EXISTENCE OF SUPERSINGULAR REPRESENTATIONS OF $p$-ADIC REDUCTIVE GROUPS

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Let $G$ be a connected simple adjoint $p$-adic group not isomorphic to a projective linear group $PGL_m(D)$ of a division algebra $D$ ($m \geq 2$), or an adjoint ramified unitary group $PU(h)$ of a split hermitian form $h$ in 3 variables. We prove that $G$ admits an irreducible admissible supercuspidal (supersingular) representation over any field of characteristic $p$.

1. Introduction

Throughout this paper, $F$ is a local non-archimedean field of characteristic $a$ and residue field of characteristic $p$ with $q$ elements, $G = G(F)$ where $G$ is a connected reductive $F$-group, $C$ a field (of coefficients) of characteristic $c$ and $C^{alg}$ an algebraic closure of $C$.

Recent applications of automorphic forms to number theory have imposed the study of smooth representations of $G$ on $C$-vector spaces for $C$ not algebraically closed and often finite with $c = p$. Indeed one expects a strong relation, à la Langlands, with $C$-representations of the Galois group of $F$ - the only established case, however, is that of $GL(2, \mathbb{Q}_p)$.

An irreducible admissible $C$-representation $\pi$ of $G$ is called supercuspidal if it is not isomorphic to a subquotient of a representation parabolically induced from an irreducible admissible $C$-representation of a Levi subgroup. In the established cases of the Langlands correspondence, they correspond to the irreducible continuous $C$-representations of the Galois group of $F$.

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of $F$. All irreducible admissible $C$-representations of $G$ are constructed from the irreducible admissible supercuspidal $C$-representations of the Levi subgroups of $G$ using parabolic induction.

When $c = p$, the finite group analogue $H(k)$ of $G$, where $H$ is a connected reductive group over a finite field $k$ of characteristic $p$, admits no irreducible supercuspidal $C$-representation [CE04 Thm.6.12].

For $C = \mathbb{F}_p^{alg}$, irreducible admissible supercuspidal $C$-representations have been constructed for some rank 1 groups:

- $\text{PGL}(2, \mathbb{Q}_p)$: Breuil [Bre03] after the pioneer work of Barthel-Livne [BL94]; this was the starting point of the Langlands $p$-adic correspondance (Colmez and al.),
- $\text{PGL}(2, F)$, Paskunas [Pas04], [BP12], using $G$-equivariant coefficient systems on the adjoint Bruhat-Tits building of $G$,
- $\text{SL}(2, \mathbb{Q}_p)$ [Abd14], [Che13],
- $U(1,1)(\mathbb{Q}_p)$ unramified [Koz16],
- $U(1,2)(F)$ unramified and $p \neq 2$ [KX15].

**Theorem 1.1.** Assume $(a,c) = (0,p)$ and $G$ absolutely simple adjoint not isomorphic to the projective linear group $\text{PGL}_m$ of a $d^2$-dimensional central division $F$-algebra ($m \geq 2, d \geq 1$), or to the adjoint unitary group of a split hermitian form in 3 variables over a ramified quadratic extension of $F$.

Then, $G$ admits an irreducible admissible supercuspidal $C$-representation.

The irreducible admissible supercuspidal $C$-representation of $G$ will be a subquotient of a subrepresentation $\rho^C := C(\Gamma \backslash G)\infty$ (smooth induction of the trivial $C$-representation) for a discrete cocompact subgroup $\Gamma$ of $G$ [BH78]. The proof is local and uses a criterion of supercuspidality proved with R. Ollivier for $C$ algebraically closed [OV, Thm 3] and that we extend to any $C$:

**Proposition 1.2.** [Supercuspidality criterion] Assume $c = p$. An irreducible admissible $C$-representation $\pi$ of $G$ is supercuspidal if and only if it contains a non-zero pro-$p$-Iwahori invariant supersingular element if and only if $\pi$ is supersingular (all pro-$p$-Iwahori invariant elements of $\pi$ are supersingular).

The equivalence supercuspidal $\Leftrightarrow$ supersingular follows also from the classification [HV17 Thm.9]. Let $\mathcal{B}$ denote an Iwahori subgroup of $G$ and $H_C(G, \mathcal{B})$ the Iwahori Hecke $C$-algebra.

**Proposition 1.3.** Assume $(a,c) = (0,p)$ and $G$ as in Thm[1.1] Then, there exists a discrete cocompact subgroup $\Gamma$ of $G$ such that the $H_C(G, \mathcal{B})$-module $C[\Gamma \backslash G/\mathcal{B}]$ contains a non-zero supersingular element.

To prove the theorem, we pick a non-zero supersingular element $v$ in $C[\Gamma \backslash G/\mathcal{B}]$ and an irreducible quotient $\pi$ of the subrepresentation of $\rho^C$ generated by $v$. Then $\pi$ is admissible ($\rho^C$ is admissible and $a = 0$) and contains a non-zero $\mathcal{B}$-invariant supersingular element, hence is supercuspidal by the supercuspidality criterion.

We explain now the meaning of supersingular, why we exclude $\text{PGL}(m,D)$ and $\text{PU}(h)$ and how we prove Prop[1.3]

We choose a minimal parabolic $F$-subgroup $\mathcal{B}$ of $G$ containing a maximal $F$-subtorus $T$. Bruhat and Tits associated to them an affine Coxeter system $(W, S)$, parameters $q_s = q^{d(s)}$
for $s \in S$, where $q$ is the residue field of $F$ and the $d(s)$ are integers $\geq 1$, and a commutative group $\Omega$ acting on the Dynkin diagram $Dyn$ of $(W,S)$ decorated with the parameters $d(s)$. The diagram $Dyn$ is the completed Dynkin diagram of a reduced root system $\Sigma$.

The Iwahori Hecke ring $H(G^{sc}, \mathcal{B}^{sc})$ of an Iwahori subgroup $\mathcal{B}^{sc}$ of the simply connected cover $G^{sc}$ of the derived group of $G$ is isomorphic to the Hecke ring $H := H(W,S,q_s)$ associated to $(W,S),(q_s) \in S$, and the Iwahori Hecke ring $H(G, \mathcal{B})$ is isomorphic to $H := \mathbb{Z}[\Omega] \rtimes H(W,S,q_s)$. To each cocharacter $\mu \in X_*(T)$ of $T$ is associated a central element $z_\mu$ of $H$ [Vig17]. Write $H_C := C \otimes H$. When $c = p$, an element $v$ in a right $H_C$-module is called supersingular if $vz_\mu^n = 0$ for all $\mu \in X_*(T)$ dominant but $\mu^{-1}$ not dominant and $n \in \mathbb{N}$ large.

A simple right $H_C$-module $M_C$ is called irreducible if $H_C$ has a natural $\mathcal{B}$-invariant of an irreducible admissible square integrable modulo center $C$-representation of $G$ is called discrete. If $G$ is semi-simple, $M_C$ is discrete if and only if the simple components of its restriction to $H_C^{sc}$ are discrete; the equivalence uses Casselman’s criterion of square integrability [Cas, §2.5] and the Bernstein Hecke elements [Vig16 Cor. 5.28].

Assume that $G$ is absolutely simple adjoint. If the type of $\Sigma$ is $A_\ell$ ($\ell \geq 2$), the parameters are equal and $G = PGL_m(D)$ for a $d^2$-dimensional central division $F$-algebra $D$ ($m \geq 2, d \geq 1$). If the type of $\Sigma$ is $A_1$ and the two parameters are equal, then $G = PU(h)$ is the adjoint unitary group of a split hermitian form $h$ in 3 variables over a ramified quadratic extension of $F$ [Tit79 §4].

When the type of $\Sigma$ is different from $A_\ell$ ($\ell \geq 1$) if the parameters are equal, we find a simple discrete $H_C$-module $M_C$ with an $H^{\mathbb{Z}[q^{1/2}]}$-integral structure $M$ of reduction, modulo a maximal ideal $\mathfrak{P}$ of $\mathbb{Z}[q^{1/2}]$, a supersingular $H_{F_p}$-module $M_{F_p}$:

- When the type is $B_\ell$ ($\ell \geq 3$), $C_\ell$ ($\ell \geq 2$), $F_4$, $G_2$, or $A_1$ with two distinct parameters, then $H_C^{sc}$ admits a discrete character which is not the special character [Bor76 5.9]. By extending or inducing such a character, we construct a simple discrete right $H_C$-module of dimension $\leq 2$ with a natural $H$-integral structure. As we avoided the Steinberg character, the reduction modulo $p$ of the $H$-integral structure is a supersingular $H_{F_p}$-module. See Prop. 5.2.

- When the type is $D_\ell$ ($\ell \geq 3$), $E_6$, $E_7$, $E_8$, the group $G$ is $F$-split [Tit79 §4]. The image by a natural involution of $H_C$ of the reflection right $H_C$-module of $C$-dimension $|S|$ is discrete [Lus83 4.23], has a natural $H^{\mathbb{Z}[q^{1/2}]}$-integral structure of reduction modulo a maximal ideal $P$ of $\mathbb{Z}[q^{1/2}]$ a supersingular $H_{F_p}$-module. See Prop. 5.3.

Now, we find $\Gamma$ such that $C[\Gamma \setminus G/\mathcal{B}]$ contains a non-zero supersingular element. The existence of discrete cocompact subgroups $\Gamma$ in $G$ is ensured by $a = 0$ [Mar91]. By the $p$-adic version of the de George-Wallach limit multiplicity formula ([DKV84 Appendix3, Prop.] plus [Kaz86 Thm.K]), the simple discrete $H_C$-module $M_C$ embeds in $C[\Gamma \setminus G/\mathcal{B}]$ for some discrete cocompact subgroup $\Gamma$ of $G$. All the $H^{\mathbb{Z}[q^{1/2}]}$-integral structures of $M_C$ have the same semi-simplification hence have supersingular reduction modulo $P$, and $M' := M_C \cap \mathbb{Z}[q^{1/2}][\Gamma \setminus G/\mathcal{B}]$ is another $H^{\mathbb{Z}[q^{1/2}]}$-integral structure of $M_C$ of reduction modulo $P$ a $H_{F_p}$-submodule $M_{F_p}'$ of $\mathbb{F}_p[\Gamma \setminus G/\mathcal{B}]$. See Prop. 6.3. The scalar extension from $\mathbb{F}_p$ to $C$ of the supersingular $H_{F_p}$-module $M_{F_p}'$ is a non-zero supersingular $H_{C}^{\mathbb{Z}[q^{1/2}]}$-submodule of $C[\Gamma \setminus G/\mathcal{B}]$. This ends the proof of Prop 4.3.

