Corrections to the Newtonian potential in the two-brane Randall-Sundrum model

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We calculate the Newtonian potential in the two-brane Randall-Sundrum model, emphasizing the effect of the finite distance between the two branes. The result obtained is quite natural: When the distance in the potential is small compared to the brane separation the two-brane model is indistinguishable from the one-brane model, whereas when the distance is large the bulk dimension behaves like an ordinary compact dimension, with an exponentially decreasing correction to the four-dimensional potential. The contribution from the radion is also included, and is found to give only a multiplicative factor of order 1 in the correction.

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I. INTRODUCTION

In this paper we study the Randall-Sundrum I model [1], which consists of two 3-branes embedded in five-dimensional anti de Sitter space. Our main focus will be the static gravitational potential between two point particles on the visible brane, and in particular the effect the finite gap between the two branes has on the potential compared to the Randall-Sundrum II model [2] with only a single brane.

The two-brane model is mostly used in connection with particle physics, where it gives a natural explanation of the hierarchy between the electroweak scale and the Planck scale, without the need to introduce very large (or very small) dimensionless parameters in the theory [1]. For a survey of the phenomenology of theories with extra dimensions, see e.g. [3] and references therein. As for the problem of graviton localization and the derivation of the gravitational potential, most of the recent activity has focused on the one-brane model [3,4]. However, some work has also been done with two branes in this context, see e.g. [5,6], and [7] where induced curvature terms in the actions for the two branes are considered. The cosmology of the two-brane model is studied in e.g. [8], and a mechanism for stabilizing the brane separation is discussed in [9]. In this paper we will simply assume that the branes are static and fixed at a constant distance.

Before doing any calculations, we should be able to give a qualitative description of what the potential should look like. The deciding factor must be the ratio between the distance \( r \) in the potential and the separation between the two branes. For short distances the second brane is far away, relatively speaking, and should therefore be invisible to the potential. The second brane could just as well be sent away to infinity, meaning that the potential must be the same as in the one-brane model for short distances. Of course, if the distance \( r \) is very short we can also ignore the entire five-dimensional curvature, meaning that \( V(r) \sim 1/r^2 \) when \( r \to 0 \). On the other hand, for large distances the more detailed structure of the fifth dimension must be invisible to the potential, and we should get the same result as with an ordinary, compact extra dimension. That is, the potential should essentially be four-dimensional, \( V(r) \sim 1/r \), with an exponentially decreasing correction from the fifth dimension.

However, it is not the physical distance \( y \), to the second brane (as measured with standard rods) that sets this scale, but rather the conformal distance \( z_r \sim e^{ryr} \). This can be understood from the fact that all particle masses as measured on the visible brane are reduced by a factor \( e^{-yr} \), as compared to the masses on the hidden brane [1]. The exponential factor originates from the warped geometry of the fifth dimension. Since the gravitational force is mediated by massive gravitons when observed in four dimensions, and massive propagators result in a factor \( e^{-mr} \), the distance scale is determined by \( r \sim m^{-1} \sim e^{ryr} \sim z_r \), i.e. the conformal rather than the physical brane separation.

This paper is organized as follows: First, the model is presented in section II. In section III we summarize the governing equations for graviton propagation in this particular background, using the results of II. We then proceed to find the spectrum of graviton masses in section IV, allowing us to finally derive approximate expressions for the gravitational potential in the limit of short and large distances, respectively.

II. THE MODEL

We assume that the five-dimensional spacetime can be parametrized by the metric

\[ ds^2 = A^2(y) \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \]

(1)

where \( A(y) \) is some function of the coordinate \( y \) of the fifth dimension, called the warp factor. The signature
we use is \( \eta_{\mu \nu} = (+1, -1, -1, -1) \). The two branes are located at \( y = 0 \) and \( y = y_r \), with an orbifold symmetry \( S^1/\mathbb{Z}_2 \) in the \( y \)-direction, and we assume that the only source of gravity is a cosmological constant \( \Lambda_B \) in the bulk, and a tension \( \lambda \) and \( \lambda_r \) on the two branes. The warp factor \( A(y) \) is determined from the Einstein equation just as in the one-brane case. With the fine-tuning \( \Lambda_B = -M^{-6}\lambda^2/6 \), \( M \) being the five-dimensional Planck mass, the solution is

\[
A(y) = e^{-\mu|y|},
\]

where \( \mu = \sqrt{|\Lambda_B/6|} = M^{-3}\lambda/6 \). The bulk space is anti de Sitter since \( \Lambda_B < 0 \), and the fine-tuning means that the effective four-dimensional cosmological constant vanishes. This configuration is often called a critical brane.

