Centrally Essential Endomorphism Rings of Abelian Groups

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Abstract. We study Abelian groups $A$ with centrally essential endomorphism ring $\text{End} \ A$. If $A$ is a such group which is either a torsion group or a non-reduced group, then the ring $\text{End} \ A$ is commutative. We give examples of Abelian torsion-free groups of finite rank with non-commutative centrally essential endomorphism rings.

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1 Introduction

All rings considered are associative rings with non-zero identity element. A ring $R$ is said to be centrally essential if for any its non-zero element $a$, there exist two non-zero central elements $x, y \in R$ with $ax = y$. Centrally essential rings are studied, for example, in [9], [10], [11], [12], [13], [14], [15].

It is clear that any commutative ring is centrally essential. In addition, every centrally essential semiprime ring is commutative; see [9, Proposition 3.3]. Therefore, in the study of centrally essential rings, we are only interested in non-commutative non-semiprime centrally essential rings.

Examples of non-commutative group algebras over fields are given in [9]. For example, if $Q_8$ is the quaternion group of order 8, then its group algebra over the field of order 2 is a non-commutative centrally essential finite local ring of order 256. In addition, in [10], it is proved that the Grassmann algebra of three-dimensional vector space over the field of order 3 is a finite non-commutative centrally essential ring, as well. In [12], there is an example of a centrally essential ring whose factor ring with respect to its prime radical is not a PI ring.

In Theorem 1.2(3) of this paper, we give an example of an Abelian torsion-free group of finite rank with centrally essential non-commutative endomorphism

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³It is clear that the ring $R$ with center $C$ is centrally essential if and only if the module $R_C$ is an essential extension of the module $C_C$. 
ring. In Example 3.9, we give additional examples of non-commutative centrally essential endomorphism rings of some Abelian torsion-free groups of infinite rank.

1.1. Remark. Let $A$ be an Abelian group which is either torsion group, or non-reduced group and let the endomorphism ring $\text{End} A$ be centrally essential. In Section 2 of this paper, we prove that the ring $\text{End} A$ is commutative. Therefore, when studying Abelian groups with non-commutative centrally essential endomorphism rings, only reduced torsion-free groups and reduced mixed groups are of interest.

Let $A$ be an Abelian torsion-free group with endomorphism ring $\text{End} A$ and let $\mathbb{Q} \text{End} A = \mathbb{Q} \otimes \text{End} A$ be the quasi-endomorphism ring\textsuperscript{4} of the group $A$. If the group $A$ has no a non-trivial quasi-decomposition\textsuperscript{5} then it is called strongly indecomposable. The pseudo-socle $\text{PSoc} A$ of the group $A$ is the pure subgroup of the group $A$ generated by all its minimal pure fully invariant subgroups.

The main result of this paper is Theorem 1.2.

1.2. Theorem. Let $A$ be a strongly indecomposable torsion-free Abelian group of finite rank, $\mathbb{Q} \text{End} A$ the quasi-endomorphism ring, and $A \neq \text{PSoc} A$.

1. If $\mathbb{Q} \text{End} A$ is a centrally essential ring, then the ring $\mathbb{Q} \text{End} A/\text{J}(\mathbb{Q} \text{End} A)$ is commutative and $C(\mathbb{Q} \text{End} A) \cap M \neq 0$ for every minimal right ideal $M$ of $\mathbb{Q} \text{End} A$.

2. If the ring $\mathbb{Q} \text{End} A/\text{J}(\mathbb{Q} \text{End} A)$ is commutative, $\text{Soc}(\mathbb{Q} \text{End} A_{\mathbb{Q} \text{End} A}) = \text{Soc}(\mathbb{Q} \text{End} A_{C(\mathbb{Q} \text{End} A)})$ and $C(\mathbb{Q} \text{End} A) \cap M \neq 0$ for every minimal right ideal $M$ of $\mathbb{Q} \text{End} A$, then $\mathbb{Q} \text{End} A$ is a centrally essential ring.

3. Let $n > 1$ be an odd positive integer. There exists a strongly indecomposable Abelian torsion-free group $A(n)$ of rank $2^n$ such that its endomorphism ring is a non-commutative centrally essential ring.

