The Constant Mean Curvature Einstein flow and the Bel-Robinson energy

Martin Reiris

Math. Dep. Massachusetts Institute of Technology

We give an extensive treatment of the Constant Mean Curvature (CMC) Einstein flow from the point of view of the Bel-Robinson energies. The article, in particular, stresses on estimates showing how the Bel-Robinson energies and the volume of the evolving states control intrinsically the flow along evolution. The treatment is for flows over compact three-manifolds of arbitrary topological type, although the form of the estimates may vary depending on the Yamabe invariant of the manifold. We end up showing well posedness of the CMC Einstein flow with $H^3 \times H^2$ regularity, and proving a criteria for a flow to be a long-time flow on manifolds with non-positive Yamabe invariant in terms only of the first order Bel-Robinson energy.

Contents

1 Introduction. 2

2 Background. 7

2.1 The (vacuum) Einstein flow. 7

2.2 The CMC gauge. 8

2.3 The CMC flow Equations in a general material setting. 11

2.4 Manifolds and Sobolev spaces. 12

2.5 The Bel-Robinson energy and the space-time curvature. 17

2.6 Scaling and Cosmological scaling. 20

3 The CMC flow, the volume and the BR-energy. 20

3.1 The reduced volume. 21

3.2 The Friedman-Lemaître equations and a cosmological interpretation of the reduced volume monotonicity. 22

3.3 Controlling the states $(g, K)$ at a given time. 24

3.3.1 Intrinsic estimates. 24

3.3.2 Estimates of elliptic type. 30

3.4 Controlling the flow $(g, K)$ along evolution. 47

4 Applications. 62

4.1 The initial value formulation in the CMC gauge. 62

4.2 Long time flows. 68

Bibliography 69

1\textsuperscript{e-mail: reiris@math.mit.edu.}

1
1 Introduction.

The aim of this article is to provide an intrinsic treatment of the Constant Mean Curvature (or simply CMC) gauge, entirely in terms of the space-time Bel-Robinson energy and the space-like volume. Roughly speaking, the CMC gauge foliates the space-time in such a way that, at every leaf, the local rate of (normal) volume expansion is constant (i.e. independent of the point). The space-like volume, in turn, is linked to the space-like scalar curvature and (though it) to the Yamabe invariant of the three-manifold. As it turns out, the link between the space-like volume and the spatial geometry is strengthened if the Bel-Robinson energy associated with the Riemann tensor of the space-time is taken into the picture. We will display precise estimates showing how the volume and the Bel-Robinson energies control the geometry of the states. This, in particular, makes the Bel-Robinson energies and the space-like volume a set of appealing variables to control the CMC flow along evolution. We will give a detailed discussion of intrinsic as well as elliptic estimates.

The CMC gauge and Bel-Robinson energies have been used together several times in the past. In [13], for instance, Christodoulou and Klainerman approached the stability of the Minkowski space-time using a maximal foliation and a elaborated control of the Bel-Robinson energies of appropriate Weyl fields. In their work, Weyl fields were used in particular as the fundamental variables from which to reconstruct the space-time metric. On the other hand, in [3], Andersson and Moncrief gave a proof of the stability of flat cones following essentially the same argumental lines. This case, which can be considered as a compact version of [13], is however greatly simplified thanks to the expansion in volume (and the compactness of the CMC Cauchy slices). In the context of the initial value formulation of the Einstein theory, Weyl fields were used by Friedrich in [19]. Friedrich included the conformal Weyl tensor of the space-time as a variable, and by doing so he obtained different hyperbolic reductions of the Einstein equations from which to launch initial value formulations. We will take, in spirit, several elements from these works. Namely, we will study how to control the space-time metric from the Bel-Robinson energies (of suitable Weyl fields) and use that knowledge to give a treatment of the Cauchy problem in General Relativity entirely inside the framework of Weyl fields.

To step further in the description of the contents, let us introduce some terminology first and then state, to exemplify, some of the main estimates that will proved. The reader can consult the background section for a detailed account on notions such as Bel-Robinson energy, or harmonic radius that we will mention below. Consider a cosmological solution \((\mathcal{M}, g)\) and a Cauchy slice \(\Sigma\). Denote by \(g\) the spatial three-metric and by \(K\) the second fundamental form of \(\Sigma\). As is well known \(K = -\frac{1}{2}g\) where \(g\) is the time derivative of \(g\) in the normal direction to \(\Sigma\). Thus it will be natural to call the pair \((g, K) = (g, -\frac{1}{2}g)\) a state. A state is in particular a CMC state if the constant mean curvature \(k = \text{tr}_g K\) is constant on \(\Sigma\). Consider the space-time Riemann tensor \(R_m\). If the space-time solution \(g\) is a vacuum solution the curvature \(R_m\) is a Weyl field as are its covariant derivatives.
\[ \nabla^j T \text{ in the normal direction } T \text{ to } \Sigma. \] Consider their associated Bel-Robinson energies and denote them by \( Q_j \). One of the main results will be to show

**Lemma 1** (Sobolev norms vs. Bel-Robinson norms) Say \( \bar{I} \geq 0 \) and say \( \Sigma \) is a compact three-manifold. Then the functional (defined over states \((g, K)\) with \( k = -3\))

\[
\|(g, K)\|_{BR} = \frac{1}{\nu} + V + \sum_{j=0}^{\bar{I}} Q_j = \frac{1}{\nu} + V + \mathcal{E}_I,
\]

controls the \( H^{I+2} \)-harmonic radius \( r_{I+2} \) and the \( H^{I+1}_g \)-norm of \( K \).

Above, \( V = (-k)^3 \text{Vol}_g(\Sigma) \) is the so called reduced volume (see later), which is essentially the usual volume if \( k \) was fixed and equal to \(-3\). \( \nu \) is the so called volume radius which is a measure of the local “flatness” (see later). \( H^{I+1}_g \) is the \( \bar{I} + 1 \)-Sobolev space and the subindex \( g \) indicates that the Sobolev-norm is the natural constructed out of the metric \( g \) (see the background section). The proof of this Lemma emerges basically as a corollary of a series of partial results which have, however, independent value. Consider for instance the case \( \bar{I} = 0 \) in Lemma 1. We show in Proposition 6 that the \( H^1_g \)-norm of \( K \), namely,

\[
\|K\|_{H^1_g}^2 = \int_{\Sigma} \left| \nabla K \right|^2 + \left| \hat{K} \right|^2 dv_g,
\]

is controlled only by \( V \) and \( Q_0 \). This will follow from the explicit intrinsic estimate

\[
\int_{\Sigma} 2|\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \leq C(|k|V + Q_0),
\]

in the case the Yamabe invariant \( Y(\Sigma) \) is \( Y(\Sigma) > 0 \), and

\[
\int_{\Sigma} 2|\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \leq C(|k|V - \mathcal{V}_{inf} + Q_0),
\]

if \( Y(\Sigma) \leq 0 \), where \( \hat{K} \) is the traceless part of \( K \), \( C \) is a numeric constant and \( \mathcal{V}_{inf} \) is the infimum of the reduced volume \( V \) among all CMC states \((g, K)\). It turns out that \( \mathcal{V}_{inf} = (-\frac{1}{6}Y(\Sigma))^\frac{1}{2} \) (see the background section). Being formal, an intrinsic estimate is an inequality (or equality) involving intrinsic Sobolev norms \( H^s_g \) but not requiring the volume radius \( \nu \). The intrinsic estimates on \( K \) above, imply, as we prove in Proposition 8, an intrinsic estimate on the \( L^2_g \)-norm of the Ricci curvature of the three-metric \( g \). Namely we show

\[
\|\text{Ric}\|_{L^2_g}^2 \leq C(|k|V + Q_0).
\]

It follows from the fundamental theorem of convergence of Riemannian manifolds (see the background section) that a priori control on \( V \) and \( Q_0 \), and in addition a priori control on \( \nu \), implies control on the \( H^2 \)-harmonic radius \( r_2 \). This is the way Lemma 1 is proved when \( \bar{I} = 0 \). The situation when \( \bar{I} > 0 \) requires the use of elliptic estimates and therefore a priori control on \( \nu \). For instance when \( \bar{I} = 1 \), it is true that
\( \mathcal{V}, \nu \) and \( \mathcal{E}_1 \) control \( \|K\|_{H^2} \) but through an inequality that involves implicitly the use of \( \nu \), and which arises through the use of suitable elliptic estimates. To handle elliptic estimates when the background metric \( g \) is one of the variables, we use Sobolev norms defined with respect to atlas composed of harmonic coordinates. Being more precise, we define \( H^1 \)-canonic harmonic atlas, and measure Sobolev norms with respect to them. A chart \( \{x_\alpha\} \) in a \( H^1 \)-canonic harmonic atlas, has the property that if we scale \( g \) as \( \tilde{g} = \frac{1}{r^2} g \) and \( \{x_\alpha\} \) as \( \tilde{x}_\alpha = \frac{1}{r} x_\alpha \) then \( \tilde{x}_\alpha \) gets defined over a ball of radius one in the metric \( \tilde{g} \) and moreover the usual \( H^1_{\tilde{g}}\) -norm of \( \tilde{g} \) is bounded above by a fixed (but arbitrary) constant. Supplied with some additional technical requirements, \( H^1 \)-canonic harmonic atlas provide us with a more or less standardized way to define Sobolev norms with respect to atlas made of harmonic coordinates. We will denote by \( H^\star_A \) Sobolev spaces defined with respect to an atlas \( A \) and in particular with respect to a canonic harmonic atlas \( A \). Once an estimate, or a certain inequality is proved using the Sobolev spaces \( H^\star_A \), one can guarantee that a similar inequality or estimate holds with respect to the intrinsic norms \( H^\star_{g} \), but the given estimate or inequality involves necessarily \( \nu \). For instance, in Proposition 14 we prove that \( \nu, V \) and \( Q_0 \) control the \( H^2_{g} \)-norm of \( 1/N \) where \( N \) is the lapse function in the CMC gauge when one take \( k \) as a choice of time. This (not self-evident) result, which in particular implies that \( N \) is never zero (even for states \((g,K)\) with \( H^2 \times H^1 \) regularity) and controlled from below by \( \nu, V \) and \( Q_0 \), requires the intermediate use of the norms \( H^\star_A \), where \( A \) is a \( H^2 \)-canonic harmonic atlas.

Lemma 1 naturally points towards a proof of well-posedness of the CMC Einstein flow, entirely in terms of the intrinsic quantities \( \nu, V \) and \( \mathcal{E}_1 \). In this respect we will prove the following version of well-posedness of the Einstein CMC-flow.

**Theorem 1** Let \((\Sigma, A)\) be a \( H^1 \)-three-manifold and say \((g_0,K_0)\) is an initial state in \( H^3 \times H^2 \) with \( k_0 < 0 \). Then

1. There is a unique \( H^3 \)-flow solution over an interval \( I = (k_{-1}, k_1) \) with \(-\infty \leq k_{-1} < k_0 < k_1 \leq 0 \). Moreover the size \( \inf \{|k_{-1} - k_0|, |k_0 - k_1|\} \) of the time interval on which the solution is guaranteed to exist is controlled from below by \( 1/\nu(k_0), \ln|k_0|, V(g_0,K_0) \) and \( \mathcal{E}_1(k_0) \).

2. There is continuity with respect to the initial conditions if we measure the space of initial conditions with the \( H^3_A \times H^2_A \) norm and the space of flow solutions with the BR-norm.

3. Because of item 1 above, we have the following continuity principle: a flow solution \(((g,K),(N,X))(k)\) is defined until past of \( k^* < 0 \) (or before \( k_* < 0 \) if the flow is running in the past direction) iff \( \limsup_{k \rightarrow k^*} 1/\nu + V(k) + \mathcal{E}_1(k) < \infty \).

The BR-norm (Bel-Robinson norm) of a flow in item 2 of the Theorem above is
The space of continuous functions over an interval \( I \) with values in the \( H^j_\Lambda \)-Sobolev space of tensors of a fixed rank. In particular in [12] Choquet-Bruhat and York gave an exposition of the Cauchy problem applicable to the CMC gauge and for initial states with \( H^3 \times H^2 \) regularity. In [5], Andersson and Moncrief considered the CMC gauge over hyperbolic-three manifolds, in the spatially \( H^2 \)-harmonic radius and therefore (see the background section) the \( H^j_\Lambda \)-norm of \( g \) where \( A_h \) is a \( H^3 \)-canonic harmonic atlas. As \( \|g\|_{H^2_\Lambda} \) controls the \( r_3 \)-harmonic radius and therefore (see the background section) the \( H^j_\Lambda \)-norm \( g \) is given from the manifold \( (\Sigma, A) \). The space \( C(I, j)(H\Lambda) \) is defined as usual as the space of continuous functions over an interval \( I \) with values in the \( H^j_\Lambda \)-Sobolev space of tensors of a fixed rank.

Consider a \( C^\infty \) CMC initial state \((g_0, K_0)\) with \( k_0 \neq 0 \) over a three-manifold that we label as \( \Sigma_0 \). It is well known the state \((g, k)\) gives rise to a \( C^\infty \) space-time \((M, g)\). By the method of barriers it is shown in the background section that there is a unique \( C^\infty \) CMC foliation of \( M \) around the slice \( \Sigma_0 \) where \( k \) varies strictly monotonically. Thus, for \( C^\infty \) space times arising from initial \( C^\infty \) CMC states, the CMC gauge is well defined at least on a neighborhood of the initial slice. The strategy we will follow to prove Theorem 1 is to show that the completion of the space of \( C^\infty \) CMC flow solutions under the BR-norm (1) is precisely the space of \( H^3 \)-CMC flow solutions (see Definition 1 in Section 3.4). Note that this is not the traditional approach to prove well-posedness of a PDE but it illustrates the applicability of the estimates in Section 3.

There is an underlying reason of why to describe the evolution in terms of the intrinsic quantities \( \mathcal{V}, \nu, \mathcal{E}_1 \) and this has to do with the continuity principle as in item 3 of Theorem 1. A common strategy to analyze the long-time evolution of

\[
\|g\|_{BR} = \|g\|_{C^0(I, 2)(H\Lambda)} + \|K\|_{C^0(I, 1)(H\Lambda)} + \sum_{k=0}^{k=1} \|(E_k, B_k)\|_{C^0(I, 0)(H\Lambda)},
\]

where \((E_0, B_0)\) and \((E_1, B_1)\) are the electric and magnetic components of the Weyl tensors \( W_0 = Rm \) and \( W_1 \) respectively. The atlas \( \mathcal{A} \) used in (1) is the atlas that is given from the manifold \( (\Sigma, A) \). The space \( C(I, j)(H\Lambda) \) is defined as usual as the space of continuous functions over an interval \( I \) with values in the \( H^j_\Lambda \)-Sobolev space of tensors of a fixed rank.

Note from (1) that the flow of three-metrics \( g(k) \), is measured in \( H^2_\Lambda \) and not in \( H^3_\Lambda \). In our approach we cannot guarantee that the metrics \( g(k) \) will lie in \( H^2_\Lambda \) (see however the claims in [12]). To circumvent this problem we included the terms \((E_0, B_0)\) and \((E_1, B_1)\) in (1). Indeed, by Lemma 1 \( \frac{1}{\rho} + \mathcal{V} + Q_0 + Q_1 \) controls the \( r_3 \)-harmonic radius and therefore (see the background section) the \( H^2_\Lambda \)-norm of \( g \) where \( A_h \) is a \( H^3 \)-canonic harmonic atlas. As \( \|g\|_{H^2_\Lambda} \) controls \( \frac{1}{\rho} \) and \( \mathcal{V} \) (when \(|k|\) is bounded) it follows that the “norm” \( \|g\|_{H^2_\Lambda} + \|(E_0, B_0)\|_{L^2_\Lambda} + \|(E_1, B_1)\|_{L^2_\Lambda} \) controls the norm \( \|g\|_{H^2_\Lambda} \). To transform this fact into a technical tool we introduce in Section 3.4 a dynamical \( H^3 \)-harmonic atlas \( \mathcal{A}(k) \). The BR-norm acquires then special relevance when \( \mathcal{A}(k) \) is taken into account. The dynamical atlas will be a fundamental piece in all our treatment of the Einstein CMC flow.

We would like to thank the referee for pointing out this reference to us.

This fact is well known (see the discussion in [22]).
flows is to design a suitable continuity principle in terms of appropriate “continuity” variables that could “describe" the flow when at least one of the variables breaks down. In other words, and being slightly ambiguous, if one defines the notion of singularity as equivalent to the blow up of the “continuity” variables, then no extra information other than the one provided by the variables themselves, would be required to analyze the structure of the singularities. In still vague but heuristic terms, one would like to have a “complete set of continuity variables”. The right choice of the “continuity” variables is thus of central importance. Indeed this is one of the major obstacles to understand the long-time evolution of the Einstein flow. Although it is unlikely that the quantities \( \nu, V \) and \( E_t \) are best adapted for this purpose, we believe they conform a interesting set, deserving to be considered with care and in depth. A related problem of considerably technical as well as conceptual difficulty, is whether \( E_0 = Q_0 \) can be used instead of \( E_1 = Q_0 + Q_1 \) in Theorem 1. This problem is known (with variations in the formulation) as the \( H^2 \)-conjecture. Its difficulty lies however well beyond the scope and the techniques developed in this article. Still some of the estimates here developed may be of tangential interest in this goal.

We require, as part of the definition of flow solution (Definition 4 in Section 3.4), that a \((H^3)\) flow \((g, N, X)\) (of three-metrics, lapse and shift) is in fact the flow induced by a CMC foliation on a \((H^3)\) space-time \((M, g)\) (see Definition 3 in Section 3.4). It follows from Theorem 1 that an initial data \((g_0, K_0)\) in \(H^3 \times H^2\) gives rise to a unique flow solution in a given shift \(X\) or, equivalently, a unique space-time \((M, g)\) up to space-time diffeomorphisms. We note that the equivalence between flow and space-time solutions is not always imposed as a requirement in some treatments of the initial value formulation in General Relativity.

As the variables \(\nu, V\) and \(Q_0\) are intrinsic to the space-time, we can see that Theorem 1 is in fact independent of which shift \(X\) we take. Indeed, our approach to the Einstein flow is independent of the shift. For this reason we have introduced a general notion of admissible shift, made up (in some sense) out of the minimum requirements that a function \(X(g, K)\) must satisfy to be the shift of an Einstein CMC flow. We may well take \(X = 0\) (which is an admissible shift) all though the article.

Let us give now an overview of the contents of the article. In the background section we provide a detailed account on the notions of: Einstein flows (Section 2.1), the CMC gauge (Section 2.2), Weyl fields and Bel-Robinson energies (Section 2.5) and scaling (Section 2.6). Although most of the article is based on standard functional analysis, we introduce some new terminology (as canonic atlas) that needs to be presented with care. This is done in Section 2.4 Section 3 contains the core of the article. In Sections 3.1 and 3.2 we discuss the notion of reduced volume and give a cosmological interpretation of its monotonicity. The fact that the reduced volume is monotonically decreasing in the expanding direction has

\[4\text{Note that “describe” in GR means “describe the space-time” an not only the flow. This fact makes GR a particularly difficult theory to treat only from the point of view of dynamical flows.}\]
important consequences for flows over three-manifolds $\Sigma$ with non-positive Yamabe invariant. One can read an easy implication of the monotonicity in item 3 of Theorem 1. In fact, because $V$ is decreasing, it remains bounded to the future, and therefore we can dispense with it in the continuity criteria. In Section 3.3 we show how the variables $\nu$, $V$ and $E_I$ control CMC states $(g, K)$. To clarify the difference between intrinsic estimates and estimates of elliptic type, we have divided the section into two subsections, Sections 3.3.1 and Section 3.3.2. In Section 3.3.1 we introduce the intrinsic estimates. These estimates are in terms of $V$ and $Q_0$ only. In Section 3.3.2 we introduce the estimates of elliptic type. This necessitates a laborious treatment of the currents $J(W_i)$ associated to the Weyl fields $W_i$. It ends up with a proof of Lemma 1. Section 3.4 is essentially an extrapolation of the analysis done in Section 3.3 for single states $(g, K)$ but now for flows $(g, K)(k)$. We introduce a BR-norm and show how it controls the flow and the space-time. In Section 3.4 too, we introduce the notion of a dynamical harmonic atlas. We spend much of Section 3.4 analyzing this concept. The dynamical harmonic atlas is the necessary ingredient to show that flow solutions are actually space-time solutions. Section 4.1 is devoted to the initial value formulation. All the treatment is entirely in terms of Bel-Robinson energies and Weyl fields. Finally, we present in Section 4.2 some partial results on long time CMC flows over manifolds with non-positive Yamabe invariant. A long time CMC flow is one for which the range of $k$ contains an interval of the form $(k_0, 0)$. What we prove is the following

**Theorem 2** Say $Y(\Sigma) \leq 0$ and $(M \sim \Sigma \times \mathbb{R}, g)$ a smooth $C^\infty$ maximally globally hyperbolic space-time. Say $(g, K)(k)$ with $k \in [a, b)$ is a CMC flow where $b$ is the lim sup of the range of $k$ and say $E_1 \leq \Lambda$. Then if $(M, g)$ is future geodesically complete the CMC flow is a long time flow i.e. $b = 0$.

Except in a few exceptions, all the article deals with the vacuum Einstein theory. In those cases where matter is considered, we will do so assuming that the energy-momentum tensor satisfies the dominant energy condition.

## 2 Background.

### 2.1 The (vacuum) Einstein flow.

All manifolds and tensors in this section and the next (Section 2.2) are considered to be smooth i.e. $C^\infty$. We will denote space-times (i.e. formally a four manifold $M$ and a Lorentz $(-+++)$ metric $g$) by $(M, g)$. All through, space-time tensors will be boldfaced. We will think the Einstein equations in vacuum

$$\text{Ric} - \frac{1}{2}Rg = 0, \text{ or } \text{Ric} = 0,$$

\[5\]It is conjectured that any flow over a manifold $\Sigma$ with non-positive Yamabe invariant is in effect a long time flow \cite{23} (see also \cite{24} and references therein).
not as global equations on \( (a \text{ given}) \ M \) but rather as evolution equations on manifolds of the form \( M \sim \Sigma \times \mathbb{R} \), where \( \Sigma \) is an orientable and compact three-manifold.

Say \( M \) is \( 3+1 \)-split by the diffeomorphism \( \phi : \Sigma \times \mathbb{R} \rightarrow M \), in a way that \( \phi^*(g)\mid_{\Sigma \times \{t\}} = g(t) \) is Riemannian for every \( t \) \((t : M = \Sigma \times \mathbb{R} \rightarrow \mathbb{R} \) is the time function coming from projecting into the second factor). Writing \( \partial_t = NT + X \) with \( T \) a normal to the time foliation and \( X \) tangential to it, the space-time metric splits into \( \Sigma \) and \( \partial_t \) components as

\[
(2) \quad \phi^* g = -(N^2 - |X|^2)dt^2 + dt \otimes X^* + X^* \otimes dt + g.
\]

In the formula above \( g \) was extended to \( M \) to be zero along \( \partial_t \) (i.e. zero when one of the entrances is \( \partial_t \) and \( X^*_a = X^b g_{ab} \). \( N \) is called the lapse and \( X \) the shift vector. As seen in a 3+1-splitting, a space-time metric \( g \) is characterized uniquely by a flow (a path) of three-metrics \( g(t) \) and lapse-shift \( (N, X)(t) \) on a fixed manifold \( \Sigma \). We will denote \( g_\phi(t) = ((g), (N, X))(t) \) and call it the Einstein flow in the \( \phi \)-splitting.

It is convenient to include the second fundamental form of the time foliation as part of the definition of the Einstein flow, thus \( g_\phi = ((g), (K), (N, X))(t) \). The Einstein flow equations in vacuum are

\[
(3) \quad R = |K|^2 - k^2,
\]

\[
(4) \quad \nabla K - dk = 0,
\]

\[
(5) \quad \dot{g} = -2NK + \mathcal{L}_X g,
\]

\[
(6) \quad \dot{K} = -\nabla \nabla N + N(Ric + kK - 2KK) + \mathcal{L}_X K.
\]

Equations (3) and (4) are the constraint equations and equations (5, 6) the Hamilton-Jacobi equations of the flow. Contracting (6) and using the constraints we get the lapse equation

\[
(7) \quad -\Delta N + |K|^2 N = \dot{k} - X(k) = N \nabla_T k.
\]

This equation can be seen too as the second variation of volume at the leaves of the time foliation.

---

6Solutions in this kind of topology are called cosmological solutions. As usual, hypersurfaces diffeomorphic to \( \Sigma \) having Riemannian inherited metrics \( g \) are called Cauchy hypersurfaces.

7From a Lagrangian point of view one studies the flow \( g(t) = ((g), (N, X))(t) \) of position \( g \), velocity \( v = \dot{g} = \mathcal{L}_{\partial_t} g = -2NK + \mathcal{L}_X g \) \((K \) is the second fundamental form of the time foliation, note the sign convention) and lapse-shift \( (N, X) \) as a solution to the Euler-Lagrange equations of the Einstein-Hilbert action (see [24]) \( S = \int_{\Sigma \times I} Rdv_g + 2\int_{\Sigma \times I} kdv_g \). alternatively, from a Hamiltonian point of view one studies the evolution of the flow \( g_\phi = ((g), (N, X)), \) of position \( g \), momentum \( P = (K - kg)dv_g \) and lapse-shift \( (N, X) \) as a solution to the Hamilton-Jacobi equations of the Hamiltonian (make \( P = (K - kg) \)) \( H = \int_{\Sigma} N(|P|^2 - \frac{\dot{P}^2}{2} - R) - 2X(P, \nabla P)dv_g \). generated by the energy and momentum constraints functions \( E = |K|^2 - k^2 - R \) and \( P = -2\nabla_T (K - kg) \), which are well known to be conserved along the flow and identically zero (in vacuum).
It is well known that the constraint equations comprise the necessary and sufficient conditions on initial states \((g, K)\) to have solutions to the Cauchy problem for the Einstein flow equations. Given a solution \((M, g)\) of the Einstein equations and a diffeomorphism \(\phi\), then the flow \(g_\phi\) is a solution of the Einstein flow equations, in particular two different diffeomorphisms \(\phi_1\) and \(\phi_2\) that coincide and have same differential over an initial Cauchy hypersurface and therefore inducing the same initial states \(((g_0, K_0), (N_0, X_0))\) give rise to two different solutions \(g_{\phi_1}\) and \(g_{\phi_2}\) on \(\Sigma \times \mathbb{R}\). Their associated space-time metrics from equation (2) are however isometric. This is the only freedom, i.e. module diffeomorphisms, solutions are unique. Note that we can always adjust the diffeomorphism over an initial Cauchy hypersurface to make \(N\) and \(X\) have any prescribed values (with \(N^2 > |X|^2\)). We now state the well known theorem of well posedness of the \(C^\infty\) Cauchy problem (see for instance [18], [24] and references therein).

**Theorem 3** (Well posedness of the \((C^\infty)\) Cauchy problem) (Existence) Given and initial state \((g_0, K_0)\) satisfying the energy and momentum constraints and arbitrary initial lapse-shift \((N_0, X_0)\), there is a solution to the Einstein flow equations (3)-(6) over a short period of a parametric time. (Uniqueness) Two globally hyperbolic solutions \(g_1, g_2\) to the Einstein flow equations with the same initial state \((g_0, K_0)\) and same initial lapse and shift \((N_0, X_0)\) are isometric i.e. there exists \((M, g)\) and diffeomorphisms \(\phi_1\) and \(\phi_2\) such that \(g = \phi_1^*(g_1) = \phi_2^*(g_2)\).

The freedom in the choice of \(\phi\), i.e. the invariance of solutions under diffeomorphisms is called gauge freedom. A choice of \(\phi\) from \(g\) in such a way that if \(g_1\) is isometric to \(g_2\) then \(\phi(g_1) = \phi(g_2)\) is said a choice of gauge.

### 2.2 The CMC gauge.

A state \((g, K)\) satisfying the energy and momentum constraints with \(k = \text{tr}_g K\) constant is called a CMC state. A cosmological solution on \(\Sigma \times \mathbb{R}\) is in the CMC gauge if the extrinsic mean curvature is constant restricted to the leaves \(\Sigma \times \{t\}\) of the time foliation.

In itself the CMC gauge is only temporal, i.e. it fixes only the time foliation but does not fix the freedom by diffeomorphisms leaving the CMC foliation invariant.

Assume that a vacuum cosmological solution \((M, g)\) has a CMC Cauchy hypersurface. Then, unless the solution is flat and of the form \(g = -dt^2 + g_F\) where \(g_F\) is a flat metric over a three-manifold \(\Sigma\), it must happen: a) there is a CMC foliation in at least a small neighborhood around it and b) if two CMC Cauchy hypersurfaces have different constant mean curvatures then they are disjoint and there is a CMC foliation inside the enclosed region on which \(k\) varies monotonically (for a proof see for instance [17], [8], for a discussion see [23]).

Let us explain how property a) is proved as that is relevant for the rest of the article. The basic tool is the technique of barriers [3], it guarantees the existence of a CMC slice of constant mean curvature \(k\) between two slices with mean curvature functions \(k_1\) and \(k_2\) with \(\sup k_1 < k < \inf k_2\). This property settles the existence
of a CMC foliation around a CMC slice with \( k \neq 0 \). In fact starting at the given CMC slice \( \Sigma \) consider the foliation around it provided by the Gauss gauge with zero shift, i.e. with \( N = 1, X = 0 \). From equation (17) one gets \( \dot{k} = |K|^2 > \frac{k^2}{3} \). Thus \( k \) is strictly monotonic and we can conclude the existence of two barriers having mean curvatures \( k_1 \) and \( k_2 \) with \( \sup k_1 < k < \inf k_2 \). To show the existence of barriers around a state \( \Sigma \) with mean curvature zero proceed as follows. If \( \int_\Sigma |K|^2 dv_g \neq 0 \) at the initial slice then we can solve the lapse equation \(-\Delta N + |K|^2 N = 1\). In this case consider the normal field \( \tilde{T} = NT \) and consider the flow of geodesics starting at the slice and having velocities \( \tilde{T} \) there. This provides a gauge at least on a small time interval. The key fact is that at the initial slice we have \( \dot{k} = 1 \) and therefore \( k \) is strictly monotonic on a small time interval around the initial slice and in that gauge. This provides suitable barriers and therefore a CMC foliation around the initial CMC slice. If instead \( K = 0 \) at the initial slice consider again the Gauss gauge starting from it. Let us restrict to the future direction. Unless \( K = 0 \) over a small time interval (and in the Gauss gauge) we have from the equation \( \dot{k} = |K|^2 \) that any other slice except the initial it is \( K \neq 0 \) and \( k \geq 0 \). We can solve therefore the lapse equation at a time after the initial time and proceeding as in the previous case get a slice with \( \inf k > 0 \). Doing the same in the opposite time direction we guarantee the existence of barriers and therefore the existence of a CMC foliation around the initial state. If it happens that \( K = 0 \) on a small time interval (in the Gauss gauge) then from equation (6) we get that \( \text{Ric} = 0 \) on a small time interval and therefore the solution is flat and of the form \( g = -dt^2 + g_F \) where \( g_F \) is a flat metric over a three-manifold \( \Sigma \).

