UNIFORM BOUNDS FOR RATIONAL POINTS ON HYPERELLIPTIC FIBRATIONS

DANTE BONOLIS AND TIM BROWNING

Abstract. We apply a variant of the square-sieve to produce an upper bound for the number of rational points of bounded height on a family of surfaces that admit a fibration over \( \mathbb{P}^1 \) whose general fibre is a hyperelliptic curve. The implied constant does not depend on the coefficients of the polynomial defining the surface.

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1. Introduction

This paper is concerned with the density of rational points on surfaces \( S \) of the shape

\[
Y^2 = X^n + Xf(U_1, U_2) + g(U_1, U_2),
\]

for appropriate binary forms \( f, g \in \mathbb{Z}[U_1, U_2] \), such that \( \deg(f) = 2n - 2 \) and \( \deg(g) = 2n \). We shall view \( S \) as a degree \( 2n \) surface in the weighted projective space \( \mathbb{P}(n, 2, 1, 1) \), with variables \( (Y, X, U_1, U_2) \). The goal of this paper is to study the counting function

\[
N(S; B) = \# \left\{ (x, y, u_1, u_2) \in \mathbb{Z}^4 : y^2 = x^n + xf(u_1, u_2) + g(u_1, u_2), |x| \leq B^2, |y| \leq B^n, |u_1|, |u_2| \leq B \right\},
\]

which can be interpreted in terms of counting rational points of bounded height in \( S(\mathbb{Q}) \) with respect to the standard exponential height on \( \mathbb{P}(n, 2, 1, 1)(\mathbb{Q}) \). Assuming that \( S \) is smooth, the surface \( S \) admits a fibration \( S \to \mathbb{P}^1 \) whose general fibre is a hyperelliptic curve of genus \( \lceil n/2 - 1 \rceil \). The following is our main result.

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Theorem 1.1. Let $S \subset \mathbb{P}(n, 2, 1, 1)$ be a smooth surface given by the equation (1.1). Assume, furthermore, that $n \geq 3$ is odd and that $g$ is separable. Then

$$N(S; B) \ll_n B^{3-1/20}(\log B)^2,$$

where the implied constant is only allowed to depend on $n$.

Consideration of the partial derivatives of the polynomial in (1.1) shows that $S$ is smooth precisely when there are no roots of the equation

$$n^n g^{n-1} = (n - 1)^{n-1}(-f)^n,$$

for which $\nabla f$ and $\nabla g$ are proportional. In our work it is also necessary to assume that $g$ is separable, which we note is equivalent to the smoothness of $S$ when $f$ is identically zero. The restriction on the parity of $n$ is an artefact of the proof and can be traced to a certain exponential sum estimate (Lemma 2.3), which can fail when $n$ is even.

One way to approach $N(S; B)$ is through work of Bombieri and Pila [2]. For any $a, b \in \mathbb{Z}$ and any choice of $\varepsilon > 0$, this yields

$$\#\{(x, y) \in (\mathbb{Z} \cap [-R, R]^2 : y^2 = x^n + ax + b) = O_{\varepsilon, n}(R^{1/n+\varepsilon}), \quad (1.3)$$

for any $R \geq 1$, where the implied constant is only allowed to depend on the degree $n$ and the choice of $\varepsilon$. An application of this with $R = B^n$ leads to the conclusion that

$$N(S; B) = O_{\varepsilon, n}(B^{3+\varepsilon}). \quad (1.4)$$

Thus our main result saves $1/20$ over this approach.

Theorem 1.1 appears to be new for $n \geq 5$, but a sharper exponent is available when $n = 3$ by using better uniform bounds for counting integer points on elliptic curves. Under a suitable hypothesis on the rank growth of elliptic curves, as explained by Heath-Brown [3], it is possible to conclude that the number of $x, y$ contributing to $N(S; B)$ is $O_{\varepsilon}(B^n)$, for any $\varepsilon > 0$, with an implied constant that only depends on $\varepsilon$. In this way one obtains a conditional upper bound

$$N(S; B) = O_{\varepsilon, S}(B^{2+\varepsilon}),$$

when $n = 3$. While we we don’t yet have access to the desired conjecture on rank growth, it has recently been shown by Bhargava et al [1] Thm. 1.2] that there exists an absolute constant $c > 0$ such that

$$\text{rank}(E) \leq (0.2785) \log_2(|\text{disc}(E)|) + c,$$

for any elliptic curve $E$ in Weierstrass form with integral coefficients. For fixed integers $|u_1|, |u_2| \leq B$, the elliptic curve one gets in (1.1) has discriminant $O(B^{12})$. Once inserted into a bound of Helfgott and Venkatesh [12 Cor. 3.9] for the number of integer points in a box that lie on an elliptic curve of given rank, this yields the estimate

$$N(S; B) = O_{S}(B^{2.87}),$$
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when \( n = 3 \). This is sharper than Theorem 1.1 but has the defect that it depends on the coefficients of \( S \), whereas our result is uniform in the coefficients of \( f \) and \( g \). Prior to this, Mendes Da Costa [17, §8] enacted a similar strategy to achieve the estimate \( N(S; B) = O(B^{3-\delta}) \) for an unspecified \( \delta > 0 \), but with an absolute implied constant. As discussed by Helfgott [11, §3], it seems difficult to extend this strategy to any instance of the surface (1.1) with \( n > 3 \), since the required uniformity in the coefficients of the hyperelliptic curve is harder to come by.

The upper bound in Theorem 1.1 is expected to be very far from the truth. We always have a lower bound \( N(S; B) \gg B \) coming from solutions with \( u_1 = u_2 = 0 \). When \( n = 3 \) the surface \( S \) is a smooth del Pezzo surface of degree 1 over \( \mathbb{Q} \) and Manin’s conjecture [5] predicts an upper bound of the form

\[
N(S; B) = O_S(B^2).
\]

This is best possible when one of the exceptional curves that lie on \( S \), of which there are 240 [16, Chap. IV], is defined over \( \mathbb{Q} \). However, we actually expect linear growth outside the set of such curves, by the Manin conjecture [5].

Our proof of Theorem 1.1 relies on a variant of the square sieve worked out by Pierce [18], which allows for an application of Heath-Brown’s \( q \)-analogue of van der Corput differencing. This approach was already put to use by Heath-Brown and Pierce [10] to study cyclic covers of \( \mathbb{P}^n \) and our proof is inspired by their work. Ultimately, for suitable primes \( p \), the proof of Theorem 1.1 is reduced to estimating a certain 4-variable exponential sum \( W_p = W_p(\lambda, h, \mu) \) defined over \( \mathbb{F}_p \). This sum is found in (2.14). It is fairly easy to get some cancellation in the sum, getting \( W_p = O(p^3) \). In order to improve (1.4) it is critical to get further cancellation, for generic choices of parameters \( \lambda, h, \mu \). While the sum is amenable to an application of work by Katz [15] on singular exponential sums, this doesn’t appear to yield any direct improvement. Instead, by adopting a method of moments expounded by Hooley [13], which we describe in the appendix, we can show that \( W_p = O(p^{5/2}) \) if \( \lambda \neq 0 \) and \((h, \mu) \neq (0, 0)\). It would be very interesting to gauge whether the sum \( W_p \) actually satisfies square-root cancellation, for we would then arrive at a version of Theorem 1.1 in which 1/20 is replaced by 1/8, which would be the limit of our approach.

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## 2. The square sieve

2.1. **Reducing the height of the coefficients.** The implied constant in Theorem 1.1 does not depend on the coefficients of \( f \) or \( g \). In fact we shall follow
the convention that all of the implied constants in the remainder of our paper are only allowed to depend on $n$, unless explicitly indicated otherwise with an appropriate subscript. One important step in achieving uniformity arises through an application of the following result, in which $\|h\|$ is used to mean the maximum of the absolute values of the coefficients of a form $h \in \mathbb{Z}[U_1, U_2]$.

**Lemma 2.1.** Let $S \subset \mathbb{P}(n, 2, 1, 1)$ be given by (1.1). Then either

$$\max\{|f|, |g|\} \ll B^{8n^2 + 4n}$$

or $N(S; B) = O_\varepsilon(B^{2+\varepsilon})$ for any $\varepsilon > 0$.

**Proof.** This argument is a variant of one given by Heath-Brown [9, Thm. 4], but we include full details for the sake of completeness. We shall be interested in polynomials formed from linear combinations of monomials belonging to the set

$$\mathcal{E} = \{Y^2\} \cup \{X^n\} \cup \{XU_1^eU_2^f : e_1 + e_2 = 2n - 2\} \cup \{U_1^{e_1}U_2^{e_2} : e_1 + e_2 = 2n\}.$$

We clearly have $\#\mathcal{E} = 4n + 2$. Let $v = (y, x, u_1, u_2)$ and let $\{v_1, \ldots, v_N\}$ be the set of all points that are counted in $N(S; B)$. We construct the $N \times (4n + 2)$ matrix

$$\mathbf{C} = (v_1^r)_{1 \leq r \leq N, e \in \mathcal{E}},$$

whose $i$th row consists of the $4n + 2$ possible monomials in $\mathcal{E}$ in the variables $x_i, y_i, u_{1,i}, u_{2,i}$. The matrix $\mathbf{C}$ has rank at most $4n + 1$, since the vector $a \in \mathbb{Z}^{4n+2}$ whose entries correspond to the coefficients of (1.1) is such that $\mathbf{C}a = 0$. We observe that $a$ is a primitive vector since its first entry is $\pm 1$.

Since $\mathbf{C}$ is not of full rank, the equation $\mathbf{C}b = 0$ has a non-zero solution constructed from the sub-determinants of $\mathbf{C}$. In particular, $|b| = O(B^{8n^2 + 4n})$ since each entry of $\mathbf{C}$ has modulus $O(B^{2n})$. There are two cases to consider. Suppose first that $b$ and $a$ are proportional. Then $|a| \leq |b| \ll B^{8n^2 + 4n}$, since $a$ is a primitive vector. Alternatively, if $b$ is not a multiple of $a$, we let $T \subset \mathbb{P}(n, 2, 1, 1)$ be the surface $B(Y, X, U_1, U_2) = 0$, say, corresponding to the vector $b$. Then $S \cap T$ has dimension 1 and we claim that it has at most $4n^2$ irreducible components. To see this we introduce the morphism $\mathbb{P}^3 \to \mathbb{P}(n, 2, 1, 1)$, given by $[z_0, z_1, z_2, z_3] \mapsto [z_0^2, z_1^2, z_2, z_3]$. Then the number of irreducible components of $S \cap T$ is bounded by the number of components of the intersection of

$$Z_0^{2n} = Z_1^{2n} + Z_1^2 f(Z_2, Z_3) + g(Z_2, Z_3)$$

with $B(Z_0^n, Z_1^2, Z_2, Z_3) = 0$. But this has at most $4n^2$ irreducible components, on applying the version of Bézout’s Theorem in Fulton [6, Example 8.4.6].

