THE IMAGES OF GELFAND-SHILOV SPACES UNDER THE BARGMANN TRANSFORM

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Abstract. We characterize the images of the Gelfand-Shilov spaces and their distribution spaces under the Bargmann transform.

0. Introduction

The aim of the paper is to deduce mapping properties of the Bargmann transform between Gelfand-Shilov spaces and their dual spaces, spaces of Gelfand-Shilov distributions of certain degrees, to convenient sets of entire functions. We also deduce certain duality properties between these dual spaces. We apply these results to characterize the Gelfand-Shilov spaces and their distribution spaces, in terms of short-time Fourier transforms and to show that all these function and distribution spaces are obtained by suitable unions or intersections of the broad family of modulation spaces, introduced in [14].

We remark that less general versions of the results here can be found in [14] and to some extent in [3], where similar investigations can be found. Especially we note that our results here include situations valid for the Gelfand-Shilov space $S_t^s$ and their distribution space $(S_t^s)'$ for $s, t \geq 1/2$, while the largest part of the analysis in [3, 14] require that $s, t > 1/2$, which do not include the limit cases $s = 1/2$ or $t = 1/2$.

We also use the result to characterize the Gelfand-Shilov spaces in terms of the short-time Fourier transform, in a similar way as in [14]. In fact, let $s, t > 0$ be such that

\[ s + t \geq 1 \quad \text{and} \quad (s, t) \neq (1/2, 1/2), \quad (0.1) \]

let $\phi \in \Sigma_t^s(\mathbb{R}^d) \setminus 0$, and let $f$ be an appropriate ultra-distribution. Then it is proved in [14] that $f \in (S_t^s)'(\mathbb{R}^d)$, if and only if the estimate

\[ (V_{\phi}f)(x, \xi) \lesssim e^{r(|x|^{1/t} + |\xi|^{1/s})}, \quad (0.2) \]

for every constant $r > 0$. Here and in the sequel, $A \lesssim B$ means that $A \leq cB$ for a suitable constant $c > 0$.

By the computations in [14] it also follows that if $s, t > 0$ satisfies (0.1) or $s, t = 1/2$, $\phi \in \Sigma_t^s(\mathbb{R}^d)$ and $f \in (S_t^s)'(\mathbb{R}^d)$, then (0.2) holds.

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for some \( r > 0 \). In Section 2 we prove that if \( f \in \mathcal{S}_{1/2}(\mathbb{R}^d) \) and \( \phi \) is Gaussian, then (1.2) holds for every \( r > 0 \).

1. Preliminaries

In this section we recall some basic properties on the Bargmann transform. We shall often formulate these results in the framework of the Gelfand-Shilov space \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) and its dual \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \) (see e.g. [7]). The reader who is not interested in this general situation may replace \( \mathcal{S}_{1/2}(\mathbb{R}^d) \) and \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \) by \( \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{S}'(\mathbb{R}^d) \) respectively. Here \( \mathcal{S}(\mathbb{R}^d) \) is the set of Schwartz functions on \( \mathbb{R}^d \), and \( \mathcal{S}'(\mathbb{R}^d) \) is the set of tempered distributions on \( \mathbb{R}^d \); see for example [3].

1.1. Gelfand-Shilov spaces. We start by recalling some facts about Gelfand-Shilov spaces. Let \( 0 < h, s, t \in \mathbb{R} \) be fixed. Then \( \mathcal{S}_{s,h}(\mathbb{R}^d) \) consists of all \( f \in C^\infty(\mathbb{R}^d) \) such that

\[
\|f\|_{\mathcal{S}_{s,h}^t} \equiv \sup_{x \in \mathbb{R}^d} \left| \frac{x^\beta \partial^\alpha f(x)}{h^{\alpha+|\beta|} \alpha! \beta!} \right|
\]

is finite. Here the supremum should be taken over all \( \alpha, \beta \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d \).

Obviously \( \mathcal{S}_{s,h}^t \) is a Banach space which increases with \( h, s \) and \( t \) and \( \mathcal{S}_{s,h}^t \hookrightarrow \mathcal{S} \). Here and in what follows we use the notation \( A \hookrightarrow B \) when the topological spaces \( A \) and \( B \) satisfy \( A \subseteq B \) with continuous embeddings. Furthermore, if \( s + t \geq 1 \) and \( (s, t) \neq (1/2, 1/2) \), or \( s = t = 1/2 \) and \( h \) is sufficiently large, then \( \mathcal{S}_{s,h}^t \) contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in \( \mathcal{S} \) and in \( \mathcal{S}_{s,h}^t \), it follows that the dual \( (\mathcal{S}_{s,h}^t)'(\mathbb{R}^d) \) of \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \) is a Banach space which contains \( \mathcal{S}'(\mathbb{R}^d) \).

The Gelfand-Shilov spaces \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \) and \( \Sigma_{s,h}^t(\mathbb{R}^d) \) are defined as the inductive and projective limits respectively of \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \). This implies that

\[
\mathcal{S}_{s,h}^t(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}^t(\mathbb{R}^d) \quad \text{and} \quad \Sigma_{s,h}^t(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}^t(\mathbb{R}^d), \quad (1.1)
\]

and that the topology for \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \) is the strongest possible one such that the inclusion map from \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \) to \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \) is continuous, for every choice of \( h > 0 \). The space \( \Sigma_{s,h}^t(\mathbb{R}^d) \) is a Fréchet space with seminorms \( \| \cdot \|_{\mathcal{S}_{s,h}^t} ; h > 0 \). Moreover, \( \Sigma_{s,h}^t(\mathbb{R}^d) \neq \{0\} \), if and only if \( s + t \geq 1 \) and \( (s, t) \neq (1/2, 1/2) \), and \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \neq \{0\} \), if and only if \( s + t \geq 1 \). From now on we assume that the Gelfand-Shilov parameter pair \( (s, t) \) are admissible, or GS-admissible, that is, \( s + t \geq 1 \) and \( (s, t) \neq (1/2, 1/2) \) when considering \( \Sigma_{s,h}^t(\mathbb{R}^d) \), and \( s + t \geq 1 \) when considering \( \mathcal{S}_{s,h}^t(\mathbb{R}^d) \).

