BERNOULLI SUMS AND RÉNYI ENTROPY INEQUALITIES

MOKSHAY MADIMAN, JAMES MELBOURNE, AND CYRIL ROBERTO

ABSTRACT. We investigate the Rényi entropy of sums of independent integer-valued random variables through Fourier theoretic means, and give sharp comparisons between the variance and the Rényi entropy for sums of independent Bernoulli random variables. As applications, we prove that a discrete “min-entropy power” is superadditive with respect to convolution modulo a universal constant, and give new bounds on an entropic generalization of the Littlewood-Offord problem that are sharp in the “Poisson regime”.

1. Introduction

For a countable set $A$, $|A|$ will denote its cardinality. The notation $\mathbb{P}$ will be reserved for a probability measure and the probability of an event $A$ will be denoted $\mathbb{P}(A)$. For a discrete random variable $X$, with values in a countable set $\mathcal{X}$, we will denote its density function with respect to the counting measure as $f_X$ so that $f_X(x) = \mathbb{P}(X = x)$, $x \in \mathcal{X}$ (when $\mathcal{X} = \mathbb{Z}$, we may use the notation $p_n := f_X(n)$ for simplicity). We will denote for $f$ a function on a countable set $\mathcal{X}$, and $\alpha \in (0, \infty)$

$$\|f\|_\alpha := \left( \sum_{x \in \mathcal{X}} |f|^\alpha(x) \right)^{\frac{1}{\alpha}}.$$ 

By continuous limits we define $\|f\|_\infty := \sup_{x \in \mathcal{X}} |f(x)|$, and $\|f\|_0 = |\{x \in \mathcal{X} : f(x) \neq 0\}|$. We will be primarily interested in the case that $\mathcal{X} = \mathbb{Z}$, the integers. The subset of the integers $\{a, a+1, \ldots, b-1, b\}$ will be denoted by $[a, b]$. When $a = 0$ we will abbreviate $[0, b]$ by $[b]$.

**Definition 1.1** (Rényi Entropy [59]). For $X$ a random variable taking values $x \in \mathcal{X}$, such that $f_X(x) = \mathbb{P}(X = x)$, define for $\alpha \in (0, 1) \cup (1, \infty)$, the $\alpha$-Rényi entropy of $X$,

$$H_\alpha(X) := (1 - \alpha)^{-1} \log \sum_{x \in \mathcal{X}} f_X^\alpha(x).$$

For $\alpha \in \{0, 1, \infty\}$ the Rényi entropy is defined through continuous limits;

- $H_0(X) := \log |\{x \in \mathcal{X} : f_X(x) > 0\}|$
- $H_1(X) := -\sum_{x \in \mathcal{X}} f_X(x) \log f_X(x)$
- $H_\infty(X) := -\log \|f_X\|_\infty$.

Note that $H_1(X)$ agrees with the usual Shannon entropy. As such we will employ the conventional notation $H(X) := H_1(X)$. Note that for $\alpha \in (0, 1) \cup (1, \infty)$ and $\alpha' = \alpha/(\alpha - 1)$ we have the expression $H_\alpha(X) = -\alpha' \log \|f\|_\alpha$. We will also use the notation $H_\alpha(f_X)$ in place of $H_\alpha(X)$ when it is more convenient to express the entropy as a function of the densities rather than variables. We take log as the natural logarithm.

2020 Mathematics Subject Classification. 94A17 (Primary), 60E15 (Secondary).

Key words and phrases. entropy, variance, log-concave random variables.

The last author was supported by the Labex MME-DII funded by ANR, reference ANR-11-LBX-0023-01 and ANR-15-CE40-0020-03 - LSD - Large Stochastic Dynamics, and the grant of the Simone and Cino Del Duca Foundation, France.
The Rényi entropy has a well known analog in the continuous setting, when $X$ is a random variable taking values in $\mathbb{R}^d$ and its distribution has a density function with respect to the usual $d$-dimensional Lebesgue measure. The (differential or continuous) Rényi entropy is defined as

$$h_\alpha(X) := (1 - \alpha)^{-1} \log \int_{\mathbb{R}^d} f_X^\alpha(x) dx$$

for $\alpha \in (0, 1) \cup (1, \infty)$. It is extended through continuous limits to $\alpha \in \{0, 1, \infty\}$.

Superadditivity properties of the Rényi entropy connect anti-concentration results in Probability [61, 9, 41], the seminal entropy power inequality due to Shannon and Stam [64, 66], and the Brunn-Minkowski inequality of convex geometry, see [23] for further background. Such connections can be traced back to [18] where the analogy between the Brunn-Minkowski inequality and the Entropy Power Inequality was first described. In [20], proofs of the Brunn-Minkowski inequality [15] and entropy power inequality [37] based on the sharp Young inequality (see [4]) were synthesized to prove Rényi entropy inequalities connecting the two results; an alternate unification using rearrangements was given in [69]. We direct the reader to [40] for further background. There has been significant recent interest and progress in understanding the behavior of the differential Rényi entropies on independent summation.

In analogy with the Shannon entropy power $N(X) := N_1(X) := e^{2h_1(X)/d}$, define for $\alpha \in [0, \infty]$, $N_\alpha(X) = e^{2h_\alpha(X)/d}$.

**Theorem 1.2** ([8, 9, 58, 47]). For $\alpha \in [1, \infty]$, there exists $c(\alpha) \geq 1/e$ such that, for independent $\mathbb{R}^d$-valued random variables $X_i$,

\[(1) \quad N_\alpha(X_1 + \cdots + X_n) \geq c(\alpha) \sum_{i=1}^n N_\alpha(X_i).\]

Further, for $\alpha \in [0, 1)$, there exists $c(\alpha) > 0$, such that if $X_i$ are further assumed to be log-concave\(^2\), then (1) holds.

Note that $c(\alpha)$ can be given independent of $n$. When $\alpha = 1$ the Entropy Power Inequality is the fact that one may take $c(\alpha) = 1$, while the Brunn-Minkowski inequality\(^3\) can be written as

\[(2) \quad N_0^{1/2}(X_1 + \cdots + X_n) \geq \sum_{i=1}^n N_0^{1/2}(X_i).\]

For all other $\alpha$, necessarily $c(\alpha) < 1$. In fact without concavity assumptions on the $X_i$, (1) fails (see [36]) for any $\alpha \in (0, 1)$. Variants of the Rényi entropy power inequalities were studied in [11, 35] and connections with optimal transport theory can be found in [60, 19].

In the discrete setting, it is natural to wonder if a parallel interplay exists, especially in light of the fruitful analogy between additive combinatorics and convex geometry, already

---

1Explicitly, $h_0(X) := \log |\{x \in \mathbb{R}^d : f_X(x) > 0\}|_d$, where $|\cdot|_d$ denotes the Lebesgue volume, $h_1(X) = h(X) := -\int_{\mathbb{R}^d} f_X(x) \log f_X(x) dx$ corresponding to the differential Shannon entropy, and $h_\infty(X) := -\log \|f_X\|_\infty$. Where $\|f_X\|_\infty$ denotes the essential suprema of $f_X$ with respect to the Lebesgue measure in this case.

2We recall that an $\mathbb{R}^d$-valued random variable is log-concave when it has a density $f$ such that $t \in [0, 1]$ and $x, y \in \mathbb{R}^d$ implies $f((1-t)x + ty) \geq f^t(x)f^t(y)$.

3The Brunn-Minkowski inequality states that $|A + B|^d \geq |A|^d + |B|^d$ for non-empty Borel measurable $A, B \subseteq \mathbb{R}^d$, with addition of sets given by Minkowski addition. For independent Borel random variables $X$ and $Y$, the support of $X + Y$, $\text{supp}(X + Y)$, is exactly the Minkowski sum of the supports, $\text{supp}(X) + \text{supp}(Y)$. Thus, applying this observation and Brunn-Minkowski, $N_\alpha^d(X + Y) = |\text{supp}(X + Y)|^d \geq \|\text{supp}(X)\|_d^d + \|\text{supp}(Y)\|_d^d = N_\alpha^d(X) + N_\alpha^d(Y)$. One obtains (2) by induction. Observe that (2) is stronger than the inequality $N_0(X_1 + \cdots + X_n) \geq \sum_{i=1}^n N_0(X_i)$.
well known, see [68]. There has been considerable interest in developing discrete versions of the entropy power inequality, see [27, 24, 33, 44, 70]. General superadditivity properties of the Rényi entropy on independent summation have proved elusive in the discrete setting, and the mentioned results only succeed in the $\alpha = 1$ case, for special classes of variables. Even in the $\alpha = 2$ case, sometimes referred to as the collision entropy in the literature, see [57], little seems to be known.

**Definition 1.3.** For $X$ a discrete random variable on $\mathbb{Z}$, and $\alpha \in [0, \infty]$ set

$$\Delta_\alpha(X) := e^{2H_\alpha(X)} - 1.$$  

Note that just as the Brunn-Minkowski inequality was written in (2) as the superadditivity of the functional $N_0^{1/2}$ (which is stronger than that of $N_0$), the Cauchy-Davenport theorem (on $\mathbb{Z}$) can be written as the superadditivity$^4$ of the functional $\Delta_{CD}(X) = (\Delta_0(X) + 1)^{1/2} - 1$ (which is stronger than that of $\Delta_0$).

**Definition 1.4 (Bernoulli Sum).** The class $B_n$ of Bernoulli $n$-sums is defined to be the set of distributions of all random variables $Y_n = X_1 + \cdots + X_n$, where $X_i$ are independent Bernoulli random variables (i.e., $P(X_i = 1) = p_i = 1 - P(X_i = 0)$ for $p_i \in [0, 1]$).

The class $\mathcal{B}$ of finite Bernoulli sums is given by

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} B_n,$$

and the class $\bar{\mathcal{B}}$ of Bernoulli sums is the closure of $\mathcal{B}$ in the weak-* topology (convergence in distribution).

As is common, we identify random variables with their distributions for ease of discussion: thus $Y$ is a Bernoulli sum (we abuse notation to write $Y \in \mathcal{B}$ instead of saying that the law of $Y$ lies in $\mathcal{B}$) if there exists a sequence of finite Bernoulli sums $Y_n$ converging in distribution to $Y$. We will show that if $S$ is a Bernoulli sum, and $\alpha \in [2, \infty]$, then

$$2\alpha' \text{Var}(S) \leq \Delta_\alpha(S).$$

This inequality is sharp for Bernoulli random variables with parameter $p$ tending to 0, and for Poisson random variables with parameter $\lambda$ tending to 0 as well. In fact for Bernoulli sums, the functional $\Delta_\alpha(X)$ is, up to absolute constants, equal to the variance when $\alpha \geq 2$.

This should be compared to the continuous setting where it is known that the entropy power is proportional to variance for Gaussian random variables. The connection between variance and entropy power has since been extended to more general log-concave random variables and Rényi entropies, and this connection has proved quite useful (see, e.g., [5, 6, 7] for a connection to Bourgain’s hyperplane conjecture [13], or [43] for connections to capacities of communication channels).

Further we will prove a “min-entropy power inequality”: for $\alpha = \infty$, we will prove that, without qualification, independent random variables $X_i$ satisfy the following Rényi entropy inequality:

$$\Delta_\infty(X_1 + \cdots + X_n) \geq c \sum_{i=1}^n \Delta_\infty(X_i),$$

for a universal $c > 0$ independent of $n$, the number of summands. As will be shown, one can take $c = \frac{1}{20}$ and we will observe below that necessarily $c \leq \frac{1}{2}$.

$^4$In fact, $N_0^{1/2}$ and $\Delta_{CD}$ turn out to satisfy the stronger property of fractional superadditivity on $\mathbb{R}$ and $\mathbb{Z}$ respectively, as recently shown by [3].
As an application, we develop new bounds for a generalized version of the Littlewood-Offord problem. Furthermore, we give sharp bounds for a Rényi entropic generalization of the Littlewood-Offord problem with $\alpha \geq 2$.

