)\). For any \(D \subset D_g\), the inclusion \(g_D \subset g\), defined in the previous section, lifts to an inclusion of Manin triples

\[
g_D \oplus h_D \subset g \oplus h
\]
We denote by \( \tilde{\mathfrak{g}}_D = (\mathfrak{g}_D \oplus \mathfrak{h}_D, \mathfrak{b}_{D,+}, \mathfrak{b}_{D,-}) \) the Manin triple attached to \( \mathfrak{g}_D \), for any \( D \subseteq D_\Phi \).

**Theorem.** There exists an equivalence of braided \( D_\Phi \)-monoidal categories from

\[
\left\{ \left( \left( D_\Phi B_B (\mathfrak{g}_B[[h]]), \otimes_B, \Phi_B, \sigma R_B \right), \left( (\Gamma_{BB'})_h, (J_{BB'})_h \right) \right) \right\}
\]

to

\[
\left\{ \left( \left( D_\Phi (\mathfrak{g}_B[[h]]), \otimes_B, \Phi_B, \sigma R_B \right), \left( (\Gamma_{BB'})_h, (J_{BB'})_h \right) \right) \right\}
\]

given by \( (\bar{F}_B, \{ \gamma_{BB'} \} ) \).

**Proof.** The natural transformations \( \gamma_{BB'} \) constructed in Section 8, define, by vertical composition, a natural transformation

\[
\gamma_{BB'} \in \text{Nat}_{\otimes} (\Gamma_{BB'}^h, (\Gamma_{BB'})_h \circ (J_{BB'})_h)
\]

for any chain of maximal length

\[
\mathbf{C} : B = C_0 \subset C_1 \subset \cdots \subset C_r = B'
\]

Any chain of maximal length defines uniquely a maximal nested set \( \mathcal{F}_C \in \text{Mns}(B, B') \), but this is not a one to one correspondence. For example, for \( D = A_3 \), the maximal nested set

\[
\mathcal{F} = \{ \{ \alpha_1 \}, \{ \alpha_3 \}, \{ \alpha_1, \alpha_2, \alpha_3 \} \}
\]

corresponds to two different chains of maximal length

\[
\mathbf{C}_1 : \{ \alpha_1 \} \subset \{ \alpha_1 \} \sqcup \{ \alpha_3 \} \subset A_3 \quad \mathbf{C}_2 : \{ \alpha_3 \} \subset \{ \alpha_1 \} \sqcup \{ \alpha_3 \} \subset A_3
\]

In order to prove that the natural transformations \( \gamma \) define a morphism of braided \( D_\Phi \)-monoidal categories, we need to prove that the transformation \( \gamma_{BB'}^\mathbf{C} \) depend only on the maximal nested set corresponding to \( \mathbf{C} \).

In particular, we have to prove that, for any \( B_1 \perp B_2 \) in \( I(D) \), the construction of the fiber functor

\[
\mathcal{C}_{B_1 \sqcup B_2} \left[ \begin{array}{c} F_{B_2, B_1 \sqcup B_2} \\ \mathcal{C}_{B_1} \end{array} \right] \rightarrow \left[ \begin{array}{c} F_{B_1, B_1 \sqcup B_2} \\ \mathcal{C}_{B_2} \end{array} \right]
\]

is independent of the choice of the chain. In our case,

\[
\mathcal{C}_{B_1 \sqcup B_2} = D(\mathfrak{g}_B [h] \otimes \mathfrak{g}_B [h])
\]

and the braided tensor structure is given by product of the braided tensor structures on

\[
\mathcal{C}_{B_1} = D_{\Phi_{B_1}} (\mathfrak{g}_B [h]) \quad \mathcal{C}_{B_2} = D_{\Phi_{B_2}} (\mathfrak{g}_B [h])
\]
Similarly, the tensor structure on the forgetful functor
\[ C_{B_1 \sqcup B_2} \to C_{B_i} \quad i = 1, 2 \]
is obtained killing the tensor structure on \( C_{B_i} \), \( i = 1, 2 \), i.e., applying the tensor structure on \( FC_{B_1, B_1 \sqcup B_2} \) and \( FC_{B_2, B_1 \sqcup B_2} \) coincide, since \([\tilde{g}_{B_1}, \tilde{g}_{B_2}] = 0\).

