Abstract

We propose a stochastic variance reduced optimization algorithm for solving a class of large-scale nonconvex optimization problems with cardinality constraints, and provide sufficient conditions under which the proposed algorithm enjoys strong linear convergence guarantees and optimal estimation accuracy in high dimensions. We further extend our analysis to an asynchronous variant of the approach, and demonstrate a near linear speedup in sparse settings. Numerical experiments demonstrate the efficiency of our method in terms of both parameter estimation and computational performance.

Index terms—Stochastic optimization, variance reduction, iterative hard thresholding, nonconvex optimization, sparse learning

1 Introduction

High dimensionality is challenging from both the statistical and computational perspectives. To make the analysis manageable, we usually assume that only a small number of variables are relevant for modeling the response variable. In the past decade, a large family of $\ell_1$ regularized or $\ell_1$-constrained sparse estimators have been proposed including Lasso[1], Logistic Lasso [2], Group Lasso [3], Graphical Lasso [4, 5], and more. The $\ell_1$ regularization serves as a convex surrogate for controlling the cardinality of the parameters, and a large family of algorithms such as proximal gradient algorithms [6] have...
been developed for finding $\ell_1$ regularized estimators in polynomial time. However, techniques based on convex relaxation, using $\ell_1$ norm as a surrogate for $\ell_0$ constraint, often incur large estimation bias, and attain worse empirical performance than those based on the cardinality constraint [7, 8]. This motivates us to study a family of cardinality constrained M-estimators. Formally, we consider solving the following nonconvex optimization problem:

$$\min_{\theta \in \mathbb{R}^d} F(\theta) \quad \text{subject to} \quad \|\theta\|_0 \leq k, \quad (1.1)$$

where $F(\theta)$ is a smooth and nonstrongly convex loss function, and $\|\theta\|_0$ denotes the number of nonzero entries in $\theta$ [9, 10].

To solve (1.1), a gradient hard thresholding (GHT) algorithm has been studied in the statistics as well as the machine learning community over the past few years [9–12]. GHT involves performing a gradient update followed by a hard thresholding operation. Let $\mathcal{H}_k(\theta)$ denote a hard thresholding operator that keeps the largest $k$ entries in magnitude and sets the other entries equal to zero. Then, given a solution $\theta^{(t+1)}$ at the $t$-th iteration, GHT performs the following update:

$$\theta^{(t+1)} = \mathcal{H}_k\left(\theta^{(t+1)} - \eta \nabla F(\theta^{(t+1)})\right),$$

where $\nabla F(\theta^{(t+1)})$ is the gradient of the objective at $\theta^{(t+1)}$ and $\eta$ is a step size. Existing literature has shown that under suitable conditions, the GHT algorithm attains linear convergence to an approximately global optimum with optimal estimation accuracy with high probability [9, 10].

The GHT algorithm, though enjoying good convergence rates, is not suitable for solving large-scale problems. The computational bottleneck stems from the fact that the GHT algorithm evaluates the (full) gradient at each iteration; its computational complexity therefore depends linearly on the number of samples. GHT algorithm, therefore, becomes computationally expensive for high-dimensional problems with large sample size.

To address the scalability issue, [13] considers a scenario that is typical in machine learning wherein the objective function decomposes over samples, i.e. where the objective function $F(\theta)$ takes an additive form over many smooth component functions:

$$F(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) \quad \text{and} \quad \nabla F(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\theta),$$

and each $f_i(\theta)$ is associated with a few samples of the entire data set (i.e., the mini-batch setting). In such settings, we can exploit the additive nature of $F(\theta)$ and consider a stochastic gradient hard thresholding (SGHT) algorithm based on unbiased estimates of
the gradient rather than computing the full gradient. In particular, the SGHT algorithm estimates the full gradient $\nabla F(\theta^{t+1})$ by a stochastic gradient $\nabla f_i(\theta^{t+1})$, where $f_i(\theta)$ is uniformly randomly sampled from all $n$ component functions with equal probability at each iteration. Though the SGHT algorithm greatly reduces the computational complexity in each iteration, it can only obtain an estimator with suboptimal estimation accuracy, owing to the variance of the stochastic gradient introduced by random sampling. Moreover, the theoretical analysis in [13] requires $F(\theta)$ to satisfy the Restricted Isometry Property (RIP) with parameter $1/7$, i.e., the restricted condition number of the Hessian matrix $\nabla^2 F(\theta)$ cannot exceed $4/3$ (see more details in Section 3). Taking sparse linear regression as an example, such an RIP condition requires the design matrix to be nearly orthogonal, which is not satisfied by many simple random correlated Gaussian designs [14].

To address the suboptimal estimation accuracy and the restrictive requirement on $F(\theta)$ in the stochastic setting, in this paper, we propose a stochastic variance reduced gradient hard thresholding (SVR-GHT) algorithm. More specifically, we exploit a semi-stochastic optimization scheme to reduce the variance introduced by the random sampling [15, 16]. The SVR-GHT algorithm contains two nested loops: On each iteration of the outer loop, SVR-GHT calculates the full gradient. In the subsequent inner loops, on each iteration the stochastic gradient update is adjusted by the full gradient followed by hard thresholding. This simple modification enables the algorithm to attain linear convergence to an approximately global optimum with optimal estimation accuracy, and meanwhile the amortized computational complexity remains similar to that of conventional stochastic optimization. Moreover, our theoretical analysis is applicable to an arbitrarily large restricted condition number of the Hessian matrix $\nabla^2 F(\theta)$. We provide a summary of comparison between our proposed SVR-GHT algorithm with GHT [10] and SGHT [13] in Table 1. To further boost the computational performance, we extend the SVR-GHT algorithm to an asynchronous variant via a lock-free approach for parallelization [17, 18]. Theoretically, we demonstrate that near linear speedup is achieved for asynchronous SVR-GHT\(^1\).

Several existing algorithms are closely related to our proposed algorithm, including the proximal stochastic variance reduced gradient algorithm [19], stochastic averaging gradient algorithm [20] and stochastic dual coordinate ascent algorithm [21]. However, these algorithms guarantee global linear convergence only for strongly convex optimization problems. Several statistical methods in existing literature are also closely related to cardinality constrained M-estimators, including nonconvex constrained M-estimators [22], and nonconvex regularized M-estimators [23]. These methods usually require somewhat complicated computational formulation and often involve many tuning parameters.

\(^1\)Our main effort in this paper is to demonstrate both theoretical and empirical performances of SVR-GHT. For its asynchronous extension, we only provide its theoretical guarantees and leave the empirical examination as a future investigation.
Table 1: Comparison with GHT[10] and SGHT[13]. Our contributions are manifold: (1) Improving the restrictions on RSC and RSS over SGHT; (2) Improving the iteration complexity and computation complexity over GHT; (3) Improving the statistical performance over SGHT (for sparse linear regression). $\kappa_s$ is the restricted condition number (defined in Section 3), $\sigma$ is the variance of noise and $k^*$ is the number of non-zero entries of the true parameter.

| Method       | Restrictions on $\kappa_s$ | Iteration Complexity | Computation Complexity | Statistical Error (Sparse Linear Regression) |
|--------------|----------------------------|----------------------|------------------------|---------------------------------------------|
| GHT[10]      | No: $\kappa_s$ bounded     | $O(\kappa_s \log(1/\epsilon))$ | $O(n \kappa_s \cdot \log(1/\epsilon))$ | $O(\sigma \sqrt{k^* \log d/(nb)})$         |
| SGHT[13]     | Yes: $\kappa_s \leq \frac{4}{3}$ | $O(\log(1/\epsilon))$ | $O(\log(1/\epsilon))$ | $O(\sigma \sqrt{k^* \log d/b})$            |
| SVR-GHT      | No: $\kappa_s$ bounded     | $O(\log(1/\epsilon))$ | $O([n + \kappa_s] \cdot \log(1/\epsilon))$ | $O(\sigma \sqrt{k^* \log d/(nb)})$         |

Remark: Though the computation complexity of SGHT may seem lower than SVR-GHT, its condition on the RSC and RSS is very restrictive, and it generally converges much slower than SVR-GHT in practice.

We discuss these methods in more details in Section 7.

2 Algorithm

Before we proceed with the proposed algorithm, we introduce some notation. Given an integer $n \geq 1$, we define $[n] = \{1, \ldots, n\}$. Given a vector $v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d$, we define vector norms: $\|v\|_1 = \sum_j |v_j|$, $\|v\|_2^2 = \sum_j v_j^2$, and $\|v\|_\infty = \max_j |v_j|$. Given an index set $I \subseteq [d]$, we define $I^C$ as the complement set of $I$, and $v_I \in \mathbb{R}^d$, where $[v_I]_j = v_j$ if $j \in I$ and $[v_I]_j = 0$ if $j \notin I$. We use $\text{supp}(v)$ to denote the index set of nonzero entries of $v$. Given two vectors $v, w \in \mathbb{R}^d$, we use $\langle v, w \rangle = \sum_{i=1}^d v_i w_i$ to denote the inner product of two vectors. Given a matrix $A \in \mathbb{R}^{d \times d}$, we use $A^\top$ to denote the transpose of $A$, and use $A_{i,\cdot}$ and $A_{\cdot,j}$ to denote the $i$-th row and $j$-th column of $A$. Given an index set $I \subseteq [d]$, we denote the submatrix of $A$ with all row indices in $I$ by $A_{I,\cdot}$, and denote the submatrix of $A$ with all column indices in $I$ by $A_{\cdot,I}$. Moreover, we use the common notations of $\Omega(\cdot)$, $O(\cdot)$, and $\Theta(\cdot)$ to characterize the asymptotics of two real sequences. For logarithmic functions, we denote $\log(\cdot)$ as the natural logarithm when we do not specify the base.

We summarize the proposed stochastic variance reduced gradient hard thresholding (SVR-GHT) algorithm in Algorithm 1. Different from the stochastic gradient hard thresholding algorithm proposed in [13], our SVR-GHT algorithm adopts the semi-stochastic optimization scheme proposed in [15], which can guarantee that the variance introduced
Algorithm 1 Stochastic Variance Reduced Gradient Hard Thresholding Algorithm. $H_k(\cdot)$ is the hard thresholding operator, which keeps the largest $k$ (in magnitude) entries and sets the other entries equal to zero.

**Input:** update frequency $m$, step size parameter $\eta$, sparsity $k$, and initial solution $\tilde{\theta}^{(0)}$.

for $r = 1, 2, \ldots$ do
  $\tilde{\theta} = \tilde{\theta}^{(r-1)}$
  $\tilde{\mu} = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\tilde{\theta})$
  $\theta^{(0)} = \tilde{\theta}$
  for $t = 0, 1, \ldots, m-1$ do
    (S1) Randomly sample $i_t$ from $[n]$
    (S2) $\bar{\theta}^{(t+1)} = \theta^{(t)} - \eta \left( \nabla f_{i_t}(\theta^{(t)}) - \nabla f_{i_t}(\tilde{\theta}) + \tilde{\mu} \right)$
    (S3) $\theta^{(t+1)} = H_k(\bar{\theta}^{(t+1)})$
  end for
  $\tilde{\theta}^{(r)} = \theta^{(m)}$
end for

by stochastic sampling over component functions diminishes with the optimization error.

3 Analysis

Throughout the analysis, we assume that the objective function $F(\theta)$ satisfies the restricted strong convexity (RSC) condition, and the component functions $\{f_i(\theta)\}_{i=1}^{n}$ satisfy the restricted strong smoothness (RSS) condition, which are defined as follows.

