A TWO WEIGHT INEQUALITY FOR CALDERÓN–ZYGMUND OPERATORS ON SPACES OF HOMOGENEOUS TYPE WITH APPLICATIONS

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Abstract. Let \((X, d, \mu)\) be a space of homogeneous type in the sense of Coifman and Weiss, i.e., \(d\) is a quasi-metric on \(X\) and \(\mu\) is a positive measure satisfying the doubling condition. Suppose that \(u\) and \(v\) are two locally finite positive Borel measures on \((X, d, \mu)\). Subject to the pair of weights satisfying a side condition, we characterize the boundedness of a Calderón–Zygmund operator \(T\) from \(L^2(u)\) to \(L^2(v)\) in terms of the \(A_2\) condition and two testing conditions. For every cube \(B \subset X\), we have the following testing conditions, with \(1_B\) taken as the indicator of \(B\)

\[
\|T(u1_B)\|_{L^2(B,v)} \leq T\|1_B\|_{L^2(u)},
\]

\[
\|T^*(v1_B)\|_{L^2(B,u)} \leq T\|1_B\|_{L^2(v)}.
\]

The proof uses stopping cubes and corona decompositions originating in work of Nazarov, Treil and Volberg, along with the pivotal side condition.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The two weight conjecture for Calderón–Zygmund operators \(T\) was first raised by Nazarov, Treil and Volberg on finding the necessary and sufficient conditions on the two weights \(u\) and \(v\) so that \(T\) is bounded from \(L^2(u)\) to \(L^2(v)\). The third author, in [Saw1] first introduced the testing conditions (which are referred to as the Sawyer-type testing conditions) into the two weight setting on the maximal function, and later [Saw2] on the fractional and Poisson integral operators; serving as motivation for the investigation by Nazarov, Treil and Volberg, see for example [NTV3].

This conjecture in the special case that the pair of weights \(u\) and \(v\) do not share a common point mass was completely solved only recently when \(T\) is the Hilbert transform on \(\mathbb{R}\). This was supplied in two papers,
Lacey–Sawyer–Shen–Uriarte-Tuero [LaSaShUr], and Lacey [Lac] that built on pioneering work of Nazarov, Treil, and Volberg [NTV3]. The central question is providing a real-variable characterisation of the inequality
\[ \sup_{0<\alpha<\beta<\infty} \|H_{\alpha,\beta}(f \ast v)\|_{L^2(\mu)} \leq \mathcal{N}\|f\|_{L^2(\mu)}, \]
where \(\mathcal{N}\) is the best constant such that the above inequality holds, \(H_{\alpha,\beta}(f \ast v)(x)\) is the standard truncation of the usual Hilbert transform, and \(u, v\) are non-negative Borel locally finite measures on \(\mathbb{R}\). The full solution is as follows.

**Theorem A.** Suppose that for all \(x \in \mathbb{R}\), \(u(\{x\}) \cdot v(\{x\}) = 0\) for the pair of weights \(u\) and \(v\). Define two positive constants \(A_2\) and \(T\) as the best constants in the inequalities below, uniform over intervals \(I\):
\[
P(u,I) \cdot P(v,I) \leq A_2; \\
\int_I H(1_Iu)^2 dv \leq T^2 u(I), \quad \int_I H(1_Iv)^2 du \leq T^2 v(I).
\]
Then (1.1) holds if and only if both (1.2) and (1.3) hold, moreover, \(\mathcal{N} \approx A_2^{1/2} + T\).

In the theorem above, the term \(P(u,I)\) is the Poisson integral with respect to the measure \(u\) at the scaling level \(|I|\) and centred at \(x_I\), that is,
\[
P(u,I) := \int_{\mathbb{R}} \frac{|I|}{(|I| + \text{dist}(x,I))^2} du(x).
\]
The restriction regarding common point masses was removed by Hytönen in [Hyt].

The aim of this paper is to provide sufficient conditions for the two-weight inequality for general Calderón–Zygmund operators on spaces of homogeneous type. Since we are working in a very general setting, the best we can hope for at the moment is to provide a collection of sufficient conditions on the weights that guarantee two weight estimates for Calderón–Zygmund operators on spaces of homogeneous type. Our main approach is a suitable version of stopping cubes and corona decompositions originating in work of Nazarov, Treil and Volberg [NTV3], along with the pivotal side condition.

Spaces of homogeneous type were introduced by Coifman and Weiss\(^1\) in the early 1970s, in [CW1], see also [CW]. We say that \((X,d,\mu)\) is a space of homogeneous type in the sense of Coifman and Weiss if \(d\) is a quasi-metric on \(X\) and \(\mu\) is a nonzero measure satisfying the doubling condition. A quasi-metric \(d\) on a set \(X\) is a function \(d : X \times X \rightarrow [0,\infty)\) satisfying
\[
(i) \ d(x,y) = d(y,x) \geq 0 \quad \text{for all } x, y \in X; \\
(ii) \ d(x,y) = 0 \quad \text{if and only if } x = y; \quad \text{and} \\
(iii) \ \text{the quasi-triangle inequality: there is a constant } A_0 \in [1,\infty) \quad \text{such that for all } x, y, z \in X,
\]
\[
d(x,y) \leq A_0[d(x,z) + d(z,y)].
\]

We say that a nonzero measure \(\mu\) satisfies the doubling condition if there is a constant \(C_\mu\) such that for all \(x \in X\) and \(r > 0\),
\[
\mu(B(x,2r)) \leq C_\mu \mu(B(x,r)) < \infty,
\]
where \(B(x,r)\) is the quasi-metric ball defined by \(B(x,r) := \{y \in X : d(x,y) < r\}\) for \(x \in X\) and \(r > 0\).

Recall that the doubling condition (1.4) implies that there exists a positive constant \(n\) (the upper dimension of \(\mu\)) such that for all \(x \in X, m \geq 1\) and \(r > 0\),
\[
\mu(B(x,mr)) \leq C_\mu m^n \mu(B(x,r)).
\]
Throughout this paper we assume that \(\mu(X) = \infty\) and that \(\mu(\{x_0\}) = 0\) for every \(x_0 \in X\).

We now recall the definition of Calderón–Zygmund operators on spaces of homogeneous type.

\(^1\)As Yves Meyer remarked in his preface to [DH], “One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1990s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis.”
\textbf{Definition 1.} We say that $T$ is a Calderón–Zygmund operator on $(X,d, \mu)$ if $T$ is bounded on $L^2(X)$ and has an associated kernel $\mathcal{R}(x,y)$ such that $T(f)(x) = \int_X \mathcal{R}(x,y)f(y)d\mu(y)$ for any $x \notin \text{supp } f$, and $\mathcal{R}(x,y)$ satisfies the following estimates: for all $x \neq y$,

\begin{equation}
|\mathcal{R}(x,y)| \leq \frac{C}{V(x,y)},
\end{equation}

and for $d(x,x') \leq (2A_0)^{-1}d(x,y)$,

\begin{equation}
|\mathcal{R}(x,y) - \mathcal{R}(x',y)| + |\mathcal{R}(y,x) - \mathcal{R}(y,x')| \leq \frac{C}{V(x,y)}\omega \left(\frac{d(x,x')}{d(x,y)}\right),
\end{equation}

where $V(x,y) := \mu(B(x,d(x,y)))$, $\omega : [0,1] \rightarrow [0, \infty)$ is continuous, increasing, subadditive, and $\omega(0) = 0$.

Note that by the doubling condition we have that $V(x,y) \approx V(y,x)$. In the theorem below we have taken $\omega(t) = t^\kappa$ for some $\kappa \in (0,1)$ for the kernel estimate (1.6) and we refer to $\kappa$ as the smoothness parameter for the kernel $\mathcal{R}(x,y)$.

The main result of this paper provides sufficient conditions on a pair of weights $u$ and $v$ so that the following two-weight norm inequality

\begin{equation}
\|T(f \cdot u)\|_{L^2(v)} \leq N\|f\|_{L^2(u)}
\end{equation}

holds for a Calderón–Zygmund operator $T$ on $(X,d, \mu)$, where $N$ is the best constant (understood as the operator norm).

We also define $l(Q)$ (in Section 3.1) to be the side-length of a dyadic cube $Q$, and for all $Q = Q_k^i \in \mathcal{D}_k$ where $\{D_k\}_{k \in \mathbb{Z}}$ is a system of dyadic cubes with the parameter $\delta$ as given in Definition 8 below. In particular we have $l(Q) = 2^k$.

We say that a set $Q$ is a \textit{cube}, or more precisely a $(c_1, C_1, \delta)$-cube, if there is $z \in X$ such that

\[ B\left(z, c_1 \delta^k\right) \subset Q^k_\omega(\omega) \subset B\left(z, C_1 \delta^k\right), \]

where $c_1, C_1$ and $\delta$ are positive constants. The cubes appearing in the main theorem below are those with $c_1, C_1$ and $\delta$ as in (3) of Theorem 7 below. Also for all cubes $Q$ and $x \in X \setminus Q$ we define $\text{dist}(x,Q) := \inf \{d(q,x) : q \in Q\}$ where $d$ is the quasi-metric on $X$. Recall that a measure $\mu$ is locally finite if for any point in $X$ there exists a neighborhood $B$ about that point so that $\mu(B) < \infty$.

We need two quantities that control certain quantities in the proof that control certain information uniformly over cubes $Q \subset X$. The first is a version of an $A_2$ condition. Suppose the pair of weights $u, v$ satisfies the following $A_2$ condition for all cubes $Q$:

\begin{equation}
\left(\frac{u(Q)}{l(Q)^\alpha}K(Q,v)\right)^{\frac{1}{2}} \lesssim A_2
\end{equation}

with $K(Q,v) := \int_X \left(\frac{l(Q)}{(l(Q) + \text{dist}(y,Q))}\right)^{\kappa} \frac{1}{\mu(B(x,y,l(Q) + \text{dist}(y,Q)))} dv(y)$, as well as the dual condition

\begin{equation}
\left(\frac{v(Q)}{l(Q)^\alpha}K(Q,u)\right)^{\frac{1}{2}} \lesssim A_2,
\end{equation}

where $A_2$ is the best constant such that the above inequalities hold. Recall here that $\kappa$ is the smoothness parameter associated to the Calderón–Zygmund kernel in Definition 1.

The second is the pivotal condition:

\begin{equation}
\sup_{Q = \cup_{i \geq 1}S_i} \sum_{i \geq 1} \Phi(S_i,1_Q u) \leq \mathcal{V}^2 u(Q),
\end{equation}

where $\Phi(Q,1_{E\mu}) := v(Q)K(Q,1_{E\mu})^2$, as well as the dual version, in which $u$ and $v$ are interchanged. Here $\mathcal{V}$ is the best constant, and the supremum is over all $r$-good subpartitions $\{S_i\}_{i \geq 1}$ of $Q$ where $r$ is defined in Definition 16. An $r$-good subpartition consists of $Q$-dyadic subcubes $\{S_i\}$ of $Q$ such that $S_i$ is $r$-good in any dyadic grid containing $Q$.

With these preliminaries, our main result is the following:
\textbf{Theorem 2.} Let $T$ be a Calderón–Zygmund operator with smoothness parameter $\kappa$. Let $u$ and $v$ be two locally finite, positive Borel measures on $X$. Suppose that $u(\{x\}) \cdot v(\{x\}) = 0$ for $x \in X$ and that they satisfy the two weight condition with constant $A_2$ and the pivotal condition with constant $\nu$. Then $T : L^2(u) \rightarrow L^2(v)$ is bounded if and only if the following testing conditions hold: for every cube $Q \subset X$, we have the following testing conditions, with $1_Q$ taken as the indicator of $Q$

\begin{equation}
\|T(u 1_Q)\|_{L^2(Q, v)} \leq \mathcal{T}\|1_Q\|_{L^2(u)},
\end{equation}

\begin{equation}
\|T^*(v 1_Q)\|_{L^2(Q, u)} \leq \mathcal{T}\|1_Q\|_{L^2(v)}.
\end{equation}

Moreover, we have that $\mathcal{N} \lesssim A_2 + \mathcal{T} + \mathcal{V}$.

\textbf{Remark 3.} We would like to point out that we introduce a new version of a Poisson-type integral $K(Q, v)$ in the main Theorem that plays the role of the standard Poisson integral as in (1.2) in Theorem A. The main reason for providing such a condition is that, for some Calderón–Zygmund operators in certain particular setting, the typical Poisson integral in that setting is not linked directly to the study of two weight inequality for the Calderón–Zygmund operator. We refer to Section 2.1 for a concrete example in the Bessel setting introduced and studied by Muckenhoupt–Stein [MuSt].

We also remark that the choice of $\kappa$ is flexible, but dictated by the smoothness of the Calderón–Zygmund kernel. If one has a different kernel possessing a different smoothness, but satisfies the appropriate $A_2$, testing and pivotal conditions, then one can have a version of Theorem 2. The choice of $\kappa$ can be dictated by the particular example at hand.

It is immediate that the testing conditions are necessary and that $\mathcal{T} \lesssim \mathcal{N}$. The forward condition follows by testing (1.7) on an indicator function of a cube and restricting the region of integration. The dual condition follows by testing the dual inequality (1.7) (obtained by interchanging the roles of $u$ and $v$) on the indicator of a cube and then again restricting the integration. In the remainder of the paper we address how to show that these testing conditions are sufficient to prove (1.7) under the additional $A_2$ and pivotal hypothesis. In the course of the proof we will also demonstrate that $\mathcal{N} \lesssim A_2 + \mathcal{T} + \mathcal{V}$. Throughout the paper, we use the notation $X \lesssim Y$ to denote that there is an absolute constant $C$ so that $X \leq CY$, where $C$ may change from one occurrence to another. If we write $X \approx Y$, then we mean that $X \lesssim Y$ and $Y \lesssim X$. And, := means equal by definition.

\section{Extension to Hilbert space valued operators.} Following [Ste, Chapter II, Section 5], we consider two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and replace the scalar-valued expressions in the definition of singular integral

$$Tf(x) = \int_X \mathcal{R}(x, y)f(y)d\mu(y),$$

with the appropriate Hilbert space valued expressions, namely with $f : X \rightarrow \mathcal{H}_1$ and $K : X \times X \rightarrow B(\mathcal{H}_1, \mathcal{H}_2)$, so that $Tf : X \rightarrow \mathcal{H}_2$. Here $B(\mathcal{H}_1, \mathcal{H}_2)$ is the Banach space of bounded linear operators $L : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ equipped with the usual operator norm. We refer to such an operator $T$ as an $H_1 \rightarrow H_2$ Calderón–Zygmund operator if its kernel satisfies the usual size and smoothness conditions

\begin{equation}
|\mathcal{R}(x, y)|_{B(\mathcal{H}_1, \mathcal{H}_2)} \leq C_{\mathcal{CZ}}V(x, y)^{-1},
\end{equation}

\begin{equation}
|\mathcal{R}(x, y) - \mathcal{R}(x', y)|_{B(X \times X, B(\mathcal{H}_1, \mathcal{H}_2))} \leq C_{\mathcal{CZ}}\left(\frac{d(x, x')}{d(x, y)}\right)^\kappa V(x, y)^{-1}, \quad \frac{d(x, x')}{d(x, y)} \leq \frac{1}{2A_0},
\end{equation}

and if $T$ is bounded from unweighted $L^2_{\mathcal{H}_1}$ to unweighted $L^2_{\mathcal{H}_2}$. In the Appendix, Section 9, to this paper we will fix two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and describe in detail the definition and interpretation of standard fractional singular integrals, the weighted norm inequality, Poisson integrals and Muckenhoupt conditions, Haar bases and pivotal conditions, which are for the most part routine.

\textbf{Theorem 4.} Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. Let $T$ be a Calderón–Zygmund operator taking $L^2_{\mathcal{H}_1}$ to $L^2_{\mathcal{H}_2}$. Let $u$ and $v$ be two locally finite positive Borel measures on a space of homogeneous type $(X, d, \mu)$. Suppose that $u(\{x\}) \cdot v(\{x\}) = 0$ for $x \in X$. Suppose the above $A_2$ and pivotal conditions hold. Suppose the following testing conditions: for every cube $Q \subset X$, we have the following testing conditions, with $1_Q$ taken as the indicator of $Q$:

$$\|T(e_1 1_Q u)\|_{L^2_{\mathcal{H}_2(v)}} \leq \mathcal{T}\|1_Q\|_{L^2(u)}, \quad \text{for all unit vectors } e_1 \text{ in } \mathcal{H}_1$$
Then there holds $\mathcal{N} \lesssim A_2 + \mathcal{T} + \mathcal{V}$.

To see how this theorem follows from the scalar-valued Theorem 2, consider the scalar operators $T_{e_1, e_2}$ associated with $T$ for every pair of unit vectors $(e_1, e_2) \in \mathcal{H}_1^{\text{unit}} \times \mathcal{H}_2^{\text{unit}}$ whose kernels $K_{e_1, e_2}(x, y)$ are given by

$$K_{e_1, e_2}(x, y) := \langle \mathbf{R}(x, y)e_1, e_2 \rangle_{\mathcal{H}_2}. $$

It is easy to see that

$$\|T\|_{L^2(\mathcal{H}_1^{\text{unit}}) \rightarrow L^2(\mathcal{H}_2^{\text{unit}})} = \sup_{(e_1, e_2) \in \mathcal{H}_1^{\text{unit}} \times \mathcal{H}_2^{\text{unit}}} \|T_{e_1, e_2}\|_{L^2(\mathcal{H}_1^{\text{unit}}) \rightarrow L^2(\mathcal{H}_2^{\text{unit}})} ,$$

with a similar equality for the testing conditions. Theorem 4 now follows immediately from Theorem 2.

2. Applications of the Main Theorem

In this section we provide several typical examples of Calderón–Zygmund operators arising from different backgrounds (including several complex variables, stratified Lie groups, and differential equations), which are not the standard Euclidean setting, but fall into the scope of spaces of homogeneous type.

2.1. Bessel Riesz transforms. As an application, we have a two-weight inequality for the Bessel Riesz transform, which is a Calderón–Zygmund operator [MuSt]. In 1965, Muckenhoupt and Stein in [MuSt] introduced a notion of conjugacy associated with this Bessel operator $\Delta_\lambda$, $\lambda > 0$, which is defined by

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - 2\lambda \frac{d}{dx} f(x), \quad x > 0.$$

They developed a theory in the setting of $\Delta_\lambda$ which parallels the classical one associated to standard Laplacian. For $\lambda \in [1, \infty)$, $R_+ := (0, \infty)$ and $dm_\lambda(x) := x^{2\lambda} \, dx$ results on $L^p(R_+, dm_\lambda)$-boundedness of conjugate functions and fractional integrals associated with $\Delta_\lambda$ were obtained. Since then, many problems based on the Bessel context were studied; see, for example, [AnKe, BeRuFaRo, BeHaNoVi, Ker, Vil]. In particular, the properties and $L^p$ boundedness (1 < $p$ < $\infty$) of Riesz transforms

$$R_{\Delta_\lambda} : = \partial_x (\Delta_\lambda)^{-\frac{1}{2}} f$$

related to $\Delta_\lambda$ have been studied extensively (see for example [AnKe, BeRuFaRo, BeFaBuMaTo, MuSt, Vil]).