The mathematical underpinning of the min-EPI is an identification of extreme points in the space of probability measures with a fixed upper bound on their density functions that was proven in [41], and a rearrangement inequality from [44]. A main technical contribution is an $L^p$-norm bound on the characteristic function of a Bernoulli random variable. Recall that $\hat{f}_X(t) = \mathbb{E}(e^{itX})$, $t \in \mathbb{R}$, denotes the Fourier transform of the (discrete) random variable $X$. For $q \geq 1$ we set $\|\hat{f}_X\|_q^q := \frac{1}{2\pi} \int_{-\pi}^{\pi} \|e^{itX}\|^q dt$. We prove that when $X$ is a Bernoulli, with variance $\sigma^2$, $\|\hat{f}_X\|_q^q \leq (6\sigma^2q)^{-1/2} \int_0^{6\sigma^2q} e^{-t^2/2} dt$, the constant 6 being optimal.

Let us outline the contents of the paper. In Section 2 we derive $L^p$ bounds for the characteristic function of a Bernoulli random variable in terms of its variance using a distributional argument. Then, we give a general theorem for the extension of such inequalities to independent sums. In Section 3 we demonstrate how the characteristic function bounds of Section 2 can be used to deliver sharp comparisons between the Rényi entropy and the variance for variables with distributions in the closure of the Bernoulli sums. It is also demonstrated that such bounds cannot be achieved for the case that $\alpha = 1$ by a counter example. In Section 4 we develop the functional analytic tools to reduce the problem of a min-EPI for general random variables to a min-EPI of variables consisting only of Bernoulli and uniform distributions. In addition, reversals and sharpenings of the min-EPI in the case that the $X_j$ are Bernoulli sums. In Section 5, the Littlewood-Offord problem is introduced, and its reduction to Bernoulli sums is given. The bounds for the min-Entropy of a Bernoulli sum in terms of its variance are applied, and then it is shown that these results can be extended to deliver Rényi bounds on an entropic Littlewood-Offord problem. Some proofs are suppressed to the appendix.

2. Bernoulli Sums

For fixed $m$, $B_m$ sums corresponds to the normalized Polya frequency sequences of length $m$, whose probabilistic behavior was studied by Pitman [56]. The class $\bigcup_{n=1}^{m} B_n$ is a subset of the ultra log-concave distributions of order $m$, studied by Pemantle [55] (see also [38]) and written ULC($m$). The ULC($m$) variables can be understood as all variables with distribution log-concave with respect to a binomial distribution. See [14, 1, 16] for important connections with combinatorics.

Bernoulli sums of unbounded length have distributions log-concave with respect to a Poisson random variable. This class often simply called “ultra log-concave”, and arise naturally in the study of intrinsic volumes (see [2, 63]) and in connection with entropic limit theorems. For example, Harremöes [25] showed that the Poisson distribution has maximum entropy among all distributions in $\bar{B}$ with fixed mean; see also [31, 71, 26, 32, 48] for other related work. See [28, 29, 67, 49, 51, 62] for recent results and further background on Bernoulli sums as well as log-concave probability sequences.

We will first derive $L^p$ bounds on the characteristic functions of Bernoulli random variables via a comparison with Gaussians. This argument will be distributional. We use $V$, the capitalization of a non-negative function $v$ to denote its distribution function. Explicitly,
Definition 2.1. For measurable \( v : \mathbb{R} \to [0, \infty) \), its distribution function \( V : [0, \infty) \), is defined by
\[
V(t) = \left| \{ x : v(x) > t \} \right|.
\]

Integrals of functions can be easily computed from integrals of their distribution functions using Fubini-Tonelli, and the following formula
\[
\int v = \int_0^\infty |\{ x : v(x) > \lambda \}|d\lambda = \int_0^\infty V(\lambda)d\lambda.
\]

If we denote \([x]_+ := \max\{x, 0\}\) then Fubini-Tonelli admits a slight generalization of (3), for \( t \geq 0 \)
\[
\int [v(x) - t]_+dx = \int_0^\infty |\{ x : v(x) > t+\}|-\lambda|d\lambda = \int_t^\infty V(\lambda)d\lambda.
\]

Lemma 2.2. For \( w \) and \( v \) non-negative functions in \( L^1 \), such that \( \int w \geq \int v \), with distribution functions \( W \) and \( V \) respectively, then, if \( W - V \leq 0 \) on \([0, t_0] \) and \( W - V \geq 0 \) on \([t_0, \infty) \), it holds
\[
\int w^p \geq \int v^p
\]
for \( p \geq 1 \).

Proof. For \( \varphi(x) \) convex and smooth, and \( x \geq 0 \), Taylor expansion gives,
\[
\varphi(x) = \varphi(0) + x\varphi'(0) + \int_0^\infty [x - t]_+\varphi''(t)dt.
\]
For \( \varphi(0) = \varphi'(0) = 0 \), applying the Taylor expansion and then (4),
\[
\int \varphi(w(y))dy = \int_0^\infty \left( \int [w(y) - t]_+dy \right) \varphi''(t)dt = \int_0^\infty \left( \int_0^\infty W(\lambda)d\lambda \right) \varphi''(t)dt.
\]
Thus to prove \( \int \varphi(w) \geq \int \varphi(v) \), it suffices to prove \( \Psi(t) = \int_t^\infty W(\lambda) - V(\lambda)d\lambda \geq 0 \), for \( t \geq 0 \).
Since \( \Psi(0) = \int w - \int v \geq 0 \), by assumption, and \( \Psi'(t) = W(t) - V(t) \geq 0 \) on \([0, t_0] \) so that \( \Psi(t) \geq 0 \) on \([0, t_0] \). For \( t \geq t_0 \) the result is immediate from \( W(\lambda) - V(\lambda) \geq 0 \) for \( \lambda \geq t \geq t_0 \). Taking \( \varphi(x) = x^p \) gives the result.

The connection between “hockey stick” integrands and general convex functions is well known in the theory of majorization [17, 30]. The result above can also be obtained through the lemma of Nazarov-Podkorytov [54]. See [53] for more on the connections between the lemma of Nazarov-Podkorytov and majorization, and see [52] where the lemma is used to derive a quantitative min-entropy power in the continuous setting. This lemma will be used to derive the following main theorem.

Theorem 2.3. For \( X \) a Bernoulli with variance \( \sigma^2 \), then \( q \geq 1 \) implies
\[
\frac{1}{2\pi} \int_\pi^{-\pi} |e^{itX}|^q \leq \frac{1}{\sqrt{6\sigma^2q}} \int_0^{\sqrt{6\sigma^2q}} e^{-t^2/2}dt.
\]

The constant 6 in inequality in Theorem 2.3, is the best possible\(^6\), and allows the derivation of several sharp inequalities in the sequel. However, at a very small loss, a completely elementary argument, for which we thank an anonymous reviewer, yields the following.

\(^6\)Note it can be shown that the function \( \Phi(x) = \frac{(t^2 - 2)^{1/2}}{x} \) is decreasing in \( x \), so that larger constants reflect a stronger inequality.
Proposition 2.4. For \( X \) a Bernoulli with variance \( \sigma^2 \), then \( q \geq 1 \) implies
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathbb{E}e^{itX}|^q dt \leq \frac{1}{\sqrt{4\sigma^2 q}} \int_0^{\sqrt{4\sigma^2 q}} e^{-t^2/2} dt.
\]

Proof. Observe that Bernoulli \( X \) with variance \( \sigma^2 \), the norm squared of its characteristic function can be expressed as a convex combination of the cosine function and the constant function one, namely
\[
|\mathbb{E}e^{itX}| = \sqrt{(1 - \lambda) + \lambda \cos(t)},
\]
where \( \lambda = 2\sigma^2 \in [0, 1/2] \). Using \( 1 - x \leq e^{-x} \) together with \( \sin s \geq \frac{2}{\pi} s \) for \( 0 \leq s \leq \pi/2 \), we have
\[
1 - \lambda + \lambda \cos t \leq e^{-\lambda(1 - \cos t)} = e^{-2\lambda \sin^2(t/2)} \leq e^{-2\lambda t^2/\pi^2}
\]
Hence with \( \lambda = 2\sigma \)
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathbb{E}e^{itX}|^q dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 - \lambda + \lambda \cos t]^{q/2} dt
\]
\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\lambda qt^2/\pi^2} dt
\]
\[
= \frac{1}{\sqrt{2\lambda q}} \int_0^{\sqrt{2\lambda q}} e^{-t^2/2} dt = \frac{1}{\sqrt{4\sigma^2 q}} \int_0^{\sqrt{4\sigma^2 q}} e^{-t^2/2} dt.
\]
\( \square \)

To pursue the sharp inequality, the expression (5) and Lemma 2.2 motivate the following definitions for \( \lambda \in [0, 1/2] \)
\[
v_{\lambda}(t) := \sqrt{(1 - \lambda) + \lambda \cos t} \quad w_{\lambda}(t) := \exp\left(-\frac{3\lambda t^2}{2\pi^2}\right).
\]
We first claim that \( \int_0^\pi v_{\lambda}^2(t) dt < \int_0^\pi w_{\lambda}^2(t) dt \) holds when \( s = 1 \).

Lemma 2.5. When \( s = 1 \), and \( \lambda \in (0, 1/2] \), \( \int_0^\pi v_{\lambda}^2(t) dt > \int_0^\pi w_{\lambda}^2(t) dt \), or
\[
\int_0^\pi \sqrt{(1 - \lambda) + \lambda \cos(t)} dt < \int_0^\pi \exp\left(-\frac{3\lambda t^2}{2\pi^2}\right) dt.
\]

Proof. Using the inequality \( \sqrt{1 - x} < 1 - \frac{x}{2} \) for \( x \in (0, 1] \),
\[
\int_0^\pi \sqrt{(1 - \lambda) + \lambda \cos(t)} dt < \int_0^\pi 1 - \lambda \frac{1 - \cos(t)}{2} dt = \pi(1 - \lambda/2).
\]
Meanwhile the inequality and \( e^x \geq 1 + x \), gives
\[
\int_0^\pi \exp\left(-\frac{3\lambda t^2}{2\pi^2}\right) dt \geq \int_0^\pi 1 + \frac{-3\lambda t^2}{2\pi^2} dt = \pi(1 - \lambda/2),
\]
so that \( \int_0^\pi w_{\lambda}^2(t) dt > \int_0^\pi v_{\lambda}^2(t) dt \) as claimed. \( \square \)

Lemma 2.6. For \( \lambda > 0 \), the function \( t \mapsto w_{\lambda}(t) - v_{\lambda}(t) \) has no more than one zero on \( (0, \pi] \).

The proof is calculus computations and we leave it to an appendix.

Proof of Theorem 2.3. As functions of \( t \), \( v_{\lambda} \) and \( w_{\lambda} \) are both strictly decreasing on \([0, \pi]\). Thus, their respective distribution functions \( V_{\lambda} \) and \( W_{\lambda} \) are just their strictly decreasing inverse functions on \([0, 1]\). Since \( w_{\lambda} > v_{\lambda} \) for small \( t \), \( W_{\lambda} > V_{\lambda} \) for \( y \) close to 1, and since \( w_{\lambda} \) and \( v_{\lambda} \) cross at no more than one point (by Lemma 2.6), \( V_{\lambda} \) and \( W_{\lambda} \) cross at no more than
one point. We consider two cases. If \( w_\lambda(\pi) - v_\lambda(\pi) \geq 0 \), then \( w_\lambda - v_\lambda \geq 0 \) on \([0, \pi]\) and the theorem holds immediately. If \( w_\lambda(\pi) - v_\lambda(\pi) < 0 \), then \( w_\lambda - v_\lambda \) has exactly one zero, and hence \( W_\lambda - V_\lambda \) has exactly one zero. This in concert with Lemma 2.5 shows that \( W_\lambda \) and \( V_\lambda \) satisfy the conditions of Lemma 2.2. Thus
\[
\int_0^\pi w_\lambda^s(t) dt > \int_0^\pi v_\lambda^s(t) dt
\]
for all \( s \geq 1 \). This is equivalent to our conclusion since,
\[
\frac{1}{2\pi} \int_{-\pi}^\pi |\mathbb{E} \exp(itX)|^q dt = \frac{1}{\pi} \int_0^\pi v_\lambda^q,
\]
and
\[
\frac{1}{\pi} \int_0^\pi w_\lambda^q dt = \frac{1}{\pi} \int_0^\pi e^{-\frac{3q\pi^2}{2\pi^2}} dt = \frac{1}{\sqrt{3q\pi}} \int_0^\sqrt{3q\pi} e^{-u^2/2} du = \frac{1}{\sqrt{6\pi^2q}} \int_0^\sqrt{6\pi^2q} e^{-u^2/2} du
\]
where the second inequality follows from the change of variable \( u = \sqrt{3q\pi}t/\pi \) and the last one from the fact that \( \lambda = 2\sigma^2 \).

