MODULI OF TWISTED SHEAVES

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Abstract. We study moduli of semistable twisted sheaves on smooth proper morphisms of algebraic spaces. In the case of a relative curve or surface, we prove results on the structure of these spaces. For curves, they are essentially isomorphic to spaces of semistable vector bundles. In the case of surfaces, we show (under a mild hypothesis on the twisting class) that the spaces are asymptotically geometrically irreducible, normal, generically smooth, and l.c.i. over the base. We also develop general tools necessary for these results: the theory of associated points and purity of sheaves on Artin stacks, twisted Bogomolov inequalities, semistability and boundedness results, and basic results on twisted Quot-schemes on a surface.

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1. Introduction

We have only recently begun to understand the role played in algebraic and arithmetic geometry by twisted sheaves. Originally studied by mathematical physicists, most research to date has focused on their derived category. In his thesis [4], Căldăraru extended the classical construction of the Fourier-Mukai transform to study equivalences of derived categories of twisted sheaves on elliptic fibrations. This was further extended in [16] to the case of genus 1 fibrations without sections which appear as elements of the Tate-Shafarevich group of a fixed elliptic fibration. These twisted Fourier-Mukai transforms arise in the presence of non-fine moduli spaces; instead of having a universal sheaf, one has a universal twisted sheaf.

Implicit in these constructions is a theory of moduli for twisted sheaves. Recent work (apparently roughly simultaneous with the work described here) has produced such a theory tailored to special cases (in particular, K3 surfaces) [60], and this theory has been used to prove a general conjecture of Căldăraru about the relationship between Hodge isometries and twisted Fourier-Mukai transforms for K3 surfaces [31]. The construction of the moduli space in [60], using a definition of stability proposed in [3], is easily seen to be equivalent to the classical construction of Simpson in the case of moduli of modules.

We develop in this paper a general theory of moduli of twisted sheaves on algebraic spaces, and then apply it to study twisted sheaves on curves and surfaces. Applications to geometry and arithmetic are taken up in [43] and [46]. The crucial observation is that by thinking of twisted sheaves as sheaves on certain stacks \( \mathcal{X} \), one can carry out an analysis very similar to the analysis of sheaves on varieties. Using the rational Chow theory of \( \mathcal{X} \) and Riemann-Roch theorems for representable morphisms of Deligne-Mumford stacks, one can define a notion of semistability for twisted sheaves. With this in hand, it is possible to “twist” classical tools such as Harder-Narasimhan filtrations, elementary transforms, and determinants, and arrive at constructions and results familiar from the classical case. This is most interesting in the case of surfaces, where we prove the following main structure theorem. Let \( X \to S \) be a projective relative surface with smooth connected geometric fibers, and let \( \mathcal{X} \to X \) be a \( \mu_n \)-gerbe. (A review of gerbes and twisted sheaves may be found in sections 2.1.1 and 2.1.2 below.)

**Theorem.** The stack of semistable \( \mathcal{X} \)-twisted sheaves is an Artin stack locally of finite presentation over \( S \). If \( \mathcal{X} \) is optimal then the substack of twisted sheaves of fixed determinant and sufficiently large discriminant is generically smooth, normal, geometrically irreducible, and l.c.i. along the stable locus.

The condition of “optimality” is added to make the results characteristic-free; it is relatively clear from our methods of proof that this hypothesis can be eliminated in characteristic 0, and recent work of Langer in the untwisted case should help eliminate this hypothesis in arbitrary characteristic (although the methods involved are slightly different from our own). Since we have not carried out either exercise, we will only give the proof here in the optimal case.

This theorem lives at the junction of several roads in algebraic geometry. First, it is a “case study” in the moduli of sheaves on tame smooth Deligne-Mumford stacks. In fact, many of the techniques of this paper – the “geometric Hilbert polynomial,” boundedness results, etc. – can be carried over to the general case (in preparation); the case of gerbes is then seen as the most
tractable case of a more general theory. Second, by a rigidification mechanism, one can use our methods to study compactified moduli spaces of $\text{PGL}_n$-bundles (or Azumaya algebras, for the arithmetically minded). The resulting statements give a first-order algebraic approximation to results of Taubes on the stable topology of the space of connections on a fixed smooth bundle. These matters will be taken up in detail in [43].

The fact that $\text{PGL}_n$-bundles are the same thing as Azumaya algebras relates the constructions of this paper to classical problems about the Brauer groups of function fields. In particular, our techniques permit more efficient and conceptual proofs of the basic facts about Brauer groups of schemes, including the results of Gabber’s thesis concerning the relation between the Brauer group and the cohomological Brauer group. (In fact, de Jong’s recent proof of Gabber’s theorem on the cohomological Brauer group of a quasi-projective scheme uses techniques similar to those of this paper.) Moreover, the structure theory for moduli spaces of twisted sheaves gives new results about the period-index problem for surfaces over finite and local fields, generalizing well-known recent results of de Jong for surfaces over algebraically closed fields. In particular, one can prove the following.

**Theorem.** Let $X$ be a smooth projective geometrically connected surface over a field $k$ and $\alpha \in \text{Br}(X)$ a Brauer class of order prime to the characteristic of $k$.

1. If $k$ is algebraically closed, then $\text{per}(\alpha) = \text{ind}(\alpha)$.
2. If $k$ is finite, then $\text{ind}(\alpha) \mid \text{per}(\alpha)^3$. If in addition $\alpha$ is unramified, then $\text{ind}(\alpha) = \text{per}(\alpha)$.
3. If $k$ is local, $\alpha$ is unramified, and $X$ has smooth reduction, then $\text{ind}(\alpha) \mid \text{per}(\alpha)^2$. If in addition $\alpha$ is unramified on a smooth model of $X$ over the integers of $k$, then $\text{ind}(\alpha) = \text{per}(\alpha)$.

The first statement is the well-known result of de Jong; the others are new. These ideas are discussed in [46].

The results described here stand as yet another example of how the use of stack-theoretic methods can clarify and extend classical results, and suggest new approaches and connections. This is the overarching philosophy of this work, and we hope that the reader will take this, if nothing else, away from this paper.

As this is a young field, we provide relatively complete foundations for the abstract theory of twisted sheaves. We devote all of section 2 to a development of the algebraic theory of twisted sheaves, including general nonsense on their deformation theory, as well as a theory of associated points and purity for sheaves on Artin stacks. We end the section with a discussion of Riemann-Roch theorems for gerbes and the basic properties of Quot spaces. In section 2.3 we show that the stack of semistable twisted sheaves is an Artin stack by using Artin’s representability theorem. Finally, section 3 is devoted to studying the resulting stacks of twisted sheaves on curves and surfaces. In the last subsection, we prove twisted analogues of O’Grady’s results on asymptotic properties of the moduli spaces, including the first theorem above.

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It has come to my attention that Stuhler and Hoffmann have recently obtained some of the results of this paper independently [29].
Notation

Following standard conventions, we use = for canonical isomorphisms.

Every locally free sheaf is assumed to have finite rank everywhere.

As should be universal by now, “algebraic stack” will mean “algebraic stack in the sense of Artin.” Deligne-Mumford stacks will be called “DM stacks.” All algebraic stacks are quasi-separated, as is any base scheme appearing in this paper.

In order to prevent psychological problems, when given a topos $X$, we will write $U$ in place of $X/U$ to stand for the restriction of $X$ to the object $U \in X$. For the sake of intuition, we will also interchangeably refer to “sheaves on $X$” and “objects in $X$” depending upon the context.

Following Huybrechts and Lehn [30], we use the notation “hom” for “dim Hom” and “ext” for “dim Ext.” In general, we have tried to keep notations in common with their book when treating the twisted analogues of classical theorems so beautifully discussed there.

We will occasionally call a stack quasi-proper if it is universally closed over Noetherian base spaces but is not necessarily separated. This arises quite often in the theory of stacks, e.g. when dealing with strictly semistable objects – GIT stacks for example can only reasonably be expected to be separated along the stable locus.

There is one pedantic grammatical convention we adopt which we hope will spread: a number with mathematical meaning is always written as a numeral, occasionally in contradiction to accepted rules of grammar (e.g., “rank 1”, “characteristic 0”).

2. Twisted sheaves

2.1. Preliminaries: twisted sheaves on ringed topoi. In this section, we lay the foundations for the theory of twisted sheaves on algebraic spaces and stacks. The reader will note that much of the first three sections is written in the language of ringed topoi. We encourage those uncomfortable with this notion to substitute “ringed site” or even “ringed space” for “ringed topos”; the exposition will remain more or less the same after this substitution (but the reader should note that sites larger than the Zariski site of a scheme are essential for the theory to actually be interesting). One reason to write in this degree of generality is to make the theory apply to algebraic stacks, where one can only really understand the theory of sheaves from the topos-theoretic point of view.

In order to link Giraud’s ideas with subsequent developments in the theory of algebraic stacks [42], we review foundations on the sites associated to a stack, sheaves on those sites, and classifying topoi associated to gerbes on (ringed) topoi. We only consider stacks in groupoids here; the task of extending the results to stacks in arbitrary (small) categories is left to the (odd) reader. (It will primarily consist of adding the word “Cartesian” in a few places.)

2.1.1. Sheaves and gerbes on stacks. Let $X$ be a topos and $F : \mathcal{S} \to X$ a stack on $X$. The topology on $X$ naturally induces a topology on $\mathcal{S}$.

Definition 2.1.1.1. The site of $\mathcal{S}$, denoted $\mathcal{S}^s$, has as underlying category

Objects: morphisms $f : S \to \mathcal{S}$ of fibered categories over $X$, where $S$ ranges over all sheaves on (=objects of) $X$

Morphisms: a morphism from $f : S \to \mathcal{S}$ to $g : S' \to \mathcal{S}$ is a pair $(\varphi, \psi)$ where

$\varphi : S \to S'$ is a morphism in $X$ and $\psi : f \sim g \circ \varphi$ is a 2-isomorphism.

A covering is given by a morphism $(\varphi, \psi)$ with $\varphi$ a covering.

Definition 2.1.1.2. The classifying topos of $\mathcal{S}$, denoted $\mathcal{F}$, is the topos of sheaves on the site of $\mathcal{S}$.

There is a morphism of topoi $\pi : \mathcal{F} \to X$: given a sheaf $\mathcal{F}$ on $X$, one gets a sheaf $\pi^* \mathcal{F}$ on $\mathcal{F}$ by assigning to $f : S \to \mathcal{S}$ the object $\mathcal{F}(S)$. The obvious exactness properties show that
this is the pullback of a morphism of topoi. In particular, when \( X \) is ringed, say by \( \mathcal{O} \), \( \mathcal{F} \) is naturally ringed by \( \pi^* \mathcal{O} \).

Remark 2.1.1.3. The reader can easily check that the formation of the classifying topos is functorial, and that our description agrees with Giraud’s original definition: \( \mathcal{F} = \text{Cart}(\mathcal{I}, \mathcal{F}(\text{Fl}(X))) \) [23, §5.1].

2.1.1.4. Given a stack \( \mathcal{I} \to X \), there is an associated stack \( \mathcal{I}(\mathcal{I}) \to \mathcal{I} \) called the inertia stack. To give the stack \( \mathcal{I}(\mathcal{I}) \) it is enough to describe its sheaf of sections.

Definition 2.1.1.5. The assignment \( (f : S \to \mathcal{I}) \mapsto \text{Aut}(f) \) is a sheaf. The corresponding stack is denoted \( \mathcal{I}(\mathcal{I}) \to \mathcal{I} \) and called the inertia stack of \( \mathcal{I} \).

Lemma 2.1.1.6. For any stack \( \mathcal{I} \), \( \mathcal{I}(\mathcal{I}) = \mathcal{I} \times \mathcal{I} \times \mathcal{I} \).

Thus, when \( X \) is the topos of sheaves on the big étale topology on affine schemes over a fixed base \( B \) and \( \mathcal{I} \) is an algebraic stack, then one easily sees that \( \mathcal{I}(\mathcal{I}) \) is also an algebraic stack and the morphism \( \mathcal{I}(\mathcal{I}) \to \mathcal{I} \) is representable, quasi-compact, and separated. We leave the following standard lemmas to the reader.

Lemma 2.1.1.7. Given a 1-morphism \( f : S \to \mathcal{I} \) of stacks, there is an induced map \( \mathcal{I}(\mathcal{I}) \to f^* \mathcal{I}(\mathcal{I}) \) in \( \mathcal{T} \).

Lemma 2.1.1.8. Given any sheaf \( F \) on \( \mathcal{T} \), there is a natural right group action \( \mu : F \times \mathcal{I}(\mathcal{I}) \to F \).

2.1.1.9. We will be concerned throughout this paper with gerbes.

Definition 2.1.1.10. The stack \( \mathcal{I} \) is a gerbe on \( X \) if

1. For any \( U \in X \) there exists a covering \( U' \to U \) such that \( \mathcal{I}_{U'} \neq \emptyset \).
2. For any \( U \in X \) and any \( s, s' \in \mathcal{I}_U \), there exists a covering \( U' \to U \) such that \( s|_{U'} \) is isomorphic to \( s'|_{U'} \).

In looser language, \( \mathcal{I} \) has local sections everywhere and any two sections are locally isomorphic. There is a “moduli-theoretic” interpretation of this definition.

Definition 2.1.1.11. The sheaf associated to \( \mathcal{I} \), denoted \( \text{Sh}(\mathcal{I}) \), is the sheafification of the presheaf whose sections over \( U \in X \) are isomorphism classes of objects in the fiber category \( \mathcal{I}_U \).

Lemma 2.1.1.12. The stack \( \mathcal{I} \) is a gerbe on \( X \) if and only if the natural map \( \text{Sh}(\mathcal{I}) \to e_X \) is an isomorphism in \( X \).

Here \( e_X \) denotes the final object of the topos \( X \). This will often be written as \( X \) by abuse of notation.

Proof. Suppose \( \mathcal{I} \) is a gerbe. By functoriality of the natural map, it is enough to demonstrate the claim when \( \mathcal{I} \) has a global section \( \sigma \) over \( X \). But then every local section is locally isomorphic to \( \sigma \), hence \( \text{Sh}(\mathcal{I}) \) is a singleton and the natural map is an isomorphism.

Suppose conversely that \( \text{Sh}(\mathcal{I}) \to e_X \) is an isomorphism. In particular,

\[
\text{Sh}(\mathcal{I})(U) = \{\emptyset\}
\]

for any \( U \in X \). By the definition of \( \text{Sh}(\mathcal{I}) \) and of sheafification, this says precisely that conditions 1 and 2 of 2.1.1.10 is satisfied.

Lemma 2.1.1.13. If \( \mathcal{I} \to X \) is a gerbe and \( F \) is a sheaf on \( \mathcal{I} \) such that the inertia action \( F \times \mathcal{I}(\mathcal{I}) \to F \) is trivial, then \( F \) is naturally the pullback of a unique sheaf on \( X \) up to isomorphism.
Definition 2.1.2.1. An \( \chi \) and a \( \chi \)-gerbe on \( m \) action objects. This is of course described in great detail in [23].

\[ \pi' \pi s F \to F \] is an isomorphism. To verify this, it suffices to work locally on \( X \), so we may assume that \( G \) has a section. One can then check using the hypothesis on the action that the pullback of \( F \) along this section equals the pushforward of \( F \) along the structure morphism. The result follows.

Remark 2.1.14. This holds more generally when \( X \) is the coarse moduli space of a Deligne-Mumford stack \( \mathcal{S} \) (with the action of inertia being studied in the big étale topology), but the proof is slightly more difficult: it follows without too much difficulty from the étale local structure of the stack as a finite group quotient stack [42, §6], [33].

Lemma 2.1.1.5. If \( \pi : \mathcal{S} \to X \) is a gerbe and \( \mathcal{S}(\mathcal{S}) \) is an abelian sheaf on \( \mathcal{S} \), then there is an abelian sheaf \( A \) on \( X \) and an isomorphism \( \pi' A \cong \mathcal{S}(\mathcal{S}) \) as objects of \( \mathcal{S} \).

Proof. This is an application of 2.1.1.13.

2.1.2. Twisted sheaves. Let \( (X, \mathcal{O}) \) be a ringed topos, \( A \) a sheaf of commutative groups on \( X \), and \( \chi : A \to G_m \) a character.

Definition 2.1.2.1. An \( A \)-gerbe on \( X \) is a gerbe \( \mathcal{S} \to X \) along with an isomorphism \( A_{\mathcal{S}} \cong \mathcal{S}(\mathcal{S}) \) in \( \mathcal{S} \).

When \( A \) is non-commutative, this definition is not correct. One must instead choose any isomorphism as in the definition in the category of liens on \( \mathcal{S} \) rather than the category of sheaves. The basic reason for this may be seen by thinking about the gerbe \( B_G \) for a non-commutative group \( G \). In general, the automorphism group of a left \( G \)-torsor is an inner form of \( G \), not \( G \) itself. (The stack of liens on a topos is the universal stack receiving a 1-morphism from the stack of groups on that topos such that two inner forms naturally map to isomorphic objects.) This is of course described in great detail in [23].

Given a cohomology class \( \alpha \in \text{H}^2(X, A) \), there is a corresponding equivalence class of \( A \)-gerbes on \( X \). We will fix such a class \( \alpha \) and an \( A \)-gerbe \( \mathcal{S} \to X \). The goal of this section is to single out a subcategory of sheaves on \( \mathcal{S} \) which will play a fundamental role in what follows.

Given an \( \mathcal{O}_{\mathcal{S}} \)-module \( \mathcal{F} \), the module action \( m : G_m \times \mathcal{F} \to \mathcal{F} \) yields an associated right action \( m' : \mathcal{F} \times G_m \to \mathcal{F} \) with \( m'(s, \varphi) = m(\varphi^{-1}, s) \). This will always be called the associated right action.

Definition 2.1.2.2. A \( d \)-fold \( \chi \)-twisted sheaf on \( \mathcal{S} \) is an \( \mathcal{O}_{\mathcal{S}} \)-module \( \mathcal{F} \) such that the natural action \( \mu : \mathcal{F} \times A \to \mathcal{F} \) given by the \( A \)-gerbe structure makes the diagram

\[
\begin{array}{ccc}
\mathcal{F} \times A & \longrightarrow & \mathcal{F} \\
\chi^d \downarrow & & \downarrow \text{id} \\
\mathcal{F} \times G_m & \longrightarrow & \mathcal{F}
\end{array}
\]

commute, where \( \chi^d(s) = \chi(s)^d \). A 1-fold twisted sheaf will be called simply a twisted sheaf.

We will see below that if \( \mathcal{S}^{(d)} \) is a gerbe representing \( d \cdot \alpha \in \text{H}^2(X, A) \), then twisted sheaves on \( \mathcal{S}^{(d)} \) are equivalent to \( d \)-fold twisted sheaves on \( \mathcal{S} \). Note that \( \mathcal{S} \)-twisted sheaves naturally form a fibered subcategory of the classifying topos \( \mathcal{S} \), viewed as a fibered category over \( X \) via the natural map \( \mathcal{S} \to X \) of topoi.

Proposition 2.1.2.3. The fibered category of \( \mathcal{S} \)-twisted sheaves is a naturally a stack \( X \).

Proof. This follows from the fact that the condition that a sheaf on \( \mathcal{S} \) be an \( \mathcal{S} \)-twisted sheaf is local on the site of \( \mathcal{S} \) along with the fact that any morphism \( Y \to X \) of topoi defines a natural stack on \( X \) by restriction.
Notation 2.1.2.4. Given a \((\mathbb{G}_m, \mu_n)\)-gerbe \(\mathcal{X}\) on a morphism of ringed topoi \(X \to S\), the stack of \(S\)-flat quasi-coherent \(\mathcal{X}\)-twisted sheaves locally of finite presentation will be denoted \(\mathcal{T}_{\mathcal{X}/S}\).

An attentive reader may have noticed that the morphism \(\chi\) yields a “change of structure group” for the gerbe \(\mathcal{X}\).

**Proposition 2.1.2.5.** Let \(f : A \to B\) be a morphism of abelian sheaves and \(\alpha \in H^2(X, A)\) a cohomology class with direct image \(f_*(\alpha) = \beta \in H^2(X, B)\). Given gerbes \(\mathcal{X}_\alpha\) and \(\mathcal{X}_\beta\) representing \(\alpha\) and \(\beta\), there is a 1-morphism \(F : \mathcal{X}_\alpha \to \mathcal{X}_\beta\) over \(X\) such that for any section \(\sigma : S \to \mathcal{X}_\alpha\), the induced morphism \(A_S = \text{Aut}(\sigma) \to \text{Aut}(F(\sigma)) = B_S\) is \(f_S\). Given a character \(\chi : B \to \mathbb{G}_m\), this induces an identification of the stack of \(\chi\)-twisted sheaves with the stack of \(\chi \circ f\)-twisted sheaves.

**Proof.** The existence of the morphism of stacks \(\mathcal{X}_\alpha \to \mathcal{X}_\beta\) is part of Giraud’s theory of non-abelian cohomology. One can also see this explicitly by re-expressing the gerbe as a stack of twisted torsors and using the natural contraction of torsors along a group homomorphism to define the morphism of gerbes. This is done in [45]. To see that the stacks of twisted sheaves are identified, it is enough to prove that for the morphism \(\chi : B \to \mathbb{G}_m\), pullback along \(\chi\) yields a 1-isomorphism from the stack of id-twisted sheaves to the stack of \(\chi\)-twisted sheaves. Since these are both stacks, it suffices to prove this locally on \(X\), so we may assume that in fact both gerbes admit global sections, say \(\sigma_\alpha\) and \(\sigma_\beta\). It is easy to see that there is an invertible sheaf \(\mathcal{L}\) on \(\mathcal{X}\) such that for any \(\chi\)-twisted sheaf \(\mathcal{F}\) on \(\mathcal{X}_\beta\), there is a natural isomorphism \(\sigma_{\alpha}^*F \sim \mathcal{L} \otimes \sigma_{\beta}^*\mathcal{F}\). It is therefore enough to prove that for any \(A\)-gerbe \(\mathcal{X}\) with a section \(\sigma\) and any character \(\chi : A \to \mathbb{G}_m\), the pullback functor \(\sigma^*\) defines an equivalence of the stack of \(\chi\)-twisted sheaves with the stack of modules on \(\mathcal{X}\). But any such \(\mathcal{X}\) is isomorphic to \(BA\), so the stack of all modules on \(\mathcal{X}\) is equivalent to the stack of \(\mathcal{O}_X\)-modules with a right \(A\)-action, and the stack of \(\chi\)-twisted sheaves is equivalent to the stack of \(A\)-equivariant \(\mathcal{O}_X\)-modules such that \(A\) acts via \(\chi\). It is clear that this last category is equivalent to the category of modules by simply forgetting the \(A\)-action, which is precisely the pullback functor.

Under this identification, the stack of \(\chi\)-twisted sheaves is identified with the stack of \(\iota\)-twisted sheaves, where \(\iota : \text{im} \chi \hookrightarrow \mathbb{G}_m\) is the natural inclusion of the image. Thus, we have gained very little but canonicality by our formalism. However, one might in the future try something similar when \(\mathcal{F}\) is not a gerbe and the inertia stack \(\mathcal{F}(\mathcal{X})\) is not constant, in which case this setup is the correct one. Such a study is related to moduli of ramified Azumaya algebras and the ramified period-index problem. These issues will be explored in future work.

### 2.1.3. Comparison with the formulation of Căldăraru

We explain in this section how our definition of twisted sheaves squares with that used by Căldăraru in [4]. The reader will note that his formulation seems more “user-friendly.” We hope to make clear below, especially in our discussion of deformations and obstructions, why the more abstract approach is essential.

Throughout, we retain the notation of the previous section: \((X, \mathcal{O})\) is a ringed topos and \(\chi : A \to \mathbb{G}_m\) is a character of a sheaf of commutative groups. Let \(\alpha \in H^2(X, A)\) be a fixed cohomology class. By a theorem of Verdier [2, Exposé V.7], there is a hypercovering \(U_* \to X\) and a cocycle \(\alpha \in \Gamma(U_2, A)\) which represents \(\alpha\) in cohomology. We fix such a representative in this section. We also fix a choice of \(A\)-gerbe \(\mathcal{X}\) representing the cohomology class \(\alpha\).

**Definition 2.1.3.1** (Căldăraru). A \(\chi\)-twisted sheaf on \(X\) is a pair \((F, g)\), where \(F\) is an \(\mathcal{O}_{U_0}\)-module and \(g : (\text{pr}_1^{U_1})^*F \sim (\text{pr}_0^{U_0})^*F\) is a gluing datum on \(U_1\) such that \(\delta g \in \text{Aut}((\text{pr}_0^{U_2})^*F)\) equals the cocycle \(\chi(\alpha)\).

We will (temporarily) call such an object a Căldăraru-\(\chi\)-twisted sheaf.
Example 2.1.3.2. Suppose $A = \mathbb{G}_m$, $\chi = \text{id}$, and $X$ is a complex analytic space. We may take the hypercovering $U_\bullet$ to be the Čech hypercovering generated by an open covering of $X$, i.e., we may replace $U_\bullet$ by an open covering $\{U_i\}$ of $X$. Then a $\chi$-twisted sheaf on $X$ is given by

1. a sheaf of modules $\mathcal{F}_i$ on each $U_i$
2. for each $i$ and $j$ an isomorphism of modules $g_{ij} : \mathcal{F}_j|_{U_{ij}} \sim \mathcal{F}_i|_{U_{ij}}$

subject to the requirement that on $U_{ijk}$, $g^{-1}_{ik}g_{ij}g_{jk} : \mathcal{F}_k|_{U_{ijk}} \sim \mathcal{F}_k|_{U_{ijk}}$ is equal to multiplication by the scalar $a \in \mathbb{G}_m(U_{ijk})$ giving the 2-cocycle. $\diamond$

Proposition 2.1.3.3. There is a natural equivalence of fibered categories between $\chi$-twisted sheaves and Căldăruşu-\(\chi\)-twisted sheaves.

Surprisingly, the proof is not obvious. It is written in gory detail in section 2.1.3 of [45]. It can at times be useful to know that these two categories are equivalent, as certain statements are completely obvious in one and completely mysterious in the other. The canonical example of this is furnished by quasi-coherent and coherent twisted sheaves on a Noetherian gerbe. One sees easily using the stack-theoretic language that any quasi-coherent twisted sheaf is the colimit of its coherent subsheaves, whereas from the Căldăruşu point of view this is far from obvious. We will summarize the important properties of quasi-coherent twisted sheaves below; the proofs are exercises and have been omitted or briefly sketched.

It is obvious that the Căldăruşu-$\chi$-twisted sheaves are equivalent to twisted sheaves for the cocycle in $\mathbb{G}_m$ induced by $\chi$. (This is an example of a property which is easier to detect using the Căldăruşu formalism – witness the proof of 2.1.2.5.) Let $\mathcal{X}_\alpha \to \mathcal{X}_{\chi(\alpha)}$ be as in 2.1.2.5, where $\chi(\alpha) \in H^2(X, \mathbb{G}_m)$.

Definition 2.1.3.4. If $\chi : A \to \mathbb{G}_m$ is the natural inclusion of a subsheaf (e.g., $\mu_n$ or $\mathbb{G}_m$), a $\chi$-twisted sheaf on $\mathcal{X}$ will be called an $\mathcal{X}$-twisted sheaf.

Corollary 2.1.3.5. The map $\mathcal{X}_\alpha \to \mathcal{X}_{\chi(\alpha)}$ induces by pullback a 1-isomorphism of the stack of $\mathcal{X}_{\chi(\alpha)}$-twisted sheaves with the stack of $\chi$-twisted sheaves.

2.2. The case of a scheme.

2.2.1. Quasi-coherent twisted sheaves. Let $X$ be a scheme, $A$ a group scheme which is faithfully flat and locally of finite presentation over $X$, $\alpha \in H^2(X_{\text{fppf}}, A)$ a flat cohomology class, and $\chi : A \to \mathbb{G}_m$ an algebraic character. Fix a gerbe $\mathcal{X}$ representing $\alpha$ in the big fppf topology on $X$. When $A$ is smooth, a theorem of Grothendieck [27, Appendix] says that the restriction of $\mathcal{X}$ to the (big or small) étale topos of $X$ is an $A$-gerbe (and this defines an isomorphism $H^2(X_{\text{fppf}}, A) \sim H^2(X_{\text{ét}}, A)$). (In fact, Grothendieck’s theorem holds for the cohomology in all degrees.) We recall for the reader some of the basic facts about quasi-coherent twisted sheaves.

Lemma 2.2.1.1. The gerbe $\mathcal{X}$ is an algebraic stack locally of finite presentation over $X$. If $X$ is quasi-separated and $A$ is finitely presented then $\mathcal{X}$ is finitely presented. The scheme $X$ is (locally) Noetherian if and only if $\mathcal{X}$ is (locally) Noetherian.

Remark 2.2.1.2. In our study of twisted sheaves on surfaces (when we actually want to say something!), we will take $A = \mu_n$ with $n$ prime to the characteristics of $X$. In this case, the reader will immediately verify that any $A$-gerbe is in fact a DM stack.

When $X$ is Noetherian we can define quasi-coherent and coherent twisted sheaves.

Lemma 2.2.1.3. The obvious morphism $F : \mathcal{X}_{\text{fppf}} \to \mathcal{X}_{\text{ét}}$ induces by pullback an equivalence of the stacks of quasi-coherent sheaves. These stacks are naturally equivalent to the stack of twisted sheaves in the lisse-étale topos of $\mathcal{X}$. 

The notion of quasi-coherence is also independent of the group chosen.

**Lemma 2.2.1.4.** Under the equivalence of 2.1.3.5, quasi-coherent sheaves are taken to quasi-coherent sheaves.

**Proof.** This follows from the fact that a sheaf on an algebraic stack $\mathcal{X}$ is quasi-coherent if and only if it pulls back to a quasi-coherent big étale sheaf on any scheme mapping to $\mathcal{X}$. Thus, if $T \rightarrow \mathcal{X}_{\alpha}$, the compatibility of pullbacks shows that given any $F$ on $\mathcal{X}_{\alpha}$, we have that $F|_{\mathcal{X}_{\alpha}|T}$ is naturally isomorphic to the pullback along the induced morphism $T \rightarrow \mathcal{X}_{\alpha}$, whence it is quasi-coherent by the assumption on $F$. □

**Proposition 2.2.1.5.** Suppose $X$ is Noetherian and $A$ is group scheme faithfully flat of finite presentation over $X$. A quasi-coherent $\chi$-twisted sheaf is the colimit of its coherent $\chi$-twisted subsheaves.

This proposition turns out to be quite useful in re-proving the basic facts about the Brauer group of a scheme. We refer the interested reader to [46].

In fact, when $A$ is diagonalizable we can split up the category of quasi-coherent sheaves into pieces indexed by characters. Suppose $D$ is a diagonalizable affine group scheme (i.e., the Cartier dual of $D$ is a constant finitely generated abelian group). Write $C$ for the dual group of $D$, which is the group of homomorphisms $D \rightarrow \mathbb{G}_m$. Let $\mathcal{X}$ be a $D$-gerbe and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{X}$. Given $\chi \in C$, there is a $\chi$-eigensheaf $\mathcal{F}_\chi \subset \mathcal{F}$. The following proposition follows easily from the representation theory of diagonalizable group schemes (and reduction to the case of a trivial gerbe).

**Proposition 2.2.1.6.** Suppose $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{X}$. The natural maps induce an isomorphism

$$\bigoplus_{\chi \in C(X)} \mathcal{F}_\chi \cong \mathcal{F}.$$ 

The eigensheaves $\mathcal{F}_\chi$ are quasi-coherent.

Let $Y \rightarrow X$ be a quasi-compact morphism of schemes and $\mathcal{X}$ a $D$-gerbe on $X$. Define $\mathcal{Y} := Y \times_X \mathcal{X}$; this is naturally a $D$-gerbe on $Y$. Denote the morphism $\mathcal{Y} \rightarrow \mathcal{X}$ by $\pi$.

**Lemma 2.2.1.7.** If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{Y}$, then the natural map $\pi_*(\mathcal{F}_\chi) \rightarrow \pi_*\mathcal{F}$ identifies $\pi_*(\mathcal{F}_\chi)$ with $(\pi_*\mathcal{F})_\chi$.

