ON PAIRS OF ONE PRIME, FOUR PRIME CUBES AND POWERS OF 2

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Abstract. In this paper, we consider the simultaneous representation of pairs of sufficiently large integers. We prove that every pair of large positive odd integers can be represented in the form of a pair of one prime, four cubes of primes and 231 powers of 2.

1. Introduction

The Goldbach conjecture is one of the most famous problems and there are many variations derived from the conjecture. In 1951, Linnik [9] proved under the assumption of the Generalized Riemann Hypothesis (GRH) that every large even integer \( N \) can be written as the sum of two primes and finite number of powers of 2, and later in 1953 [10] he proved this conjecture unconditionally: that is

\[
N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}, \quad (1.1)
\]

The explicit value of the number \( k \) was first obtained by Liu, Liu and Wang [13], in which \( k = 54000 \) is acceptable. Afterwards, several researchers improved the value of \( k \) and in 2002 Heath-Brown and Puchta [2] showed that \( k = 13 \) and, under the GRH, \( k = 7 \).

In 1999, Liu, Liu and Zhan [12] proved every large even integer \( N \) can be written as a sum of four squares of primes and \( k_1 \) powers of 2,

\[
N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_1}}, \quad (1.2)
\]

In 2001, Liu and Liu [11] proved that every large even integer \( N \) can be written as a sum of eight cubes of primes and \( k_2 \) powers of 2,

\[
N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_{k_2}}, \quad (1.3)
\]

Other problems with hybrid powers of primes of (1.1-3) have been studied by Liu, Liu and Zhan [12], Li [8], Liu and Lü [14] and Liu and Lü [16]. The detail is omitted.

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In 2013, Kong [5] studied a simultaneous version of the Goldbach-Linnik problem. Kong [5] proved that the simultaneous equations
\[
\begin{align*}
N_1 &= p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \\
N_2 &= p_3 + p_4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}
\end{align*}
\tag{1.4}
\]
are solvable for every pair of sufficiently large positive even integers \(N_1, N_2\) satisfying \(N_2 \gg N_1 \geq N_2\) for \(k = 63\) in general and for \(k = 31\) under the GRH. Then the result was improved by Kong [6] in 2017, which showed that the simultaneous equations (1.4) can be solvable for \(k = 34\) unconditionally and \(k = 18\) under the GRH. In 2013, Liu [15] first considered the result on simultaneous representation of pairs of positive odd integers \(N_2 \gg N_1 \geq N_2\), in the form
\[
\begin{align*}
N_1 &= p_1 + p_2^2 + p_3^3 + p_4^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \\
N_2 &= p_5 + p_6^2 + p_7^3 + p_8^4 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}
\end{align*}
\tag{1.5}
\]
He proved that the equations (1.5) are solvable for \(k = 332\), and then the consequence was advanced by Hu and Yang [3] for \(k_2 = 128\). In 2017, Hu and Yang [4] proved every pair of large integers with \(N_2 \gg N_1 \geq N_2\) can be represented in the form
\[
\begin{align*}
N_1 &= p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \\
N_2 &= p_6 + p_7^2 + p_8^3 + p_9^4 + p_{10}^5 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}
\end{align*}
\tag{1.6}
\]
for \(k_2 = 455\). Cai and Hu [1] improved \(k_2\) to 187 in 2020.

In this paper, we consider a simultaneous representation of the problem studied by Liu and Lü in [16]. We will show that each pair of large odd integers with \(N_2 \gg N_1 \geq N_2\) can be written in the form
\[
\begin{align*}
N_1 &= p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k} \\
N_2 &= p_6 + p_7^2 + p_8^3 + p_9^4 + p_{10}^5 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}
\end{align*}
\tag{1.7}
\]
where \(p_i\) is prime and \(k\) is a positive integer. My result is stated as follows.

**Theorem 1.1.** For \(k = 231\), the simultaneous equations (1.7) are solvable for every pair of sufficiently large positive odd integers \(N_2 \gg N_1 \geq N_2\).

