Wigner Functions for harmonic oscillator in noncommutative phase space

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Abstract

We study the Wigner Function in non-commutative quantum mechanics. By solving the time independent Schrödinger equation both on a non-commutative (NC) space and a non-commutative phase space, we obtain the Wigner Function for the harmonic oscillator on NC space and NC phase space respectively.

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1 Introduction

The study of physics effects on a NC space and a NC phase space has attracted much attention in recent years. Because the effects of the space non-commutativity may become significant in the string scale. Besides the field theory, there are many papers devoted to the study of various aspects of quantum mechanics on a NC space and a NC phase space with usual (commutative) time coordinate [1]-[11]. For example, the Aharonov-Bohm phase on a NC space and a NC phase space has been studied in Refs. [1]-[3]. The Aharonov-Casher phase for a spin-1/2 and spin-1 particle on a NC space and a NC phase space has been studied in Refs.[4]-[5]. Landau problem and HMW effect both on NC space and NC phase space were studied in Refs.[6][7]. There were some studies concerning the quantum Hall effect on NC space [8] and NC phase space[9]. Ref.[12] studied Wigner function for the non-Hamiltonian systems on a NC space. Wigner function is a very important function not only because it is equivalent to the Schrödinger wave function in quantum mechanics, but also it relates to the quantum observation, so the further study of NC wigner function is very important and useful. In this paper we study the effect of the noncommutativity via Wigner function for the harmonic oscillator. The article is organized as follows: In section 2, we review the Wigner distribution function, as an example we calculate Wigner Function for two dimensional Harmonic oscillator. In section 3, we study the Wigner functions for the Harmonic oscillator in NC space. In section 4, by using a generalized Bopp’s
shift, we deduce Wigner functions for the Harmonic oscillator on NC phase space. Conclusions are given in the last section.

## 2 Wigner functions and Harmonic oscillator

There are three types of formulation quantum mechanics. Namely, standard operator quantization, which was developed by Schrodinger, Dirac, and Heisenberg. The second is the path integral quantization, which was constructed by Feynman. The last one is the phase space formulation of quantum mechanics (also known as the Moyal quantization or deformation quantization), which was due to Wigner[13], which is less well known, but which is useful in many areas of physics. For example, it is useful in describing quantum transport process in phase space, and has importance in quantum optics, nuclear physics, condensed matter, M-theory, noncommutative geometry, and matrix models. There are no operators in this formulation of quantum mechanics. Observables and transition amplitudes are phase space integrals of classical number functions, which compose via the star product, and they weighted by the Wigner function, as in statistical mechanics. Wigner constructed a distribution function, which is real, but not everywhere positive, from the quantum-mechanical wave function. Moyal then gave the evolution equation for this distribution, introducing his famous bracket[14].

The definition of Wigner probability function of the simultaneous values of $x$ for the coordinates and $p$ for the momenta in 2d-dimensional phase space in terms of the wave function $\psi(x)$ of Schrödinger equation $\hat{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}})\psi(\hat{\mathbf{x}}) = E\psi(\hat{\mathbf{x}})$, ($\hat{H}$ is the Hamiltonian operator, where the coordinates $\mathbf{x}$, and momenta $\mathbf{p}$ satisfy standard commutation relation: $[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$) is

$$W(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} d\mathbf{y} e^{-i\mathbf{y} \cdot \mathbf{p}} \psi^*(\mathbf{x} - \frac{\hbar}{2}\mathbf{y})\psi(\mathbf{x} + \frac{\hbar}{2}\mathbf{y}),$$  

(1)

where $\psi^*(x)$ stands for complex conjugate of $\psi(x)$. One can also obtain time-independent pure state Wigner function by solving directly star-genvalue equation [16]

$$H(\mathbf{x}, \mathbf{p}) \ast_\hbar W(\mathbf{x}, \mathbf{p}) = W(\mathbf{x}, \mathbf{p}) \ast_\hbar H(\mathbf{x}, \mathbf{p}) = EW(\mathbf{x}, \mathbf{p}),$$  

(2)

where the associative star-product is

$$\ast_\hbar \equiv e^{\frac{\hbar}{2}(\overrightarrow{\partial_x} \cdot \overrightarrow{\partial_p} - \overrightarrow{\partial_p} \cdot \overrightarrow{\partial_x})},$$  

