BASIC TRIGONOMETRIC POWER SUMS WITH APPLICATIONS

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Abstract. We present the transformation of several sums of positive integer powers of the sine and cosine into non-trigonometric combinatorial forms. The results are applied to the derivation of generating functions and to the number of the closed walks on a path and in a cycle.

1. Introduction

Over the last half-century there has been widespread interest in finite sums involving powers of trigonometric functions. In 1966 Quoniam posed an open problem in which the following result was conjectured

\[(1.1) \sum_{k=1}^{\lfloor n/2 \rfloor} 2^{2m} \cos^{2m} \left( \frac{k\pi}{n+1} \right) = (n+1) \left( \frac{2m-1}{m-1} \right) - 2^{2m-1},\]

where \( m \) is a positive integer and subject to \( m < n + 1 \) \([28]\). In this equation \( \lfloor n/2 \rfloor \) denotes the floor function of \( n/2 \) or the greatest integer less than or equal to \( n/2 \). A solution to the above problem was presented shortly after by Greening et al. in \([18]\). Soon afterwards, there appeared a problem involving powers of the secant proposed by Gardner \([15]\), which was solved partially by Fisher \([11, 22]\) and completely, only recently, in \([13, 23]\). The activity subsided somewhat until such series arose in string theory in the early 1990’s with the work of Dowker \([10]\). Subsequently, a surge occurred with studies of related sums and identities as evidenced by the work of: (1) Berndt and Yeap \([3]\) on reciprocity theorems and (2) Cvijović and Srivastava \([7, 8]\) on Dowker and related sums, while \([5, 6, 31]\) were motivated by the intrinsic fascination these sums possess and derived formulas where the summand was a power of the secant, e.g.,

\[\sum_{k=0}^{n-1} \sec^{2p} \left( \frac{k\pi}{n} \right) = \sum_{k=1}^{2p-1} \left( \frac{2p-1 + kn}{2p-1} \right) \sum_{j-k}^{2p-1} \left( \frac{2p}{j+1} \right).\]

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Here, \( B_j \) denotes the Bernoulli number with index \( j \geq 0 \). As described in Appendix A, Berndt and Yeap use contour integration to derive this result, although more recently it has been studied with the aid of sampling theorems [1]. Unfortunately, (1.2) is, if not incorrect, confusing or misleading because it states that the \( j_i \) cannot equal zero. Yet, some of them are required to be zero in order to evaluate the polynomials on the rhs. In addition, it does not matter if any of the \( j_i \) are zero since \( B_0 \) is equal to unity anyway. A more precise statement of (1.2) is

\[
\frac{1}{k} \sum_{k=1}^{k-1} \cot^{2n} \left( \frac{r\pi}{k} \right) = (-1)^n - (-1)^n 2^n \sum_{j_1, j_2, \ldots, j_{2n}=0}^{n, n-j_1, \ldots, n-j_s+j_{2n}} k^{2j_{2n}-1} \prod_{r=1}^{2n} B_{2j_r} \frac{B_{2(n-j_s)}}{(2j_r)! (2(n-j_s))} ! ,
\]

where \( j_s = \sum_{i=1}^{2n} j_i \). The main difference between the two equations is that there is now an upper limit for each sum over the \( j_i \), which is dependent on the number of sums preceding it. As a consequence, there is no requirement for the \( j_0 \) index to appear as a sum as in (1.2). Instead, it has been replaced by \( n - j_s \) in (1.3). Although the above material is incidental to the material presented in this paper, because of its importance, the derivation of (1.3) is presented in Appendix A together with a description of how it is to be implemented when determining specific powers of the sum, which is another issue overlooked in [3]. In doing so, we give the \( n = 3 \) and \( n = 4 \) forms for the sum, thereby demonstrating to the reader just how intricate the evaluation of trigonometric power sums can be.

It should also be mentioned that although the arguments inside the trigonometric power of the finite sums discussed above are composed of rational numbers multiplied by \( \pi \), the actual sequence of numbers has a profound effect on the final value for the trigonometric power sum. For example, by using a recursive approach, Byrne and Smith [4] have derived the following result for the same finite sum over powers of cotangent:

\[
\sum_{r=1}^{k} \cot^{2n} \left( \frac{r - 1/2 \pi}{2k} \right) = (-1)^k k + 2 \sum_{j=1}^{n} b_{n,j} k^{2j},
\]

where the coefficients are given by

\[
b_{n,j} = \frac{1}{2^{(n-j)-1}} \sum_{\ell=1}^{n-j} (-1)^\ell \binom{2n}{\ell} b_{n-\ell,j} \quad \text{and} \quad \sum_{j=1}^{n} b_{n,j} = 1 + (-1)^{n-1}, \quad \text{for} \quad j < n.
\]

Moreover, Byrne and Smith express the \( b_{n,j} \) in terms of the odd-indexed Euler numbers as opposed to the Bernoulli numbers appearing in (1.2) and (1.3). Hence we see that the sum yields completely different results when the argument inside the cotangent is altered to \( (r - 1/2) \pi/2k \) as opposed to \( r \pi/k \) in (1.2) and (1.3).

Although there has been a greater interest in sums with inverse powers of sine or cosine (including powers of cotangent and tangent), there are still many basic trigonometric power sums that have not been solved. By a basic trigonometric power sum, we mean a finite sum involving positive powers of a cosine or sine whose arguments are rational multiples of \( \pi \). Such series can be as difficult to evaluate as their “inverse power” counterparts even though they tend to yield combinatorial solutions directly rather than involve the Riemann zeta function or related quantities (Bernoulli numbers), as in (1.2) or (1.3), before ultimately reducing to the simple polynomial solutions in Appendix A. As we shall see, just like their inverse power counterparts, the closed form expressions for
basic trigonometric power sums depend greatly, not only on the power of the trigonometric function, but also on their limits and the values of the rationals multiplying $\pi$ in the argument.

In response to the situation concerning basic trigonometric power sums, Merca recently derived formulas for various basic cosine power sums including

\begin{equation}
\sum_{k=1}^{\lfloor n/2 \rfloor} \cos^2 \left( \frac{(k - 1/2)\pi}{n} \right) = \frac{n}{2^{2p+1}} \sum_{k=-[p/n]}^{[p/n]} (-1)^k \left( \frac{2p}{p + kn} \right),
\end{equation}

\begin{equation}
\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \cos^2 \left( \frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{2p+1}} \sum_{k=-[p/n]}^{[p/n]} \left( \frac{2p}{p + kn} \right),
\end{equation}

where $n$ and $p$ represent positive integers. By using these results he was able to derive several new combinatorial identities involving finite sums over $k$ of the binomial coefficient $\binom{2m}{n - r_k}$, where $r$ is an integer. Following this work, two of us derived a formula for the basic trigonometric power sum with the alternating phase factor $(-1)^k$ in the summand and an extra factor of 2 in the denominator of the argument. The formula was also found to be combinatorial in nature, but its actual form was different when both the power of the sine or cosine and its upper limit were varied. Consequently, a computer program was required to evaluate the series in rational form for each set of these values.

In a more recent work using Dickson and Chebyshev polynomials Barbero initially obtained the following result:

\begin{equation}
R_{m,n} = 2^{2m} \sum_{k=1}^{n+1} \cos^{2m} \left( \frac{k\pi}{2n + 3} \right) = \left( n + \frac{3}{2} \right) \binom{2m}{m} - 2^{2m-1}, \quad m \geq 1,
\end{equation}

with $R_{n,0} = n + 1$ and $R_{0,m} = 1$. Though elegant, this result was found not to be entirely correct. E.g., for $m = 12$ and $n = 3$, the representation in terms of the trigonometric power sum for $R_{m,n}$ gives a value of 3 798 310, while its combinatorial form on the rhs gives a value of 3 780 094. Yet, if one replaces $n$ and $p$, respectively, by $2n + 3$ and $m$ in (1.4), then after multiplying by $2^{2m}$ one finds that the combinatorial form on the rhs yields the correct value of 3 798 310. Apparently, the discrepancy between the lhs and rhs of (1.6) has been brought to Barbero’s attention, since he has amended the original result to

\begin{equation}
2^{2m} \sum_{k=1}^{n+1} \cos^{2m} \left( \frac{k\pi}{2n + 3} \right) = \begin{cases} 
\left( n + \frac{3}{2} \right) \binom{2m}{m} - 2^{2m-1}, & 1 \leq m < (2n + 3), \\
\left( n + \frac{3}{2} \right) \binom{2m}{m} - 2^{2m-1} \\
& + (2n + 3) \sum_{i=1}^{\lfloor m/(2n+3) \rfloor} \binom{2m}{m - (2n + 3)i}), \\
& m \geq 2n + 3,
\end{cases}
\end{equation}

with $R_{n,0}$ and $R_{0,m}$ as given above. Consequently, we find that the extra sum on the rhs of the second result in (1.7) yields the discrepancy of 18216 on the rhs of (1.6) when $m = 12$ and $n = 3$. This highlights the necessity for conducting numerical checks on results rather than solely relying on proofs, where small terms such as the extra sum in the second result of (1.7) can often be neglected.
In this paper we aim to continue with the derivation of combinatorial forms for basic trigonometric power sums possessing different arguments and limits than those calculated previously. Typically, the basic trigonometric power sums studied here will be of the form:

\[ S = \sum_{k=0}^{g(n)} (\pm 1)^k f(k) \left( \frac{\cos^{2m}}{\sin^{2m}} \right) \left( \frac{qk\pi}{n} \right), \]

where \( m, q \) and \( n \) are positive integers, \( g(n) \) depends upon \( n \), e.g., \( n - 1 \) or \( \lfloor m/n \rfloor \), and \( f(k) \) is a simple function of \( k \), e.g., unity or \( \cos(k\pi/p) \) with \( p \), an integer. Surprisingly, the results presented here will be required when we study more complicated sums with inverse powers of trigonometric functions, such as the general or twisted Dowker \([10]\) and related sums \([7]\), in the future. Furthermore, we shall apply the results of Section 2 in the derivation of generating functions and finally consider an application to spectral graph theory by determining the number of closed walks of a specific length on a path and in a cycle.

