SCHUR-WEYL DUALITY FOR HEISENBERG COSETS

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ABSTRACT. Let $V$ be a simple vertex operator algebra containing a rank $n$ Heisenberg vertex algebra $H$ and let $C = \text{Com}(H, V)$ be the coset of $H$ in $V$. Assuming that the representation categories of interest are vertex tensor categories in the sense of Huang, Lepowsky and Zhang, a Schur-Weyl type duality for both simple and indecomposable but reducible modules is proven. Families of vertex algebra extensions of $C$ are found and every simple $C$-module is shown to be contained in at least one $V$-module. A corollary of this is that if $V$ is rational and $C_2$-cofinite and CFT-type, and $\text{Com}(C, V)$ is a rational lattice vertex operator algebra, then so is $C$. These results are illustrated with many examples and the $C_1$-cofiniteness of certain interesting classes of modules is established.

1. Introduction

Let $V$ be a vertex operator algebra. If $G$ is a subgroup of the automorphism group of $V$, then the invariants $V^G$ form a vertex operator subalgebra called the $G$-orbifold of $V$. If $W$ is any vertex operator subalgebra of $V$, then the $W$-coset of $V$ is the commutant $C = \text{Com}(W, V)$. Both the orbifold and coset constructions provide a way to construct new vertex operator algebras from known ones. Unfortunately, few general results concerning the structure of the resulting vertex operator subalgebras are known, but it is believed that many nice properties of $V$ are inherited by its orbifolds and cosets. We remark that while most of the literature is primarily concerned with completely reducible representations of vertex operator algebras, we are also interested in the logarithmic case in which the vertex operator algebra admits indecomposable but reducible representations.

We begin by recalling some important results in the invariant theory of vertex operator algebras that are connected to the questions that we address in this work.

1.1. From classical to vertex-algebraic invariant theory. It is valuable to view invariant-theoretic results about vertex operator algebras as generalizations of the classical results, à la Howe and Weyl [Ho, We], concerning Lie algebras and groups. For example, a well-known result of Dong, Li and Mason [DLM1] amounts to a type of Schur-Weyl duality for orbifolds, stating that for a simple vertex operator algebra $V$ and a compact subgroup $G$ of $\text{Aut} V$ (acting continuously and faithfully), the following decomposition holds as a $G \times V^G$-module:

$$V = \bigoplus_{\lambda} \lambda \otimes V_{\lambda}.$$ \hspace{1cm} (1.1)

Here, the sum runs over all the simple $G$-modules $\lambda$ and is multiplicity-free in the sense that $V_\lambda \ncong V_\mu$ if $\lambda \neq \mu$. They moreover prove that the $V_\lambda$ are simple modules for the orbifold vertex operator algebra $V^G$. Similar results have also been obtained by Kac and Radul [KR] (see Section 2.4).

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1We mention that much of this discussion generalizes immediately to vertex operator superalgebras. However, we shall generally state results for vertex operator algebras for simplicity, leaving explicit mention of the super-case to exceptions and examples.
Invariant theory for the classical groups \cite{We} can be used to obtain generators and relations for orbifold vertex operator algebras $V_G$, provided that $V$ is of free field type (meaning that the only field appearing in the singular terms of the operator product expansions of the strong generators is the identity field). Interestingly, the relations can be used to show that these vertex operator algebras are strongly finitely generated and, in many cases, explicit minimal strong generating sets can be obtained \cite{CL3,L2,L6,L5,L6}. Questions concerning cosets are usually more involved than their orbifold counterparts. However, the notion of a deformable family of vertex operator algebras \cite{CL2} can sometimes be used to reduce the identification of a minimal strong generating set for a coset to a known orbifold problem for a free field algebra \cite{CL1}.

It is of course desirable to understand the representation theory of coset vertex operator algebras. An important first question to ask is if there is also a Schur-Weyl type duality, as in the orbifold case. Let $V$ be a simple vertex operator algebra that is self-contragredient and let $A, B \subseteq V$ be vertex operator subalgebras satisfying

$$A = \text{Com}(B, V) \quad \text{and} \quad B = \text{Com}(A, V).$$

(1.2)

Under the further assumption that $A$ and $B$ are both simple, self-contragredient, regular and of CFT-type,

$$V = \bigoplus_i M_i \otimes N_i$$

(1.3)
as an $A \otimes B$-module, where each $M_i$ is a simple $A$-module and each $N_i$ is a simple $B$-module. Under further conditions, Lin finds \cite{Lin} that this decomposition is multiplicity-free and the argument relies on knowing that the representation categories of $A$ and $B$ are both semisimple modular tensor categories.

We are aiming for similar results, but generalised to include decompositions of modules that are not necessarily semisimple. Our setup is that $V$ is a simple vertex operator algebra containing a Heisenberg vertex operator subalgebra $H$. We then study the commutant $C = \text{Com}(H, V)$. For this, we assume that $C$ has a module category $\mathcal{C}$ that is a vertex tensor category in the sense of Huang, Lepowsky and Zhang \cite{HLZ} and that the $C$-modules obtained upon decomposing $V$ as an $H \otimes C$-module belong to $\mathcal{C}$. In Section 6, we establish the $C_1$-cofiniteness of Heisenberg coset modules in two families of examples.

1.2. **Rational parafermion vertex operator algebras.** Heisenberg cosets of rational affine vertex operator algebras are usually called parafermion vertex operator algebras. They first appeared in the form of the $\mathbb{Z}$-algebras discovered by Lepowsky and Wilson in \cite{LW1,LW2,LW3,LW4}, see also \cite{LP}. In physics, parafermions first appeared in the work of Fateev and Zamolodchikov \cite{FZ} where they were given their standard appellation. The relation between parafermion vertex operator algebras and $\mathbb{Z}$-algebras was subsequently clarified in \cite{DL}.

Parafermions are surely among the best understood coset vertex operator algebras and there has been substantial recent progress towards establishing a complete picture of their properties. Key results include $C_2$-cofiniteness \cite{ALY}, see also \cite{DLY,DW1}, and rationality \cite{DR}, using previous results on strong generators \cite{DLWY}. In principle, strong generators can now also be determined as in \cite{CL1}, where this was detailed for the parafermions related to $\mathfrak{sl}_3$. We remark that $C_2$-cofiniteness also follows from a recent result of Miyamoto on orbifold vertex operator algebras \cite{M4}. These powerful results also allow one, for example, to compute fusion coefficients \cite{DW2}.

It has, for some time, been believed that if a simple, rational, $C_2$-cofinite, self-contragredient vertex operator algebra of CFT-type contains a lattice vertex operator subalgebra (corresponding to an even positive-definite lattice), then the corresponding coset vertex operator algebra will also be rational. For example,
this was recently shown using indirect methods for the rational Bershadsky-Polyakov vertex operator algebras [ACL]. We prove this statement in general (see Theorem 4.12).

1.3. Results. This work is, at least in part, a continuation of our previous work on simple current extensions of vertex operator algebras [CKL]. In this vein, we start by proving some properties of simple currents (Proposition 2.5), in particular that fusing with a simple current defines an autoequivalence of any suitable category of modules. As further preparation, we also prove (Theorem 3.1) that if \( V \) is simple, \( \mathfrak{g} \) is an abelian group of \( V \)-automorphisms acting semisimply on \( V \), and

\[
V = \bigoplus_{\lambda \in \mathfrak{L} \subset \mathfrak{g}} V_{\lambda},
\]

then \( V_{\lambda} \) is a simple current for every \( \lambda \) in \( \mathfrak{L} \). The proof essentially amounts to adding details to a very similar result of Miyamoto [M2, Sec. 6], [CaM].

1.3.1. Schur-Weyl duality. We then prove a Schur-Weyl duality for Heisenberg cosets \( C = \text{Com}(H, V) \). The set-up is as follows. Let \( V \) be a simple vertex operator algebra, \( H \subseteq V \) be a Heisenberg vertex operator subalgebra that acts semisimply on \( V \), \( C = C_0 \) be the commutant of \( H \) in \( V \) and \( \mathfrak{L} \) be the lattice of Heisenberg weights of \( V \). Here \( V \) is regarded as an \( H \)-module. Then \( W = \text{Com}(C, V) \) is an extension of \( H \) by an abelian intertwining algebra. Of course, it might happen that the extension is trivial, that is, \( H = W \). Eq. (1.4) translates into

\[
V = \bigoplus_{\lambda \in \mathfrak{L}} F_{\lambda} \otimes C_{\lambda}.
\]  

(1.5)

Let \( N \) be the sublattice of all \( \lambda \in \mathfrak{L} \) for which \( C_{\lambda} \cong C \). Theorem 3.5 now says that the abelian group \( \mathfrak{L}/N \) controls the decomposition of \( V \) as a \( W \otimes C \)-module:

\[
V = \bigoplus_{[\lambda] \in \mathfrak{L}/N} W_{[\lambda]} \otimes C_{[\lambda]}.
\]  

(1.6)

Moreover, the \( C_{[\lambda]} \), \( \lambda \in \mathfrak{L}/N \) are simple currents for \( C \) whose fusion products include

\[
C_{[\lambda]} \boxtimes_C C_{[\mu]} = C_{[\lambda + \mu]}.
\]

This decomposition is multiplicity free in the sense that \( C_{[\lambda]} \not\cong C_{[\mu]} \) if \( [\lambda] \neq [\mu] \). The vertex operator algebra

\[
W = \bigoplus_{\lambda \in N} F_{\lambda}
\]

is a simple current extension of \( H \) and the \( W_{[\lambda]} \), \( [\lambda] \in \mathfrak{L}/N \), are simple currents for \( W \) with fusion products

\[
W_{[\lambda]} \boxtimes_W W_{[\mu]} = W_{[\lambda + \mu]}.
\]

We note that Li has proven [Li] that \( \bigoplus_{\lambda \in \mathfrak{L}/N} C_{[\lambda]} \) is a generalized vertex algebra.

The main Schur-Weyl duality result is then a similar decomposition for vertex operator algebra modules, see Theorem 3.8. For this let \( V, H, C, W, \mathfrak{L} \) and \( N \) be as above and let \( M \) be a \( V \)-module upon which \( H \) acts semisimply. Then, \( M \) decomposes as

\[
M = \bigoplus_{\mu \in \mathcal{M}} M_\mu = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes D_\mu = \bigoplus_{[\mu] \in \mathfrak{L}/N} W_{[\mu]} \otimes D_{[\mu]},
\]

(1.7)

where \( \mathcal{M} \) is a union of \( \mathfrak{L} \)-orbits and the \( D_\mu = D_{[\mu]} \) are \( C \)-modules satisfying \( C_{[\lambda]} \boxtimes_C D_\mu = D_{\lambda + \mu}, \) for all \( \lambda \in \mathfrak{L} \) and \( \mu \in \mathcal{M} \). Next, in Theorem 3.8 we show that each of the \( D_\mu \) have the same decomposition structure as that of \( M \). One example of this is if \( 0 \to M' \to M \to M'' \to 0 \) is exact, with \( M' \) and \( M'' \) non-zero, then \( M' \)
and \( M'' \) decompose as in (1.7):

\[
M' = \bigoplus_{\mu \in M} M'_\mu = \bigoplus_{\mu \in M} F_\mu \otimes D'_\mu, \\
M'' = \bigoplus_{\mu \in M} M''_\mu = \bigoplus_{\mu \in M} F_\mu \otimes D''_\mu.
\]

(1.8)

Moreover, \( 0 \to D'_\mu \to D_\mu \to D''_\mu \to 0 \) is also exact, for all \( \mu \in M \).

However, in general, multiplicity-freeness does not hold, for example, the parafermion coset of \( L_2(\mathfrak{sl}_2) \) yields an example of a coset module that appears twice in the decomposition of a simple \( L_2(\mathfrak{sl}_2) \)-module.

We give three criteria to guarantee that a given decomposition is multiplicity-free. One based on characters, one based on the signature of the lattice \( L \), and one based on open Hopf link invariants following [CG1, CG2].

1.3.2. Extensions of vertex operator algebras. Let \( E \) be a sublattice of \( L \). We would like to know if \( C_E = \bigoplus_{\lambda \in E} C_\lambda \) (1.9)

carries the structure of a vertex operator algebra extending that of \( C = C_0 \). Theorem 4.1, which itself follows immediately from [Li], implies that this is the case provided that

\[
W_E = \bigoplus_{\lambda \in E} W_\lambda
\]

(1.10)

is a vertex operator algebra. If \( E \) is a rank one subgroup, then this conclusion also follows from [CKL].

1.3.3. Lifting Modules. Let \( D \) be a \( C \)-module. We would like to know if it lifts to a \( C_E \)-module and also if there exists a \( H \)-module \( F_\beta \) such that \( F_\beta \otimes D \) lifts to a \( V \)-module.

This question is decided by the monodromy (composition of braidings)

\[
M_{C_\lambda, D} : C_\lambda \boxtimes D \to C_\lambda \boxtimes D.
\]

(1.11)

We have (Theorem 4.3): Let \( D \) be a generalized \( C \)-module that appears as a subquotient of the fusion product of some finite collection of simple \( C \)-modules. Let \( L' \) be the dual lattice of \( L \) and let \( U = L \otimes \mathbb{Z} \mathbb{R} \). Then, there exists \( \alpha \in U \) such that

\[
M_{C_\lambda, D} = e^{-2\pi i (\alpha, \lambda)} \text{Id}_{C_\lambda \boxtimes D}
\]

(1.12)

and \( F_\beta \otimes D \) lifts to a \( V \)-module if and only if \( \beta \in \alpha + L' \). Moreover, the lifted module is \( V \boxtimes_{H \otimes C} (F_\beta \otimes D) \). Note that the lifting problem when all involved vertex operator algebras are regular was treated in [KGM].

Further, \( D \) lifts to a \( C_E \)-module if and only if \( \alpha \) is in a certain lattice associated to \( E \) (see Corollary 4.4) and every \( C_E \)-module is a quotient of a lifted module (this follows essentially from [Lam]). The lifted module is then \( C_E \boxtimes_C D \).

1.3.4. Rationality. Miyamoto [M4] has proven that \( C \)-is \( C_2 \)-cofinite provided \( W \) is a lattice vertex operator algebra of a positive definite even lattice and provided \( V \) is \( C_2 \)-cofinite. Together with our ability of controlling modules that lift to \( V \)-modules and exactness of fusion with simple currents this implies a rationality theorem (Theorem 4.12):

Let \( V \) be simple, rational, \( C_2 \)-cofinite of CFT-type. Then, every grading-restricted generalized \( C \) module is completely reducible.

Especially, we thus have an alternative proof of the rationality of the parafermion cosets [DR, CaM] as well as of the Heisenberg cosets of the rational Bershadsky-Polyakov algebras [ACL].
1.3.5. Examples. We illustrate our results with various examples, both rational and non-rational ones, though our main interest are applications to vertex operator algebras of logarithmic conformal field theory, that is especially to indecomposable but reducible modules. Schur-Weyl duality is exemplified in the well-known rational example of \( L(\mathfrak{sl}_2) \) (Example 1) and then in much detail in the case of \( L_{-4/3}(\mathfrak{sl}_2) \) (Example 2). We especially explain how Schur-Weyl duality works for the projective covers of simple modules. Extensions of the Heisenberg cosets of \( L_k(\mathfrak{g}) \) for rational and non-zero \( k \) are discussed in Example 3. Example 4 then deals with the relation via Heisenberg cosets of various archetypical logarithmic vertex operator algebras, most notably the \( l(2) \)-singlet algebra and \( V_k(\mathfrak{gl}(1|1)) \). Especially we give the decomposition of the projective indecomposable modules of \( V_k(\mathfrak{gl}(1|1)) \) in terms of projective \( H \otimes H \otimes l(2) \)-modules. The triplet algebra \( W(2) \) is then an example of an extended vertex operator algebra that is \( C_2 \)-cofinite. The lifting of modules is illustrated in Example 5 for the modules of the \( N=2 \) super Virasoro algebra. Finally, we use the opportunity to prove that \( L_{-1}(\mathfrak{sl}(m|n)) \) appears as a Heisenberg coset of appropriate \( \beta \gamma \) and \( bc \)-vertex operator algebras. This generalizes the case \( n=0 \) of [AP]. Also the case \( m=2 \) and \( n=0 \) is exceptional and identified with a rectangular \( W \)-algebra of \( \mathfrak{sl}_4 \).

1.3.6. On \( C_1 \)-cofiniteness. Our results rely on the applicability of the vertex tensor theory of [HLZ]. Our belief is that the key criterion for this applicability is the \( C_1 \)-cofiniteness of the modules with finite composition length, see also [CMR] Sec. 6. In Section 8 we prove a few \( C_1 \)-cofiniteness results for modules of Heisenberg cosets of the affine vertex operator algebras of type \( \mathfrak{sl}_2 \) as well as those of the Bershadsky-Polyakov algebras.

1.3.7. Outlook on fusion. The main concern of this work is the relationship between the modules of the Heisenberg coset vertex operator algebra \( C \) and those of its parent algebra \( V \). A valid question is then if there is also a clear relation between the fusion product of the \( C \)-modules and the corresponding \( V \)-modules. This question is work in progress and here we announce that one can prove that the induction functor is a tensor functor under appropriate assumptions on the module category [CKM]. This rigorously establishes the connection between fusion and extended algebras that has been proposed in the physics literature [RW2].

1.4. Application: Towards new \( C_2 \)-cofinite logarithmic vertex operator algebras. Presently, there are very few known examples of \( C_2 \)-cofinite non-rational vertex operator algebras; these include the triplet algebras [AM2]TW1TW2 and their close relatives [AB]. In order to gain more experience with such logarithmic \( C_2 \)-cofinite vertex operator algebras, new examples are needed. The main application we have in mind for the work reported here is the construction of new examples of this type.

The idea is a two-step process illustrated as follows:

\[
V \xrightarrow{\text{H-coset}} C \xrightarrow{\text{extension}} C_C.
\]

A series of examples that confirms this idea were explored in [CRW], see also Example 5. There, the \( l(p) \) singlet algebras of Kausch [Ka] were (conjecturally) obtained as Heisenberg cosets of the Feigin-Semikhatov algebras [FS], see also [Gen]. The extension in the above process is then an infinite order simple current extension and the results [CM1]RW1 are the best understood \( C_2 \)-cofinite logarithmic vertex operator algebras, the \( W(p) \) triplet algebras.

New examples may be obtained by taking \( V \) to be the simple affine vertex operator algebra associated to the simple Lie algebra \( \mathfrak{g} \) at admissible, but negative, level \( k \) and \( H \) to be the Heisenberg vertex operator subalgebra generated by the affine fields associated to the Cartan subalgebra of \( \mathfrak{g} \). Here, \( \mathcal{K} \) is a subgroup of \( \mathfrak{g} \) of maximal rank. The module categories of such admissible level affine vertex operator algebras remain...
quite mysterious despite strong results concerning category \(\mathcal{O}\) \([KW1, Ar1]\). Beyond category \(\mathcal{O}\), detailed results are currently only known for \(g = \mathfrak{sl}_2\) \([AM1, Ga, CR3, CR5, RW3, R1–R3]\) and \(g = \mathfrak{sl}_3\) \([AFR]\). A first feasible task here would be to compute the characters of coset modules appearing in the decomposition of modules in \(\mathcal{O}\). We expect the appearance of Kostant false theta functions \([CM2]\) as they are the natural generalization of ordinary false theta functions that appear in the case of the admissible level parafermion coset of \(L_k(\mathfrak{sl}_2)\) \([ACR]\).

In \([ACR]\), we will study \(C_\beta\) when \(g = \mathfrak{sl}_2\) and \(k\) is negative and admissible. Under the assumption that the tensor theory of Huang-Lepowsky-Zhang applies to \(C_\beta\), we can prove that there are only finitely many inequivalent simple \(C_\beta\)-modules. It is thus natural to conjecture that \(C_\beta\) is \(C_2\)-cofinite. A consequence of \(C_2\)-cofiniteness is modularity of characters (supplemented by pseudotrace functions) \([M1]\). In \([ACR]\), we can also demonstrate this modularity of characters (plus pseudotraces) for all modules that are lifts of \(C\)-modules. We will prove the \(C_2\)-cofiniteness of \(C_\beta\), for various choice of \(\beta\) in subsequent works.

A third family of examples that fit this idea concern simple minimal \(W\)-algebras in the sense of Kac and Wakimoto \([KW2]\). These are quantum Hamiltonian reductions that are strongly generated by fields in conformal dimension one and \(3/2\) together with the Virasoro field. For certain levels, these \(W\)-algebras have a one-dimensional associated variety and they contain a rational affine vertex operator subalgebra. The Heisenberg coset of the coset of the minimal \(W\)-algebra by the rational affine vertex operator algebra thus seems to be another candidate for new \(C_2\)-cofinite algebras as infinite order simple current extensions. These cosets are explored in \([ACKL]\).

1.5. Organization. We start with a background section. There we review the vertex tensor theory of Huang, Lepowsky and Zhang and especially discuss it in the case of the Heisenberg vertex operator algebra. Next, we prove various properties of simple currents and then discuss vertex operator algebra orbifolds following Kac and Radul. Section 3 is then on Schur-Weyl duality. Section 4 is concerned with extended algebras, lifting of modules and as a special application proves our rationality theorem. In Section 5, we give a short proof that \(L_{-1}(\mathfrak{sl}(m|n))\) is a Heisenberg coset of appropriate \(\beta\gamma\) and \(bc\)-vertex operator algebras. In Section 6, we prove \(C_1\)-cofiniteness of modules appearing in Heisenberg cosets of Bershadsky-Polyakov algebras and \(L_{\ell}(\mathfrak{sl}_2)\).

