Complex Pythagorean uncertain linguistic group decision-making model based on Heronian mean aggregation operator considering uncertainty, interaction and interrelationship

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Abstract
To effectively solve the mixed problem of considering the uncertainty of individuals and groups, the interaction between membership degree (MD) and non-membership (ND), and the interrelationship between attribute variables in complicated multiple attribute group decision-making (MAGDM) problems, in this paper, a concept of complex Pythagorean uncertain linguistic (CPUL) set (CPULS) is introduced, the interaction operational laws (IOLs) of CPUL variables (CPULVs) are defined. The CPUL interaction weighted averaging and geometric operators are presented. A new concept of CPUL rough number (CPULRN) is further constructed. The CPUL rough interaction weighted averaging and geometric aggregation operators (AOs) are extended. The ordering rules of any two CPULRNs are defined. The CPUL rough interaction Heronian mean (HM) (CPULRIHM) operator and its weighted form are advanced, related properties and special cases are explored. An MAGDM model based on CPUL rough interaction weighted HM (CPULRIWHM) operator is built. Lastly, we conduct a case study of location selection problem for logistics town project to show the applicability of the proposed methodology. The sensitivity and methods comparison are analyzed to verify the effectively and superiority.

Keywords Complex Pythagorean uncertain linguistic set · Multiple attribute group decision-making · Complex Pythagorean uncertain linguistic rough number · Interaction operational laws · Heronian mean

Introduction
An attribute evaluation value usually embraces ambiguous and impermeable information in MAGDM problems. However, as the actual group decision-making (GDM) problem becomes more and more complex, scholars are faced with significant challenges in the expression of attribute variables. As an information representation method, fuzzy set (FS) [1] is widely applied to solve information modeling problems with vague and uncertain information in many fields, but the FS has only MD $M(x)$ and ND $N(x)$ ($0 \leq N(x) \leq 1$). The characteristic of IFS is $M(x) + N(x) \leq 1$. To overcome the shortcoming that the IFS cannot be used in decision scenarios where the sum of $M(x)$ and $N(x)$ greater than one, Yager [3] originated the idea of Pythagorean fuzzy set (PFS), and meets $M^2(x) + N^2(x) \leq 1$. Therefore, the nature of vagueness and uncertainty of assessment information can be depicted by PFS, which is better than IFS. However, they are neglected to manage the time-periodic troubles and two-dimensional information together in a single set. To overcome this limitation, Ramot et al. [4] first extended the classical FSs to the complex plane unite circle and advanced the concept of complex fuzzy sets (CFSs), where the MD is represented by the complex exponential form $M = \alpha e^{i\phi}$. It is composed of the module $\alpha$ (real part) and the amplitude angle $\phi$ (imaginary part) of $M$, and satisfies the conditions $\alpha \in [0,1]$ and $\phi \in [0,2\pi]$, respectively. The amplitude angle $\phi$ represents the periodicity of the evaluation value, which is also the essential
difference from the FSs. Alkouri et al. [5] extended the complex IFSs (CIFSs), where MD, N = αe^iφ and ND, N = βe^iδ provided a wider decision-making range for DMs. However, the limitation of CIFSs is that the sum of MD and ND cannot be greater than one in the real and imaginary parts. For this reason, Ullah et al. [6] developed the CPFSs consisting of the MD, M = αe^iφ and ND, N = βe^iδ, which satisfies the α^2 + β^2 ≤ 1 and (\frac{2}{\pi})^2 + (\frac{2}{\pi})^2 ≤ 1.

At present, many scholars have studied the AOs and decision technologies in CPFS context. Some researchers have paid attention to the aggregation methods for complex Pythagorean fuzzy numbers, and developed operational laws including Algebraic [7], Hamacher [8], Einstein [9], Dombi [10], Yager’s operations [11] and operational laws based on confidential levels [12], which are the core of aggregation methods. Other scholars have studied ranking methods, such as TOPSIS and ELECTRE [13, 14]. However, it is difficult for decision-makers (DMs) to express the “good” or “bad” evaluation degree of an object with MD and ND in CPFS in actual decision-making. In fact, DMs are more inclined to use linguistic variable set or uncertain linguistic variable set (ULVS) to evaluate objects [15–17], but uncertain linguistic terms have not additional phase term, and the change of the phase term cannot be described. To overcome their disadvantages and broaden the scope of the supporting grade reaching out from the unit plate in the form of complex number belonging to unit disc in a complex plane, the new concept of CPULS is proposed, which combines CPFS with ULVS. The CPULV is divided into uncertain linguistic and complex Pythagorean fuzzy parts, in which the former represents the linguistic level of DM’s qualitative evaluation of the object, and the latter describes the MD and ND to the uncertain linguistic part. The CPULS contains more information than CPFS and ULVS, so it is more suitable to deal with practical complex decision-making problems containing a large amount of uncertain, vague information with phase changes, such as medical research, government work reports, biological and face recognition, digital and image processing [18].

There may be various potential characteristics and correlations among variables in GDM. To better fit the objectivity and systematicness of actual decision-making problems, we need to carry out systematic research on the following three challenges in combination with the GDM problems in the literature [12–16]: (1) the interaction between MD and ND is not considered. (2) Individual uncertainty and group uncertainty cannot be dealt with simultaneously. (3) The interrelationship between attributes is ignored. Therefore, how to develop aggregation operators to deal with interaction, uncertainty and interrelationship is a meaning work, which is the main motivation of this paper. Specifically, for challenge (1), not matter in IFS, PFS, CIFS or CPFS environment, existing AOs adopted basic Algebraic, Einstein and Hamacher operational rules. When the variable’s MD or ND is zero, the decision results may be counterintuitive [6, 12]. However, there is no research on IOLs in CIFS and CPFS at present. Therefore, by considering the advantages of IOLs in IFS and PFS [19–21], we develop some new IOLs to use in modeling a problem involving CPULV. For challenge (2), in actual GDM process, the evaluation values given by a single DM can only express the vagueness and hesitation (it is called individual uncertainty) of the DM’s individual evaluation, but cannot deal with the imprecision and subjectivity (it is called group uncertainty) of the DM’s group evaluation. Recently, the integration of fuzzy theory and rough set theory can produce a more flexible and expressive potential framework for processing fuzzy evaluation information from a global perspective [22–24], which can reflect the integrity and rationality of the views of DMs, and also provides a research idea for this paper. For challenge (3), the interrelationship between attributes can affect the final decision result [25]. Among many existing AOs, the HM operator [26] is one of the information fusion tools capturing the interrelationship between input arguments, and it can overcome the disadvantage of computational redundancy and lower computational complexity compared with Bonferroni mean operator [27, 28]. Recently, some scholars have done a lot of research on HM with PFS [29], q-rung orthopair fuzzy set (QROFS) [30], linguistic term set [31], neutrosophic ULVS [32] and complex q-rung orthopair linguistic (CQROL) set [33]. However, no scholar has extended HM operator to CPUL context. In addition, few scholars have studied the integration of IOLs and HM operator for picture fuzzy set [34, 35] and QROFS [36], which can provide feasible ideas for this paper.

According to the above analysis, the main target of this work is to propose a GDM methodology in the CPUL environment, which comprehensively considers the uncertainty of individuals and groups, the interaction effect between membership functions and the interrelationship between input variables, and applies to solve the site selection of logistic town project. Thus, motivated by the above analysis, this paper has the following contributions:

1. We propose the CPULS, and develop the IOLs of CPULVs and the CPUL interaction AOs, such as the CPUL interactive weighted averaging (CPULIWA) and CPUL interactive weighted geometric (CPULIWG) operators.

2. We define the concept of CPULRN, and extend the IOLs of CPULRNs and the CPUL rough interaction AOs. The ordering rules of any two CPULRNs are defined.

3. We develop the CPULRIHM AOs integrating uncertainty, interaction and interrelationship, and discuss their desirable properties and some special cases.
(4) The MAGDM model based on CPULRIWHM operator is constructed to solve the location problem of logistics town project. The sensitivity and comparative analysis are performed.

To this end, several basic notions are briefly described in “Preliminaries”, that is, CPFSs, ULVSs and HM operator. In the subsequent section, the new concept of CPULS is defined, the IOLs of CPULRs are developed followed by which the novel concept of CPULRN is defined, the IOLs of CPULRs and CPULR interaction AOs are extended. In “CPULRIHM AOs”, the CPULRIHM and CPULRIWHM operators are developed. Then the MAGDM model is established based on the CPULRIWHM operator with CPULVs. In “Case study”, the MAGDM model is utilized to obtain the optimal location of logistic town project. The sensitivity and comparative analyses are performed. The conclusion is summarized in the last section.

**Preliminaries**

We briefly describe several related definitions in this section, such as the CPFSs, ULVSs and HM operator.

**Definition 1** [6, 12] . Suppose $X$ is a finite universe set, the mathematical form of CPFS $\tilde{A}$ on $X$ can be given as:

$$\tilde{A} = \left\{ x, \left( M_{\tilde{A}}(x)e^{2\pi \psi_{M_{\tilde{A}}(x)}}, N_{\tilde{A}}(x)e^{2\pi \psi_{N_{\tilde{A}}(x)}} \right) | x \in X \right\},$$

(1)

where $i = \sqrt{-1}$ and $M_{\tilde{A}}(x), N_{\tilde{A}}(x), \varphi_{M_{\tilde{A}}(x)}, \varphi_{N_{\tilde{A}}(x)} \in [0, 1]$, satisfy $0 \leq M_{\tilde{A}}^2(x) + N_{\tilde{A}}^2(x) \leq 1.0 \leq \varphi_{M_{\tilde{A}}(x)} + \varphi_{N_{\tilde{A}}(x)} \leq 1$. The real part functions $M_{\tilde{A}}(x)$ and $N_{\tilde{A}}(x)$ represent the modules of the membership and non-membership functions of $x$ to $X$, respectively, the imaginary part functions $2\pi \varphi_{M_{\tilde{A}}(x)}$ and $2\pi \varphi_{N_{\tilde{A}}(x)}$ indicate the amplitude angles of the membership and non-membership functions of $x$ to $X$, respectively. Hesitation degree is described as

$$\xi_{\tilde{A}}(x) = \sqrt{1 - M_{\tilde{A}}^2(x) - N_{\tilde{A}}^2(x)}e^{2\pi \sqrt{1 - M_{\tilde{A}}^2(x) - N_{\tilde{A}}^2(x)}}.$$  

(2)

For convenience, the complex Pythagorean fuzzy number (CPFN) form is $\tilde{a} = (M_{\tilde{a}}e^{2\pi \varphi_{M_{\tilde{a}}}}, N_{\tilde{a}}e^{2\pi \varphi_{N_{\tilde{a}}}})$.

**Definition 2** [12] . Suppose $\tilde{a} = (M_{\tilde{a}}e^{2\pi \varphi_{M_{\tilde{a}}}}, N_{\tilde{a}}e^{2\pi \varphi_{N_{\tilde{a}}}})$ is any CPFN, its score function $sc(\tilde{a})$ and accuracy function $ac(\tilde{a})$ are given as:

$$sc(\tilde{a}) = \frac{1}{\lambda} \left( M_{\tilde{a}}^2 - N_{\tilde{a}}^2 + \varphi_{M_{\tilde{a}}}^2 - \varphi_{N_{\tilde{a}}}^2 \right) \in [-1, 1].$$

(3)

$$ac(\tilde{a}) = \frac{1}{2} \left( M_{\tilde{a}}^2 + N_{\tilde{a}}^2 + \varphi_{M_{\tilde{a}}}^2 + \varphi_{N_{\tilde{a}}}^2 \right) \in [0, 1].$$

(4)

Based on the above $sc(\tilde{a})$ and $ac(\tilde{a})$ of CPFN, the size relationship between any two CPFNs $\tilde{a}_1 = (M_1e^{2\pi \psi_{M_1}}, N_1e^{2\pi \psi_{N_1}})$ and $\tilde{a}_2 = (M_2e^{2\pi \psi_{M_2}}, N_2e^{2\pi \psi_{N_2}})$ can be defined as

1. If $sc(\tilde{a}_1) > sc(\tilde{a}_2)$, then $\tilde{a}_1$ is better than $\tilde{a}_2$.
2. If $sc(\tilde{a}_1) = sc(\tilde{a}_2)$, then $ac(\tilde{a}_1) > ac(\tilde{a}_2)$, then $\tilde{a}_1$ is better than $\tilde{a}_2$; ii) if $ac(\tilde{a}_1) = ac(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$.

**Definition 3** [12] . Let $\tilde{a} = (M_0e^{2\pi \psi_{M_0}}, N_0e^{2\pi \psi_{N_0}})$, $\tilde{a}_1 = (M_1e^{2\pi \psi_{M_1}}, N_1e^{2\pi \psi_{N_1}})$ and $\tilde{a}_2 = (M_2e^{2\pi \psi_{M_2}}, N_2e^{2\pi \psi_{N_2}})$ be any three CPFNs, then the basic Algebraic operational laws (AOLs) of CPFNs are defined as

1. $\tilde{a}_1 \oplus \tilde{a}_2 = \left( \sqrt{M_1^2 + M_2^2 - M_1M_2e^{2\pi \psi_{M_1M_2}}}, N_1N_2e^{2\pi \psi_{N_1N_2}} \right)$;
2. $\tilde{a}_1 \otimes \tilde{a}_2 = \left( M_1M_2e^{2\pi \psi_{M_1M_2}}, N_1N_2e^{2\pi \psi_{N_1N_2}} \right) + \left( \sqrt{N_1^2 + N_2^2 - N_1N_2e^{2\pi \psi_{N_1N_2}}}, \frac{M_1M_2e^{2\pi \psi_{M_1M_2}}}{N_1N_2} \right)$;
3. $\lambda \tilde{a} = \left( M_0e^{2\pi \psi_{M_0}}, N_0e^{2\pi \psi_{N_0}} \right) + \sqrt{1 - (1 - M_0^2)e^{2\pi \psi_{M_0}}, N_0^2e^{2\pi \psi_{N_0}}}$
4. $\sqrt{\lambda} \tilde{a} = \left( M_0e^{2\pi \psi_{M_0}}, N_0e^{2\pi \psi_{N_0}} \right) + \sqrt{1 - (1 - N_0^2)e^{2\pi \psi_{N_0}}, M_0^2e^{2\pi \psi_{M_0}}}$

(1) $S$ is ordered, that is, if $j \geq t$, then $s_j \geq s_t$;
(2) Nonnegative operator: $Neg(s_j) = s_j$, where $t = z-1$;
(3) If $j \leq t$, then $\max(s_j, s_t) = s_j$, $\min(s_j, s_t) = s_t$.

