MODELS FOR CLASSIFYING SPACES FOR $\mathbb{Z} \rtimes \mathbb{Z}$

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Abstract. We construct two models for the classifying space for the family of infinite cyclic subgroups of the fundamental group of the Klein bottle. These examples do not fit in general constructions previously done, for example, for hyperbolic groups.

1. Introduction

Let $G$ be a discrete group. A family, $\mathcal{F}$, of subgroups of $G$ is a nonempty set of subgroups of $G$ which is closed under conjugation and taking subgroups. A model, $E_\mathcal{F}(G)$, for the classifying space of the family $\mathcal{F}$ is a $G$-CW-complex $X$, such that all of its isotropy groups belong to $\mathcal{F}$ and if $Y$ is a $G$-CW-complex with isotropy groups belonging to $\mathcal{F}$, there is precisely one $G$-map $Y \to X$ up to $G$-homotopy. Classifying spaces for families appear frequently in mathematics, notably, in various Assembly Isomorphism conjectures such as the Baum–Connes and the Farrell–Jones Conjectures, see [4].

A model for $E_{\text{VCY}}(\mathbb{Z} \rtimes \mathbb{Z})$, the classifying space for the family of virtually cyclic subgroups of $\mathbb{Z} \rtimes \mathbb{Z}$, may be constructed as countable join $\ast_{i \in \mathbb{Z}} \mathbb{R}_i$ with each $\mathbb{R}_i \simeq \mathbb{R}$ see [3], with a suitable action of $\mathbb{Z} \rtimes \mathbb{Z}$. This space is built by using the fact that each nontrivial virtually cyclic subgroup $H$ of $\mathbb{Z} \times \mathbb{Z}$ is normal in $\mathbb{Z} \times \mathbb{Z}$. We cannot build a model for $E_{\text{VCY}}(\mathbb{Z} \rtimes \mathbb{Z})$ in the same way because this is not the case in $\mathbb{Z} \times \mathbb{Z}$. In this note, we present two models for $E_{\text{VCY}}(\mathbb{Z} \times \mathbb{Z})$ (which are $\mathbb{Z} \rtimes \mathbb{Z}$-homotopy equivalent). Our principal results are Theorems 3.6 and 4.1, and Proposition 5.1.

The constructions follow the work of W. Lück and M. Weiermann in [5], and of D. Farley in [1]. The latter uses the fact that $\mathbb{Z} \times \mathbb{Z}$ is a $\text{CAT}(0)$ group as it acts by isometries on the plane, and the former follows a general construction.

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2. Classifying Spaces for Families

Let $G$ be a discrete group. A family $\mathcal{F}$ of subgroups of $G$ is a nonempty set of subgroups of $G$ which is closed under conjugation and taking subgroups. Some examples are: $\{1\}$, the family consisting of the trivial subgroup in $G$, $\mathcal{FIN}$, the family of finite subgroups of $G$, $\mathcal{VCY}$ the family of virtually cyclic subgroups of $G$ and $\mathcal{ALL}$, the family of all subgroups of $G$.

Let $H$ be a subgroup of $G$, and $\mathcal{F}$ be a family of subgroups of $G$, then $\mathcal{F}$ defines a family of $H$ as follows

$$\mathcal{F}(H) = \{K \subseteq H | K \in \mathcal{F}\}.$$ 

**Definition 2.1.** Let $\mathcal{F}$ be a family of subgroups of $G$. A model for the classifying space $E_{\mathcal{F}}(G)$ is a $G$-CW-complex $X$, such that all of whose isotropy groups belong to $\mathcal{F}$. If $Y$ is a $G$-CW-complex with isotropy groups belonging to $\mathcal{F}$, there is precisely one $G$-map $Y \to X$ up to $G$-homotopy. We denote by $B_{\mathcal{F}}(G)$ be the quotient of $X$ by the action of $G$.

In other words, $X$ is a terminal object in the category of $G$-CW complexes with isotropy groups belonging to $\mathcal{F}$. In particular, two models for $E_{\mathcal{F}}(G)$ are $G$-homotopy equivalent, and then we denote $X$ by $E_{\mathcal{F}}(G)$.

**Remark 2.2.** Given two families $\mathcal{F}_1 \subseteq \mathcal{F}_2$ of subgroups of $G$, since $E_{\mathcal{F}_2}(G)$ is a terminal object in the category of $G$-CW complexes with isotropy groups belonging to $\mathcal{F}_2$, there exists precisely one $G$-map up to $G$-homotopy

$$E_{\mathcal{F}_1}(G) \to E_{\mathcal{F}_2}(G).$$

**Definition 2.3.** Let $G$ be a group, $H \subseteq G$ and $X$ a $G$-set. The fixed point set $X^H$ is defined as

$$X^H = \{x \in X | \forall h \in H, hx = x\}.$$ 

**Theorem 2.4.** [4, Thm. 1.9] A $G$-CW-complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if the $H$-fixed point set $X^H$ is contractible for $H \in \mathcal{F}$ and is empty for $H \notin \mathcal{F}$.

A model for $E_{\mathcal{ALL}}(G)$ is $G/G$, a model for $E_{\{1\}}(G)$ is the same as a model for $EG$, the total space of the universal $G$-principal bundle $EG \to BG$, [6].

