Nonlocal Cahn-Hilliard-Brinkman System with Regular Potential: Regularity and Optimal Control

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Abstract
In this paper, we study an optimal control problem for nonlocal Cahn-Hilliard-Brinkman system, which models phase separation of binary fluids in porous media. The system evolves with regular potential in a two-dimensional bounded domain. We extend the existence of weak solution results for the system to prove the existence of strong solution under extra assumptions on the forcing term and initial datum. Further, using our regularity results, we study the tracking type optimal control problem. We prove the existence of optimal control and establish the first-order optimality condition. Lastly, we characterise optimal control in terms of the solution of the corresponding adjoint system. The existence of the solution for the adjoint system is also established.

Keywords Brinkman equation · Cahn-Hilliard equation · Strong solution · Optimal control · Nonlocal models · Phase separation

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1 Introduction
The Brinkman equation was proposed in [4] by H. C. Brinkman. It is a modified Darcy’s law to describe the flow through porous media. In recent studies, a diffuse interface variant of the Brinkman equation is proposed to model the phase separation of the incompressible binary fluids in a porous medium. The idea is to couple the Brinkman equation with the Cahn-Hilliard equation, which describes the phase separation phenomenon. Let \( \Omega \subset \mathbb{R}^2 \) be a...
bounded smooth domain with boundary $\partial \Omega$. Consider the nonlocal Cahn-Hilliard-Brinkman system (see [8]) given by,

\begin{align*}
\varphi_t + \nabla \cdot (u \varphi) &= \Delta \mu, \quad \text{in } \Omega \times (0, T), \quad (1.1) \\
\mu &= a \varphi - J \ast \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, T), \quad (1.2) \\
-\nabla \cdot (\nu(\varphi) \nabla u) + \eta u + \nabla \pi &= \mu \nabla \varphi + h, \quad \text{in } \Omega \times (0, T), \quad (1.3) \\
\text{div}(u) &= 0, \quad \text{in } \Omega \times (0, T), \quad (1.4)
\end{align*}

We endow this system with the following boundary and initial conditions,

\begin{align*}
\frac{\partial \mu}{\partial n} &= 0, \quad \text{on } \partial \Omega, \quad (1.5) \\
u = 0, \quad \text{on } \partial \Omega, \quad (1.6) \\
\varphi(0) &= \varphi_0, \quad \text{on } \Omega, \quad (1.7)
\end{align*}

where $\varphi$ denotes difference in concentrations of the two fluids, and $u$ is the average fluid velocity. The viscosity coefficient, which may depend on $\varphi$, is denoted by $\nu > 0$, permeability is denoted by $\eta > 0$ and $\pi$ is the pressure exerted on the fluid. Let $J : \mathbb{R}^d \to \mathbb{R}$ be a suitable interaction kernel, $a(x) = \int_{\Omega} J(x-y)dy$ and the spatial convolution $J \ast \varphi$ be defined by

\[(J \ast \varphi)(x) := \int_{\Omega} J(x-y)\varphi(y)dy, \quad x \in \Omega.
\]

The above system is called nonlocal because of the presence of the J term. The external forcing is denoted by $h$, and $F$ is a double-well potential accounting for phase separation, which can be singular (typically logarithmic potential) or regular (e.g. $F(s) = (s^2 - 1)^2$). In this paper, we consider only the case of regular potential. If $\nu = 0$, the system (1.1)–(1.7) becomes the so-called Cahn-Hilliard-Hele-Shaw system (also referred to as Cahn-Hilliard-Darcy system in the context of a multi-phase fluid mixture in nonporous medium) (see [25]) and is used in modelling tumour growth dynamics. There is a surge of papers in recent years that study existence, uniqueness, numerics and optimal control problems for coupled systems; Cahn-Hilliard equation coupled with other equations like Navier-Stokes, Brinkman, Darcy law or Hele-Shaw system (see [7, 10, 20, 21, 28]).

The local Cahn-Hilliard-Brinkman system is obtained by replacing $\mu$ equation in (1.1)–(1.7) by $\mu = -\Delta \varphi + F'(\varphi)$. Well-posedness and some convergence results for the local Cahn-Hilliard-Brinkman system with regular potential are studied in [3]. For the local system, the optimal control problem is studied in [31] and numerical results are obtained in [6]. Existence, optimal control and singular limits are studied for other similar models based on local Cahn-Hilliard-Brinkman systems in [11–14].

The nonlocal Cahn-Hilliard equation can be justified as it is physically relevant (see [22–24]). The nonlocal Cahn-Hilliard equation has been analysed theoretically and numerically (e.g. [1, 9, 19, 26]) under various assumptions on the potential $F$. The nonlocal version of the Cahn-Hilliard equation coupled with the Navier-Stokes system has been studied in various works for example; results about the existence of weak solution, strong solution, long time behaviour and optimal control problems are studied in [2, 5, 15–18].

The existence of a weak solution and uniqueness for the coupled nonlocal Cahn-Hilliard-Brinkman system (1.1)–(1.7) is studied in [8] for two- and three-dimensional bounded domain. We are interested in studying the optimal control problems related to the above system. But an optimal control problem requires a higher regularity of the solutions. In this work, we first address this challenging issue. We prove the existence of a strong solution for the Cahn-Hilliard-Brinkman system in a two-dimensional bounded domain. We follow the
work of [17], the existence and uniqueness of strong solution results for the Cahn-Hilliard-
Navier-Stokes system, to extend the regularity proved in [8]. We employ strong regularity
to further study the optimal control problem for the Cahn-Hilliard-Brinkman system. To
the best of our knowledge, such a result is not available in the literature to date. Our aim
is to prove differentiability of control-to-state map and existence of optimal control, and
characterise it in terms of adjoint variables.

The structure of the paper is as follows. In the next section, we recall the existence
results of the system (1.1)–(1.7) obtained in [8]. Further, we consider the system (1.1)–
(1.7) with constant viscosity $\nu$ and $\eta = 1$. We prove the existence of a strong solution and
obtain corresponding difference estimates. In Section 4, we derive the first-order
necessary optimality condition. We further study the existence of a solution for the adjoint
and the differentiability of control-to-state operator. In Section 4, we derive the first-order
necessary optimality condition. We further study the existence of a solution for the adjoint
system. Finally, we characterise the optimal control in terms of the adjoint variable.

2 Existence of Strong Solution

2.1 Functional Setting and Preliminary Results

We first explain the functional spaces needed to obtain our main results. Let us define
\[ \mathcal{D} := \{ u \in C_0^\infty(\Omega)^n : \text{div}(u) = 0 \}, \]
\[ \mathbb{G}_{\text{div}} := \text{closure of } \mathcal{D} \text{ in } L^2(\Omega; \mathbb{R}^n), \]
\[ \mathbb{V}_{\text{div}} := \text{closure of } \mathcal{D} \text{ in } H^1_0(\Omega; \mathbb{R}^n), \]
\[ H := L^2(\Omega; \mathbb{R}), \quad V := H^1(\Omega; \mathbb{R}), \]
\[ L^2 := L^2(\Omega; \mathbb{R}^n), \quad H^s := H^s(\Omega; \mathbb{R}^n), \]
\[ L^\infty(\Omega \times (0, T)) := L^\infty(\Omega \times (0, T); \mathbb{R}^n), \]
where $n = 2, 3$. Let us denote $\| \cdot \|$ and $(\cdot, \cdot)$, the norm and the scalar product, respectively,
on both $H$ and $\mathbb{G}_{\text{div}}$. The duality between any Hilbert space $X$ and its dual $X^*$ is denoted by
$X^*\langle \cdot, \cdot \rangle_X$. We know that $\mathbb{V}_{\text{div}}$ is endowed with the scalar product
\[ (u, v)_{\mathbb{V}_{\text{div}}} = (\text{div}u, \text{div}v), \quad \text{for all } u, v \in \mathbb{V}_{\text{div}}. \]
The norm on $\mathbb{V}_{\text{div}}$ is given by $\|u\|_{\mathbb{V}_{\text{div}}}^2 := \int_\Omega |\text{div}u(x)|^2 \, dx = \|\text{div}u\|^2$. We introduce the Stokes
operator $A$ with no-slip boundary condition. The operator $A : D(A) \subset \mathbb{G}_{\text{div}} \to \mathbb{G}_{\text{div}}$
is defined as $A := -P \Delta$ with domain $D(A) = H^2 \cap \mathbb{V}_{\text{div}}$, where $P : L^2 \to \mathbb{G}_{\text{div}}$ is the
Leray projector. Moreover, $A^{-1} : \mathbb{G}_{\text{div}} \to \mathbb{G}_{\text{div}}$ is a self-adjoint compact operator in $\mathbb{G}_{\text{div}}$. Therefore,
$A$ has a sequence of eigenvalues $\{\lambda_k\}$ with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_k \to \infty$ and
a family $\{v_k\} \subset D(A)$ of corresponding eigenfunctions which is orthonormal in $\mathbb{G}_{\text{div}}$. We recall Poincaré’s inequality
\[ \|u\|^2 \leq \frac{1}{\lambda_1} \|\text{div}u\|^2 \quad \forall u \in \mathbb{V}_{\text{div}}. \]
For every $f \in V'$, we define $\bar{f}$ by $\bar{f} := |\Omega|^{-1} V' \langle f, 1 \rangle_V$, which is the average of $f$ over
$\Omega$ whenever $f \in V$. Finally, we also define the operator $\mathcal{B} := -\Delta + I$ with homogeneous
Neumann boundary condition. It is well known that $\mathcal{B} : D(\mathcal{B}) \subset H \to H$ is an unbounded
linear operator in $H$ with domain $D(\mathcal{B}) = \{ v \in H^2(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega \}$ and that $B^{-1} : H \to H$ is a self-adjoint compact operator on $H$. By a classical spectral theorem, there
exists a sequence of eigenvalues $\mu_k$ with $0 < \mu_1 \leq \mu_2 \leq \ldots$ and $\mu_k \to \infty$ and a family of associated eigenfunctions $\eta_k \in D(B)$ such that $B\eta_k = \mu_k \eta_k$ for all $k \in \mathbb{N}$. The family $\{\eta_k\}$ forms an orthonormal basis in $H$ and orthogonal basis in $V$ and $D(B)$.

We need the following assumptions to deduce well-posedness of Cahn-Hilliard-Brinkman system under consideration:

(H1) Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded and connected domain with a smooth boundary.

(H2) $J \in W^{1,1}(\mathbb{R}^d)$ satisfies $J(x) = J(-x)$, and

$$a(x) := \int_{\Omega} J(x - y) dy \geq 0, \text{ a.e. } x \in \Omega.$$

(H3) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $C_0 > 0$ such that

$$F''(s) + a(x) \geq C_0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \quad (2.1)$$

(H4) There exist $C_1 > 0$, $C_2 > 0$ and $q > 0$ if $d = 2$, $q \geq \frac{1}{2}$ if $d = 3$ such that

$$F''(s) + a(x) \geq C_1 |s|^{2q} - C_2 \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

(H5) There exist $C_3 > 0$ and $p \in (1, 2]$ such that

$$|F'(s)|^p \leq C_3 (|F(s)| + 1) \quad \forall s \in \mathbb{R}.$$

(H6) $v$ is Lipschitz on $\mathbb{R}$ and there exist $v_0$, $v_1 > 0$ such that

$$v_0 \leq v(s) \leq v_1, \quad \forall s \in \mathbb{R},$$

and $\eta \in L^\infty(\Omega)$ is such that $\eta(x) \geq 0$, a.e. $x \in \Omega$.

Now, we summarise few results from [8] regarding well-posedness and the uniqueness of weak solutions of the system:

**Definition 2.1** Let $T > 0$ be given and let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$. Then, $(\varphi, u)$ is a weak solution of (1.1)–(1.7) if

$$\varphi \in C([0, T]; H) \cap L^2(0, T; V)$$

$$\varphi_t \in L^2(0, T; V')$$

$$\mu = a\varphi - J * \varphi + F'(\varphi) \in L^2(0, T; V)$$

$$u \in L^2(0, T; V_{div})$$

and it satisfies

$$\langle \varphi_t, \psi \rangle + \langle \nabla \mu, \nabla \psi \rangle = \langle u\varphi, \nabla \psi \rangle, \quad \forall \psi \in V, \text{ a.e. in } (0, T),$$

$$\langle v(\varphi) \nabla u, \nabla v \rangle + (\eta u, v) = \langle \mu \nabla \varphi, v \rangle + (h, v), \quad \forall v \in G_{div}, \text{ a.e. in } (0, T),$$

$$\varphi(0) = \varphi_0, \quad \text{a.e. in } \Omega.$$

**Theorem 2.2** ([8], Theorem 2.2) Suppose that (H1)–(H5) are satisfied. Let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$ and $h \in L^2(0, T; \nabla v_{div}')$. Then, there exists a weak solution $(\varphi, u)$ of (1.1)–(1.7). Furthermore, $F(\varphi)$ is in $L^\infty(0, T; L^1(\Omega))$ and setting

$$\mathcal{E}(\varphi(t)) = \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x - y)(\varphi(x) - \varphi(y))^2 dx dy + \int_{\Omega} F(\varphi(x)) dx.$$
the following energy equality holds for almost every $t \in (0, T)$
\[
\frac{d}{dt} E(\varphi(t)) + \|\nabla \mu\|^2 + v \|\nabla u\|^2 + \|u\| = \langle h, u \rangle.
\]

**Theorem 2.3** ([8], Proposition 2.1) Let assumptions of Theorem 2.2 hold. If $\varphi_0 \in L^\infty(\Omega)$ then any solution $(\varphi, u)$ to the problem on $[0, T]$ corresponding to $\varphi_0$ satisfies

$\varphi, \mu \in L^\infty(\Omega \times (0, T))$.