The Gelfand-Shilov distribution spaces \( (\mathcal{S}_{s,h}^t)'(\mathbb{R}^d) \) and \( (\Sigma_{s,h}^t)'(\mathbb{R}^d) \) are the projective and inductive limit respectively of \( (\mathcal{S}_{s,h}^t)'(\mathbb{R}^d) \). This means
We remark that in \([7, 10]\) it is proved that \(S^s_t(R^d)\) is the dual of \(S^s_0(R^d)\), and \(\Sigma_t^s(R^d)\) is the dual of \(\Sigma_t^s(R^d)\) (also in topological sense). For convenience we set
\[
S_s = S^s_0, \quad S'_s = (S^s_0)' , \quad \Sigma_s = \Sigma^s_t, \quad \Sigma'_s = (\Sigma^s_t)'.
\]

For every admissible \(s, t > 0\) and \(\varepsilon > 0\) we have
\[
\Sigma^s_t(R^d) \hookrightarrow S^s_t(R^d) \hookrightarrow \Sigma^{s+\varepsilon}_{t+\varepsilon}(R^d)
\]
and
\[
(\Sigma^{s+\varepsilon}_{t+\varepsilon})'(R^d) \hookrightarrow (S^s_t)'(R^d) \hookrightarrow (\Sigma^s_t)'(R^d).
\]

The Gelfand-Shilov spaces possess several convenient mapping properties, and in the case \(s = t\) they are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and under (partial) Fourier transformations.

From now on we let \(\mathcal{F}\) be the Fourier transform which takes the form
\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx
\]
when \(f \in L^1(R^d)\). Here \(\langle \cdot, \cdot \rangle\) denotes the usual scalar product on \(R^d\). The map \(\mathcal{F}\) extends uniquely to homeomorphisms on \(\mathcal{F}'(R^d)\), from \((S^s_0)'(R^d)\) to \((S^s_0)'(R^d)\) and from \((\Sigma^s_t)'(R^d)\) to \((\Sigma^s_t)'(R^d)\). Furthermore, \(\mathcal{F}\) restricts to homeomorphisms on \(\mathcal{F}(R^d)\), from \(S^s_0(R^d)\) to \(S^s_0(R^d)\) and from \(\Sigma^s_t(R^d)\) to \(\Sigma^s_t(R^d)\), and to a unitary operator on \(L^2(R^d)\).

It follows from the following lemma that elements in Gelfand-Shilov spaces can be characterized by estimates of the form
\[
|f(x)| \lesssim e^{-\varepsilon|x|^{1/t}} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-\varepsilon|\xi|^{1/s}}.
\]

The proof is omitted, since the result can be found in e.g. [4][9].

**Lemma 1.1.** Let \(s, t > 0\) and \(f \in S^t_{1/2}(R^d)\). Then the following is true:

1. if \(s + t \geq 1\), then \(f \in S^s_0(R^d)\), if and only if \((1.3)\) holds for some \(r > 0\);
2. if \(s + t \geq 1\) and \((s, t) \neq (1/2, 1/2)\), then \(f \in \Sigma^s_t(R^d)\), if and only if \((1.3)\) holds for any \(r > 0\).

Gelfand-Shilov spaces and their distribution spaces can also, in some sense more convenient ways, be characterized by means of estimates of short-time Fourier transforms, (see e.g. [11][12]). We recall here the details and start by recalling the definition of the short-time Fourier transform.
Let $\phi \in \mathcal{S}'(\mathbb{R}^d)$ be fixed. Then the short-time Fourier transform $V_\phi f$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window function $\phi$ is the Gelfand-Shilov distribution on $\mathbb{R}^{2d}$, defined by

$$V_\phi f(x, \xi) \equiv (\mathcal{F}_2(U(f \otimes \phi)))(x, \xi) = \mathcal{F}(f \phi(\cdot - x))(\xi),$$

where $(UF)(x, y) = F(y, y - x)$. If $f, \phi \in \mathcal{S}_s(\mathbb{R}^d)$, then it follows that

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int f(y)\overline{\phi(y - x)}e^{-i(y, \xi)} \, dy.$$

The next two results show that both spaces of Gelfand-Shilov functions and Gelfand-Shilov distributions can be completely identified with growth and decay properties of the short-time Fourier transforms for the involved functions and distributions. The conditions are of the forms

$$|V_\phi f(x, \xi)| \lesssim e^{-r(|x|^{1/s} + |\xi|^{1/s})},$$

(1.4)

$$|(\mathcal{F}(V_\phi f))(\xi, x)| \lesssim e^{-r(|x|^{1/s} + |\xi|^{1/s})}$$

(1.5)

and

$$|V_\phi f(x, \xi)| \lesssim e^{r(|x|^{1/s} + |\xi|^{1/s})}.$$  \(1.4)'

**Proposition 1.2.** Let $s, t, s_0, t_0 > 0$ be such that $s_0 + t_0 \geq 1$, $s_0 \leq s$ and $t_0 \leq t$, and let $\phi \in \mathcal{S}_{s_0}^{\alpha_0}(\mathbb{R}^d) \setminus 0$ and $f \in (\mathcal{S}_{s_0}^{\alpha_0})'(\mathbb{R}^d)$. Then the following is true:

1. $f \in \mathcal{S}_s^t(\mathbb{R}^d)$, if and only if (1.4) holds for some $r > 0$;
2. if in addition $\phi \in \Sigma_{\alpha_0}^t(\mathbb{R}^d)$, then $f \in \Sigma_s^t(\mathbb{R}^d)$, if and only if (1.4) holds for every $r > 0$.