A similar argument for an irreducible admissible supercuspidal complex representation of $G$ produces an integral structure with admissible reduction (Cor. 6.5).

\[1\] In Borel, the Iwahori group is the fixer $\tilde{Z}_0\mathcal{B}$ of an alcove, where $\tilde{Z}_0$ is the maximal compact subgroup of a minimal Levi subgroup; if $G$ is $F$-split, $\tilde{Z}_0\mathcal{B} = \mathcal{B}$.
Returning to $G$ general, Kret proved that $G$ admits irreducible admissible supercuspidal complex representations \cite{Kre12} (if $a = 0$ \cite{BP16}). We extend this to any field $C$ of characteristic 0 in \S8.

**Proposition 1.4.** [Change of coefficient field] If $G$ admits an irreducible admissible supercuspidal representation over some field of characteristic $c$, supposed to be finite if $c = p$, then $G$ admits an irreducible admissible supercuspidal representation over any field of characteristic $c$.

We reduce the construction of an irreducible admissible supercuspidal representation of $G$ to the case of $G$ absolutely simple and adjoint in \S9.

**Proposition 1.5.** [Reduction to a simple adjoint group] Assume $a = 0$ and $C$ algebraically closed or finite. If any connected absolutely simple adjoint $p$-adic group admits an irreducible admissible supercuspidal $C$-representation, then any connected reductive $p$-adic group admits an irreducible admissible supercuspidal $C$-representation.

Summarizing our results, we obtain:

**Theorem 1.6.** Assume $(a, c) = (0, p)$. Then $G$ admits an irreducible admissible supercuspidal $C$-representation, if:

- $PGL(m, D)$ and $PU(h)$ (as in Thm.1.7) admit an irreducible admissible supercuspidal representation over some finite field of characteristic $p$.

When $(a, c) = (0, p)$, $PGL(m, D)$ and $PU(h)$ should also admit an irreducible admissible supercuspidal $C$-representation. We miss $PGL(m, D)$ and $PU(h)$ because we do not know the integrality properties of the unramified irreducible admissible complex representations of $G$ corresponding to the integral reflections modules of the generalized affine Hecke algebras of supersingular reduction modulo $p$. The missing cases will probably be completed by Herzig with a global method for $PGL(m, D)$ and by Koziol with coefficient systems on the tree for $PU(h)$.

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2. **Iwahori Hecke ring**

We review in this section the parts of the theory of Iwahori Hecke algebras \cite{Vig16, Vig14, Vig17} appearing in the proof of the key proposition 1.3.

Let $G$ be a reductive connected $F$-group, $T$ a maximal $F$-split subtorus of $G$, $B$ a minimal $F$-parabolic subgroup of $G$ containing $T$, and $x_0$ a special point of the apartment of the
adjoint Bruhat-Tits building defined by $T$. Associated to the triple $(G, T, B, x_0)$ are: the center $Z(G)$ of $G$, the root system $Φ$, the set of simple roots $Δ$, the $G$-centralizer $Z$ of $T$, the normalizer $N$, the unipotent radical $U$ of $B$, (hence $B = ZU$), the triples $(G^s, T^s, B^s)$ and $(G^{ad}, T^{ad}, B^{ad})$ for the simply connected covering and the adjoint group of the derived subgroup of $G$, the natural homomorphisms $G^s := G^s(F) \xrightarrow{i_{sc}} G = G(F) \xrightarrow{ad} G^{ad} = G^{ad}(F)$, an alcove $C$ of the apartment with vertex $x_0$, the Iwahori subgroups $B, B^sc, B^{ad}$ of $G, G^s, G^{ad}$ fixing $C$, the Iwahori Hecke ring $H(G, B) := \text{End}_{Z \G} \Z \G$ and similarly for the simply connected and adjoint semi-simple groups.

The natural ring homomorphism $H(G^s, B^sc) \to H(G, B)$ induced by $i_{sc}$ is injective and we identify $H(G^s, B^sc)$ with a subring of $H(G, B)$. There is a canonical isomorphism $H(G^s, B^sc) \xrightarrow{j^sc} H(W, S, q_s)$ where $H := H(W, S, q_s)$ is the Hecke ring of an affine Coxeter system $(W, S)$ with parameters $(q_s = q^{d(s)})_{s \in S}$ where $q$ is the residue field of $F$ and the $d(s)$ are integers $\geq 1$. The Dynkin diagram of $(W, S)$ is the completed Dynkin diagram $Dyn$ of a reduced root system $Σ$. The image $Ω := Ω_G$ of the Kottwitz morphism of $G$ acts on the Dynkin diagram $Dyn$ decorated with the parameters $(d_s)_{s \in S}$, and the isomorphism $j^sc$ extends to an isomorphism

\[(2.1) \quad H(G, B) \xrightarrow{j^sc} \Z[Ω] \times H(W, S, q_s),\]

The Iwahori Hecke ring of $G$ is determined by the type of $Σ$, the parameters and the group $Ω$ acting on the decorated Dynkin diagram $Dyn$. The quotient of $Ω$ by the fixer of $C$ in $Ω$ is isomorphic to a subgroup $Ψ$ of the group of automorphisms $\text{Aut}(W, S, d_s)$ of the decorated Dynkin diagram. The generalized affine Hecke ring $\tilde{H} := \Z[Ω] \times H(W, S, q_s)$ is a free $\Z$-module of basis $T_w$ for $w \in \tilde{W} := W \times Ω$ satisfying the braid and quadratic relations:

\[(2.2) \quad T_w T_{w'} = T_{ww'} \text{ for } w, w' \in \tilde{W}, \quad \ell(w) + \ell(w') = \ell(ww'),\]

where $\ell$ is the length on $W$ associated to $S$ extended to $\tilde{W}$ by $\ell(ww) = \ell(wu) = \ell(w)$ for $u \in Ω, w \in W$. The linear map $T_w \mapsto (-1)^{\ell(w)} T_w$ for $w \in \tilde{W}$, where for $w = u s_1 \ldots s_n$ with $u \in Ω, s_i \in S, n = \ell(w)$,

\[(2.3) \quad T_{w^{-1}} = q_w, \text{ where } q_w := q_{s_1} \ldots q_{s_n}, \quad T_s := T_s - q_s + 1, \quad T_w := T_u T_{s_1} \ldots T_{s_n},\]

is an automorphism of $\tilde{H}$.

The unique parahoric subgroup $Z_0$ of $Z := \Z(F)$ is contained in the maximal compact subgroup $\tilde{Z}_0$. When $G$ is $F$-split or semi-simple simply connected, $Z_0 = \tilde{Z}_0$. The group $N/Z_0$ is isomorphic to $\tilde{W}$, acts on the apartment and forms a system of representatives of the double classes of $G$ modulo $B$. The subgroup $Λ := Z/Z_0$ of $N/Z_0$ is commutative finitely generated of torsion $\tilde{Z}_0/Z_0$ acting by translation on the apartment, the quotient map $N/Z_0 \to N/Z$ splits, identifying the (finite) Weyl group $W_0$ of $Σ$ with the fixer in $W$ of a special vertex of the alcove $C$. The semi-direct product $Λ \times W_0$ is equal to $W$ and $Λ^{sc} \times W_0 = W$ where $Λ^{sc} := Λ \cap W$. An element $λ \in Λ$ is called dominant (and $λ^{-1}$ anti-dominant), if $z(U \cap B) z^{-1} \subset (U \cap B)$ for $z \in Z$ lifting $λ$. The dominant monoid $Λ^+$ consists of dominant elements of $Λ$.

The cocharacter group $X_α(T)$ of $T$ is isomorphic to $T/T_0 \simeq T Z_0/Z_0 = Λ_T$ by the $W_0$-equivariant map $μ \mapsto λ_μ := μ(p_F)Z_0/Z_0$ for a fixed an uniformizer $p_F$ of $F$. A cocharacter $μ$ is dominant if $λ_μ$ is. For $λ \in Λ$ and $n \in \N$ large, $λ^n \in Λ_T$, and $(χ, λ^n)$ is defined for any character $χ \in X^*(T)$. The fixer of the alcove $C$ in $Λ$ is $Λ_{Z(G)}\tilde{Z}_0 = Z(G)\tilde{Z}_0/Z_0$ and
Lemma 2.1. The fixer of the alcove $C$ in $\Omega$ is $\Lambda_{Z(G)\hat{Z}_0}$.

Proof. Write $u \in \Omega$ as $u = \lambda w_0$ with $\lambda \in \Lambda, w_0 \in W_0$. As $w_0$ fix a special point of $C$, $\lambda$ does also. As $\lambda$ acts by translation on the apartment, $\lambda$ fixes $C$. As $\lambda$ and $u$ fix $C$, $w_0$ fixes also $C$ hence $w_0 = 1$. □

The action of $\Omega$ on the alcove $C$ and the embeddings of $\Lambda$ and $\Omega$ into $\hat{W}$ induce isomorphisms
\begin{equation}
\Omega/\Lambda_{Z(G)\hat{Z}_0} \to \Psi, \quad \Lambda/(\Lambda_{Z(G)\hat{Z}_0} \times \Lambda^{sc}) \to \hat{W}/(\Lambda_{Z(G)\hat{Z}_0} \times W) \to \Omega/\Lambda_{Z(G)\hat{Z}_0}.
\end{equation}

Lemma 2.2. The subgroup $\Lambda_{Z(G)} \times \Lambda^{sc}$ of $\Lambda$ is finitely generated of finite index.

The submonoid $\Lambda_{Z(G)} \times \Lambda^{sc}$ of the dominant monoid $\Lambda^+$ is finitely generated of finite index.

Proof. The commutative group $\Lambda_{Z(G)} \times \Lambda^{sc}$ is finitely generated and a finite index subgroup of $\Lambda$, as $\hat{Z}_0/Z_0$ is finite and (2.4) implies that $\Lambda/(\Lambda_{Z(G)\hat{Z}_0} \times \Lambda^{sc}) \cong \Psi$. Gordan’s lemma implies the second assertion (as in the proof of [HV15, 7.2 Lemma]). □

The $\hat{W}$-conjugacy class of $\lambda \in \Lambda$ is the $W_0$-orbit of $\lambda$. A basis of the center of $\hat{H}$ [Vig14, Thm.1.2] is
\[ \sum_{\lambda \in \mathcal{O}} E_\lambda \quad \text{for the } W_0\text{-orbits } \mathcal{O} \subset \Lambda, \]
where $E_\lambda$ for $\lambda \in \Lambda$ are the integral Bernstein elements of $\hat{H}$ [Vig16, Cor. 5.28, Ex.5.30]:
\begin{equation}
E_\lambda = \begin{cases} T_\lambda & \text{if } \lambda \text{ is anti-dominant} \\ T_\lambda^* & \text{if } \lambda \text{ is dominant} \end{cases},
E_{\lambda_1} E_{\lambda_2} = (q_{\lambda_1 \lambda_2} q_{\lambda_2 \lambda_1}^{-1})^{1/2} E_{\lambda_1 \lambda_2} \text{ for } \lambda_1, \lambda_2 \in \Lambda.
\end{equation}

When $\lambda_1, \lambda_2$ are both dominant (or anti-dominant), $E_{\lambda_1} E_{\lambda_2} = E_{\lambda_1 \lambda_2}$. For $\mu \in X_*(T)$, let $\mathcal{O}_\mu$ denote the $W_0$-orbit of $\mu$ and write $z_\mu := \sum_{\lambda \in \mathcal{O}_\mu} E_\lambda$. Any $W_0$-orbit $\mathcal{O}_\mu$ contains a unique dominant (resp. anti-dominant) cocharacter $\mu^+$ (resp. $\mu^-$), and $z_\mu = z_{\mu^+} = z_{\mu^-}$.

