Abstract. For \( \mathcal{C} \) a finite tensor category we consider four versions of the central monad, \( A_1, \ldots, A_4 \) on \( \mathcal{C} \). Two of them are Hopf monads, and for \( \mathcal{C} \) pivotal, so are the remaining two. In that case all \( A_i \) are isomorphic as Hopf monads. We define a monadic cointegral for \( A_i \) to be an \( A_i \)-module morphism \( 1 \to A_i(D) \), where \( D \) is the distinguished invertible object of \( \mathcal{C} \).

We relate monadic cointegrals to the categorical cointegral introduced by Shimizu (2019), and, in case \( \mathcal{C} \) is braided, to an integral for the braided Hopf algebra \( \mathcal{L} = \int^{X} X^{\vee} \otimes X \) in \( \mathcal{C} \) studied by Lyubashenko (1995).

Our main motivation stems from the application to finite dimensional quasi-Hopf algebras \( H \). For the category of finite-dimensional \( H \)-modules, we relate the four monadic cointegrals (two of which require \( H \) to be pivotal) to four existing notions of cointegrals for quasi-Hopf algebras: the usual left/right cointegrals of Hausser and Nill (1994), as well as so-called \( \gamma \)-symmetrised cointegrals in the pivotal case, for \( \gamma \) the modulus of \( H \).

For (not necessarily semisimple) modular tensor categories \( \mathcal{C} \), Lyubashenko gave actions of surface mapping class groups on certain Hom-spaces of \( \mathcal{C} \), in particular of \( SL(2, \mathbb{Z}) \) on \( \mathcal{C}(\mathbb{C}, \mathbb{1}) \). In the case of a factorisable ribbon quasi-Hopf algebra, we give a simple expression for the action of \( S \) and \( T \) which uses the monadic cointegral.

Keywords: Finite tensor categories, Hopf monads, quasi-Hopf algebras, cointegrals.

MSC: 16T99, 18M05, 18M15.

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1. Introduction

1.1. Monadic cointegrals. Let $H$ be a finite-dimensional Hopf algebra over an algebraically closed field $k$. There are two notions of cointegrals for $H$, namely left cointegrals and right cointegrals. These are elements $\lambda \in H^*$ which satisfy, for all $h \in H$,

$$
2) \quad (\lambda \otimes \text{id}) \circ \Delta(h) = \lambda(h) 1 \quad \text{(right cointegral)},
$$

$$
3) \quad (\text{id} \otimes \lambda) \circ \Delta(h) = \lambda(h) 1 \quad \text{(left cointegral)}.
$$

The unusual numbering will be explained below.

If $H$ is a pivotal Hopf algebra, that is, if it is equipped with a group-like element $g$ expressing the square of the antipode via conjugation, one can introduce two more notions of cointegrals, so-called $\gamma$-symmetrised left and right cointegrals [BBGa, FOG]. Here $\gamma \in H^*$ denotes the modulus of $H$, which encodes the difference between left and right integrals for $H$. The defining equation for $\lambda \in H^*$ to be a $\gamma$-symmetrised left/right cointegral is

$$
1) \quad (\lambda \otimes \text{id}) \circ \Delta(h) = \lambda(h) g^{-1} \quad \text{(right $\gamma$-symmetrised cointegral)},
$$

$$
4) \quad (\text{id} \otimes \lambda) \circ \Delta(h) = \lambda(h) g \quad \text{(left $\gamma$-symmetrised cointegral)}.
$$

Such $\gamma$-symmetrised cointegrals, and in particular the special case where $\gamma$ is given by the counit, have applications to modified traces and to quantum invariants of links and three-manifolds [BBGa, BBGe, DGP]. Furthermore, $\gamma$-symmetrised cointegrals are an example of $g$-cointegrals for a group-like $g$ as introduced in [Ra1].

The category $\mathcal{H}^\mathcal{M}$ of finite-dimensional left $H$-modules is a finite tensor category in the sense of [EGNO], in particular it has left and right duals. If $H$ is in addition pivotal, then $\mathcal{H}^\mathcal{M}$ becomes a pivotal tensor category [AAGTV]. One can now ask if it is possible to describe the above two (or four, in the pivotal case) notions of cointegrals in terms of the category $\mathcal{H}^\mathcal{M}$ in a such a way that it generalises to arbitrary (pivotal) finite tensor categories $\mathcal{C}$. This is indeed possible, and has been done for left cointegrals in [Sh3] using Hopf comonads on $\mathcal{C}$. In the present paper we define analogous notions for all four variants of cointegrals, working instead with Hopf monads.

Hopf monads on a rigid monoidal category $\mathcal{C}$ were introduced in [BV1]. These are monads $M$ on $\mathcal{C}$, equipped with extra structure: First, the functor $M : \mathcal{C} \to \mathcal{C}$ is equipped with a lax comonoidal structure, i.e. there are morphisms

$$
\begin{align*}
&\quad (\text{comultiplication}) \quad M(X \otimes Y) \to M(X) \otimes M(Y), \quad \text{natural in } X, Y, \\
&\quad (\text{counit}) \quad M(1) \to 1,
\end{align*}
$$

satisfying certain conditions. The multiplication and the unit of the monad $M$ have to be comonoidal as well. Finally, $M$ has (unique) left and right antipodes, again given by certain natural transformations, see Section 2.1 for details.

A module over a monad $M$ is an object $V \in \mathcal{C}$ together with an action $M(V) \to V$ of the monad. For a Hopf monad, the category $\mathcal{C}_M$ of $M$-modules is again a rigid monoidal category [BV1].

Let now $\mathcal{C}$ be a finite tensor category. The four monads we are interested in are all given in terms of coends. Namely, for $V \in \mathcal{C}$ we set

$$
A_1(V) = \int^{X \in \mathcal{C}} X \otimes (V \otimes X), \quad A_2(V) = \int^{X \in \mathcal{C}} X^\vee \otimes (V \otimes X),
$$
\[ A_3 (V) = \int_{X \in \mathcal{C}} (X \otimes V) \otimes ^\vee X, \quad A_4 (V) = \int_{X \in \mathcal{C}} (X \otimes V) \otimes X^\vee. \]

Here \(^\vee X\) denotes the right dual and \(X^\vee\) the left dual of an object \(X \in \mathcal{C}\) (see Section 1.5 below for our conventions). Finiteness of \(\mathcal{C}\) ensures existence of these coends. The index 1, \ldots, 4 indicates the “position” of the duality symbol \(^\vee\).

\(A_2\) and \(A_3\) are Hopf monads [BV2, Sec. 5.4], called central monads, and are isomorphic as Hopf monads. If \(\mathcal{C}\) is pivotal, then \(A_1\) and \(A_4\) are also Hopf monads, and all four \(A_i\) are isomorphic as Hopf monads, see Proposition 2.4. In the following, if we discuss properties of \(A_1\) and \(A_4\), we will implicitly assume that \(\mathcal{C}\) is in addition pivotal.

We define four types of monadic cointegrals as follows. Denote by \(D \in \mathcal{C}\) the distinguished invertible object of \(\mathcal{C}\). The tensor unit \(1 \in \mathcal{C}\) is an \(A_i\)-module via the counit. \(A_i(D)\) is an \(A_i\)-module by the monad multiplication. A monadic cointegral for \(A_i\) is an intertwiner of \(A_i\)-modules \(\lambda_i : 1 \to A_i(D)\). This means that \(\lambda_i\) is a morphism in \(\mathcal{C}\) which satisfies

\[
\begin{array}{ccc}
A_i(1) & \xrightarrow{A_i(\lambda_i)} & A_i^2(D) \\
\epsilon_i & & \mu_i(D) \\
1 & \xrightarrow{\lambda_i} & A_i(D)
\end{array}
\]

where \(\epsilon_i\) and \(\mu_i\) denote the counit and multiplication of \(A_i\), respectively.

In the case \(D = 1\), this definition of monadic cointegral agrees with the definition of a cointegral for a Hopf monad that first appeared in [BV1, Sec. 6.3]. For \(i = 2\) (and general \(D\)), the above definition is related by an isomorphism to the notion of “categorical cointegral” defined in [Sh3] (Corollary 2.13).

**Theorem 1.1.** For a finite tensor category \(\mathcal{C}\), non-zero monadic cointegrals for \(A_2, A_3\) (and for \(A_1, A_4\) if \(\mathcal{C}\) is pivotal) exist and are unique up to scalar multiples.

This follows from a corresponding result in [Sh3], see Proposition 2.10.

We now come to a key observation, which explains the reason for introducing four slightly different Hopf monads \(A_i\), even though they are all isomorphic. Namely, for \(H\) a finite-dimensional Hopf algebra and \(\mathcal{C} = H\mathcal{M}\), each monad \(A_i\) has a particularly natural realisation (Example 2.5). With respect to this realisation one finds, firstly, that as a vector space \(A_i(D) = H^*\), so that a \(\lambda\) as in (1.2) is an element of \(H^*\), and, secondly, the equivalences

\[
\lambda \text{ is a mon. coint. for } \begin{cases}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{cases} \iff \lambda \text{ is a right } \gamma\text{-sym.}
\]

\[
\begin{cases}
\text{right } \gamma\text{-sym.} \\
\text{right} \\
\text{left } \gamma\text{-sym.}
\end{cases}
\text{ coint. for } H,
\]

(1.3)

see Example 2.9. As before, in cases 1 and 4 we require \(H\) to be pivotal.

The universal properties of the coends \(A_i\) give isomorphisms between the various spaces of monadic cointegrals, and hence also between the various spaces of cointegrals for \(H\). This will be useful in our application to quasi-Hopf algebras.

---

1For \(\mathcal{C} = H\mathcal{M}\), the object \(D\) is the one-dimensional representation with the \(H\)-action given by \(\gamma^{-1}\), and \(\gamma\) is the modulus of \(H\).
4 Monadic cointegrals and applications to quasi-Hopf algebras

If the quasi-Hopf cointegral \( \lambda \) is . . . then the element of \( H^* \) given by . . . is a monadic cointegral for . . .

| Type     | Description                                                                 | Example       |
|----------|----------------------------------------------------------------------------|---------------|
| right \( \gamma \)-sym. | \( \lambda \left( S(\beta)^{-1}(\vartheta) \right) \) | \( A_1 \)     |
| right    | \( \lambda \left( S(\beta)^{-1}(\xi) \right) \)                          | \( A_2 \)     |
| left     | \( \lambda \left( S^{-2}(\beta)S(\xi) \right) \)                        | \( A_3 \)     |
| left \( \gamma \)-sym. | \( \lambda \left( \beta S(\vartheta) \right) \)                       | \( A_4 \)     |

Table 1. Relation between various notions of cointegrals. Here, “?” is a place holder for the function argument, \( \beta \) is the coevaluation element and \( \vartheta, \xi, \hat{\xi}, \hat{\vartheta} \) are certain elements of \( H \) defined in Section 4. For cases 1 and 4, \( H \) is required to be pivotal. If \( H \) is a (non-quasi) Hopf algebra, then \( \beta = \vartheta = \xi = \hat{\xi} = \hat{\vartheta} = 1 \) and one recovers the simple relation in (1.3).

1.2. Application to quasi-Hopf algebras. For Hopf algebras, the relation between integrals and cointegrals is very simple: passing to the dual Hopf algebra exchanges the two notions. For quasi-Hopf algebras, this is no longer the case as the definition of a quasi-Hopf algebra is not symmetric under duality. While integrals for a quasi-Hopf algebra \( H \) are defined in the same way as for Hopf algebras, cointegrals \( \lambda \in H^* \) are markedly more complicated.

The definition of left/right cointegrals for a quasi-Hopf algebra \( H \) is given in [HN2], and that of \( \gamma \)-symmetrised left/right cointegrals in [SS, BGR] and in Section 4.2 below.

Fix a finite-dimensional quasi-Hopf algebra \( H \) over an algebraically closed field. As for Hopf algebras, \( _H \mathcal{M} \) is also a finite tensor category, and each of the monads \( A_i \) on \( _H \mathcal{M} \) has a natural realisation such that the underlying vector space of \( A_i(D) \) is \( H^* \) in all four cases, cf. Section 3.3. With these realisations, we describe the monadic cointegrals for \( H \) via equations involving quasi-Hopf data. For example, an element \( \lambda \in H^* \) which is an \( H \)-module intertwiner \( 1 \to A_2(D) \) is a monadic cointegral for \( A_2 \) if and only if the equation (3.26) holds.

Our main result is a generalisation of the relations in (1.3) for quasi-Hopf algebras, namely the precise relation between the monadic cointegrals and the quasi-Hopf cointegrals from [HN2, SS, BGR]:

**Theorem 1.2.** We have the bijections from the various types of cointegrals \( \lambda \in H^* \) for the finite-dimensional quasi-Hopf algebra \( H \) to the corresponding types of monadic cointegrals in \( _H \mathcal{M} \) as shown in Table 1.

This is shown in Theorems 4.1 and 4.4.

**Remark 1.3.** Analogous to [Sh3] one can introduce monadic integrals by noting that \( A_i(1) \) is a coalgebra and defining left/right monadic integrals for \( A_i \) to be left/right comodule morphisms in \( \mathcal{C}(A_i(D), 1) \). In the quasi-Hopf case, monadic integrals are elements of \( H^{**} \cong H \), and one finds that the left/right monadic integrals for \( A_i \) are the
usual left/right integrals for the quasi-Hopf algebra $H$ (for $i = 1, 2$), and the right/left integrals (for $i = 3, 4$). In the Hopf case, a similar result was also noted in [Sh3].

A slightly different notion of integrals for Hopf monads has already appeared earlier in the literature [BV1]. If $C$ is braided, then by [BV1, Ex. 5.4] this notion agrees with the definition of cointegrals for Hopf algebras in braided categories as in [KL]. By arguments analogous to those given in Section 6, it in turn also agrees with the monadic integrals from the beginning of this remark. Here we will not go into the details of either of these points and focus instead on the study of monadic cointegrals.

1.3. Application to braided tensor categories. Let $C$ now be a braided finite tensor category. In [LM, Ly1] the coend $L = \int_{X \in C} X \otimes X$ was studied and shown to be a Hopf algebra in $C$. One can use the braiding of $C$ to construct a natural family of isomorphisms

$$\xi_V : A_2(V) \rightarrow L \otimes V.$$ 

This provides an isomorphism $A_2 \cong L \otimes ?$ of Hopf monads.

For Hopf algebras $H$ in braided categories, one can define integrals and cointegrals just as for Hopf algebras over a field, up to one additional subtlety. Namely, integrals are certain morphisms from a so-called object of integrals $\text{Int} H \in C$ to $H$. Conversely, cointegrals are certain morphisms $H \rightarrow \text{Int} H$, see [KL] and Section 6. For example, a left integral for $H$ is a morphism $\Lambda : \text{Int} H \rightarrow H$ such that

$$H \otimes \text{Int} H \xrightarrow{\text{id} \otimes \Lambda} H \otimes H \xrightarrow{\varepsilon \otimes \text{id}} 1 \otimes \text{Int} H \xrightarrow{\sim} \text{Int} H \xrightarrow{\Lambda} H$$

commutes. Here $\varepsilon$ and $m$ denote the counit and multiplication of $H$.

We find that monadic cointegrals for $A_2$ are related to left integrals for $L$, and that the object of integrals for $L$ is $\text{Int} L = \vee D$, the right dual of the distinguished invertible object of $C$ (Proposition 6.1):

**Proposition 1.4.** Let $C$ be a braided finite tensor category. Then $\Lambda : \vee D \rightarrow L$ is a left integral for $L$ if and only if

$$\lambda = [1 \xrightarrow{\text{coev}_D} \vee D \otimes D \xrightarrow{\Lambda \otimes \text{id}} L \otimes D \xrightarrow{\xi_D^{-1}} A_2(D)]$$

is a monadic cointegral for $A_2$.

Proposition 6.1 also contains a similar statement for the relation of right integrals of $L$ with monadic cointegrals for $A_2$.

An important application of $L$ and its integrals arises in the case that $C$ is modular, that is, a (not necessarily semisimple) finite ribbon category whose braiding satisfies a non-degeneracy condition called factorisability (see Section 7).

In this case, one can define a projective representation of the genus-$g$ surface mapping class group on the Hom-space $C(C^{\otimes g} \otimes 1)$ [Ly2], as well as a non-semisimple variant of the Reshetikhin-Turaev topological field theory [DGP, DGGPR]. The integral for $L$ (which is two-sided for $C$ modular) enters in both constructions.
Let us specialise to the case that $C = H\mathcal{M}$ for $H$ a finite-dimensional quasi-triangular quasi-Hopf algebra which is in addition ribbon (and so in particular pivotal). In Section 6.3 we explicitly relate left integrals for $\mathcal{L}$, right monadic cointegrals in $H\mathcal{M}$, and right cointegrals for $H$. In Section 7.2 we assume furthermore that $H$ is factorisable (as defined in [BT1]), which is equivalent to $H\mathcal{M}$ being factorisable [FGR1]. In this case, $D = 1$ and both integrals and cointegrals for $\mathcal{L}$ are two-sided. We may take $A_2(1) = \mathcal{L}$ and by the above proposition, integrals for $\mathcal{L}$ are precisely the same as monadic cointegrals for $A_2$.

We present the projective representation of $SL(2,\mathbb{Z})$ on $C(\mathcal{L}, 1)$ as an example for the mapping class group actions mentioned above by giving the action of the $S$ and $T$ generators, simplifying the corresponding expressions in [FGR1]. Denote by $Z(H)$ the centre of $H$ and write $\alpha Z = \{\alpha z \mid z \in Z(H)\}$, where $\alpha$ is the evaluation element of $H$. Recall that $\mathcal{L} = H^*$ as a vector space. One finds that via the natural isomorphism $H \cong H^{**}$ one has $\alpha Z \cong C(\mathcal{L}, 1)$. In Proposition 7.1 we compute the action of $S$ and $T$ on $\alpha Z$ to be, for $z \in Z(H)$,

$$S \cdot (\alpha z) = (\lambda | \tilde{\omega}_1 z) \tilde{\omega}_2, \quad T \cdot (\alpha z) = v^{-1} \alpha z.\$$

Here, $\lambda$ is a monadic cointegral for $A_2$, $\tilde{\omega}_1, 2$ are the components of the Hopf-pairing $\omega : \mathcal{L} \otimes \mathcal{L} \rightarrow 1$, and $v$ is the ribbon element of $H$, see Section 7 for details.

Finally, let us note that the construction in [DGGPR] of a three-dimensional topological field theory from a modular category $C$ uses the integral for $\mathcal{L}$ and the modified trace on (the projective ideal in) $C$. For $C = H\mathcal{M}$ with $H$ a factorisable ribbon quasi-Hopf algebra, monadic cointegrals for $A_2, A_3$ provide the integral for $\mathcal{L}$, and monadic cointegrals for $A_1, A_4$ provide the modified trace via the construction using symmetrised cointegrals in [SS, BGR]. An important class of factorizable quasi-Hopf algebras as inputs for such topological field theories comes from the fundamental examples of logarithmic conformal field theories [GR, FGR2, CGR, GLO, Ne].

The fact that monadic cointegrals provide integrals for $\mathcal{L}$ and modified traces in a uniform setting was one of the motivations to carry out the present investigation. Another motivation was that because of their direct categorical interpretation, in certain situations monadic cointegrals for quasi-Hopf algebras may be easier to use than those in the left column of Table 1.

1.4. **Comparison to the approach of Shibata-Shimizu.** For quasi-Hopf algebras, a relation between right cointegrals for $H$ and categorical cointegrals in the sense of [Sh3] was derived in [SS]. The main theorem in the present paper (Theorem 1.2) is an analogous result for the four types of monadic cointegrals. Comparing to [SS], we work in a dual setting that uses central Hopf monads instead of comonads. This last choice of monads over comonads is merely conventional. However, our approach to the application of monadic cointegrals to quasi-Hopf cointegrals is quite different from the one in [SS]: the latter uses a detour via the category of Yetter-Drinfeld modules, while we follow a more direct route. In our approach, we found the monadic setting better suited to make the connection to [HN2] than the comonadic picture.

A more conceptual relation between these two pictures is described in Remark 2.14.

**Outline of the paper.** In Section 2 we start by reviewing the definition of a Hopf monad. Then, the central monads and their Hopf structures are described. We define
four versions of monadic cointegrals, and show that they are related by isomorphisms to the categorical cointegral considered in [Sh3].

Section 3 contains our conventions for (pivotal) quasi-Hopf algebras. We specialise the definition of the various monadic cointegrals to the category of modules over a quasi-Hopf algebra, and we review the definition and some properties of left and right cointegrals for quasi-Hopf algebras from [HN2, BC].

Our main theorem showing that quasi-Hopf cointegrals are equivalent to monadic cointegrals is formulated in Section 4. The main ideas of the proof are outlined, while technical details are deferred to Appendix A.

Examples of monadic cointegrals for quasi-Hopf algebras are given in Section 5.

In Section 6 we consider integrals for Hopf algebras in braided finite tensor categories, and we relate left and right integrals for \( \mathcal{L} \) to monadic cointegrals for \( A_2 \). As an example, we treat finite-dimensional quasi-triangular quasi-Hopf algebras.

With \( \mathcal{C} = \mathcal{H} \mathcal{M} \) the category of finite-dimensional modules over a finite-dimensional factorisable ribbon quasi-Hopf algebra \( H \), in Section 7 we express the \( SL(2, \mathbb{Z}) \)-action on the centre of \( H \) using the monadic and the quasi-Hopf cointegral.

1.5. Conventions. Throughout this paper we fix an algebraically closed field \( k \). Following [EGNO], by a finite tensor category we mean a \( k \)-linear abelian category that

- has finite-dimensional Hom-spaces, and every object is of finite length,
- possesses a finite set of isomorphism classes of simple objects,
- has enough projectives,
- is rigid monoidal, such that the tensor product functor is \( k \)-bilinear and the monoidal unit \( 1 \) is simple.

We denote the left and the right dual of an object \( X \) by \( X^\vee \) and \( \vee X \), respectively. The corresponding evaluations and coevaluations are

\[
\begin{align*}
\ev_X &: X^\vee \otimes X \to 1, & \coev_X &: 1 \to X \otimes X^\vee, \\
\ev_X &: X \otimes \vee X \to 1, & \coev_X &: 1 \to \vee X \otimes X,
\end{align*}
\]

(1.4) satisfying the familiar zig-zag equalities. We do not assume that \( \mathcal{C} \) is strict monoidal, and (compositions of) coherence isomorphisms will be indicated.

