Multi-Boson Correlation Sampling with Multi-mode Thermal Sources

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The determination of the computational complexity of the boson sampling problem with single boson sources [1–3] has opened a novel research direction in the quantum computing field [4–7]. Some research effort has also been devoted towards the use of different input sources where the sampling from the output distribution of a linear interferometer is still a complex task not achievable by a classical device [8–11]. In this letter, we discuss the use of multi-mode thermal sources, based on a generalization of the boson sampling problem which relies on time-correlation measurements. We derive, for any possible sample, two equivalent formulations for the respective multi-boson correlation functions in terms of matrix permanents. The properties of these permanents emerging from the physics of multi-boson interference raise essential questions in the field of complexity theory. Our results are a manifestation of the physics behind correlated measurements in either photonic or atomic interferometers with bosonic thermal sources. This fascinating physics paves the way towards important applications not only in quantum computing but also in quantum metrology and quantum communication, by taking advantage of different kinds of bosonic sources.

I. MOTIVATION

Recently, the so-called boson sampling problem (BSP) [1–3] has drawn a lot of attention in the research community. This problem consists of sampling the probability distribution associated with \( N \) bosons at the output of a linear \( M \)-port interferometer, where the condition \( M \gg N \) ensures that the probability for the bunching of bosons is reasonably small [1]. The BSP is hard to solve with a classical computer since it involves an exponential number of possible samples, each connected with the permanent of a random complex matrix. Indeed, the calculation of these permanents is a \#P-problem [12] and therefore not efficiently computable with a classical computer. Several proof-of-principle demonstrations have been realized towards the solution of the BSP [4–7]. However, the difficulty of the production of a high number of single bosons, e.g. photons [13], currently prevents the implementation of boson sampling networks hard to simulate with a classical computer. Therefore, it is worthwhile to analyze the BSP for different types of sources [8–11].

In this letter, we discuss the use of independent multimode thermal sources at the input of a linear interferometer with \( M \) ports. Moreover, we analyze the joint probability rate associated with the correlated detection of \( N \) bosons in a given sample of \( N \) output ports, independently of the remaining ports. Here, differently from the original formulation of the BSP, the sampling problem relies on the times at which the bosons are detected. This leads to the more general problem of multi-boson correlation sampling (MBCSP) [14, 15].

After giving a general definition of the MBCSP in section II, we perform a full analysis for the case of thermal sources in section III. In sections III A 1 and III A 2, we derive two equivalent, yet interestingly different formulations of the \( N \)-order correlation functions in terms of matrix permanents depending on the interferometer evolution. Finally, we analyze the sampling over approximately equal detection times of the correlation measurements in section III B, address the trivial case of thermal sources with equal average boson numbers in section III C and conclude with final remarks in section IV.

II. THE MULTI-BOSON CORRELATION SAMPLING PROBLEM (MBCSP)

The formulation of the MBCSP of a given order \( N \) is the following (see Fig. 1): First, we prepare a random linear interferometer with \( M \gg N \) ports and at least \( N \) independent sources; secondly, using time-resolving detectors, we sample from all possible correlated detection events in which \( N \) single bosons are detected in only \( N \) of the \( M \) output modes at joint detection times \( \{ t_d \} _d = 1 \).

We consider here the case of photonic sources, although our results can be easily extended to atomic interferometers with bosonic sources. The probability rate for an \( N \)-fold joint detection event only in a given sample

![FIG. 1. Implementation of the MBCSP of order \( N \) with a random linear interferometer with \( M \gg N \) ports and at least \( N \) independent sources. After the evolution in the interferometer, described by a unitary random matrix \( U(M \times M) \), correlated detection events are recorded in \( N \) of the \( M \) output ports, labeled as \( d = 1, \ldots, N \), independently of the remaining ports.](image-url)
where $\hat{E}_d^{(\pm)}(t_d)$ denotes the positive/negative frequency parts of the field operator $\hat{E}_d(t_d)$ at the $d$th detector. These field operators are connected with the field operators at the input ports by a unitary operator $U$ that describes the evolution of the field in the linear interferometer. Equivalently, the interferometer can be described by a random unitary $M \times M$ matrix $U^{(M \times M)}$, which we assume for simplicity to be frequency independent (any unitary matrix $U^{(M \times M)}$ can be implemented by using only passive linear optical elements [19]). For a specific set of $N$ output modes where a joint detection occurs, the $N \times M$ submatrix

