Quantum functionalities via feedback amplification

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Feedback amplification is a key technique for synthesizing various important functionalities, especially in electronic circuits involving op-amps. This paper presents a quantum version of this methodology, where the general phase-preserving quantum amplifier and coherent (i.e., measurement-free) feedback are employed to construct systems that produce several useful functionalities; quantum versions of differentiator, integrator, self-oscillator, and active filters. The class of active filters includes the Butterworth filter, which can be used to enhance the capacity of an optical quantum communication channel, and the non-reciprocal amplifier, which enables back-action-free measurement of a superconducting qubit. A particularly detailed investigation is performed on the unstable active filter for realizing a broadband gravitational-wave detector; that is, the feedback amplification method is used to construct an active filter that compensates the phase delay of the signal and eventually recovers the sensitivity in the high frequency regime.

I. INTRODUCTION

Amplifier is an essential component in modern technology, and it is usually involved in the systems in some feedback form. Let us consider a classical amplification process \( y = Gu \) where \( u \) and \( y \) are the input and output signals, and \( G > 1 \) is the gain of the amplifier. Then by feeding a fraction of the output back to the input through the “controller” \( K \), as depicted in Fig. 1, the input-output relation is modified to

\[
y = G^{(fb)}u, \quad G^{(fb)} = \frac{G}{1+GK}. \tag{1}
\]

Then by making the gain \( G \) large, we find \( y = (1/K)u \); hence if \( K \) is a passive device such as a resistor, the entire system works as a robust amplifier which is insensitive to the parameter change in \( G \). Now, the importance of this feedback amplification technique \cite{11, 12} is not limited to the realization of such a robust amplifier. That is, by combining high-gain amplifiers (op-amps in the electrical circuits) with several passive devices such as resistors and capacitors, one can devise a variety of functional systems; e.g., integrators, active filters, switches, and self-oscillators \cite{3}.

Therefore developing the quantum version of feedback amplification theory will be of particular importance to make the existing quantum technological devices robust and further to engineer systems with new functionalities. In fact this idea has been implicitly employed in some specific systems such as \cite{4, 5}. The explicit research direction was addressed in \cite{6}, showing a general quantum analogue to the above-described robust amplification method; more precisely, it is shown that a coherent (i.e., measurement-free) feedback control \cite{7, 13, 14} of a high-gain phase-preserving amplifier \cite{5, 15, 16} and a passive device (e.g., a beam splitter) yields a robust phase-preserving amplifier.

In this paper, we first extend the above quantum feedback amplification scheme (Sec. III) with a basic stability analysis (Sec. IV), and then apply the theory to construct systems having several useful functionalities: quantum versions of differentiator and integrator (Sec. V), self-oscillator (Sec. VI), and active filters (Sec. VII). As for the quantum integrator, it will be proven useful for improving the detection efficiency of itinerant fields. The usefulness of self-oscillator is also clear; as in the classical case, it can be applied to analogue quantum memory device and frequency converter \cite{18, 19}. Active filtering is a typical application of feedback amplification, which in our case includes the quantum version of Butterworth filter \cite{20} and non-reciprocal amplifier; the former is used to realize the steep roll-off characteristic in frequency, enabling the enhancement of the capacity of a quantum communication channel \cite{21}, and the latter enables measurement of a superconducting qubit while protecting it from the back-action noise generated in the amplification process \cite{22, 23}.

In particular, we show a detailed investigation on the quantum unstable filter, which is an active filter that can compensate the delayed phase of an incoming signal for the purpose of enhancing the bandwidth of the gravitational-wave detector. All the quantum unstable filter investigated in the literature \cite{26, 27} are based on an opto-mechanical implementation of the filter, but it requires an extremely low environmental temperature.

![FIG. 1: Feedback structure of the classical amplifier.](image-url)
The proposed unstable filter based on the feedback amplification method, on the other hand, can be optically implemented in the room temperature. To see how much the filter may broaden the bandwidth in a practical setting, we carry out a detailed numerical simulation in Sec. VIII.

II. PRELIMINARIES

A. Phase preserving linear amplifier

In this paper we consider a general phase preserving linear amplifier [16, 30]. A typical realization of this system is given by the non-degenerate parametric amplifier (NDPA) [3]. In optics case, as depicted in Fig. 2, the NDPA is an optical cavity having two orthogonally polarized modes inside the cavity. Also, the mode \( a_1 \) and \( a_2 \) couples with an input field \( b_1 \) at the mirror with transmissivity proportional to \( \gamma \). The Hamiltonian of the NDPA is given by

\[
H_{\text{NDPA}} = \hbar \omega_1 a_1^{\dagger} a_1 + \hbar \omega_2 a_2^{\dagger} a_2 + i \hbar \lambda (a_1^{\dagger} a_2 e^{-2i\omega_p t} - a_1 a_2 e^{2i\omega_p t}),
\]

with \( \omega_k \) the resonant frequencies of \( a_k \), \( \lambda \in \mathbb{R} \) the coupling strength between \( a_1 \) and \( a_2 \), and \( 2\omega_p \) the pump frequency. Here we assume that \( \omega_1 = \omega_2 = \omega_p \). Then, in the rotating frame at frequency \( \omega_p \), the dynamics of the NDPA is given by the following Langevin equation [32]:

\[
\begin{bmatrix}
\dot{a}_1 \\
\dot{a}_2
\end{bmatrix} = \begin{bmatrix}
-\gamma/2 & \lambda \\
\lambda & -\gamma/2
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} - \sqrt{\gamma} \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}.
\] (2)

Note that the canonical commutation relation of input fields is given by \([b(t), b^\dagger(t')] = \delta(t-t')\), with \(\delta(t-t')\) the Dirac delta function. The output equations are given by

\[
\begin{align*}
\dot{b}_1 &= \sqrt{\gamma} a_1 + b_1, \\
\dot{b}_2 &= \sqrt{\gamma} a_2 + b_2.
\end{align*}
\] (3)

Hence, from Eqs. (2) and (3), the transfer function of the NDPA is represented as

\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s^*)
\end{bmatrix} = \frac{1}{(s + \gamma/2)^2 - \lambda^2} \begin{bmatrix}
b_1(s) \\
b_2^*(s^*)
\end{bmatrix}.
\]

Note that the operator \(b(s)\) is related to \(b(t)\) via the Laplace transformation:

\[
b(s) = \int_0^\infty e^{-st} b(t) dt, \quad b^\dagger(s) = [b(s)]^\dagger = \int_0^\infty e^{-st} b^\dagger(t) dt.
\]

From Eq. (4) we find that \(\gamma > 2\lambda\) is necessary to make the amplifier stable. The output mode \(\tilde{b}_1\) at \(s = 0\) is given by

\[
\tilde{b}(0) = \frac{\gamma^2 + 4\lambda^2}{\gamma^2 - 4\lambda^2} b_1(0) + \frac{-4\gamma\lambda}{\gamma^2 - 4\lambda^2} b_2^\dagger(0),
\]

which diverges as \(\gamma \to 2\lambda + 0\). Hence, in this parameter limit, the signal with \(s\) satisfying \(|s| \ll \gamma\) is largely amplified.

In this paper we consider the general phase-preserving linear amplifier with the following input-output relation:

\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s^*)
\end{bmatrix} = G(s) \begin{bmatrix}
b_1(s) \\
b_2^*(s^*)
\end{bmatrix},
\]

\[
G(s) = \begin{bmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s)
\end{bmatrix}.
\] (5)

The condition on \(G(s)\) is represented in the Fourier domain where the field operators are defined as

\[
b(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} b(t) dt, \quad b^\dagger(i\omega) = [b(i\omega)]^\dagger = \int_{-\infty}^{\infty} e^{i\omega t} b^\dagger(t) dt,
\]

which satisfy \([b(i\omega), b^\dagger(i\omega')] = 2\pi d(\omega - \omega')\); actually this commutation relation requires \(G(s)\) to satisfy

\[
|G_{11}(i\omega)|^2 - |G_{12}(i\omega)|^2 = |G_{22}(i\omega)|^2 - |G_{21}(i\omega)|^2 = 1, \quad G_{21}(i\omega)G_{12}^*(i\omega) - G_{22}(i\omega)G_{11}^*(i\omega) = 0, \quad \forall \omega,
\]

where \(G_{ij}^*(i\omega) = [G_{ij}(i\omega)]^*\) is the complex conjugate of \(G_{ij}(i\omega)\).

B. Passive systems

The general form of passive linear system from the inputs \((b_3, b_4)\) to the outputs \((b_5, b_6)\) in the Laplace domain is represented as

\[
\begin{bmatrix}
\tilde{b}_5(s) \\
\tilde{b}_6(s^*)
\end{bmatrix} = K(s) \begin{bmatrix}
b_3(s) \\
b_4(s^*)
\end{bmatrix},
\]

\[
K(s) = \begin{bmatrix}
K_{11}(s) & K_{12}(s) \\
K_{21}(s) & K_{22}(s)
\end{bmatrix},
\] (8)

where the creation-mode representation is used to make the notation simple, as will be shown later. The transfer function \(K(s)\) satisfies \(|K_{11}(i\omega)|^2 + |K_{12}(i\omega)|^2 = 1,\)
same equation holds for the annihilation mode operators: only pass through the cavity, and hence this cavity is an integrator for the transmitting field from the transfer function is given by

\[
\det \left[ \begin{array}{cc}
-k_1 & 1 \\
-k_2 & 1 \\
\end{array} \right] = \frac{1}{s + (\kappa_1 + \kappa_2)/2 - i\Delta}.
\]

For small detuning, the cavity works as a differentiator for the reflected field from \(a_3\) to \(a_4\). Also it works as a high-pass filter with bandwidth \(\kappa\); that is, the optical components of \(a_3\) in the domain \(|s| \gg \kappa\) can only pass through the cavity. We also call this cavity as a MCC.

III. QUANTUM FEEDBACK AMPLIFICATION

In this paper we consider the feedback-connected system shown in Fig. 4 composed of the high-gain symmetric quantum phase-preserving amplifier \(G\) and a passive system \(K\). The feedback structure is made by

\[
\begin{align*}
\tilde{b}_4 &= b_3, \\
\tilde{b}_2 &= b_2, \quad b_2 &= \tilde{b}_4,
\end{align*}
\]

which are of course the same as \(\tilde{b}_2 = b_3\) and \(b_2 = \tilde{b}_4\). The entire system has the inputs \((b_1, b_1^\dagger)\) and the outputs \((\tilde{b}_1, \tilde{b}_1^\dagger)\). From Eqs. 13 and 14, the input-output relation of this system is given by

\[
\begin{bmatrix}
\tilde{b}_1(s) \\
\tilde{b}_1^\dagger(s^*)
\end{bmatrix} = G^{(fb)}(s) \begin{bmatrix}
b_1(s) \\
b_1^\dagger(s^*)
\end{bmatrix},
\]

where

\[
\begin{align*}
G^{(fb)}(s) &= \begin{bmatrix}
G^{(fb)}_{11}(s) & G^{(fb)}_{12}(s) \\
G^{(fb)}_{21}(s) & G^{(fb)}_{22}(s)
\end{bmatrix}, \\
G^{(fb)}_{11} &= \frac{G_{11} - K_{21} \det [G]}{1 - K_{12} G_{22}}, \\
G^{(fb)}_{12} &= \frac{G_{12} K_{22}}{1 - K_{12} G_{22}},
\end{align*}
\]

FIG. 4: Feedback structure of the quantum amplifier.
These matrix entries satisfy $|G_{11}^{(fb)}(i\omega)|^2 - |G_{12}^{(fb)}(i\omega)|^2 = 1$, $\forall \omega$, etc, meaning that it also functions as a phase-preserving amplifier.