**Definition 3.1** (Restricted Strong Convexity Condition). A differentiable function $F$ is restricted $\rho_s^-$-strongly convex at sparsity level $s$ if there exists a constant $\rho_s^- > 0$ such that for any $\theta, \theta' \in \mathbb{R}^d$ with $\|\theta - \theta'\|_0 \leq s$, we have

$$F(\theta) - F(\theta') - \langle \nabla F(\theta'), \theta - \theta' \rangle \geq \frac{\rho_s^-}{2} \|\theta - \theta'\|_2^2. \quad (3.1)$$

**Definition 3.2** (Restricted Strong Smoothness Condition). For any $i \in [n]$, a differentiable function $f_i$ is restricted $\rho_s^+$-strongly smooth at sparsity level $s$ if there exists a uniform constant $\rho_s^+ > 0$ such that for any $\theta, \theta' \in \mathbb{R}^d$ with $\|\theta - \theta'\|_0 \leq s$, we have

$$f_i(\theta) - f_i(\theta') - \langle \nabla f_i(\theta'), \theta - \theta' \rangle \leq \frac{\rho_s^+}{2} \|\theta - \theta'\|_2^2. \quad (3.2)$$

Moreover, we define the restricted condition number $\kappa_s = \rho_s^+/\rho_s^-$. Note that RSS constant $\rho_s^+(i)$ may be different for each component function $f_i$, where we can simply take $\rho_s^+ = \max\{\rho_s^+(i) : i \in [n]\}$.
3.1 Computational Theory

We first present our main result characterizing the error of the objective value and estimation error of parameters at the \( r \)-th outer iteration.

**Theorem 3.3.** Let \( \theta^* \) be a sparse vector of the true model parameter such that \( \| \theta^* \|_0 \leq k^* \). Suppose \( \mathcal{F}(\theta) \) satisfies RSC condition and \( \{ f_i(\theta) \}_{i=1}^n \) satisfy RSS condition with \( s = 2k + k^* \), and Algorithm 1 is invoked with \( k \geq C_1 \kappa_\varepsilon^2 k^* \), \( \eta \rho^+_s \in [C_2, C_3] \) and \( m \geq C_4 \kappa_\varepsilon \) for some constants \( C_1, C_2, C_3 \) and \( C_4 \). Define

\[
\tilde{I} = \operatorname{supp}(H_{2k}(\nabla \mathcal{F}(\theta^*))) \cup \operatorname{supp}(\theta^*).
\]

Then, given \( \alpha = 1 + \frac{2\sqrt{r}}{\sqrt{k-k^*}} \), we have that \( \frac{a^m(a-1)}{\eta \rho_s (1-6\eta \rho_s) (a^{m-1})} + \frac{6\eta \rho^+_s}{1-6\eta \rho_s} \leq \frac{3}{4} \) holds, and Algorithm 1 returns

\[
\mathbb{E}[\mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*)] \leq \left( \frac{3}{4} \right)^r \left[ \mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*) \right] + \frac{6\eta}{(1-6\eta \rho^+_s)} \| \nabla \mathcal{F}(\theta^*) \|_2^2,
\]

and

\[
\mathbb{E}[\| \tilde{\theta}^{(r)} - \theta^* \|_2^2] \leq \sqrt{2 \left( \frac{3}{4} \right)^r \left[ \mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*) \right] \rho^+_s} + \frac{2\sqrt{3}}{\rho^+_s} \| \nabla \mathcal{F}(\theta^*) \|_\infty + \| \nabla \mathcal{F}(\theta^*) \|_2 \sqrt{\frac{12\eta}{(1-6\eta \rho^+_s)} \rho^+_s}.
\]

Moreover, given a constant \( \delta \in (0, 1) \) and a pre-specified accuracy \( \epsilon > 0 \), we need at most

\[
r = \left\lceil 4 \log \left( \frac{\mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*)}{\epsilon \delta} \right) \right\rceil
\]

outer iterations such that with probability at least \( 1-\delta \), we have simultaneously

\[
\mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*) \leq \epsilon + \frac{6\eta}{(1-6\eta \rho^+_s)} \| \nabla \mathcal{F}(\theta^*) \|_2^2,
\]

and

\[
\| \tilde{\theta}^{(r)} - \theta^* \|_2 \leq \sqrt{2 \epsilon \rho^+_s} + \frac{2\sqrt{3}}{\rho^+_s} \| \nabla \mathcal{F}(\theta^*) \|_\infty + \| \nabla \mathcal{F}(\theta^*) \|_2 \sqrt{\frac{12\eta}{(1-6\eta \rho^+_s)} \rho^+_s}.
\]

**Remark 3.4.** Theorem 3.3 has three important implications: (I) Our analysis for SVR-GHT allows \( \kappa_\varepsilon \) to increase with \( (n, d, k^*) \) as long as \( \mathcal{F}(\theta) \) and all \( f_i(\theta) \)'s satisfy RSC and RSS with \( s = \Omega(\kappa_\varepsilon^2 k^*) \). In contrast, the theoretical analysis for SGHT in [13] requires \( \kappa \) not to exceed \( 4/3 \), which can be restrictive; (II) Existing literature has shown that directly calculating \( \min_{\| \theta \|_0 \leq k^*} \mathcal{F}(\theta) \) is NP-hard [24]. But with a suitably chosen \( k = \Omega(\kappa_\varepsilon^2 k^*) \), we can always obtain a good approximation of \( \theta^* \) by SVR-GHT; (III) To get \( \tilde{\theta}^{(r)} \) satisfying (3.5) and (3.6), we need \( \mathcal{O}(\log(1/\epsilon)) \) outer iterations. Since within each outer iteration, we need to calculate a full gradient and \( m \) stochastic variance reduced gradients, the overall computational complexity is

\[
\mathcal{O}(\lceil n + \kappa_\varepsilon \rceil \cdot \log(1/\epsilon)).
\]

In contrast, the overall computational complexity of GHT is \( \mathcal{O}(\kappa_\varepsilon n \log(1/\epsilon)) \). Thus SVR-GHT gains a significant improvement over GHT when \( \kappa_\varepsilon \) is large.
3.2 Statistical Theory

We next present statistical theory of constrained M-estimators obtained by the proposed SVR-GHT algorithm. Our analysis is applicable to a large family of statistical learning problems including low-rank matrix estimation (where the cardinality constraint would be replaced by a rank constraint) and sparse precision matrix estimation. In this paper, we focus here on the two most popular examples: sparse linear regression and generalized linear models, and leave the exploration of other models to future investigation.

**Sparse Linear Regression (I):** We estimate the unknown sparse regression coefficient vector \( \theta^* \in \mathbb{R}^d \) from a noisy observation vector \( y \in \mathbb{R}^{nb} \) of linear measurements: \( y = A\theta^* + z \), where \( A \in \mathbb{R}^{nb \times d} \) is a design matrix, and \( z \in \mathbb{R}^{nb} \) is the random noise vector sampled from a Gaussian distribution with mean 0 and covariance \( \sigma^2 I \). Under the stochastic optimization setting, we divide \( A \) into \( n \) submatrices such that each submatrix contains \( b \) rows of \( A \). For notational simplicity, we define the \( i \)-th submatrix as \( A_{S_i} \), where \( S_i \) denotes the corresponding row indices of \( A \) with \( |S_i| = b \) for all \( i = 1, \ldots, n \). Then, the corresponding optimization problem is

\[
\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b} \| y_{S_i} - A_{S_i} \theta \|_2^2 \quad \text{subject to} \quad \| \theta \|_0 \leq k. \tag{3.7}
\]

We assume that for any \( v \in \mathbb{R}^d \) with \( \| v \|_0 \leq s \), the design matrix \( A \) satisfies

\[
\frac{\| Av \|_2^2}{nb} \geq \psi_1 \| v \|_2^2 - \varphi_1 \frac{\log d}{nb} \| v \|_2^2, \quad \text{and} \quad \frac{\| A_{S_i} v \|_2^2}{b} \leq \psi_2 \| v \|_2^2 + \varphi_2 \frac{\log d}{b} \| v \|_2^2, \quad \forall i \in [n], \tag{3.8}
\]

where \( \psi_1, \psi_2, \varphi_1 \) and \( \varphi_2 \) are constants that do not scale with \( (n, b, k^*, d) \). Existing literature has shown that (3.8) is satisfied by many common examples of sub-Gaussian random design [14, 25]. The next lemma shows that (3.8) implies the restricted strong convexity and smoothness.

**Lemma 3.5.** Assume that the design matrix \( A \) satisfies (3.8). Then, given large enough \( n \) and \( b \), there exist a constant \( C_5 \) and an integer \( k \) such that \( F(\theta) \) and \( \{ f_i(\theta) \}_{i=1}^n \) satisfy the RSC and RSS conditions with \( s = 2k + k^* \), where

\[
k = C_5 k^* \geq C_1 k_s^2 k^*, \quad \rho^- \geq \psi_1 / 2, \quad \text{and} \quad \rho^+ \leq 2\psi_2.
\]

A proof of Lemma 3.5 is provided in Appendix 8.3. Since Lemma 3.5 guarantees that \( F(\theta) \) and \( \{ f_i(\theta) \}_{i=1}^n \) satisfy the RSC and RSS conditions, Theorem 3.3 is applicable to the SVR-GHT algorithm for solving (3.7). This allows us to establish the following statistical guarantee for the obtained estimator.
Theorem 3.6. Suppose that the design matrix $A$ satisfies (3.8), and $k, \eta$ and $m$ are specified as in Theorem 3.3. Then given a constant $\delta \in (0, 1)$, a sufficiently small accuracy $\varepsilon > 0$, and large enough $n$ and $b$, we need at most $r = \left\lceil 4\log\left(\frac{\mathcal{F}(\hat{\theta}(0)) - \mathcal{F}(\theta^*)}{\varepsilon}\right)\right\rceil$ outer iterations such that with high probability, Algorithm 1 returns

$$
\|\hat{\theta}(r) - \theta^*\|_2 = \mathcal{O}\left(\sigma \sqrt{\frac{k^* \log d}{nb}}\right). \tag{3.9}
$$

A proof of Theorem 3.6 is given in Appendix 4.2. Theorem 3.6 guarantees that the obtained estimator attains the optimal rate of convergence in parameter estimation [26], regardless of whether $n$ is allowed to scale with $(b, k^*, d)$ or not. In contrast [13] only considers a fixed $n$ setting, and shows that the estimator obtained by the SGHT algorithm only attains $O(\sigma \sqrt{k^* \log d/b})$ with high probability (See Corollary 5 in [13]). Hence their result is suboptimal w.r.t. $n$, while ours allow $n$ to scale with $(b, k^*, d)$.

Generalized Linear Models (II): We next consider generalized linear models (GLM), which are characterized by the conditional distribution

$$
P(y_i|A_{i*}, \theta, \sigma) = \exp\left\{\frac{y_i A_{i*} \theta - h(A_{i*} \theta)}{a(\sigma)}\right\},
$$

where $a(\sigma)$ is a fixed and known scale parameter and $h(\cdot)$ is the cumulant function, which satisfies [27]

$$
h'(A_{i*} \theta) = \mathbb{E}[y_i|A_{i*}, \theta, \sigma].
$$

Besides, it is also assumed that there exists some constant $c_u$ such that $h''(x) \leq c_u$ for all $x \in \mathbb{R}$. Such boundedness assumption is necessary to establish RSS condition for GLM [23] and it holds under various settings, including linear regression, logistic regression and multinomial regression, but not for Poisson regression.

