Convergence and Supercloseness in a Balanced Norm of Finite Element Methods on Bakhvalov-Type Meshes for Reaction-Diffusion Problems

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Abstract
In convergence analysis of finite element methods for singularly perturbed reaction–diffusion problems, balanced norms have been successfully introduced to replace standard energy norms so that layers can be captured. In this article, we focus on the convergence analysis in a balanced norm on Bakhvalov-type rectangular meshes. In order to achieve our goal, a novel interpolation operator, which consists of a local $L^2$ projection operator and the Lagrange interpolation operator, is introduced for a convergence analysis of optimal order in the balanced norm. The analysis also depends on the stabilities of the $L^2$ projection and the characteristics of Bakhvalov-type meshes. Furthermore, we obtain a supercloseness result in the balanced norm, which appears in the literature for the first time. This result depends on another novel interpolant, which consists of the local $L^2$ projection operator, a vertices-edges-element operator and some corrections on the boundary.

Keywords Singular perturbation · Reaction-diffusion equation · Bakhvalov-type mesh · Finite element method · Balanced norm · Supercloseness

Mathematics Subject Classification 65N30 · 65N50

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1 Introduction

In this article we consider the singularly perturbed reaction-diffusion equation

\[-\varepsilon^2 \Delta u + bu = f \quad \text{in } \Omega := (0, 1)^2,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

(1)

where \(\varepsilon\) is a positive parameter, \(b > 0\) is (for simplicity) a positive constant and \(f\) is sufficiently smooth. Under these conditions on the data in problem (1), there exists a unique solution in \(H^1_0(\Omega) \cap H^2(\Omega)\) for all \(f \in L^2(\Omega)\). The solution to problem (1) typically exhibits boundary layers of width \(O(\varepsilon |\ln \varepsilon|)\) along all of \(\partial \Omega\) in the singularly perturbed case \(0 < \varepsilon \ll 1\) of interest.

For singularly perturbed problems, it is popular to introduce layer-adapted meshes \([14,16,20]\) to fully resolve layers. Then uniform convergence with respect to singular perturbation parameter can be achieved for standard numerical methods. There are two kinds of layer-adapted meshes widely used in the literature, which are Bakhvalov-type mesh (B-mesh) and Shishkin-type mesh (S-mesh) (see \([14]\)). There are a lot of research results on convergence theories of finite element methods on S-meshes; see \([5,6,15,16,22–25,29,30]\) and references therein.

Usually B-meshes have better performances than S-meshes. For example, Linß \([14]\) presented error estimations \(O(N^{-1})\) and \(O(N^{-1} \ln N)\) in the maximum norm for a first-order upwind difference scheme on Bakhvalov grid \([1]\) and Shishkin grid \([19]\), respectively, where \(N\) is the number of mesh intervals in each coordinate direction. Zhang and Liu \([27]\) and Linß \([14]\) proved error bounds \(O(N^{-1})\) and \(O(N^{-1} \ln N)\) in the energy norm for linear finite element method on Bakhvalov-type mesh and Shishkin mesh, respectively. However, there are very few articles on uniform convergence of finite element methods on B-meshes. One main reason is that B-meshes have specific transition points between the fine and coarse parts, which are independent of mesh parameter. In the meantime, they bring great difficulties to convergence analysis. For example, Lagrange interpolant does not work well for B-meshes. Recently, Zhang and Liu \([27,28]\) proposed an variant of Lagrange interpolant for finite element methods on B-meshes in the case of convection-diffusion equations and succeeded to obtain uniform convergence of optimal order.

For reaction-diffusion problems, they also proved optimal order of uniform convergence in the natural energy norm in \([26]\). However, the energy norm is not strong enough to capture layers as the singular perturbation parameter tends to zero. Thus balanced norms, which are stronger than standard energy norms and characterize layers in an more appropriate way, were introduced in \([13]\) for a mixed finite element method and \([18]\) for a finite element method. The authors \([18]\) introduced an \(L^2\)-projection to obtain desired estimations by \(L^\infty\)-stability of the \(L^2\)-projection on Shishkin mesh. To improve estimations in \([18]\), the authors \([7]\) introduced a new interpolation, which consists of a local \(L^2\) projection defined on the uniform part of S-meshes. Unfortunately, unlike S-meshes, there is little development on convergence theories in balanced norms on B-meshes.

In this manuscript we analyzes convergence theories in the balanced norm introduced in \([18]\) for the finite element method of any order on Bakhvalov-type rectangular meshes. For this purpose, we propose a novel interpolant according to the structures of B-meshes and layer functions. This interpolant consists of a local \(L^2\) projection defined on a proper mesh subdomain and the Lagrange interpolant. To prove the convergence of optimal order in the balanced norm, we must take into account the scales of the meshes and different stabilities of the \(L^2\) projection. The optimal order convergence is also supported by our numerical
experiments. Furthermore, we propose another novel interpolant operator, which consists of the local $L^2$ projection operator, a vertices-edges-element operator and some corrections on the boundary. By careful derivations, we obtain a supercloseness result, which appears in the literature for the first time. Here “supercloseness” means that the convergence order for the error between some interpolation of the solution $u$ and the numerical solution $u^N$ in some norm is greater than the order for $u - u^N$ in the same norm.

The rest of the paper is organized as follows. In Sect. 2 we present a priori information of the solution to (1), then introduce Bakhvalov-type meshes, finite element methods and some preliminary results. In Sect. 3 we give a new interpolant and prove uniform convergence of optimal order in the balanced norm. Supercloseness result is given in Sect. 4 by means of another novel interpolant. In Sect. 5, numerical results illustrate our theoretical results.

Let $D \subset \Omega$. In this article, we will write $(\cdot, \cdot)_D$ for the inner product in $L^2(D)$, $\| \cdot \|_D$, $\| \cdot \|_{\infty,D}$, $\| \cdot \|_1,D$ and $| \cdot |_{1,D}$ for the standard norms in $L^2(D)$, $L^\infty(D)$, $L^1(D)$ and the standard seminorm in $H^1(D)$, respectively. If $D = \Omega$, the subscript will be omitted from the above norm designations. Throughout the paper, all constants $C$ and $C_i$ are independent of $\varepsilon$ and $N$; the constants $C$ are generic while subscripted constants $C_i$ are fixed.

2 Finite Element Method on Bakhvalov-Type Mesh

2.1 Regularity Results

To construct layer-adapted meshes and analyze uniform convergence, we need a priori information of the solution $u$ to (1), such as pointwise estimations of the derivatives of the solution, the locations and widths of layers. For this aim, we give the following assumption on the solution $u$ to (1).

Assumption 1 Assume $b \geq 2\beta^2 > 0$ with a positive constant $\beta$. For any fixed $k \in \mathbb{N}$, assume that $f$ satisfies sufficient compatibility conditions (see [8]). The solution $u$ of (1) can be decomposed as

$$u = v_0 + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i \quad \forall x \in \bar{\Omega},$$

where $v_0$ is the regular part, each $w_i$ is a boundary layer function and each $z_i$ is a corner layer function. There exists a constant $C$ such that

$$\left| \frac{\partial^m x^n}{\partial x^m \partial y^n} u_0(x, y) \right| \leq C(1 + \varepsilon^{(k+1) - m - n}) \quad \text{for } 0 \leq m + n \leq k + 3,$$

$$\left| \frac{\partial^m x^n}{\partial x^m \partial y^n} w_1(x, y) \right| \leq C(1 + \varepsilon^{(k+1) - m}) e^{-n} e^{-\beta y/\varepsilon} \quad \text{for } 0 \leq m + n \leq k + 2,$$

$$\left| \frac{\partial^m x^n}{\partial x^m \partial y^n} z_1(x, y) \right| \leq C \varepsilon^{-(m+n)} e^{-\beta(x+y)/\varepsilon} \quad \text{for } 0 \leq m + n \leq k + 2,$$

and similarly for the remaining terms. Here denote $\frac{\partial^{m+n} u}{\partial x^m \partial y^n}$ by $\frac{\partial^m x^n}{\partial x^m \partial y^n} u$.

In the following analysis, we will denote $\sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i$ by $w$.