2. Main result

The main result in this paper is presented in the following theorem:

**Theorem 2.1.** Let \( m \) and \( n \) be positive integers in the basic trigonometric power sums

\[ C(m, n) := \sum_{k=0}^{n-1} \cos^{2m} \left( \frac{k\pi}{n} \right) \quad \text{and} \quad S(m, n) := \sum_{k=0}^{n-1} \sin^{2m} \left( \frac{k\pi}{n} \right). \]

Then it can be shown that

\[ C(m, n) = \begin{cases} 2^{1-2m} n \left( \frac{2m-1}{m-1} \right) + \sum_{p=1}^{\lfloor m/n \rfloor} \left( \frac{2m}{m-pn} \right), & m \geq n, \\ 2^{1-2m} n \left( \frac{2m-1}{m-1} \right), & m < n, \end{cases} \]

and

\[ S(m, n) = \begin{cases} 2^{1-2m} n \left( \frac{2m-1}{m-1} \right) + \sum_{p=1}^{\lfloor m/n \rfloor} (-1)^p \left( \frac{2m}{m-pn} \right), & m \geq n, \\ 2^{1-2m} n \left( \frac{2m-1}{m-1} \right), & m < n. \end{cases} \]

**Remark 2.1.** After preparing the manuscript, it came to our attention that \( C(m, n) \) appears in a more unwieldy form as \((18.1.5)\) in \([19]\), while the second result of \((2.1)\) appears as No. 4.4.2.11 in \([27]\). This suggests our proof is entirely different from these references. Moreover, we shall adapt the proof to determine other results not given in these references.

**Proof.** We begin by stating well-known trigonometric power sums, which appear as No. 4.4.2.1 in \([27]\). These are

\[ \sum_{k=1}^{n} \cos^{2m}(kx) = 2^{1-2m} \sum_{k=1}^{m} \left( \frac{2m}{m-k} \right) \frac{\sin(nkx)}{\sin(kx)} \cos((n+1)kx) + 2^{-2m} n \left( \frac{2m}{m} \right) \]

and

\[ \sum_{k=1}^{n} \sin^{2m}(kx) = 2^{1-2m} \sum_{k=1}^{m} (-1)^k \left( \frac{2m}{m-k} \right) \frac{\sin(nkx)}{\sin(kx)} \cos((n+1)kx) + 2^{-2m} n \left( \frac{2m}{m} \right). \]
For $x = \pi/n$, (2.3) becomes

\begin{equation}
C(m, n) = 2^{-2m} n \left( \frac{2m}{m} \right) + 2^{1-2m} \sum_{k=1}^{m} (-1)^k \left( \frac{2m}{m - k} \right) R(k) \cos(k\pi/n),
\end{equation}

where

\[ R(k) = \lim_{y \to \pi} \left\{ \frac{\sin(ky)}{\sin(ky/n)} \right\} = \lim_{\epsilon \to 0} \left\{ \frac{\sin(k(\pi + \epsilon))}{\sin(k(\pi + \epsilon)/n)} \right\}. \]

The quotient of sines given above by $R(k)$ vanishes for all values of $k$ except when $k$ is a multiple of $n$, i.e., when $k = pn$, where $p = 1, 2, \ldots, \lfloor m/n \rfloor$. For these values of $k$, we find that $R(k) = (-1)^{(n-1)p}$. The phase factor in $R(k)$ cancels $(-1)^k \cos(k\pi/n)$ in the summand of $C(m, n)$. Moreover, since $\left( \frac{2m}{m} \right) = 2 \left( \frac{2m}{m} - 1 \right)$, (2.5) becomes

\begin{equation}
C(m, n) = 2^{1-2m} n \left( \frac{2m - 1}{m} \right) + 2^{1-2m} n \sum_{p=1}^{\lfloor m/n \rfloor} \left( \frac{2m}{m - pn} \right).
\end{equation}

Re-arranging the terms on the rhs of the above result then yields the first result in (2.1). For $m < n$, the sum over $p$ vanishes and we are left with the second result in (2.1). In addition, for $m = 0$, the second term not only vanishes, but also the first term yields $n$. Finally, adopting the same approach to (2.4) yields both results in (2.2). This completes the proof.

From Theorem 2.1, we can obtain further results beginning with the following corollary:

**Corollary 2.2.** For $q \equiv 0 \pmod{n}$, where $n$ is a positive integer, the following generalizations of the above basic trigonometric power sums are given by

\begin{equation}
\sum_{k=0}^{q-1} \cos^2 m \left( \frac{k\pi}{n} \right) = \frac{q}{n} C(m, n),
\end{equation}

and

\begin{equation}
\sum_{k=0}^{q-1} \sin^2 m \left( \frac{k\pi}{n} \right) = \frac{q}{n} S(m, n),
\end{equation}

where $C(m, n)$ and $S(m, n)$ are obtained from (2.1) and (2.2) respectively.

**Proof.** From the condition on $q$, we let $q = sn$, where $s$ is a positive integer. Then we note that the basic trigonometric power sum in (2.6) can be subdivided according to

\begin{equation}
\sum_{k=1}^{q-1} \cos^2 m \left( \frac{k\pi}{n} \right) = \sum_{j=0}^{s-1} \sum_{k=jn+1}^{(j+1)n-1} \cos^2 m \left( \frac{k\pi}{n} \right) + \sum_{j=1}^{s-1} \cos^2 m \left( \frac{j\pi}{n} \right),
\end{equation}

while (2.7) can be expressed as

\begin{equation}
\sum_{k=0}^{q-1} \sin^2 m \left( \frac{k\pi}{n} \right) = \sum_{j=0}^{s-1} \sum_{k=jn+1}^{(j+1)n-1} \sin^2 m \left( \frac{k\pi}{n} \right) + \sum_{j=1}^{s-1} \sin^2 m \left( \frac{j\pi}{n} \right).
\end{equation}

The second sum on the rhs of (2.8) represents a sum over unity and hence, yields $s - 1$, while the second sum on the rhs of (2.9) vanishes. In the first sum on the rhs of both
equations we replace \( k \) by \( k + nj \), where \( k \) now ranges from unity to \( n - 1 \). Then (2.8) and (2.9) become

\[
\sum_{k=0}^{q-1} \cos^{2m} \left( \frac{k\pi}{n} \right) = s \sum_{k=1}^{n-1} \cos^{2m} \left( \frac{k\pi}{n} \right) + s ,
\]

and

\[
\sum_{k=0}^{q-1} \sin^{2m} \left( \frac{k\pi}{n} \right) = s \sum_{k=0}^{n-1} \sin^{2m} \left( \frac{k\pi}{n} \right) .
\]

From the definitions in Theorem 2.1, the sums on the rhs of (2.10) and (2.11) are \( C(m, n) \) and \( S(m, n) \), respectively. Moreover, by replacing \( s \) by \( q/n \), we arrive at the results in the corollary, which completes the proof.

**Corollary 2.3.** If we define the basic power sums

\[
C(m, n, q) := \sum_{k=0}^{n-1} \cos^{2m} \left( \frac{qk\pi}{n} \right) , \quad \text{and} \quad S(m, n, q) := \sum_{k=0}^{n-1} \sin^{2m} \left( \frac{qk\pi}{n} \right) ,
\]

where \( n \) and \( q \) are co-prime, then

\[
C(m, n, q) = C(m, n) , \quad \text{while} \quad S(m, n, q) = S(m, n) .
\]

**Proof.** Returning to the proof of Theorem 2.1, we now introduce \( x = q\pi/n \), where \( q \) is co-prime to \( n \), into (2.3). (If \( q \) is a negative integer, then we take its absolute value in what follows.) Hence (2.3) becomes

\[
C(m, n, q) = 2^{-2m} n \left( \frac{2m}{m} \right) - 1 + 2^{1-2m} \sum_{k=1}^{m} (-1)^k \left( \frac{2m}{m-k} \right) R(kq) \cos(qk\pi/n) ,
\]

where

\[
R(kq) = \lim_{\epsilon \to 0} \left\{ \frac{\sin(k(q\pi + \epsilon))}{\sin(k(q\pi + \epsilon)/n)} \right\} .
\]

That is, the argument of \( R(k) \) has been replaced by \( kq \), while the cosine is now dependent upon \( qk\pi/n \) instead of \( k\pi/n \). This means that the sum on the rhs of (2.14) is, once again, non-zero for all integer values, where \( k = pn \) with \( p \) ranging from unity to \( \lfloor m/n \rfloor \), provided \( n \) and \( q \) are co-prime. If \( m < n \), the sum of the rhs of (2.14) vanishes and we are left with the second result in (2.1). Hence we find that

\[
(-1)^{npq} R(kq) \cos(qp\pi/n) = (-1)^{pq} (-1)^{p(n+1)} n (-1)^{pq} = n .
\]

Introducing the above result into (2.14) yields the first result in the corollary for \( C(m, n, q) \).

