Geometric zeta functions for higher rank \( p \)-adic groups

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Abstract: The higher rank Lefschetz formula for \( p \)-adic groups is used to prove rationality of a several-variable zeta function attached to the action of a \( p \)-adic group on its Bruhat-Tits building. By specializing to certain lines one gets one-variable zeta functions, which then can be related to geometrically defined zeta functions.

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Introduction

Introduction to be written.

1 The Lefschetz formula

Let $F$ be a nonarchimedean local field with valuation ring $\mathcal{O}$ and uniformizer $\varpi$. Let $| \cdot |$ be the absolute value on $F$ normalized by the rule $\mu(xA) = |x|\mu(A)$, where $\mu$ is any additive Haar measure on $F$. Denote by $G$ a semisimple linear algebraic group over $F$. Let $K \subset G$ be a good maximal compact subgroup. Choose a parabolic subgroup $P = LN$ of $G$ with Levi component $L$. Let $A = A_L$ denote the largest split torus in the center of $L$. Then $A$ is called the split component of $P$. There exists a reductive subgroup $M = M_L$ of $L$, containing the derived group $L_{\text{der}}$, such that $AM$ has finite index in $L$. Let $\Phi = \Phi(G,A)$ be the root system of the pair $(G,A)$, i.e. $\Phi$ consists of all homomorphisms $\alpha : A \to \text{GL}_1$ such that there is $X$ in the Lie algebra of $G$ with $\text{Ad}(a)X = a^\alpha X$ for every $a \in A$. Given $\alpha$, let $n_\alpha$ be the Lie algebra generated by all such $X$ and let $N_\alpha$ be the closed subgroup of $N$ corresponding to $n_\alpha$. Let $\Phi^+ = \Phi(P,A)$ be the subset of $\Phi$ consisting of all positive roots with respect to $P$. Let $\Delta \subset \Phi^+$ be the subset of simple roots. Let $A^- \subset A$ be the set of all $a \in A$ such that $|a^\alpha| < 1$ for any $\alpha \in \Delta$.

An element $g$ of $G$ is called elliptic if it is contained in a compact torus. Let $M_{\text{ell}}$ denote the set of elliptic elements of $M$.

Let $X^*(A) = \text{Hom}(A, \text{GL}_1)$ be the group of all homomorphisms as algebraic groups from $A$ to $\text{GL}_1$. This group is isomorphic to $\mathbb{Z}^r$ with $r = \text{dim } A$. Likewise let $X^*_s(A) = \text{Hom}(\text{GL}_1, A)$. There is a natural $\mathbb{Z}$-valued pairing

$$X^*(A) \times X^*_s(A) \to \text{Hom}(\text{GL}_1, \text{GL}_1) \cong \mathbb{Z}$$

$$(\alpha, \eta) \mapsto \alpha \circ \eta.$$  

For every root $\alpha \in \Phi(A,G) \subset X^*(A)$ let $\bar{\alpha} \in X^*_s(A)$ be its coroot. Then $(\alpha, \bar{\alpha}) = 2$. The valuation $v$ of $F$ gives a group homomorphism $\text{GL}_1(F) \to \mathbb{Z}$. Let $A_c$ be the unique maximal compact subgroup of $A$. Let $\tilde{A} = A/A_c$; then $\tilde{A}$ is a $\mathbb{Z}$-lattice of rank $r = r(P) = \text{dim } A$. By composing with the valuation $v$ the group $X^*(A)$ can be identified with

$$\tilde{A}^* = \text{Hom}(\tilde{A}, \mathbb{Z}).$$
Let

\[ a^*_0 = \text{Hom}(\tilde{A}, \mathbb{R}) \cong X^*(A) \otimes \mathbb{R} \]

be the real vector space of all group homomorphisms from \( \tilde{A} \) to \( \mathbb{R} \) and let \( a^* = a^*_0 \otimes \mathbb{C} = \text{Hom}(\tilde{A}, \mathbb{C}) \cong X^*(A) \otimes \mathbb{C} \). For \( a \in A \) and \( \lambda \in a^* \) let

\[ a^\lambda = q^{-\lambda(a)}, \]

where \( q \) is the number of elements in the residue class field of \( F \). In this way we get an identification

\[ a^*/2\pi i \log q \tilde{A}^* \cong \text{Hom}(\tilde{A}, \mathbb{C}^\times). \]

A quasicharacter \( \nu : A \to \mathbb{C}^\times \) is called \textit{unramified} if \( \nu \) is trivial on \( A_c \). The set \( \text{Hom}(\tilde{A}, \mathbb{C}^\times) \) can be identified with the set of unramified quasicharacters on \( A \). Any unramified quasicharacter \( \nu \) can thus be given a unique real part

\[ \text{Re}(\nu) \in a^*_0. \]

This definition extends to not necessarily unramified quasi characters \( \chi : A \to \mathbb{C}^\times \) as follows. Choose a splitting \( s : \tilde{A} \to A \) of the exact sequence

\[ 1 \to A_c \to A \to \tilde{A} \to 1. \]

Then \( \nu = \chi \circ s \) is an unramified character of \( A \). Set

\[ \text{Re}(\chi) = \text{Re}(\nu). \]

This definition does not depend on the choice of the splitting \( s \). For quasicharacters \( \chi, \chi' \) and \( a \in A \) we will frequently write \( a^\lambda \) instead of \( \chi(a) \) and \( a^{\lambda+\lambda'} \) instead of \( \chi(a)\chi'(a) \). Note that the absolute value satisfies \( |a^\lambda| = a^{\text{Re}(\lambda)} \) and that a quasicharacter \( \chi \) actually is a character if and only if \( \text{Re}(\chi) = 0 \).