As in the case of sparse linear regression, we divide $A$ into $n$ submatrices, and each submatrix is denoted by $A_{S_i*}$, where $S_i$ denotes the corresponding $b$ row indices of $A$ and $|S_i| = b$ for all $i = 1, \ldots, n$. Then the generic form of our objective function corresponds to the empirical negative log likelihood, and the corresponding optimization problem is

$$
\min_{\theta \in \mathbb{R}^d} \mathcal{F}(\theta) = \frac{1}{a(\sigma) n} \sum_{i=1}^n \frac{1}{b} \sum_{\ell \in S_i} (h(A_{\ell*} \theta) - y_i A_{\ell*} \theta) \quad \text{subject to } ||\theta||_0 \leq k, \; ||\theta||_2 \leq \tau \tag{3.10}
$$

for some $\tau > 0$. The additional constraint $||\theta||_2 \leq \tau$ in (3.10) may not be necessary in practice, but essential to theoretical analysis; we further expand on this later in this section. Assume that for any $v \in \mathbb{R}^d$ with $||v||_0 \leq s$ and $||v||_2 \leq 2\tau$, the design matrix $A$ satisfies

$$
\max_{j} \frac{\|A_{j*}\|_2}{\sqrt{nb}} \leq 1, \; \mathcal{F}(\theta) \; \text{and} \; \{f_i(\theta)\}_{i=1}^n \text{ satisfy}
$$

$$
v^T \nabla^2 \mathcal{F}(\theta) v \geq \psi_1 ||v||_2^2 - \varphi_1 \frac{\log d}{nb} ||v||_1 ||v||_2, \; \text{and} \; v^T \nabla^2 f_i(\theta) v \leq \psi_2 ||v||_2^2 + \varphi_2 \frac{\log d}{b} ||v||_1 ||v||_2, \tag{3.11}
$$

\[8\]
where $\psi_1, \psi_2, \varphi_1$ and $\varphi_2$ are constants that do not scale with $(n, b, k^*, d)$. Existing literature has shown that (3.11) are satisfied by many common examples of sub-Gaussian random design [23]. The next lemma shows that (3.11) implies the restricted strong convexity and smoothness over an $\ell_2$ ball centered at $\theta^*$ with radius $2\tau$.

**Lemma 3.7.** Suppose that $F(\theta)$ and $\{f_i(\theta)\}_{i=1}^n$ satisfy (3.11). Then given large enough $n$ and $b$, there exist a constant $C_6$ and an integer $k$ such that $F(\theta)$ and $\{f_i(\theta)\}_{i=1}^n$ satisfy the RSC and RSS conditions with $s = 2k + k^*$ for any $\theta$ such that $\|\theta - \theta^*\|_2 \leq 2\tau$, where

$$k = C_6 k^* \geq C_1 \kappa_d^2 k^*, \quad \rho^- \geq \psi_1/2, \quad \text{and} \quad \rho^+ \leq 2\psi_2.$$ 

The proof of Lemma 3.7 is analogous to the proof of Lemma 3.5, thus we omit it here. Lemma 3.7 guarantees that the RSC and RSS conditions hold over a neighborhood of $\theta^*$. Therefore, to guarantee that $\|\theta - \theta^*\|_2 \leq 2\tau$, we require that $\|\theta\|_2 \leq \tau$ and $\|\theta^*\|_2 \leq \tau$. This implies that for any $\theta \in \mathbb{R}^d$, we have $\|\theta - \theta\|_2 \leq \|\theta\|_2 + \|\theta^*\|_2 \leq 2\tau$, which motivated us to consider an additional $\ell_2$-norm constraint in (3.10).

**Remark 3.8 (SVR-GHT with projection).** Due to the additional $\ell_2$-norm constraint, we need a projection step in the SVR-GHT algorithm. In particular, step (S3) would be replaced with $\theta^{(t+1)} = \Pi_r(H_k(\tilde{\theta}^{(t+1)}))$, where $\Pi_r(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an $\ell_2$-norm projection operator defined as $\Pi_r(v) = \max(\|v\|_2, \tau) \cdot v/\|v\|_2$ for any $v \in \mathbb{R}^d$. Since $\Pi_r(\cdot)$ is strictly contractive, i.e., $\|\Pi_r(\theta) - \theta\|_2 \leq \|\theta - \theta\|_2$, our results (i.e., (3.3)-(3.6)) still hold\(^2\) for the SVR-GHT algorithm with this additional projection step, whose details we omit here.

We next establish the statistical properties of the obtained estimator for GLM in the following theorem.

**Theorem 3.9.** Suppose that $A_i$’s have i.i.d. sub-Gaussian rows, $F(\theta)$ and $\{f_i(\theta)\}_{i=1}^n$ satisfy (3.11), $k$, $\eta$ and $m$ are specified as in Theorem 3.3, and $\|\theta^*\|_2 \leq \tau$. Then, given a constant $\delta \in (0, 1)$, a sufficiently small accuracy $\varepsilon > 0$, and large enough $n$ and $b$, we need at most $r = \left\lceil 4\log \left( \frac{F(\theta^*) - F(\theta^*)}{\varepsilon \delta} \right) \right\rceil$ outer iterations such that with high probability, Algorithm 1 (with an additional projection step) returns

$$\|\tilde{\theta}^{(r)} - \theta^*\|_2 = O\left( \sqrt{\frac{k^* \log d}{nb}} \right). \quad (3.12)$$

A proof of Theorem 3.9 is provided in Appendix 4.3. The statistical rate of convergence in Theorem 3.9 matches the state-of-the-art result in parameter estimation for GLM, which is established in [23].

For concreteness, we exemplify GLM using the sparse logistic regression. We estimate the unknown sparse regression coefficient vector $\theta^*$ from $nb$ independent responses

\(^2\)The gap the true objective $F(\theta^*)$ is also contractive after projection due to convexity of $F$. 

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\[ y_\ell \sim \text{Bernoulli}(\pi_\ell(\theta^*)), \ \ell \in [nb], \] where \( \pi_\ell(\theta^*) = \frac{\exp(A_\ell^\top \theta^*)}{1 + \exp(A_\ell^\top \theta^*)} \). Then, the corresponding optimization problem is

\[
\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b} \sum_{\ell \in S_i} (\log[1 + \exp(A_\ell^\top \theta)] - y_\ell A_\ell^\top \theta) \quad \text{subject to } \|\theta\|_0 \leq k, \|\theta\|_2 \leq \tau.
\]

4 Proofs of Main Results

We first present two key technical lemmas that will be instrumental in developing the computational theory for our proposed algorithm and throughout the rest of the paper.

**Lemma 4.1.** Let \( \theta^* \in \mathbb{R}^d \) be a sparse vector such that \( \|\theta^*\|_0 \leq k^* \), and \( \mathcal{H}_k(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \) be the hard thresholding operator, which keeps the largest \( k \) entries (in magnitude) and sets the other entries equal to zero. Given \( k > k^* \), for any vector \( \theta \in \mathbb{R}^d \), we have

\[
\|\mathcal{H}_k(\theta) - \theta^*\|_2^2 \leq \left( 1 + \frac{2\sqrt{k^*}}{\sqrt{k - k^*}} \right) \|\theta - \theta^*\|_2^2. \tag{4.1}
\]

A proof of Lemma 4.1 is provided in Appendix 8.1. Lemma 4.1 shows that the hard thresholding operator is nearly non-expansive when \( k \) is much larger than \( k^* \) such that \( \frac{2\sqrt{k^*}}{\sqrt{k - k^*}} \) is very small.

**Remark 4.2.** Though Lemma 4.1 looks similar to Lemma 1 in [10], they are essentially different. We provide a ratio of the distances between \( \theta \) and a \( k^* \)-sparse vector \( \theta^* \), before and after hard thresholding operation on \( \theta \), which is more intuitive than what is presented in [10] that gives a ratio of the distance between \( \theta \) and \( \mathcal{H}_k(\theta) \), and the distance between \( \theta \) and \( \theta^* \). Besides, Lemma 4.1 is a also a key property that allow us to tolerate a large condition number \( \kappa_s \), compared with a small bounded \( \kappa_s \) in [13], as long as the hard thresholding parameter \( k \) is large enough.

For notational simplicity, we denote the full gradient and the stochastic variance reduced gradient by

\[
\widetilde{\mu} = \nabla \mathcal{F}(\overline{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\overline{\theta}) \quad \text{and} \quad \mathbf{g}^{(t)}(\theta^{(t)}) = \nabla f_i(\theta^{(t)}) - \nabla f_i(\overline{\theta}) + \widetilde{\mu}. \tag{4.2}
\]

The next lemma shows that \( \mathbf{g}^{(t)}(\theta^{(t-1)}) \) is an unbiased estimator of \( \nabla \mathcal{F}(\theta^{(t-1)}) \) with a well controlled second order moment over a sparse support.

**Lemma 4.3.** Suppose that \( \mathcal{F}(\theta) \) and all \( f_i(\theta) \)'s satisfy the RSC and RSS conditions with \( s = 2k + k^* \). Let \( \theta^* \in \mathbb{R}^d \) be a sparse vector with \( \|\theta^*\|_0 \leq k^*, I^* = \text{supp}(\theta^*) \), and \( \theta^{(t)} \) be a sparse vector with \( \|\theta^{(t)}\|_0 \leq k, I^{(t)} = \text{supp}(\theta^{(t)}) \). Then conditioning on \( \theta^{(t)} \), for any \( I \supseteq (I^* \cup I^{(t)}) \), we have \( \mathbb{E}[\mathbf{g}^{(t)}(\theta^{(t)})] = \nabla \mathcal{F}(\theta^{(t)}) \) and

\[
\mathbb{E}\|\mathbf{g}^{(t)}_I(\theta^{(t)})\|_2^2 \leq 12\rho_s^+ \left[ \mathcal{F}(\theta^{(t)}) - \mathcal{F}(\theta^*) + \mathcal{F}(\overline{\theta}) - \mathcal{F}(\theta^*) \right] + 3\|\nabla I \mathcal{F}(\theta^*)\|_2^2. \tag{4.3}
\]
A proof of Lemma 4.3 is given in Appendix 8.2. When \( \theta^* \) is a global minimizer, for convex problems, we have \( \nabla F(\theta^*) = 0 \) (or the differential of \( F(\theta^*) \) contains 0 in composite minimization settings). However, we are working on a nonconvex optimization problem without such a convenience. This eventually results in this additional \( \|\nabla_I F(\theta^*)\|^2_2 \) on the R.H.S of (4.3), which is different from existing variance reduction results in [15, 16].

Remark 4.4. \( \theta^* \) can be arbitrary \( k^* \) sparse vector (1.1) in our analysis. While in establishing the statistical properties of the obtained estimator \( \tilde{\theta}(\tau) \), if \( \theta^* \) is the true model parameter, we have the estimate of expected \( F(\tilde{\theta}(\tau)) \) within the \( \varepsilon + c\|\nabla_I F(\theta^*)\|^2_2 \) distance to the expected \( F(\theta^*) \), which results in that our estimator \( \tilde{\theta}(\tau) \) is within the optimal statistical accuracy to the true model parameter \( \theta^* \).

### 4.1 Proof of Theorem 3.3

Now we are ready to provide the proof of our first main result.

