APPLICATION OF DATA ASSIMILATION FOR PARAMETER CORRECTION IN SUPER CAVITY MODELLING

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ABSTRACT

On the imperfect water entry, a high speed slender body moving in the forward direction rotates inside the cavity. The super cavity model describes the very fast motion of body in water. In the super cavity model the drag coefficient plays important role in body’s motion. In some references the drag coefficient is simply chosen by different values in the interval 0.8-1.0. In some other references this drag coefficient is written by the formula

\[ k = C_{D0}(1 + \sigma) \cos^2 \alpha \]

where \( \sigma \) is the cavity number, \( \alpha \) is the angle of body axis and flow direction, \( C_{D0} \) is a parameter chosen from the interval 0.6-0.85. In this paper the drag coefficient \( k = k_1 C_{D0}(1 + \sigma) \cos^2 \alpha \) is written with fixed \( C_{D0} = 0.82 \) and the parameter \( k_1 \) is corrected so that the simulation body velocities are closer to observation data. To find the convenient drag coefficient the data assimilation method by differential variation is applied. In this method the observing data is used in the cost function. The data assimilation is one of the effected methods to solve the optimal problems by solving the adjoin problems and then finding the gradient of cost function.

Keywords: data assimilation, optimal, Runge-Kutta methods.

1. INTRODUCTION

When slender body running very fast under water (velocity is higher than 50 m/s) the cavity phenomena is happened. Cavity may have a variety of cause. The most common example is boiling water, where the vapor pressure is increased by raising the water temperature. In hydrodynamics applications cavitation is the appearance of vapor bubbles and pockets inside homogeneous liquid medium. This phenomenon occurs because the pressure is reduced to the vapor pressure limit. In this paper we will study super cavity appearing by the very fast
movement of slender body in water that makes uncontrolled gun-launched slender body. Except the body head called by cavitator is directly touching with water, the gas layer can be covered partial or full body depending on the design of body form. The body rotates about its nose. The form of body's nose can be differently chosen such as: sharp, hemisphere, plate disk... For simple calculation we choose cavitator formed by the plate disk with diameter $d_c$ (Figure 1).

The body is consisted of two parts: the cone top and cylinder part with the diameter $d$.
- $L$ is the length of the slender body;
- $L_2$ is the body's length of cylinder part
- $L_1$ is the body's length of cone top part
- $d$ is the body's diameter
- $d_c$ the body's nose diameter

**Figure 1.** Slender body geometer.

In the super cavity model the following assumptions are ([1, 2]):
- The motion of the projectile is confined to a plane;
- The slender body rotates about its nose ([1 - 4]);
- The effect of gravity on the dynamics of this body is negligible;
- The motion of the slender body is not influenced by the presence of gas, water vapor or water drops in the cavity;

The super cavity problems are studied in [1, 2, 5 - 11]. To study the motion problems of slender body running under water there are basic approaches:
- The experimental approach consisting in observing and measuring motion by remote sensing.
- The modeling approach based on mathematical models of the flow and of the body motion.
- The models of body's motion under water include some parameters that have not a clear physical meaning because they are a synthetic representation of several physical effects such as sub-grid turbulence that can't be explicit in the model because of a necessary truncation for numerical purposes.

None of these approaches is sufficient to predict the evolution of body motion. They have to be combined to retrieve the body motion under water. All the techniques used to combine the information provided by observations and the information provided by models are named by Data Assimilation methods and have known an important development during these last decades. The Data Assimilation method using differential variation is based on the theory of optimal control for partial differential equation by Lions et al. [12, 13] and Marchuk et al. [14]. This method is applied to correct coefficients, solve the inverse problems, simulate the air and fluid pollution processes ([14 - 21]).

In this paper we will concentrate the study on the identification coefficient parameter $k_1$ of the drag coefficient $k = k_1 C_{D0} (1 + \sigma \cos^2 \alpha)$ ($C_{D0} = 0.82$). In the second section we will describe the abstract definition of an inverse problem via variation methods. The unknown coefficient is defined as the solution of an optimization problem. In the third section we will formulate the
model of the problem of body's fast motion under water problem. The 4-th section is devoted to
the application of optimal control to the identification of model's coefficient.

