MAXIMAL VECTORS IN HILBERT SPACE
AND QUANTUM ENTANGLEMENT

WILLIAM ARVESON

Abstract. Let \( V \) be a norm-closed subset of the unit sphere of a Hilbert space \( H \) that is stable under multiplication by scalars of absolute value 1. A maximal vector (for \( V \)) is a unit vector \( \xi \in H \) whose distance to \( V \) is maximum

\[
d(\xi, V) = \sup_{\|\eta\|=1} d(\eta, V),
\]

\( d(\xi, V) \) denoting the distance from \( \xi \) to the set \( V \). Maximal vectors generalize the maximally entangled unit vectors of quantum theory.

In general, under a mild regularity hypothesis on \( V \), there is a norm on \( H \) whose restriction to the unit sphere achieves its minimum precisely on \( V \) and its maximum precisely on the set of maximal vectors. This “entanglement-measuring norm” is unique. There is a corresponding “entanglement-measuring norm” on the predual of \( \mathcal{B}(H) \) that faithfully detects entanglement of normal states.

We apply these abstract results to the analysis of entanglement in multipartite tensor products \( H = H_1 \otimes \cdots \otimes H_N \), and we calculate both entanglement-measuring norms. In cases for which \( \dim H_N \) is relatively large with respect to the others, we describe the set of maximal vectors in explicit terms and show that it does not depend on the number of factors of the Hilbert space \( H_1 \otimes \cdots \otimes H_{N-1} \).

1. Introduction

Let \( H = H_1 \otimes \cdots \otimes H_N \) be a finite tensor product of separable Hilbert spaces. In the literature of physics and quantum information theory, a normal state \( \rho \) of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \) called separable or classically correlated if it belongs to the norm closed convex set generated by product states \( \sigma_1 \otimes \cdots \otimes \sigma_N \), where \( \sigma_k \) denotes a normal state of \( \mathcal{B}(H_k) \). Normal states that are not separable are said to be entangled. The notion of entanglement is a distinctly noncommutative phenomenon, and has been a fundamental theme of quantum physics since the early days of the subject. It has received increased attention recently because of possible applications emerging from quantum information theory.

In the so-called bipartite case in which \( N = 2 \), several numerical measures of entanglement have been proposed that emphasize various features (see [HHHH07], [HGBL05], [Per96], [WG07]). Despite the variety of proposed measures, only one we have seen (the projective cross norm introduced in
[Rud00], [Rud01] is capable of distinguishing between entangled mixed states and separable mixed states of bipartite tensor products. Of course, the bipartite case has special features because vectors in $H_1 \otimes H_2$ can be identified with Hilbert-Schmidt operators from $H_1$ to $H_2$, thereby allowing one to access operator-theoretic invariants – most notably the singular value list of a Hilbert-Schmidt operator – to analyze vectors in $H_1 \otimes H_2$. On the other hand, that tool is much less effective for higher order tensor products, and perhaps that explains why the higher order cases $N \geq 3$ are poorly understood. For example, there does not appear to be general agreement as to what properties a “maximally entangled” vector should have in such cases; and in particular, there is no precise definition of the term.

In this paper we propose such a definition and introduce two numerical invariants (one for vectors and one for states) that faithfully detect entanglement, in a general mathematical setting that includes the cases of physical interest. We start with a separable Hilbert space $H$ and a distinguished set of unit vectors that satisfies the following two conditions:

$V_1$: $\lambda \cdot V \subseteq V$, for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

$V_2$: For every $\xi \in H$, $\langle \xi, V \rangle = \{0\} \implies \xi = 0$.

By replacing $V$ with its closure if necessary, we can and do assume that $V$ is closed in the norm topology of $H$. A normal state $\rho$ of $\mathcal{B}(H)$ is said to be $V$-correlated if for every $\epsilon > 0$, there are vectors $\xi_1, \ldots, \xi_n \in V$ and positive numbers $t_1, \ldots, t_n$ with sum 1 such that

$$\sup_{\|x\| \leq 1} |\rho(x) - \sum_{k=1}^{n} t_k \langle x \xi_k, \xi_k \rangle| \leq \epsilon.$$ 

A normal state that is not $V$-correlated is called $V$-entangled - or simply entangled. The motivating examples are those in which $H = H_1 \otimes \cdots \otimes H_N$ is an $N$-fold tensor product of Hilbert spaces $H_k$ and

$$V = \{\xi_1 \otimes \cdots \otimes \xi_n : \xi_k \in H_k, \|\xi_1\| = \cdots = \|\xi_n\| = 1\}$$

is the set of decomposable unit vectors. In such cases the $V$-correlated states are the separable states, and when $H$ is finite dimensional, the $V$-correlated states are the simply the convex combinations of vector states $x \mapsto \langle x \xi, \xi \rangle$ with $\xi$ a unit vector of the form $\xi = \xi_1 \otimes \cdots \otimes \xi_n$, $\xi_k \in H_k$, $k = 1, \ldots, n$. Of course, there are many other examples that have less to do with physics.

In general, given such a set $V \subseteq H$, a maximal vector is defined as a unit vector $\xi \in H$ whose distance to $V$ is maximum

$$d(\xi, V) = \sup_{\|\eta\|=1} d(\eta, V),$$

$d(\eta, V)$ denoting the distance from $\eta$ to $V$. While it is not obvious from this geometric definition, it is a fact that in the case of bipartite tensor products
MAXIMAL VECTORS

$H = H_1 \otimes H_2$, maximal vectors turn out to be exactly the “maximally entangled” unit vectors of the physics literature (see (1.2) below). Sections 2 through 4 are devoted to an analysis of the geometric properties of maximal vectors in general. We introduce a numerical invariant $r(V)$ of $V$ (the inner radius) and show that when $r(V) > 0$, there is a uniquely determined “entanglement measuring norm” $\| \cdot \|_V$ on $H$ with the property that $\xi \in V$ iff $\| \xi \|_V = 1$ and $\xi$ is maximal iff $\| \xi \|_V = r(V)^{-1}$ (see Proposition 2.1 and Theorem 4.2).

In Section 5 we introduce an extended real-valued function $E(\rho)$ of normal states $\rho$ that takes values in the interval $[1, +\infty]$. This “entanglement” function $E$ is convex, lower semicontinuous, and faithfully detects generalized entanglement in the sense that $\rho$ is entangled iff $E(\rho) > 1$ (Theorems 5.3 and 6.2). We also show that under the same regularity hypothesis on the given set $V$ of unit vectors (namely $r(V) > 0$), $E$ is a norm equivalent to the ambient norm of $B(H)_* \cong L^1(H)$, and it achieves its maximum on vector states of the form $\omega(A) = \langle A\xi, \xi \rangle$, $A \in B(H)$ precisely when $\xi$ is a maximal vector (Theorem 9.1).

In the third part of the paper (Sections 8–13), we apply these abstract results to cases in which $V$ is the set of decomposable unit vectors in an $N$-fold tensor product $H = H_1 \otimes \cdots \otimes H_N$. We assume that all but one of the $H_k$ are finite dimensional, arranged so that the dimensions $n_k = \dim H_k$ weakly increase with $k$ and $n_{N-1} < \infty$. We identify the vector norm $\| \cdot \|_V$ that measures entanglement as the greatest cross norm on the projective tensor product of Hilbert spaces

$$H_1 \hat{\otimes} H_2 \hat{\otimes} \cdots \hat{\otimes} H_N$$

in general - see Theorem 8.2. Similarly, we identify the entanglement function of mixed states as the restriction to density operators of the greatest cross norm of the projective tensor product of Banach spaces

$$L^1(H_1) \hat{\otimes} L^1(H_2) \hat{\otimes} \cdots \hat{\otimes} L^1(H_N),$$

$L^1(H)$ denoting the Banach space of trace class operators on a Hilbert space $H$ (Theorem 9.1). Note that in the bipartite case $N = 2$, the latter reduces to the norm introduced in a more ad hoc way by Rudolph in [Rud00], [Rud01].

We are unable to identify the maximal vectors in this generality, and our sharpest results for multipartite tensor products require an additional hypothesis, namely that one of the spaces $H_k$ should be significantly larger than the others in the sense that $n_N \geq n_1 \cdots n_{N-1}$. In every case, of course, the entanglement measuring norm $\| \cdot \|_V$ depends strongly on relative dimensions $n_1, \ldots, n_N$ of the factors of the decomposition $H_1 \otimes \cdots \otimes H_N$, because the “shape” of its unit ball $\{ \xi : \| \xi \|_V \leq 1 \}$ depends strongly on these relative dimensions. What is interesting is that when $n_N \geq n_1 \cdots n_{N-1}$, the set of maximal vectors does not depend on that finer structure. Indeed, we show that in such cases the maximal vectors are precisely the vectors in
$H_1 \otimes \cdots \otimes H_N$ that can be represented

$$\xi = \frac{1}{\sqrt{n_1 n_2 \cdots n_{N-1}}} \sum_{k=1}^{n_1 \cdots n_{N-1}} e_k \otimes f_k$$

where $e_1, \ldots, e_{n_1 \cdots n_{N-1}}$ is an orthonormal basis for $H_1 \otimes \cdots \otimes H_{N-1}$ and $f_1, \ldots, f_{n_1 \cdots n_{N-1}}$ is an orthonormal set in $H_N$ (see Theorem 12.1). The simplest case is $N=2$, where our hypotheses reduce to $n_1 \leq n_2 \leq \infty$ with $n_1$ finite, and the expression (1.1) becomes a familiar representation of “maximally entangled” vectors of bipartite tensor products that is commonly found in the physics literature.

To make the point in somewhat more physical terms, let $H$ and $K$ be finite dimensional Hilbert spaces with $n = \dim H \leq m = \dim K$. The maximal vectors of the bipartite tensor product $H \otimes K$ are those of the form

$$\xi = \frac{1}{\sqrt{n}}(e_1 \otimes f_1 + \cdots + e_n \otimes f_n)$$

where $(e_k)$ is an orthonormal basis for $H$ and $(f_k)$ is an orthonormal set in $K$. On the other hand, if the Hilbert space $H$ represents a composite of several subsystems in the sense that it can be decomposed into a tensor product $H = H_1 \otimes \cdots \otimes H_r$ of Hilbert spaces, then the set of maximal vectors relative to the more refined decomposition $H_1 \otimes \cdots \otimes H_r \otimes K$ is precisely the same set of unit vectors (1.2).

This unexpected stability of the set of maximal vectors is established by showing that the states associated with maximal vectors $\xi$ are characterized by the following requirement on their “marginal distributions”. The algebra $A = \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1})$ can be viewed as a matrix algebra with tracial state $\tau$, and we show that a unit vector $\xi$ in $H_1 \otimes \cdots \otimes H_N$ is maximal if and only if

$$\langle (A \otimes 1_{H_N})\xi, \xi \rangle = \tau(A), \quad A \in A,$$

see Theorem 11.1. We do not know if there is a useful characterization of the marginal states of maximal vectors in the remaining cases for which $n_N < n_1 \cdots n_{N-1}$, and that is an issue calling for further research.

Of course, it was also necessary to calculate the geometric invariant $r(V)$ for these examples, see Theorems 10.1 and 10.2. A more precise and more complete summary of our main results for multipartite tensor products is presented at the end of the paper in Theorem 14.1 (also see Remark 13.3).

The idea of measuring entanglement of vectors in terms of their distance to the decomposable vectors appears in [WG03], and calculations are carried out for several examples. While a related measure was also introduced for states, it is different from the one below, and there appears to be no further overlap with this paper. Also see formula (22) of [GRW08]. A related operator-theoretic notion of entanglement for bipartite tensor products was introduced in [BNT02], where it is shown essentially that a density operator that is maximally far from the separable ones relative to the Hilbert-Schmidt norm provides a maximal violation of the Bell inequalities. Perhaps it is
also relevant to point out that the recent paper [PGWP+08] establishes unbounded violations of the Bell inequalities for tripartite tensor products using quite different methods.

This is the third of a series of papers that relate to entangled states on matrix algebras [Arv07], [Arv08]. However, while the results below certainly apply to matrix algebras, many of them also apply to the context of infinite dimensional Hilbert spaces. Finally, I thank Mary Beth Ruskai for calling my attention to some key results in the physics literature, and Yoram Gordon for helpful comments.

**Part 1. Vectors in Hilbert spaces**

2. **Detecting membership in convex sets**

Let $H$ be a Hilbert space and let $V \subseteq \{ \xi \in H : \|\xi\| = 1 \}$ be a norm-closed subset of the unit sphere of $H$ that satisfies V1 and V2. Recall that since the weak closure and the norm closure of a convex subset of $H$ are the same, it is unambiguous to speak of the closed convex hull of $V$.

In this section we show that there is a unique function $u : H \to [0, +\infty]$ with certain critical properties that determines membership in the closed convex hull of $V$, and more significantly for our purposes, such a function determines membership in $V$ itself. While the proof of Proposition 2.1 below involves some familiar ideas from convexity theory, it is not part of the lore of topological vector spaces, hence we include details.

We begin with a preliminary function $\| \cdot \|_V$ defined on $H$ by

$$\| \xi \|_V = \sup_{\eta \in V} |\langle \xi, \eta \rangle|, \quad \xi \in H. \quad (2.1)$$

Axiom V2 implies that $\| \cdot \|_V$ is a norm, and since $V$ consists of unit vectors we have $\|\xi\|_V \leq \|\xi\|$. The associated unit ball $\{ \xi \in H : \|\xi\|_V \leq 1 \}$ is a closed convex subset of $H$ that contains the unit ball $\{ \xi \in H : \|\xi\| \leq 1 \}$ of $H$ because $\|\xi\|_V \leq \|\xi\|$, $\xi \in H$.

Now consider the function $\| \cdot \|_V : H \to [0, +\infty]$ defined by

$$\| \xi \|_V = \sup_{\|\eta\|_V \leq 1} \mathbb{R}\langle \xi, \eta \rangle = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle|, \quad \xi \in H. \quad (2.2)$$

Since $\|\eta\|_V \leq \|\eta\|$, the right side of (2.2) is at least $\|\xi\|$, hence

$$\| \xi \|_V \leq \|\xi\| \leq \|\xi\|_V, \quad \xi \in H. \quad (2.3)$$

Significantly, it is possible for $\|\xi\|_V$ to achieve the value $+\infty$ when $H$ is infinite dimensional; an example is given in Proposition 8.3 below.

An extended real-valued function $u : H \to [0, +\infty]$ is said to be weakly lower semicontinuous if for every $r \in [0, +\infty)$, the set $\{ \xi \in H : u(\xi) \leq r \}$ is closed in the weak topology of $H$.

