THE BIFURCATIONS OF SOLITARY AND KINK WAVES DESCRIBED BY THE GARDNER EQUATION

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ABSTRACT. In this paper, we investigate the bifurcations of nonlinear waves described by the Gardner equation \( u_t + auu_x + bu^2u_x + \gamma u_{xxx} = 0 \). We obtain some new results as follows: For arbitrary given parameters \( b \) and \( \gamma \), we choose the parameter \( a \) as bifurcation parameter. Through the phase analysis and explicit expressions of some nonlinear waves, we reveal two kinds of important bifurcation phenomena. The first phenomenon is that the solitary waves with fractional expressions can be bifurcated from three types of nonlinear waves which are solitary waves with hyperbolic expression and two types of periodic waves with elliptic waves with trigonometric expression respectively. The second phenomenon is that the kink waves can be bifurcated from the solitary waves and the singular waves.

1. Introduction. It is well known that some phenomena in physics and engineering can be described by nonlinear partial differential equations. When we try to study the physical mechanism of the phenomena described by nonlinear partial differential equations, nonlinear wave solutions and their bifurcations often are investigated. Through study of bifurcation for an equation containing some parameters, we may know some varying process of the solutions when the parameters vary. Thus the study of bifurcation may give a good insight into the physical aspects of the problems.

In this paper, we are interested at the Gardner equation

\[
  u_t + auu_x + bu^2u_x + \gamma u_{xxx} = 0, \tag{1}
\]

where \( u(x, t) \) is the amplitude of the relevant wave mode, the dispersion coefficient \( \gamma \) is always positive, but the nonlinear coefficients \( a \) and \( b \) are positive or negative [7]. Eq. (1) is widely applied in various branches of physics, such as fluid physics and plasma physics, and attract many scholars’ attention [20], [21] and [9]. This equation arose as an auxiliary mathematical equation in the derivation of the infinite set of conservation laws of the KdV equation [15] and [16]. In [7], the authors pointed out the relationship between Eq. (1) and the mKdV equation \( v_t + bv^2v_y + \gamma v_{yyy} = 0 \) on the basis of an invertible transformation \( v = u + a/2b, y = x + a^2t/2b^2 \). Furthermore, by means of the Mirura transformation the mKdV equation can be reduced to the KdV equation [17]. The exact solutions of Eq. (1) have been investigated by many
researchers and some of powerful methods have been presented, such as the series expansion method [4], the mapping method [13], the Hirota methods [22], and so on [23], [14], [10], [6], [5] and [8]. The authors in [1] and [18] acquired some exact solution under the special parameters. However, the bifurcations of the nonlinear waves for Eq. (1) has been few discussed and understood. Hence it is the main investigation of our paper by using phase analysis which was used in some lectures, for instance [12], [19], [11], [2] and [3].

In this paper, for arbitrary given parameters \( b \) and \( \gamma \), choosing \( a \) as bifurcation parameter, we show that in Eq. (1) there exist two kinds of important bifurcation phenomena. One of the bifurcation phenomena is that the solitary waves with fractional expressions can be bifurcated from three types of nonlinear waves which are solitary waves with hyperbolic expression and two types of periodic waves with trigonometric expression and elliptic expression respectively. Another phenomenon is that the kink waves can be bifurcated from the solitary waves and the singular waves.

This paper is organized as follows. In Sec. 2, our main results are stated in a proposition. In Sec. 3, we give proof to our main results. A short conclusion is put in Sec. 4.

2. Main results. In this section, we state our main results. For given parameter \( b \) and constant \( c \), let

\[
\xi = x - ct,
\]
and

\[
\begin{align*}
    a_1 &= \sqrt{\frac{6}{|bc|}}, \\
    a_2 &= \sqrt{\frac{4}{3} |bc|}, \\
    a_3 &= -\sqrt{\frac{4}{3} |bc|}, \\
    a_4 &= -\sqrt{\frac{6}{|bc|}}.
\end{align*}
\]

(3)

Obviously, \( a_1, a_2, a_3 \) and \( a_4 \) satisfy inequality

\[
a_4 < a_3 < 0 < a_2 < a_1.
\]

In the following proposition, we will show that \( a_2, a_3 \) are two bifurcation parametric values for solitary wave bifurcation, and \( a_1, a_4 \) are two bifurcation parametric values for kink wave bifurcation.

**Proposition 1.** For given parameters \( b, \gamma \) (\( \gamma > 0 \)) and constant \( c \), let \( \xi \) and \( a_i \) (\( i = 1, 2, 3, 4 \)) be in (2) and (3). Choosing \( a \) as bifurcation parameter, then in Eq. (1), there exist solitary wave and kink wave bifurcations as follow:

**A. If** \( b > 0 \) **and** \( c < 0 \), **then there is the following solitary wave bifurcation.**

(\( A_1 \)) When \( a_2 < a < a_1 \) and \( a \to a_2 + 0 \), the peak solitary wave with fractional expression

\[
u_1(\xi) = \sqrt{\frac{3}{b}} \frac{c \gamma + 2 c \xi^2}{9 \gamma - 2 c \xi^2}.
\]

(5)

\( \) can be bifurcated from the following three types of nonlinear waves:

(\( A_1 \)) \( a \) Solitary wave with hyperbolic expression

\[
u_1(\xi) = \varphi_1 + \frac{2 \alpha_1}{\sqrt{|\Delta_3|} \cosh(\sqrt{|\alpha_1|} \xi) - \beta_1}.
\]