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2. BACKGROUND

In this section, we give a brief exposition of the results of Huang, Lepowsky and Zhang regarding the vertex tensor categories that we shall use. We mention the case of Heisenberg vertex operator algebras separately in detail. Then, we present our new results regarding properties of simple currents under fusion. After that, we review a useful result of Kac and Radul on the simplicity of orbifold models.

2.1. Conditions and assumptions regarding the theory of Huang-Lepowsky-Zhang. We begin with a quick glossary of the terminology that we shall use.

- By a generalized module of a vertex operator algebra, we shall mean a module that is graded by generalized eigenvalues of \(L_0\). A generalized module need not satisfy any of the other restrictions mentioned below regarding grading. For \(n \in \mathbb{C}\) and a generalized module \(W\), we let \(W_{[n]}\) denote the generalized \(L_0\)-eigenspace with generalized eigenvalue \(n\).
A generalized module $W$ is called lower truncated if $W_{[n]} = 0$ whenever the real part of $n$ is sufficiently negative.

A generalized module $W$ is called grading-restricted if it is lower truncated and if, moreover, for all $n$, $\dim(W_{[n]}) < \infty$.

A generalized module $W$ is called strongly graded if $\dim(W_{[n]}) < \infty$ and, for each $n \in \mathbb{C}$, $W_{[n+k]} = 0$ for all sufficiently negative integers $k$. This notion is slightly more general than that of being grading-restricted.

In the definitions above, we shall replace the qualifier “generalized” with “ordinary” if the module is graded by eigenvalues of $L_0$ as opposed to generalized eigenvalues.

Henceforth, by “module”, without qualifiers, we shall mean a grading-restricted generalized module. For convenience in the applications to follow, we shall also assume that every vertex operator algebra module is of at most countable dimension. This implies, of course, that the dimension of all vertex operator algebras will also be at most countable.

We will sometimes need broader analogues of the concepts above, wherein the restrictions pertain to doubly-homogeneous spaces with respect to Heisenberg zero modes and $L_0$. The actual statements in [HLZ] pertain to such broader situations. However, the theorems in [Hu3], that guarantee that [HLZ] may be applied in specific scenarios, assume the definitions that we have recalled above. We expect that the theorems and concepts in [Hu3] may be generalized to the broader setting we require.

Recall the notion [HLZ, Def. 3.10] of a (logarithmic) intertwining operator among a triple of modules. When the formal variable in a logarithmic intertwining operator is carefully specialized to a fixed $z \in \mathbb{C}^\times$, one gets the notion of a $P(z)$-intertwining map. [HLZ, Def. 4.2]. These maps form the backbone of the logarithmic tensor category theory developed in [HLZ]. There, tensor products (fusion products) of modules are defined via certain universal $P(z)$-intertwining maps $\boxtimes_{P(z)}$ and the monoidal structure on the module category is obtained by fixing $z \in \mathbb{C}^\times$, typically chosen to be $z = 1$ for convenience. We remark that the products $\boxtimes_{P(z)}$, for different values of $z$, together form a structure richer than that of a braided monoidal category, called vertex tensor category. This richer structure is exploited in the proofs of many important theorems, see [HKL] for some examples.

For convenience, and especially with a view towards the proof of Proposition 3.3 below, we give a definition of the fusion product of two modules, equivalent to that of [HLZ], using intertwining operators instead of intertwining maps.

**Definition 2.1.** Given modules $W_1$ and $W_2$, the fusion product $W_1 \boxtimes W_2$ is the pair $(W_1 \boxtimes W_2, {\gamma^{\boxtimes}})$, where $W_1 \boxtimes W_2$ is a module and $\gamma^{\boxtimes}$ is an intertwining operator of type $(W_1 \boxtimes W_2, W_1 \boxtimes W_2)$, that satisfies the following universal property: Given any other “test module” $W$ and an intertwining operator $\gamma$ of type $(W_1 \boxtimes W_2, W)$, there exists a unique morphism $\eta: W_1 \boxtimes W_2 \rightarrow W$ such that $\gamma = \eta \circ \gamma^{\boxtimes}$.

Note that the universal intertwining operator $\gamma^{\boxtimes}$ will often be clear from the context and hence we shall often refer to the fusion product by its underlying module.

Now, let $V$ be a vertex operator algebra and let $\mathcal{C}$ be a category of generalized $V$-modules that satisfies the following properties:

1. $\mathcal{C}$ is a full abelian subcategory of the category of all strongly graded generalized $V$-modules.

\footnote{We mention that the same notation is generally used to denote both the fusion product operation and the universal $P(z)$-intertwining map corresponding to said fusion product.}
(2) \( C \) is closed under taking contragredient duals and the \( P(z) \)-tensor product \( \boxtimes_{P(z)} \) (recall [HLZ, Def. 4.15]).

(3) \( V \) is itself an object of \( C \).

(4) For each object \( W \) of \( C \), the (generalized) \( L_0 \)-eigenvalues are real and the size of the Jordan blocks of \( L_0 \) is bounded above (the bound may depend on \( W \)).

(5) Assumption 12.2 of [HLZ] holds.

Then, \( C \) is a vertex tensor category in the sense of Huang-Lepowsky [HLZ, Thm. 12.15]. In particular, it is an additive braided monoidal category. A precise formulation of (5) may be found in [HLZ]. In essence, this assumption guarantees the convergence of products and iterates of intertwining operators in a specific class of multivalued analytic functions. It, moreover, guarantees that products of intertwining operators can be written as iterates and vice versa.

**Theorem 2.2** ([Hu3]). Let \( V \) be a vertex operator algebra satisfying the following conditions:

- \( V \) is \( C_{\text{alg}} \)-cofinite, meaning that the space spanned by
  \[
  \{ \text{Res}_z z^{-1} Y(u, z)v \mid u, v \in V_{[n]} \text{ with } n > 0 \} \cup L_{-1} V
  \]
  has finite codimension in \( V \).

- There exists a positive integer \( N \) that bounds the differences between the real parts of the lowest conformal weights of the simple \( V \)-modules and is such that the \( N \)-th Zhu algebra \( A_N(V) \) (see [DLM2]) is finite-dimensional.

- Every simple \( V \)-module is \( \mathbb{R} \)-graded and \( C_1 \)-cofinite.

Then, the category of grading-restricted generalized modules of \( V \) satisfies the conditions (1)–(5) given above, hence is a vertex tensor category.

If \( V \) is \( C_2 \)-cofinite, has no states of negative conformal weight, and the space of conformal weight 0 states is spanned by vacuum, then these conditions are satisfied [Hu3] and so the theory of vertex tensor categories may be applied to the grading-restricted generalized \( V \)-modules.

As is amply clear from Theorem 2.2 [M3] and [HLZ, Rem. 12.3], \( C_1 \)-cofiniteness already takes us a long way towards establishing that a given category of \( V \)-modules is a vertex tensor category. Our hope is that, in the future, \( C_1 \)-cofiniteness will be, along with other minor conditions (such as conditions on the eigenvalues and Jordan blocks of \( L_0 \)), essentially enough to invoke the theory developed by Huang, Lepowsky and Zhang. With this hope in mind, we shall prove several useful \( C_1 \)-cofiniteness results in Section 6.

We would also like to remark that there are still many examples of vertex operator algebras, some quite fundamental, which do not meet the known conditions that guarantee the applicability of the vertex tensor theory of [HLZ]. It is an important problem to analyse the module categories of these examples and bring them “into the fold”, as it were. Not only will this make the theory more wide-reaching, but we expect that accommodating these new examples will lead to further crucial insights into the true nature of vertex operator algebra module categories.

### 2.2. Vertex tensor categories for the Heisenberg algebra

For Heisenberg vertex operator algebras, there exist simple modules with non-real conformal weights and, therefore, one can not invoke Theorem 2.2. In this section, we shall deal with general Heisenberg vertex operator algebras, bypassing Theorem 2.2 and instead relying (mostly) on the results in [DL]. For related discussions, including self-extensions of simple modules (which are not relevant for our purposes), see [Mi, CMR, Ru].
We shall verify that a certain semi-simple category \( \mathcal{C}_R \) of modules with real conformal weights (see \([4]\) below) is closed under fusion and satisfies the associativity requirements for intertwining operators, by invoking results in \([DL]\). Once this is done, it is straightforward to verify that \( \mathcal{C}_R \) satisfies the assumptions for being vertex tensor category as in \([HLZ\text{ Sec. 12}]\).

Let \( \mathfrak{h} \) be a finite-dimensional abelian Lie algebra over \( \mathbb{C} \), equipped with a symmetric non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \). We shall identify \( \mathfrak{h} \) and its dual \( \mathfrak{h}^* \) via this form. As in \([LL\text{ Ch. 6}]\), let \( \hat{\mathfrak{h}} \) denote the Heisenberg Lie algebra and \( \mathbb{H} \) the corresponding Heisenberg vertex operator algebra (of level 1, for convenience). Given \( \alpha \in \mathfrak{h} \), we denote the (simple) Fock module of \( \mathbb{H} \), with highest weight \( \lambda \in \mathfrak{h} \), by \( F_\lambda \).

It is known (see \([LW2]\)), as an algebraic analogue of the Stone-von Neumann theorem, that these simple Fock modules exhaust the isomorphism classes of the simple \( \mathbb{H} \)-modules. Let \( \mathcal{C} \) be the semisimple abelian category of \( \mathbb{H} \)-modules generated by these simple \( \mathbb{H} \)-modules and let \( \mathcal{C}_R \) be the full subcategory generated by the Fock modules with real highest weights.

**Theorem 2.3.** The subcategory \( \mathcal{C}_R \) can be given the structure of a vertex tensor category.

**Proof.** The proof splits into the following steps. Let \( \lambda, \mu, \nu \in \mathfrak{h}^* \).

1. Using \([DL\text{ Eq. (12.10)}]\), the fusion coefficient \( (\frac{W}{f_\mu F_\nu}) \) is zero if \( W \) does not have \( F_{\mu+\nu} \) as a direct summand.

2. Proceeding exactly as in \([DL\text{ Lem. 12.6–Prop. 12.8}]\), we see that the fusion coefficient \( (\frac{F_{\mu+\nu}}{f_\mu F_\nu}) \) is either 0 or 1.

3. Let \( \mathcal{L} \) be the lattice spanned by \( \mu \) and \( \nu \). One can check that the (generalised) lattice vertex operator algebra \( V_\mathcal{L} \) satisfies the Jacobi identity given in \([DL\text{ Thm. 5.1}]\), even though \( \mathcal{L} \) is not necessarily rational. This implies that the vertex map \( Y \) of \( V_\mathcal{L} \) furnishes explicit (non-zero) intertwining operators of type \( (\frac{F_{\mu+\nu}}{f_\mu F_\nu}) \), thereby implying that the fusion coefficient \( (\frac{F_{\mu+\nu}}{f_\mu F_\nu}) \) is always 1.

4. We conclude that \( \mathcal{C} \) is closed under \( \boxtimes_{P(\mathbb{Z})} \) (recall \([HLZ\text{ Def. 4.15}]\)). In general, if \( M \) is a subgroup of \( \mathfrak{h} \), regarded as an additive abelian group, and if \( \mathcal{C}' \) is the semi-simple category generated by the Fock modules with highest weights in \( M \), then \( \mathcal{C}' \) is closed under \( \boxtimes_{P(\mathbb{Z})} \). In particular, the subcategory \( \mathcal{C}_R \) is closed under \( \boxtimes_{P(\mathbb{Z})} \).

5. Given \( \mu_1, \ldots, \mu_j \in \mathfrak{h}_R \), let \( \mathcal{L} \) be the lattice that they span. Then, \( V_\mathcal{L} \) again satisfies the Jacobi identity \([DL\text{ Thm. 5.1}]\) and the duality results of \([DL\text{ Ch. 7}]\) also go through. As a consequence, the expected convergence and associativity properties of intertwining operators among Fock modules in \( \mathcal{C}_R \) hold.

6. Since the conformal weights of all modules in \( \mathcal{C}_R \) are real, the associativity of the intertwining operators yields a natural associativity isomorphism for \( \mathcal{C}_R \) \([HLZ]\).

7. Finally, one can proceed as in \([HLZ\text{ Sec. 12}]\) to verify the remaining properties satisfied by the braiding and associativity isomorphisms. Thus, \( \mathcal{C}_R \) forms a vertex tensor category in the sense of Huang-Lepowsky and, in particular, is a braided tensor category. \( \square \)

### 2.3. Simple Currents

An important concept in the theory of vertex operator algebras is the simple current extension, wherein a given algebra \( V \) is embedded in a larger one \( W \) that is constructed from certain \( V \)-modules called simple currents. The utility of this construction is that, unlike general embeddings, the representation theories of \( V \) and \( W \) are very closely related.

**Definition 2.4.** A simple current \( J \) of a vertex operator algebra \( V \) is a \( V \)-module that possesses a fusion inverse: \( J \boxtimes J^{-1} \cong V \cong J^{-1} \boxtimes J \).
Simple currents and simple current extensions were introduced by Schellekens and Yankielowicz in [SY]. We note that more general definitions of a simple current exist, see [DLM2] for example, but that the one adopted above will suffice for the vertex operator algebras that we treat below. Pertinent examples of simple currents are the Heisenberg Fock modules $F_{\lambda}$ discussed in Section 2.2: the fusion inverse of $F_{\lambda}$ is $F_{-\lambda}$.

The great advantage of requiring invertibility is that each simple current $J$ gives rise to a functor $J \boxtimes -$ which is an autoequivalence of any $V$-module category that is closed under $\boxtimes$. The following proposition gives some consequences of this; we provide proofs in order to prepare for the similar, but more subtle arguments of the next section. We remark that the isomorphism classes of the simple currents naturally form a group, sometimes called the Picard group of the category.

**Proposition 2.5.** Let $J$ be a simple current of a vertex operator algebra $V$.

1. If $M$ is a non-zero $V$-module, then $J \boxtimes M$ is non-zero.
2. If $M$ is an indecomposable $V$-module, then $J \boxtimes M$ is indecomposable.
3. If $M$ is a simple $V$-module, then $J \boxtimes M$ is simple. In particular, $J$ is simple if $V$ is.
4. The covariant functor $J \boxtimes -$ is exact (hence, so is $- \boxtimes J$).
5. If $M$ has a composition series with composition factors $S_i$, $1 \leq i \leq n$, then $J \boxtimes M$ has a composition series with composition factors $J \boxtimes S_i$, $1 \leq i \leq n$.
6. If $M$ has a radical or socle, then so does $J \boxtimes M$. Moreover, the latter radical or socle is then given by $J \boxtimes \text{rad} M \cong \text{rad}(J \boxtimes M)$ or $J \boxtimes \text{soc} M \cong \text{soc}(J \boxtimes M)$.
7. If $M$ has a radical or socle series, then so does $J \boxtimes M$. In particular, the corresponding Loewy diagrams of $J \boxtimes M$ are obtained by replacing each composition factor $S_i$ of $M$ by $J \boxtimes S_i$.

**Proof.** If $J \boxtimes M = 0$, then $0 = J^{-1} \boxtimes J \boxtimes M \cong V \boxtimes M \cong M$. Thus, (1) follows:

\[
M \neq 0 \implies J \boxtimes M \neq 0.
\] (2.1)

Similarly, if $J \boxtimes M \cong M' \oplus M''$, then $M \cong J^{-1} \boxtimes J \boxtimes M \cong (J^{-1} \boxtimes M') \oplus (J^{-1} \boxtimes M'')$. In other words, $M$ indecomposable implies that $J \boxtimes M$ is indecomposable, which is (2).

Suppose now that $M$ is simple, but that $J \boxtimes M$ has a proper submodule $M'$. Then,

\[
0 \rightarrow M' \rightarrow J \boxtimes M \rightarrow M'' \rightarrow 0
\] (2.2)

is exact, for $M'' \cong (J \boxtimes M)/M' \neq 0$. But, fusion is right-exact [HLZ, Prop. 4.26], so

\[
J^{-1} \boxtimes M' \rightarrow M \rightarrow J^{-1} \boxtimes M'' \rightarrow 0
\] (2.3)

is exact. However, $M'' \neq 0$ implies that $J^{-1} \boxtimes M''$ is a non-zero quotient of $M$, by (1) so we must have $J^{-1} \boxtimes M'' \cong M$, as $M$ is simple. Fusing with $J$ now gives $J \boxtimes M \cong M''$, so we conclude that $M' = 0$ and that $J \boxtimes M$ is simple. The simplicity of $J \cong J \boxtimes V$ now follows from that of $V$, completing the proof of (3).

To prove (4) note that applying right-exactness to the short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ results in

\[
J \boxtimes \frac{M}{M'} \cong \frac{J \boxtimes M}{(J \boxtimes M')/\ker f},
\] (2.4)

where $f$ is the induced map from $J \boxtimes M'$ to $J \boxtimes M$ that might not be an inclusion. Fusing with $J^{-1}$ and applying (2.4), we arrive at

\[
\frac{M}{M'} \cong J^{-1} \boxtimes \frac{J \boxtimes M}{(J \boxtimes M')/\ker f} \cong \frac{J \boxtimes M}{J \boxtimes M'/\ker g},
\] (2.5)
where \( g : J^{-1} \boxtimes ((J \boxtimes M')/\ker f) \to M \) might not be an inclusion. Thus,
\[
M' \cong \frac{J^{-1} \boxtimes M'}{\ker f} \cong \frac{J \boxtimes M'/\ker f}{\ker h},
\]
where \( h : J^{-1} \boxtimes \ker f \to M' \) might not be an inclusion. We conclude that \( \ker g = 0 \) and \( \ker h = J^{-1} \boxtimes \ker f \).

But, both require that
\[
M' \cong \frac{J^{-1} \boxtimes J \boxtimes M'}{\ker f} \implies J \boxtimes M' \cong \frac{J \boxtimes M'/\ker f}{\ker h} \implies \ker f = 0.
\]

\[
f : J \boxtimes M' \to J \boxtimes M \text{ is therefore an inclusion, hence } J \boxtimes - \text{ is exact.}
\]

Suppose now that \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \) is a composition series for \( M \), so that each \( S_i = M_i/M_{i-1} \) is simple. By (4) applying \( J \boxtimes - \) to each exact sequence \( 0 \to M_{i-1} \to M_i \to S_i \to 0 \) gives another exact sequence \( 0 \to J \boxtimes M_{i-1} \to J \boxtimes M_i \to J \boxtimes S_i \to 0 \). Moreover, \( J \boxtimes S_i \) is simple, by (3). Assembling all of these exact sequences gives (5).

For (6), first recall that \( \text{rad } M \) is the intersection of the maximal proper submodules of \( M \) and that \( M_1 \subseteq M \) is maximal proper if and only if \( M/M_1 \) is simple. In this case, (3) and (4) imply that \( J \boxtimes (M/M_i) \) is simple and isomorphic to \( (J \boxtimes M)/J \boxtimes M_i) \), whence \( J \boxtimes M_i \) is maximal proper in \( J \boxtimes M \). Applying \( J^{-1} \boxtimes - \) gives the converse. Second, given a collection \( M_i \subseteq M \), (4) also implies that \( J \boxtimes (\cap_i M_i) \) is a submodule of each \( J \boxtimes M_i \), hence of \( \cap_i (J \boxtimes M_i) \). But now, \( \cap_i (J \boxtimes M_i) \cong J \boxtimes J^{-1} \boxtimes (\cap_i (J \boxtimes M_i)) \subseteq J \boxtimes (\cap_i M_i) \), hence we have \( J \boxtimes (\cap_i M_i) \cong \cap_i (J \boxtimes M_i) \). These two conclusions together give \( J \boxtimes \text{rad } M \cong \text{rad } (J \boxtimes M) \). A similar, but easier, argument establishes \( J \boxtimes \text{soc } M \cong \text{soc } (J \boxtimes M) \).

Finally, (7) follows by combining (6) with slight generalisations of the arguments used to prove (5). □

This proposition has a simple summary: fusing with a simple current preserves module structure. We remark, obviously, that a simple current \( J \) need not be simple if the vertex operator algebra \( V \) is not simple.

### 2.4. Orbifold modules

Here, we review a result of Kac and Radul [KR] on the simplicity of orbifold modules. For a very similar result see [DLM1],

Let \( A \) be an associative algebra, for example the mode algebra of a vertex operator algebra, and let \( \mathcal{G} \) be a subgroup of \( \text{Aut} A \) acting semisimply on \( A \). We consider \( A \)-modules \( M \) which admit a semisimple \( \mathcal{G} \)-action that is compatible with the \( \mathcal{G} \)-action on \( A \) and which decompose as a countable direct sum of finite-dimensional simple \( \mathcal{G} \)-modules. This compatibility means that
\[
g(am) = (ga)(gm), \quad \text{for all } g \in \mathcal{G}, a \in V \text{ and } m \in M. \tag{2.8}
\]
If we now define \( A_0 \) to be the space of \( \mathcal{G} \)-invariants \( a \in A \), so \( ga = a \) for all \( g \in \mathcal{G} \), then the actions of each \( g \in \mathcal{G} \) and each \( a \in A_0 \) commute on every such module \( M \).

Choose an \( M \) satisfying (2.8) and let \( N \) be a simple \( \mathcal{G} \)-module. Then, we may define the \( \mathcal{G} \)-module
\[
M_N = \sum \{ N_i \subseteq M : N_i \cong N \}. \tag{2.9}
\]
As the action of \( A_0 \) commutes with that of \( \mathcal{G} \) every \( a \in A_0 \) maps a given \( N_i \) to some \( N_j \) or 0, by Schur’s lemma. Thus, \( M_N \) is an \( A_0 \)-module.

