Signatures of Lifshitz transition in the optical conductivity of tilted Dirac materials

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Lifshitz transition is a kind of topological phase transition in which the Fermi surface is reconstructed. It can occur in the two-dimensional (2D) tilted Dirac materials when the energy bands change between the type-I phase (0 < t < 1) and the type-II phase (t > 1) through the type-III phase (t = 1), where different tilts are parameterized by the values of t. In order to characterize the Lifshitz transition therein, we theoretically investigate the longitudinal optical conductivities (LOCs) in type-I, type-II, and type-III Dirac materials within linear response theory. In the undoped case, the LOCs are independent of the tilt parameter in both type-I and type-II phases, but is determined by the tilt parameter in the type-II phase. In the doped case, the LOCs are anisotropic and share two resonance peaks determined by ω = ω₁(t) and ω = ω₂(t). The tilt parameter and chemical potential can be extracted from optical experiments by measuring the positions of these two peaks and their separation (ω₂(t) − ω₁(t)). With increasing the tilt, the separation goes larger in the type-I phase whereas smaller in the type-II phase. The LOCs in the asymptotical region are exactly the same as that in the undoped case. The type of 2D tilted Dirac bands can be determined by the asymptotic background values, resonance peaks and their separation in the LOCs. These can therefore be taken as signatures of Lifshitz transition therein. The results of this work are expected to be qualitatively valid for a large number of monolayer tilted Dirac materials such as 8-Pmmn borophene tuned by gate voltage and different compounds of T′-phase transition metal dichalcogenides.

I. INTRODUCTION

Graphene has triggered extremely active research in two-dimensional (2D) Dirac materials characterized by linear and/or hyperbolic energy dispersions around Dirac points in momentum space [1, 2], such as α-(BEDT-TTF)₂I₃ [3], silicene [4–8], graphene under uniaxial strain [9], 8-Pmmn borophene [10–13], transition metal dichalcogenides [14–16], partially hydrogenated graphene [17], α-SnS₂ [18], TaCoTe₅ [19], and TaIrTe₅ [20]. Among them, 2D tilted Dirac materials host tilted dispersions along a certain direction of wave vector and have been attracting increasing interests theoretically and experimentally [3, 9, 13, 16, 20]. They exhibit many significant qualitative differences in physical behaviors compared to their untilted counterparts, including plasmons [21, 22], optical conductivities [21, 23], Weiss oscillation [23], Klein tunneling [24, 25], Kondo effects [26], RKKY interactions [27, 28], planar Hall effect [29], and thermoelectric effects [30].

Lifshitz transition is a kind of topological phase transition in which the Fermi surface is reconstructed [31], which is crucial for understanding the novel states of matter and physical properties around the transition. It accounts for the huge magnetoresistance in black phosphorus [32], superconductivity in iron-based superconductors [33, 34], and abnormal transport behavior in heavy fermion materials and topological quantum materials [35–41]. In 2D tilted Dirac materials, the Lifshitz transition occurs when the energy bands change between the type-I phase (under-tilted, 0 < t < 1) and the type-II phase (over-tilted, t > 1) through the type-III phase (critical-tilted, t = 1) where t is the tilt parameter [31]. For example, 8-Pmmn borophene undergoes the Lifshitz transition under the control of tunable vertical electrostatic field [33]; the different compounds in 1T′-transition metal dichalcogenides correspond to different phases of the Lifshitz transition [15]; the one phase (1T′-MoS₂ and 1T′-MoSe₂), type-II phase (1T′-MoTe₂ and 1T′-WTe₂). These 2D tilted Dirac materials provide an excellent material platform for investigating the Lifshitz transition. To this end, a very important issue is how to most effectively characterize the Lifshitz transition in 2D tilted Dirac materials.

The optical conductivity (OC) provides a powerful method for extracting the information of energy band structure, and has been extensively investigated theoretically and experimentally in the 2D untilted Dirac bands [31, 42] and under-tilted Dirac bands [26, 29]. Especially, the exotic behaviors of longitudinal optical conductivity (LOC) can be used to characterize the topological phase transitions in both silicene [38] and 1T′-MoS₂ [29]. However, the energy bands of the above-mentioned 2D Dirac materials are restricted to either the untilted or the under-tilted (type-I phase), leaving the impact of type-II and type-III energy bands on the LOCs unexplored. To characterize the Lifshitz transition of tilted Dirac materials, we perform a comprehensive study of the LOCs in the type-I, type-II, and type-III phases. It is the purpose

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of this work to investigate the characteristics of LOC in different phases of Lifshitz transition. In particular, we focus our theoretical study of LOCs for both undoped and doped situations.

The rest of the paper is organized as follows. In Sec. II, we briefly describe the model Hamiltonian and theoretical formalism to calculate the LOC. Our analytical expressions of the interband conductivity and intraband (Drude) conductivity are presented in Sec. III and Sec. IV, respectively. In addition, the joint density of states (JDOS) are analytically calculated to confirm the interband conductivities. The results and discussion are shown in Sec. V. The conclusions are summarized in Sec. VI. Finally, we present three appendices to show detailed calculation.

II. THEORETICAL FORMALISM

We begin with the Hamiltonian in the vicinity of one of two valleys for 2D tilted Dirac materials

$$
\mathcal{H}_\kappa(k_x, k_y) = \kappa \hbar v_t k_y \tau_0 + \hbar (v_x k_x \tau_1 + v_y k_y \tau_2),
$$

(1)

where $\kappa = \pm$ labels two valleys, $k = (k_x, k_y)$ strands for the wave vector, $\tau_0$ and $\tau_i$ denote the $2 \times 2$ unit matrix and Pauli matrices, respectively. Hereafter, we define the tilt parameter $t = v_t/v_y$ and set $\hbar = 1$ for simplicity. It is noted that this system keep invariant under the valley transformation $(\kappa, k_y) \leftrightarrow (-\kappa, -k_y)$, indicating $\mathcal{H}_{-\kappa}(k_x, -k_y) = \mathcal{H}_\kappa(k_x, k_y)$. The eigenvalue evaluated from the Hamiltonian reads

$$
\varepsilon^\lambda_{\kappa}(k_x, k_y) = \kappa t v_y k_y + \lambda \mathcal{Z}(k_x, k_y),
$$

(2)

where $\mathcal{Z}(k_x, k_y) = \sqrt{v_x^2 k_x^2 + v_y^2 k_y^2}$, and $\lambda = \pm 1$ denotes the conduction and valence bands, respectively. The energy bands and the Fermi surfaces for the n-doped case at the $\kappa = +$ valley, are schematically shown in Fig. 1. For the untilted case ($t = 0$), the only Fermi surface, contributed completely by the electron pocket with a closed area [see Figs. 1(a) and 1(e)], is an ellipse obeying the equation

$$
k_x^2/v_x^2 + k_y^2/v_y^2 = 1,
$$

(3)

which reduces to a circle when $v_x = v_y$. For the type-I phase ($0 < t < 1$), the only Fermi surface, also contributed completely by the closed electron pocket [see Figs. 1(b) and 1(f)], is an ellipse obeying the equation

$$
\frac{k_x^2}{v_x^2} + \frac{(k_y + \frac{\mu t}{v_y} \kappa \tau_1 + \frac{\kappa \tau_2}{v_y} \tau_0 v^x v^y (1 - t^2)}{v_y^2} = 1,
$$

(4)

which remains an ellipse even when $v_x = v_y$. The tilt parameter $t$ moves the center of ellipse along the $k_y$ axis and changes the major axis and minor axis of ellipse. For the type-III phase ($t = 1$), the only Fermi surface, contributed entirely by the electron pocket with a open border [see Figs. 1(c) and 1(g)], is a parabola satisfying

$$
k_y = \frac{\mu}{2 \kappa v_y} - \frac{v_x^2 k_x^2}{2 \kappa \mu v_y}.
$$

(5)
Interestingly, for the type-II phase \((t > 1)\), the Fermi surface is a couple of hyperbola [see Figs. 1(d) and 1(h)] whose equation reads
\[
\left(\frac{k_y - \frac{\kappa t\mu}{v_y(1-t)}}{\frac{\mu^2}{v_y^2(1-t)^2}}\right)^2 - \frac{k_x^2}{\frac{\mu^2}{v_y^2(1-t)^2}} = 1, \tag{6}
\]
which is contributed not only by the electron pocket but also by the hole pocket.

This indicates that the Fermi surface is reconstructed when the energy band changes between the type-I phase and the type-II phase, corresponding to a Lifshitz transition [see Figs. 1(e) and 1(h)]. On the other hand, the edges of electron pocket and hole pocket determine the boundaries of interband transition of the LOCs. As a consequence, the Lifshitz transition can be characterized by the peaks of interband conductivity. It is the purpose of this work to characterize such Lifshitz transition in the tilted Dirac materials via the LOC.

Within linear response theory, the LOC \(\sigma_{jj}(\omega)\) at finite photon frequency \(\omega\) is given by
\[
\sigma_{jj}(\omega) = g_s \sum_{\kappa=\pm 1} \sigma_{jj}^\kappa(\omega), \tag{7}
\]
where \(j = x, y\) stands for the spatial component, \(g_s = 2\) represents the spin degeneracy, and \(\sigma_{jj}^\kappa(\omega)\) denotes the LOC at given valley \(\kappa\), whose explicit expression is provided in the Appendix [A] Interestingly, \(\sigma_{jj}(\omega)\) possesses the particle-hole symmetry (see Appendix [A] for details) such that we can safely replace \(\mu\) by \(|\mu|\) in all of \(\sigma_{jj}(\omega)\), \(\sigma_{jj}^\kappa(\omega)\), and \(f(x)\) because we only concern the final result of \(\sigma_{jj}(\omega)\). It can be proven that \(\sigma_{jj}^\kappa(\omega) = \sigma_{jj}^{-\kappa}(\omega)\) by considering \(\mathcal{H}_{-\kappa}(k_x, -k_y) = \mathcal{H}_{\kappa}(k_x, k_y)\), such that we are allowed to focus on \(\kappa = +\) or \(\kappa = -\) valley. Hereafter, we restrict our analysis to the n-doped case \((\mu > 0)\) and \(\kappa = +\) valley for convenience.

After some standard algebra, the real part of the LOCs can be divided into interband part and intraband part as
\[
\text{Re} \sigma_{jj}^\kappa(\omega) = \begin{cases} 
\text{Re} \sigma_{jj}^{\kappa}(\omega) + \Theta[\mu]\text{Re} \sigma_{jj}^{\kappa+}(\omega) + \Theta[-\mu]\text{Re} \sigma_{jj}^{\kappa-}(\omega), & 0 \leq t \leq 1, \\
\text{Re} \sigma_{jj}^{\kappa}(\omega) + \text{Re} \sigma_{jj}^{\kappa+}(\omega) + \text{Re} \sigma_{jj}^{\kappa-}(\omega), & t > 1,
\end{cases}
\tag{8}
\]
where \(\Theta(x)\) is the Heaviside step function satisfying \(\Theta(x) = 0\) for \(x \leq 0\) and \(\Theta(x) = 1\) for \(x > 0\), \(\mu\) denotes the chemical potential measured with respect to the Dirac point, and the interband and intraband conductivities are given respectively as
\[
\text{Re} \sigma_{jj}^{\kappa}(\omega) = \pi \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \text{Re} \sigma_{jj-\kappa}^{\kappa}(k_x, k_y) \frac{f [\varepsilon^{-\kappa}_{\omega}(k_x, k_y)] - f [\varepsilon^{+\kappa}_{\omega}(k_x, k_y)]}{\omega} \delta \left[\omega - 2\mathcal{Z}(k_x, k_y)\right],
\tag{9}
\]
\[
\text{Re} \sigma_{jj}^{\kappa,\lambda}(\omega) = \pi \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \text{Re} \sigma_{jj-\kappa,\lambda}^{\kappa,\lambda}(k_x, k_y) \left[ -\frac{df [\varepsilon^{-\lambda}_{\omega}(k_x, k_y)]}{d\varepsilon^{-\lambda}_{\omega}(k_x, k_y)} \right] \delta(\omega),
\tag{10}
\]
with \(\delta(x)\) the Dirac \(\delta\)-function and \(f(x) = \{1 + \exp([x - \mu]/k_B T)\}^{-1}\) the Fermi distribution function in which \(k_B\) is the Boltzmann constant and \(T\) represents the temperature.

