THE GEOMETRY OF HIDA FAMILIES II: Λ-ADIC (φ, Γ)-MODULES AND Λ-ADIC HODGE THEORY

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To Haruzo Hida, on the occasion of his 60th birthday.

ABSTRACT. We construct the Λ-adic crystalline and Dieudonné analogues of Hida’s ordinary Λ-adic étale cohomology, and employ integral p-adic Hodge theory to prove Λ-adic comparison isomorphisms between these cohomologies and the Λ-adic de Rham cohomology studied in [Cai14] as well as Hida’s Λ-adic étale cohomology. As applications of our work, we provide a “cohomological” construction of the family of (φ, Γ)-modules attached to Hida’s ordinary Λ-adic étale cohomology by [Dee01], and we give a new and purely geometric proof of Hida’s finiteness and control theorems. We also prove suitable Λ-adic duality theorems for each of the cohomologies we construct.

1. INTRODUCTION

1.1. Motivation. In a series of groundbreaking papers [Hid86a] and [Hid86b], Hida constructed p-adic analytic families of p-ordinary Galois representations interpolating the Galois representations attached to p-ordinary cuspidal Hecke eigenforms in integer weights $k \geq 2$ by Deligne [Del71], [Car86]. At the heart of Hida’s construction is the p-adic étale cohomology $H^1_{\text{ét}} := \varprojlim H^1_{\text{ét}}(X_1(Np^r)_Q, \mathbb{Z}_p)$ of the tower of modular curves over $\mathbb{Q}$, which is naturally a module for the “big” p-adic Hecke algebra $\mathcal{H}_p := \varprojlim_p \mathcal{H}_p$, which is itself an algebra over the completed group ring $\Lambda := \mathbb{Z}_p[\{\Delta\}] \simeq \mathbb{Z}_p[T]$ via the diamond operators $\Delta_r := 1 + p^r \mathbb{Z}_p$. Writing $e^* \in \mathcal{H}_p$ for the idempotent attached to the (adjoint) Atkin operator $U_p^*$, Hida proves (via explicit computations in group cohomology) that the ordinary part $e^*H^1_{\text{ét}}$ of $H^1_{\text{ét}}$ is finite and free as a module over $\Lambda$, and that the resulting Galois representation $ho : G_\mathbb{Q} \longrightarrow \text{Aut}_\Lambda(e^*H^1_{\text{ét}}) \simeq \text{GL}_m(\mathbb{Z}_p[T])$ p-adically interpolates the representations attached to ordinary cuspidal eigenforms.

By analyzing the geometry of the tower of modular curves, Mazur and Wiles [MW86] showed that both the inertial invariants and covariants of the the local (at $p$) representation $\rho_p$ are free of the finite same rank over $\Lambda$, and hence that the ordinary filtration of the Galois representations attached to ordinary cuspidal eigenforms interpolates in Hida’s p-adic family. As an application, they gave examples of cuspforms $f$ and primes $p$ for which the specialization of the associated Hida family of Galois representations to weight $k = 1$ is not Hodge–Tate, and so does not arise from a weight one cuspform via the construction of Deligne-Serre [DS74]. Shortly thereafter, Tilouine [Til87] clarified the geometric underpinnings of [Hid86a] and [MW86].
In [Oht95], [Oht99] and [Oht00], Ohta initiated the study of the $p$-adic Hodge theory of Hida’s ordinary $\Lambda$-adic (local) Galois representation $\rho_p$. Using the invariant differentials on the tower of $p$-divisible groups over $R_\infty := \mathbb{Z}_p[\mu_p^\infty]$ attached to the “good quotient” modular abelian varieties introduced in [MW84] and studied in [MW86] and [Til87], Ohta constructs a certain $\Lambda_{R_\infty} := R_\infty[\Delta_1]$-module $e^*H^1_{\text{Hdg}}$, which is the Hodge cohomology analogue of $e^*H^1_{\text{et}}$. Via an integral version of the Hodge–Tate comparison isomorphism [Tat67] for ordinary $p$-divisible groups, Ohta establishes a $\Lambda$-adic Hodge–Tate comparison isomorphism relating $e^*H^1_{\text{Hdg}}$ and the semisimplification of the “semilinear representation” $\rho_p \otimes \Theta_{C_p}$. Using Hida’s results, Ohta shows that $e^*H^1_{\text{Hdg}}$ is free of finite rank over $\Lambda_{R_\infty}$ and specializes to finite level exactly as one expects. As applications of his theory, Ohta provides a construction of two-variable $p$-adic $L$-functions attached to families of ordinary cuspforms differing from that of Kitagawa [Kit94], and, in a subsequent paper [Oht00], provides a new and streamlined proof of the theorem of Mazur–Wiles [MW84] (Iwasawa’s Main Conjecture for $\mathbb{Q}$) from that of Kitagawa [Kit94], and, in a subsequent paper [Oht00], provides a new and streamlined proof of the theorem of Mazur–Wiles [MW84] (Iwasawa’s Main Conjecture for $\mathbb{Q}$) with semilinear $\Gamma := \text{Gal}(K_{\infty}/K_0)$-action and commuting linear $\Phi^*$-action
\begin{align*}
0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{\text{dR}} \longrightarrow e^*H^1(\Phi, \mathcal{O}_{X_r}) \longrightarrow 0.
\end{align*}

The main result of [Cai14] is that (1.1.2) is the correct de Rham analogue of Hida’s ordinary $\Lambda$-adic étale cohomology and Ohta’s ordinary $\Lambda$-adic Hodge cohomology (see [Cai14, Theorem 3.2.3]):

**Theorem 1.1.1.** Let $d = \sum_{k=1}^{p+1} d_k$ for $d_k := \dim_{\mathbb{F}_p} S_k(\Gamma_1(N); \mathbb{F}_p)^{\text{ord}}$ the $\mathbb{F}_p$-dimension of the space of mod $p$ weight $k$ ordinary cuspforms for $\Gamma_1(N)$. Then (1.1.2) is a short exact sequence of free $\Lambda_{R_\infty}$-modules of ranks $d, 2d,$ and $d$, respectively. Applying $\otimes_{\Lambda_{R_\infty}} R_\infty[\Delta_1/\Delta_r]$ to (1.1.2) recovers the ordinary part of the scalar extension of (1.1.1) to $R_\infty$.

The natural cup-product auto-duality of (1.1.1) over $R'_p := R_p[\mu_N]$ induces a canonical $\Lambda_{R_\infty}$-linear and $\Phi^*$-equivariant auto-duality of (1.1.2) which intertwines the dual semilinear action of $\Gamma \times \text{Gal}(K_0'/K_0) \simeq \text{Gal}(K_\infty'/K_0)$ with a certain $\Phi^*$-valued twist of its standard action; see [Cai14, Proposition 3.2.4] for the precise statement. We moreover proved that, as one would expect, the $\Lambda_{R_\infty}$-module $e^*H^0(\omega)$ is canonically isomorphic to the module $eS(N, \Lambda_{R_\infty})$ of ordinary $\Lambda_{R_\infty}$-adic cusp forms of tame level $N$; see [Cai14, Corollary 3.3.5].

**1.2. Results.** In this paper, we complete our study of the geometry and $\Lambda$-adic Hodge theory of Hida families begun in [Cai14] by constructing the crystalline counterpart to Hida’s ordinary $\Lambda$-adic étale cohomology, Ohta’s $\Lambda$-adic Hodge cohomology, and our $\Lambda$-adic de Rham cohomology. Via a careful study of the geometry of modular curves and abelian varieties and comparison isomorphisms in integral $p$-adic cohomology, we prove the appropriate control and finiteness theorems, and a suitable $\Lambda$-adic...
version of every integral comparison isomorphism one could hope for. In particular, we are able to recover the entire family of \(p\)-adic Galois representations \(p^\ast\) (and not just its semisimplification) from our \(\Lambda\)-adic crystalline cohomology. A remarkable byproduct of our work is a *cohomological construction* of the family of étale \((\varphi, \Gamma)\)-modules attached to \(e^\ast H_{et}^1\) by Dee [Dee01]. As an application of our theory, we give a new and purely geometric proof of Hida’s freeness and control theorems for \(e^\ast H_{et}^1\).

In order to survey our main results more precisely, we introduce some notation. Throughout this paper, we fix a prime \(p > 2\) and a positive integer \(N\) with \(Np > 4\). Fix an algebraic closure \(\overline{\mathbb{Q}}_p\) of \(\mathbb{Q}_p\) as well as a \(p\)-power compatible sequence \(\{e^{(r)}\}_{r \geq 0}\) of primitive \(p^r\)-th roots of unity in \(\overline{\mathbb{Q}}_p\). As above, we set \(K_r := \mathbb{Q}_p(\mu_{p^r})\) and \(K'_r := \mathbb{Q}_p(\mu_{p^r})\), and we write \(R_r\) and \(R'_r\) for the rings of integers in \(K_r\) and \(K'_r\), respectively. Denote by \(\mathcal{O}_{Q_p} := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\) the absolute Galois group and by \(\mathcal{H}\) the kernel of the \(p\)-adic cyclotomic character \(\chi : \mathcal{O}_{Q_p} \to \mathbb{Z}_p^\ast\). Using that \(K'_0/\mathbb{Q}_p\) is unramified, we canonically identify \(\Gamma = \mathcal{O}_{Q_p}/\mathcal{H}\) with \(\text{Gal}(K_\infty'/K_0')\). We will denote by \(\langle u \rangle\) (respectively \(\langle v \rangle\)) the diamond operator\(^1\) in \(\mathfrak{S}_p^\ast\) attached to \(u^{-1} \in \mathbb{Z}_p^\ast\) (respectively \(v^{-1} \in (\mathbb{Z}/N\mathbb{Z})^\ast\)) and write \(\Delta_r\) for the image of the restriction of \(\langle \cdot \rangle : \mathbb{Z}_p^\ast \to \mathfrak{S}_p^\ast\) to \(1 + p^r\mathbb{Z}_p \subseteq \mathbb{Z}_p^\ast\). For convenience, we put \(\Delta := \Delta_1\), and for any ring \(A\) we write \(A_\Delta := \varprojlim_r A[\Delta/\Delta_r]\) for the completed group ring on \(\Delta\) over \(A\); if \(\varphi\) is an endomorphism of \(A\), we again write \(\varphi\) for the induced endomorphism of \(A_\Delta\) that acts as the identity on \(\Delta\). For any ring homomorphism \(A \to B\), we will write \(\langle \cdot \rangle_B := (\cdot) \otimes_A B\) and \(\langle \cdot \rangle'_B := \text{Hom}_B((\cdot) \otimes_A B, B)\) for these functors from \(A\)-modules to \(B\)-modules.\(^2\) If \(G\) is any group of automorphisms of \(A\) and \(M\) is an \(A\)-module with a semilinear action of \(G\), for any “crossed” homomorphism \(\psi : G \to A^\times\) we will write \(M(\psi)\) for the \(A\)-module \(M\) with “twisted” semilinear \(G\)-action given by \(g \cdot m := \psi(g)gm\). Finally, we denote by \(X_r := X_1(Np^r)\) the usual modular curve over \(\mathbb{Q}\) classifying (generalized) elliptic curves with a \([\mu_{Np^r}]\)-structure, and by \(J_r := J_1(Np^r)\) its Jacobian.

We analyze the tower of \(p\)-divisible groups attached to the “good quotient” modular abelian varieties introduced by Mazur-Wiles [MW84]. To avoid technical complications with logarithmic \(p\)-divisible groups, following [MW86] and [Oht95], we will henceforth remove the trivial tame character by working with \(p\)-divisible groups over \(\mathbb{F}_p\). To avoid technical complications with logarithmic \(M\)-fields, following Mazur-Wiles [MW84, Chapter 3, §2], the \(p\)-divisible group \(G_r := e^{\psi'} J_r[p^\infty]\) over \(\mathbb{Q}\) extends to a \(p\)-divisible group \(\mathfrak{S}_r\) over \(R_r\), and we write \(\mathfrak{S}_r := \mathfrak{S}_r \times_{R_r} \mathbb{F}_p\) for its special fiber. Denoting by \(D(\cdot)\) the contravariant Dieudonné module functor on \(p\)-divisible groups over \(\mathbb{F}_p\), we form the projective limits

\[
D_\infty^\ast := \varprojlim_r D(\mathfrak{S}_r^\ast) \quad \text{for} \quad \ast \in \{\text{ét}, m, \text{null}\},
\]

taken along the mappings induced by \(\mathfrak{S}_r \to \mathfrak{S}_{r+1}\). Each of these is naturally a \(\Lambda\)-module equipped with linear (!) Frobenius \(F\) and Verschiebung \(V\) morphisms satisfying \(VF = VF = p\), as well as a linear action of \(\mathfrak{S}_p^\ast\) and a “geometric inertia” action of \(\Gamma\), which reflects the fact that the generic fiber of \(\mathfrak{S}_r\) descends to \(Q_p\). The \(\Lambda\)-modules (1.2.1) have the expected structure (see Theorem 3.2.2):

**Theorem 1.2.1.** There is a canonical split short exact sequence of finite and free \(\Lambda\)-modules

\[
0 \longrightarrow D_\infty^\ast \longrightarrow D_\infty \longrightarrow D_{\infty m} \longrightarrow 0
\]

\(^1\)Note that \(\langle u^{-1} \rangle = \langle u \rangle^\ast\) and \(\langle v^{-1} \rangle_N = \langle v \rangle_N^\ast\), where \(\langle \cdot \rangle^\ast\) and \(\langle \cdot \rangle_N^\ast\) are the adjoint diamond operators; see [Cai14, §2.2].

\(^2\)This convention is unfortunately somewhat at odds with our notation \(\Lambda_\Delta\), which (as an \(\Lambda\)-module) is in general neither the tensor product \(\Lambda \otimes_{\mathbb{Z}_p} A\) nor (unless \(A\) is a complete \(\mathbb{Z}_p\)-algebra) the completed tensor product \(\Lambda\hat{\otimes}_{\mathbb{Z}_p} A\); we hope that this small abuse causes no confusion.

\(^3\)That is, \(\psi(\sigma \tau) = \psi(\sigma) \cdot \sigma \psi(\tau)\) for all \(\sigma, \tau \in \Gamma\),
with linear $\mathfrak{S}^*$ and $\Gamma$-actions. As a $\Lambda$-module, $D_\infty$ is free of rank $2d'$, while $D_\infty^\text{ét}$ and $D_\infty^m$ are free of rank $d'$, where $d' := \sum_{k=3}^p \dim_{\mathbb{F}_p} S_k(\Gamma(N); \mathbb{F}_p)^\text{ord}$. For $* \in \{\text{ét}, \text{fil}, \text{null}\}$, there are canonical isomorphisms

$$D_\infty^* \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] \simeq D^*(\mathfrak{S}_r)$$

which are compatible with the extra structures. Via the canonical splitting of (1.2.2), $D_\infty^*$ for $* = \text{ét}$ (respectively $* = \text{fil}$) is identified with the maximal subspace of $D_\infty$ on which $F$ (respectively $V$) acts invertibly. The Hecke operator $U_p^* \in \mathfrak{S}^*$ acts as $F$ on $D_\infty^\text{ét}$ and as $(p)_N V$ on $D_\infty^m$, while $\Gamma$ acts trivially on $D_\infty^\text{ét}$ and via $(\chi(\cdot))^{-1}$ on $D_\infty^m$.

The short exact sequence (1.2.2) is very nearly $\Lambda$-adically auto-dual (see Proposition 3.2.3):

**Theorem 1.2.2.** There is a canonical $\mathfrak{S}^*$-equivariant isomorphism of exact sequences of $\Lambda_{R_0'}$-modules

$$0 \rightarrow D_\infty^\text{ét}((\chi(\cdot)N)\Lambda_{R_0'}) \rightarrow D_\infty((\chi(\cdot)N)\Lambda_{R_0'}) \rightarrow D_\infty^m((\chi(\cdot)N)\Lambda_{R_0'}) \rightarrow 0$$

that is $\Gamma \times \text{Gal}(K_0'/K_0)$-equivariant, and intertwines $F$ (respectively $V$) on the top row with $V^\vee$ (respectively $F^\vee$) on the bottom.\footnote{Here, $F^\vee$ (respectively $V^\vee$) is the map taking a linear functional $f$ to $\varphi^{-1} \circ f \circ F$ (respectively $\varphi \circ f \circ V$), where $\varphi$ is the Frobenius automorphism of $R_0' = \mathbb{Z}_p[\mu_N]$.}

In [MW86], Mazur and Wiles relate the ordinary-filtration of $e^*H^1_{\text{ét}}$ to the étale cohomology of the Igusa tower studies in [MW83]. We can likewise interpret the slope filtration (1.2.2) in terms of the crystalline cohomology of the Igusa tower as follows. For each $r$, let $I_r^\infty$ and $I_r^0$ be the two “good” irreducible components of $\mathcal{X}_r \times_{R_0'} \mathbf{F}_r$ (see the discussion preceding Proposition 3.1.18), each of which is isomorphic to the Igusa curve $\text{Ig}(p^r)$ of tame level $N$ and $p$-level $p^r$. For $* \in \{0, \infty\}$ we form the projective limit

$$H^1_{\text{cris}}(I^*) := \lim_{\longrightarrow r} H^1_{\text{cris}}(I^*_r/\mathbb{Z}_p);$$

with respect to the trace mappings on crystalline cohomology induced by the canonical degeneracy maps on Igusa curves. Then $H^1_{\text{cris}}(I^*)$ is naturally a $\Lambda$-module with linear Frobenius $F$ and Verschelbing $V$ endomorphisms, and we write $H^1_{\text{cris}}(I^*)^\text{ord}$ (respectively $H^1_{\text{cris}}(I^*)^\text{F_\text{ord}}$) for the maximal $V$- (respectively $F$-) stable submodule on which $V$ (respectively $F$) acts invertibly. Letting $f^*$ be the idempotent of $\Lambda$ corresponding to projection to the part where $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$ acts nontrivially via the diamond operators, we prove (see Theorem 3.2.4):

**Theorem 1.2.3.** There is a canonical isomorphism of $\Lambda$-modules, compatible with $F$ and $V$, with $*$

$$D_\infty = D_\infty^m \oplus D_\infty^\text{ét} \simeq f^*H^1_{\text{cris}}(I^0)^\text{ord} \oplus f^*H^1_{\text{cris}}(I^\infty)^\text{F_\text{ord}}$$

which preserves the direct sum decompositions of source and target. This isomorphism is Hecke and $\Gamma$-equivariant, with $U_p^*$ and $\Gamma$ acting as $(p)_N V \oplus F$ and $(\chi(\cdot))^{-1} \oplus \text{id}$, respectively, on each direct sum.

We note that our “Dieudonné module” analogue (1.2.3) is a significant sharpening of its étale counterpart [MW86, §4], which is formulated only up to isogeny (i.e. after inverting $p$). From $D_\infty$, we can recover the $\Lambda$-adic Hodge filtration (1.1.2), so the latter is canonically split (see Theorem 3.2.7):
Theorem 1.2.4. There is a canonical $\Gamma$ and $\mathcal{S}^\ast$-equivariant isomorphism of exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & e^\ast H^0(\omega) & \rightarrow & e^\ast H^1_{\text{dR}} & \rightarrow & e^\ast H^1(\Theta) & \rightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \rightarrow & D_{\infty}^m \otimes \Lambda_{R_{\infty}} & \rightarrow & D_{\infty} \otimes \Lambda_{R_{\infty}} & \rightarrow & D_{\infty}^{\text{et}} \otimes \Lambda_{R_{\infty}} & \rightarrow & 0
\end{array}
\] (1.2.4)

where the mappings on bottom row are the canonical inclusion and projection morphisms corresponding to the direct sum decomposition $D_{\infty} = D_{\infty}^m \oplus D_{\infty}^{\text{et}}$. In particular, the Hodge filtration exact sequence (1.1.2) is canonically split, and admits a canonical descent to $\Lambda$.

We remark that under the identification (1.2.4), the Hodge filtration (1.1.2) and slope filtration (1.2.2) correspond, but in the opposite directions. As a consequence of Theorem 1.2.4, we deduce (see Corollary 3.2.8 and Remark 3.2.9):

Corollary 1.2.5. There is a canonical isomorphism of finite free $\Lambda$ (respectively $\Lambda_{R_0}$)-modules

\[
e^\ast S(N, \Lambda) \simeq D_{\infty}^m \quad \text{respectively} \quad e^\ast \mathcal{S} \otimes \Lambda_{R_0} \simeq D_{\infty}^{\text{et}} (\langle a \rangle_N) \otimes \Lambda_{R_0}
\]

that intertwines $T \in \mathcal{S} : = \varprojlim \mathcal{S}_p$ with $T^* \in \mathcal{S}^\ast$, where we let $U^p$ act as $\langle p \rangle N$ on $D_{\infty}^m$ and as $F$ on $D_{\infty}^{\text{et}}$. The second of these isomorphisms is in addition $\text{Gal}(K_0'/K_0)$-equivariant.

We are also able to recover the semisimplification of $e^\ast H^1_{\text{et}}$ from $D_{\infty}$. Writing $\mathcal{S} \subset \mathcal{G}_{Q_p}$ for the inertia subgroup at $p$, for any $\mathbb{Z}_p[\mathcal{G}_{Q_p}]$-module $M$, we denote by $M^{\mathcal{S}}$ (respectively $M^{\mathcal{S}} := M/M^{\mathcal{S}}$) the sub (respectively quotient) module of invariants (respectively covariants) under $\mathcal{S}$.

Theorem 1.2.6. There are canonical isomorphisms of $\Lambda_{W(\mathbb{F}_p)}$-modules with linear $\mathcal{S}^\ast$-action and semilinear actions of $F$, $V$, and $\mathcal{G}_{Q_p}$

\[
\begin{align*}
D_{\infty}^{\text{et}} \otimes \Lambda_{W(\mathbb{F}_p)} & \simeq (e^\ast H^1_{\text{et}})^{\mathcal{S}} \otimes \Lambda_{W(\mathbb{F}_p)} \\
D_{\infty}^m (-1) & \otimes \Lambda_{W(\mathbb{F}_p)} \simeq (e^\ast H^1_{\text{et}})^{\mathcal{S}} \otimes \Lambda_{W(\mathbb{F}_p)} 
\end{align*}
\] (1.2.5a)

and

\[
\begin{align*}
D_{\infty}^m (-1) & \otimes \Lambda_{W(\mathbb{F}_p)} \simeq (e^\ast H^1_{\text{et}})^{\mathcal{S}} \otimes \Lambda_{W(\mathbb{F}_p)} 
\end{align*}
\] (1.2.5b)

Writing $\sigma$ for the Frobenius automorphism of $W(\overline{\mathbb{F}}_p)$, the isomorphism (1.2.5a) intertwines $F \otimes \sigma$ with $\text{id} \otimes \sigma$ and $id \otimes g$ with $g \otimes g$ for $g \in \mathcal{G}_{Q_p}$, whereas (1.2.5b) intertwines $V \otimes \sigma^{-1}$ with $\text{id} \otimes \sigma^{-1}$ and $g \otimes g$ with $g \otimes g$, where $g \in \mathcal{G}_{Q_p}$ acts on the Tate twist $D_{\infty}^m (-1) := D_{\infty}^m \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1)$ as $\langle \chi(g)^{-1} \rangle \otimes \chi(g)^{-1}$. Theorem 1.2.6 gives the following “explicit” description of the semisimplification of $e^\ast H^1_{\text{et}}$:

Corollary 1.2.7. For any $T \in (e^\ast \mathcal{S}^\ast)^{\times}$, let $\lambda(T) : \mathcal{G}_{Q_p} \rightarrow e^\ast \mathcal{S}^\ast$ be the unique continuous (for the $p$-adic topology on $e^\ast \mathcal{S}^\ast$) unramified character whose value on (any lift of) Frob$_p$ is $T$. Then $\mathcal{G}_{Q_p}$ acts on $(e^\ast H^1_{\text{et}})^{\mathcal{S}}$ through the character $\lambda(U_p^{\ast -1})$ and on $(e^\ast H^1_{\text{et}})^{\mathcal{S}}$ through $\chi^{-1} : (\chi^{-1}) \lambda(U_p^{\ast -1})$. We remark that Corollary 1.2.5 and Theorem 1.2.6 combined give a refinement of the main result of [Oht95]. We are furthermore able to recover the main theorem of [MW86] (that the ordinary filtration of $e^\ast H^1_{\text{et}}$ interpolates $p$-adic analytically):
Corollary 1.2.8. Let $d'$ be the integer of Theorem 1.2.1. Then each of $(e^*H^1_{\text{et}})_{\mathcal{H}}$ and $(e^*H^1_{\text{et}})_{\mathcal{H}}$ is a free $\Lambda$-module of rank $d'$, and for each $r \geq 1$ there are canonical $\mathcal{H}^*$ and $\mathcal{G}_p$-equivariant isomorphisms of $\mathbb{Z}_p[\Delta/\Delta_r]$-modules

\begin{align}
(e^*H^1_{\text{et}})_{\mathcal{H}} \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] &\simeq e^*H^1_{\text{et}}(X_{r \mathcal{G}_p}, \mathbb{Z}_p)_{\mathcal{H}} \\
(e^*H^1_{\text{et}})_{\mathcal{H}} \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] &\simeq e^*H^1_{\text{et}}(X_{r \mathcal{G}_p}, \mathbb{Z}_p)_{\mathcal{H}}
\end{align}

To recover the full $\Lambda$-adic local Galois representation $e^*H^1_{\text{et}}$, rather than just its semisimplification, it is necessary to work with the full Diedonné crystal of $\mathcal{G}_r$ over $R_r$. Following Faltings [Fal99] and Breuil (e.g. [Bre00]), this is equivalent to studying the evaluation of the Diedonné crystal of $\mathcal{G}_r \times_{R_r} R_r/pR_r$ on the “universal” divided power thickening $S_r \to R_r/pR_r$, where $S_r$ is the $p$-adically completed PD-hull of the surjection $\mathbb{Z}_p[u_r] \to R_r$ sending $u_r$ to $e^{(r)} - 1$. As the rings $S_r$ are too unwieldy to directly construct a good crystalline analogue of Hida’s ordinary étale cohomology, we must functionally descend the “filtered $S_r$-module” attached to $\mathcal{G}_r$ to the much simpler ring $\mathcal{G}_r := \mathbb{Z}_p[u_r]$. While such a descent is provided (in rather different ways) by the work of Breuil–Kisin and Berger–Wach, neither of these frameworks is suitable for our application: it is essential for us that the formation of this descent to $\mathcal{G}_r$ commute with base change as one moves up the cyclotomic tower, and it is not at all clear that this holds for Breuil–Kisin modules or for the Wach modules of Berger. Instead, we use the theory of [CL14], which works with frames and windows à la Lau and Zink to provide the desired functorial descent to a “$(\varphi, \Gamma)$-module” $\mathcal{M}(\mathcal{G}_r)$ over $\mathcal{G}_r$. We view $\mathcal{G}_r$ as a $\mathbb{Z}_p$-subalgebra of $\mathcal{G}_{r+1}$ via the map sending $u_r$ to $\varphi(u_{r+1}) := (1 + u_{r+1})^p - 1$, and we write $\mathcal{G}_\infty := \varinjlim \mathcal{G}_r$ for the rising union of the $\mathcal{G}_r$, equipped with its Frobenius automorphism $\varphi$ and commuting action of $\Gamma$ determined by $\gamma u_r := (1 + u_r)^{\chi(\gamma)} - 1$. We then form the projective limits

$$\mathcal{M}_r^* := \varprojlim(\mathcal{M}(\mathcal{G}_r) \otimes_{\mathcal{G}_r} \mathcal{G}_\infty) \quad \text{for } * \in \{\text{ét}, m, \text{null}\}$$

taken along the mappings induced by $\mathcal{G}_r \times_{R_r} R_{r+1} \to \mathcal{G}_{r+1}$ via the functoriality of $\mathcal{M}_r(\cdot)$ and its compatibility with base change. These are $\Lambda_{\mathcal{G}_\infty}$-modules equipped with a semilinear action of $\Gamma$, a linear and commuting action of $\mathcal{H}^*$, and a $\varphi$ (respectively $\varphi^{-1}$) semilinear endomorphism $F$ (respectively $V$) satisfying $FV = \omega$ and $VF = \varphi^{-1}(\omega)$, for $\omega := \varphi(u_1)/u_1 = u_0/\varphi^{-1}(u_0) \in \mathcal{G}_\infty$, and they provide our crystalline analogue of Hida’s ordinary étale cohomology (see Theorem 3.3.2):

Theorem 1.2.9. There is a canonical short exact sequence of finite free $\Lambda_{\mathcal{G}_\infty}$-modules with linear $\mathcal{H}^*$-action, semilinear $\Gamma$-action, and semilinear endomorphisms $F, V$ satisfying $FV = \omega, VF = \varphi^{-1}(\omega)$

\begin{align}
0 &\longrightarrow \mathcal{M}_r^{\text{ét}}_{\mathcal{H}} \longrightarrow \mathcal{M}_\infty \longrightarrow \mathcal{M}_r^{m}_{\mathcal{H}} \longrightarrow 0 .
\end{align}

Each of $\mathcal{M}_r^*$ for $* \in \{\text{ét}, m\}$ is free of rank $d'$ over $\Lambda_{\mathcal{G}_r}$, while $\mathcal{M}_\infty$ is free of rank $2d'$, where $d'$ is as in Theorem 1.2.1. Extending scalars on (1.2.7) along the canonical surjection $\Lambda_{\mathcal{G}_\infty} \to \mathcal{G}_\infty[\Delta/\Delta_r]$ yields the short exact sequence

\begin{align}
0 &\longrightarrow \mathcal{M}_r(\mathcal{G}_r^{\text{ét}}) \otimes_{\mathcal{G}_r} \mathcal{G}_\infty \longrightarrow \mathcal{M}_r(\mathcal{G}_r) \otimes_{\mathcal{G}_r} \mathcal{G}_\infty \longrightarrow \mathcal{M}_r(\mathcal{G}_r^{m}) \otimes_{\mathcal{G}_r} \mathcal{G}_\infty \longrightarrow 0
\end{align}

compliedly with $\mathcal{H}^*$, $\Gamma$, $F$ and $V$. The Frobenius endomorphism $F$ commutes with $\mathcal{H}^*$ and $\Gamma$, whereas the Verscheibung $V$ commutes with $\mathcal{H}^*$ and satisfies $V \gamma = \varphi^{-1}(\omega/\gamma \omega) \cdot \gamma V$ for all $\gamma \in \Gamma$.

\footnote{As explained in Remark 2.3.4, the $p$-adic completion of $\mathcal{G}_\infty$ is actually a very nice ring: it is canonically and Frobenius equivariantly isomorphic to $W(F_p[u_0]^{\text{rad}})$, for $F_p[u_0]^{\text{rad}}$ the perfect closure of the $F_p$-algebra $F_p[u_0]$.}
Again, in the spirit of Theorem 1.2.2 and [Cai14, Proposition 3.2.4], there is a corresponding “auto-duality" result for \( M_\infty \) (see Theorem 3.3.4). To state it, we must work over \( S'_\infty := \lim_{\to r} Z_p[\mu_N][u_r] \), with the inductive limit taken along the \( Z_p \)-algebra maps sending \( u_r \) to \( \varphi(u_{r+1}) \).

