Rational points of elliptic surfaces and Zariski $N$-ples for cubic-line, cubic-conic-line arrangements

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Abstract

In this paper, we continue the study of the relation between rational points of rational elliptic surfaces and plane curves. As an application, we give first examples of Zariski pairs of cubic-line arrangements that do not involve inflectional tangent lines.

Introduction

In this article, we study the arithmetic of rational points of certain rational elliptic surfaces from a geometric point of view in order to construct arrangements of plane curves of low degree which give rise to candidates for Zariski pairs. A pair of reduced plane curves $(B_1, B_2)$ is said to be a Zariski pair if it satisfies the following conditions:

(i) For each $i$, there exists a tubular neighborhood $T(B^i)$ of $B^i$ such that $(T(B^1), B^1)$ is homeomorphic to $(T(B^2), B^2)$.

(ii) There exists no homeomorphism from $(\mathbb{P}^2, B_1)$ to $(\mathbb{P}^2, B_2)$.

The first condition can be replaced by the combinatorics (or the combinatorial type) of $B^i$. For the precise definition of the combinatorics, see [2] (It can also be found in [21]). Since the combinatorics is easier to treat with, we always consider that of $B^i$. The study of Zariski pairs was originated by Zariski in [23]. Since the ‘90’s there have been a lot of results on Zariski pairs by many mathematicians via various methods (see the reference [2], for example). As we noted in [2], there are two main ingredients in the study of Zariski pairs. Namely,

(I) To find reduced curves $B^1, B^2$ with the same combinatorics so that $B^1, B^2$ have certain different features, and

(II) To prove $(\mathbb{P}^2, B^1)$ is not homeomorphic to $(\mathbb{P}^2, B^2)$ based on the different feature as above.

A Zariski $N$-ple is a natural generalization, where the number of curves is increased. One of our new feature of this article concerns (I): Construction of plane curves via geometry and arithmetic of sections for certain rational elliptic surfaces. This basic idea can be found in [21] by the first author and in [5] by the first and second authors. In this article, however, we make use of the arithmetic of sections more intensively than previous papers.

For (II), in order to distinguish $(\mathbb{P}^2, B^1)$ and $(\mathbb{P}^2, B^2)$, our tool is Galois covers branched along $B^i$ developed in [5, 21]. Note that there are various other tools, for example, the fundamental group $\pi_1(\mathbb{P}^2 \setminus B^i, \ast)$, braid monodromy and Alexander invariants. Recently two more new tools, the linking set and the connected number, are introduced by J.-B. Meilhan, B. Guerville-Ballé [7] and T. Shirane [17], respectively.
We explain our object more concretely. The first and second authors have studied Zariski pairs (or $N$-ples) for arrangements of curves whose irreducible components are of low degree, less than or equal to 4, via geometry of sections and bisections of rational elliptic surfaces ([5, 21]). In this article, we also continue to study such objects. More precisely, we consider a reducible curve whose irreducible components consist of

(i) one irreducible cubic and lines, and

(ii) one irreducible cubic, smooth conics and lines.

In [1], a Zariski pair of sextics consisting of a smooth cubic and its three inflectional tangents was given. This example was also considered in [19] from a different approach. In [7], a Zariski pair of octics consisting of a smooth cubic and its 5 inflectional tangents is given. In [4], that of reducible curves consisting of a smooth cubic and $k$ of its inflectional tangents ($k = 4, 5, 6$) are considered. Note that there exist no Zariski pair of a smooth cubic and $k$ of its inflectional tangents for $k = 1, 2, 7, 8, 9$. Also, in [3], E. Artal Bartolo and the second author studied a Zariski pair of sextics whose irreducible components are a nodal cubic, a smooth conic and an inflectional tangent line. All of these examples contain inflectional tangents of a cubic as irreducible components. As for another new feature of this article, we focus on Zariski pairs for cubic-line or cubic-conic-line arrangements without inflectional tangents. Also, since Zariski pairs for sextic curves have already been intensively studied by many mathematicians, e.g., [13, 14, 15], we consider the case of degree 7 as follows:

**Combinatorics 1.** Let $\mathcal{E}$, $\mathcal{L}_o$ and $\mathcal{L}_i$ ($i = 1, 2, 3$) be as below and we put $B = \mathcal{E} + \mathcal{L}_o + \sum_{i=1}^3 \mathcal{L}_i$:

(i) $\mathcal{E}$: a smooth or nodal cubic curve.

(ii) $\mathcal{L}_o$: a transversal line to $\mathcal{E}$ and we put $\mathcal{E} \cap \mathcal{L}_o = \{p_1, p_2, p_3\}$.

(iii) $\mathcal{L}_i$ ($i = 1, 2, 3$): a line through $p_i$ and tangent to $\mathcal{E}$ at a different point from $p_i$. We denote the tangent point of $\mathcal{L}_i$ by $r_i$.

(iv) $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are not concurrent.

**Combinatorics 2.** Let $\mathcal{E}$, $\mathcal{C}$, $\mathcal{L}_o$ and $\mathcal{L}$ as below and we put $B = \mathcal{E} + \mathcal{C} + \mathcal{L}_o + \mathcal{L}$:

(i) $\mathcal{E}$: a nodal cubic curve.

(ii) $\mathcal{L}_o$: a transversal line to $\mathcal{E}$ and we put $\mathcal{E} \cap \mathcal{L}_o = \{p_1, p_2, p_3\}$.

(iii) $\mathcal{L}_i$: a line connecting the node of $\mathcal{E}$ and one of $p_i$, ($i = 1, 2, 3$).

(iv) $\mathcal{C}$: a contact conic to $\mathcal{E} + \mathcal{L}_o$ and intersecting $\mathcal{L}$ transversely.

Here we call a smooth conic $\mathcal{C}$ a contact conic to a reduced plane curve $B$ if the following condition is satisfied: $(\ast)$ Let $I_x(\mathcal{C}, B)$ denotes the intersection multiplicity at $x \in \mathcal{C} \cap B$. For $\forall x \in \mathcal{C} \cap B$, $I_x(\mathcal{C}, B)$ is even and $B$ is smooth at $x$.

In both combinatorics, no inflectional tangents are involved and this is the new feature compared to previous examples. Let us explain more precisely. In the following, we use the notation and terminology used in [3].
Put $Q = L_o + E$ and choose a smooth point $z_o \in E$. Consider the minimal resolution $S_Q$ of a double cover of $\mathbb{P}^2$ branched along $Q$ and blow up $S_Q$ twice at the preimage of $z_o$. Then we obtain a rational elliptic surface $\varphi_{Q,z_o}: S_Q \rightarrow \mathbb{P}^1$ and its generic fiber is denoted by $E_{Q,z_o}$. (see §1 for a more precise description). We denote the induced generically 2-to-1 morphism from $S_{Q,z_o}$ to $\mathbb{P}^2$ by $f_{Q,z_o}$. Let $E_{Q,z_o}$ be the generic fiber of $\varphi_{Q,z_o}: S_{Q,z_o} \rightarrow \mathbb{P}^1$. The group of sections of $\varphi_{Q,z_o}$, $\text{MW}(S_{Q,z_o})$, can be canonically identified with the group of $\mathbb{C}(t)$-rational points $E_{Q,z_o}(\mathbb{C}(t))$. For $s \in \text{MW}(S_{Q,z_o})$, we denote the corresponding rational point by $P_s$. Conversely, for $P \in E_{Q,z_o}(\mathbb{C}(t))$, we denote the corresponding section by $s_P$. Now, since $s \in \text{MW}(S_{Q,z_o})$ can be considered a curve, $f_{Q,z_o}(s)$ gives rise to a plane curve in $\mathbb{P}^2$. In our construction of plane curves with Combinatorics 1 and 2, we make use of lines and conics of the form $f_{Q,z_o}(s)$ for some $s \in \text{MW}(S_{Q,z_o})$. In our particular cases, it can be explained more explicitly as follows (We use the notation in [12] in order to describe the structure of $E_{Q,z_o}(\mathbb{C}(t))$):

**Combinatorics 1.** (a) $E$: a smooth cubic. If we choose $z_o \in E \setminus \{p_1, p_2, p_3\}$, $E_{Q,z_o}(\mathbb{C}(t)) \cong D^*_1 \oplus \mathbb{Z}/2\mathbb{Z}$. As we see in §3, we choose generators $P_0, P_1, P_2, P_3$ for the $D^*_1$-part suitably and the 2-torsion $P_\tau$. We also put $P_4 := P_2 + P_3 + P_\tau$, where $+,-$ denote the addition and subtraction on $E_{Q,z_o}(\mathbb{C}(t))$. Let $s_{P_i}$ be the sections corresponding to $P_i$ ($i = 1, \ldots, 4$), respectively. Then $L_i := f_{Q,z_o}(s_{P_i})$ ($i = 1, \ldots, 4$) are tangent lines to $E$ such that each of them passes through $E \cap L_o$. If $p_i$ ($i = 1, 2, 3$) are not inflection points and any three of them are not concurrent, then both $B^1 := Q + \sum_{i=1}^3 L_i$ and $B^2 := Q + \sum_{i=2}^4 L_i$ have Combinatorics 1-(a).