$$U^{(N \times M)} \equiv \left[U_{d,s}\right]_{d=1, \ldots, N, s=1, \ldots, M}$$

allows us to express the electric field operators at the detectors as linear combinations

$$\hat{E}_d^{(+)}(t_d) = \sum_{s=1}^{M} U_{d,s} \hat{E}_s^{(+)}(t_d)$$

of the field operators $\hat{E}_s^{(+)}(t_d)$ at the sources. Equivalent expressions hold for the conjugate fields $\hat{E}_d^{(-)}(t_d)$. In the next section we address the MBCSP for multi-mode thermal states, while we refer to [20] for the case of multi-mode Fock states.

### III. MBCSP FOR THERMAL INPUT STATES

We consider the state

$$\hat{\rho}_{\text{th}} \equiv \bigotimes_{s=1}^{M} \hat{\rho}_s$$

of $M$ independent multi-mode thermal states

$$\hat{\rho}_s = \int \frac{d^2 \alpha_s(\omega)}{\omega} P_{s,\text{th}}(\{\alpha_s(\omega)\}) \otimes |\alpha_s(\omega)\rangle \langle \alpha_s(\omega)|$$

at each of the input ports $s = 1, \ldots, M$, with Glauber-Sudarshan $P$-representation [16]

$$P_{s,\text{th}}(\{\alpha_s(\omega)\}) \equiv \prod_{\omega} \frac{1}{\pi \bar{n}_s(\omega)} \exp \left( -\frac{|\alpha_s(\omega)|^2}{\bar{n}_s(\omega)} \right).$$

Here, the distribution $\bar{n}_s(\omega) \equiv \bar{n}_s(\omega)$ of the mean number of photons for the source $s$ is defined by the normalized spectral distribution $\xi_s(\omega)$ and the mean number of photons $\bar{n}_s$. For simplicity, we assume equal Gaussian spectral distributions

$$\xi(\omega) = \frac{1}{\sqrt{2\pi} \Delta \omega} \exp \left( -\frac{(\omega - \omega_0)^2}{2 \Delta \omega^2} \right)$$

with central frequency $\omega_0$ and bandwidth $\Delta \omega$, and their respective Fourier transform

$$\chi(u) = \int_{-\infty}^{\infty} d\omega \xi(\omega) e^{-i\omega u} = e^{-i\omega_0 u} \exp \left( -\frac{u^2 \Delta \omega^2}{2} \right).$$

For suitable values of the average numbers of photons $\bar{n}$ and detection time resolutions much smaller than the coherence time $1/\Delta \omega$, the events corresponding to more than one photon at any of the output ports of the interferometer can be neglected.

For the state (4), Eq. (1) can be rewritten in terms of first order correlation functions

$$G^{(1)}(t_d, t_{d'}) \equiv \langle \hat{E}_{d}^{(-)}(t_d) \hat{E}_{d'}^{(+)}(t_{d'}) \rangle_{\hat{\rho}_{\text{th}}}$$

as [16]

$$G^{(N)}(\{t_d\}, \{t_d'\}) = \sum_{\sigma \in \Sigma_N} \prod_{d=1}^{N} G^{(1)}(t_d, t_{\sigma(d)}),$$

where $\sigma$ is an element of the symmetric group $\Sigma_N$ of order $N$.

Since the different sources $s$ are independent, by defining

$$G^{(s)}(t_d, t_{d'}) \equiv U_{d,s}^{*} U_{d',s} \langle \hat{E}_{s}^{(-)}(t_d) \hat{E}_{s}^{(+)}(t_{d'}) \rangle_{\hat{\rho}_s} = K^2 U_{d,s}^{*} U_{d',s} \bar{n}_s \chi_s(t_d - t_{d'}),$$

we use the narrow bandwidth approximation $\Delta \omega \ll \omega_0$ [1], Eq. (9) becomes

$$G^{(1)}(t_d, t_{d'}) = \sum_{s=1}^{M} G^{(1)}(t_d, t_{d'}).$$

We point out that the $N$th order correlation function $G^{(N)}$ in Eq. (10) corresponds to the permanent of the matrix $[G^{(1)}(t_d, t_{d'})]_{d,d'}$ with elements defined by Eqs. (12) and (11). In the following sections, we derive two equivalent formulations of $G^{(N)}$ in terms of matrix permanents depending on the entries of $U^{(N \times M)}$ in Eq. (2) and emphasize the underlying physics of multi-photon interference.