Our conventions for string diagrams are as follows. We read them from bottom to top, and coherence isomorphisms will usually not be drawn.

Left and right coevaluation and evaluation for the object \( X \in \mathcal{C} \) are drawn as

\[
\begin{align*}
\begin{array}{c}
X \\
\Uparrow
\end{array} \quad \begin{array}{c}
X^\vee \\
\Downarrow
\end{array} \quad \begin{array}{c}
\vee X \\
\Downarrow
\end{array} \quad \begin{array}{c}
X \\
\Uparrow
\end{array} \\
\begin{array}{c}
X^\vee \\
\Downarrow
\end{array} \quad \begin{array}{c}
X \\
\Uparrow
\end{array} \quad \begin{array}{c}
X \\
\Uparrow
\end{array} \quad \begin{array}{c}
\vee X \\
\Downarrow
\end{array} \\
\begin{array}{c}
X^\vee \\
\Downarrow
\end{array} \quad \begin{array}{c}
X \\
\Uparrow
\end{array} \quad \begin{array}{c}
X \\
\Uparrow
\end{array} \quad \begin{array}{c}
\vee X \\
\Downarrow
\end{array}
\end{align*}
\]

(1.5)

respectively, so that in our conventions for duals and string diagrams, arrows on the duality maps for left (right) duals point to the left (right).
A functor \( F : \mathcal{C} \to \mathcal{D} \) between monoidal categories is \emph{lax comonoidal} if there is a natural transformation \( F_2 \) and a morphism \( F_0 \),

\[
F_2(X, Y) : F(X \otimes Y) \to FX \otimes FY, \quad F_0 : F1 \to 1,
\]

satisfying certain coherence conditions so that coalgebras in \( \mathcal{C} \) are mapped to coalgebras in \( \mathcal{D} \). For that reason we will commonly refer to \( F_2 \) and \( F_0 \) as the comultiplication and the counit of the lax comonoidal functor \( F \). If \( F_2 \) and \( F_0 \) are isomorphisms then \( F \) is called a strong comonoidal functor.

Similarly, a functor \( F : \mathcal{C} \to \mathcal{D} \) between monoidal categories is \emph{lax/strong monoidal} if \( F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}} \) is lax/strong comonoidal. The corresponding natural transformation \( F_2 \) and the morphism \( F_0 \) we call the multiplication and the unit, respectively.

A natural transformation \( \varphi : F \Rightarrow G \) between two comonoidal functors is called \emph{comonoidal} if it commutes with the comonoidal structures. That is, if

\[
G_2(X, Y) \circ \varphi_{X \otimes Y} = (\varphi_X \otimes \varphi_Y) \circ F_2(X, Y) \quad \text{and} \quad F_0 = G_0 \circ \varphi_1
\]

is true for all objects \( X, Y \).

Monoidal natural transformations between monoidal functors are defined similarly.

A rigid category \( \mathcal{C} \) is called \emph{pivotal} if there is a monoidal natural isomorphism \( \delta : \text{id}_{\mathcal{C}} \Rightarrow (\_\, \_)^{\vee\vee} \), i.e. from the identity functor on \( \mathcal{C} \) to the double dual functor. The monoidal structure of the double dual is given in terms of the natural isomorphism

\[
\gamma_{V, W} : V^{\vee} \otimes W^{\vee} \to (W \otimes V)^{\vee}, \quad \gamma_{V, W} = \begin{array}{c}
\text{id} \\
\gamma_{V, W}^{-1}
\end{array}, \quad \text{and} \quad \gamma_{V, W} = \begin{array}{c}
\text{id} \\
\gamma_{V, W}^{-1}
\end{array}
\]

as the composition

\[
V^{\vee\vee} \otimes W^{\vee\vee} \xrightarrow{\gamma_{V, W}} (W \otimes V)^{\vee} \xrightarrow{(\gamma_{W, V}^{-1})^{\vee}} (V \otimes W)^{\vee\vee}.
\]

Note that the existence of the pivotal structure \( \delta \) is equivalent to requiring that the left and the right dual functor be isomorphic as monoidal functors. Indeed, given \( \delta \) we can form the isomorphism

\[
\begin{array}{c}
\delta_X \\
\text{id}
\end{array}
\]

Conversely, given a natural monoidal isomorphism \( ^\vee X \cong X^{\vee} \), we have

\[
X^{\vee\vee} \cong (\,^\vee X)^{\vee} \cong X
\]
where the second isomorphism is

\[ \omega_X = \left[ (\vee X)^\vee \sim (\vee X)^\vee \otimes 1 \xrightarrow{\text{id} \otimes \text{coev}_X} (\vee X)^\vee \otimes (\vee XX) \xrightarrow{\sim} ((\vee X)^\vee \otimes \vee X)X \right. \]

\[ \xrightarrow{\text{ev}_{\vee X} \otimes \text{id}} 1X \xrightarrow{\sim} X \].

We will suppress some of the tensor product symbols to shorten expressions, e.g. in the above expression we only left those tensor symbols between objects necessary to make the assignment of duals unambiguous. As a string diagram (1.12) simply reads

\[ \omega_X = \]

2. Monadic cointegrals

This section contains the main definition of this paper, namely that of the four types of monadic cointegrals (Definition 2.7). To state the definition we first briefly review Hopf monads [BV1], the central Hopf monad, and the distinguished invertible object. Finally we realise monadic cointegrals via Hopf comonads to establish existence and uniqueness via results in [Sh3].

2.1. Hopf monads. We first recall the basic notions from the theory of Hopf monads on rigid categories. Throughout, our conventions will closely follow [BV1].

Monads. Recall [Mac, Sec. VI] that a monad \( M \) on a category \( \mathcal{C} \) is an algebra in \( \text{End}(\mathcal{C}) \), the category of endofunctors of \( \mathcal{C} \), which is a monoidal category under composition. This means that there are natural transformations

\[ \mu : M^2 \Rightarrow M, \quad \eta : \text{id}_\mathcal{C} \Rightarrow M, \]

the multiplication and unit of \( M \), respectively, satisfying

\[ M^3V \xrightarrow{M\mu_V} M^2V \xrightarrow{\mu_V} M^2V \quad \text{and} \quad M^2V \xleftarrow{\eta_MV} MV \xrightarrow{M\eta_V} M^2V, \]

for each \( V \in \mathcal{C} \).

A module over a monad \( M \) is a tuple \((V, \rho)\), consisting of an object \( V \in \mathcal{C} \) together with a morphism \( \rho : MV \to V \), called the action, such that

\[ M^2V \xrightarrow{M\rho} MV \xrightarrow{\mu_V} MV \quad \text{and} \quad V \xrightarrow{\eta_V} MV \xrightarrow{\mu_V} MV, \]
commute. A morphism of $M$-modules from $(V, \rho)$ to $(W, \sigma)$ is a morphism $f : V \to W$ in $C$ which commutes with the action, i.e.

$$\sigma \circ Mf = f \circ \rho.$$  \hfill (2.4)

The category of $M$-modules is denoted by $C_M$. The forgetful functor from $C_M$ to $C$ has a left adjoint which sends an object $V \in C$ to the free $M$-module $(MV, \mu_V)$.

A morphism of monads is a morphism $\phi : M \Rightarrow M'$ of algebras in $\text{End}(C)$. It therefore induces a functor $\phi^* : C_{M'} \to C_M$ via pullback, cf. [BV1, Lem. 1.6].

If $C$ is monoidal, and $A \in C$ is an algebra, then $A \otimes ?$ is a monad. A comonad on $C$ is a monad on $C^{\text{op}}$.

**Bimonads.** If $C$ is a monoidal category, then a *bimonad* on $C$ is a monad $M$ such that the functor $M$ is lax comonoidal and the multiplication and unit of $M$ are comonoidal natural transformations.

The name “bimonad” is in analogy to algebras and bialgebras: The category of modules over a bimonad $(M, M_0, M_2)$ is monoidal, and a lax comonoidal structure on $M$ is the same as a monoidal structure on $C_M$ such that the forgetful functor to $C$ is strong monoidal, cf. [Mo, Thm. 7.1]. Given two $M$-modules $(V, \rho), (W, \sigma)$, their tensor product is defined by

$$(V, \rho) \otimes (W, \sigma) = (V \otimes W, (\rho \otimes \sigma) \circ M_2(V, W)), \hfill (2.5)$$

and the monoidal unit of $C_M$ is the $M$-module $(1, M_0)$, which we will also denote by $1$.

A morphism of bimonads is a comonoidal natural transformation which is a morphism of the underlying monads. We will later need the following lemma.

**Lemma 2.1** ([BV1, Lem. 2.7]). Let $M, M'$ be bimonads on $C$. Then there is a one-to-one correspondence between morphisms $f : M \Rightarrow M'$ of bimonads and strict monoidal functors $F : C_{M'} \to C_M$ whose underlying functor on $C$ is the identity functor.

**Remark 2.2.** Given a bimonad morphism $f$, the corresponding functor determined via Lemma 2.1 is just the pullback of the $M'$-module structure along $f$, i.e. $F = f^*$. Conversely, the monad morphism $f : M \Rightarrow M'$ corresponding to the functor $F : C_{M'} \to C_M$ is given as follows. Let $(M'V, F\mu'_V) \in C_{M'}$ be the image of the free $M'$-module on $V$ under $F$. Then $f_V = F\mu'_V \circ M\eta'_V$. In particular, $f^* = F$.

If $C$ is braided monoidal, and $B \in C$ is a bialgebra, then $B \otimes ?$ is a bimonad. A bicomonad on $C$ is a bimonad on $C^{\text{op}}$.

**Hopf monads.** A bimonad $M$ on a rigid category $C$ is called a *Hopf monad* if $C_M$ is rigid, following again the familiar nomenclature of algebras, bialgebras, and Hopf algebras. For a Hopf algebra, the rigid structure of its category of modules is encoded in the antipode. For Hopf monads, the corresponding concept is as follows, cf. [BV1]. A natural transformation $S^L$ with components

$$S^L_U : M(((MV)^\vee)) \to V^\vee \hfill (2.6)$$

3For any $M$-module $(B, \nu)$ and any object $A$ in $C$, the adjunction can be seen from the natural isomorphism $C_M((MA, \mu_A), (B, \nu)) \cong C(A, B)$, given by sending an $M$-module morphism $f$ to $f \circ \eta_A$, and conversely, a morphism $g : A \to B$ is sent to $\nu \circ Mg$.
is called a left antipode for $M$ if it satisfies

$$M((MV)^\vee \otimes V) \xrightarrow{M_2} M((MV)^\vee) \otimes MV \xrightarrow{M\mu_V \otimes \text{id}} M((M^2V)^\vee) \otimes MV \xrightarrow{S_{MV}^\vee \otimes \text{id}} (MV)^\vee \otimes MV \xleftarrow{\text{ev}_{MV}}$$

(2.7)

and

$$M1 \xrightarrow{M\eta_V \otimes \text{id}} M(V \vee V) \xrightarrow{M \epsilon_{MV}} M1 \xrightarrow{M_0} 1$$

$$M(MV \otimes (MV)^\vee) \xrightarrow{M_2} M^2V \otimes M((MV)^\vee) \xrightarrow{\mu_V \otimes S_{MV}^V} MV \otimes V^\vee$$

(2.8)

Given an $M$-module $(V, \rho)$, the antipode allows us to define a morphism

$$\tilde{\rho} = \left[ M(V^\vee) \xrightarrow{M(\rho^\vee)} M((MV)^\vee) \xrightarrow{S_V^\vee} V^\vee \right],$$

(2.9)

which turns $(V^\vee, \tilde{\rho})$ into an $M$-module [BV1, Thm. 3.8]. The evaluation and coevaluation are those in $C$,

$$\xleftarrow{\text{ev}_{(V, \rho)}} = \xleftarrow{\text{ev}_V}, \quad \xleftarrow{\text{coev}_{(V, \rho)}} = \xleftarrow{\text{coev}_V},$$

(2.10)

and that they are indeed $M$-module intertwiners is guaranteed by the two commuting diagrams (2.7) and (2.8). Right duals via the right antipode are defined similarly. It was also shown in [BV1, Thm. 3.8] that $C_M$ is rigid if and only if the left and right antipodes exist, and that the antipodes are unique.

A morphism of Hopf monads is a morphism of the underlying bimonads. It automatically commutes with the antipodes, [BV1, Lem. 3.13].

A Hopf comonad on $C$ is a Hopf monad on $C^{\text{op}}$.

If $C$ is braided rigid monoidal, and $H \in C$ is a Hopf algebra with invertible antipode, then $H \otimes ?$ is a Hopf monad, see [BV1, Ex. 3.10]. This example will be important in Section 6.

2.2. The central Hopf monad. Throughout the rest of this section $C$ will denote a finite tensor category. Recall the notion of a coend from e.g. [Mac, KL] or [FS, Sec. 4.2]. It follows from [KL, Cor. 5.1.8] that the coends

$$A_1(V) = \int^{X \in C} X^\vee (V^X), \quad A_2(V) = \int^{X \in C} X^\vee (VX)$$

$$A_3(V) = \int^{X \in C} (XV)^\vee X, \quad A_4(V) = \int^{X \in C} (XV) X^\vee$$

(2.11)
exist for all \( V \in \mathcal{C} \). A different and more detailed proof of existence is given in [Sh1, Thm. 3.6]. Note that the subscript \( i \) indicates the ‘position’ of the dual symbol \( \vee \). We write \( \iota_i(V) \) for the universal dinatural transformation of \( A_i(V) \) so that for example

\[
\iota_2(V)_X : X^\vee (VX) \to A_2(V) .
\] (2.12)

In particular, \( A_i : V \mapsto A_i(V) \) is an endofunctor, and the universal dinatural transformations \( \iota_i(V) \) are natural in \( V \in \mathcal{C} \). In our graphical notation the dinatural transformation \( \iota_2(V) \) is drawn as

\[ A_2(V) \]
\[ \downarrow \]
\[ X^\vee \]
\[ \downarrow \]
\[ X \]
\[ \iota_2(V)_X \]

for all \( V, X \in \mathcal{C} \). Functors like \( A_i \), and in particular the functor \( A_2 \), were already studied in e.g. [BV2]. The latter is known as the \textit{central Hopf monad} [Sh1].

We will now describe the monad structures in more detail, restricting our exposition to the case \( i = 2 \). The monad structure is similar for all other cases.

Recall the natural isomorphism \( \gamma_{X,Y} : (X^\vee)(Y^\vee) \to (YX)^\vee \) from (1.8). The multiplication \( \mu_2 : (A_2)^2 \Rightarrow A_2 \) with components \( \mu_2(V) : A_2A_2(V) \to A_2(V) \) is determined by the universal property of coends via

\[
\mu_2(V) := \gamma_{Y,X} \circ \iota_2(V)_Y \circ \iota_2(V)_X \circ \iota_2(A_2V)_Y .
\] (2.14)

Here we used what is known as the ‘Fubini theorem’ for ends and coends, cf. [Mac, Sec. IX.8], see also [Lo, Rem. 1.9], to express the dinatural transformation of the iterated coend \( A_2A_2(V) \) in terms of \( \iota_2(V) \) and \( \iota_2(A_2V) \).

The unit of \( A_2 \), i.e. the natural transformation \( \eta_2 : \text{id}_C \Rightarrow A_2 \), is defined as

\[
\eta_2(V) := \left[ V \sim 1^\vee (V1) \xrightarrow{\iota_2(V)_1} A_2(V) \right] .
\] (2.15)
For $i = 2, 3$, $A_i$ is always a Hopf monad \cite[Sec. 5.4]{BV2}. As an example we again consider $i = 2$, the other case is similar. The lax comonoidal structure is defined by

$$
A_2(U) \otimes A_2(V) \xrightarrow{\Delta_2(U, V), \iota_2} A_2(U) \otimes A_2(V) \xrightarrow{\iota_2} A_2(U) \otimes A_2(V) \xrightarrow{\iota_2, \epsilon_2} X^\vee \otimes X
$$

These are the comultiplication and counit of $A_2$. The left antipode of $A_2$ is defined by

$$
X^\vee \xrightarrow{\iota_2, \epsilon_2} X^\vee \otimes X \xrightarrow{\iota_2(U) \otimes X} X \xrightarrow{X^\vee} X
$$

following \cite[Thm. 5.6]{BV2}. Here, by $\sim$ we mean the canonical isomorphism $X \cong (X^\vee)^\vee$, defined similarly to $\omega_X : (X^\vee)^\vee \to X$ from (1.12). The right antipode is obtained analogously.

For $i = 1, 4$, the above definition of a bimonad structure on $A_i$ does not work. If $\mathcal{C}$ is pivotal, however, the natural isomorphism $X^\vee \cong \vee(X^\vee)$ from (1.10) can be used when the duals do not match up in the comultiplication and counit. For example, the counit of $A_1$ is given by

$$
X^\vee \xrightarrow{\iota_1} \vee X \xrightarrow{\epsilon_1} X
$$

One checks that in this way one obtains a Hopf monad structure on $A_1$ and $A_4$.

We summarise the preceding discussion in the following proposition.

**Proposition 2.3.** The functors $A_2$ and $A_3$ are Hopf monads. If $\mathcal{C}$ is pivotal, then $A_1$ and $A_4$ are also Hopf monads.

The following proposition will now show that the canonical natural isomorphisms $\kappa_{i,j} : A_i \Rightarrow A_j$ defined by

$$
(\kappa_{i,j} \circ \iota_i(V))_X = \iota_j(V)_X \circ (\text{isomorphism of components}),
$$

Here and in similar places below, we often omit spelling out all components and arguments of the dinatural transformations, e.g. on the LHS we have $\iota_2(U \otimes V)_X$, etc.
are isomorphisms of Hopf monads. Here, $X'$ stands for $X$, $X^\vee$ or $\vee X$ as appropriate, and the ‘isomorphisms of components’ consist of coherence isomorphisms and the isomorphisms $\vee(X^\vee) \cong X$ and, in the pivotal case, $X^\vee \cong \vee X$. For example,

$$
\begin{align*}
& \left[ X^\vee (VX) \xrightarrow{\iota_2(VX)} A_2(V) \xrightarrow{(\kappa_{2,3})_V} A_3(V) \right] \\
= & \left[ X^\vee (VX) \xrightarrow{\sim} (X^\vee V)(\vee(X^\vee)) \xrightarrow{id \otimes \omega_X^{-1}} (X^\vee V)^{\vee(X^\vee)} \xrightarrow{(\iota_3(V)^{X^\vee})} A_3(V) \right].
\end{align*}
$$

(2.20)

**Proposition 2.4.** The natural isomorphism $\kappa_{2,3}$ is an isomorphism of Hopf monads. If $\mathcal{C}$ is pivotal, then $\kappa_{i,j}$ are isomorphisms of Hopf monads for all $i,j$.

**Proof.** We first claim that the pullback $F = (\kappa_{2,3})^* : \mathcal{C}_{A_3} \to \mathcal{C}_{A_2}$ is a well-defined functor. Namely, on an $A_3$-module $(V,\rho)$ the functor acts as $F(V,\rho) = (V,F\rho)$, where we define $F\rho : A_2V \to V$ by

$$
F\rho \circ \iota_2(VX)_X = \left[ X^\vee (VX) \xrightarrow{\sim} (X^\vee V)(\vee(X^\vee)) \xrightarrow{(\iota_3(V)^{X^\vee})} A_3(V) \xrightarrow{\rho} V \right],
$$

(2.21)

so that indeed $F\rho = \rho \circ \kappa_{2,3}$ by (2.20). A calculation shows that $F\rho$ is an $A_2$-action.

Next we check the conditions in Lemma 2.1. As $F$ is given by pullback, the underlying functor is the identity on $\mathcal{C}$. To verify strict monoidality, one checks that for $(V,\rho), (W,\sigma) \in \mathcal{C}_{A_3}$ one has

$$
F(\rho \otimes \sigma \circ \Delta_3(VW)) = (F\rho \otimes F\sigma) \circ \Delta_2(VW)
$$

(2.22)

and $F(\epsilon_3) = \epsilon_2$, which is easy to see.

Thus from Lemma 2.1 we obtain a morphism of bimonads (and hence of Hopf monads) $A_2 \Rightarrow A_3$. Since $F = (\kappa_{2,3})^*$, by Remark 2.2 this morphism is given by $\kappa_{2,3}$. As $\kappa_{2,3}$ is an isomorphism, we finally get $A_2 \cong A_3$.

If $\mathcal{C}$ is pivotal, then e.g. the equivalence $G : \mathcal{C}_{A_2} \to \mathcal{C}_{A_1}$ is given by $G(V,\rho) = (V,G\rho)$ with

$$
G\rho \circ \iota_1(V)_X = \left[ \vee(XVX) \xrightarrow{\sim} X^\vee (VX) \xrightarrow{\iota_2(VX)} A_2(V) \xrightarrow{\rho} V \right],
$$

(2.23)

where the first isomorphism is given by the inverse to the one in (1.10). It is straightforward to check that $G = (\kappa_{1,2})^*$ and that it is strict monoidal and it is the identity on objects from $\mathcal{C}$. Hence $A_2 \cong A_1$ as Hopf monads.

It is not hard to see that $\mathcal{C}_{A_2} \cong \mathcal{Z}(\mathcal{C}) \cong \mathcal{C}_{A_3}$ as monoidal categories, where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld centre of $\mathcal{C}$, cf. [BV1, Sec. 9.3]. This was the reason to introduce central monads, and also explains the name.

