A Note on S. Weinberg, “Massless Particles in Higher Dimensions”

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Abstract

In [1], Weinberg made a conjecture about the little-group representations of massless particles that can be created out of the vacuum by the action of a local operator in \(d\) dimensions, generalizing his old result [2] in \(d = 4\). In this note, I prove his conjecture and extend it to arbitrary irreps of \(so(1, d - 1)\).
In [1], Steven Weinberg posed the following question. Consider a local operator, \( O(x) \), transforming in some irreducible unitary (infinite-dimensional) representation of the Poincaré algebra \( \text{iso}(1, d - 1) \). Assume that \( O(x) \) has a nonzero matrix element between a massless 1-particle state and the vacuum

\[
0 \neq \langle 0 | O(x) | k, R' \rangle
\]

By translational invariance, this matrix element is nonvanishing if and only if

\[
0 \neq \langle 0 | O(0) | k, R' \rangle
\]

We may therefore assume that \( O(0) \) transforms as some (nonunitary) finite-dimensional irreducible representation, \( R \), of the Lorentz algebra \( \text{so}(1, d - 1) \) which leaves the point \( x = 0 \) fixed. Weinberg’s question is:

- Given \( R \), what representation, \( R' \), of \( \text{so}(d - 2) \subset \text{iso}(d - 2) \) is compatible with a nonzero matrix element (1)?

Weinberg computed some examples, and conjectured an answer for (tensorial) representations \( R \), given by Young Tableaux. Here I prove his conjecture and extend it to all finite-dimensional irreps, \( R \).

The problem reduces to one in Lie theory. As an irrep of \( \text{so}(1, d - 1) \), \( R \) is a fortiori a representation of the little algebra, \( \text{iso}(d - 2) \subset \text{so}(1, d - 1) \). As a representation of \( \text{iso}(d - 2) \), \( R \) is reducible but indecomposable.

Writing \( \text{iso}(d - 2) = \text{so}(d - 2) \ltimes K \), where \( K \) is the \((d - 2)\)-dimensional abelian subalgebra of \( \text{so}(1, d - 1) \) which (along with \( \text{so}(d - 2) \)) leaves fixed a particular null momentum. The irreducible subrepresentation, \( R' \subset R \) is such that \( K \) restricts to zero on \( R' \). An alternative characterization of \( R' \) is that it is simultaneously an irrep of \( \text{iso}(d - 2) \) and of \( \text{so}(1, 1) \times \text{so}(d - 2) \), where the \( \text{so}(d - 2) \) is the common subalgebra of these two maximal subalgebras of \( \text{so}(1, d - 1) \).

With this reformulation, we can ask, “If \( R' \) is an irrep of \( \text{so}(1, 1) \times \text{so}(d - 2) \), which irrep is it?”

To answer that, we note that the highest-weight of \( R \) is contained in \( R' \).

**Proof:** Since \( K \) raises the \( \text{so}(1, 1) \) weight, it necessarily annihilates the highest weight of \( R \). Acting with \( \text{so}(d - 2) \) does not change the \( \text{so}(1, 1) \) weight and the commutator of an element of \( \text{so}(d - 2) \) with \( K \) lies in \( K \). Hence, acting on the highest weight of \( R \) with the generators of \( \text{iso}(d - 2) \), we get an irrep \( R' \) of \( \text{iso}(d - 2) \) with \( K \) represented by 0. By construction, \( R' \) is also an irrep of \( \text{so}(1, 1) \times \text{so}(d - 2) \).

With that in mind, let us decompose \( R \) under \( \text{so}(1, 1) \times \text{so}(d - 2) \)

\[
R = \bigoplus_i (\lambda_i) \otimes R_i
\]

where \( R_i \) is an irrep of \( \text{so}(d - 2) \) and \( \lambda_i \) is the \( \text{so}(1, 1) \) weight labeling the corresponding 1-dimensional irrep of \( \text{so}(1, 1) \). Without loss of generality, we can order

\[
\lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots
\]

\footnote{For our conventions for the \( \text{so}(1, 1) \) weights, see [3].}
The embedding $so(1, 1) \times so(d - 2) \hookrightarrow so(1, d - 1)$ is the one obtained by omitting the left-most node of the Dynkin diagram.

The remaining nodes are the simple roots of $so(d - 2)$. The highest weight of $R$ under $so(1, d - 1)$ is also the highest weight under the $so(1, 1) \times so(d - 2)$ subalgebra. That is, $R'$ is the representation $(\lambda_1) \otimes R_1$ in (2).

