CONGRUENCES BETWEEN HILBERT MODULAR FORMS OF WEIGHT 2, AND THE IWASA\textsc{w}A }\lambda\text{-INVARIA\textsc{nts}}

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Abstract. The purpose of this paper is to prove the equality between the algebraic Iwasawa \( \lambda \)-invariant and the analytic Iwasawa \( \lambda \)-invariant from a congruence between a Hilbert cusp form of parallel weight 2 and a Hilbert Eisenstein series of parallel weight 2. Our result is a generalization of the work of R. Greenberg and V. Vatsal \[\text{Gre–Val}\] for elliptic modular forms, which is not covered by the advanced work of C. Skinner and E. Urban \[\text{Ski–Ur}\] who assume absolute residual irreducibility.

In this paper, we study a way to obtain the equality between the algebraic Iwasawa \( \lambda \)-invariant and the analytic Iwasawa \( \lambda \)-invariant for a Hilbert cusp form of parallel weight 2 and a Hilbert Eisenstein series of parallel weight 2. This is a generalization of a result of R. Greenberg and V. Vatsal.

0. Introduction

0.1. Introduction. In this paper, we study a way to obtain the equality between the algebraic Iwasawa \( \lambda \)-invariant and the analytic Iwasawa \( \lambda \)-invariant from a congruence between a Hilbert cusp form of parallel weight 2 and a Hilbert Eisenstein series of parallel weight 2. This is a generalization of a result of R. Greenberg and V. Vatsal \[\text{Gre–Val}\] for elliptic modular forms, which is not covered by the advanced work of C. Skinner and E. Urban \[\text{Ski–Ur}\] who assume absolute residual irreducibility.

Let \( F \) be a totally real number field of degree \( n \) with narrow class number \( h_F^+ = 1 \). Let \( \mathfrak{o}_F \) denote the ring of integers of \( F \), and let \( \Delta_F \) denote the discriminant of \( F \). Let \( \mathfrak{n} \) be a non-zero ideal of \( \mathfrak{o}_F \) such that \( \mathfrak{n} \) is prime to \( 6\Delta_F \). Let \( p \) be a prime number such that \( p \geq n + 2 \) and \( p \) is prime to \( 6n\Delta_F \). We fix algebraic closures \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \) and \( \overline{\mathbb{Q}}_p \) of \( \mathbb{Q}_p \), and embeddings \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C} \). Let \( \mathcal{O} \) be the ring of integers of a finite extension \( K \) of the field \( \mathbb{Q} \) defined in \([1,4] \cong, \mathfrak{O}, \text{and} \kappa \text{the residue field of} \mathcal{O}. \text{Let} F_\infty \text{denote the cyclotomic} \mathbb{Z}_p\text{-extension of} F. \text{We put} \Gamma = \text{Gal}(F_\infty/F). \text{Let} \Lambda \text{denote the Iwasawa algebra} \mathcal{O}[\Gamma]. \text{Let} S_2(n, \mathcal{O}) \text{denote the space of Hilbert cusp forms of parallel weight 2 and level} \mathfrak{n} \text{with coefficients in} \mathcal{O} \text{(see \[\text{[1,13]}\). Let} Y(n) \text{be the Shimura variety} \Gamma_1(\mathfrak{d}_F|t_1|, \mathfrak{n})\backslash\mathcal{Y}\text{Hom}(F, \mathbb{R}) \text{defined by \[\text{[1,1]}, \text{and let} Y(n)^\text{BS} \text{be the Borel–Serre compactification of} Y(n) \text{(cf. \[\text{[3,2]}\). Let} C \text{denote the set of all cusps of} Y(n), \text{and let} D_s \text{denote the boundary of} Y(n)^\text{BS} \text{at} s \in C. \text{Let} C_\infty \text{be the subset of} C \text{consisting of cusps} \Gamma_0(\mathfrak{d}_F|t_1|, \mathfrak{n})\text{-equivalent to the cusp} \infty \text{where} \Gamma_0(\mathfrak{d}_F|t_1|, \mathfrak{n}) \text{is the congruence subgroup defined in \[\text{[1,1]}\). Let} D_{C_\infty}(n) \text{denote the union of} D_s \text{for all} s \in C_\infty. \text{Let} f \in S_2(n, \mathcal{O}) \text{be a normalized Hecke eigenform for all Hecke operators with character} \varepsilon \text{and} p\text{-ordinary. We assume that the residual Galois representation} \bar{\rho}_f (= \rho_f \mod \varpi) : G_F \to \text{GL}_2(\kappa) \text{associated to} f \text{is reducible and of the form} \bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}. \text{Theorem 0.1. Let} f \in S_2(n, \mathcal{O}) \text{as above. We assume that the above} \varphi \text{(resp.} \psi \text{)} \text{is ramified (resp. unramified) at every prime ideal of} \mathfrak{o}_F \text{lying above} p, \text{totally even (resp. totally odd)}, \text{and the associated algebraic Iwasawa} \mu\text{-invariant is 0 (for the definition, see \[\text{[2,4]}(\mu = 0)\). Let} n' \text{be the least common multiple of} n^2 \text{and} m_r m_s^2, \text{where} m_r \text{is the conductor of} * \text{. We put} g = (f \otimes 1_n) \otimes 1_{m_r} \in S_2(n', \mathcal{O}) \text{(cf. \[\text{Shim} \text{ Proposition 4.4, 4.5]}, \text{where, for a non-zero} \text{July 6, 2017.}
ideal $a$ of $\mathfrak{a}_F$, $1_a$ is the trivial character modulo $a$. Let $\chi$ be an $\mathcal{O}$-valued narrow ray class character of $F$, whose conductor is the level $n'$, such that $\chi$ is of order prime to $p$, totally even, and the algebraic Iwasawa $\mu$-invariants of $\phi_\chi$ and $\psi_\chi$ are 0. Let $f \otimes \chi$ be the Hecke eigenform twisted by $\chi$ (defined in [Shi, Proposition 4.4, 4.5]). We assume the following two conditions:

(a) the local components $H^n(\partial(Y(n')^B\mathcal{O}), \mathcal{O})_m$ and $H_{c}^{n+1}(Y(n'), \mathcal{O})_m$ are torsion-free, where $m$ is the maximal ideal of $\mathcal{H}_2(n', \mathcal{O})$ defined at the beginning of §5.1.

(b) the local component $H^n(D_{c_{\infty}}(n'), \mathcal{O})_{m'_g}$ is torsion-free, where $m'_g$ is the maximal ideal of $\mathbb{H}_2(n', \mathcal{O})'$ defined before Proposition 7.7.

Then we have the following:

1. (Theorem 2.11) the Selmer group $\text{Sel}(F, \mathcal{A}_f \otimes \chi)$ for $f \otimes \chi$ (for the definition, see §2.4, 2.5) is finitely generated $\Lambda$-cotorsion, and

$$\lambda_{\text{alg}}^\Lambda = \lambda_{\text{alg}}^\Lambda, \Sigma_0 + \lambda_{\text{alg}}^\Lambda, \Sigma_0.$$

Here $\lambda_{\text{alg}}^\Lambda$ is the algebraic Iwasawa $\lambda$-invariant for $f \otimes \chi$ defined by (2.5), and $\lambda_{\text{alg}}^{\Lambda, \Sigma_0}$ and $\lambda_{\text{alg}}^{\Lambda, \Sigma_0}$ are the classical algebraic Iwasawa $\lambda$-invariants defined by (2.6) and (2.7), respectively.

2. (Theorem 7.7) if $\chi$ is of type $S$ (that is, $\ker(\chi) \cap F_\infty = F$), then there exists a $p$-adic $L$-function $L_p(f, \chi, T) \in \mathcal{O}[T]$ satisfying the following interpolation property: for each finite order character $\rho$ of $\Gamma$ with conductor $p^n\nu$,

$$L_p(f, \chi, \rho(\gamma) - 1) = \alpha^{-\nu_p} \frac{\tau(\chi^{-1}p^{-1})D(1, f, \chi \rho)}{(-2\pi \sqrt{-1})^n \Omega^\prime_{g}} \in \mathcal{O}(\rho).$$

Here $\alpha^{-\nu_p} = \prod_{\mathfrak{p} \mid p} \alpha_{\mathfrak{p}}^{-\nu_p}$ for $p^n\nu = \prod_{\mathfrak{p} \mid p} p^n\nu_p$, where $\alpha_p$ is a unit root of the polynomial defined by (3.3), $\tau(\chi^{-1}p^{-1})$ denotes the Gauss sum of $\chi^{-1}p^{-1}$ defined by Shi (3.9), $D(1, f, \chi \rho)$ is given by the Dirichlet series in the sense of G. Shimura (see (1.10)), $\Omega^\prime_g \in \mathbb{C}^\times$ is the canonical period (defined in (1.3) of $g \in S_2(n', \mathcal{O})$ and the trivial character 1 of the Weyl group $W_G$, and $\mathcal{O}(\rho)$ denotes the ring of integers of the field generated by $\im(\rho)$ over $K$.

3. (Theorem 7.7) the algebraic Iwasawa $\lambda$-invariant $\lambda_{\text{alg}}^\Lambda$ above is equal to the analytic Iwasawa $\lambda$-invariant $\lambda_{\text{alg}}^\Lambda$ for $f \otimes \chi$ defined by (5.7):

$$\lambda_{\text{alg}}^\Lambda = \lambda_{\text{alg}}^\Lambda = \lambda_{\text{alg}}^\Lambda, \Sigma_0 + \lambda_{\text{alg}}^{\Lambda, \Sigma_0}.$$

This result is a generalization of the result of Greenberg and Vatsal [Gre–Vat] in the case where $F = \mathbb{Q}$ and weight $k = 2$. However, the methods to prove Theorem 0.1 (2) and (3) have some limitations, such as the need for the torsion-freeness of the compactly supported cohomology and the boundary cohomology. In the case where $F$ is a real quadratic field, the torsion-freeness is satisfied under some conditions (Proposition 6.3 and 6.5). We also give examples satisfying all the assumptions of the main theorem (Example 6.5).

Remark 0.2. The assumption on the algebraic Iwasawa $\mu$-invariant of a narrow ray class character $\eta$ is satisfied if $\overline{\mathbb{Q}}^{\ker(\eta)}$ is an abelian extension over $\mathbb{Q}$ by Ferrero–Washington theorem (see, for example, [Was, §7.5, Theorem 7.15]).

Our proof based on the method of Greenberg and Vatsal requires new ingredients; a criterion for the $\Lambda$-cotorsionness of Selmer groups (2.4), a construction of a $\mathbb{C}$-valued distribution interpolating special values of the $L$-functions (3.3), a criterion for the integrality of special
values of the \((p\text{-adic})\) \(L\)-functions divided by the canonical period defined in terms of parabolic cohomology \((\text{4.8})\), and a congruence between our \(p\text{-adic}\) \(L\)-function and the product of two Deligne–Ribet \(p\text{-adic}\) \(L\)-functions \((\text{3.2})\). We give an outline of the proof of the main theorem (Theorem 1.1) below in order to clarify its complicated structure, the methods used, and the places where the assumptions are necessary. The proof consists of three steps.

**Step 1.** To prove Theorem 1.1 (1).

Since \(\mathfrak{f} \otimes \chi\) is reducible, we have an exact sequence of residual Galois representations

\[ 0 \to A_{\varphi \chi}[\varpi] \to A_{\varphi \chi}[\varpi] \to A_{\psi \chi}[\varpi] \to 0. \]

Here \(A_{\ast}\) denotes a torsion quotient of the Galois representation associated to \(* \in \{ f \otimes \chi, \varphi \chi, \psi \chi \}\) (for the definition, see \((2.1)\), which is an \(\mathcal{O}\)-module, and \(A_{\ast}[\varpi]\) denotes the \(\varpi\)-torsion of \(A_{\ast}\). By the parity conditions on \(\varphi \chi\) and \(\psi \chi\), this sequence induces an exact sequence of the \(\varpi\)-torsion of (non-primitive) Selmer groups

\[ (0.1) \quad 0 \to \text{Sel}^{\Sigma}(F_{\infty}, A_{\varphi \chi})[\varpi] \to \text{Sel}(F_{\infty}, A_{\varphi \chi})[\varpi] \to \text{Sel}^{\Sigma}(F_{\infty}, A_{\psi \chi})[\varpi] \to 0 \]

(\text{Theorem 2.11 (2.1)}). It follows from the isomorphism \(\text{Sel}(F_{\infty}, A_{\varphi \chi}) = \text{Sel}^{\Sigma}(F_{\infty}, A_{\varphi \chi})\) (\text{cf. Proposition 2.6}), the vanishing results of \(H^{0}(F_{\infty}, A_{\varphi \chi}[\varpi])\) (Lemma 2.8) and \(H^{2}(F_{\infty}, A_{\varphi \chi}[\varpi])\) (Lemma 2.10), and the isomorphism \(\text{Sel}^{\Sigma}(F_{\infty}, A[\varpi]) \cong \text{Sel}^{\Sigma}(F_{\infty}, A)[\varpi]\) for \(A = A_{\varphi \chi}, A_{\chi}, A_{\psi \chi}\), or \(A_{\chi}\) (Proposition 2.3). Now, by the assumption on the algebraic Iwasawa \(\mu\)-invariants of \(\varphi \chi\) and \(\psi \chi\), the group \(\text{Sel}(F_{\infty}, A_{\varphi \chi})[\varpi]\) is finite, and hence \(\text{Sel}(F_{\infty}, A_{\varphi \chi})\) is finitely generated \(\Lambda\)-cotorsion (Theorem 2.11).

