Ground state energy in a wormhole space-time.

Nail R. Khusnutdinov∗†, Sergey V. Sushkov†§

∗Department of Physics, †Department of Mathematics,
Kazan State Pedagogical University, Mezhlaik 1, Kazan 420021, Russia

The ground state energy of the massive scalar field with non-conformal coupling ξ on the short-throat flat-space wormhole background is calculated by using zeta renormalization approach. We discuss the renormalization and relevant heat kernel coefficients in detail. We show that the stable configuration of wormholes can exist for ξ > 0.123. In particular case of massive conformal scalar field with ξ = 1/6, the radius of throat of stable wormhole a ≈ 0.16/m. The self-consistent wormhole has radius of throat a ≈ 0.0141l_p and mass of scalar boson m ≈ 11.35m_p (l_p and m_p are the Planck length and mass, respectively).

I. INTRODUCTION

Wormholes are topological handles in space-time linking widely separated regions of a single universe, or “bridges” joining two different space-times. Interest in these configurations dates back at least as far as 1916 [1] with punctuated revivals of activity following both the classic work of Einstein and Rosen in 1935 [2] and the later series of works initiated by Wheeler in 1955 [3]. More recently, a fresh interest in the topic has been rekindled by the works of Morris and Thorne [4] and Morris, Thorne, and Yurtsever [5]. These authors constructed and investigated a class of objects they referred to as “traversable wormholes”. Their work led to a flurry of activity in wormhole physics [6].

The central feature of wormhole physics is the fact that traversable wormholes are accompanied by unavoidable violations of the null energy condition, i.e., the matter threading the wormhole’s throat has to be possessed of “exotic” properties. The classical matter does satisfy the usual energy conditions, hence wormholes cannot arise as solutions of classical relativity and matter. If they exist, they must belong to the realm of semi-classical or perhaps quantum gravity. In the absence of the complete theory of quantum gravity, the semi-classical approach begins to play the most important role for examining wormholes. However, there are not so much results concerning quantized fields on the wormhole background. Recently the self-consistent wormholes in the semi-classical gravity were studied numerically in our works [8]. As well some arguments in favor of the possibility of existence of self-consistent wormhole solutions to the semi-classical Einstein equations have been given in works of Khatysmovsky in Ref. [9].

Note that all the mentioned results were obtained within the framework of various approximations, whereas no one up to now has succeeded in exact calculations of vacuum expectation values on the wormhole background. The reason for this state of affairs consists in considerable mathematical difficulties which one faces with trying to quantize a physical field on the wormhole background. To overcome these difficulties, in this work we will consider a simple model of the wormhole space-time: the short-throat flat-space wormhole. The model represents two identical copies of Minkowski space with excised from each copy spherical regions, and with boundaries of those regions are to be identified. The space-time of this model is everywhere flat except a two-dimensional singular spherical surface. Due to this fact it turns out to be possible to construct the complete set of wave modes of the massive scalar field and calculate the ground state energy.

The aim of our work is to calculate the ground state energy of the scalar field on the short-throat flat-space wormhole background using the zeta function regularization approach [10,11] which was developed in Refs. [12–14]. In framework of this approach, the ground state energy of scalar field φ is given by

\[ E(s) = \frac{1}{2} \mu^2 \zeta_L \left( s - \frac{1}{2} \right), \]

where

\[ \zeta_L(s) = \sum_{n} \left( \lambda^2_{n} + m^2 \right)^{-s} \]

PACS number(s): 04.62.+v, 04.70.Dy, 04.20.Gz
is the zeta function of the corresponding Laplace operator. To make the eigenvalues $\lambda^2_{(n)}$ discrete we assume the field $\phi$ to be put into a large ball with Dirichlet boundary condition. The $\lambda^2_{(n)}$ are eigenvalues of the three dimensional Laplace operator $\mathcal{L}$

$$(\triangle - \xi R)\phi_{(n)} = \lambda^2_{(n)}\phi_{(n)},$$

where $R$ is the curvature scalar.

The expression (1) is divergent in the limit $s \to 0$ which we are interesting in. For renormalization we subtract from (1) the divergent part of it

$$E^{\text{ren}} = \lim_{s \to 0} \left( E(s) - E^{\text{div}}(s) \right),$$

where

$$E^{\text{div}}(s) = \lim_{m \to \infty} E(s).$$

By virtue of the heat kernel expansion of zeta function is the asymptotic expansion for large mass, the divergent part has the following form

$$E^{\text{div}}(s) = \frac{1}{2} \left( \frac{\mu}{m} \right)^{2s} \frac{1}{(4\pi)^{3/2}\Gamma(s - \frac{1}{2})} \times \left\{ B_0 m^4 \Gamma(s - 2) + B_1 / 2 m^3 \Gamma(s - \frac{3}{2}) + B_1 m^2 \Gamma(s - 1) + B_3 / 2 m \Gamma(s - \frac{1}{2}) + B_2 \Gamma(s) \right\},$$

where $B_\alpha$ are the heat kernel coefficients.

In this case the renormalized ground state energy (4) obeys the normalization condition

$$\lim_{m \to \infty} E^{\text{ren}} = 0.$$

The organization of the paper is as follows. In Sec. II we describe a space-time of wormhole in the short-throat flat-space approximation and analyze the solution of equation of motion for massive scalar field. In Sec. III we obtain close expression for zeta function and ground state energy and calculate corresponding heat kernel coefficients. We analyze also the expression for ground state energy for different radius of throat. In Sec IV we discuss our results. The Appendixes A and B contain some technical details of calculations.

We use units $\hbar = c = G = 1$ (except Sec IV). The signature of the space-time, the sign of the Riemann and Ricci tensors, is the same as in the book by Hawking and Ellis [15].

II. A TRAVERSABLE WORMHOLE: THE SHORT-THROAT FLAT-SPACE APPROXIMATION

In this section we consider a simple model of a traversable wormhole. Assume that the throat of the wormhole is very short, and that curvature in the regions outside the mouth of the wormhole is relatively weak. An idealized model of such a wormhole can be constructed in the following manner: Consider two copies of Minkowski space, $M_+$ and $M_-$, with the spherical coordinates $(t, r_\pm, \theta_\pm, \varphi_\pm)$ [Notice: $M_+$ and $M_-$ have a common time coordinate $t$. One may interpret this fact as the identification $t_+ \leftrightarrow t_-]$; excise from each copy the spherical region $r_\pm < a$, where $a$ is a radius of sphere; and then identify the boundaries of those regions: $(t, a, \theta_+, \varphi_+) \leftrightarrow (t, a, \theta_-, \varphi_-)$. The Riemann tensor for this model is identically zero everywhere except at the wormhole mouths where the identification procedure takes place. Generically, there will be an infinitesimally thin layer of exotic matter present at the mouth of the wormhole.

Such an idealized geometry can be described by the following metric

$$ds^2 = -dt^2 + d\rho^2 + r^2(\rho) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),$$

where $\rho$ is a proper radial distance, $-\infty < \rho < \infty$, and the shape function $r(\rho)$ is

$$r(\rho) = |\rho| + a.$$
It is easily to see that in two regions $R_+: \rho > 0$ and $R_-: \rho < 0$ separately, one can introduce a new radial coordinate $r_\pm = \pm \rho + a$ and rewrite the metric (8) in the usual spherical coordinates:

$$ds^2 = -dt^2 + dr_\pm^2 + r_\pm^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

This form of the metric explicitly indicates that the regions $R_+: \rho > 0$ and $R_-: \rho < 0$ are flat. However, note that such the change of coordinates $r = |\rho| + a$ is not global, because it is ill defined at the throat $\rho = 0$. Hence, as was expected, the space-time is curved at the wormhole throat. To illustrate this we calculate the scalar curvature $R(\rho)$ in the metric (8):

$$R(\rho) = -8a^{-1}\delta(\rho).$$

Let us now consider a scalar field $\phi$ in the space-time with the metric (8). The equation of motion of the scalar field is

$$\square m^2 - \xi R)\phi = 0,$$ (11)

where $m$ is a mass of the scalar field, and $\xi$ is an arbitrary coupling with the scalar curvature $R$. In the metric (8), a general solution to the equation (11) can be found in the following form:

$$\phi(t, \rho, \theta, \phi) = e^{-i\omega t}u(\rho)Y_{ln}(\theta, \phi),$$ (12)

where $Y_{ln}(\theta, \phi)$ are spherical functions, $l = 0, 1, 2, \ldots$, $n = 0, \pm1, \pm2, \ldots, \pm l$, and a function $u(\rho)$ obeys the radial equation

$$u'' + 2\frac{r'}{r}u' + \left(\omega^2 - \frac{l(l + 1)}{r^2} - m^2 - \xi R\right)u = 0,$$ (13)

where a prime denotes the derivative $d/d\rho$. In the flat regions $R_\pm$, where $r(\rho) = \pm \rho + a$, $r'(\rho) = \pm 1$, and $R(\rho) = 0$, Eq.(13) reads

$$u'' + \frac{2}{\rho \pm a}u' + \left(\omega^2 - m^2 - \frac{l(l + 1)}{(\rho \pm a)^2}\right)u = 0.$$ (14)

A general solution of this equation can be written as

$$u^\pm_l[\lambda(\rho \pm a)] = A^\pm_l h^+_l[\lambda(\rho \pm a)] + B^\pm_l h^-_l[\lambda(\rho \pm a)],$$ (15)

where

$$\lambda = \sqrt{\omega^2 - m^2}, \quad |\omega| > m,$$

$h^{(i)}_l[z]$ are spherical Hankel functions, and $A^\pm_l, B^\pm_l$ are arbitrary constants.