To be more explicit, we need some notation. Highest weight representations, with highest weight $\Lambda$, will be denoted by their Dynkin labels,

$$n_i = \frac{2(\Lambda, \alpha_i)}{\alpha_i, \alpha_i}$$

where the $\alpha_i$ are the simple roots. Our convention for the $so(1, 1)$ weights will be that the adjoint representation of $so(1, d - 1)$ decomposes as

$$[0, 1, 0, 0, \ldots, 0] = (2) \otimes [1, 0, 0, \ldots, 0] \oplus (0) \otimes [0, 1, 0, \ldots, 0] \oplus (0) \otimes [0, 0, 0, \ldots, 0] \oplus (-2) \otimes [1, 0, 0, \ldots, 0] \quad (3)$$

Here

- $K = (2) \otimes [1, 0, 0, \ldots, 0]$. I.e., the generators of $K$ transform as a vector of $so(d - 2)$ and with weight +2 under $so(1, 1)$.
- $(0) \otimes [0, 1, 0, \ldots, 0]$ is the adjoint of $so(d - 2)$ and
- $(0) \otimes [0, 0, 0, \ldots, 0]$ is the generator of $so(1, 1)$.

The normalization of the $so(1, 1)$ weights $(\lambda)$ is such that tensorial representations have $\lambda$ even and spinorial representations have $\lambda$ odd.

Let

$$R = [n, n_1, n_2, \ldots, n_r] \quad (4)$$

be our chosen highest weight representation of $so(1, d - 1)$. The simple roots of $so(d - 2)$ were obtained by omitting the first simple root. The corresponding Dynkin labels are obtained by omitting the first Dynkin label of $R$. The highest-weight of the $so(1, d - 1)$ irrep $R$, with Dynkin labels (4), is the highest weight of the $so(d - 2)$ irrep with Dynkin labels $[n_1, n_2, \ldots, n_r]$. That is, our sought-after representation of $so(d - 2)$ is

$$R_1 = [n_1, n_2, \ldots, n_r] \quad (5)$$

Though we don’t need it, the $so(1, 1)$ weight is also determined:

$$\lambda_1 = \begin{cases} 
2n + n_r + 2 \sum_{i=1}^{r-1} n_i & d = 2r + 3 \\
2n + n_r + n_{r-1} + 2 \sum_{i=1}^{r-2} n_i & d = 2r + 2 
\end{cases} \quad (6)$$
Finally, let us translate Weinberg’s Young diagrams into the corresponding Dynkin labels of irreps. Consider a Young diagram, whose rows have lengths \( l_0, l_1, \ldots l_r \), where \( r = (d - 2)/2 \) for \( d \) even and \( (d - 3)/2 \) for \( d \) odd.

For \( d \) odd, the corresponding Dynkin labels for \( R \) are

\[
\begin{align*}
    n &= l_0 - l_1 \\
    n_i &= l_i - l_{i+1}, \quad i = 1, \ldots, r - 1 \\
    n_r &= 2l_r
\end{align*}
\]

For \( d \) even,

\[
\begin{align*}
    n &= l_0 - l_1 \\
    n_i &= l_i - l_{i+1}, \quad i = 1, \ldots, r - 2 \\
    n_{r-1} + n_r &= 2l_{r-1} \\
    |n_{r-1} - n_r| &= 2l_r
\end{align*}
\]

Note that (of course) we only get tensorial representations this way (\( n_r = \text{even for } d \text{ odd} \) or \( n_{r-1} + n_r = \text{even for } d \text{ even} \)). Moreover, when \( d \) is even and \( l_r > 0 \), the Young diagram corresponds to a reducible representation, decomposing into two irreps whose Dynkin labels differ by exchanging \( n_{r-1} \leftrightarrow n_r \).

For tensorial representations, dropping the first Dynkin label in passing from \( R \) to \( R_1 \) is precisely the same as Weinberg’s conjectured “decapitation” procedure: removing the first row of the Young diagram. But it extends naturally to spinorial representations as well. And, for \( d \) even, it takes care of the reducibility of Young diagrams with \( l_r > 0 \). Finally, it gives an interpretation of the \( so(1,1) \) weight in (6): \( \lambda_1 = 2l_0 \), where \( l_0 \) is the length of the row that he removes.

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References

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