Existence of weak solutions to stochastic heat equations driven by truncated $\alpha$-stable white noises with non-Lipschitz coefficients

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Abstract

We consider a class of stochastic heat equations driven by truncated $\alpha$-stable white noises for $\alpha \in (1, 2)$ with noise coefficients that are continuous but not necessarily Lipschitz continuous. We prove the existence of weak solution in probabilistic sense, taking values in two different forms under different conditions, to such an equation using a weak convergence argument on solutions to the approximating stochastic heat equations. More precisely, for $\alpha \in (1, 2)$ there exists a measure-valued weak solution. However, for $\alpha \in (1, 5/3)$ there exists a function-valued weak solution, and in this case we further show that for $p \in (\alpha, 5/3)$ the uniform $p$-th moment in $L^p$-norm of the weak solution is finite, and that the weak solution is uniformly stochastic continuous in $L^p$ sense.

Keywords: Non-Lipschitz noise coefficients; Stochastic heat equations; Truncated $\alpha$-stable white noises; Uniform $p$-th moment; Uniform stochastic continuity.

MSC Classification (2020): Primary: 60H15; Secondary: 60F05, 60G17

1 Introduction

In this paper we study the existence of weak solution in probabilistic sense to the following non-linear stochastic heat equation

$$
\begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u(t, x)}{\partial x^2} + \varphi(u(t-, x))\dot{L}_\alpha(t, x), & (t, x) \in (0, \infty) \times (0, L), \\
\frac{\partial u(t, 0)}{\partial x} = \frac{\partial u(t, L)}{\partial x} = 0, & x \in \{0, L\}, \\
u(0, x) = u_0(x), & t \in [0, \infty),
\end{cases}
$$

where $L$ is an arbitrary positive constants, $\dot{L}_\alpha$ denotes a truncated $\alpha$-stable space-time white noise on $[0, \infty) \times [0, L]$ with $\alpha \in (1, 2)$, the noise coefficient $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis given below and the initial function $u_0$ is random and measurable.

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Before studying the equation of particular form (1.1), we first consider an SPDE

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u(t, x)}{\partial x^2} + G(u(t, x)) + H(u(t, x)) \dot{F}(t, x), \quad t \geq 0, x \in \mathbb{R},$$  \hspace{1cm} (1.2)

in which $G : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, $H : \mathbb{R} \to \mathbb{R}$ is continuous and $\dot{F}$ is a space-time white noise.

When $\dot{F}$ is a Gaussian white noise, there is a growing literature on SPDEs related to (1.2) such as the stochastic heat (parabolic) equations (see, e.g., Walsh [48] and Dalang et al. [14]), stochastic Burgers type equations (see, e.g., Bertini and Cancrini [8], Da Prato et al. [16]), SPDEs with reflection (see, e.g., Zhang [54]), Parabolic Anderson Model (see, e.g., Gärtner and Molchanov [26]), dissipative stochastic systems or reaction-diffusion stochastic equations (see, e.g., Bertini and Cancrini [8], Da Prato et al. [16]), SPDEs in Hilbert space (see, e.g., Da Prato and Zabczyk [18], Liu and Röckner [32–34] and references therein), etc. In particular, such a SPDE arises from super-processes (see, e.g., Konno and Shiga [29], Dawson [19], Perkins [40] and Ren et al. [44, 45] and references therein). For $G \equiv 0$ and $H(u) = \sqrt{u}$, the solution to (1.2) is the density field of a one-dimensional super-Brownian motion. For $H(u) = \sqrt{u(1-u)}$ (stepping-stone model in population genetics), Bo and Wang [7] considered a stochastic interacting model consisting of equations (1.2) and proved the existence of weak solution in probabilistic sense to the system by using a weak convergence argument.

In the case that $\dot{F}$ is a Gaussian colored noise that is white in time and colored in space, for continuous function $H$ satisfying the linear growth condition, Sturm [46] proved the existence of a pair $(u, F)$ satisfying (1.2), the so-called weak solution in probabilistic sense, by first establishing the existence and uniqueness of lattice systems of SDEs driven by correlated Brownian motions with non-Lipschitz diffusion coefficients that describe branching particle systems in random environment in which the motion process has a discrete Laplacian generator and the branching mechanism is affected by a colored Gaussian random field, and then applying an approximation procedure. Xiong and Yang [52] proved the existence of weak solution $(u, F)$ to (1.2) in a finite spatial domain with different boundary conditions by considering the weak limit of a sequence of approximating SPDEs of (1.2). They further proved the existence and uniqueness of the strong solution under additional Hölder continuity assumption on $H$.

If $\dot{F}$ is a Lévy noise with Lipschitz continuous coefficient $H$, there is also a growing literature on existence and uniqueness of solutions to SPDEs related to (1.2). See, e.g., Albeverio et al. [1], Hausenblas [27] and Röckner and Zhang [39] for Poisson white noise; Applebaum and Wu [2] for a Lévy space-time white noise that extended the results in [1]; Peszat and Zabczyk [42] for infinite dimensional Lévy processes; Truman and Wu [47], Dong and Xu [21], Wu and Xie [50] and Hausenblas and Giri [28] for stochastic Burgers type equations; Bo and Wang [6] for Stochastic Cahn-Hilliard partial differential equations; Brzeźniak et al. [9], Dong et al. [22] and Zhu et al. [54] for stochastic Navier-Stokes equations; Duñst et al. [23] for numerical approximations. Brzeźniak et al. [10] studied general SPDEs driven by Lévy processes in Hilbert space (excluding the Lévy space-time white noise) with locally monotone coefficients, and extended the results in [32]. Brzeźniak et al. [11] studied the existence of weak solution to stochastic reaction-diffusion equations driven by real-valued and general Banach-space-valued Lévy processes.

In particular, when $\dot{F}$ is an $\alpha$-stable white noise for $\alpha \in (0, 1) \cup (1, 2)$, Balan [4] studied SPDE (1.2) with $G \equiv 0$ and Lipschitz coefficient $H$ on a bounded domain in $\mathbb{R}^d$ with zero initial condition and Dirichlet boundary, and proved the existence of strong random field solution. The approach in [4] is to first solve the equation with truncated noise (by
removing the big jumps, the jumps size exceeds a fixed value $K$, from $\dot{F}$), yielding a solution $u_K$, and then show that for $N \geq K$ the solutions $u_N = u_K$ on the event $t \leq \tau_K$, where $\{\tau_K\}_{K \geq 1}$ is a sequence of stopping times which tends to infinity as $K$ tends to infinity. Such a localization method which is also applied in Peszat and Zabczyk [41] to show the existence of weak Hilbert-space-valued solution. For the stochastic heat equations driven by additive $l^2$-valued $\alpha$-stable processes, we refer to Priola and Zabczyk [43] and references therein.

For $\alpha \in (1, 2)$, Wang et al. [49] studied the existence and pathwise uniqueness of strong function-valued solution of (1.2) with Lipschitz coefficient $H$ using a localization method, and showed a comparison principle of solutions to such equation with different initial functions and drift coefficients. Yang and Zhou [53] found sufficient conditions on pathwise uniqueness of solutions to a class of SPDEs (1.2) driven by $\alpha$-stable white noise without negative jumps and with non-decreasing Hölder continuous noise coefficient $H$. But the existence of weak solution to (1.2) with general non-decreasing Hölder continuous noise coefficient is left open. For stochastic heat equations driven by general heavy-tailed noises with Lipschitz noise coefficients, we refer to Chong [13] and references therein.

When $G = 0$, $H(u) = u^\beta$ with $0 < \beta < 1$ (non-Lipschitz continuous) in (1.2) and $\dot{F}$ is an $\alpha$-stable ($\alpha \in (1, 2)$) white noise on $[0, \infty) \times \mathbb{R}$ without negative jumps, it is shown in Mytnik [38] that there exists a weak solution $(u, F)$ satisfying (1.2) by constructing a sequence of approximating processes that is tight with its limit solving the associated martingale problem, and that in the case of $\alpha \beta = 1$ the weak uniqueness of solution to (1.2) holds. The martingale problem approach in [38] depends primarily on the Laplace transform of $\alpha$-stable noise and super-process theory which requires the assumption of non-negativity of jumps. The pathwise uniqueness is shown in [53] for $\alpha \beta = 1$ and $1 < \alpha < \sqrt{5} - 1$.

For $\alpha$-stable colored noise $\dot{F}$ without negative jumps and with Hölder continuous coefficient $H$, Xiong and Yang [51] proved the existence of weak solution $(u, F)$ to (1.2) by showing the weak convergence of solutions to SDE systems on rescaled lattice with discrete Laplacian and driven by common stable random measure, which is similar to [46]. In both [46] and [51] the dependence of colored noise helps with establishing the existence of weak solution.

Inspired by work in the above mentioned literature, we are interested in the stochastic heat equation (1.1) in which the noise coefficient $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies the following more general hypothesis:

**Hypothesis 1.1.** $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function with globally linear growth.

We consider two types of weak solutions in probabilistic sense that are measure-valued and function-valued, respectively. In addition, we also study the uniform $p$-moment and uniform stochastic continuity of the weak solution to equation (1.1). In the case that $\varphi$ is Lipschitz continuous, the existence of the strong solution can be usually obtained by standard Picard iteration [12,48] or Banach fixed point principle [6,17,49]. We thus mainly consider the case that $\varphi$ is non-Lipschitz continuous. Since the classical approaches of Picard iteration and Banach fixed point principle fail for SPDE (1.1) with non-Lipschitz $\varphi$, to prove the existence of a weak solution $(u, L_\alpha)$ to (1.1), we first construct an approximating SPDE sequence with Lipschitz continuous noise coefficients $\varphi^n$ by using a convolution approximation, and give the existence and uniqueness of strong solutions to the approximating SPDEs. We then proceed to show that the sequence of solution is tight in appropriate spaces. Finally, we prove that there exists a weak solution of (1.1) by using a weak convergence procedure.

The main contribution of this paper is proving the existence and regularity of weak solutions to equation (1.1) under general continuity assumption on noise coefficients, which
Our results also generalize the result in [38] under truncated $\alpha$ setting to consider general non-Lipschitz continuous noise coefficients and noise with negative jumps. Form of noise coefficient and the assumption of non-negativity of jumps, such that we can directly construct a sequence of approximating SPDEs that does not rely on the specific assumption. In contrast to the martingale problem approach used in [38], our method aims to generalize those in many previous works [1, 2, 4, 6, 7, 13, 14, 41, 49] under Lipschitz continuity under further assumption $\alpha \in (1, 3/5)$. To the best of our knowledge, this result is new for the stochastic heat equation driven by truncated $\alpha$-stable space-time white noise with non-Lipschitz noise coefficients.

The rest of this paper is organized as follows. In the next section, we introduce some notation and the main theorems on the existence, uniform $p$-moment and uniform stochastic continuity of weak solution to (1.1). Section 3 is devoted to the proof of the existence of measure-valued weak solution to (1.1). In Section 4 for $\alpha \in (1, 5/3)$ we prove that there exists a weak solution to (1.1) as an $L^p$-valued process with $p \in (\alpha, 5/3)$, and that the weak solution has the finite uniform $p$-th moment and the uniform stochastic continuity in the $L^p$ norm with $p \in (\alpha, 5/3)$.