Let $Y$ be an irreducible component of $S \cap T$. Suppose there exists a primitive vector $(\mu_1, \mu_2) \in \mathbb{Z}^2$ such that $Y$ is contained in the plane $\mu_2 U_1 = \mu_1 U_2$. We fix such a vector and then simply count how many vectors counted by $N(S; B)$ also satisfy $\mu_2 u_1 = \mu_1 u_2$. Assume that $\mu_1 \neq 0$. Then this quantity is bounded by the
number of vectors \((y, x, u) \in \mathbb{Z}^3\) with
\[
\mu_1^{2n} y^2 = \mu_1^{2n} x^n + \mu_1^2 x f(\mu_1, \mu_2) u_1^{2n-2} + g(\mu_1, \mu_2) u_2^{2n}.
\]

For each \(u_1\), we may appeal to the Bombieri–Pila bound (1.3) to get \(O(\epsilon (B_1 + \epsilon))\) possibilities for \(x, y\), for any \(\epsilon > 0\). This case therefore gives an overall contribution of \(O(\epsilon (B_2 + \epsilon))\). Next, we may suppose that \(Y \nsubseteq P\) for every plane \(P \subseteq \mathbb{P}(n, 2, 1, 1)\) with equation \(\mu_2 U_1 = \mu_1 U_2\), as \((\mu_1, \mu_2) \in \mathbb{Z}^2\) runs over primitive vectors. In particular \(#(Y \cap P) = O(1)\) by Bézout’s theorem. Since any non-zero vector \((u_1, u_2) \in \mathbb{Z}^2\) with \(|u_1|, |u_2| \leq B\) satisfies the equation defining \(P\) for at least one primitive vector \((\mu_1, \mu_2) \in \mathbb{Z}^2\) with norm at most \(B\), we easily obtain a contribution of \(O(B^2)\) in this case. This completes the proof of the lemma.

The surface \(S\) has a discriminant \(D_S\) that is an integer polynomial in the coefficients of \(f\) and \(g\), and which vanishes precisely when \(S\) is singular. \(D_S\) can be calculated using elimination theory, by following the arguments in [7]. Let \(\Delta f, g\) be the absolute value of the product of \(D_S\) and the discriminant of the binary form \(g\).

Then \(\Delta f, g\) is an integer, which is a polynomial in the coefficients of \(f\) and \(g\), and which vanishes precisely when the surface \(S\) is singular, or when \(g\) has a repeated root. Taken together with our hypotheses in Theorem 1.1, Lemma 2.1 allows us to proceed under the assumption that \(\Delta f, g\) is a positive integer such that
\[
\log \Delta f, g = O(\log B),
\]
where we recall our convention that the implied constant in any estimate is allowed to depend on \(n\). For any prime \(p\), if \(p \nmid \Delta f, g\) then the reduction of \(S\) modulo \(p\) is smooth and the reduction modulo \(p\) of \(g\) has no repeated roots.

2.2. Application of the sieve. We shall prove Theorem 1.1 using the variant of Heath-Brown’s square sieve introduced in [18]. This sieve offers a great deal of flexibility in the sieving set of primes, which we shall take advantage of here. We shall estimate \(N(S; B)\) by sieving for squares in the non-negative sequence
\[
\omega(m) = \# \left\{ (x, u_1, u_2) \in \mathbb{Z}^3 : m = x^n + x f(u_1, u_2) + g(u_1, u_2) \mid \frac{x}{B^2}, |u_1|, |u_2| \leq B \right\}.
\]

Let \(P, Q \geq 1\) be parameters depending on \(B\) which are to be determined in due course. For now we shall merely assume that
\[
Q \leq \sqrt{B} \leq P \leq B, \quad PQ \geq B. \tag{2.2}
\]

It is now time to reveal the sieve. Let
\[
\mathcal{P} = \{ p \text{ prime} : p \equiv 2 \mod n, p \nmid \Delta f, g \text{ and } P \leq p \leq 2P \},
\]
\[
\mathcal{Q} = \{ q \text{ prime} : Q \leq q \leq 2Q \}
\]
and
\[
\mathcal{A} = \{ p \cdot q : (p, q) \in \mathcal{P} \times \mathcal{Q} \}.
\]
On assuming $B \geq 4n^2$ we clearly have $p \nmid 2n$ for any $p \in \mathcal{P}$. According to Pierce [18, Lemma 2.1], we have

$$\sum_m \omega(m^2) \ll \frac{1}{\# \mathcal{S}} \sum_m \omega(m) + \frac{1}{\# \mathcal{S}^2} \sum_{p, p' \in \mathcal{P}} \sum_{q, q' \in \mathcal{Q}} \left| \sum_m \omega(m) \left( \frac{m}{pq} \right) \left( \frac{m}{p'q'} \right) \right|$$

$$+ \frac{\# \mathcal{Q}}{\# \mathcal{S}^2} \sum_{p, p' \in \mathcal{P}} \left| \sum_m \omega(m) \left( \frac{m}{pp'} \right) \right| + \frac{1}{\# \mathcal{S}^2} |E(\mathcal{P})|$$

$$+ \frac{\# \mathcal{P}}{\# \mathcal{S}^2} \sum_{q, q' \in \mathcal{Q}} \left| \sum_m \omega(m) \left( \frac{m}{qq'} \right) \right| + \frac{1}{\# \mathcal{S}^2} |E(\mathcal{Q})|,$$

where

$$E(\mathcal{P}) = \sum_{q \in \mathcal{Q}} \sum_{p, p' \in \mathcal{P}} \sum_m \omega(m) \left( \frac{m}{pp'} \right), \quad E(\mathcal{Q}) = \sum_{p \in \mathcal{P}} \sum_{q, q' \in \mathcal{Q}} \sum_m \omega(m) \left( \frac{m}{qq'} \right). \tag{2.3}$$

For the first sum on the right hand side we trivially have $\sum_m \omega(m) \ll B^4$. Next, we clearly have

$$\# \mathcal{P} \gg \#\{ p \text{ prime} : p \equiv 2 \mod n \text{ and } P \leq p \leq 2P \} - \omega(\Delta_{f,g})$$

$$\gg \frac{P}{\log B},$$

since $P \leq B^2$ and (2.1) ensures that $\omega(\Delta_{f,g}) \ll \log \Delta_{f,g} \ll \log B$. Moreover, $\# \mathcal{Q} \gg Q/\log B$, since $Q \leq \sqrt{B}$. Hence, in view of (2.2), we have

$$N(S; B) \ll \frac{B^4 (\log B)^2}{PQ} + \frac{1}{\# \mathcal{S}^2} \sum_{p, p' \in \mathcal{P}} \sum_{q, q' \in \mathcal{Q}} |C(pp'qq')|$$

$$+ \frac{\log B}{Q\# \mathcal{S}^2} \sum_{p, p' \in \mathcal{P}} |C(pp')| + \frac{1}{\# \mathcal{S}^2} |E(\mathcal{P})|$$

$$+ \frac{\log B}{P\# \mathcal{S}^2} \sum_{q, q' \in \mathcal{Q}} |C(qq')| + \frac{1}{\# \mathcal{S}^2} |E(\mathcal{Q})|, \tag{2.4}$$
where
\[
C(r) = \sum_{m \text{ square-free}} \omega(m) \left( \frac{m}{r} \right)
\]
\[
= \sum_{\frac{|u_1|, |u_2| \leq B}{|x| \leq B^2}} \left( \frac{x^n + xf(u_1, u_2) + g(u_1, u_2)}{r} \right),
\]
for any square-free \( r \in \mathbb{N} \).

2.3. The main oscillatory sum. This section is devoted to bounding the sum \( C(r) \), as defined in (2.5), for various choices of square-free \( r \in \mathbb{N} \). We have
\[
C(r) = \sum_{\frac{|u_1|, |u_2| \leq B}{|x| \leq B^2}} \sum_{\alpha \mod r} \left( \frac{\alpha^n + \alpha f(u_1, u_2) + g(u_1, u_2)}{r} \right)
\]
\[
\times \sum_{\frac{|x| \leq B^2}{|c| \leq 1}} \frac{1}{r} \sum_{c=1}^{r} e_r(c(\alpha - x)),
\]
on using additive characters to detect the congruence. Define
\[
S(r, c, u_1, u_2) = \sum_{\alpha \mod r} \left( \frac{\alpha^n + \alpha f(u_1, u_2) + g(u_1, u_2)}{r} \right) e_r(c\alpha)
\]
(2.6)
and
\[
U(r, c, B) = \sum_{\frac{|u_1|, |u_2| \leq B}{}} S(r, c, u_1, u_2).
\]

Then we deduce that
\[
C(r) \ll \frac{1}{r} \sum_{c=1}^{r} \min \left( B^2, \left\| \frac{c}{r} \right\|^{-1} \right) \left| U(r, c, B) \right|. \tag{2.7}
\]
The exponential sum \( S(r, c, u_1, u_2) \) in (2.6) satisfies the following basic multiplicativity property.

**Lemma 2.2.** Assume that \( r = r_0 r_1 \) with \( \gcd(r_0, r_1) = 1 \). Then
\[
S(r, c, u_1, u_2) = S(r_0, c\overline{r}_1, u_1, u_2) S(r_1, c\overline{r}_0, u_1, u_2),
\]
where \( r_1 \overline{r}_1 \equiv 1 \mod r_0 \) and \( r_0 \overline{r}_0 \equiv 1 \mod r_1 \).

**Proof.** This result follows easily on using the Chinese remainder theorem to note that \( \alpha_1 r_0 + \alpha_0 r_1 \) runs through all residue classes modulo \( r \) as \( \alpha_0 \) runs through
residue classes modulo $r_0$ and $\alpha_1$ runs through residue classes modulo $r_1$. Thus

$$S(r, c, u_1, u_2) = \left( \sum_{\alpha_0 = 1}^{r_0} \left( \frac{(\alpha_0 r_1)^n + (\alpha_0 r_1) f(u_1, u_2) + g(u_1, u_2)}{r_0} \right) e_{r_0}(\alpha_0) \right) \times \left( \sum_{\alpha_1 = 1}^{r_1} \left( \frac{(\alpha_1 r_0)^n + (\alpha_1 r_0) f(u_1, u_2) + g(u_1, u_2)}{r_1} \right) e_{r_1}(\alpha_1) \right)$$

$$= S(r_0, c r_1, u_1, u_2) S(r_1, c r_0, u_1, u_2),$$
on making a change of variables.

We shall need to bound the exponential sum $S(r, c, u_1, u_2)$ in (2.6), for given $u \in \mathbb{Z}^2$. Lemma 2.2 ensures that it suffices to look at prime values of $r$, in which setting the following result demonstrates that square-root cancellation occurs.

**Lemma 2.3.** Let $p$ be a prime and let $n$ be odd. For any $a, b, c \in \mathbb{F}_p$ there is a constant $C_n > 0$ such that

$$\left| \sum_{x \in \mathbb{F}_p} \left( \frac{x^n + ax + b}{p} \right) e_p(cx) \right| \leq C_n \sqrt{p}.$$

**Proof.** On adjusting $C_n$ we can assume that $p$ is an odd prime. We start by observing that if $\theta \in \mathbb{F}_p$ is a root $T^n + aT + b$ of multiplicity $r$, then

$$T^n + aT + b = \Phi_{\theta}(T)^r \Psi(T),$$

where $\Phi_{\theta}(T)$ is the minimal polynomial of $\theta$ over $\mathbb{F}_p$ and $\Psi(T) \in \mathbb{F}_p[T]$ is such that $\Psi(\theta) \neq 0$. We claim that there exist polynomials $h_1, h_2 \in \mathbb{F}_p[T]$ such that $T^n + aT + b = h_1 h_2^2$. If $r$ is even this is obvious with $h_1 = \Psi$ and $h_2 = \Phi_{\theta}^{r/2}$. If $r = 2k + 1$ is odd, then we take $h_1 = \Psi \Phi_{\theta}$ and $h_2 = \Phi_{\theta}^k$. It is clear that $h_1$ is non-constant, since we are assuming $n$ to be odd. Moreover, we can assume that $h_1$ separable, since any square factors can be absorbed into the term $h_2^2$. For any $x$ such that $h_2(x) \neq 0$, we have

$$\left( \frac{x^n + ax + b}{p} \right) = \left( \frac{h_1(x) h_2(x)^2}{p} \right) = \left( \frac{h_1(x)}{p} \right).$$

Thus

$$\sum_{x \in \mathbb{F}_p} \left( \frac{x^n + ax + b}{p} \right) e_p(cx) = \sum_{x \in \mathbb{F}_p, h_2(x) \neq 0} \left( \frac{h_1(x)}{p} \right) e_p(cx)$$

$$= \sum_{x \in \mathbb{F}_p} \left( \frac{h_1(x)}{p} \right) e_p(cx) + O(1)$$

$$\leq C_n \sqrt{p},$$
thanks to Theorems 2B and 2G in Schmidt [19, Chapter II]. □

Note that if \( n \) were even the left hand side would be \( p \) if \( a = b = c = 0 \). Thus it is crucial to assume that \( n \) is odd in Lemma 2.3 in order to have a result that applies to all \( a, b, c \in \mathbb{F}_q \). We are now ready to record our first result for \( U(r, c, B) \).