Let \( \Delta_P : P \to \mathbb{R}_+ \) be the modular function of the group \( P \). Then the element \( \rho_P = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) satisfies \( \Delta_P(a) = |a^{2\rho_P}|. \) For \( \nu \in a^* \) and a root \( \alpha \) let

\[ \nu_\alpha = (\nu, \hat{\alpha}) \in X^*(\text{GL}_1) \otimes \mathbb{C} \cong \mathbb{C}. \]

Note that \( \nu \in a^*_0 \) implies \( \nu_\alpha \in \mathbb{R} \) for every \( \alpha \). For \( \nu \in a^*_0 \) we say that \( \nu \) is positive, \( \nu > 0 \), if \( \nu_\alpha > 0 \) for every positive root \( \alpha \).

**Example.** Let \( G = \text{GL}_n(F) \) and let \( \varpi_j \in G \) be the diagonal matrix \( \varpi_j = \text{diag}(1, \ldots, 1, \varpi, 1, \ldots, 1) \) with the \( \varpi \) on the \( j \)-th position. Let \( \nu \in a^* \) and let

\[ \nu_j = \nu(\varpi_j A_c) \in \mathbb{C}. \]
Let α be a root, say α(diag(a_1, ..., a_n)) = a_i / a_j. Then
\[ \nu_\alpha = \nu_i - \nu_j. \]

Hence \( \nu \in a_0^* \) is positive if and only if \( \nu_1 > \nu_2 > \cdots > \nu_n \).

We will fix Haar-measures of \( G \) and its reductive subgroups as follows. For \( H \subset G \) being a torus there is a unique maximal compact subgroup \( U_H \) which is open. Then we fix a Haar measure on \( H \) such that \( \text{vol}(U_H) = 1 \). If \( H \) is connected semisimple with compact center then we choose the unique positive Haar-measure which up to sign coincides with the Euler-Poincaré measure \[ \mathcal{E}. \] So in the latter case our measure is determined by the following property: For any discrete torsionfree cocompact subgroup \( \Gamma_H \subset H \) we have
\[ \text{vol}(\Gamma_H \setminus H) = (-1)^{r(H)} \chi(\Gamma_H, \mathbb{Q}), \]
where \( r(H) \) is the \( k \)-rank of \( H \) and \( \chi(\Gamma_H, \mathbb{Q}) \) the Euler-Poincaré characteristic of \( H^*(\Gamma_H, \mathbb{Q}) \). For the applications recall that centralizers of tori in connected groups are connected \[ \mathbf{1}. \]

Assume we are given a discrete subgroup \( \Gamma \) of \( G \) such that the quotient space \( \Gamma \setminus G \) is compact. Let \( (\omega, V_\omega) \) be a finite dimensional unitary representation of \( \Gamma \) and let \( L^2(\Gamma \setminus G, \omega) \) be the Hilbert space consisting of all measurable functions \( f : G \to V_\omega \) such that \( f(\gamma x) = \omega(\gamma) f(x) \) and \( |f| \) is square integrable over \( \Gamma \setminus G \) (modulo null functions). Let \( R \) denote the unitary representation of \( G \) on \( L^2(\Gamma \setminus G, \omega) \) defined by right shifts, i.e. \( R(g) \varphi(x) = \varphi(xg) \) for \( \varphi \in L^2(\Gamma \setminus G, \omega) \). It is known that as a \( G \)-representation this space splits as a topological direct sum:
\[ L^2(\Gamma \setminus G, \omega) = \bigoplus_{\pi \in \hat{G}} N_{\Gamma, \omega}(\pi) \pi \]
with finite multiplicities \( N_{\Gamma, \omega}(\pi) < \infty \).

Suppose \( \gamma \in \Gamma \) is \( G \)-conjugate to some \( a_\gamma b_\gamma \in A^{-M_{ell}} \). We want to compute the covolume
\[ \text{vol}(\Gamma_\gamma \setminus G_\gamma). \]

An element of \( \text{GL}_n(F) \) is called neat, if the subgroup of \( \bar{F}^* \) generated by its eigenvalues, is torsion-free. An element \( x \) of \( G \) is called neat if for some injective representation \( \rho : G \to \text{GL}_n(F) \) of \( G \) the matrix \( \rho(x) \) is neat. It is easy to check that in this case the same holds for every representation \( \rho \), injective or not. A subset \( A \) of \( G \) is called neat if each element of it is. If the
characteristic of $F$ is zero, then every arithmetic group $\Gamma$ has a finite index subgroup which is neat [2].

We suppose that $\Gamma$ is neat. Since $\Gamma$ is cocompact, this implies that for every $\gamma \in \Gamma$ the Zariski closure of the group generated by $\gamma$ is a torus. It then follows that $G_{\gamma}$ is a connected reductive group [1].

An element $\gamma \in \Gamma$ is called primitive if $\gamma = \sigma^n$ with $\sigma \in \Gamma$ and $n \in \mathbb{N}$ implies $n = 1$. It is a property of discrete cocompact torsion free subgroups $\Gamma$ of $G$ that every $\gamma \in \Gamma$, $\gamma \neq 1$ is a positive power of a unique primitive element. In other words, given a nontrivial $\gamma \in \Gamma$ there exists a unique primitive $\gamma_0$ and a unique $\mu(\gamma) \in \mathbb{N}$ such that

$$\gamma = \gamma_0^{\mu(\gamma)}.$$

Let $\Sigma$ be a group of finite cohomological dimension $cd(\Sigma)$ over $\mathbb{Q}$. We write

$$\chi(\Sigma) = \chi(\Sigma, \mathbb{Q}) := \sum_{p=0}^{cd(\Sigma)} (-1)^p \dim H^p(\Sigma, \mathbb{Q}),$$

for the Euler-Poincaré characteristic. We also define the higher Euler characteristic as

$$\chi_r(\Sigma) = \chi_r(\Sigma, \mathbb{Q}) := \sum_{p=0}^{cd(\Sigma)} (-1)^{p+r} \binom{p}{r} \dim H^p(\Sigma, \mathbb{Q}),$$

for $r \in \mathbb{N}$ as $\Gamma$ acts freely on the Bruhat-Tits building $B$ of $G$, which is contractible, the quotient $\Gamma \backslash B$ is a classifying space for $\Gamma$, hence the cohomological dimension of $\Gamma$ is bounded by the dimension of $B$, hence finite.

