SCATTERING THEORY FOR SEMILINEAR SCHRÖDINGER EQUATIONS WITH AN INVERSE-SQUARE POTENTIAL VIA ENERGY METHODS

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ABSTRACT. We solve the scattering problems for nonlinear Schrödinger equations with an inverse-square potential by applying the energy methods. The methods are optimized to the abstract semilinear Schrödinger evolution equations with nonautonomous terms.

1. Introduction. In this article we consider the scattering problems for following nonlinear Schrödinger equations with an inverse-square potential

\[
\begin{cases}
    i \frac{\partial u}{\partial t} = \left(-\Delta + \frac{a}{|x|^2}\right)u + g(u) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
    u(0) = u_0,
\end{cases}
\]

(NLS)_a

where \( i = \sqrt{-1}, \ N \geq 3, \) and

\[
a \geq a(N) := -\frac{(N-2)^2}{4}.
\]

(1.1)

\( g(u) \) is a nonlinear term, for example, \( g(u) = |u|^{p-1}u \) (pure power nonlinearity; local type) and \( g(u) = u(|x|^{-\gamma} * |u|^2) \) (usual Hartree nonlinearity; nonlocal type). The contraction methods are the simple and useful methods for showing the global existence of nonlinear Schrödinger equations. But Okazawa–Suzuki–Yokota [15] showed the global unique existence with unsatisfactory conditions for \( (\text{NLS})_a \) with local nonlinearities.

Proposition 1.1 (Contraction methods: [15]). Let \( N \geq 3, \ u_0 \in H^1(\mathbb{R}^N), \) and

\[
g(u) := |u|^{p-1}u \left( 1 \leq p < \frac{N+2}{N-2} \right), \ \ a > a(N) + \frac{N(p-1)^2}{2(p+1)}.
\]

Then there exists a unique solution \( u \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N)) \) to \( (\text{NL-S})_a. \)

The unsatisfactory condition of \( a \) is removed Okazawa–Suzuki–Yokota [16] by applying the energy methods:
Hence there appears the threshold of nonnegativity and selfadjointness for

\[ \text{Critical case of energy methods still work. See Suzuki [22].} \]

\[ \text{Energy methods: [16]} \]

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Ginibre–Velo [7, 8], and Nakanishi [14, 13]). To solve (1.4) we apply the pseudo-

\[ \text{modification for the final conditions (the latter of (1.4); see e.g. Ginibre–Ozawa [6],} \]

\[ \langle \gamma \rangle \]

Assume that

\[ \text{Assume that} \]

\[ g \]

\[ \text{not exist in} \]

\[ L \]

\[ u \]

\[ \text{u} \]

\[ \text{and the limit} \]

\[ \text{u} \]

\[ \text{u} \]

\[ \text{NLS} \]

\[ \text{u} \]

\[ \text{problems.} \]

\[ \text{u} \]

\[ \text{These are the problems for the relation between the global solution} \]

\[ \text{Asymptotic behavior of global solution; (sp1)} \]

\[ \text{The existence of wave operators.} \]

\[ \text{These are the problems for the relation between the global solution} \]

\[ \lim_{t \to \pm \infty} \exp(itP_a)u(t) = u_{\pm}. \]

Now we summarize the studies of scattering theory only for the usual Hartree non-

\[ \text{and the limit} \]

\[ \lim_{t \to \infty} \exp(-it\Delta)u(t) = u_+, \]

where \( u(t) \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N)) \) is a unique and global solution to

\[ \text{(NLS)}_a \]

If \( 1 < \gamma < \min\{N, 4\} \), then the strong limit (1.3) exists in \( L^2(\mathbb{R}^N) \) for any

\[ u_0 \in \Sigma^1(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap D(|x|); \text{ if } 0 < \gamma \leq 1, \text{ then the strong limit (1.3) does} \]

\[ \text{not exist in} \]

\[ L^2(\mathbb{R}^N) \text{ except } u_0 = 0 = u_+ \text{ (see e.g. Hayashi–Tsutsumi [10, Theorem} \]

\[ 3.1 \text{ (2))}. \]

On the other hand, we show (sp2) by solving the following final-value problems.

\[ \begin{cases}
  i u_t = -\Delta u + u (|x|^{-\gamma} * |u|^2) & \text{in } [0, \infty), \\
  u_+ = \lim_{t \to \infty} \exp(-it\Delta)u(t) & \text{strongly in } \Sigma^1(\mathbb{R}^N).
\end{cases} \]

If \( 1 < \gamma < \min\{N, 4\} \), then (1.4) is well-posed, that is, for every \( u_+ \in \Sigma^1(\mathbb{R}^N) \) there uniquely exists \( u \in C(\mathbb{R}; \Sigma^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N)) \) of solution to (1.4) (see Hayashi–Tsutsumi [10]). If \( 0 < \gamma \leq 1 \) (the long range cases), we need more modification for the final conditions (the latter of (1.4); see e.g. Ginibre–Ozawa [6], Ginibre–Velo [7, 8], and Nakanishi [14, 13]). To solve (1.4) we apply the pseudo-

\[ \exp \left( \frac{i|t|^2}{4t} \right) v(t^{-1}, t^{-1}x). \]

\[ \text{conformal transform} \]

\[ u(t, x) = C v(t, x) := (it)^{-N/2} \]
Then (1.4) is converted into
\[
\begin{cases}
  i v_t = -\Delta v + t^{\gamma-2} v (|x|^{-\gamma} + |v|^2) & \text{in } [0, \infty), \\
  v(0) = v_+ := i^{-N/2} \exp(-i\Delta) e^{i|x|^2/4} \exp(-i\Delta) v_+ & \text{on } \Sigma^1(\mathbb{R}^N).
\end{cases}
\]

This is nothing but a nonautonomous Cauchy problem. Hayashi–Tsutsumi [10] solved the final-valued problems by practicing the contraction methods to the associated integral equations of (1.6). Later, Hayashi–Ozawa [9] solved the problems (1.4) directly via the contraction methods. Latterly, Suzuki [23] studied the scattering problems for (NLS)$_a$ with the usual Hartree type for $a > a(N)$. Especially in $L^2$-supercritical case ($2 < \gamma < \min\{4, N\}$), it occurs on an unsatisfactory restriction of $a$. If we solve the scattering problems (sp1) and (sp2) for (NLS)$_a$ in weighted energy class ($\Sigma^1(\mathbb{R}^N)$) or $X^1(\mathbb{R}^N) \cap D(|x|)$ with general $a$, we need to apply the pseudo-conformal transform (1.5) and solve the nonautonomous Cauchy problems. The transform (1.5) still works, however, the contraction methods are not suitable way to solve. Thus we need another method like the energy methods which established by Okazawa–Suzuki–Yokota [16] for the self-excited systems and applied Propositions 1.2 and 1.3. Our goals are to establish new methods and to apply the scattering problems in this paper.

Recently, the scattering problems for (NLS)$_a$ with the pure power type are partially solved in $H^1(\mathbb{R}^N)$, not in $\Sigma^1(\mathbb{R}^N)$. Zhang–Zheng [26] proved the Morawetz inequality and constructed the wave operators in $a \geq 0$ if $N = 3$, and in
\[ a > a(N) + \frac{4}{(p+1)^2}, \quad 1 + \frac{4}{N} < p < 1 + \frac{4}{N-2} \quad \text{if } N \geq 4. \]

On the one hand, Lu–Miao–Murphy [11] considered the profile decomposition and constructed the wave operators in $a > a(N)$ and $7/3 < p \leq 3$ if $N = 3$, and in
\[ a > a(N) + \left(\frac{N-2}{2} - \frac{1}{p+1}\right)^2, \quad \max\left\{1 + \frac{4}{N}, 1 + \frac{2}{N-2}\right\} < p < 1 + \frac{4}{N-2} \]
if $3 \leq N \leq 6$. Note that both of studies applied the contraction methods without the pseudo-conformal transform (1.5) and excluded the case $a = a(N)$ and near of $a(N)$ for higher dimension $N$. Our approach in this paper guarantees that the topology of limits is stronger (not $H^1(\mathbb{R}^N) = D((1+P_0)\frac{1}{2})$ ($a > a(N)$) but $D((1+P_0)\frac{1}{2}) \cap D(|x|)$) but the restriction of $a$ and $p$ is weaker, and more suitable for scattering problems (not as in [26, 11] but $a \geq a(N)$ and $1 + 4/N \leq p < 1 + 4/(N-2)$).

This paper is divided into four sections. In Section 2 we establish new methods (the energy methods) for nonautonomous semilinear Schrödinger equations. In Section 3 we consider the scattering problems for (NLS)$_a$; especially, nonlocal nonlinearity. In Section 4 we have some comments as concluding remarks.

2. Energy methods for abstract nonautonomous semilinear Schrödinger equations. In this section we consider following Cauchy problems for abstract nonautonomous semilinear Schrödinger equations
\[
\begin{cases}
  \frac{du}{dt} = Su + g(t, u) & \text{in } [-T, T] \times X_S^*, \\
  u(0) = u_0 & \text{in } X_S,
\end{cases}
\]
where $S$ is nonnegative and selfadjoint in the (complex) Hilbert space $X$. $X_S := D((1+S)^{1/2})$ and $X_S^* := D((1+S)^{-1/2})$ is the dual space of $X_S$. Here we see
the triplet \(X_S \subset X = X^* \subset X_S^*\). We give a typical example. Let \(S = -\Delta\) in \(X = L^2(\mathbb{R}^N)\). Then we see that \(X_S = H^1(\mathbb{R}^N)\) and \(X_S^* = H^{-1}(\mathbb{R}^N)\).

Now we present new methods. First we give assumption for the nonlinearity \(g : [-T, T] \times X_S \to X_S^*\). For the simple notation we use \(B_M := \{u \in X_S; \|u\|_{X_S} \leq M\}\).

(A1) **Existence of energy functional:** there exists \(G \in C([-T, T] \times X_S^*; \mathbb{R})\) whose real Fréchet differential \(d_{X_S} G(t, u)\) is exactly \(g(t, u)\), that is, given \(u \in X_S\) and \(t \in [-T, T]\), for every \(\varepsilon > 0\) there exists \(\delta = \delta(u, \varepsilon) > 0\) such that

\[
|G(t, u + v) - G(t, u) - \langle g(t, u), v \rangle_{X_S^*, X_S}| \leq \varepsilon \|v\|_{X_S} \quad \forall v \in B_\delta;
\]

(A2) **Lipschitz continuity of** \(g\) **in** \(u\):

\[
\|g(t, u) - g(t, v)\|_{X_S^*} \leq C(M)\|u - v\|_{X_S} \quad \forall t \in [-T, T], \forall u, v \in B_M;
\]

(A3) **Hölder-like continuity of** \(g\) **in** \(t\): there exists \(\varphi \in L^1(-T, T)\) with \(\varphi(t) \geq 0\) such that

\[
\|g(t, u) - g(s, u)\|_{X_S^*} \leq C(M) \left| \int_s^t \varphi(\sigma) d\sigma \right| \quad \forall t, s \in [-T, T], \forall u \in B_M;
\]

(A4) **Hölder-like continuity of** \(G\): given \(M > 0\), for all \(\delta > 0\) there exists a constant \(C_{1, \delta}(M) > 0\) such that

\[
|G(t, u) - G(t, v)| \leq \delta + C_{1, \delta}(M)\|u - v\|_X \quad \forall t \in [-T, T], \forall u, v \in B_M;
\]

(A5) **Hölder type continuity of** \(G_1\): \(G(t, u)\) is partially differentiable in \(t\) for every \(u \in X_S\). Moreover, there exists \(\varphi \in L^1(-T, T)\), and for any \(M > 0\) and \(\delta > 0\) there exists a constant \(C_{2, \delta}(M) > 0\) such that

\[
|G_1(t, u) - G_1(t, v)| \leq \varphi(t) [\delta + C_{2, \delta}(M)\|u - v\|_X] \quad \text{a.a.} \; t \in (-T, T), \forall u \in B_M;
\]

(A6) **Gauge condition:**

\[
\text{Re} \langle g(t, u), i u \rangle_{X_S^*, X_S} = 0 \quad \forall t \in [-T, T], \forall u \in X_S;
\]