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1 In this case, the field operators can be approximated as [21]

$$\hat{E}_{s}^{(+)}(t) = iK \int_{-\infty}^{+\infty} d\omega \hat{a}_s(\omega) e^{-i\omega t}$$

with the annihilation operators $\hat{a}_s(\omega)$ and a constant $K$. 


A. Nth-order correlation functions and permanents

1. First formulation

A compact expression of $G^{(N)}(\{t_d\}, \{t_d\})$ in Eq. (10) can be obtained by defining the Hermitian matrix

$$
B^{(N\times N)} \equiv \left[ A_{d,d'} \chi(t_d - t_{d'}) \right]_{d=1,\ldots,N, \; d'=1,\ldots,N},
$$

where $A_{d,d'}$ are elements of the Hermitian positive semidefinite matrix

$$
A^{(N\times N)} \equiv \mathcal{U}^{(N\times M)} \text{diag}(\bar{n}_1, \ldots, \bar{n}_M) \mathcal{U}^{(M\times N)},
$$

Here, the presence of both $\mathcal{U}^{(N\times M)}$ and $\mathcal{U}^{(M\times N)}$ is evidence of the multi-photon interference occurring in the optical network, as becomes clearer later. When we apply these definitions together with Eqs. (11) and (12), Eq. (10) becomes

$$
G^{(N)}(\{t_d\}, \{t_d\}) = K^{2N} \text{per} B^{(N\times N)}. \tag{15}
$$

Thus, we find that the probability rate for an $N$-fold detection in a given sample of output modes with thermal sources is mainly given by a single permanent of a Hermitian $N \times N$ matrix $B^{(N\times N)}$.

We point out that, according to Eq. (8), the contribution to $G^{(N)}$ in Eq. (15) by a given pair of detection events at detectors $d \neq d'$ vanishes if $|t_d - t_{d'}| \Delta \omega \gg 1$. In the extreme case where $|t_d - t_{d'}| \Delta \omega \gg 1$ for all $d, d'$, the only contributions to $G^{(N)}$ are the ones for which $d = d'$. In this case, Eq. (15) trivially reduces to $G^{(N)}(|t_d - t_{d'}| \Delta \omega \gg 1) = K^{2N} \prod_{d=1}^{N} A_{d,d}$, where clearly the detection measurements are physically independent of each other and no multi-photon interference occurs.

2. Second formulation

We just demonstrated that the correlation function $G^{(N)}$ for a given sample of output modes is proportional to the permanent of a $N \times N$ matrix $B^{(N\times N)}$. We notice that $B^{(N\times N)}$ is not a submatrix of the unitary matrix $U^{(M\times M)}$ as in the case of the BSP with single photon sources. We now show that $G^{(N)}$ can also be expressed as a weighted sum of modulus squared permanents of matrices only built from columns of the interferometer submatrix $U^{(N\times M)}$ in Eq. (2).

By substituting Eq. (12) in Eq. (10), and interchanging the order of the product over $d$ and the summation over $s$, we obtain

$$
G^{(N)}(\{t_d\}, \{t_d\}) = \sum_{s_1, \ldots, s_N=1}^{M} \sum_{d=1}^{N} \prod_{s} G_{s_d}^{(1)}(t_d, t_{s(d)}). \tag{16}
$$