If we choose a one-dimensional subspace \( \mathbb{C} \subseteq N \), then Schur’s lemma picks out a one-dimensional subspace \( \mathbb{C}_i \subseteq N_i \), for each \( i \). Then, each \( a \in A_0 \) maps each \( \mathbb{C}_i \) to some \( \mathbb{C}_j \) or to 0, hence
\[
M^N = \sum_{N_i \cong N} \mathbb{C}_i. \tag{2.10}
\]
is an $A_0$-module. But, because $N_i \cong N \cong N \otimes C_i$, we may write

$$M_N \cong \sum_{N_i \in N} N \otimes C_i = N \otimes M^N$$

(2.11)

as a $C \otimes A_0$-module. The semisimplicity of $M$, as a $G$-module, now gives us the decomposition

$$M \cong \bigoplus_{[N]} M_N \cong \bigoplus_{[N]} N \otimes M^N,$$

(2.12)

again as a $C \otimes A_0$-module. Here, $[N]$ denotes the isomorphism class of the simple $G$-module $N$.

The result of Kac and Radul gives conditions under which the $A_0$-modules $M^N$, appearing in (2.12), are guaranteed to be simple.

**Theorem 2.6** ([KR, Thm. 1.1 and Rem. 1.1]). With the above setup, the (non-zero) $M^N$ appearing in (2.12) will be simple $A_0$-modules provided that $M$ is a simple $A$-module.

3. Schur-Weyl duality

In this section, we state and prove results concerning the decomposition of a vertex operator algebra and its modules into modules over a Heisenberg vertex operator subalgebra and its commutant. We regard this decomposition as a vertex-algebraic analogue of the well known Schur-Weyl duality familiar for symmetric groups and general linear Lie algebras. These results are enhanced by deducing sufficient conditions for the decompositions, and their close relations, to be multiplicity-free. Finally, we illustrate our results with several carefully chosen examples.

3.1. Heisenberg cosets. Let $\mathcal{G}$ be a finitely generated abelian subgroup of the automorphism group of a simple vertex operator algebra $V$. We assume that $\mathcal{G}$ grades $V$, meaning that the actions of these automorphisms may be simultaneously diagonalised, hence that $V$ decomposes into a direct sum of $\mathcal{G}$-modules:

$$V = \bigoplus_{\lambda \in \mathcal{L}} V_\lambda.$$ 

(3.1)

Here, the $\lambda$ are elements of the (abelian) dual group $\hat{\mathcal{G}}$ of inequivalent (complex, not necessarily unitary) one-dimensional representations of $\mathcal{G}$ (recall that addition is tensor product and negation is contragredient dual), $V_\lambda$ denotes the simultaneous eigenspace upon which each $g \in \mathcal{G}$ acts as multiplication by $\lambda(g) \in \mathbb{C}$, and $\mathcal{L}$ is the subset of $\lambda \in \hat{\mathcal{G}}$ for which $V_\lambda \neq 0$. Note that the cardinality of $\mathcal{L}$ is at most countable.

The action of $V$ on itself restricts to an action of each $V_\lambda$ on each $V_\mu$. For $\lambda = \mu = 0$, where 0 denotes the trivial $\mathcal{G}$-module, this implies that $V_0$ is a vertex operator subalgebra of $V$; for $\lambda = 0$, this implies that each $V_\mu$ is a $V_0$-module. From the simplicity of $V$, it now easily follows that $\mathcal{L}$ is a subgroup of $\hat{\mathcal{G}}$: closure under addition follows from annihilating ideals being trivial [LL Cor. 4.5.15] and closure under negation follows similarly, see [LL] Prop. 3.6.

Applying Theorem 2.6, with $M = V$ and $A$ being the mode algebra of $V$, we can now improve upon (3.1). Indeed, in this setting, (2.12) becomes

$$V = \bigoplus_{\lambda \in \mathcal{L}} C_\lambda \otimes V_\lambda,$$

(3.2)

where $C_\lambda$ denotes the one-dimensional module upon which $g \in \mathcal{G}$ acts as multiplication by $\lambda(g)$, and we learn that the $V_\lambda$ are simple as $V_0$-modules. In particular, $V_0$ is a simple vertex operator algebra.

If we assume that $V_0$ satisfies the conditions required to invoke the tensor category theory of Huang, Lepowsky and Zhang (Section 2.1), then more is true. As Miyamoto has shown, the $V_\lambda$ are then simple currents for $V_0$ see [CaM, M2]. It should be noted that the proofs in [CaM, M2] assume that the group of
automorphisms under consideration is finite, however, the proof works more generally under the assumption that tensor category theory for the fixed-point algebra can be invoked. For completeness, we include an exposition of their proof in our slightly more general setting in Appendix A.

**Theorem 3.1** ([M2] Sec. 6). Assume the above setup and that \( V_0 = V^G \) satisfies conditions sufficient to invoke Huang, Lepowsky and Zhang’s tensor category theory, for example those of Theorem 2.2. Then, the \( V_\lambda \) are simple currents for \( V_0 \) with \( V_\lambda \boxtimes V_0 \mu \cong V_{\lambda + \mu} \), for all \( \lambda, \mu \in \mathcal{L} \).

Let us now restrict to vertex operator algebras \( V \) that contain a Heisenberg vertex operator subalgebra \( H \), generated by \( r \) fields \( h_i(z), i = 1, \ldots, r \), of conformal weight 1. We will assume throughout that the action of \( H \) on \( V \) is semisimple\(^3\) and that the eigenvalues of the zero modes \( h_i^0, i = 1, \ldots, r \), are all real. Let \( C \) denote the commutant vertex operator algebra of \( H \) in \( V \) and let \( \mathcal{G} \cong \mathbb{Z}^r \) be the lattice generated by the \( h_i^0 \). Each \( V_\lambda \) of the \( \mathcal{G} \)-decomposition (3.1) is a module for \( H \) since the fields of \( H \) commute with the zero modes of \( \mathcal{G} \). As \( \mathcal{G} \) acts semisimply on \( V_\lambda \) and the only simple \( H \)-module with \( h_i^0 \)-eigenvalues \( \lambda = (\lambda^1, \ldots, \lambda^r) \) is the Fock module \( F_\lambda \), we must have the following \( H \odot C \)-module decomposition:

\[
V_\lambda = F_\lambda \otimes C_\lambda, \quad \text{for all } \lambda \in \mathcal{L}.
\] (3.3)

In this setting, we may take \( \mathcal{L} \) to be the lattice of all \( \lambda \in \mathbb{R}^r \) for which \( V_\lambda \neq 0 \). Moreover, the \( C \)-module \( C_\lambda \) is simple because \( V_\lambda \) and \( F_\lambda \) are. In particular, the commutant \( C = C_0 \) is a simple vertex operator algebra.

We summarise this as follows.

**Proposition 3.2.** Let \( V \) be a simple vertex operator algebra with a Heisenberg vertex operator subalgebra \( H \) that acts semisimply on \( V \). Then, the coset vertex operator algebra \( C = \text{Com}(H, V) \) is likewise simple.

From here on, we make the following natural assumption:

We assume that we are working with categories of (generalized) \( V_0 \)- and \( C \)-modules for which the tensor category theory of Huang, Lepowsky and Zhang [HLZ] may be invoked.

Of course, we have confirmed in Section 2.2 that this theory may be invoked for semisimple \( H \)-modules with real weights. In general, we would like to apply our results to vertex operator algebras for which we are not currently able to verify this assumption. Such illustrations should therefore be regarded as conjectural. However, we view the results in these cases as strong evidence that the conditions required to invoke Huang-Lepowsky-Zhang are, in fact, significantly weaker than those that were given in Section 2.1.

Given now the fusion rules \( F_\lambda \boxtimes F_\mu \cong F_{\lambda + \mu} \) and \( V_\lambda \boxtimes V_\mu \cong V_{\lambda + \mu} \), which imply that

\[
(F_\lambda \otimes C_\lambda) \boxtimes V_0 (F_\mu \otimes C_\mu) \cong F_{\lambda + \mu} \otimes C_{\lambda + \mu},
\] (3.4)

one is naturally led to suppose that \( C_\lambda \boxtimes C_\mu \cong C_{\lambda + \mu} \). Proving this, however, is a little subtle because we are not assuming that the corresponding representation categories are semisimple. We therefore present a technical result that we shall use to confirm this supposition and other similar assertions. We remark that this result can be greatly strengthened when one of the vertex operator algebras involved is of Heisenberg or lattice type, or when the vertex operator algebras involved are rational (see [Lin]).

**Proposition 3.3.** Let \( A \) and \( B \) be vertex operator algebras and let \( A_i \) and \( B_i \), for \( i = 1, 2, 3 \), be \( A \)-modules and \( B \)-modules, respectively. Suppose that

\[
((A_1 \otimes B_1) \otimes (A_2 \otimes B_2), \Lambda_{A_1 \otimes B_1}) = (A_3 \otimes B_3, \Lambda_{A_3 \otimes B_3}).
\] (3.5)
Also assume that either of the fusion coefficients \((A_{A_1, A_2})\) or \((B_{B_1, B_2})\) is finite. Then, \((A_3 \otimes B_3, Y^{\otimes 2}_{A_3 \otimes B_3})\) may be taken to be \(((A_1 \boxtimes A_2) \otimes (B_1 \boxtimes B_2), Y^{\otimes 2}_A \otimes Y^{\otimes 2}_B)\). In particular,

\[
A_1 \boxtimes A_2 \cong A_3 \quad \text{and} \quad B_1 \boxtimes B_2 \cong B_3. \tag{3.6}
\]

**Proof.** The key here is [ADL] Thm. 2.10 which, as stated, applies to non-logarithmic intertwining operators but in fact also holds in when logarithmic intertwiners are present. Using this, we may write

\[
Y^{\otimes 2}_{A \otimes B} = \sum_{j=1}^{N} \tilde{y}^{(j)}_A \otimes \tilde{y}^{(j)}_B, \tag{3.7}
\]

for some \(N\), where each \(\tilde{y}^{(j)}_A\) is an intertwiner for \(A\) of type \((A_{A_1, A_2})\) and each \(\tilde{y}^{(j)}_B\) is of type \((B_{B_1, B_2})\) for \(B\). The universality of the fusion product now guarantees \(A\)-modules, the existence of (unique) \(A\)-module morphisms \(\mu_A^{(j)} : A_1 \boxtimes A_2 \to A_3\), such that \(\mu_A^{(j)} \circ \tilde{y}^{\otimes 2}_A = \tilde{y}^{(j)}_A\), and \(B\)-module morphisms \(\mu_B^{(j)} : B_1 \boxtimes B_2 \to B_3\), such that \(\mu_B^{(j)} \circ \tilde{y}^{\otimes 2}_B = \tilde{y}^{(j)}_B\). Setting \(\mu = \sum_{j=1}^{N} \mu_A^{(j)} \otimes \mu_B^{(j)}\), we obtain

\[
\eta \circ (Y^{\otimes 2}_A \otimes Y^{\otimes 2}_B) = \sum_{j=1}^{N} (\mu_A^{(j)} \otimes \mu_B^{(j)}) \circ (Y^{\otimes 2}_A \otimes Y^{\otimes 2}_B) = \sum_{j=1}^{N} \tilde{y}^{(j)}_A \otimes \tilde{y}^{(j)}_B = Y^{\otimes 2}_{A \otimes B}. \tag{3.8}
\]

Now, let \(X\) be a “test” \(A \otimes B\)-module and let \(\eta\) be an intertwining operator of type \((X_{A_1 \otimes B_1, A_2 \otimes B_2})\). By the universal property satisfied by \((A_3 \otimes B_3, Y^{\otimes 2}_{A_3 \otimes B_3})\), there exists a (unique) \(\eta : A_3 \otimes B_3 \to X\) such that \(\eta \circ Y^{\otimes 2}_{A \otimes B} = \eta\). It follows that

\[
(\eta \circ \mu) \circ (Y^{\otimes 2}_A \otimes Y^{\otimes 2}_B) = \eta \circ Y^{\otimes 2}_{A \otimes B} = \eta. \tag{3.9}
\]

It remains to prove that \(\eta \circ \mu : (A_1 \boxtimes A_2) \otimes (B_1 \boxtimes B_2) \to X\) is the unique \(A \otimes B\)-module morphism satisfying (3.9). However, \(Y^{\otimes 2}_A\) and \(Y^{\otimes 2}_B\) are surjective intertwining operators — this surjectivity goes hand-in-hand with the “uniqueness” requirement in the universal property, see [HLZ] Prop. 4.23 — and so, therefore, is \(Y^{\otimes 2}_A \otimes Y^{\otimes 2}_B\). This means that equation (3.9) uniquely specifies the morphism \(\eta \circ \mu\), completing the proof. \(\square\)

From this proposition, we immediately obtain the following corollary.

**Corollary 3.4.** If \(A\) and \(B\) are simple vertex operator algebras and \(M \otimes N\) is a simple current for \(A \otimes B\), then \(M\) and \(N\) are simple currents for \(A\) and \(B\), respectively. Moreover, the inverse of \(M \otimes N\) is \(M^{-1} \otimes N^{-1}\).

**Proof.** Because \(A \otimes B\) is assumed to be simple, \(M \otimes N\) and its inverse are simple \(A \otimes B\)-modules, by Proposition 2.4.3. Moreover, this simplicity hypothesis also guarantees that the inverse has the form \(\tilde{M} \otimes \tilde{N}\) [FHL] Thm. 4.7.4. Applying Proposition 3.3 to \((\tilde{M} \otimes \tilde{N}) \boxtimes A \otimes B\), we obtain \(\tilde{M} \boxtimes \tilde{A} \cong A\) and \(\tilde{N} \boxtimes B \cong B\), hence \(\tilde{M} \cong M^{-1}\) and \(\tilde{N} \cong N^{-1}\). \(\square\)

In any case, (3.4) and Proposition 3.3 give the desired conclusion:

\[
C_{\lambda} \boxtimes C_{\mu} \cong C_{\lambda + \mu}. \tag{3.10}
\]

In particular, the \(C_{\lambda}\) are simple currents for all \(\lambda \in \mathcal{L}\). We have therefore arrived at the following decomposition of \(V\) into simple currents of \(H\) and \(C\):

\[
V = \bigoplus_{\lambda \in \mathcal{L}} F_{\lambda} \otimes C_{\lambda}. \tag{3.11}
\]

However, this may be further refined if \(\lambda \neq \mu\) in \(\mathcal{L}\) does not imply that \(C_{\lambda} \neq C_{\mu}\) (this implication is obviously true for Fock modules). Suppose that \(C_{\lambda} = C_{\lambda + \mu}\) for some \(\lambda, \mu \in \mathcal{L}\). Then, we must have
$C_\mu = \mathbb{C}$ and hence $C_{n\mu} = \mathbb{C}$ for all $n \in \mathbb{Z}$. More generally, let $\mathcal{N}$ denote the sublattice of $\mu \in \mathcal{L}$ for which $C_\mu = \mathbb{C}$. Then, we may define

$$W[\lambda] = \bigoplus_{\mu \in \mathcal{N}} F_{\lambda + \mu}$$

(3.12)

and note that $W = W(0)$ will be a lattice vertex operator algebra if the conformal weights of the fields of $F_\mu$, with $\mu \in \mathcal{N}$, are all integers.\footnote{If the conformal weights are not all integers, then $W$ is a vertex operator superalgebra, or another type of generalised vertex operator algebra. This does not significantly affect the following analysis.} The decomposition (3.11) then becomes a decomposition as a $W \otimes \mathbb{C}$-module:

$$V = \bigoplus_{[\lambda] \in \mathcal{L}/\mathcal{N}} W[\lambda] \otimes C_{[\lambda]}.$$  

(3.13)

Now the $C_{[\lambda]} \equiv C_{\lambda}$, with $[\lambda] \in \mathcal{L}/\mathcal{N}$, are mutually inequivalent: $[\lambda] \neq [\mu]$ implies that $C_{[\lambda]} \ncong C_{[\mu]}$. We remark that $\mathcal{L}/\mathcal{N}$ may still be infinite because the rank of $\mathcal{N}$ may be smaller than that of $\mathcal{L}$.

We summarise these results as follows.

**Theorem 3.5.** Let:

- $V$ be a simple vertex operator algebra.
- $\mathcal{H} \subseteq V$ be a Heisenberg vertex operator subalgebra that acts semisimply on $V$.
- $\mathbb{C} = C_{0}$ be the commutant of $\mathcal{H}$ in $V$.
- $\mathcal{L}$ be the lattice of Heisenberg weights of $V$ ($V$ being regarded as an $\mathcal{H}$-module).

Then the decompositions (3.11) and (3.13) hold, where:

- The $C_{\lambda}$, $\lambda \in \mathcal{L}$, are simple currents for $\mathbb{C}$ whose fusion products include $C_{\lambda} \boxtimes_{\mathbb{C}} C_{\mu} = C_{\lambda + \mu}$.
- $W = \bigoplus_{[\lambda] \in \mathcal{L}/\mathcal{N}} F_{[\lambda]}$ is a simple current extension of $\mathcal{H}$ ($\mathcal{N}$ is the sublattice of $\lambda \in \mathcal{L}$ for which $C_{\lambda} \cong \mathbb{C}$).
- The $W[\lambda]$, $[\lambda] \in \mathcal{L}/\mathcal{N}$, are simple currents for $W$ with fusion products $W[\lambda] \boxtimes_{W} W[\mu] = W[\lambda + \mu]$.

In particular, the $C_{[\lambda]}$, $[\lambda] \in \mathcal{L}/\mathcal{N}$, of (3.13) are mutually non-isomorphic.

**Remark 3.6.** Note that we may instead choose $\mathcal{N}$ to be any subgroup of $\mathcal{L}$ in which every $\lambda \in \mathcal{N}$ satisfies $C_{\lambda} \cong \mathbb{C}$. In particular, we may take $\mathcal{N} = 0$, in which case the decomposition (3.13) reduces to that of (3.11). Obviously, the conclusion that the $C_{[\lambda]}$ are mutually non-isomorphic will only hold if $\mathcal{N}$ is taken to be maximal.

The corresponding decomposition for $V$-modules proceeds similarly. Let $M$ be a non-zero $V$-module upon which $\mathcal{H}$ acts semisimply. The $\mathcal{H}$-weight space decomposition of $M$ then gives $M = \bigoplus_{\mu \in \mathcal{M}} M_{\mu}$, where $\mathcal{M} = \{ \mu \in \mathbb{R}^r : \mu_\mathcal{H} \neq 0 \}$ is countable. Using the triviality of annihilating ideals [LL, Cor. 4.5.15] as before, we see that $\mathcal{M}$ is closed under the additive action of $\mathcal{L}$, meaning that $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$ imply that $\lambda + \mu \in \mathcal{M}$. It follows that each $M_\mu$ is a $V_0$-module. Decomposing as an $\mathcal{H} \otimes \mathbb{C}$-module, we get $M_\mu = F_\mu \otimes D_\mu$, for some $\mathbb{C}$-module $D_\mu$. The key step towards proving a decomposition theorem for modules is now to establish certain fusion products involving the $M_\mu$ and $D_\mu$.

**Proposition 3.7.** Let $V$, $\mathcal{H}$, $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{L}$ be as in Theorem 3.5 and let $M$, $\mathcal{M}$ and $M_\mu = F_\mu \otimes D_\mu$ be as in the previous paragraph. Then, the following fusion rules hold for all $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$:

$$V_\lambda \boxtimes_{V_0} M_\mu \cong M_{\lambda + \mu},$$  

(3.14a)

$$C_\lambda \boxtimes_{\mathbb{C}} D_\mu \cong D_{\lambda + \mu}.$$  

(3.14b)
We mention that when $M = V$, the fusion rule (3.14a) is precisely the result of Miyamoto reported in Theorem 3.1. However, we cannot use Miyamoto’s proof in this more general setting because it would amount to assuming the simplicity of the $M_\mu$ as $V_0$-modules.

**Proof.** We will detail the proof of the fusion rule (3.14a), noting that (3.14b) will then follow immediately by applying Proposition 3.3.

To prove (3.14a), let $\tilde{M}$ denote the $V$-submodule of $M$ generated by $M_\mu$. Then, $(M/\tilde{M})_\mu = 0$. If $v \in V_{-\lambda}$ is non-zero, for some $\lambda \in \mathcal{L}$, and $w \in (M/\tilde{M})_{\lambda + \mu}$, then it follows that $v$ must annihilate $w$, hence that $w = 0$ by the triviality of annihilating ideals [LL, Cor. 4.5.15]. We conclude that $(M/\tilde{M})_{\lambda + \mu} = 0$, that is $\tilde{M}_{\lambda + \mu} = M_{\lambda + \mu}$, for all $\lambda \in \mathcal{L}$.

The action of $V$ on $M$ now restricts to an action of $V_\lambda$ on $M_\mu$. The space generated by the latter action is therefore precisely $M_{\lambda + \mu}$ [LL, Prop. 4.5.6]. It now follows from the universal property of fusion products that there exists a surjection

$$V_\lambda \boxtimes_{V_0} M_\mu \twoheadrightarrow M_{\lambda + \mu}, \quad (3.15)$$

for each $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$. Fusing with the simple current $V_{-\lambda}$ therefore gives

$$M_\mu \cong V_{-\lambda} \boxtimes_{V_0} (V_\lambda \boxtimes_{V_0} M_\mu) \twoheadrightarrow V_{-\lambda} \boxtimes_{V_0} M_{\lambda + \mu} \twoheadrightarrow M_\mu, \quad (3.16)$$

the second surjection just being (3.15) with $(\lambda, \mu)$ replaced by $(-\lambda, \lambda + \mu)$. Since these surjections preserve conformal weights and the dimensions of the generalised eigenspaces of $L_0$ are finite, by hypothesis, it follows that $V_{-\lambda} \boxtimes_{V_0} M_{\lambda + \mu} = M_\mu$, for all $\lambda \in \mathcal{L}$, proving (3.14a). \qed

If $\lambda \in \mathcal{L}$, then the fusion rules (3.14b) imply that $D_{\lambda + \mu} = D_\mu$, hence that the $D_{[\mu]} \equiv D_\mu$ are well defined. The decomposition of $M$ as a $W \otimes C$-module now follows as before. Before stating this formally, it is convenient to observe that if $M = M^1 \cup \cdots \cup M^n$ is a disjoint union of orbits under the action of $\mathcal{L}$, then $M = M^1 \oplus \cdots \oplus M^n$ as a $V$-module, where $M^i = \bigoplus_{\mu \in \mathcal{M}^i} M^{i}_{\mu}$. While the $M_i$ need not be indecomposable as $V$-modules, several of the arguments to come will be simplified if we assume that $M$ consists of a single $\mathcal{L}$-orbit. Conclusions about more general $M$ then follow immediately from the properties of direct sums.

