An Introduction to Motivic Feynman Integrals

Claudia Rella

Department of Mathematics, University of Geneva
7 Route de Drize, 1227 Carouge, Switzerland

Abstract

This article gives a short step-by-step introduction to the representation of parametric Feynman integrals in scalar perturbative quantum field theory as periods of motives. The application of motivic Galois theory to the algebro-geometric and categorical structures underlying Feynman graphs is reviewed up to the current state of research. The example of primitive log-divergent Feynman graphs in scalar massless $\phi^4$ quantum field theory is analysed in detail.
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Introduction

In Section 1, we describe the graph-theoretic framework for the investigation of the algebraic information contained in the topology of scalar Feynman diagrams. Perturbative quantum field theories possess an inherent algebraic structure, which underlies the combinatorics of recursion governing renormalisation theory, and are thus deeply connected to the theory of graphs.

In Section 2, we broadly review preliminary notions in algebraic geometry and algebraic topology. An algebraic variety over $\mathbb{Q}$ is endowed with two distinct rational structures via algebraic de Rham cohomology and Betti cohomology, which are compatible only after complexification. The coexistence of these two cohomologies and their peculiar compatibility are linked to a specific class of complex numbers, known as numeric periods. The cohomology of an algebraic variety is equipped with two filtrations, and the mixed Hodge structure arising from their interaction constitutes the bridge between the theory of numeric periods and the theory of motives.

In Section 3, we introduce the set of periods, lying between $\mathbb{Q}$ and $\mathbb{C}$, among which are the numbers that come from evaluating parametric Feynman integrals, and we briefly review their remarkable properties. Suitable cohomological structures are exploited to derive non-trivial information about these numbers.

In Section 4, we describe how Feynman integrals are promoted to periods of motives. Technical issues arising from the presence of singularities are tackled by blow up. We adopt the category-theoretic Tannakian formalism where motivic periods, and motivic Feynman integrals in particular, reveal their most intriguing properties. We present an overview of the current progress of research towards the general understanding of the structure of scattering amplitudes via the theory of motivic periods, giving particular attention to recent results in massless scalar $\phi^4$ quantum field theory.

1 Scalar Feynman Graphs

1.1 Perturbative Quantum Field Theory

A quantum field theory encodes in its Lagrangian every admissible interaction among particles, but it does it in a way that makes decoding this information a difficult task. Fixed the initial and final states, an interaction process is associated to a probability amplitude, called its Feynman amplitude, which is determined by the set of kinematic and interaction terms in the Lagrangian. However, individual Lagrangian terms correspond to propagators and interaction vertices which can be linked together in infinitely many distinct ways to connect the same pair of initial and final states. Each of these admissible realisations of the same interaction process has to be accounted for in an infinite sum of contributions to the probability amplitude. We associate to each of these possibilities a graphical representation, called its Feynman diagram, whose individual contribution to the probability amplitude is explicitly written in the form of a Feynman integral by applying the formal correspondence between Lagrangian terms and graphical components, which is established by convention through the set of Feynman rules of the theory. It is only the sum of the contributing Feynman integrals to a given process that has a physical meaning and not the individual integrals, which are themselves interrelated by the gauge symmetry of the Lagrangian.

In perturbative quantum field theory, the sum of individual Feynman integrals is a perturbative expansion in some small parameter of the theory, typically a suitable coupling constant. Thus, the Feynman amplitude can be expanded in a formal power series, which has been shown to be divergent by Dyson. The divergence does not, however, undermine the accuracy of predictions that can be made with the theory. Indeed, although a Feynman amplitude receives contributions to any order in perturbation theory, practical calculations are made by truncating the sum at a certain order and directly evaluating only the remaining finitely many terms. Moreover, the explicit calculation of a Feynman amplitude only includes those diagrams which are one-particle irreducible, or 1PI, that is, diagrams which cannot be divided in two by cutting through a single propagator. See Fig. 1. The contribution from a non-1PI diagram at some given order can be expressed as a combination of lower-order 1PI contributions, which have already been accounted for in the formal series.

The leading order terms in the perturbative expansion of a Feynman amplitude are called tree-level contributions. Higher order diagrams are obtained from tree-level diagrams by adding internal loops. Each independent loop in a diagram is associated to an unconstrained momentum and integrals over unconstrained loop momenta are the origin of singularities in Feynman integrals. We distinguish two classes of singularities. The ultraviolet (UV) divergences arise in the limit of infinite loop momentum, a regime that is far beyond the energy scale that we have currently experimental access to and where we expect new physical phenomena to become relevant and corresponding new
terms to enter the Lagrangian. Sensitivity to the high loop momentum region is treated by means of renormalisation theory. For a renormalizable theory, a suitable adjustment of the Lagrangian parameters allows to systematically re-express the predictions of the theory in terms of renormalized physical couplings, so that they decouple from UV physics. Thus, the theory gives a finite and well-defined relation between physical observables. The infrared (IR) divergences only arise in theories with massless particles as they originate in the limit of infinitesimal loop momentum. They cannot be removed by renormalisation and introduce numerous subtleties in the evaluation of Feynman integrals which we are not dealing with in the present text. For a detailed and comprehensive presentation of perturbative quantum field theory we refer to Zee \[64\] and Srednicki \[58\].

Evaluating Feynman integrals over loop momenta has been of fundamental concern since the early days of perturbative quantum field theory. Smirnov \[57\] summarised more than fifty years of advancements in the field, providing an overview of the most powerful, successful and well-established methods for evaluating Feynman integrals in a systematic way, and at the same time showing how the problem of evaluation has become more and more critical. What could be easily evaluated has, indeed, already been evaluated years ago. Since the first insights into the problem of UV divergences in a quantum field theory presented by Dyson \[28\], \[29\], Salam \[51\], \[50\] and Weinberg \[63\], our understanding has vastly improved. Elvang and Huang \[30\] give a recent overview of the subject, including unitarity methods, BCFW recursion relations, and the methods of leading singularities and maximal cuts. Overlapping divergences can be treated iteratively, thus revealing in the first place the recursive nature of renormalisation theory. However, this combinatorics of subdivergences is only the first hint to a more fundamental algebraic structure inherent in all renormalizable quantum field theories and deeply connected to the theory of graphs.\(^2\)

### 1.2 Feynman Parametrisation

We consider a scalar quantum field theory in an even number \(D\) of space-time dimensions with Euclidean metric\(^3\) and allow different propagators to have different mass. A Feynman diagram is a connected directed graph where each edge represents a propagator and is assigned a momentum and a mass and each vertex stands for a tree-level interaction. External half-edges, also known as external legs, represent incoming or outgoing particles, while internal edges are the internal propagators of the diagram. We define the loop number to be the number of independent cycles of the graph. We adopt the convention for which all external legs have arrows pointing inwards, and consequently distinguish incoming and outgoing particles depending on the momentum being positive or negative, respectively. Momentum is positive when it points in the same direction of the arrow of the corresponding directed edge, and it is negative otherwise. We fix momenta on external lines and for each internal loop we choose an arbitrary orientation of the edges which is consistent with momentum conservation at each vertex of the graph and globally. Momentum conservation leaves one unconstrained free momentum variable for each loop. Thus, the loop number is equal to the number of independent loop momentum vectors. We only consider graphs that are one-particle irreducible.

Let \(G\) be such a Feynman graph with \(m\) external legs, \(n\) internal edges and \(l\) independent loops. Its Feynman integral, up to numerical prefactors, is

\[
I_G = (\mu^2)^{n-lD/2} \int \prod_{r=1}^{l} \frac{d^D k_r}{i \pi^{D/2}} \prod_{j=1}^{n} \frac{1}{-q_j^2 + m_j^2}
\]

where \(\mu\) is a scale introduced to make the expression dimensionless, \(k_1, \ldots, k_l\) are the independent loop momenta, \(m_1, \ldots, m_n\) are the masses of the internal lines and \(q_1, \ldots, q_n\) are the momenta flowing through them. These can be expressed as

\[
q_j = \sum_{i=1}^{l} \lambda_{ji} k_i + \sum_{i=1}^{m} \sigma_{ji} p_i
\]

\(^2\)A first discussion about the appearance of transcendental numbers in Feynman integrals and its relation to the topology of Feynman graphs is presented by Kreimer \[13\] in the framework of knot theory and link diagrams.

\(^3\)It is common practice to compute amplitudes in Euclidean space. Moving to Minkowski space involves performing an extension by analytic continuation known as Wick rotation.
where \( p_1, \ldots, p_m \) are the external momenta and \( \lambda_{ji}, \sigma_{ji} \in \{-1, 0, 1\} \) are constants depending on the particular graph structure.

Feynman [31] introduced the well-known manipulation consisting of defining a set of parameters \( x_1, \ldots, x_n \), called \textit{Feynman parameters}, one for each internal edge of the graph, and applying the formula

\[
\prod_{j=1}^{n} \frac{1}{P_j} = \Gamma(n) \int_{\{x_j \geq 0\}} d^n x \delta \left( 1 - \sum_{j=1}^{n} x_j \right) \frac{1}{\left( \sum_{j=1}^{n} x_j P_j \right)^n}
\]

(3)

with the choice \( P_j = -q_j^2 + m_j^2 \) for \( j = 1, \ldots, n \). Here, \( \Gamma \) is the Euler Gamma function and \( \delta \) is the Dirac Delta distribution. Indeed, we can write

\[
\sum_{j=1}^{n} x_j (-q_j^2 + m_j^2) = -\sum_{r=1}^{l} \sum_{s=1}^{l} k_r \cdot (M_{rs}k_s) + \sum_{r=1}^{l} 2k_r \cdot Q_r + J
\]

(4)

where \( M \) is a \( l \times l \)-matrix with scalar entries, \( Q \) is a \( l \)-vector with momentum vectors as entries and \( J \) is a scalar. \( M, Q \) and \( J \) can be suitably expressed in terms of the graph parameters \( \{x_j, q_j, m_j\}_{j=1}^{n} \). Applying Feynman parametrisation to \([\mathbb{I}]\), the \( l \)-dimensional integral over the loop momenta becomes an \((n-1)\)-dimensional integral over the Feynman parameters

\[
I_G = \Gamma \left( n - \frac{D}{2} \right) \int_{\{x_j \geq 0\}} d^n x \delta \left( 1 - \sum_{j=1}^{n} x_j \right) \frac{U^{n-(l+1)/D/2}}{F^{n-l(D/2)}}
\]

(5)

which is characterised by the polynomials \( U = \det(M) \) and \( F = \det(M)(J + QM^{-1}Q)/\mu^2 \), called \textit{first and second Symanzik polynomials} of the Feynman graph, respectively. Notice that the dimension \( D \) of space-time, entering the exponents in the integrand of \([\mathbb{I}]\), acts as regularisation. We use dimensional regularisation with \( D = 4 - 2\epsilon \) and \( \epsilon \) small parameter. A detailed description of Feynman parametrisation can be found in Srednicki [58].

\textit{Example 1.} Consider the generic one-loop diagram with \( m = n \) external legs. Its Symanzik polynomials are

\[
\begin{align*}
U_{1,\text{loop}} &= \sum_{j=1}^{n} x_j \\
F_{1,\text{loop}} &= U_{1,\text{loop}} \sum_{j=1}^{n} \frac{m_j^2}{\mu^2} x_j + \sum_{j=1}^{n} \sum_{i<j} \frac{(q_i - q_j)^2}{\mu^2} x_i x_j
\end{align*}
\]

(6)

where the internal momenta are given\(^4\) by \( q_1 = k, q_i = k + p_1 + \ldots + p_{i-1} \) for \( 1 < i < n \) and \( q_n = k - p_n \). Here, \( k \) is the unique loop momentum of the graph.

### 1.3 Graph Polynomials

Re-expression of Feynman integrals in parametric form shows that the correspondence between scalar Feynman diagrams and Feynman integrals can be reformulated in different terms. The information contained in a Feynman graph is shared out among its multiple components, which can be identified as the underlying graph structure, the directionality of edges and the various edge labels. If we destructure a Feynman graph in these layers and momentarily neglect the extra information apart from the graph structure, we observe that its integral is insensitive to changes of the graph which leave its topology unaltered. Focusing on the underlying graph topology, the Symanzik polynomials can be suitably re-interpreted and they are commonly called \textit{graph polynomials} in this context.

Let \( G \) be a finite graph without isolated vertices. \( G \) is specified by the pair \((V_G, E_G)\), where \( V_G \) is the collection of vertices and \( E_G \) is the collection of edges. We choose an arbitrary orientation of its edges and define the map

\[
\begin{align*}
\mathbb{Z}^{E_G} &\rightarrow \mathbb{Z}^{V_G} \\
e &\mapsto t(e) - s(e)
\end{align*}
\]

(7)

where \( e \in E_G \) is an edge and \( s(e), t(e) \in V_G \) are its source and sink endpoints with respect to the edge orientation. Let us extend this map to the following exact sequence

\[
0 \rightarrow H_1(G, \mathbb{Z}) \rightarrow \mathbb{Z}^{E_G} \rightarrow \mathbb{Z}^{V_G} \rightarrow H_0(G, \mathbb{Z}) \rightarrow 0
\]

(8)

\(^4\)By global momentum conservation, we have \( p_1 + \ldots + p_m = 0 \).
where \( H_0(G, \mathbb{Z}) \) and \( H_1(G, \mathbb{Z}) \) are the zeroth and first homology groups of the graph. As a consequence, the graph loop number \( l_G \) is related to the number of edges \( n_G \), the number of vertices \( v_G \) and the number of connected components \( c_G \) by

\[
l_G = \text{rank}(H_1(G, \mathbb{Z})) = |E_G| - |V_G| + \text{rank}(H_0(G, \mathbb{Z})) = n_G - v_G + c_G
\]

Assume \( G \) is a graph of Feynman type, that is, finite, connected and one-particle irreducible. Let the *valency* of a vertex be the number of edges attached to it. Being interested in the braid pattern of Feynman graphs, we omit both vertices of valency one, corresponding to the source endpoints of external legs, and vertices of valency two, corresponding to mass insertions. To such a graph \( G \) we wish to assign an integral \( l_G \) which corresponds to the one previously defined in \( \mathcal{U} \) when the neglected extra information is re-inserted. We start by associating a variable \( x_e \) to every internal edge \( e \) of the graph. These variables are known as *Schwinger parameters* and they are the graph-theoretic analogues of Feynman parameters. Let \( T \) be the set of spanning trees of \( G \). The first graph polynomial of \( G \) is defined as

\[
\Psi_G = \sum_{T \in T_1} \prod_{e \in E_T} x_e
\]

It is a homogeneous polynomial of degree \( l_G \) in the Schwinger parameters. Note that each monomial of \( \Psi_G \) has coefficient one, and \( \Psi_G \) is linear in each Schwinger parameter.

**Example 2.** The first graph polynomial of the Feynman graph shown in Fig. 2 is \( \Psi_G = x_1 \cdot \ldots \cdot x_n \left( \frac{1}{x_1} + \ldots + \frac{1}{x_n} \right) \).

![Figure 2: Example of a scalar Feynman graph with \( n \) internal propagators.](image)

By construction, the first Symanzik polynomial \( \mathcal{U} \) of a Feynman graph \( G \) does not depend on momenta and masses involved in the diagram, but is only dependent on the graph topology. Indeed, it explicitly identifies with the first graph polynomial \( \Psi_G \) of the corresponding pure graph structure. The same is not true for the second Symanzik polynomial \( \mathcal{F} \), which is a function of external momenta and internal masses. However, we can re-express \( \mathcal{F} \) in a way that clearly separates the contribution to \( \mathcal{F} \) coming from the graph topology from its other dependences. To this end, momenta and masses edge labels must re-enter our discussion. Let \( T_2 \) be the set of spanning 2-forests of \( G \) and \( P_T \) be the set of external momenta of \( G \) attached to its tree \( T \). The second graph polynomial of \( G \) is defined as

\[
\Xi_G(\{p_j, m_x\}) = \left( \sum_{e \in E_G} \frac{m_x^2}{\mu^2} x_e \right) \Psi_G - \sum_{(T_1, T_2) \in T_2} \left( \prod_{e \in E_{T_1} \cup E_{T_2}} x_e \right) \left( \sum_{p_j \in E_{T_1}} \frac{p_j \cdot p_k}{\mu^2} \right)
\]

It is a homogeneous polynomial of degree \( l_G + 1 \) in the Schwinger parameters. Note that, if all internal masses are zero, then \( \Xi_G \) is linear in each Schwinger parameter. It follows from their definitions that the second Symanzik polynomial and the second graph polynomial of a Feynman graph are, indeed, the same. Moreover, having fixed the momenta of external particles and the masses of internal propagators, we are left with the explicit dependence of \( \mathcal{F} \) on the graph structure given in terms of spanning 2-forests.

**Example 3.** To explicitly see how the individual terms in the graph polynomials arise from the knot structure of the diagram, we look closer at the one-loop Feynman graph with \( m = 4 \) external legs, also called *box diagram*, which is shown in Fig. 3.

---

5 The loop number is equivalently defined as the rank of the first homology group of the graph, while the number of connected components corresponds to the rank of the zeroth homology group of the graph.

6 A graph of zero loop number with \( k \) connected components is called a \( k \)-forest. When \( k = 1 \), the forest is called a tree. Given an arbitrary connected graph \( G \), a spanning \( k \)-forest of \( G \) is a subgraph \( T \subseteq G \) such that \( V_T = V_G \) and \( T \) is a \( k \)-forest. A spanning \( k \)-forest of \( G \) is usually denoted by the collection of its trees. A connected graph has always at least one spanning tree.

7 This gives a next-to-leading order contribution to the two-to-two particle scattering process. Srednicki [2] gives a detailed discussion of two particles elastic scattering at one-loop using standard methods in perturbative quantum field theory.
Its Symanzik polynomials are

\[ U_{\text{box}} = x_1 + x_2 + x_3 + x_4 \]

\[ F_{\text{box}} = \frac{1}{\mu^2} \left[ (x_1 + x_2 + x_3 + x_4)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4) + x_1 x_2 p_1^2 \right. \\
\left. + x_2 x_3 p_2^2 + x_3 x_4 p_3^2 + x_4 x_1 p_1^2 + x_1 x_3 (p_1 + p_2)^2 + x_2 x_4 (p_2 + p_3)^2 \right] \]

Neglecting mass terms, the remaining monomials correspond to the spanning forests shown in Fig. 4 and Fig. 5.

Figure 3: Box diagram with four legs.

Figure 4: Spanning trees in the box diagram with four legs and corresponding terms in \( U_{\text{box}} \).

Figure 5: Spanning 2-forests in the box diagram with four legs and corresponding terms in \( F_{\text{box}} \).