(A7) **Closedness condition:** let \(I \subset (-T, T)\) be an open interval. Assume that \(\{w_n\}_n\) is any bounded sequence in \(L^\infty(-T, T; X_S)\) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
w_n(t) \rightharpoonup w(t) \; (n \to \infty) \quad \text{weakly in } X_S \; \text{a.a.} \; t \in I, \\
g(t, w_n) \rightharpoonup f \; (n \to \infty) \quad \text{weakly}\ast \; \text{in } L^\infty(I; X_S^*).
\end{array} \right.
\end{align*}
\tag{2.1}
\]

Then

\[
\text{Re} \int_I \langle f(t), i \langle w(t) \rangle_{X_S^*, X_S} \rangle dt = \lim_{n \to \infty} \text{Re} \int_I \langle g(t, w_n(t)), i \langle w_n(t) \rangle_{X_S^*, X_S} \rangle dt.
\tag{2.2}
\]

Here \(f(t) = g(t, w(t))\) is guaranteed if

\[
w_n(t) \to w(t) \; (n \to \infty) \quad \text{strongly in } X \; \text{a.a.} \; t \in I.
\tag{2.3}
\]

**Theorem 2.1.** Assume that (A1)–(A7) and \(u_0 \in X_S\). Then there exist \(T_0 \in (0, T]\) (dependent on \(\|u_0\|_{X_S}\)) and \(u \in C_w([-T_0, T_0]; X_S) \cap W^{1, \infty}(-T_0, T_0; X_S^*)\) such that \(u\) is a solution to (ACP). The solution satisfies also the pseudo-conservation laws:

\[
\|u(t)\|_X = \|u_0\|_X \quad \forall t \in [-T_0, T_0],
\tag{2.4}
\]

\[
E(t, u(t)) - E(0, u_0) \leq \int_0^t G_1(s, u(s)) ds \quad \forall t \in [-T_0, T_0],
\tag{2.5}
\]

where \(E(t, \varphi) = (1/2) \|(1 + S)^{1/2} \varphi\|^2_X + G(t, \varphi)\).
Remark 2.2. We can also solve the unilateral problems of (ACP)
\[
\begin{cases}
  i \frac{du}{dt} = Su + \bar{g}(t,u) & \text{in } [0, T] \times X_S^\delta, \\
  u(0) = u_0 & \text{in } X_S.
\end{cases}
\] (2.6)
Assume that \( \bar{g} \) and \( \bar{G} \) satisfy (A1)–(A7) with replacing \( (-T, T) \) by \( (0, T) \). We define \( g \) and \( G \) as the even extension of \( \bar{g} \) and \( \bar{G} \): \( g(t,u) := \bar{g}(|t|, u) \) and \( G(t,u) := \bar{G}(|t|, u) \) for \( t \in (-T, T) \). Then \( g \) and \( G \) just satisfy (A1)–(A7). Hence Theorem 2.1 yields the local existence \( u \in C_w([0, T_0]; X_S) \cap W^{1,\infty}(0, T_0; X_S^\delta) \) of solution to (2.6).

Remark 2.3. If \( g(t, u) \) is the form of separation of variables \( h_0(t)g_0(u) \), then we assume that \( h_0 \in W^{1,\infty}(-T, T) \) and that \( g \) satisfies five conditions (G1)–(G5) (see below) as in the energy methods of self-excited system established by Okazawa–Suzuki–Yokota [16]:

(G1) Existence of energy functional: there exists \( G_0 \in C^1(X_S; \mathbb{R}) \) such that \( d_{X_S}G_0 = g_0 \);

(G2) Local Lipschitz continuity: for all \( M > 0 \) there exists \( C(M) > 0 \) such that
\[
\|g_0(u) - g_0(v)\|_{X_S^\delta} \leq C(M)\|u - v\|_{X_S} \quad \forall \ u, v \in B_M;
\]

(G3) Hölder-like continuity of energy functional: given \( M > 0 \), for all \( \delta > 0 \) there exists a constant \( C_\delta(M) > 0 \) such that
\[
|G_0(u) - G_0(v)| \leq \delta + C_\delta(M)\|u - v\|_X \quad \forall \ u, v \in B_M;
\]

(G4) Gauge condition: \( \text{Re} \left( g_0(u), iu \right)_{X_S^\delta, X_S} = 0 \quad \forall \ u \in X_S \);

(G5) Closedness condition: given a bounded open interval \( I \subset \mathbb{R} \), let \( \{w_n\}_n \) be any bounded sequence in \( L^\infty(I; X_S) \) such that
\[
\begin{cases}
  w_n(t) \rightarrow w(t) \ (n \rightarrow \infty) & \text{weakly in } X_S \ \text{a.a. } t \in I, \\
  g_0(w_n) \rightarrow f \ (n \rightarrow \infty) & \text{weakly* in } L^\infty(I; X_S^\delta).
\end{cases}
\] (2.7)

Then
\[
\text{Re} \int_I \langle f(t), iw(t) \rangle_{X_S^\delta, X_S} dt = \lim_{n \rightarrow \infty} \text{Re} \int_I \langle g_0(w_n(t)), iw_n(t) \rangle_{X_S^\delta, X_S} dt.
\] (2.8)

Here \( f = g_0(w) \) is guaranteed if
\[
w_n(t) \rightarrow w(t) \ (n \rightarrow \infty) \ \text{strongly in } X \ \text{a.a. } t \in I.
\] (2.9)

Now we verify the conditions (A1)–(A7) under the above settings. Here we see that \( \varphi(t) := |h_0(t)|, G(t,u) := h_0(t)G_0(u), \) and \( G_n(t,u) = h_0(t)G_0(u) \). Thus we omit to check (A1)–(A6). So we only verify (A7). Let \( \{w_n\}_n \) be any bounded sequence in \( L^\infty(I; X_S) \) such that
\[
\begin{cases}
  w_n(t) \rightarrow w(t) \ (n \rightarrow \infty) & \text{weakly in } X_S \ \text{a.a. } t \in I, \\
  h_0(t)g_0(w_n(t)) \rightarrow f(t) \ (n \rightarrow \infty) & \text{weakly* in } L^\infty(I; X_S^\delta).
\end{cases}
\] (2.7)

Since \( w_n(t) \) is bounded in \( X_S \), there exist \( f_0 \in L^\infty(I; X_S^\delta) \) and a subsequence \( \{n_k\}_k \) such that \( g_0(w_{n_k}(t)) \rightarrow f_0(t) \ (k \rightarrow \infty) \) weakly* in \( L^\infty(I; X_S^\delta) \); also we see \( h_0(t)g_0(w_{n_k}(t)) \rightarrow h_0(t)f_0(t) \ (k \rightarrow \infty) \) weakly* in \( L^\infty(I; X_S^\delta) \) since \( h_0 \in L^\infty(I) \).
Thus \( f(t) = h_0(t) f_0(t) \). Here \((G4)\) and \((G5)\) yield \( \text{Re} \langle f_0(t), i w(t) \rangle_{X^*_S, X_S} = 0 \) a.a. \( t \in I \). This implies the former half of \((A7)\):

\[
\int_I \text{Re} \langle f(t), i w(t) \rangle_{X^*_S, X_S} \, dt = \int_I h_0(t) \text{Re} \langle f_0(t), i w(t) \rangle_{X^*_S, X_S} \, dt = 0
\]

Next assume further that \( w_n(t) \rightarrow w(t) \) \((n \rightarrow \infty)\) strongly in \( X \) a.a. \( t \in I \). \((G5)\) implies \( f_0(t) = g_0(w(t)) \) and hence \( f(t) = h_0(t) g_0(w(t)) \). Thus \((A7)\) has been fully proved.

**Remark 2.4.** Here we can simplify and relax the condition \((G5)\) by virtue of \((G4)\); see [16, Lemma 5.3].

### 2.1. Preliminaries

Let \( I \subset \mathbb{R} \) be an open interval, and let \( Y \) be a reflexive Banach space. Then \( C(\bar{I}; Y) \) is a family of the continuous \( Y \)-valued function on \( \bar{I} \). On the other hand, the vector-valued Lebesgue space \( L^p(I; Y) \) is equipped with norm

\[
\|u\|_{L^p(I; Y)} := \left\|\|u(\cdot)\|_Y\right\|_{L^p(I)} < \infty.
\]

Moreover, the vector-valued Sobolev space \( W^{1,p}(I; Y) \) can be also defined:

\[
W^{1,p}(I; Y) := \{u \in L^p(I; Y); \|u'\|_{L^p(I; Y)} < \infty\}.
\]

Here \( u' \) denotes the weak derivative of \( u \) respect to time variable \( t \in I \). It is well-known that \( W^{1,p}(I; Y) \subset C(\bar{I}; Y) \) \((1 \leq p \leq \infty)\) is continuous (see [4, Corollary 1.4.36]).

Now let \( A \) be a linear and maximal monotone operator in \( X \), that is, \( R(1+A) = X \) and \( \text{Re} \langle u, Au \rangle_X \geq 0 \). Then \(-A\) generates contraction \( C_0\)-semigroups \( \{e^{-tA}|t \geq 0\} \subset B(X) \), the family of bounded linear operators on \( X \). Now we solve

\[
\begin{cases}
\frac{du}{dt} + Au + g_0(t, u) = 0 & \text{in } [0, T] \times X, \\
u(0) = u_0.
\end{cases}
\tag{2.10}
\]

We use \( B_{M,X} := \{u \in X; \|u\|_X \leq M\} \) for simple notation. Assume that \( g_0 \) satisfies

**\((H1)\) Lipschitz continuity of \( g_0 \) in \( u \):**

\[
\|g_0(t, u) - g_0(t, v)\|_X \leq C(M)\|u - v\|_X \quad \forall t \in [0, T], \forall u, v \in B_{M,X};
\]

**\((H2)\) Hölder-like continuity of \( g_0 \) in \( t \):** there exists \( \varphi \in L^1(0, T) \) with \( \varphi(t) \geq 0 \) such that

\[
\|g_0(t, u) - g_0(s, u)\|_X \leq C(M) \left|\int_s^t \varphi(\sigma) \, d\sigma\right| \quad \forall t, s \in [0, T], \forall u \in B_{M,X}.
\]

In a way similar to Cazenave–Haraux [4, Propositions 4.3.2 and 4.3.9] we can show the unique existence of solution to (2.10):

**Lemma 2.5.** Assume that \((H1)\) and \((H2)\). Let \( u_0 \in D(A) \). Then there exists \( u \in C([0, T_0]; D(A)) \cap C^1([0, T_0]; X) \) such that \( u \) is the local and unique solution to (2.10). Here \( T_0 \in (0, T] \) is determined by \( \|u_0\|_X \).

Next let \( S \) be a nonnegative and selfadjoint operator in the complex-valued Hilbert space \( X \). Then \( \pm iS \) are maximal monotone in \( X \). Hence we can solve
the following semilinear Schrödinger evolution equations:

\[ \begin{cases} 
  i \frac{d u}{dt} = S u + g_0(t, u) & \text{in } [-T, T] \times X, \\
  u(0) = u_0. 
\end{cases} \tag{2.11} \]

Assume (H1), (H2), with replacing \((0, T)\) by \((-T, T)\), and

(H3) **Existence of energy functional**: there exists \(G_0 \in C([-T, T] \times X; \mathbb{R})\) such that \(d_X G_0(t, u) = g_0(t, u)\), that is, given \(u \in X\) and \(t \in [-T, T]\), for every \(\varepsilon > 0\) there exists \(\delta = \delta(u, \varepsilon) > 0\) such that

\[ |G_0(t, u + v) - G_0(t, u) - (g_0(t, u), v)_X| \leq \varepsilon \|v\|_X \quad \forall \ v \in B_{\delta, X}; \]

(H4) **Hölder type continuity of** \(G_0\): \(G_0(t, u)\) is partially differentiable in \(t\) for any \(u \in X\). Moreover, \(|G_0(t, u) - G_0(t, v)| \leq \varphi(t)\delta + C_\delta(M)\|u - v\|_X\) a.a. \(t \in (-T, T), \forall u \in B_M, X;\)

(H5) **Gauge type condition**: \(\text{Re} \langle g_0(t, u), i u \rangle_X = 0 \quad \forall \ t \in [-T, T], \forall u \in X_S.\)

In a way similar to Cazenave [3, Theorem 3.3.1] for the self-excited case we can show the following:

**Lemma 2.6.** Assume that (H1)–(H5). Let \(u_0 \in X_S\). Then there uniquely exists \(u \in C([-T, T]; X_S) \cap C^1([-T, T]; X_S^2)\) such that \(u\) is a global solution to (2.11).