Our next aim is to extend the previous comparison to a finite or infinite sum of independent Bernoulli random variables. We will use the following lemma.

**Lemma 2.7.** Fix \( \Phi: (0, \infty) \to [0, \infty) \). Suppose that \( X_i \) are independent random variables such that
\[
\|\hat{f}_{X_i}\|_q \leq \Phi(c_i q)
\]
holds for all \( q \geq 1 \) and some \( c_i > 0 \). Then
\[
\|\hat{f}_{\sum_i X_i}\|_q \leq \Phi(c q)
\]
holds for all \( q \geq 1 \) as well, with \( c = \sum_i c_i \).

**Proof.** By independence and Hölder’s inequality for \( \sum_i \frac{1}{q_i} = 1 \),
\[
\|\hat{f}_{\sum_i X_i}\|_q = \left\| \prod_i \hat{f}_{X_i} \right\|_1 \leq \prod_i \|\hat{f}_{X_i}\|_{q_i} = \prod_i \left( \|\hat{f}_{X_i}\|_{q_i} \right)^{\frac{1}{q_i}}.
\]
Applying the hypothesis, and taking \( q_i = \frac{q}{c_i} \),
\[
\|\hat{f}_{\sum_i X_i}\|_q \leq \prod_i \Phi\left( \frac{1}{c_i} \right) = \Phi(c q).
\]

Thanks to the previous Lemma, the bound of Theorem 2.3 transfers to Bernoulli sums.

**Theorem 2.8.** Let \( Y \) be a Bernoulli sum with variance \( \sigma^2 \). Then
\[
\|\hat{f}_Y\|_q \leq \frac{1}{\sqrt{6\pi^2q}} \int_0^\sqrt{6\pi^2q} e^{-t^2/2} dt.
\]

**Proof.** Since \( Y \) is a Bernoulli sum, there exist a sequence of \( Y_n \) converging weakly to \( Y \). Given a \( Y_n \) of the sequence, there exists \( X_i \) independent Bernoulli with variance \( \sigma_i^2 \), such that \( \sum_{i=1}^{m(n)} X_i \). Taking \( c_i = \sigma_i^2 \), \( \Phi(x) = \frac{1}{x} \int_0^x e^{-t^2/2} dt \), the hypothesis of Lemma 2.7 is satisfied for \( X_i \) thanks to Theorem 2.3, and the conclusion of the Lemma is exactly (6).

Since the bound holds for each \( Y_n \), it is enough to observe \( \|\hat{f}_{Y_n}\|_q \to \|\hat{f}_Y\|_q \) and \( \sigma_{Y_n}^2 \to \sigma_Y^2 < \infty \). Note, that \( Y \) is necessarily log-concave since is the limit of log-concave variables \( Y_n \) (see Definition 4.5 for a definition). As a consequence there exists \( C, c > 0 \) such that
\( f_{Y_n}(k) \leq Ce^{-c|k|} \) and \( f_Y(k) \leq Ce^{-c|k|} \) holds for all \( n \), and all \( k \in \mathbb{Z} \). Thus all moments exist and there is no difficulty passing limits, and by Lévy’s continuity theorem \( \| \hat{f}_{Y_n} \|_q^q \to \| \hat{f}_Y \|_q^q \) to complete the result. 

Let us remark on the nature of the function \( z \mapsto \frac{1}{z} \int_0^z e^{-t^2/2} dt \), as it will be useful to have simple upper bounds for our applications.

**Lemma 2.9.** For \( z \in (0, \infty) \),

\[
\frac{1}{z} \int_0^z e^{-t^2/2} dt \leq \min \left\{ \frac{1}{\sqrt{1 + (z^2/3)}}, \sqrt{\frac{\pi}{2z^2}} \right\}.
\]

The proof is computational and included in the appendix [42].

**Remark 2.10.** Note that \( \sqrt{\pi/2} z^{-2} \leq \left( 1 + \frac{z^2}{3} \right)^{-1/2} \) exactly when \( z \geq \sqrt{\frac{3\pi}{6-\pi}} \approx 1.8158 \).

### 3. Rényi Entropy Inequalities

In this section we prove Rényi entropy inequalities for Bernoulli sums and their limits. We will use a less orthodox formulation of the well-known Hausdorff-Young’s inequality to translate \( L^{\alpha'} \) bounds on the characteristic functions of Bernoulli sums into \( \alpha \)-Rényi entropy bounds.

**Theorem 3.1** (Hausdorff-Young). For \( p \in [2, \infty] \), and a random variable \( X \) on \( \mathbb{Z} \) with probability mass function \( f \) then \( \| f \|_p \leq \| \hat{f} \|_q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Remark 3.2.** We observe that, in contrast with the continuous setting, the inequality \( \| f \|_p \leq \| \hat{f} \|_q \) is sharp for random variables on \( \mathbb{Z} \). To see this it is enough to consider a Dirac mass at zero \( f(0) = 1 \) and \( f(n) = 0 \) for all \( n \neq 0 \) for which \( \hat{f} \equiv 1 \) so that \( \| f \|_p = \| \hat{f} \|_p = 1 \) for all \( p \).

**Proof.** The inequality follows by the Riesz-Thorin interpolation Theorem since \( \| f \|_2 = \| \hat{f} \|_2 \) and \( \| f \|_{\infty} \leq \| \hat{f} \|_1 \). 

**Theorem 3.3.** When \( Y \) is a Bernoulli sum with variance \( \sigma^2 \) and \( \alpha \in [2, \infty] \) then

\[
H_{\alpha}(Y) \geq \log \left[ \sqrt{6\sigma^2 \alpha'} \int_0^{\sqrt{6\sigma^2 \alpha'}} e^{-t^2/2} dt \right].
\]

In particular,

\[
H_{\alpha}(Y) \geq \frac{1}{2} \max \left\{ \log \left( 1 + 2\alpha' \sigma^2 \right), \log \left( \frac{12\alpha' \sigma^2}{\pi} \right) \right\}.
\]

**Remark 3.4.** Notice that \( H_{\alpha}(Y) \geq \frac{1}{2} \log \left( 1 + 2\alpha' \sigma^2 \right) \) reads

\[
\Delta_{\alpha}(Y) \geq 2\alpha' \text{Var}(Y).
\]

Note that if \( Y_p \) is a Bernoulli random variable with parameter \( p \), Taylor expansion gives

\[
\Delta_{\alpha}(Y_p) = 2\alpha' p + o(p).
\]

Thus the constant \( 2\alpha' \) is optimal in (7).
Proof. Recall that $H_\alpha(Y) = -\alpha' \log \| f_Y \|_\alpha = -\log \| f_Y \|^{\alpha'}_{\alpha'}$. Therefore, the Hausdorff-Young Inequality $\| f_Y \|^\alpha_{\alpha'} \leq \| \hat{f}_Y \|^\alpha_{\alpha'}$ together with Theorem 2.8 guarantee that

$$H_\alpha(Y) \geq -\log \| \hat{f}_Y \|^{\alpha'}_{\alpha'} \geq \log \left[ \frac{\sqrt{6\sigma^2\alpha'}}{\int_0^{\sqrt{6\sigma^2\alpha'}} e^{-t^2/2} dt} \right].$$

The last bound follows from Lemma 2.9 applied with $z = \sqrt{6\sigma^2\alpha'}$. \hfill \Box

Our last ingredient in the proof of Theorem 3.7 is the following lemma.

Lemma 3.5. For $\alpha \geq 2$, if $X$ is a Bernoulli random variable, then

$$(8) \quad \Delta_\alpha(X) \leq 12\text{Var}(X),$$

with equality when the Bernoulli parameter $\theta = \frac{1}{2}$.

Remark 3.6. Note that if $\theta = 1/2$, $\Delta_\alpha(X) = 12\text{Var}(X)$ independent of $\alpha$ and cannot be improved for any Rényi parameter $\alpha \geq 2$. Considering the Shannon entropy for small $\theta$ shows an analogous result fails for all $\alpha \leq 1$, as $\lim_{\theta \to 0} \frac{\Delta_\alpha(X)}{\text{Var}(X)} = \infty$. It can be shown for $\alpha \in (1, 2)$, there exists $C(\alpha)$ such that $\Delta_\alpha(X) \leq C(\alpha)\text{Var}(X)$. However by investigation about $\theta$ close to $0$, $C(\alpha)$ is necessarily larger than 12 for $\alpha < \frac{6}{\pi}$.\hfill \Box

Proof of Lemma 3.5. By the monotonicity of Rényi entropy $H_\alpha(X) \leq H_2(X)$, so it suffices to prove $\Delta_2(X) \leq 12\text{Var}(X)$. This is equivalent that for $t \in [0, 1]$,

$$(t^2 + (1-t)^2)^{-2} - 1 \leq 12t(1-t).$$

This is equivalent to proving $P(t) \geq 0$, where $P$ is the polynomial,

$$P(t) = (12t(1-t) + 1)(t^2 + (1-t)^2)^2 - 1 \geq 0$$

on $[0, 1]$. However, $P$ can be factored to

$$4(1-t)(2t-1)^2(3t^2 - 3t + 2),$$

from which the non-negativity of $P$ on the interval is obvious. \hfill \Box

The main theorem is the following.

Theorem 3.7. For $\alpha \geq 2$, $X_i$ a Bernoulli sum, then

$$(9) \quad \Delta_\alpha(X_1 + \cdots + X_n) \geq \frac{\alpha'}{6} \sum_{i=1}^n \Delta_\alpha(X_i),$$

independent of the number of summands.

Remark 3.8. Note that $c(\alpha) := \frac{\alpha}{6(\alpha-1)} = \frac{\alpha'}{6} \geq \frac{1}{6}$. Further, inequality (9) fails for the Shannon entropy, as can be seen by considering $X_i$ to be iid with parameter $\theta$. Indeed, the sum $X_1 + \cdots + X_n$ has a binomial distribution, whose entropy has the well known asymptotic formula,

$$H(X_1 + \cdots + X_n) = \frac{1}{2} \log_2(2\pi e\theta(1-\theta)) + O(1/n).$$

Therefore,

$$\frac{\Delta_1(\sum_i X_i)}{\sum_i \Delta_1(X_i)} = \frac{2^{\log_2(2\pi e\theta(1-\theta))} - 1}{n \Delta_1(X_1)} = \frac{2\pi e\theta(1-\theta)2^{O(1/n)} - \frac{1}{n}}{\Delta_1(X_1)} \to 0,$$

with $\theta \to 0$. This precludes a summand independent entropy power inequality in the sense of Theorem 3.7 for the Shannon entropy.
Proof of Theorem 3.7. By Theorem 3.3, it holds
\[ \Delta_\alpha \left( \sum_i X_i \right) = e^{2H_\alpha \left( \sum_i X_i \right)} - 1 \geq 2\alpha' \text{Var} \left( \sum_i X_i \right). \]
Using that additivity of the variance of independent variables and Lemma 3.5, we conclude that
\[ 2\alpha' \text{Var} \left( \sum_i X_i \right) = 2\alpha' \sum_i \text{Var}(X_i) \geq 2\alpha' \sum_i \Delta_\alpha(X_i) = \frac{\alpha}{6} \sum_i \Delta_\alpha(X_i). \]
\[ \square \]

4. Min-Entropy Power

Given independent integer-valued random variables \( X_1, \ldots, X_n \). We investigate minimizers of the quantity \( H_\infty(X_1 + \cdots + X_n) \) on the set \( H_\infty(X_j) \geq \log C \) where \( C > 1 \). We note that \( H_\infty(X) \geq \log C \) corresponds to \( \| f \|_\infty \leq \frac{1}{C} \).