It was shown in [1] that $|G_{11}^{(fb)}(i\omega)| \approx 1/|K_1(i\omega)|$ holds in the high-gain amplification limit $|G_{11}(i\omega)| \to \infty$; because the characteristic change in the passive transfer function $K(s)$ is usually very small, this realizes the robust quantum amplification, which is the quantum analogue to the classical feedback amplification technique mentioned in the first paragraph in Sec. I. We now extend this idea to the Laplace domain. The point to derive the result is that, from Eq. (7), we have

$$\det [G(i\omega)] = G_{11}(i\omega)G_{22}(i\omega) - G_{12}(i\omega)G_{21}(i\omega)$$

and thus

$$\frac{\det [G(i\omega)]}{G_{22}(i\omega)} = \frac{1}{G_{11}^*(i\omega)} \to 0,$$

in the high-gain limit $|G_{11}(i\omega)| \to \infty$. Also again from Eq. (7), $|G_{11}(i\omega)| = |G_{22}(i\omega)|$ and $|G_{12}(i\omega)| = |G_{21}(i\omega)|$ hold. Then in the same limit, Eq. (7) leads to

$$\frac{1 - G_{12}(i\omega)}{G_{11}(i\omega)} \to 0 \implies \frac{G_{12}(i\omega)}{G_{22}(i\omega)} \to 1,$$

$$1 - \frac{G_{21}(i\omega)}{G_{22}(i\omega)} \to 1 \implies \frac{G_{21}(i\omega)}{G_{22}(i\omega)} \to 1.$$

These are equivalent to

$$\frac{G_{12}(i\omega)}{G_{22}(i\omega)} \to e^{i\theta}, \quad \frac{G_{21}(i\omega)}{G_{22}(i\omega)} \to e^{i\varphi},$$

where $\theta(\omega)$ and $\varphi(\omega)$ are certain real functions of $\omega$. We now extend the above result to assume that

$$\frac{\det [G(s)]}{G_{22}(s)} \to 0, \quad \frac{G_{12}(s)}{G_{22}(s)} \to 1, \quad \frac{G_{21}(s)}{G_{22}(s)} \to 1, \quad (20)$$

in the domain $s \in \mathbb{C}$ such that $|G_{11}(s)| \to \infty$. Moreover, we assume $G_{11}(s) = G_{22}(s)$ for all $s \in \mathbb{C}$. These conditions are indeed satisfied in the case of NDPA shown in Sec. II A for $s$ satisfying $|s| \ll \gamma$, where the high-gain limit is realized by taking $\gamma \to 2\lambda + 0$. Under the above assumptions, the transfer function matrix of the entire closed-loop system can be approximated by

$$G^{(fb)}(s) \approx \frac{-1}{K_{21}(s)} \left[ \begin{array}{c} K_{22}(s) \\ K_{11}(s) \det [K(s)] \end{array} \right]. \quad (21)$$

**FIG. 5:** System structure for the stability test of the closed-loop system

in the domain $s \in \mathbb{C}$ such that $|G_{11}(s)| \to \infty$. Hence, we now have a new quantum system that, as will be proven later, generates several interesting, robust, and useful functionalities available only in the feedback amplification setting.

The proof of Eq. (21) is as follows. First,

$$G_{11}^{(fb)} = \frac{G_{11} - K_2 G_{22}}{1 - K_2 G_{22}} \frac{1 - K_2 \det [G]/G_{22}}{(1/G_{22}) - K_2}.$$

Next,

$$G_{12}^{(fb)} = \frac{G_{12} K_{22}}{1 - K_2 G_{22}} \frac{(G_{12}/G_{22}) K_{22}}{(1/G_{22}) - K_2} \to \frac{K_{22}}{K_2}.$$

Finally,

$$G_{22}^{(fb)} = \frac{K_{11} G_{22}}{1 - K_2 G_{22}} \frac{(G_{21}/G_{22}) K_{11}}{(1/G_{22}) - K_2} \to \frac{K_{11}}{K_2}.$$

**IV. STABILITY ANALYSIS METHOD**

In the engineering viewpoint it is important to guarantee the stability of any controlled system before activating it (more precisely, before closing the loop for control). In the classical case the seminal Nyquist method [2] is often used for this purpose; here we show the quantum version of this method, particularly for the quantum feedback-controlled system with transfer function matrix [14]; note that, hence, the stability must be guaranteed for the system with finite amplification gain.

Let us represent the matrix entries of $G(s)$ and $K(s)$ as $G_{ij}(s) = g_{ij}(s)/g(s)$ and $K_{ij}(s) = k_{ij}(s)/k(s)$, respectively, where $g(s), g_{ij}(s), k(s), k_{ij}(s)$ are the polynomial functions. Then, it is easy to see that $G^{(fb)}(s)$ has the following form:

$$G^{(fb)}(s) = \frac{1}{g^2(s)k^2(s)[1 - K_{21}(s)G_{22}(s)]} \left[ \begin{array}{c} \ast \ast \\ \ast \ast \end{array} \right].$$
where for simplicity the matrix entries, the polynomial functions denoted by $\star$, are not explicitly shown. Here we assume that the original systems $G(s)$ and $K(s)$ are stable; then because $k(s)$ and $q(s)$ are the stable polynomial functions (meaning that the zeros of $k(s)$ and $q(s)$ lie in the left side plane in $\mathbb{C}$), the stability of the closed-loop system is completely characterized by the zeros of $1 - K_1(s)G_{22}(s)$.

In particular, we can now apply the classical Nyquist method to test the stability of this closed-loop system. As in the classical case let us define the open-loop transfer function:

$$L(s) = -K_1(s)G_{22}(s).$$

Then, from the Nyquist theorem, the simplest stability criterion is given as follows: if the point $-1$ lies outside the Nyquist plot, i.e., the trajectory of $L(i\omega)$ for $\omega \in (-\infty, +\infty)$ in the complex plane, then the closed-loop system is stable; otherwise, it is unstable. The point is that this stability test can be carried out for an open-loop system illustrated in Fig. 5, which is constructed via simply cascading the amplifier and the controller. In fact the input-output relation of this open-loop system is given by

$$G^{(open)}(s) = \begin{bmatrix} b_1(s) \\ b_1^*(s) \\ b_2^*(s) \end{bmatrix} = G^{(open)}_1(s) \begin{bmatrix} b_1(s) \\ b_2(s) \\ b_2^*(s) \end{bmatrix},$$

where $G^{(open)}_1(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & 0 \\ K_{11}(s)G_{21}(s) & K_{12}(s)G_{22}(s) & K_{12}(s) \\ K_{21}(s)G_{21}(s) & K_{21}(s)G_{22}(s) & K_{22}(s) \end{bmatrix}$.

Therefore, the Nyquist plot can be obtained by setting $b_i$ and $b_4$ to the vacuum fields and injecting the coherent field $(\alpha e^{i\omega})$ in the $b_2$ port with several frequencies $\omega$ in fact measuring the amplitude of the output field $b_4$ gives us the Nyquist plot in the form $L(i\omega) = -b_4^*(i\omega)/\alpha^*$. Note that, unlike the classical case, the measurement result of $b_4$ must be probabilistic, hence the Nyquist plot fluctuates, meaning that the stability margin should be discussed.

V. FUNCTIONALITIES 1: QUANTUM PID

We now start describing several functionalities realized by the proposed quantum feedback amplification method. The first functionality is the quantum PID [34]. That is, we show that, via the proper choice of the controller $K$, the ideal closed-loop system (21) functions as a differentiator (D) or integrator (I) on the input itinerant field $b_1$; hence together with the proportional component (P), which simply attenuates or amplifies the amplitude of the input, now P, I, and D components are available to us. These three are clearly the most basic components involved in almost all electrical circuits and used for constructing several useful systems such as a PID feedback controller and an analogue computer; although establishing the quantum analogue of such useful devices are not addressed in this paper, we will show a simple application of the quantum integrator at the end of this section.

A. Differentiator

Let us take the symmetric cavity [10] as the controller. In this case, the transfer function of the ideal closed-loop system (21) is given by

$$G^{(fb)}(s) = \frac{1}{\kappa} \begin{bmatrix} s + \kappa & s \\ s & s - \kappa \end{bmatrix}.$$ 

Hence, the output $\tilde{b}_3^*(s)$ is given by

$$\tilde{b}_3^*(s^*) = \frac{s}{\kappa} b_1(s) + \frac{s - \kappa}{\kappa} b_4(s^*),$$

or in the time-domain it is

$$\tilde{b}_3(t) = \frac{1}{\kappa} \frac{d}{dt} b_1(t) + \frac{1}{\kappa} \frac{d}{dt} b_4(t) - b_4(t),$$

meaning that the closed-loop system works as a differentiator for the itinerant field $b_1(t)$.

As discussed in Sec. III, the approximation is valid only in a specific $s$-region such that the high-gain limit is
effective. To see this region, we study an actual feedback controlled system composed of the optical NDPA and the control cavity, depicted in Fig. 6(a). Recall that, in the case of NDPA, the high-gain limit is achieved in the regime $|s| \ll \gamma \approx 2\lambda$, which becomes wider as $\lambda$ increases. Actually this can be seen in Fig. 6(b), showing the gain plot of the transfer function $\text{18}$ of this optical system with parameters $\gamma = 2.0\lambda$ and the ideal limit $|i\omega/\kappa|$; that is, the frequency range such that the optical system effectively approximates the ideal differentiator becomes wider as $\lambda$ gets bigger.

![Nyquist plot](image)

**FIG. 7:** (a) Nyquist plot of the quantum differentiator, where the parameters are set as $\kappa = 1$, $\lambda = 2\kappa$, and $\gamma = 2.0\lambda$. Figure (b) is a zoom-up of (a) at around $s = 0$.

Note that, as in the classical case, the differentiator itself is an unstable system, and thus this system should be used together with other components such that the entire combined system is stable. This instability can be readily seen using the method addressed in Sec. 4; the open-loop transfer function in this case is

$$L(s) = \frac{\kappa(s^2 - \gamma^2/4 - \lambda^2)}{(s + \kappa)(s^2 + \gamma s + \gamma^2/4 - \lambda^2)},$$

and the Nyquist’s plot is given by Figs. 7(a) and (b), showing that the point $-1$ lies inside the trajectory of $L(i\omega)$ and thus the system is unstable. Note that the actual Nyquist’s plot fluctuates along the curve shown in the figure, with variance $\langle (\Delta b(t)i\omega)\rangle^2$.

**B. Integrator**

Next we take the high-pass filtering cavity $\text{13}$ as the controller, where in this case the reflected field is fed back to the amplifier, as shown in Fig. 8(a) in optics case. Then the transfer function matrix $\text{21}$ of the ideal closed-loop system is given by

$$G^{(i\omega)}(s) = \frac{1}{s} \begin{bmatrix} -s - \kappa & \kappa \\ \kappa & s - \kappa \end{bmatrix}.$$

Hence the output $b_3(s^*)$ is connected to the input $b_1(s)$ as

$$b_3(s^*) = \frac{\kappa}{s} b_1(s) + \frac{s - \kappa}{s} b_4(s^*),$$

or in the time-domain it is

$$b_3(t) = \kappa \int_0^t b_1(\tau) d\tau + b_4(t) - \kappa \int_0^t b_1(\tau) d\tau.$$

This means that in a specific $s$-region where the high-gain limit of the amplifier is effective ($|s| \ll \gamma$ in the NDPA case, as discussed in Sec. 3), the closed-loop system works as an integrator for the itinerant field $b_1(t)$.