**Theorem 3.8.** Let $V$, $H$, $C$, $W$, $\mathcal{L}$ and $\mathcal{N}$ be as in Theorem 3.5 and let $M$ be a $V$-module upon which $H$ acts semismply. Then, $M$ decomposes as

$$M = \bigoplus_{\mu \in \mathcal{M}} M_\mu = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes D_\mu = \bigoplus_{[\mu] \in \mathcal{M}/\mathcal{N}} W_{[\mu]} \otimes D_{[\mu]}, \quad (3.17)$$

where $\mathcal{M}$ is a union of $\mathcal{L}$-orbits and the $D_\mu = D_{[\mu]}$ are $C$-modules satisfying $C_\lambda \boxtimes_{C} D_\mu = D_{\lambda + \mu}$, for all $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{M}$. Moreover, if we assume (for convenience) that $\mathcal{M}$ is a single $\mathcal{L}$-orbit, then:

1. If $M$ is a non-zero $V$-module, then all of the $D_\mu$ are non-zero.
2. If $M$ is a simple $V$-module, then all of the $D_\mu$ are simple.
3. If $M$ is an indecomposable $V$-module, then all of the $D_\mu$ are indecomposable.
4. If $0 \to M' \to M \to M'' \to 0$ is exact, with $M'$ and $M''$ non-zero, then $M'$ and $M''$ decompose as in (3.17):

$$M' = \bigoplus_{\mu \in \mathcal{M}} M'_\mu = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes D'_\mu, \quad M'' = \bigoplus_{\mu \in \mathcal{M}} M''_\mu = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes D''_\mu. \quad (3.18)$$

Moreover, $0 \to D'_\mu \to D_\mu \to D''_\mu \to 0$ is also exact, for all $\mu \in \mathcal{M}$. \qed
(5) If $M$ has a composition series with composition factors $S^i$, $1 \leq i \leq n$, then each $S^i$ decomposes into an $\mathbf{H} \otimes \mathbf{C}$-module as $S^i = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes T_{\mu}^i$, where the $T_{\mu}^i$, $1 \leq i \leq n$, are the composition factors of $D_{\mu}$, for each $\mu \in \mathcal{M}$. In particular, each $D_{\mu}$ has the same composition length as $M$.

(6) If $M$ has a socle, then so do the $D_{\mu}$ and $\text{soc} M = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes \text{soc} D_{\mu}$.

If $M$ has a radical, then so do the $D_{\mu}$. If, in addition, $M$ has no subquotient isomorphic to the direct sum of two isomorphic simple $\mathbf{V}$-modules, then $\text{rad} M = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes \text{rad} D_{\mu}$.

(7) If $M$ has a socle series, then so do the $D_{\mu}$ and the corresponding Loewy diagram is obtained by replacing each composition factor $S^i$ by $T_{\mu}^i$, where $S^i = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes T_{\mu}^i$.

If $M$ has a radical series, then so do the $D_{\mu}$. If, in addition, $M$ has no subquotient isomorphic to the direct sum of two isomorphic simple $\mathbf{V}$-modules, then the corresponding Loewy diagram is obtained by replacing each composition factor $S^i$ by $T_{\mu}^i$, where $S^i = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes T_{\mu}^i$.

**Proof.** We have already proven the non-numbered statements. For (1) suppose that $D_{\mu} = 0$, for some $\mu \in \mathcal{M}$. Then, $M_{\mu} = F_{\mu} \otimes D_{\mu}$ would be 0, contradicting the definition of $M$. The argument for (2) is likewise short: $M$ simple implies that each $M_{\mu}$, with $\mu \in \mathcal{M}$, is simple, by Theorem 2.4, which forces each of the $D_{\mu}$ to be simple. To prove (3) note that if some $D_{\nu}$, $\nu \in \mathcal{M}$, were decomposable, then every $D_{\mu}$, $\mu \in \mathcal{M}$, would be decomposable because $\mu - \nu \in \mathcal{L}$, hence $D_{\mu} \cong C_{\mu - \nu} \otimes_{\mathbf{C}} D_{\nu}$. But then, every $M_{\mu}$ would be decomposable, hence so would $M$, a contradiction.

Given the exact sequence in (4), it is clear that $\mathbf{H}$ acts semisimply on both $M'$ and $M''$, hence that we have the decompositions (3.18) except that some of the $M_{\mu}'$ or $M_{\mu}''$ might be zero, for some $\mu \in \mathcal{M}$. However, $\mathcal{M}$ is assumed to consist of a single $\mathcal{L}$-orbit, so either all the $M_{\mu}'$ are zero or none of them are (and the same for the $M_{\mu}''$). But, either being zero would imply that the corresponding module is zero, which is ruled out by hypothesis. Thus, the $M_{\mu}'$ and $M_{\mu}''$ are non-zero, for all $\mu \in \mathcal{M}$.

Since restricting to a $V_0$-module and projecting onto the (simultaneous) eigenspaces of the $h_0$ (which commute with $V_0 = \mathbf{H} \otimes \mathbf{C}$) are exact functors, the sequence $0 \to F_{\mu} \otimes D_{\mu}' \to F_{\mu} \otimes D_{\mu} \to F_{\mu} \otimes D_{\mu}'' \to 0$ is exact, for all $\mu \in \mathcal{M}$. However, $\text{End}_{\mathbf{F}} F_{\mu} \cong \mathbf{C}$ implies that each non-trivial map in this exact sequence has the form $id_{\mu}' \otimes d_{\mu}$, where $d_{\mu}$ is a $\mathbf{C}$-module homomorphism. The required exactness of the sequence of $\mathbf{C}$-modules thus follows, proving (4).

For (5) let $0 = M^0 \subset M^1 \subset \cdots \subset M^{n-1} \subset M^n = M$ be a composition series, so that $S^i = M^i/M^{i-1}$ is simple, for all $1 \leq i \leq n$. Then, $0 \to M^{i-1} \to M^i \to S^i \to 0$ is exact, hence so is $0 \to D_{\mu}' \to D_{\mu} \to D_{\mu}'' \to 0$, for all $1 \leq i \leq n$ and $\mu \in \mathcal{M}$, by (4). Here, we have decomposed each $M_{\mu}'$ as $M_{\mu}' = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes D_{\mu}'$, so that $D_{\mu}' = 0$ and $D_{\mu}'' = D_{\mu}$, and each $S^i$ as $S^i = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes T_{\mu}^i$. Since the $T_{\mu}^i$ are non-zero and simple, by (1) and (2) they are the composition factors of $D_{\mu}$.

We turn to (6). Let $\{M_{\mu}'\}_{i \in I}$ be the set of all simple submodules of $M$ so that $\text{soc} M = \sum_{i \in I} M_{\mu}'$. Then, each $M_{\mu}'$ decomposes as $M_{\mu}' = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes D_{\mu}'$, where $D_{\mu}'$ is a simple submodule of $D_{\mu}$, for each $i \in I$ and $\mu \in \mathcal{M}$, by (2) and (4). As sums distribute over tensor products, we have

$$\text{soc} M = \sum_{i \in I} \left( \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes D_{\mu}' \right) = \bigoplus_{\mu \in \mathcal{M}} F_{\mu} \otimes \left( \sum_{i \in I} D_{\mu}' \right).$$

(3.19)

It remains to show that for each $\mu \in \mathcal{M}$, every simple submodule of $D_{\mu}$ is one of the $D_{\mu}'$.

Consider therefore a simple submodule $E_{\mu} \subseteq D_{\mu}$, for some given $\mu \in \mathcal{M}$. Form $E_{\nu} = C_{\nu - \mu} \otimes_{\mathbf{C}} E_{\mu}$, for all $\nu \in \mathcal{M}$ (so that $\nu - \mu \in \mathcal{L}$), and note that each $E_{\nu}$ is a simple submodule of $D_{\nu}$, by parts (3) and (4) of Proposition 2.5. Tensoring over $\mathbf{C}$ is exact, so $\bigoplus_{\nu \in \mathcal{M}} F_{\nu} \otimes E_{\nu}$ is a submodule of $\bigoplus_{\nu \in \mathcal{M}} F_{\nu} \otimes D_{\nu} = M$. 

Moreover, it is a simple submodule because it has the same number of composition factors as $E_\mu$, by \[5\]. It is therefore one of the $M'$, hence $E_\mu$ is one of the $D_\mu$. It follows that $\sum_{\mu \in \mathcal{M}} D_\mu = \text{soc} D_\mu$, as required.

The same argument works for the radical, which we recall is the intersection of the maximal proper submodules, except that intersections need not distribute over sums. The additional condition on $\mathcal{M}$ guarantees this \[Ben\]. The proof of \[6\] is thus complete and the proof of \[7\] now follows similarly to that of \[5\].

**Remark 3.9.** It is not clear if the condition imposed on $\mathcal{M}$ in the radical parts \[6\] and \[7\] is required. However, if $\text{rad} \mathcal{M}$ decomposes as $\text{rad} \mathcal{M} = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes R_\mu$, then without this condition, the argument used in the proof only establishes that $R_\mu \subseteq \text{rad} D_\mu$, for each $\mu \in \mathcal{M}$.

Unlike the $C_{[\mu]}$ in \[3.13\], the coset modules $D_{[\mu]}$, $[\mu] \in \mathcal{M}/\mathcal{N}$, appearing in \[3.17\] need not be mutually non-isomorphic. We shall illustrate this with a simple example in Section 3.3. In the following section, we first give three useful criteria which guarantee that the $D_{[\mu]}$ are all non-isomorphic.

### 3.2. Criteria for being multiplicity-free

In this section, we discuss whether the decomposition \[3.17\] is multiplicity-free or not. In other words, we investigate when one can assert that the $D_\mu$ or the $D_{[\mu]}$ are mutually non-isomorphic, in the notation of Theorem 3.8.

#### 3.2.1. Criterion based on conformal weights

It may so happen that the conformal weights of the highest-weight vectors of the Heisenberg subalgebra $H$ immediately rule out multiplicities. For example, consider the case of an affine vertex operator algebra $V$ of negative level $k$ and a $V$-module $M$ whose conformal weights are bounded below. We shall assume, as in Theorem 3.8, that the corresponding set $\mathcal{M}$ is a single orbit of $L$. Suppose that the decomposition of $M$ is not multiplicity-free, so that $D_{\mu + \lambda} = D_\mu$, for some $\lambda \in L$. Then, $C_\lambda \subseteq C_{[\mu]}$, $D_\mu = D_{[\mu]}$ and so $D_{\mu + n \lambda} = D_\mu$, for all $n \in \mathbb{Z}$. However, the conformal weight of the highest-weight vector of $F_{\mu + n \lambda}$ is $\frac{1}{k} \| \mu + n \lambda \|^2$, which becomes arbitrarily negative for $|n|$ large, because $k < 0$. It follows that the conformal weights of $F_{\mu + n \lambda} \otimes D_{\mu + n \lambda} = F_{\mu + n \lambda} \otimes D_\mu$ would become arbitrarily negative, for all $\mu \in \mathcal{M}$. This contradicts the hypothesis that the conformal weights of $M = \bigoplus_{\mu \in \mathcal{M}} F_\mu \otimes D_\mu$ are bounded below, hence the $D_\mu$, with $\mu \in \mathcal{M}$, must all be mutually non-isomorphic.

#### 3.2.2. Criterion based on symmetries of characters

We can also derive a simple test to rule out multiplicities using the characters

$$
\text{ch}[F_\mu](z;q) = \text{tr}_{F_\mu} z^{h_0} q^{L_0} e^{-c/24} = z^{h_0} q^{\frac{|\mu|^2}{2}} \frac{\eta(q)}{\eta(q)}
$$

of the Fock modules. This relies on the fact that the characters of the $D_\mu$ appearing in \[3.17\] will not depend on $z$. We remark that the factors $z^{h_0}$ and $z^\mu$ should be interpreted here as $z^{h_0} \cdots z^{h_0}$ and $z^{\mu_1} \cdots z^{\mu_r}$, respectively, where $r$ is the rank of the Heisenberg vertex operator algebra $H$.

Suppose, for simplicity, that $\mathcal{M}$ consists of a single $L$-orbit, as in Theorem 3.8. Define $\mathcal{N}'$ to be the sublattice of Heisenberg weights $\lambda$ such that $D_\mu = D_{\lambda + \mu}$, for every $\mu \in \mathcal{M}$, so that $\mathcal{N} \subseteq \mathcal{N}' \subseteq L$. It follows that for every $\lambda \in \mathcal{N}'$, the character of the decomposition \[3.17\] must satisfy

$$
\text{ch}[M](z;q; \ldots) = \sum_{\mu \in \mathcal{M}} z^{\lambda} q^{\frac{|\lambda|^2}{2}} \frac{\text{ch}[D_\mu]}{\eta(q)} \sum_{\mu \in \mathcal{M}} z^{\lambda + \mu} q^{\frac{|\lambda + \mu|^2}{2}} \frac{\text{ch}[D_\mu]}{\eta(q)} = z^{\lambda} q^{\frac{|\lambda|^2}{2}} \sum_{\mu \in \mathcal{M}} z^{\lambda} q^{\frac{|\lambda + \mu|^2}{2}} \frac{\text{ch}[D_\mu]}{\eta(q)} = z^{\lambda} q^{\frac{|\lambda|^2}{2}} \frac{\text{ch}[M](q; \ldots)}{\eta(q)}.
$$

\[3.21\]
where \( q^A \) acts on a Heisenberg weight \( \mu \) to give \( q^{(A, \mu)} \). If the character of \( M \) only satisfies this equation when \( A \in \mathbb{N} \), then we may conclude that the \( D_{[\mu]} \), with \([\mu] \in M/\mathbb{N}\), are mutually non-isomorphic. In the case that \( N = 0 \), this conclusion gives the mutual inequivalence of the \( D_{[\mu]} \) for all \( \mu \in M \).

### 3.2.3. Criterion based on open Hopf links

In the case of rational vertex operator algebras, the closed Hopf links are, up to normalization, the same as the entries of the modular S-matrix [Hu2]. There is also a close connection between Hopf links and properties of characters for non-rational vertex operator algebras [CG1, CG2, CMR]. We will now explain how Hopf links give a criterion for the existence of fixed points under the action of fusing with a simple current. For this subsection, we assume that we are working in a ribbon category \( \mathcal{C} \) of vertex operator algebra modules [EGNO], such categories allow us to take (partial) traces of morphisms.

Let \( J \in \mathcal{C} \) be a simple current and fix a module \( X \in \mathcal{C} \). Assume that there exists a positive integer \( s \) such that \( J^s \boxtimes X \cong X \), so that \( X \) is a fixed point of \( J^s \). Recall that the monodromy of two modules \( A \) and \( B \) is defined by \( M_{A,B} = R_{B,A} \circ R_{A,B} \), where \( R \) denotes their braiding. Recall the notion [EGNO, Def. 8.10.1] of categorical twist, \( \theta \), which is a system of natural isomorphisms. The monodromy satisfies the following balancing for any two modules \( A, B \):

\[
\theta_{A \boxtimes B} = M_{A,B} \circ (\theta_A \boxtimes \theta_B).
\]

In vertex-tensor-categorical setup, \( \theta \) is given by \( e^{2\pi i \theta_0} \). We will also need the open Hopf link operators from [CG1, CG2]. These are defined as the partial traces \( \Phi_{A,B} = \text{ptr}(M_{A,B}) \in \text{End}(B) \) and have the important property that they define a representation of the fusion ring on \( \text{End}(B) \). In particular, it follows that \( \Phi_{X\boxtimes P} = \Phi_{J,P} \circ \Phi_{X,P} \), for any module \( P \in \mathcal{C} \), and hence that

\[
\Phi_{X,P} = \Phi_{J,P} \circ \Phi_{X,P} = \Phi_{J,P} \circ \Phi_{X,P} = \Phi_{J,P} \circ \Phi_{X,P}.
\]

We shall assume now that \( P \) is indecomposable with a finite number of composition factors, so that every endomorphism of \( P \) has a single eigenvalue, and that \( M_{J,P}, \Phi_{J,P} \) are semi-simple endomorphisms of \( J \boxtimes P \) and \( P \), respectively. The latter assumption will be automatically satisfied if \( J \) is a simple current of finite order and both \( \text{End}(P) \) and \( \text{End}(J \boxtimes P) \) are finite-dimensional [CKL, Lem. 2.13]. It will also be satisfied if \( P \) may be identified with a subquotient of an iterated fusion product of simple modules [CKL, Lem. 3.19].

With these assumptions on \( P \), Eq. (3.22) shows that the image of \( \Phi_{X,P} \) is contained in the eigenspace of \( \Phi_{J,P} \) with eigenvalue 1 and that this eigenspace is either 0 or \( P \) itself. We therefore have two possible conclusions: \( \Phi_{X,P} = 0 \) or \( \Phi_{J,P} \circ \Phi_{X,P} = \text{Id}_P \).

Following [CG1], we say that a full subcategory \( \mathcal{P} \) of \( \mathcal{C} \) is a left ideal if for all \( Q \in \mathcal{P} \), we have both \( D \boxtimes Q \in \mathcal{P} \), for all \( D \in \mathcal{C} \), and that \( D \in \mathcal{P} \) whenever the composition \( D \to Q \to D \) is the identity. We shall assume that \( \mathcal{P} \) is equipped with a modified trace \( t_{\mathcal{C}} \) ([CG1, GKP] for \( \mathcal{P} = \mathcal{C} \), the modified trace is just the ordinary trace \( t = tr \)) and a modified dimension \( d(\bullet) = t_{\mathcal{C}}(\text{Id}_\bullet) \). We also let \( \text{dim}(\bullet) = \text{tr}(\text{Id}_\bullet) \) denote the ordinary trace of the identity morphism.

We now assume that \( P \), as introduced above, belongs to a left ideal \( \mathcal{P} \) of \( \mathcal{C} \). For any object \( D \in \mathcal{C} \), the properties of the modified trace imply that

\[
\begin{align*}
t_{\mathcal{C}}(\text{Id}_D) &= t_{\mathcal{P}}(\text{Id}_D) \\
t_{\mathcal{C}}(D \boxtimes P) &= t_{\mathcal{P}}(\text{ptr}(D \boxtimes P)) = t_{\mathcal{P}}(\text{tr}(P) \boxtimes 1) \\
&= \text{dim}(D) t_{\mathcal{P}}(\text{Id}_P) = \dim(D) d(P)
\end{align*}
\]

and hence that

\[
\begin{align*}
t_{\mathcal{C}}(\Phi_{J,P}) &= t_{\mathcal{P}}(\text{ptr}(J_{J,P})) = t_{\mathcal{P}}(\text{ptr}(M_{J,P})) \\
&= t_{\mathcal{P}}(\Phi_{J,P} \circ \Phi_{X,P}) = t_{\mathcal{P}}(\Theta_{J,P} \circ (\Phi_{J,P} \circ (\Theta_{J,P}^{-1} \boxtimes \Theta_{J,P}^{-1})))
\end{align*}
\]
Here, we have used the balancing property of monodromy and have identified \( \theta_{\ell^2} \) with the scalar by which it acts. In the case that \( \Phi_{\ell^2} = \text{Id}_p \), so \( t_p(\Phi_{\ell^2}) = t_p(\text{Id}_p) = d(P) \), it follows that \( \dim(J)^\ell(\theta_{\ell^2} \otimes (\theta_{\ell^2}^{-1} \otimes \theta_{\ell^2}^{-1})) = 1 \), whenever \( d(P) \neq 0 \). We summarize this as follows.

**Proposition 3.10.** Let \( \mathcal{C} \) be a ribbon category, \( J \in \mathcal{C} \) be a simple current and \( X \in \mathcal{C} \) be a fixed point of \( J^\ell \) so that \( J^\ell \otimes X \cong X \), for some \( s \in \mathbb{Z}_{>0} \). Let \( \mathcal{P} \) be a left ideal of \( \mathcal{C} \), equipped with a modified trace \( t_\bullet \) and modified dimension \( d(\bullet) \). Let \( P \in \mathcal{P} \) be indecomposable such that \( d(P) \neq 0 \) and \( M_{J,P} \), \( \Phi_{J,P} \in \text{End}(P) \) are semisimple endomorphisms. Then, one of the following must hold:

1. \( \Phi_{X,P} = 0 \), which in turn implies that \( t_p(\Phi_{X,P}) = 0 \). If \( \mathcal{C} \) is a modular tensor category, then this implies that the corresponding modular S-matrix entry is zero.
2. \( \dim(J)^\ell(\theta_{\ell^2} \otimes (\theta_{\ell^2}^{-1} \otimes \theta_{\ell^2}^{-1})) = 1 \), where we have identified \( \theta_{\ell^2} \otimes (\theta_{\ell^2}^{-1} \otimes \theta_{\ell^2}^{-1}) \) with the scalar by which it acts.