We recall that there is a standard Poisson integral in the Bessel setting. Let $\{e^{t\lambda \sqrt{\pi}}\}_{t > 0}$ be the Poisson semigroup defined by

$$P^{[\lambda]}_t f(x) := \int_0^\infty P^{[\lambda]}_t(x, y) y^{2\lambda} \, dy,$$

where

$$P^{[\lambda]}_t(x, y) = \int_0^\infty e^{-tz}(xz)^{-\lambda+rac{1}{2}} J_{\lambda-rac{1}{2}}(xz)J_{\lambda-rac{1}{2}}(yz) z^{2\lambda} \, dz$$

and $J_\nu$ is the Bessel function of the first kind and of order $\nu$. Weinstein [Wei] established the following formula for $P^{[\lambda]}_t(x, y)$: $t, x, y \in R_+$,

$$P^{[\lambda]}_t(x, y) = \frac{2\lambda}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} \, d\theta. $$

The two weight inequality for this Poisson operator was just established recently in [LiWi], that is, for a measure $\mu$ on $R_+^{2,+} := (0, \infty) \times (0, \infty)$ and $\sigma$ on $R_+$:

$$\|P^{[\lambda]}_\sigma(f)\|_{L^2(R_+^{2,+}; \mu)} \lesssim \|f\|_{L^2(R_+^{2,+}; \sigma)},$$

if and only if testing conditions hold for the Poisson operator and its adjoint. However, this two weight Poisson inequality does not relate directly to the two weight inequality for $R_{\Delta_\lambda}$. We have to link it to the Poisson type condition introduced in (1.8).
2.2. Bergman Projection. As another application we look at the Bergman projection. Let $D := \{ z \in \mathbb{C}^n : |z| < 1 \}$ be the unit ball in $\mathbb{C}^n$; $d\mu_a(z) := (1 - |z|^2)^{a-1}d\mu(z)$ where $a > 0$ and $\mu$ is Lebesgue measure on $\mathbb{C}^n$. Let us denote by $L^p(d\mu_a)$ the Lebesgue space related to $\mu_a$, $1 \leq p \leq \infty$. The Bergman projection $T_0f$ for $f \in L^2(d\mu_a)$ is given by

$$T_0f(z) := \int_D \frac{f(\zeta)}{(1 - z \cdot \bar{\zeta})^{n+a}}d\mu(\zeta).$$

Here we have $z \cdot \bar{\zeta} := z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$ where $z := (z_1, \ldots, z_n)$ and $\zeta := (\zeta_1, \ldots, \zeta_n)$. The operator $T_0$ extends continuously on $L^p(d\mu_a)$ for $1 < p < \infty$ and weakly continuously on $L^1(d\mu_a)$. If $D$ is provided with the pseudo-distance $d$ then the triple $(D, d, \mu_a)$ is a space of homogeneous type. Note that $K_a(z, \zeta) := \frac{1}{(1 - z \cdot \zeta)^{n+a}}$ is the kernel associated to the operator $T_0$. We can observe that $K_a(z, \zeta)$ satisfies the following smoothness and size estimates: there are constants $\beta, c_B$ such that

$$|K_a(z, \zeta) - K_a(z, \zeta^0)| \lesssim \frac{|d(\zeta, \zeta^0)|^\beta}{|d(z, \zeta^0)|^{n+a+\beta}}$$

for $z, \zeta, \zeta^0$ such that $d(z, \zeta^0) > c_Bd(\zeta, \zeta^0)$. Here the pseudo-distance $d$ is defined by

$$d(z, \zeta) := ||z| - ||\zeta|| + \left| 1 - \frac{z \cdot \zeta}{|z||\zeta|} \right|.$$

Also the following size estimate holds: for $z, \zeta \in D$ with $z \neq \zeta$, we have

$$|K_a(z, \zeta)| \lesssim \frac{1}{d(z, \zeta)^{n+a}}.$$

So $T_0$ is a singular integral operator on $(D, d, \mu_a)$. Using Theorem 2 we deduce a two weight inequality for the Bergman projection using $A_2$, testing conditions and the pivotal condition associated with the kernel $K_a$.

2.3. The Szegő Projection on a Family of Unbounded Weakly Pseudoconvex Domains. Recall the family of weakly pseudoconvex domains $\{\Omega_k\}_{k=1}^\infty$ defined in Greiner and Stein [GrSt] by

$$\Omega_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_2 > |z_1|^{2k} \right\},$$

$$\partial \Omega_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : \text{Im } z_2 = |z_1|^{2k} \right\},$$

which are naturally parameterized by $z_1$ and $\text{Re } z_2$. We consider the points $\zeta, \omega, \nu$ in $\partial \Omega_k$ given by

$$\zeta := (z_1, \text{Re } z_2) := (z, t), \quad z = z_1 \in \mathbb{C} \text{ and } t \in \mathbb{R},$$

$$\omega := (u_1, \text{Re } u_2) := (w, s), \quad w = w_1 \in \mathbb{C} \text{ and } s \in \mathbb{R},$$

$$\nu := (u_1, \text{Re } u_2) := (u, r), \quad u = u_1 \in \mathbb{C} \text{ and } r \in \mathbb{R}.$$
In [Dia], Diaz defined and analyzed a pseudometric \( d(\zeta, \omega) \), globally suited to the complex geometry of \( \partial \Omega_k \), which was arrived at by a study of the Szegö kernel. This allows the treatment of the Szegö kernel as a singular integral kernel:

\[
d(\zeta, \omega) := \left( \frac{2}{i[s-t]} + \frac{|z_1|^{2k} + |w_1|^{2k}}{2} + \frac{\mu + \eta}{2} \right)^{-\frac{1}{2}} - z_1 w_1^{-\frac{1}{2}}.
\]

Then the pseudometric balls are defined as \( B_\zeta(\delta) = B_\zeta^d(\delta) := \{ \omega \in \partial \Omega_k : d(\zeta, \omega) < \delta \} \) and the volume of the associated ball is

\[
V(B_\zeta(\delta)) = 4\pi\delta^2 \left( \frac{\sin^{2k-2}(\frac{\pi}{r})}{4} |z|^{2k-2} \delta^2 + \frac{1}{2} \delta^{2k} \right),
\]

and it is shown that this volume measure is doubling. Thus \( S \) is a standard Calderón–Zygmund operator in the setting of the space of homogeneous type \( (\partial \Omega_k, d, V) \). We can again deduce a two-weight inequality in this setting from Theorem 2.

2.4. Riesz Transforms Associated with the sub-Laplacian on Stratified Nilpotent Lie Groups.

Recall that a connected, simply connected nilpotent Lie group \( G \) is said to be stratified if its left-invariant Lie algebra \( g \) (which is assumed real and of finite dimension) admits a direct sum decomposition

\[
g = \oplus_{i=1}^k V_i,
\]

where \( [V_i, V_j] = V_{i+j} \) for \( i \leq k \).

One identifies \( G \) and \( g \) via the exponential map \( : g \rightarrow G \), which is a diffeomorphism. We fix once and for all a (bi-invariant) Haar measure \( dg \) on \( G \) (which is just the lift of Lebesgue measure on \( g \) via exp). There is a natural family of dilations on \( g \) defined for \( r > 0 \) as follows:

\[
d_r \left( \sum_{i=1}^k v_i \right) = \sum_{i=1}^k r^i v_i, \quad \text{with } v_i \in V_i.
\]

This permits the definition of a dilation on \( G \), which we continue to denote by \( d_r \). We choose a basis \( \{X_1, ..., X_n\} \) for \( V_1 \) and consider the sub-Laplacian \( \Delta := \sum_{j=1}^n X_j^2 \). Observe that \( X_j, 1 \leq j \leq n, \) is homogeneous of degree 1, and that \( \Delta \) is homogeneous of degree 2, with respect to dilations in the sense that \( X_j (f \circ \delta_r) = r (X_j f) \circ \delta_r, 1 \leq j \leq n, \) and \( \delta_r \circ \Delta \circ \delta_r = r^2 \Delta \) for all \( r > 0 \).

Let \( Q \) denote the homogeneous dimension of \( G \), namely \( Q = \sum_{i=1}^k i \dim V_i \). Let \( p_h \) for \( h > 0 \) be the heat kernel, i.e., the integral kernel of \( e^{h\Delta} \) on \( G \). For convenience we set \( p_h (g) = p_h (g, o) \), which means that we identify the integral kernel with the convolution kernel, and we set \( p (g) = p_1 (g) \).

Recall that (c.f. Folland and Stein [FoSt]) \( p_h (g) = h^{-\frac{Q}{2}} p \left( \delta_{\frac{Q}{2h}} (g) \right) \) for all \( h > 0 \) and \( g \in G \). The kernel of the \( j^{th} \) Riesz transform \( R_j = X_j (\Delta)^{-\frac{1}{2}} \), \( 1 \leq j \leq n, \) is written simply as \( K_j (g, g') = K_j \left( (g')^{-1} g \right) \) where

\[
K_j (g) = \frac{1}{\sqrt{\pi}} \int_0^\infty h^{-\frac{j}{2}} X_j p_h (g) \, dh = \frac{1}{\sqrt{\pi}} \int_0^\infty h^{-\frac{j}{2} - 1} (X_j p) \left( \delta_{\frac{Q}{2h}} g \right) \, dh.
\]

The standard metric \( d \) on \( G \) is defined as \( d(g, g') := \rho \left( (g')^{-1} g \right) \) where \( \rho \) is the homogeneous norm on \( G \) ([FoSt, Chapter 1, Section A1]). The measure \( dq \) is then a doubling measure. It is well known that \( S \) is a standard Calderón–Zygmund operator in the setting of the space of homogeneous type \( (G, d, dq) \), and once more we can deduce a two weight inequality from Theorem 2.

2.5. Area Functions. In the Euclidean homogeneous space \( (\mathbb{R}^n, |\cdot|, dx) \), the Littlewood-Paley \( g \)-function and the Lusin area function are both examples of \( \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) Calderón–Zygmund operators with \( \mathcal{H}_1 = \mathbb{C} \). Indeed they are given by

\[
\begin{align*}
g (f) (x) &= \left( \int_0^\infty |\nabla (P_t * f) (x)|^2 t \, dt \right)^{\frac{1}{2}} = |G f (x)|_{\mathcal{H}_n}, \\
S f (x) &= \left( \int_\Gamma_0 \int |\nabla (P_t * f) (x - y)|^2 \frac{dt \, dy}{t^{n-1}} \right)^{\frac{1}{2}} = |S f (x)|_{\mathcal{H}_{n+1}}.
\end{align*}
\]
where

\begin{align}
Gf(x)(t) & := \nabla (P_t \ast f)(x), \quad t \in (0, \infty), \\
Sf(x)(t, y) & := \nabla (P_t \ast f)(x - y), \quad (t, y) \in \Gamma_0,
\end{align}

and \(G_n\) and \(H_{n+1}\) are Hilbert spaces with norms

\[|g|_{G_n} := \sqrt{\int_0^\infty |g(t)|^2 \, dt} \quad \text{and} \quad |h|_{H_{n+1}} := \sqrt{\int\int_{\Gamma_0} |h(t, y)|^2 \, t^{1-n} \, dt \, dy} < \infty,\]

and \(\Gamma_0\) is a fixed cone with vertex at the origin in \(\mathbb{R}^{n+1}\) opening upward into the upper half space \(\mathbb{R}^{n+1}_+\).

In a general space of homogeneous type \((X, d, \mu)\), the notion of a Poisson kernel can be approached in several different ways. First, if \(D\) is a dyadic grid on \(X\) and \(\{h_Q : Q \in D\}\) is the collection of Haar wavelets constructed in [KLPW], then with the one-dimensional projection \(\triangle_Q\) defined by \(\triangle_Q f := \langle f, h_Q \rangle \mu h_Q\), we have

\[f = \sum_{Q \in D} \triangle_Q f = \sum_{Q \in D} \triangle_Q \triangle_Q f, \quad f \in L^2(\mu),\]

where the orthonormal property and the self-adjointness of Haar projections gives the second sum \(\sum_{Q \in D} \triangle_Q \triangle_Q f\), usually called a Calderón reproducing formula for \(f\). We can then define a discrete Poisson operator by

\[P_k f := \sum_{Q \in D; i(Q) \geq 2^k} \triangle_Q \triangle_Q f.\]

However, the kernel of \(P_k\) is not Lipschitz continuous, and thus fails to be a Calderón–Zygmund kernel as defined above. The following smoother construction of a Poisson kernel had been introduced much earlier by R. Coifman, see [DaJoSe] where this first appears.

2.5.1. \textit{Coifman’s Construction of a Calderón Reproducing Formula.} Start with a smooth function \(h : (0, \infty) \to (0, \infty)\) that equals 1 on \([0, \frac{1}{2}]\) and 0 on \([2, \infty]\). Let \(T_k\) be the operator with kernel \(2^k h(2^k d(x, y))\) so that \(\frac{1}{k} \leq T_k 1 \leq C\) for some positive constant \(C\). Let \(M_k\) be the operator of multiplication by \(\frac{1}{T_k(2^k)}\), and let \(W_k\) be the operator of multiplication by \(\frac{1}{T_k(2^k)}\). Then the operator

\[S_k := M_k T_k W_k T_k M_k,\]

has kernel \(S_k(x, y)\) that satisfies

\begin{align}
S_k(x, y) &= 0 \text{ if } d(x, y) \geq C \frac{1}{2^k}, \quad \|S_k\|_{\infty} \leq C 2^k, \\
|S_k(x, y) - S_k(x', y)| + |S_k(y, x) - S_k(y, x')| &\leq C 2^k \left[2^k d(x, x')\right]^z, \\
\int_X S_k(x, y) \, d\mu(y) &:= 1 \quad \text{and} \quad \int_X S_k(x, y) \, d\mu(x) := 1.
\end{align}

Thus with \(D_k := S_{k+1} - S_k\) we have

\[f = \sum_{k \in \mathbb{Z}} D_k f, \quad f \in L^2(\mu).\]

Next fix \(N \in \mathbb{N}\) so large that if \(T^N := \sum_{k \in \mathbb{Z}} D^N_k D_k\) where \(D^N_k := \sum_{|j| \leq N} D_{k+j}\), then \(T^N\) is invertible on \(L^2(\mu)\). It follows that

\[f = T^N T^{-N} f = \sum_{k \in \mathbb{Z}} D^N_k D_k T^{-N} f = \sum_{k \in \mathbb{Z}} E_k \bar{E}_k f, \quad f \in L^2(\mu),\]

where \(E_k := D^N_k\) and \(\bar{E}_k := D_k T^{-N}\). This latter formula is usually called a Calderón reproducing formula, and substitutes for the orthonormal wavelet formula (2.5).
2.5.2. Discrete $g$-functions and Area functions. The function

$$g (f) (x) := \sum_{k=-\infty}^{\infty} |D_k f (x)|^2 = |G f (x)|_{L^2 (\mathbb{Z})}, \quad G f (x) := \{D_k f (x)\}_{k \in \mathbb{Z}}$$

plays the role of a $g$-function in a homogeneous space $X$. We could also replace $D_k$ by $E_k := D_k^N$ in this formula giving an alternative $g$-function. Property (2.6), together with an application of the Cotlar–Stein lemma, shows that in either case $g$ is a Hilbert space valued Calderon–Zygmund operator on $X$. The function

$$S f (x) := \sum_{k=-\infty}^{\infty} \overline{E_k f (x)}^2 = |S f (x)|_{L^2 (\mathbb{Z})}, \quad S f (x) := \{\overline{E_k f (x)}\}_{k \in \mathbb{Z}},$$

plays the role of an area function in $X$, and [HaSa, Theorem 3.4] shows that the kernel of $S$ satisfies (1.12), and the boundedness on $L^2 (\mu)$ is proved in [DaJoSe]. Thus $S$ is a Hilbert space valued Calderon–Zygmund operator on $X$, and Theorem 4 yields the following two weight norm inequality - a stronger result was obtained in Euclidean space by Lacey and Li [LaLi], stronger in the sense that neither the dual testing condition nor the dual pivotal condition was needed and they didn’t need to test over all unit vectors.

**Theorem 5.** Let $u$ and $v$ be two locally finite positive Borel measures on $X$. Suppose that $u(\{x\}) \cdot v(\{x\}) = 0$ for $x \in X$. Suppose the above $A_2$ and pivotal conditions hold. Suppose also that the following testing conditions hold for the area function $S$ as defined above: for every ball $B \subset X$, we have the following testing conditions, with $1_Q$ taken as the indicator of $Q$:

$$\|S (u 1_Q)\|_{L^2_{h_2} (v)} \leq T \|1_Q\|_{L^2 (u)},$$

$$\|S^* (ve_2 1_Q)\|_{L^2 (u)} \leq T \|1_Q\|_{L^2 (v)}$$

for all unit vectors $e_2$ in $H_2$.

Then there holds $N \lesssim A_2 + T + V$.

2.6. Classical Riesz transforms. As an application to known work we can look at the two weight inequality for the Hilbert Transform. We define our Hilbert operator as below for a signed measure $v$ on $\mathbb{R}$:

$$Hv(x) := p.v. \int_{\mathbb{R}} \frac{1}{x-y} v(dy).$$

For two weights $u, v$, the inequality we are interested in is

$$\|H (uf)\|_{L^2 (v)} \lesssim \|f\|_{L^2 (u)}.$$ 

Along with $A_2$ for the pair of weights $u, v$ given we have the following testing conditions holding uniformly over intervals $I$:

$$\int_I |H (1_I u)|^2 v(dx) \leq \mathcal{H}^2 u(I),$$

$$\int_I |H (1_I v)|^2 u(dx) \leq \mathcal{H}^2 v(I).$$

Here $\mathcal{H}$ denotes the smallest constants for which these inequalities are true uniformly over all intervals $I$.

In beautiful series of papers, Nazarov, Treil and Volberg, [NTV1, NTV2, NTV3] have developed a sophisticated approach towards proving the sufficiency of these testing conditions combined with the improvement of the two weight $A_2$ condition. The improvement is described below using a variant of Poisson integral. For an interval $I$ and measure $v$,

$$P(I, v) := \int_{\mathbb{R}} \frac{|I|}{|I| + \text{dist}(x, I)^2} \omega(dx),$$

$$\sup_I P(I, v) \cdot P(I, u) := A_2^2 < \infty.$$ 

We will refer to the last line above as the $A_2$ condition. In [NTV3, Theorem 2.2] Nazarov, Treil and Volberg proved the sufficiency of the $A_2$ and testing conditions above for the two weight inequality of the Hilbert transform in the presence of the pivotal condition given by

$$\sum_{r=1}^{\infty} v(I_r) P(I_r, 1_{I_r}, u)^2 \leq V^2 u(I_0),$$
and its dual, where the inequality is required to hold for all intervals $I_0$ and decompositions $\{I_r : r \geq 1\}$ of $I_0$ into disjoint intervals $I_r \subseteq I_0$. We have taken inspiration from this proof and condition in providing the proof of our main result. In [Lac, Lac1, LaSaUr, LaSaShUr] the pivotal condition was removed as a side condition and replaced with an energy condition, ultimately yielding a characterization of the two weight inequality for the Hilbert transform.

In higher dimension we look at two weight inequalities for Riesz transforms. Earlier work appears in [LaWi, SaShUr] where certain two weight inequalities for the Riesz transforms (and fractional versions) were studied. Namely for two weights, nonnegative locally finite Borel measures $u, v$ on $\mathbb{R}^n$, we are interested in the following inequality for the $d$ dimensional Riesz transform

$$\left\| \int_{\mathbb{R}^n} f(y) \frac{x - y}{|x - y|^{d+1}} u(dy) \right\|_{L^2(u)} \leq \mathcal{N}\|f\|_{L^2(u)}.$$

Here we take $0 < d \neq n - 1 \leq n$ and $\mathcal{N}$ is the best constant in the inequality above.

The $A_2$ type condition is expressed in terms of a Poisson type operator. For a cube $Q \subset \mathbb{R}^n$, we take

$$P(u, Q) := \int_{\mathbb{R}^n} \frac{|Q|^{d/n}}{|Q|^{2d/n} + \text{dist}(x, Q)^{2d}} u(dx).$$

Using the $A_2$, testing conditions and pivotal condition we can obtain sufficient conditions for the two weight inequality for the $d$ dimensional Riesz transform by the above Theorem 2 on $(\mathbb{R}^n, dx, |\cdot|_n)$, viewed as space of homogeneous type where $|\cdot|_n$ is a standard metric on $\mathbb{R}^n$.

### 2.7. Riesz Transform Associated with Certain Schrödinger Operators

Consider $L := -\Delta + \mu$, which is a Schrödinger operator with a non-negative Radon measure $\mu$ on $\mathbb{R}^n$ for $n \geq 3$. We assume that $\mu$ satisfies the following conditions: there exists a positive constant $\sigma_0 \in (1, \infty)$ such that

$$\mu(B(x, r)) \lesssim \left(\frac{r}{R}\right)^{n-2+\sigma_0} \mu(B(x, R)) \quad (2.7)$$

and

$$\mu(B(x, 2r)) \lesssim \{\mu(B(x, r)) + r^{n-2}\} \quad (2.8)$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$, where $B(x, r)$ denotes the open ball centered at $x$ with radius $r$. As pointed in [Shen], condition (2.7) may be regarded as scale-invariant Kato-condition, and (2.8) says that the measure $\mu$ is doubling on balls satisfying $\mu(B(x, r)) \geq cr^{n-2}$. We will also assume that $\mu \not\equiv 0$. When $d\mu = V(x)dx$ and $V \geq 0$ is in the reverse Hölder class $(RH)_n$, i.e.,

$$\left(\frac{1}{|B(x, r)|} \int_{B(x, r)} V(y)^n dy \right)^{1/n} \leq \frac{C}{|B(x, r)|} \int_{B(x, r)} V(y) dy,$$

then $\mu$ satisfies the conditions (2.7) and (2.8) for some $\sigma_0 > 1$. However, in general, measures which satisfy (2.7) and (2.8) need not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. For instance, when $d\mu = d\sigma(x_1, x_2)dx_3 \cdots dx_n$, where $\sigma$ is a doubling measure on $\mathbb{R}^2$, then $\mu$ satisfies (2.7) and (2.8) for some $\sigma_0 > 1$.

It is well-known that the Riesz transform $\nabla L^{-\frac{n}{2}}$ is bounded on $L^2(\mathbb{R}^n)$. Moreover, let $K(x, y)$ be the kernel of $\nabla L^{-\frac{n}{2}}$, the Riesz transforms associated to $L$. Then it was proved in [Shen] that

$$|K(x, y)| \lesssim \frac{1}{|x - y|^{n}} \quad (2.10)$$

and that for $|x - x'| \leq \frac{1}{4}|x - y|$, $|x - y'| \leq \frac{1}{4}|x - y|$, $n \geq 3$,

$$|K(x, y) - K(x', y)| \lesssim \left(\frac{|x - x'|}{|x - y|}\right)^{\sigma_0 - 1} \frac{1}{|x - y|^{n}}. \quad (2.11)$$

Here the implicit constants are independent of $x$ and $y$.