(b) $E$: a nodal cubic. If we choose $z_o \in E \setminus \{p_1, p_2, p_3\}$, $E_{Q,z_o}(\mathbb{C}(t)) \cong (A_1)^{\oplus 3} \oplus \mathbb{Z}/2\mathbb{Z}$. As we see in §3, we choose generators $P_1, P_2, P_3$ of $E_{Q,z_o}$ so that the Gram matrix of $P_i$ ($i = 1, 2, 3$) is $[[P_i, P_j]] = 1/2\delta_{ij}$ and the 2-torsion $P_\tau$. We put

$$P_4 := P_2 + P_3 + P_\tau, \quad P_5 := P_1 + P_2 + P_\tau, \quad P_6 := P_2 + P_3 + P_\tau, \quad P_7 := P_1 + P_2 + P_\tau.$$

Then we infer that $L_i := f_{Q,z_o}(s_{P_i})$ ($i = 4, 5, 6, 7$) are tangent lines to $E$ such that each of them passes through $E \cap L_o$. If any three of them are not concurrent, then both $B^1 := Q + \sum_{i=4,5,6} L_i$ and $B^2 := Q + \sum_{i=5,6,7} L_i$ have Combinatorics 1-(b).

**Combinatorics 2.** We keep our notation in Combinatorics 1. Our construction is similar to that in [21]. We first note that $L_i := f_{Q,z_o}(s_{P_i})$ ($i = 1, 2, 3$) are lines connecting the node of $E$ and $p_i$. Plane curves given by $f_{Q,z_o}(s_{2P_i})$ ($i = 1, 2, 3$) are contact conics by a similar argument to that in [21] p. 633. Suppose that $C_i$ meets $L_j$ transversely for any $i,j$. Then $B^1 := Q + L_i + C_i$ and $B^2 := Q + L_i + C_j$ ($i \neq j$) have Combinatorics 2.

Now our statement is

**Theorem 0.1** Let $B^1$ and $B^2$ be as above. Then $(B^1, B^2)$ is a Zariski pair.

The organization of this paper as follows: In §1, we give a summary on facts and previous results, which we need later. We prove Theorem 0.1 in §2. A Zariski triple and 4-ple for cubic-conic-line or cubic-conic arrangement are considered in §3 and explicit examples are given in §4.

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1 Preliminaries

1.1 Elliptic surfaces

1.1.1 Generalities

As for basic references about elliptic surfaces, we refer to [9, 10]. We also refer to [16] for general facts on the Mordell Weil lattices. In particular, for those on rational elliptic surfaces, we refer to [12]. We also use the notation and terminology used in [5, 21] freely.

In this article, by an elliptic surface, we always mean a smooth projective surface $S$ with a fibration $\varphi : S \to C$ over a smooth projective curve $C$ as follows:

(i) $\varphi$ has a section $O : C \to S$ (we identify $O$ with its image).

(ii) There exists a non-empty finite subset, $\text{Sing}(\varphi)$, of $C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 (resp. a singular curve) for $v \in C \setminus \text{Sing}(\varphi)$ (resp. $v \in \text{Sing}(\varphi)$). Note that there exist no multiple fibers since $\varphi$ has the section $O$.

(iii) $\varphi$ is minimal, i.e., there is no exceptional curve of the first kind in any fiber.

For $v \in \text{Sing}(\varphi)$, we put $F_v = \varphi^{-1}(v)$. We denote its irreducible decomposition by

$$ F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i} \Theta_{v,i}, $$

where $m_v$ is the number of irreducible components of $F_v$ and $\Theta_{v,0}$ is the unique irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ the identity component. The classification of singular fibers is well-known ([9]). We use the Kodaira Notation to denote the types of singular fibers.

For $v \in \text{Sing}(\varphi)$, we also denote the subset of $\text{Sing}(\varphi)$ consisting of points giving reducible singular fibers by $\text{Red}(\varphi) := \{ v \in \text{Sing}(\varphi) \mid F_v \text{ is reducible} \}$. Let $\text{MW}(S)$ be the set of sections of $\varphi : S \to C$. $\text{MW}(S) \neq \emptyset$ as $O \in \text{MW}(S)$. By [9, Theorem 9.1], $\text{MW}(S)$ is an abelian group with the zero element $O$. We call $\text{MW}(S)$ the Mordell-Weil group. We also denote the multiplication-by-$m$ map ($m \in \mathbb{Z}$) on $\text{MW}(S)$ by $[m]s$ for $s \in \text{MW}(S)$. On the other hand, the generic fiber $E := S_\eta$ of $S$ is as a curve of genus 1 over $\mathbb{C}(C)$, the rational function field of $C$. The restriction of $O$ to $E$ gives rise to a $\mathbb{C}(C)$-rational point of $E$, and one can regard $E$ as an elliptic curve over $\mathbb{C}(C)$, $O$ being the zero element. $\text{MW}(S)$ can be identified with the set of $\mathbb{C}(C)$-rational points $E(\mathbb{C}(C))$ canonically. For $s \in \text{MW}(S)$, we denote the corresponding rational point by $P_s$. Conversely, for an element $P \in E(\mathbb{C}(C))$, we denote the corresponding section by $s_P$. The abelian group $G_{\text{Sing}(\varphi)}$ and the homomorphism $\gamma : \text{MW}(S) \to G_{\text{Sing}(\varphi)}$ are those defined in [20, p. 83]. For $s \in \text{MW}(S)$, $\gamma(s)$ describes at which irreducible component $s$ meets on $F_v$.

Let $\text{NS}(S)$ be the Néron-Severi group of $S$ and let $T_\varphi$ be the subgroup of $\text{NS}(S)$ generated by $O$, a fiber $F$ and $\Theta_{v,i}$ ($v \in \text{Red}(\varphi)$, $1 \leq i \leq m_v - 1$). Then we have the following theorems:

**Theorem 1.1 ([16, Theorem 1.2, 1.3])** Under our assumptions,

(i) $\text{NS}(S)$ is torsion free, and
There is a natural map \( \tilde{\psi} : \text{NS}(S) \to \text{MW}(S) \) which induces an isomorphism of groups

\[
\psi : \text{NS}(S)/T_\varphi \cong \text{MW}(S) \cong E(\mathbb{C}(C)).
\]

In particular, \( \text{MW}(S) \) is a finitely generated abelian group.

Theorem 1.2 ([16, Theorem 1.3]) Under our assumptions, there is a natural map \( \tilde{\psi} : \text{NS}(S) \to \text{MW}(S) \) which induces an isomorphism of groups

\[
\psi : \text{NS}(S)/T_\varphi \cong \text{MW}(S).
\]

In particular, \( \text{MW}(S) \) is a finitely generated abelian group.

For a divisor on \( S \), we put \( s(D) = \tilde{\psi}(D) \).

In [16], a lattice structure on \( E(\mathbb{C}(C))/E(\mathbb{C}(C))_{\text{tor}} \) is defined by using the intersection pairing on \( S \) through \( \psi \). We use the terminologies, notation and results in [16], freely. In particular, \( \langle , \rangle \) denotes the height pairing and \( \text{Contr}_v \) denotes the contribution term given in [16] in order to compute \( \langle , \rangle \).

1.1.2 Double cover construction of an elliptic surface

We refer to [10, Lectures III and IV] for details. Let \( \varphi : S \to \mathbb{P}^1 \) be an elliptic surface over a smooth projective curve \( \mathbb{P}^1 \). As we see in [5] [21], \( S \) can be represented as the minimal resolution of a double cover of a Hirzebruch surface \( \Sigma_d \) as follows. The inversion of \( E \) with respect to the group law induces an involution \([-1]_\varphi \) on \( S \). Let \( S/[-1]_\varphi \) be the quotient by \([-1]_\varphi \). It is known that \( S/[-1]_\varphi \) is smooth and we can blow down \( S/[-1]_\varphi \) to its relatively minimal model \( W \). We denote the morphisms involved by

- \( f : S \to S/[-1]_\varphi \): the quotient morphism,
- \( q : S/[-1]_\varphi \to W \): the blow down, and
- \( S \to S' \to W \): the Stein factorization of \( q \circ f \).