2 Notation and main results

2.1 Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, and let $N(dt, dx, dz) : [0, \infty) \times [0, L] \times \mathbb{R} \times \{0\} \to \mathbb{N} \cup \{0\} \cup \{\infty\}$ be a Poisson random measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with intensity measure $dt dx \nu_\alpha(dz)$, where $dt dx$ denotes the Lebesgue measure on $[0, \infty) \times [0, L]$ and the jump size measure $\nu_\alpha(dz)$ for $\alpha \in (1, 2)$ is given by

$$\nu_\alpha(dz) := (c_+ z^{-\alpha-1} 1_{[0,K]}(z) + c_- (z) - z) - \alpha - 1_{[-K,0]}(z)) dz,$$

where $c_+ + c_- = 1$ and $K > 0$ is an arbitrary constant. Define

$$\tilde{N}(dt, dx, dz) := N(dt, dx, dz) - dt dx \nu_\alpha(dz).$$

Then $\tilde{N}(dt, dx, dz)$ is the compensated Poisson random measure (martingale measure) on $[0, \infty) \times [0, L] \times \mathbb{R} \setminus \{0\}$. As in Balan [4, Section 5], define a martingale measure

$$L_\alpha(dt, dx) := \int_{\mathbb{R} \setminus \{0\}} z \tilde{N}(dt, dx, dz)$$

for $(t, x) \in [0, \infty) \times [0, L]$. Then the corresponding distribution-valued derivative $\{\tilde{L}_\alpha(t, x) : t \in [0, \infty), x \in [0, L]\}$ is a truncated $\alpha$-stable space-time white noise. Write $\mathcal{G}^\alpha$ for the class of almost surely $\alpha$-integrable random functions defined by

$$\mathcal{G}^\alpha := \left\{ f \in \mathbb{B} : \int_0^t \int_0^L |f(s, x)|^\alpha dx ds < \infty, \mathbb{P}\text{-a.s. for all } t \in [0, \infty) \right\},$$

where $\mathbb{B}$ is the space of progressively measurable functions on $[0, \infty) \times [0, L] \times \Omega$. Then it holds by [38, Section 5] that the stochastic integral with respect to $\{L_\alpha(dx, ds)\}$ is well defined for all $f \in \mathcal{G}^\alpha$. 

4
Throughout this paper, $C$ denotes the arbitrary positive constant whose value might vary from line to line. If $C$ depends on some parameters such as $p, T$, we denote it by $C_{p, T}$.

Let $G_t(x, y)$ be the fundamental solution of heat equation $\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$ on the domain $[0, \infty) \times [0, L] \times [0, L]$ with Dirichlet boundary conditions (the subscript $t$ is not a derivative but a variable). Its explicit formula (see, e.g., Feller [25, Page 341]) is given by

$$G_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left\{ \exp \left( -\frac{(y - x + 2k L)^2}{2t} \right) - \exp \left( -\frac{(y + x + 2k L)^2}{2t} \right) \right\}$$

for $t \in (0, \infty), x, y \in [0, L]$; and $\lim_{t \searrow 0} G_t(x, y) = \delta_y(x)$, where $\delta$ is the Dirac delta distribution. Moreover, it holds that for $s, t \in [0, \infty)$ and $x, y, z \in [0, L]$

$$G_t(x, y) = G_t(y, x), \quad \int_0^L |G_t(x, y)| \, dy + \int_0^L |G_t(x, y)| \, dx \leq C, \quad (2.3)$$

$$\int_0^L G_s(x, y) G_t(y, z) \, dy = G_{t+s}(x, z), \quad (2.4)$$

$$\int_0^L |G_t(x, y)|^p \, dy \leq C t^{-\frac{p+1}{2}}, \quad p \geq 1. \quad (2.5)$$

Given a topological space $V$, let $D([0, \infty), V)$ be the space of càdlàg paths from $[0, \infty)$ to $V$ equipped with the Skorokhod topology. For any $p \geq 1$, we denote by $v_t \equiv \{v(t, \cdot), t \in [0, \infty)\}$ the $L^p([0, L])$-valued process equipped with norm

$$||v_t||_p = \left( \int_0^L |v(t, x)|^p \, dx \right)^{\frac{1}{p}}.$$

For any $p \geq 1$ and $T > 0$, let $L^p_{loc}([0, \infty) \times [0, L])$ be the space of measurable functions $f$ on $[0, \infty) \times [0, L]$ such that

$$||f||_{p,T} = \left( \int_0^T \int_0^L |f(t, x)|^p \, dx \, dt \right)^{\frac{1}{p}} < \infty, \quad \forall \ 0 < T < \infty.$$

Let $\mathcal{S}([0, L])$ be the Schwartz space (the space of rapidly decreasing functions) on $[0, L]$. Let $B([0, L])$ be the space of all Borel functions on $[0, L]$, and let $\mathcal{M}([0, L])$ be the space of finite Borel measures on $[0, L]$ equipped with the weak convergence topology. For any $f \in B([0, L])$ and $\mu \in \mathcal{M}([0, L])$ define $\langle f, \mu \rangle := \int_0^L f(x) \mu(dx)$ whenever it exists. With a slight abuse of notation, for any $f, g \in B([0, L])$ we also denote by $\langle f, g \rangle = \int_0^L f(x) g(x) \, dx$.

**Remark 2.1.** There are two different approaches of in the study of SPDEs: the approach of martingale measure (see, e.g., Walsh [48]) and the approach of infinite dimensional process (see, e.g., Da Prato and Zabczyk [18] and Peszat and Zabczyk [42]). For a comparison of two approaches under the Gaussian setting, we refer to Dalang and Quer-Sardanyons [13] and references therein. In this paper, we adopt Walsh’s martingale measure treatment to the Lévy space-time white noises. The noise can also be represented by the time derivative of impulsive cylindrical Lévy process (infinite dimensional Lévy process) in Peszat and Zabczyk [42, Section 7.2].
2.2 Main results

By a solution to equation (1.1) we mean a process \( u_t \equiv \{u(t, \cdot), t \in [0, \infty)\} \) satisfying the following analytical weak form equation:

\[
\langle u_t, \psi \rangle = \langle u_0, \psi \rangle + \frac{1}{2} \int_0^t \langle u_s, \psi'' \rangle ds + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varphi(u(s-, x)) \psi(x) z \tilde{N}(ds, dx, dz) \quad (2.6)
\]

for all \( t \in [0, \infty) \) and for any \( \psi \in \mathcal{D}([0, L]) \) with \( \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0 \) or equivalently satisfying the following mild form equation:

\[
u(t, x) = \int_0^L G_t(x, y) u_0(y) dy + \int_0^t \int_0^L G_{t-s}(x, y) \varphi(u(s-, y)) z \tilde{N}(ds, dy, dz) \quad (2.7)
\]

for all \( t \in [0, \infty) \) and for a.e. \( x \in [0, L] \), where the last terms in above equations follow from (2.2). For the equivalence between the weak form equation (2.6) and mild form equation (2.7), we refer to Yang and Zhou [53, Proposition 2.2] or Li [30, Theorem 7.26] and references therein. We first give the definition of a weak solution in probabilistic sense to equation (1.1).

**Definition 2.2.** Stochastic heat equation (1.1) has a weak solution in probabilistic sense with initial function \( u_0 \) if there exists a pair \((u, L_\alpha)\) defined on some filtered probability space such that \( L_\alpha \) is a truncated \( \alpha \)-stable martingale measure on \([0, \infty) \times [0, L]\) and \((u, L_\alpha)\) satisfies either equation (2.6) or equation (2.7).

We now state the main theorems in this paper. The first theorem is on the existence of weak solution in \( D([0, \infty), \mathbb{M}([0, L])) \cap L^p_{loc}([0, \infty) \times [0, L]) \) with \( p \in (\alpha, 2] \) that was first considered in Mytnik [38].

**Theorem 2.3.** If the initial function \( u_0 \) satisfies \( \mathbb{E}[||u_0||_p] < \infty \) for some \( p \in (\alpha, 2] \), then under Hypothesis 1.1 there exists a weak solution \((\hat{u}, L_\alpha)\) to equation (1.1) defined on a filtered probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})\) such that

(i) \( \hat{u} \in D([0, \infty), \mathbb{M}([0, L])) \cap L^p_{loc}([0, \infty) \times [0, L]); \)

(ii) \( L_\alpha \) is a truncated \( \alpha \)-stable martingale measure with the same distribution as \( L_\alpha \).

Moreover, for any \( T > 0 \) we have

\[
\hat{\mathbb{E}} \left[ ||\hat{u}||_{p,T}^p \right] = \hat{\mathbb{E}} \left[ \int_0^T ||\hat{u}_t||_p^p dt \right] < \infty. \quad (2.8)
\]

The proof of Theorem 2.3 is deferred to Section 3.

Under additional assumption on \( \alpha \), we can show that there exists a weak solution in \( D([0, \infty), L^p([0, L])) \), \( p \in (\alpha, 5/3) \) with better regularity.

**Theorem 2.4.** Suppose that \( \alpha \in (1, 5/3) \). If the initial function \( u_0 \) satisfies \( \mathbb{E}[||u_0||_p] < \infty \) for some \( p \in (\alpha, 5/3) \), then under Hypothesis 1.1 there exists a weak solution \((\hat{u}, L_\alpha)\) to equation (1.1) defined on a filtered probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})\) such that

(i) \( \hat{u} \in D([0, \infty), L^p([0, L])); \)

(ii) \( L_\alpha \) is a truncated \( \alpha \)-stable martingale measure with the same distribution as \( L_\alpha \).
Furthermore, for any $T > 0$ we have the following uniform $p$-moment and uniform stochastic continuity, that is,

$$
\hat{E} \left[ \sup_{0 \leq t \leq T} \| \hat{u}_t \|^p \right] < \infty,
$$

and that for each $0 \leq h \leq \delta$

$$
\lim_{\delta \to 0} \hat{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \| \hat{u}_{t+h} - \hat{u}_t \|^p \right] = 0.
$$

The proof of Theorem 2.4 is deferred to Section 4.

**Remark 2.5.** Regarding the regularity of the weak solution obtained in Theorem 2.4, we note that it is stochastically continuous or continuous in probability, even though it is not pathwise continuous.

Finally, we end up with this section with discussions of the main results in the following remarks.

**Remark 2.6.** Note that the globally linear growth of $\varphi$ in Hypothesis 1.1 guarantees the global existence of weak solutions. One can remove this condition if one only needs the existence of a weak solution up to the explosion time. On the other hand, the uniqueness of the solution to equation (1.1) is still an open problem because $\varphi$ is non-Lipschitz continuous and the driven noise is white.

**Remark 2.7.** The weak solutions of equation (1.1) in Theorems 2.3 and 2.4 are proved by showing the tightness of the approximating solution sequence $(u^n)_{n \geq 1}$ of equation (3.1); see Propositions 3.7 and 4.7 in Sections 3 and 4, respectively. To show that the equation (1.1) has a function-valued weak solution, it is necessary to restrict $\alpha \in (1, 5/3)$ due to a technical reason that Doob’s maximal inequality cannot be directly applied to show the uniform $p$-moment estimate of $(u^n)_{n \geq 1}$ that is key to the proof of the tightness for $(u^n)_{n \geq 1}$. To this end, we apply the factorization method in Lemma 3.3 for transforming the stochastic integral such that the uniform $p$-moment of $(u^n)_{n \geq 1}$ can be obtained. In order to remove this restriction and consider the case of $\alpha \in (1, 2)$, we apply another tightness criteria, i.e., Lemma 3.5, to show the tightness of $(u^n)_{n \geq 1}$. However, the weak solution of equation (1.1) is a measure-valued process in this situation. We also note that the existence of function-valued weak solution of equation (1.1) in the case of $\alpha \in [5/3, 2)$ is still an unsolved problem.

**Remark 2.8.** If we remove the restriction of the bounded jumps for the $\alpha$-stable white noise $\hat{L}_\alpha$ in equation (1.1), the jump size measure $\nu_\alpha(dz)$ in (2.7) becomes

$$
\nu_\alpha(dz) = (c_+ z^{-\alpha-1})_{0, \infty}(z) + (c_- (-z)^{-\alpha-1})_{(-\infty, 0)}(z)dz
$$

for $\alpha \in (1, 2)$ and $c_+ + c_- = 1$. As in Wang et al. [49] Lemma 3.1 we can construct a sequence of truncated $\alpha$-stable white noise $\hat{L}_\alpha^K$ with the jump size measure given by (2.7) and a sequence of stopping times $(\tau_K)_{K \geq 1}$ such that

$$
\lim_{K \to +\infty} \tau_K = \infty, \; \mathbb{P}\text{-a.s.}.
$$

Similar to equation (1.1), for given $K \geq 1$, we can consider the following non-linear stochastic heat equation

$$
\begin{align*}
\frac{\partial u_K(t, x)}{\partial t} &= \frac{1}{2} \frac{\partial^2 u_K(t, x)}{\partial x^2} + \varphi(u_K(t-, x)) \hat{L}_\alpha^K(t, x), \quad (t, x) \in (0, \infty) \times (0, L), \\
u_K(0, x) &= u_0(x), \quad x \in [0, L], \\
u_K(t, 0) &= u_K(t, L) = 0, \quad t \in [0, \infty).
\end{align*}
$$

(2.12)
If \( \varphi \) is Lipschitz continuous, similar to the proof of Proposition 3.3 in Wang et al. [43; Proposition 3.2] one can show that there exists a unique strong solution \( u_K = \{u_K(t, \cdot), t \in [0, \infty)\} \) to equation (2.12) by using the Banach fixed point principle. On the other hand, by Wang et al. [43, Lemma 3.4], it holds for each \( K \leq N \) that

\[
u_K = u_N \text{ \( \mathbb{P} \)-a.s. on} \{t < \tau_K\}.
\]

By setting

\[
u = u_K, \quad 0 \leq t < \tau_K,
\]

and by the fact (2.11), we obtain the strong (weak) solution \( \nu \) to equation (1.1) with noise

of unbounded jumps via letting \( K \uparrow +\infty \).

