A ZETA FUNCTION FOR MULTICOMPLEX ALGEBRA

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Abstract. In this paper we define and study a Dedekind-like zeta function for the algebra of multicomplex numbers. By using the idempotent representations for such numbers, we are able to identify this zeta function with the one associated to a product of copies of the field of Gaussian rationals. The approach we use is substantially different from the one previously introduced by Rochon (for the bicomplex case) and by Reid and Van Gorder (for the multicomplex case).

1. Introduction

In this paper we build on the resurgent interest for the theory of bicomplex and multicomplex numbers, to develop a definition (and discuss the fundamental properties) of a Dedekind-like zeta function for the spaces of multicomplex numbers. Our approach is significantly different from the one recently used by Rochon [19], and Reid and Van Gorder [17]. We should note that zeta functions play a significant role in a variety of fields, ranging from number theory to statistical mechanics, from quantum field theory (where they are used to regularize divergent series and divergent integrals) to dynamical systems, and finally to the theory of crystals and quasi-crystals (see e.g. [3]). We believe that zeta functions for multicomplex algebras will play an important role in a similar range of applications. From a mathematical point of view, we observe that the study of the case of multicomplex algebras represents only a first step towards the understanding of the seminal work of Hey [9] and Artin [2], within the larger context of quotient polynomial algebras. We plan to return to these issues in future papers.

To begin with, and without pretense of completeness, we recall that the space of bicomplex numbers arises when considering the space $\mathbb{C}$ of complex numbers as a real bidimensional algebra, and then complexifying it. With this process one obtains a four dimensional algebra usually denoted by $\mathbb{BC}$. The key point of the theory of functions on this algebra is that (despite the problems posed by the existence of zero-divisors in $\mathbb{BC}$) the classical notion of holomorphicity can be extended from one complex variable to this algebra, and one can therefore develop a new theory of (hyper)holomorphic functions. Modern references on this topic are [1], and [13].

The algebra $\mathbb{BC}$ is therefore four dimensional over the reals, just like the skew-field of quaternions, but while in the space of quaternions we have three anti-commutative imaginary units, in the case of bicomplex numbers one considers two imaginary units $i, j$ which commute, and so the third unit $k = ij$ ends up being a “new” root of 1; such units are usually called hyperbolic. Indeed, every bicomplex
number $Z$ can be written as $Z = z_1 + jz_2$, where $z_1$ and $z_2$ are complex numbers of the form $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. There are several ways to represent bicomplex numbers, see [13], but the one that will be central in this paper is called the idempotent representation of bicomplex numbers, and will be described in section 4.

One can then use a similar process to define the space $\mathbb{B}C_n$ of multicomplex numbers, namely the space generated over the reals by $n$ commuting imaginary units. When $n = 2$, the space of multicomplex numbers is simply the space of bicomplex numbers. The history of bicomplex numbers is not devoid of interest, and we refer the reader to the recent [5].

In [19], the author introduced and studied the properties of a holomorphic Riemann zeta function of two complex variables in the context of the bicomplex algebra. Similarly, in [17] the authors defined a multicomplex Riemann zeta function in the setup of multicomplex algebras. Both these studies generalize the Riemann zeta function to several complex variables, in the sense that in the definition of the original Riemann zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

the complex variable $s$ is replaced by a bicomplex, respectively a multicomplex variable. Our approach, in this paper, is very different. As it is well known, Dedekind generalized the Riemann zeta function by considering an algebraic number field $K$, and defining the associated Dedekind zeta function by

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s},$$

where the sum ranges through all the non-zero ideals $I$ in the ring of integers $\mathcal{O}_K$ of $K$, and (see Section 3 for the full detail) $N(I)$ denotes the norm of the ideal $I$. When $K = \mathbb{Q}$, the Dedekind zeta function reduces to the Riemann zeta function.

Thus, it is natural to look at the Dedekind approach for quadratic fields, and concurrently the Hey [9] approach for hypercomplex algebras, as a way to define a Dedekind-like zeta function in the context of the bicomplex and multicomplex vector spaces $\mathbb{B}Q$, respectively $\mathbb{B}Q_n$. As the reader will see, the crucial point in being able to explicitly calculate this type of zeta function (sometimes called the Hey zeta function in the literature) for multicomplex numbers is the existence of the idempotent representations of bicomplex and multicomplex numbers. This representation will allow us to identify the Hey zeta functions for products of copies of $\mathbb{Q}(i)$, the field of Gaussian rationals, and as a result we will have an explicit formula for such a function.

The architecture of the paper is as follows. Section 2 gives a self-contained review of quadratic fields, so that all necessary results are available to the reader. In particular, we will define the quadratic $L$-function of a quadratic field $K$, and we will calculate the value of its analytic extension for $s = 1$. Section 3 provides all the necessary background on the Dedekind’s zeta function. In Section 4 we give all the necessary information on bicomplex and multicomplex algebras. The core of the paper is Section 5 where we utilize the instruments introduced previously to calculate explicitly the Dedekind-like (Hey) zeta function for the algebra $\mathbb{B}Q$ of bicomplex numbers with rational coefficients. We finally show how this results
extends to the multicomplex case, and we explicitly write the functional equation satisfied by these functions.

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2. Review on Quadratic Fields

For the convenience of the reader and in order to make the paper self-contained, we summarize the main definitions and results for quadratic fields. Notations, definitions, and results here follow [10, 14] and the references therein.

A number field is a finite degree field extension \( K \) over \( \mathbb{Q} \). We denote by \( \mathcal{O}_K \) the ring of integers of \( K \), i.e. the ring of elements \( \alpha \in K \) that are roots of monic polynomials in \( \mathbb{Z}[X] \).

A quadratic field is a degree two extension of \( \mathbb{Q} \). It has the form \( K = \mathbb{Q}(\sqrt{d}) \), where \( d \) is a square free integer different than 1. The ring of integers in this case is

\[
\mathcal{O}_d := \mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} 
\mathbb{Z} + \mathbb{Z}\sqrt{d}, & \text{if } d \not\equiv 1 \mod 4 \\
\mathbb{Z} + \mathbb{Z}\frac{1 + \sqrt{d}}{2}, & \text{if } d \equiv 1 \mod 4
\end{cases}
\]

The norm of an element \( \alpha = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d}) \) is the (non-necessarily positive) integer defined by

\[
N(\alpha) := (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2.
\]

The norm of an ideal \( I \) of \( \mathcal{O}_K \) is defined by

\[
N(I) := |\mathcal{O}_K/I|,
\]

where we note that the quotient ring \( \mathcal{O}_K/I \) is always of finite cardinality for each number field \( K \). A particular case occurs when \( I = (\alpha) \) is a principal ideal, where \( \alpha = a + b\sqrt{d} \). Then

\[
N(I) = N((\alpha)) = |a^2 - db^2|.
\]

For an odd prime \( p \in \mathbb{Z} \), the Legendre symbol is defined by:

\[
\left( \frac{a}{p} \right) := \begin{cases} 
1, & \text{if } a^{\frac{p-1}{2}} \equiv 1 \mod p \\
-1, & \text{if } a^{\frac{p-1}{2}} \equiv -1 \mod p \\
0, & \text{if } a \equiv 0 \mod p
\end{cases}
\]

