Snakes and Ladders

M. A. Martín-Delgado, R. Shankar and G. Sierra

1 Departamento de Física Teórica I. Universidad Complutense. 28040-Madrid, Spain
2 Sloane Physics Laboratories, Yale University, New Haven CT 06520
3 Instituto de Matemáticas y Física Fundamental. C.S.I.C. Serrano 123, 28006-Madrid, Spain
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We map spin ladders with $n_l$ legs and couplings $J'$ across all rungs and $J(1 \pm \gamma)$ along the legs, staggered in both directions, to a sigma model. Setting its $\theta = (2m + 1)\pi$ (where it is known to be gapless), we locate the critical curves in the $\gamma$ versus $\frac{J'}{J}$ plane at each $n_l$, and spin $S$. The phase diagram is rich and has some surprises: when two gapped chains are suitably coupled, the combination becomes gapless. With $n_l, \gamma$ and $J'/J$ to control, the prospects for experimentally observing any one of these equivalent transitions seems bright. We discuss the order parameters and the behavior of holes in the RVB description.

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With this paper we contribute to the explosive growth in the theoretical and experimental studies of antiferromagnetic spin chains and ladders. Let begin with Haldane’s mapping of the spin-$S$ Heisenberg chain with Hamiltonian

$$H = J \sum_n S(n) \cdot S(n+1)$$

via spin coherent states on to the nonlinear sigma model with euclidean action

$$S = \int dxd\tau \left[ -\frac{1}{2g}(\nabla \Phi)^2 + \frac{i\theta}{4\pi} \Phi \cdot \partial_x \Phi \times \partial_t \Phi \right].$$

Here $\Phi$ is a three component unit vector, and $\theta$, which multiplies $i$ the integer valued winding number $W$, is

$$\theta = 2\pi S.$$  

There is a Hamiltonian derivation of this result due to Affleck, which we shall allude to later.

Since $\theta$ enters the path integral via $e^{i\theta W}$, it matters only mod $2\pi$ and when $\theta = 0, \pi$, the path integral is invariant under $x \rightarrow -x$ (parity) under which $W \rightarrow -W$. One now argues that integer spin chains have a gap and half-integer spin chains do not. First, all integer (half-integer) spin chains essentially have $\theta = 0 (\pi)$. The sigma model with $\theta = 0$ is known to have exponentially decaying correlations. As for $\theta = \pi$, since the spin-$\frac{1}{2}$ Bethe chain is gapless, so must be the $\theta = \pi$ sigma model, provided the mapping to the sigma model (most reliable for large $S$) is valid down to spin $\frac{1}{2}$. Since so much in this paper hinges on the masslessness of the $\theta = \pi$ model, we point out that Shankar and Read have shown independently that the spin model at $\theta = \pi$ is massless by considering the $\tau$-continuum Hamiltonian of the lattice regulated model.

It is also accepted that nonstaggered ladders with $n_l$ legs are gapless only if $n_l S$ is half-integer. This is most transparent when the interleg coupling $J'$ is much larger than the intraleg coupling $J$, for we can first solve the problem of $n_l$ spins along a rung, take the lowest energy multiplet in each rung, and then couple them with $J$, thereby getting a single chain, about which everything is known. One then verifies that nothing changes as $J'$ is lowered. Equivalently, one can show that the topological terms for the chains are additive, giving $\theta = 2\pi n_l S$. Different modifications of spin ladders can also be considered by adding next-to-nearest neighbor couplings. Coupled spin chains have been studied by combining mean-field theory techniques with exact results for one chain.

The spin systems to be considered here have a very important feature: they have staggered weights. Let us then begin with a single chain for which

$$J(n) = J(1 + (-1)^n \gamma)$$

is the coupling between sites $n$ and $n + 1$. Notice that $\gamma \rightarrow -\gamma$ amounts to sublattice exchange $n \rightarrow n + 1$ and that the restriction $|\gamma| < 1$, keeps the interaction antiferromagnetic.

