MOTIVIC INVARIANTS OF \( p \)-ADIC FIELDS

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ABSTRACT. I provide a complete analysis of the motivic Adams spectral sequences converging to the bigraded coefficients of the 2-completions of the motivic spectra \( \text{BPGL} \) and \( \text{kgl} \) over \( p \)-adic fields, \( p > 2 \). The former spectrum is the algebraic Brown-Peterson spectrum at the prime 2 (and hence is part of the study of algebraic cobordism), and the latter is a certain \( \text{BPGL} \)-module that plays the role of a “connective” motivic algebraic \( K \)-theory spectrum. This is the first in a series of two papers investigating motivic invariants of \( p \)-adic fields, and it lays the groundwork for an understanding of the motivic Adams-Novikov spectral sequence over such base fields.

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References

1. INTRODUCTION

This paper initiates a project to determine algebro-geometric invariants of \( p \)-adic fields via the methods of stable homotopy theory; it is also the first part of my thesis [Orm10]. The technology for such an endeavor resides in the Morel-Voevodsky motivic homotopy theory [MV99], and in the stabilizations thereof [Voe98, Hu03, Jar00]. The techniques here are natural generalizations of those used over an algebraically closed field in [HKO10], but the phenomena observed are far more nuanced because of the arithmetically richer input.

Presently, I will concern myself with the bigraded coefficients of \( \text{BPGL} \widehat{\otimes}^2 \) (the 2-complete algebraic Brown-Peterson spectrum at the prime 2) and \( \text{kgl} \widehat{\otimes}^2 \) (the 2-completion of a “connective” motivic algebraic \( K \)-theory spectrum introduced in [3] over a \( p \)-adic field, \( p > 2 \). Sequels to this work will use these results to provide information about a motivic Adams-Novikov

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spectral sequence converging to stable motivic homotopy groups of the 2-complete sphere spectrum over a \(p\)-adic field [Orm, Orm10] and, in work with Paul Arne Østvær, motivic invariants of the rational numbers [OØ]. My main computational tool in all cases is the motivic Adams spectral sequence, whose construction is outlined below and whose convergence properties are detailed in [HKO].

My grading conventions will follow those in [HKO10], where the \((m + n\alpha)\)-sphere \(S^{m+n\alpha}\) is the smash product \((S^1)^\wedge m \wedge (\mathbb{A}^1 \setminus 0)^\wedge n\). The wildcard \(*\) will refer to bigradings of the form \(m + n\alpha, m, n \in \mathbb{Z}\), in all motivic contexts, and if \(E\) is a motivic spectrum then its (bigraded) coefficients are \(E_* = \pi_*E\).

In §3, I define and establish basic properties of \(\text{BPGL}\) and \(kgl\) and identify their mod 2 motivic homology as comodules over the dual motivic Steenrod algebra \(A_*\). In §4, I run appropriate filtration spectral sequences that determine the \(E_2\)-terms of the motivic Adams spectral sequences converging to 2-complete coefficients. I then analyze these spectral sequences in §5 in order to fully determine the bigraded coefficient rings \((\text{BPGL}^\wedge 2)_*\) and \((kgl^\wedge 2)_*\). The former computation, combined with motivic Landweber exactness, permits a description of the \(\text{BPGL}^\wedge 2\) Hopf algebroid producing a computation of the \(E_2\)-term of the motivic Adams-Novikov spectral sequence over a \(p\)-adic field; see Theorem 5.11.

In order to follow this program, I use the rest of this introduction to review fundamental input from motivic cohomology and sketch the construction of the motivic \(E\)-Adams spectral sequence, \(E\) a motivic ring spectrum; complete detail on the latter of these topics is in [HKO]. In §2 I describe my conventions for \(p\)-adic fields and review arithmetic input making explicit computations possible.

**Motivic homology and the dual motivic Steenrod algebra over a field.** I now review Voevodsky’s fundamental computations of mod 2 motivic (co)homology and the (dual) motivic Steenrod algebra. By the Milnor conjecture, these are computable in terms of Milnor \(K\)-theory. In §2 I will produce explicit computations of these objects over \(p\)-adic fields using the definitions and theorems below.

Milnor defines and determines the basic properties of what is now called Milnor \(K\)-theory in his seminal paper [Mil70]. The Milnor \(K\)-theory of a field \(k\) is the graded algebra

\[
K_*^M(k) = T(k^\times)/(a \otimes (1-a) | a \neq 0, 1)
\]

where \(T(k^\times)\) is the tensor algebra on the Abelian group \(k^\times\). For obvious reasons, it is easy to confuse the group operation on \(k^\times\) and the tensor operation in \(K_*^M(k)\). Hence it is convenient to consider the \(n\)-th Milnor \(K\)-group to be generated by “Milnor symbols” \(\{a_1, \ldots, a_n\}, a_i \in k^\times\). Here \(\{} : k^\times \to K_1^M(k)\) is an isomorphism such that \(\{ab\} = \{a\} + \{b\}\) and \(\{a_1, \ldots, a_n\} = a_1 \otimes \cdots \otimes a_n\). Following common usage, let \(k_*^M(k)\) denote mod 2 Milnor \(K\)-theory \(K_*^M(k)/2\).
The main result of [Voe03a] is the following theorem.

**Theorem 1.1** ([Voe03a]). Mod 2 motivic cohomology of Spec(\(k\)) takes the form

\[ H^*({\text{Spec}(k)}; \mathbb{Z}/2) = k^M_*(k)[\tau] \]

where \(|k^M_1(k)| = \alpha\) and \(|\tau| = -1 + \alpha\).

In [Voe03b] and [Voe], Voevodsky determines the stable operations on mod 2 motivic cohomology. Let \(H\) denote the mod 2 motivic cohomology \(k\)-spectrum, where \(k\)-spectra are the stable objects of [Voe98] over the base field \(k\).

**Definition 1.2.** The motivic Steenrod algebra is the algebra of stable operations on \(H\),

\[ A^* := H^*H. \]

Voevodsky shows in [Voe03b] that \(A^*\) contains the Bockstein \(\beta\) and power operations \(P^i\), but he does not prove that they generate all of \(A^*\) in that paper. [Voe] fills the gap and proves the following structure theorem about \(A^*\).

**Theorem 1.3** ([Voe03b, Voe]). The motivic Steenrod algebra is generated by \(\beta\) and \(P^i, i \geq 0\).

\(A^*\) has the structure of a Hopf algebroid over \(H^*\). (See [Rav86, Appendix A1] for the theory Hopf algebroids.) In this paper, I will be more concerned with the dual to the motivic Steenrod algebra, \(A_* = H_*H\) which is a Hopf algebroid over \(H_* = H^{-*}\).

**Theorem 1.4** ([Voe03b, Voe]). The dual motivic Steenrod algebra is a commutative free \(H_*\)-algebra isomorphic to

\[ H_*[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_1^2 - \tau_0 \xi_1 + \tau_0 \xi_{i+1} - \rho(\tau_{i+1} + \tau_0 \xi_{i+1})). \]

Here, by a standard abuse of notation, \(\tau \in H_{1-\alpha}\) is the dual of \(\tau \in H^{-1+\alpha}\), \(\rho\) is the class of \(-1\) in \(H_{-\alpha} = k_1^M(k) = k^\times/(k^\times)^2, \ |\tau| = (2^i - 1)(1 + \alpha) + 1,\) and \(|\xi_i| = (2^i - 1)(1 + \alpha)|.\)

We will also need to know the Hopf algebroid structure of \(A_*\) which is given in the following theorem.

**Theorem 1.5** ([Voe03b, Voe]). In the Hopf algebroid \(A_*,\) the \(\xi_i\) and \(\tau_i\) support the same structure maps as in topology, elements of \(H_{0+\alpha} = k_1^M(k)\) are primitive, and \(\tau\) is not primitive in general. In particular, \(A_*\) has the following structure:

\[ \eta_L \tau = \tau \]
\[ \eta_R \tau = \tau + \rho \tau_0 \]

\[ \Delta \xi_k = \sum_{i=0}^{k} \xi_{k-i} \otimes \xi_i \]

(1)

\[ \Delta \tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i} \otimes \tau_i. \]
Remark 1.6. It follows that $\mathcal{A}^*$ has a Milnor basis of elements of the form $Q_f(r_1, \ldots, r_n)$ as in topology with the degree shift $|Q_n| = 2^n(1 + \alpha) - \alpha$; see [Bor03].