A proof of Theorem 1.2 can be found in e.g. [6] (cf. [6, Theorem 2.7]). The corresponding result for Gelfand-Shilov distributions is the following, which is essentially a restatement of [14, Theorem 2.5].

**Proposition 1.3.** Let $s, t, s_0, t_0 > 0$ be such that $s_0 + t_0 \geq 1$, $s_0 \leq s$, $t_0 \leq t$ and $(s_0, t_0) \neq (1/2, 1/2)$, and let $\phi \in \Sigma_{\alpha_0}^t(\mathbb{R}^d) \setminus 0$, and let $f \in (\Sigma_{\alpha_0}^t)'(\mathbb{R}^d)$. Then the following is true:

1. $f \in (\mathcal{S}_s^t)'(\mathbb{R}^d)$, if and only if (1.4)' holds for every $r > 0$;
2. $f \in (\Sigma_s^t)'(\mathbb{R}^d)$, if and only if (1.4)' holds for some $r > 0$.

We note that in (2) in [14, Theorem 2.5] it should stay $(\Sigma_s^t)'(\mathbb{R}^d)$ instead of $\Sigma_s^t(\mathbb{R}^d)$.

**Remark 1.4.** The short-time Fourier transform can also be used to identify the elements in $\mathcal{S}(\mathbb{R}^d)$ and in $\mathcal{S}'(\mathbb{R}^d)$. In fact, if $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ and $f \in (\mathcal{S}_{1/2})'(\mathbb{R}^d)$, then the following is true:

1. $f \in \mathcal{S}(\mathbb{R}^d)$, if and only if for every $N \geq 0$, it holds
   $$|V_\phi f(x, \xi)| \lesssim \langle x, \xi \rangle^{-N};$$

2. $f \in \mathcal{S}'(\mathbb{R}^d)$, if and only if for every $N \geq 0$, it holds
   $$\langle x, \xi \rangle^N |V_\phi f(x, \xi)| \lesssim 1.$$
(2) $f \in S'(\mathbb{R}^d)$, if and only if for some $N \geq 0$, it holds

$$|V_\phi f(x, \xi)| \lesssim \langle x, \xi \rangle^N,$$

(Cf. [3, Chapter 12].)

1.2. The Bargmann transform. Next we recall some facts about the Bargmann transform and start by recalling some function spaces in [14].

Let $p \in [1, \infty]$ and let $\omega \in L^\infty_{\text{loc}}(\mathbb{C}^d)$ be a positive function. Then the set $B^p_\omega(\mathbb{C}^d)$ is the set of all $F \in L^p_{\text{loc}}(\mathbb{C}^d)$ such that

$$\|F\|_{B^p_\omega(\mathbb{C}^d)} \equiv \|F \cdot e^{-|\cdot|^2/2} \cdot \omega\|_{L^p}$$

is bounded. We also let

$$A^p_\omega(\mathbb{C}^d) = B^p_\omega(\mathbb{C}^d) \cap A(\mathbb{C}^d),$$

$$\|F\|_{A^p_\omega(\mathbb{C}^d)} \equiv \|F\|_{B^p_\omega(\mathbb{C}^d)}$$

when $F \in A^p_\omega(\mathbb{C}^d)$, and let the topologies of $A^p_\omega(\mathbb{C}^d)$ and $B^p_\omega(\mathbb{C}^d)$ be defined through these norms. We also use the notations $A^p$ and $B^p$ instead of $A^p_\omega$ and $B^p_\omega$, respectively, when $\omega = 1$.

By letting $d\mu(z) = \pi^{-d} e^{-|z|^2} d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}^d$, it follows that $B^2(\mathbb{C}^d)$ is a Hilbert space with scalar product

$$(F, G)_{B^2} \equiv \int_{\mathbb{C}^d} F(z) \overline{G(z)} \, d\mu(z), \quad F, G \in B^2(\mathbb{C}^d). \quad (1.6)$$

We also equip $A^2(\mathbb{C}^d)$ with the scalar product

$$(F, G)_{A^2} \equiv (F, G)_{B^2}, \quad F, G \in A^2(\mathbb{C}^d). \quad (1.7)$$

For every $f \in S_{1/2}^1(\mathbb{R}^d)$, the Bargmann transform $\mathfrak{B}_d f$ is the entire function on $\mathbb{C}^d$, defined by

$$(\mathfrak{B}_d f)(z) = \langle f, \mathfrak{A}_d(z, \cdot) \rangle, \quad (1.8)$$

where the Bargmann kernel $\mathfrak{A}_d$ is given by

$$\mathfrak{A}_d(z, y) = \pi^{-d/4} \exp \left(-\frac{1}{2} \langle z, z \rangle + |y|^2 + 2^{1/2} \langle z, y \rangle \right).$$