For convenience, we give some definitions and notation used in the paper. The necessary ring-theoretical information not listed in the paper can be found in \cite{17}. The necessary information on Abelian groups not listed in the paper can be found in \cite{6} and \cite{8}.

If $R$ is a ring, then we denote by $C(R)$ and $J(R)$ the center and the Jacobson radical of the ring $R$, respectively. We use the additive form for Abelian groups. We denote by $\text{End} A$ the endomorphism ring of the Abelian group $A$. If $A = \bigoplus_{p \in P} A_p$ is a decomposition of the torsion Abelian group $A$ into

\textsuperscript{4}See 3.1 below.

\textsuperscript{5}see 3.1 below.
the direct sum of $p$-components, then $\text{supp } A = \{p \in P \mid A_p \neq 0\}$. We use the following notation: $\mathbb{Z}_{p^n}$ (resp., $\mathbb{Z}_{p^n}$) is the residue ring (resp., the additive group) modulo $p^n$; $\mathbb{Q}$ (resp., $\mathbb{Q}$) is the ring (resp., the additive group) of rational numbers; $\mathbb{Z}_{p^\infty}$ is a quasi-cyclic Abelian group; $\mathbb{Z}_p$ is the ring of $p$-adic integers. If $A$ is an Abelian torsion-free group, then $\mathbb{Q}\text{End } A$ and $\text{PSoc } A$ are the quasi-endomorphism ring and the pseudo-socle of the group $A$, respectively.

A ring $R$ is said to be local if the factor ring $R/J(R)$ is a division ring.

For a module $M$, the socle $\text{Soc } M$ is the sum of all simple submodules of $M$; if $M$ does not contain a simple submodule, then $\text{Soc } M = 0$ by definition.

An Abelian group $A$ is said to be divisible if $nA = A$ for any positive integer $n$. An Abelian group is said to be reduced if it does not contain a non-zero divisible subgroup and non-reduced, otherwise.

A subgroup $B$ of the Abelian group $A$ is said to be pure if the equation $nx = b \in B$, which has a solution in the group $A$, has a solution in $B$.

\section{Non-Reduced Abelian Groups with Centrally Essential Endomorphism Rings}

\textbf{2.1. Lemma.} Let $A$ be a module and $A = \bigoplus_{i \in I} A_i$ a direct decomposition of the module $A$. The endomorphism ring $\text{End } A$ is centrally essential if and only if for every $i \in I$ the following conditions hold.

1) $A_i$ is a fully invariant submodule in $A$;

2) the ring $\text{End } A_i$ is centrally essential.

\textbf{Proof.} Let $\text{End } A = E$ be a centrally essential ring. If condition 1) does not hold and $A_i$ is not a fully invariant submodule for some $i \in I$, then there exists a subscript $j \in I$, $j \neq i$, such that $\text{Hom } (A_i, A_j) = e_j E e_i \neq 0$, where $e_i$ and $e_j$ are the projections from the module $A$ onto the submodules $A_i$ and $A_j$, respectively. In addition,

$$e_i \cdot e_j E e_i = 0 \neq e_j E e_i = e_j E e_i \cdot e_i,$$

i.e., the idempotent $e_i$ is not central; this contradicts to \cite[Lemma 2.3]{[9]}.

If every $A_i$ is a fully invariant submodule in $A$, $i \in I$, then $\text{End } A \cong \text{End } A_i \times \text{End } \overline{A}_i$, where $\overline{A}_i$ is a complement direct summand of $A_i$. It is obvious that if $\text{End } A_i$ is not centrally essential ring, then and $\text{End } A$ is not a centrally essential ring.
If conditions 1) and 2) hold, then $\text{End} A \cong \prod_{i \in I} \text{End} A_i$ and each of the ring $\text{End} A_i$ is centrally essential. It is clear that the ring $\text{End} A$ is centrally essential, as well.

\[ \square \]

2.2. Lemma. The endomorphism ring of a divisible Abelian group $A$ is centrally essential if and only if either $A \cong Q$ or $A \cong \mathbb{Z}_{p^\infty}$.