To obtain (2.13), we put \( x = q\pi/n \) in (2.4). Then we arrive at

\[
S(m, n, q) = 2^{-2m} n \left( \frac{2m}{m} \right) + 2^{1-2m} \sum_{k=1}^{m} (-1)^{k(q+1)} \left( \frac{2m}{m-k} \right) R(kq) \cos(qk\pi/n) .
\]

As indicated above, the sum will only contribute when \( R(kq) \) is non-zero, which occurs when \( k \) is an integer multiple of \( n \). By multiplying (2.15) throughout with the phase factor of \( (-1)^{npq} \), we obtain the value of \( R(kq) \cos(qk\pi/n) \), which equals \( (-1)^{npq} n \). Introducing this value into (2.16) yields (2.13). This completes the proof.
Although it was stated that \( n \) and \( q \) need to be co-prime for (2.12) to hold, let us now assume that they are both even numbers, but are co-prime once the factor of 2 has been removed. If we let \( q = 2s \) and \( n = 2\ell \), then we have

\[
C(m, n, q) = \sum_{k=0}^{n-1} \cos^{2m} \left( \frac{sk\pi}{\ell} \right) = 2 \sum_{k=0}^{\ell-1} \cos^{2m} \left( \frac{sk\pi}{\ell} \right).
\]

Since \( s \) and \( \ell \) are co-prime, we can apply Corollary 2.3. If, however, there was another factor of 2 before \( s \) and \( \ell \) became co-prime, i.e. \( n = 4\ell \) and \( q = 4s \), then we find that

\[
C(m, n, q) = 4 \sum_{k=0}^{\ell-1} \cos^{2m} \left( \frac{sk\pi}{\ell} \right).
\]

Moreover, if \( r \) represents the product of all the common factors of \( n \) and \( q \), then we find that

\[
\sum_{k=0}^{n-1} \left\{ \cos^{2m} \left( \frac{qk\pi}{n} \right) \right\} \left( \frac{qk\pi}{n} \right) = r \sum_{k=0}^{\ell-1} \left\{ \cos^{2m} \left( \frac{sk\pi}{\ell} \right) \right\},
\]

where the curly brackets have been introduced to signify that the above results apply to either a cosine or sine power. Therefore, for \( n = r\ell \) and \( q = rs \), \( C(m, n, q) = rC(m, \ell, s) = rC(m, \ell) \) and \( S(m, n, q) = rS(m, \ell, s) = rS(m, \ell) \) according to Corollary 2.3.

3. Extensions

In this section we shall use the results of the previous section to derive solutions for more advanced basic trigonometric power sums. As stated in the introduction, Merca [26] has evaluated several basic trigonometric power sums by deriving (1.4) and (1.5) via the multisection series method. However, these results can also be derived via Theorem 2.1.

To demonstrate this, we express \( C(p, n) \) as

\[
C(p, n) = 2^{\lfloor (n-1)/2 \rfloor} \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \cos^{2p} \left( \frac{k\pi}{n} \right) + 1,
\]

where for the terms between \( k = \lfloor (n-1)/2 \rfloor \) and \( k = n-1 \), we have replaced \( k \) by \( n-k \), thereby obtaining twice the first sum via symmetry. In other words, the finite sum studied by Merca in [26] is just half of \( C(p, n) \). Dividing through by 2 yields Merca’s result when one realises that: (1) the sum over negative values of \( k \) in (1.4) is identical to that over of positive values of \( k \), and (2) the \( k = 0 \) term produces the combinatorial term on the rhs of (2.1).

We can determine formulas for other basic trigonometric power sums by manipulating (2.1) and (2.2). First we express \( C(m, n) \) as

\[
C(m, n) = \sum_{k=0}^{n-1} \cos^{2m} \left( \frac{2k\pi}{2n} \right) = \sum_{k=0,2,4,...}^{2n-2} \cos^{2m} \left( \frac{k\pi}{2n} \right).
\]

This result can be written alternatively as

\[
C(m, n) = \frac{1}{2} \sum_{k=0}^{2n-1} \left( 1 + (-1)^k \right) \cos^{2m} \left( \frac{k\pi}{2n} \right).
\]
The first sum on the rhs is simply \( C(m, 2n) \). Therefore, (3.1) becomes

\[
(3.2) \quad \sum_{k=0}^{2n-1} (-1)^k \cos^{2m}(k\pi/2n) = 2C(m, n) - C(m, 2n) .
\]

In fact, the above result can be extended to yield

\[
\sum_{k=0}^{2pm-1} (-1)^k \cos^{2m}(k\pi/2pn) = 2C(m, pn) - C(m, 2pn) ,
\]

where \( p \) is a positive integer. Hence the alternating form of \( C(m, n) \) is given by

\[
(3.3) \quad \sum_{k=0}^{n-1} (-1)^k \cos^{2m}(k\pi/n) = 2C(m, n/2) - C(m, n) ,
\]

which is only valid for even values of \( n \).

If we introduce (2.1) into (3.2), then we obtain three distinct cases depending on whether \( m < n \), \( n \leq m < 2n \) and \( m \geq 2n \). Hence we arrive at

\[
(3.4) \quad \sum_{k=0}^{2n-1} (-1)^k \cos^{2m}(k\pi/2n) = \begin{cases} 
2^{2-2m} n \left( \sum_{p=1}^{[m/n]} \left( \frac{2m}{m-pn} \right) - \sum_{p=1}^{[m/2n]} \left( \frac{2m}{m-2pn} \right) \right), & m \geq 2n , \\
2^{2-2m} \sum_{p=1}^{[m/n]} \left( \frac{2m}{m-pn} \right), & n \leq m < 2n , \\
0, & m < n .
\end{cases}
\]

In similar fashion, we can obtain the corresponding result when \( \cos^{2m}(k\pi/2n) \) is replaced by \( \sin^{2m}(k\pi/2n) \). Therefore, repeating the above calculation, we obtain

\[
\sum_{k=0}^{2n-1} (-1)^k \sin^{2m}(k\pi/2n) = 2S(m, n) - S(m, 2n) ,
\]

which yields after the introduction of (2.2)

\[
(3.5) \quad \sum_{k=0}^{2n-1} (-1)^k \sin^{2m}(k\pi/2n) = \begin{cases} 
2^{2-2m} n \left( \sum_{p=1}^{[m/n]} \left( -1 \right)^p \left( \frac{2m}{m-pn} \right) - \sum_{p=1}^{[m/2n]} \left( \frac{2m}{m-2pn} \right) \right), & m \geq 2n , \\
2^{2-2m} \sum_{p=1}^{[m/n]} \left( \frac{2m}{m-pn} \right), & n \leq m < 2n , \\
0, & m < n .
\end{cases}
\]

We can also express (3.2) as

\[
(3.6) \quad \sum_{k=0,2,4,...}^{2n-2} \cos^{2m}(k\pi/2n) - \sum_{k=1,3,5,...}^{2n-1} \cos^{2m}(k\pi/2n) = 2C(m, n) - C(m, 2n) .
\]

Alternatively, the above can be written as

\[
\sum_{k=0}^{n-1} \cos^{2m}(k\pi/n) - \sum_{k=0}^{n-1} \cos^{2m}((k + 1/2)\pi/n) = 2C(m, n) - C(m, 2n) .
\]
The above result can be simplified further to yield

\[(3.7) \quad \sum_{k=0}^{n-1} \cos^{2m}\left((k+1/2)\pi/n\right) = C(m,2n) - C(m,n).\]

Before we can combine the terms on the rhs, we need to relate the upper upper limit in the sum for \(C(m,2n)\) in (3.7), viz. \([m/2n]\), with that for \(C(m,n)\), which is \([m/n]\). If we let \(m = rn + b\), where \(0 < b < n\), then \([m/2n]\) = \(r\), while \([m/2n]\) = \([r/2 + b/2n]\).