Analogously we have an equality of natural transformation
\[ \gamma_{B_1} \circ \gamma_{B_1, B_1 \sqcup B_2} = \gamma_{B_2} \circ \gamma_{B_2, B_1 \sqcup B_2} \]
Therefore, for any maximal nested set \( F \in \text{Mns}(B, B') \), it is well defined a natural transformation
\[ \gamma_{B'} \in \text{Nat}(\Gamma_{BB'} \circ \bar{F}_{B'}, \bar{F}_B \circ \Gamma_{BB'}) \]
so that the data \( \{\bar{F}_{B}\}, \{\gamma_{BB}\} \) define an isomorphism of \( D \)-categories from \( \{D(U\tilde{g}_B[[h]])\} \) to \( \{D(Uh\tilde{g}_B)\} \).

9.4. **Extension to Levi subalgebras.** In analogy with [TL4, Thm. 9.1], we want to show that the relative twists and the Casimir associators are weight zero elements. This corresponds to show that the corresponding tensor functors \( \Gamma \) and the natural transformations \( \gamma \) lift to the level of Levi subalgebras:
\[ g_D \subset l_D = n_{D,+} \oplus h \oplus n_{D,-} \subset g \]

**Proposition.** The relative twists and the Casimir associators are weight zero elements.

**Proof.** For \( D = \emptyset \), the statement reduces to prove that the Etingof–Kazhdan functor preserves the \( h \)-action [EK6, Thm. 4.3]. The result is a consequence of Proposition 8.4 applied to Levi subalgebras.

9.5. **Reduction to category \( O^{\text{int}} \).** The Etingof–Kazhdan functor gives rise, by restriction, to an equivalence of categories
\[ \bar{F} : O_g[[h]] \to O_{Ug} \]
We will show now that this equivalence can be further restricted to integrable modules in category \( O \), i.e., modules in category \( O \) with a locally nilpotent action of the elements \( \{e_i, f_i\}_{i \in I} \) (respectively \( E_i, F_i \)).

**Proposition.** The Etingof–Kazhdan functor restricts to an equivalence of braided tensor categories
\[ \bar{F} : O^{\text{int}}_g[[h]] \to O^{\text{int}}_{Ug} \]
which is isomorphic to the identity functor at the level of \( h \)-graded \( k[[h]] \)-modules.
Proof. Let \( V \in O^\text{int}_g \). Then, the elements \( e_i, f_i \) for \( i \in I \) act nilpotently on \( V \). Then, by [Ka], for all \( \lambda \in P(V) \), there exist \( p, q \in \mathbb{Z}_{\geq 0} \) such that
\[
\{ t \in \mathbb{Z} \mid \lambda + t\alpha_i \in P(V) \} = [-p, q]
\]
Since the Cartan subalgebra \( \mathfrak{h} \) is not deformed by the quantization, the functor \( \tilde{F} \) preserves the weight decomposition. In \( U_\hbar g \), for any \( h \in \mathfrak{h} \) and \( i \in I \), we have
\[
[h, E_i] = \alpha_i(h)E_i
\]
Therefore the action of the \( E_i \)'s on \( V \) is locally nilpotent. The action of the \( F_i \)'s is always locally nilpotent, since
\[
\mathcal{P}(V) \subset \bigcup_{s=1}^r D(\lambda_s)
\]
The result follows. \( \square \)

Corollary.
(i) There exists an equivalence of braided \( D_g \)-monoidal categories between
\[
O := \{ (O^\text{int}_g, \otimes_B, \Phi_B, \sigma R_B) \}, \{ (\Gamma_{BB'}, J_{\tilde{F}'}^B) \}
\]
and
\[
O_h := \{ (O^\text{int}_{U_\hbar g_B}, \otimes_B, id, \sigma R^h_B) \}, \{ (\Gamma^h_{BB'}, id) \}
\]
(ii) There exists an isomorphism of \( D_g \)-algebras
\[
\Psi_{\tilde{F}} : \hat{U}_\hbar g \rightarrow \hat{U}_g[[\hbar]]
\]
such that \( \Psi_{\tilde{F}}(\hat{U}_\hbar g_{D_i}) = U_g D_i[[\hbar]] \) for any \( D_i \in \mathcal{F} \), where the completion is taken with respect to the integrable modules in category \( O \).