Affleck and Haldane showed that in this case

$$\theta = 2\pi S(1 + \gamma)$$

so that when $\gamma$ is varied from $-1$ to $+1$, $\theta$ passes an odd multiple of $\pi$ i.e., the system is critical, exactly $2S$ times. It is instructive to interpret these transitions in the valence bond terminology of Affleck et al. (AKLT), wherein each spin-$S$ is viewed as a symmetrized product of $2S$ spinors. As $\gamma$ is raised from $-1$, the chain goes from being fully dimerized with all the valence bonds (spinor contractions) on odd-$n$ links, to being dimerized with all valence bonds on the even-$n$ links. As each spinor index switches loyalty, it necessarily reaches a point when it can equally well go in either way, producing a nonstaggered i.e., a gapless spin-$\frac{1}{2}$ chain. (It is important to realize that the effective interaction of these spin-$\frac{1}{2}$ degrees of
freedom can be nonstaggered even though the original Heisenberg interaction is.)

Consider staggered chains shown in Fig. 2 with horizontal couplings on the $a$th-leg ($a = 1, \ldots, n_l$) obeying

$$J_a(n) = J(1 + (-1)^{n+a+1} \gamma)$$

(6)
i.e., staggered in both directions. We now show that such systems have a rich phase structure in the $\gamma$ versus $J'/J$ plane at each $n_l$ and $S$.

Recall Affleck’s derivation of the sigma model hamiltonian from the spin chain by pairing spins, forming their difference and sum, and turning these into the sigma model field and its conjugate momentum respectively, in the limit of large $S$. This method was generalized by Sierra to uniform ladders. The main difference was the $n_l$-fold increase in the number of degrees of freedom due to the transverse label $a = 1, \ldots, n_l$ for the legs. A low energy analysis indicated that only one of these modes remained low in energy and defined the effective sigma model, while the rest had a gap of order $J'$. This effective model had $\theta = 2m_l S$, (independent of couplings), yielding the previously quoted result for nonstaggered ladders, namely that only an odd number of half-integer chains were massless.

We extended this derivation to the staggered ladders and found

$$\theta = 2\pi S n_l (1 + \gamma f_{n_l}(J'/J))$$

(7)

where

$$f_{n_l}(J'/J) = \frac{1}{n_l^2} \sum_{m=1,3,\ldots,n_l-1} \frac{1}{\sin^2(\frac{2\pi m}{2n_l})} \frac{1 + J'/J \cos^2(\frac{2\pi m}{2n_l})}{1 + J'/J \cos^2(\frac{2\pi m}{2n_l})}$$

(8)

with $\delta_{n_l,\text{odd}}$ equal to 1 if $n_l$ is odd and zero otherwise. We refer the reader to Sierra for a very similar derivation in the uniform case.

The critical points follow from setting this $\theta$ equal to odd multiples of $\pi$.

We analyze two cases: $n_l = 2$, $S = \frac{1}{2}$, 1 which should convince the reader of the soundness of this method, and facilitate the discussion of the cases with larger $S$ and $n_l$.

Consider Fig. 2. On the $\gamma$ axis, where the chains decouple, there is just one (which equals $2S$) critical point corresponding to the nonstaggered spin-$\frac{1}{2}$ chain. Our theory predicts that as we turn on $J'$, this becomes two critical points that move towards the walls $|\gamma| = 1$. It also tells us that although staggering or interchain coupling are individually bad for criticality, a certain combination can sustain criticality. Can we believe this? Let us go to $\gamma = -1$. Now each chain breaks up into disconnected pairs, but the disconnected pairs of one chain do not lie opposite those of the other, but displaced by one unit. When these get coupled by $J'$, we have a “snake” chain that winds through the lattice. It is a spin-$\frac{1}{2}$ chain with staggered weights $2J$ and $J'$. Clearly at $J' = 2J$, it becomes critical as predicted by the theory. Thus the vertical $J'$ axis is seen to play the role of an effective $\gamma$ for the snake. We display this by showing three snakes on the left margin of Fig. 2, with vertical bonds which are stronger than, equal to and weaker than the horizontal ones ($2J$). As expected, the same thing happens on $\gamma = +1$, with $n_l \to n + 1$.

Although it is not so easy to understand criticality as we go into the rectangle, by continuity of $\theta$, the critical curve must exist. There is however one caveat: the phase diagram in Fig. 2 does not strictly follow from the equation for $\theta$ when $J' \to 0$: the two critical curves coming down from $J'/J = 2$ on $|\gamma| = 1$ will cut the $\gamma$ axis at distinct points on either side of the origin instead of meeting there. But we know that the sigma model mapping is doomed to fail as $J' \to 0$: we will get not one low energy field, (the putative sigma model field) but two, since the gap that separated the sigma model field from the other, of order $J'$, vanishes. Fortunately, on the $\gamma$ axis, where the chains decouple, we know everything: there is only one transition at $\gamma = 0$ which the two chains undergo simultaneously. Thus the final phase diagram was obtained by combining what we know on the $\gamma$ axis (about decoupled chains) with what we know off the $\gamma$ axis (from the sigma model). Even off the $\gamma$ axis the sigma model is only to be taken as a guide to the topology of the phase diagram and not for the exact location of the critical curves. This is because the formula for $\theta$ is generally not exact except when $\gamma = 0$ and $\theta = 2\pi n_l S$, in which case the sigma model is invariant under parity and you cannot alter $\theta$ by a small amount (say of order $1/S$), without violating parity.

