The incidence comodule bialgebra of the Baez–Dolan construction

Joachim Kock
Universitat Autònoma de Barcelona
kock@mat.uab.cat

Abstract

Starting from any operad $P$, one can consider on one hand the free operad on $P$, and on the other hand the Baez–Dolan construction on $P$. These two new operads have the same space of operations, but with very different notions of arity and substitution. The main result of this paper is that the incidence bialgebras of the two-sided bar constructions of the two operads constitute together a comodule bialgebra. The result is objective: it concerns comodule-bialgebra structures on groupoid slices, and the proof is given in terms of equivalences of groupoids and homotopy pullbacks. Comodule bialgebras in the usual sense are obtained by taking homotopy cardinality. The simplest instances of the construction cover several comodule bialgebras of current interest in analysis. If $P$ is the identity monad, then the result is the Faà di Bruno comodule bialgebra (dual to multiplication and substitution of power series). If $P$ is any monoid $\Omega$ (considered as a one-coloured operad with only unary operations), the resulting comodule bialgebra is the dual of the near-semiring of $\Omega$-moulds under product and composition, as employed in Écalle’s theory of resurgent functions in local dynamical systems. If $P$ is the terminal operad, then the result is essentially the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees, dual to composition and substitution of B-series in numerical analysis (Chartier–Hairer–Vilmart). The full generality is of interest in category theory. As it holds for any operad, the result is actually about the Baez–Dolan construction itself, providing it with a new algebraic perspective.
# Contents

0 **Introduction**  
0.1 Background and motivation ........................................ 3  
0.2 Present contributions ............................................. 6  

1 **Preliminaries**  
1.1 Objective combinatorics and linear algebra ...................... 9  
1.2 Groupoids and homotopy pullbacks ................................ 10  
1.3 Simplicial groupoids, decomposition spaces, and incidence coalgebras .................................. 12  
1.4 Polynomial functors .................................................. 15  
1.5 Trees ........................................................................... 17  
1.6 Polynomial monads and operads ..................................... 19  

2 **Free monads, Baez–Dolan construction, and two-sided bar construction**  
2.1 The free monad $P^*$ ................................................... 21  
2.2 The Baez–Dolan construction $P^\circ$ ................................ 23  
2.3 Two-sided bar construction (of one operad relative to another) ......  27  
2.4 Two-sided bar construction on $P^*$ ................................ 31  
2.5 Two-sided bar construction on $P^\circ$ ................................. 32  

3 **Incidence comodule bialgebra**  
3.1 Comodule bialgebras ..................................................... 34  
3.2 Main theorem, general form .......................................... 36  

4 **Locally finite version of the incidence comodule-bialgebra construction**  
4.1 Finiteness conditions and the reduced Baez–Dolan construction $P^\circ$ .......................... 41  
4.2 Further bar constructions and a general comodule construction ....... 44  
4.3 Main theorem, locally finite version ................................ 47  

5 **Examples**  
5.1 Baez–Dolan construction on categories .............................. 49  
5.2 Faà di Bruno ............................................................... 50  
5.3 Mould calculus ............................................................. 52  
5.4 B-series, and the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra ................................ 52  
5.5 Other examples ........................................................... 55  
5.6 Non-examples and outlook ............................................. 58
0 Introduction

The main result of this paper is the construction of a comodule bialgebra from the Baez–Dolan construction on any operad. Before explaining this result, we outline the motivation behind it, recall the Baez–Dolan construction itself, and say a word about objective combinatorics and homotopy linear algebra, the setting in which the results are staged.

0.1 Background and motivation

0.1.1. Contexts of comodule bialgebras. For $B$ a commutative bialgebra, a comodule bialgebra over $B$ is a bialgebra object in the braided monoidal category of (left) $B$-comodules [1], [81]. Although comodule bialgebras go back to the genesis of Hopf algebra theory in algebraic topology — the homology of any $H$-space is a comodule bialgebra over Milnor’s dual Steenrod bialgebra [83], [96] — and also have a certain history in quantum algebra [85], the recent burst in interest in them essentially comes from analysis.

— B-series [58]. First it was discovered in numerical analysis by Chartier, E. Hairer, and Vilmart [20] that Butcher series admit a second important operation after composition, namely substitution, and that the two operations interact in an interesting way. The interaction was recast in the language of combinatorial Hopf algebras by Calaque, Ebrahimi-Fard, and Manchon [11], who made explicit the comodule-bialgebra structure.

— Mould calculus [28]. Second, it was realised that the same structure plays an important role in mould calculus, Écalle’s extensive theory for dealing with ‘functions in a variable number of variables’ in local dynamical systems [38] (see also Écalle–Vallet [39] and Ebrahimi-Fard–Fauvet–Manchon [33]).

The examples from these two application areas will be recovered as special cases of the incidence comodule bialgebra of the Baez–Dolan construction.

— Regularity structures [60]. More recently, comodule bialgebras have surfaced in the algebraic renormalisation of regularity structures [59], a powerful framework for analysing singular stochastic PDEs. While so-called structure Hopf algebras (shuffle Hopf algebras and Butcher–Connes–Kreimer-like Hopf algebras) have played an important role for some time (see for example [21], [77], [57], [59]), a second bialgebra structure, the renormalisation bialgebra, was introduced recently in a landmark paper by Bruned, M. Hairer, and Zambotti [9]. A key point in their theory is that the structure Hopf algebra is a comodule bialgebra over the renormalisation bialgebra, in close analogy with the Calaque–Ebrahimi-Fard–Manchon situation. (See Manchon [80] and Foissy [47] for further algebraic treatment.)

— Moment-cumulant relations [88]. Finally, in Voiculescu’s theory of free probability [100], Speicher [93], [88] had discovered a beautiful moment-cumulant formula in terms of Möbius inversion in the incidence algebra of the noncrossing partitions lattice (analogous to Rota’s classical moment-cumulant relations in terms of the ordinary partition lattice). Ebrahimi-Fard and Patras [36], [37], inspired by methods of quantum field theory, gave a very different approach to the same formula, in terms of a time-ordered exponential coming from a half-shuffle in the double tensor algebra. Recently, Ebrahimi-Fard, Foissy, Kock, and Patras [34] uncovered the relationship between the two constructions in terms of a comodule-bialgebra structure on noncrossing partitions.

The topic is now under scrutiny in combinatorics; for a survey, see Manchon [81]. Fauvet, Foissy, and Manchon [40], motivated by mould calculus, gave a beautiful construction
of comodule bialgebras from finite topological spaces, building on [48], with important connections to quasi-symmetric functions. Comodule bialgebras play a prominent role in Foissy’s treatise [46] where it is shown how comodule bialgebras are induced by certain actions of the brace operad. In a different line of development, Carlier [13] showed that every hereditary species (in the sense of Schmitt [91]) induces a comodule bialgebra, and found a link with the Batanin–Markl operadic categories [7].

The present contribution provides a rather general construction of comodule bialgebras, from the Baez–Dolan construction on any operad.

0.1.2. The Baez–Dolan construction. In their seminal 1998 paper Higher-dimensional algebra III: n-categories and the algebra of opetopes [3], Baez and Dolan introduced the opetopes, a new family of shapes for defining weak higher categories in a uniform way, overcoming coherence problems by exploiting universal properties. The opetopes are defined inductively by a remarkable construction on operads $P \mapsto P^\circ$, now called the Baez–Dolan construction (reviewed below in Subsection 2.2). The colours of $P^\circ$ are the operations of $P$, and the algebras for $P^\circ$ are the operads over $P$. Although the construction may appear innocuous, it leads to a notable combinatorial richness: starting from nothing but the identity monad $\text{Id}$ on $\text{Set}$, it produces first the natural numbers as the operations of $\text{Id}^\circ$, then the planar rooted tree as operations of $(\text{Id}^\circ)^\circ$, and after that certain trees of trees, and trees of trees of trees... The $(n+1)$-dimensional opetopes are by definition the operations of the $n$th iterate of this construction.

The operations-to-colours shift can be described combinatorially in terms of trees as a shift from grafting onto leaves to substituting into nodes. This is the aspect of the Baez–Dolan construction explored in the present paper.

The Baez–Dolan construction has interesting connections with logic. It was recast in terms of a function replacement by Hermida, Makkai, and Power [61], [62], motivated by first-order logic with dependent sorts. Cheng [22], [23] made important contributions cleaning up the relationships between the different variants of the construction; see also Leinster [75]. Kock, Joyal, Batanin, and Mascari [71] exploited the formalism of polynomial functors to give a purely combinatorial description of the Baez–Dolan construction, which turned out to have a homological interpretation [94], and more recently found further applications to logic and computer science [43], [44], [29], [97]. The polynomial approach to operads has proven useful in homotopy theory. Batanin and Berger [6] used it to construct left-proper Quillen model structures for algebras for a large class of operads called tame polynomial monads. They show that for any polynomial monad, the Baez–Dolan construction is tame.

The present contribution exploits the polynomial formalism to add a new algebraic perspective to the landscape of the Baez–Dolan construction, by studying it in the context of objective algebraic combinatorics.

0.1.3. Objective combinatorics and homotopy linear algebra. The idea of objective combinatorics — the term is due to Lawvere — is to work with the algebra of sets instead of numbers, in order to obtain native bijective proofs, and reveal and exploit functorialities and universal properties that cannot exist at the numerical level. Joyal’s theory of species [64] is the starting point for the theory. For the sake of dealing seamlessly with symmetries of objects, it is often fruitful to upgrade further from sets to groupoids [4], [49], dealing directly with groupoids of combinatorial objects rather than with their sets of iso-classes. The numerical level is then recovered by taking homotopy cardinality.
Homotopy linear algebra [5], [50] serves as a general tool in this context by systematically replacing vector spaces by slice categories, and linear maps by linear functors. If \( I \) is a groupoid of combinatorial objects, classical algebraic combinatorics starts with the vector space \( \mathbb{Q}_{\pi_0 I} \) spanned by iso-classes of objects; the objective approach considers instead the slice category \( \text{Grpd}_{/I} \). Linear maps are replaced by spans of groupoids, and matrix multiplication by certain homotopy pullbacks.

At the objective level, everything is encoded with groupoids and maps of groupoids. Of course these groupoids and maps should not come out of the blue, they should organise themselves into nice configurations. In this respect simplicial structures are particularly nice, for their importance in homotopy theory and category theory. In a recent series of papers (starting with [51]), Gálvez, Kock, and Tonks show how certain simplicial groupoids called decomposition spaces admit the construction of incidence (co)algebras at the groupoid-slice level (and a Möbius inversion principle). This construction is a common generalisation of classical constructions of incidence (co)algebras of posets [63], [92], monoids [16], and Möbius categories [76], [27], [74], but reveals also most other coalgebras in combinatorics to be of incidence type (see for example [53] and [54]). Decomposition spaces are the same thing as the 2-Segal spaces of Dyckerhoff and Kapranov [32] (see [41] for the last piece of this equivalence), of importance in homological algebra and representation theory, notably in connection with Hall algebras and the Waldhausen \( S \)-construction [32], [31], [107], [89], [90]. The theory is now being developed in many directions.

The incidence comodule bialgebras of the present work will be constructed at the level of slice categories using the decomposition-space machinery.

0.1.4. Incidence bialgebras of operads. Central to this paper is the now-standard construction of a bialgebra from an operad (see [98], [99], and [17] for related constructions). Given an operad \( R \) (satisfying finiteness conditions, cf. 4.1.1 below), the associated bialgebra is free as an algebra on the set of iso-classes of operations. The comultiplication of an operation \( r \) is given by summing over all ways \( r \) can arise by operad substitution from a collection of operations fed into a single operation:

\[
\Delta(r) = \sum_{r = b \circ (a_1, \ldots, a_n)} a_1 \cdots a_n \otimes b.
\]

It is fruitful to break the construction into three steps [72], each of which is important in its own right, routing it through simplicial methods and homotopy linear algebra. The first step is to take the two-sided bar construction on the operad [106], a classical construction in algebraic topology and homological algebra [82]. This produces a simplicial groupoid \( X = \text{Bar}_S(R) \), which is a symmetric monoidal Segal space [72]. The second step constructs from any monoidal Segal groupoid \( X \) a bialgebra structure on the slice \( \text{Grpd}_{/X_1} \), using the machinery of decomposition spaces [51, 52]. The third step is to take homotopy cardinality [50], to recover usual bialgebras at the vector-space level.

This three-step realisation of the incidence-bialgebra construction is a main ingredient in this work, where there will be two of them in interaction.

0.1.5. Polynomial functors and trees. The operads used in this work are more precisely finitary polynomial monads over groupoids.

The notion of polynomial functor has origins in topology, representation theory, combinatorics, logic, and computer science, but the task of unifying these developments has only recently begun [55], [104]. As a first approximation, polynomial functors objectify
polynomial functions and formal power series. More generally, the theory turns out to
be a powerful toolbox for studying substitution and recursion. Upgraded to groupoids,
it can subsume the notions of species and operads [67], [56]. The more specific reason
polynomial functors are useful in the present context is that, since they are represented
by diagrams of groupoids
\[ I \leftarrow E \rightarrow B \rightarrow I, \]
many of the calculations performed at the objective level can be expressed naturally in
terms of these representing groupoids. Secondly, trees — the combinatorial substrate for
the theory of operads — can be represented by the same shape of diagram as polynomial
endofunctors [66]: I is then the set of edges, B the set of nodes, and E the edge-node
incidence — see 1.5. Altogether, the polynomial formalism helps articulate the interplay
between algebra and combinatorics.

0.2 Present contributions

0.2.1. The incidence comodule bialgebra construction, general form. The
present paper connects the developments outlined above to provide a rather general cat-
ergorical construction of comodule bialgebras, namely as the incidence bialgebras of the
Baez–Dolan construction for operads (polynomial monads over groupoids).

For any operad \( P \) one can construct two new operads: the free operad on \( P \), denoted
\( P^* \), and the Baez–Dolan construction \( P^\circ \). These operads have the same space of opera-
tions, namely the groupoid of \( P \)-trees, but behave very differently, with different notions
of colours, arity, and substitution. The free operad \( P^* \) is about grafting of trees, whereas
the Baez–Dolan construction \( P^\circ \) is about substituting trees into nodes of other trees.

Applying the two-sided bar construction to get simplicial groupoids
\( Y = \text{Bar}_S(P^*) \)
\( Z = \text{Bar}_S(P^\circ) \)
respectively, followed by the incidence bialgebra construction, one obtains
two different bialgebra structures on the same groupoid slice \( \text{Grpd}/Y_1 \simeq \text{Grpd}/Z_1 \).

The constructions are categorical and rather formal — no ad hoc choices are involved — but
at the same time all the groupoids and maps involved have compelling combinatorial
interpretations in terms of trees. These two bialgebras have the same underlying algebra,
but different comultiplications. The first version of the main result of this paper states
that these two bialgebras form a comodule bialgebra:

**Theorem 3.2.1.** For any operad \( P \), the two-sided bar constructions \( Y := \text{Bar}_S(P^*) \) and
\( Z := \text{Bar}_S(P^\circ) \) together endow the slice \( \text{Grpd}/S(\text{tr}(P)) \) with the structure of a comodule
bialgebra. Precisely, the incidence bialgebra of \( \text{Bar}_S(P^*) \) is a left comodule bialgebra over
the incidence bialgebra of \( \text{Bar}_S(P^\circ) \).

0.2.2. Proofs: homotopy pullbacks, trees, and active-inert factorisations. At
the objective level, the comodule-bialgebra axioms state that certain linear functors are
isomorphic. The linear functors are given by composites of spans of groupoids, extracted
naturally from the bar constructions. The spans are composed by pullback — this always
means homotopy pullback. In the end the proof amounts to establishing an equivalence
of groupoids between two different pullbacks, or more precisely, exhibiting a groupoid
which serves as pullback for both.

While all this is rather formal, exploiting basic properties of pullbacks, the actual
checks must of course use features specific to the situation. In the present case, the
proofs end up relying on properties of the category of trees, and in particular its active-inert factorisation system, recalled in 2.1.3. The groupoid central to the proof is formally described as having objects pairs of composable active maps of $P$-trees

$$W \to K \to T$$

where $W$ is a 2-level tree. At the same time, this groupoid has the clean combinatorial interpretation as a groupoid of ‘blobbed’ trees with a compatible cut, like this:

0.2.3. Finiteness conditions. At the objective level, no finiteness conditions are required, which is pleasant in terms of elbow room. However, in the end it is interesting to be able to take homotopy cardinality, to make contact with all the interesting mathematics taking place at the level of ordinary vector spaces. To this end it is necessary to impose finiteness conditions, and in fact some adjustments to the constructions are required, in order to keep things finite. For the incidence coalgebra of a decomposition space $X$ to admit a homotopy cardinality, it must be assumed locally finite [52], which means that the two maps $X_0 \to X_1$ and $X_1 \leftarrow X_2$ have finite homotopy fibres. For two-sided bar constructions, this in turn happens for locally finite operads [72]. For example, the terminal operad (the operad for commutative (unital) monoids) is not locally finite, whereas the terminal reduced operad (the operad for commutative not-necessarily-unital monoids is locally finite.

0.2.4. Locally finite version of the main theorem. While the free operad $P^*$ is always locally finite (cf. [72] and 4.1.2 below), the Baez–Dolan construction $P^\circ$ is never locally finite (4.1.3), and as a result its two-sided bar construction is not locally finite. Therefore one cannot just take homotopy cardinality of the general main theorem 3.2.1 to get a traditional comodule bialgebra. The second part of the paper explains how to fix this in a canonical way. The main point is to use the reduced Baez–Dolan construction $P^\circ := \overline{P^\circ}$, which is simply the Baez–Dolan construction followed by operad reduction, whereby all nullary operations are excluded. This tweak is completely analogous to subtleties regarding the most classical comodule bialgebras: for example, while one can multiply arbitrary formal power series, in order to substitute one power series into another, it must be required to have no constant term.

The two-sided bar construction on the reduced Baez–Dolan construction $P^\circ$ is locally finite (Lemma 4.1.6), and therefore admits a homotopy cardinality giving a traditional bialgebra in vector spaces. This is the motivation for replacing $P^\circ$ with $P^\circ$, but we continue to work at the objective level. The interaction between $P^*$ and $P^\circ$ is a bit more complicated now, since the two bialgebras are no longer supported at the same groupoid, and we need to consider comodules more general than just those underlying a bialgebra itself. The objective account of comodules is given by certain culf simplicial maps [101], [107], [12]. We need a third simplicial groupoid which is a comodule over $Bars(P^\circ)$ and
having the same base groupoid as $\text{Bar}_S(P^*)$. This is achieved through another relative two-sided bar construction (of $P^*$ relative to $P^\circ$), which also has another nice description (Proposition 4.3.3): it is the symmetrisation of the fat nerve of the opposite of the category of $P$-trees and active injections:

$$\text{Bar}_{P^\circ}(P^\circ) \simeq SN_\text{act,inj}(P)^{op}.$$ 

This comodule structure is shown to meet the finiteness requirements, and is the main ingredient in the proof of the following locally finite version of the main theorem 3.2.1.

**Theorem 4.3.6.** For any operad $P$, the two-sided bar constructions $Y = \text{Bar}_S(P^*)$ and $Z = \text{Bar}_S(P^\circ)$ together endow the slice $\text{Grpd}_{/S}(\text{tr}(P))$ with the structure of a locally finite comodule bialgebra. Precisely, the incidence bialgebra of $Y$ is a locally finite left comodule bialgebra over the incidence bialgebra of $Z$.

**0.2.5. Examples.** In the final section we work out some examples.

Subsection 5.1 deals with the special case where the operad is just a category (regarded as an operad with only unary operations). Then there is an interesting simplification, namely that the comodule bialgebra is actually free on a comodule coalgebra. This feature is shared by the examples of comodule bialgebras constructed by Carlier [13] from hereditary species.

For the more specific examples, the first is simply the incidence comodule bialgebra of the Baez–Dolan construction on the identity functor. This is shown (in Subsection 5.2) to be the Faà di Bruno comodule bialgebra, dual to composition and multiplication of power series.

The next class of examples is that of monoids, considered as one-colour operads with only unary operations. In this case we recover the comodule bialgebras of mould calculus [38].

In Subsection 5.4 we come to the example that prompted this work: the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra of rooted trees [11] from the theory of B-series [20]. In this case there is an extra step involved, namely taking core [68]. This is the passage from operadic trees to combinatorial trees, consisting simply in omitting decorations and shaving off leaves and root.

Subsection 5.5 deals with variations on the class of examples where the base operad $P$ is free on a linear order, on a quiver, or on a polynomial endofunctor. This leads to certain comodule bialgebras of words, paths in quivers, and trees.

In the final subsection 5.6, some non-examples are treated. The first is rather close to be a Baez–Dolan construction, namely the comodule bialgebra of noncrossing partitions of Ebrahimi-Fard–Foissy–Kock–Patras [34]. It is outlined here how the Baez–Dolan construction should be modified in order to cover this example. To finish, it is briefly discussed why the comodule bialgebras of Bruned, Hairer, and Zambotti [9] do not come from a Baez–Dolan construction.

## 1 Preliminaries

Objective combinatorics works with objects instead of numbers. A systematic way of achieving this is to use slice categories instead of vector spaces [50]; the most efficient version uses groupoids instead of sets, in order to take symmetries into account automatically. After a brief introduction to homotopy linear algebra, we recall the basic notions
of decomposition spaces and polynomial functors, the two main toolboxes employed in this work.

1.1 Objective combinatorics and linear algebra

In this first subsection we work with sets instead of groupoids, only to emphasise the main ideas in their simplest form. In Subsection 1.2 we upgrade to groupoids, as actually needed in this work.

1.1.1. Vector spaces versus slice categories. For $S$ a set, we denote by $Q_S$ the vector space spanned by $S$, and by $\delta_s$ the basis vector indexed by $s \in S$. A general vector in $Q_S$ is a (finite) linear combination $\sum_{s \in S} \lambda_s \delta_s$, essentially the same thing as a (finite) $S$-indexed family of scalars $\lambda_s$.

We want to consider $S$-indexed families of sets (or groupoids) $(X_s \mid s \in S)$, encoded as a single map $p : X \to S$; the members of the family are then the fibres $X_s := p^{-1}(s)$, defined formally as the pullback

$$
\begin{array}{ccc}
X_s & \longrightarrow & X \\
\downarrow & & \downarrow \\
1 & \longrightarrow & S.
\end{array}
$$

Here 1 denotes a singleton set, and $\lceil s \rceil : 1 \to S$ is the map that picks out $s$.

The families over $S$ form the objects of the slice category $\text{Set}/S$; its morphisms are commutative triangles. The slice category $\text{Set}/S$ plays the role of $Q_S$. Just as $Q_S$ is the free vector space on $S$, the slice $\text{Set}/S$ can be characterised as the sum completion of $S$.

1.1.2. Linear maps and linear functors. Instead of linear maps $Q_S \to Q_T$, we have linear functors $\text{Set}/S \to \text{Set}/T$, which means functors that preserve sums. One can prove (cf. [50]) that every such functor is given by a span

$$
\begin{array}{ccc}
S & \leftarrow & M \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
T & \to & T
\end{array}
$$

by pullback along $p$ followed by composing along $q$. This is denoted $q \circ p^*$. The basic fact in linear algebra that composition of linear maps is given in coordinates by matrix multiplication has its objective analogue in the following fundamental lemma, which ensures that composition of linear functors is given by pullback composition of spans:

**Lemma 1.1.3** (‘Beck–Chevalley’). For any pullback square

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S,
\end{array}
$$

the canonical natural transformation of functors

$$
p'_! \circ g^* \Rightarrow f^* \circ p_!
$$

is invertible.
1.1.4. Cardinality. The objective level works independently of finiteness conditions. The passage from the slice-category level to the vector-space level consists in taking cardinality; this requires some finiteness conditions.