**Theorem 2.4** ([8], Corollary 2.1) Let $(H1)$–$(H6)$ hold. If $h \in L^\infty(0, T; \mathbb{V}_\text{div}')$ for some $T > 0$. Then, any weak solution $(\varphi, u)$ to $(1.1)$–$(1.7)$ is such that

$\varphi \in L^4(0, T; L^4(\Omega)), \quad u \in L^\infty(0, T; \mathbb{V}_\text{div}).$

The following result can be proved using [[8], Proposition 2.2].

**Theorem 2.5** Let hypotheses $(H1)$–$(H6)$ hold. Suppose $h_1, h_2 \in L^\infty(0, T; \mathbb{V}_\text{div}')$. Consider two weak solutions to $(1.1)$–$(1.7)$, namely $(\varphi_1, u_1)$ and $(\varphi_2, u_2)$, corresponding to the initial data $\varphi_{1,0}$ and $\varphi_{2,0}$ such that $\varphi_{i,0} \in L^2(\Omega)$ and $F(\varphi_{i,0}) \in L^1(\Omega), i = 1, 2$. Then, there exists $N = N(T) > 0$ such that, for any $t \in [0, T]$,

\[
\|\varphi_1(t) - \varphi_2(t)\|_{\mathbb{V}'_V}^2 + \int_0^t \|u_1(t) - u_2(t)\|_{\mathbb{V}_\text{div}}^2 \leq N(\|\varphi_{1,0} - \varphi_{2,0}\|_{\mathbb{V}'_V}^2 + |\bar{\varphi}_{1,0} - \bar{\varphi}_{2,0}|)\|h_1 - h_2\|_{L^2(0,T;\mathbb{V}_\text{div}')} (2.2)
\]

In particular, $(2.3)$–$(2.8)$ have a unique solution.

**Proof** Let us denote $\varphi = \varphi_1 - \varphi_2, u = u_1 - u_2$ and $h = h_1 - h_2$. Arguing exactly as in the proof of Proposition 2.2 in [8], we can arrive at $(2.2)$. \qed

### 2.2 Strong Solution

Let us consider the Cahn-Hilliard-Brinkman system

\[
\begin{align*}
\varphi_t + u \cdot \nabla \varphi &= \Delta \mu, & &\text{in} \quad \Omega \times (0, T), \\
\mu &= a\varphi - J * \varphi + F'(\varphi), & &\text{in} \quad \Omega \times (0, T), \\
-\nu \Delta u + u + \nabla \pi &= \mu \nabla \varphi + h, & &\text{in} \quad \Omega \times (0, T), \\
\text{div} (u) &= 0, & &\text{in} \quad \Omega \times (0, T), \\
\frac{\partial \mu}{\partial n} &= 0, & &\text{on} \quad \partial \Omega \times (0, T), \\
\varphi(0) &= \varphi_0(x), & &\text{in} \quad \Omega.
\end{align*}
\]

which is obtained by assuming $\eta = 1$ and $\nu$ is independent of $\varphi$ in $(1.1)$–$(1.7)$. We are now going to prove the main theorem of this section, namely the existence of a strong solution of the system $(2.3)$–$(2.8)$ in dimension two. We consider the space

$\mathcal{U} := \{ h \in L^\infty(0, T; \mathbb{G}_\text{div}) \mid h_t \in L^2(0, T; \mathbb{V}_\text{div}') \}$.

We observe that $\mathcal{U}$ is a Banach space with the norm (see Chapter 7 in [27])

$$\|h\|_{\mathcal{U}} := \|h\|_{L^\infty(0,T;\mathbb{G}_\text{div})} + \|h_t\|_{L^2(0,T;\mathbb{V}_\text{div}')}.$$
We need the following extra assumption to prove the existence of strong solution, 
\((H7)\) \(F \in C^3(\mathbb{R}), J \in W^{2,1}(\mathbb{R}^d)\).

**Theorem 2.6** Let \(h \in \mathcal{U}\) and \(\varphi_0 \in H^2(\Omega) \cap L^\infty(\Omega)\) and hypotheses \((H1)–(H5)\) and \((H7)\) are satisfied. Then, there exists a unique strong solution for the system \((2.3)–(2.8)\) on \([0, T]\) in the following sense

\[
\begin{align*}
\varphi & \in L^\infty(0, T; H^2(\Omega)), \\
\varphi_t & \in L^\infty(0, T; H) \cap L^2(0, T; V), \\
uu & \in L^2(0, T; \mathbb{H}^2).
\end{align*}
\] (2.9) (2.10) (2.11)

**Proof** We shall carry out the proof by providing some formal higher-order estimates. The argument can be made rigorous by means of Faedo-Galerkin approximation technique (see Theorem 2.2 in[8]). Note that by Theorems 2.2, 2.4 and 2.5, there exists a unique weak solution of \((2.3)–(2.8)\) under the given assumptions. To prove higher regularity given by \((2.9)\), we take inner product of \((2.5)\) with \(-\Delta u\) to get,

\[
\nu \|\Delta u\|^2 + \|\nabla u\|^2 = - (\mu \nabla \varphi, \Delta u) - (h, \Delta u). \tag{2.12}
\]

Henceforth, we denote by a positive constant \(C = C(J, F, \Omega, \nu)\) and may vary from line to line. Now we estimate the right-hand side of \((2.12)\). Observe that

\[
\mu \nabla \varphi = (a \varphi - J * \varphi + F'(\varphi)) \nabla \varphi
\]

\[
= \nabla \left( F(\varphi) + a \frac{\varphi^2}{2} \right) - \Delta a \frac{\varphi^2}{2} - (J * \varphi) \nabla \varphi.
\]

Hence, we have

\[- (\mu \nabla \varphi, \Delta u) = \left( \nabla a \frac{\varphi^2}{2}, \Delta u \right) + ((J * \varphi) \nabla \varphi, \Delta u).\]

Then, using integration by parts, Hölder’s inequality and Young’s inequality for convolution (by extending the functions trivially outside the bounded domain), we get

\[
| (\mu \nabla \varphi, \Delta u) | \leq \frac{1}{2} \| \nabla a \|_{L^\infty} \| \varphi \|^2_{L^4} \| \Delta u \| + \| \nabla J \|_{L^1} \| \varphi \|^4_{L^4} \| \Delta u \| \tag{2.13}
\]

\[
\leq \frac{\nu}{4} \| \Delta u \|^2 + C \| \varphi \|^4_{L^4}, \tag{2.14}
\]

and

\[
| (h, \Delta u) | \leq \| h \| \| \Delta u \| \leq \frac{\nu}{4} \| \Delta u \|^2 + C \| h \|^2. \tag{2.15}
\]

Substitute \((2.14)\) and \((2.15)\) in \((2.12)\), to get,

\[
\frac{\nu}{2} \| \Delta u \|^2 \leq C (\| \varphi \|^4_{L^4} + \| h \|^2). \tag{2.16}
\]

Integrate \((2.16)\) from 0 to \(T\) and using Theorem 2.4, we get

\[
\frac{\nu}{2} \int_0^T \| \Delta u(t) \|^2 dt \leq C \left( \int_0^T \| \varphi(t) \|^4_{L^4} dt + \frac{3}{2\nu} \int_0^T \| h(t) \|^2 dt \right) < \infty.
\]

This proves \(\Delta u \in L^2(0, T; \mathbb{L}^2)\). From Lemma III.3.7 in [29], we get \((2.11)\). Since \(u \in L^2(0, T; \mathbb{H}^2)\), from step 1 of the proof of Theorem 2 in [17], we have

\[
\nabla \mu \in L^\infty(0, T; H), \ \varphi \in L^\infty(0, T; V), \ \varphi_t \in L^2(0, T; H), \ \forall T > 0. \tag{2.17}
\]
Now we prove that $\varphi_t \in L^\infty(0, T; H)$. We first differentiate (2.3) and (2.5) with respect to $t$, take $L^2$ inner product of the obtained equations with $\mu_t$ and $u_t$, respectively, and add them to get

$$(\varphi_{tt}, \mu_t) + v\|\nabla u_t\|^2 + \|u_t\|^2 + \|\nabla \mu_t\|^2 + (u_t \cdot \nabla \varphi_t, \mu_t) = (\mu \nabla \varphi_t, u_t) + \langle h_t, u_t \rangle. \quad (2.18)$$

Using integration by parts and H"older inequality, we get

$$|\langle h_t, u_t \rangle| \leq \|h_t\|_{L^p} \|u_t\|_{W^{1,p}} \leq \frac{v}{2} \|\nabla u_t\|^2 + C \|h_t\|^2_{L^p}. \quad (2.20)$$

From estimates 3.23, 3.24 and 3.27 derived in the proof of Theorem 2 [17], we have that:

$$\|\nabla \varphi\|_{L^p} \leq C(1 + \|\nabla \varphi_t\|^{1-2/p}), \quad \|\nabla \mu\|_{L^p} \leq (1 + \|\nabla \varphi_t\|^{1-2/p}), \quad (2.21)$$

and

$$\|\nabla \varphi_t\|^2 \leq \frac{4}{C_0^2} \|\nabla \mu_t\|^2 + C \|\varphi_t\|^2 + C \|\varphi_t\|^4 + C. \quad (2.22)$$

Using Hölder inequality, Gagliardo-Nirenberg inequality, (2.21) and (2.22), we get

$$|(\mu \nabla \varphi_t, u_t)| = |(\nabla \mu_t, (\varphi_t)\nabla \varphi_t)| \leq \|\nabla \mu_t\|_{L^p} \|\varphi_t\|_{W^{1,p}} \|u_t\| \leq C(1 + \|\varphi_t\|^{2/3}/3 \|\nabla \varphi_t\|^{1/3} + |\varphi_t|) \|u_t\|$$

$$\leq C(\|\varphi_t\|^{2/3}/3 \|\nabla \varphi_t\|^{1/3} + \|\varphi_t\|^{4/3}/3 \|\nabla \varphi_t\|^{1/3} + |\varphi_t| + \|\varphi_t\|^2 + |\varphi_t|^{10/3})$$

$$\leq \frac{1}{6} \|u_t\|^2 + C \|\varphi_t\|^2 + C \|\varphi_t\|^4 + C \|\varphi_t\|^4 + C. \quad (2.23)$$

Now observe that

$$(\varphi_{tt}, a \varphi_t - J \ast \varphi_t + F''(\varphi)\varphi_t) = \left( \frac{1}{2} \frac{d}{dt} \int_{\Omega} a \varphi_t^2 \right) - (a \varphi_t \cdot \nabla \varphi_t - a \cdot \nabla \varphi_t + \Delta \mu_t, J \ast \varphi_t) + \int_{\Omega} F''(\varphi)\varphi_t \varphi_{tt}$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( a + F''(\varphi) \right) \varphi_t^2 + (u_t \cdot \nabla \varphi_t, J \ast \varphi_t) + (u \cdot \nabla \varphi_t, J \ast \varphi_t)$$

$$- (\Delta \mu_t, J \ast \varphi_t) + \frac{1}{2} \int_{\Omega} F'''(\varphi)\varphi_t^3. \quad (2.24)$$

By extending the function $\varphi_t$ trivially on $\mathbb{R}^d$, we can use Young’s inequality for convolution to get

$$|\langle \nabla (J \ast \varphi_t, u_t, \varphi_t) \rangle| \leq \|\nabla (J \ast \varphi_t)\|_{L^p} \|u_t\|\|\varphi_t\|_{L^\infty} \leq \frac{1}{6} \|u_t\|^2 + C \|\varphi_t\|^2 \|\varphi_t\|^2, \quad (2.25)$$

$$|\langle \nabla (J \ast \varphi_t, u_t) \varphi_t \rangle| \leq \|\nabla (J \ast \varphi_t)\|_{L^p} \|u_t\|\|\varphi_t\| \leq C \|u_t\|_{L^2} \|\varphi_t\|^2, \quad (2.26)$$

$$|\langle \nabla (J \ast \varphi_t, \nabla \mu_t) \varphi_t \rangle| \leq \|\nabla \mu_t\| \|\nabla (J \ast \varphi_t)\| \|\varphi_t\| \leq \frac{1}{4} \|\nabla \mu_t\|^2 + C \|\varphi_t\|^2. \quad (2.27)$$