### 2.2.2. Gabber’s theorem and Morita equivalence

Using twisted sheaves, de Jong [14] has recently proven the following result of Gabber (vastly generalizing a result of his thesis [21]).

**Theorem 2.2.2.1.** If $X$ is a quasi-compact separated scheme admitting an ample invertible sheaf then $\text{Br}(X) = \text{Br}^\prime(X)$.

For the reader unfamiliar with the Brauer group, this result is equivalent to the following corollary.

**Corollary 2.2.2.2.** Given $X$ as in 2.2.2.1 and a $\mu_n$-gerbe $\mathcal{X} \rightarrow X$, there is a locally free $\mathcal{X}$-twisted sheaf $\mathcal{V}$ of constant non-zero rank.

In general, the question of when there exists a locally free $\mathcal{X}$-twisted sheaf is delicate and interesting question. It is equivalent to $\mathcal{X}$ being a quotient stack; the study of this question for gerbes is closely related the question of when an arbitrary (tame) Deligne-Mumford stack is a quotient stack. This has been studied by Vistoli, Kresch, Hassett, Edidin and others (see [37], [18], [58] and the references therein).

As a consequence of 2.2.2.2, we can rewrite the theory of twisted sheaves on such a scheme $X$ in terms of modules over the algebra $\mathcal{E}nd(\mathcal{V})$. 

**Proposition 2.2.2.3.** Given the notation of 2.2.2.2, the functor $\mathcal{W} \mapsto \mathcal{H}om(\mathcal{V}, \mathcal{W})$ establishes an equivalence of fibered categories between $\mathcal{X}$-twisted sheaves and right $\mathcal{E}nd(V)$-modules.

**Sketch of proof.** See Theorem 1.3.7 of [4]. This is a special case of “fibered Morita equivalence” which is studied in gory generality (in an arbitrary topos with sufficiently many points) in [45]. □

2.2.3. **Deformations and obstructions.** Since twisted sheaves are modules in a topos, we can try to apply the deformation theory of Illusie to study deformations and obstructions of twisted sheaves. The condition that a deformation preserve the character of the inertial action and that an obstruction take this into account makes the situation slightly more complicated than in Illusie's bare theory. We present an alternative approach to the deformation theory of twisted sheaves, parallel to the approach of Grothendieck sketched in [32]. Since extensions of quasi-coherent twisted sheaves in the category of all sheaves on a gerbe are well-behaved (and stay twisted!), it is easy to see that Illusie’s theory (with its attendant functorialities) applies perfectly. Thus, the reader who trusts that one can develop the deformation theory of twisted sheaves “from scratch” in the more complicated cases can skip ahead to the study of Artin’s conditions for this deformation theory 2.2.3.6. We note that the non-Illusian approach described here may have wider applicability. An example of this is furnished by recent work [44] on constructing an algebraic stack of complexes; one can approach the deformation theory of objects in the derived category using a derived version of Grothendieck’s approach which is not clearly related to Illusie’s approach.

2.2.3.1. In this section, we work in the Abelian category of twisted sheaves. Thus, all Ext groups are computed in this category and not in the larger category of all sheaves on $\mathcal{X}$. When we specialize to quasi-coherent twisted sheaves, this will no longer matter, as both Ext spaces are naturally isomorphic. For the moment, fix a topos $X$ and a $\mathbb{G}_m$-gerbe $\mathcal{X}$ on $X$.

**Lemma 2.2.3.2.** The category of $\mathcal{X}$-twisted sheaves contains enough injectives and enough flat objects.

**Proof.** Let $U \in X$ be an object over which $\mathcal{X}$ splits and let $\mathcal{U} = \mathcal{X} \times_X U$, with natural map $f : \mathcal{U} \to \mathcal{X}$. Then $\mathcal{U} \cong \mathbb{B} \mathbb{G}_m, U$ as $U$-stacks. The usual abstract nonsense shows that there are enough twisted injectives and flat objects on $\mathcal{U}$. Taking $f_!$ of injectives and $f^*$ of flat objects yields the desired result. The details are left to the reader (or see [4]). □

In fact, recent work [7] yields the existence of $K$-injective and $K$-flat resolutions in the homotopy category of complexes of twisted sheaves. This justifies the use of all of the usual derived functors for twisted sheaves: $\mathcal{R} \mathcal{H}om, \mathcal{R}Hom, \mathcal{L} f^*, \mathcal{L} f_!$, etc., as well as allowing proofs of the assertions below. We refer the reader unfamiliar with these ideas to [56] and [7].

We will use the “cher à Cartan” isomorphism to produce a naïve deformation and obstruction theory for twisted sheaves (without making use of the whole topos of sheaves on the gerbe). First, we recall (without proof) what the cher à Cartan isomorphism is, in our context.

**Proposition 2.2.3.3.** Let $B \to B_0$ be a morphism of rings in $X$ and $\mathcal{X} \to X$ a $\mathbb{G}_m$-gerbe. Given a complex of $\mathcal{X}$-twisted $B$-modules $M$ and a complex of $\mathcal{X}$-twisted $B_0$-modules $J$, there is a natural isomorphism in the derived category

$$\mathcal{R}Hom_B(M, J) \cong \mathcal{R}Hom_{B_0}(\mathcal{L} f_! M \otimes_{B_0} B_0, J).$$

This is the derived adjointness of $\otimes_B B_0$ and (derived) restriction of scalars to $B$. (Note that this also applies when the gerbe $\mathcal{X}$ is trivial, so in particular in any topos.) We will use this below in 2.2.3.11 to deduce localization and constructibility properties for the deformation and obstruction theory of twisted sheaves.
Corollary 2.2.3.4 (cher à Cartan). Let \( B \to B_0 \) be a surjection of rings in \( X \) and \( \mathcal{X} \to X \) a \( \mathbb{G}_m \)-gerbe. Given two \( \mathcal{X} \)-twisted \( B_0 \)-modules \( M_0 \) and \( J \), there is a natural isomorphism

\[
\mathbf{R} \text{Hom}_B(M_0, J) \to \mathbf{R} \text{Hom}_{B_0}(M_0 \otimes B_0, J)
\]

in the derived category of \( \mathcal{X} \)-twisted sheaves.

Let \( B \to B_0 \) be a square-zero extension of rings in \( X \) with kernel \( I \). Suppose \( M_0 \) and \( J \) are twisted \( B_0 \)-modules. We wish to know when there exists an \( I \)-flat extension of \( M_0 \) by \( J \).

Given any such extension, there is a naturally resulting morphism \( I \otimes B_0 M_0 \to J \), which is an isomorphism if and only if the extension is \( I \)-flat. Fix a morphism \( u : I \otimes B_0 M_0 \to J \). As in §IV.3.1 of [32], we have the following proposition.

Proposition 2.2.3.5. There is an exact sequence

\[
0 \to \text{Ext}^1_{B_0}(M_0, J) \to \text{Ext}^1_B(M_0, J) \to \text{Hom}_{B_0}(I \otimes M_0, J) \xrightarrow{\partial} \text{Ext}^2_{B_0}(M_0, J)
\]

with the property that there exists an extension with associated morphism \( u \) if and only if \( \partial(u) = 0 \). The space of all such extensions is a torsor under \( \text{Ext}^1_{B_0}(M_0, J) \).

Proof. The exact sequence is the sequence of low degree terms arising from 2.2.3.4 and the composition of functors spectral sequence. That the maps agree with the interpretation given is checked carefully in [32, p. 252ff]. Note that Illusie’s proof works in the derived category of twisted sheaves; it is not necessary to work in the category of all modules in the topos. (The Ext groups are different, but the functorialities are the same.) \( \square \)

2.2.3.6. We now have enough information to describe an obstruction theory for the problem of twisted sheaves on a scheme. In this section, \( f : X \to S \) will be a proper morphism of Noetherian excellent algebraic spaces and \( \mathcal{X} \to X \) will be a fixed \( \mathbb{G}_m \)-gerbe. (An algebraic space is excellent if every étale chart is excellent. In 18.7.7 of [26], the reader will find an example of a non-excellent scheme with an excellent finite étale cover. We thank Brian Conrad for pointing out this example.) We will develop the deformation-theoretic tools necessary to apply Artin’s Existence Theorem. Let \( A_0 \) be a reduced Noetherian ring. We recall some terminology from Artin’s paper [9].

Definition 2.2.3.7. A deformation situation is a commutative diagram of Noetherian rings \( A' \to A \to A_0 \) such that

1. \( A \to A_0 \) and \( A' \to A \) are infinitesimal extensions (i.e., they have nilpotent kernels)
2. \( \ker(A' \to A) = M \) is a finite \( A_0 \)-module.

In the classical study of versal deformations, one often takes \( A_0 \) to be a field and \( A, A' \) to be local Artinian rings with residue field \( k \).

Let \( F \) be a stack on \( S \).

Definition 2.2.3.8. An obstruction theory for \( F \) consists of two parts.

(i) For each infinitesimal extension \( A \to A_0 \) and element \( a \in F(A) \), a functor \( \text{Ob}_a : \text{Mod}_{A_0}^{\text{finite}} \to \text{Mod}_{A_0}^{\text{finite}} \)

(ii) For each deformation situation and \( a \in F(A) \), there is an element \( o_a(A') \in \text{Ob}_a(M) \) which vanishes if and only if there is an element \( F(A') \) whose reduction to \( A \) is isomorphic to \( a \).

These data are subject to two further constraints:
Given a diagram

\[
\begin{array}{ccc}
B & \xrightarrow{f} & A_0 \\
\downarrow & & \downarrow \\
A & \xrightarrow{g} & A_0 \\
\end{array}
\]

of infinitesimal extensions, one has \( \text{Ob}_a = \text{Ob}_{f(a)} \) as functors \( \text{Mod}^\text{finite}_{A_0} \to \text{Mod}^\text{finite}_{A_0} \).

For any diagram of deformation situations

\[
\begin{array}{ccc}
A' & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
B' & \xrightarrow{f} & B \\
\end{array}
\]

giving rise to an \( A_0 \)-linear map of kernels \( M_A \to M_B \), we get for any \( a \in F(A) \) an \( A_0 \)-linear map \( \text{Ob}_a(M_A) \to \text{Ob}_a(M_B) \) taking \( o_a(A') \) to \( o_a(B') \).

We will call \( F \) functoriality and \( L \) linearity of the obstruction theory.

Let \( F \) be the stack which assigns to any Noetherian affine scheme \( \text{Spec} A \to S \) the groupoid of \( A \)-flat families of coherent \( X \)-twisted sheaves \( \mathcal{F} \) on \( X \otimes_S A \).

**Lemma 2.2.3.9.** If \( \mathcal{F} \) and \( \mathcal{G} \) are \( A \)-flat coherent \( X \otimes A \)-twisted sheaves then \( \text{Ext}^i(\mathcal{F}, \mathcal{G}) \) is a finite \( A \)-module.

**Proof.** This follows from the local to global spectral sequence for \( \text{Ext} \) and the finiteness of coherent cohomology for a proper morphism. (Coherence of the sheaf \( \text{Ext} \) is a local computation in the big étale topology of \( X \), hence follows from the corresponding fact for locally Noetherian schemes.) \( \square \)

**Proposition 2.2.3.10.** The following give an obstruction theory for \( F \).

1. Given an infinitesimal extension \( A \to A_0 \) and \( \mathcal{F} \in F(A) \),
   \[
   \text{Ob}_\mathcal{F}(M) = \text{Ext}^2_{X \otimes A_0}(\mathcal{F}_0, M \otimes_{A_0} \mathcal{F}_0) = \text{Ext}^2_{X \otimes A}(\mathcal{F}, M \otimes_A \mathcal{F}).
   \]

2. Given a deformation situation \( A' \to A \to A_0 \) with kernel \( M \), \( o_\mathcal{F}(A') = \partial(\text{id} : M \otimes_A \mathcal{F} \to M \otimes_A \mathcal{F}) \) in 2.2.3.5

The equality of (1) above is a simple consequence of the cher à Cartan isomorphism and the fact that \( \mathcal{F} \otimes_A A_0 = \mathcal{F}_0 \) (by flatness).

**Proof.** Using 2.2.3.5, it suffices to check \( F \) and \( L \) and prove that \( \text{Ext}^2(\mathcal{F}_0, M \otimes_{A_0} \mathcal{F}_0) \) is a finite \( A_0 \)-module. \( F \) follows from the description of the obstruction group in terms of \( A_0 \) and \( L \) follows from the naturality of 2.2.3.5. The finiteness is 2.2.3.9. \( \square \)

We can use this formalism to prove several “localization and constructibility” results about the deformation theory of coherent twisted sheaves. These are the “conditions (4.1)” of Artin’s famous [9].
Proposition 2.2.3.11. Let $A_0$ be a reduced Noetherian ring, $A_0 \to B_0$ a flat ring extension, \( f : X \to \text{Spec} \, A_0 \) a proper morphism, \( \mathcal{X} \to X \) a \( G_m \)-gerbe, \( \mathcal{F} \) an \( A_0 \)-flat family of coherent \( \mathcal{X} \)-twisted sheaves, and \( M \) an \( A_0 \)-module. For any \( i \geq 0 \) the following hold.

1. \( \text{Ext}_X^i(\mathcal{F}, M \otimes \mathcal{F}) \otimes_{A_0} B_0 \cong \text{Ext}_X^i(\mathcal{F}_{B_0}, M_{B_0} \otimes \mathcal{F}_{B_0}) \).

2. If \( m \subset A_0 \) is a maximal ideal then
\[
\text{Ext}_X^i(\mathcal{F}, M \otimes \mathcal{F}) \otimes \hat{A}_0 \cong \varprojlim \text{Ext}_X^i(\mathcal{F}, M/m^nM \otimes \mathcal{F}),
\]
the completion of \( A_0 \) being taken with respect to \( m \).

3. There is a dense open set of points (of finite type) \( p \in \text{Spec} \, A_0 \) such that
\[
\text{Ext}_X^i(\mathcal{F}, M \otimes \mathcal{F}) \otimes \kappa(p) \cong \text{Ext}_X^i(\mathcal{F}_{\kappa(p)}, M_{\kappa(p)} \otimes \mathcal{F}_{\kappa(p)}).
\]

Proof. The proof of 1 is immediate. To prove 2, we work in Căldăraru form. This makes it clear that one can easily understand formal twisted sheaves on the formal completion of a scheme along a closed subscheme. We wish to prove that if \( X \to \text{Spec} \, A \) is a proper scheme over a complete Noetherian local ring and \( \mathcal{F} \) and \( \mathcal{G} \) are coherent twisted sheaves on \( X \) then
\[
\text{Ext}_X^i(\mathcal{F}, \mathcal{G}) \cong \varprojlim \text{Ext}_X^i(\mathcal{F} \otimes A/m^n, \mathcal{G} \otimes A/m^n).
\]
This works just as in [24, 4.5]: one shows that the completion of the sheaf \( \text{Ext}_X^i(\mathcal{F}, \mathcal{G}) \) along the closed fiber is naturally isomorphic to the sheaf \( \text{Ext}_X^i(\mathcal{F}, \mathcal{G}) \) of extensions over the formal scheme. The rest comes by taking the local-to-global Ext spectral sequence and using the finiteness of coherent cohomology to make an Artin-Rees argument. The interested reader should consult §III.4.5 of [24] for further details.

The proof of 3 is slightly subtle. Suppose first that \( A_0 \) is a finite type \( \mathbf{Z} \)-algebra. Localizing, we may suppose that \( A_0 \) is a regular Noetherian ring of finite Krull dimension \( d \) and that \( X \) and \( M \) are \( A_0 \)-flat. Thus, any \( A_0 \)-module has homological dimension at most \( d \). Let \( \mathcal{C} = R \mathcal{H} \text{om}(\mathcal{F}, M \otimes \mathcal{F}) \in D(X) \). It is easy to see (using the bound on the homological dimension) that one has
\[
\mathcal{C} \otimes \kappa(p) = R \mathcal{H} \text{om}(\mathcal{F}, M \otimes \mathcal{F} \otimes \kappa(p)).
\]
Furthermore, the bound on homological dimension implies that \( \tau_{\leq i+d} \mathcal{C} \to \mathcal{C} \) remains a quasi-isomorphism (of bounded-below complexes) in degrees \( \leq i \) upon any base change. Thus, we see that we are concerned with the base change properties of \( R f_* \tau_{\leq i+d} \mathcal{C} \). Localizing \( A_0 \), we may suppose that the (coherent) cohomology sheaves of \( \tau_{\leq i+d} \mathcal{C} \) are all \( A_0 \)-flat. By the standard cohomology and base change argument, we then see that the formation of \( R f_* \tau_{\leq i+d} \mathcal{C} \) is compatible with base change. Passing to the open subscheme of \( \text{Spec} \, A_0 \) over which the cohomology sheaves of \( R f_* \tau_{\leq i+d} \mathcal{C} \) are flat yields the open subset we seek.

In the case of arbitrary reduced \( A_0 \), note that since \( X \) is of finite type (hence of finite presentation over \( A_0 \), everything being Noetherian) and \( \mathcal{F} \) is coherent, we can descend \( X \), \( \mathcal{F} \), and \( M \) to a finite type \( \mathbf{Z} \)-subalgebra \( B \subset A_0 \). Localizing, we may take \( B \) to be regular of finite Krull dimension. Applying the previous paragraph, upon shrinking \( \text{Spec} \, B \) enough, we find a perfect complex \( \mathcal{P} \) whose cohomology universally computes the Ext spaces in question. Thus, the question reduces to the case already treated. \( \square \)

2.2.4. Determinants and equideterminantal deformations. Given a perfect object of the derived category of twisted sheaves on a topos, one can use the construction of Mumford and Knudsen [35] to define its determinant. Given a ringed topos \( X \) and a \( G_m \)-gerbe \( \mathcal{X} \to X \), let \( D^r(\mathcal{X}) \) denote the derived category of \( \mathcal{X} \)-twisted sheaves. There is a natural map \( D^r(\mathcal{X}) \to D(\mathcal{X}) \), but it is not clear what properties this map has. Of course, if \( X \) is an algebraic space and once
Consider only quasi-coherent cohomologies, then the natural functor is an equivalence onto a direct summand triangulated category.)

**Definition 2.2.4.1.** Let $X$ be a topos, $\mathcal{X} \to X$ a $G_m$-gerbe, and $\mathcal{F}$ an $\mathcal{X}$-twisted sheaf on $X$ such that $\mathcal{F}$ is perfect as an object of $D^+(\mathcal{X})$. The determinant of $\mathcal{F}$, denoted $\det \mathcal{F}$, is the Knudsen-Mumford determinant of the complex $\mathcal{F} \in D^+(\mathcal{X})$.

It is clear from the construction that $\det \mathcal{F}$ can be computed in either $D(\mathcal{X})$ or $D^+(\mathcal{X})$. More generally, if $X$ is an algebraic space then it is clear that the restriction of the functor $D^+(\mathcal{X}) \to D(\mathcal{X})$ to the sub-triangulated category $D^+(\mathcal{X})_{\text{perf}}$ of perfect complexes induces an equivalence with a triangulated direct summand of the triangulated category $D(\mathcal{X})_{\text{perf}}$. This will not be of any use to us.

Our goal in this section is to study deformations of a twisted sheaf which fix its determinant. Let $I \to A \to A_0$ be a small extension of Noetherian rings over $S$ and $X \to S$ a proper algebraic space of finite presentation. We assume that $n$ is invertible in $A_0$ in what follows. Fix a $\mu_n$-gerbe $\mathcal{X}$ on $X_A$. (By standard results in étale cohomology, $\mathcal{X}$ is in fact determined by the $\mu_n$-gerbe structure on $\mathcal{X}_{A_0}$; this fact is relevant to the study of deformations of Azumaya algebras and their generalizations. The reader is referred to [43] for details.)

**Lemma 2.2.4.2.** Let $(X, \mathcal{O})$ be a ringed topos and $A$, $B$, and $C$ complexes of $\mathcal{O}$-modules. There is a natural isomorphism

$$R\mathcal{H}om(A \otimes B, C) \cong R\mathcal{H}om(A, R\mathcal{H}om(B, C))$$

and a natural isomorphism

$$\text{RHom}(A \otimes B, C) \cong \text{RHom}(A, \text{RHom}(B, C)).$$

*Sketch of proof.* For further details see e.g. [56]. This is a close relative of 2.2.3.3 and can proven similarly using the techniques of Neeman and Spaltenstein: replace $A$ and $B$ by $K$-flat resolutions $F_A$, $F_B$, and $C$ by a $K$-injective resolution $I_C$. Then $\mathcal{H}om(F_B, I_C)$ is weakly $K$-injective, hence $R\mathcal{H}om(A, R\mathcal{H}om(B, C))$ is computed by

$$\mathcal{H}om(F_A, \mathcal{H}om(F_B, I_C)).$$

Using the hom-tensor adjunction on modules, this is naturally isomorphic to

$$\mathcal{H}om(F_A \otimes F_B, I_C),$$

which computes $R\mathcal{H}om(A \otimes B, C)$ as usual. The last formula follows upon taking derived global sections of the sheafified version. □

If $\mathcal{F}$ is perfect, then there is a natural isomorphism $\mathcal{F} \cong \mathcal{F}^{\vee \vee}$. Applying 2.2.4.2 we see that to the identity in End($\mathcal{F}$) corresponds some morphism $\text{Hom}_D(\mathcal{F} \otimes \mathcal{F}^\vee, \mathcal{O})$. This gives rise to a morphism $R\mathcal{H}om(\mathcal{F}, \mathcal{F}) \to \mathcal{O}$, called the trace morphism, which we will denote $\text{Tr}$. In what follows, we will let $\mathcal{A} = R\mathcal{H}om(\mathcal{F}, \mathcal{F})$; this is an example of a generalized Azumaya algebra, the collection of which may be used to compactify moduli of Azumaya algebras. This is discussed in [43].

**Definition 2.2.4.3.** The homotopy fiber of $\text{Tr} : \mathcal{A} \to \mathcal{O}$ in $D(X)$ is the traceless part of $\mathcal{A}$ and denoted $\text{s}\mathcal{A}$.

**Lemma 2.2.4.4.** Under the natural isomorphisms $\mathcal{A}^\vee \cong \mathcal{A}$ and $\mathcal{O}^\vee \cong \mathcal{O}$, the trace is dual to the unit $\mathcal{O} \to \mathcal{A}$.

*Proof.* By functoriality, we can localize and assume that $\mathcal{F}$ is a strict perfect complex, where this is just a computation. □
**Lemma 2.2.4.5.** If $\mathcal{F}$ is a perfect complex of $\mathcal{O}$-modules, the composition

$$\mathcal{O} \to \mathcal{R} \mathbb{H}\text{om}(\mathcal{F}, \mathcal{F}) \xrightarrow{\text{Tr}} \mathcal{O}$$

is equal to multiplication by the rank of $\mathcal{F}$.

**Proof.** If $\mathcal{F}$ is a strict perfect complex, i.e., there is a quasi-isomorphism $\mathcal{V} \sim \mathcal{F}$ with $\mathcal{V}$ a finite complex of locally free modules, this comes down to checking that the adjunction is induced by the obvious maps. As every perfect complex is locally quasi-isomorphic to such a complex, this will prove the general case by functoriality. □

**Definition 2.2.4.6.** The reduced trace of $A$ is the map $\tau = \frac{1}{\text{rk} F} \text{Tr} : \mathcal{R} \mathbb{E}\text{nd}(\mathcal{F}) \to \mathcal{O}$.

**Proposition 2.2.4.7.** Let $f : A \to B$ be a map in the derived category $D(\mathbb{C})$ of an abelian category. If $f$ has a section $g : B \to A$ then there is an isomorphism $\text{holim}(g) \cong \text{hocolim}(f)$.

**Proof.** In other words, the homotopy fiber of $g$ is isomorphic to the homotopy cofiber (“mapping cone”) of $f$. This is a straightforward exercise which works in any triangulated category. □

**Corollary 2.2.4.8.** The third vertex $p_{A}$ of the unit $\mathcal{O} \to A$ is isomorphic to the traceless part $s_{A}$.

**Proof.** This is an application of 2.2.4.7 to 2.2.4.5 and 2.2.4.4. □

The main result of this section is that the traceless part of $\mathcal{R} \mathbb{H}\text{om}(\mathcal{F}, \mathcal{F})$ governs the equideterminantal deformation theory of $\mathcal{F}$ (as long as $\det \mathcal{F}$ is unobstructed). (In fact, one can also see that the traceless part governs the deformation theory of the derived algebra $\mathcal{R} \mathbb{H}\text{om}(\mathcal{F}, \mathcal{F})$ in a precise manner. This is discussed in [43] and in [45] in great detail.) In the general case (when the rank is not invertible on the base), a more subtle analysis is called for. It is not especially difficult, and may be found in [10, §8.4], but we will not make use of it here.

For the sake of concreteness, we assume that the determinant is trivialized. In general, one need only know that the determinant is unobstructed to make the following proposition valid as stated.

**Proposition 2.2.4.9.** Let $\mathcal{F}$ be an $A_{0}$-flat $\mathcal{X}_{A_{0}}$-twisted coherent sheaf with torsion free fibers of rank $n$ and trivial determinant $\mathcal{O} \xrightarrow{\sim} \det \mathcal{F}$. Let $\mathcal{A} = \mathcal{R} \mathbb{H}\text{om}(\mathcal{F}, \mathcal{F})$.

1. The obstruction to deforming $\mathcal{F}$ while preserving the determinant lies in the hypercohomology $\mathbb{H}^{2}(I \otimes s\mathcal{A}) = \text{Ext}^{2}(\mathcal{F}, I \otimes \mathcal{F})_{0}$.
2. The isomorphism classes of equideterminantal deformations of $\mathcal{F}$ are a principal homogeneous space under the hypercohomology $\mathbb{H}^{1}(I \otimes s\mathcal{A}) = \text{Ext}^{1}(\mathcal{F}, I \otimes \mathcal{F})_{0}$.
3. The determinant-preserving infinitesimal automorphisms of a deformation are equal to $\mathbb{H}^{0}(I \otimes s\mathcal{A}) = \text{Hom}(\mathcal{F}, I \otimes \mathcal{F})_{0}$.

**Proof.** According to Illusie’s standard deformation theory of sheaves in topoi (which applies verbatim as $\mathcal{F}_{0}$ is coherent), we have only to show that the trace of the obstruction of $\mathcal{F}$ is the obstruction of $\det \mathcal{F}$. As we will only use this in the case where $X/A_{0}$ is smooth and projective, we may assume that $\mathcal{F}$ has enough locally free twisted sheaves (perhaps this should be called “twisted locally factorial”) and that every coherent twisted sheaf admits a finite locally free resolution. (When $X$ is projective, the existence of locally free twisted sheaves follows from 2.2.2.2. Note that it is not known if being regular and separated ensures the existence of sufficiently many locally free twisted sheaves.) The argument one can use to prove
this is practically identical to the argument of Artamkin [8] and proceeds by induction on the homological dimension of \( \mathcal{F} \). If \( \mathcal{F} \) is locally free, the statement is quite easy. The inductive step works as follows: choose a surjection \( 0 \to \mathcal{K} \to \mathcal{V} \to \mathcal{F} \to 0 \) with \( \mathcal{V} \) a locally free twisted sheaf whose deformation is unobstructed. Then the obstruction to deforming \( \det \mathcal{F} \) is the same as the obstruction to deforming \( \det \mathcal{K} \). Furthermore, \( \mathcal{K} \) has smaller homological dimension, hence the obstruction of \( \det \mathcal{K} \) is the trace of the obstruction of \( \mathcal{K} \). A simple argument shows that the trace of the obstruction of \( \mathcal{F} \) equals the trace of the obstruction of \( \mathcal{K} \).

The second statement works in a similar way and uses 2.2.4.8. The last statement is left to the reader. 

2.2.5. Optimality. The following notion will appear from time to time throughout this paper, so we honor it with its own subsubsection.

**Definition 2.2.5.1.** Given a \( \mu_n \)-gerbe \( \mathcal{X} \to X \), the index of \( \mathcal{X} \), denoted \( \text{ind}(\mathcal{X}) \) is the minimal rank of a locally free \( \mathcal{X} \)-twisted sheaf over the generic scheme of \( X \). The period of \( \mathcal{X} \), denoted \( \text{per}(\mathcal{X}) \), is the order of the image of \([\mathcal{X}]\) in \( \text{Br}(X) \).

**Definition 2.2.5.2.** A \( \mu_n \)-gerbe \( \mathcal{X} \) is optimal if the period is \( n \).

When \( X \) is regular, one immediately seems upon taking determinants that \( \text{per}(\mathcal{X}) | \text{ind}(\mathcal{X}) \): any \( \mathcal{X} \)-twisted sheaf of rank \( m \) over the generic scheme of \( X \) extends to a coherent \( \mathcal{X} \)-twisted sheaf of rank \( m \), and forming \( \det \mathcal{X} \) yields an \( m \)-fold \( \mathcal{X} \)-twisted invertible sheaf, which yields \( m[\mathcal{X}] = 0 \in H^2(X, G_m) \). (For details the reader is referred to [46,]). Using Galois cohomology at generic points, one also sees that \( \text{ind}(\mathcal{X}) | \text{per}(\mathcal{X})^m \) for some \( m \) (see [38]). When \( X \) is a surface (over an algebraically closed field), a theorem of de Jong (which is re-proven in [46] using the theory developed here and the theorem of Graber-Harris-Starr-de Jong [15]) shows that \( \text{per}(\mathcal{X}) = \text{ind}(\mathcal{X}) \). Thus, on a surface, the index of a \( \mu_n \)-gerbe divides \( n \). It is easy to show that the rank of any locally free \( \mathcal{X} \)-twisted sheaf is divisible by \( \text{ind}(\mathcal{X}) \).

Studying moduli of twisted sheaves of rank \( n \) on an optimal \( \mu_n \)-gerbe \( \mathcal{X} \to X \) is a non-commutative analogue of the Picard scheme, in the sense that such sheaves are essentially rank 1 right modules over an Azumaya algebra on \( X \). Thus, the stability condition which we define in section 2.3.2 becomes vacuous, making certain proofs technically easier. While we have not yet written out the general proofs, we strongly feel that this is a non-essential distinction for the theorems of 3.2.4 to hold. This deficit will be addressed in future work.

2.2.6. Purity of sheaves on Artin stacks. In this section, we will study purity of twisted sheaves as a precursor to 2.3.2, where we will study various stability conditions on twisted sheaves. The ultimate goal is to produce a tractable algebraic stack parametrizing a well-behaved collection of twisted sheaves. We develop most of this section in much greater generality for coherent sheaves on algebraic stacks.

**2.2.6.1. Support of twisted sheaves.** Twisted sheaves may be viewed both as objects on \( X \) and as objects on a \( \mu_n \)-gerbe \( \mathcal{X} \) over \( X \). This leads to two natural definitions of support for a twisted sheaf, which coincide. (In the sequel, when the gerbe is understood we will often refer to “twisted sheaves on \( X \)” for the sake of notational simplicity.)

**Definition 2.2.6.2.** Given a \( \mu_n \)-gerbe \( \mathcal{X} \to X \) and an \( \mathcal{X} \)-twisted coherent sheaf \( \mathcal{F} \), the support of \( \mathcal{F} \) is the closed substack of \( \mathcal{X} \) defined by the kernel of the map \( \mathcal{O}_X \to \mathcal{End}_\mathcal{X}(\mathcal{F}) \), which is a quasi-coherent sheaf of ideals. The schematic support of \( \mathcal{F} \) is the scheme-theoretic image in \( X \) of the support of \( \mathcal{F} \).

Since \( \mathcal{End}_\mathcal{X}(\mathcal{F}) \) is the pullback to \( \mathcal{X} \) of a coherent \( \mathcal{O}_X \)-algebra, it is immediate that the support of \( \mathcal{F} \) is the preimage of the schematic support. In particular, a twisted sheaf \( \mathcal{F} \) with
schematic support \( Y \subset X \) is naturally a \( \mathcal{X} \times_X Y \)-twisted sheaf with full schematic support (on \( Y \)). Thus, considering the support of a sheaf does not nullify its “twistedness.”

