UPPER BOUND ON THE TOTAL NUMBER OF KNOT $n$-MOSAICS

KYUNGPYO HONG, HO LEE, HWA JEONG LEE, AND SEUNGSANG OH

Abstract. Lomonaco and Kauffman introduced a knot mosaic system to give a definition of a quantum knot system which can be viewed as a blueprint for the construction of an actual physical quantum system. A knot $n$-mosaic is an $n \times n$ matrix of 11 kinds of specific mosaic tiles representing a knot. $D_n$ is the total number of all knot $n$-mosaics. Already known is that $D_1 = 1$, $D_2 = 2$, and $D_3 = 22$. In this paper we find the exact number of $D_4 = 2594$ and establish lower and upper bounds on $D_n$ for $n \geq 3$ which is;

$$2^{275} \left(9 \cdot 6^{a-3} + 1\right)^2 \cdot 2^{(a-3)^2} \leq D_n \leq 2^{275} \left(9 \cdot 6^{a-2} + 1\right)^2 \cdot (4.4)^{(a-3)^2}.$$ 

1. Introduction

Throughout this paper we will frequently use the term “knot” to mean either a knot or a link for simplicity of exposition. Lomonaco and Kauffman introduced a knot mosaic system to give a precise and workable definition of a quantum knot system which can be viewed as a blueprint for the construction of an actual physical quantum system in [3].

Let $T$ denote the set of the following 11 symbols which are called mosaic tiles;

For a positive integer $n$, we define an $n$-mosaic as an $n \times n$ matrix $M = (M_{ij})$ of mosaic tiles with rows and columns indexed from 1 to $n$. We denote the set of all $n$-mosaics by $M^{(n)}$. Obviously $M^{(n)}$ has $11^n$ elements. A connection point of a tile is defined as the midpoint of a mosaic tile edge which is also the endpoint of a curve drawn on the tile. Then each tile has zero, two or four connection points as follows;

We say that two tiles in a mosaic are contiguous if they lie immediately next to each other in either the same row or the same column. A mosaic tile within a mosaic is said to be suitably connected if each of its connection points touches a connection point of a contiguous tile. A knot $n$-mosaic is an $n$-mosaic in which all tiles are suitably connected. Then this knot $n$-mosaic represents a specific knot.

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Three examples of mosaics in Figure 1 are a 4-mosaic, the Hopf link 4-mosaic and the trefoil knot 4-mosaic.

Figure 1. Three examples of 4-mosaics

One natural question concerning knot mosaics is how many knot \( n \)-mosaics are there. We let \( K^{(n)} \) denote the subset of \( M^{(n)} \) of all knot \( n \)-mosaics. Let \( D_n \) denote the total number of elements of \( K^{(n)} \). The main theme in this paper is to established an upper bound on \( D_n \). Already known is that \( D_1 = 1 \), \( D_2 = 2 \) and \( D_3 = 22 \). You can find a complete table of \( K^{(3)} \) in Appendix A in [3].

As an analog to the planar isotopy moves and the Reidemeister moves for standard knot diagrams, Lomonaco and Kauffman created for knot mosaics the 11 mosaic planar isotopy moves and the mosaic Reidemeister moves in [3]. They conjectured that for any two tame knots (or links) \( K_1 \) and \( K_2 \), and their arbitrary chosen mosaic representatives \( M_1 \) and \( M_2 \), respectively, \( K_1 \) and \( K_2 \) are of the same knot type if and only if \( M_1 \) and \( M_2 \) are of the same knot mosaic type. This means that tame knot theory and knot mosaic theory are equivalent. Recently Kuriya proved that Lomonaco-Kauffman conjecture is true in [1].

In their paper Lomonaco and Kauffman also proposed several open questions related to knot mosaics. Define the mosaic number \( m(K) \) of a knot \( K \) as the smallest integer \( n \) for which \( K \) is representable as a knot \( n \)-mosaic. One question is the following; Is this mosaic number related to the crossing number of a knot? Recently Lee, Hong, Lee and Oh established an upper bound on the mosaic number as follows in [2]; If \( K \) be a nontrivial knot or a non-split link except the Hopf link and \( 6^3_3 \), then \( m(K) \leq c(K) + 1 \). Moreover if \( K \) is prime and non-alternating, then \( m(K) \leq c(K) - 1 \).