### 3. Preliminaries on Spaces of Homogeneous Type

Let $(X, d, \mu)$ be a space of homogeneous type as in Section 1.
3.1. A System of Dyadic Cubes. We will recall from [HyKa] a construction of dyadic cubes, which is a deep elaboration of work by M. Christ [Chr], as well as that of Sawyer–Wheeden [SaWh]. We summarize the dyadic construction of random dyadic systems from [HyMa] and [HyKa] in the following theorem. First we need to define an appropriate notion of ‘reference points’ or ‘lattice points’ in $X$.

**Definition 6.** A set of points $\{x^k_\alpha\}_{k \in \mathbb{Z}, \alpha \in A_k} \subset X$ is said to be a set of reference points if there exist constants $0 < c_0 \leq C_0 < \infty$ and $0 < \delta < 1$ such that $12A_0^3C_0\delta \leq c_0$ and

$$d(x_\alpha^k, x_\beta^k) \geq \min_{\alpha \neq \beta} c_0 \delta^k, \quad \min_{\alpha} d(x, x_\alpha^k) \leq C_0 \delta^k, \quad x \in X.$$ 

The following construction is from [HyKa, Theorems 5.1 and 5.6].

**Theorem 7.** Given a set of reference points $\{x^k_\alpha\}_{k \in \mathbb{Z}, \alpha \in A_k}$ with parameters $c_0, C_0$ and $\delta$, and sufficiently small $0 < \delta < 1$ (e.g. $144A_0^3\delta \leq 1$), there exists a probability space $(\Omega, \mathcal{F})$ such that every $\omega \in \Omega$ defines a dyadic system $D(\omega) := \{Q^k_\alpha(\omega)\}_{k \in \mathbb{Z}, \alpha \in A_k}$ related to new dyadic points $\{z^k_\alpha(\omega)\}_{k \in \mathbb{Z}, \alpha \in A_k}$ with the following geometric properties: for some $c_1$ and $C_1$ depending on $c_0, C_0, A_0$ and $\delta$,

1. If $\ell \geq k$, then either $Q^k_\alpha(\omega) \subset Q^\ell_\alpha(\omega)$ or $Q^\ell_\alpha(\omega) \cap Q^k_\alpha(\omega) = \emptyset$;
2. $X = \bigcup Q^k_\alpha(\omega)$ for all $k \in \mathbb{Z}$;
3. $B(z^k_\alpha(\omega), c_1\delta^k) \subset Q^k_\alpha(\omega) \subset B(z^k_\alpha(\omega), C_1\delta^k) =: B(Q^k_\alpha(\omega))$;
4. If $\ell \geq k$ and $Q^\ell_\alpha(\omega) \subset Q^k_\alpha(\omega)$ then $B(Q^\ell_\alpha(\omega)) \subset B(Q^k_\alpha(\omega))$,

and the following probabilistic property: There are positive constants $C_2, \eta > 0$ such that for every $x \in X, \tau > 0$ and $k \in \mathbb{Z}$,

$$\mathbb{P}\left(\omega \in \Omega : x \in \bigcup_{\alpha} \partial_{\tau, \delta^k} Q^k_\alpha(\omega)\right) \leq C_2 \tau^n.$$ 

We also have the following containment property:

$$B(x^k_\alpha, c_1\delta^k) \subset Q^k_\alpha \subset B(x^k_\alpha, C_1\delta^k) =: B(Q^k_\alpha).$$

**Definition 8.** We say that $D(\omega) = \{Q^k_\alpha(\omega)\}_{k \in \mathbb{Z}, \alpha \in A_k}$ is a system of dyadic cubes if (1)—(4) hold in Theorem 7. Given a dyadic cube $Q^k_\alpha(\omega)$, we denote the quantity $\delta^k$ by $l(Q^k_\alpha)$, by analogy with the side length of a Euclidean cube.

3.2. An Explicit Haar Basis on Spaces of Homogeneous Type. Next we recall the explicit construction in [KLPW] of a Haar basis $\{h^\epsilon_Q : Q \in D, \epsilon = 1, \ldots, M_Q - 1\}$ for $L^p(X, \mu)$, $1 < p < \infty$, associated to the dyadic cubes $Q \in D$ as follows. Here $M_Q := \#(\mathcal{H}(Q)) = \#\{R \in D_{k+1} : R \subseteq Q\}$ denotes the number of dyadic sub-cubes (which we will refer to as the “children”) the cube $Q \in D_k$ has; namely $\mathcal{H}(Q)$ is the collection of dyadic children of $Q$. It is known in [KLPW] that $\sup_{Q \in D} M_Q < \infty$.

**Theorem 9.** Let $(X, d, \mu)$ be a space of homogeneous type and suppose $\mu$ is a positive locally finite Borel measure on $X$. For $1 < p < \infty$, for each $f \in L^p(X, \mu)$, we have

$$f(x) = \sum_{Q \in D} \sum_{\epsilon = 1}^{M_Q - 1} \langle f, h^\epsilon_Q \rangle \mu h^\epsilon_Q(x),$$

where the sum converges (unconditionally) both in the $L^p(X, \mu)$-norm and pointwise $\mu$-almost everywhere.

The following theorem collects several basic properties of the functions $h^\epsilon_Q$.

**Theorem 10.** The Haar functions $h^\epsilon_Q$, $Q \in D$, $\epsilon = 1, \ldots, M_Q - 1$, have the following properties:

1. $h^\epsilon_Q$ is a simple Borel-measurable real function on $X$;
2. $h^\epsilon_Q$ is supported on $Q$;
3. $h^\epsilon_Q$ is constant on each $R \in \mathcal{H}(Q)$;
4. $f_Q h^\epsilon_Q d\mu = 0$ (cancellation);
(5) \( \langle h^\epsilon_Q, h'^\epsilon_Q \rangle = 0 \) for \( \epsilon \neq \epsilon', \epsilon' \in \{1, \ldots, M_Q - 1\} \);

(6) the collection \( \{\mu(Q)^{-1/2}e_Q \} \cup \{h^\epsilon_Q : \epsilon = 1, \ldots, M_Q - 1\} \) is an orthogonal basis for the vector space \( V(Q) \) of all functions on \( Q \) that are constant on each sub-cube \( R \in \mathcal{H}(Q) \);

(7) if \( h^\epsilon_Q \neq 0 \) then \( \|h^\epsilon_Q\|_{L^p(X,\mu)} \approx \mu(Q)^{1-p/2} \) for \( 1 \leq p \leq \infty \);

(8) \( \|h^\epsilon_Q\|_{L^1(X,\mu)} \cdot \|h^\epsilon_Q\|_{L^\infty(X,\mu)} \approx 1 \).

We denote \( h^0_Q := \mu(Q)^{-1/2}e_Q \), which is a non-cancellable Haar function. Moreover, the martingale associated with the Haar functions are as follows: for \( Q \in \mathcal{D}_k \),

\[
E_Q f := \langle f, h^0_Q \rangle_{\mu} h^0_Q \quad \text{and} \quad D_Q f := \sum_{c=1}^{M_Q-1} D_c^Q f,
\]

where \( D^Q_c f := \langle f, h^c_Q \rangle_{\mu} h^c_Q \) is the martingale operator associated with the \( c \)-th subcube of \( Q \). Also we have

\[
E_k f = \sum_{Q \in \mathcal{D}_k} E_Q f \quad \text{and} \quad D_k f = E_{k+1} f - E_k f.
\]

Hence, based on the construction of Haar system \( \{h^c_Q\} \) in [KLPW] we obtain that for each \( R \in \mathcal{D} \) and \( \eta = 1, \ldots, M_R - 1 \),

\[
\sum_{Q: R \subset Q} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_{\mu} h^\eta_Q h^c_R = \sum_{Q: R \subset Q} D_Q f \cdot h^\eta_R = E_R f \cdot h^\eta_R = \langle f, h^0_R \rangle_{\mu} h^0_R h^\eta_R.
\]

### 3.3. The Carleson Embedding Theorem in Spaces of Homogeneous Type

Now we will describe the familiar Carleson Embedding theorem, which will be crucial in the control of certain paraproduct terms.

**Theorem 11.** Fix a weight \( u \) and consider nonnegative constants \( \{a_Q : Q \in \mathcal{D}\} \). The following two inequalities are equivalent:

\[
\sum_{Q \in \mathcal{D}} a_Q ||E^u_Q f||^2 \leq C \|f\|^2_{L^2(u)}, \quad \text{where} \quad E^u_Q f := \langle f, h^0_Q \rangle_u h^0_Q;
\]

\[
\sum_{Q \in \mathcal{D} : Q \subset S} a_Q \leq Cu(S).
\]

Taking \( c \) and \( C \) to be the best constants in these inequalities, we have \( c \approx C \).

## 4. First Reduction in the Proof of the Two-Weight Inequality

We now begin to prove the two weight inequality in our setting. In this section we reduce to showing that it suffices to prove Theorem 2 under the hypothesis that \( f \) and \( g \) are ‘good’ functions (as explained below).

Let \( f \in L^2(u) \) and \( g \in L^2(v) \) be two functions. Without loss of generality, we can assume that these two functions have compact support. Moreover, it is sufficient to assume that \( f \) and \( g \) are supported on a common (large) cube \( Q_0 \), see for example [Vol]. From Theorem 9 we have

\[
f(x) = \sum_{Q \in \mathcal{D}} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u h^c_Q(x).
\]

We write this sum in two parts as follows:

\[
f(x) = \sum_{Q \subset Q_0} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u h^c_Q(x) + \sum_{Q: Q_0 \subset Q} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u h^c_Q(x).
\]

Based on Theorem 10 and (3.3) we have that

\[
E_{Q_0} f \cdot h^\eta_{Q_0} = \langle f, h^0_{Q_0} \rangle_u h^0_{Q_0} h^\eta_{Q_0} = \sum_{Q: Q_0 \subset Q} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u h^c_Q,
\]

\[
f = E_{Q_0} f + \sum_{Q \subset Q_0} \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u h^c_Q =: f_1 + f_2.
\]
Similarly we write for $g \in L^2(v)$:

$$g = E_{Q_0} g + \sum_{Q \subset Q_0} \sum_{c=1}^{M_Q-1} \langle h_Q^c, h_Q^c \rangle_v =: g_1 + g_2.$$ 

Here we have $\|f\|^2_{L^2(u)} = \|f_1\|^2_{L^2(u)} + \|f_2\|^2_{L^2(u)}$; similar formulas hold for the function $g$.

Let $T$ be a Calderón–Zygmund Operator on $(X, d, \mu)$. Given these decompositions of $f$ and $g$, let us begin the proof by looking at their inner product

$$\langle (T(u f), g)_v = \langle (T(u f_1), g_1)_v + \langle (T(u f_2), g_2)_v + \langle (T(u f_2), g_2)_v =: I_1 + I_2 + I_3 + I_4.$$ 

It is enough to obtain good estimates on each of the $I_j$ above. The first three terms are easy to control just using the testing condition assumed on the operator $T$, the last term will then require substantial analysis.

We now show how to control $I_1$, $I_2$ and $I_3$ just using the testing condition. First observe that

$$I_1 = \langle (T(u f_1), g_1)_v = \int_{Q_0} f du \cdot \int_{Q_0} g dv \int_{Q_0} \langle T(u 1_{Q_0}), 1_{Q_0} \rangle_v.$$

By Cauchy-Schwarz, applied to the function $f$ and the function $g$, and in the inner product, and then using the testing conditions assumed on the operator $T$ we have:

$$|I_1| \leq T \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$ 

The terms $I_2$ and $I_3$ are symmetric

$$I_2 = \langle (T(u f_1), g_2)_v = \int_{Q_0} f du \cdot \int_{Q_0} g dv \int_{Q_0} \langle T(u 1_{Q_0}), 1_{Q_0} \rangle_v.$$

Using Cauchy-Schwarz, the testing conditions and the fact that $\|g_2\|_{L^2(v)} \leq \|g\|_{L^2(v)}$ we get the following:

$$|I_2| \leq T \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$ 

An identical argument works for $I_3$.

By the above it suffices to prove

$$|\langle (T(u f), g)_v \|f\|_{L^2(u)} \|g\|_{L^2(v)}$$

when $f$ and $g$ have compact support in $Q_0$ and $\int_{Q_0} f du = 0$ and $\int_{Q_0} g dv = 0$. We will now decompose the inner product $(T(u f), g)_v$ using good-bad decomposition.

**Remark 12.** In order to use surgery to remove weak boundedness, we will need to work in the world of two independent systems of random grids.

### 4.1. The Good and Bad Parts of Functions

We use the good-bad decomposition of test functions to simplify the proof even further. Fix a number $\epsilon$, $0 < \epsilon < 1$. Later the choice of $\epsilon$ will be dictated by the Calderón–Zygmund properties of the operator $T$ and the underlying measure $\mu$. Also fix a sufficiently large integer $r$. The choice of $r$ will be made in this section. Finally, we consider two grids $\mathcal{D} = \{Q_k^\omega(\omega)\}_{k, \alpha}$ and $\mathcal{D}' = \{Q_k^\omega(\omega')\}_{k, \alpha}$ for $\omega, \omega' \in \Omega$.

**Definition 13.** Take a dyadic cube $Q \in \mathcal{D}$. We say that $Q$ is $r$-good in $\mathcal{D}'$ for an integer $r$, if for every cube $Q_1 \in \mathcal{D}'$ such that if $\delta^k \leq \delta^r$ with $k \geq n + r$, then either

$$\text{dist}(Q, Q_1) \geq \delta^k \delta^{n(1-\epsilon)} \text{ or } \text{dist}(Q, X \setminus Q_1) \geq \delta^k \delta^{n(1-\epsilon)}.$$ 

Above we are letting $l(Q) := \delta^k$ and $l(Q_1) := \delta^n$. If $Q$ is not $r$-good we call it $r$-bad.

We can now decompose $f$ into good and bad parts as below.

$$f = f_{\text{good}} + f_{\text{bad}}$$

$$f_{\text{bad}} := \sum_{Q \in \mathcal{D}, Q \text{ is bad}} \Delta_Q f.$$ 

**Theorem 14.** [Vol, Theorem 17.1] There holds on $(X, d, \mu)$ for $f \in L^2(u)$

$$E(\|f_{\text{bad}}\|_{L^2(u)}) \leq \varepsilon(r) \|f\|_{L^2(u)}$$

where $\varepsilon(r) \to 0$ as $r \to \infty$. A similar estimate holds for $g_{\text{bad}} \in L^2(v)$. 

Proposition 15. Consider the decompositions of $f$ and $g$ into bad and good parts on $(X,d,\mu)$, where the support cubes of the Haar projections of $f$ are good with respect to $\mathcal{G}$, and the support cubes of the Haar projections of $g$ are good with respect to $\mathcal{D}$. Let $u$ and $v$ be pairs of weights and suppose there holds uniformly over all dyadic grids $\mathcal{D}$ and $\mathcal{G}$ for some finite constant $C$

\begin{equation}
\mathbb{E}(|\langle T(uf_{\text{good}}), g_{\text{good}} \rangle_v|) \leq C \|f\|_{L^2(u)} \|g\|_{L^2(v)},
\end{equation}

where $\mathbb{E}$ refers to expectation over the product probability space $\Omega \times \Omega$. Then

$$\langle T(uf), g \rangle_v \leq 2C \|f\|_{L^2(u)} \|g\|_{L^2(v)},$$

that is

$$\|T\|_{L^2(u) \to L^2(v)} \leq 2C.$$

Proof. Note that

$$\langle T(uf), g \rangle_v = \langle T(uf_{\text{good}}), g_{\text{good}} \rangle_v + \langle T(uf_{\text{bad}}), g \rangle_v.$$

So

$$\mathbb{E}(|\langle T(uf), g \rangle_v|) \leq \mathbb{E}(|\langle T(uf_{\text{good}}), g_{\text{good}} \rangle_v|) + \mathbb{E}(|\langle T(uf_{\text{bad}}), g \rangle_v|).$$

Using (4.1) and by Theorem 14 we have

$$|\langle T(uf), g \rangle_v| \leq C \|f\|_{L^2(u)} \|g\|_{L^2(v)} + 2 \|T\|_{L^2(u) \to L^2(v)} \varepsilon(r) \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$

Notice that $\|Tf\|_{L^2(u) \to L^2(v)} = \sup \|Tf, g\|_v$. Choose $f$, $g$ and $r$ sufficiently large such that

\begin{equation}
|\langle T(uf), g \rangle_v| \geq \frac{1}{2} \|T\|_{L^2(u) \to L^2(v)} \|f\|_{L^2(u)} \|g\|_{L^2(v)} \text{ and } \varepsilon(r) < \frac{1}{8}.
\end{equation}

Then by absorbing the second term in (4.2) to left hand side, we get

$$\|T\|_{L^2(u) \to L^2(v)} \leq 2C.$$ 

\[\Box\]

Definition 16. We fix $r > 0$ with $\varepsilon(r) < \frac{1}{8}$ and throughout the paper and abbreviate $r$-good as simply good.

The upshot of the above is that if we manage to prove that for all $f \in L^2(u)$ and $g \in L^2(v)$ we have

$$\mathbb{E}(|\langle T(uf_{\text{good}}), g_{\text{good}} \rangle_v|) \leq C \|f\|_{L^2(u)} \|g\|_{L^2(v)}$$

then we obtain (4.1). The remainder of the paper is devoted to proving (4.1). We remind the reader that $\mathbb{E}$ refers to expectation taken over the product probability space $\Omega \times \Omega$. In fact, we will prove that (4.1) holds for all dyadic grids and so that statement regarding the expectation will then follow.

5. Main Decomposition

Fix a cube $Q_0$ for the rest of the paper; this is the support of the functions $f$ and $g$. Our goal is to demonstrate (4.1). This will require several additional reductions.

Throughout the rest of this paper we assume the $A_2$ conditions, the testing conditions and the pivotal conditions.

5.1. Global to Local Reduction. We will now try to control the bilinear form:

$$\langle T(uf), g \rangle_v = \sum_{Q \in \mathcal{D}, S \notin \mathcal{G}} \sum_{i=1}^{M_Q-1} \sum_{j=1}^{M_S-1} \langle f, h^i_Q \rangle_u \langle T(uh^i_Q), h^j_S \rangle_v \langle g, h^j_S \rangle_v$$

$$= \sum_{Q \in \mathcal{D}, S \notin \mathcal{G}} \sum_{i=1}^{M_Q-1} \sum_{j=1}^{M_S-1} \langle f, h^i_Q \rangle_u \langle T(uh^i_Q), h^j_S \rangle_v \langle g, h^j_S \rangle_v$$

$$+ \sum_{Q \in \mathcal{D}, S \notin \mathcal{G}} \sum_{i=1}^{M_Q-1} \sum_{j=1}^{M_S-1} \langle f, h^i_Q \rangle_u \langle T(uh^i_Q), h^j_S \rangle_v \langle g, h^j_S \rangle_v$$

$$=: A_1^1 + A_2^1.$$
Set $A_1^1 := \{(Q, S) \in D \times G : l(Q) \geq l(S)\}$ and $A_2^1 := \{(Q, S) \in D \times G : l(S) > l(Q)\}$ and

$$A_1^i := \sum_{(Q, S) \in A_j^i} \sum_{\ell = 1}^{M_Q - 1} \sum_{k = 1}^{M_S - 1} \langle f, h_Q^\ell \rangle \langle T(u h_Q^\ell), h_S^k \rangle \langle g, h_S^k \rangle.$$ 

Here $A_2^1$ is complementary to the sum $A_1^1$. The sums are estimated symmetrically. Hence it is enough to prove (4.1) for $A_1^1$. We will further decompose this term into a number of other bilinear forms $A_j^i$. The superscript $i$ denotes the generation and subscript $j$ counts the number of decompositions. To help understand these decompositions we can look at the flow chart where all the terms are listed.

(1) The flow chart starts at the bilinear form in (4.1).

(2) The hypothesis we used in controlling each bilinear form $A_2$, $T$, surgery and/or $\mathcal{V}$ is written on the edges of the chart.

(3) To control the terms $A_1^6$, $A_3^6$, $A_3^7$ and $A_4^7$ we use the stopping cube arguments given in Section 6.

(4) The edge leading to $A_3^2$ has been labelled paraproduct as all the estimates below that use paraproduct arguments to control them.