**Theorem 1.2.10.** Let \( \mu : \Gamma \to \Lambda^\infty_{S_\infty} \) be the crossed homomorphism given by \( \mu(\gamma) := \frac{\mu}{\gamma u_1} \chi(\gamma) \langle \chi(\gamma) \rangle \). There is a canonical \( \delta^* \) and Gal\((K'_\infty/K_0)\)-compatible isomorphism of exact sequences

\[
0 \longrightarrow M_{\infty}(\mu(a)N)_{\Lambda_{e'_\infty}} \longrightarrow M_{\infty}(\mu(a)N)_{\Lambda_{e'_\infty}} \longrightarrow M_{\infty}(\mu(a)N)_{\Lambda_{e'_\infty}} \longrightarrow 0
\]

intertwining \( F \) and \( V \) on the top row with \( V^\vee \) and \( F^\vee \), respectively, on the bottom. The action of Gal\((K'_\infty/K_0)\) on the bottom row is the standard one \( \gamma \cdot f := \gamma f \gamma^{-1} \) on linear duals.

The \( \Lambda_{S_\infty} \)-modules \( M_{\infty}^{\text{et}} \) and \( M_{\infty}^m \) have a particularly simple structure (see Theorem 3.3.5):

**Theorem 1.2.11.** There are canonical \( \delta^* \), \( \Gamma \), \( F \) and \( V \)-equivariant isomorphisms of \( \Lambda_{S_\infty} \)-modules

(1.2.8a) \[
M_{\infty}^{\text{et}} \simeq D_{\infty}^{\text{et}} \otimes \Lambda_{S_\infty},
\]

intertwining \( F \) and \( V \) with \( F \otimes \varphi \) and \( F^{-1} \otimes \varphi^{-1}(\omega) \cdot \varphi^{-1} \), respectively, and \( \gamma \in \Gamma \) with \( \gamma \otimes \gamma \), and

(1.2.8b) \[
M_{\infty}^m \simeq D_{\infty}^m \otimes \Lambda_{S_\infty},
\]

intertwining \( F \) and \( V \) with \( V^{-1} \otimes \omega \cdot \varphi \) and \( V \otimes \varphi^{-1} \), respectively, and \( \gamma \) with \( \gamma \otimes \chi(\gamma)^{-1} \). 

In particular, \( F \) (respectively \( V \)) acts invertibly on \( M_{\infty}^{\text{et}} \) (respectively \( M_{\infty}^m \)).

From \( M_{\infty} \), we can recover \( D_{\infty} \) and \( e^{s'}H^{1}_{\text{dR}} \), with their additional structures (see Theorem 3.3.6):

**Theorem 1.2.12.** Viewing \( \Lambda \) as a \( \Lambda_{S_\infty} \)-algebra via the map induced by \( u_r \mapsto 0 \), there is a canonical isomorphism of short exact sequences of finite free \( \Lambda \)-modules

\[
0 \longrightarrow M_{\infty}^{\text{et}} \otimes \Lambda_{\infty} \longrightarrow M_{\infty} \otimes \Lambda_{\infty} \longrightarrow M_{\infty}^m \otimes \Lambda_{\infty} \longrightarrow 0
\]

which is \( \Gamma \) and \( \delta^* \)-equivariant and carries \( F \otimes 1 \) to \( F \) and \( V \otimes 1 \) to \( V \). Viewing \( \Lambda_{R_{\infty}} \) as a \( \Lambda_{S_\infty} \)-algebra via the map \( u_r \mapsto (e^{(r)})^p - 1 \), there is a canonical isomorphism of short exact sequences of \( \Lambda_{R_{\infty}} \)-modules

(1.2.4) \[
0 \longrightarrow M_{\infty}^{\text{et}} \otimes \Lambda_{R_{\infty}} \longrightarrow M_{\infty} \otimes \Lambda_{R_{\infty}} \longrightarrow M_{\infty}^m \otimes \Lambda_{R_{\infty}} \longrightarrow 0
\]

that is \( \Gamma \) and \( \delta^* \)-equivariant, where \( i \) and \( j \) the splittings given by Theorem 1.2.4.
To recover Hida’s ordinary étale cohomology from $\mathcal{M}_\infty$, we introduce the “period” ring of Fontaine\(^6\) $\mathcal{E}^+ := \lim_{\leftarrow} \mathcal{O}_{\mathcal{C}_p}/(p)$, with the projective limit taken along the $p$-power mapping; this is a perfect valuation ring of characteristic $p$ equipped with a canonical action of $\mathcal{G}_{Q_p}$ via “coordinates”. We write $\mathcal{E}$ for the fraction field of $\mathcal{E}^+$ and $\mathcal{A} := W(\mathcal{E})$ for its ring of Witt vectors, equipped with its canonical Frobenius automorphism $\varphi$ for the fraction field of $E$. A compatible sequence who prove (under certain technical hypotheses of “deformation-theoretic nature”) that if the $\Lambda$-adic $\varepsilon$-isomorphism of $F$-modules, compatible with the actions of $\mathcal{G}_{Q_p}$-action induced by Witt functoriality. Our fixed choice of $p$-power compatible sequence $\{\varepsilon^{(r)}\}$ determines an element $\varepsilon := (\varepsilon^{(r)} \bmod p)_{r \geq 0}$ of $\mathcal{E}^+$, and we $\mathbb{Z}_p$-linearly embed $\mathcal{E}_\infty$ in $\mathcal{A}$ via $u_r \mapsto \varphi^{-1}(\varepsilon) - 1$ where $[\cdot]$ is the Teichmüller section. This embedding is $\varphi$ and $\mathcal{G}_{Q_p}$-compatible, with $\mathcal{G}_{Q_p}$ acting on $\mathcal{E}_\infty$ through the quotient $\mathcal{G}_{Q_p} \to \Gamma$.

**Theorem 1.2.13.** Twisting the structure map $\mathcal{E}_\infty \to \mathcal{A}$ by the Frobenius automorphism $\varphi$, there is a canonical isomorphism of short exact sequences of $\Lambda_\mathcal{A}$-modules with $\mathcal{G}_\varphi^e$-action

$$0 \to \mathcal{M}_\infty^\text{ét} \otimes_{\Lambda_\mathcal{E}_\infty, \varphi} \Lambda_\mathcal{A} \to \mathcal{M}_\infty \otimes_{\Lambda_\mathcal{E}_\infty, \varphi} \Lambda_\mathcal{A} \to \mathcal{M}_\infty^\text{m} \otimes_{\Lambda_\mathcal{E}_\infty, \varphi} \Lambda_\mathcal{A} \to 0$$

(1.2.9)

$$0 \to (e^s H^1_\text{ét})^\varphi \otimes_{\Lambda} \Lambda_\mathcal{A} \to e^s H^1_\text{ét} \otimes_{\Lambda} \Lambda_\mathcal{A} \to (e^s H^1_\text{ét})^\varphi \otimes_{\Lambda} \Lambda_\mathcal{A} \to 0$$

that is $\mathcal{G}_{Q_p}$-equivariant for the “diagonal” action of $\mathcal{G}_{Q_p}$ (with $\mathcal{G}_{Q_p}$ acting on $\mathcal{M}_\infty$ through $\Gamma$) and intertwines $F \otimes \varphi$ with $\text{id} \otimes \varphi$ and $V \otimes \varphi^{-1}$ with $\text{id} \otimes \varphi^{-1}$. In particular, there is a canonical isomorphism of $\Lambda$-modules, compatible with the actions of $\mathcal{G}_\varphi^e$ and $\mathcal{G}_{Q_p}$,

$$e^s H^1_\text{ét} \cong \left( \mathcal{M}_\infty \otimes_{\Lambda_\mathcal{E}_\infty, \varphi} \Lambda_\mathcal{A} \right)^{F \otimes \varphi = 1}.$$

(1.2.10)

Theorem 1.2.13 allows us to give a new proof of Hida’s finiteness and control theorems for $e^s H^1_\text{ét}$.

**Corollary 1.2.14 (Hida).** Let $d'$ be as in Theorem 1.2.1. Then $e^s H^1_\text{ét}$ is free $\Lambda$-module of rank $2d'$. For each $r \geq 1$ there is a canonical isomorphism of $\mathbb{Z}_p[\Delta/\Delta_r]$-modules with linear $\mathcal{G}_\varphi^e$ and $\mathcal{G}_{Q_p}$-actions

$$e^s H^1_\text{ét} \otimes_{\Lambda} \mathbb{Z}_p[\Delta/\Delta_r] \cong e^s H^1_\text{ét}(X_r \varphi, \mathbb{Z}_p)$$

which is moreover compatible with the isomorphisms (1.2.6a) and (1.2.6b) in the evident manner.

We also deduce a new proof of the following duality result [Oht95, Theorem 4.3.1] (cf. [MW86, §6]):

**Corollary 1.2.15 (Ohta).** Let $\nu : \mathcal{G}_{Q_p} \to \mathcal{G}_\varphi^e$ be the character $\nu := \chi(\chi) \lambda(p) N$. There is a canonical $\mathcal{G}_\varphi^e$ and $\mathcal{G}_{Q_p}$-equivariant isomorphism of short exact sequences of $\Lambda$-modules

$$0 \to \text{Hom}_{\Lambda}(e^s H^1_\text{ét}, \Lambda) \to \text{Hom}_{\Lambda}(e^s H^1_\text{ét}, \Lambda) \to \text{Hom}_{\Lambda}(e^s H^1_\text{ét}, \Lambda) \to 0$$

(1.2.6b)

The $\Lambda$-adic splitting of the ordinary filtration of $e^s H^1_\text{ét}$ was considered by Ghate and Vatsal [GV04], who prove (under certain technical hypotheses of “deformation-theoretic nature”) that if the $\Lambda$-adic family $\mathcal{F}$ associated to a cuspidal eigenform $f$ is primitive and $p$-distinguished, then the associated

\(^6\)Though we use the notation introduced by Berger and Colmez.
1.3. Overview of the article. Section 2 is preliminary: we first review in §2.1–2.2 some background material on Dieudonné modules and crystals, as well as the integral p-adic cohomology theories of [Cai09] and [Cai10]. In §2.3, we summarize the theory developed in [CL14], which uses Dieudonné crystals of p-divisible groups to provide a “cohomological” construction of the (ϕ, Γ)-modules attached to potentially Barsotti–Tate representations. We then specialize these results to the case of ordinary p-divisible groups in §2.4; it is precisely this theory which allows us to construct our crystalline analogue of Hida’s ordinary Λ-adic étale cohomology.

Section 3 constitutes the main body of this paper, and the reader who is content to refer back to §2.1–2.4 as needed should skip directly there. In section 3.1, we study the tower of p-divisible groups whose cohomology allows us to construct our Λ-adic Dieudonné and crystalline analogues of Hida’s étale cohomology in §3.2 and §3.3, respectively. We establish Λ-adic comparison isomorphisms between each of these cohomologies using the integral comparison isomorphisms of [Cai10] and [CL14], recalled in §2.2 and §2.3–2.4, respectively. This enables us to give a new proof of Hida’s freeness and control theorems and of Ohta’s duality theorem in §3.3. A key technical ingredient in our proofs is the commutative algebra formalism developed in [Cai14, §3.1] for dealing with projective limits of cohomology and establishing appropriate “freeness and control” theorems by reduction to characteristic p.

As remarked in §1.2, and following [Oht95] and [MW86], our construction of the Λ-adic Dieudonné and crystalline counterparts to Hida’s étale cohomology excludes the trivial eigenspace for the action of µp−1 ⊆ Zp× so as to avoid technical complications with logarithmic p-divisible groups. In [Oht00], Ohta uses the “fixed part” (in the sense of Grothendieck [Gro72, 2.2.3]) of Néron models with semiabelian reduction to extend his results on Λ-adic Hodge cohomology to allow trivial tame nebentype character.

We are confident that by using Kato’s logarithmic Dieudonné theory [Kat89] one can appropriately generalize our results in §3.2 and §3.3 to include the missing eigenspace for the action of µp−1.

1.4. Notation. If ϕ : A → B is any map of rings, we will often write MB := M ⊗AB for the B-module induced from an A-module M by extension of scalars. When we wish to specify ϕ, we will write M ⊗ϕA B. Likewise, if ϕ : T′ → T is any morphism of schemes, for any T-scheme X we denote by Xϕ the base change of X along ϕ. If f : X → Y is any morphism of T-schemes, we will write fϕ : XT′ → YT′ for the morphism of T′-schemes obtained from f by base change along ϕ. When T = Spec(R) and T′ = Spec(R′) are affine, we abuse notation and write XR′ or X ×R R′ for XT′.

We frequently work with schemes over a discrete valuation ring R, and will write X, Y, . . . for schemes over Spec(R), reserving X, Y, . . . (respectively X, Y, . . .) for their generic (respectively special) fibers. As this article is a continuation of [Cai14], we will freely use the notation and conventions therein.

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Hida has recently revised and expanded these very interesting notes in his preprint [Hid13], which includes new proofs and refinements of some of the background material in §3.1 of the present paper.
After a preliminary version of this paper was submitted for publication, we learned of the preprint [Wak13], in which Wake obtains a new proof of Ohta's Eichler–Shimura isomorphism as well as some refinements of Hida's finiteness and control theorems. Wake's methods are largely different from ours, as he works with the $p$-adic étale cohomology of the special fibers of modular curves, while we instead use their mod $p$ de Rham cohomology in [Cai14] and the theory of [CL14] in the present article to compare our $\Lambda$-adic de Rham cohomology with Hida's $\Lambda$-adic étale cohomology. Our cohomological construction via Dieudonné crystals of the family of étale $(\phi, \Gamma)$-modules attached to Hida’s ordinary $\Lambda$-adic étale cohomology by [Dec01] appears to be entirely new.

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2. Dieudonné Crystals and Dieudonné Modules

This section is devoted to recalling the geometric background we will need in our constructions. Much (though not all) of this material is contained in [Cai09], [Cai10], and [CL14].

2.1. Dieudonné modules and de Rham cohomology. Let $k$ be a perfect field of characteristic $p$ and $X$ a smooth and proper curve over $k$. We begin by recalling the relation between the de Rham cohomology of $X$ over $k$ and the Dieudonné module of the $p$-divisible group of the Jacobian of $X$. Let us write $H(X/k)$ for the three-term "Hodge filtration" exact sequence

$$0 \longrightarrow H^0(X, \Omega^1_{X/k}) \longrightarrow H^1_{\text{dR}}(X/k) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0.$$ 

Pullback by the absolute Frobenius gives an endomorphism of $F : H(X/k) \rightarrow H(X/k)$ that is semilinear over the $p$-power Frobenius automorphism $\varphi$ of $k$. Under the canonical cup-product autoduality of $H(X/k)$, we obtain $\varphi^{-1}$-semilinear endomorphism

$$(2.1.1) \quad V := F_* : H^1_{\text{dR}}(X/k) \longrightarrow H^1_{\text{dR}}(X/k)$$
whose restriction to $H^0(X, \Omega^1_{X/k})$ coincides with the Cartier operator [Cai14, §2.3]. Let $A$ be the “Dieudonné ring”, i.e. the (noncommutative if $k \neq \mathbb{F}_p$) ring $A := W(k)[F, V]$, where $F, V$ satisfy $FV = VF = p$, $F\alpha = \varphi(\alpha)F$, and $V\alpha = \varphi^{-1}(\alpha)V$ for all $\alpha \in W(k)$. We view $H^1_{dR}(X/k)$ as a left $A$-module in the obvious way. By Fitting’s Lemma [Cai14, Lemma 2.3.3], for $f = F$ or $V$, any finite left $A$-module $M$ admits a canonical direct sum decomposition

$$M = M^{\text{f.d.}} \oplus M^{\text{f.t.}}$$

where $M^{\text{f.d.}}$ (respectively $M^{\text{f.t.}}$) is the maximal $A$-submodule of $M$ on which $f$ is bijective (respectively $p$-adically topologically nilpotent).

**Proposition 2.1.1 (Oda).** Let $J := \text{Pic}^0_{X/k}$ be the Jacobian of $X$ over $k$ and $G := J[p^\infty]$ its $p$-divisible group. Denote by $D(\cdot)$ the contravariant Dieudonné crystal (see (2.2.6) below), so the Dieudonné module $D(G)_W$ is naturally a left $A$-module, finite and free over $W := W(k)$.

1. There are canonical isomorphisms of left $A$-modules

$$H^1_{dR}(X/k) \simeq D(J[p])_k \simeq D(G)_k.$$  

2. For any finite morphism $\rho : Y \to X$ of smooth and proper curves over $k$, the identification of (1) intertwines $\rho_*$ with $D(\text{Pic}^0(\rho))$ and $\rho^*$ with $D(\text{Alb}(\rho))$.  

3. Let $G = G^{\text{et}} \times G^{\text{mr}} \times G^{\text{dl}}$ be the canonical direct product decomposition of $G$ into its maximal étale, multiplicative, and local-local subgroups. Via the identification of (1), the canonical mappings in the exact sequence $H(X/k)$ induce natural isomorphisms of left $A$-modules

$$H^0(X, \Omega^1_{X/k})^{\text{ord}} \simeq D(G^{\text{mr}})_k \quad \text{and} \quad H^1(X, \mathcal{O}_X)^{\text{ord}} \simeq D(G^{\text{et}})_k.$$  

4. The isomorphisms of (3) are dual to each other, using the perfect duality on cohomology induced by the cup-product pairing [Cai14, Remark 2.3.3] and the identification $D(G)_k \simeq D(G)_k$ resulting from the compatibility of $D(\cdot)_k$ with duality and the autoduality of $J$.

**Proof.** Using the characterizing properties of the Cartier operator defined by Oda [Oda69, Definition 5.5] and the explicit description of the autoduality of $H^1_{dR}(X/k)$ in terms of cup-product and residues, one checks that the endomorphism of $H^1_{dR}(X/k)$ in [Oda69, Definition 5.6] is adjoint to $F^*$, and therefore coincides with the endomorphism $V := F_*$ in (2.1.1); cf. the proof of [Ser58, Proposition 9].

We recall that one has a canonical isomorphism

$$H^1_{dR}(X/k) \simeq H^1_{dR}(J/k)$$

which is compatible with Hodge filtrations and duality (using the canonical principal polarization to identify $J$ with its dual) and which, for any finite morphism of smooth curves $\rho : Y \to X$ over $k$, intertwines $\rho_*$ with $\text{Pic}^0(\rho)^*$ and $\rho^*$ with $\text{Alb}(\rho)^*$; see [Cai10, Proposition 5.4], noting that the proof given there works over any field $k$, and cf. Proposition 2.2.5. It follows from these compatibilities and the fact that the Cartier operator as defined in [Oda69, Definition 5.5] is functorial that the identification (2.1.3) is moreover an isomorphism of left $A$-modules, with the $A$-structure on $H^1_{dR}(J/k)$ defined as in [Oda69, Definition 5.8].

Now by [Oda69, Corollary 5.11] and [BBM82, Theorem 4.2.14], for any abelian variety $B$ over $k$, there is a canonical isomorphism of left $A$-modules

$$H^1_{dR}(B/k) \simeq D(B)_k.$$


Using the definition of this isomorphism in Proposition 4.2 and Theorem 5.10 of [Oda69], it is straightforward (albeit tedious\textsuperscript{8}) to check that for any homomorphism $h : B' \to B$ of abelian varieties over $k$, the identification (2.1.4) intertwines $h^*$ and $D(h)$. Combining (2.1.3) and (2.1.4) yields (1) and (2).

Now since $V = F_\ast$ (respectively $F = F_\ast$) is the zero endomorphism of $H^1(X, \mathcal{O}_X)$ (respectively $H^0(X, \mathcal{O}_X)$), the canonical mapping

$$H^0(X, \Omega^1_{X/k}) \longrightarrow H^1_{dR}(X/k) \simeq D(G)_k \quad \text{respectively} \quad D(G)_k \simeq H^1_{dR}(X/k) \longrightarrow H^1(X, \mathcal{O}_X)$$

induces an isomorphism on $V$-ordinary (respectively $F$-ordinary) subspaces. On the other hand, by Dieudonné theory one knows that for any $p$-divisible group $H$, the semilinear endomorphism $V$ (respectively $F$) of $D(H)_W$ is bijective if and only if $H$ is of multiplicative type (respectively étale). The (functorial) decomposition $G = G^{et} \times G^m \times G^I$ yields a natural isomorphism of left $A$-modules

$$D(G)_W \simeq D(G^{et})_W \oplus D(G^m)_W \oplus D(G^I)_W,$$

and it follows that the natural maps $D(G^m)_W \to D(G)_W$, $D(G)_W \to D(G^{et})_W$ induce isomorphisms (2.1.5)

$$D(G^m)_W \simeq D(G)^{Vord}_W \quad \text{and} \quad D(G)^{Ford}_W \simeq D(G^{et})_W,$$

respectively, which gives (3). Finally, (4) follows from Proposition 5.3.13 and the proof of Theorem 5.1.8 in [BBM82], using Proposition 2.5.8 of \textit{op. cit.} and the compatibility of the isomorphism (2.1.3) with duality (for which see [Col98, Theorem 5.1] and cf. [Cai10, Lemma 5.5]). \hfill $\blacksquare$

2.2. 	extbf{Universal vectorial extensions.} We now study the mixed characteristic analogue of the situation considered in §2.1. Fix a discrete valuation ring $R$ with field of fractions $K$ of characteristic zero and perfect residue field $k$ of characteristic $p$. Recall [Cai14, §2.1] that by a curve over $S := \text{Spec} R$ we mean a flat finitely presented local complete intersection $f : X \to S$ of relative dimension one with geometrically reduced fibers. Let $f : X \to S$ be a normal and proper curve over $S$ with smooth and geometrically connected generic fiber $X_K$, and write $\omega_{X/S}$ for the relative dualizing sheaf of $f$. The hypercohomology $H^i(X/R)$ of the two-term complex $\mathcal{O}_X \to \omega_{X/S}$ provides a canonical integral structure on the algebraic de Rham cohomology of the generic fiber $X_K$:

\textbf{Proposition 2.2.1} ([Cai14, 2.1.11]). \textit{Let $f : X \to S$ be a normal curve that is proper over $S = \text{Spec}(R)$. There is a canonical short exact sequence of finite free $R$-modules, which we denote $H(X/R)$,

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X/R) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0$$

that recovers the Hodge filtration of $H^1_{dR}(X_K/K)$ after extending scalars to $K$ and is canonically $R$-linearly self-dual via the cup-product pairing on $H^1_{dR}(X_K/K)$. The exact sequence $H(X/R)$ is functorial in finite morphisms $\rho : Y \to X$ of normal and proper $S$-curves via pullback $\rho^*$ and trace $\rho_*$; these morphisms recover the usual pullback and trace mappings on Hodge filtrations after extending scalars to $K$ and are adjoint with respect to the cup-product autoduality of $H(X/R)$.

There is an alternate description of the short exact sequence $H(X/R)$ of Proposition 2.2.1 in terms of Lie algebras and Néron models of Jacobians that will allow us to relate this cohomology to Dieudonné modules. To explain this description and its connection with crystals, we first recall some facts from [MM74] and [Cai10].}

\textsuperscript{8}Alternately, one could appeal to [MM74], specifically to Chapter I, 4.1.7, 4.2.1, 3.2.3, 2.6.7 and to Chapter II, §13 and §15 (see especially Chapter II, 13.4 and 1.6). See also §2.5 and §4 of [BBM82].
Fix a base scheme $T$, and let $G$ be an fppf sheaf of abelian groups over $T$. A vectorial extension of $G$ is a short exact sequence (of fppf sheaves of abelian groups)

\[
\begin{array}{c}
0 \to V \to E \to G \to 0.
\end{array}
\]  

with $V$ a vector group (i.e. an fppf abelian sheaf which is locally represented by a product of $\mathbb{G}_a$'s).

Assuming that $\text{Hom}(G, V) = 0$ for all vector groups $V$, we say that a vectorial extension (2.2.1) is universal if, for any vector group $V'$ over $T$, the pushout map $\text{Hom}_T(V, V') \to \text{Ext}^1_T(G, V')$ is an isomorphism. When a universal vectorial extension of $G$ exists, it is unique up to canonical isomorphism and covariantly functorial in morphisms $G' \to G$ with $G'$ admitting a universal extension.

**Theorem 2.2.2.** Let $T$ be an arbitrary base scheme.

1. If $A$ is an abelian scheme over $T$, then a universal vectorial extension $\mathcal{E}(A)$ of $A$ exists, with $V = \omega_A$, and is compatible with arbitrary base change on $T$.
2. If $p$ is locally nilpotent on $T$ and $G$ is a $p$-divisible group over $T$, then a universal vectorial extension $\mathcal{E}(G)$ of $G$ exists, with $V = \omega_{G^t}$, and is compatible with arbitrary base change on $T$.
3. If $p$ is locally nilpotent on $T$ and $A$ is an abelian scheme over $T$ with associated $p$-divisible group $G := A[p^\infty]$, then the canonical map of fppf sheaves $G \to A$ extends to a natural map

\[
\begin{array}{c}
0 \to \omega_G \to \mathcal{E}(G) \to G \to 0
\end{array}
\]

which induces an isomorphism of the corresponding short exact sequences of Lie algebras.

**Proof.** For the proofs of (1) and (2), see [MM74, I, §1.8 and §1.9]. To prove (3), note that pulling back the universal vectorial extension of $A$ along $G \to A$ gives a vectorial extension $\mathcal{E}'$ of $G$ by $\omega_A$. By universality, there then exists a unique map $\psi : \omega_{G^t} \to \omega_A$ with the property that the pushout of $\mathcal{E}(G)$ along $\psi$ is $\mathcal{E}'$, and this gives the map on universal extensions. That the induced map on Lie algebras is an isomorphism follows from [MM74, II, §13].

For our applications, we will need a generalization of the universal extension of an abelian scheme to the setting of Néron models; in order to describe this generalization, we first recall the explicit description of the universal extension of an abelian scheme in terms of rigidified extensions.

For any commutative $T$-group scheme $F$, a rigidified extension of $F$ by $\mathbb{G}_m$ over $T$ is a pair $(E, \sigma)$ consisting of an extension (of fppf abelian sheaves)

\[
\begin{array}{c}
0 \to \mathbb{G}_m \to E \to F \to 0
\end{array}
\]

and a splitting $\sigma : \text{Inf}^1(F) \to E$ of the pullback of (2.2.2) along the canonical closed immersion $\text{Inf}^1(F) \to F$. Two rigidified extensions $(E, \sigma)$ and $(E', \sigma')$ are equivalent if there is a group homomorphism $E \to E'$ carrying $\sigma$ to $\sigma'$ and inducing the identity on $\mathbb{G}_m$ and on $F$. The set $\text{Extrig}_T(F, \mathbb{G}_m)$ of equivalence classes of rigidified extensions over $T$ is naturally a group via Baer sum of rigidified extensions[MM74, I, §2.1], so the functor on $T$-schemes $T' \mapsto \text{Extrig}_T(F_T, \mathbb{G}_m)$ is naturally a group functor that is contravariant in $F$ via pullback (fibered product). We write $\mathcal{E}xtrig_T(F, \mathbb{G}_m)$ for the fppf sheaf of abelian groups associated to this functor.
Proposition 2.2.3 (Mazur-Messing). Let $A$ be an abelian scheme over an arbitrary base scheme $T$. The fppf sheaf $\mathcal{E}\text{xt}_{T}(A, G_{m})$ is represented by a smooth and separated $T$-group scheme, and there is a canonical short exact sequence of smooth group schemes over $T$

\begin{equation}
0 \longrightarrow \omega_{A} \longrightarrow \mathcal{E}\text{xt}_{T}(A, G_{m}) \longrightarrow A^{t} \longrightarrow 0.
\end{equation}

Furthermore, (2.2.3) is naturally isomorphic to the universal extension of $A^{t}$ by a vector group.

Proof. See [MM74], I, §2.6 and Proposition 2.6.7.

In the case that $T = \text{Spec } R$ for $R$ a discrete valuation ring of mixed characteristic $(0, p)$ with fraction field $K$, we have the following generalization of Proposition 2.2.3:

Proposition 2.2.4. Let $A$ be an abelian variety over $K$, with dual abelian variety $A^{t}$, and write $A$ and $A^{t}$ for the Néron models of $A$ and $A^{t}$ over $T = \text{Spec}(R)$. Then the fppf abelian sheaf $\mathcal{E}\text{xt}_{T}(A, G_{m})$ on the category of smooth $T$-schemes is represented by a smooth and separated $T$-group scheme. Moreover, there is a canonical short exact sequence of smooth group schemes over $T$

\begin{equation}
0 \longrightarrow \omega_{A} \longrightarrow \mathcal{E}\text{xt}_{T}(A, G_{m}) \longrightarrow A^{t0} \longrightarrow 0
\end{equation}

which is contravariantly functorial in $A$ via homomorphisms of abelian varieties over $K$. The formation of (2.2.4) is compatible with smooth base change on $T$; in particular, the generic fiber of (2.2.4) is the universal extension of $A^{t}$ by a vector group.

Proof. Since $R$ is of mixed characteristic $(0, p)$ with perfect residue field, this follows from Proposition 2.6 and the discussion following Remark 2.9 in [Cai10].

In the particular case that $A$ is the Jacobian of a smooth, proper and geometrically connected curve $X$ over $K$ which is the generic fiber of a normal proper curve $\mathcal{X}$ over $R$, we can relate the exact sequence of Lie algebras attached to (2.2.4) to the exact sequence $H(\mathcal{X}/R)$ of Proposition 2.2.1:

Proposition 2.2.5. Let $\mathcal{X}$ be a proper relative curve over $T = \text{Spec}(R)$ with smooth generic fiber $X$ over $K$. Write $J := \text{Pic}^{0}_{X/R}$ for the Jacobian of $X$ and $J^{t}$ for its dual, and let $\jmath$, $\jmath^{t}$ be the corresponding Néron models over $R$. There is a canonical homomorphism of exact sequences of finite free $R$-modules

\begin{equation}
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Lie } \omega_{J} & \longrightarrow & \text{Lie } \mathcal{E}\text{xt}_{T}(J, G_{m}) & \longrightarrow & \text{Lie } \jmath^{t0} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^{0}(\mathcal{X}, \omega_{\mathcal{X}/T}) & \longrightarrow & H^{1}(\mathcal{X}/R) & \longrightarrow & H^{1}(\mathcal{X}, \mathcal{O}_{X}) & \longrightarrow & 0 \\
\end{array}
\end{equation}

that is an isomorphism when $\mathcal{X}$ has rational singularities.\textsuperscript{9} For any finite morphism $\rho : J \to \mathcal{X}$ of $S$-curves satisfying the above hypotheses, the map (2.2.5) intertwines $\rho_{*}$ (respectively $\rho^{*}$) on the bottom row with $\text{Pic}(\rho)^{*}$ (respectively $\text{Alb}(\rho)^{*}$) on the top.

Proof. See Theorem 1.2 and (the proof of) Corollary 5.6 in [Cai10].

\textsuperscript{9}Recall that $\mathcal{X}$ is said to have rational singularities if it admits a resolution of singularities $\rho : X^{'} \to X$ with the natural map $R^{1}\rho_{*}\mathcal{O}_{X^{'}} = 0$. Trivially, any regular $\mathcal{X}$ has rational singularities.
Remark 2.2.6. Let \( X \) be a smooth and geometrically connected curve over \( K \) admitting a normal proper model \( \mathcal{X} \) over \( R \) that is a curve having rational singularities. It follows from Proposition 2.2.5 and the Néron mapping property that \( H(\mathcal{X}/R) \) is a canonical integral structure on the Hodge filtration of \( H^1_{\mathrm{dR}}(X/K) \): it is independent of the choice of proper model \( \mathcal{X} \) that is normal with rational singularities, and is contravariantly (respectively covariantly) functorial by pullback (respectively trace) in finite morphisms \( \rho : Y \to X \) of proper smooth curves over \( K \) which admit models over \( R \) satisfying these hypotheses. These facts can be proved in greater generality by appealing to resolution of singularities for excellent surfaces and the flattening techniques of Raynaud–Gruson [RG71]; see [Cai09, Theorem 5.11] for details.