Thus, the Symanzik or graph polynomials capture the algebraic information contained in the topology of a Feynman diagram and they prove to be the first tool to be used in the tentative investigation of renormalisation...
theory via the algebraic manipulation of concatenated one-loop integrals. For a more detailed overview of the properties of Feynman graph polynomials we refer to Bogner and Weinzierl.\footnote{Among other contexts, the feature of no-scaling also occurs in the evaluation of Feynman diagrams concerning the anomalous magnetic moment of the electron, as presented by Laporta and Remiddi.}

1.4 Primitive Log-Divergent $\phi^4$ Graphs

The parametric Feynman integral in (5) can be written in a slightly different notation, which turns out to be particularly useful henceforth. Neglecting prefactors and assuming $D = 4$, it is equivalent to the projective integral

$$I_G(p_j, m_e) = \int_{\sigma} \frac{\Omega}{\Psi_G} \left( \frac{\Psi_G}{\Xi_G(p_j, m_e)} \right)^{n_G-2l_G}$$

(13)

where $\sigma$ is the real projective simplex given by

$$\sigma = \{ [x_1 : \ldots : x_{n_G}] \in \mathbb{P}^{n_G-1}(\mathbb{R}) \mid x_e \geq 0, e = 1, \ldots, n_G \}$$

(14)

and $\Omega$ is the top-degree differential form on $\mathbb{P}^{n_G-1}$ expressed in local coordinates as

$$\Omega = \sum_{e=1}^{n_G} (-1)^e x_e dx_1 \wedge \ldots \wedge dx_{e-1} \wedge dx_{e+1} \wedge \ldots \wedge dx_{n_G}$$

(15)

One can check that the integrand is homogeneous of degree zero, so that the integral in projective space is well-defined and equivalent, under the affine constraint $x_{n_G} = 1$, to the previous parametric integral in affine space. Integral (13) is in general divergent, as singularities may arise if the zero sets of the graph polynomials $\Psi_G$ and $\Xi_G$ intersect the domain of integration.

Graphs satisfying the condition $n_G = 2l_G$ are called logarithmically divergent and constitute a particularly interesting class of graphs. In fact, their Feynman integral simplifies to

$$I_G = \int_{\sigma} \frac{\Omega}{\Psi_G^2}$$

(16)

where the dependence on the second Symanzik polynomial, and consequently on momenta and masses, has vanished. Being uniquely sensitive to the graph topology, such a Feynman graph describes a so-called single-scale process. For a logarithmically divergent graph $G$, we define the graph hypersurface as the zero set of its first Symanzik polynomial

$$X_G = \{ [x_1 : \ldots : x_{n_G}] \in \mathbb{P}^{n_G-1} \mid \Psi_G(x_1, \ldots, x_{n_G}) = 0 \}$$

(17)

which describes the singularities of its Feynman integral $I_G$. The following theorem on the convergence of logarithmically divergent graphs is proven by Bloch, Esnault and Kreimer.\footnote{Broadhurst and Kreimer, Schnetz. Some of the simplest $\phi^4$ graphs are shown in Fig. 6 along with the values of the associated Feynman integrals.}

**Theorem 1.** Let $G$ be logarithmically divergent. The integral $I_G$ converges if and only if every proper subgraph $\emptyset \neq \gamma \subset G$ satisfies the condition $n_\gamma > 2l_\gamma$.

A logarithmically divergent graph $G$ such that $I_G$ is convergent is called primitive log-divergent, or simply primitive. We give particular attention to the class of primitive log-divergent graphs in scalar massless $\phi^4$ quantum field theory. They are called $\phi^4$-graphs, and have vertices with valency at most four. Feynman amplitudes in $\phi^4$ theory have been computed to much higher loop orders than most other quantum field theories thanks to the work of Broadhurst and Kreimer,\footnote{Broadhurst, Kreimer, Schnetz. Some of the simplest $\phi^4$ graphs are shown in Fig. 6 along with the values of the associated Feynman integrals.} and Schnetz.\footnote{Some of the simplest $\phi^4$ graphs are shown in Fig. 6 along with the values of the associated Feynman integrals.} Some of the simplest $\phi^4$ graphs are shown in Fig. 6 along with the values of the associated Feynman integrals.

Figure 6: Examples of $\phi^4$ graphs with 3, 4, 5 and 6 loops.

Here, $\zeta$ is the Riemann zeta function, and $P_{3,5} = -\frac{216}{5} \zeta(3,5) - 81 \zeta(5) \zeta(3) + \frac{22}{5} \zeta(8)$.
1.5 Multiple Zeta Values

The Riemann zeta function is defined, on the half-plane of complex numbers $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and extended to a meromorphic function on the whole complex plane with a single pole at $s = 1$. The first tentative attempts to find polynomial relations among zeta values by multiplying terms of the form (18) have led to a generalisation of the notion of Riemann zeta value. Multiple zeta values, or MZVs, are the real numbers

$$\zeta(s_1, \ldots, s_l) = \sum_{n_1 > n_2 > \cdots > n_l \geq 1} \frac{1}{n_1^{s_1} \cdots n_l^{s_l}}$$

associated to tuples of integers $s = (s_1, \ldots, s_l)$, called multi-indices. To guarantee the convergence of the infinite series, only multi-indices such that $s_i \geq 1$ for $i = 1, \ldots, l$ and $s_1 \geq 2$ are considered. They are called admissible multi-indices. The integers $\text{wt}(s) = s_1 + \cdots + s_l$ and $l$ are called weight and length of the multi-index $s$, respectively.

Following the early observations that products of two zeta values are $\mathbb{Q}$-linear combinations of zeta and double zeta values, and that products of more than two zeta values are analogously expressed in terms of multiple zeta values of higher length, linear relations among MZVs have been the object of a more and more extensive investigation by many mathematicians, including Brown, Cartier, Deligne, Drinfeld, Écalle, Goncharov, Hain, Hoffman, Kontsevich, Terasoma, Zagier, Broadhurst and Kreimer. Indeed, the $\mathbb{Q}$-linear relations among multiple zeta values directly provide insights on the widely sought-after algebraic relations among Riemann zeta values.

The $\mathbb{Q}$-vector space generated by multiple zeta values forms an algebra under the so-called shuffle product. Analytic methods, like partial fraction expansions, provide only a few of the known relations among MZVs. Many more are obtained, although conjecturally, by performing extensive numerical experiments, as described by Blümlein et al [6].

However, enormous progress followed the analytic discovery of a crucial feature of multiple zeta values, that is, beside their representation as infinite series, they admit an alternative representation as iterated integrals over simplices of weight-dimension. Let $\Delta^p = \{(t_1, \ldots, t_p) \in \mathbb{R}^p \mid 1 \geq t_1 \geq t_2 \geq \cdots \geq t_p \geq 0\}$ and define the following measures on the open interval $(0, 1)$

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}$$

If $s$ is an admissible multi-index, write $r_i = s_1 + \cdots + s_i$ for each $i = 1, \ldots, l$ and set $r_0 = 0$. Define the measure $\omega_s$ on the interior of the simplex $\Delta^\text{wt}(s)$ by

$$\omega_s = \prod_{i=1}^{l} \underbrace{\omega_0(t_{r_{i-1}+1}) \cdots \omega_0(t_{r_i-1}) \omega_1(t_{r_i})}_{s_i-1 \text{ times}}$$

The theorem below is due to Kontsevich.

**Theorem 2.** Let $s = (s_1, \ldots, s_l)$ be an admissible multi-index. The multiple zeta value $\zeta(s)$ can be obtained by the convergent improper integral

$$\zeta(s) = \zeta(s_1, \ldots, s_l) = \int_{\Delta^\text{wt}(s)} \omega_s$$

This different way of writing multiple zeta values yields a new algebra structure associated with the so-called shuffle product. Many other linear relations among MZVs have been obtained systematically in this alternative framework. However, it is the comparison of the two coexisting fundamental representations, given by [19] and [22] which contemporarily endow the $\mathbb{Q}$-vector space of MZVs of two distinct algebraic structures, expressed by the stuffle and shuffle products, to be the most productive source of information about these numbers. Relations among MZVs are also and most interestingly derived by such a comparison. For a more detailed discussion of the classical theory of multiple zeta values we refer to the survey article by Fresán and Gil [32].

We observe the remarkable fact that $\mathbb{Q}$-linear combinations of multiple zeta values are ubiquitous in the evaluation of Feynman amplitudes in perturbative quantum field theories. It was conjectured by Broadhurst and Kreimer [10] and then proved by Brown and Schnetz [17] that Feynman integrals of the infinite family of zig-zag graphs (see Fig. 7) in $\phi^4$ theory are certain known rational multiples of the odd values of the Riemann zeta function.

**Theorem 3.** Let $Z_l$ be the zig-zag graph with $l$ loops. Its Feynman integral is

$$I_{Z_l} = 4 \frac{(2l - 2)!}{l!(l-1)!} \left(1 - \frac{1}{2^{2l-3}}\right) \zeta(2l-3)$$
Another example is given by the anomalous magnetic moment of the electron in quantum electrodynamics. The tree level Feynman diagram representing a slow-moving electron emitting a photon is depicted in Fig. 8 along with its one-loop correction.

![Figure 8: Up to one-loop Feynman diagrams contributing to the anomalous magnetic moment of the electron.](image)

Figure 8: Up to one-loop Feynman diagrams contributing to the anomalous magnetic moment of the electron.

The two-loop correction comes from the contributions of seven distinct two-loop diagrams. The total two-loop Feynman amplitude has been evaluated by Petermann [48], giving $\frac{197}{144} + \frac{1}{2}\zeta(2) - 3\zeta(2)\log(2) + \frac{3}{4}\zeta(3)$, which involves the logarithm of 2 and again values of the Riemann zeta function.

Many more examples are given by Broadhurst [9]. Due to a vast amount of evidence, it was believed for a long time that all primitive amplitudes of the form (16) in massless $\phi^4$ theory should be $\mathbb{Q}$-linear combinations of MZVs. Only recently this conjectural statement was proved false. Explicit examples of $\phi^4$-amplitudes at high loop orders not expressible in terms of multiple zeta values have been found by Panzer and Schnetz [47]. In the same work, explicit computation of all $\phi^4$-amplitudes with loop order up to 7 suggests that not all MZVs appear among them. For example, no $\phi^4$-graph is known to evaluate to $\zeta(2)$ or $\zeta(2)^2$. Remarkably, the integral representation of MZVs partially clarify the presence of these numbers in perturbative calculations in quantum field theory. Indeed, both expressions (16) and (22) are suitably interpreted as periods of algebraic varieties.

# 2 Cohomology Theory in Algebraic Geometry

## 2.1 Singular Homology

We follow the expositions by Weibel [62] and Hartshorne [35]. Let $M$ be a topological space. For each integer $n \geq 0$, the standard $n$-simplex is

$$\Delta^m_n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0, \ i = 0, ..., n\}$$

For each $i = 0, ..., n$, the face map $\delta^n_i : \Delta^{n-1}_m \to \Delta^n_m$ is defined by

$$\delta^n_i (t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$

A singular $n$-chain in $M$ is a continuous9 map $\sigma : \Delta^n_m \to M$. For each $n \geq 0$, let

$$C_n(M) = \bigoplus_\sigma \mathbb{Z}\sigma$$

9If $M$ is a differentiable manifold, we can assume the singular chains to be piecewise smooth, or smooth, without altering the homology groups.
be the free abelian group generated by singular \( n \)-chains. Elements of \( C_n(M) \) are finite \( \mathbb{Z} \)-linear combinations of the continuous maps \( \sigma : \Delta^n \to M \). For each \( n \geq 1 \), the boundary map \( \partial_n : C_n(M) \to C_{n-1}(M) \) is defined by

\[
\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta^n_i)
\]

where the alternating signs in the sum guarantee that boundary maps satisfy the condition \( \partial_{n-1} \circ \partial_n = 0 \). The pair \( (C_\bullet(M), \partial_\bullet) \) is called a homological chain complex and is graphically represented as

\[
\cdots \xrightarrow{\partial_{n+1}} C_n(M) \xrightarrow{\partial_n} C_{n-1}(M) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0
\]

**Definition 1.** The singular homology of the topological space \( M \) is the homology of the complex \( (C_\bullet(M), \partial_\bullet) \), that is

\[
H^n_n(M, \mathbb{Z}) = \begin{cases} 
C_0(M)/\text{Im}(\partial_1) & n = 0 \\
\text{Ker}(\partial_n)/\text{Im}(\partial_{n+1}) & n \geq 1
\end{cases}
\]

In degree \( n \), chains in the kernel of the boundary map \( \partial_n \) are called (closed) cycles and chains in the image of the boundary map \( \partial_{n+1} \) are called (exact) boundaries.

**Example 4.** Let \( M = \mathbb{C}^* \) be the punctured complex plane. The singular chains

\[
\sigma_0 : \Delta^0_n \to \mathbb{C}^*, \quad 1 \mapsto 1 \\
\sigma_1 : \Delta^1_n \to \mathbb{C}^*, \quad (t, 1-t) \mapsto e^{2\pi it}
\]

generate the singular homology groups \( H_0^0(\mathbb{C}^*, \mathbb{Z}) \) and \( H_1^0(\mathbb{C}^*, \mathbb{Z}) \), respectively. These are both free groups of rank one. All the other homology groups vanish.

For each \( n \geq 0 \), the free abelian group of singular \( n \)-cochains is defined by

\[
C^n(M) = \text{Hom}(C_n(M), \mathbb{Z})
\]

Analogously, applying vector duality, the coboundary maps \( d^n : C^n(M) \to C^{n+1}(M) \), which satisfy the condition \( d^{n+1} \circ d^n = 0 \), are introduced. This gives a cohomological chain complex \( (C^\bullet(M), d^\bullet) \), graphically represented as

\[
\cdots \xleftarrow{d^{n+1}} C^{n+1}(M) \xleftarrow{d^n} C^n(M) \xleftarrow{d^{n-1}} \cdots \xleftarrow{d^1} C^1(M) \xleftarrow{d^0} C^0(M) \xleftarrow{d^0} 0
\]

**Definition 2.** The singular cohomology of the topological space \( M \) is the cohomology of the complex \( (C^\bullet(M), d^\bullet) \), that is

\[
H^n_s(M, \mathbb{Z}) = \begin{cases} 
\text{Ker}(d^n) & n = 0 \\
\text{Ker}(d^n)/\text{Im}(d^{n-1}) & n \geq 1
\end{cases}
\]

Definitions 1 and 2 of singular homology and cohomology, given here with respect to \( \mathbb{Z} \), extend to other coefficient rings. For our purposes, we almost exclusively work with rational coefficients. This allows us to identify singular cohomology groups with the vector duals of the corresponding singular homology groups, that is

\[
H^n_s(M, \mathbb{Q}) \simeq \text{Hom}(H^n_s(M, \mathbb{Z}), \mathbb{Q})
\]

Thus, classes in a cohomology group can be interpreted as classes of linear functionals on the corresponding homology group. The singular cohomology of a topological space given by the complex points of an algebraic variety defined over a subfield of \( \mathbb{C} \) has a name of its own.

**Definition 3.** Let \( \mathbb{K} \) be a subfield of \( \mathbb{C} \) and let \( X \) be an algebraic variety over \( \mathbb{K} \). The Betti cohomology of \( X \) is the singular cohomology of the underlying topological space of complex points \( X(\mathbb{C}) \) equipped with the analytic topology, that is

\[
H^n_B(X) = H^n_s(X(\mathbb{C}), \mathbb{K})
\]

**Example 5.** Let \( \mathbb{G}_m = \text{Spec} \mathbb{Q}[x, 1/x] \) be the multiplicative group. \( \mathbb{G}_m \) is an algebraic variety over \( \mathbb{Q} \) and its underlying topological space of complex points is \( \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* \). For each \( n \geq 0 \), the \( n \)-th Betti cohomology group of \( \mathbb{G}_m \) is

\[
H^n_B(\mathbb{G}_m) = H^n_s(\mathbb{C}^*, \mathbb{Q})
\]

\(^{10}\)This isomorphism is true for real or complex coefficients as well, while it does not hold for integer coefficients.
2.1.1 Some Properties of Homology

We briefly recall some properties of singular homology and cohomology, assuming the ring of coefficients to be $\mathbb{Q}$.

1) **Homotopy invariance.** If $M_1$ and $M_2$ are homotopically equivalent topological spaces, then $H^*_n(M_1, \mathbb{Q}) \cong H^*_n(M_2, \mathbb{Q})$ for each $n \geq 0$. An analogous statement holds for singular cohomology.

2) **Mayer-Vietoris sequences.** For any two open subspaces $U, V \subseteq M$ such that $M = U \cup V$, there is a long exact sequence of the following form

$$
\ldots \rightarrow H^*_n(U \cap V, \mathbb{Q}) \rightarrow H^*_n(U, \mathbb{Q}) \oplus H^*_n(V, \mathbb{Q}) \rightarrow H^*_n(M, \mathbb{Q}) \rightarrow H^*_{n-1}(U \cap V, \mathbb{Q}) \rightarrow \ldots
$$

An analogous statement holds for singular cohomology.

3) **K"unneth formula.** For any two topological spaces $M_1, M_2$, for each $n \geq 0$, there is a natural isomorphism

$$
H^*_n(M_1 \times M_2, \mathbb{Q}) \cong \bigoplus_{i+j=n} H^*_i(M_1, \mathbb{Q}) \otimes H^*_j(M_2, \mathbb{Q})
$$

An analogous statement holds for singular cohomology.

4) **Push-forward.** Let $f : M_1 \to M_2$ be a continuous map between two topological spaces $M_1, M_2$. Then, $f$ induces a morphism of chain complexes

$$
f_* : \mathcal{C}_*(M_1) \to \mathcal{C}_*(M_2)
$$

called *push-forward*, sending $\sigma_1 \in \mathcal{C}_n(M_1)$ to $\sigma_2 = f \circ \sigma_1 \in \mathcal{C}_n(M_2)$. Equivalently, the following diagram

$$
\Delta^n_{st} \xrightarrow{\sigma_1} M_1 \\
\downarrow \sigma_2 \quad \downarrow f \\
M_2
$$

commutes. Hence, $f$ induces also a group homomorphism between the corresponding singular homology groups

$$
f_* : H^*_n(M_1, \mathbb{Q}) \to H^*_n(M_2, \mathbb{Q})
$$

for each $n \geq 0$.