We denote by $\mathcal{E}_p(\Gamma)$ the set of all conjugacy classes $[\gamma]$ in $\gamma$ such that $\gamma$ is in $G$ conjugate to an element $a_{\gamma}m_{\gamma} \in AM$, where $m_{\gamma}$ is elliptic and $a_{\gamma} \in A^-$. Let $\gamma \in \mathcal{E}_p(\Gamma)$. To simplify the notation let’s assume that $\gamma = a_{\gamma}m_{\gamma} \in A^-Md_l$. Let $C_{\gamma}$ be the connected component of the center of $G_{\gamma}$ then $C_{\gamma} = AB_{\gamma}$, where $B_{\gamma}$ is the connected center of $M_{m_{\gamma}}$ the latter group will also be written as $M_{\gamma}$. Let $M_{\gamma}^{der}$ be the derived group of $M_{\gamma}$. Then $M_{\gamma} = M_{\gamma}^{der}B_{\gamma}$.

Let $\Gamma_{\gamma,A} = A \cap \Gamma_{\gamma}B_{\gamma}$ and $\Gamma_{\gamma,M} = M_{\gamma}^{der} \cap \Gamma_{\gamma}AB_{\gamma}$. Similar to the proof of Lemma 3.3 of [7], one shows that $\Gamma_{\gamma,A}$ and $\Gamma_{\gamma,M}$ are discrete cocompact
subgroups of $A$ and $M_{\gamma}^{\text{der}}$ resp. Let

$$\lambda_{\gamma} \overset{\text{def}}{=} \text{vol}(\Gamma_{\gamma} \backslash A).$$

**Proposition 1.1. (a)** Assume $\Gamma$ neat and let $\gamma \in \Gamma$ be $G$-conjugate to an element of $A^{-}M_{\text{ell}}$. Then we get

$$\text{vol}(\Gamma_{\gamma} \backslash G_{\gamma}) = \lambda_{\gamma} |\chi_r(\Gamma_{\gamma})|,$$

where $r = \dim A$.

(b) Let $\Gamma, \Lambda$ be of finite cohomological dimension over $\mathbb{Q}$. Let $C_r$ be a group isomorphic to $\mathbb{Z}^r$ and assume there is an exact sequence

$$1 \to C_r \to \Gamma \to \Lambda \to 1.$$

Assume that $C_r$ is central in $\Gamma$. Then

$$\chi(\Lambda, \mathbb{Q}) = \chi_r(\Gamma, \mathbb{Q}).$$

**Proof.** [4]

For a representation $\pi$ of $G$ let $\pi^{\infty}$ denote the subrepresentation of smooth vectors, i.e., $\pi^{\infty}$ is the representation on the space $\bigcup_{H \subset G} \pi^H$, where $H$ ranges over the set of all open subgroups of $G$. Further let $\pi_N$ denote the Jacquet module of $\pi$. By definition $\pi_N$ is the largest quotient $MAN$-module of $\pi^{\infty}$ on which $N$ acts trivially. One can achieve this by factoring out the vector subspace consisting of all vectors of the form $v - \pi(n)v$ for $v \in \pi^{\infty}$, $n \in N$. It is known that if $\pi$ is an irreducible admissible representation, then $\pi_N$ is a admissible $MA$-module of finite length. For a smooth $M$-module $V$ let $H^*_c(M, V)$ denote the continuous cohomology with coefficients in $V$ as in [3].

Let $\sigma$ be an element of a group $S$ acting on a finite dimensional $F$ vector space $V$. Then we write $\lambda_{\min}(\sigma|V)$ for the minimal norm of an eigenvalue of $\sigma$ in the algebraic closure $\overline{F}$ of $F$. Likewise, $\lambda_{\max}(\sigma|V)$ is the maximal norm of such an eigenvalue. The Lie algebra $\mathfrak{g}$ of $G$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}$ is the Lie algebra of $M$ and $\mathfrak{n}$ is the Lie algebra of $N$ as well as $\mathfrak{n}$ is the Lie algebra of the opposite of $N$. Then let $\tilde{M}$ denote the set of all $m \in M$ such that

$$\lambda_{\min}(m|\mathfrak{m}) > \lambda_{\max}(m|\mathfrak{m} + \mathfrak{n}).$$
Theorem 1.2. (Lefschetz Formula)
Let $\Gamma$ be a neat discrete cocompact subgroup of $G$. Let $\varphi$ be a uniformly smooth function on $A$ with support in $A^\times$. Suppose that the function $a \mapsto \varphi(a)|a^{2\rho}|$ is integrable on $A$. Let $\sigma$ be a finite dimensional unitary representation of $M$. Let $q$ be the $F$-splitrank of $G$ and $r = \dim A$. Then

$$
\sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) \sum_{q=0}^{\dim M} (-1)^q \int_{A^-} \varphi(a) \operatorname{tr}(a|H^q_c(M, \pi_N \otimes \sigma)) \, da
$$

equals
$$
\sum_{[\gamma] \in E_P(\Gamma)} \lambda_\gamma |\chi_\Gamma(\Gamma_\gamma)| \operatorname{tr} \omega(\gamma) \operatorname{tr} \sigma(m_\gamma) a^{2\rho}_\gamma \varphi(a_\gamma).
$$

Both outer sums converge absolutely and the sum over $\pi \in \hat{G}$ actually is a finite sum, i.e., the summand is zero for all but finitely many $\pi$. For a given compact open subgroup $U$ of $A$ both sides represent a continuous linear functional on the space of all functions $\varphi$ as above which factor over $A/U$, where this space is equipped with the norm $||\varphi|| = \int_A |\varphi(a)| a^{2\rho} \, da$.