**Lemma 2.4.** Assume that \( n \) is odd and let \( r \in \mathbb{N} \) be square-free. There exists a constant \( C_n > 0 \) depending only on \( n \) such that

\[
U(r, c, B) \leq C_n \omega(r) B^2 r^{1/2}.
\]

**Proof.** This is an easy consequence of Lemmas 2.2 and 2.3. □

The previous estimate will be enough to handle all but the second term in (2.4). To handle the case \( r = pp'qq' \) for distinct primes \( p, p' \in \mathcal{P} \) and \( q, q' \in \mathcal{Q} \), it will be convenient to set \( r_0 = pp' \) and \( r_1 = qq' \). We observe that \( r_0 \asymp P^2 \) and \( r_1 \asymp Q^2 \). Since \( PQ \leq B^{3/2} \) in (2.2) we deduce that the range of summation for \( x \) is \( B^2 \gg \sqrt{r_0r_1} \). Hence it makes sense to complete the summation over \( x \) to all the classes modulo \( r_0r_1 \), as we have done here. The following estimate for \( U(r_0r_1, c, B) \) is obtained using the \( q \)-analogue of the van der Corput inequality.

**Lemma 2.5.** We have

\[
U(r_0r_1, c, B) \ll \begin{cases} 
Br_0^{1/2}r_1^{3/2} + Br_0^{5/4}r_1^{1/2} \log r_0 & \text{if } \gcd(c, r_0) = 1, \\
B^2(r_0r_1)^{1/2} & \text{if } \gcd(c, r_0) > 1.
\end{cases}
\]

The proof of this result will occupy the remainder of this subsection. We start the proof by defining

\[
A(u_1, u_2) = \begin{cases} 
S(r, c, u_1, u_2) & \text{if } |u_1|, |u_2| \leq B, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
A_0(u_1, u_2) = \begin{cases} 
S(r_0, r_1c, u_1, u_2) & \text{if } |u_1|, |u_2| \leq B, \\
0 & \text{otherwise}.
\end{cases}
\]

We define \( A_1(u_1, u_2) \) similarly and we introduce the parameter

\[
H = \left\lfloor \frac{4B}{r_1} \right\rfloor.
\]
Since \( r_1 = qq' \leq 4Q^2 \leq 4B \) by (2.2), we see that \( H \in \mathbb{N} \). We will follow the \( q \)-analogue of the van der Corput method. First of all, we find that

\[
H^2 U(r, c, B) = \sum_{\mathbf{h} \in [1, H]^2} \sum_{\mathbf{u} \in \mathbb{Z}^2} A(\mathbf{u} + h_1 r_1)
\]

\[
= \sum_{\mathbf{u} \in \mathbb{Z}^2} \sum_{\mathbf{h} \in [1, H]^2} A_0(\mathbf{u} + h_1 r_1)A_1(\mathbf{u} + h_1 r_1)
\]

\[
= \sum_{\mathbf{u} \in \mathbb{Z}^2} S(r_1, r_0 c, u_1, u_2) \sum_{\mathbf{h} \in [1, H]^2} A_0(\mathbf{u} + h_1 r_1),
\]

where \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{h} = (h_1, h_2) \). Let \( |\cdot| \) be the sup norm on \( \mathbb{R}^2 \). By the Cauchy–Schwarz inequality we get

\[
H^2|U(r, c, B)| \leq \sqrt{\Sigma_1 \Sigma_2},
\]

where

\[
\Sigma_1 = \max_{\mathbf{h} \in [1, H]^2} \sum_{\mathbf{u} \in \mathbb{Z}^2 \atop |\mathbf{u} + h_1 r_1| \leq B} |S(r_1, r_0 c, u_1, u_2)|^2
\]

and

\[
\Sigma_2 = \sum_{\mathbf{u} \in \mathbb{Z}^2} \left| \sum_{\mathbf{h} \in [1, H]^2} A_0(\mathbf{u} + h_1 r_1) \right|^2.
\]

Moreover,

\[
\Sigma_2 = \sum_{\mathbf{h} \in [1, H]^2} \sum_{\mathbf{j} \in [1, H]^2} \sum_{\mathbf{u} \in \mathbb{Z}^2} A_0(\mathbf{u} + h_1 r_1)A_0(\mathbf{u} + j_1 r_1)
\]

\[
= \sum_{\mathbf{h} \in [1, H]^2} \sum_{\mathbf{j} \in [1, H]^2} \sum_{\mathbf{u} \in \mathbb{Z}^2} A_0(\mathbf{u} + (\mathbf{h} - \mathbf{j}) r_1)A_0(\mathbf{u})
\]

\[
\leq 2H^2 \sum_{\mathbf{h} \in \mathbb{Z}^2 \atop |h| \leq H} \left| \sum_{\mathbf{u} \in \mathbb{Z}^2} A_0(\mathbf{u} + h_1 r_1)A_0(\mathbf{u}) \right|.
\]

We have

\[
\Sigma_2 \leq 2H^2(\Sigma_{2,A} + \Sigma_{2,B}),
\]

where

\[
\Sigma_{2,A} = \sum_{\mathbf{u} \in \mathbb{Z}^2} |A_0(\mathbf{u})|^2
\]

and

\[
\Sigma_{2,B} = \sum_{\mathbf{h} \in \mathbb{Z}^2 \atop 0 < |h| \leq H} \left| \sum_{\mathbf{u} \in \mathbb{Z}^2} A_0(\mathbf{u} + h_1 r_1)A_0(\mathbf{u}) \right|.
\]
The following estimate is enough to complete the treatment of $\Sigma_1$ and $\Sigma_{2,A}$ in (2.9) and (2.11), respectively.

**Lemma 2.6.** We have $\Sigma_1 = O(B^2r_1)$ and $\Sigma_{2,A} = O(B^2r_0)$. 

**Proof.** Appealing to Lemma 2.3 and the multiplicativity property in Lemma 2.2, we deduce that 

$$\Sigma_1 \ll (B + Hr_1)^2r_1 \quad \text{and} \quad \Sigma_{2,A} \ll B^2r_0.$$ 

The lemma follows on noting that $Hr_1 = [B/r_1]r_1 \leq B$. $\Box$ 

We now turn to the estimation of $\Sigma_{2,B}$, as defined in (2.12). We may write 

$$\Sigma_{2,B} = \sum_{h \in \mathbb{Z}/2} |T(r_0, h)|, \quad (2.13)$$ 

where 

$$T(r_0, h) = \sum_{-B \leq u_1 \leq B - h_1r_1} \sum_{-B \leq u_2 \leq B - h_2r_1} S(r_0, c\bar{r}_1, u + hr_1)\overline{S(r_0, c\bar{r}_1, u)}$$ 

$$= \sum_{s_1, s_2 \mod r_0} S(r_0, c\bar{r}_1, s + hr_1)\overline{S(r_0, c\bar{r}_1, s)}$$ 

$$\times \left( \sum_{-B \leq u_1 \leq B - h_1r_1} \frac{1}{r_0} \sum_{k_1 = 1}^{r_0} e_{r_0}(k_1(s_1 - u_1)) \right)$$ 

$$\times \left( \sum_{-B \leq u_2 \leq B - h_2r_1} \frac{1}{r_0} \sum_{k_2 = 1}^{r_0} e_{r_0}(k_2(s_2 - u_2)) \right).$$ 

It follows that 

$$T(r_0, h) \leq \frac{1}{r_0^2} \sum_{k_1, k_2 = 1}^{r_0} \min \left( B, \left\| \frac{k_1}{r_0} \right\|^{-1} \right) \min \left( B, \left\| \frac{k_2}{r_0} \right\|^{-1} \right) |W(k)|,$$ 

where $k = (k_1, k_2)$ and 

$$W(k) = \sum_{\alpha, \beta, s_1, s_2 \mod r_0} \left( \frac{\alpha^n + \alpha f(s + hr_1) + g(s + hr_1)}{r_0} \right)$$ 

$$\times \left( \frac{\beta^n + \beta f(s) + g(s)}{r_0} \right) e_{r_0}(c\bar{r}_1(\alpha - \beta) + k \cdot s).$$ 

Define the exponential sum 

$$W_p(\lambda, h, \mu) = \sum_{\alpha, \beta, s_1, s_2 \mod p} \left( \frac{\alpha^n + \alpha f(s + h) + g(s + h)}{p} \right)$$ 

$$\times \left( \frac{\beta^n + \beta f(s) + g(s)}{p} \right) e_p(\lambda(\alpha - \beta) + \mu \cdot s), \quad (2.14)$$
for $\lambda \in \mathbb{Z}$ and $\mathbf{h}, \mathbf{\mu} \in \mathbb{Z}^2$. It now follows from the Chinese remainder theorem that
\[
W(k) = W_p(\mathbf{c}^T \mathbf{r}, \mathbf{h}^T k \mathbf{r}) W'_p(\mathbf{c}^T \mathbf{r}, \mathbf{h}^T k \mathbf{r}) W_p(\mathbf{c}^T \mathbf{r}, \mathbf{h}^T k \mathbf{r})
\]
since $r_0 = pp'$. Thus our attention shifts to estimating $W_p(\lambda, \mathbf{h}, \mathbf{\mu})$. The trivial bound is $O(p^4)$. The bound $O(p^3)$ follows rather easily from Lemma 2.3. Any non-trivial saving over this bound will yield an improvement over the bound (1.4). Unfortunately we are not able to achieve full square-root cancellation for $W_p(\lambda, \mathbf{h}, \mathbf{\mu})$. The following result summarises our analysis and will be established in Section 3.

**Proposition 2.7.** Let $p \equiv 2 \mod n$ be a prime such that $p \nmid \Delta_{f,g}$. Let $\lambda \in \mathbb{F}_p^\times$. Then
\[
W_p(\lambda, \mathbf{h}, \mathbf{\mu}) \ll p^{5/2} \gcd(p, h_1, h_2, \mu_1, \mu_2)^{1/2},
\]
where the implied constant depends at most on $n$.

We are now ready to produce our final estimate for $\Sigma_{2,B}$, as defined in (2.13).