**Proposition 2.1.** The extended real-valued function $\| \cdot \|_V : H \to [0, +\infty]$ has the following properties:
(i) \( \|\xi + \eta\|_V \leq \|\xi\|_V + \|\eta\|_V, \quad \xi, \eta \in H. \)
(ii) \( \|\lambda \cdot \xi\|_V = |\lambda| \cdot \|\xi\|_V, \quad 0 \neq \lambda \in \mathbb{C}, \quad \xi \in H. \)
(iii) It is weakly lower semicontinuous.
(iv) The closed convex hull of \( V \) is \( \{ \xi \in H : \|\xi\|_V \leq 1 \} \).

This function is uniquely determined: If \( u : H \to [0, +\infty] \) is any function that satisfies (ii) and (iv), then \( u(\xi) = \|\xi\|_V, \xi \in H. \)

The proof rests on the following result.

**Lemma 2.2.** Let \( K \) be the closed convex hull of \( V \). Then

(2.4) \[ K = \{ \xi \in H : \|\xi\|_V \leq 1 \}, \]

and in particular,

(2.5) \[ \|\xi\|_V = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle|, \quad \xi \in H. \]

**Proof of Lemma 2.2.** For the inclusion \( \subseteq \) of (2.4), note that if \( \xi \in V \) and \( \eta \) is any vector in \( H \), then \( |\langle \xi, \eta \rangle| \leq \|\eta\|_V \), so that

\[ \|\xi\|_V = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle| \leq \sup_{\|\eta\|_V \leq 1} \|\eta\|_V \leq 1. \]

For the other inclusion, a standard separation theorem implies that it is enough to show that for every continuous linear functional \( f \) on \( H \) and every \( \alpha \in \mathbb{R} \),

\[ \sup_{\xi \in V} \Re f(\xi) \leq \alpha \implies \sup_{\|\eta\|_V \leq 1} \Re f(\eta) \leq \alpha. \]

Fix such a pair \( f, \alpha \) with \( f \neq 0 \). By the Riesz lemma, there is a vector \( \zeta \in H \) such that \( f(\xi) = \langle \xi, \zeta \rangle, \xi \in H, \) and the first inequality above implies

\[ 0 < \|\zeta\|_V = \sup_{\xi \in V} |\langle \xi, \zeta \rangle| = \sup_{\xi \in V} \Re f(\xi) \leq \alpha. \]

Hence \( \|\alpha^{-1}\zeta\|_V \leq 1. \) By definition of \( \| \cdot \|_V \) we have \( |\langle \eta, \alpha^{-1}\zeta \rangle| \leq \|\eta\|_V \), therefore \( |\langle \eta, \zeta \rangle| \leq \alpha \|\eta\|_V \), and finally

\[ \sup_{\|\eta\|_V \leq 1} \Re f(\eta) \leq \sup_{\|\eta\|_V \leq 1} |\langle \eta, \zeta \rangle| \leq \alpha, \]

which is the inequality on the right of the above implication.

To deduce the formula (2.5), use (2.4) to write

\[ \|\xi\|_V = \sup_{\eta \in V} |\langle \xi, \eta \rangle| = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle| = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle| \]

and (2.5) follows. \( \square \)

**Proof of Proposition 2.1.** Properties (i) and (ii) are obvious from the definition (2.2) of \( \| \cdot \|_V \), lower semicontinuity (iii) also follows immediately from the definition (2.2), and property (iv) follows from Lemma 2.2.

Uniqueness: Property (iv) implies that for \( \xi \in H \),

\[ u(\xi) \leq 1 \iff \|\xi\|_V \leq 1. \]
Using \( u(r \cdot \xi) = r \cdot u(\xi) \) for \( r > 0 \), we conclude that for every positive real number \( r \) and every \( \xi \in H \), one has
\[ u(\xi) \leq r \iff \|\xi\|^V \leq r, \]
from which it follows that \( u(\xi) = \|\xi\|^V \) whenever one of \( u(\xi), \|\xi\|^V \) is finite, and that \( u(\xi) = \|\xi\|^V = +\infty \) whenever one of \( u(\xi), \|\xi\|^V \) is \( +\infty \). Hence \( u(\xi) = \|\xi\|^V \) for all \( \xi \in H \).

What is more significant is that the function \( \|\cdot\|^V \) detects membership in \( V \) itself:

**Theorem 2.3.** The restriction of the function \( \|\cdot\|^V \) of (2.2) to the unit sphere \( \{\xi \in H : \|\xi\| = 1\} \) of \( H \) satisfies
\[ (2.6) \quad \|\xi\|^V \geq 1, \quad \text{and} \quad \|\xi\|^V = 1 \iff \xi \in V. \]

**Proof.** (2.3) implies that \( \|\xi\|^V \geq 1 \) for all \( \|\xi\| = 1 \).

Let \( K \) be the closed convex hull of \( V \). The description of \( K \) given in (2.4) and the properties (i) and (ii) of Proposition 2.1 imply that the extreme points of \( K \) are the vectors \( \xi \in H \) satisfying \( \|\xi\|^V = 1 \). Since \( V \) consists of extreme points of the unit ball of \( H \), it consists of extreme points of \( K \), hence \( \|\xi\|^V = 1 \) for every \( \xi \in V \).

Conversely, if \( \xi \) satisfies \( \|\xi\| = \|\xi\|^V = 1 \), then the preceding remarks show that \( \xi \) is an extreme point of \( K \), so that Milman’s converse of the Krein-Milman theorem implies that \( \xi \) belongs to the weak closure of \( V \). But if \( \xi_n \) is a sequence in \( V \) that converges weakly to \( \xi \) then
\[ \|\xi_n - \xi\|^2 = 2 - 2\Re\langle\xi_n, \xi\rangle \to 2 - 2\langle\xi, \xi\rangle = 0 \]
as \( n \to \infty \). We conclude that \( \xi \in \overline{V}^\text{norm} = V \). \( \square \)

### 3. The geometric invariant \( r(V) \)

In this section we introduce a numerical invariant of \( V \) that will play a central role.

**Definition 3.1.** The **inner radius** \( r(V) \) of \( V \) is defined as the largest \( r \geq 0 \) such that \( \{\xi \in H : \|\xi\| \leq r\} \) is contained in the closed convex hull of \( V \).

Obviously, \( 0 \leq r(V) \leq 1 \). The following result and its corollary imply that \( r(V) > 0 \) when \( H \) is finite dimensional. More generally, they imply that whenever the inner radius is positive, both \( \|\cdot\|^V \) and \( \|\cdot\|^V \) are norms that are equivalent to the ambient norm of \( H \). We write \( d(\xi, V) \) for the distance from a vector \( \xi \in H \) to the set \( V \), \( d(\xi, V) = \inf\{\|\xi - \eta\| : \eta \in V\} \).

**Theorem 3.2.** The inner radius of \( V \) is characterized by each of the following three formulas:
\[ (3.1) \quad \inf_{\|\xi\| = 1} \|\xi\|^V = r(V), \]
\begin{equation}
\sup_{\|\xi\|=1} \|\xi\|^V = \frac{1}{r(V)},
\end{equation}

\begin{equation}
\sup_{\|\xi\|=1} d(\xi, V) = \sqrt{2(1 - r(V))}.
\end{equation}

**Proof.** Let \( K \) be the closed convex hull of \( V \). If \( K \) contains the ball of radius \( r \) about \( 0 \), then for every \( \xi \in H \) we have

\[
\sup_{\|\xi\|=1} |\langle \xi, \eta \rangle| = \sup_{\|\xi\|=1, \|\eta\| \leq 1} |\langle \xi, \eta \rangle| \geq \sup_{\|\eta\| \leq 1} |\langle \xi, \eta \rangle| = r \cdot \|\xi\|.
\]

Hence

\[
\inf_{\|\xi\|=1} \|\xi\| = \inf_{\|\xi\|=1} \|\xi\|^V = \sup_{\|\|\|\| \leq 1} \|\xi\|^V = \sup_{\|\xi\|=1} \|\xi\| = r(V).
\]

Then for every \( \xi \in H \) satisfying \( \|\xi\| = 1 \), we have

\[
\sup_{\|\eta\| \leq 1} |\langle \xi, \eta \rangle| = r \cdot \sup_{\|\eta\| \leq 1} |\langle \xi, \eta \rangle| = r \cdot \|\xi\| = \sup_{\eta \in V} |\langle \xi, \eta \rangle|,
\]

and after rescaling \( \xi \) we obtain

\[
\sup_{\|\eta\| \leq 1} \Re\langle \xi, \eta \rangle \leq \sup_{\eta \in V} \Re\langle \xi, \eta \rangle, \quad \xi \in H.
\]

At this point, a standard separation theorem implies that \( \{ \eta \in H : \|\eta\| \leq r \} \) is contained in the closed convex hull of \( V \), hence \( r \leq r(V) \).

(3.2) follows from (3.1), since by definition of the norm \( \|\xi\|^V \)

\[
\sup_{\|\xi\|=1} \|\xi\|^V = \sup_{\|\xi\|=1, \|\eta\|_V = 1} |\langle \xi, \eta \rangle| = \sup_{\|\eta\|_V = 1} \|\eta\| \sup_{\eta \neq 0} \frac{\|\eta\|}{\|\eta\|_V} = \sup_{\|\eta\|_V = 1} \left( \inf_{\|\eta\|_V = 1} \|\eta\|_V \right)^{-1} = r(V)^{-1}.
\]

To prove (3.3), the distance \( d(\xi, V) \) from \( \xi \) to \( V \) satisfies

\[
d(\xi, V)^2 = \inf_{\eta \in V} \|\xi - \eta\|^2 = \inf_{\eta \in V} (2 - 2\Re\langle \xi, \eta \rangle) = 2 - 2 \sup_{\eta \in V} \Re\langle \xi, \eta \rangle
\]

\[
= 2 - 2 \sup_{\eta \in V} |\langle \xi, \eta \rangle| = 2 - 2 \|\xi\|^V,
\]

and (3.3) follows after taking square roots. \( \square \)

**Corollary 3.3.** If the inner radius \( r(V) \) is positive, then \( \|\cdot\|^V \) is a norm on \( H \) satisfying

\[
\|\xi\| \leq \|\xi\|^V \leq \frac{1}{r(V)} \|\xi\|, \quad \xi \in H.
\]

If \( H \) is finite dimensional, then \( r(V) > 0 \).
Proof. The first sentence follows from (2.3) and (3.2). If \( H \) is finite dimensional, all norms on \( H \) are equivalent, and \( r(V) > 0 \) follows from (3.1). □

**Corollary 3.4.** In general, for any closed set \( V \) of unit vectors that satisfies V1, the following five assertions about \( V \) are equivalent:

(i) The closed convex hull of \( V \) has nonempty interior.
(ii) The inner radius of \( V \) is positive.
(iii) The seminorm \( \| \cdot \|_V \) is equivalent to the ambient norm of \( H \).
(iv) The function \( \| \cdot \|_V \) is a norm equivalent to the ambient norm of \( H \).
(v) The function \( d(\cdot, V) \) is bounded away from \( \sqrt{2} \) on the unit sphere:

\[
\sup_{\| \xi \|=1} d(\xi, V) < \sqrt{2}.
\]

Proof. The equivalences (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) are immediate consequences of the formulas of Theorem 3.2. Since the implication (ii) ⇒ (i) is trivial, it suffices to prove (i) ⇒ (ii).

For that, let \( U \) be a nonempty open set contained in the closed convex hull \( K \) of \( V \). The vector difference \( U - U \) is an open neighborhood of 0 that is contained in \( K - K \). By axiom V1, \( K - K \) is contained in \( 2 \cdot K \), so that \( 2^{-1} \cdot (U - U) \) is a subset of \( K \) that contains an open ball about 0. □

### 4. Maximal vectors

Throughout this section, \( V \) will denote a norm-closed subset of the unit sphere of a Hilbert space \( H \) that satisfies V1 and V2. For every unit vector \( \xi \in H \), the distance from \( \xi \) to \( V \) satisfies \( 0 \leq d(\xi, V) \leq \sqrt{2} \); and since \( V \) is norm-closed, one has \( d(\xi, V) = 0 \) iff \( \xi \in V \).

**Definition 4.1.** By a **maximal vector** we mean a vector \( \xi \in H \) satisfying \( \| \xi \| = 1 \) and

\[
d(\xi, V) = \sup_{\| \eta \|=1} d(\eta, V).
\]

When \( H \) is finite dimensional, an obvious compactness argument shows that maximal vectors exist; and they exist for significant infinite dimensional examples as well (see Sections 8–14). Maximal vectors will play a central role throughout the remainder of this paper. In this section we show that whenever \( r(V) > 0 \), the restriction of the function \( \| \cdot \|_V \) to the unit sphere of \( H \) detects maximality as well as membership in \( V \). Indeed, in Theorem 3.2 we calculated the minimum of \( \| \cdot \|_V \) and the maximum of \( \| \cdot \|_V \) over the unit sphere of \( H \). What is notable is that when either of the two extremal values is achieved at some unit vector \( \xi \) then they are both achieved at \( \xi \); and that such vectors \( \xi \) are precisely the maximal vectors.

**Theorem 4.2.** If \( r(V) > 0 \), then for every unit vector \( \xi \in H \), the following three assertions are equivalent:

(i) \( \| \xi \|_V = r(V) \) is minimum.
(ii) \( \| \xi \|_V = r(V)^{-1} \) is maximum.
(iii) \( d(\xi, V) = \sqrt{2(1 - r(V))} \) is maximum.

**Proof.** Choose a unit vector \( \xi \). We will prove the implications (i) \( \iff \) (iii), (i) \( \implies \) (ii) and (ii) \( \implies \) (i).

(i) \( \iff \) (iii): Theorem 3.2 implies that the minimum value of \( \|\xi\|_V \) is \( r(V) \), the maximum value of \( d(\xi, V) \) is given by (iii), and that \( d(\xi, V) \) is maximized at \( \xi \) iff \( \|\xi\|_V \) is minimized at \( \xi \).

(i) \( \implies \) (ii): If \( \|\xi\|_V = r(V) \) then \( \|r(V) - 1\xi\|_V = 1 \), so that
\[
\|\xi\|_V = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle| \leq \frac{1}{r(V)}.
\]
Since (3.2) implies \( \|\xi\|_V \leq r(V) - 1 \), we conclude that \( \|\xi\|_V = r(V) - 1 \).