Let $A^*$ denote the set of all continuous group homomorphisms $\lambda: A \to \mathbb{C}^\times$, which we write in the form $a \mapsto a^\lambda$. For $\lambda \in A^*$ and an $A$-module $V$ let $V_\lambda$ denote the generalized $\lambda$-eigenspace, i.e,

$$
V_\lambda \defeq \bigcup_{k=1}^{\infty} \{ v \in V \mid (a - a^\lambda)^k v = 0 \ \forall a \in A \}.
$$

Then

$$
\int_{A^-} \varphi(a) \operatorname{tr}(a|H^q_c(M, \pi_N \otimes \sigma)) \, da
$$

equals
$$
\sum_{\lambda \in A^*} \dim H^q_c(M, \pi_N \otimes \sigma)_\lambda \int_{A^-} \varphi(a) a^\lambda \, da.
$$

For $\lambda \in A^*$ define

$$
m^{\sigma,\omega}_\lambda \defeq \sum_{\pi \in \hat{G}} N_{\Gamma,\omega}(\pi) \sum_{q=0}^{\dim M} (-1)^q \dim H^q_c(M, \pi_N \otimes \sigma)_\lambda.
$$

The sum is always finite. The theorem is equivalent to the following Corollary.
Corollary 1.3. (Lefschetz Formula)
As an identity of distributions on $A^-$ we have
$$\sum_{\lambda \in \tilde{A}^*} m^{\sigma, \omega}_\lambda \lambda = \sum_{[\gamma] \in E_P(\Gamma)} \lambda_\gamma |\chi_\gamma(\Gamma_\gamma)| a_\gamma^{2\rho} \text{tr} \omega(\gamma) \text{tr} \sigma(m_\gamma) \delta_{u_\gamma}.$$ 

2 The zeta function

Let $q$ denote the residue field cardinality and let $r = r(P)$ and $\alpha_1, \ldots, \alpha_r$ be the simple roots. Then $\rho_P = \alpha_1 + \cdots + \alpha_r$ is the modular weight. For $a \in \tilde{A}^-$ we write
$$l_j(a) = -\log_q (a^{\alpha_j}), \quad j = 1, \ldots, r.$$ 
Then $l_j(a)$ is an integer which is equal to or bigger than zero. For $\gamma \in E_P(\Gamma)$ we also write $l_j(\gamma) = l_j(\alpha_{\gamma})$. For $u \in \mathbb{C}^n$ we write
$$u^{l(a)} = u_1^{l_1(a)} \cdots u_r^{l_r(a)}$$
and likewise $u^{l(\gamma)}$. For $u \in \mathbb{C}^r$ consider the series
$$S_{\Gamma}(u) = S_{\Gamma,P,\omega,\sigma}(u) = \sum_{[\gamma] \in E_P(\Gamma)} \lambda_\gamma |\chi_\gamma(\Gamma_\gamma)| \text{tr} \omega(\gamma) \text{tr} \sigma(m_\gamma) u^{l(\gamma)}.$$ 

Theorem 2.1. The series $S_{\Gamma}(u)$ converges locally uniformly in the set
$$\{u \in \mathbb{C}^r : |u_j| < 1, \quad j = 1, \ldots, r\}.$$ 
It is a rational function in $u$. More precisely, there exists a finite subset $F \subset \tilde{A}$, elements $a_1, \ldots, a_r \in \tilde{A}$ and natural numbers $k_1, \ldots, k_r$ as well as $n_1(v), \ldots, n_r(v)$ for each $v \in F$ such that
$$S_{\Gamma}(u) = \sum_{\lambda \in \tilde{A}^*} m^{\sigma, \omega}_\lambda \prod_{v \in F} u_1^{n_1(v)} \cdots u_r^{n_r(v)} \frac{1}{1 - a_1^{\lambda} u_1^{k_1}} \cdots \frac{1}{1 - a_r^{\lambda} u_r^{k_r}}.$$ The outer sum is finite, i.e., the coefficient $m^{\sigma, \omega}_\lambda$ is zero for almost all $\lambda \in \tilde{A}^*$. 

Proof. For \( u \in \mathbb{C} \) consider the function \( \varphi_u : A \to \mathbb{C} \) defined by

\[
\varphi_u(a) = \begin{cases} 
(uq^2)^{l(a)} = u^{l(a)}a^{-2\rho P} & a \in A^-, \\
0 & a \not\in A^-.
\end{cases}
\]

The function \( \varphi \) factors over \( \bar{A} \), therefore is uniformly smooth. It is easy to see that \( \varphi(a)|a^{2\rho P} \) is integrable on \( A \) if and only if \( |u_j| < 1 \) for every \( j = 1, \ldots, r \). Assume this, then \( \varphi_u \) satisfies the Lefschetz formula, the geometric side of which equals \( S_T(u) \). The spectral side is

\[
\sum_{\lambda \in A^*} m_{\sigma, \omega}^{\lambda} \int_A \varphi_u(a) a^\lambda \, da.
\]

Note that \( a \mapsto u^{l(a)}a^{-2\rho P} \) is the restriction of a character on \( A \) to \( A^- \) which we write as \( a \mapsto a^{s_j} \). Also, we write \( a^s \) for \( a^{s_1} \cdots a^{s_r} \).

Lemma 2.2. Let \( V \) denote a \( \mathbb{Q} \) vector space of dimension \( r \in \mathbb{N} \). Let \( V_\mathbb{R} = V \otimes \mathbb{R} \) and let \( C \subset V_\mathbb{R} \) be an open rational sharp cone with \( r \) sides, i.e., its closure \( \overline{C} \) does not contain a line and there exist \( \alpha_1, \ldots, \alpha_r \in \text{Hom}(V, \mathbb{Q}) \) such that

\[
C = \{ v \in V_\mathbb{R} : \alpha_1(v) > 0, \ldots, \alpha_r(v) > 0 \}.
\]

Let \( \Sigma \subset V \) be a lattice, i.e., a finitely generated subgroup which spans \( V \). Then there exists a finite subset \( F \subset \Sigma \) and elements \( a_1, \ldots, a_r \in \Sigma \) such that \( C \cap \Sigma \) is the set of all \( v \in V \) of the form

\[
v = v_0 + k_1a_1 + \cdots + k_ra_r,
\]

where \( v_0 \in F \) and \( k_1, \ldots, k_r \in \mathbb{N}_0 \). The vector \( v_0 \) and the numbers \( k_j \in \mathbb{N}_0 \) are uniquely determined by \( v \).