Part 1: We first demonstrate (3.3) and (3.4). let \( v = \theta(t) - \eta g(t) \theta(t) \) and \( I = I^* \cup I(t) \cup I(t+1) \), where \( I^* = \text{supp}(\theta^*) \), \( I(t) = \text{supp}(\theta(t)) \) and \( I(t+1) = \text{supp}(\theta(t+1)) \). Conditioning on \( \theta(t) \), we have the following expectation that

\[
E\|v - \theta^*\|^2_2 = E\|\theta(t) - \eta g(t) \theta(t) - \theta^*\|^2_2
\]

\[
= E\|\theta(t) - \theta^*\|^2_2 + \eta^2 E\|g(t) \theta(t)\|^2_2 - 2\eta \langle \theta(t) - \theta^*, g(t) \theta(t) \rangle
\]

\[
= E\|\theta(t) - \theta^*\|^2_2 + \eta^2 E\|g(t) \theta(t)\|^2_2 - 2\eta \langle \theta(t) - \theta^*, \nabla F(\theta(t)) \rangle
\]

\[
\leq E\|\theta(t) - \theta^*\|^2_2 + \eta^2 E\|g(t) \theta(t)\|^2_2 - 2\eta \left[ F(\theta(t)) - F(\theta^*) \right]
\]

\[
\leq E\|\theta(t) - \theta^*\|^2_2 - 2\eta \left[ F(\theta(t)) - F(\theta^*) \right] + 12\eta^2 \rho^+_s \left[ F(\theta(t)) - F(\theta^*) \right] + 3\eta^2 \|\nabla_I F(\theta^*)\|^2_2
\]

\[
\leq E\|\theta(t) - \theta^*\|^2_2 - 2\eta (1 + 6\eta \rho^+_s) \left[ F(\theta(t)) - F(\theta^*) \right] + 3\eta^2 \|\nabla_I F(\theta^*)\|^2_2,
\]

(4.4)

where the first inequality follows from the RSC condition (3.2) and the second inequality follows from Lemma 4.3.

Since \( \theta(t+1) = \tilde{\theta}(\tau) = v_k \), i.e. \( \theta(t+1) \) is the best \( k \)-sparse approximation of \( v \), then we have from Lemma 4.1,

\[
\|\theta(t+1) - \theta^*\|^2_2 \leq \left( 1 + \frac{2\sqrt{k}}{\sqrt{k-2}} \right) \cdot \|v - \theta^*\|^2_2.
\]

(4.5)

Let \( \alpha = 1 + \frac{2\sqrt{k}}{\sqrt{k-2}} \). Combining (4.4) and (4.5), we have

\[
E\|\theta(t+1) - \theta^*\|^2_2 \leq \alpha E\|\theta(t) - \theta^*\|^2_2 - 2\alpha \eta (1 + 6\eta \rho^+_s) \left[ F(\theta(t)) - F(\theta^*) \right] + 12\alpha \eta^2 \rho^+_s \left[ F(\tilde{\theta}) - F(\theta^*) \right] + 3\alpha \eta^2 \|\nabla_I F(\theta^*)\|^2_2.
\]

(4.6)
Notice that $\tilde{\theta} = \theta^{(0)} = \tilde{\theta}^{(r-1)}$. By summing (4.6) over $t = 0, 1, \ldots, m - 1$ and taking expectation with respect to all $t$'s, we have

$$
\mathbb{E}\|\theta^{(m)} - \theta^*\|^2 + \frac{2\eta(1 - 6\eta\rho^+_s)(\alpha^m - 1)}{\alpha - 1} \mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*)\right]
\leq \alpha^m \mathbb{E}\|\tilde{\theta}^{(r-1)} - \theta^*\|^2 + \frac{12\eta^2 \rho^+_s (\alpha^m - 1)}{\alpha - 1} \mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r-1)}) - \mathcal{F}(\theta^*)\right] + \frac{3\eta^2 (\alpha^m - 1)}{\alpha - 1} \mathbb{E}\|\nabla \mathcal{F}(\theta^*)\|^2,
\quad (4.7)
$$

where the last inequality follows from the RSC condition (3.1) and the definition of $\tilde{I}$. It further follows from (4.7),

$$
\mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*)\right] \leq \left(\frac{\alpha^m (\alpha - 1)}{\eta \rho^+_s (1 - 6\eta \rho^+_s) (\alpha^m - 1)} + \frac{6\eta \rho^+_s}{1 - 6\eta \rho^+_s}\right) \mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r-1)}) - \mathcal{F}(\theta^*)\right] + \frac{3\eta}{2(1 - 6\eta \rho^+_s)} \mathbb{E}\|\nabla \mathcal{F}(\theta^*)\|^2.
\quad (4.8)
$$

Let $\beta = \frac{\alpha^m (\alpha - 1)}{\eta \rho^+_s (1 - 6\eta \rho^+_s) (\alpha^m - 1)} + \frac{6\eta \rho^+_s}{1 - 6\eta \rho^+_s}$ and apply (4.8) recursively, then we have the desired bound (3.3) when $\beta \leq \frac{3}{4} < 1$.

We then demonstrate (3.4). It follows from RSC condition that

$$
\mathcal{F}(\theta^*) \leq \mathcal{F}(\tilde{\theta}^{(r)}) + \langle \nabla \mathcal{F}(\theta^*), \theta^* - \tilde{\theta}^{(r)} \rangle - \frac{\rho^+_s}{2} \|\tilde{\theta}^{(r)} - \theta^*\|^2.
\quad (4.9)
$$

Let $\zeta = \left(\frac{3}{4}\right) \mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*)\right] + \frac{6\eta}{1 - 6\eta \rho^+_s} \mathbb{E}\|\nabla \mathcal{F}(\theta^*)\|^2$. Combining (3.3) and (4.9), we have

$$
\mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r)}) - \zeta\right] \leq \mathcal{F}(\theta^*) \leq \mathbb{E}\left[\mathcal{F}(\tilde{\theta}^{(r)}) + \langle \nabla \mathcal{F}(\theta^*), \theta^* - \tilde{\theta}^{(r)} \rangle - \frac{\rho^+_s}{2} \|\tilde{\theta}^{(r)} - \theta^*\|^2\right].
\quad (4.10)
$$

Using the duality of norms, we have

$$
\mathbb{E}\|\nabla \mathcal{F}(\theta^*), \theta^* - \tilde{\theta}^{(r)}\| \leq \|\nabla \mathcal{F}(\theta^*)\|_\infty \mathbb{E}\|\tilde{\theta}^{(r)} - \theta^*\|_1 \leq \sqrt{s} \|\nabla \mathcal{F}(\theta^*)\|_\infty \mathbb{E}\|\tilde{\theta}^{(r)} - \theta^*\|_2.
\quad (4.11)
$$

Combining (4.10), (4.11) and $\mathbb{E}[x]^2 \leq \mathbb{E}[x^2]$, we have

$$
\frac{\rho^+_s}{2} \left(\mathbb{E}\|\tilde{\theta}^{(r)} - \theta^*\|^2\right) \leq \sqrt{s} \|\nabla \mathcal{F}(\theta^*)\|_\infty \mathbb{E}\|\tilde{\theta}^{(r)} - \theta^*\|_2 + \zeta.
\quad (4.12)
$$

Let $a = \mathbb{E}\|\tilde{\theta}^{(r)} - \theta^*\|_2$. From (4.12), we solve the following quadratic function of $a$,

$$
\frac{\rho^+_s}{2} a^2 - \sqrt{s} \|\nabla \mathcal{F}(\theta^*)\|_\infty a - \zeta \leq 0,
$$

which yields the bound (3.4).
Now we show that with \( k, \eta \) and \( m \) specified in the theorem, we have the guarantee that \( \beta \leq 1 \) provided appropriate choices of constants. More specifically, let \( \eta \leq \frac{C_3}{\rho_s^*} \leq \frac{1}{18\rho_s^*} \), then we have

\[
\frac{6\eta \rho_s^+}{1 - 6\eta \rho_s^+} \leq \frac{6C_3}{1 - 6C_3} \leq \frac{1}{2}.
\]

If \( k \geq C_1 \kappa_s^2 k^* \) and \( \eta \geq \frac{C_1}{\rho_s^*} \) with \( C_2 \leq C_3 \), then we have that \( \alpha \leq 1 + \frac{2}{\sqrt{C_1 - 1} \kappa_s} \) and

\[
\frac{\alpha^m (\alpha - 1)}{\eta \rho_s^-(1 - 6\eta \rho_s^+)(\alpha^m - 1)} \leq \frac{2}{\sqrt{C_1 - 1} \kappa_s} \left(1 - (1 + \frac{2}{\sqrt{C_1 - 1} \kappa_s})^{-m}\right) = \frac{3}{C_2 \sqrt{C_1 - 1} \left(1 - (1 + \frac{2}{\sqrt{C_1 - 1} \kappa_s})^{-m}\right)}.
\]

Then (4.13) is guaranteed to be strictly smaller than \( \frac{1}{7} \), i.e. \( \beta < 1 \), if we have

\[
m \geq \log_{1 + \frac{2}{\sqrt{C_1 - 1} \kappa_s}} \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6}.
\]

Using the the fact that \( \ln(1 + x) > x/2 \) for \( x \in (0, 1) \), it follows that

\[
\log_{1 + \frac{2}{\sqrt{C_1 - 1} \kappa_s}} \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6} = \log \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6} \leq \log \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6} \leq \log \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6} \cdot \sqrt{C_1 - 1} \cdot \kappa_s.
\]

Then (4.14) holds if \( m \) satisfies

\[
m \geq \log \frac{C_2 \sqrt{C_1 - 1}}{C_2 \sqrt{C_1 - 1} - 6} \cdot \sqrt{C_1 - 1} \cdot \kappa_s.
\]

If we choose \( C_1 = 161^2, C_2 = \frac{1}{20}, C_3 = \frac{1}{18} \) and \( C_4 = 222 \), then we have \( \beta \leq \frac{3}{4} \).

Part 2: Next, we demonstrate (3.5) and (3.6). It follows from (3.3),

\[
\mathbb{E} \left[ \mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*) \right] - \frac{6\eta}{1 - 6\eta \rho_s^+} \| \nabla \mathcal{F}(\theta^*) \|_2^2 \leq \left(\frac{3}{4}\right)^r \left[ \mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*) \right].
\]

Let \( \xi_1, \xi_2, \xi_3, \ldots \) be a non-negative sequence of random variables, which is defined as

\[
\xi_r = \max \left\{ \mathcal{F}(\tilde{\theta}^{(r)}) - \mathcal{F}(\theta^*) - \frac{6\eta}{1 - 6\eta \rho_s^+} \| \nabla \mathcal{F}(\theta^*) \|_2^2, 0 \right\}.
\]

For a fixed \( \varepsilon > 0 \), it follows from Markov inequality and (4.15),

\[
\mathbb{P}(\xi_r \geq \varepsilon) \leq \frac{\mathbb{E}[\xi_r]}{\varepsilon} \leq \frac{\left(\frac{3}{4}\right)^r \left[ \mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*) \right]}{\varepsilon}.
\]

For a given \( \delta \in (0, 1) \), let the RHS of (4.16) be no greater than \( \delta \), which requires

$$r \geq \log_{\left(\frac{3}{4}\right)^{-1}} \frac{\mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*)}{\varepsilon \delta}.$$ 

Therefore, we have that if \( r = \left[4 \log \left(\frac{\mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*)}{\varepsilon \delta}\right)\right] \), then (3.5) holds with probability at least \( 1 - \delta \). Finally, (3.6) holds consequently via combining (3.4) and (3.5).
4.2 Proof of Theorem 3.6

