The Alon-Tarsi number of planar graphs without cycles of lengths 4 and \( l \)

Huajing Lu\(^*\)  Xuding Zhu\(^{\dag\ddagger}\)

October 29, 2019

Abstract

This paper proves that if \( G \) is a planar graph without 4-cycles and \( l \)-cycles for some \( l \in \{5, 6, 7\} \), then there exists a matching \( M \) such that \( \text{AT}(G - M) \leq 3 \). This implies that every planar graph without 4-cycles and \( l \)-cycles for some \( l \in \{5, 6, 7\} \) is 1-defective 3-paintable.

Keywords: planar graph; choice number; paint number; Alon-Tarsi number.

1 Introduction

Assume \( G \) is a graph and \( d \) is a non-negative integer. A \( d \)-defective coloring of \( G \) is a coloring of the vertices of \( G \) such that each color class induces a subgraph of maximum degree at most \( d \). A 0-defective coloring of \( G \) is also called a proper coloring of \( G \). In a coloring of the vertices of \( G \), we say an edge \( e \) is a fault edge if the end vertices of \( e \) receive the same color. A coloring of \( G \) is 1-defective if and only if the set of fault edges is a matching.

A \( k \)-list assignment of a graph \( G \) is a mapping \( L \) which assigns to each vertex \( v \) a set \( L(v) \) of \( k \) permissible colors. Given a \( k \)-list assignment \( L \) of \( G \), a \( d \)-defective \( L \)-coloring of \( G \) is a \( d \)-defective coloring \( c \) of \( G \) with \( c(v) \in L(v) \) for every vertex \( v \) of \( G \). A graph \( G \) is \( d \)-defective \( k \)-choosable if for any \( k \)-list assignment \( L \) of \( G \), there exists a \( d \)-defective \( L \)-coloring of \( G \). We say \( G \) is \( k \)-choosable if \( G \) is 0-defective \( k \)-choosable. The choice number \( \text{ch}(G) \) of \( G \) is the minimum \( k \) for which \( G \) is \( k \)-choosable.

Defective list coloring of planar graphs has been studied in a few papers. Eaton and Hall [4], and Škrekovski [12] proved independently that every planar graph is 2-defective 3-choosable. Cushing and Kierstead [2] proved...
that every planar graph is 1-defective 4-choosable. The above results can be reformulated as follows:

Assume $G$ is a planar graph. (1) For every 3-list assignment $L$ of $G$, there is a subgraph $H$ of $G$ with $\Delta(H) \leq 2$ and $G - E(H)$ is $L$-colorable. (2) For every 4-list assignment $L$ of $G$, there is a subgraph $H$ of $G$ with $\Delta(H) \leq 1$ and $G - E(H)$ is $L$-colorable.

In the proofs of [2], [4] and [12], the subgraph $H$ depends on the list assignment $L$. A natural question is whether there is a subgraph $H$ that does not depend on $L$. In other words, we ask the following questions:

(1) Is it true that every planar graph $G$ has a subgraph $H$ with $\Delta(H) \leq 2$ such that $G - E(H)$ is 3-choosable?

(2) Is it true that every planar graph $G$ has a subgraph $H$ with $\Delta(H) \leq 1$, such that $G - E(H)$ is 4-choosable?

It turns out that the answer to (1) is negative and the answer to (2) is positive. Very recently, it is shown in [9] that there is a planar graph $G$ such that (this number 3 is not a typo), $G - E(H)$ is not 3-choosable. On the other hand, as a consequence of the main result in [9], every planar graph $G$ has a matching $M$ such that $G - M$ is 4-choosable.

The main result in [9] is about the Alon-Tarsi number of $G - M$. We associate to each vertex $v$ of $G$ a variable $x_v$. The graph polynomial $P_G(x)$ of $G$ is defined as $P_G(x) = \prod_{v \in V(G)} (x_v - x_u)$, where $x = \{x_v : v \in V(G)\}$ and $<$ is an arbitrary fixed ordering of the vertices of $G$. It is easy to see that a mapping $\phi : V \to R$ is a proper coloring of $G$ if and only if $P_G(\phi) \neq 0$, where $P_G(\phi)$ means to evaluate the polynomial at $x_v = \phi(v)$ for $v \in V(G)$. Thus to find a proper coloring of $G$ is equivalent to find an assignment of $x$ so that the polynomial evaluated at the assignment is non-zero.

For a mapping $\eta : V(G) \to \{0,1,\ldots\}$, let $c_{P_G,\eta}$ be the coefficient of the monomial $\prod_{v \in V(G)} x_v^{\eta(v)}$ in the expansion of $P_G$. It follows from the Combinatorial Nullstellensatz that if $c_{P_G,\eta} \neq 0$, and $L$ is a list assignment of $G$ for which $|L(v)| = \eta(v) + 1$, then $G$ is $L$-colorable. (Note that $P_G$ is a homogeneous polynomial, and all the monomials with nonzero coefficient are of highest degree.) In particular, if $c_{P_G,\eta} \neq 0$ and $\eta(v) < k$ for all $v \in V(G)$, then $G$ is $k$-choosable. Jensen and Toft [8] defined the Alon-Tarsi number of $G$ as

$$AT(G) = \min\{k : c_{P_G,\eta} \neq 0 \text{ for some } \eta \text{ with } \eta(v) < k \text{ for all } v \in V(G)\}.$$ 

Thus for any graph $G$, $\text{ch}(G) \leq AT(G)$. The following is the main result in [9].

**Theorem 1.1** Every planar graph $G$ has a matching $M$ such that $AT(G - M) \leq 4$. 

2
This theorem actually implies the online version of 1-defective 4-choosability of planar graphs. The online version of $d$-defective $k$-choosable is called $d$-defective $k$-paintable and is defined through a two-person game.

