ON THE MAXIMAL NUMBER OF COLUMNS OF A \(\Delta\)-MODULAR MATRIX

GENNADIY AVERKOV AND MATTHIAS SCHYMURA

Abstract. We study the maximal number of pairwise distinct columns in a \(\Delta\)-modular integer matrix with \(m\) rows. Recent results by Lee et al. provide an asymptotically tight upper bound of \(O(m^2)\) for fixed \(\Delta\). We complement this and obtain an upper bound of the form \(O(\Delta)\) for fixed \(m\), and with the implied constant depending polynomially on \(m\).

1. Introduction

Full row rank integer matrices with minors bounded by a given constant \(\Delta\) in the absolute value have been extensively studied in integer linear programming as well as matroid theory. The motivation in optimization comes from the fact that, on the one hand, algorithms for integer programming tuned to the case of a constraint matrix with minors bounded in terms of \(\Delta\) and, on the other hand, proximity relations of integer programs and their linear relaxation formulated in terms of \(\Delta\) are available. More information on the background from optimization can be found in [1, 4].

For a matrix \(A \in \mathbb{R}^{m \times n}\) and for \(1 \leq k \leq \min\{m, n\}\), we write
\[
\Delta_k(A) := \max\{|\det(B)| : B \text{ is a } k \times k \text{ submatrix of } A\}
\]
for the maximal absolute value of a \(k \times k\) minor of \(A\). A matrix \(A \in \mathbb{R}^{m \times n}\) of rank \(m\) is said to be \(\Delta\)-modular and \(\Delta\)-submodular, for \(\Delta \in \mathbb{N}\), if \(\Delta_m(A) = \Delta\) and \(\Delta_m(A) \leq \Delta\), respectively.\(^1\) Moreover, a matrix \(A \in \mathbb{R}^{m \times n}\) is said to be totally \(\Delta\)-modular and totally \(\Delta\)-submodular, if \(\max_{k \in [m]} \Delta_k(A) = \Delta\) and \(\max_{k \in [m]} \Delta_k(A) \leq \Delta\), respectively, where \([m] := \{1, 2, \ldots, m\}\).\(^2\)

Our object of studies is the generalized Heller constant, which we define as
\[
h(\Delta, m) := \max\{n \in \mathbb{N} : A \in \mathbb{Z}^{m \times n} \text{ has pairwise distinct columns and } \Delta_m(A) = \Delta\}.
\]
The value \(h(\Delta, m)\) is directly related to the value \(c(\Delta, m)\) studied in [1, 4] and defined as the maximum number \(n\) of columns in a \(\Delta\)-submodular integer matrix \(A\) with \(m\) rows with the properties that \(A\) has no zero columns and for any two distinct columns \(A_i\) and \(A_j\) with \(1 \leq i < j \leq n\) one has \(A_i \neq A_j\) and \(A_i \neq -A_j\). It is clear that
\[
c(\Delta, m) = \frac{1}{2}(\max\{h(1, m), \ldots, h(\Delta, m)\} - 1)
\]
\(^1\)The authors of [1, 4] use the term \(\Delta\)-modular for what we call \(\Delta\)-submodular.
\(^2\)An integer matrix \(A \in \mathbb{Z}^{m \times n}\) of rank \(m\) is unimodular if and only if \(UA\) is totally unimodular for some \(U \in \text{GL}_m(\mathbb{Z})\).
holds, showing that \( e(\Delta, m) \) and \( h(\Delta, m) \) are “equivalent” in many respects. However, our proofs are more naturally phrased in terms of \( h(\Delta, m) \) rather than \( e(\Delta, m) \), as we prefer to prescribe \( \Delta_m(A) \) rather than providing an upper bound on \( \Delta_m(A) \) and we do not want to eliminate the potential symmetries within \( A \) coming from taking columns \( A_i \) and \( A_j \) that satisfy \( A_i = -A_j \).

Upper bounds on the number of columns in \( \Delta \)-modular integer matrices with \( m \) rows have been gradually improved over time as described in the introduction of [1]. By now it is known that for each fixed \( \Delta \), \( h(\Delta, m) \) is quadratic in \( m \) [1] and that, for each fixed \( m \), \( h(\Delta, m) \) is linear in \( \Delta \), see the comment on page 24 in [4]. However, so far there has not been any bound that is polynomial in \( m \) and linear in \( \Delta \). The authors of [4, p. 24] ask if there exists a bound of the form \( O(m^d)\Delta \) for some constant \( d \in \mathbb{N} \). We answer this question in the affirmative by showing that a bound of order \( O(m^4)\Delta \) exists.

**Theorem 1.1.** Let \( \Delta \in \mathbb{Z}_{>0} \) and \( m \geq 5 \). Then,

\[
h(\Delta, m) \leq m(m + 1) + 1 + 2(\Delta - 1) \cdot \sum_{i=0}^{m} \binom{m}{i}.
\]

It remains an open question whether our bound of order \( O(m^4)\Delta \) can be improved to a bound of order \( O(m^d)\Delta \) with \( d \in \{2, 3\} \).

