Hole Properties In and Out of Magnetization Plateaus in 2-d Antiferromagnet

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We study the signatures of magnetization plateaus and the presence or absence of Goldstone modes in terms of their effects on the physics of holes in hole-doped two-dimensional antiferromagnet defined on square lattice. Holes with quadratic dispersion around Fermi point existing at infinitesimally small doping and linear dispersion around nearly circular Fermi surface at finite but low doping are investigated. They are coupled to an effective gauge field, generated by the spin sector, which subsequently mediates interaction between the holes. We find that out-of-plateaus case leads to algebraically decaying long-range interaction between fermionic holes with both Coulombic and dipolar forms, whereas in-plateaus case leads to short-range (local) interaction. We show that the spectral peak is significantly broadened in the out-of-plateaus case, while the spectral weight is still sharply-peaked in the in-plateau case. This conclusion holds in both infinitesimally small doping limit and in the more realistic finite doping case. We also extend the result obtained for 1-D system where finite hole doping gives rise to a shift in the magnetization value of the plateaus.

Introduction. — Antiferromagnets under magnetic field have been known to display magnetization plateaus and the theory of this magnetization plateaus has been an important problem in magnetism. The explanation for the theory of this magnetization plateau has been an important problem in magnetism. The explanation for one of the central questions that theories try to answer. Even more intriguing question is what happens if we remove some spins from the antiferromagnet and thus introduce holes to it. Hole doped antiferromagnet was found important in the context of high $T_c$ superconductivity in Cuprates where superconducting state emerges upon hole doping the parent compound antiferromagnet. The physics of magnetization plateaus should have immediate consequences on the properties of holes and this is what we investigate in this work. Specifically, we will show that in-plateaus and out-of-plateaus states of antiferromagnet give rise to clear distinct types of interaction between holes and the resulting spectral function.

The theory of magnetization plateaus, without hole doping, can be relatively well understood with a spin path-integral approach [1]. In one dimension it gives rise to plateaus quantization condition derived long time ago based on Lieb-Schultz-Mattis theorem [2,3] as shown first in [5,6]. The presence of holes in 1-d can also be treated easily with bosonization [7,8,9,10] and and spin path integral[11,12]. However, generalization of the theory to two and higher dimensions remains a challenge. In this work, we will show that one can gain important insights into the physics of magnetization plateau in higher dimensions by working out the boson-fermion field theory of hole-doped antiferromagnet. To be precise, we will use field theoretical framework to investigate the physics of magnetization plateau in two dimensions by studying the signatures of magnetization plateaus in terms of hole properties.

Field Theory. — We start with Euclidean space-time effective action of 2-d antiferromagnet in the presence of holes

$$S_\phi = \int d^2 x \int dt \frac{K_\tau}{2} (\partial_\tau \phi)^2 + \frac{K_\sigma}{2} (\nabla \phi)^2 + \frac{S - m}{a^2} \partial_\tau \phi$$

(1)

$$S_{\bar{\psi}, \psi} = \int \bar{\psi} (\partial_\tau - ieA_\tau) \psi + \int \bar{\psi}_k \epsilon_{k \prime s} (\phi_{k \prime}) \psi_{k s} \delta(\sum_{k \prime s})$$

(2)

describing low-energy long-distance fluctuations around classical ground state specified by $S = (\sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0, \cos \theta_0)$ with spin $S$ and $z$ magnetization $m = S \cos \theta_0$, where $\phi$ is the phase angle fluctuation field around $\phi_0$ [1]. The $K_\tau, K_\sigma$ are stiffness coefficients which can be determined from microscopic spin model [13], giving boson velocity $v_b = \sqrt{K_\sigma/K_\tau}$. The $\bar{\psi}, \psi$ represent the creation and annihilation operator fields of the (spinless) fermionic holes. The $\epsilon_{k s} (\phi_{k \prime})$ is fermion energy dispersion that couples the holes to the spin sector represented by $\phi$ field [14]. Our theory will be very generic, but it is aimed to be a paradigm for spin systems well described by Heisenberg model with strong anisotropy and is under magnetic field,

$$H = J \sum_{i \neq j} S_i \cdot S_j + D \sum_i (S^z_i)^2 - h \sum_i S^z_i$$

(3)

with classical ground state characterized by $\cos \theta_0 = \frac{h}{25(4J + D)}$ [11], such as those systems with $S = 3/2$ where $1/3$ plateau is expected to occur at large enough $D$ [15].