Recall that to say that \( a \) is a quadratic residue modulo \( p \) means that the equation \( x^2 = a \mod p \) has a solution. We can therefore reformulate the definition of Legendre symbol as follows:

\[
\left( \frac{a}{p} \right) := \begin{cases} 
1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \mod p \\
-1, & \text{if } a \text{ is not a quadratic residue modulo } p \\
0, & \text{if } a \equiv 0 \mod p
\end{cases}
\]

An extension of the Legendre symbol, due to Kronecker, is the following. Each integer \( n \) has a prime factorization

\[
n = u \prod_{i=1}^{k} p_i^\ell_i,
\]

where \( u \) is a unit in \( \mathbb{Z} \) and \( p_i \) are distinct primes.
where \( u = \pm 1 \) and each \( p_i, 1 \leq i \leq k \) is prime. The Kronecker-Legendre symbol is defined by:

\[
\left( \frac{a}{n} \right) := \left( \frac{a}{u} \right)^k \prod_{i=1}^{k} \left( \frac{a}{p_i} \right)^{\ell_i},
\]

where:

1. for odd prime \( p \), \( \left( \frac{a}{p} \right) \) is the Legendre symbol;
2. for \( p = 2 \), we define
   \[
   \left( \frac{a}{2} \right) := \begin{cases} 0, & \text{if } a \text{ is even} \\ 1, & \text{if } a \equiv \pm 1 \mod 8 \\ -1, & \text{if } a \equiv \pm 3 \mod 8. \end{cases}
   \]
3. for \( u = \pm 1 \) we define
   \[
   \left( \frac{a}{1} \right) = 1, \quad \left( \frac{a}{-1} \right) = \begin{cases} 1, & \text{if } a \geq 0 \\ -1, & \text{if } a < 0. \end{cases}
   \]
4. we define
   \[
   \left( \frac{a}{0} \right) = \begin{cases} 1, & \text{if } a = \pm 1 \\ 0, & \text{otherwise.} \end{cases}
   \]

A fundamental result for our study of the Dedekind zeta function is the following

**Theorem 2.1.** Every non-zero ideal of \( \mathcal{O}_d \) can be written as a product of prime ideals. The decomposition is unique up to the order of the factors.

The discriminant of the quadratic number field \( K = \mathbb{Q}(\sqrt{d}) \) is defined by

\[
\Delta = \Delta_K = \begin{cases} d, & \text{if } d \equiv 1 \mod 4 \\ 4d, & \text{if } d \equiv 2, 3 \mod 4. \end{cases}
\]

Some primes in \( \mathbb{Z} \) are not prime elements in \( \mathcal{O}_d \): for example, in the case \( d = -1 \), we have:

\[
2 = i(1 - i)^2, \\
5 = (2 + i)(2 - i).
\]

This situation is described precisely by the Legendre symbol. If \( p \) is a rational prime (i.e. prime in \( \mathbb{Z} \)), then the ideal \((p) = p\mathcal{O}_d\) of \( \mathcal{O}_d \) has the following form:

\[
(p) = \begin{cases} pp'(\text{where } p \neq p'), & \text{if } \left( \frac{\Delta}{p} \right) = 1 \\ p, & \text{if } \left( \frac{\Delta}{p} \right) = -1 \\ p^2, & \text{if } \left( \frac{\Delta}{p} \right) = 0, \end{cases}
\]

where \( p, p' \) are prime ideals of \( \mathcal{O}_d \). We respectively say that in these cases the ideal \((p)\) splits, stays inert, or ramifies in \( \mathcal{O}_d \).
A subset $F$ of $\mathcal{O}_d$ is called a fractional ideal of $\mathcal{O}_d$ if there exists $\beta \in \mathcal{O}_d$, $\beta \neq 0$, such that $\beta F$ is an ideal of $\mathcal{O}_d$. Then we have, for some ideal $I$ of $\mathcal{O}_d$,

$$F = \left\{ \frac{\alpha}{\beta} \middle| \alpha \in I \right\},$$

The set $F_{\mathbb{Q}(\sqrt{d})}$ of fractional ideals of $\mathcal{O}_d$ can be equipped with an abelian group structure, as follows. If $I_1, I_2$ are ideals of $\mathcal{O}_d$, and

$$F_1 = \left\{ \frac{\alpha_1}{\beta_1} \middle| \alpha_1 \in I_1 \right\}, \quad F_2 = \left\{ \frac{\alpha_2}{\beta_2} \middle| \alpha_2 \in I_2 \right\},$$

where $\beta_1, \beta_2 \in \mathcal{O}_d$, we define:

$$F_1 F_2 := \left\{ \frac{\alpha}{\beta_1 \beta_2} \middle| \alpha \in I_1 I_2 \right\},$$

where $I_1 I_2$ is the ideal generated by all products $\alpha_1 \alpha_2$, with $\alpha_1 \in I_1, \alpha_2 \in I_2$.

The identity element is $\mathcal{O}_d$ and the inverse of a fractional ideal $F$ is given by

$$F^{-1} = \left\{ \alpha \in \mathbb{Q}(\sqrt{d}) \middle| \alpha F \subset \mathcal{O}_d \right\}.$$  

The set of principal fractional ideals $B_{\mathbb{Q}(\sqrt{d})} \subset F_{\mathbb{Q}(\sqrt{d})}$ is a subgroup of $F_{\mathbb{Q}(\sqrt{d})}$, and the quotient

$$H_d := F_{\mathbb{Q}(\sqrt{d})} / B_{\mathbb{Q}(\sqrt{d})}$$

is called the ideal class group of $\mathbb{Q}(\sqrt{d})$. Its order $h_d$ is the class number of $\mathbb{Q}(\sqrt{d})$. This notion measures how far the ring $\mathcal{O}_d$ is from being principal.

More precisely, if $h_d = 1$ then there is only one equivalence class in $H_d$, and each fractional ideal is equivalent to the principal ideal $(1) = \mathcal{O}_d$ modulo multiplication by principal ideals. That is, for each fractional ideal $A$ there exists $\alpha \in \mathcal{O}_d$ such that

$$(\alpha) = \alpha A = (1) = \mathcal{O}_d,$$

then

$$A = \left( \frac{1}{\alpha} \right).$$

Hence each fractional ideal and, therefore, each ideal is principal.

There exist nine imaginary quadratic fields of class number one. They are $\mathbb{Q}(\sqrt{d})$ for

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163.$$

If $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, all norms $a^2 - db^2$ are non-negative and the unit group in $\mathcal{O}_d$ is

$$\mathcal{O}_d^\times = \begin{cases} \{ \pm 1, \pm i \} & \text{if } d = -1 \\ \{ \pm 1, \pm \zeta_3, \pm \zeta_3^2 \} & \text{if } d = -3 \\ \{ \pm 1 \} & \text{otherwise} \end{cases},$$

where $\zeta_3$ is the principal cubic root of unity. Therefore, the order of the group $\mathcal{O}_d^\times$ of units is:

$$w_d = w = 4, 6, 2$$

according to the values $d = -1, d = -3$, or $d \neq -1, -3$, respectively.
For real quadratic fields $\mathbb{Q}(\sqrt{d})$, $d > 0$, the situation is very different. In this case, the group of units is infinite and has the form

$$O_K^\times = \{ \pm u_d^n \mid n \in \mathbb{Z} \} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z},$$

where $u_d > 1$ is the so-called fundamental unit. It is a difficult problem to find $u_d$.