Given a graph $G$ and non-negative integers $k, d$, the $d$-defective $k$-painting game on $G$ is played by two players: Lister and Painter. Initially, each vertex has $k$ tokens and is uncolored. In each round, Lister selects a nonempty set $M$ of uncolored vertices and takes away one token from each vertex in $M$. Painter colors a subset $X$ of $M$ such that the induced subgraph $G[X]$ has maximum degree at most $d$. If at the end of a certain round, there is an uncolored vertex with no tokens left, then Lister wins. Otherwise, at the end of some round, all vertices are colored, Painter wins. We say $G$ is $d$-defective $k$-paintable if Painter has a winning strategy in the $d$-defective $k$-painting game. The 0-defective $k$-painting game is also called the $k$-painting game, and we say $G$ is $k$-paintable if it is 0-defective $k$-paintable. The paint number $\chi_P(G)$ of $G$ is the minimum $k$ such that $G$ is $k$-paintable.

It follows from the definition that $d$-defective $k$-paintable implies $d$-defective $k$-choosable. The converse is not true. Indeed, although every planar graph is 2-defective 3-choosable, it was shown in [5] that there are planar graphs that are not 2-defective 3-paintable.

On the other hand, it was proved by Schauz [11] that for any graph $G$, $\chi_P(G) \leq AT(G)$. So for any graph $G$, $ch(G) \leq \chi_P(G) \leq AT(G)$. Both gaps $\chi_P(G) - ch(G)$ and $AT(G) - \chi_P(G)$ can be arbitrarily large [3]. Thus Theorem 1.1 implies that every planar graph is 1-defective 4-paintable. We observe that “having a matching $M$ so that $AT(G - M) \leq 4$” is much stronger than “being 1-defective 4-paintable”. One may compare this to the following results: It is shown in [5] that every planar graph is 3-defective 3-paintable. However, as mentioned earlier, there are planar graphs $G$ such that for any subgraph $H$ of $G$ with $\Delta(H) \leq 3$, $G - E(H)$ is not 3-choosable [9] (and hence $AT(G - E(H)) \geq 4$).

In this paper, we are interested in the Alon-Tarsi number of some subgraphs of planar graphs without cycles of lengths 4 and $l$ for some $l \in \{5, 6, 7\}$. We denote by $\mathcal{P}_{k,l}$ the family of planar graphs $G$ which contains no cycles of length $k$ or $l$. It was proved in [10] that for $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{P}_{4,l}$ is 1-defective 3-choosable. We strengthen this result and prove that for $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{P}_{4,l}$ has a matching $M$ such that $G - M$ has Alon-Tarsi number at most 3. As discussed above, this implies that for $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{P}_{4,l}$ is 1-defective 3-paintable.

For a plane graph $G$, we denote its vertex set, edge set and face set by $V(G), E(G)$ and $F(G)$, respectively. For a vertex $v$, $d_G(v)$ (or $d(v)$ for short) is the degree of $v$. A vertex $v$ is called a $k$-vertex (respectively, a $k^+$-vertex or a $k^-$-vertex) if $d(v) = k$ (respectively, $d(v) \geq k$ or $d(v) \leq k$). For $e = uv \in E(G)$, we say $e$ is an $(a, b)$-edge if $d(u) = a$ and $d(v) = b$. For $f \in F(G)$, we denote $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \cdots, u_n$ are the boundary
vertices of $f$ in cyclic order. A 3-face $[u_1u_2u_3]$ is a $(d_1, d_2, d_3)$-face if $d(u_i) = d_i$ for $i = 1, 2, 3$.

2 The main result

The following is the main result of this paper.

**Theorem 2.1** For $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{P}_{4,l}$ has a matching $M$ such that $AT(G - M) \leq 3$.

For the proof of Theorem 2.1, we use an alternate definition of Alon-Tarsi number. A digraph $D$ is Eulerian if $d_D^+(v) = d_D^-(v)$ for every vertex $v$. Note that an Eulerian digraph needs not be connected. In particular, a digraph with no arcs is an Eulerian digraph. Assume $G$ is a graph and $D$ is an orientation of $G$. Let $\mathcal{E}_e(D)$ (respectively, $\mathcal{E}_o(D)$) be the set of spanning Eulerian sub-digraphs of $D$ with an even (respectively, an odd) number of arcs. Let

$$\text{diff}(D) = |\mathcal{E}_e(D)| - |\mathcal{E}_o(D)|.$$  

An orientation $D$ of $G$ is Alon-Tarsi (AT) if $\text{diff}(D) \neq 0$. Alon and Tarsi [1] proved that if $D$ is an orientation of $G$, and $\eta(x) = d_D^+(x)$, then $c_{PG,\eta} = \pm \text{diff}(D)$. Hence the Alon-Tarsi number of $G$ can be defined alternatively as

$$AT(G) = \min\{k : G \text{ has an AT orientation } D \text{ with } \Delta_D^+(v) < k\}.$$  

The proof of Theorem 2.1 is by induction. For the purpose of using induction, instead of proving Theorem 2.1 directly, we shall prove a stronger and more technique result.

**Definition 2.2** Assume $G$ is a plane graph and $v_0$ is a vertex on the boundary of $G$. A valid matching of $(G, v_0)$ is a matching $M$ which does not cover $v_0$.

**Definition 2.3** Let $G$ be a plane graph and $v_0$ be a vertex on the boundary of $G$. An orientation $D$ of $G$ is good, if $D$ is AT with $\Delta_D^+(v) < 3$ and $d_D^+(v_0) = 0$.

We shall prove the following result, which obviously implies Theorem 2.1.

**Theorem 2.4** Assume $l \in \{5, 6, 7\}$, $G \in \mathcal{P}_{4,l}$, and $v_0$ is a vertex on the boundary of $G$. Then $(G, v_0)$ has a valid matching $M$ such that there is a good orientation $D$ of $G - M$. 

4
The proof of Theorem 2.4 uses discharging method. We shall first describe a family of reducible configurations, i.e., configurations that cannot be contained in a minimum counterexample of Theorem 2.4. Then describe a discharging procedure that leads to a contradiction to the Euler’s formula.

We shall frequently use the following lemma in the later proofs.

**Lemma 2.5** Assume $D$ is a digraph with $V(D) = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. If all the arcs between $X_1$ and $X_2$ are from $X_1$ to $X_2$. Then $D$ is AT if and only if $D[X_1]$ and $D[X_2]$ are both AT.