Proof. For \( j = 1, \ldots, r \) let \( a_j \in \Sigma \) be the unique element such that \( \alpha_i(a_j) = 0 \) for \( i \neq j \) and \( \alpha_j(a_j) \) is \( > 0 \) and minimal. Then \( a_1, \ldots, a_r \) is a basis of \( V \) inside \( \Sigma \), hence it generates a sublattice \( \Sigma' \subset \Sigma \). Let \( F \) be a set of representatives of \( \Sigma/\Sigma' \) which may be chosen such that each \( v_0 \in F \) lies in \( C \), but for every \( j = 1, \ldots, r \) the vector \( v_0 - a_j \) lies outside \( C \). It is clear that every \( v \) of the form given in the lemma is in \( C \cap \Sigma \).

For the converse, let \( v \in C \cap \Sigma \). Then there are uniquely determined \( v_0 \in F \), \( k_1, \ldots, k_r \in \mathbb{Z} \) such that \( v = v_0 + k_1a_1 + \cdots + k_ra_r \). We have to show that \( k_1, \ldots, k_r \geq 0 \). Assume that \( k_j < 0 \). Then

\[
0 < \alpha_j(v) = \alpha_j(v_0) + k_j\alpha_j(a_j) \leq \alpha_j(v_0) - \alpha_j(a_j) = \alpha_j(v_0 - a_j)
\]
and the latter is \( \leq 0 \), as \( v_0 - a_j \) lies outside \( C \), a contradiction! \( \square \)
We apply this lemma to $V = \bar{A} \otimes \mathbb{Q}$, the lattice $\bar{A}$ and the cone $A^-$. Writing the groups multiplicatively, we get

$$\int_{A^-} \varphi_u(a) a^\lambda \, da = \int_{A^-} a^{\lambda+s} \, da$$

$$= \sum_{y \in F} \sum_{k_1, \ldots, k_r = 0}^\infty \left( \frac{ya_1^{k_1} \cdots a_r^{k_r}}{1-a_1^{\lambda+s_1}} \cdots \frac{1}{1-a_r^{\lambda+s_r}} \right).$$

Writing $v^{\alpha_j} = q^{-n_j(v)}$ and $a_j^{\alpha_j} = q^{-k_j}$ we get the theorem. \qed

3 Geometric zeta functions

Let $G$ be a reductive linear group over a nonarchimedean local field and let $\Gamma \subset G$ be a torsion-free uniform lattice.

**Proposition 3.1.** (a) Every $\gamma \in \Gamma \setminus \{1\}$ closes a geodesic in $\mathcal{B}$.

(b) This sets up a bijection

$$\psi : (\Gamma \setminus \{1\})/\text{conjugation} \to \{\text{closed geodesics}\}/\text{homotopy}$$

with the property that

$$\psi([\gamma^n]) = \psi([\gamma])^n$$

for every $\gamma \in \Gamma \setminus \{1\}$ and every $n \in \mathbb{N}$.

(c) If two closed geodesics $c, c'$ in $\Gamma \setminus \mathcal{B}$ are homotopic, then there are preimages $\tilde{c}, \tilde{c}'$ in $\mathcal{B}$ which are closed by the same $\gamma \in \Gamma$.

(d) For a given $\gamma \in \Gamma$ let

$$P_\gamma = \{ x \in \mathcal{B} : d(x, \gamma x) \text{ is minimal} \}.$$  

Then $P_\gamma$ is a convex subset of the building $\mathcal{B}$ which is a union of parallel geodesics and $\gamma$ acts by translation along these geodesics. The set $P_\gamma$ equals the set of all geodesics in $\mathcal{B}$ which are closed by $\gamma$. Consequently, the closed geodesics closed by a given $\gamma$ all have the same length.
Proof. (a) Let $\Gamma \in \Gamma \setminus \{1\}$. As $\Gamma$ is torsion-free, the element $\gamma$ has no fixed point in $B$. As $\gamma$ preserves the simplicial structure on $B$, the function $p \mapsto d(p, \gamma p)$ attains a minimal value $m > 0$. The set $P = P_{\gamma}$ defined above therefore is well-defined and non-empty. We first claim that $P$ is a union of $\gamma$-stable geodesic lines on each of which $\gamma$ acts by a translation. So let $p \in P$ and let $z$ be in the line segment between $p$ and $\gamma p$. Then we have

$$d(z, \gamma z) \leq d(z, \gamma p) + d(\gamma p, \gamma z)$$
$$= d(z, \gamma p) + d(p, z)$$
$$= d(p, \gamma p) = m.$$  

As $m$ is minimal, we have equality and the geodesic from $z$ to $\gamma z$ is the composite of $\overline{z, \gamma p}$ and $\overline{\gamma p, \gamma z}$, which means that the line segment $\overline{p, \gamma z}$ is geodesic. We repeat this construction with $z$ in place of $p$ and in this way extend $\overline{p, \gamma p}$ to a geodesic line which is preserved by $\gamma$ and on which $\gamma$ acts by translation. This proves (a) and parts of (d).

(b) As $\Gamma$ is the fundamental group of $B_{\Gamma} = \Gamma \setminus B$ we have a natural bijection

$$\Gamma / \text{conjugation} \rightarrow [S^1, B_{\Gamma}],$$

where the right hand side is the set of free homotopy classes of loops. Also, there is a trivial injection

$$\{\text{closed geodesics} \}/\text{homotopy} \rightarrow [S^1, B_{\Gamma}].$$

These maps compose to give the desired injective map

$$\psi : \{\text{closed geodesics} \}/\text{homotopy} \hookrightarrow \Gamma / \text{conjugation}.$$

By the first part, the image of this map is $\Gamma \setminus \{1\} / \text{conjugation}$.