The **cardinality** of a family \( p : X \to S \) in \( \text{Set}_{/S} \) with \( X \) finite is by definition the vector

\[
|p| := \sum_{s \in S} |X_s| \delta_s \in \mathbb{Q}_S,
\]

where \( |X_s| \) is usual cardinality of the finite set \( X_s \), and \( \delta_s \) is the basis vector in \( \mathbb{Q}_S \) corresponding to \( s \). The cardinality of pullback composition is matrix multiplication.

1.1.5. Example: small categories are linear monads. Given a small category with object set \( X_0 \) and set of arrows \( X_1 \), the span

\[
X_0 \leftarrow X_1 \rightarrow X_0
\]

(source and target) induces a linear endofunctor

\[
P : \text{Set}_{/X_0} \longrightarrow \text{Set}_{/X_0}
\]

\[
A \to X_0 \quad \mapsto \quad t_! s^* (A \to X_0).
\]

The composite \( P \circ P \) is given by

\[
\begin{array}{ccc}
X_2 & \xleftarrow{d_2} & X_1 \\
\downarrow & \searrow & \downarrow \\
X_1 & \xleftarrow{d_1} & X_0 \\
\end{array}
\]

with \( X_2 \simeq X_1 \times_{X_0} X_1 \) the set of pairs of composable arrows. Composition of arrows (which is the face map \( d_1 : X_2 \to X_1 \)) defines a monad multiplication \( \mu : P \circ P \Rightarrow P \). The assignment of identity arrows to each object \( s_0 : X_0 \to X_1 \) is precisely the unit for the monad, \( \eta : \text{Id} \Rightarrow P \). The monad axioms amount precisely to category axioms. Conversely, a linear monad defines a category.

1.2 Groupoids and homotopy pullbacks

To account for symmetries of combinatorial objects, it is convenient to upgrade the theory from sets to groupoids. This is mostly straightforward, provided that all notions are taken in their homotopy sense: pullback will thus mean homotopy pullback, fibre will mean homotopy fibre, and so on. Introductions to this machinery can be found in preliminary sections of [49], or the appendix of [54]. A fuller treatment (in the case of \( \infty \)-groupoids) can be found in [50]. In due time, the book manuscript [15] should become a suitable reference.

Recall that a groupoid \( X \) is a small category in which all arrows are invertible. A map of groupoids is just a functor. An equivalence of groupoids is just an equivalence of categories. Topology provides valuable intuition and terminology: an arrow in a groupoid is also called a path; a natural transformation is also called a homotopy. We write \( \pi_0(X) \) for the set of connected components of a groupoid \( X \).
We are interested in groupoids up to equivalence. For this reason, the usual notions of pullback, fibres, slices, adjoints, and so on, are not appropriate — they are not homotopy invariant. We shall need the homotopy notions. If just they are used consistently, they behave very much like the ordinary notions do for sets.

1.2.1. Homotopy cardinality of groupoids \([5], [50]\). A groupoid \(X\) is called finite if \(\pi_0(X)\) is a finite set, and all \(\text{Aut}(x)\) are finite groups. The homotopy cardinality of a groupoid \(X\) is defined to be

\[
|X| = \sum_{x \in \pi_0 X} \frac{1}{|\text{Aut}(x)|}.
\]

More generally, the homotopy cardinality of a family \(X \to B\) is

\[
\sum_{b \in \pi_0 B} \frac{|X_b|}{|\text{Aut}(b)|} \cdot \delta_b \in \mathbb{Q}_B,
\]

where \(\delta_b\) is the basis vector corresponding to \(b\).

1.2.2. Homotopy pullbacks. A homotopy pullback is a homotopy commutative square

\[
\begin{array}{ccc}
P & \longrightarrow & Y \\ \downarrow & & \downarrow q \\ X & \longrightarrow & S \\
\end{array}
\]

satisfying a universal property among all such squares with common \(p\) and \(q\). As such it is determined uniquely up to equivalence. There are different (but equivalent) models for homotopy pullback. In this work, the only homotopy pullbacks needed will be computed using the following three practical lemmas.

1.2.3. Fibrations. A map of groupoids \(p : X \to B\) is a fibration when it satisfies the path lifting property from topology: for each \(x \in X\) and \(\beta : b \to p(x)\) in \(B\), there exists an arrow \(\alpha : x' \to x\) such that \(p(\alpha) = \beta\).

Lemma 1.2.4. An ordinary (strict) pullback is also a homotopy pullbacks if one of the two maps \(p\) and \(q\) is a fibration.

Lemma 1.2.5 (Prism Lemma). Given a prism diagram of groupoids

\[
\begin{array}{ccc}
X'' & \longrightarrow & X' \longrightarrow X \\ \downarrow & & \downarrow j \\ Y'' & \longrightarrow & Y' \longrightarrow Y \\
\end{array}
\]

in which the right-hand square is a (homotopy) pullback, then the outer rectangle is a (homotopy) pullback if and only of the left-hand square is a (homotopy) pullback.

1.2.6. Homotopy fibre. Given a map of groupoids \(p : X \to S\) and an element \(s \in S\), the (homotopy) fibre \(X_s\) of \(p\) over \(s\) is the (homotopy) pullback

\[
\begin{array}{ccc}
X_s & \longrightarrow & X \\ \downarrow j & & \downarrow p \\ 1 & \longrightarrow & S \\
\end{array}
\]
Lemma 1.2.7 (Fibre Lemma). A square of groupoids

\[
\begin{array}{ccc}
P & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
X & \rightarrow & S
\end{array}
\]

is a (homotopy) pullback if and only if for each \(x \in X\) the induced comparison map \(u_x : P_x \to Y_{f_x}\) is an equivalence.

1.2.8. Linear functors. For a groupoid \(B\), the slice category \(\text{Grpd}_B\) has objects families \(X \to B\), but in contrast to the set case, the morphisms in \(\text{Grpd}_B\) are allowed to be triangles commuting only up to an isomorphism. This slack is required to obtain a notion invariant under equivalence, but in practice many triangles commute strictly.

Every map \(f : B \to A\) defines a functor \(f^* : \text{Grpd}_A \to \text{Grpd}_B\) by homotopy pullback along \(f\). This functor has both adjoints (which means adjoints up to homotopy): the left adjoint \(f_! : \text{Grpd}_B \to \text{Grpd}_A\) is just postcomposition with \(f\).

A linear functor \(\text{Grpd}_I \to \text{Grpd}_J\) is one of the form \(q^! \circ p_*\) for a span \(I \xleftarrow{p} M \xrightarrow{q} J\). We get in this way a category \(\text{LIN}\) whose objects are slices \(\text{Grpd}_I\) and whose morphisms are linear functors. (In reality this should be a 2-category, but this subtlety plays no role in this paper.) The category \(\text{LIN}\) features a tensor product, defined as

\[
\text{Grpd}_I \otimes \text{Grpd}_J := \text{Grpd}_{(I \times J)},
\]

in analogy with the familiar fact for vector spaces \(Q_I \otimes Q_J = Q_{I \times J}\). The neutral object is \(\text{Grpd} \simeq \text{Grpd}_1\).

1.2.9. Homotopy quotients \([4]\). Given a group action \(G \times X \to X\) (for \(G\) a group and \(X\) a set or groupoid), instead of the naive quotient (set of orbits) it is better to use the homotopy quotient (also called weak quotient or action groupoid), obtained from \(X\) by sewing in a path from \(x\) to \(g.x\) for each \(x \in X\) and \(g \in G\). It has much better properties than the naive quotient regarding interaction with the other basic construction such as homotopy cardinality and homotopy pullbacks \([50]\), and it will be denoted simply \(X/G\). The naive quotient (which would be \(\pi_0(X/G)\)) is never employed. In particular, we have

\[
|X/G| = |X| / |G|
\]

(where \(|G|\) is the order of the group \(G\)).

1.2.10. Discrete and finite maps. A map of groupoids is called discrete if all its homotopy fibres are discrete. It is called finite if all its homotopy fibres are finite.

From now on, the words pullback, fibre, quotient will always refer to the homotopy notions.

1.3 Simplicial groupoids, decomposition spaces, and incidence coalgebras

1.3.1. The simplex category and the active-inert factorisation system. The simplex category \(\Delta\) is the category of nonempty finite linear orders

\[
[n] := \{0 < 1 < \cdots < n\}
\]
and monotone maps. It features an active-inert factorisation system: An arrow in $\Delta$ is \textit{active}, written $a : [m] \to [n]$, when it preserves end-points, $a(0) = 0$ and $a(m) = n$; and it is \textit{inert}, written $a : [m] \rTo [n]$, if it preserves distance, meaning $a(i + 1) = a(i) + 1$ for $0 \leq i \leq m - 1$. The active maps are generated by the codegeneracy maps $s^i : [n+1] \to [n]$ and by the \textit{inner} coface maps $d^i : [n-1] \to [n]$, $0 < i < n$, while the inert maps are generated by the \textit{outer} coface maps $d^0$ and $d^n$. Every morphism in $\Delta$ factors uniquely as an active map followed by an inert map. Furthermore, it is a basic fact \cite{51} that active and inert maps in $\Delta$ admit pushouts along each other, and the resulting maps are again active and inert.

\subsection*{1.3.2. Simplicial groupoids.} A simplicial groupoid is a functor $X : \Delta^{\text{op}} \to \text{Grpd}$. Since $\Delta$ is generated by coface and codegeneracy maps, a simplicial groupoid is conveniently described by indicating the face and degeneracy maps, picturing it as

$$
\begin{array}{cccccc}
X_0 & \overset{d_0}{\leftarrow} & X_1 & \overset{d_0}{\leftarrow} & X_2 & \overset{d_0}{\leftarrow} & \cdots \\
\overset{d_1}{\leftarrow} & \overset{d_1}{\leftarrow} & \overset{d_1}{\leftarrow} & \overset{d_1}{\leftarrow} & \overset{d_1}{\leftarrow} & \\
\overset{s_0}{\leftarrow} & \overset{s_0}{\leftarrow} & \overset{s_0}{\leftarrow} & \overset{s_0}{\leftarrow} & \overset{s_0}{\leftarrow} & \\
\end{array}
$$

The subscripts on the $d_i$ refer to which vertex of a simplex is omitted. The subscripts on the $s_i$ refer to which vertex is repeated. (The simplicial identities, such as for example $d_0 \circ d_2 = d_1 \circ d_0$, are not captured by the picture.)

\subsection*{1.3.3. Segal spaces.} A simplicial groupoid $X : \Delta^{\text{op}} \to \text{Grpd}$ is called a \textit{Segal space} when for all $n \geq 0$ the square

$$
\begin{array}{ccc}
X_{n+2} & \xrightarrow{d_{n+2}} & X_{n+1} \\
\downarrow{d_0} & & \downarrow{d_0} \\
X_{n+1} & \xrightarrow{d_{n+1}} & X_n \\
\end{array}
$$

is a (homotopy) pullback.

\subsection*{1.3.4. Fat nerve.} The base case $n = 0$ of the Segal condition can be interpreted as saying that the 2-simplices are pairs of ‘composable 1-simplices’. Indeed the Segal condition is motivated by categories. For $C$ a small category, the \textit{fat nerve} of $C$ is the simplicial groupoid

$$
\text{NC} : \Delta^{\text{op}} \longrightarrow \text{Grpd} \\
[n] \longmapsto \text{Fun}([n], C)^{\text{iso}}
$$

(strings of composable arrows). The fat nerve of a category is always a Segal space. In fact, up to homotopy, the Segal condition characterises the simplicial groupoids that are fat nerves of categories. In this work, Segal spaces arise mainly from the two-sided bar construction on an operad, cf. Section 2.

\subsection*{1.3.5. Decomposition spaces \cite{51}/2-Segal spaces \cite{32}.} A \textit{decomposition space} is a simplicial groupoid $X : \Delta^{\text{op}} \to \text{Grpd}$ that takes active-inert pushouts in $\Delta$ to pullbacks in $\text{Grpd}$. Decomposition spaces are the same thing as the 2-Segal spaces of Dyckerhoff and Kapranov \cite{32}. Every Segal groupoid is a decomposition space \cite{32}, \cite{51}.
1.3.6. The incidence coalgebra of a decomposition space [51]. The motivating property of decomposition spaces is that they admit the incidence-coalgebra construction, meaning that the functor

\[ \Delta : \text{Grpd}_{/X_1}^{(d_2,d_0)\circ d_1^*} \to \text{Grpd}_{/X_1} \otimes \text{Grpd}_{/X_1} \]

given by the canonical span

\[ X_1 \leftarrow^{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1 \]

(1)
defines a (homotopy-coherent) coassociative comultiplication [32], [51]. Note that only simplicial degree 1 and 2 are needed in order to define the comultiplication, but \( X_3 \) enters to express coassociativity, and (in the general homotopical setting) all the higher \( X_k \) are needed to express coherence [51], [32], [89].

1.3.7. Examples: posets, monoids, categories. If \( \mathcal{C} \) is a poset, a monoid, or more generally a category, then the nerve or fat nerve \( X = \mathbf{N} \mathcal{C} \) is a Segal space and hence a decomposition space. The incidence coalgebra construction on \( X \) now recovers the classical notions in the case where \( \mathcal{C} \) is a locally finite poset [63], a decomposition-finite monoid [16], or more generally a Möbius category in the sense of Leroux [76], after taking homotopy cardinality. Indeed, Leroux’s formula for comultiplication on the vector space spanned by the arrows of \( \mathcal{C} \) is

\[ \Delta(f) = \sum_{b o a = f} a \otimes b. \]

(2)

This is precisely what comes out of the general construction in 1.3.6: the sum in (2) is taken over the fibre (that is, pullback along \( d_1 \))

\[
\begin{array}{ccc}
(X_2)_f & \longrightarrow & X_2 \\
\downarrow & & \downarrow d_1 \\
1 & \longrightarrow & X_1
\end{array}
\]

(the set of pairs of composable arrows whose composite is \( f \)), and returning the two constituents of the decomposition is precisely to apply \( (d_2,d_0) \).

Many coalgebras in combinatorics arise as the homotopy cardinality of the incidence coalgebra of a decomposition space which is not a poset, monoid or category [53], [54].

1.3.8. Example [54]. Pertinent to the present undertakings is the decomposition-space realisation of the Butcher–Connes–Kreimer Hopf algebra of rooted trees [10, 30, 73]. As an algebra it is free commutative on the set of iso-classes of rooted trees \( T \). The comultiplication is defined by summing over certain admissible cuts \( c \):

\[ \Delta(T) = \sum_{c \in \text{adm. cuts}(T)} P_c \otimes R_c. \]

(3)

An admissible cut \( c \) partitions the nodes of \( T \) into two subsets or ‘layers’
One layer must form a rooted subtree $R_c$ (or be empty), and its complement forms the ‘crown’, a subforest $P_c$ regarded as a monomial of trees. To realise this from a decomposition space (cf. [54]), let $H_k$ denote the groupoid of forests with $k - 1$ compatible admissible cuts, partitioning the forest into $k$ layers (which may be empty). The $H_k$ assemble into a simplicial groupoid $H$, where degeneracy maps repeat a cut (that is, insert an empty layer), and face maps forget a cut (joining adjacent layers) or discard the top or bottom layer.

The comultiplication (3) arises from this simplicial groupoid by the general pull-push formula: for a tree $T \in H_1$, take the homotopy sum over the homotopy fibre $d^{-1}_1(T) \subset H_2$, and for each element $c$ in this fibre return the pair $(d_2c, d_0c)$ consisting of the two layers. Finally take homotopy cardinality to arrive at $P_c \otimes R_c$. This simplicial groupoid is not a Segal groupoid, since it is not possible to reconstruct a tree from the layers of a cut. One readily checks that it is a decomposition space [53], [54]. We shall come back to this example in 5.4.

1.3.9. Culf maps [51]. A simplicial map is *culf* if, considered as a natural transformation of functors $\Delta^{op} \to \text{Grpd}$, it is cartesian on active maps. Culf maps induce coalgebra homomorphisms [51]. In the present paper, the main use of this concept is that monoidal structures on decomposition spaces are required to be culf: this is what ensures that the multiplication resulting from taking cardinality is comultiplicative so as to yield a bialgebra. The second use of culfness is that the objective analogue of comodules is given by certain culf maps [101], [107], [12], cf. 3.1.2 below.

1.4 Polynomial functors

A standard reference for polynomial functors is [55]; the long manuscript [65] aims at eventually becoming a unified reference. While polynomial functors have often been studied in the context of sets, for the present purposes it is necessary to deal with polynomial functors over groupoids [67], which constitute a convenient language for operads [105]. The upgrade from sets to groupoids is essentially straightforward, as long as all notions are taken to be the homotopy notions. The full $\infty$-categorical theory is developed in [56]. For the present needs, the introductions in [67] and [70] should suffice.

A small category can be seen as a linear monad (1.1.5), or as an operad with only unary operations. The multi aspect of general operads can be accounted for by passing from linear functors to polynomial functors [55], [65]. Where linear functors are given by spans $I \leftarrow M \rightarrow J$, polynomial functors incorporate a nonlinear aspect by means of an extra ‘middle map’:

1.4.1. Polynomial functors. A *polynomial* is a diagram of groupoids

$$I \leftarrow E \xrightarrow{p} B \xrightarrow{t} J.$$

The associated *polynomial functor* is the composite

$$\text{Grpd}_I \xrightarrow{s*} \text{Grpd}_E \xrightarrow{p*} \text{Grpd}_B \xrightarrow{t*} \text{Grpd}_J.$$

(Here of course we are talking about homotopy slices, upperstar is homotopy pullback, and lowerstar and lowershriek are the homotopy adjoints to homotopy pullback, [67], [70], [56].)

1\text{standing for conservative and with unique lifting of factorisations}
We shall be concerned only with finitary polynomial functors. Abstractly this means polynomial functors that preserve sifted colimits. In terms of the representing diagrams, they are those for which the middle map has finite discrete fibres \([\ast, \ast]\). In this case, there is the following explicit formula for the polynomial functor \([\ast, \ast]\):

\[
(X_i \mid i \in I) \mapsto \left( \sum_{b \in \pi_0(B_j) \in \mathcal{E}_b} \prod_{e \in \mathcal{E}_b} X_{se} / \text{Aut}(b) \mid j \in J \right),
\]

where the quotient is of course a homotopy quotient (of the canonical action of the group \(\text{Aut}(b)\)), as in 1.2.9.

1.4.2. Examples. (The monad structures on these examples will be dealt with in 1.6.) The identity monad \(\text{Id} : \text{Grpd} \to \text{Grpd}\) is represented by \(1 \leftarrow 1 \to 1 \to 1\).

The free-monoid monad (also called the word monad)

\[
\mathcal{M} : \text{Grpd} \to \text{Grpd}
\]

\[
X \mapsto X^* = \sum_{n \in \mathbb{N}} X^n
\]

is represented by the polynomial diagram

\[
1 \leftarrow \mathbb{N}' \to \mathbb{N} \to 1,
\]

where \(\mathbb{N}'\) denotes the set \(\{(n, i) \mid i \leq n\}\) so that the fibre over \(n\) is an \(n\)-element set. The elements in \(\mathcal{M}(X)\) are finite words in \(X\).

The free-symmetric-monoidal-category monad

\[
\mathcal{S} : \text{Grpd} \to \text{Grpd}
\]

\[
X \mapsto \sum_{\mu \in \pi_0 \mathbb{B}} X^\mu / \text{Aut}(\mu)
\]

is polynomial, represented by

\[
1 \leftarrow \mathbb{B}' \to \mathbb{B} \to 1.
\]

Here \(\mathbb{B} = \mathcal{S}1\) is the groupoid of finite sets and bijections, and \(\mathbb{B}'\) is the groupoid of finite pointed sets and basepoint-preserving bijections. To be specific we take \(\mathbb{B}\) to mean the skeleton consisting of the finite sets \(\underline{n} := \{1, 2, \ldots, n\}\). Then \(\mathcal{S}X\) is the groupoid whose objects are the words in \(X\), and whose morphisms from \((x_i)_{i \in \underline{n}}\) to \((y_i)_{i \in \underline{n}}\) are given by decorated permutations, meaning pairs \((\rho, (f_i)_{i \in \underline{n}})\) where \(\rho \in \mathcal{S}_n\) is a permutation of \(\underline{n}\) and \(f_i : x_i \to y_{\rho i}\) is an arrow in \(X\) for each \(i \in \underline{n}\).

The objects of \(\mathcal{S}X\) are called monomials of objects in \(X\).

1.4.3. Morphisms of polynomial functors [55]. Morphisms of polynomial functors are essentially cartesian natural transformations, but involving also change of colours. Precisely, they are given by diagrams

\[
P' : \quad I' \leftarrow E' \longrightarrow B' \longrightarrow I'
\]

\[
P : \quad I \leftarrow E \longrightarrow B \longrightarrow I,
\]

16
where the middle square is cartesian (expressing arity preservation) \[55\]. These correspond to cartesian natural transformations

\[
\begin{array}{c}
\text{Grpd}_{/I'} \xrightarrow{p'} \text{Grpd}_{/I'} \\
F_i \Downarrow \theta \Downarrow F_i \\
\text{Grpd}_{/I} \xrightarrow{P} \text{Grpd}_{/I},
\end{array}
\]

1.4.4. Graphical interpretation of polynomial endofunctors [65]. Given a polynomial endofunctor \( P \) represented by

\[
I \xleftarrow{E} B \xrightarrow{I},
\]

\( I \) is interpreted as the groupoid of \textit{colours} (or \textit{objects}), and \( B \) as the groupoid of \textit{operations}. The \textit{arity} of an operation \( b \in B \) is (the size of) the fibre \( E_b \), and each operation is typed: the output colour of \( b \) is \( t(b) \), and the input colours are the \( s(e) \) for \( e \in E_b \). The groupoid \( E \) is thus the groupoid of operations with a marked input slot. The map \( p \) just forgets the mark. We may picture an element in \( B \) as a corolla with node labelled by \( b \in B \), and with leaves and root decorated by \( I \) according to this scheme. This is called a \( P \)-corolla.

Evaluation of \( P \) on an object \( X \rightarrow I \) has the following combinatorial interpretation [65]. \( P(X) \) is the groupoid of \( P \)-corollas as before, but furthermore with leaves decorated in \( X \) (subject to a compatibility condition: an element \( x \in X \) may decorate leaf \( \ell \) only if the colour of \( x \) (under \( X \rightarrow I \)) matches the colour of \( \ell \) (under \( E \rightarrow I \))).

In particular, the endofunctor \( P \circ P \) has as operations \( P \)-corollas whose input edges are decorated by other \( P \)-corollas, with compatible colours. This data is naturally interpreted as a ‘2-level \( P \)-tree’:

\[
\quad
\]

We now formalise this in terms of trees.

1.5 Trees

1.5.1. Trees. The trees relevant to the present context are \textit{operadic} trees, i.e. admitting open-ended edges for leaves and root, such as the following:

\[
\quad
\]

It was observed in [66] that operadic trees can be conveniently encoded by diagrams of the same shape as polynomial endofunctors. By definition, a \textit{(finite, rooted) tree} is a diagram of finite sets

\[
A \xleftarrow{s} M \xrightarrow{p} N \xrightarrow{t} A
\]

satisfying the following three conditions:

(1) \( t \) is injective

(2) \( s \) is injective with singleton complement (called the \textit{root} and denoted 1).
With \( A = 1 + M \), define the walk-to-the-root function \( \sigma : A \to A \) by \( 1 \mapsto 1 \) and \( e \mapsto t(p(e)) \) for \( e \in M \).