(3)

and $H(\mathbf{x}, \mathbf{p})$ is the classical Hamiltonian function corresponding to $\hat{H}$. The star-product encodes the entire quantum mechanical action. Recalling the action of a translation operator, the star-product induces ”Bopp” shifts

$$f(\mathbf{x}, \mathbf{p}) \ast_\hbar g(\mathbf{x}, \mathbf{p}) = f\left(\mathbf{x} + \frac{i\hbar}{2} \overrightarrow{\partial_p} , \mathbf{p} - \frac{i\hbar}{2} \overrightarrow{\partial_x}\right) g(\mathbf{x}, \mathbf{p})$$

2
\[ f(x, p - i\hbar \frac{\partial}{\partial x}) g(x, p + i\hbar \frac{\partial}{\partial x}). \] (4)

Note that \( f(x, p) \ast \hbar g(x, p) \) denotes quantum deformation of a usual commutative product of functions \( f \cdot g \). In conclusion there are two equivalent ways to get Wigner functions, namely equation (1) or (2).

To illustrate the approach of equation (2), we look at the 2-dimensional harmonic oscillator described by the following Hamiltonian (with \( m = 1, \omega = 1 \))

\[ H(x, p) = \frac{1}{2} \left[ (p_1^2 + x_1^2) + (p_2^2 + x_2^2) \right]. \] (5)

Now, let us study the corresponding eigenvalue problem of (2). Equation \( H \ast W = EW \) gives

\[ \left[ x_1^2 + p_1^2 - \frac{\hbar^2}{4} (\partial_{x_1}^2 + \partial_{p_1}^2) + x_2^2 + p_2^2 - \frac{\hbar^2}{4} (\partial_{x_2}^2 + \partial_{p_2}^2) + i\hbar \frac{1}{2} (x_1 \partial_{p_1} - p_1 \partial_{x_1} + x_2 \partial_{p_2} - p_2 \partial_{x_2}) - 2E \right] W = 0, \] (6)

whereas \( W \ast H = EW \)

\[ \left[ x_1^2 + p_1^2 - \frac{\hbar^2}{4} (\partial_{x_1}^2 + \partial_{p_1}^2) - p_1 \partial_{x_1} + x_2 \partial_{p_2} - p_2 \partial_{x_2} \right] W = 0, \] (7)

Therefore

\[ (x_1 \partial_{p_1} - p_1 \partial_{x_1} + x_2 \partial_{p_2} - p_2 \partial_{x_2}) W = 0, \] (8)

which means that \( W \) is a zero-mode of the Koopman operator \( L_{H_o} [17] \). Taking into account eq.(6) and eq.(7) we obtain

\[ \left[ x_1^2 + p_1^2 - \frac{\hbar^2}{4} (\partial_{x_1}^2 + \partial_{p_1}^2) + x_2^2 + p_2^2 - \frac{\hbar^2}{4} (\partial_{x_2}^2 + \partial_{p_2}^2) - 2E \right] W = 0, \] (9)

Introducing two new variables \( \xi \) and \( \eta \)

\[ \xi := \frac{2}{\hbar} (x_1^2 + p_1^2), \quad \eta := \frac{2}{\hbar} (x_2^2 + p_2^2), \] (10)

equation (9) may be rewritten as follows

\[ \left[ \frac{\xi}{4} - \xi \partial_{\xi}^2 - \partial_{\xi} + \frac{\eta}{4} - \eta \partial_{\eta}^2 - \partial_{\eta} \right] W(\xi, \eta) = EW(\xi, \eta), \] (11)

Let \( W(\xi, \eta) = W(\xi)W(\eta), E = E_1 + E_2 \), we have

\[ \left[ \frac{\xi}{4} - \xi \partial_{\xi}^2 - \partial_{\xi} - E_1 \right] W(\xi) = 0, \] (12)
and
\[ \left[ \frac{\eta}{4} - \eta \partial_\eta^2 - \partial_\eta - E_2 \right] W(\eta) = 0. \]
(13)

By defining \( W(\xi) \) as
\[ W(\xi) =: e^{-\xi/2} L(\xi), \]
(14)
we rewrite equation (12) as
\[ \left[ \xi \partial_\xi^2 + (1 - \xi) \partial_\xi + \frac{E - \hbar}{\hbar} - \frac{1}{2} \right] L(\xi) = 0, \]
(15)
solutions of the above equation are the Laguerre’s polynomials
\[ L_m(\xi) = \frac{1}{m!} e^{\xi} \partial_\xi (e^{-\xi} \xi^m), \]
(16)
for \( m = E_1/\hbar - 1/2 = 0, 1, \ldots \). The corresponding Wigner functions \( W_m \) are
\[ W_m = \frac{(-1)^m}{\pi \hbar} e^{-\xi/2} L_m(\xi). \]
(17)