5) **Pull-back.** Let $f : M_1 \to M_2$ be a continuous map between two topological spaces $M_1, M_2$. Then, $f$ induces a morphism of cochain complexes

$$
f^* : \mathcal{C}^*(M_2) \to \mathcal{C}^*(M_1)
$$

called *pull-back*, sending $\omega_2 \in \mathcal{C}^n(M_2)$ to $\omega_1 = \omega_2 \circ f_* \in \mathcal{C}^n(M_1)$. Equivalently, the following diagram

$$
H^*_n(M_1, \mathbb{Q}) \xrightarrow{\omega_1} \mathbb{Q} \\
\downarrow f_* \\
H^*_n(M_2, \mathbb{Q})
$$

commutes. Hence, $f$ induces also a group homomorphism between the corresponding singular cohomology groups

$$
f^* : H^n_* (M_2, \mathbb{Q}) \to H^n_* (M_1, \mathbb{Q})
$$

for each $n \geq 0$. 
2.1.2 Relative Singular Homology

Let $M$ be a topological space and $\iota : N \hookrightarrow M$ the canonical inclusion of a topological subspace $N \subseteq M$. Denote $(C_\bullet(N), \partial_\bullet^N)$ and $(C_\bullet(M), \partial_\bullet^M)$ their homological chain complexes. The morphism of complexes $\iota_\bullet : C_\bullet(N) \to C_\bullet(M)$, obtained via push-forward, is injective. Thus, for each $n \geq 1$, we define the double chain complex

$$C_n(M,N) = C_{n-1}(N) \oplus C_n(M)$$

and the differential $\partial_n : C_n(M,N) \to C_{n-1}(M,N)$ acting as

$$\partial_n(\sigma_N, \sigma_M) = (-\partial_n^N(\sigma_N), -\iota_\bullet(\sigma_N) + \partial_n^M(\sigma_M))$$

where $(\sigma_N, \sigma_M) \in C_n(M,N)$.

**Definition 4.** The **relative homology** of the pair of topological spaces $(M, N)$ is the homology of the double chain complex $(C_\bullet(M,N), \partial_\bullet)$. For $n \geq 1$, we denote the relative singular homology groups as $H_n^\bullet(M,N,Q)$.

Relative homology satisfies the following long exact sequence

$$\ldots \longrightarrow H_n^0(M,Q) \longrightarrow H_n^0(M,N,Q) \longrightarrow H_n^0(N,Q) \longrightarrow H_{n-1}^0(M,N,Q) \longrightarrow \ldots$$

(45)

where the connecting morphisms are the push-forward maps $\iota_\bullet : H_n(M,N,Q) \to H_n(M,Q)$ induced by the inclusion $\iota : N \hookrightarrow M$. Consider an element of the relative homology group $H_n^\bullet(M,N,Q)$. This is represented by a pair $(\sigma_N, \sigma_M)$ of singular chains $\sigma_N \in C_{n-1}(N)$ and $\sigma_M \in C_n(M)$ satisfying

$$\partial_n^N \sigma_N = 0, \quad \partial_n^M \sigma_M = -\iota_\bullet \sigma_N$$

(46)

Note that, since $\iota_\bullet$ is injective, the latter condition implies the former. Thus, relative homology classes are represented by chains in $M$ whose boundary is contained in $N$. Relative cohomology groups $H^n_\bullet(M,N,Q)$ are defined similarly.

**Example 6.** Let $M = \mathbb{C}^*$ be the punctured complex plane and let $N = \{p, q\} \subset M$ be the subspace consisting of the two distinct points $p, q \in \mathbb{C}^*$. Let $\sigma_2 : \Delta_1^0 \to M$ be any continuous map such that $\sigma_2(0,1) = p$ and $\sigma_2(1,0) = q$, such as the oriented segment starting at $p$ and ending at $q$. Then

$$\partial_1^1 \sigma_2 = p - q \in C_0(N)$$

(47)

Consequently, $\sigma_2$ defines a relative chain. It follows from the long exact sequence (45) that the only non-trivial relative homology group is $H_1^1(M,N,Q)$. A basis of this group is given by the chain $\sigma_2$ and the chain $\sigma_1$, introduced in Example 4 consisting of a counterclockwise circle containing the origin. Such a basis is graphically represented in Fig. 9.

![Figure 9: Basis of $H_1^1(\mathbb{C}^*, \{p, q\}, \mathbb{Q})$.](image)

2.2 De Rham Cohomology

Let $M$ be a differentiable manifold of dimension $n$. A **differential $p$-form** on $M$ is written in local coordinates as

$$\omega = \sum_{1 \leq i_1 \leq \ldots \leq i_p \leq n} f_{i_1, \ldots, i_p}(x_1, \ldots, x_n) \, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

(48)

where $f_{i_1, \ldots, i_p}(x_1, \ldots, x_n)$ are $C^\infty$-functions. Let $\Omega^p(M)$ denote the $\mathbb{R}$-vector space of differential $p$-forms on $M$ and define the space of differential forms on $M$ as

$$\Omega(M) = \bigoplus_{p=0}^n \Omega^p(M)$$

(49)

The **exterior derivative** $d : \Omega(M) \to \Omega(M)$ is the unique $\mathbb{R}$-linear map which sends $p$-forms into $(p + 1)$-forms and satisfies the following axioms:
a) If $f$ is a smooth function, $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ is the ordinary differential of $f$.

b) $d \circ d = 0$.

c) Let $\alpha$ be a $p$-form on $M$ and $\beta$ any differential form in $\Omega(M)$. Denote $\alpha \wedge \beta$ their exterior product. Then, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.

The associated cochain complex is

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

whose cohomology is denoted $H^\bullet_{dR}(M, \mathbb{R})$ and is called the (smooth) de Rham cohomology of $M$. A differential $p$-form $\omega$ is closed if $d\omega = 0$ and it is exact if there exists a differential $(p-1)$-form $\eta$ such that $\omega = d\eta$. A classical theorem by De Rham \cite{DeRham} asserts that the singular cohomology $H^\bullet_{dR}(M, \mathbb{R})$ can be computed using differential forms \cite{DeRham}.

**Theorem 4.** Let $0 \leq k \leq n$. The map

$$H^k_{dR}(M, \mathbb{R}) \to H^k_{dR}(M, \mathbb{R}) \cong \text{Hom}(H^k_{dR}(M, \mathbb{R}), \mathbb{R})$$

which sends the class of a differential form $\omega$ to the integration functional

$$\int \omega: H^k_{dR}(M, \mathbb{R}) \to \mathbb{R}$$

is an isomorphism.

### 2.2.1 Algebraic de Rham Cohomology

Assume $X$ is an affine variety over $\mathbb{Q}$ of dimension $n$ and write $X = \text{Spec}R$ where $R$ is the ring of regular functions on $X$, i.e. $R = \mathcal{O}(X)$. The algebraic $p$-forms on $X$ are the smooth differential $p$-forms on $X$ with $R$-coefficients. In local coordinates

$$\omega = \sum_{1 \leq i_1, \ldots, i_p \leq n} f_{i_1, \ldots, i_p}(x_1, \ldots, x_n) \, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where $f_{i_1, \ldots, i_p}(x_1, \ldots, x_n)$ are regular functions on $X$. The space of algebraic $p$-forms is denoted $\Omega^p_{alg-dR}(X)$. Define the space of algebraic forms on $X$ as

$$\Omega_{alg-dR}(X) = \bigoplus_{p=0}^{n} \Omega^p_{alg-dR}(X)$$

The exterior derivative $d: \Omega_{alg-dR}(X) \to \Omega_{alg-dR}(X)$, defined as in Section 2.2, canonically yields a cochain complex

$$0 \to \Omega^0_{alg-dR}(X) \xrightarrow{d} \Omega^1_{alg-dR}(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{alg-dR}(X) \to 0$$

called de Rham complex, whose associated cohomology, denoted $H^\bullet_{alg-dR}(X, \mathbb{Q})$ and called the algebraic de Rham cohomology of $X$, was first introduced by Grothendieck \cite{Grothendieck}.

**Example 7.** Consider $X = \mathbb{G}_m = \text{Spec} \mathbb{Q}[x, 1/x]$. The only non-vanishing spaces of algebraic forms are

$$\Omega^0_{alg-dR}(\mathbb{G}_m) = \mathbb{Q}[x, 1/x]$$

$$\Omega^1_{alg-dR}(\mathbb{G}_m) = \mathbb{Q}[x, 1/x] \cdot dx$$

Consequently, the following two groups

$$H^0_{alg-dR}(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}$$

$$H^1_{alg-dR}(\mathbb{G}_m, \mathbb{Q}) = \mathbb{Q}[x, 1/x] \cdot dx = \mathbb{Q} \left[ \frac{dx}{x} \right]$$

are the only non-trivial cohomology groups of $X$.

\footnote{De Rham’s theorem was first presented in his PhD thesis, published in 1931, when cohomology groups had not been introduced yet. He did not state the theorem in the way it is described today, but gave an equivalent statement involving Betti numbers and integration of closed differential forms over cycles.}

\footnote{We refer to Bott and Tu \cite{BottTu} for a comprehensive investigation of differential forms in algebraic topology.}

\footnote{The algebraic substitute for the smooth differential form is rigorously defined through the notions of Kähler differential and exterior power. Also, the proper construction of the algebraic de Rham cohomology requires the notions of sheaf cohomology and hypercohomology that we do not use here. For details on these topics we refer to Kashiwara and Schapira \cite{KashiwaraSchapira}.}
The following fundamental theorem is proven by Grothendieck [34].

**Theorem 5.** Let $X$ be a smooth affine variety defined over $\mathbb{Q}$ of dimension $n$. The following *comparison isomorphism* holds

\[
\text{comp} : H^k_{\text{alg} - \text{dR}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^k_{\text{dR}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}
\]

for $0 \leq k \leq n$.

Combining Grothendieck’s and De Rham’s theorems, an important remark follows.

**Remark.** Let $X$ be a smooth affine variety over $\mathbb{Q}$ and let $M$ be the differentiable manifold obtained as the space of complex points of $X$. Then, the smooth de Rham cohomology of $M$, equivalent to its singular cohomology, is isomorphic to the algebraic de Rham cohomology of $X$, i.e. the former can be computed considering algebraic forms only. Thus, a purely algebraic definition of cohomology is obtained.

### 2.2.2 Relative de Rham Cohomology

Let $X$ be a smooth affine $\mathbb{Q}$-variety. Denote $\Omega^p(X) \rightarrow \Omega^1(X) \rightarrow \Omega^2(X) \rightarrow \ldots$ its de Rham complex. Let $D \subseteq X$ be a simple normal crossing divisor and let $D_i$, for $i = 1, \ldots, r$, be its smooth irreducible components. For simplicity, assume that each $D_i$ is defined over $\mathbb{Q}$. For each $I \subseteq \{0, \ldots, r\}$, set

\[
D_I = \bigcap_{i \in I} D_i
\]

and define

\[
D^p = \begin{cases} X & p = 0 \\ \bigsqcup_{|I| = p} D_I & p \geq 1 \end{cases}
\]

The associated double cochain complex of $\mathbb{Q}$-vector spaces $K^{p,q} = \Omega^p(D^p)$ is graphically represented as

\[
\begin{array}{ccccccccc}
\ldots & d & \ldots & d & \ldots \\
\Omega^2(X) & \rightarrow & \bigoplus_{I} \Omega^2(D_I) & \rightarrow & \bigoplus_{i < j} \Omega^2(D_i \cap D_j) & \rightarrow & \ldots \\
\ldots & d & \ldots & d & \ldots \\
\Omega^1(X) & \rightarrow & \bigoplus_{I} \Omega^1(D_I) & \rightarrow & \bigoplus_{i < j} \Omega^1(D_i \cap D_j) & \rightarrow & \ldots \\
\ldots & d & \ldots & d & \ldots \\
\Omega^0(X) & \rightarrow & \bigoplus_{I} \Omega^0(D_I) & \rightarrow & \bigoplus_{i < j} \Omega^0(D_i \cap D_j) & \rightarrow & \ldots \\
\end{array}
\]

where the vertical differentials $d^{\text{ver}}$ are $(-1)^pd$ for each $p \geq 0$ and the horizontal differentials $d^{\text{hor}}$ are linear combinations, with coefficients equal to $\pm 1$, of restriction maps $d_{|I|} : \Omega^p(D_I) \rightarrow \Omega^p(D_J)$. Note that, thanks to the factor $(-1)^p$ in the definition of $d^{\text{ver}}$, the vertical and horizontal differentials anticommute. Let $(\Omega^* (X, D), \delta)$ denote the total cochain complex associated to $K^{p,q}$, that is

\[
\Omega^* (X, D) = \bigoplus_{p+q=\bullet} K^{p,q}, \quad \delta = (d^{\text{ver}}, d^{\text{hor}})
\]

For each $n \geq 0$, the space $\Omega^n(X, D)$ corresponds to the direct sum of the spaces on the $n$-th diagonal of the double cochain complex $K^{p,q}$ represented in (61). The total complex is in fact explicitly written down as

\[
\Omega^n(X, D) \simeq \Omega^0(X) \xrightarrow{\delta_0} \Omega^1(X, D) \simeq \Omega^1(X) \oplus \bigoplus_i \Omega^0(D_i) \xrightarrow{\delta_1} \ldots
\]

The relative algebraic de Rham cohomology $H^*_{\text{alg} - \text{dR}}(X, D)$ is the cohomology of the total cochain complex $\Omega^* (X, D)$, that is

\[
H^n_{\text{alg} - \text{dR}}(X, D) = \begin{cases} \text{Ker}(\delta^n) & n = 0 \\ \text{Ker}(\delta^n)/\text{Im}(\delta^{n-1}) & n \geq 1 \end{cases}
\]

\[\text{D} \text{ looks locally like a collection of coordinate hypersurfaces.}\]
Example 8. Let $X = \mathbb{G}_m = \text{Spec} \mathbb{Q}[x, 1/x]$ and $D = \{1, z\}$ with $z \in \mathbb{Q}$, $z \neq 1$. The corresponding double de Rham complex is

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\rightarrow & \rightarrow & \\
\mathbb{Q} \left[ x, \frac{1}{x} \right] dx & \rightarrow & 0 \\
\rightarrow & \rightarrow & \\
\mathbb{Q} \left[ x, \frac{1}{x} \right] & \rightarrow & \mathbb{Q} \oplus \mathbb{Q} \\
\rightarrow & \rightarrow & 0
\end{array}
$$

(65)

where the only non-trivial horizontal differential is the evaluation map

$$
\mathbb{Q} \left[ x, \frac{1}{x} \right] \rightarrow \mathbb{Q} \oplus \mathbb{Q}
$$

(66)

The corresponding total cochain complex is

$$
\begin{array}{ccc}
\mathbb{Q} \left[ x, \frac{1}{x} \right] & \rightarrow & \mathbb{Q} \oplus \mathbb{Q} \\
\mathbb{Q} \left[ x, \frac{1}{x} \right] dx & \rightarrow & 0 \\
\mathbb{Q} \left[ x, \frac{1}{x} \right] dx & \rightarrow & \mathbb{Q} \oplus \mathbb{Q} \\
\rightarrow & \rightarrow & \\
\rightarrow & \rightarrow & \\
\rightarrow & \rightarrow & \\
\rightarrow & \rightarrow & 0
\end{array}
$$

(67)

where the only non-trivial differential is explicitly written. The non-trivial relative algebraic de Rham cohomology groups are

$$
H^0_{\text{alg-deR}}(X, D) = \text{Ker}(\delta_0) = 0
$$

$$
H^1_{\text{alg-deR}}(X, D) = \text{coKer}(\delta_0) = \mathbb{Q} \left[ x, \frac{1}{x} \right] dx \oplus \mathbb{Q} \oplus \mathbb{Q} / \text{Im}(\delta_0)
$$

(68)

A basis of $H^1_{\text{alg-deR}}(X, D)$ is given by the classes $\left[ \left( \frac{dx}{x}, 0, 0 \right) \right]$ and $\left[ \left( \frac{dx}{z-1}, 0, 0 \right) \right]$.

### 2.3 Pure Hodge Structures

As a consequence of Theorem 5, the Betti cohomology of an algebraic variety is endowed with a richer structure than the singular cohomology of a generic topological space. Recall the following definition.

**Definition 5.** Let $K$ be a field and $(V, F)$, $(V', F')$ be filtered $K$-vector spaces. A morphism $f : V \rightarrow V'$ is called filtered if $f(F^p V) \subseteq F^p V'$ for each $p \geq 0$.

Let $M$ be a compact Kähler manifold of dimension $d$. For each pair of integers $p, q$, let

$$
H^{p,q}(M) \subseteq H^{p+q}(M, \mathbb{C}) = H^{p+q}_{dR}(M, \mathbb{C})
$$

(69)

be the subspace of smooth de Rham cohomology classes that can be represented by a $C^\infty$-closed differential $(p+q)$-form of type $(p, q)$, i.e. that can be locally expressed as

$$
\sum_{i,j} f_{I,J}(z_1, \ldots, z_d) d\bar{z}_i \wedge \ldots \wedge d\bar{z}_p \wedge d\bar{z}_j \wedge \ldots \wedge d\bar{z}_q
$$

(70)

where the sum runs over the index subsets $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ of $\{1, \ldots, d\}$ and $f_{I,J}$ are $C^\infty$-functions.

The following theorem by Hodge [37] marks the beginning of what is currently known as Hodge theory.

**Theorem 6.** Let $M$ be as above. The following direct sum decomposition holds

$$
H^n(M, \mathbb{Q}) \otimes \mathbb{Q} = \bigoplus_{p+q=n} H^{p,q}(M)
$$

(71)

for $n \leq d$.

---

\[15\] Recall that a Kähler manifold is a manifold with a complex structure, a Riemannian structure, and a symplectic structure which are mutually compatible.
We note that complex conjugation acts on the right-hand side of (71) through the action on the complex coefficients of the left-hand side, that is,
\[ \sigma \otimes \bar{\omega} = \sigma \otimes \omega \]
where \( \sigma \in H^n(M, \mathbb{Q}) \) and \( \omega \in \mathbb{C} \). Thus, the complex conjugate of \( H^{p,q}(M) \) is precisely \( H^{q,p}(M) \). This property is often called Hodge symmetry.

**Definition 6.** Let \( H \) be a finite-dimensional \( \mathbb{Q} \)-vector space and let \( H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} \). Assume that \( H_{\mathbb{C}} \) possesses a bigrading

\[ H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q} \]

satisfying \( \overline{H^{p,q}} = H^{q,p} \). Then, \( H \) is called a pure Hodge structure of weight \( n \) and the given direct sum decomposition of its complexification \( H_{\mathbb{C}} \) is called Hodge decomposition.

An equivalent definition of pure Hodge structure is obtained by observing that the data encoded in the Hodge decomposition is equivalent to a finite decreasing filtration \( H^\bullet \) of \( X \) as a consequence of Theorem 6, the Hodge decomposition can be easily referred to the algebraic de Rham cohomology of its complexification \( H_{\mathbb{C}} \) satisfying bigrading \( \bigrading \) of Definition 6.