As these quantities are computable, in principle, we can rule out fixed points for \( J = C_\lambda \) or \( W_{[\lambda]} \) and thereby deduce a multiplicity-free decomposition. We shall illustrate this proposition below in a rational example.

### 3.3. Examples

Here, we give a selection of simple examples involving the so-called parafermion cosets to illustrate the theory developed in this section. Let \( L_k(\hat{g}) \) denote the simple vertex operator algebra of level \( k \) associated with the affine Kac-Moody (super)algebra \( \hat{g} \). Given a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \), let \( H \subset L_k(\hat{g}) \) be the corresponding Heisenberg vertex operator subalgebra. The commutant \( C = \text{Com}(H, L_k(\hat{g})) \) is called the level \( k \) parafermion vertex operator algebra of type \( \mathfrak{g} \).

**Example 1.** For \( \mathfrak{g} = sl_2 \) and \( k = 2 \), the parafermion coset is the Virasoro minimal model \( M(3,4) \), also known as the Ising model. The decompositions (3.11) and (3.13) become

\[
L_2(sl_2) = \bigoplus_{\lambda \in 4\mathbb{Z}} \left[ F_\lambda \otimes K_0 \oplus F_{\lambda+2} \otimes K_{1/2} \right] = W_{[0]} \otimes K_0 \oplus W_{[2]} \otimes K_{1/2},
\]

(3.25)

where \( K_\lambda \) denotes the simple \( M(3,4) \)-module of highest weight \( h \), the lattice of \( H \)-weights of \( L_2(sl_2) \) is \( \mathcal{L} = 2\mathbb{Z} \), and the sublattice of \( H \)-weights giving isomorphic coset modules is \( \mathcal{N} = 4\mathbb{Z} \). The convention here for \( F_\lambda \) is that \( \lambda \) indicates the \( sl_2 \)-weight so that the conformal dimension of this Heisenberg module is \( \lambda^2 \). The lattice vertex operator algebra \( W \) is thus obtained by extending \( H \) by the group of simple currents generated by \( F_4 \).

The representation theory of \( L_2(sl_2) \) is semisimple and it has three simple modules \( M^0 \), \( \omega = 0,1,2 \), which are distinguished by the Dynkin labels \( k - \omega, \omega \) of their highest weights. \( L_2(sl_2) \) is identified with \( M^0 \) and the decomposition corresponding to (3.25) for \( M^2 \) is obtained by swapping \( K_0 \) with \( K_{1/2} \). In particular, the \( \mathcal{L} \)-orbit for \( M^2 \) is also \( \mathcal{M} = 2\mathbb{Z} \). The situation for \( M^1 \) is, however, slightly different:

\[
M^1 = \bigoplus_{\mu \in 2\mathbb{Z}+1} F_\mu \otimes K_{1/16} = W_{[1]} \otimes K_{1/16} \oplus W_{[-1]} \otimes K_{1/16}.
\]

(3.26)

Here, \( \mathcal{M} = 2\mathbb{Z} + 1 \) and \( \mathcal{N}' = 2\mathbb{Z} \neq \mathcal{N} \) (the non-isomorphic lattice modules are paired with isomorphic coset modules). In other words, this decomposition fails to be multiplicity-free.

To see that this is consistent with the criterion of Section 3.2.2, recall that \( sl_2 \) admits a family \( \sigma^\ell, \ell \in \mathbb{Z} \), of spectral flow automorphisms that lift to automorphisms of the corresponding affine vertex algebras. The latter may be used to twist the action on an \( L_k(sl_2) \)-module \( M \) and thereby construct new modules \( \sigma^\ell(M) \).
Using the conventions of [11], the characters of \( M \) and \( \sigma(M) \) are related by

\[
\text{ch} [\sigma(M)](z; q) = z^{k} q^{3k/4} \text{ch} [M](z^{1/2}; q).
\] (3.27)

For \( k = 2 \), spectral flow acts on the simple modules as \( \sigma(M^0) = M^{2-\omega} \), \( \omega = 0, 1, 2 \). Identifying the weight space of \( sl_2 \) with \( \mathbb{C} \) and noting that the scalar product on this space is then \( \langle \lambda, \mu \rangle = \frac{1}{2} \lambda \mu \), the criterion of Section 3.2.2 asks us to check which \( \lambda \in \mathbb{C} \) satisfy the relation

\[
\text{ch} [M^0](z; q) = z^{\lambda} q^{\lambda^2/8} \text{ch} [M^0](z^{1/2}; q) = \text{ch} [\sigma(M^0)](z; q),
\] (3.28)

for a given \( M^0 \). Since \( \sigma^2 \) acts as the identity, this relation holds for each \( \omega \) if \( \lambda \in \mathbb{N} = 4 \mathbb{Z} \). If \( \omega \neq 1 \), then it does not hold for \( \lambda = 2 \), hence \( \mathcal{N} = 4 \mathbb{Z} \) and both \( M^0 \) and \( M^2 \) have multiplicity-free decompositions in terms of lattice modules. However, this relation does hold for \( \omega = 1 \) and \( \lambda = 2 \), so we cannot conclude that the lattice decomposition of \( M^1 \) is multiplicity-free (consistent with our explicit calculation that it is not).

With a little more work, we can also see how this failure is consistent with the criterion of Section 3.2.3. Let \( X = K_{1/16} \) and let \( J \) be the simple current \( K_{1/2} \), so that \( X \) is a fixed point for \( J \): \( J \otimes X \cong X \). Since \( L_2(sl_2) \) is a unitary vertex operator algebra, \( \dim(J) = 1 \). Also, as recalled above, \( \theta \) is given by \( e^{\beta \pi \theta} \), hence, in our notation, it acts on \( K_4 \) by \( e^{2 \beta \pi i} \), where \( t = 0, 1/2, 1/16 \). Further, it is easy to check that the category \( \mathscr{C} \) of \( M(3,4) \)-modules has no non-trivial ideals except for \( \mathscr{C} \) itself.

We now verify that for every indecomposable \( P \) in \( \mathscr{C} \), either condition 1 or 2 of our Hopf link criterion is satisfied.

\( P = K_0 \): In this case, \( \theta_{\otimes \otimes P} \circ (\theta_{J}^{-1} \boxtimes \theta_{Y}^{-1}) = \theta_{K_{1/2}} \circ (\theta_{K_{1/2}}^{-1} \boxtimes \theta_{K_{0}}^{-1}) = 1. \)

\( P = K_{1/2} \): In this case, \( \theta_{\otimes \otimes P} \circ (\theta_{J}^{-1} \boxtimes \theta_{Y}^{-1}) = \theta_{K_{0}} \circ (\theta_{K_{1/2}}^{-1} \boxtimes \theta_{K_{1/2}}^{-1}) = 1. \)

\( P = K_{1/16} \): In this case, \( \theta_{\otimes \otimes P} \circ (\theta_{J}^{-1} \boxtimes \theta_{Y}^{-1}) = \theta_{K_{1/16}} \circ (\theta_{K_{1/2}}^{-1} \boxtimes \theta_{K_{1/16}}^{-1}) = -1 \), but the modular \( S \)-matrix of \( M(3,4) \) has entry \( S_{K_{1/16},K_{1/16}} = 0. \)

So we see that in the first two cases condition 2 is satisfied while condition 1 holds in the last. This is, of course, consistent with the fact that the decomposition is not multiplicity-free. As an aside, we remark that if we had only known that \( K_{1/16} \) was a fixed-point of the simple current (which implies that the decomposition is not multiplicity-free), then we could have instead deduced that \( S_{K_{1/16},K_{1/16}} \) must vanish, as above.

**Example 2.** A more interesting example is the parafermion coset with \( g = sl_2 \) at level \( k = -\frac{4}{3} \). In [Ad3], Adamović showed that the resulting coset vertex operator algebra is the (simple) singlet algebra \( l(1,3) \) of central charge \( c = -7 \). This is strongly generated by the energy-momentum tensor and a single conformal primary of weight 5. We can revisit and extend this study using the results of this section. However, we stress that the parent vertex operator algebra \( L_{-4/3}(sl_2) \) does not satisfy the conditions of Section 3.1 that would allow us to apply the theory of Huang-Lepowsky-Zhang. Nevertheless, we shall proceed with the analysis, assuming that this theory may be applied. The results suggest that this assumption is, in this case, not unreasonable.

Let \( \Lambda_0 \) and \( \Lambda_1 \) denote the fundamental weights of \( sl_2 \). The vertex operator algebra \( L_{-4/3}(sl_2) \) admits precisely three highest-weight modules, namely the simple modules \( M^0 \) whose highest weights have the form \( (k - \omega)\Lambda_0 + \omega \Lambda_1 \), where \( \omega \in \{0, -\frac{4}{3}, -\frac{4}{3} \} \), as well as an uncountable number of simple non-highest-weight modules [AM1, Ga, RW1]. In particular, \( l(1,3) \) is not a rational vertex operator algebra. As the level is negative and these highest-weight modules have conformal weights that are bounded below, the criterion of Section 3.2.1 applies and we conclude that their decompositions are multiplicity-free.
Explicitly, the decomposition (3.11) takes the form
\[
L_{-4/3}(\mathfrak{s}l_2) = \bigoplus_{\lambda \in 2\mathbb{Z}} F_{\lambda} \otimes C_{\lambda},
\] (3.29)
where \(C_{\lambda}\) is a simple highest-weight \(l(1,3)\)-module whose highest-weight vector has conformal weight
\[
\Delta_{\lambda} = \frac{3}{16}|\lambda|(3|\lambda| + 8).
\]
The convention here for \(F_{\lambda}\) is again that \(\lambda\) indicates the \(\mathfrak{s}l_2\)-weight so that the conformal dimension of this Heisenberg module is \(-\frac{1}{16} \lambda^2\). Of course, \(C_{\lambda}\) and \(C_{-\lambda}\) are not isomorphic for \(\lambda \neq 0\) because the decomposition (3.29) is multiplicity-free — they must therefore be distinguished by the action of the zero mode of the weight 5 conformal primary.

The theory of Section 3.1 shows that the \(C_{\lambda}\), with \(\lambda \in 2\mathbb{Z}\), are all (non-isomorphic) simple currents. This had been previously deduced [RW1,CM1] from the (conjectural) standard Verlinde formula of [CR4,RW2] for non-rational vertex operator algebras. Noting that \(C_{\pm 4} = 5\), we remark [CRW,RW1] that the simple current extension of \(l(1,3)\) by the \(C_{\lambda}\), with \(\lambda \in 4\mathbb{Z}\), is the triplet algebra \(W(1,3)\) of Kausch [Ka].

Consider now the \(L_{-4/3}(\mathfrak{s}l_2)\)-modules \(\sigma^{-2}(M^{-2/3})\) and \(\sigma(M^{-2/3})\), obtained by twisting the action on \(M^{-2/3}\) by the spectral flow automorphisms \(\sigma^\ell\), \(\ell \in \mathbb{Z}\). Whilst both these modules have conformal weights that are unbounded below, their decompositions into \(H \otimes l(1,3)\)-modules are nevertheless multiplicity-free:
\[
\sigma^{-2}(M^{-2/3}) = \sum_{\mu \in 2\mathbb{Z}} F_\mu \otimes D_\mu^{(-2)}, \quad \sigma(M^{-2/3}) = \sum_{\mu \in 2\mathbb{Z}} F_\mu \otimes D_\mu^{(1)}. \quad (3.30)
\]
Here, the \(D_\mu^{(-2)}\) and \(D_\mu^{(1)}\) are simple highest-weight \(l(1,3)\)-modules whose highest-weight vectors have conformal weights given by
\[
\Delta_\mu^{(-2)} = \begin{cases} \frac{1}{16}(\mu + 3\mu + 8) & \text{if } \mu \leq -2, \\ \frac{1}{16}(\mu + 4)(3\mu + 4) & \text{if } \mu \geq -2 \end{cases}
\]
and
\[
\Delta_\mu^{(1)} = \begin{cases} \frac{1}{16}(\mu - 4)(3\mu - 4) & \text{if } \mu \leq 2, \\ \frac{1}{16}(\mu)(3\mu - 8) & \text{if } \mu \geq 2, \end{cases}
\]
respectively.

The interesting thing about the \(L_{-4/3}(\mathfrak{s}l_2)\)-modules \(\sigma^{-2}(M^{-2/3})\) and \(\sigma(M^{-2/3})\) is that they appear, together with two copies of the vacuum module \(M^0\), as the composition factors of an indecomposable \(L_{-4/3}(\mathfrak{s}l_2)\)-module \(P^0\). This module was first constructed as a fusion product in [Ga] and was structurally characterised in [CR3] (see [AM3] for a construction and characterisation of a different indecomposable \(L_{-4/3}(\mathfrak{s}l_2)\)-module). The action of the Virasoro zero mode \(L_0\) on \(P^0\) is non-semisimple. The Loewy diagram for \(P^0\) has the form
\[
\begin{array}{c}
\bullet \\
\sigma^{-2}(M^{-2/3}) & P^0 & \sigma(M^{-2/3}) \\
\bigcirc \\
\end{array}
\]
where our convention is that the socle appears at the bottom. An immediate consequence of Theorem 3.8 is that there exists a countably-infinite number of mutually non-isomorphic indecomposable \(l(1,3)\)-modules \(P^0_\mu, \mu \in 2\mathbb{Z}\), on which the \(l(1,3)\) Virasoro zero mode acts non-semisimply. The Loewy diagrams of these
The existence of such $I(1,3)$-modules was predicted in [RW1] from the fact that similar indecomposables have been constructed [AM3, TW1] for a simple current extension, the triplet algebra $W(1,3)$.

4. Properties of Heisenberg Cosets

Recall from the introduction that one of our main applications for Heisenberg cosets is to construct new, potentially $C_2$-cofinite, vertex operator algebras as extensions:

$$V \xrightarrow{H\text{-coset}} C \xrightarrow{\text{extension}} E.$$ 

So far, we understand how $V$-modules decompose as $H \otimes C$-modules. The remaining tasks are to identify when $C$ may be extended by certain abelian intertwining algebras to a larger algebra $E$. This will be stated in Theorem 4.1. Since abelian intertwining algebra extensions are mild generalizations of simple current extensions, analogous arguments to [CKL] allow us to give precise criteria for the lifting of $H \otimes C$-modules to $V$-modules, see Theorem 4.3. An analogous criterion for the lifting of $C$-modules to $E$-modules is given in Corollary 4.4.

4.1. Extended Algebras. If certain Fock modules involved in the vertex operator algebra decomposition yield a lattice (super) vertex operator algebra, then the corresponding coset modules form a (super) vertex operator algebra as well. Thus, we get extensions of the coset.

**Theorem 4.1.** Let

$$V = \bigoplus_{\lambda \in L} F_\lambda \otimes C_\lambda.$$ 

If $E$ is a sub-lattice of $L$, such that $\bigoplus_{\lambda \in E} F_\lambda$ forms a lattice vertex operator algebra, then $E = \bigoplus_{\lambda \in E} C_\lambda$ has a natural vertex operator algebra structure.

**Proof.** This result is an immediate corollary of [Li, Thm. 3.1, 3.2] with $\ell = 1$, see also [DL, Li, Thm. 3.1, 3.2] in fact guarantee a generalized vertex algebra structure on $\bigoplus_{\lambda \in L} C_\lambda$. Note that no restrictions with regards to vertex tensor category theory are needed on $V$ or $C$. 

For a more general scenario involving mirror extensions, see [Lin].

**Example 3.** Let $\mathfrak{g}$ be a simple simply laced Lie algebra and let $k = \frac{p}{q} \neq 0$ be a rational number ($p, q$ co-prime). We do not require it to be an admissible level. Then $L_k(\mathfrak{g})$ is graded by $\frac{\sqrt{2}}{\sqrt{p}} Q = \sqrt{2} Q$ with $Q$ the root lattice, that is

$$L_k(\mathfrak{g}) = \bigoplus_{\lambda \in \sqrt{2} Q} F_\lambda \otimes C_\lambda.$$ 

The sublattice $p \sqrt{\frac{2}{p}} Q = \sqrt{pq} Q$ is an even sublattice so that

$$V_{\sqrt{pq} Q} = \bigoplus_{\lambda \in \sqrt{pq} Q} F_\lambda$$
is a lattice vertex operator algebra. It follows by Theorem 4.1 that
\[ E_{k,g} := \bigoplus_{\lambda \in \sqrt{pq} \mathbb{Q}} C_{\lambda} \]
is also a vertex operator algebra.

We believe that these extended vertex operator algebras have a good chance to be \( C_2 \)-cofinite. The main outcome of [ACR] is that in the case \( g = sl_2 \) and \( k + 2 \in \mathbb{Q} \setminus \{ \frac{1}{n} | n \in \mathbb{Z}_{>0} \} \) the characters of modules of the extended vertex operator algebra are modular if supplemented by pseudotraces.

In two specific examples \( C_2 \)-cofiniteness is already known. One of them is \( L_{-4/3}(sl_2) \). This is then a continuation of Example 2. Recall that \( L_{-4/3}(sl_2) = \bigoplus_{\lambda \in 2 \mathbb{Z}} F_{\lambda} \otimes C_{\lambda} \) (4.1)
is the lattice vertex operator algebra of the lattice \( \mathcal{L} = \sqrt{-6} \mathbb{Z} \) and hence
\[ W(1,3) = \bigoplus_{\lambda \in 4\mathbb{Z}} C_{\lambda} \] (4.2)
is also a vertex operator algebra. It is actually the \( W(1,3) \)-triplet that is well-known to be \( C_2 \)-cofinite [AM2]. This relation between singlet vertex operator algebra and \( L_{-4/3}(sl_2) \) has been first realized by Adamović [Ad3] and has a nice generalization to a relation between singlet vertex operator algebras and certain \( W \)-algebras [CRW].

**Example 4.** \( I(2) \)-singlet algebra and \( V_k(gl(1|1)) \)

We first illustrate how well-known somehow archetypical logarithmic VOAs are related via simple current extensions and Heisenberg cosets thus nicely illustrating the picture advocated in this work together with [CKL]. The picture is as follows:

\[ V_k(gl(1|1)) \] extension \[ L_{-1/2}(sl_2) \] extension \[ \beta \gamma \otimes V_Z \]
\[ H \otimes I(2) \] extension \[ L_{-1/2}(sl_2) \] extension \[ \beta \gamma \]
\[ I(2) \] extension \[ W(2) \]

\( I(2) \) is the \( p = 2 \) singlet VOA [AM1,CMI] and \( W(2) \) is its \( C_2 \)-cofinite but non-rational infinite order simple current extensions, called the triplet. See e.g. [AM2].

These and other extensions have been worked out in [CR1,CR2,AC] while the coset picture has been part of [CR1,CRo,CRW]. Here, the situation of the singlet algebra \( I(2) \) is that \( C_1 \)-cofiniteness of all known admissible modules is established [CMR], fusion coefficients are known [AM4] and the category of \( C_1 \)-cofinite modules is a vertex tensor category in the sense of [HLZ] provided that every \( C_1 \)-cofinite \( \mathbb{N} \)-gradable module is of finite length [CMR Thm. 17].
For reference on $l(2)$-modules we refer to [AM1][CM1]. As reference on $V_k(\mathfrak{gl}(1|1))$ we refer to [CR1]. $l(2)$ has simple typical modules $F_\lambda$ of conformal weight $\frac{1}{2}(\lambda - 1)$ for $\lambda \in \mathbb{R} \setminus \mathbb{Z}$. For $\lambda = 1 - r$ integer, we have

$$0 \to M_{r,1} \to F_{1-r} \to M_{r+1,1} \to 0$$

for simple atypical modules $M_{r,1}$ and $r$ integer. Similarly, $V_k(\mathfrak{gl}(1|1))$ has simple highest-weight modules $V_{e,n}$ where the real numbers $e, n$ are the weight labels, and $e/k$ not integer. If $e/k$ is integer say $\ell$, then the highest-weight-module decomposes as

$$0 \to A_{n-1,k} \to V_{n,\ell k} \to A_{n,\ell k} \to 0$$

with simple atypical modules $A_{n,\ell k}$ parameterized by real $n$ and integers $\ell$. The projective covers $P_{n,\ell k}$ have the form

$$0 \to V_{n+1,\ell k} \to P_{n,\ell k} \to V_{n,\ell k} \to 0.$$ 

The commutant of $l(2)$ in $V_k(\mathfrak{gl}(1|1))$ is a rank two Heisenberg vertex operator algebra, and we denote their Fock-modules by $F_{e,n}$ where we take the notation of [CRo]. Using the explicit realization of $V_k(\mathfrak{gl}(1|1))$-modules of [CRo] we can compute the decomposition of modules. The answer is as follows

$$V_k(\mathfrak{gl}(1|1)) = \bigoplus_{m \in \mathbb{Z}} F_{0,m} \otimes M_{m+1,1}, \quad A_{n,\ell k} = \bigoplus_{m \in \mathbb{Z}} F_{-\ell,m-n} \otimes M_{m,1} \quad \text{and} \quad V_{-e,-n+1} = \bigoplus_{m \in \mathbb{Z}} F_{\frac{e}{k},n+m} \otimes F_{\frac{e}{k}-m}. \quad (4.3)$$

It follows with Theorem 3.8 that

$$P_{n,\ell k} = \bigoplus_{m \in \mathbb{Z}} F_{-\ell,m+n} \otimes S_m,$$

where $S_m$ is an indecomposable $l(2)$-module that has non-split short-exact sequence

$$0 \to F_{1-m} \to S_m \to F_{2-m} \to 0.$$ 

In terms of Loewy diagrams, we have the following:

$$P_{n,\ell k} = \begin{array}{c}
A_{n,\ell k} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
A_{n-1,\ell k} \quad = \quad \bigoplus_{m \in \mathbb{Z}} F_{-\ell,m-n} \otimes \left[ \begin{array}{c}
M_{m+1,1} \\
M_{m,1} \\
M_{m+1,1} \\
M_{m,1} \\
M_{m+1,1} \\
M_{m,1}
\end{array} \right]
\end{array}.$$ 

The triplet algebra $W(2)$ is known to be $C_2$-cofinite but non-rational. It is a simple current extension of $l(2)$, namely

$$W(2) = \bigoplus_{m \in \mathbb{Z}} M_{1+2m,1}.$$ 

4.2. Lifting Coset Modules. In this subsection, we show that whether certain generalized $C$-modules $D$ could be tensored with appropriate Fock modules so that the product can be induced (lifted) to a $V$-module is essentially decided by the monodromy

$$M_{C_\lambda,D} = R_{D,C_\lambda} \circ R_{C_\lambda,D} : C_\lambda \boxtimes D \to C_\lambda \boxtimes D.$$ 

For properties of the monodromy used here, we refer to [CKL].