9.6. Quasi–Coxeter structure. The previous equivalence of braided \( D_g \)-monoidal categories induces on
\[
O = \{ (O^\text{int}_g[[\hbar]], \otimes_B, \Phi_B, \sigma R_B) \}, \{ (\Gamma_{BB'}, J_{\tilde{F}'}^B) \}
\]
a structure of quasi–Coxeter category of type \( D_g \), given by the Casimir associators \( \Phi_{\mathcal{G}\mathcal{F}} \in \text{Nat}_{\otimes}(\Gamma_{\mathcal{F}}, \Gamma_{\mathcal{G}}) \) and the local monodromies \( S_i \in \text{End}(\Gamma_i) \) defined for any \( \mathcal{G}, \mathcal{F} \in \text{Mns}(B, B') \) and \( i \in I(D) \) by
\[
\tilde{F}_B(\Phi_{\mathcal{G}\mathcal{F}}) = (\gamma_{BB'}^F)^{-1} \circ \gamma_{BB'}^G \quad S_i = \Psi_{\tilde{F}}^{\text{EK}}(S^h_i)
\]
where \( \Psi_{\tilde{F}}^{\text{EK}} : \hat{U}_\hbar \mathfrak{sl}_2 \rightarrow \hat{U}_\mathfrak{sl}_2[[\hbar]] \) is the isomorphism induced at the \( \mathfrak{sl}_2 \) level by the Etingof–Kazhdan functor.

Proposition. The equivalence of braided \( D_g \)-monoidal categories \( O \rightarrow O_h \) induces a structure of quasi–Coxeter category on \( O \).
Proof. In order to prove the proposition, we have to prove the compatibility relations of the elements $\Phi_{G,F}, S_i$ with the underlying structure of braided $D_6$–monoidal category on $\mathcal{O}$.

The element $S_i$’s satisfy the relation
\[
\Delta_F(S_i) = (R_i)^{12}_F \cdot (S_i \otimes S_i)
\]
since $\Psi_F$ is given by an isomorphism of braided $D$–monoidal categories and therefore
\[
\Psi_F((R_i^h)_F) = (R_i)_F
\]
Similarly, the braid relations are easily satisfied, since
\[
\text{Ad}(\Phi_{G,F})\Psi_F = \Psi_G
\]
The elements $\Phi_{F,G}$ defined above satisfy all the required properties:

(i) **Orientation** For any elementary pair $(F,G)$ in $\text{Mns}(B,B')$
\[
\tilde{F}_B(\Phi_{F,G}) = (\gamma_{BB'}^F)^{-1} \circ \gamma_{BB'}^G = (\tilde{F}_B(\Phi_{G,F}))^{-1}
\]
(ii) **Coherence** For any $F, G, H \in \text{Mns}(B,B')$
\[
\tilde{F}_B(\Phi_{F,G}) = (\gamma_{BB'}^F)^{-1} \gamma_{BB'}^H \circ (\gamma_{BB'}^H)^{-1} \circ \gamma_{BB'}^G = \\
= \tilde{F}_B(\Phi_{F,H}) \circ H_B(\Phi_{H,G})
\]
This property implies the coherence.

(iii) **Factorization.** Clear by construction.

Finally, the elements $\Phi_{F,G}$ satisfy
\[
\Delta(\Phi_{F,G}) \circ J_F = J_G \circ \Phi_{G,F}^2
\]
because they are given by composition of invertible natural tensor transformations. □

9.7. **Normalized isomorphisms.** In the completion $\hat{U}_{\mathfrak{sl}_2[[h]]}$ with respect to category $\mathcal{O}$ integrable modules, there are preferred element $S_{i,C}$
\[
S_{i,C} = \tilde{s}_i \exp\left(\frac{h}{2} C_i\right)
\]
where
\[
\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i)
\]
\[
C_i = \frac{(\alpha_i, \alpha_i)}{2}(e_i f_i + f_i e_i + \frac{1}{2} h_i^2)
\]

**Proposition.** There exists an equivalence of quasi–Coxeter categories of type $D_6$ between
\[
\mathcal{O} := \left\{ \left\{ C_{\mathfrak{g}_B}^{\text{int}} \otimes B, \Phi_B, \sigma R_B \right\}, \left\{ (\Gamma_{BB'}, J_{BB'}^F) \right\}, \left\{ \Phi_{G,F} \right\}, \left\{ S_{i,C} \right\} \right\}
\]
and
\[ O_h := \left\{ (\mathcal{O}_{U_h g B}^{\text{int}}, \otimes B, \Phi B, \sigma R_B^h), \{ (\Gamma_{BB'}, \id), \{ \id \}, \{ S_i^h \} \} \right\} \]