**Proof.** Denote by $D_1$ and $D_2$ the sub-digraphs $D[X_1]$ and $D[X_2]$ of $D$, respectively. Note that the set of arcs of an Eulerian digraph can be decomposed into arc disjoint union of directed cycles. Since all the arcs between $X_1$ and $X_2$ are from $X_1$ to $X_2$, and hence none of them is contained in a directed cycle, we conclude that none of these arcs is contained in an Eulerian sub-digraphs of $D$. Hence each Eulerian sub-digraph $H$ of $D$ is the arc disjoint union of an Eulerian sub-digraph $H_1$ of $D_1$ and an Eulerian sub-digraph $H_2$ of $D_2$. Now $H$ is even if and only if $H_1, H_2$ have the same parity. Hence $|E_e(D)| = |E_o(D)| = |E_o(D_1)| \times |E_o(D_2)| + |E_o(D_1)| \times |E_o(D_2)|$. This implies that $\text{diff}(D) = \text{diff}(D_1) \times \text{diff}(D_2)$. Thus, $D$ is AT if and only if $D_1$ and $D_2$ are both AT.

Assume Theorem 2.4 is not true and $G$ is a counterexample with minimum number of vertices. Let $f_0$ denote the outer face of $G$.

**Lemma 2.6** $G$ is 2-connected. Moreover, $d_G(v) \geq 3$ for all $v \in V(G) - \{v_0\}$.

**Proof.** Assume $G$ is not 2-connected. Let $B$ be a block of $G$ that contains a unique cut vertex $z^*$ and does not contain $v_0$. Let $G_1 = G - (B - \{z^*\})$. By the minimality, $(G_1, v_0)$ has a valid matching $M_1$ and there is a good orientation $D_1$ of $G_1 - M_1$. $(B, z^*)$ has a valid matching $M_2$ and there is a good orientation $D_2$ of $B - M_2$. Let $M = M_1 \cup M_2$ and $D = D_1 \cup D_2$. Applying Lemma 2.5 (with $X_1 = V(B) - \{z^*\}$ and $X_2 = V(G_1) - \{v_0\}$), $D$ is AT. So $M$ is a valid matching of $(G, v_0)$, and $G - M$ has a good orientation, a contradiction.

For the moreover part, assume to the contrary that $v \in V(G) - \{v_0\}$ and $d_G(v) \leq 2$. By induction hypothesis, $G' = G - \{v\}$ has a valid matching $M$ such that $G' - M$ has a good orientation $D'$. Extend $D'$ to an orientation $D$ of $G - M$ in which $v$ is a source vertex. It is obvious that $D$ is a good orientation of $G - M$.

**Lemma 2.7** $G - \{v_0\}$ does not contain two adjacent 3-vertices.
Proof. Assume to the contrary that $uv \in E(G)$ with $d(u) = d(v) = 3$ and $u, v \neq v_0$. Let $G^* = G - \{u, v\}$. Then $(G^*, v_0)$ has a valid matching $M^*$ such that there exists a good orientation $D^*$ of $G^* - M^*$. Let $M = M^* \cup \{uv\}$. Then $M$ is a valid matching of $(G, v_0)$. Extend $D^*$ to an orientation $D$ of $G - M$ in which $u, v$ are sources. Then $D$ is a good orientation of $G - M$. \hfill $\square$

Definition 2.8 A 3-face $f$ is called a minor triangle if $f$ is a $(3, 4, 4)$-face and $v_0$ is not on $f$. A 3-vertex $v$ is called a minor 3-vertex if $v$ is incident to a triangle and $v \neq v_0$.

Definition 2.9 A triangle chain in $G$ of length $k$ is a subgraph of $G - \{v_0\}$ consisting of vertices $w_1, w_2, \ldots, w_{k+1}, u_1, u_2, \ldots, u_k$ in which $[w_iw_{i+1}ui]$ is a $(4, 4, 4)$-face for $i = 1, 2, \ldots, k$, as depicted in Figure 1(a). We denote by $T_i$ the triangle $[w_iw_{i+1}ui]$ and denote such a triangle chain by $T_1T_2 \ldots T_k$. For convenience, a single 4-vertex is a triangle chain with 0 triangles. We say a triangle $T$ intersects a triangle chain $T_1T_2 \ldots T_k$, if $T$ has one common vertex with $T_1$.

Lemma 2.10 If a minor triangle $T_0$ intersects a triangle chain $T_1T_2 \ldots T_k$, then no vertex of $T_k$ is adjacent to a 3-vertex, except possibly $v_0$. In particular, the $k = 0$ case implies that no vertex of a minor triangle $T_0$ is adjacent to a 3-vertex $v \in V(G) - (V(T_0) \cup \{v_0\})$.

Proof. Assume to the contrary that $T_0 = [w_0w_1u_0]$ is a minor triangle that intersects a triangle chain $T_1T_2 \ldots T_k$, with $T_i = [w_iw_{i+1}ui]$ (1 ≤ $i$ ≤ $k$), and $w_{k+1}$ has a neighbour $x$ with $d(x) = 3$, as in Figure 1(b). Assume $w_0$ is a 3-vertex. Let $X = \bigcup_{i=0}^k V(T_i) \cup \{x\}$ and $G' = G - X$. By the minimality of $G$, $(G', v_0)$ contains a valid matching $M'$ and there is a good orientation $D'$ of $G' - M'$.

Let $M = M' \cup \{w_0u_0, w_1u_1, \ldots, w_ku_k, w_{k+1}x\}$. Then $M$ is a valid matching of $(G, v_0)$. Let $D$ be an orientation of $G - M$ obtained from $D'$ by adding arcs $(w_i, w_{i+1})$ and $(w_{i+1}, u_i)$ for $i = 0, 1, \ldots, k$, and all the edges between $X$ and $V - X$ are oriented from $X$ to $V - X$, as depicted in Figure 1(c). Since $D[X]$ is acyclic, $D[X]$ is AT. By Lemma 2.5, $D$ is AT. It is easy to see that $\Delta_D(v) < 3$ and $d^D_D(v_0) = 0$. Thus $D$ is a good orientation of $G - M$. \hfill $\square$

Lemma 2.11 If a triangle chain $T_1T_2 \ldots T_k$ intersects a minor triangle $T_0$, then the distance between $T_k$ and another minor triangle is at least 2. In particular, the $k = 0$ case implies that any two minor triangles have distance at least 2.