(3) \( \forall x \in A : \exists k \in \mathbb{N} : \sigma^k(x) = 1. \)

The elements of \( A \) are called edges. The elements of \( N \) are called nodes. For \( b \in N \), the edge \( t(b) \) is called the output edge of the node. That \( t \) is injective is just to say that each edge is the output edge of at most one node. For \( b \in N \), the elements of the fibre \( M_b := p^{-1}(b) \) are called input edges of \( b \). Hence the whole set \( M = \sum_{b \in N} M_b \) can be thought of as the set of nodes-with-a-marked-input-edge, i.e.

pairs \( (b, e) \) where \( b \) is a node and \( e \) is an input edge of \( b \). The map \( s \) returns the marked edge. Condition (2) says that every edge is the input edge of a unique node, except the root edge. Condition (3) says that if you walk towards the root, in a finite number of steps you arrive there. The edges not in the image of \( t \) are called leaves.

The tree \( 1 \leftarrow 0 \rightarrow 0 \rightarrow 1 \) is the trivial tree. A corolla is a tree of the form \( n+1 \leftarrow n \rightarrow 1 \rightarrow n+1 \) (one node and \( n \) input edges).

1.5.2. Inert maps of trees (tree embeddings). An inert map of trees is a diagram

\[
\begin{array}{cccc}
A' & \xleftarrow{\alpha} & M' & \xrightarrow{\beta} & N' & \xrightarrow{\alpha} & A' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xleftarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\alpha} & A,
\end{array}
\]

where the middle square is a pullback. It is a consequence of the tree axioms that inert maps of trees are necessarily injective \([66]\). The pullback condition means (in view of the Fibre Lemma 1.2.7) that a node must be mapped to a node of the same arity. In conclusion, the inert maps are precisely the full subtree inclusions (called tree embeddings in \([66]\)).

The category of trees and inert maps has nice geometric features, including a Grothendieck topology \([66]\), useful to formalise notions of gluing. Presently, it is of interest that grafting of trees is expressed as colimits in this category: every tree is canonically the colimit of its one-node subtrees. We shall later need more general maps of trees, which will be generated by the free-monad monad, cf. 2.1.3 below.

1.5.3. \( n \)-level trees. The height of an edge \( x \in A \) is defined as

\[ h(x) := \min \{ k \in \mathbb{N} : \sigma^k(x) = 1 \}, \]

(with reference to the walk-to-the-root function \( \sigma \) of Axiom 3). In particular, the root edge has height 0. An \( n \)-level tree is a tree where all edges have height \( \leq n \) and all leaves have height precisely \( n \). The trivial tree is thus a 0-level tree, and a corolla is a 1-level tree. Note that a tree without leaves is an \( n \)-level tree for all sufficiently big \( n \).

To give an \( n \)-level tree is equivalent to giving a sequence of \( n \) maps of finite sets \( A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 = 1 \). The corresponding tree is given by

\[
\sum_{i=0}^{n} A_i \leftarrow \sum_{i=1}^{n} A_i \rightarrow \sum_{i=0}^{n-1} A_i \rightarrow \sum_{i=0}^{n} A_i,
\]

readily checked to satisfy the tree axioms, and being \( n \)-levelled by construction — its set of leaves is \( A_n \).
1.5.4. **P-trees.** Having trees and polynomials on the same footing makes it easy to deal with decorations of trees [66] (see also [68, 67, 71]). With a polynomial endofunctor $P$ fixed, given by a diagram $I \leftarrow E \rightarrow B \rightarrow I$, a $P$-tree is by definition a diagram

$$
\begin{array}{c}
A \\
\downarrow ^\alpha
\end{array}
\begin{array}{c}
M
\end{array}
\begin{array}{c}
\downarrow ^j
\end{array}
\begin{array}{c}
N
\end{array}
\begin{array}{c}
\downarrow ^a
\end{array}
\begin{array}{c}
A
\end{array}
\begin{array}{c}
I
\end{array}
\begin{array}{c}
\downarrow ^\alpha
\end{array}
\begin{array}{c}
E
\end{array}
\begin{array}{c}
\downarrow ^j
\end{array}
\begin{array}{c}
B
\end{array}
\begin{array}{c}
\downarrow ^a
\end{array}
\begin{array}{c}
I
\end{array}
,$$

where the top row is a tree. Hence nodes are decorated by elements in $B$, and edges are decorated by elements in $I$, subject to obvious compatibilities. That the middle square is a pullback expresses that $n$-ary nodes of the tree have to be decorated by $n$-ary operations, and that a specific bijection is given.

1.5.5. **Examples of P-trees.** With reference to Example 1.4.2, $\text{Id}$-trees are linear trees, $\text{M}$-trees are planar trees, and $\text{S}$-trees are abstract trees (naked trees). More exotically, if $P(X) = 1 + X^2$ then $P$-trees are planar binary trees.

Denote by $\text{tr}(P)$ the groupoid of $P$-trees, by $\text{cor}(P)$ the groupoid of $P$-corollas, and by $\text{triv}(P)$ the groupoid of trivial $P$-trees.

**Lemma 1.5.6 ([70]).** There are canonical equivalences

$I \simeq \text{triv}(P)$ and $B \simeq \text{cor}(P)$.

### 1.6 Polynomial monads and operads

1.6.1. **Polynomial monads.** A polynomial monad is a polynomial endofunctor $P$ equipped with cartesian natural transformations $\text{Id} \Rightarrow P \Leftarrow P \circ P$, required to satisfy the associative and unital laws [78]. The multiplication law $\mu : P \circ P \Rightarrow P$ on a polynomial endofunctor $P$ can be seen as a rule prescribing how to contract each 2-level $P$-tree to a $P$-corolla, preserving arities. The unit law $\eta : \text{Id} \Rightarrow P$ assigns to each colour a special unary corolla; we may think of this as contraction of trivial $P$-trees $\downarrow$ to $\downarrow$. More generally, iteration of these laws gives the unbiased view on monads, which can be seen as a law prescribing how to obtain from a whole tree configuration of $P$-operations (that is, a $P$-tree) a single $P$-operation (a $P$-corolla), called the residue of the $P$-tree.

1.6.2. **Example: the free-monoid monad.** The free-monoid endofunctor from Example 1.4.2, $M(X) = \sum_{n \in \mathbb{N}} X^n$, represented by

$$1 \leftarrow \mathbb{N} \xrightarrow{d} \mathbb{N} \rightarrow 1,$$

has a canonical monad structure. For $X$ a set (or more generally a groupoid), $M(X)$ is the set of words in $X$, and $M(M(X))$ is the set of words of words in $X$. The monad multiplication is concatenation of words (that is, removal of parentheses). The unit interprets an element in $X$ as a word of length 1.

The element $n \in \mathbb{N}$ can be represented as a planar $n$-corolla. Then the composite endofunctor $M \circ M$ has as operations 2-level planar trees, and the monad multiplication simply contracts such a 2-level tree to the corolla with the same number of leaves. Preservation of leaf number is precisely to say that the natural transformations $\text{Id} \Rightarrow M \Leftarrow M \circ M$ are cartesian.
The following example is fundamental to our undertakings.

1.6.3. The free-symmetric-monoidal-category monad. See [104] for details. The free-symmetric-monoidal-category endofunctor \( S : \text{Grpd} \to \text{Grpd} \) of Example 1.4.2, represented by

\[
1 \leftarrow \mathbb{B}' \to \mathbb{B} \to 1,
\]

is naturally a monad. For \( X \) a groupoid, \( SX \) is the groupoid of monomials of objects in \( X \). Explicitly these are finite sequences of objects \( (x_i)_{i \in \mathbb{N}} \), with morphisms permutations decorated by arrows in \( X \) (see 1.4.2). The multiplication \( \mu_X : SSX \to SX \) is given by concatenation (disjoint union). The unit \( \eta_X : X \to SX \) interprets an object as a length-1 sequence. These natural transformations are readily seen to be cartesian.

1.6.4. Ordinary coloured symmetric operads as polynomial monads. By operad we mean coloured symmetric operad in \( \text{Set} \). We will not reproduce the standard definition here, because in this work we shall only consider operads in the form of polynomial monads. The basic result in this direction is Weber’s theorem:

**Theorem 1.6.5** (Weber [105], Theorem 3.3). Operads with colour set \( I \) are essentially the same thing as polynomial monads \( P : \text{Grpd}/I \to \text{Grpd}/I \) cartesian over the free-symmetric-monoidal-category monad, as in

\[
\begin{array}{ccc}
I & \xrightarrow{E} & B \\
\downarrow & & \downarrow \\
1 & \xleftarrow{\mathbb{B}'} & \mathbb{B} \to 1,
\end{array}
\]

for which \( (I \text{ is a set and } B \to \mathbb{B} \) is a discrete fibration.

Briefly, the equivalence goes as follows [72]. For an operad \( P \) with colour set \( I \), the symmetric groups \( \mathfrak{S}_n \) act on the sets of \( n \)-ary operations. Let \( B \) be the disjoint union of the homotopy quotients of these actions. There is a canonical projection map to \( \mathbb{B} \), itself the disjoint union of the 1-object groupoids \( \mathfrak{S}_n \). This is a discrete fibration, whose fibre over \( n \) is the set \( P_n \) of \( n \)-ary groupoids. The groupoid \( E \) is given by (strict) pullback. The fibre of \( E \to B \) over an operation \( r \) is the set of its input slots. The monad structure on the polynomial endofunctor comes precisely from the substitution law of the operad \( P \).

Conversely, given a polynomial monad cartesian over \( S \) as above, the discrete fibration \( B \to \mathbb{B} \) induces a \( \mathfrak{S} \)-set which is the set of operations of an operad. The set of \( n \)-ary operations is the (homotopy) fibre of \( B \to \mathbb{B} \) over \( n \). The operad substitution law comes from the monad multiplication.

1.6.6. Polynomial monads and operads as needed in this paper. It is often convenient to work with operads allowed to have a groupoid of colours rather than just a set of colours. In this paper, the reason is simple: we shall see that the Baez–Dolan construction naturally produces operads with a groupoid of colours, even if given as input an operad with a set of colours. We also need to give up the requirement that the classifying map \( B \to \mathbb{B} \) be a discrete fibration. Accordingly we define an operad (with groupoids of colours \( I \)) to be a finitary polynomial monad \( P : \text{Grpd}/I \to \text{Grpd}/I \) represented by a diagram of groupoids

\[
I \leftarrow E \xrightarrow{P} B \to I.
\]
1.6.7. Example. The free-monoid monad \( \mathcal{M} \) (see 1.4.2 and 1.6.2) is an operad in the sense of 1.6.6, by means of the diagram

\[
\begin{align*}
\mathcal{M} : & \quad 1 \xleftarrow{} N' \xrightarrow{J} N \xrightarrow{} 1 \\
\mathcal{S} : & \quad 1 \xleftarrow{} B' \xrightarrow{} B \xrightarrow{} 1.
\end{align*}
\]

Note that the map \( N \to B \) is not a fibration, but it could easily be replaced by a fibration, by letting \( N \) denote the equivalent groupoid of all finite linear orders and monotone maps.

A nonsymmetric operad is a polynomial monad over \( \mathcal{M} \). Thereby it is also over \( \mathcal{S} \); this is its symmetrisation. Note that in the polynomial formalism, the polynomial monad itself does not change under symmetrisation.

2 Free monads, Baez–Dolan construction, and two-sided bar construction

2.1 The free monad \( P^* \)

For any polynomial endofunctor \( P \), one can construct the free monad on \( P \): it is the (least) solution to the fixpoint equation of endofunctors

\[
Q \simeq \text{Id} + P \circ Q,
\]

and it exists for general categorical reasons. More importantly, there is a neat explicit polynomial representation:

**Theorem 2.1.1** ([66], [70], [56]). For a finitary polynomial endofunctor \( P \) represented by \( I \leftarrow E \rightarrow B \rightarrow I \), the free monad \( P^* \) is represented by

\[
I \leftarrow \text{tr}'(P) \rightarrow \text{tr}(P) \rightarrow I,
\]

where \( \text{tr}(P) \) is the groupoid of \( P \)-trees, and \( \text{tr}'(P) \) is the groupoid of \( P \)-trees with a marked leaf.

Graphically:

\[
\begin{align*}
\{ & \quad \} & \text{forget mark} & \{ & \quad \} \\
\{ & \quad \} & \text{marked leaf} & \{ & \quad \}
\end{align*}
\]

\[
\begin{align*}
\text{marked leaf} & \quad \text{root edge} \\
P^* & \quad \{ & \quad \}
\end{align*}
\]

(The trees in the picture are \( P \)-trees, but the \( P \)-decorations have been suppressed to avoid clutter.)
2.1.2. Example: free monad on a tree. Since a tree $A \leftarrow M \rightarrow N \rightarrow A$ is in particular a polynomial endofunctor $T$, one can consider the free monad on it. But according to 1.5.2, $T$-trees are subtrees in $T$, so that the free monad on $T$ is given by

$$A \leftarrow \text{sub}'(T) \rightarrow \text{sub}(T) \rightarrow A$$

(with evident notation).

2.1.3. Active maps and the Moerdijk–Weiss category of trees $\Omega$. Moerdijk and Weiss [84] defined the category $\Omega$ of operadic trees to be the full subcategory of the category of operads spanned by the free operads on trees. Formally, a map $T' \rightarrow T$ in $\Omega$ is a morphism of polynomial endofunctors $T' \rightarrow T^*$, that is, a diagram

$$A' \leftarrow M' \rightarrow N' \rightarrow A' \quad \quad \quad \quad \quad \quad \quad \quad \quad A \leftarrow \text{sub}'(T) \rightarrow \text{sub}(T) \rightarrow A.$$

In addition to the inert maps $\rightarrow$ already described in 1.5.2, $\Omega$ has active maps, denoted $\rightarrow$, formally generated by the free-monad monad [103]. In elementary terms they are node refinements [66]: for each corolla there is an active map to any tree with the same number of leaves:

![Diagram](image)

(The second picture here illustrates the important special case where a unary node is refined into a trivial tree.) General active maps $K \rightarrow T$ are described by the following lemma. For $K$ a tree, denote by $\text{Act}(K)$ the groupoid of all active maps out of $K$.

**Lemma 2.1.4** ([56], Lemma 5.3.8). If $C_1, \ldots, C_n$ are the nodes of a tree $K$, then there is a canonical equivalence

$$\text{Act}(K) \simeq \prod_{i=1}^{n} \text{Act}(C_i).$$

In other words, an active map out of $K$ is given by refining each node of $K$ into a tree of matching arity as above, and then gluing together all these refinement maps according to the same gluing recipe that describes the tree $K$ as a colimit of its nodes, like this:
General maps in $\Omega$ look essentially the same but can land in bigger trees. More precisely, one is allowed to postcompose with an inclusion (inert map) of trees of $T$ into a bigger tree. This leads to:

**2.1.5. The active-inert factorisation system in $\Omega$ [66].** Every arrow in $\Omega$ factors as an active map followed by an inert map, constituting the active-inert factorisation system. Restricted to the subcategory of linear trees $\Delta \subset \Omega$, this gives the active-inert factorisation system on $\Delta$ already described in 1.3.1.

In particular, we shall need to refactor as follows. Given an inert map followed by an active map (drawn as solid arrows):

there is a unique way of refactoring it into an active map followed by an inert map, as indicated.

**2.1.6. Remark.** This active-inert factorisation system is a special case of the general notion of generic-free factorisation system introduced and studied deeply by Weber [102], [103]. Many of the important properties in the abstract theory of operads can be described in terms of this factorisation system. In fact, recently Chu and Haugseng [24] have taken this viewpoint to the extreme, developing an axiomatic theory of operad-like structures based on this notion.

**2.1.7. Induction of $P$-structure along maps in $\Omega$.** Let $P$ be any polynomial endofunctor (not required to have monad structure). For an inert map of trees $S \to T$, if $T$ is has $P$-tree structure, then there is induced a $P$-tree structure on $S$, simply by composition $S \to T \to P$ (which is composition of diagrams as in 1.4.3). This defines the category $\Omega_{\text{inert}}(P)$ of $P$-trees and inert maps (for any polynomial endofunctor $P$).

If $P$ is furthermore a monad, then this functoriality extends to active maps $K \to T$. Indeed, by construction (2.1.5) an active map $K \to T$ is a morphism of polynomials $K \to T^*$, and since $P$ is a monad, the $P$-tree structure $T \to P$ gives by adjunction a morphism of polynomials $T^* \to P$. The composite $K \to T^* \to P$ is the induced $P$-structure on $K$. Altogether this defines, for any polynomial monad $P$, a category of $P$-trees $\Omega(P)$ (which in fancier terms can be described as the category of elements of the dendroidal nerve of $P$, cf. [66].)

**2.2 The Baez–Dolan construction $P^\circ$**

We now come to the central notion of this work, the Baez–Dolan construction [3]. Curiously, Baez and Dolan defined the construction for operads with categories of colours, but used it only for operads with sets of colours. The set version is not optimal for the
purposes of opetope theory. It was adjusted by Cheng [22], simply by taking the original definition seriously and allowing a groupoid of colours. An alternative adjustment was provided by Kock, Joyal, Batanin and Mascari [71], using polynomial monads over \textit{Set}. These cannot account for general operads, only for so-called sigma-cofibrant operads [66], but that is enough for the purpose of defining opetopes. The following version of the Baez–Dolan construction is the polynomial version from [71], but upgraded from sets to groupoids, in the spirit of Cheng [22].

2.2.1. \textbf{Set-up.} Let P be a polynomial monad, represented by a diagram of groupoids

\[ I \leftarrow E \rightarrow B \rightarrow I. \]

Consider the category \textit{PolyEnd}/P of polynomial endofunctors of \textit{Grpd}/I cartesian over P. This category is monoidal under composition: given \( Q \Rightarrow P \) and \( Q' \Rightarrow P \), the composite is \( Q \circ Q' \Rightarrow P \circ P \Rightarrow P \), using the monad multiplication of P. The neutral object for the composition is \( \eta : \text{Id} \Rightarrow P \) (the monad unit). Consider also the category \textit{PolyMnd}/P of polynomial monads cartesian over P. The forgetful functor U admits a left adjoint, the free-monad-over-P functor:

\[ \begin{array}{c}
\text{PolyMnd}/P \\
\uparrow \eta \downarrow U \\
\text{PolyEnd}/P.
\end{array} \]

Denote by T the monad generated by this adjunction. Now the key point is that there is a natural equivalence

\[ \text{PolyEnd}/P \cong \text{Grpd}/B. \quad (5) \]

This is because polynomial endofunctors cartesian over P are given by diagrams

\[ \begin{array}{c}
I \quad \downarrow \eta \quad V \\
E \quad \downarrow \quad B \quad \Rightarrow \quad I,
\end{array} \]

but clearly this data is completely determined by the single map \( V \rightarrow B \).

2.2.2. \textbf{The Baez–Dolan construction, polynomial version.} With notation as in 2.2.1, the \textit{Baez–Dolan construction} on P is by definition the monad

\[ \text{Grpd}/B \xrightarrow{P^\circ} \text{Grpd}/B \]

obtained by transporting the monad T along the key equivalence \textit{PolyEnd}/P \cong \textit{Grpd}/B.

2.2.3. \textbf{Note.} In the literature the Baez–Dolan construction on P is usually denoted \( P^+ \), and sometimes called the plus-construction [3], [75], [71]. The change in notation here is motivated by the fact that \( P^\circ \) will relate to \( P^* \) in the same way as substitution relates to multiplication (for power series), as will become clear.

One can now follow through the explicit description of the free-monad-over-P monad and the key equivalence (5), to obtain the following.
Theorem 2.2.4 ([71]). The Baez–Dolan construction \( P^\circ \) is polynomial, represented by

\[
B \leftarrow \operatorname{tr}(P) \longrightarrow \operatorname{tr}(P) \longrightarrow B.
\]

Here \( \operatorname{tr}(P) \) is the groupoid of \( P \)-trees, and \( \operatorname{tr}^*(P) \) is the groupoid of \( P \)-trees with a marked node. The last map takes a \( P \)-tree and contracts it to a \( P \)-corolla (i.e. an object of \( B \), cf. Lemma 1.5.6) using the original monad structure on \( P \) (cf. 1.6.1). The leftmost map returns the marked node of the \( P \)-tree, considered as a \( P \)-corolla. The middle map simply forgets the mark.

Graphically:

```
\[
\begin{array}{c}
\{ \} \\
\text{marked node}
\end{array}
\xrightarrow{\text{forget mark}}
\begin{array}{c}
\{ \} \\
\text{monad mult.}
\end{array}
\]
```

(The trees in the picture are \( P \)-trees, but the \( P \)-decorations have been suppressed to avoid clutter.)

2.2.5. Blobbed trees. An operation of the endofunctor \( P^\circ \circ P^\circ \) is a \( P \)-tree \( K \) whose nodes are decorated by \( P \)-trees \( R_i \) in a compatible way: the tree that decorates a node with local structure \( b \in B \) must have \( b \) as its residue 1.6.1 (note that the notion of residue for \( P \)-trees involves the monad multiplication of \( P \), to compose the whole tree configuration of operations to a single operation). An operation of \( P^\circ \circ P^\circ \) is thus an active map of \( P \)-trees

\[
K \rightarrow T,
\]

mapping each node in \( K \) to a subtree \( R_i \subset T \), as described in 2.1.3. A pictorial way of encoding this is to draw each node of \( K \) as a big blob and draw its decorating tree inside the blob, so that its leaves and root match the boundary of the blob. Altogether the configuration can be seen as a blobbed \( P \)-tree (called constellation in [71]) — a \( P \)-tree \( T \) with some blobs partitioning the tree into smaller trees:

```
\[
\begin{array}{c}
\text{blobbed tree}
\end{array}
\]
```

(6)

Each blob contains a tree, and each node is contained in precisely one blob. It must be stressed that there may be blobs containing only a trivial tree. This occurs naturally because the tree \( K \) may contain unary nodes \( \updownarrow \), and the decorating tree of such a unary node may be the trivial tree \( \uparrow \). For details, see [71]. Let us stress that \( \uparrow \) and \( \phi \) are examples of blobbed trees, corresponding, respectively, to the active maps \( \uparrow \rightarrow T \rightarrow P \) and \( \downarrow \rightarrow T \rightarrow P \). The combinatorial description is attractive and useful for grasping constructions, but for the actual proofs below, it is the active maps \( K \rightarrow T \rightarrow P \) we shall actually work with.
2.2.6. Monad structure on $P^o$ and operad interpretation. The monad structure is canonically given by the adjunction. The monad multiplication $\mu : P^o \circ P^o \Rightarrow P^o$ simply takes the tree of $P$-trees and returns the total $P$-tree obtained by gluing together all the decorating $P$-trees $R_i$ according to the recipe provided by $K$, to get a big $P$-tree $T$. In terms of blobbed trees, the monad multiplication consists in erasing the blobs (not their content) and retaining the total $P$-tree $T$. The unit for the monad $P^o$ is given by regarding a $P$-corolla as a $P$-tree.

We consider $P^o$ as a coloured operad, in the sense of 1.6.6: its colours are the operations of $P$. The operations of $P^o$ are the $P$-trees, and the arity of such a $P$-tree is its set of nodes. One can substitute a whole $P$-tree (that is, a $P^o$-operation) into a node of another $P$-tree if the residue of the tree matches the local structure of the node, as pictured here:

![Diagram](image)

The neutral operation (of colour $b \in B$) is the tree consisting of just that corolla $b$.

Note that for each original colour $i \in I$, the trivial tree of colour $i$ is a nullary operation of $P^o$.