2. GENERAL VARIATION APPROACH

Because In the model's parameters are a synthetic representation of several physical effects,
they can't be directly estimated. They depend both on the model and on the data. They will be
evaluated as the solution of an "Inverse Problem", basically as the solution of an optimization
problem. The advantage is that there exist many efficient algorithms for solving these problems.
Most of them require to compute the gradient of the function to be minimized. The cost function
is done by solving an "Adjoin Model". The method is described in many papers together with
the computational developments ([14 - 21]). It can be summarized as follows:

Let $X(t)$ the state vector describing the evolution of a system governed by the abstract equation:

$$
\begin{align*}
\frac{dX}{dt} &= F(X, E_1, \ldots, E_n) \\
X(0) &= X_0
\end{align*}
$$

(2.1)

where: $E_1, \ldots, E_n$ are the equation's parameters with $n$ is the number of parameters; $X(t)$ is a
unknown state vector belonging for any $t$ to a Hilbert space $\mathcal{S}$, $X_0 \in \mathcal{S}$; $F$ is a nonlinear
operator mapping $Y \times Y_P$ to $Y$ with $Y = L^2(0, T, \mathcal{S})$, $\|Y\| = (\ldots)^{1/2}$, $Y_P$ is Hilbert space (the
space of model's parameters); Suppose that for given initial value $X(0) = X_0 \in \mathcal{S}$ and
$(E_1, \ldots, E_n) \in Y_P$ there exists a unique solution $X \in \mathcal{S}$ to (2.1). In case the values of
$E = (E_1, \ldots, E_n)$ are unknown and there are some observation data $X_{obs} \in \mathcal{S}_{obs}$ with
$\mathcal{S}_{obs}$ is a Hilbert space (observation space) we introduce the functional called cost function:

$$
J(E) = \frac{1}{2} \int_{0}^{T} \left\langle H(CX - X_{obs}), CX - X_{obs} \right\rangle_{\mathcal{S}_{obs}} dt + \frac{1}{2}(E - E_0)^2
$$

(2.2)

where $(E_{0,1}, \ldots, E_{0,n})$ are priori approximation evaluations of $E_1, \ldots, E_n$; $C : \mathcal{S} \rightarrow \mathcal{S}_{obs}$ is a
linear bounded operator, $H : \mathcal{S}_{obs} \rightarrow \mathcal{S}_{obs}$ is symmetric positive definite operator; The
problem is to determine $E^* = \left( E_1^*, \ldots, E_n^* \right)$ by minimizing $J$. The second and the third terms in
$J$ are a regularization term in the sense of Tykhonov, have a well posed problem (see [15, 17]).
The optimal solutions are characterized by $\nabla J(E_1^*, \ldots, E_n^*)$, where $\nabla J$ is the gradient of $J$. To
compute this gradient we introduce $e_i$ $(i = 1, 2, \ldots, n)$, the directions in the space $Y_P$. We will
compute the Gateaux derivative of the cost function $J$ by $E = (E_1, \ldots, E_n)$ in the directions of
$e = (e_1, \ldots, e_n)$. The Gateaux derivative of the cost function $J$ in the directions of
$e = (e_1, \ldots, e_n)$ will be:
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\[
\dot{J}(E_1, ..., E_n) = \sum_{i=1}^{n} \int_{0}^{T} \left( C^T H \left( CX - X_{\text{obs}} \right), \dot{X}^{(i)} \right)_3 \, dt + \sum_{i=1}^{n} \langle E_i - E_{i,0}, e_i \rangle \\
= \sum_{i=1}^{n} \int_{0}^{T} \left( C^T H \left( CX - X_{\text{obs}} \right), \dot{X}^{(i)} \right)_3 \, dt + \sum_{i=1}^{n} \langle E_i - E_{i,0}, e_i \rangle \\
= \left( \dot{J}_{E_1}(E_1, ..., E_n), ..., \dot{J}_{E_n}(E_1, ..., E_n) \right)(e_1, ..., e_n)^T 
\]

where: \( \dot{X}^{(i)} \), \( \dot{J}_{E_i}(E_1, ..., E_n) \) respectively are the Gateaux derivatives of \( X \) and \( J \) with respect to \( E_i \) in the directions \( e_i \). Here \( <,> \) is the dot product associated with the norm operator \( \| \| \).

The optimal solution of problem is characterized by \( \dot{J}(E_1, ..., E_n) = \nabla J(e_1, ..., e_n)^T = 0 \) where \( \nabla J = \left( \dot{J}_{E_1}, ..., \dot{J}_{E_n} \right) \) is the gradient of \( J \) with respect to \( E_1, ..., E_n \); The superscript \( T \) indicates the transpose of the vector.

The Gateaux derivative equations of (2.1) by \( E_i \) in the directions of \( e_i \) (\( i = 1, 2, ..., n \)) are:

\[
\frac{d\dot{X}}{dt} = \frac{\partial F(X, E_1, ..., E_n)}{\partial X} \cdot \dot{X}^{(i)} + \frac{\partial F}{\partial E_i} \cdot e_i \\
\dot{X}^{(i)}(0) = 0 
\]

Let us introduce \( P^{(i)} \), the adjoin variable in the same space as \( X \). Multiplying equation (2.4) by \( P^{(i)} \) in space \( \mathcal{I} \) we integrate by time between 0 and \( T \). It comes:

\[
\int_{0}^{T} \left( \frac{d\dot{X}^{(i)}}{dt}, P^{(i)} \right)_3 \, dt = \int_{0}^{T} \left( \frac{dF}{dX} \cdot \dot{X}^{(i)}, P^{(i)} \right)_3 \, dt + \int_{0}^{T} \left( \frac{dF}{dE_i} - e_i, P^{(i)} \right)_3 \, dt \\
\text{or} \left( \dot{X}^{(i)}(T), P^{(i)}(T) \right)_3 - \left( \dot{X}^{(i)}(0), P^{(i)}(0) \right)_3 = \int_{0}^{T} \dot{X}^{(i)}(t) \cdot \frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \cdot P^{(i)} \right] \, dt + e_i \left[ \frac{dF}{dE_i} \right] \cdot P^{(i)}(t) \, dt 
\]

\( i = 1, 2, ..., n \)

The superscript \( ^t \) indicates the transpose of the matrix.