If \( \varphi \) is non-Lipschitz continuous, for any \( K \geq 1 \), Theorem 2.3 or Theorem 2.4 shows that there exists a weak solution \((\hat{u}_K, \hat{L}^K_\alpha)\) to equation (2.12) defined on a filtered probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})_K\). However, we can not show that for each \( K \leq N \)

\[
(\hat{u}_K, \hat{L}^K_\alpha) = (\hat{u}_N, \hat{L}^N_\alpha) \text{ \( \mathbb{P} \)-a.s. on} \{t < \tau_K\}
\]
due to the non-Lipschitz continuity of \( \varphi \). Therefore, we do not know whether there exists a common probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})\) on which all of the weak solutions \((\hat{u}_K, \hat{L}^K_\alpha))_{K \geq 1}\) are defined. Hence, the localization method in Wang et al. [49] becomes invalid, and the existence of the weak solution to equation (1.1) with untruncated \( \alpha \)-stable noise remains an unsolved problem.

**Remark 2.9.** It is not difficult to consider the stochastic heat equation (1.1) with Dirichlet boundary conditions on a bounded domain, which is given as follows:

\[
\left\{ \begin{array}{ll}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u(t, x)}{\partial x^2} + \varphi(u(t-, x))\hat{L}_\alpha(t, x), & (t, x) \in (0, \infty) \times \partial, \\
u(0, x) = u_0(x), & x \in \partial, \\
u(t, x) = 0, & (t, x) \in [0, \infty) \times \partial \partial,
\end{array} \right.
\]

where \( \partial \) is a bounded domain of \( \mathbb{R}^d (d \geq 2) \) with boundary \( \partial \partial \) of class \( C^3 \). The mild form of the above equation is given by

\[
u(t, x) = \int_\partial \hat{G}_t(x, y)\nu_0(y)dy + \int_0^t \int_\partial \hat{G}_{t-s}(x, y)\varphi(u(s-, y))\hat{L}_\alpha(ds, dy)
\]

for all \( t \geq 0 \) and for a.e. \( x \in \partial \), where \( \hat{G}_t(x, y) \) is the heat kernel on \( (0, \infty) \times \partial \times \partial \) with Dirichlet boundary conditions and \( \lim_{t \downarrow 0} \hat{G}_t(x, y) = \delta_x(y) \). In this situation, we can still construct a sequence of approximating SPDEs with noise coefficients \( \varphi^n \) given by Lemma 3.1 in Section 3. Under the assumption \( \alpha \in (1, \min\{2, 1+2/d\}) \) and \( p \in (\alpha, \min\{2, 1+2/d\}) \), one can show that for each \( n \geq 1 \) there exists a pathwise unique strong solution \( \nu^n \) of the approximating SPDE, see, e.g., Balan [4] and Wang et al. [49] for more details. With the approximating solution sequence in hand, one can follow the same procedure in this paper to prove that Theorem 2.3 holds for equation (2.13) under the assumption \( \alpha \in (1, \min\{2, 1+2/d\}) \) and \( p \in (\alpha, \min\{2, 1+2/d\}) \), and that Theorem 2.4 holds for equation (2.14) under the assumption \( \alpha \in (1, 1+2/(2+d)) \) and \( p \in (\alpha, 1+2/(2+d)) \). We omit the details because they are the same as the proof in this paper for \( d = 1 \), except replacing the heat kernel estimates (2.3)-(2.5) by

\[
\hat{G}_t(x, y) = \hat{G}_t(y, x), \quad \int_\partial |\hat{G}_t(x, y)|dy + \int_\partial |\hat{G}_t(x, y)|dx \leq C,
\]
\[
\int_{\mathcal{O}} \hat{G}_s(x,y)\hat{G}_t(y,z) dy = \hat{G}_{t+s}(x,z),
\]
and
\[
\int_{\mathcal{O}} |\hat{G}_t(x,y)|^p dy \leq C t^{-\frac{d(p-1)}{2}}, \quad p \geq 1, \quad d \geq 2.
\]

For more general kernel estimates, we refer to Peszat and Zabczyk [42, Theorem 2.6] and references therein.

3 Proof of Theorem 2.3

The proof of Theorem 2.3 proceeds in the following three steps. We first construct a sequence of the approximating SPDEs with globally Lipschitz continuous noise coefficients \((\varphi^n)_{n \geq 1}\) using a convolution approximation; see Lemma 3.1, and show that for each fixed \(n \geq 1\) there exists a unique strong solution \(u^n\) in \(D([0, \infty), L^p([0, L])\) with \(p \in (\alpha, 2]\) of the approximating SPDE; see Proposition 3.3. We then prove that the approximating solution sequence \((u^n)_{n \geq 1}\) is tight in both \(D([0, \infty), M([0, L]))\) and \(L^p_{\text{loc}}([0, \infty) \times [0, L])\) for all \(p \in (\alpha, 2]\); see Proposition 3.7. Finally, we proceed to show that there exists a weak solution \((\hat{u}, \hat{L}_\alpha)\) to equation (1.1) defined on another probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}})\) by applying a weak convergence argument.

To construct a sequence of the approximating SPDEs with Lipschitz continuous noise coefficients, we first use the heat kernel
\[
P_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t > 0, \ x \in \mathbb{R}
\]
to smooth the continuous function \(\varphi\) in the following lemma. This technique is also applied in Xiong and Yang [52] under the Gaussian colored noise setting.

Lemma 3.1. For any \(n \geq 1\) define
\[
\varphi^n(x) := \int_{\mathbb{R}} P_{1/n}(x - y) \left[ (\varphi(y) \wedge n) \vee (-n) \right] dy, \quad x \in \mathbb{R}.
\]

Then \(\lim_{n \to \infty} \varphi^n(x) = \varphi(x)\) for all \(x \in \mathbb{R}\), and \(\varphi^n\) satisfies Lipschitz condition for any fixed \(n \geq 1\), i.e., there exists a constant \(C_n\) such that
\[
|\varphi^n(x_1) - \varphi^n(x_2)| \leq C_n |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}.
\]

Proof. The first conclusion follows from the fact \(\lim_{n \to \infty} P_{1/n} = \delta_x\) and the second one is obtained by the fact \(\varphi^n \in C^\infty(\mathbb{R})\). \(\square\)

For each fixed \(n \geq 1\), we construct the approximate SPDE of the form
\[
\begin{cases}
\frac{\partial u^n(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2 u^n(t, x)}{\partial x^2} + \varphi^n(u^n(t-, x)) \hat{L}_\alpha(t, x), & (t, x) \in (0, \infty) \times (0, L), \\
u^n(0, x) = u_0(x), & x \in [0, L], \\
u^n(t, 0) = u^n(t, L) = 0, & t \in [0, \infty),
\end{cases}
\]
where the noise coefficient \(\varphi^n\) is given by Lemma 3.1.
Given $n \geq 1$, by a solution to equation (3.1) we mean a process $u^n \equiv \{u^n(t, \cdot), t \in [0, \infty)\}$ satisfying the following analytical weak form equation:

$$
\langle u^n_t, \psi \rangle = \langle u_0, \psi \rangle + \frac{1}{2} \int_0^t \langle u^n_s, \psi'' \rangle ds + \int_0^{t+} \int_{\mathbb{R}\setminus\{0\}} \varphi^n(u^n(s-, x))\psi(x)z\tilde{N}(ds,dx,dz)
$$

(3.2)

for all $t \in [0, \infty)$ and for any $\psi \in \mathcal{D}([0, L])$ with $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0$ or equivalently satisfying the following mild form equation:

$$
u^n(t, x) = \int_0^L G_t(x, y)u_0(y)dy + \int_0^{t+} \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(x, y)\varphi^n(u^n(s-, y))z\tilde{N}(ds,dy,dz)
$$

(3.3)

for all $t \in [0, \infty)$ and for a.e. $x \in [0, L]$.

We now present the definition (see also in Wang et al. [49]) of a strong solution to stochastic heat equation (3.1).

**Definition 3.2.** Given $p \geq 1$, the stochastic heat equation (3.1) has a strong solution in $D([0, \infty), L^p([0, L]))$ with initial function $u_0$ if for a given truncated $\alpha$-stable martingale measure $L_\alpha$ there exists a process $u^n \equiv \{u^n(t, \cdot), t \in [0, \infty)\}$ in $D([0, \infty), L^p([0, L]))$ such that either equation (3.2) or equation (3.3) holds.

Note that for each $n \geq 1$ the noise coefficient $\varphi^n$ is not only Lipschitz continuous but also of globally linear growth. Indeed, for a given $\epsilon > 0$ and $n_0 \in \mathbb{N}$ large enough, Lemma 3.1 and the globally linear growth of $\varphi$ imply that

$$
|\varphi^n(x)| \leq |\varphi^n(x) - \varphi(x)| + |\varphi(x)| \leq \epsilon + C(1 + |x|), \quad \forall n \geq n_0, \forall x \in \mathbb{R}.
$$

(3.4)

Therefore, we can use the classical Banach fixed point principle to show the existence and pathwise uniqueness of the strong solution to equation (3.1). Since the proof is standard, we just state the main result in the following proposition. For more details of the proof, we refer to Wang et al. [49] Proposition 3.2] and references therein. Also note that the same method was applied in Truman and Wu [47] and in Bo and Wang [6] where the stochastic Burgers equation and the stochastic Cahn-Hilliard equation driven by Lévy space-time white noise were studied, respectively.

**Proposition 3.3.** For a given $n \geq 1$, if the initial function $u_0$ satisfies $\mathbb{E}[||u_0||_p^p] < \infty$ for some $p \in (\alpha, 2]$, then there exists a pathwise unique strong solution $u^n_0 \equiv \{u^n(t, \cdot), t \in [0, \infty)\}$ to equation (3.1) such that for any $T > 0$

$$
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E} \left[||u^n_t||_p^p \right] < \infty.
$$

(3.5)

**Remark 3.4.** By Lemma 3.1, (3.4) and estimate (3.3), the stochastic integral on the right-hand side of (3.3) is well defined.
Lemma 3.5. Given a complete and separable metric space $E$, let $(X^n = \{X^n(t), t \in [0, \infty) \})_{n \geq 1}$ be a sequence of stochastic processes with sample paths in $D([0, \infty), E)$, and let $C_a$ be a subalgebra and dense subset of $C_b(E)$ (the bounded continuous functions space on $E$). Then the sequence $(X^n)_{n \geq 1}$ is tight in $D([0, \infty), E)$ if both of the following conditions hold:

(i) For every $\varepsilon > 0$ and $T > 0$ there exists a compact set $\Gamma_{\varepsilon,T} \subset E$ such that

$$\inf_{n \geq 1} \mathbb{P}[X^n(t) \in \Gamma_{\varepsilon,T} \text{ for all } t \in [0, T]] \geq 1 - \varepsilon. \quad (3.6)$$

(ii) For each $f \in C_a$, there exists a process $g_n \equiv \{g_n(t), t \in [0, \infty) \}$ such that

$$f(X^n(t)) - \int_0^t g_n(s)ds$$

is an $({\mathcal F}_t)$-martingale and

$$\sup_{0 \leq t \leq T} \mathbb{E} \| f(X^n(t)) + |g_n(t)| \| < \infty \quad (3.7)$$

and

$$\sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T |g_n(t)|^q dt \right)^{\frac{1}{q}} \right] < \infty \quad (3.8)$$

for each $T > 0$ and $q > 1$.