2.2.7. Example. Let $P$ be the identity monad $Id : Grpd \rightarrow Grpd$, which is represented by the diagram

$$1 \leftarrow 1 \rightarrow 1 \rightarrow 1.$$ 

The ones are all singleton; the two middle ones have been underlined just to stress that the play a different role than the non-underlined. According to the general graphical interpretation (1.4.4), we picture the unique operation (unique element in $B = 1$) as a $\mathbb{1}$. The free monad on $Id$ is represented by

$$1 \leftarrow N \rightarrow N \rightarrow 1,$$

where $N$ is regarded as the set of linear trees (including the trivial tree, corresponding to $0 \in N$). The Baez–Dolan construction $Id^o$ is instead represented by

$$1 \leftarrow N^* \rightarrow N \rightarrow 1.$$ 

Here $N^*$ denotes the set of (iso-classes of) linear trees with a marked node. The monad multiplication is given by substituting linear trees into the nodes of linear trees. Since $Id^o$ has one operation in each degree, it is easily seen that $Id^o$ is the free-monoid monad from Example 1.6.2, which is thus exhibited as the Baez–Dolan construction on the identity monad. We shall return to this example in 5.2.

2.2.8. Example (continued from Example 1.6.2). Let $P = M$ be the free-monoid monad on $Grpd$, $X \mapsto X^*$, represented by

$$1 \leftarrow N' \rightarrow N \rightarrow 1.$$
We regard $\mathbb{N}$ as the set of corollas (one for each arity $n \in \mathbb{N}$) and $\mathbb{N}'$ as the set of corollas with a marked leaf. The monad law now deals with grafting of corollas onto leaves. (This is a different graphical interpretation than the one just produced in the previous example with $M = \text{Id}^\circ$.)

The free monad on $M$ is represented by

$$1 \leftarrow \text{tr}'(M) \rightarrow \text{tr}(M) \rightarrow 1,$$

having as operations the set of all $M$-trees, which means planar trees. The Baez–Dolan construction $M^\circ$ is represented by

$$\mathbb{N} \leftarrow \text{tr}^\bullet(M) \rightarrow \text{tr}(M) \rightarrow \mathbb{N},$$

where $\text{tr}^\bullet(M)$ is the set of planar trees with a marked node. The monad multiplication is given by substituting trees into nodes. This monad $M^\circ$ is the free-nonsymmetric-single-coloured-operad monad [75].

2.2.9. Aside: opetopes. The original motivation [3] for the Baez–Dolan construction was to iterate it and use it to define the opetopes. These are shapes parametrising higher-dimensional many-in/one-out operations, in turn devised by Baez and Dolan to define weak $n$-categories for all $n$ (see Leinster’s book [75] for a detailed development of the theory).

The $n$-dimensional opetopes are by definition the colours of the $n$th iteration of the Baez–Dolan construction starting with $\text{Id}$:

$$\text{Op}_0 = 1 \quad \text{Op}_1 = \mathbb{1} \quad \text{Op}_2 = \mathbb{N} \quad \text{Op}_3 = \text{tr}(\text{Id}^\circ) = \text{tr}(M),$$

the set of planar trees. In dimension 4 one finds the set of blobbed planar trees, and after that it becomes increasingly complicated. An elementary combinatorial description was given in [71] in terms of something called zoom complexes, certain trees of trees of trees.

2.3 Two-sided bar construction (of one operad relative to another)

The two-sided bar construction is a classical construction in algebraic topology and homological algebra (see [82]), where it serves, among many other things, to construct classifying spaces, approximations and resolutions, and deloopings. In 2-dimensional category theory it serves to model the PROP envelope of an operad, and more generally internal algebra classifiers via codescent objects [106]; see also [6].

The relative polynomial version used here is due to Weber [106]; it concerns the situation where one polynomial monad is cartesian over another. The most important case is that of polynomial monads cartesian over $S$, the free-symmetric-monoidal-category monad 1.6.3, because these are essentially symmetric operads. We describe the constructions in that case, but $S$ could be replaced by any other polynomial monad (as we shall do in Subsection 4.2).

2.3.1. Operad morphisms (monad opfunctors). The natural notion of morphism of coloured operads does not fix the colours, and therefore at the monad level must be a bit
more involved than just natural transformations. To say that a monad \( R : \text{Grpd}_{/I} \to \text{Grpd}_{/I} \) is cartesian over \( S \) means there is a diagram of polynomial monads

\[
\begin{array}{c@{\quad}c@{\quad}c@{\quad}c}
I & E & B & I \\
\downarrow F & \downarrow \eta & \downarrow F & \downarrow F \\
1 & B' & B & 1
\end{array}
\]

intertwining the two monads \( R \) and \( S \) by means of the functor \( F \) and a natural transformation

\[ \phi : F \circ R \Rightarrow SF, \]

forming together a monad opfunctor in the sense of Street [95], a monad adjunction in the sense of Weber [106], or a vertical morphism of horizontal monads in the double-category setting [45]. This means we have the following two compatibilities with the monad structures (monad opfunctor structure on \( F \)): for any object \( A \) in \( \text{Grpd}_{/I} \) we have commutative diagrams

\[
\begin{array}{c}
F_1 A & F_1 A \\
F_1 \eta_1 & \downarrow \eta_1 \\
F_1 R A & SF_1 A \\
\downarrow \phi_1 & \downarrow \phi_1 \\
F_1 R A & SF_1 A
\end{array}
\]

It is a general fact that in the polynomial setting the natural transformation \( \phi \) is cartesian. This follows because its ingredients are the unit and counit of lowershriek-upperstar adjunctions and an instance of the Beck-Chevalley isomorphism. See [106, §3.3] for details.

### 2.3.2. One-sided bar construction.

Let \( 1 := \text{id}_I : I \to I \) denote the terminal object in \( \text{Grpd}_{/I} \), and let \( \alpha : R^1 \to 1 \) denote the unique map from \( R^1 \) in \( \text{Grpd}_{/I} \). (Note that \( \alpha \) has underlying map of groupoids \( B \to I \).) The pair \((1, \alpha)\) is the terminal \( R \)-algebra.

In \( \text{Grpd}_{/I} \) there is induced a natural simplicial-object-with-missing-top-face-maps

\[
\begin{array}{c}
1 & \cdots & 1 \\
\downarrow d_0 & \cdots & d_0 \\
R^1 & \cdots & R^1 \\
\end{array}
\]

The bottom face maps come from the action \( \alpha : R^1 \to 1 \), and the remaining face and degeneracy maps come from the monad structure. The lowest-degree maps are

\[
\begin{array}{c}
1 & \cdots & \eta_1 & \cdots & 1 \\
\downarrow \alpha & \cdots & \eta_1 & \cdots & \downarrow \alpha \\
R^1 & \cdots & R^1 \\
\end{array}
\]

and in general, \( s_k : R^n 1 \to R^{n+1}1 \) is given by \( R^{n-k} \eta_{R^1} \) and \( d_k : R^{n+1}1 \to R^n 1 \) is given by \( R^{n-k} \mu_{R^k} \), with the convention that \( \mu_{R^1} = \alpha \).

### 2.3.3. Two-sided bar construction.

We now apply \( SF \) to the diagram (9) above, to obtain inside \( \text{Grpd} \) a genuine simplicial object: the diagram now acquires the missing top face maps (written \( \cdots \)), and constitutes altogether a simplicial object in \( \text{Grpd} \) ([106], Lemma 4.3.2) which is the two-sided bar construction on \( R \), denoted \( \text{Bar}_S(R) \):

\[
\begin{array}{c}
\text{Bar}_S(R) : & SF_1 & \cdots & SF_1 R & \cdots & SF_1 RR & \cdots \\
\downarrow d_1 & \cdots & \downarrow d_1 & \cdots & \downarrow d_1 & \cdots & \downarrow d_1 \\
SF_1 & \cdots & SF_1 R & \cdots & SF_1 RR & \cdots & SF_1 RR
\end{array}
\]
The new top face maps are given by

\[ d_T := \mu^5 \circ S(\phi). \]

For example, the lowest-degree top face map is given as

\[ SS F_1 \leftarrow \mu^5_{F_1} \quad S(\phi_1) \quad SF_1 \rightarrow SF_1 R_1. \] (10)

2.3.4. **Remarks.** Note that \( F_1 \) only serves to pass from \( Grpd/I \) to \( Grpd \), so that the simplicial object really lives in the category \( Grpd \). It should further be noted that the terminal object in \( Grpd/I \) (as always denoted \( 1 \)) is the identity map \( \text{id} : I \rightarrow I \), so that we have

\[ F_1 1 = I \quad \text{and} \quad F_1 R_1 = B. \]

Furthermore, \( F_1 R R_1 \) is the groupoid of 2-level \( R \)-trees.

The \( S \) in front of everything means we are talking about monomials of objects. Monomials of degree 1 (that is, original objects) are called connected. In the cases of interest, the objects will be various kinds of trees, and it makes sense also to say forest for monomials (disjoint unions) of trees. All the face and degeneracy maps — except the top face maps — are \( S \) of a map, meaning that they send connected objects to connected objects. In contrast, the top face maps generally send connected objects to non-connected ones, which is what the wavy arrows indicate.

It should be noted that the standard notation in the literature for this two-sided bar construction is

\[ B(S F_1, R, 1). \]

2.3.5. **Interpretation of the groupoids.** The two-sided bar construction is completely formal. At the same time, all the constituents of \( X = \text{Bar}_S(R) \) have clean combinatorial interpretations. For the groupoids:

\[ X_k = S tr_k(R) \text{ is the groupoid of monomials of } k\text{-level } R \text{-trees}. \]

In particular,

- \( X_0 = S tr_0(R) = S I \) is the groupoid of monomials of colours;
- \( X_1 = S tr_1(R) = S \text{cor}(R) = S B \) is the groupoid of monomials of operations of \( R \);
- \( X_2 = S tr_2(R) = SR B \) is the groupoid of monomials of 2-level \( R \)-trees.

As to the maps, they are all about joining or deleting levels (or inserting trivial levels), using the monad structure. In the lowest degree

\[ \begin{align*}
X_0 & \xleftarrow{d_0} X_1 \\
\xrightarrow{d_1} & X_2
\end{align*} \]

- the bottom face map \( d_0 \) returns the root edges (of a monomial of corollas);
- the top face map \( d_1 \) returns the monomial of leaves.
- The degeneracy map \( s_0 \) sends a colour \( i \) to the identity operation on \( i \).

In the next degree

\[ \begin{align*}
X_1 & \xleftarrow{d_0} X_2 \\
\xrightarrow{d_1} & X_3
\end{align*} \]

- \( d_0 \) returns the bottom level of nodes;
- \( d_1 \) composes the 2-level forest to a 1-level forest using monad multiplication;
- \( d_2 \) returns the monomial of corollas resulting from deleting the bottom nodes;
• $s_0$ sends a corolla to the 2-level tree obtained by grafting identity corollas onto all leaves;
• $s_1$ sends a corolla to the 2-level tree obtained by grafting that corolla onto a single identity corolla.

And so on. The beginning of the simplicial groupoid $\text{Bar}_S(R)$ can thus be pictured like this:

$$
\begin{array}{ccc}
S\left\{ \begin{array}{c}
\text{leaves} \\
\text{root}
\end{array} \right\} & \xleftarrow{\text{leaves}} & S\left\{ \begin{array}{c}
\text{delete bottom corolla} \\
\text{compose} \\
\text{delete top forest}
\end{array} \right\} \\
\end{array} \quad \cdots
$$

(11)

(The corollas in the picture are $R$-operations. The degeneracy maps are not rendered in the picture, to avoid clutter.)

**Proposition 2.3.6** (Weber [106], Prop. 4.4.1). For any operad $R$, the simplicial groupoid $\text{Bar}_S(R)$ is a strict category object and a Segal space.

**Proposition 2.3.7** ([72], Prop. 3.7). For any operad $R$, the two-sided bar construction $X = \text{Bar}_S(R)$ is an $S$-algebra in Segal spaces. Furthermore, the structure map $SX \to X$ is culf, and hence altogether $X$ is a symmetric monoidal decomposition space.

Note that $S$ preserves (both strict and) homotopy pullbacks, so if $X$ is Segal then so is $SX$. The $S$-algebra structure is a consequence of the fact that degree-wise, $X$ is $S$ of something. The $S$-algebra structure $SX \to X$ is given in degree $k$ by monad multiplication $SFR^k_1 \xrightarrow{\mu^S_{F^k_1}} SSF^k_1$. Culfness is a consequence of the fact that $S$ is cartesian.

2.3.8. Incidence bialgebra. Since the two-sided bar construction $\text{Bar}_S(R)$ is a Segal groupoid, and in particular a decomposition space [51], one can now apply the incidence coalgebra construction (1.3.5) to obtain a coalgebra structure on the slice $\text{Grpd}_{/X_1}$, with comultiplication given by the span

$$
\begin{array}{ccc}
X_1 & \xleftarrow{d_1} & X_2 \\
& \xrightarrow{(d_2,d_0)} & X_1 \times X_1.
\end{array}
$$

Recall that $X_1$ is the groupoid of monomials of operations of $R$, and $X_2$ is groupoid of monomials of 2-level trees in $R$. The symmetric monoidal structure is disjoint union. Thanks to its culfness property, the induced multiplication functor is compatible with the comultiplication so as to give altogether a bialgebra [51].

Shortly we shall impose the finiteness conditions necessary to take homotopy cardinality (4.1.1), which will finally give an ordinary bialgebra in $\mathbb{Q}$-vector spaces, where an $R$-operation $r$ is comultiplied as

$$
\Delta(r) = \sum_{r = b\circ(a_1, \ldots, a_n)} a_1 \cdots a_n \otimes b.
$$

(Because of the bialgebra structure, it is enough to specify the comultiplication on connected elements, i.e. the operations themselves, rather than monomials of operations.)
2.4 Two-sided bar construction on $P^*$

2.4.1. Set-up. We fix an operad in the form of a polynomial monad $P$, represented by $I \leftarrow E \rightarrow B \rightarrow I$. Then, as we have seen in 2.1.1, the free monad $P^*$ is represented by $I \leftarrow \text{tr}'(P) \rightarrow \text{tr}(P) \rightarrow I$. We denote by $F : I \rightarrow 1$ the unique map to the terminal groupoid.

Let $Y = \text{Bar}_S(P^*)$ denote the two-sided bar construction of $P^*$. We have

$$Y = \text{Bar}_S(P^*) : \quad SF_1 \xlongleftarrow{d_1} SF_i P^* \xlongleftarrow{d_2} SF_i P^* P^* \xlongleftarrow{d_0} SF_i P^* P^* \quad \ldots$$

Thus $Y_0 = SF_1 = SI$ is the groupoid of monomials of colours, and $Y_1 = SF_i P^* = S \text{tr}(P)$ is the groupoid of monomials of $P$-trees (which we may interpret as $P$-forests).

2.4.2. Interpretation of the groupoids $Y_n$. The general description of the two-sided bar construction in 2.3 tells us that $Y_n$ is the groupoid of (monomials of) $n$-level trees $W$ decorated by $P^*$-trees. This is the same as (monomials of) active maps $W \rightarrow \text{Maps from char}^T$, where $W$ is an $n$-level tree and $T$ is any $P$-tree. (Note that a 1-level $P$-tree is a corolla, and to give an active map from a corolla to a $P$-tree is the same thing as giving the $P$-tree (each $P$-tree receives a unique active map from a $P$-corolla (namely its residue 1.6.1))).

2.4.3. Inner face maps as precomposition. A (connected) $n$-simplex of $\text{Bar}_S(P^*)$ is an active map of $P$-trees $W \rightarrow \# T$, where $W$ is an $n$-level tree. The inner face maps $Y_{n-1} \xlongleftarrow{d_i} Y_n$ ($0 < i < n$) are described as follows. Let $W'$ be the $(n-1)$-level tree obtained from $W$ by contracting all the edges between levels $n - 1$ and $n$, then there is a unique active map $W' \rightarrow W$. Now $d_i(W \rightarrow T)$ is the composite $W' \rightarrow W \rightarrow T$.

The same kind of description works for the degeneracy maps.

2.4.4. Outer face maps in terms of active-inert factorisation in $\Omega$. The outer face maps of $Y$ involve active-inert factorisation in $\Omega$ (cf. 2.1.5). We describe the top face maps. Given a (connected) $n$-simplex $W \rightarrow \# T$, consider the $(n-1)$-level forest $W'$ obtained by deleting the root node of $W$, with the canonical inert map of forests $W' \rightarrow W$. The top face map $Y_{n-1} \xlongleftarrow{d_i} Y_n$ returns the $(n - 1)$-simplex $W' \rightarrow T'$ appearing in the active-inert factorisation

$$W' \rightarrow \# T' \quad \gamma \downarrow \quad W \rightarrow \# T.$$

The bottom face maps $Y_{n-1} \xlongleftarrow{d_i} Y_n$ are defined similarly, by deleting the leaf level instead of the root level.

2.4.5. Layerings and cuts. We define an $n$-layering of a $P$-tree $T$ to be an active map from an $n$-level tree $W \rightarrow \# T$.

We also call it a $P$-tree with $n - 1$ compatible cuts. For the actual computations, we shall stick with active maps, so in a sense layerings and cuts are just redundant terminology.
However, they serve to strengthen the combinatorial intuition with further pictures, and to relate to the notion of directed restriction species \[49\] and the Butcher–Connes–Kreimer bialgebra of Example 1.3.8, as we shall formalise in 5.4.

In particular, an active map \( W \rightarrow T \) from a 2-level tree encodes equivalently the following data: a bottom tree (the image of the root-level node of \( W \)) whose leaves are decorated with other trees (the images of the leaf-level nodes of \( W \)). This is conveniently interpreted as a tree with a cut, as illustrated here:

2.4.6. **Layer interpretation of the face and degeneracy maps.** Trees (and forests) should be read from leaves to root, in accordance with the operad interpretation, where leaves are inputs and root is output. The face maps are best understood from this viewpoint: in the lowest degree \( Y_0 \)
- \( d_0 \) deletes the leaf level, retaining only the root colour;
- \( d_1 \) deletes the root level, retaining only the monomial of leaf colours.

In the next degree, \( Y_1 \)
- \( d_0 \) deletes the leaf layer, leaving only a tree containing the root;
- \( d_1 \) joins the two layers, keeping the underlying \( P \)-tree fixed;
- \( d_2 \) deletes the root layer, leaving only the crown forest.

The degeneracy maps (not pictured) insert empty layers:
- \( s_0 : Y_0 \rightarrow Y_0 \) sends a colour (trivial tree) to the trivial tree of that colour;
- \( s_0 : Y_1 \rightarrow Y_2 \) adds the cut of all the leaves;
- \( s_1 : Y_1 \rightarrow Y_2 \) adds the cut of the root edge.

The top face maps generally take connected objects to non-connected ones, as exemplified by \( Y_1 \xrightarrow{d_2} Y_2 \) which takes a tree with a cut to the top forest. All the non-top face maps as well as the degeneracy maps send connected objects to connected objects.

2.5 **Two-sided bar construction on \( P^o \)**

Let \( P \) be any operad, represented by \( I \xleftarrow{E} B \xrightarrow{I} \). We have seen in 2.2.4 that the Baez–Dolan construction \( P^o \) is represented by \( B \xleftarrow{\text{tr}^* (P)} \xrightarrow{\text{tr} (P)} B \). We denote by \( F : B \rightarrow 1 \) the unique map (apologising for the notation clash: previously \( F \) denoted the map \( I \rightarrow 1 \), relevant for the free-monad construction).

\[\text{(12)}\]
Denote by $Z := \text{Bar}_S(P^\circ)$ the two-sided bar construction of $P^\circ$. The beginning of $Z$ looks like this ( picturing $P$-trees as plain trees, and omitting degeneracy maps):

\[
\begin{array}{c}
S \{ \ \} \quad \xleftarrow{\text{nodes}} \quad S \{ \ \} \quad \xleftarrow{\text{residue}} \quad S \{ \ \} \\
\quad \xleftarrow{\text{blobs}} \quad S \{ \ \} \quad \xleftarrow{\text{forget blobs}} \quad S \{ \ \} \quad \xleftarrow{\text{contract blobs}} \\
\end{array}
\]

as we now formalise.

**2.5.1. Description of the face maps of $Z = \text{Bar}_S(P^\circ)$**. All the non-top face maps, as well as the degeneracy maps, send connected configurations to connected configurations. The top face maps generally take a connected configuration to a non-connected one (as indicated with wavy arrows), exemplified by $Z_1 \overset{d_2}{\to} Z_2$ which takes a blobbed tree to the forest consisting of the individual blobs. More formally, given an element $K \to T$ in $Z_2$, there is a canonical cover of $K$ by corollas (one for each node), more precisely a bijective-on-nodes inert map of forests

$$\sum_i C_i \to K.$$

For each node $C_i$, take active-inert factorisation as in the diagram

$$
\begin{array}{c}
C_i \quad \xrightarrow{\text{active}} \quad S_i \\
\downarrow \quad \downarrow \\
K \quad \to \quad T.
\end{array}
$$

Altogether

$$d_2(K \to T) = S_1 \cdots S_k.$$

The general case is described as follows. $Z_n$ is the groupoid of sequences of active maps of $P$-trees

$$K^{(n)} \to K^{(n-1)} \to \cdots \to K^{(1)} \to K^{(0)} = T \to P,$$

with the condition that $K^{(n)}$ is a forest of corollas. The last map $T \to P$ is just to say that $T$ is a $P$-tree. By 2.1.7, this induces $P$-structure on all trees $K^{(i)}$ in the sequence.

The bottom face map $d_0$ consists in deleting $T$.

The middle face maps $d_i$ ($0 < i < n$) consist in composing two consecutive maps. The degeneracy maps $s_j$ just insert identity maps.

The top face map $d_n$ cannot just delete $K^{(n)}$, because of the requirement that the sequence begins with a forest of corollas. What it does instead is to ‘look into the nodes of $K^{(n)}$ and return the remaining sequence seen through that node’. More precisely, a node of $K^{(n)}$ defines an inert map from a corolla $C \to K^{(n)}$; let $C^{(n)}$ be the forest of all nodes in $K^{(n)}$, so that $C^{(n)} \to K^{(n)}$ is an inert map with the further properties that it is bijective on nodes and its domain is a forests of corollas. Now the top face map is defined by returning the sequence obtained by active–inert factorising the whole configuration:

$$
\begin{array}{c}
C^{(n)} \to K^{(n)} \\
\downarrow \\
K^{(n-1)} \to K^{(n-1)} \\
\downarrow \\
\cdots \\
\downarrow \\
K^{(0)} \to K^{(0)}.
\end{array}
$$
3 Incidence comodule bialgebra

So far we have seen that the simplicial groupoids given by two-sided bar construction on \( \mathcal{P}^* \) and \( \mathcal{P}^\circ \), respectively, have the same groupoid in degree 1, namely the groupoid \( S(\text{tr}(\mathcal{P})) \) of monomials of \( \mathcal{P} \)-trees, and therefore define two different bialgebra structures on \( \text{Grpd}_{/Y_1} \cong \text{Grpd}_{/Z_1} \). In this section we establish the first version of the Main Theorem (3.2.1), showing that these two bialgebras form a comodule bialgebra. We first need to set up the language required.