The invertible elements in the dominant monoids $\Lambda^+$ and $X_*(T)^+$ are the subgroups $\Lambda_{Z(G)\hat{Z}_0}$ and $\cap_{\alpha \in \Delta} \text{Ker } \alpha$. Only the trivial element is invertible in the dominant monoid $\Lambda^{sc}$. Write $\Lambda_{Z(G)} := Z(G)\hat{Z}_0/Z_0$.

The generalized affine ring $\hat{H}$ contains the commutative subring $\mathcal{A}$ of $Z$-basis $(E_\lambda)_{\lambda \in \Lambda}$. When $G = Z$, the Bernstein elements are simply the classical elements $T_\lambda^Z$, and the Iwahori Hecke ring $H(Z, Z_0)$ is isomorphic to $Z[Z]$. In general, $\mathcal{A}$ is not isomorphic to $\mathbb{Z}[\Lambda]$ but the subring $A^+$ of basis $(E_\lambda)_{\lambda \in \Lambda^+}$ is isomorphic to $Z[\Lambda^+]$. Denote by $A^{sc}, A_T, A_T^{sc}, A_T^+$ the subrings of respective bases $(E_\lambda)_{\lambda \in \Lambda^{sc}}, (E_\mu)_{\mu \in X_*(T)}, (E_\lambda)_{\lambda \in \Lambda^{sc}_T}, (E_\mu)_{\mu \in X_*(T)^+}$.

Lemma 2.3. $\mathcal{A}$ is finitely generated as an $A_T$-module and as a $Z[\Lambda_{Z(G)}] \times \mathcal{A}_{sc}$-module.

$A^+$ is finitely generated as an $A_T^+$-module and as a $Z[\Lambda_{Z(G)}] \times A_{sc}^+$-module.

Proof. For $T$ [Vig14, Lemma 2.14, 2.15]. Otherwise, Lemma 2.2 □

3. Supersingular and Discrete Modules

Let $C$ be a field of characteristic $c$ and the Iwahori Hecke $C$-algebras $H_C(G^{sc}, \mathcal{B}^{sc}) := C \otimes_Z H(G^{sc}, \mathcal{B}^{sc})$ and $H_C(G, \mathcal{B}) := C \otimes_Z H(G, \mathcal{B})$, isomorphic to the affine and generalized affine $C$-algebras $H_C = C \otimes_Z H(W, S, q_s)$ and $\hat{H}_C = C[\Omega] \otimes_Z H(W, S, q_s)$.

This section introduces the supersingular right $\hat{H}_C$-modules when $c = p$ (there are none when $c \neq p$) and the discrete simple right $H_C(G, \mathcal{B})$ modules.
**Definition 3.1.** Let \( M \) be a non-zero right \( \tilde{H}_C \)-module. An element \( v \in M \) is called supersingular if and only if \( vz^\mu_n = 0 \) for all dominant \( \mu \in X_\ast(T) \) with \( \mu^{-1} \) not dominant, and some large positive integer \( n \). The \( \tilde{H}_C \)-module \( M \) is called supersingular when all its elements are supersingular\(^2\).

**Remark 3.2.** \( v \) is supersingular if and only if \( vz^\lambda_n = 0 \) for any \( \lambda \in \Lambda \setminus \Lambda_{Z(G)Z_0} \) and large \( n \). We can restrict to \( \mu \) (or \( \lambda \)) dominant, or anti-dominant.

**Fact 3.3.** - A simple \( \tilde{H}_C \)-module \( M \) is finite dimensional, and is semi-simple as an \( H_C \)-module.
- A \( \tilde{H}_C \)-module is supersingular if and only if its restriction to \( H_C \) is supersingular \([\text{Vig17, Cor.6.13}]\).
- The simple supersingular \( H_C \)-modules are \([\text{Vig17, Cor.6.13}]\) the characters which are not special or trivial (see the next section) when \( C \) is “large” of characteristic \( c = p \) \([\text{Vig17, Cor.6.13}]\).

We denote by \( \text{Mod}_C(G, \mathfrak{B}) \) and \( \text{Mod}_C(H(G, \mathfrak{B})) \) the categories of \( C \)-representations of \( G \) generated by their \( \mathfrak{B} \)-invariant vectors and of right \( H_C(G, \mathfrak{B}) \)-modules. The \( \mathfrak{B} \)-invariant functor \( \pi \mapsto \pi^{\mathfrak{B}} : \text{Mod}_C(G, \mathfrak{B}) \to \text{Mod}_C(H(G, \mathfrak{B})) \) has a left adjoint \( \Sigma : \tau \mapsto \tau \otimes_{H_C(G, \mathfrak{B})} C[\mathfrak{B}\setminus G] \).

**Fact 3.4.** When \( c \neq p \), the functor \( \pi \mapsto \pi^{\mathfrak{B}} \) induces a bijection between the isomorphism classes of the irreducible \( C \)-representations \( \pi \) of \( G \) with \( \pi^{\mathfrak{B}} \neq 0 \) and of the simple right \( H_C(G, \mathfrak{B}) \)-modules \([\text{Vig96, I.6.3}]\). When \( C = \mathbb{C} \), the functors are inverse equivalences of categories (Bernstein-Borel-Casselman).

Let \( \pi \) be an irreducible complex representation of \( G \) with \( \pi^{\mathfrak{B}} \neq 0 \). We recall the classical properties of \( \pi \), including the Casselman’s criterion of square integrability modulo center, before giving the definition of a discrete simple right \( H_C(G, \mathfrak{B}) \)-module.

**Fact 3.5.** - a) \( \pi \) is isomorphic to a subrepresentation of \( \text{ind}_B^G \sigma \) where \( \sigma \) is a \( \mathbb{C} \)-character of \( Z \) trivial on \( Z_0 \).
- b) The representation of \( Z \) on the \( U \)-coinvariants \( \text{ind}_B^G \sigma \big|_U \) is semi-simple, trivial on \( Z_0 \) and contains a subrepresentation isomorphic to \( \pi_U \).
- c) The quotient map \( f : \pi \mapsto \delta_B^{-1/2} \pi_U \) induces an \( H_{Z[q^{-1/2}]}(Z, Z_0) \)-equivariant isomorphism \( \pi^{\mathfrak{B}} \to \pi_U^{\mathfrak{B}}(= \pi_U) \) for the Bernstein \( \mathbb{Z}[q^{-1/2}] \)-algebra embedding
  \[
  H_{Z[q^{-1/2}]}(Z, Z_0) \overset{\text{tr}}{\longrightarrow} H_{Z[q^{-1/2}]}(G, \mathfrak{B}) \quad \text{tr}_B(T \lambda^Z) = \theta_\lambda := q^{-1/2}_\lambda E(\lambda) \quad \text{for } \lambda \in \Lambda,
  \]
  that is, \( f(v \theta_\lambda) = f(v) T \lambda^Z \) for \( \lambda \in \Lambda, v \in \pi^{\mathfrak{B}} \) \([\text{Vig98, II.10.1}]\). Note that \( q_\lambda = \delta_B(z) \) where \( \delta_B \) is the modulus of \( B \) and \( z \in Z \) of image \( \lambda \in \Lambda \) and \( \theta_\lambda \theta_\lambda^2 = \theta_{\lambda_1 \lambda_2} \) for \( \lambda_1, \lambda_2 \in \Lambda \).
- d) Casselman’s criterion: \( \pi \) is square integrable modulo center \([\text{Cas} \S 2.5] \) if and only if its central character is unitary and
  \[
  |\chi(\mu(pF))| \leq 1 \quad \text{for all anti-dominant } \mu \in X_\ast(T) \text{ but } \mu^{-1} \text{ not anti-dominant},
  \]
for any character \( \chi \) of \( Z \) contained in \( \delta_B^{-1/2} \pi_U \) \([\text{Cas} \text{ Thm. 6.5.1}]\).

\(^2\text{In [Vig17, Def. 6.10] there is a different definition: there exists } n > 0 \text{ with } Mz^\mu_n = 0 \text{ for all } \mu \text{ not invertible in } X_\ast(T)\)

\(^3\text{There are no non-zero supersingular modules if } c \neq p\)
**Definition 3.6.** A simple right \( H_C(G, \mathfrak{B}) \)-module is called discrete when it is isomorphic to \( \pi^\mathfrak{B} \) for an irreducible admissible square integrable modulo center \( \mathbb{C} \)-representation \( \pi \) of \( G \).

**Proposition 3.7.** A simple right \( H_C(G, \mathfrak{B}) \)-module \( M \) is discrete if and only if any complex character \( \chi \) of \( A \) contained in \( M \) satisfies: the restriction of \( \chi \) to \( \Lambda_{Z(G)} \) is unitary and
\[
|\chi(\theta_\mu)| \leq 1 \text{ for any dominant } \mu \in X_s(T) \text{ but } \mu^{-1} \text{ not dominant, } \theta_\mu := \theta_{\lambda_\mu}.
\]

**Proof.** Let \( \chi : Z \to \mathbb{C}^* \) be a character trivial on \( Z_0 \). Writing \( \chi(z) = \chi(\lambda) \) for \( z \in Z \) of image \( \lambda \in \Lambda \) and noting \( f(v)T^Z_\lambda = z^{-1}f(v) \), we have in \( \mathcal{C} \)
\[
z^{-1}f(v) = \chi(z^{-1})f(v) \iff f(v\theta(\lambda)) = \chi(\lambda^{-1})f(v) \iff v\theta(\lambda) = \chi(\lambda^{-1})v.
\]
Hence \( \chi \) is contained in \( \delta_B^{-1/2} \pi_U \) if and only \( \chi^{-1} : A \to \mathbb{C}, \chi^{-1}(\theta_\lambda) := \chi^{-1}(\lambda) \), is contained \( \pi^\mathfrak{B} \). Apply Casselman’s criterion (Fact 3.3 (d)) (the inverse of an anti-dominant element is dominant).

**Remark 3.8.** Some authors see \( \pi^\mathfrak{B} \) as a left \( H_C(G, \mathfrak{B}) \)-module. One exchanges “left” and “right” by putting \( T_wv = vT_{w^{-1}} \) for \( w \in \tilde{W}, v \in \pi^\mathfrak{B} \). The left or right \( H_C(G, \mathfrak{B}) \)-module \( \pi^\mathfrak{B} \) is called discrete if \( \pi \) is square integrable modulo center. For left modules, the proposition holds true with anti-dominant instead of dominant.