4.1. Extreme points. Let us denote the set of probability density functions supported on a finite set \( M \), with density \( f \) bounded by \( \frac{1}{C} \) by
\[ \mathcal{P}_C(M) = \left\{ f : M \to [0,1] \text{ such that } 0 \leq f \leq \frac{1}{C} \text{ and } \sum_{i \in M} f(i) = 1 \right\}. \] (10)
Note that \( \mathcal{P}_C(M) \) is a convex compact subset\(^7\) of \( \mathbb{R}^{|M|} \), and hence is the closure of the convex hull of its extreme points. Let us recall the necessary definitions. The extreme points \( \mathcal{E}(K) \) associated to a convex set \( K \) are defined as
\[ \mathcal{E}(K) = \left\{ k \in K : k = \frac{k_1 + k_2}{2} \text{ for } k_1, k_2 \in K \text{ implies } k_1 = k_2 \right\}. \]
The convex hull of a set \( K \) is
\[ co(K) = \left\{ x : \exists \lambda_i > 0 \text{ and } k_i \in K \text{ such that } \sum_{i=1}^n \lambda_i = 1, \sum_{i=1}^n \lambda_i k_i = x \right\}. \]
For \( x \in \mathbb{R} \), we write \( \lfloor x \rfloor = \max\{ n \in \mathbb{Z} : n \leq x \} \) for the entire part of \( x \).

Theorem 4.1. For \( \mathcal{P}_C(M) \) defined as in (10) with \( C \leq |M| \),
\[ \mathcal{E}(\mathcal{P}_C(M)) = \left\{ f : f = \frac{1_A}{C} + \left( 1 - \frac{|C|}{C} \right) 1_{\{x\}, |A| = |C|, x \notin A} \right\}. \]

Note that when \( C \) is chosen to be a natural number, \( \mathcal{E}(\mathcal{P}_C(M)) \) is the uniform distributions on sets of size \( C \) contained in \( M \), else the extremal distributions are “nearly uniform” representing an appropriately scaled convex combination of a uniform distribution on a set of size \( |C| \) and a disjoint point mass. When \( 1 < C \leq 2 \), the extreme points of \( \mathcal{P}_C(M) \) are probability mass functions supported on exactly two points. A more general proof of this result is given in [41]. As we will not have use for the generality, we provide a simpler proof of this result and others of this subsection in the appendix [42] to allow this article to be more self-contained.

\(^7\)We assume \( C \leq M \) else \( \mathcal{P}_C(M) = \emptyset \).
Theorem 4.2. Let $m$ be a natural number and recall that $[m] = \{0, 1, \ldots, m\}$. For $\alpha \in [0, \infty]$, a natural number $n$, constants $C_1, \ldots, C_n \leq m + 1$, independent random variables $X_i$ with probability mass functions $f_{X_i} \in \mathcal{P}_{C_i}([m])$, it holds

$$H_\alpha(X_1 + \cdots + X_n) \geq \min_{Z \in \mathcal{E}} H_\alpha(Z_1 + \cdots + Z_n),$$

where $\mathcal{E}$ is the collection of all $Z = (Z_1, \ldots, Z_n)$ such that $Z_i$ are independent variables with density $f_{Z_i} \in \mathcal{E}(\mathcal{P}_{C_i}([m])).$

Remark 4.3. We stress that the minimum in the right hand side of (11) is indeed a minimum and therefore is achieved.

We restate the case that $\alpha = \infty$ below.

Corollary 4.4. For $X_1, \ldots, X_n$ independent random variables taking values in a finite set $M$ such that $H_\infty(X_i) \geq \log C_i$, there exists $U_1, \ldots, U_n$ independent such that $f_{U_i} \in \mathcal{E}(\mathcal{P}_{C_i}(M))$ and

$$H_\infty(X_1 + \cdots + X_n) \geq H_\infty(U_1 + \cdots + U_n).$$

4.2. Rearrangement. In this section we define the notions of log-concavity and of rearrangement of functions on the integers $\mathbb{Z}$, to be used later on.

Definition 4.5. A function $f : \mathbb{Z} \to [0, \infty)$ is log-concave when

$$f^2(n) \geq f(n+1)f(n-1).$$

and $f(i)f(j) > 0$ for $i < j$ implies $f(k) > 0$ for $k \in [i,j]$.

Definition 4.6. For a function $f : \mathbb{Z} \to [0, \infty)$ with finite support,

$$f = \sum_{i=0}^{n} a_i \mathbb{1}_{\{x_i\}}$$

with $x_1 < x_2 < \cdots < x_n$ denote

$$f^# = \sum_{i=0}^{n} a_i \mathbb{1}_{\{i\}}.$$  

When $X_i$ are independent random variables with densities $f_i$, we denote by $X_i^#$ a collection of random variables such that $X_i^#$ has density $f_i^#$.

4.3. Integers. In this section, we will make use of a result due to Madiman-Wang-Woo [44] (see also [45]) that shows that the Rényi entropy power is somehow decreasing under rearrangement. More precisely, these authors prove that $f_1 \ast \cdots \ast f_n$ is majorized by $f_1^# \ast \cdots \ast f_n^#$. We refer to [46] for background on majorization.

The next theorem follows from [44], by applying the Schur concavity of Rényi entropy.

Theorem 4.7 (Theorem 1.4 [44]). For $\alpha \in [0, \infty]$, and $f_i$ are such that $f_i^#$ are log-concave then

$$H_\alpha(f_1 \ast \cdots \ast f_n) \geq H_\alpha(f_1^# \ast \cdots \ast f_n^#).$$

The significance of the theorem for our pursuits here, is that it will reduce our investigations of a min-entropy power inequality to Bernoulli and uniform distributions.

Corollary 4.8. For $X_i$ independent variables with $\|f_{X_i}\|_\infty \leq 1/C_i$ and with $C_i \in (1, 2] \cup \cup_{i=3}^\infty \{i\}$ then

$$H_\infty(X_1 + \cdots + X_n) \geq H_\infty(Z_1 + \cdots + Z_n)$$

where the variables $Z_i$ are independent and $f_{Z_i} \in \mathcal{E}(\mathcal{P}_{C_i}([\lceil C_i \rceil]))$. Moreover, $Z_i$ is Bernoulli when $C_i \in (1, 2]$ and Uniform on $\{0,1,\ldots,C_i-1\}$ when $C_i \in \cup_{i=3}^\infty \{i\}$. 

The proof is given in the appendix [42].

4.4. Min Entropy Inequality. The aim of this section is to prove the following Min Entropy power inequality which constitutes one of our main theorems.

**Theorem 4.9** (Min-EPI). For independent integer-valued random variables $X_i$, the following holds

$$\Delta_{\infty}(X_1 + \cdots + X_n) \geq \frac{1}{22} \sum_{i=1}^{n} \Delta_{\infty}(X_i).$$

In order to prove Theorem 4.9, we will use a comparison between the min-entropy and the variance (in both directions). Such a comparison is essentially known in the literature. The next result holds for all random variables and is sharp for uniform distributions.

**Theorem 4.10** (Bobkov-Chistyakov [10, 12]). For a discrete random variable $X$,

$$\Delta_{\infty}(X) \leq 12\text{Var}(X).$$

To state the other direction, we need to introduce two definitions. First we say that an integer-valued random variable $X$ is log-concave if its probability mass function $f_X$ is log-concave, in the sense of Definition 4.5. In other words, $f_X(n) \geq f_X(n-1)f_X(n+1)$ holds for all $n$, and $f_X(k)f_X(n) > 0$ for $k < m < n$ implies $f_X(m) > 0$. Next we define the notion of symmetry.

**Definition 4.11** (Symmetric Random variable). A real-valued random variable $X$ is symmetric when there exists $a$, such that

$$P(X = a + x) = P(X = a - x).$$

holds for all $x \in \mathbb{R}$.

Note that since we consider only $X$ integer-valued, $a \in \frac{1}{2}\mathbb{Z}$. Also recall that both symmetry and log-concavity are preserved under independent summation.

**Theorem 4.12** (Bobkov-Marsigletti-Melbourne [12]). For an integer-valued, symmetric, log-concave random variable $X$,

$$\Delta_{\infty}(X) \geq 2\text{Var}(X).$$

Strictly speaking, the result in [12] only covered the case that $X$ was symmetric about an integer point. The argument used a majorization result from [50] to reduce to a distribution on $\mathbb{Z}$ of the form $n \mapsto Cp^{[n]}$ (which is resolved through direct computation), and the fact proven in [12], that the variance is Schur-concave on the space of symmetric distributions. For the convenience of the reader we cover the (missing) case that $X$ is symmetric about a point belonging to $\frac{1}{2} + \mathbb{Z}$ in the appendix [42].

We are now in position to prove Theorem 4.9.

**Proof of Theorem 4.9.** Given $X_1, X_2, \ldots, X_n$ independent, we assume without loss of generality that for $i \leq k$, $\|f_{X_i}\|_{\infty} \geq 1/2$ and $i > k$ implies $\|f_{X_i}\|_{\infty} < 1/2$. By Corollary 4.4,

$$\Delta_{\infty}(X_1 + \cdots + X_n) \geq \Delta_{\infty}(B_1 + \cdots + B_k + Z_{k+1} + \cdots + Z_n)$$

where $B_i$ and $Z_j$ are all independent, the $B_i$ are Bernoulli satisfying $M(B_i) = M(X_i)$ and $Z_i$ is uniformly distributed on $\{1, 2, \ldots, n_i\}$ where $n_i$ is uniquely determined by $M(X_j) \in \left(\frac{1}{n_i+1}, \frac{1}{n_i}\right]$. Note that trivially,

$$\Delta_{\infty}(X_1 + \cdots + X_n) \geq \max\{\Delta_{\infty}(B_1 + \cdots + B_k), \Delta_{\infty}(Z_1 + \cdots + Z_n)\}.$$
Using the variance-min entropy comparisons above (i.e. Inequalities (12) and (13)) for Bernoulli random variables,
\[ \Delta_\infty \left( \sum_i B_i \right) \geq 2 \text{Var} \left( \sum_i B_i \right) = 2 \sum_i \text{Var} (B_i) \geq \frac{1}{6} \sum_i \Delta_\infty (B_i) = \frac{1}{6} \sum_i \Delta_\infty (X_i). \]

Similarly, since \( Z_1 + \cdots + Z_n \) is an independent sum of symmetric (about the point \( \frac{n-1}{2} \)) log-concave variables, using the variance-min entropy comparisons, this time for symmetric log-concave variables,
\[ \Delta_\infty (Z_{k+1} + \cdots + Z_n) \geq 2 \sum_j \text{Var} (Z_j) \geq \frac{1}{6} \sum_j \Delta_\infty (Z_j). \]

Since \( n_j \geq 2 \)
\[ \frac{\Delta_\infty (Z_j)}{\Delta_\infty (X_j)} \geq \frac{n_j^2 - 1}{(n_j + 1)^2 - 1} \geq \frac{2^2 - 1}{(2 + 1)^2 - 1} = \frac{3}{8}. \]

Thus it follows that
\[ \Delta_\infty (Z_{k+1} \cdots Z_n) \geq \frac{1}{16} \sum_{j=k+1}^n \Delta_\infty (Z_j). \]

Finally,
\[ \Delta_\infty (X_1 + \cdots + X_n) \geq \max \left\{ \frac{1}{6} \sum_{i=1}^k \Delta_\infty (X_i), \frac{1}{16} \sum_{j=k+1}^n \Delta_\infty (X_j) \right\} \geq \frac{1}{22} \sum_{i=1}^n \Delta_\infty (X_i). \]

where we use the fact that \( \max(\alpha a, \beta b) \geq \frac{\alpha \beta}{\alpha + \beta} (a + b) \), valid for any non-negative \( \alpha, \beta, a, b \). □

4.5. **Min-Entropy inequalities: Tightenings, and Reversals.** For the min-entropy we may also give a reversal of min-EPI for Bernoulli sums.