The solutions $u^\pm_l[\lambda(\rho \pm a)]$ have been obtained in the flat regions $R_\pm$ separately. To find a solution in the whole space-time we must impose matching conditions for $u^\pm_l[\lambda(\rho \pm a)]$ at the throat $\rho = 0$. The first condition demands that the solution has to be continuous at $\rho = 0$. This gives

$$u^-_l[-\lambda a] = u^+_l[\lambda a],$$

or

$$A^-_l h^+_l[-\lambda a] + B^-_l h^-_l[-\lambda a] - A^+_l h^+_l[\lambda a] - B^+_l h^-_l[\lambda a] = 0.$$ (16)

To obtain the second condition we integrate Eq.(13) within the interval $(-\epsilon, \epsilon)$ and then go to the limit $\epsilon \to 0$. Taking into account the following relations

$$r(\rho) = |\rho| + a, \quad r'(\rho) = \text{sign}\rho, \quad r''(\rho) = 2\delta(\rho),$$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} f(\rho)\delta(\rho)d\rho = f(0),$$

the second boundary condition can be written as

$$\int_{-\epsilon}^{\epsilon} f(\rho)\delta(\rho)d\rho = f(0).$$ (16)
we find
\[
\left. \frac{du^{-}_i[x]}{dx} \right|_{x=-\lambda a} = \left. \frac{du^{+}_i[x]}{dx} \right|_{x=\lambda a} + \frac{8\xi}{\lambda a} u^+_i[\lambda a],
\] (17)

Substituting Eq. (13) into (17) gives
\[
A_i^{-} h^{(1)}[-\lambda a] + B_i^{-} h^{(2)}[-\lambda a] - \left( h^{(1)}_i[\lambda a] + \frac{8\xi}{\lambda a} h^{(1)}_i[\lambda a] \right) A_i^+ - \left( h^{(2)}_i[\lambda a] + \frac{8\xi}{\lambda a} h^{(2)}_i[\lambda a] \right) B_i^+ = 0,
\] (18)

where \( h^{(i)}_i[\pm \lambda a] = (dh^{(i)}_i[x]/dx)_{x=\pm \lambda a} \).

In addition to two matching conditions (14) and (18) we must demand regular behavior of the scalar field at the infinity. For this aim, we will consider a “box approximation”, i.e., we will assume, in an intermediate stage of calculations, that the wormhole space-time has an finite radius \( R \), so that \(|\rho| \leq R\), and we will go, in the end, to the limit \( R \to \infty \). In the framework of the box approximation, we demand that the scalar field becomes to be equal zero at the space-time bounds \( \rho = \pm R \). Taking into account Eq. (15) gives
\[
u_i^{+}[-\lambda(R+a)] = 0, \quad u_i^{+}[\lambda(R+a)] = 0,
\] or
\[
A_i^{-} h^{(1)}_i[-\lambda(R+a)] + B_i^{-} h^{(2)}_i[-\lambda(R+a)] = 0,
\] (19)
\[
A_i^{+} h^{(1)}_i[\lambda(R+a)] + B_i^{+} h^{(2)}_i[\lambda(R+a)] = 0.
\] (20)

The four conditions (14), (15), (17) and (18) obtained represent a homogeneous system of linear algebraic equations for four coefficients \( A_i^{+}, B_i^{-} \). As is known, such a system has a nontrivial solution if and only if the matrix of coefficients is degenerate. Hence we get
\[
\begin{vmatrix}
 h^{(1)}_i[-\lambda a] & h^{(2)}_i[-\lambda a] & -h^{(1)}_i[\lambda a] & -h^{(2)}_i[\lambda a] \\
 h^{(1)}_i[-\lambda a] & h^{(2)}_i[-\lambda a] & -h^{(1)}_i[\lambda a] - \frac{8\xi}{\lambda a} h^{(1)}_i[\lambda a] & -h^{(2)}_i[\lambda a] - \frac{8\xi}{\lambda a} h^{(2)}_i[\lambda a] \\
 h^{(1)}_i[-\lambda(R+a)] & h^{(2)}_i[-\lambda(R+a)] & 0 & 0 \\
 0 & 0 & h^{(1)}_i[\lambda(R+a)] & h^{(2)}_i[\lambda(R+a)]
\end{vmatrix} = 0.
\] (21)

After some algebra one can show that the determinant in the above formula is factorized, and so Eq. (21) can be reduced to the following two relations:
\[
\Psi^+_1[\lambda] = 0,
\] (22)
\[
\Psi^+_2[\lambda] = 0,
\] (23)
where the functions \( \Psi_1[\lambda], \Psi_2[\lambda] \) are defined as follows:
\[
\Psi^+_1[\lambda] \equiv \frac{i\lambda}{2} \sqrt{a(a+R)} \left\{ h^{(1)}_i[\lambda(R+a)] h^{(2)}_i[\lambda a] - h^{(2)}_i[\lambda(R+a)] h^{(1)}_i[\lambda a] \right\},
\] (24)
\[
\Psi^+_2[\lambda] \equiv \frac{i\lambda^2 a}{8} \sqrt{a(a+R)} \left\{ h^{(1)}_i[\lambda(R+a)] \left( \frac{4\xi}{\lambda a} h^{(2)}_i[\lambda a] + h^{(2)}_i[\lambda a] \right) - h^{(2)}_i[\lambda(R+a)] \left( \frac{4\xi}{\lambda a} h^{(1)}_i[\lambda a] + h^{(1)}_i[\lambda a] \right) \right\}.
\] (25)

We introduced additional factors in order to simplify formulas that follow. These factors do not change the relations (22), (23). A significance of Eqs. (22) and (23) is that they determine a set of possible values of the wave number \( \lambda \), i.e., a spectrum for scalar field modes. Resolving Eq. (22) and Eq. (23) we can obtain two families, respectively:
\[
\lambda^{(1)}_{p_1}(a,R,\xi), \quad p_1 = 1, 2, 3, \ldots,
\] (26a)
\[
\lambda^{(2)}_{p_2}(a,R,\xi), \quad p_2 = 1, 2, 3, \ldots.
\] (26b)
III. GROUND STATE ENERGY AND HEAT KERNEL COEFFICIENTS.

A ground state energy is given by

\[ E = \frac{1}{2} \sum_{\alpha=1}^{\infty} \sum_{l=0}^{\infty} \sum_{p=1}^{\infty} (2l + 1) \sqrt{\lambda^{(\alpha)^2} + m^2}, \]  

which is, in fact, the zero point energy of massive scalar field. This expression is divergent. In the framework of the zeta function regularization method \cite{10,11}, the ground state energy is expressed in terms of the zeta function

\[ E(s) = \frac{1}{2} \mu^2 \zeta_L \left( s - \frac{1}{2} \right), \]  

where

\[ \zeta_L \left( s - \frac{1}{2} \right) = \sum_{\alpha=1}^{\infty} \sum_{l=0}^{\infty} (2l + 1) \left( \lambda^{(\alpha)^2} + m^2 \right)^{1/2-s} \]  

is the zeta function associated with Laplace operator \( \hat{L} = \triangle - m^2 - \xi R \). The parameter \( \mu \), with dimension of mass, has been introduced in order to have the correct dimension for the energy. For simplicity we represent Eq. (28) in slightly different form

\[ E(s) = \frac{1}{2} \left( \frac{\mu}{m} \right)^2 \zeta \left( s - \frac{1}{2} \right), \]  

where we introduced the function with dimension energy

\[ \zeta \left( s - \frac{1}{2} \right) = m^{2s} \zeta_L \left( s - \frac{1}{2} \right) \]  

which we shall call the zeta function, too.

The solutions \( \lambda_{lp}^{(\alpha)}(a, R, \xi) \) of Eqs. (22), (23) cannot be found in closed form. For this reason we use the method suggested in Ref. \cite{12}, which allows us to express the zeta function in terms of the eigenfunctions. The sum over \( p \) may be converted into a contour integral in a complex \( \lambda \)-plane using the principal of argument, namely

\[ \zeta \left( s - \frac{1}{2} \right) = \frac{m^{2s}}{2\pi i} \sum_{\alpha=1,2} \sum_{l=0}^{\infty} (2l + 1) \int_{\gamma} d\lambda \left( \lambda^2 + m^2 \right)^{1/2-s} \frac{\partial}{\partial \lambda} \ln \Psi_\nu^\alpha[\lambda], \]  

where the contour \( \gamma \) runs counterclockwise and must enclose all solutions of Eqs. (24), (25). Shifting the contour to the imaginary axis, we obtain the following formula for the zeta function

\[ \zeta \left( s - \frac{1}{2} \right) = -m^{2s} \frac{\cos \pi s}{\pi} \sum_{\alpha=1,2} \sum_{l=0}^{\infty} (2l + 1) \int_m^{\infty} dk \left( k^2 - m^2 \right)^{1/2-s} \frac{\partial}{\partial k} \ln \Psi_\nu^\alpha[ik], \]  

where the functions (24), (25) in the imaginary axis \( \lambda = ik \) read

\[ \Psi_1^\nu[ik] = I_\nu[k(R + a)]K_\nu[ka] - K_\nu[k(R + a)]I_\nu[ka], \]  

\[ \Psi_2^\nu[ik] = \left( \xi - \frac{1}{8} \right) \Psi_1^\nu[ik] + \frac{ka}{4} \{ I_\nu[k(R + a)]K_\nu'[ka] - K_\nu[k(R + a)]I_\nu'[ka] \}, \]  

with \( \nu = l + \frac{1}{2} \).