By Proposition 2.3 we have \(\text{cok}_0(\text{Sel}(F_{\infty}, A_{\varphi \chi})) = \dim_\kappa(\text{Sel}(F_{\infty}, A_{\varphi \chi}[\varpi]))\) and \(\text{cok}_0(\text{Sel}^{\Sigma}(F_{\infty}, A)) = \dim_\kappa(\text{Sel}^{\Sigma}(F_{\infty}, A)[\varpi])\) for \(A = A_{\varphi \chi}\) and \(A_{\psi \chi}\). It follows from the fact obtained by T. Weston \(\text{We}\) that, for \(A = A_{\varphi \chi}, A_{\varphi \chi}, A_{\psi \chi}\), and \(\text{Sel}(F_{\infty}, A)\) has no proper \(\Lambda\)-submodules of finite index. Now, again using the exact sequence (0.1), we obtain the equality \(\lambda_{\varphi \chi}^{alg} = \lambda_{\varphi \chi}^{\varpi} + \lambda_{\psi \chi}^{\varpi}\) (Theorem 2.11 (2.3)). The proof is based on the method of Greenberg and Vatsal \(\text{Gre–Vat, \S 2}\).

**Step 2.** To prove Theorem 1.1 (2).

We first construct a \(\mathbb{C}\)-valued distribution attached to a Hilbert cusp form \(f\) of parallel weight 2 (Proposition 3.3). This is a generalization of results of Y. Amice and J. Vélu \(\text{Ami–Ve}\), M. Vishik \(\text{Vi}\), and B. Mazur, J. Tate, and J. Teitelbaum \(\text{MTT}\). The proof is based on Mellin transforms for the compactly supported cohomology class \([\omega_\xi]\) in \(H^n_c(Y(n), \mathbb{C})\) (Proposition 3.3 and 3.6), which implies that the special values of the associated \(L\)-functions are expressed as a linear combination of the images of \([\omega_\xi]\) under the evaluation maps with integral coefficients.

Next we prove the integrality of the \(p\text{-adic}\) \(L\)-function. Let \(\omega_\xi\) (resp. \(\omega_\xi^{\text{rel}}\)) denote the image of \([\omega_\xi]\) in the parabolic cohomology \(H^n_{\text{par}}(Y(n'), \mathbb{C})\) (resp. the relative cohomology \(H^n(Y(n')^{BS}, D_{C_{\infty}}(n'); \mathbb{C}))\). By the Eichler–Shimura–Harder isomorphism \((\text{1.1})\) and the \(q\)-expansion principle over \(\mathbb{C}\), there exists \(\Omega_\xi \in \mathbb{C}^\times\) such that the class \([\omega_\xi] / \Omega_\xi\) belongs to the torsion-free part \(H^n_{\text{par}}(Y(n'), \mathbb{C}) / (\mathcal{O}\text{-torsion})\) of the parabolic cohomology, and its reduction modulo \(\varpi\) does not vanish. By using a vanishing result on \(H^{n-1}(D_{C_{\infty}}(n'), \mathbb{C})\) \(\text{[Hira, Proposition 5.3]}\), we prove that the class \([\omega_\xi] / \Omega_\xi\) belongs to the torsion-free part \(H^n(Y(n')^{BS}, D_{C_{\infty}}(n'); \mathbb{C}) / (\mathcal{O}\text{-torsion})\) of the relative cohomology (Proposition 4.1), where we use the assumptions (b), \(b_{\varphi \chi} = 1\), and weight \(k = 2\). This is a generalization of an argument of Manin–Drinfeld in the case where \(F = \mathbb{Q}\) (see, for example, \(\text{Gre–Sic, Lemma 6.9 (b)}\)). Now the integrality of the \(p\text{-adic}\) \(L\)-function follows from the fact \(D(s, f, \chi) = D(s, \xi, \chi \rho)\), the Mellin transforms for the relative cohomology class \([\omega_\xi]^{\text{rel}}\) \(\text{[Hira, Proposition 2.5 and \text{Hira, Proposition 2.6}]}\).
and the integrality of $[\omega_g]_{\text{rel}}/\Omega_g$ (Theorem 4.2), where we use the assumption that the conductor of $\chi$ is the level $n'$.

**Step 3.** To prove Theorem 0.1 (3).

Since $\lambda_{\text{alg}} f \otimes \chi = \lambda_{\phi} \chi + \Sigma_0$ (mentioned in Step 1), Theorem 0.1 (3) follows from the equality $\lambda_{\text{an}} f \otimes \chi = \lambda_{\phi} \chi + \Sigma_0$ (Theorem 5.2 (5.3)). The equality is obtained by the Iwasawa main conjecture for totally real number fields (proved by A. Wiles [Wil90]) and a congruence between our $p$-adic $L$-function and the product of two Deligne–Ribet $p$-adic $L$-functions (Theorem 5.2 (5.2)). The latter follows from congruences between special values of $L$-functions obtained from a congruence between the Hilbert cusp form $g \in S_2(n', O)$ and a Hilbert Eisenstein series $G \in M_2(n', O)$ induced by $\psi$ and $\varepsilon \psi^{-1}$ (by [Hira] Theorem 6.1), where we use the assumptions (a), (b). The proof is based on the method of Greenberg and Vatsal [Gre–Vat] Theorem 3.11 by using the result [Hira] Theorem 6.1 instead of the result of Vatsal [Vat] Theorem 2.10.

The organization of this paper is as follows.

In §1, we summarize results on Hilbert modular varieties and Hilbert modular forms in the analytic and algebraic settings.

In §2, we generalize results of Greenberg and Vatsal [Gre–Vat] on the algebraic side. We prove that the Selmer group for a Hilbert cusp form is finitely generated $\Lambda$-cotorsion, and the associated Iwasawa $\lambda$-invariant is equal to the sum of classical Iwasawa $\lambda$-invariants under some assumptions (Theorem 2.11).

In §3, we construct a $\mathbb{C}$-valued distribution attached to a Hilbert cusp form (Proposition 3.7), which interpolates special values of the associated $L$-functions (Proposition 3.8).

In §4, we prove the integrality of the $p$-adic $L$-function attached to a Hilbert cusp form divided by the canonical period (Theorem 4.2).

In §5, we generalize results of Greenberg and Vatsal [Gre–Vat] on the analytic side. We prove that the analytic Iwasawa $\lambda$-invariant for a Hilbert cusp form is equal to the sum of classical Iwasawa $\lambda$-invariants under some assumptions (Theorem 5.2 (5.3)). The key ingredient of our proof is the congruence between special values of the $L$-functions obtained from a congruence between a Hilbert cusp form of parallel weight 2 and a Hilbert Eisenstein series of parallel weight 2, which is the main theorem of [Hira].

In §6, we give examples satisfying all the assumptions of the main theorem (Example 6.6).

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**0.2. Notation.** Let $\hat{\mathbb{Z}}$ denote $\prod_l \mathbb{Z}_l$, where $l$ runs over all rational primes. We abbreviate $\mathbb{A}_Q$, the ring of adeles of $\mathbb{Q}$, to $A$. We fix a rational prime number $p > 3$. We fix algebraic
closures \( \overline{Q} \) of \( Q \) and \( \overline{Q}_p \) of the field of \( p \)-adic numbers \( \mathbb{Q}_p \), and embeddings \( \iota_p : \overline{Q} \to \overline{Q}_p \) and \( \overline{Q}_p \to \mathbb{C} \), where \( \mathbb{C} \) denotes the field of complex numbers.

Let \( F \) be a totally real number field unramified at \( p \), \( n \) the degree \([ F : \mathbb{Q} ]\) of the extension \( F/\mathbb{Q} \), and \( \mathfrak{o}_F \) the ring of integers of \( F \). For a place \( v \) of \( F \) (resp. a non-zero prime ideal \( q \) of \( \mathfrak{o}_F \)), let \( F_v \) (resp. \( \mathfrak{q}_v \)) denote the completion at \( v \) (resp. the \( q \)-adic completion) of \( F \). Let \( \mathfrak{o}_F \) denote the ring of integers of \( F \), and \( \widehat{\mathfrak{o}}_F \) the product of \( \mathfrak{o}_F \) over all non-zero prime ideals \( q \) of \( F \). Let \( J_F \) denote the set of embeddings of \( F \) into \( \mathbb{R} \). For \( a \in F \) and \( t \in J_F \), let \( a^t \) denote \( t(a) \). We have \( F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{J_F} \), and write \( (F \otimes_{\mathbb{Q}} \mathbb{R})_+^\times \) for the subgroup of \((F \otimes_{\mathbb{Q}} \mathbb{R})_+^\times \) corresponding to \((\mathbb{R}^\times)^{J_F} \), where \( \mathbb{R}^\times \) denotes the multiplicative group of positive real numbers.

As usual, \( \mathbb{A}_F \) denotes the ring of adeles of \( F \), which is the product of the finite part \( \mathbb{A}_{F,f}(\simeq \mathfrak{o}_F \otimes_{\mathfrak{o}_F} F) \) and the infinite part \( \mathbb{A}_{F,\infty}(\simeq F \otimes_{\mathbb{Q}} \mathbb{R}) \). For \( x \in \mathbb{A}_F \) and a place \( v \) of \( F \), we denote the finite component \( x_v \), the infinite component \( \hat{x}_v \), and the \( v \)-component of \( x \), respectively. For \( x \in \mathbb{A}_F \), a subset \( X \) of \( \mathbb{A}_F \), and a non-zero ideal \( \mathfrak{n} \) of \( \mathfrak{o}_F \), we write \( x_\mathfrak{n} \) and \( X_\mathfrak{n} \) for the images of \( x \) and \( X \) in \( \prod_{q \mid \mathfrak{n}} F_q \), where \( q \) denotes a non-zero prime ideal of \( \mathfrak{o}_F \). Let \( N \) denote the norm map \( N_{F/\mathbb{Q}} \) of the extension \( F/\mathbb{Q} \), \( \mathfrak{d}_F \subset \mathfrak{o}_F \) the different of \( F \), and \( \Delta_F \) the discriminant \( N(\mathfrak{d}_F) \) of \( F \), which is prime to \( p \) by assumption. Let \( \text{Cl}_F^+ \) denote the narrow ideal class group of \( F \). We have an isomorphism \( F^\times/\mathbb{A}_{F,f}^+ \simeq \mathbb{A}_{F,\infty}^+ \to \text{Cl}_F^+ \), sending the class of \( x \in \mathbb{A}_F^+ \) to the class of the fractional ideal \([x] := \prod_q q^{\text{ord}_q(x_q)} \), where \( q \) runs over the set of all non-zero prime ideals of \( \mathfrak{o}_F \).

For a non-zero ideal \( \mathfrak{b} \) of \( \mathfrak{o}_F \), let \( \text{Cl}_F^+(\mathfrak{b}) \) denote the narrow ray class group of \( F \) modulo \( \mathfrak{b} \). By a narrow ray class character of \( F \) modulo \( \mathfrak{b} \), we mean a homomorphism \( \chi : \text{Cl}_F^+(\mathfrak{b}) \to \mathbb{C}^\times \). The conductor of \( \chi \) is the smallest divisor \( \mathfrak{m}_\chi \) of \( \mathfrak{b} \) such that \( \chi \) factors through \( \text{Cl}_F^+(\mathfrak{m}_\chi) \). For a narrow ray class character \( \chi \) of \( F \) modulo \( \mathfrak{b} \), there exists an \( r = (r_i)_{i \in J_F} \in (\mathbb{Z}/2\mathbb{Z})^{J_F} \) such that \( \chi((\alpha)) = \text{sgn}(\alpha)^r \) for all \( \alpha \in F^\times \) satisfying \( \alpha \equiv 1 \pmod{\mathfrak{b}} \).

Here \( \text{sgn}(x) \) for \( x \in \mathbb{R}^\times \) denotes the sign of \( x \) and \( \text{sgn}(\alpha)^r = \prod_{i \in J_F} \text{sgn}(\alpha^i)^{r_i} \), where we identify \( J_F \) with the set of infinite places of \( F \). We call \( r \) the sign of \( \chi \). We say that \( \chi \) is totally even (resp. totally odd) if \( r_i = 0 \) (resp. \( r_i = 1 \)) for all \( i \in J_F \).

For an algebraic group \( H \) defined over \( \mathbb{Q} \), \( H(\mathbb{R}) \) is abbreviated to \( H_\infty \) and \( H_{\infty,+} \) denotes the connected component of \( H_\infty \) containing the unit. Let \( G \) denote the reductive algebraic group \( \text{Res}_{F/\mathbb{Q}} \text{GL}_2 \) over \( \mathbb{Q} \), where \( \text{Res}_{F/\mathbb{Q}} \) denotes the Weil restriction of scalars. We have \( G_\infty = \text{GL}_2(\mathbb{R})^{J_F} \), \( G_{\infty,+} = \text{GL}_2(\mathbb{R})^{J_F} \), and \( G(\mathbb{A}) = \text{GL}_2(\mathbb{A}_F) \). Let \( B \) denote the Borel subgroup of \( G \) consisting of upper triangular matrices, and let \( U \) denote its unipotent radical.