Fact a) above shows that any CMC state (with \( k \neq 0 \)) induces a (at least short time) solution of the CMC flow, while b) shows that the CMC flow is unique and the maximal range of \( k \) forms a connected open interval over which \( k \) varies monotonically. The range of \( k \) depends on the topology of \( \Sigma \) and more in particular on the Yamabe invariant \( Y(\Sigma) \) of \( \Sigma \). It is conjectured \cite{23} that if \( Y(\Sigma) \leq 0 \) then the range of \( k \) is \((-\infty, 0]\) while if \( Y(\Sigma) > 0 \) then it is \((-\infty, +\infty)\). The CMC foliation is conjectured to cover the maximal globally hyperbolic solution if \( Y(\Sigma) > 0 \) while if \( Y(\Sigma) \leq 0 \) it is conjectured to cover the maximal globally hyperbolic solution towards the past but not necessarily towards the future. In this last case the CMC foliation should avoid the singularities but nothing else than that (every point lying outside the CMC foliation should have all its future inextendible time-like geodesics extinct in a uniform interval of proper time).

Over a CMC foliation the function \( k \) is smooth except possibly at a maximal slice (i.e. a slice with \( k = 0 \)). The mean curvature \( k \) is a smooth function at a maximal slice if \( \int_\Sigma |K|^2 dv_g \neq 0 \) at the slice. If we take the mean curvature \( k \) as

\footnote{The Yamabe invariant \( Y(\Sigma) \) of a three-manifold \( \Sigma \) is defined as the supremum of the scalar curvatures of unit volume Yamabe metrics, where Yamabe metrics are metrics of constant scalar curvature minimizing the Yamabe functional. The Yamabe invariant is also called the \textit{sigma constant} \cite{15}, \cite{5}.}
time (when \( k \neq 0 \)) we get the lapse equation

\[
- \Delta N + |K|^2 N = 1.
\]

which is independent of the shift.

Finally a note about the set of CMC solutions. Not every cosmological solution \((M, g)\) admits a Cauchy hypersurface of constant mean curvature \([11]\) although there are sufficient conditions for that to be \([9]\). On the other hand every three-manifold admits so called Yamabe CMC initial states, i.e. states \((g_Y, K_Y)\) with \(R_{gY} = -6\), \(K_Y = -g\). Therefore there are CMC solutions in any \(\Sigma \times \mathbb{R}\) topology.

A CMC solution over a manifold with non-positive Yamabe invariant and with the range of \(k\) containing an interval of the form \((a, 0)\) is said a long time CMC solution (towards the future \(k\) exhausts its potential range).

2.3 The CMC flow Equations in a general material setting.

When matter is present, the Einstein equations

\[
\text{Ric} - \frac{1}{2} \text{Rg} = 8\pi G \mathbf{T},
\]

in flow form are

\[
R - |K|^2 + k^2 = 16\pi G \rho,
\]

\[
\nabla K = -8\pi G J,
\]

\[
\dot{g} = -2NK + L_X g,
\]

\[
\dot{K} = -\nabla \nabla N + N(Ric + kK - 2KK) + L_X K - N8\pi G(T - \frac{1}{2} \text{tr}_g \mathbf{T} g),
\]

where \(\mathbf{T}(T, T) = \rho\) and \(\mathbf{T}(T, .) = J\) and \(p\) is the average of the principal pressures (thus \(-\rho + 3p = \text{tr}_g \mathbf{T}\)). The last term in the right hand side (RHS) of equation \([12]\) (involving \(\mathbf{T}\)) should be restricted to \(\Sigma\). The lapse equation for the CMC time \(t = k\) is

\[
-\Delta N + (4\pi G(\rho + 3p) + |K|^2)N = 1.
\]

Some useful terminology.

We have been using informally the notion of “control”. It is convenient to use this terminology indeed, and we will appeal to it many times in the rest of the article. In an informal manner, “A” “controls” “B” if B cannot degenerate when A is not degenerate. For instance, \(\nu\), \(V\) and \(Q_0\) control the “geometry” of CMC states. Quantitatively, (and being precise), a set of quantities \(A_1, \ldots A_n\) control a quantity B if for every constant \(M\) there is \(N\) such that \(|B| \leq N\) if
$|A_1| \leq M, \ldots, |A_n| \leq M$. Another more familiar way to mean control is through the expression $|B| \leq C(A_1, \ldots, A_n)$. Still we will abuse slightly this definition and use it in a more flexible way. For instance we may say: “$\nu$, $\mathcal{V}$, $Q_0$ control $r_2$”, although properly speaking one must say: “$1/\nu$, $\mathcal{V}$ and $Q_0$ control $r_2 + 1/r_2$. The reader should find evident the meaning of each expression.

2.4 Manifolds and Sobolev spaces.

Given a subset $\Omega$ of $\mathbb{R}^n$ covered by a chart \( \{x^k\} \), we denote by $H^s_{\{x\}}(\Omega)$ the $s$-Sobolev space of distributions with derivatives in $L^2$ until order $s$, and where the derivatives are the standard partial derivatives in the coordinate system \( \{x^k\} \).

We say that the pair $(\Sigma, \mathcal{A})$ is a $H^{i+1}$-manifold if and only if $\mathcal{A}$ is an atlas covering $\Sigma$ with transition functions in $H^{i+1}$ (more precisely if \( \{x_\alpha\} \) is a chart from on a domain $\Omega$ and $x_\alpha$ is a chart from a domain $\Omega'$ then $x_\alpha'(x_{\alpha'}(s))$ are functions on $H^{i+1}(\Omega' \cap \Omega)$). We say that $(\Sigma, \mathcal{A}, g)$ is a $H^{i+1}$-Riemannian manifold if the entries $g_{ij}$ in any coordinate chart $\{x_\alpha\}$ are in $H^1_{\{x_\alpha\}}(\Omega)$ where $\Omega$ is the domain of the chart.

Let $\{B(o_\alpha, r), \alpha = 1, \ldots, m\}$ be a covering of $\Sigma$ such that every one of the balls $B(o_\alpha, r)$ is inside one of the charts in $\mathcal{A}$. Say $\{\xi_\alpha, \alpha = 1, \ldots, m\}$ is a partition of unity subordinate to the covering $\{B(o_\alpha, r)\}$ (i.e. $\xi_\alpha(p) = 1$ if $p \in B(o_\alpha, r/2)$, $\xi_\alpha(p) = 0$ if $p \notin B(o_\alpha, r)$, $\xi_\alpha \geq 0$ and $\sum_\alpha \xi_\alpha(p) = 1$ for all $p \in \Sigma$). Recall that in $\mathbb{R}^n$ the Sobolev space $H^s_{\mathbb{R}^n}$ is defined as those distributions for which the Fourier transform $\hat{u}(\xi)$ satisfies that $\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$ is finite. Thinking that the charts $\{x_\alpha\}$ are coordinates in (a subset of) $\mathbb{R}^n$, consider the tensors $U$ of rank $(l, l')$ and having entrances in $H^s$ for $s \leq i$ and $i \geq 2$. Then the Sobolev space $H^s_\mathcal{A}$ of tensors of rank $(l, l')$ is defined as the set of $U = \sum_{\alpha=1}^{\alpha=m} \xi_\alpha U_\alpha$ where $U_\alpha$ is a tensor as mentioned before. The $H^s_\mathcal{A}$ norm is defined by $\|U\|_{H^s_\mathcal{A}} = \sum_{\alpha=1}^{\alpha=m} \|\xi_\alpha U_\alpha\|_{H^s(\Sigma_{\{x_\alpha\}})}$.

As topological spaces they are independent of the atlas $\mathcal{A}$ (as long as the atlas are compatible).

The $H^i$-harmonic radius $r_i(o)$ at $o$ in a $H^{i+1}$-Riemannian three-manifold $(\Sigma, \mathcal{A}, g)$, $i \geq 2$, is defined as the supremum of the radius $r$ for which there is a coordinate chart $\{x\}$ covering $B(o, r)$ and satisfying

$$3 \frac{1}{4} \delta_{jk} \leq g_{jk} \leq \frac{4}{3} \delta_{jk},$$

$$\sum_{\alpha=1}^{\alpha=m} r^{2\alpha - 3} \left( \sum_{|I|=\alpha, j,k} \int_{B(o,r)} \frac{\partial^I}{\partial x^I} g_{jk} \right|^{2} d\nu_x) \leq 1,$$

where in the sum above $I$ is the multindex $I = (\alpha_1, \alpha_2, \alpha_3)$, and as usual $\partial^I/\partial x^I = (\partial_1)^{\alpha_1} (\partial_2)^{\alpha_2} (\partial_3)^{\alpha_3}$. Both expressions above are invariant under the simultaneous scaling $\tilde{g} = \lambda^2 g$, $\tilde{x}^\mu = \lambda x^\mu$ and $\tilde{r} = \lambda r$. Observe that if $j > i \geq 2$ then

---

9$\Omega$ will always be an open set with smooth boundary.
implement the criteria is to consider functions $\xi$ canonic harmonic atlas $A$. Pick a $C^\infty$ non-negative function $\chi$ of one variable, being one inside the interval $[0, 3/4]$ and zero inside the interval $[1, \infty]$. Given a chart $\{x_\alpha\}$ in a $H^i$-canonic harmonic atlas $A$ define $\chi_\alpha(x) = \chi(\frac{d}{dr}|x|)$, where $|x|$ is the radial coordinate from the center $o_\alpha$. Finally define

$$\xi_\alpha(x) = \frac{\chi_\alpha(x)}{\sum_\beta \chi_\beta(x)}.$$  

Let $(\Sigma, g)$ be a $H^{i+3}$-Riemannian manifold, $i \geq 0$. As it is usual one can define the $H^j$-Sobolev norm of a tensor field $U$ intrinsically using the covariant derivative $\nabla$ associated to $g$ by $\|U\|_{H^j_{\Sigma}}^2 = \int_{\Sigma} \sum_{m=0}^{\infty} |\nabla^m U|^2dv_g$. The $H^j_A$ and the $H^j_g$ Sobolev spaces are equivalent for $i \geq 0$.

All through the article we will consider pointwise products of tensors lying in different Sobolev spaces and whose product we would like to think in a third Sobolev space. All the mathematics we will need in this respect is the following fact (see [5] page 5 and references therein). Let $U$ and $V$ be two tensors over a $H^{i+3}$-Riemannian three-manifold $(\Sigma, g)$. Assume $U$ is in $H^j_A$ and $V$ is in $H^2_A$, with $i + 2 \geq s_1 \geq 0$, $i + 2 \geq s_2 \geq 1$ and one of the $s_1$ or $s_2$ is strictly greater than zero. Say $s \leq \min(s_1, s_2, s_1 + s_2 - 3/2)$. Then

$$\|U \star V\|_{H^s_A} \leq C(r_{i+2}, Vol)\|U\|_{H^j_A}\|V\|_{H^2_A},$$  

where $A$ is a $H^{i+2}$-canonic harmonic atlas. Thus for example if $s_1 \geq 2$ we have independently of $s_1$, that the pointwise product $U \star V$ is in $H^s_A$ and its norm can be estimated from above by (15). On the other hand if $s_1 = s_2 = 1$ the pointwise product of $U$ and $V$ lies in $H^0_A$ and its norm can be estimated by (15). The reader should be aware these product properties are used intensively (specially inside the Section 3.3.2), and most of the time without explicit mention.

---

10 The case $i = 0$ needs a bootstrapping argument to show that if $\nabla U = \partial U + \Gamma * U$ and $U$ are in $L^2$ then $\partial U$ is also in $L^2$.

11 The condition that $s_k \leq i + 2$ for $k = 0, 1$ is to guarantee that the Sobolev norms $H^s_A$ are well defined globally over $\Sigma$.
We will recur to elliptic estimates in a number of occasions. Except for standard elliptic estimates \([16]\) and the elliptic estimates stated in Proposition 28 later in the article, all the rest will fall under the following proposition.

**Proposition 1** Say \(r < r_{i+2}(o)\), \(i \geq 0\), and let \(\{x\}\) be a harmonic coordinate chart covering \(B(o, r_{i+2}(o))\) and satisfying \([13]\) and \([14]\). We will consider elliptic estimates for first and second order differential equations separately.

I. (Second order elliptic operators). Consider the differential equation

\[ g^{ij} \partial_i \partial_j U^m + A^m_n \partial_n U^n + B^m_n U^n = f^m. \]

with \(A \in H^{i+1}_{\{x\}}(B(o, r_{i+2}(o)))\), \(B \in H^{j}_{\{x\}}(B(o, r_{i+2}(o)))\) and \(f \in H^{j}_{\{x\}}(B(o, r_{i+2}(o)))\) and with \(j \leq i\). Suppose in addition that \(\|A\|_{H^{i+1}_{\{x\}}(B(o, r_{i+2}(o))} + \|B\|_{H^{j}_{\{x\}}(B(o, r_{i+2}(o))} \leq C(r)\). Under these hypothesis we have that if \(U\) is a \(H^{i+2}\)-strong solution of \([16]\) then

\[ \|U\|_{H^{i+2}_{\{x\}}(B(o, r/2))} \leq C(r, C(r))(\|U\|_{H^{i+1}_{\{x\}}(B(o, r))} + \|f\|_{H^{j}_{\{x\}}(B(o, r)))}. \]

If \(B = 0\) we can get an estimate of one degree higher and \([17]\) is valid for \(j = i + 1\) as well. When \(i \geq 2\) the norm \(\|U\|_{H^{i}}\) on the RHS of equation \([17]\) can be replaced by the \(L^2\)-norm of \(U\).

II. (First order elliptic operators). Consider the differential equation

\[ G^{mi} \partial_i U^m + A^m_n \partial_n U^n = f^m. \]

Assume \(G^{mi} \partial_i U^m\) is a first order elliptic operator with coefficients \(G^{mi}\) in \(H^{i+2}_{\{x\}}(B(o, r_{i+2}(o)))\) and ellipticity constants controlled by \(r\). Assume also \(A\) is in \(H^{i+1}_{\{x\}}(B(o, r_{i+1}(o)))\) and \(f\) is in \(H^{j}_{\{x\}}(B(o, r_{i+2}(o)))\) with \(j \leq i + 1\) and finally that \(\|G\|_{H^{i+2}_{\{x\}}(B(o, r_{i+2}(o)))} + \|A\|_{H^{i+1}_{\{x\}}(B(o, r_{i+2}(o)))} \leq C(r)\). Under these conditions we have that if \(U\) is a \(H^{i+1}\)-strong solution, then

\[ \|U\|_{H^{i+1}_{\{x\}}(B(o, r/2))} \leq C(r, C(r))(\|U\|_{H^{i+1}_{\{x\}}(B(o, r))} + \|f\|_{H^{j}_{\{x\}}(B(o, r)))}. \]

If \(i \geq 1\) then the norm \(\|U\|_{H^{i}}\) on the RHS of equation \([19]\) can be replaced with the \(L^2\)-norm of \(U\).

**Remark 1** The elliptic estimates above can be made global in a concrete way. Indeed let \((\Sigma, A, g)\) be a \(H^{i+3}\)-Riemannian three-manifold, where \(A\) is a \(H^{i+2}\)-canonic harmonic atlas. Then, if we take Sobolev norms with respect to the atlas \(A\) we have, for instance the elliptic estimates

\[ \|U\|_{H^{i+2}_{A}} \leq C(r_{i+2}, C, Vol)(\|U\|_{H^{i+2}_{A}} + \|f\|_{H^{i+1}_{A}}). \]

for second order elliptic equations of the form \([16]\). The constant \(C\) bounds the norms of the coefficients \(A\) and \(B\) in \(H^{i+2}_{A}\) and \(H^{i+1}_{A}\) respectively. Proving that the constant \(C\) in the elliptic estimate above can be made dependent only on \(C\).
\( r_{i+2} \) and \( Vol \) control the number of charts in a canonical atlas and the \( H^{i+3}_{\{x_\alpha\}} \)-norms of \( \xi_\alpha \). It follows from the definition of \( H^s_A \) given above that the global elliptic estimates are a direct consequence of the local elliptic estimates inside each one of the harmonic charts.

**Proof:**

1. Say \( i \geq 0 \). If we prove the case \( j = 0 \) we can proceed by induction to prove the proposition for any \( j \leq i \). Indeed assuming it is proved for \( j = j_0 \leq i-1 \), differentiate (16) with respect to \( \partial_k \) for \( k = 1, 2, 3 \), use the result for \( j = j_0 \) and with \( \partial_k U \) as \( U \). Similarly the estimate when \( B = 0 \) can be obtained from the estimate for \( j = i \) by differentiating the equation (16) with respect to \( \partial_k \), \( k = 1, 2, 3 \) and applying the result when \( j = i \) to the function \( \partial_k U \) as \( U \).

2. We proceed with the proof when \( j = 0 \). Let us divide the proof according to the cases \( i \geq 2 \), \( i = 1 \) and \( i = 0 \). We will forget including the domains where norms are defined. When it corresponds, for example when using standard elliptic estimates, one must restrict domains to proper subsets. We will also forget about the subindex \( \{x\} \) in the norms. We write for instance \( L^2 \) instead of \( L^2_{\{x_\alpha\}} \). Finally \( C(r) \) will denote a generic function depending on \( r \).

**Case 1, \( i \geq 2 \):** inspecting the coefficients \( A \) and \( B \) the result follows from the standard elliptic estimates (16), Theorem 9.11. In this case we can replace the \( H^1 \)-norm of \( U \) on the RHS of equation (16) by its \( L^2 \)-norm.

**Case 2, \( i = 1 \):** note that
\[
\|A \ast \partial U\|_{L^2} \leq C(r)\|A\|_{H^2}\|U\|_{H^1} \quad \text{and} \quad \|B \ast U\|_{L^2} \leq C(r)\|B\|_{H^1}\|U\|_{H^1}.
\]
Now apply standard elliptic estimates to the equation
\[
g^{ij}\partial_i \partial_j U = -A^{mi}_n \partial_i U^n - B^{mn}_n U^n + f,
\]
with the RHS as a non-homogeneous term and use the estimates before. This gives
\[
\|U\|_{H^2} \leq C(r, C(r)) (\|U\|_{H^1} + \|f\|_{L^2}),
\]
as desired.

**Case 3, \( i = 0 \):** this case follows by bootstrapping. First note the following three facts

1. If \( U \) is in \( L^\alpha \) (\( \alpha > 2 \)) then Hölder inequalities give
\[
\|B \ast U\|_{L^\beta} \leq \|B\|_{L^2}\|U\|_{L^\alpha}, \quad \beta = \frac{2\alpha}{2 + \alpha},
\]
\[
\|A \ast \partial U\|_{L^\beta} \leq \|A\|_{L^\alpha}\|\partial U\|_{L^\alpha}, \quad \beta = \frac{6\alpha_1}{6 + \alpha_1}.
\]

2. Standard \( L^p \) elliptic estimates applied to the equation
\[
g^{ij}\partial_i \partial_j U^m = -A^{mi}_n \partial_i U^n - B^{mn}_n U^n + f,
\]
with the RHS as a non-homogeneous term, give
\[
\|U\|_{H^2, \beta} \leq C(r)\|B \ast U\|_{L^\beta} + \|A \ast \partial U\|_{L^\beta} + \|U\|_{L^\beta} + \|f\|_{L^\beta}).
\]
3. Sobolev embeddings give [10]:

i) if $2\beta' < 3$ then $\|U\|_{L^{\alpha}} \leq C(r)\|U\|_{H^{2,\beta'}}$ with $\alpha = \frac{3\beta'}{3 - 2\beta'}$,

ii) if $3 + \beta > 2\beta > 3$ then $\|U\|_{C^{0,1/2}} \leq C(r)\|U\|_{H^{2,\beta}}$,

iii) $\|\partial U\|_{L^{\alpha}} \leq C(r)\|U\|_{H^{2,\beta}}$ with $\alpha = \frac{3\beta}{(3-\beta)}$.

Start applying 1 and 2 with $\alpha = 6$: from 1 we get $\beta = 3/2$ and with that $\beta$ we get $\alpha_1 = 2$, using those values in 2 we get

$$\|U\|_{H^{2,3/2}} \leq C(r)(\|B\|_{L^2} \|U\|_{L^6} + \|A\|_{L^6} \|\partial U\|_{L^2} + \|U\|_{L^{3/2}} + \|f\|_{L^{3/2}}) \leq C(r)(\|B\|_{L^2} + \|A\|_{L^6} + \|U\|_{H^1} + \|f\|_{L^2}).$$

Now apply 3 i) with $\beta' = \beta - 1/6 = 4/3$ to get $\alpha = 12$ and 3 iii) with $\beta = 3/2$ to get $\alpha_1 = 3$. Returning to 1 and plugging in those values we get $\beta = 12/7$ for $\alpha = 12$ and $\beta = 2$ for $\alpha_1 = 3$. We use next 2 with the minimum of those $\beta$, i.e. $\beta = 12/7$ to get

$$\|U\|_{H^{2,12/7}} \leq C(r)(\|B\|_{L^2} \|U\|_{L^{12}} + \|A\|_{L^6} \|\partial U\|_{L^{12/5}} + \|U\|_{H^1} + \|f\|_{L^2}),$$

and in turn

$$\|U\|_{H^{2,12/7}} \leq C(r)(\|B\|_{L^2} + \|A\|_{L^6} + \|U\|_{H^{2,4/3}} + \|U\|_{H^{1}} + \|f\|_{L^2}),$$

giving thus

$$\|U\|_{H^{2,12/7}} \leq C(r, \tilde{C}(r))(\|U\|_{H^1} + \|f\|_{L^2}).$$

From 3 ii) we get $\|U\|_{C^{0,1/2}} \leq C(r)\|U\|_{H^{2,12/7}}$. With that we get $\|B \ast U\|_{L^2} \leq C(r)\|B\|_{L^2} \|U\|_{H^{2,12/7}}$. We use this in 2 with $\beta = 2$ to get

$$\|U\|_{H^{2,2}} \leq C(r)(\|B\|_{L^2} \|U\|_{H^{2,12/7}} + \|A\|_{L^6} \|\partial U\|_{L^3} + \|U\|_{H^1} + \|f\|_{L^2}),$$

giving

$$\|U\|_{H^{2,2}} \leq C(\alpha_1, \tilde{C}(r)) \|U\|_{H^1} + \|f\|_{L^2},$$

as desired. The elliptic estimate on first order elliptic operators follows exactly along the same lines as we did for second order elliptic operators. \hfill \Box

To illustrate the use of the estimates in Proposition 1 let us prove a well known fact that will be needed later.

**Proposition 2** At any point $o$ on a compact $H^{i+3}$-Riemannian three-manifold $\Sigma$ ($i \geq 0$), $r_{i+2}(o)$ and $\|Ric\|_{H^{i+1}(B(o, r_{i+2}(o)))}$ control $r_{i+3}(o)$ from below. As a consequence, for any compact $H^{i+3}$-Riemannian three-manifold $\Sigma$, $\|Ric\|_{H^{i+1}(\Sigma)}$ and $r_{i+2}$ control $r_{i+3}$ from below.

**Proof:**

Pick a harmonic coordinate chart $\{x\}$ covering $B(o, r_{i+2}(o))$ and having the properties (13)-(14). In harmonic coordinates the Ricci tensor $Ric$ has the expression

$$Ric_{\alpha\beta} = -\frac{1}{2}g^{ij}\partial_i\partial_j g_{\alpha\beta} + \partial_\alpha g_\beta + \partial_\beta g_\alpha + g_\star g^*.$$
The Elliptic estimates in Proposition 1 (where \( A = \partial^{\ast} g^{\ast} g^{\ast} g^{\ast} \) and \( B = 0 \)) show that \( \| g_{\alpha \beta} \|_{H^{i+3}((B(0, r_{i+2}(0)/2)))} \) is controlled by \( \| \text{Ric} \|_{H^{i+1}((B(o, r_{i+2}(o)))} \) and \( r_{i+2}(o) \), therefore \( r_{i+3}(o) \) is controlled from below by them too.

Finally let us state the fundamental theorem of convergence in Riemannian geometry. Recall that the volume radius \( \nu \) at a point \( o \) is defined as

\[
\nu(o) = \sup \{ r/\text{vol}(B(p, s))/s^3 \geq \mu, \text{ for all } B(p, s) \subset B(o, r) \},
\]

where \( \mu \) is an arbitrary small numeric constant. The volume radius of \( \Sigma \) is defined as the infimum of \( \nu(o) \) for all \( o \) in \( \Sigma \).

**Theorem 4** (Fundamental theorem of convergence (see [21] and ref. therein)). Say \((\Sigma, A, g)\) is a compact \( H^{i+3} \)-riemannian three-manifold. Then the \( H^{i+2} \)-harmonic radius is controlled from below by \( \text{Vol}_g(\Sigma), \| \text{Ric} \|_{H^{i+1}}, \) and \( \nu \). Any sequence of metrics with \( \text{Vol}_g(\Sigma) \) and \( \| \text{Ric} \|_{H^{i}} \) uniformly bounded from above and \( \nu \) uniformly bounded from below has a subsequence converging (up to diffeomorphisms) to a \( H^{i} \)-metric in the weak \( H^{i} \)-topology.

**Remark 2** The use of the volume radius is not strictly necessary and it may well be substituted with other notions of "local flatness" (scaling as a distance) better adapted to use in General Relativity and more specifically in its dynamics.

### 2.5 The Bel-Robinson energy and the space-time curvature.

In this section we introduce Weyl fields and Bel-Robinson energies. We do so without further explanations or proofs. The reader can rely in [13] for a detailed account. We will follow it in the presentation below.

A Weyl field is a traceless \((4, 0)\) space-time tensor field having the symmetries of the curvature tensor \( \text{Rm} \). We will denote them by \( W_{abcd} \) or simply \( W \). As an example, the Riemann tensor in a vacuum solution of the Einstein equations is a Weyl field that we will be denoting by \( \text{Rm} = W_0 \) (we will use indistinctly either \( \text{Rm} \) or \( W_0 \)). The covariant derivative of a Weyl field \( \nabla_X W \) for an arbitrary vector field \( X \) is also a Weyl field. In particular \( \nabla_X W_0 = W_X \) are Weyl fields. We will be using the Weyl fields \( W_X \) with \( X = T \), where \( T \) is the future pointing unit normal field to the CMC foliation.

Given a Weyl tensor \( W \) define the left and right duals by \( * W_{abcd} = \frac{1}{2} \epsilon_{ablm} W^{lm} \) and \( W^{*}_{abcd} = W_{abcd} \frac{1}{2} \epsilon_{lmcd} \) respectively. It is \( * W = W^\ast \) and \( * (W^\ast) = - W \). Define the current \( J \) and its dual \( J^\ast \) by

\[
\nabla^\nu W_{abcd} = J_{abc},
\]

\[
\nabla^\nu W^*_{abcd} = J^*_{abc}.
\]

When \( W \) is the Riemann tensor in a vacuum solution of the Einstein equations the currents \( J \) and \( J^\ast \) are zero due to the Bianchi identities. This fact will be of fundamental importance latter.
The $L^2$-norm with respect to the foliation will be defined through the Bel-Robinson tensor. Given a Weyl field $\mathbf{W}$ define the Bel-Robinson tensor by

$$Q_{abcd}(\mathbf{W}) = \mathbf{W}_{alc \mu} \mathbf{W}^{l \mu}_{\ b \ d} + \mathbf{W}^{*}_{alc \mu} \mathbf{W}^{* l \mu}_{\ b \ d}.$$ 

The Bel-Robinson tensor is symmetric and traceless in all pairs of indices and for any pair of timelike vectors $T_1$ and $T_2$, the quantity $Q(T_1 T_1 T_1 T_2)$ is positive (provided $\mathbf{W} \neq 0$).

The electric and magnetic components of $\mathbf{W}$ are defined as

(21) \hspace{1cm} E_{ab} = \mathbf{W}_{ac \beta d} T^{\beta} T^{d},

(22) \hspace{1cm} B_{ab} = \ast \mathbf{W}_{ac \beta d} T^{\beta} T^{d}.

$E$ and $B$ are symmetric, traceless and null in the $T$ direction. It is also the case that $\mathbf{W}$ can be reconstructed from them (see [13], page 143). If $\mathbf{W}$ is the Riemann tensor in a vacuum solution we have

(23) \hspace{1cm} E_{ab} = \text{Ric}_{ab} + k K_{a b} - K^c_a K^c_b,

(24) \hspace{1cm} \epsilon_{ab}^l B_{lc} = \nabla^a K_{bc} - \nabla^b K_{ac}.

The components of a Weyl field with respect to the CMC foliation are given by $(i, j, k, l$ are spatial indices)

(25) \hspace{1cm} \mathbf{W}_{ijkT} = -\epsilon_{ij}^m B_{mk}, \ast \mathbf{W}_{ijkT} = \epsilon_{ij}^m E_{mk},

(26) \hspace{1cm} \mathbf{W}_{ijkl} = \epsilon_{ijm} \epsilon_{kln} E^{mn}, \ast \mathbf{W}_{ijkl} = \epsilon_{ijm} \epsilon_{kln} B^{mn}.

We also have

$$Q(TTTT) = |E|^2 + |B|^2,$$

$$Q_{TTTT} = 2(E \wedge B)_i,$$

$$Q_{TjTT} = -(E \times E)_i j - (B \times B)_i j + \frac{1}{3} (|E|^2 + |B|^2) g_{ij}.$$ 

The operations $\times$ and $\wedge$ are provided explicitly later. The divergence of the Bel-Robinson tensor is

$$\nabla^a Q(\mathbf{W})_{abcd} = \mathbf{W}_{b \ d}^m \mathbf{J}(\mathbf{W})_{mc n} + \mathbf{W}_{b \ c}^m \mathbf{J}(\mathbf{W})_{md n} + \ast \mathbf{W}_{b \ d}^m \mathbf{J}^{*}(\mathbf{W})_{mc n} + \ast \mathbf{W}_{b \ c}^m \mathbf{J}^{*}(\mathbf{W})_{md n}.$$

We have therefore

$$\nabla^a Q(\mathbf{W})_{aTTT} = 2E^{ij}(\mathbf{W})_{j} \mathbf{J}(\mathbf{W})_{iT j} + 2B^{ij} \mathbf{J}^{*}(\mathbf{W})_{iT j}.$$ 

18
From that we get the *Gauss equation* which will be used several times during the article

\[
\dot{Q}(W) = - \int_\Sigma 2NE^{ij}(W)J(W)_{iTj} + 2NB^{ij}(W)J^\tau(W)_{iTj} + 3NQ_{abT}\Pi^{ab}d\gamma.
\]

\(\Pi_{ab} = \nabla_a T_b\) is the *deformation tensor* and plays a fundamental role in the space-time tensor algebra. In components it is

\[
\Pi_{ij} = -K_{ij}, \quad \Pi_{iT} = 0, \quad \Pi_{T i} = \nabla_i N, \quad \Pi_{TT} = 0.
\]

The next equations are essential when it comes to get elliptic estimates of Weyl fields.

\[
div E(W)_a = (K \wedge B(W))_a + J_{TaT}(W),
\]

\[
div B(W)_a = -(K \wedge E(W))_a + J^\ast_{TaT}(W),
\]

\[
curl B_{ab}(W) = E(\nabla_T W)_{ab} + \frac{3}{2}(E(W) \times K)_{ab} - \frac{1}{2}kE_{ab}(W) + J_{aTb}(W),
\]

\[
curl E_{ab}(W) = B(\nabla_T W)_{ab} + \frac{3}{2}(B(W) \times K)_{ab} - \frac{1}{2}kB_{ab}(W) + J^\ast_{aTb}(W).
\]

The operations \(\wedge, \times\) and the operators \(Div\) and \(Curl\) are defined through

\[
(A \times B)_{ab} = \epsilon_a^{cd}\epsilon_b^{ef} A_{ce}B_{df} + \frac{1}{3}(A \circ B)_{ab} - \frac{1}{3}(trA)(trB)g_{ab},
\]

\[
(A \wedge B)_a = \epsilon_a^{bc}A^d_bB_{dc},
\]

\[
(div A)_a = \nabla_b A^b_a,
\]

\[
(curl A)_{ab} = \frac{1}{2}(\epsilon_a^{lm}\nabla_l A_{mb} + \epsilon_b^{lm}\nabla_l A_{ma}).
\]

Finally we mention a couple of formulas that will be used. If \(V\) a symmetric traceless \((2,0)\) tensor with

\[
(div V)_a = \nabla_b V^b_a = \rho,
\]

\[
(curl V)_{ab} = \frac{1}{2}(\epsilon_a^{lm}\nabla_l V_{mb} + \epsilon_b^{lm}\nabla_l V_{ma}) = \sigma,
\]

then

\[
\int_\Sigma |\nabla V|^2 + 3 < Ric, V \circ V > - \frac{1}{2}R|V|^2 = \int_\Sigma |\sigma|^2 + \frac{1}{2}|\rho|^2.
\]
We have also

\[(33) \quad d^\ast d^\ast (A) + 2 \text{div}^\ast \text{div}(A) = 2 \nabla^\ast \nabla A + \mathcal{R}(A).\]

where \(\mathcal{R}\) has the expression \(\mathcal{R}(A) = \text{Ric} \circ A + A \circ \text{Ric} - 2 \text{Rm} \circ A\). The expression \(\text{Rm} \circ A\) is defined as \((\text{Rm} \circ A)_{ab} = \text{Rm}_{aabd} A^{cd}\).