(6)
where

\[ \varphi_1 = -\frac{3a + \sqrt{9a^2 + 48bc}}{4b}, \quad (7) \]

\[ \alpha_1 = \frac{b}{6\gamma}(\varphi_1 - \varphi_1^+)(\varphi_1^- - \varphi_1), \quad (8) \]

\[ \beta_1 = -\frac{b}{3\gamma}\varphi_1 + \frac{b}{6\gamma}(\varphi_1^+ + \varphi_1^-), \quad (9) \]

\[ \Delta_1 = \beta_1^2 - \frac{2b}{3\gamma}\alpha_1, \quad (10) \]

\[ \varphi_1^+ = -\frac{1}{2b}\left(\frac{a}{2} + 3\sqrt{\frac{a^2}{4} + \frac{4bc}{3}} + \sqrt{a\left(a + \sqrt{9a^2 + 48bc}\right)}\right), \quad (11) \]

\[ \varphi_1^- = -\frac{1}{2b}\left(\frac{a}{2} + 3\sqrt{\frac{a^2}{4} + \frac{4bc}{3}} - \sqrt{a\left(a + \sqrt{9a^2 + 48bc}\right)}\right). \quad (12) \]

\[ (A_1)_b \text{ Periodic wave with trigonometric expression} \]

\[ u_2^2(\xi) = \varphi_2 + \frac{2\alpha_2}{\sqrt{\Delta_2}} \cos\left(\sqrt{\frac{1}{\alpha_2}} \xi - \beta_2\right), \quad (13) \]

where

\[ \varphi_2 = -\frac{3a + \sqrt{9a^2 + 48bc}}{4b}, \quad (14) \]

\[ \alpha_2 = \frac{b}{6\gamma}(\varphi_2 - \varphi_2^+)(\varphi_2^- - \varphi_2), \quad (15) \]

\[ \beta_2 = -\frac{b}{3\gamma}\varphi_2 + \frac{b}{6\gamma}(\varphi_2^+ + \varphi_2^-), \quad (16) \]

\[ \Delta_2 = \beta_2^2 - \frac{2b}{3\gamma}\alpha_2, \quad (17) \]

\[ \varphi_2^+ = -\frac{1}{2b}\left(\frac{a}{2} + 3\sqrt{\frac{a^2}{4} + \frac{4bc}{3}} + \sqrt{a\left(a - \sqrt{9a^2 + 48bc}\right)}\right), \quad (18) \]

\[ \varphi_2^- = -\frac{1}{2b}\left(\frac{a}{2} + 3\sqrt{\frac{a^2}{4} + \frac{4bc}{3}} - \sqrt{a\left(a - \sqrt{9a^2 + 48bc}\right)}\right). \quad (19) \]

\[ (A_1)_c \text{ Periodic wave with elliptic expression} \]

\[ u_3^2(\xi) = \frac{\mu_3(\mu_2 - \mu) + \mu(\mu_3 - \mu_2)\text{sn}^2(\eta_1\xi, k_1)}{(\mu_2 - \mu) + (\mu_3 - \mu_2)\text{sn}^2(\eta_1\xi, k_1)}, \quad (20) \]

where

\[ \mu \in (\varphi_1^+, \varphi_1^-), \quad (21) \]

\[ \eta_1 = \sqrt{\frac{b}{24\gamma}(\mu_3 - \mu_1)(\mu_2 - \mu)}, \quad (22) \]

\[ k_1 = \sqrt{\frac{(\mu_3 - \mu_2)(\mu_1 - \mu)}{(\mu_3 - \mu_1)(\mu_2 - \mu)}}, \quad (23) \]
\[\mu_1 = -\frac{2a + b\mu}{3b} - \left(\frac{\omega}{6b}\right) \left(\frac{2}{\theta_1}\right) \left(1 + i\sqrt{3}\right) - \frac{1}{6b} \left(\frac{\theta_4}{2}\right) \left(1 - i\sqrt{3}\right),\]  
(24)

\[\mu_2 = \frac{1}{2b} \left[-2a - b(\mu_1 + \mu) - \vartheta\right],\]  
(25)

\[\mu_3 = \frac{1}{2b} \left[-2a - b(\mu_1 + \mu) + \vartheta\right],\]  
(26)

\[\vartheta = \sqrt{\left|4a^2 + 24bc - 4ab\mu_1 - 3b^2\mu_1^2 - 4ab\mu - 2b^2\mu_1 - 3b^2\mu^2\right|},\]  
(27)

\[\omega = 4a^2 + 18bc - 2ab\mu - 2b^2\mu^2,\]  
(28)

\[\theta_1 = -16a^3 - 108abc + 12a^2b\mu - 108b^2\mu_1 - 30ab^2\mu - 20b^3\mu^3 + \sqrt{|\theta_2|},\]  
(29)

\[\theta_2 = 4 \left(-4a^2 - 18bc + 2ab\mu + 2b^2\mu^2\right)^3 + \theta_3^2,\]  
(30)

\[\theta_3 = 16a^3 + 108abc - 12a^2b\mu - 108b^2\mu_1 + 30ab^2\mu - 20b^3\mu^3.\]  
(31)

For the varying processes of \(u_2^\omega(\xi)\) and \(u_2^\phi(\xi)\), see Fig. 1.

\((A_2)\) When \(0 < a < a_2\) and \(a \to a_2 - 0\), the peak solitary wave with fractional expression \(u_1(\xi)\) (see (5)) can be bifurcated from only one type of periodic wave which has elliptic expression

\[u_4^\phi(\xi) = \frac{\alpha_3 \varphi_{01} + \varphi_{02}^\phi \varphi_{03} + (\alpha_3 \varphi_{01} - \varphi_{02}^\phi \varphi_{03}) \text{cn}(\eta_2 \xi, k_2)}{\alpha_3 + \beta_3 + (\alpha_3 - \beta_3) \text{sn}(\eta_2 \xi, k_2)},\]  
(32)

where

\[\Omega = a^4 + 5 \frac{a^2bc}{2} - 8b^2c^2,\]  
(33)

\[\lambda = \frac{5}{1152} a^8 b^2 c^2 + \frac{85}{1728} a^6 b^3 c^3 + \frac{7}{72} a^4 b^4 c^4 + \frac{32}{27} b^6 c^6,\]  
(34)

\[\delta = \left(a^6 + \frac{15}{4} a^4 b c + 3 a^2 b^2 c^2 + 80 b^3 c^3 - 54 \sqrt{2|\lambda|}\right)^\frac{3}{2},\]  
(35)