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## 1. Hilbert modular varieties and Hilbert modular forms

### 1.1. Analytic Hilbert modular varieties

In this subsection, we recall the definition of analytic Hilbert modular varieties. For more detail, refer to \[\text{[Dim]} \ §1.1\].

Let \( \mathcal{H} \) be the upper half plane \( \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \). The group \( \text{GL}_2(\mathbb{R})_+ \) acts on \( \mathcal{H} \) by linear fractional transformations. We can extend the action to \( \text{GL}_2(\mathbb{R}) \) by defining the
action of \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) on \( \tilde{H} \) by \( z \mapsto -\bar{z} \). We define the action of \( G_\infty = \text{GL}_2(\mathbb{R})^{J_F} \) on \( \tilde{H}^{J_F} \) by \((g_i)_{i \in J_F} : (z_i)_{i \in J_F} = (g_i z_i)_{i \in J_F} \). Let \( \mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \tilde{H}^{J_F} \). Let \( K_\infty \) and \( K_\infty^{++} \) be the stabilizers of \( i \) in \( G_\infty \) and \( G_\infty^{++}, \) respectively.

For a non-zero ideal \( n \) of \( \mathfrak{o}_F \), we define the open compact subgroup \( K_1(n) \) of \( G(\mathfrak{A}_F) \) by

\[
K_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\hat{\mathbb{Z}}) \bigg| c \in n, d -1 \in n \right\}.
\]

The adelic Hilbert modular variety \( Y(n) \) of level \( K_1(n) \) is defined by

\[
Y(n) = G(\mathbb{Q}) \backslash G(\mathfrak{A}) / K_1(n) K_\infty^{++},
\]

where \( G(\mathfrak{A}) = G(\mathfrak{A}_F) G_\infty^{++} \) and \( G(\mathbb{Q}) = G(\mathbb{Q}) \cap G_\infty^{++} \). Then \( Y(n) \) is a disjoint union of finitely many arithmetic quotients of \( \tilde{H}^{J_F} \) as follows. Let \( a \) be a fractional ideal of \( F \). We consider the following congruence subgroups of \( G(\mathbb{Q})_+ \):

\[
\Gamma_0(a, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathfrak{o}_F a^{-1} \mathfrak{o}_F) \bigg| ad - bc \in \mathfrak{o}_F^{\times} \right\},
\]

\[
\Gamma_1(a, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(a, n) \bigg| d \equiv 1 (\text{mod} n) \right\},
\]

\[
\Gamma_1^n(a, n) = \Gamma_1(a, n) \cap \text{SL}_2(F),
\]

where \( \mathfrak{o}_F^{\times} \) denotes the subgroup of \( \mathfrak{o}_F^{\times} \) consisting of totally positive units. Let \( \text{Cl}_F^+ \) be the narrow ideal class group of \( F \) and \( \mathfrak{h}_F^+ \) the narrow class number of \( \text{Cl}_F^+ \) of \( F \). Choose and fix \( t_1, \cdots, t_{h_F^+} \in \mathfrak{A}_F^{\times} \) such that \( t_{1, \infty} = 1 \) and the corresponding fractional ideals \([t_1], \cdots, [t_{h_F^+}]\) form a complete set of representatives of \( \text{Cl}_F^+ \). Throughout the paper, we assume that

\[
[t_i] \text{ is prime to } p \text{ for each } i.
\]

We put

\[
x_i = \begin{pmatrix} D^{-1} t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in G(\mathfrak{A}_F).
\]

We define the analytic Hilbert modular varieties \( Y_i \) by

\[
Y_i = \Gamma_1(\mathfrak{o}_F[t_i], n) \backslash \tilde{H}^{J_F},
\]

where \( \Gamma \) denotes \( \Gamma / (\Gamma \cap F^{\times}) \) for a congruence subgroup \( \Gamma \) of \( G(\mathbb{Q})_+ \). Then, by the strong approximation theorem, we have the following description of \( Y(n) \):

\[
Y(n) \simeq \prod_{1 \leq i \leq h_F^+} Y_i
\]

given by sending the class of \( x_i g \in Y(n) \) to the class of \( g \mathbf{i} \in Y_i \) for \( g \in G_\infty^{++} \).

1.2. Analytic Hilbert modular forms. In this subsection, we fix notation concerning the spaces of Hilbert modular forms, following [Dim] §1.2, [Shi] §2.

Let \( k \) be an integer \( \geq 2 \) and \( n \) a non-zero ideal of \( \mathfrak{o}_F \). Let \( t = \sum_{i \in J_F} t_i \in \mathbb{Z}[J_F] \). For each subset \( J \) of \( J_F \), let \( M_{k,J}(n, \mathbb{C}) \) (resp. \( S_{k,J}(n, \mathbb{C}) \)) denote the \( \mathbb{C} \)-vector space \( G_{ht,J}(K_1(n)) \) (resp. \( S_{ht,J}(K_1(n)) \)) of Hilbert modular (resp. cusp) forms of weight \( kt \), level \( K_1(n) \), and type \( J \) at infinity defined in [Dim] Definition 1.2. Let \( \chi \) be a Hecke character of \( F \) of type \(-(k - 2)t\) whose conductor divides \( n \). Let \( M_{k,J}(n, \chi, \mathbb{C}) \) (resp. \( S_{k,J}(n, \chi, \mathbb{C}) \)) denote
the subspace $G_{kl,J}(K_1(n), \chi)$ (resp. $S_{kl,J}(K_1(n), \chi)$) of $G_{kl,J}(K_1(n))$ (resp. $S_{kl,J}(K_1(n))$) of elements with character $\chi$ defined in [Dim, Definition 1.3]. If $J = J_F$, then we simply write $M_k(n, \mathbb{C})$ and $S_k(n, \mathbb{C})$ (resp. $S_k(n, \chi, \mathbb{C})$ for $M_{k, J_F}(n, \mathbb{C})$ and $M_{k, J_F}(n, \chi, \mathbb{C})$ (resp. $S_{k, J_F}(n, \chi, \mathbb{C})$ and $S_{k, J_F}(n, \chi, \mathbb{C})$), respectively. We note that the spaces $M_k(n, \mathbb{C})$ and $M_k(n, \chi, \mathbb{C})$ are embedded into the space of Hilbert modular forms defined in [Shi, §2] (see, for example, [Ge–Go, §5.5]).

For a fractional ideal $a$ of $F$, let $M_k(\Gamma_1(a, n), \mathbb{C})$ (resp. $S_k(\Gamma_1(a, n), \mathbb{C})$) denote the space $G_{kl,J_F}(\Gamma_1(a, n); \mathbb{C})$ (resp. $S_{kl,J_F}(\Gamma_1(a, n); \mathbb{C})$) of holomorphic Hilbert modular (resp. cusp) forms of weight $kt$ and of level $\Gamma_1(a, n)$ defined in [Dim, Definition 1.4]. Then we have canonical isomorphisms (cf. [Hida91, p.323] and [Hida88, Proposition 2.6a]):

$$M_k(n, \mathbb{C}) \cong \bigoplus_{1 \leq i \leq h_F^+} M_k(\Gamma_1(\mathfrak{o}_F[t_i], n), \mathbb{C}), \quad S_k(n, \mathbb{C}) \cong \bigoplus_{1 \leq i \leq h_F^+} S_k(\Gamma_1(\mathfrak{o}_F[t_i], n), \mathbb{C}).$$

1.3. Dirichlet series associated to Hilbert modular forms. In this subsection, we recall the definition and properties of the Dirichlet series associated to Hilbert modular forms, following [Shi, §2].

Let $\mathbf{h} \in M_k(n, \mathbb{C})$ and $h_i \in M_k(\Gamma_1(\mathfrak{o}_F[t_i], n), \mathbb{C})$ such that $\mathbf{h} = (h_i)_{1 \leq i \leq h_F^+}$ under the isomorphism (1.6). Then $h_i$ has the Fourier expansion of the form

$$h_i(z) = c_\infty([t_i]^{-1}, \mathbf{h})N([t_i])^{k/2} + \sum_{0 < \xi \in [t_i]} c(\xi[t_i]^{-1}, \mathbf{h})N(\xi)^{k/2}e_{\mathfrak{f}}(\xi z)$$

given by [Shi, (2.18)] and [Hida88, Proposition 4.1]. Here the notion $\gg 0$ means totally positive, $m \mapsto c(m, \mathbf{h})$ is a function on the set of all fractional ideals of $F$ vanishing outside the set of integral ideals, and $e_{\mathfrak{f}}$ denotes the additive character of $F\backslash \hat{\mathbb{A}}_F$ characterized by $e_{\mathfrak{f}}(x_\infty) = \exp(2\pi \sqrt{-1}x_\infty)$ for $x_\infty \in \hat{\mathbb{A}}_F$. We put

$$a_\infty(0, h_i) = c_\infty([t_i]^{-1}, \mathbf{h})N([t_i])^{k/2}$$

and $a_\infty(\xi, h_i) = c(\xi[t_i]^{-1}, \mathbf{h})N(\xi)^{k/2}$ for any $0 < \xi \in [t_i]$. We also put

$$C_{\infty,i}(0, \mathbf{h}) = N([t_i])^{-k/2}a_\infty(0, h_i),$$

$$C(m, \mathbf{h}) = N(m)^{k/2}c(m, \mathbf{h})$$

for all non-zero ideals $m$ of $\mathfrak{o}_F$. Let $\eta$ be a narrow ray class character of $F$. The Dirichlet series in the sense of G. Shimura [Shi, (2.25)] is defined by

$$\sum_m C(m, \mathbf{h})\eta(m)N(m)^{-s} \quad \text{for} \quad s \in \mathbb{C},$$

where $m$ runs over the set of all non-zero ideals of $\mathfrak{o}_F$. It converges absolutely if $\Re(s)$ is sufficiently large and extends to a meromorphic function on the complex plane (see, for example, [Hira, §2.3]). For each $\mathbf{h} \in M_k(n, \mathbb{C})$, let $D(s, \mathbf{h}, \eta)$ denote this meromorphic function. If $\eta$ is the trivial character, we simply write $D(s, \mathbf{h})$ for $D(s, \mathbf{h}, \eta)$.

1.4. Hecke operators on analytic Hilbert modular forms. Let $n$ be a non-zero ideal of $\mathfrak{o}_F$. In this subsection, we recall the definition of the Hecke operators acting on $M_2(n, \mathbb{C})$ and $S_2(n, \mathbb{C})$, following [Dim, §1.10].

Let $\Delta(n)$ be the following semigroup:

$$\Delta(n) = G(\hat{\mathbb{A}}_F) \cap \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathfrak{o}_F) \middle| c \in n\mathfrak{o}_F, \quad d_q \in \mathfrak{o}_F^\times \quad \text{whenever} \quad q|n \right\},$$
where \( q \) is a non-zero prime ideal of \( \mathfrak{o}_F \). For \( y \in \Delta(n) \), we define the action of the double coset \( K_1(n)yK_1(n) \) on \( M_2(n, \mathbb{C}) \) (resp. \( S_2(n, \mathbb{C}) \)) by
\[
(1.11) \quad f([K_1(n)yK_1(n)])(x) = \sum_i f(xy_i^{-1}),
\]
where \( K_1(n)yK_1(n) = \prod_i K_1(n)y_i. \) By the definition of \( M_2(n, \mathbb{C}) \) and \( S_2(n, \mathbb{C}) \), the right-hand side is independent of the choice of the representative set \( \{y_i\}_i \).

We define the Hecke operator \( T(q^\epsilon) \) (resp. \( S(q^\epsilon) \)) for a non-negative integer \( \epsilon \), a non-zero prime ideal \( q \) of \( \mathfrak{o}_F \) (resp. prime ideal \( q \) of \( \mathfrak{o}_F \) prime to \( n \)), and a uniformizer \( \varpi_q \) of \( \mathfrak{o}_{F_q} \) by the action of the double coset \( K_1(n) \left( \begin{array}{cc} q^\epsilon & 0 \\ 0 & 1 \end{array} \right) K_1(n) \) (resp. \( K_1(n) \left( \begin{array}{cc} 0 & q^\epsilon \\ 0 & \varpi_q \end{array} \right) K_1(n) \)). We note that these operators are independent of the choice of \( \varpi_q \). We put \( T(q^\epsilon) = T(q^\epsilon) \) and \( S(q^\epsilon) = S(q^\epsilon) \) (resp. \( U(q^\epsilon) = T(q^\epsilon) \)) for a non-negative integer \( \epsilon \) and a non-zero prime ideal \( q \) prime to \( n \) (resp. prime ideal \( q \) dividing \( n \)). We define \( T(m) = \prod_{q|n} T(q^\epsilon(q)) \) and \( S(m) = \prod_{q|n} S(q^\epsilon(q)) \) for all non-zero ideal \( m = \prod_{q|n} q^\epsilon(q) \) of \( \mathfrak{o}_F \) prime to \( n \) and \( U(m) = \prod_{q|n} U(q^\epsilon(q)) \) for all non-zero ideal \( m = \prod_{q|n} q^\epsilon(q) \) of \( \mathfrak{o}_F \), where \( q \) is a non-zero prime ideal.