(c) Let $\gamma$ and $\gamma'$ be elements of $\Gamma$ closing some preimages $\tilde{c}$ and $\tilde{c}'$ of $c$ and $c'$. By (b), the elements $\gamma$ and $\gamma'$ must be conjugate, which means that the preimages $\tilde{c}$ and $\tilde{c}'$ can be chosen in a way that $\gamma = \gamma'$.

(d) We already know that $P_{\gamma}$ is a union of geodesic lines. By construction, for $p \in P$, the convex hull $L_p$ of the set $\gamma^x p$ is the unique geodesic line closed by $\gamma$ and containing $p$. Now let $q$ be another point of $P$, then the distance of point on the geodesic line $L_q$ to the line $L_p$ is bounded, which can only happen if the two geodesics $L_p$ and $L_q$ lie in a common apartment and are parallel in that apartment. The convex hull of these two lines is preserved.
by \( \gamma \) and as \( \gamma \) is a translation on both lines, it is a translation on this convex hull. This proves the convexity of \( P_\gamma \).

Now finally, let \( L \) be any geodesic which is closed by \( \gamma \) and let \( z \) be a point of \( L \). Let \( p \) be a point of \( P_\gamma \), then again the lines \( L \) and \( L_p \) are parallel and thus lie in the same apartment, \( \gamma \) must act by the same translation and thus \( L \) belongs to \( P_\gamma \).

**Lemma 3.2.** Assume that \( \Gamma \) is torsion-free and let \( \gamma \in \Gamma \). Let \( S \subset B \) be a \( \Gamma \)-stable affine subset. Then there can be found an origin \( 0 \) in \( S \), a linear orthogonal transformation \( T : S \to S \) and a point \( b \in S \setminus \{0\} \) with \( Tb = b \) such that \( \gamma x = Tx + b \).

**Proof.** As \( \gamma \) fixes the euclidean structure on \( S \), it acts, after choosing an arbitrary origin, as \( \gamma x = Tx + b \) for some linear orthogonal \( T \) and some \( b \in S \). Let \( U \) be the eigenspace of the eigenvalue 1 for \( T \) and let \( V \) be its orthocomplement. We have the orthodecomposition \( b = Bu + bv \). As \( 1 - T : V \to V \) is surjective, there exists \( v_0 \in V \) with \( (1 - T)v_0 = bv \), or \( \gamma v_0 - b = Tv_0 = v_0 - bv \), which amounts to \( \gamma v_0 = v_0 + bv \). Since \( \Gamma \) is torsion-free, \( \gamma \) fixes no point in \( B \) and so \( bv \neq 0 \). Relocating the zero to the point \( v_0 \) gives the claim. \( \square \)

An element \( g \) of \( G \) is called **admissible**, if there exists a parabolic group \( P = LN \) defined over \( F \), such that \( g \) lies in \( A^\text{reg}_L M_L \). As the group \( A^\text{reg}_L M_L \) has finite index in \( L \) and there are only finitely many conjugacy classes of parabolic subgroups, there exists \( N \in \mathbb{N} \) such that \( g^N \) is admissible for every semisimple, non-elliptic element \( g \). A subgroup \( \Gamma \subset G \) is called admissible, if every \( \gamma \in \Gamma \setminus \{1\} \) is.

For simplicity of exposition, we will now assume that \( G \) is simple, which implies that the Bruhat-Tits building \( B \) is a simplicial complex. Let \( r \in \mathbb{N} \). By an \( r \)-dimensional path we understand a sequence \( \ldots, S_{-1}, S_0, S_1, \ldots \) of \( r \)-dimensional simplices such that \( S_j \) and \( S_{j+1} \) have a common face of dimension \( r - 1 \) for each \( j \in \mathbb{Z} \). We say that the path is **geodesic**, if there exists a geodesic line \( L \) with \( L \cup S_j \neq \emptyset \) for every \( j \in \mathbb{Z} \). Here \( S \) denotes the interior of the simplex \( S \). If this is the case, then all \( S_j \) lie in a common apartment \( A \). We say that a given \( \gamma \in \Gamma \) **closes** the path \( (S_j) \) if \( \gamma S_j = S_{j+n} \) holds for all \( j \in \mathbb{Z} \) and some \( n \in \mathbb{N} \). If this is the case, then \( \gamma \) stabilizes the union of the \( S_j \). This union lies in a common apartment, so it carries an euclidean structure, so, after fixing an origin in \( S \), the element \( \gamma \) acts as \( \gamma x = Tx + b \), where \( T \) is linear orthogonal and \( b \in S \setminus \{0\} \). Actually,
$T$ fixes $b$ and thus can be considered an orthogonal transformation of the orthogonal space of $b$.

4 PGL$_3$

The vertices of the building of $G = \text{PGL}_3(F)$ are parametrized by homothety classes of $O$-lattices in $F^3$. The group $G$ acts transitively on the latter, but the index three subgroup $G'$ of all $g \in G$ with $v_F(\det(g)) \equiv 0 \mod (3)$ has three orbits, which are given by the representatives

$$L_0 = \langle e_1, e_2, e_3 \rangle$$
$$L_1 = \langle e_1, e_2, \pi e_3 \rangle$$
$$L_2 = \langle e_1, \pi e_2, \pi e_3 \rangle$$

We say a vertex $v$ is of type $j \mod (3)$, if it is in the $G'$-orbit of $L_j$. We assume from now on, that $\Gamma$ is contained in $G'$, so that $\Gamma$ preserves types of vertices.

A geodesic $c$ in $B$ or $\Gamma \backslash B$ is called rational, if it contains a point of the zero skeleton and is called integral, if it is contained in the 1-skeleton of $B$ or $\Gamma \backslash B$. Every integral geodesic is rational. The vertices on an integral geodesic either have consecutive types $0, 1, 2$ or $2, 1, 0$. In the first case, the geodesic is called positive in the latter it is negative. The inverse of a positive geodesic is negative and vice versa. A geodesic parallel to an integral positive geodesic is also called positive.

An element $\gamma \in \Gamma \setminus \{1\}$ is called positive, if it closes a positive geodesic, i.e., if for one and thus every point $p$ in $P_{\gamma}$ the geodesic line through $\gamma^p$ is positive.