\[\varphi_1^\phi = \frac{4c}{a},\]  
(36)

\[\varphi_{01} = -\frac{2}{3ab} \left(a^2 + 2bc + \frac{\Omega}{\delta} + \delta\right),\]  
(37)

\[\varphi_{02} = -\frac{2}{3ab} \left(a^2 + 2bc - \frac{\Omega}{2\delta} - \frac{\delta}{2}\right),\]  
(38)

\[\varphi_{03} = -\frac{2\sqrt{3}}{3ab} \left(\frac{\Omega}{2\delta} - \frac{\delta}{2}\right),\]  
(39)

\[\alpha_3 = \sqrt{|(\varphi_{01}^\phi - \varphi_{02})^2 + \varphi_{03}^2|},\]  
(40)

\[\beta_3 = \sqrt{|(\varphi_{01} - \varphi_{02})^2 + \varphi_{03}^2|},\]  
(41)

\[\eta_2 = \sqrt{\frac{b}{6\gamma} \alpha_3 \beta_3},\]  
(42)

\[k_2 = \sqrt{\frac{(\varphi_{01}^\phi - \varphi_{01})^2 - (\alpha_3 - \beta_3)^2}{4\alpha_3 \beta_3}}.\]  
(43)
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\[(a) a = a_2 + 10^{-1} \quad (b) a = a_2 + 10^{-2} \quad (c) a = a_2 + 10^{-5} \]
\[(d) a = a_2 + 10^{-1} \quad (e) a = a_2 + 10^{-2} \quad (f) a = a_2 + 10^{-5} \]

Fig. 1. When \( a \to a_2 + 0 \), the varying process for the figures of \( u_2^2(\xi) \) is in (a), (b), (c) and the one of \( u_3^2(\xi) \) is in (d), (e), (f), where \( c = -1, b = 27 \) and \( \gamma = 2 \).

For the varying process of \( u_4^2(\xi) \), see Fig. 2.

(A) When \( a_3 < a < 0 \) and \( a \to a_3 + 0 \), the valley solitary wave with fractional expression

\[ u_2^2(\xi) = \sqrt{\frac{3\,|c|}{b}} \left( 3\gamma + 2\,c\xi^2 \right) \] (44)

can be bifurcated from the periodic wave with elliptic expression \( u_4^2(\xi) \) (see (32)). For the varying process of \( u_3^2(\xi) \), see Fig. 3.

(A) When \( a_4 < a < a_3 \) and \( a \to a_3 - 0 \), the valley solitary wave with fractional expression \( u_2^2(\xi) \) (see (44)) can be bifurcated from following three types of nonlinear waves.

(A)\( a \) Solitary wave with hyperbolic expression

\[ u_5^2(\xi) = \varphi_2 - \frac{2\alpha_2}{\sqrt{\Delta_2} \cosh(\sqrt{\alpha_2}\xi)} + \beta_2 \] (45)

where \( \varphi_2, \alpha_2, \beta_2 \) and \( \Delta_2 \) are in (14)-(17).

(A)\( b \) Periodic wave with trigonometric expression

\[ u_6^0(\xi) = \varphi_1 - \frac{2\alpha_1}{\sqrt{|\Delta_1|} \cos(\sqrt{|\alpha_1|}\xi)} + \beta_1 \] (46)

where \( \varphi_1, \alpha_1, \beta_1 \) and \( \Delta_1 \) are in (7)-(10).

(A)\( c \) Periodic wave with elliptic expression

\[ u_7^2(\xi) = \frac{\tau_3(\tau - \tau_2) + \tau(\tau_2 - \tau_3)\text{sn}^2(\eta_3\xi, k_3)}{(\tau - \tau_2) + (\tau_2 - \tau_3)\text{sn}^2(\eta_3\xi, k_3)} \] (47)

where \( \tau_2, \tau_3, \tau_4 \) are in (18)-(21),
where
\[ \tau \in (\varphi_1, \varphi_3^+), \]
\[ \eta_3 = \sqrt{\frac{b}{24\gamma}}(\tau - \tau_2)(\tau_1 - \tau_3), \quad (48) \]
\[ k_3 = \sqrt{\frac{(\tau - \tau_1)(\tau_2 - \tau_3)}{(\tau - \tau_2)(\tau_1 - \tau_3)}}, \quad (50) \]
\[ \tau_1 = \frac{1}{2b}[-2a - b(\tau + \tau_3) + \vartheta_1], \quad (51) \]
\[ \tau_2 = \frac{1}{2b}[-2a - b(\tau + \tau_3) - \vartheta_1], \quad (52) \]
\[ \tau_3 = -\frac{2a + b\tau}{3b} - \left(\frac{\omega_1}{6b}\right) \left(\frac{2}{\sigma_1}\right) \left(1 + i\sqrt{3}\right) - \frac{1}{6b} \left(\frac{\sigma_1}{2}\right) \left(1 - i\sqrt{3}\right), \quad (53) \]
\[ \vartheta_1 = \sqrt{\left|4a^2 + 24bc - 4a\beta_2 - 3b^2\gamma \sqrt{3} - 4ab\tau - 2b^2\tau - 3b^2\tau^2\right|}, \quad (54) \]
\[ \omega_1 = 4a^2 + 18bc - 2ab\tau - 2b^2\tau^2, \quad (55) \]
\[ \sigma_1 = -16a^3 - 108abc + 12a^2b\tau + 108b^2c\tau - 30ab^2\tau^2 - 20b^3\tau^3 + \sqrt{\sigma_2}, \quad (56) \]
\[ \sigma_2 = 4\left(-4a^2 - 18bc + 2ab\tau + 2b^2\tau^2\right)^3 + \sigma_3, \quad (57) \]
\[ \sigma_3 = 16a^3 + 108abc - 12a^2b\tau - 108b^2c\tau + 30ab^2\tau^2 + 20b^3\tau^3. \quad (58) \]

For the varying processes of \( u_6^*(\xi) \) and \( u_7^*(\xi) \), see Fig. 4.