Summing \( n \) equations of (2.6) we have

\[
\sum_{i=1}^{n} \left[ \left( \dot{X}^{(i)}(T), P^{(i)}(T) \right)_3 - \left( \dot{X}^{(i)}(0), P^{(i)}(0) \right)_3 \right] \\
= \sum_{i=1}^{n} \int_{0}^{T} \dot{X}^{(i)} \cdot \frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \right] \cdot P^{(i)} \, dt + e_i \left[ \frac{dF}{dE_i} \right] \cdot P^{(i)}(t) \, dt 
\]

If \( P^{(i)} \) is the solution of:

\[
\frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \right]^T \cdot P^{(i)} = C^T H \left( CX - X_{\text{obs}} \right) \\
P^{(i)}(T) = 0 
\]

then (2.7) becomes:

\[
\sum_{i=1}^{n} \left[ \left( \dot{X}^{(i)}(T), P^{(i)}(T) \right)_3 - \left( \dot{X}^{(i)}(0), P^{(i)}(0) \right)_3 \right] \\
= \sum_{i=1}^{n} \int_{0}^{T} \dot{X}^{(i)} \cdot \frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \right] \cdot P^{(i)} \, dt + e_i \left[ \frac{dF}{dE_i} \right] \cdot P^{(i)}(t) \, dt 
\]

\[
\frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \right]^T \cdot P^{(i)} = C^T H \left( CX - X_{\text{obs}} \right) \\
P^{(i)}(T) = 0 
\]
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\[
\sum_{i=1}^{n} \int_{0}^{T} \left( \dot{X}^{(i)} \cdot \frac{dP^{(i)}}{dt} + \left[ \frac{dF}{dX} \right] \cdot P^{(i)} \right) dt = \sum_{i=1}^{n} \int_{0}^{T} \left( \dot{X}^{(i)} \cdot C^T H \left( CX - X_{obs} \right) \right) dt
\]

\[
= -\sum_{i=1}^{n} e_i \int_{0}^{T} \left[ \frac{dF}{dE_i} \right] \gamma^T P^{(i)} dt
\]

Therefore, from (2.3), (2.9), we have

\[
\dot{J} \left( E_1, \ldots, E_n \right) = \sum_{i=1}^{n} \left( -\int_{0}^{T} \left[ \frac{dF}{dE_i} \right] \cdot P^{(i)} dt + E_i - E_{i,0} \right) e_i
\]

\[
= \nabla J \left( e_1, \ldots, e_n \right)^T
\]

(2.10)

with

\[
\nabla J = \left( J_{E_1} \left( E_1, \ldots, E_n \right), \ldots, J_{E_n} \left( E_1, \ldots, E_n \right) \right)
\]

(2.11)

where: \( J_{E_i} \left( E_1, \ldots, E_n \right) = -\int_{0}^{T} \frac{dF}{dE_i} \left[ \gamma^T P^{(i)} dt + E_i - E_{i,0} \right] \).

Equations 2.1 - 2.9 and the condition for the gradient (2.11) to be null are the Optimality System (O.S). The adjoin model will be run back word to get the gradient which are used to carry out an algorithm of optimization [14 - 21].

3. MATHEMATICAL MODEL FOR THE BODY MOTION

To describe the motion of body, a body fixed coordinate system as shown in Figure 2 is chosen. \((X_0,Y_0,Z_0)\) is the inertial reference frame with origin at O and \((X_1,Y_1,Z_1)\) is the non-inertial reference frame with origin at A, the tip of the slender body. The \(X_1\)-axis coincides with the longitudinal axis of the slender body. The components of velocity of point A along \(X_1\) and \(Z_1\) direction are \(U\) and \(W\) respectively. The components of velocity of point A along \(X_0\) and \(Z_0\) direction are \(U_F\) and \(W_F\) respectively. The angular velocity and rotating angular about \(Y_0\) axis are \(Q\) and \(\Phi\) respectively.