Another question proposed by Lomonaco and Kauffman is the following; Find \( D_n \) for \( n \geq 4 \). And they gave a very loose upper bound 11\( n^2 \). In this paper we establish lower and upper bounds on \( D_n \) for \( n \geq 3 \) and also give the exact number of \( D_4 \).

**Theorem 1.** \[ \frac{2}{275} (9 \cdot 6^{n-2} + 1)^2 \cdot 2^{(n-3)^2} \leq D_n \leq \frac{2}{275} (9 \cdot 6^{n-2} + 1)^2 \cdot (4.4)^{(n-3)^2} \] for \( n \geq 3 \).

**Theorem 2.** \( D_4 = 2594 \).

The upper bound in Theorem 1 is close to the actual number of \( D_n \) for small \( n \). For example, this bound says that \( D_3 = 22 \) and \( 1537 \leq D_4 \leq 3380 \).

2. **Proof of Theorem 1**

For \( n \geq 3 \), \( K^{(n)} \) is the set of all knot \( n \)-mosaics, so each mosaic is filled by suitably connected \( n^2 \) mosaic tiles entirely, and \( D_n \) is the total number of its elements. We let \( K_1^{(n)} \) denote the set of so called \( n \)-quasimosaics each of which is filled by suitably connected \( 2n - 3 \) mosaic tiles only at \( M_{1j} \) and \( M_{i1} \), \( i, j = 1, 2, \cdots, n - 1 \). Let \( d_1 \) denote the total number of elements of \( K_1^{(n)} \). Similarly let \( K_2^{(n)} \) denote the sets of all \( n \)-quasimosaics each of which is filled by suitably connected \( 4n - 8 \) tiles only at...
Lemma 3. Let $d$ denote the total number of elements of $\mathbb{K}_1^{(n)}$, $\mathbb{K}_2^{(n)}$, and $\mathbb{K}_3^{(n)}$ respectively. See three examples of elements of $\mathbb{K}_1^{(n)}$, $\mathbb{K}_2^{(n)}$, and $\mathbb{K}_3^{(n)}$ in Figure 2.

![Figure 2. Three elements of $\mathbb{K}_1^{(n)}$, $\mathbb{K}_2^{(n)}$, and $\mathbb{K}_3^{(n)}$](image)

For simplicity of exposition, a mosaic tile is called t-cp if it has a connection point on its top edge, and similarly b, l or r-cp when on its bottom, left or right edge, respectively. Sometimes we use two letters, for example, tl-cp in the case of both t-cp and l-cp. Also we use the sign \( \hat{\cdot} \) for negation so that, for example, l-cp means not t-cp, tl-cp means both l-cp and t-cp, and \( \hat{t}l \)-cp (which is differ from tl-cp) means not tl-cp, i.e. tl, tl or tl-cp.

First we will figure out $K_1^{(n)}$ and determine the number $d_1$.

**Lemma 3.** $d_1 = 2^{2n-3}$.

**Proof.** We will use inductions. The first mosaic tile $M_{11}$ has 2 choices whether $T_0$ or $T_2$. The next tile $M_{12}$ has always 2 choices in any choices of $M_{11}$ as follows; if $M_{11} = T_0$, then $M_{12}$ is tl-cp, so $M_{12}$ is either $T_0$ or $T_2$, and if $M_{11} = T_2$, then $M_{12}$ must be tl-cp to be suitably connected, so $M_{12}$ is either $T_1$ or $T_5$. By the same reason each $M_{ij}$, $j = 3, \cdots, n - 1$, has always 2 choices; if $M_{i(j-1)}$ is $\hat{r}$-cp, then $\hat{l}$-cp $M_{ij}$ is either $T_0$ or $T_2$, and if $M_{i(j-1)}$ is r-cp, then l-cp $M_{ij}$ is either $T_1$ or $T_3$. We can follow the same argument when we choose mosaic tiles $M_{ij}$, $i = 2, \cdots, n - 1$. Thus if $M_{i(i-1)}$ is b-cp, then l-cp $M_{ij}$ is either $T_0$ or $T_2$, and if $M_{i(i-1)}$ is b-cp, then t-cp $M_{ij}$ is either $T_3$ or $T_6$. Therefore every tile has exactly 2 choices. Since each quasimosaic of $K_1^{(n)}$ consists of $2n - 3$ mosaic tiles, $d_1 = 2^{2n-3}$. \( \square \)

**Fact 1.** For any $j = 2, \cdots, n - 1$, exactly the half of $K_1^{(n)}$ have b-cp $M_{ij}$‘s and the rest half have b-cp $M_{ij}$‘s. Similarly for any $i = 2, \cdots, n - 1$, exactly the half of $K_1^{(n)}$ have r-cp $M_{ij}$‘s and the rest half have $\hat{r}$-cp $M_{ij}$‘s.