We now define the sets of ascending elements

$$
\{s_d\}_{c=1}^{N} = \left\{ 1, \ldots, 1, \ldots, s_d, \ldots, s, \ldots, M, \ldots, M \right\}, \tag{17}
$$

where $N_s \geq 0$ and $\sum_{s=1}^{M} N_s = N$, with the associated weighting factors

$$
\mathcal{N}(\{s_d\}) \equiv \prod_{s=1}^{M} \frac{1}{N_s}. \tag{18}
$$

These definitions allow us to write Eq. (16) as

$$
G^{(N)}(\{t_d\}, \{t_d\}) = \sum_{\{s_d\}} \mathcal{N}(\{s_d\}) \sum_{\sigma, \sigma' \in \Sigma_N} \prod_{d=1}^{N} G_{s_{\sigma'(d)}}^{(1)}(t_d, t_{\sigma(d)}). \tag{19}
$$

By using Eq. (11) together with the matrices

$$
C_{\{s_d\}, \sigma}^{(N\times N)} \equiv \left[ \mathcal{U}^{d,s_d} \mathcal{U}_s(t_d, t_{s(d)}), \mathcal{U}_c(t_{s(d)}, t_d) \right]_{d=1,\ldots,N}, \tag{20}
$$

obviously characterized by interference-like elements, Eq. (19) can be expressed as

$$
G^{(N)}(\{t_d\}, \{t_d\}) = K^{2N} \sum_{\{s_d\}} \left\{ \mathcal{N}(\{s_d\}) \prod_{d=1}^{N} \bar{n}_{s_d} \sum_{\sigma \in \Sigma_N} \prod_{d=1}^{N} \chi(t_{\sigma(d)} - t_d) \right\} \text{per} C_{\{s_d\}, \sigma}^{(N\times N)}. \tag{21}
$$

Furthermore, each possible configuration $\{s_d\}$ is associated with a weighted sum over $\sigma$ (with weighting factors $\prod_{d=1}^{N} \chi(t_{\sigma(d)} - t_d)$) of the permanents of the corresponding “interference” matrices $C_{\{s_d\}, \sigma}^{(N\times N)}$. 

The correlation function $G^{(N)}$ in Eq. (21) contains all contributions from the possible configurations $\{s_d\}$ in Eq. (17) of ways the $N$ detected photons can originate from the $M$ sources. In particular, each configuration has a weighting factor depending on the product of the respective average number of photons $\bar{n}_{s_d}$. 


B. Sampling over approximately equal detection times

We now consider the condition \( |t_d - t_{d'}| \Delta \omega \ll 1 \) for the sampling over approximately equal detection times. Here, the intimate connection between the permanent structure of the \( N \)th order correlation function and the interference between all possible multi-path \( N \)-photon contributions to a joint detection becomes clear.

In particular, by using the resulting condition \( \prod_{d=1}^{N} \chi(t_{\sigma(d)} - t_d) = 1 \ \forall \sigma \in \Sigma_N \), Eq. (15) simplifies to
\[
G^{(N)}(|t_d - t_{d'}| \Delta \omega \ll 1) = K^{2N} \text{per } A^{(N \times N)},
\]
which only depends on the mean numbers of photons in each source and the interferometer transformation\(^2\).

In an analogous way, the equivalent expression of \( G^{(N)} \) in Eq. (21) simplifies to the incoherent sum
\[
G^{(N)}(|t_d - t_{d'}| \Delta \omega \ll 1) \approx K^{2N} \sum_{\{s_d\}} N(\{s_d\}) \prod_{d=1}^{N} \bar{n}_{s_d} |\text{per } U_{\{s_d\}}^{(N \times N)}|^2
\]
(23)
of weighted modulus squared permanents of the matrices
\[
U_{\{s_d\}}^{(N \times N)} \equiv [U_{d,s_c}]_{d=1,...,N, s_c=1,...,N}.
\]

Each matrix corresponds to a configuration \( \{s_d\} \) defining the number \( N_{s} \) of photons each source contributes to the \( N \)-fold detection and can be obtained by repeating each column \( s \) of the matrix \( U^{(N \times M)} \) in Eq. (2) \( N_{s} \) times. The terms interfering in the modulus square of \( \text{per } U_{\{s_d\}}^{(N \times N)} \) correspond to all possible indistinguishable \( N \)-photon paths which connect the \( N \) sources \( \{s_d\} \) with the \( N \) detectors of a given sample, as illustrated in Fig. 2 in the case \( N = 2 \).