**Definition 5** [16] . Let the uncertain linguistic variable (ULV) $\tilde{S} = [s_\alpha, s_\beta]$, $s_\alpha, s_\beta \in \tilde{S}$, and $0 \leq \alpha \leq \beta$, where $s_\alpha$ and $s_\beta$ are the lower limit and upper limit respectively. Suppose $\tilde{S}_1 = [s_\alpha_1, s_\beta_1]$ and $\tilde{S}_2 = [s_\alpha_2, s_\beta_2]$ are any two ULVs, and the operational properties are as below:

1. $\tilde{S}_1 \oplus \tilde{S}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} + s_{\beta_2} - \frac{s_{\beta_1} - s_{\alpha_2}}{2}];$
2. $\tilde{S}_1 \otimes \tilde{S}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1 \cdot \alpha_2}, s_{\beta_1 \cdot \beta_2}]$. 

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\begin{align*}
(3) \; \lambda \hat{S}_1 &= \lambda [s_{a_1}, s_{b_1}] = \left[ s_{x_1 \times 1 - \lambda \hat{M}_1}, s_{x_1 \times 1 - \lambda \hat{N}_1} \right], \\
\lambda &> 0, \\
(4) \; \hat{S}_1^2 &= [s_{a_1}, s_{b_1}]^2 = \left[ s_{x_1 \times 1 - \lambda \hat{M}_1}, s_{x_1 \times 1 - \lambda \hat{N}_1} \right], \quad \lambda > 0.
\end{align*}

Definition 6 [37]. Suppose the parameters $\eta, \rho \geq 0, \eta$ and $\rho$ are not 0 at the same time, $x_1 (\zeta = 1, 2, \ldots, \kappa)$ be any family of non-negative real number, if

\[ HM^{\eta, \rho}(x_1, x_2, \ldots, x_\kappa) = \left( \frac{2}{\kappa (\kappa + 1)} \sum_{\zeta = 1}^{\kappa} x_\zeta^\eta \right)^{1/\rho} \tag{5} \]

The $HM^{\eta, \rho}$ is known as Heronian mean operator.

**Complex Pythagorean uncertain linguistic sets**

This article holds that CPULS is the combination of CPFS and ULVS, which can be used to handle vague, indeterminate, unreliable and periodic assessment information in the actual world. Therefore, we define the concept of CPULS, IOLs of CPULVs and related AOs in this section.

**CPULS**

Definition 7 Suppose $X$ is the finite universe set, the mathematical form of CPULS $\hat{P}$ on $X$ is given as follows:

\[ \hat{P} = \left\{ x_j, (s_{a(x_j)}, s_{b(x_j)}), (M_\hat{P}(x_j)e^{2\pi \mathcal{P}(x_j)}, N_\hat{P}(x_j)e^{2\pi \mathcal{P}(x_j)} \right\} | x_j \in X \right\}, \tag{6} \]

where $s_{a(x_j)}, s_{b(x_j)} \in \hat{S}$; the real part function $M_\hat{P}(x_j), N_\hat{P}(x_j)$, respectively, represent the modules that $x_j$ belongs to and does not belong to $[s_{a(x_j)}, s_{b(x_j)}]$ degree, $M_\hat{P}(x_j), N_\hat{P}(x_j) : X \rightarrow [0, 1]$; the imaginary functions $2\pi \varphi_\mathcal{P}(x_j), 2\pi \varphi_{\mathcal{N}(x_j)}$, respectively, represent the amplitude angles that $x_j$ belongs to and does not belong to $[s_{a(x_j)}, s_{b(x_j)}]$ degree, $\varphi_\mathcal{P}(x_j), \varphi_{\mathcal{N}(x_j)} : X \rightarrow [0, 1]$; and meet $0 \leq M_{\hat{P}(x_j)}^2 + N_{\hat{P}(x_j)}^2 \leq 1$, $0 \leq \varphi_\mathcal{P}(x_j) + \varphi_{\mathcal{N}(x_j)} \leq 1, \forall x_j \in X$. The hesitant degree that $x_j$ belongs to $[s_{a(x_j)}, s_{b(x_j)}]$.

For convenience, we define $\hat{P}$ as the set of CPULV $\hat{p}$, that is, $\hat{p} = \left\{ (s_1, s_2), (M_\hat{P}e^{2\pi \mathcal{P}}, N_\hat{P}e^{2\pi \mathcal{P}}) \right\}$.

**Remark 1** When different values are taken for each part in CPULV, it has the following special cases:

1. If $\mathcal{M}_\hat{P} = \mathcal{N}_\hat{P} = 0, \hat{p}$ is reduced to the ULP [15, 16];
2. If $\varphi_{\mathcal{M}_\hat{P}} = \varphi_{\mathcal{N}_\hat{P}} = 0, \hat{p}$ is reduced to the Pythagorean uncertain linguistic (PUL) variable [38, 39];
3. If $\alpha = \beta = 0, \hat{p}$ is reduced to the complex Pythagorean fuzzy number [6];
4. If $\alpha = \beta = 0, \varphi_{\mathcal{M}_\hat{P}} = \varphi_{\mathcal{N}_\hat{P}} = 0, \hat{p}$ is reduced to the Pythagorean fuzzy number [3];
5. If $\alpha = \beta = 0, \mathcal{N}_\hat{P} = 0, \hat{p}$ is reduced to the fuzzy number [4];
6. If $\alpha = \beta = 0, \mathcal{N}_\hat{P} = 0, \varphi_{\mathcal{M}_\hat{P}} = 0, \hat{p}$ is reduced to the fuzzy number [1];
7. If $\alpha = \beta, \mathcal{N}_\hat{P} = 0, \hat{p}$ is reduced to the complex Pythagorean linguistic variable;
8. If $\alpha = 0, \mathcal{M}_\hat{P} = \varphi_{\mathcal{N}_\hat{P}} = 0, \hat{p}$ is reduced to the Pythagorean fuzzy linguistic variable [40].

Definition 8 Let $\hat{P}_j$ be any CPULV, its score function $sc(\hat{p}_j)$ and accuracy function $ac(\hat{p}_j)$ are defined as:

\[
sc(\hat{p}_j) = \frac{\alpha + \beta_j}{1 + M_{\hat{P}_j}^2 + \varphi_{\mathcal{N}_\hat{P}_j}^2 - \varphi_{\mathcal{M}_\hat{P}_j}^2}, \tag{7}
\]

\[
ac(\hat{p}_j) = \frac{\alpha + \beta_j}{1 + M_{\hat{P}_j}^2 + \varphi_{\mathcal{M}_\hat{P}_j}^2 + \varphi_{\mathcal{N}_\hat{P}_j}^2}. \tag{8}
\]

Based on Eqs. (7–8), the size relationship between $\hat{p}_1$ and $\hat{p}_2$ is defined as:

1. If $sc(\hat{p}_1) > sc(\hat{p}_2)$, then $\hat{p}_1$ is better than $\hat{p}_2$.
2. If $sc(\hat{p}_1) = sc(\hat{p}_2)$, then i) if $ac(\hat{p}_1) > ac(\hat{p}_2)$, then $\hat{p}_1$ is better than $\hat{p}_2$; ii) if $ac(\hat{p}_1) = ac(\hat{p}_2)$, then $\hat{p}_1$ is equal to $\hat{p}_2$.

Definition 9 Suppose $\hat{P}_j (j = 1, 2)$ are any two CPULVs, the Hamming distance measure between $\hat{p}_1$ and $\hat{p}_2$ is defined as:

\[
D(\hat{p}_1, \hat{p}_2) = \frac{1}{2\pi} \left( |(a_1 + b_1)(1 + M_1^2 + \varphi_{\mathcal{M}_1}^2 - \varphi_{\mathcal{N}_1}^2) - (a_2 + b_2)(1 + M_2^2 + \varphi_{\mathcal{M}_2}^2 - \varphi_{\mathcal{N}_2}^2) | \right). \tag{9}
\]

We can easily prove that the Hamming distance measure between $\hat{p}_1$ and $\hat{p}_2$ meets the below conditions:

1. $0 \leq D(\hat{p}_1, \hat{p}_2) \leq 1$;
2. $D(\hat{p}_1, \hat{p}_2) = D(\hat{p}_2, \hat{p}_1)$;
3. $D(\hat{p}_1, \hat{p}_2) = 0$, iff $\hat{p}_1 = \hat{p}_2$;
4. $D(\hat{p}_1, \hat{p}_2) = 1$, iff $\hat{p}_1 = \{s_1, s_2\}, (1e^{2\pi 0}, 0e^{2\pi 0})$ and $\hat{p}_2 = \{s_0, s_1\}, (0e^{2\pi 0}, 1e^{2\pi 0})$. 
Interaction operational laws of CPULVs

The operations of CPFNs in the literature [7] are AOLS. However, some counterintuitive phenomena may occur in these calculation results, because when the MD or ND in CPFN is zero, the zero can also appear in the calculation results. Therefore, the IOLs of CPULVs is proposed based on the literature [19–21].

**Definition 10** Suppose \( \tilde{p}_j (j = 1, 2) \) are any two CPULVs, \( \lambda > 0 \), then the IOLs of CPULVs can be defined as:

\[
\tilde{p}_1 \oplus \tilde{p}_2 = \left[ \frac{s_{1+\alpha_2} - 2s_{1/2}}{s_{p_1+p_2} - \epsilon_{p_1/p_2}} \right] \cdot \left( \frac{1 - \prod_{j=1}^{2} (1 - M_j^2)}{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)} \right) e^{2\pi \sqrt{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}}.
\]

\[
\tilde{p}_1 \odot \tilde{p}_2 = \left[ \frac{s_{1+\alpha_2} - 2s_{1/2}}{s_{p_1+p_2} - \epsilon_{p_1/p_2}} \right] \cdot \left( \frac{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)} \right) e^{2\pi \sqrt{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}}.
\]

\[
\tilde{p}^\lambda_1 = \left[ \frac{s_{z \times \left( \frac{\alpha_2}{2} \right)}^{z \times \left( \frac{\alpha_2}{2} \right)}} \right] \cdot \left( \frac{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)} \right) e^{2\pi \sqrt{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}}.
\]

\[
\tilde{p}^\lambda_2 = \left[ \frac{s_{z \times \left( \frac{\alpha_2}{2} \right)}^{z \times \left( \frac{\alpha_2}{2} \right)}} \right] \cdot \left( \frac{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)} \right) e^{2\pi \sqrt{1 - \prod_{j=1}^{2} (1 - M_j^2 \cdot N_j^2)}}.
\]

**Theorem 1** Suppose \( \tilde{p}_j (j = 1, 2, 3) \) are any three CPULVs, then they have the following operation properties:

1. \( \tilde{p}_1 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_1; \)
2. \( (\tilde{p}_1 \oplus \tilde{p}_2) \oplus \tilde{p}_3 = \tilde{p}_1 \oplus (\tilde{p}_2 \oplus \tilde{p}_3); \)
3. \( \tilde{p}_3 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_3; \)
4. \( (\tilde{p}_1 \oplus \tilde{p}_2) \oplus \tilde{p}_3 = \tilde{p}_1 \oplus (\tilde{p}_2 \oplus \tilde{p}_3); \)
5. \( \lambda (\tilde{p}_1 \oplus \tilde{p}_2) = \lambda \tilde{p}_1 \oplus \lambda \tilde{p}_2, \lambda > 0; \)
6. \( \tilde{p}^\lambda_1 \odot \tilde{p}^\lambda_2 = \tilde{p}^\lambda_1 \odot \tilde{p}^\lambda_2, \lambda_1, \lambda_2 > 0. \)

**CPUL interaction aggregation operators**

We proposed the CPULIWA and CPULIWG operators based on the IOLs in this subsection, and we discuss their effective properties.

**Definition 11** Suppose \( \tilde{p}_j = \{ [s_{u_j}, s_{\varphi_j}], (M_j e^{2\pi \varphi_j M_j}, N_j e^{2\pi \varphi_j N_j}) \} (j = 1, 2, \ldots, n) \) is a set of CPULVs, \( W = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n \), and satisfies \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \). We define the below CPUL interaction AOs:

\[
CPULIWA(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}_1 \oplus \sum_{j=1}^{n} w_j \tilde{p}_j.
\]

\[
CPULIWG(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}_1 \odot \sum_{j=1}^{n} (\tilde{p}_j)^{w_j}.
\]

**Theorem 2** Suppose \( \tilde{p}_j (j = 1, 2, \ldots, n) \) is a set of CPULVs, the weight vector of \( \tilde{p}_j (j = 1, 2, \ldots, n) \) is \( W = (w_1, w_2, \ldots, w_n)^T \), satisfies \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \). The aggregation results of CPULIWA and CPULIWG operators are still CPUL, and even

\[
C P U L I W A (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \left[ \frac{s_{z \times \left( \frac{\alpha_2}{2} \right)}^{z \times \left( \frac{\alpha_2}{2} \right)}} \right] \cdot \left( \frac{1 - \prod_{j=1}^{n} (1 - M_j^2 \cdot N_j^2)}{1 - \prod_{j=1}^{n} (1 - M_j^2 \cdot N_j^2)} \right) e^{2\pi \sqrt{1 - \prod_{j=1}^{n} (1 - M_j^2 \cdot N_j^2)}}.
\]

Proof is given in “Appendix A”.

\[\square\]
Proof It is easy to prove that the aggregation results of CPULIWA and CPULIWG operators are still \( \text{CPULV}s \), the proof process is omitted here. The next proof is that Eqs. (9–10) are true. First, the Eq. (9) is proved by mathematical induction.

When \( n = 2 \), according to Definition 10, we have

\[
\begin{align*}
  w_1 \tilde{p}_1 & = \left[ \begin{array}{c}
  s_{-z-a_1} z_{w_1}, \\
  s_{-z-b_1} z_{w_1}
\end{array} \right], \\
  \sqrt{1 - (1 - \mathbf{M}_j^2) w_1 e^{12\pi \sqrt{1 - (1 - \varphi_{\lambda_j}) w_1}},} \\
  \sqrt{1 - (1 - \mathbf{M}_j^2) w_1 - (1 - \mathbf{M}_j^2 - \mathbf{N}_j^2) w_1 e^{12\pi \sqrt{(1 - \varphi_{\lambda_j}) w_1 - (1 - \varphi_{\lambda_j} - \varphi_{\lambda_i}) w_1}}}.
\end{align*}
\]

Then

\[
\begin{align*}
  w_2 \tilde{p}_2 & = \left[ \begin{array}{c}
  s_{-z-a_2} z_{w_2}, \\
  s_{-z-b_2} z_{w_2}
\end{array} \right], \\
  \sqrt{1 - (1 - \mathbf{M}_j^2) w_2 e^{12\pi \sqrt{1 - (1 - \varphi_{\lambda_2}) w_2}},} \\
  \sqrt{1 - (1 - \mathbf{M}_j^2) w_2 - (1 - \mathbf{M}_j^2 - \mathbf{N}_j^2) w_2 e^{12\pi \sqrt{(1 - \varphi_{\lambda_2}) w_2 - (1 - \varphi_{\lambda_2} - \varphi_{\lambda_i}) w_2}}}.
\end{align*}
\]

So, Eq. (9) holds for \( n = 2 \).