Finally, we will need to relate the universal extension of a \( p \)-divisible group as in Theorem 2.2.2 (2) to its Dieudonné crystal. In order to explain how this goes, we begin by recalling some basic facts from crystalline Dieudonné theory, as discussed in [BBM82].

Fix a perfect field \( k \) and set \( \Sigma := \text{Spec}(W(k)) \), considered as a PD-scheme via the canonical divided powers on the ideal \( pW(k) \). Let \( T \) be a \( \Sigma \)-scheme on which \( p \) is locally nilpotent (so \( T \) is naturally a PD-scheme over \( \Sigma \)), and denote by \( \text{Cris}(T/\Sigma) \) the big crystalline site of \( T \) over \( \Sigma \), endowed with the fppf topology (see [BM79, §2.2]). If \( \mathscr{F} \) is a sheaf on \( \text{Cris}(T/\Sigma) \) and \( T' \) is any PD-thickening of \( T \), we write \( \mathscr{F}_{T'} \) for the associated fppf sheaf on \( T' \). As usual, we denote by \( i_{T/\Sigma} : T_{\text{fppf}} \to (T/\Sigma)_{\text{Cris}} \) the canonical morphism of topoi, and we abbreviate \( G := i_{T/\Sigma*}G \) for any fppf sheaf \( G \) on \( T \).

Let \( G \) be a \( p \)-divisible group over \( T \), considered as an fppf abelian sheaf on \( T \). As in [BBM82], we define the (contravariant) Dieudonné crystal of \( G \) over \( T \) to be

\[
(2.2.6) \quad D(G) := \text{ext}^1_{T/\Sigma}(G, \mathcal{O}_{T/\Sigma}).
\]

It is a locally free crystal in \( \mathcal{O}_{T/\Sigma} \)-modules, which is contravariantly functorial in \( G \) and of formation compatible with base change along PD-morphisms \( T' \to T \) of \( \Sigma \)-schemes thanks to 2.3.6.2 and Proposition 2.4.5 (ii) of [BBM82]. If \( T' = \text{Spec}(A) \) is affine, we will simply write \( D(G)_A \) for the finite locally free \( A \)-module associated to \( D(G)_{T'} \).

The structure sheaf \( \mathcal{O}_{T/\Sigma} \) is canonically an extension of \( G_a \) by the PD-ideal \( \mathfrak{J}_{T/\Sigma} \subseteq \mathcal{O}_{T/\Sigma} \), and by applying \( \mathbb{H}om_{T/\Sigma}(G, \cdot) \) to this extension one obtains (see Propositions 3.3.2 and 3.3.4 as well as Corollaire 3.3.5 of [BBM82]) a short exact sequence (the Hodge filtration)

\[
(2.2.7) \quad 0 \to \text{ext}^1_{T/\Sigma}(G, \mathfrak{J}_{T/\Sigma}) \to D(G) \to \text{ext}^1_{T/\Sigma}(G, G_a) \to 0
\]

that is contravariantly functorial in \( G \) and of formation compatible with base change along PD-morphisms \( T' \to T \) of \( \Sigma \)-schemes. The following “geometric” description of the value of (2.2.7) on a PD-thickening of the base will be essential for our purposes:

**Proposition 2.2.7.** Let \( G \) be a fixed \( p \)-divisible group over \( T \) and let \( T' \) be any \( \Sigma \)-PD thickening of \( T \). If \( G' \) is any lifting of \( G \) to a \( p \)-divisible group on \( T' \), there is a natural isomorphism

\[
0 \to \omega_{G'} \to \mathcal{L}ie(G') \to \mathcal{L}ie(G) \to 0
\]

that is moreover compatible with base change in the evident manner.

**Proof.** See [BBM82, Corollaire 3.3.5] and [MM74, II, Corollary 7.13].
Remark 2.2.8. In his thesis [Mes72], Messing showed that the Lie algebra of the universal extension of $G$ is “crystalline in nature” and used this as the definition\(^{10}\) of $D(G)$. (See chapter IV, §2.5 of [Mes72] and especially 2.5.2). Although we prefer the more intrinsic description (2.2.6) of [MM74] and [BBM82], it is ultimately Messing’s original definition that will be important for us.

2.3. Dieudonné crystals and $(\varphi, \Gamma)$-modules. In this section, we summarize the main results of [CL14], which provides a classification of $p$-divisible groups by certain semi-linear algebra structures. These structures—which arise naturally via the Dieudonné crystal functor—are cyclotomic analogues of Breuil and Kisin modules, and are closely related to Wach modules.\(^{11}\)

Fix a perfect field $k$ of characteristic $p$. Write $W := W(k)$ for the Witt vectors of $k$ and $K$ for its fraction field, and denote by $\varphi$ the unique automorphism of $W(k)$ lifting the $p$-power map on $k$. Fix an algebraic closure $\overline{K}$ of $K$, as well as a compatible sequence $\{\varepsilon^{(r)}\}_{r \geq 1}$ of primitive $p$-power roots of unity in $\overline{K}$, and set $\mathcal{G}_K := \text{Gal}(\overline{K}/K)$. For $r \geq 0$, we put $K_r := K(\mu_{p^r})$ and $R_r := W[\mu_{p^r}]$, and we set $\Gamma_r := \text{Gal}(K_{\infty}/K_r)$, and $\Gamma := \Gamma_0$.

Let $\mathcal{S}_r := W[[u_r]]$ be the power series ring in one variable $u_r$ over $W$, viewed as a topological ring via the $(p, u_r)$-adic topology. We equip $\mathcal{S}_r$ with the unique continuous action of $\Gamma$ and extension of $\varphi$ determined by

$$\gamma u_r := (1 + u_r)^{\gamma} - 1 \quad \text{for } \gamma \in \Gamma \quad \text{and} \quad \varphi(u_r) := (1 + u_r)^p - 1. \tag{2.3.1}$$

We denote by $\mathcal{O}_{\mathcal{S}_r} := \overline{\mathcal{S}_r}_{\varphi^{(r)}}$ the $p$-adic completion of the localization $\mathcal{S}_r(p)$, which is a complete discrete valuation ring with uniformizer $p$ and residue field $k((u_r))$. One checks that the actions of $\varphi$ and $\Gamma$ on $\mathcal{S}_r$ uniquely extend to $\mathcal{O}_{\mathcal{S}_r}$.

For $r > 0$, we write $\theta : \mathcal{S}_r \to R_r$ for the continuous and $\Gamma$-equivariant $W$-algebra surjection sending $u_r$ to $\varepsilon^{(r)} - 1$, whose kernel is the principal ideal generated by the Eisenstein polynomial $E_r := \varphi^*(u_r)/\varphi^{r-1}(u_r)$, and we denote by $\tau : \mathcal{S}_r \to W$ the continuous and $\varphi$-equivariant surjection of $W$-algebras determined by $\tau(u_r) = 0$. We lift the canonical inclusion $R_r \hookrightarrow R_{r+1}$ to a $\Gamma$- and $\varphi$-equivariant $W$-algebra injection $\mathcal{S}_r \hookrightarrow \mathcal{S}_{r+1}$ determined by $u_r \mapsto \varphi(u_{r+1})$; this map uniquely extends to a continuous injection $\mathcal{O}_{\mathcal{S}_r} \hookrightarrow \mathcal{O}_{\mathcal{S}_{r+1}}$, compatibly with $\varphi$ and $\Gamma$. We will frequently identify $\mathcal{S}_r$ (respectively $\mathcal{O}_{\mathcal{S}_r}$) with its image in $\mathcal{S}_{r+1}$ (respectively $\mathcal{O}_{\mathcal{S}_{r+1}}$), which coincides with the image of $\varphi$ on $\mathcal{S}_{r+1}$ (respectively $\mathcal{O}_{\mathcal{S}_{r+1}}$). Under this convention, we have $E_r(u_r) = E_1(u_1) = u_0/u_1$ for all $r > 0$, so we will simply write $\omega := E_r(u_r)$ for this common element of $\mathcal{S}_r$ for $r > 0$.

Definition 2.3.1. We write $B^\mathcal{S}_r$ for the category of Barsotti-Tate modules over $\mathcal{S}_r$, i.e. the category whose objects are pairs $(\mathcal{M}, \varphi_{\mathcal{M}})$ where

- $\mathcal{M}$ is a free $\mathcal{S}_r$-module of finite rank.
- $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear map whose linearization has cokernel killed by $\omega$,

and whose morphisms are $\varphi$-equivariant $\mathcal{S}_r$-module homomorphisms. We write $B^\mathcal{S}_r$ for the subcategory of $B^\mathcal{S}_r$ consisting of objects $(\mathcal{M}, \varphi_{\mathcal{M}})$ which admit a semilinear $\Gamma$-action (in the category $B^\mathcal{S}_r$) with the property that $\Gamma_r$ acts trivially on $\mathcal{M}/u_r\mathcal{M}$. Morphisms in $B^\mathcal{S}_r$ are $\varphi$ and $\Gamma$-equivariant morphisms of $\mathcal{S}_r$-modules. We often abuse notation by writing $\mathcal{M}$ for the pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ and $\varphi$ for $\varphi_{\mathcal{M}}$.

If $(\mathcal{M}, \varphi_{\mathcal{M}})$ is any object of $B^\mathcal{S}_r$, then $1 \otimes \varphi_{\mathcal{M}} : \varphi^*\mathcal{M} \to \mathcal{M}$ is injective with cokernel killed by $\omega$, so there is a unique $\mathcal{S}_r$-linear homomorphism $\psi_{\mathcal{M}} : \mathcal{M} \to \varphi^*\mathcal{M}$ with the property that the composition

\(^{10}\)Noting that it suffices to define the crystal $D(G)$ on $\Sigma$-PD thickenings $T'$ of $T$ to which $G$ admits a lift.

\(^{11}\)See [CL14] for the precise relationship.
of $1 \otimes \varphi_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}}$ (in either order) is multiplication by $\omega$. Clearly, $\varphi_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}}$ determine each other. We warn the reader that the action of $\Gamma$ does not commute with $\psi_{\mathfrak{M}}$; instead, for any $\gamma \in \Gamma$, one has

$$(\gamma \otimes \gamma) \circ \psi_{\mathfrak{M}} = (\gamma \omega / \omega) \cdot \psi_{\mathfrak{M}} \circ \gamma.$$  

**Definition 2.3.2.** Let $\mathfrak{M}$ be an object of $\mathcal{B}T_{\mathfrak{S}_r}^{\varphi, \Gamma}$. The dual of $\mathfrak{M}$ is the object $(\mathfrak{M}^t, \varphi_{\mathfrak{M}}^t)$ of $\mathcal{B}T_{\mathfrak{S}_r}^{\varphi, \Gamma}$ whose underlying $\mathfrak{S}_r$-module is $\mathfrak{M}^t := \text{Hom}_{\mathfrak{S}_r}(\mathfrak{M}, \mathfrak{S}_r)$, equipped with the $\varphi$-semilinear endomorphism

$$\varphi_{\mathfrak{M}}^t : \mathfrak{M}^t \xrightarrow{1 \otimes \text{id}_{\mathfrak{M}^t}} \varphi^* \mathfrak{M}^t \simeq (\varphi^* \mathfrak{M})^t \xrightarrow{\psi_{\mathfrak{M}}^t} \mathfrak{M}^t$$

and the commuting action of $\Gamma$ given for $\gamma \in \Gamma$ by

$$(\gamma f)(m) := \chi(\gamma)^{-1} \varphi^{-1}(\gamma u_r / u_r) \cdot \gamma(f(\gamma^{-1} m)).$$

There is a natural notion of base change for Barsotti–Tate modules. Let $k'/k$ be an algebraic extension (so $k'$ is automatically perfect), and write $W' := W(k')$, $R'_r := W'[\mu_{p^r}]$, $\mathfrak{S}'_r := W'[u_r]$, and so on. The canonical inclusion $W \hookrightarrow W'$ extends to a $\varphi$ and $\Gamma$-compatible $W$-algebra injection $\iota_r : \mathfrak{S}_r \hookrightarrow \mathfrak{S}'_{r+_1}$, and extension of scalars along $\iota_r$ yields a canonical canonical base change functor $\iota_{r*} : \mathcal{B}T_{\mathfrak{S}_r}^{\varphi, \Gamma} \rightarrow \mathcal{B}T_{\mathfrak{S}'_{r+1}}^{\varphi, \Gamma}$ which one checks is compatible with duality.

Let us write $\mathcal{D}^{\Gamma}_{R_r}$ for the subcategory of $p$-divisible groups over $R_r$ consisting of those objects and morphisms which descend (necessarily uniquely) to $K = K_0$ on generic fibers. By Tate’s Theorem, this is of course equivalent to the full subcategory of $p$-divisible groups over $K_0$ which have good reduction over $K_r$. Note that for any algebraic extension $k'/k$, base change along the inclusion $\iota_r : R_r \hookrightarrow R'_{r+1}$ gives a covariant functor $\iota_{r*} : \mathcal{D}^{\Gamma}_{R_r} \rightarrow \mathcal{D}^{\Gamma}_{R'_{r+1}}$.

The main result of [CL14] is the following:

**Theorem 2.3.3.** For each $r > 0$, there is a contravariant functor $\mathcal{M}_r : \mathcal{D}^{\Gamma}_{R_r} \rightarrow \mathcal{B}T_{\mathfrak{S}_r}^{\varphi, \Gamma}$ such that:

1. The functor $\mathcal{M}_r$ is an exact equivalence of categories, compatible with duality.
2. The functor $\mathcal{M}_r$ is of formation compatible with base change: for any algebraic extension $k'/k$, there is a natural isomorphism of composite functors $\iota_{r*} \circ \mathcal{M}_r \simeq \mathcal{M}_{r+1} \circ \iota_{r*}$ on $\mathcal{D}^{\Gamma}_{R_r}$.
3. For $G \in \mathcal{D}^{\Gamma}_{R_r}$, put $\overline{G} := G \times_{R_r} k$ and $G_0 := G \times_{R_r} R_r / p R_r$.
   a. There is a functorial and $\Gamma$-equivariant isomorphism of $W$-modules
      $$\mathcal{M}_r(G) \otimes_{\mathfrak{S}_r, \varphi \circ \iota} W \simeq D(\overline{G})_W,$$
      carrying $\varphi_{\mathfrak{M}} \otimes \varphi$ to $F : D(\overline{G})_W \rightarrow D(\overline{G})_W$ and $\psi_{\mathfrak{M}} \otimes 1$ to $V \otimes 1 : D(\overline{G})_W \rightarrow \varphi^* D(\overline{G})_W$.
      b. There is a functorial and $\Gamma$-equivariant isomorphism of $R_r$-modules
      $$\mathcal{M}_r(G) \otimes_{\mathfrak{S}_r, \theta \circ \varphi} R_r \simeq D(G_0)_{R_r}.$$

We wish to explain how to functorially recover the $G_K$-representation afforded by the $p$-adic Tate module $T_p G_K$ from $\mathcal{M}_r(G)$. In order to do so, we must first recall the necessary period rings; for a more detailed synopsis of these rings and their properties, we refer the reader to [Col08, §6–§8].

As usual, we put

$$\overline{E}^+ := \lim_{\xrightarrow{\longrightarrow} \times p} \mathcal{O}_{C_K} / (p),$$

$^{12}$As one checks using the intertwining relation (2.3.2).

$^{13}$Here we use the notation introduced by Berger and Colmez; in Fontaine’s original notation, this ring is denoted $R$. 
equipped with its canonical $\mathcal{G}_K$-action via “coordinates” and $p$-power Frobenius map $\varphi$. This is a perfect (i.e. $\varphi$ is an automorphism) valuation ring of characteristic $p$ with residue field $\overline{K}$ and fraction field $\overline{E} := \operatorname{Frac}(\overline{E}^+)$ that is algebraically closed. We view $\overline{E}$ as a topological field via its valuation topology, with respect to which it is complete. Our fixed choice of $p$-power compatible sequence \{$\varepsilon(r)$\}, $r \geq 0$ induces an element $\overline{\varepsilon} := (\varepsilon(r) \mod p), r \geq 0$ of $\overline{E}^+$ and we set $E_K := k(\overline{\varepsilon} - 1)$, viewed as a topological\(^\text{14}\) subring of $\overline{E}$; note that this is a $\varphi$- and $\mathcal{G}_K$-stable subfield of $\overline{E}$ that is independent of our choice of $\overline{\varepsilon}$. We write $E := E_K^{\operatorname{sep}}$ for the separable closure of $E_K$ in the algebraically closed field $\overline{E}$. The natural $\mathcal{G}_K$-action on $\overline{E}$ induces a canonical identification $\Gamma := \operatorname{Gal}(\overline{E}/E_K) = \mathcal{H} := \ker(\chi) \subseteq \mathcal{G}_K$, so $E^\mathcal{H} = E_K$. If $E$ is any subring of $\overline{E}$, we write $E^+ := E \cap \overline{E}^+$ for the intersection (taken inside $\overline{E}$).

We now construct Cohen rings for each of the above subrings of $\overline{E}$. To begin with, we put

$$\overline{\mathbb{A}}^+ := W(\overline{E}^+), \quad \text{and} \quad \overline{\mathbb{A}} := W(\overline{E});$$

each of these rings is equipped with a canonical Frobenius automorphism $\varphi$ and action of $\mathcal{G}_K$ via Witt functoriality. Set-theoretically identifying $W(\overline{E})$ with $\prod_{i=0}^\infty \overline{E}$ in the usual way, we endow each factor with its valuation topology and give $\overline{\mathbb{A}}$ the product topology.\(^\text{15}\) The $\mathcal{G}_K$ action on $\overline{\mathbb{A}}$ is then continuous and the canonical $\mathcal{G}_K$-equivariant $W$-algebra surjection $\theta : \overline{\mathbb{A}}^+ \to \mathcal{O}_{C_K}$ is continuous when $\mathcal{O}_{C_K}$ is given its usual $p$-adic topology. For each $r \geq 0$, there is a unique continuous $W$-algebra map $j_r : \mathcal{O}_{C_K} \to \overline{\mathbb{A}}$, determined by $j_r(u_r) := \varphi^{-r}(\overline{\varepsilon} - 1)$. These maps are moreover $\varphi$ and $\mathcal{G}_K$-equivariant, with $\mathcal{G}_K$ acting on $\mathcal{O}_{C_K}$ through the quotient $\mathcal{G}_K \to \Gamma$, and compatible with change in $r$. We define $A_{K,r} := \operatorname{im}(j_r : \mathcal{O}_{C_K} \to \overline{\mathbb{A}})$, which is naturally a $\varphi$ and $\mathcal{G}_K$-stable subring of $\overline{\mathbb{A}}$ that is independent of our choice of $\overline{\varepsilon}$. We again omit the subscript when $r = 0$. Note that $A_{K,r} = \varphi^{-r}(A_K)$ inside $\overline{\mathbb{A}}$, and that $A_{K,r}$ is a discrete valuation ring with uniformizer $p$ and residue field $\varphi^{-r}(E_K)$ that is purely inseparable over $E_K$. We define $A_{K,\infty} := \bigcup_{r \geq 0} A_{K,r}$ and write $\hat{A}_{K}$ (respectively $\hat{A}_{K,r}$) for the closure of $A_{K,\infty}$ in $\overline{\mathbb{A}}$ with respect to the weak (respectively strong) topology.

Let $A_{K,r}^{sh}$ be the strict Henselization of $A_{K,r}$ with respect to the separable closure of its residue field inside $\overline{E}$. Since $\overline{\mathbb{A}}$ is strictly Henselian, there is a unique local morphism $A_{K,r}^{sh} \to \overline{\mathbb{A}}$ recovering the given inclusion on residue fields, and we henceforth view $A_{K,r}^{sh}$ as a subring of $\overline{\mathbb{A}}$. We denote by $A_{r}$ the topological closure of $A_{K,r}^{sh}$ inside $\overline{\mathbb{A}}$ with respect to the strong topology, which is a $\varphi$ and $\mathcal{G}_K$-stable subring of $\overline{\mathbb{A}}$, and we note that $A_{r} = \varphi^{-r}(A_K)$ and $A_{K,r}^{\mathcal{H}} = A_{K,r}$ inside $\overline{\mathbb{A}}$. We note also that the canonical map $Z_P \to \overline{\mathbb{A}}^{\varphi = 1}$ is an isomorphism, from which it immediately follows that the same is true if we replace $\overline{\mathbb{A}}$ by any of its subrings constructed above. If $A$ is any subring of $\overline{\mathbb{A}}$, we define $A^+ := A \cap \overline{\mathbb{A}}^+$, with the intersection taken inside $\overline{\mathbb{A}}$.

\begin{remark}
We will identify $\mathcal{G}_{r}$ and $\mathcal{O}_{C_K}$ with their respective images $A_{K,r}^{sh}$ and $A_{K,r}$ in $\overline{\mathbb{A}}$ under $j_{r}$. Writing $\mathcal{G}_{\infty} := \lim_{r \to \infty} \mathcal{G}_{r}$ and $\mathcal{O}_{\infty} := \lim_{r \to \infty} \mathcal{O}_{r}$, we likewise identify $\mathcal{G}_{\infty}$ with $A_{K,\infty}^+$ and $\mathcal{O}_{\infty}$ with $A_{K,\infty}$. Denoting by $\mathcal{S}_{\infty}$ (respectively $\hat{\mathcal{S}}_{\infty}$) the $p$-adic (respectively $(p,u_0)$-adic) completions, one has

$$\mathcal{S}_{\infty} = \hat{A}_{K}^+ = W(E_K^{\operatorname{rad}^+}) \quad \text{and} \quad \hat{\mathcal{S}}_{\infty} = \hat{A}_{K}^+ = W(\overline{E}_K^+)$$

\end{remark}

\(^{14}\) The valuation $\varepsilon(r)$ on $\overline{E}$ induces the usual discrete valuation on $E_K$, with the unusual normalization $1/p^{r-1}(p-1)$.

\(^{15}\) This is what is called the weak topology on $\mathbb{A}$. If each factor of $\overline{E}$ is instead given the discrete topology, then the product topology on $\overline{\mathbb{A}} = W(\overline{E})$ is the familiar $p$-adic topology, called the strong topology.
for \( E_{\text{rad}} := \bigcup_{r \geq 0} \varphi^{-r}(E_K) \) the radiciel (=perfect) closure of \( E_K \) in \( \bar{E} \) and \( \bar{E}_K \) its topological completion. Via these identifications, \( \omega := u_0/u_1 \in A^+_K,1 \) is a principal generator of \( \ker(\theta : \tilde{A}^+ \to \mathcal{O}_{C_K}) \).

We can now explain the functorial relation between \( \mathcal{M}_r(G) \) and \( T_pG_K \):

**Theorem 2.3.5.** Let \( G \in \text{pdiv}_{R_e}^\Gamma \), and write \( H^1_{\et}(G_K) := (T_pG_K)^{\vee} \) for the \( \mathbb{Z}_p \)-linear dual of \( T_pG_K \). There is a canonical mapping of finite free \( \mathbb{A}^+_r \)-modules with semilinear Frobenius and \( \mathcal{G}_K \)-actions

\[
\mathcal{M}_r(G) \otimes_{\mathcal{G}_r,\varphi} \mathbb{A}^+_r \longrightarrow H^1_{\et}(G_K) \otimes_{\mathbb{Z}_p} \mathbb{A}^+_r
\]

that is injective with cokernel killed by \( u_1 \). Here, \( \varphi \) acts as \( \varphi_{\mathcal{M}_r(G)} \otimes \varphi \) on source and as \( 1 \otimes \varphi \) on target, while \( \mathcal{G}_K \) acts diagonally on source and target through the quotient \( \mathcal{G}_K \to \Gamma \) on \( \mathcal{M}_r(G) \). In particular, there is a natural \( \varphi \) and \( \mathcal{G}_K \)-equivariant isomorphism

\[
\mathcal{M}_r(G) \otimes_{\mathcal{G}_r,\varphi} \mathbb{A}_r \simeq H^1_{\et}(G_K) \otimes_{\mathbb{Z}_p} \mathbb{A}_r.
\]

These mappings are compatible with duality and with change in \( r \) in the obvious manner.

**Corollary 2.3.6.** For \( G \in \text{pdiv}_{R_e}^\Gamma \), there are functorial isomorphisms of \( \mathbb{Z}_p[\mathcal{G}_K] \)-modules

\[
(2.3.5a) \quad T_pG_K \simeq \text{Hom}_{\mathcal{G}_r,\varphi}(\mathcal{M}_r(G), \mathbb{A}_r)
\]

\[
(2.3.5b) \quad H^1_{\et}(G_K) \simeq (\mathcal{M}_r(G) \otimes_{\mathcal{G}_r,\varphi} \mathbb{A}_r)^{\varphi_{\mathcal{M}_r(G)}^{\otimes} = 1}.
\]

which are compatible with duality and change in \( r \). In the first isomorphism, we view \( \mathbb{A}^+_r \) as a \( \mathcal{G}_r \)-algebra via the composite of the usual structure map with \( \varphi \).

**Remark 2.3.7.** By definition, the map \( \varphi^r \) on \( \mathcal{O}_{\mathcal{G}_r} \) is injective with image \( \mathcal{O}_{\mathcal{G}_e} := \mathcal{O}_{\mathcal{G}_0} \), and so induces a \( \varphi \)-semilinear isomorphism of \( W \)-algebras \( \varphi^r : \mathcal{O}_{\mathcal{G}_e} \overset{\cong}{\longrightarrow} \mathcal{O}_{\mathcal{G}_e} \). It follows from (2.3.5b) of Corollary 2.3.6 and Fontaine’s theory of \( (\varphi, \Gamma) \)-modules over \( \mathcal{O}_{\mathcal{G}_e} \) that \( \mathcal{M}_r(G) \otimes_{\mathcal{G}_r,\varphi^r} \mathcal{O}_{\mathcal{G}_e} \) is the \( (\varphi, \Gamma) \)-module functorially associated to the \( \mathbb{Z}_p[\mathcal{G}_K] \)-module \( H^1_{\et}(G_K) \).

For the remainder of this section, we recall the construction of the functor \( \mathcal{M}_r \), both because we shall need to reference it in what follows, and because we feel it is enlightening. For details, including the proofs of Theorems 2.3.3–2.3.5 and Corollary 2.3.6, we refer the reader to [CL14].

Fix \( G \in \text{pdiv}_{R_e}^\Gamma \) and set \( G_0 := G \times_{R_e} R_r/pR_r \). The \( \mathcal{G}_r \)-module \( \mathcal{M}_r(G) \) is a functorial descent of the evaluation of the Dieudonné crystal \( D(G_0) \) on a certain “universal” PD-thickening of \( R_e/pR_e \), which we now describe. Let \( S_r \) be the \( p \)-adic completion of the PD-envelope of \( \mathcal{G}_r \) with respect to the ideal \( \ker \theta \), viewed as a (separated and complete) topological ring via the \( p \)-adic topology. We give \( S_r \) its PD-filtration: for \( q \in \mathbb{Z} \) the ideal \( \text{Fil}^q S_r \) is the topological closure of the ideal generated by the divided powers \( \{ \alpha^{[n]} \} \) for \( \alpha \in \ker \theta \) and \( n \geq q \). By construction, the map \( \theta : \mathcal{G}_r \to R_r \) uniquely extends to a continuous surjection of \( \mathcal{G}_r \)-algebras \( S_r \twoheadrightarrow R_r \) (which we again denote by \( \theta \)) whose kernel \( \text{Fil}^1 S_r \) is equipped with topologically PD-nilpotent\(^{16} \) divided powers. Similarly, the continuous \( W \)-algebra map \( \tau : \mathcal{G}_r \to W \) determined by \( \tau(u_r) = 0 \) uniquely extends to a continuous, PD-compatible \( W \)-algebra surjection \( \tau : S_r \to W \) whose kernel we denote by \( I := \ker(\tau) \). One shows that there is a unique continuous endomorphism \( \varphi \) of \( S_r \) extending \( \varphi \) on \( \mathcal{G}_r \), and that \( \varphi(\text{Fil}^1 S_r) \subseteq pS_r \); in particular, we may define \( \varphi_1 : \text{Fil}^1 S_r \to S_r \) by \( \varphi_1 := \varphi/p \), which is a \( \varphi \)-semilinear homomorphism of \( S_r \)-modules. Note that \( v_r := \varphi_1(E_r) \) is a unit of \( S_r \), so the image of \( \varphi_1 \) generates \( S_r \) as an \( S_r \)-module.

\(^{16}\)Here we use our assumption that \( p > 2 \).
Since the action of $\Gamma$ on $\mathcal{G}_r$ preserves $\ker \theta$, it follows from the universal mapping property of divided power envelopes and $p$-adic continuity considerations that this action uniquely extends to a continuous and $\varphi$-equivariant action of $\Gamma$ on $S_r$, which is compatible with the PD-structure and the filtration. Similarly, the transition map $\mathcal{G}_r \to \mathcal{G}_{r+1}$ uniquely extends to a continuous $\mathcal{G}_r$-algebra homomorphism $S_r \to S_{r+1}$ which is moreover compatible with filtrations (because $E_r(u_r) = E_{r+1}(u_{r+1})$ under our identifications), and for nonnegative integers $s < r$ we view $S_r$ as an $S_s$-algebra via these maps.

Put $\lambda := \log(1 + u_0)/u_0$, where $\log(1 + X) : \Fil^1 S_r \to S_r$ is the usual (convergent for the $p$-adic topology) power series and $u_0 := \varphi^r(u_r) \in S_r$. One checks that $\lambda$ admits the convergent product expansion $\lambda = \prod_{i \geq 0} \varphi^i(v_r)$, so $\lambda \in S_r^\infty$ and

$$\lambda = \varphi(E_r)/p = v_r \quad \text{and} \quad \frac{\lambda}{\gamma \lambda} = \chi(\gamma)^{-1}\varphi^r(\gamma u_r/u_r) \quad \text{for} \gamma \in \Gamma.$$}

**Definition 2.3.8.** Let $\BT^\varphi_{S_r}$ be the category of triples $(\mathcal{M}, \Fil^1 \mathcal{M}, \varphi_{\mathcal{M}, 1})$ where

- $\mathcal{M}$ is a finite free $S_r$-module and $\Fil^1 \mathcal{M} \subseteq \mathcal{M}$ is an $S_r$-submodule.
- $\Fil^1 \mathcal{M}$ contains $(\Fil^1 S_r)\mathcal{M}$ and the quotient $\mathcal{M}/\Fil^1 \mathcal{M}$ is a free $S_r/\Fil^1 S_r = R_r$-module.
- $\varphi_{\mathcal{M}, 1} : \Fil^1 \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear map whose image generates $\mathcal{M}$ as an $S_r$-module.