Now let us begin by proving the estimate in (4.1) for the term $A_1^1$. We decompose $A_1^1$ into the following sets. Denote by

$$A_1^2 := \{(Q, S) \in A_1^1 : \delta' l(Q) \leq l(S) \leq l(Q), \text{dist}(Q, S) \leq l(Q)\};$$

$$A_2^2 := \{(Q, S) \in A_1^1 : l(S) \leq l(Q), \text{dist}(Q, S) \geq l(Q)\};$$
\[ A_2^2 := \{(Q, S) \in A_1^1 : l(S) \leq \delta^\ast l(Q), \text{dist}(Q, S) \leq l(Q)\}. \]

We will show the following estimates below:

\[ |A_1^2| \lesssim \left(C_\ast \sqrt{A_2^2 + T + \tau^2N} \right) \|f\|_{L^2(u)} \|g\|_{L^2(v)} \]

\[ |A_2^2| \lesssim A_2 \|f\|_{L^2(u)} \|g\|_{L^2(v)} \]

The term \( A_1^2 \) is easily controlled using a ‘weak boundedness property’, which we recall (even though it will not be needed):

**Definition 17.** *Weak Boundedness Condition:* For a constant \( C_{\text{WBP}} > 1 \), we have \( C_{\text{WBP}} \) as the best constant in the inequality

\[ \left| \int_S T(u1_Q) dv \right| \leq C_{\text{WBP}} u(Q)^{\frac{3}{2}} v(S)^{\frac{1}{2}}, \]

where \( Q, S \) are cubes such that \( \delta^\ast l(Q) \leq l(S) \leq l(Q) \) and \( \text{dist}(Q, S) \leq l(Q) \).

We can avoid the weak boundedness property if we instead use ‘surgery’ to control the *average over grids* in terms of only the testing and \( A_2 \) conditions, and a small multiple of the operator norm. It is clear that the lemma below provides control on the term \( A_1^2 \) as desired with a small multiple of the norm that can be absorbed by choosing the parameter \( \tau \) appropriately small.

**Lemma 18.** The following estimate holds:

\[ E_{S \in \Omega} \sum_{(Q, S) \in A_1^2} \sum_{\epsilon=1}^{M_Q-1} \sum_{k=1}^{M_S-1} \langle f, h_Q^\epsilon \rangle_u \langle Th_Q^\epsilon, h_S^k \rangle_v \langle g, h_S^k \rangle_v \lesssim \left(C_\ast \sqrt{A_2^2 + T + \tau^2N} \right) \|f\|_{L^2(u)} \|g\|_{L^2(v)}. \]

For the proof of Lemma 18 we recall the surgery estimate (3.1) for the random dyadic systems constructed in [HyMa] and [HyKa] with parameter \( \delta > 0 \). There are positive constants \( C_3, \eta > 0 \) such that for every \( x \in X, \tau > 0 \) and \( k \in \mathbb{Z} \),

\[ \mathbb{P} \left( \left\{ \omega \in \Omega : x \in \bigcup_{\alpha} \partial_{\tau, \delta^k} Q_\alpha^k(\omega) \right\} \right) \leq C_3 \tau^\eta. \]

We can now prove an extension to spaces of homogeneous type of the surgery lemma of Lacey and Wick in [LaWi, Lemma 8.5], by repeating their argument with obvious modifications. In the estimate of Lemma 18, the explicit Haar functions are used in the decompositions, whereas in the surgery lemma it is convenient to use instead the associated Haar projections

\[ \Delta_Q^\epsilon f = \sum_{\epsilon=1}^{M_Q-1} \langle f, h_Q^\epsilon \rangle_u h_Q^\epsilon \] and \[ \Delta_S^k g = \sum_{k=1}^{M_S-1} \langle g, h_S^k \rangle_v h_S^k. \]

We say that two cubes \( Q \) and \( S \) are \( \rho \)-close if \( \frac{l(Q)}{l(S)} \leq \delta^\rho \) and \( d(Q, S) \leq \max\{l(Q), l(S)\} \).

**Lemma 19 (Surgery Lemma).** For \( 0 < \tau \leq 1 \) and sufficiently large \( r \) we have

\[ E_{S \in \Omega} \sum_{(Q, S) \in D_{\text{good}} \times D_{\text{good}}} \left| \langle T(u \Delta_Q^\epsilon f), \Delta_S^k g \rangle_v \right| \lesssim \left(C_\ast \sqrt{A_2^2 + T + \tau^2N} \right) \|f\|_{L^2(u)} \|g\|_{L^2(v)}. \]

Note that we can choose \( \rho \) to be \( r \) so Lemma 18 follows immediately from Lemma 19 since we are summing on a larger collection of cubes and we have pulled the absolute value inside the sum.

**Proof.** In order to prove (5.2), we invoke the surgery estimate using (5.1). Given \( 0 < \lambda < \frac{1}{2} \), define

\[ S_\lambda := \{ x \in S : \text{dist}(x, \partial S) > \lambda l(S) \}. \]

Then we write

\[ \langle T(u \Delta_Q^\epsilon f), \Delta_S^k g \rangle_v = \left\langle T \left( u \sum_{Q' \in H(Q)} 1_{Q'} \Delta_Q^\epsilon f \right), \sum_{S' \in H(J)} 1_{S'} \Delta_S^k g \right\rangle_v. \]
then the inequality

\[ \sum_{Q \in \mathcal{D}_{good} \times Q \in \mathcal{H}(Q)} \left| \langle T(u \Delta_Q^u f), \Delta_Q^v g \rangle \right| \]

and so

\[ \sum_{(Q,S) \in \mathcal{D}_{good} \times \mathcal{G}_{good} \times \mathcal{H}(Q)} \sum_{S' \in \mathcal{H}(S)} \left| \langle T(u \Delta_Q^u f), \Delta_Q^v g \rangle \right| \]

\[ \leq \sum_{(Q,S) \in \mathcal{D}_{good} \times \mathcal{G}_{good} \times \mathcal{H}(Q)} \sum_{S' \in \mathcal{H}(S)} \left| \langle \mathbf{E}_{Q'}^u \Delta_Q^u f, \Delta_Q^v g \rangle \right| \]

\[ + \sum_{Q \in \mathcal{D}_{good} \times \mathcal{G}_{good} \times \mathcal{H}(Q)} \sum_{S' \in \mathcal{H}(S)} \left| \langle \mathbf{E}_{Q'}^u \Delta_Q^u f, \Delta_Q^v g \rangle \right| \]

\[ =: \text{Term}_1 + \text{Term}_2. \]

Now for convenience of notation and to shorten some displays we write

\[ \sum_\star := \sum_{Q,S} \sum_{Q' \in \mathcal{H}(Q)} \sum_{S' \in \mathcal{H}(S)} \]

then the inequality

\[ \left| \langle T(u \mathbf{1}_{Q'}), \mathbf{1}_{S' \cap Q'} \rangle \right| \leq \sqrt{\int_{Q'} |T(u \mathbf{1}_{Q'})|^2 dv \sqrt{\nu(S' \cap Q')}} \leq T \sqrt{u(Q')} \sqrt{v(S')} \]

shows that

\[ \text{Term}_1 \leq \mathbf{T} \sum_\star \left( |\mathbf{E}_{Q'}^u \Delta_Q^u f| \sqrt{u(Q')} \right) \left( |\mathbf{E}_{S'}^v \Delta_Q^v g| \sqrt{v(S')} \right) \]

\[ \leq \mathbf{T} \sqrt{\sum_\star |\mathbf{E}_{Q'}^u \Delta_Q^u f|^2 u(Q')} \left( \sum_\star |\mathbf{E}_{S'}^v \Delta_Q^v g|^2 v(S') \right) \leq \mathbf{T} \|f\|_{L^2(u)} \|g\|_{L^2(v)}. \]

To control \( \text{Term}_2 \) we further decompose \( S' \setminus Q' \) into small and large parts,

\[ S' \setminus Q' := \{(S' \setminus Q') \cap \partial_Q Q'\} \bigcup \{(S' \setminus Q') \setminus \partial_Q Q'\} := E \bigcup F, \]

and estimate \( \text{Term}_2 \) accordingly,

\[ \text{Term}_2 \leq \sum_\star |\langle \mathbf{E}_{Q'}^u \Delta_Q^u f, \langle T(u \mathbf{1}_{Q'}), \mathbf{1}_E \rangle \rangle | \sum_\star |\langle \mathbf{E}_{S'}^v \Delta_Q^v g, \langle T(u \mathbf{1}_{Q'}), \mathbf{1}_F \rangle \rangle | := \text{Term}_{21} + \text{Term}_{22}. \]

Now \( F = (S' \setminus Q') \setminus \partial_Q Q' \) is contained in \( S' \) and has distance at least \( cl(Q) \) from the cube \( Q' \), so we can control \( \text{Term}_{22} \) by the \( A_2 \) condition using Cauchy-Schwarz as above,

\[ \text{Term}_{22} \leq \sum_\star |\langle \mathbf{E}_{Q'}^u \Delta_Q^u f, \langle T(u \mathbf{1}_{Q'}), \mathbf{1}_E \rangle \rangle | \left( \mathbf{A}_2 \sqrt{u(Q')} \sqrt{v(S')} \right) \sum_\star |\mathbf{E}_{S'}^v \Delta_Q^v g| \leq \mathbf{A}_2 \|f\|_{L^2(u)} \|g\|_{L^2(v)}. \]

Finally, we use the operator norm to control the average of \( \mathbf{B}_1 \) by

\[ \mathbf{E}_{D \in \Omega} \text{Term}_{21} \leq \mathbf{E}_{D \in \Omega} \sum_\star |\mathbf{E}_{Q'}^u \Delta_Q^u f| \left( \mathbf{N} \sqrt{u(Q')} \sqrt{v((S' \setminus Q') \cap \partial_Q Q')} \right) |\mathbf{E}_{S'}^v \Delta_Q^v g| \]

\[ \leq \mathbf{N} \mathbf{E}_{D \in \Omega} \|f\|_{L^2(u)} \left( \sum_\star |\mathbf{E}_{S'}^v \Delta_Q^v g|^2 v((S' \setminus Q') \cap \partial_Q Q') \right) \]

\[ \leq \mathbf{N} \mathbf{E}_{D \in \Omega} \left( \sum_\star |\mathbf{E}_{S'}^v \Delta_Q^v g|^2 v((S' \setminus Q') \cap \partial_Q Q') \right) \]
Now we use (5.1) to obtain
\[
E_{\mathcal{D} \in \Omega} \left( \sum_{Q \in \mathcal{D}_{\text{good}}, \ Q' \in \mathcal{H}(Q)} v(S \cap \partial_{c} Q') \right) \lesssim \tau^{\nu} v(S),
\]
which altogether gives
\[
E_{\mathcal{D} \in \Omega} \text{Term}_{21} \lesssim N \|f\|_{L^{2}(u)} \sqrt{\sum_{Q \in \mathcal{D}_{\text{good}}} \left| \Delta_{S}^{\nu} g \right|^{2} \tau^{\nu} v(S)} \lesssim \tau^{\frac{\nu}{2}} N \|f\|_{L^{2}(u)} \|g\|_{L^{2}(v)}.
\]
The proof of Lemma 19 is complete.

The next lemma controls \( A_{2}^{2} \).

**Lemma 20.** The following estimate holds:
\[
\left| \sum_{(Q, S) \in A_{2}^{2}} \sum_{\ell = 1}^{M_{Q}-1} \sum_{k = 1}^{M_{S}-1} \langle f, h_{Q}^{\ell}\rangle u \langle T(uh_{Q}^{\ell}), h_{S}^{k}\rangle v \langle g, h_{S}^{k}\rangle v \rangle \right| \lesssim \| f \|_{L^{2}(u)} \| g \|_{L^{2}(v)}.
\]

Before we proceed to prove Lemma 20, let us collect a couple of auxiliary lemmas.

**Lemma 21.** Let \( S \subset Q' \subset Q \) be three cubes with \( \text{dist}(\partial Q', S) \geq l(S) \). Let \( H_{S} \) be a function supported on \( S \) and with \( v \) integral zero. Then we have
\[
\left| \langle T(u1_{Q \setminus Q'}), H_{S} \rangle \right| \lesssim \| H_{S} \|_{L^{2}(v)} \Phi(S, 1_{Q \setminus Q'} u)^{\frac{1}{2}}.
\]

Here \( \Phi(S, 1_{Q \setminus Q'} u) := v(S) K \left( S, 1_{Q \setminus Q'} u \right)^{2} \) where
\[
K(S, 1_{Q \setminus Q'} u) := v(S) \int_{Q \setminus Q'} \frac{l(S)}{l(S) + \text{dist}(y, S)} \frac{1}{\mu(B(x_{S}, l(S) + \text{dist}(y, S)))} du(y),
\]
and \( \kappa \) is as in Definition 1.

The \( L^{2} \) formulation of (5.3) proves useful in many estimates below, in particular in several Carleson Embedding Theorem estimates, Theorem 29. We will apply (5.3) in the dual formulation. Namely, we have
\[
\| T(u1_{Q \setminus Q'}) - E_{\mathcal{D}} T(u1_{Q \setminus Q'}) \|_{L^{2}(v)} \lesssim \| H_{S} \|_{L^{2}(v)} \Phi(S, 1_{Q \setminus Q'} u)^{\frac{1}{2}}.
\]

**Proof.** This proof uses the standard computation in the Calderón–Zygmund theory. We use cancellation of a function to pull additional information onto the kernel of the operator. Using Fubini and the fact that \( H_{S} \) has \( v \) integral zero we get:
\[
\left| \langle T(u1_{Q \setminus Q'}), H_{S} \rangle \right| = \left| \int_{S} H_{S}(x) T(u1_{Q \setminus Q'}(x)) dv(x) \right| = \left| \int_{S} H_{S}(x) \int_{Q \setminus Q'} \tilde{R}(x, y) du(y) dv(x) \right| = \left| \int_{Q \setminus Q'} \int_{S} \left( \tilde{R}(x, y) - \tilde{R}(x_{S}, y) \right) H_{S}(x) dv(x) du(y) \right| \leq \left| \int_{Q \setminus Q'} \int_{S} \left( \frac{\text{dist}(x, x_{S})}{\text{dist}(x_{S}, y)} \right)^{\kappa} \frac{1}{\mu(B(x_{S}, \text{dist}(x_{S}, y)))} H_{S}(x) dv(x) du(y) \right| \leq \int_{Q \setminus Q'} \left( \frac{l(S)}{\text{dist}(x_{S}, y)} \right)^{\kappa} \frac{1}{\mu(B(x_{S}, \text{dist}(x_{S}, y)))} \int_{S} H_{S}(x) dv(x) du(y).
\]

Here \( x_{S} \in S \), is the center of \( S \) and \( y \in \hat{Q} \setminus Q' \) and so \( \text{dist}(x_{S}, y) \approx \text{dist}(y, S) + l(S) \). By the doubling property of the measure \( \mu \) we have \( \mu(B(x_{S}, \text{dist}(x_{S}, y))) \approx \mu(B(x_{S}, l(S) + \text{dist}(y, S))) \). These estimates can then be used to give:
\[
\left| \langle T(u1_{Q \setminus Q'}), H_{S} \rangle \right|
\]
Let \( \lambda = \frac{\kappa}{n+\kappa} \). If \( S \subset Q \subset \hat{Q} \) and \( \text{dist}(S, e(Q)) \geq \frac{1}{2} l(S)^\lambda l(Q)^{1-\lambda} \) where \( e(S) := \partial S \cup \{\text{(center of} S)\} \) then

\[
l(S)^{\sigma_0} K \left( S, 1_{\hat{Q} \setminus Q} u \right) \leq l(Q)^{\sigma_0} K \left( Q, 1_{\hat{Q} \setminus Q} u \right).
\]

Here \( \sigma_0 := \lambda(n + \kappa) - \kappa \) with \( \kappa \) as in Definition 1.

Proof. To begin with, recall that for each \( Q \in \mathcal{D} \), the containment in \( (3.2) \) holds and the outer ball that contains \( S \) is denoted by \( B(S) \).

By decomposing the space \( X \) into annuli based on \( B(S) \), we have that

\[
K \left( S, 1_{\hat{Q} \setminus Q} u \right) \\
= \int_{B(S)} \frac{l(S)}{l(S) + \text{dist}(y, S)}^{\kappa} \frac{1}{\mu(B(x_S, l(S) + \text{dist}(y, S)))} 1_{\hat{Q} \setminus Q} \left( y \right) du(y) \\
+ \sum_{k=1}^{\infty} \int_{\delta^{-k}B(S) \setminus \delta^{1-k}B(S)} \frac{l(S)}{l(S) + \text{dist}(y, S)}^{\kappa} \frac{1}{\mu(B(x_S, l(S) + \text{dist}(y, S)))} 1_{\hat{Q} \setminus Q} \left( y \right) du(y) \\
= \sum_{k=\kappa_1}^{\kappa_1} \int_{\delta^{-k}B(S) \setminus \delta^{1-k}B(S)} \frac{l(S)}{l(S) + \text{dist}(y, S)}^{\kappa} \frac{1}{\mu(B(x_S, l(S) + \text{dist}(y, S)))} 1_{\hat{Q} \setminus Q} \left( y \right) du(y).
\]

Here \( \kappa_0 \) and \( \kappa_1 \) are determined by the conditions that for \( k < \kappa_0 \)

\[
\delta^{-k}B(S) \cap (\hat{Q} \setminus Q) = \emptyset,
\]

and for \( k > \kappa_1 \), we have

\[
\delta^{1-k}B(S) \cap (\hat{Q} \setminus Q) = \emptyset.
\]

Hence, for \( \kappa_0 \leq k \leq \kappa_1 \), we have

\[
\frac{1}{2} l(S)^\lambda l(Q)^{1-\lambda} \leq \text{dist}(S, e(Q)) \lesssim \delta^{-k} l(S),
\]

and thus

\[
\delta^k \lesssim \left( \frac{l(S)}{l(Q)} \right)^{1-\lambda}.
\]

We now estimate \( K \left( S, 1_{\hat{Q} \setminus Q} u \right) \). To begin with, we give a partition between \( \kappa_0 \) and \( \kappa_1 \) to create a new collection of integers \( \tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_L \) such that \( \tilde{k}_0 = \kappa_0 \), \( \tilde{k}_1 = \kappa_1 \) and that \( \tilde{k}_1 \) satisfies \( \delta^{-\tilde{k}_1} l(S) \approx l(Q) \), \( \tilde{k}_2 \) satisfies \( \delta^{-\tilde{k}_2} l(S) \approx \delta l(Q) \), \( \ldots \), \( \tilde{k}_L \) satisfies \( \delta^{-\tilde{k}_L} l(S) \approx \delta^{-L} l(Q) \). In fact, we have \( \tilde{k}_0 = \kappa_0 \), \( \tilde{k}_1 > \tilde{k}_0 \), \( \tilde{k}_2 = \tilde{k}_1 + 1 \), and so on. Then we get that

\[
\sum_{k=\kappa_0}^{\kappa_1} = \sum_{\ell=1}^{L} \sum_{k=\tilde{k}_{\ell-1}}^{\tilde{k}_{\ell}}.
\]

Hence,

\[
K \left( S, 1_{\hat{Q} \setminus Q} u \right) \\
\lesssim \sum_{\ell=1}^{L} \sum_{k=\kappa_{\ell-1}}^{\kappa_{\ell}} \int_{\delta^{-k}B(S) \setminus \delta^{1-k}B(S)} \frac{l(S)}{l(S) + \text{dist}(y, S)}^{\kappa} \frac{1}{\mu(B(x_S, l(S) + \text{dist}(y, S)))} 1_{\hat{Q} \setminus Q} \left( y \right) du(y).
\]

completing the proof. \( \square \)
For the first term, by using the doubling property, we have that

\[
\chi \lesssim \sum_{l=1}^{L} \sum_{k=k_0}^{k_1} \int_{(\delta^{-k}B(S)\setminus \delta^{-k-1}B(S)) \cap (\hat{Q} \setminus Q)} \left( \frac{l(S)}{\delta^{1-k}l(S)} \right)^{\kappa} \frac{1}{\mu(B(x_S, \delta^{1-k}l(S)))} du(y)
\]

\[
\chi \lesssim \sum_{k=k_0}^{k_1} \delta^{nk} \int_{(\delta^{-k}B(S)\setminus \delta^{-k-1}B(S)) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_S, \delta^{1-k}l(S)))} du(y)
\]

\[+ \sum_{\ell=2}^{L} \delta^{n(k_1+\ell)} \int_{(\delta^{-(k_1+\ell)}B(S)\setminus \delta^{1-k_1-\ell}B(S)) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_S, \delta^{1-k_1-\ell}l(S)))} du(y)
\]

\[= \text{Term}_1 + \text{Term}_2.
\]

For the first term, we get

\[
\text{Term}_1 \lesssim \sum_{k=k_0}^{k_1} \delta^{nk} \frac{\mu(B(x_S, l(Q)))}{\mu(B(x_S, \delta^{1-k}l(Q))) \mu(B(x_S, l(Q)))} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_S, l(Q)))} du(y)
\]

\[
\lesssim \sum_{k=k_0}^{k_1} \delta^{nk} \left( \frac{l(Q)}{\delta^{1-k}l(S)} \right)^{\kappa} \frac{1}{\mu(B(x_Q, l(Q)))} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_Q, l(Q)))} du(y)
\]

\[
\lesssim \sum_{k=k_0}^{k_1} \delta^{nk} \left( \frac{l(Q)}{l(S)} \right)^{\lambda \kappa} \frac{1}{\mu(B(x_Q, l(Q)))} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_Q, l(Q)))} du(y)
\]

\[
\lesssim \left( \frac{l(Q)}{l(S)} \right)^{\lambda \kappa} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_Q, l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y).
\]

For the second term, we get

\[
\text{Term}_2 \lesssim \delta^{n\tilde{k}_1} \sum_{\ell=2}^{L} \delta^{n\ell} \frac{1}{\mu(B(x_S, \delta^{-\ell}l(Q)))} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_S, \delta^{-\ell}l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y)
\]

\[
\lesssim \delta^{n\tilde{k}_1} \sum_{\ell=2}^{L} \delta^{n\ell} \frac{1}{\mu(B(x_Q, \delta^{-\ell}l(Q)))} \int_{B(Q) \cap (\hat{Q} \setminus Q)} \frac{1}{\mu(B(x_Q, \delta^{-\ell}l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y)
\]

\[
\lesssim \delta^{n\tilde{k}_1} \int_{B(Q) \setminus \delta^{-\ell}B(Q)} \frac{1}{\delta^{-\ell}l(Q)} \frac{l(Q)}{\delta^{-\ell}l(Q)}^{\kappa} \frac{1}{\mu(B(x_Q, \delta^{-\ell}l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y).
\]

Next, we note that

\[
K(Q, 1_{\hat{Q} \setminus Q} u)
\]

\[
= \int_{B(Q)} \left( \frac{l(Q)}{l(Q) + \text{dist}(y, Q)} \right)^{\kappa} \frac{1}{\mu(B(x_Q, l(Q) + \text{dist}(y, Q)))} 1_{\hat{Q} \setminus Q}(y) du(y)
\]

\[
+ \sum_{j=1}^{j_1} \int_{B(Q) \setminus \delta^{-j}B(Q)} \left( \frac{l(Q)}{\delta^{-j}l(Q)} \right)^{\kappa} \frac{1}{\mu(B(x_Q, \delta^{-j}l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y)
\]

\[
\approx \int_{B(Q)} \frac{1}{\mu(B(x_Q, l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y)
\]

\[
+ \sum_{j=1}^{j_1} \int_{B(Q) \setminus \delta^{-j}B(Q)} \left( \frac{l(Q)}{\delta^{-j}l(Q)} \right)^{\kappa} \frac{1}{\mu(B(x_Q, \delta^{-j}l(Q)))} 1_{\hat{Q} \setminus Q}(y) du(y).
\]
Combining the estimates of Term1 and Term2, and the equality above, we see that
\[ K \left( S, 1_{Q \backslash Q^u} \right) \lesssim \left( \frac{l(Q)}{l(S)} \right)^{(n+\kappa) - \kappa} K \left( Q, 1_{Q \backslash Q^u} \right). \]

The proof of Lemma 22 is complete. \qed

Let us begin with the proof of Lemma 20.