Then (i) \( W = \Sigma_d \) where \( d = 2\chi(\mathcal{O}_S) \) and (ii) the branch locus \( \Delta_{f'} \) of \( f' \) is of the form \( \Delta_0 + \mathcal{T} \), where \( \Delta_0 \) is a section with \( \Delta_0^2 = -d \) and \( \mathcal{T} \sim 3(\Delta_0 + df) \), \( f \) being a fiber of the ruling \( \Sigma_d \to \mathbb{P}^1 \). Moreover, singularities of \( \mathcal{T} \) are at most simple singularities (see [22] Chapter II, §8 for simple singularities and their notation).

Conversely, if \( \Delta_0 \) and \( \mathcal{T} \) on \( \Sigma_d \), \( d \): even, satisfy the above conditions, we obtain an elliptic surface \( \varphi : S \to \mathbb{P}^1 \), as the canonical resolution of a double cover \( f' : S' \to \Sigma_d \) with \( \Delta_{f'} = \Delta_0 + \mathcal{T} \), and the following diagram (see [3] for the canonical resolution):

\[
\begin{array}{ccc}
S' & \xrightarrow{\mu} & S \\
\downarrow f' & & \downarrow f \\
\Sigma_d & \xrightarrow{q} & \hat{\Sigma}_d.
\end{array}
\]
Here, \( q \) is a composition of blowing-ups so that \( \hat{\Sigma}_d = S/([−1]_ϕ) \). Hence any elliptic surface is obtained in this way. In the case when \( S \) is rational, \( d = 2 \). In the following, we call the diagram above the double cover diagram for \( S \).

Moreover, if \( S \) is rational and has a reducible singular fiber, \( \hat{\Sigma}_2 \) can be blown down to \( \mathbb{P}^2 \), as we remark in [5, 1.2.2]. \( T \) is mapped to a reduced quartic \( Q \), which in not concurrent four lines, and \( Δ_0 \) is mapped to a smooth point \( z_0 \) on \( Q \). Conversely, given a reduced quartic \( Q \) (≠ concurrent four lines) and a point \( z_0 \in Q \), we obtain \( S \) as above, which we denote by \( S_{Q,z_0} \), which is nothing but the surface described in the introduction. The induced generically 2-1 morphism from \( S_{Q,z_0} \) to \( \mathbb{P}^2 \) is \( f_{Q,z_0} \).

### 1.1.3 \( S_{Q,z_0} \) for the case when \( Q \) is \( \mathcal{E} + \mathcal{L}_0 \) in the combinatorics 1 and 2

Let \( z_0 \) be a smooth point on \( Q \). The tangent line \( l_{z_0} \) to \( Q \) at \( z_0 \) gives rise to a singular fiber of \( \varphi_{Q,z_0} \) whose type is determined by how \( l_{z_0} \) intersects with \( Q \) as follows:

| (i) | \( I_2 \) | \( l_{z_0} \) meets \( Q \) with two other distinct points. |
| (ii) | \( III \) | \( z_0 \) is an inflection point of \( \mathcal{E} \). |
| (iii) | \( I_0^* \) | \( l_{z_0} = \mathcal{L}_0 \). |
| (iv) | \( I_4 \) | \( l_{z_0} \) passes through a point in \( Q \cap \mathcal{L}_0 \). |

Hence by [11, Table 6.2], possible configurations of reducible singular fibers of \( S_{Q,z_0} \) are as follows:

**Case 1: \( \mathcal{E} \) is a smooth cubic**

| singular fibers |
|------------------|
| (i), (ii) \{a \( I_2 \), b \( III \)\}, \( a + b = 4 \), \( a, b \geq 0 \), \( b \neq 3 \) |
| (iii) \{\( I_0^* \)\} |
| (iv) \{\( I_4, 2I_2 \)\} |

**Case 2: \( \mathcal{E} \) is a nodal cubic**

| case | configuration of singular fibers |
|------|---------------------------------|
| (i), (ii) | \{a \( I_2 \), b \( III \)\}, \( a + b = 5 \), \( 0 \leq b \leq 2 \) |
| (iii) | \{\( I_0^*, I_2 \), \( I_0^*, III \)\} |
| (iv) | \{\( I_1, 3I_2 \)\} |

For our proof of Theorem [11], we choose \( z_0 \) satisfying (i) or (ii). In these cases, since the difference between fibers of type III and \( I_2 \) do not affect the structure of \( E_{Q,z_0}(\mathbb{C}(t)) \) and since \( \mathcal{L}_0 \) gives rise to a 2-torsion section, by [12], we infer that \( E_{Q,z_0}(\mathbb{C}(t)) \cong D^*_1 \oplus \mathbb{Z}/2\mathbb{Z} \) (resp. \( (A^*_1)^3 \oplus \mathbb{Z}/2\mathbb{Z} \)) for Case 1 (resp. Case 2).

### 1.2 Galois covers

For the notation and terminology on Galois covers, we use those in [2] freely.
1.2.1 $D_{2p}$-covers

We here introduce notation for dihedral covers which we use frequently. For details, see [18]. Let $D_{2p}$ be the dihedral group of order $2p$, where $p$ is an odd prime. In order to present $D_{2p}$, we use the notation

$$D_{2p} = \langle \sigma, \tau \mid \sigma^2 = \tau^p = (\sigma \tau)^2 = 1 \rangle.$$

Given a $D_{2p}$-cover, we obtain a double cover $D(X/Y)$ of $Y$ canonically by considering the $\mathbb{C}(X)$-normalization of $Y$, where $\mathbb{C}(X)$ denotes the fixed field of the subgroup of $D_{2p}$ generated by $\tau$. Then $X$ is a $p$-fold cyclic cover of $D(X/Y)$ and we denote the covering morphisms by $\beta_1(\pi): D(X/Y) \to Y$ and $\beta_2(\pi): X \to D(X/Y)$, respectively.

1.2.2 Elliptic $D_{2p}$-covers

Let $\varphi: S \to \mathbb{P}^1$ be a rational elliptic surface and let $f: S \to \hat{\Sigma}_d$ denote the one int the double cover diagram. For our criterion to distinguish the topology of plane curves, we make use of the existence/non-existence of $D_{2p}$-covers $\pi_p: X_p \to \hat{\Sigma}_d$ satisfying (i) $D(X_p/\hat{\Sigma}_d) = S$ and (ii) $\beta_1(\pi_p) = f$. Following [21], we call such a $D_{2p}$-cover an elliptic $D_{2p}$-cover. We denote the covering transformation of $f$ by $\sigma_f$. As we remark in [21, §3], the branch locus $\Delta_{\beta_2(\pi_p)}$ of $\beta_2(\pi_p)$ is the form

$$D + \sigma_f^*D + \Xi + \sigma_f^*\Xi,$$

where

(i) no irreducible component of $D$ and $\sigma_f^*D$ is contained in any fiber (we call such a divisor horizontal), and there exist no common components between $D$ and $\sigma_f^*D$, and

(ii) all irreducible components $\Xi$ and $\sigma_f^*\Xi$ are fiber components of $\varphi$ and there exist no common components between $\Xi$ and $\sigma_f^*\Xi$.

Remark 1.1 Possible irreducible components of $\Xi$ and $\sigma_f^*\Xi$ can be determined by [21, Remark 3.1]. In particular, if singular fibers of $\varphi$ are of types I$_1$, I$_2$, II, III only, $\Xi_f = \emptyset$.

2 Proof of Theorem 0.1

We first recall the double cover diagram for $S_{Q,z_0}$. In our case, $\hat{\Sigma}_2$ can be successively blown down to $\mathbb{P}^2$. We denote it by $\overline{\eta}: \hat{\Sigma}_2 \to \mathbb{P}^2$. We then have the following combined diagram:

$$\begin{array}{ccc}
S' & \xleftarrow{\mu} & S_{Q,z_0} \\
\downarrow f' & & \downarrow f \\
\Sigma_2 & \xleftarrow{q} & \hat{\Sigma}_2 \xrightarrow{\overline{\eta}} \mathbb{P}^2.
\end{array}$$

Note that $f_{Q,z_0} = \overline{\eta} \circ f$.

Theorem 0.1 will be proved based on [21, Theorem 3.2] and the following lemma.

Lemma 2.1 Let $P_1, P_2, P_3 \in E_{Q,z_0}(\mathbb{C}(t))$ be rational points such that $f_{Q,z_0}(s_{P_i}) = \mathcal{L}_i$.