Certain quotient Hopf algebroids of $\mathcal{A}$ will be useful in my analysis of $\text{MGL}$ (see §3). The following definition is due to Mike Hill.

Definition 1.7 ([Hill]). Let $\mathcal{E}(n), 0 \leq n < \infty$, denote the quotient Hopf algebroid

$$\mathcal{E}(n) := \mathcal{A}_*/((\xi_1, \xi_2, \ldots, \tau_{n+1}, \tau_{n+2}, \ldots)) = H_*[\tau_0, \ldots, \tau_n]/(\tau_i^2 - \rho \tau_{i+1} \mid 0 \leq i < n) + (\tau_1^2).$$

If $n = \infty$, let

$$\mathcal{E}(\infty) := \mathcal{A}_*/((\xi_1, \xi_2, \ldots)) = H_*[\tau_0, \tau_1, \ldots]/(\tau_i^2 - \rho \tau_{i+1} \mid 0 \leq i).$$

The Hopf algebroid $\mathcal{E}(n)$ is dual to the sub-Hopf algebroid of $\mathcal{A}^*$ generated by the Milnor primitives $Q_i, i \leq n$.

Motivic Adams-type spectral sequences. I now introduce my main computational tools, motivic analogues of Adams-type spectral sequences from topology. The content here will only sketch their construction and properties; see [HKO] for complete detail and [HKO10, D1] for work over algebraically closed fields with these spectral sequences. For the background and applications of the Adams spectral sequence in topology, see [Ada74, Rav86].

Fix a (not necessarily highly structured) ring $k$-spectrum $E$ and a $k$-spectrum $X$. The canonical $E$-Adams resolution of $X$ is

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

in which $K_s = E \wedge X_s$ and $X_{s+1}$ is the homotopy fiber of the map $X_s \to K_s$. The associated fibration long exact sequences patch together to produce the exact couple inducing the spectral sequence.

To be precise, the fiber sequences $X_{s+1} \to X_s \to K_s$ induce long exact sequences in stable motivic homotopy groups

$$\cdots \to \pi_{m+1+n\alpha}K_s \to \pi_{m+n\alpha}X_{s+1} \to \pi_{m+n\alpha}X_s \to \pi_{m+n\alpha}K_s \to \cdots.$$
Let \( D_{s,m}^{s,m+n\alpha} = \pi_{m+n\alpha-s} X_s \) and let \( E_{s,m}^{s,m+n\alpha} = \pi_{m+n\alpha-s} K_s \). An exact couple

\[
\begin{array}{c}
D_1 \\
↓ i_1 \\
↓ k_1 \\
E_1 \\
\end{array}
\begin{array}{c}
\rightarrow D_1 \\
↑ j_1 \\
↓ k_1 \\
\end{array}
\]

is formed by letting \( i_1 \) be induced by \( X_{s+1} \rightarrow X_s \), \( j_1 \) be induced by \( X_s \rightarrow K_s \), and \( k_1 \) be induced by \( \partial \).

**Definition 1.8.** The spectral sequence induced by the above exact couple is the *motivic E-Adams spectral sequence*, or motivic E-ASS, for \( X \).

Note that the motivic E-ASS is tri-graded. I will denote the \( r \)-th page of the E-ASS by \( E_r^{s,*} \), where the first * is an integer called the *homological degree*, and the second * is a bigrading of the form \( m + n\alpha \) called the *motivic degree*. For a tri-grading \( (s, m + n\alpha) \), I call the bigrading \( m + n\alpha - s = (m - s) + n\alpha \) the *total motivic degree* or *Adams grading*; sometimes Adams grading will also refer to the tri-degree \( (s, m + n\alpha - s) \). The differentials in the motivic E-ASS take the form

\[ d_r : E_r^{s,m+n\alpha} \rightarrow E_r^{s+r,m+n\alpha+r-1} \]

In other words, the \( r \)-th differential increases homological degree by \( r \) and decreases Adams grading by 1.

In order to describe the convergence properties of the motivic E-ASS, I first must digress and introduce the concept of completion in \( Spt(k) \) and \( SH(k) \). Experts will recognize the completion I am interested in as a type of Bousfield localization. The machinery of Bousfield localization for topological spectra was introduced in [Bou79]. Hirschhorn expands on the theory and generalizes it to cellular model categories in [Hir03]. The application of this theory in the motivic context has been developed by Hornbostel in [Hor06].

Let \( \mathbb{1} \) denote the \( S^{1+\alpha} \)-stable motivic sphere spectrum, and let \( \mathbb{1}/2 \) denote the motivic mod 2 Moore spectrum, i.e., the cofiber of \( 2 : \mathbb{1} \rightarrow \mathbb{1} \).

**Definition 1.9.** The 2-completion of a k-spectrum \( X \), \( X_{2} \), is the Bousfield localization \( L_{1/2} X \) of \( X \) with respect to the class of stable cofibrations of k-spectra \( E \rightarrow F \) which become stable motivic equivalences after smashing with \( 1/2 \).

The homotopy groups of a 2-completion are related to the homotopy groups of the original k-spectrum by the following short exact sequence.

**Theorem 1.10 ([HKO10]).** If \( X \) is a k-spectrum and \( X_{2} \) its 2-completion, then the motivic stable homotopy groups of \( X \) fit into split short exact sequences

\[ 0 \rightarrow \text{Ext}(\mathbb{Z}/2^\infty, \pi_{m+n\alpha}X) \rightarrow \pi_{m+n\alpha}X_{2} \rightarrow \text{Hom}(\mathbb{Z}/2^\infty, \pi_{m-1+n\alpha}) \rightarrow 0 \]

for all \( m, n \in \mathbb{Z} \). \( \square \)
Using the language of completion, I can state the following analogue of [Rav86, Theorem 2.2.3].

**Theorem 1.11 ([HKO]).** Fix a $p$-adic field $F$ (see [2], let $E = H$ or BPGL, and let $X$ be a cell spectrum of finite type. Then the $E_2$-term of the motivic $E$-ASS is

$$\text{Ext}_{E_* E}(E_* X)$$

and the motivic $E$-ASS converges to $\pi_* X_{\hat{2}}$ where permanent cycles in tri-degree $(s, m + n\alpha)$ represent elements of $\pi_{m+n\alpha-s} X_{\hat{2}}$. \qed

**Remark 1.12.** If $E = H$, the mod 2 motivic cohomology spectrum, then the motivic $H$-Adams spectral sequence for $X$ will simply be called the the motivic Adams spectral sequence, or motivic ASS, for $X$. For $X$ cell of finite type, the motivic ASS for $X$ takes the form

$$\text{Ext}_{A_*}(H_* , H_* X) \implies \pi_* X_{\hat{2}}.$$  

If $E = BPGL$, the motivic Brown-Peterson spectrum at the prime 2, then the motivic BPGL-Adams spectral sequence for the sphere spectrum 1 will be called the motivic Adams-Novikov spectral sequence, or motivic ANSS. The motivic ANSS takes the form

$$\text{Ext}_{BPGL_* BPGL}(BPGL_* , BPGL_*) \implies \pi_* 1_{\hat{2}}.$$  

This paper concerns itself with motivic ASS computations of $\text{BPGL}_{\hat{2}}$ and $\text{kgl}_{\hat{2}}$ over $p$-adic fields. Its sequel [Orm] (which is also the second part of my thesis [Orm10]) analyzes the motivic ANSS over $p$-adic fields, in particular a motivic analogue of the alpha family in that setting. Future work with Østvær will use these computations and certain local-global principles to enact a similar program over the rational numbers [OØ].

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2. **ARITHMETIC INPUT FROM $p$-ADIC FIELDS**

The computations in this paper all occur over $p$-adic fields, $p > 2$ (see Convention 2.2), and I will use this section to survey arithmetic invariants of $p$-adic fields important to my computations in motivic homotopy theory over $p$-adic fields.