**Lemma 2.8.** Assume that $\gcd(c, r_0) = 1$. Then $\Sigma_{2,B} = O(H^2 r_0^{5/2} (\log r_0)^2)$.

**Proof.** The assumption $\gcd(c, r_0) = 1$ brings us in line for an application of Proposition 2.7, since $p, p' \equiv 2 \mod n$ for any $p, p' \in \mathcal{P}$. Hence
\[
T(r_0, \mathbf{h}) \ll r_0^{1/2} \sum_{k_1, k_2=1}^{r_0} \min \left( B, \left\| \frac{k_1}{r_0} \right\|^{-1} \right) \min \left( B, \left\| \frac{k_2}{r_0} \right\|^{-1} \right) \gcd(r_0, h_1, h_2, k_1, k_2)^{1/2}.
\]
Inserting this into (2.13), we obtain
\[
\Sigma_{2,B} \ll B^2 r_0^{1/2} \sum_{\substack{\mathbf{h} \in \mathbb{Z}^2 \\ 0 < ||\mathbf{h}|| < H}} \gcd(r_0, h_1, h_2)^{1/2} + B r_0^{1/2} \sum_{\substack{\mathbf{h} \in \mathbb{Z}^2 \\ 0 < ||\mathbf{h}|| < H}} \sum_{k=1}^{r_0-1} \left\| \frac{k}{r_0} \right\|^{-1} \gcd(r_0, h_1, h_2, k)^{1/2} + r_0^{1/2} \sum_{\substack{\mathbf{h} \in \mathbb{Z}^2 \\ 0 < ||\mathbf{h}|| < H}} \sum_{k_1, k_2=1}^{r_0-1} \left\| \frac{k_1}{r_0} \right\|^{-1} \left\| \frac{k_2}{r_0} \right\|^{-1} \gcd(r_0, h_1, h_2, k_1, k_2)^{1/2}.
\]
The third term is plainly
\[ \ll r_0^{1/2} \sum_{k_1,k_2=1}^{r_0/2} \frac{r_0^2}{k_1 k_2} \sum_{0 < |h| \leq H} \gcd(r_0, h_1, k_1, k_2)^{1/2} \]
\[ \ll \sum_{k_1,k_2=1}^{r_0/2} \frac{r_0^{5/2}}{k_1 k_2} \sum_{0 < |h| \leq H} \sum_{d|\gcd(r_0,k_1,k_2)} d^{1/2} \{ h \in \mathbb{Z}^2 : 0 < |h| \leq H \text{ and } d \mid h \} \]
\[ \ll H^2 r_0^{5/2} (\log r_0)^2. \]

Using a similar argument for the remaining two terms, we deduce that
\[ \Sigma_{2,B} \ll H^2 B^2 r_0^{1/2} + H^2 B r_0^{3/2} \log r_0 + H^2 r_0^{5/2} (\log r_0)^2. \]
The lemma follows since \( r_0 \asymp P^2 \geq B \), by (2.2). \( \square \)

We now have everything in place to estimate \( U(r_0 r_1, c, B) \) and so complete the proof of Lemma 2.5. If \( \gcd(c, r_0) > 1 \) we merely apply Lemma 2.4. On the other hand, if \( \gcd(c, r_0) = 1 \) we return to (2.8) and (2.10), in order to deduce that
\[ U(r, c, B) \ll H^{-1} \Sigma_{1,A}^{1/2} (\Sigma_{2,1} + \Sigma_{2,2})^{1/2}. \]
Inserting the bounds for \( \Sigma_{1,A}, \Sigma_{2,1} \) and \( \Sigma_{2,2} \) from Lemmas 2.6 and 2.8
\[ U(r, c, B) \ll H^{-1} \cdot B r_1^{1/2} \cdot \left( B r_0^{1/2} + H r_0^{5/4} \log r_0 \right) \]
\[ \ll \frac{B^2 (r_0 r_1)^{1/2}}{H} + B r_0^{5/4} r_1^{1/2} \log r_0. \]
This therefore completes the proof of Lemma 2.5 since \( H = [B/r_1] \gg B/r_1 \).

### 2.4. Completion of the proof of Theorem 1.1
It is now time to return to the upper bound for \( N(S; B) \) in (2.1). The following lemmas are devoted to dealing with the various terms that appear in this expression.

**Lemma 2.9.** Assume that \( P, Q \) satisfy (2.2). Then
\[ \frac{1}{\# \mathcal{E}^2} \sum_{p,p',q,q' \in \mathcal{E}} \sum_{p,p' \neq q,q'} \# C(pp'qq') \ll \left( B P^3 + B P^{5/2} Q + \frac{B^4}{P P} \right) (\log B)^2. \]

**Proof.** Applying Lemma 2.5 in (2.7), we obtain
\[ C(r_0 r_1) \ll \frac{1}{r_0 r_1} \sum_{c=1}^{r_0 r_1} \min_{\gcd(c, r_0) = 1} \left( B^2, \left\| \frac{c}{r_0 r_1} \right\|^{-1} \right) \left( B r_0^{1/2} r_1^{3/2} + B r_0^{5/4} r_1^{1/2} \log r_0 \right) \]
\[ + \frac{1}{r_1 r_0} \sum_{c=1}^{r_0 r_1} \min_{\gcd(c, r_0) > 1} \left( B^2, \left\| \frac{c}{r_0 r_1} \right\|^{-1} \right) \left( \frac{1}{r_0 r_1} \right)^{1/2} \]

The proof of Lemma 2.5 in (2.7) follows.
The first term is
\[
\ll \sum_{c=1}^{r_0r_1-1} \frac{1}{c_0} \left( Br_0^{1/2}r_1^{3/2} + Br_0^{5/4}r_1^{1/2} \log r_0 \right)
\ll \left( r_0^{1/2}r_1^{3/2} + r_0^{5/4}r_1^{1/2} \right) B(\log B)^2,
\]

since \( PQ \leq B^2 \) by (2.2). The second term is
\[
\ll \frac{B^4}{(r_0r_1)^{1/2}} + B^2(r_0r_1)^{1/2} \sum_{\substack{c=1 \\gcd(c,r_0) > 1}}^{r_0r_1-1} \frac{1}{c}.
\]

But \( r_0 = pp' \) and so
\[
\sum_{\substack{c=1 \\gcd(c,r_0) > 1}}^{r_0r_1-1} \frac{1}{c} \ll \frac{1}{p'} \sum_{c'=1}^{r_1-1} \frac{1}{c'} + \frac{1}{p} \sum_{c''=1}^{r_1-1} \frac{1}{c''} \ll \frac{(\log r_0r_1)^2}{r_0^{1/2}}.
\]

We conclude that
\[
C(r_0r_1) \ll \left( r_0^{1/2}r_1^{3/2} + r_0^{5/4}r_1^{1/2} + \frac{B^3}{(r_0r_1)^{1/2}} + Br_1^{1/2} \right) B(\log B)^2
\]

We now recall that \( r_0 \approx P^2 \) and \( r_1 \approx Q^2 \). This readily yields
\[
C(r_0r_1) \ll \left( BPQ^3 + BP^{5/2}Q + \frac{B^4}{PQ} + B^2Q \right) (\log B)^2.
\]

When \( P, Q \) are constrained to satisfy (2.2) it is clear that \( B^2Q \leq BPQ^3 \). The statement of the lemma is now obvious. \( \square \)

**Lemma 2.10.** Assume that \( P, Q \) satisfy (2.2). Then
\[
\frac{\log B}{Q\# \mathcal{P}^2} \sum_{p, p' \in \mathcal{P}, \ p \neq p'} |C(pp')| \ll \frac{B^4(\log B)^2}{PQ}
\]

and
\[
\frac{\log B}{P\# \mathcal{P}^2} \sum_{q, q' \in \mathcal{Q}, \ q \neq q'} |C(qq')| \ll \frac{B^4(\log B)^2}{PQ}.
\]

**Proof.** We apply Lemma 2.4 in (2.7) to obtain
\[
C(pp') \ll \frac{B^2(p'p)^{1/2}}{pp'} \left( B^2 + \sum_{c=1}^{p'-1} \frac{pp'}{c} \right) \ll \frac{B^4}{P} + B^2P \log P,
\]

since \( p, p' \approx P \). Since \( P \leq B \) in (2.2) we see that the \( B^2P \log P \ll (B^4 \log B)/P \) and the first part of the lemma easily follows. The second part is similar. \( \square \)
Lemma 2.11. Assume that \( P, Q \) satisfy (2.2). Then
\[
\frac{1}{\# \mathcal{F}} |E(\mathcal{P})| \ll \frac{\log B}{Q} \left( \frac{B^4}{PQ} + B^2 P^2 \right)
\]
and
\[
\frac{1}{\# \mathcal{F}} |E(\mathcal{Q})| \ll \frac{\log B}{P} \left( \frac{B^4}{PQ} + B^2 Q^2 \right).
\]

Proof. We prove the first estimate, the second following by symmetry. Recall from (2.3) that
\[
E(\mathcal{P}) = \sum_{q \in \mathcal{D}} \sum_{p, p' \in \mathcal{D}} \sum_{m} \omega(m) \left( \frac{m}{pp'} \right)
\]
\[
= \sum_{q \in \mathcal{D}} \sum_{p, p' \in \mathcal{D}} \sum_{|x| \leq B, |u_1|, |u_2| \leq B} \sum_{\alpha \in \mathbb{F}_q} \frac{\left( x^n + xf(u_1, u_2) + g(u_1, u_2) \right)}{pp'}.
\]
Breaking the \( x \)-sum into residue classes modulo \( q \), we obtain
\[
E(\mathcal{P}) = \sum_{q \in \mathcal{D}} \sum_{p, p' \in \mathcal{D}} \sum_{|x| \leq B, |u_1|, |u_2| \leq B} \sum_{\alpha \in \mathbb{F}_q} D(\alpha, u),
\]
where
\[
D(\alpha, u) = \sum_{m \in \mathbb{Z}} \left( \frac{(\alpha + mq)^n + (\alpha + mq)f(u) + g(u)}{pp'} \right).
\]
Breaking the \( m \)-sum into residue classes modulo \( pp' \), we obtain
\[
D(\alpha, u) = \sum_{\beta = 1}^{pp'} \left( \frac{(\alpha + \beta q)^n + (\alpha + \beta q)f(u) + g(u)}{pp'} \right) L(\beta),
\]
where \( L(\beta) \) is the number of \( m \in \mathbb{Z} \) for which \( \beta \equiv \alpha \mod pp' \). Clearly
\[
L(\beta) = \frac{B^2}{pp'q} + O(1).
\]
Observing that \( q \) is coprime to \( pp' \), it therefore follows from Lemma 2.3 that
\[
D(\alpha, u) \ll \frac{B^2}{(pp')^{1/2}q} + pp'.
\]
Since \( pp' \approx P^2 \) and \( q \approx Q \) it now easily follows
\[
E(\mathcal{P}) \ll \sum_{q \in \mathcal{D}} \sum_{p, p' \in \mathcal{D}} \left( \frac{B^4}{PQ} + B^2 P^2 \right).
\]
The statement of the lemma is now obvious. \( \square \)
It is finally time to combine Lemmas 2.9–2.11 in (2.4) and optimise our choice of parameters $P, Q$, in order to complete the proof of Theorem 1.1. Recalling that $Q \leq P$ in (2.2), we deduce that

$$N(S; B) \ll \left( \frac{B^4}{PQ} + \frac{B^2P^2}{Q} + BPQ^3 + BP^{5/2}Q \right) (\log B)^2.$$ 

The statement of Theorem 1.1 follows on taking $P = B^{3/5}$ and $Q = B^{9/20}$, and noting that these values clearly satisfy the constraints outlined in (2.2).

3. Estimation of the key character sum

Let $p$ be a prime such that $p \nmid 2n\Delta_{f,g}$. For any $\lambda \in \mathbb{F}_p$ and $h, \mu \in \mathbb{F}_p^2$ we recall that the exponential sum in (2.14) is defined to be

$$W_p(\lambda, h, \mu) = \sum_{\alpha, \beta, s_1, s_2 \mod p} \left( \frac{\alpha^n + \alpha f(s + h) + g(s + h)}{p} \right) \times \left( \frac{\beta^n + \beta f(s) + g(s)}{p} \right) e_p(\lambda(\alpha - \beta) + \mu \cdot s).$$

Here $f, g \in \mathbb{Z}[S_1, S_2]$ are two homogeneous polynomials of degrees $2n - 2$ and $2n$ respectively, such that $n$ is odd and $g$ is separable, and such that

$$Y^2 = X^n + X f(S_1, S_2) + g(S_1, S_2)$$

defines a smooth surface in $\mathbb{P}(n, 2, 1, 1)$. Our assumption that $p \nmid 2n\Delta_{f,g}$ ensures that the reduction modulo $p$ is also smooth and that the reduction modulo $p$ of $g$ is separable. Our task in this section is to establish Proposition 2.7. The estimate

$$W_p(\lambda, h, \mu) = O(p^3)$$

is an easy consequence of Lemma 2.3 which therefore handles the case $h = \mu = 0$. It remains to prove the following result.