![Graphs showing varying processes of \( u_6^*(\xi) \) and \( u_7^*(\xi) \)](image-url)

**Fig. 2.** When \( a \to a_2 - 0 \), the varying process of \( u_6^*(\xi) \) where \( c = -1, b = 1 \) and \( \gamma = 2 \).

![Graphs showing varying processes of \( u_6^*(\xi) \) and \( u_7^*(\xi) \)](image-url)

**Fig. 3.** When \( a \to a_3 + 0 \), the varying process of \( u_6^*(\xi) \) where \( c = -1, b = 3 \) and \( \gamma = 2 \).
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\[ a = a_3 - 10^{-1}, \quad \text{Fig. 4.} \]

When \( a \to a_3 - 0 \), the varying process for the figures of \( u^0_\alpha(\xi) \) is in (a), (b), (c) and the one of \( u^\xi_\alpha(\xi) \) is in (d), (e), (f), where \( c = -1, b = 27 \) and \( \gamma = 3 \).

**B. If** \( b < 0 \) **and** \( c > 0 \), **then there is the following kink wave bifurcation.**

\((B_1)\) When \( a > a_1 \) **and** \( a \to a_1 + 0 \), **two kink waves** with the expressions

\[ u_{3+}^\pm(\xi) = \sqrt{\frac{3c}{2|b|}} \left[ 1 \pm \tanh \left( \sqrt{\frac{c}{4\gamma}} \xi \right) \right], \quad (59) \]

**can be bifurcated from the following two types of nonlinear waves:**

\((B_1)_a\) **Peak solitary waves** with the expressions

\[ u_1^*(\xi) = \frac{12\sqrt{6c}\lambda_1 e^{\sqrt{\xi}}}{2 + 2\sqrt{6a}\lambda_1 e^{\sqrt{\xi}} + (3a^2 + 18bc)\lambda_1^2 e^{2\sqrt{\xi}}}, \quad (60) \]

and

\[ u_5^*(\xi) = \frac{12\sqrt{6c}\lambda_1 e^{-\sqrt{\xi}}}{2 + 2\sqrt{6a}\lambda_1 e^{-\sqrt{\xi}} + (3a^2 + 18bc)\lambda_1^2 e^{-2\sqrt{\xi}}}, \quad (61) \]

where

\[ \varpi_1 = 6c - 2a\varphi_1 - b\varphi_1^2, \quad (62) \]

\[ \lambda_1 = \frac{2\varphi_1}{6\sqrt{6c} - \sqrt{6a\varphi_1} + \sqrt{|c\varpi_1|}}. \quad (63) \]

\((B_1)_b\) **Singular wave solutions** with the expressions

\[ u_3^*(\xi) = \frac{\varphi_2 + (2B_0\varphi_2 - 4A_0)\lambda_2 e^{2\varpi_3} + (B_0^2 - 4A_0)\varphi_2 \lambda_2^2 e^{2\varpi_3}}{1 + 2B_0\lambda_2 e^{2\varpi_3} + (B_0^2 - 4A_0)\lambda_2^2 e^{2\varpi_3}}, \quad (64) \]

and

\[ u_4^*(\xi) = \frac{\varphi_2 + (2B_0\varphi_2 - 4A_0)\lambda_2 e^{-2\varpi_3} + (B_0^2 - 4A_0)\varphi_2 \lambda_2^2 e^{-2\varpi_3}}{1 + 2B_0\lambda_2 e^{-2\varpi_3} + (B_0^2 - 4A_0)\lambda_2^2 e^{-2\varpi_3}}, \quad (65) \]
where
\[ \eta_4 = \sqrt{\frac{bA_0}{6\gamma}}, \]  
(66)  
\[ A_0 = \frac{9a^2 + 48bc + 3a\sqrt{9a^2 + 48bc}}{4b^2}, \]  
(67)  
\[ B_0 = -\frac{(a + \sqrt{9a^2 + 48bc})}{b}, \]  
(68)  
\[ \omega_2 = \sqrt{|A_0 (\phi_1^2 - B_0\phi_1 + A_0)|}, \]  
(69)  
\[ \zeta = \frac{2A_0 - B_0\phi_1 + 2\omega_2}{\phi_1}, \]  
(70)  
\[ \lambda_2 = \frac{1}{\sqrt{|A_0|}}. \]  
(71)

For the varying processes of \( u_1^*(\xi) - u_4^*(\xi) \), see Fig. 5.  
(B2) When \( a_2 < a < a_1 \) and \( a \to a_1 - 0 \), two kink waves with the expressions \( u_3^*(\xi) \) can be bifurcated from two singular waves with the expressions \( (u_1^*(\xi) \) and \( u_2^*(\xi) \) and two valley solitary waves with the expressions \( (u_3^*(\xi) \) and \( u_4^*(\xi) \). For the varying processes of \( u_1^*(\xi) - u_4^*(\xi) \), see Fig. 6.  
(B3) When \( a_4 < a < a_3 \) and \( a \to a_4 + 0 \), two kink wave solutions  
\[ u_4^*(\xi) = -\sqrt{\frac{3c}{2|b|}} \left[ 1 \pm \tanh \left( \sqrt{\frac{c}{4\gamma}}\xi \right) \right], \]  
(72)  
can be bifurcated from following two types of special waves:  
(B3a) Singular waves with the expressions  
\[ u_3^*(\xi) = \frac{12\sqrt{6c}\lambda_3 e^{\sqrt{\frac{c}{4\gamma}}\xi}}{3a^2 + 18bc + 2\sqrt{6a}\lambda_3 e^{\sqrt{\frac{c}{4\gamma}}\xi} + 2\lambda_3^2 e^{\sqrt{2\sqrt{\frac{c}{4\gamma}}\xi}}}, \]  
(73)
and
\[ u_4^*(\xi) = \frac{12\sqrt{6c}\lambda_3 e^{-\sqrt{\frac{c}{4\gamma}}\xi}}{3a^2 + 18bc + 2\sqrt{6a}\lambda_3 e^{-\sqrt{\frac{c}{4\gamma}}\xi} + 2\lambda_3^2 e^{-2\sqrt{\frac{c}{4\gamma}}\xi}}, \]  
(74)
where
\[ \omega_3 = 6c - 2a\varphi_2 - b\varphi_2^2, \]  
(75)
\[ \lambda_3 = \frac{6\sqrt{6c}-2\sqrt{6a}\varphi_2+6\sqrt{|c\omega_3|}}{\varphi_2}. \]  
(76)
(B3b) Peak solitary waves with the expressions
\[ u_5^*(\xi) = \frac{\varphi_1 - (2B_{01}\varphi_1 - 4A_{01})\lambda_4 e^{\eta_4\xi} + (B_{01}^2 - 4A_{01})\varphi_1\lambda_4^2 e^{2\eta_4\xi}}{1 + 2B_{01}\lambda_4 e^{\eta_4\xi} + (B_{01}^2 - 4A_{01})\lambda_4^2 e^{2\eta_4\xi}}, \]  
(77)
and
\[ u_6^*(\xi) = \frac{\varphi_1 - (2B_{01}\varphi_1 - 4A_{01})\lambda_4 e^{-\eta_4\xi} + (B_{01}^2 - 4A_{01})\varphi_1\lambda_4^2 e^{-2\eta_4\xi}}{1 + 2B_{01}\lambda_4 e^{-\eta_4\xi} + (B_{01}^2 - 4A_{01})\lambda_4^2 e^{-2\eta_4\xi}}, \]  
(78)
where