For real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, the logarithm of the fundamental unit is called the regulator. For imaginary quadratic fields, the regulator is 1, as it is for the ring of (rational) integers $\mathbb{Z}$. For example, if $d = 2$, $u_2 = 1 + \sqrt{2}$ and the regulator $R_2$ of $\mathbb{Q}(\sqrt{2})$ is

$$R_2 = \log(1 + \sqrt{2}),$$

where $1 + \sqrt{2}$ is the fundamental solution of the Pell equation

$$x^2 - 2y^2 = 1.$$

Let $K$ be a quadratic field with discriminant $\Delta$, so that $\Delta = d$, if $d \equiv 1 \pmod{4}$ or $4d$ if $d \equiv 2, 3 \pmod{4}$. The quadratic character of $K$ is the morphism

$$\chi_K : (\mathbb{Z}, +) \to \mathbb{C}, \quad \chi_K(m) := \left( \frac{\Delta}{m} \right).$$

It is a fundamental property that $\chi_K$ is periodic with period $|\Delta|$.

For $K = \mathbb{Q}(\sqrt{d})$, we can also define the Dirichlet character, also denoted by $\chi_K$, by

$$\chi_K : \left( \mathbb{Z}/|\Delta|\mathbb{Z} \right)^\times \to \mathbb{C}^\times,$$

where

$$\chi_K(m + |\Delta|\mathbb{Z}) = \begin{cases} \left( \frac{m}{d} \right), & \text{if } d \equiv 1 \pmod{4} \\ (-1)^{\frac{m-1}{2}} \left( \frac{m}{d} \right), & \text{if } d \equiv 3 \pmod{4} \\ (-1)^{\frac{m^2-1}{8}} \left( \frac{m}{d} \right), & \text{if } d \equiv 2 \pmod{8} \\ (-1)^{\frac{(m-1)(m+1)}{8}} \left( \frac{m}{d} \right), & \text{if } d \equiv 6 \pmod{8} \end{cases}$$

In the case $K = \mathbb{Q}(i)$, $O_{\mathbb{Q}(i)} = \mathbb{Z}[i]$, the ring of Gaussian integers, and the Dirichlet character is

$$\chi_{\mathbb{Q}(i)} : \left( \mathbb{Z}/4\mathbb{Z} \right)^\times \to \mathbb{C}^\times, \quad \chi_{\mathbb{Q}(i)}(m) = (-1)^{\frac{m-1}{2}}.$$

We can formulate the decomposition of rational primes as follows. Let $p$ be a rational prime. The decomposition of $(p)$ in $O_K$ is given by:

$$pO_K = \begin{cases} pp', & \text{if } N(p) = N(p') = p, \quad \text{if } \chi_K(p) = 1 \\ (p), & \text{if } N((p)) = p^2, \quad \text{if } \chi_K(p) = -1 \\ (p)^2, & \text{if } N((p)) = p, \quad \text{if } \chi_K(p) = 0 \end{cases}$$

The quadratic $L$-function of a quadratic field $K$ with discriminant $\Delta$ is given by:

$$L(\chi_K, s) = \sum_{n \geq 1} \chi_K(n)n^{-s} = \prod_{p \text{ prime}} \left( 1 - \chi_K(p)p^{-s} \right)^{-1}.$$
The sum is defined for $\text{Re}(s) > 1$, but actually the quadratic $L$-function extends analytically to $\text{Re}(s) > 0$. The value of $L(\chi_K, 1)$ is a remarkable one (see e.g. [6, 10, 11]).

**Proposition 2.2.** (i) For a real quadratic field $K$:

$$L(\chi_K, 1) = -\frac{1}{\sqrt{|\Delta|}} \sum_{r=1}^{|\Delta|-1} \chi_K(r) \log \left( \frac{\pi r}{|\Delta|} \right).$$

(ii) For an imaginary quadratic field $K$:

$$L(\chi_K, 1) = -\frac{\pi}{|\Delta|^\frac{3}{2}} \sum_{r=1}^{|\Delta|-1} \chi_K(r)r.$$

### 3. The Dedekind zeta function

The Dedekind zeta function of a quadratic field $K$ is given by:

$$\zeta_K(s) = \sum_a N(a)^{-s} = \prod_p (1 - N(p)^{-s})^{-1}. \quad (3.1)$$

The sum is over all non-zero ideals $a$ of $\mathcal{O}_K$ and the product is over all prime ideals of $\mathcal{O}_K$. If denote by

$$a_n = \# \{a \mid N(a) = n \},$$

then

$$\zeta_K(s) = \sum_{n>0} \frac{a_n}{n^s}.$$  

The fundamental property of the Dedekind zeta function is the following factorization:

**Theorem 3.1.** For $\text{Re}(s) > 1$, we have

$$\zeta_K(s) = \zeta(s)L(\chi_K, s),$$

where $s \mapsto \zeta(s)$ is the Riemann zeta function.

We give below an idea of the proof. For $\text{Re}(s) > 1$ we can write:

$$\zeta_K(s) = \prod_p \prod_{p|(p)} (1 - N(p)^{-s})^{-1}, \quad (3.2)$$

where $(p) = p\mathcal{O}_K$ is the ideal generated by $p$ in $\mathcal{O}_K$. Furthermore, for $\text{Re}(s) > 1$ we have:

$$\zeta(s)L(\chi_K, s) = \prod_p \left(1 - p^{-s}\right)^{-1}(1 - \chi(p)p^{-s})^{-1}$$

**Lemma 3.2.**

$$\prod_{p|(p)} (1 - N(p)^{-s}) = (1 - p^{-s})(1 - \chi_K(p)p^{-s}).$$
Proof. Indeed, for a given rational prime $p$, if $\chi_K(p) = 1$, then

$$(p) = p\mathcal{O}_K = pp', \quad N(p) = N(p') = p,$$

and both sides are $(1 - p^{-s})^2$. If $\chi_K(p) = -1$, then $p$ is inert and $p = (p)$, with $N(p) = p^2$, and both sides are

$$(1 - p^{2(-s)}) = (1 - p^{-s})(1 + p^{-s}).$$

Finally, if $\chi_K(p) = 0$, then both sides are $1 - p^{-s}$. \hfill \square

From this lemma, we obtain

$$\zeta_K(s) = \prod_p \prod_{p\mid(p)} (1 - N(p)^{-s})^{-1}$$

$$= \prod_p (1 - p^{-s})^{-1} (1 - \chi_K(p)p^{-s})^{-1}$$

$$= \zeta(s)L(\chi_K, s).$$

A beautiful consequence of what we have seen is the Dirichlet class number formula for imaginary quadratic fields. It says that:

$$\frac{2\pi h}{w\sqrt{|\Delta|}} = L(\chi_K, 1),$$

where $h$ is the ideal class number of $K$, and $w$ is the number of roots of unity in $K$.