**Fact 2.** For any $i, j = 2, \cdots, n - 1$, $M_{ij}$ is one of $T_4, T_7, T_8, T_9$ or $T_{10}$ if it is tl-cp, either $T_1$ or $T_5$ if tl-cp, either $T_3$ or $T_6$ if tl-cp, and either $T_0$ or $T_2$. Therefore each $M_{ij}$ has 5 choices of mosaic tiles if it is tl-cp, and 2 choices if it is tl-cp.

Next we will figure out $K_2^{(n)}$ and determine the number $d_2$.

**Lemma 4.** $d_2 = \frac{2^3}{2}\left(9 \cdot 6^{n-2} + 1\right)^2$.

**Proof.** Similar to the definitions of $K_1^{(n)}$ and $d_1$, let $\mathbb{K}_2^{(n)}$, $j = 2, \cdots, n - 1$, denote the set of all n-quasimosaics each of which is filled by suitably connected mosaic
tiles as in $K_1^{(n)}$ and more tiles at $M_{2k}$, $k = 2, \cdots, j$. Let $d_{2j}$ denote the total number of elements of $K_2^{(n)}$.

First we fill the first mosaic tile $M_{22}$. By Fact 1, exactly $\left(\frac{1}{6}\right)^2 d_1$ elements of $K_1^{(n)}$ have $tl$-cp $M_{22}$’s to be suitably connected, and the rest $\frac{2}{5} d_1$ elements have $\tilde{t}$-cp $M_{22}$’s. By Fact 2, $d_{22} = \frac{1}{4} d_1 \cdot 5 + \frac{2}{5} d_1 \cdot 2 = \frac{11}{2} d_1$. Note that among all $d_{22}$ elements of $K_2^{(n)}$, $\frac{1}{4} d_1 \cdot 4 + \frac{2}{5} d_1 \cdot 1 = \frac{7}{2} d_1 = \frac{7}{11} d_{22}$ elements have $r$-cp $M_{22}$’s. Let $p_2 = \frac{7}{11}$.

Now we use induction again. For any $j = 3, \cdots, n - 1$, the same argument above guarantees that exactly $\frac{1}{4} p_{j-1} \cdot d_{2(j-1)}$ elements of $K_2^{(n)}$ can be suitably connected with $tl$-cp $M_{2j}$’s, and the rest elements with $\tilde{t}$-cp $M_{2j}$’s. Thus $d_{2j} = \frac{1}{4} p_{j-1} \cdot d_{2(j-1)} \cdot 5 + (1 - \frac{1}{4} p_{j-1}) \cdot d_{2(j-1)} \cdot 2 = (2 + \frac{3}{4} p_{j-1}) \cdot d_{2(j-1)}$. Then among all $d_{2j}$ elements of $K_2^{(n)}$, $\frac{1}{4} p_{j-1} \cdot d_{2(j-1)} \cdot 4 + (1 - \frac{1}{4} p_{j-1}) \cdot d_{2(j-1)} \cdot 1 = (1 + \frac{3}{4} p_{j-1}) \cdot d_{2(j-1)} = \frac{2 + 3 p_{j-1}}{4 + 3 p_{j-1}} \cdot d_{2j}$ elements have $r$-cp $M_{2j}$’s. Let $p_j = \frac{2 + 3 p_{j-1}}{4 + 3 p_{j-1}}$.

Therefore $d_{2(n-1)} = d_1 \cdot \frac{1}{4} + (2 + \frac{3}{4} p_2) \cdot (2 + \frac{3}{4} p_{n-2})$. Since $p_j = \frac{2^j - 2}{2^j + 2}$, the series $p_j$ satisfies the equation $2 + \frac{3}{4} p_j = \frac{1}{2} \cdot \frac{2^j - 2}{2^j + 2}$. To fill all the tiles (especially on the second row and the second column) of elements of $K_2^{(n)}$:

$$d_2 = d_1 \cdot \frac{1}{4} + (2 + \frac{3}{4} p_2) \cdot (2 + \frac{3}{4} p_{n-2}) = \frac{2}{2^2} (9 \cdot 6^{n-2} + 1)^2.$$ 

Now we will figure out $K_3^{(n)}$ and find an upper bound on $d_3$.