We will consider a model for holes which in the realistic case of finite doping has linear energy dispersion around Fermi surface. The hole doping itself will give feedback effect to the spin sector. In such linear fermion dispersion, a sea of occupied negative energy states arises due to linearization and must be removed by applying projection operator [11,12]: $P_j = 1 - \psi_j^+ \psi_j / (2S)$ at each site $j$ on the microscopic lattice model. The doping in turn modifies the plateau quantization condition via normal ordering of the fermion bilinear operator: $\psi_j^+ \psi_j = \delta^+ : \psi_j^+ \psi_j :$.
where \( \delta = \langle \psi_d^+ \psi_j \rangle \) is the doping level. We find that with hole doping \( \delta \), plateau occurs at

\[
1 - \frac{\delta}{2S} (S \pm m) \in \mathbb{Z}
\]

indicating a shift in magnetization plateau, proportional to doping level \( \delta \), compared to the zero doping case, confirming the result in 1-d [12].

As was shown in [1], the presence of the Berry phase term plays a crucial role in the large scale physics of the spin sector. If the factor in front of it is an arbitrary real number, field configuration with vortices are forbidden by quantum interference and the Goldstone field \( \phi \) is protected and the system does show long range order and gapless behavior with no plateau. On the contrary, when the Berry phase factor is an integer, vortex configurations are allowed and, for some values of the spin field stiffness, the system may disorder and acquire a gap. This is the plateau situation which can phenomenologically represented by an effective mass term in the Goldstone field, writable as \( m_{\phi}^{2}/2 \), into the effective action Eq. [1].

We describe holes in antiferromagnet as follows. For concreteness, we consider a simple model with holes hopping on square lattice with nearest-neighbor tight-binding dispersion \( \epsilon^{0}_k = -2t(\cos k_x + \cos k_y) - \mu \). This gives Fermi surface with shape which depends on the chemical potential (and thus filling factor); at chemical potential \( \mu = -4t \) we get a Fermi point (corresponding to zero or thermodynamically small number of hole doping), at \(-4t < \mu < 0 \) we get roughly circular Fermi surface that can be described by \( k^2_x + k^2_y = k_F^2 = 4 + \mu/t \) and at half filling \( \mu = 0 \), we get a square-shaped Fermi surface described by \( k_{Fy} = \pm k_{Fx} \pm \pi/2 \).

The fermionic holes will be coupled to gauge field generated by spin sector. An effective action for hole with such coupling can be derived by by considering tight-binding hopping Hamiltonian [12] with hopping integral which involves the overlap of the spin coherent states at the neighboring sites between which the hole hops [11], giving the spatial part of gauge field \( A_x, A_y \), plus applying projection operator that represents the process of doping holes [12], giving the temporal part of gauge field \( A_t \). The result is equivalent to a minimal coupling \(-i\partial_{\mu} \rightarrow -i\partial_{\mu} - eA_{\mu} \) between the spin sector’s gauge field and the hole. Considering nearest-neighbor tight binding Hamiltonian on square lattice and applying this minimal coupling to the free hole dispersion \( \epsilon^{0}_k \) gives \( \epsilon^{0}_{k'}(\phi_{k'}) = -2t(\cos k_x - ie_\phi k'_{x_\phi} + \cos k_y - ie_\phi k'_{y_\phi}) \)

where we have used \( eA_{\mu} = e_\phi \partial_{\mu} \phi \) with \( e \equiv e_\phi \) the effective gauge charge of the \( U(1) \) gauge theory [16]. Performing Taylor expansion to the two cosines terms around the minimum of the band, and doing the Euclidean space-time functional integral, we obtain

\[
Z = \int D\bar{\psi} D\psi e^{-\sum_{k'} \bar{\psi}_{k'}^c \psi_{k'} - \delta S_{\psi, \psi}}
\]

where

\[
\delta S_{\psi, \psi}^{\text{quadratic}} = \frac{1}{2} \int_{k's} \bar{\psi}_{k''} \psi_{k''} \left[ \frac{\epsilon^{0}_k}{2} \frac{k_x^2 k_y^2}{4} + \frac{k_x^2 k_y^2}{4} + \frac{1}{2} m_{\phi}^2 \right] \frac{\epsilon^{0}_{k'}}{2} \frac{k_x'^2 k_y'^2}{4} \delta(\sum_{k'} (k_x - k'_x))^2 (k_y - k'_y)^2)
\]