\textbf{Proof.} Let $A = F(A) \bigoplus T(A)$, where $0 \neq F(A)$ is the torsion-free part and $0 \neq T(A)$ is the torsion part of the group $A$. Then $F(A)$ is not a fully invariant subgroup in $A$ (see [6, Theorem 7.2.3]) and, by Lemma 2.1, the ring $\text{End} A$ is not centrally essential. Hypothetically $F(A)$ or $T(A)$ is a direct sum of $\mathbb{Z}_{p^\infty}$ or $Q$. Clearly, if the number of terms is $> 1$, the ring $\text{End} A$ has a noncentral idempotent which gives a contradiction.

\[ \square \]

Let $A = \bigoplus_{p \in P} A_p$ be the decomposition of the torsion Abelian group $A$ into the direct sum of its primary components. It follows from Lemma 2.1 that $\text{End} A$ is a centrally essential ring if and only if each of the ring $\text{End} A_p$ is centrally essential.

\[ \square \]

2.3. Lemma. The endomorphism ring of a primary Abelian group $A_p$ is centrally essential if and only if $A_p \cong \mathbb{Z}_{p^k}$ or $A_p \cong \mathbb{Z}_{p^\infty}$.

\textbf{Proof.} If the group $A_p$ is not indecomposable, then it has a co-cyclic direct summand; see [6, Corollary 5.2.3]. By considering [6, Theorem 7.1.7, Example 1.3.2 and Theorem 7.2.3], this summand or summands complement to it are not fully invariant in $A$. Consequently, $A_p \cong \mathbb{Z}_{p^k}$ or $A_p \cong \mathbb{Z}_{p^\infty}$. The converse is obvious, since rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_p$ are commutative.

\[ \square \]

2.4. Theorem. Let $A = D(A) \bigoplus R(A)$ be a non-reduced Abelian group, where $0 \neq D(A)$ and $0 \neq R(A)$ are the divisible part and the reduced part of the group $A$, respectively. The endomorphism ring of the group $A$ is centrally essential if and only if $A = D(A) \bigoplus R(A)$, where $R(A) = \bigoplus_{p \in P'} \mathbb{Z}_{p^k}$ and $D(A) \cong Q$ or $D(A) \cong \bigoplus_{p \in P''} \mathbb{Z}_{p^\infty}$; $P'$, $P''$ are the subsets of different primes with $P' \cap P'' = \emptyset$.

\textbf{Proof.} Let $\text{End} A$ be a centrally essential ring. We verify that $D(A)$ and $R(A)$ are fully invariant subgroups in $A$. Indeed, it is well known that $\text{Hom} (D(A), R(A)) = 0$. Next, if $R(A)$ is a torsion-free group, then $\text{Hom} (R(A), D(A)) \neq 0$ (see [6, Theorem 7.2.3]); this contradicts to Lemma 2.1. It is also clear that $\text{Hom} (R(A), D(A)) \neq 0$ if $R(A)$, $D(A)$ are torsion groups and $\text{supp} R(A) \cap \text{supp} D(A) \neq \emptyset$. It follows from Lemma 2.3 that $R(A)$ is the direct sum of its cyclic $p$-components and it follows from Lemma 2.2 that $D(A) \cong Q$ or $D(A) \cong \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$.

The converse assertion directly follows from Lemmas 2.1, 2.2 and 2.3.  

\[ \square \]
2.5. Corollary. The endomorphism ring of a non-reduced Abelian group is centrally essential if and only if the ring is commutative.

Proof. Indeed, it follows from Theorem 2.4 that any centrally essential endomorphism ring of a non-reduced Abelian group is the direct product of rings whose components can be only the rings $\mathbb{Z}_{p^k}, \mathbb{Q}$ and $\hat{\mathbb{Z}}_p$. \qed

It follows from Corollary 2.5 that only reduced Abelian groups can have non-commutative centrally essential endomorphism rings.