(ii) \( \implies \) (i): Assuming (ii), we have
\[
r(V) - 1 = \|\xi\|_V = \sup_{\|\eta\|_V = 1} |\langle \xi, \eta \rangle| = \sup_{\|\eta\|_V = 1} \frac{|\langle \xi, \eta \rangle|}{\|\eta\|_V},
\]
the last equality holding because the function
\[
\eta \in \{ \eta \in H : \eta \neq 0 \} \mapsto \frac{|\langle \xi, \eta \rangle|}{\|\eta\|_V}
\]
is homogeneous of degree zero. After taking reciprocals, we obtain
\[
(4.1) \quad r(V) = \inf_{\|\eta\|_V = 1} \|\eta\|_V.
\]
Now (4.1) implies that there is a sequence of unit vectors \( \eta_n \) such that
\[
(4.2) \quad \lim_{n \to \infty} \frac{\|\eta_n\|_V}{|\langle \xi, \eta_n \rangle|} = r(V).
\]
Since
\[
\frac{\|\eta_n\|_V}{|\langle \xi, \eta_n \rangle|} \geq \|\eta_n\|_V \geq r(V), \quad n = 1, 2, \ldots,
\]
it follows that \( \langle \xi, \eta_n \rangle \neq 0 \) for large \( n \); moreover, since the left side converges to \( r(V) \) we must have
\[
\lim_{n \to \infty} \|\eta_n\|_V = r(V), \quad \text{and} \quad \lim_{n \to \infty} |\langle \xi, \eta_n \rangle| = 1.
\]
Since \( \xi \) and \( \eta_n \) are unit vectors for which \( |\langle \xi, \eta_n \rangle| \) converges to 1, there is a sequence \( \lambda_n \in \mathbb{C} \), \( |\lambda_n| = 1 \), such that \( \lambda_n \langle \xi, \eta \rangle = \langle \lambda_n \cdot \xi, \eta_n \rangle \) is nonnegative and converges to 1. It follows that
\[
\lim_{n \to \infty} \|\lambda_n \cdot \xi - \eta_n\|^2 = \lim_{n \to \infty} 2 - 2\Re\langle \lambda_n \cdot \xi, \eta \rangle = 0,
\]
hence \( \lambda_n \cdot \eta_n \) converges in norm to \( \xi \). By continuity of the norm \( \|\cdot\|_V \),
\[
\|\xi\|_V = \lim_{n \to \infty} \|\lambda_n \cdot \eta_n\|_V = \lim_{n \to \infty} \|\eta_n\|_V = r(V),
\]
and (i) follows. \( \square \)
Corollary 4.3. If \( r(V) > 0 \) then \( \| \cdot \|_V \) restricts to a bounded norm-continuous function on the unit sphere of \( H \) with the property that for every unit vector \( \xi \), \( \| \xi \|_V = 1 \) iff \( \xi \in V \) and \( \| \xi \|_V = r(V)^{-1} \) iff \( \xi \) is maximal.

Part 2. Normal states and normal functionals on \( \mathcal{B}(H) \).

Let \( H \) be a Hilbert space and let \( V \) be a norm-closed subset of the unit sphere of \( H \) that satisfies axioms V1 and V2. We now introduce a numerical function of normal states of \( \mathcal{B}(H) \) that faithfully measures “generalized entanglement”, and we develop its basic properties in general. When the inner radius of \( V \) is positive, this function is shown to be the restriction of a norm on the predual \( \mathcal{B}(H)_* \) to the space of normal states, or equivalently, the restriction of a norm on the Banach space \( L^1(H) \) of trace class operators to the space of density operators.

5. Generalized entanglement of states

Fix a Hilbert space \( H \). The Banach space \( \mathcal{B}(H)_* \) of normal linear functionals on \( \mathcal{B}(H) \) identifies naturally with the dual of the \( C^* \)-algebra \( \mathcal{K} \) of compact operators on \( H \), and we may speak of the weak* topology on \( \mathcal{B}(H)_* \). Similarly, \( \mathcal{B}(H) \) identifies with the dual of \( \mathcal{B}(H)_* \), and we may speak of the weak* topology on \( \mathcal{B}(H) \). Thus, a net of normal functionals \( \rho_n \) converges weak* to zero iff

\[
\lim_{n \to \infty} \rho_n(K) = 0, \quad \forall K \in \mathcal{K},
\]

and a net of operators \( A_n \in \mathcal{B}(H) \) converges weak* to zero iff

\[
\lim_{n \to \infty} \rho(A_n) = 0, \quad \forall \rho \in \mathcal{B}(H)_*.
\]

There is a natural involution \( \rho \mapsto \rho^* \) defined on \( \mathcal{B}(H)_* \) by

\[
\rho^*(A) = \rho(A^*), \quad A \in \mathcal{B}(H),
\]

and we may speak of self adjoint normal functionals \( \rho \). Of course, \( \mathcal{B}(H)_* \) identifies naturally with the Banach *-algebra of trace class operators, but that fact is not particularly useful for our purposes.

Our aim is to introduce a measure of “generalized entanglement” for normal states. It will be convenient to define it more generally as a function (5.3) defined on the larger Banach space \( \mathcal{B}(H)_* \). For every \( X \in \mathcal{B}(H) \), define

\[
\|X\|_V = \sup_{\xi, \eta \in V} |\langle X\xi, \eta \rangle|.
\]

Axiom V2 implies that \( \| \cdot \|_V \) is a norm, and obviously \( \|X\|_V \leq \|X\| \) and \( \|X^*\| = \|X\| \) for every \( X \). Consider the \( C^* \)-algebra \( \mathcal{A} \) obtained from the compact operators \( \mathcal{K} \subseteq \mathcal{B}(H) \) by adjoining the identity operator

\[
\mathcal{A} = \{ K + \lambda \cdot 1 : K \in \mathcal{K}, \lambda \in \mathbb{C} \}.
\]

Operators in \( \mathcal{A} \) serve as “test operators” for our purposes. The \( V \)-ball in \( \mathcal{A} \)

\[
(5.2) \quad \mathcal{B} = \{ X \in \mathcal{A} : \|X\|_V \leq 1 \}
\]
is a norm-closed convex subset of \( \mathcal{A} \) that is stable under the \( * \)-operation, stable under multiplication by complex scalars of absolute value 1, and it contains the unit ball of \( \mathcal{A} \). Thus we can define an extended real-valued function \( E : \mathcal{B}(H)_* \to [0, +\infty] \) by

\[
E(\rho) = \sup_{X \in \mathcal{B}} \Re \rho(X) = \sup_{X \in \mathcal{B}} |\rho(X)|, \quad \rho \in \mathcal{B}(H)_*.
\]

**Remark 5.1 (Self adjoint elements of \( \mathcal{B}(H)_* \)).** Note that if \( \rho = \rho^* \) is self adjoint functional in \( \mathcal{B}(H)_* \), then \( E(\rho) \) can be defined somewhat differently in terms of self adjoint operators:

\[
E(\rho) = \sup \{|\rho(X)| : X \in \mathcal{A}, \|X\|_V \leq 1\}.
\]

Indeed, every \( Z \in \mathcal{B} \) has a cartesian decomposition \( Z = X + iY \) where \( X \) and \( Y \) are self adjoint with \( X = (Z + Z^*)/2 \), and we have

\[
\Re \rho(Z) = \frac{1}{2}(\rho(Z) + \rho(Z^*)) = \frac{1}{2}\rho(Z + Z^*) = \rho(X),
\]

where \( X = X^* \in \mathcal{B} \). After noting \( |\rho(X)| = \max(\rho(X), \rho(-X)) \), we obtain

\[
E(\rho) \leq \sup \{|\rho(X)| : X \in \mathcal{K}, \|X\|_V \leq 1\}.
\]

The opposite inequality is obvious.

In general, \( E(\rho) \) can achieve the value \( +\infty \) (see Remark 7.4). We first determine when the set \( \mathcal{B} \) is bounded.

**Proposition 5.2.** Let \( r(V) \) be the inner radius of \( V \) and let \( \mathcal{B}_0 \) be the set of all positive rank-one operators in \( \mathcal{B} \). Then

\[
\sup_{X \in \mathcal{B}} \|X\| = \sup_{X \in \mathcal{B}_0} \|X\| = \frac{1}{r(V)^2}.
\]

Consequently, for every normal linear functional \( \rho \in \mathcal{B}(H)_* \),

\[
\|\rho\| \leq E(\rho) \leq r(V)^{-2} \cdot \|\rho\|.
\]

**Proof.** To prove (5.4), it suffices to show that for every positive number \( M \), the following are equivalent:

(i) \( \|X\| \leq M \cdot \|X\|_V \) for every rank one projection \( X \in \mathcal{K} \).

(ii) \( \|X\| \leq M \cdot \|X\|_V \) for every \( X \in \mathcal{B}(H) \).

(iii) \( M \geq r(V)^{-2} \).

Since the implication (ii) \( \implies \) (i) is trivial, it is enough to prove (i) \( \implies \) (iii) and (iii) \( \implies \) (ii).

(i) \( \implies \) (iii): Choose a unit vector \( \zeta \in H \) and let \( X \) be the rank one projection \( X = \langle \xi, \zeta \rangle \zeta, \xi \in H \). Then (i) implies

\[
1 = \|X\| \leq M \cdot \sup \{|\langle X\xi, \eta \rangle| : \xi, \eta \in V\}
\]

\[
= M \cdot \sup \{|\langle \xi, \zeta \rangle| : \langle \zeta, \eta \rangle : \xi, \eta \in V\}
\]

\[
= M \cdot \sup \{|\langle \zeta, \xi \rangle|^2 : \xi \in V\},
\]

(5.5) The opposite inequality is obvious.
from which it follows that
\[ \sqrt{M} \cdot \sup_{\xi \in V} \Re \langle \xi, \xi \rangle = \sqrt{M} \cdot \sup_{\xi \in V} \|\langle \xi, \xi \rangle\| \geq 1. \]

Let \( K \) be the closed convex hull of \( V \). After multiplying through by \( \|\xi\| \) for more general nonzero vectors \( \xi \in H \), the preceding inequality implies
\[ \sqrt{M} \cdot \sup_{\xi \in K} \Re \langle \xi, \xi \rangle = \sqrt{M} \cdot \sup_{\xi \in V} \Re \langle \xi, \xi \rangle \geq \|\xi\| = \sup_{\|\eta\| \leq 1} \Re \langle \xi, \eta \rangle. \]

Since every bounded real-linear functional \( f : H \to \mathbb{R} \) must have the form \( f(\xi) = \Re \langle \xi, \xi \rangle \) for some vector \( \xi \in H \), a standard separation theorem implies that the unit ball of \( H \) is contained in \( \sqrt{M} \cdot K \), namely the closed convex hull of \( \sqrt{M} \cdot V \). Hence \( r(V) \geq M^{-1/2} \).

(iii) \( \Rightarrow \) (ii): Fix \( X \in \mathcal{B}(H) \) and let \( \xi_0, \eta_0 \in H \) satisfy \( \|\xi_0\| \leq 1, \|\eta_0\| \leq 1 \).

By definition of \( r(V) \), hypothesis (iii) implies that both \( \xi_0 \) and \( \eta_0 \) belong to the closed convex hull of \( \sqrt{M} \cdot V \), and hence
\[ |\langle X\xi_0, \eta_0 \rangle| \leq \sup\{|\langle X\xi, \eta \rangle| : \xi, \eta \in \sqrt{M} \cdot V\} = M \cdot \sup_{\xi, \eta \in V} |\langle X\xi, \eta \rangle| = M \cdot \|X\|. \]

After taking the supremum over \( \xi_0, \eta_0 \), we obtain \( \|X\| \leq M \cdot \|X\|_V \).

The estimates (5.5) follow immediately from (5.7).

\[ \square \]

The basic properties of the function \( E \) are summarized as follows.

**Theorem 5.3.** The function \( E : \mathcal{B}(H)_* \to [0, +\infty) \) satisfies:

(i) For all \( \rho_1, \rho_2 \in \mathcal{B}(H)_* \), \( E(\rho_1 + \rho_2) \leq E(\rho_1) + E(\rho_2) \).

(ii) For every nonzero \( \lambda \in \mathbb{C} \) and every \( \rho \in \mathcal{B}(H)_* \), \( E(\lambda \cdot \rho) = |\lambda| \cdot E(\rho) \).

(iii) \( E \) is lower semicontinuous relative to the weak* topology of \( \mathcal{B}(H)_* \).

(iv) If \( r(V) > 0 \), then \( E \) is a norm equivalent to the norm of \( \mathcal{B}(H)_* \).

Moreover, letting \( \Sigma \) be the set of all normal states of \( \mathcal{B}(H) \), we have

\[ \sup_{\rho \in \Sigma} E(\rho) = \sup_{X \in \mathcal{B}} \|X\| = \frac{1}{r(V)^2}, \]

the right side being interpreted as \( +\infty \) when \( r(V) = 0 \).

**Proof.** (i), (ii) and (iii) are immediate consequences of the definition (5.3) of \( E \) after noting that a supremum of continuous real-valued functions is lower semicontinuous, and (iv) follows from (5.5).

To prove (5.6), let \( \mathcal{B}_1 = \{ X = X^* \in \mathcal{A} : \|X\|_V \leq 1 \} \) be the set of self adjoint operators in \( \mathcal{B} \). Remark 5.1 implies that
\[ \sup_{\rho \in \Sigma} E(\rho) = \sup_{\rho \in \Sigma} \sup_{X \in \mathcal{B}_1} \rho(X) = \sup_{X \in \mathcal{B}_1} \sup_{\rho \in \Sigma} \rho(X). \]

Noting that \( \mathcal{B}_1 = -\mathcal{B}_1 \) and that the norm of a self adjoint operator agrees with its numerical radius, the right side can be replaced with
\[ \sup_{X \in \mathcal{B}_1} \sup_{\rho \in \Sigma} |\rho(X)| = \sup_{X \in \mathcal{B}_1} \|X\|. \]
Formula (5.6) now follows from (5.4) of Proposition 5.2.