![Figure 2. Axes of body and inertial frames.](image)

The relationships between body and inertial fixed velocities are described by the following formulas:

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The mathematic cavity model [1] is used to describe the motion of slender body under water in cavity. The motion of slender body in both phases is written by the following equations:

Phase 1: For \( U^2 \gg W^2 \) and \( \rho A_k (U, W, h) U^2 \gg 2mQ^2 \) the equation can be written as:

\[
\frac{\partial U}{\partial t} = -\frac{1}{2m} \rho k (U, W, h) A_k U^2
\]

\[
\frac{\partial W}{\partial t} = QU
\]

\[
\frac{\partial Q}{\partial t} = 0
\]

\[
\frac{\partial h}{\partial t} = -U \sin \vartheta + W \cos \vartheta
\]

\[
\frac{\partial \vartheta}{\partial t} = Q
\]

\[
U(0) = U_0; W(0) = W_0; Q(0) = Q_0; Q(0) = Q_0; h(0) = h_0; \Box(0) = \Box_0
\]

Phase 2: For \( U^2 \gg W^2 \) and \( \rho A_k (U, W, h) U^2 \gg 2mQ^2 \) the equation can be written as:

\[
\frac{\partial U}{\partial t} = -\frac{1}{2m} \rho k (U, W, h) F(A_k, r, l, \theta) U^2
\]

\[
\frac{\partial W}{\partial t} = KW^2 \left[ M_1 l_k + M_2 l_k x_m (L - x_m) \right] + 2KW \left[ QM_1 L_x l_k (L - x_m) \right] + QU
\]

\[
\frac{\partial Q}{\partial t} = -KM_2 \left[ W^2 l_x x_m + 2WQU l_x x_m \right],
\]

\[
\frac{\partial h}{\partial t} = -U \sin \vartheta + W \cos \vartheta
\]

\[
\frac{\partial \vartheta}{\partial t} = Q
\]

where:

- \( \vartheta \) is the angle of slender body during impact with the cavity boundary,
  \[ \tan \vartheta = \frac{W}{U} \text{ or } \vartheta = \arctan \frac{W}{U} \]

- \( M_1 = -\frac{\rho d}{m}, M_2 = \frac{\rho d}{I} \)

- \( F \left( A_k, r, l, \theta \right) = A_k + r^2 \cos^{-1} \left( \frac{r - l_k \tan \theta}{r} \right) - \left( r - l_k \tan \theta \right) \sqrt{dl_k \tan \theta} \)

- \( k \left( U, W, h \right) = k_i C_{d_0} \left( 1 + \sigma \right) \cos^2 \alpha \)

- \( C_{d_0} = 0.82 \)
- $\alpha$ is the angle between flow direction and body's direction in moving

$$ \cos \alpha = \frac{U}{\sqrt{U^2 + W^2}} $$

- $p_\infty = \rho gh + P_{\text{atm}}$ - Ambient pressure
- $l_k$ is the wetted length of the body
- $k$, $K$ are parameters; For the circular section $K = 2\pi$ ([1])
- $h$ is the water depth between the body's position and water free surface
- $\rho$ is the mass density of water
- $x_{vm}$ is the distance between body's tail and its centre of mass;
- $m$ is the mass of the slender body
- $\sigma$ is the cavitation number

$$ \sigma = \frac{p_\infty - p_c}{0.5(U^2 + W^2)} $$

- $l$ is the moment of inertia of the body about an axis parallel to the $Y_1$ axis and passing through its centre of mass
- $r = d/2$ is the radius of slender body
- $A_c = \pi d_c^2 / 4$ is the area of the cavitator
- $r_c = d_c / 2$ is the cavitator radius
- $g = 9.81$ m/s is the gravity acceleration
- $p_c$ is the vapour pressure of water

To get the above equations the following condition is needed: $\frac{l}{L} << 1$

The geometry of the cavity is given by ([1, 2, 8]):

$$ \frac{(x-l/2)^2}{(l/2)^2} + \frac{y^2}{(D_c/2)^2} = 1 $$

where the maximum diameter $D_c$ and length $l$ of the cavity shape are given by the following formulas:

$$ D_c = d_c \sqrt{k_1 C_{\text{dis}} (1+\sigma) / \sigma}, l = \frac{d_c}{\sigma} \sqrt{\log \frac{1}{\sigma}} $$

The equation (3.1) - (3.2) can be rewritten as follows:

$$ \begin{cases} \frac{\partial X}{\partial t} = A(X) \\ X(0) = X_0 \end{cases} $$ (3.3)
where: \( X = (U, W, Q, h, \vartheta) \)  
\[
\begin{align*}
X &= \left[ A_1(X), A_2(X), A_3(X), -U \sin \vartheta + W \cos \vartheta, Q \right]^T
\end{align*}
\] 

is an unknown state function vector of the equations (3.1)-(3.2) and 
\[
\begin{align*}
X_0 &= \left[ U_0, W_0, Q_0, h_0, \vartheta_0 \right]^T
\end{align*}
\]

\[
A(X) = \left[ A_1(X), A_2(X), A_3(X), -U \sin \vartheta + W \cos \vartheta, Q \right]^T
\]