\[
\eta_5 = \sqrt{\frac{b A_{01}}{6 \gamma}}, \tag{79}
\]

\[
A_{01} = \frac{9a^2 + 48bc - 3a\sqrt{9a^2 + 48bc}}{4b^2}, \tag{80}
\]

\[
B_{01} = -\frac{a - \sqrt{9a^2 + 48bc}}{b}, \tag{81}
\]

\[
\varphi_4 = \sqrt{|A_{01}(\varphi_2^2 - B_{01} \varphi_2 + A_{01})|}, \tag{82}
\]

\[
\zeta_1 = \frac{2A_{01} - B_{01} \varphi_2 + 2\varphi_4}{\varphi_2}, \tag{83}
\]

\[
\lambda_4 = \zeta_1 \sqrt{|A_{01}|}. \tag{84}
\]

The varying processes of \(u_5^\ast(\xi) - u_8^\ast(\xi)\) are similar to Fig. 5.

(B) When \(a < a_4\) and \(a \to a_4 - 0\), two kink waves with the expressions \(u_1^\ast(\xi)\) can be bifurcated from two valley solitary waves with the expressions \((u_5^\ast(\xi)\) and \(u_6^\ast(\xi)\)) and two singular waves with the expressions \((u_7^\ast(\xi)\) and \(u_8^\ast(\xi)\)). The varying processes of \(u_5^\ast(\xi) - u_8^\ast(\xi)\) are similar to Fig. 6.

Fig. 5. When \(a \to a_1 + 0\), the varying processes of \(u_1^\ast(\xi) - u_8^\ast(\xi)\) where \(c = 2, b = -27\) and \(\gamma = 1\).
3. The derivation to Proposition 1. In this section, we give derivation to our main results. Under the transformations

\[ \xi = x - ct, \quad u(x, t) = \varphi(\xi), \]  

Eq. (1) is reduced to

\[ -c\varphi' + a\varphi \varphi' + b\varphi^2 \varphi' + \gamma \varphi''' = 0. \]  

Integrating Eq. (86) once and letting integral constant be zero, it follows that

\[ -c\varphi + \frac{a}{2} \varphi^2 + \frac{b}{3} \varphi^3 + \gamma \varphi'' = 0. \]  

Letting \( \varphi'(\xi) = y \), yields the planar system

\[ \begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{c}{\gamma} \varphi - \frac{a}{2\gamma} \varphi^2 - \frac{b}{3\gamma} \varphi^3. \end{cases} \]  

Obviously, system (88) has the first integral

\[ y^2 + \frac{b}{6\gamma} \varphi^4 + \frac{a}{3\gamma} \varphi^3 - \frac{c}{\gamma} \varphi^2 = h. \]  

For given \( \gamma, b \) and \( c \), choosing \( a \) as bifurcation parameter, we acquire the bifurcation phase portraits of system (88) as Fig. 7 and Fig. 9. Employing some orbits of the bifurcation phase portraits, we derive the results of proposition 1 as follows.

3.1. The derivation to Proposition 1.A.

When \( a \in (a_2, a_1) \) or \( a \in (a_4, a_3) \), we give marks to some orbits as Fig. 8.
The derivation to Proposition 1.A is rely on Figs. 7 and 8. Firstly, we derive the expressions of \( u_i(\xi) \) \((i = 1, 2, ..., 7)\).

(1) When \( b > 0, c < 0 \) and \( a \in (a_2, a_1) \) (as Fig. 8(a)), the homoclinic orbit \( l_1 \) and the closed orbits \( l_2, l_3 \) possess respectively the following expressions

\[
l_1: \quad y^2 = \frac{b}{6\gamma} (\phi - \phi_1)^2 (\phi_1 - \phi) (\phi - \phi_1^+), \quad (\phi_1 < \phi < \phi_1^+) \tag{90}
\]

\[
l_2: \quad y^2 = \frac{b}{6\gamma} (\phi - \phi_2)^2 (\phi_2 - \phi) (\phi - \phi_2^+), \quad (\phi_2 < \phi < \phi_2^+) \tag{91}
\]

and

\[
l_3: \quad y^2 = \frac{b}{6\gamma} (\mu_3 - \phi) (\phi - \mu_2) (\phi - \mu_1) (\phi - \mu), \quad (\mu < \mu_1 < \mu_2 < \phi < \mu_3) \tag{92}
\]

where \( \phi_1, \phi_1^+, \phi_1^-, \phi_2, \phi_2^+, \phi_2^-, \mu, \mu_1, \mu_2 \) and \( \mu_3 \) are in \((7), (11), (12), (14), (18), (19), (21)-(26)\).