The definition of the Hecke operators acting on \( M_2(\Gamma_1(a, n), \mathbb{C}) \) and \( S_2(\Gamma_1(a, n), \mathbb{C}) \) and their relation (via \( (1.6) \)) to the adelic ones recalled above are explicitly given in \([\text{Shi}, (2.23)]\).

By \([\text{Shi}, (2.23)]\), we have a relation between the Hecke operators and the Fourier expansion of the following form: for \( f \in M_2(n, \mathbb{C}) \) and \( V(m') = T(m') \) or \( U(m') \),
\[
(1.12) \quad C(m, f)V(m') = \sum_{m+m' \subset \xi} N(c)C(c^{-2}mm', fS(c)).
\]

For a subalgebra \( A \) of \( \mathbb{C} \), let \( \mathbb{H}_2(n, A) \) (resp. \( \mathbb{H}_2(n, A) \)) be the commutative \( A \)-subalgebra of \( \text{End}_\mathbb{C}(M_2(n, \mathbb{C})) \) (resp. \( \text{End}_\mathbb{C}(S_2(n, \mathbb{C})) \)) generated by \( T(m), S(m) \) for all non-zero ideal \( m = \prod_{q|n} q^\epsilon(q) \) of \( \mathfrak{o}_F \) prime to \( n \) and \( U(m) \) for all non-zero ideal \( m = \prod_{q|n} q^\epsilon(q) \) of \( \mathfrak{o}_F \).

Let \( \bar{F} \) be the Galois closure of \( F \) in \( \overline{\mathbb{Q}}_p \), and let \( F' \) be the field generated by elements \( \sqrt[t]{\xi} \) for all \( \xi \in \mathfrak{o}_F \). Let \( \mathfrak{o}_F \) be the composite field of \( \mathcal{O}|(\sqrt[t]{T}) \) in \( \overline{\mathbb{Q}}_p \) for all \( t \in F \), where \( \mathcal{O} \) is the ring of integers of a finite extension of \( \Phi_p \). For \( h = (h_i)_{1 \leq i \leq h} \in M_2(n, \mathbb{C}), h_i \in M_2(\Gamma_1(\mathfrak{o}_F[t_i], n), \mathbb{C}) \) has the Fourier expansion of the form \( (1.7) \).

For a subalgebra \( A \) of \( \mathbb{C} \), we put
\[
M_2(\Gamma_1(\mathfrak{o}_F[t_i], n), A) = M_2(\Gamma_1(\mathfrak{o}_F[t_i], n)) \cap A[\{e_\mathfrak{o}_F(\xi) : \xi = 0 \text{ or } 0 < \xi \in F\}],
\]
\[
S_2(\Gamma_1(\mathfrak{o}_F[t_i], n), A) = S_2(\Gamma_1(\mathfrak{o}_F[t_i], n)) \cap A[\{e_\mathfrak{o}_F(\xi) : \xi = 0 \text{ or } 0 < \xi \in F\}],
\]
\[
(1.13) \quad M_2(n, A) = \bigoplus_{1 \leq i \leq h} M_2(\Gamma_1(\mathfrak{o}_F[t_i], n), A), \quad S_2(n, A) = \bigoplus_{1 \leq i \leq h} S_2(\Gamma_1(\mathfrak{o}_F[t_i], n), A).
\]
The space \( M_2(n, \mathcal{O}) \) (resp. \( S_2(n, \mathcal{O}) \)) is stable under \( \mathbb{H}_2(n, \mathcal{O}) \) (resp. \( \mathbb{H}_2(n, \mathcal{O}) \)) (see, for example, [Hida88, Theorem 4.11], [Hida91, Theorem 2.2 (ii)]).

2. Selmer groups

2.1. Definition of Selmer groups. In this subsection, we recall the notation and definitions of Selmer groups for a nearly ordinary Galois representation with Selmer weights in the sense of T. Weston \([\text{We}]\).

Let \( F \) be a finite extension of \( \mathbb{Q} \). Let \( F_\infty \) denote the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \). Let \( G_F \) denote the absolute Galois group of \( F \). We put \( \Gamma = \text{Gal}(F_\infty/F) \). Let \( K_p \) be a finite extension of \( \mathbb{Q}_p \) and \( V \) a finite dimensional \( K_p \)-vector space endowed with a continuous \( K_p \)-linear action
of $G_F$. We put $n = \dim_{K_p} V$. For a real place $v$ of $F$, let $d^+_v(V)$ denote the dimension of the $(\pm 1)$-eigenspaces of a complex conjugation in $G_F$, acting on $V$, respectively. Let $\mathcal{O}$ denote the ring of integers of $K_p$. We choose a $G_F$-stable $\mathcal{O}$-lattice $T$ in $V$. We put $A = V/T$.

We call $A$ a torsion quotient of $V$. Then $A$ is a discrete $\mathcal{O}$-module with the action of $G_F$, which is isomorphic to $(K_p/\mathcal{O})^n$ as an $\mathcal{O}$-module. We say that $V$ is a nearly ordinary Galois representation of $G_F$ ([We, §1.2]) if, for every place $v$ of $F$ lying above $p$, $V$ has a $G_{F_v}$-stable complete flag of $K_p$-subspaces

$$0 = V^0_v \subset V^1_v \subset \cdots \subset V^n_v = V.$$ 

A set $\mathcal{W}$ of Selmer weights for $V$ in the sense of Weston ([We, §1.2]) is a choice of integers $c_v(V)$ for every place $v$ of $F$ lying above $p$ such that $0 \leq c_v(V) \leq n$ and

$$\sum_{v \mid p} [F_v : \mathbb{Q}_p] \cdot c_v(V) = \sum_{v : \text{real places}} d^+_v(V) + \sum_{v : \text{complex places}} n.$$ 

For every place $w$ of $F_{\infty}$ lying above $p$, let $A^W_w$ denote the image of $V^{n-c_v(V)}_v$ in $A$ under the canonical morphism $V \rightarrow A$ with $v$ the restriction of $w$ to $F$:

$$\begin{array}{ccc}
0 & \longrightarrow & V^{n-c_v(V)}_v \\
& & \downarrow \\
0 & \longrightarrow & A^W_w
\end{array} \quad A = V/T \quad A/A^W_w \quad 0.$$ 

For every place $w$ of $F_{\infty}$, the local Galois cohomology $H^1_{s,\mathcal{W}}(F_{\infty,w}, A)$ is defined by

$$H^1_{s,\mathcal{W}}(F_{\infty,w}, A) = \begin{cases} 
H^1(F_{\infty,w}, A) & \text{if } w \nmid p, \\
\text{im}(H^1(F_{\infty,w}, A) \rightarrow H^1(I_{F_{\infty,w}}, A/A^W_w)) & \text{if } w \mid p.
\end{cases}$$ 

Here $I_{F_{\infty,w}}$ denotes the inertia subgroup of $G_{F_{\infty,w}}$.

Let $\Sigma(F)$ denote the set of all places of $F$. Let $\Sigma_p(F)$ denote the set of all places of $F$ lying above $p$, and let $\Sigma_{\infty}(F)$ denote the set of all infinite places of $F$. We put

$$\text{Ram}(A) = \{ v \in \Sigma(F) \mid v \notin \Sigma_p(F) \cup \Sigma_{\infty}(F) \text{ and the action of } G_F \text{ on } A \text{ is ramified} \}.$$ 

We say that a finite subset $\Sigma$ of $\Sigma(F)$ is sufficient large for $A$ ([We, §1.3]) if $\Sigma$ contains $\Sigma_p(F) \cup \Sigma_{\infty}(F) \cup \text{Ram}(A)$.

Let $A$ be a torsion quotient of a nearly ordinary Galois representation $V$ with Selmer weights $\mathcal{W}$. Let $\Sigma$ be a finite subset of $\Sigma(F)$ such that $\Sigma$ is sufficient large for $A$. We define the Selmer group $\text{Sel}_{\mathcal{W}}(F_{\infty}, A)$ of $A$ in the sense of Weston [We, §1.3] by

$$\text{Sel}_{\mathcal{W}}(F_{\infty}, A) = \ker \left( H^1(F_{\Sigma}/F_{\infty}, A) \rightarrow \prod_{w \mid p \in \Sigma} H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \right).$$ 

Here $F_{\Sigma}$ denotes the maximal extension of $F$ which is unramified outside $\Sigma$.

Next we define the non-primitive Selmer groups in the sense of R. Greenberg [Gre–Vat, §2]. Let $\Sigma_0$ be a subset of $\Sigma \setminus \{ \Sigma_p(F) \cup \Sigma_{\infty}(F) \}$. We define the non-primitive Selmer group $\text{Sel}_{\mathcal{W}}^{\Sigma_0}(F_{\infty}, A)$ of $A$ and $\Sigma_0$ by

$$\text{Sel}_{\mathcal{W}}^{\Sigma_0}(F_{\infty}, A) = \ker \left( H^1(F_{\Sigma}/F_{\infty}, A) \rightarrow \prod_{w \mid p \in \Sigma \setminus \Sigma_0} H^1_{s,\mathcal{W}}(F_{\infty,w}, A) \right).$$ 

We have $\text{Sel}_{\mathcal{W}}(F_{\infty}, A) \subset \text{Sel}_{\mathcal{W}}^{\Sigma_0}(F_{\infty}, A)$ by the definition.
Let $\Lambda$ denote the Iwasawa algebra $O[[T]]$. We know that the groups $H^1(F_\Sigma/F_\infty, A)$, $H^2(F_\Sigma/F_\infty, A)$, $\prod_{w|v}(H^1_{s,W}(F_\infty,w,A))$, and $\text{Sel}_W(F_\infty, A)$ are discrete $O$-modules with a natural continuous action of $\Gamma$. Hence these groups are regarded as $\Lambda$-modules and are known to be cofinitely generated, that is, their Pontryagin duals are finitely generated $\Lambda$-modules. The following is conjectured by [We, Conjecture 1.7]:

**Conjecture 2.1.** For every nearly ordinary Galois representation $V$ with Selmer weights $W$, the Selmer group $\text{Sel}_W(F_\infty, A)$ of a torsion quotient $A$ of $V$ is finitely generated $\Lambda$-cotorsion.

In order to confirm Conjecture 2.1 in some special cases, we will need the Selmer groups and the non-primitive Selmer groups of the residual Galois representation $A[\varpi]$ defined as follows. Let $\varpi$ be a uniformizer of $O$. Let $A[\varpi]$ denote the $\varpi$-torsion of $A$. For every place $w$ of $F_\infty$, the local Galois cohomology $H^1_{s,W}(F_\infty,w,A[\varpi])$ is defined by

$$H^1_{s,W}(F_\infty,w,A[\varpi]) = \left\{ \begin{array}{ll} H^1(F_\Sigma/F_\infty, A[\varpi]) & \text{if } w \nmid p, \\ \text{im}(H^1(F_\infty,w,A[\varpi]) \to H^1(I_{F_\infty,w}, (A/A[\varpi]_w)[\varpi])) & \text{if } w | p. \end{array} \right.$$  

We define the Selmer group $\text{Sel}_W(F_\infty, A[\varpi])$ of $A[\varpi]$ by

$$\text{Sel}_W(F_\infty, A[\varpi]) = \ker\left( H^1(F_\Sigma/F_\infty, A[\varpi]) \to \prod_{w|v \in \Sigma} H^1_{s,W}(F_\infty,w,A[\varpi]) \right).$$

We also define the non-primitive Selmer group $\text{Sel}^{\Sigma_0}_W(F_\infty, A[\varpi])$ of $A[\varpi]$ and $\Sigma_0$ by

$$\text{Sel}^{\Sigma_0}_W(F_\infty, A[\varpi]) = \ker\left( H^1(F_\Sigma/F_\infty, A[\varpi]) \to \prod_{w|v \in \Sigma \setminus \Sigma_0} H^1_{s,W}(F_\infty,w,A[\varpi]) \right).$$

We have $\text{Sel}_W(F_\infty, A[\varpi]) \subset \text{Sel}^{\Sigma_0}_W(F_\infty, A[\varpi])$ by the definition.

### 2.2. Structure of Selmer groups.

Let $A$ be a torsion quotient of a nearly ordinary Galois representation $V$ with Selmer weights $W$. Let $\Sigma$ be a finite subset of $\Sigma(F)$ such that $\Sigma$ is sufficient large for $A$. Let $\Sigma_0$ be a subset of $\Sigma\setminus\{\Sigma_p(F) \cup \Sigma_\infty(F)\}$ such that $\Sigma_0$ contains $\text{Ram}(A)$. We put $A^* = \text{Hom}(T, \mu_{p^\infty})$. This is also a discrete $O$-module equipped with the action of $\text{Gal}(F_\Sigma/F)$. For a locally compact $\mathbb{Z}_p$-module $M$, let $M^{\text{PD}}$ denote the Pontryagin dual of $M$.