Morphisms in $\BT^\varphi_{S_r}$ are $S_r$-module homomorphisms which are compatible with the extra structures. As per our convention, we often write $\mathcal{M}$ for a triple $(\mathcal{M}, \Fil^1 \mathcal{M}, \varphi_{\mathcal{M}, 1})$, and $\varphi_1$ for $\varphi_{\mathcal{M}, 1}$ when it can cause no confusion. We denote by $\BT^\varphi_{S_r}$ the subcategory of $\BT^\varphi_{S_r}$ consisting of objects $\mathcal{M}$ that are equipped with a semilinear action of $\Gamma$ which preserves $\Fil^1 \mathcal{M}$, commutes with $\varphi_{\mathcal{M}, 1}$, and whose restriction to $\Gamma_r$ is trivial on $\mathcal{M}/1.\mathcal{M}$; morphisms in $\BT^\varphi_{S_r}$ are $\Gamma$-equivariant morphisms in $\BT^\varphi_{S_r}$.

The kernel of the surjection $S_r/p^n S_r \to R_r/pR_r$ is the image of the ideal $\Fil^1 S_r + pS_r$, which by construction is equipped topologically PD-nilpotent divided powers. We may therefore define

$$\mathcal{M}_r(G) = \mathcal{D}(G_0)s_r := \lim_{\longrightarrow n} \mathcal{D}(G_0)_{S_r/p^n S_r},$$

which is a finite free $S_r$-module that depends contravariantly functorially on $G_0$. We promote $\mathcal{M}_r(G)$ to an object of $\BT^\varphi_{S_r}$ as follows. As the quotient map $S_r \to R_r$ induces a PD-morphism of PD-thickenings of $R_r/pR_r$, there is a natural isomorphism of free $R_r$-modules

$$\mathcal{M}_r(G) \otimes_{S_r} R_r \simeq \mathcal{D}(G_0)_{R_r}.$$  

By Proposition 2.2.7, there is a canonical “Hodge” filtration $\omega_G \subseteq \mathcal{D}(G_0)_{R_r}$, which reflects the fact that $G$ is a $p$-divisible group over $R_r$ lifting $G_0$, and we define $\Fil^1 \mathcal{M}_r(G)$ to be the preimage of $\omega_G$ under the composite of the isomorphism (2.3.8) with the natural surjection $\mathcal{M}_r(G) \to \mathcal{M}_r(G) \otimes_{S_r} R_r$; note that this depends on $G$ and not just on $G_0$. The Dieudonné crystal is compatible with base change (see, e.g. [BM79, 2.4]), so the relative Frobenius $F_{G_0} : G_0 \to G_0^{(p)}$ induces an canonical morphism of $S_r$-modules

$$\varphi^*(\mathcal{D}(G_0)_{S_r}) \simeq \mathcal{D}(G_0^{(p)})_{S_r} \xrightarrow{\mathcal{D}(F_{G_0})} \mathcal{D}(G_0)_{S_r},$$

which we may view as a $\varphi$-semilinear map $\varphi_{\mathcal{M}_r(G)} : \mathcal{M}_r(G) \to \mathcal{M}_r(G)$. As the relative Frobenius map $\omega_{G_0^{(p)}} \to \omega_{G_0}$ is zero, it follows that the restriction of $\varphi_{\mathcal{M}_r(G)}$ to $\Fil^1 \mathcal{M}_r(G)$ has image contained in $p\mathcal{M}_r(G)$, so we may define $\varphi_{\mathcal{M}_r(G), 1} := \varphi_{\mathcal{M}_r(G)}/p$, and one proves as in [Kis06, Lemma A.2] that the image of $\varphi_{\mathcal{M}_r(G), 1}$ generates $\mathcal{M}_r(G)$ as an $S_r$-module.
It remains to equip $\mathcal{M}_r(G)$ with a canonical semilinear action of $\Gamma$. Let us write $G_K$, for the generic fiber of $G$ and $G_K$ for its unique descent to $K = K_0$. The existence of this descent is reflected by the existence of a commutative diagram with cartesian square

$$
\begin{array}{ccc}
G_K \times K_r & \xrightarrow{1 \times \gamma} & G_K \times K_r \\
\downarrow{\rho_1} & & \downarrow{\rho_2} \\
\text{Spec}(K_r) & \xrightarrow{\gamma} & \text{Spec}(K_r)
\end{array}
$$

(2.3.9)

for each $\gamma \in \Gamma$, compatibly with change in $\gamma$; here, the subscript of $\gamma$ denotes base change along the map of schemes induced by $\gamma$. Since $G$ has generic fiber $G_{K_r} = G_K \times_K K_r$, Tate’s Theorem ensures that the dotted arrow above uniquely extends to an isomorphism of $p$-divisible groups over $R_r$

$$
G \xrightarrow{\gamma} G_{\gamma},
$$

compatibly with change in $\gamma$.

By assumption, the action of $\Gamma$ on $S_r$ commutes with the divided powers on $\text{Fil}^1 S_r$ and induces the given action on the quotient $S_r \rightarrow R_r$; in other words, $\Gamma$ acts by automorphisms on the object $(\text{Spec}(R_r/p R_r) \rightarrow \text{Spec}(S_r/p^n S_r))$ of $\text{Cris}((R_r/p R_r)/W)$. Again using the compatibility of $D(G_0)$ with base change, we therefore see that each $\gamma \in \Gamma$ gives an $S_r$-linear map

$$
\gamma^*(D(G_0)_{S_r}) \simeq D((G_0)_{\gamma} S_r) \longrightarrow D(G_0)_{S_r}
$$

and hence an $S_r$-semilinear (over $\gamma$) endomorphism $\gamma$ of $\mathcal{M}_r(G)$. One easily checks that the resulting action of $\Gamma$ on $\mathcal{M}_r(G)$ commutes with $\varphi, \varphi_1$ and preserves $\text{Fil}^1 \mathcal{M}_r(G)$. By the compatibility of $D(G_0)$ with base change and the obvious fact that the $W$-algebra surjection $\tau : S_r \rightarrow W$ is a PD-morphism over the canonical surjection $R_r/p R_r \rightarrow k$, there is a natural isomorphism

$$
\mathcal{M}_r(G) \otimes_{S_r} W \simeq D(\mathcal{G})_W.
$$

(2.3.10)

It follows easily from this and the diagram (2.3.9) that the action of $\Gamma_r$ on $\mathcal{M}_r(G)/\mathcal{M}_r(G)$ is trivial.

To define $\mathcal{M}_r(G)$, we functorially descend the $S_r$-module $\mathcal{M}_r(G)$ along the structure morphism $\alpha_r : S_r \rightarrow S_r$. More precisely, for $\mathfrak{M} \in \text{BT}^{\varphi, \Gamma}_{S_r}$, we define $\alpha_{r*}(\mathfrak{M}) := (M, \text{Fil}^1 M, \Phi_1) \in \text{BT}^{\varphi, \Gamma}_{S_r}$ via:

$$
M := \mathfrak{M} \otimes_{S_r} S_r \quad \text{with diagonal $\Gamma$-action}
$$

(2.3.12)

$$
\text{Fil}^1 M := \{ m \in M : (\varphi_{\mathfrak{M}} \otimes \text{id})(m) \in \mathfrak{M} \otimes_{S_r} \text{Fil}^1 S_r \subseteq \mathfrak{M} \otimes_{S_r} S_r \}
$$

$$
\Phi_1 : \text{Fil}^1 M \xrightarrow{\varphi_{\mathfrak{M}} \otimes \text{id}} \mathfrak{M} \otimes_{S_r} \text{Fil}^1 S_r \xrightarrow{\text{id} \otimes \varphi_{\mathfrak{M}}} \mathfrak{M} \otimes_{S_r, \varphi} S_r = M.
$$

The following is the key technical point of [CL14], and is proved using the theory of windows:

**Theorem 2.3.9.** For each $r$, the functor $\alpha_{r*} : \text{BT}^{\varphi, \Gamma}_{S_r} \rightarrow \text{BT}^{\varphi, \Gamma}_{S_r}$ is an equivalence of categories, compatible with change in $r$. 

Definition 2.3.10. For $G \in \text{pdiv}^\Gamma_{R_r}$, we write $\mathcal{M}_r(G)$ for the functorial descent of $\mathcal{M}_r(G)$ to an object of $\text{BT}^{\varphi, \Gamma}_{\mathcal{E}_r}$ as guaranteed by Theorem 2.3.9. By construction, we have a natural isomorphism of functors $\alpha_r \circ \mathcal{M}_r \simeq \mathcal{M}_r$ on $\text{pdiv}^\Gamma_{R_r}$.

Example 2.3.11. Using Messing’s description of the Dieudonné crystal of a $p$-divisible group in terms of the Lie algebra of its universal extension (cf. remark 2.2.8), one calculates that for $r \geq 1$

\[
\begin{align*}
\mathcal{M}_r(\mathbb{Q}_p/\mathbb{Z}_p) &= \mathcal{E}_r, & \varphi_{\mathcal{M}_r(\mathbb{Q}_p/\mathbb{Z}_p)} &= \varphi, & \gamma := \gamma \\
\mathcal{M}_r(\mu_{p^\infty}) &= \mathcal{E}_r, & \varphi_{\mathcal{M}_r(\mu_{p^\infty})} &= \omega \cdot \varphi, & \gamma := \chi(\gamma)^{-1} \varphi^{-1}(\gamma u_r/u_r) \cdot \gamma
\end{align*}
\]

with $\gamma \in \Gamma$ acting as indicated. Note that both $\mathcal{M}_r(\mathbb{Q}_p/\mathbb{Z}_p)$ and $\mathcal{M}_r(\mathbb{G}_m[p^\infty])$ arise by base change from their incarnations when $r = 1$, as follows from the fact that $\omega = \varphi(u_1)/u_1$ and $\varphi^{-1}(\gamma u_r/u_r) = \gamma u_1/u_1$ via our identifications.

2.4. The case of ordinary $p$-divisible groups. When $G \in \text{pdiv}^\Gamma_{R_r}$ is ordinary, one can say significantly more about the structure of the $\mathcal{E}_r$-module $\mathcal{M}_r(G)$. To begin with, we observe that for arbitrary $G \in \text{pdiv}^\Gamma_{R_r}$, the formation of the maximal étale quotient of $G$ and of the maximal connected and multiplicative-type sub $p$-divisible groups of $G$ are functorial in $G$, so each of $G^{\text{et}}$, $G^0$, and $G^\infty$ is naturally an object of $\text{pdiv}^\Gamma_{R_r}$ as well. We thus (functorially) obtain objects $\mathcal{M}_r(G^*)$ of $\text{BT}^{\varphi, \Gamma}_{\mathcal{E}_r}$ which admit particularly simple descriptions when $* = \text{ét}$ or $m$, as we now explain.

As usual, we write $\mathcal{G}^*$ for the special fiber of $G^*$ and $\mathcal{D}(\mathcal{G}^*)_W$ for its Dieudonné module. Twisting the $W$-algebra structure on $\mathcal{E}_r$ by the automorphism $\varphi^{-1}$ of $W$, we define objects of $\text{BT}^{\varphi, \Gamma}_{\mathcal{E}_r}$

\[
\begin{align*}
\mathcal{M}_r^{\text{et}}(G) := \mathcal{D}(\mathcal{G}^{\text{et}})_W \otimes_{\mathcal{E}_r} \mathcal{E}_r, & \quad \varphi_{\mathcal{M}_r^{\text{et}}} := F \otimes \varphi, & \gamma := \gamma \otimes \gamma \\
\mathcal{M}_r^{\text{m}}(G) := \mathcal{D}(\mathcal{G}^{\text{m}})_W \otimes_{\mathcal{E}_r} \mathcal{E}_r, & \quad \varphi_{\mathcal{M}_r^{\text{m}}} := V^{-1} \otimes E_r \cdot \varphi, & \gamma := \gamma \otimes \chi(\gamma)^{-1} \varphi^{-1}(\gamma u_r/u_r) \cdot \gamma
\end{align*}
\]

with $\gamma \in \Gamma$ acting as indicated. Note that these formulae make sense and do indeed give objects of $\text{BT}^{\varphi, \Gamma}_{\mathcal{E}_r}$ as $V$ is invertible\textsuperscript{17} on $\mathcal{D}(\mathcal{G}^{\text{m}})_W$ and $\gamma u_r/u_r \in \mathcal{E}_r$. It follows easily from these definitions that $\varphi_{\mathcal{M}_r^{\text{m}}}^*$ linearizes to an isomorphism when $* = \text{ét}$ and has image contained in $\omega \cdot \mathcal{M}_r^{\text{et}}(G)$ when $* = m$. Of course, $\mathcal{M}_r(G)$ is contravariantly functorial in—and depends only on—the closed fiber $\mathcal{G}^*$ of $G^*$.

Proposition 2.4.1. Let $G$ be an object of $\text{pdiv}^\Gamma_{R_r}$ and let $\mathcal{M}_r^{\text{et}}(G)$ and $\mathcal{M}_r^{\text{m}}(G)$ be as in (2.4.1a) and (2.4.1b), respectively. The map $F^r : G_0 \rightarrow G_0^{(p^r)}$ (respectively $V^r : G_0^{(p^r)} \rightarrow G_0$) induces a natural isomorphism in $\text{BT}^{\varphi, \Gamma}_{\mathcal{E}_r}$

\[
\begin{align*}
\mathcal{M}_r(G^{\text{et}}) \simeq \mathcal{M}_r^{\text{et}}(G) & \quad \text{respectively} \quad \mathcal{M}_r(G^{\text{m}}) \simeq \mathcal{M}_r^{\text{m}}(G).
\end{align*}
\]

\textsuperscript{17}A $\varphi^{-1}$-semilinear map of $W$-modules $V : D \rightarrow D$ is invertible if there exists a $\varphi$-semilinear endomorphism $V^{-1}$ whose composition with $V$ in either order is the identity. This is easily seen to be equivalent to the invertibility of the linear map $V \otimes 1 : D \rightarrow \varphi^* D$, with $V^{-1}$ the composite of $(V \otimes 1)^{-1}$ and the $\varphi$-semilinear map $\text{id} \otimes 1 : D \rightarrow \varphi^* D$.}
These identifications are compatible with change in \( r \) in the sense that for \( \star = \text{ét} \) (respectively \( \star = \text{m} \)) there is a canonical commutative diagram in \( \text{BT}_{\mathcal{E}_r}^{\Gamma} \)

\[
\begin{array}{ccc}
\mathcal{M}_r(G^* \times R_r \ R_{r+1}) (2.4.2) & \simeq & \mathcal{M}_r(G \times R_r \ R_{r+1}) \\
\mathcal{D}(G^*)_{W} \otimes \mathcal{E}_{r+1} & \simeq & \mathcal{D}(G^*)_{W} \otimes \mathcal{E}_{r+1}
\end{array}
\] (2.4.3)

where the left vertical isomorphism is deduced from Theorem 2.3.3 (2).

**Proof.** For ease of notation, we will write \( \mathcal{M}_r^\text{ét} \) and \( \mathcal{M}_r^\text{m} \) for \( \mathcal{M}_r^\text{ét}(G) \) and \( \mathcal{D}(G^*)_{W} \), respectively. Using (2.3.12), one finds that \( \mathcal{M}_r^\text{ét} := \alpha_r^*(\mathcal{M}_r^{\text{ét}}) \in \text{BT}_{\mathcal{E}_r}^{\Gamma} \) is given by the triple

\[
\mathcal{M}_r^\text{ét} := (D^\text{ét} \otimes_{W,\varphi^r} S_r, \ D^\text{ét} \otimes_{W,\varphi^r} \text{Fil}^1 S_r, \ F \otimes \varphi_1)
\]

with \( \Gamma \) acting diagonally on the tensor product. Similarly, \( \alpha_r^*(\mathcal{M}_r^{\text{m}}) \) is given by the triple

\[
(D^\text{m} \otimes_{W,\varphi^r} S_r, \ D^\text{m} \otimes_{W,\varphi^r} \text{Fil}^1 S_r, \ V^{-1} \otimes \varphi)
\]

where \( \nu_r = \varphi(E_r)/p \) and \( \gamma \in \Gamma \) acts on \( D^\text{m} \otimes_{W,\varphi^r} S_r \) as \( \gamma \otimes \chi(\gamma)^{-1} \varphi^r(\gamma u_r/u_r) \cdot \gamma \). It follows from (2.3.6) that the \( S_r \)-module automorphism of \( D^\text{m} \otimes_{W,\varphi^r} S_r \) is given by multiplication by \( \lambda \) carries (2.4.4b) isomorphically onto the object of \( \text{BT}_{\mathcal{E}_r}^{\Gamma} \) given by the triple

\[
\mathcal{M}_r^\text{m} := (D^\text{m} \otimes_{W,\varphi^r} S_r, \ D^\text{m} \otimes_{W,\varphi^r} \text{Fil}^1 S_r, \ V^{-1} \otimes \varphi)
\]

with \( \Gamma \) acting *diagonally* on the tensor product.

On the other hand, since \( G^\text{ét}_0 \) (respectively \( G^\text{m}_0 \)) is étale (respectively of multiplicative type) over \( R_r/pR_r \), the relative Frobenius (respectively Verschwindung) morphism of \( G_0 \) induces isomorphisms

\[
G^\text{ét}_0 \overset{F^r}{\simeq} (G^\text{ét}_0)^{(p^r)} \simeq \varphi^r G^\text{ét}_0 \times_k R_r/pR_r
\]

respectively

\[
G^\text{m}_0 \overset{V^r}{\simeq} (G^\text{m}_0)^{(p^r)} \simeq \varphi^r G^\text{m}_0 \times_k R_r/pR_r
\]

of \( p \)-divisible groups over \( R_r/pR_r \), where we have used the fact that the map \( x \mapsto x^{p^r} \) of \( R_r/pR_r \) factors as \( R_r/pR_r \rightarrow k \rightarrow R_r/pR_r \) in the final isomorphisms of both lines above. Since the Dieudonné crystal is compatible with base change and the canonical map \( W \rightarrow S_r \) extends to a PD-morphism \( W \rightarrow (S_r, pS_r + \text{Fil}^1 S_r) \) over \( k \rightarrow R_r/pR_r \), applying \( D(\cdot)_{S_r} \) to (2.4.6a)–(2.4.6b) yields natural isomorphisms \( D(G^\ast_0)_{S_r} \simeq D^\ast \otimes_{W,\varphi^r} S_r \) for \( \star = \text{ét}, \text{m} \) which carry \( F \) to \( F \otimes \varphi \). It is a straightforward exercise using the construction of \( \mathcal{M}_r(G^\ast) \) explained in §2.3 to check that these isomorphisms extend to give isomorphisms \( \mathcal{M}_r(G^\text{ét}) \simeq \mathcal{M}_r^\text{ét} \) and \( \mathcal{M}_r(G^\text{m}) \simeq \mathcal{M}_r^\text{m} \) in \( \text{BT}_{\mathcal{E}_r}^{\Gamma} \). By Theorem 2.3.9, we conclude that we have natural isomorphisms in \( \text{BT}_{\mathcal{E}_r}^{\Gamma} \) as in (2.4.2). The commutativity of (2.4.3) is straightforward, using the definitions of the base change isomorphisms.

Now suppose that \( G \) is ordinary. As \( \mathcal{M}_r \) is exact by Theorem 2.3.3 (1), applying \( \mathcal{M}_r \) to the connected-étale sequence of \( G \) gives a short exact sequence in \( \text{BT}_{\mathcal{E}_r}^{\Gamma} \)

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{M}_r(G^\text{ét}) & \rightarrow & \mathcal{M}_r(G) & \rightarrow & \mathcal{M}_r(G^\text{m}) & \rightarrow & 0
\end{array}
\]

(2.4.7)
which is contravariantly functorial and exact in $G$. Since $\varphi_{\mathcal{M}}$ linearizes to an isomorphism on $\mathcal{M}_r(G^{\text{et}})$ and is topologically nilpotent on $\mathcal{M}_r(G^m)$, we think of (2.4.7) as the “slope filtration” for Frobenius acting on $\mathcal{M}_r(G)$. On the other hand, Proposition 2.2.7 and Theorem 2.3.3 (3b) provide a canonical “Hodge filtration” of $\mathcal{M}_r(G) \otimes R_r \simeq D(G_0)_{R_r}$:

$$(2.4.8) \quad 0 \longrightarrow \omega_G \longrightarrow D(G_0)_{R_r} \longrightarrow \text{Lie}(G^t) \longrightarrow 0$$

which is contravariant and exact in $G$. Our assumption that $G$ is ordinary yields (cf. [Kat81]):

**Lemma 2.4.2.** With notation as above, there are natural and $\Gamma$-equivariant isomorphisms

$$(2.4.9) \quad \text{Lie}(G^t) \simeq D(G_0^{\text{et}})_{R_r} \quad \text{and} \quad D(G_0^m)_{R_r} \simeq \omega_G.$$

Composing these isomorphisms with the canonical maps obtained by applying $D(\cdot)_{R_r}$ to the connected-étale sequence of $G_0$ yield functorial $R_r$-linear splittings of the Hodge filtration (2.4.8). Furthermore, there is a canonical and $\Gamma$-equivariant isomorphism of split exact sequences of $R_r$-modules

$$(2.4.10) \quad 0 \longrightarrow \omega_G \longrightarrow D(G_0^{\text{et}})_{R_r} \longrightarrow \text{Lie}(G^t) \longrightarrow 0$$

with $i, j$ the inclusion and projection mappings obtained from the canonical direct sum decomposition $D(G^m)_W \simeq D(G_{\text{et}}^m)_W \oplus D(G^{\text{et}}^t)_W$.

**Proof.** Applying $D(\cdot)_{R_r}$ to the connected-étale sequence of $G_0$ and using Proposition 2.2.7 yields a commutative diagram with exact columns and rows

$$(2.4.11) \quad 0 \longrightarrow \omega_G \longrightarrow \omega_{G^{\text{et}}} \longrightarrow \omega_{G^m} \longrightarrow 0$$

$$0 \longrightarrow D(G_0^{\text{et}})_{R_r} \longrightarrow D(G_0)_{R_r} \longrightarrow D(G_0^m)_{R_r} \longrightarrow 0$$

$$0 \longrightarrow \text{Lie}(G^{\text{et}}) \longrightarrow \text{Lie}(G^t) \longrightarrow 0$$

where we have used the fact that that the invariant differentials and Lie algebra of an étale $p$-divisible group (such as $G^{\text{et}}$ and $G^m \simeq G^{\text{et}}$) are both zero. The isomorphisms (2.4.9) follow at once. We likewise immediately see that the short exact sequence in the center column of (2.4.11) is functorially and $R_r$-linearly split. Thus, to prove the claimed identification in (2.4.10), it suffices to exhibit natural
isomorphisms of free $R_r$-modules with $\Gamma$-action
\begin{equation}
D(G_0^{\text{et}})_{R_r} \simeq D(G_t^{\text{et}})_{W, \varphi} \otimes R_r \quad \text{and} \quad D(G_0^{\text{m}})_{R_r} \simeq D(G_m^{\text{m}})_{W, \varphi'} \otimes R_r,
\end{equation}
both of which follow easily by applying $D(\cdot)_{R_r}$ to (2.4.6a) and (2.4.6b) and using the compatibility of the Dieudonnée crystal with base change as in the proof of Proposition (2.4.1).

From the slope filtration (2.4.7) of $\mathcal{M}_r(G)$ we can recover both the (split) slope filtration of $D(G)_{W}$ and the (split) Hodge filtration (2.4.8) of $D(G_0)_{R_r}$:

**Proposition 2.4.3.** There are canonical and $\Gamma$-equivariant isomorphisms of short exact sequences
\begin{equation}
0 \longrightarrow \mathcal{M}_r(G_t^{\text{et}}) \otimes_{\mathcal{E}_r, \varphi_\otimes} W \longrightarrow \mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \varphi_\otimes} W \longrightarrow \mathcal{M}_r(G_m^{\text{m}}) \otimes_{\mathcal{E}_r, \varphi_\otimes} W \longrightarrow 0
\end{equation}
\begin{equation}
0 \longrightarrow D(G_t^{\text{et}})_{W} \longrightarrow D(G)_{W} \longrightarrow D(G_m^{\text{m}})_{W} \longrightarrow 0
\end{equation}
\begin{equation}
0 \longrightarrow \mathcal{M}_r(G_t^{\text{et}}) \otimes_{\mathcal{E}_r, \theta_\otimes \varphi} R_r \longrightarrow \mathcal{M}_r(G) \otimes_{\mathcal{E}_r, \theta_\otimes \varphi} R_r \longrightarrow \mathcal{M}_r(G_m^{\text{m}}) \otimes_{\mathcal{E}_r, \theta_\otimes \varphi} R_r \longrightarrow 0
\end{equation}
\begin{equation}
0 \longrightarrow \text{Lie}(G^t) \longrightarrow i : D(G_0)_{R_r} \longrightarrow j : D(G_0)_{R_r} \longrightarrow \omega_G \longrightarrow 0
\end{equation}

Here, $i : \text{Lie}(G^t) \hookrightarrow D(G_0)_{R_r}$ and $j : D(G_0)_{R_r} \twoheadrightarrow \omega_G$ are the canonical splittings of Lemma 2.4.2, the top row of (2.4.13b) is obtained from (2.4.7) by extension of scalars, and the isomorphism (2.4.13a) intertwines $\varphi_{\mathcal{M}_r(\cdot)} \otimes \varphi$ with $F$.

**Proof.** This follows immediately from Theorem 2.3.3 (3a) and Lemma 2.4.2. $lacksquare$

3. Ordinary $\Lambda$-adic Dieudonnée and $(\varphi, \Gamma)$-modules

In this section, we will state and prove our main results as described in §1.2. Throughout, we will use the notation of §1.2 and of [Cai14, §2.2], which we now briefly recall.

For $r \geq 1$, we write $X_r := X_1(Np^r)$ for the canonical model over $\mathbb{Q}$ with rational cusp at $i \infty$ of the modular curve arising as the quotient of the extended upper-halfplane by the congruence subgroup $\Gamma_1(Np^r)$ (cf. [Cai14, Remark 2.2.4]). There are two natural degeneracy mappings $\rho, \sigma : X_{r+1} \twoheadrightarrow X_r$ of curves over $\mathbb{Q}$ induced by the self-maps of the upper-halfplane $\rho : \tau \mapsto \tau$ and $\sigma : \tau \mapsto p \tau$; see [Cai14, Remark 2.2.5]. Denote by $J_r := \text{Pic}^0_{X_r/\mathbb{Q}}$ the Jacobian of $X_r$ over $\mathbb{Q}$ and write $\mathcal{H}_r(\mathbb{Z})$ for the $\mathbb{Z}$-subalgebra of $\text{End}_\mathbb{Q}(J_r)$ generated by the Hecke operators $\{T_l\}_{l \mid Np^r}$, $\{U_l\}_{l \mid Np}$ and the Diamond operators $\{(u)\}_{u \in \mathbb{Z}_p^\times}$. We define $\mathcal{H}_r(\mathbb{Z})^\ast$ similarly, using instead the “transpose” Hecke and diamond operators, and set $\mathcal{H}_r := \mathcal{H}_r(\mathbb{Z}) \otimes \mathbb{Z}_p$ and $\mathcal{H}_r^\ast := \mathcal{H}_r(\mathbb{Z})^\ast \otimes \mathbb{Z}_p$; see [Cai14, 2.2.21–2.2.23]. As usual, we write $e_r \in \mathcal{H}_r$ and $e_r^\ast \in \mathcal{H}_r^\ast$ for the idempotents of these semi-local $\mathbb{Z}_p$-algebras corresponding to the Atkin operators $U_p$ and $U_p^\ast$, respectively, and we put $e := (e_r)_r$ and $e^\ast := (e_r^\ast)_r$ for the induced idempotents of the “big” $p$-adic Hecke algebras $\mathcal{H} := \varprojlim_r \mathcal{H}_r$ and $\mathcal{H}^\ast := \varprojlim_r \mathcal{H}_r^\ast$; here, the maps in these projective limits are induced by the natural transition mappings on Jacobians $J_r \cong J_{r'}$ for $r' \geq r$ arising (via Picard functoriality) from $\sigma$ and $\rho$, respectively. Let $w_r$ be the Atkin–Lehner “involution” of $X_r$ over $\mathbb{Q}(\mu_{Np^r})$ corresponding to a choice of primitive $Np^r$-th root of unity as in the discussion preceding [Cai14, Proposition 2.2.6]; following the conventions of [Cai14, §2.2], we simply write $w_r$,
for the automorphism \(\text{Alb}(w_r)\) of \(J_r\) over \(\mathbb{Q}(\mu_{Np'})\) induced by Albanese functoriality. We note that for any Hecke operator \(T \in \mathcal{S}_r(\mathbb{Z})\), one has the relation \(w_r T = T^* w_r\) as endomorphisms of \(J_r\) over \(\mathbb{Q}(\mu_{Np'})\) [Cai14, Proposition 2.2.24].

3.1. \(\Lambda\)-adic Barsotti-Tate groups. In order to construct a crystalline analogue of Hida’s ordinary \(\Lambda\)-adic étale cohomology, we will apply the theory of \(\S 2.3\) to a certain “tower” \(\{\mathcal{G}_r\}_{r \geq 1}\) of \(p\)-divisible groups (a \(\Lambda\)-adic Barsotti Tate group in the sense of Hida [Hid13, Hid05a, Hid05b]) whose construction involves artfully cutting out certain \(p\)-divisible subgroups of \(J_r[p^\infty]\) over \(\mathbb{Q}\) and the “good reduction” theorems of Langlands-Carayol-Saito. The construction of \(\{\mathcal{G}_r\}_{r \geq 1}\) is certainly well-known (e.g. [MW86, §1], [MW84, Chapter 3, §1], [Til87, Definition 1.2] and [Oht95, §3.2]), but as we shall need substantially finer information about the \(\mathcal{G}_r\) than is available in the literature, we devote this section to recalling their construction and properties.

As in [Cai14, §3.3], for a ring \(A\), a nonegative integer \(k\), and a congruence subgroup \(\Gamma\) of \(\text{SL}_2(\mathbb{Z})\), we write \(S_k(\Gamma; A)\) for the space of weight \(k\) cuspforms for \(\Gamma\) over \(A\), and for ease of notation we put \(S_k(\Gamma) := S_k(\Gamma; \mathbb{Q})\). If \(\Gamma', \Gamma\) are congruence subgroups, then associated to any \(\gamma \in \text{GL}_2(\mathbb{Q})\) with \(\gamma^{-1} \Gamma' \gamma \subseteq \Gamma\) is an injective pullback mapping \(\iota_\gamma : S_k(\Gamma) \leftarrow S_k(\Gamma')\) given by \(\iota_\gamma(f) := f|_{\gamma^{-1}}\), as well as a surjective “trace” mapping

\[
\text{tr}_\gamma : S_k(\Gamma') \longrightarrow S_k(\Gamma) \quad \text{given by} \quad \text{tr}_\gamma(f) := \sum_{\delta \in \gamma^{-1} \Gamma' \gamma} (f|_{\gamma^{-1}}) |_{\delta}
\]

with \(\text{tr}_\gamma \circ \iota_\gamma = \delta\) multiplication by \([\Gamma : \gamma^{-1} \Gamma' \gamma]\) on \(S_k(\Gamma)\). If \(\Gamma' \subseteq \Gamma\), then unless specified to the contrary, we will always view \(S_k(\Gamma)\) as a subspace of \(S_k(\Gamma')\) via \(\iota_{id}\).

For nonnegative integers \(i \leq r\) we set \(\Gamma^*_r := \Gamma_1(Np^i) \cap \Gamma_0(p^r)\) for the intersection (taken inside \(\text{SL}_2(\mathbb{Z})\)), and put \(\Gamma_r := \Gamma^*_r\). We will need the following fact (cf. [Til87, pg. 339], [Oht95, 2.3.3]) concerning the trace mapping \((3.1.1)\) attached to the canonical inclusion \(\Gamma_r \subseteq \Gamma_i\) for \(r \geq i\); for notational clarity, we will write \(\text{tr}_{r,i} : S_k(\Gamma_r) \rightarrow S_k(\Gamma_i)\) for this map.