The absorptive part of the total LOC can be generally written as
\[
\frac{\text{Re} \sigma_{jj}(\omega)}{\sigma_0} = \frac{v_x}{v_y} \Gamma_{xx}(\omega) \delta_{jj} + \frac{v_y}{v_x} \Gamma_{yy}(\omega) \delta_{jj}, \tag{11}
\]
where \(\sigma_0 = e^2/4\hbar\) (we restore \(h\) for explicitness) and
\[
\Gamma_{xx}(\omega) = \Gamma_{XX}^{\text{IB}}(\omega) + \Gamma_{XX}^{\text{D}}(\omega), \tag{12}
\]
\[
\Gamma_{yy}(\omega) = \Gamma_{YY}^{\text{IB}}(\omega) + \Gamma_{YY}^{\text{D}}(\omega). \tag{13}
\]

The relation between the LOCs and the ratio of Fermi velocities \(v_x/v_y\) has also been reported in Refs. 28 and 29. For convenience, we introduce \(\Gamma_{jj}^{\text{IB/D}}(\omega)\) with \(j = x, y\), whose explicit definitions are independent of the ratio \(v_x/v_y\) (see Appendix [B] for details). Obviously, the only difference between the isotropic case \((v_x = v_y)\) and anisotropic case \((v_x \neq v_y)\) is the magnitude of the LOC.

In the next two sections, we will analytically calculate the interband and intraband LOCs by assuming zero temperature \(T = 0\) such that the Fermi distribution function \(f(x)\) can be replaced by the Heaviside step function \(\Theta[\mu - x]\). To better analyze the physics of interband LOCs, we also evaluate the joint density of states (JDOS) defined by
\[
\mathcal{J}(\omega) = g_s \sum_{\kappa=\pm 1} \mathcal{J}_\kappa(\omega), \tag{14}
\]
where
\[
\mathcal{J}_\kappa(\omega) = \int' \frac{d^2k}{(2\pi)^2} \delta [\varepsilon^+_{\kappa}(k_x, k_y) - \varepsilon^-_{\kappa}(k_x, k_y) - \omega], \tag{15}
\]
with the prime indicating that for different tilts the boundaries of integration region in \(k\) space are determined by the corresponding Fermi surface at a given
chemical potential $\mu$ in Eqs. (3)–(6). The detailed analytical calculations of LOCs and JDOS are found in Appendix B and Appendix C.

III. INTERBAND CONDUCTIVITY AND JDOS

In this section, the analytical results of the interband LOC and JDOS are listed for different tilts. Firstly, the LOCs in the undoped case ($\mu = 0$) are completely contributed by the interband transition, which are given as

$$
\Gamma_{xx}^{IB}(\omega) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
G_-(\frac{1}{t}) - G_-(\frac{1}{t}), & t > 1,
\end{cases}
$$

(16)

and

$$
\Gamma_{yy}^{IB}(\omega) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
G_+(\frac{1}{t}) - G_+(\frac{1}{t}), & t > 1,
\end{cases}
$$

(17)

where two auxiliary functions

$$
G_{\pm}(x) = \frac{1}{2} + \frac{\arcsin x}{\pi} \pm \frac{x\sqrt{1-x^2}}{\pi}
$$

(18)

are introduced for simplicity. It is interesting to note that in the undoped case the LOCs are constant in frequency. Specifically, these constant conductivities depend on the tilt parameter only in the type-II Dirac materials, but are independent of the tilt parameter in both the type-I and type-III Dirac materials.

Hereafter, we focus on the interband LOCs in the doped case for different tilts. In the untilted case ($t = 0$), we recover the result of the ordinary Dirac cone and get

$$
\Gamma_{xx}^{IB}(\omega) = \Gamma_{yy}^{IB}(\omega) = \Theta(\omega - 2\mu),
$$

which are consistent with Ref. [5]. The corresponding JDOS is given as

$$
\mathcal{J}(\omega) = \frac{1}{J_0\omega} \begin{cases} 
0, & 0 < \omega < 2\mu, \\
1, & \omega \geq 2\mu,
\end{cases}
$$

(19)

with $J_0 = \frac{\sigma_s}{4\pi v_F v_y}$.

When the Dirac cone is tilted ($t > 0$), in order to simplify our results we introduce three compacted notations

$$
\xi_\pm = \frac{2\mu \pm \omega \Theta(t)}{\omega t}, \\
\omega_1(t) = 2\mu \frac{\Theta(t)}{1+t}, \\
\omega_2(t) = 2\mu \Theta(t) \left[ \frac{\Theta(1-t)}{1-t} + \frac{\Theta(t-1)}{t-1} \right].
$$

In the subsequent three subsections, we list the analytical expressions of the interband conductivity for the type I, type II, and type III Dirac materials in sequence.

A. For type-I Dirac materials

For the type-I phase ($0 < t < 1$), the interband conductivities defined in Eq. (11) can be written by

$$
\Gamma_{xx}^{IB}(\omega) = \begin{cases} 
0, & 0 < \omega < \omega_1(t), \\
1 - G_-(\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\
1, & \omega \geq \omega_2(t),
\end{cases}
$$

(20)

and

$$
\Gamma_{yy}^{IB}(\omega) = \begin{cases} 
0, & 0 < \omega < \omega_1(t), \\
1 - G_+(\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\
1, & \omega \geq \omega_2(t),
\end{cases}
$$

(21)

where $\xi_\pm = (2\mu \pm \omega)/\omega t$. The corresponding JDOS $\mathcal{J}(\omega)$ is given by

$$
\frac{\mathcal{J}(\omega)}{J_0\omega} = \begin{cases} 
0, & 0 < \omega < \omega_1(t), \\
\arccos \xi_+, & \omega_1(t) \leq \omega < \omega_2(t), \\
1, & \omega \geq \omega_2(t).
\end{cases}
$$

(22)

It is noted that there are two tilt-dependent peaks at $\omega = \omega_1(t) = 2\mu/(1 + t)$ and $\omega = \omega_2(t) = 2\mu/(1 - t)$ in the interband LOCs, which are also confirmed by the JDOS. These expressions agree exactly with the analytical results in Ref. [29]. After substituting $v_x = 0.86v_F$, $v_y = 0.69v_F$, and $v_t = 0.32v_F$ with $v_F = 10^6$ m/s into Eq. (11), these expressions give rise to the numerical results reported in Ref. [29]. Furthermore, these results are also valid in the untilted limit ($t \to 0^+$) and/or undoped case ($\mu = 0$). In the regime of large photon energy where $\omega \gg \text{Max}[\omega_2(t), 2\mu]$ which leads to $\xi_\pm = \pm 1/t$, these interband conductivities approach to the asymptotic background values $\text{Re} \sigma_{xx}^{asy}(\omega) = \frac{\nu_y}{\nu_x} \sigma_0$ and $\text{Re} \sigma_{yy}^{asy}(\omega) = \frac{\nu_x}{\nu_y} \sigma_0$, and they satisfies $\text{Re} \sigma_{xx}^{asy}(\omega) \times \text{Re} \sigma_{yy}^{asy}(\omega) = \sigma_0^2$, all of which are independent of the tilt parameter, the same as reported in Ref. [29].

B. For type-II Dirac materials

For the type-II phase ($t > 1$), the interband conductivities defined in Eq. (11) take the form as

$$
\Gamma_{xx}^{IB}(\omega) = \begin{cases} 
0, & 0 < \omega < \omega_1(t), \\
1 - G_-(\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\
\sum_{\chi = \pm 1} \chi G_-(\xi_\chi), & \omega \geq \omega_2(t),
\end{cases}
$$

(23)
and
\[
\Gamma_{yy}^{IB}(\omega) = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\ 1 - G_+ (\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\ \sum_{\chi = \pm 1} \chi G_+ (\xi_\chi), & \omega \geq \omega_2(t), \end{cases}
\]
where \(\xi_\pm = (2\mu \pm \omega)/t\omega\). The corresponding JDOS \(\mathcal{J}(\omega)\) reads
\[
\frac{\mathcal{J}(\omega)}{\mathcal{J}_{0\omega}} = \begin{cases} \frac{\arccos \xi_-}{\pi}, & 0 < \omega < \omega_1(t), \\ \frac{\arcsin \xi_+ - \arcsin \xi_-}{\pi}, & \omega \geq \omega_2(t). \end{cases}
\]

There are also two tilt-dependent peaks at \(\omega = \omega_1(t) = 2\mu/(t + 1)\) and \(\omega = \omega_2(t) = 2\mu/(t - 1)\) in the interband LOCs, which are also confirmed by the JDOS. In the regime of large photon energy where \(\omega \gg \text{Max}\{\omega_2(t), 2\mu\}\) which leads to \(\xi_\pm = \pm 1/t\), the asymptotic background values can be obtained as
\[
\text{Re} \sigma_{xx}^{\text{asy}}(t) = \frac{v_x}{v_y} \sigma_0 \left[ G_- \left( \frac{1}{t} \right) - G_+ \left( -\frac{1}{t} \right) \right], \quad (26)
\]
\[
\text{Re} \sigma_{yy}^{\text{asy}}(t) = \frac{v_y}{v_x} \sigma_0 \left[ G_+ \left( \frac{1}{t} \right) - G_- \left( -\frac{1}{t} \right) \right], \quad (27)
\]
which is a straightforward consequence of the LOCs in the undoped case. In addition, they satisfy
\[
\text{Re} \sigma_{xx}^{\text{asy}}(t) \times \text{Re} \sigma_{yy}^{\text{asy}}(t) = \frac{4}{\pi^2} \left[ \frac{1}{t^2} - 1 + \arcsin^2 \left( \frac{1}{t} \right) \right] \sigma_0^2, \quad (28)
\]
which, different from that in the type-I phase, is tilt-dependent.

C. For type-III Dirac materials

For the type-III phase \((t = 1)\), the interband conductivities defined in Eq. \ref{c} are given as
\[
\Gamma_{xx}^{IB}(\omega) = \begin{cases} 0, & 0 < \omega < \mu, \\ 1 - G_-(\xi_-), & \omega \geq \mu, \end{cases}
\]
and
\[
\Gamma_{yy}^{IB}(\omega) = \begin{cases} 0, & 0 < \omega < \mu, \\ 1 - G_+(\xi_-), & \omega \geq \mu, \end{cases}
\]
where \(\xi_\pm = (2\mu \pm \omega)/\omega\). The corresponding JDOS \(\mathcal{J}(\omega)\) is given by
\[
\frac{\mathcal{J}(\omega)}{\mathcal{J}_{0\omega}} = \begin{cases} \arccos \xi_- / \pi, & 0 < \omega < \mu, \\ \omega \geq \mu. \end{cases}
\]

It is remarked that there is only one finite peak at \(\omega = \mu = \omega_1(1)\) in the interband LOCs, which is also confirmed by the corresponding JDOS. In the regime of large photon energy where \(\omega \gg 2\mu\), these interband conductivities approach to the asymptotic background values \(\text{Re} \sigma_{xx}^{\text{asy}}(\omega) = \frac{v_x}{v_y} \sigma_0\) and \(\text{Re} \sigma_{yy}^{\text{asy}}(\omega) = \frac{v_y}{v_x} \sigma_0\). The results for the interband LOCs, the JDOS, the asymptotic background values, and their product \(\text{Re} \sigma_{xx}^{\text{asy}}(\omega) \times \text{Re} \sigma_{yy}^{\text{asy}}(\omega)\) can also be obtained from that of type-I phase in the limit \(t \to 1^-\) or from that of type-II phase in the limit \(t \to 1^+\). As a consequence, the interband conductivities \(\text{Re} \sigma_{xx}(\omega)\) and \(\text{Re} \sigma_{yy}(\omega)\) are continuous when the tilt parameter change from \(t < 1\) to \(t > 1\) through \(t = 1\).