If \(r\) is even, then we find that \([m/n]\) = \(2[m/2n]\), but if it is odd, then we find that \([m/n]\) = \(2[m/2n] + 1\). In other words, we require the following identity:

\[\left\lfloor m/n \right\rfloor = 2\left\lceil m/2n \right\rceil + (1 - (-1)^{\left\lfloor m/n \right\rfloor})/2.\]

We now introduce (2.1) into (3.7), which yields

\[\sum_{k=0}^{n-1} \cos^{2m}\left((k+1/2)\pi/n\right) = 2^{1-2m}n\left(\binom{2m-1}{m-1} + 2\sum_{p=2,4,\ldots}^{2[m/2n]} \binom{2m}{m - pn} - \sum_{p=1}^{[m/n]} \binom{2m}{m - pn}\right).\]

At this stage we require the identity given above. Then we arrive at

\[(3.8) \quad \sum_{k=0}^{n-1} \cos^{2m}\left((k+1/2)\pi/n\right) = 2^{1-2m}n\left(\binom{2m-1}{m-1} + \sum_{p=1}^{[m/n]} (-1)^p \binom{2m}{m - pn}\right).\]

This is basically twice Merca’s result, which has been given here as (1.5). By carrying out a similar calculation with the cosine power in (3.2) replaced by a sine power and using (2.2) instead, one finds that

\[(3.9) \quad \sum_{k=0}^{n-1} \sin^{2m}\left((k+1/2)\pi/n\right) = 2^{1-2m}n\left(\binom{2m-1}{m-1} + \sum_{p=1}^{[m/n]} \left(1 + (-1)^p - (-1)^np\right) \binom{2m}{m - pn}\right).\]

For odd values of \(n\), (3.9) reduces to

\[\sum_{k=0}^{n-1} \sin^{2m}\left((k+1/2)\pi/n\right) = 2^{1-2m}n\left(\binom{2m-1}{m-1} + \sum_{p=1}^{[m/n]} \binom{2m}{m - pn}\right),\]

while for even values of \(n\), it becomes

\[\sum_{k=0}^{n-1} \sin^{2m}\left((k+1/2)\pi/n\right) = 2^{1-2m}n\left(\binom{2m-1}{m-1} + \sum_{p=1}^{[m/n]} (-1)^p \binom{2m}{m - pn}\right).\]

It was mentioned that Merca was able to evaluate finite sums involving the binomial coefficient in a few corollaries by fixing \(n\) to small values ranging from unity to 5 or 6 in (1.4) and (1.5) and directly evaluating the sums. These results can be verified by carrying out the same procedure with the results in Theorem 2.1 and by using (3.8).

The preceding results, given by (3.3) and (3.5), have had a factor \(\ell = 2\) introduced into the denominators of the cosine and sine powers in the basic trigonometric power sums of Theorem 2.1. We can develop other interesting results by multiplying and dividing the argument in the trigonometric power by integers. For example, if we multiply and divide the argument in the cosine power by 3, then \(C(m,n)\) can be expressed as

\[(3.10) \quad C(m,n) = \sum_{k=0,3,6,\ldots}^{3n-3} \cos^{2m}\left(\frac{k\pi}{3n}\right).\]
The same applies to $S(m, n)$ when we multiply and divide the argument by 3. To obtain a sum over all values of $k$ from 1 to $3n - 1$, we need to write the above sum as

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \cos^2m\left(\frac{k\pi}{3n}\right). 
$$

Consequently, we arrive at the following interesting result

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \cos^2m\left(\frac{k\pi}{3n}\right) = \frac{1}{2} \left(3C(m, n) - C(m, 3n)\right).
$$

The corresponding result for $S(m, n)$ is

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \sin^2m\left(\frac{k\pi}{3n}\right) = \frac{1}{2} \left(3S(m, n) - S(m, 3n)\right).
$$

Next, by introducing the results in Theorem 2.1 we obtain explicit expressions for both sums, which are

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \cos^2m\left(\frac{k\pi}{3n}\right) = \begin{cases} 
\frac{3n}{2m} \left(\sum_{p=1}^{\lfloor m/n \rfloor} (-1)^{pn} \left( \frac{2m}{m - pn} \right) - \sum_{p=1}^{\lfloor m/3n \rfloor} (-1)^{3pn} \left( \frac{2m}{m - 3pn} \right) \right), & m \geq 3n, \\
\frac{3n}{2m} \sum_{p=1}^{\lfloor m/n \rfloor} \left( \frac{2m}{m - pn} \right), & n \leq m < 3n, \\
0, & m < n,
\end{cases}
$$

and

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \sin^2m\left(\frac{k\pi}{3n}\right) = \begin{cases} 
\frac{3n}{2m} \left(\sum_{p=1}^{\lfloor m/n \rfloor} (-1)^{pn} \left( \frac{2m}{m - pn} \right) - \sum_{p=1}^{\lfloor m/3n \rfloor} (-1)^{3pn} \left( \frac{2m}{m - 3pn} \right) \right), & m \geq 3n, \\
\frac{3n}{2m} \sum_{p=1}^{\lfloor m/n \rfloor} (-1)^{pn} \left( \frac{2m}{m - pn} \right), & n \leq m < 3n, \\
0, & m < n.
\end{cases}
$$

In the above results we see that the final expressions for the sums are now dependent on whether $m$ is greater or less than either $n$ or $3n$, rather than $n$ and $2n$ in the previous example. Furthermore, the series on the lhs can be written as

$$
3n - 1 \sum_{k=0}^{3n-1} \cos\left(\frac{2k\pi}{3}\right) \left\{ \cos^2m\left(\frac{k\pi}{3n}\right) \right\} = \sum_{k=0}^{3n-1} (-1)^k \cos\left(\frac{k\pi}{3}\right) \left\{ \cos^2m\left(\frac{k\pi}{3n}\right) \right\}.
$$

Nevertheless, we are unable to obtain the corresponding sum with $\cos(k\pi/3)$ in the summand instead of $\cos(2k\pi/3)$. To accomplish that, we need to examine the situation when we multiply and divide by $\ell = 4$ inside the trigonometric power.

For the $\ell = 4$ situation the corresponding form of (3.14) becomes

$$
C(m, n) = \sum_{k=0, 4, 8, \ldots}^{4n - 4} \cos^2m\left(\frac{k\pi}{4n}\right).
$$
In this case the sum over all values of $k$ becomes

$$C(m, n) = \frac{1}{4} \sum_{k=1}^{4n-1} \left( 2 \cos \left( \frac{k\pi}{2} \right) + 1 + (-1)^k \right) \cos^{2n} \left( \frac{k\pi}{4n} \right).$$

The term involving unity in the above equation yields $C(m, 4n)$, while the term with the oscillating phase is simply (3.2) with $n$ replaced by $2n$ or $2C(m, 2n) - C(m, 4n)$. Therefore, we find that the $C(m, 4n)$ contributions cancel each other and we are left with

$$\sum_{k=0}^{4n-1} \cos \left( \frac{k\pi}{2} \right) \cos^{2n} \left( \frac{k\pi}{4n} \right) = 2C(m, n) - C(m, 2n),$$

which is just another derivation of (3.2). That is, we do not obtain a new basic trigonometric power sum as we did when we divided and multiplied by 3 inside the cosine power.

In fact, the same reducibility arises when we divide and multiply by 8. Therefore, we conjecture that multiplying and dividing by $2^n$ in either $C(m, n)$ or $S(m, n)$ will not produce new formulas.

So let us now turn our attention to when we multiply and divide by 6 in inside the cosine power of $C(m, n)$. Since we are dealing with an even number, we expect some reducibility to occur. Then $C(m, n)$ becomes

$$C(m, n) = \sum_{k=0,6,12,...}^{6n-6} \cos^{2m} \left( \frac{k\pi}{6n} \right).$$

With the aid of the identity

\begin{equation}
2 \cos \left( \frac{k\pi}{3} \right) + 2 \cos \left( \frac{2k\pi}{3} \right) + 1 - (-1)^{k+1} = \begin{cases} 6, & k \equiv 0 \pmod{6}, \\ 0, & \text{otherwise}, \end{cases}
\end{equation}

the above equation can be written alternatively as

\begin{equation}
C(m, n) = \frac{1}{6} \sum_{k=0}^{6n-1} \left( 2 \cos \left( \frac{k\pi}{3} \right) + 2 \cos \left( \frac{2k\pi}{3} \right) + (-1)^k + 1 \right) \cos^{2m} \left( \frac{k\pi}{6n} \right).
\end{equation}

Expressing the sum of cosines as a product, we find that (3.16) becomes

$$C(m, n) = \frac{1}{6} \sum_{k=0,2,4,...}^{6n-2} \left( 4 \cos \left( \frac{k\pi}{2} \right) \cos \left( \frac{k\pi}{6} \right) + (-1)^k + 1 \right) \cos^{2m} \left( \frac{k\pi}{6n} \right).$$

Replacing $2k$ by $k$ in the above result yields

$$C(m, n) = \frac{1}{3} \sum_{k=1}^{3n-1} \left( 2(-1)^k \cos \left( \frac{k\pi}{3} \right) + 1 \right) \cos^{2m} \left( \frac{k\pi}{3n} \right).$$

Hence we arrive at

$$\sum_{k=0}^{3n-1} (-1)^k \cos \left( \frac{k\pi}{3} \right) \cos^{2m} \left( \frac{k\pi}{3n} \right) = \frac{1}{2} \left( 3C(m, n) - C(m, 3n) + 3 \right).$$