in Euclidean space-time [15]. We see that the main effects of the spin sector manifest in the form of 4-fermion interaction term (scattering between two fermions) with kernel which is massless for long-range interaction between vortex loops but gapped for short-range interaction between vortex loops. We note that outside the plateau where \( m_{\phi} \rightarrow 0 \), as \( |k| \rightarrow 0 \) the kernel goes as \( k_{\alpha} G(k) k_\beta \sim 1/K_{r, r} \), while in the plateau where \( m_{\phi} \rightarrow \infty \), the kernel goes as \( k_{\alpha} G(k) k_\beta \rightarrow 0 \). This implies that within the plateau, we have true short-range interaction between fermionic holes whereas outside the plateau, we have nonlocal algebraically decaying interaction between fermionic holes [19]. This 2-fermion scattering action is best illustrated by the Feynman diagram in Fig. [1].

An important result of this work is the final form of this 4-fermion interaction term in the out-of-plateau and in-plateau cases, which in Euclidean space-time can be written as

\[
\delta S_{\psi, \psi}^{\text{quadratic}} = \int \bar{\psi}_{k''} \psi_{k''} V(k's) \bar{\psi}_{k'}^c \psi_{k'} \delta(\sum_{k'} (k_x - k'_x))^2 (k_y - k'_y)^2)
\]
\[ V^{\text{out-of-plat}}(k') = \frac{2}{K_s} g_2^2 \left[ \frac{[2(\mathbf{q} \cdot \mathbf{k}''') \mathbf{q} \cdot \mathbf{k}'''']}{q_0^2 + q^2 + m_s^2} \right] ^2 \]

\[ V^{\text{in-plat}}(k') = \frac{2}{K_s} A k^2 \left[ (k'''' - k^{''''}) (k''' - k''') \right] \]

\[ \gamma = -\frac{1}{4} [k''''] (k''') - k_0''] \]

for the out-of-plateau \((m_\phi \to 0)\) and in-plateau \((m_\phi \to \infty)\) cases represented in Fig. 1(a) and 1(b), respectively. We have rescaled \(K_T = K_T = K_s\) (equivalent to setting the boson velocity to unity \(v_b = 1\)) and the constants are \(m_s^2 = m_s^2 / K_s\) and \(A = 1/m_s^2\) \cite{13}. Interestingly, \(V^{\text{out-of-plat}}(k')\) contains algebraically decaying interaction with dipolar form in real space in addition to the more conventional density-density interaction term,

\[ \delta S_{\text{dip}} = \frac{2}{K_s} t^2 e_3^2 \int_{x_\mu, x_\mu'} \frac{3(d_1 \Delta x)(d_2 \Delta x) - (d_1 \cdot d_2) |\Delta x|^2}{4\pi |\Delta x|^5} \]

\[ \delta S_{\text{dens}} = -\frac{1}{2K_s} e_3^2 \int_{x_\mu, x_\mu'} \rho(x) \frac{3(|\Delta x|^2 - |\Delta x|^2)}{4\pi |\Delta x|^5} \rho(x') \]

with dipole moments \(d_1 = [\nabla \bar{\psi}(x)] \psi(x)\), \(d_2 = [\nabla \bar{\psi}(x')] \psi(x')\), \(\Delta x = x' - x\), \(|\Delta x| = |x' - x|\), \(\rho(x) = \bar{\psi}(x) \psi(x)\). In each of Eqs. (9) and (10), the spatial momentum part represents the dipole interaction whereas the frequency \((k_0'')\) part represents the density-density interaction. Surprising as it is, dipolar interaction intuitively originates from spatial nonuniformity of the hole density distribution, which gives rise to nonzero effective dipole moment, corresponding to nonzero Fourier wavevectors \(k'\) \neq 0. Such dipolar term will vanish for spatially uniform distribution of holes, where only \(k' = 0\) remains. The presence of both space and time distances in Eqs. (11) and (12), corresponding to the presence of both momentum and frequency dependences in the kernel Eq. (10), reflects the fact that the long-range interaction is not instantaneous as it is mediated by Goldstone bosons with low speed \(v_b \ll c\) in reality. \(S_{\text{dens}}\) has asymptotic spatial dependence \(V(r) \sim 1/r^3\) at large distances and is repulsive \cite{13}. This unexpected result arises from the peculiarity of the gauge field with its origin from the spin sector’s physics and its couplings to holes.