The two branes impose the boundary conditions

\[
\frac{[A^I]}{A} \bigg|_{\ y=0} = -\frac{1}{3}M^{-3}\lambda, \quad \frac{[A^I]}{A} \bigg|_{\ y=y_r} = -\frac{1}{3}M^{-3}\lambda_r
\]

(3)
on the warp factor, where \( [A^I]_{y=0} = A'(0^+) - A'(0^-) = 2A'(0^+ \) and \( [A^I]_{y=y_r} = A'(y_r^+) - A'(y_r^-) = -2A'(y_r^-) \) are the jump discontinuities of \( A(y) \) across the two branes. The boundary condition at \( y = 0 \) simply fixes the solution (2), whereas the condition at \( y = y_r \) imposes a constraint on the tension \( \lambda_r \) of the second brane. With \( A(y) = e^{-\mu|y|} \) for \( |y| \leq y_r \) we get \( \lambda_r = -6\mu M^3 = -\lambda \), which is the well-known result that the two branes have tensions with the same magnitude but opposite sign. The physical brane where we are supposed to be living, is usually taken to be the one with a positive tension, i.e. \( \lambda > 0 \) and \( \lambda_r < 0 \).

A preliminary expression for the four-dimensional Planck mass, ignoring the contribution from the radion, can be found by integrating over the \( y \)-dimension in the five-dimensional gravitational action \( S = -\frac{1}{2}M^3 \int d^4x dy \sqrt{\mathcal{g}} R \). All five-dimensional quantities are here denoted with a hat. The five-dimensional curvature scalar \( \hat{R} \) is related to the four-dimensional one through \( \hat{R} = (1/A^2)R + \ldots \) (replacing for a moment the flat four-dimensional metric \( \eta_{\mu \nu} \) by \( g_{\mu \nu} \)). Using \( \sqrt{\hat{g}} = A^4/\sqrt{\mathcal{g}} \) and demanding that \( S = -\frac{1}{2}M^3 \int d^4x \sqrt{\mathcal{g}} \hat{R} + \ldots \), we then get

\[
M^2 = \frac{M^3}{\mu} \int_{-y_r}^{y_r} A^2(y) dy = \frac{M^3}{\mu} \left( 1 - e^{-2\mu y_r} \right).
\]

The four-dimensional Planck mass thus depends only weakly on \( y_r \), as discussed in [1], and the parameters \( M \) and \( \mu \) can be chosen to be of the same order of magnitude as \( M^2 \). The presence of the radion changes this expression slightly, see eq. (22).

### III. GRAVITON PROPAGATION

We study the propagation of gravitons in the model by looking at a perturbation

\[
ds^2 = A^2(y) (\eta_{\mu \nu} + h_{\mu \nu}) \ dx^\mu dx^\nu - dy^2
\]

to the metric (12). This analysis is exactly the same for two branes as for a single brane, only that the regulator brane in [3] now remains at a finite distance. We will therefore only summarize the main results here, referring to [3] for the more detailed calculations.

Imposing the usual gauge conditions \( \partial^\mu h_{\mu \nu} = 0 \) and \( \eta^{\mu \nu} h_{\mu \nu} = 0 \), the wave equation for \( h_{\mu \nu} \) reduces to

\[
\left( \frac{1}{A^2} \nabla^2_{\alpha} - \partial^2_y - 4\frac{A'}{A} \partial_y \right) h_{\mu \nu} = 0,
\]

(6)where \( \nabla^2_{\alpha} \) is the four-dimensional Laplacian, \( \nabla^2_{\alpha} = \partial_{\alpha} \partial_{\alpha} \) in flat space. The separation of variables \( h_{\mu \nu}(x, y) = \Phi(y) \) yields

\[
(\nabla^2_{\alpha} + m^2) G_{\mu \nu}(x) = 0,
\]

\[
\Phi''(y) + \frac{4A'}{A} \Phi'(y) + m^2 \Phi(y) = 0.
\]