Let \( \sigma \in H^n(M, \mathbb{Q}) \) and \( \omega \in \mathbb{C} \). Then, by De Rham’s and Grothendieck’s theorems, the smooth de Rham cohomology of \( \mathbb{Q} \), \( \mathbb{C} \), \( \mathbb{Q} \)-vector space and let \( H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} \). Assume that \( H_{\mathbb{C}} \) possesses a bigrading

\[ H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q} \]

The relation between the two equivalent descriptions is given by

\[ H^{p,q} = F^p H_{\mathbb{C}} \cap F^q H_{\mathbb{C}} \]
\[ F^p H_{\mathbb{C}} = \bigoplus_{i \geq p} H^{i,n-i} \]

Let \( X \) be a smooth projective variety defined over \( \mathbb{Q} \) and take \( M = X(\mathbb{C}) \) to be the space of complex points of \( X \). Then, by De Rham’s and Grothendieck’s theorems, the smooth de Rham cohomology of \( M \) is isomorphic to the algebraic de Rham cohomology of \( X \) after complexification, i.e.

\[ H^n(M, \mathbb{C}) = H^n_{\text{alg-dR}}(X, \mathbb{Q}) \otimes \mathbb{C} \]

As a consequence of Theorem 6, the Hodge decomposition can be easily referred to the algebraic de Rham cohomology of \( X \) and analogously the Hodge filtration \( F^\bullet \) is defined on \( H^n_{\text{alg-dR}}(X, \mathbb{Q}) \). To keep track of these additional structures, we define the triple

\[ H^n(X) = (H^n_B(X, \mathbb{Q}), (H^n_{\text{alg-dR}}(X, \mathbb{Q}), F^\bullet), \text{comp}) \]

and call it a pure Hodge structure of weight \( n \) over \( \mathbb{Q} \). The comparison isomorphism induces a corresponding Hodge filtration on the Betti cohomology which is still denoted by \( F^\bullet \).

**Definition 7.** Let \( H \) and \( H' \) be two pure Hodge structures over \( \mathbb{Q} \), where we are omitting the weight for simplicity. A morphism between them \( f : H \to H' \) is a pair \( f = (f_B, f_{\text{alg-dR}}) \) consisting of two \( \mathbb{Q} \)-linear maps \( f_B : H_B \to H'_B \) and \( f_{\text{alg-dR}} : H_{\text{alg-dR}} \to H'_{\text{alg-dR}} \) such that the following two conditions hold

1. \( f_{\text{alg-dR}} \) is filtered with respect to the Hodge filtration, i.e.

\[ f_{\text{alg-dR}}(F^\bullet H_{\text{alg-dR}}) \subseteq F^\bullet H'_{\text{alg-dR}} \]

2. The following diagram commutes

\[ \begin{array}{ccc}
H_{\text{alg-dR}} \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\text{comp}} & H_B \otimes_{\mathbb{Q}} \mathbb{C} \\
\downarrow f_{\text{alg-dR}} \otimes \text{Id}_\mathbb{C} & & \downarrow f_B \otimes \text{Id}_\mathbb{C} \\
H'_{\text{alg-dR}} \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\text{comp}} & H'_B \otimes_{\mathbb{Q}} \mathbb{C}
\end{array} \]

The definition implies that, if \( H \) and \( H' \) have different weights, then every morphism of Hodge structures between them is zero. The following variant of Theorem 6 implies that pure Hodge structures are functorial for morphisms of algebraic varieties.

**Theorem 7.** Let \( X, Y \) be smooth projective varieties defined over \( \mathbb{Q} \) and let \( H^n(X), H^n(Y) \) be the corresponding pure Hodge structures of weight \( n \). For any morphism \( f : X \to Y \) of smooth projective varieties, the induced map on cohomology \( f^* : H^n(Y) \to H^n(X) \) is a morphism of pure Hodge structures.
Example 9. Let \( \mathbb{K} \) be a subfield of \( \mathbb{C} \). For each \( n \in \mathbb{Z} \), we define

\[
\mathbb{Q}(n) = (\mathbb{Q}, (\mathbb{K}, F^\bullet), \text{comp})
\]

where the filtration yields \( \mathbb{K} = F^{-n} \mathbb{K} \supseteq F^{-n+1} \mathbb{K} = 0 \) and the isomorphism \( \text{comp} : \mathbb{C} \to \mathbb{C} \) is given by multiplication by \( (2\pi i)^{-n} \). \( \mathbb{Q}(n) \) is a one-dimensional pure Hodge structure of weight \(-2n\) over \( \mathbb{K} \) and is called a Tate-Hodge structure. As an example, \( \mathbb{Q}(-1) \) is isomorphic to \( H^1(\mathbb{G}_m) = (H^1_B(\mathbb{G}_m), (H^1_{\text{alg-dR}}(\mathbb{G}_m), F^\bullet), \text{comp}) \) where \( F^\bullet \) is the trivial filtration concentrated in degree 1.

2.4 Mixed Hodge Structures

The cohomology in degree \( n \) of a smooth projective complex variety \( X \) carries along a pure Hodge structure of weight \( n \). However, this is no longer true when \( X \) fails to be smooth or projective. The generalisation of the notion of pure Hodge structure to any quasi-projective complex variety is due to Deligne [22], [23], [24], who proved that the cohomology of quasi-projective varieties over \( \mathbb{Q} \) are iterated extensions of pure Hodge structures.

Theorem 8. Let \( X \) be a quasi-projective variety over \( \mathbb{Q} \).

1. There exist a finite increasing filtration, called weight filtration

\[
W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \ldots \subseteq W_{2n} = H^n(X, \mathbb{Q})
\]

and a finite decreasing filtration, called Hodge filtration

\[
F^0 = H^n(X, \mathbb{C}) \supseteq F^1 \supseteq \ldots \supseteq F^n \supseteq F^{n+1} = 0
\]

such that \( F^\bullet \) induces a pure Hodge structure of weight \( m \) on each graded piece

\[
\text{Gr}_m^W H^n(X, \mathbb{Q}) = W_m/W_{m-1}
\]

2. If \( f : X \to Y \) is a morphism of quasi-projective varieties, the induced maps on cohomology \( f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q}) \) and \( f^\bullet : H^n(Y, \mathbb{C}) \to H^n(X, \mathbb{C}) \) are filtered morphisms with respect to the two filtrations, i.e.

\[
\begin{align*}
f^* (W_m H^n(Y, \mathbb{Q})) &\subseteq W_m H^n(X, \mathbb{Q}) \\
f^\bullet (F^n H^n(Y, \mathbb{C})) &\subseteq F^n H^n(X, \mathbb{C})
\end{align*}
\]

3. If \( X \) is smooth, then \( \text{Gr}_m^W H^n(X, \mathbb{Q}) = 0 \) for all \( m < n \). If \( X \) is projective, then \( \text{Gr}_m^W H^n(X, \mathbb{Q}) = 0 \) for all \( m > n \).

The following definition generalises the notion of pure Hodge structure.

Definition 8. Let \( X \) be a quasi-projective variety over \( \mathbb{Q} \). Define the triple

\[
H^n(X) = ((H^n_B(X, \mathbb{Q}), W^B), (H^n_{\text{alg-dR}}(X, \mathbb{Q}), F^\bullet, W^\text{alg-dR}), \text{comp})
\]

where \( W^B \) and \( W^\text{alg-dR} \) are the increasing filtrations associated to the Betti and algebraic de Rham cohomologies, respectively. Require that the comparison isomorphism is filtered with respect to the weight filtration, that is,

\[
\text{comp}(W^\text{alg-dR} \otimes \mathbb{Q} \mathbb{C}) = W^B \otimes \mathbb{Q} \mathbb{C}
\]

and that for each integer \( m \)

\[
\text{Gr}_m^W H^n = (\text{Gr}_m^W H^n_B, (\text{Gr}_m^W H^n_{\text{alg-dR}}, F^\bullet), \text{comp})
\]

is a pure Hodge structure over \( \mathbb{Q} \) of weight \( m \). Then, \( H^n(X) \) is called a mixed Hodge structure over \( \mathbb{Q} \).

Definition 9. Given two mixed Hodge structures \( H \) and \( H' \) over \( \mathbb{Q} \), a morphism \( f : H \to H' \) between them is a pair \( f = (f_B, f_{\text{alg-dR}}) \) consisting of two \( \mathbb{Q} \)-linear maps \( f_B : H_B \to H'_B \) and \( f_{\text{alg-dR}} : H_{\text{alg-dR}} \to H'_{\text{alg-dR}} \) such that \( f_B \) is filtered with respect to the weight filtration, while \( f_{\text{alg-dR}} \) is filtered with respect to the weight and Hodge filtrations, and both maps are compatible with the comparison isomorphism. In other words, we have

\[
\begin{align*}
f_B(W^B H_B) &\subseteq W^B H'_B \\
f_{\text{alg-dR}}(F^\bullet H_{\text{alg-dR}}) &\subseteq F^\bullet H'_{\text{alg-dR}} \\
f_{\text{alg-dR}}(W^\text{alg-dR} H_{\text{alg-dR}}) &\subseteq W^\text{alg-dR} H'_{\text{alg-dR}} \\
f_{\text{alg-dR}} \circ \text{comp} &\circ (f_B \otimes \text{Id}_{\mathbb{C}})
\end{align*}
\]
We denote by \( \text{MHS}(\mathbb{Q}) \) the category of mixed Hodge structures over \( \mathbb{Q} \). Deligne \cite{Deligne} proved that \( \text{MHS}(\mathbb{Q}) \) is an abelian category. Moreover, \( \text{MHS}(\mathbb{Q}) \) is naturally endowed with two forgetful functors
\[
\omega_B : \text{MHS}(\mathbb{Q}) \to \text{Vec}_\mathbb{Q}, \quad \omega_{dR} : \text{MHS}(\mathbb{Q}) \to \text{Vec}_\mathbb{Q}
\]
sending the mixed Hodge structure \( H \) into the \( \mathbb{Q} \)-vector spaces \( H_B \) and \( H_{dR} \), respectively. These functors are called the Betti and de Rham functors.

### 3 Periods of Motives

#### 3.1 Numeric Periods

The following elementary definition was introduced by Kontsevich and Zagier \cite{KontsevichZagier}.

**Definition 10.** A numeric period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of the form
\[
\int_\sigma f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n
\]
where the integrand \( f \) is a rational function with rational coefficients and the domain of integration \( \sigma \subseteq \mathbb{R}^n \) is defined by finite unions and intersections of domains of the form \( \{ g(x_1, \ldots, x_n) \geq 0 \} \) with \( g \) a rational function with rational coefficients.

If rational functions and coefficients are replaced in Definition 10 by algebraic functions and coefficients, the same set of numbers is obtained. On the other hand, algebraic functions in the integrand can be substituted by rational functions by enlarging the set of variables. Note that, because the integral of any real-valued function is equivalent to the volume subtended by its graph, any period admits a representation as the volume of a domain defined by polynomial inequalities with rational coefficients. Thus, the integrand can always be assumed to be the constant function 1. However, this extremely simplified framework does not prove to be particularly useful. Quite the opposite, in what follows, we mostly work with an even more general description of periods than the one given in Definition 10. We denote by \( \mathcal{P} \) the set of periods. Being \( \bar{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C} \), periods are generically transcendental numbers and nonetheless they contain only a finite amount of information, which is captured by the integrand and domain of integration of its integral representation as in (90). Indeed, just like \( \bar{\mathbb{Q}} \), \( \mathcal{P} \) is countable. Many famous numbers belong to the class of periods. Here are some examples:

(a) Algebraic numbers are periods, e.g.
\[
\sqrt{2} = \int_{2x^2 \leq 1} dx
\]

(b) Logarithms of algebraic numbers are periods, e.g.
\[
\log 2 = \int_1^2 \frac{dx}{x}
\]

(c) The transcendental number \( \pi \) is a period, as given by
\[
\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} = \int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} = \int_{x^2 + y^2 \leq 1} dx dy
\]
and alternatively by
\[
2\pi i = \oint_{\gamma_0} \frac{dz}{z}
\]
where \( \gamma_0 \) is a closed path encircling the origin in the complex plane.

(d) Values of the Gamma function at rational arguments satisfy
\[
\Gamma\left( \frac{p}{q} \right) \in \mathcal{P}, \quad p, q \in \mathbb{N}
\]
The elliptic integral
\[2 \int_{-b}^{b} \sqrt{1 + \frac{a^2 x^2}{b^4 - b^2 x^2}} \, dx\]  \hspace{1cm} (96)

representing the perimeter of an ellipse with radii \(a\) and \(b\), is a period. Note that it is not an algebraic function of \(\pi\) for \(a \neq b, a, b \in \mathbb{Q}_{>0}\).

Values of the Riemann zeta function at integer arguments \(s \geq 2\) are periods, e.g.
\[\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^3} = \iiint_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}\]  \hspace{1cm} (97)

(g) Multiple zeta values are periods by means of Kontsevich’s integral representation [22].

Conjecture 1. If a period has two different integral representations, then one expression can be transformed into the other by application of the three integral transformation rules of additivity, change of variables and Stokes’ formula, in which all integrands and domains of integration are algebraic with algebraic coefficients.

We note that even a proof of Conjecture 1 would not solve the additional problem of finding an algorithm to determine whether two given numbers in \(\mathcal{P}\) are equal, or whether a given real number, known numerically to some accuracy, is equal within that accuracy to some period. Another fundamental open problem in the theory of periods is to explicitly exhibit one number which does not belong to \(\mathcal{P}\). Such numbers must exist, because \(\mathcal{P}\) is a countable subset of \(\mathbb{C}\), but the concrete identification of one of such numbers has only been proposed conjecturally.

Conjecturally, the basis of natural logarithms \(e\) and the Euler-Mascheroni constant \(\gamma\) are not periods.

Before moving to a more sophisticated definition of periods written in the language of algebraic geometry, which is essential to subsequent developments, we mention the fruitful interplay between the theory of periods and the theory of linear differential equations. When the integrands or the domains of integration depend on some set of parameters, the integrals, as functions of these parameters, usually satisfy linear differential equations with algebraic coefficients. The solutions of these differential equations generate periods when evaluated at algebraic arguments. The differential equations occurring in this way are called Picard-Fuchs differential equations. The relation between periods and Picard-Fuchs equations has proved to be particularly productive in the case of elliptic curves, hypergeometric functions, modular forms and \(L\)-functions.

### 3.2 Algebra of Motivic Periods

The theory of periods can be alternatively developed within the formalism of algebraic geometry. We refer to Huber and Müller-Stach [38].

**Definition 11.** Let \(X\) be a smooth quasi-projective variety defined over \(\overline{\mathbb{Q}}\), \(Y \subset X\) a subvariety, \(\omega\) a close algebraic differential \(n\)-form on \(X\) vanishing on \(Y\) and \(\gamma\) a singular \(n\)-chain on the complex manifold \(X(\mathbb{C})\) with boundary contained in \(Y(\mathbb{C})\). The integral \(\int_\gamma \omega \in \mathbb{C}\) is a numeric period.

\[\int_0^1 \frac{x}{\log \frac{1}{1-x}} \, dx\]  \hspace{1cm} (98)

Indeed, there seems to be no general principle able to predict if a certain infinite sum or integral of a transcendental function is a period according to Definition 10 or able to determine whether two periods, given by explicit integrals, are equal or different. A number in \(\overline{\mathbb{Q}}\) also admits apparently different expressions, but those same techniques that work for checking the equality of algebraic numbers do not in general work for periods. In fact, two different periods may be numerically very close and yet be distinct\(^1\). However, the following conjecture is presented by Kontsevich and Zagier [42].

An example of two distinct periods which agree numerically to more than 80 digits is given by Shanks [56].

\(16\) An example of two distinct periods which agree numerically to more than 80 digits is given by Shanks [56].
The equivalence of Definitions 11 and 10 follows from the observation that the algebraic chain $\gamma$ can be deformed to a semi-algebraic chain and then broken up into small pieces, which can be bijectively projected onto open domains in $\mathbb{R}^n$ with algebraic boundary. Without loss of generality, we work with coefficients in $\mathbb{Q}$ instead of $\mathbb{Q}$. We note that, like Definition 10, Definition 11 also contains redundancy. The integral $\int_{\gamma} \omega$ can be formally destructured into the quadruple

$$(X,Y,\omega,\gamma) \quad (99)$$

and different quadrupoles can give the same resulting number. To get rid of this redundancy, the various forms of topological invariance of the integral must be suitably accounted for. Following Stokes’ theorem, the integral is insensitive to the individual cycle and form, being instead determined by the homology and cohomology classes of these. Let us associate to $\omega$ its cohomology class in the $n$-th algebraic de Rham cohomology group of $X$ relative to $Y$ and to $\gamma$ its homology class in the $n$-th Betti homology group of $X$ relative to $Y$. Then, the first step towards a unique algebraic description of periods consists of the following substitutions

$$\omega \rightarrow [\omega] \in H^n_{alg-dR}(X,Y)$$
$$\gamma \rightarrow [\gamma] \in H^n_{al}(X,Y) \quad (100)$$

into the quadrupole $(X,Y,\omega,\gamma)$. The problem of the coexistence of distinct, but similarly behaved, cohomologies associated to the same variety, which seems to imply an arbitrary choice here and in many other situations, has been tackled by Grothendieck\[17\] with the introduction of the theory of motives. He suggested that there should exist a universal cohomology theory taking values in a $\mathbb{Q}$-category of motives $\mathcal{M}$. Thus, the notion of a motive is proposed to capture the intrinsic cohomological essence of a variety. Without delving into the category-theoretic details of the theory of motives, we give now an intuitive idea of its origin and fundamental features as necessary to review its application to the theory of periods. A more rigorous discussion of motives is presented in Section 4.5. We recall from Grothendieck’s Theorem 5 that there is a comparison isomorphism $\text{comp} : H^n_{alg-dR}(X,Y) \rightarrow H^n_{al}(X,Y)$ induced by the pairing

$$H^n_{alg-dR}(X,Y) \times H^n_{sing}(X(\mathbb{C}), Y(\mathbb{C})) \rightarrow \mathbb{C}$$

$$([\omega], [\gamma]) \mapsto \int_{\gamma} \omega \quad (102)$$

Neglecting the presence of filtrations, the Hodge structure of $X$ relative to $Y$ is expressed as

$$H^n(X,Y) = (H^n_{al}(X,Y), H^n_{alg-dR}(X,Y), \text{comp}) \quad (103)$$

In the same way that the cohomology class of a differential form singles out its cohomological behaviour, the Hodge structure of an algebraic variety intuitively selects the content shared by its different coexisting cohomologies and filters out everything else. It is, therefore, the first approximate realisation of Grothendieck’s idea of a motive. We define the motivic version of the period $\int_{\gamma} \omega$ as the triple

$$[H^n(X,Y), [\omega], [\gamma]]^m \quad (104)$$

where $m$ in the superscript stands for motivic. We call a period in this guise a motivic period. This has proved to be the most profitable reformulation of the original notion of a period. However, a second source of redundancy in the description of periods via the integral formulation in Definition 11, corresponding to the same transformation rules in Conjecture 1, has yet to be factored out.