Note that, unlike the differentiator, the integrator forms a circulating field in the feedback loop between the amplifier and the controller cavity, as depicted in Fig. 8(b) in optics case and Fig. 8(c) in a microwave system case. Therefore, we regard this loop as another cavity with the mode $a_4$. In fact, for the model depicted in Fig. 8(a) where $b_2$, $b_3$, and $b_4$ are treated as the itinerant fields, they violate the Ito rule such as...
where $\omega |_k$.

First, the Hamiltonian of the system is given by

$$H_{\text{Ham}} = \sum_{k=1}^{4} \hbar \omega_k a_k^\dagger a_k + i \hbar \kappa (a_1^\dagger a_4^\dagger e^{-i \omega_p t} - a_1 a_4 e^{i \omega_p t})$$

where $\omega_3$ and $\omega_4$ are the resonant frequencies of $a_3$ and $a_4$, respectively. $g_{24}$ ($g_{34}$) describes the coupling between $a_2$ and $a_4$ ($a_3$ and $a_1$), which are given by

$$g_{24} = \sqrt{c_2 / \lambda}, \quad g_{34} = \sqrt{c_3 / \lambda},$$

with $\lambda$ the round trip length of the loop cavity and $c = 3 \times 10^8$ m/s the speed of light. Here we assume $\omega_k = \omega_p (k = 1, \ldots, 4)$. Together with the coupling to the external fields, we find that, in the rotating frame at frequency $\omega_p$, the dynamical equations are given by

$$\dot{a}_1 = -\frac{1}{2} a_1 + \lambda a_2 - \sqrt{\gamma} a_1, \quad \dot{a}_2 = \lambda a_1 + i g_{24} a_1^\dagger,$$

$$\dot{a}_3 = -\frac{1}{2} a_3 + \lambda i g_{34} a_4 - \sqrt{\gamma} a_3, \quad \dot{a}_4 = i g_{24} a_2 + i g_{34} a_3^\dagger,$$

$$\dot{b}_1 = b_1 + \sqrt{\gamma} a_1, \quad \dot{b}_3 = b_3 + \sqrt{\gamma} a_3^\dagger.$$}

The input-output equation of this system in the Laplace domain is of the form

$$\begin{pmatrix} \dot{b}_1(s) \\ \dot{b}_3(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & \tilde{G}_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} b_1(s) \\ b_3(s) \end{pmatrix},$$

where particularly $G_{21}(s)$ is given by

$$G_{21}(s) = \frac{\alpha_0}{s^4 + \beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0},$$

with

$$\alpha_0 = \sqrt{\gamma} \kappa g_{24} g_{34}, \quad \beta_0 = \gamma \kappa g_{24}^2 / 4 - \lambda^2 g_{34}^2,$$

$$\beta_1 = (\gamma g_{24}^2 + \kappa g_{24} + \gamma g_{34}^2 - \kappa \lambda^2) / 2,$$

$$\beta_2 = \gamma \kappa / 4 - \lambda^2 + g_{24}^2 + g_{34}^2, \quad \beta_3 = (\gamma + \kappa) / 2.$$}

Here, in the limit $\gamma \to 2 \lambda + 0$, together with Eq. (23), the above coefficients are approximated as

$$\alpha_0 \approx 2 \kappa \lambda^2 / \lambda, \quad \beta_0 \approx 0,$$

$$\beta_1 \approx 2 \kappa (\kappa + \lambda) / 2 - \lambda^2 / 2,$$

$$\beta_2 \approx \kappa / 2 - \lambda^2 + c(\kappa + 2 \lambda) / \lambda, \quad \beta_3 \approx \kappa / 2 + \lambda.$$}

Furthermore, we assume $|s| \ll \gamma$, so that the higher-order term of $s$ can be neglected. Then the transfer function $[23]$ can be approximated by

$$G_{21}(s) \approx \frac{\alpha_0}{\beta_2 s} \approx \frac{2 \kappa \lambda^2 / \lambda}{(2 \kappa (\kappa + \lambda) / \lambda - \kappa \lambda^2 / 2) s} \approx \frac{\kappa}{1 + \kappa \lambda / 4 \lambda}.$$

Thus, if $\kappa \ll \lambda$ and $L_{\text{loop}} \kappa / (4 \kappa) \ll 1$, this system becomes the integrator which we wish to obtain:

$$G_{21}(s) \approx \frac{\kappa}{s}.$$}

In Fig. (c), the blue dashed line is the plot of the ideal gain $|\kappa / i \omega|$, while the solid lines represent $|G_{21}(\omega)|$ in Eq. (24) with parameters $\gamma = 2 \kappa \lambda$, $c / \lambda$ loops $= 10^3 \kappa$, and $\lambda = n \kappa$ ($n = 3, 5, 7, 9$). Clearly, $|G_{21}(\omega)|$ well approximates $|\kappa / i \omega|$ in a specific $s$ region where the high-gain limit of the NDPA is effective, which is now given by $|s| \ll \gamma \approx 2 \lambda$. Hence, $\lambda$ should be relatively large to guarantee that the integrator works in a wider region in $s$; this can be actually seen in the figure, although making $\lambda$ large does not make a big difference in the parameter regime considered in the simulation.

We here give a specific set of parameters taken in Fig. (c) in the case $\lambda = 9 \kappa$. Let us take $\lambda = 3 \times 10^6$, leading to $\kappa = 1 / 3 \times 10^6$ Hz. Then $c / \lambda$ loops $= 10^4 \kappa$ leads to $L_{\text{loop}} = 0.9$ m. Also, since $\lambda$ is given by $\kappa = c T_{\text{MCC}} / L_{\text{MCC}}$ with $T_{\text{MCC}}$ the transmission of the mirrors of the MCC and $L_{\text{MCC}}$ the round trip length of the MCC, we have $T_{\text{MCC}} / L_{\text{MCC}} = \kappa = 1 / 9 = \kappa / 10^{-2}$ m$^{-1}$. Hence, if $T_{\text{MCC}} = 0.01$, the round trip length of the MCC is $L_{\text{MCC}} = 9$ m, which is ten times longer than $L_{\text{loop}}$.

The stability of the modified model depicted in Fig. (b) cannot be investigated via the stability test discussed in Sec. [14], only if the Nyquist’s theorem, we use the Routh-Hurwitz method [23]. In our case, the system is stable if and only if every root of the characteristic polynomial function in the denominator in Eq. (24), has a negative real part; the Routh-Hurwitz method systematically leads to the stability condition as follows:

$$\beta_3 > 0, \quad \frac{\beta_2 \beta_3 - \beta_1}{\beta_3} > 0, \quad \frac{\beta_2^2 \beta_0}{\beta_1 - \beta_2 \beta_3} + \beta_1 > 0.$$}

Note that $\beta_3 > 0$ is already satisfied.

C. Application to qubit detection

Here we give an application of the integrator, that can be used in a stand-alone fashion unlike the differentiator. The system of interest (not a “system” combined with the amplifier via feedback) is an open qubit that is dissipatively coupled to the external itinerant field $b_0(t)$, such as the transmon qubit coupled to the superconducting resonator; the Langevin equation of the system variable $\sigma_x(t)$ is given by

$$\frac{d}{dt} \sigma_x(t) = -\frac{\Gamma}{2} \sigma_x(t) + \sqrt{\Gamma} \sigma_x(t) (b_0(t) + b_0^\dagger(t)).$$
where $\Gamma$ is the strength of the dissipative coupling. The output field is given by $b_1(t) = \sqrt{\Gamma} \sigma_-(t) + b_0(t)$: the quadrature $q_1(t) = b_1(t) + b_1^\dagger(t)$ thus follows

$$q_1(t) = \sqrt{\Gamma} \sigma_x(t) + b_0(t) + b_0^\dagger(t).$$

Now the field state is set to the vacuum $|0\rangle_F$. Then we find

$$F(|q_1(t)|0_F) = \sqrt{\Gamma} e^{-\Gamma t/2} \sigma_x.$$

(Note that, for a system-field operator $X$, $F(|q_1(t)|0_F)$ is an operator living in the system Hilbert space.) This means that, for a very short time interval $\Gamma t \ll 1$, the above quantity becomes $F(|q_1(t)|0_F) \approx \sqrt{\Gamma} \sigma_x$, which thus takes $\pm \sqrt{\Gamma}$ as the measurement result. In other words, to measure the qubit state we need a high-speed detector.

Using the integrator changes this condition. Let us place it along the output of the qubit. That is, the output $b_1(t)$ is taken as the input to the integrator, and we measure its output $\tilde{b}_3(t)$. The quadrature $q_3(t) = \tilde{b}_3(t) + \tilde{b}_3^\dagger(t)$ then satisfies

$$F(|q_3(t)|0_F) = \sqrt{\Gamma} e^{-\Gamma t/2} \sigma_x.$$

Therefore, in the long time limit $\Gamma t \gg 1$, this output itinerant field is given by

$$F(|q_3(t)|0_F) \approx \frac{2\kappa}{\sqrt{\Gamma}} \sigma_x.$$

Hence the measurement result is $\pm 2\kappa/\sqrt{\Gamma}$, which equals to $\pm \sqrt{\Gamma}$ if the parameter in the integrator is set to $\kappa = \Gamma/2$. This means that, even if the given detector is slow, it can capture the same level of measurement signal as that obtained by a fast detector, by using the integrator.

**VI. FUNCTIONALITIES 2: SELF OSCILLATION**

Self-oscillation is also an important functionality realized with the feedback amplification method, which is indeed widely used in a variety of electrical circuit systems such as a clock. To realize a stable oscillation, of course, some nonlinearities such as a voltage saturation are necessary to be involved, but here we only focus on the linear part.

Let us consider the cavity (12) with $\kappa_1 = \kappa_2 = \kappa$:

$$K(s) = \frac{1}{s + \kappa - i\Delta} \begin{bmatrix} -\kappa & s - i\Delta \\ s - i\Delta & -\kappa \end{bmatrix}.$$

Then the transfer function of the closed-loop system, Eq. (21), is given by

$$G^{(th)}(s) = \frac{1}{s - i\Delta} \begin{bmatrix} s + \kappa - i\Delta & -\kappa \\ -\kappa & s - \kappa - i\Delta \end{bmatrix}.$$

Therefore the output $\tilde{b}_3$ is given by

$$\tilde{b}_3(s^*) = \frac{\kappa}{s - i\Delta} b_1(s) + \frac{s - \kappa - i\Delta}{s - i\Delta} b_4(s^*).$$

Because the pole is on the imaginary axis, this represents a self-oscillation of $\tilde{b}_3$. In fact, if both $b_1$ and $b_4$ are set to the vacuum and $\langle \tilde{b}_3(0) \rangle \neq 0$, then in the time domain $\tilde{b}_3$ satisfies

$$\langle \tilde{b}_3(t) \rangle = e^{-i\Delta t} \langle \tilde{b}_3(0) \rangle,$$

hence it oscillates with frequency $-\Delta$ (in the rotating-frame). Also, the spectral broadening of this oscillation can be seen from

$$\langle \tilde{b}_3(-i\omega)\tilde{b}_3(-i\omega) \rangle = \frac{\kappa^2}{(\omega - \Delta)^2} 2\pi \delta(0).$$

In practice, the cavity parameter $\kappa_1 - \kappa_2$ is set to a small positive number, which makes the system oscillating almost with a fixed frequency yet with growing amplitude; but the amplitude is saturated by some nonlinearities, and as a result a sustained oscillation can be realized.