(7)(8)The tensor \( G_{\mu \nu}(x) \) describes massive spin-2 particles in four dimensions, and the allowed values of the mass \( m \) are determined from (3). Changing to the conformal coordinate \( z = \eta y/\sqrt{\xi} = A(y) \), meaning that \( 1 + |z| = e^{\mu|y|} \), and writing \( \Phi(y) = A^{-3/2} u(z) \), (8) is reduced to

\[
[-\partial^2 + V(z)] u(z) = m^2 u(z),
\]

(9)where

\[
V(z) = \frac{9}{4} (A')^2 + 3 AA''
\]

\[
= \frac{15\mu^2}{4(1+|z|)^2} - 3\mu \delta(z) + \frac{3\mu}{1+\mu z_r} \delta(z - z_r).
\]

(10)(A' still means the derivative of \( A \) with respect to \( y \).) The two delta functions in the Schrödinger potential result in the boundary conditions

\[
2u'(0) + 3\mu u(0) = 0,
\]

(11)\[
2u'(z_r) + \frac{3\mu}{1+\mu z_r} u(z_r) = 0.
\]

(12)The solutions to (9) with the first boundary condition (11) imposed are

\[
u_0(z) = N_0(1+|z|)^{-3/2},
\]

(13)\[
u_m(z) = N_m \sqrt{1+|z|}
\]

\[
\times \left\{ Y_2[\frac{m}{\mu}(1+|z|)] - Y_1[\frac{m}{\mu}] J_1[\frac{m}{\mu}(1+|z|)] J_2[\frac{m}{\mu}(1+|z|)] \right\},
\]

(13)where \( N_0 \) and \( N_m \) are normalization constants, and \( J_n(x) \) and \( Y_n(x) \) are Bessel functions of order \( n \). The second boundary condition (12) determines the allowed mass eigenvalues, and can be simplified to

\[
\frac{Y_1[\frac{m}{\mu}(1+\mu z_r)]}{J_1[\frac{m}{\mu}(1+\mu z_r)]} = \frac{Y_1[\frac{m}{\mu}(1+\mu z_r)]}{J_1[\frac{m}{\mu}(1+\mu z_r)]}
\]

(14)for the massive modes. The massless mode will always be present.
IV. CORRECTIONS TO NEWTON’S LAW

As shown in [5], the gravitational potential between two point particles on the physical brane at \( y = 0 \), due to the exchange of virtual gravitons, can be written

\[
V(k) = \frac{1}{M^3} \sum_m |u(m,0)|^2 \frac{T_{\mu\nu}^{(m)} P_{\mu\nu\alpha\beta}^{(m)} T_{\alpha\beta}^{(m)}}{k^2 - m^2} .
\]

(15)

The energy momentum tensors are \( T^{\mu\nu} = m_1 u^\mu u^\nu = m_1 \delta^\mu_0 \delta^\nu_0 \) and \( T^{\alpha\beta} = m_1 \delta^\alpha_0 \delta^\beta_0 \), and \( P_{\mu\nu\alpha\beta}^{(m)} \) is the polarization tensor of a spin-2 particle in four dimensions with mass \( m \) and momentum \( k \). The normalization constants in \( T^{(m)} \) are found by requiring \( \int_{-z_c}^{z_c} u(z) u(z) dz = \delta_{jj} \), which means that

\[
N_j^2 = \left[ \int_{-z_c}^{z_c} \frac{dz}{(1 + |z|)^2} \right]^{-1} = \frac{\mu}{1 - e^{-2\mu y_c}} .
\]

(16)

Using \( P^{(0)}_{0000} = \frac{1}{2} \) and \( P^{(m>0)}_{00} = \frac{2}{3} \), together with \( M_{\text{Pl}}^2 = 8\pi G \) where \( G \) is the Newtonian gravitational constant, we get after Fourier inverting \( V(r) \)

\[
V(r) = -\frac{G m_1 m_2}{r} \times \left\{ 1 + \frac{4(1 - e^{-2\mu y_c})}{3\mu} \sum_{m>0} |u(m,0)|^2 e^{-mr} \right\} .
\]