The expression (33) may be simplified in the large box limit \( R \gg a \), which we are interesting in. Let us rewrite \( \Psi_1^\nu[ik] \) in the following form
\[ \Psi_1^{[ik]} = I_\nu[k(R + a)]K_\nu[ka] \left( 1 - \frac{K_\nu[k(R + a)]I_\nu[ka]}{I_\nu[k(R + a)]K_\nu[ka]} \right). \] (35)

In the large box limit, the second term in brackets obeys the inequality
\[ \frac{K_\nu[k(R + a)]I_\nu[ka]}{I_\nu[k(R + a)]K_\nu[ka]} < e^{-2mR} \] (36)
and brings exponentially small contribution for ground state energy.

Therefore, in the limit of large box we have
\[ \Psi_1^{[ik]} \approx I_\nu[k(a + R)]K_\nu[ka], \] (37a)
\[ \Psi_1^{[ik]} \approx I_\nu[k(a + R)] \left( \left( \xi - \frac{1}{8} \right) K_\nu[ka] + \frac{ka}{4} K'_{\nu-1}[ka] \right). \] (37b)

In this moment we have to make comment on above formulas. Our approach is valid if the functions Ψ
in the imaginary axis do not have zeros in the domain of integration in Eq. (33). It gives the restriction for ξ. The function Ψ_1^{[ik]} has no zeros in the imaginary axis, but function Ψ_2^{[ik]} has simple zeros if ξ > \frac{1}{4}. Indeed, by using recurrent formulas for Bessel’s function, let us represent the function Ψ_2^{[ik]} in the following form
\[ \Psi_2^{[ik]} = I_\nu[k(a + R)] \left( \left( \xi - \frac{1}{8} - \frac{\nu}{4} \right) K_\nu[ka] - \frac{ka}{4} K_{\nu-1}[ka] \right). \] (38)

Since the Bessel’s functions K_\nu are positive, the expression in brackets may change sign and therefore the function Ψ_2^{[ik]} may have zeros, if
\[ \xi - \frac{1}{8} - \frac{\nu}{4} > 0. \] (39)

The lowest boundary for ξ is 1/4 for l = 0. More precisely, in this case we have
\[ \Psi_0^{[2]} = \frac{1}{2k} \sqrt{\frac{1}{a(a + R)}} e^{kR} (1 - e^{-2k(a+R)}) \left( \xi - \frac{1}{4} - \frac{ka}{4} \right). \] (40)

As far as k > m, the function Ψ_0^{[2]} has simple zero at point k = (4ξ - 1)/a if
\[ \xi > \frac{1}{4} + \frac{ma}{4}. \] (41)

For this reason in the paper we will consider ground state energy for ξ < 1/4. In opposite case we have to modify our approach.

Taking into account these formulas we may divide the zeta function, as well as ground state energy [33], into two parts
\[ \zeta \left( s - \frac{1}{2} \right) = \zeta_{\text{ext}} \left( s - \frac{1}{2} \right) + \zeta_{\text{int}} \left( s - \frac{1}{2} \right), \] (42)
where
\[ \zeta_{\text{ext}} \left( s - \frac{1}{2} \right) = \left( \frac{2\beta_\nu^2 \cos \pi s}{\pi a(a + R)} \right) \sum_{l=0}^{\infty} \nu l^{2-2s} \int_{\beta_R/\nu}^{\infty} dx \left( x^2 - \frac{\beta_\nu^2}{\nu^2} \right)^{1/2-s} \frac{\partial}{\partial x} \ln \left\{ x^\nu K_\nu[\nu x] \right\}, \] (43)
\[ \zeta_{\text{int}} \left( s - \frac{1}{2} \right) = \left( \frac{2\beta_\nu^2 \cos \pi s}{\pi a} \right) \sum_{l=0}^{\infty} \nu l^{2-2s} \int_{\beta_a/\nu}^{\infty} dx \left( x^2 - \frac{\beta_a^2}{\nu^2} \right)^{1/2-s} \times \frac{\partial}{\partial x} \left( \ln \left\{ x^\nu K_\nu[\nu x] \right\} + \ln \left\{ x^\nu (\delta K_\nu[\nu x] + \frac{x\nu}{4} K'_{\nu}[\nu x]) \right\} \right). \] (44)

Here, \( \beta_R = m(a + R), \ \beta_a = ma, \ \delta = \xi - \frac{1}{8} \) and \( \nu = l + \frac{1}{2} \).

The first part of the zeta function [33] depends only on the size of box with throat \( R' = R + a \) and the asymptotic structure of the space time. It is exactly twice the expression in the flat Minkowski space time without throat [12].
calculated for a massive scalar field inside a ball of radius \( R \) with the Dirichlet boundary condition. The factor two is very easy explained: we consider scalar field living in the double-sided plane. The second part \((44)\) does not depend on a boundary and it depends only on the radius of throat \( a \) and non minimal coupling \( \xi \). It contains information about the space-time under consideration. The same division of zeta function into two parts has been already observed for space-time of the thick cosmic string \([13]\) and the space-time of a point-like global monopole \([4]\). Because the first part of zeta function \((13)\) has already been analyzed in great details, we proceed now to consideration the second part \((44)\).

Both expressions \((13)\) and \((44)\) and ground state energy \((30)\) are divergent in the limit \( s \to 0 \) which we are interesting in. According with renormalization procedure, we have to subtract from regularized expression for ground state energy \((34)\) all terms which survive in the limit \( m \to \infty \). This procedure corresponds to the subtraction five (three without boundary) first terms of the DeWitt-Schwinger expansion \([12\, 13]\).

Our goal now is to find in closed form the expansion of zeta function \((14)\) at the point \((-\frac{1}{2})\) as power series over \( s \) (for arbitrary mass). For this reason we use the uniform asymptotic series over inverse index for Bessel functions of large index and argument given in Ref. \([14]\). We subtract from and add to the integrand of Eq. \((44)\) the uniform expansion of it up to terms proportional to \( s^{-3} \). After subtraction we may tend \( s \to 0 \). Second part, which is the uniform expansion of integrand, gives us the pole structure of zeta function. Going in this way (see details in Appendix A) we obtain the following series for zeta function at the point \((-\frac{1}{2})\)

\[
\zeta_{\text{int}}^\alpha \left(s - \frac{1}{2}\right)_{s \to 0} = \frac{1}{(4\pi)^{3/2}\alpha\Gamma(s - \frac{1}{2})} \left\{ b_0^a \beta_3 \Gamma(s - 2) + b_{3/2}^a \beta_3^3 \Gamma(s - \frac{3}{2}) + b_1^a \beta_3^2 \Gamma(s - 1) \
+ b_3^a \beta_3 \Gamma(s - \frac{1}{2}) + b_2^a \Gamma(s) \right\} - \frac{1}{16\pi^2 a} \left\{ b^a \ln \beta_3^a + \Omega^a[\beta_3] \right\},
\]

\[
\zeta_{\text{ext}}^\alpha \left(s - \frac{1}{2}\right)_{s \to 0} = \frac{1}{(4\pi)^{3/2}(a + R)\Gamma(s - \frac{1}{2})} \left\{ b_0^{R \alpha} \beta_3^4 \Gamma(s - 2) + b_{3/2}^{R \alpha} \beta_3^3 \Gamma(s - \frac{3}{2}) + b_1^{R \alpha} \beta_3^2 \Gamma(s - 1) \
+ b_3^{R \alpha} \beta_3 \Gamma(s - \frac{1}{2}) + b_2^{R \alpha} \Gamma(s) \right\} - \frac{1}{16\pi^2(a + R)} \left\{ b^R \ln \beta_3^R + \Omega^R[\beta_R] \right\}.
\]

Here

\[
b_0^a = -\frac{8\pi}{3}, \quad b_{1/2}^a = 0, \quad b_1^a = 32\pi \left[ \frac{\xi - \frac{1}{6}}{6} \right], \quad b_{3/2}^a = 64\pi^{3/2} \left[ \frac{\xi - \frac{1}{8}}{8} \right]^2,
\]

\[
b_2^a = \frac{8\pi}{3} \left[ 128\xi^3 - 64\xi^2 + \frac{56\xi - 68}{15} \right], \quad b_{5/2}^a = \frac{16}{3} \pi^{3/2} \left[ 96\xi^4 - 72\xi^3 + 21\xi^2 - \frac{45}{16} \xi + \frac{93}{640} \right],
\]

\[
b^a = \frac{1}{2} b_0^a \beta_3^4 - b_1^a \beta_3^2 + b_2^a.
\]

and

\[
b_0^{R \alpha} = \frac{8\pi}{3}, \quad b_{1/2}^{R \alpha} = -4\pi^{3/2}, \quad b_1^{R \alpha} = \frac{16}{3} \pi, \quad b_{3/2}^{R \alpha} = -\frac{1}{3} \pi^{3/2},
\]

\[
b_2^{R \alpha} = -\frac{32}{315} \pi, \quad b_{5/2}^{R \alpha} = -\frac{1}{60} \pi^{3/2},
\]

\[
b^{R \alpha} = \frac{1}{2} b_0^{R \alpha} \beta_3^4 - b_1^{R \alpha} \beta_3^2 + b_2^{R \alpha}.
\]

Above expressions \((45)\) and \((46)\) contain all terms which survive in the limit \( s \to 0 \). The details of calculation and a closed form for \( \Omega^a[\beta_3] \) are outlined in the Appendix A. The function \( \Omega^a[\beta_3] \) tends to a constant for \( \beta_3 \to 0 \) and \( \Omega^a[\beta_3] = -b^a \ln \beta_3^a + \sqrt{\pi} \Gamma_{5/2}/\beta_3 + O(1/\beta^3_3) \) for \( \beta_3 \to \infty \) (\( a = a, R \)).