Before showing the tightness of solution sequence $(u^n)_{n \geq 1}$, we first find a uniform moment estimate in the following lemma.

Lemma 3.6. For each $n \geq 1$ let $u^n$ be the strong solution to equation (3.2) given by Proposition 3.3. Then for given $T > 0$ and $\psi \in \mathcal{S}([0, L])$ with $\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0$, we have for $p \in (\alpha, 2]$ that

$$\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^L u^n(t, x)\psi(x)dx \right|^p \right] < \infty. \quad (3.9)$$

Proof. By (3.2), it holds that for each $n \geq 1$

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^L u^n(t, x)\psi(x)dx \right|^p \right] \leq C_p(A_1 + A_2 + A_3),$$

where

$$A_1 = \mathbb{E} \left[ \left| \int_0^L u_0(x)\psi(x)dx \right|^p \right],$$

$$A_2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_0^L u^n(s, x)\psi''(x)dxds \right|^p \right],$$

$$A_3 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^{t+} \int_0^L \varphi^n(u^n(s, x))\psi(x)dz d\bar{N}(ds, dx, dz) \right|^p \right].$$
For \( p \in (\alpha, 2] \) we separately estimate \( A_1, A_2 \) and \( A_3 \) as follows. For \( A_1 \), it holds by Hölder’s inequality that
\[
A_1 \leq C_p \left( \int_0^L |\psi(x)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \mathbb{E} \left[ \int_0^L |u_0(x)|^p \, dx \right] \leq C_p \mathbb{E}[||u_0||_p^p] \leq C_p
\]
due to \( \psi \in \mathcal{S}([0, L]) \) and \( \mathbb{E}[||u_0||_p^p] < \infty \).

For \( A_2 \), it holds by \( \psi \in \mathcal{S}([0, L]) \) and Hölder’s inequality that
\[
\begin{align*}
A_2 & \leq C_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \int_0^L u^n(s, x) \psi(x) \, dx \, ds \right|^{\frac{p}{2}} \right] \\
& \leq C_p \mathbb{E} \left[ \sup_{0 \leq t \leq s} \left| \int_0^T \int_0^L u^n(r, x) \psi(x) \, dx \, dr \right|^{\frac{p}{2}} \right] ds.
\end{align*}
\]

For \( A_3 \), the Doob maximal inequality and the Burkholder-Davis-Gundy inequality imply that
\[
\begin{align*}
A_3 & \leq C_p \mathbb{E} \left[ \int_0^T \int_0^L \int_{\mathbb{R} \setminus \{0\}} |\varphi^n(u^n(s, \cdot)) \psi(x) z|^{\frac{p}{2}} N(ds, dx, dz) \right] \\
& \leq C_p \mathbb{E} \left[ \int_0^T \int_0^L \int_{\mathbb{R} \setminus \{0\}} |\varphi^n(u^n(s, \cdot)) \psi(x) z|^{\frac{p}{2}} N(ds, dx, dz) \right] \\
& = C_p \mathbb{E} \left[ \int_0^T \int_0^L \int_{\mathbb{R} \setminus \{0\}} |\varphi^n(u^n(s, \cdot)) \psi(x) z|^{\frac{p}{2}} ds dx \nu_\alpha(dz) \right],
\end{align*}
\]
where the second inequality follows from the fact that
\[
\left| \sum_{i=1}^k a_i \right|^{\frac{p}{2}} \leq \sum_{i=1}^k |a_i|^q
\]
for \( a_i \in \mathbb{R}, k \geq 1, \) and \( q \in (0, 2] \). By (2.1), it holds that for \( p > \alpha \)
\[
\int_{\mathbb{R} \setminus \{0\}} |z|^p \nu_\alpha(dz) = c_+ \int_0^K z^{p-\alpha-1} \, dz + c_- \int_{-K}^0 (-z)^{p-\alpha-1} \, dz = \frac{K^{p-\alpha}}{p-\alpha},
\]
then there exists a constant \( C_{p,K,\alpha} \) such that
\[
A_3 \leq C_{p,K,\alpha} \mathbb{E} \left[ \int_0^T \int_0^L |\varphi^n(u^n(s, x)) \psi(x)|^{p} \, ds \, dx \right].
\]

By (3.4), \( \psi \in \mathcal{S}([0, L]) \) and (3.5) in Proposition 3.3 it is easy to see that
\[
A_3 \leq C_{p,K,\alpha, T} \left( 1 + \sup_{0 \leq s \leq T} \mathbb{E}[||u^n||_p^p] \right) \leq C_{p,K,\alpha, T}.
\]

Combining the estimates \( A_1, A_2 \) and \( A_3 \), we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^L u^n(t, x) \psi(x) \, dx \right|^p \right] \leq C_{p,K,\alpha, T} + C_{p,T} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \left| \int_0^L u^n(r, x) \psi(x) \, dx \right|^p \right] ds
\]
Therefore, it holds by Gronwall’s lemma that for \( p \in (\alpha, 2] \)
\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^L u^n(t, x) \psi(x) \, dx \right|^p \right] < \infty,
\]
which completes the proof. \( \square \)
Note that for any function \( v \in L^q([0, L]) \) with \( q \geq 1 \) one can identify \( L^q([0, L]) \) as a subset of \( \mathcal{M}([0, L]) \) (the space of finite Borel measures on \([0, L]\)) using the following correspondence
\[
v(x) \mapsto v(x) dx.
\]
Then for each \( n \geq 1 \) one can identify the solution \( u^n \) of equation (3.1) as a \( \mathcal{M}([0, L]) \)-valued solution (still denoted by \( u^n \)). Let \( \mathcal{S}([0, L]) \) be the Schwartz space (the space of rapidly decreasing functions) on \([0, L]\). For each \( n \geq 1 \) and \( \psi \in \mathcal{S}([0, L]) \) with \( \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0 \), let \( \langle u^n, \psi \rangle = \{ (u^n_t, \psi, t \geq 0) \} \) be a real-valued process. Then by (3.2) we have for any \( t \geq 0 \) that
\[
\langle u^n_t, \psi \rangle = \langle u_0, \psi \rangle + \frac{1}{2} \int_0^t \langle u^n_s, \psi'' \rangle ds + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varphi^n (u^n(s, x)) \psi(x) z \tilde{N}(ds, dx, dz).
\]
Since the last term of the above equation is a martingale having a càdlàg modification, it is not difficult to see that \( \langle (u^n, \psi) \rangle_{n \geq 1} \) is a sequence of processes whose sample paths are in \( D([0, \infty), \mathbb{R}) \), which implies that \( (u^n)_{n \geq 1} \) is a sequence of processes whose sample paths are in \( D([0, \infty), \mathcal{M}([0, L])) \), see, e.g., Li [30, Section 12.3] and references therein. We now show the tightness of \( (u^n)_{n \geq 1} \) in the following proposition.

**Proposition 3.7.** The solution sequence \( (u^n)_{n \geq 1} \) to equation (3.1) given by Proposition 3.3 is tight in both \( D([0, \infty), \mathcal{M}([0, L])) \) and \( L^p_{\text{loc}}([0, \infty) \times [0, L]) \) for \( p \in (\alpha, 2] \). Let \( u \) be an arbitrary limit point of \( u^n \). Then
\[
u \in D([0, \infty), \mathcal{M}([0, L])) \cap L^p_{\text{loc}}([0, \infty) \times [0, L]) \tag{3.12}
\]
for \( p \in (\alpha, 2] \).

**Proof.** We first prove that the sequence \( \langle (u^n, \psi) \rangle_{n \geq 1} \) is tight in \( D([0, \infty), \mathbb{R}) \) by using Lemma 3.5. It is easy to see that the condition (i) in Lemma 3.5 can be verified by Lemma 3.6. In the following we mainly verify the condition (ii) in Lemma 3.5.

For each \( f \in C_b^2(\mathbb{R}) \) \( (f, f', f'') \) are bounded and uniformly continuous) with compact supports, it holds by (3.2) and Itô’s formula (see, e.g., [3] Theorem 4.4.7) that
\[
f(\langle u^n, \psi \rangle) = f(\langle u_0, \psi \rangle) + \int_0^t f'(\langle u^n_s, \psi \rangle) \langle u^n_s, \psi'' \rangle ds
\]
\[+ \int_0^t \int_{\mathbb{R} \setminus \{0\}} \mathcal{D}(\langle u^n_s, \psi \rangle, \varphi^n (u^n(s, x)) \psi(x) z) ds dx \mu_{\alpha}(dz)
\]
\[+ \int_0^t \int_{\mathbb{R} \setminus \{0\}} \{ f(\langle u^n_s, \psi \rangle) + \varphi^n (u^n(s, x)) \psi(x) z \} - f(\langle u^n_s, \psi \rangle) \tilde{N}(ds, dx, dz),
\]
where \( \mathcal{D}(u, v) = f(u + v) - f(u) - vf'(u) \) for \( u, v \in \mathbb{R} \) and the last term is a martingale.

Since \( f, f', f'' \) are bounded and \( \psi \in \mathcal{S}([0, L]) \), then
\[
|f'(\langle u^n_s, \psi \rangle) \langle u^n_s, \psi'' \rangle| \leq C \left| \int_0^L u^n(s, x) \psi(x) dx \right|.
\]

By Taylor’s formula, one can show that \( |\mathcal{D}(u, v)| \leq C(|v| \wedge |v|^2) \), which also implies that \( |\mathcal{D}(u, v)| \leq C(|v| \wedge |v|^p) \) for \( p \in (\alpha, 2] \). Thus we have for \( p \in (\alpha, 2] \),
\[
\int_0^L \int_{\mathbb{R} \setminus \{0\}} |\mathcal{D}(\langle u^n_s, \psi \rangle, \varphi^n (u^n(s, x)) \psi(x) z)| dx \mu_{\alpha}(dz)
\]
\[ \leq C \left( \int_{\mathbb{R} \setminus \{0\}} |z| \wedge |z|^p \nu_\alpha(dz) \right) \int_0^L (|\varphi^n(u^n(s,x))\psi(x)| + |\varphi^n(u^n(s,x))\psi(x)|^p)dx \]
\[ \leq C_{p,K,\alpha} \int_0^L (|\varphi^n(u^n(s,x))\psi(x)| + |\varphi^n(u^n(s,x))\psi(x)|^p)dx, \tag{3.15} \]

where by (2.1),

\[ \int_{\mathbb{R} \setminus \{0\}} |z| \wedge |z|^p \nu_\alpha(dz) = c_+ \int_0^1 z^{p-\alpha-1}dz + c_- \int_{-1}^0 (-z)^{p-\alpha-1}dz + c_+ \int_1^K z^{-\alpha}dz \]
\[ + c_- \int_{-K}^{-1} (-z)^{-\alpha}dz \]
\[ = \frac{1}{p - \alpha} + \frac{1 - K^{1-\alpha}}{\alpha - 1} \leq C_{p,K,\alpha}. \]

For given \( n \geq 1 \) let us define

\[ g_n(s) := f'(\langle u^n_s, \psi \rangle) \langle u^n_s, \psi' \rangle) + \int_0^L \int_{\mathbb{R} \setminus \{0\}} \mathcal{D}(\langle u^n_s, \psi \rangle, \varphi^n(u^n(s,x))\psi(x)s)dx \nu_\alpha(dz). \]

By (3.14), it is easy to see that

\[ f(\langle u^n_t, \psi \rangle) - \int_0^t g_n(s)ds \]

is an \((\mathcal{F}_t)\)-martingale.