(3.4)

\[
A_1(X) = \begin{cases} 
-\frac{1}{2 \rho k} \rho k (U, W, h) A_c U^2 \text{ in the first phase} \\
-\frac{1}{2 \rho k} \rho k (U, W, h) F (A_c, r, \vargamma_k, \vartheta) U^2 \text{ in the second phase}
\end{cases}
\]

(3.5)

\[
A_2(X) = \begin{cases} 
QU \text{ in the first phase} \\
KC_1 W^2 + KC_2 W + QU \text{ in the second phase}
\end{cases}
\]

\[
A_3(X) = \begin{cases} 
QU \text{ in the first phase} \\
C_3 W^2 + C_4 WQ \text{ in the second phase}
\end{cases}
\]

\[
C_1 = M_1 L_k + M_2 L_e (L - x_m); C_2 = 2 M_2 L_x L_k (L - x_m); C_3 = -M_2 L_k x_m; C_4 = -M_2 L_k x_m
\]

The equation 3.3 is solved by Runge-Kutta method.

4. CORRECTION OF \( k_1 \) COEFFICIENT

We have priori approximations \( k_{i_0} \) of \( k_i \) and measurement \( X_{obs} = (U_{obs}, W_{obs}, Q_{obs}, h_{obs}, \vartheta_{obs}) \) of the motion velocity of body. Using the cost function (see formula 4.1) the continuous problem is to determine \( k_1^* \) minimizing \( J \): 

\[
J(k_1) = \frac{1}{2} \int_0^T \left( CX - X_{obs} \right) \left( CX - X_{obs} \right) dt + \frac{1}{2} (k_1 - k_{1,0})^2
\]

(4.1)

\( C \) is an operator, that is Dirac’s matrix, from the space of the variable \( X \) to the space of observation with pointwise measurement. Therefore, we have an optimal control problem with respect to the coefficient \( k_1 \). The first step is to exhibit the Euler-Lagrange equation - necessary equation for an optimum in order to exhibit the gradient of \( J \) with respect to \( k_1 \). Then, we will be able to carry out some optimization algorithm.

The data assimilation problem is written in the form:

\[
\left\{ \begin{array}{l}
\frac{\partial X}{\partial t} = A(X) \\
X(0) = X_0 \\
J(k_1^*) = \inf_{k_1} J(k_1)
\end{array} \right.
\]

(4.2)
where $X = (U, W, Q, h, \vartheta)^T$, $A(X)$ is the vector function defined by the formula (3.4)-(3.5), and the cost function $J(k_i)$ is defined by the formula (4.1). To solve the problem (4.2) we will define the formula of function $J'_i(k_i)$ in the next subsection.

### 4.1. Computation of Gateaux derivative for the cost function $J$

Let $k_1$ being a value in the space of the control. Let us introduce the Gateau derivative $\dot{X} = (\dot{U}, \dot{W}, \dot{Q}, \dot{h}, \dot{\vartheta})^T$ of $X = (U, W, Q, h, \vartheta)^T$ by $k_1$ in the directions of $\vec{k}_i$ as follows ([22]):

$$\dot{X} = \lim_{\alpha \to 0} \frac{X(k_1 + \alpha \vec{k}_1) - X(k_1)}{\alpha}$$

Then the Gateaux derivative of the cost function $J$ with respect to $k_1$ in the directions of $\vec{k}_i$ will be:

$$\dot{J}(k_1) = \int_0^T \left( C^T (CX - X_{ab}) \right) \dot{X} dt + \left( k_1 - k_{1,0} \right) \vec{k}_i$$

(4.3)

Firstly, we will compute Gateaux derivatives $\dot{J}_i(k_i)$ of the cost function $J$ with respect to $k_i$ in the directions of $\vec{k}_i$.

The Gateau derivative equations of (3.3) with respect to $k_i$ in the direction of $\vec{k}_i$ are written as follows:

$$\frac{\partial \dot{X}}{\partial t} = N(X) \dot{X} + B(X) \vec{k}_i$$

$$\dot{X}(0) = 0$$

(4.4)

where:

$$N(X) = \begin{bmatrix}
N_{11}(X) & N_{12}(X) & 0 & N_{14}(X) & 0 \\
N_{21}(X) & N_{22}(X) & N_{23}(X) & 0 & 0 \\
0 & N_{32}(X) & N_{33}(X) & 0 & 0 \\
-\sin \vartheta & \cos \vartheta & 0 & 0 & -U \cos \vartheta - W \sin \vartheta \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

$$N_{ij}^{(1)} = \begin{cases}
N_{ij}^{(1)} & \text{in the first phase} \\
N_{ij}^{(2)} & \text{in the second phase}
\end{cases} (i = 1..3; j = 1..4)$$