Obviously, when \( n = k \), Eq. (9) is also true, that is,

\[
\begin{align*}
  \text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k) & = w_1 \tilde{p}_1 \oplus w_2 \tilde{p}_2 \\
  & = \left[ s_{-z-c} z_{\varphi_{j}^{w_1}}, s_{-z-d} z_{\varphi_{j}^{w_1}} \right], \\
  & \quad \sqrt{1 - (1 - \mathbf{M}_j^2) w_1 e^{12\pi \sqrt{1 - (1 - \varphi_{\lambda_j}) w_1}},} \\
  & \quad \sqrt{1 - (1 - \mathbf{M}_j^2) w_1 - (1 - \mathbf{M}_j^2 - \mathbf{N}_j^2) w_1 e^{12\pi \sqrt{(1 - \varphi_{\lambda_j}) w_1 - (1 - \varphi_{\lambda_j} - \varphi_{\lambda_i}) w_1}}}.
\end{align*}
\]

When \( n = k + 1 \), then we have

Thus, Eq. (9) has been proved to be true. Similarly, Eq. (10) can be proved to be true.

Therefore, the proof of Theorem 2 is completed.
Theorem 3 Suppose \( \tilde{p}_j(j = 1, 2, \ldots, n) \) is a set of CPULVs, 
\( W = (w_1, w_2, \ldots, w_n) \) is the weight vector of \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n \), and satisfies \( w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1 \). The CPULIWG and CPULIWG operators have the below properties:

1. **Idempotency:** suppose \( \tilde{p}_j = \tilde{p} = (s_{\alpha}, s_{\beta}) \), 
\( (Me^{2\pi i\phi_M}, Ne^{2\pi i\phi_N}) \), \( j = 1, 2, \ldots, n \), then

\[
CPULIWAW(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\} = \tilde{p}.
\]

2. **Monotonicity:**

   2.1. Suppose \( \tilde{p}_{\alpha j}(j = 1, 2, \ldots, n) \) is another family of CPULVs, if CPULWA operator satisfies \( s_{\alpha j} \leq s_{\alpha j'}, s_{\beta j} \leq s_{\beta j'}, \omega_{M_{\alpha j}} \leq M_{\alpha j}, M_j + N_{\alpha j} \geq M_j + N_{\alpha j}, \phi_{M_{\alpha j}} \leq \phi_{M_{\alpha j}}, \) and \( \phi_{M_{\alpha j}} + \phi_{N_{\alpha j}} \geq \phi_{M_{\alpha j}} + \phi_{N_{\alpha j}} \), then

\[
CPULIWAW(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq CPULIWAW(\tilde{p}_a, \tilde{p}_2, \ldots, \tilde{p}_n).
\]

   2.2. Suppose \( \tilde{p}_{\alpha j}(j = 1, 2, \ldots, n) \) is another family of CPULVs, if CPULWG operator satisfies \( s_{\alpha j} \leq s_{a_{\alpha j}}, s_{\beta j} \leq s_{\beta j}, \omega_{M_{\alpha j}} \leq M_{\alpha j}, M_j + N_{\alpha j} \geq M_j + N_{\alpha j}, \phi_{M_{\alpha j}} \leq \phi_{M_{\alpha j}}, \phi_{N_{\alpha j}} \geq \phi_{N_{\alpha j}} \), and \( \phi_{M_{\alpha j}} + \phi_{N_{\alpha j}} \geq \phi_{M_{\alpha j}} + \phi_{N_{\alpha j}} \), then

\[
CPULIWGW(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq CPULIWGW(\tilde{p}_a, \tilde{p}_2, \ldots, \tilde{p}_n).
\]

3. **Boundedness:** the CPULIW and CPULIW operators satisfy:

\[
\tilde{p}^- \leq CPULIWAW(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+,
\]

\[
\tilde{p}^- \leq CPULIWGW(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+,
\]

where \( \tilde{p}^- = \left( \left[ \min_{\alpha} \omega_{M_{\alpha j}}, \min_{\beta} \omega_{N_{\alpha j}} \right], \left( \min M_{\alpha j} \right) e^{i2\pi \phi_{M_{\alpha j}}} \right), \)

\[
\tilde{p}^+ = \left( \left[ \max_{\alpha} \omega_{M_{\alpha j}}, \max_{\beta} \omega_{N_{\alpha j}} \right], \left( \max M_{\alpha j} \right) e^{i2\pi \phi_{M_{\alpha j}}} \right).
\]

Proof is given in “Appendix B”.

Concept of complex Pythagorean uncertain linguistic rough number

As an effective tool to deal with subjective and imprecise information, the rough number can mine the potential knowledge hidden under the data surface, and convert each value into the form of rough number composed of lower and upper limit. To effectively deal with the individual uncertainty and group uncertainty in DMS’ assessment information in GDM, we combine the concepts of CPULV and rough number to construct a new concept of CPULRN.

CPULRN

Suppose that the finite non-empty set \( \mathbb{N} \) is the universe, \( \forall h \in \mathbb{N} \), the definition of CPULRN assume that \( \tau \) class is equivalent to CPULV class to form a family of CPULV topological sets, expressed as \( \mathbb{N} = \{\tilde{p}_j\}j = 1, 2, \ldots, l \), there is a certain dominant ordering relationship between each equivalent CPULV class, namely, \( \tilde{p}_1 < \tilde{p}_2 < \ldots < \tilde{p}_n \), then for any class \( \tilde{p}_j \in \mathbb{N}, 1 \leq j \leq l \), \( Y \subseteq \mathbb{N}, X \subseteq \mathbb{N} \), the upper approximation of \( \tilde{p}_j \) can be defined:

\[
\overline{AP}(\tilde{p}_j) = \{Y \in \mathbb{N} : \mathbb{N}(Y) \geq \tilde{p}_j, Y \subseteq X\}.
\]

And the lower approximation of \( \tilde{p}_j \):

\[
\underline{AP}(\tilde{p}_j) = \{Y \in \mathbb{N} : \mathbb{N}(Y) \leq \tilde{p}_j, Y \cap X \neq \phi\},
\]

where \( (\mathbb{N}, \mathbb{R}) \) forms approximate CPUL rough space.

Based on the classical rough number construction, any CPUL class \( \tilde{p}_j \) can be expressed by CPULRN, which consists of the CPUL rough lower limit (CPULRL) \( CPULRN(\tilde{p}_j) \) and the CPUL rough upper limit (CPULRU) \( \overline{CPULRN}(\tilde{p}_j) \), and can be expressed as follows:

\[
CPULRN(\tilde{p}_j) = CPULIWAW(\mathbb{N}(Y_1), \mathbb{N}(Y_2), ..., \mathbb{N}(Y_{QL})) | Y \in \overline{AP}(\tilde{p}_j),
\]

\[
\overline{CPULRN}(\tilde{p}_j) = CPULIWGW(\mathbb{N}(Y_1), \mathbb{N}(Y_2), ..., \mathbb{N}(Y_{QL})) | Y \in \underline{AP}(\tilde{p}_j).
\]

\( CPULRN(\tilde{p}_j) \) is obtained by utilizing the CPULWA (Eq. (9)) or CPULWG (Eq. (10)) operator to aggregate CPULVs of \( Y_1, Y_2, ..., Y_{QL} \) as \( \mathbb{N}(Y_1), \mathbb{N}(Y_2), ..., \mathbb{N}(Y_{QL}) \), and their weights are equal, namely, \( w_1 = w_2 = \ldots = w_{QL} = 1/Q_L \). Similarly, \( \overline{CPULRN}(\tilde{p}_j) \) can be obtained.

Definition 12 Based on the CPULRL \( CPULRN(\tilde{p}_j) \) and CPULRU \( CPULRN(\tilde{p}_j) \) of CPUL class \( \tilde{p}_j \), \( CPULRN(\tilde{p}_j) \) is defined as.

\[
CPULRN(\tilde{p}_j) = CPULRN(\tilde{p}_j), CPULRN(\tilde{p}_j) = \left\{ \left( \left[ \min_{\alpha} \omega_{M_{\alpha j}}, \min_{\beta} \omega_{N_{\alpha j}} \right], \left( \min M_{\alpha j} \right) e^{i2\pi \phi_{M_{\alpha j}}} \right), \left( \left[ \max_{\alpha} \omega_{M_{\alpha j}}, \max_{\beta} \omega_{N_{\alpha j}} \right], \left( \max M_{\alpha j} \right) e^{i2\pi \phi_{M_{\alpha j}}} \right) \right\}.
\]

Theorem 4 Let \( \tilde{p}_1 \), \( \tilde{p}_2 \) be any two CPULRN, and they have the following operation properties:

\( [\tilde{p}_1] + [\tilde{p}_2] = [\overline{\tilde{p}_1} \oplus \overline{\tilde{p}_2}], \overline{\tilde{p}_1} \oplus \overline{\tilde{p}_2} \).
(2) \[ [\hat{p}_1] \times [\hat{p}_2] = [\hat{p}_1 \otimes \hat{p}_2, \overline{\hat{p}_1 \otimes \hat{p}_2}], \]
(3) \[ \lambda[\hat{p}_1] = [\lambda \hat{p}_1, \lambda \overline{\hat{p}_1}], \lambda > 0, \]
(4) \[ [\hat{p}_1]^\lambda = [\hat{p}_1^\lambda, \overline{\hat{p}_1^\lambda}], \lambda > 0. \]

**Proof** Since \( \text{CPULV}(\hat{p}_j) \) \((1 \leq j \leq t)\) appears in the form of interval, its operation property is the same as the general interval number, so it is easy to prove Theorem 4.

**Example 1.** Suppose that four DMs to evaluate an attribute, and the variables are expressed by CPULVs: \( \hat{p}_1 = ([s_4,s_5],[0.4e^{2\pi(0.2)},0.7e^{2\pi(0.8)})], \hat{p}_2 = ([s_3,s_4],[0.6e^{2\pi(0.7)},0.6e^{2\pi(0.4)})], \hat{p}_3 = ([s_1,s_3],[0.7e^{2\pi(0.96)},0.5e^{2\pi(0.3)})], \hat{p}_4 = ([s_2,s_3],[0.8e^{2\pi(0.9)},0.2e^{2\pi(0.2)})], \). We can use Eq. (7) in Definition 8 and we can get:

\[
\begin{align*}
\text{sc}(\hat{p}_1) &= \frac{1}{4} \left((4 \times 5)(1 + 0.4^2 + 0.2^2 - 0.7^2 - 0.8^2)\right) = 0.158; \\
\text{sc}(\hat{p}_2) &= \frac{1}{4} \left((2 \times 4)(1 + 0.6^2 + 0.7^2 - 0.6^2 - 0.4^2)\right) = 1.995; \\
\text{sc}(\hat{p}_3) &= \frac{1}{4} \left((1 \times 3)(1 + 0.7^2 + 0.6^2 - 0.5^2 - 0.3^2)\right) = 1.510; \\
\text{sc}(\hat{p}_4) &= \frac{1}{4} \left((2 \times 3)(1 + 0.8^2 + 0.9^2 - 0.2^2 - 0.2^2)\right) = 2.963.
\end{align*}
\]

So, according to Definition 8, these CPULVs are ranked as \( \hat{p}_1 < \hat{p}_3 < \hat{p}_2 < \hat{p}_4 \).

Taking \( \hat{p}_1 \) as an example, it can be seen from the CPULRN structure that there are \( \text{Apr}(\hat{p}_1) = \{\hat{p}_1\} \) and \( \overline{\text{Apr}}(\hat{p}_1) = \{\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4\} \), then it can be obtained by CPULWA operation (Eq. (9)):

\[
\text{CPULRN}(\hat{p}_1) = ([s_4,s_5],[0.4e^{2\pi(0.2)},0.7e^{2\pi(0.8)})], \]

\[
\text{CPULRN}(\hat{p}_2) = \text{CPULWA} (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4) = ([s_2,3.94],[0.667e^{2\pi(0.711)},0.51e^{2\pi(0.429)}]).
\]

Then, \( \hat{p}_1 = [[s_4,s_5],[0.4e^{2\pi(0.2)},0.7e^{2\pi(0.8)})], \]

\( ([s_2,3.94],[0.667e^{2\pi(0.711)},0.51e^{2\pi(0.429)}]). \)

Similarly,

\[
\hat{p}_2 = [[s_2,2.757],[0.592e^{2\pi(0.566)},0.596e^{2\pi(0.533)}],
([s_2,1.687],[0.721e^{2\pi(0.83)},0.425e^{2\pi(0.287)}])];
\]

\[
\hat{p}_3 = [[s_2,3.94],[0.588e^{2\pi(0.465)},0.594e^{2\pi(0.604)}],
([s_1,1.687],[0.714e^{2\pi(0.777)},0.452e^{2\pi(0.298)}])];
\]

\[
\hat{p}_4 = [[s_2,1.757],[0.663e^{2\pi(0.429)}],
([s_2,1.687],[0.714e^{2\pi(0.777)},0.452e^{2\pi(0.298)}])].
\]

The significance of the CPULVs conversion into the CPULRNs is: The initial CPULV is only given by individual DM, which ignored the interaction between DMs and cannot accurately express the group opinions of DMs. However, the CPULRN is derived from a holistic perspective and can reflect the integrity and rationality of the DMs’ opinions. For example, the third DM’s evaluation of the attribute \( \hat{p}_3 = ([s_1,s_3],[0.7e^{2\pi(0.6)},0.5e^{2\pi(0.3)})], \) but from overall perspective, the attribute value should be \( \hat{p}_3 = ([s_2,3.94],[0.588e^{2\pi(0.465)},0.594e^{2\pi(0.604)})],(s_1,1.687],[0.714e^{2\pi(0.777)},0.452e^{2\pi(0.298)})]. \)

Thus, the CPULRN should be all CPULVs between the lower limit and the upper limit. In addition, the CPULVs are transformed into CPULRNs by applying the CPULWA or CPULIWG operator, which taking IOLs of CPULVs into account, so that the CPULRN is more reasonable, and it can avoid the counter-intuitive dilemmas. Therefore, the CPULRNs can not only reflect the uncertainty of individuals and groups, but also avoid the loss or attenuation of information.