**The connection coefficients.**

Say \(\mathcal{F}\) is the CMC foliation. Say \(X\) and \(Y\) are vector fields tangent to \(\mathcal{F}\) and commuting with \(NT\). In this setting we have

\[K(X,Y) = - \langle \nabla_X T, Y \rangle,\]

\[\nabla_X Y^b = \nabla_X Y^b - K(X,Y) T^b,\]

\[\nabla_X T^b = -X^a K_a^b,\]

\[\nabla_T T^b = - P_{ba} \frac{\nabla_a N}{N},\]

where \(P_{ab} = g_{ab} + T_a T_b\) is the horizontal projector (it projects into \(\mathcal{F}\)),

\[\nabla_T X^b = -X^a K_a^b + \frac{\nabla_X N}{N} T^b,\]

and

\[[X, T] = -\frac{\nabla_X N}{N} T.\]

2.6 Scaling and Cosmological scaling.

Given a solution \(g\) of the Einstein equations, we say that \(\lambda^2 g\) is the solution \(g\) at the scale of \(\frac{1}{\lambda}\) and we call \(\lambda\) the scale factor. We say that a CMC state \((g, K)\) is *cosmologically scaled or normalized* if \(k = -3\). This condition has a cosmological interpretation. We will see later that the Hubble parameter \(\mathcal{H}\) can be identified with \(-\frac{k}{3}\) and in this sense a cosmological normalized state is one for which its Hubble parameter is equal to one. In general, space-time tensors, scale as \(\lambda^s U\) for some weight \(s\). The table below shows the scaling rules for some common tensors.

| \(g\) | Ric | \(T\) | \(g\) | \(K\) | \(k\) | \(N\) | \(\rho\) | \(W_i\) | \(Q_i\) |
|-------|-----|-------|-------|-----|-----|-----|-----|-----|-----|
| \(\lambda^2 g\) | Ric | \(\lambda^2 g\) | \(\lambda K\) | \(\frac{2}{3}\lambda^2 N\) | \(\lambda^{-2} \rho\) | \(\lambda^{-s-2} W_i\) | \(\lambda^{-(2s+1)} Q_i\) |

3 The CMC flow, the volume and the BR-energy.

*Assumption:* from now and for the rest of the article we will assume CMC states \((g, K)\) have \(k \neq 0\) and that CMC flows are defined on a range of \(k\) of the form \((a, b)\) with \(-\infty \leq a < b \leq 0\). Also the *future* will be the direction in which the volume expands.
3.1 The reduced volume.

In this section we assume that $Y(\Sigma) \leq 0$. The reduced volume that we will use was introduced and used systematically in [15] as a reduced Hamiltonian after passing to conformal variables. It is defined (up to a constant from [15]) as

$$\mathcal{V}(g, K) = -(\frac{k}{3})^3 Vol_g(\Sigma).$$

Recall that in any flat cone the volume of the CMC slices grow as $1/(−k)^3$. With that in mind we may interpret the reduced volume as a comparison between the volume of the particular solution and the volume of a flat cone at a given $k$. We enumerate below a list of properties of $\mathcal{V}$ [15].

1. The derivative of $\mathcal{V}$ with respect to $k$ is

$$\frac{d\mathcal{V}}{dk} = -(\frac{k}{3})^2 \int_\Sigma N|\hat{K}|^2 dv_g = -(\frac{k}{3})^2 \int_\Sigma (1 - \frac{Nk^2}{3})dv_g.$$

When $\mathcal{V}' = 0$ then $N = \frac{k}{2\pi}$, $\mathcal{V}'' = 0$ and $\mathcal{V}''' = -k^4 / 9 \int_\sigma N^3|Ric|^2 dv_g$.

2. From the property before we get that $\mathcal{V}$ is strictly monotonically decreasing along the future CMC flow unless is constant in which case the solution is a flat cone.

3. $\mathcal{V}$ is scale invariant.

4. The infimum $\mathcal{V}_{inf}$ of $\mathcal{V}$ in the phase space of all CMC states is given by

$$\mathcal{V}_{inf} = \inf \{\mathcal{V}(g, K)/(g, K) \text{ is CMC} \} = (-\frac{1}{6}Y(\Sigma))^\frac{3}{2}.$$

where $Y(\Sigma)$ is the Yamabe invariant of $\Sigma$. Under zero shift we have the remarkable fact that

$$\frac{1}{4} \int_\Sigma |\dot{g}|^2 dv_g \leq \int_\Sigma \frac{|\dot{g}|^2}{4N} dv_g = \int_\Sigma \tilde{N}|\hat{K}|^2 dv_g$$

$$= \int_\Sigma \tilde{N}R + \tilde{N}k^2 dv_g = \frac{k^2}{3} Vol_g = \frac{-9}{k} \mathcal{V}$$

$$\leq 3\mathcal{V}.$$ 

\[12\] In [15] the reduced volume is named the reduced Hamiltonian.

\[13\] In the context of the long time a quantity similar to the reduced volume was also used in [1]. More precisely, the “reduced volume” in [1] was defined dynamically: say $(g, K, (N, X))$ is a solution to the CMC flow and $t_k = dist(\Sigma_{k0}, \Sigma_k)$ where $dist$ is the Lorentzian distance between the initial CMC Cauchy hypersurface $\Sigma_{k0}$ and $\Sigma_k$, then define $\mathcal{V} = \frac{Vol_g(\Sigma)}{t_k^2}$. It is proved [1] that it is a strictly monotonic quantity unless is constant in which case the solution is a flat cone. The reduced volume here defined has the advantage that it is defined on the set of all CMC states whereas the one in [A] is defined along solutions.

\[14\] The flat cones are the solutions with maximal rate of volume expansion.

21
where \( \tilde{N} = \frac{N^2}{3} \) is the lapse associated to the *cosmological time* \( t = -\frac{3}{k} \) and \( \dot{g} = \partial g/\partial t \). To obtain the first inequality we have used the fact that \( \tilde{N} \leq 1 \) and to get the equality after the fourth term we have used the lapse equation (8) multiplied by \( k^2/3 \). As \( V \) is monotonic along the future direction in a CMC solution of the Einstein vacuum equations, the RHS of equation (35) is bounded above by \( 3t V(t_0) \) where \( t_0 \) is some initial time. As for the traceless part of the time derivative of \( g \) we have
\[
\frac{1}{4} \int_\Sigma |\dot{g}|^2 dv_g \leq -\frac{3}{k} \frac{dV}{dt},
\]
implying
\[
\frac{1}{4} \int_{t_0}^{t_1} \frac{||\dot{g}||^2}{t^2} dt \leq V(t_0) - V(t_1).
\]

### 3.2 The Friedman-Lemaître equations and a cosmological interpretation of the reduced volume monotonicity.

The Robertson-Walker cosmologies implement the *cosmological principle* in its perfect form. As such it characterizes the universe by a radius \( a \), matter density \( \rho \) and the pressure \( p \). The space-time is modeled by a metric
\[
g = -dt^2 + a^2(t) g_K,
\]
on a manifold \( \Sigma \times \mathbb{R} = \mathbb{R}^3 \times \mathbb{R} \) and where \( g_K \) is a metric of constant sectional curvature equal to \( K \). It follows that the stress energy-momentum tensor is of the form \( T_{ab} = (\rho + p)T_aT_b + pg_{ab} \) with \( \rho \) and \( p \) depending only on time. The compact FLRW models are simply space-like compactification of the models above. In the \( K = 0 \) case compact Cauchy hypersurfaces obtained are flat while in the \( K = -1 \) case are hyperbolic. The Einstein equations are equivalent to the Friedman-Lemaître equations that we present below in integral form to make a closer connection with the Friedman-Lemaître equations for arbitrary solutions that we describe later
\[
\mathcal{H}^2 = \frac{\int_\Sigma (16\pi G\rho)dv_\Sigma}{6V} + -K(V_0)^2,
\]
\[
a'' = -\frac{\int_\Sigma (4\pi G(\rho + 3p))dv_\Sigma}{3V}.
\]

In the formulas above \( V_0 \) is the volume of the manifold \( \Sigma \) when it is given the unique hyperbolic metric or some flat metric according to the type of the model. In a general cosmological solution (not perfectly homogeneous and isotropic) the cosmological parameters are defined in volume average \([22], [10]^{15}\). For instance the radius \( a \) is defined as
\[
a = \left( \frac{V}{V_0} \right)^{\frac{2}{3}},
\]

\[^{15}\text{Although the averages in [22] and [10] both average in volume, they are not precisely the same.}\]
and the proper time by
\[ \frac{d\tau}{dk} = \frac{\int_{\Sigma} N \, dv_g}{V}. \]

The Hubble parameter \( \mathcal{H} \) is computed as
\[ \mathcal{H} = \frac{1}{V} \frac{dV^{1/2}}{d\tau} = \frac{1}{V} \frac{dV^{1/2}}{dk} \frac{dk}{d\tau} = \frac{1}{V} \frac{1}{3} V^{-2/3} \left( \int_{\Sigma} -N \, k \, dv_g \right) \frac{V}{\int_{\Sigma} N \, dv_g} = -\frac{k}{3}. \]

This inspires to give the name of “Hubble-gauge” to the CMC gauge, as observers at the same “instants of time” would measure the same Hubble parameter \( \mathcal{H} \) (i.e. it is constant along leaves of the CMC foliation). Define \( \rho = T_{ab} g^{ab} \) (note that \( p \) is the average of the principal pressures). Then the Friedman-Lemaître equations in this general setting are

\[ \mathcal{H}^2 = -\frac{\int_{\Sigma} R \, dv_g}{6V} + \frac{\int_{\Sigma} (16\pi G \rho + |\hat{K}|^2) \, dv_g}{6V}, \tag{36} \]

\[ \frac{a''}{a} = -\frac{1}{3} \frac{\int_{\Sigma} N (4\pi G (\rho + 3p) + |\hat{K}|^2) \, dv_g}{3V}. \tag{37} \]

Where \( N = \bar{N} \) (bar denotes volume-average) and has average equal to one. All the derivatives above are with respect to the averaged proper time \( \tau \). The first FL equation (equation (36)) is just the average of the energy constraint

\[ 16\pi \rho = R - |\hat{K}|^2 + \frac{2}{3} k^2. \]

To get the second FL equation (equation (37)) we observe that

\[ \left( \frac{a'}{a} \right)' = a'' - \left( \frac{a'}{a} \right)^2 = a'' - \mathcal{H}^2, \tag{38} \]

and

\[ \left( \frac{a'}{a} \right)' = \frac{d\mathcal{H}}{d\tau} = -\frac{1}{3} \frac{dk}{d\tau} = -\frac{V}{3 \int_{\Sigma} N \, dv_g}. \tag{39} \]

On the other hand from the lapse equation we get after integration

\[ \int_{\Sigma} N (4\pi G (\rho + 3p) + |\hat{K}|^2) \, dv_g = V - 3\mathcal{H}^2 \int_{\Sigma} N \, dv_g. \tag{40} \]

Equations (38), (39) and (40) together give the equation (37). Up to a constant the reduced volume is \( V = (3aH)^3 \) and we see that

\[ (V^{1/3})' = (3aH)' = 3a'' - a \frac{\int_{\Sigma} N (4\pi G (\rho + 3p) + |\hat{K}|^2) \, dv_g}{V}. \tag{31a} \]

If there is no matter present, i.e. \( \rho = p = 0 \), the previous equation is directly equivalent to the monotonicity formula (34) for the reduced volume. This implies that the monotonicity of the reduced volume and the well known universe deceleration are equivalent properties. Note that the reduced volume is monotonic in presence of matter at least if \( \rho + 3p \geq 0 \). That is the case if the average of the principal pressures is positive or if the mass density dominates over the (possible negative) pressure.
3.3 Controlling the states \((g, K)\) at a given time.

This section is devoted to study the intrinsic quantities \(\nu, V, E \bar{I} = \sum_{i=0}^{\bar{I}} Q(W_i)\) and \(\rho\) (if matter satisfying the dominant energy condition is present) as variables controlling the states \((g, K)\) at a given time \(t = k\).

Assumptions. All the results presented in this Section (Section 3.3), except for some exceptions explicitly indicated, will be stated in the context of vacuum solutions of the Einstein equations. Also, all the analysis will be for general three-manifolds \(\Sigma\), except when we indicate that a result is restricted to manifolds with \(Y(\Sigma) \leq 0\). We will assume that all quantities (for instance \(g, K\) or \(W_i\)) come from a \(C^\infty\) solution \(g\). In this way we avoid justifying certain operations on tensors that would require justification in the case of low regularity. Still the results of this section extend to the natural regularity of each statement or proposition. For instance Proposition 1 is valid for states \((g, K)\) with \(H^2 \times H^1\) regularity. The same holds in Proposition 14, where to exemplify, a footnote was added on how to prove the result for states with \(H^2 \times H^1\) regularity. All the time \(C\) will represent a generic constant (the same \(C\) can represent different constants in different lines). If the constant \(C\) is numeric it will be stated explicitly.

The main result in vacuum will be the following.

**Lemma 2** (Sobolev norms vs. Bel-Robinson norms) Say \(\bar{I} \geq 0\) and say \(\Sigma\) is a compact three-manifold. Then the functional (defined over cosmological normalized states \((g, K)\))

\[
\|(g, K)\|_{BR} = \frac{1}{\nu} + V + E_I,
\]

controls the \(H^{\bar{I}+2}\)-harmonic radius \(r_{\bar{I}+2}\) and the \(H^{\bar{I}+1}_g\)-norm of \(K\).

The proof of this lemma (at the end of this section) is conceptually divided into several propositions many of them having however independent value. As it turns out there are a number of intrinsic estimates, namely estimates involving only intrinsic norms \(H^s_g\), worth to be mentioned separately. Intrinsic estimates give more information than what elliptic estimates give. The elliptic estimates are stated in terms of the norms \(H^s_A\) where \(A\) are harmonic atlas. To highlight the difference between them, we break the section into two subsections.

### 3.3.1 Intrinsic estimates.

We start studying the control of the first order Bel-Robinson energy \(Q_0\) over states \((g, K)\).

**Proposition 3** Say \(\Sigma\) is a compact three-manifold. Then \(Q_0\) and \(|k|^2 \|\hat{K}\|_{L^2_g}^2\) control \(\|\nabla \hat{K}\|_{L^2_g}^2\), \(\|\hat{K}\|_{L^4_g}^4\) and \(\|\hat{Ric}\|_{L^2_g}^2\). More in particular we have

\[
(41) \quad \left( \int_M 2|\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \right)^{1/2} \leq C(|k| \|\hat{K}\|_{L^2_g} + Q_0^{1/2}),
\]
where $C$ is a numeric constant.

Observe the absence of the volume in equation (41) and that all norms involved are intrinsic.

**Proof:** Substituting $Ric = E - kK + K \circ K$, $K = \hat{K} + \frac{k}{3}g$ and $V = \hat{K}$ in equation (32) we get

$$\int_{\Sigma} |\nabla \hat{K}|^2 + \frac{5}{2} |\hat{K}|^4 - k < \hat{K}, \hat{K} \circ \hat{K} > - \frac{k^2}{3} |\hat{K}|^2 + 3 < E, \hat{K} \circ \hat{K} > dv_g = \int_{\Sigma} |B|^2 dv_g.$$ This equation gives the bound

$$\int_{\Sigma} |\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \leq C \int_{\Sigma} (|k|^2 |\hat{K}|^2 + |k||\hat{K}|^3 + |\hat{K}|^2 |E| + |B|^2) dv_g,$$

Observe now that the inequalities

$$\int_{\Sigma} |\hat{K}|^2 (|E|^2 + |B|^2) \frac{1}{2} dv_g \leq \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right) \frac{1}{2} Q_0^\frac{1}{2},$$

$$\int_{\Sigma} |\hat{K}|^3 dv_g \leq \left( \int_{\Sigma} |\hat{K}|^2 dv_g \right) \frac{1}{4} \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right) \frac{1}{2},$$

transform equation (42) into

$$2 \| \nabla \hat{K} \|_{L^2_g}^2 + \| \hat{K} \|_{L^4_g}^4 - C (|k| \| \hat{K} \|_{L^2_g} + Q_0^\frac{1}{2}) \| \hat{K} \|_{L^2_g}^2 - C (|k|^2 \| \hat{K} \|_{L^2_g}^2 + Q_0) \leq 0.$$ Now make $x^2 = 2 \int_{\Sigma} |\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g$, $a = (|k| \| \hat{K} \|_{L^2_g} + Q_0^\frac{1}{2})$ in the last equation. We get

$$x^2 - C a x - C a^2 \leq 0.$$ Solving for $x$ in the inequality above we get equation (41) which finishes the proof. □

In presence of matter satisfying the dominant energy condition (see [18] for a discussion) we have the following modification of the last proposition.

**Proposition 4** Say $\Sigma$ is a compact three-manifold and say $(g, K)$ is a cosmological normalized state over a Cauchy-slice on a solution $(M, g)$ of the Einstein equations and with matter satisfying the dominant energy condition. Then $\| \hat{K} \|_{L^2_g}$, $Q_0$ and $\| G \rho \|_{L^2_g}$ control $\| \nabla \hat{K} \|_{L^2_g}$, $\| \hat{K} \|_{L^2_g}$ and $\| Ric \|_{L^2_g}$. More precisely we have

$$\left( \int_{\Sigma} 2 |\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \right) \frac{1}{4} \leq C (\| \hat{K} \|_{L^2_g} + Q_0^\frac{1}{2} + \| G \rho \|_{L^2_g})$$

where $C$ is a numeric constant.
Proof:

If the energy-momentum tensor of matter satisfies the dominant energy condition then \( T_{ij} \geq |T_{ij}| \) for any \( i, j = 1, 2, 3 \). Recall that the space-time curvature \( Rm \) is decomposed in terms of \( W_0 \) and \( Ric \) (with an expression linear on each) and that \( Ric = 8\pi G(T - \frac{1}{2} tr g T g) \). We have then that the \( L^2_g \) norm of the components of \( Rm \) are controlled by \( Q_0 + \|G\rho\|^2_{L^2_g} \). Using these facts the proof goes exactly parallel to the one of Proposition 3, but this time using instead the identities

\[
Rm_{iTjT} = Ric_{ij} - K_{im}K^m_{j} - kK_{ij},
\]
\[
Rm_{miTj} = (d^R K)_{mij},
\]
\[
R = 16\pi G\rho + |\hat{K}|^2 - \frac{2}{3}k^2,
\]
\[
\nabla K = -8\pi GJ.
\]

The next proposition relates \( \|\hat{K}\|_{L^2_g} \) with \( V - V_{inf} \) or \( V \) depending on the signature of the Yamabe invariant \( Y(\Sigma) \).

**Proposition 5**. Say \( \Sigma \) is a compact three-manifold. Then

i) if \( Y(\Sigma) > 0 \), \( |k|^2 \int_{\Sigma} |\hat{K}|^2 dv_g \leq C|k|^\frac{1}{2} \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right)^{\frac{1}{2}} \),

ii) if \( Y(\Sigma) = 0 \), \( |k|^2 \int_{\Sigma} |\hat{K}|^2 dv_g \leq C|k|^\frac{1}{2} \left( (V - V_{inf})^2 \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right)^{\frac{1}{2}} \right) \),

iii) if \( Y(\Sigma) < 0 \), \( |k|^2 \int_{\Sigma} |\hat{K}|^2 dv_g \leq C(|k|(V - V_{inf}) + \left| k \right|^\frac{1}{2} \left( (V - V_{inf})^2 \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right)^{\frac{1}{2}} \right) \).

where \( C \) is a numeric constant.

**Proof:**

i) and ii) \( Y(\Sigma) > 0 \) or \( Y(\Sigma) = 0 \). This is immediate from the formula

\[
|k|^2 \int_{\Sigma} |\hat{K}|^2 dv_g \leq |k|^\frac{1}{2} \left( |k|^3 Vol(\Sigma) \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\hat{K}|^4 dv_g \right)^{\frac{1}{2}}.
\]

iii) \( Y(\Sigma) < 0 \). Assume \( k = -3 \) and let \( g_Y \) be the unique Yamabe metric of constant scalar curvature \( R_Y = -6 \) in the conformal class of \( g \). If \( g = \phi^4 g_Y \) then \( \phi \) is determined by

\[
- \Delta_{g_Y} \phi + \frac{R_Y}{8} \phi - \frac{1}{8} \phi^{-3} |\hat{K}|^2_Y + \frac{1}{12} k^2 \phi^5 = 0,
\]

where \( \Delta = \nabla^2 \). The maximum principle implies (putting the values \( R_Y = -6 \) and \( k = -3 \)) that

\[
6(\phi_{min}^5 - \phi_{min}) \geq \phi_{min}^{-3} |\hat{K}|^2_Y \geq 0,
\]

which makes \( \phi \geq 1 \). Then observe that

\[
-Y(\Sigma) \leq -R_Y \left( \int_{\Sigma} 1 dv_Y \right)^{\frac{1}{2}},
\]

26
where \( dv_Y = dv_g \). This gives
\[
0 \leq 6^\frac{3}{2} \left( \int_{\Sigma} \phi^6 - 1 \ dv_Y \right) \leq 6^\frac{3}{2} \int_{\Sigma} \phi^6 dv_Y - \left( -Y(\Sigma) \right)^\frac{3}{2} = \left( \frac{2}{3} k^2 Vol(\Sigma)^\frac{3}{2} \right)^\frac{3}{2} - \left( -Y(\Sigma) \right)^\frac{3}{2}.
\]
Therefore
\[
\int_{\Sigma} (\phi - 1)^k dv_Y \leq C(V - V_{inf}),
\]
for \( k = 1, \ldots, 6 \). Integrating equation (44), we get
\[
(45) \quad 6 \int_{\Sigma} (\phi^5 - \phi) dv_Y = \int_{\Sigma} \phi^{-3} |\hat{K}|_V^2 dv_Y.
\]
Observe that
\[
\int_{\Sigma} \phi^{-2} |\hat{K}|_V^2 dv_Y = \int_{\Sigma} \phi \phi^{-3} |\hat{K}|_V^2 dv_Y = \int_{\Sigma} \phi^{-3} |\hat{K}|_V^2 dv_Y + \int_{\Sigma} (\phi - 1) \phi^{-3} |\hat{K}|_V^2 dv_Y,
\]
and
\[
(46) \quad \int_{\Sigma} (\phi - 1) \phi^{-3} |\hat{K}|_V^2 dv_Y = \int_{\Sigma} (\phi - 1) \phi^2 \phi^{-5} |\hat{K}|_V^2 dv_Y \leq \left( \int_{\Sigma} (\phi - 1)^2 \phi^4 dv_Y \right)^\frac{1}{2} \left( \int_{\Sigma} \phi^{-10} |\hat{K}|_V^4 dv_Y \right)^\frac{1}{2}.
\]
On the other hand note that
\[
(47) \quad | \int_{\Sigma} (\phi - 1)^2 \phi^4 dv_Y | \leq \int_{\Sigma} |\phi^6 - 1| + 2|\phi^5 - 1| + |\phi^4 - 1| dv_Y \leq C(V - V_{inf}).
\]
Putting together equations (45), (46) and (47) we get
\[
(48) \quad \| \hat{K} \|^2_{L^2_\phi} \leq C((V - V_{inf}) + (V - V_{inf})^\frac{3}{2} \| \hat{K} \|^2_{L^2_\phi}),
\]
which after scaling finishes the proof. 

Combining Propositions 3 and 5 we get

**Proposition 6** Say \( \Sigma \) is a compact three-manifold. Then if \( Y(\Sigma) > 0 \) we have
\[
\int_{\Sigma} 2|\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \leq C(|k|V + Q_0),
\]
while if \( Y(\Sigma) \leq 0 \) we have
\[
\int_{\Sigma} 2|\nabla \hat{K}|^2 + |\hat{K}|^4 dv_g \leq C(|k|(V - V_{inf}) + Q_0),
\]
where \( C \) is a numeric constant.
We also get

**Proposition 7** Say \( \Sigma \) is a compact three-manifold. If \( Y(\Sigma) > 0 \) we have

\[
\int_\Sigma |k|^2 |\hat{K}|^2 \, dv_g \leq C(|k| \mathcal{V} + (|k| \mathcal{V} \mathcal{Q}_0)^{\frac{2}{3}}),
\]

while if \( Y(\Sigma) \leq 0 \) (same for \( Y(\Sigma) = 0 \) than for \( Y(\Sigma) < 0 \))

\[
\int_\Sigma |k|^2 |\hat{K}|^2 \, dv_g \leq C(|k|(\mathcal{V} - \mathcal{V}_{inf}) + (|k|(\mathcal{V} - \mathcal{V}_{inf})\mathcal{Q}_0)^{\frac{2}{3}}),
\]

where \( C \) is a numeric constant.

**Proof:**
Combine equations in Proposition 5 and equation (41). Making \( x = |k||\hat{K}|_{L^2} \)
and \( a = |k|^{\frac{2}{3}}(\mathcal{V} - \mathcal{V}_{inf})^{\frac{1}{3}} \) if \( Y(\Sigma) \leq 0 \) or \( a = |k|^{\frac{2}{3}} \mathcal{V} \) if \( Y(\Sigma) > 0 \) we arrive at the inequality

\[
x^2 - Cax - Ca^2 - CaQ_0 \leq 0.
\]
From it we get

\[
x^2 \leq C(a^2 + aQ_0). \quad \Box
\]

A direct consequence of the propositions above is

**Proposition 8** Say \( \Sigma \) is a compact three-manifold, then \( \mathcal{V} \), \( |k| \) and \( Q_0 \) control \( \|\hat{\text{Ric}}\|_{L^2_g} \). In particular we have

\[
\|\hat{\text{Ric}}\|_{L^2_g}^2 \leq C(|k| \mathcal{V} + Q_0),
\]

where \( C \) is a numeric constant.

**Proof:**
Use \( \hat{\text{Ric}} = E - \frac{4}{3} \hat{K} + \hat{K} \circ \hat{K} - \frac{1}{3} |\hat{K}|^2 g \) together with the Propositions 6 and 7 \( \Box \)

Using the energy constraint \( R = |\hat{K}|^2 - \frac{4}{3} k^2 \) and Proposition 6 we get

**Proposition 9** Let \( \Sigma \) be a compact three-manifold. Then, \( \mathcal{V} \), \( |k| \) and \( Q_0 \) control the scalar curvature in the following way

\[
\int_\Sigma |\nabla R|^\frac{2}{3} + R^2 \, dv_g \leq C(|k| \mathcal{V} + Q_0),
\]

where \( C \) is a numeric constant.

Note that \( |\nabla R|^\frac{2}{3} \) and \( R^2 \) scale as a distance\(^{-4}\).

**Proof:**
Squaring the energy constraint and integrating we obtain

\[
\int_\Sigma R^2 \, dv_g \leq \int_\Sigma |\hat{K}|^4 + \frac{4}{9} k^4 \, dv_g \leq C(|k| \mathcal{V} + Q_0).
\]
where in the last inequality we have used Proposition 6 \( \Box \)

On the other hand, differentiating the energy constraint we have \( |\nabla R|^\frac{2}{3} \leq C |\nabla \hat{K}|^\frac{2}{3} |\hat{K}|^\frac{2}{3} \). Integrating and applying the Hölder inequality we obtain

\[
\int_\Sigma |\nabla R|^\frac{2}{3} \, dv_g \leq C (\int_\Sigma |\nabla \hat{K}|^2 \, dv_g)^{\frac{2}{3}} (\int_\Sigma |\hat{K}|^4 \, dv_g)^{\frac{1}{3}},
\]
and if we apply Proposition 6 over each one of the factors on the RHS of the last equation we obtain
\[ \int_\Sigma |\nabla R|^4 dv_g \leq C(|k|V + Q_0), \]
as desired. \qed

Because of the monotonicity of the reduced volume in the future direction we have

**Proposition 10** Say \( Y(\Sigma) \leq 0 \). Then along the future CMC evolution \( Q_0 \) controls \( \|\text{Ric}\|_{L_2^g}, \|\tilde{K}\|_{H_1^g} \) and \( \|\tilde{K}\|_{L_4^g} \).

In the presence of matter satisfying the dominant energy condition we have

**Proposition 11** Say \( \Sigma \) is a compact three-manifold. Then (over cosmological normalized states) \( Q_0, \|G\rho\|_{L_2^g} \) and \( V - V_{\text{inf}} \) control \( \|\tilde{K}\|_{H_1^g}, \|\tilde{K}\|_{L_4^g} \). In particular if \( Y(\Sigma) \leq 0 \) we have the formula (same for \( Y(\Sigma) = 0 \) than for \( Y(\Sigma) < 0 \))
\[ \int_\Sigma |\tilde{K}|^2 dv_g \leq C((V - V_{\text{inf}}) + (V_{\text{inf}})^{1/2}(Q_0^{1/2} + \|G\rho\|_{L_2^g})) \]
while if \( Y(\Sigma) > 0 \) we have
\[ \int_\Sigma |\tilde{K}|^2 dv_g \leq C(V + V_{\text{inf}}^{1/2}(Q_0^{1/2} + \|G\rho\|_{L_2^g})), \]
where \( C \) is a numeric constant.

**Proof:**
\( Y(\Sigma) \leq 0 \): the proof proceeds in parallel to the proof of Proposition 5 but this time instead using the equation
\[ -\Delta \phi + \frac{3}{4} (\phi^5 - \phi) = \frac{1}{8}(|\tilde{K}|^2 + 16\pi G\rho)\phi^5, \]
and then the equation (43).