Substituting (91) and (92) into \( \frac{dx}{d\xi} = y \) and integrating them along \( l_1, l_2 \) and \( l_3 \) respectively, it follows that

\[
\int_{\phi}^{\phi_1^-} \frac{ds}{(s - \phi_1) \sqrt{(\phi_1^+ - s)(s - \phi_1^-)}} = \sqrt{\frac{b}{6\gamma}} |\xi|, \quad \text{(along } l_1\text{)}, \tag{93}
\]

\[
\int_{\phi}^{\phi_2^-} \frac{ds}{(s - \phi_2) \sqrt{(\phi_2^+ - s)(s - \phi_2^-)}} = \sqrt{\frac{b}{6\gamma}} |\xi|, \quad \text{(along } l_2\text{)}, \tag{94}
\]

and

\[
\int_{\phi}^{\mu_3} \frac{ds}{\sqrt{(\mu_3 - s)(s - \mu_2)(s - \mu_1)(s - \mu)}} = \sqrt{\frac{b}{6\gamma}} |\xi|, \quad \text{(along } l_3\text{)}. \tag{95}
\]

Completing the integrals above and solving the equations for \( \phi \), we obtain the solutions \( u_1^*(\xi) \) (see (6)), \( u_2^*(\xi) \) (see (13)) and \( u_3^*(\xi) \) (see (20)).
(2) When \( b > 0, c < 0 \) and \( a \in (a_3, 0) \cup (0, a_2) \) (as Fig. 7), the closed orbit connected with \((\varphi_1^+, 0)\) and \((\varphi_{01}, 0)\) possesses the following expressions

\[
y^2 = \frac{b}{6\gamma}(\varphi - \varphi^0)(\varphi_01 - \varphi)(\varphi - \varphi_{02} - \varphi_{03i})(\varphi - \varphi_{02} + \varphi_{03i}),
\]

(\text{where } \varphi^0 < \varphi \leq \varphi_{01} \text{ and } 0 < a < a_2), \quad (96)

or

\[
y^2 = \frac{b}{6\gamma}(\varphi - \varphi_{01})(\varphi_1^+ - \varphi)(\varphi - \varphi_{02} - \varphi_{03i})(\varphi - \varphi_{02} + \varphi_{03i}),
\]

(\text{where } \varphi_{01} \leq \varphi < \varphi_1^+ \text{ and } a_3 < a < 0), \quad (97)

where \( \varphi^0, \varphi_{01}, \varphi_{02} \) and \( \varphi_{03} \) are in (36)-(39).

Substituting (96) and (97) into \( \frac{dx}{ds} = y \) and integrating them along the corresponding orbit respectively, it follows that

\[
\int_{\varphi_01}^{\varphi} \frac{ds}{\sqrt{(s - \varphi_1^+)(\varphi_01 - s)(s - \varphi_{02} - \varphi_{03i})(s - \varphi_{02} + \varphi_{03i})}} = \sqrt{\frac{b}{6\gamma}|\xi|},
\]

where \( \varphi_1^+ < \varphi \leq \varphi_{01} \text{ and } 0 < a < a_2 \), \quad (98)

and

\[
\int_{\varphi_01}^{\varphi} \frac{ds}{\sqrt{(s - \varphi_01)(\varphi_1^+ - s)(s - \varphi_{02} - \varphi_{03i})(s - \varphi_{02} + \varphi_{03i})}} = \sqrt{\frac{b}{6\gamma}|\xi|},
\]

where \( \varphi_{01} \leq \varphi < \varphi_1^+ \text{ and } a_3 < a < 0 \). \quad (99)

Completing the integrals above and solving the equations for \( \varphi \), we obtain the solution \( u^1_1(\xi) \) (see (32)).

(3) When \( b > 0, c < 0 \) and \( a \in (a_4, a_3) \) (as Fig. 8(b)), the homoclinic orbit \( l_4 \) and the closed orbits \( l_5, l_6 \) possess the following expressions

\[
l_4: \quad y^2 = \frac{b}{6\gamma}(\varphi_2 - \varphi)^2(\varphi_2 - \varphi)(\varphi - \varphi_2^+), \quad (\varphi_2^+ \leq \varphi < \varphi_2 < \varphi_2^+), \quad (100)
\]

\[
l_5: \quad y^2 = \frac{b}{6\gamma}(\varphi_1 - \varphi)^2(\varphi_1^+ - \varphi)(\varphi - \varphi_1^+), \quad (\varphi_1^+ \leq \varphi \leq \varphi_1^1 < \varphi_1), \quad (101)
\]

\[
l_6: \quad y^2 = \frac{b}{6\gamma}(\varphi - \tau_3)(\tau_2 - \varphi)(\tau_1 - \varphi)(\tau - \varphi), \quad (\tau_3 \leq \varphi \leq \tau_2 < \tau_1 < \tau), \quad (102)
\]

where \( \varphi_1, \varphi_1^+, \varphi_2, \varphi_2^+, \varphi_2^+, \tau, \tau_1, \tau_2 \) and \( \tau_3 \) are in (7), (11), (12), (14), (18), (19), (48)-(53).

Substituting (100), (101) and (102) into \( \frac{dx}{ds} = y \) and integrating them along \( l_4, l_5 \) and \( l_6 \) respectively, it follows that

\[
\int_{\varphi_1^+}^{\varphi} \frac{ds}{(\varphi_2 - s)(\varphi_2^- - s)(\varphi_2^+)} = \sqrt{\frac{b}{6\gamma}|\xi|}, \quad (\text{along } l_4), \quad (103)
\]

\[
\int_{\varphi_1}^{\varphi} \frac{ds}{(\varphi_1 - s)(\varphi_1^- - s)(\varphi_1^+)} = \sqrt{\frac{b}{6\gamma}|\xi|}, \quad (\text{along } l_5), \quad (104)
\]

and

\[
\int_{\tau_3}^{\varphi} \frac{ds}{(\tau - s)(\tau_1 - s)(\tau_2 - s)(\tau_3)} = \sqrt{\frac{b}{6\gamma}|\xi|}, \quad (\text{along } l_6). \quad (105)
\]
Completing the integrals above and solving the equations for \( \varphi \), we obtain the solutions \( u_0^e(\xi) \) (see (45)), \( u_0^f(\xi) \) (see (46)) and \( u_1^f(\xi) \) (see (47)).