3.1 Comodule bialgebras

3.1.1. Classical comodule bialgebras. Recall first the definition (see for example Abe [1, § 3.2] and Manchon [81]). Fix a bialgebra \( B \) (over \( \mathbb{Q} \)). A comodule bialgebra over \( B \) is a bialgebra in the (braided) monoidal category of (left) \( B \)-comodules. Note that the notion of \( B \)-comodule uses only the coalgebra structure of \( B \), not the algebra structure, but that it is the algebra structure of \( B \) that endows the category of \( B \)-comodules with a monoidal structure is given as follows. If \( M \) and \( N \) are left \( B \)-comodules, then there is a left \( B \)-comodule structure on \( M \otimes N \) given by the composite map

\[
M \otimes N \to B \otimes M \otimes B \otimes N \xrightarrow{\mu_{13}} B \otimes M \otimes N.
\]

Here \( \mu_{13} \) is the map that first swaps the two middle tensor factors and then use the multiplication of \( B \) in the two now adjacent \( B \)-factors. It follows from the bialgebra axioms that this is a valid left \( B \)-comodule structure, giving the monoidal structure on the category of left \( B \)-comodules; the unit object is the \( B \)-comodule \( \mathbb{Q} \) (with structure map the unit of \( B \)). Furthermore, one checks that the braiding on the underlying category of vector spaces lifts to the category of \( B \)-comodules — this depends on the bialgebra \( B \) being commutative.

We now have a braided monoidal structure on \( B\text{-Comod} \), and it makes sense to consider bialgebras in here. A bialgebra in \( B\text{-Comod} \) is a \( B \)-comodule \( M \) together with structure maps

\[
\Delta_M : M \to M \otimes M \quad \quad \varepsilon_M : M \to \mathbb{Q} \quad \quad \eta_M : \mathbb{Q} \to M
\]

all required to be \( B \)-comodule maps and to satisfy the usual bialgebra axioms. We shall be concerned in particular with the requirement that \( \Delta \) and \( \varepsilon \) be \( B \)-comodule maps, which is to say that they are compatible with the coaction \( \gamma : M \to B \otimes M \):

\[
\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & M \otimes M \\
\downarrow{\gamma} & & \downarrow{\gamma \otimes \gamma} \\
B \otimes M \otimes B \otimes M & & B \otimes M \otimes B \otimes M \\
\downarrow{\mu_{13}} & & \downarrow{\mu_{13}} \\
B \otimes M & \xrightarrow{B \otimes \Delta_M} & B \otimes M \otimes M
\end{array}
\]

In the main theorem, the two other axioms will automatically be satisfied, because it will be the case that

- as a comodule, \( M \) coincides with \( B \) itself (with coaction = comultiplication),
• the algebra structure of $M$ coincides with that of $B$ (both will be free commutative).

We now turn to the objective version of these structures.

### 3.1.2. Comodule configurations.

Comodules arising in combinatorics are often the cardinality of certain slice-level comodules given by so-called comodule configurations of simplicial groupoids, first studied by Walde [101] and Young [107]. We follow the terminology of Carlier [12], [13]. For $X$ a decomposition space, a left $X$-comodule configuration is a culf map

$$u : M \to X,$$

where $M$ is a Segal space. It is useful to stare at the diagram:

\[
\begin{array}{cccccc}
X_0 & \xleftarrow{d_1} & X_1 & \xleftarrow{d_2} & X_2 & \cdots \\
\downarrow u & & \downarrow u & & \downarrow u & \\
M_0 & \xleftarrow{d_1} & M_1 & \xleftarrow{d_2} & M_2 & \cdots \\
\end{array}
\]

The actual comodule is then $\text{Grpd}_{/M_0}$, and the coaction by $\text{Grpd}_{/X_1}$ is the linear functor

$$\gamma : \text{Grpd}_{/M_0} \to \text{Grpd}_{/X_1} \otimes \text{Grpd}_{/M_0}$$

given by the span

$$M_0 \xleftarrow{d_1} M_1 \xrightarrow{(u,d_0)} X_1 \times M_0.$$

A comodule configuration $u : M \to X$ is called locally finite when $X$ is locally finite, and the face map $M_0 \xrightarrow{d_1} M_1$ is finite. This is the map whose fibres are summed over, so the condition is necessary and sufficient to be able to take homotopy cardinality to arrive at a comodule at the level of vector spaces, called the incidence comodule.

### 3.1.3. Example: decalage.

Recall that for any simplicial groupoid $X$, the decalage $\text{Dec}^\top X$ (also called the path-space construction [31]) is the simplicial groupoid obtained from $X$ by shifting all the groupoids down one degree, and omitting the top face and the top degeneracy maps. The original top face maps serve to give a simplicial map $u : \text{Dec}^\top X \to X$ called the dec map. It is a general fact ([51, Prop. 4.9]) that if $X$ is a decomposition space then $\text{Dec}^\top X$ is a Segal space and the dec map $u : \text{Dec}^\top X \to X$ is culf. This is to say that $\text{Dec}^\top X$ is a left comodule configuration over $X$. The corresponding incidence comodule is simply the incidence coalgebra of $X$ considered as a left comodule over itself.

### 3.1.4. Comodule bialgebras, objectively.

Given a symmetric monoidal decomposition space $Z$, there is induced a symmetric bialgebra structure on $\text{Grpd}_{/Z_1}$. To provide comodule-bialgebra structure on some slice $\text{Grpd}_{/A}$ we need to make the groupoid $A$ appear simultaneously as $A = Y_1$ for a monoidal decomposition space $Y$, and as $A = M_0$ for a left $Z$-comodule configuration $u : M \to Z$. Then we need to check the axioms. In the first case of interest, the underlying comodule, as well as its monoidal structure, will be $Z$ again. As a comodule configuration, this is more precisely the upper dec $u : \text{Dec}^\top Z \to Z$ from 3.1.3.
3.2 Main theorem, general form

Theorem 3.2.1. For any operad $P$, the two-sided bar constructions $\text{Bar}_S(P^*)$ and $\text{Bar}_S(P^\circ)$ together endow the slice $\text{Grpd}_{S(\text{tr}(P))}$ with the structure of a comodule bialgebra. Precisely, the incidence bialgebra of $\text{Bar}_S(P^*)$ is a left comodule bialgebra over the incidence bialgebra of $\text{Bar}_S(P^\circ)$.

3.2.2. Set-up. As usual, we assume $P$ is represented by $\xymatrix{I \ar[r]^-E & B \ar[r]^-I & }$, and employ the following notation:

- $Y = \text{Bar}_S(R^*)$ is the two-sided bar construction on $P^*$;
- $Z = \text{Bar}_S(P^\circ)$ is the two-sided bar construction on $P^\circ$.

These simplicial objects have the same groupoid in degree 1, which we give a special name:

$$A := Z_1 = Y_1 = S(\text{tr}(P)),$$

the ‘basis’ for the comodule bialgebra $\text{Grpd}_{/A}$. In all diagrams following, comultiplication in $Z$ (as well as the coaction) is written vertically, whereas comultiplication in $Y$ is written horizontally.

Proof of Theorem 3.2.1. We show below in 3.2.3 that the comultiplication of $Y$ is a $Z$-comodule map, and in 3.2.8 that the counit of $Y$ is a $Z$-comodule map. The two remaining axioms, that the algebra structure maps are $Z$-comodule maps, are automatically satisfied since the algebra structure is the same as that of $Z$, which is compatible with the comodule structure by the bialgebra axioms for $Z$.

Lemma 3.2.3. The comultiplication of $Y$ is a $Z$-comodule map.

Proof. We must show that the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc}
\text{Grpd}_{/A} & \xrightarrow{\Delta_Y} & \text{Grpd}_{/A} \otimes \text{Grpd}_{/A} \\
\downarrow \gamma & & \downarrow \gamma \otimes \gamma \\
\text{Grpd}_{/A} \otimes \text{Grpd}_{/A} & \xrightarrow{id \otimes \Delta_Y} & \text{Grpd}_{/A} \otimes \text{Grpd}_{/A} \otimes \text{Grpd}_{/A} \otimes \text{Grpd}_{/A}
\end{array}$$

(15)

Spelling out the spans that define these functors, we are faced with the solid diagram

$$\begin{array}{ccc}
A & \xleftarrow{d_Y^1} & Y_2 \\
\downarrow d_Y^2 & & \downarrow (d_Y^2, d_Y^0) & \downarrow d_Y^2 \times d_Y^1 \\
Z_2 & \xleftarrow{\gamma} & Z_2 \times Z_2 \\
\downarrow (d_Y^2, d_Y^0) & & \downarrow (d_Y^2, d_Y^0) \times (d_Y^2, d_Y^0) & \downarrow \mu_{13} \\
A \times A & \xleftarrow{id \times d_Y^1} & A \times Y_2 \\
\downarrow \mu_{13} & & \downarrow \mu_{13} \\
A \times A & \xleftarrow{d_Y^1} & A \times Y_2 \\
\downarrow \mu_{13} & & \downarrow \mu_{13} \\
A \times A & \xleftarrow{id \times (d_Y^2, d_Y^0)} & A \times A \times A \times A
\end{array}$$

(16)
To establish that Diagram (15) commutes, the standard technique (see in particular [13]) is to fill the diagram (16) with commutative squares, such that furthermore the lower left-hand and upper right-hand squares are pullbacks, as indicated. Then the Beck–Chevalley isomorphisms 1.1.3 will deliver the required natural isomorphism in (15).

We need to exhibit the middle groupoid $Q$ and the dotted maps. As is typical for an argument of this kind, these have formal categorical definitions and at the same time clean combinatorial interpretations. The groupoid $Q$ is the pullback

$$\begin{array}{ccc} Z_2 & \leftarrow & Q \\ d_2 \downarrow & & \downarrow L \\ A & \leftarrow & Y_2. \end{array}$$

In other words, the (connected) objects of $Q$ consist of an active map $K \rightarrow T$ (element in $Z_2$), and an active map $W \rightarrow K$ with $W$ a 2-level tree (element in $Y_2$), with the same $K$ (that’s the condition of being fibred over $A$). This can be taken as a strict pullback, because $d_1^Y$ is easily seen to be a fibration. In conclusion, $Q$ is the groupoid of (monomials of) active maps

$$W \rightarrow K \rightarrow T,$$

where $T$ and $K$ are $P$-trees, and $W$ is a 2-level $P$-tree. Such configurations in turn can be interpreted as (monomials of) blobbed $P$-trees with a compatible cut:

$$Q = S \left\{ \begin{array}{c} \end{array} \right\}$$

Here $T$ is the total tree, $K$ is the tree of blobs, and the 2-level tree $W$ is represented by the cut, as explained in 2.4.5.

It remains to exhibit the maps, and check that the squares are commutative and pullbacks as indicated. These checks occupy 3.2.4–3.2.7 below.

3.2.4. The upper left-hand square. The maps constituting the upper left-hand square are clear from the descriptions:

$$\begin{array}{ccc} A & \leftarrow & Y_2 \\ d_1 \downarrow & & \downarrow \text{forget } W \\ Z_2 & \leftarrow & Q. \end{array} \quad \begin{array}{ccc} \{T\} & \leftarrow & \{W \rightarrow T\} \\ \text{forget } K \uparrow & & \uparrow \text{forget } K \\ \{K \rightarrow T\} & \leftarrow & \{W \rightarrow K \rightarrow T\}, \end{array}$$

and it is obvious that the square commutes (but it is not a pullback).
3.2.5. The lower left-hand square is a pullback. The lower left-hand square is given by

\[
\begin{array}{c}
Z_2 \\
\downarrow \quad (\text{id}, d_0) \\
Z_2 \times A \\
\downarrow \quad d_2 \times \text{id} \\
A \times A
\end{array}
\longrightarrow
\begin{array}{c}
Q \\
\downarrow \\
Z_2 \times A \\
\downarrow \quad (\text{id} \times d_1) \\
A \times Y_2
\end{array}
\]

which intuitively is

\[
\{ K \rightarrow T \} \xleftarrow{\text{forget } W} \{ W \rightarrow K \rightarrow T \} \quad \text{(forest of image-trees , } K) \\
\downarrow \\
\{ S_1 \cdots S_k, K \} \xleftarrow{\text{forget } W} \{ S_1 \cdots S_k, W \rightarrow K \} \quad \text{(forest of image-trees , } W \rightarrow K)
\]

The left-hand component of the vertical maps takes \( K \rightarrow T \), interpret it as a blobbed tree, and return the forest of trees seen in the blobs. Formally this is given by active-inert factorising the maps \( C_i \rightarrow K \rightarrow T \), where \( C_1 \cdots C_k \) are the nodes of \( K \), as explained in 2.5.1. It is clear the square commutes. To see it is a pullback, compose vertically with the projection onto the second factor:

\[
\begin{array}{c}
\{ K \rightarrow T \} \xleftarrow{\text{forget } W} \{ W \rightarrow K \rightarrow T \} \quad \text{(forest of image-trees , } K) \\
\downarrow \\
\{ S_1 \cdots S_k, K \} \xleftarrow{\text{forget } W} \{ S_1 \cdots S_k, W \rightarrow K \} \\
\downarrow \quad \text{pr}_2 \\
\{ K \} \xleftarrow{\text{forget } W} \{ W \rightarrow K \}
\end{array}
\]

Now the outer rectangle is a pullback (it is the pullback defining \( Q \)). The bottom square is also a pullback, since projecting away an identity map is always a pullback. Therefore, by the Prism Lemma 1.2.5, also the top square is a pullback, which is the square of interest.

3.2.6. The upper right-hand square is a pullback. The upper right-hand square is

\[
Y_2 \xrightarrow{(d_Y^2, d_0^2)} A \times A \\
\uparrow \\
Q \longrightarrow Z_2 \times Z_2 \\
\leftarrow d_2^2 \times d_1^2 \\
\{ W \rightarrow T \} \xrightarrow{\text{return layers}} \{ (T', T'') \} \\
\uparrow \quad \text{forget } K \\
\{ W \rightarrow K \rightarrow T \} \xrightarrow{\text{return layers}} \{ (K' \rightarrow T'', K'' \rightarrow T''') \}.
\]

The horizontal map \( Y_2 \xrightarrow{(d_Y^2, d_0^2)} A \times A \) is described as follows (cf. 2.4.4). A (connected) element in \( Y_2 \) is an active map \( W \rightarrow T \) where \( W \) is a 2-level tree. By being a 2-level tree, it has a leaf-preserving inert forest inclusion \( W' \rightarrow W \) (where \( W' \) is a forest of corollas), and a root-preserving inert tree inclusion \( W'' \rightarrow W \) (where \( W'' \) is just a corolla), as in
the solid part of the diagram

\[
\begin{array}{c}
W'' \longrightarrow T' \\
| \quad | \\
\downarrow \quad \downarrow \\
W' \longrightarrow T' \\
| \quad | \\
\downarrow \quad \downarrow \\
W \longrightarrow T \\
| \quad | \\
\downarrow \quad \downarrow \\
W'' \longrightarrow T''.
\end{array}
\]

The map \((d_2, d_0) : Y_2 \to A \times A\) returns the pair \((T', T'')\) consisting of the forest \(T'\) and the tree \(T''\) appearing in the active-inert factorisation of the two maps to \(T\).

The other horizontal map

\[Q \longrightarrow Z_2 \times Z_2\]

is of the same nature. A (connected) element in \(Q\) is of the form \(W \rightarrow K \rightarrow T\) where \(W\) is a 2-level tree. Again we have the solid part of the diagram

\[
\begin{array}{c}
W'' \longrightarrow K' \longrightarrow T' \\
| \quad | \\
\downarrow \quad \downarrow \\
W' \longrightarrow K' \longrightarrow T' \\
| \quad | \\
\downarrow \quad \downarrow \\
W \longrightarrow K \longrightarrow T \\
| \quad | \\
\downarrow \quad \downarrow \\
W'' \longrightarrow K'' \longrightarrow T''.
\end{array}
\]

The map \(Q \to Z_2 \times Z_2\) returns the pair \((K' \rightarrow T', K'' \rightarrow T'')\).

The vertical maps in (17) simply forget the trees \(K\) (and \(K'\) and \(K''\)). More formally, let \(\text{Act}(\mathcal{P})\) denote the groupoid whose objects are the active maps of \(\mathcal{P}\)-trees. The right-hand map in (17) is a product of two copies of \((S\) of) \(\text{codom} : \text{Act}(\mathcal{P}) \to \Omega_{\text{iso}}(\mathcal{P})\), which is clearly a fibration: given an active map \(K \rightarrow T\) and \(T \cong S\), just compose to get an active map \(K \rightarrow S\). To see that the square (17) is a pullback (which is the main part of the proof of the theorem), we use the Fibre Lemma 1.2.7, applied to the strict fibres of these maps. The strict fibre of the map \(Q \to Y_2\) over an element \(W \rightarrow T\) is the groupoid

\[
\text{Fact}(W \rightarrow T)
\]

of all ways of factoring into \(W \rightarrow K \rightarrow T\). On the other hand, the fibre over the corresponding \((W'_1 \cdot \cdot \cdot W'_k, W'')\) \(\in A \times A\) is

\[
\left( \prod \text{Fact}(C'_i \rightarrow T'_i) \right) \times \text{Fact}(C'' \rightarrow T'').
\]

Altogether, the product is over all the nodes of \(W\), so now the equivalence of these two groupoids follows from the basic equivalence

\[
\text{Act}(W) \simeq \left( \prod \text{Act}(C'_i)\right) \times \text{Act}(C'')
\]

of Lemma 2.1.4.

In intuitive terms, the equivalence of the fibres says that to blob a tree \(T\) compatibly with a 2-layering (the active map \(W \rightarrow T\)) is the same as blobbing the root-layer tree and all the trees in the leaf-layer forest.
3.2.7. The lower right-hand square. We finally look at the bottom right square:

\[
\begin{array}{ccc}
\{W \to \# K \to \# T\} & \xrightarrow{\text{return layers}} & \{(K' \to \# T', K'' \to \# T'')\} \\
\downarrow & & \downarrow \text{disjoint union of forests of image trees, } K', K'' \\
\{S_1 \cdots S_k, W \to \# K\} & \xrightarrow{\text{keep the } S\text{-forest, return layers}} & \{S_1 \cdots S_k, K', K''\}
\end{array}
\]

where \(S_1 \cdots S_k = S'_1 \cdots S'_{k'} \cdot S''_1 \cdots S''_{k''}\). It is clear that the trees appearing in these two expressions are the same, but one may worry that they do not come in the same order. But in fact the monomials are not indexed by linear orders (that is only for notational convenience) — in reality they are indexed by the nodes of \(W\), and as such the two monomials are literally the same.

Now for the counit compatibility.

Lemma 3.2.8. The counit structure of \(Y\) is a \(Z\)-comodule map.

Proof. We must show that the following diagram commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\text{Grpd}_{/A} & \xrightarrow{\epsilon_Y} & \text{Grpd}_{/1} \\
\downarrow{\gamma} & & \downarrow{\eta} \\
\text{Grpd}_{/A} \otimes \text{Grpd}_{/A} & \xrightarrow{id \otimes \epsilon_Y} & \text{Grpd}_{/A} \otimes \text{Grpd}_{/1}.
\end{array}
\]

Spelling out the spans that define these functors, we are faced with the solid diagram

\[
\begin{array}{ccc}
A & \xleftarrow{s^Y_0} & Y_0 \xrightarrow{=} 1 \\
\downarrow{d^Y_1} & & \downarrow{=} \\
Z_2 & \leftarrow \text{?} \xrightarrow{\gamma} & 1 \\
\downarrow{(d^Y_2, d^Y_0)} & & \downarrow{\eta} \\
A \times A & \xleftarrow{\text{id} \times s^Y_0} & A \times Y_0 \xrightarrow{\eta} A \times 1.
\end{array}
\]

This time, as middle object we are forced to take simply \(Y_0\), in order to make the upper right-hand square a pullback. It remains to exhibit the other maps, and check that the squares are commutative and pullbacks as indicated. These checks occupy 3.2.9–3.2.11 below.

3.2.9. Upper left-hand square (of the counit compatibility check). This square

\[
\begin{array}{ccc}
A & \xleftarrow{s^Y_0} & Y_0 \\
\downarrow{d^Y_1} & & \downarrow{=} \\
Z_2 & \leftarrow & Y_0
\end{array}
\]

commutes because the map \(Z_2 \leftarrow Y_0\) sends a forest \(U\) of trivial trees to the identity map \(U\to U\). So both way around the square, the result is just \(U\) again.
3.2.10. Lower left-hand square (of the counit compatibility check). The square is

\[
\begin{array}{ccc}
Z_2 & \xleftarrow{\text{id} \times \alpha_0} & Y_0 \\
\downarrow_{(d^2_0, d^0_0)} & & \downarrow \\
A \times A & \xleftarrow{\text{id} \times s_0} & A \times Y_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
(U \to U) & \xleftarrow{\text{id}} & U \\
\downarrow & & \downarrow \\
(\emptyset, U) & \xleftarrow{\text{id}} & (\emptyset, U).
\end{array}
\]

The vertical map on the right takes a forest \( U \) of trivial trees, interpreted as the trivially blobbed trivial forest (hence having no blobs) to the pair \((\emptyset, U)\) consisting of the forest of all trees seen in the blobs (there are none), and the forest of all the trivial trees. It is clear this commutes. To see that it is also a pullback, paste below with the square projecting away the identity map:

\[
\begin{array}{ccc}
Z_2 & \xleftarrow{\text{id} \times \alpha_0} & Y_0 \\
\downarrow_{(d^2_0, d^0_0)} & & \downarrow \\
A \times A & \xleftarrow{\text{id} \times s_0} & A \times Y_0 \\
\downarrow_{\text{pr}_2} & & \downarrow_{\text{pr}_2} \\
A & \xleftarrow{s_0^0} & Y_0.
\end{array}
\]

We first show that the composite square is a pullback. The right-hand composite is the identity map. The left-hand composite, is the map dom : \( \text{Act}(P) \to \Omega_{\text{iso}}(P) \) sending an active map of \( P \)-trees to its domain, clearly a fibration. To check that the composite square is a pullback, we compare the fibres of the vertical maps over an element \( U \in Y_0 \) (\( U \) is a trivial forest). The fibre of the identity map is of course singleton. The fibre of dom : \( \text{Act}(P) \to \Omega_{\text{iso}}(P) \) is contractible by Lemma 2.1.4. So the composite square is a pullback by the Fibre Lemma 1.2.7. But the project-away-the-identity square is also a pullback. Therefore, by the Prism Lemma 1.2.5, also the top is square is a pullback, as required.

3.2.11. Lower right-hand square (of the counit compatibility check). Both ways around send a forest \( U \) of trivial trees to the empty forest \( \emptyset \) (of all the trees seen in the zero blobs).

This finishes the proof of Theorem 3.2.1.

4 Locally finite version of the incidence comodulebialgebra construction

4.1 Finiteness conditions and the reduced Baez–Dolan construction \( P^\circ \)

At the objective level, any poset, category, or decomposition space defines a coalgebra [51]. However, in order to take cardinality to arrive at an ordinary coalgebra in vector spaces, it is necessary to impose the finiteness condition that the poset, category, or decomposition space be locally finite [52]. This condition says that the two maps

\[
X_0 \xrightarrow{s_0} X_1, \quad X_1 \xleftarrow{d_1} X_2
\]

(18)
have finite (homotopy) fibres. Indeed, the formulae for the counit and comultiplication amount to summing over these fibres.

4.1.1. Locally finite operads. An operad \( P \) is called locally finite \([72]\) if its two-sided bar construction \( \text{Bar}_S(P) \) is locally finite. This is equivalent to demanding directly on the monad that the structure maps \( \mu : P \circ P \Rightarrow P \) and \( \eta : \text{Id} \Rightarrow P \) be finite. (Here, by definition, a map in \( \text{Grpd}/_I \) is finite if its image in \( \text{Grpd} \) is finite.)

Lemma 4.1.2. For any operad \( P \), the free operad \( P^* \) is locally finite.