Moreover, the energy equalities are available.

\[ \|u(t)\|_X = \|u_0\|_X \quad \forall \ t \in [-T, T], \tag{2.12} \]

\[ E(t, u(t)) = E(0, u_0) = \int_0^t G_0(s, u(s)) \, ds \quad \forall \ t \in [-T, T], \tag{2.13} \]

where \(E(t, v) := (1/2) \|(1 + S)^{1/2}v\|_X^2 + G_0(t, v).\)

### 2.2. Local existence (Proof of Theorem 2.1).

First, we consider the approximated problems of (ACP):

\[ \begin{cases} 
  i \frac{d u_\varepsilon}{dt} = S u_\varepsilon + (1 + \varepsilon S)^{-1}g(t, (1 + \varepsilon S)^{-1}u_\varepsilon) & \text{in } [-T, T] \times X_S^\varepsilon, \quad (ACP)_\varepsilon \\
  u_\varepsilon(0) = u_0 & \text{in } X_S.
\end{cases} \]

Here \(g_\varepsilon(t, u) := (1 + \varepsilon S)^{-1}g(t, (1 + \varepsilon S)^{-1}u_\varepsilon)\) maps \([-T, T] \times X\) to \(X\). Moreover, we define \(G_\varepsilon(t, u) := G(t, (1 + \varepsilon S)^{-1}u_\varepsilon)\) and \(E_\varepsilon(t, u) := (1/2) \|(1 + S)^{1/2}u\|_X^2 + G(t, (1 + \varepsilon S)^{-1}u_\varepsilon).\)

Now we divide the proof of Theorem 2.1 into 5 steps. The story is a similar way to Okazawa–Suzuki–Yokota [16]. Here it is unknown that \(P_a\) is \(m\)-accretive in \(L^p(\mathbb{R}^N)\) \((p \neq 2\) and \(a\) is near to \(a(N))\). Thus the Cazenave approach [3, Theorem 3.3.5] cannot be applicable.

**Step 1.** Solve \((ACP)_\varepsilon\) globally in time \(t\);

**Step 2.** Evaluate \(\|S^{1/2}u_\varepsilon(t)\|_X\) uniformly in \(t\) and \(\varepsilon\);

**Step 3.** Confirm the weak convergence of \((ACP)_\varepsilon\) to \((ACP)\);

**Step 4.** Derive the charge conservation and make a solution;

**Step 5.** Derive the energy pseudo-conservation.

**Proof of Theorem 2.1.** We divide the proof into 5 steps as above.
Step 1. Solve \((ACP)_\varepsilon\) globally in time \(t\). \((H1)-(H5)\) are followed by \((A2), (A3), (A1), (A5),\) and \((A6)\), respectively. Thus Lemma 2.6 ensures the global unique solution to \((ACP)_\varepsilon\).

Step 2. Evaluate \(\|S^{1/2}u_\varepsilon(t)\|_X\) uniformly in \(t\) and \(\varepsilon\). First let \(M := \|u_0\|_{X^S}\) and define

\[
\tau(\varepsilon) := \sup\{T_1 \in [0,T]; \|S^{1/2}u_\varepsilon(t)\|_X \leq 2M \text{ on } [-T_1,T_1]\}. \tag{2.14}
\]

Note that if \(\tau(\varepsilon) = T\) for all \(\varepsilon > 0\), this step has been finished, thus we assume \(\tau(\varepsilon) < T\). Since we see the continuity of \(u_\varepsilon\), we know \(\|S^{1/2}u_\varepsilon(\tau(\varepsilon))\|_X = 2M\) or \(\|S^{1/2}u_\varepsilon(-\tau(\varepsilon))\|_X = 2M\). Since \(u_\varepsilon\) satisfy (2.13), we have

\[
\|S^{1/2}u_\varepsilon(t)\|_X^2 - \|S^{1/2}u_\varepsilon(0)\|_X^2 = 2\left[G_\varepsilon(0,u_0) - G_\varepsilon(t,u_\varepsilon(t)) + \int_0^t G_{s\varepsilon}(s,u_\varepsilon(s)) \, ds\right].
\]

We evaluate for \(t \in [-\tau(\varepsilon),\tau(\varepsilon)]\) applying \((A4)\) and \((A5)\) as follows:

\[
G_\varepsilon(0,u_0) - G_\varepsilon(t,u_\varepsilon(t)) + \int_0^t G_{s\varepsilon}(s,u_\varepsilon(s)) \, ds \leq \delta + C_{1,\delta}(2M)\|u_\varepsilon(0) - u(t)\|_X + \left[\int_0^t \varphi(s) \left[\delta + C_{2,\delta}(M)\|u_\varepsilon(t) - u_\varepsilon(s)\|_X\right] \, ds\right].
\]

Applying \((A2)\) and \((A3)\) we calculate

\[
\|u_\varepsilon'(t)\|_{X^S} \leq \|Su_\varepsilon(t)\|_{X^S} + \|g_\varepsilon(t,u_\varepsilon(t)) - g_\varepsilon(t,0)\|_{X^S} + \|g_\varepsilon(t,0) - g_\varepsilon(0,0)\|_{X^S} + \|g_\varepsilon(0,0)\|_{X^S} \leq \tilde{C}(M) \quad \forall \varepsilon \in \{-\tau(\varepsilon),\tau(\varepsilon)\}.
\]

Now [4, Lemma 7.4.5] implies that (2.14) and (2.16) yield

\[
\|u_\varepsilon(t) - u_\varepsilon(s)\|_X \leq \sqrt{2L}|t-s|^{1/2} \quad \forall \varepsilon \in \{-\tau(\varepsilon),\tau(\varepsilon)\},
\]

where \(L := \max\{\|u_\varepsilon\|_{L^\infty(-\tau(\varepsilon),\tau(\varepsilon);X^S)}, \|u_\varepsilon'\|_{L^\infty(-\tau(\varepsilon),\tau(\varepsilon);X^S)}\}\) is uniformly bounded in \(\varepsilon > 0\). Here (2.15) and (2.17) ensure

\[
G_\varepsilon(0,u_0) - G_\varepsilon(t,u_\varepsilon(t)) - \int_0^t G_{s\varepsilon}(s,u_\varepsilon(s)) \, ds \leq \delta + \sqrt{2L}C_{1,\delta}(2M)\sqrt{t} + \left[\int_0^t \varphi(s) \left[\delta + \sqrt{2L}C_{2,\delta}(M)\sqrt{|t-s|}\right] \, ds\right]
\]

\[
\leq \delta + \sqrt{2L}C_{1,\delta}(2M)\sqrt{t} + \left[\delta + \sqrt{2L}C_{2,\delta}(2M)\sqrt{|t|}\right]\varphi_{L^1(-T,T)}.
\]

Hence (2.18) implies that

\[
\|S^{1/2}u_\varepsilon(t)\|_X^2 - \|S^{1/2}u_\varepsilon(0)\|_X^2 \leq \delta (1 + \varphi_{L^1(-T,T)}) + \sqrt{2L}(C_{1,\delta}(2M) + C_{2,\delta}(2M))\varphi_{L^1(-T,T)}\sqrt{|t|}.
\]
Putting \( t = \pm \tau(\varepsilon) \) and choosing \( \delta \) as \( 0 < \delta < \sqrt[3]{3M}/(1 + \| \varphi \|_{L^1(-T,T)}) \), we obtain

\[
\tau_\varepsilon \geq \frac{3M^2 - \delta^2(1 + \| \varphi \|_{L^1(-T,T)})^2}{2L^2(C_1,\delta(2M) + C_2,\delta(2M))\| \varphi \|_{L^1(-T,T)})^2} := T_M
\]

Hence \( \| S^{1/2}u_\varepsilon(t) \|_X \leq 2M \) on \( [-T_M, T_M] \) and \( \varepsilon > 0 \). Note that since (A6) are imposed, we have \( \| u_\varepsilon(t) \|_X = \| u_0 \|_X \). Thus \( \| u_\varepsilon(t) \|_X \) is uniformly bounded in \( [-T_M, T_M] \) and \( \varepsilon > 0 \).

**Step 3.** Confirm the weak convergence of (ACP)\(_\varepsilon\) to (ACP). First note that Step 2 implies the weak convergence to some function \( u \). Next we see from (2.16) that

\[
\| u_\varepsilon \|_{L^\infty(-T_M, T_M; X_S)} \leq 2M, \quad \| u_\varepsilon' \|_{L^\infty(-T_M, T_M; X_S)} \leq C(M), \quad \| u_\varepsilon \|_{L^\infty(-T_M, T_M; X_S^*)} \leq \varepsilon > 0.
\]

We also know that (see [21, Step 2 in Proof of Theorem 2.2])

\[
\begin{align*}
\{ u_\varepsilon(k) \} & \subset \{ u_\varepsilon \} \quad \text{and} \quad \{ u_\varepsilon(k) \} \subset \mathcal{C}_2(\varepsilon(k)) \quad \text{for any open interval} \quad \varepsilon > 0 \quad \text{that} \\
& \quad \quad \quad \text{imposed, we have} \quad \| u_\varepsilon(t) \|_X = \| u_0 \|_X. \\
& \quad \quad \quad \text{Thus} \quad \| u_\varepsilon(t) \|_X \text{ is uniformly bounded in} \quad [-T_M, T_M] \quad \text{and} \quad \varepsilon > 0.
\end{align*}
\]

We show \( v = u' \). Let \( \psi \in \mathcal{C}_c^\infty(-T_M, T_M; X_S) \). Then we see from integration by parts that

\[
\int_{-T_M}^{T_M} \langle u_\varepsilon', \psi(t) \rangle_{X_S^*, X_S} dt = -\int_{-T_M}^{T_M} \langle u_\varepsilon, \psi'(t) \rangle_{X_S^*, X_S} dt.
\]

This is nothing but \( u' = v \).

Since \( g_\varepsilon(t, u_\varepsilon) = i u_\varepsilon' - S u_\varepsilon \), there exists \( f \in L^\infty(-T_M, T_M; X_S^*) \) such that

\[
g_\varepsilon(j, t, u_\varepsilon(j))(t) \to f(j) \quad \text{weakly* in} \quad L^\infty(-T_M, T_M; X_S^*).
\]

**Step 4.** Derive the charge conservation and make a solution. First we see that

\[
(1 + \varepsilon(j))S^{-1}u_\varepsilon(j)(t) \to u(t) \quad \text{weakly in} \quad X_S, \quad \text{a.a.} \quad t \in (-T_M, T_M),
\]

\[
g(t, (1 + \varepsilon(j))S^{-1}u_\varepsilon(j)(t)) \to f(t) \quad \text{weakly* in} \quad L^\infty(-T_M, T_M; X_S^*).
\]

(A7) and (A6) yield that for any open interval \( I \subset (-T_M, T_M) \)

\[
\int_I \Re \langle f(t), i u(t) \rangle_{X_S^*, X_S} dt = \lim_{j \to \infty} \int_I \Re \langle g(t, (1 + \varepsilon(j))S^{-1}u_\varepsilon(j)(t)), i (1 + \varepsilon(j))S^{-1}u_\varepsilon(j)(t) \rangle_{X_S^*, X_S} dt = 0.
\]
Putting $I = (0, t)$ (or $I = (t, 0)$), we see (2.4):

$$\frac{1}{2}\|u(t)\|_X^2 - \frac{1}{2}\|u(0)\|_X^2 = \int_0^t \text{Re}\langle u'(s), u(s)\rangle_{X^*_\delta, X_\delta} ds$$

$$= \int_0^t \text{Re}\langle \frac{1}{2}[Su(s) + f(s)], u(s)\rangle_{X^*_\delta, X_\delta} ds = 0.$$

Next we prove $f(t) = g(t, u)$. In a view of the latter half of (A7), it remains to show that

$$u_{\varepsilon_j}(t) \to u(t) \ (j \to \infty) \quad \text{strongly in } X \quad \forall \ t \in [-T_M, T_M]. \quad (2.24)$$

We see from the conservation laws (2.12) and (2.4) that

$$\|u(t)\|_X = \|u_0\|_X = \|u_{\varepsilon_j}(t)\|_X \quad \forall \ t \in [-T_M, T_M].$$

This together with the weak convergence (2.21) yields (2.24). This concludes that $f(u) = g(t, u)$ in $L^\infty(-T_M, T_M; X^*_\delta)$ (more precisely, $u \in C_w([-T_M, T_M]; X_\delta) \cap W^{1, \infty}(-T_M, T_M; X^*_\delta)$). Thus we have constructed a local solution to (ACP) in $[-T_M, T_M]$.