IV. DRUDE CONDUCTIVITY

In this section, we turn to the Drude conductivities contributed by the intraband transition around the Fermi surface. At zero temperature, the derivative of the Fermi distribution function in Eq. \ref{c} can be replaced by \(\delta [\mu - \varepsilon(k_x, k_y)]\), and consequently the Drude conductivity reads
\[
\mathcal{R}_{\lambda(j)\lambda(i)}^{\text{D}}(\omega) = \frac{\delta(\omega)}{4\pi} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \mathcal{F}_{\lambda(j)\lambda(i)}^{\text{D}}(k_x, k_y) \times \delta [\mu - \varepsilon(k_x, k_y)]. \quad (32)
\]
In the untitled case \((t = 0)\), we recover the result of the ordinary Dirac cone. Following the convention of Eq. \ref{c}, the result takes
\[
\Gamma_{xx}^{\text{D}}(\omega) = \Gamma_{yy}^{\text{D}}(\omega) = 4\mu \delta(\omega), \quad (33)
\]
which is exactly the same as that in Ref. \ref{c}.

For the type-I phase \((0 < t < 1)\), the Drude conductivities defined in Eq. \ref{c} can be given by
\[
\Gamma_{xx}^{\text{D}}(\omega) = \frac{8\mu}{t^2} \frac{1 - \sqrt{1 - t^2}^2}{t^2 \sqrt{1 - t^2}} \delta(\omega), \quad (33)
\]
\[
\Gamma_{yy}^{\text{D}}(\omega) = \frac{8\mu}{t^2} \frac{1 - \sqrt{1 - t^2}^2}{t^2} \delta(\omega). \quad (34)
\]
Four remarks are in order here. First, these results are also valid in the untitled limit \((t \to 1^-)\). Second, in the critical-tilted limit \((t \to 1^-)\), \(\Gamma_{yy}^{\text{D}}(\omega) = 8\mu \delta(\omega)\) but \(\Gamma_{xx}^{\text{D}}(\omega)\) is divergent besides \(\delta(\omega)\). Third, the ratio \(\Gamma_{xx}^{\text{D}}(\omega)/\Gamma_{yy}^{\text{D}}(\omega)\) is always \(1/\sqrt{1 - t^2}\). Fourth, these expressions of Drude conductivity are always convergent when \(0 < t < 1\), and yield the numerical results of
Drude weight in Ref. [26], namely, \( N_1 = 4.686 \) and \( N_2 = 2.673 \) after substituting the parameters of 8-Pmmn borophene (\( v_x = 0.86 v_F, v_y = 0.69 v_F \) and \( v_z = 0.32 v_F \) with \( v_F = 10^6 \text{ m/s} \)) into Eq. (11).

For the type-II and type-III phases (\( t \geq 1 \)), we introduce a momentum cutoff \( \Lambda \) to account for the limitation of integration interval, which is a measure of the density of states due to electron and hole Fermi pockets [55]. For the type-II phase (\( t > 1 \)), the Drude conductivities defined in Eq. (11) can be given by

\[
\begin{align*}
\Gamma_{xx}^D(\omega) &= \frac{8\mu}{\pi} \left[ A(\mu, t, \Lambda) \frac{2\Lambda}{\mu} + B(\mu, t, \Lambda) \sqrt{t^2 - 1} \right. \\
& \quad \left. - C(\mu, t, \Lambda) \right] \delta(\omega), \\
\Gamma_{yy}^D(\omega) &= \frac{8\mu}{\pi} \left[ (t^2 - 1) A(\mu, t, \Lambda) \frac{2\Lambda}{\mu} + \sqrt{t^2 - 1} B(\mu, t, \Lambda) \\
& \quad + C(\mu, t, \Lambda) \right] \delta(\omega),
\end{align*}
\]

where

\[
\begin{align*}
A(\mu, t, \Lambda) &= \sum_{\chi = \pm} \frac{1}{2t} \sqrt{1 - \left( \frac{\mu - \chi \Lambda}{t\Lambda} \right)^2}, \\
B(\mu, t, \Lambda) &= \frac{1}{t^2} \ln \frac{t^2 + \frac{\mu - \Lambda}{\Lambda} + \sqrt{t^2 - 1} \sqrt{t^2 - \left( \frac{\mu - \Lambda}{\Lambda} \right)^2}}{t^2 - \frac{\mu + \Lambda}{\Lambda} + \sqrt{t^2 - 1} \sqrt{t^2 - \left( \frac{\mu + \Lambda}{\Lambda} \right)^2}}, \\
C(\mu, t, \Lambda) &= \sum_{\chi = \pm} \frac{1}{2t} \arccos \left( \frac{\mu - \chi \Lambda}{t\Lambda} \right).
\end{align*}
\]

Keeping up to \( O(1) \) of \( \Lambda \), we have the approximate expressions

\[
\begin{align*}
\Gamma_{xx}^D(\omega) &= 8\mu \left[ \frac{\sqrt{t^2 - 1} 2\Lambda}{\pi t^2} - \frac{1}{t^2} \right] \delta(\omega), \\
\Gamma_{yy}^D(\omega) &= 8\mu \left[ \frac{\sqrt{(t^2 - 1)^2}}{\pi t^2} 2\Lambda + \frac{1}{t^2} \right] \delta(\omega),
\end{align*}
\]

which indicates that both \( \Gamma_{xx}^D(\omega) \) and \( \Gamma_{yy}^D(\omega) \) are divergent when the cutoff \( \Lambda \) is taken to be infinity.

For the type-III phase (\( t = 1 \)), the Drude conductivities defined in Eq. (11) can be given by

\[
\begin{align*}
\Gamma_{xx}^D(\omega) &= 16\mu \left[ \frac{2\Lambda - \mu}{\mu} - \arccos \left( \frac{\mu}{2\Lambda} \right) \right] \delta(\omega), \\
\Gamma_{yy}^D(\omega) &= 8\mu \arccos \left( \frac{\mu - \Lambda}{\Lambda} \right) \delta(\omega),
\end{align*}
\]

which in the limit \( \Lambda \to \infty \) reduce to be

\[
\begin{align*}
\Gamma_{xx}^D(\omega) &= 8\mu \left[ \frac{2}{\pi} \frac{2\Lambda}{\mu} - 1 \right] \delta(\omega), \\
\Gamma_{yy}^D(\omega) &= 8\mu \delta(\omega),
\end{align*}
\]

indicating that \( \Gamma_{xx}^D(\omega) \) is divergent but \( \Gamma_{yy}^D(\omega) \) is convergent. By analyzing the \( \Gamma_{xx}^D(\omega) \) and \( \Gamma_{yy}^D(\omega) \) near \( t = 1 \), it can be seen that when tilt parameter changes from \( t < 1 \) to \( t = 1 \) and \( t > 1 \), the Drude conductivity \( \text{Re}^D_{yy}(\omega) \) is continuous, whereas \( \text{Re}^D_{xx}(\omega) \) changes from convergent to divergent.

![FIG. 2: The dependence of \( \Gamma_{xx}(\omega) \) and \( \Gamma_{yy}(\omega) \) on the tilt parameter. In the numerical calculation, the chemical potential is set to be \( \mu = 0.2 \text{ eV} \).](image-url)

V. RESULTS AND DISCUSSION

Utilizing the analytical expressions listed in the previous two sections, we plot the dependence of the real part of LOCs on the tilt parameter \( t \) in Fig. 2 and the explicit comparisons among the tilted Dirac bands, the LOCs and the corresponding JDOS in Fig. 3. For numerical evaluation, we replace the Dirac \( \delta \)-function in the Drude conductivity with Lorentzians according to \( \delta(x) \to (\eta/\pi)/(x^2 + \eta^2) \) with \( \eta \to 0^+ \). We at first present three general results. First, the interband conductivities in the region \( 0 < \omega < \omega_1(t) \) always vanish, namely, \( \Gamma_{ij}^D(\omega) = 0 \). Second, the results are valid for both the n-doped and p-doped cases. Third, all of the analytical results are confirmed by numerical evaluation.

For the type-I phase (\( 0 < t < 1 \)), due to the band tilting and Pauli blocking, the interband transition boundary exhibits an asymmetry. The JDOS possesses two van Hove singularities at \( \omega = \omega_1(t) \) and \( \omega = \omega_2(t) \), leading to two peaks in the LOC, as shown in Fig. 2. It can be seen from Fig. 2 and Fig. 3(c) that \( \Gamma_{jj}^D(\omega) \) increases monotonically from zero at \( \omega = \omega_1(t) \) to unity at \( \omega = \omega_2(t) \) in the region \( \omega_1(t) \leq \omega < \omega_2(t) \), and that \( \Gamma_{jj}^D(\omega) = 1 \) when \( \omega \geq \omega_2(t) \). In addition, when \( 0 \leq t < 1 \), \( \Gamma_{jj}^D(\omega) = \Gamma_{bb}^D(2\mu) = 1/2 \), which can be proved from Eqs. [20] and [21].

For the type-II phase (\( t > 1 \)), there are two van Hove singularities at \( \omega = \omega_1(t) \) and \( \omega = \omega_2(t) \) in the JDOS due to the presence of one electron pocket and one hole pocket. As shown in Fig. 3, the van Hove singularity at \( \omega = \omega_1(t) \) behaves similarly as that in the type-I phase (\( 0 < t < 1 \)). As a result, in the region \( \omega_1(t) \leq \omega < \omega_2(t) \), \( \Gamma_{jj}^D(\omega) \) increases monotonically from zero at \( \omega = \omega_1(t) \)
to unity at $\omega = \omega_2(t)$. However, the van Hove singularity at $\omega = \omega_2(t)$, showing as a dip in the JDOS, is different from that in the type-I phase ($0 < t < 1$). Consequently, $\Gamma_{xx}^{IB}(\omega)$ drops dramatically in magnitude when $\omega \geq \omega_2(t)$, whereas $\Gamma_{yy}^{IB}(\omega)$ goes larger smoothly with the increasing of photon energy. It is seen from Figs. 2, 3(g) and 3(h) that $\Gamma_{xx}^{IB}(\omega) = \Gamma_{yy}^{IB}(\omega)$ at specific point, which can also be proved from Eqs. (23) and (24) that when $1 < t < 2$,

$$
\Gamma_{xx}^{IB}(2\mu) = \Gamma_{yy}^{IB}(2\mu) = \frac{1}{2}
$$

(44)

and when $t \geq 2$,

$$
\Gamma_{xx}^{IB}(2\mu\sqrt{2/(t^2-2)}) = \Gamma_{yy}^{IB}(2\mu\sqrt{2/(t^2-2)}) = \frac{\arcsin[\zeta_+(t)] - \arcsin[\zeta_-(t)]}{\pi}
$$

(45)

with $\zeta_{\pm}(t) = \frac{1}{2}(\sqrt{1 + \frac{2}{t^2}} \pm 1)$.