Next, we consider finite but low doping levels at \(-4t < \mu < 0\), where we have roughly a circular Fermi surface. In this case, we obtain linearized dispersion \(\epsilon_k = 2t[(k_x - k_F) \sin k_x F + (k_y - k_F) \sin k_y F]\) where \(k_x^2 + k_y^2 = k^2 = 4 + \mu / t\) derived using Taylor series expansion of nearest-neighbor tight-binding energy dispersion \(\epsilon_k = -2t(\cos k_x F + \cos k_y F) - \mu\) around Fermi surface satisfying \(-2t(\cos k_x F + \cos k_y F) - \mu = 0\) \cite{17}. Using the same functional integral formalism as employed previously, we obtain 4-fermion interaction from integrating out the scalar field \(\phi\) similar to Eq. (9), but with the integral over fermion momenta constrained to be near Fermi surface only. The resulting kernels for 4-fermion interaction term in the out-of-plateau and in-plateau cases are the same as those in Eqs. (9) and (10) but with such constraint on the integral over \(k'\) \cite{13}.

We observe that the distinction of the physics of the spin sector inside and outside plateau manifests in the form of distinct characters of the resulting fermion-fermion interaction between holes in hole-doped antiferromagnet. The origin of this distinction as we noted earlier is clear. Namely, inside the plateau \(U(1)\) symmetry is preserved and there is no Goldstone mode which can mediate long-range interaction so that the resulting fermion-fermion interaction is short-ranged. Outside the plateau, \(U(1)\) symmetry is broken and as a result we have Goldstone modes which mediate and give rise to the long-range interaction between fermionic holes.

\[ A(k, \omega) = -2\text{sgn}(\omega) \text{Im} G(k, \omega) \]

where \(G(k, \omega)\) is renormalized Green’s function which embodies the effects of interaction of fermions with each other and with other degrees of freedom. We will consider approximation where we geometric sum a particular family of diagrams involving series of one-loop fermion self-energy diagrams and obtain the familiar result, \(G^{-1}(k, \omega) = G_0^{-1}(k, \omega) - \Sigma(k, \omega)\) where in this case \(\Sigma(k, \omega)\) is the one-loop self-energy correction to free fermion Green’s function \(G_0^{-1}(k, \omega) = \omega - \epsilon_k + i\eta \text{sgn}(|k| - k_F)\) with \(\eta\) an infinitesimally small positive number to be taken to zero at the end of calculation \cite{21}. We then obtain for the spectral function

\[ A(k, \omega) = -2\text{sgn}(\omega) \frac{\Sigma_f(k, \omega)}{\omega - \epsilon_k - \Sigma_R(k, \omega)} + (\Sigma_I(k, \omega))^2 \]

where \(\Sigma_R, \Sigma_I\) represent the real and imaginary parts of the self-energy respectively. The result for spectral function \(A(k, \omega)\) is expected to be qualitatively different between the in-plateau and out-of-plateau cases. This distinction is what we expect to be the prospective experimental signatures that distinguish the physics of antiferromagnet between within and outside of plateau.

For the in-plateau case, where we have local interaction, we compute the one-loop fermion self-energy diagram shown in the Fig. 2 with 4-fermion vertex given in Eq. (10) from which we obtain for the one-loop self-energy

\[ \rho(x) = \bar{\psi}(x) \psi(x) \]
We show the resulting profile of $A(\omega)$ at a fixed $|\mathbf{k}|$ in Fig. 4(b) for this off-plateau case with quadratic hole dispersion. We notice that, due to the algebraically decaying long-range fermion-fermion interaction, the spectral weight is heavily broadened compared to that of free noninteracting fermions which has hallmark delta function peak. The spectral peak broadening increases with the strength of the coupling to gauge field represented by gauge charge $e_g$ and also the Goldstone mode’s total energy bandwidth $\delta \epsilon_k \sim 2v_h \Lambda$, where $2\Lambda$ is the total momentum bandwidth. In the original microscopic spin model Eq. (3), this is achieved for large $J, D \gg \hbar$. Comparing the two cases, it can be seen that the hole spectral function in the out-of-plateau state is much more significantly broadened and suppressed compared to that of the in-plateau state. This broadening reflects the effects of Goldstone bosons which survive outside the plateau and mediate the long-range interaction. Technically speaking, the broadening arises because the frequency dependence of the self-energy derived from the long-range interaction kernel Eq. (9) out of plateau creates numerous additional poles in the renormalized Green’s function $G(\mathbf{k}, \omega)$.