Then there exists a $D_{2p}$-cover $\pi_p: X_p \to \hat{\Sigma}_2$ such that
\begin{itemize}
  \item $D(X_p/\hat{\Sigma}_2) = S_{Q,z_0}$ and
  \item $\Delta_{\beta_2}(\pi_p) = \sum_{i=1}^3 s_{\pi_p} + \sigma_f^i(\sum_{i=1}^3 s_{\pi_p})$
\end{itemize}

if and only if there exists a $D_{2p}$-cover $\pi_p : X_p \to \mathbb{F}^2$ such that

\begin{itemize}
  \item $\Delta_{\pi_p} = Q + \sum_{i=1}^3 \mathcal{L}_i$ and
  \item the branch locus of $\beta_1(\pi_p) = Q$.
\end{itemize}

\textbf{Proof.} Assume that there exists a $D_{2p}$-cover of $\hat{\Sigma}_2$ described as above. Consider the Stein factorizaiton $\pi_p : X_p \to \mathbb{F}^2$ of $\overline{\pi}_p$. As the branch locus of $\pi_p$ is $\overline{\pi}(\Delta_{\pi_p})$, $\Delta_{\pi_p} = Q + \sum_{i=1}^3 \mathcal{L}_i$ and the covering $\beta_1(\pi_p)$ is branched along $Q$. Conversely, suppose that there exists a $D_{2p}$-cover $\pi_p : X_p \to \mathbb{F}^2$ satisfying the above condition. Take the $\mathbb{C}(\overline{X}_p)$-normalization $\pi_p : X_p \to \hat{\Sigma}_2$ of $\hat{\Sigma}_2$. Since $\Delta_{\beta_1}(\pi_p) = Q$, $\overline{\pi}(\Delta_{\beta_1}(\pi_p)) = Q$. Hence we infer that $D(X_p/\hat{\Sigma}_2)$ is a double cover of $\hat{\Sigma}_2$ branched along $f(\mathcal{O})$ and the proper transform of $Q$, i.e., $D(X_p/\hat{\Sigma}_2) = S_{Q,z_0}$ and $\beta_1(\pi_p) = f$. Let $\beta_1(\pi_p) : X_p \to S_{Q,z_0}$ be the $p$-cyclic cover determined by $\pi_p$. By \cite{21}, Remark 3.1, (i), any irreducible component of singular fibers can not be contained in the branch locus of $\beta_2(\pi_p)$. Hence $\overline{\pi}(f(\Delta_{\beta_2}(\pi_p))) = \sum_{i=1}^3 \mathcal{L}_i$.

By the above lemma, we will be able to choose sections appropriately in constructing our configurations so that dihedral covers exist or do not exist. The difference in (non-)existence allows us to distinguish our configurations topologically.

\textbf{Proof for Combinatorics 1-(a) $\mathcal{E}$:} a smooth cubic. Choose $z_0 \in \mathcal{E} \setminus \{p_1,p_2,p_3\}$ and let $S_{Q,z_0}$ be the rational elliptic surface as in the Introduction. By Section 1, $E_{Q,z_0}(\mathbb{C}(t)) \cong D_4^* \oplus \mathbb{Z}/2\mathbb{Z}$. We choose generators $P_0, P_1, P_2, P_3$ of the $D_4^*$ part such that

$$\langle \{P_i, P_j\} \rangle = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$ 

We denote the 2-torsion point by $P_\tau$. Let $s_{P_i}$ be the corresponding section for each $P_i$ ($i = 0, 1, 2, 3, \tau$). Since $\langle P_i, P_j \rangle$ is determined by the intersection numbers, $s_{P_i}s_{P_j}$, $s_{P_i}O$, and $\text{Contr}_v(s_{P_i})$, $\text{Contr}_v(s_{P_i}, s_{P_j})$, i.e., at which component of each singular fiber $s_{P_i}$ intersects, we have may assume that the following:

(i) By \cite{16} Theorem 10.8, we may assume that $s_{P_i}O = 0$ ($i = 0, 1, 2, 3$).

$$\sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_{P_i}), \sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_{P_i}, s_{P_j}) = 0, 1/2, 1, 3/2, or 2.$$

(ii) For $P_i$ ($i = 1, 2, 3$), $\langle P_0, P_i \rangle = 1$ implies that $\sum_v \text{Contr}_v(s_{P_i}) = 1$. Also for $\{i, j\} \subset \{1, 2, 3\}, i \neq j$, $\langle P_i, \pm P_j \rangle = 1/2$ implies $s_{P_i}s_{P_j} = 0$, $\sum_v \text{Contr}_v(s_{P_i}, s_{P_j}) = 1/2$.

(iii) For $P_0$, $\sum_v \text{Contr}_v(s_{P_0}) = 0$.

(iv) For $P_\tau$, as $P_\tau$ is a torsion, $\langle P_\tau, P_i \rangle = 0$ and we have $\sum_v \text{Contr}_v(s_{P_\tau}) = 2$. 

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From the facts as above, we have that

\[
\begin{align*}
\gamma_{Q,z_0}(P_r) &= (1,1,1,1) & \gamma_{Q,z_0}(P_0) &= (0,0,0,0) & \gamma_{Q,z_0}(P_1) &= (1,1,0,0) \\
\gamma_{Q,z_0}(P_2) &= (1,0,1,0) & \gamma_{Q,z_0}(P_3) &= (1,0,0,1) or (0,1,1,0) \\
\end{align*}
\]

By replacing \( P_3 \) by \( P_3+P_r \), if necessary, we may assume that \( \gamma_{Q,z_0}(P_3) = (1,0,0,1) \). Put \( P_4 := P_2-P_3+P_r \). Then \( \gamma_{Q,z_0}(P_4) = (1,1,0,0) \). We now label the irreducible components of the singular fibers as in the following figure:

![Figure 1](image-url)

(Note that we label singular fibers of type III similarly, if they exist.) We now blow down smooth rational curves \( f(\Theta_{0,0}), f(\Theta_{1,1}), f(\Theta_{2,1}), f(\Theta_{3,1}) \) in this order. The resulting surface is \( \mathbb{P}^2 \) and this is the morphism \( \overline{q} \) in this case. Note that \( f_{Q,z_0} = \overline{q} \circ f \) and \( \overline{q} \circ f(O \cup \Theta_{0,0}) = z_o \).

**Lemma 2.2**

(i) The image of the fixed locus of \([-1]_\varphi\) is a smooth cubic \( \mathcal{E} \) and a transversal line \( \mathcal{L}_o \) to \( \mathcal{E} \). Moreover, \( \overline{q} \circ f(s_{P_i}) = \mathcal{L}_o \).

(ii) \( \{ f_{Q,z_0}(\Theta_{1,1}), f_{Q,z_0}(\Theta_{2,1}), f_{Q,z_0}(\Theta_{3,1}) \} = \mathcal{E} \cap \mathcal{L}_o \). We denote \( p_i = f_{Q,z_0}(\Theta_{i,1}) \).

(iii) Put \( \mathcal{L}_i := f_{Q,z_0}(s_{P_i}) \). Then \( f_{Q,z_0}(s_{P_i}) (i = 1, 2, 3, 4) \) passes through \( p_i \) \( (i = 1, 2, 3) \), respectively. Also \( \mathcal{L}_4 \) passes through \( p_1 \).

(iv) Furthermore, \( (a) \mathcal{L}_i \) is tangent to \( \mathcal{E} \) at a point other than \( p_i \) \((p_1 for \mathcal{L}_4)\) or \( p_i \) \((resp. \ p_1)\) is an inflection point of \( \mathcal{E} \) and \( \mathcal{L}_i \) \((resp. \ 4)\) is an inflectional tangent line.

**Proof.** The statements (i) and (ii) are immediate from our construction of \( S_{Q,z_0} \). We show that the statements (iii) and (iv) hold for \( \mathcal{L}_1 = f_{Q,z_0}(s_{P_1}) \) only, as our proof for the remaining sections are the same. Since \( \gamma_{Q,z_0}(P_1) = (1,1,0,0) \), \( s_{P_1}\Theta_{2,1} = 1 \). This shows that \( f_{Q,z_0}(\Theta_{2,1}) \in \mathcal{L}_1 \). We now go on to (iv). Since \( \gamma_{Q,z_0}(P_1) = (1,1,0,0) \), \( s_{P_1}\Theta_{1,0} = 0 \) and \( z_o \notin \mathcal{L}_1 \). As general fiber \( F \) to \( \varphi_{Q,z_0} \) is mapped to a line \( l \) through \( z_o \) and \( s_{P_1}F = 1 \), \( z_o \notin f_{Q,z_0}(s_{P_1}) \) implies that \( \mathcal{L}_1 \cap l \) consists of only one point. This shows that \( \mathcal{L}_1 \) is a line. If \( \mathcal{L}_1 \) is not the line described in (iv), the closure of \( f_{Q,z_0}^{-1}(\mathcal{L}_1 \setminus (\mathcal{L}_1 \cap \mathcal{E})) \) is irreducible. On the other hand, it must contains both \( s_{P_1} \) and \([−1]_{\varphi_{Q,z_0}}(s_{P_1})\), which is a contradiction. \( \square \)
Remark 2.1 Conversely, from the proof of Lemma 2.2 we observe that:

(i) Any tangent line to $\mathcal{E}$ through $p_i$ gives rise a point $P \in E_{Q,z_o}(\mathbb{C}(t))$ with $\langle P, P \rangle = 1$.