(17)

Using the identity \( J_n(x) Y_{n+1}(x) - Y_n(x) J_{n+1}(x) = -\frac{2}{x} \), the expression for \( |u(m,0)|^2 \) can be simplified to

\[
|u(m,0)|^2 = \frac{4\mu^2}{\pi^2 m^2} \frac{N_j^2}{J_1^2(m)} \times \frac{2\mu^3}{\pi^2 m^3} \left[ \int_0^{\mu z_c} (1 + x) \times \left\{ J_1(m) Y_2(m)(m+2) - Y_1(m) J_2(m)(m+2) \right\}^2 dx \right]^{-1} .
\]

(18)

In order to proceed further we must find the allowed mass eigenvalues by solving \( \text{[14]} \), at least approximately.

If we also include the radion, which has been ignored until now, the potential changes to \( \text{[11]} \)

\[
V(r) = -\frac{G m_1 m_2}{r} \times \left\{ 1 + \frac{4}{3\mu} \frac{1 - e^{-2\mu y_c}}{1 + \frac{4}{3} e^{-2\mu y_c}} \sum_{m>0} |u(m,0)|^2 e^{-mr} \right\} ,
\]

(19)

and the four-dimensional Planck mass is then

\[
M_{\text{Pl}}^2 = \frac{M^3}{\mu} \frac{1 - e^{-2\mu y_c}}{1 + \frac{4}{3} e^{-2\mu y_c}} \equiv \frac{1}{8\pi G} .
\]

(20)

Comparing eq. \( \text{[19]} \) and \( \text{[20]} \) with \( \text{[17]} \) and \( \text{[14]} \), respectively, we see that the whole effect of the radion is the extra factor \( (1 + \frac{4}{3} e^{-2\mu y_c}) \).

A. Finding the mass eigenvalues

So far, the conformal distance \( z_r \) to the second brane has been considered a free parameter of arbitrary magnitude. However, the original idea of Randall and Sundrum \( \text{[1]} \) was that the two-brane model should explain the hierarchy between the electroweak scale \( M_{\text{EW}} \sim 1 \text{ TeV} \) and the Planck scale \( M_{\text{Pl}} \sim 10^{18} \text{ GeV} \), which is obtained by setting

\[
e^{\mu y_c} = 1 + \mu z_r \sim 10^{15} .
\]

(21)

In the following, we will therefore use as a simplifying assumption that \( \mu z_r \) is a very large number. The general features of the results we will obtain in the end, however, are independent of this assumption. In appendix \( \text{[A]} \) the other extreme case \( \mu z_r \ll 1 \) is studied, showing that the only thing that affects whether the two-brane model can be distinguished from the one-brane model is the ratio \( r/z_r \).

The function \( f(x) = Y_1(x)/J_1(x) \) is almost periodic, and approaches \( \tan(x - \frac{3\pi}{4}) \) fast for large \( x \). From \( \text{[14]} \) we therefore get \( m_n z_r \approx n \pi \) as a first approximation. However, we can find a much more accurate expression than this for the lowest eigenvalues in the limit \( \mu z_r \gg 1 \). Since \( \frac{m_n}{\mu} \approx \frac{\pi}{\mu z_r} \ll 1 \) we expand \( f(x) \) for small arguments:

\[
f\left( \frac{m_n}{\mu} \right) = -\frac{4\mu^2}{\pi mn^2} + \mathcal{O}(\ln \frac{m_n}{\mu}) .
\]

(22)

The pole at \( x = 0 \) must coincide with a pole at \( x = c_n \), that is, the \( n \)-th zero of the Bessel function, \( J_1(c_n) = 0 \). With the expansion \( f(c_n + \epsilon_n) = Y_1(c_n)/\epsilon_n J_1(c_n) + \mathcal{O}(1) \) and demanding that \( f\left( \frac{m_n}{\mu} \right) = f\left( \frac{m_n}{\mu} (1 + \mu z_r) \right) = f(c_n + \epsilon_n) \) we then get

\[
\epsilon_n = -\frac{\pi n^2 Y_1(c_n)}{4\mu^2 J_1(c_n)} + \mathcal{O}\left( \frac{m_n^2}{\mu^3} \ln \frac{m_n}{\mu} \right) ,
\]