Comparing the above expression with that obtained by the Mellin transformation taking the trace of heat kernel (in three dimensions),

\[
\zeta \left(s - \frac{1}{2}\right)_{m \to \infty} = \frac{m^{2s}}{\Gamma(s - \frac{1}{2})} \int_0^\infty dt t^{s-3/2} K[t]_{t \to 0}
\]

\[
= \frac{1}{(4\pi)^{3/2}\Gamma(s - \frac{1}{2})} \left\{ B_0 m^4 \Gamma(s - 2) + B_{1/2} m^3 \Gamma(s - \frac{3}{2}) + B_1 m^2 \Gamma(s - 1) \
+ B_{3/2} m \Gamma(s - \frac{1}{2}) + B_2 \Gamma(s) + \cdots \right\},
\]

7
we obtain the heat kernel coefficients:

\[
B_0 = \frac{8\pi}{3} \left[ (a + R)^3 - a^3 \right],
\]
\[
B_{1/2} = -4\pi^{3/2}(a + R)^2,
\]
\[
B_1 = 32\pi \left[ \xi - \frac{1}{6} a + \frac{16}{3} \pi (a + R) \right],
\]
\[
B_{3/2} = 64\pi^{3/2} \left[ \xi - \frac{1}{8} \right]^2 - \frac{1}{3} \pi^{3/2},
\]
\[
B_2 = \frac{8\pi}{3a} \left[ 128\xi^3 - 64\xi^2 + \frac{56}{5} \xi - \frac{68}{105} \right] - \frac{32}{315} \frac{\pi}{a(a + R)},
\]
\[
B_{5/2} = \frac{16 \pi^{3/2}}{3a^2} \left[ 96\xi^4 - 72\xi^3 + 21\xi^2 - \frac{45}{16} \xi + \frac{93}{640} \right] - \frac{1}{60} \frac{\pi^{3/2}}{(a + R)^2} .
\]  

Using above scheme we calculated also the coefficient \(B_{5/2}\) which we will need later for analysis. We should like to note the difference between Eqs. (43), (44) and Eq. (51). The Eq. (51) is an asymptotic expansion of zeta function over inverse mass \(m \to \infty\) but the formulas (43), (44) are correct for arbitrary mass \(m\) and small \(s \to 0\). In fact we extracted an asymptotic (for \(m \to \infty\)) part of zeta function in the form (51) and saved a finite part of it. In the limit \(m \to \infty\) the finite part tends to zero and both formulas are in agreement. This is the reason that the function \(\Omega^\alpha [\beta_a] = -b^\alpha \ln \beta_a^2 + \sqrt{\pi} b_{5/2}^\alpha / \beta_a + O(1/\beta_a^2)\) for \(\beta_a \to \infty\) (\(\alpha = a, R\)).

As far as the space-time under consideration has singular two-dimensional surface \(\Sigma\) with codimension one, we cannot use standard formulas obtained for smooth background and we have to utilize formulas obtained by Gilkey, Kirsten and Vassilevich in Ref. [17]. The heat kernel coefficients (52) coincide exactly with that obtained from general formulas in three dimensions given in Ref. [17]. We have to take into account that extrinsic curvature tensor of surface \(\Sigma\) is obtained as covariant derivative of the outward unit normal vector \(N_a\):

\[
K_{\alpha\beta} = \nabla_\alpha N_\beta.
\]  

For this reason this vector has coordinates \(N_a = (0, \pm 1, 0, 0)\) on the spheres \(\rho = \pm R\), and

\[
\text{tr} K = \frac{2}{R + a}
\]  

in both cases. In Appendix A we found general formulas for arbitrary heat kernel coefficients and traced out them in manifest form up to \(b_5\).

To obtain the ground state energy we have to subtract from our expressions (30), (43), (44) all terms which will survive in the limit \(m \to \infty\). Then we set \(s = 0\) and radius of box \(R \to \infty\). Therefore we arrive to the following expression

\[
E^{\text{ren}} = -\frac{1}{32\pi^2 a} \left\{ b^\alpha \ln \beta_a^2 + \Omega^\alpha [\beta_a] \right\} .
\]  

A similar general structure for the ground state energy in massless case has been obtained firstly by Blau, Visser and Wipf [10] using dimensional consideration only and it was confirmed by detail calculations in Refs. [13, 14].

Using above-mentioned behavior of the \(\Omega^\alpha [\beta_a]\), the ground state energy tends to zero for large radius of throat

\[
E^{\text{ren}} \approx -\frac{b_5^\alpha}{32\pi^{3/2}ma_2}, \ a \to \infty,
\]  

and it is divergent for small radius of throat:

\[
E^{\text{ren}} \approx -\frac{b_2^\alpha}{16\pi^2a} \ln (ma), \ a \to 0.
\]  

The numerical calculations of ground state energy \(E^{\text{ren}} / m\) (53) as a function of \(\beta_a = ma\) is depicted in Fig. 1 and 2 for \(\xi = \frac{1}{3}\) and \(\xi = 0\), respectively. The details of numerical calculations is analyzed in Appendix B.

8
FIG. 1. The ground state energy $E = E^{\text{ren}}/m$ as a function of $ma$ for fixed mass $m$ and $\xi = \frac{1}{6}$. The energy has minimum at point $ma \approx 0.16$ with depth $E_{\text{min}}/m \approx -0.0025$.

FIG. 2. The ground state energy $E = E^{\text{ren}}/m$ as a function of $ma$ for fixed mass $m$ and $\xi = 0$. There is no minimum energy; it is always negative.

IV. DISCUSSION

We have calculated the ground state energy of the massive scalar field on the short-throat flat-space wormhole background (see Eq. (55)). It can be written down in the form

$$E^{\text{ren}} = -\frac{hc}{a} f(\beta_a), \quad (58)$$

where $\beta_a = mca/\hbar$, and $f(\beta_a)$ is a function of $\beta_a$ which has the asymptotic

$$f(\beta_a) \approx \frac{b_2}{16\pi^2} \ln \beta_a, \quad \beta_a \to 0,$$

$$f(\beta_a) \approx \frac{b_5/2}{32\pi^{3/2} \beta_a}, \quad \beta_a \to \infty.$$ 

To characterize the behavior of the ground state energy as a function of $\xi$ we note that the coefficient $b_{5/2}$ is positive for all values $\xi$ and hence the ground state energy tends to $-0$ while as $\beta_a \to \infty$. In the limit $\beta_a \to 0$, the behavior of the ground state energy is determined by the sign of $b_2$ (see Eq. (57)) and depends on $\xi$. For $\xi < \xi_* \approx 0.123$, the $b_2$ is negative and the ground state energy tends to minus infinity, otherwise it tends to plus infinity. This difference in an asymptotical behavior at $\beta_a \to 0$ results in two qualitatively different pictures describing the behavior of the ground state energy. In the first case $\xi < \xi_*$, the ground state energy is monotonically increasing from $-\infty$ to 0 and has no extremum (see Fig. 2); while in the second case $\xi > \xi_*$, it has a global minimum. For example in Fig. 1 the

\footnote{In this section we use dimensional units $G$, $c$, and $\hbar$.}
graph of $E^{\text{ren}}/m$ versus $\beta_a$ is shown for $\xi = \frac{1}{6}$. It is seen that the ground state energy has the minimum at $\beta_a \approx 0.16$ with depth $E_{\text{min}}/m \approx -0.0025$.

Let us now speculate about the result obtained. Suppose that the quantum massive scalar field plays the role of the “exotic” matter maintaining an existence of the short-throat flat-space wormhole in a self-consistent manner. This means that the semiclassical Einstein equations have to be satisfied,

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle T_{\mu\nu} \rangle^{\text{ren}},$$

(59)

where $G_{\mu\nu}$ is the Einstein tensor, and $\langle T_{\mu\nu} \rangle$ is the renormalized vacuum expectation values of the stress-energy tensor of the scalar field. The total energy in a static space-time is given by

$$E = \int_V \varepsilon \sqrt{g(3)} d^3x,$$

(60)

where $\varepsilon = -\langle T^t_t \rangle^{\text{ren}} = -G^t_t c^4/8\pi G$ is energy density, and the integral is calculated over the whole space. In the spherically symmetric metric (8) we obtain

$$E = -\frac{c^4}{2G} \int_{-\infty}^{\infty} G^t_t r^2(\rho) d\rho.$$  

(61)

Using the relations $G^t_t = 2r''/r + (r'^2 - 1)/r^2$ and $r(\rho) = |\rho| + a$ we can calculate

$$E = -\frac{2c^4a}{G}.$$  

(62)

Note that the total energy is negative.