(ii) Any contact conic to $Q$ through $z_o$ gives rise a point $P \in E_{Q,z_o}(\mathbb{C}(t))$ with $\langle P, P \rangle = 2$.

Now let

$$B^1 = \mathcal{E} + \mathcal{L}_0 + \sum_{i=1}^{3} \mathcal{L}_i, B^2 = \mathcal{E} + \mathcal{L}_0 + \sum_{i=2}^{4} \mathcal{L}_i.$$ 

Then by [21] Theorem 3.2 and Lemma 2.1 a $D_{2p}$-cover branched at $2(\mathcal{E} + \mathcal{L}_0) + p(\sum_{i=2}^{4} \mathcal{L}_i)$ exists, but does not exist for $2(\mathcal{E} + \mathcal{L}_0) + p(\sum_{i=1}^{3} \mathcal{L}_i)$. Hence, $(B^1, B^2)$ is a Zariski pair if their combinatorics are the same.

Proof for Combinatorics 1-(b) $\mathcal{E}$: a nodal cubic. Choose $z_0 \in \mathcal{E} \setminus \{p_1, p_2, p_3\}$ and let $S_{Q,z_0}$ be the rational elliptic surface as in the Introduction. By Section 1, $E_{Q,z_0}(\mathbb{C}(t)) \cong (A_1^4)^{\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$. We choose generators $P_1, P_2, P_3$ of the $(A_1^4)^{\oplus 2}$ part such that

$$\langle (P_i, P_j) \rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We denote the 2-torsion point by $P_\tau$. Let $s_{P_i}$ be the corresponding section for each $P_i$ ($i = 1, 2, 3, \tau$)

(i) By [16] Theorem 10.8, we may assume that $s_{P_i}O = 0$, $(i = 0, 1, 2, \tau)$ and

$$\sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_{P_i}), \sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_{P_i}, s_{P_j}) = 0, 1/2, 1, 3/2, \text{or } 2, 5/2.$$

(ii) For $P_i$ ($i = 1, 2, 3$), $\langle P_i, P_i \rangle = 1/2$ implies that $\sum_v \text{Contr}_v(s_{P_i}) = 3/2$. Also for $\{i, j\} \subset \{1, 2, 3\}, i \neq j$, $\langle P_i, P_j \rangle = 0$ implies $s_{P_i}s_{P_j} = 0, \sum_v \text{Contr}_v(s_{P_i}, s_{P_j}) = 1$

(iii) For $P_\tau$, as $P_\tau$ is a torsion section, $\langle P_\tau, P_\tau \rangle = 0$ and we have $\sum_v \text{Contr}_v(s_{P_\tau}) = 2$.

(iv) Furthermore, $\langle P_i, P_\tau \rangle = 0$ implies $\sum_v \text{Contr}_v(s_{P_i}, s_{P_\tau}) = 1$.

From the above facts, we can assume that

$$\gamma_{Q,z_0}(P_1) = (1, 1, 1, 1, 0) \quad \gamma_{Q,z_0}(P_2) = (1, 0, 1, 0, 1) \quad \gamma_{Q,z_0}(P_3) = (1, 1, 0, 0, 1)$$

By replacing $P_3$ by $P_3 + P_\tau$ if necessary, we may assume that $\gamma_{Q,z_0}(P_3) = (1, 0, 0, 1, 1)$. We now label the irreducible components of the singular fibers as in the following figure. (cf. [20] No. 24, p. 90). Note that we use different labelings.)

Now consider $P_{i,j}^\pm = P_i \pm P_j \pm P_\tau$ ($\{i, j\} \subset \{1, 2, 3\}, i \neq j$). Then, since $\gamma_{Q,z_0}$ is a group homomorphism, $s_{P_{i,j}^\pm} \Theta_{1,1} = s_{P_{i,j}^\pm} \Theta_{k,1} = 1, k \neq i, j$. Also, since $\sum_v \text{Contr}(s_{P_{i,j}^\pm}) = 1$, and $\langle P_{i,j}^\pm, P_{i,j}^\pm \rangle = \langle P_{i,j}, P_{i,j} \rangle = 1$, we have $s_{P_{i,j}^\pm}O = 0$. 

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Let \( f : S_{Q,z} \to \Sigma_2 \) be the one in the double cover diagram. We now blow down smooth rational curves \( f(\Theta_0,0), f(O), f(\Theta_1,1), f(\Theta_2,1), f(\Theta_3,1), f(\Theta_4,1) \) in this order. The resulting surface is \( \mathbb{P}^2 \) and this morphism coincides with \( \mathcal{F} \) as in the previous case.

\[ \begin{align*}
    s_{P_1} & \quad s_{P_2} \\
    \Theta_0,0 & \quad \Theta_1,0 \\
    \Theta_0,1 & \quad \Theta_1,1 \\
    \Theta_2,0 & \quad \Theta_2,1 \\
    \Theta_3,0 & \quad \Theta_3,1 \\
    \Theta_4,0 & \quad \Theta_4,1 \\
    O & \\
\end{align*} \]

**Figure 2**

**Lemma 2.3**

(i) The image of the fixed locus of \([-1]_z\) is a nodal cubic \( \mathcal{E} \) and a transversal line \( \mathcal{L}_o \) to \( \mathcal{E} \). Moreover, \( \mathcal{F} \circ f(s_{P_i}) = \mathcal{L}_o \) and the node is the image of \( \Theta_{4,1} \).

(ii) \( \{ f_{Q,z_{o}}(\Theta_{1,1}), f_{Q,z_{o}}(\Theta_{2,1}), f_{Q,z_{o}}(\Theta_{3,1}) \} = \mathcal{E} \cap \mathcal{L}_o \). We denote \( p_i = f_{Q,z_{o}}(\Theta_{i,1}) \)

(iii) For \( \{ i,j,k \} = \{ 1,2,3 \} \), \( f_{Q,z_{o}}(s_{P_{i,j}}^\pm) (i \neq j) \) passes through \( p_k, k \neq i,j \ (i = 1,2,3) \), respectively.

(iv) Put \( \mathcal{L}_k^\pm := f_{Q,z_{o}}(s_{P_{i,j}}^\pm) \). Then either (a) \( \mathcal{L}_k^\pm \) is tangent to \( \mathcal{E} \) at a point distinct to \( p_k \) or \( p_k \) is an inflection point of \( \mathcal{E} \) and \( \mathcal{L}_k^\pm \) is an inflectional tangent line.

**Proof.** We prove the part about the node in statement (i). Since \( s_{P_i}, \Theta_{4,0} = 1 \), the remaining component(s) of the ramification locus must intersect \( \Theta_{4,1} \) at two distinct points. Hence the image of \( f(\Theta_{4,1}) \) gives rise to a node on the branch locus that is not on \( \mathcal{L}_o \). The remaining statements can be proved in a similar way as that in Lemma 2.2. \( \square \)

Finally, as in the Introduction, we put

\[
\begin{align*}
    p_4 := P_{1,3}^+ = P_1 + P_3 + P_7 & \quad p_5 := P_{1,2}^+ = P_1 + P_2 + P_7 \\
    p_6 := P_{2,3}^+ = P_2 + P_3 + P_7 & \quad p_7 := P_{3,1}^- = P_3 - P_1 + P_7 \\
\end{align*}
\]

Now let

\[
\mathcal{B}^1 = \mathcal{E} + \mathcal{L}_0 + \sum_{i=4}^{6} \mathcal{L}_i, \quad \mathcal{B}^2 = \mathcal{E} + \mathcal{L}_0 + \sum_{i=5}^{7} \mathcal{L}_i.
\]