**Lemma 3.1.1.** Fix integers \(i \leq r\) and let \(\text{tr}_{r,i} : S_k(\Gamma_r) \rightarrow S_k(\Gamma_i)\) be the trace mapping \((3.1.1)\) attached to the inclusion \(\Gamma_r \subseteq \Gamma_i\). For \(\alpha := \begin{pmatrix} 1 & 0 \\ 0 & p^r \end{pmatrix}\), we have an equality of \(\mathbb{Q}\)-endomorphisms of \(S_k(\Gamma_r)\)

\[
\iota_{\alpha^{-i}} \circ \text{tr}_{r,i} = (U_{p^r}^*)^{-i} \sum_{\delta \in \Delta_i/\Delta_r} \langle \delta \rangle.
\]

**Proof.** We have index \(p^{-i}\) inclusions of groups \(\Gamma_r \subseteq \Gamma_i^* \subseteq \Gamma_i\) with \(\Gamma_r\) normal in \(\Gamma_i^*\), as it is the kernel of the canonical surjection \(\Gamma_i^* \rightarrow \Delta_i/\Delta_r\). For each \(\delta \in \Delta_i/\Delta_r\), we fix a choice of \(\sigma_\delta \in \Gamma_i^*\) mapping to \(\delta\) and calculate that

\[
\Gamma_i = \prod_{\delta \in \Delta_i/\Delta_r} \prod_{j=0}^{p^{-i}-1} \Gamma_i \sigma_\delta \theta_j \quad \text{where} \quad \theta_j := \begin{pmatrix} 1 & 0 \\ jNp^i & 1 \end{pmatrix}.
\]

On the other hand, for each \(0 \leq j < p^{-i}\) one has the equality of matrices in \(\text{GL}_2(\mathbb{Q})\)

\[
p^{-i} \theta_j \alpha^{-(r-i)} = \tau_r \begin{pmatrix} 1 & 0 \\ -j & p^{-i} \end{pmatrix} \tau_r^{-1} \quad \text{for} \quad \tau_r := \begin{pmatrix} 0 & -1 \\ -p^r & 0 \end{pmatrix}.
\]

The claimed equality \((3.1.2)\) follows easily from \((3.1.3)\) and \((3.1.4)\), using the equalities of operators \((\cdot)|_{\sigma_\delta} = \langle \delta \rangle\) and \(U_{p^r}^* = w_r U_p w_r^{-1}\) on \(S_k(\Gamma_r)\) [Cai14, Proposition 2.2.24].
Perhaps the most essential “classical” fact for our purposes is that the Hecke operator $U_p$ acting on spaces of modular forms “contracts” the $p$-level, as is made precise by the following:

**Lemma 3.1.2.** If $f \in S_k(\Gamma_r^i)$ then $U_p^d f$ is in the image of the canonical map $\iota_{d!} : S_k(\Gamma_{r-d}^i) \rightarrow S_k(\Gamma_r^i)$ for each integer $d \leq r-i$. In particular, $U_p^{r-i} f$ is in the image of $S_k(\Gamma_i) \rightarrow S_k(\Gamma_r^i)$.

Certainly Lemma 3.1.2 is well-known (e.g. [Til87], [Oht99], [Hid13]); because of its importance in our subsequent applications, we sketch a proof (following the proof of [Oht99, Lemma 1.2.10]; see also [Hid13] and [Hid05a, §2]). We note that $\Gamma_r \subseteq \Gamma_r^i$ for all $i \leq r$, and the resulting inclusion $S_k(\Gamma_i^r) \hookrightarrow S_k(\Gamma_r)$ has image consisting of forms on $\Gamma_r$ which are eigenvectors for the diamond operators and whose associated character has conductor with $p$-part dividing $p^i$.

**Proof of Lemma 3.1.2.** Fix $d$ with $0 \leq d \leq r-i$ and let $\alpha := \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)$ be as in Lemma 3.1.1; then $\alpha^d$ is an element of the commeasurator of $\Gamma_r^i$ in $\SL_2(\Q)$. Consider the following subgroups of $\Gamma_{r-d}^i$:

\[
H := \Gamma_{r-d}^i \cap \alpha^{-d} \Gamma_r^i \alpha^d \\
H' := \Gamma_{r-d}^i \cap \alpha^{-d} \Gamma_{r-d}^i \alpha^d,
\]

with each intersection taken inside of $\SL_2(\Q)$. We claim that $H = H'$ inside $\Gamma_{r-d}^i$. Indeed, as $\Gamma_r \subseteq \Gamma_{r-d}^i$, the inclusion $H \subseteq H'$ is clear. For the reverse inclusion, if $\gamma := \left( \begin{smallmatrix} * & * \\ * & * \end{smallmatrix} \right) \in \Gamma_{r-d}^i$, then we have $\alpha^{-d} \gamma \alpha^d = \left( \begin{smallmatrix} a & b \\ -ap & a \end{smallmatrix} \right)$, so if this lies in $\Gamma_{r-d}^i$ we must have $x \equiv 0 \mod p^r$ and hence $\gamma \in \Gamma_r^i$. We conclude that the coset spaces $H \backslash \Gamma_{r-d}^i$ and $H' \backslash \Gamma_{r-d}^i$ are equal. On the other hand, for any commeasurable subgroups $\Gamma, \Gamma'$ of a group $G$ and any $g$ in the commeasurator of $\Gamma$ in $G$, an elementary computation shows that we have a bijection of coset spaces

\[(\Gamma' \cap g^{-1} \Gamma g) \backslash \Gamma' \simeq \Gamma \backslash \Gamma' \]

via $(\Gamma' \cap g^{-1} \Gamma g) \gamma \mapsto \Gamma g \gamma$. Applying this with $g = \alpha^d$ in our situation and using the decomposition

\[
\Gamma_{r-d}^i \alpha^d \Gamma_{r-d}^i = \prod_{j=0}^{p^d-1} \Gamma_{r-d}^i \left( \begin{array}{cc} 1 & j \\ 0 & p^d \end{array} \right)
\]

(see, e.g. [Shi94, proposition 3.36]), we deduce that we also have

\[
\Gamma_r^i \alpha^d \Gamma_{r-d}^i = \prod_{j=0}^{p^d-1} \Gamma_r^i \left( \begin{array}{cc} 1 & j \\ 0 & p^d \end{array} \right).
\]

Writing $U : S_k(\Gamma_r^i) \rightarrow S_k(\Gamma_{r-d}^i)$ for the “Hecke operator” given by (e.g. [Oht99, §3.4]) $\Gamma_r^i \alpha^d \Gamma_{r-d}^i$, an easy computation using 3.1.5 shows that the composite

\[
S_k(\Gamma_r^i) \xrightarrow{U} S_k(\Gamma_{r-d}^i) \xrightarrow{\iota_{d!}} S_k(\Gamma_r^i)
\]

coincides with $U_p^d$ on $q$-expansions. By the $q$-expansion principle, we deduce that $U_p^d$ on $S_k(\Gamma_r^i)$ indeed factors through the subspace $S_k(\Gamma_{r-d}^i)$, as desired. \qed

For each integer $i$ and any character $\varepsilon : (\Z/Np^i\Z)^\times \rightarrow \overline{\Q}^\times$, we denote by $S_2(\Gamma_i, \varepsilon)$ the $\mathcal{H}_r$-stable subspace of weight 2 cusp forms for $\Gamma_i$ over $\overline{\Q}$ on which the diamond operators act through $\varepsilon(\cdot)$. Define

\[
\nabla_r := \bigoplus_{i=1}^{r} \bigoplus_{\varepsilon} S_2(\Gamma_i, \varepsilon)
\]
where the inner sum is over all Dirichlet characters defined modulo \( Np^j \) whose \( p \)-parts are primitive (i.e. whose conductor has \( p \)-part exactly \( p^j \)). We view \( \text{V}_r \) as a \( \mathbb{Q} \)-subspace of \( S_2(\Gamma_r) \) in the usual way (i.e. via the embeddings \( \iota_{4d} \)). We define \( \text{V}_r^* \) as the direct sum (3.1.6), but viewed as a subspace of \( S_2(\Gamma_r) \) via the \text{“nonstandard”} embeddings \( \iota_{4d-1} : S_2(\Gamma_1) \rightarrow S_2(\Gamma_r) \).

As in [Cai14, 2.5.17], we write \( f' \) for the idempotent of \( \mathbb{Z}(p)[\mathbb{F}_p^2] \) corresponding to “projection away from the trivial \( \mathbb{F}_p^2 \)-eigenspace;” explicitly, we have

\[
f' := 1 - \frac{1}{p-1} \sum_{g \in \mathbb{F}_p^2} g.
\]

We set \( h := (p-1)f' \), so that \( h^2 = (p-1)h \) and define endomorphisms of \( S_2(\Gamma_r) \):

\[
U_r^* := h' \circ (U_p^*)^{r+1} = (U_p^*)^{r+1} \circ h \quad \text{and} \quad U_r := h \circ (U_p)^{r+1} = (U_p)^{r+1} \circ h'.
\]

**Corollary 3.1.3.** As subspaces of \( S_2(\Gamma_r) \) we have \( w_r(\text{V}_r) = \text{V}_r \). The space \( \text{V}_r \) (respectively \( \text{V}_r^* \)) is naturally an \( \mathcal{H}_r \) (resp. \( \mathcal{H}_r^* \))-stable subspace of \( S_2(\Gamma_r) \), and admits a canonical descent to \( \mathbb{Q} \). Furthermore, the endomorphisms \( U_r \) and \( U_r^* \) of \( S_2(\Gamma_r) \) factor through \( \text{V}_r \) and \( \text{V}_r^* \), respectively.

**Proof.** The first assertion follows from the relation \( w_r \circ \iota_{4d-1} = \iota_{4d} \circ w_1 \) as maps \( S_2(\Gamma_1) \rightarrow S_2(\Gamma_r) \), together with the fact that \( w_1 \) on \( S_2(\Gamma_1) \) carries \( S_2(\Gamma_1, \varepsilon) \) isomorphically onto \( S_2(\Gamma_1, \varepsilon^{-1}) \). The \( \mathcal{H}_r \)-stability of \( \text{V}_r \) is clear as each of \( S_2(\Gamma_1, \varepsilon) \) is an \( \mathcal{H}_r \)-stable subspace of \( S_2(\Gamma_r) \); that \( \text{V}_r^* \) is \( \mathcal{H}_r^* \)-stable follows from this and the comutation relation \( T^*w_r = w_rT \) [Cai14, Proposition 2.2.4]. That \( \text{V}_r \) and \( \text{V}_r^* \) admit canonical descents to \( \mathbb{Q} \) is clear, as \( \mathcal{H}_r \)-conjugate Dirichlet characters have equal conductors. The final assertion concerning the endomorphisms \( U_r \) and \( U_r^* \) follows easily from Lemma 3.1.2, using the fact that \( h' : S_2(\Gamma_r) \rightarrow S_2(\Gamma_r) \) has image contained in \( \bigoplus_{i=1}^{r+1} \mathfrak{S}_i(\Gamma_r^*) \). \( \square \)

**Definition 3.1.4.** We denote by \( V_r \) and \( V_r^* \) the canonical descents to \( \mathbb{Q} \) of \( \text{V}_r \) and \( \text{V}_r^* \), respectively.

Following [MW84, Chapter III, §1] and [Til87, §2], we recall the construction of certain “good” quotient abelian varieties of \( J_r \) whose cotangent spaces are naturally identified with \( V_r \) and \( V_r^* \). In what follows, we will make frequent use of the following elementary result:

**Lemma 3.1.5.** Let \( f : A \rightarrow B \) be a homomorphism of commutative group varieties over a field \( K \) of characteristic 0. Then:

1. The formation of Lie and Cot commutes with the formation of kernels and images: the kernel (respectively image) of \( \text{Lie}(f) \) is canonically isomorphic to the Lie algebra of the kernel (respectively image) of \( f \), and similarly for cotangent spaces at the identity. In particular, if \( A \) is connected and \( \text{Lie}(f) = 0 \) (respectively \( \text{Cot}(f) = 0 \)) then \( f = 0 \).
2. Let \( i : B' \hookrightarrow B \) be a closed immersion of commutative group varieties over \( K \) with \( B' \) connected. If \( \text{Lie}(f) \) factors through \( \text{Lie}(i) \) then \( f \) factors (necessarily uniquely) through \( i \).
3. Let \( j : A \rightarrow A'' \) be a surjection of commutative group varieties over \( K \) with connected kernel. If \( \text{Cot}(f) \) factors through \( \text{Cot}(j) \) then \( f \) factors (necessarily uniquely) through \( j \).

**Proof.** The key point is that because \( K \) has characteristic zero, the functors \( \text{Lie}(\cdot) \) and \( \text{Cot}(\cdot) \) on the category of commutative group schemes are exact. Indeed, since \( \text{Lie}(\cdot) \) is always left exact, the exactness of \( \text{Lie}(\cdot) \) follows easily from the fact that any quotient mapping \( G \rightarrow H \) of group varieties in characteristic zero is smooth (as the kernel is a group variety over a field of characteristic zero and hence automatically smooth), so the induced map on Lie algebras is a surjection. By similar reasoning one shows that the right exact \( \text{Cot}(\cdot) \) is likewise exact, and the first part of (1) follows easily. In
particular, if $\text{Lie}(f)$ is the zero map then $\text{Lie}(\text{im}(f)) = 0$, so $\text{im}(f)$ is zero-dimensional. Since it is also smooth, it must be étale. Thus, if $A$ is connected, then $\text{im}(f)$ is both connected and étale, whence it is a single point; by evaluation of $f$ at the identity of $A$ we conclude that $f = 0$. The assertions (2) and (3) now follow immediately by using universal mapping properties.

To proceed with the construction of good quotients of $J_r$, we write $Y_r := X_1(Np^r; Np^{r-1})$ for the canonical model over $\mathbb{Q}$ with rational cusps at $\infty$ of the modular curve corresponding to the congruence subgroup $\Gamma_{r+1}^r$ (cf. \cite[Remark 2.2.18]{Cai14}), and consider the diagrams of “degeneracy mappings” of curves over $\mathbb{Q}$ for $i = 1, 2$

\[ X_r \xrightarrow{\pi} Y_{r-1} \xrightarrow{\pi_1} X_{r-1} \]

where $\pi$ and $\pi_2$ are induced by the canonical inclusions of subgroups $\Gamma_r \subseteq \Gamma_{r-1} \subseteq \Gamma_r$ via the upper-halfplane self map $\tau \mapsto \tau$, and $\pi_1$ is induced by the inclusion $\alpha^{-1}\Gamma_{r}^{-1}\alpha \subseteq \Gamma_{r-1}$ via the mapping $\tau \mapsto p\tau$ where $\alpha$ is as in Lemma 3.1.1; see \cite[2.9]{Cai14} for a moduli-theoretic description of these maps. We note that the compositions $\pi \circ \pi_2$ and $\pi \circ \pi_1$ coincide with the degeneracy maps $\rho$ and $\sigma$, respectively \cite[Remark 2.2.18]{Cai14}.

These mappings covariantly (respectively contravariantly) induce mappings on the associated Jacobians via Albanese (respectively Picard) functoriality. Writing $JY_r := \text{Pic}^0_Y/Q$ and setting $K^i_r := JY^i_1$ for $r = 1, 2$ we inductively define abelian subvarieties $i^i_r : K^i_r \hookrightarrow JY_r$ and abelian variety quotients $\alpha^i_r : J_r \rightarrow B^i_r$ as follows:

\[ B_{r-1}^i := J_{r-1}/\text{Pic}^0(\pi)(K^i_{r-1}) \quad \text{and} \quad K^i_r := \ker(JY_r \xrightarrow{\alpha^i_{r-1} \circ \text{Alb}(\pi_1)} B_{r-1}^i) \]

for $r \geq 2$, $i = 1, 2$, with $i^i_{r-1}$ and $i^i_r$ the obvious mappings; here, $(\cdot)^0$ denotes the connected component of the identity of $\cdot$. As in [Oht95, §3.2], we have modified Tilouine’s construction [Til87, §2] so that the kernel of $\alpha^i_r$ is connected; i.e. is an abelian subvariety of $J_r$ (cf. Remark 3.1.8). Note that we have a commutative diagram of abelian varieties over $\mathbb{Q}$ for $i = 1, 2$

\[ J_{r-1} \xrightarrow{\alpha^i_{r-1}} B_{r-1}^i \]

\[ \text{Alb}(\pi_1) \]

\[ K^i_r \xrightarrow{i^i_r} JY_r \xrightarrow{\alpha^i_{r-1} \circ \text{Alb}(\pi_1)} B_{r-1}^i \]

\[ \text{Pic}^0(\pi) \]

\[ K^i_r \xrightarrow{\text{Pic}^0(\pi) \circ \alpha^i_r} J_r \xrightarrow{\alpha^i_r} B^i_r \]

with bottom two horizontal rows that are complexes.

\textbf{Warning 3.1.6.} While the bottom row of (3.1.11) is exact in the middle by definition of $\alpha^i_r$, the central row is not exact in the middle: it follows from the fact that $\text{Alb}(\pi_i) \circ \text{Pic}^0(\pi_i)$ is multiplication by $p$ on $J_{r-1}$ that the component group of the kernel of $\alpha^i_{r-1} \circ \text{Alb}(\pi_1) : JY_r \rightarrow B_{r-1}^i$ is nontrivial with order divisible by $p$. Moreover, there is no mapping $B_{r-1}^i \rightarrow B^i_r$ which makes the diagram (3.1.11) commute.

In order to be consistent with the literature, we adopt the following convention:
Definition 3.1.7. We set $B_r := B_r^2$ and $B_r^i := B_r^i$, with $B_r^i$ defined inductively by (3.1.10). We likewise set $\alpha_r := \alpha_r^2$ and $\alpha_r^i := \alpha_r^i$. 

Remark 3.1.8. We briefly comment on the relation between our quotient $B_r$ and the “good” quotients of $J_r$ considered by Ohta [Oht95], by Mazur-Wiles [MW84], and by Tilouine [Til87]. Recall [Til87, §2] that Tilouine constructs an abelian variety quotient $\alpha_r' : J_r \to B_r'$ via an inductive procedure nearly identical to the one used to define $B_r = B_r^1$: one sets $K_1^r := \text{Jac}^1$, and for $r \geq 2$ defines

$$B_{r-1}^r := J_{r-1}/\text{Pic}^0(\pi)(K_{r-1}^r) \quad \text{and} \quad K_r^r := \ker(JY_r \xrightarrow{\alpha_r' \circ \text{Alb}(\pi_1)} B_{r-1}^r).$$

Using the fact that the formation of images and identity components commutes, one shows via a straightforward induction argument that $\alpha_r : J_r \to B_r$ identifies $B_r$ with $J_r/(\ker \alpha_r')^0$; in particular, our $B_r$ is the same as Ohta’s [Oht95, §3.2] and Tilouine’s quotient $\alpha_r' : J_r \to B_r'$ uniquely factors through $\alpha_r$ via an isogeny $B_r \to B_r'$ which has degree divisible by $p$ by Warning 3.1.6. Due to this fact, it is essential for our purposes to work with $B_r$ rather than $B_r'$. Of course, following [Oht95, 3.2.1], we could have simply defined $B_r$ as $J_r/(\ker \alpha_r')^0$, but we feel that the construction we have given is more natural. On the other hand, we remark that $B_r$ is naturally a quotient of the “good” quotient $J_r \to A_r$ constructed by Mazur-Wiles in [MW84, Chapter III, §1], and the kernel of the corresponding surjective homomorphism $A_r \to B_r$ is isogenous to $J_0 \times J_0$.

Proposition 3.1.9. Over $F := \mathbb{Q}(\mu_{Np^r})$, the automorphism $w_r$ of $J_{rF}$ induces an isomorphism of quotients $B_{rF} \simeq B_{rF}^*$. The abelian variety $B_r$ (respectively $B_r^*$) is the unique quotient of $J_r$ by a $\mathbb{Q}$-rational abelian subvariety with the property that the induced map on cotangent spaces

$$\text{Cot}(B_r) \xrightarrow{\text{Cot}(\alpha_r)} \text{Cot}(J_r) \simeq S_2(\Gamma_r; \mathbb{Q}) \quad \text{respectively} \quad \text{Cot}(B_r^*) \xrightarrow{\text{Cot}(\alpha_r^*)} \text{Cot}(J_r^*) \simeq S_2(\Gamma_r; \mathbb{Q})$$

has image precisely $V_r$ (respectively $V_r^*$). In particular, there are canonical actions of the Hecke algebras $\mathfrak{H}_r(\mathbb{Z})$ on $B_r$ and $\mathfrak{H}_r^*(\mathbb{Z})$ on $B_r^*$ for which $\alpha_r$ and $\alpha_r^*$ are equivariant.

Proof. By the construction of $B_r^i$ and the fact that $\rho w_r = w_{r-1} \sigma$ as maps $X_{rF} \to X_{r-1F}$ [Cai14, Proposition 2.2.6] the automorphism $w_r$ of $J_{rF}$ carries $\ker(\alpha_r)$ to $\ker(\alpha_r^*)$ and induces an isomorphism $B_{rF} \simeq B_{rF}^*$ over $F$ that intertwines the action of $\mathfrak{H}_r$ on $B_r$ with $\mathfrak{H}_r^*$ on $B_{rF}^*$. The isogeny $B_r \to B_{rF}^*$ of Remark 3.1.8 induces an isomorphism on cotangent spaces, compatibly with the inclusions into $\text{Cot}(J_r)$. Thus, the claimed identification of the image of $\text{Cot}(B_r)$ with $V_r$ follows from [Til87, Proposition 2.1] (using [Til87, Definition 2.1]). The claimed uniqueness of $J_r \to B_r$ follows easily from Lemma 3.1.5 (3). Similarly, since the subspace $V_r$ of $S_2(\Gamma_r)$ is stable under $\mathfrak{H}_r$, we conclude from Lemma 3.1.5 (3) that for any $T \in \mathfrak{H}_r(\mathbb{Z})$, the induced morphism $J_r \xrightarrow{T} J_r \to B_r$ factors through $\alpha_r$, and hence that $\mathfrak{H}_r(\mathbb{Z})$ acts on $B_r$ compatibly (via $\alpha_r$) with its action on $J_r$. \[\]
Lemma 3.1.10. There exist unique morphisms $B^*_r \equiv B^*_{r-1}$ of abelian varieties over $\mathbb{Q}$ making

\[
\begin{array}{c}
J_r \xrightarrow{\alpha^*_r} B^*_r \\
\text{Alb}(\sigma) \downarrow \quad \downarrow \\
J_{r-1} \xrightarrow{\alpha^*_{r-1}} B^*_{r-1}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
J_r \xrightarrow{\alpha^*_r} B^*_r \\
\text{Pic}^0(\rho) \downarrow \quad \downarrow \\
J_{r-1} \xrightarrow{\alpha^*_{r-1}} B^*_{r-1}
\end{array}
\]

commute; these maps are moreover $\mathcal{F}^*_r(\mathbb{Z})$-equivariant. By a slight abuse of notation, we will simply write $\text{Alb}(\sigma)$ and $\text{Pic}^0(\rho)$ for the induced maps on $B^*_r$ and $B^*_{r-1}$, respectively.

Proof. Under the canonical identification of $\text{Cot}(J_r) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ with $S_2(\Gamma_r)$, the mapping on cotangent spaces induced by $\text{Alb}(\sigma)$ (respectively $\text{Pic}^0(\rho)$) coincides with $\iota_{\alpha} : S_2(\Gamma_{r-1}) \to S_2(\Gamma_r)$ (respectively $\text{tr}_{r,r-1} : S_2(\Gamma_r) \to S_2(\Gamma_{r-1})$). As the kernel of $\alpha^*_r : J_r \to B^*_r$ is connected by definition, thanks to Lemma 3.1.5 (3) it suffices to prove that $\iota_{\alpha}$ (respectively $\text{tr}_{r,r-1}$) carries $V^*_r \equiv V^*_{r-1}$ to $V^*_r$ (respectively $V^*_r$ to $V^*_{r-1}$). On one hand, the composite $\iota_{\alpha} \circ \text{tr}_{r,r-1} : S_2(\Gamma_r, \varepsilon) \to S_2(\Gamma_r)$ coincides with the embedding $\iota_{\alpha} \circ \text{tr}_{r,r-1}$, and it follows immediately from the definition of $V^*_r$ that $\iota_{\alpha}$ carries $V^*_r$ into $V^*_r$. On the other hand, an easy calculation using (3.1.2) shows that one has equalities of maps $S_2(\Gamma_r, \varepsilon) \to S_2(\Gamma_r)$

\[\iota_{\alpha} \circ \text{tr}_{r,r-1} \circ \iota_{\alpha(r-1)} = \begin{cases} 
\iota_{\alpha(r-1)} \cdot \iota_{\alpha(r)} & \text{if } i < r \\
0 & \text{if } i = r.
\end{cases}\]

Thus, the image of $\iota_{\alpha} \circ \text{tr}_{r,r-1} : V^*_r \to S_2(\Gamma_r)$ is contained in the image of $\iota_{\alpha} : V^*_r \to S_2(\Gamma_r)$; since $\iota_{\alpha}$ is injective, we conclude that the image of $\text{tr}_{r,r-1} : V^*_r \to S_2(\Gamma_{r-1})$ is contained in $V^*_{r-1}$ as desired. 

For $f'$ as in (3.1.7), we write $e^{*'} := f'_* e^* \in \mathcal{F}^*_r$ and $e' := f'_* e \in \mathcal{F}$ the sub-idempotents of $e^*$ and $e$, respectively, corresponding to projection away from the trivial eigenspace of $\mu_{p-1}$.

Proposition 3.1.11. The maps $\alpha_r$ and $\alpha^*_r$ induce isomorphisms of $p$-divisible groups over $\mathbb{Q}$

\[
(3.1.12) \quad e^{*'} J_r[p^\infty] \simeq e' \cdot B^*_r[p^\infty] \quad \text{and} \quad e' J_r[p^\infty] \simeq e \cdot B^*_r[p^\infty],
\]

respectively, that are $\mathcal{F}^*_r$ (respectively $\mathcal{F}$) equivariant and compatible with change in $r$ via $\text{Alb}(\sigma)$ and $\text{Pic}^0(\rho)$ (respectively $\text{Alb}(\rho)$ and $\text{Pic}^0(\sigma)$).

We view the maps (3.1.8) as endomorphisms of $J_r$ in the obvious way, and again write $U_r^*$ and $U_r$ for the induced endomorphism of $B^*_r$ and $B_r$, respectively. To prove Proposition 3.1.11, we need the following geometric incarnation of Corollary 3.1.3:

Lemma 3.1.12. There exists a unique $\mathcal{F}^*_r(\mathbb{Z})$ (respectively $\mathcal{F}_r(\mathbb{Z})$)-equivariant map $W^*_r : B^*_r \to J_r$ (respectively $W_r : B_r \to J_r$) of abelian varieties over $\mathbb{Q}$ such that the diagram

\[
(3.1.13) \quad \begin{array}{c}
J_r \xrightarrow{\alpha^*_r} B^*_r \\
\downarrow \text{tr}_r \quad \downarrow \text{tr}_r \\
J_r \xrightarrow{\alpha_r} B_r
\end{array}
\quad \text{and} \quad
\begin{array}{c}
J_r \xrightarrow{\alpha^*_r} B^*_r \\
\downarrow \text{tr}_r \quad \downarrow \text{tr}_r \\
J_r \xrightarrow{\alpha_r} B_r
\end{array}
\]

commutes.
Proof. Consider the endomorphism of $J_r$ given by $U_r$. Due to Corollary 3.1.3, the induced mapping on cotangent spaces factors through the inclusion $\text{Cot}(B_r) \hookrightarrow \text{Cot}(J_r)$. Since the kernel of the quotient mapping $\alpha_r : J_r \to B_r$ giving rise to this inclusion is connected, we conclude from Lemma 3.1.5 (3) that $U_r$ factors uniquely through $\alpha_r$ via an $\mathcal{H}_r$-equivariant morphism $W_r : B_r \to J_r$. The corresponding statements for $B_r^*$ are proved similarly. ■

Proof of Proposition 3.1.11. From (3.1.13) we get commutative diagrams of $p$-divisible groups over $\mathbb{Q}$

$\begin{align*}
\begin{array}{ccc}
\text{e}^{st} J_r[p^\infty] & \xrightarrow{\alpha_r^*} & \text{e}^{st} B_r[p^\infty] \\
\text{e}^{st} J_r[p^\infty] & \xrightarrow{\alpha_r^*} & \text{e}^{st} B_r[p^\infty]
\end{array}
\end{align*}$

(3.1.14)

in which all vertical arrows are isomorphisms due to the very definition of the idempotents $\text{e}^{st}$ and $e'$. An easy diagram chase then shows that all arrows must be isomorphisms. ■

As in the introduction, we put $K_r = \mathbb{Q}_{\mu(p^r)}$, $K'_r := K_r(\mu_N)$ and write $R_r$ and $R'_r$ for the valuation rings of $K_r$ and $K'_r$, respectively. We set $\Gamma := \text{Gal}(K_\infty/K_0)$, and write $a : \text{Gal}(K'_0/K_0) \to (\mathbb{Z}/N\mathbb{Z})^\times$ the character giving the tautological action of $\text{Gal}(K_0/K_0)$ on $\mu_N$.

Proposition 3.1.13. The abelian varieties $B_r$ and $B_r^*$ acquire good reduction over $K_r$.

Proof. See [MW84, Chap III, §2, Proposition 2] and cf. [Hid86a, §9, Lemma 9]. ■

We will write $\mathcal{B}_r$, $\mathcal{B}_r^*$, and $\mathcal{J}_r$, respectively, for the Néron models of the base changes $(B_r)_{K_r}$, $(B_r^*)_{K_r}$ and $(J_r)_{K_r}$ over $T_r := \text{Spec}(R_r)$; due to Proposition 3.1.11, both $\mathcal{B}_r$ and $\mathcal{B}_r^*$ are abelian schemes over $T_r$. By the Néron mapping property, there are canonical actions of $\mathcal{J}_r(\mathbb{Z})$ on $\mathcal{B}_r$, $\mathcal{J}_r$ and of $\mathcal{J}_r^*(\mathbb{Z})$ on $\mathcal{B}_r^*$, $\mathcal{J}_r$ over $R_r$ extending the actions on generic fibers as well as “semilinear” actions of $\Gamma$ over the $\Gamma$-action on $R_r$ (cf. (2.3.9)). For each $r$, the Néron mapping property further provides diagrams

$\begin{align*}
\begin{array}{ccc}
\mathcal{J}_r \times T_r & \xrightarrow{\alpha_r^*} & \mathcal{B}_r^* \times T_r \\
\mathcal{J}_r \times T_r & \xrightarrow{\alpha_r^*} & \mathcal{B}_r^* \\
\mathcal{J}_r+1 & \xrightarrow{\alpha_{r+1}^*} & \mathcal{B}_{r+1}^*
\end{array}
\end{align*}$

(3.1.15)

of smooth commutative group schemes over $T_{r+1}$ in which the inner and outer rectangles commute, and all maps are $\mathcal{J}_{r+1}^*(\mathbb{Z})$ (respectively $\mathcal{J}_{r+1}(\mathbb{Z})$) and $\Gamma$ equivariant.