Here

$$\langle z, w \rangle = \sum_{j=1}^d z_j w_j, \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d, \quad w = (w_1, \ldots, w_d) \in \mathbb{C}^d,$$

and otherwise $\langle \cdot, \cdot \rangle$ denotes the duality between test function spaces and their corresponding duals. We note that the right-hand side in (1.8) makes sense when $f \in S_{1/2}^1(\mathbb{R}^d)$ and defines an element in the set $A(\mathbb{C}^d)$ of all entire functions on $\mathbb{C}^d$. In fact, $y \mapsto \mathfrak{A}_d(z, y)$ can be interpreted as an element in $S_{1/2}(\mathbb{R}^d)$ with values in $A(\mathbb{C}^d)$. 
If in addition $f$ is an integrable function, then $\mathcal{V}_d f$ takes the form
\[
(\mathcal{V}_d f)(z) = \int \mathcal{A}_d(z, y) f(y) \, dy,
\]
or
\[
(\mathcal{V}_d f)(z) = \pi^{-d/4} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} (\langle z, z \rangle + |y|^2) + 2^{1/2} (\langle z, y \rangle) \right) f(y) \, dy.
\]

Several properties for the Bargmann transform were established by Bargmann in \cite{1,2}. For example, in \cite{1} it is proved that $A^2(\mathbb{C}^d)$ is a Hilbert space under the scalar product (1.7), and that $f \mapsto \mathcal{V}_d f$ is a bijective and isometric map from $L^2(\mathbb{R}^d)$ to the Hilbert space $A^2(\mathbb{C}^d)$. Furthermore, if $\mathcal{V}_d^*$ is the adjoint of $\mathcal{V}_d$, i.e. $\mathcal{V}_d$ and $\mathcal{V}_d^*$ should fulfill
\[
(\mathcal{V}_d f, G)_{B^2} = (f, \mathcal{V}_d^* G)_{L^2},
\]
when $f \in L^2(\mathbb{R}^d)$ and $G \in B^2(\mathbb{C}^d)$, then the inverse of $\mathcal{V}_d : L^2(\mathbb{R}^d) \rightarrow A^2(\mathbb{C}^d)$ is given by $\mathcal{V}_d^*$. For future references we note that $\mathcal{V}_d^*$ is given by
\[
(\mathcal{V}_d^* G)(x) = \pi^{-d/4} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} (\langle z, z \rangle + |x|^2) + 2^{1/2} \langle z, x \rangle \right) G(z) \, d\mu(z),
\]
when $G \in B^2(\mathbb{C}^d)$.

Later on we need the following result, based on \cite[Theorem 3.2]{14}.

**Proposition 1.5.** Let $p_1, p_2 \in [1, \infty]$, $h_1, h_2 \in \mathbb{R}$ be such that $h_2 < h_1$, and let
\[
M_{s,t}(x + i\xi) = |x|^{1/t} + |\xi|^{1/s},
\]
\[
v_{s,t,h}(z) = e^{hM_{s,t}(z)}, \quad x, \xi \in \mathbb{R}^d.
\]
Then
\[
A^{p_2}_{(v_{s,t,h})}(\mathbb{C}^d) \subseteq A^{p_1}_{(v_{s,t,h})}(\mathbb{C}^d),
\]
and
\[
\|F\|_{A^{p_2}_{(v_{s,t,h})}} \lesssim \|F\|_{A^{p_1}_{(v_{s,t,h})}}, \quad F \in A(\mathbb{C}^d).
\]

**Proof.** Let $\Omega$ be the set of all weights $z \mapsto v_{s,t,h}(z) \langle z \rangle^r$, where $r \in \mathbb{R}$. Then $\Omega$ is an admissible family of weights in the sense of \cite[Definition 1.4]{14}. By Theorem 3.2 in \cite{14} we get
\[
\|F\|_{A^{p_2}_{(v_{s,t,h})}} \lesssim \|F\|_{A^{p_1}_{(v)}}, \quad F \in A(\mathbb{C}^d),
\]
where $v(z) = v_{s,t,h}(z) \langle z \rangle^N$, for some $N \geq 0$. Since $v \lesssim v_{s,t,h_1}$, we get
\[
\|F\|_{A^{p_1}_{(v)}} \lesssim \|F\|_{A^{p_1}_{(v_{s,t,h})}}, \quad F \in A(\mathbb{C}^d),
\]
and the result follows from these estimates. \qed
In [1] it is also proved that the Hermite functions are mapped by the Bargmann transform into convenient monomials. More precisely, for any multi-index $\alpha \in \mathbb{N}^d$, the Hermite function $h_\alpha$ of order $\alpha$ is defined by
\[
h_\alpha(x) = \pi^{-d/4}(-1)^{|\alpha|}(2^{|\alpha|} \alpha!)^{-1/2}e^{|x|^2/2}(\partial^\alpha e^{-|x|^2}).
\]
It follows that
\[
h_\alpha(x) = \frac{1}{(2\pi)^{d/2}}e^{-|x|^2/2}p_\alpha(x),
\]
for some polynomial $p_\alpha$ on $\mathbb{R}^d$, which is called the Hermite polynomial of order $\alpha$.

The set $\{h_\alpha\}_{\alpha \in \mathbb{N}^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. It is also a basis for any of the Gelfand-Shilov spaces and their distribution spaces at above.