(23)

and hence

\[
\frac{m_n}{\mu} = \frac{c_n + \epsilon_n}{1 + \mu z_r} = \frac{c_n}{1 + \mu z_r} - \frac{\pi Y_1(c_n)c_n^2}{4J_1(c_n)(1 + \mu z_r)^3} + \mathcal{O}\left( \frac{c_n^4}{(1 + \mu z_r)^5} \ln \frac{c_n}{1 + \mu z_r} \right) .
\]

(24)

For large values of \( n \) we use \( J_1(c_n) \sim \cos(c_n - \frac{3\pi}{4}) = 0 \), which means that \( c_n \approx (n + \frac{1}{4})\pi \).

B. Short distances, \( \mu r \ll 1 \)

For very short distances we should expect to get the same result as for flat, five-dimensional Minkowski space, i.e. we should expect the relative correction \( \Delta(r) \) to the Newtonian potential to be proportional to \( 1/r \), where \( V(r) \sim V_0(r)(1 + \Delta) \), \( V_0(r) = -G m_1 m_2/r \). The full
potential will then behave like \( V(r) \sim 1/r^2 \) when \( r \to 0 \), since \( \Delta \gg 1 \) in this limit.

From (14) we see that \( r \to 0 \) corresponds to letting \( m \to \infty \). More precisely, from the assumption \( \mu r \ll 1 \) we get \( m \sim r^{-1} \gg \mu \), allowing us to use the asymptotic expressions for the Bessel functions in (18). The integral thus reduces to

\[
I_m \simeq \frac{4\mu^2}{\pi^2 m^2} \int_0^{\mu r} u^{m-1} e^{-\mu r} \frac{\sin \left( \frac{\pi u}{m} \right)}{m} du \simeq \frac{2\mu^3 \mu r}{\pi^2 m^2} \frac{\sin \left( \frac{\pi u}{m} \right)}{m},
\]

which means that \(|u(m,0)|^2 \simeq 1/\mu r\). By also using \( m_n \simeq n\pi/\mu r \), we then get

\[
\Delta \simeq \frac{4}{3\mu r} \left( 1 - e^{-2\mu r} \right) \sum_{n=1}^{\infty} \frac{e^{-n\pi r}}{\mu r}, \quad \mu r \ll 1.
\]

The only dependence on the brane separation \( y_r \) lies in the factor \((1 - e^{-2\mu r})/(1 + \frac{1}{2} e^{-2\mu r})\), which is exactly the factor that enters into the five-dimensional Planck mass \( M \) from (20). To be more specific, since \( \Delta \gg 1 \) the complete potential is

\[
V(r) \simeq \frac{4G}{3\pi\mu} \left( 1 - e^{-2\mu r} \right) \frac{m_1 m_2}{r^2} = -\frac{m_1 m_2}{6\pi^2 M^3 r^2}, \quad \mu r \ll 1.
\]

by using (20) and \( G = (8\pi M^3)^{-1} \). The natural way of defining the gravitational constant \( G_D \) in \( D \) spacetime dimensions is by requiring the gravitational force to be \( F = G_D m_1 m_2/r^{D-2} \). Since the force is given by the derivative of the potential we thus get \( G_D = (3\pi^2 M^3)^{-1} \), and hence \( V(r) \simeq -G_D m_1 m_2/2r^2 \).

From (21) we also notice, of course, that we reproduce the one-brane result by taking the limit \( y_r \to \infty \).

### C. Large distances, \( \mu r \gg 1 \)

The limit of very large distances, \( r \to \infty \), corresponds to \( m \to 0 \) (or more precisely \( m \ll \mu \)). Things will therefore be a little more complicated, since we can’t use the asymptotic expressions for the Bessel functions in (18) anymore. Instead, the integral must be evaluated more or less analytically as it stands. Changing to the variable \( u = \frac{m}{\mu} (1 + x) \) we get

\[
I_m \simeq \frac{\mu^2}{m^2} \int_0^{\mu r} u \left[ J_1\left( \frac{m}{\mu} \right) Y_2(u) - Y_1\left( \frac{m}{\mu} \right) J_2(u) \right] du
\]

\[
\simeq \frac{\mu^2}{m^2} \left[ J_1^2\left( \frac{m}{\mu} \right) I_1 - 2J_1\left( \frac{m}{\mu} \right) Y_1\left( \frac{m}{\mu} \right) I_2 + Y_1^2\left( \frac{m}{\mu} \right) I_3 \right],
\]