A classical result of Heller [3] shows that the maximal number of pairwise distinct columns in a unimodular integer matrix with \( m \) rows is \( m^2 + m + 1 \), that is, \( h(1, m) = m^2 + m + 1 \). The previously best bounds on the generalized Heller constant \( h(\Delta, m) \) are those of Lee, Paat, Stallknecht & Xu [4]:

**Theorem 1.2 ([4, Thm. 2 & Prop. 1 & Prop. 2]).** Let \( \Delta, m \in \mathbb{Z}_{>0} \). Then,

\[
h(\Delta, m) = m^2 + m + 1 + 2m(\Delta - 1) \quad \text{if} \quad \Delta \leq 2 \quad \text{or} \quad m \leq 2,
\]

and, for all other cases \((\Delta, m)\), one has the bounds

\[
m^2 + m + 1 + 2m(\Delta - 1) \leq h(\Delta, m) \leq (m^2 + m)\Delta^2 + 1.
\]

Note that the case \( \Delta = 1 \) is Heller’s original result. As a conjecture Lee et al. formulate that the lower bound in (1) is actually the correct value of \( h(\Delta, m) \), for any choice of \( \Delta, m \in \mathbb{Z}_{>0} \). A \( \Delta \)-modular integer matrix with \( m \) rows and that many columns has the difference set of

\[
\{e_1, e_2, \ldots, e_m\} \cup \{2e_1, 3e_1, \ldots, \Delta e_1\}
\]

as its columns, where \( e_i \) denotes the \( i \)th coordinate unit vector.

2. **Counting by residue classes**

Our main idea is to count the columns of a \( \Delta \)-modular integer matrix by residue classes of a certain lattice. This is the geometric explanation for the linearity in \( \Delta \) of our upper bound in Theorem 1.1.

To be able to count in the non-trivial residue classes, we need to extend the Heller constant \( h(1, m) \) to a shifted setting. Given a translation vector \( t \in \mathbb{R}^m \) and a matrix \( A \in \mathbb{R}^{m \times n} \), the shifted matrix \( t + A := t1^\top + A \) has columns \( t + A_i \), where \( A_1, \ldots, A_n \) are the columns of \( A \), and \( 1 \) denotes the all-one vector.
Definition 2.1. For any \( m \in \mathbb{Z}_{>0} \), we define the shifted Heller constant \( h_t(m) \) as the maximal number \( n \) such that there exists a translation vector \( t \in \{0, 1\}^m \setminus \{0\} \) and a matrix \( A \in \{-1, 0, 1\}^{m \times n} \) with pairwise distinct columns such that \( t + A \) is totally 1-submodular, that is, \( \max_{k \in [m]} \Delta_k(t + A) \leq 1 \).

Note that, in contrast to the generalized Heller constant \( h(\Delta, m) \), we do not necessarily require \( t + A \) to have full rank, but we restrict \( A \) to have entries in \( \{-1, 0, 1\} \) only. Moreover, the reason for restricting the non-zero translation vectors to the half-open unit cube \( \{0, 1\}^m \) becomes apparent in the proof of the following crucial estimate.

Lemma 2.2. For every \( \Delta, m \in \mathbb{Z}_{>0} \), we have

\[
h(\Delta, m) \leq h(1, m) + (\Delta - 1) \cdot h_0(m).
\]

Proof. Let \( A \in \mathbb{Z}^{m \times n} \) be a matrix with \( \Delta_m(A) = \Delta \) and pairwise distinct columns and let \( X_A \subseteq \mathbb{Z}^m \) be the set of columns of \( A \). Without loss of generality, we may assume that \( X_A = -X_A \) and that \( 0 \in X_A \), since this does not affect that \( \Delta_m(A) = \Delta \) and only possibly increases the number of its columns. Further, let \( b_1, \ldots, b_m \in X_A \) be such that \( |\text{det}(b_1, \ldots, b_m)| = \Delta \) and consider the parallelepiped

\[
P_A := [-b_1, b_1] + \cdots + [-b_m, b_m] = \left\{ \sum_{i=1}^m \alpha_i b_i : -1 \leq \alpha_i \leq 1, \forall i \in [m] \right\}.
\]

Observe that \( X_A \subseteq P_A \). Indeed, assume to the contrary that there exists an \( x = \sum_{i=1}^m \alpha_i b_i \in X_A \), with, say \( |\alpha_j| > 1 \). Then, we obtain that

\[
|\text{det}(b_1, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_m)| = |\alpha_j| \Delta > \Delta,
\]

which contradicts that \( A \) was chosen to be \( \Delta \)-modular.

Now, consider the sublattice \( \Lambda := \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_m \) of \( \mathbb{Z}^m \), whose index in \( \mathbb{Z}^m \) equals \( \Delta \). We seek to bound the number of elements of \( X_A \) that fall into a fixed residue class of \( \mathbb{Z}^m \) modulo \( \Lambda \). To this end, let \( x \in \mathbb{Z}^m \) and consider the residue class \( x + \Lambda \). Every element \( z \in (x + \Lambda) \cap P_A \) is of the form \( z = \sum_{i=1}^m \alpha_i b_i \), for some \( \alpha_1, \ldots, \alpha_m \in [-1, 1] \) and can be written as

\[
z = \sum_{i=1}^m (\alpha_i - \{\alpha_i\}) b_i + \sum_{i=1}^m \{\alpha_i\} b_i,
\]

where \( \{\alpha_i\} = \alpha_i - \lfloor \alpha_i \rfloor \in [0, 1) \) is the fractional part of \( \alpha_i \), and where \( \bar{x} := \sum_{i=1}^m \{\alpha_i\} b_i \) is the unique representative of \( x + \Lambda \) in the half-open parallelepiped \( [0, b_1) + \cdots + [0, b_m) \), and in particular, is independent of \( z \).