**Proof of Lemma 20.** Recall that the pairs of cubes \((Q, S) \in A^2_2\) satisfy \(l(S) \leq l(Q)\) and \(\text{dist}(Q, S) \geq l(Q)\). We can apply Lemma 21 to \(\langle Th_\delta'', h_S'' \rangle_v\). To see this note that \(h_Q''\) is constant on each child \(Q_\epsilon\) where \(\epsilon \in \{1, 2, \ldots, M_Q-1\}\).

Take a child \(Q_\epsilon\) and apply Lemma 21 with the largest cube \(\hat{Q}\) taken to be \(\hat{Q} := \text{hull} \left[ Q_\epsilon, \left( \frac{l(Q)}{l(S)} \right)^{1-\kappa} S \right]\). Here \(\rho S\) means the cube with same centre as \(S\) and length equal to \(\rho l(S)\) and by hull above we mean \(\hat{Q}\) is the smallest cube containing both cubes \(Q_\epsilon\) and \(\left( \frac{l(Q)}{l(S)} \right)^{1-\kappa} S\).

The two cubes \(Q_\epsilon\) and \(\left( \frac{l(Q)}{l(S)} \right)^{1-\kappa} S\) are disjoint. We take \(Q' \subset \hat{Q}\) so that \(\hat{Q} \setminus Q' = Q_\epsilon\). Then using Lemma 21 we get the following estimate
\[ \beta(Q, S) := \left| \sum_{\epsilon} \langle T(1_{Q_\epsilon} h_\epsilon Q', u), h^k_{S}\rangle_v \right| \leq \sum_{\epsilon} |E_{Q_\epsilon}^n(h_Q'')| |\langle T(1_{Q_\epsilon}, u), h^k_{S}\rangle_v|. \]

Here \(E_{Q_\epsilon}^n(h_Q'') := \frac{1}{m(Q_\epsilon)} \int_{Q_\epsilon} h_Q''(x) du(x)\). Observe \(|E_{Q_\epsilon}^n(h_Q'')| \leq \frac{1}{u(Q_\epsilon)^x}\).

We have \(K(S, 1_{Q_u}) \lesssim \frac{l(S)^n}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} u(Q_\epsilon)\), where \(n\) is the upper dimension of \(\mu\) and we have used the doubling property of the measure \(\mu\). Hence we get
\[ \beta(Q, S) \lesssim \sum_{\epsilon} v(S)^{1/2} u(Q_\epsilon)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} v(S)^{1/2} \langle g, h^k_{S}\rangle_v. \]

To continue, we may assume that \(\|f\|_{L^2(u)} = \|g\|_{L^2(v)} = 1\). We then estimate
\[ |A^2_2| = \left| \sum_{\epsilon=1}^{M_Q-1} \sum_{k=1}^{M_S-1} \sum_{(Q, S) \in A^2_2} \langle f, h_Q'', T(uh_\epsilon Q', h^k_{S}), (g, h^k_{S}) \rangle_v \right| \]
which can be written as follows
\[ |A^2_2| \leq \sum_{Q} \sum_{S: l(S) \leq l(Q) \\text{dist}(Q, S) \geq l(Q)} \sum_{\epsilon=1}^{M_Q-1} \sum_{k=1}^{M_S-1} |\langle f, h_Q'', u \rangle v(S)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} v(S)^{1/2} \langle g, h^k_{S}\rangle_v| \]

\[ \lesssim \sum_{Q} \sum_{S: l(S) \leq l(Q) \\text{dist}(Q, S) \geq l(Q)} \sum_{\epsilon=1}^{M_Q-1} \sum_{k=1}^{M_S-1} |\langle f, h_Q'', u \rangle v(S)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} v(S)^{1/2} \langle g, h^k_{S}\rangle_v| \]

\[ \lesssim \sum_{Q} \sum_{\epsilon=1}^{M_Q-1} |\langle f, h_Q'' \rangle u| v(S)^{1/2} \sum_{S: l(S) \leq l(Q) \\text{dist}(Q, S) \geq l(Q)} \left( \frac{l(S)}{l(Q)} \right)^{-\sigma} u(Q)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} v(S)^{1/2} \langle g, h^k_{S}\rangle_v \]

\[ + \sum_{S} \sum_{k=1}^{M_S-1} |\langle g, h^k_{S}\rangle v|^{2} \sum_{Q: l(Q) \leq l(S) \\text{dist}(Q, S) \geq l(Q)} \left( \frac{l(S)}{l(Q)} \right)^{\sigma} u(Q)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa + n}} v(S)^{1/2} \]

\[ =: A^2_{21} + A^2_{22}, \]

where in the last inequality we have inserted the gain and loss term \(\left( \frac{l(S)}{l(Q)} \right)^{\pm \sigma}\) with \(0 < \sigma < 1\).
We first consider the term $A_{22}$. For each fixed $Q$ we have

$$\sum_{S: l(S) \geq l(Q)} \left( \frac{l(S)}{l(Q)} \right)^{\sigma} u(Q)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa+n}} v(S)^{1/2}$$

$$\lesssim u(Q)^{1/2} \sum_{i=0}^\infty \delta_i^{i\sigma} \left( \sum_{S: l(S) = \delta_i l(Q)} \frac{l(S)^\kappa}{(\text{dist}(Q, S))^{\kappa+n}} v(S) \right)^{1/2} \left( \sum_{S: l(S) = \delta_i l(Q)} \frac{l(S)^\kappa}{(\text{dist}(Q, S))^{\kappa+n}} \right)^{1/2}$$

$$\lesssim \sum_{i=0}^\infty \delta_i^{i\sigma} \left( \frac{u(Q)}{l(Q)^n} K(Q, v) \right)^{1/2} \lesssim A_2,$$

where the last inequality follows from the fact that $\sigma > 0$. Consider the term $A_{21}$. For each fixed $S$ we have

$$\sum_{Q: l(S) \geq l(Q)} \left( \frac{l(S)}{l(Q)} \right)^{-\sigma} u(Q)^{1/2} \frac{l(S)^\kappa}{(l(S) + \text{dist}(Q, S))^{\kappa+n}} v(S)^{1/2}$$

$$\lesssim v(S)^{1/2} \sum_{j=0}^\infty \delta_j^{(1-\sigma)} \sum_{Q: l(S) = \delta_j l(Q)} \frac{l(Q)^\kappa}{(\text{dist}(Q, S))^{\kappa+n}} u(Q)^{1/2}$$

$$\lesssim v(S)^{1/2} \sum_{j=0}^\infty \delta_j^{(1-\sigma)} \left( \sum_{Q: l(S) = \delta_j l(Q)} \frac{l(Q)^\kappa}{(\text{dist}(Q, S))^{\kappa+n}} u(Q) \right)^{1/2} \left( \sum_{Q: l(S) = \delta_j l(Q)} \frac{l(Q)^\kappa}{(\text{dist}(Q, S))^{\kappa+n}} \right)^{1/2},$$

which is bounded by

$$v(S)^{1/2} \sum_{j=0}^\infty \delta_j^{(1-\sigma)} K(\delta_j^{1/n} S, u) \left( \frac{1}{l(\delta_j^{1/n} S)^n} \right)^{1/2} \lesssim A_2,$$

where the last inequality follows from the fact that $\sigma < 1$. Thus, with any fixed $0 < \sigma < 1$ we have from the above inequalities that

$$|A_2^2| \lesssim A_2 \sum_Q \sum_{e=1}^{M_Q-1} |\langle f, h^e_Q \rangle_u|^2 + A_2 \sum_S \sum_{k=1}^{M_g-1} |\langle g, h^k_S \rangle_v|^2$$

$$= \left( ||f||_{L^2(u)}^2 + ||g||_{L^2(v)}^2 \right) A_2 = 2A_2.$$

The proof of Lemma 20 is complete. \qed

We have now reduced matters to the case of considering $A_3^2$. We further decompose $A_3^2$ into

$$A_3^1 := \{(Q, S) \in A_3^1 : l(S) \leq \delta^* l(Q), Q \cap S = \emptyset, \text{dist}(Q, S) \leq l(Q)\},$$

$$A_3^2 := \{(Q, S) \in A_3^1 : l(S) \leq \delta^* l(Q), Q \cap S \neq \emptyset, \text{dist}(Q, S) \leq l(Q)\}.$$

We are now going to prove in this section that

$$|A_3^1| \leq A_2 \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$
Fix $p \geq r$. Observe that

$$A_3^2(p) = \left| \sum_{Q} \sum_{c=1}^{M_Q-1} \sum_{k=1}^{M_S-1} \langle f, h^c_Q \rangle_u \langle T(uh^c_Q), h^k_S \rangle_v \langle g, h^k_S \rangle_v \right|$$

$$\lesssim \left[ \sum_{Q} \left( \sum_{c=1}^{M_Q-1} \langle f, h^c_Q \rangle_u \right)^2 \right]^\frac{1}{2} \left[ \sum_{Q} \sum_{c=1}^{M_Q-1} \sum_{k=1}^{M_S-1} \langle (uh^c_Q), h^k_S \rangle_v \langle g, h^k_S \rangle_v \right]^\frac{1}{2} \lesssim \Lambda(p) \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$  

The last two inequalities follow from the Cauchy-Schwarz inequality and we use Fubini to get $\|g\|_{L^2(v)}$ above and $\delta^{-p}$ in the expression below. Here

$$\Lambda(p)^2 := \delta^{-p} \sup_{Q} \sum_{S: \delta^{-p}(l(S)) = l(Q)} \sum_{Q: \dist(Q, S) \leq l(Q)} \sum_{c=1}^{M_Q-1} \sum_{k=1}^{M_S-1} \langle (uh^c_Q), h^k_S \rangle_v^2.$$  

But $S$ is good, so that Lemma 21 applies to each child of $Q$ and $S$ yielding

$$\Lambda(p)^2 \lesssim \sup_{Q} \delta^{-p} \sum_{c} \sum_{k} \sum_{S: \delta^{-p}(l(S)) = l(Q)} \sum_{Q: \dist(Q, S) \leq l(Q)} \frac{v(S_k)}{u(Q_c)} K(S_k, 1_Q, u)^2.$$  

Hence we have the following, using Lemma 22

$$\Lambda(p)^2 \lesssim \sup_{Q} \delta^{-p} \sum_{c} \sum_{k} \sum_{S: \delta^{-p}(l(S)) = l(Q)} \sum_{Q: \dist(Q, S) \leq l(Q)} \frac{v(S_k)}{u(Q_c)} \left( \frac{l(S)}{l(Q)} \right)^{-2\sigma_0} \cdot K(Q, 1_Q, u)^2$$

$$\lesssim \sup_{Q} \delta^{-p}\sum_{c} \sum_{k} \sum_{S: \delta^{-p}(l(S)) = l(Q)} \sum_{Q: \dist(Q, S) \leq l(Q)} \frac{u(Q_c)}{l(Q)^{2n}} \sum_{Q^c} v(S_k)$$

$$\lesssim \delta^{-p(1+2\sigma_0)} A_2.$$  

Above, we have used that

$$K(S, 1_Q, u) \leq \frac{l(S)^n}{(l(S) + \dist(Q, S))^{c+n}} u(Q).$$  

This is clearly summable in $p \geq r$ as $-\sigma_0 > \frac{1}{2}$ (as we can fix $\lambda \in (0, 1)$ such that $\lambda < \frac{2}{n+1}$) so the proof is complete.

### 6. Stopping Cubes and Corona Decompositions

Our focus is now on the “short range terms” given by $A^2_3$. Here we will use our pivotal condition. In this section we will decompose $A^2_3$ further and estimate each piece.

Recall

$$A_2^3 := \{(Q, S) \in A_1^2 : l(S) \leq \delta^r l(Q), Q \cap S \neq \emptyset, \text{dist}(Q, S) \leq l(Q)\}$$
and

\[ A^2_3 := \sum_{(Q,S) \in A^3_3} \sum_{i=1}^{M_Q-1} \sum_{k=1}^{M_S-1} (f, h^i_Q)_u(T(uh^i_Q), h^i_S)_v(g, h^i_S)_v. \]

Denote \( \Delta^n_Q f = \sum_{i=1}^{M_Q-1} (f, h^i_Q)_u h^i_Q \). Then observe that:

\[
\langle T(\Delta^n_Q f), \Delta^n_S g \rangle_v = \langle T((1_Q - 1_{Q_S}) \Delta^n_Q f), \Delta^n_S g \rangle_v + \langle T(1_{Q_S} \Delta^n_Q f), \Delta^n_S g \rangle_v
\]

(6.1) \[ = \langle T((1_Q \setminus 1_{Q_S}) \Delta^n_Q f), \Delta^n_S g \rangle_v + \langle T(1_{Q_S} \Delta^n_Q f), \Delta^n_S g \rangle_v \]

(6.2) \[ + E^{n}_{Q_S}(\Delta^n_Q f)(T(1_{Q_S} \setminus 1_Q), \Delta^n_S g)_v \]

(6.3) \[ - E^{n}_{Q_S}(\Delta^n_Q f)(T(1_{Q_S} \setminus 1_Q), \Delta^n_S g)_v. \]

First observe that \( 1_Q - 1_{Q \setminus Q_S} = 1_{Q_S} \) and secondly

\[ 1_{Q_S} \Delta^n_Q f = \sum_{i=1}^{M_Q-1} (f, h^i_Q)_u h^i_Q \cdot 1_{Q_S} \]

and so

\[ E^{n}_{Q_S}(\Delta^n_Q f) = \frac{1}{u(Q_S)} \int_{Q_S} \Delta^n_Q f(x) du(x) \]

\[ = \frac{1}{u(Q_S)} \int_X \sum_{i=1}^{M_Q-1} (f, h^i_Q)_u h^i_Q(x) \cdot 1_{Q_S}(x) du(x) \]

\[ = \frac{1}{u(Q_S)} \sum_{i=1}^{M_Q-1} \int_X h^i_Q(x) \cdot 1_{Q_S}(x) du(x). \]

Here \( Q_S \) is child of \( Q \) containing \( S \) and \( Q \) the parent of \( Q_S \).

6.1. The Decomposition of the Short Range Term. To estimate \( A^3_3 \) and conclude this section, we combine the splitting into (6.1), (6.2), (6.3) and the following Corona decomposition. Namely select the cubes \( \tilde{Q} \) that appear in (6.1)-(6.3) according to the stopping rule below. Recall the set \( A^3_2 \). Using the fact that \( S \) is good we can make this set more explicit, i.e. \( S \subset Q \) and \( l(S) < \delta l(Q) \)

\[ A^3_3 := \{(Q,S) : A^2_3, l(S) \leq \delta l(Q), S \subset Q \}. \]

6.1.1. The Corona Decomposition. We are going to define the ‘stopping cubes’ and the ‘Corona decomposition’. Let us first define the following functionals:

\[ \Phi(Q, 1_{E}u) := v(Q)K(Q, 1_{E}u)^2, \]

\[ \Psi(Q, 1_{E}u) := \sup_{Q = \cup_{i=1}^{r}Q_i} \sum_{i \geq 1} \Phi(Q_i, 1_{E}u), \]

where \( Q_i \) are dyadic subcubes of \( Q \), hence lie in some dyadic grid as \( Q \) and where the supremum is over all \( r \)-good dyadic subpartitions \( \{Q_i \}_{i \geq 1} \) of \( Q \). We have the following pivotal condition

\[
\sum_{r \geq 1} \Phi(Q_r, 1_{Q}u) \leq V^2 u(Q).
\]

(6.4)

We will use certain key properties of \( \Psi \) to estimate the term \( A^3_3 \) defined below.

Definition 23. Given any cube \( Q_o \), we will set \( S(Q_o) \) to be the maximal \( D^u \) strict subcubes \( S \subset Q_o \) such that

\[
\Psi(S, 1_{Q_o}u) \geq 4V^2 u(S).
\]

(6.5)

The collection \( S(Q_o) \) can be empty.
We will be able to now recursively define $S_1 := \{Q_0\}$ and $S_{j+1} := \cup_{S \in S_j} S(S)$. The collection of $S := \cup_{j=0}^{\infty} S_j$ is the collection of stopping cubes. Let us define $\rho : S \mapsto \mathbb{N}$ by $\rho(S) := j$ for all $S \in S_j$, so that $\rho(S)$ denotes the generation in which $S$ occurs in the construction of $S$.

Let us now discuss the associated Corona Decomposition.

**Definition 24.** For $S' \in S$, we are going to set $P(S')$ to be all the pairs of cubes $(Q, S)$ such that

1. $Q \in D^u$, $S \in D^v$, $S \subset Q$ and $l(S) \leq \delta l(Q)$;
2. $S'$ is the $S$ parent of $Q$ which is the child of $Q$ containing $S$.

Observe that we can write $A^2_3 = \cup_{S' \in S} P(S')$, where $A^2_3$ is as defined above. We next define the Coronas associated to $f$ and $g$.

**Definition 25.** Let $C^u(S')$ be all those $Q \in D^u$ such that $S'$ is a minimal member of $S$ that contains a $D^u$ child of $Q$. The definition of $C^u(S')$ is similar but not symmetric: all those $S \in D^v$ such that $S'$ is the smallest member of $S$ that contains $S$ and satisfies $l(S) \leq \delta l(S')$ together with all those $S \in D^v$ such that for some $S'' \in S(S)$ we have $S \in C^v(S'')$ with $S \subset S''$ with $l(S) \geq \delta l(S'')$. The collections $\{C^u(S') : S' \in S\}$ and $\{C^v(S') : S' \in S\}$ are referred to as the Corona Decompositions (the collection $C^v$ is called the shifted Corona in the literature).

We will now define the projection operators associated to these Coronas

$$P^u_S f := \sum_{Q \in C^u(S')} \sum_{c=1}^{M_Q - 1} \langle f, h^u_Q \rangle_S h^u_Q.$$ 

Similarly we can define $P^v_S g$. Observe that $P^u_S g$ projects only cubes $S$ with $l(S) \leq \delta l(S')$.

We have the estimate below which we will use in the proofs below

$$\sum_{S \in S} \|P^u_S f\|^2_{L^2(u)} \leq \sup_{Q \in D^u} M_Q \|f\|^2_{L^2(u)}$$

where $M_Q$ is the number of children of $Q$ in the grid $D^u$. Recall that we also assume throughout the paper

$$\sup_{Q \in D^u} M_Q < \infty.$$ 

Observe in the definition of stopping cubes we are using the functional $\Psi$ associated with hypothesis (6.4). So the stopping cubes can be viewed as the enemy of verifying (6.4).