Now we verify the moment estimates (3.7) and (3.8) of the condition (ii) in Lemma 3.5. For each \( t \in [0, T] \), it holds by the boundedness of \( f \), estimates (3.14), (3.15) and (3.4) that

\[ \mathbb{E} [|f(\langle u^n_t, \psi \rangle) + |g_n(t)|] \leq C \left( 1 + \mathbb{E} \left[ \left| \int_0^L u^n(t,x)\psi(x)dx \right| \right] \right) \]
\[ + C_{p,K,\alpha} \mathbb{E} \left[ \int_0^L (|\psi(x)| + |u^n(t,x)\psi(x)|)dx \right] \]
\[ + C_{p,K,\alpha} \mathbb{E} \left[ \int_0^L (|\psi(x)|^p + |u^n(t,x)\psi(x)|^p)dx \right]. \]

Since \( \psi \in \mathcal{D}([0, L]) \) implies that \( \psi \) is bounded, then it holds by Hölder’s inequality and (3.9) that for \( p \in (\alpha, 2] \)

\[ \mathbb{E} [|f(\langle u^n_t, \psi \rangle) + |g_n(t)|] \leq C_{p,K,\alpha} \left( 1 + \left( \sup_{0 \leq t \leq T} \mathbb{E}[||u^n_t||_p^p] \right)^{\frac{1}{p}} + \sup_{0 \leq t \leq T} \mathbb{E}[||u^n_t||_p^p] \right), \]

and so by (3.5),

\[ \sup_{0 \leq t \leq T} \mathbb{E} [|f(\langle u^n_t, \psi \rangle) + |g_n(t)|] < \infty, \]

which verifies the estimate (3.7).

To verify (3.8), it suffices to show that for each \( n \geq 1 \)

\[ \mathbb{E} \left[ \int_0^T |g_n(t)|^q dt \right] < \infty \]

14
for some \( q > 1 \). By the estimates (3.14) - (3.15) and (3.4), we have
\[
\mathbb{E} \left[ \int_0^T |g_n(t)|^q dt \right] \leq C_q \mathbb{E} \left[ \int_0^T \left( \int_0^L u^n(t, x)\psi(x)dx \right)^q dt \right] \\
+ C_{p,q,K,\alpha} \mathbb{E} \left[ \int_0^T \left( \int_0^L (|\psi(x)| + |u^n(t, x)\psi(x)|)dx \right)^q dt \right] \\
+ C_{p,q,K,\alpha} \mathbb{E} \left[ \int_0^T \left( \int_0^L (|\psi(x)|^p + |u^n(t, x)\psi(x)|^p)dx \right)^q dt \right].
\]

Taking \( 1 < q < 2/p \), the Hölder inequality and boundedness of \( \psi \) imply that
\[
\mathbb{E} \left[ \int_0^T |g_n(t)|^q dt \right] \leq C_{p,q,K,\alpha,T} \left( 1 + \sup_{0 \leq t \leq T} \mathbb{E}||u^n_t||_q^q + \sup_{0 \leq t \leq T} \mathbb{E}||u^n_t||_{pq}^q \right),
\]
and so by (3.35),
\[
\sup_{n \geq 1} \mathbb{E} \left[ \int_0^T |g_n(t)|^q dt \right] < \infty,
\]
which verifies the estimate (3.8).

Therefore, for each \( \psi \in \mathcal{S}'([0, L]) \) with \( \psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0 \), the sequence \( (u^n, \psi^n)_{n \geq 1} \) is tight in \( D([0, \infty), \mathbb{R}) \), and so it holds by Mitoma’s theorem (see, e.g., Mitoma 3.7 and Walsh 4.8 pp.361–365) that \( (u^n)_{n \geq 1} \) is tight in \( D([0, \infty), \mathbb{M}([0, L])) \).

On the other hand, by (3.35) we have for each \( T > 0 \)
\[
\sup_{n \geq 1} \mathbb{E} \left[ \int_0^T \int_0^L |u^n(t, x)|^p dx dt \right] \leq C_T \sup_{n \geq 1} \mathbb{E}[||u^n_t||_p^p] < \infty
\]
for \( p \in (\alpha, 2) \). The Markov’s inequality implies that for each \( \varepsilon > 0, T > 0 \) there exists a constant \( C_{\alpha,T} \) such that
\[
\sup_{n \geq 1} \mathbb{P} \left[ \int_0^T \int_0^L |u^n(t, x)|^p dx dt > C_{\alpha,T} \right] < \varepsilon
\]
for \( p \in (\alpha, 2) \). Therefore, the sequence \( (u^n)_{n \geq 1} \) is also tight in \( L^p_{loc}([0, \infty) \times [0, L]) \) for \( p \in (\alpha, 2) \), and the conclusion (3.12) holds.

**Proof of Theorem 2.3** We are going to prove Theorem 2.3 by applying weak convergence arguments. For each \( n \geq 1 \), let \( u^n \) be the strong solution of equation (3.1) given by Proposition 3.3. It can also be regarded as an element in \( D([0, \infty), \mathbb{M}([0, L])) \) with the Skorokhod topology. By Proposition 3.7 there exists a \( D([0, \infty), \mathbb{M}([0, L])) \cap L^p_{loc}([0, \infty) \times [0, L]) \)-valued random variable \( u \) such that \( u^n \) converges to \( u \) in distribution in \( D([0, \infty), \mathbb{M}([0, L])) \cap L^p_{loc}([0, \infty) \times [0, L]) \) for \( p \in (\alpha, 2) \). On the other hand, the Skorokhod Representation Theorem (see, e.g., Either and Kurtz 2.17 Theorem 3.1.8) yields that there exists another filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and on it a further subsequence \( (\hat{u}^n)_{n \geq 1} \) and \( \hat{u} \) which have the same distribution as \( (u^n)_{n \geq 1} \) and \( u \), so that \( \hat{u}^n \) almost surely converges to \( \hat{u} \) in \( D([0, \infty), \mathbb{M}([0, L])) \cap L^p_{loc}([0, \infty) \times [0, L]) \) for \( p \in (\alpha, 2) \).

For each \( t \geq 0, n \geq 1 \) and any test function \( \psi \in \mathcal{S}'([0, L]) \) with \( \psi(0) = \psi(L) = 0 \) and \( \psi'(0) = \psi'(L) = 0 \), let us define
\[
\hat{M}_t^n(\psi) := \int_0^L \hat{u}^n(t, x)\psi(x)dx - \int_0^L \hat{u}_0(x)\psi(x)dx - \frac{1}{2} \int_0^t \int_0^L \hat{u}^n(s, x)\psi''(x)dxds.
\]
Since \( \hat{u}^n \) almost surely converges to \( \hat{u} \) in the Skorokhod topology as \( n \to \infty \), then

\[
\hat{M}_t^n(\psi) \xrightarrow{P.a.s.} \int_0^L \hat{u}(t,x)\psi(x)dx - \int_0^L \hat{u}_0(x)\psi(x)dx - \frac{1}{2} \int_0^t \int_0^L \hat{u}(s,x)\psi''(x)dxds
\]

(3.16)
in the Skorokhod topology as \( n \to \infty \).

By (3.2) and the fact that \( \hat{u}^n \) has the same distribution as \( u^n \) for each \( n \geq 1 \), we have

\[
\hat{M}_t^n(\psi) \overset{D}{=} \int_0^L u^n(t,x)\psi(x)dx - \int_0^L u_0(x)\psi(x)dx - \frac{1}{2} \int_0^t \int_0^L u^n(s,x)\psi''(x)dxds
\]

\[
= \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \psi(x)\varphi^n(u^n(s,x))z\tilde{N}(ds,dx,dz),
\]

where \( \overset{D}{=} \) denotes the identity in distribution. The Burkholder-Davis-Gundy inequality, (3.10) - (3.11) and (3.4) imply that for \( p \in (\alpha, 2] \)

\[
\mathbb{E}[||\hat{M}_t^n(\psi)||_p^p] = \mathbb{E} \left[ \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \psi(x)\varphi^n(u^n(s,x))z\tilde{N}(ds,dx,dz) \right]^p
\]

\[
\leq C_p\mathbb{E} \left[ \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} |\psi(x)|^p(1+|u^n(s,x)|)^p|z|^pdsdx\nu_{\alpha}(dz) \right]
\]

\[
\leq C_{p,K,\alpha,T} \left( \int_0^t |\psi(x)|^pdx + \sup_{x \in [0,L]} |\psi(x)| \sup_{0 \leq t \leq T} \mathbb{E} \left[ ||u^n_t||_{\alpha}^p \right] \right).
\]

Then by \( \psi \in \mathcal{S}([0,L]) \) and (3.5), we have for each \( T > 0 \)

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}[||\hat{M}_t^n(\psi)||_p] < \infty.
\]

Therefore, it holds by (3.16) that there exists an \( (\tilde{\mathcal{F}}_t) \)-martingale \( \hat{M}_t(\psi) \) such that \( \hat{M}_t^n(\psi) \) converges weakly to \( \hat{M}_t(\psi) \) as \( n \to \infty \), and for each \( t \geq 0 \)

\[
\hat{M}_t(\psi) = \int_0^L \hat{u}(t,x)\psi(x)dx - \int_0^L \hat{u}_0(x)\psi(x)dx - \frac{1}{2} \int_0^t \int_0^L \hat{u}(s,x)\psi''(x)dxds.
\]

(3.17)

By Hypothesis 1.1 and Lemma 3.1 the quadratic variation of \( \{\hat{M}_t^n(\psi), t \in [0, \infty)\} \) satisfies that

\[
\langle \hat{M}^n(\psi), \hat{M}^n(\psi) \rangle_t = \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \varphi^n(u^n(s,x))^2\psi(x)^2z^2dsdx\nu_{\alpha}(dz)
\]

\[
\overset{D}{=} \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \varphi^n(\hat{u}^n(s,x))^2\psi(x)^2z^2dsdx\nu_{\alpha}(dz)
\]

\[
\overset{P \rightarrow a.s.}{\underset{n \to \infty}{\longrightarrow}} \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \varphi(\hat{u}(s,x))^2\psi(x)^2z^2dsdx\nu_{\alpha}(dz), \ t \in [0,T],
\]

as \( n \to \infty \). We define by \( \{\langle \hat{M}(\psi), \hat{M}(\psi) \rangle_t, t \in [0, \infty)\} \) the quadratic variation process

\[
\langle \hat{M}(\psi), \hat{M}(\psi) \rangle_t := \int_0^t \int_0^L \int_{\mathbb{R}\{0\}} \varphi(\hat{u}(s,x))^2\psi(x)^2z^2dsdx\nu_{\alpha}(dz), \ t \geq 0.
\]
As in Konno and Shiga [29, Lemma 2.4] or Mytnik [38, Lemma 5.7], \((\hat{M}(\psi), \hat{M}(\psi))_t\) corresponds to an orthonormal martingale measure \(\hat{M}(dt, dx, dz)\) defined on the filtered probability space \((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{P})\) in the sense of Walsh [48, Chapter 2] whose quadratic measure is given by

\[
\varphi(\hat{u}(t, x)) z^2 dt dx \nu_\alpha(dz).
\]

Let \(\{\hat{L}_\alpha(t, x) : t \in [0, \infty), x \in [0, L]\}\) be another truncated \(\alpha\)-stable white noise, defined possibly on \((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{P})\), independent of \(\hat{M}(dt, dx, dz)\) and define

\[
\hat{L}_\alpha(t, \psi) := \int_0^{t+} \int_0^{L} \int_{\mathbb{R} \setminus \{0\}} \frac{1}{\varphi(\hat{u}(s-, x))} 1_{\{\varphi(\hat{u}(s-, x)) \neq 0\}} \psi(x) z \hat{M}(ds, dx, dz)
\]

\[
+ \int_0^{t+} \int_0^{L} \psi(x) 1_{\{\varphi(\hat{u}(s-, x)) = 0\}} \hat{L}_\alpha(ds, dx).
\]

Then \(\{\hat{L}_\alpha(t, x) : t \in [0, \infty), \psi \in \mathcal{S}([0, L]), \psi(0) = \psi(L) = 0, \psi'(0) = \psi'(L) = 0\}\) determines a truncated \(\alpha\)-stable white noise \(\hat{L}_\alpha(t, x)\) on \((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{P})\) with the same distribution as \(\hat{L}_\alpha(t, x)\) such that

\[
\hat{M}_0(\psi) = \int_0^{t+} \int_0^{L} \varphi(\hat{u}(s-, x)) \psi(x) \hat{L}_\alpha(ds, dx) = \int_0^{t+} \int_0^{L} \int_{\mathbb{R} \setminus \{0\}} \varphi(\hat{u}(s-, x)) \psi(x) z \hat{N}(ds, dx, dz),
\]

where \(\hat{N}(dt, dx, dz)\) denotes the compensated Poisson random measure associated to the truncated \(\alpha\)-stable martingale measure \(\hat{L}_\alpha(t, x)\). Hence, it holds by (3.17) that \((\hat{u}, \hat{L}_\alpha)\) is a weak solution in probabilistic sense to equation (1.1) defined on \((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{P})\).