In the self-consistent case the total energy must coincide with the ground state energy of the scalar field. Equating Eqs. (58) and (62) gives

$$\frac{2c^4a}{G} = \frac{\hbar c}{a} f(\beta_a),$$

or

$$a = l_P \sqrt{\frac{f(\beta_a)}{2}},$$  

(63)

where $l_P = \sqrt{\hbar G/c^3}$ is the Planck length. To make further estimations we take into account that in order to be stable a quantum system should be in the state with minimum of ground state energy. This requirement can be fulfilled in case $\xi > \xi_*$. In particular, for $\xi = 1/6$ the minimum $E_{\text{min}}/mc^2 \approx -0.0025$ is achieved at $\beta_a = mca/\hbar \approx 0.16$. This gives $f(\beta_a) \approx 4 \times 10^{-4}$, so that

$$a \approx 0.0141l_P,$$

(64)

and

$$m \approx 11.35 m_P,$$  

(65)

where $m_P = (\hbar c/G)^{1/2}$ is the Planck mass.

Thus, our estimations have revealed that the self-consistent semiclassical wormhole, if exists, should possess the throat of sub-Planckian radius, and the quantum scalar field maintaining the wormhole’s existence should have super-Planckian mass. Of course, it should be noted that our consideration has been restricted by a toy model of the short-throat flat-space wormhole, and so one may expect that in more realistic models results would slightly be changed.

Let us emphasize that the result obtained in this work for the wormhole configuration can be generalized. Really, the behavior of ground state energy for small (57) and large (56) values of the throat’s radius $a$ depends only on two dimensionless heat kernel coefficients $b_2$ and $b_{5/2}$, respectively. Instead of the radius $a$, we could use a typical size of system $\lambda$ (throat) and calculate the coefficients $b_2$ and $b_{5/2}$ on the corresponding background. Now let us consider the dimensionless ground state energy $E^{\text{ren}}/m$. Obviously, it will only depend on the dimensionless combination $m\lambda$, and hence the limit of large (small) mass will correspond to limit of the large (small) size of the system. Since for
renormalization we have to subtract the first five terms (up to $b_2$) of expansion for large mass the ground state energy
in this limit should be proportional to the next non-vanishing term of expansion
\[ E^{ren} \approx \frac{1}{2} \frac{1}{(4\pi)^{3/2}} \frac{b_5/2}{(m\lambda)^2} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-\frac{1}{2})} \bigg|_{s \to 0} = -\frac{b_5/2}{32\pi^{3/2}(m\lambda)^2}, \]  
which coincides with Eq. (56). We would like to note that the coefficient $b_5/2$ is non-zero in the limit $R \to \infty$
for background with singular scalar curvature as it was shown in Ref. [17]. For smooth, non-singular geometrical
characteristics of background it is zero and we have to take into account the next non-vanishing coefficient which is $\tilde{b}_3$. In this case we have the following expression in the limit $m\lambda \to \infty$:
\[ E^{ren} \approx -\frac{b_3}{32\pi^2(m\lambda)^3}. \]  
The origin of logarithmic term, as well as the behavior for small size of system is following. The structure of poles
of zeta function does not depend on the parameters of system $m$ and $\lambda$. The subtraction of the asymptotics for great
mass brings us the following contribution to the ground state energy
\[ \frac{(m\lambda)^2s - 1}{2(\lambda m)(4\pi)^{3/2}\Gamma(s-\frac{1}{2})} \times \left\{ b_0(\lambda m)^4\Gamma(s-2) + b_1(\lambda m)^3\Gamma(s-\frac{3}{2}) 
+ b_1(\lambda m)^2\Gamma(s-1) + b_3(\lambda m)^2\Gamma(s-\frac{1}{2}) + b_2\Gamma(s) \right\}_{s \to 0}, \]  
where $b_\alpha$ are dimensionless heat kernel coefficients. Taking the limit in this formula we observe that the heat kernel
coefficients with integer indices will be survived:
\[ -\frac{1}{32\pi^2(\lambda m)} \left( \frac{1}{2} b_0(\lambda m)^4 - b_1(\lambda m)^2 + b_2 \right) \ln(\lambda m)^2. \]
Therefore in the limit $\lambda \to 0$ one has
\[ E^{ren} \approx -\frac{b_2 \ln(\lambda m)}{16\pi^2(\lambda m)}, \]  
in agreement with Eq. (71).
Therefore the necessary condition that the ground state energy will possess a minimum is following: the coefficients
$b_2$ and the next non-vanishing coefficient ($b_5/2$ for singular curvature and $b_3$ for non-singular) must be positive. If it
is so, the discussion above is valid and the self-consistent semi-classical wormhole exist. The radius of throat of stable
wormhole and the mass of scalar boson in this case depend on the model of wormhole and value of non-conformal
coupling $\xi$.

ACKNOWLEDGMENT

The authors would like to thank M. Bordag, J.S. Dowker, K. Kirsten and D. Vassilevich for helpful comments on
some items of paper. This work was supported by the Russian Foundation for Basic Research grant No 99-02-17941.

APPENDIX A:

The uniform asymptotic expansion of the modified Bessel’s functions have the form below
\[ K_\nu [\nu x] = \sqrt{\frac{\pi t}{2\nu}} e^{-\nu \eta} \sum_{k=0}^{\infty} \frac{u_k[t]}{(-\nu)^k}, \quad I_\nu [\nu x] = \sqrt{\frac{t}{2\pi \nu}} e^{\nu \eta} \sum_{k=0}^{\infty} \frac{v_k[t]}{\nu^k}, \quad (A1) \]
\[ K'_\nu [\nu x] = -\sqrt{\frac{\pi}{2\nu x^2 t}} e^{-\nu \eta} \sum_{k=0}^{\infty} \frac{u_k[t]}{(-\nu)^k}, \quad I'_\nu [\nu x] = \sqrt{\frac{1}{2\pi \nu x^2 t}} e^{\nu \eta} \sum_{k=0}^{\infty} \frac{v_k[t]}{\nu^k}, \]
where
\[ t = \frac{1}{\sqrt{1 + x^2}}, \quad \eta = \sqrt{1 + x^2} + \ln \frac{x}{1 + \sqrt{1 + x^2}}. \]

\[ u_{k+1}[t] = \frac{1}{2} \int_0^1 (1 - 5t^2) u_k[t] \, dt, \quad u_0[t] = 1 \]
\[ v_{k+1}[t] = u_{k+1}[t] + t(t^2 - 1) \left\{ \frac{1}{2} u_k[t] + tu_k'[t] \right\}, \quad v_0[t] = 1. \]

Taking into account these formulas in Eq. \[44\] we obtain power series over \( s \) for zeta function. The uniform asymptotic expansion \([A3]\) up to \( \nu^{-n} \) allows us to take into account terms up to \( m^{3-n} \). Because we need for all terms which survive in limit \( m \to \infty \) we use uniform expansion up to \( n = 3 \).

Therefore we have the following expression for the zeta function

\[ \zeta_{int}^a \left( s - \frac{1}{2} \right) = -\frac{2 \beta^2 a \cos \pi s}{\pi a} \sum_{l=0}^{\infty} \nu^{2-2s} \int_{\beta_a/\nu}^\infty dx \left( x^2 - \frac{\beta^2}{\nu^2} \right)^{1/2-s} \left( \ln \{ x^\nu K_p[nx] \} + \ln \{ x^\nu (\delta K_p[nx] + \frac{x^\nu}{4} K_p'[nx]) \} \right) - 3 \sum_{k=1}^3 (-\nu)^{-k} N_k \]

where functions \( N_p \) may be found in closed form for arbitrary index \( p \) using simple program in package Mathematica. For \( p \geq 0 \) they are polynomial of 3\( p \) degree and has the following form

\[ N_p[t] = \sum_{k=0}^{p} a_{p,k} t^{p+2k}. \]

The first five \( N_p \) are listed below

\[ N_0 = 0, \quad N_{-1} = 2\eta, \]
\[ N_1 = \left[ 4\delta - \frac{1}{4} \right] t + \frac{1}{12} t^3, \]
\[ N_2 = -8 \left[ \delta - \frac{1}{8} \right] t^2 - 2 \left[ \delta - \frac{1}{8} \right] t^4 - \frac{1}{8} t^6, \]
\[ N_3 = \frac{1}{3} \left[ 64\delta^3 - 24\delta^2 + \frac{9}{2} \left( -19 \delta + \frac{169}{64} \right) t^3 + \frac{1}{5} \left( 40\delta^2 - 25\delta + \frac{169}{64} \right) t^5 + \frac{1}{7} \left( -329 \delta - \frac{179}{576} \right) t^7 \right]. \]

We should like to note that the expression \([A3]\) is identical to original one \([14]\). First term is finite in the limit \( s \to 0 \); all divergences are contained into the second part.