**Interaction operational laws of CPULRNs**

The following IOLs of CPULRNs can be extended based on the Definition 10 in this subsection.

**Definition 13.** Suppose \( \{\hat{p}_j\} (j = 1, 2) \) are two any CPULRNs, \( \lambda > 0 \), then the IOLs of CPULRNs are defined as:
\[
\begin{align*}
(2) & \quad [\tilde{p}_1] \oplus [\tilde{p}_2] = \left[ \begin{array}{c}
\frac{s_{z_1}(1-M_j^2)}{1-M_j^2} \\
\frac{s_{z_2}(1-M_j^2)}{1-M_j^2} \\
\frac{s_{z_3}(1-M_j^2)}{1-M_j^2}
\end{array} \right] \cdot \left[ \begin{array}{c}
\frac{\Pi_{j=1}^2(1-M_j^2)}{1-M_j^2} e^{2\pi i (\Pi_{j=1}^2(1-M_j^2) - \Pi_{j=1}^2(1-M_j^2))} \\
\frac{\Pi_{j=1}^2(1-M_j^2)}{1-M_j^2} e^{2\pi i (\Pi_{j=1}^2(1-M_j^2) - \Pi_{j=1}^2(1-M_j^2))} \\
\frac{\Pi_{j=1}^2(1-M_j^2)}{1-M_j^2} e^{2\pi i (\Pi_{j=1}^2(1-M_j^2) - \Pi_{j=1}^2(1-M_j^2))}
\end{array} \right] \cdot [\tilde{p}_1] \oplus [\tilde{p}_2] = \left[ \begin{array}{c}
\frac{s_{z_1}(1-M_j^2)}{1-M_j^2} \\
\frac{s_{z_2}(1-M_j^2)}{1-M_j^2} \\
\frac{s_{z_3}(1-M_j^2)}{1-M_j^2}
\end{array} \right]
\end{align*}
\]

\[
\begin{align*}
(3) & \quad [\tilde{p}_1] \star [\tilde{p}_2] \star [\tilde{p}_3] = [\tilde{p}_1] \star ([\tilde{p}_2] \star [\tilde{p}_3]), \\
(4) & \quad \lambda([\tilde{p}_1] \otimes [\tilde{p}_2]) = \lambda([\tilde{p}_1] \otimes [\tilde{p}_2]), \\
(5) & \quad [\tilde{p}_1] \otimes [\tilde{p}_1] = [\tilde{p}_1], \\
(6) & \quad [\tilde{p}_1] \star [\tilde{p}_1] = [\tilde{p}_1].
\end{align*}
\]

Theorem 5 Suppose \([\tilde{p}_j](j = 1, 2, 3)\) are any three CPUL-RNs, \(\lambda, \lambda_1, \lambda_2 > 0\), then we have the below operational properties:

1. \([\tilde{p}_1] \oplus [\tilde{p}_2] = [\tilde{p}_2] \oplus [\tilde{p}_1].
2. \(((\tilde{p}_1] \oplus [\tilde{p}_2] \oplus [\tilde{p}_3]), [\tilde{p}_1] \oplus [\tilde{p}_2] \oplus [\tilde{p}_3]).
3. \([\tilde{p}_1] \oplus [\tilde{p}_2] = [\tilde{p}_2] \oplus [\tilde{p}_1].
4. \(((\tilde{p}_1] \oplus [\tilde{p}_2] \oplus [\tilde{p}_3]), [\tilde{p}_1] \oplus [\tilde{p}_2] \oplus [\tilde{p}_3]).
5. \(\lambda([\tilde{p}_1] \otimes [\tilde{p}_2]) = \lambda([\tilde{p}_1] \otimes [\tilde{p}_2]).
6. \([\tilde{p}_1] \otimes [\tilde{p}_1] = [\tilde{p}_1], [\tilde{p}_1] \star [\tilde{p}_1] = [\tilde{p}_1].
\]

Then, we can also easily define the CPULRIWA and CPULRIWG operators based on the Definition 13 and Sect. “CPUL interaction aggregation operators”.

\[
\begin{align*}
\text{CPULRIWA}([\tilde{p}_1], [\tilde{p}_2], \ldots, [\tilde{p}_n]) & = \sum_{j=1}^{n} \lambda_j \cdot [\tilde{p}_j], \\
\text{CPULRIWG}([\tilde{p}_1], [\tilde{p}_2], \ldots, [\tilde{p}_n]) & = \sum_{j=1}^{n} \lambda_j \cdot [\tilde{p}_j].
\end{align*}
\]

Definition 14 Suppose \([\tilde{p}_j](j = 1, 2, \ldots, n)\) is a family of CPUL-RNs, \(W = (w_1, w_2, \ldots, w_n)^T\) is the weight vector of \(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\), and meets \(w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1\). We define the below CPUL rough interaction weighted AOs (CPULRIWA and CPULRIWG) based on Theorem 2.
Similar to Theorem 2, we can easily prove that the Definition 14 is true according to the IOLs of CPULRNs in Definition 13, and the CPULRIWA and CPULRIWG operators satisfy Idempotence, Boundedness and Monotonicity, thus the proof is omitted here.

The ordering of CPURNs

Before discussing the comparison of the size relationship between two CPURNs, we introduce the expected value \( \text{Exp}(\tilde{p}_j) \) and the distance measure \( \text{Exp}(\tilde{p}_j) \) of CPURN \( \tilde{p}_j \).

**Definition 15** Let \( \tilde{p}_j \) be any CPURN, then its expect value is.

\[
\text{Exp}(\tilde{p}_j) = \frac{1}{4} \left( u_j + \frac{\beta_j}{z} + \frac{z}{\beta_j} + \frac{\beta_j}{u_j} - \frac{\beta_j}{z} - \frac{z}{\beta_j} \right) \\
\left( 1 + \Delta_j^2 + \Delta_j^2 \right)
\]

\[
\text{Exp}(\tilde{p}_j) = \frac{1}{4} \left( u_j + \frac{\beta_j}{z} + \frac{z}{\beta_j} + \frac{\beta_j}{u_j} - \frac{\beta_j}{z} - \frac{z}{\beta_j} \right) \\
\left( 1 + \Delta_j^2 + \Delta_j^2 \right)
\]

**Definition 16** Let \( \text{Exp}(\tilde{p}_j) \) and \( \text{Dis}(\tilde{p}_j) \) indicate the expect value and distance measure of \( \tilde{p}_j \), respectively, where \( \text{Dis}(\tilde{p}_j) \) means the Hamming distance between the upper limit CPULV and the lower limit CPULV of \( \tilde{p}_j \), which is calculated by the Definition 8 (Eq. (6)). Suppose \( \tilde{p}_j \) \( (j = 1, 2) \) are any two CPURNs, then the size relation comparison rule is defined as:

1. If \( \text{Exp}(\tilde{p}_1) > \text{Exp}(\tilde{p}_2) \), then \( \tilde{p}_1 \) is better than \( \tilde{p}_2 \);
2. If \( \text{Exp}(\tilde{p}_1) = \text{Exp}(\tilde{p}_2) \), \( \text{Dis}(\tilde{p}_1) = \text{Dis}(\tilde{p}_2) \), then \( \tilde{p}_1 \) and \( \tilde{p}_2 \) express the same information;
3. If \( \text{Exp}(\tilde{p}_1) = \text{Exp}(\tilde{p}_2) \), \( \text{Dis}(\tilde{p}_1) < \text{Dis}(\tilde{p}_2) \), then \( \tilde{p}_1 \) is better than \( \tilde{p}_2 \).

Complex Pythagorean uncertain linguistic rough interaction Heronian mean aggregation operators

Considering uncertainty, interactively and interrelationship, we define the CPURRIHM and CPURRIWHM operators based on the Definition 13 and HM operator, and then their some effective properties and special cases are explored.

**Definition 17** Suppose \( \tilde{p}_j \) \( (j = 1, 2, \ldots, n) \) is a family of CPURNs. For any \( \sigma, \tau \geq 0 \), with \( \sigma + \tau > 0 \), if.

\[
\text{CPURRIH}M_{\sigma, \tau}([\tilde{p}_1], [\tilde{p}_2], \cdots, [\tilde{p}_n]) = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} \sum_{k=1}^{n} [\tilde{p}_j]^{\sigma} \phi [\tilde{p}_k]^{\tau} \right)^{-\frac{1}{n+1}}
\]

then, \( \text{CPURRIH}M_{\sigma, \tau} \) is named the complex Pythagorean uncertain linguistic rough interaction HM (CPURRIHM) operator.

**Theorem 6** Suppose \( \tilde{p}_j \) \( (j = 1, 2, \ldots, n) \) is a family of CPURNs, the aggregation result of above Eq. (24) is a CPURN, that is,

\[
\text{CPURRIH}M_{\sigma, \tau}([\tilde{p}_1], [\tilde{p}_2], \cdots, [\tilde{p}_n])
\]

\[
= \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} [\tilde{p}_j]^{\sigma} \phi [\tilde{p}_k]^{\tau} \right]^{-\frac{1}{n+1}}
\]

where

\[
A = \frac{1}{\pi^{n+1}} \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{u_j}{\beta_j} \right)^{\sigma} \left( \frac{z_j}{\beta_j} \right)^{\tau} \right) \right) \right)^{\frac{1}{n+1}}
\]

\[
B = \frac{1}{\pi^{n+1}} \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{u_j}{\beta_j} \right)^{\sigma} \left( \frac{z_j}{\beta_j} \right)^{\tau} \right) \right) \right)^{\frac{3}{n+1}}
\]
\[
C = \left\{ \frac{1}{\sqrt{\pi^*}} \left[ 1 - \prod_{j=1}^{n} \left( 1 - (1 - \lambda_j^2)\gamma (1 - \lambda_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right]^\frac{1}{2} \right. \\
\left. \prod_{j=1}^{n} \left( 1 - (1 - \lambda_j^2)\gamma (1 - \lambda_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right\}
\]

\[
\Gamma_C = \left\{ \frac{1}{\sqrt{\pi^*}} \left[ 1 - \prod_{j=1}^{n} \left( 1 - (1 - \beta_j^2)\gamma (1 - \beta_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right]^\frac{1}{2} \\
\prod_{j=1}^{n} \left( 1 - (1 - \beta_j^2)\gamma (1 - \beta_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right\}
\]

\[
D = \left\{ \frac{1}{\sqrt{\pi^*}} \left[ 1 - \prod_{j=1}^{n} \left( 1 - (1 - \alpha_j^2)\gamma (1 - \alpha_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right]^\frac{1}{2} \\
\prod_{j=1}^{n} \left( 1 - (1 - \alpha_j^2)\gamma (1 - \alpha_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right\}
\]

\[
\Gamma_D = \left\{ \frac{1}{\sqrt{\pi^*}} \left[ 1 - \prod_{j=1}^{n} \left( 1 - (1 - \phi_j^2)\gamma (1 - \phi_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right]^\frac{1}{2} \\
\prod_{j=1}^{n} \left( 1 - (1 - \phi_j^2)\gamma (1 - \phi_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right\}
\]

**Proof** According to Definition 13, we can get.

\[
[p_j]^\gamma = \left[ \begin{array}{c}
\left( \left[ s_j \left( \psi_j^2 \right) \gamma \right] + \left( \left[ s_j \left( \psi_j^2 \right) \gamma \right] \right) \gamma \left( 1 - (1 - \lambda_j^2)\gamma (1 - \lambda_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right)^{\frac{1}{2}} \\
\left( \left[ s_j \left( \psi_j^2 \right) \gamma \right] + \left( \left[ s_j \left( \psi_j^2 \right) \gamma \right] \right) \gamma \left( 1 - (1 - \lambda_j^2)\gamma (1 - \lambda_j^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right)^{\frac{1}{2}}
\end{array} \right]
\]

\[
[p_k]^\gamma = \left[ \begin{array}{c}
\left( \left[ s_k \left( \psi_k^2 \right) \gamma \right] + \left( \left[ s_k \left( \psi_k^2 \right) \gamma \right] \right) \gamma \left( 1 - (1 - \lambda_k^2)\gamma (1 - \lambda_k^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right)^{\frac{1}{2}} \\
\left( \left[ s_k \left( \psi_k^2 \right) \gamma \right] + \left( \left[ s_k \left( \psi_k^2 \right) \gamma \right] \right) \gamma \left( 1 - (1 - \lambda_k^2)\gamma (1 - \lambda_k^2)\gamma \right)^{\frac{2}{\pi(\pi^*)}} \right)^{\frac{1}{2}}
\end{array} \right]
\]
Then,

\[
[p_j]^{\tau} \otimes [p_k]^{\tau} = \begin{bmatrix}
\left[\frac{S_j}{(\tau \zeta_j)^{\alpha_j}} \right] & \left[\frac{S_k}{(\tau \zeta_k)^{\alpha_k}} \right] \\
\left[\frac{S_j}{(\tau \zeta_j)^{\beta_j}} \right] & \left[\frac{S_k}{(\tau \zeta_k)^{\beta_k}} \right]
\end{bmatrix},
\]

where

\[
\sum_{j=1}^{n} [p_j]^{\tau} \otimes [p_k]^{\tau} = \left[\left[\frac{s_{\tilde{A}}}{s_{\tilde{B}}}, \left(\frac{\hat{C} e^{i2\pi \Gamma_c}}{\hat{D} e^{i2\pi \Gamma_d}}\right)\right]\right],
\]

Further,

\[
\sum_{j=1}^{n} [p_j]^{\tau} \otimes [p_k]^{\tau} = \left[\left[\frac{s_{\tilde{A}}}{s_{\tilde{B}}}, \left(\frac{\hat{C} e^{i2\pi \Gamma_c}}{\hat{D} e^{i2\pi \Gamma_d}}\right)\right]\right],
\]

where

\[
\hat{A} = \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\alpha_j}{\tau \zeta_j}\right)^{\tau} \left(\frac{\alpha_k}{\tau \zeta_k}\right)^{\tau}\right)\right),
\]

\[
\hat{B} = \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\beta_j}{\tau \zeta_j}\right)^{\tau} \left(\frac{\beta_k}{\tau \zeta_k}\right)^{\tau}\right)\right),
\]

\[
\hat{C} = \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \mathcal{N}_j^{2\tau} (1 - \mathcal{N}_k^{2\tau})^{\tau} + (1 - \mathcal{M}_j^{2\tau} (1 - \mathcal{M}_k^{2\tau})^{\tau})^{\tau}\right)\right)^{\frac{1}{2}},
\]

\[
\Gamma_c = \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\mathcal{N}_j}^{2\tau} (1 - \varphi_{\mathcal{N}_k}^{2\tau})^{\tau} + (1 - \varphi_{\mathcal{M}_j}^{2\tau} (1 - \varphi_{\mathcal{M}_k}^{2\tau})^{\tau})^{\tau}\right)\right)^{\frac{1}{2}},
\]

\[
\hat{D} = \left(\prod_{j=1}^{n} \left(1 - (1 - \mathcal{N}_j^{2\tau} (1 - \mathcal{N}_k^{2\tau})^{\tau} + (1 - \mathcal{M}_j^{2\tau} (1 - \mathcal{M}_k^{2\tau})^{\tau})^{\tau}\right)\right)^{\frac{1}{2}},
\]