We may conclude that when the inner radius is positive, $E(\cdot)$ is uniformly continuous on the unit ball of $\mathcal{B}(H)_*$:

**Corollary 5.4.** Assume that $r(V) > 0$. Then for $\rho, \sigma \in \mathcal{B}(H)_*$ we have

(5.7) \[ |E(\rho) - E(\sigma)| \leq r(V)^{-2} \|\rho - \sigma\|. \]

**Proof.** Theorem 5.3 (iv) implies that $E(\cdot)$ is a norm on $\mathcal{B}(H)_*$, hence

\[ |E(\rho) - E(\sigma)| \leq E(\rho - \sigma) \leq r(V)^{-2} \|\rho - \sigma\|, \]

the second inequality following from (5.5). □

6. $V$-correlated states and faithfulness of $E$

Given two unit vectors $\xi, \eta \in H$, we will write $\omega_{\xi,\eta}$ for the linear functional

\[ \omega_{\xi,\eta}(A) = \langle A\xi, \eta \rangle, \quad A \in \mathcal{B}(H). \]

One has $\|\omega_{\xi,\eta}\| = \|\xi\| \cdot \|\eta\| = 1$, and $\omega_{\eta,\xi}^* = \omega_{\xi,\eta}$. We begin by recalling two definitions from the introduction.

**Definition 6.1.** A normal state $\rho$ of $\mathcal{B}(H)$ is said to be $V$-correlated if for every $\epsilon > 0$, there is an $n = 1, 2, \ldots$, a set of vectors $\xi_1, \ldots, \xi_n \in V$ and a set of positive reals $t_1, \ldots, t_n$ satisfying $t_1 + \cdots + t_n = 1$ such that

\[ \|\rho - (t_1 \omega_{\xi_1,\xi_1} + \cdots + t_n \omega_{\xi_n,\xi_n})\| \leq \epsilon. \]

A normal state $\rho$ that is not $V$-correlated is said to be entangled.

By (5.5), $E(\rho) \geq 1$ for every normal state $\rho$. The purpose of this section is to prove the following result that characterizes entangled states by the inequality $E(\rho) > 1$. We assume that $H$ is a perhaps infinite dimensional Hilbert space, that $V \subseteq \{\xi \in H : \|\xi\| = 1\}$ satisfies hypotheses V1 and V2, but we make no assumption about the inner radius of $V$.

**Theorem 6.2.** A normal state $\rho$ of $\mathcal{B}(H)$ is $V$-correlated iff $E(\rho) = 1$.

The proof of Theorem 6.2 requires some preparation that is conveniently formulated in terms of the state space of the unital $C^*$-algebra

\[ \mathcal{A} = \mathcal{K} + \mathbb{C} \cdot 1 = \{K + \lambda \cdot 1 : K \in \mathcal{K}, \ \lambda \in \mathbb{C}\}, \]

which of course reduces to $\mathcal{B}(H)$ when $H$ is finite dimensional. After working out these preliminaries, we will return to the proof of Theorem 6.2 later in the section. The state space of $\mathcal{A}$ is compact convex in its relative weak*-topology, not to be confused with the various weak*-topologies described in the previous section. We write $\Sigma_V$ for the set of all states $\rho$ of $\mathcal{A}$ that satisfy

(6.1) \[ |\rho(X)| \leq \|X\|_V = \sup_{\xi,\eta \in V} |\langle X\xi, \eta \rangle|, \quad X \in \mathcal{A}. \]
Theorem 6.3. Every state of $\Sigma_V$ is a weak* limit of states of $\mathcal{A}$ of the form
\[ t_1 \cdot \omega_{\xi_1, \xi_1} + \cdots + t_n \cdot \omega_{\xi_n, \xi_n} \]
where $n = 1, 2, \ldots$, $\xi_1, \ldots, \xi_n \in V$ and the $t_k$ are positive reals with sum 1.

Proof. Since (6.1) exhibits $\Sigma_V$ as an intersection of weak* closed subsets of the state space of $\mathcal{A}$, it follows that $\Sigma_V$ is weak* compact as well as convex. The Krein-Milman theorem implies that $\Sigma_V$ is the weak* closed convex hull of its extreme points, hence it suffices to show that for every extreme point $\rho$ of $\Sigma_V$, there is a net of vectors $\xi_n \in V$ such that
\[ \rho(X) = \lim_{n \to \infty} \langle X \xi_n, \xi_n \rangle, \quad X \in \mathcal{A}. \]

To that end, consider the somewhat larger set $\Omega_V$ of all bounded linear functionals $\omega$ on $\mathcal{A}$ that satisfy
\[ |\omega(X)| \leq \sup_{\xi, \eta \in V} |\langle X \xi, \eta \rangle| = \|X\|_V, \quad X \in \mathcal{A}. \]

Since $\|X\|_V \leq \|X\|$, $\Omega_V$ is contained in the unit ball of the dual of $\mathcal{A}$, and it is clearly clearly convex and weak* closed, hence compact. We claim that
\[ \Omega_V = \overline{\text{conv}}^\text{weak*} \{\omega_{\xi, \eta} \mid \mathcal{A} : \xi, \eta \in V\}, \]

conv denoting the convex hull. Indeed, the inclusion $\supseteq$ is immediate from the definition of $\Omega_V$. For the inclusion $\subseteq$, choose an operator $X \in \mathcal{A}$ and a real number $\alpha$ such that $\Re \omega_{\xi, \eta}(X) = \Re \langle X \xi, \eta \rangle \leq \alpha$ for all $\xi, \eta \in V$. By axiom V1, this implies that for fixed $\xi, \eta \in V$ we have
\[ |\langle X \xi, \eta \rangle| = \sup_{|\lambda| = 1} \Re \langle X \lambda \cdot \xi, \eta \rangle = \sup_{|\lambda| = 1} \Re \langle X \lambda \xi, \eta \rangle \leq \alpha \]
and after taking the supremum over $\xi, \eta$ on the left side we obtain $\|X\|_V \leq \alpha$. It follows that for every $\omega \in \Omega_V$,
\[ |\omega(X)| \leq \|X\|_V \leq \alpha \]
and (6.4) now follows from a standard separation theorem.

Now let $\rho$ be an extreme point of $\Sigma_V$. Then $\rho \in \Omega_V$, and we claim that in fact, $\rho$ is an extreme point of $\Omega_V$. Indeed, if $\omega_1, \omega_2 \in \Omega_V$ and $0 < t < 1$ are such that $\rho = t \cdot \omega_1 + (1 - t) \cdot \omega_2$, then
\[ 1 = \rho(1) = t \cdot \omega_1(1) + (1 - t) \cdot \omega_2(1). \]
Since $|\omega_k(1)| \leq \|\omega_k\| \leq 1$ and 1 is an extreme point of the closed unit disk, it follows that $\omega_1(1) = \omega_2(1) = 1$. Since $\|\omega_k\| \leq 1 = \omega_k(1)$, this implies that both $\omega_1$ and $\omega_2$ are states of $\mathcal{A}$, hence $\omega_k \in \Sigma_V$. By extremality of $\rho$, we conclude that $\omega_1 = \omega_2 = \rho$, as asserted.

Finally, since $\rho$ is an extreme point of $\Omega_V$ and $\Omega_V$ is given by (6.4), Milman’s converse of the Krein-Milman theorem implies that there is a net of pairs $\xi_n, \eta_n \in V$ such that $\omega_{\xi_n, \eta_n}$ converges to $\rho$ in the weak* topology.
It remains to show that we can choose \( \eta_n = \xi_n \) for all \( n \), and for that consider \( \omega_{\xi_n, \eta_n}(1) = \langle \xi_n, \eta_n \rangle \), which converges to \( \rho(1) = 1 \) as \( n \to \infty \). This implies that
\[
\| \xi_n - \eta_n \|^2 = 2(1 - R(\xi_n, \eta_n)) \to 0,
\]
as \( n \to \infty \), so that \( \| \omega_{\xi_n, \xi_n} - \omega_{\xi_n, \eta_n} \| \leq \| \xi_n - \eta_n \| \to 0 \) as \( n \to \infty \). Hence \( \omega_{\xi_n, \xi_n} \) converges weak* to \( \rho \), and the desired conclusion (6.2) follows. \( \square \)

**Proof of Theorem 6.2.** It is clear from the definition (5.3) that \( E(\rho) \geq 1 \) in general. We claim first that \( E(\rho) = 1 \) for every \( V \)-correlated normal state \( \rho \). Indeed, since \( E(\cdot) \) is a convex function that is lower semicontinuous with respect to the norm topology on states, the set \( \mathcal{C} \) of all normal states \( \rho \) for which \( E(\rho) \leq 1 \) is norm closed and convex. It contains every state of the form \( \omega_{\xi, \xi} \) for \( \xi \in V \) since for every \( X \in \mathcal{A} \) we have
\[
\omega_{\xi, \xi}(X) = |\langle X \xi, \xi \rangle| \leq \sup_{\eta, \zeta \in V} |\langle X \eta, \zeta \rangle| = \|X\|_V
\]
so that \( E(\omega_{\xi, \xi}) \leq 1 \). Hence \( \mathcal{C} \) contains every \( V \)-correlated state.

Conversely, let \( \rho \) be a normal state for which \( E(\rho) = 1 \), or equivalently,
\[
|\rho(X)| \leq \|X\|_V, \quad X \in \mathcal{A}.
\]

Theorem 6.3 implies that there is a net of normal states \( \rho_n \) of \( \mathcal{B}(H) \), each of which is a finite convex combination of states of the form \( \omega_{\xi, \xi} \) with \( \xi \in V \), such that
\[
\rho(X) = \lim_{n \to \infty} \rho_n(X), \quad X \in \mathcal{A},
\]
and in particular
\[
\rho(K) = \lim_{n \to \infty} \rho_n(K), \quad K \in \mathcal{K}.
\]

It is well known that if a net of normal states converges to a normal state pointwise on compact operators, then in fact \( \|\rho - \rho_n\| \to 0 \) as \( n \to \infty \) (for example, see Lemma 2.9.10 of [Arv03]). We conclude from the latter that \( \rho \) is \( V \)-correlated. \( \square \)

**Remark 6.4.** In the special case where \( H \) is a tensor product of Hilbert spaces \( H = H_1 \otimes H_2 \) and \( V = \{ \xi_1 \otimes \xi_2 : \xi_k \in H_k, \|\xi_1\| = \|\xi_2\| = 1 \} \), Holevo, Shirokov and Werner showed [HSW05] that when \( H_1 \) and \( H_2 \) are infinite dimensional, there are normal states that can be norm approximated by convex combinations of vector states of the form \( \omega_{\xi, \xi} \), \( \xi \in V \), but which cannot be written as a discrete infinite convex combination
\[
\rho = \sum_{k=1}^{\infty} t_k \cdot \omega_{\xi_k, \xi_k}
\]
with \( \xi_k \in V \) and with nonnegative numbers \( t_k \) having sum 1. On the other hand, they also show that every such \( \rho \) can be expressed as an integral
\[
(6.5) \quad \rho(X) = \int_S \langle X \xi, \xi \rangle \, d\mu(\xi), \quad X \in \mathcal{B}(H)
\]
where $\mu$ is a probability measure on the Polish space

$$S = \{\xi = \eta \otimes \zeta : \|\eta\| = \|\zeta\| = 1\}.$$ 

It seems likely that an integral representation like (6.5) should persist for $V$-correlated states in the more general setting of Theorem 6.2, where of course $S$ is replaced with $V$—though we have not pursued that issue.

7. Maximally entangled states

The entanglement of a normal state $\rho$ satisfies $1 \leq E(\rho) \leq r(V)^{-2}$, and the minimally entangled states were characterized as the $V$-correlated states in Theorem 6.2. In this section we discuss states at the opposite extreme.

**Definition 7.1.** A normal state $\rho$ satisfying $E(\rho) = r(V)^{-2}$ is said to be maximally entangled.

We now calculate the entanglement of (normal) pure states in general, and we characterize the maximally entangled pure states in cases where the inner radius of $V$ is positive.

**Theorem 7.2.** Let $V$ be a norm-closed subset of the unit sphere of $H$ satisfying $V1$ and $V2$, let $\xi$ be a unit vector in $H$ and let $\omega$ the corresponding vector state $\omega(X) = \langle X\xi, \xi \rangle$, $X \in \mathcal{B}(H)$. Then

\begin{equation}
E(\omega) = (\|\xi\|^V)^2.
\end{equation}

Assuming further that $r(V) > 0$, then $\omega$ is maximally entangled iff $\xi$ is a maximal vector. More generally, let $\rho$ be an arbitrary maximally entangled normal state, and decompose $\rho$ into a perhaps infinite convex combination of vector states

\begin{equation}
\rho(X) = t_1 \cdot \omega_1 + t_2 \cdot \omega_2 + \cdots
\end{equation}

where the $t_k$ are positive numbers with sum 1 and each $\omega_k$ has the form $\omega_k(X) = \langle X\xi_k, \xi_k \rangle$, $X \in \mathcal{B}(H)$, with $\|\xi_k\| = 1$. Then each $\omega_k$ is maximally entangled.

The proof of Theorem 7.2 makes use of the following basic inequality:

**Lemma 7.3.** For every $\xi, \eta \in H$ and every $A \in \mathcal{B}(H)$,

\begin{equation}
|\langle A\xi, \eta \rangle| \leq \|A\|_V \|\xi\|^V \|\eta\|^V.
\end{equation}

**Proof of Lemma 7.3.** After rescaling both $\xi$ and $\eta$, it is enough to show that

\begin{equation}
\|\xi\|^V \leq 1, \quad \|\eta\|^V \leq 1 \implies |\langle A\xi, \eta \rangle| \leq \|A\|_V.
\end{equation}

To that end, assume first that $\xi, \eta \in V$. Then

$$|\langle A\xi, \eta \rangle| \leq \sup_{\xi, \eta \in V} |\langle A\xi, \eta \rangle| = \|A\|_V.$$ 

Since $\langle A\xi, \eta \rangle$ is sesquilinear in $\xi, \eta$, the same inequality $|\langle A\xi, \eta \rangle| \leq \|A\|_V$ persists if $\xi$ and $\eta$ are finite convex combinations of elements of $V$, and by passing to the norm closure, $|\langle A\xi, \eta \rangle| \leq \|A\|_V$ remains true if $\xi$ and $\eta$ belong
to the closed convex hull of $V$. By Lemma 2.2, the closed convex hull of $V$

is $\{\zeta \in H : \|\zeta\|^V \leq 1\}$, and (7.4) follows.

Proof. Let $\xi \in H$ be a unit vector with associated vector state $\omega$ and let

$A = \mathcal{K} + \mathbb{C}\cdot 1$. Then for every $A \in \mathcal{A}$ satisfying $\|A\|^V \leq 1$, (7.3) implies

$$|\omega(A)| = |\langle A\xi, \xi \rangle| \leq (\|\xi\|^V)^2,$$

and $E(\omega) \leq (\|\xi\|^V)^2$ follows from the definition (5.3) after taking the supre-

num over $A$.