The following lemma could be easily proved as in [CKL] and will be used frequently below. For a vertex operator algebra $V$, and its vertex tensor category $\mathcal{C}$. Let $\text{Pic}_\mathcal{C}(V)$ denote the Picard groupoid
(see [Ca, FRS]). That is, Pic\(_{\mathcal{C}}(V)\) is the full subcategory of simple currents. Clearly Pic\(_{\mathcal{C}}(V)\) is closed under tensor product.

**Lemma 4.2.** Let \(X \in \mathcal{C}\) be such that for \(J_i \in \text{Pic}_{\mathcal{C}}(V)\), \(M_{J_i,X} = \lambda_i \cdot X \cdot \text{Id}_{\mathcal{S}X}\) where \(\lambda_i \cdot X \in \mathcal{C}\) for \(i = 1, 2\). Then, \(\lambda_{J_1} \cdot X \cdot \lambda_{J_2} \cdot X = \lambda_{J_1 \otimes J_2} \cdot X\).

**Theorem 4.3.** Let \(V, H, C, \mathcal{L}\) be as in Theorem 3.5, let \(\mathcal{L}'\) be the dual lattice, \(U = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}\) and let \(D\) be a generalized \(C\)-module that appears as a subquotient of fusion product of some simple \(C\)-modules. Then, there exists \(\alpha\) in \(U\), such that for all \(\lambda \in \mathcal{L}\),

\[
M_{C, D} = e^{-2\pi i \langle \alpha, \lambda \rangle} \text{Id}_{C, \otimes D}
\]

and \(F_\beta \otimes D\) lifts to a \(V\)-module if and only if \(\beta \in \alpha + \mathcal{L}'\).

**Proof.** Recall that we are working with categories of \(C\) and \(H\) that have real weights for the respective \(L_0\)s. Additionally, recall that we are working over semi-simple category for \(H\) and a category for \(C\) each object of which has globally bounded \(L_0\)-Jordan blocks.

We know that \(\mathcal{L}\) is equipped with a symmetric nondegenerate bilinear form \(\langle \cdot, \cdot \rangle\), and this form takes real values since the conformal weights with respect to the Heisenberg are real. By non-degeneracy of \(\langle \cdot, \cdot \rangle\), given a homomorphism \(f : \mathcal{L} \to S^1\), there exists an \(\alpha \in U\) such that

\[
f(\lambda) = e^{2\pi i \langle \alpha, \lambda \rangle}
\]

for all \(\lambda \in \mathcal{L}\). Moreover, \(\beta \in U\) satisfies Eq. \((4.4)\) if and only if \(\beta \in \alpha + \mathcal{L}'\).

Since each of the \(C_\lambda\) is a simple current, by results in [CKL], we know that the monodromy \(M_{C_\lambda, D} = M_\lambda \cdot \text{Id}_{C_\lambda \otimes D}\) for some scalar, say \(M_\lambda \in \mathbb{C}^\times\). Since \(M_{C_\lambda, D}\) is semi-simple and \(C_\lambda \otimes D\) have globally bounded \(L_0\)-Jordan blocks, proceeding as in the proof of \([\text{CKL}, \text{Eq. } (3.10)]\), we gather that \(M_{C_\lambda, D} = (\theta_{C_\lambda \otimes D})_{ss} \circ ((\theta^{-1}_{C_\lambda})_{ss} \otimes (\theta^{-1}_{D})_{ss})\), where \(ss\) denotes the semi-simple part. Since each of the modules involved has real conformal weights, we get that \(M_\lambda = e^{2\pi r_\lambda}\) for some \(r_\lambda \in \mathbb{R}\). So, \(M_\lambda \in S^1\) for all \(\lambda \in \mathcal{L}\). Using Lemma 4.2 we deduce that \(\lambda \mapsto M_\lambda\) is a homomorphism \(\mathcal{L} \to S^1\) and so is \(\lambda \mapsto M_\lambda^{-1}\) since \(S^1\) is abelian.

Now, in Eq. \((4.4)\), we take \(f(\lambda) = M_\lambda^{-1}\) and we get an \(\alpha \in U\) such that \(M_\lambda^{-1} = e^{2\pi i \langle \alpha, \lambda \rangle} = M_\alpha \cdot \text{Id}_\mathcal{C} \otimes D\). Using Proposition 3.3 we conclude that \((F_\alpha \otimes C_\lambda) \otimes (F_\alpha \otimes D) \cong F_{\alpha + \lambda} \otimes (C_\lambda \otimes D)\), and therefore, monodromy factors over the \(\otimes\) tensors. We conclude that \(M_{F_\alpha \otimes C_\lambda, \otimes D} = M_{F_\alpha \otimes C_\lambda} \otimes M_{C_\lambda, D} = 1\). It now follows that \(F_\alpha \otimes D\) lifts. Moreover, from the arguments above we can conclude that \(F_\beta \otimes D\) lifts if and only if \(\beta \in \alpha + \mathcal{L}'\). \(\square\)

We now combine this with extensions of \(C\) as in Theorem 4.1 to deduce the following.

**Corollary 4.4.** Assume the setup of Theorem 4.3. Let \(\mathcal{E}\) be a sublattice of \(\mathcal{L}\) such that \(E = \oplus_{\lambda \in \mathcal{E}} C_\lambda\) has a vertex operator algebra structure inherited from \(V\) exactly as in Theorem 4.1. Then \(D\) lifts to a \(E\)-module \(\oplus_{\lambda \in \mathcal{E}} C_\lambda \otimes D\) iff \(\alpha \in \mathcal{E}'\), where \(\mathcal{E}'\) is the dual lattice of \(\mathcal{E}\).

**Proof.** Recall that each \(C_\lambda\) is a simple current for \(C\). Therefore, using \([\text{HKL}]\) (for the “if” direction) and \([\text{CKL}]\) (for the “only if” direction), we know that \(\oplus_{\lambda \in \mathcal{E}} C_\lambda \otimes D\) is an \(E\)-module iff \(M_{C_\lambda, C_\mu \otimes D} = \text{Id}_{C_\lambda \otimes (C_\mu \otimes D)}\), for all \(\lambda, \mu \in \mathcal{E}\). Since \(E\) is a vertex operator algebra, we know that \(M_{C_\lambda, C_\mu} = \text{Id}_{C_\lambda \otimes C_\mu}\) for all \(\lambda, \mu \in \mathcal{E}\). By properties of monodromy, we gather that \(M_{C_\lambda, C_\mu \otimes D} = \text{Id}_{C_\lambda \otimes (C_\mu \otimes D)}\) for \(\lambda, \mu \in \mathcal{E}\) iff \(M_{C_\lambda, D} = \text{Id}_{C_\lambda \otimes D}\) for all \(\lambda \in \mathcal{E}\), which in turn holds iff \(\alpha \in \mathcal{E}'\). \(\square\)

**Remark 4.5.** Since \(E\) is a simple current extension of \(C\), we can utilize arguments similar to \([\text{Lam}]\) Thm. 4.4] in order to analyze certain simple \(E\)-modules. Let \(X\) be a simple \(E\)-module such that there
exists a simple $C$-module $X_0 \subset X$. (In the notation of $[\text{Lam}]$, the role of group $G$ is played by $P$ and the $\mathcal{V}^X$ are $C_\lambda$ for $\lambda \in P$.) Then, $\mathcal{T}(X_0) = \oplus_{\lambda \in P} C_\lambda \otimes X_0$ has a natural structure of an (induced) $E$-module and it inherits onto $X$.

**Example 5.** We now illustrate the lifting properties with unitary minimal models of the $N = 2$ super Virasoro algebra. We refer the reader to $[\text{Ad1}, \text{Ad2}, \text{DPYZ}]$ and $[\text{S}]$ for additional information on these minimal models. 

We start with some well-known results whose proofs can be found e.g. in $[\text{CL1}]$. Let $k$ be a positive integer, then $L_k(sL_2)$ contains the lattice vertex operator algebra $V_{\mathcal{L}_\mu}$ with $\mathcal{L}_\mu = \alpha \mathbb{Z}$ and $\alpha^2 = 2k$, so $\mathcal{L}_\mu \cong \sqrt{2k}\mathbb{Z}$. The $b\mathbb{c}$-ghost vertex operator algebra $E(1)$ is isomorphic to $V_{\mathcal{L}_\beta}$ with $\mathcal{L}_\beta = \beta \mathbb{Z}$ and $\beta^2 = 1$, so $\mathcal{L}_\beta \cong \mathbb{Z}$. Then the lattice $\mathcal{L}_\mu \oplus \mathcal{L}_\beta$ contains the lattice $\mathcal{L}_\gamma = \gamma \mathbb{Z}$ with $\gamma = \alpha + k\beta$ as sublattice. The orthogonal complement is $N = \alpha \mathbb{Z}$ with $\alpha = \alpha - 2\beta$. Note, that $\gamma^2 = k(k+2)$ and $\mu^2 = 2(k+2)$. In $[\text{CL1}, \text{Sec. 8}]$ it is proved that

$$S_k := \text{Com} \left( V_{\mathcal{L}_\mu}, L_k(sL_2) \otimes E(1) \right)$$

is the simple and rational $N = 2$ super Virasoro algebra at central charge $c = 3k/(k+2)$.

We will now explain how to obtain simple $S_k$-modules. For this let $\lambda$ be an integer with $0 \leq \lambda \leq k$. Further let $\Lambda_0$ and $\Lambda_1$ be the usual fundamental weights of $sL_2$. Then the simple $L_k(sL_2)$-modules are the integrable highest weight modules $L(\lambda)$ of weight $(k-\lambda)\Lambda_0 + \lambda \Lambda_1$. $V_{\mathcal{L}_\alpha + \mathcal{L}_\mu}$ appears in $L(\lambda)$ if and only if $\lambda + n$ is even. This follows directly since $V_{\mathcal{L}_\alpha + \mathcal{L}_\mu}$ appears in the decomposition of $L_k(sL_2)$ if and only if $n$ is even. We now express lattice vectors of $L'_\alpha \oplus L'_\beta$ in terms of those of $L'_\gamma \oplus L'_\mu$, namely,

$$\frac{a}{2k} \alpha + b \beta = (a + bk) \frac{\gamma}{k(k+2)} + (a - 2b) \frac{\mu}{2(k+2)}$$

with $a, b \in \mathbb{Z}$.

It follows that $V_{\mathcal{L}_\alpha + \mathcal{L}_\beta} \otimes N^\mu$ is contained in $L(\lambda) \otimes V_{\mathcal{L}_\beta}$ if and only if $\lambda + n$ is even as well. We thus get

$$\mathcal{L}(\lambda) \otimes V_{\mathcal{L}_\beta} \cong \begin{cases} \bigoplus_{\nu \in 2\mathbb{Z}/\mathcal{L}} V_{\mathcal{L}_\nu} \otimes M(\lambda, \nu) & \text{if } \nu + \lambda \text{ is even} \\ \bigoplus_{\nu \in 2\mathbb{Z}/\mathcal{L}} V_{\mathcal{L}_\nu} \otimes M(\lambda, \nu) & \text{if } \nu + \lambda \text{ is odd} \end{cases}$$

as $V_{\mathcal{L}_\mu} \otimes S_k$-modules. By Theorem 3.8 (2) all $M(\lambda, \nu)$ are simple $S_k$-modules. On the other hand, by Theorem 4.3 for every $\mathcal{L}_k$-module $M$ there exists a $V_N$-module $V_{\nu+N}$ such that

$$V_{\nu+N} \otimes M$$

lifts to a $V_N \otimes S_k$-module if and only if $\rho \in (2\mathbb{Z})^0/\mathbb{N} = \frac{1}{2}\mathbb{N}/\mathbb{N}$. Finally, we announce that the relation between the tensor category of a vertex operator algebra and its extensions can be made quite explicit $[\text{CKM}]$ and that these results imply that every simple $S_k$-module appears in the decomposition of at least one of the $\mathcal{L}(\lambda) \otimes V_{\mathcal{L}_\beta}$ and moreover

$$M(\lambda, \nu) \cong M(\lambda', \nu')$$

if and only if $\lambda' = k - \lambda$ and $\nu' = \nu + \frac{\mu}{2} \mod \mathcal{L}_\mu$.

### 4.3. Rationality

In this section we prove an interesting rationality result: Let $V$ be simple, rational, CFT-type (that is, conformal weights of $V$ are non-negative and the zeroth weight space is spanned by vacuum) and $C_2$-cofinite. Then, Theorem 4.12 states that every grading-restricted generalized $C$-module is completely reducible.

We work with the following setup: Let $C = \text{Com}(H, V)$. Assume that $\text{Com}(C, V) = V_{\mathcal{L}}$, where $(\mathcal{L}, \langle , , \rangle)$ is a positive definite even lattice. With this, $(V_{\mathcal{L}}, C)$ form a commuting pair and $C$ is simple. We now collect
well-known results from the literature that guarantee that we can invoke vertex tensor category theory for 
C, under suitable assumptions on V.

**Lemma 4.6.** If V is C₂-cofinite then so is C. In particular, if V = Lₖ(\(\hat{g}\)) with k ∈ \(\mathbb{N}\) then C is C₂-cofinite.

**Proof.** The proof of the most general statement can be found in [M4]. For the case of V = Lₖ(\(\hat{g}\)) with k ∈ \(\mathbb{N}\), see [ALY]. □

**Lemma 4.7.** If V is simple and CFT-type, then so is C.

**Proof.** Firstly, since \(V_{\mathcal{L}}\) and C form a commuting pair, there exists a non-zero map \(V_{\mathcal{L}} \otimes C \rightarrow V\). Since \(V_{\mathcal{L}} \otimes C\) is both simple, so is \(V_{\mathcal{L}} \otimes C\) and hence this map is an injection. Now, \(1 \otimes C_n \subset V_n\) for any n, in particular, we conclude that \(C_n = 0\) for \(n < 0\) and \(C_0 = \mathbb{C}1\). □

**Lemma 4.8.** If V is simple, C₂-cofinite and self-contragredient, then so is C.

**Proof.** Note that C is simple and we have an injection \(V_{\mathcal{L}} \otimes C \hookrightarrow V\). Since \(V' \cong V\), there exists an invariant bilinear form on V [FHL]. Any invariant form on V is automatically symmetric, by [Li] Prop. 2.6] (see also [FHL]). Moreover, the space of symmetric invariant forms on V is naturally isomorphic to \((V_0/L_1V_1)^*\) [Li] Thm. 3.1]. Since \(V_0 = \mathbb{C}1\), we conclude that \(L_1V_1 = 0\). Now, \(L_1V_1 = 0\) implies that \(L_1(1 \otimes C_1) = 1 \otimes ((L_{\mathcal{L}})_{\mathcal{C}}C_1) = 0\). This implies that \((L_{\mathcal{L}})_{\mathcal{C}}C_1 = 0\). This implies that \(C_0/(L_{\mathcal{L}})_{\mathcal{C}}C_1 \neq 0\), and hence there exists a symmetric invariant bilinear form on C, by [Li] Cor. 3.2]. In other words, \(C' \cong C\). □

**Lemma 4.9.** If V is simple, C₂-cofinite and CFT-type, then

1. The category of grading-restricted generalized modules for V and C satisfy the conditions needed to invoke Huang, Lepowsky and Zhang’s tensor category theory.

2. Denoting the finite abelian group \(\mathcal{L}'/\mathcal{L}\) by \(\mathcal{S}\), there exists a subgroup \(\mathcal{H}\) of \(\mathcal{S}\) such that

\[V = \bigoplus_{\lambda \in \mathcal{H}} V_{\lambda} \otimes C_{\lambda}\.

3. Each \(C_{\lambda}\) appearing above is a simple current for C.

**Proof.** (1) follows from [Hu3] and previous lemmas. (2) and (3) follow from our results above. □

**Lemma 4.10.** Let \((\mathcal{L},\langle \cdot,\cdot\rangle)\) be a positive definite even lattice, \(\mathcal{L}'\) be the dual lattice and let \(\mathcal{S} = \mathcal{L}'/\mathcal{L}\). Then, \(f : \mu \mapsto Q_\mu\) where \(Q_\mu(v) = \exp(2\pi i\langle \mu, v\rangle)\) for \(\mu, v \in \mathcal{S}\) is an isomorphism \(\mathcal{S} \cong \hat{\mathcal{S}}\).

**Proof.** It is clear that the image of f is in \(\hat{\mathcal{S}}\). Let \(\lambda\) be in the kernel of f. Then, we see that \(\langle \lambda, \mathcal{L}'\rangle \subset \mathbb{Z}\), therefore, \(\lambda \in \mathcal{L}' = \mathcal{L}\), hence \(\lambda = 0\) in \(\mathcal{S}\). □

**Lemma 4.11.** Let C be C₂-cofinite and CFT-type. Then the endomorphism space of any grading-restricted generalized module for C is finite dimensional. Moreover, each grading-restricted generalized module has finite length and has \(L_0\)-Jordan blocks of bounded length.

**Proof.** These are the results [Hu3] Thm. 3.24, Prop. 4.1 and Prop. 4.7]. In fact, the conclusions hold under weaker hypotheses. □

**Theorem 4.12.** Let V be simple, rational, C₂-cofinite and CFT-type. Then, every grading-restricted generalized C-module is completely reducible.
Proof. We shall freely use the lemmas above. Let $W$ be a grading-restricted generalized $C$-module. We know that $W$ decomposes as a finite direct sum of indecomposable modules. Therefore, without loss of generality, let $W$ be indecomposable.

Since $W$ is indecomposable and $C_\lambda$ are finite order simple currents for every $\lambda \in \mathcal{H}$, by [CKL] Lem. 3.17, we know that $M_{C_\lambda, W}$ is a scalar multiple, say $M_\lambda \in \mathbb{C}^\times$, of identity morphism. Let us assume that $W$ is such that for some non-zero $C$-modules $R$ and $S$, we have an exact sequence:

$$0 \rightarrow R \rightarrow W \rightarrow S \rightarrow 0.$$

We know from [CKL] Lem. 3.19(b) that $M_{C_\lambda, R} = M_\lambda \text{id}_{C_\lambda \boxtimes R}$ and $M_{C_\lambda, S} = M_\lambda \text{id}_{C_\lambda \boxtimes S}$. From Lemma 4.3, we know that $\lambda \mapsto M_\lambda^{-1}$ are homomorphisms $\mathcal{H} \rightarrow \mathbb{S}^1$.

We now seek a $\mu \in \mathcal{L}'$ such that for the $V_\mathcal{L}$ module $V_{\mu + \mathcal{L}}$, the monodromy of $V_{\lambda + \mathcal{L}} \otimes C_\lambda$ with $V_{\mu + \mathcal{L}} \otimes X$ is trivial, for $X = R, S, W$ and for all $\lambda \in \mathcal{H}$. In other words, we want to find a $\mu$ such that for all $\lambda \in \mathcal{H}$,

$$M_{V_{\lambda + \mathcal{L}}, V_{\mu + \mathcal{L}}} = M_\lambda^{-1}.$$

Since $\mathcal{H} \leq \mathfrak{g}$ are finite abelian groups, every character of $\mathcal{H}$ can be extended to a character of $\mathfrak{g}$. Pick a $\chi \in \mathfrak{g}$ that extends $\lambda \mapsto M_\lambda^{-1}$. We will be done if we can find a $\mu$ such that for each $\lambda \in \mathfrak{g} = \mathcal{L}' / \mathcal{L}$,

$$\exp(2\pi i(\mu, \lambda)) = M_{V_{\mu + \mathcal{L}}, V_{\lambda + \mathcal{L}}} = \chi(\lambda).$$

By Lemma 4.10, we know that there indeed exists a $\mu \in \mathcal{L}'$ such that $Q_\mu = \chi$.

For $X = R, S, W$, denote $V_{\mu + \mathcal{L}} \otimes X$ by $\tilde{X}$ and let

$$\tilde{X}_e = \bigoplus_{\lambda \in \mathcal{H}} (V_{\mu + \mathcal{L}} \otimes V_{\lambda + \mathcal{L}}) \otimes (C_\lambda \boxtimes X) = \bigoplus_{\lambda \in \mathfrak{g}} V_{\lambda + \mathcal{L}} \otimes (C_\lambda \boxtimes X).$$

We now invoke [HKL] Thm. 3.4] to get that $\tilde{X}_e$ is indeed a generalized (untwisted) module for $V$ when $X = R, S, W$.

Using flatness of simple currents, we deduce the exact sequence of $V$-modules

$$0 \rightarrow \tilde{R}_e \rightarrow \tilde{W}_e \rightarrow \tilde{S}_e \rightarrow 0.$$

However, every such exact sequence splits by rationality of $V$. Note that any morphism of $V$-modules must preserve Heisenberg weights. Hence, we get that $0 \rightarrow R \rightarrow W \rightarrow S \rightarrow 0$ splits.

Now we can combine our results with those of [Hu2] to obtain the following corollary.