Remark 1 In [8], sufficient compatibility conditions on $f$ are presented to ensure Assumption 1 hold true in the case of constant $b$. 

[Springer]
2.2 Bakhvalov-Type Meshes

Two Bakhvalov-type meshes will be discussed. Let $N \in \mathbb{N}$ be divisible by 4. The first Bakhvalov-type mesh is introduced in [17] and defined by

$$x_i = y_i = \psi(i/N) = \begin{cases} \frac{-\sigma \varepsilon}{\beta} \ln(1 - 4(1 - \varepsilon)i/N) & \text{for } i = 0, \ldots, N/4, \\ d_1(i/N - 1/4) + d_2(i/N - 3/4) & \text{for } i = N/4, \ldots, 3N/4, \\ 1 + \frac{\sigma \varepsilon}{\beta} \ln(1 - 4(1 - \varepsilon)(1 - i/N)) & \text{for } i = 3N/4, \ldots, N, \end{cases}$$

where $\sigma$ will be defined later and $d_1$, $d_2$ are used to ensure the continuity of $\psi(t)$ at $t = 1/4$ and $t = 3/4$. The second Bakhvalov-type mesh is introduced in [9,10] and its mesh generating function is

$$\varphi(t) = \begin{cases} \frac{-\sigma \varepsilon}{\beta} \ln(1 - 4t) & \text{for } t \in [0, \vartheta], \\ d_3(t - \vartheta) + d_4(t - 1 + \vartheta) & \text{for } t \in (\vartheta, 1 - \vartheta), \\ 1 + \frac{\sigma \varepsilon}{\beta} \ln(1 - 4(1 - t)) & \text{for } t \in [1 - \vartheta, 1] \end{cases}$$

where $\sigma$ will be specified later, $\vartheta = 1/4 - C_1 \varepsilon$ with some positive constant $C_1$ independent of $\varepsilon$ and $N$, $d_3$ and $d_4$ are chosen so that $\varphi(t)$ is continuous at $t = \vartheta$ and $t = 1 - \vartheta$. The original Bakhvalov mesh [1] can be recovered from (7) by setting $\vartheta = 1/4 - \mathcal{O}(\varepsilon)\varepsilon$ with $0 < C_2 \leq \mathcal{O}(\varepsilon) \leq C_3$.

**Assumption 2** Assume that $\varepsilon \leq \min\{\frac{\beta}{4\sigma}, 1\}N^{-1}$ in our analysis. In practice it is not a restriction.

For technical reasons, we also assume

$$\frac{\sigma}{4e\beta} \leq C_1 \leq \frac{1}{2} \max\{\frac{\sigma}{\beta}, \frac{1}{4}\}. \quad (8)$$

Assume $N \geq \max\{8, 2\ln\frac{\sigma}{\beta}\}$. Under Assumption 2 and (8), we have $1/4 - N^{-1} \leq \vartheta < 1/4$ and $x_{N/4} \leq 1/4$ for meshes (6) and (7). The location of $\vartheta$ and conditions imposed on $N$ and $C_1$ will simplify our later analysis without changing the essential difficulties in our analysis.

The mesh points are $x_i = y_i = \psi(i/N)$ or $x_i = y_i = \varphi(i/N)$ for $i = 0, 1, \ldots, N$. By drawing lines parallel to the axis through mesh points $\{(x_i, y_j)\}$, we obtain a Bakhvalov-type rectangular mesh with equidistant cells in the coarse region $\Omega_0 = (x_{N/4}, x_{3N/4})^2$ and anisotropic cells in the layer region $\Omega \setminus \Omega_0$. The triangulation is denoted by $T^N$. Denote by $\tau_{i,j}$ for the element $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and by $\tau$ for a generic rectangular element, which dimensions are written as $h_{x,\tau}$ and $h_{y,\tau}$. Define $h_i := x_{i+1} - x_i = y_{i+1} - y_i$ for $0 \leq i \leq N - 1$.

In the following lemma, we collect some important properties possessed by Bakhvalov-type meshes, which are important for convergence analysis. The reader is referred to [27] for the detailed proof.
Lemma 1  Let Assumption 2 hold true. On Bakhvalov-type mesh (6) or (7), one has
\begin{align}
    h_0 & \leq h_1 \leq \ldots \leq h_{N/4-2}, \\
    \frac{\sigma}{4\beta} \epsilon & \leq h_{N/4-2} \leq \frac{\sigma}{\beta} \epsilon, \\
    C \epsilon & \leq h_{N/4-1} \leq CN^{-1}, \\
    C_4N^{-1} & \leq h_i \leq C_5N^{-1} \quad N/4 \leq i \leq N/2, \\
    x_{N/4-1} & = 1 - x_{3N/4+1} \geq C \epsilon \ln N, \quad x_{N/4} = 1 - x_{3N/4} \geq C \sigma \epsilon \ln(1/\epsilon).
\end{align}

Note \( h_i = h_{N-1-i} \) for \( i = 0, 1, \ldots, N/2 \).

Let \( i^* = N/4 - 2 \). Then one has
\begin{align}
    h_i^\mu e^{-\beta x_i/\epsilon} & \leq Ce^{\mu}N^{-\mu} \quad \text{for} \ 0 \leq i \leq i^* \text{ and } 0 \leq \mu \leq \sigma. \quad (14)
\end{align}

Similar bounds hold for the variable \( y \).

2.3 Finite Element Method

Now we present the finite element method for problem (1). First, the weak form of problem (1) is written as
\begin{align}
    \begin{cases}
    \text{find } u \in V \text{ such that for all } v \in V \\
    a(u, v) := \epsilon^2(\nabla u, \nabla v) + (bu, v) = (f, v),
    \end{cases}
\end{align}

with \( V := H^1_0(\Omega) \). The natural energy norm derived from \( a(\cdot, \cdot) \) is
\[
    \|v\|_\epsilon := (\epsilon^2|v|^2 + \|v\|^2)^{1/2}.
\]

The bilinear form \( a(\cdot, \cdot) \) is coercive with respect to this energy norm, i.e.,
\[
    a(v, v) \geq \min\{2\beta^2, 1\}\|v\|_\epsilon^2 \ \forall v \in V. \quad (16)
\]

From the Lax-Milgram lemma, the weak formulation (15) has a unique solution.

Let \( Q_k(\tau) = \left\{ \sum_j c_j p_j(x)q_j(y) : p_j, q_j \text{ polynomials of degree } \leq k, (x, y) \in \tau \right\} \). We introduce the finite element spaces
\[
    V^N := \{ v \in H^1(\Omega) : v|_\tau \in Q_k(\tau) \ \forall \tau \in T^N \} \quad \text{and} \quad V^N_0 := V^N \cap H^1_0(\Omega). \]

Clearly, \( V^N_0 \subset V \). When we replace the infinite dimensional space \( V \) with the finite dimensional space \( V^N_0 \), we get the finite element method
\begin{align}
    \begin{cases}
    \text{find } u^N \in V^N_0 \text{ such that for all } v^N \in V^N_0 \\
    a(u^N, v^N) = (f, v^N).
    \end{cases}
\end{align}

Also, it is easy to verify the coercivity
\[
    a(v^N, v^N) \geq C\|v^N\|_\epsilon^2 \ \forall v^N \in V^N_0, \quad (18)
\]

and the Galerkin orthogonality
\[
    a(u - u^N, v^N) = 0 \ \forall v^N \in V^N_0, \quad (19)
\]
where (15) and (17) have been used. Furthermore, we introduce the balanced norm (see [18]), which is defined by

$$\|v\|_b := (\varepsilon |v|^2 + \|v\|^2)^{1/2}. \quad (20)$$

Clearly, the balanced norm $\| \cdot \|_b$ is stronger than the energy norm $\| \cdot \|_\varepsilon$ in the case of $0 < \varepsilon \ll 1$. Furthermore, the former is better suited to capture of layers. For example, for a typical layer function $e^{-x/\varepsilon}$, $\|e^{-x/\varepsilon}\|_\varepsilon$ and $\|e^{-x/\varepsilon}\|_b$ are of order $O(\varepsilon^{1/2})$ and of order $O(1)$, respectively. These orders imply that the balanced norm $\| \cdot \|_b$ is more appropriate to capture layers than the energy norm $\| \cdot \|_\varepsilon$ when $0 < \varepsilon \ll 1$. The reader is also referred to [3,18] for discussions on these two norms.