The tilt parameter $t$ and the chemical potential $\mu$ can be extracted from optical experiments by measuring two resonance peaks at $\omega = \omega_1(t)$ and $\omega = \omega_2(t)$. As shown in Fig. 4(b), the first peak $\omega = \omega_1(t)$ decreases monotonically with the increasing of $t$ no matter in the type-I phase ($0 < t < 1$) or type-II phase ($t > 1$), however, the second peak $\omega = \omega_2(t)$ increases in the type-I phase ($0 < t < 1$) and decreases in the type-II phase ($t > 1$) as $t$ increases. Explicitly, the tilt parameter $t$ satisfies the relation

$$
t = \frac{\omega_2(t) - \omega_1(t)}{\omega_2(t) + \omega_1(t)}.
$$

(46)

Combined with the separation between two peaks

$$
\Delta \omega(t) = \omega_2(t) - \omega_1(t) = \begin{cases} 
\frac{4\mu}{1-t^2}, & 0 < t < 1, \\
\frac{4\mu}{t^2-1}, & t > 1,
\end{cases}
$$

(47)

one can further determine the chemical potential $\mu$. Interestingly, it is shown in Fig. 4(c) that with increasing the tilt parameter $t$ this separation goes larger in the type-I phase ($0 < t < 1$) whereas smaller in the type-II phase ($t > 1$). In experiments, the positions of these two peaks and their separation can also be used to determine whether the Lifshitz transition occurs and which phase the Dirac materials belong to.

For the untilted case ($t = 0$), the JDOS exhibits only one van Hove singularity at $\omega = 2\mu$, so the LOC behaves as a step function. For the type-III phase ($t = 1$), there is one van Hove singularity at $\omega = \omega_2(t) = \mu$ and one van Hove singularity at $\omega = \omega_2(t) = \infty$. As a result, there is only one peak at finite frequency in the LOCs, as shown in Fig. 3(f). In addition, $\Gamma_{yy}^{IB}(\omega)$ approaches asympotically to unity at $\omega \to \infty$, which means $\Gamma_{xx}^{asymp} = \Gamma_{yy}^{asymp} = 1$. 

FIG. 3: The interband transitions, JDOS and LOCs. Two van Hove singularities appear at $\omega = \omega_1$ and $\omega = \omega_2$ in the type-I and type-II phases, but there is only one van Hove singularity at $\omega = \omega_1$ in the type-III phase. $\Gamma_{xx}^{IB}(\omega)$ and $\Gamma_{yy}^{IB}(\omega)$ pass through a fixed point either at $\omega = 2\mu$ for $0 < t \leq 2$ or determined by the tilted parameter for $t > 2$. In the numerical calculation, the chemical potential is set to be $\mu = 0.2$ eV.
VI. CONCLUSIONS

In this work, we theoretically investigated the LOCs in type-I, type-II, and type-III phases of 2D tilted Dirac energy bands. In the undoped case, the interband LOCs are constants either independent of the tilt parameter in both type-I and type-III phases, or determined by the tilt parameter in the type-II phase. In the doped type-I or type-II phase, the interband LOCs are anisotropic and share two peaks at $\omega = \omega_1(t)$ and $\omega = \omega_2(t)$, which are also confirmed by the JDOS. The tilt parameter $t$ and chemical potential $\mu$ can be extracted from optical experiments by measuring the positions of these two peaks and their separation $\Delta \omega(t) = \omega_2(t) - \omega_1(t)$. With increasing the tilt parameter $t$ this separation goes larger in the type-I phase whereas smaller in the type-II phase. At large photon energy regime, the interband LOCs decay to certain asymptotic values which are exactly the same as that in the undoped case.

Through the shapes, asymptotic background values, resonance peaks and their separation in the LOCs, the type of tilted Dirac bands can be determined. These quantities can be taken as signatures of the Lifshitz transition in the 2D tilted Dirac materials. The results of this work are expected to be qualitatively valid for a large number of monolayer tilted Dirac materials such as $8\text{-Pmmn}$ borophene tuned by gate voltage and different compounds of $T'$-phase transition metal dichalcogenides.

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Appendix A: Definition and Particle-hole symmetry of LOC and JDOS

In this appendix, we give the definition of LOC and JDOS, and prove their particle-hole symmetry.

1. Definition and Particle-hole symmetry of LOC

Within linear response theory, the LOC for the photon frequency $\omega$ and chemical potential $\mu$ is given by

$$\sigma_{jj}(\omega, \mu) = \sum_{\kappa=\pm} \sigma_{jj}^{\kappa}(\omega, \mu),$$

(A1)
where

$$\sigma_{jj}^\kappa(\omega, \mu) = \frac{i}{\omega} \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \sum_{\lambda = \pm} \sum_{\lambda' = \pm} F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) f\left[\varepsilon^\lambda_\kappa(k_x, k_y), \mu\right] - f\left[\varepsilon^{\lambda'}_\kappa(k_x, k_y), \mu\right] + i\eta, \tag{A2}$$

Here we keep the explicit dependence of \( \mu \) temporarily for the sake of proving the particle-hole symmetry. In these two equations, \( j = x, y \) refer to spatial coordinates, \( \lambda \) and \( \lambda' \) are the band indices with the conduction band \((\lambda, \lambda' = +)\) and valence band \((\lambda, \lambda' = -)\), \( \eta \) denotes a positive infinitesimal, and \( f(x, \mu) \) is the Fermi distribution function. In addition, \( f(x, \mu), \varepsilon^\lambda_\kappa(k_x, k_y) \) and \( F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) \) are given as

$$f(x, \mu) = \frac{1}{1 + \exp[\beta(x - \mu)]}, \tag{A3}$$

$$\varepsilon^\lambda_\kappa(k_x, k_y) = \kappa v_y k_y + \lambda \mathcal{Z}(k_x, k_y), \tag{A4}$$

$$F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) = \frac{e^2}{2} v_x^2 \left\{ 1 + \lambda \lambda' \frac{v^2_x k_x^2 - v^2_y k_y^2}{[\mathcal{Z}(k_x, k_y)]^2} \right\}, \tag{A5}$$

$$F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) = \frac{e^2}{2} v_x^2 \left\{ 2(1 + \lambda \lambda') + 1 + \lambda \lambda' \frac{v^2_x k_x^2 + v^2_y k_y^2}{[\mathcal{Z}(k_x, k_y)]^2} + 2(\lambda + \lambda') \frac{\kappa v_y k_y}{\mathcal{Z}(k_x, k_y)} \right\}, \tag{A6}$$

where \( \beta = 1/k_B T \) and \( \mathcal{Z}(k_x, k_y) = \sqrt{v^2_x k_x^2 + v^2_y k_y^2} \). It can be verified that

$$f(x, \mu) = 1 - f(-x, -\mu), \tag{A7}$$

$$\varepsilon^\lambda_\kappa(k_x, k_y) = -\varepsilon^{-\lambda}_\kappa(k_x, k_y), \tag{A8}$$

$$F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) = F_{\lambda', -\lambda}^{\kappa;jj}(k_x, k_y) = F_{\lambda', -\lambda'}^{\kappa;jj}(k_x, k_y). \tag{A9}$$

By utilizing these relations, we have

$$\sigma_{jj}(\omega, \mu) = \frac{i}{\omega} \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \sum_{\kappa = \pm} \sum_{\lambda = \pm} \sum_{\lambda' = \pm} F_{\lambda, \lambda'}^{\kappa;jj}(k_x, k_y) f\left[\varepsilon^\lambda_\kappa(k_x, k_y), \mu\right] - f\left[\varepsilon^{\lambda'}_\kappa(k_x, k_y), \mu\right] + i\eta \tag{A10}$$

which indicates \( \sigma_{jj}(\omega, \mu) \) respects the particle-hole symmetry, namely,

$$\sigma_{jj}(\omega, \mu) = \sigma_{jj}(\omega, -\mu) = \sigma_{jj}(\omega, |\mu|). \tag{A11}$$

Keeping this property of \( \sigma_{jj}(\omega, \mu) \) in mind, we can safely replace \( \mu \) in all of \( \sigma_{jj}(\omega, \mu), \sigma_{jj}^\kappa(\omega, \mu), \) and \( f(x, \mu) \) by \( |\mu| \) since we only concern the final result of \( \sigma_{jj}(\omega, \mu) \). Hereafter, we restrict our analysis to the n-doped case \((\mu > 0)\).
For simplicity, we further denote \( \sigma_{jj}(\omega, \mu) \equiv \alpha_{jj}(\omega) \), \( \sigma'_{jj}(\omega, \mu) \equiv \alpha'_{jj}(\omega) \), and \( f(x, \mu) \equiv f(x) \). Consequently, we have

\[
\sigma_{jj}(\omega) = i \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \sum_{\kappa = \pm} \sum_{\lambda = \pm \lambda' = \pm} F_{\kappa,\lambda,\lambda'}^{\pm,j}(k_x, k_y) \frac{f[\varepsilon^\kappa_{\lambda}(k_x, k_y)] - f[\varepsilon^\kappa_{\lambda}(k_x, k_y)]}{\omega + \varepsilon^\kappa_{\lambda}(k_x, k_y) - \varepsilon^\kappa_{\lambda}(k_x, k_y) + i\eta}. \tag{A12}\]

\[
\sigma'_{jj}(\omega) = i \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \sum_{\lambda = \pm \lambda' = \pm} F_{\kappa,\lambda,\lambda'}^{\pm,j}(k_x, k_y) \frac{f[\varepsilon^\kappa_{\lambda}(k_x, k_y)] - f[\varepsilon^\kappa_{\lambda}(k_x, k_y)]}{\omega + \varepsilon^\kappa_{\lambda}(k_x, k_y) - \varepsilon^\kappa_{\lambda}(k_x, k_y) + i\eta}. \tag{A13}\]

2. Definition and Particle-hole symmetry of JDOS

The joint density of states (JDOS) for the photon frequency \( \omega \) and chemical potential \( \mu \) is given by

\[
\mathcal{J}(\omega, \mu) = \sum_{\kappa = \pm} \mathcal{J}_\kappa(\omega, \mu), \tag{A14}\]

where

\[
\mathcal{J}_\kappa(\omega, \mu) = g_0 \int' \frac{dk_x dk_y}{(2\pi)^2} \delta \left[ \varepsilon^\kappa_{\lambda}(k_x, k_y) - \varepsilon^\kappa_{\lambda}(k_x, k_y) - \omega \right] = g_0 \int \frac{dk_x dk_y}{(2\pi)^2} \delta \left[ 2\epsilon(k_x, k_y) - \omega \right], \tag{A15}\]

in which the prime indicates that for different tilts the boundaries of integration regions in \( k \) space are determined by the corresponding Fermi surface at a given chemical potential \( \mu \). Here we keep the explicit dependence of \( \mu \) temporarily for the sake of proving the particle-hole symmetry.

After introducing \( \tilde{k}_x = v_x k_x \) and \( \tilde{k}_y = v_y k_y \), the JDOS can be written as

\[
\mathcal{J}_\kappa(\omega, \mu) = \frac{g_0}{4\pi^2 v_x v_y} \int' \frac{dk_x dk_y}{(2\pi)^2} \delta \left[ \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) - \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) - \omega \right] = \frac{g_0}{\pi} \int \frac{dk_x dk_y}{(2\pi)^2} \delta \left[ \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) - \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) - \omega \right], \tag{A16}\]

where \( g_0 = g_0/(4\pi v_x v_y) \) and \( \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) = \kappa \tilde{k}_y + \lambda \sqrt{\tilde{k}_x^2 + \tilde{k}_y^2} \).