Let $C_{\text{int}}(\Gamma)$ denote the set of all integral geodesics in $\Gamma \backslash B$. Then every element of $C_{\text{int}}(\Gamma)$ is actually closed, as we show below.

For $G = \text{PGL}_3(F)$ there are three different classes of proper parabolics, $P_0$ is the group of all upper triangular matrices, then there is

$$P_1 = \begin{pmatrix} * & \; & \; \\ 0 & 0 & \; \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & * \\ 0 & \; \end{pmatrix}.$$ 

We write $P_j = L_j N_j$ for the Levi decomposition and we fix subgroups $M_j A_j \subset L_j$ as in Section [4] We choose $A_0$ to be the group of all diago-
nal matrices, $A_1$ to be the subgroup of all matrices of the form diag$(a, a, b)$ with $a, b \in F$, and $A_2$ to consist of all matrices diag$(a, b, b)$.

An element diag$(a, b, c)$ of $A_0$ is called strongly regular, if the absolute values $|a|, |b|, |c|$ are all different.

**Definition 4.1.** Let

$$Z_{1,+}(u) = \prod_c \left(1 - u^{l(c)}\right),$$

where the product is extended over all closed integral positive primitive geodesics in $\Gamma \backslash \mathcal{B}$. Here a closed geodesic $c$ is called primitive, if it is not a power of a shorter one.

**Lemma 4.2.** The infinite product $Z_{1,+}(u)$ actually is a polynomial in $u$.

*Proof.* A standard calculation shows that $Z_{1,+}(u) = \det(1 - uT)$, where $T$ is the operator on the free complex vector space generated by the edges of $\Gamma \backslash \mathcal{B}$ which is defined by $T(e) = \sum_{e'} e'$, where the sum ranges over all edges connected to $e$ such that the path $ee'$ is positive. \hfill $\square$

**Definition 4.3.** Let

$$Z_{2,+}(u) = \prod_p \left(1 - u^{l(p)}\right),$$

where the product ranges over all positive primitive closed geodesic paths in $\Gamma \backslash \mathcal{B}$ of dimension 2 and the length is the number of chambers such a path contains.

Similar to the above, it can be shown that $Z_{2,+}(u)$ is a polynomial.

**Theorem 4.4.** After replacing the group $\Gamma$ with a finite index subgroup, we have the identity of rational functions,

$$\frac{Z_{2,+}(u)}{Z_{1,+}(u^2)} = \exp \left(- \int_0^u S_{\Gamma,P_1}(z) \, dz\right).$$

Or, otherwise stated, $S_{\Gamma,P_1}(u) = \frac{F'}{F}(u)$, where $F(u) = \frac{Z_{1,+}(u^2)}{Z_{2,+}(u)}$.

For the proof of the theorem, we will need the following lemma.
Lemma 4.5. After replacing the group $\Gamma$ with a finite index subgroup, we can assume $\Gamma$ to be regular in the sense that every $\gamma \in \Gamma \setminus \{1\}$ lies in the regular set $G^{\text{reg}}$.

Proof. By Margulis’s arithmeticity result we know that $\Gamma$ is arithmetic, so there exists a global field $\kappa$, of which $F$ is a local completion, and a division algebra $M$ over $\kappa$ of degree 3, which splits at $F$, such that $\Gamma$ is commensurable with the image of $M(\mathcal{O})^\times$ in $G(F)$, where $\mathcal{O}$ is some order in $\kappa$. Replacing $\Gamma$ by a finite index subgroup, we may assume that $\Gamma$ lies in that image. For a given $\gamma \in \Gamma \setminus \{1\}$ fix a preimage $\tilde{\gamma} \in M(\mathcal{O})$. The centralizer $M_{\tilde{\gamma}}$ of $\tilde{\gamma}$ in $M$ is a proper subalgebra, whose degree must divide the degree of $M$, which is a prime, therefore the degree of $M_{\tilde{\gamma}}$ is one, so $M_{\tilde{\gamma}}$ is a field, hence commutative and so is $G_{\gamma}$ which is the image of $M_{\tilde{\gamma}}(F)$. Therefore $\gamma$ is regular. $\square$

Proof of the theorem. By the lemma we can assume $\Gamma$ to be regular. In this case, each centralizer $G_{\gamma}$ is a torus, so $\Gamma_{\gamma}$ will be isomorphic to $\mathbb{Z}$, so that $\chi_1(\Gamma_{\gamma}) = 1$. By the normalizations of Haar measures we see that $\lambda_{\gamma} = l(\gamma_0)$, where $\gamma_0$ is the underlying primitive element. Thus the Selberg zeta function equals

$$S_{\Gamma, P_1}(u) = \sum_{[\gamma] \in E_{P_1}(\Gamma)} l(\gamma_0) u^{l(\gamma_)}.$$ 

So that for small enough $u$,

$$\exp \int_0^u S_{\Gamma, P_1}(z) \, dz = \exp \sum_{[\gamma]} l(\gamma_0) \frac{1}{l(\gamma)} u^{l(\gamma)}$$

$$= \exp \sum_{[\gamma_0]} \sum_{n=1}^{\infty} \frac{u^{l(\gamma_0)n}}{n}$$

$$= \exp \left( - \sum_{\gamma_0} \log(1 - u^{l(\gamma_0)}) \right)$$

$$= \prod_{[\gamma_0] \in E_{P_1, \text{prim}}(\Gamma)} \left( 1 - u^{l(\gamma_0)} \right)^{-1},$$

where the product extends over all primitive elements in $E_{P_1}(\Gamma)$. Taking inverses, it remains to show

$$\frac{Z_{1,+}(u^2)}{Z_{2,+}(-u)} = \prod_{[\gamma_0] \in E_{P_1, \text{prim}}(\Gamma)} \left( 1 - u^{l(\gamma_0)} \right).$$
To prove this, we will make use of the following phenomenon: If \( p \) is a closed gallery path in \( \Gamma \backslash B \), then the boundary of \( p \) consists of two or one closed integral geodesics, depending on whether \( p \) is orientable or not. In the orientable case, the length of \( p \) will be twice the length of either of the geodesics, so the contribution of \( p \) to the product \( Z_{1,+}(u) \) will equal the contribution of either of the two geodesics in \( Z_{1,+}(u^2) \). The minus sign will not play a role as the length of the gallery path is even. In the non-orientable case, one gets only one closed geodesic and this has the same length as \( p \), which is an odd number and one gets the contribution \( \frac{1-u^{2l(\gamma)}}{1+u^{l(\gamma)}} = 1-u^{l(\gamma)} \).