Similarly, for \( n = E_2/\hbar - 1/2 = 0, 1, \ldots \). The corresponding Wigner functions \( W_n \) are
\[ W_n = \frac{(-1)^n}{\pi \hbar} e^{-\eta/2} L_n(\eta). \]
(18)

Thus we have
\[ W_{mn} = \frac{(-1)^{m+n}}{(\pi \hbar)^2} e^{-(\xi+\eta)/2} L_m(\xi) L_n(\eta). \]
(19)

Substituting equation (10) into equation (19), we may rewrite the Wigner functions \( W_{mn} \) for 2-dimensional harmonic oscillator as follows
\[ W_{mn}(x_1, p_1, x_2, p_2) = \frac{(-1)^{m+n}}{(\pi \hbar)^2} e^{-(x_1^2 + p_1^2 + x_2^2 + p_2^2)/\hbar} L_m \left[ \frac{2}{\hbar} \right] L_n \left[ \frac{2}{\hbar} \right]. \]
(20)

When \( n = 0, m = 0 \), we have
\[ W_{00} = \frac{1}{(\pi \hbar)^2} e^{-(x_1^2 + p_1^2 + x_2^2 + p_2^2)/\hbar}, \]
(21)

The Wigner distribution function is non-negative Gaussian distribution function. However, in the classical limit \( \hbar \longrightarrow 0 \) all Wigner functions \( W_{mn} \) tend to well defined classical probability distributions. For example
\[ W_{00}(x_1, p_1, x_2, p_2) \longrightarrow \delta(x_1)\delta(p_1)\delta(x_2)\delta(p_2). \]
(22)
3 Wigner functions for Harmonic oscillator on NC space

On a NC plane coordinates $\hat{x}_{nc}^i$ and momenta $\hat{p}_{nc}^i$ ($i = 1, 2$) operators satisfy the following commutation relations

$$[\hat{x}_{nc}^i, \hat{x}_{nc}^j] = i\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_{nc}^i, \hat{p}_j] = i\hbar\delta_{ij}. \quad (23)$$

The Schrödinger equation on a NC space is

$$\hat{H}(\hat{x}, \hat{p}) *_{\theta} \psi^{nc}(x) = E \psi^{nc}(x), \quad (24)$$

where the Moyal-Weyl (or star) product is defined as

$$*_{\theta} = e^{i\frac{\theta}{2}(\overrightarrow{\partial}_{x_1} \overrightarrow{\partial}_{x_2} - \overrightarrow{\partial}_{x_2} \overrightarrow{\partial}_{x_1})}, \quad (25)$$

After obtaining $\psi^{nc}(x)$ from (24), the Wigner function on a NC space is

$$W^{nc}(x, p) = \frac{1}{(2\pi)^2} \int dy \ e^{-iy \cdot p} \ \psi^{nc}(x + \frac{\hbar}{2} y) *_{\theta} \psi^{nc}(x - \frac{\hbar}{2} y), \quad (26)$$

Alternatively one can also get the NC space Wigner function by solving the following stargenvalue equation

$$H(x, p) * W^{nc}(x, p) = W^{nc}(x, p) * H(x, p) = E W^{nc}(x, p), \quad (27)$$

where

$$* = *_{h} *_{\theta} = \exp \left( \frac{i\hbar}{2} \sum_{i=1}^{2} \left( \overrightarrow{\partial}_{x_i} \overrightarrow{\partial}_{p_i} - \overrightarrow{\partial}_{p_i} \overrightarrow{\partial}_{x_i} \right) + \frac{i\theta}{2} \left( \overrightarrow{\partial}_{x_1} \overrightarrow{\partial}_{x_2} - \overrightarrow{\partial}_{x_2} \overrightarrow{\partial}_{x_1} \right) \right) \quad (28)$$

Instead of solving the NC space Schrödinger equation by using the star product procedure, we use a Bopp’s shift method, that is, we replace the $*_{\theta}$-product in Schrödinger equation with usual product by making a Bopp’s shift

$$\hat{x}_{nc}^i = \hat{x}_i - \frac{1}{2\hbar} \theta_{ij} \hat{p}_j, \quad \hat{p}_{nc}^i = \hat{p}_i, \quad i = 1, 2. \quad (29)$$

where $\theta_{ij} = \theta \epsilon_{ij}$. Then the equation (24) takes the following form

$$\hat{H}(\hat{x}_{nc}, \hat{p}_{nc}) \psi^{nc}(x) = E \psi^{nc}(x), \quad (30)$$