**Proposition 3.1.** Let $p \equiv 2 \mod n$ and let $p \nmid 2n\Delta_{f,g}$. Assume that $\lambda \in \mathbb{F}_p^\times$ and $h, \mu \in \mathbb{F}_p^2$, with $(h, \mu) \neq (0, 0)$. Then there exists a constant $C_n > 0$ such that

$$|W_p(\lambda, h, \mu)| \leq C_n p^{5/2}.$$

The same estimate holds for $W_p(\lambda, h, \mu)$ for any prime $p$, but the restriction $p \equiv 2 \mod n$ makes the proof notationally less cumbersome. We have not been able to apply existing results in the literature to deduce Proposition 3.1. However, after some preliminary manoeuvres we shall bring the sum into a form that can be handled by work of Katz [15, Theorem 4]. Unfortunately, as we shall discuss in Section 3.1, the relevant varieties are too singular to extract any improvement over the bound $O(p^3)$ from Katz. Thus we shall adopt an alternative course of action to arrive at Proposition 3.1.
Since \( \lambda \neq 0 \), our first move is to observe that
\[
W_p(\lambda, h, \mu) = \sum_{x=(u,v,x,y,s_1,s_2) \in \mathbb{F}_p^6 \atop G_1(x)=G_2(x)=0} e_p(\lambda(x - y) + \mu \cdot s),
\]
for polynomials \( G_1, G_2 \in \mathbb{F}_p[U, V, X, Y, S_1, S_2] \) given by
\[
\begin{align*}
G_1 &= -U^2 + X^n + X f(S_1, S_2) + g(S_1, S_2), \\
G_2 &= -V^2 + Y^n + Y f(S_1 + h_1, S_2 + h_2) + g(S_1 + h_1, S_2 + h_2).
\end{align*}
\]
It will be more convenient to transform \( W_p(\lambda, h, \mu) \) into a sum in which the monomials involving \( U, V, X, Y \) have degree 2. This is achieved in the following result.

**Lemma 3.2.** Assume that \( p \equiv 2 \mod n \) and let \( \gamma \in \mathbb{F}_p^\times \) be a non-square. Then we have
\[
W_p(\lambda, h, \mu) = \frac{1}{4} \sum_{i,j \in \{0,1\}} W_{p,i,j}(\lambda, h, \mu),
\]
where
\[
W_{p,i,j}(\lambda, h, \mu) = \sum_{x \in \mathbb{F}_p^6 \atop G_1^{(i)}(x)=G_2^{(j)}(x)=0} e_p(\lambda(\gamma^i x^2 - \gamma^j y^2) + \mu \cdot s),
\]
for \( i, j \in \{0,1\} \), with
\[
\begin{align*}
G_1^{(i)} &= -U^{2n} + \gamma^{ni} X^{2n} + \gamma^i X^2 f(S_1, S_2) + g(S_1, S_2), \\
G_2^{(j)} &= -V^{2n} + \gamma^{nj} Y^{2n} + \gamma^j Y^2 f(S_1 + h_1, S_2 + h_2) + g(S_1 + h_1, S_2 + h_2).
\end{align*}
\]

**Proof.** Since \( p \equiv 2 \mod n \) we have that \( \gcd(n, p-1) = 1 \) and then every element of \( \mathbb{F}_p \) is a \( n \)-th power in \( \mathbb{F}_p \). Next, recall that \( \mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} = \{\pm 1\} \) and let \( \gamma \in \mathbb{F}_p^\times \) be a non-square. If \( \alpha \in \mathbb{F}_p \), then there exists \( a \in \mathbb{F}_p \) such that either \( \alpha = a^2 \) or \( \alpha = \gamma a^2 \). The statement of the lemma is now clear. \( \square \)

### 3.1. Comparison with work of Katz.

In this short section we take a moment to check what comes out of applying general work by Katz \[15\] on singular exponential sums. We can recognise our exponential sum \( W_{p,i,j}(\lambda, h, \mu) \) as the exponential sum considered in \[15\] Theorem 4. Let \( X \subset \mathbb{F}_p^6 \) be the geometrically integral complete intersection
\[
\begin{align*}
0 &= -U^{2n} + \gamma^{ni} X^{2n} + \gamma^i X^2 f(S_1, S_2) + g(S_1, S_2), \\
0 &= -V^{2n} + \gamma^{nj} Y^{2n} + \gamma^j Y^2 f(S_1 + h_1, S_2 + h_2) + g(S_1 + h_1, S_2 + h_2).
\end{align*}
\]
Let \( L \) be the hyperplane \( T = 0 \) and let \( H \) be the hypersurface
\[
\lambda(\gamma^i X^2 - \gamma^j Y^2) + (\mu \cdot S)T = 0.
\]
Then \( W_{p,i,j}(\lambda, \mathbf{h}, \mathbf{\mu}) \) precisely matches the exponential sum considered in \[15, \text{Theorem 4}] , with \( V = X[1/T] \) and \( f \) being given by the function 
\[
(\lambda(\gamma^i X^2 - \gamma^j Y^2) + (\mathbf{\mu} \cdot \mathbf{S})T)/T^2.
\]

Let \( \delta = \dim \text{Sing}(X \cap L \cap H) \). In the most favourable situation, when \( \delta \geq \dim \text{Sing}(X \cap L) \), it follows from \[15, \text{Theorem 4(1)}\] that
\[
W_{p,i,j}(\lambda, \mathbf{h}, \mathbf{\mu}) \ll p^{(5+\delta)/2}.
\]

But \( X \cap L \cap H \) is cut out by the system
\[
\begin{aligned}
0 &= -U^{2n} + X^{2n} + X^2 f(S_1, S_2) + g(S_1, S_2), \\
0 &= -V^{2n} + Y^{2n} + Y^2 f(S_1, S_2) + g(S_1, S_2), \\
0 &= \gamma^i X^2 - \gamma^j Y^2, \\
0 &= T.
\end{aligned}
\]

The Jacobian matrix of this system is
\[
\begin{pmatrix}
-2nU^{2n-1} & 0 & 0 & 0 \\
0 & -2nV^{2n-1} & 0 & 0 \\
2nX^{2n-1} + 2X f & 0 & 2\gamma^i X & 0 \\
0 & 2nY^{2n-1} + 2Y f & -2\gamma^j Y & 0 \\
\frac{\partial g}{\partial S_1} + X^2 \frac{\partial f}{\partial S_1} & \frac{\partial g}{\partial S_2} + Y^2 \frac{\partial f}{\partial S_2} & 0 & 0 \\
\frac{\partial g}{\partial S_1} + X^2 \frac{\partial f}{\partial S_1} & \frac{\partial g}{\partial S_2} + Y^2 \frac{\partial f}{\partial S_2} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The set of points such that the third column vanishes satisfies the system
\[
\begin{aligned}
U^{2n} &= g(S_1, S_2), \\
V^{2n} &= g(S_1, S_2), \\
X &= Y = T = 0,
\end{aligned}
\]
which has dimension 1. Thus \( \delta \geq 1 \) and \[15, \text{Theorem 4}\] will not yield an improvement over the bound \( O(p^3) \).

3.2. Strategy for proving Proposition 3.1. The goal of this section is to prove Proposition 3.1 subject to an estimate for the dimension of the singular locus of a certain variety that will be examined in the next section. We shall always assume that \( \lambda \neq 0 \) and \( (\mathbf{h}, \mathbf{\mu}) \neq (0, 0) \).

We assume that \( p \equiv 2 \mod n \) and \( p \nmid 2n\Delta_{f,g} \). Let \( \gamma \in \mathbb{F}_p^\times \) be a non-square. Our first move is an application of Lemma 3.2 rendering it sufficient to study \( W_{p,i,j}(\lambda, \mathbf{h}, \mathbf{\mu}) \), for \( i, j \in \{0, 1\} \). We shall apply a method of Hooley \[13\] to do so, the outcome of which we have recorded as Theorem A.1 in the appendix. This requires us to estimate
\[
N(\tau) = \# \left\{ \mathbf{x} = (u, v, x, y, s_1, s_2) \in \mathbb{F}_q^6 : G_1^{(i)}(\mathbf{x}) = G_2^{(j)}(\mathbf{x}) = 0, \lambda(\gamma^ix^2 - \gamma^jy^2) + \mathbf{\mu} \cdot \mathbf{s} = \tau \right\},
\]
where \( q = p^r \). Since \( Y \) only appears to even degree in \( G_2^{(j)} \), we can eliminate \( y \) from \( G_2^{(j)}(x) \) by writing \( \gamma^j y^2 = \gamma^i x^2 + \lambda \mu \cdot s - \lambda t \), where \( \lambda \) is the inverse of \( \lambda \) in \( \mathbb{F}_q^\times \). This yields

\[
N(\tau) = \# \left\{ y = (u, v, x, s_1, s_2) \in \mathbb{F}_q^5 : G_1^{(i)}(y) = G_2^{(i)}(y) = 0 \right\},
\]

where

\[
G_2^{(i)} = -V^{2n} + (\gamma^i X^2 + \lambda(\mu \cdot S) - \lambda \tau)^n \\
+ (\gamma^i X^2 + \lambda(\mu \cdot S) - \lambda \tau)f(S_1 + h_1, S_2 + h_2) + g(S_1 + h_1, S_2 + h_2).
\]

The polynomial \( G_1^{(i)} \) is homogenous of degree \( 2n \). We shall need to homogenise the polynomial \( G_2^{(i)} \), which we do by introducing a sum over \( i \in \mathbb{F}_q^\times \) and making an obvious change of variables. The resulting polynomial \( H_2^{(i)} \in \mathbb{F}_q[V, X, S_1, S_2, T] \) is given by

\[
H_2^{(i)} = -V^{2n} + (\gamma^i X^2 + \lambda T(\mu \cdot S) - \lambda \tau T^2)^n \\
+ (\gamma^i X^2 + \lambda T(\mu \cdot S) - \lambda \tau T^2)f(S_1 + h_1 T, S_2 + h_2 T) \\
+ g(S_1 + h_1 T, S_2 + h_2 T).
\]

To ease notation we henceforth suppress the index \( i \) from our notation, setting \( G_1^{(i)} = G \) and \( H_2^{(i)} = H_\tau \). We are now led to the expression

\[
N(\tau) = \frac{1}{q - 1} (N_1(\tau) - N_2(\tau)),
\]

where

\[
N_1(\tau) = \# \left\{ (y, t) \in \mathbb{F}_q^6 : G(y, t) = H_\tau(y, t) = 0 \right\},
\]

and

\[
N_2(\tau) = \# \left\{ y \in \mathbb{F}_q^5 : G(y) = H_\tau(y, 0) = 0 \right\},
\]

where \( y = (u, v, x, s_1, s_2) \), as before.

In order to estimate \( N_1(\tau) \) we shall need to know about the singular locus of the complete intersection cut out by the two polynomials \( G \) and \( H_\tau \). This is summarised in the following result.

**Proposition 3.3.** Assume that \( \lambda \neq 0 \) and \( (h, \mu) \neq (0, 0) \). For all but at most \( 32n^5 \) choices of \( \tau \in \mathbb{F}_q \), the equations \( G = H_\tau = 0 \) cut out a complete intersection of codimension \( 2 \) in \( \mathbb{F}_q^5 \), with isolated singularities.