### 3 Uniform Convergence

For convergence analysis in the balanced norm, we will present an interpolation operator, which components will be introduced at first.

Set $\Omega_0^{\ast} := (x_{N/4-1}, x_{3N/4+1})^2$ and $\Omega_0^{**} := (x_{N/4-2}, x_{3N/4+2})^2$. Introduce an $L^2$-projection $\pi$ as follows: for $s \in L^2(\Omega_0^{\ast})$, find $\pi s \in W_N$ such that

$$((s - \pi s), v^N)_{\Omega_0^{\ast}} = 0 \quad \forall v^N \in W_N,$$

where $W_N := \{v|_{\Omega_0^{\ast}} : v \in V_N\}$. Of course, one has the $L^2$-stability

$$\|\pi v\|_{\Omega_0^{\ast}} \leq C \|v\|_{\Omega_0^{\ast}}. \quad (21)$$

Denote by $\mathcal{I}$ the Lagrange interpolation operator from $C^0(\overline{\Omega})$ to $V_N$. Furthermore, define $\chi \in V_0^N$ by

$$\chi(s_l, t_m) = \begin{cases} 1 & (s_l, t_m) \in \partial \Omega_0^{\ast}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{s_l, t_m\}$ are the interpolation points of the Lagrange interpolation.

Recall $w = \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_j$. Then the interpolation used in convergence analysis is defined by

$$P_{\varepsilon} u = P_1 v_0 + P_2 w, \quad (22)$$

where

$$P_1 v_0 := \begin{cases} \pi v_0 & \text{in } \Omega_0^{\ast}, \\ \mathcal{I}[(1 - \chi)v_0 + \chi \pi v_0] & \text{in } \Omega \setminus \Omega_0^{\ast}, \end{cases} \quad (23)$$

and

$$P_2 w := \begin{cases} 0 & \text{in } \Omega_0^{\ast}, \\ \mathcal{I}[(1 - \chi)w] & \text{in } \Omega \setminus \Omega_0^{\ast}. \end{cases} \quad (24)$$

Clearly, $P_{\varepsilon} u \in V_0^N$.

The following lemma provides some pointwise bounds for errors between the Lagrange interpolant and the $L^2$ projection.
Lemma 2 For $v_0$ introduced in Assumption 1, one has
\[
\|I v_0 - \pi v_0\|_{\infty, \partial \Omega_0^*} + \|I v_0 - \pi v_0\|_{\infty, \Omega_0^*} + \|v_0 - \pi v_0\|_{\infty, \Omega_0^*} \leq CN^{-(k+1)},
\]
\[
\|\nabla (I v_0 - \pi v_0)\|_{\Omega_0} \leq CN^{-k}, \quad \|\nabla (I v_0 - \pi v_0)\|_{\Omega_0^* \setminus \Omega_0} \leq C\varepsilon^{-1/2}N^{-(k+1)}.
\]

Proof Standard interpolation theory and Lemma 1 imply $\|v_0 - I v_0\|_{\infty, \Omega_0^*} + \|v_0 - I v_0\|_{\Omega_0^*} \leq CN^{-(k+1)}$. From the $L^\infty$-stability of the $L^2$-projection $\pi$ [4, Theorem 1], one has
\[
\|I v_0 - \pi v_0\|_{\infty, \Omega_0^*} = \|\pi (I v_0 - v_0)\|_{\infty, \Omega_0^*} \leq C\|I v_0 - v_0\|_{\infty, \Omega_0^*} \leq CN^{-(k+1)},
\]
\[
\|I v_0 - \pi v_0\|_{\infty, \Omega_0^*} \leq \|I v_0 - v_0\|_{\infty, \Omega_0^*} \leq CN^{-(k+1)},
\]
\[
\|I v_0 - \pi v_0\|_{\infty, \Omega_0^*} \leq \|I v_0 - v_0\|_{\infty, \Omega_0^*} + \|I v_0 - v_0\|_{\Omega_0^*} \leq CN^{-(k+1)}.
\]

Hölder inequalities, inverse inequalities, the $L^\infty$-stability of the $L^2$-projection [4, Theorem 1] and Lemma 1 yield
\[
\|\nabla (I v_0 - \pi v_0)\|_{\Omega_0^* \setminus \Omega_0} \leq \text{meas}(\Omega_0^* \setminus \Omega_0)^{1/2}\|\nabla (I v_0 - \pi v_0)\|_{\infty, \Omega_0^* \setminus \Omega_0}
\]
\[
\leq Ch^{1/2}_{N/4-1} \|I v_0 - \pi v_0\|_{\infty, \Omega_0^* \setminus \Omega_0}
\]
\[
=Ch^{1/2}_{N/4-1} \|\pi (I v_0 - v_0)\|_{\infty, \Omega_0^*}
\]
\[
\leq Ch^{1/2}_{N/4-1} \|I v_0 - v_0\|_{\infty, \Omega_0^*}
\]
\[
\leq C\varepsilon^{-1/2}N^{-(k+1)}.
\]

Inverse inequalities and the $L^2$-stability of the $L^2$-projection (21) yield
\[
\|\nabla (I v_0 - \pi v_0)\|_{\Omega_0} \leq CN\|I v_0 - \pi v_0\|_{\Omega_0} \leq CN\|I v_0 - v_0\|_{\Omega_0^*}
\]
\[
\leq CN\|I v_0 - v_0\|_{\Omega_0^*} \leq CN^{-k}.
\]

\[\square\]

Remark 2 Define
\[
V_h = \{v \in C[x_{N/4-1}, x_{3N/4+1}]; \ v|_{(s_j, s_{j+1})} \in P_k, \ j = 0, \ldots, N/2 + 1\}
\]
with $s_j = x_{N/4-1+j}$ for $j = 0, \ldots, N/2 + 2$. In the same way as the proof of [4, Theorem 1], we could easily prove the $L^\infty$-stability of the $L^2$-projection $\pi_h^\infty: L^2(x_{N/4-1}, x_{3N/4+1}) \rightarrow V_h$, that is
\[
\|\pi_h^\infty v\|_{\infty, (x_{N/4-1}, x_{3N/4+1})} \leq C\|v\|_{\infty, (x_{N/4-1}, x_{3N/4+1})} \forall v \in L^\infty(x_{N/4-1}, x_{3N/4+1}).
\]

Furthermore, from tensor product we could easily obtain
\[
\|\pi v\|_{\infty, \Omega_0^*} \leq C\|v\|_{\infty, \Omega_0^*} \forall v \in L^\infty(\Omega_0^*).
\]

The error $u - u^N$ is split as follows:
\[
u = u - u^N = (u - P_c u) + (P_c u - u^N) =: \eta + \xi.
\]
From the coercivity (18) and the Galerkin orthogonality (19) we have
\[
C\|\xi\|_{\varepsilon}^2 \leq a(P_c u - u, \xi) \leq \varepsilon^{1/2}\|\eta\|_{b}\|\xi\|_{\varepsilon} + |(b\eta, \xi)|.
\]
(25)

In the following analysis, we give the estimations on each term in the right-hand side of (25).
Lemma 3  Let $\sigma \geq k + 1$. Under Assumptions 1 and 2 we have

$$|(b\eta, \xi)| \leq C\varepsilon^{1/2}N^{-(k+1)}\ln^{1/2}N\|\xi\|.$$  

Proof Our arguments are based on the following splitting

$$(b\eta, \xi) = (b(v_0 - P_1v_0), \xi) + (b(w - P_2w), \xi)$$

$$= (b(v_0 - \pi v_0), \xi)\Omega_0^* + (b(\overline{I}(v_0), \xi)\Omega_0^* + (b(\chi(v_0 - \pi v_0)], \xi)\Omega_0^{**}\setminus\Omega_0^*$$

$$+ (bw, \xi)\Omega_0^* + (b(w - \overline{I}w), \xi)\Omega_0^* + (b(\chi w), \xi)\Omega_0^{**}\setminus\Omega_0^*$$

$$=: I + II + III + IV + V + VI.$$  

From the definition of $\pi$, we obtain

$$I = 0.$$  

(26)