This is just (3.12) again except for the term of 3/2. Even though we have demonstrated the reducible nature of basic trigonometric power sums, we have not been able to produce a result with $\cos(k\pi/3)$ in the summand instead of $\cos(2k\pi/3)$. However, it can be seen
that $\cos(k\pi/3)$ does appear in the identity given by (3.15). Therefore, let us construct a situation involving the identity and the cosine power together, viz.

\[
\frac{1}{3} \sum_{k=0}^{3n-1} \left( \cos \left( \frac{\pi k}{3} \right) + \cos \left( \frac{2\pi k}{3} \right) + \frac{1 - (-1)^{k+1}}{2} \right) \cos^{2m} \left( \frac{k\pi}{3n} \right) = \sum_{k=0,6,12,...}^{6\lfloor (3n-1)/6 \rfloor} \cos^{2m} \left( \frac{k\pi}{3n} \right).
\]

The first series on the lhs of the above equation is the result we wish to determine, while the second series is given by (3.12). The next term with $1/2$ is simply $C(m, 3n)/2$. Thus, we are left with two series to evaluate. The first of these can be determined by replacing $n$ with $3n/2$ in (3.2), which yields

\[
\sum_{k=0}^{3n-1} (-1)^k \cos^{2m} \left( \frac{k\pi}{3n} \right) = 2C(m, 3n/2) - C(m, 3n).
\]

Inserting the results for $C(m, n)$ in Theorem 2.1 yields

\[
\sum_{k=0}^{3n-1} (-1)^k \cos^{2m} \left( \frac{k\pi}{3n} \right) = \begin{cases} 
6m \sum_{p=1}^{\lfloor 2m/3n \rfloor} \left( \frac{2m}{m - 3pn/2} \right) - \sum_{p=1}^{\lfloor m/3n \rfloor} \left( \frac{2m}{m - 3pn} \right), & m \geq 3n, \\
6m \sum_{p=1}^{\lfloor 2m/3n \rfloor} \left( \frac{2m}{m - 3pn/2} \right), & 3n/2 \leq m < 3n, \\
0, & m < 3n/2.
\end{cases}
\]

For the above result to be valid, $n$ must also be even. Since $n$ is even, $\lfloor (3n - 1)/6 \rfloor = n/2 - 1$. Then the series on the rhs of (3.17) can be expressed as

\[
\sum_{k=0,6,12,...}^{6\lfloor (3n-1)/6 \rfloor} \cos^{2m} \left( \frac{k\pi}{3n} \right) = \sum_{k=0}^{n/2-1} \cos^{2m} \left( \frac{k\pi}{n/2} \right).
\]

In other words, the above sum is equal to $C(m, n/2)$ provided $n$ is even. If we introduce (3.18) and (3.20) into (3.17) together with the other previously mentioned results, then after a little algebra we find that

\[
\sum_{k=0}^{3n-1} \cos \left( \frac{k\pi}{3} \right) \cos^{2m} \left( \frac{k\pi}{3n} \right) = 3C(m, n/2) - 3C(m, n)/2 + C(m, 3n)/2 - C(m, 3n/2),
\]
where \( n \) is an even positive integer. Introducing the results from Theorem 2.1 into the above equation yields

\[
\sum_{k=0}^{3n-1} \cos \left( \frac{k\pi}{3} \right) \cos^2 m \left( \frac{k\pi}{3n} \right) = \begin{cases} 
\frac{3n}{2^{2m}} \left( \sum_{p=1}^{\left\lfloor 2m/n \right\rfloor} \left( \frac{2m}{m - pn/2} \right) - \sum_{p=1}^{\left\lfloor m/n \right\rfloor} \left( \frac{2m}{m - pn} \right) 
- \sum_{p=1}^{\left\lfloor 2m/3n \right\rfloor} \left( \frac{2m}{m - 3pn/2} \right) + \sum_{p=1}^{\left\lfloor m/3n \right\rfloor} \left( \frac{2m}{m - 3pn} \right) \right), & m \geq 3n, \\
\frac{3n}{2^{2m}} \left( \sum_{p=1}^{\left\lfloor 2m/n \right\rfloor} \left( \frac{2m}{m - pn/2} \right) - \sum_{p=1}^{\left\lfloor m/n \right\rfloor} \left( \frac{2m}{m - pn} \right) 
- \sum_{p=1}^{\left\lfloor 2m/3n \right\rfloor} \left( \frac{2m}{m - 3pn/2} \right) \right), & 3n/2 \leq m < 3n, \\
\frac{3n}{2^{2m}} \left( \sum_{p=1}^{\left\lfloor m/n \right\rfloor} \left( \frac{2m}{m - pn/2} \right) - \sum_{p=1}^{\left\lfloor m/3n \right\rfloor} \left( \frac{2m}{m - pn} \right) \right), & n \leq m < 3n/2, \\
\frac{3n}{2^{2m}} \left( \sum_{p=1}^{\left\lfloor 2m/n \right\rfloor} \left( \frac{2m}{m - pn/2} \right) \right), & n/2 \leq m < n, \\
0, & m < n/2.
\end{cases}
\]

The sine version of the above basic trigonometric power sum can be obtained in a similar manner with the various \( C(m, n) \) terms replaced by their corresponding \( S(m, n) \) terms. In Appendix B we examine the case when the argument of the cosine power is multiplied and divided by 5. In this case two different prefactors of the cosine powers, viz., \( \cos(2\pi k/5) \) and \( \cos(4\pi k/5) \), arise, which are not easily decoupled from one another. This implies that it is not possible without additional information to obtain elegant results such as those above when we multiply and divide by numbers that possess prime number factors greater than or equal to 5.

4. Generating functions

In this section we use the results of Section 2 to determine several generating functions. We begin by defining the exponential generating function

\[
G_1(n; z) := \sum_{m=0}^{\infty} \frac{z^m}{m!} C(m/2, n).
\]

After some algebra we eventually arrive at

\[
G_1(n; z) = \sum_{k=0}^{n-1} e^{z \cos(k\pi/n)} = \sum_{k=0}^{n-1} \cosh \left( z \cos \left( \frac{k\pi}{n} \right) \right) + \sinh z
\]

\[
= 2n \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^{2k} \sigma_k(n) + nI_0(z) + \sinh z,
\]

where

\[
\sigma_k(n) := \frac{1}{(2k)!} \sum_{p=1}^{\left\lfloor k/n \right\rfloor} \left( \frac{2k}{k + pn} \right).
\]

and \( I_0(z) \) represents the modified Bessel function of zeroth order. Note that for \( k < n \), \( \sigma_k(n) = 0 \).
The result given by (4.2) can also be regarded as the generating function for \( \sigma_k(n) \). For fixed small values of \( n \), the \( \sigma_k(n) \) can be determined by the direct evaluation of the series \( C(m, n) \) in Theorem 2.1 or its variants in the corollaries. Moreover, it was mentioned in the introduction that Merca has evaluated several combinatorial identities involving the binomial coefficient in a series of corollaries in [26]. In fact, Corollary 6 of this reference presents specific values of the \( \sigma_k(n) \) for \( n \) ranging from unity to six. These results have been determined via (1.4), which we have seen follows from Theorem 2.1.

We can extend \( G_1(n; z) \) to \( G_1(n, q; z) \) by introducing the result for \( C(m/2, n, q) \) as given in Corollary 2.3 into (4.1). Then for odd values of \( q \), the generating function \( G_1(n, q; z) \) becomes

\[
G_1(n, q; z) = \sum_{k=0}^{n-1} e^{z \cos(qk\pi/n)} = \sum_{k=0}^{n-1} \cosh \left( z \cos \left( \frac{qk\pi}{n} \right) \right) + \sinh z
\]

(4.4)

The intermediate member involving the summation over the hyperbolic cosine has been obtained by: (1) expanding the exponential as a series in the first sum, (2) splitting the resultant sum into two equal parts, (3) substituting \( \cos m \left( \frac{qk\pi}{n} \right) \) by \( (-1)^m \cos m \left( \frac{\pi q}{n} \right) (n - k)/n \) in one of the parts and (4) replacing \( n - k \) by \( k \). Moreover, the intermediate member does not apply for even values of \( q \), although the first and third members are still equal to one another.