$$N_{11}^{(1)} = \frac{1}{2m} \rho k C_{J} \left( 1 + \frac{p_m - p_i}{0.5 \rho (U^2 + W^2)} \right) \frac{U^2 + 3U^2 W^2}{(U^2 + W^2)^{3/2}} A + \frac{1}{m} k \rho C_{J} \frac{p_m - p_i}{0.5 \rho (U^2 + W^2)^{3/2}} U A$$

$$N_{11}^{(2)} = \frac{1}{2m} \rho k C_{J} \left( 1 + \frac{p_m - p_i}{0.5 \rho (U^2 + W^2)} \right) \frac{U W (U^2 + W^2)^{1/2}}{(U^2 + W^2)^{3/2}} A + \frac{1}{m} k \rho C_{J} \frac{p_m - p_i}{0.5 \rho (U^2 + W^2)^{3/2}} W U A$$
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\[ N_{14}^{(1)} = \frac{\rho}{2m} k_1 C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( \frac{g}{U^2 + W^2} \right) U^3 A \]

\[ N_{21}^{(1)} = Q \; ; \; N_{22}^{(1)} = 0 \; ; \; N_{31}^{(1)} = U \; ; \; N_{32}^{(1)} = 0 \; ; \; N_{33}^{(1)} = 0 \]

\[ N_{11}^{(2)} = -\frac{1}{2m} k_1 \rho C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( \frac{2(U^4 + 3U^2 W^2)}{(U^2 + W^2)^{3/2}} \right) U F + \frac{1}{m} k_1 \rho C_{d_0} \left( \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) U^4 F_c \]

\[ + \frac{\rho}{2m} k_1 C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( \frac{r \left( r - l_k \tan \theta \right)}{r} \right) \left( \frac{l_k}{r} \right) \left( \frac{\sqrt{l_k \tan \theta}}{2} \right) \frac{U_W}{\sqrt{U^2 + W^2}} \]

\[ N_{12}^{(2)} = \frac{1}{2m} k_1 \rho C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( U^2 W F + \frac{1}{m} k_1 \rho C_{d_0} \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} W U^3 F_c \right) \]

\[ - \frac{\rho}{2m} k_1 C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( r^2 \cos^2 \left( \frac{r - l_k \tan \theta}{r} \right) \left( \frac{l_k}{r} \right) \left( \frac{\sqrt{l_k \tan \theta}}{2} \right) \frac{U^2}{\sqrt{U^2 + W^2}} \right) \]

\[ N_{14}^{(2)} = \frac{\rho}{2m} k_1 C_{d_0} \left( 1 + \frac{p_e - p_f}{0.5 \rho (U^2 + W^2)} \right) \left( \frac{g}{U^2 + W^2} \right)^{3/2} U^3 F_c \]

\[ N_{21}^{(2)} = Q \; ; \; N_{22}^{(2)} = 2KC_W + KC_2 Q \; ; \; N_{31}^{(2)} = KC_2 W + U \; ; \; N_{32}^{(2)} = 2KC_W + KC_2 Q \]

\[ N_{33}^{(2)} = KC_2 W \]

\[ B = (B_1, B_2, B_3, 0, 0) \]

\[ B_1 = \begin{cases} 
-\frac{1}{2m} \rho C_{d_0} (1 + \sigma) \frac{U^4}{U^2 + W^2} A \; \text{for the first phase} \\
-\frac{1}{2m} \rho C_{d_0} (1 + \sigma) \frac{U^4}{U^2 + W^2} F_e - \frac{1}{2m} k_1 (U, W, h) U^2 F_{c_1} l_1' \; \text{for the second phase} 
\end{cases} \]

\[ F_{c_{1,2}} = \begin{cases} \left( \frac{r - l_k \tan \theta}{r} \right) \left( \frac{l_k}{r} \right) \left( \frac{\sqrt{l_k \tan \theta}}{2} \right) \frac{U^2}{\sqrt{U^2 + W^2}} 
\end{cases} \]

\[ B_2 = \begin{cases} 
0 \; \text{for the first phase} \\
C_{1,2} W^2 + C_{1,2}' W Q \; \text{for the second phase} 
\end{cases} \]

\[ B_3 = \begin{cases} 
0 \; \text{for the first phase} \\
C_{3,4} W^2 + C_{3,4}' W Q \; \text{for the second phase} 
\end{cases} \]

\[ C_{1,2,3,4}' \] are the derivatives of those functions with respect to parameter \( k_1 \).

Multiplying the equation (4.4) by adjoin variable \( P = (P_1, P_2, P_3, P_4, P_5)^T \) in the same space as \( X \) and then integrating by \( t \) between 0 and \( T \) we have:

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\[
\left( \hat{X}(T), P(T) \right) - \left( \hat{X}(0), P(0) \right) = \int_0^T \left( \hat{X}, \frac{dP}{dt} + F(X, P) \right) dt + \sum_{i=1}^n \int_0^T B_i P^i dt \quad (4.6)
\]

where: \( F(X, P) = N^T P \) with \( N(X) \) is defined by the formula (4.5).