To prove the opposite inequality $E(\omega) \geq (\|\xi\|^V)^2$, consider

$$\|\xi\|^V = \sup_{\|r\|^V = 1} |\langle \xi, \zeta \rangle|.$$

Let $\zeta_n$ be a sequence of vectors in $H$ satisfying satisfying $\|\zeta_n\|^V = 1$ for all

$n = 1, 2, \ldots$ and $|\langle \xi, \zeta_n \rangle| \uparrow \|\xi\|^V$ as $n \to \infty$. Consider the sequence of rank

one operators $A_1, A_2, \ldots$ defined by $A_n(\eta) = \langle \eta, \zeta_n \rangle \zeta_n$, $\eta \in H$, and note

that $\|A_n\|^V = 1$. Indeed, we have

$$\|A_n\|^V = \sup_{\eta_1, \eta_2 \in V} |\langle A_n\eta_1, \eta_2 \rangle| = \sup_{\eta_1, \eta_2 \in V} |\langle \eta_1, \zeta_n \rangle \langle \zeta_n, \eta_2 \rangle|$$

$$= (\sup_{\eta \in V} |\langle \zeta_n, \eta \rangle|^2) = \|\zeta_n\|^V = 1.$$

So by (5.3), $E(\rho) \geq |\rho(A_n)|$ for every $n = 1, 2, \ldots$. But since

$$\rho(A_n) = \langle A_n\xi, \xi \rangle = |\langle \xi, \zeta_n \rangle|^2 \uparrow (\|\xi\|^V)^2$$

as $n \to \infty$, it follows that $E(\rho) \geq (\|\xi\|^V)^2$.

For the second paragraph, assume that $r(V) > 0$. Theorem 4.2 implies

that $\|\xi\|^V = r(V)^{-1}$ iff $\xi$ is a maximal vector; and from (7.1) we conclude

that $\omega$ is a maximally entangled state iff $\xi$ is a maximal vector.

Let $\rho$ be a maximally entangled state of the form (7.2). By symmetry

and since all the $t_k$ are positive, it suffices to show that $\omega_1$ is maximally

entangled. For that, consider the normal state

$$\sigma = \frac{t_2}{1 - t_1} \omega_2 + \frac{t_3}{1 - t_1} \omega_3 + \cdots.$$

We have $\rho = t_1 \cdot \omega_1 + (1 - t_1) \cdot \sigma$, and since $E$ is a convex function,

$$\frac{1}{r(V)^2} = E(\rho) \leq t_1 E(\omega_1) + (1 - t_1) E(\sigma).$$

Since $E(\omega_1)$ and $E(\sigma)$ are both $\leq r(V)^{-2}$, it follows that $E(\omega_1) = E(\sigma) =

r(V)^{-2}$, hence $\omega_1$ is a maximally entangled pure state. \qed

Remark 7.4 (Infinitely entangled states). Consider the case $H = H_1 \otimes H_2$

with $V$ the set of decomposable unit vectors $\eta_1 \otimes \eta_2$, with $\eta_k$ a unit vector

in $H_k$, $k = 1, 2$. When $\dim H_1 = \dim H_2 = \infty$, infinitely entangled normal

states exist. Indeed, Proposition 8.3 below implies that there are unit vectors

$\xi$ satisfying $||\xi||^V = +\infty$ in this case, and by Theorem 7.2, such a $\xi$ gives

rise to a vector state $\omega$ for which $E(\omega) = +\infty$. 

Part 3. \(N\)-fold tensor products

In the remaining sections we consider Hilbert spaces presented as \(N\)-fold tensor products

\[
H = H_1 \otimes \cdots \otimes H_N
\]

in which at most one of the factors \(H_k\) is infinite-dimensional. We can arrange that the dimensions \(n_k = \dim H_k\) increase \(n_1 \leq \cdots \leq n_N\), so that \(n_{N-1} < \infty\). The set \(V\) of distinguished vectors is the set of all decomposable unit vectors

\[
V = \{\xi_1 \otimes \cdots \otimes \xi_N : \xi_k \in H_k, \|\xi_1\| = \cdots = \|\xi_N\| = 1\}.
\]

The general results above imply that we will have a rather complete understanding of separable states and entanglement once we determine the inner radius of \(V\), have an explicit description of the maximal vectors, and identify the entanglement norm of states. In the remaining sections we present our progress in carrying out those calculations. We calculate the vector norms \(\| \cdot \|_V\) and \(\| \cdot \|_V^\gamma\) and the entanglement measuring norm \(E\) of normal states in general. In order to determine the maximal vectors one must first calculate the inner radius \(r(V)\). While we are unable to obtain an explicit formula in general, we do obtain such a formula under the assumption that \(H_N\) is “large” in the sense that \(n_N \geq n_1 \cdots n_{N-1}\) and we characterize maximal vectors as those unit vectors that purify the tracial state of the subalgebra \(\mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \otimes 1_{H_N}\). Of course, a natural setting in which all of the results of this section are valid is that in which exactly one of the factors of \(H_1 \otimes \cdots \otimes H_N\) is infinite dimensional.

8. Calculation of the vector norms \(\| \cdot \|_V\) and \(\| \cdot \|_V^\gamma\)

Remark 8.1 (Projective tensor products). We begin by reviewing the definition and universal property of the projective tensor product \(E_1 \hat{\otimes} \cdots \hat{\otimes} E_N\) of complex Banach spaces \(E_1, \ldots, E_N\). We require these results only when at most one of \(E_1, \ldots, E_N\) is infinite dimensional and we confine the discussion to such cases, with the \(E_k\) arranged so that their dimensions \(n_k = \dim E_k\) weakly increase with \(k\) and satisfy \(n_{N-1} < \infty\). Every vector \(z\) of the algebraic tensor product of vector spaces \(E_1 \odot \cdots \odot E_N\) can be expressed as a sum of elementary tensors

\[
(8.1) \quad z = \sum_{k=1}^{n} x_1^k \otimes \cdots \otimes x_N^k,
\]

in many ways, with \(1 \leq n \leq n_1n_2 \cdots n_{N-1}\), \(x_j^k \in E_j\), \(j = 1, \ldots, N\). The projective norm (or greatest cross norm) \(\|z\|_\gamma\) is defined as

\[
\|z\|_\gamma = \inf \sum_{k=1}^{n} \|x_1^k\| \|x_2^k\| \cdots \|x_N^k\|
\]
the infimum extended over all representations of \( z \) of the form (8.1). It is a fact that the norm \( \| \cdot \|_\gamma \) makes the algebraic tensor product into a Banach space - the projective tensor product - denoted \( E_1 \otimes \cdots \otimes E_N \). The projective norm is a cross norm \( (\| x_1 \otimes \cdots \otimes x_N \|_\gamma = \|x_1\| \cdots \|x_N\|) \) that dominates every cross norm on \( E_1 \otimes \cdots \otimes E_N \).

It is characterized by the following universal property: For every Banach space \( F \) and every bounded multilinear mapping \( B : E_1 \times \cdots \times E_N \to F \), there is a unique bounded linear operator \( L : E_1 \otimes \cdots \otimes E_N \to F \) with the property \( L(x_1 \otimes \cdots \otimes x_N) = B(x_1, \ldots, x_N) \) for all \( x_j \in E_j \), \( 1 \leq j \leq N \), and the norm of the linearizing operator \( L \) is given by

\[
\|L\| = \sup\{\|B(x_1, \ldots, x_N)\| : \|x_j\| \leq 1, j = 1, \ldots, N\}.
\]

In particular, the norm of a linear functional \( F : E_1 \otimes \cdots \otimes E_N \to \mathbb{C} \) is

\[
\|F\| = \sup_{\|x_1\|=\cdots=\|x_N\|=1} |F(x_1 \otimes x_2 \otimes \cdots \otimes x_N)|.
\]

Moreover, every bounded linear functional \( F \) on \( E_1 \otimes \cdots \otimes E_N \) can be written as a finite linear combination of decomposable functionals

\[
F = \sum_{k=1}^{n_1 n_2 \cdots n_N - 1} F_k^1 \otimes \cdots \otimes F_k^N,
\]

where for each \( j = 1, \ldots, N \), \( F_j^k \) is a bounded linear functional on \( E_j \).

We now calculate the vector norms \( \| \cdot \|_V \) and \( \| \cdot \|_V \) for cases in which \( V \) is the set of decomposable unit vectors in \( N \)-fold tensor products \( H_1 \otimes \cdots \otimes H_N \) where the dimensions \( n_k = \dim H_k \) weakly increase with \( n_{N-1} < \infty \). The space \( H_N \) is allowed to be infinite dimensional.

**Theorem 8.2.** For every \( \xi \in H_1 \otimes \cdots \otimes H_N \), let \( F_\xi \) be the element of the dual of the projective tensor product \( H_1 \otimes \cdots \otimes H_N \) defined by

\[
F_\xi(\eta_1 \otimes \cdots \otimes \eta_N) = \langle \eta_1 \otimes \cdots \otimes \eta_N, \xi \rangle.
\]

Then the norms \( \| \cdot \|_V \) and \( \| \cdot \|_V \) are given by

\[
\|\xi\|_V = \|F_\xi\|, \quad \|\xi\|_V = \|\xi\|_\gamma, \quad \xi \in H_1 \otimes \cdots \otimes H_N.
\]

**Proof.** The first formula of (8.4) is an immediate consequence of the definition of \( \|\xi\|_V \) and the formula (8.2), since

\[
\|\xi\|_V = \sup_{\|\eta_1\|=\cdots=\|\eta_N\|=1} |\langle \eta_1 \otimes \cdots \otimes \eta_N, \xi \rangle| = \sup_{\|\eta_1\|=\cdots=\|\eta_N\|=1} |F_\xi(\eta_1 \otimes \cdots \otimes \eta_N)| = \|F_\xi\|.
\]

For the second formula, write

\[
\|\xi\|_V = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta \rangle| = \sup_{\|\eta\|_V \leq 1} |F_\eta(\xi)|.
\]

The formula just proved asserts that \( \|\eta\|_V = \|F_\eta\| \), so the preceding formula can be written
\[
\|\xi\|_V = \sup_{\|\eta\| \leq 1} |F_\eta(\xi)| \leq \|\xi\|_\gamma.
\]

For the opposite inequality, we use the Hahn-Banach theorem to find a linear functional \( F \) of norm 1 in the dual of \( H_1 \otimes \cdots \otimes H_N \) such that \( \|\xi\|_\gamma = F(\xi) \). By the Riesz lemma there is a unique vector \( \eta \in H_1 \otimes \cdots \otimes H_N \) such that \( F(\zeta) = \langle \zeta, \eta \rangle \) for all \( \zeta \), and in particular \( \|\xi\|_\gamma = F(\xi) = \langle \xi, \eta \rangle = F_\eta(\xi) \). By the first part of the proof we have \( \|\eta\|_V = \|F_\eta\| = 1 \). Hence
\[
\|\xi\|_\gamma = \langle \xi, \eta \rangle \leq \sup_{\|\eta\|_V \leq 1} \|\xi\|_\gamma = \|\xi\|_V,
\]
and \( \|\xi\|_\gamma = \|\xi\|_V \) follows. ☐

The following observation implies that \( r(V) \) can be zero and infinitely entangled vectors can exist. While the physics literature contains examples of infinitely entangled states (e.g., see [KSW02]), it seems worthwhile to present concrete examples of that phenomenon in this context.

**Proposition 8.3.** Consider the case \( N = 2 \), and let \( H = H_1 \otimes H_2 \) where \( H_1 \) and \( H_2 \) are both infinite dimensional. Then there are vectors \( \xi \in H \) satisfying \( \|\xi\| = 1 \) and \( \|\xi\|_V = +\infty \).

**Proof.** Let \( \theta_1, \theta_2, \ldots \) be positive numbers with sum 1, such as \( \theta_k = 2^{-k} \), let \( n_1, n_2, \ldots \) be positive integers such that \( \theta_k n_k \to \infty \) as \( k \to \infty \), and let \( e_1, e_2, \ldots \) and \( f_1, f_2, \ldots \) be orthonormal sets in \( H_1 \) and \( H_2 \) respectively. Partition the positive integers into disjoint subsets \( S_1, S_2, \ldots \) such that \( |S_k| = n_k \) for \( k = 1, 2, \ldots \). For every \( k = 1, 2, \ldots \), let \( \xi_k \) be the vector
\[
\xi_k = \sum_{j \in S_k} e_j \otimes f_j.
\]

Obviously, \( \|\xi_k\|^2 = |S_k| = n_k \), and we claim that \( \|\xi_k\|_V = 1 \). Indeed,
\[
\|\xi_k\|_V = \sup_{\|\eta\| = |\zeta| = 1} |\langle \xi_k, \eta \otimes \zeta \rangle| = \sup_{\|\eta\| = |\zeta| = 1} \left| \sum_{j \in S_k} \langle e_j, \eta \rangle \langle f_j, \zeta \rangle \right| = 1,
\]
where the last equality is achieved with unit vectors \( \eta, \zeta \) of the form
\[
\eta = n_k^{-1/2} \sum_{k \in S_k} e_k, \quad \zeta = n_k^{-1/2} \sum_{j \in S_k} f_j.
\]
The vectors \( \xi_1, \xi_2, \ldots \) are mutually orthogonal, so that
\[
\xi = \sum_k \frac{\sqrt{\theta_k}}{\|\xi_k\|} \xi_k = \sum_k \frac{\sqrt{\theta_k}}{\sqrt{n_k}} \xi_k
\]
defines a unit vector in \( H \).
We claim that \( \|\xi\|_V = +\infty \). To see that, fix \( k = 1, 2, \ldots \) and use \( \|\xi_k\|_V = 1 \) to write
\[
\|\xi\|_V = \sup_{\|\eta\|_V = 1} |\langle \xi, \eta \rangle| \geq |\langle \xi, \xi_k \rangle| = \frac{\sqrt{\theta_k}}{\sqrt{n_k}} \|\xi_k\|^2 = \sqrt{\theta_k n_k}.
\]
By the choice of \( n_k \) the right side is unbounded, hence \( \|\xi\|_V = +\infty \). \( \Box \)

9. Calculation of the entanglement norm \( E \)

Continuing in the context of the previous section, we now calculate the entanglement norm \( E(\rho) \) of normal states \( \rho \) on \( \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \). We write \( \mathcal{L}^1(H) \) for the Banach space of trace class operators on a Hilbert space \( H \), with trace norm
\[
\|A\| = \text{tr} |A|, \quad A \in \mathcal{L}^1(H),
\]
\(|A|\) denoting the positive square root of \( A^*A \). Every normal linear functional \( \rho \) on \( \mathcal{B}(H) \) has a density operator \( A \in \mathcal{L}^1(H) \), defined by
\[
\rho(B) = \text{tr}(AB), \quad B \in \mathcal{B}(H),
\]
and the identification of \( \rho \) with its density operator \( A \) is a linear isometry.