Definition 3.1.14. We define $\mathcal{G}_r := \text{e}^{st}(\mathcal{B}_r^*[p^\infty])$ and we write $\mathcal{G}_r' := \mathcal{G}_r^\vee$ for its Cartier dual, each of which is canonically an object of $\text{pdive}_R$. For each $r \geq s$, noting that $U_r^*$ is an automorphism of $\mathcal{G}_r$, we obtain from (3.1.16) canonical morphisms

$\begin{align*}
\rho_{r,s} : \mathcal{G}_s \times T_s \xrightarrow{\text{Pic}(\rho)^{r-s}} \mathcal{G}_r \\
\rho_{r,s} : \mathcal{G}_s \times T_s \xrightarrow{(U_r^s \text{Alb}(\rho))^{r-s}} \mathcal{G}_r
\end{align*}$

(3.1.16)

in $\text{pdive}_R$, where $(\cdot)^i$ denotes the $i$-fold composition, formed in the obvious manner. In this way, we get towers of $p$-divisible groups $\{\mathcal{G}_r, \rho_{r,s}\}$ and $\{\mathcal{G}_r', \rho_{r,s}'\}$; we will write $G_r$ and $G'_r$ for the unique descents
Let $J_r$ act on $\mathcal{G}_r$ through the action of $\mathcal{H}_r$ on $\mathcal{B}_r^\ast$, and on $\mathcal{G}_r^\ast = \mathcal{G}_r^\vee$ by duality (i.e. as $(T^\ast)^\vee$). The maps (3.1.16) are then $\mathcal{H}_r^\ast$-equivariant.

By Proposition 3.1.11, $G_r$ is canonically isomorphic to $e^{s'}J_r[p^{\infty}]$, compatibly with the action of $\mathcal{H}_r^\ast$. Since $J_r$ is a Jacobian—hence principally polarized—one might expect that $\mathcal{G}_r$ is isomorphic to its dual in $\text{pdiv}^\Gamma_{R_r}$. However, this is not quite the case as the canonical isomorphism $J_r \simeq J_r^\vee$ intertwines the actions of $\mathcal{H}_r$ and $\mathcal{H}_r^\ast$, thus interchanging the idempotents $e_s'\psi$ and $e_s$. To describe the precise relationship between $\mathcal{G}_r^\ast$ and $\mathcal{G}_r$, we proceed as follows. For each $\gamma \in \text{Gal}(K'_\infty/K_0) \simeq \Gamma \times \text{Gal}(K'_0/K_0)$, let us write $\phi_\gamma : G_{r,K'_1} \xrightarrow{\sim} \gamma^\ast(G_{r,K'_1})$ for the descent data isomorphisms encoding the unique $Q_p = K_0$-descent of $G_{r,K'_1}$ furnished by $G_r$. We “twist” this descent data by the $\text{Aut}_{Q_p}(G_r)$-valued character $\langle \chi \rangle \langle a \rangle_N$ of $\text{Gal}(K'_\infty/K_0)$ explicitly, for $\gamma \in \text{Gal}(K'_0/K_0)$ we set $\psi_\gamma := \phi_\gamma \circ \langle \chi \rangle \langle a \rangle_N$ and note that since $\langle \chi \rangle \langle a \rangle_N$ is defined over $Q_p$, the map $\gamma \mapsto \psi_\gamma$ really does satisfy the cocycle condition. We denote by $G_r(\langle \chi \rangle \langle a \rangle_N)$ the unique $p$-divisible group over $Q_p$ corresponding to this twisted descent datum. Since the diamond operators commute with the Hecke operators, there is a canonical induced action of $\mathcal{H}_r$ on $G_r(\langle \chi \rangle \langle a \rangle_N)$. By construction, there is a canonical $K'_r$-isomorphism $G_r(\langle \chi \rangle \langle a \rangle_N) \simeq G_{r,K'_1}$. Since $G_r$ acquires good reduction over $K_r$ and the $\mathcal{H}_r$-representation afforded by the Tate module of $G_r(\langle \chi \rangle \langle a \rangle_N)$ is the twist of $T_pG_r$ by the unramified character $\langle a \rangle_N$, we conclude that $G_r(\langle \chi \rangle \langle a \rangle_N)$ also acquires good reduction over $K_r$, and we denote the resulting object of $\text{pdiv}^\Gamma_{R_r}$ by $\mathcal{G}_r(\langle \chi \rangle \langle a \rangle_N)$.

**Proposition 3.1.15.** There is a natural $\mathcal{H}_r^\ast$-equivariant isomorphism of $p$-divisible groups over $Q_p$

\[
G_r^\ast \simeq G_r(\langle \chi \rangle \langle a \rangle_N)
\]

which uniquely extends to an isomorphism of the corresponding objects in $\text{pdiv}^\Gamma_{R_r}$ and is compatible with change in $r$ using $\rho_{r,s}$ on $G_r^\ast$ and $\rho_{r,s}$ on $G_r$.

**Proof.** Let $\varphi_r : J_r \rightarrow J_r^\vee$ be the canonical principal polarization over $Q_p$; one then has the relation $\varphi_r \circ T = (T^\ast)^\vee \circ \varphi_r$ for each $T \in \mathcal{H}_r(Z)$. On the other hand, the $K'_r$-automorphism $w_r : J_{r,K'_1} \rightarrow J_{r,K'_1}$ intertwines $T \in \mathcal{H}_r(Z)$ with $T^s \in \mathcal{H}_r^s(Z)$. Thus, the $K'_r$-morphism

\[
\psi_r : J_{r,K'_1} \xrightarrow{(U^{s,r})^\vee} J_{r,K'_1} \xrightarrow{\varphi_r^{-1}} J_{r,K'_1} \xrightarrow{w_r} J_{r,K'_1}
\]

is $\mathcal{H}_r^s(Z)$-equivariant. Passing to the induced map on $p$-divisible groups and applying $e^{s'}$, we obtain from Proposition 3.1.11 an $\mathcal{H}_r^s$-equivariant isomorphism of $p$-divisible groups $\psi_r : G_{r,K'_1}^s \simeq G_{r,K'_1}$. As

\[
J_{r,K'_r} \xrightarrow{\langle \chi(\gamma) \rangle \langle a \rangle_N w_r} J_{r,K'_r} \xrightarrow{1 \times \gamma} (J_{r,K'_r})_{\gamma} \xrightarrow{\gamma^s(w_r)} (J_{r,K'_r})_{\gamma}
\]

commutes for all $\gamma \in \text{Gal}(K'_r/K_0)$ [Cai14, Proposition 2.2.6], the $K'_r$-isomorphism $\psi_r$ uniquely descends to an $\mathcal{H}_r^s$-equivariant isomorphism (3.1.17) of $p$-divisible groups over $Q_p$. By Tate’s Theorem, this identification uniquely extends to an isomorphism of the corresponding objects in $\text{pdiv}^\Gamma_{R_r}$. The asserted

\[20\text{Of course, } G_r^\ast = G_r^\vee. \text{ Our non-standard notation } \mathcal{G}_r^\ast \text{ for the Cartier dual of } \mathcal{G}_r \text{ is preferable, due to the fact that } \rho_{r,s} \text{ is not simply the dual of } \rho_{r,s}; \text{ indeed, these two mappings go in opposite directions!} \]
compatibility with change in \( r \) boils down to the commutativity of the diagrams

\[
\begin{array}{c}
\begin{aligned}
\bigwedge J_{r}[p^{\infty}]^\vee \overset{(U^{\ast})_{r}^\vee}{\longrightarrow} \bigwedge J_{s}[p^{\infty}]^\vee
\end{aligned}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{aligned}
\bigwedge J_{r}[p^{\infty}]^\vee \overset{\varphi^{-1}}{\longleftarrow} \bigwedge J_{s}[p^{\infty}]^\vee
\end{aligned}
\end{array}
\]

for all \( s \leq r \). The commutativity of the first diagram is clear, while that of the second follows from [Cai14, Proposition 2.6] and the fact that for any finite morphism \( f : Y \rightarrow X \) of smooth curves over a field \( K \), one has \( \varphi_{Y} \circ \text{Pic}^{0}(f) = \text{Alb}(f)^{\vee} \circ \varphi_{X} \), where \( \varphi_{\ast} : J_{\ast} \rightarrow J_{\ast}^\vee \) is the canonical principal polarization on Jacobians for \( \ast = X, Y \) (see, for example, the proof of Lemma 5.5 in [Cai10]).

We now wish to study the special fiber of \( \mathcal{X}_{r} \), and relate it to the special fibers of the integral models of modular curves studied in [Cai14, 2.2]. To that end, let \( \mathcal{X}_{r} \) be the Katz–Mazur integral model of \( X_{r} \) over \( R_{r} \) defined in [Cai14, 2.2.3]; it is a regular scheme that is proper and flat of pure relative dimension 1 over \( \text{Spec} R_{r} \) with smooth generic fiber naturally isomorphic to \( X_{r} \). According to [Cai14, Proposition 2.2.10], the special fiber \( \mathcal{X}_{r}^{n} := \mathcal{X}_{r} \times_{R_{r}} \mathbb{F}_{p} \) is the ”disjoint union with crossings at the supersingular points” [KM85, 13.1.5] of smooth and proper Igusa curves \( I_{(a,b,u)} := I_{g_{\text{max}}(a,b)} \) indexed by triples \((a,b,u)\) with \( a, b \) running over nonnegative integers that sum to \( r \) and \( u \in (\mathbb{Z}/p_{\text{min}}(a,b)\mathbb{Z})^\times \); in particular, \( \mathcal{X}_{r} \) is geometrically reduced. We write \( \mathcal{X}_{r}^{n} \) for the normalization of \( \mathcal{X}_{r} \), which is a disjoint union of Igusa curves \( I_{(a,b,u)} \). The canonical semilinear action of \( \Gamma \) on \( \mathcal{X}_{r} \) that encodes the descent data of the generic fiber to \( \mathbb{Q}_{p} \) [Cai14, 2.2.3] induces, by base change, an \( \mathbb{F}_{p} \)-linear ”geometric inertia action” of \( \Gamma \) on \( \mathcal{X}_{r}^{n} \); in this way the \( p \)-divisible group \( \text{Pic}^{0}_{\mathcal{X}_{r}^{n}/\mathbb{F}_{p}}[p^{\infty}] \) of the Jacobian of \( \mathcal{X}_{r}^{n} \) over \( \mathbb{F}_{p} \) is equipped with an action of \( \Gamma \) over \( \mathbb{F}_{p} \) and (via the Hecke correspondences [Cai14, 2.2.21]) canonical actions of \( \mathcal{G}_{r} \) and \( \mathcal{G}_{r}^{\ast} \).

**Definition 3.1.16.** Define \( \Sigma_{r} := e_{r}^{\ast} \text{Pic}^{0}_{\mathcal{X}_{r}/\mathbb{F}_{p}}[p^{\infty}] \), equipped with the induced actions of \( \mathcal{G}_{r} \) and \( \Gamma \).

Since \( \mathcal{X}_{r} \) is regular, and proper flat over \( R_{r} \) with (geometrically) reduced special fiber, \( \text{Pic}^{0}_{\mathcal{X}_{r}/R_{r}} \) is a smooth \( R_{r} \)-scheme by §8.4 Proposition 2 and §9.4 Theorem 2 of [BLR90]. By the Néron mapping property, we thus have a natural mapping \( \text{Pic}^{0}_{\mathcal{X}_{r}/R_{r}} \rightarrow \mathcal{G}_{r}^{0} \) that recovers the canonical identification on generic fibers, and is in fact an isomorphism by [BLR90, §9.7, Theorem 1]. Composing with the map \( \alpha_{\ast}^{\ast} : \mathcal{G}_{r} \rightarrow \mathcal{G}_{r}^{\ast} \) and passing to special fibers yields a homomorphism of smooth commutative algebraic groups over \( \mathbb{F}_{p} \)

\[
\begin{array}{c}
\begin{aligned}
\text{Pic}^{0}_{\mathcal{X}_{r}/\mathbb{F}_{p}} \longrightarrow \mathcal{G}_{r}^{0} \longrightarrow \mathcal{G}_{r}^{\ast}
\end{aligned}
\end{array}
\]

Due to [BLR90, §9.3, Corollary 11], the normalization map \( \mathcal{X}_{r}^{n} \rightarrow \mathcal{X}_{r} \) induces a surjective homomorphism \( \text{Pic}^{0}_{\mathcal{X}_{r}^{n}/\mathbb{F}_{p}} \rightarrow \text{Pic}^{0}_{\mathcal{X}_{r}/\mathbb{F}_{p}} \) with kernel that is a smooth, connected linear algebraic group over \( \mathbb{F}_{p} \). As any homomorphism from an affine group variety to an abelian variety is zero, we conclude that (3.1.18) uniquely factors through this quotient, and we obtain a natural map of abelian varieties:

\[
\begin{array}{c}
\begin{aligned}
\text{Pic}^{0}_{\mathcal{X}_{r}^{n}/\mathbb{F}_{p}} \longrightarrow \mathcal{G}_{r}^{\ast}
\end{aligned}
\end{array}
\]
that is necessarily equivariant for the actions of $\mathcal{H}_r^e(\mathbb{Z})$ and $\Gamma$. The following Proposition relates the special fiber of $\mathcal{H}_r$ to the $p$-divisible group $\Sigma_r$ of Definition 3.1.16, and will allow us in Corollary 3.1.21 to give an explicit description of the special fiber of $\mathcal{H}_r$.

**Proposition 3.1.17.** The mapping (3.1.19) induces an isomorphism of $p$-divisible groups over $\mathbb{F}_p$

$$\mathcal{G}_r := e^{r}\mathcal{T}_r[p^\infty] \simeq e^{r'}\text{Pic}_0^{0} \mathcal{X}_r/F_p[p^\infty] =: \Sigma_r$$

that is $\mathcal{H}_r^e$ and $\Gamma$-equivariant and compatible with change in $r$ via the maps $\rho_{r,s}$ on $\mathcal{G}_r$ and the maps $\text{Pic}^0(\rho)^{r-s}$ on $\Sigma_r$.

**Proof.** The diagram (3.1.13) induces a corresponding diagram of Néron models over $R_r$ and hence of special fibers over $\mathbb{F}_p$. Arguing as above, we obtain a commutative diagram of abelian varieties

$$\begin{array}{ccc}
\text{Pic}_0^{0} \mathcal{X}_r/F_p & \xrightarrow{\overline{\pi}_r} & \mathcal{T}_r \times F_p \\
U_r^* & \swarrow & \downarrow W_r^* \\
\text{Pic}_0^{0} \mathcal{X}_r/F_p & \xrightarrow{\overline{\pi}_r} & \mathcal{T}_r \times F_p
\end{array}$$

over $\mathbb{F}_p$. The proof of 3.1.11 now goes through *mutatis mutandis* to give the claimed isomorphism (3.1.20).

In [Cai14, §2.5], we analyzed the the structure of the de Rham cohomology of the smooth and proper curve $\mathcal{X}_r^m$ over $\mathbb{F}_p$; we now apply this analysis and Oda’s description (Proposition 2.1.1) of Dieudonné modules in terms of de Rham cohomology to understand the structure of $\Sigma_r$. For each $r$, as in [Cai14, Remark 2.2.12] we write $I_r^\infty := I_{(r,0,1)}$ and $I_r^0 := I_{(0,r,1)}$ for the two “good” irreducible components of $\mathcal{X}_r$; by [Cai14, Proposition 2.5.6], the ordinary part of the de Rham cohomology of $\mathcal{X}_r^m$ is entirely captured by the de Rham cohomology of these two good components. Writing $i_r^*: I_r^* \hookrightarrow \mathcal{X}_r^m$ for the canonical closed immersions, we reinterpret this fact in the language of Dieudonné modules:

**Proposition 3.1.18.** For each $r$, there is a natural isomorphism of $A := \mathbb{Z}_p[F,V]$-modules

$$\text{D}(\Sigma_r)_{\mathbb{F}_p} \simeq e^{r'}H^1_{\text{dR}}(\mathcal{X}_r^m/F_p) \simeq f'H^0(I_r^\infty, \Omega^1)^{\text{ord}} \oplus f'H^1(I_r^0, \Omega^1)^{\text{ord}}$$

which is compatible with $\mathcal{H}_r^e$, $\Gamma$, and change in $r$ and which carries $\text{D}(\Sigma_r^m)_{\mathbb{F}_p}$ (respectively $\text{D}(\Sigma_r^\text{et})_{\mathbb{F}_p}$) isomorphically onto $f'H^0(I_r^0, \Omega^1)^{\text{ord}}$ (respectively $f'H^1(I_r^\infty, \Omega^1)^{\text{ord}}$). In particular, $\Sigma_r$ is ordinary.

**Proof.** The identifications of [Cai14, Proposition 2.5.6] are induced by the closed immersions $i_r^*$ and are therefore compatible with the natural actions of Frobenius and the Cartier operator. The isomorphism (3.1.22) is then an immediate consequence of Proposition 2.1.1 and [Cai14, Proposition 2.5.6]. Since this isomorphism is compatible with $F$ and $V$, we have

$$\text{D}(\Sigma_r^m)_{\mathbb{F}_p} \simeq \text{D}(\Sigma_r)_{\mathbb{F}_p}^{\text{ord}} \simeq f'H^0(I_r^0, \Omega^1)^{\text{ord}}$$

and

$$\text{D}(\Sigma_r^\text{et}) \otimes \mathbb{Z}_p \mathbb{F}_p \simeq \text{D}(\Sigma_r)_{\mathbb{F}_p}^{\text{ord}} \simeq f'H^1(I_r^\infty, \Omega^1)^{\text{ord}}$$

and we conclude that the canonical inclusion $\text{D}(\Sigma_r^m)_{\mathbb{Z}_p} \oplus \text{D}(\Sigma_r^\text{et})_{\mathbb{Z}_p} \hookrightarrow \text{D}(\Sigma_r)_{\mathbb{Z}_p}$ is surjective, whence $\Sigma_r$ is ordinary by Dieudonné theory.
With Proposition 3.1.18 as a starting point, we can now completely describe the structure of $\Sigma_r$ in terms of the two good components $I_r^*$. Since $\overline{I}_0$ is the disjoint union of proper smooth and irreducible Igusa curves $I(a,b,u)$, we have a canonical identification of abelian varieties over $F_p$

\[
(3.1.24) \quad \text{Pic}_0^{\overline{I}_r/F_p} = \prod_{(a,b,u)} \text{Pic}_0^{I(a,b,u)/F_p}.
\]

For $*=0,\infty$ let us write $j_r^*:=\text{Pic}_0^{I_r/F_p}$ for the Jacobian of $I_r$ over $F_p$. The canonical closed immersions $i_r^*:I_r^*\hookrightarrow\overline{I}_r^0$ yield (by Picard and Albanese functoriality) homomorphisms of abelian varieties over $F_p$

\[
(3.1.25) \quad \text{Alb}(i_r^*):j_r^* \longrightarrow \text{Pic}_0^{\overline{I}_r/F_p} \quad \text{and} \quad \text{Pic}^0(i_r^*):\text{Pic}_0^{\overline{I}_r/F_p} \longrightarrow j_r^*.
\]

Via the identification (3.1.24), we know that $j_r^*$ is a direct factor of $\text{Pic}_0^{\overline{I}_r/F_p}$; in these terms $\text{Alb}(i_r^*)$ is the unique mapping which projects to the identity on $j_r^*$ and to the zero map on all other factors, while $\text{Pic}^0(i_r^*)$ is simply projection onto the factor $j_r^*$. As $\Sigma_r$ is a direct factor of $f^\prime\text{Pic}_0^{\overline{I}_r/F_p}[p^{\infty}]$, these mappings induce homomorphisms of $p$-divisible groups over $F_p$

\[
(3.1.26a) \quad f^\prime j_r^0[p^{\infty}] \longrightarrow \text{Alb}(i_r^*) \longrightarrow f^\prime \text{Pic}_0^{\overline{I}_r/F_p}[p^{\infty}] \longrightarrow \text{proj} \longrightarrow \Sigma_r
\]

\[
(3.1.26b) \quad \Sigma_{\text{et}} \longrightarrow \text{incl} \longrightarrow f^\prime \text{Pic}_0^{\overline{I}_r/F_p}[p^{\infty}] \longrightarrow \text{Pic}^0(i_r^*) \longrightarrow f^\prime j_r^0[p^{\infty}] \longrightarrow \Sigma_r
\]

which we (somewhat abusively) again denote by $\text{Alb}(i_r^*)$ and $\text{Pic}^0(i_r^*)$, respectively. The following is a sharpening of [MW84, Chapter 3, §3, Proposition 3] (see also [Til87, Proposition 3.2]):

**Proposition 3.1.19.** The mappings (3.1.26a) and (3.1.26b) are isomorphisms. They induce a canonical split short exact sequences of $p$-divisible groups over $F_p$

\[
(3.1.27) \quad 0 \longrightarrow f^\prime j_r^0[p^{\infty}] \longrightarrow \text{Alb}(i_r^*) \longrightarrow \Sigma_r \longrightarrow \text{Pic}^0(i_r^*) \longrightarrow f^\prime j_r^0[p^{\infty}] \longrightarrow 0
\]

which is:

1. $\Gamma$-equivariant for the geometric inertia action on $\Sigma_r$, the trivial action on $f^\prime j_r^0[p^{\infty}]$, and the action via $\chi(\cdot)$^{-1} on $f^\prime j_r^0[p^{\infty}]$.
2. $\mathfrak{S}_r^*$-equivariant with $U_r^*$ acting on $f^\prime j_r^0[p^{\infty}]$ as $F$ and on $f^\prime j_r^0[p^{\infty}]$ as $(p)_{N_V}$.
3. Compatible with change in $r$ via the mappings Pic$^0(p)$ on $j_r^*$ and $\Sigma_r$.

**Proof.** It is clearly enough to prove that the sequence (3.1.27) induced by (3.1.26a) and (3.1.26b) is exact. Since the contravariant Dieudonné module functor from the category of $p$-divisible groups over $F_p$ to the category of $A$-modules which are $\mathcal{Z}_p$ finite and free is an exact anti-equivalence, it suffices to prove such exactness after applying $\mathbf{D}(\cdot)_{\mathcal{Z}_p}$. As the resulting sequence consists of finite free $\mathcal{Z}_p$-modules, exactness may be checked modulo $p$ where it follows immediately from Proposition 3.1.18 by using [Cai14, Proposition 2.5.6]. The claimed compatibility with $\Gamma$, $\mathfrak{S}_r^*$, and change in $r$ follows easily from Propositions 2.2.14, 2.2.20 and 2.2.13 of [Cai14], respectively.

**Remark 3.1.20.** It is possible to give a short proof of Proposition 3.1.19 along the lines of [MW84] or [Til87] by using [Cai14, Proposition 2.2.20] directly. We stress, however, that our approach via Dieudonné modules gives more refined information, most notably that the Dieudonné module of $\Sigma_r[p]$ is free as an $F_p[\Delta/\Delta_r]$-module. This fact will be crucial in our later arguments.
Together, Propositions 3.1.17 and 3.1.19 give the desired description of the special fiber of $\mathcal{G}_r$ (cf. §3 and §4, Proposition 1 of [MW86] and pgs. 267–274 of [MW84]):

**Corollary 3.1.21.** For each $r$, the $p$-divisible group $\mathcal{G}_r/R_r$ is ordinary, and there is a canonical exact sequence, compatible with change in $r$ via $\rho_{r,s}$ on $\mathcal{G}_r$ and $\text{Pic}^0(p)^{r-s} \to j_r^*[p^\infty]$.

\[
\begin{array}{cccc}
0 & \longrightarrow & f^r_j\mathcal{O}_r[p^\infty]^m & \text{Alb}(\mathcal{O}_r)V^r & \mathcal{G}_r & \text{Pic}^0(p)^{r}\to & 0 \\
\end{array}
\]

where $i^*_r : I^*_r \to \overline{\mathcal{G}_r}$ are the canonical closed immersions for $* = 0, \infty$. Moreover, (3.1.28) is compatible with the actions of $\mathcal{H}^*$ and $\mathcal{H}$, with $U^*_p$ (respectively $\gamma \in \Gamma$) acting on $f^r_j\mathcal{O}_r[p^\infty]^m$ as $\langle p \rangle_N V$ (respectively $\langle \chi(\gamma) \rangle^{-1}$) and on $f^r_j\mathcal{O}_r[p^\infty]^{\text{ét}}$ as $F$ (respectively id).

3.2. Ordinary families of Dieudonné modules. Let $\{\mathcal{G}_r/R_r\}_{r \geq 1}$ be the tower of $p$-divisible groups given by Definition 3.1.14. From the canonical morphisms $\rho_{r,s} : \mathcal{G}_s \times_T T_r \to \mathcal{G}_r$ we obtain a map on special fibers $\overline{\mathcal{G}_s} \to \overline{\mathcal{G}_r}$ over $\mathbb{F}_p$ for each $r \geq s$; applying the contravariant Dieudonné module functor $\mathcal{D}(\cdot) := \mathcal{D}(\cdot)_{Z_p}$ yields a projective system of finite free $\mathbb{Z}_p$-modules $\{\mathcal{D}(\overline{\mathcal{G}_r})\}_r$ with compatible linear endomorphisms $F, V$ satisfying $FV = VF = p$.

**Definition 3.2.1.** We write $\mathcal{D}_\infty := \varprojlim_r \mathcal{D}(\overline{\mathcal{G}_r})$ for the projective limit of the system $\{\mathcal{D}(\overline{\mathcal{G}_r})\}_r$. For $* \in \{\text{ét}, m\}$ we write $\mathcal{D}_\infty^* := \varprojlim_r \mathcal{D}(\overline{\mathcal{G}_r}^*)$ for the corresponding projective limit.

Since $\mathcal{H}_p^*$ acts by endomorphisms on $\overline{\mathcal{G}_r}$, compatibly with change in $r$, we obtain an action of $\mathcal{H}^*$ on $\mathcal{D}_\infty$ and on $\mathcal{D}_\infty^*$. Likewise, the “geometric inertia action” of $\mathcal{H}$ on $\overline{\mathcal{G}_r}$ gives an action of $\mathcal{H}$ on $\mathcal{D}_\infty$ and $\mathcal{D}_\infty^*$. As $\overline{\mathcal{G}_r}$ is ordinary thanks to Corollary 3.1.21, applying $\mathcal{D}(\cdot)$ to the (split) connected-étale sequence of $\overline{\mathcal{G}_r}$ gives, for each $r$, a functorially split exact sequence

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{D}(\mathcal{G}_r^{\text{ét}}) & \longrightarrow & \mathcal{D}(\mathcal{G}_r) & \longrightarrow & \mathcal{D}(\mathcal{G}_r)^m & \longrightarrow & 0 \\
\end{array}
\]

with $\mathbb{Z}_p$-linear actions of $\mathcal{H}, F, V,$ and $\mathcal{H}_p^*$. Since projective limits commute with finite direct sums, we obtain a split short exact sequence of $\Lambda$-modules with linear $\mathcal{H}_p^*$ and $\mathcal{H}$-actions and commuting linear endomorphisms $F, V$ satisfying $FV = VF = p$.

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{D}_\infty^\text{ét} & \longrightarrow & \mathcal{D}_\infty & \longrightarrow & \mathcal{D}_\infty^m & \longrightarrow & 0 \\
\end{array}
\]

**Theorem 3.2.2.** Let $d' := \sum_{k=3}^p d_k$ for $d_k := \dim_{\mathbb{F}_p} S_k(G_1(N); \mathbb{F}_p)^{\text{ord}}$ the $\mathbb{F}_p$-dimension of the space of $p$-ordinary mod $p$ cusps of weight $k$ of level $N$. Then:

1. $\mathcal{D}_\infty$ is a free $\Lambda$-module of rank $2d'$, and $\mathcal{D}_\infty^*$ is free of rank $d'$ over $\Lambda$ for $* \in \{\text{ét}, m\}$.
2. For each $r \geq 1$, applying $\otimes_{\mathbb{Z}_p}[\Delta/\Delta_r]$ to (3.2.2) yields the short exact sequence (3.2.1), compatibly with $\mathcal{H}_p^*, \mathcal{H}, F$ and $V$.
3. Under the canonical splitting of (3.2.2), $\mathcal{D}_\infty^\text{ét}$ is the maximal subspace of $\mathcal{D}_\infty$ on which $F$ acts invertibly, while $\mathcal{D}_\infty^m$ corresponds to the maximal subspace of $\mathcal{D}_\infty$ on which $V$ acts invertibly.
4. The Hecke operator $U^*_p$ acts as $F$ on $\mathcal{D}_\infty^\text{ét}$ and as $\langle p \rangle_N V$ on $\mathcal{D}_\infty^m$.
5. $\mathcal{H}$ acts trivially on $\mathcal{D}_\infty^\text{ét}$ and via $\langle \chi \rangle^{-1}$ on $\mathcal{D}_\infty^m$.

**Proof.** In [Cai14, §3.1], we established a general commutative algebra formalism for dealing with projective limits of modules and proving structural and control theorems as in (1) and (2), respectively. In order to apply the main result of our formalism to the present situation, we take (in the notation of [Cai14, Lemma 3.1.2]) $A_r = \mathbb{Z}_p, I_r = \langle p \rangle$, and $M_r$ each one of the terms in (3.2.1), and we must check that the hypotheses
(3.2.3a) $\overline{M}_r := M_r/pM_r$ is a free $\mathbf{F}_p[\Delta/\Delta_r]$-module of rank $d'$
(3.2.3b) For all $s \leq r$ the induced transition maps $\overline{\rho}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s$ are surjective.

By Propositions 3.1.17 and 3.1.18, there is a natural isomorphism of split short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbf{D}(\mathbb{G}_r^\text{et})_{\mathbf{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{G}_r)_{\mathbf{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{G}_r^m)_{\mathbf{F}_p} & \longrightarrow & 0 \\
\downarrow & & \approx & & \downarrow & & \approx & & \downarrow & \\
0 & \longrightarrow & f' H^1(I_0^0, \mathcal{O})^{\text{Frob}} & \longrightarrow & f' H^0(I_0^0, \Omega^1)^{\text{Frob}} \oplus f' H^1(I_0^0, \mathcal{O})^{\text{Frob}} & \longrightarrow & f' H^0(I_0^0, \Omega^1)^{\text{Frob}} & \longrightarrow & 0
\end{array}
\]

that is compatible with change in $r$ using the trace mappings attached to $\rho : I_*^0 \rightarrow I_*^0$ and the maps on Dieudonné modules induced by $\overline{\rho}_{r,s} : \overline{\mathbb{G}}_s \rightarrow \overline{\mathbb{G}}_r$. The hypotheses (3.2.3a) and (3.2.3b) are therefore satisfied thanks to Proposition 2.4.1 and Lemma 2.5.5 of [Cai14]. It follows that the conclusions of [Cai14, Lemma 3.1.12] hold. By Propositions 3.1.17 and 3.1.18, there is a natural isomorphism of split short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbf{D}(\mathbb{G}_r^\text{et})_{\mathbf{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{G}_r)_{\mathbf{F}_p} & \longrightarrow & \mathbf{D}(\mathbb{G}_r^m)_{\mathbf{F}_p} & \longrightarrow & 0 \\
\downarrow & & \approx & & \downarrow & & \approx & & \downarrow & \\
0 & \longrightarrow & f' H^1(I_0^0, \mathcal{O})^{\text{Frob}} & \longrightarrow & f' H^0(I_0^0, \Omega^1)^{\text{Frob}} \oplus f' H^1(I_0^0, \mathcal{O})^{\text{Frob}} & \longrightarrow & f' H^0(I_0^0, \Omega^1)^{\text{Frob}} & \longrightarrow & 0
\end{array}
\]

The short exact sequence (3.2.2) is very nearly “auto dual”:

**Proposition 3.2.3.** There is a canonical isomorphism of short exact sequences of $\Lambda_{R_0'}$-modules

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbf{D}_\infty((\chi)\langle a \rangle_N)_{\Lambda_{R_0'}} & \longrightarrow & \mathbf{D}_\infty((\chi)\langle a \rangle_N)_{\Lambda_{R_0'}} & \longrightarrow & \mathbf{D}_\infty((\chi)\langle a \rangle_N)_{\Lambda_{R_0'}} & \longrightarrow & 0 \\
\downarrow & & \approx & & \downarrow & & \approx & & \downarrow & \\
0 & \longrightarrow & (\mathbf{D}_\infty^\vee)_{\Lambda_{R_0'}} & \longrightarrow & (\mathbf{D}_\infty^\vee)_{\Lambda_{R_0'}} & \longrightarrow & (\mathbf{D}_\infty^\vee)_{\Lambda_{R_0'}} & \longrightarrow & 0
\end{array}
\]

that is $\mathfrak{g}^*$ and $\Gamma \times \text{Gal}(K_0'/K_0)$-equivariant, and intertwines $F$ (respectively $V$) on the top row with $V^\vee$ (respectively $F^\vee$) on the bottom.