In [1] it is then proved that
\[(Vh_\alpha)(z) = z^\alpha \sqrt{\alpha!}, \quad z \in \mathbb{C}^d. \tag{1.11}\]

Next we recall the links between the Bargmann transform and the short-time Fourier transform, when the window function $\phi$ is given by
\[
\phi(x) = \pi^{-d/4}e^{-|x|^2/2}. \tag{1.12}
\]
More precisely, let $S$ be the dilation operator given by
\[(SF)(x, \xi) = F(2^{-1/2}x, -2^{-1/2}\xi), \tag{1.13}\]
when $F \in L^1_{\text{loc}}(\mathbb{R}^{2d})$. Then it follows by straightforward computations that
\[
(V_d f)(z) = (\mathfrak{M} h_\alpha)(z) = \frac{z^\alpha}{\sqrt{\alpha!}}, \quad z \in \mathbb{C}^d.
\]
\[V_\phi f(x, \xi) = (2\pi)^{-d/2}e^{-|x|^2/4}e^{-i(x, \xi)}V_d f(2^{-1/2}x, -2^{-1/2}\xi) = (2\pi)^{-d/2}e^{-|x|^2/2}e^{-i(x, \xi)}(S^{-1}(V_\phi f))(x, \xi), \tag{1.14}\]
or equivalently,
\[
V_\phi f(x, \xi) = (2\pi)^{-d/2}e^{-|x|^2/2}e^{-i(x, \xi)/2}V_d f(2^{-1/2}x, -2^{-1/2}\xi).
\]
(1.15)

For future references we observe that (1.14) and (1.15) can be formulated into
\[
\mathfrak{M} = U_\phi \circ V_\phi, \quad \text{and} \quad U_\phi^{-1} \circ \mathfrak{M} = V_\phi, \tag{1.16}\]
where $U_\phi$ is the linear, continuous and bijective operator on $\mathfrak{S}'(\mathbb{R}^{2d}) \simeq \mathfrak{D}'(\mathbb{C}^d)$, given by
\[
(U_\phi F)(x, \xi) = (2\pi)^{d/2}e^{(|x|^2 + |\xi|^2)/2}e^{-i(x, \xi)}F(2^{1/2}x, -2^{1/2}\xi). \tag{1.17}
\]
2. Mapping properties of Gelfand-Shilov spaces and their distribution spaces, under the Bargmann transform

In this section we discuss the image of the Bargmann transform on Gelfand-Shilov and tempered function spaces, and their distribution spaces. A part of the analysis is based on dual properties of these spaces. In the end we also apply our results to deduce continuity properties of Short-time Fourier transform with Gaussians as window functions. We also use the results to show that the Gelfand-Shilov spaces of functions or distributions can be obtained by appropriate unions or intersections of certain modulation spaces, introduced in [14].

For every \( s, t \geq 1/2 \), we consider the sets

\[
A_{0,t}^s(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2} - rM_{s,t}(z)} \text{ for every } r > 0 \},
\]

\[
A_s(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2} - rM_{s,t}(z)} \text{ for some } r > 0 \},
\]

\[
A_{\infty}(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2}(z)^{-N}} \text{ for every } N > 0 \},
\]

\[
A_{\infty}'(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2}N} \text{ for some } N > 0 \},
\]

\[
(A_s)'(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2+2rM_{s,t}(z)}} \text{ for every } r > 0 \},
\]

\[
(A_{0,t})'(C^d) \equiv \{ F \in A(C^d) ; |F(z)| \lesssim e^{\frac{|z|^2}{2+2rM_{s,t}(z)}} \text{ for some } r > 0 \},
\]

with canonical topologies. Here \( M_{s,t} \) is given by (1.10). We also set

\[
A_s = A_s^s \quad \text{and} \quad A_s' = (A_s')',
\]

of entire functions.

By Proposition [15] we have the following relations between the spaces here above and \( A_{p(v, s, t, h)}^p \).

**Proposition 2.1.** Let \( p \in [1, \infty] \), \( s, t \geq 1/2 \) and let \( v_{s,t,h} \) be the same as in Proposition [1,15]. Then

\[
\bigcup_{h > 0} A_{p(v_{s,t,h})}^p(C^d) = A_s(C^d)
\]

The next result concerns convenient estimates for the reproducing kernel operator \( \Pi_A \) in [1,14]. More precisely, let \( \Omega \) be the set of all \( F \in L^1_{\text{loc}}(C^d) \) such that

\[
\int_{C^d} |F(z)||e^{-\frac{|z|^2}{2} + N|z|}| d\lambda(z) < \infty \quad (2.1)
\]

for every \( N \geq 1 \). Then \( \Pi_A \) is the operator from \( \Omega \) to \( \mathcal{D}'(C^d) \), defined by the formula

\[
(\Pi_A F)(z) \equiv \int_{C^d} F(w)e^{z,w} d\mu(w).
\]
We note that \( \Pi_A F = F \) when \( F \in \Omega \cap A(C^d) \). In view of Lemma 4.1 in [14].