Further,

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} [p_j]^{\tau} \otimes [p_k]^{\tau} = \left[\left[\frac{s_{\tilde{A}}}{s_{\tilde{B}}}, \left(\frac{\hat{C} e^{i2\pi \Gamma_c}}{\hat{D} e^{i2\pi \Gamma_d}}\right)\right]\right].
\]
where

\[ \tilde{A} = z \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{\alpha_j}{z} \right) \left( \frac{\alpha_k}{z} \right)^\sigma \right) \right), \]

\[ \tilde{B} = z \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{\beta_j}{z} \right) \left( \frac{\beta_k}{z} \right)^\sigma \right) \right), \]

\[ \tilde{C} = \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{1}{z} \right)^\sigma \left( \frac{1}{z} \right)^\sigma \right) \right), \]

\[ \Gamma_{\tilde{C}} = \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{1}{z} \right)^\sigma \left( \frac{1}{z} \right)^\sigma \right) \right), \]

\[ \tilde{D} = \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{1}{z} \right)^\sigma \left( \frac{1}{z} \right)^\sigma \right) \right), \]

\[ \Gamma_{\tilde{D}} = \left( 1 - \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( \frac{1}{z} \right)^\sigma \left( \frac{1}{z} \right)^\sigma \right) \right). \]
\[ \Gamma_B = \left( \prod_{j=1}^{n} \prod_{j=k}^{n} \left( 1 - (1 - \varphi_{M_j}^a)^p (1 - \varphi_{N_j}^a)^p \right) \right)^{\frac{2}{n(n+1)}} - \prod_{j=1}^{n} \left( 1 - \varphi_{M_j}^a - \varphi_{N_j}^a \right) \left( 1 - \varphi_{M_k}^a - \varphi_{N_k}^a \right) \frac{1}{2} \]

And then we obtain,

\[ CPULRIHM([\tilde{\phi}_1], [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} \sum_{j=k}^{n} (p_j)^\tau \otimes [p_k]^\tau \right) \frac{1}{\tau} \]

where the special forms of \( A, B, C, \Gamma_C, D \) and \( \Gamma_D \) can be seen from the Eq. (25) in Theorem 6.

Therefore, the proof is completed.

Based on Definition 4, we can easily prove that the CPULRIHM operator has the below properties.

**Theorem 7** Suppose \([\tilde{\phi}_j](j = 1, 2, \ldots, n)\) is a family of CPULRNs, the CPULRIHM operator has the below properties:

1. **Idempotency**: suppose \([\tilde{\phi}_j] = \tilde{\phi}_j, j = 1, 2, \ldots, n\), then
   \[ CPULRIHM^{\sigma, \tau}(\tilde{\phi}_1, [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) = \tilde{\phi} \]

2. **Boundedness**: the CPULRIHM operator satisfies:
   \[ [\tilde{\phi}]^- \leq CPULRIHM^{\sigma, \tau}(\tilde{\phi}_1, [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) \leq [\tilde{\phi}]^+ \]
   \[ \text{where } [\tilde{\phi}]^- = \min_{1 \leq j \leq n}([\tilde{\phi}_j]), \quad [\tilde{\phi}]^+ = \max_{1 \leq j \leq n}([\tilde{\phi}_j]) \]

3. **Monotonicity**: suppose \([\tilde{\phi}_j](j = 1, 2, \ldots, n)\) is another family of CPULRNs, if the CPULRIHM operator meets
   \[ s_{M_j} \leq s_{M_j}, s_{N_j} \leq s_{N_j}, \varphi_{M_j} \leq \varphi_{M_j}, \varphi_{N_j} \leq \varphi_{N_j}, \alpha_j \leq \alpha_j, \beta_j \leq \beta_j \]
   then
   \[ CPULRIHM^{\sigma, \tau}(\tilde{\phi}_1, [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) \leq CPULRIHM^{\sigma, \tau}(\tilde{\phi}_1, [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) \]

In the following, Eq. (25) reduces several special cases when the parameters \( \sigma, \tau \) take different values:

**Case 1.** If \( \sigma = \tau = 0.5 \), then the CPULRIHM reduces to CPUL rough interaction basic HM operator.

**Case 2.** If \( \sigma = \tau = 1 \), then the CPULRIHM reduces to CPUL rough interaction line HM operator.

**Case 3.** If \( \sigma \neq 0, \tau \neq 0 \), then the CPULRIHM reduces to CPUL rough interaction generalized linear ascending weighted operator, that is,

\[ \lim_{\sigma, \tau \to \infty} CPULRIHM^{\sigma, \tau}(\tilde{\phi}_1, [\tilde{\phi}_2], \ldots, [\tilde{\phi}_n]) \]

\[ = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} (p_j)^\tau \otimes [p_k]^\tau \right) \frac{1}{\tau} \]

\[ = \left( [s_{\Delta A}, s_{\Delta B}], \left( C e^{\frac{\pi}{2} \Gamma} C, D e^{\frac{\pi}{2} \Gamma} D \right), [s_{\Delta A}, s_{\Delta B}], \left( C e^{\frac{\pi}{2} \Gamma} C, D e^{\frac{\pi}{2} \Gamma} D \right) \right) \]

\[ \text{where } A = z \left( 1 - \left( \prod_{j=1}^{n} \left( 1 - \left( \frac{\alpha_j}{z} \right)^\tau \right)^\tau \right) \right) \frac{1}{\tau}, \quad B \]

\[ = z \left( 1 - \left( \prod_{j=1}^{n} \left( 1 - \left( \frac{\beta_j}{z} \right)^\tau \right)^\tau \right) \right) \frac{1}{\tau}, \quad C \]
\[ C = \left\{ 1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\lambda_j}^2)^{1/2} + (1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2)^{1/2}\right) \right\}^{\frac{1}{2}} \\
- \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 \right)^{1/2} \right\}^{\frac{1}{2}} \\
\Gamma_C = \left\{ 1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\lambda_j}^2)^{1/2} + (1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2)^{1/2}\right) \right\}^{\frac{1}{2}} \\
- \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 \right)^{1/2} \right\}^{\frac{1}{2}} \\
D = \left\{ 1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\lambda_j}^2)^{1/2} + (1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2)^{1/2}\right) \right\}^{\frac{1}{2}} \\
- \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 \right)^{1/2} \right\}^{\frac{1}{2}} \\
\Gamma_D = \left\{ 1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\lambda_j}^2)^{1/2} + (1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2)^{1/2}\right) \right\}^{\frac{1}{2}} \\
- \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 \right)^{1/2} \right\}^{\frac{1}{2}} \\
\]

**Case 4.** If \( \sigma \neq 0, r, 0 \), then the CPULRIHM reduces to CPUL rough interaction generalized linear descending weighted operator, that is

\[ \lim_{r \to 0} CPULRIHM^{\sigma, r}(\{\vec{p}_1\}, \{\vec{p}_2\}, \ldots, \{\vec{p}_n\}) = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} (1 - j) \right)^{\frac{1}{\sigma}} \]

\[ = \left[ \left\{ sA, sB \right\}, \left( C_{e^{i2\pi \Gamma_0}}, D_{e^{i2\pi \Gamma_0}} \right) \right], \left[ \left\{ sA, sB \right\}, \left( C_{e^{i2\pi \Gamma_0}}, D_{e^{i2\pi \Gamma_0}} \right) \right], \]

where \( A = z \left( 1 - \prod_{j=1}^{n} \left(1 - \left( \frac{\alpha_j}{z} \right)^{(n+1-j)} \right)^{\frac{1}{n(n+1)}} \right) \), \( B = z \left( 1 - \prod_{j=1}^{n} \left(1 - \left( \frac{\beta_j}{z} \right)^{(n+1-j)} \right)^{\frac{1}{n(n+1)}} \right) \),

\[ C = \left\{ 1 - \prod_{j=1}^{n} \left(1 - (1 - \varphi_{\lambda_j}^2)^{(n+1-j)} + (1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2)^{(n+1-j)} \right)^{\frac{2}{n(n+1)}} + \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 - \varphi_{\lambda_j}^2 \right)^{\frac{2(n+1-j)\sigma}{n(n+1)}} \right\}^{\frac{1}{\sigma}} \]

\[ - \prod_{j=1}^{n} \left(1 - \varphi_{\lambda_j}^2 \right)^{\frac{2(n+1-j)}{n(n+1)}} \right\}^{\frac{1}{2}} \]
\[ \Gamma_C = \left\{ \left( 1 - \prod_{j=1}^{n} \left[ 1 - \left( 1 - \varphi_{\mathcal{N}_j}^{2}(n+1-j)\sigma + (1 - \varphi_{\mathcal{M}_j}^{2} - \varphi_{\mathcal{N}_j}^{2})^{(n+1-j)\sigma} \right) \right] \right) \right\}^{1/2} \]

\[ D = \left\{ \left( 1 - \prod_{j=1}^{n} \left[ 1 - \left( 1 - \varphi_{\mathcal{N}_j}^{2}(n+1-j)\sigma + (1 - \varphi_{\mathcal{M}_j}^{2} - \varphi_{\mathcal{N}_j}^{2})^{(n+1-j)\sigma} \right) \right] \right) \right\}^{1/2} \]

**Case 5.** If \( \sigma + \infty, \tau = 0 \), then the CPULRIHM reduces to

\[
\lim_{\sigma \to +\infty} C P U L R I H M^{\sigma,0}([\tilde{p}_1], [\tilde{p}_2], \ldots) = \max_j ([\tilde{p}_j]) \quad (31)
\]

**Case 6.** If \( \sigma 0, \tau = 0 \), then the CPULRIHM reduces to CPUL rough interaction geometric averaging operator, that is

\[
\lim_{\sigma \to 0} C P U L R I H M^{\sigma,0}([\tilde{p}_1], [\tilde{p}_2], \ldots, [\tilde{p}_n]) = \left( \frac{1}{n} \prod_{j=1}^{n} (1 - [\tilde{p}_j]) \right) = \left( \left( \frac{1}{n} \prod_{j=1}^{n} [\tilde{p}_j] \right) \right)^{1/2} \quad (32)
\]

Notably, the importance of each input CPULVs of the CPULRIHM operator in Definition 18 is not considered. However, the weight of an attribute, as an important input parameter, has a crucial part in the aggregation process of attribute variables and can affect the result of AOs in many actual MAGDM problems. Therefore, we further develop the CPULRIWHM operator.

**Definition 18** Suppose \([\tilde{p}_j] (j = 1, 2, \ldots, n)\) is a family of CPULRN. For any \(\sigma, \tau \geq 0\) with \(\sigma + \tau > 0\), if

\[
C P U L R I W H M^{\sigma,\tau}([\tilde{p}_1], [\tilde{p}_2], \ldots, [\tilde{p}_n]) = \left( \frac{2}{n(n+1)} \sum_{j=1}^{n} \sum_{k=j}^{n} (w_j [\tilde{p}_j])^\sigma \otimes (w_k [\tilde{p}_k])^\tau \right) \right\}^{\frac{1}{\sigma + \tau}}
\]

Then, \(C P U L R I W H M^{\sigma,\tau}\) is named the complex Pythagorean uncertain linguistic rough interaction weighted HM (CPULRIWHM) operator, where the weight vector of \([\tilde{p}_j] (j = 1, 2, \ldots, n)\) is \(W = (w_1, w_2, \ldots, w_n)^T\), meets \(w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1\).

**Theorem 8** Suppose \([\tilde{p}_j] (j = 1, 2, \ldots, n)\) is a family of CPULRN, For any \(\sigma, \tau \geq 0\) with \(\sigma + \tau > 0\), then the value of above Eq. (33) is still a CPULRN, and even.
\[ CPULRIWHM^{\alpha,\tau}([p_1], [p_2], \ldots, [p_n]) = \left[ \left[ s_{\alpha}, s_{\beta}\right], \left( C' e^{i 2\pi \Gamma_{\alpha} \tau}, D' e^{i 2\pi \Gamma_{\tau} \sigma} \right) \right] \left[ \left[ s_{\alpha}, s_{\beta}\right], \left( C' e^{i 2\pi \Gamma_{\alpha} \tau}, D' e^{i 2\pi \Gamma_{\tau} \sigma} \right) \right], \]

where

\[
A' = \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( 1 - \frac{\alpha_j}{z} \right)^{w_j} \left( 1 - \frac{\alpha_k}{z} \right)^{w_k} \right)^{\sigma} \right) \right)^{\frac{2}{n(n+1)}} \\
B' = \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( 1 - \frac{\beta_j}{z} \right)^{w_j} \left( 1 - \frac{\beta_k}{z} \right)^{w_k} \right)^{\sigma} \right) \right)^{\frac{1}{n(n+1)}} \\
C' = \left\{ \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( 1 - M_j^2 - N_j^2 \right)^{w_j} \left( 1 - M_k^2 - N_k^2 \right)^{w_k} \right)^{\sigma} \right) \right)^{\frac{2}{n(n+1)}} \frac{1}{\sigma+\tau} \right\}^{1/2} \\
\Gamma_{C'} = \left\{ \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( 1 - \varphi_m \right)^{w_j} \left( 1 - \varphi_n \right)^{w_k} \right)^{\sigma} \right) \right)^{\frac{2}{n(n+1)}} \frac{1}{\sigma+\tau} \right\}^{1/2} \\
D' = \left\{ \left( 1 - \left( \prod_{j=1}^{n} \prod_{k=j}^{n} \left( 1 - \left( 1 - M_j^2 - N_j^2 \right)^{w_j} \left( 1 - M_k^2 - N_k^2 \right)^{w_k} \right)^{\sigma} \right) \right)^{\frac{2}{n(n+1)}} \frac{1}{\sigma+\tau} \right\}^{1/2}.
\]
Step 1: The individual CPUL evaluation matrix $\tilde{D}^c$ is constructed. It is necessary to guarantee that the types of attributes remain consistent in decision process, so the cost attribute is converted into benefit attribute. We can apply the conversion technology (Eq. (35)) to get the normalized individual CPULDM $\tilde{R}^c = \left[\tilde{r}_{ij}^c\right]_{m \times n}$:

$$\tilde{r}_{ij}^c = \begin{cases} 
\tilde{d}_{ij}^c = [w_1 \cdot \tilde{s}_{ij}^c, w_2 \cdot \tilde{s}_{ij}^c, \ldots, w_n \cdot \tilde{s}_{ij}^c], \\
\left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right), \\
\left[\tilde{s}_{1ij}^c, \tilde{s}_{2ij}^c, \ldots, \tilde{s}_{nj}^c\right], \\
\left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right), \\
\end{cases}
$$

where $l_1$ and $l_2$ represent benefit and cost attribute respectively.