**Proof.** Using the result of Proposition 9.6, it is enough to prove that the natural transformation \( \gamma_i \)
\[ \mathcal{O}_{i, h}^{\text{int}} \xrightarrow{\gamma_i} \mathcal{O}_{i, h}^{\text{int}} \]

can be modified in such a way that the induced isomorphism at the level of endomorphism algebras \( \widehat{U_h \mathfrak{sl}_2^i} \to \hat{U_{\mathfrak{sl}_2^i}}[[\hbar]] \) maps \( S_i^h \) to \( S_{i, C} \). The natural transformation used in Corollary 9.5 induces the Etingof–Kazhdan isomorphism
\[ \Psi_{i, EK} : \widehat{U_h \mathfrak{sl}_2^i} \to \hat{U_{\mathfrak{sl}_2^i}}[[\hbar]] \]
which is the identity mod \( \hbar \) and the identity on the Cartan subalgebra. As above, we denote by \( S_i \) the element \( \Psi_{i, EK}(S_i^h) \). Then \( S_i \equiv S_{i, C} \mod \hbar \) and, by [TL4, Proposition 8.1, Lemma 8.4], we have
\[ S_i^2 = S_{i, C}^2 \quad S_i = \text{Ad}(x)(S_{i, C}) \]
on the integrable modules in category \( O \), for \( x = (S_{i, C} \cdot S_i^{-1})^{1/2} \). Therefore, the modified isomorphism
\[ \Psi_i := \text{Ad}(x) \circ \Psi_{i, EK} \]
maps \( S_i^h \) to \( S_{i, C} \). Moreover, \( \Psi_i \) correspond with the natural transformation given by the composition of \( \gamma_i \) with \( x \in \hat{U_{\mathfrak{sl}_2^i}}[[\hbar]] = \text{End}(f) \)
\[ \mathcal{O}_{i}^{\text{int}} \xrightarrow{x} \mathcal{O}_{i, h}^{\text{int}} \]

The result follows substituting \( \gamma_i \) with \( x \circ \gamma_i \) in Proposition 9.6. \( \square \)

### 9.8. The main theorem

We now state in more details the main theorem of the paper and summarize the proof outlined in the previous results.

**Theorem.** Let \( \mathfrak{g} \) be a symmetrizable Kac–Moody algebra with a fixed \( D_\mathfrak{g} \)–structure and \( U_h \mathfrak{g} \) the corresponding Drinfeld–Jimbo quantum group with the analogous \( D_\mathfrak{g} \)–structure. For any choice of a Lie associator \( \Phi \), there exists an equivalence of quasi–Coxeter categories between
\[ O := \left\{ (\mathcal{O}_{U_h g B}^{\text{int}}, \otimes B, \Phi B, \sigma R_B), \{ (\Gamma_{BB'}, \id), \{ \id \}, \{ S_{i, C} \} \} \right\} \]
and
\[ O_h := \left\{ (\mathcal{O}_{U_h g B}^{\text{int}}, \otimes B, \Phi B, \sigma R_B^h), \{ (\Gamma_{BB'}^{BB'}, \id), \{ \id \}, \{ S_i^h \} \} \right\} \]
where $\otimes_B$ denotes the standard tensor product in $O_{B}^{\text{int}}$ and
\[
S_{i,C} = \tilde{s}_{i} \exp \left( \frac{h}{2} C_{i} \right),
\]
\[
\Phi_B = 1 \mod h^2,
\]
\[
R_B = \exp \left( \frac{h}{2} \Omega_D \right),
\]
\[
\text{Alt}_2 J^B_B = \frac{h}{2} \left( r_{B'}^2 - r_{B}^{21} \right) - \frac{r_{B'}^2 - r_{B}^{21}}{2}
\]
and $\Phi_{G,F}, J^B_B$ are weight zero elements.

**Proof.** The existence of an equivalence is a consequence of the constructions of Section 8 and proved in Theorem 9.3 and Proposition 9.6, 9.7, concerning the local monodromies $S_{i,C}$.

The properties of associators $\Phi_B$ and $R$–matrices $R_B$ are direct consequences of the construction in Section 4,5. The relation satisfied by the relative twists $J^B_B$ is proven by a simple application of Proposition 5.20 and Corollary 5.20. It is easy to check that the 1–jet of the twist $J^B_B$ differs from the 1–jet of the twist $J^B_B'$ (as defined in Section 5) by a symmetric element that cancels out computing the alternator. Therefore, Corollary 5.20 holds for $J^B_B'$ as well.

Finally, as previously explained, the weight zero property of the relative twists $J^B_B$ and the Casimir associators $\Phi_{G,F}$ is proved in Proposition 8.4, 9.4. This complete the proof of Theorem 9.8.

□

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