Taking this result on faith for the moment, let us see how it suffices to complete the proof of Proposition 3.1. Appealing to Theorem 1 in the appendix by Katz to [14], it follows from Proposition 3.3 that there exists a set \( U \subset \mathbb{F}_q \), with \( \#U \leq 32n^5 \), such that

\[
\frac{N_1(\tau)}{q - 1} = q^3 + O(q^2)
\]
for all \( \tau \notin U \). When \( \tau \in U \) we invoke the Lang–Weil estimate to deduce that
\[
\frac{N_1(\tau)}{q-1} = q^3 + O(q^{5/2}).
\]
The implied constants in both of these estimates depend only on \( n \). On the other hand, the variety \( G = H_\tau = T = 0 \) has codimension 3 in \( \mathbb{P}^{5}_{\mathbb{F}_q} \). Thus
\[
\frac{N_2(\tau)}{q-1} = O(q^2),
\]
by the Lang–Weil estimate, for a further implied constant that depends only on \( n \). Hence
\[
\sum_{\tau \in \mathbb{F}_q} |N(\tau) - q^3|^2 \leq \sum_{\tau \notin U} q^4 + \sum_{\tau \in U} q^5 \ll q^5,
\]
for an implied constant depending only on \( n \). Theorem A.1 now yields
\[
W_{p,i,j}(\lambda, h, \mu) \ll p^{5/2},
\]
for \( i, j \in \{0, 1\} \). Once inserted into Lemma 3.2, this therefore completes the proof of Proposition 3.1 subject to a verification of Proposition 3.3.

4. The singular locus

This section is devoted to proving Proposition 3.3. Since we are working over \( \mathbb{F}_q \), without loss of generality we may assume that \( i = 0 \). Thus
\[
G = -U^{2n} + X^{2n} + X^2 f(S_1, S_2) + g(S_1, S_2)
\]
and
\[
H_\tau = -V^{2n} + (X^2 + \overline{X}T(\mu \cdot S) - \overline{X}_\tau T^2)^n
\]
\[
+ (X^2 + \overline{X}T(\mu \cdot S) - \overline{X}_\tau T^2)f_h(S_1, S_2, T) + g_h(S_1, S_2, T),
\]
where
\[
f_h(S_1, S_2, T) = f(S_1 + h_1 T, S_2 + h_2 T),
\]
and similarly for \( g_h \). Let us denote by \( V_\tau \subset \mathbb{P}^{5}_{\mathbb{F}_q} \) the variety cut out by the equations \( G = H_\tau = 0 \). It is clearly a complete intersection of codimension 2. Our task is to show that \( \dim \text{Sing}(V_\tau) = 0 \), for all but at most \( 32n^5 \) choices of \( \tau \in \mathbb{F}_q \).

The Jacobian \( J_\tau \) of \( V_\tau \) is given by the matrix
\[
\begin{pmatrix}
-2nU^{2n-1} & 0 & 2nX^{2n-1} + 2X f & X^2 \frac{\partial f}{\partial S_1} + \frac{\partial g}{\partial S_1} & X^2 \frac{\partial f}{\partial S_2} + \frac{\partial g}{\partial S_2} & 0 \\
0 & -2nV^{2n-1} & \frac{\partial H_\tau}{\partial X} & \frac{\partial H_\tau}{\partial S_1} & \frac{\partial H_\tau}{\partial S_2} & \frac{\partial H_\tau}{\partial T}
\end{pmatrix},
\]
where
\[
\frac{\partial H_\tau}{\partial T} = n\overline{X}(\mu \cdot S - 2\tau T)(X^2 + \overline{X}T(\mu \cdot S) - \overline{X}_\tau T^2)^{n-1} + \overline{X}(\mu \cdot S - 2\tau T)f_h
\]
\[
+ (X^2 + \overline{X}T(\mu \cdot S) - \overline{X}_\tau T^2) \frac{\partial f_h}{\partial T} + \frac{\partial g_h}{\partial T},
\]
A point \([y, t] \in \mathbb{P}_F^5\) belongs to \(\text{Sing}(V_\tau)\) if and only if \([y, t] \in V_\tau\) and the matrix \(J_{\tau}\) has rank at most 1 when evaluated at the vector \((y, t)\), where we recall that \(y = (u, v, x, s_1, s_2)\). Note that any point \([y, t] \in \mathbb{P}_F^5\) with \(u = x = s_1 = s_2 = 0\) lies in \(\text{Sing}(V_\tau)\) if and only if
\[
v^{2n} = (-\lambda^n - \lambda f(h) + g(h))t^{2n}.\]

Thus \(\dim \text{Sing}(V_\tau) \geq 0\). It is sufficient to show that \(\dim \text{Sing}(V_\tau) \leq 0\) for all but at most \(32n^5\) values of \(\tau\).

We proceed by breaking \(\text{Sing}(V_\tau) \subset \mathbb{P}_F^5\) into three subsets \(K^{(1)}_\tau \cup K^{(2)}_\tau \cup L_\tau\). Here, \(K^{(1)}_\tau\) is the set of \([y, t] \in \text{Sing}(V_\tau)\) for which the first row in \(J_{\tau}\) vanishes. In other words \(K^{(1)}_\tau\) is cut out by the system of equations
\[
\begin{cases}
G = H_\tau = 0 \\
U = nX^{2n-1} + Xf = X^2\frac{\partial f}{\partial S_1} + \frac{\partial g}{\partial S_1} = X^2\frac{\partial f}{\partial S_2} + \frac{\partial g}{\partial S_2} = 0.
\end{cases}
\]

Likewise, \(K^{(2)}_\tau\) is the set of \([y, t] \in \text{Sing}(V_\tau)\) for which the second row in \(J_{\tau}\) vanishes, so that it is cut out by the system of equations
\[
\begin{cases}
G = H_\tau = 0 \\
V = \frac{\partial H_\tau}{\partial x} = \frac{\partial H_\tau}{\partial S_1} = \frac{\partial H_\tau}{\partial S_2} = \frac{\partial H_\tau}{\partial T} = 0.
\end{cases}
\]

We shall prove the following result.

**Lemma 4.1.** We have \(\dim(K^{(i)}_\tau) \leq 0\) for \(i = 1, 2\).

Finally, let \(L_\tau\) be the set of \([y, t] \in \text{Sing}(V_\tau)\) for which neither row vanishes in \(J_{\tau}\) vanishes. Thus it is cut out by the system of equations
\[
\begin{align*}
0 &= G = H_\tau, \\
0 &= \frac{\partial H_\tau}{\partial T}, \\
0 &= \frac{\partial G}{\partial S_1} - \frac{\partial G}{\partial S_2} \cdot \frac{\partial H_\tau}{\partial S_1}, \\
0 &= U = V.
\end{align*}
\]

We shall prove the following result.

**Lemma 4.2.** We have \(\dim(L_\tau) \leq 0\), for all but at most \(32n^5\) choices of \(\tau \in \mathbb{F}_q\).

Taken together, Lemmas 4.1 and 4.2 complete the proof of Proposition 3.3.

**Proof of Lemma 4.1.** We start by considering \(K^{(2)}_\tau\), which we wish to show has dimension at most 0. To this end it is enough to show that it does not intersect
the hyperplane $T = 0$. In view of (4.2), the point $[y, 0]$ lies on $K^{(2)}_\tau$ if and only if
\[
\begin{align*}
0 &= v = t, \\
0 &= -u^{2n} + x^{2n} + x^2f(s_1, s_2) + g(s_1, s_2) \\
0 &= x^{2n} + x^2f(s_1, s_2) + g(s_1, s_2), \\
0 &= 2nx^{2n-1} + 2xf(s_1, s_2), \\
0 &= x^2 \frac{\partial f}{\partial s_1}(s_1, s_2) + \frac{\partial g}{\partial s_1}(s_1, s_2), \\
0 &= x^2 \frac{\partial f}{\partial s_2}(s_1, s_1) + \frac{\partial g}{\partial s_2}(s_1, s_2), \\
0 &= n\lambda(\mu \cdot s)x^{2n-2} + \lambda(\mu \cdot s)f_1(s_1, s_2, 0) + x^2\frac{\partial f}{\partial T}(s_1, s_2, 0) + \frac{\partial g}{\partial T}(s_1, s_2, 0).
\end{align*}
\]
Since $p \nmid 2n\Delta_{f,g}$, it also follows that the plane curve
\[
X^{2n} + X^2f(S_1, S_2) + g(S_1, S_2) = 0
\]
in $\mathbb{P}_q^2$ is smooth. Hence the 3rd, 4th, 5th and 6th equation together imply that $x = s_1 = s_2 = 0$. But then the 2nd equation implies that $u = 0$ and this proves that the intersection of $K^{(2)}_\tau$ with the hyperplane $T = 0$ is empty, whence $\dim(K^{(2)}_\tau) \leq 0$. We obtain $\dim(K^{(1)}_\tau) \leq 0$ by repeating the same argument with (4.1) and switching the role of $u$ and $v$. \qed

**Proof of Lemma 4.2.** We begin by showing that $L_\tau$ has finitely many points with $t = 0$. When $T = 0$, the system (4.3) becomes
\[
\begin{align*}
0 &= X^{2n} + X^2f(S_1, S_2) + g(S_1, S_2), \\
0 &= n\lambda(\mu \cdot S)X^{2n-2} + \lambda(\mu \cdot S)f + X^2\frac{\partial f}{\partial T}(S_1, S_2, 0) + \frac{\partial g}{\partial T}(S_1, S_2, 0), \\
0 &= T = U = V.
\end{align*}
\]
Note that for any binary form $F \in \mathbb{F}_q[S_1, S_2]$ we have
\[
\frac{\partial G}{\partial T}(S_1 + h_1T, S_2 + h_2T), = \sum_{i=1,2} h_i \frac{\partial G}{\partial S_i}(S_1 + h_1T, S_2 + h_2T),
\]
\[
= (\mathbf{h} \cdot \nabla G)(S_1 + h_1T, S_2 + h_2T).
\]
Thus the system becomes
\[
\begin{align*}
0 &= X^{2n} + X^2f(S_1, S_2) + g(S_1, S_2), \\
0 &= n\lambda(\mu \cdot S)X^{2n-2} + \lambda(\mu \cdot S)f + X^2(\mathbf{h} \cdot \nabla f)(S_1, S_2) + (\mathbf{h} \cdot \nabla g)(S_1, S_2), \\
0 &= T = U = V.
\end{align*}
\]
If $\mu \neq 0$ the monomial $(\mu \cdot S)X^{2n-2}$ does not vanish identically. If $\mu = 0$, then $\mathbf{h} \cdot \nabla g$ does not vanish identically, since then $\mathbf{h} \neq 0$. Thus the 2nd equation involves a non zero polynomial of degree $2n - 1$ in $X, S_1, S_2$. On the other hand, the 1st equation defines an irreducible form of degree $2n$, and so the system meets in at most $4n^2 - 2n$ points by Bézout’s theorem.
Let us now count the solutions of system (4.3) with \( t \neq 0 \). We shall introduce a further variable \( Z = \overline{\lambda}(\mu \cdot S - \tau T) \), leading us to study the system

\[
\begin{aligned}
0 &= X^{2n} + X^2 f(S_1, S_2) + g(S_1, S_2), \\
0 &= (X^2 + T Z)^{n} + (X^2 + T Z) f_h + g_h, \\
0 &= n(2Z - \overline{\lambda}(\mu \cdot S))(X^2 + T Z)^{n-1} + (2Z - \overline{\lambda}(\mu \cdot S)) f_h \\
&\quad + (X^2 + T Z) \frac{\partial f_h}{\partial T} + \frac{\partial g_h}{\partial T}, \\
0 &= (X^2 \frac{\partial f}{\partial S_1} + \frac{\partial g}{\partial S_1})((X^2 + T Z) \frac{\partial f_h}{\partial S_2} + \frac{\partial g_h}{\partial S_2}) \\
&\quad - (X^2 \frac{\partial f}{\partial S_2} + \frac{\partial g}{\partial S_2})((X^2 + T Z) \frac{\partial f_h}{\partial S_1} + \frac{\partial g_h}{\partial S_1}), \\
0 &= Z - \overline{\lambda}(\mu \cdot S - \tau T).
\end{aligned}
\] (4.4)

Let \( V \subset \mathbb{P}^4_{\mathbb{F}_q} \) be the zero set of the first four equations. This variety does not depend on \( \tau \). We shall prove below in Proposition 4.4 that \( \dim(V) = 1 \).