**Example 2.5.** Let $\mathcal{C} = \mathcal{H}_\mathcal{M}$ be the category of finite-dimensional modules over a finite-dimensional Hopf algebra $H$, and let $i = 2,3$. As vector spaces, the $A_i(V)$ are isomorphic to $H^* \otimes V$, and we choose the module structures as follows. With $h \in H$, $f \in H^*$, and $v \in V$, the action $\hat{\wedge}$ of $H$ on $A_i(V)$ is

$$
h \hat{\wedge} (f \otimes v) = \langle f \mid S(h(1)) \otimes h(3) \rangle \otimes h(2)v,
$$

where $\hat{\wedge}$ has the structure we want to use throughout the remainder of the text. Thus we prefer to present the canonical structure in the four cases first and then establish the isomorphisms afterwards.
Here we use the sumless Sweedler-notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ etc., see Section 3.1 for details. Note that $A_2(1)$ is the coadjoint representation of $H$, cf. [FGR1, Sec. 7]. The universal dinatural transformations are defined as

$$\iota_2(V) = \sum_i \langle f | e_i \cdot x \rangle e_i \otimes v,$$

$$\iota_3(V) = \sum_i \langle f | e_i \cdot x \rangle e_i \otimes v,$$

where $f \in X^*$, $v \in V$, $x \in X$, and $\{e_i\}$ is a basis of $H$ with dual basis $\{e^i\}$. In string diagram notation, these read

$$\iota_2(V)_X = \begin{array}{c}
\xymatrix{
H^* & V \\
X^* & V & X \\
}
\end{array}, \quad \iota_3(V)_X = \begin{array}{c}
\xymatrix{
H^* & V \\
X & V & X \\
}
\end{array},$$

where the boxed $\text{Vect}_k$ signifies that this is to be read in the category of $k$-vector spaces. The actions in (2.24) are uniquely determined by requiring $\iota_2$ and $\iota_3$ to be morphisms in $\mathcal{C}$.

The units are

$$\eta_2(V) = \eta_3(V) = \varepsilon \otimes \text{id}_V,$$

where $\varepsilon$ is the counit of $H$, and with $\Delta$ the comultiplication of $H$, the multiplications are given by\footnote{We use $\langle \cdot | \cdot \rangle : V^* \otimes V \to k$ to denote the canonical pairing in vector spaces.}

$$\mu_2(V) = (\Delta^\text{op})^* \otimes \text{id}_V, \quad \mu_3(V) = \Delta^* \otimes \text{id}_V.$$

The comultiplication of $A_i$ is given by linear maps

$$\Delta_i(V,W) : H^* \otimes V \otimes W \to H^* \otimes V \otimes H^* \otimes W$$

for all $V, W \in \mathcal{C}$ explicitly as follows. We have

$$\Delta_2(V,W)(f \otimes v \otimes w) = \sum_{i,j} \langle f | e_i \cdot e_j \rangle e_i^i \otimes v \otimes e_j^j \otimes w$$

and

$$\Delta_3(V,W)(f \otimes v \otimes w) = \sum_{i,j} \langle f | e_j \cdot e_i \rangle e_i^i \otimes v \otimes e_j^j \otimes w$$

for $f \in H^*$. The counits, being morphisms $A_i(1) \to 1$, can be identified with elements in $H$, and we find that they are given by the unit of $H$,

$$\varepsilon_2 = \varepsilon_3 = 1.$$

\footnote{The convention we use for the dual map $\Delta^*$ on $H^* \otimes H^*$ is as follows: for $f, g \in H^*$ and $b \in H$ we set $\langle \Delta^*(f \otimes g)(b) \rangle := \langle f \otimes g(\Delta(b)) \rangle$. Ditto for $\Delta^\text{op}$.}
The left antipode is given by linear maps
\[ S^l_i(V) : H^* \otimes (H^* \otimes V)^* \to V^*, \]
for \( V \in C \), and, denoting by \( \tilde{S}^l_i(V) \) the corresponding endomorphism of \( H \otimes V \), we have
\[ \tilde{S}^l_2(V) = S \otimes \text{id}_V \quad \text{and} \quad \tilde{S}^l_3(V) = S^{-1} \otimes \text{id}_V. \] (2.34)

Assume now that \( H \) is a pivotal Hopf algebra, i.e. that it contains a grouplike element \( g \), called the pivot, satisfying \( S^2(a) = gag^{-1} \) for all \( a \in H \) [AAGTV, BBGa]. The two remaining actions on \( A_i(V) \) for \( i = 1, 4 \) can be chosen as
\[ h \mapsto (f \otimes v) = \langle f | S^{-1}(h(1))h(3) \rangle \otimes h(2)v \]
\[ h \mapsto (f \otimes v) = \langle f | S(h(3))h(1) \rangle \otimes h(2)v. \] (2.35)
With this definition, the corresponding universal dinatural transformations are the same linear maps as before:
\[ \iota_1(V)_X = \iota_2(V)_X \quad \text{and} \quad \iota_4(V)_X = \iota_3(V)_X. \] (2.36)
The counits are
\[ \epsilon_1 = g^{-1} \quad \text{and} \quad \epsilon_4 = g. \] (2.37)

Rather than determining the Hopf monad structure on each \( A_i \) separately as stated before Proposition 2.3, it may be easier to work out only one, say \( A_2 \), and then to transport the structure via the isomorphisms \( \kappa_{ij} \). By Proposition 2.4, this gives the same result. The \( \kappa_{ij} \) take a simple form in the Hopf case:
\[ (\kappa_{12})_V(f \otimes v) = \langle f | g^{-1} ? \rangle \otimes v, \]
\[ (\kappa_{23})_V(f \otimes v) = \langle f \circ S | \otimes v, \]
\[ (\kappa_{43})_V(f \otimes v) = \langle f | g ? \rangle \otimes v, \] (2.38)
for all \( f \in H^*, v \in V \). Note that then e.g. \( \epsilon_1 = \epsilon_2 \circ (\kappa_{12})_1 \).

2.3. The distinguished invertible object. Since \( C \) is a finite tensor category, all projective objects are injective [EGNO, Prop.6.1.3]. In particular the socle (i.e. the maximal semisimple subobject) of the projective cover \( P_U \) of a simple object \( U \) is again simple. One can show that the socle \( D \) of \( P_U \) is an invertible object [EGNO, Lem. 6.4.1], and we call it the distinguished invertible object of \( C \).\(^8\) This is equivalent to saying that
\[ P^U_Y \cong P^D_Y. \] (2.39)
We call \( C \) unimodular if \( D \cong 1 \).

Example 2.6. Let \( H \) be a finite-dimensional Hopf algebra. A left integral in \( H \) is an element \( c^l \in H \) such that \( hc^l = \varepsilon(h)c^l \) for all \( h \in H \). It can be shown that left integrals are unique up to scalar, and thus there is a unique algebra morphism \( \gamma : H \to k \) such that
\[ c^l h = \gamma(h)c^l \quad \text{for all} \ h \in H. \] (2.40)
This algebra morphism is called the modulus of \( H \), and by abuse of notation we denote the associated one-dimensional \( H \)-module again by \( \gamma \).

\(^8\)This means that our \( D \) is in fact dual to the distinguished invertible object of [EGNO, Sec. 6.4]. However, our definition agrees with the one given in [ENO, Sec. 6].
In [EO, Prop. 2.13] it is shown that the distinguished invertible object $D$ of the category $\mathcal{H}_M$ of finite-dimensional left $H$-modules is precisely the one-dimensional $H$-module with action given by $\gamma^{-1} = \gamma \circ S$.

2.4. Monadic cointegrals for finite tensor categories. Consider the free $A_i$-module $(A_i(D), \mu_i(D))$.

**Definition 2.7.** For $i = 2$ (resp. $i = 3$), a morphism

$$\lambda_i : 1 \to A_i(D)$$

is called a right (resp. left) monadic cointegral of $C$ if it intertwines the trivial $A_i$-action on $1$ and the free action on $A_i(D)$. If $C$ is pivotal, then for $i = 1$ (resp. $i = 4$) such a morphism is called a right (resp. left) $D$-symmetrised monadic cointegral of $C$.

We denote the subspace of monadic cointegrals in $C(1, A_i(D))$ by:

$$i = 1 : \int^{r,D-sym}_C \quad i = 2 : \int^{r,mon}_C \quad i = 4 : \int^{l,D-sym}_C \quad i = 3 : \int^{l,mon}_C$$

(2.42)

**Remark 2.8.**

1. The $A_i$-module intertwining condition from (2.41) for a morphism $\lambda_i : 1 \to A_i(D)$ in $C$ is equivalent to the commutativity of the diagram

$$\begin{array}{ccc}
A_i(1) & \xrightarrow{A_i(\lambda_i)} & A_i^2(D) \\
\epsilon_i \downarrow & & \downarrow \mu_i(D) \\
1 & \xrightarrow{\lambda_i} & A_i(D)
\end{array}$$

(2.43)

that is

$$\lambda_i \circ \epsilon_i = \mu_i(D) \circ A_i(\lambda_i).$$

2. In [BV1, Eq. (45)], a cointegral of a bimonad $T$ was defined as an intertwiner of $T$-modules from $(1, T_0)$ to $(T(1), \mu_1)$. Thus, if $C$ is unimodular, a right (resp. left) monadic cointegral of $C$ is just a cointegral of the bimonad $A_2$ (resp. $A_3$).

(2.44)

(2) It follows immediately from Proposition 2.4 and from the diagram (2.43) that $\lambda_i$ is a monadic cointegral for $A_i$ if and only if $(\kappa_{i,j})_{D} \circ \lambda_i$ is a monadic cointegral for $A_j$.

The names for the monadic cointegrals are chosen because of the relation to cointegrals for Hopf algebras, as we will see in the following example.9

---

9According to our convention of calling the invariants under the regular actions of a Hopf algebra integrals, one could also call e.g. the right $D$-symmetrised monadic cointegral simply an integral for the Hopf monad $A_1$. This would follow more closely the nomenclature of [BV1] (who, however, call the invariants under the regular actions of a Hopf algebra “cointegrals”, which is opposite to our convention). It would also fit to Corollary 2.13, which roughly states that monadic cointegrals are dual to the categorical cointegrals of [Sh3].

However, as we explain in Example 2.9 and Section 4, the reason for keeping these names is that the four versions of monadic cointegrals automatically correspond to the four versions of cointegrals for $H$ if $C = H$-mod for a pivotal (quasi) Hopf algebra $H$. 

---
Example 2.9. Let \( C = H^\mathcal{M} \) be as in Example 2.5. By Example 2.6, the distinguished invertible object \( D \) is just the ground field \( k \) with action given by the algebra morphism \( \gamma^{-1} \), where \( \gamma \) is the modulus of \( H \). Thus, a morphism \( 1 \to A_i(D) \) is the same as an element in \( H^* \) intertwining some specific \( H \)-actions.

Let us first look at the linear condition coming from diagram (2.43). Using the Hopf monad structure as given in Example 2.5, we see that a right (resp. left) monadic cointegral is, as a linear form, a solution to

\[
(\lambda_2 \otimes \text{id})(\Delta(h)) = \lambda_2(h)1, \quad \text{resp.} \quad (\text{id} \otimes \lambda_3)(\Delta(h)) = \lambda_3(h)1. \tag{2.45}
\]

That is, it is a right (resp. left) cointegral for the Hopf algebra \( H \) in the usual sense, cf. [Ra2, Def. 10.1.2]. Conversely, e.g. a solution to the first equation in (2.45) is a right monadic cointegral, provided it is in addition an intertwiner \( 1 \to A_2(D) \) of \( H \)-modules. However, by [Ra2, Thm. 10.5.4(e)] a right cointegral \( \lambda \) satisfies

\[
\gamma^{-1}(\lambda_2(b))\lambda(S(\lambda_3)b) = \lambda_2(S(b_2)a). \tag{2.46}
\]

That is, the intertwining condition is automatically satisfied.

If \((H, g)\) is a pivotal Hopf algebra, then diagram (2.43) can, as a linear equation, be evaluated for \( i = 1, 4 \), and it gives the equations

\[
(\lambda_1 \otimes \text{id})(\Delta(h)) = \lambda_1(h)g^{-1}, \quad (\text{id} \otimes \lambda_4)(\Delta(h)) = \lambda_4(h)g. \tag{2.47}
\]

According to [FOG, Sec. 4.4], solutions to these equations are precisely \( \gamma \)-symmetrised cointegrals for \( H \) (where we regard \( H \) as a Hopf \( G \)-coalgebra for \( G \) the trivial group), see also [BBGa] for the unimodular case. As above, in the converse direction, solutions to e.g. the first equation in (2.47) are automatically intertwiners of \( H \)-modules from \( 1 \) to \( A_1(D) \).

Finally, let us note that \( \gamma \)-symmetrised cointegrals are an example of \( g \)-cointegrals for a group-like \( g \) as introduced in [Ra1] (and called \( g \)-integrals there), see [BGR, Rem. 3.10].

Let us stress a point already made in the introduction. As we just saw, via the very natural realisation of each monad \( A_i \) given in Example 2.5, the monadic cointegrals for \( A_1, \ldots, A_4 \) reduce to four known versions of cointegrals for finite dimensional (pivotal) Hopf algebras. This is an important motivation to keep all four of the \( A_i \), even though they are all isomorphic. Indeed, also in the Hopf case one can easily give explicit isomorphisms between the four spaces of cointegrals, but in practice it is important to have all four notions available, rather than singling one out arbitrarily.

The preceding example shows that for \( C = H^\mathcal{M} \) with \( H \) a finite-dimensional (pivotal) Hopf algebra, left/right (\( D \)-symmetrised) monadic cointegrals exist and are unique up to scalar. The next proposition states that this remains true for any (pivotal) finite tensor category.

Proposition 2.10. Let \( C \) be a finite tensor category.

1. Non-zero left/right monadic cointegrals exist and are unique up to scalar multiples.
2. Suppose \( C \) is in addition pivotal. Then non-zero left/right \( D \)-symmetrised monadic cointegrals exist and are unique up to scalar multiples.

---

To see this, note that by [FOG, Prop. 4.18] the linear form \( \lambda_1 \) lies in the space \( X_1 \) from (A.39). This space is isomorphic to \( C(1, A_1(D)) \), and the isomorphism (A.40) is the identity in the Hopf case.
The proof will follow from results in [Sh3], after we relate monadic cointegrals to the categorical cointegral of [Sh3], and is given at the end of the next subsection.

2.5. Relation to the categorical cointegral. Define functors $Z^i$ via the ends

$$
Z^1(V) = \int_{X \in C} \mathcal{V}(VX), \quad Z^2(V) = \int_{X \in C} X^\vee (VX)
$$

$$
Z^3(V) = \int_{X \in C} (XV)^\vee X, \quad Z^4(V) = \int_{X \in C} (XV) X^\vee
$$

with corresponding universal dinatural transformations $\pi_i(V)$, so that for example

$$
\pi_4(V)_X : Z^4(V) \to (XV) X^\vee.
$$

Below we will give an adjunction between $Z^4$ and $A_2$. One can formulate such adjunctions in the three other cases, too, but we will not need this and will only consider $Z^4$ in the following.

Similarly to how the $A_i$, $i = 2, 3$ became Hopf monads, $Z^4$ becomes a Hopf comonad and we denote the comultiplication, counit, multiplication, and unit by

$$
\Delta^4(V) : Z^4(V) \to Z^4 Z^4(V), \quad \varepsilon^4(V) : Z^4(V) \to V,
$$

$$
\mu^4(V,W) : Z^4(V) \otimes Z^4(W) \to Z^4(V \otimes W), \quad u^4 : 1 \to Z^4(1),
$$

respectively. $Z^4$ is precisely the central comonad of [Sh3], where also a detailed description of the structure maps (2.50) can be found.

We can now recall the definition of the categorical cointegral from [Sh3, Def. 4.3]: It is a $Z^4$-comodule morphism

$$
\lambda^{\text{Sh}} : (Z^4(D^\vee), \Delta^4(D^\vee)) \to 1
$$

from the cofree comodule on $D^\vee$ to the tensor unit considered as the trivial comodule.\footnote{Although Shimizu’s definition is not explicitly stated this way, it is easy to see that [Sh3, Def. 4.3] and (2.51) are equivalent. This is also mentioned in the proof of [Sh3, Thm. 4.8].}

To relate the two notions categorical cointegral and monadic cointegral, we observe that there is an adjunction $A_2 \dashv Z^4$, i.e. the central Hopf monad $A_2$ is left adjoint to $Z^4$. Indeed,

$$
\mathcal{C}(A_2(V), W) \cong \text{Dinat}(-^\vee (V -), W)
$$

$$
\cong \text{Dinat}(V, (-W)^{-\vee}) \cong \mathcal{C}(V, Z^4(W)).
$$

(2.52)

We denote the counit and unit of this adjunction by

$$
\overline{\epsilon} : A_2 Z^4 \Rightarrow \text{id}_C, \quad \overline{\eta} : \text{id}_C \Rightarrow Z^4 A_2
$$

(2.53)

respectively. They can easily be deduced from (2.52); for example

$$
\begin{array}{cc}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Z4V) at (0,1) {$Z^4V$};
  \node (V) at (1,0) {$V$};
  \node (Z4) at (0,2) {$Z^4$};
  \draw[->] (V) to (Z4);
  \draw[->] (X) to (Z4V);
  \draw[->] (V) to (X);
  \draw[->] (Z4) to (X);
\end{tikzpicture}
\end{array} = \begin{array}{cc}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Z4V) at (0,1) {$Z^4V$};
  \node (V) at (1,0) {$V$};
  \node (Z4) at (0,2) {$Z^4$};
  \draw[->] (V) to (Z4);
  \draw[->] (X) to (Z4V);
  \draw[->] (V) to (X);
  \draw[->] (Z4) to (X);
\end{tikzpicture}
\end{array}
$$

(2.54)

determines the counit.
For a comonad $M$ on $C$, the category of comodules is denoted by $C^M$.

**Lemma 2.11.** The functor $F : C^{Z^4} \to C_{A_2}$, given on objects and morphisms by

$$F(V, \rho) = (V, \bar{\epsilon}_V \circ A_2(\rho)), \quad Ff = f,$$

is an equivalence.

**Proof.** This statement follows immediately from the fact that $A_2$ is left adjoint to $Z^4$. The inverse equivalence $G : C_{A_2} \to C^{Z^4}$ is given on objects and morphisms by

$$G(V, \nu) = (V, Z^4(\nu) \circ \bar{\eta}_V), \quad Gf = f,$$

and a simple check using the adjunction triangles proves the claim. □

We make the following observation.

**Proposition 2.12.** Let $F$ be as in Lemma 2.11. There is an isomorphism

$$(A_2(V), \mu_2(V)) \cong (F(Z^4(\wp V), \Delta^4(\wp V)))^\wp$$

of $A_2$-modules, natural in $V \in C$.

**Proof.** Abbreviate

$$\bar{V} = Z^4(\wp V) \quad \text{and} \quad \rho_{\bar{V}} = \Delta^4(\wp V) : \bar{V} \to Z^4(\bar{V}).$$

Under the equivalence from Lemma 2.11 we have

$$F(\bar{V}, \rho_{\bar{V}}) = (\bar{V}, \sigma_{\bar{V}})$$

where $\sigma_{\bar{V}} = \bar{\epsilon}_{\bar{V}} \circ A_2(\rho_{\bar{V}}) : A_2(\bar{V}) \to \bar{V}$ is the $A_2$-action corresponding to the free coaction.

Define the natural isomorphism $E_V : A_2(V) \to \bar{V}^\wp$ by

$$E_V \circ \iota_2(X)_V = \bar{V}^\wp$$

for $V \in C$. Here $\omega_X : (\wp X)^\wp \to X$ denotes the natural isomorphism from (1.12).

We want to show that $E_V$ is an $A_2$-module map, that is

$$E_V \circ \mu_2(V) = S^2_\wp(\bar{V}) \circ A_2 \left( \sigma^\wp_{\bar{V}} \circ E_V \right),$$

where we also used the action (2.9) on the dual $A_2$-module. To check that this equality holds we establish that both sides of (2.61) satisfy the same universal property for the iterated coend $A_2A_2$. For the left hand side of (2.61) we get
The right hand side of (2.61) composed with the same dinatural transformation immediately yields

\[
E_V \circ \mu_2(V) \circ \iota_2(A_2V)_Y \\
\circ (\text{id}_{Y^\vee} \otimes (\iota_2(V)_X \otimes \text{id}_Y)) = \\
\tilde{V}^\vee.
\] (2.62)

A simple calculation shows

\[
\pi_4(\tilde{V}_{X^\vee}^\vee \circ \sigma_{\tilde{V}} \circ \iota_2(\tilde{V})_{Y^\vee}) = \\
\tilde{V}.
\] (2.64)
and we thus get

\begin{equation}
(2.63) = (2.65)
\end{equation}

Let $\gamma^r$ be the analogue of $\gamma$ (defined in (1.8)) for right duals. One checks that the diagram

\begin{equation}
(\gamma Y \otimes Y)^{\vee} \xrightarrow{(\gamma^r)^{-1}_{Y,X}} (\gamma Y)^{\vee} \otimes (\gamma X)^{\vee} \xrightarrow{\omega_{Y \otimes X}} \omega_{Y \otimes X}
\end{equation}

commutes.

Dinaturality of $\pi_4$ then implies

\begin{equation}
\begin{array}{ccc}
\gamma Y & \gamma Y (\gamma X)^{\vee} & \gamma Y (\gamma X)^{\vee} \\
& \omega & \omega \\
\pi_4 & (\gamma^r)^{-1} & (\gamma^r)^{-1} \\
\omega & \omega & \omega \\
Z^4(V) & Z^4(V) & Z^4(V)
\end{array}
\end{equation}

After plugging this into (2.65) and substituting the definitions of $\gamma^r, \gamma$, and $\omega$, we see that this agrees with (2.62).

Combining Lemma 2.11 and Proposition 2.12 and using $\gamma D \cong D^\gamma$ (this holds for all invertible objects), we get

**Corollary 2.13.** There is an isomorphism

\begin{equation}
\mathcal{C}_{A_2}(1, (A_2(D), \mu_2(D))) \cong \mathcal{C}Z^4(\Delta^4(D^\gamma), 1).
\end{equation}

After these preparations, we can show the existence and uniqueness (up to scalar) of monadic cointegrals.
Proof of Proposition 2.10. By the preceding corollary, the right monadic cointegral is equivalent to the categorical cointegral (2.51) of [Sh3]. Existence and uniqueness of categorical cointegrals were established in [Sh3, Thm. 4.8]. The claim then follows from Remark 2.8 (2). □

Remark 2.14. Recall that the definition of integrals and cointegrals in the Hopf case is symmetric under duality. More precisely, if \( H \) is a finite-dimensional Hopf algebra, then a left cointegral for \( H \) is the same as a morphism \( \lambda^l : H \rightarrow 1 \) in the category of left \( H \)-comodules, where we regard \( H \) as the coregular comodule. Equivalently, we can consider the dual Hopf algebra \( H^* \) (with the structure given by transposition of that of \( H \), cf. footnote 7). In this case it is a morphism \( \lambda^l : 1 \rightarrow H^* \) in the category of left \( H^* \)-modules, where we regard \( H^* \) as the regular module.