On the other hand, since \(\hat{u}^n\) has the same distribution as \(u^n\) for each \(n \geq 1\), then the moment estimates (3.5) in Proposition 3.3 can be replaced by

\[
\sup_{n \geq 1} \sup_{0 \leq t \leq T} \hat{E} \left[||\hat{u}_t^n||^p_p\right] < \infty
\]

for \(p \in (\alpha, 2]\). For moment estimate (2.8), the Fatou’s Lemma implies that for \(p \in (\alpha, 2]\)

\[
\hat{E} \left[||\hat{u}_T^n||^p_p\right] = \hat{E} \left[\int_0^T ||\hat{u}_t^n||^p_p dt\right] \leq \liminf_{n \to \infty} C_T \sup_{0 \leq t \leq T} \hat{E} \left[||\hat{u}_t^n||^p_p\right] < \infty,
\]

which completes the proof.

\[\square\]

4 Proof of Theorem 2.4

The proof of Theorem 2.4 is similar to that of Theorem 2.3. The main difference between them is that in the current proof we need to prove the solution sequence \((u^n)_{n \geq 1}\) to equation (3.1), obtained from Proposition 3.3, is tight in \(D([0, \infty), L^p([0, L]))\) for \(p \in (\alpha, 5/3]\). To this end, we need the following tightness criteria; see, e.g., Ethier and Kurtz [24, Theorem 3.8.6 and Remark (a)]. Note that the same criteria was also applied in Sturm [46] with Gaussian colored noise setting.

Lemma 4.1. Given a complete and separable metric space \((E, \rho)\), let \((X^n)\) be a sequence of stochastic processes with sample paths in \(D([0, \infty), E)\). The sequence is tight in \(D([0, \infty), E)\) if the following conditions hold:
(i) For every $\varepsilon > 0$ and rational $t \in [0, T]$, there exists a compact set $\Gamma_{\varepsilon, T} \subset E$ such that

$$\inf_n \mathbb{P}[X^n(t) \in \Gamma_{\varepsilon, T}] \geq 1 - \varepsilon. \quad (4.1)$$

(ii) There exists $p > 0$ such that

$$\limsup_{\delta \to 0} \mathbb{E} \left[ \sup_n \sup_{0 \leq t \leq T} (\rho(X^n_{t+\delta}, X^n_t) \wedge 1)^p \right] = 0. \quad (4.2)$$

To verify condition (i) of Lemma 4.1, we need the following characterization of the relatively compact set in $L^p([0, L])$, $p \geq 1$; see, e.g., Sturm [46, Lemma 4.3].

Lemma 4.2. A subset $\Gamma \subset L^p([0, L])$ for $p \geq 1$ is relatively compact if and only if the following conditions hold:

(a) $\sup_{f \in \Gamma} \int_0^L |f(x)|^p dx < \infty$,

(b) $\lim_{y \to 0} \int_0^L |f(x + y) - f(x)|^p dx = 0$ uniformly for all $f \in \Gamma$,

(c) $\lim_{\gamma \to \infty} \int_{(L - \frac{\gamma}{2}, L]} |f(x)|^p dx = 0$ for all $f \in \Gamma$.

The proof of the tightness of $(u^n)_{n \geq 1}$ is accomplished by verifying conditions (i) and (ii) in Lemma 4.1. To this end, we need some estimates on $(u^n)_{n \geq 1}$, that is, the uniform bound estimate in Lemma 4.3, the temporal difference estimate in Lemma 4.5, and the spatial difference estimate in Lemma 4.6, respectively.

Lemma 4.3. Suppose that $\alpha \in (1, 5/3)$ and for each $n \geq 1$ $u^n$ is the solution to equation (3.1) given by Proposition 3.3. Then for given $T > 0$ there exists a constant $C_{p,K,\alpha,T}$ such that

$$\sup_n \mathbb{E} \left[ \sup_{0 \leq t \leq T} ||u^n_t||_p \right] \leq C_{p,K,\alpha,T}, \text{ for } p \in (\alpha, 5/3). \quad (4.3)$$

Proof. For each $n \geq 1$, by (3.3) it is easy to see that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} ||u^n_t||_p \right] \leq C_p(A_1 + A_2),$$

where

$$A_1 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^L G_t(\cdot, y) u_0(y) dy \right|^p \right], \quad A_2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^{t+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} G_{t-s}(\cdot, y) \varphi_n(u^n(s-, y)) z \tilde{N}(ds, dy, dz) \right|^p \right].$$

We separately estimate $A_1$ and $A_2$ as follows. For $A_1$, it holds by Young’s convolution inequality and (2.3) that

$$A_1 \leq C \mathbb{E} \left[ \int_0^L \sup_{0 \leq t \leq T} \left( \int_0^L |G_t(x, y)| dx \right) |u_0(y)|^p dy \right] \leq C_T \mathbb{E}[||u_0||_p^p].$$

By Proposition 3.3, we have $\mathbb{E}[||u_0||_p^p] < \infty$ for $p \in (\alpha, 2]$, and so there exists a constant $C_{p,T}$ such that $A_1 \leq C_{p,T}$. 

18
For $A_2$, we use the factorization method; see, e.g., Da Prato et al.\cite{17}, which is based on the fact that for $0 < \beta < 1$ and $0 \leq s \leq t$,

$$
\int_s^t (t - r)^{\beta - 1} (r - s)^{-\beta} dr = \frac{\pi}{\sin(\beta \pi)}.
$$

For any function $v : [0, \infty) \times [0, L] \to \mathbb{R}$ define

$$
\mathcal{J}^\beta v(t, x) := \frac{\sin(\beta \pi)}{\pi} \int_0^t \int_0^L (t - s)^{\beta - 1} G_{t-s}(x, y) v(s, y) dy ds,
$$

$$
\mathcal{J}^\beta_n v(t, x) := \int_0^{t+} \int_0^L (t - s)^{-\beta} G_{t-s}(x, y) \varphi^n(v(s, y)) z \tilde{N}(ds, dy, dz).
$$

By the stochastic Fubini Theorem and (2.4), we have

$$
\mathcal{J}^\beta \mathcal{J}^\beta_n u^n(t, x) = \frac{\sin(\beta \pi)}{\pi} \int_0^t \int_0^L (t - s)^{\beta - 1} G_{t-s}(x, y) \left( \int_0^{s+} \int_0^L (s - r)^{-\beta} \right.
$$

$$
\times G_{s-r}(y, m) \varphi^n(u^n(r-, m)) z \tilde{N}(dr, dm, dz) \left) dy ds
$$

$$
= \frac{\sin(\beta \pi)}{\pi} \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} \left( \int_0^L G_{t-s}(x, y) G_{s-r}(y, m) dy \right) ds \right. \varphi^n(u^n(r-, m)) z \tilde{N}(dr, dm, dz)
$$

$$
= \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} G_{t-s}(x, y) \varphi^n(u^n(s-, y)) z \tilde{N}(ds, dy, dz).
$$

Thus,

$$
A_2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \mathcal{J}^\beta \mathcal{J}^\beta_n u^n_t \right\|^p \right].
$$

Until the end of the proof we fix a $0 < \beta < 1$ satisfying

$$
1 - \frac{1}{p} < \beta < \frac{3}{2p} - \frac{1}{2}, \quad \text{(4.4)}
$$

which requires that

$$
\frac{3}{2p} - \frac{1}{2} - \left(1 - \frac{1}{p}\right) > 0.
$$

Therefore, we need the assumption $p < 5/3$ for this lemma.

Back to our main proof, to estimate $A_2$ we first estimate $\mathbb{E}[\left\| \mathcal{J}^\beta_n u^n_t \right\|^p]$. For $p \in (\alpha, 5/3)$, the Burkholder-Davis-Gundy inequality (for fixed $t$ and varying $s$), (3.10) + (3.11) and (3.4) imply that

$$
\mathbb{E}[\left\| \mathcal{J}^\beta_n u^n_t \right\|^p] = \int_0^L \mathbb{E} \left[ \left\| \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} (t - s)^{\beta} G_{t-s}(x, y) \varphi^n(u^n(s-, y)) z \tilde{N}(ds, dy, dz) \right\|^p \right] dx
$$

$$
\leq C_p \int_0^L \int_0^t \int_0^L \int_{\mathbb{R} \setminus \{0\}} \mathbb{E} \left[ \left\| (t - s)^{-\beta} G_{t-s}(x, y) \varphi^n(u^n(s, y)) z \right\|^p \nu_\alpha(dz) dy ds dx
$$

$$
\leq C_{p,K,\alpha} \int_0^L \int_0^t \int_0^L \mathbb{E} \left[ 1 + |u^n(s, y)|^p \right] (t - s)^{-\beta p} G_{t-s}(x, y) \nu_\alpha(dz) dy ds dx.
$$
Therefore, it follows from (4.6), (4.7), and the Hölder inequality that

\[ \mathbb{E}[||J_n^p u^n||_p^p] \leq C_{p,K,a} \left( L + \mathbb{E} \left[ \sup_{0 \leq s \leq t} ||u^n_s||_p^p \right] \right) \int_0^T s^{-(\frac{p-1}{2} + \beta)p} ds. \]

For \( p < 5/3 \), by (4.5) we have

\[ \int_0^T s^{-(\frac{p-1}{2} + \beta)p} ds < \infty. \]

Therefore, there exists a constant \( C_{p,K,a,T} \) such that

\[ \mathbb{E} \left[ ||J_n^p u^n||_p^p \right] \leq C_{p,K,a,T} \left( 1 + \mathbb{E} \left[ \sup_{0 \leq t \leq T} ||u^n_t||_p^p \right] \right). \] (4.5)

We now estimate \( A_2 = \mathbb{E}[\sup_{0 \leq t \leq T} ||J_n^p u^n||_p^p] \). The Minkowski inequality implies that

\[
A_2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{\sin(\pi \beta)}{\pi} \left( \int_0^t \int_0^L (t - s)^{\beta-1} G_{t-s}(\cdot, y) J_n^p u^n(s, y) dy ds \right)^p \right] \\
\leq \frac{\sin(\pi \beta)}{\pi} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t (t - s)^{\beta-1} \left( \int_0^L G_{t-s}(\cdot, y) J_n^p u^n(s, y) dy \right)^p ds \right)^{\frac{1}{p}} \right]. \] (4.6)

By the Hölder inequality and (2.3), we have

\[
\left\| \int_0^L G_{t-s}(\cdot, y) J_n^p u^n(s, y) dy \right\|_p^p \\
= \left( \int_0^L \left( \int_0^L |G_{t-s}(x, y)|^{\frac{p}{\beta}}} |G_{t-s}(x, y)|^{\frac{1}{\beta}}} J_n^p u^n(s, y) dy \right)^{\frac{p}{\beta}}} dx \right)^{\frac{1}{\beta}}} \\
\leq \left( \int_0^L \left( \int_0^L |G_{t-s}(x, y)| dy \right)^{\frac{p}{\beta}}} \left( \int_0^L G_{t-s}(x, y) |J_n^p u^n(s, y)|^{\beta} dy \right)^{\frac{1}{\beta}}} dx \right)^{\frac{1}{\beta}}} \\
\leq \left( \sup_{x \in [0, L]} \int_0^L |G_{t-s}(x, y)| dy \right)^{\frac{p}{\beta}}} \left( \int_0^L \int_0^L \left| G_{t-s}(x, y) \right| J_n^p u^n(s, y) dy \right)^{\frac{1}{\beta}}} dx dy \right)^{\frac{1}{\beta}}} \\
\leq C_T ||J_n^p u^n||_p. \] (4.7)