Such that equation (27) can be rewritten as

$$H(x^{nc}, p^{nc}) *_{h} W^{nc}(x, p) = W^{nc}(x, p) *_{h} H(x^{nc}, p^{nc}) = E W^{nc}(x, p), \quad (31)$$
where
\[ x_{nc}^i = x_i - \frac{1}{2\hbar} \theta_{ij} p_j, \quad p_{nc}^i = p_i, \quad i = 1, 2. \] (32)

By comparing equation (2) with (31) we obtain
\[ W_{mn}^{nc}(x_1, p_1, x_2, p_2) = W(x_1 \rightarrow x_1^{nc}, p_1 \rightarrow p_1^{nc}, x_2 \rightarrow x_2^{nc}, p_2 \rightarrow p_2^{nc}) \] (33)

Therefore
\[ W_{mn}^{nc}(x_1, p_1, x_2, p_2) = \frac{(-1)^{m+n}}{(\pi\hbar)^2} e^{-((x_1^{nc})^2 + (p_1^{nc})^2 + (x_2^{nc})^2 + (p_2^{nc})^2) / \hbar} \]
\[ L_m \left[ \frac{2}{\hbar} ((x_1^{nc})^2 + (p_1^{nc})^2) \right] L_n \left[ \frac{2}{\hbar} ((x_2^{nc})^2 + (p_2^{nc})^2) \right]. \] (34)

Inserting (32) into (34), and neglecting term with \( \theta^2 \), we have
\[ W_{mn}^{nc}(x_1, p_1, x_2, p_2) = \frac{(-1)^{m+n}}{(\pi\hbar)^2} e^{-[(x_1^2 + p_1^2 + x_2^2 + p_2^2) - \frac{\theta}{\hbar}(x_1 p_2 - x_2 p_1)] / \hbar} \]
\[ L_m \left[ \frac{2}{\hbar} (x_1^2 + p_1^2 - \frac{\theta}{\hbar} x_1 p_2) \right] L_n \left[ \frac{2}{\hbar} (x_2^2 + p_2^2 - \frac{\theta}{\hbar} x_2 p_1) \right], \] (35)

this is the Wigner functions for 2-dimensional Harmonic oscillator in NC space. For \( n = 0, m = 0 \), one has
\[ W_{00}^{nc}(x_1, p_1, x_2, p_2) = \frac{1}{(\pi\hbar)^2} e^{-[(x_1^2 + p_1^2 + x_2^2 + p_2^2) - \frac{\theta}{\hbar}(x_1 p_2 - x_2 p_1)] / \hbar} \] (36)

### 4 Wigner functions for Harmonic oscillator in NC phase space

The case of both space-space and momentum-momentum noncommuting [10][11] is different from the case of only space-space noncommuting. Thus on a NC phase space, not only the coordinate operators are noncommutative as in (23), but also the momentum operators in equation (23) satisfy the following commutation relations
\[ [\hat{p}_{nc}^i, \hat{p}_{nc}^j] = i\theta_{ij}, \quad i, j = 1, 2. \] (37)

Here \( \{\theta_{ij}\} \) is a totally antisymmetric matrices which represent the noncommutative property among the momenta on a NC phase space, and play analogous role to \( \hbar \) in the usual quantum mechanics. The Schrödinger equation on a NC phase space is written as
\[ \hat{H}(\hat{x}, \hat{p}) \ast_{\theta} \psi^{ncps}(x) = E \psi^{ncps}(x), \] (38)
where the $\star \bar{\theta}$-product in Eq. (38), for NC phase space, is defined by
\[
\star \bar{\theta} = e^{i\bar{\theta} \left( \frac{\partial}{\partial p_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_2} \frac{\partial}{\partial p_1} \right)}.
\] (39)

Wigner function on a NC phase space is written as
\[
W^{ncps}(x, p) = \frac{1}{(2\pi)^2} \int dy \ e^{-iyp} \psi^{ncps}(x + \frac{\hbar}{2}y) \star \bar{\theta} \psi^{*ncps}(x - \frac{\hbar}{2}y),
\] (40)

The $\star$-genvalue equation on a NC phase space is
\[
H(x, p) \star W(x, p) = W(x, p) \star H(x, p) = E \ W(x, p),
\] (41)

where $\star = \star \bar{\theta} \star \bar{\theta}$.