3 Proof of Theorem 1.2

3.1. Quasi-decompositions and strongly indecomposable groups.
Let $A$ and $B$ be two Abelian torsion-free groups. One says that $A$ is quasi-contained in $B$ if $nA \subseteq B$ for some positive integer $n$. If $A$ is quasi-contained in $B$ and $B$ is quasi-contained in $A$ (i.e., if $nA \subseteq B$ and $mB \subseteq A$ for some $n, m \in \mathbb{N}$), then one says that $A$ is quasi-equal to $B$ (we write $A \sim B$). A quasi-equality $A \sim \bigoplus_{i \in I} A_i$ is called a quasi-decomposition (or a quasi-direct decomposition) of the Abelian group $A$; these subgroups $A_i$ are called quasi-summands of the group $A$. If the group $A$ does not have non-trivial quasi-decompositions, then $A$ is said to be strongly indecomposable. A ring $Q \otimes \text{End} A$ is called the quasi-endomorphism ring of the group $A$; we denote it by $Q \text{End} A$; see details in [8, Chapter I, §5]. We note that

$$Q \text{End} A = \{ \alpha \in \text{End}_Q(Q \otimes A) \mid (\exists n \in \mathbb{N})(na \in \text{End} A) \}.$$  

It is well known (e.g., see [8, Proposition 5.2]) that the correspondence

$$A \sim e_1 A \bigoplus \ldots \bigoplus e_k A \rightarrow Q \text{End} A = Q \text{End} Ae_1 \bigoplus \ldots \bigoplus Q \text{End} Ae_k$$

between finite quasi-decompositions of the torsion-free group $A$ and finite decompositions of the ring $Q \text{End} A$ in to a direct sum of left ideals, where $\{e_i \mid i = 1, \ldots, k\}$ is a complete orthogonal system of idempotents of the ring $Q \text{End} A$, is one-to-one.

3.2. Proposition. The endomorphism ring $E$ of an Abelian torsion-free group $A$ is centrally essential if and only if the quasi-endomorphism ring $QE$ of $A$ is centrally essential.

Proof. Let $0 \neq \hat{a} \in QE$. For some $n \in \mathbb{N}$, we have $n\hat{a} = a \in E$ and there exist $x, y \in C(E)$ with $ax = y \neq 0$. In this case, $\hat{a}\hat{x} = \hat{y}$, where $\hat{x} = x$, $\hat{y} = \frac{1}{n} \cdot y \in C(QE)$, i.e., $QE$ is a centrally essential ring.

\hspace{1cm} \text{c.f. [8, Proposition 2.2]}
Conversely, for every $0 \neq a \in E$, there exist non-zero $\tilde{x}, \tilde{y} \in C(\mathbb{Q}E)$ with $a\tilde{x} = \tilde{y}$. In addition, there exist $n, m \in \mathbb{N}$ such that $n\tilde{x} \in C(E)$ and $m\tilde{y} \in C(E)$. Then $ax = y$, where $x = mn\tilde{x}, y = mn\tilde{y} \in C(E)$. □

Let $A = \bigoplus_{i=1}^{n} A_i = A'$ be a decomposition of the Abelian torsion-free group $A$ of finite rank into a quasi-direct sum of strongly indecomposable groups (e.g., see [8, Theorem 5.5]). By considering Lemma 2.1 and Proposition 3.2, we obtain that the ring $\text{End } A$ is centrally essential if and only if all subgroups $A_i$ are fully invariant in $A'$, and every ring $\text{End } A_i$ is centrally essential. Therefore, the problem of describing Abelian torsion-free groups of finite rank with centrally essential endomorphism rings is reduced to the similar problem for strongly indecomposable groups.

3.3. Proposition. Let $A$ be a strongly indecomposable Abelian group and $A = \text{PSoc } A$. The ring $\text{End } A$ is centrally essential if and only if $\text{End } A$ is a commutative ring.

Proof. If $A = \text{PSoc } A$, then $\text{End } A$ is a semiprime ring (e.g., see [8, Theorem 5.11]). It follows from [11, Proposition 3.3] that the ring $\text{End } A$ is commutative. The converse is obvious. □

3.4. Proposition. Let $R$ be a local Artinian ring which is not a division ring and $C(R) = C$.

1. If $R$ is a centrally essential ring, then the ring $R/J(R)$ is commutative and $C \cap M \neq 0$ for every minimal right ideal $M$.