For sparse linear model, we have $\nabla F(\theta^*) = A^T z/(nb)$. Since $z$ has i.i.d. $N(0, \sigma^2)$ entries, then $A^T_j z/(nb) \sim N(0, \sigma^2 ||A^*_j||^2/(nb)^2)$ for any $j \in [d]$. Using the Mill’s inequality for tail bounds of Normal distribution, we have

$$P \left( \left| \frac{A^T_j z}{nb} \right| > 2\sigma \sqrt{\frac{\log d}{nb}} \right) = P \left( \left| \frac{A^T_j z}{\sigma ||A^*_j||_2} \right| > 2 \frac{\sqrt{nb \log d}}{||A^*_j||_2} \right) \leq ||A^*_j||_2 \sqrt{\frac{1}{2\pi nb \log d}} \exp \left( -\frac{4 nb \log d}{||A^*_j||_2^2} \right).$$

This implies, using union bound and the assumption $\frac{\max_j ||A^*_j||_2}{\sqrt{nb}} \leq 1$,

$$P \left( \left\| \frac{A^T_j z}{nb} \right\|_\infty > 2\sigma \sqrt{\frac{\log d}{nb}} \right) \leq d^{-4} \frac{d^{-4}}{\sqrt{2\pi \log d}}.$$

Then we have the following result holds with probability at least $1 - \frac{1}{\sqrt{2\pi \log d}} \cdot d^{-4}$

$$\|\nabla F(\theta^*)\|_\infty \leq \left\| \frac{A^T z}{nb} \right\|_\infty \leq 2\sigma \sqrt{\frac{\log d}{nb}}. \quad (4.17)$$

Conditioning on (4.17), it follows consequently that

$$\|\nabla \tilde{F}(\theta^*)\|_2^2 \leq s \|\nabla F(\theta^*)\|_\infty^2 \leq \frac{4\sigma^2 s \log d}{nb}. \quad (4.18)$$

We have from Lemma 3.5 that $s = 2k + k^* = (2C_5 + 1)k^*$ for some constant $C_5$ when $n$ and $b$ are large enough. For a given $\varepsilon > 0$ and $\delta \in (0,1)$, if

$$r \geq 4\log \left( \frac{\mathcal{F}(\tilde{\theta}^{(0)}) - \mathcal{F}(\theta^*)}{\varepsilon \delta} \right),$$

then with probability at least $1 - \delta - \frac{1}{\sqrt{2\pi \log d}} \cdot d^{-4}$, we have from (3.4), (4.17) and (4.18) that

$$\|\tilde{\theta}^{(r)} - \theta^*\|_2 \leq c_3 \sigma \sqrt{\frac{k^* \log d}{nb}}, \quad (4.19)$$

for some constant $c_3$, which completes the proof.

4.3 Proof of Theorem 3.9

The only difference between the proof of Theorem 3.9 and the proof of Theorem 3.6 is the upper bounds of $\|\nabla F(\theta^*)\|_\infty$ and $\|\nabla \tilde{F}(\theta^*)\|_2^2$. When $A_1$’s are sub-Gaussian, it follows from [23] that for some constants $c_4, c_5$ and $c_6$, with probability at least $1 - c_4 d^{-c_5}$, we have

$$\|\nabla F(\theta^*)\|_\infty \leq c_6 \sqrt{\frac{\log d}{nb}}. \quad (4.20)$$
Conditioning on (4.20), it follows consequently that
\[ \|\nabla \tilde{I} F(\theta^*)\|_2^2 \leq s \|\nabla F(\theta^*)\|_\infty^2 \leq \frac{c_6^2 s \log d}{nb}. \] (4.21)

The rest of the proof follows immediately from the proof of Theorem 3.6.

5 Asynchronous SVR-GHT

We provide an extension of SVR-GHT to its asynchronous variant. More specifically, we assume a parallel processing architecture, where each processor makes stochastic gradient updates of a global parameter \( \theta \) stored in shared memory, via a lock-free manner. We make two important assumptions, following the notions in [17, 18]. The first assumption is that the time delay in between the evaluation \( t' \) and updating \( t \) is bounded above by a positive integer \( \varsigma \), i.e., \( t - t' \leq \varsigma \), which captures the degree of parallelism. The second assumption is the sparsity of data. Formally, we assume \( \mathbb{E} \|\theta_e\|_2^2 \leq \Delta \|\theta\|_2^2 \), where \( e \subseteq [d] \) and \( \theta_e \) denotes the non-zero entries of \( \theta \) indexed by \( e \). Our particular interest is when \( \Delta \ll 1 \). We only discuss the computational theory for asynchronous SVR-GHT. The statistical analysis follows analogously to SVR-GHT.

5.1 Computational Theory

We present our main result characterizing the error of objective value and estimation error of parameters at the \( r \)-th outer iteration of asynchronous SVR-GHT.

**Theorem 5.1.** Suppose that \( F(\theta) \) satisfies RSC condition and \( \{f_i(\theta)\}_{i=1}^n \) satisfy RSS condition with \( s = 2k + k^* \), and there exist constants \( C_1, C_2, C_3, C_4, \) and \( C_5 \) such that \( k \geq C_1 \kappa^2 s^* \), \( \eta \rho_+ \in [C_2, C_3] \), \( m \geq C_4 \kappa s \) and \( \Delta \varsigma^2 \leq C_5 \). Define \( \theta^* \) to be the true model parameter and \( \tilde{I} = \text{supp}(H_{2k}(\nabla F(\theta^*))) \cup \text{supp}(\theta^*) \).

Then, given some \( \alpha = 1 + \frac{2\sqrt{s}}{\sqrt{k} - k} \) satisfying \( \left( \frac{1}{\eta \rho_+}, \frac{a^m(a-1)}{1-12\eta \rho_+ \Gamma}, \frac{1 - 12\eta \rho_+ \Gamma}{1 - 12\eta \rho_+ \Gamma} \right) \leq \frac{5}{6} \), where \( \Gamma = \frac{1 + \rho_+ \Delta \varsigma^2 \eta}{1 - 2\rho_+ \Delta \varsigma^2 \eta^2} \), asynchronous version of Algorithm 1 returns

\[ \mathbb{E} \left[ F(\tilde{\theta}^{(r)}) - F(\theta^*) \right] \leq \left( \frac{5}{6} \right)^r \left[ F(\tilde{\theta}^{(0)}) - F(\theta^*) \right] + \frac{18\eta \Gamma}{1 - 12\eta \rho_+ \Gamma} \|\nabla \tilde{I} F(\theta^*)\|_2^2, \]

and

\[ \mathbb{E} \|\tilde{\theta}^{(r)} - \theta^*\|_2 \leq \sqrt{\frac{2\left( \frac{5}{6} \right)^r \|\nabla F(\tilde{\theta}^{(0)}) - F(\theta^*)\|_\infty}{\rho_s}} + \frac{2\sqrt{s}\|\nabla F(\theta^*)\|_\infty}{\rho_s} + \|\nabla \tilde{I} F(\theta^*)\|_2 \sqrt{\frac{36\eta \Gamma}{(1 - 12\eta \rho_+ \Gamma)\rho_s^2}}. \]
Moreover, given a constant $\delta \in (0, 1)$ and a pre-specified accuracy $\varepsilon > 0$, we need at most
\[
r = \left\lceil 4 \log \left( \frac{\mathcal{F}(\hat{\theta}^{(0)}) - \mathcal{F}(\theta^*)}{\varepsilon \delta} \right) \right\rceil
\]
outer iterations such that with probability at least $1 - \delta$, we have simultaneously
\[
\mathcal{F}(\hat{\theta}^{(r)}) - \mathcal{F}(\theta^*) \leq \varepsilon + \frac{18\eta \Gamma}{1 - 12\eta \rho_s^4} \| \nabla_{\mathcal{I}} \mathcal{F}(\theta^*) \|_2^2,
\]
and
\[
\|\hat{\theta}^{(r)} - \theta^*\|_2 \leq \sqrt{\frac{2\varepsilon}{\rho_s^2}} + \frac{2\sqrt{3} \| \nabla \mathcal{F}(\theta^*) \|_\infty}{\rho_s^2} + \| \nabla_{\mathcal{I}} \mathcal{F}(\theta^*) \|_2 \sqrt{\frac{36\eta \Gamma}{(1 - 12\eta \rho_s^4) \rho_s^6}}.
\]

Theorem 5.1 indicates that the sufficient numbers of iterations are of the same order for SVR-GHT and asynchronous SVR-GHT to achieve the same order of errors for objective values and parameter estimations. Therefore, when $\Delta \zeta^2 = \mathcal{O}(1)$, asynchronous SVR-GHT can be $\zeta$ times faster than SVR-GHT due to the parallelism (if we ignore the time for data transmission and so forth). This is analogous to the conclusion for stochastic variance reduced gradient descent algorithm for convex problems [18].

### 5.2 Proof of Theorem 5.1

Recall from (4.2) that $g^{(t)}(\theta^{(t)}) = \nabla f_i(\theta^{(t)}) - \nabla f_i(\hat{\theta}) + \nabla \mathcal{F}(\hat{\theta})$. We also denote $u = \theta^{(t)} - \eta h^{(t)}(\theta^{(t)})$, where $h^{(t)}(\theta^{(t)}) = \nabla f_i(\theta^{(t)}) - \nabla f_i(\hat{\theta}) + \nabla \mathcal{F}(\hat{\theta})$ for the actual evaluation at $t$-th iteration. Then we have

\[
\mathbb{E}\|u - \theta^*\|_2^2 = \mathbb{E}\|\theta^{(t)} - \eta h^{(t)}(\theta^{(t)}) - \theta^*\|_2^2
\]
\[
= \mathbb{E} \left[ \|\theta^{(t)} - \theta^*\|_2^2 + \eta^2 \|h^{(t)}(\theta^{(t)})\|_2^2 - 2\eta \langle \theta^{(t)} - \theta^*, h^{(t)}(\theta^{(t)}) \rangle \right] \tag{5.1}
\]

Following the result in [18] (Theorem 2)\(^3\), we have

\[
\mathbb{E}\langle \theta^{(t)} - \theta^*, h^{(t)}(\theta^{(t)}) \rangle \geq \mathbb{E} \left[ \mathcal{F}(\theta^{(t)}) - \mathcal{F}(\theta^*) - \rho_s^2 \Delta \zeta^2 \eta^2 \sum_{j=1}^{t-1} \|h^{(j)}(\theta^{(j)})\|_2^2 \right]. \tag{5.2}
\]

and

\[
\sum_{t=0}^{m-1} \mathbb{E}\|h^{(t)}(\theta^{(t)})\|_2^2 \leq \frac{2}{1 - 2\rho_s^2 \Delta \zeta^2 \eta^2} \sum_{t=0}^{m-1} \mathbb{E}\|g^{(t)}(\theta^{(t)})\|_2^2. \tag{5.3}
\]

\(^3\)The hard thresholding operation onto the set $\mathcal{I}$ does not affect the result since it contains all non-zero entries of $\theta^{(t)} - \theta^*$ and $\theta^{(t+1)} - \theta^{(t)}$. Also note that the RSS condition used in [18] is $\|f_i(\theta) - f_i(\theta^*)\|_2 \leq \rho_s^2 \|\theta - \theta^*\|_2$, which is equivalent to our RSS condition in Definition 3.2 by [28]. We assume this version of RSS for the analysis of asynchronous SVR-GHT.
Combing (5.1) and (5.2), we have
\[ E\|u - \theta^*\|^2_2 \leq E \left[ \|\theta(t) - \theta^*\|^2_2 + \eta^2 \|h_t(\theta(t))\|^2_2 - 2\eta \left( F(\theta(t)) - F(\theta^*) \right) + \rho_s^+ \Delta c\eta^3 \sum_{j=t'}^{t-1} \|h(j)(\theta(j))\|^2_2 \right] \]
\[ (5.4) \]