2.2.6.3. Associated points on Artin stacks. We can now define a torsion filtration on a twisted sheaf. To do this properly, we will briefly develop the theory of associated points and torsion subsheaves on an arbitrary Noetherian algebraic stack. (When trying to generalize these results to the non-Noetherian case, certain equivalences will fail, making the theory developed here only one possibility.) Throughout, we systematically work with the underlying topological space \(|\mathcal{X}|\) of a Noetherian algebraic stack. The support of a sheaf will be taken to mean simply the underlying set of points of \(|\mathcal{X}|\), or the reduced closed substack structure on that set when it is closed (e.g., if \( \mathcal{F} \) is coherent). We will not require (as is typical) that \( \text{Supp}(\mathcal{F}) \) is the closure of the set of points where \( \mathcal{F} \) is supported.

Let \( \mathcal{F} \) be a quasi-coherent sheaf on \( \mathcal{X} \).

Definition 2.2.6.4. A point \( p \in |\mathcal{X}| \) is an associated point of \( \mathcal{F} \) if there is a quasi-coherent subsheaf \( \mathcal{G} \) such that \( p \in \text{Supp}(\mathcal{G}) \subset \{ p \} \). The set of associated points of \( \mathcal{F} \) will be written \( \text{Ass}(\mathcal{F}) \).

If \( \mathcal{F} \) is coherent, this is the same as requiring that \( \text{Supp}(\mathcal{G}) = \overline{\{ p \}} \). In general, this is not the case, as supports need not be closed for quasi-coherent sheaves.

Remark 2.2.6.5. When \( \mathcal{X} \) is a Noetherian scheme, this is the same as the usual notion (essentially because one can extend quasi-coherent subsheaves off of generic points). More generally, if \( \mathcal{X} \) is a Noetherian DM stack, one can say that a geometric point \( p \to \mathcal{X} \) is associated to \( \mathcal{F} \) if \( p \) is an associated point for the stalk of \( \mathcal{F} \) at \( p \) (as a module over \( \mathcal{O}_{p,\mathcal{X}}^{\text{sh}} \)). By an argument similar to 2.2.6.6 below, a point of \(|\mathcal{X}|\) is associated iff some (and hence any) geometric point lying over it is associated, so this also yields the same notion as 2.2.6.4.

Proposition 2.2.6.6. Let \( f : X \to \mathcal{X} \) be a flat surjection, with \( X \) a Noetherian scheme. If \( \mathcal{F} \) is a quasi-coherent sheaf on \( \mathcal{X} \), then \( \text{Ass}(\mathcal{F}) = f(\text{Ass}(\mathcal{F}|_X)) \).

Proof. Write \( \mathcal{F}' = \mathcal{F}|_{X'} \). Given a point \( p \in \text{Ass}(\mathcal{F}) \), it is easy to see that a generic point of \( f^{-1}(\{ p \}) \) will be in \( \text{Ass}(\mathcal{F}') \). Conversely, let \( q \in \text{Ass}(\mathcal{F}') \) and let \( Y = f^{-1}(\{ f(q) \}) \) as a reduced closed subscheme of \( X \). Let \( \mathcal{G} \subset \mathcal{F}' \) be the maximal quasi-coherent subsheaf supported on \( Y \). (It is not true that \( \text{Supp}(\mathcal{G}) = \overline{\{ q \}} \), but we at least know that \( q \in \text{Supp}(\mathcal{G}) \).) We claim that \( \mathcal{G} \) descends to a subsheaf of \( \mathcal{F} \) with support containing \( f(q) \) and contained in \( \{ f(q) \} \). To see this, it is enough to show that the two pullbacks of \( \mathcal{G} \) to \( X \times_X Y \) are equal as subsheaves. In fact, by colimit considerations, we may assume that \( \mathcal{F} \) is coherent. We are reduced to the following situation: given a flat surjection \( g : Z \to W \) of Noetherian algebraic stacks with \( W \) an affine scheme, a closed subspace \( Y \subset W \), and a coherent sheaf \( \mathcal{F} \) on \( W \), let \( \mathcal{G}_Y \subset \mathcal{F} \) denote the maximal subsheaf \( \mathcal{F} \) with \( \text{Supp}(\mathcal{G}) = Y \). It suffices to show that \( g^*(\mathcal{G}_Y) \) is the maximal subsheaf of \( \mathcal{F} \) with support on \( f^{-1}(Y) \). To prove this, let \( \mathcal{I} \) be the ideal cutting out the reduced structure on \( Y \). By flatness, \( \mathcal{J} = g^*(\mathcal{I}) \) is a sheaf of ideals cutting out a subsheaf of \( \mathcal{F} \) supported on \( g^{-1}(Y) \). To say that \( \mathcal{G}_Y \) is maximal is the same as saying that the sheaf \( \mathcal{H}om(\mathcal{O}/\mathcal{J}^n, \mathcal{F}/\mathcal{G}_Y) \) vanishes for all \( n > 0 \). By flat pullback, we conclude that \( \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Z/\mathcal{J}^n, \mathcal{F}_Z/g^*(\mathcal{G}_Y)) = 0 \), whence \( g^n(\mathcal{G}_Y) \) is maximal.

Corollary 2.2.6.7. If \( f : \mathcal{X}' \to \mathcal{X} \) is a flat surjection of Noetherian algebraic stacks and \( \mathcal{F} \) is a quasi-coherent \( \mathcal{O}_{\mathcal{X}} \)-module, then \( \text{Ass}(\mathcal{F}) = f(\text{Ass}(\mathcal{F}|_{\mathcal{X}'}) \).

Proof. Choosing a smooth cover of \( \mathcal{X}' \) reduces this to 2.2.6.6.

Corollary 2.2.6.8. If \( \mathcal{F} \) is a coherent sheaf on \( \mathcal{X} \) then \( \text{Ass}(\mathcal{F}) \) is finite.
Proof. The stack \( \mathcal{X} \) has a smooth cover by a Noetherian scheme \( X' \). By 2.2.6.6, we are reduced to the case of a scheme, where this is a classical result [49, 6.5].

Points of a stack are subject to the relations of specialization and generalization in the usual way. This gives \( \text{Ass}(\mathcal{F}) \) the structure of partially ordered set. By 2.2.6.8 there are well-defined minimal elements of \( \text{Ass}(\mathcal{F}) \).

It is easy to check that \( \text{Ass}(\mathcal{F}) = \text{Supp}(\mathcal{F}) \) and that the minimal points of \( \text{Ass}(\mathcal{F}) \) coincide with the minimal points of \( \text{Supp}(\mathcal{F}) \).

Lemma 2.2.6.9. Suppose \( \mathcal{F}, \mathcal{G}, \mathcal{H} \) are three coherent sheaves on \( \mathcal{X} \) fitting into an exact sequence \( 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \).

1. \( \text{Ass}(\mathcal{F}) \subset \text{Ass}(\mathcal{G}) \subset \text{Ass}(\mathcal{F}) \cup \text{Ass}(\mathcal{H}) \). If the sequence is split exact, the second inclusion is a bijection.
2. The minimal points of \( \text{Ass}(\mathcal{H}) \) are contained in \( \text{Ass}(\mathcal{G}) \).
3. If \( \mathcal{G} \neq 0 \), then \( \text{Ass}(\mathcal{G}) \neq \emptyset \).

Proof. This is precisely analogous to the classical proof [49, §6].

Definition 2.2.6.10. A torsion subsheaf of \( \mathcal{F} \) is a subsheaf \( \mathcal{F}' \subset \mathcal{F} \) with the property that none of the minimal points of \( \text{Ass}(\mathcal{F}) \) are contained in \( \text{Ass}(\mathcal{F}') \).

Note that any minimal point of \( \text{Ass}(\mathcal{F}) \) which is also associated to a subsheaf \( \mathcal{F}' \) will be minimal in \( \text{Ass}(\mathcal{F}') \).

Lemma 2.2.6.11. The sum of any two torsion subsheaves of \( \mathcal{F} \) is a torsion subsheaf. There is a unique maximal coherent torsion subsheaf of \( \mathcal{F} \).

Proof. Suppose \( \mathcal{F}' \) and \( \mathcal{F}'' \) are torsion subsheaves of \( \mathcal{F} \). By 2.2.6.9, the minimal points of \( \text{Ass}(\mathcal{F}' + \mathcal{F}'') \) are contained in \( \text{Ass}(\mathcal{F}') \cup \text{Ass}(\mathcal{F}'') \). This proves the first statement. The second follows by taking the sum of all torsion subsheaves of \( \mathcal{F} \) (which is allowable because they form a set).

The maximal torsion subsheaf of \( \mathcal{F} \) will be called the torsion subsheaf of \( \mathcal{F} \) and denoted \( T(\mathcal{F}) \).

Lemma 2.2.6.12. Any non-minimal point of \( \text{Ass}(\mathcal{F}) \) is contained in \( \text{Ass}(T(\mathcal{F})) \).

Proof. Immediate from the definition!

Remark 2.2.6.13. When \( \mathcal{X} \) is a gerbe bound by a diagonalizable group scheme, the decomposition 2.2.1.6 respects torsion subsheaves, so we see that we have also developed a good theory of torsion subsheaves for twisted sheaves.

Definition 2.2.6.14. A coherent sheaf \( \mathcal{F} \) is pure if \( T(\mathcal{F}) = 0 \).

Remark 2.2.6.15. By 2.2.6.12, we see that \( \mathcal{F} \) is pure if and only if \( \text{Ass}(\mathcal{F}) \) consists solely of minimal points, i.e., the partial ordering on \( \text{Ass}(\mathcal{F}) \) is trivial.

Lemma 2.2.6.16. Given any coherent sheaf \( \mathcal{F} \) on \( \mathcal{X} \), the sheaf \( \mathcal{F}/T(\mathcal{F}) \) is pure.

Proof. There is an exact sequence \( 0 \to T(\mathcal{F}) \to \mathcal{I} \to T(\mathcal{F}/T(\mathcal{F})) \to 0 \). It follows from 2.2.6.9 and the definition of the torsion subsheaf that \( \mathcal{I} \subset T(\mathcal{F}) \). Thus, \( T(\mathcal{F}/T(\mathcal{F})) = 0 \).

Lemma 2.2.6.17. If \( X \to \mathcal{X} \) is a smooth cover, then \( \mathcal{F} \) is pure if and only if \( \mathcal{F}|_{X} \) is pure.

Proof. As in 2.2.6.6, it suffices to show that if \( Z \to W \) is a smooth map of schemes then the pullback of a torsion free sheaf is torsion free. As this is a local property on the source and target and is obviously true for arbitrary quasi-finite flat morphisms (hence for étale morphisms), we see that it suffices to prove that the pullback of a torsion free sheaf on \( W \) to \( \mathbf{A}_{W}^{n} \) is torsion free.
Again by 2.2.6.6, we see that any torsion subsheaf on $A^d_W$ must have all associated points lying over minimal (generic) points of $W$. Thus, we may assume $W$ is the spectrum of an Artinian local ring $R$ and we wish to show that the pullback of any finite $R$-module to $A^d_R$ cannot have torsion. Taking a composition series, we may assume that $R$ is a field. The result follows from the fact that $A^d_K$ is Cohen-Macaulay.

Let $\pi : X' \to X$ be a flat surjection of Noetherian algebraic stacks representable by an open immersion into an integral ring extension and $\mathcal{F}$ a coherent sheaf on $X$.

**Proposition 2.2.6.18.** $\mathcal{F}$ is pure if and only if $\mathcal{F}|_{X'}$ is pure.

**Proof.** By the going-up lemma [49, 9.4] and flatness (which implies the going-down lemma [49, 9.5]), a morphism such as $\pi$ has the property that $\pi(p)$ is minimal if and only if $p$ is minimal. The result now follows from 2.2.6.6.

**Corollary 2.2.6.19.** If $X$ is over a field $k$, then the purity of a coherent sheaf is invariant under finite extensions of $k$. If $X$ is finite type over $k$ then purity is geometric.

The finite type hypothesis in the second statement serves only to ensure that $X \otimes K$ is Noetherian for any extension $K \supset k$, so that our theory applies.

**Remark 2.2.6.20.** When $X$ is an integral universally catenary scheme of finite Krull dimension (for example, a projective variety) and $\mathcal{F}$ is an $X$-twisted sheaf with support of dimension $d$, we can filter $T(\mathcal{F})$ by the dimension of support: let $T_e(\mathcal{F})$ be the maximal subsheaf of $\mathcal{F}$ whose support is of dimension at most $e$. Then $T(\mathcal{F}) = T_{d-1}(\mathcal{F}) \supset T_{d-2}(\mathcal{F}) \supset \cdots \supset T_0(\mathcal{F})$. This filtration can be useful when considering various notions of semistability, as in §1.6 of [30]; it will not come up in the sequel.

**2.2.6.21.** We will now show that the property of being pure is open in flat families of coherent sheaves on a proper algebraic stack.

**Proposition 2.2.6.22.** Let $\pi : X \to S$ be a proper morphism of finite presentation from an algebraic stack to an algebraic space. Suppose $\mathcal{F}$ is an $S$-flat family of coherent sheaves. The locus of points $s \in S$ such that $\mathcal{F}_s$ is pure is open.

**Proof.** We may reduce to the case where $S$ is affine, Noetherian, and even excellent (in fact, affine of finite type over $\mathbb{Z}$). Indeed, we may present the stack $X$ as a groupoid $X_1 \to X_0 \times X_0$ of finite presentation between two schemes of finite presentation over $S$. Thus, we may descend $X$ to a Noetherian base (using the results of §8 of [25]). Having done this, note that a coherent sheaf on $X$ is given by a coherent sheaf on $X_0$ with an action of the groupoid, i.e., an isomorphism of the pullbacks to $X_1$ which is compatible with the groupoid structure. By Grothendieck’s theory of limits, we can descend these data to a finite level.

Consider the set $\Xi$ of points $x \in |X|$ with the property “$x$ is contained in the support of $T(\mathcal{F}_{\pi(x)})$.” It suffices to show that $\pi(\Xi)$ is constructible and that $\pi(\Xi)$ is closed under specialization when $\mathcal{F}$ is proper over $S$ (and $S$ is Noetherian).

The second statement is immediate: it suffices to check this when $S$ is the spectrum of a discrete valuation ring. By flatness, the minimal points of $\text{Ass}(\mathcal{F})$ all lie in the generic fiber for any coherent subsheaf $\mathcal{F} \subset \mathcal{F}$. Thus, the torsion subsheaf $T(\mathcal{F})$ of the total family $\mathcal{F}$ is non-zero if and only if the torsion subsheaf of the generic fiber is non-zero. On the other hand, since $S$ is Dedekind, $T(\mathcal{F})$ is also a flat family, and in particular has constant fiber dimension, like $\mathcal{F}$. Finally, the cokernel $\mathcal{F}/T(\mathcal{F})$ is pure, hence $S$-flat, by 2.2.6.16. These facts combine to yield the statement about specialization. (Properness is not necessary for this, as long as we assume that the specialization on the base is contained in the image of $\pi$.)

The first statement (that $\pi(\Xi)$ is constructible) is more subtle. It is easily reduced to showing that if $S$ is an integral and Noetherian affine scheme and the generic fiber of $\Xi$ is
non-empty, then $\Xi$ is non-empty over an open subscheme of $S$. By 2.2.6.6, we may assume that $\mathcal{X}$ is in fact a scheme and that $\pi$ is surjective. The argument for schemes is classical, and is left to the reader. (It is also written out in full detail as Proposition 4.1.2.21 of [45].) □

As a consequence of the Proposition, when $\mathcal{X}$ is a $\mathbb{G}_m$-gerbe, there is an open substack of $\mathcal{T}_{\mathcal{X}/S}$ (see 2.1.2.4) representing families of pure twisted sheaves. Note that since the support of a flat family over a dvr is itself flat over the dvr, the dimension of the fibers of a flat family of coherent $\mathcal{X}$-twisted sheaves over a locally Noetherian base scheme is locally constant.

Corollary 2.2.6.23. Let $X \to S$ be a proper flat morphism of finite presentation with geometrically integral fibers. There is an open substack of $\mathcal{T}_{\mathcal{X}/S}$ consisting of families of torsion free sheaves, i.e., pure sheaves of maximal dimension.

Definition 2.2.6.24. If $X \to S$ is a proper flat morphism of finite presentation and $\mathcal{X} \to X$ is a $\mu_n$-gerbe, the open substack parametrizing families with torsion free fibers is denoted $\text{Tw}_{\mathcal{X}/S}$.

2.2.6.25. Suppose $X$ is a smooth projective variety over a field $k$ and $\mathcal{X} \to X$ is a $\mu_n$-gerbe with $n \in k^{\times}$. Let $\text{Tw}_{\mathcal{X}/k}(n)$ denote the stack parametrizing torsion free twisted sheaves of rank $n$. Since $\mathcal{X}$ is smooth, any $S$-flat family of twisted sheaves $\mathcal{F}$ on $\mathcal{X} \times S$ has finite homological dimension everywhere. In other words, $\mathcal{F}$ is perfect as an object of the derived category. Using the constructions of section 2.2.4, we can thus define the determinant of $\mathcal{F}$, which will be the pullback to $\mathcal{X} \times S$ of an invertible sheaf on $X \times S$ (as $\mathcal{F}$ has rank $n$). This yields a morphism of algebraic stacks

$$\det : \text{Tw}_{\mathcal{X}/k}(n) \to \text{Pic}_{X/k}.$$ Given an invertible sheaf $\mathcal{L}$ on $X$, one can form the fiber of $\det$ over the resulting $k$-point of $\text{Pic}_{X/k}$.

Definition 2.2.6.26. $\text{Tw}_{\mathcal{X}/k}(n, \mathcal{L}) := \text{Tw}_{\mathcal{X}/k} \times \text{Pic}_{X/k} k$

Chasing through the definition of the natural 1-fiber product of stacks shows that the objects of $\text{Tw}_{\mathcal{X}/k}(n, \mathcal{L})$ are pairs $(\mathcal{F}, \varphi)$ consisting of a torsion free twisted sheaf $\mathcal{F}$ of rank $n$ and a chosen isomorphism $\det \mathcal{F} \sim \mathcal{L}$. The deformation theory for $\text{Tw}(n, \mathcal{L})$ was developed above in section 2.2.4; as we show in [43], it is this deformation theory which governs the stack of Azumaya algebras (PGL$_n$-bundles) and its compactification by “generalized Azumaya algebras.”

2.2.7. Riemann-Roch, Hilbert polynomials, and Quot spaces. In this section, we develop a notion of Hilbert polynomial for twisted sheaves which we will ultimately use to define semistability. In later sections, we will show that on a surface (and more generally on a variety which carries a twisted sheaf with sufficiently many vanishing Chern classes), our notion agrees with Simpson’s notion [55, §3] and thus yields a GIT quotient corepresenting the stack of semistable twisted sheaves. For higher dimensional ambient varieties, it will still be possible to show that the stack of stable twisted sheaves is a gerbe over an algebraic space, but dealing with properly semistable points is difficult in the absence of a GIT description of the moduli problem.

2.2.7.1. In order to define our semistability condition, and for future reference, we briefly recall the basic facts about rational Chow rings of DM stacks over a field. Vistoli [59] and Gillet [22] have defined Chow theories which only work rationally but which are formally identical to the usual Chow theory: in Vistoli’s approach, one takes the Chow groups to be generated by integral closed substacks modulo rational equivalence (suitably defined). There is a refined theory due to Edidin and Graham [17] which applies to quotient stacks to yield an integral Chow theory which agrees with Vistoli’s theory when tensored with $\mathbb{Q}$. A further refinement of
the integral theory for algebraic stacks stratified by quotient stacks was developed by Kresch in his thesis [36]. We will denote the rational Chow groups (which are the same in all of these theories) by \( A_{\mathbb{Q}} \), and we will write \( A_{\mathbb{Q}}^n \) for the group generated by cycles of codimension \( n \). When the underlying stack is smooth, the graded group \( \oplus A_{\mathbb{Q}}^n \) has a commutative ring structure. As usual, there is a theory of Chern classes and a splitting principle. The theory admits proper pushforwards, flat pullbacks, and Gysin maps \([59]\). It is useful to note that since the splitting principle uses only the construction of projective bundles, one need never leave the category of smooth tame DM stacks with (quasi-)projective coarse moduli spaces, if one so desires.

Given a proper DM stack \( \mathcal{X} \) with moduli space \( X \), one can show that the proper pushforward \( A_{\mathbb{Q}}(\mathcal{X}) \to A_{\mathbb{Q}}(X) \) is an isomorphism which respects the ring structure when both are smooth (see [59]). In particular, when \( \mathcal{X} \) is of dimension \( n \), there is a rational degree function \( \deg : A^n(\mathcal{X})_{\mathbb{Q}} \to \mathbb{Q} \). Given any element \( \alpha \) of the graded group \( A_{\mathbb{Q}}^n(\mathcal{X}) \), we will let \( \alpha_n \) denote the part in degree \( n \). Given a class \( \beta \in A^*(\mathcal{X})_{\mathbb{Q}} \), we will let \( \deg \beta \) denote the degree of \( \beta_n \).

Let \( \mathcal{X} \) be a smooth proper DM stack of dimension \( n \) over a field \( k \) with projective moduli space \( X \). Recall that \( K^0(\mathcal{X}) \) is the Grothendieck group of vector bundles on \( \mathcal{X} \), while \( K_0(\mathcal{X}) \) is the Grothendieck group of coherent sheaves. When every coherent sheaf on \( \mathcal{X} \) admits a finite resolution by locally free sheaves, it is easy to see that \( K^0 \cong K_0 \). In general, \( K^0 \) is a ring and \( K_0 \) is a \( K^0 \)-module (via tensor product). One of the basic problems for arbitrary DM stacks is the fact that \( K^0 \) and \( K_0 \) are not isomorphic even on smooth DM stacks. For smooth quotient stacks, they are naturally the same, which makes it easier to prove theorems. (This is yet another place where 2.2.2.2 and its corollaries have a large impact.)

Let \( \alpha \in K^0(\mathcal{X}) \). We will write \( T_{\mathcal{X}/k} \) for the Todd class of the tangent sheaf \( T_{\mathcal{X}/k} \) of \( \mathcal{X} \).

**Definition 2.2.7.2.** The geometric Euler characteristic of \( \alpha \) is

\[
\chi^g(\alpha) := [\mathcal{I}(\mathcal{X}) : \mathcal{X}] \deg(\text{ch}(\alpha) \cdot T_{\mathcal{X}/k}).
\]

When \( X \) is projective with chosen polarization \( \mathcal{O}(1) \), the geometric Hilbert polynomial of \( \alpha \) is the function

\[
n \mapsto P^g_\alpha(n) = \chi^g(\alpha(n)),
\]

where \( \alpha(n) := \alpha \otimes \mathcal{O}(n) \).

To verify that \( P^g_\alpha \) is a polynomial, it suffices to prove it when \( \alpha = [\mathcal{E}] \), \( \mathcal{E} \) a locally free sheaf on \( \mathcal{X} \). This then follows by a simple splitting principle calculation left to the reader.

**Remark 2.2.7.3.** The geometric Euler characteristic and Hilbert function are clearly additive functions on the category of (perfect) coherent sheaves. When \( \mathcal{F} \) is the pullback to \( \mathcal{X} \) of a coherent sheaf on \( X \), they agree with the usual Euler characteristic and Hilbert function by the Grothendieck-Hirzebruch-Riemann-Roch theorem. However, for sheaves which are not pullbacks, they do not agree with the usual cohomologically defined functions. For a trivial example, consider the case of a gerbe \( \mathcal{X} \) over an algebraic curve \( X \). In this case, there is an invertible sheaf \( \mathcal{L} \) on \( \mathcal{X} \) which is \( \mathcal{X} \)-twisted whose \( n \)th tensor power \( \mathcal{L}^\otimes n \) is the pullback of an invertible sheaf \( \mathcal{M} \) on \( X \). The geometric Euler characteristic of \( \mathcal{L} \) is easily seen to be

\[
\chi^g(\mathcal{L}) = \deg c_1(\mathcal{L}) + \chi(\mathcal{O}_X) = \frac{1}{n} \deg c_1(\mathcal{M}) + \chi(\mathcal{O}_X).
\]

Thus, one can easily produce gerbes \( \mathcal{X} \) and \( \mathcal{X} \)-twisted sheaves with non-zero \( \chi^g \). On the other hand, if we use coherat cohomology to compute the cohomological Euler characteristic, we find \( \chi(\mathcal{L}) = 0 \) when \( \mathcal{L} \) has non-trivial stabilizer action. (For an even more trivial example, let \( \mathcal{X} \) be a gerbe over a point!) There are ways of rectifying this difference, due to Toën [57], by instead working with the Chow theory of the inertia stack paying more careful attention to the representations of inertia on fibers of vector bundles. It is interesting to note that in many
cases the “correct” Riemann-Roch formula yields 0, whereas the seemingly blunt instrument wielded here produces non-zero answers, thus somehow capturing geometric information about sheaves which is not cohomological and which is not visible on non-stacky varieties.

Recent results of Vistoli-Kresch [37], Edidin-Hassett-Kresch-Vistoli [18], and Gabber/de Jong [14] (stated by Gabber and proven by Gabber and independently by de Jong) show that any separated smooth generically tame DM stack over a field with quasi-projective moduli space is a quotient stack, and that such a stack has the “resolution property”: any coherent sheaf is a quotient of a locally free sheaf. In these cases, the natural map $K^0 \to K_0$ is thus an isomorphism. We will denote it simply by $K(X)$.

**Proposition 2.2.7.4.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a projective l.c.i. morphism of DM stacks of finite type over a field such that $\mathcal{Y}$ admits a finite flat cover $\pi : Y \to \mathcal{Y}$. Given any class $\alpha \in K^0(\mathcal{X})$,

there is a natural equality

$$\text{ch}(f_* \alpha) = f_* (\text{ch}(\alpha) \cdot Td_f)$$
in $A(\mathcal{Y})_\mathbb{Q}$.

**Proof.** Form the Cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & \mathcal{X} \\
\downarrow{\tilde{f}} & & \downarrow{f} \\
Y & \xrightarrow{\pi} & \mathcal{Y}
\end{array}
$$

The morphism $\tilde{f}$ is a projective l.c.i. morphism of schemes; hence the formula holds for $\tilde{f}$. Furthermore, we have that $\pi_* \pi^*$ and $\tilde{\pi}_* \tilde{\pi}^*$ are both identified with multiplication by $\deg \pi$ (which follows immediately from the definitions of the pushforward and pullback functors just as in the classical situation); thus, to show an equality in rational Chow groups, it suffices to show equality after pulling back by $\pi$. Furthermore, the formation of $\text{ch}(\alpha)$ commutes with flat pullback and $\tilde{\pi}^* Td_f = Td_{\tilde{f}}$. Finally, we know that flat pullback commutes with proper pushforward (3.9 of [59]). Combining these statements yields

$$\pi^* f_* (\text{ch}(\alpha) \cdot Td_f) = \tilde{f}_* (\text{ch}(\tilde{\pi}^* \alpha) \cdot Td_{\tilde{f}}) = \text{ch}(\tilde{f}_* \tilde{\pi}^* \alpha) = \pi^* \text{ch}(f_* \alpha).$$

\[\square\]

**Corollary 2.2.7.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a projective morphism of smooth pseudo-projective DM stacks. Then for all $\alpha \in K(\mathcal{X})$,

$$\text{ch}(f_* \alpha) \cdot Td_\mathcal{Y} = f_* (\text{ch}(\alpha) \cdot Td_\mathcal{X})$$
in $A(\mathcal{Y})_\mathbb{Q}$.

**Proof.** Any such morphism must be l.c.i., so we can apply 2.2.7.4, once we note that any smooth generically tame DM stack over a field admits a finite flat cover by a smooth scheme (apply the main result of [14] to Theorem 2.2 of [37] and use this as input into Theorem 2.1 of [37]). \[\square\]

**Corollary 2.2.7.6.** Let $\iota : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a closed immersion of smooth pseudo-projective DM stacks and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$. Then $\chi^g(\mathcal{X}, \mathcal{F}) = \chi^g(\mathcal{Y}, \iota_* \mathcal{F})$.

**Remark 2.2.7.7.** When $\mathcal{X}$ is a $\mu_n$-gerbe over a smooth projective variety $X$ and there is a locally free $\mathcal{X}$-twisted sheaf with sufficiently many vanishing Chern classes (e.g., $X$ is a surface), then the formula in the last sentence of the proof of 2.3.2.8 below gives a much more concrete proof of 2.2.7.6.
We fix a smooth pseudo-projective DM stack \( \mathcal{X} \) with moduli space \( \pi : \mathcal{X} \to X \) in what follows. For the moment, we assume that the base is a field; we will see in a moment that the geometric Hilbert function is constant in flat families and invariant under extension of base field. We also assume that the moduli space \( X \) is quasi-projective over \( k \), with fixed very ample \( O(1) \). We prove our results under the following hypothesis.

**Hypothesis 2.2.7.8.** For any sufficiently large integer \( n > 0 \), a general section of \( O(n) \) has smooth vanishing locus on \( \mathcal{X} \).

This is clearly satisfied when \( \mathcal{X} \to X \) is a \( \mu_n \)-gerbe (or, more generally, a \( G \)-gerbe with \( G \) a smooth group scheme). In fact, 2.2.7.8 holds whenever \( \mathcal{X} \) is smooth tame DM stack (a stacky Bertini theorem), a fact which will appear in a paper currently in preparation [47].

**Lemma 2.2.7.9.** The geometric Hilbert function is geometric: if \( k \subset K \) is an extension of fields and \( X \) is a smooth geometrically connected projective variety over \( k \), then for any coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \), \( P^g_\mathcal{F} = P^g_\mathcal{F} \otimes_K \) as functions on \( \mathbb{Z} \).

**Proof.** This follows from the fact that Chern classes of arbitrary (perfect) coherent sheaves pull back under Tor-independent maps. \( \square \)

**Notation 2.2.7.10.** Following the conventions of Huybrechts and Lehn [30, §1.2], we write \( P^g_\mathcal{F}(m) = \dim \mathcal{F} \sum_{i=0}^{\text{dim} \mathcal{F}} \alpha_i(\mathcal{F}) \frac{m^i}{i!} \).

With this definition the coefficients \( \alpha_i \) need not be integers (contrary to [30, p. 10]).

**Definition 2.2.7.11.** Given a coherent sheaf \( \mathcal{F} \) of dimension \( d \) on \( \mathcal{X} \), the geometric rank of \( \mathcal{F} \) is defined to be \( \text{rk} \mathcal{F} := \frac{\alpha_d(\mathcal{F})}{\alpha_d(O_\mathcal{X})} \).

The geometric degree of \( \mathcal{F} \) is defined to be \( \deg \mathcal{F} = \alpha_{d-1}(\mathcal{F}) - \text{rk}(\mathcal{F}) \cdot \alpha_{d-1}(O_\mathcal{X}) \).

**Lemma 2.2.7.12.** Suppose \( k \) is infinite. Given a coherent sheaf \( \mathcal{F} \) on \( \mathcal{X} \), for any integer \( n > 0 \) there is a global section \( \sigma \) of \( O(n) \) such that \( \sigma : \mathcal{F}(-n) \to \mathcal{F} \) is injective.

**Proof.** The set \( \text{Ass} \mathcal{F} \) is finite and determined by its image in \( X \). Since \( O(n) \) is very ample, there is a section missing these finitely many points. It is easy to see that any associated point of the kernel of \( \sigma \) must then be contained in the zero locus of \( \sigma \), contradicting the choice of \( \sigma \) and 2.2.6.9. \( \square \)

**Lemma 2.2.7.13.** For any coherent twisted sheaf, \( \deg P^g_\mathcal{F} = \dim \mathcal{F} \). In particular, if \( \mathcal{F} \) is torsion free then geometric rank of \( \mathcal{F} \) is non-zero.

**Proof.** Using 2.2.7.8, this is clear by induction and the previous lemma, once we have verified it when \( \dim X = 0 \). In this case, the geometric Euler characteristic is just the dimension of the fiber of \( \mathcal{F} \) over any geometric point of \( \mathcal{X} \) divided by the degree of the inertia stack \( \mathcal{I}(\mathcal{X}) \to \mathcal{X} \). Indeed, the pushforward map \( \mathcal{X} \to \text{Spec}(k) \) sends 1 to \( 1/\deg(\mathcal{I}(\mathcal{X})/\mathcal{X}) \).