We can also derive the generating function for the case when \( C(m/2, n, q) \) is replaced by \( S(m/2, n, q) \) in the preceding analysis. That is, by defining the exponential generating function, \( H_1(n, q; z) \), as

\[
H_1(n, q; z) := \sum_{m=0}^{\infty} \frac{z^m}{m!} S(m/2, n, q) = \sum_{k=0}^{n-1} e^{z \sin(qk\pi/n)}
\]

we can obtain a similar closed-form solution to (4.4), provided \( q \) is an even integer. In this instance we introduce the result in Corollary 2.3 and split the resulting sum. Then we replace \( \sin^m(k\pi/n) \) by \( \sin^m(\pi - k\pi/n) \) and proceed by combining the summations. Hence we find that

\[
H_1(n, q; z) = nI_0(z) + 2n \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^{2k} \sigma_k^-(n)
\]

(4.5)

where \( \sigma_k^-(n, q) \) has replaced \( \sigma_k(n) \) and is defined as

\[
\sigma_k^-(n, q) := \frac{1}{(2k)!} \sum_{p=1}^{[k/n]} (-1)^p \binom{2k}{k + pn}.
\]

In obtaining (4.5) we have split the basic sine power sum and replaced \( \sin^{2m}(\pi k/n) \) in one of the resulting sums by \( \sin^{2m}(\pi + \pi(n - k)/n) \). If \( n \) is even, then \( \sigma_k^-(n) \) reduces to \( \sigma_k(n) \), while if it is odd, then (4.5) becomes

\[
\sigma_k(n, q) := \frac{1}{(2k)!} \sum_{p=1}^{[k/n]} (-1)^p \binom{2k}{k + pn}.
\]

As mentioned in the introduction Merca [26] obtains identities for the above result by fixing \( n \) and evaluating the series in (1.3) directly.
The above results are not the only examples, where the results of Section 2 can be used to obtain generating functions. For example, when \(|z| < 1\), we can expand the denominator in the sum \(\sum_{k=1}^{n-1} \frac{1}{1 - z \cos^2(k \pi/n)}\) and introduce (2.1), thereby obtaining

\[
\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{1 - z \cos^2(k \pi/n)} = \sum_{k=0}^{\infty} \left(\frac{z}{4}\right)^k \left(\frac{(2k)!}{(k)!^2} + 2 \sum_{p=1}^{[k/n]} \frac{2k}{k - pm}\right) - \frac{1}{n(1-z)}.
\]

The last term on the rhs, which arises from removing the \(k = 0\) term in \(C(m, n)\), can be incorporated as the \(k = 0\) term in the sum on the lhs. In addition, by introducing the duplication formula for the gamma function, viz., No. 8.335(1) in [16], we find that the first term on the rhs reduces to the binomial series for \(1/\sqrt{1 - z}\), while we introduce the definition for \(\sigma_k(n)\) or \(\text{[4.3]}\) into the second term on the rhs. Consequently, we arrive at

\[
\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{1 - z \cos^2(k \pi/n)} = \frac{1}{\sqrt{1 - z}} + 2 \sum_{k=0}^{\infty} \left(\frac{z}{4}\right)^k (2k)! \sigma_k(n).
\]

Similarly, one can consider the analogous sum where \(\cos(k \pi/n)\) is replaced by \(\sin(k \pi/n)\). In this instance one employs (2.2) in the analysis, which finally yields

\[
\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{1 - z \sin^2(k \pi/n)} = \frac{1}{\sqrt{1 - z}} + 2 \sum_{k=0}^{\infty} \left(\frac{z}{4}\right)^k (2k)! \sigma_k(n).
\]

5. Closed walks

Here we demonstrate that the main result of Section 2 can be used in calculating closed walks on a path and also in a cycle. We begin by recalling that the adjacency matrix \(A\) of a graph \(G\) is the binary matrix with rows and columns indexed by the vertices of \(G\), such that the \((i, j)\)-entry is equal to 1 if \(i\) and \(j\) are adjacent, and zero otherwise. Since loops are not allowed in the graphs under consideration, the diagonal entries of \(A\) are all zero. A walk of length \(r\) on \(G\) represents a sequence along \(r + 1\) adjacent vertices (not necessarily different) and hence, possesses \(r\) edges. A walk is said to be closed if the first and terminal vertices or endpoints are the same. A circuit is known as a closed walk when it has no repeating edges, while a closed walk with repeating vertices is referred to as a cycle.

Evaluating the number of closed walks on a graph has been an active topic of research that spans across combinatorics, graph theory, and linear algebra (cf. [9 20 24 29 30]). Although our result for the number of closed walks will be general, when we turn to cycles, we will need to restrict the closed walks to even length and the cycles to odd order.

With the aid of (2.1) we can now prove the following theorem:

**Theorem 5.1.** The number of closed walks of length \(2m\) on a path \(P_{n-1}\) is given by

\[
p(2m) = \begin{cases} 
2n \left(\binom{2m-1}{m-1} + \sum_{k=1}^{[m/n]} \binom{2m}{m - kn}\right) - 2^{2m}, & m \geq n, \\
2n \binom{2m-1}{m-1} - 2^{2m}, & m < n,
\end{cases}
\]

**Proof.** It is well-known that the \((i, j)\) entry of \(A^k\) represents the number of walks on \(G\) of length \(k\) with endpoints \(i\) and \(j\). Furthermore, if \(\lambda\) is an eigenvalue of \(A\), then \(\lambda^k\) is an eigenvalue of \(A^k\). Hence the trace of \(A^k\) is equal to the sum of the \(k\)th powers of the eigenvalues of \(A\), which, in turn, equals the total number of closed walks of length
on $G$, which we denote here by $p(k)$ (cf. [9, p.14]). Because the adjacency matrix can be represented by a tridiagonal matrix with ones on the sub- and super-diagonals and zeros elsewhere, its eigenvalues for a path $P_{n-1}$ with $n-1$ vertices are $2\cos(\ell \pi/n)$, where $\ell = 1, \ldots, n-1$, (cf., e.g., [12]). The result in the theorem follows by summing over all values of $\ell$ and then by applying (2.1). This completes the proof.

It is interesting to notice that with Theorem 5.1 we get the sequence A198632 in [25].

We now turn our attention to closed walks in a cycle by presenting the following theorem.

**Theorem 5.2.** The number of closed walks of length $2m$ on the cycle $C_n$, where $n$ is odd, is given by

$$p(2m) = \begin{cases} 
2n \binom{2m-1}{m-1} + \sum_{r=1}^{[m/n]} \binom{2m}{m-rn}, & m \geq n, \\
2n \binom{2m-1}{m-1}, & m < n.
\end{cases}$$

**Proof.** In this instance the eigenvalues of an $n$-cycle are equal to $2\cos(2\ell \pi/n)$, for $\ell = 0, 1, \ldots, n-1$. Because of this, we can follow the previous proof except that we use (2.12) instead of (2.1). Consequently, we arrive at the result in the theorem. This completes the proof.

6. Conclusion

In this paper we have presented combinatorial forms for the two main basic trigonometric power sums $C(m, n)$ and $S(m, n)$ in Theorem 2.1. We have been able to extend these results to derive combinatorial forms for other basic trigonometric power sums, where either the arguments in the trigonometric power and/or their limits have been altered. Where possible we have been able to relate our results to existing solutions such as those appearing in [26]. In addition, we have demonstrated that our main results can be applied to generating functions, but even more interesting, is that they were used to determine the number of closed walks on a path and in a cycle. In the future we intend to apply the results presented here when we study the general or twisted Dowker [10] and related sums [7].

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8. Appendix A

In the introduction it was stated that Berndt and Yeap’s result for the sum over even powers of cotangent, viz., (1.2), was imprecise and that a better formulation was given by (1.3). Here, we prove this by referring to [3]. To enable the reader to develop an understanding of how polynomials in $k$ arise when evaluating finite sums of powers of the cotangent, we shall also show how the formula is implemented for specific values of $n$, which is also lacking in [3].
Broadly speaking, Berndt and Yeap derive \(^{(13)}\) via the third case considered in \(^{[3]}\) Theorem 2.1. The other two cases will be discussed in a future work. The theorem deals with the contour integration of the function

\[
f(z) = \cot^m(\pi z) \cot(\pi(hz - a)) \cot(\pi(kz - b))
\]

over a positively oriented rectangle with vertices at \(\pm iR\) and \(1 \pm iR\), where \(R > \epsilon\), and possessing semi-circular indentations at 0 and 1 of radius \(\epsilon\), where \(\epsilon < \min\{(h - 1 + a)/h, (k - 1 + b)/k\}\). The third case is represented by \(a = b = 0\) and hence \(f(z)\) becomes

\[(8.1) \quad f(z) = (hk)^{-1}(\pi z)^{-m-2} \left( \sum_{j=0}^{\infty} a(j)x^j \right)^m \sum_{\mu=0}^{\infty} a(\mu)(h'x)^\mu \sum_{\nu=0}^{\infty} a(\nu)(k'x)^\nu ,
\]

where \(a_j = (-1)^j 2^{2j} B_{2j}/(2j)!\), \(x = (\pi z)^2\), \(h' = h^2\) and \(k' = k^2\). From \((8.1)\) we see that there is a pole of order \(m + 2\) at \(z = 0\). Therefore, the aim is to evaluate the residue of \(f(z)\) at \(z = 0\), which is given by

\[(8.2) \quad \text{Res } f(z) \bigg|_{z=0} = \frac{1}{(m+1)!} \frac{d^{m+1}}{dz^{m+1}} \left( z^{m+2} f(z) \right) \bigg|_{z=0}.
\]

Before we can evaluate this, we need to evaluate the product of the infinite series on the rhs of \((8.1)\). Berndt and Yeap proceed by introducing coefficients \(C(j_1, \ldots, j_m, \mu, \nu)\), which are not given explicitly, but that they arise when a sum over all \((m + 2)\)-tuples \((j_1, \ldots, j_m, \mu, \nu)\) is evaluated under the condition that \(2 \left( \sum_{i=1}^{m} j_i + \mu + \nu \right) = m + 1\). This, however, leads to the imprecision in \((13)\). Here we adopt a different approach based on extending the Cauchy product formula \([32]\).