The rest follows the analysis of Theorem 3.3. Specifically, by summing (5.4) over \( t = 0, 1, \ldots, m - 1 \), taking expectation with respect to all \( t \)'s, and combining Lemma 4.1, Lemma 4.3 and (5.3), we have
\[ E\|\theta^{(m)} - \theta^*\|^2_2 + \frac{2\eta (1 - 12\rho_s^+ \eta \Gamma)(\alpha^m - 1)}{\alpha - 1} E\left[ F(\bar{\theta}(r)) - F(\theta^*) \right] \leq \left( \frac{2\alpha^m}{\rho_s^+} + \frac{24\rho_s^+ \eta^2 \Gamma (\alpha^m - 1)}{\alpha - 1} \right) E\left[ F(\bar{\theta}^{(r-1)}) - F(\theta^*) \right] + \frac{6\eta^2 \Gamma (\alpha^m - 1)}{\alpha - 1} \|\nabla_{\theta} F(\theta^*)\|^2_2, \]
\[ (5.5) \]
where \( \Gamma = \frac{1+\rho_s^+ \Delta c^2 \eta}{1-2\rho_s^+ \Delta c^2 \eta^2} \). It further follows from (5.5)
\[ E\left[ F(\bar{\theta}(r)) - F(\theta^*) \right] \leq \left( \frac{\alpha^m (\alpha - 1)}{\eta \rho_s^+ (1 - 12\eta \rho_s^+ \Gamma)(\alpha^m - 1)} + \frac{12\eta \rho_s^+ \Gamma}{1 - 12\eta \rho_s^+ \Gamma} \right) E\left[ F(\bar{\theta}^{(r-1)}) - F(\theta^*) \right] \]
\[ + \frac{3\eta \Gamma}{1 - 12\eta \rho_s^+ \Gamma} \|\nabla_{\theta} F(\theta^*)\|^2_2. \]
\[ (5.6) \]
Finally, \( \frac{\alpha^m (\alpha - 1)}{\eta \rho_s^+ (1 - 12\eta \rho_s^+ \Gamma)(\alpha^m - 1)} + \frac{12\eta \rho_s^+ \Gamma}{1 - 12\eta \rho_s^+ \Gamma} \leq \frac{5}{6} \) holds with the same choices of \( C_1 \sim C_4 \) as in Theorem 3.3 and \( C_5 = \frac{1}{2} \).

6 Experiments

We compare the empirical performance of the SVR-GHT algorithm with two other candidate algorithms: GHT proposed in [10] and SGHT proposed in [13] on both synthetic data and real data. We also discuss the performance of parameter estimation between cardinality constrained problem (1.1) and \( \ell_1 \) regularized problem solved by the stochastic variance reduced gradient (SVRG) type of method.

6.1 Simulated Data

We consider a sparse linear regression problem. We generate each row of the design matrix \( A_i \) independently from a \( d \)-dimensional Gaussian distribution with mean 0 and covariance matrix \( \Sigma \in \mathbb{R}^{d \times d} \). The response vector is generated from the linear model \( y = A\theta^* + \epsilon \), where \( \theta^* \in \mathbb{R}^d \) is the regression coefficient vector, and \( \epsilon \) is generated from an \( n \)-dimensional Gaussian distribution with mean 0 and covariance matrix \( \sigma^2 \mathbf{I} \). We set
\(nb = 10000, d = 25000, k^* = 200\) and \(k = 500\). For \(\Sigma\), we set \(\Sigma_{ii} = 1\) and \(\Sigma_{ij} = c\) for some constant \(c \in (0, 1)\) for all \(i \neq j\). The nonzero entries in \(\theta^*\) are sampled independently from a uniform distribution over the interval \((-2, +2)\). We divide 10000 samples (rows of the design matrix) into \(n\) mini batches evenly, and each mini batch contains \(b = 10000/n\) samples.

Figure 1 illustrates the computational performance of the GHT, SGHT, and SVR-GHT algorithms for eight different settings. First four settings are noiseless, i.e., \(\sigma = 0\) with (1) \((n, b) = (10000, 1), \Sigma_{ij} = 0.1\); (2) \((n, b) = (10000, 1), \Sigma_{ij} = 0.5\); (3) \((n, b) = (200, 50), \Sigma_{ij} = 0.1\); (4) \((n, b) = (200, 50), \Sigma_{ij} = 0.5\). For simplicity, we choose \(m = n\) throughout our experiments\(^4\). The last four settings are noisy with \(\sigma = 1\) and identical choices of \((n, b), \Sigma_{ij}\) and \(m\) as in (1)-(4). Since the SGHT and SVR-GHT algorithms are stochastic, we plot the objective values averaged over 50 different runs. We illustrate step sizes \(\eta = 1/256, 1/512\) and \(1/1024\). The horizontal axis corresponds to the number of passes over the entire dataset; computing a full gradient is counted as 1 pass, while computing a stochastic gradient is counted as \(1/n\)-th of a pass. The vertical axis corresponds to the ratio of current objective value over the objective value using \(\tilde{\theta}^{(0)} = 0\). We further provide the optimal relative estimation error \(\|\tilde{\theta}^{(10^6)} - \theta^*\|_2 / \|\theta^*\|_2\) after \(10^6\) effective passes of the entire dataset for each setting of three algorithms. The optimal estimation error is obtained by averaging over 50 different runs, each of which is chosen from a sequence of step sizes \(\eta \in \{1/2^5, 1/2^6, \ldots, 1/2^{14}\}\).

| Method | \(\sigma = 0\) | \(\sigma = 1\) |
|--------|----------------|-----------------|
|        | \(\Sigma_{ij} = 0.1\) | \(\Sigma_{ij} = 0.5\) | \(\Sigma_{ij} = 0.1\) | \(\Sigma_{ij} = 0.5\) |
| \((n, b)_1\) | \(< 10^{-20}\) | \(< 10^{-20}\) | 0.00851 | 0.02940 |
| \((n, b)_2\) | \(< 10^{-20}\) | \(< 10^{-20}\) | 0.13885 | 0.02490 |
| GHT    | \(< 10^{-20}\) | \(< 10^{-20}\) | 0.00968 | 0.00970 |
| SGHT   | \(< 10^{-20}\) | \(< 10^{-20}\) | 0.13885 | 0.06412 |
| SVR-GHT | \(< 10^{-20}\) | \(< 10^{-20}\) | \(< 10^{-20}\) | 0.02823 |

We see that SVR-GHT uniformly outperforms the other two candidate algorithms in terms of the convergence rate under all eight settings. While GHT also enjoys linear con-

\(^4\)Larger \(m\) results in slightly increasing number of effective passes of the entire dataset required to achieve the same decrease of objective values, which is also observed in a closed related proximal gradient method with variance reduction [19]
Figure 1: Comparison among the three candidate algorithms under eight different settings on simulated data sets. The horizontal axis corresponds to the number of passes over the entire dataset. The vertical axis corresponds to the ratio of current objective value over the objective value using $\tilde{\theta}^{(0)} = 0$. For each algorithm, option 1, 2 and 3 correspond to $\eta = 1/256,1/512$ and $1/1024$ respectively. It is evident from the plots that SVR-GHT uniformly outperforms the other candidate algorithms in terms of the convergence rate in each of the eight settings.
verge guarantees, its computational complexity within each iteration is \( n \) times larger than SVR-GHT; consequently, its performance is much worse than that of SVR-GHT. Besides, we also see that SGHT converges worse than SVR-GHT in all settings. This is perhaps because the largest eigenvalue of any 500 by 500 submatrix of the covariance matrix is large (larger than 50 or 250) such that the underlying design matrix violates the Restricted Isometry Property (RIP). This might explain the poor performance of SGHT. On the other hand, the optimal estimation error of SVR-GHT is comparable to GHT, both of which outperform SGHT uniformly, especially in noisy settings. It is important to note that with the optimal step size, the estimation of GHT usually becomes stable after \( > 10^5 \) passes, while the estimation of SVR-GHT usually becomes stable within a few dozen to a few hundred passes, which validates the significant improvement of computational cost of SVR-GHT over GHT.

### 6.2 Real Data

We adopt a subset of RCV1 dataset with 9625 documents and 29992 distinct words, including the classes of “C15”, “ECAT”, “GCAT”, and “MCAT” [29]. We apply logistic regression to perform binary classification for all classes, each of which uses 5000 documents for training, i.e., \( nb = 5000 \) and \( d = 29992 \), with the same proportion of documents from each class, and the rest for testing. We illustrate the computational performance of the GHT, SGHT, and SVR-GHT algorithms with two different settings for each class: Setting (1) has \((n, b) = (5000, 1)\); Setting (2) has \((n, b) = (100, 50)\). We choose \( k = 200 \) and \( m = n \) for both settings of all classes. For all three algorithms, we plot their objective values and provide the optimal classification errors averaged over 10 different runs using different data separations. Figure 2 demonstrates the result for “C15”, and the other classes have analogous performance. The horizontal axis corresponds to the number of passes over the entire dataset. The vertical axis corresponds to the misclassification rate on the test dataset. Similar to the synthetic numerical evaluations, SVR-GHT uniformly outperforms the other two candidate algorithms in terms of the convergence rate under both settings. We further provide the optimal misclassification rates of all classes for three algorithms in Table 3, where the optimal step size \( \eta \) for each algorithm is chosen from a sequence of values \( \{1/2^5, 1/2^6, \ldots, 1/2^{14}\} \). Similar to the simulated data sets again, the optimal misclassification rate of SVR-GHT is comparable to GHT, both of which outperform SGHT uniformly. The estimation of GHT generally requires \( > 10^6 \) passes to become stable, while the estimation of SVR-GHT generally requires a few hundred to a few thousand passes to be stable, which validates the significant improvement of computational cost of SVR-GHT over GHT for this real dataset.
Figure 2: Comparison among the three candidate algorithms under two different settings on the RCV1 data set for class C15, ECAT, GCAT and MCAT. The horizontal axis corresponds to the number of passes over the entire dataset. The vertical axis corresponds to misclassification rate on the test data. It is evident from the plots that SVR-GHT uniformly outperforms the other candidate algorithms in both settings.