We turn now to the case $K = \mathbb{Q}(i)$, which is our main interest. The Dedekind zeta function of $\mathbb{Q}(i)$ is given by:

$$\zeta_{\mathbb{Q}(i)} = \zeta(s)L(\chi_{-4}, s)$$

$$= \frac{1}{1 - 2^{-s}} \prod_{p \equiv 1 \pmod{4}} \frac{1}{(1 - p^{-s})^2} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - p^{-2s}},$$

where, in order to simplify the notations, we write $\chi_{-4}$ instead of $\chi_{\mathbb{Q}(i)}$, since $-4$ is the discriminant of the field $\mathbb{Q}(i)$. The Dirichlet $L$-series is then:

$$L(\chi_{-4}, s) = \sum_{n \geq 1} \frac{\chi_{-4}(n)}{n^s}.$$ 

Note that $\chi_{-4}$ is the character of $\mathbb{Z}$, of period 4, given by:

$$\chi_{-4}(n) = \begin{cases} 
0, & \text{if } n \equiv 0 \pmod{4} \\
1, & \text{if } n \equiv 1 \pmod{4} \\
0, & \text{if } n \equiv 2 \pmod{4} \\
-1, & \text{if } n \equiv 3 \pmod{4}
\end{cases}.$$ 

This implies that

$$L(\chi_{-4}, s) = \sum_{n \geq 0} \frac{1}{(4n + 1)^s} - \sum_{n \geq 0} \frac{1}{(4n + 3)^s}.$$ 

To study the analytic continuation of $L(\chi_{-4}, s)$ we relate it to the Hurwitz zeta function, defined by:

$$\zeta(s, \alpha) = \sum_{n \geq 0} \frac{1}{(n + \alpha)^s}, \quad \text{Re}(s) > 1, \quad \alpha > 0.$$
This function reduces to the Riemann zeta function for $\alpha = 1$. The general theory says that $s \mapsto \zeta(s, \alpha)$ admits a meromorphic continuation to the whole plane, with a simple pole at $s = 1$ with residue 1. We need the special value

$$
\lim_{s \to 1} \left( \zeta(s, \alpha) - \frac{1}{s - 1} \right) = -\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}
$$

where $\Gamma$ is the Euler function (see e.g. [4]). An immediate consequence of this fact is that $s \mapsto L(\chi_{-4}, s)$ is analytically continuable to the whole plane, according to the equality:

$$
L(\chi_{-4}, s) = 4^{-s} \left( \zeta \left( s, \frac{1}{4} \right) - \zeta \left( s, \frac{3}{4} \right) \right).
$$

Another representation of $L(\chi_{-4}, s)$ is

$$
L(\chi_{-4}, s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s} = \beta(s),
$$

where $\beta$ is the Dirichlet Beta function.

The function $s \mapsto \zeta(s) L(\chi_{-4}, s)$ extends meromorphically to all $\mathbb{C}$, with a simple pole at $s = 1$. We then have, for $0 < |s - 1| < \infty$:

$$
\zeta(s) L(\chi_{-4}, s) = \frac{C_{-1}}{s - 1} + C_0 + C_1(s - 1) + C_2(s - 1)^2 + \cdots
$$

We give below the values of $C_{-1}$ and $C_0$. From the expansions:

$$
\zeta(s) = \frac{1}{s - 1} + \gamma + \gamma_1(s - 1) + \cdots
$$

where $\gamma$ is the (small) Euler constant, and

$$
L(\chi_{-4}, s) = L(\chi_{-4}, 1) + L'(\chi_{-4}, 1)(s - 1) + \cdots
$$

we obtain

$$
\zeta(s) L(\chi_{-4}, s) = \frac{L(\chi_{-4}, 1)}{s - 1} + L'(\chi_{-4}, 1) + \gamma L(\chi_{-4}, 1) + \cdots
$$

Hence $L(\chi_{-4}, 1)$ is the residue of $\zeta(s) L(\chi_{-4}, s)$ at $s = 1$ and

$$
\gamma_{\mathbb{Q}(i)} := L'(\chi_{-4}, 1) + \gamma L(\chi_{-4}, 1)
$$

is what we may call the Euler constant $\gamma_{\mathbb{Q}(i)}$ of the field $\mathbb{Q}(i)$ (as $\gamma = \gamma_{\mathbb{Q}}$).

$$
L(\chi_{-4}, 1) = \lim_{s \to 1} 4^{-s} \left( \zeta \left( s, \frac{1}{4} \right) - \zeta \left( s, \frac{3}{4} \right) \right) = \frac{1}{4} \left( \lim_{s \to 1} \left( \zeta \left( s, \frac{1}{4} \right) - \frac{1}{s - 1} \right) - \lim_{s \to 1} \left( \zeta \left( s, \frac{3}{4} \right) - \frac{1}{s - 1} \right) \right) = \frac{1}{4} \left( \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \frac{\Gamma'(\frac{3}{4})}{\Gamma(\frac{3}{4})} \right).
$$

The logarithmic derivative $\frac{\Gamma'(z)}{\Gamma(z)}$ of the $\Gamma$-function is a remarkable function. We only need to know it satisfies the functional equation

$$
\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(1 - z)}{\Gamma(1 - z)} = \pi \cot(\pi z).
$$
Hence

\[ L(\chi_{-4}, 1) = -\frac{1}{4} \left( \frac{\Gamma' \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} - \frac{\Gamma' \left( \frac{3}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right) = \frac{\pi}{4}. \]

To find the Euler constant \( \gamma_Q \) of the field \( Q(i) \), we observe that:

\[ \gamma_Q = \frac{L'(1)}{L(1)} = \gamma + \frac{L'(\chi_{-4}, 1)}{L(\chi_{-4}, 1)}, \]

which is what it is called the Sierpinski constant (see e.g. [8]):

\[
\gamma + \frac{L'(\chi_{-4}, 1)}{L(\chi_{-4}, 1)} = \log \left( \frac{2\pi e^{2\gamma}}{\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right)} \right) = \frac{\pi}{3} \log 4 + 2\gamma - 4 \sum_{k=1}^{\infty} \log \left( 1 - e^{-2\pi k} \right) = 0.8228252 \ldots
\]

Hence the analogue of the Euler constant for \( Q(i) \) is:

\[ \gamma_{Q(i)} = \frac{L'(\chi_{-4}, 1)}{L(\chi_{-4}, 1)} = \frac{\pi}{4} \log \left( \frac{2\pi e^{2\gamma}}{\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right)} \right) = \frac{\pi}{2} \left( \gamma + \log 2 + \frac{3}{2} \log \pi - 2 \log \Gamma \left( \frac{1}{4} \right) \right), \]

where we used the classical complement formula for the \( \Gamma \)-function:

\[ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \ldots \]

**Remark 3.3.** We would like to give an idea on how to calculate the coefficients of the Dirichlet series of \( \zeta_{Q(i)}(s) \). First, for \( \Re(s) > 1 \), we have:

\[
\zeta_{Q(i)}(s) = \frac{1}{4} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^s} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s},
\]

where \( r_2(n) = \left| \{(p,q) \in \mathbb{Z} \times \mathbb{Z}, \quad p^2 + q^2 = n\} \right| \)

is the number of representation of \( n \) as a sum of two squares.