**Step 5. Derive the energy pseudo-conservation.** We prove (2.5). Fix $t \in [-T_M, T_M]$ arbitrarily. We see from (2.24) that

$$(1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t) \to u(t) \ (j \to \infty) \quad \text{strongly in } X \quad \forall \ t \in [-T_M, T_M]. \quad (2.25)$$

(A4) implies that for every $\delta > 0$,

$$|G(t, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t)) - G(t, u(t))|$$

$$\leq \delta + C_{1, \delta}(2M) \| (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t) - u(t) \|_X.$$

Since $C_{1, \delta}(2M)$ is independent of $j$, we obtain from (2.25) that

$$\limsup_{j \to \infty} |G(t, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t)) - G(t, u(t))| \leq \delta$$

for any $\delta > 0$ and for all $t \in [-T_M, T_M]$. Thus we have

$$G(t, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t)) \to G(t, u(t)) \ (j \to \infty). \quad (2.26)$$

On the other hand, (A5) implies that for any $\delta > 0$ and a.a. $t \in (-T_M, T_M)$

$$|G_t(t, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t)) - G_t(t, u(t))|$$

$$\leq \varphi(t) \| \delta + C_{2, \delta}(2M) \| (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t) - u(t) \|_X|.$$

Since $C_{2, \delta}(2M)$ is independent of $j$, it follows from (2.25) that

$$\limsup_{j \to \infty} |G_t(t, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(t)) - G_t(t, u(t))| \leq \delta \varphi(t).$$

Hence we have

$$\int_0^t G_t(s, (1 + \varepsilon(j)S)^{-1}u_{\varepsilon(j)}(s)) ds \to \int_0^t G_t(s, u(s)) ds \quad (j \to \infty). \quad (2.27)$$

(2.21) implies that

$$\|(1 + S)^{1/2}u(t)\|_X \leq \liminf_{j \to \infty} \|(1 + S)^{1/2}u_{\varepsilon(j)}(t)\|_X \quad \forall \ t \in [-T_M, T_M]. \quad (2.28)$$
Thus we see from (2.28) and (2.26) that for all $t \in [-T_M, T_M]$

\[
E(t, u(t)) \leq \liminf_{j \to \infty} E_{\varepsilon(j)}(t, u_{\varepsilon(j)}(t))
\]

\[= \liminf_{j \to \infty} \left( E_{\varepsilon(j)}(0, u_0) + \int_0^t G_t(s, u_{\varepsilon(j)}(s)) \, ds \right). \]

In the last equality we have used (2.13). Since $(1 + \varepsilon(j)S)^{-1} u_0 \to u_0 \ (j \to \infty)$ strongly in $X_S$ and (2.27), we see that

\[
\liminf_{j \to \infty} E_{\varepsilon(j)}(0, u_0) = \lim_{j \to \infty} E_{\varepsilon(j)}(0, u_0) = E(0, u_0).
\]

Therefore we conclude (2.5). This completes the proof of Theorem 2.1. \qed

2.3. Remarks on the global existence.

**Theorem 2.7.** Assume (A1)–(A7) and

(A8) there exists $\varepsilon > 0$ such that

\[
G(t, u) \geq -[(1 - \varepsilon)/2] \|(1 + S)^{1/2}u\|^2_X - C(\|u\|_X) \quad \forall t \in [-T, T], \forall u \in X_S;
\]

(A9) there exists $\psi \in L^1(-T, T)$ with $\psi(t) \geq 0$ such that

\[\text{sgn}(t) G_t(t, u) \leq \psi(t) \|[(1 + S)^{1/2}u]\|^2_X + C(\|u\|_X) \quad \text{a.a.} \ t \in (-T, T), \forall u \in X_S.
\]

Then the local solution $u \in C_{w}([-T_0, T_0]; X_S) \cap W^{1,\infty}(-T_0, T_0; X_S^*)$ to (ACP) can be extended globally in time $t \in [-T, T]$.

**Proof.** Combine (2.5) into (2.4), (A8), and (A9). It follows that

\[
\|(1 + S)^{1/2}u(t)\|^2_X \leq \frac{2A_0}{\varepsilon} + \int_0^T \frac{2}{\varepsilon} \psi(s) \|S^{1/2}u(s)\|^2_X \, ds,
\]

where

\[A_0 := \frac{1}{2} \|(1 + S)^{1/2}u(0)\|^2_X + G(0, u(0)) + C(\|u(0)\|_X) (1 + \|\psi\|_{L^1(-T, T)}).
\]

Hence the Gronwall lemma implies that the uniform boundedness of $\|S^{1/2}u(t)\|^2_X$:

\[
\|S^{1/2}u(t)\|^2_X \leq \frac{2A_0}{\varepsilon} \exp\left( \int_0^T \frac{2}{\varepsilon} \psi(s) \, ds \right) = \frac{2A_0}{\varepsilon} \exp\left( \int_{-T}^T \frac{2}{\varepsilon} \psi(s) \, ds \right).
\]

Extending the interval step by step, we get the global solution $u \in C_{w}([-T, T]; X_S) \cap W^{1,\infty}(-T, T; X_S^*)$. \qed

**Remark 2.8.** If the local solution $u \in C_{w}([-T_0, T_0]; X_S) \cap W^{1,\infty}(-T_0, T_0; X_S^*)$ to (ACP) is unique, then the pseudo-conservation inequality (2.5) is exact equality. Moreover, we have

\[
E(t_2, u(t_2)) - E(t_1, u(t_1)) = \int_{t_1}^{t_2} G_t(s, u(s)) \, ds \quad \forall t_1, t_2 \in [-T_0, T_0].
\]

Hence we can confirm that $u$ belongs also to $C([-T_0, T_0]; X_S) \cap C^1([-T_0, T_0]; X_S^*)$. 
3. Scattering problems for \((\text{NLS})_a\) with nonlocal term. Now we consider the scattering problems for the Hartree type equations

\[
\begin{aligned}
\begin{cases}
  i \frac{\partial u}{\partial t} = P_a u + u K(|u|^2) \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \\
  u(0, x) = u_0(x),
\end{cases}
\end{aligned}
\]  

(HE)

where \(K\) is the integral operator whose kernel is \(k\):

\[K(f)(x) := \int_{\mathbb{R}^N} k(x, y) f(y) \, dy.\]

If \(k(x, y) = W(x - y)\) (convolution type), then \((\text{HE})_a\) is a Hartree equation. Especially, if \(W(x) = |x|^{-\gamma} (k(x, y) = |x-y|^{-\gamma})\), then \((\text{HE})_a\) is a usual Hartree equation and analyzed by many people. In our setting, we can also consider general kernels, for example, \(k(x, y) = W(x + y)\) and \(k(x, y) = U(x)W(x - y)U(y)\). Suzuki [20] proved the global existence of solutions to \((\text{HE})_a\) under the generalized kernels for \(a > a(N)\). Thus we have the following results for global existence. Here we denote

\[
\mathcal{D} := X_{P_a} = \begin{cases} 
  H^1(\mathbb{R}^N) & \text{if } a > a(N), \\
  X^1(\mathbb{R}^N) & \text{if } a = a(N),
\end{cases}
\]

\[
\Sigma := X_{P_a} \cap D(|x|) = \begin{cases} 
  \Sigma^1(\mathbb{R}^N) = H^1(\mathbb{R}^N) \cap D(|x|) & \text{if } a > a(N), \\
  \Sigma^1(\mathbb{R}^N) = X^1(\mathbb{R}^N) \cap D(|x|) & \text{if } a = a(N).
\end{cases}
\]

**Proposition 3.1** ([20, 22]). Let \(N \geq 3\) and \(a \geq a(N)\). Assume that \(k\) satisfies

(K1) \(k\) is real-valued and symmetric: \(k(y, x) = k(x, y) \in \mathbb{R}\) \(a.a.x, y \in \mathbb{R}^N\);

(K2) \(k\) can be divided into several symmetric kernels \(k_j\) so that

\[k_j \in L^2_{\beta_j} (L^0_y), \quad 0 \leq \alpha_j^{-1} + \beta_j^{-1} < \frac{4}{N}, \quad 1 \leq \alpha_j \leq \beta_j \leq \infty;\]

(K3) \(k_{\cdots} (x, y) := \max\{0, -k(x, y)\}\) can be divided into several symmetric kernels \(k_{\cdots j}\) so that

\[k_{\cdots j} \in L^2_{\beta_{\cdots j}} (L^0_y), \quad 0 \leq \alpha_{\cdots j}^{-1} + \beta_{\cdots j}^{-1} < \frac{2}{N}, \quad 1 \leq \alpha_{\cdots j} \leq \beta_{\cdots j} \leq \infty.\]

Then for every \(u_0 \in \mathcal{D}\) there exists a unique global weak solution \(u\) to \((\text{HE})_a\). Moreover, \(u\) belongs to \(C(\mathbb{R}; \mathcal{D}) \cap C^1(\mathbb{R}; \mathcal{D}^*)\) and satisfies conservation laws

\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall \ t \in \mathbb{R},
\]  

(3.1)

where the “energy” \(E\) is defined as

\[
E(\varphi) := \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla \varphi|^2 + \frac{a|\varphi|^2}{|x|^2} \right] \, dx + \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x, y)|\varphi(x)|^2|\varphi(y)|^2 \, dx dy.
\]  

(3.2)

If assume further that \(u_0 \in \Sigma\), then \(u\) belongs also to \(C(\mathbb{R}; \Sigma)\).

We can also obtain the virial identity for \((\text{HE})_a\) (see [21, 24]).

**Proposition 3.2.** Let \(N \geq 3\) and \(a \geq a(N)\). Assume that \(k\) satisfies (K1), (K2), and

(K4) \(\overline{k}(x, y) := 2k(x, y) + x \cdot \nabla_x k(x, y) + y \cdot \nabla_y k(x, y)\) can be divided into several symmetric kernels \(\overline{k}_j\) so that

\[\overline{k}_j \in L^2_{\overline{\beta}_j} (L^0_y), \quad 0 \leq \overline{\alpha}_j^{-1} + \overline{\beta}_j^{-1} < \frac{4}{N}, \quad 1 \leq \overline{\alpha}_j \leq \overline{\beta}_j \leq \infty.\]
Then for every $u_0 \in \Sigma$ the local weak solution $u \in C([-T_1,T_2];\Sigma) \cap C^1([-T_1,T_2];\mathcal{D}^*)$ satisfies the virial identity

$$\frac{d^2}{dt^2} \|xu(t)\|^2_{L^2} = 16 E(u(t)) - 2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \tilde{k}(x,y)|u(t,x)|^2|u(t,y)|^2 \, dx \, dy.$$ 

Since the global well-posedness for (HE)$_a$ is well-studied, we consider the scattering problems for (HE)$_a$. First we construct wave operators for (HE)$_a$, that is, we solve

$$\begin{cases}
i u_t = P_a u + u K(|u|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
\lim_{t \to \infty} \exp(itP_a) u(t) = u_+ \text{ strongly in } \Sigma.
\end{cases} \quad (FV)_a$$

Here applying the pseudo-conformal transform (1.5), $(FV)_a$ can be converted into the following Cauchy problems for nonautonomous semilinear Schrödinger equations.

$$\begin{cases}
i v_t = P_a v + v K_{1/t}(|v|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
v(0) = v_+ := i^{-N/2} \exp(itP_a) e^{1/4} \exp(itP_a) u_+ & \text{in } \Sigma,
\end{cases} \quad (NV)_a$$

where $K_{1/t}$ is the integral operator whose kernel is $t^{-2}k(t^{-1}x,t^{-1}y)$.