Table 3: Comparison of optimal classification errors among the three candidate algorithms for both settings of all four classes. We denote \((n,b)_1 = (5000,1)\) and \((n,b)_2 = (100,50)\). SVR-GHT achieves comparable result with GHT, both of which uniformly outperforms SGHT in each of the eight settings.

|        | C15         | ECAT        | GCAT        | MCAT        |
|--------|-------------|-------------|-------------|-------------|
|        | \((n,b)_1\) | \((n,b)_2\) | \((n,b)_1\) | \((n,b)_2\) |
| \hline
| GHT    | 0.02844     | 0.05581     | 0.03028     | 0.05703     |
| SGHT   | 0.03259     | 0.03361     | 0.06851     | 0.07179     | 0.06263     | 0.09142     | 0.07638     | 0.08228     |
| SVR-GHT| 0.02826     | 0.02867     | 0.05628     | 0.05631     | 0.03354     | 0.03444     | 0.05877     | 0.05927     |

### 6.3 \(\ell_0\) Norm vs. \(\ell_1\) Norm via SVRG

We further discuss the empirical performance of sparsity induced problems using \(\ell_0\) norm and \(\ell_1\) norm respectively, solved via SVRG type of method. For the purpose of illustration, we consider the sparse linear regression problem (3.7) for the \(\ell_0\) norm problem and the following constrained minimization for the \(\ell_1\) norm problem,

\[
\min_{\theta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{b} \| y_{S_i} - A_{S_i} \theta \|_2^2 + \lambda \| \theta \|_1, \tag{6.1}
\]

where \(\lambda > 0\) is a regularization parameter. The \(\ell_0\) norm problem (3.7) is solved by SVR-GHT, and the \(\ell_1\) norm problem (6.1) is solved by proximal SVRG method (Prox-SVRG) [19]. We follow the same settings as in Section 6.1 for data generation and choice of parameters for SVR-GHT. For \(\ell_1\) norm problem (6.1), we choose the optimal regularization...
parameter \( \lambda \) from a sequence \( \{1/2^2, 1/2^4, 1/2^6, \ldots, 1/2^{20}\} \) that returns the optimal relative estimation error \( \| \tilde{\theta}^{(10^6)} - \theta^* \|_2 / \| \theta^* \|_2 \).

Table 4: Comparison of optimal relative estimation errors between (3.7) and (6.1) under eight different settings on simulated data sets. We denote \((n, b)_1 = (10000, 1)\) and \((n, b)_2 = (200, 50)\).

| Method | \( \Sigma = 0.1 \) | \( \Sigma = 0.5 \) | \( \Sigma = 0.1 \) | \( \Sigma = 0.5 \) |
|--------|----------------|----------------|----------------|----------------|
| \( \ell_0 \) norm | \( < 10^{-20} \) | \( < 10^{-20} \) | \( < 10^{-20} \) | \( 0.00968 \) |
| \( \ell_1 \) norm | \( \approx 10^{-6} \) | \( \approx 10^{-7} \) | \( \approx 10^{-6} \) | \( 0.01715 \) |

Table 4 provides the evaluation results for all settings, each of which is averaged over 50 different runs. We have that \( \ell_0 \) norm problem outperforms \( \ell_1 \) norm problem in terms of statistical accuracy. Besides, it is important to note that we only need to tune the step size \( \eta \), which is insensitive in sparse linear regression, for \( \ell_0 \) norm problem and the sparsity parameter \( k \) is fixed in our setting. On the other hand, for \( \ell_1 \) norm problem, we need to tune both step size \( \eta \) and regularization parameter \( \lambda \) to obtain the optimal estimation, which require much more tuning efforts. Moreover, we observe empirically that SVR-GHT converges faster than Prox-SVRG in all settings, where SVR-GHT typically requires a few dozen to a few hundred passes of data to converge, while Prox-SVRG requires a few thousand passes to converge in general.

7 Discussion

The SVR-GHT algorithm presented in this paper is closely related to some recent work on stochastic optimization algorithms, including Prox-SVRG algorithm [19], stochastic averaging gradient (SAG) algorithm [20] and stochastic dual coordinate ascent algorithm (SDCA, [21]). However, the focus in these previous works has been on establishing global linear convergence for optimization problems involving strongly convex objective with a convex constraint, whereas SVR-GHT guarantees linear convergence for optimization problems involving a nonstrongly convex objective with nonconvex cardinality constraint.

Other related work includes nonconvex regularized M-estimators proposed by Loh and Wainwright [23]. In particular, Loh and Wainwright [23] consider the following non-
convex optimization problem:

$$\min_{\theta} \mathcal{F}(\theta) + \mathcal{P}_{\lambda,\gamma}(\theta) \quad \text{subject to } \|\theta\|_1 \leq R,$$

(7.1)

where $\mathcal{P}_{\lambda,\gamma}(\theta)$ is a nonconvex regularization function with tuning parameters $\lambda$ and $\gamma$; Popular choices for $\mathcal{P}_{\lambda,\gamma}(\theta)$ are the SCAD and MCP regularization functions studied in [7, 8]. Loh and Wainwright [23] show that under restricted strong convexity and restricted strong smoothness conditions, similar to those studied here, the proximal gradient algorithm attains linear convergence to approximate global optima with optimal estimation accuracy. Accordingly, one could adopt the Prox-SVRG algorithm to solve (7.1) in a stochastic fashion, and trim the analyses in [19] and [23] to establish similar convergence guarantees. We remark, however, that Problem (7.1) involves three tuning parameters, $\lambda$, $\gamma$, and $R$ which, in practice, requires a large amount of tuning effort to attain good empirical performance. In contrast, Problem (1.1) involves a single tuning parameter, $k$, which makes tuning more efficient.

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8 Appendix

8.1 Proof of Lemma 4.1

For notational convenience, define \( \theta' = H_k(\theta) \). Let \( \text{supp}(\theta^*) = I^* \), \( \text{supp}(\theta) = I \), \( \text{supp}(\theta') = I' \), and \( \theta''' = \theta - \theta' \) with \( \text{supp}(\theta''') = I'' \). Clearly we have \( I' \cup I'' = I \), \( I' \cap I'' = \emptyset \), and \( \|\theta\|_2^2 = \|\theta'\|_2^2 + \|\theta''\|_2^2 \). Then we have that

\[
\|\theta' - \theta^*\|_2^2 - \|\theta - \theta^*\|_2^2 = \|\theta''\|_2^2 - 2\langle \theta', \theta^* \rangle - \|\theta\|_2^2 + 2\langle \theta, \theta^* \rangle = 2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2. \tag{8.1}
\]

If \( 2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 \leq 0 \), then (4.1) holds naturally. From this point on, we will discuss the situation when \( 2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 > 0 \).

Let \( I^* \cap I' = I^{\ast 1} \) and \( I^* \cap I'' = I^{\ast 2} \), and denote \((\theta^*)_I^{\ast 1} = \theta^{\ast 1}, (\theta^*)_I^{\ast 2} = \theta^{\ast 2}, (\theta^*)_I^{\prime 1} = \theta^{\prime 1}, \) and \((\theta^*)_I^{\prime 2} = \theta^{\prime 2}. \) Then we have that

\[
2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 = 2\langle \theta^{\ast 2}, \theta^{\ast 2} \rangle - \|\theta''\|_2^2 \leq 2\langle \theta^{\ast 2}, \theta^{\ast 2} \rangle - \|\theta^{\ast 2}\|_2^2 \leq 2\|\theta^{\ast 2}\|_2 \|\theta^{\ast 2}\|_2 - \|\theta^{\ast 2}\|_2^2. \tag{8.2}
\]

Let \( |\text{supp}(\theta^{\ast 2})| = |I^{\ast 2}| = k^{**} \) and \( \theta_{2, \text{max}} = \|\theta^{\ast 2}\|_\infty \), then consequently we have \( \|\theta^{\ast 2}\|_2 = m \cdot \theta_{2, \text{max}} \) for some \( m \in [1, \sqrt{k^{**}}] \). Notice that we are interested in \( 1 \leq k^{**} \leq k^* \), because (4.1) holds naturally if \( k^{**} = 0 \). In terms of \( \|\theta^{\ast 2}\|_2 \), the RHS of (8.2) is maximized when:

Case 1: \( m = 1 \), if \( \|\theta^{\ast 2}\|_2 \leq \theta_{2, \text{max}} \);

Case 2: \( m = \|\theta^{\ast 2}\|_2 / \theta_{2, \text{max}}, \) if \( \theta_{2, \text{max}} < \|\theta^{\ast 2}\|_2 < \sqrt{k^{**}} \theta_{2, \text{max}} \);

Case 3: \( m = \sqrt{k^{**}} \), if \( \|\theta^{\ast 2}\|_2 \geq \sqrt{k^{**}} \theta_{2, \text{max}} \).

Case 1: If \( \|\theta^{\ast 2}\|_2 \leq \theta_{2, \text{max}} \), then the RHS of (8.2) is maximized when \( m = 1 \), i.e. \( \theta^{\ast 2} \) has only one nonzero element \( \theta_{2, \text{max}} \). By (8.2), we have

\[
2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 \leq 2\theta_{2, \text{max}} \|\theta^{\ast 2}\|_2 - \theta_{2, \text{max}} ^2 \leq 2\theta_{2, \text{max}} \theta_{2, \text{max}}^2 - \theta_{2, \text{max}} ^2 = \theta_{2, \text{max}} ^2. \tag{8.3}
\]

Denote \( \theta_{1, \text{min}} \) as the smallest element of \( \theta^{\ast 1} \) (in magnitude), which indicates that \( |\theta_{1, \text{min}}| \geq |\theta_{2, \text{max}}| \) as \( \theta' \) contains the largest \( k \) entries and \( \theta'' \) contains the smallest \( d - k \) entries of \( \theta \).

For \( \|\theta - \theta^*\|_2^2 \), we have that

\[
\|\theta - \theta^*\|_2^2 = \|\theta' - \theta^{\ast 1}\|_2^2 + \|\theta'' - \theta^{\ast 2}\|_2^2 \\
= \|\theta_{(I^{\ast 1})^C}\|_2^2 + \|\theta_{I^{\ast 1}} - \theta^{\ast 1}\|_2^2 + \|\theta^{\ast 2}\|_2^2 - (2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2) \geq (k - k^* + k^{**}) \theta_{1, \text{min}} ^2 - \theta_{2, \text{max}} ^2 \tag{8.4}
\]

where the last inequality follows from the fact that \( \theta_{(I^{\ast 1})^C} \) has \( k - k^* + k^{**} \) entries larger than \( \theta_{1, \text{min}} \) (in magnitude). Combining (8.1), (8.3) and (8.5), we have that

\[
\frac{\|\theta' - \theta^*\|_2^2 - \|\theta - \theta^*\|_2^2}{\|\theta - \theta^*\|_2^2} \leq \frac{\theta_{2, \text{max}} ^2}{(k - k^* + k^{**}) \theta_{1, \text{min}} ^2 - \theta_{2, \text{max}} ^2} \leq \frac{\theta_{2, \text{max}} ^2}{(k - k^* + k^{**}) \theta_{2, \text{max}} ^2 - \theta_{2, \text{max}} ^2} \leq \frac{1}{k - k^*}. \tag{8.6}
\]
Case 2: If $\|\theta^*\|_2 < \sqrt{k^{**}\theta_{2,\text{max}}}$, then the RHS of (8.2) is maximized when $m = \frac{\|\theta^*\|_2}{\theta_{2,\text{max}}}$. By (8.2), we have that

$$2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 \leq 2\sqrt{k^{**}\theta_{2,\text{max}}} \cdot m\theta_{2,\text{max}} - \theta_{2,\text{max}}^2 \leq k^{**}\theta_{2,\text{max}}^2. \quad (8.7)$$

By (8.4), we have that

$$\|\theta - \theta^*\|_2^2 \geq (k - k^* + k^{**})\theta_{1,\text{min}}^2 + m^2\theta_{2,\text{max}}^2 - \theta_{2,\text{max}}^2 \geq (k - k^* + k^{**})\theta_{1,\text{min}}^2. \quad (8.8)$$

Combining (8.1), (8.7) and (8.8), we have that

$$\frac{\|\theta' - \theta^*\|_2^2 - \|\theta - \theta^*\|_2^2}{\|\theta - \theta^*\|_2^2} \leq \frac{k^{**}\theta_{2,\text{max}}^2}{(k - k^* + k^{**})\theta_{1,\text{min}}^2} \leq \frac{k^{**}}{k - k^* + k^{**}}. \quad (8.9)$$