Therefore, it follows from (4.6), (4.7), and the Hölder inequality that

\[
A_2 \leq \frac{\sin(\pi \beta)}{\pi} C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t (t - s)^{\beta-1} ||J_n^p u^n||_p^p ds \right)^p \right] \\
\leq \frac{\sin(\pi \beta)}{\pi} C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t s^{\frac{p-1}{\beta} - 1} ds \right)^p \left( \int_0^t (t - s)^{\beta-1} ||J_n^p u^n||_p^p ds \right)^p \right] \\
\leq \frac{\sin(\pi \beta)}{\pi} C_{p,T} \int_0^T (T - s)^{(\beta-1)p} \mathbb{E} ||J_n^p u^n||_p^p ds. \]

By (4.5), it also holds that

\[
A_2 \leq \frac{\sin(\pi \beta)}{\pi} C_{p,K,a,T} \int_0^T (T - s)^{(\beta-1)p} \left( 1 + \mathbb{E} \left[ \sup_{0 \leq t \leq T} ||u^n_t||_p^p \right] \right) ds. \]
\[
\lesssim \frac{\sin(\pi \beta)C_{p,K_0,T}}{\pi} \left(1 + \int_0^T (T-s)^{(\beta-1)p} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \| u^n_t \|^p \right] ds \right). \tag{4.8}
\]

Combining (4.8) and the estimate for \( A_1 \), we have for each \( T > 0 \),
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \| u^n_t \|^p \right] \leq C_{p,T} + \frac{\sin(\pi \beta)C_{p,K_0,T}}{\pi} \int_0^T (T-s)^{(\beta-1)p} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \| u^n_r \|^p \right] ds.
\]

Since \( \beta > 1 - 1/p \), applying a generalized Gronwall’s Lemma (see, e.g., Lin [31] Theorem 1.2]), we have
\[
\sup_n \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| u^n_t - u^n \|^p \right] \leq C_{p,K_0,T}, \quad \text{for } p \in (\alpha, 5/3),
\]
which completes the proof.

\[\square\]

**Remark 4.4.** Note that one can estimate term \( A_2 \) in the proof of Lemma 4.3 using the Kotelenez inequality or similar maximal inequalities. For more details, we refer to Marinelli et al. [33] and references therein.

**Lemma 4.5.** Suppose that \( \alpha \in (1, 5/3) \) and for each \( n \geq 1 \) \( u^n \) is the solution to equation (3.1) given by Proposition 3.3. Then for given \( T > 0 \), \( 0 \leq h \leq \delta \) and \( p \in (\alpha, 5/3) \)
\[
\lim_{\delta \to 0} \sup_n \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| u^n_{t+h} - u^n_t \|^p \right] = 0. \tag{4.9}
\]

**Proof.** For each \( n \geq 1 \), by the factorization method in the proof of Lemma 4.3, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \| u^n_{t+h} - u^n_t \|^p \right] \leq C_p(B_1 + B_2),
\]
where
\[
B_1 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_0^L (G_{t+h}(\cdot - y) - G_t(\cdot - y)) u_0(y) dy \right\|^p \right],
\]
\[
B_2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \| \mathcal{J}_\beta f_\beta u^n_{t+h} - \mathcal{J}_\beta f_\beta u^n_t \|^p \right].
\]

For \( B_1 \), Young’s convolution inequality and (2.3) imply that
\[
B_1 \leq \mathbb{E} \left[ \int_0^L \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_0^L |G_{t+h}(x, y) + |G_t(x, y)| |dx \right) u_0(y) |dy \right] \leq C_T \mathbb{E} \left[ \| u_0 \|^p \right] < \infty.
\]

Therefore, it holds by Lebesgue’s dominated convergence theorem that \( B_1 \) converges to 0 as \( \delta \to 0 \).

For \( B_2 \), it is easy to see that
\[
B_2 \leq \frac{\sin(\pi \beta)C_p}{\pi} (B_{2,1} + B_{2,2} + B_{2,3}),
\]
where
\[
B_{2,1} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_0^t \int_0^L (t-s)^{\beta-1}(G_{t+h-s}(\cdot, y) - G_{t-s}(\cdot, y)) f_\beta u^n(s, y) dy ds \right\|^p \right],
\]
\[
B_{2,2} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_0^L (G_{t+h}(\cdot - y) - G_t(\cdot - y)) f_\beta u^n_t(y) dy \right\|^p \right],
\]
\[
B_{2,3} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_0^L (G_{t+h}(\cdot - y) - G_t(\cdot - y)) f_\beta u^n_t(y) dy \right\|^p \right].
\]
By the Hölder inequality and (4.10) we have for $B_{2,2} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_0^t \int_0^L ((t + h - s)^{\beta - 1} - (t - s)^{\beta - 1}) G_{t+h-s}(\cdot, y) \mathcal{J}_\beta^n u^n(s, y) dy ds \right\|_p \right]^p$, and $B_{2,3} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left\| \int_t^{t+h} \int_0^L (t + h - s)^{\beta - 1} G_{t+h-s}(\cdot, y) \mathcal{J}_\beta^n u^n(s, y) dy ds \right\|_p \right]^p$.

By the assumption $p \in (\alpha, 5/3)$ of this lemma we can choose a $0 < \beta < 1$ satisfying $1 - 1/p < \beta < 3/2p - 1/2$. By Lemma 4.3 and (4.5), there exists a constant $C_{p,K,\alpha,T}$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E} \left[ ||\mathcal{J}_\beta^n u^n||_p^p \right] \leq C_{p,K,\alpha,T}. \quad (4.10)$$

To estimate $B_{2,1}$, we set $G_t^h(x, y) = G_{t+h}(x, y) - G_t(x, y)$. Similar to the estimates for (4.6) and (4.7) in the proof of Lemma 4.3, we have

$$B_{2,1} \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_0^t (t - s)^{\beta - 1} \left( \sup_{x \in [0,L]} \int_0^L G_{t-s}^h(x, y) dy \right) \right)^{p-1} \right].$$

It also follows from the Hölder inequality and (4.10) that

$$B_{2,1} \leq C_{p,T} \sup_{0 \leq t \leq T} \mathbb{E} \left[ ||\mathcal{J}_\beta^n u^n||_p^p \right] \sup_{0 \leq h \leq \delta} \left( \int_0^T s^{(\beta - 1)p} \left( \sup_{x \in [0,L]} \int_0^L G_{t-s}^h(x, y) dy \right) ds \right)^{p-1} \leq C_{p,K,\alpha,T} \sup_{0 \leq h \leq \delta} \left( \int_0^T s^{(\beta - 1)p} \left( \sup_{x \in [0,L]} \int_0^L G_{t-s}^h(x, y) dy \right) ds \right)^{p-1} \cdot$$

Moreover, since $\beta > 1 - 1/p$, it holds by (2.3) that

$$\int_0^T s^{(\beta - 1)p} \left( \sup_{x \in [0,L]} \int_0^L G_{t-s}^h(x, y) dy \right) ds \leq \int_0^T s^{(\beta - 1)p} \left( \sup_{x \in [0,L]} \left( \int_0^L |G_{t+h-s}(x, y)| dy + \int_0^L |G_{t-s}(x, y)| dy \right) \right) \leq C_{p,T} \int_0^T s^{(\beta - 1)p} ds < \infty.$$

Thus, Lebesgue’s Dominated Convergence Theorem implies that $B_{2,1}$ converges to 0 as $\delta \to 0$.

For $B_{2,2}$, the Minkowski inequality and Young’s convolution inequality imply that

$$B_{2,2} \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_0^t ((t + h - s)^{\beta - 1} - (t - s)^{\beta - 1}) \left| \int_0^L G_{t+h-s}(\cdot, y) \mathcal{J}_\beta^n u^n(s, y) dy \right| ds \right)^p \right] \leq C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_0^t ((t + h - s)^{\beta - 1} - (t - s)^{\beta - 1}) ||\mathcal{J}_\beta^n u^n||_p ds \right)^p \right].$$

By the Hölder inequality and (4.10) we have for $\beta > 1 - 1/p$,

$$B_{2,2} \leq C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \int_0^t ((t + h - s)^{\beta - 1} - (t - s)^{\beta - 1}) ||\mathcal{J}_\beta^n u^n||_p ds \right]$$
\begin{align*}
&\leq C_{p,T} \sup_{0 \leq t \leq T} \mathbb{E}[\|\mathcal{F}_s^n u_t^n\|^p] \int_0^T |(s + \delta)^{\beta - 1} - s^{(\beta - 1)}|^p ds \\
&\leq C_{p,K,a,T} \int_0^T |(s + \delta)^{\beta - 1} - s^{(\beta - 1)}|^p ds < \infty.
\end{align*}

Therefore, by Lebesgue’s dominated convergence theorem, we know that $B_{2,2}$ converges to 0 as $\delta \to 0$.

For $B_{2,3}$, similar to $B_{2,2}$, we get

\begin{align*}
B_{2,3} &\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_t^{t+h} \left( t + h - s \right)^{\beta - 1} \left| \int_0^L G_{t+h-s}(\cdot, y) \mathcal{F}_s^n u_t^n(s,y) dy \right| ds \right)^p \right] \\
&\leq C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_t^{t+h} \left| (t + h - s)^{\beta - 1} \int_0^L \mathcal{F}_s^n u_t^n(s,y) dy \right| ds \right)^p \right] \\
&\leq C_{p,T} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq h \leq \delta} \left( \int_t^{t+h} \left| (t + h - s)^{\beta - 1} \int_0^L \mathcal{F}_s^n u_t^n(s,y) dy \right| ds \right)^p \right] \\
&\leq C_{p,T} \mathbb{E}[\|\mathcal{F}_s^n u_t^n\|^p] \sup_{0 \leq h \leq \delta} \int_t^{t+h} |(t + h - s)^{\beta - 1}| ds \leq C_{p,K,a,T} \int_0^{\delta} s^{(\beta - 1)} ds. \tag{4.11}
\end{align*}

Since $\beta > 1 - 1/p$, we can conclude that the right-hand side of (4.11) converges to 0 as $\delta \to 0$. Therefore, by the estimates of $B_{2,1}, B_{2,2}, B_{2,3}$ and $B_1$, the desired result (4.9) holds, which completes the proof.

\textbf{Lemma 4.6.} For each $n \geq 1$ let $u^n$ be the solution to equation (3.1) given by Proposition 3.3. Then for given $t \in [0, \infty)$, $0 \leq |x_1| \leq \delta$ and $p \in (\alpha, 2]$

\begin{equation}
\lim_{\delta \to 0} \sup_{n} \mathbb{E} \left[ \sup_{|x_1| \leq \delta} \|u^n(t, \cdot + x_1) - u^n(t, \cdot)\|^p \right] = 0. \tag{4.12}
\end{equation}

\textbf{Proof.} Since the shift operator is continuous in $L^p([0, L])$, then for each $n \geq 1$ and $\delta > 0$ there exists a pathwise $x_1^{n,\delta}(t) \in \mathbb{R}$ such that $|x_1^{n,\delta}(t)| \leq \delta$ and

\begin{equation*}
\sup_{|x_1| \leq \delta} \|u^n(t, \cdot + x_1) - u^n(t, \cdot)\|^p = \|u^n(t, \cdot + x_1^{n,\delta}(t)) - u^n(t, \cdot)\|^p.
\end{equation*}

As before, it is easy to see that

\begin{equation*}
\mathbb{E}[\|u^n(t, \cdot + x_1^{n,\delta}(t)) - u^n(t, \cdot)\|^p] \leq C_p(C_1 + C_2),
\end{equation*}

where

\begin{align*}
C_1 &= \mathbb{E} \left[ \left| \int_0^L (G_t(\cdot + x_1^{n,\delta}(t), y) - G_t(\cdot, y)) u_0(y) dy \right|^p \right], \\
C_2 &= \mathbb{E} \left[ \left| \int_0^{t+} \int_0^L \int_{\mathbb{R}\setminus\{0\}} (G_{t-s}(\cdot + x_1^{n,\delta}(t), y) - G_{t-s}(\cdot, y)) y \mathcal{F}_s^n u^n(s-, y)) \mathcal{F}_s^n (\cdot, y) dy ds \right|^p \right].
\end{align*}

For $C_1$, Young’s convolution inequality and (2.3) imply that

\begin{equation*}
C_1 \leq \mathbb{E} \left[ \left( \int_0^L |G_t(x + x_1^{n,\delta}(t), y)| + |G_t(x, y)| \right) u_0(y)^p dy \right] \leq C_T \mathbb{E}[\|u_0\|^p] < \infty.
\end{equation*}
Thus, the Lebesgue dominated convergence theorem implies that $C_1$ converges to 0 as $\delta \to 0$.