Secondly, we have:

\[
\zeta_{Q(i)}(s) = \zeta(s)L(\chi_{-4}, s) = \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s} \right) = \sum_{n_1, n_2=1}^{\infty} \frac{\chi_{-4}(n_2)}{n_1^s n_2^s} = \sum_{n=1}^{\infty} \left( \sum_{d|n} \frac{\chi_{-4}(d)}{n^s} \right),
\]
so that
\[ \zeta_Q(i)(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \]
where
\[ a_n = \sum_{d|n} \chi_{-4}(d) = \frac{1}{4} r_2(n) = \frac{1}{4} \left| \{(p, q) \in \mathbb{Z} \times \mathbb{Z}, \ p^2 + q^2 = n \} \right|. \]

For \( \text{Re}(s) > 1 \) we obtain:
\[ \zeta_Q(i)(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{2}{10^s} + \frac{1}{13^s} + \frac{1}{16^s} + \frac{2}{17^s} + \frac{1}{18^s} + \frac{2}{20^s} + \frac{3}{25^s} + \cdots \]

A nice application of the formula for \( r_2(n) \) given above is the following (see [16]). Let \( \sigma_0^{(1)}(n) \) and \( \sigma_0^{(3)}(n) \) be the number of divisors of \( n \) congruent to 1 and 3 modulo 4, respectively. Then we have:
\[ \sum_{d|n} \chi_{-4}(d) = \sigma_0^{(1)}(n) - \sigma_0^{(3)}(n), \]
and so
\[ r_2(n) = \left( \sigma_0^{(1)}(n) - \sigma_0^{(3)}(n) \right). \]

If follows that for each positive integer \( n \), we have \( \sigma_0^{(1)}(n) \geq \sigma_0^{(3)}(n) \) (it means that there are more divisors of \( n \) congruent to 1 (mod 4) than congruent to 3 (mod 4)). Furthermore, if \( p \) is prime, \( p \equiv 1 \mod 4 \), then \( r_2(p) = 8 \), which is a famous theorem of Fermat (see e.g. [16]).

4. Multicomplex Algebras

Without giving many details (for which we refer the reader to the fairly comprehensive recent references [12, 13, 15, 20, 21]), we will simply say that the space \( \mathbb{BC}_n \) of multicomplex numbers is the space generated over the reals by \( n \) commuting imaginary units. The algebraic properties of this space and analytic properties of multicomplex valued functions defined on \( \mathbb{BC}_n \) has been studied in [21].

In the case of only one imaginary unit, denoted by \( i_1 \), the space \( \mathbb{BC}_1 \) is the usual complex plane \( \mathbb{C} \). Since, in what follows, we will have to work with different complex planes, generated by different imaginary units, we will denote such a space also by \( \mathbb{C}(i_1) \), in order to clarify which imaginary unit is used in the space itself.

The next case occurs when we have two commuting imaginary units \( i_1 \) and \( i_2 \). This yields the bicomplex space \( \mathbb{BC}_2 \), or \( \mathbb{B} \). For simplicity of notation, we will relabel the units as
\[ i := i_1, \quad j := i_2, \quad k := ij = i_1i_2. \]

Note that \( k \) is a hyperbolic unit, i.e. it is a unit which squares to 1. Because of these various units in \( \mathbb{B} \), there are several different conjugations that can be defined naturally. We will not make use of these conjugations in this paper, but we refer the reader to [13].
Moreover, the inverse of an invertible bicomplex number is realized component-wise in the idempotent representation above. Specifically, if

$$e := \frac{1+k}{2}, \quad e^\dagger := \frac{1-k}{2},$$

$$e \cdot e^\dagger = 0, \quad e^2 = e, \quad (e^\dagger)^2 = e^\dagger,$$

$$e + e^\dagger = 1, \quad e - e^\dagger = k.$$

Just like \(\{1,j\}\), they form a basis of the complex algebra \(BC\), which is called the idempotent basis. If we define the following complex variables in \(C(i)\):

$$\beta_1 := z_1 - i_1 z_2, \quad \beta_2 := z_1 + i_1 z_2,$$

the \(C(i)\)-idempotent representation for \(Z = z_1 + i_2 z_2\) is given by

$$Z = \beta_1 e + \beta_2 e^\dagger.$$

The \(C(i)\)-idempotent is the only representation for which multiplication is component-wise, as shown in the next proposition.

**Proposition 4.1.** The addition and multiplication of bicomplex numbers can be realized component-wise in the idempotent representation above. Specifically, if \(Z = a_1 e + a_2 e^\dagger\) and \(W = b_1 e + b_2 e^\dagger\) are two bicomplex numbers, where \(a_1, a_2, b_1, b_2 \in C(i)\), then

$$Z + W = (a_1 + b_1) e + (a_2 + b_2) e^\dagger,$$

$$Z \cdot W = (a_1 b_1) e + (a_2 b_2) e^\dagger,$$

$$Z^n = a_1^n e + a_2^n e^\dagger.$$

Moreover, the inverse of an invertible bicomplex number \(Z = a_1 e + a_2 e^\dagger\) (in this case \(a_1 \cdot a_2 \neq 0\)) is given by

$$Z^{-1} = a_1^{-1} e + a_2^{-1} e^\dagger,$$

where \(a_1^{-1}\) and \(a_2^{-1}\) are the complex multiplicative inverses of \(a_1\) and \(a_2\), respectively.

One can see this also by computing directly which product on the bicomplex numbers of the form

$$x_1 + ix_2 + jx_3 + kx_4, \quad x_1, x_2, x_3, x_4 \in \mathbb{R}$$

is component wise, and one finds that the only one with this property is given by the mapping:

$$x_1 + ix_2 + jx_3 + kx_4 \mapsto ((x_1 + x_4) + i(x_2 - x_3), (x_1 - x_4) + i(x_2 + x_3)), \quad (4.1)$$

which corresponds exactly with the idempotent decomposition

$$Z = z_1 + jz_2 = (z_1 - i z_2)e + (z_1 + i z_2)e^\dagger,$$

where \(z_1 = x_1 + ix_2\) and \(z_2 = x_3 + ix_4\).

The principal ideals \(\langle e \rangle\) and \(\langle e^\dagger \rangle\) generated by \(e\) and \(e^\dagger\) in \(BC\) have the following properties:

$$\langle e \rangle \cdot \langle e^\dagger \rangle = \{0\}, \quad \langle e \rangle \cap \langle e^\dagger \rangle = \{0\}, \quad \langle e \rangle + \langle e^\dagger \rangle = BC,$$

so they are coprime ideals in \(BC\).
We now turn to the definition of the multicomplex spaces, $\mathbb{B}C_n$, for values of $n \geq 2$. These spaces are defined by taking $n$ commuting imaginary units $i_1, i_2, \ldots, i_n$ i.e. $i_2 = -1$, and $i_ai_b = i_bi_a$ for all $a, b = 1, \ldots, n$. Since the product of two commuting imaginary units is a hyperbolic unit, and since the product of an imaginary unit and a hyperbolic unit is an imaginary unit, we see that these units will generate a set $\mathfrak{A}_n$ of $2^n$ units, $2^n - 1$ of which are imaginary and $2^n - 1$ of which are hyperbolic units. Then the algebra generated over the real numbers by $\mathfrak{A}_n$ is the multicomplex space $\mathbb{B}C_n$ which forms a ring under the usual addition and multiplication operations. As in the case $n = 2$, the ring $\mathbb{B}C_n$ can be represented as a real algebra, so that each of its elements can be written as $Z = \sum_{I \in \mathfrak{A}_n} Z_I I$, where $Z_I$ are real numbers.