**Proposition 2.2.** Assume that $H^0(F_\infty, A^* \otimes_O K_p/O) = 0$ and $\text{Sel}_W(F_\infty, A)$ is finitely generated $\Lambda$-cotorsion. Then we have

$$\text{Sel}^{\Sigma_0}_W(F_\infty, A)/\text{Sel}_W(F_\infty, A) \simeq \prod_{w|v \in \Sigma_0} H^1_{s,W}(F_\infty,w,A).$$

In particular, $\text{Sel}^{\Sigma_0}_W(F_\infty, A)$ is finitely generated $\Lambda$-cotorsion, and the following equalities hold:

$$\text{corank}_O(\text{Sel}^{\Sigma_0}_W(F_\infty, A)) = \text{corank}_O(\text{Sel}_W(F_\infty, A)) + \sum_{w|v \in \Sigma_0} \text{corank}_O(H^1_{s,W}(F_\infty,w,A)), $$

$$\mu(\text{Sel}^{\Sigma_0}_W(F_\infty, A)^{\text{PD}}) = \mu(\text{Sel}_W(F_\infty, A)^{\text{PD}}).$$

**Proof.** By our assumptions and [We, §1.4, Proposition 1.8], the sequence

$$0 \to \text{Sel}_W(F_\infty, A) \to H^1(F_\Sigma/F_\infty, A) \to \prod_{w|v \in \Sigma} H^1_{s,W}(F_\infty,w,A) \to 0$$


is exact. Then our first assertion follows from the snake lemma for

\[ 0 \rightarrow \text{Sel}_W(F_\infty, A) \rightarrow H^1(F_\Sigma/F_\infty, A) \rightarrow \prod_{w | v \in \Sigma} H^1_s(W(F_\infty, w), A) \rightarrow 0 \]

0 \rightarrow \text{Sel}^{\Sigma_0}_W(F_\infty, A) \rightarrow H^1(F_\Sigma/F_\infty, A) \rightarrow \prod_{w | v \in \Sigma \setminus \Sigma_0} H^1_s(W(F_\infty, w), A) \rightarrow 0.

We fix a place \( v \in \Sigma_0 \). For every place \( w \) of \( F_\infty \) lying above \( v \), by \([\text{NSW}, \text{Chapter VII, Theorem } 7.1.8 \,(i)]\), the exact sequence \( 0 \rightarrow A[\overline{\omega}] \rightarrow A \xrightarrow{\times \overline{\omega}} A \rightarrow 0 \) implies that \( H^1(F_\infty, w), A) \) is divisible. Moreover, since \( w \nmid p \), by \([\text{Gre89}, \text{§3, Proposition } 2]\), \( \prod_{w | v} H^1_s(W(F_\infty, w), A) = \prod_{w | v} H^1_s(W(F_\infty, w), A) \) is finitely generated \( \Lambda \)-cotorson. Now \( \prod_{w | v} H^1_s(W(F_\infty, w), A) \) is a finitely generated \( \mathcal{O} \)-comodule and hence we obtain the equalities as desired. \( \square \)

**Proposition 2.3.** Let \( p \) be an odd prime number. Assume that, for every place \( w \) of \( F_\infty \) lying above \( p \), \( I_{F_\infty, w} \) acts trivially on \( A/A_w^W \) and \( H^0(F, A[\overline{\omega}]) = 0 \). Then we have

\[ \text{Sel}^{\Sigma_0}_W(F_\infty, A[\overline{\omega}]) \simeq \text{Sel}^{\Sigma_0}_W(F_\infty, A)[\overline{\omega}] \]

**Proof.** The assumption \( H^0(F, A[\overline{\omega}]) = 0 \) implies that \( H^0(F_\infty, A[\overline{\omega}]) = 0 \) because \( \Gamma \) is a pro-\( p \) group. Then we have \( H^0(F_\infty, A) = 0 \) and hence the exact sequence \( 0 \rightarrow A[\overline{\omega}] \rightarrow A \xrightarrow{\times \overline{\omega}} A \rightarrow 0 \) induces an isomorphism

\[ H^1(F_\Sigma/F_\infty, A[\overline{\omega}]) \simeq H^1(F_\Sigma/F_\infty, A)[\overline{\omega}] \]

We have the following commutative diagram:

\[ \begin{array}{cccc}
\text{Sel}^{\Sigma_0}_W(F_\infty, A[\overline{\omega}]) & \xrightarrow{\sim} & \text{Sel}^{\Sigma_0}_W(F_\infty, A)[\overline{\omega}] & \xrightarrow{\sim} & \text{Sel}^{\Sigma_0}_W(F_\infty, A) \\
H^1(F_\Sigma/F_\infty, A[\overline{\omega}]) & \xrightarrow{\sim} & H^1(F_\Sigma/F_\infty, A)[\overline{\omega}] & \xrightarrow{\sim} & H^1(F_\Sigma/F_\infty, A) \\
\prod_{w | v \in \Sigma \setminus \Sigma_0} H^1_s(W(F_\infty, w), A[\overline{\omega}]) & \xrightarrow{\ast} & \prod_{w | v \in \Sigma \setminus \Sigma_0} H^1_s(W(F_\infty, w), A) \\
\end{array} \]

Thus, for the proof, it suffices to show the injectivity of the morphism \( \ast \). In order to do it, we consider the following commutative diagram:

\[ \begin{array}{cccc}
\prod_{w | v \in \Sigma \setminus \Sigma_0} H^1_s(W(F_\infty, w), A[\overline{\omega}]) & \xrightarrow{\ast} & \prod_{w | v \in \Sigma \setminus \Sigma_0} H^1(I_{F_\infty, w}, A[\overline{\omega}]) \times \prod_{w | v \in \Sigma p(F)} H^1(I_{F_\infty, w}, D[\overline{\omega}]) \\
\prod_{w | v \in \Sigma \setminus \Sigma_0} H^1_s(W(F_\infty, w), A) & \xrightarrow{\ast} & \prod_{w | v \in \Sigma \setminus \Sigma_0} H^1(I_{F_\infty, w}, A) \times \prod_{w | v \in \Sigma p(F)} H^1(I_{F_\infty, w}, D) \\
\end{array} \]

where we write \( D \) for \( A/A_w^W \) to simplify the notation. Here the injectivity of the horizontal morphisms follows from that, for every place \( w \) of \( F_\infty \) such that \( w \nmid p \), \( G_{F_\infty, w}/I_{F_\infty, w} \) has profinite order prime to \( p \). Thus, it suffices to show the injectivity of the morphism \( \ast \ast \).
For a place $w \mid v \in \Sigma \setminus \Sigma_0$ such that $w \mid p$, by the definition of $\Sigma_0$, we have $H^0(I_{F_{\infty, w}}, A) = A$, which is divisible, and hence the exact sequence $0 \to A[v] \to A \to 0$ implies that

$$H^1(I_{F_{\infty, w}}, A[v]) \to H^1(I_{F_{\infty, w}}, A)$$

is injective.

For a place $w|v \in \Sigma_p(F)$, by the assumption on $D$, we have $H^0(I_{F_{\infty, w}}, D) = D$, which is divisible, and hence the exact sequence $0 \to D[v] \to D \to 0$ implies that

$$H^1(I_{F_{\infty, w}}, D[v]) \to H^1(I_{F_{\infty, w}}, D)$$

as desired. □

**Proposition 2.4.** Assume that $Sel^\Sigma_W(F_\infty, A[v])$ is finite. Then, under the same assumptions as Proposition 2.3, $Sel^\Sigma_W(F_\infty, A)$ and $Sel_W(F_\infty, A)$ are finitely generated $\Lambda$-cotorsion.

**Proof.** By our assumptions and Proposition 2.3, $Sel^\Sigma_W(F_\infty, A[v])$ is finite. Since we have $(Sel^\Sigma_W(F_\infty, A)[v])^{PD} \simeq Sel^\Sigma_W(F_\infty, A)^{PD}/v$, which is finite, the structure theorem of finitely generated $\Lambda$-modules (see, for example, [Was, Chapter 13, Theorem 13.12]) implies that $Sel^\Sigma_W(F_\infty, A)$ is finitely generated $\Lambda$-cotorsion. Since $Sel_W(F_\infty, A) \subset Sel^\Sigma_W(F_\infty, A)$, we have $Sel^\Sigma_W(F_\infty, A)^{PD} \to Sel_W(F_\infty, A)^{PD}$ and hence $Sel_W(F_\infty, A)$ is also finitely generated $\Lambda$-cotorsion. □

Let $\kappa$ denote the residue field of $O$.

**Proposition 2.5.** Assume that $H^0(F_\infty, A^* \otimes_O K_p/O) = 0$. Then, under the same assumptions as Proposition 2.3 and Proposition 2.4, we have

$$\text{corank}_O(Sel^\Sigma_W(F_\infty, A)) = \dim_\kappa(Sel^\Sigma_W(F_\infty, A[v]))$$

**Proof.** By Proposition 2.4, $Sel^\Sigma_W(F_\infty, A)^{PD}$ is a finitely generated $O$-module and hence, by Proposition 2.3, it suffices to show that $Sel^\Sigma_W(F_\infty, A)$ has no proper $\Lambda$-submodules of finite index. By Proposition 2.4, $Sel_W(F_\infty, A)$ is finitely generated $\Lambda$-cotorsion and hence Proposition 2.2 implies that $Sel^\Sigma_W(F_\infty, A)/Sel_W(F_\infty, A) \simeq \prod_{v|w \in \Sigma_0} H^1_{\Lambda}(F_{\infty, w}, A)$, which is divisible (by the proof of Proposition 2.2). Now the assertion follows from the fact obtained by [We, Proposition 1.8 (2)] that $Sel_W(F_\infty, A)$ has no proper $\Lambda$-submodule of finite index. □

We can compare $Sel^\Sigma_W(F_\infty, A)$ with $Sel_W(F_\infty, A)$ by the following proposition:

**Proposition 2.6.** Let $v$ be a finite place of $F$ prime to $p$, and let $P_v(X) = \det((1 - \text{Frob}_v X)|_{V_{F_v}}) \in O[X]$. Let $P_v = P_v(N(q_v)^{-1} \gamma_v) \in \Lambda$, where $q_v$ denotes the prime ideal of $\mathfrak{p}_F$ corresponding to $v$, and $\gamma_v$ denotes the Frobenius automorphism corresponding to $v$ in $\Gamma$. Then, for each place $w$ of $F_\infty$ lying above $v$, the characteristic ideal of the $\Lambda$-module $\prod_{w|v} H^1_{\Lambda}(F_{\infty, w}, A)^{PD}$ is generated by $P_v$.

**Proof.** We fix a place $w$ of $F_\infty$ lying above $v$. Let $\Gamma_v$ denote the decomposition subgroup of $\Gamma$ for $w|v$. Since $H^1_{\Lambda}(F_{\infty, w}, A) \otimes O[\Gamma_v] \simeq \prod_{w|v} H^1_{\lambda}(F_{\infty, w}, A)^{PD}$ is generated by $P_v$. Let $R_{F_v}$ denote the ramification subgroup of $G_{F_v}$, which has profinite order prime to $p$. Since $I_{F_v}/R_{F_v} \simeq \prod_{(\ell, q_v) = 1} \mathbb{Z}_{\ell}(1)$ (see, for example, [SE, Chapter IV, §2, Exercise 2]), there exists a
unique subgroup $J_{F_v}$ of $I_{F_v}$ such that $J_{F_v}$ has profinite order prime to $p$ and $I_{F_v}/J_{F_v} \simeq \mathbb{Z}_p(1)$. We put $\bar{T}_{F_v} = I_{F_v}/J_{F_v}$ and $G = G_{F_{\infty,w}}/J_{F_v}$. Therefore we obtain

$$H^1_{\text{et}}(F_{\infty,w}, A) \simeq H^1(G, A^J_{F_v}) \simeq H^1(G, A_{J_{F_v}}) \simeq H^1(\bar{T}_{F_v}, A_{J_{F_v}})^{G/\bar{T}_{F_v}}.$$

Here the first isomorphism is obtained from the inflation-restriction sequence for $1 \to J_{F_v} \to G_{F_{\infty,w}} \to G \to 1$ because $J_{F_v}$ has profinite order prime to $p$, the second isomorphism is obtained from that the canonical morphism $A^J_{F_v} \hookrightarrow A \to A_{J_{F_v}}$ induces an isomorphism $A^J_{F_v} \simeq A_{J_{F_v}}$ because $J_{F_v}$ has profinite order prime to $p$, and the last isomorphism is obtained from the inflation-restriction sequence for $1 \to \bar{T}_{F_v} \to G \to G/\bar{T}_{F_v} \to 1$ because $G/\bar{T}_{F_v}$ has profinite order prime to $p$. Note that

$$H^1(\bar{T}_{F_v}, A_{J_{F_v}}) \simeq A_{I_{F_v}}(-1)$$

as $G_{F_{\infty,w}}/I_{F_v}$-modules. Indeed, if $e_v$ is a topological generator of $\bar{T}_{F_v} \simeq \mathbb{Z}_p(1)$, then we have $H^1(\bar{T}_{F_v}, A_{J_{F_v}}) \simeq H^1(\bar{T}_{F_v}, A_{J_{F_v}}/(e_v - 1)A_{J_{F_v}}) \simeq \text{Hom}(\mathbb{Z}_p(1), A_{I_{F_v}}) \simeq A_{I_{F_v}}(-1)$. Let $\alpha_1, \ldots, \alpha_e$ denote the eigenvalues of Frobenius $e_v \in G_{F_{\infty,w}}/I_{F_v}$ acting on $A_{J_{F_v}}$. Then the eigenvalues of Frobenius acting on $(A_{I_{F_v}}(-1))^{PD}$ are $N(q_v)^{-1}\alpha_1^{-1}, \ldots, N(q_v)^{-1}\alpha_e^{-1}$ and hence, again using the fact that $G/\bar{T}_{F_v}$ has profinite order prime to $p$, those eigenvalues $N(q_v)^{-1}\alpha_i^{-1}$ which are principal units are the eigenvalues of $\gamma_v$ acting on $(A_{I_{F_v}}(-1))^{G/\bar{T}_{F_v}}$. Therefore, our assertion follows from that $1 - N(q_v)^{-1}\alpha_i^{-1} \in \mathcal{O}[[\Gamma_v]]$ is a unit if and only if $N(q_v)^{-1}\alpha_i^{-1}$ is a principal unit. \hfill $\Box$

2.3. Selmer groups for 1-dimensional representations. In this subsection, we compute the Selmer groups for 1-dimensional representations under some assumptions.