Write \( \alpha(z) = (\bar{\alpha}_1, \ldots, \bar{\alpha}_m) \in \{-1, 0, 1\}^m \) for the (integral) coefficients in the representation (2) of \( z \), and \( \alpha(\bar{x}) = (\{\alpha_1\}, \ldots, \{\alpha_m\}) \in [0, 1)^m \) for those of \( \bar{x} \).

In symbols, we have \( z = B(\alpha(\bar{x}) + \alpha(z)) \), for \( B = (b_1, \ldots, b_m) \) of \( \mathbb{Z}^m \times m \).

Because the vectors \( (x + \Lambda) \cap X_A \) constitute a \( \Delta \)-submodular system and since \( |\text{det}(b_1, \ldots, b_m)| = \Delta \), the set of vectors \( \{\alpha(z) + \alpha(\bar{x}) : z \in (x + \Lambda) \cap X_A\} \) are 1-submodular systems. For the residue class \( \Lambda \), this system is given by \( \{\alpha(z) : z \in \Lambda \cap X_A\} \subseteq \{-1, 0, 1\}^m \) and moreover has full rank as it contains \( e_1, \ldots, e_m \), and we are thus in the setting of the classical Heller constant \( h(1, m) \).

For the \( \Delta - 1 \) non-trivial residue classes \( x + \Lambda, x \notin \mathbb{Z}^m \), we are in the setting of the shifted Heller constant \( h_0(m) \). Indeed, as the set of vectors
\{b_1, \ldots, b_m\} \cup (x + \Lambda) \cap X_A \subseteq X_A \text{ is } \Delta\text{-submodular, the set } \\
\{e_1, \ldots, e_m\} \cup \{\alpha(\vec{z}) + \alpha(\vec{z}) : z \in (x + \Lambda) \cap X_A\} \\
has all its minors, of any size, bounded by 1 in absolute value. By the \definition of \(h_n(m)\) the second set in this union has at most \(h_n(m)\) elements.

As a consequence, we get \(n = |X_A| \leq h(1, m) + (\Delta - 1) \cdot h_n(m)\), as desired.

\textbf{Remark 2.3.} The proof above shows that we actually want to bound the number of columns \(n\) of a matrix \(A \in \{-1, 0, 1\}^{m \times n}\) such that the system \\
\{e_1, \ldots, e_m\} \cup \{t + A_1, \ldots, t + A_n\} \\
is \(1\)-submodular, for some \(t \in [0, 1]^m \setminus \{0\}\). However, \(t + A\) is totally \(1\)-submodular if and only if \(\{e_1, \ldots, e_m\} \cup (t + A)\) is \(1\)-submodular.

\textbf{Remark 2.4.} As any matrix \(A \in \{-1, 0, 1\}^{m \times n}\) with pairwise distinct columns can have at most \(3^n\) columns, one gets the bound \(h_n(m) \leq 3^n\).

Thus, Lemma 2.2 directly implies the estimate \(h(\Delta, m) \leq 3^m \cdot \Delta\).

2.1. Small dimensions and lower bounds in the shifted setting. Recall that the original Heller constant is given by \(h(1, m) = m^2 + m + 1\). The following exact results for dimensions two and three show the difference between this original (unshifted) and the shifted setting grasped by \(h_n(m)\).

\textbf{Proposition 2.5.} We have \(h_n(2) = 6\) and \(h_n(3) = 12\).

\textit{Proof.} First, we show that \(h_n(2) = 6\). Let \(A \in \{-1, 0, 1\}^{2 \times n}\) have distinct columns and let \(t \in [0, 1]^2 \setminus \{0\}\) be such that \(t + A\) is totally \(1\)-submodular.

Since \(t \neq 0\), it has a non-zero coordinate, say \(t_1 > 0\). As the \(1 \times 1\) minors of \(t + A\), that is, the entries of \(t + A\), are bounded in absolute value by 1, we get that the first row of \(A\) can only have entries in \(\{-1, 0\}\). This shows already that \(n \leq 6\), as there are simply only 6 options for the columns of \(A\) respecting this condition.

An example attaining this bound is given by \\
\[
A = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.
\]

One can check that (up to permutations of rows and columns) this is actually the unique example \((A, t)\) with 6 columns in \(A\).

Now, we turn our attention to proving \(h_n(3) = 12\). The lower bound follows by the existence of the following matrix and translation vector \\
\[
A = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}.
\]

Checking that \(t + A\) is indeed totally \(1\)-submodular is a routine task that we leave to the reader.