In the polar coordinate, \( \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) = [\lambda + \kappa t \sin \phi] \tilde{k} \) and \( \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) - \tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) = 2\tilde{k} \). Solving the equation

\[
\tilde{\varepsilon}^\kappa_{\lambda}(\tilde{k}_x, \tilde{k}_y) = [\lambda + \kappa t \sin \phi] \tilde{k}_F(\kappa, \lambda, \mu) = \mu, \tag{A17}\]

we have the Fermi wave vectors

\[
\tilde{k}_F(\kappa, \lambda, \mu) = \frac{\mu}{\lambda + \kappa t \sin \phi} = \frac{\text{sgn}(\mu) |\mu|}{\lambda + \kappa t \sin \phi} = \frac{|\mu|}{\text{sgn}(\mu) \lambda + \text{sgn}(\mu) \kappa t \sin \phi} = \tilde{k}_F^{\kappa,\lambda}(\mu) [\Theta(1 - t) \delta_{\lambda,\text{sgn}(\mu)} + \Theta(t - 1) (\delta_{\lambda,\text{sgn}(\mu)} + \delta_{\lambda, -\text{sgn}(\mu)})], \tag{A18}\]

satisfying the relation

\[
\tilde{k}_F^{\kappa,\lambda}(-\mu) = \tilde{k}_F^{\kappa,\lambda}(\mu). \tag{A19}\]
From the above two relations, the JDOS can be rewritten as

\[
\mathcal{J}(\omega, \mu) = \tilde{\Theta}(t) \left[ \tilde{\Theta}(1-t) + \Theta(t-1) \right] \mathcal{J}(\omega, \mu) = \tilde{\Theta}(t) \left[ \tilde{\Theta}(1-t) + \Theta(t-1) \right] \sum_{\kappa=\pm} \mathcal{J}_\kappa(\omega, \mu)
\]

\[
= \tilde{\Theta}(t) \left[ \tilde{\Theta}(1-t) + \Theta(t-1) \right] \frac{J_0}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{d}k_x \tilde{d}k_y \delta \left[ 2\tilde{k} - \omega \right]
\]

\[
= \tilde{\Theta}(t) \tilde{\Theta}(1-t) \frac{J_0}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{d}k_x \tilde{d}k_y \delta \left[ 2\tilde{k} - \omega \right] \sum_{\kappa=\pm} \sum_{\lambda=\pm} \Theta \left[ \omega - 2\tilde{k}_F(\kappa, \lambda, \mu) \right]
\]

\[
+ \tilde{\Theta}(t) \tilde{\Theta}(1-t) \frac{J_0}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{d}k_x \tilde{d}k_y \delta \left[ 2\tilde{k} - \omega \right] \sum_{\kappa=\pm} \sum_{\lambda=\pm} \Theta \left\{ \text{sgn}(\mu) \left[ \tilde{k}^{\kappa, \lambda}_{F_0}(-\mu) - \tilde{k}^{\kappa, +}_{F_0}(\mu) \right] \right\} \Theta \left[ \omega - 2\tilde{k}_F(\kappa, \lambda, \mu) \right]
\]

\[
\times \left\{ \Theta \left[ \omega - 2\tilde{k}_F^{\kappa, \lambda}(\mu)\delta_{\lambda, \text{sgn}(\mu)} \right] - \Theta \left[ \omega - 2\tilde{k}_F^{\kappa, -}(\mu)\delta_{\lambda, \text{sgn}(-\mu)} \right] \right\}.
\]

(A20)

Consequently, we have

\[
\mathcal{J}(\omega, -\mu) = \tilde{\Theta}(t) \tilde{\Theta}(1-t) \frac{J_0}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{d}k_x \tilde{d}k_y \delta \left[ 2\tilde{k} - \omega \right] \sum_{\kappa=\pm} \sum_{\lambda=\pm} \Theta \left[ \omega - 2\tilde{k}_F^{\kappa, -}(\mu)\delta_{\lambda, \text{sgn}(-\mu)} \right]
\]

\[
+ \tilde{\Theta}(t) \tilde{\Theta}(1-t) \frac{J_0}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{d}k_x \tilde{d}k_y \delta \left[ 2\tilde{k} - \omega \right] \sum_{\kappa=\pm} \sum_{\lambda=\pm} \Theta \left\{ \text{sgn}(\mu) \left[ \tilde{k}^{\kappa, -}_{F_0}(\mu) - \tilde{k}^{\kappa, +}_{F_0}(\mu) \right] \right\} \Theta \left[ \omega - 2\tilde{k}_F(\kappa, \lambda, \mu) \right]
\]

\[
\times \left\{ \Theta \left[ \omega - 2\tilde{k}_F^{\kappa, \lambda}(\mu)\delta_{\lambda, \text{sgn}(\mu)} \right] - \Theta \left[ \omega - 2\tilde{k}_F^{\kappa, -}(\mu)\delta_{\lambda, \text{sgn}(-\mu)} \right] \right\}.
\]

(A21)

Obviously, \( \mathcal{J}(\omega, \mu) \) respects the particle-hole symmetry, namely,

\[
\mathcal{J}(\omega, \mu) = \mathcal{J}(\omega, -\mu) = \mathcal{J}(\omega, |\mu|).
\]

(A22)

Keeping this property of \( \mathcal{J}(\omega, \mu) \) in mind, we can safely replace \( \mu \) in all of \( \mathcal{J}(\omega, \mu), \mathcal{J}_\kappa(\omega, \mu), \) and \( \Theta(x, \mu) \) by \( |\mu| \) since we only concern the final result of \( \mathcal{J}(\omega, \mu) \). Hereafter, we are allowed to restrict our analysis to the n-doped case (\( \mu > 0 \)). For simplicity, we further denote \( \mathcal{J}(\omega, \mu) \equiv \mathcal{J}(\omega), \mathcal{J}_\kappa(\omega, \mu) \equiv \mathcal{J}_\kappa(\omega), \) and \( \Theta(x, \mu) \equiv \Theta(x). \)

Appendix B: Detailed calculation of LOC

In the following, we present the detailed calculation of them in the n-doped case (\( \mu > 0 \)). Before doing that, we factor the ratio \( v_y/v_x \) or \( v_x/v_y \) out from the original expressions in order to simplify the calculation. This way, the
original expressions in the “anisotropic model” \((v_x \neq v_y\) and \(v_t \geq 0\)) are converted to be the rescaled forms in the “isotropic model” \((v_x = v_y\) and \(v_t \geq 0\)). Next we focus on the interband part and intraband part of LOC, respectively.

1. Calculation of Interband LOC

After substituting \(\tilde{k}_x = v_x k_x\) and \(\tilde{k}_y = v_y k_y\) into Eq.\((9)\) and expressing it in terms of \(\tilde{k} = \sqrt{k_x^2 + k_y^2}\) and \(\phi = \arctan(k_y/k_x)\), the original expressions in the “anisotropic model” are converted to be the rescaled forms in the “isotropic model” as

\[
\text{Re} \sigma^{\kappa}_{jj(IB)}(\omega) = \int \frac{d\tilde{k}_x d\tilde{k}_y}{4\pi v_x v_y} \hat{F}^{\kappa:jj}_{-+}(\tilde{k}_x, \tilde{k}_y) f \left[ (\kappa \tilde{k}_y - \tilde{k}) - f \left[ (\kappa \tilde{k}_y + \tilde{k}) \right] \right] \delta \left[ \omega - 2\tilde{k} \right],
\]

(B1)

where

\[
\hat{F}^{\kappa:xx}_{-+}(\tilde{k}_x, \tilde{k}_y) = 4\sigma_0 v_x^2 \frac{k_x^2}{k^2} = 4\sigma_0 v_x^2 \sin^2 \phi,
\]

\[
\hat{F}^{\kappa:yy}_{-+}(\tilde{k}_x, \tilde{k}_y) = 4\sigma_0 v_y^2 \frac{k_y^2}{k^2} = 4\sigma_0 v_y^2 \cos^2 \phi.
\]

As a consequence, the ratio \(v_y/v_x\) or \(v_x/v_y\) can be factored out from the original expressions as

\[
\text{Re} \sigma^{\kappa}_{xx(IB)}(\omega) = \sigma_0 \frac{v_x}{v_y} \Gamma^{\kappa}_{xx(IB)}(\omega),
\]

(B2)

\[
\text{Re} \sigma^{\kappa}_{yy(IB)}(\omega) = \sigma_0 \frac{v_y}{v_x} \Gamma^{\kappa}_{yy(IB)}(\omega),
\]

(B3)

where two dimensionless auxiliary functions

\[
\Gamma^{\kappa}_{xx(IB)}(\omega) = \int_0^{+\infty} \frac{dk}{\omega} \int_0^{2\pi} \frac{\sin^2 \phi d\phi}{\pi} \delta \left[ \omega - 2\tilde{k} \right] \left[ f \left[ (\kappa \sin \phi - 1) \tilde{k} \right] - f \left[ (\kappa \sin \phi + 1) \tilde{k} \right] \right],
\]

(B4)

\[
\Gamma^{\kappa}_{yy(IB)}(\omega) = \int_0^{+\infty} \frac{dk}{\omega} \int_0^{2\pi} \frac{\cos^2 \phi d\phi}{\pi} \delta \left[ \omega - 2\tilde{k} \right] \left[ f \left[ (\kappa \sin \phi - 1) \tilde{k} \right] - f \left[ (\kappa \sin \phi + 1) \tilde{k} \right] \right],
\]

(B5)

are introduced for convenience. The ratio \(v_x/v_y\) in \(\text{Re} \sigma^{\kappa}_{xx(IB)}(\omega)\) and \(v_y/v_x\) in \(\text{Re} \sigma^{\kappa}_{yy(IB)}(\omega)\) are totally different in the “anisotropic model” but the same in the “isotropic model”. Therefore, the calculation of \(\text{Re} \sigma^{\kappa}_{xx(IB)}(\omega)\) and \(\text{Re} \sigma^{\kappa}_{yy(IB)}(\omega)\) boils down to calculating \(\Gamma^{\kappa}_{xx(IB)}(\omega)\) and \(\Gamma^{\kappa}_{yy(IB)}(\omega)\). Integrating over \(\tilde{k}\) leads us to

\[
\Gamma^{\kappa}_{xx(IB)}(\omega) = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 \phi d\phi}{\pi} \left[ f \left[ (\kappa \sin \phi - 1) \frac{\omega}{2} \right] - f \left[ (\kappa \sin \phi + 1) \frac{\omega}{2} \right] \right],
\]

and hence

\[
\Gamma^{IB}_{xx}(\omega) = g_x \left[ \Gamma^{+}_{xx(IB)}(\omega) + \Gamma^{-}_{xx(IB)}(\omega) \right]
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 \phi d\phi}{\pi} \left[ f \left[ (t \sin \phi - 1) \frac{\omega}{2} \right] - f \left[ (t \sin \phi + 1) \frac{\omega}{2} \right] + f \left[ (-t \sin \phi - 1) \frac{\omega}{2} \right] - f \left[ (-t \sin \phi + 1) \frac{\omega}{2} \right] \right],
\]

which includes the contribution of different valleys, where \(g_x = 2\) is the degeneracy parameter of spin. Parallel procedures give rise to

\[
\Gamma^{IB}_{yy}(\omega) = g_x \left[ \Gamma^{+}_{yy(IB)}(\omega) + \Gamma^{-}_{yy(IB)}(\omega) \right]
\]