We assume that $F$ is a totally real number field. Let $\Sigma$ be a finite subset of $\Sigma(F)$ such that $\Sigma$ contains $\Sigma_p(F) \cup \Sigma_{\infty}(F)$. Let $\theta : \text{Gal}(F_{\Sigma}/F) \to \mathcal{O}^\times$ be a continuous homomorphism. Let $V_{\theta}$ denote the space $\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the action of $\text{Gal}(F_{\Sigma}/F)$ via $\theta$. Let $A_{\theta}$ denote the cofree $\mathcal{O}$-module $\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ of corank 1 with the action of $\text{Gal}(F_{\Sigma}/F)$ via $\theta$. Let $K$ denote the extension of $F$ such that $\text{ker}(\theta) = \text{Gal}(F_{\Sigma}/K)$, and let $K_{\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $K$. We put $G = \text{Gal}(K_{\infty}/F_{\Sigma})$ and $\Delta = \text{Gal}(K_{\infty}/F_{\infty})$. We assume that

$$(p, \sharp\Delta) = 1.$$ 

Then we can identify $\Gamma$ with a subgroup of $G$ such that $G = \Delta \times \Gamma$. Moreover, the restriction map induces an isomorphism:

$$H^1(F_{\Sigma}/F_{\infty}, A_{\theta}) \simeq H^1(F_{\Sigma}/K_{\infty}, A_{\theta})^\Delta.$$

Since $\text{Gal}(F_{\Sigma}/K_{\infty})$ acts trivially on $A_{\theta}$, we have

$$H^1(F_{\Sigma}/K_{\infty}, A_{\theta}) = \text{Hom}(\text{Gal}(M^{\Sigma}_{\infty}/K_{\infty}), A_{\theta}),$$

where $M^{\Sigma}_{\infty}$ denotes the maximal abelian pro-$p$ extension of $K_{\infty}$ unramified outside $\Sigma$. Let $M^{\infty}_{\Sigma}$ denote the maximal abelian pro-$p$ extension of $K_{\infty}$ unramified outside $\Sigma_p(F) \cup \Sigma_{\infty}(F)$. Let $L_{\infty}$ denote the maximal abelian pro-$p$ extension of $K_{\infty}$ unramified everywhere. We put $X_{\infty} = \text{Gal}(M^{\Sigma}_{\infty}/K_{\infty})$ and $Y_{\infty} = \text{Gal}(L_{\infty}/K_{\infty})$.

If $\theta$ is totally even (resp. totally odd), then, for every place $v \in \Sigma_{\infty}(F)$, we have $d_v(V_{\theta}) = 0$ (resp. $d_v(V_{\theta}) = 1$). Then $V_{\theta}$ is a nearly ordinary Galois representation with Selmer weights $c_v(V_{\theta}) = 0$ (resp. $c_v(V_{\theta}) = 1$) for every place $v \in \Sigma_{\infty}(F)$. Now, for $B = A_{\theta}$ and $A_{\theta}[\varpi]$, we
can define the Selmer groups $\text{Sel}(F_\infty, B)$ and the non-primitive Selmer groups $\text{Sel}^\Sigma_0(F_\infty, B)$ as $\Sigma_0$.

**Proposition 2.7.** Let $\xi$ be the restriction $\theta|_\Delta$ of $\theta$ to $\Delta$ and $\Sigma_0 = \Sigma \backslash \{\Sigma_p(F) \cup \Sigma_\infty(F)\}$. Assume that $p$ is odd and $(p, \sharp \Delta) = 1$.

(1) \[ \text{Sel}(F_\infty, A_\theta) \simeq \begin{cases} \text{Hom}_\mathcal{O}((X_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A_\theta) & \text{if } \theta \text{ is totally even,} \\ \text{Hom}_\mathcal{O}((Y_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A_\theta) & \text{if } \theta \text{ is totally odd.} \end{cases} \]

In particular, $\text{Sel}(F_\infty, A_\theta)$ is finitely generated $\Lambda$-cotorsion.

(2) For $B = A_\theta$ and $A_\theta[\varpi]$, \[ \text{Sel}^\Sigma_0(F_\infty, B) \simeq \begin{cases} H^1(F\Sigma/F_\infty, B) & \text{if } \theta \text{ is totally even,} \\ \ker\left( H^1(F\Sigma/F_\infty, B) \to \prod_{w | \nu \in \Sigma_p(F)} H^1(I_{F_\infty, w}, B) \right) & \text{if } \theta \text{ is totally odd.} \end{cases} \]

(3) Assume that $\xi$ is non-trivial if $\theta$ is totally even, and $\xi \neq \omega$ if $\theta$ is totally odd. Then \[ H^0(F, A_\theta \otimes_\mathcal{O} K_p/\mathcal{O}) = 0. \]

Moreover, $\text{Sel}^\Sigma_0(F_\infty, A_\theta)$ is finitely generated $\Lambda$-cotorsion.

**Proof.** (1), (2) First we treat the case where $\theta$ is totally even. Let $\theta$ denote the set of the Selmer weights for $V_\theta$ as explained before this proposition. Then we have $A_{\theta, w} = A_\theta$ for every place $w$ of $F_\infty$ lying above $v \in \Sigma_p(F)$ and hence, for $B = A_\theta$ and $A_\theta[\varpi]$, we have \[ H^1_{s, 0}(F_\infty, w, B) = 0, \]
which proves (2). Moreover the assumption $(p, \sharp \Delta) = 1$ implies (1): \[ \text{Sel}(F_\infty, A_\theta) = \text{Hom}(X_\infty, A_\theta)^\Delta = \text{Hom}_\mathcal{O}((X_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A_\theta). \]

Next we treat the case where $\theta$ is totally odd. Let $1$ denote the set of the Selmer weights for $V_\theta$ as explained before this proposition. Then we have $A_{\theta, w}^1 = 0$ for every place $w$ of $F_\infty$ lying above $v \in \Sigma_p(F)$ and hence, for $B = A_\theta$ and $A_\theta[\varpi]$, we have \[ H^1_{s, 1}(F_\infty, w, B) = \text{im}\left( H^1(F_\infty, w, B) \to H^1(I_{F_\infty, w}, B) \right), \]
which proves (2). Moreover the assumption $(p, \sharp \Delta) = 1$ implies (1): \[ \text{Sel}(F_\infty, A_\theta) = \text{Hom}(Y_\infty, A_\theta)^\Delta = \text{Hom}_\mathcal{O}((Y_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A_\theta). \]

According to a well-known theorem of K. Iwasawa ([Iwa59], [Iwa73]), $X_\infty$ and $Y_\infty$ are finitely generated $\Lambda$-torsion, and hence $\text{Sel}(F_\infty, A_\theta)$ is finitely generated $\Lambda$-cotorsion as desired.

(3) We note that the vanishing of $H^0(F, A_\theta \otimes_\mathcal{O} K_p/\mathcal{O})$ is equivalent to the vanishing of its $\varpi$-torsion $H^0(F, \text{Hom}(A_\theta[\varpi], \kappa(1)))$. Thus, the first assertion follows from our assumption on $\xi$. Now, by combining with Proposition 2.2 and (1), $\text{Sel}^\Sigma_0(F_\infty, A_\theta)$ is finitely generated $\Lambda$-cotorsion as desired. \qed
2.4. Selmer groups for Hilbert modular forms. In this subsection, we apply the general theory above to the Iwasawa main conjecture of Hilbert modular forms.

Let \( \mathfrak{f} \in S_2(n, \mathcal{O}) \) be a normalized Hecke eigenform for all \( T(q) \) and \( U(q) \) with character \( \varepsilon \). We assume that \( \mathfrak{f} \) is \( p \)-ordinary, that is, for every prime ideal \( p \) of \( \sigma_F \) lying above \( p \), the Hecke eigenvalue \( C(p, \mathfrak{f}) \) of \( T(p) \) is prime to \( p \). Let

\[
\rho_F: G_F \rightarrow \text{GL}(T_f) \cong \text{GL}_2(\mathcal{O})
\]
denote the associated Galois representation, which satisfies the following (1), (2), (3), (4):

1. \( \rho_F \) is unramified at every prime ideal \( q \) of \( \sigma_F \) such that \( q \nmid np \);
2. \( \text{Tr}(\rho_F(\text{Frob}_q)) = C(q, \mathfrak{f}) \) for every prime ideal \( q \) of \( \sigma_F \) such that \( q \nmid np \);
3. \( \text{det}(\rho_F(\text{Frob}_q)) = \varepsilon(q)N(q) \) for every prime ideal \( q \) of \( \sigma_F \) such that \( q \nmid np \);
4. \( \rho_F \) is totally odd.

Let \( V_f \) denote the space \( T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) with the action of \( G_F \) via \( \rho_F \). Let \( A_f \) denote the cofree \( \mathcal{O} \)-module \( T_f \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \) of corank 2 with the action of \( G_F \) via \( \rho_F \).

From the result of [Wili88, Theorem 2.1.4], for every prime ideal \( p \) of \( \sigma_F \) lying above \( p \), the restriction of \( \rho_F \) to the decomposition group \( G_{F, p} \) is of the form

\[
\rho_F|_{G_{F, p}} \sim \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix},
\]

where \( \rho_2: G_{F, p} \rightarrow O^\times \) is unramified such that \( \rho_2 \) sends the arithmetic Frobenius to a unit-root of \( X^2 - C(q, \mathfrak{f})X + \varepsilon(p)N(p) = 0 \). Then, for every place \( w \) of \( F_{\infty} \) lying above \( p \), \( A_{f, w} \) is defined by the following exact sequence of \( \mathcal{O}[G_{F, p}] \)-modules:

\[
0 \rightarrow A_{f, w} \rightarrow A_f \rightarrow A_f/A_{f, w} \rightarrow 0,
\]

where \( v \) denotes the restriction of \( w \) to \( F \) and \( p_v \) denotes the prime ideal of \( \sigma_F \) corresponding to \( v \). Here \( G_{F, p} \) acts on \( A_{f, w} \) via the character \( \rho_1: G_{F, p} \rightarrow O^\times \), and on \( A_f/A_{f, w} \) via the character \( \rho_2: G_{F, p} \rightarrow O^\times \). Thus, \( V_f \) is a nearly ordinary Galois representation with Selmer weights \( c_v(V_f) = 1 \) for every place \( v \in \Sigma_p(F) \).

Let \( \Sigma_0 \) be the set of all finite places \( v \) of \( F \) such that \( q_v \) divides \( n \), where \( q_v \) denotes the prime ideal of \( \sigma_F \) corresponding to \( v \). We put \( \Sigma = \Sigma_0 \cup \Sigma_p(F) \cup \Sigma_{\infty}(F) \). Now, for \( B = A_f \) and \( A_f[\varpi] \), we can define the Selmer groups \( \text{Sel}(F_{\infty}, B) \) and the non-primitive Selmer groups \( \text{Sel}^{n}\!(F_{\infty}, B) \) as \( 2.1 \).

Let \( \kappa \) denote the residue field of \( \mathcal{O} \). We assume that

- (RR) the residual representation \( \bar{\rho}_F: G_F \rightarrow \text{GL}_2(\kappa) \) is reducible and of the form

\[
\bar{\rho}_F \sim \begin{pmatrix} \bar{\varphi} & 0 \\ 0 & \bar{\psi} \end{pmatrix},
\]

that is, there exists an exact sequence of \( \kappa[G_F] \)-modules

\[
0 \rightarrow \Phi \rightarrow A_f[\varpi] \rightarrow \Psi \rightarrow 0,
\]

where \( G_F \) acts on \( \Phi \) via the character \( \bar{\varphi} : G_F \rightarrow \kappa^\times \), and on \( \Psi \) via the character \( \bar{\psi} : G_F \rightarrow \kappa^\times \).

Let \( (A_{\varphi}, \varphi) \) (resp. \( (A_{\psi}, \psi) \)) be an \( \mathcal{O} \)-module \( A_{\varphi} \simeq K_p/\mathcal{O} \) (resp. \( A_{\psi} \simeq K_p/\mathcal{O} \)) with the action of \( G_F \) via the character \( \varphi = \chi_{\text{cyc}} \varepsilon^{-1} : G_F \rightarrow \kappa^\times \hookrightarrow O^\times \) (resp. \( \psi : G_F \rightarrow \kappa^\times \hookrightarrow O^\times \)),

where \( \chi_{\text{cyc}} \) is the \( p \)-adic cyclotomic character of \( G_F \). Note that \( A_{\varphi}[\varpi] = \Phi \) and \( A_{\psi}[\varpi] = \Psi \).