Using Gagliardo-Nirenberg inequality and (2.22), we also have

$$\int_{\Omega} F'''(\varphi)\varphi_t^3 \leq C \|\varphi_t\|^3 \leq C(\|\nabla \varphi_t\| \|\varphi_t\|^2 + \|\varphi_t\|^3) \leq \frac{1}{8} \|\nabla \mu_t\|^2 + C \|\varphi_t\|^4 + C. \quad (2.28)$$
Substituting (2.19), (2.20), and (2.23)-(2.28) in (2.18), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \int_\Omega (a + F''(\varphi)) \varphi_t^2 \right) + \nu \| \nabla \varphi_t \|^2 + \frac{1}{2} \| \varphi_t \|^2 + \frac{1}{4} \| \nabla \mu_t \|^2 \leq f(t) \| \varphi_t \|^2 + C \| \varphi_t \|^4 + C \| h_t \|^2 + C.
\]
(2.29)
where \( f(t) = C(\| u \|^2_{\Omega^2} + 1) \). Hence, using the technique from Theorem 2 in [17], we deduce that
\[
\varphi_t \in L^\infty(0, T; H), \quad \forall \, T > 0.
\]
(2.30)
From (2.21), we also have
\[
\nabla \varphi, \nabla \mu \in L^\infty(0, T; L^p(\Omega)) \quad \forall \, T > 0, \quad 2 \leq p < \infty.
\]
(2.31)
From (2.3), using Hölder inequality, we get
\[
\| \Delta \mu \| \leq \| \varphi_t \| + \| \varphi_t \| L^4. (2.32)
\]
The above estimate along with (2.30) and (2.31) leads to the regularity
\[
\Delta \mu \in L^\infty(0, T; L^2(\Omega)). (2.33)
\]
Using the smoothness of the boundary \( \partial \Omega \) and the conditions \( \frac{\partial \mu}{\partial n} = 0 \), we conclude that
\[
\mu \in L^\infty(0, T; H^2(\Omega)). (2.34)
\]
Now to prove regularity of \( \varphi \), let us denote by \( \partial^2_{ij} := \frac{\partial^2}{\partial x_i \partial x_j} \). Using (H3) and Hölder inequality, we get
\[
\int_\Omega \partial^2_{ij} \mu \partial^2_{ij} \varphi = \int_\Omega (\varphi \partial^2_{ij} a + \partial_i \partial_j \varphi + \partial_i \varphi \partial_j a + a \partial^2_{ij} \varphi - \partial^2_{ij} (J \ast \varphi) + F''(\varphi) \partial_i \varphi \partial_j \varphi
\]
\[
+ F''(\varphi) \partial^2_{ij} \varphi \partial^2_{ij} \varphi \geq C_0 \| \partial^2_{ij} \varphi \|^2 + \int_\Omega (\varphi \partial^2_{ij} a - \partial^2_{ij} (J \ast \varphi)) \partial^2_{ij} \varphi + \int_\Omega (\partial_i \partial_j \varphi + \partial_i \varphi \partial_j a) \partial^2_{ij} \varphi
\]
\[
+ \int_\Omega F''(\varphi) \partial_i \varphi \partial_j \varphi \partial^2_{ij} \varphi \geq C_0 \| \partial^2_{ij} \varphi \|^2 - C(\| \varphi \|^2 + \| \partial_j \varphi \|^2 + \| \partial_j \varphi \|^2 + \| \partial_i \varphi \|^2 + \| \partial_j \varphi \|^2 \| \varphi \|^2_L^4 \| \varphi \|^2_L^4). (2.35)
\]
Observe
\[
| (\partial^2_{ij} \mu, \partial^2_{ij} \varphi) | \leq \| \partial^2_{ij} \mu \| \| \partial^2_{ij} \varphi \| \leq C_0 \frac{1}{4} \| \partial^2_{ij} \varphi \|^2 + C \| \partial^2_{ij} \mu \|^2. (2.36)
\]
Using (2.35) and (2.36) and thanks to (2.31), we have
\[
\| \partial^2_{ij} \varphi \|^2 \leq C \| \partial^2_{ij} \mu \|^2 + C.
\]
Taking summation over \( i, j \) and using (2.34), we get
\[
\varphi \in L^\infty(0, T; H^2(\Omega)).
\]
\[\square\]

**Remark 2.7** Substituting (2.13) and (2.15) in (2.12), we get
\[
v \| \Delta u \|^2 + \| \nabla u \|^2 \leq \frac{1}{2} \| \nabla a \|_{\Omega^\infty} \| \varphi \|^2_{L^4} \| \Delta u \| + \| \nabla J \|_{L^1} \| \varphi \|^2_{L^4} \| \Delta u \| + \| h \| \| \Delta u \|.
\]
Sobolev inequality implies
\[
v \| \Delta u \| \leq C \| \varphi \|^2_{\Omega^2} + \| h \|.
\]
Using (2.17), we infer that
\[ \Delta u \in L^\infty(0, T; L^2), \]
and using Theorem 2.4, we get
\[ u \in L^\infty(0, T; H^2). \]

**Theorem 2.8** Suppose hypotheses (H1)–(H5) and (H7) are fulfilled. Let \( h_1, h_2 \in \mathcal{U} \) and let \( [\varphi_1, u_1] \) and \([\varphi_2, u_2]\) be two strong solutions (in the sense of (2.9)–(2.11)) of the system (2.3)–(2.8) corresponding to \( h_1 \) and \( h_2 \), respectively, with the same initial data \( \varphi_0 \in H^2(\Omega) \cap L^\infty(\Omega) \). Then, there exists a constant \( C > 0 \) such that
\[ \| \varphi_1 - \varphi_2 \|^2_{L^\infty([0, t]; H)} + \| \nabla(\varphi_1 - \varphi_2) \|^2_{L^2((0, t); H)} + \| u_1 - u_2 \|^2_{L^2((0, t); \nabla}) \leq C \| h_1 - h_2 \|^2_{L^2(0, t; \nabla}), \]
for every \( t \in [0, T] \).

**Proof** Set \( \varphi = \varphi_1 - \varphi_2, u = u_1 - u_2 \) and \( h = h_1 - h_2 \). We can write the equations satisfied by \( \varphi \) and \( u \). The weak formulation of equations for \( \varphi \) and \( u \) can be written as
\[ (\varphi_t, \psi) + (\nabla \mu, \nabla \psi) = (u \varphi_1, \nabla \psi) + (u_2 \varphi, \nabla \psi), \quad \forall \psi \in V, \]
\[ (\nu, \nabla u, \nabla v) + (u, v) = (\mu \nabla \varphi_1, v) + (\mu_2 \nabla \varphi, v) + (h, v), \quad \forall v \in \nabla \}
where \( \mu = a \varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2) \).

Let us take \( \psi = \varphi \) and \( v = u \) in (2.38) and (2.39), respectively, to obtain
\[ \frac{1}{2} \frac{d}{dt} \| \varphi(t) \|^2 + (u \nabla \varphi_1, \varphi) + (u_2 \nabla \varphi, \varphi) = (\Delta \mu, \varphi), \]
\[ \nu \| \nabla u \|^2 + \| u \|^2 = (\mu \nabla \varphi_1, u) + (\mu_2 \nabla \varphi, u) + (h, u). \]

Now we estimate terms in (2.40). We denote by \( C = C(J, F, \nu, \Omega, C_0) \). Using Hölder and Ladyzhenskaya inequalities
\[ |(u \nabla \varphi_1, \varphi)| \leq \| u \|_L^4 \| \nabla \varphi_1 \|_L^4 \| \varphi \| \leq \frac{\nu}{2} \| \nabla u \|^2 + C \| \nabla \varphi_1 \|_L^2 \| \varphi \|^2. \]

Observe that,
\[ \int_{\Omega} u_2(\nabla \varphi) = \int_{\Omega} u_2 \nabla \left( \frac{\varphi^2}{2} \right) = -\int_{\Omega} \text{div}(u) \frac{\varphi^2}{2} = 0. \]

Using (H3), we get
\[ - (\Delta \mu, \varphi) = (\nabla \mu, \nabla \varphi) = (\nabla (a \varphi - J * \varphi + F(\varphi_1) - F(\varphi_2)), \nabla \varphi) \]
\[ = ((a + F''(\varphi_2)) \nabla \varphi, \nabla \varphi) + (\varphi \nabla a - \nabla J * \varphi, \nabla \varphi) + ((F''(\varphi_1) - F''(\varphi_2)) \nabla \varphi_1, \nabla \varphi) \]
\[ \geq C_0 \| \nabla \varphi \|^2 + (\varphi \nabla a - \nabla J * \varphi, \nabla \varphi) + ((F''(\varphi_1) - F''(\varphi_2)) \nabla \varphi_1, \nabla \varphi). \]

Right-hand side terms of (2.43) can be estimated as follows
\[ |(\varphi \nabla a - \nabla J * \varphi, \nabla \varphi)| \leq \| \varphi \| \| \nabla a \| \| \nabla \varphi \| + \| \nabla J \| \| \varphi \| \| \nabla \varphi \| \leq \frac{C_0}{4} \| \nabla \varphi \|^2 + C \| \varphi \|^2. \]
Using H"older inequality and Gagliardo-Nirenberg inequality, we get
\[ |(F'(\varphi_1) - F'(\varphi_2))\nabla \varphi_1, \nabla \varphi| \leq C \|\varphi\|_{L^4} \|\nabla \varphi_1\|_{L^4} \|\nabla \varphi\| \leq C (\|\varphi\| + \|\varphi\|^{1/2} \|\nabla \varphi\|^{1/2}) \|\nabla \varphi_1\|_{L^4} \|\nabla \varphi\| \leq \frac{C_0}{4} \|\nabla \varphi\|^2 + C (\|\nabla \varphi_1\|_{L^4}^2 + \|\nabla \varphi_1\|_{L^4}^4) \|\varphi\|^2, \] (2.45)

Substituting (2.44) and (2.45) in (2.43), we get
\[ -(\Delta \mu, \varphi) \geq \frac{C_0}{2} \|\nabla \varphi\|^2 - C (1 + \|\nabla \varphi_1\|_{L^4}^2 + \|\nabla \varphi_1\|_{L^4}^4) \|\varphi\|^2. \] (2.46)

Using (2.42) and (2.46) in (2.40), we get
\[ \frac{1}{2} \frac{d}{dt} \|\varphi(t)\|^2 + \frac{C_0}{2} \|\nabla \varphi\|^2 \leq \frac{\nu}{5} \|\nabla u\|^2 + C (1 + \|\nabla \varphi_1\|_{L^4}^2 + \|\nabla \varphi_1\|_{L^4}^4) \|\varphi\|^2. \] (2.47)

Now we estimate the terms in (2.41). Observe that using Taylor series we get
\[ \|\mu\| = |a \varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2)| \leq C \|\varphi\|. \] (2.48)

Using (2.48), we get
\[ |(\mu \nabla \varphi_1, u)| \leq \|\mu\| \|\nabla \varphi_1\|_{L^4} \|u\|_{L^4} \leq \frac{\nu}{5} \|\nabla u\|^2 + C \|\nabla \varphi_1\|^2 \|\varphi\|^2, \] (2.49)

and using integration by parts
\[ |(\mu_2 \nabla \varphi, u)| = |(\varphi \nabla \mu_2, u)| \leq \|\varphi\| \|\nabla \mu_2\|_{L^4} \|u\|_{L^4} \leq \frac{\nu}{5} \|\nabla u\|^2 + C \|\nabla \mu_2\|^2 \|\varphi\|^2, \] (2.50)

also
\[ |(h, u)| \leq \|h\|_{\nabla \div} \|\nabla u\| \leq \frac{\nu}{5} \|\nabla u\|^2 + C \|h\|^2_{\nabla \div}. \] (2.51)

Combining (2.49), (2.50) and (2.51), we get
\[ \frac{2\nu}{5} \|\nabla u\|^2 + \|u\|^2 = C (\|\nabla \varphi_1\|^2 + \|\nabla \mu_2\|^2_{L^4}) \|\varphi\|^2 + C \|h\|^2_{\nabla \div}. \] (2.52)

Adding (2.47) and (2.52), we get
\[ \frac{1}{2} \frac{d}{dt} \|\varphi(t)\|^2 + \frac{C_0}{2} \|\nabla \varphi\|^2 + \frac{\nu}{5} \|\nabla u\|^2 + \|u\|^2 \leq C (1 + \|\nabla \varphi_1\|^2_{L^4} + \|\nabla \varphi_1\|^4_{L^4} + \|\nabla \mu_2\|^2_{L^4}) \|\varphi\|^2 + C \|h\|^2_{\nabla \div}. \] (2.53)

By integrating (2.53) from 0 to t, we get
\[ \|\varphi(t)\|^2 + C_0 \int_0^t \|\nabla \varphi(s)\|^2 ds + \frac{\nu}{5} \int_0^t \|\nabla u(s)\|^2 ds + \int_0^t \|u(s)\|^2 ds \leq C \int_0^t \alpha(s) \|\varphi(s)\|^2 ds + C \int_0^t \|h(s)\|^2_{\nabla \div} ds, \]

where \( \alpha(t) = C (1 + \|\nabla \varphi_1(t)\|^2_{L^4} + \|\nabla \varphi_1(t)\|^4_{L^4} + \|\nabla \mu_2(t)\|^2_{L^4}) \in L^1(0, T) \). By applying Grönwall’s inequality, we get
\[ \|\varphi(t)\|^2 + \frac{C_0}{2} \int_0^t \|\nabla \varphi\|^2 + \frac{\nu}{5} \int_0^t \|\nabla u\|^2 + \int_0^t \|u\|^2 \leq C \exp \left(\int_0^t \alpha(s) ds\right) \|h\|_{L^2(0,T; \nabla \div)}, \]
which, for every $t \in [0, T]$ gives

$$\|u_1 - u_2\|_{L^2(0, t; \mathcal{V}_{\text{div}})}^2 + \|\varphi_1 - \varphi_2\|_{C^0(0, t; \mathcal{V})}^2 + \|\nabla(\varphi_1 - \varphi_2)\|_{L^2(0, t; H)}^2 \leq C \|h_1 - h_2\|_{L^2(0, t; \mathcal{V}_{\text{div}})},$$

(2.54)

Since for any $h \in \mathcal{U}$ we have $u \in L^\infty(0, T; \mathbb{H}^2)$, we can prove the following theorem for a higher-order estimate of $\varphi$ using the same techniques as in Lemma 2.6 in [18] and estimates used in the proof of Theorem 2.8.