Taking duals provides a (contravariant) equivalence,

\[
\text{H-comod} \cong \text{H}^*-\text{mod}
\]

and in particular we have an isomorphism

\[
\text{(H-comod)}(H, 1) \cong \text{(H}^*-\text{mod)}(1, H^*)
\]

of vector spaces. We will need one more observation. Abbreviate \( C = H\text{-mod} \) and recall the computation (2.46). This showed that there is an \( H \)-module structure on \( H^* = A_3(D) \) and similarly on \( H = Z^4(D^\vee) \) (which are not the (co)regular) such that:

\[
\text{(H-comod)}(H, 1) \cong \text{(H}^*-\text{mod)}(1, H^*)
\]

\[
\cap C(Z^4(D^\vee), 1) \cap C(1, A_3(D))
\]

(2.71)

For quasi-Hopf algebras \( H \) the corresponding line of reasoning to relate integrals and cointegrals fails at the outset, as \( H^* \) is not again a quasi-Hopf algebra. Instead, there is the following categorical version of it. We have a (contravariant) equivalence given by the composition of equivalences

\[
C_{Z^4} \cong C_{A_2} \cong C_{A_2} \cong C_{A_3} \cong C_{A_3}
\]

(2.72)

Corollary 2.13 then implies

\[
C_{Z^4} \left( \left( Z^4(D^\vee), \Delta^4(D^\vee) \right), 1 \right) \cong C_{A_3} \left( 1, \left( A_3(D), \mu_3(D) \right) \right)
\]

(2.73)

To relate the right hand side of (2.73) in the Hopf case to that of (2.71), one uses the explicit form of the monad multiplication in (2.28). For the left hand side, one correspondingly uses the coproduct of \( Z^4 \), we omit the details.

The categorical version (2.73) of the isomorphism (2.71) provides a more conceptual reason for the relation between the left monadic cointegral and the categorical cointegral of [Sh3].

2.6. Rewriting the monadic cointegral via \( Z^4 \). In Corollary 2.13 we saw one way to rewrite the definition of monadic cointegrals in terms of Hom-spaces in \( C_{Z^4} \). We will later need the more direct relation we present here.

Under the equivalence from Lemma 2.11, specifically (2.56), we can map the free \( A_2 \)-module on any \( U \in C \) to its corresponding \( Z^4 \) comodule

\[
(A_2(U), R_U) \quad \text{with} \quad R_U = \left[ A_2(U) \xrightarrow{\tilde{\eta}_{A_2(U)}} Z^4A_2(U) \xrightarrow{Z^4(\mu_2(U))} Z^4A_2(U) \right].
\]

(2.74)
Note that this assignment is in fact natural in $U$, i.e. we have a natural transformation
\[ R : A_2 \Rightarrow Z^4 A_2, \]
which we call the \textit{categorical coaction}.\textsuperscript{12} Since we have equivalence (2.55), we can see the equality
\[ \mathcal{C}_{A_2}(1, (A_2(D), \mu_2(D))) = \mathcal{C}^{Z^4}(1, (A_2(D), R_D)) \]
of subspaces of $\mathcal{C}(1, A_2(D))$. An element in the subspace on the left hand side is by definition a right monadic cointegral. Spelling out the condition to be in the subspace on the right hand side proves the following lemma (recall from (2.50) the notation $u^4$ for the unit of $Z^4$):

\textbf{Lemma 2.15.} A morphism $\lambda : 1 \rightarrow A_2(D)$ in $\mathcal{C}$ is a right monadic cointegral if and only if
\[ R_D \circ \lambda = Z^4(\lambda) \circ u^4. \]

\textbf{3. Cointegrals for quasi-Hopf algebras}

In this section we first set out our conventions for quasi-Hopf algebras, which agree with those used in [FGR1, BGR]. Then, we specialise monadic cointegrals to the category of modules over a quasi-Hopf algebra, and we recall the definition of cointegrals from [HN2, BC].

Throughout this section, let $H$ be a finite-dimensional quasi-Hopf algebra over $k$.

\textbf{3.1. Conventions and definitions.} The antipode of $H$ is denoted by $S$, and the coassociator and its inverse by $\Phi$ and $\Psi$, respectively. For the coproduct, the counit, and the unit we use the standard notations $\Delta$, $\varepsilon$, $1$. The evaluation and coevaluation elements are $\alpha$ and $\beta$, and without loss of generality we assume that $\varepsilon(\alpha) = 1 = \varepsilon(\beta)$.

For elements in tensor powers of $H$, e.g. $a \in H \otimes H$, we write
\[ a = a_1 \otimes a_2, \quad a_{21} = \tau(a) = a_2 \otimes a_1, \]
(3.1)
where $\tau$ is the tensor flip of vector spaces. This notation is extended to higher tensor powers of $H$.

The components of the coassociator and its inverse will be written with upper-case Latin letters and lower-case Latin letters like
\[ \Phi = X_1 \otimes X_2 \otimes X_3 \quad \text{and} \quad \Psi = x_1 \otimes x_2 \otimes x_3, \]
(3.2)
respectively. For further copies of $\Phi$ and $\Psi$ in an expression different letters like $Y, y$ are used. We remark that our conventions for $\Phi$ differ from those of e.g. [HN2, BC], in that we use the quasi-coassociativity condition
\[ (\Delta \otimes \text{id})(\Delta(h)) \cdot \Phi = \Phi \cdot (\text{id} \otimes \Delta)(\Delta(h)) \]
(3.3)
for all $h \in H$. Thus, the $\Phi$ in [HN2, BC] is our $\Psi$. The pentagon axiom is
\[ (\Delta \otimes \text{id} \otimes \text{id})(\Phi) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\Phi) = (\Phi \otimes \text{id}) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \cdot (\text{id} \otimes \Phi). \]
(3.4)
A sumless Sweedler-notation for (iterated) coproducts is used, i.e.
\[ \Delta(h) = h_{(1)} \otimes h_{(2)}, \quad (\Delta \otimes \text{id})(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_{(2)} \]
(3.5)
for $h \in H$.

\textsuperscript{12}The categorical coaction turns $A_2$ into a $Z^4$-comodule in $\text{End}(\mathcal{C})$.\textsuperscript{13}
With the above notation we can write the antipode axioms as
\[ S(h(1))\alpha h(2) = \varepsilon(h)\alpha, \quad h(1)\beta S(h(2)) = \varepsilon(h)\beta, \quad h \in H, \]
and
\[ S(X_1)\alpha X_2\beta S(X_3) = 1, \quad x_1\beta S(x_2)\alpha x_3 = 1. \]
The latter are also referred to as zig-zag axioms.

By considering either the opposite multiplication or the opposite comultiplication we get new quasi-Hopf algebras \( H^{\text{op}} \) and \( H^{\text{cop}} \), with the quasi-Hopf structure given by \( S^{\text{op}} = S^{\text{cop}} = S^{-1}, \Phi^{\text{op}} = \Psi, \alpha^{\text{op}} = S^{-1}(\beta), \beta^{\text{op}} = S^{-1}(\alpha), \Phi^{\text{cop}} = \Psi_{321}, \alpha^{\text{cop}} = S^{-1}(\alpha), \) and \( \beta^{\text{cop}} = S^{-1}(\beta) \).

We will make frequent use of the hook notation, for \( h \in H, f \in H^* \),
\[ h \rightarrow f = \langle f \mid ?h \rangle, \quad f \leftarrow h = \langle f \mid h? \rangle \]
and
\[ f \rightarrow h = h(1)f(h(2)), \quad h \leftarrow f = f(h(1))h(2). \]

3.2. Modules. Finite-dimensional left \( H \)-modules and their intertwiners form the category \( _H\mathcal{M} \), and we now recall its structure, following [BC]. This category is a finite tensor category; the tensor product of two objects \( V, W \in _H\mathcal{M} \) is the vector space \( V \otimes_k W \) equipped with the diagonal \( H \)-action. The left dual \( V^\vee \) and the right dual \( ^\vee V \) of a module \( V \) are both modelled on the dual vector space \( V^* \), and \( h \in H \) acts on \( v^* \in V^\vee \) via
\[ \langle h.v^*, w \rangle = \langle v^* - S(h) \mid w \rangle = \langle v^*, S(h)w \rangle \]
and on \( v^* \in ^\vee V \) via
\[ \langle h.v^*, w \rangle = \langle v^* - S^{-1}(h) \mid w \rangle = \langle v^*, S^{-1}(h)w \rangle. \]
The left and right evaluations are
\[ \overleftarrow{ev}_V : V^\vee \otimes V \rightarrow 1, \quad \overleftarrow{ev}_V(v^* \otimes v) = \langle v^*, \alpha w \rangle \]
\[ \overrightarrow{ev}_V : V \otimes ^\vee V \rightarrow 1, \quad \overrightarrow{ev}_V(w \otimes v^*) = \langle v^*, S^{-1}(\alpha)w \rangle, \]
and the left and right coevaluations are
\[ \overleftarrow{coev}_V : 1 \rightarrow V \otimes V^\vee, \quad \overleftarrow{coev}_V(1) = \sum_i v_i \otimes v^i \]
\[ \overrightarrow{coev}_V : 1 \rightarrow ^\vee V \otimes V, \quad \overrightarrow{coev}_V(1) = \sum_i v^i \otimes S^{-1}(\beta)v_i, \]
which we expressed in terms of a basis \( \{v_i\} \) of \( V \), with dual basis \( \{v^i\} \).

Recall the natural isomorphism \( \gamma_{V;W} \) from (1.8). There is a unique invertible element \( f \in H \otimes H \), called the Drinfeld twist, such that
\[ \gamma_{V;W}(f \otimes g)(w \otimes v) = g(f_1w)f(f_2v) \]
for all \( f \in V^*, v \in V, g \in W^*, w \in W \), see [Dr] and e.g. [FGR1, Lem. 6.7]. The statement that \( \gamma_{V;W} \) be an intertwiner translates to the equation
\[ f \cdot \Delta(S(a)) \cdot f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(a)). \]
Analogously, we can express the natural isomorphism $\gamma^r_{V,W} : \forall V \otimes \forall W \to \forall (WV)$, obtained by mirroring the diagram in (1.8) at the vertical axis, by using what we call the Drinfeld twist for right duals $f^r$:

$$\gamma^r_{V,W}(f \otimes g)(w \otimes v) = g(f_1^r w)f(f_2^r v)$$

for all $f \in V^*, v \in V, g \in W^*, w \in W$.

Both $f$ and $f^r$ can be written in terms of the defining data of the quasi-Hopf algebra $H$, and we will give their explicit form in Section 3.4.

If $H\mathcal{M}$ is pivotal, the quasi-Hopf algebra $H$ is called pivotal. The pivotal structure on $H\mathcal{M}$ corresponds uniquely to an element $g \in H$ called the pivot satisfying

$$\Delta(g) = f^{-1} \cdot (S \otimes S)(f_{21}) \cdot (g \otimes g), \quad \varepsilon(g) = 1,$$

and $S^2(h) = ghg^{-1}$, for all $h \in H$ [BCT, Prop. 3.2]. As for Hopf algebras, the inverse of the pivot is $g^{-1} = S(g)$, see [BT2, Prop. 3.12].

### 3.3. Monadic cointegrals for quasi-Hopf algebras

Left and right integrals for quasi-Hopf algebras are defined in the same way as for Hopf algebras, see Example 2.6, and it was shown in [HN2] that the space of left (resp. right) integrals of a quasi-Hopf algebra is one-dimensional. The modulus $\gamma$ of a quasi-Hopf algebra is defined as in the Hopf case, cf. (2.40). As for Hopf algebras, the distinguished invertible object $D$ of $H\mathcal{M}$ is the one-dimensional module with action given by $\gamma^{-1} = \gamma \circ S$, see [EGNO, Prop. 6.5.5], and we have $\gamma^{-1} = \gamma^\vee$ as $H$-modules. We call $H$ unimodular if $\gamma = \varepsilon$, or equivalently, if every left integral is also right.

We now want to describe the monadic cointegrals for $H$. To this end, let us first give our realizations of the central Hopf monads $A_i$ on the category $H\mathcal{M}$. As for ordinary Hopf algebras, the objects $A_i(V), V \in H\mathcal{M}$, have $H^* \otimes V$ as underlying vector space, and the same dinatural transformations $\iota_i(V)$ as in (2.25) and (2.36). These uniquely determine the $H$-action on $A_i(V)$ (denoted by $\cdot$) to be

$$h \overset{1}{\cdot} (f \otimes v) = \langle f | S^{-1}(h_{(1)})?h_{(2,2)} \rangle \otimes h_{(2,1)} \cdot v,$$

$$h \overset{2}{\cdot} (f \otimes v) = \langle f | S(h_{(1)})?h_{(2,2)} \rangle \otimes h_{(2,1)} \cdot v,$$

$$h \overset{3}{\cdot} (f \otimes v) = \langle f | S^{-1}(h_{(2)})?h_{(1,1)} \rangle \otimes h_{(1,2)} \cdot v,$$

$$h \overset{4}{\cdot} (f \otimes v) = \langle f | S(h_{(2)})?h_{(1,1)} \rangle \otimes h_{(1,2)} \cdot v.$$  

Let us also record here the Hopf monad isomorphisms from Proposition 2.4 for (pivotal) quasi-Hopf algebras, using our realizations of the monads.

**Proposition 3.1.** The canonical Hopf monad isomorphisms

$$A_1(V) \overset{(\kappa_{1,2})_V}{\longrightarrow} A_2(V) \overset{(\kappa_{2,3})_V}{\longrightarrow} A_3(V) \overset{(\kappa_{3,4})_V}{\longrightarrow} A_4(V)$$

from Proposition 2.4 (for the maps $(\kappa_{1,2})_V, (\kappa_{3,4})_V$, we require a pivot $g$) are given by the linear maps

$$((\kappa_{1,2})_V(f \otimes v) = (f \leftarrow g^{-1}) \otimes v,$$

$$((\kappa_{2,3})_V(f \otimes v) = \langle f | S(?X_1)X_3 \rangle \otimes X_2 \cdot v,$$

$$((\kappa_{3,4})_V(f \otimes v) = (f \leftarrow g^{-1}) \otimes v).$$
for $f \in H^*$, $v \in V$.

Proof. The proof is a straightforward computation. For example, for $\kappa_{2,3}$ one needs to check the commutativity of

$$
\begin{aligned}
X^V \otimes (V \otimes X) &\xrightarrow{\sim} (X^V \otimes V) \otimes X \\
&\xrightarrow{\sim} (X^V \otimes V) \otimes ^V(X^V) \\
&\xrightarrow{(\kappa_{2,3})_V} A_2(V) \\
&\xrightarrow{(\kappa_{2,3})_V} A_3(V)
\end{aligned}
$$

(3.21)

Note here that the isomorphism $X \cong ^V(X^V)$ is the same linear map as in $\text{Vect.}$ □

Define $\tau \in H^{\otimes 5}$ by

$$
\tau = X_1 \otimes X_2 y_1 \otimes x_1 (X_{3(1)} y_2)_{(1)} \otimes x_2 (X_{3(1)} y_2)_{(2)} \otimes x_3 X_{3(2)} y_3
$$

(3.22)

The multiplication of $A_2$ can be computed explicitly from (2.14). For $f, g \in H^*$ and $v \in V$, the image $\mu_2(V)(f \otimes g \otimes v) \in H^* \otimes V$ under multiplication can be identified with the linear map

$$
H \ni h \mapsto (g \otimes f)((S \otimes S)(\tau_{21})f \Delta(h)\tau_{45})\tau_{3, v} \in V.
$$

(3.23)

The counit $\epsilon_2 : A_2(1) \to 1$ is easily computed from (2.16), and we identify it with the element $\epsilon_2 = \alpha \in H$. Let us recall the right monadic cointegral equation from (2.44):

$$
\lambda \circ \epsilon_2 = \mu_2(D) \circ A_2(\lambda).
$$

(3.24)

This is an equality of (linear) endomorphisms of $H^*$, and evaluating it on $f \in H^*$, we immediately get

$$
f(\alpha)\lambda = \gamma^{-1}(\tau_3)(\lambda \otimes f)((S \otimes S)(\tau_{31})f \Delta(h)\tau_{45})\epsilon^i.
$$

(3.25)

This is clearly equivalent to

$$
\lambda(h)\alpha = \gamma^{-1}(\tau_3)(\lambda \otimes \text{id})((S \otimes S)(\tau_{31})f \Delta(h)\tau_{45})
$$

(3.26)

for all $h \in H$.

Altogether, an element $\lambda \in H^*$ is a right monadic cointegral if and only if it satisfies (3.26) and is an $H$-module intertwiner $1 \to A_2(\gamma^{-1})$ (see (A.19) to see this written out as a linear equation).\textsuperscript{13}

Similarly, with $f^r$ the Drinfeld twist for right duals and $\sigma = (\tau^{\text{cop}})_{54321}$ given explicitly by

$$
\sigma = x_{1(1,1)} Y_{1} X_1 \otimes x_{1(1,2)} Y_{2} X_{2(1)} \otimes x_{1(2)} Y_3 X_{2(2)} \otimes x_2 X_3 \otimes x_3,
$$

(3.27)

one obtains necessary conditions for the three remaining types of monadic cointegrals. Namely, if $\lambda \in H^*$ is a

1. right $D$-symmetrised monadic cointegral then it satisfies

$$
\lambda(h)g^{-1}\alpha = \gamma^{-1}(\tau_3)(\lambda \otimes \text{id})((S^{-1} \otimes S)^{-1}(\tau_{21})f^r \Delta(h)\tau_{45})
$$

(3.28)

2. left monadic cointegral then it satisfies

$$
\lambda(h)S^{-1}(\alpha) = \gamma^{-1}(\sigma_3)(\text{id} \otimes \lambda)((S^{-1} \otimes S)^{-1}(\sigma_{54})f^r \Delta(h)\sigma_{12})
$$

(3.29)

\textsuperscript{13}The $H$-intertwiner condition is automatic for monadic cointegrals for Hopf algebras, and an analogous condition is automatic for cointegrals of quasi-Hopf algebras as defined in Definition 3.2 below. The corresponding statement remains to be shown in the monadic setting for quasi-Hopf algebras, but it is not needed for the present paper.
(4) left $D$-symmetrised monadic cointegral then it satisfies

$$\lambda(h)g S^{-1}(\alpha) = \gamma^{-1}(\sigma_3)(\text{id} \otimes \lambda)((S \otimes S)(\sigma_{\tau_4})f \Delta(h)\sigma_{12})$$

(3.30)

for all $h \in H$.

3.4. Special elements and relations. Here we want to give the closed form of the
Drinfeld twist and its inverse. We closely follow [BC], but note that our conventions
are slightly different, i.e. our $\Phi$ is their $\Phi^{-1}$.

We will need the four elements $q^R, p^R, q^L, p^L$ in $H \otimes H$, given by

$$q^R = x_1 \otimes S^{-1}(\alpha x_3)x_2, \quad p^R = X_1 \otimes X_2\beta S(X_3),$$

$$q^L = S(X_1)\alpha X_2 \otimes X_3, \quad p^L = x_2 S^{-1}(x_1 \beta) \otimes x_3. \quad (3.31)$$

These satisfy the identities

$$\Delta(q^R_1)p^R_1[1 \otimes S(q^R_2)] = 1 \otimes 1, \quad [1 \otimes S^{-1}(p^R_2)]q^R \Delta(p^R_1) = 1 \otimes 1,$$

$$\Delta(q^L_2)p^L[S^{-1}(q^L_1) \otimes 1] = 1 \otimes 1, \quad [S(p^L_1) \otimes 1]q^L \Delta(p^L_2) = 1 \otimes 1, \quad (3.32)$$

and, for all $a \in H$,

$$[1 \otimes S^{-1}(a_{(2)})]q^R \Delta(a_{(1)}) = [a \otimes 1]q^R,$$

$$[S(a_{(1)}) \otimes 1]q^L \Delta(a_{(2)}) = [1 \otimes a]q^L,$$

$$\Delta(a_{(1)})p^R[1 \otimes S(a_{(2)})] = p^R[a \otimes 1],$$

$$\Delta(a_{(2)})p^L[S^{-1}(a_{(1)}) \otimes 1] = p^L[1 \otimes a]. \quad (3.33)$$

These elements and relations are well-known in the representation theory of quasi-Hopf
algebras. For their interpretation in terms of natural transformations and categorical
identities see [HN1] and e.g. [BGR, Sec. 3].

Next, let us also define $\varepsilon, \delta \in H \otimes H$ by

$$\varepsilon = S(x_2)q^L_1 x_{3(1)} \otimes S(x_1)\alpha q^L_2 x_{3(2)} = S(q^R_2 X_{1(2)})X_2 \otimes S(q^R_1 X_{1(1)})\alpha X_3 \quad (3.34)$$

and

$$\delta = x_{1(1)}p^R_1 \beta S(x_3) \otimes x_{1(2)}p^R_2 S(x_2) = X_1 \beta S(X_{3(2)})p^L_2 \otimes X_2 S(X_{3(1)})p^L_1. \quad (3.35)$$

Then the Drinfeld twist is given by

$$f = (S \otimes S)(\Delta^\text{cop}(p^R_1))\varepsilon \Delta(p^R_2) \quad (3.36)$$

and its inverse is

$$f^{-1} = \Delta(q^R_1)\delta(S \otimes S)(\Delta^\text{cop}(q^L_2)). \quad (3.37)$$

The explicit form of the Drinfeld twist for right duals can similarly be given as

$$f^r = (S^{-1} \otimes S^{-1})(\varepsilon_{21} \Delta^\text{cop}(p^L_1))\Delta(p^L_2),$$

$$(f^r)^{-1} = \Delta(q^R_2)(S^{-1} \otimes S^{-1})(\Delta^\text{cop}(q^R_1)\delta_{21}). \quad (3.38)$$

We end this subsection by stating some technical properties of the Drinfeld twist
which we will need later on (see [BC]). The Drinfeld twist satisfies the identity

$$(1 \otimes f) \cdot (\text{id} \otimes \Delta)(f) \cdot \Psi = (S \otimes S \otimes S)(\Psi_{321}) \cdot (f \otimes 1) \cdot (\Delta \otimes \text{id})(f), \quad (3.39)$$

or, written in Sweedler-notation,

$$f_{1}x_{1} \otimes f_{1}f_{2(1)}x_{2} \otimes f_{2}f_{2(2)}x_{3}.$$
\[ S(\alpha) \tilde{f} \cdot S(\beta) f = S(\alpha) \cdot S(\beta) f, \]  
(3.40)

where we use the symbol \( \tilde{f} \) to denote another copy of the Drinfeld twist.