**Lemma 3.9.** For a character \( \chi : A \to \mathbb{C} \), the following properties are equivalent:
1. \( \chi|\Lambda_{Z(G)} \) is unitary and \( |\chi(\theta_\mu)| \leq 1 \) for any dominant \( \mu \in X_s(T) \) with \( \mu^{-1} \) not dominant.
2. \( |\chi(\theta_\lambda)| \leq 1 \) for any dominant \( \lambda \in \Lambda \).
3. \( |\chi(\theta_\lambda)| = 1 \) for any \( \lambda \in \Lambda_{Z(G)} \) and \( |\chi(\theta_\lambda)| \leq 1 \) for any dominant \( \lambda \in \Lambda_{sc} \).

**Proof.** (ii) implies (i) because \( |\chi(\theta_\lambda)| \leq 1 \) for any \( \lambda \in \Lambda^+ \), implies \( |\chi(\theta_\lambda)| = 1 \) when \( \lambda \) is invertible in \( \Lambda^+ \), i.e. \( \lambda \in \Lambda_{Z(G)}Z_0 \). Conversely, (i) implies that for \( \lambda \in \Lambda^+ \) and \( n \in \mathbb{N} \) large with \( \lambda^n \in \Lambda^+ \) we have \( |\chi(\theta_\lambda^n)| \leq 1 \). This implies \( |\chi(\theta_\lambda)| \leq 1 \) hence (ii). The arguments of the equivalence between (i) and (iii) are similar using Lemma 2.2. \( \square \)

**Proposition 3.10.** A simple right \( H_C(G, \mathfrak{B}) \)-module \( M \) is discrete if and only if \( \Lambda_{Z(G)} \) acts on \( M \) by a unitary character and the simple components of \( M \) restricted to \( H_C(G^{sc}, \mathfrak{B}^{sc}) \) are discrete.

**Proof.** Lemma 3.9 (iii). \( \square \)

4. Characters

In this section, \( C \) is a field of characteristic \( c \) and \( G \) is absolutely simple. We give the characters \( H \to C \) which extend to \( \tilde{H} \). This is an exercice, already in the litterature when \( C = \mathbb{C} \) is the complex field [Bor76].

For distinct \( s, t \in S \), the order \( n_{s,t} \) of \( st \) is finite except if the type is \( A_1 \). In the finite case,
\[
(T_sT_t)^r = (T_tT_s)^r \text{ if } n_{s,t} = 2r, \quad (T_sT_t)^rT_t = T_t(T_sT_t)^r \text{ if } n_{s,t} = 2r + 1.
\]
The \( T_s \) for \( s \in S \) and the relations (2.2), (4.1) give a presentation of \( H \). A presentation of \( \tilde{H} \) is given by the \( T_u, T_s \) for \( u \in \Omega, s \in S \) and the relations (2.2), (4.1) and
\[
T_uT_u' = T_{uu'}, T_uT_s = T_{u(s)}T_u \text{ for } u \in \Omega, s \in S.
\]
Proof. When \( W \) is a \( p \)-adic reductive group and \( \Sigma \) is equal to \( C_\ell(\ell \geq 2) \) and different parameters \( d_2 \neq d_3 \), where \( \text{Aut}(W,S,d_i) = \{1\} \), \( \text{Aut}(W,S) = \mathbb{Z}/2\mathbb{Z} \).

Lemma 4.2. A map \((T_s)_{s \in S} \rightarrow C\) is the restriction of a character \( \chi : H \rightarrow C \) if and only if:

- when \( c \neq p \), it is constant and equal to \( \chi_i = -1 \) or \( q^{d_i} \) on each \( S_i \).
- when \( c = p \), its values are \(-1\) or \(0\). There are \( 2^{|S|} \) characters.

Proof. This follows from the presentation of \( H \). When \( c \neq p \), the \( T_w \) are invertible. When \( c = p \) [Vig17] Prop.2.2].

The unique character \( \chi : H \rightarrow C \) with \( \chi(T_s) = q_s \) (resp. \( \chi(T_s) = -1 \)) for all \( s \in S \) is called the trivial (resp. special) \( C \)-character. If they are equal then \( c \neq p \), 0.

A character \( \chi : H \rightarrow C \) extends to a character of \( \tilde{H} \) if and only if \( \chi(T_s) = \chi(T_{u,v}) \) for all \( s \in S \) and \( u \in \Omega \). When the image \( \Psi \) of \( \Omega \) in \( \text{Aut}(W,S,d_i) \) is trivial or when \( m = 1 \), any character of \( H \) extends to \( \tilde{H} \). The extensions are not unique in general. The trivial and special characters extend, their extensions are also called trivial and special.

Lemma 4.3. Assume \( c \neq p \). A character \( \chi = (\chi_i)_{1 \leq i \leq m} : H \rightarrow C \) extends to a character of \( \tilde{H} \) except for

- type \( A_1 \), equal parameters \( d_1 = d_2 \), \( \Psi \simeq \mathbb{Z}/2\mathbb{Z} \), when \( \chi_1 \neq \chi_2 \).
- type \( C_\ell(\ell \geq 2) \), equal parameters \( d_2 = d_3 \), \( \Psi \simeq \mathbb{Z}/2\mathbb{Z} \), when \( \chi_2 \neq \chi_3 \).

Proof. Assume \( m > 1 \) and \( \Psi \) not trivial. Then \( \Psi \simeq \text{Aut}(W,S) \simeq \mathbb{Z}/2\mathbb{Z} \) with three cases:

Type \( A_1 \) and equal parameters \( d_1 = d_2 \), say \( d \). Then \( \Psi \) permutes \( s_1, s_2 \). Only the special and trivial characters of \( H \) extends to \( \tilde{H} \).
Type $C_\ell (\ell \geq 2)$ and equal parameters $d_2 = d_3$, say $d$. Then $\Psi$ permutes $s_2, s_3$, only the $C$-characters of $H$ with $\chi_2 = \chi_3$ extend to $\tilde{H}$.

Type $B_\ell (\ell \geq 3)$. Then, $\Psi$ stabilizes $S_1$ and $S_2$ hence all $C$-characters of $H$ extend to $\tilde{H}$. \hfill \Box

We combine these results in a proposition:

**Proposition 4.4.** (i) $H_C$ admits $2^m$ characters $\chi$, they are all $\mathbb{Z}$-integral, and their reduction modulo $p$ are supersingular except for the special and trivial character.

(ii) $\tilde{H}_C$ admits a $\mathbb{Z}$-integral simple (left or right) module of supersingular reduction modulo $p$, and restriction $\chi \oplus \overline{\chi}$ to $H_C$, then:

- if $\chi_2, \chi_3 \neq \chi_1$,
- if $\chi_2, \chi_3 = \chi_1$ and $\chi_3 \neq \chi_2$,
- if $\chi_2, \chi_3 = \chi_1$ and $\chi_3 = \chi_2$.

Hence all $\mathbb{Z}$-integral complex characters of $H_C$, except the special and trivial character, extend to $\mathbb{Z}$-integral complex characters of $\tilde{H}_C$, of supersingular reduction modulo $p$ if $\chi$ is not special or trivial.

**Proof.** (i) Lemma 4.2, Fact 5.3. The reduction modulo $p$ of a special of trivial character of $H$ is not supersingular.

(ii) When $\chi$ extends to $\tilde{H} = H \otimes \mathbb{Z}[\Omega]$ we can extend it trivially ($\Omega$ acts by the trivial character). When $\chi$ does not extend, the normal subgroup $\Lambda_{Z(G)\mathbb{Z}}$ of $\Omega$ has index 2 (2.4).

We extend $\chi$ trivially to $H \otimes \mathbb{Z}[\Lambda_{Z(G)\mathbb{Z}}]$ and then induce to $\tilde{H}$. \hfill \Box

5. Discrete simple modules with supersingular reduction

In this section, $G$ is absolutely simple and $P$ is a maximal ideal of $\mathbb{Z}[q^{1/2}]$ of residue field $\mathbb{F}_p$.

**Proposition 5.1.** [Key result] There exists a simple discrete right $\tilde{H}_C$-module $M_C$ with a $\mathbb{Z}[q^{1/2}]$-integral structure $M$ of supersingular reduction modulo $P$, except if the type of $\Sigma$ is $A_\ell (\ell \geq 1)$ and the parameters are equal.

**Proof.** A special complex character of $H_C$ extending the $\mathfrak{B}^{sc}$-invariant of the Steinberg complex representation of $G^{sc}$, is discrete and integral but its reduction modulo $p$ is not supersingular. A trivial complex character is not discrete. The discrete non-special characters complex characters are integral of supersingular reduction modulo $p$. The discrete non-special characters $\chi : H \to \mathbb{C}$ were computed by Borel [Bor76, 5.8] and those extending to $\tilde{H}$ have been described in Prop. 4.3(i). Applying Prop. 5.10, we have:

- if a discrete non-special character of $H_C$ extends to $\tilde{H}_C$, we obtain a simple discrete integral $\tilde{H}_C$-module $M_C$ of supersingular reduction modulo $p$ and of dimension 1.
- if $H_C$ admits discrete non-special characters complex characters but none extends to $\tilde{H}_C$. Using Prop. 4.3(ii), we obtain a simple discrete integral $\tilde{H}_C$-module $M_C$ of supersingular reduction modulo $p$ and of dimension 2.
- if the special character is the only discrete character of $H_C$, the reflection left $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$-module $M$, free of rank $|S|$ over $\mathbb{Z}[q^{1/2}]$ has a supersingular reduction modulo $P$. The left $\tilde{H}_C$-module $M_C$ is discrete when the type is different from $A_\ell (\ell \geq 1)$ [Lus83].

These are the main lines of the proof. We give the details in the rest of this section. \hfill \Box

Instead of $\chi = (\chi_i)$ as in section §4 we write $\chi = \chi(\epsilon_i)$ where $\epsilon_i = \begin{cases} -1 & \text{if } \chi_i = -1 \\ d_i & \text{if } \chi_i = q^{d_i} \end{cases}$ for $1 \leq i \leq m$. 


Proposition 5.2. There exists a discrete non special right $\tilde{H}_C$-module $M_C$ when the type of $\Sigma$ is $B_4(\ell \geq 3), C_6(\ell \geq 2), F_4, G_2, A_1$ with $d_1 \neq d_2$. The $\mathbb{C}$-dimension $r$ of $M_C$ is:

- $r = 1$ when $\Psi$ is trivial or when the type is $B_4(\ell \geq 3), C_6(\ell \geq 6), F_4, G_2, A_1$ with $d_1 \neq d_2$, $C_2$ with parameters $(1, 2, 2), (2, 3, 3)$, $C_3$ with parameters $(2, 1, 1)$, $C_4$ with parameters $((d, d, d), (2, 1, 1), C_5$ with parameters $(d, d, d), (2, 1, 1), (2, 3, 3)$.