Instead of solving the NC phase space Schrödinger equation, we use a Bopp’s shift method, that is, we replace the $\star \bar{\theta} \star \bar{\theta}$-product in Schrödinger equation with usual product by making the following a Bopp’s shift
\[
\hat{x}^{ncps}_i = x_i - \frac{1}{2\alpha\hbar} \theta_{ij} \hat{p}_j, \quad \hat{p}^{ncps}_i = p_i + \frac{1}{2\alpha\hbar} \bar{\theta}_{ij} \hat{x}_j, \quad i = 1, 2.
\] (42)

where $\bar{\theta}_{ij} = \bar{\theta} \epsilon_{ij}$, $\theta \bar{\theta} = 4\hbar^2 \alpha^2 (1 - \alpha^2)$, $\alpha = 1 - \frac{\hbar^2}{8\hbar^2} = 1 + O(\theta^2)$. Hereafter we choose $\alpha = 1$. Then the equation (38) takes the following form
\[
\hat{H} \left( \hat{x}^{ncps}, \hat{p}^{ncps} \right) \psi^{ncps}(x) = E \ \psi^{ncps}(x),
\] (43)

Such that equation (41) can be rewritten as
\[
H(x^{ncps}, p^{ncps}) \star_\hbar W^{ncps}(x, p) = W^{ncps}(x, p) \star_\hbar H(x^{ncps}, p^{ncps}) = E \ W^{ncps}(x, p),
\] (44)

where
\[
x^{ncps}_i = x_i - \frac{1}{2\hbar} \theta_{ij} p_j, \\
p^{ncps}_i = p_i + \frac{1}{2\hbar} \bar{\theta}_{ij} x_j, \quad i = 1, 2.
\] (45)

By comparing equation (2) with (44) we obtain
\[
W^{ncps}_{mn}(x_1, p_1, x_2, p_2) = W(x_1 \rightarrow x_1^{ncps}, p_1 \rightarrow p_1^{ncps}, x_2 \rightarrow x_2^{ncps}, p_2 \rightarrow p_2^{ncps})
\] (46)
Therefore
\[ W_{mn}^{n\text{cph}}(x_1, p_1, x_2, p_2) = \frac{(-1)^{m+n}}{(\pi \hbar)^2} e^{-[(x_1^{n\text{cph}})^2 + (p_1^{n\text{cph}})^2 + (x_2^{n\text{cph}})^2 + (p_2^{n\text{cph}})^2] / \hbar} \]
\[ L_m \left[ \frac{2}{\hbar} (x_1^{n\text{cph}})^2 + (p_1^{n\text{cph}})^2 \right] L_n \left[ \frac{2}{\hbar} (x_2^{n\text{cph}})^2 + (p_2^{n\text{cph}})^2 \right] \].
\[ (47) \]

Inserting Eq.(45) into Eq.(47), and neglecting term with \( \theta^2 \), and \( \bar{\theta}^2 \), we have
\[ W_{mn}^{n\text{cph}}(x_1, p_1, x_2, p_2) = \frac{(-1)^{m+n}}{(\pi \hbar)^2} e^{-[x_1^2 + p_1^2 + x_2^2 + p_2^2 - \frac{\theta \bar{\theta}}{\hbar} (x_1 p_2 - x_2 p_1)] / \hbar} \]
\[ L_m \left\{ \frac{2}{\hbar} (x_1^2 + p_1^2) \right\} L_n \left\{ \frac{2}{\hbar} (x_2^2 + p_2^2) \right\} \].
\[ (48) \]

this is the Wigner functions for the Harmonic oscillator on a NC phase space. Again Wigner functions which corresponds to the ground sate wave function is given by
\[ W_{00}^{n\text{cph}}(x_1, p_1, x_2, p_2) = \frac{1}{(\pi \hbar)^2} e^{-[x_1^2 + p_1^2 + x_2^2 + p_2^2 - \frac{\theta \bar{\theta}}{\hbar} (x_1 p_2 - x_2 p_1)] / \hbar}. \]
\[ (49) \]

5 Conclusion remarks

In this paper, we study the Wigner functions for the Harmonic oscillator both on a noncommutative space and a noncommutative phase space. Instead of doing tedious star product calculation, we use the ”shift” method, i.e. the star product in equations (24) and (38) can be replaced by Bopp’s shift equations (30) for NC space and (43) for NC phase space. These shifts are equal to the star product. The additional terms in (35) on a NC space and in (48) on a NC phase space are related to the non-commutativity of space and phase space. This effect is expected to be tested at a very high energy level, and the experimental observation of the effect remains to be further studied.

The method we use in this paper may also be employed to other physics problem on NC space and NC phase space. The further study on the issue will be reported in our forthcoming papers.

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