From [26, Lemma 4], one has

$$|II| + |V| \leq C\left(\|v_0 - \overline{I}v_0\|_{\infty, \Omega_0^*} + \|w - \overline{I}w\|_{\infty, \Omega_0^*}\right)\|\xi\|_{1, \Omega_0^*}$$

$$\leq CN^{-(k+1)}\operatorname{meas}^{1/2}(\Omega \setminus \Omega_0^*)\|\xi\|_{\Omega_0^*}$$

$$\leq C\varepsilon^{1/2}N^{-(k+1)}\ln^{1/2}N\|\xi\|.$$  

(27)

Lemmas 1 and 2, (4) and (5) yield

$$|III| + |VI| \leq C\left(\|\overline{I}(v_0 - \pi v_0)\|_{\infty, \partial \Omega_0^*} + \|\overline{I}w\|_{\infty, \partial \Omega_0^*}\right)\|\xi\|_{1, \partial \Omega_0^*}$$

$$\leq C\left(\|\pi(v_0 - v_0)\|_{\infty, \partial \Omega_0^*} + N^{-\sigma}\right)$$

$$\leq C\varepsilon^{1/2}N^{-(k+1)}\|\xi\|.$$  

(28)

From (4) and (5), we get

$$|IV| \leq C\|w\|_{\Omega_0^*}\|\xi\|_{\partial \Omega_0^*} \leq C\varepsilon^{1/2}N^{-\sigma}\|\xi\|.$$  

(29)

Collecting (26)–(29), we are done.  

□

Lemma 4  Let $\sigma \geq k + 1$. Under Assumptions 1 and 2 we have

$$\|\eta\|_b \leq CN^{-k}.$$  

Proof From (22), (23) and (24) one has

$$\|\eta\|_b \leq \|v_0 - \pi v_0\|_{b, \Omega_0^*} + \|\overline{I}(v_0 - \pi v_0)\|_{b, \Omega_0^*} + \|\overline{I}(\chi(v_0 - \pi v_0))\|_{b, \Omega_0^{**}\setminus\Omega_0^*}$$

$$+ \|w - \overline{I}w\|_{b, \Omega_0^*} + \|\overline{I}(\chi w)\|_{b, \Omega_0^{**}\setminus\Omega_0^*}$$

$$=: S_1 + S_2 + S_3 + S_4 + S_5.$$  

To analyze $S_1$, we need the following bounds

$$\|\nabla(v_0 - \pi v_0)\|_{\Omega_0} \leq \|\nabla(v_0 - \overline{I}v_0)\|_{\Omega_0} + \|\nabla(\overline{I}(v_0 - \pi v_0))\|_{\Omega_0} \leq CN^{-k},$$

$$\|\nabla(v_0 - \pi v_0)\|_{\Omega_0^{**}\setminus\Omega_0^*} \leq \|\nabla(v_0 - \overline{I}v_0)\|_{\Omega_0^{**}\setminus\Omega_0^*} + \|\nabla(\overline{I}(v_0 - \pi v_0))\|_{\Omega_0^{**}\setminus\Omega_0^*}$$

$$\leq CN^{-k} + Ce^{-1/2}N^{-(k+1)},$$

$$\|v_0 - \pi v_0\|_{\Omega_0^*} \leq \|v_0 - \overline{I}v_0\|_{\Omega_0^*} + \|\overline{I}v_0 - \pi v_0\|_{\Omega_0^*} \leq CN^{-(k+1)} + C\|\overline{I}v_0 - \pi v_0\|_{\infty, \Omega_0^*} \leq CN^{-(k+1)},$$

$$\leq CN^{-(k+1)}.$$
which could be derived from standard interpolation theories and Lemma 2. Then we obtain
\[ |S_1| \leq \varepsilon^{1/2} \|\nabla (v_0 - \pi v_0)\|_{\Omega_0^*} + \|v_0 - \pi v_0\|_{\Omega_0^*} \leq C \varepsilon^{1/2} N^{-k} + CN^{-(k+1)}. \]  
(30)

Similar to [26, Lemma 4], we have
\[ \|v_0 - I v_0\|_{\infty, \Omega \setminus \Omega_0^*} + \|w - I w\|_{\infty, \Omega \setminus \Omega_0^*} \leq CN^{-(k+1)}. \]

Imitating the proof of Lemma 5 in [26] and replacing [26, (3.15)] by
\[ |w_1 - I w_1|_{1, \Omega \setminus \Omega_0^*} \leq C \varepsilon^{1/2} N^{-k}, \]
and similarly have
\[ |w - I w|_{1, \Omega \setminus \Omega_0^*} \leq C \varepsilon^{1/2} N^{-k}. \]

Thus we obtain
\[ S_2 + S_4 \leq CN^{-k}. \]  
(31)

Inverse inequalities, Lemmas 1 and 2 yield
\[ S_3 \leq \varepsilon^{1/2} \|\nabla (I (\chi (v_0 - \pi v_0)))\|_{\Omega_0^*} + \|I (\chi (v_0 - \pi v_0))\|_{\Omega_0^*} \leq C \varepsilon^{1/2} h_{N/4-2}^{-1} + 1 \|I (\chi (v_0 - \pi v_0))\|_{\Omega_0^*} \]
\[ \leq C \varepsilon^{1/2} h_{N/4-2}^{-1} + 1 h_{N/4-2}^{1/2} \|I v_0 - \pi v_0\|_{\infty, \Omega_0^*} \leq CN^{-(k+1)}. \]  
(32)

Similarly, one has
\[ S_5 \leq CN^{-\sigma}. \]  
(33)

Collecting (30)–(33), we are done. \[ \square \]

Similar to [7, Theorem 2.6], we obtain the following theorem from Lemmas 3 and 4.

**Theorem 1** Let \( \sigma \geq k + 1 \). Let Assumptions 1 and 2 hold. Then for the exact solution \( u \) to (1) and the numerical solution \( u^N \) to (17) on Bakhvalov-type rectangular mesh (6) or (7), one has
\[ \|u - u^N\|_b \leq CN^{-k}. \]

**4 Supercloseness**

In order to derive the supercloseness result in the balanced norm, we need another novel interpolant, which will be described in the following.

Instead of Lagrange interpolant operator used in the previous section, we introduce a vertices-edges-element interpolation operator \( A : C^0(\hat{\Omega}) \rightarrow V_N \) (see [12,21]). This interpolant is used for superconvergence analysis of the diffusion term. First we define the interpolant operator on the reference element \( \hat{\tau} = (-1, 1)^2 \), whose vertices and edges are denoted by \( \hat{a}_i \) and \( \hat{e}_i \) respectively for \( i = 1, \ldots, 4 \). Let \( \hat{v}(\cdot, \cdot) \in C(\hat{\tau}) \). The operator
\( \hat{A} : C(\tilde{\tau}) \rightarrow Q_k(\tilde{\tau}) \) is determined by \((k+1)^2\) continuous linear functionals \( \hat{F} : C(\tilde{\tau}) \rightarrow \mathbb{R} \), which are defined by

\[
\hat{u} \rightarrow \hat{u}(\hat{a}_i) \quad i = 1, \ldots, 4,
\]

\[
\hat{u} \rightarrow \int_{\hat{e}_i} \hat{v} q ds \quad \forall q \in P_{k-2}(\hat{e}_i) \quad i = 1, \ldots, 4,
\]

\[
\hat{u} \rightarrow \int_{\tilde{\tau}} \hat{v} q dx dy \quad \forall q \in Q_{k-2}(\tilde{\tau}).
\]

From [21, Lemma 3], the operator \( \hat{A} \) is uniquely determined. Then using the affine transformation to map from \( \tilde{\tau} \) to an arbitrary \( \tau \in T^N \), one obtains the corresponding interpolation operator \( A_\tau : C(\tilde{\tau}) \rightarrow Q_k(\tau) \). At last a continuous global interpolation operator \( A : C(\bar{\Omega}) \rightarrow V^N \) is defined by setting

\[
(Av)_{|\tau} := A_\tau(v_{|\tau}) \quad \forall \tau \in T^N.
\]

Besides, we denote by \( F \) degree of freedom (DoF) of \( V^N \), which originates from the linear functional \( \hat{F} \).