### 2. Outline of the method

In this section, we give an outline for the proof of Theorem 1.1. In order to apply the circle method, we set
\[
P_i = N_i^{1/9-\varepsilon}, \quad Q_i = N_i^{8/9+\varepsilon}
\]
for \(i = 1, 2\). By Dirichlet’s lemma in [19], each \(\alpha \in [1/Q, 1 + 1/Q]\) can be written in the form
\[
\alpha = a/q + \theta, \quad |\theta| \leq 1/qQ
\]
for some integers $a, q$ with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. According to the lemma, for any integers $a_i, q_i$ satisfying

$$1 \leq a_i \leq q_i \leq P_i, \quad (a_i, q_i) = 1,$$

we define the major arcs $\mathcal{M}_i$ and minor arcs $m_i$ as usual, namely

$$\mathcal{M}_i = \bigcup_{q_i \leq P_i} \bigcup_{1 \leq a_i \leq q_i} \mathcal{M}_i(a_i, q_i), \quad m_i = \left[ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}_i,$$

where $i = 1, 2$ and

$$\mathcal{M}_i(a_i, q_i) = \left\{ \alpha_i : \left| \alpha_i - \frac{a_i}{q_i} \right| \leq \frac{1}{q_i Q_i} \right\}.$$

It follows from $2P_i \leq Q_i$ that every pair of the arcs $\mathcal{M}_i$ is mutually disjoint respectively. We further define

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = \{ (\alpha_1, \alpha_2) : \alpha_1 \in \mathcal{M}_1, \alpha_2 \in \mathcal{M}_2 \}, \quad m = \left[ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right]^2 \setminus \mathcal{M}.$$

As the value in [16], let $\delta = 10^{-4}$, and

$$U_i = \left( \frac{N_i}{16(1 + \delta)} \right)^{1/3}, \quad V_i = U_i^{5/6}, \quad L = \log_2 \left( \frac{N_i}{\log N_i} \right) \quad (2.1)$$

for $i = 1, 2$. Let $\omega$ be a small constant, we set

$$f_i(\alpha) = \sum_{\omega N_i < p \leq N_i} (\log p) e(p \alpha), \quad S_i(\alpha) = \sum_{p \sim U_i} (\log p) e(p^3 \alpha),$$

$$T_i(\alpha) = \sum_{p \sim V_i} (\log p) e(p^3 \alpha), \quad G(\alpha) = \sum_{v \leq L} e(2^v \alpha), \quad \sigma_\lambda(\alpha) = \{ \alpha \in [0, 1] : |G(\alpha)| \geq \lambda L \} \quad (2.2)$$

$$\sigma_\lambda(\alpha_1, \alpha_2) = \{ (\alpha_1, \alpha_2) \in [0, 1]^2 : |G(\alpha_1 + \alpha_2)| \geq \lambda L \},$$

where $i = 1, 2$, $e(x) := \exp(2\pi i x)$.

Let

$$R(N_1, N_2) = \sum \log p_1 \cdots \log p_{10}$$

be the weighted number of solutions of (1.7) in $(p_1, \cdots, p_{10}, v_1, \cdots, v_k)$ with

$$\omega N_1 < p_1 \leq N_1, \quad p_2, p_3 \sim U_1, \quad p_4, p_5 \sim V_1,$$

$$\omega N_2 < p_6 \leq N_2, \quad p_7, p_8 \sim U_2, \quad p_9, p_{10} \sim V_2, \quad v_j \leq L.$$
for $j = 1, 2, \ldots, k$. Then $R(N_1, N_2)$ can be written as
\[
R(N_1, N_2) = \int_0^1 \int_0^1 \left( \prod_{i=1}^2 f_i(\alpha_i)S^2(\alpha_i)T^2(\alpha_i)e(-\alpha_iN_i) \right) G^k(\alpha_1 + \alpha_2)\,d\alpha_1\,d\alpha_2
\]
\[
= \left\{ \int\int + \int\int + \int\int \right\} \cdots d\alpha_1\,d\alpha_2
\]
\[
= R_1(N_1, N_2) + R_2(N_1, N_2) + R_3(N_1, N_2).
\]

We will prove Theorem 1.1 by estimating the term $R_1(N_1, N_2)$, $R_2(N_1, N_2)$ and $R_3(N_1, N_2)$. For $\chi \mod q$, define
\[
C_1(\chi; a) = \sum_{h=1}^q \overline{\chi}(h)e\left(\frac{ah}{q}\right), \quad C_1(q, a) = C_1(\chi^0, a),
\]
\[
C_3(\chi; a) = \sum_{h=1}^q \overline{\chi}(h)e\left(\frac{ah^3}{q}\right), \quad C_3(q, a) = C_3(\chi^0, a),
\]
where $C_1(q, a)$ is the Ramanujan sum and $C_1(q, a) = \mu(q)$, if $(a, q) = 1$. If $\chi_1, \ldots, \chi_5$ are characters $\mod q$, then we write
\[
B(n, q; \chi_1, \ldots, \chi_5) = \sum_{a=1}^q C_1(\chi_1, a)C_3(\chi_2, a)\cdots C_3(\chi_5, a)e\left(-\frac{an}{q}\right)
\]
and
\[
B(n, q) = B(n, q; \chi_0, \ldots, \chi_0), \quad A(n, q) = \frac{B(n, q)}{\varphi^3(q)}, \quad \mathcal{S}(n) = \sum_{q=1}^\infty A(n, q). \tag{2.3}
\]