Now we explain about the pseudo-conformal transform more precisely. Simple calculations yield that

$$\exp(-itP_a) D_\nu = D_\nu \exp(-\nu^2 t P_a),$$
$$\exp(-itP_a) M_\nu = M_{\nu/(1+b\nu)} D_{1/(1+b\nu)} \exp\left( -\frac{it}{1+bt} P_a \right),$$
$$D_\nu M_b = M_{\nu b^2} D_\nu, \quad \nabla D_\nu = \nu D_\nu \nabla, \quad \nabla M_b = i b M_b \left( \frac{x}{2} - \frac{i}{b} \nabla \right),$$

where $(D_\nu u)(x) := \nu^{N/2} u(\nu x) \, (\nu > 0)$ and $(M_b u)(x) := \exp(ib|x|^2/4) u(x) \, (b \in \mathbb{R})$.

Thus we can rewrite the transformation (1.5) as

$$(Cv)(t,x) = i^{-N/2} D_{1/t} M_t v(t^{-1},x).$$

Operating $\exp(-i(1-t)P_a)$ we see

$$\exp(-i(1-t)P_a) u(t,x) = i^{-N/2} M_1 D_1 \exp(-i(1-t)P_a) v(t^{-1},x).$$

Letting $t \to \infty$ we conclude $\exp(-iP_a) u_+ = i^{-N/2} M_1 \exp(-iP_a) v(0,x)$. Thus if we solve $(NV)_a$, we can also solve $(FV)_a$. Note that we need to set $\Sigma$ not as $\mathcal{D}$ (usual energy space) but as $\mathcal{D} \cap \mathcal{D}(|x|)$ (weighted energy space) so that the transform $C$ works well. In fact, we have following relations

$$\|\nabla u(t)\|_{L^2} = |t|^{-1} \|\nabla (M_{-t} v(t^{-1}))\|_{L^2} = \left\| \left( \frac{x}{2} + \frac{i}{t} \nabla \right) v(t^{-1}) \right\|_{L^2},$$
$$\|\nabla v(t)\|_{L^2} = |t|^{1/2} \|\nabla v(t^{-1})\|_{L^2}.$$ 

**Theorem 3.3.** Let $N \geq 3$ and $a \geq a(N)$. Assume that (K1), (K3), and (K2a) $k$ can be divided into several symmetric kernels so that

$$k_j \in L^2_x(L^\infty_y), \quad \frac{2}{N} \leq \alpha_j^{-1} + \beta_j^{-1} < \frac{4}{N}, \quad 1 \leq \alpha_j \leq \beta_j \leq \infty;$$

(K2a) $\tilde{k}$ can be divided into several symmetric kernels so that

$$\tilde{k}_j \in L^\beta_x(L^\alpha_y), \quad \frac{2}{N} < \tilde{\alpha}_j^{-1} + \tilde{\beta}_j^{-1} < \frac{4}{N}, \quad 1 \leq \tilde{\alpha}_j \leq \tilde{\beta}_j \leq \infty.$$
Then for every \( u_+ \in \Sigma \) there uniquely exists a solution \( u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*) \) to (FV). \( \)

Next we consider the asymptotic free for (HE). Note that the existence the limit in \( L^2(\mathbb{R}^N) \) is proved by [23, Proposition 3.1] for \( a > a(N) \) and by [24, Theorem 4.2] for \( a = a(N) \) without the unsatisfactory restriction for \( \gamma \in (1, \min\{N, 4\}) \). Here we see the discrepancy of the topology between the wave operators and the scattering states. This is resolved by applying the energy methods.

**Theorem 3.4.** Let \( N \geq 3 \) and \( a \geq a(N) \). Assume that (K1), (K2a), (K4a), and (K3a) \( k \) and \( -\tilde{k} \) is nonnegative: \( k(x, y) \geq 0 \) and \( \tilde{k}(x, y) \leq 0 \) a.a. \( x, y \in \mathbb{R}^N \). Then for every global solution \( u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*) \) to (HE) there exists \( u_+ \in \Sigma \) such that

\[
\lim_{t \to \infty} \exp(itP_a)u(t) = u_+ \quad \text{strongly in } \Sigma. \quad (3.3)
\]

### 3.1. Notations and some topics related to \( P_a \)

We need some spaces of measurable functions based on \( \mathbb{R}^N \) to apply the energy methods established in Section 2 to solve the scattering problems. Let \( \Omega \subset \mathbb{R}^N \) be an open set. \( L^p(\Omega) \) is the usual Lebesgue space with norm

\[
\|u\|_{L^p(\Omega)} := \left( \int_{\Omega} |u(x)|^p \, dx \right)^{1/p}, \quad u \in L^p(\Omega) \quad (1 \leq p < \infty),
\]

\[
\|u\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |u(x)|, \quad u \in L^\infty(\Omega).
\]

If \( \Omega = \mathbb{R}^N \), we omit to denote \( \mathbb{R}^N \) from the norm: \( \|u\|_{L^p} := \|u\|_{L^p(\mathbb{R}^N)} \). Let \( p \in [1, \infty] \). Then \( p' \in [1, \infty] \) denotes the Hölder conjugate \( p' := p/(p-1) \). \( H^1(\mathbb{R}^N) \) is the usual \( L^2 \)-type Sobolev space with the norm

\[
\|u\|_{H^1} := (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}, \quad u \in H^1(\mathbb{R}^N).
\]

On the one hand, \( H^{-1}(\mathbb{R}^N) \) is the dual of \( H^1(\mathbb{R}^N) \). Note that we have a usual triplet

\[
H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),
\]

where the inclusion is continuous and dense. In particular, we have

\[
H^1(\mathbb{R}^N) \subset L^r(\mathbb{R}^N), \quad L^{r'}(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N), \quad 2 \leq r \leq \frac{2N}{N-2}, \quad N \geq 3.
\]

Another triplet is available. By virtue of the usual Hardy inequality (1.2), the energy class \( D((1 + P_a)^{1/2}) \) coincides with \( H^1(\mathbb{R}^N) \) for \( a > a(N) \). But the critical case \( a = a(N) \) implies that the energy class does not coincide with \( H^1(\mathbb{R}^N) \). So we define \( X^1(\mathbb{R}^N) := D((1 + P_a(N))^{1/2}) \). We can see the triplet as follows:

\[
X^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset X^{-1}(\mathbb{R}^N) := \text{dual space of } X^1(\mathbb{R}^N).
\]

Moreover, Suzuki [22, Theorem 3.2] ensures for \( 0 < s < 1 \) and \( N \geq 3 \) that

\[
\|(-\Delta)^{s/2} v\|_{L^2} \leq \frac{\Gamma((N + 2s)/4) \Gamma((1 - s)/2)}{\Gamma((N - 2s)/4) \Gamma((1 + s)/2)} \|P_a(N)^{s/2} v\|_{L^2}
\]

(\( \Gamma \) denotes the Gamma function). This implies that \( H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \) \((0 < s < 1)\). In particular, we have

\[
X^1(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad L^{q'}(\mathbb{R}^N) \subset X^{-1}(\mathbb{R}^N), \quad 2 \leq q < \frac{2N}{N-2}, \quad N \geq 3.
\]
We see the Sobolev embedding inequalities in both cases:

\[ \|u\|_{L^r} \leq c(r) \|u\|_{L^2}^{N(r-2)/(2r)} \| (1 + P_a)^{1/2} u \|_{L^2}^{N(r-2)/(2r)}. \] (3.4)

Note that the Rellich compactness lemma is available.

**Lemma 3.5** (see e.g. [22, Lemma 4.5]). Let \( N \geq 3, \ 0 < s \leq 1, \) and \( 2 \leq r < 2N/(N-2s). \) Assume that \( \Omega \subset \mathbb{R}^N \) is the bounded open set with smooth boundary. Then the inclusion \( H^s(\Omega) \subset L^r(\Omega) \) is compact.

We apply the so-called Strichartz estimates for the verifying the uniqueness of \((\text{HE})_a.\) Here the estimates established Burq–Planchon–Stalker–Tahvildar-Zadeh [2, Theorem 3] for \( a > a(N) \) and Suzuki [22, Proposition 4.8] for \( a = a(N).\)

**Definition 3.6.** We call \((\tau, \rho)\) the Schrödinger admissible pair if \( \tau > 2, \ \rho \geq 2, \) and

\[ \frac{2}{\tau} + \frac{N}{\rho} = \frac{N}{2}. \]

**Lemma 3.7.** Let \( N \geq 3, \ a \geq a(N) \) and \((\tau_j, \rho_j) \ (j = 0, 1, 2)\) are the Schrödinger admissible pairs. Then the following inequalities hold:

\[ \| \exp(-itP_a)v \|_{L^{\tau_0}(L^{\rho_0})} \leq C_{\tau} \|v\|_{L^2} \ \forall \ v \in L^2(\mathbb{R}^N), \] (3.5)

\[ \left\| \int_0^t \exp(-i(t-s)P_a)F(s, x) \, ds \right\|_{L^{\tau_2}(L^{\rho_2})} \leq C_{\tau_1, \tau_2} \|F\|_{L^{\tau_1}(L^{\rho_1})} \] (3.6)

for all \( F \in L^{\tau_1'}(\mathbb{R}; L^{\rho_1'}(\mathbb{R}^N)).\)

**Remark 3.8.** We exclude the pair \((2, 2N/(N-2))\) from being Schrödinger admissible in this article. We name this pair the endpoint. (3.5) with the endpoint for \( a > a(N) \) is proved by Burq–Planchon–Stalker–Tahvildar-Zadeh [2, Theorem 3]. (3.6) with the endpoint for \( a > a(N) \) is showed by Pierfelice [19]. Note that (3.5) and (3.6) are broken down in the endpoint case and \( a = a(N); \) see Mizutani [12].

Now we give the generalized Hölder-Riesz inequalities for integral operators.

**Lemma 3.9** ([20, Lemma 2.4]). Let \( k : \mathbb{R}_+^N \times \mathbb{R}_y^N \to \mathbb{C} \) be a measurable function. Assume that \( \|k\|_{L^\alpha(M^2)} < \infty \) and \( \|k\|_{L^\beta(L^2)} < \infty \) with \( \alpha \leq \beta. \) Define \( \rho, \gamma \in [1, \infty) \) so that

\[ \alpha \leq \rho \leq \beta, \ \text{and} \ \frac{1}{\rho} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}. \]

Then the mapping \( K \) is bounded from \( L^\gamma(\mathbb{R}^N) \to L^\rho(\mathbb{R}^N).\)

\[ \|Kf\|_{L^\rho} \leq \max \{\|k\|_{L^\alpha(M^2)} \|k\|_{L^\beta(L^2)} \} \|f\|_{L^\gamma}. \]

Hence we see that if \( k \in L^\beta(L^2) \) is symmetric \((k(y, x) = k(x, y)), \)

\[ \|u_1 K(u_2u_3)\|_{L^{\rho'}} \leq \|k\|_{L^\beta(L^2)} \|u_1\|_{L^\rho} \|u_2\|_{L^\rho} \|u_3\|_{L^\rho}, \] (3.7)

\[ \left\| \int_{\mathbb{R}^N} u_1 u_2 K(u_3u_4) \, dx \right\| \leq \|k\|_{L^\beta(L^2)} \|u_1\|_{L^\rho} \|u_2\|_{L^\rho} \|u_3\|_{L^\rho} \|u_4\|_{L^\rho}, \] (3.8)

where \( r := 4[2 - \alpha^{-1} - \beta^{-1}]^{-1} \in [2, \infty]. \) Note that \( H^s(\mathbb{R}^N) \subset L^r(\mathbb{R}^N) \) \((s := N(\alpha^{-1} + \beta^{-1})/4 < N/2)\) is a continuous inclusion.