**Definition 12.** The space $\mathcal{P}^m$ of motivic periods is defined as the $\mathbb{Q}$-vector space generated by the symbols $[H^*(X,Y), [\omega], [\gamma]]^m$ after factorisation modulo the following three equivalence relations:

1. **Bilinearity.** $[H^*(X,Y), [\omega], [\gamma]]^m$ is bilinear in $[\omega]$ and $[\gamma]$.
2. **Change of variables.** If $f : (X_1, Y_1) \rightarrow (X_2, Y_2)$ is a $\mathbb{Q}$-morphism of pairs of algebraic varieties, $\gamma_1 \in H^*_{al}(X_1, Y_1)$ and $\omega_2 \in H^*_{alg-dR}(X_2, Y_2)$, then

$$[H^*(X_1,Y_1), f^*[\omega_2], [\gamma_1]]^m = [H^*(X_2,Y_2), [\omega_2], f_*[\gamma_1]]^m \quad (105)$$

where $f^*$ is the pull-back of $f$ and $f_*$ is its push-forward.

---

\[17\] Grothendieck proposed the notion of a motive in a letter to Serre in 1964. He himself did not author any publication on motives, although he mentioned them frequently in his correspondence. The first formal expositions of the theory of motives are due to Demazure[27] and Kleiman[31], who based their work on Grothendieck’s lectures on the topic.
3) Stokes’ formula. Assume for simplicity that $X$ is a smooth affine algebraic variety defined over $\mathbb{Q}$ of dimension $d$ and $D \subseteq X$ is a simple normal crossing divisor. Denote by $\tilde{D}$ the normalisation of $D$. The variety $\tilde{D}$ contains a simple normal crossing divisor $\tilde{D}_1$ coming from double points in $D$. If $[\omega] \in H^d_{\text{alg}-\text{dR}}(X, D)$ and $[\gamma] \in H^d_B(X, D)$, then

$$[H^d(X, D), \delta[\omega], [\gamma]]^m = [H^{d-1}(\tilde{D}, \tilde{D}_1), [\omega|_{\tilde{D}}], \partial[\gamma]]^m$$

(106)

where $\delta : H^d_{\text{alg}-\text{dR}}(X, D) \to H^{d-1}_{\text{alg}-\text{dR}}(X, D)$ is the coboundary operator acting on the algebraic de Rham cohomology and $\partial : H^d_B(X, D) \to H^{d-1}_B(\tilde{D}, \tilde{D}_1)$ is the boundary operator acting on the Betti homology.

We observe that the space of motivic periods $P^m$ is naturally endowed with an algebra structure. Indeed, new periods are obtained by taking sums and products of known ones.

### 3.3 Period Map

We call period map the evaluation homomorphism

$$\text{per} : P^m \longrightarrow P$$

$$[H^n(X, Y), [\omega], [\gamma]]^m \longmapsto \int_\gamma \omega$$

(107)

Following the construction in Section 3.2, the period map is explicitly surjective, while injectivity is, on the other hand, not proven. Indeed, a numeric period has a unique motivic realisation only conjecturally. Conjecture 1 is equivalent to the period conjecture below.

**Conjecture 2.** The period map per : $P^m \to P$ is an isomorphism.

Let us briefly discuss the key idea underlying the period conjecture. A $\mathbb{Q}$-morphism $f : (X_1, Y_1) \to (X_2, Y_2)$ between two pairs of algebraic varieties induces a change of coordinates between the corresponding algebraic de Rham cohomologies by pull-back, that is

$$\begin{array}{ccc}
(X_1, Y_1) & \xrightarrow{f} & H^\bullet_{\text{alg}-\text{dR}}(X_1, Y_1) \\
& \downarrow f & \downarrow f^* \\
(X_2, Y_2) & \xrightarrow{f} & H^\bullet_{\text{alg}-\text{dR}}(X_2, Y_2)
\end{array}$$

(108)

The same morphism $f$ acts on the spaces of complex points underlying the given algebraic varieties and induces a change of coordinates between the corresponding singular homologies by push-forward, that is

$$\begin{array}{ccc}
(X_1(\mathbb{C}), Y_1(\mathbb{C})) & \xrightarrow{f} & H^\bullet(X_1(\mathbb{C}), Y_1(\mathbb{C})) \\
& \downarrow f & \downarrow f_* \\
(X_2(\mathbb{C}), Y_2(\mathbb{C})) & \xrightarrow{f} & H^\bullet(X_2(\mathbb{C}), Y_2(\mathbb{C}))
\end{array}$$

(109)

By means of such changes of coordinates, one can easily derive two distinct integral representations of the same numeric period. For example, taking $[\gamma_1] \in H^\bullet(X_1(\mathbb{C}), Y_1(\mathbb{C}))$ and $[\omega_2] \in H^\bullet_{\text{alg}-\text{dR}}(X_2, Y_2)$, we have

$$\int_{f_*[\gamma_1]} [\omega_2] = \int_{[\gamma_1]} f^*[\omega_2]$$

(110)

The corresponding two motivic representations of the same numeric period

$$[H^\bullet(X_1, Y_1), f^*[\omega_2], [\gamma_1]]^m, \quad [H^\bullet(X_2, Y_2), [\omega_2], f_*(\gamma_1)]^m$$

could a priori be different motivic periods. However, they are identified with each other by change of variables. Indeed, the period conjecture corresponds to the statement that, whenever different motivic representations of the same period arise, they can always be interrelated by the three equivalence relations in Definition 12.
Definition 13. Let $X$ be a smooth quasi-projective $\mathbb{Q}$-variety, $Y \subset X$ a subvariety and $H = H^\bullet(X, Y)$ the Hodge structure of $X$ relative to $Y$. Assume that $\{[\omega_i]\}_{i=1}^n$ is a basis of the algebraic de Rham cohomology $H^\bullet_{alg-dR}(X, Y)$ and that $\{[\gamma_j]\}_{j=1}^n$ is a basis of the Betti homology $H^\bullet_B(X, Y)$. Denote $\text{per}|_H$ the period map restricted to the motivic periods in $\mathcal{P}^m$ that are built on the given Hodge structure $H$. Observe that $\text{per}|_H$ is entirely determined by the values that it takes when evaluated at $[H, [\omega_j], [\gamma_j]]^m$, that is

$$\text{per}|_H([H, [\omega_j], [\gamma_j]]^m) = \int_{\gamma_j} \omega_j$$

(112)

for each pair of indices $(i, j)$ with $i, j = 1, \ldots, n$. Define the period matrix of $H$ as the $n \times n$-matrix with complex entries $(p_{ij})_{i,j=1,\ldots,n}$ given by

$$p_{ij} = \int_{\gamma_i} \omega_j$$

(113)

The period matrix expresses in a different guise the same information contained in the period map, once it has been restricted to a specific Hodge structure.

Example 10. Let $H = H^1(\mathbb{G}_m, \{1, z\})$. As shown in Examples 6 and 7, a basis of the Betti homology $H^1_B(\mathbb{G}_m, \{1, z\})$ is given by $[\gamma_0]$ and $[\gamma_1]$, and a basis of the algebraic de Rham cohomology $H^1_{alg-dR}(\mathbb{G}_m, \{1, z\})$ is given by $[\omega_0] = \left(\left(\frac{dx}{z^2}, 0, 0\right)\right)$ and $[\omega_1] = \left(\left(\frac{dx}{z-1}, 0, 0\right)\right)$. The period matrix of $H$ is then

$$\begin{pmatrix} 2\pi i & \log(z) \\ 0 & 1 \end{pmatrix}$$

(114)

3.4 Examples

3.4.1 Motivic $2\pi i$

The numeric period $2\pi i$ is given by the contour integral

$$2\pi i = \oint_{\gamma_0} \frac{dx}{x}$$

(115)

where $\gamma_0$ is a counterclockwise cycle encircling the origin in the punctured complex plane $\mathbb{C}^\times$. As observed in Example 5, the complex manifold $\mathbb{C}^\times$ is isomorphic to the topological space of complex points $\mathbb{G}_m(\mathbb{C})$ underlying the $\mathbb{Q}$-algebraic variety $\mathbb{G}_m$. As shown in Examples 4 and 7, we have that

$$H^1_B(\mathbb{G}_m) = \mathbb{Q}[\gamma_0]$$

$$H^1_{alg-dR}(\mathbb{G}_m) = \mathbb{Q}
\begin{pmatrix} dx \\ x \end{pmatrix}$$

(116)

Setting $H^1(\mathbb{G}_m) = (H^1_B(\mathbb{G}_m), H^1_{alg-dR}(\mathbb{G}_m), \text{comp})$, a motivic version of $2\pi i$ is

$$(2\pi i)^m = \left[H^1(\mathbb{G}_m), \left[\frac{dx}{x}, [\gamma_0]\right]\right]^m$$

(117)

which is alternatively represented by the pairing

$$H^1_{alg-dR}(\mathbb{G}_m) \times H^1_B(\mathbb{G}_m) \rightarrow \mathbb{C}$$

$$\left(\left[\frac{dx}{x}\right], [\gamma_0]\right) \mapsto \oint_{\gamma_0} \frac{dx}{x} = 2\pi i$$

(118)

A second integral representation of $2\pi i$ is given by

$$2\pi i = \int_{\mathbb{P}^1(\mathbb{C})} \frac{dz \wedge \bar{dz}}{(1 + z \bar{z})^2}$$

(119)

where $\frac{dz \wedge \bar{dz}}{(1 + z \bar{z})^2}$ is a closed smooth algebraic 2-form over the closed manifold $\mathbb{P}^1(\mathbb{C})$. Because $\mathbb{P}^1(\mathbb{C})$ is compact and Kähler, Theorem 8 applies, giving the Hodge decomposition

$$H^2_{alg-dR}(\mathbb{P}^1(\mathbb{C})) \otimes \mathbb{Q} = \bigoplus_{p+q=2} H^p_{alg-dR}(\mathbb{P}^1)$$

(120)
where the forms in $H_{alg-dR}^{p,q}$ contain $p$ copies of the holomorphic differential $dz$ and $q$ copies of the anti-holomorphic differential $d\bar{z}$. Therefore, $\left[ \frac{dz \wedge d\bar{z}}{(1+z \bar{z})^{2}} \right] \in H_{alg-dR}^{1,1}(\mathbb{P}^{1})$ and integral (119) corresponds to the following motivic period

$$(2\pi i)^{m} = \left[ H^{2}(\mathbb{P}^{1}), \left[ \frac{dz \wedge d\bar{z}}{(1+z \bar{z})^{2}} \right], [\mathbb{P}^{1}(\mathbb{C})] \right]^{m} \quad (121)$$

**Remark.** The two apparently different motivic periods in (117) and (121) are the same, thus preserving the period conjecture. To show this, define

$$A = \mathbb{P}^{1}(\mathbb{C}) \setminus \{ \infty \} \cong \mathbb{C} \subset \mathbb{P}^{1}(\mathbb{C}), \quad B = \mathbb{P}^{1}(\mathbb{C}) \setminus \{ 0 \} \cong \mathbb{C} \subset \mathbb{P}^{1}(\mathbb{C})$$

which satisfy the relations

$$A \cap B \cong \mathbb{C}^{*} \cong G_{m}(\mathbb{C}), \quad A \cup B = \mathbb{P}^{1}(\mathbb{C}) \quad (122)$$

By the Mayer-Vietoris theorem applied to the singular homology groups, the following long exact sequence holds

$$
\begin{array}{cccccccccc}
0 & \longrightarrow & H_{0}^{\ast}(A \cup B) & \longrightarrow & H_{0}^{\ast}(A) \oplus H_{0}^{\ast}(B) & \longrightarrow & H_{1}^{\ast}(A \cap B) & \longrightarrow & H_{1}^{\ast}(A \cup B) & \longrightarrow & H_{1}^{\ast}(A) \oplus H_{1}^{\ast}(B) & \longrightarrow & H_{2}^{\ast}(A \cup B) & \longrightarrow & H_{2}^{\ast}(A) \oplus H_{2}^{\ast}(B) & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow & \downarrow & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
& & H_{1}^{\ast}(A) \oplus H_{1}^{\ast}(B) & & H_{1}^{\ast}(A \cup B) & & H_{1}^{\ast}(A \cap B) & & H_{1}^{\ast}(A) \oplus H_{1}^{\ast}(B) & & H_{1}^{\ast}(A \cup B) & & H_{1}^{\ast}(A) \oplus H_{1}^{\ast}(B) & & H_{2}^{\ast}(A \cup B) & & H_{2}^{\ast}(A) \oplus H_{2}^{\ast}(B) & & \\
\end{array}
$$

(124)

Here, the step $H_{1}^{\ast}(A \cap B) \rightarrow H_{2}^{\ast}(A \cup B)$ is an isomorphism, giving

$$H_{1}^{\ast}(G_{m}(\mathbb{C})) \cong H_{2}^{\ast}(\mathbb{P}^{1}(\mathbb{C})) \quad (125)$$

Similarly, one can prove that the whole Hodge structures $H^{1}(G_{m})$ and $H^{2}(\mathbb{P}^{1})$ are isomorphic and that the change of coordinates occurring between them relates the cohomology classes $\left[ \frac{dz \wedge d\bar{z}}{(1+z \bar{z})^{2}} \right]$ and $\left[ \frac{dx}{x} \right]$ and the homology classes $[\gamma_{0}]$ and $[\mathbb{P}^{1}(\mathbb{C})]$ via pull-back and push-forward maps, respectively.

### 3.4.2 Motivic $\log(z)$

Recall the integral representation of $\log(z)$, $z \in \mathbb{Q}\setminus\{1\}$, given by

$$\log(z) = \int_{1}^{z} \frac{dx}{x} \quad (126)$$

As in the case of $2\pi i$, this is an integral over the punctured complex plane $\mathbb{C}^{*} = G_{m}(\mathbb{C})$. However, contrary to the case of $2\pi i$, where the integration path $\gamma_{0}$ is closed, integral (126) is performed on an open path, precisely any continuous oriented path $\gamma_{1} \subset \mathbb{C}^{*}$, starting at 1 and ending at $z$, which is contractible to the oriented segment from 1 to $z$. The integration path being open requires the framework of relative homology. Let $G_{m}$ be the ambient variety, $\mathbb{C}^{*}$ the underlying topological space and $\{1,z\}$ with $z \in \mathbb{Q}\setminus\{1\}$ a simple normal crossing divisor in $\mathbb{C}^{*}$. As shown in Examples 3 and 8, we have

$$H_{1}^{B}(G_{m}, \{1,z\}) = \mathbb{Q}[\gamma_{0}, \gamma_{1}] \quad (127)$$

$$H_{alg-dR}^{1}(G_{m}, \{1,z\}) = \mathbb{Q}\left[ \left( \frac{dx}{x}, 0, 0 \right), \left( \frac{dx}{z-1}, 0, 0 \right) \right]$$

Setting as usual $H^{1}(G_{m}, \{1,z\}) = (H_{B}^{1}(G_{m}, \{1,z\}), H_{alg-dR}^{1}(G_{m}, \{1,z\}), \text{comp})$, a motivic version of $\log(z)$ is

$$\log(z)^{m} = \left[ H^{1}(G_{m}, \{1,z\}), \left[ \frac{dx}{x}, 0, 0 \right], [\gamma_{1}] \right]^{m} \quad (128)$$

which is alternatively represented by the pairing

$$H_{alg-dR}^{1}(G_{m}, \{1,z\}) \times H_{B}^{1}(G_{m}, \{1,z\}) \rightarrow \mathbb{C}$$

$$\left( \left[ \left( \frac{dx}{x}, 0, 0 \right), [\gamma_{1}] \right] \right) \mapsto \int_{\gamma_{1}} \frac{dx}{x} = \log(z) \quad (129)$$
3.4.3 Elementary Relations

Many relations among numeric periods are simply recast in the formalism of motivic periods. In fact, Hodge structures conjecturally capture all algebraic relations between periods.

**Example 11.** For \( a, b \in \mathbb{Q} \setminus \{1\} \), we have

\[
\log(ab)^m = \log(a)^m + \log(b)^m
\]

(130)

**Example 12.** Consider \( H = H^1(\mathbb{G}_m, \{1,z\}) \) again, and let \( \gamma \) be the union of the paths \( \gamma_0 \) and \( \gamma_1 \) in the punctured complex plane, as shown in Fig. 10.

![Figure 10: The paths \( \gamma_0, \gamma_1 \) and \( \gamma \) in \( \mathbb{C}^* \).](image)

The numeric period obtained by integrating \( \omega_0 \) along \( \gamma \) is

\[
\int_\gamma \omega_0 = \int_{\gamma_0} \omega_0 + \int_{\gamma_1} \omega_0 = 2\pi i + \log(z)
\]

(131)

and in the formalism of motives we have

\[
(2\pi i + \log(z))^m = [H_1, [\omega_0], [\gamma]]^m
\]

\[
= [H, [\omega_0], [\gamma_0 \cup \gamma_1]]^m
\]

\[
= [H, [\omega_0], [\gamma_0]]^m + [H, [\omega_0], [\gamma_1]]^m
\]

\[
= (2\pi i)^m + \log(z)^m
\]

(132)

where we have used that \([H^1(\mathbb{G}_m, \{1,z\}), [\omega_0], [\gamma_0]]^m = [H^1(\mathbb{G}_m), [\omega_0], [\gamma_0]]^m\).

4 Feynman Motives

4.1 Singularities and the Blow Up

Multiple zeta values and convergent Feynman integrals are periods by means of the integral representations (22) and (16), respectively. In both cases, singularities of the integrand can be contained in the domain of integration, a feature that does not occur in the examples of \( 2\pi i \) and \( \log(z) \). Whenever singularities are present, they have to be taken care of with particular attention.