Set \( j^* = N/4 - 2 \) and \( \Omega_{w_1} := [0, 1] \times [y_0, y_{j^*+1}] \). The operator \( S_1 : C^0(\bar{\Omega}) \rightarrow V^N \) is defined by

\[
S_1 w_1 := \begin{cases}
0 & \text{in } \Omega \setminus \Omega_{w_1}, \\
A w_1 - B_1 w_1 & \text{in } \Omega_{w_1},
\end{cases}
\]

(34)

where \( B_1 w_1 \) satisfies

\[
F(B_1 w_1) = \begin{cases}
F(w_1) & \text{if } F \text{ is the DoF of } V^N \text{ attached to } [0, 1] \times \{ y = y_{j^*+1} \}, \\
0 & \text{otherwise}.
\end{cases}
\]

If \( F \) is the DoF of \( V^N \) attached to \([0, 1] \times \{ y = y_{j^*+1} \} \), then it must be one of the following forms

\[
v \rightarrow v(x_i, y_{j^*+1}) \quad \text{for } i = 0, \ldots, N,
\]

\[
v \rightarrow \frac{j+1}{h_j^{j+1}} \int_{x_i}^{x_{i+1}} v(s, y_{j^*+1}) (s - x_i)^j ds \quad \text{for } i = 0, \ldots, N - 1 \text{ and } j = 0, \ldots, k - 2.
\]

In fact, \( B_1 w_1 \) is introduced for the continuity of \( S_1 w_1 \) at \([0, 1] \times \{ y = y_{j^*+1} \} \). Set \( \Omega_{z_1} := [x_0, x_{i^*+1}] \times [y_0, y_{j^*+1}] \). The operator \( T_1 : C^0(\bar{\Omega}) \rightarrow V^N \) is defined by

\[
T_1 z_1 := \begin{cases}
0 & \text{in } \Omega \setminus \Omega_{z_1}, \\
A z_1 - C_1 z_1 & \text{in } \Omega_{z_1},
\end{cases}
\]

(35)

where the operator \( C_1 \) is defined in a similar way to \( B_1 \) except that the degrees of freedom for \( C_1 \) are attached to \([0, x_{i^*+1}] \times \{ y = y_{j^*+1} \} \cup \{ x = x_{i^*+1} \} \times [0, y_{j^*+1}] \). The operators \( S_i \) and \( T_i \) for \( i = 2, 3, 4 \) could be defined similarly. Also we give a boundary correction \( C(S_1 w_1) \in V^N \) for \( S_1 w_1 \), which is defined by

\[
F(C(S_1 w_1)) = \begin{cases}
F(w_1) & \text{if } F \text{ is the DoF of } V^N \text{ attached to } \Gamma_{w_1}, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \Gamma_{w_1} := \partial \Omega \setminus \partial \Omega_{w_1} \). With the help of this correction, we have

\[
(S_1 w_1 + C(S_1 w_1))_{|\partial \Omega} = A w_1|_{\partial \Omega}.
\]
By the same token, we could define the corrections \( C(S_i w_i) \) and \( C(T_i z_i) \) for \( S_i w_i \) and \( T_i z_i \) for \( i = 1, 2, 3, 4 \), respectively.

Introduce the discrete function \( Dv_0 \in V_N^0 \), which is defined by

\[
F(Dv_0) = \begin{cases} F(\pi v_0 - v_0) & \text{if } F \text{ is the DoF of } V^N \text{ attached to } \partial \Omega_0^* , \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
E v_0 := \begin{cases} \pi v_0 & \text{in } \Omega_0^* , \\ A v_0 + Dv_0 & \text{in } \Omega \setminus \Omega_0^* . \end{cases}
\] (36)

Note that \( (Dv_0)|_{\partial \Omega_0^*} = (\pi v_0 - A v_0)|_{\partial \Omega_0^*} \). The definition of \( Dv_0 \) ensures the continuity of \( E v_0 \) on \( \bar{\Omega} \).

Now we are in a position to propose the interpolation used for our supercloseness analysis, which is defined by

\[
P_s u = E v_0 + \sum_{i=1}^{4} S_i w_i + \sum_{i=1}^{4} T_i z_i + C(w),
\] (37)

where \( C(w) = \sum_{i=1}^{4} C(S_i w_i) + \sum_{i=1}^{4} C(T_i z_i) \). Clearly \( P_s u \in V_N^0 \).

**Remark 3** The operators \( B_1, C_1, C \) and \( D \) are defined by the degrees of freedom \( F \), which determine the operator \( A \). From the existence and uniqueness of operator \( A \), the operators \( B_1, C_1, C \) and \( D \) are uniquely defined.

**Remark 4** The Lagrange interpolant based on equidistant interpolation points is not superclose to the finite element solution of the Laplacian operator with the homogeneous Dirichlet boundary condition in \( H^1 \) norm in the case of 2-dimensional \( Q_k \) finite elements with \( k \geq 1 \); see [11]. Instead, it is easy to prove the supercloseness between the vertices-edges-element interpolation \( A u \) and the finite element solution; see [12,21]. This is why the operator \( A \) is introduced for the supercloseness analysis.

The following lemma could be found in [21, Lemma 4], which is important for analysis of the diffusion part.

**Lemma 5** Let \( \tau \in T^N \). Let \( v \in H^{k+2}(\tau) \) and \( A_\tau v \in Q_k(\tau) \) be its vertices-edges-element interpolant. Then for each \( v^N \in Q_k(\tau) \) we have

\[
\int_\tau (A_\tau v - v)_x v^N_x dxdy \leq Ch_{x,\tau}^{k+1} \left\| \frac{\partial^{k+2} v}{\partial x \partial y^{k+1}} \right\|_{\tau} \left\| v^N_x \right\|_{\tau}
\]

and

\[
\int_\tau (A_\tau v - v)_y v^N_y dxdy \leq Ch_{x,\tau}^{k+1} \left\| \frac{\partial^{k+2} v}{\partial x^{k+1} \partial y} \right\|_{\tau} \left\| v^N_y \right\|_{\tau}.
\]

For the vertices-edges-element interpolation, we have the following interpolation errors (see [21, Lemma 7]).
The following $L^\infty$-stabilities will be used later.

**Lemma 7** Let $v \in C^0(\bar{\Omega})$. There exists a constant $C$ independent of $v$ such that

\[
\|B_1 v\|_{\infty(\Omega_{w_1})} \leq C \|v\|_{\infty(\Omega_{w_1})}, \\
\|(B_1 v)_x\|_{\infty(\Omega_{w_1})} \leq C \|v_x\|_{\infty(\Omega_{w_1})}, \\
\|C_1 z_1\|_{\infty(\Omega_{z_1})} \leq C \|z_1\|_{\infty(\Omega_{z_1})}.
\]

**Proof** For $\tau_i, j^*$ with $i = 0, \ldots, N - 1$, one has

\[
B_1 v|_{\tau_i,j^*} = v(x_i, y_{j^*+1}) \varphi_i(x, y) + v(x_i+1, y_{j^*+1}) \varphi_{i+1}(x, y) + \sum_{m=0}^{k-2} \frac{m+1}{h_{m+1}} \int_{x_i}^{x_i+1} v(s, y_{j^*+1})(s-x_i)^m \varphi_{i,m}(x, y) ds.
\]

The inequality $\|B_1 v\|_{\infty(\tau_i,j^*)} \leq C \|v\|_{\infty(\tau_i,j^*)}$ follows. In a similar way, we could prove $\|C_1 z_1\|_{\infty(\Omega_{z_1})} \leq C \|z_1\|_{\infty(\Omega_{z_1})}$. Note

\[
v(s, y_{j^*+1}) = v(x, y_{j^*+1}) + \int_x^s v_t(t, y_{j^*+1}) dt \quad \text{for} \quad s, x \in [x_i, x_{i+1}],
\]

and

\[
\varphi_i(x, y) + \sum_{m=0}^{k-2} \varphi_{i,m}(x, y) + \varphi_{i+1}(x, y) \equiv 0 \quad \text{for} \quad (x, y) \in \tau_i, j^*.
\]