We write $\underline{E}G$ for $E_{\mathcal{FIN}}(G)$, this is known as the universal $G$-CW-complex for proper $G$-actions, and we write $\underline{E}G$ for $E_{\mathcal{VCY}}(G)$.
2.1. Constructing models from models for smaller families. Let $\mathcal{F}$ and $\mathcal{G}$ be two families of a group $G$, with $\mathcal{F} \subseteq \mathcal{G}$, such that we know a model for $E_{\mathcal{F}}(G)$. In [5] W. Lück and M. Weiermann build a model $E_{\mathcal{G}}(G)$ from $E_{\mathcal{F}}(G)$, and in [1] Farley builds another model for groups acting on CAT(0)-spaces. In this Section we present these models.

Let $\mathcal{F} = \mathcal{F}_N$ and $\mathcal{G} = \mathcal{VCY}$. Following [5], define an equivalence relation $\sim$ on $\mathcal{F} = \mathcal{VCY} - \mathcal{F}_N$ as

$$V \sim W \iff |V \cap W| = \infty,$$

for $V$ and $W$ in $\mathcal{F}$, where $| \star |$ denotes the cardinality of the set $\star$. Let $[\mathcal{F}]$ denote the set of equivalence classes under the above relation and let $[H] \in [\mathcal{F}]$ be the equivalence class of $H \in \mathcal{F}$.

For $H \in \mathcal{F}$

$$N_G[H] := \{g \in G \mid [g^{-1} H g] = [H]\}$$

$$= \{g \in G \mid |g^{-1} H g \cap H| = \infty\}.$$

This is the isotropy group of $[H]$ under the $G$-action on $[\mathcal{F}]$ induced by conjugation. Note that $N_G[H]$ is the commensurator of $H$ in $G$. Define a family of subgroups $\mathcal{G}[H]$ of $N_G[H]$ by

$$\mathcal{G}[H] := \{K \subseteq N_G[H] \mid K \in [\mathcal{F}], |K \cap H| = \infty\} \cup \{\mathcal{F}_N \cap N_G[H]\}.$$

The method to build a model of $E_{\mathcal{G}}(G)$ from one of $E_{\mathcal{F}}(G)$ is with the following theorem.

**Theorem 2.5.** [5, Thm. 2.3] Let $\mathcal{F} \subseteq \mathcal{G}$ and $\sim$ as above. Let $I$ be a complete system of representatives $[H]$ of the $G$-orbits in $[\mathcal{G} - \mathcal{F}]$ under the $G$-action coming from conjugation. Choose arbitrary $N_G[H]$-CW-models for $E_{\mathcal{F} \cap N_G[H]}(N_G[H])$, $E_{\mathcal{G}[H]}(N_G[H])$ and an arbitrary $G$-CW-model for $E_{\mathcal{F}}(G)$. Define $X$ a $G$-CW-complex by the cellular $G$-pushout

$$\coprod_{[H] \in I} G \times_{N_G[H]} E_{\mathcal{F} \cap N_G[H]}(N_G[H]) \xrightarrow{i} E_{\mathcal{F}}(G)$$

$$\coprod_{[H] \in I} G \times_{N_G[H]} E_{\mathcal{G}[H]}(N_G[H]) \xrightarrow{f_H} X$$

such that $f_H$ is a cellular $N_G[H]$-map for every $[H] \in I$ and $i$ is an inclusion of $G$-CW-complexes, or such that every map $f_H$ is an inclusion of $N_G[H]$-CW-complexes for every $[H] \in I$ and $i$ is a cellular $G$-map. Then $X$ is a model for $E_{\mathcal{G}}(G)$.

The maps in Theorem 2.5 are given by the universal property of classifying spaces for families and inclusions of families of subgroups (see...
Remark (2.2)).

The following is Definition 2.2, in [1].

**Definition 2.6.** Let \( F \subseteq G \) be families of subgroups of a group \( G \). We say that a \( G \)-CW complex \( X \) is an \( I_{G-F}G \)-complex if

(i) whenever \( H \in G - F \), \( X^H \) is contractible;

(ii) whenever \( H \not\in G \), \( X^H = \emptyset \).

Observe that if all isotropy groups are in \( G \) then (ii) holds. Since the trivial subgroup is not in \( G - F \), \( X \) is not necessarily contractible.

**Theorem 2.7.** [1, Prop. 2.4] If \( G \) is a group and \( F \subseteq G \) are families of subgroups of \( G \), then the join

\[ (E_F) \ast (I_{G-F}G) \]

is a model for the classifying space \( E_GG \).

3. **First model for \( E(Z \times Z) \)**

Let \( K := (Z \times Z) \setminus \mathbb{R}^2 \) be the Klein bottle. Following [1] we construct a model for \( E(Z \times Z) \) using the fact that \( Z \times Z \) acts by isometries on \( \mathbb{R}^2 \). This action is given by deck transformations of the universal covering \( p: \mathbb{R}^2 \to K \) of the Klein bottle as \( Z \times Z \cong \pi_1(K) \).

3.1. **Virtually cyclic subgroups of \( Z \times Z \).** Let \( Z \times Z \) be the *Klein bottle group* with multiplication

\[(n_1, m_1)(n_2, m_2) = (n_1 + (-1)^{m_1}n_2, m_1 + m_2),\]

inverse element

\[(n, m)^{-1} = ((-1)^{1-m}n, -m)\]

and the neutral element is \((0, 0)\).