**Theorem 4.13.** For \( X_i \) independent Bernoulli sums,
\[ \Delta_\infty (\sum_i X_i) \leq 6 \sum_i \Delta_\infty (X_i). \]

**Proof.** By Theorems 4.10 and 3.3 (see Inequality (7)),
\[ \Delta_\infty (X_1 + \cdots + X_n) \leq 12 \ \text{Var} \left( \sum_i X_i \right) = 12 \sum_i \text{Var} (X_i) \leq 6 \sum_i \Delta_\infty (X_i). \]

In the case that \( X_i \) are concentrated about a point, one can actually tighten the min-EPI beyond the \( \Delta_\infty (\sum_i X_i) \geq \frac{1}{6} \sum_i \Delta_\infty (X_i) \) that one would achieve through variance comparisons in the min-EPI reversals for Bernoulli sums, as we show in what follows. We will need the following Lemma whose proof is suppressed to the appendix.

**Lemma 4.14.** For \( X \) a Bernoulli random variable, and \( t \in [-\pi, \pi] \),
\[ |E e^{itX}| \leq e^{-\Delta_\infty (X)t^2/24}. \]

Observe that, when \( X \) is expanding the inequality at \( t = 0 \) shows that the constant 1/24 is optimal in the latter and we will show that this inequality can be used to derive a sharpening of the min-EPI for for Bernoulli sums.
Theorem 4.15. For $X_i$ independent and integer-valued such that $\|f_{X_i}\|_\infty = c_i \geq \frac{1}{2},$

$$\Delta_\infty(X_1 + \cdots + X_n) \geq \frac{\pi^2}{36} \sum_{j=1}^{n} \Delta_\infty(X_i).$$

Proof. By Corollary 4.8 it suffices to prove the result when the $X_i$ are independent Bernoulli. By Lemma 4.14,

$$\|\hat{f}_{X_i}\|_q^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\Delta_\infty(X_i)q/t^2/24} dt = \frac{\int_0^{\pi^2 \Delta_\infty(X_i)q/12} e^{-t^2/2} dt}{\sqrt{\pi^2 \Delta_\infty(X_i)q/12}}.$$ 

Thus applying Lemma 2.7 with $\Phi(q) = \frac{1}{\sqrt{\pi^2q/12}} \int_0^{\pi^2q/12} e^{-t^2/2} dt$, and $c_i = \Delta_\infty(X_i)$, we have

$$\|\hat{f}_{\sum_i X_i}\|_q^2 \leq \Phi \left( q \sum_i \Delta_\infty(X_i) \right).$$

By Hausdorff-Young, this gives

$$\|f_{\sum_i X_i}\|_\infty \leq \|\hat{f}_{\sum_i X_i}\|_1 \leq \frac{\int_0^{\pi^2 \sum_j \Delta_\infty(X_j)/12} e^{-t^2/2} dt}{\sqrt{\pi^2 \sum_j \Delta_\infty(X_j)/12}}.$$

Applying Lemma 2.9 with $z = \sqrt{\pi^2 \sum_j \Delta_\infty(X_j)/12}$ this gives

$$\|f_{\sum_j X_j}\|_\infty \leq \sqrt{\frac{1}{1 + \frac{\pi^2}{36} \sum_j \Delta_\infty(X_j)}}$$

which yields,

$$\Delta_\infty(\sum_j X_j) \geq \frac{\pi^2}{36} \sum_j \Delta_\infty(X_j).$$

□

Let us note that the largest constant $c$ such that $\Delta_\infty(\sum_j X_j) \geq c \sum_j \Delta_\infty(X_i)$ holds for any collection of independent $X_i$ is no larger than $\frac{1}{2}$, as can be seen by taking $X_1$ and $X_2$ to be iid Bernoulli with parameter $p = 1/2$ (and $\pi^2/36 \simeq 0.27 > \frac{1}{4}$). Note that one can alternatively apply Theorem 3.3 and 3.5 to obtain a similar result $\Delta_\infty(X_1 + \cdots + X_n) \geq 2 \sum_{i=1}^{n} \text{Var}(X_i) \geq \frac{1}{6} \Delta_\infty(X_i)$ at the expense of a constant. Also note that applying the Bernoulli tightening to the proof of Theorem 4.9 gives

$$\Delta_\infty(X_1 + \cdots + X_n) \geq \max \left\{ \frac{\pi^2}{36} \sum_{i=1}^{k} \Delta_\infty(X_i), \frac{1}{16} \sum_{j=k+1}^{n} \Delta_\infty(X_i) \right\} \geq \sum_{i=1}^{n} \frac{\Delta_\infty(X_i)}{16 + 36/\pi^2},$$

and an improvement to a constant $c = \frac{1}{16+36/\pi^2} > \frac{1}{20}$ in Theorem 4.9.

5. Littlewood-Offord Problem

In this section we apply the results above to an entropic generalization of the Littlewood-Offord problem.

As usual, we denote by the dot sign the usual scalar product in $\mathbb{R}^n$ so that $S_v = v \cdot B$ with $v = (v_1, \ldots, v_n)$ and $B = (B_1, \ldots, B_n)$. We first present a generalization of [65, Theorem 1.2] (see also [34]).
Lemma 5.1. Let $B = (B_1, \ldots, B_n)$ such that $B_i$ are independent Bernoulli random variables, for $\alpha \in [0, \infty]$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that $v_i \neq 0$ for all $i$, with $S_v := v_1 B_1 + \cdots + v_n B_n$, it holds that

$$H_\alpha(S_v) \geq \max_{a \in \{-1, 1\}^n} H_\alpha(S_a).$$

Proof. Considering $\mathbb{R}$ as a vector space and choose a linear function $T : \mathbb{R} \rightarrow \mathbb{Q}$ such that $T(v_i) \in \mathbb{Z} - \{0\}$. Then

$$H_p(S_v) \geq H_p(T(S_v))$$

since deterministic functions of a random variable decrease Rényi entropy. As $T(S_v) = T(v_1)B_1 + \cdots + T(v_n)B_n$, it suffices to consider integer coefficients. Assuming $v_i \in \mathbb{Z} - \{0\}$, take $a_i = \text{sign}(v_i)$ where $\text{sign}(x) = \mathbb{1}_{(0, \infty)}(x) - \mathbb{1}_{(-\infty, 0)}(x)$, then $(v_1 B_1)^\# + \cdots + (v_n B_n)^\#$ has the same distribution as $S_a + m$ where $m := \# \{i : v_i < 0\}$.

Applying Theorem 4.7,

$$H_\alpha(S_v) \geq H_\alpha((v_1 B_1)^\# + \cdots + (v_n B_n)^\#) = H_\alpha(S_a + m) = H_\alpha(S_a).$$

□

Theorem 5.2. For $S_v = v_1 B_1 + \cdots + v_n B_n$, where $v_i \neq 0$, $B_i$ independent Bernoulli of variance $\sigma_i^2$ and $\alpha \geq 2$,

$$H_\alpha(S_v) \geq \log \left[ \frac{\sqrt{6\sigma^2\alpha'}}{\int_0^{\sqrt{6\sigma^2\alpha'}} e^{-t^2/2} dt} \right] \geq \frac{1}{2} \max \left\{ \log(1 + 2\alpha'\sigma^2), \log \left( \frac{12\alpha'\sigma^2}{\pi} \right) \right\},$$

where $\sigma^2 = \sum_i \sigma_i^2$.

Proof. By Lemma 5.1, it suffices to consider $v$ with $v_i = \pm 1$, and since $v_i B_i$ is the translation of a Bernoulli $X_i$ with the same variance as $B_i$,

$$H_\alpha(v_1 B_1 + \cdots + v_n B_n) = H_\alpha(X_1 + \cdots + X_n) \geq \log \frac{\sqrt{6\sigma^2\alpha'}}{\int_0^{\sqrt{6\sigma^2\alpha'}} e^{-t^2/2} dt}.$$

The inequality follows from Theorem 3.3. Applying Lemma 2.9 completes the proof. □

When $v_i = 1$ for all $i$ and the $B_i$ are iid Bernoulli$(\lambda)$, then

$$H_\alpha(S_v) = \frac{1}{2} \log (1 + 2\alpha'\lambda n + o(\lambda),)$$

so the constant $2\alpha'$ is optimal in the inequality $H_\alpha(S_v) \geq \frac{1}{2} \log(1 + 2\alpha'\sigma^2)$, or equivalently $\Delta_\alpha(S_v) \geq 2\alpha'\sigma^2$, for every $\alpha \geq 2$.

Let us relate Theorem 5.2 to the usual Littlewood-Offord problem, that is determining upper bounds on $Q(S, 0) := \max_x P(S = x)$ where $S := \sum_{i=1}^n v_i B_i$ where $v_i \in \mathbb{R} \setminus \{0\}$ and $B_i$ iid Bernoulli of parameter $p$, classically chosen with $p = 1/2$. We recall the following question of Fox, Kwan, and Sauermann.

Question 5.1 ([22] Question 6.2). For $(v_1, \ldots, v_n) \in (\mathbb{R} \setminus \{0\})^n$ and $B_1, \ldots, B_n$ iid Bernoulli with some parameter $0 < p \leq 1/2$ and $S = v_1 B_1 + \cdots + v_n B_n$. What upper bounds (in terms of $n$ and $p$) can we give on the maximum point probability $Q(S, 0) = \max_{x \in \mathbb{R}} P(S = x)$?

---

8To find such a map, when $n = 1$ choose a Hamel Basis for $\mathbb{R}$ over $\mathbb{Q}$ that extends $(v_1)$, take $\Phi(v_1) = 1$ and all other basis elements to 0. By induction, choose a linear map such that $\Phi(v_i) \neq 0$ for $i < n$, and $\Psi$ such that $\Psi(v_i) \neq 0$. Then choose a map of the form $\lambda \Phi + \Psi$ for $\lambda \in \mathbb{Q}$ such that $\lambda \Phi(v_i) - \Psi(v_i) \neq 0$ for all $i$ and write $T = \lambda \Phi - \Psi$. Since $T(v_i) = \frac{q_i}{m_i}$ for $p_i, q_i \in \mathbb{Z} - \{0\}$, $T = gT$ for $q = \prod_i q_i$ yields such a map.
When \( p = \frac{1}{2} \) bounds this is a reformulation of the classical problem, determining the number of subsums that fall in a given location \([39]\).

**Lemma 5.3.** Let \( B = (B_1, \ldots, B_n) \) such that \( B_i \) are independent Bernoulli random variables, for \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) such that \( v_i \neq 0 \) for all \( i \), it holds

\[
Q(v \cdot B, 0) \leq \max_{a \in \{-1,1\}^n} Q(a \cdot B, 0).
\]

**Proof.** This is Lemma (5.1) in the case \( \alpha = \infty \).

Let us emphasize the following result, which follows from taking \( \alpha = \infty \) in Theorem 5.2.