If \( P \) is satisfying the following equation:
\[
\begin{cases}
\frac{dP}{dt} + F(X, P) = -C^T H(CX - X_{obs}) \\
P(T) = 0
\end{cases}
\]

Then the Gateau derivative \( \hat{J}_{k_i}(k_i) \) of the cost function \( J \) with respect to \( k_1 \) in the directions of \( k_i \) is: (see formula 4.3):
\[
\hat{J}_{k_i}(k_i) = \int_0^T \left( \hat{X}, \frac{dP}{dt} + F(X, P) \right) dt + (k_i - k_{i,0}) \Xi_i - \Xi_i \left( -\int_0^T B \cdot P^i dt + (k_i - k_{i,0}) \right) = \Xi_i J'_{k_i}
\]

Therefore, the function \( J'_{k_i}(k_i) \) is calculated by the following formula:
\[
J'_{k_i} = \int_0^T (B_1 P_1 + B_2 P_2 + B_3 P_3) dt + (k_i - k_{i,0}) \quad (4.8)
\]

### 4.2. Algorithm to solve the optimal control problem

The optimal method is based on inverse BFGS update [23 - 26]. The algorithm schema is written as follows:

a. Let \( I = 0 \): Get the initial value \( k_{i,j} = k_{i,0} \); \( H_i = 1 \); Solve equations 3.3 with the parameter \( k_{i,j} \); and the adjoin equations 4.7; Get the function \( J_{k_i}(k_{i,j}) \) by the formula 4.8

b. Calculate
\[
d_i = -H_i J_{k_i} \quad (k_{1,i})
\]

c. Calculate \( \alpha_i \) so that is satisfied the Armijo-Wolfe conditions ([25, 26]):
\[
J(k_{1,i} + \alpha_i d_i) \leq J(k_{1,i}) + \alpha_i \beta J'_{k_i}(k_{1,i})d_i
\]

where \( \beta \in (0, 1) \). Typically \( \beta \) ranges from \( 10^{-4} \) to 0.1

This \( \alpha_i \) can be found by the following schema steps ([27]):

c.1 \( \alpha_{initial} = 1 \);

c.2 Given \( \tau \in (0, 1) \). Typically \( \tau = 0.5 \);

c.3 Let \( l=0 \) then \( \alpha^l = \alpha_{initial} \);

c.4 Check:
While not \( J(k_{1,i} + \alpha^l d_i) \leq J(k_{1,i}) + \alpha^l \beta J_k^i (k_{1,i}) d_i \)

Set \( \alpha^{n+1} = \tau \alpha^l \)

Increase \( l \) by 1

End while

c.5 Set \( \alpha_i = \alpha^{(l)} \);

d. Calculate: \( \Delta k_{1,i} = s_i = -\alpha_i H_i J^i (k_{1,i}) \);

e. Calculate: \( k_{i+1} = k_i + \Delta k_{1,i} \);

f. Solve equations 3.3 with the parameter \( k_{i+1} \) and the adjoin equations 4.7.

g. Get the function \( J_{k_i} (k_{i+1}) \) by the formula 4.8.

h. Calculate \( y_i = J^i_{k_i} (k_{i+1}) - J^i_{k_i} (k_{1,i}) \);

i. Calculate \( H_{i+1} = \left( 1 - \frac{s_i y_i}{y_i s_i} \right) + \frac{s_i s_i}{y_i s_i} \);

j. Let \( i = i + 1 \)

k. Go to step b if \( J_{k_i} (k_{i+1}) \geq \epsilon \) ( \( \epsilon > 0 \) is given).

If \( J_{k_i} (k_{i+1}) = 0 \) the optimal process is stopped. Then, we have \( k_1 = k_1^* \).

4.3. Simulation experiment on correcting parameter \( k_1 \) so that \( U \) is closed to measurement

Let the body with \( m = 0.025091315 \) kg, \( L_1 = 2.5 \) cm, \( L_2 = 11.5 \) cm, \( d = 0.57 \) cm, \( d_c = 0.12 \) cm, \( U_0 = 240 \) m/s, \( W_0 = 0 \), \( Q = 1 \) rad./s, \( h_0 = 7 \) m, \( \square = 0 \), \( I_x = 1.81.10^-4 \) kgm\(^2\), \( x_m = 10.01 \) cm.

We will test the problem by considering the following experiments:

- By the same way as [16, 28] we can have the observation data \( X_{obs} = \{ U_{obs}, W_{obs}, Q_{obs}, h_{obs}, \vartheta_{obs} \} \) as follows:

  Let model run in 0.5s with values \( k_1 = 1 \) simulating the true velocity \( X = (U, W, Q, h, \vartheta) \) by solving the equations (3.1)-(3.2).

  This velocity \( X \) is used as a reference \( X_{obs} \).

  The measurement \( X_{obs} \) is obtained by the values of \( X \) in all the time period.