**Theorem 2.9** Let us assume that (H1)–(H5) and (H7) are satisfied. Let $h_1$, $h_2 \in \mathcal{U}$ and let $[\varphi_1, u_1]$ and $[\varphi_2, u_2]$ be two strong solutions of the system (2.3)–(2.8) (in the sense of (2.9)–(2.11) corresponding to $h_1$ and $h_2$, respectively, with the same initial data $\varphi_0 \in H^2(\Omega) \cap L^\infty(\Omega)$. Then, there exists a constant $C > 0$ such that for every $t \in (0, T]$.

$$\|\varphi_1 - \varphi_2\|_{L^\infty(0, t; \mathcal{V})}^2 + \|\varphi_1 - \varphi_2\|_{C^0(0, t; \mathcal{V})}^2 + \|\varphi_1 - \varphi_2\|_{L^2(0, t; H)}^2 + \|\nabla(\varphi_1 - \varphi_2)\|_{L^2(0, t; H)}^2$$

$$\leq C \|h_1 - h_2\|_{L^2(0, t; \mathcal{V}_{\text{div}})}^2,$$

(2.55)

**3 Optimal Control**

In this section, we study the optimal control problem related to (2.3)–(2.8). We define the set of admissible controls as the closed and bounded subset of $\mathcal{U}$:

$$\mathcal{U}_{\text{ad}} = \{U \in \mathcal{U} : \|U\|_{\mathcal{U}} \leq R \text{ for some } R > 0\}.$$  

(3.1)

The optimal control problem of interest is to minimise the tracking type cost functional $\mathcal{J}$

$$\mathcal{J}(\varphi, u, U) := \int_0^T \|\varphi(t) - \varphi_d(t)\|^2 dt + \int_0^T \|u(t) - u_d(t)\|^2 dt$$

$$+ \int_\Omega |\varphi(x, T) - \varphi_\Omega|^2 dx + \int_0^T \|U(t)\|^2 dt,$$

(3.2)

over the set of admissible controls $\mathcal{U}_{\text{ad}}$ subject to the system

$$\varphi_t + u \cdot \nabla \varphi = \Delta \mu, \quad \text{in } \Omega \times (0, T),$$

(3.3)

$$\mu = \alpha \varphi - \nabla \varphi + F'(\varphi), \quad \text{in } \Omega \times (0, T),$$

(3.4)

$$-\nu \Delta u + u + \nabla \pi = \mu \nabla \varphi + U, \quad \text{in } \Omega \times (0, T),$$

(3.5)

$$\text{div}(u) = 0, \quad \text{in } \Omega \times (0, T),$$

(3.6)

$$u = \frac{\partial \mu}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, T),$$

(3.7)

$$\varphi(0) = \varphi_0(x), \quad \text{in } \Omega.$$  

(3.8)

Note that the system (3.3)–(3.8) is obtained by choosing forcing term as an admissible control in (2.3)–(2.8). We assume that the desirable concentration $\varphi_d$ and desirable velocity $u_d$ belong to $L^2(\Omega \times (0, T))$ and $L^2(0, T; \mathcal{G}_{\text{div}})$, respectively, and $\varphi_\Omega \in L^2(\Omega)$. Then, the optimal control problem can be rewritten as

$$\min_{U \in \mathcal{U}_{\text{ad}}} \{\mathcal{J}(\varphi, u, U)|(\varphi, u, U) \text{ is unique strong solution of (3.3) – (3.8)}\}. \quad \text{(OCP)}$$

Let us define the control-to-state operator $S : U \rightarrow (\varphi, u)$ where $(\varphi, u)$ solves (3.3)–(3.8) with control $U$. Note that,

$$S : \mathcal{U} \rightarrow \mathcal{V} := C([0, T]; H) \cap L^2(0, T; \mathcal{V}) \times L^2(0, T; \mathcal{V}_{\text{div}}).$$
From Theorem 2.2, Theorem 2.5 and Theorem 2.6, we can say that $S$ is a well-defined map from $U$ to $V$. In fact, it is locally Lipschitz continuous.

In this section, we prove three important results. First one is to prove the existence of optimal control for the problem (OCP) defined above. Second result is to show the existence of a solution for the linearised system, linearised around the optimal state. Lastly, we prove that the control-to-state operator $S$, identified above, is differentiable and the Fréchet derivative of $S$ is given in terms of the solution of the linearised system.

3.1 Existence of Optimal Control

Theorem 3.1 Suppose hypotheses $(H1)$–$(H5)$ and $(H7)$ are satisfied. Then, the optimal control problem (OCP) admits a solution.

Proof Let us define $l = \inf_{U \in U_{ad}} J(\varphi, u, U)$. Since $0 \leq l < \infty$, there exists a minimising sequence $\{U_n\} \in U_{ad}$ for (3.2) such that

$$\lim_{n \to \infty} J(\varphi_n, u_n, U_n) = l,$$

where $[\varphi_n, u_n] = S(U_n)$ is a corresponding state solution of the system (3.3)–(3.8). Since $\{U_n\} \in U_{ad}$ is bounded in $U$, by Banach-Alaoglu theorem, there exists a subsequence again denoted by $\{U_n\}$ and a function $U^* \in U$ such that

$$U_n \rightharpoonup U^* \quad \text{in} \quad L^\infty(0, T; G_{\text{div}}),$$

$$\frac{dU_n}{dt} \rightharpoonup \frac{dU^*}{dt} \quad \text{in} \quad L^2(0, T; V'_{\text{div}}).$$

Moreover, due to the embedding $U \subset L^\infty(0, T; G_{\text{div}}) \subset L^2(0, T; G_{\text{div}})$, we have

$$U_n \rightarrow U^* \quad \text{in} \quad L^2(0, T; G_{\text{div}}).$$

By weak$^*$ lower semicontinuity of the norm on $U$, we infer that $U^* \in U_{ad}$. Using the estimates in Theorem 2.2 in [8], we get

$$\{\varphi_n\} \text{ is uniformly bounded in } L^\infty(0, T; H) \cap L^2(0, T; V),$$

$$\{\varphi'_n\} \text{ is uniformly bounded in } L^2(0, T; V'),$$

$$\{\mu_n\} \text{ is uniformly bounded in } L^2(0, T; V),$$

$$\{u_n\} \text{ is uniformly bounded in } L^2(0, T; V_{\text{div}}).$$

We can find sub-sequences (still denoted by same subscript) and $\varphi^* \in L^\infty(0, T; H) \cap L^2(0, T; V), u^* \in L^2(0, T; V)$ such that

$$\varphi_n \rightharpoonup \varphi^* \quad \text{in} \quad L^\infty(0, T; H),$$

$$\varphi_n \rightarrow \varphi^* \quad \text{in} \quad L^2(0, T; V),$$

$$\varphi'_n \rightharpoonup \varphi^*_t \quad \text{in} \quad L^2(0, T; V'),$$

$$u_n \rightharpoonup u^* \quad \text{in} \quad L^2(0, T; V_{\text{div}}).$$

By Aubin-Lions compactness lemma, we get

$$\varphi_n \xrightarrow{x} \varphi^* \quad \text{in} \quad C([0, T]; H),$$
which gives

\[
\mu_n = a\varphi_n - J * \varphi_n + F'(*) \rightharpoonup a\varphi^* - J * \varphi^* + F'(*) = \mu.
\]

Using these convergences, we pass to the limit in the weak formulation of (3.3)–(3.8), like in [8], written for each \( n \in \mathbb{N} \), then we can see that \([\varphi^*, u^*] = S(U^*)\). Since \( \mathcal{J} \) is convex and continuous functional, it follows that \( \mathcal{J} \) is weakly lower semi continuous. Hence, we have

\[
\mathcal{J}(\varphi^*, u^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(\varphi_n, u_n, U_n),
\]

which implies

\[
l \leq \mathcal{J}(\varphi^*, u^*, U^*) \leq \liminf_{n \to \infty} \mathcal{J}(\varphi_n, u_n, U_n) = \lim_{n \to \infty} \mathcal{J}(\varphi_n, u_n, U_n) = l.
\]

Hence, we conclude that \([\varphi^*, u^*] \) is the optimal state with the optimal control \( U^* \).

### 3.2 Linearised System

Let \( U^* \) be a locally optimal control and \([\varphi^*, u^*] \) be the corresponding strong solution of the system (3.3)–(3.8) (in the sense of Theorem 2.6). Let \( U \in \mathcal{U} \) be given. Let us denote by \( \psi = \varphi - \varphi^* \) and \( w = u - u^* \). Then, the following system is obtained by linearising the system (3.3)–(3.8) around an optimal state \([\varphi^*, u^*] \).

\[
\begin{align*}
\psi_t + w \cdot \nabla \varphi^* + u^* \nabla \psi &= \Delta \tilde{\mu}, \quad \text{in} \quad \Omega \times (0, T), \\
\tilde{\mu} &= a\psi - J * \psi + F''(*)\psi, \quad \text{in} \quad \Omega \times (0, T), \\
-\nu \Delta w + w + \nabla \pi_w &= \tilde{\mu} \nabla \varphi^* + \mu^* \nabla \psi + U, \quad \text{in} \quad \Omega \times (0, T), \\
\text{div} (w) &= 0, \quad \text{in} \quad \Omega \times (0, T), \\
w &= \frac{\partial \tilde{\mu}}{n} = 0, \quad \text{on} \quad \partial \Omega \times (0, T), \\
\psi(0) &= \psi_0(x), \quad \text{in} \quad \Omega,
\end{align*}
\]

where \( \mu^* = a\varphi^* - J * \varphi^* + F'(*) \).

**Remark 3.2** In the above system, we have linearised around \([\varphi^*, u^*] \); the strong solution of the system with locally optimal control \( U^* \) which is useful in the next section. However, we can linearise the system (3.3)–(3.8) around any strong solution \([\bar{\varphi}, \bar{u}] \) with control \( \bar{U} \in \mathcal{U} \), and the existence of solution result, namely Theorem 3.3, can be proved similarly for this general case.