**Proof.** As in the proof of Theorem 3.2.2, we apply the formalism of [Cai14, §3.1]. Let us write $\rho'_{r,s} : \mathbb{G}_r \rightarrow \mathbb{G}_s$ for the maps on special fibers induced by (3.1.16). Thanks to Proposition 3.1.15, the definition 3.1.14 of $\overline{\mathbb{G}}_r := \overline{\mathbb{G}}_r^\vee$, the natural isomorphism $\mathbb{G}_r \times_{R_s} R'_s \cong \mathbb{G}_r((\chi)\langle a \rangle_N) \times_{R_s} R'_s$, and the compatibility of the Dieudonné module functor with duality, there are natural isomorphisms of $R'_0$-modules

\[
\begin{array}{cccccc}
\mathbf{D}(\mathbb{G}_r)((\chi)\langle a \rangle_N) \otimes_{Z_p} R'_0 & \cong & \mathbf{D}(\mathbb{G}_r((\chi)\langle a \rangle_N)) \otimes_{Z_p} R'_0 & \cong & \mathbf{D}(\mathbb{G}_r^\vee) \otimes_{Z_p} R'_0 & \cong & \mathbf{D}(\mathbb{G}_r^\vee) \otimes_{Z_p} R'_0
\end{array}
\]

that are $\mathfrak{g}^*$-equivariant, $\text{Gal}(K_0'/K_0)$-compatible for the standard action $\sigma \cdot f(m) := \sigma f(\sigma^{-1} m)$ on the $R'_0$-linear dual of $\mathbf{D}(\mathbb{G}_r) \otimes_{Z_p} R'_0$, and compatible with change in $r$ using $\rho_{r,s}$ on $\mathbf{D}(\mathbb{G}_r)$ and $\rho'_{r,s}$ on $\mathbf{D}(\mathbb{G}_r^\vee)$. We claim that the resulting perfect “evaluation” pairings

\[
\langle \cdot, \cdot \rangle_r : \mathbf{D}(\mathbb{G}_r)((\chi)\langle a \rangle_N) \otimes_{Z_p} R'_0 \times \mathbf{D}(\mathbb{G}_r^\vee) \otimes_{Z_p} R'_0 \longrightarrow R'_0
\]

are compatible with change in $r$ via the maps $\rho_{r,s}$ and $\rho'_{r,s}$ in the sense of [Cai14, 3.1.4]; i.e. that

\[
\langle \rho_{r,s} x, \rho'_{r,s} y \rangle_s = \sum_{\delta \in \Delta_s/\Delta_r} \langle x, \delta^{-1} y \rangle_r
\]
holds for all $x, y$. Indeed, the compatibility of (3.2.5) with change in $r$ and the very definition (3.1.16) of the transition maps $p'_{r,s}$ implies that for $r \geq s$

\begin{equation}
(3.2.8) \quad (D(Pic^0(\rho)^{r-s})x, y)_s = \langle x, D(U_p^* U^r \sigma^s) y \rangle_r;
\end{equation}

on the other hand, it follows from Lemma 3.1.1 (using Lemma 3.1.5) that we have

\begin{equation}
(3.2.9) \quad \text{Pic}(\rho) \circ \text{Alb}(\sigma) = U_p^* \sum_{\delta \in \Delta_r/\Delta_{r+1}} \langle \delta^{-1} \rangle,
\end{equation}

in $\text{End}_{Q_s}(J_{r+1})$, and together (3.2.8)–(3.2.9) imply the desired compatibility (3.2.7). It follows that the hypotheses of [Cai14, Lemma 3.1.4] is verified, and we conclude that the pairings (3.2.6) give rise to a perfect $\text{Gal}(K_{\infty}/K_0)$-compatible duality pairing $\langle \cdot, \cdot \rangle : D_{\infty}(\langle \chi \rangle \langle a \rangle) \otimes \Lambda_{R_0} \times D_{\infty} \otimes \Lambda_{R_0} \to \Lambda_{R_0}$ with respect to which $T^*$ is self-adjoint for all $T^* \in \mathfrak{f}^*$ as this is true at each finite level $r$ thanks to the $\mathfrak{f}^*$-compatibility of (3.2.5). That the resulting isomorphism (3.2.4) intertwines $F$ with $V^\vee$ and $V$ with $F^\vee$ is an immediate consequence of the compatibility of the Dieudonné module functor with duality.

We can interpret $D^*_{\infty}$ in terms of the crystalline cohomology of the Igusa tower as follows. Let $I^0_r$ and $I^{\infty}_r$ be the two “good” components of $\mathcal{X}_r$ as in the discussion preceding Proposition 3.1.18, and form the projective limits

\[ H^1_{\text{cris}}(I^*) := \lim_{r} H^1_{\text{cris}}(I^*_r) \]

for $* \in \{\infty, 0\}$, taken with respect to the trace maps on crystalline cohomology (see [Ber74, VII, §2.2]) induced by the canonical degeneracy mappings $\rho : I^*_r \to I^*_s$. Then $H^1_{\text{cris}}(I^*)$ is naturally a $\Lambda$-module (via the diamond operators), equipped with a commuting action of $F$ (Frobenius) and $V$ (Verscheibung) satisfying $FV = VF = p$. Letting $U_p^*$ act as $F$ (respectively $\langle p \rangle_N V$) on $H^1_{\text{cris}}(I^*)$ for $* = \infty$ (respectively $* = 0$) and the Hecke operators outside $p$ (viewed as correspondences on the Igusa curves) act via pullback and trace at each level $r$, we obtain an action of $\mathfrak{f}^*$ on $H^1_{\text{cris}}(I^*)$. Finally, we let $\Gamma$ act trivially on $H^1_{\text{cris}}(I^*)$ for $* = \infty$ and via $\langle \chi^{-1} \rangle$ for $* = 0$.

**Theorem 3.2.4.** There is a canonical $\mathfrak{f}^*$ and $\Gamma$-equivariant isomorphism of $\Lambda$-modules

\[ D_{\infty} = D_{\infty}^{\text{m}} \oplus D_{\infty}^{\text{et}} \simeq f^* H^1_{\text{cris}}(f^0) V^{\text{ord}} \oplus f^* H^1_{\text{cris}}(I^{\infty}) V^{\text{ord}} \]

which respects the given direct sum decompositions and is compatible with $F$ and $V$.

**Proof.** From the exact sequence (3.1.28), we obtain for each $r$ isomorphisms

\begin{equation}
(3.2.10) \quad D(\mathcal{G}^{m}_r) \xrightarrow{\simeq} f^* D(j^0_{\rho^*[p^\infty]}) V^{\text{ord}} \quad \text{and} \quad f^* D(j^\infty_{\rho^*[p^\infty]}) V^{\text{ord}} \xrightarrow{\simeq} D(\mathcal{G}^{\text{et}}_r)
\end{equation}

that are $\mathfrak{f}^*$ and $\Gamma$-equivariant (with respect to the actions specified in Corollary 3.1.21), and compatible with change in $r$ via the mappings $D(\rho_{r,s})$ on $D(\mathcal{G}^*_r)$ and $D(\rho)$ on $D(j^*_r[p^\infty])$. On the other hand, for any smooth and proper curve $X$ over a perfect field $k$ of characteristic $p$, thanks to [MM74] and [Ill79, II, §3 C Remarque 3.11.2] there are natural isomorphisms of $W(k)[F, V]$-modules

\begin{equation}
(3.2.11) \quad D(J_X[p^\infty]) \simeq H^1_{\text{cris}}(J_X/W(k)) \simeq H^1_{\text{cris}}(X/W(k))
\end{equation}

that for any finite map of smooth proper curves $f : Y \to X$ over $k$ intertwine $D(\text{Pic}(f))$ and $D(\text{Alb}(f))$ with trace and pullback by $f$ on crystalline cohomology, respectively. Applying this to $X = I^*_r$ for $* = 0, \infty$, appealing to the identifications (3.2.10), and passing to inverse limits completes the proof. 

\[ \blacksquare \]
We now wish to relate the slope filtration (3.2.2) to the Hodge filtration (1.1.2) of our ordinary $\Lambda$-adic de Rham cohomology studied in [Cai14]. Applying the idempotent $f'$ of (3.1.16) to (1.1.2) yields a short exact sequence of free $\Lambda_{R_{\infty}}$-modules with semilinear $\Gamma$-action and commuting action of $\mathfrak{H}_r^*$:

\[
(3.2.12) \quad 0 \longrightarrow e^{*'} H^0(\omega) \longrightarrow e^{*'} H^1_{dR} \longrightarrow e^{*'} H^1(\mathcal{O}) \longrightarrow 0 .
\]

The key to relating (3.2.12) to the slope filtration (3.2.2) is the following comparison isomorphism:

**Proposition 3.2.5.** For each positive integer $r$, there is a natural isomorphism of short exact sequences

\[
(3.2.13) \quad 0 \longrightarrow \omega_{\mathfrak{H}_r} \longrightarrow D(\mathfrak{H}_{r,0})_{R_r} \longrightarrow \text{Lie} (\mathfrak{H}_r) \longrightarrow 0
\]

that is compatible with $\mathfrak{H}_r^*$, $\Gamma$, and change in $r$ using the mappings (3.1.16) on the top row and the maps $\rho_r$ on the bottom. Here, the bottom row—with obvious abbreviated notation—is obtained from (1.1.1) by applying $e^{*'}$ and the top row is the Hodge filtration of $D(\mathfrak{H}_{r,0})_{R_r}$ given by Proposition 2.2.7.

**Proof.** Let $\alpha_r^* : J_r \to B_r^*$ be the map of Definition 3.1.7. We claim that $\alpha_r^*$ induces a canonical isomorphism of short exact sequences of free $R_r$-modules

\[
(3.2.14) \quad 0 \longrightarrow \omega_{\mathfrak{G}_r} \longrightarrow D(\mathfrak{G}_{r,0})_{R_r} \longrightarrow \text{Lie} (\mathfrak{G}_r) \longrightarrow 0
\]

that is $\mathfrak{H}_r^*$ and $\Gamma$-equivariant and compatible with change in $r$ using the map on Néron models induced by $\text{Pic}^0(\rho)$ and the maps (3.1.16) on $\mathfrak{G}_r$. Granting this claim, the proposition then follows immediately from Proposition 2.2.5.

To prove our claim, we introduce the following notation: set $V := \text{Spec}(R_r)$, and for $n \geq 1$ put $V_n := \text{Spec}(R_r/p^n R_r)$. For any scheme (or $p$-divisible group) $T$ over $V$, we put $T_n := T \times_V V_n$. If $\mathcal{A}$ is a Néron model over $V$, we will write $H(\mathcal{A})$ for the short exact sequence of free $R_r$-modules obtained by applying Lie to the canonical extension (2.2.4) of $\mathcal{A}^0$. If $G$ is a $p$-divisible group over $V$, we similaly write $H(G_n)$ for the short exact sequence of Lie algebras associated to the universal extension of $G_n^t$ by a vector group over $V_n$ (see Theorem 2.2.2, (2)). If $\mathcal{A}$ is an abelian scheme over $V$ then we have natural and compatible (with change in $n$) isomorphisms

\[
H(\mathcal{A}_n[p^\infty]) \simeq H(\mathcal{A}_n) \simeq H(\mathcal{A})/p^n,
\]

thanks to Theorem 2.2.2, (3) and (1); in particular, this justifies our slight abuse of notation.

Applying the contravariant functor $e^{*'} H(\cdot)$ to the diagram of Néron models over $V$ induced by (3.1.13) yields a commutative diagram of short exact sequences of free $R_r$-modules

\[
(3.2.15) \quad e^{*'} H(\mathfrak{H}_r) \leftarrow e^{*'} H(\mathfrak{B}_r^*) \quad \text{and} \quad e^{*'} H(\mathfrak{H}_r) \leftarrow e^{*'} H(\mathfrak{B}_r)
\]
in which both vertical arrows are isomorphisms by definition of $e^{*'}$. As in the proofs of Propositions 3.1.11 and 3.1.17, it follows that the horizontal maps must be isomorphisms as well:

(3.2.17) \[ e^{*'}H(\mathcal{O}_r) \simeq e^{*'}H(\mathcal{B}_r^*) \]

Since these isomorphisms are induced via the Néron mapping property and the functoriality of $H(\cdot)$ by the $\mathcal{S}_r^+(\mathbb{Z})$-equivariant map $\alpha_r^*: J_r \to B_r^*$, they are themselves $\mathcal{S}_r^+$-equivariant. Similarly, since $\alpha_r^*$ is defined over $\mathbb{Q}$ and compatible with change in $r$ as in Lemma 3.1.10, the isomorphism (3.2.17) is compatible with the given actions of $\Gamma$ over $K_r$ giving the descent data of $J_{rK_r}$ and $B_{rK_r}$ to $\mathbb{Q}_p$ and change in $r$. Reducing (3.2.17) modulo $p^n$ and using the canonical isomorphism (3.2.15) yields the identifications

(3.2.18) \[ e^{*'}H(\mathcal{O}_r)/p^n \simeq e^{*'}H(\mathcal{B}_r^*)/p^n \simeq e^{*'}H(\mathcal{B}_{r,n}^*[p^\infty]) \simeq H(e^{*'}\mathcal{B}_{r,n}^*[p^\infty]) =: H(\mathcal{S}_{r,n}) \]

which are clearly compatible with change in $n$, and which are easily checked (using the naturality of (3.2.15) and our remarks above) to be $\mathcal{S}_r^*$ and $\Gamma$-equivariant, and compatible with change in $r$. Since the surjection $R_r \to R_r/pR_r$ is a PD-thickening, passing to inverse limits (with respect to $n$) on (3.2.18) and using Proposition 2.2.7 now completes the proof. □

**Corollary 3.2.6.** Let $r$ be a positive integer. Then the short exact sequence of free $R_r$-modules

(3.2.19) \[
0 \longrightarrow e^{*'}H^0(\omega_r) \longrightarrow e^{*'}H^1_{dR,r} \longrightarrow e^{*'}H^1(\mathcal{O}_r) \longrightarrow 0
\]

is functorially split; in particular, it is split compatibly with the actions of $\Gamma$ and $\mathcal{S}_r^*$. Moreover, (3.2.19) admits a functorial descent to $\mathbb{Z}_p$: there is a natural isomorphism of split short exact sequences

(3.2.20) \[
0 \longrightarrow e^{*'}H^0(\omega_r) \longrightarrow e^{*'}H^1_{dR,r} \longrightarrow e^{*'}H^1(\mathcal{O}_r) \longrightarrow 0
\]

that is $\mathcal{S}_r^*$ and $\Gamma$ equivariant, with $\Gamma$ acting trivially on $\mathcal{G}_r^{\text{ét}}$ and through $\langle \chi \rangle^{-1}$ on $\mathcal{G}_r^m$. The identification 3.2.20 is compatible with change in $r$ using the maps $\rho_*$ on the top row and the maps induced by

\[
\mathcal{G}_r = \mathcal{G}_r^m \times \mathcal{G}_r^{\text{ét}} \overset{\nu^{-1} \times F}{\longrightarrow} \mathcal{G}_r^m \times \mathcal{G}_r^{\text{ét}} \quad \overset{\bar{\eta}}{\longrightarrow} \mathcal{G}_{r+1}
\]
on the bottom row.

**Proof.** Consider the isomorphism (3.2.13) of Proposition 3.2.5. As $\mathcal{S}_r$ is an ordinary $p$-divisible group by Corollary 3.1.21, the top row of (3.2.13) is functorially split by Lemma 2.4.2, and this gives our first assertion. Composing the inverse of (3.2.13) with the isomorphism (2.4.10) of Lemma 2.4.2 gives the claimed identification (3.2.20). That this isomorphism is compatible with change in $r$ via the specified maps follows easily from the construction of (2.4.10) via (2.4.12). □

We can now prove Theorem 1.2.4. Let us recall the statement:
Theorem 3.2.7. There is a canonical isomorphism of short exact sequences of finite free $\Lambda_{R_\infty}$-modules

\[
0 \longrightarrow e'^sH^0(\omega) \longrightarrow e'^sH^1_{dR} \longrightarrow e'^sH^1(\mathcal{O}) \longrightarrow 0
\]

(3.2.21)

\[
0 \longrightarrow D^m_\infty \otimes_{\Lambda} \Lambda_{R_\infty} \longrightarrow D_\infty \otimes_{\Lambda} \Lambda_{R_\infty} \longrightarrow D^\ell_\infty \otimes_{\Lambda} \Lambda_{R_\infty} \longrightarrow 0
\]

that is $\Gamma$ and $\mathfrak{F}^*$-equivariant. Here, the mappings on bottom row are the canonical inclusion and projection morphisms corresponding to the direct sum decomposition $D_\infty = D^m_\infty \oplus D^{\ell}_\infty$. In particular, the Hodge filtration exact sequence (3.2.12) is canonically split, and admits a canonical descent to $\Lambda$.

Proof. Applying $\otimes_{R_\infty}$ to (3.2.20) and passing to projective limits yields an isomorphism of split exact sequences

\[
0 \longrightarrow \lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r^m) \otimes_{\mathbb{Z}_p} R_\infty \right) \longrightarrow \lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r) \otimes_{\mathbb{Z}_p} R_\infty \right) \longrightarrow \lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r^\ell) \otimes_{\mathbb{Z}_p} R_\infty \right) \longrightarrow 0
\]

On the other hand, the isomorphisms $\overline{\mathfrak{f}}_r^m = \overline{\mathfrak{f}}_r \times \overline{\mathfrak{f}}_r^\ell \text{ V-r \times F-r} \overline{\mathfrak{f}}_r^m \times \overline{\mathfrak{f}}_r^\ell = \overline{\mathfrak{f}}_r$ induce an isomorphism of projective limits

\[
\lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r) \otimes_{\mathbb{Z}_p} R_\infty \right) \cong \lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r^\ell) \otimes_{\mathbb{Z}_p} R_\infty \right)
\]

which is visibly compatible with the the canonical splittings of source and target. The result now follows from [Cai14, Lemma 3.1.2 (5)] and the proof of Theorem 3.2.2, which guarantee that the canonical mapping $D_\infty \otimes_{\Lambda} \Lambda_{R_\infty} \rightarrow \lim_{\Gamma} \left( D(\overline{\mathfrak{f}}_r) \otimes_{\mathbb{Z}_p} R_\infty \right)$ is an isomorphism respecting the natural splittings.

As in [Cai14, §3.3], for any subfield $K$ of $\mathbb{C}_p$ with ring of integers $R$, we denote by $eS(N; \Lambda_R)$ the module of ordinary $\Lambda_R$-adic cuspsforms of level $N$ in the sense of [Oht95, 2.5.5]. Following our convention above Proposition 3.1.11, we write $e' S(N; \Lambda_R)$ for the direct summand of $eS(N; \Lambda_R)$ on which $\mu_{p-1} \hookrightarrow \mathbb{Z}_p^\times \subseteq \mathfrak{f}$ acts nontrivially.

Corollary 3.2.8. There is a canonical isomorphism of finite free $\Lambda$-modules

\[
e' S(N; \Lambda) \simeq D^m_\infty
\]

that intertwines $T \in \mathfrak{f}$ on $e' S(N; \Lambda)$ with $T^* \in \mathfrak{f}^*$ on $D^m_\infty$, where $U^*_p$ acts on $D^m_\infty$ as $\langle p \rangle N V$.

Proof. We claim that there are natural isomorphisms of finite free $\Lambda_{R_\infty}$-modules

\[
D^m_\infty \otimes_{\Lambda} \Lambda_{R_\infty} \simeq e'^s H^0(\omega) \simeq e' S(N, \Lambda_{R_\infty}) \simeq e' S(N, \Lambda) \otimes_{\Lambda} \Lambda_{R_\infty}
\]

and that the resulting composite isomorphism intertwines $T^* \in \mathfrak{f}^*$ on $D^m_\infty$ with $T \in \mathfrak{f}$ on $e' S(N, \Lambda)$ and is $\Gamma$-equivariant, with $\gamma \in \Gamma$ acting as $\langle \chi(\gamma) \rangle^{-1} \otimes \gamma$ on each tensor product. Indeed, the first and second isomorphisms are due to Theorem 3.2.7 and [Cai14, Corollary 3.3.5], respectively, while the final isomorphism is a consequence of the definition of $e' S(N; \Lambda_R)$ and the facts that this $\Lambda_R$-module
Remark 3.2.9. Via Proposition 3.2.3 and the natural $\Lambda$-adic duality between $e\mathfrak{S}$ and $eS(N; \Lambda)$ [Oht95, Theorem 2.5.3], we obtain a canonical $\text{Gal}(K'_0/K_0)$-equivariant isomorphism of $\Lambda_{R_0}$-modules
\[ e\mathfrak{S} \otimes \Lambda_{R'_0} \simeq D^\text{et}_{\infty}(\langle a \rangle_N) \otimes \Lambda_{R_0} \]
that intertwines $T \otimes 1$ for $T \in \mathfrak{S}$ acting on $e\mathfrak{S}$ by multiplication with $T^* \otimes 1$, with $U^*_p$ acting on $D^\text{et}_{\infty}(\langle a \rangle_N)$ as $F$. From Theorem 3.2.4 and Corollary 3.2.8 we then obtain canonical isomorphisms
\[ e\mathfrak{S} \otimes \Lambda_{R'_0} \simeq f' H^1_{\text{cris}}(I^0)^{\text{ord}} \]
respectively
\[ e\mathfrak{S} \otimes \Lambda_{R'_0} \simeq f' H^1_{\text{cris}}(I^\infty)^{\text{ord}}(\langle a \rangle_N) \otimes \Lambda_{R_0} \]
that intertwine $T$ (respectively $T \otimes 1$) with $T^*$ (respectively $T^* \otimes 1$) where $U^*_p$ acts on crystalline cohomology as $\langle p \rangle_N V$ (respectively $F \otimes 1$). The second of these isomorphisms is moreover $\text{Gal}(K'_0/K_0)$-equivariant.

In order to relate the slope filtration (3.2.2) of $D_{\infty}$ to the ordinary filtration of $e^{*'}H^1_{\text{et}}$, we require:

Lemma 3.2.10. Let $r$ be a positive integer let $G_r = e^{*'}J_r[p^\infty]$ be the unique $\mathcal{Q}_p$-descent of the generic fiber of $\mathfrak{S}_r$, as in Definition 3.1.14. There are canonical isomorphisms of free $W(\overline{\mathbb{F}_p})$-modules
\[ D(\mathfrak{S}^m_{\mathcal{V}_r}) \otimes W(\overline{\mathbb{F}_p}) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p G^m_{\mathcal{V}_r}, \mathbb{Z}_p) \otimes W(\overline{\mathbb{F}_p}) \]  
(3.2.25a)
\[ D(\mathfrak{S}^m_{\mathcal{V}_r})(-1) \otimes W(\overline{\mathbb{F}_p}) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p G^m_{\mathcal{V}_r}, \mathbb{Z}_p) \otimes W(\overline{\mathbb{F}_p}) \]  
(3.2.25b)
that are $\mathfrak{S}^*_r$-equivariant and $\mathfrak{Q}_r$-compatible for the diagonal action on source and target, with $\mathfrak{Q}_r$ acting trivially on $D(\mathfrak{S}^m_{\mathcal{V}_r})$ and via $\chi^{-1}$ on $D(\mathfrak{S}^m_{\mathcal{V}_r})(-1) := D(\mathfrak{S}^m_{\mathcal{V}_r}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-1)$. The isomorphism (3.2.25a) intertwines $F \otimes \sigma$ with $1 \otimes \sigma$ while (3.2.25b) intertwines $V \otimes \sigma^{-1}$ with $1 \otimes \sigma^{-1}$.

Proof. Let $\mathfrak{S}$ be any object of $\text{pd}_{\mathcal{V}_r}$, and write $G$ for the unique descent of the generic fiber $\mathfrak{S}_{K_r}$ to $\mathcal{Q}_p$. We recall that the semilinear $\Gamma$-action on $\mathfrak{S}$ gives the $\mathbb{Z}_p[\mathfrak{Q}_r]$-module $T_p \mathfrak{S} := \text{Hom}_{\mathfrak{Q}_r}(\mathcal{Q}_p/\mathbb{Z}_p, \mathfrak{S}[\mathfrak{Q}_r])$ the natural structure of $\mathbb{Z}_p[\mathfrak{Q}_r]$-module via $g \cdot f := g^{-1} \circ f \circ g$. It is straightforward to check that the natural map $T_p \mathfrak{S} \to T_p G$, which is an isomorphism of $\mathbb{Z}_p[\mathfrak{Q}_r]$-modules by Tate’s theorem, is an isomorphism of $\mathbb{Z}_p[\mathfrak{Q}_r]$-modules as well.

For any étale $p$-divisible group $H$ over a perfect field $k$, one has a canonical isomorphism of $W(\overline{k})$-modules with semilinear $\mathfrak{S}_r$-action
\[ D(H) \otimes W(\overline{k}) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p H, \mathbb{Z}_p) \otimes W(\overline{k}) \]
that intertwines $F \otimes \sigma$ with $1 \otimes \sigma$ and $1 \otimes g$ with $g \otimes g$ for $g \in \mathfrak{S}_r$; for example, this can be deduced by applying [BM79, §4 a)] to $H^r$ and using the fact that the Dieudonné crystal is compatible with base change. In our case, the étale $p$-divisible group $\mathfrak{S}^\text{et}_r$ lifts $\mathfrak{S}^\text{et}_{\mathcal{V}_r}$ over $R_r$, and we obtain a natural isomorphism of $W(\overline{\mathbb{F}_p})$-modules with semilinear $\mathfrak{S}_r$-action
\[ D(\mathfrak{S}^\text{et}_{\mathcal{V}_r}) \otimes W(\overline{\mathbb{F}_p}) \simeq \text{Hom}_{\mathbb{Z}_p}(T_p \mathfrak{S}^\text{et}_{\mathcal{V}_r}, \mathbb{Z}_p) \otimes W(\overline{\mathbb{F}_p}) \].
By naturality in $G_r$, this identification respects the semilinear $\Gamma$-actions on both sides (which are trivial, as $\Gamma$ acts trivially on $G_r^\et$); as explained in our initial remarks, it is precisely this action which allows us to view $T_pG_r^\et$ as a $\Bbb Z_p[\mathcal{G}_p]$-module, and we deduce (3.2.25a). The proof of (3.2.25b) is similar, using the natural isomorphism (proved as above) for any multiplicative $p$-divisible group $H/k
ot\simeq$\Bbb W(k)$, \[ H^1_{\et}(X, W(k)) \simeq T_pH^t \otimes \Bbb W(k), \] which intertwines $\nu \otimes \sigma^{-1}$ with $1 \otimes \sigma^{-1}$ and $1 \otimes \psi$ with $g \otimes \theta$, for $g \in G_k$. \hfill \blacksquare

\textbf{Proof of Theorem 1.2.6 and Corollary 1.2.8.} For a $p$-divisible group $H$ over a field $K$, we will write $H^1_{\et}(H) := \Hom_{\Bbb Z_p}(T_pH, \Bbb Z_p)$; our notation is justified by the standard fact that, for $J_X$ the Jacobian of a curve $X$ over $K$, there is a natural isomorphisms of $\Bbb Z_p[\mathcal{G}_K]$-modules

\[ H^1_{\et}(J_X[p\infty]) \simeq H^1_{\et}(X, \Bbb Z_p). \] (3.2.26)

It follows from (3.2.25a)–(3.2.25b) and \textbf{Theorem 3.2.2} (1)–(2) that $H^1_{\et}(G_r^*) \otimes_{\Bbb Z_p} W(\overline{F}_p)$ is a free $W(\overline{F}_p)[\Delta/\Delta_r]$-module of rank $d'$ for $* \in \{\et, m\}$, and hence that $H^1_{\et}(G_r^*)$ is a free $\Bbb Z_p[\Delta/\Delta_r]$-module of rank $d'$ by [Cai14, Lemma 3.1.3]. In a similar manner, using the faithful flatness of $W(\overline{F}_p)[\Delta/\Delta_r]$ over $\Bbb Z_p[\Delta/\Delta_r]$, we deduce that the canonical trace mappings

\[ H^1_{\et}(G_r^*) \longrightarrow H^1_{\et}(G_r^*) \] (3.2.27)

are surjective for all $r \geq r'$. From the commutative algebra formalism of [Cai14, Lemma 3.1.2], we conclude that $H^1_{\et}(G_r^*) := \varinjlim_r H^1_{\et}(G_r^*)$ is a free $\Lambda$-module of rank $d'$ and that there are canonical isomorphisms of $\Lambda W(\overline{F}_p)$-modules

\[ H^1_{\et}(G_r^*) \otimes \Lambda W(\overline{F}_p) \simeq \varinjlim_r \left( H^1_{\et}(G_r^*) \otimes_{\Bbb Z_p} W(\overline{F}_p) \right) \]

for $* \in \{\et, m\}$. Since we likewise have canonical identifications

\[ \mathbf{D}^* \otimes \Lambda W(\overline{F}_p) \simeq \varinjlim_r \left( \mathbf{D}(G_r^*) \otimes_{\Bbb Z_p} W(\overline{F}_p) \right) \]

thanks again to [Cai14, Lemma 3.1.2] and (the proof of) \textbf{Theorem 3.2.2}, passing to inverse limits on (3.2.25a)–(3.2.25b) gives a canonical isomorphism of $\Lambda W(\overline{F}_p)$-modules

\[ \mathbf{D}^* \otimes \Lambda W(\overline{F}_p) \simeq H^1_{\et}(G_r^*) \otimes \Lambda W(\overline{F}_p) \] (3.2.28)

for $* \in \{\et, m\}$.