We now consider the more realistic finite hole doping situation for which $-4t < \mu < 0$ and we have roughly circular Fermi surface. Following similar procedure as that for quadratic dispersion, nontrivial calculation gives, for in-plateau case, the profile of spectral function as shown in Fig. 5(a) where we notice clearly that even with linear dispersion, on-plateau case’s local interaction does not affect much the free fermion’s delta function spectral weight. Equally delicate computation for the out-of-plateau case gives spectral function shown in Fig. 5(b) which suggests that off-plateau’s long-range interaction also significantly broadens out the spectral weight.

Discussion. — We have demonstrated that the fermion
spectral function of hole-doped antiferromagnet can be used as a direct probe of on-plateau vs. off-plateau physics of the spin sector. We have shown that within plateau the spin sector generates local fermion-fermion interaction while outside plateau it generates long-range fermion-fermion interaction with both density-density and dipolar contents. This difference manifests in the spectral function of the holes. In particular, our result predicts that the hole spectral function for the in-plateau case remains a sharp delta function hallmark of free fermion spectral function with negligible broadening, whereas outside plateau, the hole spectral function is significantly broadened and reduced in height, subject to an appropriate sum rule. We also predict that finite hole doping will shift the magnitude of plateaus. It is important to also mention that, with the presence of long-range algebraic interactions, there is a possibility for the formation of Wigner crystal \[23\] of holes, when the density-density interaction, which is indeed repulsive in this case, dominates over dipolar interaction and kinetic energies. In contrast to the usual Coulomb case however, based on dimensional analysis, we expect the Wigner crystal to occur at high density of holes rather than low density. This is due to the fact that the algebraic interaction decays as \(V(r) \sim 1/r^3\) rather than the usual \(V(r) \sim 1/r^2\), with kinetic energy goes as \(1/r^2\).

The spectral function is normally measured with photoemission experiment at given energy (frequency) \(\omega\) as function of momentum \(K\) or vice versa. The results presented here should be able to be directly checked by experiments. Interestingly, our result implies that in the case where there is a set of adjacent plateaus, there will be consecutive sharpening and broadening changes in the hole spectral function as one tunes parameter across consecutive plateau-off-plateau transitions. We hope that this work motivates further theoretical, numerical, and experimental research on magnetization plateaus in hole-doped antiferromagnet.

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\[
K_r = \frac{1}{2a^2 (4J + D)} ,\quad K_r = J (S^2 - m^2) \tag{16}
\]

[14] In the rest of this paper, \(k’\)s represents the set of all momenta-frequencies appearing in the expression: \(k’ = k, k’, k’’, \ldots\), \(K_{k’} = \oint d^dk’/ (2\pi)^3 \oint d^dk’/ (2\pi)^3 \int d^dk’/ (2\pi)^3 \cdot \cdot \cdot \) where \(d^dk = dkd^2k\), and \(\delta(\sum_{k’} k’) = \delta(k - k’ + k’’ \cdot \cdot \cdot)\) imposing the conservation of momentum-frequency.

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\[
g_2 = \frac{1}{2S} \left( \frac{S - m}{a^2} + \frac{2\pi m^2 S^2 - \cos kFr}{(1 - \frac{4\pi}{3r^2})(4J + D)} \right) \tag{17}
\]

at doping level \(\delta\). Lorentz invariant theory requires \(\epsilon_{g_r} = \epsilon_{g_x} = \epsilon_{g_y} = \epsilon_q\) which can be achieved by appropriate rescaling of space-time.