On the one hand, we can prove the boundedness of integral operator whose kernel is the type of \( U(x)W(x \pm y)U(y). \) It follows from the usual Hölder and Young inequalities.
Lemma 3.10. Let $U_j \in L^{p_j}(\mathbb{R}^N) \ (j = 1, 2)$, $W \in L^q(\mathbb{R}^N)$, and 
$$k(x, y) := U_1(x)W(x \pm y)U_2(y).$$
Assume that $p_j := 2p_j q_j/(p_j + 2q_j) \in [1, \infty]$. Then the integral operator $K$ is bounded from $L^{p_j}(\mathbb{R}^N)$ to $L^{p_j}(\mathbb{R}^N)$:
$$\|Kf\|_{L^{p_j}} \leq \|U_1\|_{L^{p_1}} \|U_2\|_{L^{p_2}} \|W\|_{L^q} \|f\|_{L^{p_j}}.$$

We also obtain
$$\|u_1 K(u_2 u_3)\|_{L^{p_1}_1} \leq C \|u_1\|_{L^{p_1}} \|u_2\|_{L^{p_2}} \|u_3\|_{L^{p_2}},$$
for $j = 1, 2$.

3.2. Wave operators for Hartree type equations with an inverse-square potential. Now we prove Theorem 3.3.

Proof. Step 1. Local existence for $(NV)_a$ via the energy methods. First we solve $(NV)_a$ solve $(A1)$--$(A7)$. By virtue of (3.7) and (3.8) (with $s < 1$), we can easily check $(A1)$--$(A6)$ and omit to proof. Note that 
$$\|t^{-2}k(t^{-1}x, t^{-1}y)\|_{L^2_\tau(L^2_\rho)} = |t|^{-2+\eta(a^{-1}+\beta^{-1})} \|k\|_{L^2_\tau(L^2_\rho)}$$
and $-2 + N(a^{-1} + \beta^{-1}) \geq 0$ by (K2a) (with putting $(a, \beta) = (a_j, \beta_j)$). Also 
$$\partial_t |t^{-2}k(t^{-1}x, t^{-1}y)| = -t^{-3}k(t^{-1}x, t^{-1}y),$$
$$\|\partial_t |t^{-2}k(t^{-1}x, t^{-1}y)|\|_{L^2_\tau(L^2_\rho)} = |t|^{-3+\eta(a^{-1}+\beta^{-1})} \|k\|_{L^2_\tau(L^2_\rho)}$$
and $-3 + N(a^{-1} + \beta^{-1}) \geq -1$ by (K4a) (with putting $(a, \beta) = (a_j, \beta_j)$). Now we only check (A7) since we need delicate calculations. Define $g_j(t, u) := u K_{j,1/\tau}(|u|^2)$, where $K_{j,1/\tau}$ is an integral operator whose kernel is $t^{-2}k_j(t^{-1}x, t^{-1}y)$ (where $k_j$ is defined as in (K2a)). Let $I \subset \mathbb{R}$ be an open and bounded interval and assume that 
$$\{w_n\}_n \subset L^\infty(I; \mathcal{D})$$
be a sequence satisfying
$$\begin{cases}
g_n(t) \to w(t) \ (n \to \infty) & \text{weakly in } \mathcal{D} \quad \text{a.a. } a \in I, \\
g(t, w_n(t)) \to f(t) \ (n \to \infty) & \text{weakly* in } L^\infty(I; \mathcal{D}^*). 
\end{cases}$$

Since $\{g_j(w_n)\}_n$ is bounded in $L^\infty(I; \mathcal{D}^*)$ and the Sobolev embeddings, there exist a subsequence $\{w_{n(m)}\}_m$ of $\{w_n\}_n$ and $f_j \in L^\infty(I; \mathcal{D}^*)$ such that 
$$g_j(t, w_{n(m)}(t)) \to f_j(t) \ (m \to \infty) \quad \text{weakly* in } L^\infty(I; \mathcal{D}_j^* (\mathbb{R}^N))$$
for $r_j := 4(2 - \alpha_j^{-1} - \beta_j^{-1}) \in [2N/(N - 1), 2N/(N - 2)]$. To confirm (2.2) let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open subset with $C^1$ boundary. Then 
$$\langle f_j(t), w(t)\rangle_{L^r_j(\Omega), L^s_j(\Omega)} = (f_j(t) - g_j(t, w_{n(m)}(t)), w(t))_{L^r_j(\Omega), L^s_j(\Omega)}$$
$$+ (g_j(t, w_{n(m)}(t)), w(t) - w_{n(m)}(t))_{L^r_j(\Omega), L^s_j(\Omega)} + (g_j(t, w_{n(m)}(t)), w_{n(m)}(t))_{L^r_j(\Omega), L^s_j(\Omega)} =: J_{j1}(t) + J_{j2}(t) + J_{j3}(t).$$
The weak convergence (3.11) asserts that
\[
\int_I J_{j1}(t) \, dt \to 0 \quad (m \to \infty).
\] (3.13)

Next we consider \(J_{j2}\). The Rellich compactness lemma (Lemma 3.5) implies that \(w_{n(m)}(t) \to w(t) \ (m \to \infty)\) strongly in \(L^\gamma(\Omega)\) a.a. \(t \in I\). It follows from the boundedness of \(\{g_j(w_{n(m)}(t))\}_m\) in \(L^\gamma(\Omega)\) a.a. \(t \in I\) that \(J_{j2}(t) \to 0 \ (m \to \infty)\) for a.a. \(t \in I\). We see the boundedness of \(\{w_{n(m)}\}_m\) in \(L^\infty(I; L^\gamma(\Omega))\) and \(\{g_j(w_{n(m)})\}_m\) in \(L^\infty(I; L^\gamma(\Omega))\). The dominated convergence lemma yields that
\[
\int_I J_{j2}(t) \, dt \to 0 \quad (m \to \infty).
\] (3.14)

(K1) implies that \(\text{Im } J_{j3}(t) = 0\) a.a. \(t \in I\). Integrating (3.12) over \(I\) and using (3.13) and (3.14), we obtain
\[
\text{Re} \int_I \langle f_j(t), i w(t) \rangle_{L^\gamma_t(\Omega), L^\gamma^*(\Omega)} \, dt = 0.
\]

Since \(\Omega\) is arbitrary, (A6) implies (2.2).

Next we show that \(f(t) = g(t, w(t))\) by assuming further that \(w_n(t) \to w(t)\) \((n \to \infty)\) in \(L^2(\mathbb{R}^N)\) a.a. \(t \in I\). Let \(M := \sup_n \|w_n\|_{L^\infty(I; \mathcal{D})}\). It follows from (3.7) and (3.4) that
\[
\|g(t, w_n(t)) - g(t, w(t))\|_{\mathcal{D}} \leq \sum_j c(r_j)^4 |t|^{-2 + N(1/\alpha_j + 1/\beta_j)} \|k_j\|_{L^\alpha_j(L^\beta_j)} \times \|w_n(t)\|_{\mathcal{D}}^2 + \|w_n(t)\|_{\mathcal{D}} \|w(t)\|_{\mathcal{D}} + \|w(t)\|_{\mathcal{D}}^2
\]
\[
\times \|w_n(t) - w(t)\|_{\mathcal{D}} \|N(r_j - 2)/(2r_j)\| w_n(t) - w(t)\|_{L^2}\|_{\mathcal{D}}^2 \|w_n(t) - w(t)\|_{L^2}^{N(r_j - 2)/(2r_j)}
\]
\[
\leq 6 \sum_j c(r_j)^4 M^{2 + N(r_j - 2)/(2r_j)} \|[t]^{-2 + N(1/\alpha_j + 1/\beta_j)}\| \|k_j\|_{L^\alpha_j(L^\beta_j)} \times \|w_n(t) - w(t)\|_{L^2}^{1 - N(r_j - 2)/(2r_j)}
\]
\[
\to 0 \quad (n \to \infty) \quad \text{a.a. } t \in I.
\]

Thus we see that \(g(t, w_n(t)) \to g(t, w(t)) \ (n \to \infty)\) strongly in \(L^\infty(I; \mathcal{D})\) and (A7) is verified. Thus we can conclude the local existence of (HE)\(_a\) by Theorem 2.1.

**Step 2. Uniqueness for (NV)\(_a\).** Let \(v_1\) and \(v_2\) be local solutions which belong to \(C_w([-T_0, T_0]; \mathcal{D}) \cap W^{1, \infty}(-T_0, T_0; \mathcal{D}^*)\) to (NV)\(_a\). We can convert (NV)\(_a\) into integral equations. We obtain
\[
v_1(t) - v_2(t) = -i \int_0^t \exp(-i(t - s)P_a) [v_1(s) K_{1/s}(|v_1(s)|^2) - v_2(s) K_{1/s}(|v_2(s)|^2)] \, ds.
\] (3.15)

Define \(\tau_j := 8N^{-1}(\alpha_j^{-1} + \beta_j^{-1})^{-1}\) so that \((\tau_j, r_j)\) is a Schrödinger admissible pair. It follows from (3.6) and (3.7) that for any Schrödinger admissible pair \((\tau, \rho)\)
\[
\|v_1 - v_2\|_{L_t^\tau L_x^\rho} \leq \sum_j C_{\tau, r_j} \left(\|(v_1 - v_2) K_{j, 1/t}(|v_1|^2)\|_{L_t^\tau L_x^\rho} + \|v_2 K_{j, 1/t}(|v_1|^2 - |v_2|^2)\|_{L_t^\tau L_x^\rho}\right)
\]
It is sufficient to prove (3.3) that there exists a potential. Next we prove Theorem 3.4.

3.3. Asymptotic free for Hartree type equations with an inverse-square potential. Next we prove Theorem 3.4.

Proof. Let \( u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*) \) be a unique solution to (HE)\(_a\). Then \( v = C^{-1}u \in C((0, \infty); \Sigma) \cap C^1((0, \infty); D^*) \) satisfies \( iv_1 = P_a v + v K_{1/a}(|v|^2) \) in \((0, \infty)\). It is sufficient to prove (3.3) that there exists \( v_0 \in \Sigma \) such that \( v(t) \to v_0 \) \((t \to +0)\) strongly in \( \Sigma \). Here \( v \) satisfies the energy conservation:

\[
E(t, v(t)) - E(t_0, v(t_0)) = \int_{t_0}^t G_t(s, v(s)) \, ds.
\]

Put \( 0 < t < 1 \) and \( t_0 = 1. \) The latter half of (K3a) yields \( E(t, v(t)) - E(1, v(1)) \leq 0. \) Moreover, the former half of (K3a) implies that \( \|(1 + P_a)^{1/2}v(t)\|_{L_2}^2 \leq 2E(t, v(t)) \). Hence \( \|(1 + P_a)^{1/2}v(t)\|_{L_2} \) is uniformly bounded \((t \in (0, 1))\). Also \( v(t) \) is uniformly bounded in \( \Sigma \) (see Step 3 in the proof of Theorem 3.3). This implies that there
exists $v_0 \in \Sigma$ such that $v(t) \to v_0$ weakly in $\Sigma$. It follows from Step 1 of proof in Theorem 3.3 that there uniquely exists $w \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ such that
\[
\begin{cases}
i w_t = P_\alpha w + vK_{1/2}(|w|^2) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
u(0) = v_0 & \text{in } \Sigma.
\end{cases}
\]
Uniqueness in $(1, \infty)$ implies $v = w$. Since $w$ is continuous at $t = 0$ in $\Sigma$, we conclude that $v(t) \to v_0$ ($t \to +0$) strongly in $\Sigma$. \hfill \Box

**Remark 3.12.** Remark 3.11 implies that we can also confirm that for every global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ to (HE)$_a$ there exists $u_- \in \Sigma$ which satisfies (3.16).