**Theorem 3.3** Suppose that (H1)–(H5) and (H7) are satisfied. Then, for every \( U \in \mathcal{U} \) there exists a unique weak solution for (3.9)–(3.14), such that

\[
\psi \in C([0, T]; H) \cap L^2(0, T; V) \cap H^1(0, T; V'), \quad w \in L^2(0, T; \mathcal{V}_{\text{div}}).
\]

**Proof** We prove the existence of solution for linearised system using Faedo-Galerkin approximation scheme using the method in [8]. We consider the families of functions \( (\eta_k) \) and \( (v_k) \), eigenfunctions of the operator \(-\Delta + I : \mathcal{D}(\mathcal{B}) \to H \) and of the Stokes operator, respectively. Now, we define a finite dimensional subspaces \( \Psi_n := \langle \eta_1, \ldots, \eta_n \rangle \).
and $V = \langle v_1, \ldots, v_n \rangle$ spanned by first $n$ functions of respective spaces, and orthogonal projectors on these spaces, $P_n := P_{v_n}$ and $P_n := P_{\psi_n}$. Let the functions

$$\psi_n(t) := \sum_{i=1}^{n} a_i^{(n)}(t) \eta_i, \quad w_n(t) := \sum_{i=1}^{n} b_i^{(n)}(t) v_i,$$

as a solution of the following approximation

$$(\psi_n(t), \eta_i) + (w_n(t) \cdot \nabla \psi^*(t), \eta_i) + (u^*(t) \cdot \nabla \psi_n(t), \eta_i) = - (\nabla \tilde{\mu}_n(t), \nabla \eta_i),$$

(3.15)

$$(\nabla w_n(t), \nabla v_i) + (w_n(t), v_i) = (\tilde{\mu}_n(t) \nabla \psi^*(t), v_i) + (\mu^*(t) \nabla \psi_n(t), v_i) + (U(t), v_i),$$

(3.16)

$$\psi_n(0) = 0.$$  

(3.17)

for $i = 1, \ldots, n$. This is nothing but a Cauchy problem for a system of $2n$ ordinary differential equations in the $2n$ unknowns $a_i^{(n)}$ and $b_i^{(n)}$. Using the Cauchy-Lipschitz theorem, we find a unique solution $(\psi_n, w_n)$ to the approximated system. Now, multiplying (3.15) by $a_i^{(n)}$, and (3.16) by $b_i^{(n)}$ and summing over $i = 1, \ldots, n$, we get

$$v \|\nabla w_n\|^2 + \|w_n\|^2 + \frac{d}{dt} \|\psi_n\|^2 + (w_n \cdot \nabla \phi^*, \psi_n) + (u^* \cdot \nabla \psi_n, \psi_n) + (\nabla \tilde{\mu}_n, \nabla \psi_n) = (\mu^* \nabla \psi^*, w_n) + (\mu^* \nabla \psi_n, w_n) + (U(t), w_n).$$

(3.18)

We estimate the terms in (3.18).

$$|(w_n \cdot \nabla \phi^*, \psi_n)| \leq \|w_n\|_{L^1} \|\nabla \phi^*\|_{L^1} \|\psi_n\| \leq \frac{v}{4} \|\nabla w_n\|^2 + C \|\nabla \phi^*\|^2_{L^2} \|\psi_n\|^2,$$

(3.19)

$$|(u^* \cdot \nabla \psi_n, \psi_n)| \leq \|u^*\|_{L^\infty} \|\nabla \psi_n\| \|\psi_n\| \leq \frac{C_0}{8} \|\nabla \psi_n\|^2 + C \|u^*\|^2_{L^1} \|\psi_n\|^2.$$  

(3.20)

Using (H3), we get

$$(\nabla \tilde{\mu}_n, \nabla \psi_n) = (\nabla (a \psi_n - J \ast \psi_n + F''(\psi^*) \psi_n), \nabla \psi_n)$$

$$= (a \nabla \psi_n + \psi_n \nabla a - \nabla J \ast \psi_n + F''(\psi^*) \nabla \psi_n + F'''(\psi^*) \nabla \psi^* \psi_n, \nabla \psi_n)$$

$$\geq C_0 \|\nabla \psi_n\|^2 + (\psi_n \nabla a, \nabla \psi_n) - (\nabla J \ast \psi_n, \nabla \psi_n) + (F'''(\psi^*) \nabla \psi^* \psi_n, \nabla \psi_n).$$

(3.21)

We estimate the right-hand side terms of (3.21) using Hölder’s and Young’s inequalities,

$$(\psi_n \nabla a, \nabla \psi_n) \leq \|\psi_n\| \|\nabla a\|_\infty \|\nabla \psi_n\| \leq \frac{C_0}{4} \|\nabla \psi_n\|^2 + \frac{1}{C_0} \|\psi_n\|^2 \|\nabla a\|^2,$$

(3.22)

$$(\nabla J \ast \psi_n, \nabla \psi_n) \leq \|\nabla J\|_{L^1} \|\psi_n\| \|\nabla \psi_n\| \leq \frac{C_0}{4} \|\nabla \psi^2\| + \frac{1}{C_0} \|\nabla J\|^2_{L^1} \|\psi_n\|^2,$$

(3.23)

and using Hölder’s, Young’s and Gagliardo-Nirenberg inequalities,

$$(F'''(\psi^*) \nabla \psi^* \psi_n, \nabla \psi_n) \leq C \|\nabla \psi^*\|_{L^1} \|\psi_n\| \|\nabla \psi_n\|$$

$$\leq C \|\nabla \psi^*\|_{L^1} (\|\psi_n\| + \|\psi_n\|^{1/2} \|\nabla \psi_n\|^{1/2}) \|\nabla \psi_n\|$$

$$\leq \frac{C_0}{4} \|\nabla \psi_n\|^2 + C \|\nabla \psi^*\|^2_{L^4} \|\psi_n\|^2 + C \|\nabla \psi^*\|^4_{L^4} \|\psi_n\|^2.$$  

(3.24)

Substituting (3.22)–(3.24) in (3.21), we get

$$(\nabla \tilde{\mu}_n, \nabla \psi_n) \geq \frac{C_0}{4} \|\nabla \psi_n\|^2 - C \|\psi_n\|^2.$$  

(3.25)
Using (2.48) to estimate right-hand side terms of (3.18), we get
\[ |(\bar{\mu}_n \nabla \varphi^*, w_n)| = \| \bar{\mu}_n \| \| \nabla \varphi^* \|_{L^2} \| w_n \|_{L^2} \leq C \| \psi_n \| \| \nabla \varphi^* \|_{L^2} \| \nabla w_n \| \leq \frac{v}{4} \| \nabla w_n \|^2 + C \| \varphi \|_2 \| \nabla \varphi^* \|_{L^2}^2, \]  
(3.26)

\[ |(\mu^* \nabla \psi_n, w_n)| = |(\psi_n \nabla \mu^*, w_n)| \leq \| \nabla \mu^* \|_{L^2} \| \psi_n \|_{L^2} \| w_n \|_{L^2} \leq \| \nabla \mu^* \|_{L^2} \| \psi_n \| \| \nabla w_n \| \leq \frac{v}{4} \| \nabla w_n \|^2 + \frac{1}{8} \| \nabla \mu^* \|_{L^2}^2 \| \psi_n \|_2^2, \]  
(3.27)

and
\[ |(U(t), w_n)| \leq \| U \| \| w_n \| \leq \frac{1}{2} \| w_n \|^2 + \frac{1}{2} \| U \|^2. \]  
(3.28)

Substituting (3.19), (3.20) and (3.25)–(3.28) in (3.18), we arrive at
\[ \frac{d}{dt} \| \psi_n \|^2 + \frac{v}{4} \| \nabla w_n \|^2 + \frac{1}{2} \| w_n \|^2 + \frac{C_0}{8} \| \nabla \psi_n \|^2 \leq C(1 + \| u^* \|_{H^1}) \| \psi_n \|^2 + \frac{1}{2} \| U \|^2. \]  
(3.29)

By employing the Grönwall’s lemma, we conclude that
\[ \| \psi_n(t) \|^2 \leq \exp \left( C \int_0^t (1 + \| u^*(s) \|_{H^1}) ds \right) \left( \int_0^t \| U(s) \|^2 ds \right), \]
for all \( t \in [0, T] \). Since \( u^* \in L^2(0, T; \mathbb{H}^2) \), we have
\[ \| \psi_n \|_{L^\infty(0, T; H)} \leq C \| U \|_{\mathcal{L}^1}, \]  
(3.30)

and integrating (3.29) from 0 to \( t \), we get
\[ \| \psi_n \|_{L^2(0, T; V)} \leq C \| U \|_{\mathcal{L}^1}, \]  
(3.31)
\[ \| w_n \|_{L^2(0, T; \mathcal{V}_{div})} \leq C \| U \|_{\mathcal{L}^1}. \]  
(3.32)

Moreover, from (3.15), we get
\[ \| (\psi_n)_t \|_{V'} \leq C(\| \nabla w_n \| \| \nabla \varphi^* \| + \| \nabla u^* \| \| \nabla \psi_n \| + C \| \psi_n \|_V + \| \nabla \varphi^* \|_{L^2} \| \psi_n \|_V), \]
that is
\[ \| (\psi_n)_t \|_{L^2(0, T; V')} \leq C \| U \|_{\mathcal{L}^1}. \]  
(3.33)

for every \( n \in \mathbb{N} \). From above uniform bounds, we can obtain sub-sequences of \( \{ \psi_n \} \), \( \{ (\psi_n)_t \} \) and \( \{ w_n \} \), again denoted by \( \{ \psi_n \} \), \( \{ (\psi_n)_t \} \) and \( \{ w_n \} \) and functions \( \psi \in L^\infty(0, T; H) \cap L^2(0, T; V) \), \( \psi_t \in L^2(0, T; V') \) and \( w \in L^2(0, T; \mathcal{V}_{div}) \), such that
\[ \psi_n \rightharpoonup^w \psi \quad \text{in} \quad L^\infty(0, T; H), \]
\[ \psi_n \rightharpoonup^w \psi \quad \text{in} \quad L^2(0, T; V), \]
\[ (\psi_n)_t \rightharpoonup^w \psi_t \quad \text{in} \quad L^2(0, T; V'), \]
\[ w_n \rightharpoonup^w w \quad \text{in} \quad L^2(0, T; \mathcal{V}_{div}). \]

By passing to the limit in (3.15)–(3.17), we can say that there exists a weak solution \( (w, \psi) \in L^2(0, T; \mathcal{V}_{div}) \times L^\infty(0, T; H) \cap L^2(0, T; V) \cap H^1(0, T; V'). \) By Aubin’s
compactness lemma, we have that $\psi_n \rightarrow \psi$ in $L^2(0, T; H)$ and $\psi \in C([0, T]; H)$. This gives,

$$(\psi, w) \in (C([0, T]; H) \cap L^2(0, T; V) \cap H^1(0, T; V')) \times L^2(0, T; V_{\text{div}}).$$

To prove that the solution $[\psi, w]$ of (3.9)–(3.14) is unique, let $[\psi_1, w_1]$ and $[\psi_2, w_2]$ be any two solutions of (3.9)–(3.14). Let $\psi = \psi_1 - \psi_2$ and $w = w_1 - w_2$. Then, $[\psi, w]$ satisfies

$$\begin{align*}
\psi_t + w \cdot \nabla \varphi^* + u^* \nabla \psi &= \Delta \tilde{\mu}, \\
\tilde{\mu} &= a\psi - J * \psi + F''(\varphi^*)\psi, \\
-\nu \Delta w + w + \nabla \tilde{\pi} &= \tilde{\mu} \nabla \varphi^* + \mu^* \nabla \psi, \\
\nabla \cdot (w) &= 0, \\
w|_{\partial\Omega} = \frac{\partial \tilde{\mu}}{\partial n} |_{\partial\Omega} = 0, \\
\psi(0) &= \psi_0(x).
\end{align*}$$

We take inner product of (3.34) and (3.36) with $\psi$ and $w$, respectively. Making use of the estimates derived for the terms in (3.18), we can prove that the weak solution of the system (3.9)–(3.14) is unique. From the estimates (3.30)–(3.33), we can also conclude that the mapping $U \mapsto [\psi, w]$ is a continuous linear mapping from $U$ to $L^\infty(0, T; H) \cap L^2(0, T; V) \cap H^1(0, T; V') \times L^2(0, T; V_{\text{div}})$.

### 3.3 Differentiability of Control-to-State Operator

In this section, we prove the differentiability of the control-to-state operator. We need the following assumption on $F$, namely,

$$(H8) \quad F \in C^4(\mathbb{R}).$$

**Theorem 3.4** Suppose hypotheses $(H1)$–$(H5)$ and $(H7)$–$(H8)$ are satisfied. Then, the control-to-state operator $S : U \rightarrow V$ is Fréchet differentiable. Moreover, for any $\tilde{U} \in U$, the Fréchet derivative $S'$ at $\tilde{U}$ in the direction of $U$ is given by

$$S'(\tilde{U})(U) = (\tilde{\psi}, \tilde{w}),$$

for every $U \in U$, where $[\tilde{\psi}, \tilde{w}]$ is the unique weak solution of the linearised system with control $U$, which is linearised around strong solution of the controlled system (3.3)–(3.8) with control $\tilde{U}$.