- $r = 2$ when the type is $C_2$ with parameters $(d, d, d), (2, 1, 1)$, $C_3$ with parameters $(d, d, d), (1, 2, 2), (2, 3, 3)$, $C_4$ with parameters $(1, 2, 2), (2, 3, 3), C_5$ with parameters $(1, 2, 2)$; then $M_C$ extends the $H_C$-module $\chi(-1, -1, d) \oplus \chi(-1, -1, -1)$.

Proof. When $m = 1$ the only discrete complex character is the special one $[\text{Bo}76] 5.7$. When $m > 1 H_C$ admits a discrete non-special character except for the type $A_1$ and parameters $d_1 = d_2 [\text{Bo}76] 5.8$.

Assume $m > 1$ or the type is $A_1$ and $d_1 \neq d_2$. Some discrete non-special character of $H_C$ extends to $\tilde{H}_C$ with the exception: type $C_4(\ell \geq 2)$, different $\epsilon_2 \neq \epsilon_3$, $\Psi \simeq \mathbb{Z}/2\mathbb{Z}$, parameters $d_2 = d_3$, say $d$, $- d_1 = d$ and $\ell = 2, 3$.

- $d_1 \neq d$ and $\ell = 2$ with parameters $(2, 1, 1)$, or $\ell = 3, 4$ with parameters $(1, 2, 2), (2, 3, 3)$, or $\ell = 5$ with parameters $(1, 2, 2)$.

In the exceptional case, no discrete character of $H_C$ extends to $\tilde{H}_C$; $\chi(-1, -1, d), \chi(-1, -1, -1)$ are discrete characters of $H_C$, and the simple $\tilde{H}_C$-module $\chi(-1, -1, d) \oplus \chi(-1, -1, -1)$ as a $H_C$-module (Prop. 4.4) is discrete by Prop. 3.10.

We consider now the types $D_\ell(\ell \geq 2), E_6, E_7, E_8$. Then $G$ is $F$-split and for distinct $s, t \in S$, the order $n_{s,t}$ of $st$ is 2 or 3 $[\text{Ir}79] \S 4$.

Proposition 5.3. Assume that the type of $\Sigma$ is $D_\ell(\ell \geq 2), E_6, E_7, E_8$. Then, the image $M$ by the automorphism $T_w \mapsto (-1)^{\ell(w)} T_w$ of $H(G, \mathfrak{B})$ of the reflection right $H_{F[1/2]}(G, \mathfrak{B})$-module has a supersingular reduction modulo $P$ and $M_C$ is a discrete simple right $H_C(G, \mathfrak{B})$-module of dimension $\lvert S \rvert$.

Proof. The reflection left $H_{F[1/2]}(G, \mathfrak{B})$-module is the free $\mathbb{Z}[1/2]$-module of basis $(e_s)$ for $s \in S$ with the structure of $H_{F[1/2]}(G, \mathfrak{B})$-module satisfying for $s, t \in S$, $u \in \Omega$,

$$T_u(e_t) = e_{u(t)}, \quad T_s(e_t) = \begin{cases} -e_t & \text{for } s = t, \\ qe_t & \text{for } s \neq t, n_{s,t} = 2, \\ qe_t + q^{1/2}e_s & \text{for } s \neq t, n_{s,t} = 3. \end{cases}$$

of image by the automorphism $T_w \mapsto (-1)^{\ell(w)} T_w$ for $w \in \tilde{W}$, satisfying

$$T_u(e_t) = e_{u(t)}, \quad T_s(e_t) = \begin{cases} qe_t & \text{for } s = t, \\ -e_t & \text{for } s \neq t, n_{s,t} = 2, \\ -e_t + q^{1/2}e_s & \text{for } s \neq t, n_{s,t} = 3. \end{cases}$$

The reduction modulo $P$ of this left $H_{F[1/2]}(G, \mathfrak{B})$-module, say $M$, is the $F_p$-vector space of basis $(e_s)$ for $s \in S$ with the structure of left $H(G, \mathfrak{B})$-module satisfying for $s, t \in S$, $u \in \Omega$,

$$T_u(e_t) = e_{u(t)}, \quad T_s(e_t) = \begin{cases} 0 & \text{for } s = t, \\ -e_t & \text{for } s \neq t, \end{cases}$$

4As $d_1 \neq d$, the possible parameters $(d_1, d, d)$ are $(2, 1, 1)(1, 2, 2), (2, 3, 3) [\text{Ir}79] \S 4$
The restriction to $H_{F_p}(G^\text{sc}, \mathfrak{B}^\text{sc})$ of this $H_{F_p}(G, \mathfrak{B})$-module is the direct sum of the characters

$$\chi_s(T_t) = \begin{cases} 0 & \text{for } s = t, \\ -1 & \text{for } s \neq t. \end{cases}$$

These characters are supersingular hence the reduction of $M$ modulo $P$ is supersingular (Fact 3.3). The scalar extension of $M$ from $\mathbb{Z}[q^{1/2}]$ to $\mathbb{C}$ is a discrete simple $H_{\mathbb{C}}(G, \mathfrak{B})$-module $M_{\mathbb{C}}$ [Lus83, 4.23]. The properties (discrete scalar extension of $\mathbb{C}$ and supersingular reduction modulo $P$) remain true if one sees as $M$ as a right $H_{\mathbb{Z}[q^{1/2}]}(G, \mathfrak{B})$-module $(T_w v = v T_w^{-1}$ for $w \in \hat{W}, v \in M)$.

6. Admissible integral structure via discrete cocompact subgroups

Let $E$ be a number field of ring of integers $O_E$, $P_E$ a maximal ideal of $O_E$ with residue field $k = O_E/P_E$, and $C/E$ a field extension.

**Definition 6.1.** We say that a $C$-representation $\pi$ of $G$

a) descends to $E$ if there exists an $E$-representation $V$ of $G$ and a $G$-equivariant $C$-linear isomorphism $\varphi : C \otimes_{O_E} V \to \pi$. We call $\varphi$ (and more often $V$) an $E$-structure of $\pi$.

b) is $O_E$-integral if $\pi$ contains a $G$-stable $O_E$-submodule $L$ such that, for any open compact subgroup $K$ of $G$, the $O_E$-module $L^K$ is finitely generated, and the natural map $\varphi : C \otimes_{O_E} L \to \pi$ is an isomorphism. We call $\varphi$ (and more often $L$) an $O_E$-integral structure of $\pi$ (we say integral if $O_E = \mathbb{Z}$). The $G$-equivariant map $L \to k \otimes_{O_E} L$ (and more often the $k$-representation $k \otimes_{O_E} L$ of $G$) is called the reduction of $L$ modulo $P_E$. When $k \otimes_{O_E} L$ is admissible for all $P_E$, we say that $L$ is admissible.

We give analogous definitions for a $H_{\mathbb{C}}(G, \mathfrak{B})$-module.

For any commutative ring $A$ and a discrete cocompact subgroup $\Gamma$ of $G$, let $\rho^\Gamma_A$ denote the smooth $A$-representation of $G$ acting by right translation on

$$A(\Gamma \backslash G) = \{ f : G \to A \mid f(\gamma g k) = f(g) \ (\gamma \in \Gamma, g \in G, k \in K_f) \}$$

where $K_f$ is some open compact subgroup of $G$ depending on $f$. The complex representation $\rho^\Gamma_C$ of $G$ has an admissible integral structure $\rho^\Gamma := \rho^\Gamma_{E_C}$. The reduction of $\rho^\Gamma$ modulo a prime number $c$ is the admissible representation $\rho^\Gamma_{E_c}$. The next proposition supposes $G$ semi-simple only to simplify.

**Proposition 6.2.** Assume $a = 0$ and $G$ semi-simple. If $\pi$ is a square integrable $C$-representation of $G$, then there exists a discrete cocompact subgroup $\Gamma$ of $G$ such that $\text{Hom}_G(\pi, \rho^\Gamma_C) \neq 0$.

**Proof.** As $a = 0$, there exists a strictly decreasing sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of discrete cocompact subgroups of $G$, normal of finite index in $\Gamma = \Gamma_0$, [BH78], [Mar91], IX (4.7) (D) Thm. and (4.8) Corollary (iv), [Rog90], Prop. 1.3]. The normalized multiplicity of $\pi$ in $\rho^\Gamma_C$ is

$$m_{\Gamma, d\delta}(\pi) := \text{vol}_\Gamma \text{dim}_C(\text{Hom}_G(\pi, \rho^\Gamma_C))$$

where $\text{vol}_\Gamma$ is the volume of $\Gamma \backslash G$ for a $G$-invariant measure induced by a Haar measure on $G$. By the limit multiplicity formula, the sequence $(m_{\Gamma_n, d\delta}(\pi))$ converges, and its limit is not 0 ([DKV84, Appendix 3] plus [Kaz86]).

**Proposition 6.3.** Assume $a = 0$. Let $\pi$ an irreducible $C$-representation of $G$ such that $\pi^{\mathfrak{B}} \neq 0$ and $\Gamma$ a discrete cocompact subgroup of $G$. 


1) If $\varphi : \mathbb{C} \otimes E V \to \pi$ is an $E$-structure of $\pi$, then $\varphi^{\mathfrak{g}} : \mathbb{C} \otimes E V^{\mathfrak{g}} \to \pi^{\mathfrak{g}}$ is an $E$-structure of $\pi^{\mathfrak{g}}$, and the natural map $\text{Hom}_{EG}(V, \rho_E^\Gamma) \to \text{Hom}_{CG}(\pi, \rho_E^\Gamma)$ is an isomorphism.

2) If $\psi : \mathbb{C} \otimes E W \to \pi^{\mathfrak{g}}$ is an $E$-structure of $\pi^{\mathfrak{g}}$, then $\Sigma(\psi) : \mathbb{C} \otimes E \Sigma(W) \to \pi$ is an $E$-structure of $\pi$.

3) An irreducible subrepresentation $V$ of $\rho_E^\Gamma$ admits an admissible $O_E$-integral structure $\rho_E^{\Gamma}$ of reduction modulo $P_E$ contained in $\rho_k^\Gamma$.

**Proof.** We recall a general result in algebra [Bou12 §12, n°2 Lemme 1]: Let $C/C'$ be a field extension, $A$ a $C'$-algebra. For $A$-modules $M,N$, the natural map

$$C \otimes_{C'} \text{Hom}_A(M,N) \to \text{Hom}_{C \otimes_{C'} A}(C \otimes_{C'} M, C \otimes_{C'} N)$$

is injective, and bijective if $C/C'$ is finite or the $A$-module $M$ is finitely generated.

1) Take $C/C' = \mathbb{C}/E$, $(M,N) = (E[\mathfrak{B}\setminus G], V)$ or $(V, \rho_E^\Gamma)$. Then (6.1) is an isomorphism because $E[\mathfrak{B}\setminus G]$ (resp. $V$) is an $E$-representation of $G$ generated by the characteristic function of $\mathfrak{B}$ (resp. irreducible).