**Definition 26.** Given a pair $(Q, S) \in A^2_3$, choose $\tilde{Q} \in S$ to be the unique stopping cube such that $Q_S \in C^u(\tilde{Q})$, where $Q_S$ is the $S$ child of $\tilde{Q}$. Equivalently, $\tilde{Q} \in S$ is determined by the requirement $(Q, S) \in P(Q)$.

Note that if $Q_S \notin S$, then $Q \subset \tilde{Q}$, while if $Q_S \in S$, then $\tilde{Q}$ is the child of $Q$ containing $S$. With the choice of $\tilde{Q}$ in the splitting of (6.1)-(6.3) we obtain $|A^2_3| \leq \sum_{j=1}^{\infty} |A^4_j|$ where

$$A^4_1 := \sum_{(Q, S) \in A^2_3} T(1_Q \cdot Q_S, \Delta^u_S g, \Delta^u_S g),$$

$$A^4_2 := \sum_{S' \in S} \sum_{(Q, S) \in P(S')} E^u_Q \langle \Delta^v_Q j \rangle_{(1_S \cdot Q_S, \Delta^u_S g, \Delta^u_S g)},$$

$$A^4_3 := \sum_{S' \in S} \sum_{(Q, S) \in P(S')} E^u_Q \langle \Delta^v_Q j \rangle_{(1_S \cdot Q_S, \Delta^u_S g, \Delta^u_S g)}.$$ 

The three terms above are referred to as the \textit{neighbor}, \textit{paraproduct} and \textit{stopping term} respectively.

The paraproduct term $A^4_3$ is further decomposed and estimates on this term will be handled in Section 8, while in the remainder of this section we will prove:

$$|A^4_1| \lesssim A^2_2 \|f\|_{L^2(u)} \|g\|_{L^2(v)},$$

$$|A^4_2| \lesssim V \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$
6.2. Control of the Neighbor Term $A_1^\delta$. The neighbor terms are defined in (6.6) and we are to prove (6.9). Recall we have $Q \in \mathcal{D}^\nu$, $S \in \mathcal{G}^\nu$ contained in $Q$, with $l(S) \leq \delta l(Q)$ and $Q_S$ is the child of $Q$ containing $S$

Fix a child $\theta \in \{1, 2, \ldots, M_Q - 1\}$ and an integer $s' \geq r$. Here we use that $Q \setminus Q_\theta = \bigcup_{i=1}^{M_Q-1} Q_i$.

We are now going to estimate the inner product as that in (6.1):

$$\langle T(1_Q \setminus 1_{Q_\theta} u \Delta_Q^u f), \Delta_S^u g \rangle_v = \sum_{i=0}^{M_Q-1} \langle T(1_{Q_i} u \Delta_Q^u f), \Delta_S^u g \rangle_v$$

$$= \sum_{i=0}^{M_Q-1} \mathbf{E}_{Q_i}^u (\Delta_Q^u f) \langle T(1_{Q_i}, u \Delta_Q^u f), \Delta_S^u g \rangle_v.$$ 

Here we will use $\|\Delta_S^u g\|_{L^2(v)} = \left( \sum_{k=1}^{M_S-1} |\langle g, h_S^k \rangle_v|^2 \right)^{\frac{1}{2}}$ and $\frac{l(S)}{l(Q_\theta)} = \delta^{s'}$ in Lemma 21 with $S \subset Q_\theta \subset Q$ to obtain

$$|\langle T(1_Q, u \Delta_Q^u f), \Delta_S^u g \rangle_v| \lesssim v(S)^{\frac{1}{2}} \left( \sum_{k=1}^{M_S-1} |\langle g, h_S^k \rangle_v|^2 \right)^{\frac{1}{2}} \sum_{i=0}^{M_Q-1} K(S, 1_{Q_i}, u)$$

$$\lesssim v(S)^{\frac{1}{2}} \left( \sum_{k=1}^{M_S-1} |\langle g, h_S^k \rangle_v|^2 \right)^{\frac{1}{2}} \delta^{-(s' \sigma_0)} \sum_{i=0}^{M_Q-1} K(Q_\theta, 1_{Q_i}, u).$$

Here we applied Lemma 21 and Lemma 22 to $S \subset Q \setminus \bigcup_{i=1}^{M_Q-1} Q_i$.

For the sum below we keep the lengths of the cubes $S$ fixed and we are under the assumption that $S \subset Q_\theta$. Define

$$\Lambda(Q, \theta, s')^2 := \sum_{S : l(S) = \delta^{s'} l(Q)} \sum_{S \subset Q_\theta} \left( \sum_{k=1}^{M_S-1} |\langle g, h_S^k \rangle_v|^2 \right).$$

Then we have the following estimate using Cauchy-Schwarz

$$A_1^\delta(Q, \theta, s') := \sum_{S : l(S) = \delta^{s'} l(Q)} \sum_{S \subset Q_\theta} |\langle T(1_Q \setminus 1_{Q_\theta} u \Delta_Q^u f), \Delta_S^u g \rangle_v|$$

$$\leq \delta^{-(s' \sigma_0)} \sum_{i=0}^{M_Q-1} \mathbf{E}_{Q_i}^u (\Delta_Q^u f) K(Q_\theta, 1_{Q_i}, u) \sum_{S : l(S) = \delta^{s'} l(Q)} \sum_{S \subset Q_\theta} v(S)^{\frac{1}{2}} \left( \sum_{k=1}^{M_S-1} |\langle g, h_S^k \rangle_v|^2 \right)^{\frac{1}{2}}$$

$$\leq \delta^{-(s' \sigma_0)} \sum_{i=0}^{M_Q-1} |\mathbf{E}_{Q_i}^u (\Delta_Q^u f)| K \left( Q_\theta, 1_{Q_i}, u \right) v(Q_\theta)^{\frac{1}{2}} \Lambda(Q, \theta, s').$$

We will now use the following to estimate $A_1^\delta(Q, \theta, s')$:

$$|\mathbf{E}_{Q_i}^u (\Delta_Q^u f)| \leq \left( \sum_{j=1}^{M_Q-1} |\langle f, h_Q^j \rangle_u|^2 \right)^{\frac{1}{2}} u(Q_i)^{-\frac{1}{2}}.$$ 

Substituting this into the above we find:
We can now estimate the terms in the square bracket above by

\[
A_3(S', s) := \sup_{Q \in \mathcal{C}^u(S')} \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} \frac{1}{u(Q_S)} |\langle T(1_{S' \setminus Q_S} u), h^k_S \rangle_v|^2.
\]

This completes the proof of the estimate (6.9).

6.3. Control of the Stopping Term \( A_3 \). To control (6.8) we want to prove (6.10). Here we will use the hypothesis (6.5).

We first define for \( S' \in \mathcal{S} \) and \( s \geq 0 \) an integer

\[
A_3(S', s) := \sum_{(Q, S) \in \mathcal{P}(S')} |\mathcal{E}_{Q_S}(\Delta_{Q} f)\langle T(1_{S' \setminus Q_S} u), \Delta_{S}^u g \rangle_v| \lesssim \delta^{-\sigma_0} V F(S') A(S', s),
\]

where

\[
F(S')^2 := \sum_{Q \in \mathcal{C}^u(S')} \sum_{\epsilon = 1}^{M_Q - 1} |\langle f, h^\epsilon_Q \rangle_u|^2,
\]

\[
\Lambda(S', s)^2 := \sum_{Q \in \mathcal{C}^u(S')} \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} \sum_{k = 1}^{M_S - 1} |\langle g, h^k_S \rangle_v|^2.
\]

Using Cauchy-Schwarz in the variable \( Q \) variable above, and appealing to the following inequality

\[
|\mathcal{E}_{Q_S}(\Delta_{Q} f)| \leq \left( \sum_{j = 1}^{M_Q - 1} |\langle f, h^j_Q \rangle_u|^2 \right)^{\frac{1}{2}} u(Q)^{-\frac{1}{2}},
\]

to continue the estimate for \( A_3(S', s) \),

\[
A_3(S', s) \lesssim F(S') \left[ \sum_{Q \in \mathcal{C}^u(S')} \left( \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} \frac{1}{u(Q_S)} |\langle T(1_{S' \setminus Q_S} u), \Delta_{S}^u g \rangle_v| \right) \right]^{\frac{1}{2}}.
\]

We can now estimate the terms in the square bracket above by

\[
\sum_{Q \in \mathcal{C}^u(S')} \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} |\langle g, h^k_S \rangle_v|^2 \times \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} \frac{1}{u(Q_S)} |\langle T(1_{S' \setminus Q_S} u), h^k_S \rangle_v|^2 \lesssim \Lambda(S', s)^2 A(S', s),
\]

where

\[
A(S', s) := \sup_{Q \in \mathcal{C}^u(S')} \sum_{(Q, S) \in \mathcal{P}(S')} \sum_{l(S) = \delta^l(Q)} \frac{1}{u(Q_S)} |\langle T(1_{S' \setminus Q_S} u), h^k_S \rangle_v|^2.
\]
We will now estimate the term \( A(S', s) \). We will denote the children of \( Q \) by \( Q_\theta \) for \( \theta \in \{1, 2, ..., M_Q - 1\} \) and denote the children of \( S \) by \( S_k \) for \( k \in \{1, 2, ..., M_S - 1\} \). Also observe that \( h_S^k \) is supported on \( S \) and
\[
 h_S^k = \sum_{k=1}^{M_S-1} C_{S_k} 1_{S_k} \text{ such that } \| h_S^k \|_{L^2(u)}^2 = \sum_{k=1}^{M_S-1} C_{S_k}^2 = 1. \]
So using (5.3) we have the following:
\[
 A(S', s) \lesssim \sup_{Q \in \mathcal{C}(S')} \sup_{\theta \in \{1, 2, ..., M_Q - 1\}} \sum_{(Q, S) \in \mathcal{P}(S') : Q_\theta = Q} \sum_{l(S) = \delta l(Q)} \sum_{k=1}^{M_S-1} \frac{1}{u(Q_\theta)} \Phi(S_k, 1_{S \setminus Q_\theta} u)
\]
\[
 \lesssim \sup_{Q \in \mathcal{C}(S')} \sup_{\theta \in \{1, 2, ..., M_Q - 1\}} \sum_{(Q, S) \in \mathcal{P}(S') : Q_\theta = Q} \sum_{l(S) = \delta l(Q)} \sum_{k=1}^{M_S-1} \frac{1}{u(Q_\theta)} u(S_k) K(S_k, 1_{S \setminus Q_\theta} u)^2
\]
\[
 \lesssim \sup_{Q \in \mathcal{C}(S')} \sup_{\theta \in \{1, 2, ..., M_Q - 1\}} \sum_{(Q, S) \in \mathcal{P}(S') : Q_\theta = Q} \sum_{l(S) = \delta l(Q)} \sum_{k=1}^{M_S-1} \frac{1}{u(Q_\theta)} \delta^{-\sigma_0(s+1)} \Phi(Q_\theta, 1_{S \setminus Q_\theta} u)
\]
\[
 \lesssim \sup_{Q \in \mathcal{C}(S')} \sup_{\theta \in \{1, 2, ..., M_Q - 1\}} \sum_{(Q, S) \in \mathcal{P}(S') : Q_\theta = Q} \sum_{l(S) = \delta l(Q)} \sum_{k=1}^{M_S-1} \delta^{-\sigma_0(s+1)} u(S') \nu^2
\]
\[
 \lesssim \delta^{-\sigma_0(s+1)} \nu^2.
\]
Here we used Lemma \ref{lem:22} in the third inequality, used (6.4) in the second to last line and also that the number of children of any cube is uniformly finite. Here we have also used that \((Q, S) \in \mathcal{P}(S')\), so we have that \( S' \) is the \( S \)-parent of \( Q_S \) hence \( Q_S \) is not a stopping cube, so (6.5) does not hold, hence giving the estimate above.

We can observe that
\[
 \sum_{S' \in S} \mathcal{F}(S')^2 \lesssim \| f \|^2_{L^2(u)}.
\]
And we have the following
\[
 |A_3^4| \leq \sum_{S' \in S} \sum_{s=0}^\infty A_3^4(S', s) \lesssim \nu \| f \|^2_{L^2(u)} \nu^2_{L^2(u)}.
\]

7. The Carleson Measure Estimates

In this section we will prove Carleson measure estimates useful for the analysis on the paraproduct term \( A_2^4 \). In the lemma below we use the stopping time definition. For a cube \( S \in \mathcal{D}^u \) let
\[
 \mathcal{P}_S^u(g) := \sum_{S' \in \mathcal{D}^u : S' \subset S} \sum_{k=1}^{M_{S'}} (g, h_{S'}^k) h_{S'}^k.
\]
Observe that the projection \( \mathcal{P}_S^u \) is onto the span of all Haar functions \( h_S' \) supported in the \( \mathcal{D}^u \) cube \( S \). In contrast \( \mathcal{P}_S^v \) projects onto the span of all Haar functions \( h_S \) with \( S \) in the corona \( \mathcal{C}^v(S) \) where \( S \) is a stopping cube in the \( \mathcal{D}^v \) grid.

**Lemma 27.** Fix a cube \( Q_0 \in \mathcal{D}^u \) and let \( \tilde{Q}_0 \in S \) be its \( S \)-parent. Let \( \{Q_m : m \geq 1\} \subset \mathcal{D}^v \) be a strict subpartition of \( Q_0 \). Suppose that \( Q_m \) is good for \( m \geq 1 \). Let \( \{S_m, S' : s' \geq 1\} \subset \mathcal{D}^u \) be a subpartition of \( Q_m \)
with \( l(S_{m,s'}) < \delta l(Q_m) \) for all \( m, s' \geq 1 \). We then have the following

\[
(7.1) \quad \sum_{m,s' \geq 1} \| \hat{P}_{S_{m,s'}}^u(T(1_{Q_0 \setminus Q_m} u)) \|^2_{L^2(v)} \lesssim V^2 u(Q_0).
\]

**Proof.** We are now going to use the \( L^2 \) formulation (5.3) of Lemma 21 to deduce (7.1). We begin with the \( L^2 \) formulation to obtain

\[
\sum_{m,s' \geq 1} \| \hat{P}_{S_{m,s'}}^u(T(1_{Q_0 \setminus Q_m} u)) \|^2_{L^2(v)} \lesssim \sum_{m,s' \geq 1} \Phi(S_{m,s'}, 1_{Q_0 \setminus Q_m} u) \\
\lesssim \sum_{m,s' \geq 1} \Phi(S_{m,s'}, 1_{Q_0} u) + \sum_{m,s' \geq 1} \Phi(S_{m,s'}, 1_{Q_0 \setminus Q_m} u).
\]

The last inequality follows from the definition of \( \Phi \) and

\[
K(S, 1_{Q_0} u) = K(S, 1_{Q_0 \setminus Q_0} u) + K(S, 1_{Q_0} u).
\]

If \( Q_0 \neq \hat{Q}_0 \), we estimate the sum involving \( \hat{Q}_0 \setminus Q_0 \) using the fact that \( \{S_{m,s'}\}_{m,s' \geq 1} \) is a \( m \)-good subpartition of \( Q_0 \). We use Lemma 22 to get the first inequality below and given the fact that (6.5) fails and using (6.4) we obtain the following when \( 0 < t < r \)

\[
\sum_{m,s' \geq 1} \Phi(S_{m,s'}, 1_{Q_0 \setminus Q_m} u) \lesssim \sum_{m,s' \geq 1} \left[ \frac{l(Q_0)}{l(S_{m,s'})} \right] \Phi(Q_m, 1_{Q_0} u) \\
\lesssim \sum_{m,b' \geq 1} \sum_{S_{m,b'} \ni D^{b'}} \left[ \frac{l(Q_0)}{l(S_{m,b'})} \right] \Phi(Q_m, 1_{Q_0} u) \\
\lesssim \sum_{m,b' \geq 1} \sum_{t < r} \delta^{-\sigma \delta} \Phi(Q_m, 1_{Q_0} u) \\
\lesssim V^2 u(Q_0).
\]

Then to estimate the sum involving \( Q_0 \setminus Q_m \), we use the fact that \( \{S_{m,s'}\}_{m,s' \geq 1} \) is a \( r \)-good subpartition of \( Q_m \) for each \( m \). Then using the same steps as above and using (6.4) we obtain

\[
\sum_{m,s' \geq 1} \Phi(S_{m,s'}, 1_{Q_0 \setminus Q_m} u) \lesssim \sum_{m,s' \geq 1} \left[ \frac{l(Q_m)}{l(S_{m,s'})} \right] \Phi(Q_m, 1_{Q_0} u) \lesssim V^2 u(Q_0).
\]

This last estimate can be also used to prove when \( Q_0 = \hat{Q}_0 \in S \).

**Theorem 28.** We have the following Carleson measure estimates for \( S' \in S \) and \( K \in D^u \):

\[
(7.2) \quad \sum_{J' \in S(S')} u(J') \leq \frac{1}{4} u(S') \quad \text{and} \quad \sum_{S' \in S: S' \ni K} u(S') \lesssim u(K);
\]

\[
(7.3) \quad \sum_{J \in C^v(S'): J \subset K, l(J) < \delta^r l(K)} |\langle T(1_{S'} u), h_J^v \rangle|_v^2 \lesssim (V^2 + T^2) u(K).
\]

**Proof.** For the second part of the inequality in (7.2), it suffices to verify it for \( K = S_0 \in S \). And then the case we are interested in follows from recursive application of the estimate from the first half of the inequality (7.2) to the cube \( S_0 \) and all of its children in \( S \).

We now prove the first half of (7.2). The cubes in the collection \( S(S_0) = \{S'_m : m \geq 1\} \) given in the Definition 23 are pairwise disjoint and strictly contained in \( Q_0 \). Each of them satisfies (6.5), so we can apply that estimate along with (6.4) to get

\[
\sum_{J' \in S(S_0)} u(J') = \sum_{m \geq 1} u(S'_m) \leq \frac{1}{4} V^2 \sum_{m \geq 1} \Psi(S'_m, 1_{S_0} u) \leq \frac{1}{4} u(S_0).
\]
We will now prove (7.3). First we will fix $S' \in \mathcal{S}$ and $K$, which can be assumed to be a subset of $S'$. We will apply the operator $T$ to $u_{1_K}$ as opposed to $u_{1_{S'}}$, and we can then use the testing condition for $T$ to get

$$\sum_{J \in C^u(S') : J \in K, l(J) \leq \delta^t l(K)} |T(1_{\mathcal{K}u}), h_j^u)|^2 \leq \int_K |T(1_{\mathcal{K}u})|^2 dv \leq T^2 u(K).$$

Now we will apply $T$ to $u_{1_{S' \cap K}}$ and show

$$\sum_{J \in C^u(S') : J \in K, l(J) < \delta^t l(K)} |T(1_{S' \cap K}u), h_j^u)|^2 \lesssim V^2 u(K).$$

We can assume that $K \subset S'$ and there is some $J \in C^u(S')$ with $J \subset K$. From this we can say that $K$ is not a stopping cube. Therefore the cube $K$ must fail (6.5).

Let $\mathcal{J}$ denote the maximal cubes $J \in C^u(S')$ with $J \subset K$ and $l(J) \leq \delta^t l(K)$. Using the definition of $\hat{P}_S^u(g)$, we can use (7.2), with $Q = S'$ and $J \in \mathcal{J}$. It gives

$$\sum_{J \in \mathcal{J}} \|\hat{P}_S^u T(1_{S' \cap K}u)\|_{L^2(v)}^2 \lesssim \sum_{J \in \mathcal{J}} \Phi(J, 1_{S' \cap K}u) \lesssim V^2 u(K).$$

The second inequality uses the fact $K$ fails (6.4). This proves (7.3).

The following Carleson measure estimate uses hypothesis (6.4) in the proof. It will provide the decay in the parameter $t$ in Theorem 29. For all integers $t \geq 0$, we define for $S \in \mathcal{S}$, which are not maximal

$$\alpha_t(S) := \sum_{S' : \pi_{D^u}(S') = S} \|\hat{P}_S^u T(u_{1_{\pi_{D^u}(S')}}, S)\|_{L^2(v)}^2,$$

Here $\pi_{D^u}(S)$ is the $t$-ancestor of $S$ in $D^u$. Also we are taking the projection $T(u_{1_{\pi_{D^u}(S')}}, S)$ associated to parts of the corona decomposition which are ‘far below’ $S$. We have the following off-diagonal estimate.

**Theorem 29.** The following Carleson measure estimate holds:

$$\sum_{S : \pi_{D^u}(S') \subset K} \alpha_t(S) \lesssim \delta^{-\sigma_0 t} V^2 u(K), \quad K \in D^u.$$  

The implicit constant is independent of the choices of the cube $K$ and $t \geq 1$.

In the estimate (7.4), we need to observe the fact that the dyadic parent $\pi_{D^u}(S)$ of $S$ appears. In fact the role of dyadic parents is revealed in the next proof. We use the negation of (6.5) when $\pi_{D^u}(S) \notin \mathcal{S}$, and otherwise we use (6.4).