**Lemma 2.2.** Let \( h_1, h_2 \) be such that \( 0 < 2h_1 \leq h_2 < 1/2, F \in L^1_{loc}(C^d) \) be such that (2.1) holds for every \( N \geq 1 \). Then

\[
\| (\Pi_A F) e^{-|z|^2/2 + h_2 M_{s,t}(z)} \|_{L^1} \lesssim \| F e^{-|z|^2/2 + h_1 M_{s,t}(z)} \|_{L^1}.
\]

**Proof.** Let

\[
G(z) = F(z) e^{-|z|^2/2 + h_1 M_{s,t}(z)}
\]

and

\[
H(z) = e^{-|z|^2/2 + 2h_1 M_{s,t}(z)}.
\]

Then \( H \in L^1 \) due to the assumptions. Futhermore,

\[
| (\Pi_A F)(z) e^{-|z|^2/2 + h_2 M_{s,t}(z)} | \leq \int |G(w)| e^{-|z-w|^2/2 + h_1 M_{s,t}(w) - h_2 M_{s,t}(z)} d\lambda(w) \tag{2.2}
\]

Since

\[
h_1 M_{s,t}(w) - h_2 M_{s,t}(z) \leq h_1 (M_{s,t}(w) - 2M_{s,t}(z)) \leq 2h_1 M_{s,t}(z - w),
\]

it follows from (2.2) that

\[
| (\Pi_A F)(z) e^{-|z|^2/2 + h_2 M_{s,t}(z)} | \leq (|G| * H)(z).
\]

The result now follows by applying the \( L^1 \)-norm on the last inequality, and using Young’s inequality. \( \square \)

In the following two results we deduce dual properties of \( A^s_t(C^d) \), and links between our classes of analytic functions and Gelfand-Shilov spaces.

**Proposition 2.3.** Let \( s, t \geq 1/2 \). Then the \( A^2 \)-form \( (\cdot, \cdot)_{A^2} \) on \( A_{1/2}(C^d) \) extends uniquely to a continuous sesqui-linear form from \( A^s_t(C^d) \times (A^s_t)'(C^d) \) to \( C \), and from \( (A^s_t)'(C^d) \times A^s_t(C^d) \) to \( C \). Furthermore, the dual space of \( A^s_t(C^d) \) can be identified with \( (A^s_t)'(C^d) \) through this form.

Proposition 2.3 follows in the case \( s, t > 1/2 \) from Theorems 3.4, 3.9 and 4.7 in [14]. In the following we give a proof which also covers the cases when \( s = 1/2 \) or \( t = 1/2 \).

**Theorem 2.4.** The map \( f \mapsto \mathcal{M}_d f \) from \( L^2(R^d) \) to \( \Lambda^2(C^d) \) extends uniquely to a homeomorphism from \( S^1_{1/2}(R^d) \) to \( A^1_{1/2}(C^d) \). Furthermore the following is true:

(1) if \( s, t \geq 1/2 \), then the map \( f \mapsto \mathcal{M}_d f \) restricts to homeomorphisms from \( S^s_t(R^d) \) to \( A^s_t(C^d) \), and from \( (S^s_t)'(R^d) \) to \( (A^s_t)'(C^d) \);

(2) if \( s, t > 1/2 \), then the map \( f \mapsto \mathcal{M}_d f \) restricts to homeomorphisms from \( \Sigma^s_t(R^d) \) to \( A^s_{0,t}(C^d) \), and from \( (\Sigma^s_t)'(R^d) \) to \( (A^s_{0,t})'(C^d) \);
(3) the map \( f \mapsto \Omega_d f \) restricts to homeomorphisms from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{A}_\infty(\mathbb{C}^d) \), and from \( \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{A}'_\infty(\mathbb{C}^d) \).

We shall first prove Theorem 2.4 (2), (3) and the first part of (1). Thereafter we prove Proposition 2.3 and then we prove the last part of Theorem 2.4 (1).

**Proof of Theorem 2.4** Let \( \phi \) be given by (1.16), \( s, t \geq 1/2 \) and \( s_0, t_0 > 1/2 \). Then it follows from (1.13), Proposition 1.2 and Remark 1.4 that the Bargmann transform restricts to continuous and injective mappings from \( \mathcal{S}_t^s \) to \( \mathcal{A}_t^s \), from \( \Sigma_{t_0}^{s_0} \) to \( \mathcal{A}_{0,t_0}^{s_0} \), and from \( \mathcal{S} \) to \( \mathcal{A}_\infty \). From these relations and Proposition 1.3 it also follows that the Bargmann transform extends uniquely to continuous and injective mappings from \( (\mathcal{S}_t^{s_0})' \) to \( (\mathcal{A}_t^{s_0})' \), from \( (\Sigma_{t_0}^{s_0})' \) to \( (\mathcal{A}_{0,t_0}^{s_0})' \), and from \( \mathcal{S}' \) to \( \mathcal{A}'_\infty \).

The surjectivity follows from Remark 1.7 in [13], and Theorems 3.4 and 3.9 in [14] in the case \( s, t > 1/2 \).

It remains to prove the surjectivity for (1) when \( s \) and \( t \) may attain 1/2. First we consider the map

\[
\mathcal{U}_d : \mathcal{S}_t^s(\mathbb{R}^d) \to \mathcal{A}_t^s(\mathbb{C}^d),
\]

and assume that \( F \in \mathcal{A}_{\mathbb{C}^d} \) be arbitrary.

Let \( s_0 > s \) and \( t_0 > t \). Then \( \mathcal{S}_s^t \subseteq \mathcal{S}_s^{t_0} \), and from the first part of the proof it follows that there is a unique element \( f \in \mathcal{S}_s^{t_0} \) such that \( F \mathcal{U}_d f \).

It now follows from Proposition 1.2 and (1.15) that \( f \in \mathcal{S}_t^s \), and the result follows in this case as well.

**Proof of Proposition 2.3** For any \( G \in L^1_{\text{loc}}(\mathbb{C}^d) \) which satisfies

\[
G(z) e^{N|z| - |z|^2} \in L^1(\mathbb{C}^d),
\]

the form

\[
\ell_G(F) \equiv (F, G)_{B^2} = \int_{\mathbb{C}^d} F(z) G(z) \, d\mu(z)
\]

is well-defined for every (analytic) polynomial \( F \) on \( \mathbb{C}^d \). The definition of \( \ell_G \) extends in usual ways to other situations, provided \( G \) satisfies appropriate conditions.