\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 \phi d\phi}{\pi} \left[ f \left[ (t \sin \phi - 1) \frac{\omega}{2} \right] - f \left[ (t \sin \phi + 1) \frac{\omega}{2} \right] + f \left[ (-t \sin \phi - 1) \frac{\omega}{2} \right] - f \left[ (-t \sin \phi + 1) \frac{\omega}{2} \right] \right].
\]

(B6)
In order to obtain the analytical expressions, we perform the integrations over $\phi$ at zero temperature where the Fermi distribution function $f(x)$ can be replaced by the Heaviside step function $\Theta[\mu - x]$ and consequently have

$$\Gamma_{xx}^{IB}(\omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin^2 \phi d\phi \left\{ \Theta \left[ \mu - (t \sin \phi - 1) \frac{\omega}{2} \right] - \Theta \left[ \mu - (t \sin \phi + 1) \frac{\omega}{2} \right] \right\}$$

$$= \int_{-1}^{1} \frac{2x^2}{\pi \sqrt{1 - x^2}} \left\{ \Theta \left[ \mu - (tx - 1) \frac{\omega}{2} \right] - \Theta \left[ \mu - (tx + 1) \frac{\omega}{2} \right] \right\}, \quad (B7)$$

and

$$\Gamma_{yy}^{IB}(\omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2 \phi d\phi \left\{ \Theta \left[ \mu - (t \sin \phi - 1) \frac{\omega}{2} \right] - \Theta \left[ \mu - (t \sin \phi + 1) \frac{\omega}{2} \right] \right\}$$

$$= \int_{-1}^{1} \frac{2\sqrt{1 - x^2} dx}{\pi} \left\{ \Theta \left[ \mu - (tx - 1) \frac{\omega}{2} \right] - \Theta \left[ \mu - (tx + 1) \frac{\omega}{2} \right] \right\}. \quad (B8)$$

After rewriting the Heaviside step function, we get

$$\Gamma_{xx}^{IB}(\omega) = \int_{-1}^{1} \frac{2x^2}{\pi \sqrt{1 - x^2}} \left\{ \Theta \left[ -x + \frac{2\mu/\omega + 1}{t} \right] - \Theta \left[ -x + \frac{2\mu/\omega - 1}{t} \right] \right\}$$

$$= \int_{-1}^{1} \frac{2x^2}{\pi \sqrt{1 - x^2}} \left\{ \Theta \left[ -x + \xi_+ \right] - \Theta \left[ -x + \xi_- \right] \right\}, \quad (B9)$$

and

$$\Gamma_{yy}^{IB}(\omega) = \int_{-1}^{1} \frac{2\sqrt{1 - x^2} dx}{\pi} \left\{ \Theta \left[ -x + \xi_+ \right] - \Theta \left[ -x + \xi_- \right] \right\}. \quad (B10)$$

with $\xi_{\pm} = \frac{2\mu/\omega + \Theta(t)}{t}$.

By imposing constrain on the integration interval via Heaviside step function, $\Gamma_{jj}^{IB}(\omega)$ can be obtained for the type-I, type-II, and type-III Dirac bands as follows.

**a. Interband LOC for type-I phase**

For the type-I phase ($0 < t < 1$), due to $\xi_+ = (2\mu/\omega + 1)\Theta(t) > 1$, we have $\Theta[-x + \xi_+] = 1$. The energy $\omega$ have three region determined by the Heaviside step function $\Theta[-x + \xi_-]$, which are $0 < \omega < \frac{2\mu}{t+1}$ for $\Theta[-x + \xi_-] = 1$ at $-1 < x < 1$, $\frac{2\mu}{t+1} \leq \omega < \frac{2\mu}{t-2}$ for $\Theta[-x + \xi_-] = 1$ at $-1 < x < \xi_-$, and $\omega \geq \frac{2\mu}{t-1}$ for $\Theta[-x + \xi_-] = 0$ at $-1 < x < 1$. The $\Gamma_{jj}^{IB}(\omega)$ are given by

$$\Gamma_{xx}^{IB}(\omega) = 1 - \int_{-1}^{1} \frac{2x^2}{\pi \sqrt{1 - x^2}} \Theta [-x + \xi_-] = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\
1 - G_-(\xi_+), & \omega_1(t) \leq \omega < \omega_2(t), \\
1, & \omega \geq \omega_2(t), \end{cases} \quad (B11)$$

and

$$\Gamma_{yy}^{IB}(\omega) = 1 - \int_{-1}^{1} \frac{2\sqrt{1 - x^2} dx}{\pi} \Theta [-x + \xi_-] = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\
1 - G_+(\xi_+), & \omega_1(t) \leq \omega < \omega_2(t), \\
1, & \omega \geq \omega_2(t), \end{cases} \quad (B12)$$
\[ \omega_1(t) = 2\mu \frac{\Theta(t)}{1 + t}, \]
\[ \omega_2(t) = 2\mu \Theta(t) \left[ \frac{\Theta(1-t)}{1-t} + \frac{\Theta(t-1)}{t-1} \right], \]
\[ G_\pm(x) = \frac{1}{2} \arcsin \frac{x}{\pi} \pm \frac{x\sqrt{1-x^2}}{\pi}. \]  
(B13)

b. Interband LOC for type-II phase

At the type-II phase \((t > 1)\), the Heaviside step function \(\Theta[-x + \xi_+]\) is equal to 1, when \(0 < \omega < \frac{2\mu}{t+1}\) for \(x \in (-1, 1)\), and when \(\omega \geq \frac{2\mu}{t+1}\) for \(x \in (-1, 1)\). The second Heaviside step function satisfy \(\Theta[-x + \xi_-] = 1\) when \(0 < \omega < \frac{2\mu}{t+1}\) for \(x \in (-1, 1)\), and \(\omega \geq \frac{2\mu}{t+1}\) for \(x \in (-1, 1)\). So that there have three region of the energy \(\omega\), which are \(0 < \omega < \frac{2\mu}{t+1}\), \(\frac{2\mu}{t+1} \leq \omega < \frac{2\mu}{t-1}\), and \(\omega \geq \frac{2\mu}{t-1}\). So, we have

\[ \Gamma_{xx}^{IB}(\omega) = \int_{-1}^{1} \frac{2x^2dx}{\pi \sqrt{1-x^2}} \{ \Theta[-x + \xi_+] - \Theta[-x + \xi_-] \} = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\ 1 - G_-(-\xi_+), & \omega_1(t) \leq \omega < \omega_2(t), \\ \sum_{\chi = \pm 1} \chi G_-(\xi_\chi), & \omega \geq \omega_2(t), \end{cases} \]

and

\[ \Gamma_{yy}^{IB}(\omega) = \int_{-1}^{1} \frac{2\sqrt{1-x^2}dx}{\pi} \{ \Theta[-x + \xi_+] - \Theta[-x + \xi_-] \} = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\ 1 - G_+(\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\ \sum_{\chi = \pm 1} \chi G_+(\xi_\chi), & \omega \geq \omega_2(t). \end{cases} \]  
(B14)

\[ \Gamma_{yy}^{IB}(\omega) = \int_{-1}^{1} \frac{2\sqrt{1-x^2}dx}{\pi} \{ \Theta[-x + \xi_+] - \Theta[-x + \xi_-] \} = \begin{cases} 0, & 0 < \omega < \omega_1(t), \\ 1 - G_+(\xi_-), & \omega_1(t) \leq \omega < \omega_2(t), \\ \sum_{\chi = \pm 1} \chi G_+(\xi_\chi), & \omega \geq \omega_2(t). \end{cases} \]  
(B15)

c. Interband LOC for type-III phase

For the type-III phase \((t = 1)\), due to \(\xi_+ = 2\mu/\omega + 1 > 1\), we have \(\Theta[-x + \xi_+] = 1\), and now the \(\xi_-\) is \(\xi_- = 2\mu/\omega - 1\). The energy \(\omega\) have two region by solving the Heaviside step function \(\Theta[-x + \xi_-] = 1\) or 0, which are \(0 < \omega < \mu\) and \(\omega \geq \mu\). The \(\Gamma_{ij}^{IB}(\omega)\) are given by

\[ \Gamma_{xx}^{IB}(\omega) = 1 - \int_{-1}^{1} \frac{2x^2dx}{\pi \sqrt{1-x^2}} \Theta[-x + \xi_-] = \begin{cases} 0, & 0 < \omega < \mu, \\ 1 - G_-(-\xi_-), & \omega \geq \mu, \end{cases} \]  
(B16)

and

\[ \Gamma_{yy}^{IB}(\omega) = 1 - \int_{-1}^{1} \frac{2\sqrt{1-x^2}dx}{\pi} \Theta[-x + \xi_-] = \begin{cases} 0, & 0 < \omega < \mu, \\ 1 - G_+(\xi_-), & \omega \geq \mu. \end{cases} \]  
(B17)

2. Detailed Calculation of Intraband LOC

After substituting \(\tilde{k}_x = v_x k_x\) and \(\tilde{k}_y = v_y k_y\) into Eq.\(10\) and expressing it in terms of \(\tilde{k} = \sqrt{\tilde{k}_x^2 + \tilde{k}_y^2}\) and \(\phi = \arctan(\tilde{k}_y/\tilde{k}_x)\), the original expressions in the “anisotropic model” are converted to be the rescaled forms in the
then the derivative reduces to

$$\tilde{\varepsilon}_\lambda''(\tilde{k}_x, \tilde{k}_y) = \frac{d^2 f}{d\tilde{\varepsilon}_\lambda^2(\tilde{k}_x, \tilde{k}_y)} \delta(\omega),$$

(\text{B18})

with

$$\tilde{f}^{\epsilon;xx}_{\lambda,\lambda}(\tilde{k}_x, \tilde{k}_y) = 4\sigma_0 \frac{\tilde{v}_x^2 \tilde{v}_y^2}{\tilde{k}^2} = 4\sigma_0 \tilde{v}_x^2 \cos^2 \phi,$$

$$\tilde{f}^{\epsilon;yy}_{\lambda,\lambda}(\tilde{k}_x, \tilde{k}_y) = 4\sigma_0 \tilde{v}_y^2 \left( \tilde{k}^2 + \frac{\tilde{k}_y^2}{\tilde{k}^2} + 2\frac{\lambda \kappa t \tilde{k}_y}{\tilde{k}} \right) = 4\sigma_0 \tilde{v}_y^2 (\sin \phi + \lambda \kappa t)^2,$$

where $\tilde{\varepsilon}_\lambda''(\tilde{k}_x, \tilde{k}_y) = \kappa t \tilde{k}_y + \lambda \kappa t$.