**Theorem 9.1.** Let \( \rho \) be a normal state of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \) with density operator \( A \), \( \rho(X) = \text{tr}(AX) \). The entanglement of \( \rho \) is given by
\[
E(\rho) = \|A\|_\gamma,
\]
where \( \| \cdot \|_\gamma \) is the greatest cross norm on the projective tensor product of Banach spaces \( \mathcal{L}^1(H_1) \otimes \cdots \otimes \mathcal{L}^1(H_N) \).

Before giving the proof, we first calculate the norm \( \|B\|_V \), defined on operators \( B \in \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \) as in (5.1), in the current setting in which \( V \) is the set of decomposable unit vectors of \( H_1 \otimes \cdots \otimes H_N \):

**Lemma 9.2.** For every operator \( B \in \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \), one has
\[
\|B\|_V = \sup \{ |\text{tr}(B(T_1 \otimes \cdots \otimes T_N))| : T_k \in \mathcal{L}^1(H_k), \text{tr}|T_k| \leq 1 \}.
\]

**Proof.** In this case, the definition (5.1) of the norm \( \|B\|_V \) becomes
\[
\|B\|_V = \sup \{ |\langle B(\xi_1 \otimes \cdots \otimes \xi_N, \eta_1 \otimes \cdots \otimes \eta_N)\rangle| \}
\]
the supremum extended over all pairs \( \xi_k, \eta_k \in H_k, k = 1, \ldots, N \) that satisfy \( \|\xi_k\| = \|\eta_k\| = 1 \). It follows that this formula can be written equivalently as
\[
\|B\|_V = \sup \{ |\text{tr}(B(T_1 \otimes \cdots \otimes T_N))| \}
\]
the supremum extended over all rank one operators \( T_k \in \mathcal{B}(H_k) \) having norm 1. It is well known that for every Hilbert space \( H \), the unit ball of the Banach space \( \mathcal{L}^1(H) \) of trace class operators is the closure (in the trace norm) of the set of convex combinations of rank one operators of norm at most 1. It follows that the formula (9.3) is equivalent to (9.2). \( \Box \)
Proof of Theorem 9.1. We claim first that the bounded linear functionals on the projective tensor product $\mathcal{L}^1(H_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{L}^1(H_N)$ are precisely those of the form

$$F_B(A) = \text{trace}(AB), \quad A \in \mathcal{L}^1(H_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{L}^1(H_N),$$

where $B$ is a operator in $\mathcal{B}(H_1 \otimes \cdots \otimes H_N)$. Indeed, for every operator $B \in \mathcal{B}(H_1 \otimes \cdots \otimes H_N)$, the universal property of the projective cross norm implies that there is a unique bounded linear functional $F_B$ on $\mathcal{L}^1(H_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{L}^1(H_N)$ that satisfies

$$F_B(T_1 \otimes \cdots \otimes T_N) = \text{trace}((T_1 \otimes \cdots \otimes T_N)B), \quad T_k \in \mathcal{L}^1(H_k), \ 1 \leq k \leq N.$$ 

For the opposite inclusion, by (8.3), every bounded linear functional $F$ on $\mathcal{L}^1(H_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{L}^1(H_N)$ is a finite sum of the form

$$F = \sum_{j=1}^n F_1^j \otimes \cdots \otimes F_N^j$$

where $F_k^j$ belongs to the dual of $\mathcal{L}^1(H_k)$, $1 \leq k \leq N$. Letting $B_k^j \in \mathcal{B}(H_k)$ be the operator defined by $F_k^j(T) = \text{trace}(TB_k^j)$, one sees that the operator

$$B = \sum_{j=1}^n B_1^j \otimes \cdots \otimes B_N^j \in \mathcal{B}(H_1 \otimes \cdots \otimes H_N)$$

satisfies (9.4), and the claim is proved.

Note too that by the universal property of projective tensor products, Lemma 9.2 implies that the norm of the linear functional $F_B$ associated with an operator $B$ as in (9.4) is given by

$$\|F_B\| = \|B\|_V.$$ 

Fixing $\rho(X) = \text{trace}(AX)$ as above, the Hahn-Banach theorem, together with the preceding remarks, implies that

$$\|A\|_\gamma = \sup_{\|F_B\| \leq 1} |F_B(A)| = \sup_{\|F_B\| \leq 1} |\text{trace}(AB)|.$$ 

Using (9.5), the right side becomes

$$\sup_{\|F_B\| \leq 1} |\text{trace}(AB)| = \sup_{\|B\|_V \leq 1} |\text{trace}(AB)| = \sup_{\|B\|_V \leq 1} |\rho(B)| = E(\rho),$$

and (9.1) is proved. \qed

10. Calculation of the inner radius

Let $V$ be the set of all decomposable unit vectors in a tensor product $H = H_1 \otimes \cdots \otimes H_N$ with weakly increasing dimensions $n_k = \text{dim} H_k$ and $n_{N-1} < \infty$. In this section we establish a universal lower bound on $r(V)$, we show that this lower bound is achieved when $n_N$ is sufficiently large, and we exhibit maximal vectors for those cases.
Theorem 10.1. In general, the inner radius satisfies

\begin{equation}
\tag{10.1}
r(V_N) \geq \frac{1}{\sqrt{n_1 n_2 \cdots n_{N-1}}}
\end{equation}

Proof. By formula (3.2) of Theorem 3.2, it suffices to show that for every unit vector \( \xi \in H \),

\begin{equation}
\tag{10.2}
\|\xi\|^V \leq \sqrt{n_1 n_2 \cdots n_{N-1}}.
\end{equation}

Fix orthonormal bases

\begin{equation}
\tag{10.3}
\{e^1_1, \ldots, e^1_{n_1}\}, \ldots, \{e^{N-1}_{1,\ldots,1}, \ldots, e^{N-1}_{n_N-1}\}
\end{equation}

for \( H_1, \ldots, H_{N-1} \) respectively. Every unit vector \( \xi \in H_1 \otimes \cdots \otimes H_N \) can be decomposed uniquely into a sum

\begin{equation}
\tag{10.4}
\xi = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} e^1_{i_1} \otimes \cdots \otimes e^{N-1}_{i_{N-1}} \otimes \xi_{i_1,\ldots,i_{N-1}},
\end{equation}

where \( \{\xi_{i_1,\ldots,i_{N-1}}\} \) is a set of vectors in \( H_N \) satisfying

\[ n_1, \ldots, n_{N-1} \sum_{i_1,\ldots,i_{N-1}=1}^{n_1,\ldots, n_{N-1}} \|\xi_{i_1,\ldots,i_{N-1}}\|^2 = 1. \]

Indeed, \( \xi_{i_1,\ldots,i_{N-1}} \) is the vector of \( H_N \) defined by

\[ \langle \xi_{i_1,\ldots,i_{N-1}}, \zeta \rangle = \langle \xi, e^1_{i_1} \otimes \cdots \otimes e^{N-1}_{i_{N-1}} \otimes \zeta \rangle, \quad \zeta \in H_N. \]

By Theorem 8.2, \( \| \cdot \|^V \) is a cross norm on the algebraic tensor product \( H_1 \otimes \cdots \otimes H_N \), so from (10.4) and the Schwarz inequality, we conclude that

\[ \|\xi\|^V \leq \sum_{i_1,\ldots,i_{N-1}} \|e^1_{i_1} \otimes \cdots \otimes e^{N-1}_{i_{N-1}} \otimes \xi_{i_1,\ldots,i_{N-1}}\|^V \]

\[ \leq (\sum_{i_1,\ldots,i_{N-1}} 1)^{1/2} (\sum_{i_1,\ldots,i_{N-1}} \|\xi_{i_1,\ldots,i_{N-1}}\|^2)^{1/2} = (n_1 \cdots n_{N-1})^{1/2}, \]

and (10.2) follows. \( \square \)

Assume now that \( n_N \geq n_1 n_2 \cdots n_{N-1} \), choose a set of orthonormal bases \( \{e^1_{i_1}\}, \ldots, \{e^{N-1}_{i_{N-1}}\} \) for \( H_1, \ldots, H_{N-1} \) as in (10.3), let

\[ \{f_{i_1,\ldots,i_{N-1}} : 1 \leq i_1 \leq n_1, \ldots, 1 \leq i_{N-1} \leq n_{N-1}\} \]

be an orthonormal set in \( H_N \), and consider the unit vector \( \xi \in H_1 \otimes \cdots \otimes H_N \) defined by

\begin{equation}
\tag{10.5}
\xi = \frac{1}{\sqrt{n_1 \cdots n_{N-1}}} \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} e^1_{i_1} \otimes \cdots \otimes e^{N-1}_{i_{N-1}} \otimes f_{i_1,\ldots,i_{N-1}}.
\end{equation}
Theorem 10.2. For all cases in which \( n_N \geq n_1 \cdots n_{N-1} \), we have
\[
r(V) = \frac{1}{\sqrt{n_1 \cdots n_{N-1}}}.
\]
and vectors of the form (10.5) are maximal vectors.

Proof. Let \( \xi \) be a unit vector of the form (10.5). We will show that
\[
\|\xi\|^V = \sqrt{n_1 \cdots n_{N-1}}.
\]
Once (10.7) is established, formula (3.2) of Theorem 3.2 implies that
\[
r(V) - 1 = \sup_{\|\xi\|=1} \|\xi\|^V \geq \sqrt{n_1 \cdots n_{N-1}},
\]
so that \( r(V) \leq (n_1 \cdots n_{N-1})^{-1/2} \), and (10.6) will follow after an application
of Theorem 10.1. At that point, (10.7) makes the assertion
\[
\|\xi\|^V = r(V) - 1,
\]
and Theorem 4.2 will imply that \( \xi \) is maximal.

Thus it suffices to establish (10.7). Note first that by (3.2) and (10.1),
\[
\|\xi\|^V \leq \frac{1}{r(V)} \leq \sqrt{n_1 \cdots n_{N-1}}.
\]
For the opposite inequality, Theorem 8.2 implies that \( \|\xi\|^V \) is the projective
cross norm \( \|\xi\|_\gamma \), and it suffices to exhibit a linear functional \( F \) of norm 1
on the projective tensor product \( H_1 \hat{\otimes} \cdots \hat{\otimes} H_N \) such that
\[
F(\xi) = \sqrt{n_1 \cdots n_{N-1}}.
\]
For that, consider the vector
\[
\eta = (n_1 \cdots n_{N-1})^{1/2} \cdot \xi = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} e_{i_1}^1 \otimes \cdots \otimes e_{i_{N-1}}^{N-1} \otimes f_{i_1, \ldots, i_{N-1}},
\]
and define \( F \) on \( H_1 \hat{\otimes} \cdots \hat{\otimes} H_N \) by \( F(\zeta) = \langle \zeta, \eta \rangle \). By the universal property
of the projective tensor product, the norm of \( F \) is
\[
\|F\| = \sup\{ |F(v_1 \otimes \cdots \otimes v_N)| : v_k \in H_k, \|v_k\| \leq 1 \}.
\]
Choosing \( v_k \in H_k \), we have
\[
F(v_1 \otimes \cdots \otimes v_k) = \langle v_1 \otimes \cdots \otimes v_k, \eta \rangle
\]
\[
= \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} \langle v_1, e_{i_1}^1 \rangle \cdots \langle v_{N-1}, e_{i_{N-1}}^{N-1} \rangle \langle v_N, f_{i_1, \ldots, i_{N-1}} \rangle
\]
\[
= \langle v_N, \sum_{i_1, \ldots, i_{N-1}} \langle e_{i_1}^1, v_1 \rangle \cdots \langle e_{i_{N-1}}^{N-1}, v_{N-1} \rangle f_{i_1, \ldots, i_{N-1}} \rangle.
\]
Using orthonormality of \( \{ f_{i_1, \ldots, i_{N-1}} \} \), we can write
\[
\sup_{\|v_N\| \leq 1} |F(v_1 \otimes \cdots \otimes v_N)| = \| \sum_{i_1, \ldots, i_{N-1}} \langle e_{i_1}^1, v_1 \rangle \cdots \langle e_{i_{N-1}}^{N-1}, v_{N-1} \rangle f_{i_1, \ldots, i_{N-1}} \| ^2 \\
= \sum_{i_1, \ldots, i_{N-1}} |\langle e_{i_1}^1, v_1 \rangle|^2 \cdots |\langle e_{i_{N-1}}^{N-1}, v_{N-1} \rangle|^2 \\
= \|v_1\|^2 \cdots \|v_{N-1}\|^2,
\]
so that \( \|F\| = \sup\{\|v_1\|^2 \cdots \|v_{N-1}\|^2 : \|v_k\| \leq 1 \} = 1 \). Applying this linear functional to \( \xi \), we find that
\[
F(\xi) = \langle \xi, \eta \rangle = \sqrt{n_1 \cdots n_{N-1}} \cdot \|\xi\|^2 = \sqrt{n_1 \cdots n_{N-1}}
\]
and the desired inequality \( \|\xi\|_\gamma \geq \sqrt{n_1 \cdots n_{N-1}} \) is proved. \( \square \)

11. Significance of the formula \( r(V) = (n_1 n_2 \cdots n_{N-1})^{-1/2} \)

Theorem 10.1 asserts that for \( N \)-fold tensor products \( H = H_1 \otimes \cdots \otimes H_N \) in which the dimensions \( n_k = \dim H_k \) increase with \( k \) and satisfy \( n_{N-1} < \infty \), the inner radius of the set \( V \) of decomposable vectors satisfies

\[
(11.1) \quad r(V) \geq \frac{1}{\sqrt{n_1 n_2 \cdots n_{N-1}}}.
\]

We have also seen that for fixed \( n_1 \leq \cdots \leq n_{N-1} < \infty \), equality holds in (11.1) when \( n_N \) is sufficiently large (see Theorem 10.2).

In this section we show that equality in (11.1) can be characterized in a way that is perhaps unexpected, in that \( r(V) = (n_1 \cdots n_{N-1})^{-1/2} \) iff the tracial state of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \) can be extended to a pure state of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \otimes \mathcal{B}(H_N) \). We also characterize that situation in terms of the size of \( n_N \).