In particular, following [15], it is natural to define the $n$-dimensional multicomplex space as follows:

$$\mathbb{B}C_n := \{Z_n = Z_{n-1,1} + i_n Z_{n-1,2} | Z_{n-1,1}, Z_{n-1,2} \in \mathbb{B}C_{n-1}\}$$

with the natural operations of addition and multiplication. Since $\mathbb{B}C_{n-1}$ can be defined in a similar way using the $i_{n-1}$ unit and multicomplex elements of $\mathbb{B}C_{n-2}$, we recursively obtain, at the $k$-th level:

$$Z_n = \sum_{|I|=n-k} \prod_{t=k+1}^n (i_t)^{\alpha_t-1} Z_{k,I}$$

where $Z_{k,I} \in \mathbb{B}C_k$, $I = (\alpha_{k+1}, \ldots, \alpha_n)$, and $\alpha_j \in \{1, 2\}$.

Just as in the case of $\mathbb{B}C_2$, there exist idempotent bases in $\mathbb{B}C_n$, that will be organized at each “nested” level $\mathbb{B}C_k$ inside $\mathbb{B}C_n$ as follows. Denote by

$$e_{kl} := \frac{1 + ik_j}{2}, \quad e_{kl} := \frac{1 - ik_j}{2}.$$ 

Consider the following sets:

$$S_1 := \{e_{n-1,n}, e_{n-1,n}\},$$

$$S_2 := \{e_{n-2,n-1} \cdot S_1, e_{n-2,n-1} \cdot S_1\},$$

$$\vdots$$

$$S_{n-1} := \{e_{12} \cdot S_{n-2}, e_{12} \cdot S_{n-2}\}.$$ 

At each stage $k$, the set $S_k$ has $2^k$ idempotents. It is possible to immediately verify the following

**Proposition 4.2.** In each set $S_k$, the product of any two idempotents is zero.

We have several idempotent representations of $Z_n \in \mathbb{B}C_n$, as follows.

**Theorem 4.3.** Any $Z_n \in \mathbb{B}C_n$ can be written as:

$$Z_n = \sum_{j=1}^{2^k} Z_{n-k,j} e_j,$$

where $Z_{n-k,j} \in \mathbb{B}C_{n-k}$ and $e_j \in S_k$.

Due to the fact that the product of two idempotents is 0 at each level $S_k$, we will have many zero divisors in $\mathbb{B}C_n$ organized in “singular cones”.

In particular, at the last stage, we obtain:
Proposition 4.4. Any \( Z_n \in \mathbb{BC}_n \) admits the idempotent writing:

\[
Z_n = \sum_{j=1}^{2^{n-1}} \beta_j e_j,
\]

where \( \beta_j \) are complex numbers (with respect to one fixed imaginary unit, say \( i \)) and \( e_j \in S_{n-1} \). For a chosen imaginary unit \( i \), this decomposition is unique.

As in the case of bicomplex numbers, in this representation, the multiplication of multicomplex numbers is component-wise, and yields the isomorphism:

\[
\mathbb{BC}_n \simeq \sum_{j=1}^{2^{n-1}} \mathbb{C}(i)e_j.
\]

This decomposition allows us to introduce a formula analogous to (4.1). Explicitly, for \( n = 3 \) in \( \mathbb{BC}_3 \), we have:

\[
\begin{align*}
Z_3 &= Z_1 + i_3Z_{22} = (Z_{21} - i_2Z_{22})e_{23} + (Z_{21} + i_2Z_{22})\overline{e_{23}} \\
&= ((x_1 + x_7 + x_4 - x_6) + i_1(x_2 + x_8 - x_3 + x_5)) e_1 \\
&\quad + ((x_1 + x_7 - x_4 + x_6) + i_1(x_2 + x_8 + x_3 - x_5)) e_2 \\
&\quad + ((x_1 - x_7 + x_4 + x_6) + i_1(x_2 - x_8 - x_3 - x_5)) e_3 \\
&\quad + ((x_1 - x_7 - x_4 - x_6) + i_1(x_2 - x_8 + x_3 + x_5)) e_4
\end{align*}
\]

This idempotent representation gives a component wise multiplicative structure on \( \mathbb{C}^4 \):

\[
((x_1 + i_1x_2) + i_2(x_3 + i_1x_4)) + i_3((x_5 + i_1x_6) + i_2(x_7 + i_1x_8)) \mapsto
\]

\[
[(x_1 + x_7 + x_4 - x_6) + i_1(x_2 + x_8 - x_3 + x_5), (x_1 + x_7 - x_4 + x_6) + i_1(x_2 + x_8 + x_3 - x_5),
(x_1 - x_7 + x_4 + x_6) + i_1(x_2 - x_8 - x_3 - x_5), (x_1 - x_7 - x_4 - x_6) + i_1(x_2 - x_8 + x_3 + x_5)]
\]

A tedious but straightforward computation shows that this is the only multiplication on tricomplex numbers that is component-wise.

5. Bicomplex and Multicomplex zeta functions

We consider first the case of bicomplex algebra \( \mathbb{BC} = \mathbb{BC}_2 \). Inside \( \mathbb{BC} \) we consider the vector space \( \mathbb{BQ} \) over \( \mathbb{Q} \)

\[
\mathbb{BQ} := \{ Z = x_1 + y_1i + x_2j + y_2k \mid x_\ell, y_\ell \in \mathbb{Q}, \quad \ell = 1, 2 \}.
\]

The vector space \( \mathbb{BQ} \) can be equipped with a structure of \( \mathbb{Q} \)-algebra, generated by the two variables \( i,j \), with the relations

\[
i j = ji, \quad i^2 + 1 = j^2 + 1 = 0.
\]

so that

\[
\mathbb{BQ} \simeq \mathbb{Q}[i,j] / (i^2 + 1, j^2 + 1)
\]

which is a commutative algebra. Furthermore, note that

\[
\mathbb{Q}[X,Y] / (X^2 + 1, Y^2 - 1) \simeq \mathbb{Q}[X,Y] / (X^2 + 1, Y^2 + 1)
\]
by \( X \mapsto X \) and \( Y \mapsto XY \). Indeed,

\[
(XY)^2 + 1 = X^2(Y^2 - 1) + (X^2 + 1) \in (X^2 + 1, Y^2 - 1),
\]

\[
(XY)^2 - 1 = X^2(Y^2 + 1) - (X^2 + 1) \in (X^2 + 1, Y^2 + 1).
\]

More generally, if we let \( K \) be a field and consider the \( K \)-algebra

\[
A = K[X, Y, Z]/(X^2 + 1, Y^2 + 1, Z - XY).
\]