**Proof** Note that the theorem states for any control $\tilde{U}$,

$$S'(\tilde{U})(U) = (\tilde{\psi}, \tilde{w}),$$

where $[\tilde{\psi}, \tilde{w}]$ is the unique weak solution of the linearised system (3.9)–(3.14) with control $U$. The Fréchet derivative at optimal control is going to be useful for us to characterise first-order optimality condition. Hence, we will prove the theorem for an optimal control $U^\ast$. The general statement can be proved analogously.
Let \([\varphi^*, u^*] = S(U^*)\) be the solution to the system (3.3)–(3.8) with control \(U^*\). Let \([\tilde{\varphi}, \tilde{u}]\) be solution of the system (3.3)–(3.8) with \(U^* + U\). Let \(\xi = \tilde{\varphi} - \varphi^*, \ z = \tilde{u} - u^*\). Then, \(\xi, z\) satisfies,

\[
\xi_t + z \cdot \nabla \xi + z \cdot \nabla \varphi^* + u^* \nabla \xi = \Delta \mu_{\xi},
\]

\[
\mu_{\xi} = a\dot{\xi} - J * \xi + F'(\tilde{\varphi}) - F'(\varphi^*),
\]

\[\begin{align*}
\nabla \pi_z = \mu_{\xi} \nabla \varphi^* + \mu_{\xi} \nabla \xi + \mu_{\xi} \nabla \mu_{\xi} + U, \\
\text{div} \ z = 0,
\end{align*}
\]

\[
z_{|\partial \Omega} = 0, \quad \frac{\partial \mu_{\xi}}{\partial n}_{|\partial \Omega} = 0,
\]

\[
\xi(0) = 0,
\]

where \(\pi_z = \pi_{\tilde{u}} - \pi_{u^*}\) with \(\pi_{\tilde{u}}\) and \(\pi_{u^*}\) are the pressure terms appearing in (3.5) for \(U^* + U\) and \(U^*\), respectively. Now, let us define \(\rho = \xi - \psi, \ y = z - w\) where \([\psi, w]\) is the solution of the linearised system (3.9)–(3.14), corresponding to \(U^*\). Then, \((\rho, y)\) satisfies

\[
\rho_t + y \cdot \nabla \varphi^* + u^* \cdot \nabla \rho + z \cdot \nabla \xi = \Delta \mu_{\rho},
\]

\[
\mu_{\rho} = a\rho - J * \rho + F'(\tilde{\varphi}) - F'(\varphi^*) - F''(\varphi^*)\psi,
\]

\[\begin{align*}
-\nabla \Delta y + y + \nabla \pi_y &= \mu_{\rho} \nabla \varphi^* + (a\xi - J * \xi + F' (\tilde{\varphi}) - F'(\varphi^*)) \nabla \xi \\
&+ (a\varphi^* - J * \varphi^* + F'(\varphi^*)) \nabla \rho, \\
\text{div} \ y &= 0,
\end{align*}
\]

\[
y = 0, \quad \frac{\partial \mu_{\rho}}{\partial n} = 0,
\]

\[
\rho(0) = 0,
\]

where \(\pi_y = \pi_{\tilde{u}} - \pi_{u^*} - \pi_w\) with \(\pi_w\) is the pressure term appearing in (3.11). Now, our aim is to prove that

\[
\frac{\|\rho, y\|_V}{\|U\|_{L^4}} \to 0 \quad \text{as} \quad \|U\|_{L^4} \to 0.
\]  (3.46)

For, take inner product of \(\rho\) with (3.40) and of \(y\) with (3.42), to get

\[
\frac{1}{2} \frac{d}{dt} \|\rho(t)\|^2 + (y \cdot \nabla \varphi^*, \rho) + (u^* \cdot \nabla \rho, \rho) + (z \cdot \nabla \xi, \rho)
\]

\[= (\Delta (a\rho - J * \rho + F'(\tilde{\varphi}) - F'(\varphi^*) - F''(\varphi^*)\psi), \rho),
\]

\[\nabla \|\nabla y\|^2 + \|y\|^2 = (\mu_{\rho} \nabla \varphi^*, y) + ((a\xi - J * \xi + F'(\tilde{\varphi}) - F'(\varphi^*)) \nabla \xi, y)
\]

\[+(a\varphi^* - J * \varphi^* + F'(\varphi^*)) \nabla \rho, y). \quad \]  (3.48)

We estimate right-hand side terms of (3.47) one by one,

\[
(\mu_{\rho} \nabla \varphi^*, y) = ((a\rho - J * \rho + F'(\tilde{\varphi}) - F(\varphi^*) - F''(\varphi^*)\psi) \nabla \varphi^*, y)
\]

\[= (a\rho \nabla \varphi^*, y) - ((J * \rho) \nabla \varphi^*, y) - ((F'(\tilde{\varphi}) - F(\varphi^*) - F''(\varphi^*)\psi) \nabla \varphi^*, y)
\]

\[= I_1 + I_2 + I_3.
\]
where $I_1, I_2$ and $I_3$ can be estimated as follows

$$
|I_1| \leq \|a\|_{L^\infty} \|\rho\| \|\nabla \phi^*\|_{L^4} \|y\|_{L^4} \leq \|a\|_{L^\infty} \|\rho\| \|\nabla \phi^*\|_{L^4} \|\nabla y\|
$$

$$
\leq \frac{\nu}{10} \|\nabla y\|^2 + C \|\nabla \phi^*\|^2_{L^4} \|\rho\|^2, \quad (3.49)
$$

$$
|I_2| = \|J \ast \rho\| \|\nabla \phi^*\|_{L^4} \|y\|_{L^4} \leq \|J\|_{L^1} \|\rho\| \|\nabla \phi^*\|_{L^4} \|\nabla y\|
$$

$$
\leq \frac{\nu}{10} \|\nabla y\|^2 + C \|\nabla \phi^*\|^2_{L^4} \|\rho\|^2, \quad (3.50)
$$

and notice that, since $\rho = \phi - \phi^* - \psi$, using Taylor series we can write

$$F'(\phi) - F'(\phi^*) - F''(\phi^*)\psi = F'(\phi) - F'(\phi^*)\phi + F''(\phi^*)\rho$$

$$= \frac{1}{2} F''(\theta\phi + (1 - \theta)\phi^*)\phi + F''(\phi^*)\rho, \quad (3.51)$$

for some $\theta \in (0, 1)$. Then,

$$|I_3| = |(\frac{1}{2} F''(\theta\phi + (1 - \theta)\phi^*)\xi \nabla \phi^*, y) + (F''(\phi^*)\rho \nabla \phi^*, y)|
$$

$$\leq C_F \|\xi\|^2 \|\nabla \phi^*\|_{L^4} \|y\|_{L^4} + C_F \|\rho\| \|\nabla \phi^*\|_{L^4} \|y\|_{L^4}
$$

$$\leq C_F \|\xi\|^2 \|\nabla \phi^*\|_{L^4} \|\nabla y\| + C_F \|\rho\| \|\nabla \phi^*\|_{L^4} \|\nabla y\|
$$

$$\leq \frac{\nu}{10} \|\nabla y\|^2 + C \|\xi\|^2 \|\nabla \phi^*\|^2_{L^4} + \frac{\nu}{10} \|\nabla y\|^2 + C \|\rho\|^2 \|\nabla \phi^*\|^2_{L^4}, \quad (3.52)
$$

Combining (3.49), (3.50) and (3.52), we get

$$|\langle \mu, \rho \nabla \phi^*, y \rangle| \leq \frac{2\nu}{5} \|\nabla y\|^2 + C \|\nabla \phi^*\|^2_{L^4} \|\rho\|^2 + C \|\xi\|^2 \|\nabla \phi^*\|^2_{L^4}. \quad (3.53)
$$

Using integration by parts, we estimate the second term on the right-hand side of (3.47) as follows

$$((\phi^* - J \ast \phi^* + F'(\phi^*))\nabla \rho, y) = -((a\nabla \phi^* + \phi^* \nabla a - \nabla J \ast \phi^* + F''(\phi^*)\nabla \phi^*)\rho, y),$$

$$\|a\nabla \phi^* \rho, y\| \leq \|a\|_{L^\infty} \|\nabla \phi^*\|_{L^4} \|\rho\| \|y\|_{L^4} \leq \frac{\nu}{20} \|\nabla y\|^2 + C \|\nabla \phi^*\|^2_{L^4} \|\rho\|^2, \quad (3.54)$$

$$\|\phi^* \rho \nabla a, y\| \leq \|\phi^*\|_{L^4} \|\rho\| \|\nabla a\|_{L^\infty} \|y\|_{L^4} \leq \frac{\nu}{20} \|\nabla y\|^2 + C \|\phi^*\|^2_{L^4} \|\rho\|^2, \quad (3.55)$$

$$\|((\nabla J \ast \phi^*)\rho, y\| \leq \|\nabla J\|_{L^1} \|\phi^*\|_{L^4} \|\rho\| \|y\|_{L^4} \leq \frac{\nu}{20} \|\nabla y\|^2 + C \|\phi^*\| v \|\rho\|^2, \quad (3.56)$$

Combining above four estimates gives,

$$|((\phi^* - J \ast \phi^* + F'(\phi^*))\nabla \rho, y)\| \leq \frac{\nu}{20} \|\nabla y\|^2 + C \|\phi^*\|^2_{V} + \|\nabla \phi^*\|^2_{L^4} \|\rho\|^2. \quad (3.57)$$

The third term on the right-hand side of (3.47) can be estimated as

$$\|((a\xi - J \ast \xi + F'(\phi) - F'((\phi^*)))\nabla \xi, y\|$$

$$= \|a\nabla \xi \|_2^2, y\| + |((\nabla J \ast \xi)\xi, y\| + |(F'(\phi) + (1 - \theta)\phi^*))\xi \nabla \xi, y\|$$

$$\leq \|\nabla a\|_{L^\infty} \|\xi\|^2 \|\nabla y\| + \|\nabla J\|_{L^1} \|\xi\|_{L^4} \|\xi\| \|y\|_{L^4} + C_F \|\xi\|_{L^4} \|\nabla \xi\| \|y\|_{L^4}
$$

$$\leq \frac{\nu}{5} \|\nabla y\|^2 + C \|\xi\|^2_{V}. \quad (3.58)$$
Substituting (3.53), (3.54) and (3.55) in (3.47), we get
\[
\nu \|\nabla y\|^2 + \|y\|^2 \leq \frac{2\nu}{5} \|\nabla y\|^2 + C \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2 + C \|\xi\|_{V}^4 \|\nabla \varphi^*\|_{L^4}^2
\]
\[
+ \frac{\nu}{5} \|\nabla y\|^2 + C \|\varphi^*\|_{V}^2 + \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2 + \frac{\nu}{5} \|\nabla y\|^2 + C \|\xi\|_{V}^4,
\]
which implies
\[
\frac{\nu}{5} \|\nabla y\|^2 + \|y\|^2 \leq C \|\nabla \varphi^*\|_{L^4}^2 + \|\varphi^*\|_{V}^2 + \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2 + C \|\xi\|_{V}^4 \|\nabla \varphi^*\|_{L^4}^2 + C \|\xi\|_{V}^4. \quad (3.56)
\]
We estimate the terms in (3.47) as follows
\[
|\langle y \cdot \nabla \varphi^*, \rho \rangle| \leq \|y\|_{L^4} \|\nabla \varphi^*\|_{L^4} \|\rho\| \leq \frac{\nu}{10} \|\nabla y\|^2 + C \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2. \quad (3.57)
\]
Observe that,
\[
(u^* \cdot \nabla, \rho) = \left( u^*, \nabla \left( \frac{\rho^2}{2} \right) \right) = \left( \text{div}(u^*), \frac{\rho^2}{2} \right) = 0, \quad (3.58)
\]
\[
|\langle z \cdot \nabla \xi, \rho \rangle| \leq \|z\|_{L^4} \|\nabla \xi\| \|\rho\|_{L^4} \leq C_0 \|\nabla z\| \|\nabla \xi\| \|\nabla \rho\| + \|\rho\| \leq \frac{C_0}{10} \|\nabla \rho\|^2 + C \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2 + \frac{1}{2} \|\rho\|^2 + C \|\nabla z\| \|\nabla \xi\|^2 \leq \frac{C_0}{10} \|\nabla \rho\|^2 + \frac{1}{2} \|\rho\|^2 + C \|\nabla \varphi^*\|_{L^4}^2 \|\rho\|^2. \quad (3.59)
\]
From (3.51), we can write
\[
(\Delta (a \rho - J * \rho + F'(\bar{\varphi}) - F'(\varphi^*) - F''(\varphi^*)\psi), \rho)
\]
\[
= -(\nabla (a \rho - J * \rho + \frac{1}{2} F'''(\theta \bar{\varphi} + (1 - \theta) \varphi^*) \xi^2 + F''(\varphi^*) \rho), \nabla \rho) + (\nabla J * \rho, \nabla \rho) - (F'''(\theta \bar{\varphi} + (1 - \theta) \varphi^*) \xi \nabla \rho, \nabla \rho) - (\frac{1}{2} F^{(d)}(\theta \bar{\varphi} + (1 - \theta) \varphi^*)(\nabla \bar{\varphi} + (1 - \theta) \nabla \varphi^*)) \xi^2, \nabla \rho) - (F'''(\varphi^*) \nabla \varphi^* \rho, \nabla \rho) - (F''(\varphi^*) \nabla \rho, \nabla \rho)
\leq - C_0 \|\nabla \rho\|^2 - (\rho \nabla a, \nabla \rho) + (\nabla J * \rho, \nabla \rho) - (F'''(\theta \bar{\varphi} + (1 - \theta) \varphi^*) \xi \nabla \rho, \nabla \rho) - (\frac{1}{2} F^{(d)}(\theta \bar{\varphi} + (1 - \theta) \varphi^*)(\nabla \bar{\varphi} + (1 - \theta) \nabla \varphi^*)) \xi^2, \nabla \rho) - (F'''(\varphi^*) \nabla \varphi^* \rho, \nabla \rho) - (F''(\varphi^*) \nabla \varphi^* \rho, \nabla \rho). \quad (3.60)
\]
where in the last estimate we have used (2.1). Now we estimate right-hand side terms of (3.60) using Hölder and Sobolev inequalities
\[
|\langle \rho \nabla a, \nabla \rho \rangle| \leq \|\nabla a\|_\infty \|\rho\| \|\nabla \rho\| \leq \frac{C_0}{10} \|\nabla \rho\|^2 + C \|\rho\|^2, \quad (3.61)
\]
\[
|\langle \nabla J * \rho, \nabla \rho \rangle| \leq \|\nabla J\|_{L^1} \|\rho\| \|\nabla \rho\| \leq \frac{C_0}{10} \|\nabla \rho\|^2 + C \|\rho\|^2, \quad (3.62)
\]
\[
|\langle \frac{1}{2} F^{(d)}(\theta \bar{\varphi} + (1 - \theta) \varphi^*)(\nabla \bar{\varphi} + (1 - \theta) \nabla \varphi^*)) \xi^2, \nabla \rho \rangle|
\leq C_F (\|\nabla \bar{\varphi}\|_{L^4} + \|\nabla \varphi^*\|_{L^4}) \|\nabla \rho\| \|\nabla \rho\| \leq \frac{C_0}{10} \|\nabla \rho\|^2 + C (\|\nabla \bar{\varphi}\|_{L^4} + \|\nabla \varphi^*\|_{L^4})^2 \|\xi\|_{L^8}^4 \leq \frac{C_0}{10} \|\nabla \rho\|^2 + C (\|\nabla \bar{\varphi}\|_{L^4}^2 + \|\nabla \varphi^*\|_{L^4}^2) \|\xi\|_{L^8}^4, \quad (3.63)
\]
\[(F''(\theta \phi + (1 - \theta) \phi^* ) \phi \nabla \xi, \phi \nabla \rho) \leq C F \| \nabla \xi \|_{L^2} \| \nabla \rho \| \leq C F \| \nabla \xi \|_{H^2} \| \nabla \rho \|
\]
\[
\leq \frac{C_0}{10} \| \nabla \rho \|^2 + C \| \xi \|_{L^2}^2 \| \xi \|_{H^2}^2.
\] (3.64)