**Corollary 5.4.** For \( v_i \in \mathbb{R} \setminus \{0\} \), \( S_v = v_1X_1 + \cdots + v_nX_n \) where \( X_i \) are independent Bernoulli random variables with variance \( \sigma_i \), and denoting by \( \sigma^2 = \sum_j \sigma_j^2 \), then

\[
Q(S_v, 0) \leq \frac{1}{\sqrt{6\sigma^2}} \int_0^{\sqrt{6\sigma^2}} e^{-t^2/2} dt.
\]

**Corollary 5.5.** When \( S_v = v \cdot B \) for \( B = (B_1, \ldots, B_n) \) for \( B_i \) iid Bernoulli random variables of parameter \( p \),

\[
Q(S_v, 0) \leq \frac{1}{\sqrt{1 + 2np(1-p)}}.
\]

**Proof.** Applying 2.9 to Corollary 5.4 while observing that \( \sigma^2 = np(1-p) \) gives the result.

Corollary 5.4 and Corollary 5.5 are sharp for any \( n \) as can be seen by taking small variance Bernoulli and \( v_i = 1 \). Moreover, since \( X \mapsto Q(X, 0) \) is lower semi-continuous with respect to the weak topology for \( X \) taking values on \( \mathbb{Z} \), that the inequality \( Q(S, 0) \leq \frac{1}{\sqrt{6\sigma^2}} \int_0^{\sqrt{6\sigma^2}} e^{-t^2/2} dt \) holds when \( S \) is Poisson of parameter \( \lambda \), in which case \( \sigma^2 = \lambda \) and we have the following Taylor expansions for \( \lambda \) near zero

\[
Q(S, 0) = e^{-\lambda} = 1 - \lambda + o(\lambda),
\]

while by Lemma 2.9 (with \( \sigma^2 = \lambda \) \( \frac{1}{\sqrt{6\sigma^2}} \int_0^{\sqrt{6\sigma^2}} e^{-t^2/2} dt \leq 1 - \lambda + o(\lambda) \). Thus in the “Poisson regime”, when \( S_v \) is a sum of many small variance Bernoulli, and Poisson approximation can be invoked, inequality (15) is tight. In the Gaussian regime, when the local limit theorem applies, for example for a sequence of iid Bernoulli random variables \( S = X_1 + \cdots + X_n \) gives

\[
Q(S, 0) \leq \frac{1}{\sqrt{2\pi\sigma^2_S}} + o(1/\sqrt{n}).
\]

Meanwhile by integrating on the whole \([0, \infty),\]

\[
\frac{1}{\sqrt{6\sigma^2}} \int_0^{\sqrt{6\sigma^2}} e^{-t^2/2} dt \leq \frac{1}{\sqrt{6\sigma^2}} \int_0^{\infty} e^{-t^2/2} dt = \sqrt{\frac{\pi}{12\sigma^2}},
\]

so the bounds cannot be improved by more than a constant factor in the Gaussian regime. In particular when the \( X_i \) are iid Bernoulli with parameter 1/2, then by Erdős’s sharp solution (see [21]), to the Littlewood-Offord problem, \( Q(S, 0) \leq 2^{-n} \left( \frac{n}{\lfloor n/2 \rfloor} \right) \approx \sqrt{\frac{2}{\pi n}}, \) while \( \sqrt{\frac{\pi}{12\sigma^2}} = \sqrt{\frac{\pi}{\sqrt{n}}} \) (and we are off by a factor of \( \pi/\sqrt{6} \approx 1.28 \)).
ACKNOWLEDGEMENTS

The authors thank Arnaud Marsiglietti for stimulating discussion and in particular suggesting the connection to the Littlewood-Offord question of [22], as well as an anonymous reviewer whose careful reading and suggestions have improved this article, and to whom Proposition 2.4 is to be credited.

The last author was supported by the Labex MME-DDI funded by ANR, reference ANR-11-LBX-0023-01 and ANR-15-CE40-0020-03 - LSD - Large Stochastic Dynamics, and the grant of the Simone and Cino Del Duca Foundation, France.

SUPPLEMENTAL MATERIAL

In this appendix we will prove Lemma 2.6, Lemma 2.9, Theorem 4.1 and 4.2, Corollary 4.8, Theorem 4.12, and Lemma 4.14.

Proof of Lemma 2.6. Observe that \( w_\lambda(t) - v_\lambda(t) \) has no more than one zero on \((0, \pi)\) if and only if \( H(t) := w_\lambda(t)^2 - v_\lambda(t)^2 \) has no more than one zero on \((0, \pi)\). Taking a derivative, we see that for \( \lambda > 0 \) and small enough \( t \)

\[
H'(t) = \lambda \sin t - \frac{6t \lambda e^{-3\lambda t^2/\pi^2}}{\pi^2} > 0.
\]

Further we see that for \( \lambda > 0 \), \( H'(t) = 0 \) iff \( \lambda = \frac{\pi^2 \log \left( \frac{6t}{\pi^2 \sin t} \right)}{3t^2} \). Now we claim that the function

\[
(0, \pi) \ni t \mapsto \lambda(t) = \frac{\pi^2 \log \left( \frac{6t}{\pi^2 \sin t} \right)}{3t^2}
\]

is strictly increasing and hence one to one from \((0, \pi)\) into \((-\infty, \infty)\) (since \( \lim_{t \to 0} \lambda(t) = -\infty \) and \( \lim_{t \to \pi} \lambda(t) = +\infty \)). Hence for fixed \( \lambda \), there exists exactly one \( t = t_\lambda \) such that \( H'(t) = 0 \). Computing

\[
\lambda'(t) = \frac{\pi^2}{3t^2} \left( -t \cot(t) + 2 \log \left( \frac{\pi^2 \sin(t)}{6t} \right) + 1 \right)
\]

and since \( \lim_{t \to 0} \lambda'(t) = \infty \), it suffices to show that \( \lambda'(t) \) has no zeros on \((0, \pi)\). That is that

\[
f(t) := 1 - t \cot(t) + 2 \log \left( \frac{\pi^2 \sin(t)}{6t} \right)
\]

has no zeros on \((0, \pi)\). Note that \( \lim_{t \to 0} f(t) = 2 \log \left( \frac{\pi^2}{6} \right) > 0 \), so it is enough to show that \( f \) is increasing on \((0, \pi)\). We have, for \( t \in (0, \pi) \)

\[
f'(t) = 1 + \cot(t) + \frac{t}{\sin^2(t)} - \frac{2}{t} \quad \text{and} \quad f''(t) = \frac{2}{t^2} - \frac{2t \cos(t)}{\sin^3(t)}.
\]

Now we claim that \( f''(t) > 0 \) on \((0, \pi)\) which is equivalent to saying that

\[
\sin^3(t) - t^3 \cos(t) > 0, \quad t \in (0, \pi).
\]

For \( t \geq \pi/2 \) this is immediate. For \( t < \pi/2 \) we use the Taylor series bounds \( \sin(t) \geq t - t^3/6 \) and \( \cos(t) \leq 1 - t^2/2 + t^4/24 \),

\[
\sin^3(t) - t^3 \cos(t) \geq (t - t^3/6)^3 - t^3(1 - t^2/2 + t^4/24)
= t^7(3 - t)(t + 3)/216
\]

which is clearly positive on \((0, \pi/2)\). The claim is proved and hence \( f' \) is increasing. Since \( \lim_{t \to 0} f'(t) = 1 > 0 \), we infer that \( f \) is increasing on \((0, \pi)\) as expected.
Thus $H'$ is positive for small $t$ and has at most one zero, and thus it follows that $H$ has at most one $0$ on $(0, \pi)$ since $H(0) = 0$, and $H$ is increasing and then decreasing. \hfill \Box

**Proof of Lemma 2.9.** The second term follows from $\int_0^\pi e^{-t^2/2}dt \leq \int_0^\infty e^{-t^2/2}dt = \sqrt{\pi/2}$, which implies $\int_0^\pi e^{-t^2/2}dt/z \leq \sqrt{\pi/2z^2}$. The first term is more complicated. It is enough to prove that $y \mapsto F(y) := \frac{1}{(\int_0^y e^{-t^2/2}dt)^2} - \frac{1}{y^2}$ is non-decreasing on $(0, \infty)$. Indeed, this would imply for any $y > 0$ that $F(y) \geq \lim_{y \downarrow 0} F(y) = \frac{1}{3}$ which can be rephrased as the expected bound

$$\int_0^y e^{-t^2/2}dt \leq \left(1 + \frac{z^2}{3}\right)^{-1/2}. \tag{16}$$

To prove that $F$ is non-decreasing, we take the derivative and obtain that

$$F'(y) = \frac{2}{y^3 \left(\int_0^y e^{-t^2/2}dt\right)^3} \left(-y^3 e^{-y^2/2} + \left(\int_0^y e^{-t^2/2}dt\right)^3\right), \quad y > 0.$$ 

Set $G(y) := -y^3 e^{-y^2/2} + \left(\int_0^y e^{-t^2/2}dt\right)^3, \quad y > 0$, and observe that

$$G'(y) = e^{-y^2/2} \left(-3y^2 + 3y^4 + 3 \left(y - \frac{y^3}{6}\right)^2\right).$$

Now, since $e^{-t^2/2} \geq 1 - t^2/2$, we have $\int_0^y e^{-t^2/2}dt \geq y - \frac{y^3}{6}$. Therefore, for $y \in (0, \sqrt{6})$ (so that $y - \frac{y^3}{6} \geq 0$), it holds

$$G'(y) \geq e^{-y^2/2} \left(-3y^2 + 3y^4 + 3 \left(y - \frac{y^3}{6}\right)^2\right) = \frac{y^6 e^{-y^2/2}}{12} > 0.$$ 

On the other hand, for $y \geq \sqrt{6}$, we observe that $-3y^2 + y^4 > 0$ so that $G'(y) > 0$ on $[\sqrt{6}, \infty)$ and therefore on $(0, \infty)$. As a consequence $G$ is increasing on $(0, \infty)$, and since $\lim_{y \downarrow 0} G(y) = 0$, $F$ is non-decreasing on $(0, \infty)$ as expected. The limit as $y$ tends to zero is an easy consequence of the Taylor expansion $\int_0^y e^{-t^2/2}dt = y - y^3/6 + o(y^3)$, while (16) is obtained directly from $F(y) \geq 1/3$. \hfill \Box

**Proof of Theorem 4.1.** We will prove the two inclusions $(\subset, \supset)$ of the sets.

Given $f \in \mathcal{P}_C(M)$, if there exists $i \neq j$ such that $f(i), f(j) \in \{0, \frac{1}{C}\}$ then $g_1 = f + \epsilon(\mathbb{1}_{\{i\}} - \mathbb{1}_{\{j\}})$ and $g_2 = f - \epsilon(\mathbb{1}_{\{i\}} - \mathbb{1}_{\{j\}})$ are distinct elements of $\mathcal{P}_C(M)$ (for $\epsilon$ small enough), and since $\frac{g_1 + g_2}{2} = f$, $f \notin \mathcal{E}(\mathcal{P}_C(M))$. This proves that extreme points of $\mathcal{P}_C(M)$ have at most one value in $(0, \frac{1}{C})$ and therefore proves the first inclusion.