The two integrals \( I_1 = \int u Y_2(u) du \) and \( I_3 = \int u J_2(u) du \) are easily expressed in terms of other Bessel functions, whereas the integral \( I_2 = \int u Y_2(u) J_2(u) du \) is a little more complicated and involves the Meijer \( G \)-function \( \mathbf{1} \). However, all the different terms can be expanded in powers of \( m/\mu \), and they turn out to be of completely different orders of magnitude. The dominant term can be shown to be the upper limit \( u = c_n \) in \( I_3 \), where

\[
I_m \simeq \frac{4\mu^4}{\pi^2 m^4} I_3(c_n) = \frac{2\mu^4 c_n^2 J_2^2(c_n)}{\pi^2 m^4},
\]

where we have used \( I_3(u) = \frac{1}{2} u^2 [J_2^2(u) - J_1(u) J_3(u)] \) together with \( J_1(c_n) = 0 \). Inserting this into (18) gives

\[
|u(m,0)|^2 \simeq \frac{m^2}{\mu c_n^2 J_2^2(c_n)} \simeq \frac{\mu}{(1 + \mu z_r)^2 J_2^2(c_n)}.
\]

For very large distances, \( r \gg z_r \), we get the dominant contribution from the first eigenvalue \( m_1 \),

\[
\Delta \simeq \frac{4}{3(1 + \mu z_r)^2 J_2^2(c_1)} \left( 1 - e^{-2\mu r} \right) e^{-m_1 r}
\]

\[
\simeq \frac{8.21953739}{(1 + \mu z_r)^2} e^{-3.83170597 r/z_r}, \quad r \gg z_r,
\]

where we have ignored the terms \( e^{-2\mu r} \) (since this represents a higher order correction) and inserted the value of \( c_1 \) in the last line. The correction thus decreases exponentially for large distances, which is the familiar result for a compact extra dimension. The correction is, however, strongly suppressed compared to an ordinary compact dimension, because of the factor \((1 + \mu z_r)^{-2} \sim 10^{-30} \).

For intermediate distances we must sum over all the allowed eigenvalues. Using the approximations \( J_2^2(c_n) \simeq \frac{2}{\pi c_n} \) and \( c_n \simeq (n + \frac{1}{2}) \pi \) we then get

\[
\Delta \simeq \frac{2\pi^2}{3(1 + \mu z_r)^2} \sum_{n=1}^{\infty} (n + \frac{1}{2}) e^{-(n+1/4)\pi r/z_r}
\]

\[
= \frac{\pi^2 e^{-\pi r/4z_r}}{24(1 + \mu z_r)^2} \sinh^2\left( \frac{\pi r}{4z_r} \right), \quad \mu r \gg 1.
\]

Since we have assumed \( \mu z_r \gg 1 \) this expression is also valid for \( r \ll z_r \), as long as we still have \( \mu r \gg 1 \). In this limit (33) reduces to

\[
\Delta \simeq \frac{2}{3\mu r^2}, \quad 1 \ll \mu r \ll \mu z_r,
\]

which is the same result as for a single brane (3), and in agreement with (3).

In figure (1) the two expressions (27) and (33) for short and large distances are combined into a single plot (using a simple interpolation for the transition at \( \mu r \sim 1 \)). As is evident from the figure, there are three different regions, defined by the scale of the two parameters \( \mu \) and
z_r in the theory: For short distances (μr < 1) the potential is essentially five-dimensional, just as expected, since spacetime can always be considered locally flat. For intermediate distances (μ^{-1} < r < z_r) we have the dominant term \( \Delta \simeq 2/3\mu^2r^2 \) just as in the one-brane model. Here the distance is large enough to be affected by the warped geometry of the fifth dimension, but still too short to notice the presence of the second brane. Finally, for large distances (r > z_r) the fifth dimension is manifestly compact, with an exponential cutoff in the correction to the potential.