Proof. Put \( Y = \text{Bar}_S(P^*) \). The fibre of \( Y_1 \xleftarrow{d_1} Y_2 \) over a \( P \)-tree \( T \) is the discrete groupoid of all ways to cut the tree \( T \), or more formally, the active maps \( W \to T \), where \( W \) is a 2-level tree. The map \( Y_0 \xrightarrow{\delta} Y_1 \) is even a monomorphism, and therefore in particular is finite.

Lemma 4.1.3. The Baez–Dolan construction \( P^\circ \) is never locally finite.

Proof. Put \( Z = \text{Bar}_S(P^\circ) \). The fibre of \( Z_1 \xleftarrow{d_1} Z_2 \) over a \( P \)-tree is the discrete groupoid of all ways of blobbing the tree, as in 2.2.5. Since each edge admits an arbitrary number of trivial blobs, this is an infinite set.

4.1.4. The reduced Baez–Dolan construction. For the sake of finiteness, and to be able to take cardinality, we need to work instead with the reduced Baez–Dolan construction

\[
P^\circ := P^\circ
\]

(already considered by Baez and Dolan \([3]\)). This is simply the operad \( P^\circ \) with all nullary operations removed. Recall that the operations of \( P^\circ \) are the \( P \)-trees, and that the nullary operations are the trivial \( P \)-trees; we are thus excluding trivial \( P \)-trees.

The two-sided bar construction of the reduced Baez–Dolan construction \( P^\circ \) will be denoted

\[
Z := \text{Bar}_5(P^\circ).
\]

For the rest of the paper, this will replace \( Z = \text{Bar}_5(P^\circ) \), which will no longer be considered.

Excluding nullary operations means disallowing trivial trees and trivial blobs. Intuitively, for the lowest degrees of \( Z \) we have:

- \( Z_0 \) is the groupoid of monomials of \( P \)-corollas;
- \( Z_1 \) is the groupoid of monomials of nontrivial \( P \)-trees;
- \( Z_2 \) is the groupoid of monomials of nontrivial \( P \)-trees with only nontrivial blobs.

More formally:

4.1.5. Active injections, reduced covers, spanning forests. The category of trees \( \Omega \) (and the category of \( P \)-trees \( \Omega(P) \)) has another factorisation system than the active-inert system exploited so far, namely the surjective-injective factorisation system. (The notions injective and surjective refer to the effect on edges.) All inert maps are injective, but the active maps come in two flavours: active injections which refine nodes into nontrivial trees (these are generated by the active coface maps), and active surjections which refine unary nodes into trivial trees (these are generated by the codegeneracy maps).

\[3\]In \([50]\) it is furthermore required that \( X_1 \) is homotopy finite, but this has turned out to be a superfluous requirement.
More formally we can now describe $Z_k$ as the groupoid of sequences of active injections

$$K^{(n)} \to K^{(n-1)} \to \cdots \to K^{(1)} \to K^{(0)} = T \to P,$$

with the condition that $K^{(n)}$ is a forest of corollas (just like in 2.5.1 except we now require active \textit{injections} instead of arbitrary active maps).

The opposite of the category of active injections into a fixed nontrivial tree $T$ is a preorder equivalent to the poset of reduced covers of $T$ (cf. [66, 2.3.2]). Both are equivalent to the power set of the set of inner edges in $T$. Here a reduced cover of $T$ is an inert map of forests $\sum_i R_i \to T$ which is bijective on nodes, and where the $R_i$ are nontrivial trees. It could also be called a spanning forest. The correspondence goes like this: given an active injection $K \to T$, let $R_i$ be the subtrees of $T$ arising from active-inert factorisation of composite maps $C_i \to K \to T$ as in 2.5.1 (where as usual the $C_i$ are the nodes of $K$).

With this correspondence, $Z_2$ can be described also as the groupoid of reduced covers of trees, and $Z_3$ can be described as the groupoid of reduced covers of reduced covers. The active-injections interpretation is good for describing $Z_1 \xleftarrow{d_1} \xrightarrow{d_0} Z_2$ (but not $d_2$): we have $d_0(K \to T) = K$ and $d_1(K \to T) = T$. The reduced-covers interpretation is convenient for describing $Z_1 \xleftarrow{d_2} \xrightarrow{d_1} Z_2$ (but not $d_0$): we have $d_1(\sum_i R_i \to T) = T$ and $d_2(\sum_i R_i \to T) = \sum_i R_i$.

In the following it will be practical to favour the active-injections interpretation, but we will have to convert to reduced-covers viewpoint each time we describe a top face map.

**Lemma 4.1.6.** For any operad $P$ the reduced Baez–Dolan construction $P^\otimes$ is locally finite.

**Proof.** The fibre of $Z_1 \xleftarrow{d_1} \xrightarrow{d_0} Z_2$ over a $P$-tree $T$ is the discrete groupoid of all active injections $K \to T$, this is a finite since there are only finitely many nodes in $T$. Put in other terms, the fibre is the discrete groupoid of all ways of blobbing the tree in such a way that each blob contains at least one node. The degeneracy map $Z_0 \xrightarrow{\Delta} Z_1$ assigns to a corolla the same corolla with a single blob around it. This map is even a mono, so in particular finite. \hfill \Box

**4.1.7. ‘General trees as comodule over nontrivial trees’.** We now have a locally finite simplicial groupoid $Z = Bar_S(P^\otimes)$ (which is a symmetric monoidal Segal space), so its incidence bialgebra admits a homotopy cardinality. But now it no longer has the same underlying space as the incidence bialgebra of $Y = Bar_S(P^*)$, and some further adjustments are required. It is not possible to adjust $Y$ in the same way, because $Y_0$ consists entirely of trivial trees, so these cannot just be thrown away. Instead we need a separate comodule structure on $Y_1$, which in the objective setting should be a comodule configuration $u : M \to Z$ with $M_0 = Y_1$. In rough terms we need to exhibit ‘general $P$-trees as a comodule over nontrivial $P$-trees’. (Note that it does not work simply to use $Bar_S(P^\otimes)$ as $M$, because it cannot possibly be culf over $Bar_S(P^\otimes)$: there are obviously more ways of drawing blobs on a tree than drawing nontrivial blobs.)

The simplicial groupoid $M$ is finally going to have

- $M_0$ the groupoid of (monomials of) arbitrary trees (possibly trivial);
- $M_1$ the groupoid of (monomials of) arbitrary trees (possibly trivial) with nontrivial blobs.

One can fiddle with these conditions, figure out what the higher $M_k$ should be, assemble them into a simplicial groupoid, and prove that it is culf over $Z$, so as to form
indeed a comodule configuration. Rather than doing this by hand, we shall embark on a small detour to be able to deduce these properties from a general construction: we shall define a relative two-sided bar construction \( C := F_1 \text{Bar}_P(P^{\otimes}) \) and then put \( M := SC \). General principles will then imply that \( M \) culf over \( Z \), and that \( M \) is locally finite as a comodule, as required.

### 4.2 Further bar constructions and a general comodule construction

We shall exploit further two-sided bar constructions to give an abstract construction of comodules from a pair of operads, one cartesian over the other.

#### 4.2.1 Set-up

We place ourselves in the situation of an operad map \( R \Rightarrow P \), in the form of polynomial monads related by monad opfunctors

\[
R \Rightarrow P \Rightarrow S,
\]

altogether represented by polynomial diagrams

\[
\begin{array}{ccc}
R : & J & \overset{U}{\leftarrow} V \overset{J}{\rightarrow} J \\
& G & \\
P : & I & \overset{E}{\leftarrow} B \overset{I}{\rightarrow} I \\
& F & \\
S : & 1 & \overset{B'}{\leftarrow} \overset{B}{\rightarrow} 1.
\end{array}
\]

The monad opfunctor \( R \Rightarrow P \) is given by the functor \( G ! \) and a natural transformation

\[
\theta : G_1 R \Rightarrow PG_1;
\]

the monad opfunctor \( P \Rightarrow S \) is given by the functor \( F ! \) and a natural transformation

\[
\psi : F_1 P \Rightarrow SF_1,
\]

both satisfying the axioms of 2.3.1. The composite exhibits also \( R \) as an operad, with monad opfunctor \( R \Rightarrow S \) given by the functor \( F_1 G_1 \) and the natural transformation

\[
\phi : F_1 G_1 R \Rightarrow SF_1 G_1,
\]

which is simply \( \psi G_1 \circ F_1 \theta \).

**Proposition 4.2.2.** From operads \( R \Rightarrow P \Rightarrow S \) as in 4.2.1, there is induced a simplicial map \( F_1 \text{Bar}_P(R) \rightarrow \text{Bar}_S(R) \), and it is culf. In other words, \( F_1 \text{Bar}_P(R) \) is a comodule configuration over \( \text{Bar}_S(R) \).

**Proof.** The two-sided bar constructions \( \text{Bar}_S(R) \) and \( F_1 \text{Bar}_P(R) \) are the top and bottom row of the diagram

\[
\begin{array}{ccc}
\text{Bar}_S(R) : & SF_1 G_1 & 1 \overset{d_1 \to d_0 \to s_0}{\leftarrow} SF_1 G_1 & R1 \overset{d_1 \to d_0}{\leftarrow} SF_1 G_1 & RR1 \leftarrow & \cdots \\
& \psi_{G_1} & \psi_{G_1,1} & \psi_{G_1,1} & \psi_{G_1,RR1} & \\
F_1 \text{Bar}_P(R) : & F_1 PG_1 & 1 \overset{d_1 \to d_0 \to s_0}{\leftarrow} F_1 PG_1 & R1 \overset{d_1 \to d_0}{\leftarrow} F_1 PG_1 & RR1 \leftarrow & \cdots
\end{array}
\]
The vertical comparison maps are components of the natural transformation $\psi$. We first check that this is a simplicial map. By naturality of $\psi$, it is clear that the vertical maps commute with all degeneracy and face maps except perhaps the top face maps. The top face maps are special since they involve the monad multiplication of $S$, and require a separate check: for $k \geq 0$, put $A := R \cdot A$. The compatibility with the top face between degree $k + 1$ and $k$ is commutativity of the outline of the diagram

Commutativity of the pentagon on the left is a monad-opfunctor axiom (8) for $\psi$. The triangle is (S applied to) the definition of $\phi$. Commutativity of the square is naturality of $\psi$. Furthermore, the simplicial map is cartesian on the non-wavy part because $\psi$ is a cartesian natural transformation. In particular the simplicial map is culf, and therefore makes the bottom row $F_i \text{Bar}_P(R)$ a comodule over the top row $\text{Bar}_S(R)$.

4.2.3. Finite operad maps. An operad map, in the form of a monad opfunctor $(G, \theta) : R \Rightarrow P$ represented by

$$
\begin{align*}
R : & \quad J \leftarrow U \rightarrow V \rightarrow J \\
P : & \quad I \leftarrow E \rightarrow B \rightarrow I,
\end{align*}
$$

is called finite when the maps $G$ and $H$ are finite (then the map $K$ is finite too, by pullback).

Lemma 4.2.4. If $(G, \theta) : R \Rightarrow P$ is finite, then all components of $\theta$ are finite.

Proof. It is enough to check that $\theta_1$ is finite, because $\theta$ is a cartesian natural transformation, and therefore all the other components of $\theta$ are pullbacks of $\theta_1$. The map $\theta_1$ is related to $G$ and $H$ by the diagram

$$
\begin{align*}
V = R(J) & \quad \downarrow \theta_1 \\
H & \quad P(J) \quad \downarrow P(G) \\
B = P(I) &
\end{align*}
$$

But $G$ and $H$ are finite by assumption, and $P(G)$ is too (because endofunctors underlying operads preserve finite maps). Now it follows from the next lemma that also $\theta_1$ is finite.
Lemma 4.2.5. Given maps of groupoids $A \xrightarrow{f} B \xrightarrow{g} C$, if two out of $f$, $g$, and $g \circ f$ are finite then so is the third.

Proof. With $b \in B$ and $c = g(b)$, consider the pullback diagram

$$
\begin{array}{ccc}
A_b & \longrightarrow & A_c \\
\downarrow & & \downarrow \\
1 & \longrightarrow & B_c \\
\downarrow & & \downarrow \\
1 & \longrightarrow & C.
\end{array}
$$

Here $A_b$, $B_c$, and $A_c$ are fibres of the maps $f$, $g$, and $g \circ f$, respectively. Now the result follows from the 2-out-of-3 property for finiteness \cite{50} in the fibre sequence $A_b \rightarrow A_c \rightarrow B_c$.

Lemma 4.2.6. If $R \Rightarrow P$ is finite, and $R$ is locally finite, then the simplicial groupoid $C = F_1 \text{Bar}_P(R)$ is locally finite (as a comodule configuration). This means that the face map $C_0 \xleftarrow{d_1} C_1$ is finite.

Proof. The solid diagram expresses one of the axioms (8) for the monad opfunctor $R \Rightarrow P$. The maps $\theta_1$ and $\theta_{R_1}$ are finite by Lemma 4.2.4 since the operad map is finite. Furthermore, $G_1 \mu_{R_1}^P$ is finite since $R$ is assumed locally finite. It now follows from Lemma 4.2.5 that the dotted arrow is finite. But the map we are concerned with, $C_0 \xleftarrow{d_1} C_1$, is $F_1$ applied to this dotted map (and lowershrieks preserve finiteness).

4.2.7. Free $S$-algebra on a comodule. For the desired application of this construction, we will need to pass to a comodule of monomials of $R$-operations. This is achieved in a canonical way since Bar$_S(R)$ is a symmetric monoidal decomposition space. The Bar$_S(R)$-comodule structure on $SF_1 \text{Bar}_P(R)$ is given by

$$
SF_1 \text{Bar}_P(R) \rightarrow S \text{Bar}_S(R) \rightarrow \text{Bar}_S(R),
$$

where the last map is the symmetric monoidal structure. This composite is again culf, because $S$ preserves culfness, and the structure map itself is culf (1.6.3). With shorthand notation $Z := \text{Bar}_S(R)$ and $C := F_1 \text{Bar}_P(R)$, the relevant span from this comodule configuration is

$$
SC_0 \xleftarrow{S(d_1)} SC_1 \xrightarrow{S(u, d_0)} S(Z_1 \times C_0) \xrightarrow{\sim} SZ_1 \times SC_0 \rightarrow SZ_1 \times SC_0 \rightarrow Z_1 \times SC_0
$$

(the two middle maps expressing together that $S$ is colax monoidal with respect to the cartesian product).
4.3 Main theorem, locally finite version

4.3.1. Set-up. Let $P$ be any operad, represented by $I \leftarrow E \rightarrow B \rightarrow I$, and let $F$ denote either of the maps $I \rightarrow 1$ and $B \rightarrow 1$. (In any case we use $F_i$ only to move from slices to $Grpd$.)

We now instantiate the constructions of 4.2 to the operad map $P \Rightarrow P$. This map is finite: in the notation of 4.2.3, $G$ is the identity, and $V \rightarrow B$ is the monomorphism given by inclusion of the nontrivial part, and in particular is finite too. As before, we put $Z := \text{Bar}_S(P \circ), \quad C := F_i \text{Bar}_P(P \circ), \quad M := SC$.

**Lemma 4.3.2.** The simplicial map

$$u : F_i \text{Bar}_P(P \circ) \rightarrow \text{Bar}_S(P \circ),$$

is cof, and hence constitutes a comodule configuration. This comodule configuration is furthermore locally finite.

**Proof.** That $u$ is a comodule configuration is an immediate consequence of Proposition 4.2.2. Local finiteness follows from 4.2.6, since clearly $P \Rightarrow P$ is finite. 

The comodule configuration expands to

$$Z : \quad SF_1 1 \leftrightarrow d_1 \rightarrow s_0 \leftrightarrow SF_1 P \circ 1 \leftrightarrow d_2 \rightarrow s_0 \leftrightarrow SF_1 P \circ P \circ 1 \rightarrow \ldots$$

$$C : \quad F_i P \circ 1 \leftrightarrow d_1 \rightarrow s_0 \leftrightarrow F_i P \circ P \circ 1 \leftrightarrow d_2 \rightarrow s_0 \leftrightarrow F_i P \circ P \circ P \circ 1 \rightarrow \ldots$$

In the first row, we have arbitrary corollas, then nontrivial trees, then nontrivial trees with only nontrivial blobs. In the second row, we have arbitrary trees, then arbitrary trees with nontrivial blobs, then arbitrary trees with nested nontrivial blobbings, and so on. In pictures:

(19)
Proposition 4.3.3. For any operad \( P \), we have
\[
C := F_1 \text{Bar}_P^\circ (P^\circ) \simeq N\Omega_{\text{act.inj}}(P)^{\text{op}}.
\]
Recall that \( N \) denotes the fat nerve.

Proof. The objects in \( C_0 \) are arbitrary \( P \)-trees (including the trivial trees); these are also the objects of \( \Omega_{\text{act.inj}}(P) \). The objects in \( C_1 \) are active injections \( K \rightarrow T \). The face maps go as follows: \( C_0 \xrightarrow{d_0} C_1 \) returns \( K \) (‘contract the blobs’), while \( C_0 \xrightarrow{d_1} C_1 \) returns \( T \) (‘underlying tree’). So this is precisely opposite to how the face maps work in \( N\Omega_{\text{act.inj}}(P) \). In higher degrees the checks are the same. \( \square \)

The equivalence in Proposition 4.3.3 is just the restriction of the following (and the proof the same):

Proposition 4.3.4. For any operad \( P \), we have
\[
F_1 \text{Bar}_P^\circ (P^\circ) \simeq N\Omega_{\text{active}}(P)^{\text{op}}.
\]

This result is interesting also in view of its analogy with the following:

Proposition 4.3.5. For any operad \( P \), we have
\[
F_1 \text{Bar}_P^* (P^*) \simeq N\Omega_{\text{inert,root-pres}}(P)^{\text{op}}.
\]

Here \( \Omega_{\text{inert,root-pres}}(P) \) is the category of \( P \)-trees and only root-preserving inert maps. The two propositions add a new layer to the analogy (observed in [66]) between \( \Omega_{\text{inert}} \) with the (root-preserving, leaves-preserving) factorisation system and \( \Omega \) with the (active, inert) factorisation system.

We are now ready for the locally finite version of the Main Theorem 3.2.1:

Theorem 4.3.6. For any operad \( P \), the two-sided bar constructions \( Y = \text{Bar}_S^* (P^*) \) and \( Z = \text{Bar}_S^\circ (P^\circ) \) together endow the slice \( \text{Grpd}_{/S(\text{tr}(P))} \) with the structure of a locally finite comodule bialgebra. Precisely, the incidence bialgebra of \( Y \) is a locally finite left comodule bialgebra over the incidence bialgebra of \( Z \).

Proof. The comodule structure on \( \text{Grpd}_{/S(\text{tr}(P))} \) is given by the comodule configuration
\[
u : SF_1 \text{Bar}_P^\circ (P^\circ) \rightarrow \text{Bar}_S^\circ (P^\circ)
\]

of Lemma 4.3.2, where also its finiteness as comodule is established. (Finiteness of the involved bialgebras was established in Lemmas 4.1.2 and 4.1.6.) Establishing that the bialgebra structure maps of \( Y \) are \( Z \)-comodule maps is a repetition of all the arguments in the proof of Theorem 3.2.1. The difference is that is that the comodule configuration \( M \) is no longer just \( Z \) itself, so the face maps of \( Z = \text{Bar}_S^\circ (P^\circ) \) must be replaced by the ones from \( M = SF_1 \text{Bar}_P^\circ (P^\circ) \). Writing \( A := Y_1 = M_0 \), the main diagram now takes the
and the middle object has to be $Q := M_1 \times_A Y_2$. The groupoids involved are full sub-groupoids of those in 3.2.1, so commutativity of the diagrams and the fibre calculations to establish the pullback conditions are the same again.

\[ \text{(20)} \]

## 5 Examples

### 5.1 Baez–Dolan construction on categories

Before coming to more explicit examples, we deal with the case where $P$ is a small category, or more precisely a Segal groupoid (such as for example the fat nerve of an ordinary category). We view $P$ as an operad with only unary operations. As a polynomial monad this means that it is cartesian over the identity monad

\[ P \Rightarrow \text{Id} \Rightarrow S, \]

which leads to some special features. The first of these is straightforward:

**Lemma 5.1.1.** For any small category $R$, considered as an operad with only unary operations, $R \Rightarrow \text{Id}$, we have

\[ \text{Bar}_S(R) \simeq S(\text{Bar}_{\text{Id}}(R)) \simeq S(NR). \]

(Taking fat nerve here is only to stress that it is regarded as a simplicial groupoid, not as a polynomial monad.)

If $P$ is a small category, then so is $P^*$. We thus have

\[ Y := \text{Bar}_S(P^*) = S(\text{Bar}_{\text{Id}}(P^*)). \]

It follows that the comultiplication of $Y$ is actually $S$ of a comultiplication, namely the standard incidence comultiplication of the Segal groupoid $P^*$. Recall that in Theorem 4.3.6, the comodule structure too comes from a simplicial groupoid which is $S$ of something: $M = SC$. In fact, for $P$ a category, the whole comodule bialgebra structure is just $S$ of a comodule coalgebra structure. The following result formalises this, providing a cheaper unary version of the main theorems. The proof is very similar.

**Theorem 5.1.2.** For any small category $P$, the incidence coalgebra of $\text{Bar}_{\text{Id}}(P^*) \simeq NP^*$ is a left comodule coalgebra over the incidence bialgebra of $\text{Bar}_{P^0}(P^\otimes)$. 

49
5.1.3. Remark. The comodule bialgebras of Carlier [13] defined from hereditary species are all of the form ‘S of a comodule coalgebra’, although they do not come from neither categories nor operads.

Adding an algebra structure freely on top of a coalgebra structure is important to get access to the machinery of antipodes [91] rather than just Möbius inversion (see [14] for this perspective at the objective level). One important special case is that of the cartesian envelope of a poset, which has been exploited to great effect by Aguiar and Ferrer [2].

5.2 Faà di Bruno

5.2.1. Baez–Dolan construction on the identity monad. Let $P$ be the terminal category. Considered as a polynomial monad it is the identity monad $P := \text{Id}$, represented by the polynomial $1 ← 1 → 1 → 1$ (as in Example 2.2.7). Now $P$-trees are linear trees, and the category of $\text{Id}$-trees is $\Omega(\text{Id}) \simeq \Delta$.

The free monad $P^* = \text{Id}^*$ is the one-object category whose arrows are the linear trees (with leaf as domain and root as codomain), composed by grafting. In other words, the category is just the monoid $(\mathbb{N}, +, 0)$.

The Baez–Dolan construction is the operad $P \circ \text{Id} = \text{Id} \circ$ whose only colour is $\bullet$, and whose operations are the linear trees. The input slots are the nodes, and substitution works like this:

This operad is the free-monoid operad: $\text{Id}^\circ \simeq M$. From the $\text{Id}^\circ$ viewpoint, the operations are linear trees, and the operations of $\text{Id}^\circ \circ \text{Id}^\circ$ are active maps of linear trees. From the viewpoint of the equivalent operad $M$, the operations are planar corollas, and the operations of $M \circ M$ are 2-level planar trees. For the reduced Baez–Dolan construction $\text{Id}^\circ \simeq \overline{M}$ we just exclude the trivial tree (or in the corolla interpretation, the nullary corolla).

The two-sided bar construction giving the comodule configuration is described by Proposition 4.3.3:

$$\text{Bar}_{\text{Id}^\circ}(\text{Id}^\circ) \simeq N\Delta_{\text{act.inj}}^\text{op},$$

and combined with the well-known equivalence (see [51])

$$\Delta_{\text{act.inj}}^\text{op} \simeq \Delta_{\text{surj}}^+$$

we get the interpretation from the $M$-viewpoint: $\text{Bar}_M(\overline{M})$ is the fat nerve of the category $\Delta_{\text{surj}}^+$ of finite ordinals (including the empty ordinal) and monotone surjections.