For the upper bound, let \(A \in \{-1, 0, 1\}^{3 \times n}\) and \(t \in [0, 1]^3 \setminus \{0\}\) be such that \(t + A\) is totally \(1\)-submodular. Let \(s\) be the number of non-zero entries of \(t \neq 0\). Just as we observed for \(h_n(2)\), we get that there are \(s \geq 1\) rows of \(A\) only containing elements from \(\{-1, 0\}\). Thus, if \(s = 3\) there are only \(2^3 = 8\) possible columns and if \(s = 2\), there are only \(2^2 \cdot 3 = 12\) possible columns, showing that \(n \leq 12\) in both cases.
We are left with the case that \( s = 1 \), and we may assume that \( A \) has no entry equal to 1 in the first row and that \( t_1 > 0 \). Assume for contradiction that \( n \geq 13 \). There must be \( \ell \geq 7 \) columns of \( A \) with the same first coordinate, which we subsume into the submatrix \( A' \). By the identity \( \text{h}(1,2) = 7 \) applied to the last two rows, and \( t_2 = t_3 = 0 \), we must have \( \ell = 7 \) and up to permutations, \( A' = \begin{pmatrix} a & a & a & a & a & a & a \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{pmatrix} \), for some \( a \in \{ -1, 0 \} \). Since the absolute value of the \( 2 \times 2 \) minors of \( t + A \) are bounded by 1, the remaining \( n - \ell \geq 6 \) columns of \( A \) are different from \((b, 1, 1)^\top \) and \((b, -1, -1)^\top \), where \( b \) is such that \( \{a, b\} = \{-1, 0\} \). Under these conditions, we find that \( A \) contains either \( B = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \) or \( C = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \) as a submatrix. However, the condition \( |\det(t + B)| \leq 1 \) gives \( t_1 \geq 1 \), and \( |\det(t + C)| \leq 1 \) gives \( t_1 \leq 0 \), both contradicting the assumption that \( 0 < t_1 < 1 \).

Combining Lemma 2.2, the identity \( \text{h}(1,m) = m^2 + m + 1 \), and Proposition 2.5 yields the bounds \( \text{h}(\Delta, 2) \leq 6\Delta + 1 \) and \( \text{h}(\Delta, 3) \leq 12\Delta + 1 \). The latter bound improves upon Theorem 1.2. However, as \( \text{h}(\Delta, 2) = 4\Delta + 3 \), we see that the approach via the shifted Heller constant \( \text{h}_s(m) \) cannot give optimal results.

A quadratic lower bound on \( \text{h}_s(m) \) can be obtained as follows:

**Proposition 2.6.** For every \( m \in \mathbb{Z}_{\geq 0} \), we have

\[
\text{h}_s(m) \geq \text{h}(1, m - 1) = m(m - 1) + 1.
\]

**Proof.** Let \( A' \in \{ -1, 0, 1 \}^{(m-1) \times n} \) be a totally unimodular matrix with \( n = \text{h}(1, m - 1) \) columns, and let \( A \in \{ -1, 0, 1 \}^{m \times n} \) be obtained from \( A' \) by simply adding a zero-row as the first row. Then, for the translation vector \( t = \left( \frac{1}{m}, 0, \ldots, 0 \right) \) the matrix \( t + A \) is totally 1-submodular.

Indeed, we only need to look at its \( k \times k \) minors, for \( k \leq m \), that involve the first row, as \( A' \) is totally unimodular by choice. But then, the triangle inequality combined with developing the given minor by the first row, shows that its absolute value is bounded by 1.

3. A POLYNOMIAL UPPER BOUND ON \( \text{h}_s(m) \)

An elegant and alternative proof for Heller’s result that \( \text{h}(1,m) = m^2 + m + 1 \) has been suggested by Bixby & Cunningham [2] and carried out in detail in Schrijver’s book [6, § 21.3]. They first reduce the problem to consider only the supports of the columns of a given (totally) unimodular matrix and then apply Sauer’s Lemma from extremal set theory that guarantees the existence of a large cardinality set that is shattered by a large enough family of subsets of \( [m] \).

We show that this approach can in fact be adapted for the shifted Heller constant \( \text{h}_s(m) \). The additional freedom in the problem that is introduced by the translation vectors \( t \in [0, 1]^m \setminus \{0\} \) makes the argument a bit more involved, but still gives a low degree polynomial bound. To this end, we write \( \text{supp}(y) := \{ j \in [m] : y_j \neq 0 \} \) for the support of a vector \( y \in \mathbb{R}^m \) and

\[
\mathcal{E}_A := \{ \text{supp}(A_i) : i \in [n] \} \subseteq 2^{[m]}
\]
for the family of supports in a matrix \( A \in \mathbb{R}^{m \times n} \) with columns \( A_1, \ldots, A_n \). We use the notation \( 2^Y \) for the power set of a finite set \( Y \).

Just as in the unshifted Heller setting, each support can be realized by at most two columns of \( A \), if there exists a translation vector \( t \in [0,1]^m \) such that \( t + A \) is totally 1-submodular.

**Proposition 3.1.** Let \( A \in \{-1,0,1\}^{m \times n} \) and \( t \in [0,1]^m \) be such that \( \Delta_k(t + A) \leq 1 \), for \( k \in \{1,2\} \). Then, each \( E \in \mathcal{E}_A \) is the support of at most two columns of \( A \).