**Example 13.** The period \( \zeta(2) \) is given by the following integral

\[
\zeta(2) = \int_{1 \geq x_1 \geq x_2 \geq 0} \frac{dx_1}{x_1} \frac{dx_2}{1 - x_2}
\]

(133)

over the complex manifold \( \mathbb{C}^2 \). The domain of integration is the simplex

\[
\sigma = \{(x_1, x_2) \in \mathbb{C}^2 | 1 \geq x_1 \geq x_2 \geq 0\}
\]

(134)

and the integrand is the differential 2-form

\[
\omega = \frac{dx_1}{x_1} \frac{dx_2}{1 - x_2}
\]

(135)
Observing that $\mathbb{C}^2$ is isomorphic to the topological space of complex points $\mathbb{A}^2(\mathbb{C})$, underlying the affine $\mathbb{Q}$-algebraic variety $\mathbb{A}^2 = \text{Spec}\mathbb{Q}[x_1, x_2]$, we may try to build $\zeta(2)^m$ as we did for the examples in Section 3.4. Consider the lines $l_0 = \{x_1 = 0\}$ and $l_1 = \{x_2 = 1\}$ in the affine plane $\mathbb{A}^2$. Since $L = l_0 \cup l_1$ is the locus of singular points of $\omega$, the latter is an algebraic 2-form on $X = \mathbb{A}^2\setminus L$. Thus, $[\omega]$ is a class in the second algebraic de Rham cohomology group of $X$ and, consequently, we may want to consider the integral (133) as a period of $X$ relative to some divisor containing the boundary of $\sigma$. In an attempt to do so, define the simple normal crossing divisor

$$D = \{x_1 = x_2\} \cup \{x_1 = 1\} \cup \{x_2 = 0\} \subset \mathbb{A}^2$$

(136)

containing the boundary of $\sigma$. Note that $D$ is not in $X$ because $D \cap L \neq \emptyset$. However, the divisor $D \setminus (D \cap L) \subset X$ does no longer contain $\partial \sigma$. The problem arises from the fact that $\sigma$ itself is not contained in $X$, intersecting the singular locus $L$ in two points

$$p = (0, 0) = \sigma \cap l_0 = D \cap l_0$$

$$q = (1, 1) = \sigma \cap l_1 = D \cap l_1$$

(137)

Removing the singular points $p, q$ from $D$ and considering the second relative Hodge structure $H^2(X, D \setminus (D \cap L))$ does not solve the mentioned technical issue, because $[\sigma]$ is not a class in $H^2_\beta(X, D \setminus (D \cap L))$. See Fig. 11.

![Figure 11: Construction of $\zeta(2)^m$ in the affine plane $\mathbb{A}^2$.](image)

4.2 Motivic Multiple Zeta Values

Consider $\zeta(2)$ again. The blow up of the affine plane $\mathbb{A}^2$ along the singular points $p, q$ is defined as the closed subvariety

$$Y = \text{Blow}_{p,q}(\mathbb{A}^2) \subset \mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$$

(138)

For any positive integer $n$, the $n$-dimensional affine variety over $\mathbb{Q}$ is defined as $\mathbb{A}^n = \text{Spec}\mathbb{Q}[x_1, \ldots, x_n]$. For any field extension $\mathbb{K} \supset \mathbb{Q}$, the space of $\mathbb{K}$-points of $\mathbb{A}^n$ is $\mathbb{A}^n(\mathbb{K}) = \mathbb{K}^n$. The multiplicative group $G_m = \text{Spec}\mathbb{Q}[x, x^{-1}]$ satisfies $G_m = \text{Spec}\mathbb{Q}[x_1, x_2]/(1 - x_1x_2) = \mathbb{A}^1\setminus\{0\} \subset \mathbb{A}^2$, that is, $G_m$ is an hyperbola in $\mathbb{A}^2$.  

![Figure 12: Qualitative illustration of the blow up of the singular points of $\zeta(2)$.](image)
given by the equations

\[ x_1 \alpha_1 = x_2 \beta_1 \]
\[ (x_1 - 1) \alpha_2 = (x_2 - 1) \beta_2 \]  

(139)

where \([\alpha_i : \beta_i], i = 1,2\), are homogeneous coordinates on the two copies of \(\mathbb{P}^1\). The projection of \(Y\) onto the first factor in \(\mathbb{A}^2 \times \mathbb{P}^1 \times \mathbb{P}^1\) is the proper surjective map

\[ \pi : Y \longrightarrow \mathbb{A}^2 \]
\[ (x_1, x_2) \times [\alpha_1 : \beta_1] \times [\alpha_2 : \beta_2] \longmapsto (x_1, x_2) \]  

(140)

The inverse of the projection \(\iota = \pi^{-1}\), mapping the affine plane \(\mathbb{A}^2\) into its blow up \(Y\), replaces the singular points \(p, q \in \mathbb{A}^2\) by corresponding projective lines \(E_p, E_q \subset Y\), called \textit{exceptional divisors}. Precisely, we have

\[ \iota(p) = \iota(0,0) = (0,0) \times \mathbb{P}^1 \times [1 : 1] = E_p \]
\[ \iota(q) = \iota(1,1) = (1,1) \times [1 : 1] \times \mathbb{P}^1 = E_q \]  

(141)

Moreover, the restriction of \(\iota\) to the complement in \(\mathbb{A}^2\) of the singular points \(p, q\)

\[ \iota|_{\mathbb{A}^2 \setminus \{p,q\}} : \mathbb{A}^2 \setminus \{p,q\} \longrightarrow Y \setminus (E_p \cup E_q) \]
\[ (x_1, x_2) \longmapsto (x_1, x_2) \times [1 : 1] \times [1 : 1] \]  

(142)

is an isomorphism. For any closed subset \(C \subset \mathbb{A}^2\), the image \(\iota(C)\) is called \textit{total transform} of \(C\). The \textit{strict transform} of \(C\), denoted \(\hat{C}\), is the closed subset of \(Y\) obtained by first removing the points \(p, q\) if they are in \(C\), then applying \(\iota\), and finally taking the Zariski closure, that is

\[ \hat{C} = \overline{\iota(C \setminus \{p,q\})} \subseteq \iota(C) \]  

(143)

It follows that the strict transforms of \(l_0, l_1\) are the affine lines

\[ L_0 = \hat{l}_0 = \{(0, x_2) \times [1 : 0] \times [1 - x_2 : 1] \mid x_2 \in \mathbb{A}^1\} \]
\[ L_1 = \hat{l}_1 = \{(x_1, 1) \times [1 : x_1] \times [0 : 1] \mid x_1 \in \mathbb{A}^1\} \]  

(144)

and their total transforms are

\[ \iota(l_0) = L_0 \cup E_p \]
\[ \iota(l_1) = L_1 \cup E_q \]  

(145)

We observe that \(L_0, E_p\) and \(L_1, E_q\) intersect in only one point each. Precisely

\[ L_0 \cap E_p = \{(0, 0) \times [1 : 0] \times [1 : 1]\} \]
\[ L_1 \cap E_q = \{(1, 1) \times [1 : 1] \times [0 : 1]\} \]  

(146)

Moreover, we have

\[ L_1 \cap E_p = \emptyset = L_0 \cap E_q \]
\[ L_1 \cap L_0 = \{(0, 1) \times [1 : 0] \times [0 : 1]\} \]
\[ L_0 \cap E_q = \emptyset \]  

(147)

Similarly, we define the strict transform \(\hat{\sigma}\) of the domain of integration. Observing that the closed points of \(E_p\) can be interpreted as lines passing through \(p\), and analogously that the closed points of \(E_q\) can be interpreted as lines passing through \(q\), we obtain

\[ \hat{\sigma} \cap E_p = \{(0, 0) \times [t : 1] \times [1 : 1] \mid 0 \leq t \leq 1\} \]
\[ \hat{\sigma} \cap E_q = \{(1, 1) \times [1 : t] \times [1 : 1] \mid 0 \leq t \leq 1\} \]  

(148)

which, combined with (146), imply that

\[ \hat{\sigma} \cap L_0 = \emptyset, \quad \hat{\sigma} \cap L_1 = \emptyset \]  

(149)

See Fig. [13] for a graphical representation of the blow up.
Figure 13: The strict transform of $\sigma$ in the blow up $Y$.

As the map $\iota$ is applied to the ambient variety, giving the reshaped domain $\hat{\sigma}$, the differential form $\omega$ is replaced by its pull-back $\pi^*(\omega)$, denoted by $\hat{\omega}$. Let us now show that the pull-back $\hat{\omega}$ is only singular on the strict transform $L = L_0 \cup L_1$. We use local coordinates on the blow up $Y$. In particular, consider a patch of $Y$ around the point $L_0 \cap E_p$ as shown in Fig. 14.

Figure 14: Local patch of $Y$ around the intersection of $L_0$ and $E_p$.

Here, a local system of coordinates is explicitly given by

$$t = \frac{x_1}{x_2} = \frac{\beta_1}{\alpha_1}, \quad s = x_2$$

where $L_0$ and $E_p$ have equations $t = 0$ and $s = 0$, respectively. Applying this change of variables to $\hat{\omega}$, we have

$$\hat{\omega} = \frac{d(st)}{st} \wedge \frac{ds}{1-s} = \frac{ds}{s} \wedge \frac{ds}{1-s} + \frac{dt}{t} \wedge \frac{ds}{1-s} = \frac{dt}{t} \wedge \frac{ds}{1-s}$$

It follows that $\hat{\omega}$ is singular along the strict transform $L_0$, while it is smooth along the exceptional divisor $E_p$, because it has no pole at $s = 0$. Analogously, we find that $\hat{\omega}$ is singular along $L_1$, but not along $E_q$. Then, the singular locus of $\hat{\omega}$ is $L$. Observe that the complement $Y \setminus L$ is the closed affine subvariety of $\mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{A}^1$ given by the equations

$$x_1 t = x_2$$

$$x_1 - 1 = (x_2 - 1)s$$

where $t, s$ are affine coordinates on the two copies of $\mathbb{A}^1$. Therefore, the differential form $\hat{\omega}$ determines a class in $H^2_{\text{alg-dR}}(Y \setminus L)$. Moreover, it follows from (149) that, moving from the original affine plane $\mathbb{A}^2$ to the blow up $Y$, the singular locus of the differential form $\omega$ and the domain of integration $\hat{\sigma}$ are disjoint. As usual, we may want to consider the integral (133) as a period of $Y \setminus L$ relative to some divisor containing the boundary of $\hat{\sigma}$. The blow up construction is thus successful for the period $\zeta(2)$ if $[\hat{\sigma}]$ turns out to be a class in the given relative Betti homology group. To see this, recall that $\partial \sigma$ is contained in the union $D$ of the affine lines

$$m_1 = \{x_1 = x_2\}, \quad m_2 = \{x_1 = 1\}, \quad m_3 = \{x_2 = 0\}$$

Thus, we naturally consider the normal crossing divisor $M \subset Y$ defined by

$$M = \iota(D) = \iota(m_1 \cup m_2 \cup m_3) = E_p \cup E_q \cup M_1 \cup M_2 \cup M_3$$

In principle, $\hat{\omega}$ might have singularities along the total transform of $L_0 \cup L_1$, i.e. $L_0 \cup L_1 \cup E_p \cup E_q$. However, in the case of $\zeta(2)$, it turns out that $\hat{\omega}$ has no singularities along the exceptional divisors. More generally, this condition determines whether the blow up prescription turns out to be successful or not for a given period.
Indeed, the pairing of \( \hat{H} \) Setting the Hodge structure Besides, the restriction of \( \hat{H}, i = 1, 2, 3 \) of \( M \) gives zero, implying

\[
[\hat{\sigma}] \in H^2_{alg-dR}(Y \setminus L, M \setminus (M \cap L))
\]  

(155)

Besides, the restriction of \( \hat{\omega} \) to every irreducible component \( M_i, i = 1, 2, 3 \), of \( M \) gives zero, implying

\[
[\hat{\omega}] \in H^2_{alg-dR}(Y \setminus L, M \setminus (M \cap L))
\]  

(156)

Setting the Hodge structure \( H = H^2(Y \setminus L, M \setminus (M \cap L)) \), the resulting motivic version of \( \zeta(2) \) is

\[
\zeta(2)^m = [H, [\hat{\omega}], [\hat{\sigma}]^m]
\]  

(157)

Indeed, the pairing of \([\hat{\sigma}]\) and \([\hat{\omega}]\) yields

\[
\int_{\hat{\sigma}} \hat{\omega} = \int_{\hat{\sigma}} \pi^*(\omega) = \int_{\sigma_{\hat{\sigma}}} \omega = \int_{\sigma} \omega = \zeta(2)
\]  

(158)

by the equivalence relation under change of variables in \( P^m \). We observe that the whole period matrix of \( H \) is

\[
\begin{pmatrix} (2\pi i)^2 & \zeta(2) \\ 0 & 1 \end{pmatrix}
\]  

(159)

### 4.3 Motivic Feynman Integrals

In an attempt to overcome singularity issues, the blow up procedure can be similarly applied to generic MZVs and other families of periods, such as convergent Feynman integrals. For an exposition of the general computation of the Hodge structure of a blow up we refer to Voisin [61].

Let \( G \) be a primitive log-divergent Feynman graph, \( E_G \) the collection of its edges and \( n_G = |E_G| \), as in Section 1.4. Recall that \( x_e \) denotes the Schwinger parameter associated to \( e \in E_G \), and \( \Psi_G, I_G, \) and \( X_G \) denote the first graph polynomial, the Feynman integral, and the graph hypersurface, as given in [10], [16], and [17], respectively. Denote by \( \omega_G \) and \( \sigma \) the integrand and the domain of integration of \( I_G \). Since \( \omega_G \) is a top-degree algebraic differential form on \( P^{n_G-1} \setminus X_G \), and \( \partial \sigma \) is contained in the union \( D \) of the coordinate hyperplanes \( \{x_e = 0, e \in E_G\} \), we may intuitively try to build the motive \( I_G^m \) on the relative Hodge structure

\[
H^{n_G-1}(P^{n_G-1} \setminus X_G, D \setminus (D \cap X_G))
\]  

(160)

However, this naïve attempt fails whenever the hypersurface \( X_G \) intersects the integration cycle \( \sigma \) non-trivially, implying the presence of non-negligible singularities. Whenever singularities are present, \( \sigma \) does not define an element in the corresponding naïve relative Betti homology group. To successfully build the motive \( I_G^m \) in the presence of singularities, the blow up technique is applied.

A linear subvariety \( L \subset P^{n_G-1} \) defined by the vanishing of a subset of the set of Schwinger parameters is called a coordinate linear space, while its subspace of real points with non-negative coordinates is denoted by

\[
L(\mathbb{R}_{\geq 0}) = \{(x_e)_{e \in E_G} \in L \mid x_e \in \mathbb{R}_{\geq 0}\}
\]  

(161)

Since the coefficients of \( \Psi_G \) are positive, the locus of problematic singularities is

\[
\sigma \cap X_G(\mathbb{C}) = \bigcup_{L \subset X_G} L(\mathbb{R}_{\geq 0})
\]  

(162)

where the union is taken over all coordinate linear spaces \( L \subset X_G \).

**Remark.** The coordinate linear spaces \( L \subset X_G \) are in one-to-one correspondence with the subgraphs \( \gamma \subset G \) such that \( l_\gamma > 0 \). It follows

\[
\sigma \cap X_G(\mathbb{C}) = \bigcup_{\gamma \subset G} L_\gamma(\mathbb{R}_{\geq 0})
\]  

(163)

where the union is taken over all subgraphs \( \gamma \subset G \) with \( l_\gamma > 0 \). Here, \( L_\gamma \) is the linear subvariety of \( P^{n_G-1} \) defined by the equations \( \{x_e = 0, e \in E_\gamma\} \).

The following theorem is proven, and an explicit algorithmic construction of the blow ups is given, by Bloch, Esnault and Kreimer [5].
Theorem 9. Let $G$ be a primitive log-divergent Feynman graph such that every proper subgraph of $G$ is primitive. There exists a tower

$$
\pi : P = P_r \to P_{r-1} \to \ldots \to P_1 \to P_0 = \mathbb{P}^{n_G-1}
$$

such that, for each $i = 1, \ldots, r$, $P_i$ is obtained by blowing up $P_{i-1}$ along the strict transform of a coordinate linear space $L_i \subset X_G$, and the following conditions hold:

1. The pulled-back differential $\hat{\omega}_G = \pi^*\omega_G$ has no poles along the exceptional divisors associated to the blow ups.
2. Let $B$ be the total transform of $D$ in $P$, i.e.

$$
B = \iota(D) = \pi^{-1}\left( \bigcup_{e \in E_G} \{ x_e = 0 \} \right)
$$

Then, $B \subset P$ is a normal crossing divisor such that none of the non-empty intersections of its irreducible components is contained in the strict transform $Y_G$ of $X_G$ in $P$.

3. The strict transform of $\sigma$ in $P$ does not meet $Y_G$, that is, $\hat{\sigma} \cap Y_G(\mathbb{C}) = \emptyset$.

As a consequence of Theorem 9, the motive $I^G_B$ associated to any subdivergence-free primitive log-divergent Feynman graph $G$ can be written explicitly. Being $\partial \hat{\sigma} \subset B \setminus (B \cap Y_G)$, the domain of integration defines the class

$$
[\hat{\sigma}] \in H_{n_G-1}^B(P \setminus Y_G, B \setminus (B \cap Y_G))
$$

called Betti framing, while the integrand defines the class

$$
[\hat{\omega}_G] \in H_{n_G-1}^B(\mathbb{P} \setminus Y_G, \mathbb{P} \setminus (B \cap Y_G))
$$

called de Rham framing. Brown and Doryn present a method for explicit computation of the framings on the cohomology of Feynman graph hypersurfaces. Then, the Hodge structure $H = H^{n_G-1}(P \setminus Y_G, B \setminus (B \cap Y_G))$ is called the graph Hodge structure and the motivic Feynman integral $I^G_B$ is given by

$$
I^G_B = [H, [\hat{\omega}_G], [\hat{\sigma}]]^m
$$

Indeed, the pairing of the classes $[\hat{\omega}_G]$ and $[\hat{\sigma}]$ yields the period

$$
\int_{\hat{\sigma}} \hat{\omega}_G = \int_{\hat{\sigma}} \pi^* (\omega_G) = \int_{\pi^*(\hat{\sigma})} \omega_G = \int_{\sigma} \omega_G = I_G
$$

by the equivalence relation under change of variables in $\mathcal{P}^m$.

Example 14. Adopting the following notation

$$
\mathcal{P}_{\text{log}} = \mathcal{Q}(I_G \mid G \text{ is a primitive log-divergent Feynman graph})
$$

$$
\mathcal{P}_{\phi^4} = \mathcal{Q}(I_G \mid G \text{ is a primitive log-divergent Feynman graph in $\phi^4$ theory})
$$

we observe that the sequence of inclusions $\mathcal{P}_{\phi^4} \subset \mathcal{P}_{\text{log}} \subset \mathcal{P}$ is preserved after promoting numeric periods to periods of motives, that is, $\mathcal{P}_{\phi^4}^m \subset \mathcal{P}_{\text{log}}^m \subset \mathcal{P}^m$ holds.