2) The functors $\pi \mapsto \pi^{\mathfrak{g}}$-invariants and its left adjoint $\Sigma$ commute with scalar extension from $E$ to $C$ and when $C = \mathbb{C}$, are inverse equivalences of categories (Fact 5.4). Hence $\Sigma(\psi) : \Sigma(C \otimes E W) = C \otimes E \Sigma(W) \to \Sigma(\pi^{\mathfrak{g}}) = \pi$ is an isomorphism.

3) For any open compact subgroup $K$ of $G$, the $O_E$-module $(\rho_E^\Gamma)^K$ is free and $\rho_E^\Gamma$ contains $L := V \cap \rho_E^\Gamma$ as $O_E$-representations of $G$. The $O_E$-submodule $L^K$ of $(\rho_E^\Gamma)^K$ is finitely generated because the ring $O_E$ is nowhereethere. The natural linear $G$-equivariant isomorphism $E \otimes_{O_E} \rho_E^\Gamma \to \rho_E^\Gamma$ restricts to a linear $G$-equivariant isomorphism $E \otimes_{O_E} L \to V$ and therefore $L$ is an $O_E$-integral structure of $V$. We have $P_E \cap L = P_E(\Gamma \setminus G)^\infty \cap V = P_E(\Gamma \setminus G)^\infty \cap V = P_E \cap L$, and the reduction modulo $P_E$ of $L$, i.e. $(O_E/P_E) \otimes_{O_E} L = k \otimes_{O_E} L$ is contained in the reduction modulo $P_E$ of $\rho_E^\Gamma$, i.e. $\rho_k^\Gamma$ as $k$-representations of $G$.

As $\rho_k^\Gamma$ is admissible, $k \otimes L$ is also.

Although the ring $\mathbb{Z}[q^{1/2}]$ is not always the ring of integers $O_E$ of $E = \mathbb{Q}[q^{1/2}]$, the arguments of 3) apply to this ring. The key proposition Prop. 5.1 implies:

**Corollary 6.4.** Assume $a = 0$, $G$ absolutely simple, the type of $\Sigma$ is different from $A_\ell(\ell \geq 1)$ when the parameters are equal, and $M_C$ a right $H_C(G, \mathfrak{B})$-module as in Prop. 5.7. Then the irreducible square integrable $\mathbb{C}$-representation $\Sigma(M_C)$ of $G$ admits an admissible $\mathbb{Z}[q^{1/2}]$-integral structure.

When $G$ is semi-simple, an irreducible admissible supercuspidal $\mathbb{C}$-representation of $G$ descends to a number field [Vig96 II.4.9]. Propositions 6.2 and 6.3 imply:

**Corollary 6.5.** Assume $a = 0$ and $G$ semi-simple. An irreducible admissible supercuspidal $\mathbb{C}$-representation admits an admissible $O_E$-integral structure of reduction modulo $P_E$ contained in $\rho_k^\Gamma$, for some discrete cocompact subgroup $\Gamma$ of $G$.

### 7. Supercuspidality Criterion

This section contains the proof of Proposition 1.2. We use the description of the scalar extension of an admissible irreducible representation $\pi$ of $G$ (a theorem with Henniart [IV17]).

The commutant $D$ of $\pi$ is a division algebra of finite dimension over $C$. Let $E$ denote the center of $D$, $E_\delta/C$ the maximal separable extension contained in $E/C$ and $\delta$ the reduced degree of $D/E$. Let $C_{\text{alg}}$ be an algebraic closure of $C$ containing $E$ and $\pi_{C_{\text{alg}}}$ the scalar extension of $\pi$ from $C$ to $C_{\text{alg}}$. 

Fact 7.1. [IV17] The length of $\pi_{\text{Calg}}$ is $\delta[E : C]$ and

$$\pi_{\text{Calg}} \simeq \oplus_{i \in \text{Hom}(E_{s}, C^{\text{Calg}})} V_i$$

where $V_i$ is indecomposable of commutant $C^{\text{Calg}} \otimes_i E_{s} E$, descends to a finite extension $C'$ of $C$, has length $[E : E_{s}]$ and its irreducible subquotients are all isomorphic, say to $\rho_i$. The $\rho_i$ are admissible, of commutant $C^{\text{Calg}}$, Aut$_C(C^{\text{Calg}})$-conjugate, not isomorphic each other, descend to a finite extension $C'/C$, and the descents seen as $C$-representations of $G$, are $\pi$-isotypic of finite length.

It is straightforward to deduce, using that parabolic induction commutes with scalar extension from $C$ to $C^{\text{Calg}}$, Aut$_C(C^{\text{Calg}})$, descent and finite direct sums, the following compatibility between supercuspidality and scalar extension from $C$ to $C^{\text{Calg}}$.

Proposition 7.2. $\pi$ is supercuspidal if and only if some irreducible subquotient $\rho$ of $\pi_{\text{Calg}}$ is supercuspidal if and only if any irreducible subquotient $\rho$ of $\pi_{\text{Calg}}$ is supercuspidal.

Remark 7.3. When $C \subset \mathbb{C}$, some irreducible subquotient $\rho$ of $\pi_C$ is square integrable modulo center, if and only if any irreducible subquotient $\rho$ of $\pi_C$ is square integrable modulo center.

The pro-$p$ Sylow subgroup $U$ of the Iwahori subgroup $B$ of $G$ is a pro-$p$ Iwahori subgroup of $G$. The theory of modules over the pro-$p$ Iwahori Hecke ring $H(G, U)$ is similar to the theory of the Iwahori Hecke ring $H(G, B)$ and we refer the reader to [Vig16] and to [Vig17] for the description of $H(G, U)$ and of the supersingular right $H_{C}(G, U)$-modules. It is easy to see that the $U$-invariant functor and supersingularity commute with scalar extension from $C$ to $C^{\text{Calg}}$, Aut$_C(C^{\text{Calg}})$, descent and finite direct sums.

We prove Proposition 1.2 by descending to $C$ the supercuspidal criterion proved over $C^{\text{Calg}}$ with Ollivier in [OV Thm.3].

Suppose that $\pi$ contains a non-zero $U$-invariant supersingular element. Some irreducible subquotient $\rho$ of $\pi_{\text{Calg}}$ does also. By the supercuspidality criterion over $C^{\text{Calg}}$, $\rho$ is supercuspidal and the $H_{C^{\text{Calg}}}(G, U)$-module $\rho^{U}$ is supersingular. A descent $\rho'$ of $\rho$ to a finite extension $C'/C$ is also supercuspidal and $\rho'^{U}$ is also a supersingular $H_{C'}(G, U)$-module. As $\rho'$ is $\pi$-isotypic as a $C$-representation, $\rho'^{U}$ is a direct sum of $\pi^{U}$ as a $H_{C}(G, U)$-module. Hence $\pi$ is supercuspidal and the $H_{C}(G, U)$-module $\pi^{U}$ is supersingular.

Conversely, suppose that $\pi$ is supercuspidal. By Prop 7.2 an irreducible subquotient $\rho$ of $\pi_{\text{Calg}}$ is supercuspidal. By the supercuspidality criterion over $C^{\text{Calg}}$, the $H_{C^{\text{Calg}}}(G, U)$-module $\rho^{U}$ is supersingular. As above, we deduce that the $H_{C}(G, U)$-module $\pi^{U}$ is supersingular. □

8. Change of coefficient field

This section contains the proof of Prop 1.4 divided in three steps. Let $F_c = \begin{cases} \mathbb{Q} & \text{if } c = 0 \\ F_c & \text{if } c \neq 0 \end{cases}$

be the prime field of characteristic $c$ and $C/F_c$ a field extension of algebraic closure $C^{\text{Calg}}/F^{\text{Calg}}_c$.

Step 1 shows that, if $c \neq p$ and $G$ admits an irreducible admissible supercuspidal $C$-representation $\pi$, then $G$ admits one over a finite extension of $F_c$.

Indeed, twisting by a $C^{\text{Calg}}$-character of $G$ we can suppose that the values of the central character of $\pi$ belong to $F_{c}^{\text{Calg}}$. By [Vig96] II.4.9, when $c \neq p$, $\pi$ descends to a finite extension $F'_{c}/F_{c}$. The descent preserves irreducibility, admissibility and supercuspidality. A descent of $\pi$ from $C^{\text{Calg}}$ to $F'_{c}$ is an irreducible admissible supercuspidal $F'_{c}$-representation of $G$. 


Lemma 8.1. Assume that $\tau$ is irreducible, admissible, supercuspidal and the lemma is known when $\tau$ is supercuspidal. Then $\tau$ is admissible and finitely generated as a $C'$-representation of $G$. This implies that $\pi$ contains an irreducible admissible $C'$-representation $\pi'$ ([HV12] Lemma 7.10) for $c = p$ and the next lemma for $c \neq p$). By adjunction, $\pi$ is a quotient of the scalar extension $\pi'_{C'}$ of $\pi'$ from $C'$ to $C$. We prove that if $\pi$ is supercuspidal then $\pi'$ is supercuspidal. Suppose that $\pi'$ is not supercuspidal, subquotient of $\text{ind}_P^G \tau'$ for a proper parabolic subgroup $P$ of $G$ and $\tau'$ an irreducible admissible $C'$-representation of a Levi subgroup $M$ of $P$. The parabolic induction is compatible with the the scalar extension from $C'$ to $C$, hence $\pi'_{C'}$ is a subquotient of $\text{ind}_P^C \tau'_{C'}$ and $\pi$ is also. The $C'$-representation $\tau'_{C'}$ of $M$ has finite length and its irreducible subquotients are admissible ([HV17], Cor.4). Hence $\pi$ is a subquotient of $\text{ind}_P^C \rho$ for some irreducible admissible subquotient of $\tau'_{C'}$, therefore $\pi$ is not supercuspidal.

Step 3 shows that if $G$ admits an irreducible admissible supercuspidal $F_c$-representation then $G$ does over any field of characteristic $c$.

Indeed, let $L/C$ be a field extension of algebraic closure $L^{\text{alg}}/C^{\text{alg}}$ and $\pi$ an irreducible admissible supercuspidal $C$-representation of $G$. Let $\tau$ be an irreducible subquotient of the scalar extension $\pi_L$ of $\pi$ from $C$ to $L$. It is admissible ([HV17], Cor.4). The scalar extension $\tau^{L,\text{alg}}$ of $\tau$ from $L$ to $L^{\text{alg}}$ is a subquotient of the scalar extension $\pi^{L,\text{alg}}$ of $\pi$ from $C$ to $L^{\text{alg}}$, equal to the scalar extension of $\pi_L$ from $L$ to $L^{\text{alg}}$. By [HV17], $\pi^{L,\text{alg}}$ has finite length, the irreducible subquotients of $\pi^{L,\text{alg}}$ are absolutely irreducible, admissible, supercuspidal and descend to a finite extension $C'/C$, and the descents are $\pi$-isotypic as $C$-representations of $G$. Therefore $\tau^{L,\text{alg}}$ has the same properties. We deduce that the extension $\tau_L$ of $\tau$ from $L$ to a finite extension $L'$ of $L$ has finite length, the irreducible subquotients of $\tau'$ are absolutely irreducible, admissible, supercuspidal and $\tau$-isotypic as $L$-representations of $G$. This implies that $\tau$ is supercuspidal.