This general shape of the correction \( \Delta(r) \) will be the same for all choices of the parameters \( \mu \) and \( z_r \). The only thing that will change when varying \( z_r \) (and keeping \( \mu \) fixed) is the position where the exponential cutoff occurs. Of course, if \( \mu z_r < 1 \) we will only see two different regions, as in this case the branes are so close together that the bulk space between them is essentially flat.

\[ \frac{\text{log } \Delta}{\text{log } \mu} \]

FIG. 1: The figure shows the relative correction \( \Delta \) to the Newtonian potential from (24) and (25) for short and large distances, respectively, where \( V(r) = V_0(r)(1 + \Delta) \). The conformal distance \( z_r \) to the second brane is chosen such that \( \mu z_r = 10^{15} \). The figure clearly shows the transition from the one-brane result to that of a compact extra dimension around \( \mu r \sim 10^{15} \), i.e. around \( r \sim z_r \).

V. SUMMARY

In this paper we have considered the Randall-Sundrum model with two branes. The goal has been to study the effect on the Newtonian gravitational potential due to the finite distance between the branes. As expected, when the distance \( r \) in the potential is small compared to the brane separation \( z_r \), the potential is asymptotically the same as in the one-brane model where \( z_r \to \infty \). In a sense, a short-range measurement of gravity is unable to see what is going on at a much larger scale.

It is not until the distance \( r \) becomes comparable to \( z_r \) that we can see the effect of the finite distance between the branes, and the potential changes from that of the one-brane model to that of a model with an ordinary compact extra dimension, where the relative correction to the four-dimensional potential decreases exponentially. The entire bulk-space dimension thus becomes invisible to a long-range experiment in gravity.

Both of these examples illustrate a quite general feature of almost all of physics – that a measurement done at a particular scale can only reveal information about the underlying physics at the approximately same scale. It should be noted, however, that it is not the physical distance \( y_r \) to the second brane that sets this scale, but rather the conformal distance \( z_r \simeq e^{\mu y_r}/\mu \), since all graviton masses are suppressed by the same factor \( e^{\mu y_r} \).

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APPENDIX A: THE LIMIT \( \mu z_r \ll 1 \)

When the conformal distance to the second brane is small compared to the curvature radius of the fifth dimension, we would expect the fifth dimension to behave like an ordinary compact dimension at all scales. More precisely, we should get the same potential \( V(r) \) for all distances \( r \) as with flat, five-dimensional space, where one of the dimensions is compact with an extension \( L = 2z_r \).

When \( \mu z_r \ll 1 \) the conformal distance \( z_r \) between the branes is equal to the physical distance \( y_r \), since \( 1 + \mu z_r = e^{\mu y_r} \simeq 1 + \mu y_r \). In this limit even the first mass eigenvalue will be very large, i.e. \( m_1/\mu \gg 1 \), and we can use the asymptotic expressions for all Bessel functions in (13). This yields \( \tan(\frac{\pi}{4} - \frac{\pi}{2} m z_r) \simeq \tan(\frac{\pi}{4} - \frac{\pi}{2} + m z_r) \), with the solution

\[ m_n \simeq \frac{n \pi}{z_r}, \quad \mu z_r \ll 1. \]  

This result should be compared to (24). For large values of \( n \) the two expressions are identical to the lowest order, i.e. when (24) is expanded in powers of \( 1/\mu z_r \). However, in (24) it is better to use \( \frac{m}{\mu} \simeq \frac{n}{\mu} \) than \( m_n \simeq \frac{n}{\mu} \) even when \( \mu z_r \gg 1 \), because the next correction in the latter case would be \( 1/\mu^2 z_r^2 \), whereas the next correction in (24) is \( 1/(1 + \mu z_r)^3 \ll 1/\mu^2 z_r^2 \).

The quantization (A1) is exactly what we get from a flat, compact extra dimension, where the wavefunction is \( u(z) \sim e^{imz} \) and the periodic boundary condition gives \( e^{imL} = 1 \), with \( L = 2z_r \).