(The denominator is just the cardinality of the stabilizer of a geometric point of \( \mathcal{X} \).) \( \square \)

**Remark 2.2.7.14.** In particular, the geometric Hilbert function of \( \mathcal{F} \) vanishes if and only if \( \mathcal{F} = 0 \). Furthermore, one sees that the geometric rank of \( \mathcal{F} \) is precisely the rank of \( \mathcal{F} \) as an \( O \)-module. Unfortunately, one cannot show this by arguing that \( \mathcal{F} \) and \( \mathcal{O}^\text{rk} \mathcal{F} \) agree on a dense open substack, as this is false. Instead, one must appeal directly to the Hirzebruch-Riemann-Roch formula (and the computation [20, 3.2.2] of Chern classes of a twist). We leave the details to the reader. The geometric degree of \( \mathcal{F} \) is related to the degree of \( \text{det}(\mathcal{F}) \) just
as in the case of ordinary sheaves: \( \alpha_{d-1} = \deg \det(\mathcal{F}) \) (so the geometric degree is arrived at by a linear transformation familiar from \([30, 1.6.8ff]\)). This will aid us in comparing various notions of semistability and slope-semistability to their classical counterparts (as in Simpson’s theory for semistability of modules for sheaves of algebras \([55, \S 3]\)).

2.2.7.15. For the rest of this section, we will consider only the case where \( \mathcal{X} \to X \) is a \( \mu_n \)-gerbe with \( n \in \mathcal{O}(X)^\times \) and \( X \) is a smooth projective scheme over a Noetherian affine base \( S \). Generalizations of these results and those of the following section to the case of a smooth tame DM stack will be considered in an upcoming paper \([47]\).

We start with a refinement of 2.2.2.2 better suited to the eventual study of stability.

**Proposition 2.2.7.16.** Given a \( \mu_n \)-gerbe on a smooth projective morphism \( \mathcal{X} \to X \to S \) with Noetherian affine base \( S \), there is a locally free \( \mathcal{X} \)-twisted sheaf \( \mathcal{V} \) of constant non-zero rank and trivial determinant.

**Proof.** The existence of \( \mathcal{V} \) is a non-trivial result which holds on any (separated) scheme with an ample invertible sheaf. We refer the reader to the work of de Jong \([14]\) for the (upcoming) details. To make the determinant trivial, first consider an ample invertible sheaf. We refer the reader to the work of d e Jong \([14]\) for the (upcoming) details. To make the determinant trivial, first consider an ample invertible sheaf. We refer the reader to the work of d e Jong \([14]\) for the (upcoming) details.

Recall that \( \mathcal{X} \) has the resolution property if every coherent sheaf on \( \mathcal{X} \) is the quotient of a locally free sheaf \([58]\). The present virtue of 2.2.7.16 lies in the following corollary.

**Corollary 2.2.7.17.** For any affine \( T \to S \), the stack \( \mathcal{X}_T \) has the resolution property.

**Proof.** By 2.2.1.6, the category of coherent sheaves on \( \mathcal{X} \) breaks up according to the degree of twisting. It suffices to show that coherent \( \mathcal{X} \)-twisted sheaves have the resolution property. Applying the fibered Morita equivalence \( \mathcal{H}om(\mathcal{V}, \cdot) \) reduces us to showing that \( \mathcal{H}om(\mathcal{V}) \)-modules on \( X \) have the resolution property. This follows from the fact that coherent sheaves on a projective morphism have the resolution property.

**Proposition 2.2.7.18.** If \( \mathcal{F} \) on \( \mathcal{X} \) is \( S \)-flat, then \( P^g_s \) is constant for all geometric points \( s \to S \).

**Proof.** Let \( \mathcal{G}^\bullet \to \mathcal{F} \) be a locally free resolution of \( \mathcal{F} \). As \( \mathcal{X} \to S \) is smooth, \( \mathcal{G}^\bullet \) may be taken to be a finite resolution. If \( \mathcal{F} \) is flat, then for any \( s \to S \), the complex \( \mathcal{G}_s^\bullet \) is a resolution of \( \mathcal{F}_s \). Thus, to prove that \( P^g \) is constant for \( \mathcal{F} \), it suffices by additivity to prove it when \( \mathcal{F} \) is assumed locally free. In this case, we may globally apply the splitting principle (noting that the base change which filters the sheaf produces another proper smooth \( S \)-flat family). Thus, it is enough to show that given invertible sheaves \( L_1, \ldots, L_d \) on \( \mathcal{X} \) (with \( d = \dim \mathcal{X}/S \)), the intersection product \( c_1(L_1) \cdot \cdots \cdot c_1(L_d) \) is constant in fibers. As \( A(\mathcal{X})_Q = A(X)_Q \), it suffices (by raising each \( L_i \) to the \( n \)th tensor power and using multi-linearity) to prove this for invertible sheaves on \( X \). This is now a standard calculation using the fact that Euler characteristics are constant in a flat family. (In other words, we return to Kleiman’s definition of intersection product using Snapper’s lemma \([11, \S 1], [34]\), where the intersection number appears as a coefficient in a polynomial Euler characteristic.)

Thus, given \( P \), the substack \( \mathbf{Tw}_{\mathcal{X}/S}(P) \subset \mathbf{Tw}_{\mathcal{X}/S} \) consisting of twisted sheaves with fixed geometric Hilbert polynomial \( P \) is open (in fact, a union of connected components). Since we will verify shortly that \( \mathbf{Tw}_{\mathcal{X}/S} \) is an algebraic stack, it will immediately follow that \( \mathbf{Tw}_{\mathcal{X}/S}(P) \) is an algebraic stack.

2.2.7.19. Let \( P \) be a fixed polynomial and \( \mathcal{E} \) a fixed coherent \( \mathcal{X} \)-twisted sheaf. We will briefly study the space of quotients of \( \mathcal{E} \) with a fixed geometric Hilbert polynomial.
**Definition 2.2.7.20.** Let $\text{Quot}_{X/S}^P(\mathcal{E})$ denote the functor on affine $S$-schemes which assigns to $T \to S$ the set of subsheaves $\mathcal{G} \subset \mathcal{E}_T$ such that $\mathcal{E}_T/\mathcal{G}$ is $T$-flat with geometric Hilbert polynomial $P$ in every fiber over $S$.

**Proposition 2.2.7.21.** The functor $\text{Quot}_{X/S}^P(\mathcal{E})$ is represented by locally projective scheme $\text{Quot}_{X/S}^P(\mathcal{E})$ over $S$.

**Proof.** It follows by an easy application of Artin’s representability theorem that $\text{Quot}$ is representable by an algebraic space which satisfies the valuative criterion of properness. This is checked in great detail in [52] in a slightly different context (which is sufficiently close to ours to be a complete proof in our case as well). The only fact that remains to prove is that the functor $\text{Quot}$ is bounded in the sense of [30, 1.7.5]. In other words, we need to show that there is a quasi-compact scheme surjecting onto the functor.

Let $V$ be a locally free $\mathcal{X}$-twisted sheaf. Given any $\mathcal{G} \subset \mathcal{E}_T$ as above, note that the inclusion $\mathcal{H}\text{om}(V_T, \mathcal{G}) \subset \mathcal{H}\text{om}(V_T, \mathcal{E}_T)$ has $T$-flat cokernel $\mathcal{C}$. If we knew that the (classical) Hilbert polynomial of the fibers of $\mathcal{C}$ were always the same (as $\mathcal{G}$ varies), we would be done. Unfortunately, this is highly unlikely. However, since we do know the geometric rank and geometric degree of $(\mathcal{E}_T/\mathcal{G})_s$, we know the rank and degree of $\mathcal{C}_s$. Furthermore, we know that the $\mathcal{C}_s$ are quotients of a fixed sheaf $\mathcal{U} := \mathcal{H}\text{om}(V, \mathcal{E})$. By a result of Grothendieck [30, 1.7.9], we know that the set of quotients of $\mathcal{U}_s$ with slope bounded above is bounded. A consideration of the proof of Huybrechts and Lehn [ibid.] shows that the set of Hilbert polynomials appearing in such quotients is finite and independent of $s$. Thus, as the geometric Hilbert polynomial is locally constant, we see that $\text{Quot}_{X/S}^P(\mathcal{E})$ is a union of finitely many connected components of finitely many schemes of quotients of $\mathcal{E}\text{nd}(V)$-modules which are themselves projective over $S$. This completes the proof. □

**Remark 2.2.7.22.** The proof actually works (with slight modification) for any coherent $\mathcal{E}$ on $\mathcal{X}$ (independent of twisting).

We end this section with a lemma which will be useful in section 3.2.4 and which shows some of the similarities between classical sheaves and twisted sheaves on surfaces. Let $X$ be a smooth projective surface. First, a provisional definition.

**Definition 2.2.7.23.** Given a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ of dimension 0, the length of $\mathcal{F}$ is

$$\ell(\mathcal{F}) = \chi^q(\mathcal{F}).$$

It readily follows from the definition of $\chi^q$ that if $\mathcal{F}$ is supported on a residual gerbe with rank $r$, then the length of $\mathcal{F}$ is $r$. (The salient observation is that the pushforward $A^2(\mathcal{X}) \to A^2(X)$ is naturally identified with multiplication by $1/n$. On the other hand, if one desires that $\ell_Y(\mathcal{F}_Y) = \deg(Y/\mathcal{X})\ell(\mathcal{F})$ for a finite flat map $Y \to \mathcal{X}$, then one should omit the factor of $[\mathcal{X}(\mathcal{X} : \mathcal{X}]$ in the formula. For the purposes of this paper, this is immaterial, so we have normalized everything to yield integer-valued functions. In future work [47] this distinction will be important, and we will correspondingly alter the definition.)

Given an integer $\ell > 0$, we will also write $\text{Quot}(\mathcal{E}, \ell)$ for $\text{Quot}_{X/S}^\ell(\mathcal{E})$ (to be consistent with existing notations in the untwisted category).

**Lemma 2.2.7.24.** Suppose $X$ is a smooth surface. If $\mathcal{E}$ is a locally free $\mathcal{X}$-twisted sheaf of rank $r$, then $\text{Quot}(\mathcal{E}, \ell)$ is irreducible of dimension $\ell(r + 1)$.

**Proof.** We use the highly non-trivial fact that this is true when $\mathcal{E}$ is trivial [30, 6.A.1]. First, note that there is an open subspace of the Quot corresponding to length $\ell$ quotients which are just $\ell$ distinct “twisted lines” (invertible sheaves supported on residual gerbes). This open subspace is isomorphic to an étale $(\mathbb{P}^{r-1})\ell$-bundle over $\text{Sym}^\ell(X) \setminus \Delta$, where $\Delta$ is the
multidiagonal, hence is irreducible (and has the right dimension). It is thus enough to show that the entire Quot is the closure of this open, which is the same as showing that any quotient may be deformed into a quotient with reduced support. Let $\mathcal{E} \to \mathcal{Z}$ be any quotient of length $\ell$. Write the support (with its natural scheme structure) of $\mathcal{Z}$ as $Z$ (which will be the preimage of a closed subscheme of $X$). The quotient map is the same as a quotient $\mathcal{E}_{Z} \to \mathcal{Z}$. Since $Z$ is a scheme of finite length over an algebraically closed field, we have $\text{Br}(Z) = 0$. Let $\mathcal{L}$ be a twisted invertible sheaf on $Z$; any two invertible twisted sheaves are in fact mutually isomorphic. Twisting down by $\mathcal{L}$, we see that $\mathcal{E}_{Z} \to \mathcal{Z}$ is the same thing as a surjection $\mathcal{O}_{Z}^r \to \mathcal{Q}$. (In other words, $\mathcal{E} \otimes \mathcal{L}^\vee \cong \mathcal{O}^r$.) By the irreducibility of Quot($\mathcal{O}^r, \ell$), we know that there is a complete discrete valuation ring $R$ containing $k$ and a flat family of quotients $\mathcal{O}_{X}^r \otimes_R \mathcal{Q}$ on $X \otimes_R$ whose special fiber is $\mathcal{O}^r \to \mathcal{Q}$. The support $S$ of $\mathcal{Q}$ will be finite over $R$, and hence will be strictly Henselian. Thus, $\text{Br}(S) = 0$, and we may choose an invertible twisted sheaf $\mathcal{L}$ on $S$ (for the pullback of $\mathcal{L}$ to $X \otimes R$). Since $S$ is semilocal, it follows that $\mathcal{L}^r \cong (\mathcal{E} \otimes R)_S$. Thus, twisting the quotient $\mathcal{Q}$ by $\mathcal{L}$, we find an effective deformation of $\mathcal{Z}$ into a quotient with reduced support. 

2.3. Algebraic moduli. In this section, we will show that the stack of twisted sheaves is algebraic (in the sense of Artin). In the process, we will develop a theory of semistable twisted sheaves and study the relation to Geometric Invariant Theory.

We prove that the stack of twisted sheaves on a proper morphism $X \to S$ of finite presentation over an excellent (quasi-separated) base is algebraic. This sets the stage for a study of stability of twisted sheaves (in its Mumford-Takemoto and Gieseker forms) when $X \to S$ is projective and its use in producing GIT quotient stacks and corepresenting projective schemes for stacks of semistable twisted sheaves in 2.2.7 and 2.3.2. The work on Gieseker stability will require the definition of a suitable Hilbert polynomial. We define and study this polynomial for stacks of semistable twisted sheaves (in its Mumford-Takemoto and Gieseker forms) when $\text{char}(S) = 0$ and a flat family of quotients $\mathcal{O}_{X}^r \otimes_R \mathcal{Q}$ on $X \otimes_R$ whose special fiber is $\mathcal{O}^r \to \mathcal{Q}$. The support $S$ of $\mathcal{Q}$ will be finite over $R$, and hence will be strictly Henselian. Thus, $\text{Br}(S) = 0$, and we may choose an invertible twisted sheaf $\mathcal{L}$ on $S$ (for the pullback of $\mathcal{L}$ to $X \otimes R$). Since $S$ is semilocal, it follows that $\mathcal{L}^r \cong (\mathcal{E} \otimes R)_S$. Thus, twisting the quotient $\mathcal{Q}$ by $\mathcal{L}$, we find an effective deformation of $\mathcal{Z}$ into a quotient with reduced support. 

2.3.1. Abstract existence. Let $X \to S$ be an algebraic space which is proper of finite presentation over a locally Noetherian scheme, and let $\mathcal{X} \to X$ be a fixed $\mu_n$-gerbe, where $n$ is prime to $\text{char}(X)$. Consider the $S$-groupoid $\mathcal{T}_{\mathcal{X}/S}$ which assigns to an affine scheme $\text{Spec} R \to S$ over $S$ the category whose objects are $R$-flat families of coherent $\mathcal{X}$-twisted sheaves. (We reserve the notation $\text{Tw}$ for twisted sheaves without embedded points; 2.2.6 above shows that
$\text{Tw} \subset \mathcal{T}$ is an open substack.) Our goal in this section is to apply Artin’s Theorem [9] to prove the following.

**Proposition 2.3.1.1.** $\mathcal{F}/S$ is an algebraic stack locally of finite presentation over $S$.

**Lemma 2.3.1.2.** The result of 2.3.1.1 is true if and only if it is true when $S$ is excellent and Noetherian.

*Proof.* Since 1) $X$ is of finite presentation, 2) being algebraic is local on $S$ and stable under base change, 3) the formation of étale cohomology is compatible with affine limits [19, I.4], and 4) the formation of the stack $\mathcal{T}$ is compatible with base change, we may replace $S$ with a finite type $\mathbb{Z}$-algebra.

Most of the components necessary to apply Artin’s Theorem are described in the deformation theory of 2.2.3.

**Lemma 2.3.1.3.** Let $R$ be a complete local Noetherian ring, and suppose $S = \text{Spec } R$ above. Given a compatible system of twisted sheaves $\mathcal{F}_i$ on $\mathcal{X}/R/\mathfrak{m}^{i+1}$, there is a twisted sheaf $\mathcal{F}$ on $\mathcal{X}$ whose reduction modulo $\mathfrak{m}^{i+1}$ is compatibly isomorphic to $\mathcal{F}_i$.

*Proof.* This follows directly from the result of Olsson and Starr for sheaves on DM stacks (Proposition 2.1 of [52]), generalizing earlier work of Abramovich and Vistoli (appendix to [6]). (One could also use Olsson’s general version of the existence theorem for Artin stacks, proved in [51] as a consequence of Chow’s lemma for such stacks.) If $X \to S$ is a projective morphism of schemes, then by 2.2.2.2 and Morita equivalence, the category of coherent twisted sheaves is equivalent to the category of coherent modules over $\mathcal{O}_X$. But then we are reduced to the classical form of Grothendieck’s Existence Theorem for modules over a coherent algebra [24, §5].

**Lemma 2.3.1.4** (Schlessinger). Suppose $A_1 \to A_0 \leftarrow A_2$ is a diagram of commutative rings such that $A_2 \to A_0$ is a surjection with nilpotent kernel $J$. Suppose give a diagram of flat modules $M_1 \to M_0 \leftarrow M_2$ over the diagram of rings inducing isomorphisms $M_i \otimes A_0 \cong M_0$. Let $B = A_2 \times_{A_0} A_1$ and $N = M_2 \times_{M_0} M_1$. Then $N$ is a flat $B$-module and $N \otimes A_i \cong M_i$.

*Proof.* The proof of this result given in [54] only treats a special case which does not suffice for our purposes and the reference given there for the general case is not publicly available. Thus, we give a proof which works for Noetherian rings and indicate how to generalize it to arbitrary commutative rings.

To see that $N$ is $B$-flat, we use the local criterion of flatness [49, §22]. Since $A_2 \to A_0$ is surjective (say with kernel $J$), we see that $B \to A_1$ is surjective with (nilpotent) kernel $I := J \times_{A_0} 0_{A_1}$. It is easy to see that $N/IN \cong M_1$ as $A_1$-modules. To show that $N$ is flat over $B$, it remains to show that the natural map $\varphi : I \otimes_B N \to IN$ is an isomorphism. We may assume (after filtering $J$ and proceeding inductively) that $J$ is generated by a single element $t$ of square 0. (This step of the proof only works in the Noetherian case, but the usual “equational criterion” for flatness [49, 7.6] will work in the general case. We choose to analyze this case for the sake of simplicity, and because it suffices for our purposes.) The statement that $I \otimes N \to IN$ is an isomorphism is then equivalent to the statement that if $n = m_2 \times m_1$ satisfies $(t \times 0)n = 0$ then $(t \times 0) \otimes n = 0$. But if $(t \times 0)n = 0$, then $tm_2 = 0$. As $M_2$ is flat over $A_2$, we have $m_2 = tm_2$ so that $m_2 \otimes 0 \in A_0$. Thus, $m_1 \otimes 0 \in A_0$, so $m_1 = \sum k_j m_1^{(j)}$ for some $k_j \in \text{ker}(A_1 \to A_0)$, and so $m_2 \times m_1 = (t \times k_1)(m_2 \times m_1^{(1)}) + (0 \times k_2)(m_2 \times m_2^{(2)}) + \cdots$. Plugging this in, we find that $(t \times 0) \otimes n = 0$ as required.

*Proof of 2.3.1.1.* We recall Artin’s conditions: let $F$ be the stack of twisted sheaves, $\mathcal{F}$ the associated presheaf of isomorphism classes. Given a morphism of rings $B \to A$ and an element
a ∈ F(A), we will denote F_a(B) the fiber of F(B) → F(A) over a (and similarly for \(\overline{F}\)). The first conditions which must be satisfied to apply Artin's theorem are the Schlessinger-Rim criteria (our versions are slightly more general then are necessary; see [9] for Artin's list):

(S1a) given a diagram \(A' \to A \leftarrow B\) with \(A' \to A\) surjective with nilpotent kernel, and given \(a \in F(A)\), the canonical map

\[ \overline{F}_a(A' \times_A B) \to \overline{F}_a(A') \times \overline{F}_a(B) \]

is surjective.

(S1b) If \(B \to A\) is a surjection, \(b \in F(B)\) with image \(a \in F(A)\), and \(M\) is a finite \(A\)-module then the canonical map

\[ \overline{F}_b(B \oplus M) \to \overline{F}_a(A \oplus M) \]

is bijective.

(S2) Given \(a \in F(A)\), the \(A\)-module \(\overline{F}_a(A \oplus M)\) is finite. (The module structure comes about via S1b. See [9, 1, 54] for details.)

(Aut) Given \(a \in F(A)\), the module \(\text{Aut}_a(A \oplus M)\) of infinitesimal automorphisms of \(a\) is a finite \(A\)-module.

In our case, these are easy to check. (S1a) follows from 2.3.1.4 by an argument similar to [54, 3.1]. (S1b) follows from the chére Cartan isomorphism 2.2.3.3. (S2) comes from the coherence of derived pushforwards of coherent sheaves on proper morphisms. (Aut) follows from 2.2.3.9.

In addition to the "local versality" conditions, one must check effectivezation and constructibility conditions. In particular, one must check that the map \(F(\hat{A}) \to \varprojlim F(\hat{A}/m^n)\) is a 1-isomorphism of groupoids for a local Noetherian \(A\) over \(S\). This follows from 2.3.1.3 above. The constructibility conditions are the following: the deformation and obstruction theories are compatible with étale localizations and completion (2.2.3.11(1) and (2)), and there is a dense open where they are compatible with base change to fibers (2.2.3.11(3)). One requires that similar conditions hold for the group of infinitesimal automorphisms; this is also subsumed in 2.2.3.11. The last condition to check is that given a reduced finite type \(S\)-affine \(\text{Spec} A_0 \to S\) and an element \(a_0 \in F(A_0)\), any automorphism which induces the identity in the fiber at a dense set of points \(A_0 \to k\) of finite type over \(S\) is the identity morphism. This is local on \(\mathcal{X}\), so it reduces to the case where \(\mathcal{X} = X\) is affine and \(\mathcal{F}\) is an \(S\)-flat coherent sheaf on \(X\). This reduces to showing that a section \(\sigma\) of \(\mathcal{F}\) which vanishes in fibers over a dense set of points of \(\text{Spec} A_0\) is the zero section. By flatness, the locus of points in \(\text{Spec} A_0\) over which \(\sigma\) vanishes is closed under specialization. On the other hand, one easily sees that the set is constructible. (The only non-trivial point comes in checking that if \(A_0\) is integral and \(\sigma\) does not vanish on the generic fiber, then there is an open subset of \(\text{Spec} A_0\) consisting of fibers where \(\sigma\) does not vanish. There is an open subset \(U\) of \(X\) consisting of points \(x \in X\) such that \(\sigma_{\kappa(x)} \neq 0\), as \(\mathcal{F}\) is coherent. But \(X \to \text{Spec} A_0\) is of finite type and \(A_0\) is Noetherian, so the image of \(U\) contains an open subset of \(\text{Spec} A_0\) by Chevalley’s theorem.) Thus, the set of fibers where \(\sigma\) vanishes is closed, so if it contains a dense set it is all of \(\text{Spec} A_0\), as required. \(\square\)

2.3.2. Semistability and boundedness. We wish in this section to define a reasonable stability condition for twisted sheaves on smooth projective varieties. Variations on this theme occur throughout the study of moduli of sheaves. The basic goal is to produce a condition which cuts out a well-behaved substack of the stack of pure sheaves. Historically, this has meant two things: from the differential-geometric angle, stability conditions are related to the existence of certain types of metrics on bundles; from the algebro-geometric direction, the choice of a stability condition is influenced by the use of Geometric Invariant Theory to construct the moduli of such sheaves as a quotient stack.
Using the geometric Hilbert polynomial, we define a stability condition for twisted sheaves analogous to the classical definition for untwisted sheaves. As usual, a coarsening of our relation will define \( \mu \)-semistability (also known as Mumford-Takemoto semistability and slope semistability). (In characteristic zero, this condition is probably equivalent to the existence of certain metrics on the associated analytic stack [orbifold]. We do not pursue this matter here.)

On surfaces, we relate our construction to GIT via a Morita equivalence and fundamental work of Simpson on moduli of modules for a sheaf of algebras [55]. (More generally, we make this comparison when there exists a locally free twisted sheaf with sufficiently many vanishing Chern classes.)

**Definition 2.3.2.1.** An \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of dimension \( d \) is semistable (respectively stable) if for any subsheaf \( \mathcal{G} \subset \mathcal{F} \) we have \( \alpha_d(\mathcal{F})P^g_\mathcal{F} \leq \alpha_d(\mathcal{G})P^g_\mathcal{G} \) (respectively \( \alpha_d(\mathcal{F})P^g_\mathcal{F} < \alpha_d(\mathcal{G})P^g_\mathcal{G} \)).

**Lemma 2.3.2.2.** A semistable coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) is pure.

**Proof.** If \( \mathcal{G} \subset \mathcal{F} \) is a torsion subsheaf, then \( \dim \mathcal{G} < \dim \mathcal{F} \), which means that \( P^g_\mathcal{G} \leq 0 \) (as \( \alpha_d(\mathcal{F}) \neq 0 \)). Thus, \( P^g_\mathcal{G} = 0 \), and therefore \( \mathcal{G} = 0 \) by 2.2.7.14. \( \square \)

**Definition 2.3.2.3.** The reduced geometric Hilbert polynomial of a coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of dimension \( d \) is \( p^g_\mathcal{F} := (1/\alpha_d)P^g_\mathcal{F} \).

By definition, an \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) is semistable if and only if it is pure and for any subsheaf \( \mathcal{G} \subset \mathcal{F} \) we have \( p^g_\mathcal{G} \leq p^g_\mathcal{F} \).

**Definition 2.3.2.4.** The slope of a coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of dimension \( d \) is

\[
\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\text{rk} \mathcal{F}}.
\]

**Definition 2.3.2.5.** A coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of dimension \( d \) is \( \mu \)-semistable (respectively \( \mu \)-stable) if \( \mathcal{F} \) is pure and for any subsheaf \( \mathcal{G} \subset \mathcal{F} \) we have \( \mu(\mathcal{G}) \leq \mu(\mathcal{F}) \) (respectively \( \mu(\mathcal{G}) < \mu(\mathcal{F}) \)).

**Remark 2.3.2.6.** If we define the modified slope of an \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of dimension \( d \) as \( \hat{\mu}(\mathcal{F}) := \alpha_{d-1}/\alpha_d \), then we get the same notion of slope semistability as above. We will use both notions of slope interchangeably.

**Definition 2.3.2.7.** Given a sheaf of algebras \( \mathcal{A} \) on \( X \), an \( \mathcal{A} \)-module \( \mathcal{F} \) is Simpson (semi)stable if the inequality of 2.3.2.1 holds for subsheaves \( \mathcal{G} \) which are \( \mathcal{A} \)-submodules.

**Lemma 2.3.2.8.** Let \( X \) be a smooth projective variety and \( \mathcal{X} \) a \( \mu_n \)-gerbe on \( X \) with a locally free twisted sheaf \( \mathcal{V} \) of rank \( \nu \) such that \( c_i(V) = 0 \in A^i_Q(\mathcal{X}) \) for all \( 1 \leq i < n \). Let \( \mathcal{A} = \text{End}_X(\mathcal{V}) \). Then semistability of \( \mathcal{X} \)-twisted sheaves is equivalent to Simpson-semistability of \( \mathcal{A} \)-modules via the fibered Morita equivalence \( \mathcal{W} \mapsto \text{Hom}(\mathcal{V},\mathcal{W}) \).

**Proof.** By the Riemann-Roch theorem,

\[
\chi(\text{Hom}(\mathcal{V},\mathcal{W})) = \deg(\text{ch}(\mathcal{V}^\vee) \cdot \text{ch}(\mathcal{W}) \cdot \text{Td}_X)
\]

\[
= \nu \deg(\text{ch}(\mathcal{W}) \cdot \text{Td}_X) + \deg(\text{ch}(\mathcal{V}^\vee)_1 \cdot (\text{ch}(\mathcal{W}) \cdot \text{Td}_X)_{n-1}) + \cdots
\]

\[
\cdots + \text{rk}(\mathcal{W}) \deg(\text{ch}(\mathcal{V}^\vee) \cdot \text{Td}_X)
\]

The assumption about the Chern classes of \( \mathcal{V} \) kills all of the terms but the first and the last. We see that \( \chi(\text{Hom}(\mathcal{V},\mathcal{W})) = \nu v^g(\mathcal{W}) + \text{rk}(\mathcal{W}) \cdot \text{constant} \), whence the result follows. \( \square \)

**Proposition 2.3.2.9.** Let \( X \) be a smooth projective variety and \( \mathcal{X} \) a \( \mu_n \)-gerbe on \( X \). The category of \( \hat{\mu} \)-semistable coherent \( \mathcal{X} \)-twisted sheaves with fixed geometric Hilbert polynomial is bounded.
Proof. By 2.2.7.16, there is a locally free \( \mathcal{X} \)-twisted sheaf \( \mathcal{V} \) with \( \det \mathcal{V} = \mathcal{O} \). Applying a result of Simpson [55, 3.3], one sees that if \( \mathcal{F} \) comes from a set of coherent twisted sheaves with fixed geometric Hilbert polynomial \( P \), then the \( \mathcal{A} := \mathcal{End}(\mathcal{V}) \)-module \( \mathcal{Hom}(\mathcal{V}, \mathcal{F}) \) has fixed slope and \( \mu_{\max} \) bounded above by a constant depending only upon \( P \).

To show boundedness, first consider the subset of reflexive sheaves. Given a reflexive sheaf \( F \) on \( X \), temporarily write

\[
P^g_F(m) = \sum_{i=0}^{\dim X} a_i(F) \left( \frac{m + \dim X - i}{\dim X - i} \right).
\]

By a result of Langer (proving a theorem of Maruyama in arbitrary characteristic), the set of coherent reflexive sheaves \( F \) on \( X \) with a fixed upper bound on \( \mu_{\max}(F) \), \( a_0(F) = r \), \( a_1(F) = a_1 \), and \( a_2(F) \geq a_2 \) for fixed \( a_0, a_1, a_2 \) is bounded [41, 4.3]. Thus, to apply this to our situation, it remains to show that the “codimension 2” coefficient \( a_2 \) is bounded below. Looking at the formula in 2.3.2.8 and using the formula for the Chern character, we see that if \( \det \mathcal{V} = \mathcal{O} \) then the correction to the codimension 2 term of the Hilbert polynomial of \( \mathcal{Hom}(\mathcal{V}, \mathcal{F}) \) coming from higher terms (= after the first term) is given by \( -\kappa(c_2(\mathcal{V}) \cdot c_1(\mathcal{O}(1))^{d-2})\nu^{d-2} \), where \( \kappa \) is a coefficient which depends only on \( d \) (the dimension of \( X \)) and the degree of \( \mathcal{O}(1) \) on \( X \). In particular, after dualizing \( \mathcal{V} \) if necessary, we may assume that this correction term is always non-negative. Thus, we see that the Morita equivalence we apply will yield Hilbert polynomials with bounded below codimension 2 terms. Applying Langer’s theorem [ibid.], we are done for reflexive twisted sheaves.

Given a torsion free twisted sheaf, taking its reflexive hull preserves \( \mu \)-semistability, fixes \( a_0 \) and \( a_1 \), and increases \( a_2 \). Thus, we have just shown that the set of reflexive hulls of the sheaves we are interested in is bounded. In particular, only finitely many geometric Hilbert polynomials occur. We can now apply 2.2.7.21 finitely many times to yield the desired result.

\[\square\]

Corollary 2.3.2.10. The category of semistable \( \mathcal{X} \)-twisted sheaves with fixed geometric Hilbert polynomial is bounded.

Proof. It is elementary that any semistable sheaf is \( \hat{\mu} \)-semistable. \(\square\)

Corollary 2.3.2.11. Let \( X \to S \) be a smooth projective morphism, \( \mathcal{X} \) a \( \mu_n \)-gerbe on \( X \), and \( \mathcal{F} \) an \( S \)-flat family of coherent \( \mathcal{X} \)-twisted sheaves. The locus of \( \hat{\mu} \)-semistable (resp. semistable, resp. geometrically \( \hat{\mu} \)-stable, resp. geometrically stable) fibers of \( \mathcal{F} \) is open in \( S \).