The optical realization of the self-oscillator is very similar to the model of the integrator shown in Fig. [S1a] and (b). The only difference between the self-oscillator and the integrator is that the detuning of the MCC is not zero for the case of self-oscillator, while it is zero for the integrator. Also as in the integrator, the self-oscillator forms a loop cavity between the NDPA and MCC, and thus it should be modeled as the system shown in Fig. [S1b] with non-zero detuning $\Delta$ in the mode $a_3$. Then the whole dynamical equation is given by

$$\dot{a}_1 = -\frac{\gamma}{2} a_1 + \lambda a_3^\dagger - \sqrt{\gamma} b_1, \quad \dot{a}_2 = \lambda a_1 + ig_2 a_4^\dagger,$$

$$\dot{a}_3 = -\left(\frac{\kappa}{2} - i\Delta\right) a_3^\dagger + ig_{34} a_4^\dagger - \sqrt{\kappa} b_4^\dagger,$$

$$\dot{a}_4 = ig_{24} a_2^\dagger + ig_{34} a_3^\dagger,$$

$$\dot{b}_1 = b_1 + \sqrt{\gamma} a_1, \quad \dot{b}_3 = b_4 + \sqrt{\kappa} a_3^\dagger.$$

If $\langle b_1(t) \rangle = \langle b_4(t) \rangle = 0 \ \forall t$, the mean dynamics is

$$\frac{d}{dt} \begin{bmatrix} \langle a_1 \rangle \\ \langle a_2 \rangle \\ \langle a_3 \rangle \\ \langle a_4 \rangle \end{bmatrix} = A_{osc} \begin{bmatrix} \langle a_1 \rangle \\ \langle a_2 \rangle \\ \langle a_3 \rangle \\ \langle a_4 \rangle \end{bmatrix}, \quad \langle \tilde{b}_3 \rangle = \sqrt{\kappa} \langle a_3^\dagger \rangle,$$

where

$$A_{osc} = \begin{bmatrix} -\gamma/2 & \lambda & 0 & 0 \\ \lambda & 0 & 0 & ig_{24} \\ 0 & 0 & -\kappa/2 + i\Delta & ig_{34} \\ 0 & ig_{24} & g_{34} & 0 \end{bmatrix}.$$

Figure [2] shows the mean time evolution of the quadratures of $b_3(t)$:

$$\tilde{q}_3(t) = \frac{\tilde{b}_3(t) + \tilde{b}_3^\dagger(t)}{\sqrt{2}}, \quad \tilde{p}_3(t) = \frac{\tilde{b}_3(t) - \tilde{b}_3^\dagger(t)}{\sqrt{2}i}.$$
The parameters as well as the initial conditions are set as follows: $\Delta = 1$, $\lambda = 0.01\Delta$, $\gamma = 2.01\lambda$, $c/L_{\text{loop}} = 0.1\Delta$, $\kappa = 0.1\Delta$ (black lines in the figure), $0.01\Delta$ (light blue lines), and $\langle a_1(0) \rangle = \langle a_2^\dagger(0) \rangle = \langle a_3^\dagger(0) \rangle = (a_4^\dagger(0)) = 1/\sqrt{2}$. Now the ideal oscillation (28) is represented in terms of quadrature as

$$
\begin{bmatrix}
\langle \hat{q}_3(t) \rangle \\
\langle \hat{p}_3(t) \rangle
\end{bmatrix} = 
\begin{bmatrix}
\cos(\Delta t) & \sin(\Delta t) \\
-\sin(\Delta t) & \cos(\Delta t)
\end{bmatrix}
\begin{bmatrix}
\langle \hat{q}_3(0) \rangle \\
\langle \hat{p}_3(0) \rangle
\end{bmatrix}.
$$

With the initial values mentioned above, these are given by $\langle \hat{q}_3(t) \rangle = \cos(\Delta t)$ and $\langle \hat{p}_3(t) \rangle = -\sin(\Delta t)$, meaning that there is a $\pi/2$ phase difference between the two quadratures. The figure shows that the smaller value of $\kappa$ leads to the slower attenuation of the oscillation. This is simply because the smaller $\kappa$ is, the less amount of photons leaks out from the loop cavity and accordingly the MCC with detuning $\Delta$. Thus, by setting $\kappa$ smaller, we can preserve the coherence of the light field oscillating with frequency $\Delta$ in the MCC. However, making $\kappa$ smaller also reduces the amount of photons coming from the NDPA into the MCC, and thus the amplitude of oscillation is limited. Conversely, a large value of $\kappa$ allows a flow of large amount of photons from NDPA to MCC. Therefore, there is a tradeoff between the coherence time and the amplitude of the self-oscillation.

VII. FUNCTIONALITIES 3: ACTIVE FILTERS

As we have seen in Sec. [113] the 2-input 2-output cavity works as a low-pass or high-pass filter with bandwidth $\kappa$ and maximal gain 1; here we show that, by combining the feedback amplification method, several types of filter with tunable bandwidth and gain, i.e., the quantum version of active filters, can be engineered.

A. High-Q active filter

First we show a simple first-order active filter. As in the quantum integrator, the controller is chosen as the high-pass filtering cavity $[12]$ with zero-detuning $\Delta = 0$, which in this case is set to be asymmetric (i.e., $\kappa_1 < \kappa_2$):

$$
K(s) = \frac{1}{s + (\kappa_1 + \kappa_2)/2} \times \left[ \frac{\sqrt{\kappa_1 \kappa_2}}{s + (\kappa_2 - \kappa_1)/2} - \frac{s}{\sqrt{\kappa_1 \kappa_2}} \right].
$$

Then, the closed-loop system (21) realized in the high-gain amplification limit is given by

$$
\begin{bmatrix}
\hat{b}_1(s) \\
\hat{b}_3(s^*)
\end{bmatrix} = G^{(fb)}(s) \begin{bmatrix}
\hat{b}_1(s) \\
\hat{b}_3(s^*)
\end{bmatrix},
$$

$$
G^{(fb)}(s) = \frac{1}{s + (\kappa_2 - \kappa_1)/2} \times \left[ \frac{\sqrt{\kappa_1 \kappa_2}}{s - (\kappa_1 + \kappa_2)/2} - \frac{s}{\sqrt{\kappa_1 \kappa_2}} \right].
$$

Here we focus on the output $\hat{b}_3(s^*)$:

$$
\hat{b}_3(s^*) = \frac{\sqrt{\kappa_1 \kappa_2}}{s + (\kappa_2 - \kappa_1)/2} \hat{b}_1(s) + \frac{s - (\kappa_1 + \kappa_2)/2}{s + (\kappa_2 - \kappa_1)/2} \hat{b}_1^\dagger(s^*).
$$

Hence, this system functions as a low-pass filter for $b_1(s)$ with bandwidth $(\kappa_2 - \kappa_1)/2$. In contrast to the standard low-pass filter [10] with bandwidth $\kappa$, the bandwidth of this active filter can be made very small by making $\kappa_1$ and $\kappa_2$ close to each other. As a result, the Q-factor can be largely enhanced from $Q = \omega_0/2\kappa$ to $Q' = \omega_0/(\kappa_2 - \kappa_1)$. For instance for a coherent light field with frequency $\omega_0 = 3 \times 10^{14}$ Hz, an optical cavity $\kappa = 3 \times 10^6$ leads to $Q = 5 \times 10^7$, while the active filter with $\kappa_1 = \kappa$ and $\kappa_2 = 1.01\kappa$ leads to $Q' = 1 \times 10^{10}$. Note that this active filter also functions as an amplifier with gain $2\sqrt{\kappa_1 \kappa_2}/(\kappa_2 - \kappa_1)$, which becomes large if Q-factor increases. Importantly, in this case the idler noise mode $b_1$ is also amplified with gain $(\kappa_2 + \kappa_1)/(\kappa_2 - \kappa_1)$ at $s = 0$. This means that basically the filtering makes sense only for an input field with amplitude much bigger than $(\kappa_2 + \kappa_1)/(\kappa_2 - \kappa_1)$. Also we add a remark that, as discussed in the case of integrator, the feedback loop now constructs another cavity, which should be taken into account for more precise modeling of the filter; we will give such a detailed investigation in Sec. [111] for another type of active filter discussed in the next subsection.

B. Unstable filter

The functionality provided by an active filter is not only modifying the frequency response, but changing the phase of an input field. In fact Miao et al. proposed a very interesting quantum active filter that can effectively change the phase and thereby enhance the bandwidth of the gravitational wave detector or, in a wider sense, any cavity-based quantum sensor [20]. A rough description of their idea is as follows. When a gravitational wave propagates through the cavity, then it must pick up a phase $\phi_{\text{arm}}(t) = -2\Omega L_{\text{arm}}/c$, where $\Omega$, $L_{\text{arm}}$, $$
\begin{bmatrix}
\langle q_3(t) \rangle \\
\langle p_3(t) \rangle
\end{bmatrix} = 
\begin{bmatrix}
\cos(\Delta t) & \sin(\Delta t) \\
-\sin(\Delta t) & \cos(\Delta t)
\end{bmatrix}
\begin{bmatrix}
\langle q_3(0) \rangle \\
\langle p_3(0) \rangle
\end{bmatrix}.
$$

FIG. 9: Self-oscillation of the mean quadratures of $\hat{b}_3$. 
and \( c \) are the gravitational-wave frequency, the length of the cavity, and \( c \) the speed of light, respectively. This extra phase eventually limits the bandwidth of the detector; hence constructing an auxiliary intra-cavity filter with the transfer function \( e^{-i\phi_{arm}(\Omega)} = e^{2\Omega L_{arm}/c} \) will compensate this extra phase and thus may recover the bandwidth.

Here we show that the feedback amplification method can be employed to realize such a phase-cancelling filter in a fully optical setting. We again use the closed-loop system (29) and now consider the output \( \tilde{b}_1 \):

\[
\tilde{b}_1(s) = G_{11}^{(fb)}(s) b_1(s) + G_{12}^{(fb)}(s) b_4^*(s^*)
\]

\[
= -\frac{s + (\kappa_1 + \kappa_2)/2}{s + (\kappa_2 - \kappa_1)/2} b_1(s) + \frac{\sqrt{\kappa_1 \kappa_2}}{s + (\kappa_2 - \kappa_1)/2} b_4^*(s^*).
\]

Let us then set \( \kappa_2 = 0 \):

\[
\tilde{b}_1(s) = G_{11}^{(fb)}(s) b_1(s) = -\frac{s + \kappa_1/2}{s - \kappa_1/2} b_1(s).
\]

(29)

In the frequency domain \( s = i\Omega \) this equation reduces to

\[
\tilde{b}_1(i\Omega) = G_{11}^{(fb)}(i\Omega) b_1(i\Omega) = -\frac{i\Omega + \kappa_1/2}{i\Omega - \kappa_1/2} b_1(i\Omega).
\]

Then by setting \( \Omega \ll \kappa_1 \) and \( \kappa_1 = 2c/L_{arm} \), we actually find that \( G_{11}^{(fb)} \) approximates our target filter:

\[
G_{11}^{(fb)}(i\Omega) = -\frac{\Omega^2 + \kappa_1^2/4 + i\Omega\kappa_1}{\Omega^2 + \kappa_1^2/4} \approx \frac{\kappa_1^2/4 + i\Omega\kappa_1}{\kappa_1^2/4} = e^{4i\Omega/\kappa_1} = e^{2\Omega L_{arm}/c} = e^{-i\phi_{arm}(\Omega)}.
\]

(30)

This phase-cancelling filter might be realizable in practice by carefully devising the controller cavity so that the optical loss \( \kappa_2 \) is very small. Note that in the original proposal \( 26 \) an opto-mechanical oscillator was employed to realize the same filter where in that case \( \kappa_2 \) represents the magnitude of the thermal bath added on the oscillator; hence \( \kappa_2 \approx 0 \) requires the oscillator to be in an ultra-low temperature environment.

Lastly note that the system (29) is clearly unstable; particularly the system (40) represents a phase-lead filter that violates the causality. Similar to the case of integrator, therefore, in a practical setting such an unstable filter must be incorporated in a bigger system that is totally stable. In Sec. VIII we give a detailed study on how much the unstable filter (29) could compensate the phase delay and enhance the bandwidth of the gravitational-wave detector in a practical setup.

