Saturation at low $x$ appears as an almost unavoidable consequence of the two-gluon exchange generic structure. Consistency of the ansatz for the vector part of the color dipole cross section with conventional evolution determines the energy dependence of the saturation scale.

In this written version of my talk I will restrict myself to a discussion of the empirical evidence for the concept of “saturation” at low $x$ in deep inelastic lepton-nucleon scattering (DIS) and of a consistency argument that allows one to predict the energy dependence of the saturation scale.

In the model-independent analysis of the experimental data from HERA on DIS at low $x$ carried out in the summer of the year 2000, we found that the data on the total virtual photoabsorption cross section lie on a universal curve when plotted against the dimensionless variable

$$\eta = \frac{Q^2 + m_0^2}{\Lambda_{sat}(W^2)},$$

where

$$\Lambda_{sat}(W^2) = B \left( \frac{W^2}{W_0^2} + 1 \right)^{C_2} \approx B \left( \frac{W^2}{W_0^2} \right)^{C_2}.$$ (2)

Compare fig. 1. The energy-dependent quantity, $\Lambda_{sat}(W^2)$, acts as the scale (“saturation scale” or “saturation momentum”) that determines the range of $Q^2$ in which the energy dependence (at fixed $Q^2$) is either hard ($\eta \gg 1$) or soft ($\eta \ll 1$). The model-independent analysis only rests on the assumption that $\sigma_{\gamma^*p}(W^2, Q^2)$ be a smooth function of $\eta$. The fitting procedure gave

$$m_0^2 = 0.15 \pm 0.04 GeV^2,$$

$$W_0^2 = 1081 \pm 12 GeV^2,$$

$$C_2 = 0.27 \pm 0.01.$$ (3)

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As long as only smoothness of $\sigma_{\gamma^*p}$ is assumed, the constant $B$ can be arbitrary. With the explicit form of $\sigma_{\gamma^*p}$ in the generalized vector dominance-color dipole picture (GVD-CDP), we found

$$B = 2.24 \pm 0.43 GeV^2.$$  \hspace{1cm} (4)

Figure 1: The total photoabsorption cross section as a function of the scaling variable $\eta$ from (1).

Note that the data shown in fig. 1 include all data available for $x \simeq Q^2/W^2 < 0.1$ and $0 \leq Q^2 < 1000 GeV^2$, in particular, photoproduction ($Q^2 = 0$) is included.

Since the HERA energy, $W$, is limited, for large values of $Q^2$ small values of $\eta << 1$ cannot be explored. The low-$\eta$ region in fig. 1 contains data close to photoproduction, while the large-$\eta$ region is populated by large-$Q^2$ measurements. Nevertheless, fig. 1 suggests that the “saturation” property

$$\lim_{W^2 \rightarrow \infty} \frac{\sigma_{\gamma^*p}(\eta(W^2, Q^2))}{\sigma_{\gamma p}(W^2)} = 1$$  \hspace{1cm} (5)

to be valid for any fixed $Q^2$.

In terms of the structure function

$$F_2(x, Q^2) \simeq \frac{Q^2}{4\pi^2\alpha} \sigma_{\gamma^*p}(\eta(W^2, Q^2)),$$  \hspace{1cm} (6)
where \( x \simeq Q^2/W^2 \), according to (5) we have

\[
\lim_{W^2 \to \infty} \frac{4\pi^2 \alpha}{\sigma_{\gamma p}(W^2)} \frac{F_2(x, Q^2)}{\sigma_{\gamma p}(W^2)} = Q^2.
\]

(7)

An explicit empirical test of the approach to saturation accordingly requires to plot the data for the ratio of the structure function \( F_2(x, Q^2) \) and the photoproduction cross section as a function of \( \eta \) at fixed \( Q^2 \). Saturation requires the ratio (7) to become flat and approach the value of \( Q^2 \) as a function of \( \eta \) as soon as \( \eta \) becomes small, \( \eta \ll 1 \).

The plot of the experimental data in fig. 2\[3\], for \( Q^2 < \sim 0.5 \text{GeV}^2 \), shows the expected flattening in the \( \eta \)-dependence for \( \eta \ll 1 \). For larger values of \( Q^2 \) the expected flattening for \( \eta \leq 0.1 \) cannot be verified at present due to lack of energy.

No explicit theoretical ansatz is needed for the plots in figs. 1 and 2. We have nevertheless included the theoretical curves from the GVD-CDP\[1, 2, 4\] that provides a theoretical basis for the observed scaling in \( \eta \).