Proof. It suffices to prove this when \( S \) is affine, whence we may assume that \( \mathcal{X} \) is a quotient stack (2.2.2.2 again!) and that the theory developed above applies. Now one can apply [30, 2.3.1] verbatim, with the additional remark that their proof also works for \( \hat{\mu} \)-semistability (even though they do not state this explicitly). \(\square\)

2.3.3. Applications of GIT. Let \((X, \mathcal{O}(1))\) be a (polarized) smooth projective variety over an algebraically closed field \( k \) and \( \mathcal{X} \to X \) a \( \mu_n \)-gerbe with \( n \in k^\times \). According to 2.3.2.11, there is an algebraic stack \( \text{Tw}^{ss}_{\mathcal{X}/k}(r, P) \) of semistable twisted sheaves of fixed rank \( r \) and geometric Hilbert polynomial \( P \) containing an open substack \( \text{Tw}^s_{\mathcal{X}/k}(r, P) \) of geometrically stable points (which contains a further open substack of geometrically \( \mu \)-stable points). We will make use of the algebraic Picard stack \( \mathcal{Pic}_{\mathcal{X}/k} \) parametrizing invertible sheaves on the stack \( \mathcal{X} \). The reader uncomfortable with this (in fact, the methods used in this paper show it is algebraic) may feel free to consider only sheaves of rank \( n \) in this section (in which case the determinant will in fact be a section of \( \mathcal{Pic}_{\mathcal{X}/k} \)). Recall that there is a determinant 1-morphism

\[
\text{Tw}_{\mathcal{X}/k}(n) \to \mathcal{Pic}_{\mathcal{X}/k}.
\]

Let \( L \) be an invertible sheaf on \( \mathcal{X} \), corresponding to a 1-morphism \( \varphi_L : k \to \mathcal{Pic}_{\mathcal{X}/k} \).
Definition 2.3.3.1. With the above notation, the stack of semistable twisted sheaves of rank $r$, determinant $L$ and geometric Hilbert polynomial $P$ is

$$\text{Tw}_{\mathcal{X}/k}^{ss}(r, L, P) := \text{Tw}_{\mathcal{X}/k}^{ss}(r, P) \times_{\mathcal{P}_{\mathcal{X}/k}} \varphi \times_{\mathcal{X}/k} \mathbf{k}.$$ 

The open substack of geometrically stable sheaves will be denoted $\text{Tw}_{\mathcal{X}/k}^{s}(r, L, P)$.

The usual computation [42, 2.2.2] of the 1-fiber product of stacks shows that $\text{Tw}_{\mathcal{X}/k}^{ss}(r, L, P)$ has as objects over $T \to \text{Spec } \mathbf{k}$ pairs $(\mathcal{V}', \varphi)$, where $\mathcal{V}'$ is a flat family of torsion free semistable $\mathcal{X}$-twisted sheaves parametrized by $T$, $\varphi : \det \mathcal{V}' \to \mathcal{L}T$ is an isomorphism, and for all points $t \to T$ one has $P^{r}_{\mathcal{V}'_{t}} = P$. As usual, isomorphisms in the groupoid are given by isomorphisms of the sheaves $\mathcal{V}'$ which respect the trivializations $\varphi$. Combining 2.3.1.1 with 2.3.2.11 shows that $\text{Tw}_{\mathcal{X}/k}^{ss}$ and $\text{Tw}_{\mathcal{X}/k}^{s}$ are algebraic stacks, locally of finite presentation over $k$, hence (as Pic is algebraic) the same is true for $\text{Tw}_{\mathcal{X}/k}^{ss}(r, L, P)$ and $\text{Tw}_{\mathcal{X}/k}^{s}(r, L, P)$.

Lemma 2.3.3.2. The stack $\text{Tw}_{\mathcal{X}/k}^{ss}(r, P)$ (resp. $\text{Tw}_{\mathcal{X}/k}^{s}(r, L, P)$) is quasi-compact and universally closed over $k$. The substack $\text{Tw}_{\mathcal{X}/k}^{s}(r, P)$ (resp. $\text{Tw}_{\mathcal{X}/k}^{s}(r, L, P)$) is quasi-compact and separated over $k$.

Proof. The numerical properties of the geometric Hilbert polynomial allow for a transcription of Langton’s proof [30, §2.B]. The uncomfortable reader may use the Morita equivalence of 2.3.2.8 to reduce this to [55, §4] (but only when there exists a locally free twisted sheaf with sufficiently many vanishing Chern classes, e.g., if $X$ is a surface).

Suppose $X$ is a surface. Given $L$, fixing the geometric Hilbert polynomial of $\mathcal{V}$ is the same as fixing $\text{deg } c_{2}(\mathcal{V})$ by the Riemann-Roch formula. In this case, we will often write $\text{Tw}(r, L, c)$ in place of $\text{Tw}(r, L, P)$ in order to align ourselves with the classical literature on surfaces. When all of the adornments are clear from context (or irrelevant), we will omit them from the notation.

Historically, moduli of semistable sheaves (and more generally modules) were studied using the tools of Geometric Invariant Theory, as developed in Mumford’s thesis [50]. The basic consequence of these methods is a proof that $\text{Tw}^{ss}$ is corepresented by a projective scheme; in fact, one can say quite a bit more about the corepresenting object using the full theory. The philosophy adopted here is that the stack is really a more fundamental object. (It is galling that the semistability of a sheaf still lacks a convincing explanation in intrinsic terms without recourse to GIT. However, as we remind the reader, stable sheaves do have a convincing description in terms of unitary connections in characteristic 0. In fact, these bundles arose independently of GIT and it was only discovered later that they solve a GIT problem [53].)

We will apply some of the classical results in this section to deduce GIT-like properties of our own moduli problem. When the underlying projective variety is a surface, techniques of Simpson will yield the result for all of $\text{Tw}^{ss}$. In general, even without GIT, one can show that $\text{Tw}^{s}$ has a coarse moduli space.

Lemma 2.3.3.3. Let $\mathcal{X}$ be an algebraic stack, and suppose $\mathcal{I}(\mathcal{X}) \to \mathcal{X}$ is fppf. Then the big étale sheaf $\text{Sh}(\mathcal{X})$ associated to $\mathcal{X}$ is an algebraic space and $\mathcal{X} \to \text{Sh}(\mathcal{X})$ is a coarse moduli space.

Proof. This is essentially the content of item 2 of the appendix to [9]. A similar construction may also be found in [5].

Proposition 2.3.3.4. $\text{Tw}^{s}(r, L, P) \to \text{Sh}(\text{Tw}^{s}(r, L, P))$ is a $\mu_{r}$-gerbe on an algebraic space of finite type over $k$.

The class of this $\mu_{r}$-gerbe in $\text{H}^{2}(X, G_{m})$ is the famous “Brauer obstruction” to the existence of a tautological twisted sheaf with determinant $L$ on $\text{Sh}(\text{Tw}^{s}(r, L, P) \times \mathcal{X})$. 

Definition 2.3.3.5. The algebraic space $\text{Tw}^s_{\mathcal{X}/k}(r, L, c) := \text{Sh} \text{Tw}^s_{\mathcal{X}/k}(r, L, c)$ is the moduli space of stable twisted sheaves.

Thus, that $\text{Tw}^s$ is an algebraic space, as we have seen above, is quite easy to prove using the proper abstract foundations. The interesting challenge is to show the existence of an ample invertible sheaf on $\text{Tw}^s$. This really does seem like a difficult problem. Of course, by 2.3.2.10 if we apply a Morita equivalence there are only finitely many possible Hilbert polynomials occurring, so we see that it suffices to prove quasi-projectivity under the assumption that both the geometric Hilbert polynomial and the Morita-Simpson-Hilbert polynomial is constant in fibers. Showing abstractly that there is an ample invertible sheaf is an interesting problem.

Proposition 2.3.3.6. Let $\mathcal{X} \to X$ be a $\mu_n$-gerbe on a smooth projective variety of dimension $d$. Suppose there is a twisted sheaf $\mathcal{V}$ such that all Chern classes but (possibly) $c_d(\mathcal{V})$ are zero in $A(\mathcal{X}) \mathbb{Q}$. Then $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, P)$ is a GIT quotient stack with stable sublocus $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, P)$.

Proof. By 2.3.2.8, this reduces to work of Simpson [55, §4].

Corollary 2.3.3.7. Given the hypotheses of 2.3.3.6, there is a morphism to a projective scheme

$$\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, P) \to \text{Tw}^{ss}_{\mathcal{X}/k}(r, L, P)$$

corepresenting $\text{Tw}^{ss}_{\mathcal{X}/k}(r, L, P)$ in the category of schemes and an open subscheme $U \subset \text{Tw}^{ss}$ such that the restriction of the morphism $\text{Tw}^{ss} \to \text{Tw}^{ss}$ to $U$ yields an isomorphism $\text{Tw}^s \to \text{Tw}^s \cong U$.

Question 2.3.3.8. In the absence of a $\mathcal{V}$ with sufficiently many vanishing Chern classes, is it still true that the coarse moduli space $\text{Tw}^s$ is quasi-projective? Attempting to prove this in various naïve ways always leads one back to GIT. If the space is quasi-projective, can one find a projective scheme corepresenting $\text{Tw}^{ss}$ by taking a projective closure of $\text{Tw}^s$?

Remark 2.3.3.9. The work described in section 2.3.4 gives an indication of how one might go about proving that $\text{Tw}^{ss}$ exists and has projective components as long as there is a locally free $\mathcal{X}$-twisted sheaf, say $\mathcal{V}$. Indeed, in this case one is easily led to conjecture that the space $\text{Tw}^{ss}$ is related by Mumford-Thaddeus-type flips to the space of Simpson-semistable $\mathcal{E}nd(\mathcal{V})$-modules, which is shown to be a GIT quotient in [55]. The mechanism should be very similar to the notion of “twisted stability” investigated by Matsuki-Wentworth and Yoshioka, as described below. We intend to return to this question in the future.

2.3.4. Essentially trivial gerbes. In this section we describe the situation for a $\mu_n$-gerbe $\mathcal{X}$ on a projective variety $X$ which is the gerbe of $n$th roots of an invertible sheaf. These correspond to the kernel of the natural map $H^2(X, \mu_n) \to H^2(X, G_m)$. If one chooses the “correct” polarization of $X$, then the stack of semistable twisted sheaves is canonically isomorphic to the stack of semistable sheaves on the underlying variety $X$. These spaces have essentially been studied by Ellingsrud-Göttsche, Thaddeus, Yoshioka, and Matsuki-Wentworth, in the guise of “twisted stability.” These authors did not write in terms of gerbes, but rather investigated what happens when instead of computing the Hilbert polynomial of a torsion free sheaf $F$ one computes the Hilbert polynomial of $F \otimes \mathcal{O}(\alpha)$, where $\alpha$ is some $\mathbb{Q}$-divisor (with stability now being called “$\alpha$-twisted stability”). We refer the reader to their work ([61] and the references therein) for a detailed description of the situation (in characteristic 0); we will only use a small bit of the theory in what follows. At the end of the section we will spend a few moments considering what happens when the base field is not algebraically closed. Let $X \to \text{Spec} k$ be a geometrically connected smooth projective variety over a field.

Definition 2.3.4.1. A $\mu_n$-gerbe $\mathcal{X} \to X$ is (geometrically) essentially trivial if the class $[\mathcal{X}]$ has trivial image in $H^2(X, G_m)$ (respectively, $H^2(X \otimes_k \mathbb{K}, G_m)$).
Lemma 2.3.4.2. A gerbe $\mathcal{X}$ is essentially trivial if and only if there exists an invertible $\mathcal{X}$-twisted sheaf $\mathcal{L}$.

Proof. If $\mathcal{X}$ is essentially trivial, then the associated $\mathcal{G}_m$-gerbe is isomorphic to $\mathcal{B}\mathcal{G}_m$. We can thus pullback a twisted line from $\mathcal{B}\mathcal{G}_m$. Conversely, if there is an invertible $\mathcal{X}$-twisted sheaf $\mathcal{L}$, then $\delta_*(\mathcal{L}) \cong \mathcal{O}_X|_\mathcal{X}$ is (the pullback to $\mathcal{X}$) of an Azumaya algebra with Brauer class $[\mathcal{X}]$, whence $[\mathcal{X}] = 0 \in H^2(X, \mathcal{G}_m)$. □

Given $\mathcal{L}$ as in 2.3.4.2, it follows that $\mathcal{L} \otimes n$ will be the pullback of an invertible sheaf on $X$. This observation allows to classify essentially trivial gerbes using the Kummer sequence.

Definition 2.3.4.3. Let $\mathcal{M}$ be an invertible sheaf on $X$. The gerbe of $n$th roots of $\mathcal{M}$, denoted $[\mathcal{M}]_{1/n}$, is the stack whose objects over $T$ are pairs $(\mathcal{L}, \varphi)$, where $\mathcal{L}$ is an invertible sheaf on $X \times T$ and $\varphi: \mathcal{L} \otimes n \rightarrow \mathcal{M}$ is an isomorphism.

It is immediate that $[\mathcal{M}]_{1/n}$ is a $\mathcal{G}_m$-gerbe.

Proposition 2.3.4.4. The cohomology of the Kummer sequence $1 \rightarrow \mathcal{G}_n \rightarrow \mathcal{G}_m \rightarrow \mathcal{G}_m \rightarrow 1$ yields an exact sequence

$$0 \rightarrow \text{Pic}(X) / n \text{Pic}(X) \rightarrow H^2(X, \mathcal{G}_m) \rightarrow \text{Br}(X)[n] \rightarrow 0.$$ 

Under this identification the cohomology class of any essentially trivial gerbe $\mathcal{X}$ in $H^2(X, \mathcal{G}_m)$ equals the image of the class of $\mathcal{L} \otimes n$, where $\mathcal{L}$ is an $\mathcal{X}$-twisted invertible sheaf.

Proof. The construction of the (moderately) long exact sequence in non-abelian cohomology shows that given $\mathcal{M} \in \text{Pic}(X)$, the coboundary $\delta(\mathcal{M}) \in H^2(X, \mathcal{G}_m)$ is just $[\mathcal{M}]_{1/n}$. Up to isomorphism, this gerbe depends only on the residue of $\mathcal{M}$ modulo $n \text{Pic}(X)$. The sequence shows that any essentially trivial gerbe has the form $\delta(\mathcal{M})$ for some $\mathcal{M}$. On $\delta(\mathcal{M})$, there is a universal $n$th root $\mathcal{L}$ with $\mathcal{L} \otimes n \rightarrow \mathcal{M}$. If $\mathcal{N}$ is any other invertible twisted sheaf then $\mathcal{L} \otimes \mathcal{N}^{\vee}$ is an untwisted invertible sheaf, say $\mathcal{M}'$, and one has $\mathcal{L} \otimes n \cong (\mathcal{M}') \otimes n$. Thus, the $n$th tensor power of any invertible twisted sheaf lies in the same class as $\mathcal{M}$ modulo $n \text{Pic}(X)$.

In the following proposition, we use the notion of twisted stability and from 3.2 of [48]. Given a class $\gamma \in \text{Pic}(X) \otimes \text{Q}$, the $\gamma$-twisted Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on $X$ is $n \mapsto \chi(\mathcal{F} \otimes \gamma(n))$, computed formally using the Riemann-Roch formula. (This is what Matsuki and Wentworth use to define $\gamma$-twisted (semi)stability.)

Proposition 2.3.4.5. Suppose $\mathcal{X}$ is essentially trivial. Then there exists $\gamma \in \text{Pic}(X) \otimes \text{Q}$ such that there is an isomorphism of $\text{Tw}^{ss}_{\mathcal{X}/k}(n, L, P)$ with the stack $\text{Sh}^{ss}_{\mathcal{X}/k}(n, L(n\gamma), P)$ of $\gamma$-twisted semistable sheaves on $X$ of rank $n$, determinant $L(n\gamma)$, and twisted Hilbert polynomial $P$.

Proof. There is some $\mathcal{M}$ such that $\mathcal{X} = [\mathcal{M}]_{1/n}$. Let $M$ be a universal $n$th root of $\mathcal{M}$ on $\mathcal{X}$. The functor $\mathcal{V} \mapsto \mathcal{V} \otimes M^{\vee}$ yields an equivalence of categories from $\mathcal{X}$-twisted sheaves to sheaves on $X$. It is easy to see that semistability of $\mathcal{V}$ as a twisted sheaf translates into $1/n$. $\mathcal{M}$-twisted stability of $\mathcal{V} \otimes M^{\vee}$. □

Corollary 2.3.4.6. Suppose $\mathcal{X}$ is geometrically essentially trivial. There is an isomorphism

$$\text{Tw}^{\mu}_{\mathcal{X}/k}(n, L) \otimes \overline{\mathbb{F}} \cong \text{Sh}^{\mu}_{\mathcal{X}/k}(n, L') \otimes \overline{\mathbb{F}},$$

where the superscript $\mu$ denotes the open substacks of $\mu$-stable sheaves and $\mathcal{X} = [L \otimes (L')^{\vee}]^{1/n}$. If $X$ is a surface, there is an isomorphism

$$\text{Tw}^{\mu}_{\mathcal{X}/k}(n, L, P) \otimes \overline{\mathbb{F}} \cong \text{Sh}^{\mu}_{\mathcal{X}/k}(n, L', Q) \otimes \overline{\mathbb{F}},$$

with $Q$ an appropriate polynomial.
Proof. The first part follows from the fact that slope stability is independent of a Matsuki-Wentworth twisting by a \( Q \)-divisor and is left to the reader. To see that the Hilbert polynomial is constant when \( X \) is a surface, note that the functor \( \mathcal{F} \mapsto \mathcal{F} \otimes M' \) described in 2.3.4.5 fixes the determinant and also preserves the discriminant. It follows that it sends \( \mathcal{F} \)-twisted sheaves with a given second Chern class \( c \) to ordinary sheaves with a fixed second Chern class \( c' \). Fixing the determinant and the \( c_2 \) then fixes the Hilbert polynomial. \( \square \)

3. Curves and surfaces

In this section, we develop the theory of \( \text{Tw}^{ss} \) when the underlying variety \( X \) is a curve or a surface. Over an algebraically closed field, there is a guiding meta-theorem: *Anything which happens in the theory of \( \text{Sh}^{ss} \) happens in the theory of \( \text{Tw}^{ss} \).* For curves, this is not just a meta-theorem: as we will show in section 3.1, \( \text{Sh}^{ss} \) and \( \text{Tw}^{ss} \) are isomorphic (with the proper adornments added to the symbols). For surfaces, there is not a similar direct comparison, but the classical structure theory for \( \text{Sh}^{ss} \) carries over to \( \text{Tw}^{ss} \). In particular, as the second Chern class grows, \( \text{Tw}^{ss} \) becomes irreducible. One can further compute examples on e.g. K3 surfaces, but we have unfortunately not included these examples here. (However, Yoshioka has worked out quite a bit for K3 surfaces over the complex numbers. See [60].) Despite the excessively abstract foundations, we thus have a reasonable understanding of the geometry of these moduli spaces for low-dimensional varieties over algebraically closed fields. There are many gems from the untwisted world waiting to be properly twisted which we have not been able to include here. They will hopefully appear in future work.

When the base field is allowed to be non-algebraically closed, things get more interesting, and the stacks \( \text{Tw}^{ss} \) carry arithmetic information which \( \text{Sh}^{ss} \) knows nothing about. The straightforward geometry of the moduli spaces can now be brought to bear on arithmetic problems. We will exploit this extra information in [46] when we study the Brauer group of a surface over an algebraically closed field, a finite field, and a local field. The work here also appears to be just the beginning of a possibly fruitful line of investigation.

3.1. Twisted sheaves on curves. We illustrate the theory developed up to this point with the example of semistable twisted sheaves on curves. This serves two purposes: first, twisted sheaves are easy to understand. Second, we will use the results mentioned in this section when we study semistable twisted sheaves on surfaces.

By a curve \( C \) we will always mean a proper smooth geometrically connected curve over a field \( k \).

Remark 3.1.0.7. Much of what we say here can be generalized to singular curves, but even the classical theory of sheaves has not been very well worked out in the non-smooth case. Furthermore, it would be slightly more complex to develop the theory of the geometric Hilbert polynomial in this context, but it can be done using the theory of localized Chern classes on a smooth embedding of the singular gerbe. We spare the reader the technical details in this work. The reason to consider more general curves is that in the relative case it is nice to be able to handle degenerate fibers. For example, if a surface \( X \) carries a generically nice pencil \( \bar{X} \to \mathbb{P}^1 \), it is likely (usually necessary) that there will be singular fibers in the pencil. We would still like to relate the space of semistable twisted sheaves on \( \bar{X} \) to the relative space of twisted sheaves of \( X \) viewed as a family of curves over \( \mathbb{P}^1 \). For the applications of the theory here and in [46], it will suffice to consider only smooth curves, however, so we omit the more general theory for the sake of brevity.

3.1.1. A curve over a point. Let \( C \to \text{Spec} \ k \) be a curve over an algebraically closed field, and let \( \mathcal{E} \to C \) be a \( \mu_n \)-gerbe over \( C \) with \( n \in k^\times \).
Lemma 3.1.1.1. If $X$ is a scheme of dimension at most 1 over an algebraically closed field, then $\text{Br}(X) = 0$.

Proof. We sketch the proof. One first reduces to the case where $X$ is reduced. (E.g., consider $0 \to \mathcal{F} \to \mathcal{O}_X \to \mathcal{O}_{X_{\text{red}}} \to 0$. Then $1 \to 1 + I \to \mathcal{O}_X^* \to \mathcal{O}_{X_{\text{red}}}^* \to 1$ is exact, and taking cohomology we find $0 = H^2(X, I) \to H^2(X, G_m) \to H^2(X_{\text{red}}, G_m) \to H^2(X, I) = 0$ is exact.) It then suffices to show that the Brauer group of any irreducible component vanishes (see 3.1.4.8ff for the type of reasoning used in this argument). This follows from Tsen’s theorem.

In other words, by 2.3.4.2 there exists an invertible $\mathcal{C}$-twisted sheaf, say $\mathcal{L}$. Recall that for any torsion free coherent sheaf $\mathcal{F}$ on $C$, one has $\deg(\mathcal{F}) := \chi(\mathcal{F}) - \text{rk}\mathcal{F}\chi(\mathcal{O}_C)$.

Definition 3.1.1.2. Given a $\mathcal{C}$-twisted sheaf $\mathcal{F}$, the degree of $\mathcal{F}$ is

$$\deg(\mathcal{F}) := n \deg c_1(\mathcal{F}).$$

Here the degree of $c_1(\mathcal{F})$ is computed by applying proper pushforward in rational Chow theorem from $\mathcal{C}$ to a point. It is easy to see that multiplying by $n$ is necessary in order for the pullback from $C$ to $\mathcal{C}$ to preserve degrees.

Definition 3.1.1.3. The slope of $\mathcal{F}$, denoted $\mu(\mathcal{F})$, is $\deg(\mathcal{F})/\text{rk}\mathcal{F}$. The twisted sheaf $\mathcal{F}$ is (semi-)stable if for every twisted subsheaf $\mathcal{G} \subset \mathcal{F}$, we have $\mu(\mathcal{G})(\leq)\mu(\mathcal{F})$.

It is easy to see that tensoring with $\mathcal{L}^\vee$ creates a bijection between the semistable $\mathcal{C}$-twisted sheaves of rank $r$ and degree $d$ and the semistable sheaves on $C$ with rank $r$ and degree $d - r \deg \mathcal{L}$. Note that this last number must be an integer. In fact, $\deg \mathcal{L} \in \frac{1}{n}\mathbb{Z}$, so $d \in \frac{1}{\gcd r,n}\mathbb{Z}$.

Given a $\mu_n$-gerbe $\mathcal{G} \to C$, let $\overline{\mathcal{G}}(\mathcal{G}) \in (1/n)\mathbb{Z}/\mathbb{Z}$ be the fraction corresponding to the image of $[\mathcal{G}]$ under $H^2(C, \mu_n) \to \mathbb{Z}/n\mathbb{Z} \to (1/n)\mathbb{Z}/\mathbb{Z}$.

Proposition 3.1.1.4. With the above notation, the stack of semistable $\mathcal{C}$-twisted sheaves of rank $r$ and degree $d$ is non-canonically isomorphic to the stack of semistable sheaves on $C$ of rank $r$ and degree $d - r \overline{\mathcal{G}}$.

Proof. It well-known that the stack of semistable sheaves on $C$ of rank $r$ and degree $d$ is non-canonically isomorphic to the stack of semistable sheaves of rank $r$ and degree $d + nr$ for any $n$. Using the above notation, it is straightforward to check that any invertible $\mathcal{C}$-twisted sheaf $\mathcal{L}$ will have degree $q + \overline{\mathcal{G}}$ for some integer $q$. The result follows by combining these statements.

In particular, it is a GIT stack (hence corepresented by a projective variety).

The usual structure theory for moduli spaces of semistable sheaves on smooth curves developed by Seshadri, Ramanan, Ramanathan, Narasimhan, Mumford, Newstead, etc., now carries over to the twisted setting. (See [50, Appendix 5C] for a relatively exhaustive list of references.) We omit proofs for the sake of brevity.

Corollary 3.1.1.5. If $C$ is smooth of genus $g \geq 2$, the moduli space of semistable $\mathcal{C}$-twisted sheaves of rank $r$ and any fixed determinant is unirational of dimension $(r^2 - 1)(g - 1)$. The stack of semistable $\mathcal{C}$-twisted sheaves of rank $r$ and fixed degree $d$ is irreducible, smooth, and unirational over $k$ of dimension $r^2(g - 1) + 1$ at stable points. The stable locus is a gerbe over a smooth unirational quasi-projective variety.

Note that, as usual, even though the stack is smooth, its corepresenting GIT quotient need not be smooth away from the stable locus.

Proposition 3.1.1.6. Suppose $d - r \overline{\mathcal{G}} \in \mathbb{Z}$ and $r$ are relatively prime. The open immersion $\text{Tw}^s_{\mathcal{G}/k}(r,d) \hookrightarrow \text{Tw}^s_{\mathcal{G}/k}(r,d)$ is an isomorphism. In this case, $\text{Tw}^s$ is a smooth rational projective variety isomorphic to $\text{Sh}(\text{Tw}^{ss})$. There is a tautological sheaf $\mathcal{F}$ on $\text{Tw}^{ss} \times \mathcal{G}$, and $\text{Pic}(\text{Tw}^{ss}) \cong \mathbb{Z}$. 
3.1.2. The relative case. When $C$ is allowed to move over a base (or descend over a non-algebraically closed base field) things get more interesting. In this section, we let $\pi : C \to S$ denote a proper flat morphism of finite presentation whose geometric fibers are curves as above, and we let $\mathcal{C} \to C$ be a $\mu_n$-gerbe with $n \in \mathcal{O}_S(S)^\times$. We fix a rank $r$ and a rational number $d$, the degree.

Note that the degree map on the relative Picard scheme induces a morphism
\[ \varphi : H^0(S, \mathbb{R}^2 \pi_* \mu_n) \to H^0(S, \mathbb{Z}/n\mathbb{Z}) \]
which is a relative version of the map considered in the proof of 3.1.1.4: The image over a connected component $S' \subset S$ is equal to $n$ times the constant value for the minimal degree of an invertible $\mathcal{C}$-twisted sheaf on a fiber. If $S$ is connected, write $\delta := (1/n)\varphi([\mathcal{C}])$ as above (where $\varphi([\mathcal{C}])$ is chosen to lie between 0 and $n - 1$).

**Lemma 3.1.2.1.** Let $C$ be a geometrically connected smooth proper curve over a separably closed field $k$ of characteristic exponent $p$ and $\mathcal{C} \to C$ a $\mu_n$-gerbe. If $(n,p) = 1$, then there is an invertible $\mathcal{C}$-twisted sheaf of degree $\delta(\mathcal{C})$.

**Proof.** It suffices to show the existence of an invertible twisted sheaf $\mathcal{L}$, as it is then clear (as in section 2.3.4) that $\deg \mathcal{L}$ has fractional part $\delta$, so we can tensor with an untwisted invertible sheaf to bring the degree down to $\delta$. Furthermore, showing that $\mathcal{L}$ exists is equivalent to showing that the Brauer class of $[\mathcal{C}]$ is 0. By Tsen’s theorem, this is true of $\mathcal{C} \otimes k$, hence it is true for some finite extension $L/k$. Since $k$ is separably closed, $[L:k]$ is a power of $p$. The inflation-restriction sequence in Galois cohomology over the function field of $C$ then shows that some power of $p$ kills $[\mathcal{C}] \in H^2(C, \mathbb{G}_m)$. Since $n[\mathcal{C}] = 0$ (as it comes from a $\mu_n$-gerbe), the result follows. \hfill \Box

**Proposition 3.1.2.2.** Suppose $S$ is connected. The stack of semistable $S$-flat $\mathcal{C}$-twisted sheaves of rank $r$ and degree $d$ is an étale form of the stack of semistable $S$-flat sheaves of rank $r$ and degree $d - r\delta$.

**Proof.** We simply need to note that one can étale-locally on the base find an invertible twisted sheaf $\mathcal{L}$ on $C$ of degree $\delta$. (The obstruction to the gluing of these local invertible sheaves is the image of $[\mathcal{C}]$ in $H^1(S, \mathbb{R}^1 \pi_* \mathbb{G}_m)$.) The comparison is made by tensoring with $\mathcal{L}^{-\delta}$; this will not change the $S$-flatness of the sheaf because it will not change its local structure. Applying 3.1.2.1 gives such an $\mathcal{L}$ in the fiber over a separable closure of the residue field of any point $s \in S$. Applying the deformation theory of section 2.2.3, it is easy to see that $\mathcal{L}$ extends to an étale neighborhood of $s$. \hfill \Box

If one takes $S$ to be the spectrum of a field $k$, then one can describe the space $\text{Tw}^{ss}$ in terms of Galois twists of $\text{Sh}^{ss}$. Similarly, if $X \to S$ is a surface fibered over a curve, one can use relative stacks of twisted sheaves of rank 1 (“twisted Picard spaces”) to reconstruct Artin’s isomorphism between the Brauer group of $X$ and the Tate-Shafarevich group of the Jacobian of the generic fiber $X_K(S)$. We have excluded these topics from this paper, as our main concern here is with the geometry of the moduli spaces and not arithmetic. They are discussed in detail in [45] and touched on in [46].

3.1.3. Moving twisted sheaves on curves. Given a divisor moving in a surface and a twisted sheaf on the divisor, we can push it along the moving curve. This gives us a way of connecting two stable twisted sheaves on linearly equivalent smooth divisors in a family. Throughout this section, $X$ is a smooth projective surface over an algebraically closed field $k$ and $\mathcal{X} \to X$ is a fixed $\mu_n$-gerbe on $X$.

**Proposition 3.1.3.1.** Let $C_0$ and $C_1$ be smooth curves in $X$ and let $D_i$ be the pushforward to $\mathcal{X}$ of a stable locally free twisted sheaf on $C_i \times_X \mathcal{X}$. If $C_0$ is linearly equivalent to $C_1$ and
$P^q_{\mathcal{X}, \mathcal{P}_0} = P^q_{\mathcal{X}, \mathcal{P}_1}$, then there is an irreducible $k$-variety $S$, two points $s_0, s_1 \in S(k)$, and an $S$-flat family of $\mathcal{X}$-twisted sheaves $\mathcal{F}$ on $\mathcal{X} \times S$ such that $\mathcal{F}_{s_i} \cong \mathcal{P}_i$.

**Proof.** The idea is to push $\mathcal{P}_0$ along an embedded deformation of $C_0$ into $C_1$ and then move the image through the moduli space of twisted sheaves on $C_1$. We can actually do both simultaneously (which is more likely to yield an irreducible parameter space for the family).

Since $C_0$ and $C_1$ are linearly equivalent, there is a flat Cartier divisor $\mathcal{C} \subset \mathcal{X} \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ such that $\mathcal{C}_0 = C_0$ and $\mathcal{C}_1 = C_1$. (E.g., one can take the total space of the pencil of sections of $\mathcal{O}(C_0)$ generated by $C_0$ and $C_1$.) Passing to an open subset $U \subset \mathbf{P}^1$ if necessary, we may assume $\mathcal{C} \rightarrow U$ is smooth. Consider the stack $\mathcal{M} := \text{Tw}^\text{ss}_{\mathcal{X} \times \mathcal{C}/U}(n, \mathcal{P})$. It is a classical result that the stack $\text{Sh}^\text{ss}_{C/k}(n, d)$ is an irreducible GIT quotient stack [50, Appendix 5C]. Thus, applying 3.1.2.2 and using quasi-properness, we see that $\mathcal{M}$ is irreducible and smooth over $U$ (and thus smooth over $k$).