[17] This linearized dispersion can be approximately represented by isotropic cone dispersion \(\epsilon_k = \sqrt{F^2 \left( |k| - k_F \right)}\) with uniform Fermi velocity \(\sqrt{F^2} = 2t \sin k_F\), to be used in field theoretical perturbation calculations.
[18] Please see the Supplementary Material.
[19] In this work, we define Coulomb interaction to be that derived from Gauss law \(\nabla \cdot E = -\nabla^2 V = \rho/\epsilon_0\), giving, for particles of charge \(e\)

\[
H = \int_{k_1, k_2, q} -\bar{\psi}_{k_2 - q} \psi_{k_2} \frac{4\pi e^2}{q^2} \bar{\psi}_{k_1 + q} \psi_{k_1}
\]

The (true) long-rangeness is signalled by the divergence of the kernel \(V(q) = 4\pi e^2 / q^2 \rightarrow \infty\) as \(q \rightarrow 0\). Weaker divergence, e.g. \(V(q) \rightarrow V(0)\) with \(0 < V(0) < \infty\) as \(q \rightarrow 0\) indicates faster decaying long-range interaction.
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Hole Properties In and Out of Magnetization Plateau in 2-d Antiferromagnet
Supplementary Material

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Derivation of Fermion Effective Action

The effective action for fermion obtained from integrating the spin sector scalar field is derived using functional integral formalism in Euclidean space-time. From the full action Eqs. (1) and (2), we can write the partition function as

\[ Z = \int D\bar{\psi} D\psi e^{-S_0_{\bar{\psi},\psi}} \int D\phi e^{-\int_{k,k'} \phi_k \phi_{k'} \delta(k-k') - \int f_k \phi_k} \]

First, we consider quadratic dispersion \( \epsilon_k = \frac{1}{2} m^2 \phi(k') \) for hole valid near the minimum of the band, which will be at Fermi level and forms Fermi point when chemical potential \( \mu = -4t \) corresponding to zero or thermodynamically small hole doping. That is, we can write noninteracting kinetic fermionic hole action

\[ S_{\bar{\psi},\psi} = \int_r \bar{\psi}(r) (\partial_r - t \nabla^2) \psi(r) = \int_k \bar{\psi}_k (ik_0 + tk^2) \psi_k \]

The correction (second) term in Eq. (20) vanishes as \( |k| \rightarrow 0 \), which should indeed be the case since we assumed classical ground state with broken global continuous symmetry and the associated massless Nambu-Goldstone modes. In the zeroth order approximation,

\[ f_k = -\left( \frac{S - m}{a^2} \right) k_0 - 2it e_g \int_{k',k''} k \cdot k'' \bar{\psi}_{k''} e_g \psi_{k'} \delta(-k'' + k'' + k') + \int_{k',k''} \bar{\psi}_{k''} e_g k_0 \psi_{k'} \delta(k + k'' - k'') \]

where in this derivation, \( k, k' \) are reserved for the momentum-frequency of the spin sector scalar field \( \phi \) while the other \( k' \)'s are the momenta-frequencies of the fermions. Integrating out the bosonic scalar field \( \phi \) in Eq. (18), we have

\[ Z = \int D\bar{\psi} D\psi e^{-\int \bar{\psi} \delta_h \psi_k - \frac{1}{4} \int_{k,k'} G(k) f_{k'} \delta(k-k')} \]

The inverse kernel in Eq. (18) is given by

\[ G^{-1}(k) = \left[ \frac{K_r}{2} k_0^2 + \frac{K_r}{2} k^2 + \frac{1}{2} m_2^2 \right] \]

where the last term (effective mass) will later determine whether one is in plateau or out of plateau. The hole is coupled to gauge field from spin sector which can be represented by minimal coupling \( k_\mu \rightarrow k_\mu - e A_\mu \). In the whole following derivation, we will set the boson velocity to unity \( v_b = 1 \) for brevity. We obtain action of quadratic dispersed fermion coupled to gauge field

\[ S_{\bar{\psi}, \psi} = \int_{k' s} \bar{\psi}_{k' s} (it k_0 + e_g k_\mu \phi_{k'} - t(k'' - ie_g k' \phi_{k''})^2) \psi_{k', s} \delta \sum_{k'} \]

The mass of the vortex loops receives correction from the fermion-gauge field coupling

\[ \frac{1}{2} m^2_\phi(k, k') = \frac{1}{2} m^2_\phi \delta(k - k) + t e_g \int_{k'', k'''} k^2 \bar{\psi}_{k'''} \psi_{k''} \delta(-k'' + k'' + k' - k)) \]

we can take \( m^2_\phi(k, k') = m^2_\phi \delta(k - k') \). This is justified in the low energy limit \( |k| \rightarrow 0 \) and by the observation that the correction term is of order \( \alpha \sim e_g^2 \), which is the small parameter in the perturbation expansion we are doing.