For $C_2$, it follows from the Burkholder-Davis-Gundy inequality, (3.11), (3.13) and (2.3) that for $p \in (0, \alpha]$ and $\delta > 0$,

$$C_2 = \int_0^L E \left[ \int_0^{t+} \int_0^{L-x} \int_{\mathbb{R} \setminus \{0\}} (G_{t-s}(x+x_1^n, y) - G_{t-s}(x, y) + \nu^n(u^n(t, x) - u^n(t, y))zN(ds, dy, dz) \right] dx$$

$$\leq C_p \int_0^L \int_0^t \int_0^{L-x} \int_{\mathbb{R} \setminus \{0\}} E[(1 + |u^n(s, y)|)^p][(G_{t-s}(x+x_1^n, y) - G_{t-s}(x, y))^{p|dsdydx}$$

$$\leq C_{p,K,\alpha} \int_0^L \int_0^t \int_0^{L-x} \int_{\mathbb{R} \setminus \{0\}} E[(1 + |u^n(s, y)|)^p][(G_{t-s}(x+x_1^n, y) - G_{t-s}(x, y))^{p|dsdydx}$$

$$\leq C_{p,K,\alpha} \left( \int_0^L \int_0^t |G_{t-s}(x+x_1^n, y) - G_{t-s}(x, y)|^p ds dy dx \right) \left( L + \sup_{0 \leq s \leq t} E[|u^n_s|^p] \right)$$

$$\leq C_{p,K,\alpha} \left( \int_0^L \int_0^t (|G_{t-s}(x+x_1^n, y)|^p + |G_{t-s}(x, y)|^p) ds dy dx \right) \left( L + \sup_{0 \leq s \leq t} E[|u^n_s|^p] \right)$$

$$\leq C_{p,K,\alpha} \left( \int_0^L (t-s)^{\frac{p-1}{2}} ds \right) \left( L + \sup_{0 \leq s \leq t} E[|u^n_s|^p] \right)$$

Therefore, it holds by (3.5) and Lebesgue’s dominated convergence theorem that $C_2$ converges to 0 as $\delta \to 0$. Hence, by the estimates of $C_1$ and $C_2$, we obtain

$$\lim_{\delta \to 0} \sup_n E \left[ \sup_{|x_1| \leq \delta} ||u^n(t, \cdot + x_1) - u^n(t, \cdot)||^p_p \right] = 0,$$

which completes the proof. $\square$

**Proposition 4.7.** Suppose that $\alpha \in (1, 5/3)$. The sequence of solutions $(u^n)_{n \geq 1}$ to equation (3.7) given by Proposition 3.3 is tight in $D([0, \infty), L^p([0, L]))$ for $p \in (\alpha, 5/3)$.

**Proof.** From (3.5) and Markov’s inequality, for each $\varepsilon > 0$, $p \in (\alpha, 2]$ and $T > 0$ there exists a $N \in \mathbb{N}$ such that

$$\sup_n P \left[ ||u^n_n||^p_p > N \right] \leq \varepsilon, \quad t \in [0, T].$$

Let $\Gamma_{\varepsilon, T}$ be a closed set defined by

$$\Gamma_{\varepsilon, T} := \{ u_t \in L^p([0, L]) : ||u_t||^p_p \leq N, t \in [0, T] \}. \quad (4.13)$$

By Lemma 4.6 and Markov’s inequality, it holds that for each $\varepsilon > 0$, $p \in (\alpha, 2]$ and $T > 0$

$$\lim_{\delta \to 0} \sup_n P \left[ \sup_{|x_1| \leq \delta} ||u^n(t, \cdot + x_1) - u^n(t, \cdot)||^p_p > \varepsilon \right] = 0, \quad t \in [0, T].$$

Then for $k \in \mathbb{N}$ we can choose a sequence $(\delta_k)_{k \geq 1}$ with $\delta_k \to 0$ as $k \to \infty$ such that

$$\sup_n P \left[ \sup_{|x_1| \leq \delta_k} ||u^n(t, \cdot + x_1) - u^n(t, \cdot)||^p_p > \frac{1}{k} \right] \leq \varepsilon, \quad t \in [0, T].$$
Let $\Gamma_{\varepsilon,T}^2$ be a closed set defined by

$$\Gamma_{\varepsilon,T}^2 := \bigcap_{k=1}^{\infty} \left\{ v_t \in L^p([0, L]) : \sup_{|x_1| \leq \delta_k} ||v(t, \cdot + x_1) - v(t, \cdot)||^p \leq \frac{1}{k}, t \in [0, T] \right\}. \tag{4.14}$$

We next prove that for each $\varepsilon > 0$ and $p \in (\alpha, 2]$,

$$\lim_{\gamma \to \infty} \sup_n P\left[ \int_{(L-\frac{\varepsilon}{2},L]} |u^n(t,x)|^p \, dx \, > \, \varepsilon \right] = 0. \tag{4.15}$$

It is easy to see that

$$\mathbb{E}\left[ \int_{(L-\frac{\varepsilon}{2},L]} |u^n(t,x)|^p \, dx \right] = \mathbb{E}\left[ \int_0^L |u^n(t,x)|^p 1_{(L-\frac{\varepsilon}{2},L]}(x) \, dx \right] \leq C_p(D_1 + D_2),$$

where

$$D_1 = \int_0^L \mathbb{E}\left[ \int_0^L G_t(x,y)u_0(y)dy \right]^p 1_{(L-\frac{\varepsilon}{2},L]}(x) \, dx,$$

$$D_2 = \int_0^L \mathbb{E}\left[ \int_0^{t+} \int_0^L \int_{R \setminus \{0\}} G_{t-s}(x,y)\varphi^n(u^n(s-,y))zN(ds,dy,dz) \right]^p 1_{(L-\frac{\varepsilon}{2},L]}(x) \, dx.$$

It is easy to prove that $D_1$ converges to 0 as $\gamma \to \infty$ by using Young’s convolution inequality and Lebesgue’s dominated convergence theorem. For $D_2$, it holds by the Burkholder-Davis-Gundy inequality, (3.10)-(3.11) and (3.3) that for $p \in (\alpha, 2]$,

$$D_2 \leq C_p \int_0^L \int_0^L \int_{R \setminus \{0\}} \mathbb{E}[|G_{t-s}(x,y)\varphi^n(u^n(s-,y))z|] 1_{(L-\frac{\varepsilon}{2},L]}(x)u_\alpha(dz)dydsdx
\leq C_{p,K,\alpha} \int_0^L \int_0^L \int_0^L \mathbb{E}[1 + |u^n(s,y)|]^p|G_{t-s}(x,y)|^p 1_{(L-\frac{\varepsilon}{2},L]}(x)dydsdx
\leq C_{p,K,\alpha,T} \left( L + \sup_{0 \leq t \leq T} \mathbb{E} \left[ ||u^n_t||^p \right] \right) \int_0^L \int_0^t (t-s)^{-\frac{\alpha-1}{2}} 1_{(L-\frac{\varepsilon}{2},L]}(x)dxds.$$

Since $p \leq 2$, it holds that

$$D_2 \leq C_{p,K,\alpha,T} \left( L + \sup_{0 \leq t \leq T} \mathbb{E} \left[ ||u^n_t||^p \right] \right).$$

By (3.5), $D_2$ converges to 0 as $\gamma \to \infty$. Therefore, (4.15) is obtained from the estimates of $D_1$ and $D_2$ and Markov’s inequality.

For any $k \in \mathbb{N}$ and $T > 0$ we can choose a sequence $(\gamma_k)_{k \geq 1}$ with $\gamma_k \to \infty$ as $k \to \infty$ such that

$$\sup_n P\left[ \int_{(L-\frac{\varepsilon}{2},L]} |u^n(t,x)|^p \, dx \, > \, \frac{1}{k} \right] \leq \frac{\varepsilon}{3} 2^{-k}, \quad t \in [0, T].$$

Let $\Gamma_{\varepsilon,T}^3$ be a closed set defined by

$$\Gamma_{\varepsilon,T}^3 := \bigcap_{k=1}^{\infty} \left\{ v_t \in L^p([0, L]) : \int_{(L-\frac{\varepsilon}{2},L]} |v(t,x)|^p \, dx \leq \frac{1}{k}, t \in [0, T] \right\}. \tag{4.16}$$
Combining (4.13), (4.14) and (4.16) to define

\[ \Gamma_{\varepsilon,T} := \Gamma_{\varepsilon,T}^1 \cap \Gamma_{\varepsilon,T}^2 \cap \Gamma_{\varepsilon,T}^3, \]

then \( \Gamma_{\varepsilon,T} \) is a closed set in \( L^p([0,L]), p \in (\alpha,2] \). For any function \( f \in \Gamma_{\varepsilon,T} \) the definition of \( \Gamma_{\varepsilon,T} \) implies that the conditions (a)-(c) in Lemma 4.2 hold, and so \( \Gamma_{\varepsilon,T} \) is a relatively compact set in \( L^p([0,L]), p \in (\alpha,2] \). Combining the closeness and relatively compactness, we know that \( \Gamma_{\varepsilon,T} \) is a compact set in \( L^p([0,L]), p \in (\alpha,2] \). Moreover, the definition of \( \Gamma_{\varepsilon,T} \) implies that

\[ \inf_n \mathbb{P}[u^n_t \in \Gamma_{\varepsilon,T}] \geq 1 - \frac{\varepsilon}{3} \left( 1 + 2 \sum_{k=1}^{\infty} 2^{-k} \right) = 1 - \varepsilon, \]

which verifies condition (i) of Lemma 4.1. Condition (ii) of Lemma 4.1 is verified by Lemma 4.5 with \( p \in (\alpha,5/3) \). Therefore, \( (u^n)_{n \geq 1} \) is tight in \( D([0,\infty),L^p([0,L])) \) for \( p \in (\alpha,5/3) \), which completes the proof. \( \Box \)

Proof of Theorem 2.4. According to Proposition 4.7, there exists a \( D([0,\infty),L^p([0,T])) \)-valued random variable \( u \) such that \( u^n \) converges to \( u \) in distribution in the Skorohod topology. The Skorohod Representation Theorem yields that there exists another filtered probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{\mathbb{P}}) \) and on it a further subsequence \( (\hat{u}^n)_{n \geq 1} \) and \( \hat{u} \) which have the same distribution as \( (u^n)_{n \geq 1} \) and \( u \), so that \( \hat{u}^n \) almost surely converges to \( \hat{u} \) in the Skorohod topology. The rest of the proofs, including the construction of a truncated \( \alpha \)-stable measure \( \hat{L}_\alpha \) such that \( (\hat{u}, \hat{L}_\alpha) \) is a weak solution to equation (1.1), is same as the proof of Theorem 2.3 and we omit them.

Since \( \hat{u}^n \) has the same distribution as \( u^n \) for each \( n \geq 1 \), the moment estimate (4.3) in Lemma 4.3 can be written as

\[ \sup_{n \geq 1} \hat{\mathbb{E}} \left[ \sup_{0 \leq t \leq T} ||\hat{u}^n_t||_p^p \right] \leq C_{p,K,\alpha,T}. \]

Hence, by Fatou’s Lemma,

\[ \hat{\mathbb{E}} \left[ \sup_{0 \leq t \leq T} ||\hat{u}_t||_p^p \right] \leq \liminf_{n \to \infty} \hat{\mathbb{E}} \left[ \sup_{0 \leq t \leq T} ||\hat{u}^n_t||_p^p \right] < \infty. \]

This yields the uniform \( p \)-moment estimate (2.9). Similarly, we can obtain the uniform stochastic continuity (2.10) by Lemma 4.5. \( \Box \)

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