Integrating over \( x \) with help of integral

\[ \int_{\beta/\nu}^{\infty} dx x^{1/2-s} (1 + x^2)^{-p/2} = \frac{\Gamma(\frac{3}{2} - s) \Gamma(s + \frac{p-3}{2})}{2 \Gamma(\frac{p}{2})} \left( \frac{\nu}{\beta} \right)^{p-3+2s} \left( 1 + \frac{\nu^2}{\beta^2} \right)^{-s - \frac{p-3}{2}} \]

and taking the limit \( s \to 0 \) in the first term we get

\[ \zeta_{int}^a \left( s - \frac{1}{2} \right) \to -\frac{1}{16\pi^2 a} A_f[\beta_a] + \frac{1}{(4\pi)^{3/2} a \Gamma(s - \frac{1}{2})} \sum_{k=1}^3 (-1)^k A_k[\beta_a], \]

where

\[ A_f[\beta_a] = 32\pi \sum_{l=0}^{\infty} \nu^2 \int_{\beta/\nu}^{\infty} dx \sqrt{x^2 - \frac{\beta^2}{\nu^2} \frac{\partial}{\partial x} \ln K_p(\nu x) + \ln \left[ \delta K_p(\nu x) + \frac{x^\nu}{4} K_p'[\nu x] \right] + 2\nu \eta(x) + \frac{1}{\nu} N_1 - \frac{1}{\nu^2} N_2 + \frac{1}{\nu^3} N_3}. \]
\[ A_{-1} = 4\pi \beta^2 \Gamma(s - \frac{1}{2}) \sum_{l=0}^{\infty} \frac{Z(0, l + s - 1)}{\Gamma(l + s + 1/2)}, \tag{A9} \]

\[ A_p = -8\pi^{3/2} \beta^{1-p} \sum_{k=0}^{p} \frac{a_{p,k}}{\Gamma(l + s + 1/2)} Z(2k, s + k + \frac{p-1}{2}), \tag{A10} \]

\[ Z(p, s) = \Gamma(s) \sum_{l=0}^{\infty} \frac{2\nu}{(1 + \nu^2/\beta_a^2)^s} \left( \frac{\nu}{\beta_a} \right)^p. \tag{A11} \]

The first four \( A_p \) are listed below:

\[ A_0[\beta_a] = 0, \]

\[ A_1[\beta_a] = -8\pi \left[ \left( 4\delta - \frac{1}{4} \right) Z(0, s) + \frac{1}{6} Z(2, s + 1) \right], \]

\[ A_2[\beta_a] = \frac{4\pi^{3/2}}{\beta_a^1} \left[ 16 \left( \delta - \frac{4}{5} \right)^2 Z(0, s + \frac{1}{2}) + 4 \left( \delta - \frac{1}{8} \right) Z(2, s + \frac{3}{2}) + \frac{1}{8} Z(4, s + \frac{5}{2}) \right], \tag{A12} \]

\[ A_3[\beta_a] = -\frac{16\pi}{3\beta_a^2} \left[ 64\delta^3 - 24\delta^2 + 9 \left( \delta - \frac{19}{64} \right) Z(0, s + 1) + \frac{2}{5} \left( 40\delta^2 - 25\delta + \frac{169}{64} \right) Z(2, s + 2) + \frac{4}{35} \left( \frac{49}{\delta - \frac{329}{64}} \right) Z(4, s + 3) + \frac{179}{2520} Z(6, s + 4) \right]. \]

To find the heat kernel coefficients we have to take limit \( m \to \infty \) in Eq. (A7). The asymptotic expansion of \( Z(0, q) \) over inverse power of \( \beta_a^2 \) was found in Ref. [14]:

\[ Z(0, s) = \beta_a^2 \Gamma(s - 1) + 2 \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \Gamma(l + s) \beta_a^{-2l} \zeta_H(-1 - 2l, \frac{1}{2}), \tag{A13} \]

where \( \zeta_H(s, a) \) is the Hurwitz zeta function

\[ \zeta_H(s, a) = \sum_{l=0}^{\infty} (l + a)^{-s}, \quad s > 1. \tag{A14} \]

Other functions \( Z(2k, s + k + \frac{p-1}{2}) \) in (A10) are expressed in terms of \( Z(0, q) \) by relation

\[ Z(2k, s + k + \frac{p-1}{2}) = \sum_{n=0}^{\infty} \frac{k! n!}{n!(k-n)!} \frac{\Gamma(k + \frac{p-1}{2} + s)}{\Gamma(n + \frac{p-1}{2} + s)} Z(0, n + \frac{p-1}{2} + s). \tag{A15} \]

Taking into account above formulas we obtain the following formulas for heat kernel coefficients

\[ b_n = -\frac{1}{\Gamma(s - 2 + n)} \sum_{p=0}^{n} \alpha_{n-p-1}(2p - 1), \tag{A16} \]

\[ b_{n+1/2} = \frac{1}{\Gamma(s - \frac{3}{2} + n)} \sum_{p=0}^{n} \alpha_{n-p-1}(2p), \tag{A17} \]

where \( (l, p \geq 0) \)

\[ \alpha_{-1}(-1) = \frac{8\pi}{3} \Gamma(s - 2), \]

\[ \alpha_l(-1) = 16\pi(-1)^{l+1} \frac{\zeta_H(-1 - 2l, \frac{1}{2})}{l!(1 - 2l)} \Gamma(s - 1 + l), \]

\[ \alpha_l(p) = -8\pi^{3/2} \sum_{k=0}^{p} \frac{a_{p,k}}{\Gamma(k + \frac{p}{2})} z_l(p, k). \tag{A18} \]
\[
z_{-1}(p, k) = \sum_{n=0}^{k} (-1)^n \frac{k!}{n!(k-n)!} \frac{\Gamma(k + \frac{p-1}{2} + s)}{n + \frac{p-3}{2} + s},
\]
\[
z_l(p, k) = 2 \frac{(-1)^l}{l!} \zeta_H(-1 - 2l, \frac{1}{2}) \sum_{n=0}^{k} (-1)^n \frac{k!}{n!(k-n)!} \frac{\Gamma(k + \frac{p-1}{2} + s)}{n + \frac{p-3}{2} + s} \Gamma(l + n + \frac{p-1}{2} + s).
\]

Using these formulas, one obtains the following expressions for heat kernel coefficients

\[
b_0^a = -\frac{8\pi}{3} b_{a/2}^0, \quad b_1^a = 32 \pi \left[ \xi \frac{1}{6} \right], \quad \frac{b_3^a}{2} = 64 \pi^3 / 2 \left[ \xi \frac{1}{8} \right]^2,
\]
\[
b_2^a = \frac{8\pi}{3} \left[ \frac{128\xi^3 - 64\xi^2 + 56\xi - 68}{105} \right], \quad \frac{b_5^a}{2} = \frac{16\pi^3}{3} \left[ \frac{96\xi^4 - 72\xi^3 + 21\xi^2 - 45\xi - 93}{640} \right], \quad \text{(A19)}
\]
\[
b_3^a = \frac{8\pi}{3} \left[ \frac{4096\xi^5 - 4096\xi^4 + 35584\xi^3 - 1088\xi^2 + 848\xi - 144}{385} \right].
\]

The coefficient \(b_{k/2}\) is polynomial of \((k - 1)\)-th order over \(\xi\). The coefficient \(b_2\) changes its sign at point \(\xi_s \approx 0.123\) and \(b_{5/2}\) is positive for arbitrary \(\xi\).

Our problem now is to take limit \(s \to 0\) in the second part of Eq. (A3). Because the function \(Z(p, s)\) with \(p = 2, 4, \cdots\) may be expressed in terms the \(Z(0, s)\) only, let us analyze it in details. Let us suppose for a moment that \(\beta_a < 1\) and represent \(Z(0, s)\) as power series over \(\beta_a:\)

\[
Z(0, s) = 2\beta_a^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(n + s)}{\beta_a^2 \zeta_H(2n + 2s - 1, \frac{1}{2})}.
\]

The gamma function \(\Gamma(s)\) has simple poles in points \(s = 0, -1, -2, \cdots\) and the Hurwitz zeta function \(\zeta_H(s, p)\) has simple pole only in one point \(s = 1\). They have the following expansion near their poles

\[
\Gamma(s - n)_{s \to 0} = \frac{(-1)^n}{n!} \left( \frac{1}{s} + \Psi[n + 1] \right) + O(s), \quad \zeta_H(s + 1, p)_{s \to 0} = \frac{1}{s} - \Psi[p] + O(s), \quad \text{(A21)}
\]

where \(\Psi[x]\) is the digamma function.

All divergences of the function \(Z(0, s)\) in the limit \(s \to 0\) are contained in the first two terms with \(n = 0, 1\). The rest of series is finite and we set \(s = 0\) in it. Therefore we obtain the following expression

\[
Z(0, s)_{s \to 0} = 2\beta_a^2 \left\{ \frac{1}{2} \beta_a^2 \Gamma(s - 1) + \frac{1}{24} \Gamma(s) + \frac{1}{2} \beta_a^2 \left[ 1 - 2\gamma - 4 \ln 2 \right] - \frac{1}{12} \left[ 12\zeta_H(-1) + \ln 2 \right] \right\}
\]
\[
+ 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \beta_a^{2n} \zeta_H(2n - 1, \frac{1}{2}) + O(s),
\]

where \(\zeta_H(s)\) is the Riemann zeta function and \(\gamma\) is the Euler constant.