Step 2: According to Definition 11, the normalized individual CPULDM $\tilde{R}^c$ is transformed into CPUL rough decision matrix (CPULRDM) $[\tilde{R}]^c = \left[\tilde{r}_{ij}^c\right]_{m \times n}$:

$$[r_{ij}]^c = \left\lfloor \left[\tilde{s}_{ij}^c, \tilde{s}_{ij}^c\right], \left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right)\right\rfloor, \\
\left[\tilde{s}_{ij}^c, \tilde{s}_{ij}^c\right], \left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right).
$$

The MAGDM model based on CPULRWHM operator

The CPULRWHM operator-based MAGDM model is constructed in this section. The CPUL GDM problems are described as below:

For a CPUL GDM problem, suppose $A = \{a_1, a_2, \ldots, a_m\}$ is an alternative set, $C = \{c_1, c_2, \ldots, c_n\}$ is an attribute set. The attribute weight vector is $W = (w_1, w_2, \ldots, w_n)^T$, and satisfies $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$. Let $E = \{e_1, e_2, \ldots, e_l\}$ be a group of DMs, the DM weight vector is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ meets $\lambda_c \in [0,1]$ and $\sum_{c=1}^n \lambda_c = 1$. Suppose individual CPUL decision matrix (CPULDM) given by the $\zeta$-th DM $e_c$ is expressed as $\tilde{D}^c = \left[\tilde{d}_{ij}^c\right]_{m \times n}$, where $\tilde{d}_{ij}^c = [s_{ij}^c, s_{ij}^c, \ldots, s_{ij}^c]$, $\left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right)$, meets $0 \leq \left(\tilde{M}_{ij}^c\right)^2 + \left(\tilde{N}_{ij}^c\right)^2 \leq 1$.

This CPULV $\tilde{d}_{ij}^c$ indicates that the $\zeta$-th DM $e_c$ ($\zeta = 1, 2, \ldots, \lambda$) gives the preference value of alternative $a_i$ ($i = 1, 2, \ldots, m$) with regard to the attribute $c_j$ ($j = 1, 2, \ldots, n$). The specific steps of the MAGDM method are mentioned below, and Fig. 1 shows the flowchart.

**Step 1:** The individual CPUL evaluation matrix $\tilde{D}^c$ is constructed. It is necessary to guarantee that the types of attributes remain consistent in decision process, so the cost attribute is converted into benefit attribute. We can apply the conversion technology (Eq. (35)) to get the normalized individual CPULDM $\tilde{R}^c = \left[\tilde{r}_{ij}^c\right]_{m \times n}$:

$$\tilde{r}_{ij}^c = \begin{cases} 
\tilde{d}_{ij}^c = [w_1 \cdot \tilde{s}_{ij}^c, w_2 \cdot \tilde{s}_{ij}^c, \ldots, w_n \cdot \tilde{s}_{ij}^c], \\
\left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right), \\
\left[\tilde{s}_{1ij}^c, \tilde{s}_{2ij}^c, \ldots, \tilde{s}_{nj}^c\right], \\
\left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right), \\
\end{cases}
$$

where $l_1$ and $l_2$ represent benefit and cost attribute respectively.

**Step 2:** According to Definition 11, the normalized individual CPULDM $\tilde{R}^c$ is transformed into CPUL rough decision matrix (CPULRDM) $[\tilde{R}]^c = \left[\tilde{r}_{ij}^c\right]_{m \times n}$.

$$[r_{ij}]^c = \left\lfloor \left[\tilde{s}_{ij}^c, \tilde{s}_{ij}^c\right], \left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right)\right\rfloor, \\
\left[\tilde{s}_{ij}^c, \tilde{s}_{ij}^c\right], \left(\tilde{M}_{ij}^c e^{i2\pi(\psi_{ij}^c)}, \tilde{N}_{ij}^c e^{i2\pi(\psi_{ij}^c)}\right).
$$

**Step 3:** The CPULRIWA (Eq. (21)) or CPULRIWG (Eq. (22)) operator is used to aggregate the assessment information of alternative $a_i$ under attribute $c_j$, and the individual CPULRDM $[\tilde{R}]^c$ is aggregated into the group CPULRDM.

![Fig. 1 The flowchart of the proposed approach](image-url)
Case study: site selection of logistics town project

Jiujiang is a prefecture-level city in northern Jiangxi Province, China. Located at the junction of the Yangtze River Economic Development Zone and Beijing-Kowloon Railway Economic Development Zone, Jiujiang is a shipping center city in the middle reaches of Yangtze River. It is about 240 km away from the central city of central China—Wuhan, and about 650 km away from the core city of Yangtze River Delta economic circle in eastern China—Shanghai. Jiujiang’s highway, railway, waterway and other infrastructure layout is reasonable, and the comprehensive transportation system is perfect. In 2015, Jiujiang was designated as a regional circulation node city by the Ministry of commerce and other departments of China, becoming an important node in the national backbone circulation network.

To better join the Yangtze River Delta economic circle and play to the advantage of the regional logistics node city and channel network, the government of Jiujiang city wants to accelerate the development and upgrading of regional logistics and create characteristic logistics industry, and plans to introduce the logistics town project. Taking this as the carrier, the integration of warehousing, distribution, processing, centralization, procurement and retail store can be realized by building hardware facilities such as warehouse and freight center in the town and using modern information technology such as Internet of Things and cloud computing as soft support.

One of the challenges facing the government is how to select the best location for the logistics town project. Project location has a great influence on the function and productivity of logistics town. Jiujiang government invites three experts (DMs) to form an evaluation committee $E = \{e_1, e_2, e_3\}$ to understand the basic characteristics of logistics town location, where $e_1$ is a senior executive of a consulting company, who is good at regional logistics industry planning; $e_2$ is a local government official in charge of land resource planning and utilization; $e_3$ is a senior professor in the field of logistics from Nanchang University in Jiangxi Province. After discussion, the DMs determine four sites as the alternative location for the logistics town project $A = \{a_1, a_2, a_3, a_4\}$, which are scattered in Jiujiang and depicted in Fig. 2.

According to the evaluation opinions of the DM panel, the location of the logistics town is related to the following key factors to maximize profits: project floor area ($c_1$), land cost ($c_2$), distance from port ($c_3$), distance from railway freight station ($c_4$), distance to highways ($c_5$) and impact on local economy ($c_6$). DMs give the evaluation value in the form of CPULV, with attributes $c_1$ and $c_6$ as benefit type attributes, and the linguistic term set $S = \{s_0 = \text{extremely low}, s_1 = \text{very low}, s_2 = \text{poor}, s_3 = \text{medium}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$. The set linguistic terms $S = \{s_0 = \text{extremely low}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{fair}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{extremely high}\}$ used for the cost type attributes $c_2, c_3, c_4$ and $c_5$. Let the DM’s and attribute’s weight vector be $\lambda = (0.35, 0.45, 0.20)^T$ and $W = (0.149, 0.159, 0.176, 0.171, 0.166, 0.179)^T$, respectively. Therefore, Table 1 show the evaluation matrices of individual CPULV $\tilde{D}^i (i = 1, 2, 3)$ given by DMs.

We determine the best location of logistics town according to Sect. 6, and the specific process is as below:

Step 1: Eq. (35) is used to normalize the individual CPULD $\tilde{R}^i$, then we can get the normalized individual CPULDM $\tilde{R}^i = \left[\begin{array}{cccc} l_{20} & l_{21} & l_{22} & l_{23} \end{array}\right]^T$. For example, the cost type attribute $\tilde{r}^i_{12} = \left[\begin{array}{cccc} l_{20} & l_{21} & l_{22} & l_{23} \end{array}\right]^T = \left[\begin{array}{cccc} 0.8 & 0.3 & 0.6 & 0.4 \end{array}\right]^T$ is normalized, and we can obtain $\tilde{r}^i_{12} = \left[\begin{array}{cccc} 0.3 & 0.6 & 0.4 & 0.0 \end{array}\right]^T$.

Step 2: According to Definition 11, the normalized individual CPULDM $\tilde{R}^i = \left[\begin{array}{cccc} l_{20} & l_{21} & l_{22} & l_{23} \end{array}\right]^T$ is converted into individual CPULRDM $\tilde{R}^i$ using CPULIWA operator. E.g., the normalized CPULVs of the DMs on attribute $c_2$ alternation $a_1$ are $\tilde{r}^i_{12} = \left[\begin{array}{cccc} l_{20} & l_{21} & l_{22} & l_{23} \end{array}\right]^T = \left[\begin{array}{cccc} 0.3 & 0.6 & 0.4 & 0.0 \end{array}\right]^T$; $\tilde{r}^i_{12} = \left[\begin{array}{cccc} 0.3 & 0.6 & 0.4 & 0.0 \end{array}\right]^T$; $\tilde{r}^i_{12} = \left[\begin{array}{cccc} 0.3 & 0.6 & 0.4 & 0.0 \end{array}\right]^T$.

Step 3: Using CPULRIWHM (Eq. (21)), the individual CPULRDM $[\tilde{R}]^i$ is aggregated into the group CPULRDM $\tilde{G}$, the aggregation results are shown in Table 2.
Step 4: The CPULRIWHM \((\sigma = \tau = 1)\) operator (Eq. (34)) is applied to get the comprehensive evaluation values of alternatives:

\[
\bar{x}_1 = \left[\left(0.224, 0.224, 0.172e^{i2\pi(0.158)}, 0.341e^{i2\pi(0.333)}\right), \left(0.291, 0.289e^{i2\pi(0.314)}\right)\right];
\]

\[
\bar{x}_2 = \left[\left(0.358, 0.288e^{i2\pi(0.225)}, 0.244e^{i2\pi(0.245)}\right), \left(0.404, 0.302e^{i2\pi(0.26)}, 0.245e^{i2\pi(0.203)}\right)\right];
\]

\[
\bar{x}_3 = \left[\left(0.313, 0.182e^{i2\pi(0.169)}, 0.321e^{i2\pi(0.311)}\right), \left(0.418, 0.197e^{i2\pi(0.200)}, 0.309e^{i2\pi(0.259)}\right)\right];
\]

\[
\bar{x}_4 = \left[\left(0.504, 0.214e^{i2\pi(0.240)}, 0.282e^{i2\pi(0.258)}\right), \left(0.608, 0.256e^{i2\pi(0.280)}, 0.246e^{i2\pi(0.235)}\right)\right].
\]

Step 6: The ranking of all alternatives is determined by the expected value of the alternative: \(a_4 > a_2 > a_3 > a_1\). That is, \(a_4\) is the optimal location of logistics town project.

Sensitivity analysis

To analyze the impact of parameters \(\sigma\) and \(\tau\) in the CPULRIWHM operator on alternative ranking, the expected values of alternatives are calculated to determine the ordering of alternatives when the parameters \(\sigma\) and \(\tau\) take different values, the results are seen in Table 3.

From Table 3, on the whole, the expected value of each alternative decreases gradually with the increase of parameters \(\sigma\) and \(\tau\). In this process, \(a_4\) and \(a_2\) are the optimal and second best alternatives respectively and remain unchanged, while \(a_1\) and \(a_3\) have slight changes. It can be seen that the alternative ranking is generally relatively stable in the process of parameters \(\sigma\) and \(\tau\) changes. It is worth noting that the parameter value is chosen by DMs based on their oven risk preferences. However, if the value of parameter is too small, the interrelationship between input arguments cannot be reflected. Therefore, considering the complexity...

Fig. 2 Geographical locations of the alternative sites

Step 5: The Eq. (23) is utilized to calculate the expected value of each alternative:

\[
Exp(\bar{x}_1) = 0.420; \quad Exp(\bar{x}_2) = 0.647; \quad Exp(\bar{x}_3) = 0.538; \quad Exp(\bar{x}_4) = 0.851.
\]
of decision model calculation and identification ability, the parameters $\sigma$ and $\tau$ are generally taken as one.

### Comparative study

The proposed method is compared with the existing methods to verify the effectiveness of the method presented. There are existing methods, such as the PUL weighted averaging (PULWA) operator [38], PUL weighted geometric (PULWG) operator [38], PUL-VIKOR [39], complex intuitionistic uncertain linguistic weighted arithmetic HM (CIULWAHM) operator [41], complex intuitionistic uncertain linguistic weighted geometric HM (CIULWGHM) operator [41], complex $q$-runge orthopair uncertain linguistic weighted averaged (CQROULWA) operator [42], complex $q$-runge orthopair uncertain linguistic weighted geometric (CQROULWG) operator [42], complex $q$-runge orthopair uncertain linguistic VIKOR (CQ-VIKOR) [42] and CQROL weighted HM (CQROLWHM) operator [33]. It should be noted that the imaginary parts of MD and ND of CPULVs in Table 1 are omitted in the calculation process of PULWA, PULWG and PUL-VIKOR; the compromise coefficient $\eta = 0.5$ in the calculation process of PUL-VIKOR and CQ-VIKOR methods; the CPUV is converted into complex Pythagorean linguistic number by averaging the upper and lower limits of the uncertain linguistic part of CPULVs in Table 1, and the parameters $\sigma = \tau = 1$ are also used in the CQROLWHM operator calculation process. Thus, the comparison results are shown in Table 4. An intuitive comparison of ranking results is shown in Fig. 3.