We have

\[
A = K[X, Y]/(X^2 + 1, Y^2 + 1),
\]

because we can eliminate the variable \( Z \) by using the relation \( Z = XY \). We can further write

\[
A = (K[X]/(X^2 + 1))[Y]/(Y^2 + 1),
\]

and we denote \( C := K[X]/(X^2+1) \), and by \( i \) the class of \( X \in K[X] \) in this quotient. Then we get:

\[
A = C[Y]/(Y^2 + 1) = C[Y]/(Y + i)(Y - i) \simeq C[Y]/(Y + i) \times C[Y]/(Y - i),
\]

where the isomorphism (of \( C \)-algebras and hence of \( K \)-algebras) is induced by the canonical map

\[
(\pi_1, \pi_2) : C[Y] \to C[Y]/(Y + i) \times C[Y]/(Y - i)
\]

given by the two surjective maps \( \pi_1, \pi_2 \). If we return to \( K[X, Y, Z] \), the images of \( X, Y \) and \( Z \) in \( A \) by the isomorphism just considered are \((i, i), (i, -i)\) (note that the class of \( Y \) in the first factor is \(-i\), due to the relation \( Y + i = 0 \)) and \((1, i)(-i, i) = (1, -1)\). In conclusion, for \( x_1, x_2, x_3, x_4 \in K \), the class of \( x_1 + x_2X + x_3Y + x_4Z \) in \( A \) is

\[
(x_1 + x_2i + x_3i + x_4i, x_1 - x_2 + x_3 - x_4)i,
\]

which is exactly the map \((4.1)\) corresponding to the idempotent representation of bicomplex numbers.

Therefore, we obtain the following characterization of the algebra \( \mathbb{B}Q \):

**Theorem 5.1.** The algebra \( \mathbb{B}Q \) is isomorphic to the product \( \mathbb{Q}(i) \times \mathbb{Q}(i) \). The isomorphism is given by \((5.2)\) for \( C = \mathbb{Q} \).

**Proof.** Explicitly, we use the idempotent representation of bicomplex numbers in order to get a component-wise product of ideals, which is necessary for the Chinese Remainder theorem:

\[
\mathbb{B}Q \simeq \mathbb{Q}(i)\mathfrak{e} + \mathbb{Q}(i)\mathfrak{e}^4,
\]

and recall that the ideals \( \mathfrak{I}_e = \mathbb{Q}(i)\mathfrak{e} \) and \( \mathfrak{I}_{e^i} = \mathbb{Q}(i)\mathfrak{e}^i \) are coprime in \( \mathbb{B}Q \), with \( \mathfrak{I}_e \cdot \mathfrak{I}_{e^i} = \{0\} \). The Chinese Remainder theorem yields

\[
\mathbb{B}Q \simeq \mathbb{B}Q/\mathfrak{I}_e \times \mathbb{B}Q/\mathfrak{I}_{e^i} = \mathfrak{I}_{e^i} \times \mathfrak{I}_e \simeq \mathbb{Q}(i) \times \mathbb{Q}(i),
\]

which is what we had to prove. \( \square \)

The main objective now is to define a Dedekind-like zeta function for an algebra which is the product of fields. According to Artin \([2]\) and Hey \([3]\), one can define a Dedekind-like zeta function for hypercomplex algebras (such as the bicomplex and multicomplex ones), if one considers ideals of maximal order in the defining formula \((3.1)\). Moreover, it follows also that the resulting zeta function for a product
algebra will be the product of the corresponding zeta functions of the factors, whenever the multiplication is defined component-wise.

Therefore, the plan is to find the maximal order of the algebra \( BQ \) (see Theorem 5.2 below), and to derive the ideal structure of \( BQ \) (Lemma 5.3), concluding with the main result (Theorem 5.4) defining the bicomplex zeta function.

We recall the following notions (see e.g. [18]). Let \( R \) be a commutative domain with quotient field \( K \), and let \( A \) be a finite-dimensional \( K \)-algebra. If a full \( R \)-lattice \( \mathcal{L} \) in \( A \) (i.e. \( \mathcal{L} \) is a finitely generated \( R \)-submodule such that \( KL = A \)) is a subring of \( A \), then one says that \( \mathcal{L} \) is an \( R \)-order in \( A \). If, moreover, \( \mathcal{L} \) is not properly contained in any \( R \)-order of \( A \), then it is called a maximal order.

An algebra \( A \) is called semisimple if it is isomorphic to a direct sum of simple algebras (i.e. do not have non-trivial subalgebras). Furthermore, \( A \) is called a separable \( K \)-algebra if \( A \) is semisimple and the center of each simple component of \( A \) is a separable field extension of \( K \).

We now prove the following:

**Theorem 5.2.** \( BQ \) is a semisimple separable algebra. Moreover, the maximal order of \( BQ \) is the product \( \mathbb{Z}[i] \times \mathbb{Z}[i] \).

**Proof.** The first statement follows from Theorem 5.1 above and a theorem of Weierstrass and Dedekind (see e.g. [7, Theorem 2.4.1 (pp. 38)]), which states that a commutative semisimple algebra is isomorphic to a direct product of fields, and, conversely, a direct product of fields is a semisimple algebra.

Next, we recall [18, Theorem 10.5] that in a separable algebra with a central idempotent decomposition

\[
A = A_1 + \cdots + A_t
\]

(such as the bicomplex and multicomplex algebras) every maximal order has a corresponding maximal order decomposition at each level. In particular, if one defines by \( \{e_i\} \) the central idempotents of \( A \) such that \( e_i e_j = 0 \), \( i \neq j \), \( 1 = e_1 + \cdots + e_t \), and \( A_i = Ae_i \), then each maximal order is a direct sum of the maximal orders of each component.

From Theorem 5.1 it follows that the maximal order of \( BQ \) is the product of the maximal orders of each factor, and the maximal order of \( \mathbb{Q}(i) \) is \( \mathbb{Z}[i] \).

In order to derive the ideal structure of the algebra \( BQ \), we recall a more general classical lemma on the ideal structure of a product of two unitary commutative rings.

**Lemma 5.3.** Let \( R = R_1 \times R_2 \) be the product of two unitary commutative rings. If \( \mathcal{I}(A) \) is the monoid of ideals of a ring \( A \), with the binary operation \((a, b) \mapsto a \cdot b\), then

\[
\mathcal{I}(R) = \mathcal{I}(R_1) \times \mathcal{I}(R_2).
\]

**Proof.** To begin with, if \( I_1 \) is an ideal of \( R_1 \) then \( I_1 \times \{0\} \) is an ideal of the product. Moreover

\[
R/I_1 \simeq R_1/I_1 \times R_2.
\]
Similarly, for an ideal $I_2$ of $R_2$. Moreover, if $I_1$ is an ideal of $R_1$ and $I_2$ is an ideal of $R_2$, then $I_1 \times I_2$ is an ideal of $R_1 \times R_2$, and
\[(R_1 \times R_2)/(I_1 \times I_2) \cong R_1/I_1 \times R_2/I_2.\]
Reciprocally, if $R_1$ and $R_2$ are two commutative rings and $I$ is an ideal of $R_1 \times R_2$, we define:
\[I_1 := \{ r_1 \in R_1 \mid \exists r_2 \in R_2, (r_1, r_2) \in I \},\]
\[I_2 := \{ r_2 \in R_2 \mid \exists r_1 \in R_1, (r_1, r_2) \in I \},\]
then $I_1, I_2$ are ideals of $R_1, R_2$, respectively, and $I = I_1 \times I_2$. Indeed, observe first that $I_1 \times \{0\}$ and $\{0\} \times I_2$ are ideals of $R$, contained in $I$: if $i_1 \in I_1$, then for some $r_2 \in R_2$, $(i_1, r_2) \in I$, and then
\[(i_1, 0) = (i_1, r_2)(1, 0),\]
and similarly for $I_2$. Therefore
\[I_1 \times I_2 = (I_1 \times \{0\}) + (\{0\} \times I_2) \subset I.\]
Conversely, if $(x, y) \in I$, then
\[(x, y) = (x, 0)(1, 0) + (0, y)(0, 1) \in I_1 \times I_2.\]
\[\square\]
It follows that ideals of maximal order in $BQ$ are products of ideals of maximal order in each one of the factors $Q(i)$, i.e. products of ideals of $Z[i]$.