Along an arc starting at $\gamma = -1$, $J' = 0$ and ending at $\gamma = 1$, $J' = 0$, $\theta$ rises continuously from 0 to $4\pi$. The critical behavior is the same across any of these critical curves and the gap will behave as $t^{2/3}$, where $t$ is the control parameter, as predicted by Cross and Fisher. (There will be logarithmic corrections since the $\theta = \pi$ sigma model differs from the conformally invariant WZW model by a marginally irrelevant operator.) Chitra et al avoid the log by adding a special value of nm coupling to the spin $\frac{1}{2}$ chain and find an exponent very close to $2/3$.

Let us examine Fig. 2 in terms of the RVB picture of White, Noack and Scalapino for a nonstaggered $n_l = 2$, $S = \frac{1}{2}$ system which corresponds to a point vertically above the origin in Fig. 2, with some generic $J'$. (It might help to consult their Fig. 3 for this discussion.) In the absence of defects, the bonds in each two by two square resonate between being vertical (with coupling $J'$) and horizontal (with coupling $J$). A defect forces the bonds to be horizontal, staggered, and nonresonating, till we reach the next defect. This causes a linear confining potential and restricts the excitations to spin-1. In our problem, we are free to move to the left of this point towards negative $\gamma$. Now the staggered horizontal bond configuration between the defects becomes more favorable and we soon hit the critical curve on which the staggered bonds configuration becomes degenerate with the resonant ones,
and the defects (spinons) are liberated. To the left of
the critical curve confinement resumes, for reasons best
understood if we drop vertically from the critical point
to the \( J' = 0 \) axis. Now we have decoupled staggered
chains. The bonds are dimerized in the preferred sublat-
tices. A pair of defects now forces singlets on unfavora-
ble bonds in the region in between. When \( J' \) is turned on,
between the defect, the bonds can resonate since the de-
fact has lined them up across each other. Increasing \( J' \)
improves resonance and we finally hit the critical curve.
In general all our critical curves may be characterized as
those on which the defects are unconfined.

What about the order parameter for the different
phases? Once again it is best to move up the \( \gamma = -1 \) axis,
where we see that the valence bonds go from being hori-
zontal to vertical. This is just the Affleck-Haldane trans-
fer of bonds on a chain, but along the length of the snake,
wherein even/odd bonds turn into vertical/horizontal
bonds.

Consider Fig. 3 for spin-1. Once again on \( \gamma = -1 \) we
get a spin-1 snake, which becomes gapless when its stag-
gering equals \( \pm 1/2 \) according to the sigma model \([\Box, \Box]\).
Once again the ratio of couplings \( J'/2J \) determines the
effective staggering along the snake, not to be confused
with the original \( \gamma \) for the ladder. Setting \( J'/2J \) equal to
\((1 \pm 1/2)/(1 \mp 1/2)\) we get critical values \( J'/2J = 3, 1/3 \).
It is clear that we can adapt the nonlocal order parameter
of den Nijs and Rommelse \([\Box]\) (rendered along the snake)
to describe the \( Z_2 \) symmetries. We do not discuss spin-1
further since it resembles spin-\( \frac{1}{2} \) in other respects.

For larger values of spin and \( n_l \), each single-chain tran-
sition on the \( \gamma \) axis splits into \( n_l \) transitions as we turn on
\( J' \). The critical curves bend towards the wall (\(|\gamma| = 1\))
nearest to them. The parameter \( \theta \) rises continuously from
0 to \( 4\pi n_l S \) as we follow the arc shown in Figure 3. There
are however some differences. First, we get honeycomb
ladders instead of snakes for larger \( n_l \). Next, we no
longer have an easy way to see the sigma model is even
qualitatively correct when it locates critical lines for us.
However, we expect the model to be weakest when \( n_l \) or
\( S \) is small. Having passed the test there, it seems im-
une to further jeopardy. Finally, if \( n_l S \) is half-integer,
an odd number of lines will emanate from the origin, one
of which will go straight up to \( J' = \infty \) (corresponding
to nonstaggered odd-\( n_l \) half-integer spin chains ladders,
known to be gapless).