Conversely, consider $f$ such that there exists $i_o$ with $f(j) \in \{0, 1/C\}$ for all $j \neq i_o$ and suppose that $f = \frac{g_1 + g_2}{2}$ for some $g_1, g_2 \in \mathcal{P}_C(M)$. For $j \neq i_o, f(j)$ is an extreme point of the interval $[0, \frac{1}{C}]$ and $\frac{g_1(j) + g_2(j)}{2} = f(j)$. Hence $g_1(j) = g_2(j) = f(j)$. By the constraint that $f, g_1, \text{ and } g_2$ are probability mass functions we must have $f(i_o) = g_1(i_o) = g_2(i_o)$ as well and the second inclusion is proved. \hfill \Box

**Proof of Theorem 4.2.** Note that $X_1 + \cdots + X_n$ is supported on $[nm]$, and as a function of densities, the map $f \mapsto H_\alpha(f)$ is continuous and quasi-concave in the sense that densities $f$ and $g$ satisfy $H_\alpha(f + (1/2)g) \geq \min\{H_\alpha(f), H_\alpha(g)\}$. Indeed, continuity is obvious since the probability distributions under consideration have finite support, and quasi-concavity follows
from the expression \( H_\alpha(f) = \alpha' \log \|f\|_\alpha \), and the convexity of \( f \mapsto \|f\|_\alpha \) for \( \alpha > 1 \) and its concavity for \( \alpha < 1 \). Thus the continuity of the map from \( \mathcal{P}_{C_1}([m]) \times \cdots \times \mathcal{P}_{C_n}([m]) \to [0, \infty) \), given by \((f_1, \ldots, f_n) \mapsto H_\alpha(f_1 \cdots f_n)\) is continuous since the convolution can be expressed as a polynomial of the terms of \( f_i(k) \). What is more, the map is coordinate quasi-concave, since convolution is coordinate affine, in the sense that 
\[
\frac{f_1 + f_2 \cdots + f_n}{2} = f_1 f_2 \cdots f_n + \frac{f_2 \cdots f_n}{2}.
\]
Thus the map \( g \mapsto H_\alpha(g \ast f_1 \ast \cdots \ast f_n) \) is continuous and quasi-concave on \( \mathcal{P}_{C_1}([m]) \). Since \( \mathcal{P}_{C_1}([m]) \) is a compact, convex subset of \( \mathbb{R}^{m+1} \), and since points of compact convex subsets can be written as convex combinations of their extreme points by Krein-Milman, there exists a minimizer \( g_1 \in \mathcal{E}(\mathcal{P}_{C_1}([m])) \) such that
\[
H_\alpha(f_1 \ast f_2 \ast \cdots \ast f_n) \geq H_\alpha(g_1 \ast f_2 \ast \cdots \ast f_n).
\]
Iterating the argument gives the proof.

**Proof of Corollary 4.8.** Let us first assume that the \( X_i \) are all supported on a finite set \([m]\) for some \( m \). This implies that \( C_i \leq |[m]| = m + 1 \). By Theorem 4.2 there exists densities \( g_i \in \mathcal{E}(\mathcal{P}_{C_i}([m])) \) such that
\[
H_\infty(f_1 \ast \cdots \ast f_n) \geq H_\infty(g_1 \ast \cdots \ast g_n).
\]
However, by the assumption that \( C_i \in (1, 2] \cup \{i\} \) the \( g_i \) either takes only two values, in which case \( g_i^\# \) is a Bernoulli, or \( g_i \) is a uniform distribution on \( C_i \) values, in which case \( g_i^\# \) is a uniform distribution on \( \{0, 1, \ldots, C_i - 1\} \). In either case \( g_i^\# \) is log-concave and Theorem 4.7, gives
\[
H_\infty(g_1 \ast \cdots \ast g_n) \geq H_\infty(g_1^\# \ast \cdots \ast g_n^\#).
\]
Combining the two inequalities completes the proof when the \( X_i \) have compact support. The general inequality is an approximation argument. Define an auxiliary function
\[
f_i^{(m)}(n) = \begin{cases} 
\min\{f_X(n), \frac{1}{C_i} - \frac{1}{m}\} & \text{if } n \in [-m, m], \\
0 & \text{else.}
\end{cases}
\]
and from \( f_i^{(m)} \) define a density
\[
f_x^{(m)}(n) = f_i^{(m)}(n) + \frac{1 - \sum_{k=-m}^{m} f_i^{(m)}(k)}{2m+1} 1_{[-m, m]}(n).
\]
The probability mass functions \( f_x \) converge pointwise to \( f_X \) with \( m \to \infty \), are compactly supported and \( \|f_x^{(m)}(n)\|_\infty \leq \|f_X\|_\infty \leq 1/C_i \). Further pointwise convergence coincides with weak-convergence on \( \mathbb{Z} \) by the Portmanteau theorem since all subsets of \( \mathbb{Z} \) are closed and open. Note that \( f \mapsto H_\infty(f) \) is upper semi-continuous with respect to weak convergence since for \( f_\beta \to f \) pointwise, \( f(n) = \lim_\beta f_\beta(n) \leq \liminf_\beta \|f_\beta\|_\infty \). Thus it follows that \( \limsup_\beta H_\infty(f_\beta) \to H_\infty(f) \). Taking the max over \( n \) gives \( \|f\|_\infty \leq \liminf_\beta \|f_\beta\|_\infty \) and hence
\[
\limsup_\beta H_\infty(f_\beta) \leq H_\infty(f).
\]
Since the map \((f_1, \ldots, f_n) \mapsto f_1 \ast \cdots \ast f_n \) corresponds to the mapping of a product space of measures \((\mu_1, \ldots, \mu)\) to their product measure, a weak continuous operation, and composed with pushing forward \( \mu_1 \otimes \cdots \otimes \mu_n \) under the continuous map \( T(x) = x_1 + \cdots + x_n \) a weak-continuous mapping, \((f_1, \ldots, f_n) \mapsto f_1 \ast \cdots \ast f_n \) is a composition of weakly continuous functions and hence weakly continuous as well. Thus, \((f_1, \ldots, f_n) \mapsto H_\infty(f_1 \ast \cdots \ast f_n)\) is upper semi-continuous and we have
\[
H_\infty(f_1 \ast \cdots \ast f_n) \geq \limsup_\beta H_\infty(f_{X_i}^{(m)} \ast \cdots \ast f_X^{(m)}).
\]
Since the $f_{X_i^{(m)}}$ are compactly supported $H_\infty(f_{X_1^{(m)}} \ast \cdots \ast f_{X_n^{(m)}}) \geq H_\infty(Z_1 + \cdots + Z_n)$. □

**Proof of Theorem 4.12.** Suppose that $f$ is a log-concave density symmetric about a point $n - \frac{1}{2}$. Note that if $f$ is supported on only two points, the inequality is true immediately, since by symmetry $f$ is a translation of Bernoulli with parameter $1/2$. Thus, assume that $f$ is supported on at least 4 points so that $\|f\|_\infty < 1/2$ and observe that $\|f\|_\infty = f(n-1) = f(n)$. Take $p = 1 - 2\|f\|_\infty > 0$ and define a density $g$ by $g(n+k) = \|f\|_\infty p^k$ for $k \geq 0$ and $g(n+k) = \|f\|_\infty p^{k-1}$ for $k < 0$. Note that $g$ is symmetric log-concave density, satisfying $\|g\|_\infty = \|f\|_\infty$ and $g < f$. Translating $g(k) = g(n + k - 1/2)$, and $f(k) = f(n + k - 1/2)$ we obtain densities symmetric about 0 taking values on $\frac{1}{2} + \mathbb{Z}$. It is straightforward to prove from the majorization that $X \sim g$ and $Y \sim f$ that $EX^2 \geq EY^2$ and since both variables are centered, $\text{Var}(X) \geq \text{Var}(Y)$ while by definition $\Delta_\infty(X) = \Delta_\infty(Y)$. Thus it suffices to prove the inequality for $X$ and $p \in (0, 1)$.

In this case, with integer $k \geq 0$, $\mathbb{P}(X = \pm(\frac{1}{2} + k)) = \frac{1-p^2}{2} p^k$, direct computations gives

$$
\Delta_\infty(X) = \frac{4}{(1-p)^2} - 1,
$$

$$
\text{Var}(X) = (1-p) \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) p^k = \frac{p^2 + 6p + 1}{4(1-p)^2}.
$$

Thus the inequality $\Delta_\infty(X) \geq 2\text{Var}(X)$ is equivalent to proving $5 - 2p - 3p^2 \geq 0$ for $p \in [0, 1]$. □

**Proof of Lemma 4.14.** Since both sides of (4.5) are invariant in the transformation of the parameter $p \mapsto 1 - p$, we may assume $p \in [1/2, 1]$ and compute explicitly with (2.4), it is equivalent to prove

$$(1-p)^2 + p^2 + 2p(1-p) \cos(t) \leq e^{-\left(\frac{1}{p} - 1\right)^2/12}.
$$

Setting $q = 1/p \in [1, 2]$, the desired inequality is equivalent to proving that

$$
F(q) := q^2 e^{-\frac{2q^2}{9} t^2} - 2(q-1) \cos(t) - (q-1)^2 - 1 \geq 0.
$$

Our first aim is to prove that $F$ is concave on the interval $[1, 2]$ for any given $t \in [0, \pi]$. Observe that

$$
F'(q) = \left( 2q - q^3 t^2 \right) e^{-\frac{2q^2}{9} t^2} - 2 \cos(t) - 2(q-1),
$$

and

$$
F''(q) = \left( 2 - 2e^{-\frac{2q^2}{9} t^2} + \frac{q^2 t^2}{6} \right) \left( -5 + \frac{q^2 t^2}{6} \right) e^{-\frac{2q^2}{9} t^2}.
$$

Given $t \in [0, \pi]$, set $r = \frac{q^2 t^2}{9} \in \left[\frac{t^2}{9}; \frac{2t^2}{9}\right]$ and $G(r) := 2 - 2e^{-\frac{2q^2}{9} t^2} + r(-5 + r)$ so that $F$ is concave on $[1, 2]$ reduces to proving that $G$ is negative on $[\frac{t^2}{9}; \frac{2t^2}{9}]$. Observe that

$$
G'(r) = -e^{-\frac{2q^2}{9} t^2} + \frac{2q^2 t^2}{9} + 2r - 5 \quad \text{and} \quad G''(r) = -\frac{1}{2} e^{-\frac{2q^2}{9} t^2} + \frac{2q^2 t^2}{9} + 2.
$$

We infer that $G''$ is decreasing on $[\frac{t^2}{9}; \frac{2t^2}{9}]$ and may change sign depending on the value of the parameter $t$. We need to distinguish between two cases.

1. Assume first that $t \leq \sqrt{4 \log 4}$. Then $G''(2t^2/3) = -\frac{1}{2} e^{\frac{t^2}{9}} + 2 \geq 0$. In that case we conclude that $G'' \geq 0$ on the whole interval and therefore that $G'$ is non-decreasing. Hence $G'(r) \leq G'(2t^2/3) = -e^{\frac{t^2}{9}} + \frac{4}{3} r^2 - 5$. It is easy to see that the mapping $[0, \infty) \ni t \mapsto H(t) := -e^{\frac{t^2}{9}} + \frac{4}{3} r^2 - 5$ is increasing on $[0, \sqrt{4 \log 16/9}]$ so that, for $t \in [0, \sqrt{4 \log 4}]$, $H(t) \leq H(\sqrt{4 \log 4}) = \frac{4}{3} \log 4 - 5$. □
Therefore $G'(r) \leq 0$ and hence $G(r) \leq G(t^2/6) = \frac{t^2}{6}(-5 + \frac{t^2}{9}) \leq 0$ since $t \in [0, \sqrt{4 \log 4}]$. As an intermediate conclusion we proved that $G \leq 0$ in case (1).

(2) Assume now that $t \geq \sqrt{4 \log 4}$. Then $G''$ changes sign. Namely, since $G''(t^2/6) = 3/2 \geq 0$ and $G''(2t^2/3) = -\frac{3}{4} t^2 + 2 \leq 0$, $G''$ is non-negative on $[\frac{t^2}{6}, r_0]$ and non-positive on $[r_0, 2t^2/3]$, with $r_0 := 2(\log 4 + \frac{t^2}{12})$. It follows that, for $t \leq \pi$ and $r \in \left[\frac{t^2}{6}, \frac{2t^2}{3}\right]$, $G'(r) \leq G'(r_0) = 4(\log 4 + \frac{t^2}{9}) - 9 \leq 4 \log 4 + \frac{t^2}{9} - 9 \simeq -0.16 \leq 0$. We conclude that $G$ is non-increasing and therefore that $G(r) \leq G(t^2/6) = \frac{t^2}{6}(-5 + \frac{t^2}{9}) \leq 0$ since $t \leq \pi$. As a conclusion we proved that $G \leq 0$ in case (2) and therefore in any case. This shows that $F$ is concave.