Let \( V_1, \ldots, V_k \) be the irreducible components of \( V \) that are not contained in the hyperplane \( T = 0 \). Using the form of Bézout’s theorem found in Example 8.4.6 of Fulton [6], one can bound \( k \) by the product of the degrees of the forms defining \( V \) in \( \mathbb{P}^4_{\mathbb{F}_q} \). Thus

\[ k \leq 4n^2(2n - 1)^3 \leq 32n^5. \]

Moreover, Proposition 4.3 implies that \( \dim(V_i) \leq 1 \) for \( 1 \leq i \leq k \). For any \( i \in \{1, \ldots, k\} \) let \( v_i = [x_i, z_i, s_{i,1}, s_{i,2}, t_i] \in V_i \) be such that \( t_i \neq 0 \). Then \( v_i \) lies on the hyperplane \( Z = \overline{\lambda}((\mu \cdot S - \tau T)) \) if and only if

\[ z_i = \overline{\lambda}((\mu \cdot s_i - \tau t_i)). \]

This is true for at most a single value of \( \tau \), since \( t_i \neq 0 \). Hence for all but at most \( k \leq 32n^5 \) exceptional \( \tau \), we may conclude that \( V_i \) is not contained in the hyperplane \( Z = \overline{\lambda}(\mu \cdot S - \tau T) \), for \( i \in \{1, \ldots, k\} \), which implies that the intersection of \( V_i \) with the hyperplane \( Z = \overline{\lambda}(\mu \cdot S - \tau T) \) has dimension 0. It follows that the system (4.4) has finitely many solutions with \( t \neq 0 \), for all but at most \( 32n^5 \) values of \( \tau \).

It remains to prove that the dimension of \( V \) is 1. For this we shall require the following preliminary facts.

**Lemma 4.3.** Let \( F \in \mathbb{F}_q[S_1, S_2] \) be a separable polynomial of degree \( d \) and let \( P \subset \mathbb{P}^1_{\mathbb{F}_q} \) be its zero locus. Then

(i) For any \([a, b], [c, d] \in P \) we have

\[
\frac{\partial F}{\partial S_1}(a, b) \frac{\partial F}{\partial S_2}(c, d) - \frac{\partial F}{\partial S_2}(a, b) \frac{\partial F}{\partial S_1}(c, d) = 0
\]

if and only if \([a, b] = [c, d] \).

(ii) For any \( h_1, h_2, s_1, s_2 \in \mathbb{F}_q \) with \([s_1, s_2] \neq [h_1, h_2] \), let

\[ G(T) = F(s_1 + h_1 T, s_2 + h_2 T) \].
Then $G$ is non-constant and separable, with
\[
\deg(G) = \begin{cases} 
  d & \text{if } [h_1, h_2] \notin P, \\
  d - 1 & \text{if } [h_1, h_2] \in P.
\end{cases}
\]

Furthermore, if $t, t'$ are distinct roots of $G$ then
\[
[s_1 + h_1 t, s_2 + h_2 t] \neq [s_1 + h_1 t', s_2 + h_2 t'].
\]

Proof. Without loss of generality we may assume that $P$ consists of points
\[
[1, \alpha_1], \ldots, [1, \alpha_d],
\]
for distinct $\alpha_1, \ldots, \alpha_d \in \overline{F}_q$. It follows that $F = \prod_{i=1}^{d} (S_1 \alpha_i - S_2)$, so that
\[
\frac{\partial F}{\partial S_1} = \sum_{i=1}^{d} \alpha_i \prod_{j \neq i} (S_1 \alpha_j - S_2), \quad \frac{\partial F}{\partial S_2} = -\sum_{i=1}^{d} \prod_{j \neq i} (S_1 \alpha_j - S_2).
\]

If $[1, \alpha_k], [1, \alpha_m] \in P$ then
\[
0 = \frac{\partial F}{\partial S_1}(1, \alpha_k) \frac{\partial F}{\partial S_2}(1, \alpha_m) - \frac{\partial F}{\partial S_2}(1, \alpha_k) \frac{\partial F}{\partial S_1}(1, \alpha_m)
\]
\[
= (\alpha_k - \alpha_m) \left( \prod_{j \neq k} (\alpha_j - \alpha_k) \right) \left( \prod_{j \neq m} (\alpha_j - \alpha_m) \right)
\]
if and only if $\alpha_k = \alpha_m$. This establishes part (i).

Turning to part (ii), we first assume that $[h_1, h_2] \notin P$. Then $t$ is a root of $g$ if and only if either $s_1 + h_1 t = s_2 + h_2 t = 0$ or $[s_1 + h_1 t, s_2 + h_2 t] \in P$. Hence $t$ is a root of $g$ if and only if there exists $[1, \alpha] \in P$ and $\nu_1, \nu_2 \in \overline{F}_p$ such that
\[
\begin{aligned}
  s_1 + h_1 t &= \nu_1 \\
  s_2 + h_2 t &= \nu_2 \alpha.
\end{aligned}
\]

But then
\[
t = \frac{\alpha s_1 - s_2}{h_2 - \alpha h_1}.
\]

Moreover, for $[1, \alpha_1], [1, \alpha_2] \in P$ we get that
\[
(\alpha_1 s_1 - s_2)(h_2 - \alpha_2 h_1) = (\alpha_2 s_1 - s_2)(h_2 - \alpha_1 h_1)
\]
if and only if $(\alpha_1 - \alpha_2)(s_1 h_2 - s_2 h_1) = 0$, which is if and only if $\alpha_1 = \alpha_2$, since we are assuming $[s_1, s_2] \neq [h_1, h_2]$. The result follows since $f$ is of degree $d$ and separable. Suppose next that $[h_1, h_2] \in P$. Then we repeat the same argument, but observe that the system (4.5) is not solvable in the case $[1, \alpha] = [h_1, h_2]$. The final claim in part (iii) is a direct consequence of our argument. \qed
Recall that $V \subset \mathbb{P}^4_{\mathbb{F}_q}$ is given by
\[
\begin{align*}
0 &= X^{2n} + X^2 f(S_1, S_2) + g(S_1, S_2), \\
0 &= (X^2 + TZ)^n + (X^2 + TZ)f_h + g_h, \\
0 &= n(2Z - \lambda(\mu \cdot S))(X^2 + TZ)^{n-1} + (2Z - \lambda(\mu \cdot S))f_h \\
&\quad + (X^2 + TZ)\frac{\partial f_h}{\partial T} + \frac{\partial g_h}{\partial T}, \\
0 &= (X^2 + TZ)\frac{\partial f_h}{\partial S_1} + \frac{\partial g_h}{\partial S_1}(X^2 + TZ)\frac{\partial f_h}{\partial S_2} + \frac{\partial g_h}{\partial S_2} \\
&\quad + (X^2 + TZ)\frac{\partial g_h}{\partial S_1}(X^2 + TZ)\frac{\partial f_h}{\partial S_2} + \frac{\partial g_h}{\partial S_2}. 
\end{align*}
\]

We shall prove the following result.

**Proposition 4.4.** Assume that $p \nmid 2n\Delta_{f,g}$. Then $\dim(V) = 1$.

We can rewrite the final equation in (4.6) as
\[(X^2 + TZ)U_1 = U_2, \tag{4.7}\]
where
\[
U_1 = X^2 \left( \frac{\partial f_h}{\partial S_1} \frac{\partial f}{\partial S_1} - \frac{\partial f_h}{\partial S_2} \frac{\partial f}{\partial S_2} \right) + \left( \frac{\partial f_h}{\partial S_1} \frac{\partial g}{\partial S_1} - \frac{\partial f_h}{\partial S_2} \frac{\partial g}{\partial S_2} \right), \tag{4.8}\n\]
\[
U_2 = X^2 \left( \frac{\partial g_h}{\partial S_1} \frac{\partial f}{\partial S_1} - \frac{\partial g_h}{\partial S_2} \frac{\partial f}{\partial S_2} \right) + \left( \frac{\partial g_h}{\partial S_1} \frac{\partial g}{\partial S_1} - \frac{\partial g_h}{\partial S_2} \frac{\partial g}{\partial S_2} \right).
\]

We shall first prove Proposition 4.4 in the case where $f$ is identically zero. Then $U_1 = 0$ and
\[U_2 = \frac{\partial g_h}{\partial S_2} \frac{\partial g}{\partial S_1} - \frac{\partial g_h}{\partial S_1} \frac{\partial g}{\partial S_2}.
\]
Thus, $V$ is cut out by the system of equations
\[
\begin{align*}
0 &= X^{2n} + g, \\
0 &= (X^2 + TZ)^n + g_h, \\
0 &= n(2Z - \lambda(\mu \cdot S))(X^2 + TZ)^{n-1} + \frac{\partial g_h}{\partial T}, \\
0 &= \frac{\partial g_h}{\partial S_1} \frac{\partial g}{\partial S_2} - \frac{\partial g_h}{\partial S_2} \frac{\partial g}{\partial S_1}.
\end{align*}
\]
Moreover, it is clear that the polynomial $X^{2n} + g$ does not divide the polynomial in the last equation. Hence the 1st and 4th equation meet in a curve in $\mathbb{P}^4_{\mathbb{F}_q}$.

On the other hand, the polynomial $(x^2 + tZ)^n + g_h(s_1, s_2, t)$ is not constant in $Z$ when $t \neq 0$. This implies that any component of $V$ that is not contained in the hyperplane $T = 0$ has dimension 1. Moreover, the intersection of $V$ with the hyperplane $T = 0$ cuts out the the system of equations
\[
\begin{align*}
0 &= X^{2n} + g, \\
0 &= X^{2n-2}(2Z - \lambda(\mu \cdot S)) + h \cdot \nabla g(S_1, S_2) = 0, \\
0 &= T,
\end{align*}
\]
which also has dimension $1$. Hence $\dim(V) = 1$, as desired.
Moreover, since \( U \) if we multiply the second equation of (4.6) by \( U_1^n \) we obtain

\[
0 = U_1^n \left((X^2 + TZ)^n + (X^2 + TZ)f_h + g_h\right) = U_2^n + U_2 U_1^{n-1} f_h + U_1^n g_h.
\]

It follows that the points of \( V \) are solutions to the system

\[
\begin{align*}
0 &= X^{2n} + X^2f(S_1, S_2) + g(S_1, S_2), \\
0 &= U_2^n + U_2 U_1^{n-1} f_h + U_1^n g_h, \\
0 &= n(2Z - \lambda(\mu \cdot S))(X^2 + TZ)^{n-1} + (2Z - \lambda(\mu \cdot S))f_h \\
&\quad + (X^2 + TZ)\frac{\partial f_h}{\partial T} + \frac{\partial g_h}{\partial T}, \\
U_2 &= (X^2 + TZ)U_1,
\end{align*}
\]

where \( U_1, U_2 \in \mathbb{F}_q[S_1, S_2, T, X] \) are given by (4.8). We proceed by proving the following fact.

**Lemma 4.5.** The polynomial \( X^{2n} + X^2f + g \) does not divide the polynomial \( U_2^n + U_2 U_1^{n-1} f_h + U_1^n g_h \).