What does the sigma model have to say about holes?
It was shown by Shankar \([\Box]\) that in the large \( S \) limit,
the holes in the single chain may be represented by spin-
less fermions that couple to the sigma model field via a
gauge interaction. At finite doping, the fermions render the
\( \theta \) term ineffective, wiping out the sharp distinction
between integer and half-integer chains. The extension of
this calculation to ladders will tell us if all ladders have
exponential decay upon doping.

To summarize, we have considered the phase diagram
of ladders with staggered couplings by mapping the ladd-
ers to a sigma model and setting its topological coeffi-
cient \( \theta \) to an odd multiple of \( \pi \) (when the model is
known to be massless). There were a few surprises: we
have examples here wherein coupling gapped chains leads
to gapless chains. This is because there is an interplay
between staggering and interchain coupling which sepa-
rately destroy gaplessness, but together can conspire to
keep the system gapless. Thus two spin-\( \frac{1}{2} \) chains with
small staggering and small \( J' \) can remain massless. At
all these phase transitions the gap will vanish as \( t^{2/3} \), (up
to logarithms). We expect gapped states to exhibit linear
confinement of a pair of defects and possibly pairing, if
the doping is macroscopic.

It will be worth confirming these predictions by Monte
Carlo, Density Matrix Renormalization Group, series ex-
pansions and so on. The sigma model complements these
approaches: it does not do so well numerically, but man-
gages to give at one stroke the phase diagram for any
choice of \( S \) and \( n_l \). For instance, we know that on
\( \gamma = \pm 1 \), where we have a honeycomb ladder, each transi-
tion of a single chain gets transformed into \( n_l \) transitions
as \( J' \) is varied. This accumulation of critical points (for
any spin, half-integer or otherwise) facilitates extrapolation
to the ordered state in \( d = 2 \), although we cannot
raise \( n_l \) too much.

We leave it to the ingenuity of the experimentalists to
find ladders wherein bonds alternate in both directions
\([\Box]\), and either \( \gamma \) or \( J'/J \), or both, can be varied
at least slightly. There is also the option of studying the
honeycomb ladder, an extreme case of bond alternation
(\( \gamma = \pm 1 \)). Once any such a ladder is found, it will
have many transitions, whatever be the spin. For example
a honeycomb ladder with four legs and spin 1 will have four
transitions as \( J' \) varied, say by applying pressure.

Given the many formal similarities between the \( \theta \) pa-
rameter of the Quantum Hall Effect and the one we have
here, one can expect strong parallels between coupled
chains and coupled Hall planes.

Note added in proof: The phase diagrams have a nice
extension to \( J' < 0 \), for the case where \( J_n(n) = J(1 +
\gamma(-1)^n) \), i.e., the staggering is only along the leg but not
along the rung direction: if we lower \( J' \) from 0 to \(-\infty \)
each transition point of the decoupled spin-S chain splits
into \( n_l \) lines and all \( 2n_l S \) of them flow down to \( J = -\infty \)
and terminate at the gammas corresponding to the \( n_l S \)
transitions of the spin 2S chain.

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FIG. 1. A typical staggered ladder with staggered couplings $J(1 \pm \gamma)$ along the horizontal legs and $J'$ along the vertical rungs.

FIG. 2. Phase diagram for staggered ladders with $n_l = 2$ and spin $\frac{1}{2}$. The solid lines represent critical lines corresponding to the $\theta$-parameter being an odd multiple of $\pi$, (the dashed line is just indicative for counting the number of critical lines between $\gamma = -1$ and $\gamma = +1$.) On the margins we show the snake patterns associated to criticality (non-staggered) and to gaped phases (staggered).

FIG. 3. Phase diagram for staggered ladders with $n_l = 2$ and spin 1. See Fig. 2 for similar explanations.
\begin{figure}
\centering
\begin{array}{c}
J (1+\gamma) \\
J (1-\gamma)
\end{array}
\begin{array}{cccc}
\hline
J (1+\gamma) & J (1-\gamma) \\
\hline
J' & J' & J' & J' \\
\hline
J' & J' & J' & J' \\
\hline
\end{array}
\caption{Figure 1}
\end{figure}
Figure 2
Figure 3