ON FUNCTIONS WITHOUT A NORMAL ORDER

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ABSTRACT. The method of Turán in establishing the normal order for the number of prime divisors of a number is used to show that a certain class of arithmetic functions do not have a normal order.

1. Introduction

The normal order of an arithmetic function, defined in [2, p. 356], measures the ‘usual size’ of the function: A function \( \psi(n) \geq 0 \) is said to have a normal order \( f(n) \) if, to every \( \epsilon > 0 \), the number of \( n \leq x \) for which \( |\psi(n) - f(n)| < \epsilon f(n) \) is \( o(x) \), as \( x \to \infty \). It is tacitly assumed that \( f(n) \) is increasing—otherwise, every such \( \psi(n) \) has itself as normal order.

The notion was first introduced by G. H. Hardy and S. Ramanujan [1], who proved that \( \omega(n) \), the number of distinct prime divisors of \( n \), has the normal order \( \log \log n \). Their proof was much simplified by P. Turán ([2, p. 356], [5]), who showed that the result can be established from the asymptotic formulae for the first and the second moments of \( \omega(n) \); indeed it is sometimes said that probabilistic number theory stems from [5]. By applying Turán’s method ‘in reverse’, so to speak, S. L. Segal [4] showed that Euler’s totient function \( \phi(n) \) does not have a normal order. We distil the argument used by Segal, thereby extending his result to a certain class of arithmetic functions.

2. A class of functions without a normal order

Let \( \mathcal{M} \) denote the class of arithmetic functions \( \psi \) for which there are positive constants \( A, B, C \) such that \( 0 \leq \psi(n) < Cn \) and, as \( x \to \infty \),

\[
\sum_{n \leq x} \psi(n) \sim \frac{Ax^2}{2} \quad \text{and} \quad \sum_{n \leq x} \psi^2(n) \sim \frac{Bx^3}{3}.
\]

**Theorem.** Let \( \psi \in \mathcal{M} \). If \( A^2 < B \) then \( \psi \) does not have a normal order.

**Proof.** Let \( A, B, C \) be constants associated with \( \psi \in \mathcal{M} \), and set

\[
R(x) = \sum_{n \leq x} \left( \psi(n) - An \right) = o(x^2), \quad \text{as} \quad x \to \infty.
\]

Suppose that \( \psi(n) \) has the normal order \( f(n) \); we may assume without loss that \( f(n) < 2Cn \), so that \( |\psi(n) - f(n)| \leq \max\{\psi(n), f(n)\} < 2Cn \). Making use of [2],
and \( f(n) \) being increasing, we find, by partial summation, that

\[
\left| \sum_{n \leq x} \left( \psi(n) - An \right) f(n) \right| \leq \max_{n \leq x} |R(n)| \left\{ \sum_{n \leq x-1} \left( f(n+1) - f(n) \right) + f(x) \right\} \\
= o(x^3) \quad \text{as} \quad x \to \infty.
\]

Let \( \epsilon > 0 \). Appealing to the definition of normal order and separating terms depending on whether \( |\psi(n) - f(n)| < \epsilon f(n) \), or not, we then have, as \( x \to \infty \),

\[
\sum_{n \leq x} (\psi(n) - f(n))^2 \leq 4\epsilon^2 C^2 \sum_{n \leq x} n^2 + 4C^2 x^2 o(x) = \frac{4\epsilon^2 C^2 x^3}{3} + o(x^3).
\]

From (1), (3), (4), together with

\[
\psi^2(n) = A^2 n^2 + (\psi(n) - f(n))^2 + 2(\psi(n) - An)f(n) - (f(n) - An)^2 \\
\leq A^2 n^2 + (\psi(n) - f(n))^2 + 2(\psi(n) - An)f(n),
\]

we now have, on summing over \( n \leq x \),

\[
\frac{Bx^3}{3} + o(x^3) \leq \frac{A^2 x^3}{3} + \frac{4\epsilon^2 C^2 x^3}{3} + o(x^3).
\]

If \( \epsilon = \epsilon(A, B, C) \) is sufficiently small, and \( x \) is large, then the inequality here is untenable for \( A^2 < B \). The theorem is proved.

3. Segal’s theorem on \( \phi(n) \)

**Lemma.** For Euler’s function \( \phi(n) \), we have, as \( x \to \infty \),

\[
\sum_{n \leq x} \phi(n) = \frac{Ax^2}{2} + O(x \log x)
\]

and

\[
\sum_{n \leq x} \phi^2(n) = \frac{Bx^3}{3} + O(x^2 \log^2 x),
\]

where, for primes \( p \),

\[
A = \prod_p \left( 1 - \frac{1}{p^2} \right) \quad \text{and} \quad B = \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^4} \right).
\]

Thus \( \phi \in \mathcal{M} \), and it is readily seen that \( A^2 < B \), so that \( \phi(n) \) does not have a normal order. The asymptotic formula (5) is due to F. Mertens [3], and (6) is due to Segal [4], who gave a somewhat elaborate proof. For completeness sake, we give the proof of the lemma here.