Proof. Assume to the contrary that $T_1T_2 \ldots T_k$ with $T_i = [w_iw_{i+1}ui]$ (1 ≤ $i$ ≤ $k$) is a triangle chain that intersects a minor triangle $T_0 = [w_0w_1u_0]$, and the distance between $T_k$ and another minor triangle $T'_0 = [xyz]$ with
Figure 1: (a) A triangle chain. (b) The configuration in Lemma 2.10. (c) For the proof of Lemma 2.10, where a thick line is an edge in the matching $M$.

$d(x) = 3$ is less than 2. By Lemma 2.10, we may assume $w_{k+1}y$ is a $(4,4)$-edge connecting $T_1$ and $T_0^*$, as in Figure 2(a). Let $X = \cup_{i=0}^{k}V(T_i) \cup V(T_0^*)$ and $G' = G - X$. Then $(G', v_0)$ has a valid matching $M'$ and there is a good orientation $D'$ of $G' - M'$.

Let $M = M' \cup \{w_0u_0, w_1u_1, \ldots, w_ku_k, w_{k+1}y, xz\}$. Then $M$ is a valid matching of $(G, v_0)$. Let $D$ be an orientation of $G - M$ obtained from $D'$ by adding arcs $(x,y), (y,z), (w_i, w_{i+1})$ and $(w_{i+1}, u_i)$ for $i = 0, 1, \ldots, k$, and all the edges between $X$ and $V - X$ are oriented from $X$ to $V - X$, as in Figure 2(b). Obviously, $D[X]$ is acyclic, so $D[X]$ is AT. By Lemma 2.5, $D$ is AT. Additionally, $\Delta_{D^*}(v) < 3$ and $d_{D^*}^X(v_0) = 0$. That is to say, $D$ is a good orientation of $G - M$, a contradiction.

The remainder of the proofs use a discharging procedure. The initial charge $ch$ is defined as: $ch(x) = d(x) - 4$ for $x \in V(G) \cup F(G)$. Applying equalities $\sum_{v \in V(G)} d(v) = 2|E(G)| = \sum_{f \in F(G)} d(f)$ and Euler’s formula $|V(G)| - |E(G)| + |F(G)| = 2$, we conclude that

$$\sum_{x \in V(G) \cup F(G)} ch(x) = -8.$$

In a discharging procedure, $ch(x \rightarrow y)$ denotes the charge discharged
from an element $x$ to another element $y$, $ch(x \rightarrow)$ and $ch(\rightarrow x)$ denote the charge totally discharged from or to $x$, respectively. The final charge $ch^*(x)$ of $x \in V(G) \cup E(G)$ is defined as $ch^*(x) = ch(x) - ch(x \rightarrow) + ch(\rightarrow x)$. By applying appropriate discharging rules, we shall arrive at a final charge that $ch^*(x) \geq 0$ for all $x \in V(G) \cup E(G) \setminus \{v_0, f_0\}$, and $ch^*(v_0) + ch^*(f_0) > -8$. As the total charge does not change in the discharging process, this is a contradiction.

The discharging rules for graphs $G \in P_{4,l}$ for $l \in \{5, 6, 7\}$ are different. We use three sections to discuss graphs $G \in P_{4,l}$ for $l \in \{5, 6, 7\}$, respectively.

3 Planar graphs without 4- and 5-cycles

This section considers plane graphs without 4- and 5-cycles. We first derive more properties of a minimal counterexample $G$ to Theorem 2.4, where $G \in P_{4,5}$.

Lemma 3.1 Assume $f$ is a 6-face of $G$ which is adjacent to five triangles, and none of the vertices in these triangles is $v_0$. If $f$ has one 3-vertex, then there is at least one $5^+$-vertex on the five triangles.

Proof. Let $f = [v_1v_2v_3v_4v_5v_6]$, $v_1$ be a 3-vertex and $T_i = [v_iv_{i+1}u_i]$ ($i = 1, 2, \ldots, 5$) be the five triangles (see Figure 3(a)). Assume to the contrary that there is no $5^+$-vertex on $T_i$. By Lemma 2.10, we may assume all $v_{i+1}$ and $u_i$ are 4-vertices for $i = 1, 2, \ldots, 5$. Let $X = \cup_{i=1}^5 V(T_i)$ and $G' = G - X$. Then $(G', v_0)$ has a valid matching $M'$ and there is a good orientation $D'$ of $G' - M'$.

Let $M = M' \cup \{v_1u_1, v_2u_2, \ldots, v_5u_5\}$. Then $M$ is a valid matching of $(G, v_0)$. Let $D$ be the orientation of $G - M$ obtained from $D'$ by adding arcs
(v_1, v_6) and (v_{i+1}, u_i), (v_i, v_{i+1}) for i = 1, \ldots, 5, and all the edges between X and V - X are oriented from X to V - X (see Figure 3(b)). Clearly, \( \Delta_D^+(v) < 3 \) and D is AT by Lemma 2.5, a contradiction.

The discharging rules are as follows:

R1 Assume \( f \neq f_0 \) is a 3-face. Then each face adjacent to \( f \) transfers \( \frac{1}{3} \) charge to \( f \).

R2 Assume \( v \neq v_0 \) is 3-vertex. If \( v \) is contained in a triangle, then each of the other two faces incident to \( v \) transfers \( \frac{1}{2} \) charge to \( v \); otherwise each face incident to \( v \) transfers \( \frac{1}{3} \) charge to \( v \).

R3 Assume \( u \neq v_0 \) is a 5*-vertex and \( f \neq f_0 \) is a 6-face. If \( f \) is adjacent to \( s \) triangles that are incident to \( u \), then \( u \) transfers \( \frac{s}{6} \) charge to \( f \).

R4 \( f_0 \) transfers \( \frac{1}{3} \) charge to each adjacent triangle, and \( \frac{1}{3} \) charge to each incident 3-vertex \( v \neq v_0 \). \( v_0 \) transfers \( \frac{1}{2} \) charge to each 6-face \( f \neq f_0 \) which is either incident to \( v_0 \), or is not incident to \( v_0 \) but adjacent to a triangle \( T \) which is incident to \( v_0 \).

Claim 3.2 If a 6-face \( f \) has three minor 3-vertices, then \( \text{ch}(\to f) \geq \frac{1}{2} \).