**Proof.** Observe that in view of the condition \( \Delta_1(t + A) \leq 1 \) and the assumption that \( t_i \geq 0 \), for every \( i \in [m] \), we must have \( t_r = 0 \), as soon as there is an entry equal to 1 in the \( r \)th row of \( A \).

Now, assume to the contrary that there are three columns \( A_i, A_j, A_k \) of \( A \) having the same support \( E \in \mathcal{E}_A \). Then, clearly \( |E| \geq 2 \) and the restriction of \( A \) to the rows indexed by \( E \) is a \pm 1-matrix. Also observe that there must be two rows \( r, s \in E \) so that \( A \) contains an entry equal to 1 in both of these rows. Indeed, if there is at most one such row, then the columns \( A_i, A_j, A_k \) cannot be pairwise distinct. Therefore, we necessarily have \( t_r = t_s = 0 \). Now, there are two options. Either two of the columns \( A_i, A_j, A_k \) are such that their restriction to the rows \( r, s \) give linearly independent \pm 1-vectors. This however would yield a \( 2 \times 2 \) submatrix of \( t + A \) with minor \pm 2, contradicting that \( \Delta_2(t + A) \leq 1 \). In the other case, the restriction of the three columns to the rows \( r, s \) has the form \( \pm \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \) or \( \pm \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \), up to permutation of the indices \( i, j, k \). If \( |E| = 2 \), then this cannot happen as \( A \) is assumed to have pairwise distinct columns. So, \( |E| \geq 3 \), and considering the columns, say \( A_i, A_j \), which agree in the rows \( r, s \), there must be another index \( \ell \in E \setminus \{r, s\} \) such that \( (A_i)_\ell = 1 \) and \( (A_j)_\ell = -1 \), or vice versa. In any case this means that also \( t_\ell = 0 \) and that there is a \( 2 \times 2 \) submatrix of \( t + A \) in the rows \( r, \ell \) consisting of linearly independent \pm 1-vectors. Again this contradicts that \( \Delta_2(t + A) \leq 1 \), and thus proves the claim. \( \square \)

As mentioned above this observation on the supports allows to use \textit{Sauer’s Lemma} from extremal set theory which we state for the reader’s convenience.

**Lemma 3.2** (Sauer [5]). Let \( m, k \in \mathbb{Z}_{>0} \). If \( \mathcal{E} \subseteq 2^{[m]} \) is such that \( |\mathcal{E}| > \binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{k} \), then there is a subset \( Y \subseteq [m] \) with \( k + 1 \) elements that is shattered by \( \mathcal{E} \), meaning that \( \{ E \cap Y : E \in \mathcal{E} \} = 2^Y \).

Now, the strategy to bounding the number of columns in a matrix \( A \in \{-1,0,1\}^{m \times n} \) such that \( t + A \) is totally 1-submodular for some \( t \in [0,1]^m \) is to use the inequality \( |\mathcal{E}_A| \geq \frac{1}{n} \), which holds by Proposition 3.1, and then to argue by contradiction. Indeed, if \( n > 2 \sum_{i=0}^{k-1} \binom{m}{i} \), then by Sauer’s Lemma there would be a \( k \)-element subset \( Y \subseteq [m] \) that is shattered by \( \mathcal{E}_A \). In terms of the matrix \( A \), this means that (possibly after permuting rows or columns) it contains a submatrix of size \( k \times 2^k \) which has exactly one column for each of the \( 2^k \) possible supports and where in each column the non-zero entries are chosen arbitrarily from \( \{-1,1\} \). For convenience we call any such matrix a \textit{Sauer Matrix} of size \( k \). For concreteness, a Sauer Matrix of size 3 is of the
be totally $S$ has a Sauer Matrix restriction of $\tau$.

The combinatorial proof of $h(1, m) = m^2 + m + 1$ is based on the fact that no Sauer Matrix of size 3 is totally 1-submodular (cf. Schrijver [6, §21.3] and Bixby & Cunningham [2]). In order to extend this kind of argument to the shifted setting, we need some more notation.

**Definition 3.3.** Let $S$ be a Sauer Matrix of size $k$. We say that a vector $r \in [0, 1)^k$ is feasible for $S$ if $r + S$ is totally 1-submodular. Further, we say that $S$ is feasible for translations if there exists a vector $r \in [0, 1)^k$ that is feasible for $S$, and otherwise we say that $S$ is infeasible for translations.

Moreover, the Sauer Matrix $S$ is said to be of type $(s, k - s)$, if there are exactly $s$ rows in $S$ that contain at least one entry equal to 1.

As feasibility of a Sauer Matrix of type $(s, k - s)$ is invariant under permuting rows, we usually assume that each of its first $s$ rows contains an entry equal to 1.