**Remark 3.1.** For \((t_1, t_2), (n, m) \in Z \times Z\) we have that

\[(t_1, t_2)(n, m)(t_1, t_2)^{-1} = (t_1, t_2)(n, m)((-1)^{1-t_2}t_1, -t_2) = (t_1, t_2)(n + (-1)^m(-1)^{1-t_2}t_1, m - t_2) = (t_1 + (-1)^{t_2}(n + (-1)^{m+1-t_2}t_1), m) = ((-1)^{t_2}n + t_1 + (-1)^{m+1}t_1, m) \]

(5)

In \( Z \times Z \) the families \( \mathcal{FIN} \) and \( \mathcal{VCY} \) are

\[ \mathcal{FIN} = \{1\} \]

\[ \mathcal{VCY} = \{C \subseteq Z \times Z \mid C \text{ is infinite cyclic}\} \cup \{1\}. \]
Remark 3.2. We classify all infinite cyclic subgroups of the family $\mathcal{VCY}$ in $\mathbb{Z} \rtimes \mathbb{Z}$. Let $n, m \in \mathbb{Z}, k \in \mathbb{N}$. Observe that
\[(n, m)^k = \left([1 + (-1)^m + (-1)^{2m} + \cdots + (-1)^{(k-1)m}]n, km\right);
\]
\[(n, m)^{-k} = \left([(-1)^{1-m} + (-1)^{1-2m} + \cdots + (-1)^{1-km}]n, -km\right).
\]
Therefore infinite cyclic subgroups in $\mathcal{VCY}$ are of the following form
\[(n, 2m') = \left([1 + (-1)^{m'} + (-1)^{2m'} + \cdots + (-1)^{1-km}]n, km\right);
\]
\[(n, 2m' + 1)^k = \left([0, k(2m' + 1)], \text{ if } k \text{ is even}
\right.
\)
\[\left.(n, 2m' + 1)^k = \left[0, k(2m' + 1)\right], \text{ if } k \text{ is odd.}
\]

3.2. A model for $E(\mathbb{Z} \rtimes \mathbb{Z})$. The group $\mathbb{Z} \rtimes \mathbb{Z}$ acts on $\mathbb{R}^2$ by deck transformations of the universal covering $p: \mathbb{R}^2 \to \mathcal{K}$ of the Klein bottle. Explicitly, the action is as follows: let $(n, m) \in \mathbb{Z} \rtimes \mathbb{Z}$ and $(t, r) \in \mathbb{R}^2$, then
\[(n, m)(t, r) = (n + (-1)^m t, m + r).
\]

Remark 3.3. A model for $E(\mathbb{Z} \rtimes \mathbb{Z})$ is $\mathbb{R}^2$, because $\mathbb{R}^2$ is contractible and the action (8) is free and properly discontinuous.

Let $\ell(a, b)$ denote the geodesic line in $\mathbb{R}^2$ determined by $a, b \in \mathbb{R}$ as
\[\ell(a, b) := \{(x, ax + b) \mid x \in \mathbb{R}\},
\]
and let
\[\ell(\infty, b) := \{(b, y) \mid y \in \mathbb{R}\},
\]
denote the geodesic line parallel to $y$-axis, determined by $b$. Let $\mathcal{L}$ be the space of lines in $\mathbb{R}^2$:
\[\mathcal{L} := \{\ell(a, b) \mid a \in \mathbb{R} \cup \{\infty\}, \ b \in \mathbb{R}\}.
\]
The space of lines is a metric space with the following distance
\[d(\ell_1, \ell_2) = \begin{cases} \frac{1}{1+k} & \text{if } \ell_1 \text{ and } \ell_2 \text{ bound a flat strip of width } k; \\ 1 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are not parallel.}
\end{cases}
\]
Then we have that $\mathcal{L} = \bigsqcup_{a \in \mathbb{R} \cup \{\infty\}} \mathbb{R}_a$, where $\mathbb{R}_a := \{\ell(a, b) \mid b \in \mathbb{R}\}$, and the metric in each connected component $\mathbb{R}_a$ is given by $d$.

Since the action of $\mathbb{Z} \rtimes \mathbb{Z}$ in $\mathbb{R}^2$ sends geodesic lines to geodesic lines, it induces an action of $\mathbb{Z} \rtimes \mathbb{Z}$ on $\mathcal{L}$. For $(n, m) \in \mathbb{Z} \rtimes \mathbb{Z}$ and $\ell(a, b) \in \mathcal{L}$, this action is as follows,
\[(n, m)\ell(a, b) = \ell((-1)^m a, b + m - (-1)^m an), \text{ if } a \in \mathbb{R};
\]
\[(n, m)\ell(\infty, b) = \ell(\infty, n + (-1)^m b).
\]
Definition 3.4. Let \((n, m) \neq (0, 0)\) in \(\mathbb{Z} \times \mathbb{Z}\). We say that a line \(\ell \subset \mathbb{R}^2\) is an axis for \((n, m)\) if \((n, m)\ell = \ell\) and \((n, m)\) acts by translation on \(\ell\).