(For the bar construction $\text{Bar}_{\text{Id}^\circ}(\text{Id}^\circ)$ giving the bialgebra, there is also a nerve interpretation: it is the category whose objects are finite sets and whose maps are surjections equipped with a linear order on each fibre. This is a bit exotic, and it is more convenient just to continue with tree interpretations.)
5.2.2. Incidence comodule bialgebra. Denote by $a_n$ the linear tree with $n$ nodes. The comultiplication coming from $\text{Bar}_S(\text{Id}^*) = \text{Bar}_S(\mathbb{N})$ is given (on connected elements) by

$$\Delta(a_n) = \sum_{i+j=n} a_i \otimes a_j \quad n \geq 0.$$ 

Here the sum is over nonnegative numbers summing to $n$.

The comultiplication coming from $\text{Bar}_S(\text{Id} \circ)$ is given (on connected elements) by

$$\Delta(a_n) = \sum_{n_1+\cdots+n_k=n} a_{n_1} \cdots a_{n_k} \otimes a_k \quad n \geq 1.$$ 

Here the sum is over all compositions of the natural number $n$. This composition comes about as an active map $a_k \rightarrow a_n$. Inclusion of the $k$ one-node trees into $a_k$ defines the numbers $n_i$ by active-inert factorisation:

$$[1] \quad \cdots \rightarrow \quad [n] \quad \downarrow \quad \vdash \quad \downarrow \quad [k] \quad \rightarrow \quad [n].$$

**Proposition 5.2.3.** The incidence comodule bialgebra of the Baez–Dolan construction on $\text{Id}$ is the Faà di Bruno comodule bialgebra.

**Proof.** This is clear once we describe the Faà di Bruno comodule bialgebra. Consider the ring of formal power series $\mathbb{Q}[[z]]$ under multiplication and under substitution (assuming zero constant term).

For $k \geq 0$, consider the linear functional

$$a_k : \mathbb{Q}[[z]] \rightarrow \mathbb{Q}$$

$$\sum_n f_n z^n \mapsto f_k.$$

The *Faà di Bruno bialgebra* is the polynomial ring $\mathcal{F} = \mathbb{Q}[a_1, a_2, \ldots]$, with comultiplication $\Delta : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$ given as dual to substitution of power series (we disallow constant terms):

$$\Delta(a_n)(f \otimes g) := a_n(g \circ f).$$

It follows that

$$\Delta(a_n) = \sum_{n_1+\cdots+n_k=n} a_{n_1} \cdots a_{n_k} \otimes a_k, \quad n \geq 1,$$

just as for the $Z$-comultiplication of linear trees.

On the other hand, $\mathcal{M} = \mathbb{C}[a_0, a_1, a_2, \ldots]$ is a bialgebra too, with comultiplication $\Delta : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}$ dual to multiplication of power series:

$$\Delta(a_n)(f \otimes g) := a_n(f \cdot g).$$

This expands to

$$\Delta(a_n) = \sum_{i+j=n} a_i \otimes a_j, \quad n \geq 0,$$

precisely as for the $Y$-comultiplication of linear trees.

Together, $\mathcal{F}$ and $\mathcal{M}$ form the *Faà di Bruno comodule bialgebra*: $\mathcal{M}$ is a comodule bialgebra over $\mathcal{F}$. The comodule-bialgebra axioms follow from the fact that substitution distributes over multiplication on the left.
5.2.4. Remarks. The present form of the Faà di Bruno bialgebra is how it appears in
algebraic topology, where it is called the (dual) Landweber–Novikov bialgebra (see for
example [86]). The usual form of the Faà di Bruno bialgebra (see for example [42]) uses
rather the basis
\[ A_k : \mathbb{Q}[[z]] \to \mathbb{Q} \]
\[ \sum_n f_n z^n \to f_k, \]
in which case the natural simplicial realisation is as the fat nerve of the category of finite
sets and surjections (cf. [64], [49]). This comes about as the two-sided bar construction
\( \text{Bar}_S(S) \) (see [72]), but cannot arise as a Baez–Dolan construction.

The factorial-free form of the Faà di Bruno bialgebra given here admits a noncommu-
tative variant [8] which is also called the Dynkin–Faà di Bruno bialgebra in the theory
of numerical integration on manifolds [87]; see also [35]. Objectively, the noncommuta-
tive bialgebra is the incidence bialgebra of \( \text{Bar}_{\Omega^\circ}(\text{Id}^\circ) \), but this is not so interesting in
the present context, since noncommutative bialgebras do not in general admit comodule
bialgebras.

5.3 Mould calculus

5.3.1. Mould calculus. The mould calculus was introduced by Écalle [38] as a com-
binatorial toolbox for his theory of resurgence in the theory of local dynamical systems
(see Cresson [28]). For \( \Omega \) a monoid, let \( (\Omega^*, \cdot) \) denote the free monoid on \( \Omega \). A mould
is a function \( M^* : \Omega^* \to k \) (for \( k \) a ring), taking a word \( w \in \Omega^* \) to \( M^w \). There are two
basic operations: the product is defined as
\[(M \times N)^w = \sum_{w=w''w'''} M^{w''} N^{w'''}, \quad \omega \in \Omega^* \]
(where \( \cdot \) is concatenation of words), and the composition is defined as
\[(M \circ N)^w = \sum_{k>0} \sum_{w=w_1 \cdots w_k} N^{w_1} \cdots N^{w_k} M^{||w_1|| \cdots ||w_k||}, \quad \omega \in \Omega^*. \]
Here \( ||w|| \in \Omega \) denotes the multiplication of the word \( w \in \Omega^* \) in the monoid \( \Omega \). Composition
distributes over the product, but only from the left [38].

5.3.2. Moulds via Baez–Dolan construction. Consider the monoid \( \Omega \) as an operad
with only one colour and only unary operations, and let \( \Omega^\circ \) be the Baez–Dolan construc-
tion. Its colours are the elements of \( \Omega \). There is a \( k \)-ary operation of profile \( (a_1, \ldots, a_k; b) \)
if and only if \( a_1 \cdots a_k = b \), for \( k > 0 \). There is also the operad \( \Omega^* \), the free operad on
\( \Omega \). The incidence algebra of the operad \( \Omega^* \) is the algebra of moulds under \( \times \), and the
incidence algebra of \( \Omega^\circ \) is the algebra of moulds under \( \circ \). More precisely, the incidence
bialgebra of the whole structure is a comodule bialgebra.

5.4 B-series, and the Calaque–Ebrahimi-Fard–Manchon comod-
ule bialgebra

Throughout we have considered operadic trees — trees with open ended edges. In this
subsection we are concerned with trees without open-ended edges: they are defined as
connected and simply connected graphs with a distinguished vertex called the root. In the (huge) literature employing them, they are simply called rooted trees. Below we adhere to this convention as long as no operadic trees are involved, but call them combinatorial trees when contrast with operadic trees is required (after all, operadic trees are rooted too).

5.4.1. B-series. B-series were introduced by Butcher [10] in his study of order conditions for Runge–Kutta methods, and named after him by E. Hairer (see [58]). They are formal series indexed by (combinatorial) rooted trees, of the form

\[ B(a, hf, y) = \sum_{\tau \in \mathcal{T}} \frac{h^{\vert \tau \vert}}{\tau!} a(\tau) f^\tau(y) \]

where \(\mathcal{T}\) is the set of rooted trees, \(\vert \tau \vert\) denotes the number of nodes in \(\tau\), and \(f^\tau\) is the elementary differential associated to \(\tau\) (for \(f\) a vector field), and \(h\) is a step-size parameter. \(a(\tau)\) are the coefficients, encoded as a complex-valued function on \(\mathcal{T}\). For an initial-value problem \(\dot{y} = f(y), y(0) = y_0\), the exact solution can be expanded as a B-series, but more importantly, many numerical methods, including all Runge–Kutta methods,\(^4\) can be regarded as a B-series, the coefficients \(a(\tau)\) being the weights assigned to the elementary differentials \(f^\tau\) of \(f\).

5.4.2. Composition and substitution of B-series. There are two fundamental operations one can perform on B-series: composition and substitution. Composition (due to Butcher [10]): for \(b(\emptyset) = 1\),

\[ B(a, hf, B(b, hf, y)) = B(b \cdot a, hf, y). \]

This characterises the product \(\cdot\) defining the Butcher group of B-series, which was later rediscovered as the group of characters of the Connes–Kreimer Hopf algebra of rooted trees in perturbative renormalisation [30], [25] (see Example 1.3.8).

Substitution (introduced by Chartier, E. Hairer, and Vilmart [18], [20]): if \(b(\emptyset) = 0\) then \(B(b, hf, -)\) is a vector field, so it makes sense to substitute it into another B-series in the \(hf\) slot:

\[ B(a, B(b, hf, -), y) = B(b \star a, hf, y). \]

This characterises a new product \(\star\), which can be described combinatorially in terms of contracting subtrees. They showed that \(\star\) acts on \(\cdot\) by group homomorphisms. (The substitution product is important in backward error analysis and in the more general theory of modified (preprocessed) integrators [19].)

5.4.3. The Calaque–Ebrahimi-Fard–Manchon comodule bialgebra. Calaque, Ebrahimi-Fard, and Manchon [11] gave a Hopf-algebra theoretic interpretation of composition and substitution inspired by quantum field theory, relating the substitution product with a tree version of the Connes–Kreimer Hopf algebra of Feynman graphs [73], [26].

---

\(^4\)An intrinsic characterisation of B-series methods was given only recently in terms of affine equivariance [79].
The bialgebras are both the free commutative algebra on the set of rooted trees. The comultiplications are such that their respective characters form the group structures on B-series.

The comultiplication corresponding to composition of B-series is simply the Butcher–Connes–Kreimer Hopf algebra of rooted trees of Example 1.3.8. The comultiplication corresponding to substitution of B-series is defined by summing over all ways of partitioning the set of nodes into subtrees. The left-hand tensor factor is then constituted by the forest of all these subtrees, whereas the right-hand tensor factor is obtained by contracting each subtree to a single node.

Since the trees involved are only combinatorial trees (as opposed to operadic trees), it is not possible to realise this bialgebra as the incidence bialgebra of an operad — there is not enough typing information available to make sense of substituting a tree into the node of another tree. But for operadic trees this works, and we shall see that the operadic analogue of the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra is the incidence comodule bialgebra of a Baez–Dolan construction (of the terminal operad). The precise relationship is given by taking core of an operadic tree, as we proceed to explain.

5.4.4. The core of a P-tree [68]. The core of a P-tree is the combinatorial tree constituted by its inner edges, obtained by forgetting all decorations and shaving off leaf edges and root edge. Taking core constitutes a bialgebra homomorphism from the bialgebra of P-trees (the incidence bialgebra of Bar\(S(P^\circ)\)) to the Butcher–Connes–Kreimer Hopf algebra. This bialgebra homomorphism is induced by a symmetric monoidal cuf map from Bar\(S(P^\circ)\) to the decomposition space of Example 1.3.8.

The core map compares operadic approaches with the standard combinatorial viewpoint in renormalisation. One advantage of the operadic viewpoint is that the operadic trees have a leaf grading, which cannot be seen in the core. This has been exploited in connection with BPHZ renormalisation [69] and in connection with combinatorial Dyson–Schwinger equations [70]. Existence of the leaf grading is closely related to the fact that operadic trees form a Segal object, whereas taking core destroys the Segal-ness. Combinatorial trees form instead only a decomposition space [54], as mentioned in Example 1.3.8.

We now specialise to the case where P is the terminal operad (Comm), so that P-trees are just naked trees.

5.4.5. Core of Bar\(S(P^\circ)\) and \(F_i\) Bar\(\circ\)\(P(P^\circ)\). Each of the bar constructions Bar\(S(P^\circ)\) and \(F_i\) Bar\(\circ\)\(P(P^\circ)\) admits a core version (which however cannot possibly arise as the two-sided bar construction of an operad). We describe the first one by hand, by mimicking the explicit description of the two-sided bar construction of Bar\(S(P^\circ)\).

Let \(H^\circ\) denote the simplicial groupoid with \(H^\circ_1\) the groupoid of combinatorial forests, and \(H^\circ_2\) the groupoid of blobbed combinatorial forests, such that each blob contains at least one node (and as usual, each node is contained in precisely one blob). \(H^\circ_0\) is the groupoid of forests consisting only of one-node trees. The face and degeneracy maps have the same descriptions as in Figure (19).

For the comodule \(F_i\) Bar\(\circ\)\(P(P^\circ)\) the core is in fact the upper decalage of \(H^\circ\). Just as Bar\(\circ\)\(P(P^\circ)\) is the fat nerve of the opposite of the category of nontrivial P-trees and active injections (cf. Proposition 4.3.3), also the core is a fat nerve, namely the fat nerve of the category of combinatorial trees and edge contractions.
Lemma 5.4.6. The simplicial groupoid $H^\circ$ just described is a symmetric monoidal decomposition space. Taking core defines a cuf map $\text{Bar}_5(\mathbb{P}^\circ) \to H^\circ$ (and hence also a cuf map from $F_1 \text{Bar}_{\mathbb{P}}(\mathbb{P}^\circ)$).

Proof. The proofs are standard and will not be given in detail. The symmetric monoidal structure is just ‘disjoint union’. To check the decomposition-space axiom, the first observation is that the inner face maps are discrete fibrations (the fibres being discrete sets of possible blobbings). The appropriate pullback squares are now verified by computing fibres of the inner face maps involved. Cuf-ness of the taking-core map is clear: the possible blobbings of a $\mathbb{P}$-tree depends only on its core, not on leaves, root, or $\mathbb{P}$-structure.

Proposition 5.4.7. The core of the incidence comodule bialgebra of the Baez–Dolan construction of the terminal operad is the Calaque–Ebrahimi-Fard–Manchon comodule bialgebra.

Proof. This is just a verification. For the Butcher–Connes–Kreimer comultiplication the result was already in [68]. For the Chartier–Hairer–Vilmart–Calaque–Ebrahimi-Fard–Manchon comultiplication, the point is simply that the summation is over blobbings of a given tree. (See also [72, Ex. 7.5] for this result.)

5.5 Other examples

5.5.1. Linear orders and comodule bialgebras of monotone words. Let $\mathbb{P}$ be a (countable) linear order, considered as a linearly ordered alphabet. For definiteness we shall take the linear order $\mathbb{P} = \mathbb{N}$. Since $\mathbb{P}$ is a poset and hence a category, it is a coloured operad with only unary operations. The $\mathbb{P}$-trees are the non-empty monotone words.

The operad $\mathbb{P}^\ast$ is the category whose objects are the letters, and whose morphisms from $i$ to $j$ are the monotone words that start in $i$ and end in $j$ (allowing the one-letter word for $i = j$). Two words are composed by 1-letter-overlap concatenation, as exemplified by

$$
\begin{array}{c}
2667 \\
\sim \\
\rightarrow 78899
\end{array}
\rightarrow
\begin{array}{c}
26678899
\end{array}
$$

The operad $\mathbb{P}^\circ$ has colours monotone two-letter words. The operations are monotone words of length at least 2. The output colour of a word is the pair consisting of the first and the last letter. The input slots of such a word are the gaps between letters, and the colour of an input slot is the pair of adjacent letters. A whole word can thus be substituted into a gap provided its first and last letters agree with the adjacent letters of the gap, and in the result of the substitution, these two letters are replaced by the whole word, as pictured here:

$$
\begin{array}{c}
22456 \\
\sim \\
\rightarrow 11224566899
\end{array}
\rightarrow
\begin{array}{c}
11224566899
\end{array}
\rightarrow
\begin{array}{c}
11266899
\end{array}
$$
(Had we taken $P^\circ$ instead of $P^\circ$, we would have also the one-letter words, and an allowed substitution would be, for instance, to replace a subword 44 by 4.)

The comodule bialgebra is the polynomial algebra on these monotone words. For the comultiplication corresponding to $P^\ast$, the 1-letter words are group-like, and the degree of a word is its length minus 1.

Example:

$$\Delta(2335) = 2 \otimes 2335 + 23 \otimes 335 + 233 \otimes 35 + 2335 \otimes 5.$$ 

The comultiplication corresponding to $P^\circ$ is given by summing over subwords that include the first and last letter, and then putting this subword in the right tensor factor, and putting on the right the monomial consisting of the words read within the original word from one letter in the subword to the next.

Example:

$$\Delta(35688) = 35 \cdot 56 \cdot 68 \cdot 88 \otimes 35688 \\
+ 35 \cdot 56 \cdot 688 \otimes 3568 + 35 \cdot 568 \cdot 88 \otimes 3588 + 356 \cdot 68 \cdot 88 \otimes 3688 \\
+ 35 \cdot 5688 \otimes 358 + 356 \cdot 688 \otimes 368 + 3568 \cdot 88 \otimes 388 \\
+ 35688 \otimes 38.$$ 

5.5.2. Variation: contractible groupoids. As a variation of the previous example, instead of a linear order take $P$ to be a contractible groupoid (codiscrete groupoid on an alphabet). Now $P$-trees are non-empty words in the alphabet (without any monotonicity constraints). Concatenation and substitution work exactly as before.

5.5.3. Quivers and comodule bialgebras of paths. Linear orders are free categories on linear quivers. Example 5.5.1 immediately generalises to general quivers (directed graphs). Let $Q$ be a quiver, and let $P$ be the free category on $Q$. Then a $P$-tree is a marked path in the quiver, meaning a path with ‘stations’ marked along the path, including the start and the finish. Formally a marked path is a configuration

$$\Delta^k \rightarrow \Delta^l \rightarrow Q.$$ 

(Warning: the two maps live in different categories and cannot be composed: since $Q$ is only a quiver, not a category, the two maps do not make $\Delta^k$ into a path in $Q$!)

Now $P^\ast$ is the category whose objects are the vertices of $Q$ and whose morphisms are the marked paths. Composition is just concatenation of marked paths at their endpoints.

In the operad $P^\circ$, the colours are the nontrivial paths. The operations are the non-trivial marked paths without trivial stages. The output colour of a marked path is the path itself; the input colours are the stages of the marked path, meaning the paths from one station to the next.

Substitution replaces a stage with a further marking. The formalisation of this (unpacking the general constructions) involves active-inert pushouts in $\Delta$ (1.3.1). In detail, a marked path with a chosen input slot is the configuration

$$\Delta^l \rightarrow \Delta^k \rightarrow \Delta^s \rightarrow \Delta^t \rightarrow Q.$$ 

56
with input colour $\Delta^s \to Q$ obtained by active-inert factorisation in $\Delta$, as indicated. Note that the curved map does make sense as a composite, because inert maps in $\Delta$ are quiver maps, so $\Delta^s \to Q$ is a path in $Q$. To give another operation with matching output is to give $\Delta^h \to \Delta^s \to Q$, and since $\Delta^1$ is initial in the category of active maps, this must factor the map $\Delta^1 \to \Delta^s$:

$\Delta^1 \to \Delta^h \to \Delta^s \to Q$.

The result of the substitution is obtained by taking the active-inert pushout as indicated:

$\Delta^1 \to \Delta^h \to \Delta^s \to Q,$

which finally induces $\Delta^r \to \Delta^t$ by the universal property of the pushout. The result of the substitution is thus

$\Delta^r \to \Delta^t \to Q.$

### 5.5.4. Polynomial endofunctors and subdivided trees.

The preceding examples have multi versions. The multi analogue of a linear order is a tree, and the multi analogue of a quiver is a polynomial endofunctor. Recall that a tree is a special case of a polynomial endofunctor (1.5.1). For brevity we treat the latter case.

Let $Q$ be a polynomial endofunctor, and consider $P = Q^*$, the free monad on $Q$. Then $P$-trees are subdivided $Q$-trees, or more formally: $Q$-trees $T \to Q$ equipped with an active map $K \to T$. We write

$K \to T \to Q$

(with the warning again that the two maps live in different categories and cannot be composed, and $K$ is not a $Q$-tree).

The colours of $P^*$ are the original colours of $Q$. The operad structure on $P^*$ is given by grafting, which is gluing of subdivided trees.

The Baez–Dolan construction $P \circ$ has colours the nontrivial $Q$-trees. For the operations, we now have nontrivial subdivided trees, meaning

$K \to T \to Q$

where $K \to T$ is an active injection. The output colour is the whole tree $T \to Q$. The input slots are the nodes of $K$; the colour of an input slot is the $Q$-tree $S \to Q$ obtained as active-inert factorisation (2.1.3)

$C \to S \to Q$

$K \to T \to Q.$
Note that the curved does make sense as a composite, because inert maps are maps of polynomial endofunctors (1.5.2), so $S$ becomes indeed a $Q$-tree.

Substitution amounts to further refinement of subtrees of $T$. Formally it involves again active-inert pushouts, now in $\Omega$. In the diagram

$$
\begin{array}{c}
C \xrightarrow{H} S \\
\downarrow \quad \downarrow \\
K \xrightarrow{R} T \rightarrow Q
\end{array}
$$

the solid part represents the data required for a substitution: an operation $H \xrightarrow{\Delta} S \rightarrow Q$ with output colour $S \rightarrow Q$, to be substituted into the input slot of $K \xrightarrow{\Delta} T \rightarrow Q$ corresponding to $C \rightarrow K$. The result is the operation $R \xrightarrow{\Delta} T \rightarrow Q$ obtained by first factoring $C \rightarrow S$ through $H$, then forming the active-inert pushout, and finally using its universal property.

5.6 Non-examples and outlook

The examples in the previous subsection concerned the case where $P$ is itself free. In the general case, $P$ is not required to be free, but freeness comes in since of course $P^*$ is free. The following discussion looks beyond the free case, towards more general Baez–Dolan constructions.

5.6.1. Moment-cumulant relations in free probability. Free probability, introduced by Voiculescu [100] in the 1980s, is a noncommutative analogue of classical probability, originally motivated by operator algebras. Freeness is the analogue of independence. Speicher [93] discovered that the combinatorics underlying free probability is that of noncrossing partitions, contrasting the ordinary partitions in classical probability, and established a beautiful cumulant-moment formula for free cumulants in terms of Möbius inversion in the incidence algebra of the noncrossing partitions lattice [88]. Ebrahimi-Fard and Patras [36], [37] gave a very different approach to the moment-cumulant relations, in terms of a time-ordered exponential coming from a half-shuffle in the tensor algebra.

5.6.2. The comodule bialgebra of noncrossing partitions. The link between the two constructions was found recently by Ebrahimi-Fard, Foissy, Kock, and Patras [34], in terms of a comodule bialgebra structure on noncrossing partitions. This in turn is induced by two different operad structures on noncrossing partitions: the gap-insertion operad structure on noncrossing partitions works like this:

The block-substitution operad structure on noncrossing partitions works like this:
The incidence bialgebras of the two operads together form a comodule bialgebra [34].

5.6.3. Balanced Baez–Dolan construction (tentative). The comodule bialgebra of noncrossing partitions cannot result directly from a Baez–Dolan construction, since the gap-insertion operad is not free. However, it is ‘not too far’ from being free [34].

Define a balanced operad to be a nonsymmetric operad satisfying the following equation: substituting operation $a$ into the first slot of operation $b$ equals substituting $b$ into the last slot of $a$:

For ternary operations, this is the equation for the algebraic theory of generalised pseudo-heaps (non-Mal’cev heaps), studied by Wagner in the 1950s in differential geometry.

We now claim that the forgetful functor from balanced operads to endofunctors cartesian over the free-monoid monad $M$ admits a left adjoint, the free-balanced-operad functor.