\( Y(\Sigma) > 0 \): use equation (43) in the formula
\[ k|k|^2 \int_\Sigma |\tilde{K}|^2 dv_g \leq C|k|^{1/2}V^{1/2}(\int_\Sigma |\tilde{K}|^4 dv_g)^{1/2}, \]
and after making \( x = \|\tilde{K}\|_{L_2^g} \) solve for \( x \). \qed

The lapse has important natural properties that we describe in the proposition below.

**Proposition 12** Say \( \Sigma \) is a compact three-manifold. Then, \( \|N\|_{L_\infty} \leq 3/k^2 \) and \( \|N\|_{H_2^g} \) is controlled by \( 1/|k| \) and \( V \). In particular we have the bound
\[ \int_\Sigma |\nabla N|^2 + |\tilde{K}|^2 N^2 + \frac{k^2}{3} N^2 dv_g = \int_\Sigma N \leq \frac{1}{9|k|^5} V. \]

**Proof:**
From the maximum principle we have \( N \leq 3/|k|^2 \). To get equation (49) multiply the lapse equation by \( N \), integrate, and use the estimate for \( \|N\|_{L_\infty} \). \qed
3.3.2 Estimates of elliptic type.

Assumptions in Section 3.3.2 All through this section we will assume \((g, K)\) is a cosmologically normalized state.

We turn now our attention to the study of the influence of the higher order Bel-Robinson energies over states.

We need a notion of Sobolev norm for space-time tensors and with respect to the CMC foliation. We do that in the following way. Consider a space-time tensor \(U_{a_1,\ldots,a_l}\) of rank \((l,0)\). For any subsequence \(I = (i_1,\ldots,i_n)\) \((n \leq l)\) of the sequence \((1,\ldots,l)\), and the obvious compliment subsequence \(\bar{I}\), define

\[
T_I = T_{a_1} \cdots T_{a_n}
\]

and

\[
P_I = P_{a_1}^{a'_1} \cdots P_{a_{n}}^{a'_{n}}
\]

where \(P_a^{a'} = g_a^{a'} + T_a T^{a'}\) is the horizontal projector. We can decompose \(U\) as

\[
U = \sum_{n=1,\ldots,l; \mid I \mid = n} (P_I \left< U, T_I \right>)(T_I),
\]

where \(\left< U, T_I \right>\) is the contraction of \(U\) and \(T_I\). For each summand, the factors on the left are horizontal, while the factors on the right are vertical. For instance, for \(U_{ab}\) we have

\[
U_{ab} = P_a^{a'} P_b^{b'} U_{a'b'} - P_a^{a'} (U_{a'b'} T^{b'}) T_b - P_b^{b'} (U_{a'b'} T^{a'}) T_a + (U_{a'b'} T^{a'} T^{b'}) T_a T_b.
\]

The same decomposition holds for tensors of arbitrary rank \((l,l')\). Now the \(H^*_s\)-norm of \(U\) on a slice \(\Sigma\) of a CMC foliation is defined as the sum of the \(H^*_s\)-norms of the tensors \(P_I \left< U, T_I \right>\). We will be using this convention somehow implicitly all through and without further comments.

From now on \(W_i = \nabla^{L_1} W_0\) where \(T\) is the future pointing unit normal to the CMC foliation.

During the proof of the propositions until the end of this section, we will use the notation \(H^*_s\) instead of \(H^*_{s,x_j}(B(o,r))\) which is the one used inside the statements.

**Proposition 13** Say \(\Sigma\) is a compact three-manifold. Then, (the data) \(\nu, \mathcal{V}\) and \(Q_0\) control \(\|N\|_{H^2_A}\) where \(A\) is a \(H^2\)-canonic harmonic atlas.

**Proof:**

By Proposition 12 \(\|N\|_{L^2_1}\) is controlled by \(\mathcal{V}\) and \(Q_0\). By Proposition 3 \(|\tilde{K}|^2\) and therefore \(|K|^2\) are controlled in \(L^2_0\) by the data. The result then follows by Proposition 1 (I).

\[\square\]
Remark 3 One can get an estimate for the intrinsic norm \( \|N\|_{H^2} \) in terms only of \( V \) and \( Q_0 \) (i.e. without involving the volume radius) if the Ricci curvature is bounded below. Indeed that follows from
\[
\int_{\Sigma} |\nabla \nabla N|^2 + \text{Ric}(\nabla N, \nabla N) dv_g = \int_{\Sigma} \langle \Delta N \rangle^2 dv_g,
\]
the use of the lapse equation in the RHS of the equation above and finally Propositions 12 and 6.

The proposition below shows that for cosmologically normalized states, \( \nu, V \) and \( Q_0 \) control the \( H^2_A \) norm of \( 1/N \), where \( A \) is a \( H^2 \)-canonic harmonic atlas. This implies in particular that the infimum of the lapse is never zero even for states with low regularity. As a corollary, we get that \( \nu, V \) and \( Q_0 \) control the \( H^1_A \) norm of the deformation tensor \( \Pi \).

Proposition 14 Let \( \Sigma \) be a compact three-manifold. Then \( \nu, V \) and \( Q_0 \) control \( \|1/N\|_{H^2_A} \) where \( A \) is a \( H^2 \)-canonic harmonic atlas. In particular they control \( \|\Pi\|_{H^1_A} \).

Proof:

Multiplying the lapse equation by \( 1/N^2 \) and integrating gives
\[
\int_{\Sigma} \frac{2 |\nabla N|^2}{N^3} + \frac{1}{N^2} dv_g = \int_{\Sigma} \frac{|K|^2}{N} dv_g \leq \left( \int_{\Sigma} |K|^4 dv_g \right)^{\frac{1}{4}} \left( \int_{\Sigma} \frac{1}{N^2} dv_g \right)^{\frac{3}{4}}.
\]
This shows in particular that \( \|N\|_{H^1} \) is controlled by \( V \) and \( Q_0 \). We multiply now the lapse equation by \( 1/N^3 \) and integrate, it gives
\[
\int_{\Sigma} 3 |\nabla N|^2 \frac{1}{N^4} + \frac{1}{N^3} dv_g = \int_{\Sigma} \frac{|K|^2}{N^2} dv_g \leq \left( \int_{\Sigma} |K|^4 \right)^{\frac{1}{4}} \left( \int_{\Sigma} \frac{1}{N^4} dv_g \right)^{\frac{3}{4}}.
\]
By the Sobolev embedding \( H^1_A \hookrightarrow L^6_A \), the RHS is controlled by \( \nu, V \) and \( Q_0 \). We will use this estimate below. Consider the Laplacian of \( 1/N \). We compute
\[
\Delta \frac{1}{N} = \frac{\Delta N}{N} + \frac{|\nabla N|^2}{N^2} = -\frac{1}{N} - |K|^2 + \frac{|\nabla N|^2}{N^2}.
\]
We have then the elliptic non-homogeneous equation for \( 1/N \)
\[
\Delta \frac{1}{N} + \frac{1}{N} + \frac{|\nabla N|^2}{N} \frac{1}{N} = -|K|^2,
\]

16 We will operate assuming a priori regularity of \( 1/N \). This is indeed guaranteed from the assumption made at the beginning of Section 3. To show that in this case the estimate also descend for states \( (g, K) \) with \( H^2 \times H^1 \) regularity proceed as follows. Smooth out the coefficient \( |K|^2 \) to get, from the maximum principle applied to the lapse equation, an upper bound on \( 1/N \). This gives the necessary regularity to operate. As the estimate on the \( L^\infty \) norm of \( 1/N \) that one obtains (having smoothed \( |K|^2 \)) depends only on \( \nu, V \) and \( Q_0 \) it passes to the limit when the smoothing is undone.

17 It is important here that the norm of the embedding is controlled from above by \( \nu, V \) and \( Q_0 \).
where, to the effect of applying elliptic estimates, we are thinking $|\nabla N|^2/N$ as a factor in front of the variable $1/N$. We know that $|K|^2$ is controlled in $L^2_A$. The result then follows by Proposition 1 (I), if we show that $|\nabla N|^2/N$ is controlled in $L^2_A$. We compute

$$
\int_\Sigma \left( \frac{|\nabla N|^2}{N} \right)^2 dv_g = \left( \int_\Sigma \frac{|\nabla N|^2}{N^4} dv_g \right)^{\frac{1}{2}} \left( \int_\Sigma |\nabla N|^6 dv_g \right)^{\frac{1}{2}}.
$$

As was shown above, the first factor in the RHS of the previous equation is controlled by $\nu$, $V$, and $Q_0$. The second factor is controlled by $\nu$, $V$, and $Q_0$ by the embedding $H^1_A \hookrightarrow L^6_g$ and Proposition 13.

**Proposition 15** Say $I \geq 0$. Let $\bar{r} < r < r_{I+2}(o)$, and say $\{x\}$ is a harmonic coordinate system covering $B(o, r_{I+2}(o))$ and satisfying \[13\]-\[14\]. Then (the data)

$$
\|W_0\|_{H^{I+1}_A(B(o,r))}, \|W_1\|_{H^{I+1}_A(B(o,r))}, \|\hat{K}\|_{L^2_A(B(o,r))}, \bar{r}$ and $r$ control $\|Ric\|_{H^{I+2}_A(B(o,\bar{r}))}$ and $\|\hat{K}\|_{H^{I+2}_A(B(o,\bar{r}))}$. In particular, they control $r_{I+3}(o)$ from below.

**Proof:**

The proof proceeds studying the equation \[33\] (making $\hat{K} = A$) to get the estimate on $\hat{K}$ and an appropriate elliptic system \([53]-[54]\) to get the estimate on $Ric$. From equation \[24\] we have

$$
d^\nabla (\hat{K})_{ijk} = \nabla_i \hat{K}_{jm} - \nabla_j \hat{K}_{im} = \epsilon_{ij}^l B_{lm},
$$

and therefore

\[(50)\]

$$
d^\nabla d^\nabla \hat{K} = -\epsilon_{ij}^l \nabla_i B_{tm} - \epsilon_{im}^l \nabla_i B_{lj} = -2 \text{curl}(B).
$$

From equation \[30\] we have

$$
\text{curl}(B) = E(\nabla_T W) + \frac{3}{2} (E \times K) - \frac{1}{2} kE.
$$

Equations \[50\], \[51\] and \[33\] give the elliptic equation

\[(52)\]

$$
2 \nabla^* \nabla \hat{K} = -R(\hat{K}) - 2(E(W_1) + \frac{3}{2} (E \times K) - \frac{1}{2} kE).
$$

Recall $R(\hat{K})$ is a linear expression in $\hat{K}$ with coefficients involving only $Ric$. We consider the case $I = 0$ first. This case in turn is the only one that demands a special treatment. In order to apply Proposition 1 (I) to the elliptic equation \[52\] we need first to obtain control on the $H^1_{(\bar{r}+r)/2}$-norm of $\hat{K}$. This estimate follows from standard elliptic estimates on the elliptic system

\[d^\nabla (\hat{K})_{ijk} = \epsilon_{ij}^l B_{lm},

\nabla^2 \hat{K}_{ij} = 0.\]
We apply then Proposition 11(I), on the equation (52) to get the control on the $H^2_r$-norm of $\hat{K}$. To get the estimate on the $H^1_r$-norm of $\text{Ric}$ we get first an estimate for the $H^1_{(\bar{r}+r)/2}$-norm of $E$. From it and the equation $E = \text{Ric} + k\hat{K} - K \circ K$ the estimate on $\text{Ric}$ follows. To get the estimate on $E$ apply standard\textsuperscript{18} elliptic estimates on the elliptic system

\begin{equation}
\text{curl}E = -B(W_1) - \frac{3}{2}(B \times K) + \frac{1}{2}kB,
\end{equation}

\begin{equation}
\text{div}E = (K \wedge B).
\end{equation}

Now we treat the cases $I > 0$. We note that as the harmonic chart $\{x\}$ satisfies [13]-[14] we have control on $\|\text{Ric}\|_{H^I_{\bar{r}}}$ by the data, and as was mentioned before the coefficients of $\hat{K}$ in the expression $\mathcal{R}(\hat{K})$ involve (linearly) only $\text{Ric}$. Thus we can apply Proposition 1(I) to the elliptic equation (52) to get the control on the estimates on the elliptic system (53).

We apply then Proposition 1(I), on the equation (52) to get the control on the $\|\text{Ric}\|_{H^I_{\bar{r}}}$, similarly applying Proposition 1(II) to the elliptic system (53). We get control on $\|E\|_{H^{I+1}_r}$, and therefore on $\|\text{Ric}\|_{H^{I+1}_r}$ from the equation $E = \text{Ric} + k\hat{K} - K \circ K$. \hfill $\square$

**Proposition 16** Say $I \geq 0$. Let $\bar{r} < r < r_{I+2}(o)$, and say $\{x\}$ is a harmonic coordinate system covering $B(o, r_{I+2}(o))$ and satisfying [13]-[14]. Then (the data) $\|W_0\|_{H^{I+1}_{(i+1)}(B(o,r))}$, $\|W_1\|_{H^I_{(i)}(B(o,r))}$, $\|\bar{K}\|_{L^2_2(B(o,r))}$, $\|N\|_{H^I_{(i)}(B(o,\bar{r}))}$, $\bar{r}$ and $r$ control $\|N\|_{H^{I+3}_{(i)}(B(o,\bar{r}))}$.

**Proof:**

We consider the case $I = 0$, the cases $I \geq 1$ easily follow by induction. By Proposition 12 $\|\text{Ric}\|_{H^{I+1}_{(r-r)/4}}$ is controlled by the data. Therefore $\|g\|_{H^{(r+r)/2}}$ is controlled by the data. By Proposition 12 too, $|\bar{K}|^2$ is controlled in $H^2_{(\bar{r}+r)/2}$. We can apply then standard elliptic estimates on the lapse equation

$$-\Delta N + |\bar{K}|^2 N = 1,$$

to get that $\|N\|_{H^2_r}$ is controlled by the data, as desired. \hfill $\square$

**Proposition 17** Say $\bar{r} < r \leq r_{I+1}(o)$ with $I \geq 1$ and say $\{x\}$ is a harmonic coordinate system satisfying [13]-[14] and covering $B(o, r_{I+1}(o))$. Then for any $(i,j)$ satisfying $0 \leq j \leq I$ and $1 \leq i \leq I$, $\|W_{ij}\|_{H^{I+1}_{(i)}(B(o,\bar{r}))}$ is controlled by (the data) $\|W_{ij+1}\|_{H^{I+1}_{(i+1)}(B(o,r))}$, $\|W_{ij}\|_{H^{I+1}_{(i)}(B(o,r))}$, $\|W_0\|_{H^{I+1}_{(i)}(B(o,r))}$, $\|W_1\|_{H^{I+1}_{(i)}(B(o,r))}$, $\|J(W_{ij})\|_{H^{I+1}_{(i)}(B(o,r))}$, $\bar{r}$, $r$ and $\|\bar{K}\|_{L^2_2(B(o,r))}$.

**Proof:**

\textsuperscript{18}To apply standard elliptic estimates we note that as was proved before the coefficients involving $K$ are controlled in $H^2_{(\bar{r}+3(r-r)/4)}$. 

33
Think the elliptic system \(28\)-\(31\) as a first order elliptic system of the form \(18\), with \(U = (E(W_j), B(W_j))\). By Proposition \(15\) the coefficients \(A_{\mu}^n\) which involve only \(\hat{K}\) are controlled in \(H^{1+1}_{1x}(B(o,(\bar{r}+r)/2))\) by the data. The result then follows by applying Proposition \(1\) \(2\) to the elliptic system \(28\), \(29\).

\[
\begin{align*}
\text{Remark 4}
\quad &\text{Applying Proposition \(17\) when } j = 0 \text{ we deduce that } \|W_0\|_{H^r} \text{ is controlled by } \|W_0\|_{H^{r-1}} \text{ and } \|W_1\|_{H^{r-1}}, \bar{r}, r \text{ and } \|\hat{K}\|_{L^2}. \text{ This tells essentially that one can replace the data } \|W_0\|_{H^{r-1}}, \|W_1\|_{H^{r-1}} \text{ by the data } \|W_0\|_{H^r}, \|W_1\|_{H^{r-1}} \text{ inside those statements whose hypothesis contain the data } r \text{ and } \|\hat{K}\|_{L^2}. \text{ This Remark will be used later(sometimes implicitly).}
\end{align*}
\]

We prove next, in Proposition \(18\) an inductive formula for the currents \(J(W_j)\) and then in Proposition \(19\) estimates on the time derivative of the deformation tensor \(\Pi\).

\begin{equation}
J(W_j) = \sum (\nabla_T^{m_1} \Pi)^{n_1} \cdots (\nabla_T^{m_j} \Pi)^{n_j} \Pi^l \nabla W_k
\end{equation}

(55)

\begin{equation}
+ \sum (\nabla_T^{m_1} \Pi)^{\tilde{n}_1} \cdots (\nabla_T^{m_s} \Pi)^{\tilde{n}_s} \Pi^\tilde{l} \nabla (T * \Pi^q) (W_k).
\end{equation}

(56)

where every asterisk * is some tensor product and each expression of the form \((\nabla_T^\alpha \Pi)^n\) is a *-product of \(\nabla_T \Pi\) with itself \(n\)-times. The sum on the RHS of (55) is among sequences \((m_1, n_1), \ldots, (m_s, n_s), l, k)\) with \(k \leq j-1, \; m_1 \geq 1, \ldots, m_s \geq 1\) and \(\sum n_j(1 + m_j) + l + k = j\), while the sum (56) is among sequences \((\tilde{m}_1, \tilde{n}_1), \ldots, (\tilde{m}_s, \tilde{n}_s), \tilde{l}, \tilde{q}, \tilde{k})\) with \(\tilde{m}_1 \geq 1, \ldots, \tilde{m}_s \geq 1\) and \(\sum \tilde{n}_j(1 + \tilde{m}_j) + \tilde{r} + l + q = j-1\).

\begin{equation}
J(W_j) = \Pi * \nabla W_0 + T * \Pi * W_0 \nabla W_j + \nabla_T J(W_j).
\end{equation}

(57)

\textbf{Proof:}

First note that

\begin{align*}
J(W_{j+1})_{bcd} &= \nabla^a (\nabla_T W_{j,abcd}) = (\nabla^a T^e) \nabla_e W_{j,abcd} + T^e \nabla^a \nabla_e W_{j,abcd} \\
&= \Pi * \nabla W_j + T * \Pi * W_j + \nabla_T J(W_j).
\end{align*}

Now for \(j = 1\) we have \(J(W_1) = \Pi * \nabla W_0 + T * \Pi * W_0 \nabla W_1\) which agrees with the form of the formula above (55), (56). For \(j > 1\) we proceed by induction. Assume the \(J(W_j)\) has the desired expansion. The first two terms in the RHS of equation (57) are of the desired form, so it remains to prove that \(\nabla_T J(W_j)\) is of the desired form. Terms in (55) and (56) are characterized by sequences \((m_1, n_1), \ldots, (m_s, n_s), l, k)\) with \(\sum j_n(1 + m_j) + l + k = j\) and \((\tilde{m}_1, \tilde{n}_1), \ldots, (\tilde{m}_s, \tilde{n}_s), \tilde{l}, \tilde{q}, \tilde{k})\) with \(\sum \tilde{n}_j(1 + \tilde{m}_j) + \tilde{r} + l + q = j-1\). We show that the \(\nabla_T\) derivative of any term of the form (55) or (56) gives terms of the same form characterized by sequences adding \(j + 1\) for terms of the form (55) and \(j\) for terms of the form (56). Let us consider derivatives of terms of the form (55). The derivatives \(\nabla_T (\nabla_T^j \Pi)^n = n (\nabla_T^{j+1} \Pi) * (\nabla_T^j \Pi)^{n-1}\)
transform the pairs \((m, n)\) (inside a sequence characterizing a term of the form) into pairs \((m + 1, 1), (m, n - 1)\) but leaving the rest of the sequence unaltered. As \(1(1 + 1 + m) + (n - 1)(1 + m) = 1 + n(1 + m)\) the new sequence adds \(j + 1\). Similarly the derivatives \(\nabla_T \Pi^l = \nabla_T \Pi \ast \Pi^{l-1}\) transform the \(l\) (inside a sequence) into the pair (1, 1) and the number \(l - 1\) but leaving the rest of the sequence unaltered. Again in this case as \(1(1 + 1) + l - 1 = l + 1\), the new sequence adds \(j + 1\). Finally the derivatives \(\nabla_T \nabla W_k\) are

\[
\nabla_T \nabla W_k = \nabla W_{k+1} + \Pi \ast W_k + T \ast Rm \ast W_k, \tag{58}
\]

which transform the number \(k\) (inside a sequence) into three new sequences. One with the new number \(k + 1\) and leaving the rest of the sequence unaltered, thus adding \(j + 1\). A second with the numbers \(k\) and \(l = 1\) and leaving the rest unaltered, thus adding \(j + 1\). Finally a third of the kind which has values \(l = q, 0, k = k\) and \((\tilde{m}_1, \tilde{n}_1), \ldots, (\tilde{m}_s, \tilde{n}_s) = ((m_1, n_1), \ldots, (m_s, n_s))\) adding \(j\) as desired.

The analysis of the \(\nabla_T\) derivatives for terms of the form proceeds exactly in the same fashion. \(\square\)

**Proposition 19** Say \(\bar{r} < r \leq r_{I+2}(o)\), \(I \geq 0\). Let \(\{x\}\) be a harmonic coordinate system satisfying the conditions (13) and covering \(B(o, r_{I+2}(o))\). Then, for any \((m, i)\) satisfying, \(m \geq 0, i \geq 0\) and \(2 \leq m + i \leq I + 2\), \(\|\nabla_T^m \Pi\|_{H^I_{(x)}(B(o,r))}\) is controlled by (the data) \(\|W_k\|_{H^I_{(x)}(B(o,r))}\), for \(k = 0, \ldots, m\), \(\|\nabla_T^k N\|_{H^I_{(x)}(B(o,r))}\), for \(k = 0, \ldots, m\), \(\|K\|_{L^2_{(x)}(B(o,r))}\), \(\bar{r}\) and \(r\).

**Proof:**

The proof proceeds by induction. We first observe that the cases comprising those \((m, i)\) such that \(m = 0\) and \(2 \leq i \leq I + 2\) are proved by Propositions 13 and 14. The induction process will be as follows: assume the proposition is proved for all \((m, i)\) with \(2 \leq m + i \leq \bar{I} + 2\) and \(m \leq \bar{m}\). This assumption will be referred in what follows as the inductive hypothesis (IH). Under the inductive hypothesis we will prove that the proposition is valid for \(m = \bar{m} + 1\) and all \(i\) with \(2 \leq i + \bar{m} + 1 \leq I + 2\). In this way we cover all \((m, i)\) with \(2 \leq m + i \leq I + 2\). From now on we assume \((m, i) = (\bar{m} + 1, \bar{i})\) with \(2 \leq \bar{i} + \bar{m} + 1 \leq I + 2\) and the data for \((m, i) = (\bar{m} + 1, \bar{i})\).

**Observation 1:** A crucial observation which is easily checked is the following: the hypothesis of the Proposition for \(m = \bar{m} + 1\) and \(i = \bar{i}\), contain the hypothesis of the Proposition for all \((m, i)\) with \(m \leq \bar{m}\) and \(2 \leq i + m \leq \bar{m} + \bar{i} + 1\). In this way we have that \(\|\nabla_T^k \Pi\|_{H^I_{(x)}(m-\bar{m})+1}\), with \(0 \leq \bar{m} \leq \bar{m} + 1\), is controlled by the data of the Proposition for \((m, i) = (\bar{m} + 1, \bar{i})\).

The following commutation relation will be used recursively.

\[
\nabla_T(\nabla a U_b) = \nabla a(\nabla_T U_b) + T^a U^d Rm_{cabd} - \Pi^c a \nabla a U_b. \tag{59}
\]

From it, we get

\[
\nabla_T^{\bar{m}+1} a b = \nabla_T^{\bar{m}}(\nabla a(\nabla_T T_b) + T^a T^d Rm_{cabd} - \Pi^c a \nabla a b). \tag{60}
\]
We treat each one of the three terms that appear on the RHS of the last equation separately. We treat first the last term. We make the expansion
\[ \nabla^m_T(\Pi_a^c \Pi_{cb}) = \sum_{\alpha + \beta = m} \nabla^\alpha_T \Pi_a^c \nabla^\beta_T \Pi_{cb}. \]

From Observation 1, we get that at each summand, \( \nabla^\alpha_T \Pi_a^c \) and \( \nabla^\beta_T \Pi_{cb} \), are controlled in \( H_{i+\alpha}^{\tilde{m}} \) and \( H_{i+\beta}^{\tilde{m}} \) respectively. We get therefore that the full expression is controlled in \( H_{i+\tilde{m}} \). We treat next the second kind of terms in equation (60).

We make the expansion
\[ \nabla^\tilde{m}_T(T^c T^d R_{m_{cabd}}) = \sum_{\alpha + \beta + \gamma = \tilde{m}} (\nabla^\alpha_T T^c)(\nabla^\beta_T T^d)W_{\gamma,cabd}. \]

We will treat this term using the following Fact, which, as the other Facts to be stated later, are going to be proved after the main argument is finished.

**Fact 1**: terms of the form \( \nabla^\delta_T T \) for \( 1 \leq \delta \leq \tilde{m} + 1 \) are controlled in \( H_{i+\delta}^{\tilde{m}+2} \) by the data.

We know that \( \|W_{\gamma}\|_{H_{i+\gamma}} \) is controlled by the data. It follows from this and Fact 1 that the second kind of terms in equation (60) are also controlled in \( H_{i}^{\tilde{m}} \). We discuss now the first kind of terms in equation (60), namely the terms of the form
\[ \nabla^\tilde{m}_T(\nabla_a(\nabla_T T_b)). \]

We would like to pass the \( \nabla_T \)'s on the left of this expression to the right of \( \nabla_a \). We will show that every time a \( \nabla_T \) is moved past of \( \nabla_a \) we generate a pair of terms that are seen to be controlled in \( H_{i}^{\tilde{m}} \). We write
\[ \nabla^\tilde{m}_T(\nabla_a(\nabla_T T_b)) = \nabla^{\tilde{m}-1}_T(\nabla_a(\nabla_T(\nabla_T T_b)) + T^c(\nabla_T T^d)R_{m_{cabd}} - \Pi_a^c(\nabla_c(\nabla_T T_b))). \]

We state now Fact 2 and Fact 3 that treat the second and third kind of terms appearing in the right hand side of the previous equation.

**Fact 2**: terms of the form \( \nabla^{\tilde{m}-j}_T(T^c(\nabla_j^T T^d)R_{m_{cabd}}), \) with \( 1 \leq j \leq \tilde{m} \) are controlled in \( H_{i}^{\tilde{m}+j} \) by the data.

**Fact 3**: terms of the form \( \nabla^\Gamma_T(\Pi_a^c \nabla_c(\nabla_j^T T_b)), \) with \( 1 \leq j \leq \tilde{m} \) and \( 0 \leq \Gamma \leq \tilde{m} - j \), are controlled in \( H_{i}^{\tilde{m}} \) by the data.
The first kind of terms in equation (61) is reduced again moving \( \nabla_T \) past of \( \nabla_a \). We compute
\[
\nabla_T^{\bar{m}-1}(\nabla_a(\nabla_T^2 T_b)) = \nabla_T^{\bar{m}-2}(\nabla_a(\nabla_T^3 T_b) + T^c \nabla_T^2 T^d R_{abcd} - \Pi_a \nabla_c(\nabla_T^2 T_b)).
\]

Again the second and third kind of terms in the previous equation are treated using Fact 2 and Fact 3. We keep going like this until we get a last term to be treated. This term has the form
\[
(62) \quad \nabla_a(\nabla_T^{\bar{m}+1} T_b).
\]

We must treat this term following another route. We write
\[
\nabla_a(\nabla_T^{\bar{m}+1} T_b) = \nabla_a(\nabla_T^{\bar{m}}(-\frac{\nabla_b N}{N})) = -\nabla_a(\nabla_T^{\bar{m}}(P_b^{b'} \frac{\nabla_b N}{N})),
\]
where \( P_b^{b'} \) is the horizontal projection. We make the expansion
\[
\nabla_a(\nabla_T^{\bar{m}}(P_b^{b'} \frac{\nabla_b N}{N})) = \sum_{\alpha+\beta=\bar{m}} \nabla_a((\nabla_T^{\alpha} P_b^{b'})(\nabla_T^{\beta} \frac{\nabla_b N}{N})).
\]
Let us consider the expression \( \nabla_T^{\gamma} P_b^{b'} \). We compute
\[
\nabla_T^{\gamma} P_b^{b'} = -\nabla_T^{\gamma}(T_b T^{b'}) = -\sum_{\gamma+\delta=\alpha} (\nabla_T^{\gamma} T_b)(\nabla_T^{\delta} T^{b'}).
\]

By Fact 2 we know each summand is controlled in \( H^{\bar{r}+2}_b \) and therefore the full expression is. Note by this, that if \( \nabla_a \) is applied to them, we get by writing \( \nabla_a = P_a^{a'} \nabla_a - T_a \nabla_T \), that the outcome is controlled in \( H^{\bar{r}+1}_b \). Let us consider now the expression \( \nabla_T^{\gamma} \frac{\nabla_b N}{N} \). We compute
\[
(63) \quad \nabla_T^{\gamma} \frac{\nabla_b N}{N} = \sum_{\gamma+\delta=\beta} (\nabla_T^{\gamma} \frac{1}{N})(\nabla_T^{\delta} \nabla_b N).
\]
Note that if we expand \( \nabla_T^{\gamma} 1/N \) (using the quotient rule) we get using the data that this term is controlled at least in \( H^{\bar{r}+2}_b \). We consider next the expression \( \nabla_T^{\delta} \nabla_b N \). From the identity
\[
\nabla_T \nabla_b f = \nabla_b \nabla_T f - \Pi_b \nabla_c f,
\]
we compute
\[
(64) \quad \nabla_T^{\delta} \nabla_b N = \nabla_T^{\delta-1} \nabla_b \nabla_T N - \nabla_T^{\delta-1} (\Pi_b^{b'} \nabla_c N).
\]
We use the next Fact (Fact 4) to treat the second term in the RHS of the previous equation.

Fact 4 : Say \( 0 \leq \delta - \bar{\delta} \leq \bar{m} - 1 \) and \( 0 \leq \delta \leq m - 1 \). Then, terms of the form
\[
\nabla_T^{\delta-\bar{\delta}}(\Pi_b^{b'} \nabla_c(\nabla_T^{\bar{\delta}} N)),
\]
are controlled in \( H^{\bar{r}+2}_b \) by the data.
If we keep moving $\nabla T$ past of $\nabla \psi$ in the RHS of equation (64) and applying at each time the Fact 4 we get that, except for the term

$$\nabla b' \nabla \bar{m} T N,$$

occurring when $\beta = \bar{m}$, all the rest are controlled in $H^{\bar{m}+2}_{\bar{i}+2}$. Similarly if we apply $\nabla a$ over the expression (63) we get, after writing $\nabla a = P_a a' - T_a \nabla T$, that when $\beta < \bar{m}$, the outcome is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$ and when $\beta = \bar{m}$ it can be written as the term

$$\nabla a \nabla b' \nabla \bar{m} T N,$$

plus a term controlled in $H^{\bar{m}+1}_{\bar{i}+2}$. Putting all together we conclude that the expression (62) is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$ if we can prove that the expression

$$P_b' \nabla a \nabla b' \nabla \bar{m} T N,$$

is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$. To see that write it in the form

$$P_b' \nabla \psi (P_a a' \nabla \bar{m} T N - T_a \nabla \bar{m} T N) = \nabla b \nabla a \nabla \bar{m} T N - P_b' \Pi \nabla a \nabla \bar{m} T N - T_a \nabla b \nabla \bar{m} T N,$$

and use the data. To finish the proposition it remains to prove Facts 1-4 that we do next.