With \( m_n/\mu \gg 1 \) we get \( |u(m,0)|^2 \simeq 1/z_r \), just as in section (IV.B) and the correction \( \Delta \) to the Newtonian potential is therefore

\[ \Delta \simeq \frac{4}{3\mu z_r^2} \left( 1 - e^{-2\mu y_r} \right) + \sum_{n=1}^{\infty} e^{-n\pi r/z_r} \simeq \frac{2}{e^{\pi r/z_r} - 1}, \]  

where we have used \( y_r \simeq z_r \). Again we get the same result as with a flat, extra dimension (see [11]). For short and
large distances, respectively, (A2) is simplified to
\[
\Delta \simeq \begin{cases} 
\frac{2z}{\pi r}, & r \ll z_r, \\
2e^{-\pi r/z}, & r \gg z_r.
\end{cases}
\] (A3)

Although a little more camouflaged than in the limit \( \mu z_r \gg 1 \), the result for \( r \ll z_r \) is once again the same as for a single brane, since the factor \( z_r \) is absorbed into the five-dimensional Planck mass. Alternatively, if we had kept the factor \( (1 - e^{-2\mu y_r})/(1 + \mu y_r) \) in (A2), we would get the exact same result as (27) for \( r \ll z_r \).

In any case, the conclusion is that in the limit \( r \ll z_r \) the two-brane model is indistinguishable from the one-brane model, whereas in the limit \( r \gg z_r \) the extra dimension behaves like an ordinary, flat-space compact dimension.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999), hep-ph/9905221
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999), hep-th/9906064
[3] V.A. Rubakov, Phys. Usp. 44, 871 (2001), hep-ph/0104152; Y.A. Kubyshin, hep-ph/0111027 (2001)
[4] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84, 2778 (2000), hep-th/9911035; S.B. Giddings, E. Katz and L. Randall, J. High Energy Phys. 03, 023 (2000), hep-th/0002091; D.J.H. Chung, L. Everett and H. Davoudiasl, Phys. Rev. D64, 065002 (2001), hep-ph/0010103; A. Karch and L. Randall, J. High Energy Phys. 05, 008 (2001), hep-th/0011156; N. Deruelle and T. Dolezel, Phys. Rev. D64, 105006 (2001), gr-qc/0105118; E. Kiritsis, N. Tetradis and T.N. Tomaras, J. High Energy Phys. 03, 019 (2002), hep-th/0202037; P. Singh and N. Dadhich, hep-th/0202037 (2002); S. Nojiri and S.D. Odintsov, Phys. Lett. B548, 215 (2002), hep-th/0209066; M. Ito, Phys. Lett. B554, 180 (2003), hep-th/0212108; K. Ghoroku, A. Nakamura and M. Yahiro, Phys. Lett. B571, 223 (2003), hep-th/0303068; P. Callin and F. Ravndal, Phys. Rev. D70, 104009 (2004), hep-ph/0403302
[5] M.N. Smolyakov and I.P. Volobuev, hep-th/0208025 (2002)
[6] E.E. Boos, Y.A. Kubyshin, M.N. Smolyakov and I.P. Volobuev, hep-th/0105304 (2001); Class. Quant. Grav. 19, 4591 (2002), hep-th/0202009
[7] Y. Shtanov and A. Viznyuk, hep-th/0312261 (2003); hep-th/0404077 (2004)
[8] I. Brevik, K. Børkje and J.P. Morten, Gen. Rel. Grav. 36, 2021 (2004), gr-qc/0310103
[9] W.D. Goldberger and M.B. Wise, Phys. Rev. Lett. 83, 4922 (1999), hep-ph/9907447; Phys. Lett. B475, 275 (2000), hep-ph/9911457
[10] P. Callin and F. Ravndal, hep-ph/0412109 (2004)
[11] At this point, we are simply ignoring the radion field \( \phi(x) \sim h_{44} \). The component \( h_{44} \), on the other hand, can be set to zero as a gauge choice \[ \{1, \frac{1}{2}, -1, 0, \frac{1}{2}, \frac{1}{2} \} \]. The radion will later be inserted back into the final result, using [13].
[12] One can show that
\[
I_2 = \int uY_2(u)J_1(u)du = \frac{1}{2\sqrt{\pi}} G_{3,5}^2\left[u, \frac{1}{2}, 1, 3, -1, 0, \frac{1}{2}\right].
\]
For the definition of \( G \), see e.g. the help system of Mathematica (http://www.wolfram.com/).