C. Butterworth filter

Let us move back to the problem of modifying the frequency response via a filter. A particularly useful band-pass filter, which is often used in classical electrical circuits, is the so-called Butterworth filter. The transfer function of the \( n \)-th order classical Butterworth filter is given by \( T_n(s) = g/B_n(s) \) where \( g \) is a constant and the followings are examples of polynomials \( B_n(s) \):

\[
B_1(s) = s + 1, \quad B_2(s) = s^2 + \sqrt{2}s + 1, \quad B_3(s) = (s + 1)(s^2 + s + 1).
\]

The gain of the filter is given by

\[
|T_n(i\omega)| = \frac{g}{\sqrt{(\omega/\omega_B)^{2n} + 1}}
\]

(31)

which has the steep roll-off characteristic of frequency, particularly for large \( n \), at the cut-off frequency \( \omega_B \).

A quantum version of Butterworth filter has actually been employed in the literature; in Ref. \( 21 \), a fourth-order quantum Butterworth filter was applied to enhance the channel capacity of a linear time-invariant bosonic channel. However, its physical realization has not been discussed. Here we show that, in the simple case \( n = 2 \), the feedback amplification technique can be used to realize the quantum Butterworth filter.

The controller \( K \) is chosen as the cascaded cavities, an optical case of which is depicted in Fig. 10. The left cavity with mode \( c_1 \) has two inputs \( (b_3, b_4) \) and two outputs \( (b_3', b_4') \), and the right one with mode \( c_2 \) has two inputs \( (b_3, b_4) \) and two outputs \( (b_3', b_4) \). We assume that the detuning of the left and right cavities are \( \Delta \) and \( -\Delta \), respectively. A phase shifter \( e^{i\pi} \) is placed in the path from \( b_4' \) to \( b_3' \). Then the two input and output fields are connected as follows:

\[
\begin{bmatrix}
  b_3''(s^*) \\
  b_4''(s^*)
\end{bmatrix} = K_l(s) \begin{bmatrix}
  b_3'(s^*) \\
  b_4'(s^*)
\end{bmatrix}, \quad \begin{bmatrix}
  b_3'(s^*) \\
  b_4'(s^*)
\end{bmatrix} = K_r(s) \begin{bmatrix}
  b_3''(s^*) \\
  b_4''(s^*)
\end{bmatrix},
\]

where

\[
K_l(s) = \frac{1}{s + (\kappa_1 + \kappa_2)/2 - i\Delta}
\]

\[
\times \begin{bmatrix}
  s - (\kappa_1 - \kappa_2)/2 - i\Delta & -\sqrt{\kappa_1 \kappa_2} \\
  -\sqrt{\kappa_1 \kappa_2} & s + (\kappa_1 - \kappa_2)/2 - i\Delta
\end{bmatrix}.
\]

FIG. 10: The controller \( K \) for realizing the quantum Butterworth filter in the optical setting. The detuning of the left cavity is \( \Delta \), while that of the right cavity is \( -\Delta \). This controller has the inputs \( (b_3, b_4) \) and outputs \( (b_3', b_4') \). A phase shifter is embedded between the two cavities.
Here we set \( \Delta = (\kappa_1 + \kappa_2)/2 \), then \( K(s) \) is represented as

\[
K(s) = \begin{bmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{bmatrix} = \frac{1}{k(s)} \begin{bmatrix} k_{11}(s) & k_{12}(s) \\ k_{21}(s) & k_{22}(s) \end{bmatrix},
\]

where

\[
k(s) = s^2 + (\kappa_1 + \kappa_2)s + (\kappa_1 + \kappa_2)^2/2,
\]

\[
k_{11}(s) = \{k_1 - k_2 + i(k_1 + k_2)\}/\sqrt{k_1 k_2},
\]

\[
k_{12}(s) = -s^2 + (\kappa_2 - \kappa_1)s + (\kappa_2 - \kappa_1)^2/2,
\]

\[
k_{21}(s) = s^2 + (\kappa_2 - \kappa_1)s + (\kappa_2 - \kappa_1)^2/2,
\]

\[
k_{22}(s) = \{k_1 - k_2 - i(k_1 + k_2)\}/\sqrt{k_1 k_2}.
\]

Then, from Eq. (21), the closed-loop system composed of this controller and a high-gain amplifier \( G \) has the following transfer function:

\[
G^{(\text{fb})}(s) = -\frac{1}{k_{21}(s)} \begin{bmatrix} k(s) & k_{22}(s) \\ k_{11}(s) & \{k_{11}(s)k_{22}(s) - k_{12}(s)k_{21}(s)\}/k(s) \end{bmatrix}.
\]

Hence the output \( \tilde{b}_3 \) is given by

\[
\tilde{b}_3(s) = G^{(\text{fb})}_{21}(s)\tilde{b}_1(s) + G^{(\text{fb})}_{22}(s)\tilde{b}_3(s^*)
\]

\[
= -\frac{k_{11}(s)}{k_{21}(s)} \frac{k_{11}(s)k_{22}(s) - k_{12}(s)k_{21}(s)}{k(s)} b_1(s^*)
\]

\[
= -\frac{\{k_1 - k_2 + i(k_1 + k_2)\}}{s^2 + (\kappa_2 - \kappa_1)s + (\kappa_2 - \kappa_1)^2/2} b_1(s^*)
\]

\[
-\frac{s^2 + (\kappa_2 + \kappa_1)s + (\kappa_2 + \kappa_1)^2/2}{s^2 + (\kappa_2 - \kappa_1)s + (\kappa_2 - \kappa_1)^2/2} b_3(s^*).
\]

The transfer function from \( b_1 \) to \( \tilde{b}_3 \) has a form of the second order Butterworth filter with cut-off frequency \( \omega_B = (\kappa_2 - \kappa_1)/\sqrt{2} \) and maximal gain \( g = \sqrt{2\kappa_1\kappa_2}(\kappa_1^2 + \kappa_2^2)/\omega_B^2 \). Also, it is easy to see that the transfer function from \( b_1^\dagger \) to \( \tilde{b}_3 \) has the same form of second order Butterworth filter as above. Note that, as mentioned in Sec. \( \text{VIA} \) the amplitude of the input field should be much bigger than that of the amplified idler vacuum noise.

Figure 11 shows the gain plot of the second-order Butterworth filter developed above. In this figure, the black solid line shows the gain plot of the ideal transfer function \( G^{(\text{fb})}_{21}(i\omega) = -k_{11}(i\omega)/k_{21}(i\omega) \), which corresponds to \( \gamma \to 2\lambda + 0 \), while the dotted lines show the gain plot of \( G^{(\text{fb})}_{21}(i\omega) = G_{21}(i\omega)K_{11}(i\omega)/\{1 - K_{21}(i\omega)G_{22}(i\omega)\} \) with \( \gamma = 2.01\lambda \) and several parameters \( \lambda = m\kappa_1 \) \((m = 1, 3, 5, 7)\). The other parameters are fixed to \( \kappa_2 = 1.5\kappa_1 \) and \( \Delta = (\kappa_1 + \kappa_2)/2 \). Now, as mentioned before, \( |G_{11}(i\omega)| \gg 1 \) holds in the frequency range \( \omega \ll \gamma = 2.01\lambda \). Therefore, making \( \lambda \) bigger results in broadening the frequency range where the approximation is valid, and in fact Fig. 11 shows that the dotted line approaches to the ideal solid line as \( \lambda \) gets larger.

**D. Non-reciprocal amplifier**

The last topic in this section is a proposal to construct a non-reciprocal (directional) amplifier, via the feedback amplification method. This special type of amplifier is particularly important in the field of superconducting circuit based quantum technologies [37–39]. In the microwave regime the phase-preserving amplifier follows the same equation (5), but the configuration is not like the optical case shown in Fig. 2; the input \( b_1(b_2) \) and the corresponding output \( \tilde{b}_1(\tilde{b}_2) \) propagate along the same transmission line but with opposite direction. If the purpose of the use of amplifier is to detect a small signal generated by, e.g., a superconducting qubit system, then the propagating direction of the reflected output field must be changed via e.g. a circulator, to protect the source system from the back action noise (if the output is the idler mode) or to extract the output (if the output is the amplified signal). In fact there have been a number of theoretical and experimental proposals of the non-reciprocal amplifier [22, 23]. Out scheme is similar to [23], but with a clear concept of using feedback amplification in mind to realize a robust non-reciprocal amplifier.

The proposed non-reciprocal amplifier has a form of (coherent) feedback shown in Fig. 12(a). This whole system has three inputs \( \{b_1, b_2, b_3\} \) and three outputs \( \{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\} \); particularly \( b_3 \) is the signal and \( \tilde{b}_3 \) is the am-
...mentioned above, the input controller \((K)\) and a reciprocal amplifier composed of two amplifiers \((G, \bar{G})\) and an ideal associator \(\mathcal{P}\) for the whole feedback controlled system as follows:

\[
\begin{pmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s^*) \\
\tilde{b}_3(s)
\end{pmatrix} = G(s) \begin{pmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s) \\
G_{31}(s) & G_{32}(s)
\end{pmatrix} \begin{pmatrix}
\tilde{b}_1(s) \\
\tilde{b}_2(s^*) \\
\tilde{b}_3(s)
\end{pmatrix},
\]

where

\[
G(s) = \begin{pmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s) \\
G_{31}(s) & G_{32}(s)
\end{pmatrix},
\]

and

\[
G_\text{fb}(s) = \begin{pmatrix}
G_{11}(\text{fb})(s) & G_{12}(\text{fb})(s) \\
G_{21}(\text{fb})(s) & G_{22}(\text{fb})(s) \\
G_{31}(\text{fb})(s) & G_{32}(\text{fb})(s)
\end{pmatrix},
\]

Thus, in a \(s\)-domain such that both \(G(s)\) and \(\bar{G}(s)\) have a large gain, the transfer function matrix \(G_\text{fb}(s)\) converges to

\[
G_\text{fb}(s) = \begin{pmatrix}
\frac{1}{K_{23}(s)} & 0 & 0 \\
0 & K_{12}(s)/K_{23}(s) & 0 \\
-K_{12}(s)/K_{23}(s) & 0 & -\det[K(s)]/K_{23}(s)
\end{pmatrix}.
\] (32)

The proof of Eq. \(32\) is given in Appendix A. The point of this result is that, because \(b_2\) is vacuum, the back-action field \(b_1\) propagating toward the input port (see Fig. 12(b)) is also a vacuum field in this high-gain limit; \(\tilde{b}_1(s) = -\tilde{b}_2(s)\). Therefore, as the output signal \(b_3\) contains the input signal \(b_3\) with amplification gain \(K(s)/K_{23}\), this feedback-controlled system functions as the non-reciprocal amplifier or more broadly the non-reciprocal active filter if \(K(s)\) is appropriately designed. Note that \(b_1\) is suppressed due to the destructive interference between \(b_3\) and \(b_2\), which is no more than the feedback effect.