Let $M \rightarrow \mathcal{M}$ be a smooth cover. Write $M_i, i = 1, \ldots, t$ for the connected components of $M$. Then each $M_i$ is an open irreducible subspace of $M$, hence has open image in $\mathcal{M}$. Since $\mathcal{M}$ is irreducible, there is some $i$ such that $M_i \rightarrow \mathcal{M}$ is surjective. In other words, $\mathcal{M}$ has an irreducible smooth cover. Choosing points $m_0, m_1 \in M(k)$ mapping to $\mathcal{P}_0$ and $\mathcal{P}_1$ respectively, we see that we can make a family of semistable sheaves on $\mathcal{C} \times_U M$ containing $\mathcal{P}_0$ and $\mathcal{P}_1$. Since $\mathcal{C} \subset \mathcal{X} \times U$, we see that $\mathcal{C} \times_U M \subset \mathcal{X} \times M$. Pushing forward the family yields the result. □

**Corollary 3.1.3.2.** The conclusion of 3.1.3.1 holds when $\mathcal{P}_i$ are invertible twisted sheaves, without explicit stability hypotheses.

**Proof.** This follows from the fact that any invertible sheaf is stable and the fact that $\text{Sh}^\text{ss}_{C/k}(1, d) = \text{Pic}^d_{C/k}$ is smooth and irreducible. □

### 3.1.4. Moduli of restrictions.
We use the above machinery to study what happens when restricting stable twisted sheaves on a surface $X$ to a very ample smooth curve $D$. In particular, we show that there are no positive-dimensional complete families of locally free stable twisted sheaves on $X$ which all restrict to the same stable twisted sheaf (up to isomorphism) on $D$. This will ultimately be used to show that asymptotically, the irreducible components of the stack of semistable twisted sheaves on $X$ contain both locally free and non-locally free points.

Throughout this section, $X$ is a smooth projective surface (with fixed very ample invertible sheaf $\mathcal{O}(1)$) over an algebraically closed field $k$ and $\mathcal{X} \rightarrow X$ is a $\mu_n$-gerbe on $X$, with $n \in k^\times$.

**Notation 3.1.4.1.** Whenever $\mathcal{X}$ is a stack of sheaves (on a curve or surface), we will let $\mathcal{X}_{lf}$ denote the open substack parametrizing locally free sheaves.

Let $D \in |\mathcal{O}(1)|$ be a general member.

**Situation 3.1.4.2.** Let $C$ be a smooth projective curve over $k$ and $\varphi : C \rightarrow \text{Tw}^\text{ss}_{\mathcal{X}/k, lf}$ a 1-morphism to the locally free locus corresponding to $\mathcal{F}$ on $C \times X$. Suppose that every object $\varphi(c)$, $c \in C(k)$, restricts to a fixed stable locally free $\mathcal{X}_D$-twisted sheaf $\mathcal{F}_0$.

**Definition 3.1.4.3.** A divisor $D \subset X$ is $\varphi$-sticky if there exists a simple locally free $\mathcal{X}_D$-twisted sheaf $\mathcal{F}_0$ such that for every $c \in C(k)$, the object $\varphi(c)|_D$ is isomorphic to $\mathcal{F}_0$.

We will suppress the $\varphi$ from the notation when it is clear (essentially, for the rest of this section).

**Proposition 3.1.4.4.** $\varphi$ is essentially constant (i.e., isotrivial).

(Since $\varphi$ lands in the stable locus, being isotrivial is equivalent to the map to the coarse moduli space $\text{Tw}^\text{ss}$ being constant. Indeed, if $\varphi$ is isotrivial, then there is a finite étale extension
$C' \to C$ such that the induced map $C' \to \text{Tw}^s$ is constant, whence the original map must be constant. Conversely, if $C \to \text{Tw}^s$ is constant, then $\varphi$ lands in the fiber $\mathcal{F}$ of $\text{Tw}^s \to \text{Tw}^s$ over a point. Since $\mathcal{F}$ is a $\mu_n$-gerbe, the map $pt \to \mathcal{F}$ is finite étale, and pulling back by this map yields a finite étale cover $C' \to C$ such that the restriction of $\varphi$ to $C'$ is constant. 

**Lemma 3.1.4.5.** There is an open subset of $|\mathcal{O}(1)|$ consisting of smooth sticky divisors $D'$. 

**Proof.** Write $P$ for $|\mathcal{O}(1)|$. Let $I \subset X \times P$ be the incidence correspondence of $\mathcal{O}(1)$; the fiber of the second projection over a point $p \in |\mathcal{O}(1)|$ is the divisor corresponding to $p$. The family $\mathcal{F}$ on $C \times X$ corresponding to $\varphi$ pulls back to give a flat family $\mathcal{F}$ of twisted sheaves on $C \times I \to C \times P$. (The sheaf $\mathcal{F}$ is flat by e.g. a Hilbert polynomial calculation after applying a Morita equivalence.) The stickiness condition on $D$ says that the locus $\Psi$ of stable fibers contains all of $C \times \{[D]\}$. By openness of stability and properness of $C$, we conclude that there is an open $U \subset P$ such that $C \times U \subset \Psi$. We have a map $C \times U \to \text{Tw}^s_{I_U/U}$ over $U$ such that the fiber $C \times \{[D]\}$ collapses to a point in the fiber $\text{Tw}^s_D/k$. By the usual rigidity lemma, it follows that an open subset of fibers gets collapsed in the map to $\text{Tw}^s_{I_U/U}$. By Tsen’s theorem, there is an open set of sticky divisors $D'$. (The careful reader will note that the formation of the coarse moduli space $\text{Tw}^s$ commutes with arbitrary base change in this case because $\text{Tw}^s \to \text{Tw}^s$ is a gerbe, so $\text{Tw}^s$ is equal to the sheafification of $\text{Tw}^s$ in the big étale topology, which is tautologically of formation compatible with base change.) \hfill $\square$

**Lemma 3.1.4.6.** Suppose $D$ is sticky. The twisted sheaf $\mathcal{F}_{C \times D}$ has the form $\text{pr}_1^*(\mathcal{M}) \times \text{pr}_2^*(\mathcal{F}_0)$, where $\mathcal{M}$ is an invertible sheaf of $\mathcal{O}_C$-modules and $\mathcal{F}_0$ is a stable twisted sheaf on $D$.

**Proof.** Write $\mathcal{D} := \mathcal{F} \times_X D$. Since $D$ is sticky, the family $\mathcal{F}_{C \times D}$ gives rise to a diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\varphi} & \text{Tw}^s_{\mathcal{D}/k} \\
\downarrow{\tilde{\varphi}} & & \downarrow{\pi} \\
\text{Tw}^s_{\mathcal{D}/k} & \xleftarrow{\psi} & C
\end{array}
$$

such that $\varphi$ is constant with value $[\mathcal{F}_0]$. There is also a constant lift $\psi$ of $\varphi$ given by the family $\text{pr}_2^*\mathcal{F}_0$ on $C \times D$. Since $\pi$ is a $G_m$-gerbe, we see that $\psi$ and $\tilde{\varphi}$ are identified with two sections of a trivial $G_m$-gerbe. Using one of them to trivialize the gerbe, they differ by a map $C \to B\text{G}_m$, which gives the invertible sheaf $\mathcal{M}$. \hfill $\square$

**Definition 3.1.4.7.** A (possibly singular) divisor $E$ is called $\varphi$-slippery if there is an invertible sheaf $\mathcal{M}$ on $C$ and a fixed twisted sheaf $\mathcal{F}_0$ on $E$ such that $\mathcal{F}_{C \times E} \cong \text{pr}_1^*\mathcal{M} \otimes \text{pr}_2^*\mathcal{F}_0$.

We will similarly suppress the $\varphi$ from the notation when it is clear from context. We just showed that for a smooth divisor $D$, if $D$ is sticky then it is slippery, and that if $D$ is sticky for one smooth very ample divisor, then an open set in $|D|$ parametrizes sticky points $D$. Using these two facts, we now provide an inductive procedure for enlarging a slippery divisor $D$.

**Lemma 3.1.4.8.** Let

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\psi} & D
\end{array}
$$

be a Cartesian diagram of surjections of sheaves of groups in a topos $T$. The natural map $BA \to BB \times_{BD} BC$ is a 1-isomorphism of classifying stacks.

**Proof.** The natural map $BA \to BB \times_{BD} BC$ is given by sending a right $A$-torsor $F_A$ to the triple $(F_A \times^AB, F_A \times^A C, \varphi)$, where $\varphi : (F_A \times^AB) \times^B D \to (F_A \times^AC) \times^CD$ is the natural
isomorphism arising from the associativity of the contracted product (i.e., \((F_A \times^A B) \times^B D \cong F_A \times^A (B \times^B D) \cong F_A \times^A D\) and similarly for \(C\)). There is a 1-morphism in the other direction arising as follows. An object of \(BB \times_B BD BC\) is given by a triple \((F_B, F_C, \psi)\), where \(\psi : F_B \times^B D \cong F_C \times^C D\) is an isomorphism of right \(D\)-torsors. Given such an object, one can produce a right \(A\)-torsor \(FA\) by forming the fiber square

\[
\begin{array}{ccc}
F_A & \xrightarrow{\square} & F_C \\
\downarrow & & \downarrow \\
F_B & \xrightarrow{\psi} & F_{C,D}
\end{array}
\]

where \(F_{B,D} := F_B \times^B D\), etc. That \(FA\) is in fact an \(A\)-torsor follows from the surjectivity of \(B \to D\). We leave it as an exercise to check that these maps of stacks are 1-inverse to one another. \(\square\)

We can use 3.1.4.8 to prove a (twisted) classical result about vector bundles on a union of curves meeting transversely. Let \(D\) and \(D'\) be curves with transverse intersection \(D \cap D' = \{q_1, \ldots, q_r\}\). Let \(X\) be a \(k\)-scheme. The transversality of the intersection of \(D\) and \(D'\) says that the diagram of surjections of sheaves of rings on \(X \times (D \cup D')\)

\[
\begin{array}{ccc}
\mathcal{O}_{X \times (D \cup D')} & \xrightarrow{\theta} & \mathcal{O}_{X \times D'} \\
\downarrow & & \downarrow \\
\mathcal{O}_{X \times D} & \xrightarrow{\theta} & \mathcal{O}_{X \times (D \cap D')}
\end{array}
\]

is Cartesian, where all schemes are given their reduced structures. (More generally, given a ring \(\mathcal{O}\) in a topos and two ideals \(I\) and \(I'\) such that \(I \cap I' = 0\), one has a corresponding diagram. For non-CM schemes, there can be complex information at embedded intersection points.) Here we write (by abuse of notation) \(\mathcal{O}_{D}\) for the pushforward of the structure sheaf of \(D\) and similarly for \(D'\) and \(D \cap D'\). It follows that given any \(k\)-scheme \(X\) the diagram

\[
\begin{array}{ccc}
\text{GL}_n \mathcal{O}_{X \times (D \cup D')} & \xrightarrow{\text{GL}_n \theta} & \text{GL}_n \mathcal{O}_{X \times D'} \\
\downarrow & & \downarrow \\
\text{GL}_n \mathcal{O}_{X \times D} & \xrightarrow{\text{GL}_n \theta} & \text{GL}_n \mathcal{O}_{X \times (D \cap D')}
\end{array}
\]

is a Cartesian diagram of surjections of sheaves of groups on \(X \times (D \cup D')\).

Suppose \(\mathcal{V}\) and \(\mathcal{V}'\) are locally free sheaves of rank \(n\) on \(D\) and \(D'\), respectively. Our goal is to describe the space of locally free sheaves \(\mathcal{W}\) on \(D \cup D'\) which restrict to \(\mathcal{V}\) on \(D\) and \(\mathcal{V}'\) on \(D'\).

Define a stack \(\Sigma\) on \(k\)-schemes as follows. Given a \(k\)-scheme \(X\), the fiber category \(\Sigma_X\) is the groupoid of triples \((\mathcal{W}, \alpha, \beta)\) where \(\mathcal{W}\) is locally free of rank \(n\) on \(X \times (D \cup D')\) and \(\alpha : \mathcal{W}|_{X \times D} \cong \mathcal{V}|_{X \times D}\) and \(\beta : \mathcal{W}|_{X \times D'} \cong \mathcal{V}'|_{X \times D'}\) are isomorphisms. It is easy to see that in fact this groupoid is discrete, i.e., \(\Sigma\) is the stack associated to a sheaf.

**Proposition 3.1.4.9.** With the above notation, there is an isomorphism

\[\Sigma \cong \text{Isom}_{D \cap D'}(\mathcal{V}|_{D \cap D'}, \mathcal{V}'|_{D \cap D'}).\]
Proof. By 3.1.4.8 and transversality (as discussed above), we have a 1-isomorphism of stacks
\[ \text{BGL}_n \times_{D \cup D'} \sim \text{BGL}_n \times_{D} \times \text{BGL}_n \times_{D} \text{BGL}_n \times_{D'} \text{BGL}_n \times_{D'} \].
This shows that the functor sending \((\mathcal{W}, \alpha, \beta)\) to \(\alpha \circ \beta^{-1}\) defines an isomorphism \(\Sigma \rightarrow \text{Isom}\). The details are left to the reader. \(\square\)

**Corollary 3.1.4.10.** Suppose \(\mathcal{V}\) and \(\mathcal{V}'\) are locally free simple sheaves of rank \(n\). The moduli space of locally free sheaves \(\mathcal{W}\) of rank \(n\) on \(D \cup D'\) such that \(\mathcal{W}|_D \cong \mathcal{V}\) and \(\mathcal{W}|_{D'} \cong \mathcal{V}'\) is isomorphic to \(\text{GL}_n^r / \text{G}_m\), where \(\text{G}_m\) is embedded along the diagonal. Moreover, this scheme is affine.

**Proof.** That the scheme is affine follows from the fact that the quotient is the complement of a hypersurface (cut out by the product of the determinants) in a projective space. Since \(\mathcal{V}\) and \(\mathcal{V}'\) are simple, it is easy to see that the moduli space \(\mathcal{M}\) parametrizing \(\mathcal{W}\) restricting to \(\mathcal{V}\) and \(\mathcal{V}'\) exists as an algebraic space. Furthermore, there is a surjection \(\Sigma \rightarrow \mathcal{M}\) which is a \(\text{G}_m\)-bundle, and in fact \(\mathcal{M}\) is identified with \(\Sigma / \text{G}_m\), where \(\text{G}_m\) acts in the natural way on the isomorphism \(\beta\). Applying 3.1.4.9 completes the proof. \(\square\)

**Corollary 3.1.4.11.** Suppose \(\mathcal{C} \rightarrow D \cup D'\) is a \(\mu^r_n\)-gerbe and \(\mathcal{V}\) and \(\mathcal{V}'\) are locally free simple twisted sheaves of rank \(n\) on \(\mathcal{C}_D\) and \(\mathcal{C}_{D'}\). The moduli space \(\mathcal{M}(\mathcal{V}, \mathcal{V}')\) of locally free twisted sheaves \(\mathcal{W}\) of rank \(n\) on \(\mathcal{C}\) such that \(\mathcal{W}|_{\mathcal{C}_D} \cong \mathcal{V}\) and \(\mathcal{W}|_{\mathcal{C}_{D'}} \cong \mathcal{V}'\) is (non-canonically) isomorphic to \(\text{GL}_n^r / \text{G}_m\).

**Proof.** This follows from 3.1.4.10 after twisting down by a \(\mathcal{C}\)-twisted invertible sheaf. \(\square\)

**Lemma 3.1.4.12.** Suppose \(D\) and \(D'\) are (not necessarily smooth) slippery elements of \(|\mathcal{O}(1)|\) which intersect transversely. Then \(D \cup D'\) is slippery.

**Proof.** In the decompositions \(\mathcal{F}_{C \times D} \cong \text{pr}^* \mathcal{M} \otimes \text{pr}^* \mathcal{F}_0\) and \(\mathcal{F}_{C \times D'} \cong \text{pr}_1^* \mathcal{M} \otimes \text{pr}_2^* \mathcal{F}'\), we claim that \(\mathcal{M} \cong \mathcal{M}'\). Indeed, let \(q \in D \cap D'\) be a point. Restricting \(\mathcal{F}\) to \(C \times \{q\}\) and using the two decompositions, we find that \(\mathcal{M} \otimes (\mathcal{F}_0 \otimes \kappa(q)) \cong \mathcal{M}' \otimes (\mathcal{F}'_0 \otimes \kappa(q))\). Both \(\mathcal{F}_0 \otimes \kappa(q)\) and \(\mathcal{F}'_0 \otimes \kappa(q)\) are non-zero finite-dimensional \(\kappa(q)\) = \(k\)-vector spaces of dimension \(r\). Thus, we conclude that both \(\mathcal{M} \otimes \mathcal{M}^{-1}\) and \(\mathcal{M}' \otimes \mathcal{M}'^{-1}\) have non-zero global sections, whence \(\mathcal{M} \cong \mathcal{M}'\). Choosing such an isomorphism and twisting down by \(\text{pr}_1^* \mathcal{M}\), there results a map from \(C\) to the moduli space \(\mathcal{M}(\mathcal{V}, \mathcal{V}')\) of 3.1.4.11, with \(\mathcal{V} = \mathcal{F}_0\) and \(\mathcal{V}' = \mathcal{F}'_0\). Since \(\mathcal{M}(\mathcal{V}, \mathcal{V}')\) is affine and \(\mathcal{C}\) is proper, the map \(\mathcal{C} \rightarrow \mathcal{M}(\mathcal{V}, \mathcal{V}')\) must be constant. Thus, as simple sheaves, \(\mathcal{M}_m\)-gerbe over moduli and \(\mathcal{C}\) is a curve over an algebraically closed field, Tsen’s theorem shows that the family \(\text{pr}_1^* \mathcal{M} \otimes \mathcal{F}_{C \times (D \cup D')}\) is constant. Thus, \(D \cup D'\) is slippery. \(\square\)

**Proof of 3.1.4.4.** Note that \(\text{Tw}^s\) is a \(\text{G}_m\)-gerbe over its moduli space \(\text{Tw}^s\). This means that any curve \(C\) in \(\text{Tw}^s\) admits a 1-morphism \(C \rightarrow \text{Tw}^s(\mathcal{F})\) lifting the inclusion \(C \hookrightarrow \text{Tw}^s\). Replacing \(C\) by the normalization of the lift of its image in \(T\), we may assume that the map \(C \rightarrow \text{Tw}^s\) is separably generated. Thus, to show that it is essentially constant, it suffices to show that the map on tangent spaces is the zero map, i.e., that the first-order deformations of any point in \(C\) induce the trivial deformation of the image point in moduli. We will do this by showing that they induce the trivial deformation on a sufficiently ample divisor. It is easy to see that given a locally free twisted sheaf \(\mathcal{G}\) on \(X\), the space of first-order infinitesimal deformations of \(\mathcal{G}\) which restrict to the trivial deformation on an effective divisor \(D\) is principal homogeneous under the kernel of the restriction map \(\text{H}^1(X, \mathcal{E}nd(\mathcal{G})) \rightarrow \text{H}^1(D, \mathcal{E}nd(\mathcal{G}))\); in the case where \(\mathcal{G}\) and \(\mathcal{G}_D\) are simple, this is precisely \(\text{H}^1(X, \mathcal{E}nd(\mathcal{G})(-D))\). Thus, if \(D\) is sufficiently ample, the deformations of \(\mathcal{G}\) inject into the deformations of \(\mathcal{G}_D\). By 3.1.4.5 and 3.1.4.12, we see that 1) \(R(D)\) holds for some \(D\) in \(\mathcal{O}(1)\), and 2) when \(R(D)\) holds for some \(D \in |\mathcal{O}(1)|\), there is an arbitrarily ample divisor \(D^{(n)} = D_1 \cup D_2 \cup \cdots \cup D_n \in |\mathcal{O}(n)|\) such that \(R'(D^{(n)})\) holds. But \(R'(D^{(n)})\) says precisely that the infinitesimal deformation of \(\mathcal{F}_{D^{(n)}}\)
induced by a tangent vector \( t \) of \( C \) is trivial. As \( D^{(n)} \) is arbitrarily ample, we see that the deformation of \( \mathcal{F} \) induced by \( t \) is also trivial. \( \square \)

3.2. Twisted sheaves on surfaces. In this section, we discuss the moduli of twisted sheaves on surfaces. In the process, we develop tools to reduce certain twisted statements to their classical counterparts. This should be viewed as a preliminary survey of a theory which is certainly amenable to significant further development. In particular, ongoing work of Langer ([39]) should help clarify the classical situation in positive characteristic (and therefore, in our view, in characteristic 0 as well), and we believe that his methods will ultimately prove useful in the twisted case.

Throughout, we focus on moduli of twisted sheaves of rank \( n \). This is technically simpler, as then determinants naturally take values in the Picard group of \( X \) itself. This is also the case one is naturally led to consider when approaching the classification of (generalized) Azumaya algebras of degree \( n \) in a Brauer class of order \( n \), which is the most natural (and naïve) thing to do on a surface. In general, if one wants to consider rank \( r \) twisted sheaves on a \( \mu_n \)-gerbe \( \mathcal{X} \), then there is a \( \mu_r \)-gerbe \( \mathcal{X}_r \) carrying them all with the same Brauer class as \( \mathcal{X} \). Moreover, the natural map \( \mathcal{X}_r \to \mathcal{X}_n \) serves to identify the stacks of semistable sheaves via pullback. Thus, we lose nothing by assuming that \( r = n \).

In order to orient the reader, we sketch the contents of this section: in 3.2.1 we discuss the discriminant and estimates for the dimension of \( \text{Tw}^{ss} \) (at stable points). In 3.2.2 and 3.2.3, we then discuss how (semi)stability behaves under restriction to curves in the surface, giving twisted forms of results of Langer [41]. We prove a Bogomolov inequality for twisted sheaves and use it to re-prove a result of Artin and de Jong bounding the second Chern class of an Azumaya algebra. Finally, in 3.2.4, we generalize O’Grady’s results on asymptotic smoothness and irreducibility to the space of twisted sheaves on an optimal gerbe (2.2.5.2). The restriction to optimal gerbes is made in order to have results which hold in all characteristics. As we discuss below, it is likely that this restriction is unnecessary, using arguments of O’Grady in characteristic 0 and recent work of Langer in positive characteristic. Unfortunately, we have not written out the proofs, so we must exclude those results from our treatment.

The reader will observe throughout this section evidence for our meta-theorem (“All phenomena which occur for moduli spaces of semistable sheaves on surfaces also occur for moduli spaces of semistable twisted sheaves”). Unlike the case of curves, the evidence in this case is purely behavioral and not attributable to any direct comparison of the twisted and untwisted situations.

3.2.1. Discriminants and dimension estimates.

Definition 3.2.1.1. Let \( X \) be a smooth projective surface and \( \mathcal{X} \to X \) a \( \mu_n \)-gerbe. Given a coherent \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) of rank \( r \), the discriminant of \( \mathcal{F} \) is the quantity

\[
\Delta(\mathcal{F}) := \deg(2rc_2(\mathcal{F}) - (r - 1)c_1(\mathcal{F})^2) \in \mathbb{Z}.
\]

Proof that \( \Delta(\mathcal{F}) \in \mathbb{Z} \). Since \( X \) is smooth and projective, 2.2.2.2 shows that \( \mathcal{F} \) has a finite global resolution by locally free twisted sheaves. A formal calculation (in \( K^0 \)) shows that

\[
\deg \text{ch}^\vee(\mathcal{F}) \text{ch}(\mathcal{F}) \text{Td}_X = \sum_{i=0}^{2} (-1)^i \text{ext}^i(\mathcal{X}, \mathcal{F}),
\]

where \( \text{ch}^\vee(\mathcal{F}) = (-1)^i \text{ch}(\mathcal{F})_i \). Another formal calculation shows that

\[
\deg \text{ch}^\vee(\mathcal{F}) \text{ch}(\mathcal{F}) = \text{rk}(\mathcal{F})^2 - \Delta(\mathcal{F}).
\]

We thus conclude that \( \chi(\mathcal{F}, \mathcal{F}) = \Delta(\mathcal{F}) - (\text{rk}(\mathcal{F})^2 - 1)\chi(\mathcal{O}_X) \). (Such “formal calculations” show at least that the Chern character and all related results – Grothendieck-Hirzebruch-Riemann-Roch, discriminant calculations, etc. – can be extended to \( K^0 \), hence to the homotopy
category of strict perfect complexes. When $X$ is projective, any perfect complex on $X$ admits a left resolution by a strict perfect complex. This can be used to show that in fact such formal calculations apply to objects of $D(X)_{\text{parf}}$, the derived category of perfect complexes.

When $\mathcal{F}$ is locally free, $\Delta(\mathcal{F}) = c_2(\mathcal{E}nd(\mathcal{F}))$. (More generally, using the remarks above, one has $\Delta(\mathcal{F}) = c_2(R\mathcal{E}nd(\mathcal{F}))$.) The discriminant plays an important role in the behavior of the moduli space.

**Lemma 3.2.1.2.** The discriminant is locally constant in flat families: given an $S$-flat family of coherent twisted sheaves $\mathcal{F}$ on $X \times S$ with $S$ connected, the number $\Delta(\mathcal{F}_s)$ is constant for all (geometric) points $s \in S$.

**Proof.** Implicit is the statement that $\Delta(\mathcal{F})$ may be computed after making any base field extension, which is clear. One easy way to see that $\Delta(\mathcal{F}_s)$ is locally constant in our case is to (locally on $S$) resolve $\mathcal{F}$ by a complex of locally free twisted sheaves $\mathcal{V}^\bullet \to \mathcal{F}$, use the fact that $\Delta(\mathcal{F}) = c_2(\text{Hom}^\bullet(\mathcal{V}^\bullet, \mathcal{V}^\bullet))$ and then use the fact that intersection products and geometric Hilbert polynomials are constant in a flat family (2.2.7.18). Another proof is based on the equality $\Delta(\mathcal{F}) = \chi(\mathcal{F}, \mathcal{F}) + (\text{rk}(\mathcal{F}))^2 - 1)\chi(\mathcal{O}_X)$ and the semicontinuity theorems for higher Ext (whose methods are demonstrated somewhat in 2.2.3.11(3)). $\square$

In fact, when the determinant is fixed, it is equivalent to specify $\Delta$, $P^g$, or $c_2$. Since we will usually fix a determinant in what follows, this means we can use any of these surrogates to divide the moduli problem into clusters of connected components.

Recall that the deformation theory of $\text{Tw}^{ss}_\mathcal{F}/k(n, P)$ at a point $[\mathcal{F}]$ is governed by the vector spaces $\text{Ext}^1(\mathcal{F}, \mathcal{F})$ and $\text{Ext}^2(\mathcal{F}, \mathcal{F})$, while the deformation theory with fixed determinant is determined by $\text{Ext}^1(\mathcal{F}, \mathcal{F})_0$ and $\text{Ext}^2(\mathcal{F}, \mathcal{F})_0$ (2.2.3), where the subscript 0 denotes traceless elements (2.2.4). We can use this to estimate the dimension of $\text{Tw}^{ss}/_k(n, L, P)$. We remind the reader of a well-known lemma, whose proof may be extracted from Schlessinger’s thesis [54] and which is written up explicitly in 2A.11 of [30].

**Lemma 3.2.1.3.** Let $k$ be a field and $F : \text{Art}_k \to$ Set a functor with a hull $R$. If the embedding dimension $\dim_k m_R/m_R^2 = d$ and $F$ has an obstruction theory with values in an $r$-dimensional vector space $\mathcal{O}$, then $d \geq \dim R \geq d - r$.

**Proposition 3.2.1.4.** Suppose $\mathcal{F}$ is a semistable $\mathcal{X}$-twisted sheaf of rank $n$, geometric Hilbert polynomial $P$, and determinant $L$. Given an algebraic stack $\mathcal{M}$ containing $\mathcal{F}$ as a point, write $\dim_\mathcal{M}$ for the dimension of the universal deformation space of $\mathcal{F}$.

1. $\text{ext}^1(\mathcal{F}, \mathcal{F}) \geq \dim_\mathcal{M} \text{Tw}^{ss}/_k(n, P) \geq \text{ext}^1(\mathcal{F}, \mathcal{F}) - \text{ext}^2(\mathcal{F}, \mathcal{F})$;
2. $\text{ext}^1(\mathcal{F}, \mathcal{F})_0 \geq \dim_\mathcal{M} \text{Tw}^{ss}/_k(n, L, P) \geq \text{ext}^1(\mathcal{F}, \mathcal{F})_0 - \text{ext}^2(\mathcal{F}, \mathcal{F})_0$.

In both cases, the moduli stack is a local complete intersection at $\mathcal{F}$ if the lower bound is achieved and formally smooth at $\mathcal{F}$ if and only if the upper bound is achieved.

**Proof.** This is an application of the results of 2.2.3 and 2.2.4 along with 3.2.1.3. $\square$

**Definition 3.2.1.5.** Given a semistable twisted sheaf of rank $n$, geometric Hilbert polynomial $P$, and determinant $L$, the expected dimension of $\text{Tw}^{ss}(n, L, P)$ at $\mathcal{F}$ is the quantity

$$\text{expdim}_\mathcal{M} \text{Tw}^s(n, L, P) := \text{ext}^1(\mathcal{F}, \mathcal{F})_0 - \text{ext}^2(\mathcal{F}, \mathcal{F})_0.$$

**Lemma 3.2.1.6.** The expected dimension at stable points is independent of the choice of $\mathcal{F} \in \text{Tw}^s(n, L, P)$ and is equal to $\Delta(\mathcal{F}) - (n^2 - 1)\chi(\mathcal{O}_X)$. The expected dimension jumps at properly semistable points. There is a constant $\beta_\infty$ such that for all points $\mathcal{F} \in \text{Tw}^{ss}/_k(n, L, P)_k$,

$$\text{expdim} \text{Tw}^s(n, L, P) \leq \dim_\mathcal{M} \text{Tw}^s(n, L, P) \leq \text{expdim} \text{Tw}^s(n, L, P) + \beta_\infty.$$
Proof. The formula for the expected dimension follows from the identity
\[ -\text{hom}(\mathcal{F}, \mathcal{F})_0 + \text{ext}^1(\mathcal{F}, \mathcal{F})_0 - \text{ext}^2(\mathcal{F}, \mathcal{F})_0 = \chi(\mathcal{O}_X) - \sum_{i=0}^{2} (-1)^i \text{ext}^i(\mathcal{F}, \mathcal{F}) \]
and formal calculations. One uses the fact that stable sheaves $\mathcal{F}$ are simple ($\text{End}(\mathcal{F}) = k$), which immediately implies that $\text{Hom}(\mathcal{F}, \mathcal{F})_0 = 0$, and the rest follows. (The given identity also uses the trace map splitting 2.2.4.5 and thus requires that the rank of $\mathcal{F}$ be relatively prime to the characteristic of $X$.) Details of this type of calculation may be found in [30, 4.5, 6.1, 8.3]. Since $\Delta(\mathcal{F})$ is determined by the determinant and Hilbert polynomial, we see that this is independent of the stable twisted sheaf $\mathcal{F}$.

The jumping of the expected dimension at properly semistable points comes from the fact that $\text{Hom}(\mathcal{F}, \mathcal{F})_0$ need not be zero. The identity above shows that

$$\expdim_\mathcal{F} \text{Tw}^ss_{\mathcal{F}/\mu}(n, L, P) - \text{Hom}(\mathcal{F}, \mathcal{F})_0$$

is constant, so the expected dimension jumps whenever there are traceless endomorphisms (i.e., infinitesimal automorphisms acting trivially on the determinant).