and using Gagliardo-Nirenberg inequality we get

\[|F''(\phi^*) \nabla \phi^* \rho, \phi \nabla \rho)| \leq C \| F \nabla \phi^* \|_{L^4} \| \rho \|_{L^4} \| \nabla \rho \|
\]
\[
\leq C \| \rho \|_{L^4} \| \nabla \rho \| \leq C \| \nabla \phi^* \|_{L^2} (\| \rho \|_{L^2}^{1/2} \| \nabla \rho \|_{L^2}^{1/2} + \| \rho \| \| \nabla \rho \|
\]
\[
\leq \frac{C_0}{5} \| \nabla \rho \|^2 + C (\| \nabla \phi^* \|_{L^4}^2 + \| \nabla \phi^* \|_{H^4}^2) \| \rho \|^2.
\] (3.65)

Substituting (3.61)–(3.65) in (3.60)

\[|\Delta (\mu \rho - J * \rho + F'(\phi) - F'(\phi^*) \psi), \rho)\| + \frac{4C_0}{10} \| \nabla \rho \|^2 \leq C (1 + \| \nabla \phi^* \|_{L^4}^2 + \| \nabla \phi^* \|_{H^4}^2) \| \rho \|^2 + C (\| \nabla \phi \|_{L^4} + \| \nabla \phi \|_{H^4}) \| \xi \|_{L^2} + C \| \xi \|_{L^2}^2 \| \xi \|_{H^2}^2.
\] (3.66)

Using (3.57), (3.59) and (3.66) in (3.47), we get

\[\frac{1}{2} \frac{d}{dt} \| \rho(t) \|^2 + \frac{3C_0}{10} \| \nabla \rho \|^2 \leq \frac{\nu}{10} \| \nabla y \|^2 + C (1 + \| \nabla \phi^* \|_{L^4}^2 + \| \nabla \phi^* \|_{H^4}^2) \| \rho \|^2 + C \| \nabla \phi \|_{L^4} + \| \nabla \phi \|_{H^4}) \| \xi \|_{L^2} + C \| \xi \|_{L^2}^2 \| \xi \|_{H^2}^2.
\] (3.67)

Combining (3.56) and (3.67)

\[\frac{1}{2} \frac{d}{dt} \| \rho(t) \|^2 + \frac{\nu}{10} \| \nabla y \|^2 + \| y \|^2 + \frac{3C_0}{10} \| \nabla \rho \|^2 \leq C (1 + \| \nabla \phi^* \|_{H^1}^2 + \| \nabla \phi^* \|_{H^1}^2 + \| \phi^* \|_{H^1}^2) \| \rho \|^2 + C \| \nabla \phi \|_{L^4} + \| \nabla \phi \|_{H^4}) \| \xi \|_{L^2} + C \| \xi \|_{L^2}^2 \| \xi \|_{H^2}^2
\]
\[+ C \| \xi \|_{L^4} + \| \nabla \phi^* \|_{L^4} + C \| \xi \|_{L^4}^2.
\] (3.68)

By Grönwall’s lemma (differential form), we deduce that

\[\| \rho(t) \|^2 \leq \exp \left( C \int_0^t (1 + \| \nabla \phi^* \|_{L^4}^2 + \| \nabla \phi^* \|_{H^4}^2 + \| \phi^* \|_{L^2}^2) ds \right) \left( \int_0^T \alpha(t) dt \right),
\]

where \( \alpha(t) = C \| \nabla \phi \|_{L^4}^2 + C \| \nabla \phi \|_{H^4}^2 + C \| \nabla \phi \|_{L^4}^2 + \| \phi^* \|_{L^2}^2 + C \| \xi \|_{L^2}^2 \| \xi \|_{H^2}^2 + C \| \xi \|_{L^2}^2 \| \nabla \phi^* \|_{L^4}^2 + C \| \xi \|_{L^4}^2.

Using difference estimates given in (2.37) and (2.55), we can show that

\[\int_0^T \alpha(t) dt \leq \| U \|_{L^2(0,T; V_{div})}^4,
\]

and hence

\[\| \rho(t) \|^2 \leq \| U \|_{L^2(0,T; V_{div})}^4.
\]

Integrating (3.68), we get

\[\| \rho \|_{L^4(0,T; V)}^2 + \| \rho \|_{L^2(0,T; V_{div})}^2 \leq C \| U \|_{L^2(0,T; V_{div})}^4,
\]

which gives

\[\| \rho \|_{L^2(0,T; H)}^2 + \| \rho \|_{L^2(0,T; V)}^2 + \| \rho \|_{L^2(0,T; V_{div})}^2 \leq C \| U \|_{L^2(0,T; V_{div})}^4.
\]
We use the inclusions $U \subset L^\infty(0, T; G_{\text{div}}) \subset L^2(0, T; \mathbb{V}'_{\text{div}})$,
\[ \|S(U^* + U) - S(U^*) - [w, \psi]\|_{\mathbb{V}} \leq C\|U\|_{L^4} \]
Thus, as $\|U\|_{L^4} \to 0$ left-hand side of the above equation will also tend to 0, to conclude hence, the proof of the theorem.

4 Characterisation of Locally Optimal Controls

In this section, we derive the variational inequality satisfied by the optimal control. Further, we introduce the adjoint system (4.2)–(4.6) (below) and discuss its solvability. Finally, we characterise the optimal control in terms of adjoint variables.

4.1 First-order Optimality Condition

We prove the following theorem using the result of Theorem 3.4.

**Theorem 4.1** Suppose hypotheses (H1)–(H5) and (H7)–(H8) are satisfied. Let us assume that $U^* \in U_{\text{ad}}$ is a local minimiser for (OCP) such that $S(U^*) = [\psi^*, u^*]$. Then, optimal triplet satisfies
\[
\int_0^T \int_{\Omega} (\psi^* - \psi_d) \psi \, dx \, dt + \int_0^T \int_{\Omega} (u^* - u_d) \cdot w \, dx \, dt \\
+ \int_{\Omega} (\psi^*(T) - \psi_{\Omega}) \psi(T) \, dx + \int_0^T \int_{\Omega} U^* \cdot (U - U^*) \, dx \, dt \geq 0, \quad \forall U \in U_{\text{ad}}, (4.1)
\]
where $[\psi, w]$ is the unique weak solution of the linearised system (3.9)–(3.14) with $U - U^*$ in (3.11) instead of $U$.

**Proof** Let us denote by $G(U) := J(S(U), U)$ for all $U \in U_{\text{ad}}$. Since $U_{\text{ad}}$ is a convex set, for any minimiser $U^* \in U_{\text{ad}}$, from Lemma 2.21 in [30], we deduce
\[
G'(U^*)(U - U^*) \geq 0, \quad \forall U \in U_{\text{ad}}.
\]
where $G'$ is Fréchet derivative. Since $J$ is in the quadratic functional form, using chain rule we can write the Fréchet derivative of $G$ at every $U^* \in U$ as follows
\[
G'(U^*) = J''_{\psi^*, u^*}(S(U^*), U^*) \circ S'(U^*) + J'_{U^*}(S(U^*), U^*).
\]
Gateaux derivative of $J$ in the direction $(h_1, h_2)$ can be written as
\[
J'_{\psi^*, u^*}(\psi^*, u^*, U^*)(h_1, h_2) = \int_0^T \int_{\Omega} (\psi^* - \psi_d) h_1 \, dx \, dt + \int_0^T \int_{\Omega} (u^* - u_d) \cdot h_2 \, dx \, dt \\
+ \int_{\Omega} (\psi^*(T) - \psi_{\Omega}) h_1(T) \, dx
\]
for all $[h_1, h_2] \in \mathbb{V}$ and
\[
J'_{U^*}(\psi^*, u^*, U^*)(W) = \int_0^T \int_{\Omega} U^* \cdot W \, dx \, dt \quad \forall W \in U.
\]
From Theorem 3.4, we know that $S'(U^*)(U - U^*) = [\psi, w]$. Hence, we get

$$0 \leq (J'_{\psi^*, u^*}(S(U^*), U^*) \circ S'(U^*) + J_{\psi}^\prime(S(U^*), U^*, U - U^*)$$

$$= \int_0^T \int_\Omega (\varphi - \varphi_d) \psi dx dt + \int_0^T \int_\Omega (u^* - u_d) \cdot w dx dt + \int_\Omega (\varphi(T) - \varphi_\Omega) \psi(T) dx$$

$$+ \int_0^T \int_\Omega u^* \cdot (U - U^*) dx dt.$$  

Thus, (4.1) follows. \hfill \Box

### 4.2 Adjoint System

In this subsection, we want to characterise first-order optimality condition by eliminating the linearised variables $[\psi, w]$ from (4.1). For a locally optimal control $U^*$ and corresponding state $(\varphi^*, u^*)$, let the corresponding adjoint variables be denoted by $[\eta, v]$. The adjoint system satisfied by $[\eta, v]$ is given by

$$- \eta_t + v \cdot \nabla \varphi^* + J \ast (v \cdot \nabla \varphi^*) - (\nabla J \ast \varphi^*) \cdot v - u^* \cdot \nabla \eta$$

$$- a \Delta \eta + \nabla J \ast \nabla \eta - F''(\varphi^*) \Delta \eta = \varphi^* - \varphi_d,$$  

(4.2)  
$$- v \Delta v + v + \eta \nabla \varphi^* + \nabla q = u^* - u_d,$$  

(4.3)
$$\text{div} (v) = 0,$$  

(4.4)  
$$v \cdot n|_{\partial \Omega} = \frac{\partial \eta}{\partial n}|_{\partial \Omega} = 0,$$  

(4.5)  
$$\eta(T, \cdot) = \varphi^*(T) - \varphi_\Omega.$$  

(4.6)

For every $\chi \in V$ and $z \in V_{\text{div}}$ for all $t \in [0, T]$, the weak formulation of the above system can be written as

$$- v^* \langle \eta_t, \chi \rangle_V + (v \cdot \nabla \varphi^*, \chi) + (J \ast (v \cdot \nabla \varphi^*), \chi) - ((\nabla J \ast \varphi^*) \cdot v, \chi)$$

$$- (u^* \cdot \nabla \eta, \chi) - (a \Delta \eta, \chi) + (\nabla J \ast \nabla \eta, \chi) - (F''(\varphi^*) \Delta \eta, \chi) = (\varphi^* - \varphi_d, \chi),$$  

(4.7)  
$$v(\nabla v, \nabla z) + (v, z) + (\eta \nabla \varphi^*, z) = (u^* - u_d, z).$$  

(4.8)

**Theorem 4.2** Assume hypotheses $(H1)$–$(H5)$ and $(H7)$–$(H8)$ are satisfied. Then, the adjoint system (4.2)–(4.6) has a unique weak solution $[v, \eta]$ satisfying

$$\eta \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap H^1(0, T; V'),$$

$$v \in L^2(0, T; V_{\text{div}}).$$

**Proof** We can prove the existence of a weak solution using Faedo-Galerkin approximation as in the proof of Theorem 3.3. We derive the basic estimates that a weak solution should satisfy. Let us take $\chi = \eta$ and $z = v$ in (4.7) and (4.8), respectively. This leads to

$$- \frac{1}{2} \frac{d}{dt} \|\eta\|^2 + (v \cdot \nabla \varphi^*, \eta) + (J \ast (v \cdot \nabla \varphi^*), \eta) - ((\nabla J \ast \varphi^*) \cdot v, \eta) - (u^* \cdot \nabla \eta, \eta)$$

$$- (a \Delta \eta, \eta) + (\nabla J \ast \nabla \eta, \eta) - (F''(\varphi^*) \Delta \eta, \eta) = (\varphi^* - \varphi_d, \eta),$$  