Let us consider an example. If, as shown in Fig. 12(b), \(K\) is given by a beam splitter with power transmissivity \(T\), that is,

\[
K(s) = \begin{pmatrix}
\sqrt{T} & -1 & 0 \\
0 & \sqrt{1-T} & 0 \\
0 & 0 & \sqrt{1/T-1}
\end{pmatrix},
\]

then we have

\[
G_\text{fb}(s) = \begin{pmatrix}
0 & -1 & 0 \\
-1/\sqrt{T} & 0 & 0 \\
1/\sqrt{1/T-1} & 0 & 1/\sqrt{T}
\end{pmatrix}.
\] (33)

Hence, the input signal \(b_3\) is amplified with amplification gain \(1/\sqrt{T}\); importantly, this non-reciprocal amplification is robust against the characteristic changes in the original amplifiers \((G, \bar{G})\) because the gain \(1/\sqrt{T}\) is a tunable yet static quantity.
VIII. APPLICATION TO GRAVITATIONAL-WAVE DETECTION

As emphasized several times, any functionality realized via the feedback amplification method should be evaluated in such a way that it is incorporated in a concrete setup with particular purpose, to see its actual performance under practical constraints. Here we study the unstable (phase-cancelling) filter discussed in Sec. VII B and see how much it might broaden the bandwidth of the typical gravitational-wave detector.

A. Basics of gravitational-wave detector

The most basic schematic of the gravitational-wave detector, particularly the laser interferometer gravitational-wave observatory [40, 41], is shown in Fig. 13. The input laser with frequency \( \omega_0 \) is injected to the arm cavities through the power recycling mirror (PRM). Each arm cavity is composed of two mirrors: the input test mass (ITM) and the end test mass (ETM). A tidal force of ETMs, and they satisfy \( [X(t), P(t)] = i \hbar \). \( M \) is the mass of ETMs. \( d \) is the sideband mode of the interferometer field, with detuning \( \Delta_d \), which satisfies \( [d(t), d^\dagger(t)] = 1 \). \( G_{\text{arm}} \) represents the coupling strength between \( X \) and \( d \), and it is given by \( G_{\text{arm}} = \sqrt{2P_{\text{arm}}/\hbar \omega_d} \) with \( P_{\text{arm}} \) the arm cavity power [20]. Then the dynamics of the system are given by

\[
\dot{X} = \frac{1}{M} P, \quad \dot{P} = \hbar G_{\text{arm}}(d + d^\dagger) + F_{\text{GW}},
\]

\[
d = -(i \Delta_d + \frac{\gamma_{\text{IFO}}}{2}) d + i G_{\text{arm}} X - \sqrt{\gamma_{\text{IFO}}} d_{\text{in}}, \tag{34}
\]

where \( \gamma_{\text{IFO}} \) describes the coupling between \( d \) and \( d_{\text{in}} \). Also, the output equation of the system is given by

\[
d_{\text{out}} = d_{\text{in}} + \sqrt{\gamma_{\text{IFO}}} d.
\tag{35}
\]

Note that \( [d_{\text{in}}(t), d_{\text{in}}^\dagger(t')] = [d_{\text{out}}(t), d_{\text{out}}^\dagger(t')] = \delta(t - t') \). The input-output relation in the Laplace domain, in terms of the quadratures \( \xi_{\text{out}}(s) \) and \( \xi_{\text{in}}(s) \), is given by

\[
\begin{bmatrix}
Q_{\text{out}}(s) \\
P_{\text{out}}(s)
\end{bmatrix} = J(s) \begin{bmatrix}
F_{\text{GW}}(s) \\
Q_{\text{in}}(s) \\
P_{\text{in}}(s)
\end{bmatrix},
\]

with

\[
J(s) = \begin{bmatrix}
J_{11}(s) & J_{12}(s) & J_{13}(s) \\
J_{21}(s) & J_{22}(s) & J_{23}(s)
\end{bmatrix} = \begin{bmatrix}
0 & \frac{s - \gamma_{\text{IFO}}/2}{s + \gamma_{\text{IFO}}/2} & 0 \\
\frac{\sqrt{2\gamma_{\text{IFO}} G_{\text{arm}}}}{Ms^2(s + \gamma_{\text{IFO}}/2)} & -\frac{2\gamma_{\text{IFO}} G_{\text{arm}}}{Ms^2(s + \gamma_{\text{IFO}}/2)} & \frac{s - \gamma_{\text{IFO}}/2}{s + \gamma_{\text{IFO}}/2}
\end{bmatrix},
\]

where \( \Delta_d = 0 \) is assumed. The gravitational-wave strain signal \( h \), which is defined as \( F_{\text{GW}}(t) = M L_{\text{arm}} h(t) \), can be detected by homodyne measuring \( F_{\text{d}} \). The quantum noise operator is then defined as

\[
F_N(s) = \frac{P_{\text{out}}(s)}{ML_{\text{arm}} s^2 J_{21}(s)} - h(s)
\]

\[
= \Xi_Q(s) Q_{\text{in}}(s) + \Xi_P(s) P_{\text{in}}(s),
\]

where

\[
\Xi_Q(s) = -\frac{\sqrt{2\gamma_{\text{IFO}} G_{\text{arm}}}}{ML_{\text{arm}} s^2(s + \gamma_{\text{IFO}}/2)}
\]

\[
\Xi_P(s) = \frac{s - \gamma_{\text{IFO}}/2}{\sqrt{2\gamma_{\text{IFO}} G_{\text{arm}}}}.
\]

Hence \( F_N(s) \) is composed of the radiation pressure noise \( \Xi_Q(s) Q_{\text{in}}(s) \) and the shot noise \( \Xi_P(s) P_{\text{in}}(s) \), which are dominant in the low and high frequency range, respectively. The noise magnitude of \( F_N(i\Omega) \) is quantified by the spectral density \( S(\Omega) \), which is defined by

\[
2\pi S^2(\Omega) \delta(\Omega - \Omega') = \langle F_N(i\Omega) F_N(-i\Omega') + F_N(-i\Omega') F_N(i\Omega) \rangle / 2.
\tag{36}
\]

It is now calculated as

\[
S(\Omega) = \sqrt{\langle |\Xi_Q(i\Omega)|^2 \rangle + \langle |\Xi_P(i\Omega)|^2 \rangle / 2},
\]
which is lower bounded by the standard quantum limit (SQL) [27, 42, 45, 46]:

\[ \text{SQL} = |\lambda_\text{laser}| \gamma_{\text{IFO}} = 2\pi c \times 200 \text{ Hz}. \]

Figure 14 shows \( S(\Omega) \) in the following typical setup [27, 43]: \( M = 40 \text{ kg}, L_{\text{arm}} = 4 \text{ km}, P_{\\text{arm}} = 800 \text{ kW}, \omega_0 = 2\pi c / \lambda_\text{laser}, \lambda_\text{laser} = 1064 \text{ nm}, \Delta_d = 0, \gamma_{\text{IFO}} = 2\pi \times 200 \text{ Hz}. \)

**B. Effect of the unstable filter**

As seen above, the detection sensitivity (roughly the inverse of the noise magnitude) is limited by the quantum noise. Especially, the following equality holds [47], meaning that there is a tradeoff between the bandwidth and the peak sensitivity in the high frequency range:

\[ \int_0^\infty \frac{1}{|\xi_p(i\Omega)|^2} d\Omega = 2\pi G_{\text{arm}}^2 L_{\text{arm}}^2. \]

In fact, because the integral does not depend on the bandwidth of the cavity, \( \gamma_{\text{IFO}}, \) a broad-band enhancement of the sensitivity is not allowed.

As described in Sec. VII B, the above tradeoff is attributed to the frequency-dependent propagation phase \( \phi_{\text{arm}}(\Omega) = -2\Omega L_{\text{arm}} / c \) and the idea proposed in [27] was to construct a filter with transfer function \( e^{-i\phi_{\text{arm}}(\Omega)} = e^{2\Omega L_{\text{arm}} / c} \) to compensate \( \phi_{\text{arm}}(\Omega) \). Also, what was described in Sec. VII B is that, unlike the optomechanics-based scheme proposed in [27], we can construct the same filter [30] in all-optics setup, using the feedback amplification method. Figure 15(b) shows \( \phi_{\text{arm}}(\Omega) \) and \( \phi_G = \arg(G_{12}^*(\Omega)) \) in the high-gain limit. The parameters are set as \( L_{\text{arm}} = 4 \text{ km}, \gamma = 3 \times 10^6 \text{ Hz}, \gamma = 2.01\lambda, \kappa_1 = 2c / L_{\text{arm}}, \text{ and } \kappa_2 = 0. \) We can see from this figure that the filter certainly achieves the desired phase cancelation in the frequency range where \( \Omega \ll \kappa_1 = 2c / L_{\text{arm}} \approx 2\pi \times (2.39 \times 10^4) \text{ Hz} \) is satisfied.

Now recall that the filter is realized as the feedback-controlled system shown in Fig. 15(a). As discussed in Sec. VII B, the feedback loop between the NDPA (G) and the control cavity (K) forms another cavity which we call the “loop cavity”, while in Fig. 15(b) the light field circulating in the feedback loop is regarded as an itinerant field. Then, denoting \( a_4 \) for the loop cavity mode, the Hamiltonian of the filter is given by

\[ H_{\text{famp}} = \sum_{k=1}^4 \hbar \omega_k a_k^\dagger a_k + i\hbar \lambda (a_1^\dagger a_2 e^{-i2\Omega p t} - a_1 a_2 e^{i2\Omega p t}) + h g_{24}(a_3^\dagger a_4 + a_2^\dagger a_3) + h g_{34}(a_4^\dagger a_4 + a_3 a_3^\dagger), \]

where \( \omega_3 \) and \( \omega_4 \) are the resonant frequencies of \( a_3 \) and \( a_4, \) respectively. \( g_{24} \) (\( g_{34} \)) describes the coupling between \( a_2 \) and \( a_4 \) \( (a_3 \text{ and } a_4), \) which are given by

\[ g_{24} = \sqrt{c\kappa_1 / L_{\text{loop}}}, \quad g_{34} = \sqrt{c\kappa_2 / L_{\text{loop}}}. \]

with \( L_{\text{loop}} \) the round trip length of the loop cavity. Here we assume \( \omega_k = \omega_p \) \( (k = 1, \ldots, 4). \) Then in the rotating frame at frequency \( \omega_p, \) the dynamics and output equation of the filter are given by

\[ \dot{a}_1 = -\frac{1}{2} a_1 + \lambda a_2^\dagger - \sqrt{2} b_m, \quad \dot{a}_2 = \lambda a_1 + i g_{24} a_4^\dagger, \]

\[ \dot{a}_3 = \lambda a_3 + i g_{34} a_4^\dagger, \quad \dot{a}_4 = \lambda a_4 + i g_{24} a_3^\dagger. \]
\[ \dot{a}_3 = -\frac{\kappa_2}{2}a_3 + ig_3a_4^\dagger, \quad \dot{a}_4 = ig_2a_2 + ig_3a_3, \]
\[ b_{\text{out}} = b_{\text{in}} + \sqrt{\kappa}_1. \]

The input-output relation of this system in the Laplace domain is represented as
\[ b_{\text{out}}(s) = Z(s)b_{\text{in}}(s), \tag{38} \]
where \( Z(s) \) is the rational transfer function; the exact form of \( Z(s) \) is given in Appendix B. Now we show that \( Z(i\Omega) \) approximates the target unstable filter \( e^{i2\Omega L_{\text{arm}}/c} \), under the following assumptions:
\[ |s| \ll \kappa_1 \ll \gamma, \quad \gamma \to 2\lambda + 0, \quad \kappa_1 = 2c/L_{\text{arm}}. \tag{39} \]