**Definition 2.1.** A $p$-adic field is a complete discrete valuation field of characteristic 0 with finite residue field.

It is well-known that every $p$-adic field is a finite extension of the $\ell$-adic rationals $\mathbb{Q}_\ell$ for some rational prime $\ell$. If $p$ is a specified rational prime, then the term “$p$-adic field” will refer to a finite extension of $\mathbb{Q}_p$. 
Convention 2.2. The structure of $p$-adic fields differs in the cases $p = 2$ and $p > 2$: for instance, $\mathbb{Q}_2$ has 8 square classes (i.e. $|\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2| = 8$) while $p$-adic fields have 4 square classes for every $p > 2$. In order to avoid a great many minor modifications, I will only deal with $p$-adic fields for which $p > 2$ in this paper. Henceforth, the term $p$-adic field will only refer to nondyadic $p$-adic fields, i.e., $p$-adic fields for which $p > 2$; moreover, the letter $F$ will always refer to a $p$-adic field unless stated otherwise.

Let $v : F \to \mathbb{Z} \cup \infty$ denote the valuation on $F$. $F$ has a ring of integers $O := \{x \in F \mid v(x) \geq 0\}$. It is trivial to check that $O$ is a domain, and $\tilde{F} = \text{Frac} O$, the field of fractions of $O$. Moreover, $O$ is a local ring with maximal ideal $m := \{x \in F \mid v(x) \geq 1\}$. A uniformizer of $F$ is an element $\pi \in F$ such that $v(\pi) = 1$; note that for any choice of uniformizer $\pi$, $(\pi) = m$.

The residue field of $F$ is

$$F := O/m.$$ 

Note that $F$ is necessarily a finite field; let $q := |F|$ and call $q$ the residue order of $F$. Of course, $q$ is a prime power $p^m$ where $F$ is a $p$-adic field.

At variance with the nomenclature of number theorists, I will call $\mathbb{F}_p$ the $\mathbb{Z}$-adic field with chosen uniformizer $\pi$.

As a consequence of Hensel’s lemma (see, e.g., [Cas86, Lemma 3.1]), the units of a $p$-adic field $F$ are equipped with a Teichmüller lift $\mathbb{F}_p^\times \hookrightarrow F^\times$.

Proposition 2.3. Let $F$ be a $p$-adic field with chosen uniformizer $\pi$. Identify $\mathbb{F}_p^\times$ with its image under the Teichmüller lift. Then

$$F^\times = \pi^\mathbb{Z} \times \mathbb{F}_p^\times \times (1 + m).$$

Proof. The result is entirely classical. Every element $x \in F^\times$ can be written in the form $\pi^{v(u)}u$ for $u \in O^\times$ of valuation 0. Observe that $O^\times/(1 + m) = \mathbb{F}_p^\times$, and the Teichmüller lift provides a splitting. \hfill $\Box$

Definition 2.4. For an arbitrary field $k$, the group of square classes of $k$ is $k^\times/(k^\times)^2$ where $(k^\times)^2 = \{x^2 \mid x \in k\}$.

Corollary 2.5. Let $F$ be a $p$-adic field with chosen uniformizer $\pi$ and choose $u$ to be a nonsquare in the Teichmüller lift $\mathbb{F}_p^\times$. The the square classes of $F$ are

$$F^\times/(F^\times)^2 = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$ 

When $q = |\mathbb{F}_p^\times| \equiv 3 \ (4)$, we may choose $u$ to be $-1$; when $q \equiv 1 \ (4)$, the image of $-1$ in the square classes of $F$ is zero. \hfill $\Box$

Every discretely valued field $(E, v)$ comes equipped with a tame symbol

$$\left(\frac{\cdot : \cdot}{E}\right) : E^\times \times E^\times \to \mathbb{F}_p^\times$$

defined by the formula

$$\left(\frac{x, y}{E}\right) = (-1)^{v(x)v(y)}x^{v(y)}y^{-v(x)} \mod m.$$
Lemma 2.6 ([Mil71, Lemma 11.5]). The tame symbol is a Steinberg symbol and hence induces a homomorphism $K_2^M(E) \to \mathbb{E}^\times = K_1^M(E)$. □

As a consequence of Lemma 2.6 and [Mil70, Example 1.7], I can determine the mod 2 Milnor $K$-theory of a $p$-adic field, a result presumably well-known to those who study such objects.

Proposition 2.7. Fix a $p$-adic field $F$, a uniformizer $\pi$, and a nonsquare $u \in F^\times$.

As a $\mathbb{Z}$-graded $\mathbb{Z}/2$-algebra,

$$k_*^M(F) = \begin{cases} 
\mathbb{Z}/2[\langle u \rangle, \langle \pi \rangle]/\langle \langle u \rangle^2, \langle \pi \rangle^2 \rangle & \text{if } q \equiv 1 \pmod{4}, \\
\mathbb{Z}/2[\langle u \rangle, \langle \pi \rangle]/\langle \langle u \rangle^2, \langle \pi \rangle (\langle u \rangle - \langle \pi \rangle) \rangle & \text{if } q \equiv 3 \pmod{4}
\end{cases}$$

where $|\langle \pi \rangle| = |\langle u \rangle| = 1$.

Proof. Abusing notation, I will write $x$ for $\{x\} \in K_1^M(F)$ or $k_1^M(F)$ whenever the context does not admit confusion.

Since $k_1^M(F)$ is the group of square classes of $F$, Corollary 2.5 implies

$$k_1^M(F) = \pi^{\mathbb{Z}/2} \times u^{\mathbb{Z}/2}.$$  

Moreover, by [Mil70, Example 1.7(2)], $k_2^M(F)$ has dimension 1 as a $\mathbb{Z}/2$-vector space. By the same reference, we also know that $K_1^M(F)$ is divisible for every $n \geq 3$, and it follows that $k_n^M(F) = 0$ for all $n \geq 3$.

We still must determine the multiplicative structure of $k_*^M(F)$, which amounts to determining the products $u^2, u\pi, \pi^2 \in k_2^M(F)$. First note that

$$\left( \begin{array}{c} u \\ \pi \\ F \end{array} \right) = (-1)^0 u^1 \pi^0 = u \in K_1^M(F),$$

which reduces to the nontrivial generator $u$ of $k_1^M(F)$. By Lemma 2.6, it follows that $u\pi \not= 0 \in k_3^M(F)$.

The argument above also proves that, after reduction mod 2, the tame symbol is an isomorphism $k_2^M(F) \to k_1^M(F)$. Hence to compute the products $u^2$ and $\pi^2$, it suffices to compute

$$\left( \begin{array}{c} u \\ u \\ F \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} \pi \\ \pi \\ F \end{array} \right).$$

These symbols are 1 and $-1$, respectively, so $u^2 = 0 \in k_1^M(F)$ while $\pi^2$ is nontrivial iff $q \equiv 3 \pmod{4}$. This determines the multiplicative structure given in (2).

As a corollary to this computation, Theorems 1.1 and 1.4 allow a complete description of the coefficients of mod 2 motivic cohomology and the dual Steenrod algebra over a $p$-adic field.

Theorem 2.8. Over a $p$-adic field $F$, the coefficients of mod 2 motivic homology are

$$H_* = k_*^M(F)[\tau]$$

where $|\tau| = 1 - \alpha$, $|k_1^M(F)| = -n\alpha$, and $k_*^M(F)$ has the form given in Proposition 2.7.
The dual motivic Steenrod algebra has the form

\[ A_\ast = H_\ast[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau \xi_{i+1} - \rho(\tau_{i+1} + \tau_0 \xi_{i+1})). \]

The class \( \rho \) is trivial iff \( q \equiv 1 \pmod{4} \). In this case,

\[ A_\ast = H_\ast[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau \xi_{i+1}) \cong A_\ast^C \otimes_{H_\ast^C} k_\ast^M(F) \]

where \( (H_\ast^C, A_\ast^C) \) is the dual motivic Steenrod algebra over \( \mathbb{C} \), which has the structure

\[ H_\ast^C = \mathbb{Z}/2[\tau], \]

\[ A_\ast^C = \mathbb{Z}/2[\tau, \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \tau \xi_{i+1}). \]

**Proof.** Most of the theorem is a concatenation of results in Theorems 1.1 and 1.4 and Proposition 2.7. The form of \( (H_\ast^C, A_\ast^C) \) is obvious after noting that \( k_\ast^M(\mathbb{C}) \) is trivial outside of degree 0. The class \( \rho \) is trivial iff \(-1\) is a square in \( \mathbb{F}_q^\times \); it is standard that this is the case iff \( q \equiv 1 \pmod{4} \). \( \square \)

The structure of \( H_\ast \) over \( F \) is depicted in Figure 1. Here the horizontal axis measures the \( \mathbb{Z} \)-component of the motivic bigrading, while the vertical axis measures the \( \mathbb{Z} \alpha \)-component. Each “diamond” shape is a copy of \( k_\ast^M(F) \), and the diagonal arrows of slope \(-1\) represent \( \tau \)-multiplication.