Now $F$ being concave, $F \geq 0$ is a consequence of the fact that $F(1) = 0$ and $F(2) = 4 e^{-\frac{t^2}{4}} - 2 \cos(t) - 2 \geq 0$. To see the latter, one can observe that $2 \cos(t) + 2 = 4 \cos^2(t/2)$ so that $F(2) \geq 0$ is equivalent to saying that $e^{t^2/4} \geq \cos^2(t/2)$ on $[0, \pi]$ which in turn is equivalent to saying that $e^{u^2/2} \geq \cos(u)$ for any $u = \frac{t}{2} \in [0, \frac{\pi}{2}]$. Taking the logarithm, we end up with proving that $I(u) := -\frac{u^2}{2} - \log \cos(u) \geq 0$ on $[0, \frac{\pi}{2}]$. Since $I'(u) = -u + \tan(u) \geq 0$ the desired conclusion immediately follows. This ends the proof of the Lemma. $\square$

References

[1] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials iii: Mason’s ultra-log-concavity conjecture for independent sets of matroids. arXiv preprint arXiv:1811.01600, 2018.

[2] H. Aravinda, A. Marsiglietti, and J. Melbourne. Concentration inequalities for ultra log-concave distributions. Studia Mathematica, 265(1):1–10, January 2022.

[3] F. Barthe and M. Madiman. Volumes of subset Minkowski sums and the Lyusternik region. Preprint, arXiv:2112.06518, 2021.

[4] W. Beckner. Inequalities in Fourier analysis. Ann. of Math. (2), 102(1):159–182, 1975.

[5] S. Bobkov and M. Madiman. Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures. J. Funct. Anal., 63(12):7747–7752, 2017.

[6] S. Bobkov and G. P. Chistyakov. Bounds for the maximum of the density of the sum of independent random variables. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 408(Veroyatnost i Statistika. 18):62–73, 324, 2012.

[7] S. Bobkov and G. P. Chistyakov. Entropy power inequality for the Rényi entropy. IEEE Trans. Inform. Theory, 61(2):708–714, February 2015.

[8] S. Bobkov and G. P. Chistyakov. On concentration functions of random variables. J. Theoret. Probab., 28:976–988, 2015.

[9] S. Bobkov and A. Marsiglietti. Variants of the entropy power inequality. IEEE Trans. Inform. Theory, 63(12):7747–7752, 2017.

[10] Sergey Bobkov, Arnaud Marsiglietti, and James Melbourne. Concentration functions and entropy bounds for discrete log-concave distributions. Combinatorics, Probability and Computing, 31(1):54–72, 2022.

[11] J. Bourgain. On high-dimensional maximal functions associated to convex bodies. Amer. J. Math., 108(6):1467–1476, 1986.

[12] Petter Brändén and June Huh. Lorentzian polynomials. Annals of Mathematics, 192(3):821–891, 2020.

[13] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis, 22(4):366–389, 1976.

[14] Swee Hong Chan and Igor Pak. Log-concave poset inequalities. arXiv preprint arXiv:2110.10740, 2021.

[15] K. M. Chong. Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications. Canad. J. Math., 26:1321–1340, 1974.

[16] M.H.M. Costa and T.M. Cover. On the similarity of the entropy power inequality and the Brunn-Minkowski inequality. IEEE Trans. Inform. Theory, 30(6):837–839, 1984.
[19] T. A. Courtade, M. Fathi, and A. Pananjady. Quantitative stability of the entropy power inequality. *IEEE Trans. Inform. Theory*, 64(8):5691–5703, 2018.

[20] A. Dembo, T.M. Cover, and J.A. Thomas. Information-theoretic inequalities. *IEEE Trans. Inform. Theory*, 37(6):1501–1518, 1991.

[21] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[22] Jacob Fox, Matthew Kwan, and Lisa Sauermann. Combinatorial anti-concentration inequalities, with applications. *Math. Proc. Cambridge Philos. Soc.*, 171(2):227–248, 2021.

[23] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.*), 39(3):355–405 (electronic), 2002.

[24] S. Haghighatshoar, E. Abbe, and E. Telatar. A new entropy power inequality for integer-valued random variables. *IEEE Trans. Inform. Th.*, 60(7):3787–3796, July 2014.

[25] P. Harremoës. Binomial and Poisson distributions as maximum entropy distributions. *IEEE Trans. Inform. Theory*, 47(5):2039–2041, 2001.

[26] P. Harremoës, O. Johnson, and I. Kontoyiannis. Thinning, entropy, and the law of thin numbers. *IEEE Trans. Inform. Theory*, 56(9):4228–4244, 2010.

[27] P. Harremoës and C. Vignat. An entropy power inequality for the binomial family. *J. Inequal. Pure Appl. Math.*, 4(5):Article 93, 6 pp. (electronic), 2003.

[28] E. Hillion and O. Johnson. A proof of the Shepp-Olkin entropy concavity conjecture. *Bernoulli*, 23(4B):3638–3649, 2017.

[29] E. Hillion and O. Johnson. A proof of the Shepp-Olkin entropy monotonicity conjecture. *Electron. J. Probab.*, 24:Paper No. 126, 14, 2019.

[30] H. Joe. Majorization, randomness and dependence for multivariate distributions. The Annals of Probability, pages 1217–1225, 1987.

[31] O. Johnson. An information-theoretic central limit theorem for finitely susceptible FKG systems. *Teor. Veroyatn. Primen.*, 50(2):331–343, 2005.

[32] O. Johnson, I. Kontoyiannis, and M. Madiman. Log-concavity, ultra-log-concavity, and a maximum entropy property of discrete compound Poisson measures. *Discrete Appl. Math.*, 161:1232–1250, 2013. DOI: 10.1016/j.dam.2011.08.025.

[33] O. Johnson and Y. Yu. Monotonicity, thinning, and discrete versions of the entropy power inequality. *IEEE Trans. Inform. Theory*, 56(11):5387–5395, 2010.

[34] T. Juškevičius and V. Kurauskas. On the Littlewood-Offord problem for arbitrary distributions. *Random Structures Algorithms*, 58(2):370–380, 2021.

[35] J. Li. Rényi entropy power inequality and a reverse. *Studia Math.*, 242:303–319, 2018.

[36] J. Li, A. Marsiglietti, and J. Melbourne. Further investigations of Rényi entropy power inequalities and an entropic characterization of s-concave densities. In B. Klartag and E. Milman, editors, *Geometric aspects of functional analysis: Israel Seminar (GAFA) 2017-2019*, volume 2266 of *Lecture Notes in Mathematics*, pages 95–123. Springer, 2020.

[37] E. H. Lieb. Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys.*, 62(1):35–41, 1978.

[38] Thomas M Liggett. Ultra logconcave sequences and negative dependence. *Journal of Combinatorial Theory, Series A*, 79(2):315–325, 1997.

[39] J. E. Littlewood and A. C. Offord. On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):277–286, 1943.

[40] M. Madiman, J. Melbourne, and P. Xu. Forward and reverse entropy power inequalities in convex geometry. *Convexity and Concentration*, pages 427–485, 2017.

[41] M. Madiman, J. Melbourne, and P. Xu. Rogozin’s convolution inequality for locally compact groups. 2017.

[42] M. Madiman, M. Melbourne, and C. Roberto. Supplement to bernoulli sums and Rényi entropy inequalities. *Bernoulli*.

[43] M. Madiman, P. Nayar, and T. Tkocz. Sharp moment-entropy inequalities and capacity bounds for log-concave distributions. *IEEE Trans. Inform. Theory*, 67(1):81–94, January 2021.

[44] M. Madiman, L. Wang, and J. O. Woo. Majorization and Rényi entropy inequalities via Sperner theory. *Discrete Math.*, 342(10):2911–2923, October 2019. Available at *arXiv:1712.00913*.

[45] M. Madiman, L. Wang, and J. O. Woo. Rényi entropy inequalities for sums in prime cyclic groups. *SIAM J. Discrete Math.*, 35(3):1628–1649, 2021.

[46] A. W. Marshall, I. Olkin, and B. C. Arnold. *Inequalities: theory of majorization and its applications*. Springer Series in Statistics. Springer, New York, second edition, 2011.

[47] A. Marsiglietti and J. Melbourne. On the entropy power inequality for the Rényi entropy of order [0, 1]. *IEEE Trans. Inform. Theory*, 65(3):1387–1396, 2019.
[48] A. Marsiglietti and J. Melbourne. Moments, concentration, and entropy of log-concave distributions. Preprint, arXiv:2205.08293, 2022.

[49] Arnaud Marsiglietti and James Melbourne. Geometric and functional inequalities for log-concave probability sequences. *Discrete & Computational Geometry*, pages 1–31, 2023.

[50] J. Melbourne and T. Tkocz. Reversal of Rényi entropy inequalities under log-concavity. *IEEE Transactions on Information Theory*, 67(1):45–51, 2020.

[51] James Melbourne and Cyril Roberto. Quantitative form of Ball’s cube slicing in $\mathbb{R}^n$ and equality cases in the min-entropy power inequality. *Proceedings of the American mathematical society*, 2022.

[52] James Melbourne and Cyril Roberto. Transport-majorization to analytic and geometric inequalities. *Journal of Functional Analysis*, 284(1):109717, 2023.

[53] James Melbourne and Cyril Roberto. Quantitative form of Ball’s cube slicing in $\mathbb{R}^n$ and equality cases in the min-entropy power inequality. *Proceedings of the American mathematical society*, 2022.

[54] F. L. Nazarov and A. N. Podkorytov. Ball, Haagerup, and distribution functions. In *Complex analysis, operators, and related topics*, volume 113 of *Oper. Theory Adv. Appl.*, pages 247–267. Birkhäuser, 2000.

[55] Robin Pemantle. Towards a theory of negative dependence. *Journal of Mathematical Physics*, 41(3):1371–1390, 2000.

[56] J. C. Principe. *Information theoretic learning: Rényi’s entropy and kernel perspectives*. Springer Science & Business Media, 2010.

[57] E. Ram and I. Sason. On Rényi entropy power inequalities. *IEEE Transactions on Information Theory*, 62(12):6800–6815, 2016.

[58] A. Rényi. On measures of entropy and information. In *Proc. 4th Berkeley Symp. Math. Statist. and Prob.*, Vol. I, pages 547–561. Univ. California Press, Berkeley, Calif., 1961.

[59] M. Rudelson and R. Vershynin. Small ball probabilities for linear images of high-dimensional distributions. *Int. Math. Res. Not. IMRN*, (19):9594–9617, 2015.

[60] M. Singhal. Erdos-Littlewood-Offord problem with arbitrary probabilities. arXiv preprint arXiv:1912.02886, 2019.

[61] T. Tao and V. Vu. *Additive combinatorics*, volume 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[62] L. Wang and M. Madiman. Beyond the entropy power inequality, via rearrangements. *IEEE Trans. Inform. Theory*, 60(9):5116–5137, September 2014.

[63] J. O. Woo and M. Madiman. A discrete entropy power inequality for uniform distributions. In *Proc. IEEE Int'l. Symp. Inform. Theory*, pages 1625–1629, Hong Kong, China, June 2015.

[64] K. Yu. On the entropy of compound distributions on nonnegative integers. *IEEE Trans. Inform. Theory*, 55(8):3645–3650, 2009.

University of Delaware, Department of Mathematical Sciences, 501 Ewing Hall, Newark DE 19716, USA.

Email address: madiman@udel.edu

Department of Probability and Statistics, Centro de Investigación en matemáticas (CIMAT)

Email address: james.melbourne@cimat.mx

MODAL’X, UPL, Univ. Paris Nanterre, CNRS, F92000 Nanterre France

Email address: croberto@math.cnrs.fr