These are the same as the conditions for showing \( G_{11}^{(b)}(i\Omega) \approx e^{i2\Omega L_{\text{arm}}/c} \), except \( \kappa_1 \ll \gamma \). First, from \( \gamma \to 2\lambda + 0, \kappa_1 = 2c/L_{\text{arm}}, \) and Eq. (37), we have
\[ Z(s) \approx s^4 + \alpha_3s^3 + \alpha_2s^2 + \alpha_1s^1 + \alpha_0 \]
\[ \frac{s^4 + \beta_3s^3 + \beta_2s^2 + \beta_1s + \beta_0}{s^4 + \beta_3s^3 + \beta_2s^2 + \beta_1s + \beta_0}, \tag{40} \]

where
\[ \alpha_3 = -\beta_3 = -\lambda, \quad \alpha_2 = \beta_2 = \frac{2c(c + L_{\text{arm}}\lambda)}{L_{\text{arm}}L_{\text{loop}}} - \lambda^2, \]
\[ \alpha_1 = -\beta_1 = -\frac{2c\lambda(c + L_{\text{arm}}\lambda)}{L_{\text{arm}}L_{\text{loop}}}, \quad \alpha_0 = \beta_0 = -\frac{2c^2\lambda^2}{L_{\text{arm}}L_{\text{loop}}}. \]

Next, from \(|s| \ll \gamma\), or equivalently \( |s| \ll \lambda\), we have
\[ Z(s) \approx \frac{s^4 + \alpha_1s^3 + \alpha_2s^2 + \alpha_1s + \alpha_0}{s^4 + \beta_3s^3 + \beta_2s^2 + \beta_1s + \beta_0}, \tag{41} \]

Lastly again from Eq. (39) we have \( c \ll L_{\text{arm}}\lambda \) and thus
\[ Z(s) \approx \frac{s + c/L_{\text{arm}}}{s - c/L_{\text{arm}}}. \]

This is exactly the same as the transfer function \( G_{11}^{(b)}(s) \) in Eq. (29) with \( \kappa_1 = 2c/L_{\text{arm}}\). Hence \( Z(i\Omega) \approx e^{i2\Omega L_{\text{arm}}/c} \) holds, and the system depicted in Fig. 15(a) approximates the target filter. Figure 15(c) plots the phase of the original \( Z(i\Omega) \) given in Appendix B where \( L_{\text{loop}} = 0.5\) m and the other parameters are the same as those used in Fig. 15(b). This clearly shows the exact model incorporating the loop cavity \( a_4 \) certainly has the desired phase cancelling effect. Importantly, the figures (b) and (c) are almost the same, meaning that the simplified model without \( a_4 \) is still useful.

### C. The entire system and stabilizing control

In the previous subsection we have seen that the constructed unstable filter certainly has a desired phase-cancellation property, from which we expect that the sensitivity of the gravitational-wave detector can be broadly enhanced in the high-frequency range. Here we model the entire system composed of the interferometer and the unstable filter depicted in Fig. 16. Note that the phase cancellation filter is an unstable system, and the entire system must be stabilized; here we employ the measurement-based feedback for this purpose.

Let us begin with the dynamics of the entire system without stabilization. Here we assume that \( \omega_p = \omega_k = \omega_0 \) (\( k = 1, \ldots, 4 \)). Then in the rotating frame at frequency \( \omega_0 \), the Hamiltonian of the entire system is given by
\[ H_{\text{tot}} = \frac{P^2}{2M} + \Delta d^d - hG_{\text{arm}}(d + d^\dagger)X - F_{\text{GW}}X + h\Delta_{\text{NI}}(d^d a_1 + da_2^\dagger) + ih\lambda(a_1^d a_2^\dagger - a_2a_1) + h\Delta_{\text{NI}}(a_2^d a_4 + a_4^d a_2 + a_3^d a_4 + a_4^d a_3), \]

where again \((X, P)\) are the differential (position, momentum) operators of ETMs and \( d \) is the sideband mode of the interferometer field. We assume that only \( a_1 \) couples with \( d \), with strength \( g_{\text{NI}} = \sqrt{c/(2L_{\text{arm}})} \). The signal leaks to outside through the SRM where the vacuum input \( d_{\text{in}} \) must enter. Then the dynamical equation of the entire system are given by
\[ \dot{X} = \frac{1}{M}P - i\frac{\Delta_{\text{IR}}}{2}d - hG_{\text{arm}}(d + d^\dagger) + F_{\text{GW}}, \]
\[ \dot{d} = -i\Delta_{\text{IR}}d - \frac{\gamma_{\text{IR}}}{2}d + iG_{\text{arm}}X - ig_{\Delta_{\text{NI}}}a_1 - \sqrt{\gamma_{\text{IR}}d_{\text{in}}}, \]
\[ \dot{a}_1 = -\frac{\gamma_{\text{IR}}}{2}a_1 - ig_{\Delta_{\text{NI}}}a_3 + \sqrt{\gamma_{\text{IR}}d_{\text{in}}}, \]
\[ \dot{a}_2 = \lambda a_2^d - ig_{\Delta_{\text{IR}}}a_4, \quad \dot{a}_3 = -\frac{\kappa_{\text{loss}}}{2}a_3 - ig_{\Delta_{\text{IR}}}a_4 - \sqrt{\kappa_{\text{loss}}b_{\text{loss}}}, \]
\[ \dot{a}_4 = -\frac{\kappa_{\text{loss}}}{2}a_4 - ig_{\Delta_{\text{IR}}}a_2 - ig_{\Delta_{\text{IR}}}a_4 - \sqrt{\kappa_{\text{loss}}b_{\text{loss}}}, \]

![FIG. 16: Structure of the entire controlled system.](image)
where $b_{\text{loss}}$ ($k=1, 3, 4$) are the noise field representing the optical losses of the internal modes $a_k$ with magnitude $\kappa_{\text{loss}}$. We use the quadrature representation $q_k = (a_k - a_k^*)/\sqrt{2}$, $p_k = (a_k + a_k^*)/\sqrt{2}$ ($k = d, 1, 2, 3, 4$), $Q_d^{\text{out}} = (d_{\text{in}} - d_{\text{out}}^*)/\sqrt{2}$, $P_d^{\text{out}} = (d_{\text{in}} - d_{\text{out}}^*)/\sqrt{2}$, $Q_{\text{loss}} = (b_{\text{loss}} + b_{\text{loss}}^*)/\sqrt{2}$, $P_{\text{loss}} = (b_{\text{loss}} - b_{\text{loss}}^*)/\sqrt{2}$ ($n = 1, 3, 4$). Also we define the dimensionless operators $X_M = X/\sqrt{\Omega_M}/\hbar$ and $P_M = P/\sqrt{\Omega_M}$, with $\Omega_M$ the eigenfrequency of the ETM; they satisfy $[X_M, P_M] = i$. Then the above dynamical equations are summarized to

$$
\dot{x} = Ax + B_w w, \quad y = Cx + D_w,
$$

(42)

where $x = [X_M \ P_M \ q_1 \ p_1 \ q_2 \ p_2 \ q_3 \ p_3 \ q_4 \ p_4]^T$, $w = [F_{GW} Q_d^{\text{out}} P_d^{\text{out}} Q_{\text{loss}} P_{\text{loss}} Q_{\text{loss}} P_{\text{loss}} Q_{\text{loss}} P_{\text{loss}}]^T$, and $y = [Q_{\text{out}} P_{\text{out}}]^T$. The matrices $A, B, C, D$ in $\mathbb{R}^{12 \times 12}$, $B_w \in \mathbb{R}^{12 \times 9}$, $C \in \mathbb{R}^{2 \times 12}$, $D \in \mathbb{R}^{2 \times 9}$ are shown in Appendix A. Note that $A$ has eigenvalues with positive real part, meaning that the entire system is unstable.

To stabilize the system, we apply the linear quadratic gaussian (LQG) feedback control [48]. This control is generally conducted by feeding a measurement output back to control the system. In our case we measure $Q_{\text{out}}$ or $P_{\text{out}}$ by the photodetector (note that measuring both quadratures is prohibited by quantum mechanics); the measurement result is used to construct the estimate $\hat{x}$, which is fed back to control the ETMs directly by implementing a piezo-actuator [49]. This control is modeled by adding the classical input $u = -F_u \hat{x}$ to the dynamics of the oscillator, where $F_u \in \mathbb{R}^{1 \times 12}$ is the feedback gain to be designed. In the LQG setting, the (quantum) Kalman filter is used to obtain the least squared estimate $\hat{x}$. The entire controlled system is then given by

$$
\dot{x} = Ax + B_w w + Bu, \quad y = C x + D_w,
$$

where $K_u \in \mathbb{R}^{12 \times 12}$ is the Kalman gain (shown later). $B_u = [0, 1, 0, \cdots, 0]^T \in \mathbb{R}^{12}$ (only the second entry is non-zero) represents that the actuator directly drives $X$ of the oscillator. $C_m \in \mathbb{R}^{1 \times 12}$ and $D_m \in \mathbb{R}^{1 \times 9}$ are the first or second row vectors of $C$ and $D$, respectively; for instance, if $y_m = Q_d^{\text{out}}$, then $C_m$ and $D_m$ are the first row vector of $C$ and $D$. Here we define $\epsilon = \hat{x} - x$. Then the above dynamical equation is rewritten as

$$
\begin{bmatrix}
\dot{x} \\
\dot{\epsilon}
\end{bmatrix} = A_{\text{tot}} \begin{bmatrix}
x \\
\epsilon
\end{bmatrix} + B_{\text{tot}} w, \quad y_m = C_{\text{tot}} \begin{bmatrix}
x \\
\epsilon
\end{bmatrix} + D_{\text{tot}} w,
$$

where

$$
A_{\text{tot}} = \begin{bmatrix}
A - B_u F_u & -B_u F_u \\
0 & A - K_u C_m
\end{bmatrix}, \quad C_{\text{tot}} = \begin{bmatrix}
C_m & 0
\end{bmatrix},
$$

$$
B_{\text{tot}} = \begin{bmatrix}
B_w \\
K_u D_m - B_w
\end{bmatrix}, \quad D_{\text{tot}} = D_m.
$$

The entire system becomes stable when $A_{\text{tot}}$ has no eigenvalue with positive real part. Since the eigenvalues of $A_{\text{tot}}$ are the same as those of $A - B_u F_u$ and $A - K_u C_m$, we can stabilize the system by determining appropriate $F_u$ and $K_u$. The necessary and sufficient condition for such $F_u$ and $K_u$ to exist is that the system is controllable and observable; that is, the following controllability matrix $C_u$ and observability matrix $O_{ym}$ are of full-rank:

$$
C_u = [B_u \ AB_u \cdots A^{11} B_u],
$$

$$
O_{ym} = [C_m^T \ A^T C_m^T \cdots (A^T)^{11} C_m^T]^T.
$$

(43)

In the LQG setup, $F_u$ and $K_u$ are determined from the policy to minimize the following cost function $J$ and the estimation error $\epsilon$:

$$
J = \lim_{t \to \infty} \frac{1}{t} \langle \int_0^t (x^T(\tau)Q x(\tau) + Ru^2(\tau)) d\tau \rangle,
$$

$$
\epsilon = \langle (x - \hat{x})^T(x - \hat{x}) \rangle,
$$

where $Q \in \mathbb{R}^{12 \times 12}$ and $R \in \mathbb{R}$ are the weighting parameters. These parameters are constant numbers that do not depend on time. From the separation principle, these two optimization problems can be solved separately. If the optimal solutions of $F_u$ and $K_u$ are uniquely determined, then they stabilize the entire system and given by

$$
F_u = R^{-1} B_u^T P_F, \quad K_u = (P_K C_m^T + B_u V D_m^T) (D_m V D_m^T)^{-1},
$$

where $P_F \in \mathbb{R}^{12 \times 12}$ and $P_K \in \mathbb{R}^{12 \times 12}$ are the solutions of the following algebra Riccati equations:

$$
P_F A + A^T P_F - P_F B_u R^{-1} B_u^T P_F + Q = 0,
$$

$$
P_K A^T + A P_K + B_u V D_m^T - (P_K C_m^T + B_u V D_m^T) \times (D_m V D_m^T)^{-1} (P_K C_m^T + B_u V D_m^T)^T = 0.
$$

$V$ is the covariance matrix of the noise vector $w$. Note that that $F_{GW}$ is now assumed to be a Gaussian noise with known variance; since in reality $F_{GW}$ is not a noisy Gaussian signal, this assumption means that we consider the worst case scenario which still guarantees the existence of stabilizing controller.