In general, the higher the column repetition rate in Eq. (24) is for a given configuration \( \{s_d\} \), the less complex is the computation of the matrix permanent. This physically manifests as a lower number of interfering \( N \)-photon paths. Therefore, the degree of multi-photon interference in each configuration is strictly connected with the complexity of the corresponding matrix permanent appearing in the \( N \)-fold detection probability rate (23). In particular, the only configurations where no column repetition occurs are the ones where \( N \) sources contribute to a \( N \)-fold detection (see Fig. 2 (a) for \( N = 2 \)), as in the original boson sampling formulation with single-photon sources. Indeed, these configurations, consisting of \( N! \) interfering \( N \)-photon paths, correspond to permanents of completely random matrices of complexity class \( \#P \).

C. Equal average photon numbers

We now consider the trivial case where all thermal sources have the same mean number of photons, \( \bar{n}_s = \bar{n} \ \forall s \) and derive two notable properties for the permanents of the matrices \( C_{\{s_d\},\sigma}^{(N \times N)} \) in Eq. (20) and \( U_{\{s_d\}}^{(N \times N)} \) in Eq. (24).

In this case, we easily find that the correlation function in Eq. (15) reduces to the constant expression
\[
G^{(N)}(\{t_d\}, \{t_d\}) = K^{2N} \bar{n}^N,
\]
(25)
which, as expected [23], is independent of the evolution in the interferometer. If we compare Eq. (25) with Eq. (21) in the limit of identical mean numbers of photons, we find that the property
\[
\sum_{\{s_d\}} N(\{s_d\}) \text{per } C_{\{s_d\},\sigma}^{(N \times N)} = \begin{cases} 1 & \sigma = 1 \\ 0 & \sigma \neq 1 \end{cases}
\]
(26)
holds for the matrices $\mathcal{U}^{(N\times N)}_{\{s_d\},\sigma}$ in Eq. (20). Further, since Eq. (25) is independent of the detection times $t_d$, it must also correspond to the expression (23) in the condition of equal mean numbers of photons. This yields the second property

$$\sum_{\{s_d\}} N(\{s_d\}) \left| \text{per} \mathcal{U}^{(N\times N)}_{\{s_d\},\sigma} \right|^2 = 1 \quad (27)$$

for the matrices $\mathcal{U}^{(N\times N)}_{\{s_d\},\sigma}$ in Eq. (24). These two properties, which characterize the behaviour of permanents in complexity theory, stem from the physical independence of the statistical properties of equal-temperature thermal sources from the interferometer.

**IV. FINAL REMARKS**

In this letter, we generalized the boson sampling problem to the multi-boson correlation sampling problem of order $N$. Here, $N$ correlated measurements are performed at the output of a linear optical $M$-port interferometer by sampling over all possible $N$ of $M$ output ports and their respective detection times. Moreover, we considered multi-mode thermal input sources instead of single-boson sources. For a given sample, the joint-detection probability rate turns out to be proportional to the permanent of a Hermitian $N \times N$ matrix, whose elements depend not only on the interferometer matrix but also on the average number of bosons from each source and on the detection times. Except for the trivial case of equal average numbers of bosons, the degree of complexity of this kind of permanent is not obvious. However, we demonstrated that this permanent, in the case of sampling over approximately equal detection times, can be cast as a weighted sum of modulus squared permanents, some of complexity class $\#P$. In addition, we showed how the physics of multi-boson interference with thermal sources characterizes the varying degrees of complexity of these permanents. Indeed, each permanent is associated with a possible physical configuration for the number of bosons each source contributes to the detection: the larger the number is of corresponding interfering multi-path amplitudes, the higher the permanent complexity is.

Our results provide a deeper fundamental understanding of the computational power of multi-boson interference, paving the way towards novel important applications in quantum information [24]. Moreover, new essential questions are raised at the interplay between quantum optics and complexity theory.

**Acknowledgments.** V.T. would like to thank M. Freyberger and W. P. Schleich, as well as J. Franson, T. Pittmann, and Y.H. Shih for fruitful discussions during his visit at UMBC in the summer of 2013.

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