Corollary 4.13. If $V$ is simple, rational, CFT-type and self-contragredient then we have the following:

1. Finite reductivity: Every $C$-module is completely reducible, there exist finitely many inequivalent irreducible modules, fusion coefficients amongst irreducible modules are finite.

2. Each finitely generated generalized $C$-module is a $C$-module.

3. The category $C$-modules has a structure of a modular tensor category.

Example 6. The Bershadsky-Polyakov algebra [Ber,Pol] is the quantum Hamiltonian reduction of $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ for the non-principal nilpotent embedding of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3$. This vertex operator algebra is strongly generated by four fields of conformal dimension $1, \frac{1}{2}$ and $\frac{3}{2}$. We denote its simple quotient by $\mathcal{W}_\ell$. This vertex operator algebra is rational provided $\ell$ is a positive integer [A?2]. In this case it contains the lattice vertex operator algebra $V_L$ of the lattice $L = \sqrt{6(\ell - 1)}\mathbb{Z}$ as sub vertex operator algebra [ACL]. Furthermore the
coset is rational, since it is isomorphic to the principal \( W \)-algebra \( W(\mathfrak{sl}_2) \) at level \( k = -2\ell + \frac{2\ell + 3}{2\ell + 3} \), and central charge \( c = -\frac{3(2\ell - 1)^2}{2\ell + 3} \) [ACT], but the latter is rational [Ar3]. Our results give thus another more direct proof of rationality of this coset.

5. Heisenberg Cosets inside Free Field Algebras and \( L_{-1}(\mathfrak{sl}(m|n)) \)

We use the opportunity to prove that \( L_{-1}(\mathfrak{sl}(m|n)) \) arise as certain Heisenberg cosets inside free field algebras, i.e. tensor products of \( bc \) and \( \beta \gamma \) systems. It had been known for a while that the affine vertex operator subalgebra is a sub-vertex operator algebra of the coset [KW3]. Moreover this gives a different proof to a recent result on the case \( n = 0 \) and \( m \geq 3 \) [AP]. As simple affine vertex operator subalgebras are poorly understood at present we hope that one can use this realization to clarify the structure of \( L_{-1}(\mathfrak{sl}(m|n)) \)-modules.

Let \( S \) denote the \( \beta \gamma \)-system, which has even generators \( \beta, \gamma \) and OPE relations
\[
\beta(z)\gamma(w) \sim (z-w)^{-1}, \quad \gamma(z)\beta(w) \sim -(z-w)^{-1}, \quad \beta(z)\beta(w) \sim 0, \quad \gamma(z)\gamma(w) \sim 0.
\]

Let \( H \) be the copy of the Heisenberg algebra with generator \( h = :\beta\gamma:. \) and let \( C = \text{Com}(H,S) \). By a theorem of Wang [WA], \( C \) is isomorphic to the simple Zamolodchikov \( W_3 \)-algebra with \( c = -2 \). The explicit generators, suitably normalized, are as follows:
\[
L = :\beta\gamma\gamma: + 2 :\beta\partial\gamma: - 2 : (\partial\beta)\gamma:, \quad W = :\beta\beta\gamma\gamma: + 3 :\beta(\partial\gamma)\gamma: - 6 : (\partial\beta)\beta\gamma\gamma: - 6 : (\partial\beta)\partial\gamma: + 3 : (\partial^2\beta)\gamma:
\]

Now let \( S(n) \) denote the rank \( n \) \( \beta\gamma \)-system, which has generators \( \beta^i, \gamma^j \) for \( i = 1, \ldots, n \) satisfying
\[
\beta^j(z)\gamma^i(w) \sim \delta_{ij}(z-w)^{-1}, \quad \gamma^j(z)\beta^i(w) \sim -\delta_{ij}(z-w)^{-1}, \\
\beta^j(z)\beta^i(w) \sim 0, \quad \gamma^j(z)\gamma^i(w) \sim 0.
\]

Let \( H \) be the Heisenberg algebra with generator
\[
h = \sum_{i=1}^{n} :\beta^i\gamma^i:
\]
and let \( C(n) = \text{Com}(H,S(n)) \). Note that \( C(n) \) contains \( n \) commuting copies of \( W_3 \) with generators \( L^i, W^i \), obtained from \( L \) and \( W \) above by replacing \( \beta \) and \( \gamma \) with \( \beta^i \) and \( \gamma^i \). Moreover, \( C(n) \) contains the fields
\[
X^{jk} = - :\beta^j\gamma^k:, \quad j, k = 1, \ldots, n, \quad j \neq k, \\
H^\ell = - :\beta^1\gamma^1: + :\beta^{\ell+1}\gamma^{\ell+1}:, \quad 1 \leq \ell < n,
\]
which generate a homomorphic image of the affine vertex algebra \( V^{-1}(\mathfrak{sl}_n) \).

A consequence of Theorem 7.3 of [L1] is

**Lemma 5.1.** \( C(n) \) is generated as a vertex algebra by \( \{L^i, W^i, X^{jk}, H^\ell\} \) for \( i, j, k, \ell \) as above.

**Proof.** In the notation of [L1], the lattice \( A \subset \mathbb{Z}^n \) is spanned by \( (1, 1, \ldots, 1) \) so \( A^\perp \) is precisely the root lattice of \( \mathfrak{sl}_n \). \( \square \)

By a recent theorem of Adamović and Perše [AP], for \( n \geq 3 \) \( C(n) \) is precisely the image of the map \( V_{-1}(\mathfrak{sl}_n) \to C(n) \), and is therefore isomorphic to the simple affine vertex algebra \( L_{-1}(\mathfrak{sl}_n) \). Using Lemma 5.1 we now provide a much shorter proof of this result. It suffices to show that \( L^i \) and \( W^i \) lie in the image of the map \( V_{-1}(\mathfrak{sl}_n) \to C(n) \), and by symmetry it is enough to prove this for \( L^1 \) and \( W^1 \). This is immediate
from the following calculations:

\[ L^1 = :H^1 H^2 : + :X^{12} X^{21} : + :X^{13} X^{31} : - :X^{23} X^{32} : - \partial H^1 . \]

\[ W^1 = - :H^1 H^2 H^2 : - :X^{12} X^{21} H^2 : - :X^{13} X^{31} H^1 : - :X^{13} X^{31} H^2 : + :X^{23} X^{32} H^2 : - :X^{13} X^{32} X^{21} : + \frac{1}{2} :X^{12} \partial X^{21} : - \frac{3}{2} : (\partial X^{12}) X^{21} : + \frac{7}{2} :X^{13} \partial X^{31} : - \frac{9}{2} : (\partial X^{13}) X^{31} : - \frac{1}{2} :X^{23} \partial X^{32} : + \frac{3}{2} : (\partial X^{23}) X^{32} : - \frac{1}{2} :H^1 \partial H^2 : + \frac{1}{2} : (\partial H^1) H^2 : + \frac{1}{2} \partial^2 H^1 . \]

Next, we find a minimal strong generating set for the remaining case C(2). In this case, it is readily verified that \( L^1 \) and \( W^1 \) do not lie in the affine vertex algebra generated by \( X^{12}, X^{21}, H^1 \). However, consider the following elements of C(2):

\[ P = - \frac{1}{2} L^2_{(0)} X^{12} + \frac{1}{3} :H^1 X^{12} : + \frac{2}{3} \partial X^{12} \]

\[ = :\beta^1 \partial \gamma^2 : - : (\partial \beta^1) \gamma^2 : + \frac{1}{3} :\beta^1 \beta^1 \gamma^2 : + \frac{2}{3} :\beta^1 \beta^2 \gamma^2 : , \]

\[ Q = - \frac{1}{2} L^2_{(0)} X^{21} - \frac{2}{3} :H^1 X^{21} : + \frac{1}{3} \partial X^{21} \]

\[ = :\beta^2 \partial \gamma^1 : - : (\partial \beta^2) \gamma^1 : + \frac{1}{3} :\beta^1 \beta^2 \gamma^1 : + \frac{2}{3} :\beta^2 \beta^2 \gamma^1 : , \]

\[ R = L^1 - L^2 , \]

\[ L = :X^{12} X^{21} : + \frac{1}{4} :H^1 H^1 : - \frac{1}{2} \partial H^1 . \]

Here \( L \) is the Sugawara Virasoro field of the affine vertex algebra of \( V_{-1} (\mathfrak{sl}_2) \), which has central charge 1, and \( X^{12}, X^{21}, H^1 \) are primary of weight one with respect of \( L \). It is easily verified that \( P, Q, R \) are primary of weight 2 with respect to \( L \), and that \( \{ X^{12}, X^{21}, H^1, P, Q, R \} \) close under operator product expansion, so they strongly generate a vertex subalgebra \( C'(2) \subset C(2) \). Moreover, we have

\[ L^1 = \frac{1}{2} R^+ :X^{12} X^{21} : + \frac{1}{2} :H^1 H^1 : - \frac{1}{2} \partial H^1 , \]

\[ L^2 = - \frac{1}{2} R^+ :X^{12} X^{21} : + \frac{1}{2} :H^1 H^1 : - \frac{1}{2} \partial H^1 , \]

\[ W^1 = - \frac{1}{2} R H^1 : - :P X^{21} : - \frac{1}{2} :H^1 H^1 H^1 : - \frac{5}{3} :X^{12} X^{21} H^1 : - \frac{13}{3} : (\partial X^{12}) X^{21} : + \frac{10}{3} :X^{12} \partial X^{21} : - \frac{1}{6} : (\partial H^1) H^1 : + \frac{1}{3} \partial^2 H^1 , \]

\[ W^2 = - \frac{1}{2} R H^1 : - :P X^{21} : + \frac{1}{2} :H^1 H^1 H^1 : + \frac{4}{3} :X^{12} X^{21} H^1 : + \frac{19}{6} : (\partial X^{12} X^{21} : - \frac{25}{6} :X^{12} \partial X^{21} : - \frac{5}{3} : (\partial H^1) H^1 : + \frac{3}{4} \partial R + \frac{7}{12} \partial^2 H^1 . \]

Since \( C(2) \) is generated by \( L^1, L^2, W^1, W^2, X^{12}, X^{21}, H^1 \), this shows that \( C'(2) = C(2) \). We obtain

**Theorem 5.2.** \( C(2) \) is of type \( W(1, 1, 1, 2, 2, 2) \). In fact, it is the simple quotient of an algebra of type \( W(1, 1, 1, 2, 2, 2) \) where the Virasoro field in weight 2 coincides with the Sugawara field.

**Remark 5.3.** Recall that each embedding of \( \mathfrak{sl}_2 \) inside a reductive Lie super algebra \( \mathfrak{g} \) gives an associated affine \( \mathcal{W} \)-super algebra from the affine vertex super algebra of \( \mathfrak{g} \) at level \( k \). Denote by \( \mathcal{W}^k (\mathfrak{sl}_4) \) the universal affine \( \mathcal{W} \)-algebra of \( \mathfrak{sl}_4 \) for the embedding of \( \mathfrak{sl}_2 \) such that \( \mathfrak{sl}_4 \) decomposes into four copies of the adjoint representation of \( \mathfrak{sl}_2 \) plus three copies of the trivial one. This implies that \( \mathcal{W}^k (\mathfrak{sl}_4) \) is of
Consider the Heisenberg algebra well-known to be trivial for then reveals that the simple quotient of operator algebra. A free field realization of \( W^{k}(\mathfrak{sl}_2) \) is given in Example 3.3 of [ArMo]. A computation then reveals that the simple quotient \( W_{-5/2}(\mathfrak{sl}_2) \) is isomorphic to \( \mathbb{C}(2) \).

Next we consider Heisenberg cosets inside \( bc \)-systems and \( bc\beta \gamma \)-system. First, consider the rank \( n \) \( bc \)-system \( E(n) \) with odd generators \( b^i, c^i \) satisfying
\[
 b^i(z)c^j(w) \sim \delta_{ij}(z-w)^{-1}, \quad c^j(z)b^i(w) \sim \delta_{ij}(z-w)^{-1},
\]
\[
 b^i(z)b^j(w) \sim 0, \quad c^i(z)c^j(w) \sim 0.
\]
Consider the Heisenberg algebra \( H \) with generators \( h = -\sum_{i=1}^{n} :b^i c^i: \) and let \( D(n) = \text{Com}(H, E(n)) \). It is well-known to be trivial for \( n = 1 \) and isomorphic to \( L_1(\mathfrak{sl}_n) \) for \( n \geq 2 \).

Now we consider the Heisenberg algebra \( H \) inside \( S(n) \otimes E(m) \) with generator
\[
 h = \sum_{i=1}^{n} :b^i c^i: - \sum_{j=1}^{m} :b^i c^j:.
\]
Let \( C(n,m) = \text{Com}(H, S(n) \otimes E(m)) \). It is easy to verify that \( C(n,m) \) contains the following fields:
\[
 X^{jk} = - :b^j c^k:; \quad j, k = 1, \ldots, n; \quad j \neq k, 
\]
\[
 H^j = - :b^1 c^j: + :b^{l+1} c^{j+l}:; \quad 1 \leq l < n, 
\]
\[
 R^{rs} = :b^r c^s:; \quad r, s = 1, \ldots, m; \quad r \neq s, 
\]
\[
 H^u = :b^1 c^1: - :b^{u+1} c^{u+1}:; \quad 1 \leq u < m, 
\]
\[
 f^r = :b^1 c^r:; \quad 1 \leq r < m, 
\]
\[
 \phi^{r,k} = :b^r c^k:; \quad j, k = 1, \ldots, n, \quad r, s = 1, \ldots, m.
\]

Moreover, these generate a homomorphic image of \( V_1(\mathfrak{sl}(n|m)) \). By a similar argument to the proof of Lemma 5.1 we obtain

**Lemma 5.4.** For all \( n \geq 1 \) and \( m \geq 1 \), \( C(n,m) \) is generated as a vertex algebra by \( L^i, W^i \) for \( i = 1, \ldots, n \), together with the image of the map \( V_1(\mathfrak{sl}(n|m)) \rightarrow C(n,m) \).

**Theorem 5.5.** For all \( n \geq 1 \) and \( m \geq 1 \), \( C(n,m) \) is isomorphic to the simple affine vertex superalgebra \( L_1(\mathfrak{sl}(n|m)) \).

**Proof.** Since \( C(n,m) \) is simple, it suffices to show that \( L^i, W^i \) lie in the image of the map \( V_1(\mathfrak{sl}(n|m)) \rightarrow C(n,m) \). By symmetry it is enough to show this for \( L^1 \) and \( W^1 \). Consider the following fields in the image of \( V_1(\mathfrak{sl}(n|m)) \):
\[
 f^{1,1} = :b^1 c^1:; \quad \psi^{1,1} = :b^1 c^1:; \quad \phi^{1,1} = :b^1 c^1:.
\]
A straightforward calculation shows that
\[
 L^1 = :f^{1,1} f^{1,1}: - 2 :\psi^{1,1} \phi^{1,1}: + \partial f^{1,1}, 
\]
\[
 W^1 = :f^{1,1} f^{1,1} f^{1,1}: - 3 :f^{1,1} \psi^{1,1} \phi^{1,1}: + 3 :(\partial \psi^{1,1}) \phi^{1,1}: - \frac{1}{2} \partial^2 f^{1,1}.
\]
\[\square\]
6. SOME $C_1$-COFINITENESS RESULTS

In this section, we show that the simple parafermion algebra of $\mathfrak{sl}_2$, as well as the coset of the Heisenberg algebra inside the Bershadsky-Polyakov algebra, both admit large categories of $C_1$-cofinite modules.

6.1. The $\mathfrak{sl}_2$ parafermion algebra. We work with the usual generating set $X,Y,H$ for the universal affine vertex algebra $V_k(\mathfrak{sl}_2)$. Let $l_k \subset V_k(\mathfrak{sl}_2)$ denote the maximal proper ideal graded by conformal weight, so that the simple affine vertex algebra $L_k(\mathfrak{sl}_2)$ is isomorphic to $V_k(\mathfrak{sl}_2)/l_k$. By abuse of notation, we use the same symbols $X,Y,H$ for the generators of $L_k(\mathfrak{sl}_2)$. Let $N_k(\mathfrak{sl}_2) = \text{Com}(H,L_k(\mathfrak{sl}_2))$ denote the simple parafermion algebra of $\mathfrak{sl}_2$. We will prove the following.

**Theorem 6.1.** For all $k \neq 0$, every irreducible $N_k(\mathfrak{sl}_2)$-module appearing in $L_k(\mathfrak{sl}_2)$ has the $C_1$-cofiniteness property according to Miyamoto’s definition.

In the case where $k$ is a positive integer, $N_k(\mathfrak{sl}_2)$ is rational, so the $C_1$-cofiniteness of the above modules is already known. Therefore we will assume for the rest of this discussion that $k$ is not a positive integer. Since $l_n$ is generated by either $(X^n+1)$ or $(Y^n+1)$ for any positive integer $n$, it follows that if $k$ is not a positive integer, $l_k$ does not contain $(X^n)$ or $(Y^n)$ for any $n$.

Recall that $L_k(\mathfrak{sl}_2)^{U(1)} \cong H \otimes N_k(\mathfrak{sl}_2)$ where the $U(1)$ action is infinitesimally generated by the zero mode of the field $H$. Since each irreducible $L_k(\mathfrak{sl}_2)^{U(1)}$-module $M$ appearing in $L_k(\mathfrak{sl}_2)$ is isomorphic to $H \otimes N$ where $N$ is an irreducible $N_k(\mathfrak{sl}_2)$-module, it suffices to prove the $C_1$-cofiniteness of the irreducible modules $M$.

Recall that for all $k \in \mathbb{C}$, $V_k(\mathfrak{sl}_2)^{U(1)}$ has a strong generating set

$$\{H, U_{0,i} = :X^iY: | i \geq 0\}.$$  

For all $k \neq 0$ and $i \geq 4$, there is a relation of weight $i+2$ for the form

$$U_{0,i} = P_i(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}),$$

where $P_i$ is a normally ordered polynomial in $H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}$, and their derivatives. Therefore $V_k(\mathfrak{sl}_2)^{U(1)}$ is strongly generated by $\{H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}\}$ and hence is of type $W(1,2,3,4,5)$ for all $k \neq 0$. Moreover, since the map $V_k(\mathfrak{sl}_2)^{U(1)} \rightarrow L_k(\mathfrak{sl}_2)^{U(1)}$ is surjective, the same strong generating set works for $L_k(\mathfrak{sl}_2)^{U(1)}$.

Since $U(1)$ is compact and $L_k(\mathfrak{sl}_2)$ is simple, we have a decomposition

$$L_k(\mathfrak{sl}_2) = \bigoplus_{n \in \mathbb{Z}} L_n \otimes M_n,$$

where $L_n$ is the irreducible, one-dimensional $U(1)$-module indexed by $n \in \mathbb{Z}$ and the $M_n$’s are inequivalent, irreducible $L_k(\mathfrak{sl}_2)^{U(1)}$-modules. Here $M_n$ consists of elements where $H(0)$ acts by $2n$. Since $(X^n) \neq 0$ and $(Y^n) \neq 0$ in $L_k(\mathfrak{sl}_2)$, and these elements lie in $M_n$ and $M_{-n}$ and have minimal conformal weight $n$, it follows that $M_n$ and $M_{-n}$ are generated as $L_k(\mathfrak{sl}_2)^{U(1)}$-modules by $(X^n)$ and $(Y^n)$ respectively. Note that we have a similar decomposition

$$V_k(\mathfrak{sl}_2) = \bigoplus_{n \in \mathbb{Z}} L_n \otimes \tilde{M}_n,$$

where the $\tilde{M}_n$’s are $V_k(\mathfrak{sl}_2)^{U(1)}$-modules which are no longer irreducible when $V_k(\mathfrak{sl}_2)$ is not simple.

Recall that a module $M$ for a vertex algebra $V$ is called $C_1$-cofinite if $M/C_1(M)$ is finite-dimensional, where $C_1(M)$ is spanned by

$$\{\alpha(k)m | m \in M, k < 0, \text{wt}(\alpha) > 0\}.$$
To prove the $C_1$-cofiniteness property of $M_n$ as a $L_k(\mathfrak{sl}_2)^{(1)}$-module for all $n$, it suffices to prove the $C_1$-cofiniteness of $M_{\pm 1}$. In fact, we shall prove a stronger statement: $\tilde{M}_{\pm 1}$ are $C_1$-cofinite as $V_k(\mathfrak{sl}_2)^{(1)}$-modules. Since the map $\tilde{M}_{\pm 1} \to M_{\pm 1}$ is surjective and compatible with the actions of $V_k(\mathfrak{sl}_2)^{(1)}$ and $L_k(\mathfrak{sl}_2)^{(1)}$, this implies the $C_1$-cofiniteness of $M_{\pm 1}$. We only prove the $C_1$-cofiniteness of $\tilde{M}_{-1}$; the proof for $\tilde{M}_1$ is the same.

Since $V_k(\mathfrak{sl}_2)$ is freely generated by $X, Y, H$, it has a good increasing filtration

$$V_k(\mathfrak{sl}_2)_{(0)} \subset V_k(\mathfrak{sl}_2)_{(1)} \subset \cdots, \quad V_k(\mathfrak{sl}_2) = \bigcup_{d \geq 0} V_k(\mathfrak{sl}_2)_{(d)},$$

where $V_k(\mathfrak{sl}_2)_{(d)}$ is spanned by iterated Wick products of $X, Y, H$ and their derivatives, of length at most $d$. Then $\tilde{M}_{-1}$ inherits this filtration, and $(\tilde{M}_{-1})_{(d)}$ has a basis consisting of

$$\{ \partial^i X \partial^j Y | j \geq 0 \}.$$  

Lemma 6.2. Any $\omega \in \tilde{M}_{-1}$ of weight $m > 0$ is equivalent to a scalar multiple of $\partial^{m-1} Y$, modulo $C_1(\tilde{M}_{-1})$.