Proof. Assume \( f = [v_1v_2v_3v_4v_5v_6] \). By Lemma 2.7 we may assume that \( v_1, v_3, \) and \( v_5 \) are the three minor 3-vertices. Then each of \( v_1, v_3, v_5 \) is incident to exactly one triangle. Hence at most two of the three triangles intersect each other. Thus we may assume that the three triangles adjacent to \( f \) are either \( T_1, T_2, T_4 \), or \( T_1, T_3, T_5 \), where \( T_i = [v_iv_{i+1}u_i] \).
Case 1 The three triangles incident to \( f \) are \( T_1, T_2, T_4 \).

If \( v_0 \) is a vertex of \( f \) or \( T_1, T_2 \) or \( T_4 \), then \( v_0 \) transfers \( \frac{1}{6} \) charge to \( f \) by R4. By Lemma 2.10, at least one of the three triangles have a 5-vertex \( v \neq v_0 \) which sends at least \( \frac{1}{6} \) charge to \( f \). So \( ch(\rightarrow f) \geq \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \).

Assume \( v_0 \) is not a vertex of \( f, T_1, T_2 \) or \( T_4 \). By Lemma 2.10, either \( v_2 \) is a 5-vertex or both of \( u_1 \) and \( u_2 \) are 5-vertices. In both cases, \( f \) receives \( \frac{1}{3} \) charge in total from \( v_2, u_1 \) and \( u_2 \). Moreover, by Lemma 2.10, either \( v_4 \) or \( u_4 \) is a 5-vertex, which transfers \( \frac{1}{6} \) charge to \( f \). Hence, \( ch(\rightarrow f) \geq \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \).

Case 2 The three triangles incident to \( f \) are \( T_1, T_3, T_5 \).

By Lemma 2.10, each of the three triangles has either a 5-vertex or \( v_0 \) which transfers at least \( \frac{1}{6} \) charge to \( f \). Thus, \( ch(\rightarrow f) \geq \frac{1}{2} \). \( \square \)

Claim 3.3 If a 6-face \( f \) has two 3-vertices other than \( v_0 \) and is adjacent to four triangles, then \( ch(\rightarrow f) \geq \frac{1}{3} \).

Proof. Assume \( f = [v_1v_2v_3v_4v_5v_6] \) and \( T \) is a triangle adjacent to \( f \). If \( v_0 \) is a vertex of \( f \) or \( T \), then \( v_0 \) transfers \( \frac{1}{6} \) charge to \( f \) by R4. Assume \( v_0 \) is neither a vertex of \( f \) nor a vertex of any triangle \( T \) adjacent to \( f \).

By Lemma 2.7, we may assume that either \( v_1 \) and \( v_4 \) or \( v_1 \) and \( v_3 \) are the two 3-vertices. For \( i = 1, 2, \ldots, 6 \), if \( v_iv_{i+1} \) is contained in a triangle, then let \( T_i = [v_i v_{i+1} u_i] \) be the triangle. We need to consider five cases.

Case 1 The four triangles incident to \( f \) are \( T_1, T_2, T_3, T_5 \) while the two 3-vertices are \( v_1 \) and \( v_3 \).

If at least one of \( v_2 \) and \( v_3 \) is a 5-vertex, then by R3, \( ch(\rightarrow f) \geq \frac{1}{3} \).

Assume both \( d(v_2) \) and \( d(v_3) \) are 4-vertices. By Lemma 2.10 and Lemma 2.11 at least two of \( u_1, u_2 \) and \( u_3 \) are 5-vertices each of which transfers \( \frac{1}{6} \) charge to \( f \). So \( ch(\rightarrow f) \geq \frac{1}{3} \).

Case 2 The four triangles incident to \( f \) are \( T_1, T_2, T_4, T_5 \) while the two 3-vertices are \( v_1 \) and \( v_4 \).

By Lemma 2.10 and R3, at least one of \( u_1, u_2, v_2 \) and \( v_3 \) is a 5-vertex transferring at least \( \frac{1}{6} \) charge to \( f \). By symmetry, at least one of \( u_4, u_5, v_5 \) and \( v_6 \) transfers at least \( \frac{1}{6} \) charge to \( f \). Thus, we are done.

Case 3 The four triangles incident to \( f \) are \( T_1, T_2, T_4, T_5 \) while the two 3-vertices are \( v_1 \) and \( v_3 \).

If \( v_2 \) is a 5-vertex, then \( v_2 \) transfers \( \frac{1}{6} \) charge to \( f \) by R3. Assume \( v_2 \) is a 4-vertex. By Lemma 2.10, both \( u_1 \) and \( u_2 \) are 5-vertices each of which transfers \( \frac{1}{6} \) charge to \( f \).

Case 4 The four triangles incident to \( f \) are \( T_1, T_3, T_4, T_5 \) while the two 3-vertices are \( v_1 \) and \( v_3 \).

By Lemma 2.10 at least one of \( v_2 \) and \( u_1 \) is a 5-vertex transferring \( \frac{1}{6} \) charge to \( f \). Moreover, using Lemma 2.10 again, at least one of \( v_4, v_5, v_6, u_3, u_4 \) and \( u_5 \) is a 5-vertex transferring at least \( \frac{1}{6} \) to \( f \). So \( ch(\rightarrow f) \geq \frac{1}{3} \).

Case 5 The four triangles incident to \( f \) are \( T_3, T_4, T_5, T_6 \) while the two 3-vertices are \( v_1 \) and \( v_3 \).
If one of \( v_4, v_5 \) and \( v_6 \) is a \( 5^+ \)-vertex, then such a \( 5^+ \)-vertex sends \( \frac{1}{3} \) charge to \( f \) by R3. Assume all of \( v_4, v_5, v_6 \) are \( 4 \)-vertices. Then at least two of \( u_3, u_4, u_5, u_6 \) are \( 5^+ \)-vertices each sending \( \frac{1}{6} \) to \( f \). Otherwise, it will contradict to Lemma 2.10 or Lemma 2.11. Again \( ch(\rightarrow f) \geq \frac{1}{3} \).

\[ \square \]

**Check charge on vertices** \( v \neq v_0 \)

Let \( v \) be a \( 3 \)-vertex. By R2, \( v \) gets 1 from incident \( 6^+ \)-faces. That is \( ch^*(v) = ch(v) - ch(v \rightarrow) + ch(\rightarrow v) = -1 - 0 + 1 = 0 \).