**Proof of Fact 1:**

We prove it by induction. First note that $\nabla T T a = T^c \Pi c a$ is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$ and thus the Fact 1 holds when $\delta = 1$. Assume we have shown the Fact 1 is valid until $\delta = \delta_0$, we will show it is also valid when $\delta = \delta_0 + 1$. We compute

$$\nabla \delta_0 + 1 T a = \nabla \delta_0 (T^c \Pi c a) = \sum_{\alpha + \beta = \delta_0} (\nabla \alpha T^c) (\nabla \beta \Pi c a).$$

From the IH we know $\nabla \beta \Pi$ is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$ and by the assumption $\nabla \alpha T^c$ is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$. The two estimates imply the Fact 1.

**Proof of Fact 2:**

We compute

$$\nabla \bar{m} - j (T^c \nabla \beta j T^d) R M_{c a b d} = \sum_{\alpha + \beta + \gamma = \bar{m} - j} (\nabla \alpha T^c) (\nabla \beta j T^d) \mathbf{W}_{c a b d}.$$

By hypothesis $\mathbf{W}_{c a b d}$ is controlled in $H^{\bar{m}+1}_{\bar{i}+2}$ and by Fact 4 any one of the other two factors in the previous equation is controlled at least in $H^{\bar{m}+1}_{\bar{i}+2}$. The Fact 2 gets proved from both estimates.

**Proof of Fact 3:**

The proof of this fact follows by induction. First note that the Fact 3 is valid when $\Gamma = 0$. This follows from Fact 4 and writing $\nabla c = P_c c' \nabla c' - T_c \nabla T$. Assume
the Fact 3 is valid for all \((\Gamma, j)\) satisfying \(\Gamma \leq \Gamma_0, \Gamma_0 \leq \bar{m} - 2, 0 \leq \Gamma \leq \bar{m} - j\) and \(1 \leq j \leq \bar{m}\). Then we show the Fact is valid when \(\Gamma = \Gamma_0 + 1\) as well. We compute

\[
\nabla_T^\gamma (\Pi_a^c (\nabla c \nabla_T^j T_b)) = \sum_{\alpha+\beta=\Gamma} (\nabla_T^{\alpha} \Pi_a^c) (\nabla_T^\beta \nabla c (\nabla_T^j T_b)).
\]

The factors \(\nabla_T^\gamma \Pi\) are controlled in \(H^{i+1}_{\tilde{j} + (\bar{m} - \alpha) + 1}\) and therefore controlled in \(H^{i+2}_{\tilde{j}}\). It is enough to prove then that the factors \(\nabla_T^\beta \nabla c (\nabla_T^j T_b)\) are controlled in \(H^{i}_\tilde{j}\) by the data. We compute them in the form

\[
\nabla_T^\beta (\nabla c (\nabla_T^j T_b)) = \nabla_T^{\beta-1} (\nabla c (\nabla_T^{j+1} T_b)) + T^m (\nabla_T^j T^n) \text{Rm}_{\nu m b n} - \Pi_a^m \nabla m (\nabla_T^j T_b)).
\]

The third kind of term in the RHS of the previous equation is controlled in \(H^{i}_\tilde{j}\) by the data. The second kind is controlled in \(H^{i}_\tilde{j}\) by the same argument that Fact 2 was proved. For the first kind of term, we move \(\nabla_T\) past of \(\nabla c\) again. This generates two new terms which as we have shown for the two last terms in the RHS of the previous equation, are controlled in \(H^{i}_\tilde{j}\) by the assumption and Fact 2. We keep operating like this until we get a last term \(\nabla c \nabla_T^{\beta + \delta} T_b\). Writing \(\nabla c = P_{c} c \nabla c - T_c \nabla T\), and using Fact 1 we get that this last term is also controlled in \(H^{i}_\tilde{j}\), thus finishing the proof.

\(\square\)

Proof of Fact 3:

We prove this Fact 3 by induction. First note that the Fact 3 is valid when \(\delta = \bar{\delta}\) for \(\delta = 0, \ldots, \bar{m} - 1\). This follows directly by writing \(\nabla c = P_{c} c \nabla c - T_c \nabla T\) and using the data. Assume now we have shown the Fact 3 is valid for all \((\delta, \bar{\delta})\) with \(0 \leq \bar{\delta} \leq \delta \leq \bar{m} - 1\) and \(\delta - \bar{\delta} = L\), where \(0 \leq L \leq \bar{m} - 2\). We will show the Fact 3 is valid also when \(\bar{\delta} = L + 1\). We write

\[
\nabla_T^{L+1} (\Pi_a^c (\nabla c (\nabla_T^{\tilde{j}} N)) = \sum_{\alpha+\beta=L+1} (\nabla_T^{\alpha} \Pi_a^c) (\nabla_T^\beta \nabla c (\nabla_T^{\tilde{j}} N)),
\]

The factors \(\nabla_T^\gamma \Pi\) are controlled in \(H^{i+1}_{\tilde{j} + (\bar{m} - \alpha) + 1}\). We need to show the factors \(\nabla_T^\beta (\nabla c (\nabla_T^{\tilde{j}} N))\) are controlled in \(H^{i+2}_{\tilde{j}}\). We write

\[
\nabla_T^\beta (\nabla c (\nabla_T^{\tilde{j}} N)) = \nabla_T^{\beta-1} (\nabla c (\nabla_T^{\tilde{j}+1} N - \Pi_a^m \nabla m (\nabla_T^{\tilde{j}} N)).
\]

The second kind of term in the RHS of the previous equation is controlled in \(H^{i+1}_{\tilde{j}}\) by the assumption. For the first term we move again a \(\nabla_T\) past of \(\nabla c\) and use the assumption. We keep moving \(\nabla_T\)-s past of \(\nabla c\) until getting the last term \(\nabla c \nabla_T^{\beta + \delta} N\). It follows from the data that this term is controlled in \(H^{i+2}_{\tilde{j}}\) (note that \(\beta + \delta \leq \bar{m} - 1\)). Putting all together we get that the expression (65) is controlled in \(H^{i+2}_{\tilde{j}}\) by the data.

\(\square\)

Proposition 20 Say \(\bar{r} < r \leq r_{I+2}\). For \((i, j)\) satisfying \(0 \leq i \leq I\) and \(i + j \leq I + 1\), \(|I (W_j)| |H^{i}_{\tilde{z}}(B(o, r))|\) is controlled by (the data) \(|W_j|_{H^{i+1}_{\tilde{z}}(B(o, r))}\), for \(k = 0, \ldots, j\), \(|\nabla_T^{j-k} N|_{H^{i+2+k}_{\tilde{j}}(B(o, r))}\), for \(k = 0, \ldots, j\), \(\bar{r}\), and \(|K|_{L^2(B(o, r))}\).
Proof:

The proof is based analyzing the inductive formula for the current

\[
J(W_j) = \sum (\nabla^m_i \Pi)^{n_i} \cdots (\nabla^m_i \Pi)^{n_i} \cdot \Pi^i \cdot \nabla W_k
\]

\[
+ \sum (\nabla^m_i \Pi)^{n_i} \cdots (\nabla^m_i \Pi)^{n_i} \cdot \Pi^i \cdot \nabla^q (T \ast Rm \ast W_k),
\]

where the indices satisfy \(\sum_j n_j (1 + m_j) + l + k = j\), with \(k \leq j - 1\) for summands appearing on the RHS of (80) and \(\sum_j \tilde{n}_j (1 + \tilde{m}_j) + \tilde{k} + \tilde{l} + q = j - 1\) for summands of the form (81). We treat first the summands appearing on the RHS of (80). We will show that the terms \(\nabla^m_i \Pi \Pi\) and \(\Pi\) are controlled in \(H^{1+2}\). By Proposition (19) \(\nabla^m_i \Pi\) is controlled in \(H^{2+2}\) from \(\|W_k\|_{H^{m+\epsilon-k}}\), \(k = 0, \ldots, m\) and \(\|\nabla^{m-k}_T N\|_{H^{m+\epsilon-k}}\), \(k = 0, \ldots, m\) as long as \(m + i \leq I\) (and \(m + i + 2 \geq 2\) which is satisfied trivially). We get this condition from the hypothesis: indeed we have \(1 + m \leq j\), \(i + j \leq I + 1\) so we have \(i + m \leq I\).

On the other hand we have a priori control on \(\|W_k\|_{H^{i+2-k}}\) for \(k = 0, \ldots, j\) and \(\|\nabla^{n-k}_T N\|_{H^{i+2-k}}\), \(k = 0, \ldots, j\) from the data, which covers the condition on \(W_k\) and \(N\) required before. Similarly we have control on \(\|\Pi\|_{H^{i+1}}\) and we know \(i + j + 1 \geq i + 2\). Let us consider now the factors \(\nabla W_k\) where \(k \leq j - 1\). Writing \(\nabla_a W_k = P_a^i \nabla_a W_k + T_a W_{k+1}\) we get that the expression is controlled in \(H^{1}_j\) by the data. The proof that summands of the form (81) are also controlled in \(H^{1}_j\) is direct from what we have shown and the data, after expanding \(\nabla^q (T \ast Rm \ast W_k)\) using the product rule.

We state below the global versions of Propositions (11) (15) (17) (19) and (20) and whose proof is straightforward. That will be useful in the proof of Lemma (2).

Proposition 21 Say \(\Sigma\) is a compact \(H^{1+3}\)-Riemannian three-manifold, where \(I \geq 0\). Then, \(\|Rc\|_{H^{1+3}}\), \(\|\bar{K}\|_{H^{1+2}}\), and \(r_{1+2}\) are controlled by (the data) \(\|W_0\|_{H^{1+3}}\), \(\|W_1\|_{H^{1+2}}\), \(\|\bar{K}\|_{L^5}\), \(V\) and \(r_{1+2}\), where \(A\) is a \(H^{1+2}\)-canonic harmonic atlas.

Proposition 22 Say \(\Sigma\) is a compact \(H^{1+3}\)-manifold, where \(I \geq 0\). Then, (the data) \(\|W_0\|_{H^{1+3}}\), \(\|W_1\|_{H^{1+2}}\), \(\|\bar{K}\|_{L^5}\), \(\|N\|_{L^5}\) and \(r_{1+2}\) control \(\|N\|_{H^{1+3}}\), where \(A\) is a \(H^{1+2}\)-canonic harmonic atlas.

Proposition 23 Say \(\Sigma\) is a compact \(H^{1+2}\)-Riemannian three-manifold, where \(I \geq 1\). Then, for any \((i, j)\) satisfying \(0 \leq j \leq I\) and \(1 \leq i \leq I\), \(\|W_j\|_{H^{1+3}}\) is controlled by the data \(\|W_{j+1}\|_{H^{1+2}}\), \(\|W_j\|_{H^{1+2}}\), \(\|W_0\|_{H^{1+2}}\), \(\|W_1\|_{H^{1+2}}\), \(J(W_j)\|_{H^{1+2}}\), \(\|\bar{K}\|_{L^5}\), \(V\) and \(r_{1+2}\), where \(A\) is a \(H^{1+2}\)-canonic harmonic atlas.

Proposition 24 Say \(\Sigma\) is a compact \(H^{1+3}\)-Riemannian three-manifold, \(I \geq 0\). Then, for any \((m, i)\) satisfying \(m \geq 0\), \(i \geq 0\) and \(2 \leq i + m \leq I + 2\), \(\|\nabla^m T \Pi\|_{H^{1+3}}\) is controlled by \(\|W_k\|_{H^{1+3}}\), \(\|W_j\|_{H^{1+2}}\), \(\|W_0\|_{H^{1+2}}\), \(\|W_1\|_{H^{1+2}}\), \(J(W_j)\|_{H^{1+2}}\), \(\|\bar{K}\|_{L^5}\), \(V\) and \(r_{1+2}\), where \(A\) is a \(H^{1+2}\)-canonic harmonic atlas.
Proposition 25 Say $\Sigma$ is a compact $H^{I+3}$-Riemannian three-manifold, where $I \geq 0$. Then, for any $(i, j)$ satisfying $0 \leq i \leq I$, $1 \leq j \leq I$ and $i+j \leq I+1$, $\| J(W_j) \|_{H^I_{\mathcal{A}}}$, is controlled by (the data) $\| W_{j-k} \|_{H^{i+k}}$, for $k = 0, \ldots, j$, $\| \nabla^k_{\mathcal{T}} N \|_{H^{i+k+2}}$, for $k = 0, \ldots, j$, $\| \hat{K} \|_{L^2_\mathcal{A}}$, $V$ and $r_{I+2}$, where $\mathcal{A}$ is a $H^{I+2}$-canonic harmonic atlas.

In the next proposition we make a last step before proving Lemma 2. We will denote with an upper-index $(k)$ the $k$-th Lie derivatives in the time direction $\partial_t = NT$.

Proposition 26 Say $\Sigma$ is a compact $H^{I+3}$-Riemannian three-manifold, $I \geq 0$. Then, for any $(m, i)$ satisfying $2 \leq i + m \leq I + 2$, $0 \leq i \leq m$ (the data) $V$, $r_{I+2}$, $\| W_k \|_{H^{i+1+(m-k)}}$, for $k = 0, \ldots, m$, control $\| \nabla^k_{\mathcal{T}} N \|_{H^{i+1+(m-k)}}$, $\| g^{(k)} \|_{H^{i+1+(m-k)}}$ for $k = 0, \ldots, m$, and $\| K^{(k)} \|_{H^{i+1+(m-k)}}$ for $k = 0, \ldots, m - 1$ (if $m \neq 0$), where $\mathcal{A}$ is a $H^{I+2}$-canonic harmonic atlas.

Remark 5 Note that the hypothesis made on the Weyl fields $W_k$ in Proposition 26 are the same as the hypothesis made for the Weyl fields $W_k$ inside the Proposition 24. Also note that we may have used the norms $\| \nabla^k_{\mathcal{T}} N \|_{H^{i+1}}$, instead of the norms $\| N^{(k)} \|_{H^{i+1}}$ as we can see from the identity $N = N \nabla_{\mathcal{T}} T N$ that one set of norms control the other. Finally note that the conclusions extracted on the lapse in Proposition 26 are exactly the hypothesis on the lapse inside Proposition 24.

Proof: The proof proceeds by induction. We treat first the case when $m = 0$ and $2 \leq i \leq I + 2$. Note that by Proposition 24 $\| \hat{K} \|_{L^2_\mathcal{A}}$ is controlled by $\| W_0 \|_{H^{i+1}_{\mathcal{A}}}$, and $V$. Note also that standard elliptic estimates on the elliptic system

$$d^\nabla (\hat{K})_{ijk} = \epsilon_{ij}^l B lm,$$

$$\nabla^j \hat{K}_{ij} = 0,$$

show that $\| \hat{K} \|_{H^{i+1}}$ is controlled by $\| W_0 \|_{H^{i+1}_{\mathcal{A}}}$, $r_{I+2}$ and $V$. This proves the required control on $K$. Now, use this estimate for $\hat{K}$ on equations (30) and (31) (that define $E(W_1)$ and $B(W_1)$) to show that $\| W_1 \|_{H^{i+2}_{\mathcal{A}}}$ is controlled by $\| W_0 \|_{H^{i+1}_{\mathcal{A}}}$, $r_{I+2}$ and $V$. Now use Proposition 22 to show that $\| N \|_{H^{i+1}_{\mathcal{A}}}$ is controlled by $\| W_0 \|_{H^{i+1}_{\mathcal{A}}}$, $r_{I+2}$ and $V$. Finally from Proposition 21 we get that $\| \hat{Ric} \|_{H^{i+1}_{\mathcal{A}}}$ is controlled and therefore by Theorem 3 $\| g \|_{H^{i+1}_{\mathcal{A}}}$ is controlled too.

We treat next the case when $m = 1$ and $1 \leq i \leq I + 1$. The fact that $\hat{K}$ is controlled in $H^{i+1}_{\mathcal{A}}$ follows from Proposition 21. This in turn implies the estimate on $g' = -2 N K$. It remains to show that $N$ is controlled in $H^{i+2}_{\mathcal{A}}$ and $N'$ in $H^{i+1}_{\mathcal{A}}$ by the data. The first estimate follows from Proposition 22. For the second estimate we need to differentiate the lapse equation. For convenience we write it in the form

$$-\Delta N + (R_g + k^2) N = 1.$$
Note from this form of the lapse equation that the $N$ is a function of the metric $g$ only. Differentiating with respect to time ($t = k$) we get

$$-\Delta N' + |K|^2 N' = \Delta N + (R_g' + 2k)N.$$  

(68)

For the derivative of the Laplacian we have \[7\]

$$\Delta f = g^{ab}\nabla_a\nabla_b f - g^{ab}(\nabla_a f)(\nabla^c g_{cb} + \frac{1}{2}\nabla_b(g^{de}g_{de})).$$

and for the derivative of the scalar curvature we have \[7\]

$$R_g' = -\Delta tr_g g' + \delta_g \delta g' - <\text{Ric}, g'>.$$ 

We can write then

$$\Delta N = -2NK^{ab}\nabla_a\nabla_b N - g^{ab}(\nabla_a N)(-2(\nabla^c N)K_{cb} - k\nabla_b N),$$

and

$$R_g' = 2k(-1 + N|K|^2) - 2k(\nabla_a\nabla_b N)K^{ab} + 2NkRic_{ab}K^{ab}.$$  

(69)

A direct inspection of equations (69) and (70) shows that the RHS of equation (68) is controlled in $H^i_A$ by the data. The elliptic estimates of Proposition (1) applied to the equation (68) would show that $N'$ is controlled in $H^i_A + 2$ if we can show that the $H^i_A$-norm of $N'$ is controlled by the data. To get that estimate multiply equation (68) by $N'$ and integrate (denote the right hand of equation (68) as $F$).

We get

$$\int_{\Sigma} |\nabla N|^2 + |K|^2(N)^2 dv_g = \int_{\Sigma} FN dv_g \leq \left( \int_{\Sigma} F^2 dv_g \right)^{\frac{1}{2}} \left( \int_{\Sigma} (N)^2 dv_g \right)^{\frac{1}{2}}.$$  

It is apparent from this that $\|N\|_{H^i_A}$ is controlled by the data.

Assume now that the proposition has been proved for all $(m, i)$ with $m \leq \tilde{m}$ and $2 \leq i + m \leq I + 2$. We will show it is valid when $(m, i) = (\tilde{m} + 1, \tilde{i})$ with $2 \leq \tilde{i} + \tilde{m} + 1 \leq I + 2$ as well. Observe that the estimates we want to prove for the metric $g$ follows from those we want to prove for the lapse $N$ and the second fundamental form $K$ using $g = -2NK$. Observe too, that the hypothesis when $(m, i) = (\tilde{m} + 1, \tilde{i})$ contains the hypothesis when $(m, i) = (\tilde{m}, \tilde{i} + 1)$. As a consequence of this, we have automatic control over $\|\nabla_k^m N\|_{H_{i+1}^i + k}$, for $k = 0, \ldots, \tilde{m}$ and $\|K^{(k)}\|_{H_{i+1}^i - m + 1 - k}$, for $k = 0, \ldots, \tilde{m} - 1$ from the data (for $(m, i) = (\tilde{m} + 1, \tilde{i})$). It remains to prove therefore that the data (for $(m, i) = (\tilde{m} + 1, \tilde{i})$) controls also $\|K^{(\tilde{m})}\|_{H_{i+1}^i + 1}$ and $\|\nabla^{\tilde{m} + 1} N\|_{H_{i+1}^i + 1}$. We show first the control over $K'$ and then we show it over $N$. It follows from the Remark (1) and Proposition (2) that $\|\nabla_k^m \Pi\|_{H_{i+1}^i + 1 - k}$, for $k = 0, \ldots, \tilde{m}$ are controlled by the data. If $U_{\alpha \beta}$ is a $T$-null and symmetric, space-time tensor, the time derivative of $U$ has the expression

$$U'_{\alpha \beta} = N \nabla_T U_{\alpha \beta} + N(U_{\alpha \gamma} \Pi_{\beta}^\gamma + U_{\omega \beta} \Pi_{\alpha}^\omega),$$  

(71)
It is easy to see that $U$ is $T$-null and symmetric. Choose $U_{ab} = K_{ab} = P_a^d P_b^b \Pi_{d'b'}$. Using recursively equation (71) together with the fact that $\| \nabla_T^m \Pi \|_{H^k_{i,m+1}}$ for $k = 0, \ldots, \bar{m}$ is controlled by the data, it is direct to see that $\| K^{(\bar{m})} \|_{H^k_{i+1}}$ is controlled by the data too.

We prove now that $\| \nabla_T^{\bar{m}+1} N \|_{H^k_{i+1}}$ is controlled by the data too. This will come from differentiating the lapse equation $\bar{m} + 1$-times. We will rely on the following fact whose proof will not be included and is straightforward.

**Fact 5:** the $\bar{m} + 1$-th time derivative of the lapse equation has an expression of the form

$$- \Delta N^{(\bar{m}+1)} + | (R_g + k^2) N^{(\bar{m}+1)} = F,$$

where $F$ is an expression controlled in $H^k_{i,A}$ by the data for $((m,i) = (\bar{m} + 1, \bar{i}))$.

With respect to the proof of this fact, we only mention that it can be proved differentiating equation (68) $\bar{m}$-times, using the expressions (69) and (70) for the RHS of (68). The derivative of the Ricci tensor is given by

$$\text{Ric}^t = \frac{1}{2} \Delta_L g^t - \delta^* \left( \delta g g^t \right) - \frac{1}{2} D_g d(tr g),$$

where $\Delta_L$ is the Lichnerowicz Laplacian (see [7]). The time derivative of the connection $\nabla$ is calculated by the following formula. Denote $\nabla^t$ the covariant derivative of $g(t)$ and pick an arbitrary time independent vector field $U^a$. Then

$$\nabla^t U^b = \nabla^t_a U^b + \tilde{\Gamma}^b_{ac} U^c,$$

where $\tilde{\Gamma}$ is

$$\tilde{\Gamma}^b_{ac} = \frac{1}{2} \left( \nabla^t_a g(t)_{bd} + \nabla^t_b g(t)_{ad} - \nabla^t_d g(t)_{ab} \right) g(t)^c.$$

Now $\nabla^t_a$ at $t = t^*$ is calculated by differentiating (73) with respect to time and evaluating at $t = t^*$ (note that the time derivative of the first term in (72) vanishes).

From Fact 5 and Proposition 1 we get therefore that $\| \nabla_T^{\bar{m}+1} N \|_{H^k_{i+1}}$ is controlled by the data.

We are ready to prove Lemma 2.

**Proof (of Lemma 2):**

We prove first that the $\bar{I} + 2$ harmonic radius is controlled from below by the BR-functional $\| (g,K) \|_{BR}$. We consider the case $\bar{I} = 0$ first and then the cases $\bar{I} > 0$.

**Case $\bar{I} = 0$.** This case follows by Proposition 5 and Theorem 4.

**Case $\bar{I} > 0$.** The proof in this case proceeds as follows. We prove first that $r_{\bar{I} + 2}$ is controlled by $r_{\bar{I} + 1}$ and $\| (g,K) \|_{BR}$. Indeed, the proof of that shows, more generally, that for any $1 \leq J \leq \bar{I}$, $r_{J + 1}$ is controlled by $r_{J + 1}$ and $\| (g,K) \|_{BR}$. As a consequence, we deduce that $r_{\bar{I} + 2}$ is controlled by $r_{2}$ and $\| (g,K) \|_{BR}$. By case $\bar{I} = 0$, $r_2$ is controlled by $\| (g,K) \|_{BR}$. The case $\bar{I} > 0$ would follow.
We prove now that \( r_{I+2} \) is controlled by \( r_{I+1} \) and \( \| (g, K) \|_{BR} \). Note that by Proposition 6, \( \| \tilde{K} \|_{L^2}^2 \) is controlled by the BR-functional. We can then replace \( \| \tilde{K} \|_{L^2}^2 \) by \( \| (g, K) \|_{BR} \) in the hypothesis of Propositions 21, 23 and 25 that we are going to use in what follows. By Proposition 21 we know that \( r_{I+2} \) is controlled by \( \| W_0 \|_{H^I_\Lambda}, \| W_1 \|_{H^I_\Lambda}, r_{I+1} \) and \( \| (g, K) \|_{BR} \), where \( \Lambda \) is a \( H^{I+1} \)-canonic harmonic atlas. If \( I = 0 \) we are done, as we have that \( r_3 \) is controlled by \( r_2 \) and \( \| (g, K) \|_{BR} \).

If instead \( I > 1 \) we apply consecutively Propositions 23 and 25. From Proposition 23 applied with \( I = I, \ i = I - 1 \) and \( j = 1 \) we deduce that \( \| W_0 \|_{H^I_\Lambda}, \| W_1 \|_{H^I_\Lambda} \) are controlled by \( \| W_0 \|_{H^{I-2}_\Lambda}, \| W_1 \|_{H^{I-2}_\Lambda}, \| W_2 \|_{H^{I-2}_\Lambda}, \| J W_1 \|_{H^{I-2}_\Lambda}, r_{I+1} \) and \( \| (g, K) \|_{BR} \). From Proposition 25 applied with \( I = I - 1, \ j = 1 \) and \( i = I - 2 \) we deduce that \( \| J W_1 \|_{H^{I-2}_\Lambda} \) is controlled by \( \| W_0 \|_{H^{I-2}_\Lambda}, \| W_1 \|_{H^{I-2}_\Lambda} \) and \( \| (g, K) \|_{BR} \). To organize visually the iteration, we have included the Figure 1. The norms included at a given column control the norms at the previous column (there may be more information at a given column than what is actually needed to control the norms on a previous column). The step we have just done represents the control of the second column over the first. Now, applying consecutively Propositions 23 and 25 over each new column we get at each step, we reach a last column with the norms \( \| W_0 \|_{H^I_\Lambda}, \ldots, \| W_j \|_{H^I_\Lambda} \). As a result of this we get that \( r_{I+2} \) is controlled by \( r_{I+1} \) and \( \| (g, K) \|_{BR} \) as desired.

To show control over \( \| \tilde{K} \|_{H^I_\Lambda} \) observe that as we have shown above \( \| W_0 \|_{H^{I-1}_\Lambda} \) and \( \| W_1 \|_{H^{I-1}_\Lambda} \) are controlled by \( \| (g, K) \|_{BR} \) where \( \Lambda \) is a \( H^{I-1} \)-canonic harmonic atlas. By Proposition 21 \( \| \tilde{K} \|_{H^I_\Lambda} \) is controlled by \( \| (g, K) \|_{BR} \). It follows that \( \| \tilde{K} \|_{H^I_\Lambda} \) is controlled by \( \| (g, K) \|_{BR} \) too.

---

\[ \begin{array}{cccccc}
 1 & 2 & 3 & 4 & 5 & \ldots & I + 1 & \ldots \\
 W_0 & H^{I+1} & H^I & H^I & H^I & \ldots & H^I & \ldots \\
 W_1 & H^I & H^{I-1} & H^{I-1} & H^{I-1} & \ldots & H^{I-1} & \ldots \\
 W_2 & H^{I-1} & H^{I-2} & H^{I-2} & H^{I-2} & \ldots & H^{I-2} & \ldots \\
 W_3 & H^{I-2} & H^{I-3} & H^{I-3} & \ldots & \ldots & \ldots & \ldots \\
 W_4 & H^{I-3} & H^{I-4} & \ldots & \ldots & \ldots & \ldots & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 W_{I+1} & & & & H^0 & \ldots & H^0 & \ldots \\
\end{array} \]

Figure 1: Iteration of control of the Sobolev norms of \( W_0 \) and \( W_1 \). Each column contains the kind of norms that control the norms on the previous column. In the table \( I = I - 1 \).

\[ ^{19} \text{Note that } r_{I+1}, \mathcal{V} \text{ and } \mathcal{E}_j \text{ control } \| W_j \|_{H^I_\Lambda} \text{ for } j = 0, \ldots, I \text{ and where } \Lambda \text{ is a } H^{I+1} \text{-canonic harmonic atlas.} \]
To finish this section let us prove a proposition on the structure of the $L^2_\gamma$-norm of the current $J(W_j)$ that will be of use in the initial value formulation.

**Proposition 27** Let $\Sigma$ be a compact three-manifold, $(g,K)$ a cosmological normalized state and $j \geq 2$. Then the $L^2_\gamma$-norm of the current $J(W_j)$ can be bounded as

\[
\|J(W_j)\|_{L^2_\gamma} \leq C(\mathcal{E}_{j-1}, \nu, \mathcal{V})Q_j^{\frac{1}{2}} + C(\mathcal{E}_{j-1}, \nu, \mathcal{V}).
\]

**Proof:**

The proof is better divided in two cases, when $j = 2$ and when $j \geq 3$. Before going into the analysis of these cases let us make the following observation.

**Observation 1.** As an outcome of the proof of Lemma 2 it can be seen that $\|\nabla^m_T \Pi\|_{H^2_\gamma}$ for $m \leq j-2$, $\|\nabla^{j-1}_T \Pi\|_{H^1_\gamma}$, $\|\nabla W_0\|_{H^1_\gamma}$, $\|\nabla W_k\|_{H^0_\gamma}$ for $k = 0, \ldots, j-2$ and $\|J(W_{j-1})\|_{H^0_\gamma}$ are controlled by $\mathcal{E}_{j-1}$, $\nu$ and $\mathcal{V}$.