**Example:** the free balanced operad on the terminal (reduced) nonsymmetric operad $M$ should be the gap-insertion operad of noncrossing partitions.

Now one should essentially just modify the Baez–Dolan construction to refer to this putative free-balanced-operad monad.

**Example:** the balanced Baez–Dolan construction on the terminal (reduced) nonsymmetric operad $\bar{M}$ should be the operad of noncrossing partitions with block substitution.

Assuming these claims and constructions work out, it will follow that the comodule bialgebra of noncrossing partitions of Ebrahimi-Fard–Foissy–Kock–Patras [34] is the incidence comodule bialgebra of the balanced Baez–Dolan construction on $\bar{M}$.

Verifying all the details has been postponed for future work, as indeed it would seem worthwhile developing a theory for more general Baez–Dolan constructions.

5.6.4. Regularity structures. As mentioned briefly in the introduction, Bruned, Hairer and Zambotti [9] have given an algebraic approach to renormalisation of regularity structures, where a comodule bialgebra plays a key role. The overall shape of the comultiplications involved resembles the Calaque–Ebrahimi-Fard–Manchon situation, but the tree structures are considerably more complicated (the paper [9] contains 40 pages of tree combinatorics!), because of intricate decorations required to encode the associated analytic objects. Briefly, vertices represent integration variables, edges represent integration kernels; there are additional numerical decorations, of nodes to represent Taylor remainders, and of edges to represent derivatives of the kernels. All these decorations are not just dead weight with respect to the combinatorics of the comultiplications, but transform in a non-trivial way with the contractions and extractions, as required in order to express the behaviour of the analytic objects.

It is clear that this comodule bialgebra escapes the range of examples covered by the Baez–Dolan construction in its pure form. For examples, it is easy to check that the simplicial groupoids defining the comultiplications are not Segal spaces, and therefore cannot arise as two-sided bar constructions. Another issue is that the sums involved are not finite, as would be the case in the situation of a free operad.

In spite of these discouragements, it is not unlikely that there are still relationships to be uncovered, involving passage to the core, and perhaps a Baez–Dolan relative to
something fancier than the free-monad monad. The challenge is to obtain an operadic interpretation of the decorations.

Even if the Baez–Dolan construction turns out not to be useful in this context, there may still be opportunities for the techniques of the present paper, and in particular for an objective approach. Firstly, the infinite sums appearing in the comultiplication formulae, which in the paper are handled through clever gradings and conditions ensuring that certain infinite matrices are upper-triangular, suggest that slice categories could be a natural framework. Secondly, the formulae for the comultiplications involve factorial denominators which transform according to some generalised Chu–Vandermonde identity, suggesting that these slices should be groupoid slices.

At the moment these considerations are speculative, and at the moment we list the Bruned–Hairer–Zambotti comodule bialgebra as a non-example, calling for further investigation.

Acknowledgments. This work was presented at the Workshop on comodule bialgebras (GDR Renormalisation) in Clermont-Ferrand, November 2018. I wish to thank Dominique Manchon for a wonderful conference, and for the perfect opportunity for me to expose this material. I have benefitted much from related collaborations with Imma Gálvez, Andy Tonks, Mark Weber, Louis Carlier, Kurusch Ebrahimi-Fard, Loïc Foissy, and Frédéric Patras, all of whom I thank for their influence on various parts of this work. Thanks are due also to Marcelo Fiore, Ander Murua, Pierre-Louis Curien, Paul-André Melliès, André Joyal, Gabriella Böhm, and Birgit Richter, for input and feedback.

Support from grants MTM2016-80439-P (AEI/FEDER, UE) of Spain and 2017-SGR-1725 of Catalonia is gratefully acknowledged.

References

[1] Eiichi Abe. Hopf algebras, vol. 74 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1980. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka. Cited p. 3, 34

[2] Marcelo Aguiar and Walter Ferrer Santos. Galois connections for incidence Hopf algebras of partially ordered sets. Adv. Math. 151 (2000), 71–100. Cited p. 50

[3] John C. Baez and James Dolan. Higher-dimensional algebra. III. n-categories and the algebra of opetopes. Adv. Math. 135 (1998), 145–206. ArXiv:q-alg/9702014. Cited p. 4, 23, 24, 27, 42

[4] John C. Baez and James Dolan. From finite sets to Feynman diagrams. In B. Engquist and W. Schmid, editors, Mathematics unlimited—2001 and beyond, pp. 29–50. Springer-Verlag, Berlin, 2001. ArXiv:math.QA/0004133. Cited p. 4, 12

[5] John C. Baez, Alexander E. Hoffnung, and Christopher D. Walker. Higher dimensional algebra VII: groupoidification. Theory Appl. Categ. 24 (2010), 489–553. ArXiv:0908.4305. Cited p. 5, 11

[6] Michael Batanin and Clemens Berger. Homotopy theory for algebras over polynomial monads. Theory Appl. Categ. 32 (2017), 148–253. ArXiv:1305.0086. Cited p. 4, 27

[7] Michael Batanin and Martin Markl. Operadic categories and duoidal Deligne’s conjecture. Adv. Math. 285 (2015), 1630–1687. ArXiv:1404.3886. Cited p. 4

[8] Christian Brouder, Alessandra Frabetti, and Christian Krattenthaler. Non-commutative Hopf algebra of formal diffeomorphisms. Adv. Math. 200 (2006), 479–524. ArXiv:math/0406117. Cited p. 52

[9] Yvain Bruned, Martin Hairer, and Lorenzo Zambotti. Algebraic renormalisation of regularity structures. Invent. Math. 215 (2019), 1039–1156. ArXiv:1610.08468. Cited p. 3, 8, 59

[10] John C. Butcher. An algebraic theory of integration methods. Math. Comp. 26 (1972), 79–106. Cited p. 14, 53
[11] Damien Calaque, Kurusch Ebrahimi-Fard, and Dominique Manchon. Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series. Adv. Appl. Math. 47 (2011), 282–308. ArXiv:0806.2238. Cited p. 3, 8, 53

[12] Louis Carlier. Incidence bicomodules, Möbius inversion, and a Rota formula for infinity adjunctions. Algebr. Geom. Topol. (2019). ArXiv:1801.07504, to appear. Cited p. 7, 15, 35

[13] Louis Carlier. Hereditary species as monoidal decomposition spaces, comodule bialgebras, and operadic categories. Preprint, arXiv:1903.07964. Cited p. 4, 8, 35, 37, 50

[14] Louis Carlier and Joachim Kock. Antipodes of monoidal decomposition spaces. Commun. Contemp. Math. (2018). ArXiv:1807.11858, DOI:10.1142/S0219199718500815. Cited p. 50

[15] Louis Carlier and Joachim Kock. Homotopy theory and combinatorics of groupoids. Book manuscript in preparation (2019). Cited p. 10

[16] Pierre Cartier and Dominique Foata. Problèmes combinatoires de commutation et réarrangements. No. 85 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, New York, 1969. Republished in the “books” section of the Séminaire Lotharingien de Combinatoire. Cited p. 5, 14

[17] Frédéric Chapoton and Muriel Livernet. Relating two Hopf algebras built from an operad. Int. Math. Res. Notices 2007 (2007), Art. ID rnm131, 27 pages. ArXiv:0707.3725. Cited p. 5

[18] Philippe Chartier, Ernst Hairer, and Gilles Vilmart. A substitution law for B-series vector fields. INRIA Report (2005). Cited p. 53

[19] Philippe Chartier, Ernst Hairer, and Gilles Vilmart. Numerical integrators based on modified differential equations. Math. Comp. 76 (2007), 1941–1953. Cited p. 53

[20] Philippe Chartier, Ernst Hairer, and Gilles Vilmart. Algebraic structures of B-series. Found. Comput. Math. 10 (2010), 407–427. Cited p. 3, 8, 53

[21] Kuo-Tsai Chen. Iterated integrals and exponential homomorphisms. Proc. London Math. Soc. (3) 4 (1954), 502–512. Cited p. 3

[22] Eugenia Cheng. Weak n-categories: opetopic and multitopic foundations. J. Pure Appl. Algebra 186 (2004), 109–137. ArXiv:math.CT/0304277. Cited p. 4, 24

[23] Eugenia Cheng. Weak n-categories: comparing opetopic foundations. J. Pure Appl. Algebra 186 (2004), 219–231. ArXiv:math.CT/0304279. Cited p. 4

[24] Hongyi Chu and Rune Haugseng. Homotopy-coherent algebra via Segal conditions. Preprint, arXiv:1907.03977. Cited p. 23

[25] Alain Connes and Dirk Kreimer. Hopf algebras, renormalization and noncommutative geometry. Comm. Math. Phys. 199 (1998), 203–242. ArXiv:hep-th/9808042. Cited p. 53

[26] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem. Comm. Math. Phys. 210 (2000), 249–273. ArXiv:hep-th/9912092. Cited p. 53

[27] Mireille Content, François Lemay, and Pierre Leroux. Catégories de Möbius et fonctorialités: un cadre général pour l’inversion de Möbius. J. Combin. Theory Ser. A 28 (1980), 169–190. Cited p. 5

[28] Jacky Cresson. Calcul moulien. Ann. Fac. Sci. Toulouse Math. (6) 18 (2009), 307–395. Cited p. 3, 52

[29] Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. Syntactic approaches to opetopes. Preprint, arXiv:1903.05848. Cited p. 4

[30] Arne Dür. Möbius functions, incidence algebras and power series representations, vol. 1202 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986. Cited p. 14, 53

[31] Tobias Dyckerhoff. Higher categorical aspects of Hall Algebras. In Advanced Course on (Re)emerging Methods in Commutative Algebra and Representation Theory, vol. 70 of Quaer. CRM, Barcelona, 2015. ArXiv:1505.06940. Cited p. 5, 35

61
Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal spaces*. No. 2244 in Lecture Notes in Mathematics. Springer-Verlag, 2019. ArXiv:1212.3563. Cited p. 5, 13, 14

Kurusch Ebrahimi-Fard, Frédéric Fauvet, and Dominique Manchon. *A comodule-bialgebra structure for word-series substitution and mould composition*. J. Algebra **489** (2017), 552–581. ArXiv:1609.03549. Cited p. 3

Kurusch Ebrahimi-Fard, Loïc Foissy, Joachim Kock, and Frédéric Patras. *Operads of (noncrossing) partitions, interacting bialgebras, and moment-cumulant relations*. Preprint, arXiv:1907.01190. Cited p. 3, 8, 58, 59

Kurusch Ebrahimi-Fard, Alexander Lundervold, and Dominique Manchon. *Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras*. Internat. J. Algebra Comput. **24** (2014), 671–705. ArXiv:1402.4761. Cited p. 52

Kurusch Ebrahimi-Fard and Frédéric Patras. *Cumulants, free cumulants and half-shuffles*. Proc. A. **471** (2015), 20140843, 18pp. ArXiv:1409.5664. Cited p. 3, 58

Kurusch Ebrahimi-Fard and Frédéric Patras. *The splitting process in free probability theory*. Int. Math. Res. Notices **2016** (2017), 911–945. ArXiv:1905.09580, To appear. Cited p. 5

Kurusch Ebrahimi-Fard and Bruno Vallet. *The arborification-coarborification transform: analytic, combinatorial, and algebraic aspects*. Ann. Fac. Sci. Toulouse Math. (6) **13** (2004), 575–657. Cited p. 3

Frédéric Fauvet, Loïc Foissy, and Dominique Manchon. *The Hopf algebra of finite topologies and mould composition*. Ann. Inst. Fourier (Grenoble) **67** (2017), 911–945. Cited p. 3

Matthew Feller, Richard Garner, Joachim Kock, May U. Proulx, and Mark Weber. *Every 2-Segal space is unital*. Commun. Contemp. Math. (2019). ArXiv:1905.09580, To appear. Cited p. 5

Eric Finster. *Opetopic! Online graphical proof assistant for higher category theory*, available at [http://opetopic.net/](http://opetopic.net/). Cited p. 4

Eric Finster. *Towards higher universal algebra in type theory*. Homotopy Type Theory Electronic Seminar 2018, (video recording at [https://www.youtube.com/watch?v=h1CVHVTa1qQ](https://www.youtube.com/watch?v=h1CVHVTa1qQ)). Cited p. 4

Thomas M. Fiore, Nicola Gambino, and Joachim Kock. *Monads in double categories*. J. Pure Appl. Algebra **215** (2011), 1174–1197. ArXiv:1006.0797. Cited p. 28

Loïc Foissy. *Algebraic structures associated to operads*. Preprint, arXiv:1702.05344. Cited p. 4

Loïc Foissy. *Algebraic structures on typed decorated rooted trees*. Preprint, arXiv:1811.07572. Cited p. 3

Loïc Foissy, Claudia Malvenuto, and Frédéric Patras. *Infinitesimal and $B_{\infty}$-algebras, finite spaces, and quasi-symmetric functions*. J. Pure Appl. Algebra **220** (2016), 2434–2458. ArXiv:1403.7488. Cited p. 4

Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Groupoids and Faà di Bruno formulae for Green functions in bialgebras of trees*. Adv. Math. **254** (2014), 79–117. ArXiv:1207.6404. Cited p. 4, 10, 32, 52

Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Homotopy linear algebra*. Proc. Royal Soc. Edinburgh A **148** (2018), 293–325. ArXiv:1602.05082. Cited p. 5, 8, 9, 10, 11, 12, 42, 46

Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory*. Adv. Math. **331** (2018), 952–1015. ArXiv:1512.07573. Cited p. 5, 13, 14, 15, 30, 35, 41, 50
[52] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness. Adv. Math. 333 (2018), 1242–1292. ArXiv:1512.07577. Cited p. 5, 7, 41

[53] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces and restriction species. Int. Math. Res. Notices (2019). ArXiv:1708.02570. Cited p. 5, 14, 15

[54] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces in combinatorics. Preprint, arXiv:1612.09225. Cited p. 5, 10, 14, 15, 54

[55] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. Math. Proc. Cambridge Phil. Soc. 154 (2013), 153–192. ArXiv:0906.4931. Cited p. 5, 15, 16, 17

[56] David Gepner, Rune Haugseng, and Joachim Kock. ∞-Operads as Analytic Monads. Preprint, arXiv:1712.06469. Cited p. 6, 15, 16, 21, 22

[57] Massimiliano Gubinelli. Ramification of rough paths. J. Differential Equations 248 (2010), 693–721. ArXiv:math/0610300. Cited p. 3

[58] Ernst Hairer, Christian Lubich, and Gerhard Wanner. Geometric numerical integration. Structure-preserving algorithms for ordinary differential equations, vol. 31 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, second edition, 2006. Cited p. 3, 53

[59] Martin Hairer. A theory of regularity structures. Invent. Math. 198 (2014), 269–504. ArXiv:1303.5113. Cited p. 3

[60] Martin Hairer. Introduction to regularity structures. Braz. J. Probab. Stat. 29 (2015), 175–210. Cited p. 3

[61] Claudio Hermida, Michael Makkai, and John Power. On weak higher dimensional categories. I. 1. J. Pure Appl. Algebra 154 (2000), 221–246. Cited p. 4

[62] Claudio Hermida, Michael Makkai, and John Power. On weak higher-dimensional categories. I. 2. J. Pure Appl. Algebra 157 (2001), 247–277. Cited p. 4

[63] Saï-nicole A. Joni and Gian-Carlo Rota. Coalgebras and bialgebras in combinatorics. Stud. Appl. Math. 61 (1979), 93–139. Cited p. 5, 14

[64] André Joyal. Une théorie combinatoire des séries formelles. Adv. Math. 42 (1981), 1–82. Cited p. 4, 52

[65] Joachim Kock. Notes on polynomial functors. Rough draft, 420pp. Available from http://mat.uab.cat/~kock/cat/polynomial.html, 2009. Cited p. 15, 17

[66] Joachim Kock. Polynomial functors and trees. Int. Math. Res. Notices 2011 (2011), 609–673. ArXiv:0807.2874. Cited p. 6, 17, 18, 19, 21, 22, 23, 24, 43, 48

[67] Joachim Kock. Data types with symmetries and polynomial functors over groupoids. In Proceedings of the 28th Conference on the Mathematical Foundations of Programming Semantics (Bath, 2012), vol. 286 of Electronic Notes in Theoretical Computer Science, pp. 351–365, 2012. ArXiv:1210.0828. Cited p. 6, 15, 16, 19

[68] Joachim Kock. Categorification of Hopf algebras of rooted trees. Cent. Eur. J. Math. 11 (2013), 401–422. ArXiv:1109.5785. Cited p. 8, 19, 54, 55

[69] Joachim Kock. Perturbative renormalisation for not-quite-connected bialgebras. Lett. Math. Phys. 105 (2015), 1413–1425. ArXiv:1411.3098. Cited p. 54

[70] Joachim Kock. Polynomial functors and combinatorial Dyson-Schwinger equations. J. Math. Phys. 58 (2017), 041703, 38pp. ArXiv:1512.03027. Cited p. 15, 19, 21, 54

[71] Joachim Kock, André Joyal, Michael Batanin, and Jean-François Mascari. Polynomial functors and opetopes. Adv. Math. 224 (2010), 2690–2737. ArXiv:0706.1033. Cited p. 4, 19, 24, 25, 27

[72] Joachim Kock and Mark Weber. Paù di Bruno for operads and internal algebras. J. Lond. Math. Soc. 99 (2019), 919–944. ArXiv:1609.03276. Cited p. 5, 7, 20, 30, 42, 52, 55
[73] Dirk Kreimer. *On the Hopf algebra structure of perturbative quantum field theories*. Adv. Theor. Math. Phys. 2 (1998), 303–334. ArXiv:q-alg/9707029. Cited p. 14, 53

[74] F. William Lawvere and Matías Menni. *The Hopf algebra of Möbius intervals*. Theory Appl. Categ. 24 (2010), 221–265. Cited p. 5

[75] Tom Leinster. *Higher Operads, Higher Categories*. London Math. Soc. Lecture Note Series. Cambridge University Press, Cambridge, 2004. ArXiv:math.CT/0305049. Cited p. 4, 24, 27

[76] Pierre Leroux. *Les catégories de Möbius*. Cahiers Topol. Géom. Diff. 16 (1976), 280–282. Cited p. 5, 14

[77] Terry J. Lyons. *Differential equations driven by rough signals*. Rev. Mat. Iberoamericana 14 (1998), 215–310. Cited p. 3

[78] Saunders Mac Lane. *Categories for the working mathematician, second edition*. No. 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1998. Cited p. 19

[79] Robert I. McLachlan, Klas Modin, Hans Munthe-Kaas, and Olivier Verdier. *B-series methods are exactly the affine equivariant methods*. Numer. Math. 133 (2016), 599–622. Cited p. 53

[80] Dominique Manchon. *An introduction to combinatorial Hopf algebras and renormalisation*. Lecture notes, Gabes, Tunisia (2016). http://www.fsg.rnu.tn/imgsite/cours/cours2016.pdf. Cited p. 3

[81] Dominique Manchon. *A review on comodule-bialgebras*. In Computation and Combinatorics in Dynamics, Stochastics and Control. Abel Symposium 2016., vol. 13 of Abel Symposia. Springer, Cham, 2016. Cited p. 3, 34

[82] J. Peter May. *The geometry of iterated loop spaces*, vol. 271 of Lectures Notes in Mathematics. Springer-Verlag, Berlin, 1972. Cited p. 5, 27

[83] John Milnor. *The Steenrod algebra and its dual*. Ann. of Math. (2) 67 (1958), 150–171. Cited p. 3

[84] Ieke Moerdijk and Ittay Weiss. *Dendroidal sets*. Algebr. Geom. Topol. 7 (2007), 1441–1470. ArXiv:math/0701293. Cited p. 22

[85] Richard K. Molnar. *Semi-direct products of Hopf algebras*. J. Algebra 47 (1977), 29–51. Cited p. 3

[86] Jack Morava. *Some examples of Hopf algebras and Tannakian categories*. In Algebraic topology (Oaxtepec, 1991), vol. 146 of Contemp. Math., pp. 349–359. Amer. Math. Soc., Providence, RI, 1993. Cited p. 52

[87] Hans Munthe-Kaas. *Lie-Butcher theory for Runge-Kutta methods*. BIT Numer. Math. 35 (1995), 572–587. Cited p. 52

[88] Alexandru Nica and Roland Speicher. *Lectures on the combinatorics of free probability*, vol. 335 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006. Cited p. 3, 58

[89] Mark D. Penney. *Simplicial spaces, lax algebras and the 2-Segal condition*. Preprint, arXiv:1710.02742. Cited p. 5, 14

[90] Thomas Poguntke. *Higher Segal structures in algebraic K-theory*. Preprint, arXiv:1709.06510. Cited p. 5

[91] William R. Schmitt. *Hopf algebras of combinatorial structures*. Canad. J. Math. 45 (1993), 412–428. Cited p. 4, 50

[92] William R. Schmitt. *Incidence Hopf algebras*. J. Pure Appl. Algebra 96 (1994), 299–330. Cited p. 5

[93] Roland Speicher. *Multiplicative functions on the lattice of noncrossing partitions and free convolution*. Math. Ann. 298 (1994), 611–628. Cited p. 3, 58

64
[94] Richard Steiner. *Opetopes and chain complexes.* Theory Appl. Categ. **26** (2012), 501–519. ArXiv:1204.6723. Cited p. 4

[95] Ross Street. *The formal theory of monads.* J. Pure Appl. Algebra **2** (1972), 149–168. Cited p. 28

[96] Robert M. Switzer. *Algebraic topology—homotopy and homology.* Springer-Verlag, New York-Heidelberg, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212. Cited p. 3

[97] Cédric Ho Thanh and Chaitanya Leena Subramaniam. *Opetopic algebras I: Algebraic structures on opetopic sets.* Preprint, arXiv:1911.00907. Cited p. 4

[98] Pepijn van der Laan. *Operads and the Hopf algebras of renormalisation.* Preprint, arXiv:math-ph/0311013. Cited p. 5

[99] Pepijn van der Laan and Ieke Moerdijk. *The renormalisation bialgebra and operads.* Preprint, arXiv:hep-th/0210226. Cited p. 5

[100] Dan Voiculescu. *Lectures on free probability theory.* In *Lectures on probability theory and statistics (Saint-Flour, 1998)*, vol. 1738 of Lecture Notes in Mathematics, pp. 279–349. Springer, Berlin, 2000. Cited p. 3, 58

[101] Tashi Walde. *Hall monoidal categories and categorical modules.* Preprint, arXiv:1611.08241. Cited p. 7, 15, 35

[102] Mark Weber. *Generic morphisms, parametric representations and weakly Cartesian monads.* Theory Appl. Categ. **13** (2004), 191–234. Cited p. 23

[103] Mark Weber. *Familial 2-functors and parametric right adjoints.* Theory Appl. Categ. **18** (2007), 665–732. Cited p. 22, 23

[104] Mark Weber. *Polynomials in categories with pullbacks.* Theory Appl. Categ. **30** (2015), 533–598. ArXiv:1106.1983. Cited p. 5, 20

[105] Mark Weber. *Operads as polynomial 2-monads.* Theory Appl. Categ. **30** (2015), 1659–1712. ArXiv:1412.7599. Cited p. 15, 20

[106] Mark Weber. *Internal algebra classifiers as codescent objects of crossed internal categories.* Theory Appl. Categ. **30** (2015), 1713–1792. ArXiv:1503.07585. Cited p. 5, 27, 28, 30

[107] Matthew B. Young. *Relative 2-Segal spaces.* Algebr. Geom. Topol. **18** (2018), 975–1039. ArXiv:1611.09234. Cited p. 5, 7, 15, 35

65