### 3. Comodules over the dual motivic Steenrod algebra

In this section, I work over a general characteristic 0 field \( k \). Let \( H \) denote the mod 2 motivic cohomology \( k \)-spectrum. I determine the \( A_\ast \)-comodule structure on \( H_\ast \text{BPGL} \) and \( H_\ast \text{kgl} \). For the definition of \( \text{BPGL} \), see [HK01, Vez01]; it is constructed via a motivic Quillen idempotent. The spectrum \( \text{kgl} \) is a connective variant of the algebraic \( K \)-theory \( k \)-spectrum.
KGL (see [Voe98, §6.2]) which has not been discussed in the literature previously. I begin by defining kgl as a BPGL-algebra and determining a few of its basic properties. I then use a result of Borghesi [Bor03] (a natural algebraicization of the Steenrod algebra-module structure on the mod 2 singular cohomology of the topological Brown-Peterson spectrum) to determine the \( A^* \)-comodule structure on \( H^*_{BPGL} \). The same theorem of Borghesi and techniques of Wilson [Wil75] produce the \( A^* \)-comodule structure on \( H^*_{kgl} \).

I also track similar results for a version of motivic Johnson-Wilson cohomology in a series of remarks. Throughout, I work at the prime 2.

Recall that there are canonical elements \( v_1, v_2, \ldots \in \text{BPGL}_* \) that appear in dimensions \( |v_i| = (2^i - 1)(1 + \alpha) \) [HK01]. Also let \( v_0 = 2 \in \text{BPGL}_0 \). These elements are the images of \( v_i \in \text{BP}_2(2^i - 1) \) induced by the Lazard ring isomorphism \( \text{MU}_* \to \text{MGL}_*(1+\alpha) \). As a model for “connective” algebraic \( K \)-theory, I make the following definition suggested by Mike Hopkins and communicated to me by Mike Hill.

**Definition 3.1.** Define a motivic spectrum kgl as the quotient of \( \text{BPGL} \) by \( (v_2, v_3, \ldots) \).

Note that kgl is well-defined by motivic Landweber exactness [NØS].

**Remark 3.2.** More generally, one could define the motivic Johnson-Wilson spectra \( \text{BPGL}(n) \) to be the quotient of \( \text{BPGL} \) by the ideal \( (v_{n+1}, v_{n+2}, \ldots) \), in which case kgl = \( \text{BPGL}(1) \). Note that these spectra fit into cofiber sequences

\[
\Sigma|v_n|\text{BPGL}(n) \to \text{BPGL}(n) \to \text{BPGL}(n-1).
\]

The following theorem relates the coefficients of kgl with established objects of interest, the algebraic \( K \)-groups of the ground field.

**Convention 3.3.** Throughout the rest of this section, let KGL denote the 2-localization of the full k-spectrum KGL.

**Theorem 3.4.** Let \( v_1 kgl_* \) denote the \( v_1 \)-power torsion in the coefficients of kgl, i.e., the elements \( x \in kgl_* \) such that there exists \( n \in \mathbb{N} \) such that \( v_1^n x = 0 \in kgl_* \). (I will sometimes refer to these elements simply as \( v_1 \)-torsion.) Then there is an exact sequence

\[
0 \to v_1 kgl_* \to kgl_* \to KGL_*.
\]

Moreover, if \( KGL_* \) denotes the subalgebra of \( KGL_* \) consisting of elements in degree \( m + n\alpha, m \geq 0 \), then there is a short exact sequence

\[
0 \to v_1 kgl_* \to kgl_* \to KGL_* \to 0.
\]

**Proof.** By the motivic Conner-Floyd theorem [NØS], \( KGL_* = v_1^{-1} kgl_* \). The first exact sequence is then a basic fact of localization.

Clearly, though, the map \( kgl_* \to KGL_* \) is not surjective since \( KGL_* \) is Bott = \( v_1 \)-periodic. Note, though, that \( KGL_* \) is generated by \( v_1 \) of dimension \( 1 + \alpha \) and elements of degree \( m + n\alpha, m \geq 0 \). (In fact, we could restrict the second collection of generators to degrees \( 0 + n\alpha, n \leq 0 \).) Again by the
motivic Conner-Floyd theorem, it is now a straightforward combinatorial check that $kgl_* \to KGL_*$ is surjective in dimensions $m + n\alpha$, $m \geq 0$.

Remark 3.5. The $k$-spectrum $kgl$ is connective in the sense that $kgl_{m+n\alpha} = 0$ for all $m < 0$. We will see that there is a rich class of $v_1$-torsion in $KGL_*$, so it is the case that $kgl_*$ is bigger than $KGL_*$ in its nonvanishing dimensional range. Still, producing computations of $kgl_*$ explicit enough to capture its $v_1$-torsion will determine $KGL_*$ in a meaningful dimensional range by the second exact sequence. In particular,

$$(kgl_*/v_1 kgl_*)_{m+0\alpha} = KGL_{m+0\alpha}$$

for $m \geq 0$.

I now turn to determining the $A_*$-comodule structure of $H_*BPGL$ and $H_*kgl$. To access these, I will determine the $A^*$-module structure of $H^*BPGL$ and $H^*kgl$ first.

Recall the Milnor primitives $Q_i \in A^*$, $|Q_i| = 2^i(1 + \alpha) - \alpha$ from §1. The following theorem of Borghesi should appear quite familiar to topologists.

Theorem 3.6 ([Bor03, Proposition 6]). The mod 2 motivic cohomology of $MGL$ takes the form

$$H^*MGL = (A^*/(Q_0, Q_1, \ldots))[m_i \mid i \neq 2^n - 1]$$

as an $A^*$-module where $|m_i| = i(1 + \alpha)$.

Corollary 3.7. The mod 2 motivic cohomology of $BPGL$ takes the form

$$H^*BPGL = A^*/(Q_0, Q_1, \ldots)$$

as an $A^*$-module.

Recall Definition 1.7 which defines the $A_*$-algebras $\mathcal{E}(n)$, $0 \leq n \leq \infty$. In particular, we have

$$\mathcal{E}(\infty) = A_*/(\xi_1, \xi_2, \ldots) = H_*[\tau_0, \tau_1, \ldots]/(\tau_i^2 - \rho \tau_{i+1} \mid i \geq 0),$$

$$(3)$$

$$\mathcal{E}(1) = A_*/(\xi_1, \xi_2, \ldots, \tau_2, \tau_3, \ldots) = H_*[\tau_0, \tau_1]/(\tau_0^2 - \rho \tau_1, \tau_1^2).$$

These algebras are dual to $A^*/(Q_0, Q_1, \ldots)$ and $A^*/(Q_0, Q_1)$, respectively.

There is a general yoga of passing from $A^*$-module structure on cohomologies to $A_*$-comodule structure on homologies. Applied to the above situation, I get the following theorem describing the $A_*$-comodule structure on $H_*BPGL$.

Theorem 3.8. As an $A_*$-comodule algebra,

$$H_*BPGL = A_* \square_{\mathcal{E}(\infty)} H_*.$$

□
To determine the \( \mathcal{A}^* \)-comodule structure of \( H^* \kgl \) I will first determine \( H^* \kgl \) as an \( \mathcal{A}^* \)-module and then apply the same yoga. My determination of \( H^* \kgl = H^* \BPGL \langle 1 \rangle \) is modeled on the topological calculation [Wil75]. The calculation depends on Corollary 3.7 and the following unpublished result of Mike Hopkins and Fabien Morel.