Let \( v \) be a \( 4 \)-vertex. \( ch^*(v) = ch(v) = 0 \).

Let \( v \) be a \( 5^+ \)-vertex. By R3, \( v \) only transfers charge to \( 6 \)-faces that are adjacent to a triangle incident to \( v \). Assume \( v \) is incident with \( t \) triangles, then \( 0 < t \leq \lfloor \frac{\text{deg}(v)}{2} \rfloor \). Each triangle incident with \( v \) is adjacent to at most three \( 6 \)-faces, and \( v \) transfers \( \frac{1}{6} \) to each of the three \( 6 \)-faces (note that if a \( 6 \)-face \( f \) is adjacent to two triangles that are incident to \( v \), then \( v \) transfers \( 2 \times \frac{1}{6} \) charges to \( f \)). Hence \( v \) sends out at most \( \frac{1}{3}t \) charge. So we have \( ch^*(v) = ch(v) - ch(v \rightarrow) \geq d(v) - 4 - \frac{1}{3}t \geq d(v) - 4 - \frac{1}{2} \times \lfloor \frac{\text{deg}(v)}{2} \rfloor \geq 0 \).

**Check charge on faces** \( f \neq f_0 \)

Let \( f \) be a \( 3 \)-face. R1 guarantees \( ch^*(f) \geq 0 \).

Let \( f \) be a \( 6 \)-face. Assume that \( f \) has \( s \) \( 3 \)-vertices other than \( v_0 \). Then \( s \leq 3 \) by Lemma 2.7, and \( f \) is adjacent to at most \( (6 - s) \) triangles.

If \( s = 0 \), then \( f \) sends at most \( \frac{1}{3} \) to each adjacent triangle, and hence \( ch(f \rightarrow) \leq \frac{1}{3} \times 6 = 2 \) and \( ch^*(f) \geq 0 \).

Assume \( s = 3 \). If \( f \) is adjacent to at most two triangles, then \( f \) has at most two minor \( 3 \)-vertices. So \( ch(f \rightarrow) \leq \frac{1}{2} \times 2 + \frac{1}{3} \times 1 = 2 \) and \( ch^*(f) \geq 0 \). Assume \( s = 3 \) is adjacent to at most three triangles, then all these three \( 3 \)-vertices are minor. By Claim 3.2, we have \( ch^*(f) = d(f) - 4 - ch(f \rightarrow) + ch(\rightarrow f) \geq 2 - (\frac{1}{2} \times 3 + \frac{1}{3} \times 3) + \frac{1}{2} = 0 \).

Assume \( s = 2 \). If \( f \) is adjacent to at most three triangle, then \( ch(f \rightarrow) \leq \frac{1}{2} \times 2 + \frac{1}{3} \times 3 = 2 \) and \( ch^*(f) \geq 0 \). If \( f \) is adjacent to four triangles, then by Claim 3.3, \( ch^*(f) = d(f) - 4 - ch(f \rightarrow) + ch(\rightarrow f) \geq 2 - (\frac{1}{2} \times 2 + \frac{1}{3} \times 4) + \frac{1}{3} = 0 \).

Assume \( s = 1 \). If \( f \) is adjacent to at most four triangles, then \( ch(f \rightarrow) \leq \frac{1}{2} \times 2 + \frac{1}{3} \times 4 = \frac{11}{6} < 2 \). Assume \( f \) is adjacent to five triangles. Then \( ch(f \rightarrow) = \frac{1}{2} + \frac{1}{3} \times 5 = \frac{13}{6} \). On the other hand, either at least one vertex of the five triangles is a \( 5^+ \)-vertex transferring \( \frac{1}{6} \) charge to \( f \) by Lemma 3.1, or \( v_0 \) is a vertex of the five triangles transferring \( \frac{1}{6} \) to \( f \) by R4. Hence \( ch^*(f) \geq 2 - \frac{13}{6} + \frac{1}{6} = 0 \).

Let \( f \) be a \( 7^+ \)-face. Assume \( f \) has \( s \) \( 3 \)-vertices other than \( v_0 \), then \( s \leq \lfloor \frac{\text{deg}(f)}{2} \rfloor \) and \( f \) is adjacent to at most \( (d(f) - s) \) triangles. Hence \( ch^*(f) = d(f) - 4 - \lfloor \frac{1}{2} \times s + \frac{1}{3} \times (d(f) - s) \rfloor = \frac{2}{3}d(f) - \frac{1}{3}s - 4 \geq (\frac{2}{3} - \frac{1}{12})d(f) - 4 > 0 \).

**Check charge on \( f_0 \) and \( v_0 \)**

By R4, it is clear that \( v_0 \) transfers at most \( (d(v_0) - 1) \times \frac{1}{6} \) charge to others. That is, \( ch^*(v_0) \geq d(v_0) - 4 - (d(v_0) - 1) \times \frac{1}{3} = \frac{2}{3}d(v_0) - \frac{11}{3} \geq -\frac{7}{3} \)(as \( d(v_0) \geq 2 \)).
Since $f_0$ is incident with at most $\lfloor \frac{d(f_0)}{2} \rfloor$ 3-vertices each getting $\frac{1}{3}$ charge from it, and $f_0$ is adjacent to at most $d(f_0)$ triangles each getting $\frac{1}{3}$ charge from it. We have $ch^*(f_0) \geq d(f_0) - 4 - \frac{1}{2} \lfloor \frac{d(f_0)}{2} \rfloor - \frac{1}{3}d(f_0) \geq \frac{5}{12}d(f_0) - 4 \geq - \frac{11}{4}$.

Consequently, we obtain the following contradiction, and the proof is complete.

$$0 \leq \sum_{x \in V \cup F \setminus \{v_0, f_0 \}} ch^*(x) = -8 - ch^*(v_0) - ch^*(f_0) \leq - \frac{35}{12}.$$  

4 Planar graphs without 4- and 6-cycles

This section shows plane graph without 4- and 6-cycles. We list our discharging rules as follows:

R1 Assume $f \neq f_0$ is a 3-face. Then each face adjacent to $f$ transfers $\frac{1}{3}$ charge to $f$.