**Case $j = 2$.** When $j = 2$ we can write schematically (see for instance Proposition 18)

\[
J(W_2) = \Pi \ast \Pi \ast \nabla W_0 + \nabla T \ast \nabla W_0 + \Pi \ast \nabla W_1 + T \ast \text{Rm} \ast W_1
\]

\[
+ \Pi \ast T \ast \text{Rm} \ast W_0.
\]

We need to prove that the $L^2_\gamma$-norm of any one of the terms on the RHS of the last equation can be bounded by an expression of the form (74). Let us treat them case by case. By **Observation 1** in the first term in the RHS of equation (75) the coefficients $\Pi$ of $\nabla W_0$ are controlled in $H^2_\gamma$ and the term $\nabla W_0$ itself is controlled in $H^0_\gamma$ by $\mathcal{E}_1$, $\nu$, and $\mathcal{V}$. Altogether the $L^2_\gamma$-norm can therefore be bounded by $C(\mathcal{E}_1, \nu, \mathcal{V})$. Let us treat next the term $\nabla_T \Pi \ast \nabla W_0$. We can write

\[
\|\nabla_T \Pi \ast \nabla W_0\|_{L^2_\gamma} \leq C(\mathcal{E}_1, \nu, \mathcal{V})\|\nabla_T \Pi\|_{H^1_\gamma}\|\nabla W_0\|_{H^1_\gamma}.
\]

By the **Observation 1** the norm $\|\nabla_T \Pi\|_{H^1_\gamma}$ is controlled by $\mathcal{E}_1$, $\nu$ and $\mathcal{V}$. We need to estimate the norm $\|\nabla W_0\|_{H^1_\gamma}$. Write

\[
\nabla_a W_0 = P_a \nabla_a W_0 - T_a W_0.
\]

Using formulas (21)–(26) we can write schematically

\[
W_0 = \epsilon \ast \epsilon \ast E + \epsilon \ast B \ast T + E \ast T \ast T.
\]

Using this and the connection formulas we compute (schematically)

\[
P_a \nabla_a W_0 = K \epsilon \ast \epsilon \ast E + \epsilon \ast \epsilon \ast \nabla E + K \epsilon \ast B \ast T + \epsilon \ast \nabla B \ast T + \epsilon \ast B \ast K + \nabla E \ast T \ast T + E \ast T \ast K.
\]

Note that for any $U$ an $V$ we have $\|U \ast V\|_{L^2_\gamma} \leq \|U\|_{L^4_\gamma} \|V\|_{L^4_\gamma}$, and still by Sobolev embeddings less or equal than $C(\mathcal{E}_1, \nu, \mathcal{V})\|V\|_{H^1_\gamma}\|U\|_{H^1_\gamma}$.
From this we can write

\[(76) \quad \|P_a \nabla a \cdot W_0\|_{L^2} \leq C(\mathcal{E}_1, \nu, \mathcal{V})(\|E_0\|_{H^1_0} + \|B_0\|_{H^1_0}).\]

In the same way one can prove

\[(77) \quad \|\nabla W_0\|_{H^1_0} \leq C(\mathcal{E}_1, \nu, \mathcal{V})(\|E_0\|_{H^1_0} + \|B\|_{H^1_0}).\]

From the elliptic regularity of Proposition 1 applied to the elliptic system (28)-(31)

\[(78) \quad \|E_1, B_1\|_{H^1_0} \leq C(\mathcal{E}_1, \nu, \mathcal{V})(\|E_1, B_2\|_{H^1_0} + \|E_1, B_1\|_{H^1_0} + \|J(W_0)\|_{H^1_0}),\]

and

\[(79) \quad \|E_0, B_0\|_{H^1_0} \leq C(\mathcal{E}_1, \nu, \mathcal{V})(\|E_0, B_1\|_{H^1_0} + \|E_0, B_0\|_{H^1_0}).\]

It follows from equations (77), (78) and (79) that

\[\|\nabla \Pi \ast \nabla W_0\|_{L^2} \leq C(\mathcal{E}_1, \nu, \mathcal{V})Q^2 + C(\mathcal{E}_1, \nu, \mathcal{V}).\]

Let us treat consider now the term \(\Pi \ast \nabla W_1\) in the RHS of equation (76). The coefficient \(\Pi\) is controlled in \(H^2_0\) by \(\mathcal{E}_1, \nu\) and \(\mathcal{V}\). Now use (78) to get a bound on \(\|\Pi \ast \nabla W_1\|_{L^2_0}\) of the desired form. The \(L^2_0\)-norm of the last two terms \(\ast \sum Rm \ast W_1\)

\[\text{and}\]

\[\Pi \ast T \ast Rm \ast W_0\]

are treated similarly.

**Case \(j \geq 3\).** This time we appeal to the full structure of the current \(J(W_j)\) provided by Proposition 18. Recall

\[(80) \quad J(W_j) = \sum (\nabla_t^{m_1} \Pi)^{n_1} \ast \cdots \ast (\nabla_t^{m_s} \Pi)^{n_s} \ast \Pi^l \ast \nabla W_k\]

\[(81) \quad + \sum (\nabla_t^{m_1} \Pi)^{l_1} \ast \cdots \ast (\nabla_t^{m_s} \Pi)^{l_s} \ast \Pi^l \ast \nabla_t^q (T \ast Rm \ast W_k),\]

The sum on the RHS of (80) is among sequences \((m_1, n_1), \ldots, (m_s, n_s), l, k)\) with \(k \leq j - 1, m_1 \geq 1, \ldots, m_s \geq 1\) and \(\sum_j n_j(1 + m_j) + l + k = j\), while the sum (81) is among sequences \((\hat{m}_1, \hat{n}_1), \ldots, (\hat{m}_s, \hat{n}_s), \hat{l}, \hat{q}, \hat{k})\) with \(\hat{m}_1 \geq 1, \ldots, \hat{m}_s \geq 1\) and \(\sum_j \hat{n}_j(1 + \hat{m}_j) + \hat{k} + \hat{l} + \hat{q} = j - 1\).

Let us treat the terms on the RHS of equation (80). Say \(k = 0\). In this case the coefficients in \(\nabla W_0\) are controlled in \(H^2_0\) except when there is only one coefficient of the form \(\nabla_t^{j-1} \Pi\). Now by Observation 1 either \(\nabla W_0\) or \(\nabla_t^{j-1} \Pi\) are controlled in \(H^2_0\) by \(\mathcal{E}_j-1, \nu, \mathcal{V}\). It follows that the norm \(\|\nabla_t^{j-1} \Pi \ast \nabla W_0\|_{L^2_0}\) is controlled by \(\mathcal{E}_1, \nu\) and \(\mathcal{V}\). Say now that \(k = 1, \ldots, j - 2\). In these cases the coefficients in \(\nabla W_k\) are controlled in \(H^2_0\) by \(\mathcal{E}_j-1, \nu\) and \(\mathcal{V}\). It follows from this and Observation 1 that when \(k = 0, \ldots, j - 2\) the \(L^2_0\)-norm of the expression in the RHS of equation (80) is controlled by \(\mathcal{E}_{j-1}, \nu\) and \(\mathcal{V}\). Say finally that \(k = j - 1\). In this case there is only one possibility, namely the expression on the RHS of
equation (80) is of the form $\Pi \ast \nabla W_{j-1}$. From elliptic regularity applied to the elliptic system (28)-(31) we know

$$\| (E, j-1, B_{j-1}) \|_{H^1_g} \leq C(E_1, \nu, V)(\| (E_j, B_j) \|_{H^0_g} + \| (E_{j-1}, B_{j-1}) \|_{H^0_g} + \| J(W_{j-1}) \|_{H^0_g}).$$

It follows that

$$\| \nabla W_{j-1} \|_{L^2_g} \leq C(E_1, \nu, V)(Q^j_{j-1} + C(E_{j-1}, \nu, V)),$$

as desired. This finishes the treatment of the terms of the form (80). The terms of the form (81) are treated similarly.

3.4 Controlling the flow $(g, K)$ along evolution.

In this section we introduce the functional space in which solutions to the Einstein flow equations will lie. In the same vein as Section 3.3, we introduce a BR-functional (see later)

$$\| \phi \ast g \|_{BR} = \| g \|_{C^0(I, 2)(H_A)} + \| K \|_{C^0(I, 1)(H_A)} + \sum_{k=0}^{k=i-2} \| (E_k, B_k) \|_{C^0(I, 0)},$$

and discuss how it controls: i. the set of flow solutions, ii. a dynamical smooth structure on the space-time manifold, and iii. the space-time metric as a metric over the space-time manifold with the dynamical smooth structure. The results will be used in the next section when we discuss the initial value formulation for the CMC gauge.

We start giving the definition of admissible gauge. We then give the definitions of space-time solutions and flow solutions. After that we introduce the BR-norm and present the main results of the section.

**Definition 1** Say $I$ is an interval (for the time coordinate $t = k$). Define $C(I, \alpha, \beta)(H_*) = \cap_{j=0}^{j=\alpha} C^j(I, H_*^{j-1})$ where the subindex $\ast$ indicates the structure with respect to which the Sobolev space is defined. Each space $C^j(I, H_*^{j-1})$ is provided with its usual sup norm. We may allow $\beta$ to be less than $\alpha$.

**Definition 2** Say $(\Sigma, A_\infty)$ is a $C^\infty$-manifold. A differentiable function $X : C^\infty(S(\Sigma)) \times C^\infty(TS(\Sigma)) \rightarrow C^\infty(T(\Sigma))$ which is diffeomorphism invariant i.e. $X(\phi^* g, \phi^* K) = \phi^*(X(g, K))$ for any diffeomorphism $\phi : \Sigma \rightarrow \Sigma$ (the classes $C^\infty(\ast)$ are defined with respect to $A_\infty$) is an admissible shift if

1. for any $H^{j+1}_A$ ($j \geq 2$) atlas $A$ which is compatible with $A_\infty$ at the $j+1$ level of regularity, $X$ can be extended uniquely to a differentiable function (also denoted by $X$) $X : H^{j}_A \times H^{j-1}_A \rightarrow H^j_A$ with the property that:

2. $\| X \|_{H^j_A}$ is controlled by $\| g \|_{H^j_A}$ and $\| K \|_{H^{j-1}_A}$ and
3. for every path \((g, K)(t)\), such that \((g', K')\) is in \(C(I, j - k, j)(H_A) \times C(I, j - k, j - 1)(H_A)\) (0 \(\leq k \leq j\)) and with the norms \(\| (g, K) \|_{C^0(I, j) \times C^0(I, j)}\) and \(\| (g', K') \|_{C(I,j-k,j) \times C(I,j-k,j-1)}\) bounded by \(\Lambda\), \(X'\) is in \(C(I, j - k, j)(H_A)\) with norm controlled by \(\Lambda\).

**Examples.** 1. The zero shift is an example of an admissible gauge.

2. Definition 2 of admissible gauges share many of its properties with the Andersson-Moncrief gauge \(\mathbb{G}\) on hyperbolic manifolds. Although we do not claim that the Andersson-Moncrief gauge is admissible we would like to explain how some of the characteristics of admissible gauges are actually present in the it. Let \(g_H\) be a hyperbolic metric on \(\Sigma\). For every \(g\) in \(\Sigma\) perform a diffeomorphism \(\phi\) in \(\Sigma\) in such a way that the identity is a harmonic map between \(\phi^*(g)\) and \(g_H\). The identity is harmonic iff the vector field \(V^k = g^{ij}e^k(\nabla_i e_j - \nabla^H_i e_j)\) is zero. The CMCSH (Constant mean curvature - spatially harmonic) fixes the shift \(X\) in such a way that at every time the identity map \(id : (\Sigma, g(t)) \rightarrow (\Sigma, g_H)\) is harmonic. With this condition the equation for the lapse and shift are the following [5]

\[
\begin{align*}
\Delta X^i + \text{Ric}^i_j X^j - L_X V^i &= (-2N K^{mn} + 2\nabla^m X^n)\varepsilon^i (\nabla_m e_n - \nabla^H_m e_n) \\
&+ 2\nabla^m N K^i_m - \nabla^i Nk.
\end{align*}
\]

Thus, when the identity is a harmonic map, \(X\) is defined through (82) and (83) and otherwise it is defined by making it diffeomorphism invariant. Consider the \(C^\infty\) atlas \(A_\infty\) for which \(g_H\) is \(C^\infty\). As the differential operators (82) and (83) defining \((N, X)\) are elliptic its is clear by elliptic regularity that if \((g, K)\) are \(C^\infty\) then \((N, X)\) are too. Therefore \(X : C^\infty(S(\Sigma)) \times C^\infty(TS(\Sigma)) \rightarrow C^\infty(TS(\Sigma))\). In the same way if \(A\) is a \(H^{j+1}\) atlas which is compatible with \(A_\infty\) at the \(j\)-level of regularity, by elliptic regularity \(X\) extends to differentiable functions on \(H^i_A \times H^{i-1}_A \rightarrow H^i_A\). On the other hand Lemma 3.2 in [5] shows that \(\|X\|_{H^i_A}\) is controlled by \(\|g\|_{H^{i-1}_A}\) and \(\|K\|_{H^{i-2}_A}\). This says that \(X\) satisfies item 2 in Definition 2 (in fact it represents an improvement). Also from Lemma 3.2 in [5] we have that, for a path \((g, K)(\lambda)\), the norm \(\|X'\|_{H^i_A}\) is controlled by \(\|g\|_{H^{i-1}_A}\), \(\|K\|_{H^{i-2}_A}\), \(\|g'\|_{H^{i-1}_A}\), \(\|K'\|_{H^{i-2}_A}\). The shift \(X\) satisfies therefore item 3 in Definition 2 with \(k = j\) (in fact it represents and improvement).

**Remark 6** Observe the following property of admissible gauges. Say \((\Sigma, A)\) is a \(C^\infty\)-three-manifold and suppose \(X\) is admissible. Suppose \(A_1\) is another \(C^\infty\) atlas in \(\Sigma\) compatible with \(A\) at least at the third level of regularity. It is known there is a \(C^\infty\)-diffeomorphism \(\phi : (\Sigma, A) \rightarrow (\Sigma, A_1)\) and therefore the properties of \(X\) over \(A\) pull back over \(A_1\).

**Definition 3** A \(H^i\)-space-time solution \((M, g)\) of the Einstein equations (in vacuum) is a \(H^{i+1}\)-Lorentzian four-manifold \((M, g)\) satisfying \(\text{Ric} = 0\).
Definition 4 A $H^1$-flow solution of the Einstein CMC flow equations in the admissible spatial gauge $X(g, K)$, is a space-time solution $(M, g)$ (in vacuum) for which there is a $H^1$-diffeomorphism $\phi : M \to \Sigma \times I$ where $(\Sigma, A)$ is a $H^{1+1}$-three-manifold and $\Sigma \times I$ is supplied with the product structure, such that

1. $\phi_*(g)$ is (therefore) a $H^{1-1}$-space-time solution of the Einstein equations in vacuum.

2. The time foliation is CMC.

3. The components $g, N, X$ of the $3+1$ splitting of $\phi_*g$ in $\Sigma \times I$ satisfy: $g$ is in $C(I, H^{1-1})$ and $g'$ is in $C(I, i-1, i-1)(H_A)$. $N, X$ are in $C(I, i-1, i-1)(H_A)$, and $K$ is in $C(I, i-1, i-1)(H_A)$. These fields moreover satisfy the CMC flow equations (84). (84)

4. For every $k = 0, \ldots, i-2$, $\phi_*(W_k)$ has electric-magnetic decomposition $(E_k, B_k)$ in $C(I, i - 2 - k, i - 2 - k)(H_A)$, satisfying the equations (28)-(31).

5. There is a dynamical $H^{i+1}$-atlas $A(t)$ with dynamical charts $\{x_\alpha(t)\}$ for which the set of space-time charts $\{x_\alpha = (x_\alpha(t), t)\}$ form a $H^{i+1}$-atlas of $\Sigma \times I$ making $\phi$ a $H^{i+1}$-diffeomorphism. Also the transition functions $x_\alpha(x_\beta)$ are in $C(I, i+1, i+1)(H_{x_\beta})$ and $(g, N, X_{x_\beta})$ are in $C(I, i, i)(H_{x_\beta})$, where $X_{x_\beta}$ is the shift vector of the coordinate system $x_\beta$.

A comment is in order. In item 5 of Definition 4, the time derivatives required to define the spaces $C(I, *, *)(H_{x_\beta})$ are with respect to the time coordinate of $x_\beta$.

In practice (for $i = 3$) the dynamical atlas $A(t)$ will be given out of the following construction of harmonic coordinates. Let $\{\tilde{x}^k\}$ be a coordinate chart in a $H^3$ Riemannian manifold $(\Sigma, \tilde{g})$. Say $B(o, \alpha_2) \subset B(o, \alpha_1)$ are two balls inside the chart. Pick a smooth non-negative function $\xi$ being one in $B(o, \alpha_1)$ and zero in $B(o, \alpha_2)$ and define for any given metric $g$ in $\Sigma$, the metric $\tilde{g}_{ij} = \xi g_{ij} + (1 - \xi)\delta_{ij}$. Think the coordinates $\{\tilde{x}^k\}$ as coordinates on a three-torus $T^3$ with metric $\tilde{g}$ over the chart and extended to be flat on the rest. Extend the coordinate $\tilde{x}^k$ smoothly to the rest of the torus in such a way that $\int_{T^3} \tilde{x}^k d\tilde{v}_\tilde{g}$ where $\tilde{g}$ is $\tilde{g}_{ij} = \xi \tilde{g}_{ij} + (1 - \xi)\delta_{ij}$. Define the function $h^k$ to be zero on the chart and equal to $\Delta_{\tilde{g}} \tilde{x}^k$ on the rest of the torus. Then we can solve uniquely for $x^k$ in $\Delta_{\tilde{g}} x^k = h^k$ if we impose the condition that $\int_{T^3} x^k d\tilde{v}_\tilde{g} = 0$. Now suppose $\tilde{A} = \{\tilde{x}_\alpha, \alpha = 1, \ldots, n\}$ is a $H^2$-canonic harmonic atlas for the manifold $(\Sigma, \tilde{g})$ where each $\tilde{x}_\alpha$ is defined over a ball $B(o_\alpha, r_2/2)$ ($r_2(\tilde{g})$). Pick $\delta < 1$ but close to one. Choose $\alpha_1 = \delta r_2$ and $\alpha_2 = r_2$. Then for $\epsilon$ sufficiently small, if $\|g - \tilde{g}\|_{H^2_A} \leq \epsilon$ the functions $\{x^k_\alpha, k = 1, 2, 3\}$ defined over $B(o_\alpha, r_2/2)$ are harmonic coordinates for $g$. Also if $\epsilon$ is sufficiently small, the charts $\{x_\alpha\}$ extend to $B(o_\alpha, \delta r_2)$ and satisfying

$$
\frac{\delta^3}{4} \delta_{jk} \leq g_{jk} \leq \frac{4}{\delta^3} \delta_{jk},
$$

(84)
Thus we have constructed a new harmonic atlas for the metric $g$ that we will call $\delta$-canonic. It is clear that all the results we have discussed so far in Section 3.3 hold if instead of using a canonic atlas we use $\delta$-canonic atlas for a fixed $\delta$. Now suppose $g(t)$ is a path of metrics in $C^0(I,2)(H_A)$ and suppose $\|g(0) - \tilde{g}\|_{H^2_A} \leq \epsilon/2$ (with $\epsilon$ as before), then for any $t$ in a subinterval $I' \subset I$ the harmonic atlas $\{x_\alpha(t)\} = \mathcal{A}(t)$ is well defined. We will call $\mathcal{A}(t)$ the dynamical atlas constructed out of $(\tilde{g}, \tilde{A}, g(t))$ and denote the dynamical charts by $\{x_\alpha(t)\}$.

Remark 7 As constructed the atlas $\mathcal{A}(t)$ is only an $H^3$-harmonic dynamical atlas and not, as is required, a $H^4$-harmonic dynamical atlas. We will see later (Proposition 30) that in fact this construction gives us the desired $H^4$-harmonic atlas.

We describe now a useful BR-norm (Bel-Robinson norm) on the space of metrics on $\Sigma \times I$ with the properties in Definition 4 above and then move to explain how it controls the space of flow solutions.

Definition 5 Let $\phi_*g$ be a flow metric on $\Sigma \times I$ as explained in Definition 4. Define the BR-norm of $g$ out of $\mathcal{A}$ by

$$\|\phi_*g\|_{BR} = \|g\|_{C^0(I,2)(H_A)} + \|K\|_{C^0(I,1)(H_A)} + \sum_{j=0}^{j=i-2} \|(E_j, B_j)\|_{C^0(I,0)(H_A)}$$

Note that the definition of BR-norm does not need the condition in item 4 of Definition 4 to be imposed on $\phi_*g$. The importance of the BR-norm is that it is easy to handle in the Einstein equations and is intended to measure (some) “$H^i_A \times I$-norm” of $\phi_*g$ without being in $H^i_A \times I$. At the same time as explained later it does controls (some) $H^i_A \times I$-norms with respect to the coordinates $x_\alpha$.

From now on we fix $i = 3$. We note however that the treatment we make may well be systematized to any regularity $i \geq 4$.

We will need elliptic estimates for scalar equations of the form

$$\Delta \phi = h,$$

where $h$ has an expression of the form

$$h = T_0 \circ V_0 + T_1 \circ \nabla V_1 + T_2 \circ \nabla^2 V_2,$$

for tensors $T_m$ and $V_m$, $m = 0, 1, 2$ of arbitrary rank and where the operation $\circ$ is any full contraction (like $T_2^{abc} \nabla_a \nabla_c V_2, b$). In particular we would like to obtain elliptic estimates in terms of the $L^2_A$-norm of $V_i$, $i = 1, 2, 3$. (see the hypothesis inside the Proposition). Thus the non-homogeneous term $h$ will lie, in general, in

\[\text{footnote}: \text{We have said what a } H^2-\delta\text{-canonic is. It should be clear how to define } H^i-\delta\text{-canonic.}\]
a negative Sobolev space. In this context, the elliptic estimates that we will obtain (except item 1 in the proposition below) do not follow from standard elliptic estimates or Proposition \[24\] and deserve a special treatment due to the low regularity of the metric $g$ which is assumed in $H^3$.

**Proposition 28** Let $(\Sigma, A, g)$ be a $C^\infty$-Riemannian manifold and say $A$ is a $H^3$-canonic harmonic atlas. Let $T_m, V_m, m = 0, 1, 2$ be $C^\infty$-tensors of arbitrary rank. Say $\phi$ is a solution of

$$\Delta \phi = h,$$

with $\int_\Sigma \phi dv_g = 0$. Then, we have the elliptic estimates

1. For $j = 0, 1, 2$ we have

$$\|\phi\|_{H^{j+2}} \leq C(r_3, V_\omega)\|h\|_{H^j}. $$

2. If $h = T_1 \circ \nabla V_1 + T_0 \circ V_0$, we have

$$\|\phi\|_{H^j} \leq C(r_3, V_\omega, \|T_1\|_{H^2}, \|T_0\|_{H^2}, \|V_1\|_{H^2}, \|V_0\|_{H^2}); $$

3. If $h = T_0 \circ V_0 + T_1 \circ \nabla V_1 + T_2 \circ \nabla^2 V_2$ we have

$$\|\phi\|_{H^j} \leq C(r_3, V_\omega, \|T_1\|_{H^2}, \|T_2\|_{H^2}, \|T_0\|_{H^2}, \|V_1\|_{H^2}, \|V_2\|_{H^2}, \|V_0\|_{H^2}).$$

With low regularity the estimates extend in the following sense. Say $(\Sigma, A_i, g_i)$ is a sequence converging in $H^1$ (for $A_i$) and $H^3$ (for $g_i$) to $(\Sigma, A_\infty, g_\infty)$ where $(\Sigma, A_\infty, g_\infty)$ is $H^4$-Riemannian manifold, with $A_\infty$ a $H^3$-canonic harmonic atlas. Say too $h_i, T_{m,i} \circ V_{m,i}$ are sequences with $h_i \to h_\infty$, $e^{\nabla^j} \in H^j$ for $j$ either 0, 1 or 2, $T_{m,i} \to T_{m,\infty}$ in $H^{m+1}$ (for item 2) and $H^m$ (for item 3) and $V_{m,i} \to V_{m,\infty}$ in $H^0$. Finally say $\phi_i$ is the sequence of solutions to (86). Then, (for item 1) $\phi_i \to \phi_\infty$ in $H^j$ for $j$ either 0, 1 or 2 (for item 2) $\phi_i \to \phi_\infty$ in $H^1$ and (for item 3) $\phi_i \to \phi_\infty$ in $H^0$.

**Proof:**

Recall the fact that $\Delta : H^2 \to H^0$ is Fredholm of kernel the constants and index zero. Moreover if we consider the operators $\Delta : H^2 \to H^0$ and $\Delta^{-1} : H^0 \to H^2$ as operators from the orthogonal complements of the constants into the orthogonal complement of the constants, their norms are controlled by $r_3$ and $V_\omega$.

**Item 1.** This case follows from standard elliptic estimates (or Proposition \[1\] and the observation above.

---

\[22\]We haven’t found these estimates in the standard references on PDE.

\[23\]The boundedness of $\Delta$ is direct form elliptic estimates. The boundedness of $\Delta^{-1}$ follows by first proving the image of $\Delta$ is the orthogonal complement of the constants and then using the open mapping theorem. The fact that the image of $\Delta$ is the orthogonal complement of the constants follows by expanding $H^0$ in terms of eigenfunctions of $\Delta$. 

51
Item 3. Pick $\xi$ with $\int_{\Sigma} \xi \, dv_g = 0$. Observe that in any contraction of the form $T \circ \nabla V$ or $T \circ \nabla^2 V$ we can always integrate by parts once or twice respectively. Multiply equation (86) by $\xi$ and integrate. We get

$$\int_{\Sigma} \xi \Delta \phi \, dv_g = \int_{\Sigma} \left( (\nabla (\xi T_1)) \circ V_1 + (\nabla^2 (\xi T_2)) \circ V_2 + (\xi T_0) \circ V_0 \right) \, dv_g.$$

Expanding the covariant derivatives on the RHS of the last equation and using Sobolev embeddings and Hölder inequalities we get

$$\int_{\Sigma} (\Delta \xi) \phi \, dv_g \leq C(r_3, Vol) \left( \|\xi\|_{H^2} \left( \|V_0\|_{H^2} + \|T_2\|_{H^2} \|V_2\|_{H^2} \right) + \|\xi\|_{H^2} \|T_2\|_{H^2} \|V_2\|_{H^2} + \|\xi\|_{H^2} \|T_1\|_{H^2} \|V_1\|_{H^2} \right).$$

(87)

Make $\bar{\xi} = \Delta \xi$. Then from $\|\xi\|_{H^2} \leq C(r_3, Vol) \|ar{\xi}\|_{H^2}$ we get that $\bar{\xi} \to 0$, $\phi$ is a bounded linear map from $L^2$ into $\mathbb{R}$. In particular $\|\phi\|_{L^2}$ and so $\|\phi\|_{H^2}$ are controlled by $r_3$, $Vol$ and the respective norms of $T_m$ and $V_m$ for $m = 0, 1, 2$.

Item 2. Multiply equation (86) by $\phi$ and integrate by parts. We get

$$\int_{\Sigma} |\nabla \phi|^2 \, dv_g = \int_{\Sigma} \nabla \phi \circ T_1 \circ V_1 + \phi \nabla T_1 \circ V_1 + \phi T_0 \circ V_0.$$

Using again Sobolev embeddings and Hölder inequalities we have

$$\int_{\Sigma} |\nabla \phi|^2 \, dv_g \leq C(r_3, Vol) \left( \|\phi\|_{H^2} \|T_1\|_{H^2} \|V_1\|_{H^2} + \|\phi\|_{H^2} \|T_0\|_{H^2} \|V_0\|_{H^2} \right).$$

(88)

From item 3 we have control on $\|\phi\|_{H^2}$. Using it and equation (88) we get control on $\|\phi\|_{H^2}$ from $r_3$, $Vol$ and the respective norms of $T_m$ and $V_m$ for $m = 0, 1, 2$ as desired.

We prove now (sketchily) the last part of the proposition. Assume then we have a sequence $g_i, \phi_i, h_i, V_{m,i}, T_{m,i}$ as indicated in the statement of the proposition.

Item 1 Although the proof of this case is trivial, we will follow one that can be applied in the proofs of items 2 and 3 too. For a metric $g$ and a field $\phi$ write

$$\Delta_g \phi = \Delta_{g_\infty} \phi + g_{ab} \Gamma^c_{ab} \nabla_c \phi + (g^{ab} - g^{ab}_{\infty}) \nabla_a \nabla_b \phi.$$

where $\nabla - \nabla^\infty = \Gamma$. Note that $\|\Gamma_i\|_{H^2} \to 0$ and $\|g^{ab}_{\infty} - g^{ab}\|_{H^2} \to 0$. Using this formula rewrite the subtraction of equation (86) for the fields $\phi_1, g_i, h_i$ to the equation (86) for the fields $\phi_k, g_k, h_k$, as

$$\Delta_{g_{\infty}} (\phi_l - \phi_k) = h_{lk},$$

with $\|h_{lk}\|_{H^2} \to 0$ and $\int_{\Sigma} \phi_l - \phi_k \, dv_{g_{\infty}} \to 0$. It follows that $\|\phi_l - \phi_k\|_{H^2} \to 0$, thus the sequence $\{\phi_i\}$ is Cauchy in $H^2_{\infty}$ and therefore convergent.

Items 2 and 3. This item follows in the same way as item 1 was proved. Observe that the RHS of equations (87) and (88) tends to zero (thus making the norms $\|\phi\|_{H^2}$ and $\|\phi\|_{H^2}$ tend to zero too) if, simultaneously, the (respective) norm of
one of the fields in each one of the pairs \((T_0, V_0), (T_1, V_1)\) and \((T_2, V_2)\) tends to zero. In the same way as in item 1 subtract equation (86) for the fields \(\phi_l, g_l, T_{m,l}, V_{m,l}\) for \(m = 0, 1, 2\) to the equation (86) for the fields \(\phi_k, g_k, T_{m,k}, V_{m,k}\) for \(m = 0, 1, 2\). Rearranging the result conveniently and using the observation above we get that \(\phi_l - \phi_k\) is a Cauchy sequence in \(H^1_A\) (for item 2) and in \(H^0_A\) (for item 3). The claim then follows.

Remark 8 One can obtain exactly the same kind of estimates as in items 1 and 2 in Proposition 28 for equations of the form

\[-\Delta \phi + f \phi = h,\]

where \(f\) is in \(H^2\) and \(f > f_{\text{inf}} > 0\). This time the estimates depend also on the \(H^2_A\)-norm of \(f\) and \(f_{\text{inf}}\). The estimates, in particular, will be required to estimate the \(H^0_A\)-norm of \(N^{\text{'''}}\) (the third time derivative of the lapse).

Proposition 29 Let \(\phi_* g\) be a flow solution in \(\Sigma \times I\) with BR-norm out of a \(H^4\)-atlas \(A\) bounded by \(\Lambda\). Then, the norms of \(g, g', K\) in \(C^0(I', H^2_A), C(I', 2, 2)(H^2_A)\) and \(C(I', 2, 2)(H^2_A)\) respectively and the norms of \(N, X\) and \(X'\) in \(C(I', 3, 3)(H^2_A), C^0(I', 2)(H_A)\) and \(C(I', 2, 2)(H_A)\) respectively, are controlled by \(\Lambda\), where \(I'\) is a subinterval of \(I\) with size controlled from below by \(\Lambda\).

Proof:

The idea for the proof is to look at the equations

\[g' = -2NK + \mathcal{L}_X g,\]

\[K' = -\nabla^2 N + N(E - K \circ K) + \mathcal{L}_X K,\]

\[-\Delta N + |K|^2 N = 1,\]

\[E' = N \text{Curl} B - (\nabla N \wedge B) - \frac{5}{2} N(E \times K) - \frac{2}{3} N(E.K)g - \frac{1}{2} k_N E, + \mathcal{L}_X E,\]

\[B' = -N \text{Curl} E - (\nabla N \wedge E) - \frac{5}{2} N(B \times K) - \frac{2}{3} N(B.K)g - \frac{1}{2} k_N B + \mathcal{L}_X B,\]

and observe that the time derivatives of \(g\) and \(K\) can be calculated using recursively those equations and its time derivatives. There are delicate points however that have to be addressed with care.

Without loss of generality we will prove the proposition with \(A\) a canonical atlas for the initial metric \(g(0)\). We will make that assumption on \(A\) from now on.

Observation 1. By the definition of the BR-norm, \(g\) is in \(C^0(I, 2)(H_A)\) with norm controlled by \(\Lambda\). Therefore, given \(\epsilon > 0\), there is an interval \(I' \subset I\) with a

---

24The initial metric \(g(0)\) is always assumed to be in \(H^3\)
size controlled from below by $\epsilon$ and $\Lambda$ such that the atlas $\mathcal{A}(t)$ is well defined for all $t \in I'$ and every chart satisfies $\|x^k_a(t) - x^k_a(0)\|_{C(I',\mathbb{R})} \leq \epsilon$. As a result, if $\epsilon$ is chosen sufficiently small, then for any $t \in I'$ the identity $id : (\Sigma, \mathcal{A}(0)) \rightarrow (\Sigma, \mathcal{A}(t))$ is a $H^3$-diffeomorphism.  Moreover, for any tensor field $U$, the norm $\|U\|_{H^3_A(t)}$ controls the norm $\|U\|_{H^3_A}$ and vice versa, for any $t \in I'$.