As conjectured\[5, 6\] a long time ago, DIS at low \( x \) in terms of the virtual-photon-proton Compton amplitude is to be understood in terms of diffractive forward scattering of the hadronic \((q\bar{q})^{J=1}\) (vector) states the virtual photon dissociates or fluctuates into. With the advent of QCD, the underlying Pomeron exchange became understood in terms of the coupling of two gluons\[8\] to the \((q\bar{q})^{J=1}\) state. The gauge-theory structure implies...
that the \( (q\bar{q})_{J=1}^{f=1} p \) color-dipole cross section, proportional to the imaginary part of the \( (q\bar{q})_{T,L}^{f=1} p \) forward-scattering amplitude, takes the form:

\[
\sigma_{(q\bar{q})_{J=1}^{f=1} p}^J(\vec{r}_\perp', W^2) \int d^2\vec{l}_\perp' \sigma_{(q\bar{q})_{J=1}^{f=1} p}^J(\vec{l}_\perp^2, W^2) \cdot (1 - e^{-i\vec{l}_\perp \cdot \vec{r}_\perp'})
\]

\[
\simeq \sigma^{(\infty)} \begin{cases} 
1, & \text{for } \vec{r}_\perp^2 \to \infty, \\
\frac{1}{2} \vec{r}_\perp^2 \Lambda_{\text{sat}}^2(W^2), & \text{for } \vec{r}_\perp^2 \to 0,
\end{cases}
\]

where by definition

\[
\sigma^{(\infty)} \equiv \pi \int d\vec{l}_\perp^2 \sigma_{(q\bar{q})_{J=1}^{f=1} p}^J(\vec{l}_\perp^2, W^2),
\]

and

\[
\Lambda_{\text{sat}}^2(W^2) \equiv \frac{\pi}{\sigma^{(\infty)}} \int d\vec{l}_\perp^2 \sigma_{(q\bar{q})_{J=1}^{f=1} p}^J(\vec{l}_\perp^2, W^2).
\]

The (virtual) photoabsorption cross section is obtained from (8) by multiplication with the (light-cone) photon-wave function and subsequent integration over the transverse \( q\bar{q} \) separation \( \vec{r}_\perp = \vec{r}_\perp'/\sqrt{z(1-z)} \) and the variable \( z \) with \( 0 \leq z \leq 1 \) that e.g. determines angular distribution of the quark in the \( q\bar{q} \) rest frame.

It is important to note that the two-gluon-exchange dynamical mechanism evaluated for \( x \to o \) implies the existence of the saturation scale \( \Lambda_{\text{sat}}^2(W^2) \) according to (8). The scale \( \Lambda_{\text{sat}}^2(W^2) \) is related to the effective value of the gluon transverse momentum, \( \vec{l}_\perp = \vec{l}_\perp'/\sqrt{z(1-z)} \), that enters the photoabsorption cross section as a consequence of the two-gluon-exchange mechanism. While an energy-independent scale \( \Lambda_{\text{sat}}^2 = \text{const} \) a priori cannot be strictly excluded, it appears theoretically unlikely. Among other things, constancy would mean that the effective gluon transverse momentum from (10) would be energy independent, the diffractively produced \( q\bar{q} \) mass spectrum be energy independent, the full \( W \) dependence reduced to a factorizing \( W \) dependence due to a potential (weak) energy dependence of \( \sigma^{(\infty)} \) alone, etc. The generic two-gluon-exchange structure “almost” rules out \( \Lambda_{\text{sat}}^2 = \text{const} \) and accordingly requires saturation.

Taking advantage of the fact that the \( J=1 \) part of the dipole cross section (8) is essentially determined by the quantities \( \sigma^{(\infty)} \) and \( \Lambda_{\text{sat}}^2(W^2) \) in (9) and (10), the total photoabsorption cross section becomes approximately:

\[
\sigma_{p}(W^2, Q^2) \simeq \frac{\alpha_R e^{-e^-}}{3\pi} \sigma^{(\infty)} \begin{cases} 
\ln \frac{\Lambda_{\text{sat}}^2(W^2)}{Q^2 + m_0^2}, & (Q^2 \ll \Lambda_{\text{sat}}^2(W^2)), \\
\frac{1}{2} \Lambda_{\text{sat}}^2(W^2), & (Q^2 \gg \Lambda_{\text{sat}}^2(W^2)).
\end{cases}
\]

A detailed evaluation leads to the theoretical results displayed in figs. 1 and 2.