We define
\[
\Xi(N, k) = \{(1 - \eta)N \leq n \leq N : n = N - 2^{e_1} - \cdots - 2^{e_k}\}. \tag{2.4}
\]
with $k \geq 2$ and $N \equiv 1(\mod 2)$.

**Lemma 2.1.** For $n \in \Xi(N, k)$, we have
\[
\int_{\mathfrak{M}} f_i(\alpha)S_i^2(\alpha)T_i^2(\alpha)e(-n\alpha)\,d\alpha = \frac{1}{3^i} \mathcal{S}(n)\mathfrak{J}_i(n) + O(N_i^{11/9})
\]
for $i = 1, 2$, where $\mathcal{S}(n)$ is defined in (2.3) and
\[
\mathfrak{J}_i(n) = \sum_{\substack{m_1 + \cdots + m_5 = n \\text{with} \ u_3^3 \leq m_2, m_3 \leq u_3^3, v_3^3 \leq m_4, m_5 \leq v_3^3}} (m_2 \cdots m_5)^{-2/3}
\]
with
\[
\mathcal{S}(n) \geq 0.8842495063, \quad \mathfrak{J}_1(n) \geq 2.7335671N_i^{11/9}.
\]

**Proof.** The proof of Lemma 2.1 can be found in [10].
Lemma 2.2. Let $\Xi(N, k)$ be defined in (2.4) with $2 \leq k \leq L$ and $N \equiv 1(\text{mod } 2)$. Then we have
\[
\sum_{\substack{n_1 \in \Xi(N_1, k) \
 n_2 \in \Xi(N_2, k) \
 n_1 \equiv n_2 \equiv 1(\text{mod } 2)}} 1 \geq (1 - \varepsilon)L^k.
\]

Proof. The proof of Lemma 2.2 is straightforward, so we omit the detail. \qed

Now we give several lemmas to estimate the value in $R_2(N_1, N_2)$ and $R_3(N_1, N_2)$.

Lemma 2.3. (Vinogradov). Let
\[
S_1(N, \alpha) = \sum_{p \sim N} (\log p)e(p\alpha)
\]
where $\alpha = a/q + \lambda$ subject to $1 \leq a \leq q \leq N$, $(a, q) = 1$ and $|\lambda| \leq 1/q^2$. Then
\[
S_1(N, \alpha) \ll (Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2})L^c.
\]

Lemma 2.4. (Kumchev). Let
\[
S_k(N, \alpha) = \sum_{p \sim N} (\log p)e(p^k\alpha)
\]
where $\alpha = a/q + \lambda$ subject to $1 \leq a \leq q \leq N$, $(a, q) = 1$ and $|\lambda| \leq 1/qQ$, with
\[
Q = \begin{cases} 
  x^{3/2}, & \text{if } k = 2, \\
  x^{12/7}, & \text{if } k = 3.
\end{cases}
\]
Then
\[
S_k(N, \alpha) \ll N^{1-\varepsilon^*} + \frac{q^c NL^c}{\sqrt{q(1 + |\lambda|N^k)}}.
\]
\[
\varepsilon^* = \begin{cases} 
  1/8, & \text{if } k = 2, \\
  1/14, & \text{if } k = 3.
\end{cases}
\]

The two results on exponential sums over primes can be found in [19] and [7]. On the minor arcs, we need estimates for the measure of the set $\sigma_\lambda(\alpha)$ defined in (2.2).