2. If the ring $R/J(R)$ is commutative, $\text{Soc } (R_C) = \text{Soc } (R_R)$ and $C \cap M \neq 0$ for every minimal right ideal $M$, then $R$ is a centrally essential ring.

Proof. 1. It is well known that if $R$ is an Artinian ring, then $J(R)$ is a nilpotent ideal of some index $k$. We note that if $M$ is a minimal right ideal of $R$, then $MJ(R) = 0$. Indeed, if $MJ(R) = M$, then $$M = MJ(R) = MJ^2(R) = \ldots = MJ^k(R) = 0.$$ Let $R$ be a centrally essential ring. It follows from [10, Theorem 2] that the ring $R/J(R)$ is commutative.

We assume that $C \cap M = 0$ for some minimal right ideal $M$. By assumption, for $0 \neq a \in M$, there exist $x, y \in C(R)$ with $ax = y \neq 0$. Since $x \notin J(R)$ (otherwise, $ax = 0$), we have that $x$ is invertible in $R$ and $a = x^{-1}y \in C$, a contradiction.

2. Let $C \cap M \neq 0$ for every minimal right ideal $M$ of $R$. We verify that $M \subseteq C$. Let $C \cap M = K$. By assumption, the ring $R/J(R)$ is commutative; therefore, we have $rs - sr \in J(R)$, for all $r, s \in R$. For every $k \in K$,
we have \( k(rs - sr) = 0 \). On the other hand, since \( k \in C \), we have that \((kr)s = ksr = s(kr)\) and \( kr \in C \). In addition, \( kr \in M \). Consequently, \( K \) is a right ideal. From the property that \( M \) is a minimal right ideal, we have that \( K = M \) or \( K = 0 \). However \( K \neq 0 \); therefore, \( K = M \) and \( M \subset C \). Therefore, \( \text{Soc}(R_C) = \text{Soc}(R_R) \subseteq C \). It follows from [10, Theorem 3] that \( R \) is a centrally essential ring. □

3.5. Example. We will find centrally essential endomorphism rings of strongly indecomposable Abelian torsion-free groups of rank 2 and 3.

If \( A \) is an strongly indecomposable group of rank 2, then the ring \( \text{End} A \) is commutative (e.g., see [1, Theorem 4.4.2]). Consequently, \( \text{End} A \) is a centrally essential ring. Let \( A \) be a strongly indecomposable group of rank 3. Then the algebra \( \mathbb{Q}\text{End} A \) is isomorphic to one of the following \( \mathbb{Q} \)-algebras ([1, Theorem 2]):

- \( K \cong \left\{ \begin{pmatrix} x & 0 & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x, z \in \mathbb{Q} \right\} \),
- \( R \cong \left\{ \begin{pmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x, y, z \in \mathbb{Q} \right\} \),
- \( S \cong \left\{ \begin{pmatrix} x & y & z \\ 0 & x & ky \\ 0 & 0 & x \end{pmatrix} \mid x, y, z \in \mathbb{Q}, 0 \neq k \in \mathbb{Q}, k = \text{const} \right\} \),
- \( T \cong \left\{ \begin{pmatrix} x & y & z \\ 0 & x & t \\ 0 & 0 & x \end{pmatrix} \mid x, y, z, t \in \mathbb{Q} \right\} \).

The rings \( K, R, S \) are commutative; consequently, they are centrally essential. The ring \( T \) is not commutative (in addition, \( \text{PSoc} A \) is of rank 1). We have

- \( J(T) = \left\{ \begin{pmatrix} 0 & y & z \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \mid y, z, t \in \mathbb{Q} \right\} \),
- \( C(T) = \left\{ \begin{pmatrix} x & 0 & z \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x, z \in \mathbb{Q} \right\} \),
- \( M = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \mid t \in \mathbb{Q} \right\} \),

where \( M \) is the minimal right ideal of \( T \). We note that the ring \( T/J(T) \) is commutative, but \( C(T) \cap M = 0 \). It follows from Proposition 3.4(1) that the
We denote by \( R \) the endomorphism ring of an Abelian torsion-free group of rank \( n \). It is known (e.g., see [10], Proposition 2.5.) that every \( \mathbb{Q} \)-algebra of dimension \( n \) can be realized as the quasi-endomorphism ring of an Abelian torsion-free group of rank \( n \). Therefore,