The function \( f_k \) multiplying linear term in Eq. (18) is

\[ = \int D\bar{\psi} D\psi e^{-S^0_{\bar{\psi}, \psi} + \delta S_{\text{quadratic}}_{\bar{\psi}, \psi}} \]

The first term in Eq. (21), coming from Berry phase in Eq. (11), gives rise to constant energy shift plus small \( (O(\sqrt{\alpha} \sim e_g)) \) correction to bilinear fermion action,

\[ \delta S_{\text{kinetic}} = -2 \left( \frac{S - m}{a^2} \right) e_g \int_{k' s} k_0^2 G(k') \bar{\psi}_{k', s} \psi_{k', s} \delta \sum_{k'} \]

(23)
but we are more interested in fermion-fermion interaction. Considering the zeroth order approximation mentioned previously, we obtain

$$\delta S_{\text{quadratic}}^{\text{quad}} = \int_{k's} \frac{e^2 g^2}{2 K_s} \left( \partial_{\tau} \partial_{\tau'} e^{-m_s |x'_{\mu} - x_{\mu}|} \right) \rho(x') + \frac{2 t^2 e^2 g}{K_s} \int_{x_{\mu},x'_{\mu'}} \left( \frac{\delta S_{x}}{\delta x} \right) \left( \frac{\delta S_{x}}{\delta x'} \right) \left( \frac{\delta S_{\psi}}{\delta \psi} \right) \left( \frac{\delta S_{\bar{\psi}}}{\delta \bar{\psi}} \right)$$

(24)

as given in the main text, with superscript 'quadratic' refers to quadratic dispersion (to be abbreviated as 'quad' whenever necessary). We have taken into account the fact that \( k \) and \( k' \) are both momentum-frequency of the gauge field and therefore, eventually \( k = k' \).

The real space form of the above 4-fermion interaction is found to be

$$\delta S_{\text{quad}}^{\text{quad}} = \frac{e^2 g^2}{2 K_s} \int_{x_{\mu},x'_{\mu'}} \rho(x) \left( \partial_{\tau} \partial_{\tau'} e^{-m_s |x'_{\mu} - x_{\mu}|} \right) \rho(x') + \frac{2 t^2 e^2 g}{K_s} \int_{x_{\mu},x'_{\mu'}} \left( \bar{\psi}(x) \left[ \frac{3(x' - x)^2 - |x'_{\mu} - x_{\mu}|^2}{4\pi|x'_{\mu} - x_{\mu}|^3} \right] \psi(x) \left[ \bar{\psi}(x') \right] \right)$$

(25)

which takes the form of density-density and dipole-dipole (to be explained below) interactions. Here, \( \rho(x) = \bar{\psi}(x) \psi(x) \) is fermion density operator, \( x_{\mu} = (\tau, x) \) is Euclidean space-time coordinate, \( |x'_{\mu} - x_{\mu}| = \sqrt{(x' - x)^2 + (y' - y)^2 + \delta_{\mu}(\tau' - \tau)^2} \), and the spatial derivatives \( \nabla_x, \nabla_{x'} \) in the middle act only on the kernel. We have set \( K_\tau = K_\tau = K_s \) in obtaining Eq. (25), which also means we set the boson velocity to unity \( v_b = 1 \) as mentioned earlier.

In the case of confined vortex loops out-of-plateau, we evaluate twice derivatives of the kernel and take \( m_\phi \to 0 \) in the end, giving us

$$\delta S_{\text{quadratic - out-of-plateau}} = -\frac{1}{2 K_s} e^2 g^2 \int_{x_{\mu},x'_{\mu'}} \rho(x) \left( \frac{3(\tau' - \tau)^2 - |x'_{\mu} - x_{\mu}|^2}{4\pi|x'_{\mu} - x_{\mu}|^3} \right) \rho(x')$$

$$+ \frac{t^2 e^2 g}{K_s} \int_{x_{\mu},x'_{\mu'}} \left( \bar{\psi}(x) \left[ \partial_{\tau} \psi(x) \right] \left[ \frac{3(x' - x)^2 - |x'_{\mu} - x_{\mu}|^2}{4\pi|x'_{\mu} - x_{\mu}|^3} \right] \psi(x') \left[ \partial_{\tau'} \psi(x') \right] \right)$$