A direct computation shows that \( f \) further satisfies

\[ f_1 \beta S(f_2) = S(\alpha) , \quad S(\beta) f_2 = \alpha . \]  
(3.41)

Lastly, we have

\[ \Delta(\beta) f^{-1} = \delta . \]  
(3.42)

3.5. Cointegrals via coactions. In preparation for the proof of the main theorem let us now briefly review the original definition of cointegrals for quasi-Hopf algebras from [HN2].

Following [HN2, BC], we set

\[
\begin{align*}
U &= f^{-1}(S \otimes S)(q^{R}_{21}), \\
V &= (S^{-1} \otimes S^{-1})(f^{L}_{21} p^{R}_{21}),
\end{align*}
\]  
(3.43)

Definition 3.2. A left cointegral for \( H \) is an element \( \lambda^l \in H^* \) satisfying

\[(\text{id} \otimes \lambda^l)(V \Delta(h)U) = \gamma(X_1) \lambda^l(h S(X_2)) X_3 \]  
(3.44)

for all \( h \in H \), and a right cointegral for \( H \) is a left cointegral for \( H^{\text{cop}} \). Explicitly, this means it is an element \( \lambda^r \in H^* \) satisfying

\[(\text{id} \otimes \lambda^r)(V^{\text{cop}} \Delta^{\text{cop}}(h) U^{\text{cop}}) = \gamma(x_3) \lambda^r(h S^{-1}(x_2)) x_1 . \]  
(3.45)

Cointegrals exist, and a non-zero cointegral is a non-degenerate linear form on \( H \), uniquely determined up to scalar [HN2, Thm. 4.3]. We denote the spaces of left and right cointegrals by

\[ \mathcal{I}_{\lambda^l} H \quad \text{and} \quad \mathcal{I}_{\lambda^r} H, \]  
(3.46)

respectively. Let now \( \lambda^l \) and \( \lambda^r \) be a left and a right cointegral of \( H \), respectively. These satisfy

\[ \lambda^l(S^{-1}(a) b) = \lambda^l(b S(a \leftarrow \gamma)) \quad \text{and} \quad \lambda^r(S(a) b) = \lambda^r(b S^{-1}(\gamma \rightarrow a)), \]  
(3.47)

for \( a, b \in H \), see [HN2, Lem. 5.1]. With

\[ u = (\gamma \otimes S^2)(V) \quad \text{and} \quad u^{\text{cop}} = (\gamma \otimes S^{-2})(V^{\text{cop}}) , \]  
(3.48)

left and right cointegrals can be related as follows [BC, Prop. 4.3],

\[ (\lambda^r \leftarrow u) \circ S \in \mathcal{I}_{\lambda^l} H, \quad \text{resp.} \quad (\lambda^l \leftarrow u^{\text{cop}}) \circ S^{-1} \in \mathcal{I}_{\lambda^r} H . \]  
(3.49)

The corresponding relation between left and right monadic cointegrals is given by \((\kappa_{2,3})_D\), see Remark 2.8 (2). It is worth comparing the definition of \( u \) to that of \( \kappa_{2,3} \) from Proposition 3.1, which only used a single coassociator.

The left-hand side of the left cointegral equation (3.44) has the following categorical interpretation, which we recall from [BC, Sec. 3]. Let \( _H \mathcal{M}^H \) be the category of \( H \)-bimodules, equipped with the monoidal structure of the category of modules over the quasi-Hopf algebra \( H \otimes H^{\text{cop}} \).
In $H\mathcal{M}_H$, the regular bimodule $H$ is a coalgebra, and we can thus consider the comonad $\mathcal{Y}^r : B \mapsto B \otimes H$ on $H\mathcal{M}_H$. With $H$ equipped with the obvious $\mathcal{Y}^r$-comodule structure, via the coproduct, the right dual $\mathcal{Y}^r H$ becomes a $\mathcal{Y}^r$-comodule via $\rho^r : \mathcal{Y}^r H \rightarrow \mathcal{Y}^r(\mathcal{Y}^r H)$,

\[
\rho^r = \mathcal{Y}^r H \xrightarrow{\sim} (\mathcal{Y}^r H \otimes \mathcal{Y}^r H) \xrightarrow{\text{coev} \otimes \text{id}} (\mathcal{Y}^r H)\mathcal{Y}^r H \xrightarrow{(\text{id} \otimes \Delta) \otimes \text{id}} (\mathcal{Y}^r(H \mathcal{H}H))\mathcal{Y}^r H \xrightarrow{\sim} \mathcal{Y}^r(\mathcal{Y}^r H).
\]

Here all coherence isomorphisms are those of $H\mathcal{M}_H$, so that explicitly, $\rho^r$ sends $f \in H^*$ to

\[
\rho^r(f) = f(V_2(e_i)U_2)e^i \otimes V_1(e_i)U_1.
\]

Then (3.44) can be written as

\[
\rho^r(\lambda^l) = \gamma(X_1)\lambda^l.X_2 \otimes X_3,
\]

where by the dot we mean the right $H$-action on the right dual $\mathcal{Y}^r H$, i.e. $(f.a)(h) = f(hS(a))$ for $f \in \mathcal{Y}^r H$, $a \in H$.

Similarly, the left dual $H^r$ carries a natural left $H$-comodule structure, i.e. it is a comodule of the comonad $\mathcal{Y}^l : B \mapsto H \otimes B$; call the corresponding coaction $\rho^l$. Using the explicit expressions from above, one verifies $(\rho^l)^{\text{cop}}_{21} = \rho^r$, and we obtain that $\lambda^r \in H^*$ is a right cointegral if and only if

\[
\rho^l(\lambda^r) = \gamma(x_3)x_1 \otimes \lambda^r.x_2,
\]

and the dot here denotes the action on the left dual.

For later use, we relate the comonad $Z^l$ on $H\mathcal{M}$ (recall its definition in Section 2.5) and $\mathcal{Y}^l$ on $H\mathcal{M}_H$ as follows. Consider the functor

\[
\mathcal{A} : H\mathcal{M}_H \rightarrow H\mathcal{M}
\]

sending a bimodule $B$ to the $H$-module $B$ with action

\[
h \otimes b \mapsto h(1)_1.bS(h(2)) - \gamma^{-1} = \gamma^{-1}(h(2,1))h(1)_1.bS(h(2,2)),
\]

for $h \in H$, $b \in B$. Then there is a natural isomorphism $\varphi$ making the diagram

\[
\begin{tikzcd}
H\mathcal{M}_H \ar[r, \mathcal{Y}^l] \ar[d, \mathcal{A}] & H\mathcal{M}_H \ar[d, \mathcal{A}] \\
H\mathcal{M} \ar[r, \varphi] & H\mathcal{M}
\end{tikzcd}
\]

commute.

---

The coassociativity diagram for the coalgebra $H$ in the non-strict category $\mu\mathcal{M}_H$ is precisely the quasi-coassociativity condition (3.3) on the quasi-Hopf algebra $H$.

The full meaning of (3.52) is that $\lambda^l$ is a ‘coinvariant’ of the coaction $\rho^r$. This statement is made precise using the so-called fundamental theorem of quasi-Hopf bimodules, cf. [HN2, Sec. 3], which states that there is a monoidal adjoint equivalence $(\mu\mathcal{M}_H)^{\mathcal{Y}^r} \cong \mu\mathcal{M}$. The left $H$-module of coinvariants of a $\mathcal{Y}^r$-comodule $B$ is then defined as the image of $B$ under the equivalence. By [HN2, Cor. 3.9], (3.52) is an equivalent characterization of the coinvariants of $\mathcal{Y}^r H$. 
To see this, let us also specialise the comonad $\mathbb{Z}^4$ from [Sh3] to the case $\mathcal{C} = H\mathcal{M}$. We choose the realisation such that the objects $\mathbb{Z}^4(V)$ are given by the underlying vector space $H \otimes V$ with actions
\[
h^4(a \otimes v) = h_{(1,1)} a S(h_{(2)}) \otimes h_{(1,2)} v.
\]
For later use, let us record that the unit of $\mathbb{Z}^4$ is given by the coevaluation element,
\[
u^4 = \beta.
\]
We then get the following explicit form of $\varphi$.

**Proposition 3.3.** The family of maps
\[
\varphi_B : \mathcal{A}Y^h(B) \to \mathbb{Z}^4 \mathcal{A}(B),
\]
\[
h \otimes b \mapsto \gamma^{-1}(x_3(1)X_2(1)Y_1) x_1 X_1(h_{(2)}) f_1^{-1} S(X_3 Y_3)
\]
\[
\otimes x_2 X_1(b).f_2^{-1} S(x_3(2)X_2(2)Y_2),
\]
for $B \in H\mathcal{M}_H$ defines a natural isomorphism as in (3.56).

**Proof.** That the above map is natural in $B \in H\mathcal{M}_H$ is immediate, and proving that it intertwines the corresponding $H$-actions is a straightforward calculation. For convenience we state that the action on $\mathcal{A}Y^h(B)$ is
\[
h \otimes (a \otimes b) \mapsto h_{(1,1)} a S(h_{(2)}) \gamma^{-1} (1) \otimes h_{(1,2)} b S(h_{(2)}) \gamma^{-1} (2)
\]
and the action on $\mathbb{Z}^4 \mathcal{A}(B)$ is
\[
h \otimes (a \otimes b) \mapsto h_{(1,1)} a S(h_{(2)}) \otimes h_{(1,2,1)} b S(h_{(1,2,2)}) \gamma^{-1}
\]
for $a, h \in H$, $b \in B$. Here the dot denotes the action on the bimodule $B$.

Finally, the inverse of $\varphi_B$ can be read off directly from the explicit expression in (3.59). \hfill \Box

4. Main Theorem

We are now ready to state our two main theorems, which are Theorems 4.1 and 4.4 below.

4.1. Left and right monadic cointegrals.

**Theorem 4.1.** Let $H$ be a finite-dimensional quasi-Hopf algebra with modulus $\gamma$.

(1) Define the linear map
\[
(\gamma)_{\text{mon}} : H^* \to H^*, \quad f_{\text{mon}} = \langle f \mid S(\beta)^{-1}(\xi) \rangle,
\]
where $\xi = (\text{id} \otimes \gamma)(f^{-1})$. Then $\lambda_{\text{mon}}$ is a right monadic cointegral if and only if $\lambda \in H^*$ is a right cointegral.

(2) Define the linear map
\[
\text{mon}(\gamma) : H^* \to H^*, \quad \text{mon} f = \langle f \mid S^{-2}(\beta)^{-1}(\xi) \rangle,
\]
where $\xi = \xi^{\text{cop}} = (S^{-1} \otimes \gamma^{-1})(f^{-1})$. Then $\lambda_{\text{mon}}$ is a left monadic cointegral if and only if $\lambda \in H^*$ is a left cointegral,
Let us explain the main ideas in the proof. First, we specialise the equivalent characterisation of (right) monadic cointegrals from Lemma 2.15 to the quasi-Hopf setting. The resulting equation resembles the right cointegral equation (3.53) we encountered in our discussion of quasi-Hopf algebras. Indeed, we find a nice relationship between the categorical coaction $R_D$ from (2.74) and the left coaction from [BC], cf. Proposition 4.2 below.

Using the relation between these two coactions we then show that the map (4.1) sends a right cointegral to a right monadic cointegral via a direct calculation. This establishes Part 1. Part 2 will then be inferred from Part 1 using the isomorphism $A_2 \cong A_3$.

The details follow below, with some technical steps deferred to Appendix A.2.

**Relation to quasi-Hopf cointegrals.** Let $C = H \mathcal{M}$ and recall our realization of the Hopf comonad $\mathcal{Z}_4$ from (3.57) and (3.58). In this setting, the distinguished invertible object of $C$ is $\gamma^\vee$, and we can rewrite Equation (2.77) on right monadic cointegrals as the linear equation

$$R_{\gamma^\vee} \circ \lambda = \beta \otimes_k \lambda, \quad (4.3)$$

where we identified morphisms from $1$ to $H$ with elements in $H$. We will show that this equation is equivalent to the right cointegral equation (3.53),

$$\rho^l(\lambda^r) = \gamma(x_3)x_1 \otimes \lambda^r \cdot x_2, \quad (4.4)$$

where $\lambda^r \in H^*$ satisfies $\lambda = (\lambda^r)^\mathrm{mon}$ with the $(-)^\mathrm{mon}$ defined in (4.1).

We will first relate the categorical coaction $R_{\gamma^\vee}$ and the coaction $\rho^l$. To this end, recall the functor $\mathcal{A} : H \mathcal{M}_H \to H \mathcal{M}$ from (3.54). One can check that in our realisation of the central Hopf monad we have the equality $\mathcal{A}(\gamma^\vee) = A_2(\gamma^\vee)$ of $H$-modules.

**Proposition 4.2.** With the natural isomorphism $\varphi : \mathcal{A} \mathcal{Y}^l \Rightarrow Z^4 \mathcal{A}$ from (3.56) and the left $H$-coaction $\rho^l : H^\vee \to \mathcal{Y}^l(H^\vee)$ from Section 3.5 we have that

$$R_{\gamma^\vee} = \left[\mathcal{A}(H^\vee) \xrightarrow{\mathcal{A}(\rho^l)} \mathcal{A} \mathcal{Y}^l(H^\vee) \xrightarrow{\varphi_{H^\vee}} Z^4 \mathcal{A}(H^\vee)\right]. \quad (4.5)$$

The proof of this proposition has been relegated to Appendix A.1.

Note that $\mathcal{A}$ does not do anything on morphisms; in particular, the linear maps $\mathcal{A}(\rho^l)$ and $\rho^l$ are identical. Then this proposition together with (4.3), says that $\lambda$ is a right monadic cointegral if and only if it satisfies

$$(\varphi_{H^\vee} \circ \rho^l)(\lambda) = \beta \otimes_k \lambda. \quad (4.6)$$

In Appendix A.2 we prove that this is equivalent to the right cointegral equation (4.4) using the map (4.1). This finishes the proof of the first part of Theorem 4.1.

The second part is the same as the first part for the coopposite quasi-Hopf algebra, but we follow a more direct approach using the isomorphism $A_2 \cong A_3$ from Proposition 2.4 and Proposition 3.1, see Appendix A.3.

Namely, in the appendix, we show that the diagram

$$\begin{array}{ccc}
\int_{\mathcal{C}}^r & \xrightarrow{(4.1)} & \int_{\mathcal{C}}^{r,\mathrm{mon}} \\
\downarrow^{(\ast)} & & \downarrow^{(\kappa_{2,3})_D \circ ?} \\
\int_{\mathcal{C}}^l & \xrightarrow{(4.2)} & \int_{\mathcal{C}}^{l,\mathrm{mon}}
\end{array}$$

(4.7)
commutes, where \((\ast)\) is, up to a non-zero factor, the isomorphism between left and right cointegrals \((3.49)\), and \(\kappa_{2,3}: A_2 \Rightarrow A_3\) is the Hopf monad isomorphism from \((3.20)\).

### 4.2. Left and right \(D\)-symmetrised monadic cointegrals

To prove an analogous result of Theorem 4.1 for \(D\)-symmetrised monadic cointegrals, we first need to recall the appropriate version of cointegrals on the quasi-Hopf side.

#### Definition 4.3

Let \((H, g)\) be a finite-dimensional pivotal quasi-Hopf algebra.

1. The right \(\gamma\)-symmetrised cointegrals \(\hat{\lambda}^r \in H^*\) are the solutions of \((\hat{\lambda}^r \otimes \text{id})(q^R \Delta(h) p^R) = \gamma(X_1) \hat{\lambda}^r(X_2 h) \cdot g^{-1} S(X_3)\) for all \(h \in H\). \((4.8)\)

2. The left \(\gamma\)-symmetrised cointegrals \(\hat{\lambda}^l\) are the solutions of \((\text{id} \otimes \hat{\lambda}^l)(q^L \Delta(h) p^L) = \gamma(x_3) \hat{\lambda}^l(x_2 h) \cdot g S^{-1}(x_1)\) for all \(h \in H\). \((4.9)\)

An equivalent definition has been given in [SS, Sec. 6.4], and the above equations also appear in [BGR, Lem. 3.7]. In that lemma it is shown that, given a right (resp. left) cointegral \(\lambda^r\) (resp. \(\lambda^l\)) for \(H\), the \(\gamma\)-symmetrised version can be expressed as

\[
\hat{\lambda}^r = \lambda^r \leftarrow ug \quad \text{(resp. } \hat{\lambda}^l = \lambda^l \leftarrow \text{cop} g^{-1}) \]

where \(u\) was defined in \((3.48)\). (This is the actual definition of \(\gamma\)-symmetrised cointegrals given in [SS].) From \((4.10)\) it is clear that non-zero \(\gamma\)-symmetrised cointegrals exist and are unique up to scalars, and from \((3.47)\) one easily verifies

\[
\hat{\lambda}^r(ab) = \hat{\lambda}^r((b \leftarrow \gamma)a) \quad \text{and} \quad \hat{\lambda}^l(ab) = \hat{\lambda}^l((\gamma \rightarrow b)a) \quad \text{(4.11)}
\]

for all \(a, b \in H\), see [BGR, Eqn. (3.44)].

We denote the one-dimensional spaces of right and left \(\gamma\)-symmetrised cointegral by

\[
\int_H^{r,\gamma} \quad \text{and} \quad \int_H^{l,\gamma}. \quad \text{(4.12)}
\]

Now we can extend Theorem 4.1 to the pivotal case.

#### Theorem 4.4

Let \((H, g)\) be a pivotal quasi-Hopf algebra.

- Consider the linear map

\[
(\ ?)^{\gamma\text{-sym}} : H^* \to H^*, \quad f^{\gamma\text{-sym}} = \langle f \mid S(\beta) ? S^{-1}(\vartheta) \rangle, \quad \text{(4.13)}
\]

where \(\vartheta = (\gamma^{-1} \otimes S^{-1})(p^L)\). Then \(\lambda^{\gamma\text{-sym}}\) is a right \(D\)-symmetrised monadic cointegral if and only if \(\lambda \in H^*\) is a right \(\gamma\)-symmetrised cointegral.

- Consider the linear map

\[
\gamma^{\text{sym}}(\ ?) : H^* \to H^*, \quad \gamma^{\text{sym}} f = \langle f \mid \beta ? S(\hat{\vartheta}) \rangle, \quad \text{(4.14)}
\]

where \(\hat{\vartheta} = \vartheta^{\text{cop}} = (S \otimes \gamma^{-1})(p^R)\). Then \(\gamma^{\text{sym}}\lambda\) is a left \(D\)-symmetrised monadic cointegral if and only if \(\lambda \in H^*\) is a left \(\gamma\)-symmetrised cointegral.

The proof is via monad isomorphisms as in the second part of Theorem 4.1 and can be found in Appendix A.2.
5. Examples

Here we give examples of quasi-Hopf algebras and their cointegrals. Our examples are mostly non-unimodular; some unimodular examples can be found e.g. in [BC, Ex. 3.7] and [BGR]. All examples below are considered over the complex numbers $\mathbb{C}$.

Example 5.1. This is example 2.2 and 3.4 in [BC]. Consider the unital $\mathbb{C}$-algebra generated by $g$ and $x$, obeying relations $g^2 = 1$, $x^4 = 0$ and $gxg^{-1} = -x$. Define two orthogonal idempotents $p_\pm = \frac{1}{2}(1 \pm g)$. The comultiplication and counit are given on generators by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1$$
$$\Delta(x) = x \otimes (p_+ \pm ip_-) + 1 \otimes p_+ x + g \otimes p_- x, \quad \varepsilon(x) = 0$$

(5.1)

and with $\Phi = \Psi = 1 \otimes 1 - 2p_- \otimes p_- p_-$, we obtain two 8-dimensional quasi-bialgebras, denoted $H_{\pm}(8)$, both of which admit an antipode $S(g) = g$, $S(x) = -x(p_+ \pm ip_-)$, and with evaluation and coevaluation element $\alpha = g$ and $\beta = 1$, respectively.

A right integral is given by $c' = x^3 p_+$, while $c'' = p_+ x^3 = x^3 p_-$ is a left integral. One computes that the modulus of $H_{\pm}(8)$ is $\gamma(x) = 0$, $\gamma(g) = -1$. Thus, the quasi-Hopf algebra in this example is not unimodular. Note also that $\gamma = \gamma^{-1}$.

Finally, we want to give the cointegrals of $H_{\pm}(8)$. Basis elements of $H_{\pm}(8)$ are of the form $B_{m,n} = g^m x^n$, $0 \leq m \leq 1$, $0 \leq n \leq 3$, and we denote the element dual to $B_{m,n}$ by $B_{m,n}^*$. The Drinfeld twist is given by

$$f_{\pm 1} = 2p_+ \otimes p_+ - g \otimes g$$

(5.2)

and so we obtain the right monadic cointegral

$$\lambda_{\text{mon}}^r = B_{0,3}^* \pm iB_{1,3}^*.$$  

(5.3)

Concretely, one solves (3.26) and finds that its solution space is one-dimensional, so its elements automatically are morphisms in $H_\mathcal{M}$. With the Hopf monad isomorphism $\kappa_{2,3}$ from Proposition 3.1 it is then easy to show that

$$\lambda_{\text{mon}}^l = B_{1,3}^*$$

(5.4)

is a non-zero left monadic cointegral. Using the isomorphisms from Theorem 4.1, we obtain the ‘classical’ right and left cointegrals,

$$\lambda^r = B_{0,3}^* \mp iB_{1,3}^* \quad \text{and} \quad \lambda^l = B_{0,3}^*.$$  

(5.5)

The same expression for the left cointegral was also derived in [BC, Ex. 3.9].