3.6. Example. Let \( V \) be a vector \( \mathbb{Q} \)-space with basis \( e_1, e_2, e_3 \) and let \( \Lambda(V) \) be the Grassmann algebra of the space \( V \); i.e., \( \Lambda(V) \) is an algebra with operation \( \wedge \), generators \( e_1, e_2, e_3 \) and defining relations

\[
e_i \wedge e_j + e_j \wedge e_i = 0 \quad \text{for all} \quad i, j = 1, 2, 3.
\]

Then \( \Lambda(V) \) is a \( \mathbb{Q} \)-algebra of dimension 8 with basis \( \{1, e_1, e_2, e_3, e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3, e_1 \wedge e_2 \wedge e_3\} \) and \( \Lambda(V) \) is a non-commutative centrally essential ring (see details in [10, Example 1]). We consider the regular representation of the algebra \( \Lambda(V) \). If \( x \in \Lambda(V) \),

\[
x = q_0 \cdot 1 + q_1 e_1 + q_2 e_2 + q_3 e_3 + q_1 e_1 \wedge e_2 + q_5 e_2 \wedge e_3 + q_6 e_1 \wedge e_3 + q_7 e_1 \wedge e_2 \wedge e_3,
\]

then the matrix \( A_x \in \text{Mat}_8(\mathbb{Q}) \) has the form

\[
\begin{pmatrix}
q_0 & q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 \\
0 & q_0 & 0 & 0 & -q_2 & 0 & -q_3 & q_5 \\
0 & 0 & q_0 & 0 & q_1 & -q_3 & 0 & -q_6 \\
0 & 0 & 0 & q_0 & 0 & q_2 & q_1 & q_4 \\
0 & 0 & 0 & 0 & q_0 & 0 & 0 & q_3 \\
0 & 0 & 0 & 0 & 0 & q_0 & 0 & q_1 \\
0 & 0 & 0 & 0 & 0 & 0 & q_0 & -q_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_0
\end{pmatrix}.
\]

We denote by \( R \) the corresponding subalgebra in \( \text{Mat}_8(\mathbb{Q}) \). It is clear that the radical \( J(R) \) consists of properly upper triangular matrices in \( R \) and \( A_x \in C(R) \) if and only if \( q_1 = q_2 = q_3 = 0 \). In addition, \( \text{Soc}(R_R) = \{A_x = (a_{ij}) \in R \mid a_{ij} = 0, i \neq 1, j \neq 8\} \) and \( \text{Soc}(R_C) = \{A_x = (a_{ij}) \in C(R) \mid a_{ii} = 0\} \). Since \( \text{Soc}(R_C) \neq \text{Soc}(R_R) \), the corresponding condition of Proposition 3.4(2) is not necessary. Therefore, we obtain the negative answer to [10, Open questions 4.5(2)].

3.7. The completion of the proof of Theorem 1.2.
1, 2. It is known that the ring \( \mathbb{Q}\text{End} A \) is a local Artinian ring (e.g., see [3, Corollary 5.3]). It remains to use Proposition 3.4.

3. In [10, Proposition 2.5.], it is proved that the Grassmann algebra \( \Lambda(V) \) over a field \( F \) of characteristic 0 or \( p \neq 2 \) is a centrally essential ring if and only if the dimension of the space \( V \) is odd. We set \( F = \mathbb{Q} \). It is known (e.g., see [10]) that every \( \mathbb{Q} \)-algebra of dimension \( n \) can be realized as the quasi-endomorphism ring of an Abelian torsion-free group of rank \( n \). Therefore,
by considering Example 3.6 and Proposition 3.2, we obtain the required property.

Under conditions of Theorem 1.2, if the rank of the group $A$ is square-free, then the ring $\mathbb{Q}\text{End}A/J(\mathbb{Q}\text{End}A)$ is commutative [1, Lemma 4.2.1].

By considering Proposition 3.2, we obtain

3.8. Corollary. Let $A$ be a strongly indecomposable Abelian torsion-free group of finite rank, $A \neq \text{PSoc} A$ and the rank of the group $A$ is square-free.