At zero temperature, the Fermi distribution function $f(x)$ can be replaced by the Heaviside step function $\Theta[\mu - x]$, then the derivative reduces to $\delta \left[ \mu - \tilde{\varepsilon}_\lambda''(\tilde{k}_x, \tilde{k}_y) \right]$. In the polar coordinate, we obtain

$$\text{Re} \sigma^{\epsilon,\lambda}_{xx(D)}(\omega) = \frac{\sigma_0}{\pi} \frac{v_x}{v_y} \int_0^{+\infty} \tilde{k} d\tilde{k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \phi d\phi \left[ \mu - (\kappa t \sin \phi + \lambda) \tilde{k} \right] \delta(\omega),$$

(\text{B19})

As a consequence, the ratio $v_y/v_x$ or $v_x/v_y$ can be factored out from the original expressions as

$$\text{Re} \sigma^{\epsilon,\lambda}_{xx(D)}(\omega) = g_x \left[ \text{Re} \sigma^{+,\lambda}_{xx(D)}(\omega) + \text{Re} \sigma^{-,\lambda}_{xx(D)}(\omega) \right] = \frac{\sigma_0}{v_y} \frac{v_x}{v_y} \Gamma^{D,\lambda}_{xx}(\omega),$$

(\text{B21})

$$\text{Re} \sigma^{\epsilon,\lambda}_{yy(D)}(\omega) = g_y \left[ \text{Re} \sigma^{+,\lambda}_{yy(D)}(\omega) + \text{Re} \sigma^{-,\lambda}_{yy(D)}(\omega) \right] = \frac{\sigma_0}{v_x} \frac{v_y}{v_x} \Gamma^{D,\lambda}_{yy}(\omega),$$

(\text{B22})

where two dimensionless auxiliary functions

$$\Gamma^{D,\lambda}_{xx}(\omega) = \frac{8}{\pi} \int_0^{+\infty} \tilde{k} d\tilde{k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \phi d\phi \left[ \mu - (t \sin \phi + \lambda) \tilde{k} \right] \delta(\omega),$$

(\text{B23})

$$\Gamma^{D,\lambda}_{yy}(\omega) = \frac{8}{\pi} \int_0^{+\infty} \tilde{k} d\tilde{k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \phi + \lambda t)^2 d\phi \left[ \mu - (t \sin \phi + \lambda) \tilde{k} \right] \delta(\omega),$$

(\text{B24})

are introduced for convenience. The ratio $v_x/v_y$ in $\text{Re} \sigma^{\epsilon,\lambda}_{xx(D)}(\omega)$ and $v_y/v_x$ in $\text{Re} \sigma^{\epsilon,\lambda}_{yy(D)}(\omega)$ are totally different in the "anisotropic model" but the same in the "isotropic model". Therefore, the calculation of $\text{Re} \sigma^{\epsilon,\lambda}_{xx(D)}(\omega)$ and $v_y/v_x$ in $\text{Re} \sigma^{\epsilon,\lambda}_{yy(D)}(\omega)$ boils down to calculating $\Gamma^{D,\lambda}_{xx}(\omega)$ and $\Gamma^{D,\lambda}_{yy}(\omega)$.

\textbf{a. Intraband LOC for type-I phase}

For the type-I phase ($0 < t < 1$), by replacing $\sin \phi$ with $x$ and integrating over $\tilde{k}$, we get

$$\Gamma^{D,\text{sgn}(\mu)}_{xx}(\omega) = \lambda^{D,\text{sgn}(\mu)}_{1,xx}(t) \delta(\omega),$$

(\text{B25})

$$\Gamma^{D,\text{sgn}(\mu)}_{yy}(\omega) = \lambda^{D,\text{sgn}(\mu)}_{1,yy}(t) \delta(\omega).$$

(\text{B26})
where
\[
\mathcal{N}^{D,\text{sgn}(\mu)}_{1,xx}(t) = \frac{8\mu}{\pi} \int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{[1 + \text{sgn}(\mu)tx]^2} = \frac{8\mu}{t^2}\frac{1 - \sqrt{1-t^2}}{\sqrt{1-x^2}},
\]
\[
\mathcal{N}^{D,\text{sgn}(\mu)}_{1,yy}(t) = \frac{8\mu}{\pi} \int_{-1}^{1} dx \frac{[x + \text{sgn}(\mu)t]^2}{[1 + \text{sgn}(\mu)tx]^2} = \frac{8\mu}{t^2}\frac{1 - \sqrt{1-t^2}}{\sqrt{1-x^2}}.
\]

Utilizing the following two relations
\[
\text{Re} \sigma^{\lambda}_{\text{xx}(D)}(\omega) = \sigma_0 \frac{\nu_x}{\nu_y} \mathcal{N}^{D,\text{sgn}(\mu)}_{1,xx}(t) \delta(\omega) \equiv \sigma_0 \frac{4\mu}{\pi} N_1 \delta(\omega),
\]
\[
\text{Re} \sigma^{\lambda}_{\text{yy}(D)}(\omega) = \sigma_0 \frac{\nu_y}{\nu_x} \mathcal{N}^{D,\text{sgn}(\mu)}_{1,yy}(t) \delta(\omega) \equiv \sigma_0 \frac{4\mu}{\pi} N_2 \delta(\omega),
\]
we have
\[
N_1 = 2\pi \frac{\nu_x}{\nu_y} \frac{1 - \sqrt{1-t^2}}{t^2},
\]
\[
N_2 = 2\pi \frac{\nu_y}{\nu_x} \frac{1 - \sqrt{1-t^2}}{t^2}.
\]

These expressions of Drude conductivity yield the numerical results of Drude weight in the Ref.\[26\], namely, \(N_1 = 4.686\) and \(N_2 = 2.673\) after substituting the parameters of 8-Pmmn borophene (\(\nu_x = 0.86\nu_F, \nu_y = 0.69\nu_F\) and \(\nu_t = 0.32\nu_F\) with \(\nu_F = 10^6 m/s\)).

In the limit \(t \to 0\), we recover the result for ordinary Dirac cone,
\[
\mathcal{N}^{D,\text{sgn}(\mu)}_{1,xx}(0) = \mathcal{N}^{D,\text{sgn}(\mu)}_{1,yy}(0) = 4\mu.
\]

In the limit \(t \to 1^-\), we have
\[
\mathcal{N}^{D,\text{sgn}(\mu)}_{1,xx}(1^-) \to \infty,
\]
\[
\mathcal{N}^{D,\text{sgn}(\mu)}_{1,yy}(1^-) = 8\mu.
\]

\[b.\quad \text{Intraband LOC for type-II phase}\]

For the type-II phase \((t > 1)\), there is not only an electron pocket but also a hole pocket at one valley. We take \(\kappa = +1\) and \(\mu > 0\) as an example. In the polar coordinates, there are constrains on the values of \(\tilde{\kappa}\) and \(\phi\). For \(\lambda = +1\), \(\tilde{\kappa}_{\text{min}} = \mu/(t + 1)\), \(\phi \in (\phi_1, \pi + |\phi_1|)\) where \(\phi_1\) is determined by \((t \sin \phi_1 + 1)\Lambda = \mu\), where \(\Lambda\) is the cutoff of \(\tilde{\kappa}\). The cut-off \(\Lambda\) is a measure of the density of states due to electron and hole Fermi pockets [65]. For \(\lambda = -1\), \(\tilde{\kappa}_{\text{min}} = \mu/(t - 1)\), \(\phi \in (\phi_2, \pi - |\phi_2|)\), where \(\phi_2\) is obtained by solving \((t \sin \phi_2 - 1)\Lambda = \mu\). The schematic diagrams of these constrains in the polar coordinate are explicitly shown in Fig.\[5\].

Consequently, for \(\lambda = 1\), \(\Gamma^{D,\mu}_{\text{xx}+}(\omega)\) is given by
\[
\Gamma^{D,\mu}_{\text{xx}+}(\omega) = \frac{8\mu}{\pi} \int_{\mu/(1+t)}^{\Lambda} \tilde{\kappa} d\tilde{\kappa} \int_{\phi_1}^{\pi/2} \cos^2 \phi d\phi \left[\mu - (t \sin \phi + 1)\tilde{\kappa}\right] \delta(\omega)
\]
\[
= \frac{8\mu}{\pi} \int_{\mu/(1+t)}^{\Lambda} \tilde{\kappa} d\tilde{\kappa} \int_{(\tilde{\kappa} - 1)^{-1}}^{1} \sqrt{1 - x^2} dx \left[\mu - (tx + 1)^\frac{1}{2}\right] \delta(\omega)
\]
\[
= \frac{8\mu}{\pi} \int_{(\tilde{\kappa} - 1)^{-1}}^{1} \sqrt{1 - x^2} \left[\left(\Lambda - \frac{\mu}{t + 1}\right) dx\Theta \left[\Lambda - \frac{\mu}{t + 1}\right]\right] \delta(\omega).
\]

Similarly, for \(\lambda = -1\), \(\Gamma^{D,\mu}_{\text{xx}-}(\omega)\) reads
\[
\Gamma^{D,\mu}_{\text{xx}-}(\omega) = \frac{8\mu}{\pi} \int_{(\tilde{\kappa} + 1)^{-1}}^{1} \sqrt{1 - x^2} dx\Theta \left[\Lambda - \frac{\mu}{t - 1}\right] \delta(\omega).
\]
FIG. 5: Schematic diagram of the limitation of the angle \( \phi \) imposed by the delta function in the n-doped case \((\mu > 0)\). In the \( \kappa = +1 \) valley shown in (a), the angle ranges from \( \phi_1 \) to \( \pi + |\phi_1| \) for \( \lambda = + \), but becomes \((\phi_2, \pi - \phi_2)\) for \( \lambda = - \). The case for \( \kappa = -1 \) is shown in (b) for reference. The corresponding optical transitions at \( \kappa = + \) valley are referred to (c).

As a consequence, \( \Gamma_{xx}^{D}(\omega) \) can be written as

\[
\Gamma_{xx}^{D}(\omega) = \Gamma_{xx}^{D,-}(\omega) + \Gamma_{xx}^{D,+}(\omega)
\]

\[
= \frac{8\mu}{\pi} \left\{ \int_{(\xi-1)^+}^{1} \frac{\sqrt{1-x^2}}{(tx+1)^2} dx \Theta \left[ \Lambda - \frac{\mu}{t+1} \right] + \int_{(\xi+1)^-}^{1} \frac{\sqrt{1-x^2}}{(tx-1)^2} dx \Theta \left[ \Lambda - \frac{\mu}{t-1} \right] \right\} \delta(\omega)
\]

\[
= \mathcal{A}_{H,xx}^{D,\text{sgn}(\mu)}(\Lambda, t) \delta(\omega),
\]

where

\[
\mathcal{A}_{H,xx}^{D,\text{sgn}(\mu)}(\Lambda, t) = \frac{8\mu}{\pi} \left[ A(\mu, t, \Lambda) \frac{2\Lambda}{\mu} + B(\mu, t, \Lambda) \frac{\sqrt{t^2-1}}{t^2} - C(\mu, t, \Lambda) \right],
\]

with

\[
A(\mu, t, \Lambda) = \sum_{\chi = \pm} \frac{1}{2t} \sqrt{1 - \left( \frac{\mu - \chi \Lambda}{t \Lambda} \right)^2},
\]

\[
B(\mu, t, \Lambda) = \frac{1}{t^2} \ln \frac{t^2 + \mu - \Lambda}{t^2 - \mu + \Lambda} + \frac{1}{t^2} \sqrt{t^2 - \left( \frac{\mu - \Lambda}{\Lambda} \right)^2} - \frac{1}{t^2} \sqrt{t^2 - \left( \frac{\mu + \Lambda}{\Lambda} \right)^2},
\]

\[
C(\mu, t, \Lambda) = \sum_{\chi = \pm} \frac{1}{t^2} \arccos \frac{\mu - \chi \Lambda}{t \Lambda}.
\]