**Theorem 3.9** (Hopkins-Morel). Recall that \( \BPGL \langle 0 \rangle = \BPGL / (v_1, v_2, \ldots) \). After 2-completion, this \( k \)-spectrum is the 2-complete integral motivic cohomology spectrum,

\[
\BPGL \langle 0 \rangle \hat{\otimes} = H \mathbb{Z}_2.
\]

This enables me to make the following computation of \( H^* \kgl \) based on the argument from topology in [Wil75]. (Since the first draft of this paper was written, a similar argument has appeared in [IS, §5].)

**Theorem 3.10.** As an \( \mathcal{A}^* \)-module algebra,

\[
H^* \kgl = \mathcal{A}^* // (Q_0, Q_1).
\]

**Proof.** As in Remark 3.2, there is a cofiber sequence

\[
\Sigma^{1+\alpha} \kgl / v_{1} \to \kgl \to \BPGL \langle 0 \rangle.
\]

This induces a long exact sequence in cohomology

\[
H^0 \kgl \xrightarrow{\partial} H^{2+\alpha} \BPGL \langle 0 \rangle \to H^{2+\alpha} \BPGL \langle 1 \rangle \to H^1 \BPGL \langle n \rangle \to \cdots
\]

Note that the canonical map \( \BPGL \to \kgl \) induces, in cohomology, the map

\[
H^* \kgl \to A^* // (Q_0, Q_1, \ldots)
\]

by Corollary 3.7. By a simple dimension count, the map on coefficients \( \BPGL_n \to \kgl_n \) is an isomorphism in dimensions \( m + n\alpha \) with \( n \leq 6 - m \). It follows that the map in cohomology is an isomorphism in the same dimensional range. Now \( |Q_1| = 2+\alpha \), which falls in the dimensional range, so we know that \( Q_1(1) = 0 \in H^* \kgl \).

By the above long exact sequence, it follows that \( 1 \in H^0 \kgl \) maps to \( \lambda Q_1(1) \) in \( H^{2+\alpha} \BPGL \langle 0 \rangle \). By Theorem 3.9, \( H^* \BPGL \langle 0 \rangle = A^* // (Q_0) \), so \( \lambda Q_1(1) \neq 0 \). As in [Wil75 Proposition 1.7], it follows that

\[
H^* \kgl = \mathcal{A}^* // (Q_0, Q_1),
\]

as desired. \( \square \)

**Remark 3.11.** In general, \( |Q_n| = |v_n| + 1 \), so the cofiber sequence of Remark 3.2 allows us to run the cohomology long exact sequence argument of [Wil75 Proposition 1.7] for general \( n \). A dimensional analysis similar to the one above still works, and we can conclude that

\[
H^* \BPGL \langle n \rangle = \mathcal{A}^* // (Q_0, \ldots, Q_n)
\]

as \( \mathcal{A}^* \)-modules for all \( 0 \leq n \leq \infty \).
Since $\mathcal{E}(1)$ is dual to $A^*/(Q_0, Q_1)$, the following theorem follows immediately.

**Theorem 3.12.** As an $A_*$-comodule algebra,
\[ H_* \kgl = A_* \boxtimes_{\mathcal{E}(1)} H_* . \]

\[ \square \]

**Remark 3.13.** By Remark 3.11, it also follows that
\[ H_*^{\BPGL} = A_* \boxtimes_{\mathcal{E}(n)} H_* \]
for all $0 \leq n \leq \infty$.

### 4. Motivic Ext-Algebras

Theorems 3.8 and 3.12 identify the homology of $\BPGL$ and $\kgl$ in the category of $A_*$-comodules. By Theorem 1.11, this data forms part of the input to the $E_2$-term of the motivic ASS for $\BPGL$ and $\kgl$. In fact, both $E_2$-terms take the form
\[ \Ext_{A_*}(H_*, A_* \boxtimes_{\mathcal{E}(n)} H_*) . \]

The following change of rings theorem will identify this algebra with the cohomology of $\Ext_{\mathcal{E}(n)}(H_*, H_*)$.

**Theorem 4.1** ([Rav86, Theorem A1.3.12]). For $0 \leq n \leq \infty$, the map of Hopf algebroids $(H_*, A_*) \to (H_*, \mathcal{E}(n))$ induces an isomorphism
\[ \Ext_{A_*}(H_*, A_* \boxtimes_{\mathcal{E}(n)} H_*) \cong \Ext_{\mathcal{E}(n)}(H_*, H_*) . \]

\[ \square \]

**Remark 4.2.** While Theorem 4.1 identifies the $E_2$-term of the motivic ASS for $\BPGL$ as the cohomology of $\mathcal{E}(n)$, I will only compute this $E_2$-term in the cases $n = 1, \infty$ over a $p$-adic field $F$, i.e., the cases of $\BPGL$ and $\BPGL^{\infty}$.

Fix a $p$-adic field $F$ (see §2, especially Convention 2.2) with residue order $q$. In this section, I begin by computing $\Ext_{\mathcal{E}(\infty)}(H_*, H_*)$ over $F$; this is the $E_2$-term for the motivic ASS computing $(\BPGL)^\infty$. This work was antecedent to Hill’s paper [Hill] in which he performs similar computations over the field of real numbers $\mathbb{R}$.

Recall that when $q \equiv 1 \pmod{4}$, Remark 2.8 implies that $\Ext_{\mathcal{E}(\infty)}(H_*, H_*)$ is easily computable in terms of its complex counterpart $\Ext_{\mathcal{E}(\infty)}^c(H^C_*, H^C_*)$. In fact, $\Ext_{\mathcal{E}(\infty)}^F(H^F_*, H^F_*) = (\Ext_{\mathcal{E}(\infty)}^c(H^C_*, H^C_*) \otimes_{H^*_C} H^F_*)$ as Hopf algebroids, so, by change of base,
\[ \Ext_{\mathcal{E}(\infty)}^F(H^F_*, H^F_*) = \Ext_{\mathcal{E}(\infty)}^c(H^C_*, H^C_*) \otimes_{H^*_C} H^F_* \]
when $q \equiv 1 \pmod{4}$. Moreover, since $\rho = 0$, $\mathcal{E}(\infty)^C = \mathcal{E}(\infty)^{\text{top}} \otimes_{\mathbb{Z}/2} H^C_*$. Here $\mathcal{E}(\infty)^{\text{top}}$ is the analogous quotient of the topological dual Steenrod algebra,
but degree-shifted so that elements usually in degree $2m$ appear in dimension $m(1 + \alpha)$. Hence, again by change of base, I can compute the $E_2$-term of the motivic ASS for $\text{BPGL}_C$. To be precise,

\[
\text{Ext}_{E(\infty)^C}(H^C_*, H^C_*) = \text{Ext}_{E(\infty)_{\text{top}}}(H^\text{top}_*, H^\text{top}_*) \otimes_{\mathbb{Z}/2} H^C_*
\]

\[
= \mathbb{Z}/2[v_0, v_1, \ldots] \otimes_{\mathbb{Z}/2} H^C_*.
\]

(See [Rav86, Corollary 3.1.10] for the computation in topology.)

This yields, for $q \equiv 1 \pmod{4}$, the computation

\[
\text{Ext}_{E(\infty)^F}(H^F_*, H^F_*) = H^F_*[v_0, v_1, \ldots]
\]

where $|v_i| = (1, (2^i - 1)(1 + \alpha) + 1)$.