Combining Theorems 3.3 and 3.4, we see that (HE)$_a$ is asymptotic complete in $\Sigma$ under the assumption (K1), (K2a), (K3a), and (K4a). Moreover, we can show some identities between $u_0$ and $u_+$:

**Proposition 3.13.** Let $a \geq a(N)$. Assume (K1), (K3a), (K4a) and (K2b) $k$ can be divided into several symmetric kernels so that
\[
k_j \in L^2_0(L^N_{y'}) , \quad \frac{2}{N} < \alpha_j^{-1} + \beta_j^{-1} < \frac{4}{N}, \quad 1 \leq \alpha_j \leq \beta_j \leq \infty;
\]
then one has
\[
\|u_+\|_{L^2} = \|u_0\|_{L^2} , \quad \|P_\alpha^{1/2}u_+\|_{L^2} = \frac{1}{2}\|P_\alpha^{1/2}u_0\|_{L^2} + G(1, u_0),
\]
\[
\|ux_+\|_{L^2} = \|ux_0\|_{L^2} + 8 \int_0^\infty sG_i(1, u(s)) \, ds
\]
for every global and unique solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; \mathcal{D}^*)$ to (HE)$_a$, where $u_+$ is defined in (3.3) and
\[
G(t, \varphi) := \frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} t^{-2} k(t^{-1}x, t^{-1}y) |\varphi(x)|^2 |\varphi(y)|^2 \, dx \, dy,
\]
\[
G_i(t, \varphi) := -\frac{1}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} t^{-3} k(t^{-1}x, t^{-1}y) |\varphi(x)|^2 |\varphi(y)|^2 \, dx \, dy.
\]

**Proof.** Now we define $v_+ := i^{-N/2} \exp(iP_{\alpha})e^{i|x|^2/4} \exp(iP_{\alpha})\pi \tau$. Then $v := C^{-1}u \in C([0, \infty); \Sigma) \cap C^1([0, \infty), \mathcal{D}^*)$ satisfies (NV)$_a$. Now $u(t)$ and $v(t)$ hold the following identities:
\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2} , \quad \|v(t)\|_{L^2} = \|v_+\|_{L^2},
\]
\[
E(1, u(t)) = E(1, u_0),
\]
\[
E(t, v(t)) = E(0, v_+) + \int_0^t G_i(s, v(s)) \, ds,
\]
where $E(t, \varphi) := (1/2) \|P_\alpha^{1/2}\varphi\|_{L^2}^2 + G(t, \varphi)$. Note that (3.20) is the charge conservation and (3.21) and (3.22) are the energy conservation (see Remark 2.8). Moreover, the pseudo-conformal transform (1.5) implies that $G(1, u(t)) = t^{-2}G(t^{-1}, v(t^{-1}))$.

**Step 1. Charge identity (3.17).** This is simply followed by (3.20):
\[
\|u_0\|_{L^2} = \|u(t)\|_{L^2} = \|v(t^{-1})\|_{L^2} = \|v_+\|_{L^2} = \|u_+\|_{L^2}.
\]
Step 2. Identities for some functionals. First we remark the commutation relation \([i\partial_t - P_a, C(t)]w(t) = 0\), where

\[
C(t)w(t) := \left[\frac{|x|^2}{4} w(t) + it x \cdot \nabla w(t) + \frac{iNt}{2} w(t) + t^2 P_a w(t) \right]
\]

(see e.g. [2, page 532]). Since \(M_{-1/t}(x/2 + it\nabla)f = it\nabla[M_{-1/t}f]\), we see that \(C(t)\) is nonnegative and symmetric:

\[
\int_{\mathbb{R}^N} C(t)f g\, dx = \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \left( \frac{x}{2} + it\nabla \right) f \cdot \left( \frac{x}{2} + it\nabla \right) g + \frac{at^2}{|x|^2} f g \right] dx
\]

\[
= t^2 \int_{\mathbb{R}^N} \left[ \nabla(M_{-1/t}f) \cdot \nabla(M_{-1/t}g) + \frac{a}{|x|^2} (M_{-1/t}f)(M_{-1/t}g) \right] dx;
\]

especially,

\[
\|C(t)^{1/2}f\|_{L^2}^2 = t^2 \|P_a^{1/2}[M_{-1/t}f]\|_{L^2}^2. \tag{3.23}
\]

Since \(w(t) = C(t)^{1/2} \exp(-itP_a)\varphi\) satisfies \(i w_t - P_a w(t) = 0\), the charge conservation implies

\[
\|C(t)^{1/2} \exp(-itP_a)\varphi\|_{L^2}^2 = \|C(0)^{1/2}\varphi\|_{L^2}^2 = \frac{1}{4} \|x\varphi\|_{L^2}^2. \tag{3.24}
\]

Step 3. Energy identity (3.18). We see from (3.21) and (3.23) that

\[
E(1, u_0) = E(1, u(t)) \tag{3.25}
\]

\[
= \frac{1}{2} \|P_a^{1/2}D_{1/t}M_{t}v(t^{-1})\|_{L^2}^2 + t^{-2}G(t^{-1}, v(t^{-1}))
\]

\[
= \frac{1}{2} \|P_a^{1/2}C(1/t^{1/2}v(t^{-1}))\|_{L^2}^2 + t^{-2}G(t^{-1}, v(t^{-1})).
\]

By virtue of (K2b) (not (K2a)), we have

\[
\lim_{t \to +0} G(t, v(t)) = G(0, v_+) = 0. \tag{3.26}
\]

Letting \(t \to 0\) in (3.25), we obtain \(E(1, u_0) = (1/8) \|x v_+\|_{L^2}^2\). It follows from (3.24) that we conclude (3.18):

\[
\|x v_+\|_{L^2}^2 = \|x \exp(-itP_a)M_{-1} \exp(-itP_a)u_+\|_{L^2}^2
\]

\[
= 4 \|C(-1)^{1/2}M_{-1} \exp(-itP_a)u_+\|_{L^2}^2
\]

\[
= 4 \|P_a^{1/2} \exp(-itP_a)u_+\|_{L^2}^2 = 4 \|P_a^{1/2}u_+\|_{L^2}^2.
\]

Step 4. Weight identity (3.19). Define \(f(t) := 8E(t^{-1}, v(t^{-1}))\). Note that \(f(t) = 4 \|C(t)^{1/2}u(t)\|_{L^2}^2 + 8t^2 G(1, u(t))\). By virtue of (3.22), we see \(f'(t) = -8t^{-2}G(t^{-1}, v(t^{-1})) = -8t G_1(1, u(t)). \tag{3.26}\) implies

\[
\lim_{t \to -\infty} f(t) = 8E(0, v_+) = 4 \|P_a^{1/2}v_+\|_{L^2}^2 = \|x u_+\|_{L^2}^2.
\]

Hence we obtain

\[
\|xu_+\|_{L^2}^2 - \|xu_0\|_{L^2}^2 = \lim_{t \to -\infty} [f(t) - f(0)] = \int_0^\infty f'(s) \, ds = -8 \int_0^\infty s G_1(1, u(s)) \, ds.
\]

This is nothing but (3.19).
There exists $u$ such that $u \in L^p(\mathbb{R}^N)$ and $W \in L^q(\mathbb{R}^N)$, then $k(x, y) := U(x)W(x \pm y)U(y)$ belongs to $L^p_x(L^{p/(p+1)}_y)$. Thus we can specialize the kernel $k$ as the form $U(x)W(x \pm y)U(y)$.

(L1) $U$ and $W$ are real-valued and nonnegative. If we select $W(x - y)$, we additionally impose that $W$ is even: $W(-x) = W(x)$;

(L2) $U \in L^{p_1}(\mathbb{R}^N) + L^{p_2}(\mathbb{R}^N)$ and $W \in L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$, where $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ satisfy

$$2p_1^{-1} + q_1^{-1}, 2p_2^{-1} + q_2^{-1} \in \left[\frac{2}{N}, \frac{4}{N}\right];$$

(L3) There exists $\theta \in [0, 1]$ such that $x \cdot \nabla_x U + \theta U \in L^{p_1}(\mathbb{R}^N) + L^{p_2}(\mathbb{R}^N)$ and $x \cdot \nabla_x W + 2(1 - \theta)W \in L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$, where $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ satisfy

$$2p_1^{-1} + q_1^{-1}, 2p_2^{-1} + q_2^{-1} \in \left[\frac{2}{N}, \frac{4}{N}\right];$$

(L4) There exists $\theta \in [0, 1]$ such that $x \cdot \nabla_x U + \theta U \leq 0$ and $x \cdot \nabla_x W + 2(1 - \theta)W \leq 0$.

We can solve the scattering problems for (HE)$_a$ with a specified integral kernel. The proof is completed in ways similar to Theorems 3.3 and 3.4 with replacing (3.7) and (3.8) by (3.9) and (3.10), respectively.

Corollary 3.14. Let $N \geq 3$ and $a \geq a(N)$.

(i) Assume (L1)–(L3). Then for every $u_+ \in \Sigma$ there uniquely exists a solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (FV)$_a$.

(ii) Assume (L1)–(L4). Then for every global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (HE)$_a$ there exists $u_+ \in \Sigma$ such that (3.3).

3.4. Remarks for scattering problems for nonlinear Schrödinger equations with an inverse-square potential. Scattering problems of other nonlinearities can be also solved. One is the simple nonlinear terms:

Corollary 3.15. Assume either that

(S1) $g(u) = \lambda |u|^{p-1}u$ with $\lambda > 0$ and $1 + 4/N \leq p < 1 + 4/(N - 2)$;

(S2) $g(u) = \lambda |x|^{-r} |u|^{p-1}u$ with $\lambda > 0$, $0 < r < 2$, and $1 + (4 - 2r)/N \leq p < 1 + (4 - 2r)/(N - 2)$;

(S3) $g(u) = \lambda (|x|^{-\gamma} |u|^2)u$ with $\lambda > 0$ and $2 \leq \gamma < \min\{N, 4\}$;

(S4) $g(u) = \lambda u |x|^{-\alpha} |x|^{-\beta} \ast (|x|^{-\alpha} |u|^2)$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$, and $2 \leq 2\alpha + \beta < \min\{N, 4\}$.

Then one has

(i) for every $u_+ \in \Sigma$ there uniquely exists a global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (NLS)$_a$ with (3.3);

(ii) for every unique global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (NLS)$_a$ there exists $u_+ \in \Sigma$ such that (3.3).
The energy functionals $G$ related to $g$ as above are defined as follows:

\begin{align*}
\text{(S1)} \quad G(u) &:= \frac{\lambda}{p+1} \int_{\mathbb{R}^N} |u(x)|^{p+1} \, dx; \\
\text{(S2)} \quad G(u) &:= \frac{\lambda}{p+1} \int_{\mathbb{R}^N} \frac{|u(x)|^{p+1}}{x^r} \, dx; \\
\text{(S3)} \quad G(u) &:= \frac{\lambda}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\gamma} \, dx dy; \\
\text{(S4)} \quad G(u) &:= \frac{\lambda}{4} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^2}{|x|^\alpha |x-y|^\beta |y|^\alpha} \, dx dy.
\end{align*}

Applying the pseudo-conformal transform, $v = C^{-1}u$ satisfies $iv_t = Pa_v + i^\theta g(v)$, where $\theta$ is determined as

\begin{align*}
\text{(S1)} : \quad \theta &= \frac{N(p-1)}{2} - 2; \\
\text{(S2)} : \quad \theta &= \frac{N(p-1)}{2} + r - 2; \\
\text{(S3)} : \quad \theta &= \gamma - 2; \\
\text{(S4)} : \quad \theta &= 2\alpha + \beta - 2.
\end{align*}

A local nonlinearity is another case for the solvable scattering problems. We can also consider (NLS)$_{\theta}$ with local nonlinearities as the pure power term $|u|^{p-1}u$.