The series in above formula may be analytically continued for arbitrary value of \(\beta_a\). First of all using series representation of Hurwitz zeta function (A14) we represent this series in the following form

\[
j_2(\beta_a) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \beta_a^{2n} \zeta_H(2n - 1, \frac{1}{2}) = \sum_{\nu=0}^{\infty} \nu \left[ - \ln \left( 1 + \frac{\beta_a^2}{\nu^2} \right) + \frac{\beta_a^2}{\nu^2} \right]. \quad \text{(A23)}
\]

Then, taking into account the integral representation for logarithm

\[
\ln x = \int_0^x \frac{dt}{1 + t},
\]

and the close expression for series below

\[
j_0(x^2) = \sum_{\nu=0}^{\infty} \frac{1}{\nu (\nu^2 + x^2)} = \frac{1}{2x^2} \left\{ \Psi \left( \frac{1}{2} - ix \right) + \Psi \left( \frac{1}{2} + ix \right) - 2\Psi \left( \frac{1}{2} \right) \right\}, \quad \text{(A24)}
\]

14
Here we introduced the following notations

\[ j_2(\beta) = \beta^2 \int_0^1 dx \{ \Psi \left( \frac{1}{2} - i x \beta \right) + \Psi \left( \frac{1}{2} + i x \beta \right) - 2 \Psi \left( \frac{1}{2} \right) \}. \tag{A25} \]

The function in rhs is analytical in whole complex plane and therefore it gives analytical continuation the series \( j_2(\beta_a) \) for arbitrary value of \( \beta_a \). This representation of \( j_2(\beta) \) is preferable for numerical calculations.

Using the same approach for other \( Z(p, q) \) we arrive at the following formulas for \( A_k[\beta_a] \)

\[
\begin{align*}
A_{-1}[\beta_a] &= -\frac{8\pi}{3} \beta_a^2 \left\{ \frac{7}{160} \Gamma(s) - \frac{1}{4} \beta_a^2 \Gamma(s-1) - \beta_a^4 \Gamma(s-2) \right\} + \omega_{-1}(\beta_a), \\
A_1[\beta_a] &= -\frac{8\pi}{3} \beta_a^2 \left\{ \left( \delta - \frac{1}{16} \right) \Gamma(s) + 12 \left( \delta - \frac{1}{48} \right) \beta_a^2 \Gamma(s-1) \right\} + \omega_1(\beta_a), \\
A_2[\beta_a] &= 4\pi^{3/2} \beta_a^2 \left\{ 16\delta^2 \beta_a \Gamma(s-1) + \frac{4}{3 \beta_a} \left( \delta - \frac{1}{8} \right)^2 \Gamma(s+1) \right\} + \omega_2(\beta_a), \\
A_3[\beta_a] &= \frac{8\pi}{3} \beta_a^2 \left\{ 128\delta^3 - 16\delta^2 + \frac{1}{5} \delta + \frac{71}{3360} \right\} \Gamma(s) + \omega_3(\beta_a),
\end{align*}
\]  
(A26)

where

\[
\begin{align*}
\omega_{-1}(\beta_a) &= 8\pi \left\{ \left[ \frac{7}{160} - \frac{7}{2} \zeta’(-3) + \frac{1}{240} \ln 2 \right] + \beta_a^2 \left[ 2\zeta’(-1) + \frac{1}{6} \ln 2 + \frac{1}{4} \right] \right. \\
&\left. + \beta_a^4 \left[ \frac{1}{3} \gamma - \frac{13}{36} + \frac{2}{3} \ln 2 \right] + j_3(\beta_a) \right\}, \\
\omega_1(\beta_a) &= -16\pi \left\{ \left[ \frac{1}{4} \zeta’(-1) + \frac{1}{48} \ln 2 + \frac{1}{144} \right] + \delta \left[ -4\zeta’(-1) - \frac{1}{3} \ln 2 \right] \right. \\
&\left. + \beta_a^2 \left[ \frac{1}{12} \theta - \frac{1}{8} + \frac{1}{6} \ln 2 \right] + \delta \left[ -4\gamma - 8 \ln 2 + 2 \right] + 4 \left( \delta - \frac{1}{16} \right) j_2(\beta_a) + \frac{1}{6} \beta_a^4 j_0(\beta_a) \right\}, \\
\omega_2(\beta_a) &= -8\pi^2 \left\{ -16\delta^2 \beta_a - 16 \left[ \delta - \frac{1}{8} \right]^2 j_{+,0}(\beta_a) - 2 \left[ \delta - \frac{1}{8} \right] j_{+,0}(\beta_a) - \frac{3}{32} j_{+,0}(\beta_a) \right\}, \\
\omega_3(\beta_a) &= -\frac{32}{3} \pi \left\{ \left[ \frac{71}{3360} \ln 2 + \frac{757}{20160} + \frac{71}{6720} \gamma \right] + \delta \left[ \frac{1}{10} \gamma - \frac{4}{5} + \frac{1}{5} \ln 2 \right] + \delta^2 \left[ -8\gamma + 8 - 16 \ln 2 \right] \\
&+ \delta^3 \left[ 128 \ln 2 + 64 \gamma \right] + \left[ 64\delta^3 - 24\delta^2 + \frac{9}{2} - \frac{19}{64} \right] j_{1,1}(\beta_a) + \frac{2}{5} \left[ 40\delta^2 - 25\delta + \frac{169}{64} \right] j_{2,1}(\beta_a) \right. \\
&\left. + \frac{8}{35} \left[ \delta - \frac{329}{64} \right] j_{3,1}(\beta_a) + \frac{179}{420} j_{4,1}(\beta_a) \right\}. 
\end{align*}
\]  
(A27)

Here we introduced the following notations

\[
\begin{align*}
j_3(\beta) &= \sum_{n=3}^{\infty} \frac{(-1)^n}{n(n-\frac{1}{2})(n-1)} \beta^{2n} \zeta_H \left( 2n - 3, \frac{1}{2} \right), \tag{A28} \\
j_{p,q}(\beta) &= \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(n+p)}{n! \Gamma(p)} \beta^{2n} \zeta_H \left( 2n + q, \frac{1}{2} \right). \tag{A29}
\end{align*}
\]

The functions \( j_3(\beta) \) and \( j_{p,q}(\beta) \) with integer \( p \) and \( q \) are expressed in terms the function \( j_0(\beta) \) only, by relations

\[
\begin{align*}
j_3(\beta) &= -2\beta^4 \int_0^1 dx xx(1-x)^2 \left\{ \Psi \left( \frac{1}{2} - i x \beta \right) + \Psi \left( \frac{1}{2} + i x \beta \right) - 2 \Psi \left( \frac{1}{2} \right) \right\}, \\
j_{1,1}(\beta) &= -x j_0(x) |_{x=\beta^2}, \\
j_{2,1}(\beta) &= -2x j_0(x) - x^2 j_0'''(x) |_{x=\beta^2}, \\
j_{3,1}(\beta) &= -3x j_0(x) - 3x^2 j_0''(x) - \frac{1}{2} x^3 j_0''(x) |_{x=\beta^2}, \\
j_{4,1}(\beta) &= -4x j_0(x) - 6x^2 j_0''(x) - 2x^3 j_0''(x) - \frac{1}{6} x^4 j_0''(x) |_{x=\beta^2}.
\end{align*}
\]  
(A30)
For half-integer indexes \( p \) and \( q \) in Eq. (A29) we use the integral representation for Hurwitz zeta function from Ref. [18]

\[
\zeta_H(s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(v-1)x}s^{-1}}{e^x - 1} \, dx
\]  

(A31)

and obtain the following formulas

\[
\begin{align*}
J_\frac{1}{2}, 0(\beta) &= -\beta \int_0^\infty \frac{dx}{\sinh(x)} J_1(2\beta x), \\
J_\frac{3}{2}, 0(\beta) &= -\beta \int_0^\infty \frac{dx}{\sinh(x)} \{3J_1(2\beta x) - 2\beta x J_2(2\beta x)\}, \\
J_\frac{5}{2}, 0(\beta) &= -\beta \int_0^\infty \frac{dx}{\sinh(x)} \left\{5J_1(2\beta x) - \frac{20}{3} \beta x J_2(2\beta x) + \frac{4}{3} \beta x J_3(2\beta x)\right\}.
\end{align*}
\]  

(A32)

Substituting the formulas obtained in Eq. (A7) one has

\[
\zeta_{\text{int}}^a \left(s - \frac{1}{2}\right)_{s \to 0} = \frac{\beta_a^{2s}}{(4\pi)^{3/2}\Gamma(s - \frac{1}{2})} \left\{b_2^a m^4 \Gamma(s - 2) + b_{1/2}^a m^3 \Gamma(s - \frac{3}{2}) + b_1^a m^2 \Gamma(s - 1) + b_{3/2}^a m \Gamma(s - \frac{1}{2}) + b_2^a \Gamma(s)\right\} - \frac{1}{16\pi^2a} \Omega^a(\beta_a),
\]  

(A33)

where

\[
\Omega^a(\beta_a) = A_f^a(\beta_a) + \sum_{k=1}^3 (-1)^k \omega_k(\beta_a).
\]  

(A34)

In the limit \( s \to 0 \) an additional finite contribution appears due to multiplication \( s \ln \beta_a^2 \) and gamma functions in brackets (A33). Because the gamma function has simple poles in points \( 0, -1, -2, \cdots \), the heat kernel coefficients with integer indexes will give the finite contributions and we arrive at the following expression

\[
\zeta_{\text{int}}^a \left(s - \frac{1}{2}\right)_{s \to 0} = \frac{1}{(4\pi)^{3/2}a\Gamma(s - \frac{1}{2})} \left\{b_2^a \beta_a^4 \Gamma(s - 2) + b_{1/2}^a \beta_a^3 \Gamma(s - \frac{3}{2}) + b_1^a \beta_a^2 \Gamma(s - 1) + b_{3/2}^a \beta_a \Gamma(s - \frac{1}{2}) + b_2^a \Gamma(s)\right\} - \frac{1}{16\pi^2a} \{b^a \ln \beta_a^2 + \Omega^a[\beta_a]\}.
\]  

(A35)

Therefore we divided the zeta function in two parts: the asymptotic singular part of zeta function in standard form and finite contribution.