**Proof.** We can assume that \( g \) is separable over \( \mathbb{F}_q \) since \( p \nmid \Delta_{fg} \). We argue by contradiction, by assuming that \( X^{2n} + X^2f + g \) divides the polynomial \( U_2^n + U_2 U_1^{n-1} f_h + U_1^n g_h \). Taking \( X = 0 \), this implies that the polynomial \( g \) divides

\[
F = U_2(S_1, S_2, T, 0)^n + U_2(S_1, S_2, T, 0)U_1(S_1, S_2, T, 0)^{n-1} f_h(S_1, S_2, T) \\
+ U_1(S_1, S_2, T, 0)^n g_h(S_1, S_2, T).
\]

Choose \([s_1, s_2] \in \mathbb{P}^1_{\mathbb{F}_q}\) such that \( g(s_1, s_2) = 0 \), with \( \nabla g(s_1, s_2) \) not proportional to \( \nabla g(h_1, h_2) \). This is possible by part (i) of Lemma 4.3 which shows that \( \nabla g(h_1, h_2) \) can be proportional to at most one of the vectors \( \nabla g(s_1, s_2) \). It also follows from part (i) of Lemma 4.3 that \([h_1, h_2] \neq [s_1, s_2] \). For this choice of \( s_1, s_2 \), the polynomial \( U_2(s_1, s_2, T, 0) \) has degree \( 2n - 1 \), with non-zero leading coefficient

\[
\frac{\partial g}{\partial S_2}(h_1, h_2) \frac{\partial g}{\partial S_1}(s_1, s_2) - \frac{\partial g}{\partial S_1}(h_1, h_2) \frac{\partial g}{\partial S_2}(s_1, s_2).
\]

Since \( g \mid F \) we get

\[
0 \equiv U_1(s_1, s_2, T, 0)^n g_h(s_1, s_2, T) + U_2(s_1, s_2, T, 0)^{n-1} f_h(s_1, s_2, T),
\]

identically in \( T \). In particular it follows that \( U_1(s_1, s_2, T, 0) \mid U_2(s_1, s_2, T, 0) \), so that there exists \( W \in \mathbb{F}_q[T] \) such that

\[
U_2(s_1, s_2, T, 0) = U_1(s_1, s_2, T, 0) \cdot W(T).
\]

Moreover, since \( U_2(s_1, s_2, T, 0) \) has degree \( 2n - 1 \) and \( U_1(s_1, s_2, T, 0) \) has degree at most \( 2n - 3 \), we conclude that \( \deg W \geq 2 \). Thus we have

\[
g_h(s_1, s_2, T) = W(T)^n + W(T) \cdot f_h(s_1, s_2, T),
\]
identically in $T$. This implies that $W(T)|_{gh(s_1, s_2, T)}$. On the other hand, $gh(s_1, s_2, T) = g(s_1 + h_1 T, s_2 + h_2 T)$ is a separable polynomial by part (ii) of Lemma 4.3 with degree at least $2n - 1$. Thus $W$ is a separable polynomial of degree $\geq 2$. It follows that there exists $t \neq 0$ such that

$$U_2(s_1, s_2, t, 0) = g(s_1 + h_1 t, s_2 + h_2 t) = 0.$$  

If $s_1 + h_1 t = s_2 + h_2 t = 0$ then $s_1 h_2 - s_2 h_1 = 0$, which implies that $[h_1, h_2] = [s_1, s_2]$ in $\mathbb{P}^1_{\mathbb{F}_q}$. This is impossible, by our construction of $[s_1, s_2]$.

We now put $\tilde{s}_1 = s_1 + h_1 t$ and $\tilde{s}_2 = s_2 + h_2 t$. We have already seen that $(\tilde{s}_1, \tilde{s}_2) \neq (0, 0)$. Moreover, by part (ii) of Lemma 4.3, $[s_1, s_2] \neq [\tilde{s}_1, \tilde{s}_2]$, since $t \neq 0$. Hence there exists $[\tilde{s}_1, \tilde{s}_2] \in \mathbb{P}^1_{\mathbb{F}_q}$, which is a root of $g$ distinct from $[s_1, s_2]$, such that

$$\frac{\partial g}{\partial S_2}(\tilde{s}_1, \tilde{s}_2) \frac{\partial g}{\partial S_1}(s_1, s_2) - \frac{\partial g}{\partial S_1}(\tilde{s}_1, \tilde{s}_2) \frac{\partial g}{\partial S_2}(s_1, s_2) = 0.$$  

This contradicts part (i) of Lemma 4.3 which thereby completes the proof. \hfill \Box

It follows from Lemma 4.3 that the system

$$\begin{cases}
0 = X^{2n} + X^2 f + g, \\
0 = U_2^n + U_2 U_1^{n-1} f_h + U_1^n g_h,
\end{cases}$$

defines a variety of dimension 1 in $\mathbb{P}^3_{\mathbb{F}_q}$. On the other hand, for any $s_1, s_2, x, t \in \mathbb{F}_q$, the polynomial

$$n(2Z - \lambda(\mu \cdot s))(x^2 + tZ)^2 + (2Z - \lambda(\mu \cdot s))f_h(s_1, s_2, t)$$

$$+ (x^2 + tZ) \frac{\partial f_h}{\partial T}(s_1, s_2, t) + \frac{\partial g_h}{\partial T}(s_1, s_2, t)$$

has degree 0 in $Z$ if and only if $t = f(s_1, s_2) = 0$. It follows that any components of $V$ that are not contained in the intersection of $T = 0$ with $f(S_1, S_2) = 0$ have dimension 1. On the other hand, the intersection of $V$ with the variety $T = f(S_1, S_2) = 0$ has dimension 1, since $X$ is constrained by the equation $X^{2n} + g(S_1, S_2) = 0$. This finally completes the proof of Proposition 4.4.

**Appendix A. Hooley’s method of moments for exponential sums**

Our work uses a general procedure due to Hooley [13], which allows one to estimate a very general family of exponential sums over a finite field, provided that one can control the second moment of an appropriate counting function.

**Theorem A.1 (Hooley).** Let $F, G_1, \ldots, G_k \in \mathbb{Z}[X_1, \ldots, X_m]$ be polynomials of degree at most $d$ and let

$$S = \sum_{x \in \mathbb{F}_p^n \cap \{G_1(x) = \cdots = G_k(x) = 0\}} e_p(F(x)).$$
For each \( r \geq 1 \) and \( \tau \in \mathbb{F}_p^r \), write
\[
N_r(\tau) = \# \{ x \in \mathbb{F}_p^m : G_1(x) = \cdots = G_k(x) = 0, F(x) = \tau \}.
\] (A.1)

If there exist \( N_r \in \mathbb{R} \) such that
\[
\sum_{\tau \in \mathbb{F}_p^r} |N_r(\tau) - N_r|^2 \ll_{d,k,m} p^{\kappa r},
\]
where \( \kappa \in \mathbb{Z} \) is independent of \( r \), then \( S \ll_{d,k,m} p^{s/2} \).

This result relies crucially on Deligne’s resolution of the Weil conjectures and can be extracted from work of Hooley [13]. At the recommendation of one of the anonymous referees we will give a full proof of Theorem A.1 in this appendix.

Let \( \psi : \mathbb{F}_q \to \mathbb{C} \) be any non-trivial additive character, where \( q = p^r \). Define
\[
S_r(\mu) = \sum_{x \in \mathbb{F}_q^m} \psi(\mu F(x)),
\]
for any \( \mu \in \mathbb{F}_q^* \). We clearly have \( S = S_1(\mu) \), for a suitable \( \mu \in \mathbb{F}_q^* \). The idea is to study the moment
\[
M_r = \sum_{\mu \in \mathbb{F}_q^*} |S_r(\mu)|^2.
\]

Recall the definition (A.1) of \( N_r(\tau) \). Then clearly
\[
S_r(\mu) = \sum_{\tau \in \mathbb{F}_q} N_r(\tau) \psi(\mu \tau) = \sum_{\tau \in \mathbb{F}_q} (N_r(\tau) - N_r) \psi(\mu \tau),
\]
for any \( N_r \in \mathbb{R} \). It follows from orthogonality of characters that
\[
M_r \leq \sum_{\mu \in \mathbb{F}_q} |S_r(\mu)|^2 = q \sum_{\tau \in \mathbb{F}_q} |N_r(\tau) - N_r|^2.
\]

By hypothesis, there exists \( N_r \in \mathbb{R} \) such that
\[
M_r \ll_{d,k,m,k} q^{\kappa+1},
\] (A.2)
for some \( \kappa \in \mathbb{Z} \) that is independent of \( r \).

Now fix \( \mu \in \mathbb{F}_p^* \). Associated to \( S_r(\mu) \) is a zeta function whose rationality is assured by the work of Dwork [4]. Thus there exists numbers \( \alpha_1, \ldots, \alpha_N \in \mathbb{C} \) such that
\[
S_r(\mu) = \sum_{1 \leq j \leq n} \alpha_j^r - \sum_{n < j \leq N} \alpha_j^r,
\] (A.3)
where \( N = O_{d,m,k}(1) \). Furthermore, it follows from Deligne’s resolution of the Weil conjectures [3] that there exist integers \( m_1, \ldots, m_N \geq 0 \) such that
\[
|\sigma(\alpha_j)| \leq p^{m_j/2}, \quad (1 \leq j \leq N)
\]
for any automorphism \( \sigma \) of \( \mathbb{Q} \). We claim that the integers \( m_j \) are independent of \( \mu \), for \( \mu \in \mathbb{F}_p^* \). Let \( \beta_1, \ldots, \beta_N \in \mathbb{C} \) be such that \( S_r(1) = \sum_{1 \leq j \leq n} \beta_j^r - \sum_{n < j \leq N} \beta_j^r \).
and let \( \sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}} \) be the automorphism that takes \( \psi(1) \) to \( \psi(\mu) \). Then \( S_r(\mu) = \sigma(S_r(1)) \), whence \( \alpha_j = \sigma(\beta_j) \) for \( 1 \leq j \leq N \). The claim readily follows.

Let \( H = \max_{1 \leq j \leq N} m_j \) and let \( J = \{ j \leq N : m_j = H \} \). We shall prove that \( J \) is empty when \( H \geq \kappa + 1 \). From this it will follow that \( |\alpha_j| \leq q^\kappa \) for each \( 1 \leq j \leq N \), whence

\[
S_r(\mu) \ll_{d,m,k} q^{\kappa/2},
\]

which is satisfactory for Theorem [A.1]. Assume that \( H \geq \kappa + 1 \). For each \( j \in J \), we write \( \alpha_j = \omega_j p^{H/2} \) for a root of unity \( \omega_j \) depending on \( \mu \). Then it follows from (A.3) that

\[
S_r(\mu) = q^{H/2} \sum_{j \in J} \omega_j^r + O_{d,m,k}(q^{(H-1)/2}).
\]

But then

\[
q^{-H} |S_r(\mu)|^2 = \left| \sum_{j \in J} \omega_j^r \right|^2 + O_{d,m,k}(q^{-1/2}).
\]

Hence

\[
q^{-H} M_r \geq q^{-H} \sum_{\mu \in \mathbb{F}_p^*} |S_r(\mu)|^2 = \sum_{\mu \in \mathbb{F}_p^*} \left| \sum_{j \in J} \omega_j^r \right|^2 + O_{d,m,k}(pq^{-1/2}).
\]

Combining this with (A.2), we deduce that

\[
\sum_{\mu \in \mathbb{F}_p^*} \left| \sum_{j \in J} \omega_j^r \right|^2 \ll_{d,m,k} p^{1-r/2} + 1.
\]

For each \( j \in J \), suppose that \( \omega_j = \exp(2\pi i a_j/q_j) \) for appropriate coprime integers \( a_j \) and \( q_j \). Choose \( r \in \mathbb{Z} \) such that \( r \geq 2 \) and \( q_j \mid r \), for \( j \in J \). Then it will follow that \( \omega_j^r = 1 \), for each \( j \in J \), whence

\[
(p - 1) \# J^2 = O_{d,m,k}(1).
\]

This implies that \( J \) is empty, as claimed, provided that \( p \gg_{d,m,k} 1 \).

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**IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria**

*Email address*: dante.bonolis@ist.ac.at

*Email address*: tdb@ist.ac.at