The axis space, for elements of \(\mathbb{Z} \times \mathbb{Z}\) in \(\mathbb{R}^2\), is defined as follows

\[ \mathcal{A} := \{ \ell \in \mathcal{L} \mid \ell \text{ is an axis for some } (n, m) \in \mathbb{Z} \times \mathbb{Z} - (0, 0) \}. \]

By (13) all the lines \(\ell(\infty, b)\) are axes. And by (12) we have

\[ (n, m)\ell(a, b) = \ell(a, b) \iff \ell((-1)^m a, b + m - (-1)^m a n) = \ell(a, b); \]

(14) \(m\) even and \(a = \frac{m}{n}\).

Observe that if \((n, m)\) fixes \(\ell(a, b)\), then it acts on \(\ell(a, b)\) by translation. Therefore \(\ell(a, b) \in \mathcal{A}\) if only if \(a \in \mathbb{Q} \cup \infty\), and then

\[ \mathcal{A} = \{ \ell(a, b) \in \mathcal{L} \mid a \in \mathbb{Q} \cup \infty, b \in \mathbb{R} \} \]

(15) \[ = \prod_{a \in \mathbb{Q} \cup \{\infty\}} \mathbb{R}. \]

Proposition 3.5. The axis space \(\mathcal{A}\) of \(\mathbb{R}^2\) is an \(I_{\text{VCY}-\{1\}}(\mathbb{Z} \times \mathbb{Z})\)-complex.

The proof of Proposition 3.5 is given in the next Subsection.

Theorem 3.6. A model for the classifying space for the family of virtually cyclic subgroups of \(\mathbb{Z} \times \mathbb{Z}\) is the join

\[ E(\mathbb{Z} \times \mathbb{Z}) = \mathbb{R}^2 \ast \prod_{a \in \mathbb{Q} \cup \{\infty\}} \mathbb{R}. \]

(16) Therefore, the quotient by the action is

\[ B(\mathbb{Z} \times \mathbb{Z}) = \mathcal{K} \ast \prod_{a \in \mathbb{Q} \cup \{\infty\}} S^1. \]

(17) Proof. By Proposition 3.5, the axis space of \(\mathbb{R}^2\), is an \(I_{\text{VCY}-\{1\}}(\mathbb{Z} \times \mathbb{Z})\)-complex. Thus, by Theorem 2.7 and because \(\mathbb{R}^2\) is a model for \(E(\mathbb{Z} \times \mathbb{Z})\), we have (16). On the other hand, (17) easily follows by looking the action of \(\mathbb{Z} \times \mathbb{Z}\) on \(\mathcal{K}\) and \(\mathcal{A}\), since the action of \(\mathbb{Z} \times \mathbb{Z}\) on \(\mathcal{A}\) is by translation. \(\square\)

3.3. Proof of Proposition 3.5. In this Section we will prove that the axis space \(\mathcal{A}\) is an \(I_{\text{VCY}-\{1\}}(\mathbb{Z} \times \mathbb{Z})\)-complex, which follow from Lemmas 3.7 and 3.8, below.

We denote by \(\text{Iso}(a, b)\) the isotropy subgroup of \(\ell(a, b) \in \mathcal{A}\), where \(a \in \mathbb{Q} \cup \{\infty\}\), and \(b \in \mathbb{R}\), that is,

\[ \text{Iso}(a, b) := \{ (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid (n, m)\ell(a, b) = \ell(a, b) \}. \]
Lemma 3.7. All isotropy subgroups of the action of $\mathbb{Z} \times \mathbb{Z}$ in the axis space $A$ are in the set $\mathcal{VCY} - \{1\}$.

Proof. We compute the isotropy subgroups of the action of $\mathbb{Z} \times \mathbb{Z}$ on $A$. See Remark 3.2 about the family $\mathcal{VCY}$ of $\mathbb{Z} \times \mathbb{Z}$.

(i) If $a \in \mathbb{Q} - \{0\}$ and $b \in \mathbb{R}$, by (14), we have $(n, m) \in \text{Iso}(\ell(a, b))$ iff $m$ is even and $a = \frac{m}{n}$.

Suppose $a = \frac{a_1}{a_2}$ with $\gcd(a_1, a_2) = 1$, then:

1. If $a_1$ is even then $\text{Iso}(a, b) = \langle (a_2, a_1) \rangle \in \mathcal{VCY} - \{1\}$.

2. If $a_1$ is odd then $\text{Iso}(a, b) = \langle (2a_2, 2a_1) \rangle \in \mathcal{VCY} - \{1\}$.

(ii) Now if $a = 0$, by (12) we have $\text{Iso}(0, b) = \langle (1, 0) \rangle \in \mathcal{VCY} - \{1\}$.

(iii) Lastly, if $a = \infty$, by (13) we have

$(n, m) \in \mathbb{Z} \times \mathbb{Z} \in \text{Iso}(\infty, b)$ iff $n = (1 - (-1)^m)b$.

If $m$ is even, then $n = 0$, and if $m$ is odd, $n = 2b$, but $b \in \mathbb{R}$ and $n \in \mathbb{Z}$, then we conclude that

1) if $2b \in \mathbb{Z}$ then by (7)

$\text{Iso}(\infty, b) = \langle (0, 2) \rangle \cup \{(2b, 2m') + 1 \mid m' \in \mathbb{Z}\}$

$= \langle (2b, 1) \rangle \in \mathcal{VCY} - \{1\}$;

2) if $b \in \mathbb{R}$ and $2b \notin \mathbb{Z}$ then by (6)

$\text{Iso}(\infty, b) = \langle (0, 2) \rangle \in \mathcal{VCY} - \{1\}$.