Then by [21] Theorem 3.2 and Lemma 2.2 a \( D_{2p}- \) cover branched at \( 2(\mathcal{E} + \mathcal{L}_0) + p(\sum_{i=5}^{7} \mathcal{L}_i) \) exists, but does not exist for \( 2(\mathcal{E} + \mathcal{L}_0) + p(\sum_{i=4}^{6} \mathcal{L}_i) \). Hence, \( (\mathcal{B}^1, \mathcal{B}^2) \) is a Zariski pair if their combinatorics are the same.
Definition 3.1. Let \( \varphi_{Q,z_0} : S_{Q,z_0} \to \mathbb{P}^1 \) be the rational elliptic surface corresponding to Combinatorics 1-(b). Under the same notation as before, Consider \([2]P_i \ (i = 1, 2, 3)\) and their corresponding sections \(s_{[2]P_i} \ (i = 1, 2, 3)\), respectively. Since \(\langle [2]P_i, [2]P_1 \rangle = 2\) and \(\gamma_{Q,z_0}(s_{[2]P_i}) = (0, 0, 0, 0, 0)\), we infer that \(f_{Q,z_0}(s_{[2]P_i}) \ (i = 1, 2, 3)\) are contact conics \(C_i \ (i = 1, 2, 3)\) to \(Q\) and \(z_0\) is one of the tangent points between \(C_i\) and \(Q\) by the argument in the proof of \([21]\) Theorem 5 (ii), p. 633]. Now, for example, let \(\mathcal{L} = f_{Q,z_0}(s_{P_1})\) and consider two curves

\[
B^1 := Q + \mathcal{L} + C_1, \quad B^2 := Q + \mathcal{L} + C_2.
\]

If there exists a homeomorphism \(h : (\mathbb{P}^2, B^1) \to (\mathbb{P}^2, B^2)\), it satisfies \(h(Q) = Q\). Hence by \([21]\) Proposition 4.4, there exist no homeomorphisms \(\langle \mathbb{P}^2, B^1 \rangle \to \langle \mathbb{P}^2, B^2 \rangle\). Moreover, if both \(B^1\) and \(B^2\) have the Combinatorics 2, then \((B^1, B^2)\) is a Zariski pair. In \(\S 5\), we show that such an example exists.

3 Zariski triple and 4-ple for cubic-conic-line arrangements

In \([5]\), we give examples of Zariski N-plies for conic, conic-quartic arrangements. By similar arguments to those in \([5]\), we give a Zariski triple and 4-ple for cubic-conic-line arrangements. Throughout this section, we use the terminology, notation and results in \([5]\), freely. The combinatorics considered in this section is as follows:

**Combinatorics 3.** Let \(\mathcal{E}, \mathcal{L}_o\) and \(C_i \ (i = 1, 2, 3)\) be as below. Put \(B = \mathcal{E} + \mathcal{L}_o + \sum_{i=1}^{3} C_i:\)

(i) \(\mathcal{E}\): (a) a smooth cubic or (b) a nodal cubic.

(ii) \(\mathcal{L}_o\): a transversal line to \(\mathcal{E}\) and we put \(\mathcal{E} \cap \mathcal{L}_o = \{p_1, p_2, p_3\}\).

(iii) \(C_i \ (i = 1, 2, 3)\): contact conics to \(\mathcal{E} + \mathcal{L}_o\). Each of them is tangent to \(\mathcal{E} + \mathcal{L}_o\) at four points

(iv) \(C_i\) and \(C_j\) intersect transversally for \(i < j\) and \(C_1 \cap C_2 \cap C_3 = \emptyset\).

In the construction of plane curves with Combinatorics 3-(a) and (b), how to find a contact conic \(C\) to \(\mathcal{E} + \mathcal{L}_o\) is crucial and we make use of bisections of elliptic surfaces as we did in \([5]\). Let us recall that a bisection is defined as follows:

**Definition 3.1.** Let \(\varphi : S \to C\) be an elliptic surface over \(C\). Let \(F\) be a general fiber of \(\varphi\). A bisection of \(\varphi\) is a horizontal curve \(D\) with \(FD = 2\). Here, a horizontal curve with respect to \(\varphi\) is a curve that does not contain any fiber components.

Put \(Q = \mathcal{E} + \mathcal{L}_o\) and let \(\varphi_{Q,z_0} : S_{Q,z_0} \to \mathbb{P}^1\) be the rational elliptic surface as before and let \(f_{Q,z_0} : S_{Q,z_0} \to \mathbb{P}^2\) be the generically 2-1 morphism. Likewise \([5]\), we construct a contact conic \(C\) as the image \(f_{Q,z_0}(D)\) of an irreducible bisection \(D\).
Proposition 3.3. Our proof is almost parallel to that of [5, Proposition 5]. Let $P_i (i = 1, 2, 3, 4)$ be the rational points introduced in Proof for Combinatorics 1-(a) of the previous section. Define $Q_i \in E_{Q,x} (C(t)) (i = 1, 2, 3, 4)$ by

$$ [Q_0 + P_1 Q_1 Q_2 Q_3] = [P_0 P_1 P_2 P_3] \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}. $$

Likewise [5, p. 234], we now consider six irreducible bisections $D_0, \ldots, D_5$ as follows:

(i) $s(D_0) = -s_{Q_0}, s(D_i) = -s_{Q_i} (1 \leq i \leq 3)$, $s(D_4) = -s_{Q_2}, s(D_5) = -s_{Q_0 + Q_1}$. Here $s(D_i)$ is the section determined uniquely by $D_i$ in [16, Lemma 5.1].

(ii) $f_{Q,x} (D_i) (i = 0, 1, \ldots, 5)$ are contact conics to $Q$ tangent at 4 distinct points.

(iii) $C_i$ and $C_j$ intersect transversally if $i \neq j$ and $C_1 \cap C_2 \cap C_3 = \emptyset$.

Now if we put

$B^1 := Q + C_1 + C_2 + C_3, B^2 := Q + C_0 + C_1 + C_2, B^3 := Q + C_0 + C_1 + C_4, B^4 := Q + C_0 + C_1 + C_5.$

Then by [5, Theorem 4, Corollary 3, Corollary 4], we have

**Proposition 3.1**

(i) $\text{Cov}_B (\mathbb{P}^2, 2Q + p(C_i + C_j), D_{2p}) \neq \emptyset$ if and only if $\{i, j\} \subset \{1, 2, 3\}$.

(ii) $\text{Cov}_B (\mathbb{P}^2, 2Q + p(C_i + C_j + C_k), D_{2p}) \neq \emptyset$ if and only if $\{i, j, k\} \subset \{1, 2, 3\}$ or $\{0, 1, 5\}$.

From Proposition 3.1 we have

**Proposition 3.2** If $D_0, \ldots, D_5$ as above exist for $Q$, $(B^1, B^2, B^3, B^4)$ is a Zariski 4-ple.

Combinatorics 3-(b). Our proof is almost parallel to that of [4, Theorem 1]. Let $P_i (i = 1, 2, 3)$ be the rational points introduced in Proof for Combinatorics 1-(a) of the previous section. Define $Q_i \in E_{Q,x} (C(t)) (i = 1, 2, 3)$ by $Q_i := [2]P_i (i = 1, 2, 3)$, respectively. We now consider 5 bisections as follows:

(i) $s(D_i) = -s_{Q_i} (1 \leq i \leq 3), s(D_4) = -s_{Q_2}, s(D_5) = -s_{Q_1}$. Here $s(D_i)$ is the section determined uniquely by $D_i$ in [16, Lemma 5.1].

(ii) $f_{Q,x} (D_i) (i = 1, \ldots, 5)$ are contact conics to $Q$ tangent at 4 distinct points.

(iii) $C_i$ and $C_j$ intersect transversally if $i \neq j$ and $C_1 \cap C_2 \cap C_3 = \emptyset$.

Now if we put

$B^1 := Q + C_1 + C_2 + C_3, B^2 := Q + C_1 + C_2 + C_4, B^3 := Q + C_1 + C_4 + C_5.$

Then by [5, Theorem 4], we have

**Proposition 3.3** $\text{Cov}_B (\mathbb{P}^2, 2Q + p(C_i + C_j), D_{2p}) \neq \emptyset$ if and only if $\{i, j\} \subset \{1, 2, 3\}$. 


This shows that there exist no homeomorphism \((\mathbb{P}^2, \mathcal{B}^i) \to (\mathbb{P}^2, \mathcal{B}^j)\) if \(i \neq j\) by a similar argument to the proof of [5] Proposition 3.1. Thus we have

**Proposition 3.4** If \(D_1, \ldots, D_5\) as above exist for \(\mathcal{Q}\), \((\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3)\) is a Zariski triple.

As we see in the next section, the six (resp. five) bisections as above exist for \(\mathcal{Q}\) given by an explicit equation. Thus we have

**Theorem 3.1** There exists a Zariski 4-ple (resp. triple) for Combinatorics 3-(a) (resp. (b)).

**Remark 3.1** (i) By increasing the number of contact conics, our result can be generalized to Zariski 4-plet in the same way as in [5].