We have everything necessary now to prove our main result:

**Theorem 5.4.** The Dedekind-like (Hey) zeta function of the algebra $BQ$ is
\[\zeta_{BQ}(s) = \zeta(s)^2 \cdot L(\chi_{-4}, s)^2.\]
Moreover, $\zeta_{BQ}$ has a double pole at $s = 1$, a residue equal to $\frac{\pi}{2} \gamma_{Q(i)}$, and it verifies the functional equation:
\[\pi^{-2(s-2s)} \Gamma^2(1-s) \zeta_{BQ}(1-s) = \pi^{-2s} \Gamma^2(s) \zeta_{BQ}(s).\]

**Proof.** The Hey zeta function of the algebra $BQ$ is defined as in (3.1) for all ideals of maximal order of $BQ$. According to Theorems 5.1 $BQ$ is isomorphic to the product $Q(i) \times Q(i)$, and using Theorem 5.2 and Lemma 5.3 the maximal order ideals of $BQ$ are products of maximal ideals of $Z[i]$. It follows that the zeta function of $BQ$ is the product of the two respective Dedekind zeta functions of $Q(i)$. Since we know the expression (3.3) for the Dedekind zeta function for the field $Q(i)$, this proves the first part of the theorem.

As for the second part, it follows from the functional equations of the Dirichlet beta and of the Riemann zeta functions written in the following form:
\[\beta(1-s) = \left(\frac{\pi}{2}\right)^{-s} \sin\left(\frac{\pi}{2} s\right) \Gamma(s) \beta(s),\]
\[\zeta(1-s) = \frac{1}{2^s \pi^{s-1} \sin\left(\frac{\pi}{2s}\right) \Gamma(1-s) \zeta(s)}.\]
This concludes the proof of our main result.  \[\square\]
The Dirichlet $L$-series $L(\chi_{-4}, s)$ is analytic in $\{\text{Re}(s) > 0\}$ by general principles on Dirichlet series. This follows from the fact that if $(a_n)$ is a sequence of complex numbers such that there exist $C > 0$ and $r > 0$ such that for large $n$, we have:

$$\left| \sum_{k=1}^{n} a_k \right| \leq C n^r,$$

then the Dirichlet series

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad s \in \mathbb{C}$$

is analytic on $\{\text{Re}(s) > 0\}$. Now if $K$ is a quadratic field with discriminant $\Delta$, the quadratic character $\chi_K$,

$$\chi_K(m) = \left( \frac{\Delta}{m} \right)$$

is periodic of period $|\Delta|$, so for any $n$,

$$\sum_{n=n_0}^{n_0+|\Delta|-1} \chi_K(n) = 0$$

and there exists $C > 0$ such that

$$\left| \sum_{k=1}^{n} \chi_K(k) \right| \leq C.$$

The analytic (and meromorphic) continuation of $\beta$ (and $\zeta$) can also be deduced from the integral formulas:

$$\beta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{e^t + e^{-t}} \, dt, \quad \text{Re}(s) > 0,$$

$$\zeta(s) = \frac{1}{(1 - 2^{1-s}) \Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{e^t + 1} \, dt.$$

From the decomposition in infinite products of $L(\chi_{-4}, s)$ and of the Riemann zeta function, we obtain:

$$\zeta_{\mathbb{Q}}(s) = \left( \prod_{p \equiv 1 \mod 4} (1 - p^{-s})^{-2} \prod_{p \equiv 3 \mod 4} (1 - p^{-2s})^{-2} (1 - 2^{-s})^{-1} \right)^2$$

$$= \left( \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^s} \right)^2. \quad (5.3)$$

**Remark 5.5.** We extend Remark 3.3 to the case of our bicomplex zeta function. Let $a_n$ be the Dirichlet series coefficients of $\zeta_{\mathbb{Q}(i)}(s)$ and let $A_n$ be the corresponding coefficients of $\zeta_{\mathbb{Q}}(s)$. Then

$$A_n = \sum_{p,q; pq=n} a_p a_q = \frac{1}{16} \sum_{p,q; pq=n} r_2(p)r_2(q).$$
In particular $A_1 = 1$ and if $n$ is prime
\[ A_n = \frac{1}{2} r_2(n) \]
because $r_2(1) = 4$. If moreover, $n$ is prime congruent to 1 (mod 4), by Fermat theorem we get $A_n = 4$. So we have precise information about the Dirichlet coefficients $A_n$ of our bicomplex zeta function $\zeta_{\mathbb{B}Q}(s)$, when $n$ is prime and a fixed value when $n$ is prime congruent to 1 (mod 4). The first coefficients of the Dirichlet series of the bicomplex zeta series are:
\[ A_1 = 1, A_2 = 2, A_3 = 0, A_4 = 3, A_5 = 4, \ldots \]

We now conclude with the general case of the multicomplex algebra $\mathbb{B}C_n$. The definitions and proofs follow closely the case $n = 2$ of bicomplex numbers, so, for simplicity, we just state the main result.

The corresponding “rational” subalgebras $\mathbb{B}Q_n$ are defined analogously to (5.1). The component-wise multiplication given by the idempotent representation in the last stage (see (4.2) and (4.3)) produces the splitting of $\mathbb{B}Q_n$ into $2^{n-1}$ factors of $\mathbb{Q}(i_1)$:
\[ \mathbb{B}Q_n \cong \prod_{\ell=1}^{2^{n-1}} \mathbb{B}Q_n / I_{e_{\ell}} \cong \mathbb{Q}(i_1)^{2^{n-1}}. \]

As before, the maximal order of the product above is the product of the maximal orders of $\mathbb{Q}(i_1)$ (which is $\mathbb{Z}[i_1]$). In complete analogy, we obtain the following expression for the associated zeta function of multicomplex numbers:

**Theorem 5.6.** The Dedekind-like zeta function of the algebra $\mathbb{B}Q_n$ is
\[ \zeta_{\mathbb{B}Q_n}(s) = \zeta(s)^{2^{n-1}} \cdot L(\chi_{-4},s)^{2^{n-1}}. \tag{5.4} \]

The multicomplex zeta functions above has a pole of order $2^{n-1}$ at $s = 1$ and a residue equal to $\frac{\pi}{2} \gamma_{\mathbb{Q}(i)}$. It verifies the functional equation:
\[ \pi^{-2^{n-1}(1-s)} \Gamma^{2^{n-1}}(1-s) \zeta_{\mathbb{B}Q_n}(1-s) = \pi^{-2^{n-1}} \Gamma^{2^{n-1}}(s) \zeta_{\mathbb{B}Q_n}(s). \tag{5.5} \]

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