Note that $H_{\pm}(8)$ is not pivotal. Indeed, one easily checks that already for the generator $x$, $S^2(x) = h = hx$ implies $h = 0$, so that $S^2$ is not inner.

Example 5.2. Fix $N \in \mathbb{N}$, $\beta \in \mathbb{C}$ satisfying $\beta^4 = (-1)^N$. This example is based on the symplectic fermion ribbon quasi-Hopf algebra $Q(N, \beta)$ from [FGR2, Sec. 3], $Q(N, \beta)$ is factorisable, so in particular unimodular and its cointegrals were already discussed in [BGR, Sec. 5]. We now restrict to the sub-quasi-Hopf algebra $H(N, \beta) \subset Q(N, \beta)$, which is defined as follows. As a unital $\mathbb{C}$-algebra, it is generated by $K, f_i$, $1 \leq i \leq N$, with defining relations

$$\{f_i, K\} = 0, \quad \{f_i, f_j\} = 0, \quad K^4 = 1,$$

(5.6)

where $\{a, b\} = ab + ba$ is the anticommutator.
A PBW-type basis of \( H(N, \beta) \) is\(^ {16} \)
\[
B_{j,l} = \left( \prod_{i=1}^{N} f_i^{j_i} \right) K^l \quad | \quad j \in \{0, 1\}^N, \ 0 \leq l \leq 3 \quad . 
\]
(5.7)

Elements in the corresponding dual basis are simply decorated by an asterisk.

Using the orthogonal central idempotents \( e_0 = \frac{1}{2}(1+K^2) \) and \( e_1 = 1 - e_0 \), and setting \( \omega = (e_0 + ie_1)K \), the comultiplication and the counit are
\[
\Delta(K) = K \otimes K - (1 + (-1)^N)e_1 K \otimes e_1 K , \quad \varepsilon(K) = 1 , \\
\Delta(f_i) = f_i \otimes 1 + \omega \otimes f_i , \quad \varepsilon(f_i) = 0 . \quad (5.8)
\]
The coassociator and its inverse are
\[
\Phi_{\pm} = 1 \otimes 1 \otimes 1 + e_1 \otimes e_1 \otimes \{e_0(K^N - 1) + e_1(\beta_{\pm} - 1)\} , \quad (5.9)
\]
where \( \beta_{\pm} = e_0 + \beta^2(\pm iK)^N e_1 \). The evaluation and coevaluation elements are \( \alpha = 1 \), \( \beta = \beta_+ \), and the antipode is
\[
S(K) = K^{(-1)N}, \quad S(f_i) = f_i(e_0 + (-1)^N i e_1)K . \quad (5.10)
\]
Then, with \( X = 1 + K + K^2 + K^3 \), we see that
\[
c^l = X \prod_{i=1}^{N} f_i , \quad c^r = \left( \prod_{i=1}^{N} f_i \right) X \quad (5.11)
\]
are a left and a right integral, respectively. From this, one easily computes that the modulus is the algebra homomorphism given on generators by
\[
\gamma(K) = (-1)^N, \quad \gamma(f_i) = 0 . \quad (5.12)
\]
In particular, \( H(N, \beta) \) is unimodular if and only if \( N \) is even. Note that just as in the previous example \( \gamma = \gamma^{-1} \).

Now we describe the cointegrals of \( H(N, \beta) \). The Drinfeld twist and its inverse are given by
\[
f_{\pm} = e_0 \otimes 1 + e_1 \otimes e_0 K^N + e_1 \beta_{\mp} \otimes e_1 , \quad (5.13)
\]
see also [FGR2, (3.35)]. We again find the right monadic cointegral via (3.26) and then obtain the left monadic cointegral via the isomorphism of Hopf monads from Proposition 3.1:
\[
\lambda^{r, \text{mon}} = B^*_{N,0} \quad \text{and} \quad \lambda^{l, \text{mon}} = \delta_{N,\text{even}} B^*_{N,0} + \delta_{N,\text{odd}} (B^*_{N,1} - i B^*_{N,3}) , \quad (5.14)
\]
where \( \vec{N} \) is the multi-index consisting only of 1s. In particular, the left and the right monadic cointegral do not agree unless \( \vec{N} \) is even.

With our main theorem, we obtain the right and the left quasi-Hopf cointegral
\[
\lambda^{r} = a^r_{+} B^*_{N,0} + a^r_{-} B^*_{N,2} - \delta_{N,\text{odd}}(B^*_{N,1} - B^*_{N,3}) , \quad \\
\lambda^{l} = a^l_{+} B^*_{N,0} + a^l_{-} B^*_{N,2} - \delta_{N,\text{odd}}(B^*_{N,1} + B^*_{N,3}) , \quad (5.15)
\]
where the coefficients are
\[
a^r_{\pm} = \delta_{N,\text{even}}(1 \pm \beta^2) + \delta_{N,\text{odd}}\beta^2 i \quad \text{and} \quad a^l_{\pm} = \delta_{N,\text{even}} \pm \beta^2 . \quad (5.16)
\]
\(^{16}\)We use the convention \( \prod_{i=1}^{n} a_i = a_1 \cdot \prod_{i=2}^{n} a_i \) for the non-commutative product.
Example 5.3. Fix an odd integer \( t \), let \( p \geq 2 \) be an integer, and set \( q = e^{i\pi/p} \). Here we consider the quasi Hopf modification \( \tilde{U}_q(\sl_2) \) of the restricted quantum group \( \tilde{U}_q(\sl_2) \) from [CGR, Sec. 4]. This quasi-Hopf algebra is factorisable, and as in the previous example we will consider a non-unimodular sub-quasi-Hopf algebra \( U^- \). As a \( \mathbb{C} \)-algebra, it is generated by \( F \) and \( K \), with defining relations

\[
F^p = 0, \quad K^{2p} = 1, \quad \text{and} \quad KFK^{-1} = q^{-2}F,
\]

and a natural choice of basis of \( U^- \) is therefore

\[
\{ B_{m,n} = F^m K^n \mid 0 \leq m \leq p - 1, \quad 0 \leq n \leq 2p - 1 \}.
\]

Using the two central idempotents \( e_0 = \frac{1}{2}(1 + K^p) \) and \( e_1 = 1 - e_0 \), the comultiplication and the counit are

\[
\Delta_t(F) = F \otimes 1 + (e_0 + q^{-t}e_1)K^{-1} \otimes F, \quad \varepsilon(F) = 0,
\]

\[
\Delta_t(K) = K \otimes K, \quad \varepsilon(K) = 1.
\]

The coassociator and the antipode are given by

\[
\Phi_t = 1 \otimes 1 \otimes 1 + e_1 \otimes e_1 \otimes (K^{-t} - 1)
\]

\[
S_t(F) = -KF(e_0 + q^{-t}e_1), \quad S_t(K) = K^{-1},
\]

and finally, evaluation and coevaluation element are

\[
\alpha = 1, \quad \beta_t = e_0 + K^{-t}e_1
\]

respectively.

From these data, one finds that the Drinfeld twist and its inverse are

\[
f^{\pm 1} = e_0 \otimes 1 + e_1 \otimes e_0K^{\mp t} + e_1K^{\pm t} \otimes e_1,
\]

see (3.36). This quasi-Hopf algebra is pivotal, and the pivot we choose is

\[
g_t = e_0K - e_1K^{t+1}.
\]

Set \( X = \sum_{i=0}^{2p-1} K^i \). Then one can see that \( c^r = F^{p-1}X \) and \( c^l = XF^{p-1} \) are a right and a left integral for \( U^- \), respectively. From

\[
c^lF = 0, \quad c^lK = q^{-2}c^l,
\]

\[
Fc^r = 0, \quad Kc^r = q^2c^r
\]

we can see that \( U^- \) is non-unimodular. The modulus is

\[
\gamma(F) = 0, \quad \gamma(K) = q^{-2}.
\]

The order of \( \gamma \) is \( p \), and in particular \( \gamma \neq \gamma^{-1} \) if \( p > 2 \).

Using (3.26) one verifies that the space of right monadic cointegrals is

\[
\mathcal{C}B_{p-1,0}^*.
\]

The left monadic cointegrals can then be found via the isomorphism from Proposition 3.1. Normalizing the result, we obtain

\[
\lambda^{l,\text{mon}} = (1 + q^{-t(p-1)})B_{p-1,p-1}^* + (1 - q^{-t(p-1)})B_{p-1,2p-1}^*.
\]

\[17\]Different values of \( t \) lead to twist-equivalent quasi-bialgebras, cf. [CGR, Thm. 4.1]. Explicitly, with \( J(t', t) = e_0 \otimes 1 + e_1 \otimes K^{(t'-1)/2} \) we get e.g. \( J(t', t) \cdot \Delta(h) \cdot J(t', t)^{-1} = \Delta_p(h) \) for all \( h \in U^- \).
Also from Proposition 3.1 we obtain the right $D$-symmetrised monadic cointegral
\[ \lambda^{r,D\text{-sym}} = B_{p-1,p-1}^* + B_{p-2p-1}^* + q^{2t}(B_{p-1,p-t-1}^* - B_{p-1,2p-t-1}^*) \]  
(5.28)

and the left $D$-symmetrised monadic cointegral
\[ \lambda^{l,D\text{-sym}} = B_{p-1,0}^* + B_{p-1,p}^* + q^{-t(p+1)}(B_{p-1,1-t}^* - B_{p-1,2p-t}^*) . \]  
(5.29)

The ‘classical’ $[HN2]$-cointegrals of $U^-$ can now be obtained using Theorem 4.1. The right and the left cointegral are found to be
\[ \lambda^r = B_{p-1,0}^* + B_{p-1,p}^* + B_{p-1,t}^* - B_{p-1,p+t}^* \]  
(5.30)

and
\[ \lambda^l = B_{p-1,p-1}^* + B_{p-2p-1}^* + q^{-t(p-1)}(B_{p-1,1-t}^* - B_{p-1,2p-t-1}^*) , \]  
(5.31)

respectively. Finally, we give the $\gamma$-symmetrised cointegrals of $U^-$ using the characterization (4.10). The right $\gamma$-symmetrised cointegral is given by
\[ \hat{\lambda}^r = B_{p-1,0}^* \]  
(5.32)

and the left $\gamma$-symmetrised cointegral is
\[ \hat{\lambda}^l = (1 + q^{-t(p-1)})B_{p-1,0}^* + (1 - q^{-t(p-1)})B_{p-1,p}^* . \]  
(5.33)

For $p = 2$ and $t = 1$, $U^-$ is, as a quasi-Hopf algebra, isomorphic to
\[ H(N = 1, \beta = \exp(i\pi/4)) \]  
(5.34)

from the previous example, by mapping generators according to
\[ F \mapsto if, \ K \mapsto K, \]  
(5.35)

cf. [CGR, Rem. 4.3(2)]. Note that, under this isomorphism, the cointegrals of $U^-$ agree with those of $H(N = 1, \beta = \exp(i\pi/4))$.

6. Cointegrals for the coend in the braided case

In a braided category $\mathcal{C}$ there exist notions of integrals and cointegrals for Hopf algebras internal to $\mathcal{C}$. If $\mathcal{C}$ is in addition finite tensor, then the coend $L = \int X \in \mathcal{C} X^\vee \otimes X$ is an example of such a Hopf algebra [LM, Ly1]. In this section we relate left and right integrals for $L$ to right monadic cointegrals and consider quasi-triangular quasi-Hopf algebras as an example.

In this section, let $\mathcal{C}$ be a braided finite tensor category.

6.1. Integrals and cointegrals for Hopf algebras in $\mathcal{C}$. Let $A$ be a Hopf algebra in $\mathcal{C}$ with invertible antipode, see e.g. [KL] or [FGR1, Sec. 2.2]. Then the notions of (left/right) integrals and (left/right) cointegrals are well-defined, see [KL, Prop. 4.2.4]. We repeat the definition of a left integral for $A$. It consists of an invertible object $\text{Int}_A$, the object of integrals, and a morphism $\Lambda_A : \text{Int} \rightarrow A$ making the diagram

\[
\begin{array}{ccc}
A \otimes \text{Int}_A & \xrightarrow{\text{id} \otimes \Lambda_A} & A \otimes A \\
\varepsilon \otimes \text{id} & & \downarrow m \\
1 \otimes \text{Int}_A & \sim & \text{Int}_A \\
& \Lambda_A & \rightarrow A
\end{array}
\]  
(6.1)
commute. Here, \( m \) and \( \varepsilon \) are multiplication and counit of the Hopf algebra \( A \), respectively. Right integrals are defined similarly (with the same object \( \text{Int}_A \)). It is known that non-zero (left/right) integrals \( \Lambda_A \) exists and are uniquely determined up to scalar [KL, Prop. 4.2.4]. Note that the above diagram is just the statement that a left integral for \( A \) is a morphism \( \Lambda_A : \text{Int}_A \to A \) of left \( A \)-modules, where the \( A \)-actions are given by the left and right side of the diagram (6.1), respectively.

As remarked in [BV1, Ex. 3.10], tensoring with a Hopf algebra with invertible antipode in a braided category yields a Hopf monad. The category of modules over the Hopf algebra is then the same as the category of modules over the corresponding Hopf monad.

6.2. Central Hopf monad via the coend in braided case. In the braided setting, the coend \( L = \int^{X \in C} X^\vee \otimes X \) with universal dinatural transformation \( j \) becomes a Hopf algebra [LM, Ly1], see also [FS, FGR1] for a review. It is easy to see that \( A_2 \) is in fact isomorphic to the Hopf monad obtained by tensoring with \( L \). The isomorphism \( \xi_V : A_2(V) \to L \otimes V \) we choose is obtained via

\[
\xi_V \circ \iota_2(V)_X = \left[ X^\vee (VX) \xrightarrow{\text{id} \otimes c_{X^\vee \otimes X}} X^\vee (XV) \xrightarrow{\sim} (X^\vee X) V \xrightarrow{j_X \otimes \text{id}} LV \right]. \tag{6.2}
\]

The inverse of the braiding appears to make \( \xi \) an isomorphism of bimonads, with the bimonad structure on \( L \otimes ? \) inherited from the bialgebra structure on \( L \) as defined in [FGR1, Sec. 3.3]. In the same way, \( ? \otimes L \) becomes a bimonad and we get a bimonad isomorphism \( \zeta : A_2 \Rightarrow (? \otimes L) \) via

\[
\zeta_V \circ \iota_2(V)_X = \left[ X^\vee (VX) \xrightarrow{\sim} (X^\vee V) X \xrightarrow{c_{X^\vee \otimes X}^{-1} \otimes \text{id}} (X^\vee X) V \xrightarrow{\text{id} \otimes j_X} V \mathcal{L} \right]. \tag{6.3}
\]

Again the inverse braiding is required to make \( \zeta \) a bimonad morphism.

In a finite tensor category, an invertible object has isomorphic left and right duals, and so in particular there is an up to scalars unique isomorphism \( D^\vee \xrightarrow{\sim} \vee D \). This fact is used in formulating the following proposition.

**Proposition 6.1.** Let \( C \) be a braided finite tensor category.

1. The distinguished invertible object is dual to the object of integrals for \( L \),
   \[
   D \cong (\text{Int}_L)^\vee. \tag{6.4}
   \]

2. Let \( \Lambda_L : \vee D \to L \) be non-zero. Then
   \[
   \Lambda_L \text{ is a } \begin{cases} \text{left integral for } L \text{ in the sense of (6.1), resp.} \\ \text{right integral for } L \end{cases}
   \]
   if and only if
   \[
   \lambda := \begin{cases} \begin{array}{l} 1 \\ \xrightarrow{\text{coev}_D} \vee D \otimes D \xrightarrow{\Lambda_L \otimes \text{id}} L \otimes D \xrightarrow{\xi_D^{-1}} A_2(D) \end{array} \end{cases}, \text{ resp.} \tag{6.5} \]
   is a non-zero right monadic cointegral of \( C \).

The first statement was already observed in [Sh1, Thm. 6.8].
Proof. We will only treat the case of left integrals for \( \mathcal{L} \) explicitly, the case of right integrals can be shown analogously.

Let us abbreviate \( X = \text{Int} \mathcal{L} \), so that by existence of left cointegrals we can find a non-zero morphism \( \Lambda_\mathcal{L} : X \to \mathcal{L} \) which satisfies (6.1). We now define \( \lambda \) as in part (2), but with \( X \) instead of \( \forall D \):

\[
\lambda := \left[ 1 \xleftarrow{\coev_X} X \otimes X^\vee \xrightarrow{\Lambda_\mathcal{L} \otimes \text{id}} \mathcal{L} \otimes X^\vee \xrightarrow{\xi_{X^\vee}^{-1}} A_2 \left( X^\vee \right) \right]. \tag{6.6}
\]

Note that \( \lambda \) is non-zero, too.

The somewhat lengthy computation below will establish that \( \lambda \) from (6.6) is an \( A_2 \)-intertwiner. By [Sh3, Lem. 4.1] and Corollary 2.13 the distinguished invertible object \( D \) is the unique (up to unique isomorphism) invertible object such that the space of \( A_2 \)-intertwiners from \( 1 \) to \( A_2 (D) \) is non-zero. Thus we must have \( X^\vee \cong D \), proving part (1). Together with part (1), the fact that \( \lambda \) is the unique (up to unique isomorphism) invertible object such that the space of \( A_2 \)-intertwiners from \( 1 \) to \( A_2 (D) \) is non-zero. Thus we must have \( X^\vee \cong D \), proving part (1). The direction \( \Leftarrow \) of part (2) can be verified by an analogous computation, where a right monadic cointegral \( \lambda \) gets mapped to a left integral of \( \mathcal{L} \) via

\[
\Lambda_\mathcal{L} := \left[ \forall D \xrightarrow{\sim} 1 \otimes \forall D \xrightarrow{\lambda \otimes \text{id}} A_2 (D) \otimes \forall D \xrightarrow{\xi_D \otimes \text{id}} (\mathcal{L}D) \otimes \forall D \xrightarrow{\sim} \mathcal{L}(D\forall D) \right]
\]

Note that this is indeed inverse to (6.5).

Let us now turn to the verification that \( \lambda \) in (6.6) is indeed an \( A_2 \)-intertwiner. Note that since \( \xi \) is an isomorphism of bimonoids, it satisfies

\[
\left[ \mathcal{L} (\mathcal{L} V) \xrightarrow{\sim} (\mathcal{L} \mathcal{L} V) \xrightarrow{m \otimes \text{id}_V} \mathcal{L} V \xrightarrow{\xi_{V}^{-1}} A_2 (V) \right] = \left[ \mathcal{L} (\mathcal{L} V) \xrightarrow{\xi_{V}^{-1}} A_2 (L V) \xrightarrow{A_2 (\xi_{V}^{-1})} (A_2)^{2} (V) \xrightarrow{\mu_2 (V)} A_2 (V) \right], \tag{6.8}
\]

and

\[
\left[ \mathcal{L} 1 \xrightarrow{\xi_{1}^{-1}} A_2 (1) \xrightarrow{\xi_{2}} 1 \right] = \left[ \mathcal{L} 1 \xrightarrow{\varepsilon \otimes \text{id}_1} 11 \xrightarrow{\sim} 1 \right], \tag{6.9}
\]

where \( m \) and \( \varepsilon \) are the multiplication and the counit of \( \mathcal{L} \).

For the next calculation, let us explicitly denote components of the left unitor and the associator by

\[
l_V : 1V \to V \quad \text{and} \quad \alpha_{U,V,W} : U(VW) \to (UV)W, \tag{6.10}
\]

respectively, for \( U, V, W \in \mathcal{L} \). Then

\[
\mu_2 (X^\vee) \circ A_2 (\lambda)
\]

\[
\overset{(6.6)}{=} \mu_2 (X^\vee) \circ A_2 (\xi_{X^\vee}^{-1} \circ (\Lambda_\mathcal{L} \otimes \text{id}_{X^\vee}) \circ \coev_X)
\]

\[
\overset{(6.8)}{=} \xi_{X^\vee}^{-1} \circ (m \otimes \text{id}_{X^\vee}) \circ \alpha_{\mathcal{L},\mathcal{L},X^\vee} \circ \xi_{\mathcal{L}\otimes X^\vee} \circ A_2 ((\Lambda_\mathcal{L} \otimes \text{id}_{X^\vee}) \circ \coev_X)
\]

\[
\overset{\xi_{\mathcal{L}}}{}{=} \xi_{X^\vee}^{-1} \circ (m \otimes \text{id}_{X^\vee}) \circ \alpha_{\mathcal{L},\mathcal{L},X^\vee} \circ (\text{id}_L \otimes (\Lambda_\mathcal{L} \otimes \text{id}_{X^\vee})) \circ (\text{id}_L \otimes \coev_X) \circ \xi_1
\]

\[
\overset{\alpha}{}{=} \xi_{X^\vee}^{-1} \circ ((m \circ (\text{id}_L \otimes \Lambda_\mathcal{L})) \otimes \text{id}_{X^\vee}) \circ \alpha_{\mathcal{L},\mathcal{L},X^\vee} \circ (\text{id}_L \otimes \coev_X) \circ \xi_1
\]
for the bimonad isomorphisms $\xi$ by Proposition 6.1, the object of integrals of $L$
Remark 6.2. Let $\lambda$ show that left and right integrals for $L$
coherence of braided monoidal categories that details in the same notation as used here. We denote the multiplicative inverse of the $R$
Quasi-triangular quasi-Hopf algebras.
result has also been shown by different means in [Sh1, Thm. 6.9].
\[ (6.1) \]
\[ (6.7) \]
\[ (6.9) \]
\[ (6.6) \]
\[ (6.11) \]
\[ (6.12) \]
for the bimonad isomorphisms $\xi$ and $\zeta$ from (6.2) and (6.3). Now one can show that left and right integrals for $L$ agree. Indeed, the composition
\[ \{\text{right } L\text{-integrals}\} \xrightarrow{(6.5)} C_{A_2}(1, A_2(1)) \xrightarrow{(6.7)} \{\text{left } L\text{-integrals}\} \]
of isomorphisms is proportional to the identity on the one-dimensional subspace of right $L$-integrals of $C(1, L)$. Therefore, a right integral for $L$ is also left and vice versa. This result has also been shown by different means in [Sh1, Thm. 6.9].