Then we have

\[
(B_1 v)_x|_{\tau_i,j^*} = \int_x^{x_i} v_t(t, y_{j^*+1}) dt \varphi_i(x, y) + \int_x^{x_i+1} v_t(t, y_{j^*+1}) dt \varphi_{i+1}(x, y) + \sum_{m=0}^{k-2} \frac{m+1}{h_{m+1}} \int_{x_i}^{x_i+1} \left( \int_x^s v_t(t, y_{j^*+1}) dt \right)(s-x_i)^m \varphi_{i,m}(x, y) ds,
\]

and

\[
\|(B_1 v)_x\|_{\infty(\tau_i,j^*)} \leq CM \|v_x\|_{\infty(\tau_i,j^*)} \leq C \|v_x\|_{\infty(\tau_i,j^*)}
\]

where from scaling arguments one has

\[
M = h_i \max_{l=i,i+1} \left\{ \|(\varphi_l(x, y))_x\|_{\infty(\tau_i,j^*)}, \|(\varphi_{i,m}(x, y))_x\|_{\infty(\tau_i,j^*)} \right\} \leq C.
\]

Consider $e(\Omega_{w_1}) = \bigcup_{i=0}^{N-1} \tau_i, j^*$ and we are done. \qed
The error is split as follows:

\[ u - u^N = (u - P_su) + (P_su - u^N) =: \tilde{\eta} + \tilde{\xi}. \]

From the coercivity (18), the Galerkin orthogonality (19) and (37), one has

\[
C\|\tilde{\xi}\|_{\epsilon}^2 \leq a(P_su - u, \tilde{\xi}) = \varepsilon^2 (\nabla \tilde{\eta}, \nabla \tilde{\xi}) + (b\tilde{\eta}, \tilde{\xi})
\]

\[= : \varepsilon^2 \mathcal{S}_1 + \mathcal{S}_2, \tag{38} \]

where

\[
\mathcal{S}_1 = (\nabla(v_0 - \mathcal{E}v_0), \nabla\tilde{\xi}) + \sum_{i=1}^{4} (\nabla(w_i - S_i w_i), \nabla\tilde{\xi}) + \sum_{i=1}^{4} (\nabla(z_i - T_i z_i), \nabla\tilde{\xi}),
\]

\[
\mathcal{S}_2 = (b(v_0 - \mathcal{E}v_0), \tilde{\xi}) + \sum_{i=1}^{4} (b((w_i - S_i w_i) + (z_i - T_i z_i)), \tilde{\xi}) + a(C(w), \tilde{\xi}).
\]

The terms in the right-hand side of (38) will be analyzed in the following two lemmas.

**Lemma 8** Let Assumptions 1 and 2 hold. Let \( \sigma \geq k + 3/2 \). Then one has

\[ |\mathcal{S}_1| \leq C \varepsilon^{-3/2}(\varepsilon^{1/2}N^{-k} + N^{-(k+1)})\|\tilde{\xi}\|_\epsilon. \]

**Proof** From Assumption 1, we just analyze \((\nabla(v_0 - \mathcal{E}v_0), \nabla\tilde{\xi}), (\nabla(w_1 - S_1 w_1), \nabla\tilde{\xi})\) and \((z_1 - T_1 z_1)_x, \tilde{\xi}_x\). The remaining terms can be treated in a similar manner. We split these terms as follows:

\[(\nabla(v_0 - \mathcal{E}v_0), \nabla\tilde{\xi}) + (\nabla(w_1 - S_1 w_1), \nabla\tilde{\xi}) + ((z_1 - T_1 z_1)_x, \tilde{\xi}_x) = \sum_{i=1}^{6} S_i \tag{39} \]

where

\[S_1 = (\nabla(v_0 - A v_0), \nabla\tilde{\xi})_{\Omega \setminus \Omega_0^*} + (\nabla(w_1 - A w_1), \nabla\tilde{\xi})_{\Omega w_1} + ((z_1 - A z_1)_x, \tilde{\xi}_x)_{\Omega z_1}, \]

\[S_2 = ((B_1 w_1)_x, \tilde{\xi}_x)_{\epsilon(\Omega w_1)}, \]

\[S_3 = ((B_1 w_1)_y, \tilde{\xi}_y)_{\epsilon(\Omega w_1)} + ((C_1 z_1)_x, \tilde{\xi}_x)_{\epsilon(\Omega z_1)}, \]

\[S_4 = (\nabla w_1, \nabla\tilde{\xi})_{\Omega \setminus \Omega w_1} + ((z_1)_x, \tilde{\xi}_x)_{\Omega \setminus \Omega z_1}, \]

\[S_5 = (\nabla(v_0 - \pi v_0), \nabla\tilde{\xi})_{\Omega^* \setminus \Omega_0} + (\nabla(v_0 - \pi v_0), \nabla\tilde{\xi})_{\Omega 0}, \]

\[S_6 = (\nabla(D v_0), \nabla\tilde{\xi})_{\Omega \setminus \Omega_0^*}. \]

Applying Lemmas 1 and 5 to \( S_1 \), then we have

\[
|((w_1 - A w_1)_x, \tilde{\xi}_x)_{\Omega w_1}| \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} h_{j,y}^{k+1} \left\| \frac{\partial^{k+2} w_1}{\partial x \partial^{k+1} y} \right\|_{\tau_{ij}} \|\tilde{\xi}_x\|_{\tau_{ij}}
\]

\[
\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} h_{j,y}^{k+1} e^{-(k+1)} e^{-\beta y_j/\varepsilon} h_{i,x}^{1/2} h_{j,y}^{1/2} \|\tilde{\xi}_x\|_{\tau_{ij}}
\]

\[
\leq C \varepsilon^{1/2} \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} N^{-(k+2)} \|\tilde{\xi}_x\|_{\tau_{ij}}
\]

\[
\leq C \varepsilon^{1/2} N^{-(k+1)} \|\tilde{\xi}_x\|
\]
and similarly obtain
\[
|((\nabla (v_0 - A v_0), \nabla \tilde{\xi})_{\Omega \setminus \Omega_0^*} + |((w_1 - A w_1)_y, \tilde{\xi})_{\Omega_w} | + |((z_1 - A z_1)_x, \tilde{\xi})_{\Omega_{z_1}} |
\leq C \varepsilon^{-1/2} N^{-(k+1)} \| \nabla \tilde{\xi} \|. \tag{41}
\]

From Lemma 7, one has
\[
|S_2| \leq \|(B_1 w_1)_x \|_{\infty, e(\Omega_{w_1})} \| \nabla \tilde{\xi} \|
\leq \|e(\Omega_{w_1})\|^{1/2} \|(B_1 w_1)_x \|_{\infty, e(\Omega_{w_1})} \| \nabla \tilde{\xi} \|
\leq C \varepsilon^{1/2} \|(w_1)_x \|_{\infty, e(\Omega_{w_1})} \| \nabla \tilde{\xi} \|
\leq C \varepsilon^{1/2} N^{-\sigma} \| \nabla \tilde{\xi} \|. \tag{42}
\]

Inverse inequalities yield
\[
|S_3| \leq \|(B_1 w_1)_y \|_{\infty, e(\Omega_{w_1})} + \|(C_1 z_1)_x \|_{\infty, e(\Omega_{z_1})} \| \nabla \tilde{\xi} \|
\leq C (h^{-1/2}_{N/4-1}, (B_1 w_1)_x \|_{\infty, e(\Omega_{w_1})} + h^{-1/2}_{N/4-1} \|C_1 z_1)_x \|_{\infty, e(\Omega_{z_1})} \| \nabla \tilde{\xi} \|
\leq C \varepsilon^{-1} \|(B_1 w_1)_x \|_{\infty, e(\Omega_{w_1})} + \|C_1 z_1)_x \|_{\infty, e(\Omega_{z_1})} \| \nabla \tilde{\xi} \|
\leq C \varepsilon^{-1} N^{-(k+1)} \| \nabla \tilde{\xi} \|. \tag{43}
\]