To end the proof of Prop.4 it remains to prove the lemma announced in Step 2:

**Lemma 8.1.** Assume $c \neq p$. An admissible finitely generated $C$-representation $\pi$ of $G$ has finite length.

**Proof.** The extension $\pi^{C,\text{alg}}$ of $\pi$ from $C$ to $C^{\text{alg}}$ is also admissible finitely generated. The lemma is known when $C$ is algebraically closed ([Vig96], II.5.1) hence $\pi^{C,\text{alg}}$ has finite length, implying that $\pi$ has finite length. \qed

9. **Reduction to an Absolutely Simple Adjoint Group**

As well known, the adjoint group $G^\text{ad}$ of $G$ is $F$-isomorphic to a finite product of reductive connected $F$-groups

\begin{equation}
G^\text{ad} \simeq H \times \prod_i R_{F'_i/F}(G'_i)
\end{equation}

where $H := H(F)$ is compact (hence any smooth irreducible $C$-representation of $H$ is admissible and supercuspidal), the $F'_i/F$ are finite separable extensions and $R_{F'_i/F}(G'_i)$ are scalar restrictions from $F'_i$ to $F$ of absolutely simple adjoint connected $F'_i$-groups $G'_i$.

**Proposition 9.1.** Assume that the field $C$ is algebraically closed or finite. If for any $i$, $G'_i := G'_i(F')$ admits an irreducible admissible supercuspidal $C$-representation, then $G$ admits an irreducible admissible supercuspidal $C$-representation.
For $C = \mathbb{C}$ and the finite group analogue $H(k)$ of $G$, where $H$ is a connected reductive group over a finite field $k$ of characteristic $p$, this is proved in [Kre12, Proof of Prop. 2.1].

The proposition is the combination of the Propositions 9.2, 9.3, 9.5, 9.8 corresponding to the operations: finite product, central extension, scalar restriction, with $C$ algebraically closed or finite.

1) Finite product Let $G_1, G_2$ be two connected reductive $F$-groups and $\sigma, \tau$ irreducible admissible $C$-representations of $G_1 := G_1(F), G_2 := G_2(F)$.

**Proposition 9.2.** Assume that $C$ is algebraically closed.

a) The tensor product $\sigma \otimes_C \tau$ is an irreducible admissible $C$-representation of $G_1 \times G_2$.

b) Each irreducible admissible $C$-representation of $G_1 \times G_2$ is of this form.

c) $\sigma \otimes_C \tau$ determines $\sigma, \tau$ (modulo isomorphism).

d) $\sigma \otimes_C \tau$ is supercuspidal if and only if $\sigma$ and $\tau$ are supercuspidal.

**Proof.** $\sigma \otimes \tau$ is admissible as, for open compact subgroups $K_i$ of $G_i$, we have a natural isomorphism [Bou12, §12, 2 Lemme 1]:

$$\text{Hom}_{K_1}(1, \sigma) \otimes \text{Hom}_{K_2}(1, \tau) \rightarrow \text{Hom}_{K_1 \times K_2}(1 \times 1, \sigma \otimes \tau).$$

Suppose now $C$ algebraically closed.

a) It is known that $\sigma \otimes_C \tau$ is irreducible [Bou12, §12, 2 Cor.1] (the commutant of $\sigma$ is $C$ [HV12].)

b) Let $\pi$ be an irreducible admissible $C$-representation of $G_1 \times G_2$ and let $K_1, K_2$ be any compact open subgroups of $G_1, G_2$ such that $\pi_{K_1 \times K_2} \neq 0$.

When $c = p$, the $C$-representation of $G_1$ generated by $\pi_{K_2}$ is admissible as $\pi_{K_1 \times K_2}$ is finite dimensional over $C$. It contains an irreducible admissible $C$-subrepresentation $\sigma$ [HV12, Lemma 7.10]. Let $\tau := \text{Hom}_{G_1}(\sigma, \pi)$ with the natural action of $G_2$. The representation $\sigma \otimes_C \tau$ embeds naturally in $\pi$; as $\pi$ is irreducible, it is isomorphic to $\sigma \otimes_C \tau$, and $\tau$ is irreducible. As $\pi$ is admissible, $\tau$ is admissible.

When $c \neq p$, $\pi_{K_1 \times K_2}$ is a simple right $H_C(G_1 \times G_2, K_1 \times K_2)$-module [Vig96, I.6.3]. We have $H_C(G_1 \times G_2, K_1 \times K_2) \simeq H_C(G_1, K_1) \otimes_C H_C(G_2, K_2)$. By [Bou12, §1 Proposition 2], the finite dimensional simple $H_C(G_1, K_1) \otimes H_C(G_2, K_2)$-modules are factorizable (we can also imitate with the Hecke algebras the argument above for $c = p$) hence $\pi_{K_1 \times K_2} \simeq \sigma_{K_1} \otimes \tau_{K_2}$ for irreducible admissible $C$-representations $\sigma, \tau$ of $G_1, G_2$, and $\pi \simeq \sigma \otimes_C \tau$ [Vig96, I.6.3].

c) As a $C$-representation of $G_1$, $\sigma \otimes_C \tau$ is $\sigma$-isotypic. As a representation of $G_2$, $\sigma \otimes_C \tau$ is $\tau$-isotypic. Hence c).

d) The parabolic subgroups of $G_1 \times G_2$ are product of parabolic subgroups of $G_1$ and of $G_2$. Let $P, Q$ be parabolic subgroups of $G_1, G_2$ of Levi subgroups $M, L$ and let $\pi_1$ be an irreducible admissible $C$-representation of $M \times L$. By b), $\pi_1$ is factorizable, $\pi_1 = \sigma_1 \otimes_C \tau_1$ for irreducible admissible $C$-representations $\sigma_1, \tau_1$ of $M, L$; we have a natural isomorphism $\text{ind}_{G_1}^{G_2} \sigma_1 \otimes_C \text{ind}_{Q}^{G_2} \tau_1 \rightarrow \text{ind}_{P \times Q}^{G_1 \times G_2} \pi_1$. The irreducible subquotients of $\text{ind}_{P \times Q}^{G_1 \times G_2} \pi_1$ are the tensor products of the irreducible subquotients of $\text{ind}_{G_1}^{G_2} \sigma_1$ and of $\text{ind}_{Q}^{G_2} \tau_1$. Hence d).

Until the end of 1), we assume that $C$ is a finite field. The next proposition is deduced from [HV17, Thm.1].

---

The proof is due to Henniart
Proposition 9.3. Let $\pi$ be an irreducible admissible $C$-representation of $G$. The commutant $D$ of $\pi$ is a finite extension of $C$ and the scalar extension $\pi_D$ of $\pi$ from $C$ to $D$ is isomorphic to

$$\pi_D \cong \bigoplus_{i \in \Gal(D/C)} \pi_i$$

for irreducible admissible $D$-representations $\pi_i$ of $G$ of commutant $D$, not isomorphic to each other, forming a single $\Gal(D/R)$-orbit and seen as $C$-representations, are isomorphic to $\pi$.

Proof. The commutant $D$ of $\pi$ is a division algebra of finite dimension over $C$, and a finite extension of $C$ is a finite field and is Galois. As a $C$-representation, $\pi_D$ is $C$-isotypique of length $[D : C]$. Apply [HV17, Thm.1] with $R' = D$. □

Returning to $\sigma$ and $\tau$ when $C$ is finite, the commutants $D_{\sigma}$ and $D_{\tau}$ of $\sigma$ and $\tau$ are finite extensions of $C$ of dimensions $d_{\sigma}$, $d_{\tau}$. We embed them in $C^{alg}$ and we consider the field $D$ generated by $D_{\sigma}$ and $D_{\tau}$ of $C$-dimension $\text{lcm}(d_{\sigma}, d_{\tau})$, and the intersection $D' = D_{\sigma} \cap D_{\tau}$ of $C$-dimension $\gcd(d_{\sigma}, d_{\tau})$. The fields $D_{\sigma}, D_{\tau}$ are linearly disjoint on $D'$, $D_{\sigma} \otimes_{D'} D_{\tau} \simeq D$, and

$$(9.2) \quad D_{\sigma} \otimes_{C} D_{\tau} \simeq \prod_{\tau'} \ D_{\tau'}$$

because $D_{\sigma} \otimes_{C} D_{\tau} \simeq D_{\sigma} \otimes_{D'} (D' \otimes_{C} D') \otimes_{D'} D_{\tau}$, $D' \otimes_{C} D' \simeq D' \otimes_{C} C[X]/(P[X]) \simeq D'[X]/(P[X]) \simeq \prod_{\tau'} D'_{\tau'}$ for any $P[X] \in C[X]$ irreducible of degree $[D' : C]$.

Proposition 9.4. The $C$-representation $\sigma \otimes_{C} \tau$ of $G_1 \times G_2$ is isomorphic to

$$\sigma \otimes_{C} \tau \cong \bigoplus_{i=1}^{\gcd(d_{\sigma}, d_{\tau})} \pi_j$$

for irreducible admissible $C$-representations $\pi_j$ of commutant $D$ and not isomorphic to each other. We have $\sigma$ and $\tau$ supercuspidal $\iff$ all the $\pi_j$ supercuspidal $\iff$ some $\pi_j$ supercuspidal.

Proof. From Prop 9.3 the scalar extensions $\sigma_D, \tau_D$ of $\sigma, \tau$ from $C$ to $D$ are isomorphic to

$$\sigma_D \cong \bigoplus_{i \in \Gal(D_{\sigma}/C)} \sigma_i, \quad \tau_D \cong \bigoplus_{r \in \Gal(D_{\tau}/C)} \tau_r$$

for irreducible admissible $D$-representations $\sigma_i, \tau_r$ of $G_1, G_2$ of commutant $D$, not isomorphic to each other, forming a single $\Gal(D/R)$-orbit, descending to $D_{\sigma}$ (resp. $D_{\tau}$) and seen as $C$-representations, isomorphic to $\sigma$ (resp. $\tau$). The $C$-representation $\sigma \otimes_{C} \tau$ of $G_1 \times G_2$ is admissible, of scalar extension from $C$ to $D$:

$$(9.3) \quad (\sigma \otimes_{C} \tau)_D \cong \sigma_D \otimes_{D} \tau_D \cong \bigoplus_{(i,r) \in \Gal(D_{\sigma}/C) \times \Gal(D_{\tau}/C)} \sigma_i \otimes_{D} \tau_r.$$ 

The $D$-representation $\sigma_i \otimes_{D} \tau_r$ of $G_1 \times G_2$ is admissible of commutant $D \otimes_{D} D = D$ [Bou12, §12, no 2, lemma 1]. Hence $\sigma_i \otimes_{D} \tau_r$ is absolutely irreducible and $(\sigma \otimes_{C} \tau)_D$ is semi-simple (9.3). This implies that $\sigma \otimes_{C} \tau$ is semi-simple [Bou12, §12, no 7, Prop.8]; its commutant contains $D_{\sigma} \otimes_{C} D_{\tau}$. From Prop 9.3 $\sigma \otimes_{C} \tau$ has length $d_{\sigma} d_{\tau}/\text{lcm}(d_{\sigma}, d_{\tau}) = \gcd(d_{\sigma}, d_{\tau})$; by (9.2), its commutant is $D_{\sigma} \otimes_{C} D_{\tau}$ and its irreducible components $\pi_j$ are admissible of commutant $D$ and not isomorphic to each other.