We assume the following parity condition:

- (Parity) \( \varphi \) is ramified at every prime ideal \( p \) of \( \sigma_F \) lying above \( p \) and totally even, and \( \psi \) is unramified at every prime ideal \( p \) of \( \sigma_F \) lying above \( p \) and totally odd.
Hence, for every prime ideal $p$ of $\mathfrak{p}_F$ lying above $p$, we have $\psi(\text{Frob}_p) \equiv C(p, f) \pmod{\pi}$ by (2.1). We also assume that

$$(\mu = 0) \quad \mu(\text{Sel}(F_\infty, A_\varphi)^{\text{PD}}) = 0 \text{ and } \mu(\text{Sel}(F_\infty, A_\psi)^{\text{PD}}) = 0.$$ 

In order to prove Theorem 2.11 below, we need the following three lemmas.

**Lemma 2.8.** Assume that $p$ is odd. Then we have

$$H^0(F, \Phi) = 0, \ H^0(F, \Psi) = 0, \ H^0(F_\infty, \Phi) = 0, \ H^0(F_\infty, \Psi) = 0.$$ 

**Proof.** Since $\Gamma$ is a pro-$p$ group, it suffices to show that $H^0(F, \Phi) = 0$ and $H^0(F, \Psi) = 0$. The condition (Parity) implies that $H^0(F, \Phi) = 0$ (because $\varphi$ is ramified at every prime ideal $p$ of $\mathfrak{p}_F$ lying above $p$) and $H^0(F, \Psi) = 0$ (because $\psi$ is totally odd and $p$ is odd) as desired. □

**Lemma 2.9.** Assume that $p$ is odd. Then we have

$$H^0(F, A_f[\pi]) = 0.$$ 

**Proof.** The sequence (2.2) induces an exact sequence

$$0 \to H^0(F, \Phi) \to H^0(F, A_f[\pi]) \to H^0(F, \Psi).$$

Now our assertion follows from Lemma 2.8. □

**Lemma 2.10.** Assume that $p$ is odd. Then we have

$$H^2(F_\Sigma/F_\infty, \Phi) = 0.$$ 

**Proof.** The exact sequence $0 \to \Phi \to A_\varphi \xrightarrow{\pi} A_\varphi \to 0$ induces a long exact sequence

$$H^1(F_\Sigma/F_\infty, A_\varphi) \xrightarrow{\pi} H^1(F_\Sigma/F_\infty, A_\varphi) \to H^2(F_\Sigma/F_\infty, \Phi) \to H^2(F_\Sigma/F_\infty, A_\varphi).$$

Thus, for the proof, it is enough to show the following:

(i) $H^1(F_\Sigma/F_\infty, A_\varphi)$ is divisible;

(ii) $H^2(F_\Sigma/F_\infty, A_\varphi) = 0$.

First we prove (i). Since $\varphi$ is totally even, we have $H^1(F_\Sigma/F_\infty, A_\varphi) = \text{Sel}^{\Sigma_0}(F_\infty, A_\varphi)$ by Proposition 2.7 (2). Since $\varphi$ is non-trivial, Proposition 2.7 (3) implies that $\text{Sel}^{\Sigma_0}(F_\infty, A_\varphi)$ is finitely generated $\Lambda$-cotorsion, and $\mu(\text{Sel}^{\Sigma_0}(F_\infty, A_\varphi)^{\text{PD}}) = 0$ by Proposition 2.2 and the assumption ($\mu = 0$). Moreover, $\text{Sel}^{\Sigma_0}(F_\infty, A_\varphi)$ has no proper $\Lambda$-submodules of finite index by the proof of Proposition 2.8 and the assumption ($\mu = 0$). Therefore, by combining them, $H^1(F_\Sigma/F_\infty, A_\varphi)$ is divisible as desired.

Next we prove (ii). By the above (i), we know $\text{corank}_\Lambda(H^1(F_\Sigma/F_\infty, A_\varphi)) = 0$. Then, by [Gre89 §4, Proposition 3], we have $\text{corank}_\Lambda(H^2(F_\Sigma/F_\infty, A_\varphi)) = 0$ and hence $H^2(F_\Sigma/F_\infty, A_\varphi)$ is finitely generated $\Lambda$-cotorsion. Since $H^2(F_\Sigma/F_\infty, A_\varphi)$ is $\Lambda$-cofree by [Gre89 §4, Proposition 4], we obtain $H^2(F_\Sigma/F_\infty, A_\varphi) = 0$ as desired. □

**Theorem 2.11.** Assume that $p$ is odd. Then, under the above assumptions (RR), (Parity), and ($\mu = 0$), $\text{Sel}(F_\infty, A_f)$ and $\text{Sel}^{\Sigma_0}(F_\infty, A_f)$ are finitely generated $\Lambda$-cotorsion, and the following equality holds:

$$(2.3) \quad \text{corank}_\mathcal{O}(\text{Sel}^{\Sigma_0}(F_\infty, A_f)) = \text{corank}_\mathcal{O}(\text{Sel}^{\Sigma_0}(F_\infty, A_\varphi)) + \text{corank}_\mathcal{O}(\text{Sel}^{\Sigma_0}(F_\infty, A_\psi)).$$
**Proof.** For the proof of our first assertion, it suffices to check that $\text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi])$ is finite by applying Proposition 2.4 with the help of Lemma 2.8. By the exact sequence (2.4) with the help of Lemma 2.8 and Lemma 2.10, we have an exact sequence

$$0 \to H^1(F_{\Sigma}/F_{\infty}, \Phi) \to H^1(F_{\Sigma}/F_{\infty}, A_f[\varpi]) \to H^1(F_{\Sigma}/F_{\infty}, \Psi) \to 0.$$ 

Then this sequence induces an exact sequence

$$(2.4) \quad 0 \to \text{Sel}^{\Sigma}_{\infty}(F, \Phi) \to \text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi]) \to \text{Sel}^{\Sigma}_{\infty}(F, \Psi) \to 0.$$ 

Indeed, by the definition of $\text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi])$ and the condition (Parity), we have

$$\text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi]) = \ker \left( H^1(F_{\Sigma}/F_{\infty}, A_f[\varpi]) \to \prod_{w|\varpi \in \Sigma_p(F)} H^1(I_{F_{\infty,w}}, \Psi) \right),$$

and by Proposition 2.7 (2), we have $\text{Sel}^{\Sigma}_{\infty}(F, \Phi) = H^1(F_{\Sigma}/F_{\infty}, \Phi)$ and $\text{Sel}^{\Sigma}_{\infty}(F, \Psi) = \ker \left( H^1(F_{\Sigma}/F_{\infty}, \Psi) \to \prod_{w|\varpi \in \Sigma_p(F)} H^1(I_{F_{\infty,w}}, \Psi) \right)$. Hence, for the proof, it suffices to check that $\text{Sel}^{\Sigma}_{\infty}(F, \Phi)$ and $\text{Sel}^{\Sigma}_{\infty}(F, \Psi)$ are finite. Proposition 2.7 (3) implies that $\text{Sel}^{\Sigma}_{\infty}(F, A_\varphi)$ and $\text{Sel}^{\Sigma}_{\infty}(F^2, A_\psi)$ are finitely generated $\Lambda$-cotorson. Moreover, by the assumption $(\mu = 0)$, Proposition 2.7 implies that $\mu(\text{Sel}^{\Sigma}_{\infty}(F, A_\varphi)^{PD}) = 0$ and $\mu(\text{Sel}^{\Sigma}_{\infty}(F, A_\psi)^{PD}) = 0$. Now, by combining them and Proposition 2.8, $\text{Sel}^{\Sigma}_{\infty}(F, \Phi)$ and $\text{Sel}^{\Sigma}_{\infty}(F, \Psi)$ are finite as desired.

Next we prove (2.4). Since $H^0(F, A_\varphi \otimes_{\mathcal{O}} K_p/\mathcal{O}) = 0$ and $H^0(F, A_\psi \otimes_{\mathcal{O}} K_p/\mathcal{O}) = 0$ by Proposition 2.7 (3), we have $H^0(F, A_\varphi \otimes_{\mathcal{O}} K_p/\mathcal{O}) = 0$. Therefore we obtain

$$\text{corank}_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi])) = \dim_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, A_f[\varpi]))$$

$$= \dim_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, \Phi)) + \dim_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, \Psi))$$

$$= \text{corank}_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, A_\varphi)) + \text{corank}_{\mathcal{O}}(\text{Sel}^{\Sigma}_{\infty}(F, A_\psi))$$

as desired. Here the first equality follows from Proposition 2.8 with the help of Lemma 2.9, the second equality follows from the exact sequence (2.4), and the last equality follows from Proposition 2.5 with the help of Lemma 2.8. 

### 2.5. Applications to the algebraic Iwasawa $\lambda$-invariants

In this subsection, we fix $f \in S_2(n, \mathcal{O})$ satisfying the conditions (RR) and (Parity) in (2.4) Let $n'$ be the least common multiple of $n^2$ and $m_p m_p^2$. Let $\Sigma_0$ be the set of all finite places $v$ of $F$ such that $q_v$ divides $n'$, where $q_v$ is the prime ideal of $v F$ corresponding to $v$. We put $\Sigma = \Sigma_0 \cup \Sigma_p(F) \cup \Sigma_{\infty}(F)$. Let $\chi$ be an $\mathcal{O}$-valued narrow ray class character of $F$, whose conductor is $n'$, such that $\chi$ is of order prime to $p$, totally even, and

$$\mu(\text{Sel}(F_{\infty}, A_{\varphi\chi})^{PD}) = 0, \quad \mu(\text{Sel}(F_{\infty}, A_{\psi\chi})^{PD}) = 0.$$

The assumption (RR) implies that the residual representation $\tilde{\rho}_{f \otimes \chi}$ is reducible and of the form

$$\tilde{\rho}_{f \otimes \chi} \sim \begin{pmatrix} \varphi_{\chi} & * \\ 0 & \psi_{\chi} \end{pmatrix}.$$ 

Then the triple $(f \otimes \chi, \varphi_{\chi}, \psi_{\chi})$ satisfies the conditions (RR), (Parity), and $(\mu = 0)$ in (2.4). Hence, by applying Theorem 2.11 to $(f \otimes \chi, \varphi_{\chi}, \psi_{\chi})$ instead of $(f, \varphi, \psi)$, the Selmer group $\text{Sel}(F_{\infty}, A_{f \otimes \chi})$ is finitely generated $\Lambda$-cotorson. Here we note that $\text{Sel}(F_{\infty}, A_{f \otimes \chi}) = \text{Sel}(F_{\infty}, A_f[\varpi])$.
Sel^\Sigma_0(F_\infty, A_{f, \chi}) by the assumption on \chi (cf. Proposition 2.6). Then we can define the algebraic Iwasawa \lambda-invariant \lambda^{alp}_{\chi, \Sigma_0} by

\lambda^{alp}_{\chi, \Sigma_0}(\lambda(Sel(F_\infty, A_{f, \chi}))^{PD}) = \text{corank}_O(Sel(F_\infty, A_{f, \chi})).

(2.5)

By Proposition 2.7, we can also define the algebraic Iwasawa \lambda-invariants \lambda_{\chi, \Sigma_0} and \lambda_{\psi, \Sigma_0} by

\lambda_{\chi, \Sigma_0}(\lambda(Sel^\Sigma_0(F_\infty, A_{\chi}))^{PD}) = \text{corank}_O(Sel^\Sigma_0(F_\infty, A_{\chi})),

(2.6)

\lambda_{\psi, \Sigma_0}(\lambda(Sel^\Sigma_0(F_\infty, A_{\psi}))^{PD}) = \text{corank}_O(Sel^\Sigma_0(F_\infty, A_{\psi})).

(2.7)

Therefore, by the equality (2.3), we obtain

\lambda^{alp}_{\chi, \Sigma_0} = \lambda_{\chi, \Sigma_0} + \lambda_{\psi, \Sigma_0}.

(2.8)

3. Construction of distributions

3.1. Mellin transform. In this subsection, we give a Mellin transform for a Hilbert cusp form.

Let \eta be an \eta-valued narrow ray class character of \eta with sign \eta \in (Z/2Z)^r, whose conductor is denoted by \eta, such that \eta is prime to the \eta_i for each \eta. Let (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times (resp. (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})_o) consist of elements whose conductor is \eta. We fix a non-canonical isomorphism of \eta-modules (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})_o \simeq (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times \simeq (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})_o \simeq (\eta F/\eta)\otimes \eta F/\eta. Hence we may canonically identify (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times \otimes \eta F/\eta with a subgroup of \Cl^+_F(\eta) under the canonical extension

\eta \rightarrow (\eta o \eta)^\times \otimes \eta F/\eta \rightarrow \Cl^+_F(\eta) \rightarrow \Cl_F^+ \rightarrow 1.

(3.1)

We fix a splitting of the sequence (3.1). Let \eta_\bar{b} be a map on (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times of \eta F/\eta defined by \eta_\bar{b}(\bar{b}) = \text{sgn}(\bar{b})^\times \eta(\bar{b})/d_F^{-1}[t_i]^{-1}). We note that \eta_i(\bar{b}) = \eta(\bar{b}) for any \bar{b} \in (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times of \eta F/\eta and 0 \leq \bar{b} \leq [t_i] prime to \eta.