4.4 Tannakian Formalism

We briefly introduce the fundamentals of the theory of Tannakian categories, following the more detailed and comprehensive exposition by Deligne et al. The concept of a Tannakian category was first introduced by Saavedra Rivano to encode the properties of the category $\text{Rep}_K(G)$ of the finite-dimensional $K$-linear representations of an affine group scheme $G$ over a field $K$. Let us recall some preliminary notions in category theory. In the following, $K$ is a given field.

Definition 14. A $K$-linear category $\mathcal{C}$ is an additive category such that, for each pair of objects $X, Y \in \text{Ob}(\mathcal{C})$, the group $\text{Hom}_\mathcal{C}(X, Y)$ is a $K$-vector space and the composition maps are $K$-bilinear.

---

21 The graph Hodge structure is also explicitly known in the general case of renormalised amplitudes of single-scale graphs due to the work of Brown and Kreimer, who pave the way for the rigorous investigation of divergent Feynman graphs and their renormalised amplitudes from an algebro-geometric perspective.
Definition 15. Let $\mathcal{C}$ be a $K$-linear category endowed with a $K$-bilinear functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(a) An associativity constraint for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\phi = \phi_{\cdot, \cdot} : \cdot \otimes (\cdot \otimes \cdot) \to (\cdot \otimes \cdot) \otimes \cdot$$

such that the following two conditions hold:

(a.1) For all $X, Y, Z \in \text{Ob}(\mathcal{C})$, the map $\phi_{X,Y,Z}$ is an isomorphism.

(a.2) For all $X, Y, Z, T \in \text{Ob}(\mathcal{C})$, the following diagram commutes

\[
\begin{array}{ccc}
X \otimes (Y \otimes (Z \otimes T)) & \overset{1 \otimes \phi_{Y,Z,T}}{\longrightarrow} & (X \otimes Y) \otimes (Z \otimes T) \\
\phi_{X,Y \otimes Z,T} & & \phi_{X \otimes Y,Z,T} \\
(X \otimes (Y \otimes Z)) \otimes T & \overset{\phi_{X,Y,Z} \otimes \text{Id}}{\longrightarrow} & ((X \otimes Y) \otimes Z) \otimes T \\
\end{array}
\]

(b) A commutativity constraint for $(\mathcal{C}, \otimes)$ is a natural transformation

$$\psi = \psi_{\cdot, \cdot} : \cdot \otimes \cdot \otimes \cdot \to \cdot \otimes \cdot$$

such that the following two conditions hold:

(b.1) For all $X, Y \in \text{Ob}(\mathcal{C})$, the map $\psi_{X,Y}$ is an isomorphism.

(b.2) For all $X, Y \in \text{Ob}(\mathcal{C})$, the following composition is the identity

$$\psi_{Y,X} \circ \psi_{X,Y} : X \otimes Y \to X \otimes Y$$

(c) An associativity constraint and a commutativity constraint are compatible if, for all $X, Y, Z \in \text{Ob}(\mathcal{C})$, the following diagram commutes

\[
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \overset{\phi_{X,Y,Z}}{\longrightarrow} & (X \otimes Y) \otimes Z \\
\phi_{X,Z,Y} & & \psi_{X \otimes Y,Z} \\
(X \otimes Z) \otimes Y & \overset{\psi_{X,Z,Y} \otimes \text{Id}}{\longrightarrow} & (Z \otimes X) \otimes Y \\
\end{array}
\]

(d) A pair $(U, u)$ consisting of an object $U \in \text{Ob}(\mathcal{C})$ and an isomorphism $u : U \to U \otimes X$ is an identity object if the functor $X \mapsto U \otimes X$ is an equivalence of categories.

Definition 16. A $K$-linear tensor category is a tuple $(\mathcal{C}, \otimes, \phi, \psi)$ consisting of a $K$-linear category $\mathcal{C}$, a $K$-bilinear functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and compatible associativity and commutativity constraints $\phi, \psi$ such that $\mathcal{C}$ contains an identity object.

Definition 17. An object $L \in \text{Ob}(\mathcal{C})$ is invertible if the functor $X \mapsto L \otimes X$ is an equivalence of categories. Equivalently, $L$ is invertible if and only if there exists an object $L' \in \text{Ob}(\mathcal{C})$ such that $L \otimes L' \simeq 1$. Then, $L'$ is also invertible.

Definition 18. Let $(\mathcal{C}, \otimes)$ be a $K$-linear tensor category, where we omit the constraints $\phi, \psi$ for simplicity, and let $X, Y \in \text{Ob}(\mathcal{C})$. Assume that there exists an object $Z \in \text{Ob}(\mathcal{C})$ such that, for all $T \in \text{Ob}(\mathcal{C})$, the functors $T \mapsto \text{Hom}(T, Z)$ and $T \mapsto \text{Hom}(T \otimes X, Y)$ admit a functorial isomorphism

$$\text{Hom}(T, Z) \cong \text{Hom}(T \otimes X, Y)$$

In this case, the functor $T \mapsto \text{Hom}(T \otimes X, Y)$ is said to be representable and the object $Z$ is called the internal Hom between the objects $X$ and $Y$. It is alternatively written as $\text{Hom}(X, Y)$ and it is unique up to isomorphism.

Definition 19. The dual of an object $X \in \text{Ob}(\mathcal{C})$ is defined as $X^\vee = \text{Hom}(X, 1)$. If $X^\vee$ and $(X^\vee)^\vee$ exist, then there is a natural morphism $X \mapsto (X^\vee)^\vee$, and the object $X$ is reflexive if such a morphism is an isomorphism.
Definition 20. A \( K \)-linear tensor category \((C, \otimes)\) is rigid if the following conditions hold:

1. For all \( X, Y \in \text{Ob}(C) \), \( \text{Hom}(X, Y) \) exists.
2. For all \( X_1, X_2, Y_1, Y_2 \in \text{Ob}(C) \), the natural morphism

\[
\text{Hom}(X_1, Y_1) \otimes \text{Hom}(X_2, Y_2) \rightarrow \text{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2)
\]

is an isomorphism.
3. All objects are reflexive.

Definition 21. A Tannakian category over the field \( K \) is a rigid abelian \( K \)-linear tensor category \( T \) such that \( \text{End}(1) = K \), and there exists an exact faithful \( K \)-linear tensor functor \( \omega : T \rightarrow \text{Vec}_K \), where \( \text{Vec}_K \) is the category of finite-dimensional vector spaces over \( K \). Any such functor is called a fibre functor.

Example 15. The category \( \text{Vec}_K \) of finite-dimensional \( K \)-vector spaces, together with the identity functor, is a Tannakian category over \( K \).

Example 16. The category \( \text{GrVec}_K \) of finite-dimensional graded \( K \)-vector spaces, together with the forgetful functor \( \omega : \text{GrVec}_K \rightarrow \text{Vec}_K \), sending \((V, (V_n)_{n \in \mathbb{Z}})\) to \( V \), is a Tannakian category over \( K \).

Example 17. The category \( \text{Rep}_K(G) \) of finite-dimensional \( K \)-linear representations of an abstract group \( G \), together with the functor \( \omega : \text{Rep}_K(G) \rightarrow \text{Vec}_K \) that forgets the action of \( G \), is a Tannakian category over \( K \).

Let us fix a Tannakian category \( T \) over \( K \) and a fibre functor \( \omega \) of \( T \). Let \( R \) be a \( K \)-algebra. We denote by \( \text{Aut}^\otimes(\omega)(R) \) the collection of families \((\lambda_X)_{X \in \text{Ob}(T)}\) of \( R \)-linear automorphisms

\[
\lambda_X : \omega(X) \otimes_R R \rightarrow \omega(X) \otimes_R R
\]

which are compatible with the tensor structure and functorial. Here, compatibility with the tensor structure and functoriality mean that:

1. For all \( X_1, X_2 \in \text{Ob}(T) \), the following diagram commutes

\[
\begin{array}{ccc}
\omega(X_1 \otimes X_2) \otimes R & \xrightarrow{\lambda_{X_1 \otimes X_2}} & \omega(X_1 \otimes X_2) \otimes R \\
\downarrow & & \downarrow \\
\omega(X_1) \otimes \omega(X_2) \otimes R & & \omega(X_1) \otimes \omega(X_2) \otimes R \\
\downarrow & & \downarrow \\
(\omega(X_1) \otimes R) \otimes_R (\omega(X_2) \otimes R) & \xrightarrow{\lambda_{X_1} \otimes_R \lambda_{X_2}} & (\omega(X_1) \otimes R) \otimes_R (\omega(X_2) \otimes R)
\end{array}
\]

(179)

2. The following diagram commutes

\[
\begin{array}{ccc}
\omega(1) \otimes R & \xrightarrow{\lambda_1} & \omega(1) \otimes R \\
\downarrow & & \downarrow \\
R & \xrightarrow{\text{Id}} & R
\end{array}
\]

(180)

3. For all \( X, Y \in \text{Ob}(T) \) and for every morphism \( \alpha \in \text{Hom}(X, Y) \), the following diagram commutes

\[
\begin{array}{ccc}
\omega(X) \otimes R & \xrightarrow{\lambda_X} & \omega(X) \otimes R \\
\omega(\alpha) \otimes \text{Id} & & \omega(\alpha) \otimes \text{Id} \\
\omega(Y) \otimes R & \xrightarrow{\lambda_Y} & \omega(Y) \otimes R
\end{array}
\]

(181)

Denote \( \text{Aut}^\otimes(\omega) = \text{Aut}^\otimes(\omega)(K) \) the group of \( K \)-linear automorphisms of the fibre functor \( \omega \). Deligne et al.\textsuperscript{22} proved that all Tannakian categories are categories of finite-dimensional linear representations of a pro-algebraic group.

Theorem 10. Let \( T \) be a Tannakian category over \( K \) with a fibre functor \( \omega \).

1. The functor \( R \mapsto \text{Aut}^\otimes(\omega)(R) \) is representable by an affine group scheme over \( K \), which is denoted as \( \text{Aut}^\otimes(\omega) \) or \( G^\omega \), and is called the Tannaka group of the pair \((T, \omega)\).

\textsuperscript{22}In the given diagrams, all unlabelled tensor products are over \( K \) and all unlabelled arrows are the natural isomorphisms.
(2) For every $X \in \text{Ob}(\mathcal{T})$, the group $\text{Aut}^\otimes(\omega)$ acts naturally on $\omega(X)$ and the functor

$$
\begin{align*}
\mathcal{T} & \longrightarrow \text{Rep}_K(G^\otimes) \\
X & \longmapsto \omega(X)
\end{align*}
$$

(182)

sending $X$ to the vector space $\omega(X)$ with this action of $\text{Aut}^\otimes(\omega)$, is an equivalence of categories.

Given a second fibre functor $\omega'$, we analogously define $\text{Isom}^\otimes(\omega, \omega')(R)$ to be the collection of families $(\tau_X)_{X \in \text{Ob}(\mathcal{T})}$ of $R$-linear isomorphisms

$$
\tau_X : \omega(X) \otimes_K R \longrightarrow \omega'(X) \otimes_K R
$$

(183)

which are compatible with the tensor structure and functorial. Again, we denote $\text{Isom}^\otimes(\omega, \omega') = \text{Isom}^\otimes(\omega, \omega')(K)$. Deligne et al [20] proved the following result.

**Theorem 11.** Let $\mathcal{T}$ be a Tannakian category over $K$ with two fibre functors $\omega$ and $\omega'$. The functor $R \mapsto \text{Isom}^\otimes(\omega, \omega')(R)$ is representable by an affine scheme over $K$, which is denoted as $\text{Isom}^\otimes(\omega, \omega')$, and is a right torsor under $\text{Aut}^\otimes(\omega)$ and a left torsor under $\text{Aut}^\otimes(\omega')$.

### 4.5 Motivic Galois Theory

Grothendieck’s idea of a universal cohomology theory taking values in a $\mathbb{Q}$-category of motives $\mathcal{M}$ is intimately connected to the theory of Hodge structures. Recall the rigorous notions of pure and mixed Hodge structures over $\mathbb{Q}$, given in Sections 2.3 and 2.4. On the one hand, the cohomology of a smooth projective $\mathbb{Q}$-variety is fundamentally described by a pure Hodge structure. On the other hand, applying the resolution of singularities by Hironaka [30], the cohomology of a singular quasi-projective $\mathbb{Q}$-variety can be expressed in terms of cohomologies of smooth projective varieties, and since cohomologies of different degrees get mixed in this expression, it is fundamentally described by a mixed Hodge structure. Thus, enhancing the naive description in Section 3.2 pure Hodge structures represent suitable candidates to actualise the idea of motives of smooth projective varieties proposed by Grothendieck. Similarly, mixed Hodge structures potentially represent motives of singular or quasi-projective varieties. Specifically looking at the application of Hodge theory to the theory of motivic periods, we identify the category of motives with $\mathbb{Q}$-linear representations of the motivic Galois group, that is

$$
\mathcal{M} \simeq \text{Rep}_\mathbb{Q}(G_{dR})
$$

(184)

**Remark.** We observe that the motivic Galois group can alternatively be realised via Betti cohomology as $G^B = \text{Aut}^\otimes(\omega_B)$, and the corresponding category of finite-dimensional $\mathbb{Q}$-linear representations is still the same category of motives $\mathcal{M}$.

In Tannakian formalism, the space of motivic periods $\mathcal{P}^m$ is expressed as

$$
\mathcal{P}^m = \mathbb{Q}([M, \omega, \sigma]^m \mid M \in \text{Ob}(\mathcal{M}), \omega \in \omega_{dR}(M), \sigma \in \omega_B(M)')
$$

(185)

with implicit factorisation modulo bilinearity and functoriality. Thus, an alternative but equivalent description of motivic periods is obtained. Indeed, $\mathcal{P}^m$ is isomorphic to the space of regular functions on the affine $\mathbb{Q}$-scheme $\text{Isom}^\otimes(\omega_{dR}, \omega_B)$, that is

$$
\mathcal{P}^m \simeq \mathcal{O}(\text{Isom}^\otimes(\omega_{dR}, \omega_B))
$$

(186)

The isomorphism is made explicit by

$$
\mathcal{P}^m \longrightarrow \mathcal{O}(\text{Isom}^\otimes(\omega_{dR}, \omega_B))
$$

(187)

$$
[M, \omega, \sigma]^m \longmapsto \left( (\lambda_X)_{X \in \text{Ob}(\mathcal{M})} \mapsto \sigma \circ \lambda_M \right)
$$

where $\sigma \circ \lambda_M$ gives

$$
\begin{align*}
\omega_{dR}(M) & \xrightarrow{\lambda_M} \omega_B(M) \\
\omega & \longmapsto \lambda_M(\omega) \longmapsto \sigma(\lambda_M(\omega))
\end{align*}
$$

(188)
Then, following Theorem 10, the motivic Galois group $G^{dR}$ has a natural action on $\text{Isom}^\otimes(\omega_{dR}, \omega_B)$ denoted by

$$\nabla : G^{dR} \otimes \text{Isom}^\otimes(\omega_{dR}, \omega_B) \to \text{Isom}^\otimes(\omega_{dR}, \omega_B)$$

which induces a dual coaction on the corresponding spaces of regular functions

$$\Delta : \mathcal{P}^m \longrightarrow \mathcal{O}(G^{dR}) \otimes \mathcal{P}^m$$

where $\{e_i\}$ is a basis of $\omega_{dR}(M)$ and $\{e'_i\}$ is the dual basis, called Galois coaction. We denote $\mathcal{P}^{dR} = \mathcal{O}(G^{dR})$ the dual of the motivic Galois group and call it the space of de Rham periods.

**Remark.** Note that the space of de Rham periods is naturally a Hopf algebra, while the space of motivic periods is not, although we can associate symbols to them. Thus, the Galois coaction turns the finite-dimensional $\mathbb{Q}$-vector space $\mathcal{P}^m$ into a comodule over the Hopf algebra $\mathcal{P}^{dR}$. A detailed discussion is presented by Brown [13, 12].

**Example 18.** Consider the motivic logarithm $\log(z)^m$ for $z \in \mathbb{Q}\setminus\{1\}$. Following Section 3.4.2 we have

$$\log(z)^m = \left[ H^1(G_m, \{1, z\}), \left[ \frac{dx}{x}, \gamma_1 \right]^m \right]$$

where we write $\left[ \frac{dx}{x} \right] = \left[ \left( \frac{dx}{x}, 0, 0 \right) \right]$ for simplicity. Thus, the corresponding motive is $M = H^1(G_m, \{1, z\})$, while the complete period matrix of $M$ is

$$\left( \begin{array}{c} 2\pi i \\ 0 \\ 1 \end{array} \right)$$

Direct application of the prescription in (190) gives the explicit decomposition

$$\Delta \left[ M, \left[ \frac{dx}{x} \right], \gamma_1 \right]^m = \left[ M, \left[ \frac{dx}{x} \right], \left[ \frac{dx}{x - 1} \right]^m \right] \otimes \left[ M, \left[ \frac{dx}{x - 1} \right], \gamma_1 \right]^m$$

which is equivalent to

$$\Delta \log(z)^m = \log(z)^{dR} \otimes 1^m + (2\pi i)^{dR} \otimes \log(z)^m$$

Here, $1^m$ and $\log(z)^m$ are called Galois conjugates of $\log(z)^m$.

**Example 19.** As for $\log(z)^m$, the Galois coaction of the motivic multiple zeta values $\zeta(s)^m$ can be computed explicitly. In particular, for $n \geq 1$, we have

$$\Delta \zeta(2)^m = 1^{dR} \otimes \zeta(2)^m$$

$$\Delta \zeta(2n + 1)^m = \zeta(2n + 1)^{dR} \otimes 1^m + 1^{dR} \otimes \zeta(2n + 1)^m$$

Thus, the Galois coaction is trivial on $\zeta(2)^m$, while $\zeta(2n + 1)^m$ has the non-trivial Galois conjugate $1^m$. Moreover

$$\Delta(\zeta(2)^m \zeta(2n + 1)^m) = \zeta(2n + 1)^{dR} \otimes \zeta(2)^m + 1^{dR} \otimes \zeta(2)^m \zeta(2n + 1)^m$$

### 4.6 Coaction Conjecture

We look at the example of scalar massless $\phi^4$ quantum field theory and consider the Galois coaction restricted to $\mathcal{P}^m_{\phi^4}$. This is a priori valued in the whole space $\mathcal{P}^{dR} \otimes \mathcal{P}^m$. However, after computing every known $\phi^4$-amplitude with loop order at most 7 and explicitly verifying that in each case the Galois coaction preserves the space $\mathcal{P}^m_{\phi^4}$, Panzer and Schnetz [23, 17] proposed the following conjecture, known as the coaction conjecture.