The last inequality follows immediately from the fact that there is a constant $\beta_\infty$ such that for all semistable twisted sheaves of rank $r$ with fixed discriminant (and no restrictions on Chern classes if char $X = 0$), $\text{ext}^2(\mathcal{F}, \mathcal{F})_0 \leq \beta_\infty$. In characteristic 0, this follows easily (using the methods of section 3.2.3) from the Lê Potier-Simpson estimate and the fact that the endomorphism sheaf of a semistable sheaf is semistable [30, 4.5.7], a fact which does not hold in positive characteristic. In general, this is slightly subtle (whence the restriction on the determinant, which is not present in characteristic 0) and will be proven in 3.2.3.7 below. \qed

3.2.2. Preparation for restriction theorems. In this section, we study the following question: given a $\mu_\mu$-gerbe $\mathcal{F} \to X$, how can one construct a finite flat cover $Y \to \mathcal{F}$ with $Y$ smooth and such that a general member of $|\mathcal{O}_X(1)|$ has smooth preimage on $Y$? Slight complications arise in positive characteristic, but this is nonetheless always possible. In the end of the section, we recall a result of Artin and de Jong which can be used to ensure that $\deg Y/\mathcal{F} = \text{ind } \mathcal{F}$.

Let $X$ be a smooth projective surface over an algebraically closed field $k$ and $P \to X$ a Brauer-Severi variety of relative dimension $n$. Note that the Brauer class of $P$ is split by $P$, hence by any subscheme of $P$. Choose a projective embedding of $P$. Let $D \subset X$ be a smooth divisor. We start with a lemma about generic hyperplane sections of a Brauer-Severi scheme, which is essentially a refinement of a special case of a lemma of Vistoli and Kresch [37].

**Lemma 3.2.2.1.** Let $P \to X$ be a surjective map of smooth projective varieties with fibers of equidimension $n$ which is generically smooth over $D$. Let $P \to \mathbb{P}^N$ be a closed immersion. A generic hyperplane section $P_H$ of $P$ has the following properties: $P_H$ is smooth and irreducible, $P_H \times_X D \subset P_H$ is an irreducible smooth divisor, $P_H \to X$ is surjective and generically smooth over $D$ with fibers of equidimension $n - 1$.

**Proof.** Let $\Xi$ be the projective space parameterizing hyperplane sections of $P$. The smoothness of the hyperplane section of $P$ and its intersection with the pre-image of $D$ defines an open subset $U \subset \Xi$. Let $d \in D(k)$ be a smooth point with smooth fiber $P_d \subset P$. The condition that a hyperplane $H \in \Xi$ intersect $P_d$ in a smooth variety of dimension $n - 1$ defines an open subset $V$ in $\Xi$. Let $W = U \cap V$. We claim that the hyperplane sections parametrized by $W$ have the properties of the lemma. Indeed, if $H \in W$, then $P_H$ and $P_H \times_X D$ are smooth and irreducible since $W \subset U$. Furthermore, the fiber of $P_H \to X$ over $d$ is smooth of dimension $n - 1$ (and hence also irreducible, incidentally) since $H \in V$. We claim that this forces $P_H \to X$ be surjective, generically smooth, with equidimensional fibers. Indeed, we have $\dim P - \dim X = n$, hence $\dim P_H - \dim X = n - 1$. If $\text{im } P_H = I$, then the usual
inequalities [49, §15] show that \( \dim P_H - \dim I \leq n - 1 \) (as \( \dim(P_H)_d = n - 1 \)). Thus, \( I = X \) and \( P_H \rightarrow X \) is surjective. Applying the identity once more shows that any closed fiber has dimension at least \( n - 1 \) at any closed point. Thus, every closed fiber is equidimensional of dimension \( n - 1 \).

\[ \textbox{Lemma 3.2.2.2.} \text{ Let } f : C \rightarrow \text{Spec} K \text{ be a normal curve over a field. If } S \subset C \text{ is a closed subscheme which is finite étale over } K \text{ and } f \text{ is smooth along } C \setminus S, \text{ then } f \text{ is smooth.} \]

\[ \text{Proof.} \text{ The scheme } C \text{ is Noetherian and reduced. Thus, to show that the sheaf } \Omega^1_{C/K} \text{ is locally free of rank 1, it suffices to show that for every point } P \in C, \text{ the } \kappa(P)-\text{vector space } \Omega^1_{C/K} \otimes_{C} \kappa(P) \text{ is 1-dimensional. For points } P \in C \setminus S, \text{ this holds by assumption. On the other hand, given a point } Q \in S, \text{ there is a canonical sequence} \]

\[ m_Q/m_Q^2 \rightarrow \Omega^1_{C/K} \otimes \kappa(Q) \rightarrow \Omega^1_{\kappa(Q)/K} \rightarrow 0. \]

Since \( Q \) is a Weil divisor on a normal separated scheme, it is a Cartier divisor and therefore the left-most term is 1-dimensional over \( \kappa(Q) \). Since \( \kappa(Q) \) is separable over \( K \), the right-most term vanishes.

\[ \textbox{Lemma 3.2.2.3. Suppose } Y \text{ is a smooth surface over an algebraically closed field } k \text{ and } D, D' \in |\mathcal{O}(1)| \text{ are very ample divisors such that } D \text{ is at worst nodal and } D \text{ and } D' \text{ intersect transversely. Then the general member of the pencil spanned by } D \text{ and } D' \text{ is smooth.} \]

\[ \text{Proof.} \text{ Write } \tilde{Y} \rightarrow \mathbf{P}^1 \text{ for the total space of the pencil. The non-smooth locus of } \tilde{Y} \rightarrow \mathbf{P}^1 \text{ has the property that it is unramified at } \mathbf{P}^1 \text{ at } [D]. \text{ Indeed, the fiber over } [D] \text{ is a nodal curve, so this follows from the standard construction of the scheme structure on the non-smooth locus using Fitting ideals [13, 2.21]. (Really, this just comes down to showing that the relative differentials of a node are supported precisely on the node with length 1.) Thus, all components of the non-smooth locus which intersect the generic fiber must be generically étale over } \mathbf{P}^1 \text{. This implies that any non-smooth points of the generic fiber have separable residue fields over } k(\mathbf{P}^1). \text{ The result follows by 3.2.2.2.} \]

An alternative (well-known) argument (rather than exploit the scheme structure of the non-smooth locus) comes from the miniversal deformation space of a node. Completing \( \tilde{Y} \) with respect to the uniformizing parameter of \( [D] \) at one of the nodes over \( [D] \) yields an effective formal deformation of the node over \( k[[t]] \) with the property that the total space is regular. On the other hand, the versal deformation of a node is isomorphic to \( k[[\xi, X_0, X_1]}/(X_0X_1 - \xi) \) parametrized by \( k[[\xi]] \). (In other words, given any family of curves \( \mathcal{C} \rightarrow S \) with a node \( c \) in a closed fiber \( \mathcal{G}_s \), there is a map \( k[[\xi]] \rightarrow \mathcal{G}_{s,c} \) such that \( \mathcal{G}_{s,c} = \mathcal{G}_{s,s} \otimes k[[\xi]] k[[\xi, X_0, X_1}]/(X_0X_1 - \xi). \) Thus, there is some map \( k[[\xi]] \rightarrow k[[t]] \) giving rise to \( \tilde{Y} \), and the condition of regularity forces \( \xi \) to map to \( ut \) where \( u \) is a unit of \( k[[t]] \). This shows that the generic fiber is smooth in the generalizations of the node. (Indeed, the compatibility properties of \( \Omega^1 \) allow us to assume that the base is \( k[[t]] \). Now the map from \( \tilde{Y}_{\text{node}} \) to its completion is regular as \( \tilde{Y} \) is excellent. Thus, the map from the generic fiber of \( \tilde{Y}_{\text{node}} \) to the generic fiber of the completion is regular. But given a regular map \( A \rightarrow B \) of Noetherian rings over a field, it follows that \( A \) is geometrically regular over the field if and only if \( B \) is geometrically regular over the field. This applies to our situation to show that \( \tilde{Y}_{\text{node}} \) is smooth over \( k((t)) \).)

\[ \textbox{Proposition 3.2.2.4. There exists a smooth subvariety } Y \subset P \text{ which is finite flat generically étale over } X \text{ such that for every } n, \text{ the pullback of a general member of } |\mathcal{O}_X(n)| \text{ to } Y \text{ is smooth.} \]

\[ \text{Proof.} \text{ By 3.2.2.1 and induction, we may carry this out for } n = 1. \text{ (Indeed, once a single smooth member pulls back to a smooth divisor, it will hold for a general smooth member. This follows from a consideration of the pullback of the incidence correspondence for } \mathcal{O}(1) \text{ on } X \text{ to } Y \text{ and} \]

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the standard results about generization of smoothness in a flat family.) Let $f : Y \to X$ be the restriction of the projection $P \to X$. We will show that once the result holds for $n = 1$, it holds for all $n$. Indeed, once it holds for $n = 1$, there is a dense open in $|\mathcal{O}_X(1)|$ of smooth members whose preimages in $Y$ are smooth. Given $n$, we may choose $n$ such general members which intersect transversely away from the branch curve of $f$. Call such a resulting nodal divisor $D_n$. Choose $D'_n \in |\mathcal{O}_X(n)|$ which is at worst nodal and intersects $D_n$ transversely away from the branch curve. Then the pencil generated by $D_n \times_X Y$ and $D'_n \times_X Y$ satisfies the conditions of 3.2.2.3, hence has smooth general member. (So does the pencil generated by $D_n$ and $D'_n$ on $X$.)

3.2.2.5. It is likely that the cover produced by 3.2.2.4 is not ideal, in the sense that the degree of the map $Y \to X$ is far too large. (We can know this “abstractly” because the proof of 3.2.2.4 is so easy.) In fact, if $P \to X$ is a Brauer-Severi variety of relative dimension $d - 1$ representing a Brauer class of index $d$, the lowest degree for the map $Y \to X$ arising in 3.2.2.4 will be $d^{d-1}$. On the other hand, we know by results of Artin and de Jong [10, §8.1] that there will be a finite flat surjection $Y' \to X$ from a smooth surface to $X$ of degree $d$. Moreover, using methods similar to those already shown above, one can actually find such a cover such that a general member of $|\mathcal{O}(1)|$ has smooth preimage on $Y'$. The interested reader can find a few more details in [46]. Given the Azumaya algebra $\mathcal{A}$ on $X$ of degree $d$ representing $P \to X$, the idea of Artin and de Jong’s construction is to let $Y'$ be determined by the “characteristic polynomial” of a general section of $\mathcal{A} \otimes L$, where $L$ is a sufficiently ample invertible sheaf on $X$. In other words, thinking of $\mathcal{A}$ as a form of $M_d(\mathcal{O}_X)$, one can see that for any invertible sheaf $L$, the reduced norm yields an algebraic map $\mathcal{A} \otimes L \to \text{Sym}^d L$ with image in the polynomial sections of degree $d$. The locus of zeros of such a polynomial function on $L^\vee$ gives a finite cover of $X$ of degree $d$, which in this case will factor through the gerbe $\mathcal{X}$. Taking general $L$ and a general section yields a smooth such cover. If, in addition to the arguments of Artin and de Jong, one pays attention to the generic branching behavior of such a cover (which may require making $L$ more ample), one gets the following statement.

**Proposition 3.2.2.6.** Given a smooth surface $X$ and a $\mu_n$-gerbe $\mathcal{X}$, if there is a locally free $\mathcal{X}$-twisted sheaf of rank $d$ then there is a finite flat surjection of smooth surfaces $Y \to X$ of degree $d$ such that

1. there exists an invertible $\mathcal{X} \times_X Y$-twisted sheaf, and
2. for every very ample invertible sheaf $\mathcal{O}(1)$ on $X$, a general member has smooth preimage in $Y$.

We will use this in the sequel to make better numerical estimates.

**Remark 3.2.2.7.** The method of 3.2.2.1 and 3.2.2.4 seems likely to generalize to higher dimensional varieties $X$. The only difficulty in the argument is in ensuring that general members of $\mathcal{O}(n)$ have smooth preimages once it is true for $n = 1$. For the applications envisioned, it is in fact sufficient that such divisors have normal preimages, which may be easier to arrange. In either case, it seems likely that a similar (more subtle) analysis of the behavior of a pencil with a fiber consisting of a divisor with sufficiently transverse crossings will yield a geometrically normal generic fiber, which is enough for applications. In other words, there would result a finite flat cover $Y \to X$ by a smooth variety such that the general member of $\mathcal{O}(n)$ has normal integral preimage.

On the other hand, the method of Artin and de Jong seems harder to generalize directly, because their construction can produce singularities in the cover in codimension 3. Nevertheless, if one is willing to allow $Y$ to be normal, it is conceivable that a refinement of their method could yield a finite flat covering with a better degree and all of the properties necessary to carry out analogues of our proofs below. This of course has the advantage of yielding a better
numerical answer, hence more effective bounds, but at the present time it is not clear if having a non-smooth cover $Y$ is compatible with the methods used here. We leave this investigation to future work(ers).

3.2.3. **Restriction theorems and the Bogomolov inequality.** Classically, Mehta and Ramanathan proved that the restriction of a slope-semistable sheaf to a general sufficiently ample divisor is again slope-semistable. An effective version (which specifies what “sufficiently” means) was first proven in characteristic 0 by Bogomolov; a recent paper of Langer [41] gives a much more general statement, valid in all characteristics. Using Langer’s results, we will give twisted versions of these theorems in this section. One of the (future) uses of these theorems is to construct the Uhlenbeck compactification of the space of twisted sheaves (and then, hopefully, the space of $\text{PGL}_n$-bundles). We also use the work of Langer to provide a twisted Bogomolov inequality, recovering earlier work of Artin and de Jong [10, §7.2] in the context of Azumaya algebras. Throughout, $X$ is a smooth projective surface over an algebraically closed field $k$.

**3.2.3.1.** We first study restriction theorems. Fix a $\mu_n$-gerbe $\mathcal{X} \to X$.

**Lemma 3.2.3.2.** Let $f : Y \to X$ be a finite separable morphism of smooth surfaces. A torsion free coherent twisted sheaf $F$ on $X$ is $\mu$-semistable if and only if $f^*\mathcal{F}$ is $\mu$-semistable.

**Proof.** This may be found in [30, 3.2.2].

**Lemma 3.2.3.3.** Let $f : Y \to X$ be a finite flat map of smooth surfaces of degree $d$, $\mathcal{O}_X(1)$ a very ample invertible sheaf on $X$, $\mathcal{X} \to X$ a $\mu_n$-gerbe, $n \in k^\times$. Write $Y = \mathcal{X} \times_X Y$. The diagram

$$
\begin{array}{ccc}
\mathbb{A}^2(\mathcal{X})_Q & \xrightarrow{f^*} & \mathbb{A}^2(\mathcal{Y})_Q \\
\text{deg} & & \text{deg} \\
\mathbb{Q} & \xrightarrow{d} & \mathbb{Q}
\end{array}
$$

commutes. In particular, given a torsion free $\mathcal{X}$-twisted sheaf $\mathcal{F}$, one has

$$
\mu_{f^*\mathcal{O}_X(1)}(f^*\mathcal{F}) = d\mu_{\mathcal{O}_X(1)}(\mathcal{F})
$$

and $\Delta(f^*\mathcal{F}) = d\Delta(\mathcal{F})$.

**Proof.** It suffices to show that the similar diagram with $X$ and $Y$ in place of $\mathcal{X}$ and $\mathcal{Y}$ commutes. That can be seen easily on the level of 0-cycles.

By 3.2.3.6, we may fix a finite map $f : Y \to X$ of smooth surfaces of degree $d = \text{ind}(\mathcal{X})$ with the property that a general member of any very ample linear system on $X$ has smooth preimage in $Y$, and such that there is an invertible twisted sheaf $\mathcal{L}$ on $Y$. (The salient feature of such a cover is that the ramification curve is generically unramified over the branch curve.) Fix a very ample linear system $\mathcal{O}_X(1)$ on $X$, with associated divisor class $H$. Following Langer [41], we choose a nef divisor $A$ on $Y$ such that $T_Y(A)$ is globally generated, and we set

$$
\beta_r = \left( \frac{r(r-1)}{p-1}AH \right)^2,
$$

where we assume that char $X = p$. This depends upon $A$, and it is slightly unfortunate that this fact is not recorded in the notation. (When char $X = 0$, set $\beta_r = 0$.) Our method has the perverse consequence that effective restriction theorems are easier to prove than generic restriction theorems.

**Proposition 3.2.3.4** (Twisted Langer). Let $\mathcal{E}$ be a torsion free $\mathcal{X}$-twisted sheaf of rank $r$. Let $D \in |kH|$ be a smooth divisor such that $\mathcal{E}_D$ is torsion free and $D \times_X Y$ is smooth.
Proof. After twisting by $L^\vee$, the pullback of $E$ to $Y$ is naturally identified with a torsion free coherent untwisted sheaf $F$, satisfying the stability conditions of (i) or (ii). Furthermore, $\Delta(F) = \text{ind}(\mathcal{X}) \Delta(E)$ and $\deg_{H}(Y) = \text{ind}(\mathcal{X}) \deg_{H}(X)$. The inequalities reduce to those of Langer's effective restriction theorems [41, 5.2 and 5.4], whence $k$ is effectively bounded, whereas by 3.2.2.6 we know that a general member of $|kH|$ will have smooth preimage in $Y$. It would be interesting to find a more effective version which does away with the (abstract) selection of a general member of the linear system in favor of a criterion depending upon the geometry of a member.

**Remark 3.2.3.5.** It is irritating to have to pay attention to $D \times_X Y$, as this makes the result quite a bit less effective. One might be tempted to see 3.2.3.4 (as we have proven it) as an “effective generic restriction theorem,” as the integer $k$ is effectively bounded, whereas by 3.2.2.6 we know that a general member of $|kH|$ will have smooth preimage in $Y$. It would be interesting to find a more effective version which does away with the (abstract) selection of a general member of the linear system in favor of a criterion depending upon the geometry of a member.

**Corollary 3.2.3.6 (Twisted Mehta-Ramanathan).** If $F$ is a torsion free $\mu$-semistable $\mathcal{X}$-twisted sheaf then the restriction of $F$ to a general sufficiently ample curve $C \subset X$ is $\mu$-semistable.

**Proof.** This is immediate from 3.2.3.4 and the properties of preimages of divisors ensured by 3.2.2.6 (or 3.2.2.4, which will just change the estimates in 3.2.3.4).

As promised in 3.2.1.6, we prove the existence of the universal constant $\beta_\infty$ such that $\text{ext}^2(F,F) \leq \beta_\infty$ for all $\mu$-semistable $F$ with rank $n$ and fixed discriminant $\Delta$. The notation grates slightly with the notation $\beta_r$ of this section, but we have chosen to retain the notation of both Huybrechts and Lehn ($\beta_\infty$) and Langer ($\beta_r$). In future sections, we will not return to the restriction theorems, so $\beta_r$ will vanish, which makes this annoyance temporary.

**Lemma 3.2.3.7.** There exists a constant $\beta_\infty$ depending only on $X, \mathcal{X}, Y, n, H$ and $\Delta$ such that for any $\mu$-semistable twisted sheaf $F$ of rank $n$ and discriminant $\Delta$, one has $\text{ext}^2(F,F) \leq \beta_\infty$.

**Proof.** It suffices to prove this after pulling back to $Y$. (Indeed, by the obvious twisted Serre duality, one can see that $\text{ext}^2(F,F) = \text{hom}(F,F \otimes \omega_X)_0$. Furthermore, $f^*\omega_X \hookrightarrow \omega_Y$, so

$$\text{Hom}(F,F \otimes \omega_X)_0 \leq \text{Hom}_Y(F_Y, F_Y \otimes f^*\omega_X)_0 \leq \text{Hom}_Y(F_Y, F_Y \otimes \omega_Y)_0$$

and we may apply Serre duality again on $Y$. Thus, we may assume that $F$ is a semistable untwisted sheaf. We can then suppress $Y$ from the notation; the dependence of $\beta_\infty$ on $Y$ only comes in the form of $\beta_r$ in the formula. Pushing the formulas given here back down to $X$ will result in multiplying each $\deg_{H}(X)$ and each $\Delta(F)$ by $\text{ind}(\mathcal{X})$.

In general, we have $\Delta(\text{end}(F)) \leq 2n^2 \Delta(F)$. Indeed, $F$ injects into its reflexive hull $F^{\vee\vee}$, yielding an injection $\text{end}(F) \hookrightarrow \text{end}(F^{\vee\vee})$. It is not hard to see that

$$\ell(\text{end}(F^{\vee\vee})/\text{end}(F)) \leq n\ell(F^{\vee\vee}/F).$$

On the other hand [30, 3.4.1], we have

$$\Delta(F) = \Delta(F^{\vee\vee}) + 2n\ell(F^{\vee\vee}/F)$$

(i) If $E$ is $\mu$-stable and

$$k > \frac{r-1}{r} \frac{\text{ind}(\mathcal{X}) \Delta(E)}{\text{deg}_{H}(X)} + \frac{1}{\text{deg}_{H}(X)} + \frac{(r-1)\beta_r}{\text{deg}_{H}(X)}$$

then $E_D$ is $\mu$-stable.

(ii) If $E$ is $\mu$-semistable and all of the Jordan-Hölder factors of $E$ have torsion free restrictions to $D$, and the inequality of (i) holds, then $E_D$ is $\mu$-semistable.
and similarly for $\mathcal{E}nd(\mathcal{F})$. Combining (1) with (2) for $\mathcal{F}$ and for $\mathcal{E}nd(\mathcal{F})$ shows that $\Delta(\mathcal{E}nd(\mathcal{F})) \leq 2n^2 \Delta(\mathcal{F})$. Now a theorem of Langer [41, 5.1] combined with the inequality $\Delta(\mathcal{E}nd(\mathcal{F})) \leq 2n^2 \Delta(\mathcal{F})$ and the fact that $\mu(\mathcal{E}nd(\mathcal{F})) = 0$ shows that

$$\mu_{\max}(\mathcal{E}nd(\mathcal{F})) \leq 2n \deg H(X) \Delta(\mathcal{F}) + \beta_r.$$  

Another theorem of Langer [40, 3.3] says (in the case of surfaces) that for any torsion free sheaf of rank $n$ on $X$,

$$h^0(X, E) \leq n \deg H(X)\left(\frac{\mu_{\max}(E)}{\deg H(X)} + f(n) + 2\right),$$

where $f(n) = -1 + \sum_{i=1}^{\infty} 1/i$. Combining this with the estimate for $\mu_{\max}(\mathcal{E}nd(\mathcal{F}))$ yields a bound for $\text{Hom}(\mathcal{F}, \mathcal{F})$. Similarly, we get a bound for $\text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)$ which differs from the first by a constant depending only on $X$. By Serre duality,

$$\text{ext}^2(\mathcal{F}, \mathcal{F})_0 = \text{hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X) - h^0(\omega_X),$$

so we are done. \qed

**Remark 3.2.3.8.** Note that bounding the discriminant does not suffice to bound the Hilbert polynomial when the determinant is not fixed. Thus, 3.2.3.7 is non-trivial. Of course, when working with a fixed determinant and therefore a bounded set of sheaves, some constant $\beta_\infty$ will exist by virtue of the boundedness and the usual semicontinuity theorems for Ext sheaves.

In characteristic 0 (or for strongly semistable sheaves in general, which we will briefly describe below), the dependence upon the discriminant disappears; it is not clear to me if this should still be true in positive characteristic.

**3.2.3.9.** We can also use the work of Langer and the coverings of 3.2.6 to produce a version of the Bogomolov inequality for twisted sheaves. After defining a notion of Frobenius pullback and strict semistability for twisted sheaves, we can use these methods to recover a Bogomolov-like inequality first proven by Artin and de Jong in the context of Azumaya algebras. This inequality will be important at one point during the study of asymptotic properties of the moduli spaces.

We begin by defining a Frobenius map which is appropriate for our situation. First, note that the (absolute) Frobenius can be defined for stacks of characteristic $p$. If $\mathcal{S} \to S$ is such a stack (with $\text{char}(S) = \{p\}$), which we may assume split as a fibered category, then the Frobenius 1-morphism $F_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ sends a 1-morphism $T \to \mathcal{S}$ to the composition

$$T \xrightarrow{F_T} T \xrightarrow{\mathcal{F}} \mathcal{S}$$

(and fixes all morphisms in fiber categories).

**Lemma 3.2.3.10.** If $\mathcal{X} \to X$ is any stack and $\chi : \mathcal{I}(\mathcal{X}) \to G_m$ is any character, then the Frobenius map $F_{\mathcal{X}}$ pulls back $\chi$-twisted sheaves to $p$-fold $\chi$-twisted sheaves. In particular, if $\mathcal{X} \to X$ is a $\mu_n$-gerbe, then the Frobenius map $F_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ pulls back $\mathcal{X}$-twisted sheaves to $p$-fold twisted sheaves.

**Proof.** Note that the map on the site of $\mathcal{X}$ induced by the Frobenius is the identity. In particular, there is a natural isomorphism

$$F_{\mathcal{X}}^*(\mathcal{I}(\mathcal{X})) \xrightarrow{\sim} \mathcal{I}(\mathcal{X})$$

(as this is true for any sheaf). It is not hard to see that the composition

$$\mathcal{I} \xrightarrow{\sim} F^* \mathcal{I} \xrightarrow{\sim} \mathcal{I}$$

is equal to the identity, where the left-hand map in the composition is the natural map 2.1.1.7. Under this identity, given any sheaf $\mathcal{F}$ on $\mathcal{X}$, the action $\mathcal{F} \times \mathcal{I} \to \mathcal{F}$ pulls back under $F_{\mathcal{X}}$ to be the same action $\mathcal{F} \times \mathcal{I} \to \mathcal{F}$. On the other hand, given any $\mathcal{O}_\mathcal{X}$-module $\mathcal{M}$, the
Given an $O$-structure on $F^*M$, the pullback of a section $s$ is given by $s_p$. The second statement of the lemma is just a restatement of the first one for readers who cleverly skipped section 2.1! \[ \square \]

**Definition 3.2.3.11.** Let $\ell$ be the order of $p$ in $(\mathbb{Z}/n\mathbb{Z})^\times$. The power $F^\ell_X$ is called the twisted Frobenius of $X$, denoted $F_X^{\tau}$. The resulting map $X \xrightarrow{F^\ell_X} X \times_{X,F^\ell_X} X$ is an isomorphism of $\mu_n$-gerbes which pulls back twisted sheaves to twisted sheaves.

**Definition 3.2.3.12.** An $X$-twisted sheaf $\mathcal{F}$ is strictly ($\mu$-) semistable if $(F^q_X)^* \mathcal{F}$ is ($\mu$-) semistable for all $q \geq 0$.

As with untwisted sheaves, it is the strictly $\mu$-semistable twisted sheaves which have the best properties.

**Proposition 3.2.3.13 (Twisted Langer-Bogomolov Inequality).** Let $\mathcal{E}$ be a torsion free $\mathcal{X}$-twisted sheaf, $Y \to X$ a cover as in 3.2.2.6 and $\beta_r$ as in 3.2.3.4.

(i) If $\mathcal{E}$ is $\mu$-semistable then $\text{ind}(\mathcal{X})^2 \Delta(\mathcal{E}) + \beta_r \geq 0$.
(ii) If $\mathcal{E}$ is strongly $\mu$-semistable then $\Delta(\mathcal{E}) \geq 0$
(iii) If $\text{rk}(\mathcal{E}) = \text{ind}(\mathcal{X})$ then $\Delta(\mathcal{E}) \geq 0$.

**Proof.** Parts (i) and (ii) follow immediately from Langer’s version of the Bogomolov inequality [41, 3.2] (which is our statement if $\text{ind}(\mathcal{X}) = 1$) and 3.2.3.2. Part (iii) follows from the fact that if $\text{rk}(\mathcal{E}) = \text{ind}(\mathcal{X})$, then $\mathcal{E}$ has no proper torsion free submodules of strictly smaller rank, so $\mathcal{E}$ is $\mu$-stable. Thus, since the rank of $\mathcal{E}$ is unchanged by Frobenius pullback, $\mathcal{E}$ is strongly $\mu$-stable and we may apply (ii). \[ \square \]

**Remark 3.2.3.14.** The fact that $n$ is prime to the characteristic figures essentially into part (iii). We see from (i) that in general there is still a lower bound for the second Chern class of any Azumaya algebra of class $[\mathcal{X}]$, depending only upon $\mathcal{X}$ (and possibly the choice of covering $Y \to X$).

**Corollary 3.2.3.15 (Artin, de Jong [10, 7.2.1]).** Let $X$ be a smooth projective surface with function field $K$, and let $A$ be an Azumaya algebra over $X$ such that $A_K$ is a division ring of degree prime to the characteristic. Then $c_2(A) \geq 0$.

**Proof.** Let $\text{deg} A = d$. There is a $\mu_d$-gerbe $\mathcal{X} \to X$ and a locally free $\mathcal{X}$-twisted sheaf $\mathcal{V}$ of rank $d$ and trivial determinant such that $\mathcal{E}\text{nd}(\mathcal{X}) = A$. It is easy to see that $c_2(A) = 2rc_2(\mathcal{V})$, so we are done by 3.2.3.13(iii). \[ \square \]

**Remark 3.2.3.16.** Artin and de Jong’s original proof of 3.2.3.15 is not very difficult, but in their approach positive characteristic and characteristic 0 are treated in completely different ways. Our method “explains” what is going on in a characteristic free manner. They must also bound the second Chern class from below by a different method before showing it is 0, while both things happen at once in our approach (which also applies to more general Azumaya algebras with possibly non-division generic points). Finally, our proof gives a reason for the failure of 3.2.3.15 when the characteristic divides the degree, namely the failure of strong stability of $\mathcal{V}$. We feel that this is another demonstration of the usefulness of working with twisted sheaves (and thus thinking sheaf-theoretically). \[ \diamondsuit \]
3.2.4. Asymptotic properties for optimal classes. In this section we study the behavior of $\text{Tw}_{/X}^{ss}(n, L, c^2)$ as $\Delta \to \infty$. We will always work with spaces of twisted sheaves with a fixed determinant. Due to inadequacies in the classical theory of semistable sheaves on surfaces in positive characteristic (currently being ameliorated by Langer), we only prove these theorems in the optimal case in all characteristics. For the arithmetic applications of [46], this is the only case that is needed.

The approach is essentially that of O’Grady, described beautifully by Huybrechts and Lehn in [30, Chapter 9]. The biggest difference between the approach here and their approach is 3.2.4.22, which is an alternative ending step in the proof of asymptotic irreducibility. Other than this, the rest of the proof is essentially identical to the classical proof. In the optimal case, certain better numerical estimates can be made, which we present here. Otherwise, we quote the book of [30] for certain proofs. While they were written in an untwisted context, they carry over verbatim (as indicated) to the twisted (arbitrary characteristic) context. I believe (but have not carefully checked) that in the non-optimal characteristic 0 case, one can carry out a similar transcription of the classical proofs. However, I have avoided dealing with $e$-stability and related numerical estimates in this work, so the reader should take this belief with a grain of salt. It is likely that the current characteristic-free work of Langer ([39]) will prove amenable to a twisted transcription.

Throughout this section, $\mathcal{X} \to X$ is an optimal $\mu_n$-gerbe with $n$ prime to the characteristic of the base field $k$. Thus, any rank $n$ torsion free twisted sheaf will be $\mu$-stable. We will continue to use the notation $\text{Tw}^{ss}$, even though in this case there are equalities $\text{Tw}(n, L, P) = \text{Tw}^{ss}(n, L, P)$. Furthermore, all of these stacks are Deligne-Mumford and are gerbes over their moduli spaces. We are therefore free to conflate their closed substacks and closed subspaces of their coarse moduli spaces; in particular, the dimension theory does not change.

We write $\text{Tw}$ for $\text{Tw}_{/X}$, etc. We will also use the notation $\text{Tw}(n, L, c)$, where $c = c^2$, rather than $\text{Tw}(n, L, P)$, where $P$ is the geometric Hilbert polynomial. (By the Riemann-Roch theorem, these are equivalent sets of data.) Finally, as we will always work with fixed rank and determinant, we will write $\text{Tw}^{ss} (\Delta)$ for $\text{Tw}^{ss}(n, L, c)$, where $\Delta$ is the discriminant.