Evidently, from the definitions it follows that \( \ell_G \) belongs to the dual of \( \mathcal{S}_t^s(\mathbb{C}^d) \) when \( G \in (\mathcal{S}_t^s)'(\mathbb{C}^d) \). Hence

\[
(\mathcal{S}_t^s)'(\mathbb{C}^d) \subseteq (\mathcal{S}_t^s(\mathbb{C}^d))'.
\]

We need to prove the opposite inclusion, and that the map \( G \mapsto \ell_G \) from \( (\mathcal{S}_t^s)'(\mathbb{C}^d) \) to \( (\mathcal{S}_t^s(\mathbb{C}^d))' \) is injective.

For any \( s, t \geq 1/2 \) and \( h \in \mathbb{R} \), let \( \mathcal{B}_t^s(\mathbb{C}^d) \) be the set of all \( F \in L^1_{\text{loc}}(\mathbb{C}^d) \) such that

\[
\|F\|_{\mathcal{B}_t^s} \equiv \int_{\mathbb{C}^d} |F(z)| e^{-((|z|^2/2 + hM_s)(z))} \, d\lambda(z)
\]
is finite. Also let $A_{t,h}^s(C^d) = B_{t,h}^s(C^d) \cap A(C^d)$, with topology inherited from $B_{t,h}^s(C^d)$. Then
\[ A_t^s(C^d) = \bigcup_{h>0} A_{t,h}^s(C^d), \quad (2.4) \]
with inductive limit topology. Since $A_{t,h}^s$ is decreasing with respect to $h$, we may assume that $h < h_0$ in $(2.4)$ for some small $h_0$ such that $A_{t,h_0}^s(C^d)$ is non-trivial.

Let $\ell \in (A_{t,h}^s(C^d))^\prime$. By Hahn-Banach’s theorem, $\ell$ is extendable to a continuous form on $B_{t,h}^s(C^d)$. Hence there is an element $G \in L^\infty_{loc}(C^d)$ such that
\[ \text{ess sup}_{z \in C^d} |G(z)e^{-|z|^2/2-hM_{t,s}(z)}| < \infty \quad (2.5) \]
and $\ell(F) = \ell_G(F)$.

By letting $G_0 = \Pi_A G$, it follows from Lemma 4.1 in [14], Proposition 1.5 and Lemma 2.2 that $G_0 \in A_{t,-3h}^s(C^d)$ and $\ell_G(F) = \ell_{G_0}(F)$ when $F$ is a polynomial. Since the Hermite functions are dense in $S_t^s$ and in $S_{t,h}^s$, it follows from $(1.1)$ and the first part of Theorem 2.4 (1) that polynomials are dense in $A_t^s$ and in $A_{t,h}^s$. This gives $\ell_G(F) = \ell_{G_0}(F)$ when $F \in A_{t,h}^s$. Hence
\[ (A_{t,h}^s(C^d))^\prime \subseteq A_{t,-3h}^s(C^d) \quad (2.6) \]
Since
\[ (A_t^s(C^d))^\prime = \bigcap_{h>0} (A_{t,h}^s(C^d))^\prime \quad \text{and} \quad (A_t^s(C^d))^\prime = \bigcap_{h>0} A_{t,-h}^s(C^d), \]
with projective limit topologies on the left-hand sides, $(2.6)$ shows that equality is attained in $(2.3)$. Hence the map $G \mapsto \ell_G$ is continuous and surjective from $(A_t^s(C^d))^\prime$ to $(A_t^s(C^d))^\prime$.

By Remark 4.3 in [14] it also follows that the latter map is injective. This gives the result. \hfill \square

\textbf{The end of the proof of Theorem 2.4.} It remains to prove that $\mathfrak{G}_d$ is continuous and bijective from $(S_t^s)'(R^d)$ to $(A_t^s)'(C^d)$. We shall consider the adjoint $T = \mathfrak{G}_d^*$, acting from $(A_t^s)'(C^d)$ to $(S_t^s)'(R^d)$.

Evidently, by the assumptions it follows by straight-forward computations that $T$ is well-defined and continuous map from $(A_t^s)'(C^d)$ to $(S_t^s)'(R^d)$, and that
\[ (F,G)_A^2 = (f,g)_{L^2}, \quad f = \mathfrak{G}_d^*F, \quad g = \mathfrak{G}_d^*G, \quad F \in A_t^s(C^d), \quad G \in (A_t^s)'(C^d). \]
We claim that $T$ is injective.

In fact, if $G_1, G_2 \in (A_t^s)'(C^d)$ are chosen such that $TG_1 = TG_2$ and $g_j = TG_j$, then for some positive constants $c_\alpha$ we have
\[ (G_1, z^\alpha)_A^2 = c_\alpha(g_1, h_\alpha) = c_\alpha(g_2, h_\alpha) = (G_2, z^\alpha)_A^2. \]
This implies that \((G_1, F)_{A^2} = (G_2, F)_{A^2}\) for every analytic polynomial \(F\), and since the set of such polynomials are dense in \(\mathcal{A}^s\) we get \(G_1 = G_2\), and the injectivity follow.

We need to prove that \(T\) is surjective. Therefore, let \(\ell \in (\mathcal{S}_r^s(\mathbb{R}^d))'\).