**Conjecture 3.** Galois conjugates of $\phi^4$-periods are still $\phi^4$-periods, i.e.

$$\Delta(\mathcal{P}^m_{\phi^4}) \subseteq \mathcal{P}^{dR} \otimes \mathcal{P}^m_{\phi^4}$$

Panzer and Schnetz [23, 17] explicitly computed the first examples of $\phi^4$-amplitudes which are not MZVs. Such numbers are polylogarithms at 2nd and 6th roots of unity. The coaction conjecture is verified for them as well.
Such a conjecture implies the existence of a fundamental hidden symmetry underlying the class of $\phi^4$-periods that we do not yet properly understand. Indeed, the unexpected observations by Panzer and Schnetz, and the resulting conjecture, have greatly stimulated research, motivating the search for a mathematical mechanism able to distinguish $\phi^4$-periods from periods of all graphs, and thus explain this surprising evidence.

A first advancement in this direction has already been made. Suitably enlarging the space of amplitudes under consideration, the coaction conjecture is proven by Brown [12]. Define the finite-dimensional $\phi^4$-vector space $P_{\phi^4}$ associated to a $\phi^4$-graph $G$ to be the space of motivic versions of all integrals of the form

$$I_G = \int_P \frac{P(\{x_e\})}{\Phi_G} \Omega$$

(198)

where $k \geq 1$ is an integer, and $P$ is any polynomial in $\mathbb{Q}[\{x_e\}]$ such that $I_G$ converges.

Theorem 12. $P_{\phi^4}$ is stable under the Galois coaction, i.e. $\Delta(P_{\phi^4}) \subseteq P_{dR} \otimes P_{\phi^4}$.

### 4.7 Weights and the Small Graph Principle

We apply the notions of Hodge and weight filtrations, introduced in Sections 2.3 and 2.4, to the theory of motivic periods. For $M \in \text{Ob}(\mathcal{M})$, the $\mathbb{Q}$-vector space $\omega_{dR}(M)$ is equipped with a decreasing Hodge filtration $F$ and an increasing weight filtration $W_{dR}$, while $\mathbb{Q}$-vector space $\omega_B(M)$ is provided with a weight filtration $W_B$ only. Mixed Hodge structures, contrary to pure ones, do not have a well-defined weight. However, the graded quotients with respect to the weight filtration do possess a pure Hodge structure of definite weight, as described in Definition 8. These properties are used to define a notion of weight for motivic periods.

**Definition 22.** The weight filtration on $\omega_{dR}(M)$ induces a weight filtration on the space of motivic periods by

$$W_{dR}P^m = \mathbb{Q}[\{M,\omega,\sigma\}^m | \omega \in W_{dR}\omega_{dR}(M))$$

(199)

Denote $W = W_{dR}$ for simplicity. A given motivic period $[M,\omega,\sigma]^m$ is said to have weight at most $n$ if it belongs to $W_nP^m$, and it has weight $n$ if it belongs to the graded quotient $G_nW^m = W_nP^m/W_{n-1}P^m$.

**Remark.** We observe that the weight of motivic periods can alternatively, but equivalently, be defined from the Betti side via the weight filtration induced on $P^m$ by $W_B$.

**Example 20.** Consider $M = H^1(\mathbb{G}_m,\{1,z\})$ again. Its weight filtration in de Rham realisation is

$$W_{-1} = 0 \subseteq W_0 = W_1 = \mathbb{Q}(0) \subseteq W_2 = H^1(\mathbb{G}_m,\{1,z\})$$

(200)

Observing that $0, 1 \in W_0$ and $2\pi i, \log(z) \in W_2$, the weight of each entry of the period matrix of $M$ is determined. Indeed, $0, 1$ are periods of weight zero, while $2\pi i, \log(z)$ have weight 2.

**Example 21.** The weight filtration can be used to systematically study $P_{\phi^4}$ weight by weight. For example, direct computation in low weight shows that

$$W_0P_{\phi^4} = W_1P_{\phi^4} = W_2P_{\phi^4} = \mathbb{Q}(0)$$

(201)

The following conjecture, known as small graph principle, is due to Brown [12].

**Conjecture 4.** Let $G$ be a primitive log-divergent Feynman graph in scalar massless $\phi^4$ theory. Denote by $[M_G,\omega_G,\sigma]^m$ the explicit form of its motivic Feynman integral $I_G^m$. The elements in the right-hand side of the coaction formula for $\Delta[M_G,\omega_G,\sigma]^m$ can be expressed in the form

$$\prod_{i}[M_{\gamma_i},\omega_{\gamma_i},\sigma]^m$$

(202)

where the product runs over a subset $\{\gamma_i\}$ of the set of subgraphs and quotient graphs of $G$.

The small graph principle implies that the Galois conjugates of weight at most $k$ of the motivic amplitude of a primitive Feynman graph are associated to its sub-quotient graphs with at most $k + 1$ edges. Thus, when interested in periods of weight at most $k$, it suggests to look at graphs with at most $k + 1$ edges. It follows that the topology of a given graph constrains the Galois theory of its amplitudes. The following theorem is proven by Brown [12].

**Theorem 13.** Let $G$ be a primitive log-divergent Feynman graph. If $G$ has a single vertex or a single loop, then $M_G = \mathbb{Q}(0)$. 

35
Example 22. Because \( \log(z)^m \) has weight 2, the small graph principle suggests that any \( \log(z)^m \) appearing in the right-hand side of the coaction formula for a given \( \phi^4 \)-period comes from graphs with at most three edges. Theorem 13 implies that all two-edge graphs are trivial, i.e. the associated motive is the Hodge-Tate motive \( \mathbb{Q}(0) \), which does not have \( \log(z)^m \) in its period matrix. Writing down all possible graphs with three edges, we get the graphs shown in Fig. 15 along with the associated graph polynomials in the Schwinger parameters.

![Feynman graphs with 3 edges and their first graph polynomials](image)

The two outer graphs (a) and (d) are also trivial by Theorem 13 while the two middle graphs (b) and (c) satisfy \( MG = \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \). However, \( \log(z) \) cannot be obtained as an integral with a denominator equal to either of their graph polynomial. It follows that \( \log(z)^m \) cannot be a Galois conjugate of any \( \phi^4 \)-period. By the coaction conjecture and Equation (194), we conclude that \( \log(z)^m \notin \mathcal{P}_{\phi^4} \).

Example 23. Direct computation by Panzer and Schnetz [47] shows that all \( \phi^4 \)-periods of loop order up to 6 are \( \mathbb{Q} \)-linear combinations of multiple zeta values. Following the small graph principle, we order the set of MZVs by weight

\[
\begin{align*}
1 & \quad \zeta(2) \quad \zeta(3) \quad \zeta(2)^2 \quad \zeta(5) \quad \zeta(3)^2 \quad \zeta(7) \quad \zeta(3,5) \quad \cdots \\
& \quad \zeta(2)\zeta(3) \quad \zeta(2)^3 \quad \zeta(2)\zeta(5) \quad \zeta(2)\zeta(3)^2 \quad \cdots \\
& \quad \zeta(2)^2\zeta(3) \quad \cdots
\end{align*}
\]

As a consequence of the coaction conjecture and Equation (196), \( \zeta(2)^m \notin \mathcal{P}_{\phi^4} \) implies that all elements which are linear in \( \zeta(2) \) cannot be \( \phi^4 \)-periods. Analogously, \( (\zeta(2)^2)^m \notin \mathcal{P}_{\phi^4} \) implies that all MZVs quadratic in \( \zeta(2) \) are not \( \phi^4 \)-periods. The set of MZVs that can appear as \( \phi^4 \)-periods is then reduced to

\[
\begin{align*}
1 & \quad \zeta(3) \quad \zeta(5) \quad \zeta(3)^2 \quad \zeta(7) \quad \zeta(3,5) \quad \cdots \\
& \quad \zeta(2)^3 \quad \cdots
\end{align*}
\]

From similar considerations, other highly non-trivial constraints at all loop orders in perturbation theory can be derived using the Galois coaction and weight filtrations. Indeed, whenever it is shown that a given period is not a \( \phi^4 \)-period, we automatically deduce that all periods that have the given one among their Galois conjugates cannot appear in \( \mathcal{P}_{\phi^4} \) either.

Remark. Structures even more fundamental that those captured by the coaction conjecture and the small graph principle underly the space of motivic periods of Feynman graphs. Although not being sufficiently explored in the literature, the notion of operad in the category of motives imposes strong constraints on the admissible periods and it should be the object of further investigation. The operad structure underlying the space of motivic Feynman integrals is interestingly the same structure governing the renormalisation group equation. Kaufmann and Ward [40] provide details on related notions in category theory.

Conclusions

Originally providing a framework for re-organising and re-interpreting much of the previous knowledge on Feynman integrals, the theory of motivic periods has revealed unexpected features, placing restrictions on the set of numbers which can occur as amplitudes and paving the way for a more comprehensive understanding of their general structure. Indeed, the coaction conjecture gives new constraints at each loop order, which in turn propagate to all higher loop orders because of the recursive structure inherent in perturbative quantum field theories. At the same time, the small graph principle makes finite computations at low-loop into all-order results.
Assume to deal with a Feynman integral of the form \( \int_\mathcal{P} \omega \). The general prescription for its investigation via the theory of motivic periods can be summarised as follows.

1. Associate the integral representation \( \int_\mathcal{P} \omega \) to a motivic representation \([H, \omega, \sigma]^m\), deriving explicitly the corresponding algebraic varieties and cohomology classes.

2. Use all the known information about the mixed Hodge structure \( H \) to derive explicit filtrations.

3. Write down the period matrix of \( H \).

4. Apply the Galois coaction and derive the Galois conjugates.

5. Apply the theory of weights of mixed Hodge structures to reduce the calculation of the Galois conjugates to the study of motivic periods of small graphs.

6. Analyse explicitly the few admissible small graphs and eliminate the excluded periods, sometimes called holes.

7. Possibly use other known symmetries of the specific example at hand to draw conclusions.

This picture is, however, extensively conjectural. The very first step of replacing numeric periods with their motivic version requests the validity of the period conjecture. Moreover, even disregarding the conjectural status of current results, the present state of understanding of motivic amplitudes is still far from building a theory. Although the given general prescription for the investigation of motivic Feynman integrals has been particularly fruitful for massless scalar \( \phi^4 \) quantum field theory, further advancements are needed to enlarge the reach of current results.

Speculating in full generality, consider the whole class of Feynman integrals in perturbative quantum field theory. We expect them to have a natural motivic representation and thus to generate a space \( \mathcal{H} \) of motivic periods and a corresponding coaction \( \Delta : \mathcal{H} \to \mathcal{P}^{dR} \otimes \mathcal{P}^m \). A potential coaction principle would then state that \( \Delta(\mathcal{H}) \subseteq \mathcal{A} \otimes \mathcal{H} \). Being \( \mathcal{A} \) a Hopf algebra, we could canonically introduce the group \( C \) of homomorphisms from \( \mathcal{A} \) to any commutative ring. It would follow that the coaction principle can be recast in terms of the group action \( \mathcal{C} \times \mathcal{H} \to \mathcal{H} \), that is, the space of amplitudes is stable under the action of the group \( \mathcal{C} \), often referred to as cosmic Galois group. This speculative construction, that broadly reproduces the general prescription summarised above, motivates a programme of research leading towards a systematic study of scattering amplitudes via the representation theory of groups.

Although practically harder than the \( \phi^4 \)-case, like-minded attempts are already on the way to gather information about the numbers that come from evaluating other classes of Feynman integrals.

1. Towards a general motivic description of scalar quantum field theories, Abreu et al. \cite{1}, \cite{2}, \cite{3} give evidence suggesting that scalar Feynman integrals of small graphs with non-trivial masses and momenta satisfy similar properties to \( \phi^4 \)-periods. A diagrammatic coaction for specific families of integrals appearing in the evaluation of scalar Feynman diagrams, such as multiple polylogarithms and generalised hypergeometric functions, is proposed and a connection between this diagrammatic coaction and graphical operations on Feynman diagrams is conjectured. At one-loop order, a fully explicit and very compact representation of the coaction in terms of one-loop integrals and their cuts is found. Moreover, Brown and Dupont \cite{15} investigate a rigorous theory of motives associated to certain hypergeometric integrals.

2. A subsequent generalisation arises transitioning from scalar quantum field theories to gauge theories. The problem of dealing with much more involved parametric integrands which are not explicitly expressed in terms of the Symanzik polynomials of the associated Feynman graphs has only recently been tackled. A combinatorial and graph-theoretic approach to Schwinger parametric Feynman integrals in quantum electrodynamics by Golz \cite{33} has revealed that the parametric integrands can be explicitly written in terms of new types of graph polynomials related to specific subgraphs. The tensor structure of quantum electrodynamics is given a diagrammatic interpretation. The resulting significant simplification of the integrands paves the way for a systematic motivic description of gauge theories.

3. In the same research direction, a high-precision computation of the 4-loop contribution to the electron anomalous magnetic moment \( g - 2 \) by Laporta \cite{14} shows the presence of polylogarithmic parts with fourth and sixth roots of unity. This result is conjecturally recast in motivic formalism by Schnetz \cite{54}, giving a more compact expression which explicitly reveals a Galois structure. In this work, the \( \mathbb{Q} \)-vector spaces of Galois conjugates of the \( q - 2 \) are conjectured up to weight four.

As a final remark, we mention that scattering amplitudes do not appear exclusively in perturbative quantum field theory. Among other settings, there are string perturbation theory and \( \mathcal{N} = 4 \) super Yang-Mills theory. In each of these theories, after suitably defining the space of integrals or amplitude\( ^{\text{24}} \) under consideration, a version of the

\footnote{In various modern approaches to \( \mathcal{N} = 4 \) SYM, including the bootstrap method, on-shell techniques, and the amplituhedron, the amplitude is constructed independently of the Feynman graphs. In these settings, the coaction principle operates on the entire amplitude, contrary to the case of perturbative quantum field theory, where it operates graph by graph.}
coaction principle is expected to hold and some promising preliminary results have already been found. We refer to the work of Schlotterer, Stieberger and Taylor \cite{52, 53} and subsequent developments for superstring perturbation theory and to the work of Caron-Huot et al \cite{19, 18} for the planar limit of $N=4$ super Yang-Mills theory.

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References

[1] S. Abreu, R. Britto, C. Duhr, and E. Gardi. Diagrammatic Hopf algebra of cut Feynman integrals: the one-loop case. J. High Energy Phys., (12):090, front matter+72, 2017.
[2] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew. Coaction for Feynman integrals and diagrams. PoS, LL2018:047, 2018.
[3] Samuel Abreu, Ruth Britto, Claude Duhr, Einan Gardi, and James Matthew. From positive geometries to a coaction on hypergeometric functions. JHEP, 02:122, 2020.
[4] Y. André. Une introduction aux motifs (motifs purs, motifs mixtes, pérôdes), volume 17 of Panoramas et Synthèses [Panoramas and Syntheses]. Société Mathématique de France, Paris, 2004.
[5] S. Bloch, H. Esnault, and D. Kreimer. On motives associated to graph polynomials. Comm. Math. Phys., 267(1):181–225, 2006.
[6] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren. The multiple zeta value data mine. Comput. Phys. Comm., 181(3):582–625, 2010.
[7] C. Bogner and S. Weinzierl. Feynman graph polynomials. Internat. J. Modern Phys. A, 25(13):2585–2618, 2010.
[8] R. Bott and L. W. Tu. Differential forms in algebraic topology, volume 82 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.
[9] D. Broadhurst. Multiple zeta values and modular forms in quantum field theory. In Computer algebra in quantum field theory, Texts Monogr. Symbol. Comput., pages 33–73. Springer, Vienna, 2013.
[10] D. J. Broadhurst and D. Kreimer. Knots and numbers in $\phi^4$ theory to 7 loops and beyond. Internat. J. Modern Phys. C, 6(4):519–524, 1995.
[11] D. J. Broadhurst and D. Kreimer. Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Phys. Lett. B, 393(3-4):403–412, 1997.
[12] F. Brown. Feynman amplitudes, coaction principle, and cosmic Galois group. Commun. Number Theory Phys., 11(3):453–556, 2017.
[13] F. Brown. Notes on motivic periods. Commun. Number Theory Phys., 11(3):557–655, 2017.
[14] F. Brown and D. Doryn. Framings for graph hypersurfaces. 1 2013.
[15] F. Brown and C. Dupont. Lauricella hypergeometric functions, unipotent fundamental groups of the punctured Riemann sphere, and their motivic coactions. 7 2019.
[16] F. Brown and D. Kreimer. Angles, scales and parametric renormalization. Lett. Math. Phys., 103(9):933–1007, 2013.
[17] F. Brown and O. Schnetz. Proof of the zig-zag conjecture. arXiv:1208.1890, 8 2012.
[18] S. Caron-Huot, L. J. Dixon, J. M. Drummond, F. Dulat, J. Foster, O. Gürdoğan, M. von Hippel, A. J. McLeod, and G. Papathanasiou. The Steinmann Cluster Bootstrap for $N=4$ Super Yang-Mills Amplitudes. PoS, CORFU2019:003, 2020.
[19] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod, and G. Papathanasiou. The cosmic Galois group and extended Steinmann relations for planar $n=4$ super-ynag-mills amplitudes. J. High Energy Phys., (9):061, 65, 2019.
[20] P. Colmez and J. Serre, editors. Correspondance Grothendieck-Serre, volume 2 of Documents Mathématiques (Paris) [Mathematical Documents (Paris)]. Société Mathématique de France, Paris, 2001.
[21] G. De Rham. Sur l’analyse situs des variétés à n dimensions. NUMDAM, [place of publication not identified], 1931.
[22] P. Deligne. Théorie de Hodge. I. In Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pages 425–430. 1971.
[23] P. Deligne. Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math., (40):5–57, 1971.
[24] P. Deligne. Théorie de Hodge. III. Inst. Hautes Études Sci. Publ. Math., (44):5–77, 1974.
[25] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. Ann. Sci. École Norm. Sup. (4), 38(1):1–56, 2005.
[26] P. Deligne, J. S. Milne, A. Ogus, and K. Shih. Hodge cycles, motives, and Shimura varieties, volume 900 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1982.
[27] M. Demazure. Motifs des variétés algébriques. In Séminaire Bourbaki vol. 1969/70 Exposés 364–381, pages 19–38, Berlin, Heidelberg, 1971. Springer Berlin Heidelberg.
[28] F. J. Dyson. The radiation theories of Tomonaga, Schwinger, and Feynman. Phys. Rev. (2), 75:486–502, 1949.
[29] F. J. Dyson. Divergence of perturbation theory in quantum electrodynamics. Phys. Rev. (2), 85:631–632, 1952.
[30] H. Elvang and Y. Huang. Scattering Amplitudes. arXiv:1308.1697, 8 2013.

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