From Table 4, the CIULWAHM and CIULWGHM operators are not applicable to this case, because these two operators cannot process CPUVs in Table 1 where the sum of MD and ND of real and imaginary parts is greater than one. The reason why the results of PULWA operator, PULWG operator and PUL-VIKOR method are different from the proposed method is that these methods do not consider the interaction between MD and ND in the real and imaginary parts of CPULVs shown in Table 1, the uncertainty of individuals and groups, and the interrelationship between variables. The CQROULWA and CQROULWG operators also do not consider the above interaction, uncertainty and interrelationship at the same time, but carry out aggregation of CPUVs more rigidly. Although the best alternative $a_4$ for

| C    | A        | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|------|----------|--------------|--------------|--------------|
| $c_1$ | $a_1$    | $[s_3, s_5], (0.7e^{2\pi i 0.7}, 0.5e^{2\pi i 0.6})$ | $[s_3, s_4], (0.7e^{2\pi i 0.7}, 0.5e^{2\pi i 0.6})$ | $[s_3, s_5], (0.7e^{2\pi i 0.7}, 0.5e^{2\pi i 0.6})$ |
| $c_2$ | $a_1$    | $[s_3, s_4], (0.9e^{2\pi i 0.9}, 0.2e^{2\pi i 0.3})$ | $[s_3, s_5], (0.2e^{2\pi i 0.3})$ | $[s_3, s_4], (0.9e^{2\pi i 0.9}, 0.2e^{2\pi i 0.3})$ |
| $c_3$ | $a_1$    | $[s_3, s_4], (0.8e^{2\pi i 0.8}, 0.3e^{2\pi i 0.3})$ | $[s_3, s_5], (0.3e^{2\pi i 0.4})$ | $[s_3, s_4], (0.8e^{2\pi i 0.8}, 0.3e^{2\pi i 0.4})$ |
| $c_4$ | $a_1$    | $[s_3, s_4], (0.6e^{2\pi i 0.5}, 0.5e^{2\pi i 0.3})$ | $[s_3, s_4], (0.5e^{2\pi i 0.5})$ | $[s_3, s_4], (0.5e^{2\pi i 0.5})$ |
| $c_5$ | $a_1$    | $[s_3, s_4], (0.5e^{2\pi i 0.7}, 0.6e^{2\pi i 0.3})$ | $[s_3, s_4], (0.4e^{2\pi i 0.4})$ | $[s_3, s_4], (0.5e^{2\pi i 0.7}, 0.6e^{2\pi i 0.3})$ |
| $c_6$ | $a_1$    | $[s_3, s_4], (0.7e^{2\pi i 0.4}, 0.4e^{2\pi i 0.6})$ | $[s_3, s_4], (0.3e^{2\pi i 0.3})$ | $[s_3, s_4], (0.7e^{2\pi i 0.4}, 0.4e^{2\pi i 0.6})$ |
|        | $a_2$    | $[s_3, s_4], (0.4e^{2\pi i 0.6}, 0.7e^{2\pi i 0.4})$ | $[s_3, s_4], (0.3e^{2\pi i 0.3})$ | $[s_3, s_4], (0.4e^{2\pi i 0.6}, 0.7e^{2\pi i 0.4})$ |
|        | $a_3$    | $[s_3, s_4], (0.4e^{2\pi i 0.4}, 0.7e^{2\pi i 0.5})$ | $[s_3, s_4], (0.3e^{2\pi i 0.3})$ | $[s_3, s_4], (0.5e^{2\pi i 0.7}, 0.6e^{2\pi i 0.3})$ |
|        | $a_4$    | $[s_3, s_4], (0.8e^{2\pi i 0.8}, 0.4e^{2\pi i 0.3})$ | $[s_3, s_4], (0.3e^{2\pi i 0.3})$ | $[s_3, s_4], (0.8e^{2\pi i 0.8}, 0.4e^{2\pi i 0.3})$ |
| $A$     | $c_1$                                                                 | $c_2$                                                                 | $c_3$                                                                 |
|---------|------------------------------------------------------------------------|------------------------------------------------------------------------|------------------------------------------------------------------------|
| $a_1$   | $[< [s_{2.27}^{12.98}],[0.70e^{12.700}],[0.50e^{22.060}]>,< [s_{2.30}^{12.60}],[0.70e^{12.700}],[0.50e^{22.060}>]>$ | $[< [s_{1.85}^{33.13}],[0.30e^{22.462}],[0.77e^{12.698}]>,< [s_{1.73}^{33.06}],[0.41e^{22.620}],[0.66e^{22.740}]>]$ | $[< [s_{1.05}^{23.05}],[0.37e^{22.260}],[0.75e^{12.820}]>,< [s_{1.78}^{13.16}],[0.46e^{22.440}],[0.65e^{22.600}]>]$ |
| $a_2$   | $[< [s_{4.17}^{33.81}],[0.89e^{12.820}],[0.20e^{22.283}]>,< [s_{4.71}^{33.85}],[0.84e^{12.875}],[0.21e^{22.229}]>]$ | $[< [s_{3.00}^{33.07}],[0.22e^{22.320}],[0.82e^{12.740}]>,< [s_{3.00}^{33.07}],[0.19e^{22.280}],[0.87e^{12.500}]>]$ | $[< [s_{4.17}^{33.82}],[0.71e^{22.710}],[0.33e^{12.420}]>,< [s_{4.44}^{33.10}],[0.76e^{22.760}],[0.33e^{12.350}]>]$ |
| $a_3$   | $[< [s_{4.02}^{33.95}],[0.75e^{12.750}],[0.41e^{22.360}]>,< [s_{4.60}^{33.98}],[0.70e^{12.790}],[0.40e^{22.310}]>]$ | $[< [s_{3.74}^{33.96}],[0.36e^{22.230}],[0.80e^{12.840}]>,< [s_{3.52}^{33.94}],[0.31e^{22.300}],[0.80e^{12.740}]>]$ | $[< [s_{3.21}^{33.93}],[0.58e^{22.540}],[0.54e^{12.570}]>,< [s_{3.27}^{33.44}],[0.68e^{22.590}],[0.49e^{12.520}]>]$ |
| $a_4$   | $[< [s_{1.32}^{33.27}],[0.54e^{12.850}],[0.64e^{22.290}]>,< [s_{2.02}^{33.95}],[0.59e^{12.740}],[0.53e^{22.430}]>]$ | $[< [s_{3.64}^{33.50}],[0.73e^{22.670}],[0.39e^{12.410}]>,< [s_{3.71}^{33.99}],[0.73e^{22.790}],[0.45e^{22.260}]>]$ | $[< [s_{3.97}^{33.14}],[0.74e^{22.740}],[0.36e^{12.340}]>,< [s_{4.44}^{33.63}],[0.79e^{22.790}],[0.31e^{22.300}]>]$ |

Table 2: Aggregated CPULR decision matrix $[\tilde{G}]$
Table 3 The results of alternatives with different parameters

| $\sigma$, $\tau$ | $a_1$     | $a_2$     | $a_3$     | $a_4$     | Ranking         |
|-----------------|-----------|-----------|-----------|-----------|-----------------|
| 0.5, 0.5        | 1.42531   | 1.74592   | 1.60953   | 2.14615   | $a_4 > a_2 > a_3 > a_1$ |
| 1.0, 0.5        | 0.79826   | 1.13322   | 0.97404   | 1.36662   | $a_4 > a_2 > a_3 > a_1$ |
| 0.5, 1.0        | 0.77698   | 1.02370   | 0.92133   | 1.38299   | $a_4 > a_2 > a_3 > a_1$ |
| 1.0, 1.0        | 0.41979   | 0.64689   | 0.53834   | 0.85150   | $a_4 > a_2 > a_3 > a_1$ |
| 1.0, 3.0        | 0.05874   | 0.08773   | 0.06846   | 0.14901   | $a_4 > a_2 > a_3 > a_1$ |
| 3.0, 1.0        | 0.05975   | 0.08754   | 0.07091   | 0.15356   | $a_4 > a_2 > a_3 > a_1$ |
| 1.0, 5.0        | 0.01151   | 0.01423   | 0.01054   | 0.03086   | $a_4 > a_2 > a_3 > a_1$ |
| 5.0, 1.0        | 0.01170   | 0.01414   | 0.01104   | 0.03197   | $a_4 > a_2 > a_3 > a_1$ |
| 3.0, 3.0        | 0.00494   | 0.01660   | 0.00937   | 0.02445   | $a_4 > a_2 > a_3 > a_1$ |
| 5.0, 5.0        | 0.00000   | 0.00058   | 0.00023   | 0.00101   | $a_4 > a_2 > a_3 > a_1$ |

Table 4 The results of alternatives with different methods

| Methods          | $a_1$     | $a_2$     | $a_3$     | $a_4$     | Ranking         |
|------------------|-----------|-----------|-----------|-----------|-----------------|
| CIULWAHM [41]    | Failed    | Failed    | Failed    | Failed    | Failed          |
| CIULWGHM [41]    | Failed    | Failed    | Failed    | Failed    | Failed          |
| PULWA [38]       | 1.084     | 2.270     | 1.305     | 2.257     | $a_2 > a_4 > a_3 > a_1$ |
| PULWG [38]       | 0.638     | 0.000     | 1.020     | 0.912     | $a_3 > a_4 > a_1 > a_2$ |
| PUL-VIKOR [39]   | 0.051     | 0.000     | 1.000     | 0.142     | $a_2 > a_1 > a_4 > a_3$ |
| CQROULWA [42]    | $-0.21$   | $2.103$   | $-0.19$   | 1.762     | $a_2 > a_4 > a_3 > a_1$ |
| CQROULWG [42]    | $-0.66$   | $-0.24$   | $-0.69$   | $-0.60$   | $a_2 > a_4 > a_1 > a_3$ |
| CQ-VIKOR [42]    | 1.000     | 0.394     | 0.858     | 0.000     | $a_4 > a_2 > a_3 > a_1$ |
| CQROLWHM [33] ($\sigma = \tau = 1$) | $-0.13$   | 0.684     | $-0.16$   | 0.702     | $a_4 > a_2 > a_3 > a_1$ |
| this paper ($\sigma = \tau = 1$)  | 0.420     | 0.647     | 0.538     | 0.851     | $a_4 > a_2 > a_3 > a_1$ |

Fig. 3 Comparison of alternative rankings with other methods

CQROLWHM operator which only considers the interrelationship between attributes, the results of other alternatives are slightly diverse from those of the proposed method. The CQ-VIKOR method is completely different from our method in terms of decision principle, but the obtained results are completely consistent, which demonstrates the rationality and effectiveness of the method in this work. Therefore, compared with the above-mentioned existing methods, the results of this method are more comprehensive, effective and scientific in terms of both the form expression of evaluation variables and the process of variable processing.
Table 5 Comparison of features of various AOs

| AOs                     | Operational laws | Considering periodicity of the data | Interaction between functions | Uncertainty of individual and group | Interrelationship between variables | Flexibility for DM’s preferences |
|-------------------------|------------------|-------------------------------------|------------------------------|-------------------------------------|------------------------------------|----------------------------------|
| PULWA [38]              | AOLs             | No                                  | No                           | No                                  | No                                 | No                               |
| PULWG [38]              | AOLs             | No                                  | No                           | No                                  | No                                 | No                               |
| CIULWAHM [41]           | AOLs             | Yes                                 | No                           | No                                  | Yes                                | Yes                              |
| CIULWGHM [41]           | AOLs             | Yes                                 | No                           | No                                  | Yes                                | Yes                              |
| CQROULWA [42]           | AOLs             | Yes                                 | No                           | No                                  | No                                 | No                               |
| CQROULWG [42]           | AOLs             | Yes                                 | No                           | No                                  | No                                 | No                               |
| CQROLWHM [33]           | AOLs             | Yes                                 | No                           | No                                  | Yes                                | Yes                              |
| CPULIW A                | IOLs             | Yes                                 | Yes                          | No                                  | No                                 | No                               |
| CPULIWG                 | IOLs             | Yes                                 | Yes                          | No                                  | No                                 | No                               |
| CPULRIWA                | IOLs             | Yes                                 | Yes                          | Yes                                 | No                                  | No                               |
| CPULRIWG                | IOLs             | Yes                                 | Yes                          | Yes                                 | No                                  | No                               |
| CPULRIWHM               | IOLs             | Yes                                 | Yes                          | Yes                                 | Yes                                | Yes                              |

**Advantage analysis**

The existing AOs are compared with the CPUL interactive, CPUL rough interactive and CPULRIHM AOs proposed in this paper in terms of operational laws, data periodicity, interaction between membership functions, uncertainty of individuals and groups, interrelationship between variables and flexibility of DMs’ preference. Table 5 shows the comparison of features of various AOs. It is not difficult to find that the proposed AOs are based on the IOLs and can deal with vague, uncertain and periodic evaluation data. Moreover, the CPULRIWHM operator can systematically and comprehensively consider the interaction, uncertainty and interrelationship, and can express the preferences of DMs through parameter changes.

Therefore, some superiorities of the proposed method are summarized as below:

1. The CPULS cannot only accurately characterize vagueness and uncertainty, but also characterize the periodicity of data. It can help DMs improve their freedom to represent their opinions and more accurately describe preferences. Moreover, the CPULS has a certain generality.
2. The concept of CPULRN can describe the uncertainty of individual and group evaluation information, and can preserve the uncertainty in the decision-making process to avoid partial information loss.
3. The proposed CPULIW A and CPULIWG operators can avoid the counterintuitive phenomenon. The proposed CPULRW A and CPULRWG operators can not only avoid counterintuitive phenomena, but also express and preserve individual and group uncertainties. In addition, the CPULRIHM AOs, which integrates the interaction between membership functions in real and imaginary parts, individual and group uncertainty and interrelationship between variables, is more suitable for complex realistic decision problems.

(4) The MAGDM model based on the CPULRIWHM operator can be effectively applied to settle the practical GDM problem. The proposed CPULRIWHM operator is more flexible than the existing PULWA, PULWG, CQROULWA and CQROULWG operators because its parameters $\sigma$ and $\tau$ can be changed.

**Conclusion**

The CPULS theory has the combined characteristics of the CPFS and ULVS theories, which can accurately describe the vagueness, uncertainty and periodicity of evaluation information. In this paper, we have defined the CPULS and IOLs of CPULVs, and the CPULWA and CPULWG operators have been developed. Then, we have constructed the new concept of CPULRN, and have extended the IOLs of CPULRN, the CPULRIWA and CPULRIWG operators. Further, the CPULRIHM and CPULRIWHM operators have been proposed. Additionally, we have applied the CPULRIWHM operator to handle the MAGDM issues. Finally, the plausibility, scientificity and superiority of the proposed are illustrated by a real case of location selection for logistics town project. Using the proposed model based on CPULRIWHM operator, we have integrated systematically uncertainty, interaction and interrelationship in CPULS context. The proposed operators...
not only enrich the fuzzy decision theories and have important enlightenment significance for the follow-up aggregation research, but also provide new ideas for solving practical decision-making problems.