Case 3: If $\|\theta^*\|_2 \geq \sqrt{k^{**}\theta_{2,\text{max}}}$, then the RHS of (8.2) is maximized when $m = \sqrt{k^{**}}$. Let $\|\theta^*\|_2 = \gamma\theta_{2,\text{max}}$ for some $\gamma \geq \sqrt{k^{**}}$. We have from (8.2) that

$$2\langle \theta'', \theta^* \rangle - \|\theta''\|_2^2 \leq 2\gamma\sqrt{k^{**}}\theta_{2,\text{max}}^2 - k^{**}\theta_{2,\text{max}}^2. \quad (8.10)$$

By (8.4), we have that

$$\|\theta - \theta^*\|_2^2 \geq (k - k^* + k^{**})\theta_{1,\text{min}}^2 + \gamma^2\theta_{2,\text{max}}^2 - \gamma\sqrt{k^{**}}\theta_{2,\text{max}}^2 + k^{**}\theta_{2,\text{max}}^2. \quad (8.11)$$

Combining (8.1), (8.10) and (8.11), we have

$$\frac{\|\theta' - \theta^*\|_2^2 - \|\theta - \theta^*\|_2^2}{\|\theta - \theta^*\|_2^2} \leq \frac{2\gamma\sqrt{k^{**}}\theta_{2,\text{max}}^2 - k^{**}\theta_{2,\text{max}}^2}{(k - k^* + k^{**})\theta_{1,\text{min}}^2 + \gamma^2\theta_{2,\text{max}}^2 - \gamma\sqrt{k^{**}}\theta_{2,\text{max}}^2 + k^{**}\theta_{2,\text{max}}^2} \leq \frac{2\gamma\sqrt{k^{**}} - k^{**}}{k - k^* + 2k^{**} + \gamma^2 - 2\gamma\sqrt{k^{**}}}. \quad (8.12)$$

Inspecting the RHS of (8.12) carefully, we can see that it is either a bell shape function or a monotone decreasing function when $\gamma \geq \sqrt{k^{**}}$. Setting the first derivative of the RHS in terms of $\gamma$ to zero, we have $\gamma = \frac{1}{2}\sqrt{k^{**}} + \sqrt{k - k^* + \frac{3}{4}k^{**}}$ (the other root is smaller than $\sqrt{k^{**}}$).

Denoting $\gamma_* = \max\{\sqrt{k^{**}}, \frac{1}{2}\sqrt{k^{**}} + \sqrt{k - k^* + \frac{3}{4}k^{**}}\}$ and plugging it into the RHS of (8.12), we have

$$\frac{\|\theta' - \theta^*\|_2^2 - \|\theta - \theta^*\|_2^2}{\|\theta - \theta^*\|_2^2} \leq \max\left\{ \frac{k^{**}}{k - k^* + k^{**}}, \frac{2\sqrt{k^{**}}}{2\sqrt{k - k^* + \frac{5}{4}k^{**} - \sqrt{k^{**}}}} \right\}. \quad (8.13)$$
Combining (8.6), (8.9) and (8.13), and taking $k > k^*$ and $k^* \geq k^{**} \geq 1$ into consideration, we have
\[
\max \left\{ \frac{1}{k - k^*} \frac{k^{**}}{k - k^* + k^{**}}, \frac{2\sqrt{k^{**}}}{2\sqrt{k - k^* + \frac{5}{4}k^{**}} - \sqrt{k^{**}}} \right\} \leq \frac{2\sqrt{k^{**}}}{4\sqrt{k - k^*} + \frac{5}{4}k^{**} - \sqrt{k^{**}}}
\leq \frac{2\sqrt{k^*}}{2\sqrt{k - k^*} - \sqrt{k^*}} \leq \frac{2\sqrt{k^*}}{\sqrt{k - k^*}},
\]
which proves the result.

### 8.2 Proof of Lemma 4.3

It is straightforward that the stochastic variance reduced gradient (4.2) satisfies
\[
\mathbb{E}g^{(t)}(\theta^{(t)}) = \mathbb{E}v f_i(\theta^{(t)}) - \mathbb{E}v f_i(\tilde{\theta}) + \tilde{\mu} = \nabla \mathcal{F}(\theta^{(t)}),
\]
Thus $g^{(t)}(\theta^{(t)})$ is an unbiased estimate of $\nabla \mathcal{F}(\theta^{(t)})$ and the first claim is verified.

Next, we bound $\mathbb{E}\|g^{(t)}_I(\theta^{(t)})\|_2^2$. For any $i \in [n]$ and $\theta$ with supp($\theta$) $\subseteq$ $I$, consider
\[
\phi_i(\theta) = f_i(\theta) - f_i(\theta^*) - \langle \nabla f_i(\theta^*), \theta - \theta^* \rangle.
\]
Since $\nabla \phi_i(\theta^*) = \nabla f_i(\theta^*) - \nabla f_i(\theta^*) = 0$, we have that $\phi_i(\theta^*) = \min_{\theta} \phi_i(\theta)$, which implies
\[
0 = \phi_i(\theta^*) \leq \min_{\eta} \phi_i(\theta - \eta \nabla f_i(\theta)) = \min_{\eta} \phi_i(\theta) - \eta \|\nabla f_i(\theta)\|_2^2 + \frac{\rho_+^s \eta^2}{2} \|\nabla f_i(\theta)\|_2^2
\leq \frac{1}{2\rho_+^s} \|\nabla f_i(\theta)\|_2^2,
\]
where the last inequality follows from the RSS condition and the last equality follows from the fact that $\eta = 1/\rho_+^s$ minimizes the function. By (8.14), we have
\[
\|\nabla I f_i(\theta) - \nabla f_i(\theta^*)\|_2^2 \leq 2\rho_+^s [f_i(\theta) - f_i(\theta^*) - \langle \nabla f_i(\theta^*), \theta - \theta^* \rangle].
\]
Since the sampling of $i$ from $[n]$ is uniform sampling, we have from (8.15)
\[
\mathbb{E}\|\nabla I f_i(\theta) - \nabla f_i(\theta^*)\|_2^2 = \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(\theta) - \nabla f_i(\theta^*)\|_2^2
\leq 2\rho_+^s [\mathcal{F}(\theta) - \mathcal{F}(\theta^*) - \langle \nabla \mathcal{F}(\theta^*), \theta - \theta^* \rangle]
\leq 2\rho_+^s [\mathcal{F}(\theta) - \mathcal{F}(\theta^*) + \|\nabla \mathcal{F}(\theta^*), \theta - \theta^* \|]
\leq 4\rho_+^s [\mathcal{F}(\theta) - \mathcal{F}(\theta^*)],
\]
where the last inequality is from the RSC condition of $\mathcal{F}(\theta)$. 

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By the definition of \( g_{I}^{(t)} \) in (4.2), we can verify the second claim as

\[
\mathbb{E}\|g_{I}^{(t)}(\theta^{(t)})\|_2^2 \leq 3E\|\nabla_{I} f_{i}((\tilde{\theta}) - \nabla_{I} f_{i}(\theta^{*})) - \nabla_{I} \mathcal{F}(\tilde{\theta}) + \nabla_{I} \mathcal{F}(\theta^{*})\|_2^2
\]

\[
+ 3E\|\nabla_{I} f_{i}(\theta^{(t)}) - \nabla_{I} f_{i}(\theta^{*})\|_2^2 + 3\|\nabla_{I} \mathcal{F}(\theta^{*})\|_2^2
\]

\[
\leq 3E\|\nabla_{I} f_{i}(\theta^{(t)}) - \nabla_{I} f_{i}(\theta^{*})\|_2^2 + 3E\|\nabla_{I} f_{i}(\tilde{\theta}) - \nabla_{I} f_{i}(\theta^{*})\|_2^2 + 3\|\nabla_{I} \mathcal{F}(\theta^{*})\|_2^2
\]

\[
\leq 12\rho_{s}^{+}\left[\mathcal{F}(\theta^{(t)}) - \mathcal{F}(\theta^{*}) + \mathcal{F}(\tilde{\theta}) - \mathcal{F}(\theta^{*})\right] + 3\|\nabla_{I} \mathcal{F}(\theta^{*})\|_2^2 \tag{8.17}
\]

where the first inequality follows from \( \|a + b + c\|_2^2 \leq 3\|a\|_2^2 + 3\|b\|_2^2 + 3\|c\|_2^2 \), the second inequality follows from \( \mathbb{E}\|x - \mathbb{E}x\|_2^2 \leq \mathbb{E}\|x\|_2^2 \) with \( \mathbb{E}\left[\nabla_{I} f_{i}(\tilde{\theta}) - \nabla_{I} f_{i}(\theta^{*})\right] = \nabla_{I} \mathcal{F}(\tilde{\theta}) - \nabla_{I} \mathcal{F}(\theta^{*}) \), and the last inequality follows from (8.16).

### 8.3 Proof of Lemma 3.5

For any \( \theta, \theta' \in \mathbb{R}^{d} \) in sparse linear model, we have \( \nabla^{2} \mathcal{F}(\theta) = A^{\top}A \) and

\[
\mathcal{F}(\theta) - \mathcal{F}(\theta') - \langle \nabla \mathcal{F}(\theta'), \theta - \theta' \rangle = \frac{1}{2}(\theta - \theta')^{\top}\nabla^{2} \mathcal{F}(\theta')(\theta - \theta') = \frac{1}{2}\|A(\theta - \theta')\|^{2}_2,
\]

where \( \theta'' \) is between \( \theta \) and \( \theta' \) and \( \|\theta - \theta'\|_0 \leq 2k \leq s \). Let \( v = \theta - \theta' \), then \( \|v\|_0 \leq s \) and \( \|v\|_2^2 \leq s\|v\|_2^2 \). By (3.8), we have

\[
\frac{\|Av\|_2^2}{nb} \geq \psi_1 \|v\|_2^2 - \varphi_1 \frac{s \log d}{nb} \|v\|_2^2, \quad \text{and} \quad \frac{\|AS_i v\|_2^2}{b} \leq \psi_2 \|v\|_2^2 + \varphi_2 \frac{s \log d}{b} \|v\|_2^2, \quad \forall i \in [n],
\]

which further imply

\[
\rho_{s}^{-} = \inf_{\|v\|_0 \leq s} \frac{\|Av\|_2^2}{nb} \geq \psi_1 - \varphi_1 \frac{s \log d}{nb}, \quad \text{and} \quad \rho_{s}^{+} = \sup_{\|v\|_0 \leq s, i \in [n]} \frac{\|AS_i v\|_2^2}{b} \leq \psi_2 + \varphi_2 \frac{s \log d}{b}. \tag{8.18}
\]

If \( b \geq \frac{\varphi_2 s \log d}{\psi_2} \) and \( n \geq \frac{2\psi_1 \varphi_2}{\psi_1 \varphi_2} \), then we have \( nb \geq \frac{2\psi_1 \varphi_2 s \log d}{\psi_1} \). Combining these with (8.18), we have

\[
\rho_{s}^{-} \geq \frac{1}{2} \psi_1, \quad \text{and} \quad \rho_{s}^{+} \leq 2\psi_2.
\]

By the definition of \( \kappa \), this indicates \( \kappa_{s} = \frac{\rho_{s}^{+}}{\rho_{s}^{-}} \leq \frac{4\psi_2}{\psi_1} \). Then for some \( C_5 \geq \frac{16C_1 \psi_2^2}{\psi_1^2} \), we have

\[
k = C_5 k^* \geq C_1 \kappa_{s}^2 k^*.
\]