When $q \equiv 3 \pmod{4}$, $E(\infty)$ does not split over $E(\infty)^C$. In order to deal with the extra complexity introduced by the relation $\tau^2 = \rho \tau + 1$, I filter by powers of $\rho$ and consider the associated filtration spectral sequence [Rav86, Theorem A1.3.9]. (In [Hill], Hill refers to this spectral sequence (over $\mathbb{R}$) as the “$\rho$-Bockstein spectral sequence.”) Since $E(\infty)^F/(\rho) = E(\infty)^C \oplus \pi E(\infty)^C$, this spectral sequence takes the form

\[
E_1 = \left( \begin{array}{c}
\text{Ext}_{E(\infty)^C}(H^C_*, H^C_*) \\
\text{Ext}_{E(\infty)^C}(H^C_*, H^C_*)
\end{array} \right) \left[ (\rho)/ (\rho^2) \right] \implies \text{Ext}_{E(\infty)^F}(H^F_*, H^F_*).
\]

Since $\eta_R(\tau) - \eta_L(\tau) = \rho \tau_0$ in $E(\infty)^F$, $\tau$ supports the $d_1$-differential

\[
d_1 \tau = \rho \tau_0.
\]

Since they are on the boundary, the generators $\pi, \rho, v_0, v_1, \ldots$ do not support differentials, and we have determined the $E_2$ page of the filtration spectral sequence:

\[
E_2 = \frac{k^M_* (F)[\tau^2, v_0, \ldots]}{(\rho v_0)} \oplus \frac{\rho \tau k^M_* (F)[\tau^2, v_0, \ldots]}{(\rho^2 v_0)}.
\]

A dimensional analysis shows that there are no nontrivial $d_2$ differentials. Since $\rho^2 = 0$, the spectral sequence collapses here.

In order to fully determine $\text{Ext}_{E(\infty)}(H_*, H_*)$ when $q \equiv 3 \pmod{4}$, I must address hidden extensions in $E_2 = E_{\infty}$.

**Proposition 4.3.** For $q \equiv 3 \pmod{4}$, any hidden extensions in the associated graded $E_{\infty}$ of $\text{Ext}_{E(\infty)}(H_*, H_*)$ are contained in the ideal $(\pi \rho \tau v_1 - \tau^2 v_0)$.

**Proof.** A quick glance at tri-degrees shows that we only need concern ourselves with candidate relations of the form

\[
\pi \rho \tau^{2r+1} \prod_{a \in A} v_{i_a} - \tau^{2r+2} \prod_{b \in B} v_{j_b}.
\]

Note that $\pi, \rho, \tau$ all have Ext-degree 0 while all of the $v_i$ have Ext-degree 1. It follows that $|A| = |B|$. Computing the total dimension of both sides of
the equation, we derive the equality

\[ \sum_{a \in A} 2^{i_a} = 1 + \sum_{b \in B} 2^{i_b}. \]

It follows that the formal quotient

\[ \prod_{a \in A} v_i a \prod_{b \in B} v_j b = v_1 v_0, \]

so any extant relations are of the form

\[ \rho \pi \tau^{2r} v_1 = \tau^{2r+1} v_0. \]

This, combined with the filtration spectral sequence computation, proves the following theorem.

**Theorem 4.4.** Over a \( p \)-adic field \( F \),

\[
\text{Ext}_{A_*}(H_*, H_* \text{BPGL}) = \begin{cases} 
  k_*^M(F)[\tau, v_0, \ldots] & \text{if } q \equiv 1 \pmod{4}, \\
  k_*^M(F)[\tau^2, v_0, \ldots]/(\rho v_0) \oplus \rho \tau k_*^M(F)[\tau^2, v_0, \ldots] & \text{if } q \equiv 3 \pmod{4}
\end{cases}
\]

up to possible relations in the ideal \((\pi \rho \tau v_1 - \tau^2 v_0)\) in the \( q \equiv 3 \pmod{4} \) case. \( \square \)

**Remark 4.5.** Note that by Proposition 2.7, the algebra \( k_*^M(F) \) takes the form

\[
k_*^M(F) = \begin{cases} 
  \mathbb{Z}/2[\pi, \rho]/(\rho^2, \pi^2) & \text{if } q \equiv 1 \pmod{4}, \\
  \mathbb{Z}/2[\pi, \rho]/(\rho^2, \pi(\rho - \pi)) & \text{if } q \equiv 3 \pmod{4}
\end{cases}
\]

In the next section, I will show that any candidate relations are killed by the Adams spectral sequence for BPGL, so any extra precision in our understanding of \( \text{Ext}_{E(\infty)}(H_*, H_* \text{BPGL}) \) is not completely necessary for applications to BPGL.

The same methodology employed to determine the cohomology of \( E(\infty) \) works for \( E(1) \). I record the result in the following theorem.

**Theorem 4.6.** Over a \( p \)-adic field \( F \),

\[
\text{Ext}_{A_*}(H_*, H_* \text{kgl}) = \begin{cases} 
  k_*^M(F)[\tau, v_0, v_1] & \text{if } q \equiv 1 \pmod{4}, \\
  k_*^M(F)[\tau^2, v_0, v_1]/(\rho v_0) \oplus \rho \tau k_*^M(F)[\tau^2, v_0, v_1] & \text{if } q \equiv 3 \pmod{4}
\end{cases}
\]

up to possible relations in the ideal \((\pi \rho \tau v_1 - \tau^2 v_0)\) in the \( q \equiv 3 \pmod{4} \) case. \( \square \)
Proof. This follows the exact same structure as the $\mathcal{E}(\infty)$ case. Start by noting [Rav86, Theorem 3.1.16] that the topological $\mathcal{E}(1)$ has cohomology

$$\text{Ext}_{\mathcal{E}(1)^{\text{top}}}(H_s^{\text{top}}, H_s^{\text{top}}) = \mathbb{Z}/2[v_0, v_1].$$

When $q \equiv 1 \pmod{4}$, $\mathcal{E}(1)^F$ splits as

$$\mathcal{E}(1)^F = \mathcal{E}(1)^C \otimes_{H_*^C} H_*^F$$
$$= \mathcal{E}(1)^{\text{top}} \otimes_{\mathbb{Z}/2} H_*^F.$$

Hence, when $q \equiv 1 \pmod{4}$,

$$\text{Ext}_{\mathcal{E}(1)}(H_*, H_*) = k_*^M(F)(\pi, u)[\tau, v_0, v_1]$$

by Proposition 2.7 and Theorem 2.8.

The case $q \equiv 3 \pmod{4}$ remains and I again filter by powers of $\rho$ and run a filtration spectral sequence. The $E_1$-term is

$$E_1 = \left( \begin{array}{c}
\text{Ext}_{\mathcal{E}(1)^C}(H_*^C, H_*^C) \\
\pi \text{Ext}_{\mathcal{E}(1)^C}(H_*^C, H_*^C)
\end{array} \right)[\rho]/(\rho^2) \implies \text{Ext}_{\mathcal{E}(1)^F}(H_*^F, H_*^F).$$

Since $\eta_R(\tau) - \eta_L(\tau) = \rho \tau_0$ in $\mathcal{E}(1)^F$, $\tau$ supports

$$d_1 \tau = \rho \tau_0$$

exactly as in the $\mathcal{E}(\infty)$ case. This determines the $E_2$-term of the filtration spectral sequence to be

$$E_2 = k_*^M(F)[\tau^2, v_0, v_1]/(\rho v_0) \oplus \rho \tau k_*^M(F)[\tau^2, v_0, v_1].$$

Again, $\rho^2 = 0$ so the spectral sequence collapses here. Candidate relations are handled exactly as they are in Proposition 4.3, just with fewer $v_i$s, completing the proof of the theorem. \qed

In the next section, we will see that the candidate relations in the cohomology of $\mathcal{E}(1)$ are of no consequence after running the motivic ASS for $kgl$.

5. $(\text{BPGL}_2)_*$ AND $(\text{kgl}_2)_*$ VIA THE MOTIVIC ADAMS SPECTRAL SEQUENCE

Theorem 4.4 determines the $E_2$-term of the motivic ASS for BPGL (see also Theorem 1.11). This spectral sequence takes the form

$$\text{Ext}_{A_*}(H_*, H_*\text{BPGL}) \implies (\text{BPGL}_2)_*$$

where $\text{BPGL}_2$ is the 2-completion (i.e., Bousfield localization at $1/2$) of BPGL. My main tool in analyzing this spectral sequence is the following higher Leibniz rule due to May.
Proposition 5.1 ([May70, Proposition 6.8]). If \( x \) supports a \( d_r \)-differential, then \( x^2 \) survives to \( E_{r+1} \) and

\[
d_{r+1}x^2 = xd_r(x)v_0.
\]

A small fact about the motivic ASS for \( kgl \) is needed in order to run the motivic ASS for \( BPGL \) when \( q \equiv 3 \ (4) \). I will state this as a lemma here and prove it at the end of the section.