$\square$

Lemma 3.8. If $H \in \mathcal{VCY} - \{1\}$, the fixed point set $A^H$ is contractible.

Proof. Let $H = \langle (n, m) \rangle \in \mathcal{VCY} - \{1\}$, a infinite cyclic subgroup of $\mathbb{Z} \times \mathbb{Z}$ (see (6) and (7)), we will describe

$A^H = \{ \ell \in A \mid H\ell = \ell \}$,

along the following lines.

(i) Suppose $m$ is even, and $n \neq 0$, then by (6),

$H = \{(kn, km) \mid k \in \mathbb{Z}\}$.

Observe by (13) that

$(kn, km)\ell(\infty, b) = \ell(\infty, kn + b)$. 

$\square$
Since $n \neq 0$, then $\ell(\infty, b) \not\in A^H$ for every $b \in \mathbb{R}$.

Now if $a \neq \infty$ and $b \in \mathbb{R}$, by (14), we have

$$\ell(a, b) \in A^H \iff a = \frac{km}{kn} = \frac{m}{n}.$$ 

Therefore

$$A^H = \mathbb{R}_{\frac{m}{n}},$$

which is contractible.

(ii) Let $m \neq 0$ even and $K = \langle (0, m) \rangle = \{(0, km) \mid k \in \mathbb{Z}\}$. By (12)

$$(0, km)\ell(a, b) = \ell(a, b + 2km),$$

we have that $\ell(a, b) \not\in A^K$ whenever $a \in \mathbb{Q}$.

On the other side, by (13),

$$(0, km)\ell(\infty, b) = \ell(\infty, b), \forall k \in \mathbb{Z}, \text{ and } \forall b \in \mathbb{R}.$$ 

Therefore,

$$A^K = \mathbb{R}_{\infty},$$

which is contractible.

(iii) If $m$ is odd and $R = \langle (n, m) \rangle$, recall from (7) that

$$(n, m)^k = \begin{cases} (0, km) & \text{if } k \text{ is even} \\ (n, km) & \text{if } k \text{ is odd}. \end{cases}$$

Let $a \in \mathbb{Q}$ and $b \in \mathbb{R}$. Since $m$ is odd, by (14), no $(n, m) \in R$ fixes $\ell(a, b)$.

Now, if $a = \infty$ and $b \in \mathbb{R}$, by (13) we have

$$\ell(\infty, b) \in A^R \iff b = \frac{n}{2}.$$ 

Therefore

$$A^R = \{\ell(\infty, \frac{n}{2})\},$$

the space with a single point in $\mathbb{R}_{\infty}$.

\[\square\]

4. BUILDING A SECOND MODEL FOR $E(\mathbb{Z} \times \mathbb{Z})$

In this Section we build a model for $E(\mathbb{Z} \times \mathbb{Z})$ following [5, Sec. 2.1]. We recall that subgroups in $\mathcal{VCY}$ are of the form (6) or (7), and $\mathcal{FIN} = \{1\}$. In $\mathcal{Y} = \mathcal{VCY} - \{1\}$, we have the following equivalences:

(i) Define $H := \langle (1, 0) \rangle$. Let $n \in \mathbb{Z} - \{0\}$. By (6),

$$\langle (n, 0) \rangle \subseteq \langle (1, 0) \rangle,$$

hence by (2)

$$\langle (n, 0) \rangle \sim \langle (1, 0) \rangle.$$
In fact these subgroups of \( \mathfrak{F} \) are the unique subgroups that intersect \( H \) in an infinite set. Denote this class by

\[
[H] = \langle (1, 0) \rangle = \langle (n, 0) \rangle, \quad \forall n \in \mathbb{Z} - \{0\}.
\]

(ii) Fix \( n, m \in \mathbb{Z} - \{0\} \) and

\[
R = \langle (n, 2m) \rangle.
\]

Then there is a maximal subgroup in \( \mathfrak{F} \) which contains \( R \).

Let \( s := \gcd(m, n) \), we define a subgroup \( R' \) as follows,

\[
R' = \langle \left( \frac{n}{s}, \frac{2m}{s} \right) \rangle.
\]

By (6), it is easy to see that \( R' \) is the maximal subgroup in \( \mathfrak{F} \) which contains \( R \), then by (2),

\[
R \sim R'.
\]

Observe by (6) that the only subgroups of \( \mathfrak{F} \) related to \( R' \) are the subgroups of \( \mathfrak{F} \) which are contained in \( R' \). Denote by \( [R] \) this class in \( \mathfrak{F} \).