Proof. It suffices to assume that $\omega$ is a monomial of the form (6.1) with $r + 2s > 0$, which has filtration degree $r + 2s + 1$. Let

$$\nu = \{ \partial^i X \partial^j Y | j \geq 0 \}.$$  

and observe that $\nu$ has weight $m$ and lies in $C_1(\tilde{M}_{-1})$, and $\omega - \nu$ has filtration degree $r + 2s$. Therefore by induction on filtration degree, $\omega$ is equivalent to an element of filtration degree one and weight $m$. The only such element up to scalar multiples is $\partial^{m-1} Y$. \hfill $\square$

Now we are ready to prove Theorem 6.1. By the preceding lemma, is enough to prove that

$$\partial^i Y \in C_1(\tilde{M}_{-1}),$$

for $i$ sufficiently large. For this purpose, we compute

$$(U_{0,4})_{(0)}(\partial^i Y) = (k + 2/5) \partial^{i+5} Y + \cdots,$$

where the remaining terms are of the form

$$\{ \partial^{i+4-r} Y | 0 \leq r \leq i \},$$

and hence lie in $C_1(\tilde{M}_{-1})$. Recall that for all $k \neq 0$, we have a relation

$$U_{0,4} = P_k(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}).$$

We claim that

$$P_k(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3})_{(0)}(\partial^i Y) \in C_1(\tilde{M}_{-1}).$$

To see this, let $\omega$ be a term appearing in $P_k(H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3})$ of the form $\alpha_1 \cdots \alpha_i$ where $t > 1$ and each $\alpha_j$ is one of the fields $H, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}$ or their derivatives. Then $\alpha_0(\partial^i Y) \in C_1(\tilde{M}_{-1})$ because the zero mode of such an operator cannot consist of only annihilation operators (i.e., non-negative modes.
of \( \alpha_t \). If \( t = 1 \), then \( \omega \) is a total derivative by weight considerations, so \( \omega_{i0}(\partial^iY) = 0 \). It follows that for all \( k \neq -2/5 \), \( \partial^iY \in C_i(M_{i-1}) \) for all \( i \geq 5 \).

Finally, suppose that \( k = -2/5 \). A similar computation shows that

\[
(U_{0.5})_{i0}(\partial^iY) = \frac{1}{15} \partial^{i+6}Y + \cdots,
\]

where the remaining terms are of the form

\[
:) (\partial^r H)(\partial^{i+5-r}Y):, \quad 0 \leq r \leq i,
\]

and hence lie in \( C_i(M_{i-1}) \). The same argument using the relation \( U_{0.5} = P_{5}(H, U_{0.0}, U_{0.1}, U_{0.2}, U_{0.3}) \) shows that \( \partial^iY \in C_i(M_{i-1}) \) for all \( i \geq 6 \).

6.2. Bershadsky-Polyakov algebras. Let \( W^k \) denote the universal Bershadsky-Polyakov algebra which is freely generated by fields \( J, T, G^\pm \) of weights \( 1, 2, \frac{1}{2}, \frac{3}{2} \), respectively, and whose OPE structure can be found in [FS]. This algebra appeared originally in [Ber, FS], and it coincides with the Feigin-Semikhatov algebra \( W^{(3)}_3 \) [FS] as well as the minimal \( W \)-algebra of \( W^k(sl_{3, f_{\text{min}}}) \) [KW2]. Let \( I_k \subset W^k \) denote the maximal proper ideal graded by conformal weight, and let \( W_k = W^k/I_k \) be the simple quotient.

The field \( J \) generates a Heisenberg algebra \( H \), and we define

\[
C^k = \text{Com}(H, W^k), \quad C_k = \text{Com}(H, W_k).
\]

In [ACL] it was shown that \( C^k \) is of type \( W(2, 3, 4, 5, 6, 7) \) for all \( k \) except for \( \{-1, -\frac{1}{2}\} \), and since there is a projection \( C^k \rightarrow C_k \), the generators of \( C^k \) descend to give strong generator for \( C_k \) as well.

**Theorem 6.3.** For all \( k \neq -1, -\frac{1}{2} \), every irreducible \( C_k \)-module appearing in \( W_k \) has the \( C_k \)-cofiniteness property according to Miyamoto’s definition.

The proof of this result is similar to the case of parafermion algebras above. First, suppose that \( k = p/2 - 3 \) for \( p = 5, 7, 9, \ldots \). As shown in [ACL], \( C_{p/2-3} \) is isomorphic to the simple, rational \( W(sl_{p-3}) \)-algebra with central charge \( c = \frac{-1}{p}(p-4)^2 \), and \( W_{p/2-3} \) is a simple current extension of \( C_{p/2-3} \otimes V_L \), where \( V_L \) is the lattice vertex algebra for \( L = \sqrt{3p-92} \). From this result, it is immediate that Theorem 6.3 holds in these cases, so from now on we assume that \( k \) is not of this form. Since \( I_{p/2-3} \) is generated by \( \{(G^+)^{p-2}: \) for \( p = 5, 7, 9, \ldots \), it follows that if \( k \neq p/2 - 3 \), \( I_1 \) does not contain \( (G^+)^n \) for any \( n > 0 \).

Recall that \( (W_k)^U(1) \cong H \otimes C_k \) where the \( U(1) \) action is infinitesimally generated by the zero mode of \( J \). Since each irreducible \( (W_k)^U(1) \)-module \( M \) appearing in \( W_k \) is isomorphic to \( H \otimes N \) where \( N \) is an irreducible \( C_k \)-module, it suffices to prove the \( C_k \)-cofiniteness of the irreducible modules \( M \).

By Theorem 5.3 of [ACL], for all \( k \neq -1, -\frac{1}{2} \), \( (W^k)^U(1) \) has a strong generating set

\[
\{J, L, U_{0,i} = : G^+ \partial^i G^- : \mid i \geq 0 \}.
\]

For all \( k \neq -1, -\frac{1}{2} \) and \( i \geq 5 \), there is a relation of weight \( i + 3 \) for the form

\[
U_{0,i} = P_i(J, L, U_{0.0}, U_{0.1}, U_{0.2}, U_{0.3}, U_{0.4}),
\]

where \( P_i \) is a normally ordered polynomial in \( J, L, U_{0.0}, U_{0.1}, U_{0.2}, U_{0.3}, U_{0.4} \), and their derivatives. Therefore \( (W_k)^U(1) \) is strongly generated by \( \{J, L, U_{0.0}, U_{0.1}, U_{0.2}, U_{0.3}, U_{0.4}\} \) and hence is of type \( W(1, 2, 3, 4, 5, 6, 7) \) for all \( k \neq -1, -\frac{1}{2} \). Since the map \( (W^k)^U(1) \rightarrow (W_k)^U(1) \) is surjective, the same strong generating set works for \( (W_k)^U(1) \).
We have a decomposition
\[ \mathcal{W}_k = \bigoplus_{n \in \mathbb{Z}} L_n \otimes M_n, \]
where \( L_n \) is the irreducible, one-dimensional \( U(1) \)-module indexed by \( n \in \mathbb{Z} \) and the \( M_n \)'s are inequivalent, irreducible \( (\mathcal{W}_k)^{(1)} \)-modules. Here \( M_n \) consists of elements where \( J(0) \) acts by \( n \). This contains a unique up to scalar element \( \omega_n \) of minimal weight \( \frac{3n}{2} \). Here \( \omega_0 = 1 \), \( \omega_n = (G^+)^{-n} : \) for \( n < 0 \), and \( \omega_n = (G^-)^n : \) for \( n > 0 \). It follows that so \( M_n \) is generated as a \( (\mathcal{W}_k)^{(1)} \)-module by \( \omega_n \) for all \( n \).

As usual, to prove the \( C_1 \)-cofiniteness of \( M_n \) as a \( (\mathcal{W}_k)^{(1)} \)-module for all \( n \), it suffices to prove the \( C_1 \)-cofiniteness of \( M_{n+1} \). For this purpose, it is enough to prove that \( \tilde{M}_{n+1} \) are \( C_1 \)-cofinite as \( (\mathcal{W}_k)^{(1)} \)-modules. We only prove the \( C_1 \)-cofiniteness of \( \tilde{M}_{-1} \); the proof for \( \tilde{M}_1 \) is the same.

Recall from \( \text{ACL} \) that \( \mathcal{W}_k \) has a weak filtration
\[ (\mathcal{W}_k)^{(0)} \subset (\mathcal{W}_k)^{(1)} \subset \cdots \subset (\mathcal{W}_k)^{(d)}, \]
where \( (\mathcal{W}_k)^{(d)} \) is spanned by iterated Wick products of \( J, L, G^\pm \) and their derivatives, where at most \( d \) of the fields \( G^\pm \) and their derivatives appear. Then \( \tilde{M}_{-1} \) inherits this filtration, and \( (\tilde{M}_{-1})^{(d)} \) has a basis consisting of
\[ : (\partial^{a_1} L) \cdots (\partial^{a_i} L)(\partial^{b_1} J) \cdots (\partial^{b_i} J)(\partial^{c_1} G^+) \cdots (\partial^{c_r} G^+)(\partial^{d_1} G^-) \cdots (\partial^{d_{r+1}} G^-) :, \]
where \( r \geq 0 \) and \( 0 \leq a_1 \leq \cdots \leq a_i \), \( 0 \leq b_1 \leq \cdots \leq b_j \), \( 0 \leq c_1 \leq \cdots \leq c_r \), and \( 0 \leq d_1 \leq \cdots \leq d_{r+1} \).

**Lemma 6.4.** Any \( \omega \in \tilde{M}_{-1} \) of weight \( m + \frac{3}{2} > 0 \) is equivalent to a scalar multiple of \( \partial^m G^- \), modulo \( C_1(\tilde{M}_{-1}) \).

**Proof.** By the same argument as previous, \( \omega \) is equivalent modulo \( C_1(\tilde{M}_{-1}) \) to a linear combination of terms of the form
\[ : (\partial^{a_1} L) \cdots (\partial^{a_i} L)(\partial^{b_1} J) \cdots (\partial^{b_i} J)(\partial^{c} G^-) :, \]
All such terms except possibly \( \partial^m G^- \) clearly lie in \( C_1(\tilde{M}_{-1}) \). \( \square \)

To prove Theorem 6.3, it is enough to show that
\[ \partial^i G^- \in C_1(\tilde{M}_{-1}), \]
for \( i \) sufficiently large. For this purpose, we compute
\[ (U_{0,5})_{(0)}(\partial^i G^-) = \left( k^2 + \frac{2}{21} k + \frac{1}{28} \right) \partial^{i+7} G^- + \cdots, \]
where the remaining terms lie in \( C_1(\tilde{M}_{-1}) \). Recall that for all \( k \neq -1, -\frac{3}{2} \), we have a relation
\[ U_{0,5} = P_3(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}). \]
We claim that
\[ P_3(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4})_{(0)}(\partial^i G^-) \in C_1(\tilde{M}_{-1}). \]
To see this, let \( \omega \) be a term appearing in \( P_3(J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4}) \) of the form : \( \alpha_1 \cdots \alpha_t : \) where \( t > 1 \) and each \( \alpha_j \) is one of the fields \( J, L, U_{0,0}, U_{0,1}, U_{0,2}, U_{0,3}, U_{0,4} \) or their derivatives. Then \( \omega_{(0)}(\partial^i G^-) \in C_1(\tilde{M}_{-1}) \) because the zero mode of such an operator cannot consist of only annihilation operators. If \( t = 1 \), then \( \omega \) is a total derivative by weight considerations, so \( \omega_{(0)}(\partial^i G^-) = 0 \). It follows that if \( k \) is not a root of \( x^2 + \frac{2}{21} x + \frac{1}{28} \), \( \partial^i G^- \in C_1(\tilde{M}_{-1}) \) for all \( i \geq 7 \).
In what follows, we will often make the identification \( \lambda \in \mathcal{M}_2 \). Let us think of words, we shall prove that automorphisms of simple \( \mathcal{V}_0 \), i.e., hence, there exists a logarithmic intertwining operator combining with the observation that coefficients of \( V \in (A) \), and identifying \( V = \bigoplus_{\lambda \in \mathcal{L}} V_{\lambda} \) for some subgroup \( \lambda \) of \( \mathcal{G} \). Assume that we are working with a category of \( \mathcal{V}_0 \)-modules that satisfies the conditions required to invoke the Huang, Lepowsky, Zhang’s tensor category theory.

We denote the vertex operator map of \( V \) by \( V \). Fix an \( i \in \mathcal{L} \). We shall prove that \( \mathcal{V}_i \otimes \mathcal{V}_i \cong \mathcal{V}_0 \). In other words, we shall prove that \( \mathcal{V} \) is a simple current. The proof we provide below is essentially the proof given in [M2], [CaM].

We break the proof in several steps.

1. Let us think of \( Y \) as a \( \mathcal{V} \)-intertwining operator of type \( (V, V) \). We have already assumed that \( V \) is a simple VOA, i.e., \( V \) is simple as a \( \mathcal{V} \)-module. Using Proposition 11.9 of [DL], we see that for any \( t_1, t_2 \in V \), \( Y(t_1, x) t_2 \neq 0 \). This implies that coefficients of \( Y(t_1, x) t_2 \) as \( t_1 \) runs over \( V_j \) and \( t_2 \) runs over \( V_k \) span a non-zero \( \mathcal{V}_0 \)-module of \( \mathcal{V}_{j+k} \). Since \( \mathcal{V}_{j+k} \) is a simple \( \mathcal{V}_0 \)-module, we get that coefficients of \( Y(t_1, x) t_2 \) as \( t_1 \in V_j \) and \( t_2 \in V_k \) span \( \mathcal{V}_{j+k} \).

2. Given generalized \( \mathcal{V}_0 \)-modules \( A, B \), we denote by \( y_{A, B}^{G^0} \) the “universal” intertwining operator of type \( (A, B) \) furnished by the universal property of tensor products. If \( \mathcal{V}_0 \) is a direct summand of \( A \), then we assume that \( y_{A, B}^{G^0} \) is normalized so that \( y_{A, B}^{G^0}(v_0, x) b = Y_B(v_0, x) b \) for all \( v_0 \in \mathcal{V}_0 \) and \( b \in B \), where \( Y_B \) is the module map for the \( \mathcal{V}_0 \)-module \( B \). Moreover, for finite direct sums, \( A = \bigoplus A_i \), we will assume that \( y_{A, B}^{G^0} |_{A_i, B} = y_{A_i, B}^{G^0} \).

3. In what follows, we will often make the identification \( \mathcal{V}_0 \otimes \mathcal{V}_i \cong \mathcal{V}_i \).

4. Recall that we have fixed an \( i \in \mathcal{L} \). By [HLZ], we have the associativity of intertwining operators, and hence, there exists a logarithmic intertwining operator \( y_{r, x, i} \) of type \( (V_{i+1} \otimes \mathcal{V}_i, \mathcal{V}_i \otimes \mathcal{V}_i) \) such that for complex numbers \( x, y \) with \( |x| > |y| > |x - y| > 0 \),

\[
\langle w', y_{r, x, i}^{G^0}(Y(u, x - y) u, y) v_i \rangle = \langle w', y_{r, x, i}(u, x) y_{V_i}^{G^0}(u, y) v_i \rangle,
\]

for any \( u \in \mathcal{V}_r, u_s \in \mathcal{V}_s \) and \( v \in \mathcal{V}_i, w' \in (V_{r+s} \otimes \mathcal{V}_i)' \).

5. Taking \( u \equiv 1 \), we get:

\[
\langle w', y_{V_i}^{G^0}(u, y) v_i \rangle = \langle w', y_{0, x, i}^{G^0}(1, x) y_{V_i}^{G^0}(u, y) v_i \rangle,
\]

for all \( v \in \mathcal{V}_i \). Using Jacobi identity we get that \( y_{0, x, i}(u_0, x) v^e, \) where \( u_0 \in \mathcal{V}_0 \) and \( v^e \in \mathcal{V}_i \), equals the action of \( u_0 \) by the \( \mathcal{V}_0 \)-module map.

6. Taking \( u \equiv 1 \) in (A.1), and identifying \( \mathcal{V}_0 \otimes \mathcal{V}_i \) with \( \mathcal{V}_i \), we get that:

\[
\langle w', y_{r, 0, i}(u, x) v_i \rangle = \langle w', y_{0, x, i}^{G^0}(1, x) y_{V_i}^{G^0}(u, y) v_i \rangle = \langle w', y_{V_i}^{G^0}(Y(u_1, x - y) 1) y) v_i \rangle
\]
For complex numbers $x, y$ with $|x| > |y| > |x - y| > 0$. We may now choose $y = \frac{1}{2}x$, as this satisfies the required constraints, and deduce that

$$y_{r,0,i}(u_r,x)v_i = \sum_{s,t} Y_{r,s,t}(u_r,x)^{y}_{i,s,t}(v_i)$$

where all the equalities hold for complex numbers $u, v$ in fact surjects onto $\mathbb{C}$.

(7) For complex numbers $|x| > |y| > |z| > |x - z| > |y - z| > |x - y| > 0$ we have that:

$$\langle u', y_{r,s,t}(u_r,x)y_{s,t,j}(u_s,y)\rangle = \langle u', y_{r,s,t}(u_r,x)y_{s,t,j}(u_s,y)\rangle = \langle u', y_{r,s,t}(u_r,x)y_{s,t,j}(u_s,y)\rangle = \langle u', y_{r,s,t}(u_r,x)y_{s,t,j}(u_s,y)\rangle$$

Again, since coefficients of $y_{r,s,t}(u_r,x)\sum_{s,t} Y_{r,s,t}(u_r,x)^{y}_{i,s,t}(v_i)$ span $V_i \otimes V_i$, we get that for all $u_r \in V_r$ and $u_s \in V_s$ and $v^r \in V_i \otimes V_i$,

$$y_{r,s,t}(u_r,x)y_{s,t,j}(u_s,y)v^r = y_{r,s,t}(u_r,x)v^r$$

(8) Now we consider $V_{-1} \otimes V_i$. Since the $Y$ map for the vertex operator algebra $V$ furnishes a $V_0$-intertwining operator of type $(V_0 \otimes V_i)$, by universal property of tensor products, there exists a morphism from $V_{-1} \otimes V_i$ to $V_0$. Since the coefficients of $Y(u_{-1},x)u_r$ for $u_{-1} \in V_{-1}$ and $u_r \in V_i$ span $V_0$, $V_{-1} \otimes V_i$ in fact surjects onto $V_0$. Since the latter is simple, proving simplicity of $V_{-1} \otimes V_i$ will give us that $V_{-1} \otimes V_i \cong V_0$.

(9) Let $B$ be a non-zero $V_0$ submodule of $V_{-1} \otimes V_i$ ($V_{-1} \otimes V_i$ is non-zero since it surjects onto $V_0$) and let

$$E = \text{Span}\{\text{Coefficients of } y_{r,-i,j}(u_r,x)^{b}\mid u_r \in V_r, b \in B\}.$$ 

Since the type of $y_{r,-i,j}$ is $(V_0 \otimes V_i)$. $E$ can be regarded as a $V_0$-submodule of $V_i$.

(10) $E$ is in fact a non-zero submodule of $V_i$. Indeed, if it were 0, then, the left-hand side of (A.5) with $r = t = -i, s = i$ would be 0 and hence we would get that $y_{0,-i,j}(Y(u_{-1},x-y)u_r,x)\otimes V_i$ is 0 for all $u_{-1} \in V_{-1}$, $u_r \in V_i$ and $b \in B$. However, in this case, coefficients of $Y(u_{-1},x-y)u_r$ span $V_0$ and $y_{0,-i,j}(u_r,x)\otimes V_i$ for $u_0 \in V_0$ is equal to $Y_B(u_0,x)b$ where $Y_B$ is the module map for the $V_0$-module B. Since the coefficients of the module map span the entire module, we have a contradiction.

(11) Since $0 \subset E \subset V_i$ and $V_i$ is simple, $E = V_i$.

(12) Using $E = V_i$ and using equation (A.4),

$$\text{Span}\{\text{Coefficients of } y_{r,-i,j}(v_{-1},x)\mid v_{-1} \in V_{-1}, v_i \in V_i, b \in B\}$$

$$= \text{Span}\{\text{Coefficients of } y_{r,-i,j}(v_{-1},x)\mid v_{-1} \in V_{-1}, v_i \in V_i, b \in B\}$$

$$= \text{Span}\{\text{Coefficients of } y_{r,-i,j}(v_{-1},x)\mid v_{-1} \in V_{-1}, v_i \in V_i, b \in B\}$$

$$= V_{-i} \otimes V_i$$

Now, using the right-hand side of equation (A.5),

$$\text{Span}\{\text{Coefficients of } y_{r,-i,j}(v_{-1},x)\mid v_{-1} \in V_{-1}, v_i \in V_i, b \in B\}$$

$$= \text{Span}\{\text{Coefficients of } y_{r,-i,j}(Y(v_{-1},x-y)u_r)\mid v_{-1} \in V_{-1}, v_i \in V_i, b \in B\}$$
\[
= \text{Span}\{\text{Coefficients of } Y_{b}(v_0, x - y) | v_0 \in V_0, b \in B\}
= \text{Span}\{\text{Coefficients of } Y_{b}(v_0, x - y) | v_0 \in V_0, b \in B\}
= B.
\]

This shows that \( V_{-j} \bigotimes V_i = B \) for any non-zero submodule \( B \) of \( V_{-j} \bigotimes V_i \). We conclude that \( V_{-j} \bigotimes V_i \) is simple. Hence, it equals \( V_0 \).

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