Then \(\ell \circ T \in (\mathcal{A}_r^s(\mathbb{C}^d))'\). By Proposition 2.3, there is an element \(G \in (\mathcal{A}_r^s(\mathbb{C}^d))'\) such that \(\ell(T F) = (F, G)_{B^2}\). Hence, if \(F = \mathfrak{H}_d f\) and \(g = TG\) we get
\[
\ell(f) = \ell(T F) = (F, G)_{B^2} = (f, TG)_{L^2}.
\]
This shows that \(T\) from \((\mathcal{A}_r^s(\mathbb{C}^d))'\) to \((\mathcal{S}_r^s(\mathbb{R}^d))' \cong (\mathcal{S}_r^s(\mathbb{R}^d))'\) is surjective, and the result follows. \[\square\]

By combining the links (1.14) and (1.15) between the Bargmann transform and the short-time Fourier transform with Gaussian window, and Theorem 2.4, we get the following complementary result to Proposition 1.3.

**Proposition 2.5.** Let \(s, t \geq 1/2, \phi \in C^\infty(\mathbb{R}^d) \setminus \{0\}\) be Gaussian, and let \(f \in \mathcal{S}_s^t(\mathbb{R}^d)\). Then \(f \in (\mathcal{S}_s^t(\mathbb{R}^d))'\), if and only if (1.15) holds for every \(r > 0\).

**Remark 2.6.** Especially the cases when \(s = 1/2\) or \(t = 1/2\) in Theorem 2.3 and Propositions 2.3 and 2.5 seem to be new, and are often not taken into account in e.g. [14]. In fact, in [14] it is usually assumed that the involved weights should belong to \(\mathcal{P}_r^q\) or the slightly larger class \(\mathcal{P}_r^q\). Here we note if \(\omega \in \mathcal{P}_r^q\), then
\[
e^{-r|x|^2} \lesssim \omega(x) \lesssim e^{r|x|^2}, \quad \text{for every } r > 0,
\]
which is one of the basic conditions in the definitions of the classes \(\mathcal{P}_r^q(\mathbb{R}^d)\) and \(\mathcal{P}_r^q(\mathbb{R}^d)\).

By replacing the condition (2.7) by the relaxed condition
\[
e^{-r|x|^2} \lesssim \omega(x) \lesssim e^{r_0|x|^2}, \quad \text{for every } r > 0 \text{ and some } r_0 > 0,
\]
in the definitions of \(\mathcal{P}_r^q\) and its subclasses, it follows that the results in [14], except Theorem 4.7 and Lemma 4.11, still hold in these more general situations.

The next result extends Theorem 3.9 in [14], and follows from Theorems 3.2 and 3.4 in [14], and Theorem 2.4. Here we refer to [14] for the definition of the general type of modulation spaces \(M(\omega, \mathcal{B})\).

**Proposition 2.7.** Let \(\mathcal{B}\) be a mixed quasi-norm space on \(\mathbb{R}^{2d}\), and set
\[
\omega_r(x, \xi) \equiv e^{r(|x|^{1/h} + |\xi|^{1/h})}, \quad h > 0.
\]
Then the following is true:

1. If \(s, t \geq 1/2\), then
\[
\bigcup_{r>0} M(\omega_r, \mathcal{B}) = \mathcal{S}_r^s(\mathbb{R}^d) \quad \text{and} \quad \bigcap_{r>0} M(1/\omega_r, \mathcal{B}) = (\mathcal{S}_r^s)^*(\mathbb{R}^d);
\]
(2) if \( s, t > 1/2 \), then
\[
\bigcap_{r>0} M(\omega_r, \mathcal{B}) = \Sigma^*_s(\mathbb{R}^d) \quad \text{and} \quad \bigcup_{r>0} M(1/\omega_r, \mathcal{B}) = (\Sigma^*_s)'(\mathbb{R}^d).
\]

REFERENCES

[1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, Comm. Pure Appl. Math., 14 (1961), 187–214.
[2] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory., Comm. Pure Appl. Math., 20 (1967), 1–101.
[3] (with M. Cappiello, L. Rodino) Radial symmetric elements and the Bargmann transform, Integral Transform. Spec. Funct. 25 (2014), 756–764.
[4] J. Chung, S.-Y. Chung, D. Kim, Characterizations of the Gelfand-Shilov spaces via Fourier transforms, Proc. Amer. Math. Soc. 124 (1996), 2101–2108.
[5] K. H. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[6] K. Gröchenig, G. Zimmermann, Spaces of test functions via the STFT J. Funct. Spaces Appl. 2 (2004), 25–53.
[7] I. M. Gelfand, G. E. Shilov, Generalized functions, II-III, Academic Press, New York London, 1968.
[8] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol I–III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983, 1985.
[9] F. Nicola, L. Rodino, Global pseudo-differential calculus on Euclidean spaces, Pseudo-Differential Operators. Theory and Applications 4 Birkhäuser Verlag, Basel, 2010.
[10] S. Pilipović, Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions, SIAM J. Math. Anal. 17 (1986), 477–484.
[11] M. Signahl, J. Toft, Mapping properties for the Bargmann transform on modulation spaces, J. Pseudo-Differ. Oper. Appl. 3 (2012), 1–30.
[12] N. Teofanov, Ultradistributions and time-frequency analysis. In “Pseudo-differential operators and related topics”, 173–192, Oper. Theory Adv. Appl., 164, Birkhäuser, Basel, 2006.
[13] J. Toft Continuity properties for modulation spaces with applications to pseudo-differential calculus, II, Ann. Global Anal. Geom., 26 (2004), 73–106.
[14] J. Toft, The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators, J. Pseudo-Differ. Oper. Appl. 3 (2012), 145–227.

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