Lemma 5.2. In the motivic ASS for \( kgl \) over a \( p \)-adic field \( F \) with residue order \( q \equiv 3 \ (4) \), \( \tau^2 \) supports the differential

\[
d_2\tau^2 = \rho\tau v_0^2.
\]

Via a map of spectral sequences induced by the canonical map \( BPGL \to kgl \), Lemma 5.2 implies the following result.

Lemma 5.3. In the motivic ASS for \( BPGL \) over a \( p \)-adic field \( F \) with residue order \( q \equiv 3 \ (4) \), \( \tau^2 \) supports the differential

\[
d_2\tau^2 = \rho\tau v_0^2.
\]

Proof. The canonical map of \( F \)-spectra \( BPGL \to kgl \) induces a map of Adams resolutions and hence a map of motivic ASSs,

\[f : \{\text{motivic ASS for } BPGL\} \to \{\text{motivic ASS for } kgl\}.\]

On the \( E_2 \)-term for \( BPGL \) given in Theorem 4.4, the map is simply reduction by \((v_2, v_3, \ldots)\). Since \( f \) is injective in dimension \( (2, 3 - 2\alpha) \), Lemma 5.2 implies that \( d_2\tau^2 = \rho\tau v_0^2 \) in the motivic ASS for \( BPGL \) as well (when \( q \equiv 3 \ (4) \)).

The following proposition is an immediate corollary of Proposition 5.1 and Lemma 5.3.

Proposition 5.4. If \( q \equiv 3 \ (4) \), then there are differentials

\[
d_{1+\nu}\tau^{2^\nu} = \rho\tau^{2^\nu-1}v_0^{1+\nu}
\]

for all \( \nu \geq 1 \) in the motivic ASS for \( BPGL \). By the usual Leibnitz rule, this implies that the \( \tau \)-powers support differentials

\[
d_{1+\nu(s)}\tau^s = \rho\tau^{s-1}v_0^{1+\nu(s)}
\]

where \( \nu = \nu_2 \) is the 2-adic valuation on rational integers.

In the \( q \equiv 1 \ (4) \) case the \( \tau \)-powers support a similar family of differentials. To access these via the higher Leibniz rule, I must first determine what (if any) differential \( \tau \) supports. To this end, I employ the following theorem of Morel.

Theorem 5.5 ([Mor04b]). Over an arbitrary perfect field \( k \), the coefficients of algebraic cobordism \( MGL \) in dimension \( 0 - n\alpha, n \geq 0 \), are given by

\[MGL_{-n\alpha} = K^M_n(k).\]
In particular, $\text{MGL}_{-\alpha} = \mathbb{K}^\times$. Over our $p$-adic field $F$, the connectivity of $\text{MGL}$ [Mor04a] and Theorem 1.10 imply that $(\text{MGL}_2)_{-\alpha} = \mathbb{Z}_2 \{\pi\} \oplus \mathbb{Z}/2^k \{u\}$ where $k = \nu(q - 1)$. Since $v_0$, which represents $2 + \rho\eta$ in $\pi_0\mathbb{L}_2$, has image 2 in $(\text{MGL}_2)_0$, $\tau$ must support the differential

$$d_k \tau = u v_0^k;$$

otherwise, $(\text{MGL}_2)_{-\alpha}$ will not have the appropriate torsion. Again by the higher Leibniz rule, I have proved the following proposition.

**Proposition 5.6.** If $q \equiv 1 \pmod{4}$ and $k = \nu(q - 1)$, then

$$d_{k+v} \tau_2^v = u \tau^{2v-1} v_0^{k+v}$$

for all $v \geq 0$ in the motivic ASS for BPGL. This implies that the $\tau$-powers support differentials

$$d_{k+\nu(s)} \tau^s = u \tau^{s-1} v_0^{k+\nu(s)}.$$

Proof. The generators $\pi, u, \pi u, v_0, v_1, \ldots$ do not support differentials since they are on the boundary of the $E_2$-term. Propositions 5.4 and 5.6 specify the differentials on the $\tau$-powers, and this determines the entire spectral sequence. The structure of $\Gamma'$ is a consequence of the same two propositions. □

Via Propositions 5.4 and 5.6 I derive the following unified description of the abutment of the motivic ASS for BPGL.

**Theorem 5.7.** The $E_\infty$-term of the motivic ASS for BPGL over a local field $F$ is

$$\Gamma'[v_1, v_2, \ldots]$$

where $\Gamma'$ has additive structure

$$\Gamma' = \begin{cases} 
\mathbb{Z}/2[v_0] & \text{in dimension 0}, \\
\mathbb{Z}/2[v_0] \oplus \mathbb{Z}/2[v_0]/v_0^k & \text{in dimension } -\alpha, \\
\mathbb{Z}/2[v_0]/v_0^k & \text{in dimension } -2\alpha, \\
\mathbb{Z}/2[v_0]/v_0^{k+\nu(i)} & \text{in dimension } (i-1)(1-\alpha) - \epsilon\alpha \\
0 & \text{for } i \geq 1 \text{ and } \epsilon = 1 \text{ or } 2, \\
\end{cases}$$

for $i \geq 1$ and $\epsilon = 1$ or 2.

The multiplicative structure of $\Gamma'$ is specified by the following comments: $v_0$-multiplication is already captured by the above description. The summands of $\Gamma'_{-\alpha}$ are generated by $\pi$ and $u$, satisfying the relations in Proposition 2.7, in particular, $\pi u$ generates $\Gamma'_{-2\alpha}$. Let $\gamma_i$ denote an additive generator of $\Gamma'_{i(1-\alpha)-\alpha}$. Then $\pi \gamma_i$ generates $\Gamma'_{i(1-\alpha)-2\alpha}$ while $u \gamma_i = 0$. All other products in $\Gamma'$ are trivial by dimension reasons.

Proof. The generators $\pi, u, \pi u, v_0, v_1, \ldots$ do not support differentials since they are on the boundary of the $E_2$-term. Propositions 5.4 and 5.6 specify the differentials on the $\tau$-powers, and this determines the entire spectral sequence. The structure of $\Gamma'$ is a consequence of the same two propositions. □
Remark 5.8. The behavior of this spectral sequence is depicted in Figure 2. Here elements in degree \((s, m + n\alpha)\) are depicted in total motivic degree \(m + n\alpha - s\) with the homological degree suppressed. (The horizontal axis measures \(\mathbb{Z}\) while the vertical axis measures \(\mathbb{Z}\alpha\).)

The diagonal arrows in the Figure 2 represent \(v_i\)-multiplication, \(i \geq 1\). In the dimensional range shown, \(E_{v(q-1)+3} = E_\infty\). Note that the \(v_0\)-towers in all pictures actually come out of the page, as do the \(v_i\)-multiplication arrows.
A quick inspection of tri-degrees reveals that there are no hidden extensions except those created by $v_0$-multiplication. Indeed, the lines of slope 1 originating in the nonzero dimensions of $\Gamma'$ do not overlap, so we only need to worry about $v_0$-towers. Since $v_0$-represents 2 in $(BPGL_2)_{\mathbb{Z}}$, any copies of $\mathbb{Z}/2[v_0]$ produce copies of the 2-adic integers $\mathbb{Z}_2$, and any copies of $\mathbb{Z}/2[v_0]/v_0^{k+\nu(s)}$ produce copies of $\mathbb{Z}/2^{k+\nu(s)}$. This proves the following theorem.