**Proof.** We will first show that

$$\sum_{S \in \mathcal{S}(\hat{S})} \alpha_t(S) \leq \delta^{-\sigma_0 t} V^2 u(\hat{S}), \quad \hat{S} \in \mathcal{S}.$$

For this proof, we will set $\mathcal{S}(\hat{S}) := \{S' \in \mathcal{S} : \pi_{D^u}(S') = S\}$, using this notation for $S \in \mathcal{S}(\hat{S})$. We apply the $L^2$ formulation estimate (5.4) of Lemma 21 to the expression $\alpha_t$.

$$(7.5) \quad \mathcal{S}(S') := \{J \in C^u(S) : J \text{ is maximal with } J \subset S', l(J) < \delta^t l(S')\}.$$  

From the definition above we have $l(J) < \delta^t l(S')$ for all $J \in \mathcal{S}(S')$ and as all Haar functions have mean zero, we can apply the $L^2$ formulation (5.4) of Lemma 21. Using this, we see that

$$\alpha_t(S) \lesssim \sum_{S' \in \mathcal{S}(\hat{S})} \sum_{J \in \mathcal{S}(S')} \Phi(J, 1_{S \setminus S'}u).$$

And by using (6.4) we get

$$\sum_{S \in \mathcal{S}(\hat{S})} \alpha_t(S) \lesssim \sum_{S \in \mathcal{S}(\hat{S})} \sum_{S' \in \mathcal{S}(S')} \sum_{J \in \mathcal{S}(S')} \Phi(J, 1_{S \setminus S'}u) \lesssim \delta^{-\sigma_0 t} V^2 \sum_{S \in \mathcal{S}(\hat{S})} u(S) \lesssim \delta^{-\sigma_0 t} V^2 u(\hat{S}).$$

The last inequality follows from hypothesis (6.4).

Now fix $K$ as in (7.4) and let $\hat{S} \in \mathcal{S}$ be the stopping cube such that $K \in C^u(\hat{S})$. Let $\mathcal{G}_t := \{S_t\}_t$ be the maximal cubes from $S$ that are strictly contained in $K$. Inductively we define the $(k+1)^{st}$ generation $\mathcal{G}_{k+1}$
to consist of the maximal cubes from $\mathcal{S}$ that are strictly contained in some $k^{th}$ generation cube $S \in \mathcal{G}_k$. Inequality (7.6) shows that
\[
\sum_{S \in \mathcal{G}_{k+1}} \alpha_t(S) \lesssim \delta^{-\sigma t} \mathcal{V}^2 \sum_{S \in \mathcal{G}_{k+1}} u(S).
\]
We have from (7.2) that
\[
\sum_{k=1}^{\infty} \sum_{S \in \mathcal{G}_k} u(S) \lesssim \sum_{S \in \mathcal{G}_1} u(S) \lesssim u(K).
\]
This will be all we need for the case $K = \hat{S}$. For the case $K \neq \hat{S}$ we will use Lemma 27 to control the first generation of cubes $S$ in $\mathcal{G}_1$:
\[
\sum_{S \in \mathcal{G}_1} \alpha_t(S) \lesssim \delta^{-\sigma t} u(K).
\]
Indeed we will apply Lemma 27 with $\hat{Q}_0 = \hat{S}$, $Q_0 = K$, $\{Q_r\}_{r \geq 1} = \mathcal{G}_1$ and $\{J_{r,s}\}_{s \geq 1} = \bigcup_{S' \in \mathcal{S}(S')} S(S')$.

When $K \neq \hat{S}$ we have
\[
\sum_{S \in \mathcal{S}: \pi_{\mathcal{P}_n}(S) \subseteq K} \alpha_t(S) = \sum_{S \in \mathcal{G}_1} \alpha_t(S) + \sum_{k=1}^{\infty} \sum_{S \in \mathcal{G}_{k+1}} \alpha_t(S) \lesssim \delta^{-\sigma t} u(K) + \delta^{-\sigma t} \mathcal{V} \sum_{k=1}^{\infty} \sum_{S \in \mathcal{G}_k} u(S)
\]
\[
\lesssim \delta^{-\sigma t} \mathcal{V}^2 u(K).
\]
If $K = \hat{S}$ we have $\mathcal{G}_0 = \{\hat{S}\}$ and we get the estimate
\[
\sum_{S \in \mathcal{S}: \pi_{\mathcal{P}_n}(S) \subseteq \hat{S}} \alpha_t(S) \lesssim \delta^{-\sigma t} \mathcal{V} \sum_{S \in \mathcal{G}_k} u(S)
\]
\[
\lesssim \delta^{-\sigma t} \mathcal{V}^2 u(\hat{S}).
\]

The proof of Theorem 29 is complete. \(\square\)

We need a Carleson measure estimate that is a common variant of (7.2) and (7.4). We define
\[
\beta(S) := \|P_S^v T(u\pi_{\mathcal{P}_n}(S))\|_{L^2(v)}^2.
\]

**Theorem 30.** We have the following Carleson measure estimate
\[
\sum_{S \in \mathcal{S}: \pi_{\mathcal{P}_n}(S) \subseteq K} \beta(S) \lesssim (\mathcal{T}^2 + \mathcal{V}^2) u(K).
\]

**Proof.** Using the decomposition $\pi_{\mathcal{P}_n}(S) = S \cup \{\pi_{\mathcal{P}_n}(S) \setminus S\}$, we write $\beta(S) \leq 2(\beta_1(S) + \beta_2(S))$ where
\[
\beta_1(S) := \|P_S^v T(u1_S)\|_{L^2(v)}^2 \quad \text{and} \quad \beta_2(S) := \|P_S^v T(u1_{\pi_{\mathcal{P}_n}(S) \setminus S})\|_{L^2(v)}^2.
\]

We have by the testing condition $\beta_1(S) \leq \mathcal{T}^2 u(S)$, so now by (7.2), we need only consider the Carleson measure norm of the terms $\beta_2(S)$.

Now we will fix an cube $\hat{K}$ of the form $K = \pi_{\mathcal{P}_n}(S_0)$ for some $S_0 \in \mathcal{S}$. Let $\mathcal{R}$ be the maximal cubes of the form $\pi_{\mathcal{P}_n}(S) \subseteq \hat{K}$ and for $R \in \mathcal{R}$, let $\mathcal{S}(R)$ be all cubes $S$ in $\mathcal{S}$ with $S \subseteq R$ and $S$ is maximal. Now by using the definition of $P_S^v(g)$ and (7.5), we can estimate
\[
\sum_{R \in \mathcal{R}} \sum_{S \in \mathcal{S}(R)} \beta_2(S) \lesssim \sum_{R \in \mathcal{R}} \sum_{S \in \mathcal{S}(R)} \sum_{J \in \mathcal{S}(S)} \|P_J^v T(u1_{\pi_{\mathcal{P}_n}(S) \setminus S})\|_{L^2(v)}^2 \lesssim \mathcal{V}^2 u(\hat{K}).
\]

By careful arrangement of the collections $\mathcal{R}$, $\mathcal{S}(R)$ and $\mathcal{S}(S)$ we have applied (7.1) in the last step. Here we use the same strategy as we used in the proof of Theorem 29.

We argue this inequality enough to conclude the Theorem. Suppose that $S' \in \mathcal{S}$, with $S' \subseteq K$, but $S'$ is not in any collection $\mathcal{S}(R)$ for $R \in \mathcal{R}$. It follows that $S' \subseteq S$ for some $S \in \mathcal{S}(R)$ and $R \in \mathcal{R}$. This implies that the Carleson measure estimate (7.2) completes the proof. \(\square\)
We collect one last Carleson measure estimate. Define

\[ \gamma(S) := \| P^*_S T(u \Pi^1_{S}(S) \backslash \Pi^2_{P_E}(S)) \|_{L^2(v)}. \]

**Theorem 31.** We have the estimate

\[ \sum_{S \in \mathcal{S}} \gamma(S) \lesssim \mathcal{V}^2 u(K). \]

**Proof.** We can take \( K = \pi^1_{P_E}(S_0) \) for some \( S_0 \in \mathcal{S} \), and we can assume that \( K \notin \mathcal{S} \) as otherwise we are applying the \( T \) to the zero function. We then repeat the argument as in the previous proof. We use a similar construction for the proof here as in Theorem 30. \( \square \)

8. THE PARAPRODUCT TERMS

We are going to prove bounds on the paraproduct term \( A^5_1 \) now. Before we prove the bounds, we will reorganize the sum in (6.2) according to the corona decomposition. We need to observe that for \( S \in \mathcal{C}^u(S') \) and \( S \subset Q \), we need not have \( Q \in \mathcal{C}^u(S') \). It could be the case that \( Q \in \mathcal{C}^u(\pi^1_{S}(S')) \) for some ancestor \( \pi^1_{S}(S') \) of \( S' \). Remember the ancestor \( \pi^1_{S}(S') \) is defined only for \( 1 \leq t \leq \rho(S') \).

Now we will split the sum into two parts \( A^5_1 = A^5_1 + A^5_2 \) where

\begin{align*}
A^5_1 &:= \sum_{S' \in \mathcal{S}} \sum_{(Q,S) \in \mathcal{P}(S')} \mathcal{E}^u_{Q,S}(\Delta^u_{Q,S}) T(1_{S'} u), \Delta^u_{S} g) v; \\
A^5_2 &:= \sum_{S' \in \mathcal{S}} \sum_{(Q,S) \in \mathcal{P}(S')} \sum_{t=1}^{\rho(S')} \mathcal{E}^u_{Q,S}(\Delta^u_{Q,S}) T(1_{\pi^1_{S}(S')} u), \Delta^u_{S} g) v.
\end{align*}

Observe that in \( A^5_1 \) we consider the case where both \( Q \) and \( S \) are controlled by the same stopping cube. Whereas in \( A^5_2 \), \( (Q,S) \in \mathcal{P}(\pi^1_{S}(S')) \), where \( \pi^1_{S}(S') \) is \( t \)-fold parent of \( S' \) in the grid \( \mathcal{S} \).

We will now show

\[ |A^5_1| \lesssim (T + \mathcal{V}) \| f \|_{L^2(u)} \| g \|_{L^2(v)}. \]

We will have to further decompose \( A^5_2 \).

8.1. The first Paraproduct Term \( A^5_1 \). Fix \( S' \in \mathcal{S} \) and \( S \in \mathcal{C}^u(S') \). Observe that we have the following telescoping identity:

\[ \sum_{Q : (Q,S) \in \mathcal{P}(S')} \mathcal{E}^u_{Q,S}(\Delta^u_{Q,S}) f = \mathcal{E}^u_{Q,S} f - \mathcal{E}^u_{\pi^1_{S}(S')} f. \]

Here \( Q_{S,*} \) is the minimal member of \( \mathcal{C}^u(S') \) that contains \( S \) and \( l(S) < \delta' l(Q) \). As \( S \) is good, such cubes exist. So we have

\begin{align*}
A^5_1 &= \sum_{S' \in \mathcal{S}} A^5_1(S'); \\
A^5_1(S') &= \sum_{S \in \mathcal{C}^u(S')} (\mathcal{E}^u_{Q,S} f - \mathcal{E}^u_{\pi^1_{S}(S')} f)(T(1_{S'} u), \Delta^u_{S} g) v.
\end{align*}

We now give our first paraproduct estimate.

**Lemma 32.** We have the following estimate

\[ A^5_1(S') \lesssim (T + \mathcal{V}) \| P^*_S f - 1_{S'} \mathcal{E}^u_{\pi^1_{S}(S')} f \|_{L^2(u)} \| \mathcal{P}^*_S g \|_{L^2(v)}, \quad S' \in \mathcal{S}. \]

We have defined the projections appearing on the righthand side in Section 6.

**Proof.** For \( Q \in \mathcal{C}^u(S') \), let us define

\[ L^*_Q g := \sum_{S \in \mathcal{C}^u(S') : Q_{S,*} = Q} \Delta^u_{S} g. \]

Now using Cauchy-Schwarz and the fact that \( \mathcal{E}^u_{Q,S} f = \mathcal{E}^u_{Q} \mathcal{P}^*_S f \) and \( L^*_Q g = L^*_Q \mathcal{P}^*_S g \), then we have by re-indexing
\begin{align*}
|A_1^S(S')| & = \left| \sum_{Q \in C^v(S')} (E_Q^u P^u_S f - E_{\pi_D(S')}^u f) \langle T(1_{S'}u), L_Q^v P^v_S g \rangle_v \right| \\
& \leq \left[ \sum_{Q \in C^v(S')} \left| E_Q^u P^u_S f - E_{\pi_D(S')}^u f \right|^2 \| L_Q^v T(1_{S'}u) \|_{L^2_v}^2 \right] \left( \sum_{Q \in C^v(S')} \left| L_Q^v P^v_S g \right|^2 \right)^{\frac{1}{2}} \| P^u_S g \|_{L^2_v} \\
& \leq \| P^u_S g \|_{L^2_v} \left[ \sum_{Q \in C^v(S')} \left| E_Q^u (P^u_S f - E_{\pi_D(S')}^u f) \right|^2 \| L_Q^v T(1_{S'}u) \|_{L^2_v} \right] \left( \sum_{Q \in C^v(S')} \left| L_Q^v P^v_S g \right|^2 \right)^{\frac{1}{2}} .
\end{align*}

By the Carleson Embedding Theorem, Theorem 11, this last factor is at most \( \| (P^u_S f - E_{\pi_D(S')}^u f) \|_{L^2(u)} \) times the Carleson measure norm of the coefficients \( \{ \| L_Q^v T(1_{S'}u) \|_{L^2_v} : Q \in C^u(S') \} \).

Using (7.3) in Theorem 28 we know this at most a constant multiple of \( T + V \), so the proof is complete. \( \square \)

Now using Lemma 32 we get the desired estimate on \( A_1^S \)
\[
|A_1^S| \lesssim (T + V) \sum_{S' \in S} \| P^u_{S'} f - 1_{S'} E_{\pi_D(S')}^u f \|_{L^2(u)} \| P^v_{S'} g \|_{L^2(v)} \\
\lesssim (T + V) \left( \sum_{S' \in S} \| P^u_{S'} f - 1_{S'} E_{\pi_D(S')}^u f \|_{L^2(u)}^2 \sum_{S' \in S} \| P^v_{S'} g \|_{L^2(v)}^2 \right)^{\frac{1}{2}} .
\]

That is because we have
\[
P^u_{S'} f = \sum_{Q \in C^v(S')} M^{-1}_Q \sum_{t=1}^{M_Q} \langle f, h_Q^t \rangle_u h_Q^t \quad \text{and} \quad P^v_{S'} g = \sum_{S \in C^v(S')} M^{-1}_S \sum_{k=1}^{M_S} \langle g, h_S^k \rangle_u h_S^k,
\]
\[
\| P^u_{S'} g \|_{L^2(v)} \leq \| g \|_{L^2(v)} \quad \text{and} \quad \| P^v_{S'} g \|_{L^2(u)} \leq \| f \|_{L^2(u)} .
\]

Also we have
\[
\sum_{S' \in S} \| P^u_{S'} f - 1_{S'} E_{\pi_D(S')}^u f \|_{L^2(u)}^2 \leq \sum_{S' \in S} \| P^u_{S'} f \|_{L^2(u)}^2 + \sum_{S' \in S} \| 1_{S'} E_{\pi_D(S')}^u f \|_{L^2(u)}^2
\]
and
\[
\sum_{S' \in S} \| 1_{S'} E_{\pi_D(S')}^u f \|_{L^2(u)}^2 = \sum_{S' \in S} u(S') | E_{\pi_D(S')}^u f |^2 \\
\leq \sum_{S' \in S} u(S') | E_{\pi_D(S')}^u f |^2 \\
\lesssim \| M_u f \|_{L^2(u)}^2 \\
\lesssim \| f \|_{L^2(u)}^2 .
\]

Here we are using (7.2) to conclude that the maximal function \( M_u \) dominates the sum above. This completes the proof.

8.2. Telescoping Arguments. We will now use telescoping arguments as in (8.3) for \( A_2^S \). For \( S' \in S \setminus \{Q_0\} \), fixing a \( S \in C^v(S') \) then summing over \( Q \), we get
\[
A_2^S(S') := \sum_{t=1}^{Q(S')} \sum_{(Q,S) \in P^u_{\pi_D(S')}} E_Q^u (\Delta^u_Q f) \langle T(1_{\pi_D(S')}u), \Delta^u_Q g \rangle_v
\]
\begin{equation}
\sum_{t=1}^{\rho(S')} \sum_{S \in C^t(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v).
\end{equation}

This is because with $S \in C^t(S')$ fixed, the sum over $Q$ such that $(Q, S) \in P(\pi_{S-1}^t(S'))$ is a function of $S'$ and $t$. The smallest cube that contributes to the sum is $\pi_{S-1}^t(S')$ which is the second parent of $\pi_{S}^t(S')$ and the largest cube that contributes to the sum is $\pi_{S}^t(S')$. Observe the sum over $S$ is independent of the sum over $t$ in (8.6). Below we will further decompose $A_2^2(S')$ by adding and subtracting a cancellative term

\begin{equation}
A_2^2(S') = \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v) + E_{\pi_{S}^t(S')}^u(S')(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v)
\end{equation}

where we have defined:

\begin{equation}
A_1^6(S') := \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v)
\end{equation}

\begin{equation}
A_2^2(S') := \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v).
\end{equation}

Observe here $A_2^2$ is a telescoping term in itself, so we can sum by parts to get the following

\begin{equation}
A_2^2(S') = E_{\pi_{S}^t(S')}^u(S')(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v) + \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v) = A_2^2(S') + A_3^6(S').
\end{equation}

In the sum above for the missing term $E_{\pi_{S}^t(S')}^u(S')(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v)$, where $Q_0$ is the largest cube that was fixed, we are going to assume the expectation is zero.

Combining the steps above we can now decompose $A_2^6$ as

\begin{equation}
A_2^6 = A_1^6 + A_2^2 + A_3^6 \text{ where } A_i = \sum_{S' \in S \setminus Q_0} A_i^6(S') \text{ for } i = 1, 2, 3,
\end{equation}

and

\begin{equation}
A_1^6(S') := \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v);
\end{equation}

\begin{equation}
A_2^2(S') := (E_{\pi_{S}^t(S')}^u(S')(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v);
\end{equation}

\begin{equation}
A_3^6(S') := \sum_{t=1}^{\rho(S')} (E_{\pi_{S-1}^t}^u(S') - E_{\pi_{S}^t}^u(S'))(f)(T(1_{\pi_{S}^t(S')}, u), \Delta_S^u g_v).
\end{equation}

Observe the expression $A_1^6$ has cancellative terms in both $f$ and $g$, so it is not a paraproduct term, while $A_2^2$ is a paraproduct similar to $A_1^5$. The third term is also a paraproduct term. We will now prove the following estimates for each of these terms:

\begin{equation}
|A_1^6| \leq (T + V)\|f\|_{L^2(u)}\|g\|_{L^2(v)};
\end{equation}

\begin{equation}
|A_2^2| \leq (T + V)\|f\|_{L^2(u)}\|g\|_{L^2(v)};
\end{equation}

\begin{equation}
|A_3^6| \leq V\|f\|_{L^2(u)}\|g\|_{L^2(v)}.
\end{equation}

We will begin the proof of these estimates above starting with the term $A_3^6$. 
8.3. The Paraproduct Term $A_3^6$. Let us fix $t$ and define

$$A_3^6(S', t) := (E_{\pi_{S^*}(\tau_{S^*}(S'))}f)(T(1_{\pi_{S^*}(S')} u), P_{S'}^v g)_v, \quad S' \in S, \rho(S') \geq t;$$

$$A_3^6(t) := \sum_{S' \in S, \rho(S') \geq t} A_3^6(S', t).$$

We can see that the $t$-fold parent of $S'$ is defined by imposing the restriction $\rho(S') \geq t$. We want to show

$$|A_3^6(t)| \lesssim \delta^{\sigma t} V \|f\|_{L^2(v)} \|g\|_{L^2(v)} \quad t \geq 1.$$ 

Here the constant $\sigma = -2\sigma_0 > 0$, hence we get (8.14) when we will sum over $t \geq 1$.

Since the $P_S^v$, are orthogonal, we get the following using Cauchy-Schwarz

$$|A_3^6(t)| \leq \|g\|_{L^2(v)} \left( \sum_{S' \in S, \rho(S') \geq t} |E_{\pi_{S^*}(\tau_{S^*}(S'))}f|^2 \|P_{S'}^v T(1_{\pi_{S^*}(S')} u)\|^2_{L^2(v)} \right)^{\frac{1}{2}}.$$

Using the definition of $\alpha_t(S)$ given in Section 6, the sum above is

$$\left( \sum_{S' \in S} \alpha_t(S') |E_{\pi_{S^*}(\tau_{S^*}(S'))}f|^2 \right)^{\frac{1}{2}}.$$

Now using Theorem 29 we have our desired estimate on $A_3^6(t)$ as the Carleson measure norm of the coefficients $\{\alpha_t(S') : S' \in S\}$ is at most $C\delta^{\sigma t} V$.