(ii) In Figure 2, if we blow down \(f(\Theta_{1,0}), f(O), f(\Theta_{0,1}), f(\Theta_{1,1}), f(\Theta_{2,1})\) and \(f(\Theta_{3,1})\), in this order, the the image \(\mathcal{Q}_1\) of the branch locus \(\Delta_f\) of \(f\) consists of two smooth conics intersecting 4 points. This is the one we consider in [5] Theorem 1.

## 4 Examples

We end this paper by giving explicit examples for Combinatorics 1, 2 and 3. Let us begin with explaining our method to construct explicit bisections briefly introduced in [4]. We here use notation and terminology in [5, 2.2.3] freely.

Let \(U_i \cong \mathbb{C}^2\) \((i = 1, 2)\) be affine open sets of \(\Sigma_2\) with coordinates \((t, x)\) \((i = 1)\) and \((s, x')\) \((i = 2)\) such that \(t = 1/s, x = x'/s^2\). Suppose that \(E_{\mathcal{Q}, \mathcal{Z}_o}\) is given by a Weierstrass equation

\[
E_{\mathcal{Q}, \mathcal{Z}_o} : y^2 = f_T(t, x), \quad f_T(t, x) = x^3 + b_2(t)x^2 + b_3(t)x + b_4(t),
\]

where \(b_i \in \mathbb{C}[t]\), deg \(b_i \leq 4\) and \(f_T\) defines the trisection \(T\) on \(U_i\). Let \(P = (x_P(t), y_P(t)) \in E_{\mathcal{Q}, \mathcal{Z}_o}(\mathbb{C}(t))\). Consider the line in \(\mathbb{A}^2_{\mathbb{C}(t)}\) through \(P\) defined by

\[
L_P : y = l_P(t, x), \quad l_P(t, x) = r(t)(x - x_P(t)) + y_P(t), \quad r(t) \in \mathbb{C}(t).
\]

Then \(f_T(t, x) - l_P(t, x)^2\) factors into the form \((x - x_P(t))g(t, x), g(t, x) \in \mathbb{C}(t)[x]\). Suppose that \(x_P(t), y_P(t) \in \mathbb{C}[t]\) and we choose \(r(t) \in \mathbb{C}[t]\) such that \(g(t, x) \in \mathbb{C}[t, x]\) is irreducible and the total degree of \(g\) is 2. Then the conic \(C(r(t), P)\) given by \(g(t, x) = 0\) is a contact conic to \(\mathcal{Q}\). Moreover, if we put

\[
f_{\mathcal{Q}, \mathcal{Z}_o} C(r(t), P) = C^+ + C^-,
\]

we have (i) \(C^\pm\) are bisections and (ii) (if we choose \(\pm\) suitably) \(s(C^+) = -s_P\). We construct the bisections \(D_i\) in the previous sections in this way. Now we go on to construct examples for each combinatorics.

**Example 4.1 (Combinatorics 1)** Let \([T, X, Z]\) be homogeneous coordinates of \(\mathbb{P}^2\) and let \((t, x) := (T/Z, X/Z)\) be affine coordinates for \(\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Z = 0\}\).

**Combinatorics 1-(a):** Consider \(\mathcal{E}\) and \(\mathcal{L}_o\) given by the affine equations:

\[
\mathcal{E} : x^2 - t^3 - 3t^2 - 2t = 0, \quad \mathcal{L}_o : x = 0.
\]
then we have the following coordinates:

\[ E_{Q,z_o} : y^2 = x(x^2 - t^3 - 3t^2 - 2t) \]

Put \( p_1 = [0,0,1] \), \( p_2 = [-1,0,1] \), and \( p_3 = [-2,0,1] \). Choose \([0,1,0]\) as \( z_o \). Then \( \varphi_{Q,z_o} \) has 4 singular fibers of type III and \( E_{Q,z_o}(\mathbb{C}(t)) \cong D_1^* \oplus \mathbb{Z}/2\mathbb{Z} \). \( P_\tau \) is given by \((0,0)\). We choose \( P_0, P_1, P_2, P_3 \) for a basis of \( D_1^* \)-part as follows:

- \( P_0 := \left( -\frac{1-i}{4} \right) (-t-1+i)^2, \frac{\sqrt{2} - 2i}{8} (-t-1+i)(t+1+i)^2 \),
- \( P_1 := \left( -\sqrt{2} - 1 \right) t, t (\sqrt{2} + t) \sqrt{2} - 1 \).
- \( P_2 := \left[ (1+i)(t+1), \sqrt{-1-i}(t+1)(-t-1+i) \right], \)
- \( P_3 := \left[ -i \left( \sqrt{2} + 1 \right)(t+2), \sqrt{i \left( \sqrt{2} + 1 \right)} \left( t + 2 + \sqrt{2} \right)(t+2) \right], \)

By straightforward computation, we see that \( L_i : x - x_{P_i}(t) = 0 \) are tangent lines for \( E \) through \( p_i \) \( (i = 1, 2, 3) \), while \( x - x_{P_3}(t) = 0 \) is a contact conic to \( Q \) through \( z_o \), and we infer that \( \langle P_0, P_0 \rangle = 2 \) and \( \langle P_3, P_3 \rangle = 1 \).

Also by explicit computation we have

\[ P_4 := P_2 - P_3 + P_\tau = \left[ -(\sqrt{2} + 1)t, -\frac{\sqrt{-1-i}}{2}(i + 1 + i\sqrt{2})(-\sqrt{2} + t)t \right]. \]

Let \( L_4 : x - x_{P_4}(t) = 0 \). Then since \( L_4 \) is a tangent line, \( \langle P_4, P_4 \rangle = 1 \) from Remark [2.3] which implies \( \langle P_2, P_3 \rangle = \frac{1}{2} \). We can compute the height pairing for \( \langle P_4, P_4 \rangle \) \( \{i, j \} \subset \{0, 1, 2, 3\} \) in a similar way and we infer that \( \langle P_6, P_6 \rangle = \frac{1}{2} \) and \( L_i \) and \( \mathcal{Q} + \sum_{i=1}^{4} L_i \) have Combinatorics (1-a).

**Remark 4.1** The generators as above are computed by Ms. Emiko Yorisaki in her master’s thesis [22].

**Combinatorics 1-(b)** Consider

\[ \mathcal{E} : x^2 - t^3 - t^2 = 0, \]
\[ \mathcal{L}_o : 2x - 3t - 3 = 0. \]

Note that \( \mathcal{E} \) has the node at \([0,0,1] \). Put \( p_1 = [-1,0,1] \), \( p_2 = [-3/4,3/8,1] \), and \( p_3 = [3,6,1] \), then \( \mathcal{E} \cap \mathcal{L}_o = \{ p_1, p_2, p_3 \} \). We put \( \mathcal{Q} = E + \mathcal{L}_o \) and choose \([0,1,0] \) as the distinguished point \( z_o \). As we discussed in Section 3 \( E_{Q,z_o}(\mathbb{C}(t)) \cong (A_1^\otimes 3) \oplus \mathbb{Z}/2\mathbb{Z} \) and we can assume lines \( L_i (i = 1, 2, 3) \) connecting the node of \( \mathcal{E} \) and \( p_i (i = 1, 2, 3) \) are generators of \( (A_1^\otimes 3) \)-part. Then we have the following coordinates:

\[ P_\tau = \left[ \frac{3t + 3}{2}, 0 \right], \quad P_1 = \left[ 0, \frac{\sqrt{5}t(t+1)}{2} \right], \quad P_2 = \left[ -\frac{t}{2}, \frac{\sqrt{5}t(t+3)}{4} \right], \quad P_3 = \left[ 2t, \frac{\sqrt{5}t(t-3)}{2} \right]. \]
Furthermore, by computation on $E_{Q,z_0}(C(t))$, we have $P_4, P_5, P_6, \text{ and } P_7$ as follows:

\[
P_4 := P_3 + P_2 + P_7 = \frac{(1 - 2\sqrt{3}) (2 t + 21 + 3\sqrt{3})}{52}, \frac{\sqrt{2}(\sqrt{3} + \sqrt{1})(4 t + 3)(2 t + 3 + \sqrt{3})}{16}
\]

\[
P_5 := P_1 + P_2 + P_7 = \frac{-2\sqrt{1}(t + 1), (2 + \sqrt{1})(t + 1)(t + 2)}{\sqrt{2}}
\]

\[
P_6 := P_2 + P_3 + P_7 = \frac{(1 + 2\sqrt{3}) (2 t + 21 - 3\sqrt{3})}{52}, \frac{\sqrt{2}(\sqrt{3} - \sqrt{1})(4 t + 3)(2 t + 3 - \sqrt{3})}{16}
\]

Let $f_{Q,z_0}$ be as before and $L_i := f_{Q,z_0}(s_{P_i}) (i = 4, 5, 6, 7)$. Put

\[
B^1 := Q + \sum_{i=4}^{6} L_i, \quad B^2 := Q + \sum_{i=5}^{7} L_i.
\]

Then we have a Zariski pair $(B^1, B^2)$.