**Theorem 11.1.** Let \( V \) be the decomposable unit vectors in a tensor product of finite dimensional Hilbert spaces \( H = H_1 \otimes \cdots \otimes H_N \), with \( n_k = \dim H_k \) weakly increasing with \( k \), consider the subfactor
\[
\mathcal{A} = \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1})
\]
of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \), and let \( \tau \) be the tracial state of \( \mathcal{A} \). The following assertions are equivalent:

(i) Minimality of the inner radius:

\[
(11.2) \quad r(V) = (n_1 \cdots n_{N-1})^{-1/2}.
\]

(ii) Existence of purifications: There is a unit vector \( \xi \in H_1 \otimes \cdots \otimes H_N \) such that

\[
(11.3) \quad \tau(A) = \langle (A \otimes \mathbf{1}_{H_N}) \xi, \xi \rangle, \quad A \in \mathcal{A}.
\]

(iii) Lower limit on \( \dim H_N \): \( n_N \geq n_1 n_2 \cdots n_{N-1} \).

The proof of Theorem 11.1 requires the following elementary result.
Lemma 11.2. Let $H$ and $K$ be finite dimensional Hilbert spaces and let $\omega$ be a faithful state of $B(H)$. If there is a vector $\xi \in H \otimes K$ such that

$$\omega(A) = \langle (A \otimes 1_K)\xi, \xi \rangle, \quad A \in B(H),$$

then $\dim K \geq \dim H$.

Proof. Let $\eta$ be a unit vector in $H \otimes H$ such that

$$\omega(A) = \langle (A \otimes 1_H)\eta, \eta \rangle = \langle (1_H \otimes A)\eta, \eta \rangle, \quad A \in B(H).$$

For example, setting $n = \dim H$, let $\Omega$ be the density operator of $\omega$, with eigenvalue list $\lambda_1 \geq \cdots \geq \lambda_n > 0$ and corresponding eigenvectors $e_1, \ldots, e_n$. One can take

$$\eta = \sqrt{\lambda_1} \cdot e_1 \otimes e_1 + \cdots + \sqrt{\lambda_n} \cdot e_n \otimes e_n.$$

Since $\omega$ is a faithful state, $\eta$ is a cyclic and separating vector for $B(H) \otimes 1_H$.

For every $A \in B(H)$ we have

$$\|(A \otimes 1_H)\eta\|^2 = \omega(A^* A) = \|(A \otimes 1_K)\xi\|^2,$$

hence there is an isometry $U : H \otimes H = (B(H) \otimes 1_H)\eta \to H \otimes K$ satisfying

$$U(A \otimes 1_H)\eta = (A \otimes 1_K)\xi, \quad A \in B(H).$$

It follows that $\dim H \cdot \dim K = \dim(H \otimes K) \geq \dim(H \otimes H) = (\dim H)^2$, and $\dim K \geq \dim H$ follows after canceling $\dim H$. \qed

Proof of Theorem 11.1. The implication (ii) $\implies$ (iii) follows after applying Lemma 11.2 to the case $H = H_1 \otimes \cdots \otimes H_{N-1}$ and $K = H_N$, and (iii) $\implies$ (i) is an immediate consequence of Theorem 10.2.

(i) $\implies$ (ii): Since $H_1 \otimes \cdots \otimes H_N$ is finite dimensional, maximal vectors exist. We claim that for every maximal vector $\xi$, one has

$$\langle (E_1 \otimes E_2 \otimes \cdots \otimes E_{N-1} \otimes 1_{H_N})\xi, \xi \rangle = \frac{1}{n_1n_2\cdots n_{N-1}}. \tag{11.4}$$

For the proof, choose a unit vector $e_k \in E_k H_k$, $1 \leq k \leq N - 1$ and consider the operator $U : H_N \to H_1 \otimes \cdots \otimes H_N$ defined by

$$U\zeta = e_1 \otimes \cdots \otimes e_{N-1} \otimes \zeta, \quad \zeta \in H_N.$$

$U$ is a partial isometry whose range projection is $E_1 \otimes \cdots \otimes E_{N-1} \otimes 1_{H_N}$, and since $\langle UU^*\xi, \xi \rangle = \|U^*\xi\|^2$, (11.4) is equivalent to the assertion

$$\|U^*\xi\| = \frac{1}{\sqrt{n_1\cdots n_{N-1}}}. \tag{11.5}$$
We claim first that \( \|U^*\xi\| \leq (n_1 \cdots n_{N-1})^{-1/2} \). Indeed, for every unit vector \( \zeta \in H_N \) we have

\[
|\langle U^*\xi, \zeta \rangle| = |\langle \xi, U\zeta \rangle| = |\langle \xi, e_1 \otimes \cdots \otimes e_{N-1} \otimes \zeta \rangle| \\
\leq \sup_{\|\eta_1\| = \cdots = \|\eta_{N-1}\| = 1} |\langle \xi, \eta_1 \otimes \cdots \otimes \eta_{N-1} \rangle| = \|\xi\|_V \\
= r(V) = \frac{1}{\sqrt{n_1 \cdots n_{N-1}}}.
\]

where the equality \( \|\xi\|_V = r(V) \) follows from the characterization of maximal vectors of Theorem 4.2. The asserted inequality now follows after taking the supremum over \( \|\zeta\| = 1 \).

To prove (11.5), choose orthonormal bases

\[
\{e_1^1, \ldots, e_{n_1}^1\}, \ldots, \{e_1^{N-1}, \ldots, e_{n_{N-1}}^{N-1}\}
\]

for \( H_1, \ldots, H_{N-1} \) respectively, such that \( e_1^1 = e_1, \ldots, e_1^{N-1} = e_{N-1} \). For every sequence of integers \( i_1, \ldots, i_{N-1}, 1 \leq i_k \leq n_k \), consider the operator

\[
U_{i_1, \ldots, i_{N-1}} : \zeta \in H_N \mapsto e_1^1 \otimes \cdots \otimes e_{i_{N-1}}^{N-1} \otimes \zeta \in H_1 \otimes \cdots \otimes H_N.
\]

The preceding argument implies \( \|U_{i_1, \ldots, i_{N-1}}^*\xi\|^2 \leq (n_1 \cdots n_{N-1})^{-1} \) for each \( i_1, \ldots, i_{N-1} \), hence

\[
(11.6) \quad \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} \|U_{i_1, \ldots, i_{N-1}}^*\xi\|^2 \leq \sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} \frac{1}{n_1 \cdots n_{N-1}} = 1.
\]

For each \( k = 1, \ldots, N-1 \) and every \( i = 1, \ldots, n_k \), let \( E_i^k \) be the projection onto the subspace of \( H_k \) spanned by \( e_i^k \). For each \( i_1, \ldots, i_{N-1} \), we have

\[
\|U_{i_1, \ldots, i_{N-1}}^*\xi\|^2 = \langle (E_1^1 \otimes \cdots \otimes E_{i_{N-1}}^{N-1} \otimes 1_{H_N})\xi, \xi \rangle.
\]

Since the projections occurring in the right side are mutually orthogonal and sum to the identity operator of \( H_1 \otimes \cdots \otimes H_N \), the left side of (11.6) is

\[
\sum_{i_1=1}^{n_1} \cdots \sum_{i_{N-1}=1}^{n_{N-1}} \langle (E_1^1 \otimes \cdots \otimes E_{i_{N-1}}^{N-1} \otimes 1_{H_N})\xi, \xi \rangle = \|\xi\|^2 = 1.
\]

It follows that the inequality of (11.6) is actually equality; and since each summand satisfies \( \|U_{i_1, \ldots, i_{N-1}}^*\xi\|^2 \leq (n_1 \cdots n_{N-1})^{-1} \), we must have equality throughout the summands. Formula (11.4) follows.

Let \( \mathcal{S} \) be the set of all operators \( A \in \mathcal{A} \) for which (11.3) holds. Obviously, \( \mathcal{S} \) is a linear space, and by (11.4), every tensor product \( E_1 \otimes \cdots \otimes E_{N-1} \) of rank one projections \( E_k \in \mathcal{B}(H_k) \) belongs to \( \mathcal{S} \). For fixed \( k \), the rank one projections in \( \mathcal{B}(H_k) \) span \( \mathcal{B}(H_k) \), so by multilinearity, \( \mathcal{S} \) contains all operators of the form \( A_1 \otimes \cdots \otimes A_{N-1} \) with \( A_k \in \mathcal{B}(H_k) \). Since operators of the form \( A_1 \otimes \cdots \otimes A_{N-1} \) span \( \mathcal{A} \) itself, Theorem 11.1 follows. \( \square \)
12. Homogeneity and the Case \( n_N \geq n_1 \cdots n_{N-1} \)

Continuing under the hypotheses \( n_1 \leq \cdots \leq n_{N-1} < \infty \), we show in this section that when \( n_N \geq n_1 \cdots n_{N-1} \), the set of maximal vectors in \( H_1 \otimes \cdots \otimes H_N \) is acted upon transitively by the unitary group of \( H_N \), and we draw out several consequences.

**Theorem 12.1.** Assume that \( n_N \geq n_1 \cdots n_{N-1} \) and let \( \xi_1 \) and \( \xi_2 \) be two maximal vectors in \( H_1 \otimes \cdots \otimes H_N \). Then there is a unitary operator \( U \) in \( \mathcal{B}(H_N) \) such that

\[
(12.1) \quad \xi_2 = (1_{H_1} \otimes \cdots \otimes 1_{H_{N-1}} \otimes U) \xi_1.
\]

Maximal vectors are characterized as the unit vectors \( \xi \in H_1 \otimes \cdots \otimes H_N \) that purify the tracial state of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \) as in (11.3).

Finally, the maximal vectors for \( H_1 \otimes \cdots \otimes H_N \) are simply the vectors of the form

\[
(12.2) \quad \xi = \frac{1}{\sqrt{n_1 n_2 \cdots n_{N-1}}} (e_1 \otimes f_1 + \cdots + e_{n_1 \cdots n_{N-1}} \otimes f_{n_1 \cdots n_{N-1}}),
\]

where \( \{e_k : 1 \leq k \leq n_1 \cdots n_{N-1}\} \) is an orthonormal basis for \( H_1 \otimes \cdots \otimes H_{N-1} \) and \( \{f_k : 1 \leq k \leq n_1 \cdots n_{N-1}\} \) is an arbitrary orthonormal set in \( H_N \).

We require the following elementary consequence of familiar methods associated with the GNS construction. We sketch the proof for completeness.

**Lemma 12.2.** Let \( \xi_1, \xi_2 \) be vectors in \( H_1 \otimes \cdots \otimes H_N \) such that

\[
(12.3) \quad \langle (A \otimes 1_{H_N}) \xi_1, \xi_1 \rangle = \langle (A \otimes 1_{H_N}) \xi_2, \xi_2 \rangle
\]

for all \( A \in \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \). Then there is a unitary operator \( U \in \mathcal{B}(H_N) \) such that

\[
(12.4) \quad (1_{H_1} \otimes \cdots \otimes 1_{H_{N-1}} \otimes U) \xi_1 = \xi_2.
\]

**Proof.** Consider the following subalgebra \( \mathcal{B} \) of \( \mathcal{B}(H_1 \otimes \cdots \otimes H_N) \)

\[
\mathcal{B} = \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \otimes 1_{H_N}.
\]

\( \mathcal{B} \) is a finite dimensional factor isomorphic to \( \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \) whose commutant is \( 1_{H_1 \otimes \cdots \otimes H_{N-1}} \otimes \mathcal{B}(H_N) \).

For \( k = 1, 2 \), consider the finite dimensional subspace \( H^k \) of the tensor product \( H_1 \otimes \cdots \otimes H_N \) defined by \( L_k = \{B\xi_k : B \in \mathcal{B}\} \). Since

\[
\|B\xi_k\|^2 = \langle B^* B\xi_k, \xi_k \rangle = \langle B^* B\xi_2, \xi_2 \rangle = \|B\xi_2\|^2, \quad k = 1, 2, \quad B \in \mathcal{B},
\]

Remark 11.3 (Finite dimensionality). Notice that the hypothesis \( n_N < \infty \) was used only in the proof of (i) \( \implies \) (ii), and there only to ensure the existence of maximal vectors. If maximal vectors are known to exist in a setting in which \( n_N = \infty \), then the proof of (i) \( \implies \) (ii) applies verbatim. Of course, whenever (iii) holds, maximal vectors exist by Theorem 10.2.
there is a unique partial isometry $V$ in the commutant of $\mathcal{B}$ having initial space $L_1$, final space $L_2$, such that
\[ V \xi_1 = B \xi_2, \quad B \in \mathcal{B}; \]
and in particular, this operator satisfies $V \xi_1 = \xi_2$.

Since both spaces $L_k$ are invariant under $\mathcal{B}$, they are the ranges of projections in the commutant of $\mathcal{B}$, and therefore must have the form $L_k = H_1 \otimes \cdots \otimes H_{N-1} \otimes K_k$, $k = 1, 2$, where $K_k$ is a finite dimensional subspace of $H_N$. Moreover, since $V$ belongs to the commutant of $\mathcal{B}$, it has the form $V = 1_{H_1 \otimes \cdots \otimes H_{N-1}} \otimes U_0$ where $U_0$ is a partial isometry in $\mathcal{B}(H_N)$ having initial and final spaces $K_1$ and $K_2$ respectively. Finally, since a finite rank partial isometry $U_0 \in \mathcal{B}(H_N)$ can always be extended to a unitary operator $U \in \mathcal{B}(H_N)$, we obtain a unitary operator $U \in \mathcal{B}(H_N)$ with the property asserted in (12.4).

□

Proof of Theorem 12.1. Choose an orthonormal set

\[ \{f_{i_1, \ldots, i_{N-1}} : 1 \leq i_1 \leq n_1, \ldots, 1 \leq i_{N-1} \leq n_{N-1}\} \]

in $H_N$ and let $\xi$ be the vector of the form (10.5). Theorem 10.2 implies that $\xi$ is a maximal vector.

Let $\xi'$ be another maximal vector. The proof of the implication (i) $\implies$ (ii) of Theorem 11.1 implies that

\[ \langle A \xi_1, \xi_1 \rangle = \langle A \xi_2, \xi_2 \rangle = \tau(A), \quad A \in \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1}) \otimes 1_{H_N}, \]

(see Remark 11.3), where $\tau$ is the tracial state. Lemma 12.2 implies that there is a unitary operator $U \in \mathcal{B}(H_N)$ such that $\xi' = (1_{H_1 \otimes \cdots \otimes H_{N-1}} \otimes U) \xi$. Notice that this implies that $\xi'$ also has the form (12.2), in which $\{f_{i_1, \ldots, i_{N-1}}\}$ is replaced with $\{Uf_{i_1, \ldots, i_{N-1}}\}$. It also shows that every maximal vector purifies the tracial state $\tau$.