8.4. The paraproduct term $A_2^6$. We have $\pi_{\alpha_{D^*}}(S') \subset \pi_{\alpha_{S^*}}(S')$, so we will now decompose term $A_2^6$ into two terms by writing $\pi_{\alpha_{S^*}}(S') = \pi_{\alpha_{D^*}}(S') \cup \{\pi_{\alpha_{S^*}}(S') \setminus \pi_{\alpha_{D^*}}(S')\}$ to give us,

$$|A_1^6| := \left| \sum_{S' \in S} (E_{\pi_{S^*}(S')} f)(T(1_{\pi_{S^*}(S')} u), P_{S'}^v g)_v \right| \lesssim (T + V) \|f\|_{L^2(w)} \|g\|_{L^2(v)};$$

$$|A_2^6| := \left| \sum_{S' \in S} (E_{\pi_{S^*}(S')} f)(T(1_{\pi_{S^*}(S')} \setminus \pi_{S^*}(S')) u), P_{S'}^v g)_v \right| \lesssim V \|f\|_{L^2(w)} \|g\|_{L^2(v)}.$$

Using these we get (8.13). Now we will prove these inequalities. Remembering the definition of $\beta(S)$, we can now estimate

$$|(T(1_{\pi_{S^*}(S')} u), P_{S'}^v g)_v| = |(P_{S'}^v T(1_{\pi_{S^*}(S')} u), P_{S'}^v g)_v| \leq \beta(S') \frac{1}{2} \|P_{S'}^v g\|_{L^2(v)}.$$

As the projections are mutually orthogonal we have, summing over $S'$

$$|A_1^6| \leq \left( \sum_{S' \in S} \beta(S') \|E_{\pi_{S^*}(S')} f\|^2 \right)^{\frac{1}{2}} \|g\|_{L^2(v)} \lesssim (T + V) \|f\|_{L^2(w)} \|g\|_{L^2(v)}.$$

We have used the estimate on $\beta(S)$ in Theorem 30 to get the estimate in the last step.

We can use a similar approach to prove (8.15). Using the definition of $\gamma(S)$, we have

$$|(T(1_{\pi_{S^*}(S')} \setminus \pi_{S^*}(S')) u), P_{S'}^v g)_v| = |(P_{S'}^v T(1_{\pi_{S^*}(S')} \setminus \pi_{S^*}(S')) u), P_{S'}^v g)_v| \leq \gamma(S') \frac{1}{2} \|P_{S'}^v g\|_{L^2(v)}.$$

We have the following estimate using Theorem 31,

$$|A_2^6| \leq \left( \sum_{S' \in S} \gamma(S') \|E_{\pi_{S^*}(S')} f\|^2 \right)^{\frac{1}{2}} \|g\|_{L^2(v)} \lesssim V \|f\|_{L^2(w)} \|g\|_{L^2(v)}.$$
8.5. The term $A^u_1$. Observe in the definition of $A^u_1(S')$ we can write the following for the difference of expectations,

$$E_{\pi_D^u(\tau_S^{-1}(S'))}^u f - E_{\pi_D^u(\tau_S^{-1}(S'))}^u f = -E_{\pi_D^u(\tau_S^{-1}(S'))}^u \Delta_{\pi_D^u(\tau_S^{-1}(S'))}^u f, \quad \pi_S^{-1}(S') \in S.$$ 

By re-indexing the sum $A^u_1(S')$ defined above we get

$$A^u_1(S') = \sum_{i=1}^{\rho(S')} \left( E_{\pi_D^u(\tau_S^{-1}(S'))}^u \Delta_{\pi_D^u(\tau_S^{-1}(S'))}^u f \right) (T(1_\pi^u(S') u), P^v_{S'} g) v = A^u_7(S') + A^u_4(S');$$

where

$$A^u_7(S') := \sum_{i=1}^{\rho(S')} \left( E_{\pi_D^u(\tau_S^{-1}(S'))}^u \Delta_{\pi_D^u(\tau_S^{-1}(S'))}^u f \right) (T(1_\pi^u(S') \tau_S^{-1}(S') u), P^v_{S'} g) v,$$

$$A^u_4(S') := \sum_{i=1}^{\rho(S')} \left( E_{\pi_D^u(\tau_S^{-1}(S'))}^u \Delta_{\pi_D^u(\tau_S^{-1}(S'))}^u f \right) (T(1_\pi^u(S') \tau_S^{-1}(S') u), P^v_{S'} g) v.$$

We will now show that

$$\sum_{S' \in S \setminus \{S_0\}} A^u_7(S') \lesssim V \|f\|_{L^2(u)} \|g\|_{L^2(v)};$$

$$\sum_{S' \in S \setminus \{S_0\}} A^u_4(S') \lesssim V \|f\|_{L^2(u)} \|g\|_{L^2(v)}.$$

We can prove (8.19) using a similar approach as we did for (8.14) as there is orthogonality present with Haar differences applied to $f$ in (8.17).

We will start the proof of (8.20) by re-indexing the sum. We have

$$A^u_4(S', t) := \left( E_{\pi_D^u(\tau_S^{-1}(S'))}^u \Delta_{\pi_D^u(\tau_S^{-1}(S'))}^u f \right) \sum_{J \in S} (T(1_\pi^u(S') \tau_S^{-1}(S') u), P^v_{S'} g) v$$

and we will prove that:

$$\sum_{S' \in S \setminus \{S_0\}} A^u_4(S') \lesssim \sum_{S' \in S} A^u_4(S', t) \lesssim \delta^{-\sigma_0 t} V \|f\|_{L^2(u)} \|g\|_{L^2(v)} \quad t \geq 1.$$

(Here the decay in $t$ is slightly worse in comparison to the previous estimates.) We will exploit the implicit orthogonality in the sum above. Note that we have

$$\sum_{S' \in S} \sum_{i=1}^{M_{\pi_D^u(\tau_S^{-1}(S'))}^{-1}} |(f, h_i^{\tau_S^{-1}(S')}) v|^2 \leq \|f\|^2_{L^2(u)};$$

$$\sum_{S' \in S} \left( \sum_{J \in S : \pi^{-1}(J) = S'} \|P^v_J g\|_{L^2(v)} \right)^2 \leq \|g\|^2_{L^2(v)},$$

and combining these facts we get (8.21) from the estimate

$$\left( \sum_{J \in S : \pi^{-1}(J) = S'} \|P^v_J T(1_\pi^u(S') \tau_S^{-1}(S') u)\|_{L^2(v)} \right)^2 \lesssim \delta^{-\sigma_0 t} (T^2 + V^2) u(\pi_D^u(S')) S' \in S, t \geq 1.
Let us now prove (8.22). We use the geometric decay in (7.2) and apply hypothesis (6.4). Fix a $S' \in S$ and an integer $w := \frac{1}{2}$. Let us denote by $S'_w$, all the cubes $J \in S$ with $\pi^w_S(J) = S'$. We have

$$\left\| \sum_{J \in S: \pi^{t-1}(J) = S'} T_{\pi^w_S(J)}u \right\|_{L^2(\nu)}^2 = \sum_{J \in S'_w} B(J),$$

where

$$B(J) := \left\| \sum_{J' \in S: \pi^{t-1}(J') = S'} T_{\pi^w_S(J')}u \right\|_{L^2(\nu)}^2.$$

We now decompose $B(J)$ as $B(J) = B_1(J) + B_2(J)$, where

$$B_1(J) := \left\| \sum_{J' \in S: \pi^{t-1}(J') = S'} T_{\pi^w_S(J') \setminus J}u \right\|_{L^2(\nu)}^2,$$

$$B_2(J) := \left\| \sum_{J' \in S: \pi^{t-1}(J') = S'} T_{\pi^w_S(J)}u \right\|_{L^2(\nu)}^2.$$

Using the testing condition we have

$$\sum_{J \in S'_w} B_2(J) \leq T^2 \sum_{J \in S'_w} u(J) \leq \delta^{\frac{n}{2}} T^2 u(\pi^1_D(S')).$$

Here we used the Carleson measure property of $u$ on stopping cubes (7.2) to deduce the last line. Now using the notation of (7.5) and applying (7.1) we can see that

$$\sum_{J \in S'_w} B_1(J) = \sum_{J \in S'_w} \sum_{J' \in S: \pi^{t-1}(J') = J} \sum_{J \in J(J')} \left\| T_{\pi^w_S(J)}u \right\|_{L^2(\nu)}^2 \leq V^2 \delta^{\frac{n}{2}} u(\pi^1_D(S')).$$

This completes the proof of (8.22).

9. Appendix: Hilbert Space Valued Operators

Here we make precise the definitions arising in the setting of weighted norm inequalities for Hilbert space valued singular integrals, beginning with a Calderón–Zygmund kernel. We define a standard $B(\mathcal{H}_1, \mathcal{H}_2)$-valued Calderón–Zygmund kernel $\mathcal{K}(x, y)$ to be a function $\mathcal{K} : X \times X \to B(\mathcal{H}_1, \mathcal{H}_2)$ satisfying the following fractional size and smoothness conditions of order $\delta$ for some $\delta > 0$: For $x \neq y$,

$$|\mathcal{K}(x, y)|_{B(\mathcal{H}_1, \mathcal{H}_2)} \leq \frac{C_{\mathcal{K}}}{V(x, y)},$$

$$|\nabla \mathcal{K}(x, y) - \nabla \mathcal{K}(x', y)|_{B(X \times X, B(\mathcal{H}_1, \mathcal{H}_2))} \leq C_{\mathcal{K}} \left( \frac{d(x, x')}{d(x, y)} \right)^{\delta} \frac{1}{V(x, y)}, \quad \frac{d(x, x')}{d(x, y)} \leq \frac{1}{2A_0},$$

and the last inequality also holds for the adjoint kernel in which $x$ and $y$ are interchanged.

We now turn to a precise definition of the weighted norm inequality

$$\|T_{\sigma}f\|_{L^2_{\mathcal{H}_2}(\omega)} \leq C \|f\|_{L^2_{\mathcal{H}_1}(\sigma)}, \quad f \in L^2(\sigma),$$

where $\sigma$ and $\omega$ are locally finite positive Borel measures on $X$, and $L^2_{\mathcal{H}_1}(\sigma)$ is the Hilbert space consisting of those functions $f : X \to \mathcal{H}_1$ for which

$$\|f\|_{L^2_{\mathcal{H}_1}(\sigma)} := \sqrt{\int_X |f(x)|^2_{\mathcal{H}_1} \, d\sigma(x)} < \infty,$$

equipped with the usual inner product. A similar definition holds for $L^2_{\mathcal{H}_2}(\omega)$. For a precise definition of (9.2) we suppose that $K$ is a standard $B(\mathcal{H}_1, \mathcal{H}_2)$-valued Calderón–Zygmund kernel, and we introduce
a family \( \{ \eta_{\delta,R} \}_{0<\delta<R<\infty} \) of nonnegative functions on \([0,\infty)\) so that the truncated kernels \( R_{\delta,R}(x,y) := \eta_{\delta,R} \) are bounded with compact support for fixed \( x \) or \( y \). Then the truncated operators

\[
T_{\sigma,\delta,R}f(x) := \int_X R_{\delta,R}(x,y) f(y) \, d\sigma(y), \quad x \in X,
\]

are pointwise well-defined, and we will refer to the pair \( \{ R, \{ \eta_{\delta,R} \}_{0<\delta<R<\infty} \} \) as a singular integral operator, which we typically denote by \( T \), suppressing the dependence on the truncations.

**Definition 33.** We say that a singular integral operator \( T = \{ R, \{ \eta_{\delta,R} \}_{0<\delta<R<\infty} \} \) satisfies the norm inequality (9.2) provided

\[
\| T_{\sigma,\delta,R}f \|_{L^2_{\mu,\omega}} \leq N \| f \|_{L^2_{\mu,\omega}}, \quad f \in L^2(\sigma), \quad 0 < \delta < R < \infty.
\]

It turns out that, in the presence of the Muckenhoupt conditions, the norm inequality (9.2) is essentially independent of the choice of truncations used, which justifies suppressing the dependence on the truncations.

The following cube testing conditions, dual to each other and referred to as T1 conditions, are necessary for the boundedness of \( T \) from \( L^2(\sigma) \) to \( L^2(\omega) \),

\[
T^2 := \sup_{Q \in P} \sup_{\sigma_1 \in \text{unit} \, H_1} \frac{1}{\sigma(Q)} \int_Q |T(1_Q \sigma_1)|^2_{H_2} \, d\sigma < \infty,
\]

\[
(T^*)^2 := \sup_{Q \in P} \sup_{\omega_2 \in \text{unit} \, H_2} \frac{1}{\omega(Q)} \int_Q |T^*(1_Q \omega_2)|^2_{H_1} \, d\omega < \infty,
\]

and where we interpret the right sides as holding uniformly over all truncations of \( T \).

### 9.1. Weighted Haar bases for \( L^2_{\mu}(\mu) \)

Now we turn to the definition of weighted Haar bases of \( L^2_{\mu}(\mu) \) where \( H \) is a separable Hilbert space and \( \mu \) is a locally finite positive Borel measure on \( X \). We will use a construction of a Haar basis for \( L^2_{\mu}(\mu) \) in \( X \) that is adapted to a measure \( \mu \) (c.f. [NTV2] for the scalar case). Given a dyadic cube \( Q \in D \), where \( D \) is a dyadic grid of cubes from \( \mathbb{P}^n \), let \( \Delta_{H,Q}^\mu \) denote the projection of the subspace \( L^2_{H,Q}(\mu) \) of \( L^2_{\mu}(\mu) \) that consists of \( H \)-linear combinations of the indicators of the children \( H(Q) \) of \( Q \) that have \( \mu \)-mean zero over \( Q \):

\[
L^2_{H,Q}(\mu) := \left\{ f = \sum_{Q' \in H(Q)} b_{Q'} 1_{Q'} : b_{Q'} \in H, \int_Q f \, d\mu = 0 \right\},
\]

where the expression \( b_{Q'} 1_{Q'} \) refers to the \( H \)-valued function \( b_{Q'} 1_{Q'}(x) = \begin{cases} b_{Q'} & (x \in Q') \\ 0 & (x \in H) \end{cases} \) that is constant on \( Q' \) and vanishes off \( Q' \).

If \( \{ b_m \}_{m=1}^\infty \) is any orthonormal basis for \( H \) then we define the finite-dimensional projections \( \Delta_{H,Q}^\mu b_m \) onto

\[
L^2_{H,Q}(\mu; b_m) := \left\{ f = \sum_{Q' \in H(Q)} a_{Q'} b_m 1_{Q'} = b_m \sum_{Q' \in H(Q)} a_{Q'} 1_{Q'} : a_{Q'} \in \mathbb{R}, \int_Q f \, d\mu = 0 \right\},
\]

so that \( \Delta_{H,Q}^\mu = \sum_{m=1}^\infty \Delta_{H,Q}^\mu b_m \). Then we have the important telescoping property for dyadic cubes \( Q_1 \subset Q_2 \) that arises from the martingale differences associated with the projections \( \Delta_{\mu}^\mu b_m \) for a fixed basis vector \( b_m \):

\[
1_{Q_0}(x) \left( \sum_{Q \in \{Q_1, Q_2\}} \Delta_{H,Q}^\mu b_m f(x) \right) = 1_{Q_0}(x) \left( \mathbb{P}_{Q_0}^{\mu, b_m} f - \mathbb{P}_{Q_2}^{\mu, b_m} f \right), \quad Q_0 \in H(Q_1), \quad f \in L^2(\mu),
\]

where

\[
\mathbb{P}_{Q}^{\mu, b_m} f(x) := \begin{cases} E_{Q}^{\mu, b_m} f & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases} = \mathbb{P}_{\mu,\mu} \mathbb{P}_{Q} H f,
\]

\[
E_{Q}^{\mu, b_m} f := \int_{Q} \langle f(x), b_m \rangle \, d\mu(x) = \left\langle \int_{Q} f(x) \, d\mu(x), b_m \right\rangle_{H} = \left\langle E_{Q} f, b_m \right\rangle_{H} b_m = \mathbb{P}_{\mu}^H E_{Q} f,
\]
\[ E_Q^\mu f := \int_Q f(x) \, d\mu(x) \in \mathcal{H}. \]

Taking sums over projections we obtain the more general telescoping property for any bounded projection \( P \) on \( \mathcal{H} \):
\[
1_{Q_0}(x) \left( \sum_{Q \in \{Q_1, Q_2\}} P \triangle_{H;Q}^\mu f(x) \right) = 1_{Q_0}(x) \left( P E_Q^\mu f - P E_{Q_2}^\mu f \right), \quad Q_0 \in \mathcal{H}(Q_1), \, f \in L^2(\mu). \]

It is sometimes convenient to use a fixed orthonormal basis \( \{ h_{Q}^{\mu,a,b,m} \}_{a \in \Gamma_n, \, Q \in \mathcal{D} \, \& \, m \geq 1} \) of \( L^2_\mu(\mu) \) where \( \Gamma_n := \{0, 1\}^n \setminus \{1\} \) is a convenient index set with \( 1 := (1, 1, \ldots, 1) \). Then \( \{ h_{Q}^{\mu,a,b,m} \}_{a \in \Gamma_n, \, Q \in \mathcal{D} \, \& \, m \geq 1} \) is an orthonormal basis for \( L^2(\mu) \), with the understanding that we add the constant function \( 1 \) if \( \mu \) is a finite measure. In particular we have
\[
\|f\|_{L^2_\mu(\mu)}^2 = \sum_{Q \in \mathcal{D}} \left\| \triangle_{H;Q}^\mu f \right\|_{L^2_\mu(\mu)}^2 = \sum_{Q \in \mathcal{D}} \left| \hat{f}(Q) \right|^2,
\]
\[
\left| \hat{f}(Q) \right|^2 := \sum_{a \in \Gamma_n} \left( f, h_{Q}^{\mu,a} \right)_\mu^2 = \sum_{a \in \Gamma_n} \sum_{m=1}^{\infty} \left( f, h_{Q}^{\mu,a,b,m} \right)_\mu^2,
\]
where the measure is suppressed in the notation \( \hat{f} \), along with the parameters \( a \in \Gamma_n \) and \( m \geq 1 \).

Finally, let
\[
\mathbb{E}_Q^\mu f(x) := \begin{cases} E_Q^\mu f & \text{if } x \in Q, \\ 0 & \text{if } x \notin Q \end{cases}
\]
be projection onto the subspace \( \mathcal{H}1_Q \) of constant \( \mathcal{H} \)-valued functions on \( Q \), and note that we have
\[
f = \sum_{Q \in \mathcal{D}} \triangle_{H;Q}^\mu f,
\]
with convergence in \( L^2_\mu(\mu) \) since
\[
\triangle_{H;Q}^\mu f = \sum_{Q \in H(Q)} \mathbb{E}_Q^\mu f - \mathbb{E}_Q^\mu f = \sum_{Q \in H(Q)} 1_Q \left( \mathbb{E}_Q^\mu f - \mathbb{E}_Q^\mu f \right),
\]
and the Hilbert space valued version of the dyadic Lebesgue differentiation theorem gives
\[
\lim_{Q \ni x} \mathbb{E}_Q^\mu f = f(x), \quad \text{for } \mu\text{-a.e. } x \in X.
\]

**Caution:** While the scalar identity \( 1_Q \triangle_{H;Q}^\mu f = 1_Q E_Q^\mu f \) extends readily to the Hilbert space setting, the operator identity
\[
T_\sigma \left( 1_Q E_Q^\mu f \right) \triangle_{H;Q}^\mu f = \left( E_Q^\mu f \right) T_\sigma \left( 1_Q f \right)
\]
fails in the Hilbert space setting where \( T_\sigma : L^1_{H_1} \to L^1_{H_2} \) is a vector in \( \mathcal{H}_2 \) while \( E_Q^\mu f \triangle_{H;Q}^\mu f \) is a vector in \( \mathcal{H}_1 \). Even if \( \mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2 \), an operator \( T_\sigma \) does not typically commute with an element in \( \mathcal{H} \) unless \( \mathcal{H} \) is the scalar field.

Nevertheless, we can indeed take the \( \mathcal{H} \)-norm of the element \( E_Q^\mu f \triangle_{H;Q}^\mu f \) outside the operator \( T_\sigma \), i.e.
\[
|T_\sigma \left( 1_Q E_Q^\mu f \right) \triangle_{H;Q}^\mu f|_{\mathcal{H}} = \left| E_Q^\mu f \right|_{\mathcal{H}} T_\sigma (1_Q e), \quad e := \frac{E_Q^\mu f \triangle_{H;Q}^\mu f}{\left| E_Q^\mu f \triangle_{H;Q}^\mu f \right|_{\mathcal{H}}}.\]
Note that this $e$ is a unit vector in $H$, and this motivates our use of testing conditions of the form

$$\int_Q |T_\sigma(1_Q e)|^2 \, d\omega \leq T^2 \sigma(Q), \quad Q \in \mathcal{P}^n \text{ and } e : Q \to \text{unit } H \text{ measurable.}$$

**Remark 34.** A stronger form of the testing condition, analogous to the indicator/cube testing condition arising in connection with the two weight norm inequality for the Hilbert transform [LaSaShUr], is this:

$$\int_Q |T_\sigma(1_Q e)|^2 \, d\omega \leq T^2 \sigma(Q), \quad Q \in \mathcal{P}^n \text{ and } e : Q \to \text{unit } H \text{ measurable.}$$

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