Similarly, \( \Gamma_{yy}^{D}(\omega) \) can be written as

\[
\Gamma_{yy}^{D}(\omega) = \Gamma_{yy}^{D,+}(\omega) + \Gamma_{yy}^{D,-}(\omega)
\]

\[
= \frac{8\mu}{\pi} \left\{ \int_{(\xi-1)^+}^{1} \frac{1}{(1+tx)^2} \frac{(x+t)^2}{\sqrt{1-x^2}} dx \Theta \left[ \Lambda - \frac{\mu}{1+t} \right] + \int_{(\xi+1)^-}^{1} \frac{1}{(tx-1)^2} \frac{(x-t)^2}{\sqrt{1-x^2}} dx \Theta \left[ \Lambda - \frac{\mu}{t-1} \right] \right\} \delta(\omega)
\]

\[
= \mathcal{A}_{H,yy}^{D,\text{sgn}(\mu)}(\Lambda, t) \delta(\omega),
\]

where

\[
\mathcal{A}_{H,yy}^{D,\text{sgn}(\mu)}(\Lambda, t) = \frac{8\mu}{\pi} \left[ t^2 - 1 \right] A(\mu, t, \Lambda) \frac{2\Lambda}{\mu} + \sqrt{t^2 - 1} B(\mu, t, \Lambda) + C(\mu, t, \Lambda).
\]

(B38)
Keeping the order of $O(1)$ of $\Lambda$, we have

\[ N_{\text{III},xx}^{D,\text{sgn}(\mu)}(\Lambda, t) = 8\mu \left[ \frac{\sqrt{t^2 - 1}}{\pi t^2} \frac{2\Lambda}{\mu} - \frac{1}{t^2} \right], \tag{B43} \]

\[ N_{\text{III},yy}^{D,\text{sgn}(\mu)}(\Lambda, t) = 8\mu \left[ \frac{\sqrt{(t^2 - 1)^3}}{\pi t^2} \frac{2\Lambda}{\mu} + \frac{1}{t^2} \right]. \tag{B44} \]

c. Intraband LOC for type-III phase

For the type-III phase ($t = 1$), $\Gamma_{\text{xx}}^{D}(\omega)$ and $\Gamma_{\text{yy}}^{D}(\omega)$ can be given by

\[ \Gamma_{\text{xx}}^{D,\text{sgn}(\mu)}(\omega) = N_{\text{III},xx}^{D,\text{sgn}(\mu)}(\Lambda) \delta(\omega), \tag{B45} \]

\[ \Gamma_{\text{yy}}^{D,\text{sgn}(\mu)}(\omega) = N_{\text{III},yy}^{D,\text{sgn}(\mu)}(\Lambda) \delta(\omega). \tag{B46} \]

\[ N_{\text{III,xx}}^{D,\text{sgn}(\mu)}(\Lambda) = \frac{8\mu}{\pi} \int_{\frac{1}{2} - 1}^{1} \frac{1 - x^2}{(1 + x)^2} dx \Theta \left[ \Lambda - \frac{\mu}{2} \right] = \frac{16\mu}{\pi} \left[ \frac{2\Lambda}{\mu} - 1 - \arccos \sqrt{\frac{\mu}{2\Lambda}} \right], \tag{B47} \]

and

\[ N_{\text{III,yy}}^{D,\text{sgn}(\mu)}(\Lambda) = \frac{8\mu}{\pi} \int_{\frac{1}{2} - 1}^{1} \frac{1}{(1 + x)^2} \frac{(x + 1)^2}{\sqrt{1 - x^2}} dx \Theta \left[ \Lambda - \frac{\mu}{2} \right] = \frac{8\mu}{\pi} \arccos \left( \frac{\mu}{\Lambda} - 1 \right). \tag{B48} \]

Keeping the order of $O(1)$ of $\Lambda$, we have

\[ N_{\text{III,xx}}^{D,\text{sgn}(\mu)}(\Lambda) = \frac{16\mu}{\pi} \left[ \frac{2\Lambda}{\mu} - \arccos \sqrt{\frac{\mu}{2\Lambda}} \right] = 8\mu \left[ \frac{2}{\pi} \frac{2\Lambda}{\mu} - 1 \right], \tag{B49} \]

and

\[ N_{\text{III,yy}}^{D,\text{sgn}(\mu)}(\Lambda) = \frac{8\mu}{\pi} \arccos \left( \frac{\mu}{\Lambda} \right) = 8\mu. \tag{B50} \]

Appendix C: Detailed calculation of JDOS

In the following, we present the detailed calculatoin of JDOS in the n-doped case ($\mu > 0$) (sgn($\mu$) = $+$) for the type-I, type-II, and type-III Dirac bands.

1. Calculation of JDOS for type-I phase

For the type-I phase ($0 < t < 1$), the conduction band is partially occupied by electrons, and the Fermi wave vectors is $k_F^{+}(\mu) = \mu/(\kappa t \sin \phi + 1)$. Due to the Pauli blocking, the energy of photon must excite electrons from the valence band to the conduction band above the Fermi surface, which requires $\omega \geq \tilde{\epsilon}_\kappa^+ \left[ \hat{k}_F^{\kappa, +}(\mu) \right] - \tilde{\epsilon}_\kappa^- \left[ \hat{k}_F^{\kappa, +}(\mu) \right] = 2\hat{k}_F^{\kappa, +}(\mu) =$
$2\mu/(\kappa t \sin \phi + 1)$ with $\phi \in [0, 2\pi]$. The JDOS at $+\kappa$ valley can be written as

$$
\mathcal{J}_\kappa(\omega) = \mathcal{J}_{0} \int_{0}^{+\infty} \frac{dk}{\kappa t} \int_{0}^{2\pi} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - 2\kappa_{F}^{+}(\mu) \right]
$$

$$
= \mathcal{J}_{0} \int_{0}^{+\infty} \frac{dk}{\kappa t} \int_{0}^{2\pi} d\phi \delta(2\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{\kappa t \sin \phi + 1} \right]
$$

$$
= \mathcal{J}_{0} \omega \int_{-\pi/2}^{\pi/2} d\phi \Theta \left[ \omega - \frac{2\mu}{\kappa t \sin \phi + 1} \right]
$$

$$
= \mathcal{J}_{0} \omega \int_{-\pi/2}^{\pi/2} d\phi \left\{ \Theta \left[ \omega - \frac{2\mu}{\kappa t \sin \phi + 1} \right] + \Theta \left[ \omega - \frac{2\mu}{\kappa t \sin \phi + 1} \right] \right\} = \mathcal{J}_{-\kappa}(\omega).
$$

As a result, $\mathcal{J}(\omega) = g_{v} \mathcal{J}_{\kappa}(\omega)$, where $g_{v} = 2$ denotes the valley degeneracy. After introducing $x = \sin \phi$, one can obtain

$$
\mathcal{J}(\omega) = g_{v} \mathcal{J}_{\kappa}(\omega) = \mathcal{J}_{0} \omega \pi \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} \Theta \left[ \omega - \frac{2\mu}{tx + 1} \right] = \mathcal{J}_{0} \omega \pi \begin{cases} 0, & 0 < \omega < \omega_{1}(t), \\ \frac{\arccos \xi}{\pi}, & \omega_{1}(t) \leq \omega < \omega_{2}(t), \\ 1, & \omega \geq \omega_{2}(t), \end{cases}
$$

where

$$
\xi_{\pm} = \frac{2\mu \pm \omega \Theta(t)}{\omega t},
$$

$$
\omega_{1}(t) = \frac{2\mu \Theta(t)}{1 + t},
$$

$$
\omega_{2}(t) = \frac{2\mu}{t} \left[ \frac{\Theta(1 - t)}{1 - t} + \frac{\Theta(t - 1)}{t - 1} \right].
$$

In addition, it is easy obtain for the untilded case ($t = 0$) that

$$
\mathcal{J}(\omega) = \mathcal{J}_{0} \omega \frac{\pi}{2} \int_{0}^{2\pi} d\phi \Theta \left[ \omega - 2\mu \right] = \mathcal{J}_{0} \omega \frac{\pi}{2} \begin{cases} 0, & 0 < \omega < 2\mu, \\ 1, & \omega \geq 2\mu. \end{cases}
$$

2. Calculation of JDOS for type-II phase

For the type-II phase ($t > 1$), the valence band is partially occupied by holes, and the electrons transition area will be restricted by valence band and conduction band. In order to conveniently describe the integration area of JDOS in $\tilde{k}$ space, we introduce two angle parameters $\phi_{1}$ and $\phi_{2}$, which are obtained by solving $(t \sin \phi + 1)\Lambda = \mu$ and $(t \sin \phi - 1)\Lambda = \mu$ respectively, where $\Lambda$ is the cutoff of $\tilde{k}$, as shown in Fig. The photon energy contributed to the LOC is limited to $2\kappa_{F}^{+}(\mu) \leq \omega \leq 2\kappa_{F}^{-}(\mu)$.

In the polar coordinate, the JDOS for $\kappa = +$ valley and $\kappa = -$ valley can be respectively written as

$$
\mathcal{J}_{+}(\omega) = \mathcal{J}_{0} \frac{\pi}{2} \left\{ \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{1}}^{\pi + |\phi_{1}|} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - 2\kappa_{F}^{+}(\mu) \right] - \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{2}}^{\pi - \phi_{2}} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - 2\kappa_{F}^{-}(\mu) \right] \right\}
$$

$$
= \mathcal{J}_{0} \frac{\pi}{2} \left\{ \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{1}}^{\pi + \phi_{1}} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{t \sin \phi + 1} \right] - \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{2}}^{\pi - \phi_{2}} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{t \sin \phi - 1} \right] \right\}
$$

$$
= 2\mathcal{J}_{0} \frac{\pi}{2} \left\{ \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{1}}^{\pi/2} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{t \sin \phi + 1} \right] - \int_{\phi_{1}}^{\Lambda} \frac{k\tilde{k}}{\kappa t} \int_{\phi_{2}}^{\pi/2} d\phi \delta(\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{t \sin \phi - 1} \right] \right\},
$$

(C7)
and

\[
\mathcal{J}_-(\omega) = \frac{2J_0}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} k \, dk \, \int_{-\pi/2}^{\pi/2} d\phi (2\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{\sqrt{1-t\sin \phi}} \right] - \int_{-\pi/2}^{\pi/2} k \, dk \, \int_{-\pi/2}^{\pi/2} d\phi (2\tilde{k} - \omega) \Theta \left[ \omega - \frac{2\mu}{\sqrt{1-t\sin \phi}} \right] \right\},
\]

where \( \Theta[A - \frac{\mu}{\mu+1}] \) and \( \Theta[A - \frac{\mu}{\mu+1}] \) are omitted here.

By replacing \( \sin \phi \) with \( x \) and integrating over \( \tilde{k} \), we get

\[
\mathcal{J}(\omega) = \mathcal{J}_+(\omega) + \mathcal{J}_-(\omega)
\]

\[
= J_0 \frac{\omega}{2\pi} \left\{ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{|tx+1|} \right] - \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{|tx-1|} \right] \right\}
\]

\[
= J_0 \frac{\omega}{2\pi} \left\{ \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{tx+1} \right] - \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{tx-1} \right] \right\}
\]

\[
= J_0 \frac{\omega}{\pi} \begin{cases} 
0, & \omega < \omega_1(t), \\
\arccos \xi_-, & \omega_1(t) \leq \omega < \omega_2(t), \\
\arcsin \xi_+ - \arcsin \xi_-, & \omega \geq \omega_2(t).
\end{cases}
\]

(C9)

3. Calculation of JDOS for type-III phase

For the type-III phase \( (t = 1) \), we can easily obtain that

\[
\mathcal{J}(\omega) = J_0 \frac{\omega}{2\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{x+1} \right] + J_0 \frac{\omega}{2\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \Theta \left[ \omega - \frac{2\mu}{x-1} \right]
\]

\[
= J_0 \frac{\omega}{\pi} \begin{cases} 
0, & \omega < \mu, \\
\arccos \xi_-, & \omega \geq \mu.
\end{cases}
\]

(C10)

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