Using the same approach for \( \zeta_{\text{ext}}^a(s - \frac{1}{2}) \) one has

\[
\zeta_{\text{ext}}^a \left(s - \frac{1}{2}\right)_{s \to 0} = \frac{1}{(4\pi)^{3/2}(a + R)\Gamma(s - \frac{1}{2})} \left\{b_2^R \beta_R^4 \Gamma(s - 2) + b_{1/2}^R \beta_R^3 \Gamma(s - \frac{3}{2}) + b_1^R \beta_R^2 \Gamma(s - 1) + b_{3/2}^R \beta_R \Gamma(s - \frac{1}{2}) + b_2^R \Gamma(s)\right\} - \frac{1}{16\pi^2(a + R)} \{b^R \ln \beta_R^2 + \Omega^R[\beta_R]\}.
\]  

(A36)

By virtue of the fact that in the limit \( m \to \infty \) the above formulas must give us the asymptotic expansion \([51]\), the function \( \Omega[\beta] \) has the following behavior \( (\beta_a \to \infty, \ (\alpha = a, R)) \)

\[
\Omega^a[\beta_a] = -b^a \ln \beta_a^2 - \frac{2\sqrt{\pi}}{\Gamma(s - \frac{3}{2})} \sum_{k=5}^\infty b_{k/2}^a \beta_a^4 - k \Gamma(s + \frac{k}{2} - 2) \bigg|_{s \to 0}
\]  

(A37)

\[
\begin{align*}
&= -b^a \ln \beta_a^2 + \sum_{k=5}^\infty b_{k/2}^a \beta_a^4 - k \Gamma(k/2 - 2) \\
&= -b^a \ln \beta_a^2 + \sqrt{\pi} b_{5/2}^a \beta_a^{-1} + O(\beta_a^{-2}).
\end{align*}
\]
APPENDIX B:

The main problem for numerical calculation the ground state energy is the term $A_f[\beta]$ given by Eq. (A8). The series in (A8) is low convergent. To calculate this expression let us, first of all, represent it in the following form

$$A_f = \frac{A_f}{32\pi} = \sum_{l=0}^{\infty} \sigma_\nu, \text{ where}$$

$$\sigma_\nu = \nu^2 \int_\beta/\nu^{\infty} dx \sqrt{x^2 - \beta^2} \frac{\partial}{\nu^2 \partial x} \left( \ln K_{\nu}(\nu x) + \ln \left[ \delta K_{\nu}(\nu x) + \frac{x\nu}{4} K'_{\nu}(\nu x) \right] \right)$$

and divide the series in two parts

$$A_f = \sum_{l=0}^{N} \sigma_\nu + \sum_{l=N+1}^{\infty} \sigma_\nu.$$

The first sum we calculated numerically. The calculations become lighter due to fact that the Bessel’s functions of second kind with half-integer indexes are polynomial with simple exponent factor [16]. In the second sum we use the uniform expansion of the integrand over inverse power of index $\nu$. Since we have already subtracted first three terms $N_1, N_2$ and $N_3$, the uniform expansion of integrand will start from $\nu^{-4}$ and we obtain the following expression

$$\sum_{l=N+1}^{\infty} \sigma_1 = \sum_{l=N+1}^{\infty} \nu^2 \int_\beta/\nu^{\infty} dx \sqrt{x^2 - \beta^2} \frac{\partial}{\nu^2 \partial x} \sum_{p=4}^{\infty} (-\nu)^{-p} N_p[t],$$

where $N_p[t]$ is the polynomial of $3p$ degree:

$$N_p[t] = \sum_{k=0}^{p} a_{p,k} t^{p+2k}.$$  \hspace{1cm} (B5)

The coefficients $a_{p,k}$ for $p = 1, 2, 3$ may be singled out from Eq. (A5).

Then one takes derivatives and integrals in Eq. (B4) and changes the sums over $p$ and $l$. After this we arrive at the following formula

$$\sum_{l=N+1}^{\infty} \sigma_l = \sum_{p=4}^{\infty} N_p,$$

where

$$N_p = \frac{\sqrt{\pi}}{2} (-1)^{3-p} \sum_{k=0}^{p} a_{p,k} \frac{\Gamma\left(\frac{q}{2} - \frac{1}{2} + k\right)}{\Gamma\left(\frac{q}{2} + k\right)} h[p, p + 2k, \beta, N],$$

$$h[p, q, \beta, N] = \sum_{l=N+1}^{\infty} \nu^{2-p} \left(1 + \frac{\beta^2}{\nu^2}\right)^{-(q-1)/2}.$$ \hspace{1cm} (B8)

The function $h$ may be found in closed form for integer $p$ and $q$.

The above function $h$ may be estimated by

$$h[p, q, \beta, N] \approx \left(N + \frac{3}{2}\right)^{2-p},$$

and the series (B6) is fast convergent for great $N$. We use $N = 10$ and in order to work with precision $10^{-10}$ it is enough to take expansion up to $p = 8$ (five terms). In fact, this procedure converts the low convergent series to fast convergent series over $N^{2-p}$. 

17
To illustrate above approach we reproduce in Fig. 3 the three steps of calculation of $A_f$: (i) the zero term (thick curve), (ii) the contribution of first eleven terms up to $l = 10$ (middle thickness) and (iii) exact curve (up to $p = 8$ in uniform expansion, thin curve). Ten terms $l = 1 \div 10$ gives us the correction of 36% for zero term, and the series from $l = 11$ to $\infty$ gives us 4% correction.

\[ A = \frac{A_f}{\Delta \xi} \] as a function of $ma$ for $\xi = \frac{1}{6}$. Thick curve is zero term ($l = 0$) contribution. The curve of middle thickness is the contribution of the first eleven terms up to $l = 10$. Thin curve reproduces the calculations with high precision (up to $p = 8$ of uniform expansion).

FIG. 3. The function $A = \frac{A_f}{\Delta \xi}$ as a function of $ma$ for $\xi = \frac{1}{6}$. Thick curve is zero term ($l = 0$) contribution. The curve of middle thickness is the contribution of the first eleven terms up to $l = 10$. Thin curve reproduces the calculations with high precision (up to $p = 8$ of uniform expansion).

[1] L. Flamm, Phys. Z., 17, 48 (1916).
[2] A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935).
[3] J.A. Wheeler, Phys. Rev. 97, 511 (1955).
[4] M.S. Morris and K.S. Thorne, Am. J. Phys. 56, 395 (1988).
[5] M.S. Thorne, K.S. Thorne and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).
[6] The reader can find some references on wormhole physics in the excellent book by Visser [7].
[7] M. Visser, Lorentzian Wormholes: from Einstein to Hawking, (American Institute of Physics, Woodbury, 1995).
[8] S.V. Sushkov, Phys. Lett. A 164, 33 (1992); A. Popov and S.V. Sushkov, Quantum Field Theory Under the Influence of External Conditions, ed. M. Bordag (Teubner, Leipzig, 1996); D. Hochberg, A. Popov, S. V. Sushkov, Phys. Rev. Lett. 78, 2050 (1997).
[9] V. Khatsymovsky, Phys. Lett. B 429, 254 (1998).
[10] J.S. Dowker and R. Critchley, Phys. Rev. D 13, 3224 (1976); W. Hawking, Commun. Math. Phys. 55, 133 (1977); S.K. Blau, M. Visser, and A. Wipf, Nucl. Phys. B 310, 163 (1988).
[11] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, and S. Zerbini, Zeta Regularization Techniques with Applications, (World Scientific, Singapore, 1994).
[12] M. Bordag, J. Phys. A 28, 755 (1995); M. Bordag and K. Kirsten, Phys. Rev. D 53, 5753 (1996); M. Bordag, K. Kirsten, and E. Elizalde, J. Math. Phys. 37, 895 (1996); M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, Phys. Rev. D 56, 4896 (1997); M. Bordag, K. Kirsten, and D. Vassilevich, Phys. Rev. D 59, 05011 (1999).
[13] N.R. Khusnutdinov and M. Bordag, Phys. Rev. D 59, 064017 (1999).
[14] E.R. Bezerra de Mello, V.B. Bezerra and N.R. Khusnutdinov, J. Math. Phys. 42, 562 (2001).
[15] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time, (Cambridge University Press, Cambridge, London, 1973).
[16] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, (US National Bureau of Standards, Washington, 1964).
[17] P.B. Gilkey, K. Kirsten and D.V. Vassilevich, Nucl. Phys. B 601, 125 (2001).
[18] H. Bateman and A. Erdélyi, Higher Transcendental Functions (McGraw-Hill, New York, 1953), Vol. 1.