3.2.4.1. We first outline the asymptotic properties and their proofs. The statements will be proven in 3.2.4.12 below.

**Definition 3.2.4.2.** The closed subspace in $\text{Tw}^{ss}(n, L, c)$ parametrizing non-locally free twisted sheaves is the boundary, denoted $\partial \text{Tw}^{ss}(n, L, c)$.

For any map $T \to \text{Tw}^{ss}(n, L, c)$ corresponding to a family of twisted sheaves on $T \times X$, the preimage of $\partial \text{Tw}^{ss}$ in $T$ is a closed subspace $\partial T$, which we will also call the boundary of $T$.

**Definition 3.2.4.3.** A ($\mu$-stable) point $\mathcal{F} \in \text{Tw}^{ss}$ is good if $\mathcal{F}$ is locally free and $\text{ext}^2(\mathcal{F}, \mathcal{F})_0 = 0$.

(We include the $\mu$-stability so that the reader is aware of the general definition.) In general, we will write $\beta(\mathcal{F}) = \text{ext}^2(\mathcal{F}, \mathcal{F})_0$ and $\beta(Z) = \max\{\beta(\mathcal{F}) | \mathcal{F} \in Z\}$ for a substack $Z \subset \text{Tw}^{ss}(\Delta)$. The good locus is the vanishing set for $\beta$.

**Lemma 3.2.4.4.** There is an open substack of good points $\text{Tw}^{ss}_g(\Delta) \subset \text{Tw}^{ss}(\Delta)$ which is smooth over $k$ with smooth moduli space.

**Proof.** The openness follows from the semicontinuity properties of higher Exts (see [12] and 2.2.3.11(3) for an example of the method involved in the twisted case). Smoothness of the stack is well known and comes from 3.2.1.3 (which shows that the universal deformation space of a point is formally smooth). Smoothness of the moduli space follows from the fact that $\text{Tw}^s(\Delta) \to \text{Tw}^s(\Delta)$ is a $\mu_n$-gerbe. \qed
The asymptotic properties of $\text{Tw}^{ss} (\Delta)$ come from an analysis of the substacks $\partial \text{Tw}^{ss} (\Delta)$ and $\text{Tw}^{ss}_g (\Delta)$. We can first show that sufficiently large irreducible closed substack of $\text{Tw}^{ss} (\Delta)$ must intersect $\partial \text{Tw}^{ss} (\Delta)$.

**Proposition 3.2.4.5.** There are constants $A_1$, $C_1$, and $C_2$ such that if $\Delta \geq A_1$ and if $Z$ is an irreducible closed substack of $\text{Tw}^{ss} (\Delta)$ such that

$$\dim Z > \left(1 - \frac{1}{n + 2}\right) \Delta + C_1 \sqrt{\Delta} + C_2$$

then $\partial Z \neq \emptyset$.

Using 3.2.4.5, we will then show that as $\Delta$ grows, so does the codimension of the complement of $\text{Tw}^{ss}_g (\Delta)$. More precisely, we have the following. Let $W = \text{Tw}^{ss} (\Delta) \setminus \text{Tw}^{ss}_g (\Delta)$ (as a reduced closed substack).

**Proposition 3.2.4.6.** There is a constant $C_3 \geq C_2$ and a constant $A_2 \geq A_1$ such that for all $\Delta \geq A_2$,

$$\dim W \leq \left(1 - \frac{1}{2n}\right) \Delta + C_1 \sqrt{\Delta} + C_3.$$ 

Thus, the stack will asymptotically become smooth in codimension 1 and everywhere l.c.i. of the expected dimension, hence normal.

**Proposition 3.2.4.7.** Suppose $\Delta$ satisfies

1. $\Delta > A_1$
2. $\Delta - (n^2 - 1) \chi(\mathcal{O}_X) \geq \left(1 - \frac{1}{2n}\right) \Delta + C_1 \sqrt{\Delta} + C_3 + 2.$

Then every irreducible component of $\text{Tw}^{ss} (\Delta)$ intersects $\text{Tw}^{ss}_g (\Delta)$. In particular, it is generically smooth of the expected dimension. Furthermore, $\text{Tw}^{ss} (\Delta)$ is normal and a local complete intersection.

**Proof.** The two properties and the fact that $\text{expdim} \text{Tw}^{ss} (\Delta) = \Delta - (n^2 - 1) \chi(\mathcal{O}_X)$ (at any point, hence on any irreducible component) shows that the locus of good points $\text{Tw}^{ss}_g (\Delta)$ is dense in every component of $\text{Tw}^{ss} (\Delta)$. When $\text{ext}^2 (\mathcal{F}, \mathcal{F})_0 = 0$, one then has

$$\dim \text{Tw}^{ss}_g (\Delta) = \text{ext}^1 (\mathcal{F}, \mathcal{F})_0 = \text{expdim} \text{Tw}^{ss}_g (\Delta),$$

so the stack $\text{Tw}^{ss} (\Delta)$ is generically smooth of the expected irreducible component, hence at every point. This implies by 3.2.1.3 that $\text{Tw}^{ss} (\Delta)$ is a local complete intersection. Furthermore, by condition (2) and 3.2.4.6, $\text{Tw}^{ss} (\Delta)$ is regular in codimension 1. By Serre’s theorem, $\text{Tw}^{ss} (\Delta)$ is normal. \qed

Another use of 3.2.4.5 is in proving that $\text{Tw}^{ss} (\Delta)$ is irreducible for sufficiently large $\Delta$. Suppose $\mathcal{F} \in \text{Tw}^{ss} (\Delta)$ is good. This implies that $\mathcal{F}$ lies on a unique irreducible component of $\text{Tw}^{ss} (\Delta)$. Any subsheaf $\mathcal{F}' \subset \mathcal{F}$ of finite colength $\ell$ (i.e., such that the quotient $\mathcal{F} / \mathcal{F}'$ has finite length $\ell$) must also be good. Indeed, by twisted Serre duality (which is derived from the usual Grothendieck duality for the complex $\mathbf{R} \mathcal{H} \text{Hom}(\mathcal{F}', \mathcal{F})$ on $X$) and compatibility with trace, $\text{ext}^2 (\mathcal{F}', \mathcal{F}')_0 = \text{hom}(\mathcal{F}', \mathcal{F}' \otimes \omega_X)_0$, and similarly for $\mathcal{F}$. Furthermore, taking the reflexive hull gives a natural injection $\text{Hom}(\mathcal{F}', \mathcal{F}' \otimes \omega_X)_0 \hookrightarrow \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \omega_X)_0$.

**Lemma 3.2.4.8.** $\Delta (\mathcal{F}') = \Delta (\mathcal{F}) + 2n \ell$.

**Proof.** This reduces to showing that $c_2 (\mathcal{F} / \mathcal{F}') = \ell$, which itself reduces to showing that a twisted sheaf $\mathcal{F}$ of length 1 has $c_2 (\mathcal{F}) = 1$. This follows from the twisted Hirzebruch-Riemann-Roch theorem 2.2.7.5 applied to the inclusion of $\text{Supp} (\mathcal{F})$ in $\mathcal{F}$, along with a trivial calculation when $\mathcal{F}$ is a $\mu_n$-gerbe over a geometric point. \qed
Thus, \( \mathcal{F}' \) lies on a unique irreducible component of \( \mathbf{Tw}^{ss}(\Delta + 2n\ell) \). It is trivial that every locally free twisted sheaf \( \mathcal{F} \) contains a colength 1 subsheaf \( \mathcal{F}_1 \). Let \( \Lambda_\Delta \) denote the set of irreducible components of \( \mathbf{Tw}^{ss}(\Delta) \).

**Lemma 3.2.4.9.** Suppose \( \Delta \) satisfies the conditions of 3.2.4.7. The map sending a good twisted sheaf \( \mathcal{F} \) to \( \mathcal{F}_1 \) yields a well-defined map \( \varphi : \Lambda_\Delta \to \Lambda_{\Delta+2n} \).

**Proof.** It follows from 2.2.7.24 that the irreducible component containing \( \mathcal{F}_1 \) is independent of the choice of \( \mathcal{F}_1 \). \( \Box \)

The idea behind the proof of irreducibility of \( \mathbf{Tw}^{ss}(\Delta) \) for large \( \Delta \) is to show that \( \varphi \) is eventually surjective, and that any two points are eventually brought together under an iterate of \( \varphi \).

**Proposition 3.2.4.10.** There is a constant \( A_3 \) such that for all \( \Delta \geq A_3 \), the following hold.

1. Every irreducible component of \( \mathbf{Tw}^{ss}(\Delta) \) contains a locally free good twisted sheaf.
2. Every irreducible component of \( \mathbf{Tw}^{ss}(\Delta) \) contains a point \( \mathcal{F} \) such that both \( \mathcal{F} \) and \( \mathcal{F}^\vee \) are good and \( \ell(\mathcal{F}^\vee/\mathcal{F}) = 1 \).

**Theorem 3.2.4.11.** There is a constant \( A_4 \) so that for all \( \Delta \geq A_4 \), the stack \( \mathbf{Tw}^{ss}(\Delta) \) is irreducible.

**Proof.** By 3.2.4.10(2), for \( \Delta \geq A_3 \) the map \( \varphi : \Lambda_{\Delta-2n} \to \Lambda_\Delta \) is surjective. We wish to show that this implies that \( \Lambda_\Delta \) is eventually a singleton. In the twisted case, there is a slight wrinkle, as \( c_2 \) need not be an integer. Thus, not all discriminants are congruent modulo \( 2n \). However, we do know that \( \Delta \) is always an integer. Consider the sequences of surjections

\[
\begin{array}{ccccccc}
\Lambda_\Delta & \longrightarrow & \Lambda_{\Delta+2n} & \longrightarrow & \Lambda_{\Delta+4n} & \longrightarrow & \cdots \\
\Lambda_{\Delta+1} & \longrightarrow & \Lambda_{\Delta+1+2n} & \longrightarrow & \Lambda_{\Delta+1+4n} & \longrightarrow & \cdots \\
& & \vdots & & \vdots & & \\
\Lambda_{\Delta+2n-1} & \longrightarrow & \Lambda_{\Delta+2n-1+2n} & \longrightarrow & \Lambda_{\Delta+2n-1+4n} & \longrightarrow & \cdots \\
\end{array}
\]

For any sufficiently large discriminant \( \Delta' \), one of the sequences above will contain \( \Lambda_{\Delta'} \). If we show that any two components in the first set of the sequence eventually map to the same point, then we see that each sequence is eventually singletons, and hence that any \( \Lambda_{\Delta'} \) is eventually a singleton (for large enough \( \Delta' \)).

We claim that it is enough to show that given locally free \( \mathcal{V} \) and \( \mathcal{W} \) of rank \( n \) with the same determinant and discriminant, there are finite colength subsheaves \( \mathcal{V}' \subset \mathcal{V} \) and \( \mathcal{W}' \subset \mathcal{W} \) and an irreducible flat family containing both \( \mathcal{V}' \) and \( \mathcal{W}' \). This is not obviously the same as making colength 1 subsheaves of locally free good sheaves in each stage. To see that these are the same, note that the irreducibility of the twisted Quot scheme shows that we may assume that the supports of \( \mathcal{V}/\mathcal{V}' \) and \( \mathcal{W}/\mathcal{W}' \) are finite sets of distinct reduced points. Now suppose given a family of twisted sheaves \( \mathcal{F} \) on \( X \times S \). The \( S \)-scheme of quotients of \( \mathcal{F} \) of length \( \ell \) with supports distinct reduced points disjoint from the singular locus of \( \mathcal{F} \) in each fiber is easily seen to be irreducible when \( S \) is irreducible (see e.g., the proof of 2.2.7.24). Thus, if \( S \) is irreducible, so is this scheme of quotients. So as we let a point move in it, it will end up in the same irreducible component of \( \mathbf{Tw}^{ss}(\Delta + 2n) \). Since at each stage 3.2.4.10 implies that each
successive quotient may be irreducibly connected to a locally free sheaf, we see that \( V' \rightarrow V' \) is the \( \ell(V/V') \)th iterate of \( \varphi \).

We will prove the existence of \( V' \) and \( W' \) below in 3.2.4.19.

3.2.4.12. We now prove everything! First comes 3.2.4.5.

**Lemma 3.2.4.13.** Let \( C \in \mathcal{O}(N) \) be a smooth member (for any \( N \)) and let \( \mathcal{C} = \mathcal{X} \times_X C \). Let \( Z \subset \text{Tw}^{ss}(\Delta) \) be a closed irreducible substack with \( \partial Z = \emptyset \). If \( \dim Z > \dim \text{Tw}^{ss}_{\ell/k}(n, \mathcal{O}_C) \) then there is a point of \( Z \) parametrizing an \( \mathcal{X} \)-twisted sheaf \( \mathcal{F} \) whose restriction to \( C \) is unstable.

**Proof.** By 3.1.4.4, we see that if it is defined the restriction map \( Z \rightarrow \text{Tw}^{ss}_{\ell/k}(n, \mathcal{O}_C) \) is finite. Thus, if every restriction of a point of \( Z \) to \( C \) is stable, we see that \( \dim Z \leq \dim \text{Tw}^{ss}_{\ell/k}(n, \mathcal{O}_C) \).

**Proposition 3.2.4.14.** Let \( Z \subset \text{Tw}^{ss}(\Delta) \) be a closed irreducible substack. Let \( C \in \mathcal{O}(N) \) be smooth. Suppose \( Z \) contains a point \( [\mathcal{F}] \) such that \( \mathcal{F}|_C \) is unstable. If \( \dim Z > \expdim \text{Tw}^{ss}(\Delta) + \beta_\infty + \chi(\mathcal{O}_X) - \frac{n-1}{2} C(C - K) \) then \( \partial Z \neq \emptyset \).

**Proof.** This may be copied almost verbatim from [30, 9.5.4], but omit the part about \( e \)-stability.

**Proof of 3.2.4.5.** This is an application of 3.2.4.13 and 3.2.4.14. Indeed, these show that if \( Z \) is an irreducible component such that

\[
\dim Z > \dim \text{Tw}^{ss}_{\ell/k}(n, L_C) = \frac{n^2-1}{2} (N^2 H^2 + NK H)
\]

and

\[
\dim Z > \Delta - (n^2 - 1) \chi(\mathcal{O}_X) + \beta_\infty + \frac{n^2 - 1}{2} C(C - K)
\]

then \( \partial Z \neq \emptyset \). We seek a function of \( \Delta \) which is greater than both right-hand sides for large \( \Delta \) (and some choice of \( N \)) but which is smaller than \( \Delta - (n^2 - 1) \chi(\mathcal{O}_X) \) by an amount which grows without bound as \( \Delta \) increases. (The second condition becomes necessary when trying to make the codimension of \( W \) high.) For the purposes of the present work, we do not make any attempt to be especially effective; this will make things easier. Letting \( N \sim c\sqrt{\Delta} \) and examining the resulting inequalities for that value of \( N \) leads one to choose \( c \) with

\[
c^2 < \frac{2}{(n+1)(n-1)H^2}
\]

to ensure that the “leading term” (coefficient of \( \Delta \)) of the top line is larger than that of the bottom line and less than \( \Delta \). As we let \( \Delta \) grow, this will eventually produce positive integers for \( N \), and working through the arithmetic shows that there will be a function \( f(\Delta) = C_1 \sqrt{\Delta} + C_2 \) such that for \( N \sim C_1 \sqrt{\Delta} \), the inequalities are satisfied and \( f(\Delta) < \Delta - (n^2 - 1) \chi(\mathcal{O}_X) \). Then any \( Z \) with \( \dim Z > f(\Delta) \) will satisfy both 3.2.4.13 and 3.2.4.14 and have dimension strictly smaller than the expected dimension of \( \text{Tw}^{ss}(\Delta) \). For a similar argument, see [30, pp. 209-210].

3.2.4.15. Next come 3.2.4.6 and 3.2.4.10. We begin with some preparatory lemmas.

**Lemma 3.2.4.16.** If \( \partial \text{Tw}^{ss}(\Delta) \neq \emptyset \) then \( \text{codim}(\partial \text{Tw}^{ss}(\Delta), \text{Tw}^{ss}(\Delta)) \leq n - 1 \).
Proof. The statement is local on the stack. Locally on $\text{Tw}^{ss}$, one may choose a locally free resolution of the universal object on $\text{Tw}^{ss}(\Delta) \times X$ by two sheaves $\varphi : L_1 \to L_0 \to \mathcal{F}_{\text{univ}}$ (as surfaces have homological dimension 2). The result follows from studying the locus where the rank of $\varphi$ drops, which is known from standard theorems about determinantal schemes. See [30, 9.2.2] for more details. Note that while the reference given for determinantal loci is written over $\mathbb{C}$, the estimates are independent of the characteristic.

We need one more lemma, which is well known.

**Lemma 3.2.4.17.** If $\mathcal{F}$ is an $S$-flat family of torsion free twisted sheaves then the function $s \mapsto \ell(\mathcal{F}_s^{\vee\vee}/\mathcal{F}_s)$ is upper semicontinuous. If $S$ is reduced and the function is constant than the formation of the reflexive hull commutes with base change and $\mathcal{F}^{\vee\vee}$ is locally free.

**Proof.** See e.g. [30, 9.6.1]. One uses the fact that a surface has homological dimension 2 and that there are locally free resolutions (which is true in the twisted setting as well).

**Definition 3.2.4.18.** The double-dual stratification of $\text{Tw}^{ss}(\Delta)$ is given by subsets

$$\text{Tw}^{ss}(\Delta)_\nu = \{ \mathcal{F} | \ell(\mathcal{F}^{\vee\vee}/\mathcal{F}) \geq \nu \}.$$

These are closed subsets by 3.2.4.17. For any family of torsion free twisted sheaves over $S$, there is an induced stratification $S_{\nu}$ by pullback along the classifying map $S \to \text{Tw}^{ss}(\Delta)$.

The most important fact about this stratification is that formation of the double dual induces a map

$$\partial \text{Tw}^{ss}(\Delta)_\nu \setminus \partial \text{Tw}^{ss}(\Delta)_{\nu+1}\text{\red} \to \text{Tw}^{ss}_{lf}(\Delta - 2n\ell).$$

The fiber over a (locally free) point $\mathcal{F}$ is just (set-theoretically, at least) $\text{Quot}(\mathcal{F}, \ell)$. Let $Z \subset \text{Tw}^{ss}(\Delta)$ be a closed irreducible subspace with $\partial Z \neq \emptyset$ and $\beta(Z) > 0$. Following section 9.6 of [30], we define a sequence of triples

$$Y_i \subset Z_i \subset \text{Tw}^{ss}(\Delta_i)$$

as follows: $\Delta_0 = \Delta$, $Z_0 = Z$, and $Y_i \subset \partial Z_i$ is an irreducible component of the maximal open stratum of the double-dual stratification of $\partial Z$. If $\ell$ is the constant co-length on this stratum, then, as we just remarked, there is an induced map $Y_i \to \text{Tw}^{ss}(\Delta_i - 2n\ell)$. Set $\Delta_{i+1} = \Delta_i - 2n\ell$ and $Z_{i+1}$ equal to the closure of the image of $Y_i$. There is some index $m$ such that $\partial Z_m = \emptyset$ by the twisted Langer-Bogomolov inequality $\Delta \geq 0$ 3.2.3.13(iii) (which applies since $\ind(\mathcal{F}) = n$).

Using 2.2.7.24 and 3.2.4.16, one finds $\dim Z_i \geq \dim Y_{i-1} - \ell_i(n+1)$ and $\dim Y_{i-1} \geq \dim Z_{i-1} - (n - 1)$, whence $\dim Z_i \geq \dim Z_{i-1} - (2n - 1)\ell_i - 1$. A careful analysis of when equality can hold between $\dim Z_i$ and $\dim Z_{i-1} - (2n - 1)\ell_i - 1$ (which may be found in [30, pp. 211-212]) yields an inequality

$$\dim Z_m - \left(1 - \frac{1}{2n}\right)\Delta_m \geq \dim Z - \left(1 - \frac{1}{2n}\right)\Delta - \beta_\infty.$$

It is now clear what is going to happen: if $\dim Z$ is too large, then $\dim Z_m$ is too large, i.e., satisfies 3.2.4.5, contradicting the fact that $\partial Z_m = \emptyset$. The numerical details may be found in [30, p. 212-213], where it is shown that

$$C_3 := \max\{C_2 + \beta_\infty, A_1/2n + 2\beta_\infty - (n^2 - 1)\chi(\mathcal{O}_X)\}$$

works in the statement of 3.2.4.6.

Finally, the proof of 3.2.4.10 may be copied verbatim from [30, p. 213].

**3.2.4.19.** As promised in 3.2.4.11, we show that given two (good) locally free twisted sheaves $\mathcal{V}$ and $\mathcal{W}$ with the same rank, determinant, and discriminant, there are finite co-length subsheaves $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{W}' \subset \mathcal{W}$ which belong to a common irreducible family of (good) twisted sheaves.
Lemma 3.2.4.20. A general map \( \mathcal{V} \to \mathcal{W}(N) \) is injective with cokernel supported on a divisor where it has rank 1 in every fiber.

Proof. This is a Bertini type theorem. Over any field, the space of \( n \times n \)-matrices which have rank at most \( n - 1 \) is a divisor in \( M_n(k) \) with singular locus of codimension 3 (in the divisor) given by matrices of rank at most \( n - 2 \). Thus, the cone of matrices of rank at most \( n - 2 \) has codimension 4 in each fiber, and a standard argument shows that on a surface a generic section (for \( N \) large enough that \( \mathcal{H}om(\mathcal{V}, \mathcal{W}(N)) \) is globally generated) will avoid this locus. As the rank drops on a divisor, we are done. \( \square \)

Corollary 3.2.4.21. A general map \( \mathcal{V} \to \mathcal{W}(N) \) is injective with cokernel an invertible twisted sheaf supported on a smooth curve in \( |\mathcal{O}(nN)| \).

Proof. This involves a similar Bertini argument with the second jet bundle of a matrix algebra. At a point \( p \) with local coordinates \( x \) and \( y \), an element of the fiber of this bundle is a matrix \( M_0 + xM_1 + yM_2 \). Taking the determinant yields a function \( f_0 + xf_1 + yf_2 \) (as \( x^2 = y^2 = xy = 0 \) in the jet bundle). In order for the determinant to vanish to order at least 2 at the point, all three functions \( f_i \) must vanish. This defines a “forbidden cone” of codimension 3 in every fiber (see [10, 8.1.1.6] for a verification that these conditions are independent), which is greater than the dimension of \( X \). The usual argument shows that once the jet bundle is globally generated, a general section will miss the forbidden cone in each fiber. \( \square \)

Proposition 3.2.4.22. Let \( \mathcal{V} \) and \( \mathcal{W} \) be two locally free twisted sheaves of rank \( n \) with the same determinant and discriminant. Then there exist torsion free twisted sheaves and finite colength inclusions \( \mathcal{V}' \subseteq \mathcal{V} \) and \( \mathcal{W}' \subseteq \mathcal{W} \) (of the same colength) and an irreducible flat family of twisted sheaves containing \( \mathcal{V}' \) and \( \mathcal{W}' \). If \( \mathcal{V} \) and \( \mathcal{W} \) are both (good) \((\mu-)\)(semi)stable, then there exists an irreducible family consisting of (good) \((\mu-)\)(semi)stable sheaves.

Proof. For \( N \) sufficiently large, there are extensions
\[
0 \to \mathcal{V}(-N) \to \mathcal{V} \to \mathcal{P} \to 0
\]
and
\[
0 \to \mathcal{V}(-N) \to \mathcal{W} \to \mathcal{Z} \to 0,
\]
where \( \mathcal{P} \) and \( \mathcal{Z} \) are invertible twisted sheaves on smooth curves in the linear system \( |\mathcal{O}(nN)| \). Furthermore, \( \mathcal{P} \) and \( \mathcal{Z} \) have the same geometric Hilbert polynomial. By 3.1.3.2, there is an irreducible variety \( S \) (which we may assume is affine) and an \( S \)-flat family of twisted sheaves \( \mathcal{D} \) on \( X \times S \) supported on an \( S \)-flat Cartier divisor which interpolates between \( \mathcal{P} \) (the fiber over \( s_0 \in S(k) \)) and \( \mathcal{Z} \) (the fiber over \( s_1 \in S(k) \)). The idea is to make \( \mathcal{V}' \) and \( \mathcal{W}' \) by taking the inverse image of finite colength subsheaves \( \mathcal{P}(-m) \) and \( \mathcal{Z}(-m) \) for \( m \) sufficiently large that we can connect torsion free extensions over this family.

We will use the usual Grauert semicontinuity results for Ext spaces to make a connected family interpolating between finite colength subsheaves of \( \mathcal{P} \) and \( \mathcal{Z} \). We can do this explicitly quite easily as follows. Let \( \mathcal{F}^* \to \mathcal{D} \) be a finite resolution by a complex of locally free twisted sheaves. (In fact, it will have length at most 2!) Twisting \( \mathcal{F}^* \) by a very negative power of \( \mathcal{O}(1) \), we see that the perfect complex \( \mathcal{E} = \mathcal{H}om^*(\mathcal{F}^*(-m), \mathcal{V}(-N)) \) on \( S \) universally computes relative Ext spaces. In other words, for any \( T \to S \), \( \mathcal{H}^i(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{T}) \cong \operatorname{Ext}^i_{\mathcal{O}_{X \times T}}(\mathcal{D}(-m)X \times S T, \mathcal{V}(-N)_{X \times S T}) \). Moreover, for large enough \( m \), it is the case that the function
\[
s \in S \mapsto \dim \mathcal{H}^0(\mathcal{E} \otimes \kappa(s))
\]
is constant and the function
\[
s \in S \mapsto \dim \mathcal{H}^2(\mathcal{E} \otimes \kappa(s))
\]
is the 0 function (by Serre duality). Standard methods (see [28, III.12] for example) now show that $H^1(\mathcal{O})$ is a locally free sheaf and that for all $f : T \to S$ the natural map $f^*H^1(\mathcal{O}) \to H^1(\mathcal{O} \otimes T)$ is an isomorphism. Let $V \to S$ be the vector bundle whose sections are $H^1(\mathcal{O})$. Then we have shown that $V$ represents the functor $T \to S \mapsto \text{Ext}^1(\mathcal{O}(-m), \mathcal{Y}(-N))$ (for sufficiently large $m$). (In fact, we could have easily shown that $H^2(\mathcal{O} \otimes T)$ is universally 0 to begin with, by a trivial homological dimension calculation, but the method here generalizes slightly to higher dimensional ambient varieties.) As such, there is a universal extension

$$0 \to \mathcal{Y}(-N)V \times X \to \mathcal{E} \to \mathcal{D}(-m)V \times X \to 0.$$ 

Furthermore, once the existence of a vector bundle representing $\text{Ext}^1(\mathcal{D}(-m), \mathcal{Y}(-N))$ is true for $m$, it will be true for all $m' > m$. Thus, to get $V$ to have nice properties, we can keep enlarging $m$.

We claim that for sufficiently large $m$, given any $s \in S$ there is a non-empty open subset $U_s \subset V_s$ parametrizing torsion free extensions. It is enough to prove that there is a single torsion free extension by the openness of purity in families. Furthermore, the existence of such a point is stable under increases of $m$: if $\mathcal{E}$ is torsion free element of $\text{Ext}^1(\mathcal{D}(-m), \mathcal{Y}(-N))$, then the preimage of $\mathcal{D}_s(-m - m_0)$ in $\mathcal{E}$ gives a torsion free element in $\text{Ext}^1(\mathcal{D}_s(-m - m_0), \mathcal{Y}(-N))$. Let $s$ be a point of $S$, so that we are considering extensions $\text{Ext}_X^1(\mathcal{D}(-m), \mathcal{Y}(-N))$. We are reduced to proving that if $m$ is large enough, there is a point of this space representing a torsion free twisted sheaf. Let $\mathcal{E}$ be any extension with torsion subsheaf $T(\mathcal{E})$. Since $\mathcal{Y}(-N)$ is torsion free, the intersection $\mathcal{Y}(-N) \cap T(\mathcal{E}) = 0$, so $T(\mathcal{E}) \hookrightarrow \mathcal{D}_s(-m)$. Now consider the situation generically. Over the local ring at the generic point of $\text{Supp} \mathcal{D}$ there is certainly a torsion free extension, so over the complement $W := X \setminus D$ of some sufficiently ample hyperplane section $D \in |\mathcal{O}(m)|$ there is a torsion free extension

$$0 \to \mathcal{Y}(-N)W \to \mathcal{E}_W \to \mathcal{D}_s(-m)|_W \to 0.$$ 

Since $W$ is affine, this may be realized as a map $\varphi_W : \mathcal{F}^{-1}(-m)|_W \to \mathcal{Y}(-N)|_W$ whose composition $\mathcal{F}^{-2}(-m)|_W \to \mathcal{F}^{-1}(-m)|_W \to \mathcal{Y}(-N)|_W$ is 0. Twisting by a high power of $D$ we find an extension $\varphi : \mathcal{F}^{-1}(-m - cm_0) \to \mathcal{Y}(-N)$ of $\varphi_W$. It is immediate that $\varphi$ satisfies the cocycle condition, hence gives rise to an extension $\mathcal{E}$ which restricts to $\mathcal{E}_W$ on $W$. Since $W$ contains the generic point of $C$ and $\mathcal{D}_s(-m - cm_0)$ is torsion free, the inclusion $T(\mathcal{E}) \hookrightarrow \mathcal{D}_s(-m - cm_0)$ implies that $T(\mathcal{E}) = 0$.

Therefore, by the openness of the torsion free locus and Noetherian induction, we may choose a large $m$ so that the torsion free locus of every fiber of $V \to S$ is open and dense. This implies that the locus $U \subset V$ parametrizing torsion free sheaves is irreducible.

Now consider the original points $s_0$ and $s_1$ over which lie $\mathcal{D}$ and $\mathcal{D}$. Choosing a section of $\mathcal{O}(m)$, yields an injection $\mathcal{D}(-m) \hookrightarrow \mathcal{D}$, and taking preimages of $\mathcal{D}(-m)$ and $\mathcal{D}(-m)$, we find finite colength subsheaves $\mathcal{Y}' \subset \mathcal{Y}$ and $\mathcal{W}' \subset \mathcal{W}$ parametrized by points of $U$, hence lying in an irreducible family of torsion free twisted sheaves. If $\mathcal{Y}$ and $\mathcal{W}$ are (good) $(\mu)$-(semi)stable, then the same is true of $\mathcal{Y}'$ and $\mathcal{W}'$, and we are done by the openness of these loci in families and irreducibility. 

\begin{remark}
This is the key step to proving that the stack of semistable twisted sheaves is asymptotically irreducible for non-optimal classes as well. Our proof is sufficiently general to work in the general (non-optimal) case. However, some of the other foundations (notably a study of $c$-stability) cannot be carried out in positive characteristic yet. The general characteristic 0 case is likely to work precisely as it does in the classical case, but we have not yet checked the details. 
\end{remark}

\begin{remark}
We can give a relative version of all of the constructions here. Stack-theoretically, this extension is trivial. The GIT construction of Simpson also gives a good global projective
corepresenting scheme (although in positive characteristic it is no longer clear whether or not this is universal on the base). In the case of an optimal class, all points will be stable, so $\text{Tw}^s_{X/S}$ is a gerbe over a projective scheme, which shows that in this case the formation of the coarse moduli space is universal on $S$. (Universal means “compatible with all base change.” It is always true that the GIT quotient is uniform, which means that it is compatible with flat base change.)

**Proposition 3.2.4.25.** Let $\mathcal{X} \to X \to S$ be a $\mu_n$-gerbe on a smooth proper morphism of finite presentation of algebraic spaces with geometrically connected fibers of dimension 2, and assume that $n$ is invertible on $S$. Suppose $\mathcal{X}$ has optimal geometric fibers. The stack $\text{Tw}^s_{X/S}(\Delta) \to S$ is a proper flat local complete intersection morphism for large $\Delta$.

**Proof.** (This result can also be finagled when the fibers are either geometrically optimal or geometrically essentially trivial, and is likely to hold completely generally in characteristic 0 by a simple extension of our methods. As above, the general positive characteristic case is still in progress.) This follows from 3.2.4.7 which shows that $\text{Tw}^s_{X/S}(\Delta)$ is l.c.i. over $S$ (by 3.2.1.3), combined with the local criterion of flatness. □

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