We begin by multiplying one of the series in the power by the penultimate series in \((8.1)\). If we denote this product as \(P_1\), then by the Cauchy product formula, it can be expressed as

\[P_1 = \sum_{j=0}^{\infty} x^j A_1(j) ,
\]

where \(A_1(j) = \sum_{\ell_1=0}^{j} a(\ell_1) h^{\ell_1} a(j - \ell_1)\). Now we multiply \(P_1\) by the final series in \((8.1)\), which yields

\[
P_2 = P_1 \sum_{\nu=0}^{\infty} a(\nu) k^{\nu} x^\nu = \sum_{j=0}^{\infty} x^j \sum_{\ell_1=0}^{j} a(\ell_1) k^{\ell_1} A_1(j - \ell_1) = \sum_{j=0}^{\infty} x^j A_2(j - \ell_1) ,
\]

and

\[
A_2(j) = \sum_{\ell_1=0}^{j} \sum_{\ell_2=0}^{j - \ell_1} a(\ell_1) a(\ell_2) k^{\ell_1} h^{\ell_2} a(j - \ell_1 - \ell_2) .
\]

Next we multiply \(P_2\) by another series in the power to obtain \(P_3\), obtaining

\[
P_3 = P_2 \sum_{j=0}^{\infty} a(j)x^j = \sum_{j=0}^{\infty} x^j \sum_{\ell_1=0}^{j} a(\ell_1) A_2(j - \ell_1) = \sum_{j=0}^{\infty} A_3(j)x^j,
\]

where

\[
A_3(j) = \sum_{\ell_1=0}^{j} \sum_{\ell_2=0}^{j - \ell_1} \sum_{\ell_3=0}^{j - \ell_1 - \ell_2} h^{\ell_2} k^{\ell_3} a(\ell_1) a(\ell_2) a(\ell_3) a(j - \ell_1 - \ell_2 - \ell_3) .
\]
Continuing this process until all the series have been multiplied out, we eventually arrive at
\[
\left( \sum_{j=0}^{\infty} a(j) x^j \right)^m \sum_{\mu=0}^{\infty} a(\mu) (h' x)^\mu \sum_{\nu=0}^{\infty} a(\nu) (k' x)^\nu = \sum_{j=0}^{\infty} A_{m+1}(j) x^j ,
\]
where the coefficients are given by
\[
A_{m+1}(j) = \sum_{\ell_1=0}^{j} \sum_{\ell_2=0}^{j-\ell_1-\ell_2} \ldots \sum_{\ell_{m+1}=0}^{j-\ell_1-\ell_2-\cdots-\ell_m} h' \ell_1 k' \ell_{m+1}^m \prod_{i=1}^{m+1} a(\ell_i) a(j-\ell_1-\ell_2-\cdots-\ell_m-\ell_{m+1}).
\]
(8.3)

If we let \( \ell_s = \sum_{i=1}^{m+1} \ell_i \) and replace the various terms in (8.3) by their values in \( f(z) \), then we find that
\[
f(z) = (\pi z)^{m-2} \sum_{j=0}^{\infty} (-1)^j (2\pi z)^{2j} \sum_{\ell_1, \ell_2, \ell_3, \ldots, \ell_{m+1}=0} h' 2\ell_{m+1}^m \prod_{i=1}^{2m+1} \frac{B_{2\ell_i}}{(2\ell_i)!} \frac{B_{2(j-\ell_s)}}{(2(j-\ell_s))!} .
\]
(8.4)

Hence there is a pole of order \( m+2 \) at \( z = 0 \). Introducing the above result into (8.2), we see that there is only a residue when \( m+1 \) is equal to one of the even powers of \( z \) inside the summation over \( j \). Therefore, \( m \) must be odd for \( f(z) \) to yield a residue. Introducing (8.4) into (8.2), with \( m \) replaced by \( 2n-1 \), where \( n \) is a positive integer, yields
\[
\text{Res } f(z) \bigg|_{z=0} = \frac{(-1)^n}{\pi} \sum_{\ell_1, \ell_2, \ldots, \ell_{2n}=0} h' 2\ell_{2n+1}^m \prod_{i=1}^{2n} \frac{B_{2\ell_i}}{(2\ell_i)!} \frac{B_{2n-2\ell_s}}{(2n-2\ell_s)!} ,
\]
where \( \ell_s = \sum_{i=1}^{2n} \ell_i \).

To obtain a finite sum over powers of the cotangent, we need to consider the entire contour around \( f(z) \). This means that there are simple poles at \( z = (j+a)/h \) and \( z = (r+b)/k \), where \( j \) and \( r \) are non-negative integers such that \( 0 < j + a < h \) and \( 0 < r + b < k \). Since this is the third case, where \( a = b = 0 \), these become \( z = j/h \) and \( z = r/k \). Moreover, because \( h = 1 \), we can disregard the poles at \( z = j \), while \( r \) ranges from \( 1 \) to \( k-1 \). In addition, by noting that \( \lim_{y \to \infty} \cot(c(x \pm iy) + d) = \pm i \), for \( c > 0 \) and \( d \) real, Berndt and Yeap are able to evaluate the contour integral directly, whereby obtaining
\[
\frac{1}{2\pi i} \int_C f(z) \, dz = \frac{(-1)^n}{\pi} .
\]

By applying Cauchy’s residue theorem, we finally arrive at (1.3).

To conclude this appendix, let us now discuss the implementation of (1.3) for \( n = 2 \). Then the \( j_i \) range from \( j_1 \) to \( j_4 \). For \( n = j_s \) to be non-zero, we require some of the \( j_i \) to be zero, whereas according to the Berndt-Yeap result given by (1.2), they should be greater than zero. For \( n = 2 \), \( j_0 \) can be equal to 0, 1, or 2. When \( j_0 = 2 \), all the other \( j_i \)'s must vanish and the sum in (1.3) contributes the value \( -(1)^2(2^4)k^3B_4/4! \), which in turn equals \( -k^3/45 \) since \( B_4 = -1/30 \). When \( j_0 = 1 \), either the remaining \( j_i \) equal unity or \( 2-j_s \) equals unity. Hence there are four possibilities, each yielding the same contribution. The total contribution for \( j_0 = 1 \) becomes \( 4k((-1)(2^3)B_2/2!)^2 \) or \( 4k/9 \) since \( B_2 = 1/6 \).
When $j_0 = 0$, we have two separate cases. In the first of these cases either one of the remaining $j_i$ or $2 - j_s$ equals 2. Since there are four possibilities, we obtain a contribution of $4 \cdot 2^4B_4/(4!k)$ for this case. For the second case one of the remaining $j_i$ or $2 - j_s$ is equal to unity and another one must also be equal to unity. Since there are effectively four variables including $2 - j_s$, this means there are $\binom{4}{2} = 6$ combinations. Therefore, the contribution from the second case is $6(-2^2B_2/2)^2/k$. Combining the two cases yields the total contribution for $j_0 = 0$, which is $(-4/45 + 2/3)k^{-1}$. Thus, (1.3) for $n = 2$ gives

$$\frac{1}{k} \sum_{r=1}^{k-1} \cot^4\left(\frac{\pi r}{k}\right) = 1 - \left(-k^3/45 + 4k/9 + 26/45k\right).$$

After a little algebra, one eventually obtains

$$\sum_{r=1}^{k-1} \cot^4\left(\frac{\pi r}{k}\right) = \frac{1}{45} (k-1)(k-2)(k^2 + 3k - 13),$$

which appears as Corollary 2.6a in Berndt and Yeap [3]. In a similar fashion one can calculate the results for $n = 3$ and $n = 4$, the details of which are not presented here. After a little algebra, one finds that

$$\sum_{r=1}^{k-1} \cot^6\left(\frac{\pi r}{k}\right) = \frac{1}{945} (k-1)(k-2)(2k^4 + 6k^3 - 28k^2 - 96k + 251),$$

and

$$\sum_{r=1}^{k-1} \cot^8\left(\frac{\pi r}{k}\right) = \frac{1}{14175} (k-1)(k-2)(3k^6 + 9k^5 - 59k^4 - 195k^3 + 457k^2 + 1761k - 3551).$$

By using a different method, Gessel has obtained (8.5), which, aside from a phase factor, appears as $q_6(n)$ in [17]. It should also be mentioned that beyond $n = 4$, the calculations become cumbersome due to the rapidly increasing number of combinations when the $j_i$ are summed to $n$. For these values of $n$, a computer program will be needed to evaluate (1.3).

9. Appendix B

In this appendix we consider multiplying and dividing the argument in the trigonometric powers of the sums $C(m, n)$ and $S(m, n)$ by 5 or what is referred to as the $\ell = 5$ case according to the terminology of Section 3. In so doing, the material presented here should enable the reader to consider other values of $\ell$, although we shall see that higher values of $\ell$ are not as tractable as the cases studied in Section 3.