**Proof.** By Möbius inversion, we have

\[
\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d},
\]

where \( \mu(n) \) is the Möbius function; the formula can also be verified by taking \( n \) to be a prime power, and noting that the functions involved are multiplicative. It
follows that, as \( x \to \infty \),

\[
\sum_{n \leq x} \phi(n) = \sum_{ab \leq x} a \mu(b) = \sum_{b \leq x} \mu(b) \sum_{a \leq x/b} a = \sum_{b \leq x} \mu(b) \left\{ \frac{1}{2} \left( \frac{x}{b} \right)^2 + O \left( \frac{x}{b} \right) \right\} = \frac{Ax^2}{2} + E_1(x) + E_2(x),
\]

where

\[
A = \sum_{b=1}^{\infty} \frac{\mu(b)}{b^2} = \prod_p \left( 1 - \frac{1}{p^2} \right),
\]

\[
E_1(x) = O \left( x^2 \sum_{b>x} \frac{1}{b^2} \right) = O(x), \quad E_2(x) = O \left( x \sum_{b \leq x} \frac{1}{b} \right) = O(x \log x),
\]

so that (5) is proved.

Again, from the functions involved being multiplicative, it can be checked that

\[
\left( \sum_{d \mid n} \frac{\mu(d)}{d} \right)^2 = \sum_{a \mid n} \frac{\mu(a)^2}{a^2} g(a), \quad \text{where} \quad g(a) = \prod_{p \mid a} (1 - 2p).
\]

Thus, as \( x \to \infty \),

\[
\sum_{n \leq x} \phi^2(n) = \sum_{ab \leq x} a^2 \mu^2(b) g(b) = \sum_{b \leq x} \mu^2(b) g(b) \left\{ \frac{x^3}{3b^3} + O \left( \frac{x^2}{b^2} \right) \right\} = \frac{Bx^3}{3} + E_3(x) + E_4(x)
\]

where

\[
B = \sum_{b=1}^{\infty} \frac{\mu^2(b) g(b)}{b^3} = \prod_p \left( 1 - \frac{2}{p^2} + \frac{1}{p^3} \right),
\]

\[
E_3(x) = O \left( x^3 \sum_{b>x} \frac{|g(b)|}{b^3} \right), \quad E_4(x) = O \left( x^2 \sum_{b \leq x} \frac{|g(b)|}{b^2} \right).
\]

Apply the bound \(|g(b)| \leq \prod_{p\mid b} (2p) \leq 2^{\omega(b)} b \leq d(b)b\), where \( d(n) \) is the divisor function, and consider

\[
\sum_{b>x} \frac{d(b)}{b^2} = \sum_{uv>x} \frac{1}{u^2 v^2} = \sum_{u \leq x} \frac{1}{u^2} \sum_{v>x/u} \frac{1}{v^2} + \sum_{u>x} \frac{1}{u^2} \sum_{v=1}^{\infty} \frac{1}{v^2} = O \left( \frac{1}{x} \sum_{u \leq x} \frac{1}{u} \right) + O \left( \sum_{u>x} \frac{1}{u^2} \right) = O \left( \frac{\log x}{x} \right),
\]

\[
\sum_{b \leq x} \frac{d(b)}{b} = \sum_{uv \leq x} \frac{1}{u v} = O(\log^2 x).
\]

Thus \( E_3(x) = O(x^2 \log x) \) and \( E_4(x) = O(x^2 \log^2 x) \), and the lemma is proved.

Finally, we remark that Turán’s method is more flexible than what is used to establish the theorem. Roughly speaking, the argument applies to any \( \psi(n) \) for which the second moment sum \( \sum_{n \leq x} \psi^2(n) \) is substantially larger than what ‘might
be expected’ from the bound for the first moment sum $\sum_{n \leq x} \psi(n)$. For example, from

$$\sum_{n \leq x} d(n) \sim x \log x \quad \text{and} \quad \sum_{n \leq x} d^2(n) \sim \frac{x \log^3 x}{\pi^2}, \quad \text{as} \quad x \to \infty,$$

we see that the average value for $d(n)$ is $\log n$, whereas the average value for $d^2(n)$ is $\log^3 n/\pi^2$, which is significantly larger than $\log^2 n$. The proof of the theorem can easily be adapted to show that $d(n)$ does not have a normal order.

References

[1] G. H. Hardy and S. Ramanujan, “The normal number of prime factors of a number $n$,” Quart. J. Math. 48 (1917), 76–92.
[2] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4-th edition, (Oxford), 1961.
[3] F. Mertens, “Über einige asymptotische Gesetze der Zahlentheorie,” J. für die Reine und Angew. Math., 77 (1874), 289.
[4] S. L. Segal, “A note on normal order and the Euler $\phi$-function,” J. London Math. Soc., 39 (1964), 400–404.
[5] P. Turán, “On a theorem of Hardy and Ramanujan,” J. London Math. Soc., 9 (1934), 274–276.