**Proposition 3.4.** Let $k$ be a natural number and assume that no Sauer Matrix of size $k$ is feasible for translations. Then,

$$h_s(m) \leq 2 \cdot \sum_{i=0}^{k-1} \binom{m}{i} \in \mathcal{O}(m^{k-1}).$$

**Proof.** Assume for contradiction that there is a matrix $A \in \{-1, 0, 1\}^{m \times n}$ and a translation vector $t \in [0, 1)^m$ such that $t + A$ is totally 1-submodular and $n > 2 \sum_{i=0}^{k-1} \binom{m}{i}$. By Proposition 3.1, we have $|E_A| \geq \frac{1}{2} n > \sum_{i=0}^{k-1} \binom{m}{i}$ and thus by Sauer’s Lemma (up to permuting rows or columns) the matrix $A$ has a Sauer Matrix $S$ of size $k$ as a submatrix. Writing $r \in [0, 1)^k$ for the restriction of $t$ to the $k$ rows of $A$ in which we find the Sauer Matrix $S$, we get that by the total 1-submodularity of $t + A$, the matrix $r + S$ necessarily must be totally 1-submodular as well. This however contradicts the assumption. 

In contrast to the unshifted setting, for the sizes 3 and 4, there are Sauer Matrices $S$ and vectors $r$, such that $r + S$ is totally 1-submodular. For instance,

$$S = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad r = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}, \quad r = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$


In both cases, $2(r + S)$ is a matrix all of whose entries are either 1 or $-1$. By Hadamard’s inequality, the determinant of any $\pm 1$-matrix of size $k \leq 4$ is at most $2^k$, and thus $\Delta_k(r + S) \leq 1$ for all $k \leq 4$, in the two examples above.

Our aim is to show that this pattern does not extend to higher dimensions, and that no Sauer Matrix of size 5 is feasible for translations. The proof requires a more detailed study of Sauer Matrices of special types and sizes 4 and 5.

**Proposition 3.5.**

(i) The vector $r \in [0, 1]^4$ is feasible for the Sauer Matrix of type $(0, 4)$ if and only if $r = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$.

(ii) The Sauer Matrix of type $(0, 5)$ is infeasible for translations.

(iii) No Sauer Matrix of type $(1, 4)$ is feasible for translations.

(iv) If $r \in [0, 1]^4$ is feasible for a Sauer Matrix of type $(1, 3)$, then $r = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$.

(v) No Sauer Matrix of type $(2, 3)$ is feasible for translations.

The proof of these statements is based on identifying certain full-rank submatrices of the respective Sauer Matrix for which the minor condition provides a strong obstruction for feasibility. The details are given in Section 4.

**Lemma 3.6.** There does not exist a Sauer Matrix $S$ of size 5 with the property that $r + S$ is totally 1-submodular, for some translation vector $r \in [0, 1]^5$.

**Proof.** Assume that there is a Sauer Matrix $S$ of size 5 and a vector $r \in [0, 1]^5$ such that $\Delta_k(r + S) \leq 1$, for all $k \leq 5$. Note that if in the ith row of $S$ there is an entry equal to 1, then $r_i = 0$, because of $\Delta_1(r + S) \leq 1$. So, if there are three rows in $S$ containing an entry equal to 1, then they contain a Sauer Matrix of size 3 that is itself totally 1-submodular. However, we already noted that no such Sauer Matrix exists.

Thus, we may assume that $S$ is a Sauer Matrix whose type is either $(0, 5)$, $(1, 4)$, or $(2, 3)$. We have proven in Proposition 3.5 (ii), (iii), and (v), however, that all such Sauer Matrices are infeasible for translations. □

With these preparations we are now able to prove our main result.

**Theorem 1.1.** In view of Lemma 2.2, we have $h(\Delta, m) \leq h(1, m) + (\Delta - 1) \cdot h_5(m)$. The claimed bound now follows by Heller’s identity $h(1, m) = m^2 + m + 1$ and the fact that $h_5(m) \leq 2 \sum_{i=0}^4 \binom{m}{i}$, which holds by combining Proposition 3.4 and Lemma 3.6. □

### 4. Feasibility of Sauer Matrices in low dimensions

Here, we complete the discussion from the previous section and give the proof of Proposition 3.5. Parts of the argument are based on the observation that the condition $|\det(r + M)| \leq 1$, for any $M \in \mathbb{R}^{k \times k}$, is equivalent to a pair of linear inequalities in the coordinates of $r \in \mathbb{R}^k$. This turns the question on whether a given Sauer Matrix is feasible for translations into the question of whether an associated polyhedron is non-empty.
Proposition 3.5. (i): Assume that \( r \in \{0, 1\}^4 \) is such that \( r + S \) is totally 1-submodular, and consider the following two \( 4 \times 4 \) submatrices of \( S \):

\[
M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}.
\]

By the \( 4 \times 4 \) minor condition on \( r + S \), we have

\[
|\det(r + M)| = r_1 \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2r_1 \leq 1,
\]

and hence \( r_1 \leq \frac{1}{2} \). Likewise, we have

\[
|\det(r + N)| = (1 - r_1) \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2(1 - r_1) \leq 1,
\]

and hence \( r_1 \geq \frac{1}{2} \), so that actually \( r_1 = \frac{1}{2} \). Analogous arguments for the other coordinates of \( r \), show that \( r = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T \) as claimed. The fact that \( r + S \) is totally 1-submodular has been already discussed above.