Assumption 1 and direct calculations yield
\[
|S_4| \leq \| \nabla w_1 \|_{\Omega \setminus \Omega_{w_1}} + \| \nabla z_1 \|_{\Omega \setminus \Omega_{z_1}} \| \nabla \tilde{\xi} \| \leq C \varepsilon^{-1/2} N^{-\sigma} \| \nabla \tilde{\xi} \|. \tag{44}
\]

Hölder inequalities, inverse inequalities, Lemmas 1 and 2 yield
\[
|S_5| \leq C h^{-1}_{N/4-1} \| v_0 - \pi v_0 \|_{\infty, \Omega \setminus \Omega_0} \| \nabla \tilde{\xi} \|_{\Omega \setminus \Omega_0} + C N \| v_0 - \pi v_0 \|_{\infty, \Omega_0} \| \nabla \tilde{\xi} \|_{\Omega_0}
\leq C h^{-1}_{N/4-1} N^{-(k+1)} h^{1/2}_{N/4-1} \| \nabla \tilde{\xi} \|_{\Omega \setminus \Omega_0} + C N N^{-(k+1)} \| \nabla \tilde{\xi} \|_{\Omega_0}
\leq C \varepsilon^{-3/2} (N^{-(k+1)} + \varepsilon^{1/2} N^{-k}) \| \nabla \tilde{\xi} \|. \tag{45}
\]

From the definition of \( D v_0 \), we have
\[
|S_6| \leq C \| \nabla (D v_0) \|_{\infty, \partial \Omega_0^*} \| \nabla \tilde{\xi} \|_{1, e(\Omega \setminus \Omega_0^*)}
\leq C h^{-1}_{N/4-2} \| A v_0 - \pi v_0 \|_{\infty, \partial \Omega_0^*} \cdot h^{1/2}_{N/4-2} \| \nabla \tilde{\xi} \|_{e(\Omega \setminus \Omega_0^*)}
\leq C \varepsilon^{-3/2} \| A v_0 - \pi v_0 \|_{\infty, \Omega_0^*} \| \nabla \tilde{\xi} \|_{\varepsilon}
\leq C \varepsilon^{-3/2} N^{-(k+1)} \| \nabla \tilde{\xi} \|. \tag{46}
\]

where Lemmas 2 and 6 yield
\[
\| A v_0 - \pi v_0 \|_{\infty, \Omega_0^*} \leq \| A v_0 - v_0 \|_{\infty, \Omega_0^*} + \| v_0 - \pi v_0 \|_{\infty, \Omega_0^*} \leq C N^{-(k+1)}.
\]

Substituting (40)–(46) into (39) and considering \( \sigma \geq k + 3/2 \), we are done. \( \square \)

**Lemma 9** Let Assumptions 1 and 2 hold. Let \( \sigma \geq k + 3/2 \). Then one has
\[
|S_2| \leq C \varepsilon^{1/2} N^{-(k+1)} \ln^{1/2} N \| \nabla \tilde{\xi} \|_{\varepsilon}.
\]
\[ \mathcal{J}_2 = (b(v_0 - \varepsilon v_0), \tilde{\xi}) + \sum_{i=1}^{4} (b(w_i - S_i w_i), \tilde{\xi}) + \sum_{i=1}^{4} (b(z_i - T_i z_i), \tilde{\xi}) + a(C(w), \tilde{\xi}) \] (47)

First, from (36), Hölder inequalities and Lemma 1 we obtain

\[ |(b(v_0 - \varepsilon v_0), \tilde{\xi})| \leq |(b(v_0 - A v_0), \tilde{\xi})| + |(b(D v_0), \tilde{\xi})| \]
\[ \leq C \| v_0 - A v_0 \|_{\infty, \Omega \setminus \Omega_0} \| \tilde{\xi} \|_{1, \Omega \setminus \Omega_0} + C \| A v_0 - \pi v_0 \|_{\infty, \Omega \setminus \Omega_0} \| \tilde{\xi} \|_{1, \pi v_0} \] (48)

\[ \leq C \varepsilon^{1/2} N^{-(k+1)} \ln^{1/2} N \| \tilde{\xi} \|_{\Omega \setminus \Omega_0} + C N^{-(k+1)} h_{N/4}^{1/2} \| \tilde{\xi} \|_{\pi v_0} \]

Second, Hölder inequalities and Lemma 1 yield

\[ a(C(w), \tilde{\xi}) \leq \varepsilon^2 |(\nabla C(w), \nabla \tilde{\xi})| + |(bC(w), \tilde{\xi})| \]
\[ \leq C \varepsilon^2 \| \nabla C(w) \|_{\infty, \pi(\Omega)} \| \nabla \tilde{\xi} \|_{1, \pi(\Omega)} + C \| C(w) \|_{\infty, \pi(\Omega)} \| \tilde{\xi} \|_{1, \pi(\Omega)} \]
\[ \leq C \varepsilon^2 h_0^{-1} N^{-\sigma} \cdot h_0^{1/2} \| \nabla \tilde{\xi} \|_{\pi(\Omega)} + C N^{-\sigma} \cdot h_0^{1/2} \| \tilde{\xi} \|_{\pi(\Omega)} \]
\[ \leq C \varepsilon^{1/2} N^{-(\sigma-1/2)} \| \tilde{\xi} \|_{\pi(\Omega)} + C \varepsilon^{1/2} N^{-1} \| \tilde{\xi} \|_{\pi(\Omega)} \]

where we have used \( \| C(w) \|_{\infty, \pi(\Omega)} \leq C N^{-\sigma} \) and \( C_6 \varepsilon N^{-1} \leq h_0 \leq C_7 \varepsilon N^{-1} \).

At last, we analyze \( (b(w_1 - S_1 w_1), \tilde{\xi}) \) and the remaining terms can be discussed in a similar way. Lemmas 6 and 1 yield

\[ \| w_1 - A w_1 \|_{\Omega \setminus \Omega_0}^2 = \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} \| w_1 - A w_1 \|_{\tau_{ij}}^2 \]
\[ \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} \left( \sum_{l+m=k+1} C_{l,m} h_{l,m}^k \| \frac{\partial^{k+1} w_1}{\partial x^l \partial y^m} \|_{\tau_{ij}} \right)^2 \]
\[ \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} \left( \sum_{l+m=k+1} C_{l,m} h_{l,m}^{k} e^{-m} e^{-\beta y_j / \varepsilon} h_{l,m}^{1/2} \right)^2 \]
\[ \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} \left( \sum_{l+m=k+1} N^{-(l+1/2)} e^{-m+1/2} N^{-(m+1/2)} e^{-m} \right)^2 \]
\[ \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{j^*} (e^{1/2} N^{-(k+2)})^2 \leq C \varepsilon N^{-2(k+1)} \]

Lemma 7 yields

\[ \| B_1 w_1 \|_{e(\Omega \setminus \Omega_0)} \leq C \| e(\Omega \setminus \Omega_0) \|_{1/2} \| w_1 \|_{\infty, e(\Omega \setminus \Omega_0)} \leq C h_{j^*}^{1/2} N^{-\sigma} \]

From the Cauchy-Schwarz inequality one obtains

\[ (b(w_1 - S_1 w_1), \tilde{\xi}) = (b(w_1, \tilde{\xi}) e(\Omega \setminus \Omega_0) + (b(w_1 - A w_1), \tilde{\xi}) e(\Omega \setminus \Omega_0) + (b B_1 w_1, \tilde{\xi}) e(\Omega \setminus \Omega_0) \]
\[ \leq C \| w_1 \|_{\Omega \setminus \Omega_0} \| \tilde{\xi} \|_{\Omega \setminus \Omega_0} + C \| w_1 - A w_1 \|_{\Omega \setminus \Omega_0} \| \tilde{\xi} \|_{\Omega \setminus \Omega_0} + \| B_1 w_1 \|_{e(\Omega \setminus \Omega_0)} \| \tilde{\xi} \|_{e(\Omega \setminus \Omega_0)} \]
\[ \leq C (\varepsilon^{1/2} N^{-\sigma} + \varepsilon^{1/2} N^{-(k+1)} + C h_{j^*}^{1/2} N^{-\sigma}) \| \tilde{\xi} \|_{\pi(\Omega)} \]
\[ \leq C \varepsilon^{1/2} N^{-(k+1)} \| \tilde{\xi} \|_{\pi(\Omega)} \]
Substituting (48)–(50) into (47), we are done. 