Secondly, we prove that \( u_1^e(\xi) \), \( u_2^e(\xi) \), \( u_3^e(\xi) \) and \( u_4^e(\xi) \) have the same limit function \( u_1(\xi) \) when \( a \to a_2 \). Note that when \( a \to a_2, \varphi_1, \varphi_2, \varphi_02, \varphi_0^1, \varphi_0^2, \varphi_0^3, \mu, \mu_1 \) and \( \mu_2 \) tend to \( \varphi_1 = -\sqrt{3c}/b, \varphi_01, \varphi_0^1, \varphi_2^2 \) and \( \mu_3 \) tend to \( \varphi_2 = \sqrt{|c|}/3b \) respectively. Further we have

\[ \begin{align*}
\varphi_{03} & \to 0, \quad \alpha \to 0, \quad \alpha_1 \to 0, \quad \alpha_2 \to 0, \\
\beta_1 & \to \frac{b}{6\gamma}(\varphi_2 - \varphi_1^1), \\
\beta_2 & \to \frac{b}{6\gamma}(\varphi_2 - \varphi_1^1), \\
\eta_1 & = \sqrt{\frac{b}{24\gamma}(\mu_3 - \mu_1)(\mu_2 - \mu)} \to 0, \\
\sqrt{\Delta_1} & = \sqrt{\beta_1 - \frac{2b}{3\gamma}\alpha_1} = \beta_1 - \frac{b\alpha_1}{3\gamma\beta_1} + \ldots, \\
\sqrt{\Delta_2} & = \sqrt{\beta_2 - \frac{2b}{3\gamma}\alpha_2} = \beta_2 - \frac{b\alpha_2}{3\gamma\beta_2} + \ldots, \\
\cos(\eta\xi) & = 1 - \frac{(\eta\xi)^2}{2!} + \frac{(\eta\xi)^4}{4!} + \ldots, \\
\cosh(\eta\xi) & = 1 + \frac{(\eta\xi)^2}{2!} + \frac{(\eta\xi)^4}{4!} + \ldots, \\
\cosh(\eta\xi, k) & = 1 - \frac{(\eta\xi)^2}{2!} + (1 + 4k^2)\frac{(\eta\xi)^4}{4!} + \ldots,
\end{align*} \]

and

\[ sn(\eta\xi, k) = \eta\xi - (1 + k^2)\frac{(\eta\xi)^3}{3!} + \ldots. \]

Thus we have:

\[ \begin{align*}
\lim_\substack{a \to a_2 \to 0 \to 0}{u_1^e(\xi)} = \lim_\substack{a \to a_2 \to 0}{\varphi_1 + \frac{2\alpha_1}{\beta_1 - \frac{b\alpha_1}{3\gamma\beta_1} + o(\alpha_1)} \left(1 + \frac{\alpha_1\xi^2}{2} + o(\alpha_1)\right) - \beta_1} \\
= \varphi_1^* + \frac{6b\gamma(\varphi_2^* - \varphi_1^*)}{b^2(\varphi_2^* - \varphi_1^*)^2\xi^2 - 12b\gamma} \\
= u_1(\xi) \quad \text{ (see (5))}. \quad (116)
\end{align*} \]

\[ \begin{align*}
\lim_\substack{a \to a_2 \to 0 \to 0}{u_2^e(\xi)} = \lim_\substack{a \to a_2 \to 0}{\varphi_2 + \frac{2\alpha_2}{\beta_2 - \frac{b\alpha_2}{3\gamma\beta_2} + o(\alpha_2)} \left(1 + \frac{\alpha_2\xi^2}{2} + o(\alpha_2)\right) - \beta_2} \\
= \varphi_1^* + \frac{6b\gamma(\varphi_2^* - \varphi_1^*)}{b^2(\varphi_2^* - \varphi_1^*)^2\xi^2 - 12b\gamma} \\
= u_1(\xi) \quad \text{ (see (5))}. \quad (117)
\end{align*} \]
\[
\lim_{a \to a_2^-} u_3^a(\xi) = \lim_{a \to a_2^-} \frac{\mu_3(\mu_2 - \mu) + \mu(\mu_3 - \mu_2)\sin^2(\eta_2 \xi, k_1)}{(\mu_2 - \mu) + (\mu_3 - \mu_2)\sin^2(\eta_2 \xi, k_1)}
\]

\[
= \lim_{a \to a_2^-} \frac{\mu_3(\mu_2 - \mu) + (\mu_3 - \mu_2)(\mu_2 - \mu_1)(\mu_2 - \mu)\xi^2 + o((\mu_2 - \mu)^2)}{24(\mu_2 - \mu) + (\mu_3 - \mu_2)(\mu_3 - \mu_1)(\mu_2 - \mu)\xi^2 + o((\mu_2 - \mu)^2)}
\]

\[
= \frac{24\gamma \phi_1^* + \phi_1^* (\phi_1^* - \phi_2^*)^2 b \xi^2}{24\gamma + (\phi_1^* - \phi_2^*)^2 b \xi^2}
\]

\[
= u_1(\xi) \quad \text{(see (5)).} 
\]

Similarly, we can prove that \(u_1^a(\xi), u_6^a(\xi), u_9^a(\xi)\) and \(u_7^a(\xi)\) have the same limit function \(u_2(\xi)\) when \(a \to a_3\). These complete the derivations of proposition 1.A.

Fig. 9. The bifurcation phase portraits of system (88) for given \(\gamma > 0, b < 0\) and \(c > 0\).