(26)

We note that we obtain Coulombic-like algebraically decaying long-range density-density interaction in the first term and dipolar interaction in the remaining terms. The dipolar nature of the remaining terms is indicated by the presence of derivatives on fermion fields and the form of the kernel. This is nontrivial 4-fermion interaction but can be treated with field theoretical perturbation theory.
left and right mover fermions with linear dispersion. The relevant momentum dependence to that of 1-d theory with

\[ k \]

\[ k \]

valid at finite but low doping levels, \( k_{\ell} \approx 1 \). This linearized fermion action has equivalent momentum dependence to that of 1-d theory with left and right mover fermions with linear dispersion. The 1/|\( \mathbf{k}'' \)|, 1/|\( \mathbf{k}''' \)| factors however, pose technical difficulty as their inverse Fourier transforms are not well defined. In this case, \( \mathbf{k}'' \), \( \mathbf{k}''' \) are the momenta of fermions. To handle this, we approximate the integral over the whole Fourier space with integral over Fermi surface, for which |\( \mathbf{k}'' \)| = |\( \mathbf{k}''' \)| = \( k_F \). With this, we have Eq. 31 as the final form of effective fermion action,
ably, precisely the same expression for effective action as the one obtained above using linearized dispersion.

\[ \epsilon_k = 2t[(k_x - k_{xF}) \sin k_{xF} + (k_y - k_{yF}) \sin k_{yF}] \]

Again, considering the confined \( m_\phi \to 0 \) limit out-of-plateau, we obtain

\[ \delta S_{\text{linear-out-of-plateau}} = -\frac{1}{2K_s} e_g^2 \int_{x_\mu, x'_\mu} \rho(x) \frac{3(t' - t)^2 - |x'_\mu - x_\mu|^2}{4\pi |x'_\mu - x_\mu|^5} \rho(x') \]

\[ + t^2 e_g^2 \frac{2}{K_s} \int_{x_\mu, x'_\mu} \psi(x)[\partial_x \psi(x)] \frac{3(x' - x)^2 - |x'_\mu - x_\mu|^2}{4\pi |x'_\mu - x_\mu|^5} (\partial_{x'} \psi(x')) \frac{3(y' - y)^2 - |y'_\mu - y_\mu|^2}{4\pi |y'_\mu - y_\mu|^5} (\partial_{y'} \psi(x')) \psi(x') + \int_{x_\mu, x'_\mu} \psi(x)[\partial_x \psi(x)] [\partial_{x'} \psi(x')] \psi(x) \]

where the relevant fermions contributing to the integral are implicitly constrained to live near the Fermi surface.

In the deconfined limit \( m_\phi \to \infty \) in the plateau, we should get

\[ \delta S_{\text{linear-in-plateau}} = \frac{1}{2K_s} e_g^2 \int_{x_\mu, x'_\mu} \rho(x) (\partial_{x'} \psi(x)][\partial_{x'} \psi(x)] + \int_{x_\mu, x'_\mu} \psi(x)[\partial_x \psi(x)] [\partial_{x'} \psi(x')] \psi(x) \]

The resulting net kernel is

\[ V^{\text{in-plat}}(k') = \frac{2}{K_s} A e_g^2 [-t^2(\mathbf{k}'' - \mathbf{k}'') \cdot (\mathbf{k}''' - \mathbf{k}''')] \]

\[ - \frac{1}{4} (k_{0''} - k_{0''})(k_{0'} - k_{0'}) \]

This 4-fermion effective action for linear dispersion on roughly circular Fermi surface is almost the same as that for quadratic dispersion. The sole difference being that the linear dispersion 4-fermion action involves fermions close to the circular Fermi surface whereas the quadratic one involves fermions near a Fermi point. The two cases are however smoothly connected as one slowly increases the chemical potential from \( \mu = -4t \) up. We have rechecked the above calculations using QFT's perturbation theory and obtained the same results.

[1] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Physics (Dover Publications, 1963).