(4.9)  
$$v \|\nabla v\|^2 + \|v\|^2 + (\eta \nabla \varphi^*, v) = (u^* - u_d, v).$$  

(4.10)
Terms in (4.9) and (4.10) can be estimated using Gagliardo-Nirenberg and Hölder inequalities as follows

\[ |(\mathbf{v} \cdot \nabla \varphi^*, \eta)| \leq C_J \|\mathbf{v}\|_{L^4} \|\varphi^*\|_{L^4} \|\eta\| \leq \frac{\nu}{10} \|\nabla \mathbf{v}\|^2 + C \|\varphi^*\|^2_{L^4} \|\eta\|^2, \quad (4.11) \]

\[ |(\mathbf{J} * (\mathbf{v} \cdot \nabla \varphi^*), \eta)| \leq C_J \|\mathbf{v}\|_{L^4} \|\nabla \varphi^*\|_{L^4} \|\eta\| \leq \frac{\nu}{10} \|\nabla \mathbf{v}\|^2 + C \|\nabla \varphi^*\|^2_{L^4} \|\eta\|^2, \quad (4.12) \]

\[ |((\nabla \mathbf{J} * \varphi^*) \cdot \mathbf{v}, \eta)| \leq C_J \|\varphi^*\|_{L^4} \|\mathbf{v}\|_{L^4} \|\eta\| \leq \frac{\nu}{10} \|\nabla \mathbf{v}\|^2 + C \|\varphi^*\|^2_{L^4} \|\eta\|^2. \quad (4.13) \]

Using integration by parts and divergence free condition, we get

\[ |(\mathbf{u} \cdot \nabla \eta, \eta)| = 0, \quad (4.14) \]

Observe that by integration by parts we get

\[-(a \Delta \eta, \eta) - (F''(\varphi^*) \Delta \eta, \eta) = (a + F''(\varphi^*) \nabla \eta, \nabla \eta) + (\nabla a, \eta \nabla \eta) + (F'''(\varphi^*) \nabla \varphi^*, \eta \nabla \eta) \geq C_0 \|\nabla \eta\|^2 + (\nabla a, \eta \nabla \eta) + (F'''(\varphi^*) \nabla \varphi^*, \eta \nabla \eta). \quad (4.15)\]

Right-hand side terms of (4.15) can be estimated as follows

\[ |(\nabla \eta, \eta)| \leq C_J \|\nabla \eta\| \|\eta\| \leq \frac{C_0}{6} \|\nabla \eta\|^2 + C_J \|\eta\|^2, \quad (4.16) \]

\[ |(F'''(\varphi^*) \nabla \varphi^*, \eta \nabla \eta)| \leq C_F \|\nabla \varphi^*\|_{L^4} \|\eta\| \|\nabla \eta\| \leq C_F \|\nabla \varphi^*\|_{L^4} (\|\eta\|^2/2 + \|\nabla \eta\|^2/2 + \|\eta\|) \|\nabla \eta\| \leq \frac{C_0}{12} \|\nabla \eta\|^2 + \|\nabla \varphi^*\|^4_{L^4} \|\eta\|^2 + \frac{C_0}{12} \|\nabla \eta\|^2 + \|\nabla \varphi^*\|^2_{L^4} \|\eta\|^2 \leq \frac{C_0}{6} \|\nabla \eta\|^2 + (\|\nabla \varphi^*\|^4_{L^4} + \|\nabla \varphi^*\|^2_{L^4}) \|\eta\|^2. \quad (4.17) \]

Substituting (4.16) and (4.17) in (4.15), we get

\[-(a \Delta \eta, \eta) - (F''(\varphi^*) \Delta \eta, \eta) \geq \frac{2C_0}{3} \|\nabla \eta\|^2 - C \|\eta\|^2 - (\|\nabla \varphi^*\|^4_{L^4} + \|\nabla \varphi^*\|^2_{L^4}) \|\eta\|^2. \quad (4.18) \]

We also have

\[ |(\nabla \mathbf{J} * \nabla \eta, \eta)| \leq C_J \|\nabla \eta\| \|\eta\| \leq \frac{C_0}{6} \|\nabla \eta\|^2 + C_J \|\eta\|^2, \quad (4.19) \]

and

\[ |(\varphi - \varphi_d, \eta)| \leq \|\varphi - \varphi_d\| \|\eta\| \leq \frac{1}{2} \|\varphi - \varphi_d\|^2 + \frac{1}{2} \|\eta\|^2. \quad (4.20) \]

Using (4.11)–(4.14) and (4.18)–(4.19) in (4.9), we get

\[-\frac{1}{2} \frac{d}{dt} \|\eta\|^2 \leq \frac{3\nu}{10} \|\nabla \mathbf{v}\|^2 - \frac{C_0}{2} \|\nabla \eta\|^2 + C (1 + \|\varphi^*\|^2_{L^4} + \|\nabla \varphi^*\|^2_{L^4}) \|\eta\|^2 + \frac{1}{2} \|\varphi^* - \varphi_d\|^2. \quad (4.21) \]

The right-hand side of (4.10) can be estimated as

\[ |(\eta \nabla \varphi^*, \mathbf{v})| \leq \|\eta\| \|\nabla \varphi^*\|_{L^4} \|\mathbf{v}\|_{L^4} \leq \|\eta\| \|\nabla \varphi^*\|_{L^4} \|\nabla \mathbf{v}\| \leq \frac{\nu}{10} \|\nabla \mathbf{v}\|^2 + \|\eta\|^2 \|\nabla \varphi^*\|^2_{L^4}, \quad (4.22) \]

and

\[ |(\mathbf{u} - \mathbf{u}_d, \mathbf{v})| \leq \|\mathbf{u}^* - \mathbf{u}_d\| \|\mathbf{v}\| \leq C_\Omega \|\mathbf{u}^* - \mathbf{u}_d\| \|\nabla \mathbf{v}\| \leq \frac{\nu}{10} \|\nabla \mathbf{v}\|^2 + C \|\mathbf{u}^* - \mathbf{u}_d\|^2. \quad (4.23) \]
Using (4.22) and (4.23) in (4.10), we get

\[
\frac{4\nu}{5} \|\nabla v\|^2 + \|v\|^2 \leq C\|u^* - u_d\|^2 + \|\eta\|^2 \|\nabla \varphi^*\|_{L^2}^2. \tag{4.24}
\]

Combining (4.21) and (4.24), we get

\[
- \frac{1}{2} \frac{d}{dt} \|\eta\|^2 + \frac{\nu}{5} \|\nabla v\|^2 + \|v\|^2 + \frac{C_0}{2} \|\nabla \eta\|^2 \\
\leq C(1 + \|\varphi^*\|_{L^4}^2 + \|\nabla \varphi^*\|_{L^4}^2 + \|\nabla \varphi^*\|_{L^4}^4) \|\eta\|^2 + C(\|u^* - u_d\|^2 + \|\varphi^* - \varphi_d\|^2).
\tag{4.25}
\]

Integrating (4.25) over \((t, T)\), we get

\[
\|\eta(t)\|^2 + \frac{\nu}{5} \int_t^T \|\nabla v(s)\|^2 + \int_t^T \|v(s)\|^2 + \frac{C_0}{2} \int_t^T \|\nabla \eta(s)\|^2 \\
\leq \|\eta(T)\|^2 + C \int_t^T \alpha(s) \|\eta(s)\|^2 ds + C \int_t^T \beta(s) ds,
\tag{4.26}
\]

where \(\alpha(t) = 1 + \|\varphi^*\|_{L^4}^2 + \|\nabla \varphi^*\|_{L^4}^2 + \|\nabla \varphi^*\|_{L^4}^4\) and \(u \beta(t) = \|u^* - u_d\|(t)\|^2 + \|\varphi^* - \varphi_d\|(t)\|^2\). Using Grönwall’s inequality, we deduce

\[
\|\eta(t)\|^2 \leq \left[\|\eta(T)\|^2 + C \int_0^T \beta(s) ds\right] \exp\left(\left[\frac{C}{2} \int_0^T \alpha(s) ds\right]\right).
\tag{4.27}
\]

Since \(\eta \in C^1(0, T)\), we have that

\[
\eta \in L^\infty(0, T; H).
\]

Thus, from (4.27) and (4.26), we conclude

\[
v \in L^2(0, T; \nabla v) \quad \text{and} \quad \eta \in L^2(0, T; V).
\tag{4.28}
\]

In fact, from the estimate

\[
v \|\nabla v\|^2 + \|v\|^2 = -(\eta \nabla \varphi^*, v) + (u^* - u_d, v) \leq C(\|\eta\| \|\nabla \varphi^*\|_{L^4} + \|u^* - u_d\|) \|\nabla v\|,
\]

we can also deduce

\[
v \in L^\infty(0, T; \nabla v).
\]

From (4.7), we have the following estimate,

\[
\|v\|_{H^1} \leq C(\|\varphi^*\| \|\nabla v\| + \|\nabla v\| \|\varphi^*\| + \|\nabla u^*\| \|\nabla \eta\| + \|\nabla \eta\| + \|\varphi^* - \varphi_d\|),
\]

thus using (4.28), we can conclude

\[
\eta_t \in L^2(0, T; V').
\]

To prove the uniqueness of the solution to the system (4.2)–(4.6), consider two solutions \([\eta_1, v_1]\) and \([\eta_2, v_2]\) of the system. Denoting \(\eta = \eta_1 - \eta_2, v = v_1 - v_2\) and \(q = q_1 - q_2\), we get

\[
- \eta_t + v \cdot \nabla \alpha \varphi^* + J(\varphi^*) \cdot v - u^* \cdot \nabla \eta \\
- a \Delta \eta + \nabla \varphi^* \cdot \nabla q - F''(\varphi^*) \Delta \eta = 0, \tag{4.29}
\]

\[
- v \Delta v + v + \eta \nabla \varphi^* + \nabla q = 0, \tag{4.30}
\]

\[
\text{div}(v) = 0, \tag{4.31}
\]

\[
v \cdot \nu = 0, \tag{4.32}
\]

\[
\eta(T, \cdot) = 0. \tag{4.33}
\]
Taking the inner product of (4.29) and (4.30) with $\eta$ and $v$, respectively, and recalculating estimates similar to in the existence arguments, we conclude that the solution to the system (4.2)–(4.6) must be unique.

Using the adjoint system (4.2)–(4.6), now we can eliminate $\psi, w$ from (4.1) and obtain optimal control in the form of adjoint variable. We have the following lemma.

**Lemma 4.3** Suppose (H1)–(H5) and (H7)–(H8) are satisfied. Let $U^* \in \mathcal{U}_{ad}$ be a locally optimal control for (OCP) with corresponding solution $[\phi^*, u^*]$ and the solution of adjoint system $\eta, v$. Then, we have the following variational inequality.

$$\int_0^T \int_\Omega (v + U^*) \cdot (U - U^*) \geq 0, \quad \forall U \in \mathcal{U}_{ad}. \quad (4.36)$$

**Proof** Let us take inner product of (4.2) and (4.3) with $\psi$ and $w$, respectively, and add them, where $(\psi, w)$ is the solution of the linearised system around $(\phi^*, u^*)$ with control $U^*$. We get

$$- \langle \eta_t, \psi \rangle + (v \cdot \nabla a \phi^*, \psi) + (J * (v \cdot \nabla \phi^*), \psi) - ((\nabla J * \phi^*) \cdot v, \psi) - (u^* \cdot \nabla \eta, \psi) - (a \Delta \eta, \psi) + (\nabla \psi, \nabla w) + (v, w) + (\eta \nabla \phi^*, w) = (\phi^* - \phi_d, \psi) + (u^* - u_d, w). \quad (4.34)$$

Similarly, take the inner product of (3.9) and (3.11) with $\eta$ and $v$, respectively, where $U$ in the equation for $w$ is replaced with $U - U^*$; and add them to get

$$\langle \psi_t, \eta \rangle + (w \cdot \nabla \phi^*, \eta) + (u^* \nabla \psi, \eta) + v(\nabla w, \nabla v) + (w, v) + (\nabla (\alpha \psi - J * \psi + F''(\hat{\phi}) \psi), \nabla \eta) = ((\alpha \psi - J * \psi + F''(\hat{\phi}) \psi) \nabla \phi^*, v) + ((\alpha \phi - J * \phi + F'(\phi^*)) \nabla \psi, v) + (U - U^*, v). \quad (4.35)$$

Subtract (4.34) from (4.35) and integrate from 0 to $T$. Using (4.1), we arrive at

$$\int_0^T \int_\Omega (v + U^*) \cdot (U - U^*) \geq 0, \quad \forall U \in \mathcal{U}_{ad}. \quad (4.36)$$

Since $\mathcal{U}_{ad}$ is a non empty convex closed subset of $\bar{U}$, from the first-order optimality condition (4.36), we can write the optimal control $U^*$ (see [30]), in terms of $v$, using the projection onto $\mathcal{U}_{ad}$ as

$$U^* = P_{\mathcal{U}_{ad}}(-v). \quad \square$$

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