Another application of Lemma 12.2 shows that every vector $\eta$ in the tensor product $H_1 \otimes \cdots \otimes H_N$ that purifies the tracial state $\tau$ above must have the form $\eta = (1_{H_1 \otimes \cdots \otimes H_{N-1}} \otimes U) \xi$ where $\xi$ is the vector above, therefore $\eta$ is also a maximal vector of the form (10.5). Finally, since every vector of the apparently more general form (12.2) must purify the tracial state $\tau$ as above, it follows from Lemma 12.2 that there is a unitary operator $U \in \mathcal{B}(H_N)$ such that $\eta = (1_{H_1 \otimes \cdots \otimes H_{N-1}} \otimes U) \xi$. It follows that $\eta$ can be rewritten so that it has the form (10.5), and is therefore maximal. □

Remark 12.3 (Stability of maximal vectors when $\dim H_N$ is large). It is of interest to reformulate the above results as follows. Let $H$, $K$ be finite dimensional Hilbert spaces such that $\dim H \leq \dim K$, consider the bipartite tensor product $G = H \otimes K$ with the associated set

\[ V = \{ \xi \otimes \eta \in G : \xi \in H, \eta \in K, \|\xi\| = \|\eta\| = 1\} \]

of unit decomposable vectors. Suppose we are given a further decomposition of $H$ into a tensor product $H = H_1 \otimes \cdots \otimes H_r$, with the resulting set

\[ \tilde{V} = \{ \xi_1 \otimes \cdots \otimes \xi_r \otimes \eta \in G : \xi_k \in H_k, \eta \in K, \|\xi_k\| = \|\eta\| = 1\} \]
of decomposable unit vectors in $G$. Then the preceding results show that the sets $V$ and $\tilde{V}$ give rise to the same set of maximal vectors, and their inner radii satisfy $r(V) = r(\tilde{V})$.

That fact seems remarkable, given that the entanglement measuring norms $\| \cdot \|_V$ and $\| \cdot \|_{\tilde{V}}$ are different. Indeed, recall from Theorem 8.2 that the entanglement measuring norms for $V$ and $\tilde{V}$ are, respectively, the projective cross norms on the bipartite tensor product $(H \otimes H) \otimes K$ and the tripartite tensor product $H \otimes H \otimes K$, respectively. To see that the norms are different, it suffices to exhibit a linear functional $F$ on the vector space $H \otimes H \otimes K$ with the property that its norm in the dual of $(H \otimes H) \otimes K$ is 1 but its norm in the dual of $H \otimes H \otimes K$ is $< 1$. To that end, choose a unit vector $e \in H \otimes H$ that does not decompose into a tensor product $e_1 \otimes e_2$, let $f$ be an arbitrary unit vector in $K$, and set

$$F(\xi_1 \otimes \xi_2 \otimes \eta) = \langle \xi_1 \otimes \xi_2, e \rangle \langle \eta, f \rangle, \quad \xi_k \in H, \quad \eta \in K.$$ 

If one views $F$ as a linear functional in the dual of $(H \otimes H) \otimes K$, then its norm is $\|e\| \cdot \|f\| = 1$. On the other hand,

$$\sup_{\|\xi_1\| = \|\xi_2\| = \|\eta\| = 1} |F(\xi_1 \otimes \xi_2 \otimes \eta)| = \sup_{\|\xi_1\| = \|\xi_2\| = \|\eta\| = 1} |\langle \xi_1 \otimes \xi_2, e \rangle \cdot \langle \eta, f \rangle|$$

$$= \sup_{\|\xi_1\| = \|\xi_2\| = 1} |\langle \xi_1 \otimes \xi_2, e \rangle| < 1,$$

since $e$ is not a decomposable vector. This implies that the norm of $F$ as an element of the dual of $H \otimes H \otimes K$ is $< 1$, and we conclude that $\| \cdot \|_V \neq \| \cdot \|_{\tilde{V}}$.

In a similar way, one can see that while the entanglement measuring function $E$ of states is different for the two sets $V$ and $\tilde{V}$, the set of maximally entangled states is the same for both sets $V$ and $\tilde{V}$.

13. Remarks on the case $n_N < n_1n_2 \cdots n_{N-1}$

In this section we continue the discussion of $N$-fold tensor products $H = H_1 \otimes \cdots \otimes H_N$ with increasing dimensions $n_k = \dim H_k$, with $n_{N-1} < \infty$, and with $V$ the set of decomposable unit vectors. We have discussed the cases in which $n_N \geq n_1 \cdots n_{N-1} \leq n_N$ at some length, having calculated the inner radius of $V$ and having identified the maximal vectors. The following result and its corollary address the remaining cases. The fact is that we have little information about the inner radius and the structure of maximal vectors in such cases that goes beyond the content of Corollary 13.2. Perhaps there is no simple formula for $r(V)$ in general.

**Theorem 13.1.** If $n_N < n_1 \cdots n_{N-1}$, then

$$r(V) > \frac{1}{\sqrt[n_1n_2 \cdots n_{N-1}]}.$$ 

**Proof.** By Theorem 10.1, $r(V) \geq (n_1 \cdots n_{N-1})^{-1/2}$, and we have to show that equality cannot hold. But if equality held, then the hypothesis on $n_N$ implies that $H_1 \otimes \cdots \otimes H_N$ is finite dimensional, so that maximal vectors
exist. Every maximal vector $\xi$ satisfies the criteria of Theorem 11.1 (i), but item (iii) of Theorem 11.1 contradicts the hypothesis on $n_N$. □

**Corollary 13.2.** The inner radius is given by $r(V) = (n_1 \cdots n_{N-1})^{-1/2}$ if $n_N \geq n_1 \cdots n_{N-1}$; otherwise, $r(V) > (n_1 \cdots n_{N-1})^{-1/2}$.

**Remark 13.3 (Best constants for the projective norm of $H_1 \hat{\otimes} \cdots \hat{\otimes} H_N$).** It is of interest to reformulate the information about the inner radius given by Theorems 10.2 and 13.1 in purely Banach space terms. Given finite dimensional Hilbert spaces $H_1, \ldots, H_N$, let $\| \cdot \|_\gamma$ be the projective cross norm on the tensor product $H_1 \otimes \cdots \otimes H_N$ and let $\| \cdot \|$ be its Hilbert space norm. Then one has the following information about the best constant $c$ for which $\| \xi \|_\gamma \leq c \cdot \| \xi \|$ for all $\xi \in H_1 \otimes \cdots \otimes H_N$:

$$c = \sup_{\| \xi \|=1} \| \xi \|_\gamma = \sqrt{n_1 \cdots n_{N-1}}, \quad \text{if } n_N \geq n_1 \cdots n_{N-1},$$

and

$$c = \sup_{\| \xi \|=1} \| \xi \|_\gamma < \sqrt{n_1 \cdots n_{N-1}}, \quad \text{if } n_N < n_1 \cdots n_{N-1}.$$ 

Note too that the preceding results provide no further information about the constant $c$ in cases where $n_N < n_1 \cdots n_{N-1}$, and the problem of developing sharper information is one of obvious significance for quantum information theory as well as for the local theory of Banach spaces. For example, in the case of bipartite tensor products, it is shown in [GL74] that the space $\mathcal{B}(H_1, H_2)$ (endowed with the operator norm) fails to have local unconditional structure if the dimensions of $H_1$ and $H_2$ are large, with further developments in [Gor81]. Also see [GJ99], an important paper on the local theory and the many connections with ideal norms.

**Remark 13.4 (qubit triplets).** The simplest case of tripartite tensor products to which our results do not apply is the case in which $V$ is the set of unit vectors $f \otimes g \otimes h$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We have not attempted to calculate $r(V)$ or determine the maximal vectors for this example; and if one seeks to extend the preceding calculations into the cases $n_N < n_1 \cdots n_{N-1}$, this would seem the natural place to begin. Notice that Corollary 13.2 implies $r(V) > 2$.

In a more qualitative direction, one might seek asymptotic information about the behavior of $r(V_N)$ for large $N$, where $V_N$ is the set of decomposable unit vectors of $(\mathbb{C}^2)\otimes^N$.

### 14. Summary of results for $N$-fold tensor products

We have not interpreted the main abstract results for multipartite tensor products. For the reader’s convenience, we conclude by summarizing the results of Proposition 5.2, and Theorems 4.2, 6.2, 6.3, 7.2, 8.2, 9.1, 11.1 in more concrete terms for these special cases. Let $H_1, \ldots, H_N$ be Hilbert spaces whose dimensions $n_k = \dim H_k$ are weakly increasing, with $n_{N-1} < \infty$. For brevity, we confine ourselves to the case in which $n_N \geq n_1 \cdots n_{N-1}$.
where our results are sharp; however some of the following statements remain valid in the remaining cases as well. What is missing in the remaining cases \( n_N < n_1 \cdots n_{N-1} \) is that we have only rough knowledge of the inner radius (see Theorem 13.1), and correspondingly little information about the structure of maximal vectors. Obviously, the existence of those gaps in what we know about multipartite entanglement calls for further research.

Let \( V \) be the decomposable unit vectors \( \xi_1 \otimes \cdots \otimes \xi_N \) in the tensor product of Hilbert spaces \( H = H_1 \otimes \cdots \otimes H_N \), in which \( \xi_k \in H_k \), and \( \| \xi_k \| = 1 \).

**Theorem 14.1.** Let \( \| \cdot \| \) be the ambient norm of \( H = H_1 \otimes \cdots \otimes H_N \) and let \( \| \cdot \|_\gamma \) be the norm of the projective tensor product of Hilbert spaces \( H_1 \hat{\otimes} \cdots \hat{\otimes} H_N \). The restriction of \( \| \cdot \|_\gamma \) to the unit sphere of \( H \) has these properties. Its range is the interval \( \| S \|_\gamma = [1, \sqrt{n_1 \cdots n_{N-1}}] \). For every \( \xi \in S \) one has \( \| \xi \|_\gamma = 1 \) iff \( \xi \in V \) is a decomposable vector, and

\[
\| \xi \|_\gamma = \sqrt{n_1 \cdots n_{N-1}} \iff \xi \text{ is maximal} \iff \xi \text{ has the form (12.2)}.
\]

The maximal vectors are also characterized as the unit vectors \( \xi \in H \) that purify the tracial state of \( B(H_1 \otimes \cdots \otimes H_{N-1}) \) in the sense of (11.3).

Let \( \| \cdot \|_\gamma \) be the norm of the projective tensor product of Banach spaces \( L^1(H_1) \hat{\otimes} \cdots \hat{\otimes} L^1(H_N) \), and let \( \mathcal{D} \) be the space of all density operators - positive operators in \( B(H) \) having trace 1. The range of \( \| \cdot \|_\gamma \) on \( \mathcal{D} \) is

\[
\| \mathcal{D} \|_\gamma = [1, n_1 \cdots n_{N-1}].
\]

Let \( A \in \mathcal{D} \) and let \( \rho(X) = \text{trace}(AX) \) be the corresponding normal state of \( B(H) \). Then \( \rho \) is separable \( \iff \| A \|_\gamma = 1 \), and for every rank one density operator \( A_\eta = \langle \eta, \xi \rangle \xi, \eta \in H \), \( \| A \|_\gamma = n_1 \cdots n_{N-1} \iff \xi \) is a maximal vector. If a mixed state \( \rho \) is maximally entangled in the sense that its density operator \( A \) satisfies \( \| A \|_\gamma = n_1 \cdots n_{N-1} \), then \( A \) is a convex combination of rank one projections associated with maximal vectors.

In particular, the unique entanglement measuring norms for vectors and states are identified in these cases as \( \| \xi \|_V = \| \xi \|_\gamma \) and \( E(\rho) = \| A \|_\gamma \), respectively, where \( A \) is the density operator of the state \( \rho \).

**References**

[Arv03] W. Arveson. *Noncommutative Dynamics and E-semigroups*. Monographs in Mathematics. Springer-Verlag, New York, 2003.

[Arv07] W. Arveson. The probability of entanglement. *preprint*, pages 1–31, 2007. arXiv:0712.4163.

[Arv08] W. Arveson. Quantum channels that preserve entanglement. *preprint*, pages 1–14, 2008. arXiv:0801.2531.

[BNT02] R. A. Bertlmann, H. Narnhofer, and W. Thirring. A geometric picture of entanglement and Bell inequalities. *Phys. Rev. A*, 66:032319, 2002.

[GJ99] Y. Gordon and M. Junge. Volume ratios in \( L_p \) spaces. *Studia Math.*, 136:147–182, 1999.
Y. Gordon and D. R. Lewis. Absolutely summing operators and local unconditional structures. *Acta Math.*, 133:27–48, 1974.

Y. Gordon. A note on the GL constant of $E \otimes F$. *Israel J. Math.*, 39:141–144, 1981.

O. Gühne, M. Reimpell, and R. Werner. Lower bounds on entanglement measures from incomplete information. *preprint*, 2008. arXiv:0802.1734 [quant-ph].

P. Hyllus, O. Gühne, D. Bruß, and M. Lewenstein. Relations between entanglement witnesses and Bell inequalities. *Phys. Rev. A*, 72:012321, 2005.

R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *preprint*, 2007. arXiv:quant-ph/0702225v2.

A. S. Holevo, M. E. Shirokov, and R. Werner. Separability and entanglement-breaking in infinite dimensions. *preprint*, pages 1–12, 2005. arXiv:quant-ph/0504204v1.

M. Keyl, D. Schlingemann, and R. Werner. Infinitely entangled states. *preprint*, 2002. arXiv:quant-ph/0212014.

A. Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, 77(8):1413–1415, 1996.

D. Perez-Garcia, M. Wolf, C. Palazuelos, I. Villanueva, and M. Junge. Unbounded violation of tripartite Bell inequalities. *Comm. Math. Phys.*, 279:455–486, 2008.

O. Rudolph. A separability criterion for density operators. *J. Phys. A: Math. Gen.*, 33:3951–3955, 2000.

O. Rudolph. A new class of entanglement measures. *J. Math. Phys.*, 42:2507–2512, 2001.

T.-C. Wei and P. Golbart. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. *Phys. Rev. A*, 68:042307, 2003.

X. Wang and S. Gu. Negativity, entanglement witnesses and quantum phase transition in spin-1 Heisenberg chains. *J. Phys. A: Math. Theor.*, 40:10759–10767, 2007.