R2 Assume $v \neq v_0$ is 3-vertex. If $v$ is contained in a triangle, then each of the other two faces incident to $v$ transfers $\frac{1}{2}$ charge to $v$; otherwise each face incident to $v$ transfers $\frac{1}{3}$ charge to $v$.

R3 $f_0$ transfers $\frac{1}{2}$ charge to each adjacent triangle, and $\frac{1}{2}$ charge to each incident 3-vertex $v \neq v_0$.

■ Check charge on vertices $v \neq v_0$

For $d(v) = 3$, R2 ensures that the final charge of $v$ is non-negative. For $d(v) \geq 4$, no transference on $v$, we have $ch^*(v) = ch(v) \geq 0$.

■ Check charge on faces $f \neq f_0$

Let $f$ be a 3-face. R1 guarantees $ch(\rightarrow f) = 1$. So $ch^*(f) = -1 + 1 = 0$.

Let $f$ be a 5-face. Since $G$ does not contain 6-cycle, $f$ is not adjacent to any triangle. Thus $f$ only discharges to the non-minor 3-vertices each of which gets $\frac{1}{3}$ charge from $f$. On the other hand, $f$ is incident with at most two such 3-vertices by Lemma 2.7. It concludes that $ch^*(f) = 1 - \frac{1}{3} \times 2 > 0$.

Let $f$ be a 7-face. Assume $f$ is incident with $s$ 3-vertices besides $v_0$. Then by Lemma 2.7, $s \leq \lfloor \frac{d(f)}{2} \rfloor$. By R1 and R2, $f$ transfers at most $\frac{1}{2}s$ to 3-vertices and $(d(f) - s) \times \frac{1}{3}$ to triangles. Hence, we have $ch^*(f) \geq d(f) - 4 - \frac{1}{2}s - \frac{1}{3}(d(f) - s) = \frac{2}{3}d(f) - \frac{1}{6}s - 4 \geq \frac{7}{12}d(f) - 4 > 0$.

■ Check charge on $f_0$ and $v_0$

It is obvious that $ch^*(v_0) = ch(v_0) = d(v_0) - 4 \geq -2$.

Since $f_0$ is incident with at most $\lfloor \frac{d(f_0)}{2} \rfloor$ 3-vertices each getting $\frac{1}{3}$ charge from it, and $f_0$ is adjacent to at most $d(f_0)$ triangles each getting $\frac{1}{3}$ charge from it. We have $ch^*(f_0) \geq d(f_0) - 4 - \frac{1}{2} \lfloor \frac{d(f_0)}{2} \rfloor - \frac{1}{3}d(f_0) \geq \frac{5}{12}d(f_0) - 4 \geq - \frac{11}{4}$.
Consequently, we obtain the following contradiction, and the proof is complete.

\[ 0 \leq \sum_{x \in V \cup F \setminus \{v_0, f_0\}} ch^*(x) = -8 - ch^*(f_0) - ch^*(v_0) \leq -\frac{13}{4}. \]

5 Planar graphs without 4- and 7-cycles

In this section, we consider plane graphs without 4- and 7-cycles. First we derive more properties of a minimal counterexample \( G \) to Theorem 2.4 for \( G \in \mathcal{P}_{4,7} \).

| Figure 4: (a) A special 5-cycle and an adjacent triangle. (b) For the proof of Lemma 5.2 where a thick line is an edge in the matching \( M \).

Definition 5.1 A 5-cycle \( f = [u_1 u_2 u_3 u_4 u_5] \) is called special if it is adjacent to a triangle \( T = [u_1 u_5 u_6] \) with \( u_i \neq v_0 \) (\( i = 1, 2, \ldots, 6 \)), and all the vertices are 4-vertices except that \( u_1 \) and \( u_3 \) are 3-vertices, as depicted in Figure 4(a).

Lemma 5.2 \( G \) has no special 5-cycle.

Proof. Assume \( f = [u_1 u_2 u_3 u_4 u_5] \) is a special 5-cycle and \( T = [u_1 u_5 u_6] \) is a triangle adjacent to \( f \), where \( d(u_1) = d(u_3) = 3 \) and \( d(u_i) = 4 \) for \( i = 2, 4, 5, 6 \). Let \( X = \{u_1, u_2, \ldots, u_6\} \) and \( G' = G - X \). Then, by the minimality, \( (G', v_0) \) has a valid matching \( M' \) and there is a good orientation \( D' \) of \( G' - M' \).

Let \( M = M' \cup \{u_1 u_2, u_3 u_4, u_5 u_6\} \), then \( M \) is a valid matching of \( (G, v_0) \). Let \( D \) be an orientation of \( G \) obtained from \( D' \) by adding arcs \( (u_1, u_6) \),
(u_1, u_5), (u_5, u_4) and (u_3, u_2), and all the edges between X and V − X are oriented from X to V − X, as depicted in Figure 3(b). It is obvious that \(D[X]\) is AT. Then, by Lemma 2.5, \(D\) is AT. As \(\Delta_D(v) < 3\) and \(d_D^+(v_0) = 0\), \(D\) is a good orientation of \(G − M\), a contradiction. \(\square\)

The discharging rules are defined as follows:

R1 Assume \(f \neq f_0\) is a 3-face. Then each face adjacent to \(f\) transfers \(\frac{1}{3}\) charge to \(f\).

R2 Assume \(v \neq v_0\) is 3-vertex. If \(v\) is contained in a triangle, then each of the other two faces incident to \(v\) transfers \(\frac{1}{2}\) charge to \(v\); otherwise each face incident to \(v\) transfers \(\frac{1}{3}\) charge to \(v\).

R3 Assume \(u \neq v_0\) is a \(5^+\)-vertex and \(f \neq f_0\) is a 5-face. Then \(u\) transfers \(\frac{1}{6}\) charge to \(f\) either \(f\) is incident to \(u\), or \(f\) is not incident to \(u\) but adjacent to a triangle which is incident to \(u\).