6.3. Quasi-triangular quasi-Hopf algebras. Let $H$ be a finite-dimensional quasi-triangular quasi-Hopf algebra with universal $R$-matrix $R$, see e.g. [FGR1, Sec. 6] for details in the same notation as used here. We denote the multiplicative inverse of the $R$-matrix by $\overline{R}$. The category $C = _H^H \mathcal{M}$ is a braided finite tensor category.
The coend $L$ can be realised by $H^*$ with the coadjoint action, see e.g. [FGR1, Sec. 7], and with our realization of the Hopf monad $A_2$ as in Section 3.3 we get the following formula for the Hopf monad isomorphism $A_2 \cong L$ from (6.2).

**Lemma 6.3.** The isomorphism $\xi_V : A_2(V) \to L \otimes V$ from (6.2) is given by
\[ \xi_V(f \otimes v) = \langle f \mid S(X_1)X_2\overline{R}_1 \otimes X_3\overline{R}_2.v \]
for $V \in C$, $f \in H^*$, $v \in V$.

The proof is a straightforward computation.

Next, we give the explicit formulas relating right monadic cointegrals and left integrals for the coend. As usual, we identify linear maps $k \to V$ with elements in $V$.

**Lemma 6.4.** Let $\lambda \in H^*$ be a right monadic cointegral. Then
\[ \Lambda_L := \gamma^{-1}(q_2^RX_3\overline{R}_2)\langle \lambda \mid S(q_1^R(1)X_1)q_1^R(2)X_2\overline{R}_1 \]
is a left integral for the coend $L$. 

\[ \text{nat.} \]

\[ \text{coher.} \]

\[ \text{nat.} \]

\[ (6.6) \]

\[ \text{nat.} \]

\[ \text{coher.} \]

\[ \text{nat.} \]

\[ \text{coher.} \]
The proof amounts to evaluating (6.7) in $\mathcal{HM}$ using Lemma 6.3. We arrive at the following corollary.

**Corollary 6.5.** Let $X^r \in H^*$ be a right cointegral for the quasi-Hopf algebra $H$. Then
\[
\Lambda_L := \gamma^{-1}(q^R_2 X_3 \overline{R}_2 S^{-1}(f_2^{-1}))(X^r | S(q^R_1(1)X_1\beta)q^R_1(2)X_2 \overline{R}_1 S^{-1}(f_1^{-1}))
\] (6.16)
is a left integral for the coend $\mathcal{L}$.

**Proof.** Combining Theorem 4.1 with the previous lemma immediately yields the formula. □

Using Proposition 6.1 (2), one can also write formulas similar to (6.15), resp. (6.16), for the relation between right integrals for $\mathcal{L}$ and right monadic cointegrals, resp. right cointegrals for $H$. We will skip the details.

**Remark 6.6.** Let $H$ be unimodular with right cointegral $X^r$. Observe that then $D = 1$ and $A_2(1) = \mathcal{L}$ as $H$-modules. The relationship (6.15) between integrals for $\mathcal{L}$ and right monadic cointegrals is now particularly simple:
\[
\Lambda_L = \lambda.
\] (6.17)
By Remark 6.2, left and right integrals for $\mathcal{L}$ coincide, and so (6.17) says that the right monadic cointegral and the left/right integral for $\mathcal{L}$ are given by the same linear form on $H$. The relation to right cointegrals for $H$ also simplifies: $\Lambda_L = (X^r | S(\beta))$.

### 7. Application: $SL(2, \mathbb{Z})$-action

#### 7.1. $SL(2, \mathbb{Z})$-action for modular tensor categories

In a braided finite tensor category $\mathcal{C}$, the Hopf algebra $L = \int X X^\vee \otimes X$ admits a Hopf pairing
\[
\omega : L \otimes L \to 1,
\] (7.1)
see [Ly1] for details or [FGR1, Sec. 3.3] for a review. By a *modular tensor category* we mean a ribbon finite tensor category which is *factorisable*, that is, in which the Hopf pairing (7.1) induces an isomorphism $L \cong L^\vee$ of Hopf algebras. Equivalent definitions of factorisability can be found in [Sh2].

Let for the rest of this section $\mathcal{C}$ be a modular tensor category with ribbon twist $\vartheta$.

Since $\mathcal{C}$ is factorisable it is in particular unimodular [KL, Lem. 5.2.8], and the Hopf algebra $L$ has a two-sided integral $\Lambda_L : 1 \to L$ by Remark 6.2, see also [KL, Cor. 5.2.11].

Define the morphism $Q : L \otimes L \to L \otimes L$ by
\[
Q = \begin{pmatrix}
L & L \\
\jmath_X & \jmath_Y
\end{pmatrix},
\] (7.2)
This is related to the Hopf pairing $\omega$ via $\omega = (\epsilon \otimes \epsilon) \circ Q$, denoting by $\epsilon$ the counit of $L$.

Next, define $S, \mathcal{T} \in \text{End}_\mathcal{C}(L)$ by
\[
S = (\epsilon \otimes \text{id}) \circ Q \circ (\text{id} \otimes \Lambda_L), \quad \mathcal{T} \circ \jmath_X = \jmath_X \circ (\text{id} \otimes \vartheta_X).
\] (7.3)
These endomorphisms satisfy
\[(ST)^3 = \lambda S^2 = \lambda S^{-1}_L \tag{7.4}\]
where \(\lambda\) is a non-zero constant and \(S_L\) is the antipode of \(L\) \cite{Ly1}.

Recall that \(SL(2, \mathbb{Z})\) is the group generated by \(S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and \(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) together with the relations
\[(ST)^3 = S^2, \quad S^4 = \text{id}. \tag{7.5}\]

It was shown in \cite{Ly1} that the \(k\)-vector space \(\mathcal{C}(1, L)\) carries a projective \(SL(2, \mathbb{Z})\)-action, given by
\[S.f = S \circ f, \quad T.f = T \circ f \tag{7.6}\]
for \(f : 1 \rightarrow L\), see \cite[Sec. 5]{FGR1} for a review.

Equivalently, one also obtains an action on \(\mathcal{C}(L, 1)\), given by
\[S.f = f \circ S, \quad T.f = f \circ T \tag{7.7}\]
for \(f : L \rightarrow 1\).

### 7.2. \(SL(2, \mathbb{Z})\)-action for factorisable quasi-Hopf algebras

Let now \(H\) be a finite-dimensional factorisable\(^{18}\) ribbon quasi-Hopf algebra with ribbon element \(v\) and \(R\)-matrix \(R\). As mentioned before, \(H\) being factorisable implies that it is unimodular. Thus, by Remark 6.6, the left integral for \(L\) and the right monadic cointegral of \(\mathcal{M}\) are given by the same linear form on \(H\).

The linear injection
\[\alpha Z = \{\alpha z \mid z \in Z(H)\} \rightarrow \mathcal{C}(L, 1), \quad \alpha z \mapsto \delta_{\text{Vect}}(\alpha z) = \langle ? | \alpha z \rangle \tag{7.8}\]
can be shown to be an isomorphism\(^{19}\), and so we get an action \(S_{\alpha Z}\) of the \(S\)-generator on \(\alpha Z\) by setting
\[S_{\alpha Z}(h) = \delta^{-1}_{\text{Vect}}(\langle ? | h \circ S \rangle) = \delta^{-1}_{\text{Vect}}(\langle S(?) | h \rangle), \tag{7.9}\]
for \(h \in \alpha Z\). Since \(S \in \text{End}_C(L)\) and \(L \cong_k H^*\), we can define \(\hat{S} \in \text{End}_k(H)\) via
\[\langle f | \hat{S}(h) \rangle = \langle S(f) | h \rangle \tag{7.10}\]
for all \(h \in H, f \in H^*\), and it is then immediate that
\[S_{\alpha Z} = \hat{S}|_{\alpha Z}. \tag{7.11}\]

We will express the Hopf pairing \(\omega\) from (7.1) via an element \(\hat{\omega} \in H \otimes H\) such that
\[\omega(f \otimes g) = g(\hat{\omega}_1)f(\hat{\omega}_2) \tag{7.12}\]
for all \(f, g \in H^*\). An expression of \(\hat{\omega}\) in term of quasi-Hopf data was derived in \cite[Thm. 7.3]{FGR1}. We will not need it explicitly.

\(^{18}\)A quasi-triangular quasi-Hopf algebra \(H\) is factorisable \cite{BT1} if and only if a certain linear map \(\mathfrak{M} : H^* \rightarrow H\) involving the monodromy is bijective. In \cite[Sec. 7.3]{FGR1} it was shown that this is equivalent to \(\mathcal{M}\) being factorisable. In particular, \(\mathcal{M}\) is a modular tensor category for \(H\) factorisable and ribbon.

\(^{19}\)Note also that \(\mathcal{C}(1, L) \cong \beta Z\), which is defined similarly.
Proposition 7.1. Let \( \lambda \in H^* \) be the right monadic cointegral for \( _H^M \). The \( S \)- and \( T \)-transformations on \( \alpha Z \) are given by the linear maps
\[
S_{\alpha Z}(\alpha z) = \langle \lambda | \hat{\omega}_1 z \rangle \hat{\omega}_2 \\
T_{\alpha Z}(\alpha z) = v^{-1} \alpha z
\]  
(7.13)

for \( z \in Z \).

Proof. The action of \( T \) is immediate from [FGR1, Sec. 8]. For the action of \( S \) we use (7.11) and compute
\[
\hat{S}(\alpha z) = \langle \lambda | S(X_1)\hat{\omega}_1 X_2 S(X_3(1)p_L^L)\alpha z X_3(2)p_L^L \rangle \hat{\omega}_2 \\
\quad = \langle \lambda | \hat{\omega}_1 S(p_L^L)\alpha p_L^L z \rangle \hat{\omega}_2 \\
\quad = \langle \lambda | \hat{\omega}_1 z \rangle \hat{\omega}_2 ,
\]
(7.14)

where in the first step (i) we used the form of \( \hat{S} \) as given in [FGR1, (8.15)], (ii) uses (3.6) for the underlined part and that \( \Phi \) is normalised, and (iii) follows from the definition of \( p_L^L \) in (3.31) and the zig-zag axiom (3.7).

We can express the action of \( S \) on \( \alpha Z \) using the right cointegral \( \lambda^r \) from (3.45) and Theorem 4.1 as
\[
S_{\alpha Z}(\alpha z) = \langle \lambda^r | S(\beta)\hat{\omega}_1 z \rangle \hat{\omega}_2 .
\]
(7.15)

Remark 7.2. One can also show that
\[
S_{\alpha Z}(\alpha z) = \hat{\omega}_1 \langle \lambda | \hat{\omega}_2 z \rangle ,
\]
(7.16)

where \( \lambda \) is the right monadic cointegral. To see this, one checks \( \omega \circ (\vartheta_L \otimes \text{id}) = \omega \circ c_{\varphi L}^{-1} \) using \( \vartheta_L = (S_L)^2 \), see [KL, Lem. 5.2.4]. This then readily implies
\[
\omega \circ (f \otimes \text{id}) = \omega \circ (\text{id} \otimes f)
\]
(7.17)

for any \( f \in C(1, L) \). The claim follows since \( \langle \lambda | ?z \rangle \in C(1, L) \) for \( z \) central.

Appendix A. Proofs for Section 4

A.1. Proof of Proposition 4.2. Before giving the proof we need to show some intermediate results.

Using the explicit form of the unit and the counit of the adjunction, we can give the following simple characterization of the components of \( R \).

Lemma A.1. The coactions \( R_U \) defined in (2.74) satisfy
\[
Y \quad A_2 U \quad Y^\vee \\
\pi_4 \\
R_U \\
\gamma_{Y,X} \quad \text{id} \\
X^\vee \quad U \quad X
\]
(A.1)
Proof. By definition,

\[(A_2(U), R_U) = (A_2(U), Z^4(\mu_2(U)) \circ \tilde{\eta}_{A_2U})\]  \hspace{1cm} (A.2)

as \(Z^4\)-comodules. Then

\[
\begin{array}{ccc}
Y & A_2U & Y^\vee \\
\downarrow \pi_4 & & \downarrow \pi_4 \\
Z^4(\mu_2(U)) & \tilde{\eta}_{A_2U} & \end{array} = 
\begin{array}{ccc}
Y & A_2U & Y^\vee \\
\downarrow \mu_2(U) & & \downarrow \mu_2(U) \\
\tilde{\eta}_{A_2U} & \end{array} = 
\begin{array}{ccc}
Y & A_2U & Y^\vee \\
\downarrow \iota_2 & & \downarrow \iota_2 \\
\iota_2 & \end{array}
\hspace{1cm} (A.3)
\]

together with the definition (2.14) of the multiplication of \(A_2\) proves the claim. \(\square\)

Lemma A.2. Let \(V \in H_M\), \(v \in V\), \(h^* \in H^*\), and choose the realization of the central Hopf monad \(A_2\) as given in (3.18) and the comonad \(Z^4\) in (3.57). Then

\[
R_V(h^* \otimes v) = \langle h^* \mid S(x_{2(2)}p_{21}^L X_1) f_1 \left[ e_i x_{3(2)} X_2 Y_2 \right]_{(1)} p_{1}^R \rangle \\
\times x_1 S(x_{2(1)}p_{21}^L) f_2 \left[ e_i x_{3(2)} X_2 Y_2 \right]_{(2)} p_{2}^R S(X_3 Y_3) \\
\otimes e^i \otimes x_{3(1)} X_2 Y_1 v, \hspace{1cm} (A.4)
\]

where \(\{e_i\}\) is a basis of \(H\) with corresponding dual basis \(\{e^i\}\), and summation over \(i\) is implied.

Proof. Set

\[
\Xi = (\text{id} \otimes \text{id} \otimes \Delta \otimes \text{id})(\Psi \otimes 1) \cdot (\Delta \otimes \Delta \otimes \text{id})(\Phi) \cdot (1 \otimes 1 \otimes \Phi) \\
= x_1 X_{1(1)} \otimes x_2 X_{1(2)} \otimes x_{3(1)} X_{2(1)} Y_1 \otimes x_{3(2)} X_{2(2)} Y_2 \otimes X_3 Y_3, \\
\Theta = (S \otimes S)(p_{21}^L)f, \\
\Omega = \Xi_1 \otimes S(\Xi_2) \otimes \Xi_4 \otimes S(\Xi_5) \otimes \Xi_6, \hspace{1cm} (A.5)
\]

\[
\begin{array}{ccc}
Y & A_2U & Y^\vee \\
\downarrow \iota_2 & & \downarrow \iota_2 \\
\iota_2 & \end{array}
\]

where \(\{e_i\}\) is a basis of \(H\) with corresponding dual basis \(\{e^i\}\), and summation over \(i\) is implied.

Proof. Set

\[
\Xi = (\text{id} \otimes \text{id} \otimes \Delta \otimes \text{id})(\Psi \otimes 1) \cdot (\Delta \otimes \Delta \otimes \text{id})(\Phi) \cdot (1 \otimes 1 \otimes \Phi) \\
= x_1 X_{1(1)} \otimes x_2 X_{1(2)} \otimes x_{3(1)} X_{2(1)} Y_1 \otimes x_{3(2)} X_{2(2)} Y_2 \otimes X_3 Y_3, \\
\Theta = (S \otimes S)(p_{21}^L)f, \\
\Omega = \Xi_1 \otimes S(\Xi_2) \otimes \Xi_4 \otimes S(\Xi_5) \otimes \Xi_6, \hspace{1cm} (A.5)
\]

where \(\{e_i\}\) is a basis of \(H\) with corresponding dual basis \(\{e^i\}\), and summation over \(i\) is implied.
then from Lemma A.1 one computes

\[
\pi_4(A_2 U)_Y \circ R_U \circ \iota_2(U)_X =
\]

Recall that the box with \(\text{Vect}_k\) means that these pictures are to be understood as linear maps. Specializing \(X\) and \(Y\) to the regular left module \(H\), we note that \(\pi_4(V)_H\) has a left inverse

\[
(H \otimes V) \otimes H^\vee \rightarrow H \otimes V, \quad h \otimes v \otimes f \mapsto f(1)h \otimes v ,
\]

while \(\iota_4(V)_H\) has a right inverse

\[
H^* \otimes V \rightarrow H^\vee \otimes (V \otimes H) \quad f \otimes v \mapsto f \otimes v \otimes 1 .
\]
Applying the inverses we obtain the explicit form of $R_U$,

$$\begin{align*}
R_U(h^* \otimes u) &= \langle h^* \otimes \text{id} \mid (1 \otimes \Omega_1) \Theta \Delta(\Omega_2 e_i \Omega_3) p^R(1 \otimes \Omega_4) \rangle \otimes e^i \otimes \Omega_5.v \\
&= \langle h^* \otimes \text{id} \mid (1 \otimes \Xi_1) \Theta \Delta(S(\Xi_2) e_i \Xi_4) p^R(1 \otimes S(\Xi_5)) \rangle \otimes e^i \otimes \Xi_3.v \\
&= \langle h^* \mid S(p^L_2) f_1 [S(\Xi_2) e_i \Xi_4]_1 p^R_1 \rangle \\
&\times \Xi_1 S(p^L_1) f_2 [S(\Xi_2) e_i \Xi_4]_2 p^R_2 S(\Xi_5) \otimes e^i \otimes \Xi_3.v \\
&= \langle h^* \mid S(p^L_2) f_1 [S(X_2(1,2)) e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 X_1(1) S(p^L_1) f_2 [S(X_2(1,2)) e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \\
&\otimes e^i \otimes x_3(1) X_2(1) Y_1.v \\
&= \langle h^* \mid S(x_2(2) X_1(2,2)) p^L_2 f_1 [e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 X_1(1) S(x_2(1)) X_1(1,2) p^L_1 f_2 [e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \\
&\otimes e^i \otimes x_3(1) X_2(1) Y_1.v \\
&= \langle h^* \mid S(x_2(2)) p^L_2 X_1(1) f_1 [e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 S(x_2(1)) p^L_1 f_2 [e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \\
&\otimes e^i \otimes x_3(1) X_2(1) Y_1.v \\
&= \langle h^* \mid S(x_2(2)) p^L_2 X_1(1) f_1 [e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 S(x_2(1)) p^L_1 f_2 [e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \otimes e^i \otimes x_3(1) X_2(1) Y_1.v \\
&= \langle h^* \mid S(x_2(2)) p^L_2 X_1(1) f_1 [e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 S(x_2(1)) p^L_1 f_2 [e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \otimes e^i \otimes x_3(1) X_2(1) Y_1.v \\
&= \langle h^* \mid S(x_2(2)) p^L_2 X_1(1) f_1 [e_i x_3(2) X_2(2) Y_2]_1 p^R_1 \rangle \\
&\times x_1 S(x_2(1)) p^L_1 f_2 [e_i x_3(2) X_2(2) Y_2]_2 p^R_2 S(X_3 Y_3) \otimes e^i \otimes x_3(1) X_2(1) Y_1.v
\end{align*}$$

for $h^* \in H^*$, $u \in U$. \hfill $\square$

**Proof of Proposition 4.2.** We will need the identity

$$q_2^L[f_2^{-1}]_{(2)} \otimes S(f_1^{-1})q_1^L[f_2^{-1}]_{(1)} = (S \otimes S)(p^R)f_21,$$

which can be seen as follows:

$$q_2^L[f_2^{-1}]_{(2)} \otimes S(f_1^{-1})q_1^L[f_2^{-1}]_{(1)}$$
\[(3.31)\] \[X_3[f_2^{-1}]_{(2)} \otimes S(X_1 f_1^{-1}) \alpha X_2[f_2^{-1}]_{(1)} \]
\[(3.40)\] \[f_2^{-1} S(X_1) f_2 \otimes S([f_1^{-1}]_{(1)} f_1^{-1} S(X_3)) \alpha [f_1^{-1}]_{(2)} f_2^{-1} S(X_2) f_1 \]
\[(3.6)\] \[\varepsilon(f_1^{-1}) f_2^{-1} S(X_1) f_2 \otimes S(f_1^{-1}) f_2^{-1} S(X_2) f_1 \]
\[(\ast)\] \[S(X_1) f_2 \otimes S^2(X_3) S(\beta) S(X_2) f_1 \]
\[(3.31)\] \[(S \otimes S) (p^R) \cdot f_{21} \cdot \] (A.12)

In the step labelled (\ast) one uses (3.41) (dashed underline) and that the counit applied to any leg of the inverse Drinfeld twist yields 1 (dotted underline). The identity (A.11) immediately implies
\[(A.13)\] \[p^R = S^{-1}(q^L_2 f_2^{-1}) \otimes S^{-1}(q^L_1 f_1^{-1}) f_1^{-1}. \]

For the proof of the proposition, let now \(h^* \in H^*\). Then
\[(\varphi_{H^*} \circ A(\rho))(h^*)\]
\[= \gamma^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* | S(p^L_2 f_1 [e_i]_{(1)} S^{-1}(q^L_2 f_2^{-1})) \]
\[\times x_1 X_{1(1)} S(p^L_1 f_2 [e_i]_{(2)} S^{-1}(q^L_1 f_1^{-1}) f_1^{-1} S(X_3 Y_3) \]
\[\otimes x_2 X_{2(2)} e^j \tilde{f}_2^{-1} S(x_{3(2)} X_{2(2)} Y_2) \]
\[(3.8)\] \[= \gamma^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* | S(p^L_2 f_1 [e_i]_{(1)} S^{-1}(q^L_2 f_2^{-1})) \]
\[\times x_1 X_{1(1)} S(p^L_1 f_2 [e_i]_{(2)} S^{-1}(q^L_1 f_1^{-1}) f_1^{-1} S(X_3 Y_3) \]
\[\otimes x_3 (2) X_{2(2)} Y_2 S^{-1}(\tilde{f}_2^{-1}) \rangle e^i \leftarrow S(x_{2(1)} X_{1(2)}) \]
\[(3.15)\] \[= \gamma^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* | S(p^L_2 f_1 [e_i x_{3(2)} X_{2(2)} Y_2]_{(1)} S^{-1}(q^L_2 f_2^{-1})) \]
\[\times x_1 X_{1(1)} S(p^L_1 f_2 [e_i x_{3(2)} X_{2(2)} Y_2]_{(2)} S^{-1}(q^L_1 f_1^{-1} f_1^{-1} \tilde{f}_1^{-1} S(X_3 Y_3) \]
\[\otimes e^i \leftarrow S(x_{2(1)} X_{1(2)}) \]
\[(A.13)\] \[= \gamma^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* | S(p^L_2 f_1 [e_i x_{3(2)} X_{2(2)} Y_2]_{(1)} p^R_1) \]
\[\times x_1 X_{1(1)} S(p^L_1 f_2 [e_i x_{3(2)} X_{2(2)} Y_2]_{(2)} p^R_2 S(X_3 Y_3) \]
\[\otimes e^i \leftarrow S(x_{2(1)} X_{1(2)}) \]
\[(3.15)\] \[= \gamma^{-1}(x_{3(1)} X_{2(1)} Y_1) \langle h^* | S(x_{2(2)} X_{1(2)} p^L_2 f_1 [e_i x_{3(2)} X_{2(2)} Y_2]_{(1)} p^R_1) \]
\[\times x_1 X_{1(1)} S(x_{2(1)} X_{1(2)} p^L_1 f_2 [e_i x_{3(2)} X_{2(2)} Y_2]_{(2)} p^R_2 S(X_3 Y_3) \otimes e^i \]
so that (3.18), we have

\[ H \otimes \text{category of } \text{cointegrals to a Hom-space containing monadic cointegrals. To do this, we define} \]

\[ \text{Proposition A.3.} \]

\[ \text{Let } \gamma^{-1}(x_{3(1)}X_{2(1)}Y_1) \langle h^* \mid S(x_{2(2)}p_2^{R}X_1)\gamma \mid \gamma \rangle \]

\[ \times x_1S(x_{2(1)}p_1^{R})f_2 \left[ e_i x_{3(2)}X_2(2)Y_2 \right] \left( 1 \right) p_2^{R}S(X_3Y_3) \otimes e_i . \]  

(A.14)

From Lemma A.2 we obtain

\[ R_{\gamma^\forall}(h^*) = \gamma^{-1}(x_{3(1)}X_{2(1)}Y_1) \langle h^* \mid S(x_{2(2)}p_2^{R}X_1)\gamma \mid \gamma \rangle \]

\[ \times x_1S(x_{2(1)}p_1^{R})f_2 \left[ e_i x_{3(2)}X_2(2)Y_2 \right] \left( 1 \right) p_2^{R}S(X_3Y_3) \otimes e_i , \]  

(A.15)

so that

\[ R_{\gamma^\forall} = \varphi_{H^\forall} \circ \mathcal{A}(\rho) \]  

(A.16)

indeed holds, finishing the proof.