**Observation 1.** As an application of the Observation 1, note that the norms $\|g\|_{C^0(I',2)(H^3_A(t))}$, $\|K\|_{C^0(I',2)(H^3_A(t))}$, $\|(E_0, B_0)\|_{C^0(I',0)(H^3_A(t))}$, $\|(E_1, B_1)\|_{C^0(I',0)(H^3_A(t))}$ are controlled by $\Lambda$. Therefore, by the results in Section 3.3.2 for any $t \in I'$ the norms $\|g\|_{H^3_A(t)}$, $\|K\|_{H^3_A(t)}$, $\|N\|_{H^3_A(t)}$, $\|(E_0, B_0)\|_{H^3_A(t)}$, $\|(E_1, B_1)\|_{H^3_A(t)}$ are controlled by $\Lambda$. From the Observation 1 above so are controlled the same norms but with respect to the atlas $\mathcal{A}$.

The proof goes in a ladder-like scheme. We follow the order

$$g, K, N, X \Rightarrow g', K', N', X' \Rightarrow g'', K'', N'', X''' \Rightarrow \ldots.$$ 

Namely, proceeding from left to right, once we prove that a field is controlled (in its respective norm) we use that control to prove control on the next field (in its respective norm).

1. $\|g\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. Direct from the definition of BR-norm.

2. $\|K\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. Direct from Observation 2.

3. $\|N\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. Direct from Observation 2.

4. $\|X\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. As $X$ is admissible we get, applying item 1 in the Definition 2 with $j = 3$ and with atlas $\mathcal{A}(t)$, that $\|X\|_{H^3_A(t)}$ is controlled by $\Lambda$. This item (4) then follows by Observation 1.

5. $\|g\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. From equation (90) and Observation 2 (or items 2 and 3) we see that we need to control the $H^3_A$-norm of $\mathcal{L}_X g$. This follows from the formula

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a,$$

and Observation 1.

6. $\|K\|_{C^0(I',1)(H^3_A(t))}$ controlled by $\Lambda$. From equation (90) and Observation 2 we see that we need to control the $H^1_A$-norm of $\mathcal{L}_X K$. This follows from the formula

$$(94) \quad \mathcal{L}_X U_{ab} = \nabla_X U_{ab} + U_{ac} \nabla_b X^c + U_{bc} \nabla_a X^c,$$

with $U = K$, Observation 1 and item 4.

7. $\|N\|_{C^0(I',2)(H^3_A(t))}$ controlled by $\Lambda$. Differentiating the lapse equation with respect to time we get

$$(95) \quad - \Delta N' + |K|^2 N' = -(|K|^2)' N + \Delta' N.$$

Recall that the derivative of the Laplacian has the expression

$$(96) \quad \Delta f = g^{ab} \nabla_a \nabla_b f - g^{ab} (\nabla_a f) (\nabla^c g_{cb} + \frac{1}{2} \nabla_b (g^{de} g_{de})).$$

25 The fact that $id$ is a diffeomorphism is somehow standard and can be elaborated from the following fact. Say $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a $H^3_{\mathbb{R}^3}$ map being the identity on $B(0, 1)^c$ and with $\|\phi - id\|_{H^3_{\mathbb{R}^3}} \leq \epsilon$. Then if $\epsilon$ is sufficiently small $\phi^{-1}$ exists, is in $H^3_{\mathbb{R}^3}$ and has $H^3_{\mathbb{R}^3}$ norm controlled by $\epsilon$. 

54
Using (96) (with \( f = N \)) in (95) and the previous items, we see that the RHS of (95) is controlled in \( H^1_A(t) \). The elliptic estimates of Proposition 28 (see the Remark 8) show that \( N' \) is controlled in \( H^2_A(t) \) (in fact in \( H^3_A(t) \)) by \( \Lambda \) and therefore by Observation 1 so in \( H^3_A \).

8. \( \|X\|_{C^0(I',2)(H\Lambda)} \) controlled by \( \Lambda \). As \( X \) is admissible we can apply item 2 in Definition 2 with \( j = 2, k = 2 \) and atlas \( \mathcal{A} \) to get that \( X' \) is controlled in \( H^2_A \) by \( \Lambda \).

9. \( \|g'\|_{C^0(I',1)(H\Lambda)} \) controlled by \( \Lambda \). Differentiating equation (89) in time we get

\[
g'' = -2N'K - 2N(K)' + (\mathcal{L}_X g)' .
\]

The first two terms in the previous equation are controlled in \( H^1_A \) by the items 6 and 7. Let us consider the derivative in time of the Lie derivative of \( g \) with respect to \( X \). Differentiating \( \mathcal{L}_X g_{ab} = \nabla_a X_b + \nabla_b X_a \) with respect to time we get

\[
(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a + \nabla_a X_b + \nabla_b X_a = (\mathcal{L}_X g)_{ab} .
\]

Recalling how the time derivative of \( \nabla \) was calculated in Proposition 26, a direct inspection of all the terms of \( \nabla_a X_b \) show that \( \nabla_a X_b \) is controlled in \( H^1_A \). The control in \( H^1_A \) of the last two terms in equation (98) follows from item 8.

10. \( \|K'\|_{C^0(I',0)(H\Lambda)} \) controlled by \( \Lambda \). Differentiate equation (90) with respect to time. We get

\[
K'' = - (\nabla^2)'N - \nabla^2 N' + N(E - K \circ K) + N(E - (K \circ K)) + (\mathcal{L}_X K)' .
\]

A direct inspection of this equation and use of the previous items shows that all the terms in it, except perhaps \( NE' \) and \( (\mathcal{L}_X K)' \), are controlled in \( H^1_A \) by \( \Lambda \). To check that the term \( NE' \) is controlled in \( H^0_A \) examine equation (91) and use the previous items. Recalling that \( \mathcal{L}_X K = \nabla_X K_{ab} + K_{ac} \nabla_b X^c + K_{bc} \nabla_a X^c \) and using the previous items, we see that the term \( (\mathcal{L}_X K)' \) is controlled in \( H^0_A \) by \( \Lambda \).

11. \( \|N'\|_{C^0(I',1)(H\Lambda)} \) controlled by \( \Lambda \). Differentiate (93) once more with respect to time to get

\[
- \Delta N'' + |K|^2 N'' = 2\Delta N' - 2(|K|^2)'N' + \Delta'N + (|K|^2)'N .
\]

We want to apply Proposition 28 (see Remark 8). We check that the RHS of equation (100) is controlled in \( H^1_A \) by \( \Lambda \). The first term \( \Delta N' \) in the RHS of equation (100) is calculated using equation (99) with \( f = N' \). A direct inspection of its terms using items 5 and 7 show that they are controlled in \( H^0_A \) by \( \Lambda \). Now let us consider the time derivative of \( |K|^2 \). Writing \( |K|^2 = K_{ab} K_{cd} g^{ac} g^{bd} \), the time derivative is

\[
|K|^2' = 2K_{ab} K_{cd} g^{ac} g^{bd} + 2K_{ab} K_{cd} g^{ac} g^{bd} .
\]

Differentiating once more we get

\[
|K|^2'' = 2K_{ab} K_{cd} g^{ac} g^{bd} + 2K_{ab} K_{cd} g^{ac} g^{bd} + 4K_{ab} K_{cd} g^{ac} g^{bd} + 4K_{ab} K_{cd} g^{ac} g^{bd} + 4K_{ab} K_{cd} g^{ac} g^{bd} + 2K_{ab} K_{cd} g^{ac} g^{bd} + 2K_{ab} K_{cd} g^{ac} g^{bd} .
\]
Inspecting equations (101) and (102) and using items 5, 6, 9 and 10 one gets that \(|K|^2\) and \(|K|^{2,\infty}\) are controlled in \(H^1_A\) and \(H^0_A\) respectively by \(\Lambda\). This shows that the second and fourth terms in equation (100) are controlled in \(H^0_A\). Finally, let us consider the term \(\Delta N\). For this we differentiate \(\Delta\) in equation (96) once more (leaving \(f = N\) invariant). We get

\[
\Delta N = g^{ab} \nabla_a \nabla_b N + g^{ab} \nabla_a \nabla_b N - g^{ab} \nabla_a N (\nabla^c g_{cb} + \frac{1}{2} \nabla^c (g^{de} g_{de}))
+ g^{ab} \nabla_a N (\nabla^c g_{cb} + \nabla^c g_{cb} + \frac{1}{2} \nabla_b (g^{de} g_{de} + g^{de} g_{de}')).
\]

A direct inspection of this equation using items 1, 3, 5 and 9 and the expression of \(\nabla\) found before shows that \(\Delta N\) is also controlled in \(H^0_A\) by \(\Lambda\).

12. \(\|X\|_{C^0(I, 1)(H_A)}\) controlled by \(\Lambda\). As \(X\) is admissible this item follows applying item two in Definition 2 with \(j = 2, k = 1\) and atlas \(A\).

13. To control \(\|g^{\prime \prime}\|_{C^0(I, 0)(H_A)}\), \(\|N^{\prime \prime}\|_{C^0(I, 0)(H_A)}\) and \(\|X^{\prime \prime}\|_{H^0_A}\) proceed in the same fashion as in the previous items. \(\square\)

Let \(\{x_\alpha\}\) be the \(\delta\)-canoncal dynamical as was defined after Definition 4. Note as it was defined, it is only a \(H^3\)-atlas and not a \(H^4\)-atlas as required in Definition 4. In the proposition below we discuss this and further properties of \(\mathcal{A}(t)\) in relation with Definition 4. Recall the coordinates \(x_\alpha\) are defined over \(B(o_\alpha, r_2/2)\) where \(r_2\) is the \(H^2\)-harmonic radius of the metric \(\tilde{g}\). In addition those charts can be extended to \(B(o_\alpha, \delta r_2)\) satisfying the properties (S1) and (S5).

**Proposition 30** A flow solution \(\phi_\lambda g\) in \(\Sigma \times I\) with BR-norm bounded by \(\Lambda\) has the following properties (at least on a smaller interval \(I(\Lambda)\))

1. \(\Lambda\) controls the \(C(I, 4, 4)(H_{x_\beta})\) norm of the transition functions \(x_\alpha(x_\beta)\).

2. \(\Lambda\) controls the \(C(I, 3, 3)(H_{x_\beta})\) norm of \(g = (g, N, X_{x_\beta})\) where \(X_{x_\beta}\) is the shift vector of the coordinates \(x_\beta\).

3. Let \(g(\lambda)\) be a path of metrics parameterized by \(\lambda\). Then the BR-norm of \(g' = \frac{dg}{d\lambda}\) i.e.

\[
\|g'(\phi_\lambda g)\| = \|g'\|_{C^0(I, 2)(H_{A(0)})} + \|K'\|_{C^0(I, 1)(H_{A(0)})} + \sum_{j=1}^{j=1} \|E_j', B_j'\|_{C^0(I, 0)(H_{A(0)})},
\]

controls the norms of \(x_\alpha' = dx_\alpha/d\lambda\) and \(g'^\prime\) in \(C(I, 4, 4)(H_{x_\beta})\) and \(C(I, 3, 3)(H_{x_\beta})\) respectively. We will assume without lost of generality that the BR-norm of \((\phi_\lambda g)\)' is bounded by \(\Lambda\) too (i.e. the same as the bound for the BR-norm of the solution).

**Remark 9** i. In item 1 the time derivatives of \(x_\alpha\) (needed in the norm \(C(I, 4, 4)(H_{x_\beta})\)) are with respect to the time vector field defined by the chart \(\{x_\beta(t), t\}\).
ii. Note that item 1 in Proposition 30 implies that when $\Sigma \times I$ is provided with the set of charts $\{(x_\alpha(t), t), \alpha = 1, \ldots, m\}$ it makes it a $H^4$-four-manifold.

iii. Item 2 in Proposition 30 shows that $g$ is a $H^3$-metric in $\Sigma \times I$ with the $\{(x_\alpha(t), t)\}$ atlas.

iv. In item 3 we use the convention that (as we have been assuming) the $C(1, 3, 3)(H_{x_\alpha})$-norm of $g'$ is the $C(1, 3, 3)(H_{x_\alpha})$-norm of each of its horizontal components. To find which the components are, pick two $\lambda$-independent horizontal vectors $V^a$ and $W^u$ and compute

$$g'_{ab}V^aW^b = g'_{ab}V^aW^b,$$

and

$$g'_{ab}V^aT^b = -g_{ab}V^aT^b.$$

Observing that

$$0 = \left(\frac{\partial}{\partial t}\right)' = N'T + NT' + X',$$

we get

$$g'_{ab}V^aT^b = \frac{X''}{N}g_{ab}V^a.$$

Finally

$$g'_{ab}T^aT^b = -2g_{ab}T^aT^b = \frac{2N'}{N}.$$

It follows that the components of $g'$ form a set equivalent to $(g', N', X')$. If we prove that $(g', N', X')$ are in $C(1, 3, 3)(H_{x_\alpha})$ with norm controlled by $\Lambda$ the same will be true for $g'$. We will do that when proving the proposition.

v. It is straightforward to prove from item 3 in Proposition 30 that $(g', N', x')$ is in $C(1, 3, 3)(H_A)$ and $(g', N', X')$ is in $C(1, 2, 2)(H_A)$ with norm controlled by $\Lambda$. We will use this fact when discussing the initial value formulation in the CMC gauge.

**Proof:**

*Item 1.* *Step 1.* We show first that the transition functions $x_\alpha(x_\beta)$ have $H^4_{x_\alpha}$ norm controlled by $\Lambda$. By Proposition 21 we know $\|Ric(t)\|_{H^4_{x_\alpha}}$ is controlled by $\Lambda$. For the same argument as in Proposition 2 this implies that the norm $\|g\|_{H^3_{x_\alpha}(B(\theta_\beta, \delta_{r_2}))}$ is controlled by $\Lambda$. Now, express the harmonic condition of the coordinate $x^k_\alpha$ over $\{x_\beta\}$. We have

$$g^{ij}\partial_{x^j_\beta}\partial_{x^k_\alpha}x^k_\alpha + \left(\frac{1}{\sqrt{g}}\partial_{x^j_\beta}\sqrt{gg^{ij}}\right)\partial_{x^j_\alpha}x^k_\alpha = 0.$$

We can apply the elliptic estimates of Proposition 11(I) to get that $\|x_\alpha\|_{H^4_{x_\alpha}(B(\theta_\beta, r_2/2) \cap B(\theta_\alpha, r_2/2))}$ is controlled by $\Lambda$.

*Step 2.* Next we prove that the norms $\|x_\alpha\|_{H^3_{x_\alpha}(B(\theta_\alpha, r_2/2))}$, $\|x_\alpha\|_{H^2_{x_\alpha}(B(\theta_\alpha, r_2/2))}$, $\|x_\alpha\|_{H^1_{x_\alpha}(B(\theta_\alpha, r_2/2))}$ and $\|x_\alpha\|_{H^0_{x_\alpha}(B(\theta_\alpha, r_2/2))}$ are controlled by $\Lambda$ and where the
dot means derivative with respect to $\partial_t = NT + X$ (and not the time vector field of the chart $\{(x_\beta(t), t)\}$ as we will do in Step 3). We will prove that studying the defining equation for $x^k_\alpha$ on the torus. Control will be obtained in the order

$$x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha \Rightarrow x_\alpha .$$

We have defined $x^k_\alpha$ by solving

$$\Delta_{\bar{g}(t)} x^k_\alpha = h^k,$$

subject to the condition

$$\int_{T^3} x^k_\alpha dv_{\bar{g}(t)} = 0.$$

The idea is to differentiate consecutively these equations with respect to time in each step use the control gotten in the previous step. Differentiating both equations with respect to time we get

$$\Delta(x^k_\alpha)^\prime = -\Delta x^k_\alpha,$$

and

$$\int_{T^3} (x^k_\alpha)^\prime dv_{\bar{g}(t)} = -\int_{T^3} x^k_\alpha \xi (-Nk + \nabla^a X_a) dv_{\bar{g}}.$$

Using formula (96) for the derivative of the Laplacian we see that the RHS of equation (105) is controlled in $H^2_{\alpha}$ by $\Lambda$. We split $(x^k_\alpha)^\prime = (x^k_\alpha)_0 + c$ where $c$ is a constant and $(x^k_\alpha)_0$ has average zero. From Proposition 28 we conclude that $\| (x^k_\alpha)_0 \|_{H^2_{\alpha}}$ is controlled by $\Lambda$ and using equation (106) we conclude that $c$ is controlled by $\Lambda$ too. As a result $\| x_\alpha \|_{H^2_{\alpha}(B(o_\alpha, r_2/2))}$ is controlled by $\Lambda$ too.

We show next that the norms $\| x_\alpha \|^2_{H^2_{\alpha}(B(o_\alpha, r_2/2))}, \| x_\alpha \|^2_{H^2_{\alpha}(B(o_\alpha, r_2/2))}$ and $\| x_\alpha \|^2_{H^2_{\alpha}(B(o_\alpha, r_2/2))}$ are controlled by $\Lambda$. We proceed exactly as we did before. Differentiate equations (105) and (106) and use Proposition 28. A lengthy but straightforward calculation shows that every time equation (105) is differentiated we can apply Proposition 28. In other words the RHS of the equation

$$\Delta x^{(i)}_\alpha = \sum_{n=0}^{n=i} c_n \Delta^{(n)} x^{(i-n)}_\alpha,$$

$((\ast)$ denotes the $\ast$-derivative$)$ for each $i = 1, 2, 3, 4$ has the structure of $h$ in Proposition 28. The claim the follows.

Step 3. We are now in condition to finish the proof of item 1. We will use the notation $H^*_{\alpha\beta}$ instead of $H^*_{\alpha\beta}(B(o_\beta, r_2/2) \cap B(o_{alpha}, r_2/2))$. Let us denote $x_\beta = (x_\beta(t_\beta), t_\beta)$ and $x_0 = (x_\beta(0), t_0)$ where $\{x_0 = x_\beta(0)\}$ is a time independent chart. The functions $t_\beta$ and $t_0$ are the same and just $t$ (the subindex is to differentiate $\partial_i$ from $\partial_{t_\beta}$). For any function $f$ the chain rule gives

$$\frac{\partial f}{\partial t_0} = \frac{\partial f}{\partial t_\beta} + \frac{\partial f}{\partial x^i_\beta} \frac{\partial t_\beta}{\partial t_0}.$$
From this we conclude that

\[
\frac{\partial}{\partial t} = NT + X - \frac{\partial x^i_\alpha}{\partial t_0} \frac{\partial}{\partial x^i_\beta}.
\]

Now suppose we want to show that \( \|\partial_t x^k_\alpha\|_{H^3_x} \) is controlled by \( \Lambda \). Apply \( \partial t_\beta \) using formula (107) over \( x^k_\alpha \) (for \( k = 1, 2, 3 \)). We get

\[
\frac{\partial x^k_\alpha}{\partial t_\beta} = x^k_\alpha \cdot -x^i_\beta \cdot \frac{\partial x^k_\alpha}{\partial x^i_\beta}.
\]

Steps 1 and 2 show that both terms on the RHS of the previous equation are controlled in \( H^3_x \). Showing control over the other time derivatives follows the same strategy. To illustrate the procedure let us prove that \( \|\partial_t^2 x^k_\alpha\|_{H^2_x} \) is controlled by \( \Lambda \) too. The remaining two cases are done in the same fashion. Apply the expression (107) twice over \( x^k_\alpha \). We get

\[
\frac{\partial^2 x^k_\alpha}{\partial t^2_\beta} = \frac{\partial^2 x^k_\alpha}{\partial t_0^2} - \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta} - \frac{\partial}{\partial t_0} (\frac{\partial x^i_\beta}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta}) + \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial}{\partial x^i_\beta} (\frac{\partial x^i_\beta}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta}).
\]

From Steps 1 and 2 we see that all the terms in the RHS of the previous equation except perhaps the third are controlled in \( H^2_x \). To treat the third we compute

\[
\frac{\partial}{\partial t_0} (\frac{\partial x^i_\beta}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta}) = \frac{\partial^2 x^i_\beta}{\partial t_0^2} \frac{\partial x^k_\alpha}{\partial x^i_\beta} + \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial^2 x^k_\alpha}{\partial x^i_\beta \partial t_0}.
\]

The first term on the RHS of this last equation is controlled in \( H^2_x \) by Steps 1 and 2. Let us consider the factor

\[
\frac{\partial}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta}.
\]

on the second term of the RHS of equation (108). As we know that the factor \( \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta} \) is controlled in \( H^2_x \), we would like to prove that the term (109) is also controlled in \( H^2_x \). Rewrite (109) as

\[
\frac{\partial}{\partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta} = \frac{\partial^2 x^m_\beta}{\partial x^m_0 \partial t_0} \frac{\partial x^k_\alpha}{\partial x^i_\beta} + \frac{\partial x^m_\beta}{\partial t_0} \frac{\partial}{\partial x^m_0} (\frac{\partial x^k_\alpha}{\partial x^i_\beta}).
\]

The first term in the previous equation is controlled by \( \Lambda \) in \( H^2_x \) by Steps 1 and 2. The second term is treated using

\[
\frac{\partial x^m_0}{\partial x^j_\beta} = \delta^j_0.
\]

From this we compute

\[
\frac{\partial^2 x^m_0}{\partial t_0 \partial x^i_\beta} = \frac{\partial x^m_0}{\partial x^j_\beta} \frac{\partial x^j_\beta}{\partial x^i_\beta} \frac{\partial^2 x^k_\alpha}{\partial x^i_\beta \partial t_0}.
\]
It follows from \textit{Steps 1 and 2} that this terms is controlled in $H^2_{x,\beta}$ too.

\textit{Item 2}. Checking this property is straightforward. We will use the notation $H^*_x$ instead of $H^*_x(B(o_\beta, r_2/2))$. We will only check that $\|X_{x,\beta}\|_{H^*_x}$ and $\|L_{\partial t\beta} X_{x,\beta}\|_{H^2_{x,\beta}}$ are controlled by $\Lambda$. The remaining two cases are carried out in the same fashion.

From equation (107) we get $X_{x,\beta} = X - \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial}{\partial x^i_\beta}$. We know $\|X\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$ and from \textit{item 1} we know $\|\partial_t x^i_\beta\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$. It follows that $\|X_{x,\beta}\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$ too. Let us consider now the Lie derivative of $X_{x,\beta}$ in the direction of $\partial t\beta$. We apply $L_{\partial t\beta}$ to $X_{x,\beta}$.

It follows from the equations above and \textit{item 1} that the $H^2_{x,\beta}$-norm of $L_{\partial t\beta} X_{x,\beta}$ is controlled by $\Lambda$. Consider now $L_{\partial t\beta} X$. We compute

$$L_{\partial t\beta} \frac{\partial x^i_\beta}{\partial t_0} = \left( \frac{\partial}{\partial \beta_0} \right) \frac{\partial x^i_\beta}{\partial \beta_0}.$$

and

$$\frac{\partial}{\partial t\beta} \frac{\partial x^i_\beta}{\partial t_0} = \frac{\partial^2 x^i_\beta}{\partial t^2_0} - \frac{\partial x^i_\beta}{\partial t_0} \frac{\partial}{\partial x^i_\beta} \frac{\partial}{\partial t_0}.$$

It follows from these two equations, \textit{item 1} and Proposition 29 that $\|L_{\partial t\beta} X\|_{H^2_{x,\beta}}$ is controlled by $\Lambda$.

\textit{Item 3}. We will show that $x^\prime_{\beta}$ and $g^\prime, N^\prime, X^\prime_{x,\beta}$ are controlled in $H^4_{x,\beta}$ and $H^3_{x,\beta}$ respectively. Here, prime means derivation with respect to $\lambda$. The control of their time derivatives (as $\frac{d g^\prime}{dt}$ or $N^\prime \cdot \cdot$) in their respective space, is carried out along similar lines and will not be included. We start showing that $\|x^\prime_{\beta}\|_{H^2_{x,\beta}}$ is controlled by $\Lambda$. We look at the equation defining $\{x_{\beta}\}$ in the torus. After differentiating in $\lambda$ we get

(110) \hfill $\Delta(x^k_{\beta})' = -\Delta'(x^k_{\beta}),$

and

(111) \hfill $\int_{T^3} (x^k_{\beta})' dv_{\bar{\beta}} = - \int_{T^3} x^k_{\bar{x}} \frac{trg^\prime}{2} dv_{\bar{\beta}}.$

As $g^\prime$ is controlled in $H^3_{x,\beta}$ by $\Lambda$ so is controlled the term on the right of equation (111). Using formula (96) it is seen that the RHS of equation (110) is controlled in $H^2_{T^3}$ by $\Lambda$. The elliptic estimates of Proposition 28 show that $(x^k_{\beta})'$ is controlled
in $H^3_{x,\beta}$ by $\Lambda$. Still we want to show that $\|(x_k^b)^{\prime}\|_{H^4_{x,\beta}}$ is controlled by $\Lambda$. To do this step further we prove next that $\|(g_{ij})^{\prime}\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$. As a result if we write equation (110) over the $\{x_{\beta}\}$ coordinates

\[
\partial_{x_i}^{\prime}(\sqrt{g}g^{ij}\partial_{x_j}^{\prime}(x_k^b)) = -\partial_{x_i}^{\prime}(\sqrt{g}g^{ij}^{\prime}\partial_{x_j}^{\prime}(x_k^b)),
\]

we can apply the elliptic estimates of Proposition II (I) to see that $\|(x_k^b)^{\prime}\|_{H^4_{x,\beta}}$ is controlled by $\Lambda$ too. To show that $\|g^{\prime}\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$ we proceed as follows. We look at the system of equations

\[
\frac{1}{2}g^{lm}\partial_{m}g_{ij} + Q(\partial_{*}g_{s}, g^{*})_{ij} = E_{0,ij} + K_{s}^{s}K_{sj} - kK_{ij},
\]

(114)

\[
\nabla^{a}E_{0,ab} = (K \wedge B_{0})_{b},
\]

(115)

\[
(Curl E_{0})_{ab} = B_{l,ab} + \frac{3}{2}(B_{0} \times K)_{ab} - \frac{1}{2}kB_{0,ab}.
\]

The first equation is on the $\{x_{\beta}(\lambda)\}$ coordinate system (the other two are tensorial) and for this reason we have to be specially careful when we differentiate with respect to $\lambda$. We utilize the following. For any two tensor $T$, it is

\[
T_{ij} = T\left(\frac{\partial}{\partial x^{i}(\lambda)}, \frac{\partial}{\partial x^{j}(\lambda)}\right) = T\left(\frac{\partial}{\partial x^{m}(0)}, \frac{\partial}{\partial x^{l}(0)}\right) \frac{\partial x^{m}(0)}{\partial x^{i}(\lambda)} \frac{\partial x^{l}(0)}{\partial x^{j}(\lambda)}.
\]

Thus, when we differentiate with respect to $\lambda$ we get

\[
(T_{ij})^{\prime} = (T^{i})_{ij} + T_{il}(\frac{\partial x^{l}(0)}{\partial x^{i}(\lambda)})^{\prime} + T_{mj}(\frac{\partial x^{m}(0)}{\partial x^{j}(\lambda)})^{\prime}.
\]

Observe that $(\frac{\partial x^{m}(0)}{\partial x^{i}(\lambda)})^{\prime}$ is controlled in $H^2_{x,\beta}$. Differentiating the system of equations (113)-(115) with respect to $\lambda$ and evaluate it in the $\{x_{\beta}\}$ coordinates. After a straightforward calculation we get an elliptic system of the form

\[
g^{lm}\partial_{m}(g_{ij})^{\prime} = ((E_{0})_{ij}^{\prime} + H_{ij}^{1},
\]

(118)

\[
\nabla^{i}(E_{0})_{ij}^{\prime} = H_{ij}^{0},
\]

(119)

\[
(Curl E_{0})_{lm}^{\prime} = H_{lm}^{0},
\]

(120)

where $H_{ij}^{1}$ is a term controlled in $H^1_{x,\beta}$ by $\Lambda$ and $H_{ij}^{0}$ and $H_{lm}^{0}$ are terms controlled in $H^0_{x,\beta}$ by $\Lambda$. Standard elliptic estimates show that $\|(g_{ij})^{\prime}\|_{H^3_{x,\beta}}$ is controlled by $\Lambda$.

Now that we know $x_{\beta}^{\prime}$ is controlled in $H^{4}_{x,\beta}$ we conclude after inspecting equation (117) that $g^{\prime}$ is controlled in $H^{2}_{x,\beta}$ by $\Lambda$. It is straightforward to deduce, after
differentiating equation (52) in \( \lambda \), that \( ||K'||_{H^2_{x\beta}} \) is controlled by \( \Lambda \). Differentiating
the lapse equation and using standard elliptic estimates we get that \( N' \) is controlled
in \( H^3_{x\beta} \). Using the definition of admissible gauge we get that \( X' \) is controlled in
\( H^3_{x\beta} \) too. 

Before going into the initial value formulation let us mention that the space
of flow solutions on an interval \( I \) is complete under the BR-norm. The proof is
straightforward and will not be included.

Proposition 31 Let \((\Sigma, A)\) be a \( H^4 \)-manifold. The space of \( H^3 \)-flow solutions
over an interval \( I \) is complete under the BR-norm.

4 Applications.

4.1 The initial value formulation in the CMC gauge.

Theorem 5 Let \((\Sigma, A)\) be a \( H^4 \)-manifold and say \((g_0, K_0)\) is an initial state in
\( H^3 \times H^2 \) with \( k_0 < 0 \). Then

1. There is a unique \( H^3 \)-flow solution over an interval \( I = (k_{-1}, k_1) \) with \(-\infty \leq k_{-1} < k_0 < k_1 \leq 0 \). Moreover the size \( \inf \{|k_{-1} - k_0|, |k_0 - k_1|\} \) of the time
   interval on which the solution is guaranteed to exist is controlled from below
   by \( 1/\nu(k_0) \), \( \ln|k_0| \), \( V(g_0, K_0) \) and \( E_1(k_0) \).

2. There is continuity with respect to the initial conditions if we measure the
   space of initial conditions with the \( H^3_A \times H^2_A \) norm and the space of flow
   solutions with the BR-norm.

3. Because of item 1 above, we have the following continuity principle: a flow
   solution \((g, K, (N, X))(k)\) is defined until past of \( k^* < 0 \) (or before \( k_* < 0 \) if
   the flow is running in the past direction) iff \( \limsup_{k \to k^*} 1/\nu + V(k) + E_1(k) < \infty \).