**Proposition 3.16** ([16, 21, 22]). Let $N \geq 3$ and $a \geq a(N)$. Assume that $f : \mathbb{C} \to \mathbb{C}$ satisfies (F1)–(F3). Then for every $u_0 \in D$ there uniquely exists a global solution $u \in C(\mathbb{R}; D) \cap C^1(\mathbb{R}; D^*)$ to

\begin{align*}
\begin{cases}
  iv_t = Pu + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\
  u(0,x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases} \tag{3.27}
\end{align*}

Moreover, $u$ satisfies the conservation laws: $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $E(u(t)) = E(u_0)$, where

$$E(\varphi) := \frac{1}{2} \|P^{1/2}\varphi\|_{L^2}^2 + \int_{\mathbb{R}^N} F(\varphi) \, dx.$$ 

Furthermore, if $u_0 \in \Sigma$, then $u$ belongs also to (3.27) and

$$\frac{d^2}{dt^2} \|x u(t)\|_{L^2}^2 = 16 E(u(t)) - \int_{\mathbb{R}^N} \left[ 2(N+2) F(u(t)) - N f(u(t)) u(t) \right] \, dx.$$ 

Let $f(t,x,u) : [0,\infty) \times \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ be a local nonlinearity with (F2) for $u$-variable. Then if $u$ satisfies $iv_t = Pu + f(t,x,u)$, $v = C^{-1}u$ satisfies $iv_t = Pa_v + t^{2-N/2} f(t^{-1}x, t^{N/2}v)$. Therefore we can solve the scattering problems for (3.27) in ways similar to Theorems 3.3 and 3.4.
Theorem 3.17. Let $N \geq 3$ and $a \geq a(N)$. Assume that $f$ satisfies (F2) and (F1a) There exist $C > 0$ and $p \in [1+4/N, 1+4/(N-2))$ such that for all $u_1, u_2 \in \mathbb{C}$
\[
|f(u_1) - f(u_2)| \leq C(|u_1|^{4/N} + |u_2|^{4/N} + |u_1|^{p-1} + |u_2|^{p-1})|u_2 - u_1|;
\]
(F3a) $F(x) \geq 0$ for $x > 0$ and there exist $C_1', C_2' > 0$ and $q_1, q_2$ such that $1+4/N < q_1 < q_2 < (N+2)/(N-2)$ and
\[
0 \geq 2(N+2)F(x) - Nx f(x) \geq -C_1' x^{q_1} - C_2' x^{q_2} \quad x > 0.
\]
Then one has
(i) for every $u_+ \in \Sigma$ there exists a global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (3.27) which satisfies (3.3);
(ii) for every global solution $u \in C(\mathbb{R}; \Sigma) \cap C^1(\mathbb{R}; D^*)$ to (3.27) there exists $u_+$ which satisfies (3.3).

4. Concluding remarks.

4.1. Remarks on the energy methods: Generalization for systems. We can generalize the energy methods (Theorem 2.1) for the system. Let $B : X^*_S \rightarrow X^*_S$ be a bounded linear operator with the following conditions:
- $BSu = S Bu$ for $u \in X_S$;
- $B$ is bounded and symmetric operator in $X$;
- $B$ is coercive in $X$: there exists $\varepsilon > 0$ such that $\text{Re} \langle Bu, u \rangle_X \geq \varepsilon \|u\|^2_X$.

By using $B$, (H5) is replaced with (H5a):
\[
\text{Re} \langle g_0(t, u), i Bu \rangle_X = 0 \quad \forall \ t \in [-T, T], \ \forall \ u \in X.
\]

We have the following in a way similar to Lemma 2.6.

Lemma 4.1. Assume (H1)–(H4) and (H5a). Then for any $u_0 \in X_S$ there uniquely exists the global solution of (2.6) $u \in C([-T, T]; X_S) \cap C^1([-T, T]; X^*_S)$. Moreover, $u$ satisfies the conservation laws
\[
\text{Re} \langle Bu(t), u(t) \rangle_X = \text{Re} \langle Bu_0, u_0 \rangle_X, \quad E_0(t, u(t)) = E_0(0, u_0) + \int_0^t G_0(s, u(s)) \, ds.
\]

Applying this lemma, we can generalize Theorem 2.1.

4.2. Applications to the energy methods: linear Schrödinger evolution systems with time-dependent potentials. Let $N \geq 1$. We consider
\[
\begin{cases}
  iu_t = -\Delta u + V(t)u & \text{in } (0, T), \\
  u(0) = u_0 \in H^1(\mathbb{R}^N),
\end{cases}
\]
where $V(t)$ is a real-valued measurable function.

Corollary 4.2. Let $N \geq 1$. Define the index $q(N)$ so that $q(1) = 1, q(2) > 1$, and $q(N) = N/2$ $(N \geq 3)$.
Assume that $V$ satisfies
(V1) $V(t, x) \in L^\infty(0, T; L^\infty(\mathbb{R}^N)) + L^\infty(0, T; L^{q(N)}(\mathbb{R}^N))$;
(V2) $V(t, x) \in L^1(0, T; L^\infty(\mathbb{R}^N)) + L^1(0, T; L^{q(N)}(\mathbb{R}^N))$.

Calculating the energy functional, we obtain
\[
\begin{align*}
\frac{d}{dt} \mathcal{E}(u(t)) &= \int_{\mathbb{R}^N} \left( |u(t)|^2 - \frac{1}{2^*} |V(t)u(t)|^2 \right) \, dx \\
&\leq \frac{\|u(t)\|^2}{2^*} - \frac{1}{2^*} \left( \frac{\|V(t)u(t)\|^2}{2^*} \right).
\end{align*}
\]
Then for every $u_0 \in H^1(\mathbb{R}^N)$ there uniquely exists a global solution $u \in C([0,T]; H^1(\mathbb{R}^N)) \cap C^1([0,T]; H^{-1}(\mathbb{R}^N))$ to (4.1). Moreover, $u$ satisfies for any $s, t \in [0,T]$ 
\[ \|u(t)\|_{L^2}^2 = \|u(s)\|_{L^2}^2, \]
\[ \int_{\mathbb{R}^N} ||\nabla u(t)||^2 + V(t)||u(t)||^2 \, dx = \int_{\mathbb{R}^N} ||\nabla u(s)||^2 + V(s)||u(s)||^2 \, dx \]
\[ + \int_s^t \left[ \int_{\mathbb{R}^N} \partial V(\sigma)||u(\sigma)||^2 \, dx \right] \, d\sigma. \]

This follows from Theorems 2.1 and 2.7. Since (4.1) is a linear problem, we can see the uniqueness in a simple way. Here Corollary 4.2 is the energy-class version of Yajima [25]. Moreover, we can define the family of linear operators $\{U(t,s)\}t, s \in [0,T] \subset \mathcal{B}(H^1(\mathbb{R}^N))$ such that $u(t) = U(t,s)u_s$ satisfies 
\[ \begin{cases} iu_t = -\Delta u + V(t)u & \text{in } (0,T), \\
u(s) = u_s \in H^1(\mathbb{R}^N). \end{cases} \]

Here $U(t,s)$ holds the following properties of evolution systems: 
- $U(t,s)\varphi$ belongs to $C([0,T] \times [0,T]; H^1(\mathbb{R}^N))$; 
- If $\varphi_n \rightharpoonup \varphi \ (n \to \infty)$ strongly in $H^1(\mathbb{R}^N)$, then $U(t,s)\varphi_n \to U(t,s)\varphi \ (n \to \infty)$ in $C([0,T] \times [0,T]; H^1(\mathbb{R}^N))$; 
- $U(t,t)$ is an identity in $H^1(\mathbb{R}^N)$; 
- $U(t,s)U(s,r) = U(t,r)$.

Let $N \geq 3$. In a way similar to Corollary 4.2 we can solve 
\[ \begin{cases} iu_t = P_a u + V(t)u & \text{in } (0,T), \\
u(s) = u_s \in \mathcal{D}. \end{cases} \]

Here if $a > a(N)$, then we impose that $V(t)$ satisfies $\mathbf{V1}$ and $\mathbf{V2}$; if $a = a(N)$, then we impose that $V(t)$ satisfies $\mathbf{V1}$ and $\mathbf{V2}$ with putting $g(N) > N/2$.

Hence we can construct a family of operators $\{U_a(t,s)\}t, s \in [0,T] \subset \mathcal{B}(\mathcal{D})$ such that $u(t) = U_a(t,s)u_s$ satisfies (4.4). In addition to $\mathbf{V1}$ and $\mathbf{V2}$ assume further $\mathbf{V3}$ $x \cdot \nabla V(t,x) \in L^1(0,T; L^\infty(\mathbb{R}^N)) + L^1(0,T; L^q(\mathbb{R}^N)(\mathbb{R}^N))$.

Then $U_a(t,s)$ is a bounded mapping from $\Sigma$ to $\Sigma$.

**Remark 4.3.** $V(t,x) := Z |x - c(t)|^{-1} \ (Z \in \mathbb{R})$ does not satisfy $\mathbf{V2}$ even when $c(t) \in W^{1,\infty}(0,T; \mathbb{R}^N)$; $V_c(t,x) = Z c'(t) \cdot (x - c(t))|x - c(t)|^{-3}$. Thus another argument is required to solve the Schrödinger evolution equations with Coulomb type singularity of moving-center; see Baudouin–Kavian–Puel [1], Okazawa–Yokota–Yoshii [17], and Okazawa–Yoshii [18] for strong solutions $u \in C([0,T]; H^2(\mathbb{R}^N)) \cap C^1([0,T]; L^2(\mathbb{R}^N))$. Thus our energy methods may be improved.

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**REFERENCES**

[1] L. Baudouin, O. Kavian and J.-P. Puel, Regularity for a Schrödinger equation with singular potentials and application to bilinear optimal control, *J. Differential Equations*, 216 (2005), 188–222.

[2] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, *J. Funct. Anal.*, 203 (2003), 519–549.
[3] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, 2003.

[4] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and its Applications, 13. The Clarendon Press, Oxford University Press, New York, 1998.

[5] T. Cazenave and F. B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, *Comm. Math. Phys.*, 147 (1992), 75–100.

[6] J. Ginibre and T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, *Comm. Math. Phys.*, 151 (1993), 619–645.

[7] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations. I, *Rev. Math. Phys.*, 12 (2000), 361–429.

[8] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations. II, *Ann. Henri Poincaré*, 48 (1998), 17–37.

[9] N. Hayashi and T. Ozawa, Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations, *Ann. Inst. Henri Poincaré*, 48 (1998), 619–645.

[10] N. Hayashi and Y. Tsutsumi, Scattering theory for Hartree type equations, *Ann. Inst. Henri Poincaré*, 46 (1995), 187–213.

[11] J. Lu, C. Miao, J. Murphy, Scattering in $H^1$ for the intercritical NLS with an inverse-square potential, *J. Differ. Equ.*, 264 (2018), 3174–3211.

[12] H. Mizutani, Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential, *J. Differential Equations*, 263 (2017), 3832–3853.

[13] K. Nakanishi, Modified wave operators for the Hartree equation with data, image and convergence in the same space. II, *Ann. Henri Poincaré*, 3 (2002), 503–535.

[14] K. Nakanishi, Modified wave operators for the Hartree equation with data, image and convergence in the same space, *Commun. Pure Appl. Anal.*, 1 (2002), 237–252.

[15] N. Okazawa, T. Suzuki and T. Yokota, Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials, *Appl. Anal.*, 91 (2012), 1605–1629.

[16] N. Okazawa, T. Suzuki and T. Yokota, Energy methods for abstract nonlinear Schrödinger equations, *Evol. Equ. Control Theory*, 1 (2012), 337–354.

[17] N. Okazawa, T. Yokota and K. Yoshii, Remarks on linear Schrödinger evolution equations with Coulomb potential with moving center, *SUT J. Math.*, 46 (2010), 155–176.

[18] N. Okazawa and K. Yoshii, Linear Schrödinger evolution equations with moving Coulomb singularities, *J. Differential Equations*, 254 (2013), 2964–2999.

[19] V. Pierfelice, Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces, *Math. Z.*, 260 (2008), 377–392.

[20] T. Suzuki, Energy methods for Hartree type equation with inverse-square potentials, *Evol. Equ. Control Theory*, 2 (2013), 531–542.

[21] T. Suzuki, Blowup of nonlinear Schrödinger equations with inverse-square potentials, *Differ. Equ. Appl.*, 6 (2014), 309–333.

[22] T. Suzuki, Solvability of nonlinear Schrödinger equations with some critical singular potential via generalized Hardy-Rellich inequalities, *Funct. Ekvac.*, 59 (2016), 1–34.

[23] T. Suzuki, Scattering theory for Hartree equations with inverse-square potentials, *Appl. Anal.*, 96 (2017), 2032–2043.

[24] T. Suzuki, Virial identities for nonlinear Schrödinger equations with an inverse-square potential of critical coefficient, *Differ. Equ. Appl.*, 9 (2017), 327–352.

[25] K. Yajima, Existence of solutions for Schrödinger evolution equations, *Comm. Math. Phys.*, 110 (1987), 415–426.

[26] J. Zhang and J. Zheng, Scattering theory for nonlinear Schrödinger equations with inverse-square potential, *J. Funct. Anal.*, 267 (2014), 2907–2932.

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