3.2. The derivation to Proposition 1.B.

When \(a \in (a_1, +\infty)\) or \(a \in (a_2, a_1)\), we give marks to some orbits as Fig. 10.
The derivation to Proposition 1.B is rely on Figs. 9 and 10. Firstly, we derive the expressions of $u_i^*(\xi)$ ($i = 1, 2, ..., 7$).

(1) When $b < 0$, $c > 0$ and $a \in (a_1, +\infty)$ (as Fig. 10(a)), the homoclinic orbit $\Gamma_1$ connecting with $(0,0)$ embraces the expression

$$y^2 = \varphi^2 \left( -\frac{b}{6\gamma} \varphi^2 - \frac{a}{3\gamma} \varphi + \frac{c}{\gamma} \right), \quad (0 < \varphi \leq \varphi_3),$$

(120)

where

$$\varphi_3 = -a + \sqrt{a^2 + 6bc} \frac{b}{b}.$$

(121)

Substituting (120) into $\frac{d\varphi}{d\xi} = y$ and integrating it, we have

$$\int_{\varphi_1}^{\varphi} \frac{ds}{s \sqrt{-\frac{b}{6\gamma} s^2 - \frac{a}{3\gamma} s + \frac{c}{\gamma}}} = \xi.$$

(122)

Completing the integral above and solving the equation for $\varphi$, we obtain the solution $u_1^*(\xi)$ (see (60)) and $u_2^*(\xi) = u_1^*(-\xi)$ (see (61)) and they represent two soliton waves.

(2) When $b < 0$, $c > 0$ and $a \in (a_1, +\infty)$ (as Fig. 10(a)), the orbit $\Gamma_2$ connecting with $(\varphi_2, 0)$ embraces the expression

$$y^2 = -\frac{b}{6\gamma} \varphi^2 \left( \varphi^2 + B_0 \varphi + A_0 \right), \quad (-\infty < \varphi < \varphi_2),$$

(123)

where $A_0$ and $B_0$ are in (67)-(68).

Substituting (123) into $\frac{d\varphi}{d\xi} = y$ and integrating it, we have

$$\int_{\varphi_1}^{\varphi} \frac{ds}{s \sqrt{-\frac{b}{6\gamma} (s^2 + B_0 s + A_0)}} = \xi.$$

(124)

Completing the integral above and solving the equation for $\varphi$, we obtain the solution $u_3^*(\xi)$ (see (64)) and $u_4^*(\xi) = u_3^*(-\xi)$ (see (65)) and they represent two singular waves.
When \( b < 0, c > 0 \) and \( a \in (a_2, a_1) \) (as Fig. 10(b)), the orbit \( \Gamma_5 \) connecting with \((0, 0)\) embraces the expression
\[
y^2 = \varphi^2 \left( -\frac{b}{6\gamma} \varphi^2 - \frac{a}{3\gamma} \varphi + \frac{c}{\gamma} \right), \quad (0 < \varphi < +\infty).
\] (125)

Substituting (125) into \( \frac{d\varphi}{dt} = y \) and integrating it, we have
\[
\int_{\varphi_1}^{\varphi} ds \sqrt{-\frac{b}{6\gamma} s^2 - \frac{a}{3\gamma} s + \frac{c}{\gamma}} = \xi.
\] (126)

Completing the integral above and solving the equation for \( \varphi \), we also obtain \( u_1^+ (\xi) \) (see (60)) and \( u_2^+ (\xi) = u_1^+ (-\xi) \) (see (61)) while they represent two singular waves in this case.

(4) When \( b < 0, c > 0 \) and \( a \in (a_2, a_1) \) (as Fig. 10(b)), the homoclinic orbit \( \Gamma_4 \) connecting with \((\varphi_2, 0)\) embraces the expression
\[
y^2 = -\frac{b}{6\gamma} \varphi^2 \left( \varphi^2 + B_0 \varphi + A_0 \right), \quad (\varphi_4 \leq \varphi < \varphi_2),
\] (127)

where
\[
\varphi_4 = \frac{-a + \sqrt{9a^2 + 48bc} - 2a \left( a - \sqrt{|9a^2 + 48bc|} \right)}{4b}.
\] (128)

\( A_0 \) and \( B_0 \) are in (67)-(68). Substituting (127) into \( \frac{d\varphi}{dt} = y \) and integrating it, we have
\[
\int_{\varphi_1}^{\varphi} ds \sqrt{-\frac{b}{6\gamma} (s^2 + B_0 s + A_0)} = \xi.
\] (129)

Completing the integral above and solving the equation for \( \varphi \), we also obtain \( u_3^+ (\xi) \) (see (64)) and \( u_4^+ (\xi) = u_3^+ (-\xi) \) (see (65)) while they represent two soliton waves in this case.

Secondly, we prove that \( u_1^+ (\xi) \) have the limit function \( u_1^+ (\xi) \) when \( a \to a_1 \). Note that when \( a \to a_1 \), we have \( \varphi_1 \to \sqrt{-\frac{a}{8b}}, \lambda_1 \to 6\sqrt{-bc} \) and \( 3a^2 + 18bc \to 0 \). Thus, we get
\[
\lim_{a \to a_1} u_1^+ (\xi) = u_1^+ (\xi) \quad \text{(see (59)).}
\] (130)

Similarly, we can finish the rest. Hereeto, we complete the derivation of proposition 1.

4. Conclusions. In this paper, we have studied the bifurcation of nonlinear wave described by Eq. (1). For given parameters \( b, \gamma \) and wave speed \( c \), parameter \( a \) has been chosen as bifurcation parameter. We have revealed two kinds of important bifurcation phenomena. In the first phenomenon there exist two bifurcation parametric values \( a_2 \) and \( a_3 \) given in (3). When bifurcation parameter \( a \) tends to these values, the solitary waves with fractional expressions can be bifurcated from three types of nonlinear waves listed in proposition 1.A. In the second phenomenon there also exist two bifurcation parametric values \( a_1 \) and \( a_4 \) given in (3). When
bifurcation parameter $a$ tends to these values, the kink waves can be bifurcated from the solitary waves and the singular waves listed in proposition 1.B.

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