Applying the functor $H^1_{\et}(\cdot)$ to the connected-étale sequence of $G_r$ yields a short exact sequence of $\Bbb Z_p[\mathcal{G}_p]$-modules

\[
0 \longrightarrow H^1_{\et}(G_r^\et) \longrightarrow H^1_{\et}(G_r) \longrightarrow H^1_{\et}(G_r^m) \longrightarrow 0
\]

which naturally identifies $H^1_{\et}(G_r^*)$ with the invariants (respectively covariants) of $H^1_{\et}(G_r)$ under the inertia subgroup $J_r \subseteq \mathcal{G}_p$ for $* = \et$ (respectively $* = m$). As $G_r = e^{\star'}J_r[p\infty]$ by definition, we
deduce from this and (3.2.26) a natural isomorphism of short exact sequences of $\mathbb{Z}_p[\mathcal{Q}_p]$-modules

$$0 \longrightarrow H^1_{\text{et}}(G_{\text{et}}^\dagger) \longrightarrow H^1_{\text{et}}(G_r) \longrightarrow H^1_{\text{et}}(G^m_r) \longrightarrow 0$$

(3.2.29)

where for notational ease we abbreviate $H^1_{\text{et},r} := H^1_{\text{et}}(X_r\mathcal{O}_p, \mathbb{Z}_p)$. As the trace maps (3.2.27) are surjective, passing to inverse limits on (3.2.29) yields an isomorphism of short exact sequences

$$0 \longrightarrow H^1_{\text{et}}(G_{\text{et}}^\dagger) \longrightarrow H^1_{\text{et}}(G_{\text{et}}) \longrightarrow H^1_{\text{et}}(G^m_{\text{et}}) \longrightarrow 0$$

(3.2.30)

Since inverse limits commute with group invariants, the bottom row of (3.2.30) is canonically isomorphic to the ordinary filtration of Hida’s $e'' H^1_{\text{et}}$, and Theorem 1.2.6 follows immediately from (3.2.28).

Corollary 1.2.8 is then an easy consequence of Theorem 1.2.6 and [Cai14, Lemma 3.1.3]; alternately one can prove Corollary 1.2.8 directly from [Cai14, 3.1.2], using what we have seen above.

3.3. Ordinary families of $(\varphi, \Gamma)$-modules. We now study the family of Dieudonné crystals attached to the tower of $p$-divisible groups $\{S_r/R_r\}_{r \geq 1}$. For each pair of positive integers $r \geq s$, we have a morphism $\rho_{r,s} : S_s \times_{T_s} T_r \rightarrow S_r$ in $\text{pdiv}_{R_r}^{\Gamma}$; applying the contravariant functor $\mathcal{M}_r : \text{pdiv}_{R_r}^{\Gamma} \rightarrow \text{BT}_{\mathcal{E}_r}$ studied in §2.3–2.4 to the map on connected-étale sequences induced by $\rho_{r,s}$ and using the exactness of $\mathcal{M}_r$ and its compatibility with base change (Theorem 2.3.3), we obtain morphisms in $\text{BT}_{\mathcal{E}_r}$

$$0 \longrightarrow \mathcal{M}_r(S_{\text{et}}^\dagger) \longrightarrow \mathcal{M}_r(S_r) \longrightarrow \mathcal{M}_r(S^m_r) \longrightarrow 0$$

(3.3.1)

Definition 3.3.1. Let $* = \text{et}$ or $* = m$ and define

$$\mathcal{M}_\infty := \lim_{r \rightarrow} \left( \mathcal{M}_r(S_r) \otimes \mathcal{E}_r \right)$$

$$\mathcal{M}_\infty^* := \lim_{r \rightarrow} \left( \mathcal{M}_r(S^*_r) \otimes \mathcal{E}_r \right)$$

with the projective limits taken with respect to the mappings induced by (3.3.1).

Each of (3.3.2) is naturally a module over the completed group ring $\Lambda_{\mathcal{E}_\infty}$ and is equipped with a semilinear action of $\Gamma$ and a $\varphi$-semilinear Frobenius morphism defined by $\overline{F} := \lim_{r \rightarrow} (\varphi_{\mathcal{M}_r} \otimes \varphi)$. Since $\varphi$ is bijective on $\mathcal{E}_\infty$, we also have a $\varphi^{-1}$-semilinear Verschiebung morphism defined as follows. For notational ease, we provisionally set $M_r := \mathcal{M}_r(S_r) \otimes_{\mathcal{E}_r} \mathcal{E}_\infty$ and we define

$$V_r : M_r \xrightarrow{\psi_{\mathcal{M}_r}} \varphi^* M_r \xrightarrow{\alpha_{m \rightarrow \varphi^{-1}(\alpha)m}} M_r$$

(3.3.3)

with $\psi_{\mathcal{M}_r}$ as above Definition 2.3.2. It is easy to see that the $V_r$ are compatible with change in $r$, and we put $V := \lim_{r \rightarrow} V_r$ on $\mathcal{M}_\infty$. We define Verschiebung morphisms on $\mathcal{M}_\infty^*$ for $* = \text{et}, m$ similarly. Using
that the composite of $\psi_{M_r}$ and $1 \otimes \varphi_{M_r}$ in either order is multiplication by $E_r(u_r) =: \omega$, one checks

$$FV = \omega \quad \text{and} \quad VF = \varphi^{-1}(\omega).$$

Due to the functoriality of $M_r$, we moreover have a $\Lambda_{\mathfrak{S}_\infty}$-linear action of $\mathfrak{H}^*$ on each of $(3.3.2)$ which commutes with $F$, $V$, and $\Gamma$.

**Theorem 3.3.2.** Let $d'$ be the integer specified in Theorem 3.2.2. Then $M_{\infty}$ (respectively $M^*_{\infty}$ for $\ast = \text{ét}, m$) is a free $\Lambda_{\mathfrak{S}_\infty}$-module of rank $2d'$ (respectively $d'$) and there is a canonical short exact sequence of $\Lambda_{\mathfrak{S}_\infty}$-modules with linear $\mathfrak{H}^*$-action and semi linear actions of $\Gamma$, $F$ and $V$

$$(3.3.4) \quad 0 \longrightarrow M^*_\infty \longrightarrow M_{\infty} \longrightarrow M^m_{\infty} \longrightarrow 0.$$ 

**Extension of scalars of (3.3.4) along the quotient $\Lambda_{\mathfrak{S}_\infty} \twoheadrightarrow \mathfrak{S}_\infty[\Delta/\Delta_r]$ recovers the exact sequence**

$$(3.3.5) \quad 0 \longrightarrow M_r(S^d_{\text{ét}}) \otimes \mathfrak{S}_r \longrightarrow M_r(S_r) \otimes \mathfrak{S}_r \longrightarrow M_r(S^m_r) \otimes \mathfrak{S}_r \longrightarrow 0.$$ 

for each integer $r > 0$, compatibly with $\mathfrak{H}^*$, $\Gamma$, $F$, and $V$. The Frobenius endomorphism $F$ commutes with $\mathfrak{H}^*$ and $\Gamma$, while $V$ commutes with $\mathfrak{H}^*$ and satisfies $V \gamma = \varphi^{-1}(\omega/\omega) \cdot \gamma V$ for all $\gamma \in \Gamma$.

**Proof.** Since $\varphi$ is an automorphism of $\mathfrak{S}_\infty$, pullback by $\varphi$ commutes with projective limits of $\mathfrak{S}_\infty$-modules. As the canonical $\mathfrak{S}_\infty$-linear map $\varphi^* \Lambda_{\mathfrak{S}_\infty} \rightarrow \Lambda_{\mathfrak{S}_\infty}$ is an isomorphism of rings (even of $\mathfrak{S}_\infty$-algebras), it therefore suffices to prove the assertions of Theorem 3.3.2 after pullback by $\varphi$, which will be more convenient due to the relation between $\varphi^* M_r(S_r)$ and the Dieudonné crystal of $S_r$.

Pulling back $(3.3.1)$ by $\varphi$ gives a commutative diagram with exact rows

$$0 \longrightarrow \varphi^* M_r(S^d_{\text{ét}}) \longrightarrow \varphi^* M_r(S_r) \longrightarrow \varphi^* M_r(S^m_r) \longrightarrow 0$$

and, as in the proof of Theorem 3.2.2, we apply the commutative algebra formalism of [Cai14, §3.1]. In the notation of [Cai14, Lemma 3.1.2], we take $A_r := \mathfrak{S}_r$, $I_r := (u_r)$, $B = \mathfrak{S}_\infty$, and $M_r$ each one of the terms in the top row of $(3.3.6)$, and we must check that the hypotheses

$$(3.3.7a) \quad \overline{M}_r := M_r/u_r M_r \text{ is a free } \mathbb{Z}_p[\Delta/\Delta_r]\text{-module of rank } d'$$

$$(3.3.7b) \quad \text{For all } s \leq r \text{ the induced transition maps } \overline{p}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s \text{ are surjective}$$

hold. The isomorphism $(2.4.13a)$ of Proposition 2.4.3 ensures, via Theorem 3.2.2 (1), that the hypothesis $(3.3.7a)$ is satisfied.

Due to the functoriality of $(2.4.13a)$, for any $r \geq s$, the mapping obtained from $(3.3.6)$ by reducing modulo $I_s$ is identified with the mapping on $(3.2.1)$ induced (via functoriality of $D(\cdot)$) by $\overline{p}_{r,s}$. As was shown in the proof of Theorem 3.2.2, these mappings are surjective for all $r \geq s$, and we conclude that the hypothesis $(3.3.7b)$ holds as well. Moreover, the vertical mappings of $(3.3.6)$ are then surjective by Nakayama’s Lemma, so as in the proof of Theorem 3.2.2 (and keeping in mind that pullback by $\varphi$ commutes with projective limits of $\mathfrak{S}_\infty$-modules), we obtain, by applying $\otimes \mathfrak{S}_r, \mathfrak{S}_\infty$ to $(3.3.6)$, passing to projective limits, and pulling back by $(\varphi^{-1})^*$, the short exact sequence $(3.3.4)$. The final assertion is an immediate consequence of the functorial construction of $\varphi_{\mathfrak{M}_r(\cdot)}$, the definition $(3.3.3)$ of $V$, and the intertwining relation $(2.3.2)$. \hfill \blacksquare
Remark 3.3.3. In the proof of Theorem 3.3.2, we could have alternately applied [Cai14, Lemma 3.1.2] with $A_r = \mathcal{C}_r$ and $I_r = \langle E_r \rangle$, appealing to the identifications (2.4.13b) of Proposition 2.4.3 and (3.2.13) of Proposition 3.2.5, and to Theorem 1.1.1 ([Cai14, Theorem 3.2.3]).

The short exact sequence (3.3.4) is closely related to its $\Lambda_{\mathcal{C}_\infty}$-linear dual. In what follows, we write $\mathcal{G}'_{\infty} := \lim_{\tau} Z_p[\mu_N][u_r]$, taken along the mappings $u_r \mapsto \varphi(u_{r+1})$; it is naturally a $\mathcal{G}_\infty$-algebra.

Theorem 3.3.4. Let $\mu : \Gamma \to \Lambda_{\mathcal{C}_\infty}^\otimes$ be the crossed homomorphism given by $\mu(\gamma) := \frac{m_1}{\gamma a_1} \chi(\gamma)\langle \chi(\gamma) \rangle$. There is a canonical $\mathcal{G}'_{\infty}$ and $\text{Gal}(K_{\infty}/K_0)$-equivariant isomorphism of exact sequences of $\Lambda_{\mathcal{G}_{\infty}}$-modules

$$0 \longrightarrow \mathcal{M}^\dagger_{\infty}(\mu(a)N)_{\Lambda_{\mathcal{G}'_{\infty}}} \longrightarrow \mathcal{M}_{\infty}(\mu(a)N)_{\Lambda_{\mathcal{G}'_{\infty}}} \longrightarrow \mathcal{M}_{\infty}(\mu(a)N)_{\Lambda_{\mathcal{G}'_{\infty}}} \longrightarrow 0$$

(3.3.8)

that intertwines $F$ (respectively $V$) on the top row with $V^\dagger$ (respectively $F^\dagger$) on the bottom.

Proof. We first claim that there is a natural isomorphism of $\mathcal{G}'_{\infty}[\Delta/\Delta_r]$-modules

$$\mathcal{M}_r(\mathcal{S}_r)(\langle \chi \rangle(a)N)_{\mathcal{G}'_{\infty}} \simeq \text{Hom}_{\mathcal{G}_{\infty}}(\mathcal{M}_r(\mathcal{S}_r) \otimes_{\mathcal{G}_r} \mathcal{G}_{\infty}, \mathcal{G}_{\infty})$$

(3.3.9)

that is $\mathcal{G}'_{\infty}$-equivariant and $\text{Gal}(K_{\infty}/K_0)$-compatible for the standard action $\gamma \cdot f(m) := \gamma f(\gamma^{-1}m)$ on the right side, and that intertwines $F$ and $V$ with $V^\dagger$ and $F^\dagger$, respectively. Indeed, this follows immediately from the identifications

$$\mathcal{M}_r(\mathcal{S}_r)(\langle \chi \rangle(a)N)_{\mathcal{G}'_{\infty}} \simeq \mathcal{M}_r(\mathcal{S}_r')_{\mathcal{G}_{\infty}} \times \mathcal{M}_r(\mathcal{S}_r)_{\mathcal{G}_{\infty}} :\mathcal{M}_r(\mathcal{S}_r)_{\mathcal{G}_{\infty}} \otimes \mathcal{G}'_{\infty}$$

(3.3.10)

and the definition (Definition 2.3.2) of duality in $\text{BT}_{\mathcal{G}_r}^\otimes$; here, the first isomorphism in (3.3.10) results from Proposition 3.1.15 and Theorem 2.3.3 (2), while the final identification is due to Theorem 2.3.3 (1). The identification (3.3.9) carries $F$ (respectively $V$) on its source to $V^\dagger$ (respectively $F^\dagger$) on its target due to the compatibility of the functor $\mathcal{M}_r(\cdot)$ with duality (Theorem 2.3.3 (1)).

From (3.3.9) we obtain a natural $\text{Gal}(K_{\infty}/K_0)$-compatible evaluation pairing of $\mathcal{G}'_{\infty}$-modules

$$\langle \cdot, \cdot \rangle_r : \mathcal{M}_r(\mathcal{S}_r)(\mu(a)N)_{\mathcal{G}'_{\infty}} \otimes \mathcal{G}'_{\infty} \times \mathcal{M}_r(\mathcal{S}_r)_{\mathcal{G}_{\infty}} \longrightarrow \mathcal{G}'_{\infty}$$

(3.3.11)

with respect to which the natural action of $\mathcal{G}'_{\infty}$ is self-adjoint, due to the fact that (3.3.10) is $\mathcal{G}'_{\infty}$-equivariant by Proposition 3.1.15. Due to the compatibility with change in $r$ of the identification (3.1.17) of Proposition 3.1.15 together with the definitions (3.1.16) of $\rho_{r,s}$ and $\rho'_{r,s}$, the identification (3.3.10) intertwines the map induced by Pic$^0(\rho)$ on its source with the map induced by $U_{p}^{s-1}\text{Alb}(\sigma)$ on its target. For $r \geq s$, we therefore have

$$\langle \mathcal{M}_r(\rho_{r,s})x, \mathcal{M}_r(\rho_{r,s})y \rangle_s = \langle x, \mathcal{M}_r(U_{p}^{s-s} \text{Pic}^0(\rho)^{s-s} \text{Alb}(\sigma)^{s-s})y \rangle_r = \sum_{\delta \in \Delta_r/\Delta_r} \langle x, \delta^{-1}y \rangle_r,$$

where the final equality follows from (3.2.9). Thus, the perfect pairings (3.3.11) satisfy the compatibility condition of [Cai14, Lemma 3.1.4] (as in (3.2.7) of the proof of Proposition 3.2.3) which, together with Theorem 3.3.2, completes the proof.

The $\Lambda_{\mathcal{G}_{\infty}}$-modules $\mathcal{M}^\dagger_{\infty}$ and $\mathcal{M}_{\infty}$ admit canonical descents to $\Lambda$:
Theorem 3.3.5. There are canonical $\mathcal{H}^*$, $\Gamma$, $F$ and $V$-equivariant isomorphisms of $\Lambda_{\mathcal{G}_\infty}$-modules
\begin{equation}
\mathcal{M}_\infty^\text{et} \simeq D_\infty^\text{et} \otimes \Lambda_{\mathcal{G}_\infty},
\end{equation}
interwining $F$ and $V$ with $F \otimes \varphi$ and $F^{-1} \otimes \varphi^{-1}(\omega) \cdot \varphi^{-1}$, respectively, and $\gamma \in \Gamma$ with $\gamma \otimes \gamma$, and
\begin{equation}
\mathcal{M}_\infty^m \simeq D_\infty^m \otimes \Lambda_{\mathcal{G}_\infty},
\end{equation}
interwining $F$ and $V$ with $V^{-1} \otimes \omega \cdot \varphi$ and $V \otimes \varphi^{-1}$, respectively, and $\gamma$ with $\gamma \otimes \chi(\gamma)^{-1} \gamma u_1/u_1$. In particular, $F$ (respectively $V$) acts invertibly on $\mathcal{M}_\infty^\text{et}$ (respectively $\mathcal{M}_\infty^m$).

Proof. We twist the identifications (2.4.2) of Proposition 2.4.1 to obtain natural isomorphisms
\[
\mathcal{M}_r(\mathcal{G}_r^{\text{et}}) \xrightarrow{\simeq}{F' \circ (2.4.2)} D(\mathcal{G}_r^{\text{et}})z_p \otimes z_p \mathcal{G}_r \quad \text{and} \quad \mathcal{M}_r(\mathcal{G}_r^{m}) \xrightarrow{\simeq}{V' \circ (2.4.2)} D(\mathcal{G}_r^{m})z_p \otimes z_p \mathcal{G}_r
\]
that are $\mathcal{H}_r^*$-equivariant and, Thanks to 2.4.3, compatible with change in $r$ using the maps on source and target induced by $p_{r,s}$. Passing to inverse limits and appealing again to [Cai14, Lemma 3.1.2] and (the proof of) Theorem 3.2.2, we deduce for $\ast \in \{\text{ét, m}\}$ natural isomorphisms of $\Lambda_{\mathcal{G}_\infty}$-modules
\[
\mathcal{M}_\infty^\ast \simeq \lim_{\longrightarrow \ r} \left(D(\mathcal{G}_r^\ast)z_p \otimes z_p \mathcal{G}_\infty\right) \simeq D_\infty^\ast \otimes \Lambda_{\mathcal{G}_\infty}
\]
that are $\mathcal{H}_r^*$-equivariant and satisfy the asserted compatibility with respect to Frobenius, Verscheibung, and the action of $\Gamma$ due to Proposition 2.4.1 and the definitions (2.4.1a)–(2.4.1b).

We can now prove Theorem 1.2.12, which asserts that the slope filtration (1.2.12) of $\mathcal{M}_\infty$ specializes, on the one hand, to the slope filtration (3.2.2) of $D_\infty$, and on the other hand to the Hodge filtration (3.2.12) (in the opposite direction!) of $e^{\ast}H^1_{\text{dR}}$. We recall the precise statement:

Theorem 3.3.6. Let $\tau : \Lambda_{\mathcal{G}_\infty} \twoheadrightarrow \Lambda$ be the $\Lambda$-algebra surjection induced by $u_r \mapsto 0$. There is a canonical $\Gamma$ and $\mathcal{H}_r^*$-equivariant isomorphism of split exact sequences of finite free $\Lambda$-modules
\begin{equation}
0 \longrightarrow \mathcal{M}_\infty^\text{et} \otimes \Lambda_{\mathcal{G}_\infty} \tau \longrightarrow \mathcal{M}_\infty \otimes \Lambda_{\mathcal{G}_\infty} \tau \longrightarrow \mathcal{M}_\infty^m \otimes \Lambda_{\mathcal{G}_\infty} \tau \longrightarrow 0
\end{equation}
which carries $F \otimes 1$ to $F$ and $V \otimes 1$ to $V$.

Let $\theta \circ \varphi : \Lambda_{\mathcal{G}_\infty} \twoheadrightarrow \Lambda_{R_{\mathcal{G}_\infty}}$ be the $\Lambda$-algebra surjection induced by $u_r \mapsto (e^{(r)})^p - 1$. There is a canonical $\Gamma$ and $\mathcal{H}_r^*$-equivariant isomorphism of split exact sequences of finite free $\Lambda_{R_{\mathcal{G}_\infty}}$-modules
\begin{equation}
0 \longrightarrow \mathcal{M}_\infty^\text{et} \otimes \Lambda_{R_{\mathcal{G}_\infty}} \varphi \longrightarrow \mathcal{M}_\infty \otimes \Lambda_{R_{\mathcal{G}_\infty}} \varphi \longrightarrow \mathcal{M}_\infty^m \otimes \Lambda_{R_{\mathcal{G}_\infty}} \varphi \longrightarrow 0
\end{equation}
where $i$ and $j$ are the canonical sections given by the splitting in Theorem 1.2.4.

Proof. To prove the first assertion, we apply [Cai14, Lemma 3.1.2] with $A_r = \mathcal{G}_r$, $I_r = (u_r)$, $B = \mathcal{G}_\infty$, $B' = \mathbb{Z}_p$ (viewed as a $B$-algebra via $\tau$) and $M_{\tau} = \mathcal{M}_r^\ast$ for $\ast \in \{\text{ét, m, null}\}$, and, as in the proofs of Theorems 3.2.2 and 3.3.2, we must verify the hypotheses
(3.3.15a) \( \overline{M}_r := M_r / u_r M_r \) is a free \( \mathbb{Z}_p[\Delta/\Delta_r] \)-module of rank \( d' \).

(3.3.15b) For all \( s \leq r \) the induced transition maps \( \overline{p}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s \) are surjective.

Thanks to (2.4.13a) in the case \( G = S_r \), we have a canonical identification \( \overline{M}_r := M_r / I_r M_r \simeq D(\mathcal{S}_r) \mathbb{Z}_p \) that is compatible with change in \( r \) in the sense that the induced projective system \( \{ \overline{M}_r \}_r \) is identified with that of Definition 3.2.1. It follows from this and Theorem 3.2.2 (1)–(2) that the hypotheses (3.3.15a)–(3.3.15b) are satisfied, and (3.3.13) is an isomorphism by [Cai14, Lemma 3.1.3 (5)].

In exactly the same manner, the second assertion follows by appealing to [Cai14, Lemma 3.1.2] with \( A_r = \mathcal{S}_r, I_r = (E_r), B = \mathcal{S}_\infty, B' = R_\infty \) (viewed as a \( B \)-algebra via \( \theta \circ \varphi \)) and \( M_r = \mathcal{M}_r \), using (2.4.13b) and Proposition 3.2.5 together with Theorem 1.1.1 (see [Cai14, Theorem 3.2.3]) to verify the requisite hypotheses in this setting.

**Proof of Theorem 1.2.13 and Corollary 1.2.14.** Applying Theorem 2.3.5 to (the connected-étale sequence of) \( S_r \) gives a natural isomorphism of short exact sequences

\[
0 \longrightarrow \mathcal{M}_r(S^\text{ét}_r) \otimes_{\mathcal{S}_r, \varphi} A_r \longrightarrow \mathcal{M}_r(S^\text{m}_r) \otimes_{\mathcal{S}_r, \varphi} A_r \longrightarrow 0
\]

\[
0 \longrightarrow H^1_{\text{ét}}(S^\text{ét}_r) \otimes_{\mathcal{S}_r} A_r \longrightarrow H^1_{\text{ét}}(S^\text{m}_r) \otimes_{\mathcal{S}_r} A_r \longrightarrow 0
\]

Due to Theorem 3.3.2, the terms in the top row of (3.3.16) are free of ranks \( d', 2d' \), and \( d' \) over \( \overline{A}_r[\Delta/\Delta_r] \), respectively, so we conclude from [Cai14, Lemma 3.1.3] (using \( A = \mathbb{Z}_p[\Delta/\Delta_r] \) and \( B = A_r[\Delta/\Delta_r] \) in the notation of that result) that \( H^1_{\text{ét}}(S^*_r) \) is a free \( \mathbb{Z}_p[\Delta/\Delta_r] \)-module of rank \( d' \) for \( * = \text{ét, m} \) and that \( H^1_{\text{ét}}(S^*_r) \) is free of rank \( 2d' \) over \( \mathbb{Z}_p[\Delta/\Delta_r] \). Using the fact that \( \mathbb{Z}_p \to A_r \) is faithfully flat, it then follows from the surjectivity of the vertical maps in (3.3.6) (which was noted in the proof of Theorem 3.3.2) that the canonical trace mappings \( H^1_{\text{ét}}(S^*_r) \to H^1_{\text{ét}}(S^*_r) \) for \( * = \text{ét, m} \) are surjective for all \( r \geq r' \).

Applying [Cai14, Lemma 3.1.2] with \( A_r = \mathbb{Z}_p, M_r := H^1_{\text{ét}}(S^*_r), I_r = (0), B = \mathbb{Z}_p \) and \( B' = \overline{A} \), we conclude that \( H^1_{\text{ét}}(S^*_\Lambda) \) is free of rank \( d' \) (respectively \( 2d' \)) over \( \Lambda \) for \( * = \text{ét, m} \) (respectively \( * = \text{null} \)), that the specialization mappings

\[
H^1_{\text{ét}}(S^*_\Lambda) \otimes_{\mathbb{Z}_p} \Delta/\Delta_r \longrightarrow H^1_{\text{ét}}(S^*_r)
\]

are isomorphisms, and that the canonical mappings for \( * = \{ \text{ét, m, null} \} \)

\[
H^1_{\text{ét}}(S^*_\Lambda) \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow \varprojlim_r \left( H^1_{\text{ét}}(S^*_r) \otimes_{\overline{A}} \mathbb{Z}_p \right)
\]

are isomorphisms. Invoking the isomorphism (3.3.30) gives Corollary 1.2.14. By [Cai14, Lemma 3.1.2] with \( A_r = \mathcal{S}_r, M_r = \mathcal{M}_r(S^*_r), I_r = (0), B = \mathcal{S}_\infty \) and \( B' = \overline{A} \), we similarly conclude from (the proof of) Theorem 3.3.2 that the canonical mappings for \( * \in \{ \text{ét, m, null} \} \)

\[
\mathcal{M}^*_\infty \otimes_{\mathcal{S}_\infty, \varphi} \Lambda \longrightarrow \varprojlim_r \left( \mathcal{M}_r(S^*_r) \otimes_{\mathcal{S}_r} \overline{A} \right)
\]

are isomorphisms. Applying \( \otimes_{\overline{A}} \overline{A} \) to the diagram (3.3.16), passing to inverse limits, and using the isomorphisms (3.3.17) and (3.3.18) gives (again invoking (3.3.30)) the isomorphism (1.2.9). Using the
fact that the inclusion \( \mathbb{Z}_p \hookrightarrow \tilde{A}^{e=1} \) is an equality, the isomorphism (1.2.10) follows immediately from (1.2.9) by taking \( F \otimes \varphi \)-invariants.

Using Theorems 1.2.13 and 3.3.4 we can give a new proof of Ohta’s duality theorem [Oht95, Theorem 4.3.1] for the \( \Lambda \)-adic ordinary filtration of \( e^s H^1_{\text{et}} \) (see Corollary 1.2.15): 

**Theorem 3.3.7.** There is a canonical \( \Lambda \)-bilinear and perfect duality pairing 
\[
\langle \cdot, \cdot \rangle_{\Lambda} : e^s H^1_{\text{et}} \times e^s H^1_{\text{et}} \rightarrow \Lambda
\]
determined by 
\[
\langle x, y \rangle_{\Lambda} \equiv \sum_{\delta \in \Delta/\Delta_r} (x, w_r U^s \delta^{-1} y) \delta \text{ mod } I_r
\]
with respect to which the action of \( \mathcal{S}_r^{*} \) is self-adjoint; here, \( \langle \cdot, \cdot \rangle_{\Lambda} \) is the usual cup-product pairing on \( H^1_{\text{et}, r} \) and \( I_r := \ker(\Lambda \rightarrow \mathbb{Z}_p[\Delta/\Delta_r]) \). Writing \( \nu : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathcal{S}_r^{*} \) for the character \( \nu := \chi \langle \chi \rangle \lambda(\langle p \rangle N) \), the pairing (3.3.19) induces a canonical \( \mathcal{G}_{\mathbb{Q}_p} \)- and \( \mathcal{S}_r^{*} \)-equivariant isomorphism of exact sequences

\[
0 \rightarrow (e^s H^1_{\text{et}})_{\nu} \rightarrow e^s H^1_{\text{et}} \rightarrow (e^s H^1_{\text{et}})_{\nu} \rightarrow 0
\]

\[
0 \rightarrow \text{Hom}_\Lambda((e^s H^1_{\text{et}})_{\nu}, \Lambda) \rightarrow \text{Hom}_\Lambda(e^s H^1_{\text{et}}, \Lambda) \rightarrow \text{Hom}_\Lambda((e^s H^1_{\text{et}})_{\nu}, \Lambda) \rightarrow 0
\]

**Proof.** The proof is similar to that of Proposition 3.2.3, using Corollary 1.2.14 and applying [Cai14, Lemma 3.1.4] (cf. the proofs of [Cai14, 3.2.4] and [Oht95, Theorem 4.3.1] and of [Sha11, Proposition 4.4]). Alternatively, one can prove Theorem 3.3.7 by appealing to Theorem 3.3.4 and the isomorphism (1.2.10) of Theorem 1.2.13.

**Proof of Theorem 1.2.16.** Suppose first that (3.3.4) admits a \( \Lambda_{G_{\infty}} \)-linear splitting \( \mathfrak{M}_\infty^m \rightarrow \mathfrak{M}_{\infty} \) which is compatible with \( F, V, \) and \( \Gamma \). Extending scalars along \( \Lambda \rightarrow \Lambda_{G_{\infty}} \rightarrow \Lambda_{\tilde{A}} \) and taking \( F \otimes \varphi \)-invariants yields, by Theorem 1.2.13, a \( \Lambda \)-linear and \( \mathcal{G}_{\mathbb{Q}_p} \)-equivariant map \( (e^s H^1_{\text{et}})_{\nu} \rightarrow e^s H^1_{\text{et}} \) whose composition with the canonical projection \( e^s H^1_{\text{et}} \rightarrow (e^s H^1_{\text{et}})_{\nu} \) is necessarily the identity.

Conversely, suppose that the ordinary filtration of \( e^s H^1_{\text{et}} \) is \( \Lambda \)-linearly and \( \mathcal{G}_{\mathbb{Q}_p} \)-equivariantly split. Applying \( \otimes \mathbb{Z}_p[\Delta/\Delta_r] \) to this splitting gives, thanks to Corollary 1.2.14 and the isomorphism (3.2.29), a \( \mathbb{Z}_p[\mathcal{G}_r] \)-linear splitting of

\[
0 \rightarrow T_p G^m_r \rightarrow T_p G_r \rightarrow T_p G^\text{et}_r \rightarrow 0
\]

which is compatible with change in \( r \) by construction. By \( \Gamma \)-descent and Tate’s theorem, there is a natural isomorphism

\[
\text{Hom}_{p\text{div}_{R_{\Gamma}}}(G^\Gamma_r, \mathcal{S}_r) \simeq \text{Hom}_{\mathbb{Z}_p[\mathcal{G}_r]}(T_p G^\text{et}_r, T_p G_r)
\]

and we conclude that the connected-\( \text{étale} \) sequence of \( \mathcal{S}_r \) is split (in the category \( p\text{div}_{R_{\Gamma}} \)), compatibly with change in \( r \). Due to the functoriality of \( \mathfrak{M}_r(\cdot) \), this in turn implies that the top row of (3.3.1) is split in \( B T_{G_{\infty}}^\Gamma \), compatibly with change in \( r \), which is easily seen to imply the splitting of (3.3.4).

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