**Example 4.2 (Combinatorics 2, 3-(b))** We keep the same affine equations of $E, L_o, L_1, L_2,$ and $L_3$ as Example 4.1 1-(b). Choose $[0, 1, 0]$ as the distinguished point $z_0$. As we discussed in Section 3, we have a contact conic $C$ to $Q$ such that (i) $C := f_{Q,z_0}(C^+)$ and (ii) $P_{C^+} = [2]P_1$.

We put $B^1 := Q + C + L_i (i = 1, 2, 3)$. From 0.1 we see that both of $(B^1, B^2)$ and $(B^1, B^3)$ are Zariski pairs having Combinatorics 2.

Next we will give explicit example of Zariski triple with Combinatorics 3-(b). Using the same basis and coordinates given in Example 4.1 1-(b), we obtain $Q_1, Q_2, Q_3$ by explicit calculations as follows:

- $Q_1 = [2]P_1 = \left[\frac{\sqrt{6} t (t^2 - 9 t - 9)}{36}\right]
- Q_2 = [2]P_2 = \left[\frac{1}{\sqrt{2}} (t^2 + 16 t + 16) \cdot \frac{\sqrt{2}}{32} (t^2 - 16 t - 16)\right]
- Q_3 = [2]P_3 = \left[\frac{1}{2} (t^2 + t + 1), \frac{\sqrt{2} t + b_3}{4} (t^2 - t - 1)\right]

We use the method given at the beginning of this section and construct bisections $D_1, \ldots, D_5$ and contact conics $C_1, \ldots, C_5$ corresponding to $-s_{Q_1}, -s_{Q_2}, -s_{Q_3}$. The equations can be calculated by using the data in the following table:

| conic            | rational point | $r(t)$            |
|------------------|----------------|-------------------|
| $C_j, (j = 1, 2, 3)$ | $Q_1$          | $\frac{\sqrt{2}}{4} t + b_{j}$ |
| $C_4$            | $Q_2$          | $\frac{\sqrt{2}}{4} t + b_4$ |
| $C_5$            | $Q_3$          | $\frac{\sqrt{2}}{4} t + b_5$ |

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The explicit equations for $C_j$ ($j = 1, 2, 3$), $C_4$, and $C_5$ to $\mathcal{E} + \mathcal{L}_o$ become as follows:

$$
C_j : b_j (b_j + 6\sqrt{6}) t^2 - 2\sqrt{6} b_j t x + 6 b_j + 3 b_j (3 b_j + 2\sqrt{6}) t - 6 b_j^2 x + 9 b_j^2 = 0,
$$

$$
C_4 : (2 b_4^2 + 30\sqrt{2} b_4 + 33) t^2 - 8(\sqrt{2} b_4 - 1) t x
+ 16 x^2 + 16(2 b_4^2 + 4\sqrt{2} b_4 + 3) t - 8(2 b_4^2 - 1) x + 16(2 b_4^2 + 2\sqrt{2} b_4 + 1) = 0,
$$

$$
C_5 : (b_5^2 - 6) t^2 - 2(\sqrt{2} b_5^2 - 2) t x - 2 x^2 + (b_5^2 - 4\sqrt{2} b_5 - 6) t
+ 2(b_5^2 + 2) x + b_5^2 - 2\sqrt{2} b_5 - 2 = 0.
$$

We put

$$
\mathcal{B}^1 := \mathcal{E} + \mathcal{L}_o + C_1 + C_2 + C_3, \quad \mathcal{B}^2 := \mathcal{E} + \mathcal{L}_o + C_1 + C_2 + C_4, \quad \mathcal{B}^3 := \mathcal{E} + \mathcal{L}_o + C_1 + C_4 + C_5.
$$

If we choose general $b_1, b_2, b_3, b_4, b_5 \in \mathbb{C}$, then we have a Zariski triple $\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3$.

**Example 4.3 (Combinatorics 3-(a))** Let $P_0, P_1, P_2$ and $P_3 \in E_{\mathbb{Q}, z_0}(\mathbb{C}(t))$ be the basis of $D^*_4$-part considered in Example 4.1-(a). Based on these points, we construct $Q_0, Q_1, Q_2, Q_3$ as in $\S 3$. For $Q_0, Q_1, Q_2$, we have explicit coordinates

- $Q_0 := \left[ -\left(\frac{\sqrt{2} + 1}{2}\right) \left( 6 - 2 t \sqrt{2} + t^2 - 4 \sqrt{2} + 4 t \right),
- \frac{1}{8} \sqrt{1 - i} \left( -2 + 4 i + 3 i \sqrt{2} - \sqrt{2} \right) \left( 2 t \sqrt{2} + t^2 + 4 \sqrt{2} - 2 t - 6 \right) \left( -2 + \sqrt{2} - t \right) \right]

- $Q_1 := \left[ \frac{1}{4} (-1 + i) \left( 2 i + 2 it + 2 t + t^2 \right),
- \frac{\sqrt{2}}{16} \left( i \sqrt{2} - 2 i + \sqrt{2} - 2 i \right) \left( -it^2 + t^3 - 2 it + 3 t^2 - 2 - 2 i + 4 t \right) \right]

- $Q_2 := \left[ \frac{1}{4} (1 + i) \left( -t - 1 + i \right)^2, \frac{1}{8} (-1 + i) \sqrt{1 - i} \left( -t - 1 + i \right) \left( t + 1 + i \right)^2 \right]

We use the method given at the beginning of this section and construct bisections $D_0, \ldots, D_5$ and contact conics $C_0, \ldots, C_5$ corresponding to $-s_{Q_0}, -s_{Q_1}, -s_{Q_2}$, and $-s_{Q_0} + Q_1$. The equations can be calculated by using the data in the following table.

| conic   | rational point | $r(t)$                           |
|---------|----------------|----------------------------------|
| $C_0$   | $Q_0$          | $\frac{1}{4} \sqrt{1 - i} \left( -2 i - i \sqrt{2} + \sqrt{2} \right) t + b_0$ |
| $C_j$, ($j = 1, 2, 3$) | $Q_1$          | $\frac{1}{4} \sqrt{1 - i} t + b_j$                       |
| $C_4$   | $Q_2$          | $\frac{1}{4} \sqrt{2 - 2i(i + 1)t + b_4}$                |
| $C_5$   | $Q_1 + Q_2$    | $\frac{1}{4} \left( t - i \right) \left( i \sqrt{2} + i + \sqrt{2} \right) t + b_5$ |

For $C_0$, by using $P = Q_0$ and $r(t) = \left(-2i - i\sqrt{2} + \sqrt{2} \right) \frac{\sqrt{1-i}}{4} t + b_0$ as above, we have the explicit equation
\( \mathcal{C}_0 : \left( t^2 + 4t - 2t\sqrt{2} + 6 - 4\sqrt{2} + 2\sqrt{2}x - 2x \right) b^2 \\
+ \sqrt{1-i} \left( i\sqrt{2}t^2 - i\sqrt{2}tx + 4i\sqrt{2}t - it^2 + 2\sqrt{2}t^2 - \sqrt{2}tx + 4i\sqrt{2} - 6it + 10t\sqrt{2} - 3t^2 \\
+ 2tx - 14 - 6i + 10\sqrt{2} - 14t \right) b \\
- \left( 7 - 5\sqrt{2} \right) \left( 6\sqrt{2}tx - 4x^2\sqrt{2} + 2t\sqrt{2} + 2\sqrt{2}x - t^2 + 8tx - 6x^2 + 2x - 2 \right) = 0 \)

We omit the equations of the other conics as they are rather long. We put
\( \mathcal{B}^1 := Q + C_1 + C_2 + C_3, \mathcal{B}^2 := Q + C_0 + C_1 + C_2, \mathcal{B}^3 := Q + C_0 + C_1 + C_4, \mathcal{B}^4 := Q + C_0 + C_1 + C_5. \)

It can be checked that for a general choice of \( b_0, \ldots, b_5 \), these curves have Combinatorics 3-(a), and we have a Zariski 4-ple.

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