**Theorem 5.9.** Let $w_i = 2^{k+i\nu(i)}$, $k = \nu(q - 1)$. The coefficients of the 2-complete Brown-Peterson spectrum $BPGL_2$ over a $p$-adic field $F$ are

$$(BPGL_2)_* = \Gamma[v_1, v_2, \ldots]$$

where $|v_i| = (2^i - 1)(1 + \alpha)$ and, additively,

$$\Gamma = \begin{cases} 
\mathbb{Z}_2 & \text{in dimension 0}, \\
\mathbb{Z}_2 \oplus \mathbb{Z}/w_1 & \text{in dimension } - \alpha, \\
\mathbb{Z}/w_1 & \text{in dimension } - 2\alpha, \\
\mathbb{Z}/w_i & \text{in dimension } (i - 1)(1 - \alpha) - \epsilon\alpha \text{ for } i \geq 1 \text{ and } \epsilon = 1 \text{ or } 2, \\
0 & \text{otherwise}.
\end{cases}$$

Multiplicative structure is the same as in Theorem 5.7. \hfill \Box

**Corollary 5.10.** The coefficients of the 2-complete algebraic cobordism spectrum over a $p$-adic field $F$ are

$$(MGL_2)_* = \Gamma[v_1, v_2, \ldots, u_j | j \neq 2^n - 1]$$

where $|v_i| = (2^i - 1)(1 + \alpha)$ and $|u_j| = j(1 + \alpha)$. \hfill \Box

Before moving on to the motivic ASS for $kgl$, I record here a consequence of the identification of $(BPGL_2)_*$ that will be vital to understanding the motivic ANSS over a $p$-adic field.

**Theorem 5.11.** The Hopf algebroid for $BPGL_2$ over a $p$-adic field $F$ splits as

$$(BPGL_2, BPGL_2 BPGL_2) = (BP^\wedge_2, BP^\wedge_2, BP^\wedge_2) \otimes_{\mathbb{Z}_2} \Gamma.$$ 

Moreover, the $E_2$-term of the motivic ANSS in homological degree $s$ is

$$\text{Ext}_{BPGL_2, BPGL_2}(BPGL_2, BPGL_2) = \Bigoplus_{\nu_1, \nu_2} \text{Tor}_1^{\mathbb{Z}_2} (\text{top } E_2^{s+1}, \Gamma).$$

Here $\text{top } E_2 = \text{Ext}_{BP^\wedge_2, BP^\wedge_2}(BP^\wedge_2, BP^\wedge_2)$ with elements shifted so that elements appearing in degree $(s, 2m)$ in topology appear in degree $(s, m(1+\alpha))$ motivically.

**Proof.** For typographical simplicity, I drop the 2-completion $(\ )_2$ from my notation in this proof.

The second statement is an easy consequence of the first via the cobar resolution computing $\text{Ext}_{BPGL, BPGL}(BPGL_*, BPGL_*)$ and the universal coefficient theorem.
As a consequence of motivic Landweber exactness, Naumann-Østvær-Spitzweck [NØS] deduce a splitting of the MGL Hopf algebroid as
\[(MGL_*, MGL_* MGL) = (MU_*, MU_* MU) \otimes_{MU_* MGL} MGL_*\]
where \(MU_*\) is the coefficients of (topological) complex cobordism, the Lazard ring. This splitting passes to BPGL, so
\[(BPGL_*, BPGL_* BPGL) = (BP_* BP_* BP) \otimes_{BP_* BPGL} (BP_* \otimes_{\mathbb{Z}_2} \Gamma)\]
\[= (BP_* BP_* BP) \otimes_{\mathbb{Z}_2} \Gamma.\]

This description of the \(E_2\)-term of the motivic ANSS over \(F\) already pays dividends in the form of a graded algebra with infinitely many nonzero components previously undiscovered in the stable stems \(\pi_* 1\).

**Theorem 5.12.** The algebra \(\Gamma\) is permanent and represents a copy of \(\Gamma\) in \(\pi_* 1\).

**Proof.** The elements of \(\Gamma = E_2^{0,0} \otimes \Gamma\) are on the boundary of the spectral sequence and hence permanent. A simple dimensional analysis shows that they are not the targets of any differentials. \(\square\)

We can interpret this theorem as saying that the 2-complete Hurewicz map induced by the 2-complete unit \(1_2 \to I\mathbb{Z}_2\) splits; see Remark 5.18.

I now turn to motivic ASS computations for \(kgl\).

**Proposition 5.13.** If \(q \equiv 3 \pmod{4}\), then there are differentials
\[d_{1+v} \tau^{2^v} = \rho \tau^{2^v-1} v_0^{1+v}\]
for all \(v \geq 1\) in the motivic ASS for \(kgl\). By the usual Leibniz rule, this implies that the \(\tau\)-powers support differentials
\[d_{1+\nu(s)} \tau^s = \rho \tau^{s-1} v_0^{1+\nu(s)}\]

**Proof.** This is a consequence of Lemma 5.2 and the higher Leibniz rule, Proposition 5.1. \(\square\)

**Lemma 5.14.** If \(q \equiv 1 \pmod{4}\) and \(k = \nu(q - 1)\), then \(\tau\) supports the differential
\[d_k \tau = \rho v_0^{k}\]
in the motivic ASS for \(kgl\) over \(F\).

**Proof.** This differential is the image of the analogous one in the motivic ASS for BPGL under the spectral sequence map induced by BPGL \(\to kgl\). \(\square\)

**Proposition 5.15.** If \(q \equiv 1 \pmod{4}\), then there are differentials
\[d_{k+v} \tau^{2^v} = \rho \tau^{2^v-1} v_0^{k+v}\]
for all \(v \geq 0\) in the motivic ASS for \(kgl\). By the usual Leibniz rule, this implies that the \(\tau\)-powers support differentials
\[d_{k+\nu(s)} \tau^s = \rho \tau^{s-1} v_0^{k+\nu(s)}\].
Proof. This is an immediate consequence of Lemma 5.14 and Proposition 5.1. □

Propositions 5.13 and 5.15 produce the following unified description of the abutment of the motivic ASS for \( kgl \) over \( F \).

**Theorem 5.16.** The \( E_\infty \)-term of the motivic ASS for \( kgl \) over a \( p \)-adic field \( F \) is

\[ \Gamma'[v_1] \]

where \( \Gamma' \) is as in Theorem 5.7.

**Proof.** This is a consequence of Propositions 5.13 and 5.15 along with simple dimensional accounting. □

**Theorem 5.17.** The coefficients of the 2-complete connective algebraic \( K \)-theory \( F \)-spectrum over a \( p \)-adic field \( F \) are

\[ (kgl_2^\wedge)_* = \Gamma[v_1] \]

where \( \Gamma \) is as in Theorem 5.9.

**Remark 5.18.** Note that the algebra \( \Gamma \) is the coefficients of the 2-complete integral cohomology \( F \)-spectrum \( HZ_2 \). This is a consequence of Theorem 3.9 and a simple computation with the motivic ASS for \( BPGL_\langle 0 \rangle \).

It remains to prove Lemma 5.2. This is the only part of this paper using outside calculations in higher algebraic \( K \)-theory (excluding, say, the construction of \( KGL \) via Quillen’s projective bundle formula), and it would be satisfying to be able to remove this dependancy. I have not yet discovered a method for doing so, and it is still quite interesting that these methods only require input from \( KGL_3 \) in the case \( q \equiv 3 \ 4 \).

**Proof of Lemma 5.2.** Recall from, e.g., [RW00] that \( (KGL_2)_3 = \mathbb{Z}/4 \) when \( q \equiv 3 \ 4 \). By Theorem 4.6 and dimensional accounting, \( (kgl_2^\wedge)_3 \) is generated by \( \rho \tau v_0^2 \). Since \( v_0 \) represents 2, Theorem 3.4 implies that \( \rho \tau v_1^2 v_0^2 \) is \( v_1 \)-torsion or 0, i.e., \( \rho \tau v_0^2 \) is \( v_1 \)-torsion or 0. Suppose for contradiction that \( \rho \tau v_0^2 \neq 0 \in (kgl_2^\wedge)_{1-2\alpha} \) and \( \rho \tau v_0^2 v_1^2 = 0 \). Then a differential in the motivic ASS for \( kgl \) must hit \( \rho \tau v_0^2 v_1^2 \), but by Theorem 4.6 and dimensional accounting, there are no elements of low enough homological degree in Adams grading one more than the Adams grading of \( \rho \tau v_0^2 v_1^2 \). I conclude that \( \rho \tau v_0^2 = 0 \in (kgl_2^\wedge)_{1-2\alpha} \), and the only differential capable killing \( \rho \tau v_0^2 \) is

\[ d_2 \tau^2 = \rho \tau v_0^2. \]

□

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