(iii) We have the following inclusions by (6) and (7)

\[
\langle (0, 2k) \rangle \subseteq \langle (0, 2) \rangle, \quad \text{for every } k \in \mathbb{Z} - \{0\};
\]

\[
\langle (0, 2(2s + 1)) \rangle \subseteq \langle (r, 2s + 1) \rangle, \quad \text{for every } r, s \in \mathbb{Z}; \text{ and}
\]

\[
\langle (t, 2u + 1) \rangle \subseteq \langle (t, 1) \rangle \quad \text{for every } t, u \in \mathbb{Z}.
\]

By the previous inclusions and (2) we have the following relations

\[
\langle (0, 2(2m + 1)) \rangle \sim \langle (0, 2) \rangle \sim \langle (r, 1) \rangle, \text{ and}
\]

\[
\langle (n, 2m + 1) \rangle \sim \langle (n, 1) \rangle \sim \langle (0, 2) \rangle \sim \langle (r, 1) \rangle \sim \langle (r, 2s + 1) \rangle
\]

for every \( n, m, r, s \in \mathbb{Z} \). Denote this class by

\[
[K] = \langle (n, 2m + 1) \rangle.
\]

We have in \( \mathfrak{F} \) the classes \([H], [K]\) and infinitely many countable classes of type \([R]\), as many as maximal subgroups in \( \mathfrak{F} \) of the form \( \langle (n, 2m) \rangle \) there are, with \( n, m \neq 0 \) relatively prime.

Also \( \mathbb{Z} \times \mathbb{Z} \) acts on \( \mathfrak{F} \) by conjugation. The classes \([H]\) and \([K]\) are fixed by conjugation and the classes of type \([R]\) are permuted.
4.1. **Explicit models.** We describe models for \(E(\text{Z} \rtimes \text{Z})\), \(E_{\mathcal{G}}[\star]\) and \(E(\text{N}_{\text{Z} \rtimes \text{Z}}[\star])\), with \(\mathcal{G}[\star]\) defined in (3),(4) and [\star] ∈ [\mathfrak{F}]. Let \(g = (t_1, t_2) ∈ \text{Z} \rtimes \text{Z}\).

1. A model for \(E(\text{Z} \rtimes \text{Z})\) is \(\mathbb{R}^2\).
2. A model for \(E(\text{Z} \times \text{Z})\) is \(\mathbb{R}^2\), since \(\text{Z} \times \text{Z}\) acts freely on \(\mathbb{R}^2\) by translation.
3. (a) Let \([H] ∈ [\mathfrak{F}]\), where \(H\) is as in (18), by (5) note: if \((t_1, t_2)(1, 0)(t_1, t_2)^{-1} = ((-1)^{t_2}, 0)\), then \(g^{-1}Hg = H\), and therefore
   \[
   \text{N}_{\text{Z} \rtimes \text{Z}}[H] = \{g ∈ \text{Z} \rtimes \text{Z} \mid |g^{-1}Hg ∩ H| = ∞\}
   \]
   \[\cong \text{Z} \rtimes \text{Z}\]
   So by (4): \(\mathcal{G}[H] = \mathcal{VCY}(H)\).

   (b) We claim that a model for \(E_{\mathcal{VCY}(H)}(\text{Z} \rtimes \text{Z})\) is \(\mathbb{R}\). Define the action of \(\text{Z} \rtimes \text{Z}\) on \(\mathbb{R}\) as follows
   \[
   (t_1, t_2)x = t_2 + x, \quad (t_1, t_2) ∈ \text{Z} \rtimes \text{Z}, \quad x ∈ \mathbb{R}.
   \]
   Observe that any point \(x ∈ \mathbb{R}\) is fixed by \((t_1, t_2)\) if only if \(t_2 = 0\). If \(S\) is a subgroup not in \(\mathcal{VCY}(H)\), then the \(S\)-fixed point set \(R^S\) is the empty set.

   Let \(S ∈ \mathcal{VCY}(H)\), then \(S = \langle (n, 0) \rangle\) for some \(n ∈ \text{Z} - \{0\}\). Then the \(S\)-fixed point set \(R^S = \mathbb{R}\) is contractible. Therefore, by Proposition 2.4, we have the claim.

4. (a) Let \(R = \langle (n, 2m) \rangle\), with \(n, m\) fixed in \(\text{Z} - \{0\}\), and suppose \(R\) is maximal in \(\mathfrak{F}\). By (5) we have that
   \[
   (t_1, t_2)(n, 2m)(t_1, t_2)^{-1} = ((-1)^{t_2}n, 2m).
   \]
   Then we have two possibilities,

   i) \(gRg^{-1} = R\) if only if \(t_2\) is even,

   ii) \(gRg^{-1} = \langle (-n, 2m) \rangle\) if only if \(t_2\) is odd,

   therefore \(gRg^{-1} ∩ R = \{1\}\).

   We conclude by (3) that
   \[
   \text{N}_{\text{Z} \rtimes \text{Z}}[R] = \{(t_1, 2t_2) \mid t_1, t_2 ∈ \text{Z}\} = \text{Z} × \text{Z},
   \]
   and so by (4) and (6):
   \[
   \mathcal{G}[R] = \{\langle (ln, 2lm) \rangle \mid l ∈ \text{Z}\}
   = \mathcal{VCY}(R)
   \]
   (b) A model for \(E_{\mathcal{VCY}(R)}(\text{Z} × \text{Z})\) is \(\mathbb{R}\) by Proposition 2.4. Observe that the normalizer \(\text{N}_{\text{Z} \rtimes \text{Z}}(R)\) of \(R\) is equal to \(\text{N}_{\text{Z} \rtimes \text{Z}}[R]\), therefore
$R$ is a normal subgroup of $N_{\mathbb{Z} \times \mathbb{Z}}[R] = \mathbb{Z} \times \mathbb{Z}$ and we have the following exact sequence