Lemma 2.5. We have
\[
\text{meas}(\sigma_\lambda(\alpha)) \ll N_i^{-E(\lambda)}, \quad E(0.961917) > \frac{113}{126} + 10^{-10}.
\]

Proof. The definition of $E(\lambda)$ was constructed by Heath-Brown and Puchta in [2] and we take $\lambda = 0.961917$. \qed

A crucial step is the next two lemmas which give the estimates of two integrations used in estimating $R_3(N_1, N_2)$. They are obtained by Ren [17] and Kong [6].
Lemma 2.6. Let $n$ be an even integer, and $\rho(n)$ the number of representations of $n$ in the form

$$n = p_1^3 + \cdots + p_4^3 - p_5^3 - \cdots - p_8^3, \quad 0 \leq |n| \leq N_i,$$

and subject to

$$p_1, p_2, p_5, p_6 \sim U_i, \quad p_3, p_4, p_7, p_8 \sim V_i.$$

Then we have

$$\rho(n) \leq bU_iV_4^4(\log N_i)^{-8},$$

with $b = 268096$.

Lemma 2.7. Let

$$J = \sum_{1 \leq m_1, \ldots, m_4 \leq L} \prod_{i=1}^{2} r_i(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}),$$

where

$$r_i(n) = \#\{\omega N < p_i \leq N_i : n = p_1 - p_2\}.$$

with $\omega$ is a small positive constant. Then we have

$$J \leq 305.8869 \frac{N_1N_2L^4}{(\log N_1 \log N_2)^2}.$$

3. Proof of Theorem 1.1

We begin with estimating $R_1(N_1, N_2)$.

Lemma 3.1.

$$R_1(N_1, N_2) \geq 0.00089051(1 - \varepsilon)N_1^{11}N_2^{11}L^k + O(N_1^{11}N_2^{11}L^{k-1}). \quad (3.1)$$

Proof. Subject to the definition of $\Xi(N, k)$, every

$$n_i = N_i - 2^{v_1} - \cdots - 2^{v_k}$$

with $v_i \leq L$ will be concluded in $\Xi(N_i, k)$. We apply lemma 2.1 and 2.2 to get that

$$R_1(N_1, N_2) = \frac{1}{3^8} \sum_{n_1 \in \Xi(N_1, k) \atop n_2 \in \Xi(N_2, k)} \left(\mathcal{S}(n_1)J_1(n_1) + O(N_1^{11}L^{-1})\right) \left(\mathcal{S}(n_2)J_1(n_2) + O(N_2^{11}L^{-1})\right)$$

$$\geq 0.00089051 \sum_{n_1 \in \Xi(N_1, k) \atop n_2 \in \Xi(N_2, k)} N_1^{11}N_2^{11} + O(N_1^{11}N_2^{11}L^{k-1})$$

$$\geq 0.00089051(1 - \varepsilon)N_1^{11}N_2^{11}L^k + O(N_1^{11}N_2^{11}L^{k-1}).$$

□
For \( R_2(N_1, N_2) \), we will use the untrivial estimates for \( f_i(\alpha) \) and \( S_i(\alpha) \) and the measure of \( \sigma(\lambda) \). Applying lemma 2.3 and 2.4 we have

\[
\max_{\alpha \in \mathbb{m}} |f_i(\alpha)| \ll N_1^{1-1/18+\varepsilon},
\]

\[
\max_{\alpha \in \mathbb{m}} |S_i(\alpha)| \ll N_1^{1/3-1/42+\varepsilon}. \tag{3.2}
\]

Lemma 3.2.

\[ R_2(N_1, N_2) \ll N_1^{11/11} N_2^{11/11} L^{-1}. \]

Proof. By the definition of \( \mathbb{m} \), we have

\[ \mathbb{m} \subset \{ (\alpha_1, \alpha_2) : \alpha_1 \in \mathbb{m}_1, \alpha_2 \in [0.1] \} \cup \{ (\alpha_1, \alpha_2) : \alpha_1 \in [0, 1], \alpha_2 \in \mathbb{m}_2 \} \]

Then we have

\[
R_2(N_1, N_2) \leq L^k \left( \int \int + \int \int \right) \prod_{i=1}^{2} f_i(\alpha_i) S_i^2(\alpha_i) T_i^2(\alpha_i) e(-\alpha_i N_i) d\alpha_1 d\alpha_2
\]

\[
= L^k \left( \int \int + \int \int \right), \tag{3.3}
\]

where we use the trivial bound of \( G(\alpha_1 + \alpha_2) \). Due to symmetry of \( \alpha_1 \) and \( \alpha_2 \) in the integration, we only estimate one integration on the right of (3.3). We use lemma 2.5, lemma 2.6, 2.7, the trivial bound of \( T_i(\alpha) \) and Cauchy-Schwarz inequality to get