From Prop. 9.2 over $C^{alg}$, $\sigma_i \otimes_{D} \tau_r$ is supercuspidal if and only if $\sigma_i$ and $\tau_r$ are, if and only if all $\sigma_i$ and all $\tau_r$ are. From Prop. 7.2 this is also equivalent to $\pi_j$ supercuspidal for some $j$, and to $\pi_j$ supercuspidal for all $j$. □

2) Central extension The natural surjective $F$-morphism $G \rightarrow G^{ad}$ of kernel $\mathbf{Z}(G)$ induces between the $F$-points an exact sequence

$$0 \rightarrow Z(G) \rightarrow G \rightarrow G^{ad} \rightarrow H^1(F, \mathbf{Z}(G)).$$
The image \( i(G) \) of \( G \) is a closed cocompact normal subgroup of \( G^{ad} \) and \( H^1(F, Z(G)) \) is commutative.

Until the end of 2), we assume \( a = 0 \). The group \( H^1(F, Z(G)) \) is finite \cite{PrR94}*{Thm. 6.14} implying that \( i(G) \) is an open normal subgroup of \( G^{ad} \) of finite commutative quotient.

**Proposition 9.5.** \( G^{ad} \) admits an irreducible admissible supercuspidal \( C \)-representation if and only if \( G \) does.

The inflation from \( i(G) \) to \( G \) identifies the representations of \( i(G) \) with the representations of \( G \) trivial on \( Z(G) \); it respects irreducibility and admissibility. The functor (inflation from \( i(G) \) to \( G \)) \( \circ \) (restriction from \( G^{ad} \) to \( i(G) \)) from \( C \)-representations of \( G^{ad} \) to representations of \( G \) trivial on \( Z(G) \) is denoted by \( - \circ i \).

Let \( \hat{\rho} \) be an irreducible admissible \( C \)-representation of \( G \) inflating a representation \( \rho \) of the normal open subgroup \( i(G) \) of finite index in \( G^{ad} \). The \( C \)-representation \( \rho \) of \( i(G) \) is irreducible admissible and induces a representation \( \text{ind}_{i(G)}^{G^{ad}} \rho \) of \( G^{ad} \) which is admissible of finite length. Any irreducible quotient \( \pi \) of \( \text{ind}_{i(G)}^{G^{ad}} \rho \) is admissible (when \( c = p \) this uses that \( a = 0 \)), by adjunction \( \pi|_{i(G)} \) contains a subrepresentation isomorphic to \( \rho \) and by inflation from \( i(G) \) to \( G \), \( \hat{\rho} \) is isomorphic to a subquotient of \( \pi \circ i \).

Let \( \pi \) be an irreducible admissible \( C \)-representation of \( G^{ad} \). The restriction \( \pi|_{i(G)} \) of \( \pi \) to \( i(G) \) is semi-simple of finite length, its irreducible components \( \rho \) are \( G^{ad} \)-conjugate \cite{Vig96}*{I.6.12}. The \( C \)-representation \( \pi \circ i \) of \( G \) is semi-simple of finite length, of irreducible components the inflations \( \hat{\rho} \) of the irreducible components \( \rho \) of \( \pi|_{i(G)} \). Remark that if \( \pi' \) is a smooth \( C \)-representation of \( G^{ad} \) such that some subquotient of \( \pi'|_{i(G)} \) is isomorphic to some \( \hat{\rho} \), then some subquotient of \( \pi'|_{i(G)} \) is isomorphic to \( \pi \circ i \).

Proposition 9.5 follows from:

**Proposition 9.6.** \( \pi \) is supercuspidal if and only if some \( \hat{\rho} \) is supercuspidal if and only if all \( \hat{\rho} \) are supercuspidal.

*Proof.* It suffices to prove: \( \pi \) not supercuspidal \( \Rightarrow \) all \( \hat{\rho} \) not supercuspidal, and then, some \( \hat{\rho} \) not supercuspidal \( \Rightarrow \) \( \pi \) not supercuspidal. We check first the compatibility of the parabolic induction with \( - \circ i \).

(i) The parabolic \( F \)-subgroups of \( G \) and of \( G^{ad} \) are in bijection. If the parabolic \( F \)-subgroup \( P \) of \( G \) corresponds to the parabolic \( F \)-subgroup \( Q \) of \( G^{ad} \), \( i \) restricts to an isomorphism between their unipotent radicals \cite{Bor91}*{22.6 Thm.}, sends a Levi subgroup \( M \) of \( P \) onto a Levi subgroup \( L \) of \( Q \), and induces between the \( F \)-points the exact sequence:

\[
0 \to Z(G) \to M \xrightarrow{i} L \to H^1(F, Z(G)).
\]

We have \( G^{ad} = Q i(G) \), \( Q \cap i(G) = i(P) = i(M)U \) where \( i(M) \) is an open normal subgroup of \( L \) of finite commutative quotient and \( U \) is the unipotent radical of \( Q \). If \( \sigma \) is a smooth \( C \)-representation of \( L \), then (ind\(_Q^{G^{ad}} \sigma|_{i(G)} \simeq \text{ind}_{i(P)}^{i(M)}(\sigma|_{i(M)}) \) and by inflation from \( i(G) \) to \( G \):

\[
(\text{ind}_{Q}^{G^{ad}} \sigma) \circ i \simeq \text{ind}_{i}^{G}(\sigma \circ i).
\]

(ii) Let \( \pi \) be an irreducible admissible not supercuspidal \( C \)-representation of \( G^{ad} \), isomorphic to a subquotient of \( \text{ind}_{Q}^{G^{ad}} \sigma \) for \( Q \neq G^{ad} \) and \( \sigma \) irreducible admissible \( C \)-representation

\[\text{in } \cite{Vig96}*{I.6.12} \text{ the condition that the index is invertible in } C \text{ is not necessary and not used in the proof} \]
of $L$. Therefore $\pi \circ i$ is isomorphic to a subquotient of $(\text{ind}^\text{ad}_Q \sigma) \circ i$, and by (9.4) all the $\tilde{\rho}$ are isomorphic to a subquotient of $\text{ind}^\text{ad}_G \tilde{\tau}$ for some irreducible subquotient $\tilde{\tau}$ of $\sigma \circ i$ (depending on $\rho$). As $\tilde{\tau}$ is admissible and $P \neq G$, all the $\tilde{\rho}$ are not supercuspidal.

(iii) Let $\pi$ be an irreducible admissible $C$-representation of $G^\text{ad}$ such that some irreducible component $\tilde{\rho}$ of $\pi \circ i$ is not supercuspidal, isomorphic to a subquotient of $\text{ind}^p G \tau$ for $P \neq G$ and $\tau'$ irreducible admissible $C$-representation of $M$. The central subgroup $\hat{Z}(G)$ acts trivially on $\tilde{\rho}$ hence also on $\tau'$. Therefore $\tau' = \tilde{\tau}$ for an irreducible subquotient $\tau$ of $\sigma_{i(M)}$ where $\sigma$ is an irreducible admissible $C$-representation of $L$. The representation $\hat{\rho}$ is isomorphic to a subquotient of $\text{ind}^\text{ad}_G (\sigma \circ i)$. By (9.4) and the remark above Prop. 9.6, $\pi \circ i$ is isomorphic to a subquotient of $(\text{ind}^\text{ad}_Q \sigma)_{i(M)}$. This representation is isomorphic to

$$\text{ind}^\text{ad}_i (\sigma_{i(M)}) \simeq \text{ind}^L (\text{ind}^\text{ad}_i (\sigma_{i(M)})) \simeq \text{ind}^\text{ad}_Q (\sigma \otimes_C C[i(M) \setminus L]).$$

The $C$-representation $\sigma \otimes_C C[i(M) \setminus L]$ of $L$ has finite length and its irreducible subquotients $\nu$ are admissible (Lemma 9.7 below). Therefore $\pi$ is isomorphic to a subquotient of $\text{ind}^\text{ad}_Q \nu$ for some $\nu$ and $Q \neq G^\text{ad}$, hence $\pi$ is not supercuspidal.

**Lemma 9.7.** Let $\pi$ be an irreducible admissible $C$-representation of $G$ and let $V$ be a finite dimensional smooth $C$-representation of $G$. Then the representation $\pi \otimes_C V$ of $G$ has finite length and its irreducible subquotients are admissible.

**Proof.** The scalar extension $(\pi \otimes_C V)_{\text{Calg}}$ of $\pi \otimes_C V$ to an algebraic closure $C^\text{alg}$ of $C$ is isomorphic to $\pi_{\text{Calg}} \otimes_C V_{\text{Calg}}$. The length of $\pi \otimes_C V$ is bounded above by the length of $(\pi \otimes_C V)_{\text{Calg}}$ and a $C$-representation of $G$ is admissible if and only if its scalar extension from $C$ to $C^\text{alg}$ is admissible. By Fact 3.3, the length of the $C^\text{alg}$-representation $\pi_{\text{Calg}} \otimes_C V_{\text{Calg}}$ of $G$ is the product of the finite lengths of $\pi_{\text{Calg}}$ and of $V_{\text{Calg}}$, and its irreducible subquotients are $\pi_i \otimes \chi$ for the irreducible quotients $\pi_i$ of $\pi_{\text{Calg}}$ and the $C^\text{alg}$-characters $\chi$ of $L$ trivial on $i(M)$. As $\pi_i$ is admissible, $\pi_i \otimes \chi$ is admissible, and all subquotients of $\pi_{\text{Calg}} \otimes_C V_{\text{Calg}}$ are admissible.

□

3) **Scalar restriction** Let $F'/F$ be a finite separable extension, $G'$ a connected reductive $F'$-group and $G := R_{F'/F}(G')$ the scalar restriction of $G'$ from $F'$ to $F$. As topological groups, $G' := G'(F')$ is equal to $G := G(F)$. By [BT65] 6.19. Cor.], $G'$ and $G$ have the same parabolic subgroups, hence:

**Proposition 9.8.** $G'$ admits an irreducible admissible supercuspidal $C$-representation if and only if $G$ does.

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