  Then we have \( X_{obs} \) in every time step.

- In the testing the model is running in the time period 0.5s with values \( k_i = 2, k_1 \). Then, the vector function \( X = (U, W, Q, h, \vartheta) \) is obtained by solving equations (3.1)-(3.2).
The equations (3.1)-(3.2) are solved by Runge Kutta method.

- Using the formula of function $J_k$ (4.8) the optimal control problem (4.2) is solved by the algorithm schema in subsection 4.2. Then the minimum of $J(k)$ is found by the formula (4.1) with $k_1^*$ value.

- The process finding the coefficient is shown in Figure 3. By this process the error of obtain coefficient in the end optimal process is less than 0.00001 percentage. In the Figure 4 the obtain cost function $J$ in the end of optimal process is nearly zero (less than 0.00001). The error percentages of velocities $U$ by $X_1$ direction with reference $U_{obs}$ with and without correction coefficient $k_1$ are shown in Figure 5. With the correction coefficient the percentage errors of velocities are less than 0.00016 %.

- We have done real experimental of projectile running underwater. The cavity is presented in the Picture 1. In the real measurement we have 96 measured points of velocities $U$ by $X_1$ direction with the initial velocity $U_0 = 271.2$ m/s. The other initial conditions are chosen approximately $W_0 = 0$, $Q_0 = 1$ rad./s, $h_0 = 1$ m, $\theta_0 = 0$.

- Let the model run with the beginning coefficient $k_1 = 2.5$ then the optimal coefficient $k_1^* = 0.909999046325684$ is found by the optimal program.

- The comparison between velocity measurement and the other ones of calculation with $k_1 = 2.5$ or optimal coefficient $k_1^* = 0.909999046325684$ is presented in the figure 6.

- By this figure it is easy to see that with optimal coefficient $k_1^* = 0.909999046325684$ the model is closer to measurement than the other one without correction.

![Figure 3. Correcting coefficient $k_1$ in optimal process (Left); Coefficient error percent in optimal process correcting $k_1$ (Right).](image-url)
Figure 4. Cost function $J$ in optimal process correcting $k_1$.

Figure 5. Percent error of velocity $U(t)$ with optimal correction of coefficient $k_1 = k_1^\ast$ (left); Percent error of velocity $U(t)$ with coefficient $k_1 = 2$ (Right).
4. CONCLUSIONS

In the model of slender body running very fast under water the coefficient $k_1$ strongly effects to the simulation results (the right of Figure 5). By the results presented in Figures 3,4 it is easy to see that by the data assimilation method the corrected coefficient $k_1^*$ can be nearly
equal to the reference coefficient $k$. It follows that the velocity $U(t)$ is closed to the one in the reference model (the left of the Figure 5 or Figure 6). Then the data assimilation method can be used as the good tool to correct coefficient in the model of body running fast under water.

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TÓM TÁT

ỨNG DỤNG PHƯƠNG PHÁP ĐỘNG HÓA SÓ LIỆU ĐỂ HIỆU CHỈNH THAM SÓ TRONG MÔ HÌNH SIÊU XÂM THỤC

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Trong môi trường nước, khi một vật thể có hình dạng mạnh di chuyển với vận tốc nhanh hướng về phía trước sẽ tự quay trong một khe rộng (còn gọi là khoang hơi hay túi hơi xâm thực).

Trong mô hình khe rộng hệ số cán của vật thể đóng vai trò rất quan trọng trong quá trình di chuyển. Theo Salis, Garabedian, Kiceniuk hệ số cán này được chọn bởi các giá trị thích hợp trong khoảng từ 0,8 đến 1. Theo Rand, Kirschner thì hệ số cán này được viết bởi công thức \( k = C_{p0}(1+\sigma)\cos^2\alpha \) với \( \sigma \) là số cavitation (số xâm thực), \( \alpha \) là góc giữa trực của vật thể mạnh và hướng của di chuyển. \( C_{p0} \) là tham số thường được chọn trong khoảng từ 0,6 đến 0,85. Trong bài báo này hệ số cần được viết dưới dạng \( k = k_{p}C_{p0}(1+\sigma)\cos^2\alpha \), trong tính toán hệ số \( C_{p0} \) được lấy bằng 0,82 và bằng phương pháp toàn học hệ số chửa biết \( k_{1} \) sẽ được hiệu chỉnh sao cho các vận tốc di chuyển trong mô hình gần với các số liệu quan sát được. Phương pháp toàn học được áp dụng để tìm hệ số chửa biết \( k_{1} \) là phương pháp đối hóa số liệu. Trong phương pháp này các số liệu quan sát được sử dụng trong hàm mục tiêu. Dây chính là một trong những phương pháp hữu hiệu để giải các bài toán tối ướt bằng cách giải bài toán liên hợp rơi tính gradient của hàm mục tiêu.

**Từ khóa:** đồng hóa số liệu, tối ưu, phương pháp Runge-Kutta.