R4 \(f_0\) transfers \(\frac{1}{4}\) charge to each adjacent triangle, and \(\frac{1}{2}\) charge to each incident 3-vertex \(v \neq v_0\). \(v_0\) transfers \(\frac{1}{2}\) charge to each 5-face \(f \neq f_0\) which is either incident to \(v_0\), or is not incident to \(v_0\) but adjacent to a triangle \(T\) which is incident to \(v_0\).

\[\text{\underline{Check charge on vertices} } v \neq v_0\]
Let \(v\) be a 3-vertex. By R2, \(ch^*(v) \geq 0\).
Let \(v\) be a 4-vertex. We have \(ch^*(v) = ch(v) = 0\).
Let \(v\) be a \(5^+\)-vertex. By R3, \(v\) transfers at most \(\frac{1}{6} \times d(v)\) charge to 5-faces. It follows that \(ch^*(v) \geq d(v) - 4 - \frac{1}{6}d(v) = \frac{5}{6}d(v) - 4 > 0\).

\[\text{\underline{Check charge on faces} } f \neq f_0\]
Let \(f\) be a 3-face. R1 guarantees \(ch^*(f) \geq 0\).
Let \(f\) be a 5-face. By Lemma 2.7, \(f\) has at most two 3-vertices other than \(v_0\). Since \(G\) has no 7-cycle, \(f\) is adjacent to at most one triangle. Namely, \(f\) has at most one minor 3-vertex. If \(f\) has at most one 3-vertex other than \(v_0\), then \(ch(f \rightarrow) \leq \frac{7}{3} + \frac{1}{3} < 1\). Assume \(f\) has two 3-vertices other than \(v_0\).
Firstly, if \(f\) does not have any minor 3-vertex, then \(f\) transfers at most \(\frac{1}{3}\) charge to the unique triangle and \(\frac{1}{3} \times 2\) to the non-minor 3-vertices. That is, \(ch^*(f) = ch(f) - ch(f \rightarrow) \geq 1 - 1 = 0\).
Assume \(f\) has a minor 3-vertex. Assume \(f = [v_1v_2v_3v_4v_5]\) with \(d(v_1) = 3\) and \(T = [v_1v_2v_6]\). In this case, \(ch(f \rightarrow) = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6}\). If one of \(v_i\) \((i = 2, 3, \ldots, 6)\) is \(v_0\) or a \(5^+\)-vertex, then such \(v_i\) transfers at least \(\frac{1}{6}\) charge to \(f\) by R4 and R3. Thus, \(ch^*(f) = ch(f) - ch(f \rightarrow) + ch(\rightarrow f) \geq 1 - \frac{2}{6} + \frac{1}{6} \geq 0\).
Assume \(f\) and \(T\) does not contain \(v_0\) and \(5^+\)-vertex. By Lemma 2.7 and Lemma 2.10, another 3-vertex must be \(v_4\). Thus, there is a special 5-cycle in \(G\), contradicting to Lemma 5.2.
Let $f$ be a 6-face. By Lemma 2.7, $f$ has at most three 3-vertex other than $v_0$. Since $G$ has no 4- and 7-cycles, $f$ is adjacent to at most one triangle $T$ which shares two common edges with $f$. If $f$ is not adjacent to any triangle, $f$ only sends charge to non-minor 3-vertices each getting $\frac{1}{3}$ from $f$. Hence, $ch^*(f) \geq d(f) - 4 - \frac{1}{2} \times 3 > 0$. If $f$ is adjacent to one triangle, then $f$ has at most one minor 3-vertex. Thus, $ch^*(f) \geq d(f) - 4 - \frac{1}{2} \times 2 - \frac{1}{3} > 0$.

Let $f$ be a 8+ face. If $f$ is incident with $s$ 3-vertices other than $v_0$ where $0 \leq s \leq \lfloor \frac{d(f)}{2}\rfloor$. Then $f$ transfers at most $\frac{1}{2} \times s$ charge to 3-vertices and $(d(f) - s) \times \frac{1}{3}$ charge to triangles. Hence, we have $ch^*(f) \geq d(f) - 4 - \frac{1}{2} s - \frac{1}{3}(d(f) - s) = \frac{2}{3}d(f) - \frac{1}{6}s - 4 \geq \frac{7}{12}d(f) - 4 > 0$.

Check charge on $f_0$ and $v_0$
For this checking procedure is the same as the last part in Section 3, we omit the details. That is $ch^*(v_0) \geq -\frac{7}{4}$ and $ch^*(f_0) \geq -\frac{11}{4}$.

Thus, we will have $0 \leq \sum_{x \in V \cup F \setminus \{v_0, f_0\}} ch^*(x) = -8 - ch^*(v_0) - ch^*(f_0) \leq -\frac{35}{12}$, a contradiction.

References

[1] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (2) (1992) 125-134.

[2] W. Cushing and H. A. Kierstead, Planar graphs are 1-relaxed, 4-choosable, European J. Combin. 31(5) (2010) 1385-1397.

[3] L. Duraj, G. Gutowski and J. Kozik, Chip games and paintability, Electron. J. Combin. 23 (2016), no. 3, Paper 3.3, 12 pp.

[4] N. Eaton and T. Hall, Defective list colorings of planar graphs, Bull. Inst. Combin. Appl. 25(1999) 79-87.

[5] G. Gutowski, M. Han, T. Krawczyk and X. Zhu, Defective 3-paintability of planar graphs, Electron. J. Combin. 25 (2018), no. 2, Paper 2.34, 20 pp.

[6] J. Grytczuk and X. Zhu, The Alon-Tarsi number of a planar graph minus a matching, arXiv:1811.12012.

[7] M. Han and X. Zhu, Locally planar graphs are 2-defective 4-paintable, European J. Combin. 54 (2016) 35-50.

[8] T. Jensen and B. Toft, Graph Coloring Problems, Wiley, New York, 1995.

[9] R. Kim, S-J. Kim, X. Zhu. The Alon-Tarsi number of subgraphs of a planar graph, manuscript.
[10] K-W. Lih, Z. Song, W. Wang and K. Zhang, A note on list improper coloring planar graphs, Appl. Math. Lett. 14 (2001) 269-273.

[11] U. Schauz, Flexible color lists in Alon and Tarsi’s theorem, and time scheduling with unreliable participants, Electron. J. Combin. 17 (2010):R13:1–18.

[12] R. Škrekovski, List improper colourings of planar graphs, Combin. Probab. Comput. 8 (1999) 293-299.