To investigate the $\ell = 5$ case, we require the following general identity:

$$\sum_{j=1}^{\ell} e^{2\pi ijk/\ell} = \begin{cases} \ell, & k \equiv 0 \pmod{\ell}, \\ 0, & \text{otherwise}. \end{cases}$$

Multiplying and dividing the argument in the cosine power of $C(m, n)$ as defined in Section 4 by 5, we obtain

$$C(m, n) = \sum_{k=0,5,10,...}^{5n-5} \cos^{2m}\left(\frac{k\pi}{5n}\right).$$
Next we put $\ell = 5$ in (9.1) and introduce it into the above equation. After a little algebra, we arrive at

\begin{equation}
C(m, n) = \frac{1}{5} \sum_{k=0}^{5n-1} \left( 2 \cos\left(\frac{2\pi k}{5}\right) + 2 \cos\left(\frac{4\pi k}{5}\right) + 1 \right) \cos^{2m}\left(\frac{k\pi}{5n}\right).
\end{equation}

The last sum on the rhs of (9.2) is $C(m, 5n)$. Hence we are left with two distinct sums. To isolate these sums, we need to consider an even multiple of 5, e.g., $\ell = 10$, since we observed that the basic trigonometric sums in Section 4 turned out to be reducible when $\ell$ was even.

By multiplying and dividing the argument of the cosine power in $C(m, n)$ by 10, we find that

\begin{equation}
C(m, n) = \sum_{k=0,10,20,\ldots}^{10n-10} \cos^{2m}\left(\frac{k\pi}{10n}\right).
\end{equation}

Now we introduce the $\ell = 10$ version of (9.1) into the above result, which after a little algebra yields

\begin{equation}
C(m, n) = \frac{1}{10} \sum_{k=0}^{10n-10} \left( 2 \cos\left(\frac{\pi k}{2}\right) + 2 \cos\left(\frac{2\pi k}{5}\right) + 2 \cos\left(\frac{3\pi k}{5}\right) \\
+ 2 \cos\left(\frac{4\pi k}{5}\right) + 1 + (-1)^k \right) \cos^{2m}\left(\frac{k\pi}{10n}\right).
\end{equation}

The above result can be simplified by introducing the trigonometric identity for the sum of two cosines, which is given as No. 1.314(3) in [16]. In this instance we sum the first and fourth cosines on the rhs and then the second and third cosines. Then we obtain

\begin{equation}
10 C(m, n) = \sum_{k=0}^{10n-10} \left( 4 \cos\left(\frac{\pi k}{2}\right) \cos\left(\frac{3\pi k}{10}\right) + 4 \cos\left(\frac{\pi k}{2}\right) \cos\left(\frac{\pi k}{10}\right) \\
+ 1 + (-1)^k \right) \cos^{2m}\left(\frac{k\pi}{10n}\right).
\end{equation}

In (9.3) all terms with odd values of $k$ vanish, so we can replace $k$ by $2k$, which leads to

\begin{equation}
10 C(m, n) - 2 C(m, 5n) = \sum_{k=0}^{5n-1} (-1)^k \left( 4 \cos\left(\frac{3\pi k}{5}\right) + 4 \cos\left(\frac{\pi k}{5}\right) \right) \cos^{2m}\left(\frac{k\pi}{5n}\right).
\end{equation}

Alternatively, the above result can be written as

\begin{equation}
10 C(m, n) - 2 C(m, 5n) = \sum_{k=0}^{5n-1} \left( 4 \cos\left(\frac{2\pi k}{5}\right) + 4 \cos\left(\frac{4\pi k}{5}\right) \right) \cos^{2m}\left(\frac{k\pi}{5n}\right).
\end{equation}

This, however, is twice (9.2). Therefore, the $\ell = 10$ case reduces to the $\ell = 5$ case, just as we observed in the $\ell = 3$ and $\ell = 6$ cases. Worse still, the two series involving $\cos(2\pi k/5)$ and $\cos(4\pi k/5)$ cannot be decoupled. That is, extra information is required before each of these series can be evaluated separately. However, we can express (9.2) as

\begin{equation}
C(m, n) = \frac{1}{5} \sum_{k=0}^{5n-1} \left( 2 \cos\left(\frac{2\pi k}{5}\right) + 2 \cos\left(\frac{6\pi k}{5}\right) + 1 \right) \cos^{2m}\left(\frac{k\pi}{5n}\right).
\end{equation}
By applying the identity for the sum of two cosines, we finally arrive at the following basic cosine power sum:

\[(9.6) \sum_{k=0}^{5n-1} \cos \left(\frac{2\pi k}{5}\right) \cos \left(\frac{4\pi k}{5}\right) \cos^2 \left(\frac{k\pi}{5n}\right) = \frac{1}{4} \left(5C(m, n) - C(m, 5n)\right) .\]

In actual fact the above result is not very surprising because the product of the cosines external to the cosine power is given by

\[
\cos \left(\frac{2\pi k}{5}\right) \cos \left(\frac{4\pi k}{5}\right) = \begin{cases} 1, & k \equiv 0 \pmod{5} \\ -1/4, & \text{otherwise}. \end{cases}
\]

It is also interesting to note that we cannot use the alternating version of \(C(m, n)\) to decouple the sums in (9.5). From (3.3) we have

\[n-1 \sum_{k=0}^{5n-1} (-1)^k \cos^2 \left(\frac{5k\pi}{5n}\right) = \sum_{k=0,5,...}^{5n-5} \cos \left(\frac{\pi k}{5}\right) \cos^2 \left(\frac{k\pi}{5n}\right) = 2C(m, n/2) - C(m, n) ,\]

where \(n\) can only be even. By following (9.2) we can express the above result as

\[\frac{1}{5} \sum_{k=0}^{5n-1} \cos \left(\frac{\pi k}{5}\right) \left(2 \cos \left(\frac{2\pi k}{5}\right) + 2 \cos \left(\frac{4\pi k}{5}\right) + 1\right) \cos^2 \left(\frac{k\pi}{5n}\right) = 2C(m, n/2) - C(m, n) .\]

After a little algebra we arrive at

\[\sum_{k=0}^{5n-1} \left(2 \cos \left(\frac{3\pi k}{5}\right) + 2 \cos \left(\frac{\pi k}{5}\right)\right) \cos^2 \left(\frac{k\pi}{5n}\right) = 10C(m, n/2) - 2C(m, 5n/2) + C(m, 5n) - 5C(m, n) .\]

Once again, we are unable to decouple the component sums. In fact, as one goes to higher primes, there will be more component sums appearing in the final result, which makes the task of isolating them on their own even more difficult to accomplish. Nevertheless, we can combine the cosines on the lhs, thereby obtaining

\[\sum_{k=0}^{5n-1} \cos \left(\frac{\pi k}{5}\right) \cos \left(\frac{2\pi k}{5}\right) \cos^2 \left(\frac{k\pi}{5n}\right) = \frac{1}{4} \left(10C(m, n/2) - 2C(m, 5n/2) + C(m, 5n) - 5C(m, n)\right) .\]

Comparing the above result with (9.6), we see that they are the alternating versions of one another.

We can, however, derive a result for the first sum on the rhs of (9.4), although it may not be regarded as very elegant. First, we express the sum as

\[(9.7) \sum_{k=0}^{5n-1} \cos \left(\frac{2\pi k}{5}\right) \cos^2 \left(\frac{k\pi}{5n}\right) = \sum_{k=0}^{5n-1} \cos \left(\frac{2k\pi}{5n}\right) \cos^2 \left(\frac{k\pi}{5n}\right) .\]

From [27, No. 1.1.10], we have

\[(9.8) \cos \left(\frac{2nk\pi}{5n}\right) = 2^{n-1} \cos^2 \left(\frac{k\pi}{5n}\right) + n \sum_{j=0}^{n-1} \frac{(-1)^j}{j+1} \binom{2n-j-2}{j} 2^{2n-2j-2} \cos^2 \left(\frac{k\pi}{5n}\right) .\]
Introducing (9.8) into (9.7) yields
\[
\sum_{k=0}^{5n-1} \cos \left( \frac{2k\pi}{5} \right) \cos^{2m} \left( \frac{k\pi}{5n} \right) = \sum_{k=0}^{5n-1} \left( 2^{2n-1} \cos^{2m+2n} \left( \frac{k\pi}{5n} \right) + n \sum_{j=0}^{n-1} \frac{(-1)^{j+1}}{j+1} 2^{2n-2j-2} \times \binom{2n-j-2}{j} \cos^{2m+2n-2j-2} \left( \frac{k\pi}{5n} \right) \right).
\]

Recognizing that the sum over \( k \) is the basic cosine power sum defined in Section 2, we finally arrive at
\[
\sum_{k=0}^{5n-1} \cos \left( \frac{2k\pi}{5} \right) \cos^{2m} \left( \frac{k\pi}{5n} \right) = 2^{2n-1} C(m+n, 5n) + n \sum_{j=0}^{n-1} \frac{(-1)^{j+1}}{j+1} 2^{2n-2j-2} \times \binom{2n-j-2}{j} C(m+n-j-1, 5n).
\]

(9.9)

The other sum involving \( \cos(4k\pi/5) \) instead of \( \cos(2k\pi/5) \) can be directly obtained from (9.4). Although (9.9) is cumbersome, it does nevertheless demonstrate that the basic cosine power sum given above is combinatorial in nature or rational as a consequence of Theorem 2.1.

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