$$0 \rightarrow R \xrightarrow{i} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \rightarrow 0$$

where $i$ is the inclusion and $\phi$ is the projection onto the quotient $(\mathbb{Z} \times \mathbb{Z})/R \cong \mathbb{Z}$. Let $(t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}$ and $x \in \mathbb{R}$, define the action of $\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{R}$ as follows:

$$(t_1, t_2) \cdot x = \phi(t_1, t_2) + x.$$ 

So, $\phi(t_1, t_2) + x = x$ if only if $(t_1, t_2) \in \ker \phi = R$, therefore the isotropy groups are in $\mathcal{VCY}(R)$. Furthermore, $\mathbb{R}^R = \mathbb{R}$ is contractible, and if $S \not\in \mathcal{VCY}(R)$ then $\mathbb{R}^S = \emptyset$.

5. (a) Let $[K]$ be as in (20) and $(t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}$. By equation (5), it follows that

$$(t_1, t_2)(0, 2)(t_1, t_2)^{-1} = (0, 2).$$

Since $[(0, 2)] = [K]$, we conclude by (3) that

$$N_{\mathbb{Z} \times \mathbb{Z}}[K] = \mathbb{Z} \times \mathbb{Z},$$

so by (3), (6) and (7), note:

$$\mathcal{G}[K] = \{D \subseteq \mathbb{Z} \times \mathbb{Z} \mid D \in \mathcal{G}, |D \cap K| = \infty\} \cup \{1\}$$

$$= \{\langle(n, 2m + 1)\rangle \mid n, m \in \mathbb{Z}\} \cup \{1\}.$$

(b) A model for $E_{\mathcal{G}[K]}(\mathbb{Z} \times \mathbb{Z})$ is as follows:

By (7), note that $\langle(n, 2m + 1)\rangle \subseteq \langle(n, 1)\rangle$ for all $n, m \in \mathbb{Z}$. Let $K_n = \langle(n, 1)\rangle$ and consider a point $k_n$ for each $K_n$. Observe for $g = (t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}$, $gK_ng^{-1} = K_m$ that $m = (-1)^{t_2}n + 2t_1$ by Remark 3.1. Then $\mathbb{Z} \times \mathbb{Z}$ acts by permuting the subgroups $K_n$, $n \in \mathbb{Z}$.

We define an action of $\mathbb{Z} \times \mathbb{Z}$ on $X := \{k_n \mid n \in \mathbb{Z}\}$ as follows

$$g \cdot k_n = k_m \text{ iff } gK_ng^{-1} = K_m.$$ 

By the above observation and (7), note:

$$g \cdot k_n = k_n \text{ iff } g \in N_{\mathbb{Z} \times \mathbb{Z}}(K_n) = K_n.$$ 

Therefore the $\mathbb{Z} \times \mathbb{Z}$-set $X$ is a model for $I_{\mathcal{G}[K]} - \{1\}$. Since a model for $E(\mathbb{Z} \times \mathbb{Z})$ is $\mathbb{R}^2$, by Theorem 2.7 we conclude that a model for $E_{\mathcal{G}[K]}(\mathbb{Z} \times \mathbb{Z})$ is the join $X \ast \mathbb{R}^2$. 


4.2. A second model for $E(Z \times Z)$. We are now ready to apply Theorem 2.5 and obtain a model for $E(Z \times Z)$ by the following $(Z \times Z)$-pushout. Let $G = Z \times Z$ and $A = Z \times Z$, then

$$(21) \quad G \times_G EG \coprod_{id \times_G G} G \times_A EA \xrightarrow{id \times G \coprod id \times_A f_G} G \times_G E_{VCY(H)}G \coprod_{id \times G \coprod id \times_A f_G} G \times_A E_{VCY(R)}A \xrightarrow{\coprod \{k_n\}_{n \in \mathbb{Z}}} E(Z \times Z),$$

where $I$ is a complete system of representatives $[R]$ of the $G$-orbits under conjugation over the classes of subgroups of type $[R] = [\langle (n, 2m) \rangle]$, $(n, m) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$.

Applying models given in Section 4.1 and the fact $G \times_G Y = Y$, we have

$$(22) \quad \mathbb{R}^2 \coprod \mathbb{R}^2 \coprod_{l \in I} G \times_A \mathbb{R}^2 \xrightarrow{id \coprod g} \mathbb{R} \coprod \{\{k_n\}_{n \in \mathbb{Z}} \ast \mathbb{R}^2\} \coprod_{l \in I} G \times_A \mathbb{R} \xrightarrow{\coprod [1_G, f_i]} E(Z \times Z).$$

The maps are given by the universal property of classifying spaces, applied with inclusions of families of subgroups, these are as follows:

1. $p: \mathbb{R}^2 \to \mathbb{R}$ is the projection on the $y$-axis, $(t, s) \mapsto s$. Since $G$ acts on the $y$-axis of $\mathbb{R}^2$ by translation and the action of $G$ on $\mathbb{R}$ is also by translation, then $p$ is a $G$-map, which is cellular. By Remark 2.2 $p$ is unique up to $G$-homotopy.