Several remarks are appropriate:
i) Unitarity for the hadronic \((q\bar{q})\) proton interaction requires the integral (9) to exist and \(\sigma^{(\infty)}\) to be at most weakly dependent on the energy \(W\). The fit yields \(\sigma^{(\infty)} \simeq \text{const} \simeq 30\text{mb}\).

ii) The existence of a scale, \(\Lambda^2_{\text{sat}}(W^2)\), according to (10) appears as a straightforward consequence of the two-gluon-exchange structure. This structure implies that the forward-scattering amplitude depends on the effective gluon transverse momentum.

iii) Since unitarity \((\sigma^{(\infty)} \simeq \text{const})\) cannot be disputed, and the assumed two-gluon exchange generic structure seems safe, once \(\Lambda^2_{\text{sat}} = \text{const}\) is abandoned, we must have the transition to the logarithmic behavior in (11), i.e. saturation as depicted in fig. 2 even far beyond the energy range accessible at present.

iv) The gluon structure function from (8) is given by

\[
\alpha_s(Q^2) xg(x, Q^2) = \frac{1}{8\pi^2} \sigma^{(\infty)} \Lambda^2_{\text{sat}} \left( \frac{Q^2}{x} \right),
\]

again disfavoring constancy of \(\Lambda^2_{\text{sat}}(W^2)\).

v) When saturation and the logarithmic behavior in (11) set in, the usual connection between \(F_2\) and the gluon structure function breaks down. An extensive literature (compare e.g. \[11\] and the references therein) attempts to apply (nonlinear) evolution equations for gluon distributions even in this logarithmic domain.

We examine the theoretical description of the experimental data for \(F_2(x, Q^2)\) in somewhat more detail. The ansatz for the dipole cross section underlying the results depicted in fig. 1 and fig. 2 is given by (8) with \[2, 4\]

\[
\bar{\sigma}_{(q\bar{q})}^{(J=1)}(\vec{l}^2_{\perp}, W^2) = \bar{\sigma}_{(q\bar{q})}^{(J=1)}(\vec{l}^2_{\perp}, W^2) = \sigma^{(\infty)} \frac{1}{\pi} \delta(\vec{l}^2_{\perp} - \Lambda^2_{\text{sat}}(W^2)).
\]

The total virtual photoabsorption cross section may be represented in terms of the diffractive forward production of discrete and continuum \((q\bar{q})^{J=1}\) (vector) states \[3\]. The upper limit in the mass of such states, \(M^2_{(q\bar{q})} \lesssim m^2 \simeq 484\text{Gev}^2\) at HERA energies, enters the description of \(\sigma_{\gamma\gamma} p(\eta)\) at large values of \(\eta\). In fig. 3 and fig. 4, as an example, we show \(F_2(x, Q^2)\) for several values of \(Q^2\).

In (12), we noted the connection between the dipole cross section (8) in the limit of small interquark separation and the gluon structure function of the proton. At low
x and sufficiently large values of $Q^2$, the change of the structure function $F_2(x, Q^2)$ is determined by the gluon structure function alone via

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = R_{e^+e^-} \frac{\alpha_s(Q^2)}{9\pi} x g(x, Q^2),$$

(14)

where $R_{e^+e^-} = 3 \sum Q_f^2 = 10/3$ for four flavors of quarks with charges $Q_f$. Substituting the expression for $F_2(x, Q^2)$ at large $Q^2$ from (11) and (6) on the left-hand side in (14), and the expression (12) for the gluon structure function on the right-hand side, we obtain
an interesting consistency constraint on the saturation scale that reads

$$\frac{\partial}{\partial \ln W^2} A_{sat}^2(2W^2) = \frac{1}{3} A_{sat}^2(W^2).$$

(15)
Substitution of the asymptotic power law (2) implies

\[ C_2^{\text{theory}} = \frac{1}{3} \left( \frac{1}{2} \right)^{C_2} \]  

or

\[ C_2^{\text{theory}} = 0.276. \]  

(16)  

(17)

Consistency with conventional DGLAP evolution of our ansatz (13) for the \( J = 1 \) (vector) part of the color dipole cross section implies the prediction (17) for the energy dependence of the saturation scale (2). The prediction (17) is consistent with the experimental value of \( C_2 \) in (3) deduced from the analysis of the experimental data, where \( C_2 \) was left as a free parameter. Note that this predicted energy dependence heavily relies on the choice of \( W \) as the relevant variable in the dipole cross section and the saturation scale, \( \Lambda^2_{\text{sat}} = \Lambda^2_{\text{sat}}(W^2) = \Lambda^2_{\text{sat}}(Q^2/x) \). Moreover, it relies on the assumed equality of the scattering of longitudinally and transversely polarized (q\( \bar{q} \))\( J=1 \) states (13), a constraint that is imposed beyond the generic two-gluon-exchange structure.\(^1\) The available data on the longitudinal-to-transverse ratio \( \sigma_{\gamma^*p}/\sigma_{\gamma^*p} \) are consistent with this constraint.\(^{13} \)

Further investigation on this significant result on the energy dependence of the saturation scale, e.g. its relation to the double-log approximation of the gluon structure function are in progress.\(^{13} \)

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