This kind of reduction is used in the sequel.

Start with a positive closed primitive integral geodesic \( c \), choose a preimage \( \tilde{c} \) in \( B \) and let \( \gamma \in \Gamma \) be closing \( \tilde{c} \).

**First case.** Assume that \( G_\gamma \) is a split torus. Then \( \gamma \) induces a translation on the apartment \( S \) attached to \( G_\gamma \). This apartment therefore lies in \( P_\gamma \). It follows that \( \gamma \) is primitive. We let \( \mathbb{R} \) act on \( P_\gamma \) via the geodesic action and see that \( P_\gamma / \mathbb{R} \) is a tree which contains a line \( L \). We claim that the structure of this tree is as such that \( P_\gamma / \mathbb{R} \) is a union of disjoint finite trees. This is a consequence of the fact that \( \Gamma_\gamma \backslash P_\gamma \) is compact. So, modulo geodesic gallery paths, one can reduce each of the finite trees to a point and be reduced to the apartment \( S \). The image of \( S \) in \( \Gamma \backslash B \) is a union of closed geodesics or of closed gallery paths and both occur in the same number, so that they cancel in the quotient \( \frac{Z_{1,+}(u^2)}{Z_{2,+}(u)} \).

**Second case.** If \( G_\gamma \) is a non-split torus, then \( P_\gamma \) will not contain an apartment. Then \( P_\gamma / \mathbb{R} \) is compact and contains \( P_{\gamma_0} / \mathbb{R} \), where \( \gamma_0 \) is the primitive underlying \( \Gamma \). Modulo gallery paths, one reduces to \( P_{\gamma_0} \), and we have two situations. The first is that \( P_{\gamma_0} \) contains an integral geodesic, so we can reduce to that one and get one remaining contribution of the form \( (1-u^{2l(\gamma_0)}) \). If \( P_{\gamma_0} \) does not contain an integral geodesic, this implies that \( P_\gamma \) is a single line going through the interior of a gallery path, which is not closed by \( \gamma \), but by \( \gamma^2 \). In the quotient, this is exact the non-orientable case and the argument given above proves the Theorem.

\[ \square \]

### 5 Riemann Hypothesis

Recall that \( Z_{1,+}(u) \) and \( Z_{2,+}(u) \) are polynomials so that \( Z_{1,+}(u) = \det(I - L_{E}u) \) for some parahoric Hecke operator \( L_{E} \); \( Z_{2,+}(u) = \det(I - L_{B}u) \) for
some Iwahori Hecke operator $L_B$ \[5\]. Given a smooth unramified representation $V$ of $G$, consider
\[
Q(V, u) = \frac{\det(I + L_B u)}{\det(I - L_E u^2)}
\]
where the determinant take over on parahoric and Iwahori fixed vectors of $V$ respectively. Then we have
\[
\frac{Z_{2,+}(-u)}{Z_{1,+}(u^2)} = \prod_V Q(V, u)^{m_V}
\]
where $V$ runs through all irreducible unitary Iwahori-spherical subrepresentations of $L^2(\Gamma\backslash G)$ and $m_V$ is its multiplicity. From Table 1 and Table 2 in \[5\], we have

(a) If $V$ is a principal series representation, then $Q(V, u) = 1$.

(b) If $V$ is the trivial representation twisted by a cubic unramified character $\chi$ of $F$, then $Q(V, u) = \frac{1}{1-\chi(\pi)u}$ and $m_V = 1$.

(c) If $V$ is the Steinberg representation twisted by a cubic unramified character $\chi$ of $F$, then $Q(V, u) = 1 - \chi(\pi)u$ and $m_V = \chi(X_\Gamma) - 1$.

(d) If $V$ the irreducible subrepresentation of $\text{Ind}(\chi|^{-1/2}, \chi|^{1/2}, \chi^{-2})$, where $\chi$ is an unramified unitary character of $F^\times$. Then $Q(V, u) = 11 - q^{1/2}\chi(\pi)u$. Moreover, $V$ is not tempered.

(e) The irreducible subrepresentation of $\text{Ind}(\chi|^{1/2}, \chi|^{-1/2}, \chi^{-2})$, where $\chi$ is an unramified unitary character of $F^\times$. Then we have $Q(V, u) = 1 - q^{1/2}\chi(\pi)u$.

We summarize the above in the following theorem

**Theorem 5.1.** $\frac{Z_{2,+}(-u)}{Z_{1,+}(u^2)} = \frac{(1-u^3)^{\chi-1}P_1(u)}{(1-q^3u^3)P_2(u)}$ where $P_1(u) = \prod_\alpha (1 - \alpha u)$ and $P_2(u) = \prod_\beta (1 - \beta u)$ with $|\alpha| = |\beta| = q^{1/2}$.

**Corollary 5.2.** When $X_\Gamma$ is a Ramanujan complex so that all irreducible unramified subrepresentations of $L^2(\Gamma\backslash G)$ are tempered, then
\[
\frac{Z_{2,+}(-u)}{Z_{1,+}(u^2)} = (1-u^3)^{\chi} \frac{P_1(u)}{(1-u^3)(1-q^3u^3)}
\]
where $P_1(u) = \prod_\alpha (1 - \alpha u)$ with $|\alpha| = q^{1/2}$ of degree $N_1 - 3N_0 + 6$. Here $N_i$ is the number of $i$-simplex in $X_\Gamma$. 
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