\[
\int \int \ll N_1^{1-1/18+2/3-2/42+5/9+\varepsilon} \int_0^1 |f_2(\alpha_2) S_2^2(\alpha_2) T_2^2(\alpha_2)| \left( \int \frac{d\beta}{\# \in [\alpha_2, 1+\alpha_2] \quad G(\beta) \geq \lambda L} \right) d\alpha_2
\]

\[
\ll N_1^{11/9-13/126+\varepsilon} N_1^{-E(\lambda)} \left( \int_0^1 |f_2(\alpha_2)|^2 d\alpha_2 \right)^{1/2} \left( \int_0^1 |S_2(\alpha_2)|^4 |T_2(\alpha_2)|^4 d\alpha_2 \right)^{1/2}
\]

\[
\ll N_1^{11/11} N_2^{11/11} L^{-1},
\]

where set \( \beta = \alpha_1 + \alpha_2 \) to give the integral transformation and we used the periodicity of \( G(\alpha) \) and the condition \( N_2 \gg N_1 \geq N_2 \). Similarly,

\[
\int \int \ll N_1^{11/11} N_2^{11/11} L^{-1}.
\]

This yields

\[ R_2(N_1, N_2) \ll N_1^{11/11} N_2^{11/11} L^{-1}. \]
Finally, we treat with the $R_3(N_1, N_2)$. By using lemma 2.6 and (2.1) we can get the following lemma.

**Lemma 3.3.** Let $S_i(\alpha)$ and $T_i(\alpha)$ be as in (2.2). Then
\[
\int_0^1 |S_i(\alpha)T_i(\alpha)|^4 \, d\alpha \leq 0.359127 N_1^{13}.
\]

**Lemma 3.4.**
\[
R_3(N_1, N_2) \ll 6.2809957 \lambda^{k-3} N_1^{13} N_2^{13} L^k.
\]

**Proof.** By the definition of $R_3(N_1, N_2)$ and Cauchy-Schwarz inequality, we have
\[
R_3(N_1, N_2) \leq (\lambda L)^{k-2} \int_0^1 \int_0^1 \left| \prod_{i=1}^2 f_i(\alpha_i) S_i^2(\alpha_i) T_i^2(\alpha_i) \right| G^2(\alpha_1 + \alpha_2) \, d\alpha_1 d\alpha_2
\]
\[
\leq (\lambda L)^{k-2} \left( \int_0^1 \int_0^1 |f_1^2 f_2^2 G^4| \, d\alpha_1 d\alpha_2 \right)^{1/2} \left( \int_0^1 \int_0^1 |S_1^4 T_1^4 S_2^4 T_2^4| \, d\alpha_1 d\alpha_2 \right)^{1/2}
\]
\[
= (\lambda L)^{k-2} J^{1/2} I^{1/2}.
\] (3.4)

Observe that
\[
J = \int_0^1 \int_0^1 |f_1^2(\alpha_1) f_2^2(\alpha_2) G^4(\alpha_1 + \alpha_2)| \, d\alpha_1 d\alpha_2
\]
\[
= \sum_{1 \leq m_1, \ldots, m_4 \leq L} \prod_{i=1}^2 t_i(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}),
\]

where
\[
t_i(n) = \sum_{\omega_{N_i} < p_1, p_2 < N_i \atop p_1 - p_2 = n} \log p_1 \log p_2.
\]

Applying lemma 2.7 we get
\[
J \leq (\log N_1 \log N_2)^2 \sum_{1 \leq m_1, \ldots, m_4 \leq L} \prod_{i=1}^2 r_i(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4})
\]
\[
\leq 305.8869 N_1 N_2 L^4.
\] (3.5)

For $I$ we use lemma 3.3 to get
\[
I \leq 0.359127^2 N_1^{13} N_2^{13}.
\] (3.6)

Taking (3.5) and (3.6) in (3.4), we can obtain the conclusion.  
\hfill \square
Combining lemma 3.4 with lemma 3.2 and lemma 3.1, we get
\[ R(N_1, N_2) \geq (0.00089051 - 6.2809957\lambda^{k-2})(1 - \varepsilon)N_1^{12} N_2^{12} L^k. \]
Recall that \( \lambda = 0.961917 \). When \( k \geq 231 \) and \( \varepsilon = 10^{-10} \), we have
\[ R(N_1, N_2) > 0. \]
for sufficiently large integers \( N_1 \) and \( N_2 \) satisfying the given condition.

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