Remark 10 i. As an outcome of the proof it is seen that if an initial state is \( C^\infty \)
the flow solution is \( C^\infty \). Moreover the growth of high-order Bel-Robinson energies
\( E_i \) (i \( \geq 2 \)) is at most exponential (i.e. \( E_i \leq C\exp^r \), i \( \geq 2 \)) on any compact subinterval
of \( I \) with constants \( C \) and \( r \) depending on the BR-norm of the solution on the
subinterval.

ii. The Theorem 5 does not make any a priori hypothesis on the topology of \( \Sigma \).
On manifolds with \( Y(\Sigma) \leq 0 \) we know the range of \( k \) is always a subset of \((-\infty, 0)\)
but on manifolds with \( Y(\Sigma) > 0 \) the flow may be defined until \( k_1 = 0 \) and beyond.
The Theorem 5 does not discuss a continuity criteria in this case (at \( k = k_1 = 0 \)).
One should be able to prove however that if \( ||K||_{L^2} \geq M > 0 \) all along the flow and
\( \lim_{k \to k_1 = 0} 1/\nu(k) + V(k) + E_1(k) < \infty \) then the flow is extendible behind \( k_1 = 0 \).
This situation corresponds to CMC solutions which are not time symmetric at the
admitting a unique extension to the definition of flow solutions determined from the initial data. But also, as will be proved (and as is required in a map that returns $C$ appropriate time interval). As will be shown, the map $S$ is Lipschitz and therefore admitting a unique extension to $H^3_x \times H^3$ in this sense the solutions are uniquely determined from the initial data. But also, as will be proved (and as is required in the definition of flow solutions), the flow solutions thus constructed are also space–time solutions. It follows from a well known result (see for instance [14] Theorem 4.27 and references therein) that any space-time solution with the same initial data is diffeomorphic to the space-time solution arising from the flow solution.

Proof:

We will prove Theorem [5] towards the future, the proof towards the past is naturally the same. We choose $t = k$.

Item 1. Step 1. Smooth the atlas $A$ to make it $C^\infty$ but compatible with $A$ at the 4-th level of regularity. We prove that there are $\epsilon > \Lambda$ and $I$ such that for any $C^\infty$ initial state $(g, K)$ with $\| g - g_0 \|_{H^3_x} \leq \epsilon$ and $\| K - K_0 \|_{H^3_x} \leq \epsilon$ the $C^\infty$-solution $\phi_{g, K}$ is defined on $I$ and has BR-norm bounded by $\Lambda$. For that we start proving that there are $\epsilon > \Lambda$ and $I$ such that while the $C^\infty$-flow solution is defined on $I$, the BR-norm over the interval where the solution is defined is bounded by $\Lambda$. Take $\Lambda$ equal to $C(\| g_0 \|_{H^3_x} + \| K_0 \|_{H^3_x} + \| (E_0(t_0), B_0(t_0)) \|_{H^3_x} + \| (E_1(t_0), B_1(t_0)) \|_{H^3_x})$ where $C$ is a constant greater than one to be fixed later. Suppose there exists $(g_1, K_1)$ $C^\infty$ initial states converging in $H^3_x \times H^3$ to $(g_0, K_0)$ such that the BR-norm $\|(g, K)\|_{BR}$ over a time interval $I = [t_0, t_i]$ is equal to $2\Lambda$, with $t_i \to t_0$. We can write

\begin{align*}
\| g(t) - g_0 \|_{H^3_x} &\leq \int_{t_0}^{t_i} \| \dot{g} \|_{H^3_x} dt, \\
\| K(t) - K_0 \|_{H^3_x} &\leq \int_{t_0}^{t_i} \| \dot{K} \|_{H^3_x} dt.
\end{align*}

Observe that the integrands are controlled by $\Lambda$ by Propositions [29]. Recall too that $\|J(W_i)\|_{L^p}$ and $\|\Pi\|_{H^3_x}$ are controlled by $\Lambda$. It follows from the Gauss equation that $\sup_{t \in [t_0, t_i]} \{E_1(t) - E_1(t_0)\} \to 0$ as $t_i \to t_0$. We see therefore that when $t_i - t_0$
is sufficiently small, for a constant $C$ that was fixed big enough\footnote{This constant $C$ is chosen big enough to control the BR-norm of the solution on the interval $[t_0, t_1]$}. the BR-norm of the solution on the interval $[t_0, t_1]$ is less than $\Lambda$ which is a contradiction.

Next we prove that for such $\epsilon$, $\Lambda$ and $I$ the $C^\infty$-solution $\phi, g$ is defined in all of $I$. To show that it is enough to prove that $Q_i$, $i \geq 2$ remains bounded (while defined) on $I$. In fact by Theorem\footnote{Theorem 30} and Proposition\footnote{Proposition 2} we know that: i. if the flow is defined until $t_*$ in $I$, ii. its BR-norm is bounded by $\Lambda$ and all $Q_i$ for $i \geq 0$ remain controlled by $\Lambda$, then the flow $(g, K)(t)$ must converge in $C^\infty$ to a $C^\infty$ state. Therefore the flow can be continued beyond $t_*$ which is a contradiction. Using Proposition\footnote{Proposition 27} bound the integrand involving $J(W_i)$ in the Gauss equation for $W_i$ as

$$
\int \sum |N(E_i^a J(W_i)_{a T b} + B_i^{ab} J(W_i)_{a T b})| dv_0 \leq C_1 (E_{i-1, \nu} \| \tilde{K} \|_{L^2}) Q_i^\frac{1}{2} (Q_i^\frac{1}{2}) + C_2 (E_{i-1, \nu} \| \tilde{K} \|_{L^2}).
$$

Thus, from the Gauss equation we get the inequality

$$|Q_i| \leq C_1 (E_{i-1, \Lambda})(Q_i + C_2 (E_{i-1, \Lambda}) Q_i^\frac{1}{2}).$$

Proceeding inductively in $i \geq 2$ using this inequality we get that $Q_i$ for $i \geq 2$ has at most exponential growth (with constants controlled by $\Lambda$) as desired.

**Step 2:** Until now we have a well defined map $S$ from $C^\infty$ initial states $\epsilon$-close to $(g_0, K_0)$ in $H^2 \times H^2$ into $C^\infty$ solutions defined on $I$ with uniformly bounded BR-norm. We will prove now that the map $S$ is Lipschitz if we take $\epsilon$ sufficiently small. It will follow from the completeness of the space of flow solutions that $S$ can be uniquely extended to a continuous map from $H^2 \times H^2$ initial states $\epsilon$-close to $(g_0, K_0)$ in $H^2 \times H^2$ into $H^2$-flow solutions in the sense of Definition\footnote{Definition 4}. To show that $S$ is Lipschitz we proceed as follows. First we prove that if $\epsilon$ is sufficiently small, any two states $(g_1, K_1)$ and $(g_2, K_2)$ in the ball of center $(g_0, K_0)$ and radius $\epsilon$ in $H^2 \times H^2$ can be joined by a path $(g, K)(\lambda)$, with $\lambda \in [0, 1]$, of $C^\infty$ initial states and with $\lambda$-derivative controlled in $H^2 \times H^2$ by $\| (g_0, K_0) \|_{H^2 \times H^2}$ (for any two $C^\infty$ states $(g_1, K_1)$ and $(g_2, K_2)$ as before, denote by $g(\lambda)$ the solutions with initial states $(g, K)(\lambda)$, $\lambda \in [0, 1]$). Then we prove that we can choose $\epsilon$ sufficiently small in such a way that there is a subinterval $I'$ of $I$ such that the BR-norm of $g' = \frac{dg}{d\lambda}$ on $I'$ (see Proposition\footnote{Proposition 30} item 3)

$$
\| (\phi, g)' \|_{BR} = \| g' \|_{C^0(I', 2)(H_A)} + \| K' \|_{C^0(I', 1)(H_A)} + \sum_{j=1}^{j=1} \| (E_j', B_j') \|_{C^0(I', 0)(H_A)}
$$

is as small as we want. By Proposition\footnote{Proposition 30} this is enough to guarantee that the map $S$ is Lipschitz\footnote{Lipschitz}.

Let us start with the construction of the paths $(g, K)(\lambda)$. Define $(\tilde{g}, \tilde{K})(\lambda) = \lambda (g_1, K_1) + (1-\lambda)(g_2, K_2)$. Then, for every $\lambda$ find the transverse traceless part of $\tilde{K}$...
with respect to $\tilde{g}$ and call it $\tilde{K}_{TT}$ (see [20]). Now note that if $\tilde{K}_0 \neq 0$ then for $\epsilon > 0$ sufficiently small it is $\tilde{K}_{TT} \neq 0$ (there is continuity in the York decomposition) and if $\tilde{K}_0 = 0$ then for $\epsilon$ sufficiently small $\tilde{g}$ is conformally deformable to a metric of constant negative scalar curvature. Thus, according to [20], the energy constraint is in all cases conformally solvable. It is direct to see that the path of initial states $(g, K)(\lambda)$ constructed in this way has $\|(g', K')\|_{H^2_\lambda \times H^2_\lambda}$-norm controlled by the $H^3_\lambda \times H^2_\lambda$-norm of $(g_0, K_0)$.

We next show that $\epsilon$ can be chosen small enough in such a way that at every $\lambda \in [0, 1]$ and for every path of $C^\infty$ initial states as described above, the BR-norm of $g'$ on a uniform interval $I' \subset I$ is as small as we want. We will get control on $\|g'\|_{BR}$ from a contradiction argument. We need some preliminary discussion on the $\lambda$-derivative of the Weyl fields $W = W_0$ and $W = W_1$.

We first note that the $\lambda$-derivative of a Weyl field is not necessarily a Weyl field as we have

$$(W_{abcd})' g^{bd} = - W_{abcd}(g^{bd})'.$$

Note that $(W_{abcd})' g^{ac} g^{bd} = 0$. We can construct from $W'$ a Weyl field by making it traceless according to the formula

$$(121) \quad \tilde{W}'_{abcd} = W'_{abcd} - \frac{1}{2} (g_{ac} W'_{bd} + g_{bd} W'_{ac} - g_{bc} W'_{ad} - g_{ad} W'_{bc}),$$

where we have defined $W'_{ac} = W'_{abcd} g^{bd}$, which makes it a $(2, 0)$ symmetric and traceless tensor. We conclude from this that if we control $g'$ in $H^2_\lambda$ (i.e. all of its components at every time slice) and $Q(\tilde{W}')$ then we control $\|W'\|_{H^2_\lambda}$ (at every time slice). We will get control on $Q(\tilde{W}')$ using the Gauss equation and to use the Gauss equations we need an expression for the divergence of $W'$. We compute

$$\tilde{J}_{bcd} = \nabla^a \tilde{W}'_{abcd} = \nabla^a W'_{abcd} - \frac{1}{2} (g_{ac} \nabla^a W'_{bd} + g_{bd} \nabla^a W'_{ac} - g_{bc} \nabla^a W'_{ad} - g_{ad} \nabla^a W'_{bc}).$$

that we arrange in the form

$$(122) \quad \tilde{J}_{bcd} = \nabla^a W'_{abcd} - \frac{1}{2} (\nabla_c W'_{bd} - \nabla_d W'_{bc}) - \frac{1}{2} (g_{bd} \nabla^a W'_{ac} - g_{bc} \nabla^a W'_{ad}).$$

We have therefore three different terms that we have to estimate on the RHS of the last equation. The first term can be computed by

$$(123) \quad \nabla^a W'_{abcd} = - \nabla'^a W_{abcd} + \tilde{J}'_{bcd},$$

if we compute $\tilde{J}'_{bcd}$. Contracting the last equation we get

$$(124) \quad \nabla^a W'_{ac} = \tilde{J}'_{bcd} g^{bd}.$$

which can be used instead of each summand in the third term of equation (122) if we compute $\tilde{J}'_{bcd}$. The second term $\nabla_c W'_{ba} - \nabla_a W'_{bc}$ in equation (122) is calculated by differentiating and contracting the identity

$$\nabla_a W_{bcde} + \nabla_b W_{cdae} + \nabla_c W_{abde} = \frac{1}{3} \varepsilon_{fabc} J_{de}^f.$$
Differentiating we get
\[
(\nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde})g^{bd} + \nabla_a W'_{ce} + \nabla^b W'_{beca} - \nabla_c W'_{ae} = \nabla_a W'_{ce} - \nabla_c W'_{ae} = - (\nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde})g^{bd} - \nabla^b W'_{beca} + \frac{1}{3} \epsilon_{fabc} J'^{f}_{de} + \frac{1}{3} \epsilon_{fabc} J^*_f {de}',
\]
and rearranging terms
\[
\nabla_a W'_{ce} - \nabla_c W'_{ae} = - (\nabla_a W_{bcde} + \nabla_b W_{cade} + \nabla_c W_{abde})g^{bd} - \nabla^b W'_{beca} + \frac{1}{3} \epsilon_{fabc} J'^{f}_{de} + \frac{1}{3} \epsilon_{fabc} J^*_f {de}'.
\]
\[(125)\]

Now, \( J(W_0)' = 0 \) and \( J(W_1)' \) has an expression of the form
\[
(126) \quad J(W_1)' = \Pi' \ast \nabla W_0 + \Pi' \ast \nabla' W_0 + \Pi' \ast \nabla W_0' + T' \ast \text{Rm} \ast W_0 + T' \ast \text{Rm} \ast W_0'.
\]

Recall finally that \( J^*_{abc} = \frac{1}{2} J_{alm} \epsilon^{lm}_{bc} \). Using the discussion above, let us give a schematic representation of the currents \( \tilde{J}(W_0)' \) and \( \tilde{J}(W_1)' \). The expressions will be used in the Gauss equations for \( \tilde{W}_0 \) and \( \tilde{W}_1 \). The expression for \( \tilde{J}(W_0)' \) comes from using equations (123), (124) and (125) together with \( J(W_0)' = 0 \) in equation (122). We can write (schematically)
\[
\tilde{J}(W_0)' = \nabla' \ast W_0.
\]
\[(127)\]

To get a schematic expression for \( \tilde{J}(W_1)' \) use again equations (123), (124) and (125) in equation (122). We can write (schematically)
\[
\tilde{J}(W_1)' = \nabla' \ast W_1 + J(W_1)' \ast g + \epsilon' \ast \epsilon \ast J(W_1) + \epsilon \ast \epsilon \ast J(W_1)'.
\]
with the expression for \( J(W_1)' \) borrowed from equation (126).

From Proposition 30 (item 3) it is direct to show (see also iv. and v. in Remark 9) that the BR-norms of \( g \) and \( g' \) over an interval \( I' \) control the norms
\[
\|\nabla'(t)\|_{H^2_{\lambda}}, \|\Pi'(t)\|_{H_{\lambda}^2}, \|\nabla W_0(t)\|_{H^1_{\lambda}}, \|\epsilon'(t)\|_{H^1_{\lambda}}, \|T'(t)\|_{H^1_{\lambda}}.
\]
for every \( t \) in \( I' \). Moreover, as \( \|g'(t)\|_{H^2_{\lambda}} \) is controlled by the BR-norms of \( g \) and \( g' \) (see v. in Remark 9), we have
\[
\|g'(t) - g'(t_0)\|_{H^2_{\lambda}} \leq C(t - t_0),
\]
for every \( t \) in \( I' \), where \( C \) depends on the BR-norm of \( g \) and \( g' \) on \( I' \). The Gauss equation applied to \( \tilde{W}_0 \) gives the inequality
\[
(129) \quad |Q(\tilde{W}_0')| \leq C Q(\tilde{W}_0'),
\]
where \( C \), again, depends on the BR-norms of \( g \) and \( g' \) on \( I' \). To see that, use the schematic expression (127) for the current \( \tilde{J}(W_0)' \) in the Gauss equation (equation 27)
\[
\hat{Q}(\tilde{W}_0') = - \int_{\Sigma} 2NE^{ij}(\tilde{W}_0')\tilde{J}(\tilde{W}_0')iT_j + 2NB^{ij}(\tilde{W}_0')\tilde{J}^*(\tilde{W}_0')iT_j + 3NQ_{abT}^{\Phi} dv_g,
\]
66
together with the control on the norms $\|\Phi\|_{H^2_\lambda}$ and $\|\nabla\|_{H^2_\lambda}$ mentioned before. Similarly one obtains an inequality for the time derivative of the Bel-Robinson energy associated to $W_1$

\begin{equation}
|Q(W_1')| \leq C(Q(W_1) + Q(W_1')^{1/2}).
\end{equation}

(130)

where, as above, depends on the BR-norms of $g$ and $g'$ over $I'$. Now, from the inequalities (129) and (130) we have

\begin{equation}
|Q(W_0')(t) - Q(W_0')(t_0)| \leq C(t - t_0).
\end{equation}

(131)

\begin{equation}
|Q(W_1')(t) - Q(W_1')(t_0)| \leq C(t - t_0).
\end{equation}

(132)

where, again, $C$ depends on the BR-norms of $g$ and $g'$ on $I'$. Fix two $\lambda$-independent horizontal vector fields $V^a$ and $W^b$. For any Weyl field $W = W_0$ or $W = W_1$ we have

\begin{equation}
E_{ab} V^a W^b = W'_{acbd} T^{e} T^{d} V^a W^b + W_{acbd} T^{e} T^{d} V^a W^b + W_{acbd} T^{e} T^{d} V^a W^b,
\end{equation}

(133)

and recall (see iv. in Remark 9)

\begin{equation}
T^a = -\frac{X^a}{N} - \frac{N'}{N} T^a.
\end{equation}

(134)

On the other hand

\begin{equation}
W'_{abcd} = \tilde{W}'_{abcd} + \frac{1}{2} (g_{ac} W'_{bd} + g_{bd} W'_{ac} - g_{bc} W'_{ad} - g_{ad} W'_{bc}),
\end{equation}

(135)

and

\begin{equation}
W'_{ac} = -W_{acbd} g^{bd}.
\end{equation}

(136)

From (133), (134), (135) and (136) we can write\textsuperscript{28} for $E = E(W_0)$ or $E = E(W_1)$

\begin{equation}
\|E\|_{H^0_\lambda}^2 \leq C(\Lambda) (\|\tilde{W}'\|_{H^0_\lambda}^2 + \|W\|_{H^0_\lambda}^2 \|g'\|_{H^2_\lambda}^2) \leq C(\Lambda) (Q(W') + Q(W)\|g'\|_{H^2_\lambda}).
\end{equation}

(137)

Doing the same for $B(W_0)$ or $B(W_1)$ and putting altogether gives

\begin{equation}
\|(E_0', B_0')\|_{H^0_\lambda} + \|(E_1', B_1')\|_{H^0_\lambda} \leq C(\Lambda) (Q^2(\tilde{W}_0') + Q^2(\tilde{W}_1') + (Q^2(W_0) + Q^2(W_1))\|g'\|_{H^2_\lambda}).
\end{equation}

(138)

Note finally that by Proposition\textsuperscript{30} $\|\frac{d}{dt} g'\|_{H^2_\lambda}$ and $\|\frac{d}{dt} K'\|_{H^0_\lambda}$ are controlled by the BR-norm of $g$ and $g'$ over an interval $I'$. This implies

\begin{equation}
\|g'(t) - g'(t_0)\|_{H^2_\lambda} \leq C(t - t_0),
\end{equation}

(139)

\textsuperscript{28}Note that inside expression (137) there are norms with respect to $A$ and norms with respect to $g$. We use implicitly that for any space-time tensor $U$ we have $A(\Lambda) \|U\|_{H^2_\lambda} \leq \|U\|_{H^2_\lambda} \leq B(\Lambda) \|U\|_{H^0_\lambda}$.
where $C$, as above, depends on the BR-norm of $g$ and $g'$ on $I'$.

We start the argument by contradiction. We will end up showing that the BR-norm of $g'$ over a time interval $I' \subset I$ can be made as small as we want if $\epsilon$ is chosen small enough. Suppose that there is $(g_{1,i}, K_{1,i})$ and $(g_{2,i}, K_{2,i})$ converging to $(g_0, K_0)$ in $H^3_A \times H^2_A$ and such that for some $\lambda_i \in [0, 1]$ the BR-norm of $g'_i(\lambda_i)$ over an interval $I' = [t_0, t_1]$, with $t_1 \to t_0$, is equal to a positive constant $\tilde{C}$. We note that as $(g_{j,i}, K_{j,i})$, $j = 1, 2$ converge to $(g_0, K_0)$ in $H^3_A \times H^2_A$ it is $(g'_i, K'_i)(\lambda) \to 0$ in $H^3_A \times H^2_A$ and uniformly in $\lambda \in [0, 1]$. This fact has several consequences on the $\lambda$-derivative of the fields at time $t = t_0$. Namely, for the path $(g, K)(\lambda)$ it is $\|N'(t_0)||H^3_A \to 0$ and $\|X'(t_0)||H^3_A \to 0$. From (134) we conclude that $\|T'(t_0)||H^3_A \to 0$ and therefore $\|g'(t_0)||H^3_A \to 0$. Now, decompose $W_0(t_0)$ and $W_1(t_0)$ into vertical and horizontal components using formulas (21)-(26). Differentiating with respect to $\lambda$ we get that $\|W'_0(t_0)||H^0_A \to 0$ and $\|W'_1(t_0)||H^0_A \to 0$. It follows from (121) that $\|\tilde{W}'_0(t_0)||H^0_A \to 0$ and $\|\tilde{W}'_1(t_0)||H^0_A \to 0$. In particular we have $Q(W'_0(t_0)) \to 0$ and $Q(W'_1(t_0)) \to 0$. Taking this into account, equations (128), (131), (132), (139) and (140), show that if $t_1 - t_0$ is taken small enough the BR-norm of $g'$ over $I' = [t_0, t_1]$ is less than $\tilde{C}$ which is a contradiction.

Item 2. This item follows from the fact that the map $S$ from initial states into flow solutions is Lipschitz.

Item 3. This item follows from item 1.

\[ \Box \]

4.2 Long time flows.

We give here an application showing that a $(C^\infty)$ CMC-flow over a three-manifold with non-positive Yamabe invariant is a long-time flow if the corresponding globally hyperbolic space-time is future geodesically complete and the first order Bel-Robinson energy of the CMC flow remains bounded above.

Theorem 6 Say $Y(\Sigma) \leq 0$ and $(M \sim \Sigma \times \mathbb{R}, g)$ a smooth $C^\infty$ maximally globally hyperbolic space-time. Say $(g, K)(k)$ with $k \in [a, b)$ is a CMC flow where $b$ is the lim sup of the range of $k$ and say $E_1 \leq \Lambda$. Then if $(M, g)$ is future geodesically complete the CMC flow is a long time flow i.e. $b = 0$.

A CMC slice with mean curvature $k_0$ will be denoted by $\Sigma_{k_0}$. Let $t$ be a global smooth time function on $M$. Assume the range of $t$ is $(-\infty, \infty)$. Let $T = -\nabla t/|g(\nabla t, \nabla t)|^{1/2}$ be the unit future pointing normal to the foliation induced by $t$. Observe that if $\xi : [c, d] \to M$ is a curve then $t(\xi) = dt(\xi) = g(\nabla t, \xi)$, therefore if $g(T, \xi) \neq 0$ everywhere the curve cannot be closed (i.e. $\xi(c) = \xi(d)$).

Proof: We will proceed by contradiction and assume that $b < 0$. Under that assumption we first prove that the CMC foliation lies entirely between $\{t = t_0\}$ and $\{t = t_1\}$, for some $t_0$ and $t_1$. Suppose this is not the case, then there exists a sequence of points $p_1$ in $\Sigma_{k_0}$ with $k_i \to b$ and $t(p_i) \to \infty$. For a given $i$ denote by
γi, a past directed geodesic starting at pi and maximizing the Lorentzian distance to the initial CMC slice Σi. Clearly γi are perpendicular to Σi. Also because the lapse for the CMC time k is bounded above by \(3/b^2\) the length of the geodesics γi is controlled above by b. There is a subsequence of geodesics γij converging into an inextendible geodesic γ∞ of bounded length and perpendicular to Σi. This is a contradiction as we have assumed the space-time was future geodesically complete.

Now \(E_1\) is assumed less than \(\Lambda\) and if \(b < 0\) then \(Vol_{g(k)}(\Sigma)\) is bounded above for all \(k\) in \([a, b)\), therefore for the CMC flow not to be extendable beyond \(b\) there must be a sequence of points \(p_i\) in \(\Sigma_{k_i}\) with \(k_i \to b\) such that the volume radius \(\nu_{g(k_i)}(p_i) \to 0\). Choose it in such a way that \(\nu_{g(k_i)}(p) \geq \nu_{g(k_i)}(p_i)\) for all \(p \in \Sigma_{k_i}\). We will be arguing with the rescaled space-time \(\lambda_i^2 g\) with \(\lambda_i = 1/\nu_{g(k_i)}(p_i)\).

Denote with a subindex \(\lambda_i\) any new rescaled quantity, for example \(k_{\lambda_i} = k_i/\lambda_i\), \(g_{\lambda_i} = \lambda_i^2 g(k_i)\) and so on. Noting that \(\int_{\Sigma} |K|^2 dv_g\), \(\int_{\Sigma} |Ric|^2 dv_g\) and \(Q_0\) scale as a distance\(^{-1}\) and \(Q_1\) as a distance\(^{-3}\), the sequence \((\Sigma, p_i, g_{\lambda_i}, K_{\lambda_i})\) is converging (strongly) in \(H^3 \times H^2\) into a flat (and complete) initial state \((\bar{\Sigma}, p_\infty, g_\infty, K_\infty)\) where \(\bar{\Sigma}\) is some three-manifold, \(g_\infty\) is a flat metric and \(K_\infty = 0\). In particular there is sequence of geodesic loops \(l_i\) in \(\Sigma_{g_{\lambda_i}}\) (say of length one) and based at \(p_i\). Parameterized with the arc-length \(s\) we write \(l_i : [0, 1] \to \Sigma_{g_{\lambda_i}} \subset M\). As noted above there must be for any \(i\) a \(s_i\) with \(g_{\lambda_i}(\dot{l}_i(s_i), T_{\lambda_i}) = 0\). At \(l_i(s_i)\) consider the orthonormal frame \((\dot{l}_i, e_2, e_3, T_{\lambda_i})\). Parallel transport it along \(l_i\) and call the resulting frames as \(e_\delta, \delta = 1, 2, 3, 4\). As observed above the states are becoming flatter therefore \(g_{\lambda_i}(\nabla_{\dot{l}_i} \dot{l}_i, e_\delta) \to 0\). Now think the loops \(l_i\) as inside the scaled space-time \((M, \lambda_i^2 g)\). Call \(l_\infty(s_\infty)\) a limit point of \(l_i(s_i)\) in \(M\). As the scaled space-time is becoming closer and closer to Minkowski around \(l_\infty(s_\infty)\) the sequence of loops and frames converge to a solution of the system, i. \(g_{\text{Min}}(\nabla_{l_\infty} \dot{l}_\infty, e_\delta) = 0\), ii. \(\nabla_{l_\infty} e_\delta = 0\), where \(g_{\text{Min}}\) is the Minkowski metric on \(\mathbb{R}^4\) and \(l_\infty : [0, 1] \to \mathbb{R}^4\), and \(g_{\text{Min}}(\dot{l}(s_\infty), \dot{l}(s_\infty)) = 1\). Necessarily, such solution must be a piece of segment of length one and therefore not closed.

\[\square\]

**Remark 11** The same proof works to prove that a \(C^\infty\) CMC flow solution over a three-manifold with non-positive Yamabe invariant is either a long time flow or the CMC foliation reaches the end of the maximally globally hyperbolic solution, in the sense that for any compact subset of the maximally globally hyperbolic space-time, there is a CMC slice which is disjoint from it (and to the future of a fixed CMC slice).

**References**

[1] Anderson, Michael T. On long-time evolution in general relativity and geometrization of 3-manifolds. Comm. Math. Phys. 222 (2001), no. 3, 533–567.

\(^{29}\)It is important here that \(K \to 0\) in \(H^2\). For this we need the assumption that \(E_1\) and not just \(E_0 = Q_0\) are bounded along evolution.
[2] Anderson, Michael T. Extrema of curvature functionals on the space of metrics on 3-manifolds. *Calc. Var. Partial Differential Equations* 5 (1997), no. 3, 199–269.

[3] Anderson, Michael T. Scalar curvature and geometrization conjectures for 3-manifolds. *Comparison geometry (Berkeley, CA, 1993–94)*, 49–82, Math. Sci. Res. Inst. Publ., 30, Cambridge Univ. Press, Cambridge, 1997.

[4] Andersson, Lars; Moncrief, Vincent. Future complete vacuum space-times. *The Einstein equations and the large scale behavior of gravitational fields*, 299–330, Birkhäuser, Basel, 2004.

[5] Andersson, Lars; Moncrief, Vincent. Elliptic-hyperbolic systems and the Einstein equations. *Ann. Henri Poincaré* 4 (2003), no. 1, 1–34.

[6] Andersson, Lars The global existence problem in general relativity. *The Einstein equations and the large scale behavior of gravitational fields*, 71–120, Birkhäuser, Basel, 2004.

[7] Besse, Arthur L. Einstein manifolds. *Springer-Verlag, Berlin, 2008.*

[8] Bartnik, Robert. The existence of maximal surfaces in asymptotically flat spacetimes. *Asymptotic behavior of mass and spacetime geometry (Corvallis, Ore., 1983)*, 57–60, Lecture Notes in Phys., 202, Springer, Berlin, 1984.

[9] Bartnik, Robert. Remarks on cosmological space-times and constant mean curvature surfaces. *Comm. Math. Phys. 117* (1988), no. 4, 615–624.

[10] Buchert, Thomas. Dark energy from structure: a status report *arXiv:0707.2153*.

[11] Chruściel, Piotr T; Isenberg, James; Pollack, Daniel Initial data engineering. *Comm. Math. Phys. 257* (2005), no. 1, 29–42.

[12] Choquet-Bruhat, Yvonne; York, James W. Geometrical well posed systems for the Einstein equations. *C. R. Acad. Sci. Paris Sr. I Math.* 321 (1995), no. 8, 1089–1095.

[13] Christodoulou, Demetrios; Klainerman, Sergiu The global nonlinear stability of the Minkowski space. *Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, 1993.*

[14] Fischer, A; Marsden, J. The initial value problem and the dynamical formulation fo general relativity. *General relativity. An Einstein centenary survey. Edited by S. W. Hawking and W. Israel. Cambridge University Press, Cambridge-New York, 1979.*
[15] Fischer, Arthur E.; Moncrief, Vincent The reduced hamiltonian of general relativity and the $\sigma$-constant of conformal geometry. Mathematical and quantum aspects of relativity and cosmology (Pythagorean, 1998), 70–101, Lecture Notes in Phys., 537, Springer, Berlin, 2000.

[16] Gilbarg, David; Trudinger, Neil S. Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[17] Gerhardt, Claus $H$-surfaces in Lorentzian manifolds. Comm. Math. Phys. 89 (1983), no. 4, 523–553.

[18] Hawking, S.W; Ellis, G. F.R. The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[19] Friedrich, Helmut Hyperbolic reductions for Einstein’s equations. Classical Quantum Gravity 13 (1996), no. 6, 1451–1469.

[20] Isenberg, James, Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. Classical Quantum Gravity 12 (1995), no. 9, 2249–2274.

[21] Petersen, Peter Riemannian geometry. Graduate Texts in Mathematics, 171. Springer-Verlag, New York, 1998.

[22] Reiris, Martin General $K = -1$ Friedman-Lematre models and the averaging problem in cosmology. Classical Quantum Gravity 25 (2008), no. 8, 085001, 26 pp.

[23] Rendall, Alan D. Constant mean curvature foliations in cosmological spacetimes. Journes Relativistes 96, Part II (Ascona, 1996). Helv. Phys. Acta 69 (1996), no. 4, 490–500.

[24] Wald, Robert M. General relativity. University of Chicago Press, Chicago, IL, 1984.