D. Quantum noise of the stabilized system

The quantum noise observed at the detector is calculated as follows. First we have

$$
y_m(s) = \left[ C_{\text{tot}} (sI - A_{\text{tot}})^{-1} B_{\text{tot}} + D_{\text{tot}} \right] w(s)
$$

$$
= \Psi_{Q_\epsilon}(s) Q_d^{\text{in}}(s) + \Psi_{P_\epsilon}(s) P_{\text{tot}}(s) + \Psi_h(h(s))
$$

$$
+ \sum_{k=1,3,4} \left( \Psi_{Q_k}(s) Q_{\text{loss}}(s) + \Psi_{P_k}(s) P_{\text{loss}}(s) \right),
$$
The parameters of the LQG controller are chosen as 

where the functions $\Psi_x$ are the transfer functions of the corresponding noises and the gravitational-wave strain signal $h(t)$. The quantum noise operator is defined as $F_N(i\Omega) = y_m(i\Omega)/\Psi_h(i\Omega) - h(i\Omega)$. Then from Eq. (36), we obtain the noise spectral density:

$$S^2(\Omega) = \left( |\Psi_{Q_d}|^2 + |\Psi_{P_a}|^2 + |\Psi_{Q_1}|^2 + |\Psi_{P_1}|^2 \\ + |\Psi_{Q_3}|^2 + |\Psi_{P_3}|^2 + |\Psi_{Q_4}|^2 + |\Psi_{P_4}|^2 \right) / 2 |\Psi_h|^2.$$ 

Here the parameters of the interferometer are chosen as follows: $M = 40 \text{ kg}$, $L_{\text{arm}} = 4 \text{ km}$, $P_{\text{arm}} = 800 \text{ kW}$, $\Omega_M = 1 \text{ Hz}$, $\omega_0 = 2\pi/c/\lambda_{\text{laser}}$, $\lambda_{\text{laser}} = 1064 \text{ nm}$, $\Delta_d = -63.0 \text{ Hz}$, $\gamma_{\text{IFO}} = 1062 \text{ Hz}$. Note that, unlike the setup in Fig. 17, a non-zero value of $\Delta_d$ is taken, which is necessary for the system to be controllable and observable. These non-zero value of $\Delta_d$ as well as the value of $\gamma_{\text{IFO}}$ are calculated from the scaling law [43, 50] of an GW detector with a signal recycling mirror. Also the parameters of the unstable filter are chosen as

$$\lambda = 3 \times 10^6 \text{ Hz}, \quad \gamma = 2.01\lambda, \quad \kappa_1 = \frac{2c}{L_{\text{arm}}}, \quad L_{\text{loop}} = 0.5 \text{ m},$$

$$\gamma_{\text{loss}} = 1 \text{ MHz}, \quad \kappa_{\text{loss}} = 100 \text{ Hz}, \quad \kappa_{\text{loss}} = 600 \text{ kHz}.$$ 

The parameters of the LQG controller are chosen as $Q = I$, $R = 1$, $V = \text{ diag}[0, 1/2, \cdots, 1/2]$ (all $1/2$ except the first element). We then have Fig. 17 showing that actually the proposed unstable filter can enhance the bandwidth in the high-frequency regime without sacrificing the peak-sensitivity.

We conclude this section with the discussion on the possible advantage and disadvantage of the proposed filter. Figures 18(a) and (b) show the quantum noise of the controlled system with several optical path length in (a) NDPA and (b) the loop cavity. Recall that these quantities are related to the cavity length via $\gamma_{\text{loss}} = c\Gamma_{\text{loss}}/L_{\text{NDPA}}$ and $\kappa_{\text{loss}} = c\Gamma_{\text{loss}}/L_{\text{loop}}$, where $\Gamma_{\text{loss}}$ and $L_{\text{loop}}$ denote the optical loss ratio of the corresponding cavity mode. Hence, importantly, the figure shows that the sensitivity is not largely affected by the optical loss both in NDPA and the loop cavity (i.e., the feedback loop). In particular, the loss in the loop cavity has almost no effect on the sensitivity, as expected from the fact that the feedback amplification scheme is in general robust against the imperfection in the feedback loop [8]. As for the loss in NDPA, there is certainly some impact on the sensitivity in the high frequency regime, but this can be reduced by making the length of NDPA longer.

FIG. 18: Quantum noise of the controlled system in (a) several optical path length of the NDPA, $L_{\text{NDPA}}$ and (and $\Gamma_{\text{loss}} = 0.1 \%$), (b) several optical path length of the loop cavity, $L_{\text{loop}}$ and $T_{\text{loss}} = 0.1 \%$. Also (c) is the case where one of the three damping rates of the losses is 150 kHz, while the other two are zeros. Type 1 is the case of ($\gamma_{\text{loss}}, \kappa_{\text{loss}}, \kappa_{\text{loss}} = (150, 0, 0) \text{ kHz}$). In Type 2 ($\gamma_{\text{loss}}, \kappa_{\text{loss}}, \kappa_{\text{loss}} = (0, 150, 0) \text{ kHz}$. In Type 3 ($\gamma_{\text{loss}}, \kappa_{\text{loss}}, \kappa_{\text{loss}} = (0, 0, 150) \text{ kHz}$). The other parameters are the same as those used in Fig. 17.
On the other hand, the parameter $\kappa_{\text{loss}}$, i.e., the optical loss in the control cavity with mode $a_3$, has a large impact on the sensitivity, as indicated from Fig. 18(c). In this figure, Type 1 is the case where $\gamma_{\text{loss}} = 150$ kHz and the others are zeros, Type 2 is the case where $\kappa_{\text{loss}} = 150$ kHz and the others are zeros, and Type 3 is the case where $\kappa_{\text{loss}} = 150$ kHz and the others are zeros. This figure tells why $\kappa_{\text{loss}}$ is chosen to be much smaller than $\gamma_{\text{loss}}$ and $\kappa_{\text{loss}}$ in Fig. 17. To achieve such a small loss, the optical path length of the control cavity should be long: from $\kappa_{\text{loss}} = cT_{\text{loss}}/L_{\text{MCC}}$ with $T_{\text{loss}}$ the noise transmissivity and $L_{\text{MCC}}$ the round trip length of the MCC, if $\kappa_{\text{loss}} = 100$ Hz is required, we need, e.g., $T_{\text{loss}} = 0.01 \%$ and $L_{\text{MCC}} = 300 \text{ m}$. That is, although the proposed unstable filter based on the feedback amplification method can be constructed in all-optics way in contrast to the opto-mechanical proposal [26], a very careful fabrication for the control cavity is required.

IX. CONCLUSION

In this paper, we have shown that a variety of quantum functionalities are generated under the unique concept of feedback amplification. We hope that, combined with the several established quantum information methods such as entanglement generation [51] and analogue information processing [52], those basic functionalities might be effectively applied to enhance the performance of existing quantum technological devices and moreover to create a new quantum mechanical machine.

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Appendix A: Proof of Eq. 42

We apply the proof that was used to derive Eq. 21 in Sec. III. First, similar to Eq. 20, we assume

\[
\begin{align*}
\det [G(s)] & \to 0, \\
\frac{\det [G(s)]}{G_{22}(s)} & \to 0, \\
\frac{\det [G(s)]}{G_{22}(s)} & \to 0, \\
\frac{\det [G(s)]}{G_{22}(s)} & \to 0, \\
\frac{G_{12}(s)}{G_{22}(s)} & \to 0, \\
\frac{G_{12}(s)}{G_{22}(s)} & \to 0, \\
\frac{G_{21}(s)}{G_{22}(s)} & \to 0, \\
\frac{G_{21}(s)}{G_{22}(s)} & \to 0.
\end{align*}
\]

Also, we assume $G_{11}(s) = G_{22}(s)$ and $G_{11}(s) = G_{22}(s)$ \forall s \in C$. Then in the high gain limit $|G_{11}(s)| \to \infty$ (\iff $|G_{22}(s)| \to \infty$) and $|G_{11}(s)| \to \infty$ (\iff $|G_{22}(s)| \to \infty$) the transfer functions are approximated by

\[
G_{11}^{(fb)} = \frac{G_{12} - G_{21}G_{22}\det [G]}{1 - G_{21}G_{22}} \to 0,
\]

\[
G_{12}^{(fb)} = \frac{G_{11}G_{22}G_{22}}{1 - G_{21}G_{22}} \to 1,
\]

\[
G_{13}^{(fb)} = \frac{G_{11}K_{21}}{1 - G_{21}G_{22}K_{22}} \to 0,
\]

\[
G_{21}^{(fb)} = \frac{G_{21}K_{21}}{1 - G_{21}G_{22}K_{22}} \to 0,
\]

\[
G_{22}^{(fb)} = \frac{G_{12} + G_{21}K_{22}\det [G]}{1 - G_{21}G_{22}} \to 0,
\]

\[
G_{23}^{(fb)} = \frac{G_{11}G_{21}K_{21}}{1 - G_{21}G_{22}K_{22}} \to 0.
\]

Appendix B: Exact expression of $Z(s)$

The transfer function $Z(s)$ in Eq. 48 is given by

\[
Z(s) = \frac{s^4 + \alpha_3s^3 + \alpha_2s^2 + \alpha_1s + \alpha_0}{s^4 + \beta_3s^3 + \beta_2s^2 + \beta_1s + \beta_0},
\]

where

$\alpha_3 = -\gamma/2, \alpha_2 = g_{24}^2 + g_{34}^2 - \lambda^2, \alpha_1 = -(g_{24}^2 + g_{34}^2)\gamma/2,$

$\alpha_0 = -\lambda^2 g_{34}^2, \beta_3 = \gamma/2, \beta_2 = g_{24}^2 + g_{34}^2 - \lambda^2,$

$\beta_1 = (g_{24}^2 + g_{34}^2)\gamma/2, \beta_0 = -\lambda^2 g_{34}^2.$
Appendix C: The matrix entries of $A, B_w, C, D$

\[
A = \begin{bmatrix}
0 & \Omega_M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}G_M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\gamma_{\text{IFO}}}{2} & \frac{\Delta}{2} & 0 & g_{\text{NI}} & 0 & 0 & 0 & 0 \\
\sqrt{2}G_M & 0 & -\frac{\Delta}{2} & -\frac{\gamma_{\text{IFO}}}{2} & -g_{\text{NI}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_{\text{NI}} & -\frac{\gamma_{\text{loss}}}{2} & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & -g_{\text{NI}} & 0 & 0 & -\frac{\gamma_{\text{loss}}}{2} & 0 & -\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{\text{loss}}}{2} & 0 & 0 & -\kappa_{3\text{loss}} & 2 \gamma_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{\text{loss}}}{2} & 0 & 0 & -g_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{\text{loss}}}{2} & 0 & -g_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{3\text{loss}}}{2} & \frac{\Delta}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{4\text{loss}}}{2} & 0 \\
\end{bmatrix}
\]

\[
B_w = \begin{bmatrix}
0 & -\sqrt{\gamma_{\text{IFO}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\sqrt{\gamma_{\text{IFO}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{\gamma_{1\text{loss}}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{\gamma_{1\text{loss}}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{2\text{loss}}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{2\text{loss}}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{3\text{loss}}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{3\text{loss}}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{4\text{loss}}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & \sqrt{\gamma_{\text{IFO}}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\gamma_{\text{IFO}}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
, \quad D = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

where $G_M = G_{\text{arm}}\sqrt{\hbar/(M\Omega_M)}$.  

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