**Lemma 5.** $\frac{2}{2^3} (9 \cdot 6^{n-2} + 1)^2 \cdot 2^{(n-3)^2} \leq d_3 \leq \frac{2}{2^3} (9 \cdot 6^{n-2} + 1)^2 \cdot (4.4)^{(n-3)^2}$.

**Proof.** Let $i, j = 3, \cdots, n - 1$. As a continuation of Fact 2, if $M_{ij}$ is $tl$-cp, then 4 tiles $T_7$, $T_8$, $T_9$, and $T_{10}$ among 5 choices have $r$-cp, and if $M_{ij}$ is $\tilde{t}$-cp, then one tile among 2 choices has $r$-cp. These facts guarantee that between one-half and four-fifths quasimosaics of $K_3^{(n)}$ have $r$-cp $M_{ij}$’s, and similarly for $b$-cp $M_{ij}$’s.

Unlike the argument in the proof of Lemma 4, the probabilities of $M_{ij}$ having $t$-cp and $l$-cp are not independent. To calculate $d_3$, we thus have to multiply to $d_2$ at least $0.5 + (1 - 0.5) \cdot 2 = 2$ and at most $\frac{1}{2} \cdot 5 + (1 - \frac{1}{2}) \cdot 2 = 4.4$ for each $M_{ij}$. Thus we have:

$$d_2 \cdot 2^{(n-3)^2} \leq d_3 \leq d_2 \cdot (4.4)^{(n-3)^2}.$$ 

Finally we will finish the proof of Theorem 1. For each $n$-quasimosaic of $K_3^{(n)}$, there is exactly one way to fill mosaic tiles to be suitably connected at every $M_{ij}$ or $M_{in}$ where $i, j = 1, \cdots, n$, because every tile has even numbered connection points. This implies that $D_n = d_3$.

Indeed the inequality of the upper bound appears only on Lemma 5. This means that the equality holds for $n = 3$, so $D_3 = 22$.

3. $D_4 = 2594$

In this section we will figure out $K_4^{(n)}$ and determine the number $D_4$. We let $K_4^{(n)}$ denote the set of all 4-quasimosaics each of which is filled by suitably connected 4 mosaic tiles only at $M_{ij}$, $i, j = 2, 3$. Let $d_c$ denote the total number of elements of $K_4^{(n)}$. A common edge of two $M_{ij}$’s is called a central edge. Note that there are four central edges as bold segments depicted in Figure 4.

**Fact 3.** As in Fact 2, if both central edges of $M_{ij}$ have connection points, then $M_{ij}$ has 5 choices of mosaic tiles. Otherwise, it has 2 choices.

First we will figure out $K_4^{(n)}$ and determine the number $d_c$. Since each central edge has 2 cases whether it has a connection point or not, we split into 16 cases whether each of four central edges has a connection point or not.
Among 16 cases, there is 1 case where all four central edges have connection points. By Fact 3, every \( M_{ij} \) has 5 choices, so we have \( 5^4 \) different 4-quasimosaics in \( \mathbb{K}^{(4)} \). There are 4 cases where exactly three central edges have connection points. In each case two of \( M_{ij} \)'s have 5 choices and the other two have 2 choices, and so we have \( 5^2 \cdot 2^2 \) different 4-quasimosaics. There are another 4 cases where only two perpendicular central edges have connection points. In each case only one of \( M_{ij} \)'s has 5 choices and the other three have 2 choices, and so we have \( 5 \cdot 2^3 \). In each of all the other 7 cases among 16 cases, every \( M_{ij} \) has 2 choices, so we have \( 2^4 \). Thus we have the following:

\[
d_c = 5^4 + 4 \cdot 5^2 \cdot 2^2 + 4 \cdot 5 \cdot 2^3 + 7 \cdot 2^4 = 1297.
\]

Finally we are ready to finish the proof of Theorem \( \ref{thm:mosaic} \). For each 4-quasimosaic in \( \mathbb{K}^{(4)} \), there are exactly two ways to fill mosaic tiles to be suitably connected at the rest twelve boundary \( M_{ij} \)'s. For, every tile has even numbered connection points, so the union of boundary edges of \( M_{22} \cup M_{23} \cup M_{32} \cup M_{33} \) has even number of connection points. This implies that \( D_4 = 2d_c \).

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