2. The map $g: \mathbb{R}^2 \to \{k_n\}_{n \in \mathbb{Z}} \ast \mathbb{R}^2$ is the inclusion, because the $G$-action is the same on $\mathbb{R}^2$, we have that $g$ is a $G$-map.

3. Let $l \in I$, $f_l: \mathbb{R}^2 \to \mathbb{R}$ are the quotient map of $\mathbb{R}^2$ on the line through $(n, 2m)$ and the origin. It follows the map is $G$-equivariant.

4. The map $i$ is the identity on the first two $G$-spaces corresponding to the disjoint union, and it is the natural $G$-map on the third $G$-space.

**Theorem 4.1.** Let $G = Z \times Z$, $A = Z \times Z$ and $I$ be a complete system of representatives $[R]$ of the $G$-orbits under conjugation over the classes of subgroups of type $[R] = [\langle (n, 2m) \rangle]$, $(n, m) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$. From the $G$-pushout (22) we have

$$E(Z \times Z) = \prod_{x \in \mathbb{R}^2} \{k_n\}_{n \in \mathbb{Z}} \ast \mathbb{R}^2 \coprod_{l \in I} G \times_A \mathbb{R}^2 \forall x \in \mathbb{R}^2: p(x) \sim g(x) \sim [1_G, f_i(x)],$$
MODELS FOR CLASSIFYING SPACES FOR $\mathbb{Z} \times \mathbb{Z}$

where the maps $p$, $g$ and $f_1$ are as before.

5. Homology

In this Section we compute the homology groups of the model given in Theorem 3.6, $B(\mathbb{Z} \times \mathbb{Z}) = (\bigsqcup J \mathbb{S}^1) \ast \mathcal{K}$, where $J$ is an countably infinite set. To simplify notation, denote $X = \bigsqcup J \mathbb{S}^1$.

Since the Klein bottle is path-connected, then the join $X \ast \mathcal{K}$ is simply connected, [2, Sec. 7.2]. Therefore $H_0(X \ast \mathcal{K}) = \mathbb{Z}$ and $H_1(X \ast \mathcal{K}) = 0$.

From the well known short exact sequence of the join, see [7, Ch. 8], we have:

$$0 \longrightarrow \widetilde{H}_{n+1}(X \ast \mathcal{K}) \longrightarrow \widetilde{H}_n(X \times \mathcal{K}) \xrightarrow{\pi} \widetilde{H}_n(X) \oplus \widetilde{H}_n(\mathcal{K}) \longrightarrow 0$$

where the homomorphism $\pi$ is given by $a \mapsto (\pi_X(a), -\pi_\mathcal{K}(a))$. (Here $\pi_X$ and $\pi_\mathcal{K}$ are the homomorphisms induced in homology by the projections of $X \times \mathcal{K}$ on $X$ and $\mathcal{K}$ respectively.) Then we have the following exact sequence:

$$(23) \quad 0 \longrightarrow \widetilde{H}_{n+1}(X \ast \mathcal{K}) \rightarrow \bigoplus J \widetilde{H}_n(\mathbb{S}^1 \times \mathcal{K}) \rightarrow \left( \bigoplus J \widetilde{H}_n(\mathbb{S}^1) \right) \oplus \widetilde{H}_n(\mathcal{K}) \rightarrow 0.$$

Thus, by the CW structure of $\mathcal{K}$ and $\mathbb{S}^1$, we use the Künneth theorem to obtain the homology groups of the product $\mathbb{S}^1 \times \mathcal{K}$:

$$H_i(\mathbb{S}^1 \times \mathcal{K}) = \begin{cases} \mathbb{Z}, & i = 0; \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2, & i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}_2, & i = 2; \\ 0, & i > 2. \end{cases}$$

Then for $n = 1$:

$$0 \rightarrow \widetilde{H}_{n+1}(X \ast \mathcal{K}) \rightarrow \bigoplus J (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2) \xrightarrow{\pi} \left( \bigoplus J \mathbb{Z} \right) \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \rightarrow 0,$$

and we conclude that

$$H_2(X \ast \mathcal{K}) = \bigoplus J'(\mathbb{Z} \oplus \mathbb{Z}_2),$$

where $J'$ is a countably infinite set. If $n = 2$ in (23), then

$$H_3(X \ast \mathcal{K}) = H_2(X \times \mathcal{K}) = \bigoplus J(\mathbb{Z} \oplus \mathbb{Z}_2).$$

And at last, for $i > 3$, $H_i(X \ast \mathcal{K}) = 0$. 
Proposition 5.1. Homology groups of $B(\mathbb{Z} \times \mathbb{Z}) = \left( \bigsqcup_J S^1 \right) \ast K$, where $J$ is an countably infinite set (Thm. 3.6), are the following:

$$H_i\left(\left( \bigsqcup_J S^1 \right) \ast K\right) = \begin{cases} 
\mathbb{Z}, & i = 0; \\
0, & i = 1; \\
\bigoplus_{J'} (\mathbb{Z} \oplus \mathbb{Z}_2), & i = 2; \\
\bigoplus_J (\mathbb{Z} \oplus \mathbb{Z}_2), & i = 3; \\
0, & i > 3,
\end{cases}$$

where $J'$ is a countable infinite set.

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