(ii): The argument is similar to the one for the first part. Assume for contradiction, that there is a vector \( r \in \{0, 1\}^5 \) such that \( \Delta_S(r + S) \leq 1 \). Consider the following two \( 5 \times 5 \) submatrices of \( S \):

\[
X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}.
\]

By the \( 5 \times 5 \) minor condition on \( r + S \), we have

\[
|\det(r + X)| = r_1 \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 3r_1 \leq 1,
\]

and hence \( r_1 \leq \frac{1}{3} \). Likewise, we have

\[
|\det(r + Y)| = (1 - r_1) \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 3(1 - r_1) \leq 1.
\]

Therefore, we get \( r_1 \geq \frac{2}{3} \), a contradiction.

(iii): Without loss of generality, we may assume that the first row of \( S \) contains an entry equal to 1, and we assume for contradiction that there is some \( r \in \{0, 1\}^5 \) such that \( r + S \) is totally 1-submodular. As the entries of \( r + S \) are contained in \([-1, 1]\), we get that \( r_1 = 0 \). Moreover, the last four rows of \( S \) contain a Sauer Matrix of type \((0, 4)\). By part (i), this means that \( r_2 = r_3 = r_4 = r_5 = \frac{1}{2} \), so that in summary there is only one possibility for the translation vector \( r \).

Now, as \( r_1 = 0 \), we may multiply the first row of \( S \) with \(-1\) if needed, and can assume that the vector \((-1, -1, -1, -1, -1)^T\) is a column of \( S \). If \( M \) denotes any of the four matrices

\[
\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}
\]


then the absolute value of the determinant of $r + M$ equals $3/2$. Thus, if indeed $\Delta_S(r + S) \leq 1$, then these matrices cannot be submatrices of $S$. In particular, this implies that

$$M' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

must be a submatrix of $S$. However, the determinant of $r + M'$ equals $-2$, in contradiction to $r + S$ being totally 1-submodular.

(iv): We assume that the first row of each considered Sauer Matrix $S$ of type $(1, 3)$ contains an entry equal to 1, so that $r_1 = 0$. As in (iii) we can moreover assume that $(-1, -1, -1, -1)^T$ is a column of $S$ (by possibly multiplying the first row by $-1$). We now employ a case distinction based on the signs of the entries in the first row of the columns $a = (\pm 1, -1, 0, 0)^T$, $b = (\pm 1, 0, -1, 0)^T$, and $c = (\pm 1, 0, 0, -1)^T$ of $S$.

Case 1: $a_1 = b_1 = c_1 = -1$.

Under this assumption, $S$ contains the matrix $N$ from part (i) as a submatrix and thus $r_1 \geq \frac{1}{2}$, contradicting that $r_1 = 0$.

Case 2: $a_1 = b_1 = c_1 = 1$.

In this case, $S$ contains the submatrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}.$$

The conditions $|\det(r + A)| \leq 1$ and $|\det(r + B)| \leq 1$ translate into the contradicting inequalities $r_2 + r_3 + r_4 \leq 1$ and $r_2 + r_3 + r_4 \geq 2$, respectively.

Case 3: Exactly two of the entries $a_1, b_1, c_1$ equal $-1$.

Without loss of generality, we may permute the last three rows of $S$, and assume that $a_1 = b_1 = -1$. We find that $S$ now contains the submatrices

$$C = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The conditions $|\det(r + C)| \leq 1$, $|\det(r + D)| \leq 1$ and $|\det(r + E)| \leq 1$ translate into the contradicting inequalities $r_2 + r_3 \leq 1$, $r_4 \geq \frac{1}{2}$, and $r_4 + 1 \leq r_2 + r_3$, respectively.

Case 4: Exactly two of the entries $a_1, b_1, c_1$ equal $1$.

As in Case 3, we may assume that $a_1 = b_1 = 1$. Here, the following six matrices can be found as submatrices in $S$:

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}. $$
The minor conditions for these matrices translate into the inequality system

\[
\begin{align*}
    r_4 &\leq \frac{1}{2} \\
    r_3 &\leq r_2 \\
    r_2 &\leq r_3 \\
    r_4 &\geq \frac{1}{2} \\
    r_2 + r_3 &\geq 1 \\
    r_2 + r_3 &\leq 1
\end{align*}
\]

in the same order as the matrices were given above. Solving this system of inequalities shows that necessarily \( r_2 = r_3 = r_4 = \frac{1}{2} \), and the proof is complete.

(v): Assume that there is a Sauer Matrix \( S \) of type \((2, 3)\) and a vector \( r \in [0, 1)^5 \) that is feasible for \( S \). Observe that \( S \) contains feasible Sauer Matrices of types \((1, 3)\) in its rows indexed by \( \{1, 3, 4, 5\} \) and by \( \{2, 3, 4, 5\} \). By part (iv) this means that necessarily we have \( r = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T \), and we can now argue similarly as we did in part (iii).

First of all, as \( r_1 = r_2 = 0 \), we may multiply the first or second row of \( S \) with \(-1\) if needed, and can assume that the vectors \((-1, 0, -1, -1, -1)^T\) and \((0, -1, 0, 0, 0)^T\) are columns of \( S \). We distinguish cases based on the signs of the entries in the first or second row of the columns \( a = (\pm 1, 0, -1, 0, 0)^T \), \( b = (\pm 1, 0, -1, 0)^T \), \( c = (\pm 1, 0, 0, -1)^T \), and \( a' = (0, \pm 1, -1, 0, 0)^T \), \( b' = (0, \pm 1, 0, -1)^T \), \( c' = (0, \pm 1, 0, -1)^T \) of \( S \).