Now we are in a position to present our supercloseness result in the balanced norm.

**Theorem 2** Let Assumptions 1 and 2 hold. Let $P_s u$ defined in (37) be the interpolation to the solution $u$ of (1). Let $u^N$ be the solution of (15). Then one has

$$
\|P_s u - u^N\|_b \leq C(\varepsilon^{1/2}N^{-k} + N^{-(k+1)}\ln^{1/2} N).
$$

**Proof** From (38), Lemmas 8 and 9, we obtain

$$
\|\tilde{\xi}\|_{\varepsilon} \leq C\varepsilon^{1/2}(\varepsilon^{1/2}N^{-k} + N^{-(k+1)}\ln^{1/2} N),
$$

which implies the following estimations

$$
\varepsilon^{1/2}\|\nabla(P_s u - u^N)\| \leq C(\varepsilon^{1/2}N^{-k} + N^{-(k+1)}\ln^{1/2} N),
$$

$$
\|P_s u - u^N\| \leq C\varepsilon^{1/2}(\varepsilon^{1/2}N^{-k} + N^{-(k+1)}\ln^{1/2} N).
$$

Thus we are done. 

**Remark 5** From Theorems 1 and 2, we could conclude that Theorem 2 presents a supercloseness result. It is the first supercloseness result in the balanced norm in the literature.

## 5 Numerical Experiment

In this section we present numerical experiments on Bakhvalov-type rectangular meshes that support our theoretical results. All calculations were carried out by using Intel Visual Fortran 11 and the discrete problems were solved using GMRES (see, e.g., [2]).

We present errors and convergence orders in the computed solutions for the boundary value problem

$$
-\varepsilon^2 \Delta u + 2u = f(x, y) \quad \text{in } \Omega = (0, 1)^2, \\
u = 0 \quad \text{on } \partial \Omega,
$$

(51)

where the right-hand side $f$ is chosen in such a way that $u(x, y)$ is the exact solution. For the computations we will assign values to the parameters in (6) and (7). We set $\sigma = k + 1$ when the error $\|u - u^N\|_b$ is discussed and $\sigma = k + 3/2$ when the error $\|P_s u - u^N\|_b$ is discussed. Also, we set $\beta = 1$ and $C_1 = 4\sigma/(3\beta)$ in (7).

In our numerical tests, we will consider $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, \ldots, 10^{-6}$, $k = 1, 2, 3$ and $N = 12, 24, \ldots, 768$. Our numerical experiments imply that meshes (6) and (7) have same performances. Thus we only present numerical results for mesh (6).

For a fixed $\varepsilon$ and $N$, we have evaluated the error $e_c^{N, \varepsilon} = \|u - u^N\|_b$ and $e^s_{N, \varepsilon} = \|P_s u - u^N\|_b$, where $P_s u$ is the interpolation of the exact solution $u$ to (51), which is defined in (37) and $u^N$ represents its numerical approximation. In the following we present the errors

$$
e^{N}_{c} = \max_{\varepsilon = 10^{-3}, 10^{-6}} e^{N, \varepsilon}_{c}, \quad e^{N}_{s} = \max_{\varepsilon = 10^{-3}, 10^{-6}} e^{N, \varepsilon}_{s}
$$

(52)
Fig. 1 Errors $e^N_c$, $e^N_s$, $e^N_{c,10^{-2}}$ and $e^N_{s,10^{-2}}$ when $k = 3$

Table 1 Errors $e^N_c$ and $e^N_s$ defined in (52) and rates $p^N_c$ and $p^N_s$ defined in (53) for $k = 1$ and $k = 2$

| $N$  | $e^N_c$       | $p^N_c$ | $e^N_s$       | $p^N_s$ | $e^N_{c,10^{-2}}$ | $p^N_{c,10^{-2}}$ | $e^N_{s,10^{-2}}$ | $p^N_{s,10^{-2}}$ |
|------|---------------|---------|---------------|---------|-------------------|-------------------|-------------------|-------------------|
| 12   | 0.397E0       | 1.04    | 0.101E0       | 2.05    | 0.106E0           | 2.01              | 0.524E-1          | 3.24              |
| 24   | 0.193E0       | 1.00    | 0.245E-1      | 2.11    | 0.265E-1          | 2.00              | 0.555E-2          | 3.10              |
| 48   | 0.963E-1      | 1.00    | 0.566E-2      | 2.07    | 0.661E-2          | 2.00              | 0.646E-3          | 2.86              |
| 96   | 0.481E-1      | 1.00    | 0.135E-2      | 2.04    | 0.165E-2          | 2.00              | 0.889E-4          | 2.50              |
| 192  | 0.241E-1      | 1.01    | 0.328E-3      | 2.02    | 0.413E-3          | 2.00              | 0.160E-4          | 3.16              |
| 384  | 0.120E-1      | 1.00    | 0.808E-4      | 2.01    | 0.892E-4          | —                 | 0.328E-6          | —                 |
| 768  | 0.601E-2      | —       | 0.201E-4      | —       | —                 | —                 | —                 | —                 |

Table 2 Errors and rates in the balanced norm for $k = 1$ and $k = 2$ when $\varepsilon = 10^{-2}$

| $N$  | $e^N_{c,10^{-2}}$ | order | $e^N_{s,10^{-2}}$ | order | $e^N_{c,10^{-2}}$ | order | $e^N_{s,10^{-2}}$ | order |
|------|-------------------|-------|-------------------|-------|-------------------|-------|-------------------|-------|
| 12   | 0.432E0           | 1.01  | 0.119E0           | 2.04  | 0.106E0           | 2.01  | 0.524E-1          | 3.24  |
| 24   | 0.215E0           | 1.00  | 0.289E-1          | 2.08  | 0.265E-1          | 2.00  | 0.555E-2          | 3.10  |
| 48   | 0.107E0           | 1.00  | 0.685E-2          | 2.00  | 0.661E-2          | 2.00  | 0.646E-3          | 2.86  |
| 96   | 0.533E-1          | 1.00  | 0.171E-2          | 1.87  | 0.165E-2          | 2.00  | 0.889E-4          | 2.47  |
| 192  | 0.268E-1          | 1.01  | 0.467E-3          | 1.57  | 0.413E-3          | 2.00  | 0.160E-4          | 1.91  |
| 384  | 0.134E-1          | 1.00  | 0.158E-3          | 1.02  | 0.103E-3          | —     | 0.425E-5          | —     |
| 768  | 0.669E-2          | —     | 0.779E-4          | —     | —                 | —     | —                 | —     |
and the corresponding orders of convergence
\[ p_N^c = \frac{\ln e_N^c - \ln e_{2N}^c}{\ln 2}, \quad p_N^s = \frac{\ln e_N^s - \ln e_{2N}^s}{\ln 2}. \] (53)

Numerical results are presented in Tables 1, 2 and log-log chart 1. Table 1 lists errors $e_N^c$ and $e_N^s$ on Bakhvalov-type meshes (6) in the cases of $k = 1$ and $k = 2$. Table 2 lists errors $e_{N,e}^c$ and $e_{N,e}^s$ on Bakhvalov-type meshes (6) for $\varepsilon = 10^{-2}$ in the cases of $k = 1$ and $k = 2$. Errors in the balanced norm for the case $k = 3$ are plotted in Fig. 1. The data in Tables 1, 2 and Fig. 1 show optimal uniform convergence of $\|u - u_N\|_b$ with respect to the singular perturbation parameter $\varepsilon$, which implies that Theorem 1 is sharp. Table 1 and Fig. 1 show that the supercloseness result is of order $k + 1$ for $k$th rectangular finite elements when $\varepsilon \leq N^{-2}$. From Table 2 and Fig. 1, we can observe that convergence orders decrease towards $k + 1/2$ as $N \to \infty$ when $\varepsilon \geq N^{-2}$. The data imply Theorem 2 is sharp.

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Data Availability The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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