□

A.2. Proof of Theorem 4.1 (1). The first step in the proof of the Theorem 4.1 is to map cointegrals to a Hom-space containing monadic cointegrals. To do this, we define the space

\[ \mathcal{X}_2 = \{ f \in H^\forall \mid f \leftarrow S(a) = S^{-1}(\gamma \rightarrow a) \rightarrow f \quad \forall a \in H \} \]

\[ = \{ f \in H^\forall \in H\mathcal{M}_H \mid a.f = f.(\gamma \rightarrow a) \quad \forall a \in H \}, \]  

(A.17)

where the dot denotes the action on the left dual of the regular bimodule in $H\mathcal{M}_H$, the category of $H \otimes H^{op}$-modules as introduced in Section 3.5.

Note that right cointegrals are automatically in $\mathcal{X}_2$ by (3.47):

\[ f_H^r \subset \mathcal{X}_2 . \]  

(A.18)

By (3.18), we have

\[ \mathcal{C}(1, A_2(\gamma^\forall)) \]

\[ = \{ f \in H^\forall \mid \varepsilon(h)f(a) = f(S(h_{(1)})a(h_{(2)} \leftarrow \gamma^{-1})) \quad \forall h, a \in H \} \]

\[ = \{ f \in H^\forall \in H\mathcal{M}_H \mid \varepsilon(h)f = h_{(1)}.f.S(h_{(2)} \leftarrow \gamma^{-1}) \quad \forall h \in H \}, \]  

(A.19)

where in the second line we again let the dot denote the action on the left dual of the regular bimodule.

We then have the following proposition.

**Proposition A.3.** Let $\xi = (\text{id} \otimes \gamma)(f^{-1})$. Then the map

\[ a_2 : \mathcal{X}_2 \to \mathcal{C}(1, A_2(\gamma^\forall)), \quad f \mapsto \beta.f.\xi, \]  

(A.20)

is a linear isomorphism.

**Proof.** Abbreviate $\mathcal{C}_2 := \mathcal{C}(1, A_2(\gamma^\forall))$, and let us check that $a_2(\mathcal{X}_2) \subset \mathcal{C}_2$. To this end, observe that the defining equation for $f \in H^\forall$ to be in $\mathcal{X}_2$ may be rewritten as

\[ S(a).f = f.\xi S(a \leftarrow \gamma^{-1})\xi^{-1} \]  

(A.21)

by using the definition of the Drinfeld twist. Then we compute

\[ h_{(1)}a_2(f).S(h_{(2)} \leftarrow \gamma^{-1}) = h_{(1)}\beta.f.\xi S(h_{(2)} \leftarrow \gamma^{-1}) \]

\[ = h_{(1)}\beta S(h_{(2)}).f.\xi \]  

(A.21)

\[ = \varepsilon(h)\beta.f.\xi \]

\[ = \varepsilon(h)a_2(f) . \]  

(A.22)
Next, we claim that the assignment
\[ B_2 : f \mapsto q_1 f S(q_2^L \leftarrow \gamma^{-1}) \xi^{-1} \] (A.23)
is the two-sided inverse of \( A_2 \). First of all, \( B_2(C_2) \subset X_2 \). Indeed,
\[
B_2(f) \xi S(a \leftarrow \gamma^{-1}) \xi^{-1} = q_1 f S(q_2^L \leftarrow \gamma^{-1}) \xi^{-1} \xi S(a \leftarrow \gamma^{-1}) \xi^{-1} = q_1 f S((aq_2^L) \leftarrow \gamma^{-1}) \xi^{-1} = S(a(1))q_1 f S((q_2^L a(2,1)) \leftarrow \gamma^{-1}) \xi^{-1} = S(a)q_1 f S((q_2^L) \leftarrow \gamma^{-1}) \xi^{-1} = S(a)B_2(f). \] (A.24)

Here \((*)\) uses that \( f \in C_2 \).

It is not hard to see that \( B_2 \) is a left inverse of \( A_2 \):
\[
B_2A_2(f) = B_2(\beta f \xi) = q_1 \beta f \xi S(q_2^L \leftarrow \gamma^{-1}) \xi^{-1} = q_1 \beta S(q_2^L) f = f. \] (A.25)

To see that \( A_2B_2 = \text{id} \) we need the fact that \( \beta = (S \otimes \varepsilon)(p^L) \), and the \( p^L, q^L \)-relation in (3.32). We compute
\[
A_2B_2(f) = A_2(q_1 f S(q_2^L \leftarrow \gamma^{-1}) \xi^{-1}) = \beta q_1 f S(q_2^L \leftarrow \gamma^{-1}) = S(p_1^L)q_1 f S(q_2^L \leftarrow \gamma^{-1}) = S(p_1^L)q_1 p_2^L(1) f S((q_2^L p_2^L(2)) \leftarrow \gamma^{-1}) = 1 f S(1 \leftarrow \gamma^{-1}) = f, \] (A.26)

using that \( f \in C_2 \) in \((*)\).  

We will need the following technical lemma.

**Lemma A.4.** Let \( f \in X_2 \). Then
\[ \gamma(x_3) \varphi_{H^\vee}(\beta(1) x_1 \xi(1) \otimes \beta(2). f. x_2 \xi(2)) = \beta \otimes_k \beta \cdot f. \xi \] (A.27)

**Proof.**
\[
\gamma(x_3) \varphi_{H^\vee}(\beta(1) x_1 \xi(1) \otimes \beta(2). f. x_2 \xi(2)) = \gamma^{-1}(y_{3(1)} X_{2(1)} Y_1) \gamma(x_3) y_{1(1)} X_{1(1)} \delta_1 f_1 X_{1(1)} f^{-1}_1 S(X_3 Y_3) \]
\[
\otimes y_{2(1)} X_{2(1)} \delta_2 f_2 \cdot f. x_2 \xi(2) f^{-1}_2 S(y_{3(2)} X_{2(2)} Y_2) = \gamma^{-1}(y_{3(1)} X_{2(1)} Y_1) \gamma(x_3 F_2^{-1}) y_{1(1)} X_{1(1)} \delta_1 f_1 x_1 F_1^{-1} f^{-1}_1 S(X_3 Y_3) \]
\[
\otimes y_{2(1)} X_{2(1)} \delta_2 f_2 \cdot f. x_2 F_1^{-1} f_2 S(y_{3(2)} X_{2(2)} Y_2) \].
\[ (*) \quad \gamma^{-1}(y_3(1)X_2(1)Y_1)\gamma(f_2(2)^{-1}x_3y_1X_1(1)\delta_1f_1x_1f_1^{-1}(1)f_1^{-1}S(X_3Y_3) \\
\otimes y_2X_1(2)\delta_2.f.f_2(2)^{-1}y_1X_1(1)\delta_1S(x_3)S(X_3Y_3) \\
\otimes y_2X_1(2)\delta_2.f.S(y_3(2)X_2(2)Y_2) \\
= \gamma^{-1}(y_3(1)X_2(1))y_1X_1(1)\delta_1S(X_3)\otimes y_2X_1(2)\delta_2.f.S(y_3(2)X_2(2)) \quad (3.35) \]

Now we have all the necessary ingredients and can prove our main theorem.

**Proof of Theorem 4.1 (1).** By Proposition A.3, for each \( \lambda^C \in C(1, A_2(D)) \) there is a unique \( \lambda \in \mathcal{X}_2 \) such that \( \lambda^C = \lambda_2(\lambda) = \lambda.\beta.\xi. \) Assume first that \( \lambda \) is a right cointegral. Then

\[ (\varphi_H \circ \rho(\lambda)) = \gamma(x_3)\varphi_H(\Delta(\beta).x_1 \otimes \lambda.x_2).\Delta(\xi) \quad (\ast) \quad \beta \otimes \lambda^C \quad (A.29) \]

shows that \( \lambda^C \) is a right monadic cointegral, using the equivalent characterisation (4.6). Here (\ast) uses that \( \lambda \) is a right cointegral, and (\**\) uses Lemma A.4.

Conversely, assume that \( \lambda^C \) is a right monadic cointegral. Note that for any \( f \in \mathcal{X}_2 \) we have

\[ f = q_1^L:\beta.S(q_2^L).f = q_1^L:\beta.f.(\gamma \rightarrow S(q_2^L)), \quad (A.30) \]

where the first step is the zig-zag axiom (3.7), and the second step uses that \( f \in \mathcal{X}_2 \). Then

\[ \rho(\lambda) = \rho(q_1^L:\beta.\lambda.\xi^{-1}(\gamma \rightarrow S(q_2^L))) \]

\[ (1) \Delta(q_1^L)\rho(\beta.\lambda.\xi)\Delta(\xi^{-1}(\gamma \rightarrow S(q_2^L))) \]

\[ (2) \Delta(q_1^L)\varphi_H^{-1}(\beta \otimes \beta.\lambda.\xi)\Delta(\xi^{-1}(\gamma \rightarrow S(q_2^L))) \]

\[ (3) \gamma(x_3) \Delta(q_1^L).\beta(1)x_1\xi(1) \otimes \beta(2).\lambda.x_2\xi(2).\Delta(\xi^{-1}(\gamma \rightarrow S(q_2^L))) \]

\[ = \gamma(x_3) ((q_1^L:\beta)(1)\otimes (q_1^L:\beta)(2).\lambda.x_2).\Delta(\gamma \rightarrow S(q_2^L)) \]
λcointegral, (3) follows from Lemma A.4, (4) uses that see (3.47), and analogously to Proposition A.3 one can show that are automatically contained in the space

\[ \gamma(x_3) \Delta(\gamma \to (q_1^L \beta)) \Delta(\gamma \to S(q_2^L)) = \gamma(x_3) x_1 \otimes \lambda.x_2 \]  
(A.31)

shows that $\lambda \in \mathcal{X}_2$ is a right cointegral in the sense of [BC, HN2]. The step labelled (1) uses that $\rho' \lambda$ is a bimodule morphism, (2) is the fact that $\lambda' = \beta.\lambda.\xi$ is a monadic cointegral, (3) follows from Lemma A.4, (4) uses that $\lambda \in \mathcal{X}_2$, and (5) is an application of quasi-coassociativity. □

A.3. Proof of Theorem 4.1 (2). Similarly to right cointegrals, left cointegrals for $H$ are automatically contained in the space

\[ \mathcal{X}_3 = \{ f \in H^* \mid f \leftarrow S^{-1}(a) = S(a \leftarrow \gamma) \to f \} \]

\[ = \{ f \in \gamma H \in H.M_H \mid a.f = f.(a \leftarrow \gamma) \}, \]  
(A.32)

see (3.47), and analogously to Proposition A.3 one can show that

\[ \mathcal{A}_3 = \mathcal{A}_2^\text{cop} : \mathcal{X}_3 \to \mathcal{C}(1, A_3(\gamma')) \], \hspace{1em} \mathcal{A}_3(f) = S^{-1}(\beta).f.\hat{\xi} \]  
(A.33)

is an isomorphism. Note that here the dot denotes the action on the right dual of the regular $H \otimes H^\text{cop}$-module. The Hopf monads $A_2$ and $A_3$ are canonically isomorphic via $\kappa_{2,3} : A_2 \Rightarrow A_3$, see (2.21) and Proposition 3.1. This allows us to transport right monadic cointegrals to left monadic cointegrals. Thus, upon showing that

\[ \int_H \hspace{1em} \mathcal{X}_2 \hspace{1em} \mathcal{A}_2 \hspace{1em} \mathcal{C}(1, A_2(\gamma')) \hspace{1em} \int_C^{r,\text{mon}} \]

\[ \mathcal{A}_3 \hspace{1em} \mathcal{X}_3 \hspace{1em} \mathcal{C}(1, A_3(\gamma')) \hspace{1em} \int_C^{l,\text{mon}} \]  
(A.34)

commutes, we know that $\mathcal{A}_3$ maps left cointegrals to left monadic cointegrals. Here (*) maps the right cointegral $\lambda'$ to 20

\[ \lambda' = \gamma(\alpha S(\beta))^{-1} \cdot (\lambda' \circ S \leftarrow (u^\text{cop})^{-1}). \]  
(A.35)

By [BC, Prop. 4.3] this is a left cointegral, cf. (3.49).

The right hand square in (A.34) commutes by construction. Using the explicit formula (3.20) for $\kappa_{2,3}$, one finds that the upper path of the left hand square is

\[ \lambda' \mapsto \gamma^{-1}(X_2)\langle \lambda' \mid S(\beta)S(?X_1)X_3S^{-1}(\xi) \rangle \]

\[ = \gamma^{-1}(X_2)\langle \lambda' \circ S \mid S^{-2}(\xi)S^{-1}(X_3)?X_1\beta \rangle \]

\[ = \gamma(\alpha S(\beta))\gamma^{-1}(X_2)\langle \lambda' \mid u^\text{cop}S^{-2}(\xi)S^{-1}(X_3)?X_1\beta \rangle \]

\[ \overset{(1)}{=} \gamma(\alpha S(\beta))\gamma^{-1}(X_2p_1^L)\langle \lambda' \mid S^{-1}(X_3p_2^L)?X_1\beta \rangle \]

\[ \overset{(2)}{=} \gamma(\alpha S(\beta))\gamma^{-1}(X_2p_1^L)\gamma((X_3p_2^L)_{(1)})(\lambda' ? X_1\beta S((X_3p_2^L)_{(2)})), \]  
(A.36)

the step marked (1) follows directly from the definition of $u$ and $\xi$, see (3.48) resp. Theorem 4.1, and step (2) uses $\lambda' \in \mathcal{X}_3$.

20The zig-zag axiom (3.7) implies that both $\gamma(\alpha)$ and $\gamma(\beta)$ are invertible in $k$. Therefore, the prefactor in (A.35) is well-defined.
The lower path of the left square of \((A.34)\) evaluates to
\[
\lambda^r \mapsto \langle \lambda^l | S^{-2}(\beta)^{-1} S(\xi) \rangle
\]
\[
\overset{(3.47)}{=} \langle \lambda^l | S((S^{-1}(\beta) \leftarrow \gamma) \hat{\xi}) \rangle
\]
\[
\overset{(*)}{=} \gamma^{-1}(\beta(2) \mathcal{F}^{-1}_2)(\lambda^l | S(\beta(1) \mathcal{F}^{-1}_1))
\]
\[
\overset{(3.42)}{=} \gamma^{-1}(X_2)\gamma((X_3(1)p^L_1)\langle \lambda^l | ?X_1 S((X_3(2)p^L_2)) \rangle,
\]
(A.37)
where \((*)\) uses the definition of \(\hat{\xi}\) as in Theorem 4.1.

We have
\[
\gamma(\alpha S(\beta)S(p^L_1)p^L_2(1))p^L_2(2) \overset{(3.31)}{=} \gamma(\alpha S(\beta)x_1 \beta S(x_2) x_3(1))x_3(2)
\]
\[
\overset{(3.4),(*)}{=} \gamma(\alpha S(\beta)x_1 \beta S(y_1 x_2) y_2 x_3) y_3
\]
\[
\overset{(3.7)}{=} \gamma(S(y_1 \beta) y_2) y_3
\]
\[
\overset{(**)}{=} \gamma(p^L_1)p^L_2
\]
(A.38)
where \((*)\) uses \(\gamma \otimes \gamma^{-1} \circ \Delta = \varepsilon\), and \((**)\) uses \(\gamma \circ S = \gamma \circ S^{-1}\) and \((3.31)\). Therefore the two expressions \((A.36)\) and \((A.37)\) are equal.

A.4. **Proof of the pivotal case.** The proof of Theorem 4.4 is similar to the proof of the second part above.

First, define the spaces
\[
\mathcal{X}_1 = \{ f \in H^* \mid f(ab) = f((b \leftarrow \gamma)a) \},
\]
\[
\mathcal{X}_4 = \{ f \in H^* \mid f(ab) = f((\gamma \rightarrow b)a) \}.
\]
(A.39)

By (4.11) we have \(\hat{\lambda}^r \in \mathcal{X}_1\) and \(\hat{\lambda}^l \in \mathcal{X}_4\), and similarly to Proposition A.3 one may show that
\[
\mathcal{A}_1 : \mathcal{X}_1 \rightarrow \mathcal{C}(1, \mathcal{A}_1(\gamma^\vee)) \quad \text{and} \quad f \mapsto \langle f | S^{-1}(\beta)^{-1} S(\vartheta) \rangle,
\]
\[
\mathcal{A}_4 : \mathcal{X}_4 \rightarrow \mathcal{C}(1, \mathcal{A}_4(\gamma^\vee)) \quad \text{and} \quad f \mapsto \langle f | \beta^1 S^{-1}(\hat{\vartheta}) \rangle,
\]
(A.40)
with \(\vartheta = (\gamma^{-1} \otimes S^{-1}) (p^L r)\) and \(\hat{\vartheta} = \vartheta^\text{op},\) are linear isomorphisms.

Then a simple computation shows that the diagram
\[
\begin{array}{ccc}
j^r_H & \rightarrow & \mathcal{X}_2 \xrightarrow{\mathcal{A}_2} \mathcal{C}(1, \mathcal{A}_2(\gamma^\vee)) \xrightarrow{\sim} j^{r,\text{mon}}_C \\
(*) \downarrow & & \downarrow \sim \\
j^{r,\gamma}_H & \rightarrow & \mathcal{X}_1 \xrightarrow{\mathcal{A}_1} \mathcal{C}(1, \mathcal{A}_1(\gamma^\vee)) \xrightarrow{\sim} j^{r,D\text{-sym}}_C
\end{array}
\]
(A.41)
commutes, with \((*)\) sending a right cointegral \(\lambda^r\) to the right symmetrised cointegral \(\lambda^r \leftarrow u g\), and \(\sim\) is induced by the isomorphism of Hopf monads from Proposition 2.4.

Indeed, a right cointegral \(\lambda^r\) gets mapped to the right \(D\)-symmetrised monadic cointegral
\[
\lambda^{r,D\text{-sym}} = \langle \lambda^r | S(\beta) g S^{-1}(\xi) \rangle
\]
(A.42)
by the upper path, and to
\[(\lambda^{r,D-\text{sym}})' = \langle \lambda^{r} | u g S^{-1}(\beta) S(\vartheta) \rangle\]  \hfill (A.43)
by the lower path.

The upper path, evaluated on \(S^{-1}(h), h \in H\), yields
\[
\lambda^{r,D-\text{sym}}(S^{-1}(h)) = \langle \lambda^{r} | S(\beta) g S^{-1}(\xi h) \rangle = \langle \lambda^{r} | S(\xi h \beta) g \rangle \]
\[
= \langle \lambda^{l} | u \cop g^{-1} \xi h \beta \rangle . \tag{3.49} \hfill (A.44)
\]
Evaluating the lower path on \(S^{-1}(h), h \in H\), we get
\[
(\lambda^{r,D-\text{sym}})'(S^{-1}(h)) = \langle \lambda^{r} | u g S^{-1}(h \beta) S(\vartheta) \rangle = \langle \lambda^{l} \circ S^{-1} | S(h \beta) g S(\vartheta) \rangle = \langle \lambda^{l} | \vartheta g^{-1} h \beta \rangle . \tag{3.49} \hfill (A.45)
\]
The claim then follows from
\[
u^{\cop} g^{-1} \xi g = \gamma(V_{1}^{\cop} f_{2}^{-1}) S^{-2}(V_{2}^{\cop}) g^{-1} f_{1}^{-1} g = \gamma(V_{1}^{\cop} f_{2}^{-1}) g^{-1} V_{2}^{\cop} f_{1}^{-1} g \]
\[
= \gamma(S(p_{1}^{L}) \tilde{f}_{2} f_{2}^{-1}) g^{-1} S(p_{2}^{L}) \tilde{f}_{1} f_{1}^{-1} g = \gamma^{-1}(p_{1}^{L}) S^{-1}(p_{2}^{L}) = \vartheta \tag{3.43} \hfill (A.46)
\]
A similar diagram involving left cointegrals and their symmetrised version then finishes the proof of the theorem. \(\square\)

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