**Case 1:** \( a_1 = b_1 = c_1 = 1 \) or \( a'_2 = b'_2 = c'_2 = -1 \).

Here, one of the matrices

\[
C_1 = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 1 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}
\quad \text{or} \quad
C_2 = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

must be a submatrix of \( S \), but the absolute value of the determinant of both \( r + C_1 \) and \( r + C_2 \) equals 3/2.

**Case 2:** Two of the entries \( a_1, b_1, c_1 \) equal \(-1\) or two of the entries \( a'_2, b'_2, c'_2 \) equal 1.

Without loss of generality, we may permute the last three rows of \( S \), and assume that either \( a_1 = b_1 = -1 \) or \( a'_2 = b'_2 = 1 \). Now, one of the matrices

\[
C_3 = \begin{bmatrix}
-1 & 0 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}
\quad \text{or} \quad
C_4 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

must be a submatrix of \( S \), but again the absolute value of the determinant of both \( r + C_3 \) and \( r + C_4 \) equals 3/2.

**Case 3:** Up to permuting the last three rows of \( S \) we have \( \begin{bmatrix}
a_1 & b_1 & c_1 \\
a'_2 & b'_2 & c'_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 1 \\
-1 & -1 & 1
\end{bmatrix} \).

With this assumption, one of the matrices

\[
\begin{bmatrix}
-1 & 0 & 0 & 1 & 1 \\
-1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & -1 & 1 & 1 \\
-1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 \\
-1 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1
\end{bmatrix}
\]

must be a submatrix of $S$, because one of the four vectors $(\pm 1, \pm 1, -1, -1, -1)^T$ must be a column of $S$. As before, if $F$ denotes any of these four matrices, then the absolute value of the determinant of $r + F$ equals $3/2$.

**Case 4:** Up to permuting the last three rows of $S$ we have \[
\begin{bmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    -1 & 1 & 1 \\
    -1 & -1 & 1 \\
    0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
    -1 & 1 & 0 & 0 & 0 \\
    0 & -1 & 0 & -1 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & -1
\end{bmatrix}.
\]
In this case, one of the matrices
\[
C_7 = \begin{bmatrix}
    -1 & 0 & 1 & 0 & 0 \\
    0 & -1 & 0 & -1 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & -1
\end{bmatrix} \quad \text{or} \quad C_8 = \begin{bmatrix}
    1 & -1 & 0 & 0 & 0 \\
    0 & 0 & -1 & -1 & 0 \\
    0 & 0 & -1 & -1 & 0 \\
    0 & 0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
must be a submatrix of $S$, because one of the vectors $(\pm 1, 0, 0, 0, 0)^T$ must be a column of $S$. As before, the absolute value of the determinant of both $r + C_7$ and $r + C_8$ equals $3/2$.

In conclusion, in all cases we found a $5 \times 5$ minor of $r + S$ whose absolute value is greater than 1, and thus no feasible Sauer Matrix of type $(2, 3)$ can exist.

\[\square\]

5. Discussion and open problems

The determination of the exact value of $h(\Delta, m)$ remains an open problem. Note that the bounds from other sources and the bound we prove here are incomparable when both $m$ and $\Delta$ vary. In order to understand the limits of our method for upper bounding $h(\Delta, m)$, it is necessary to determine the exact asymptotic behavior of $h_s(m)$. Finally, for (partial) verification of the conjecture by Lee et al. one could try checking this conjecture in the cases where $m$ and/or $\Delta$ are fixed to small values. The smallest choice of $\Delta$, for which the conjecture is open is $\Delta = 3$. As for the case of fixed $m$, we suspect that our upper bounds on $h(\Delta, m)$, for $m = 3$ and $m = 4$, are not tight.

References

1. Stephan Artmann, Robert Weismantel, and Rico Zenklusen, *A strongly polynomial algorithm for bimodular integer linear programming*, Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, 2017, pp. 1206–1219.
2. Robert E. Bixby and William H. Cunningham, *Short cocircuits in binary matroids*, Eur. J. Comb. 8 (1987), 213–225.
3. Isidore Heller, *On linear systems with integral valued solutions*, Pac. J. Math. 7 (1957), 1351–1364.
4. Jon Lee, Joseph Paat, Ingo Stallknecht, and Luze Xu, *Polynomial upper bounds on the number of differing columns of $\Delta$-modular integer programs*, https://arxiv.org/abs/2105.08160, 2021.
5. Norbert Sauer, *On the density of families of sets*, J. Combin. Theory Ser. A 13 (1972), 145–147.
6. Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986, A Wiley-Interscience Publication.
ON THE MAXIMAL NUMBER OF COLUMNS OF A $\Delta$-MODULAR MATRIX

BTU Cottbus-Senftenberg, Platz der Deutschen Einheit 1, 03046 Cottbus, Germany

Email address: averkov@b-tu.de

BTU Cottbus-Senftenberg, Platz der Deutschen Einheit 1, 03046 Cottbus, Germany

Email address: schymura@b-tu.de