1. If the endomorphism ring $\text{End} A$ of the group $A$ is centrally essential, then $C(\mathbb{Q}\text{End} A) \cap M \neq 0$ for every minimal right ideal $M$ of $\mathbb{Q}\text{End} A$.

2. If $\text{Soc}(\mathbb{Q}\text{End} A) = \text{Soc}(\mathbb{Q}\text{End} AC(\mathbb{Q}\text{End} A))$ and $C(\mathbb{Q}\text{End} A) \cap M \neq 0$ for every minimal right ideal $M$ of $\mathbb{Q}\text{End} A$, then $\text{End} A$ is a centrally essential ring.

3.9. Example. Let $R = \mathbb{Z}[x, y]$ be the polynomial ring in two variables $x$ and $y$. We use the construction described in [7, Proposition 7]. We consider the ring

$$ T(R) = \left\{ \begin{pmatrix} f & d_1(f) & g \\ 0 & f & d_2(f) \\ 0 & 0 & f \end{pmatrix} \mid f, g \in \mathbb{Z}[x, y] \right\}, $$

where $d_1, d_2$ are two derivations of the ring $\mathbb{Z}[x, y]$, $d_1 = \frac{\partial}{\partial x}$, $d_2 = \frac{\partial}{\partial y}$. Then $T(R)$ is a non-commutative ring with $J(R) = e_{13}R \subseteq C(T(R))$, where $e_{13}$ is the matrix unit; see [7, Corollary 8]. If $0 \neq a \in T(R) \setminus C(T(R))$, then $0 \neq ae_{13} \in C(T(R))$. Therefore, $T(R)$ is a centrally essential ring. Since $T(R)$ is a countable ring with reduced torsion-free additive group, it follows from the familiar Corner theorem (e.g., see [8, Theorem 29.2]) that for the ring $T(R)$, there exist $\mathfrak{M}$ of Abelian groups $A_i$ such that $\text{End} A_i \cong T(R)$ and $\text{Hom}(A_i, A_j) = 0$ for all $i \neq j$, where $\mathfrak{M}$ is an arbitrary preset cardinal number; see [3], [2]. We note that the endomorphism ring of the direct sum of such groups is a non-commutative centrally essential ring, as well.

4 Remarks and Open Questions

4.1. Open question. Is it true that there exist strongly indecomposable Abelian torsion-free groups of rank $< 8$ whose endomorphism rings are non-commutative centrally essential rings?

4.2. Open question. An Abelian group is said to be super-decomposable if it does not have non-zero indecomposable direct summands. Is it true that there
exist a super-decomposable Abelian group with non-commutative centrally essential endomorphism ring?

4.3. **Open question.** Is it true that the endomorphism ring of the direct sum of all the groups $A(n)$ from Theorem 1.2(3) is a non-commutative centrally essential ring with polynomial identity?

4.4. **Open question.** Is it true that there exists an Abelian group $A$ with centrally essential endomorphism ring $\text{End} A$ which is not a ring with polynomial identity?

4.5. A ring is said to be **right distributive** (resp., **right uniserial**) if the lattice its right ideals is distributive (resp., is a chain). If the endomorphism ring $\text{End} A$ of the group $A$ of finite rank is right uniserial, then $\text{End} A$ is an invariant principal right ideal domain; see [5, Proposition 3.4]. Therefore, if $A$ is an Abelian torsion-free group of finite rank and $\text{End} A$ is a centrally essential right uniserial ring, then $\text{End} A$ is commutative ([11, Proposition 2.8]). In addition, it is known that every right distributive local ring is right uniserial. Consequently, every centrally essential right distributive quasi-endomorphism ring of an Abelian torsion-free group of finite rank is commutative. We note that there exist non-commutative uniserial Artinian centrally essential rings; see [13].

In connection to 4.5, we formulate open questions 4.6 and 4.7.

4.6. **Open question.** Is it true that there exist Abelian torsion-free groups of finite rank whose endomorphism rings are non-commutative right distributive centrally essential rings?

4.7. **Open question.** Is it true that there exist Abelian groups whose endomorphism rings are non-commutative right distributive (or right uniserial) centrally essential rings?

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