Directions for future research are described below. First, the use of the proposed improved AOs of CPULSs should be investigated in the different fuzzy climates, such as simplified Neutrosophic sets [43], hesitant fuzzy sets [44], probabilistic linguistic term sets [45] and linguistic q-rung orthopair fuzzy sets [46], etc. For the second future direction, we will focus on the integration of the proposed AOs with other ranking technologies (e.g., VIKOR [47], WASPAS [48], MARCOS [49], etc.), and extend and improve these technologies in the CPULSs environment. For the third future direction, to enhance more managerial implications, we need to utilize them to solve different decision problems in real world, such as business administration, environment protection, energy resources management, etc.

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Appendix

Appendix A: Proof of Theorem 1

Proof

(1) According to Eq. (1) in Definition 10, (1) is kept.
(2) For (2), based on Eq. (1) in Definition 10, we can obtain

\[
(p_1 \oplus p_2) \oplus p_3 = \left[ \begin{array}{c}
 s_{a_1 a_2 a_3} - \frac{s_{b_1 b_2 b_3}}{2} \\
 s_{b_1 + b_2 - \frac{s_{b_1 b_2 b_3}}{2}} \\
 s_{b_3 + b_4 - \frac{s_{b_3 b_4 b_5}}{2}}
\end{array} \right] \cdot \left[ \begin{array}{c}
 \sqrt{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)} e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}}
\end{array} \right] \oplus \left[ \begin{array}{c}
 s_{a_1} \\
 s_{a_2} \\
 s_{a_3}
\end{array} \right] = \left[ \begin{array}{c}
 s_{a_1 a_2 a_3} - \frac{s_{b_1 b_2 b_3}}{2} \\
 s_{b_1 + b_2 - \frac{s_{b_1 b_2 b_3}}{2}} \\
 s_{b_3 + b_4 - \frac{s_{b_3 b_4 b_5}}{2}}
\end{array} \right] \cdot \left[ \begin{array}{c}
 \sqrt{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)} e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}}
\end{array} \right] \oplus \left[ \begin{array}{c}
 s_{a_1} \\
 s_{a_2} \\
 s_{a_3}
\end{array} \right] = \left[ \begin{array}{c}
 s_{a_1 a_2 a_3} - \frac{s_{b_1 b_2 b_3}}{2} \\
 s_{b_1 + b_2 - \frac{s_{b_1 b_2 b_3}}{2}} \\
 s_{b_3 + b_4 - \frac{s_{b_3 b_4 b_5}}{2}}
\end{array} \right] \cdot \left[ \begin{array}{c}
 \sqrt{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)} e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}} \\
 \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2) - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2) e^{\frac{12\pi i}{1 - \prod_{j=1}^{2}(1 - \mathcal{M}_{j}^2 - \mathcal{N}_{j}^2)}}
\end{array} \right] \oplus \left[ \begin{array}{c}
 s_{a_1} \\
 s_{a_2} \\
 s_{a_3}
\end{array} \right]
\]
\[ \tilde{p}_1 \oplus (\tilde{p}_2 \oplus \tilde{p}_3) \]

\[ = \left( \left[ s_{a_1}, \frac{M_1 e^{2\pi q_1 \lambda_1}}{\mathcal{N}_1 e^{2\pi q_1 \lambda_1}} \right] \right) \oplus \left( \left[ s_{a_2 \oplus a_3}, \frac{M_2 e^{2\pi q_2 \lambda_2}}{\mathcal{N}_2 e^{2\pi q_2 \lambda_2}} \right] \right) \oplus \left( \left[ s_{a_4 \oplus a_5 \oplus a_6}, \frac{M_3 e^{2\pi q_3 \lambda_3}}{\mathcal{N}_3 e^{2\pi q_3 \lambda_3}} \right] \right) \]

\[ \left( \frac{\sqrt{\prod_{j=1}^{3} (1 - \mathcal{M}_j^2) e^{2\pi \sqrt{(1 - \mathcal{M}_j^{-2})}}}}{\prod_{j=1}^{3} (1 - \mathcal{M}_j^2) - \prod_{j=1}^{3} (1 - \mathcal{M}_j^2 - \mathcal{N}_j^2) e^{2\pi \sqrt{(1 - \mathcal{M}_j^{-2})}}} \right)^{12\pi \sqrt{(1 - \mathcal{M}_j^{-2})}} \]

That is, \( (\tilde{p}_1 \oplus \tilde{p}_2) \oplus \tilde{p}_3 = \tilde{p}_1 \oplus (\tilde{p}_2 \oplus \tilde{p}_3) \), which completes the proof of (2).

(3) According the Eq. (2) in Definition 10, (3) is kept.

(4) Similar to the proof of (2), the proof of (4) is omitted here.

(5) For (5), based on Eq. (1) and Eq. (3) in Definition 10, we can obtain

\[ \lambda(\tilde{p}_1 \oplus \tilde{p}_2) \]

\[ = \lambda \left( \left[ s_{a_1 + a_2}, \frac{M_1 e^{2\pi q_1 \lambda_1}}{\mathcal{N}_1 e^{2\pi q_1 \lambda_1}} \right] \right) \oplus \left( \left[ s_{a_3 + a_4}, \frac{M_2 e^{2\pi q_2 \lambda_2}}{\mathcal{N}_2 e^{2\pi q_2 \lambda_2}} \right] \right) \oplus \left( \left[ s_{a_5 + a_6}, \frac{M_3 e^{2\pi q_3 \lambda_3}}{\mathcal{N}_3 e^{2\pi q_3 \lambda_3}} \right] \right) \]

\[ \left( \frac{\sqrt{\prod_{j=1}^{3} (1 - \mathcal{M}_j^2) e^{2\pi \sqrt{(1 - \mathcal{M}_j^{-2})}}}}{\prod_{j=1}^{3} (1 - \mathcal{M}_j^2) - \prod_{j=1}^{3} (1 - \mathcal{M}_j^2 - \mathcal{N}_j^2) e^{2\pi \sqrt{(1 - \mathcal{M}_j^{-2})}}} \right)^{12\pi \sqrt{(1 - \mathcal{M}_j^{-2})}} \]

And

(6) Similar to the proof of (5), the proof of (6) is omitted here.
Appendix B: Proof of Theorem 3

Proof  (1) Since $\tilde{p}_j = \tilde{p}$, based on Theorem 2, we have.

$$C_{\text{PU LiWA}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = C_{\text{PU LiWA}}(\tilde{p}, \tilde{p}, \ldots, \tilde{p})$$

$$= \left[ \left. \sum_{j=1}^{n} (1-\varphi_j) \right]_{0}, \left. \left. \sum_{j=1}^{n} (1-\varphi_j) \right]_{0} \right]$$

$$= \left( \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0}, \left. \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0} \right]$$

Similarly, we can get.

$$C_{\text{PU LiWG}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}.$$ 

So, $C_{\text{PU LiWA}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = C_{\text{PU LiWG}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}$, which proves the idempotency of the CPULIW A and CPULIW G operators (2).

(2.1) According to Theorem 2, we can get the aggregated result of CPULiWA being still a CPUL, that is,

$$C_{\text{PU LiWA}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \left( \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0}, \left. \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0} \right]$$

For $C_{\text{PU LiWA}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)$, we have

$$= \left( \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0}, \left. \left. \sum_{j=1}^{n} (1-\varphi_j)^{w_j} \right]_{0} \right]$$

Similarly, we can get.

$$C_{\text{PU LiWG}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}.$$ 

So, $C_{\text{PU LiWA}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = C_{\text{PU LiWG}}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}$, which proves the idempotency of the CPULIW A and CPULIW G operators (2).
(i) First, we need to prove $s_\alpha \leq s_{\alpha^*}$. Since $s_{\alpha_j} \leq s_{\alpha_j}$ for any $j$, we have:

\[
\frac{\alpha_j}{z} \leq \frac{\alpha_{\alpha_j}}{z} \Rightarrow 1 - \frac{\alpha_j}{z} \geq 1 - \frac{\alpha_{\alpha_j}}{z} \Rightarrow z \prod_{j=1}^n \left(1 - \frac{\alpha_j}{z}\right)^{w_j}
\]

\[
\geq z \prod_{j=1}^n \left(1 - \frac{\alpha_{\alpha_j}}{z}\right)^{w_j} \Rightarrow z - z \prod_{j=1}^n \left(1 - \frac{\alpha_{\alpha_j}}{z}\right)^{w_j}
\]

\[
\leq z \prod_{j=1}^n \left(1 - \frac{\alpha_{\alpha_j}}{z}\right)^{w_j}.
\]

So, $s_\alpha \leq s_{\alpha^*}$.

(ii) Similarly, we can prove that $s_\beta \leq s_{\beta^*}$.

(iii) Next, we need to prove $\mathcal{M} \leq \mathcal{M}^*$. Since $\mathcal{M}_j \leq \mathcal{M}_{\alpha_j}$ for any $j$, we have

\[
\mathcal{M}_j^2 \leq \mathcal{M}_{\alpha_j}^2 \Rightarrow 1 - \mathcal{M}_j^2 \geq 1 - \mathcal{M}_{\alpha_j}^2 \Rightarrow \prod_{j=1}^n \left(1 - \mathcal{M}_j^2\right)^{w_j}
\]

\[
\geq \prod_{j=1}^n \left(1 - \mathcal{M}_{\alpha_j}^2\right)^{w_j} \Rightarrow 1 - \prod_{j=1}^n \left(1 - \mathcal{M}_j^2\right)^{w_j}
\]

\[
\leq 1 - \prod_{j=1}^n \left(1 - \mathcal{M}_{\alpha_j}^2\right)^{w_j}
\]

Thus, $\mathcal{M} \leq \mathcal{M}^*$.

(iv) Similarly, we can prove that $\varphi_\mathcal{M} \leq \varphi_{\mathcal{M}^*}$.

(v) Then, we need to prove $\mathcal{N} \geq \mathcal{N}^*$. Since $\mathcal{M}_j \leq \mathcal{M}_{\alpha_j}, \mathcal{M}_j + \mathcal{N}_j \geq \mathcal{M}_j + \mathcal{N}_{\alpha_j}$ for any $j$, we have

\[
\mathcal{M}_j^2 + \mathcal{N}_j^2 \geq \mathcal{M}_{\alpha_j}^2 + \mathcal{N}_{\alpha_j}^2 \Rightarrow 1 - \mathcal{M}_j^2 - \mathcal{N}_j^2
\]

\[
\leq 1 - \mathcal{M}_{\alpha_j}^2 - \mathcal{N}_{\alpha_j}^2 \Rightarrow \prod_{j=1}^n \left(1 - \mathcal{M}_j^2 - \mathcal{N}_j^2\right)
\]

\[
\leq \prod_{j=1}^n \left(1 - \mathcal{M}_{\alpha_j}^2 - \mathcal{N}_{\alpha_j}^2\right)
\]

\[
\Rightarrow \sqrt[2n]{\prod_{j=1}^n \left(1 - \mathcal{M}_j^2 - \mathcal{N}_j^2\right)^{w_j}}
\]

\[
\leq \sqrt[2n]{\prod_{j=1}^n \left(1 - \mathcal{M}_{\alpha_j}^2 - \mathcal{N}_{\alpha_j}^2\right)^{w_j}}
\]

Thus, $\mathcal{N} \geq \mathcal{N}^*$.

(vi) Similarly, we can prove that $\varphi_\mathcal{N} \geq \varphi_{\mathcal{N}^*}$.

In the following, suppose $\text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \mathcal{A}$ and $\text{CPULIWA}(\tilde{p}_{\alpha_1}, \tilde{p}_{\alpha_2}, \ldots, \tilde{p}_{\alpha_n}) = \mathcal{A}^*$, then according to Eq. (7), we can obtain.

\[
sc(A) = \frac{(\alpha + \beta)(1 + \mathcal{M}^2 + \varphi_\mathcal{A} - \mathcal{N}^2 - \varphi_\mathcal{N}^2)}{4}
\]

\[
sc(A^*) = \frac{(\alpha^* + \beta^*)(1 + \mathcal{M}^* + \varphi_\mathcal{A} - \mathcal{N}^* - \varphi_\mathcal{N}^*)}{4}
\]

So, $sc(A) \leq sc(A^*)$, i.e., $\text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{CPULIWA}(\tilde{p}_{\alpha_1}, \tilde{p}_{\alpha_2}, \ldots, \tilde{p}_{\alpha_n})$, which proves the monotonicity of CPULIWA operator.

(2.2) Similar to the proof of (2.1), we can prove that $\text{CPULIWG}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{CPULIWG}(\tilde{p}_{\alpha_1}, \tilde{p}_{\alpha_2}, \ldots, \tilde{p}_{\alpha_n})$ is true if the CPULIWG operator satisfies $s_{\alpha_j} \leq s_{\alpha_{\alpha_j}}, s_{\beta_j} \leq s_{\beta_{\alpha_j}}, N_j \leq N_{\alpha_j}, M_j + N_j \geq M_{\alpha_j} + N_{\alpha_j}, \varphi_{M_j} \geq \varphi_{M_{\alpha_j}}$, and $\varphi_{M_j} + \varphi_{N_j} \geq \varphi_{M_{\alpha_j}} + \varphi_{N_{\alpha_j}}$. Thus, the monotonicity of CPULIWG operator is proved.

(3) Since $s_{\min_{\alpha_j}} \leq s_{\alpha_j} \leq s_{\max_{\alpha_j}}, s_{\min_{\beta_j}} \leq s_{\beta_j} \leq s_{\max_{\beta_j}}$, $\min M_j \leq \mathcal{M}_j \leq \max M_j, \min \varphi_{M_j} \leq \varphi_{M_j} \leq \max \varphi_{M_j}, \min N_j \leq \mathcal{N}_j \leq \max N_j$ and $\min \varphi_{N_j} \leq \varphi_{N_j} \leq \max \varphi_{N_j}$ for all $j$, based on the above proof process of the properties monotonicity and idempotency, we can further have.

\[
\text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)
\]

\[
\geq \text{CPULIWA}(\tilde{p}^-, \tilde{p}^-, \ldots, \tilde{p}^-) = \tilde{p}^-
\]

\[
\text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)
\]

\[
\leq \text{CPULIWA}(\tilde{p}^+, \tilde{p}^+, \ldots, \tilde{p}^+) = \tilde{p}^+
\]

That is, $\tilde{p}^- \leq \text{CPULIWA}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+$.

Similarly, we can get $\tilde{p}^+ \leq \text{CPULIWG}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^-$.

Thus, we fully prove the boundedness of the CPULIWA and CPULIWG operators.

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