We put \eta_{\bar{b}, \eta} = \{ e \in \eta F/\eta \mid \bar{b} \equiv 1 (mod \eta) \}. We fix a complete set \eta_{\bar{b}, \eta} of representatives of (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times of \eta F/\eta, \eta_{\bar{b}, \eta} \in (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times of \eta F/\eta, \eta_{\bar{b}, \eta} in \eta F/\eta.

For a subset \eta_{\bar{b}, \eta} of J, we put \eta_{\bar{b}, \eta} = \eta_{\bar{b}, \eta} of \eta_{\bar{b}, \eta} if \eta \in J (resp. \eta \in J \setminus J), and

(3.2)

\int_{\eta_{\bar{b}, \eta}} dz_{\bar{b}, \eta} = \bigwedge dz_{i, \eta}.

Proposition 3.1. Let \eta = (\eta_i)_{1 \leq i \leq h_F^+} \in S_2(\eta, C) be a Hecke eigfn of all T(q) and U(q), and let \eta a \eta-valued narrow ray class character of \eta, whose conductor is denoted by \eta, such that \eta is prime to the \eta_i for each \eta. Then, under the above notation, we have

\sum_{i=1}^{h_F^+} \sum_{\bar{b}_i \in \eta_{\bar{b}, \eta}} \eta_i(\bar{b}_i)^{-1} \int_{\eta_{\bar{b}, \eta}} \cdots d\eta_{\bar{b}, \eta} = \eta(\eta_i^{-1}) \int_{\eta_{\bar{b}, \eta}} \cdots d\eta_{\bar{b}, \eta}.

Here \bar{b_i} denotes the image of \bar{b_i} in (m_\eta^{-1}d_F^{-1}[t_i]^{-1}/d_F^{-1}[t_i]^{-1})^\times of \eta F/\eta under the canonical map, \eta(\eta_i^{-1}) denotes the Gauss sum (defined by Shih (3.9)), and the integrals are independent of the choice of the lift \eta_i of \bar{b_i}.
In order to prove the proposition, we compute the following zeta integral: for \( h \in S_k(n, \mathbb{C}) \),
\[
Z(h, s) = \int_{K_F^\infty/F^\infty} h(t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) |t|^s d^x t.
\]

**Proposition 3.2.** Let \( h = (h_i)_{1 \leq i \leq h_F^\infty} \in S_k(n, \mathbb{C}) \). Then we have
\[
Z(h, s) = |D|^{-s} D(s + k/2, h) \left( \frac{\Gamma(s + k/2)}{(2\pi)^{s+k/2}} \right)^n.
\]

**Proof.** Note that
\[
K_F^\infty = \prod_{1 \leq i \leq h_F^\infty} (F \otimes \mathbb{R})_i^{\infty} \hat{\omega}_{F_i} D_i^{-1} F_i^{-1} F^\infty.
\]
Then we have
\[
Z(h, s) = \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} \hat{\omega}_{F_i} D_i^{-1} F_i^{-1} F^\infty} h(t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) |t|^s d^x t
\]
\[
= \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} \hat{\omega}_{F_i} D_i^{-1} F_i^{-1} F^\infty} h(D_i^{-1} t_i^{-1} t_\infty \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) |D_i^{-1} t_i^{-1} t_\infty|^s d^x t_\infty.
\]
Since \( (F \otimes \mathbb{R})_i^{\infty} \hat{\omega}_{F_i} F_i^{-1} F^\infty = (F \otimes \mathbb{R})_i^{\infty} / \hat{\omega}_{F_i} \cdot \hat{\omega}_{F_i} \) and \( h \) is right-invariant with respect to the action of \( \hat{\omega}_{F_i} \), we have
\[
Z(h, s) = \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} / \hat{\omega}_{F_i}} h(D_i^{-1} t_i^{-1} t_\infty \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) |D_i^{-1} t_i^{-1} t_\infty|^s d^x t_\infty.
\]
Thus we obtain (3.3)
\[
Z(h, s) = \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} / \hat{\omega}_{F_i}} h_i(\sqrt{-1} t_\infty)^k / 2 |D_i^{-1} t_i^{-1} t_\infty|^s d^x t_\infty
\]
\[
= \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} / \hat{\omega}_{F_i}} \sum_{0 < |\xi| \leq |t_i|} a_\infty(\xi, h_i) e_F(\sqrt{-1} \xi t_\infty) |D_i^{-1} t_i^{-1}| s^{s+k/2} d^x t_\infty
\]
\[
= |D|^{-s} \sum_{i=1}^{h_F^\infty} \int_{(F \otimes \mathbb{R})_i^{\infty} / \hat{\omega}_{F_i}} \sum_{0 < |\xi| \leq |t_i|} a_\infty(\xi, h_i) N([t_i]^{-1})^{k/2} |N(\xi[t_i]^{-1})^{s+k/2} e_F(\sqrt{-1} \xi t_\infty) (\xi t_\infty)^{s+k/2} d^x (\xi t_\infty)
\]
\[
= |D|^{-s} \sum_{i=1}^{h_F^\infty} \frac{C(\xi[t_i]^{-1}, h)}{N(\xi[t_i]^{-1})^{s+k/2}} \int_{(F \otimes \mathbb{R})_i^{\infty}} e_F(\sqrt{-1} t_\infty) t_\infty^{s+k/2} d^x t_\infty
\]
\[
= |D|^{-s} D(s + k/2, h) \left( \frac{\Gamma(s + k/2)}{(2\pi)^{s+k/2}} \right)^n,
\]
which proves the assertion. Here the first equality follows from the definition of \( h_i \) ([Shi (2.7)]) and the fourth equality follows from the definition of \( C(\xi[t]^{-1}, h) \) ([L3]). \( \square \)

By evaluating the equation (3.3) at \( s = 0 \), we obtain the following:
Proposition 3.3. Let \( h = (h_i)_{1 \leq i \leq h_F^*} \in S_k(n, \mathbb{C}) \). Then, under the above notation, we have

\[
\sum_{i=1}^{h_F^*} \int_{\sqrt{-1}(F \otimes \mathbb{R})^*_{\eta} / \mathcal{O}_{F,+}^*} h_i(z) \text{Im}(z)^{k/2-1} dz_{J_F} = D(k/2, h) \left( \frac{\Gamma(k/2)}{2\pi^{k/2} \sqrt{1}} \right)^n.
\]

Now we prove Proposition 3.1. Let \( f \otimes \eta = ((f \otimes \eta)_i)_{1 \leq i \leq h_F^*} \in S_2(n_\eta, \mathbb{C}) \) be a Hecke eigenform defined by

\[
(f \otimes \eta)_i(z) = \sum_{0 \leq \xi \in [t_i]} a_\infty(\xi, f_i) \eta(\xi [t_i]^{-1}) e_F(\xi z)
\]

(Shi Proposition 4.4, 4.5), where \( n_\eta \) is the least common multiple of \( n, m_\eta^2 \), and \( m_\eta n \). Since \( D(s, f \otimes \eta) = D(s, f, \eta) \), by applying Proposition 3.3 to \( f \otimes \eta \) instead of \( h \), we have

\[
\tau(\eta^{-1}) \sum_{i=1}^{h_F^*} \int_{\sqrt{-1}(F \otimes \mathbb{R})^*_{\eta} / \mathcal{O}_{F,+}^*} (f \otimes \eta)_i(z) dz_{J_F} = \tau(\eta^{-1}) \frac{D(1, f, \eta)}{(2\pi)^{k/2} \sqrt{1}}.
\]

Thus it suffices to show that the left-hand side of (3.4) is equal to the left-hand side of the equation in Proposition 3.1. By [Shi (3.11)], we have

\[
\eta(\xi [t_i]^{-1}) \tau(\eta^{-1}) = \sum_{b \in (m_\eta^{-1} \mathcal{O}_F^{-1}[t_i]^{-1}/\mathcal{O}_F^{-1}[t_i]^{-1})^\times} \eta_b(b)^{-1} e_F(\xi b).
\]

Then the left-hand side of (3.4) is equal to

\[
\sum_{i=1}^{h_F^*} \int_{\sqrt{-1}(F \otimes \mathbb{R})^*_{\eta} / \mathcal{O}_{F,+}^*} \sum_{b_i \in S_i} \sum_{u \in T} a_\infty(\xi, f_i) \sum_{b_i \in S_i} \eta_b(b_i)^{-1} e_F(\xi b_i u^{-1}) e_F(\xi z) dz_{J_F}
\]

\[
= \sum_{i=1}^{h_F^*} \sum_{b_i \in S_i} \sum_{u \in T} f_i(z + b_i u^{-1}) dz_{J_F},
\]

which is the the left-hand side of the equation in Proposition 3.1 as desired. Here we may regard \( f_i(z + b_i) \) as a function on \( \sqrt{-1}(F \otimes \mathbb{R})^*_{\eta} / \mathcal{O}_{F,+}^* \) because \( f_i(uz + b_i) = f_i(z + b_i) \) for any \( u \in \mathcal{O}_F^{\times} \).

Furthermore, the integrals in the last line of this equation are independent of the choice of the lift \( b_i \) of \( \tilde{b}_i \) (because if \( \tilde{b}_i = \tilde{b}_i' \) then there is \( \gamma \in \Gamma_1(\mathcal{O}_F[t_i], n) \) such that \( b = \gamma(b') \) and \( \gamma(\infty) = \infty \)). Hence the integral depends only on the image \( b_i \) of \( b_i \) in \((m_\eta^{-1} \mathcal{O}_F^{-1}[t_i]^{-1}/\mathcal{O}_F^{-1}[t_i]^{-1})^\times / \mathcal{O}_F^{\times} \) and it shall be denoted by

\[
\int_{\sqrt{-1}(F \otimes \mathbb{R})^*_{\eta} / \mathcal{O}_{F,\eta^{-1}}^*} f_i(z + b_i) dz_{J_F}.
\]

We consider a Mellin transform in the anti-holomorphic case. Let \( W_G \) denote the Weyl group \( K_\infty / K_{\infty,+} \), which is identified with the set \( \{ w_J \mid J \subset J_F \} \), where, for each subset \( J \) of \( J_F \), \( w_J \in K_\infty \) such that \( w_{J_i} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) if \( i \in J \) and \( w_{J_i} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) if \( i \in J_F \setminus J \). Then \( W_G \) acts on the space \( \bigoplus_{J \subset J_F} S_2, J(n, \mathbb{C}) \) of Hilbert cusp forms via \( h \mapsto h_J := h[K_\infty w_J K_\infty] \).
Proposition 3.4. Under the same notation and assumptions as Proposition 3.1, we have

\[ f_{J,i}(z) = \sum_{\mu \in [t_i], \{\mu\} = J} a(\mu, f_{J,i})e_F(\mu z) \]
given by [Hida94, Theorem 6.1]. Here \( \{\mu\} = \{ \iota \in J_F | \mu^\iota > 0 \} \) Since the action of \( W_G \) is compatible with the Hecke operators \( T(q) \) and \( U(q) \), \( f_{J,i} \) is also a Hecke eigenform of all \( T(q) \) and \( U(q) \). Hence, by [Hida94, Corollary 6.2], the \( V(q) \)-eigenvalue \( a(\mu, f_{J,i})N([t_i]^{-1}) \) is equal to \( C(q, f) \), where \( q = \mu[t_i]^{-1} \) and \( V(q) = T(q) \) or \( U(q) \).

We define \( (f \otimes \eta)_J = ((f \otimes \eta)_{J,i})_{1 \leq i \leq h_F^+} \in S_{2J}(n_\eta, \mathbb{C}) \) by

\[ (f \otimes \eta)_J(z) = \sum_{\mu \in [t_i], \{\mu\} = J} a(\mu, f_{J,i})\eta(\nu_{J,i})\eta(\mu[t_i]^{-1})e_F(\mu z), \]

where \( \nu_{J,i} \in A_{F,\infty} \) such that \( \nu_{J,i} = 1 \) if \( \iota \in J \) and \( \nu_{J,i} = -1 \) if \( \iota \in J_F \setminus J \). Now the same argument as in the proof of Proposition 3.1 shows the following:

**Proposition 3.4.** Under the same notation and assumptions as Proposition 3.1, we have

\[ \sum_{i=1}^{h_F^+} \sum_{b_i \in S_i} \eta_i(b_i)^{-1} \int_{\sqrt{-1}(F \otimes \mathbb{R})_+^\infty} f_{J,i}(z + b_i)dz = \eta_{\infty}(\nu_{J,i})\tau(\eta^{-1}) \frac{D(1, f, \eta)}{(-2\pi \sqrt{-1})^n}. \]

Here \( \nu_{J,i} \in A_{F,\infty} \) such that \( \nu_{J,i} = 1 \) if \( \iota \in J \) and \( \nu_{J,i} = -1 \) if \( \iota \in J_F \setminus J \), and \( dz \) is defined by (3.5).

3.2. Relation between cohomology class and Dirichlet series. In this subsection, we give a cohomological description of the Dirichlet series defined by (1.10). We keep the notation in (3.1). We fix \( i \in \mathbb{Z} \) with \( 1 \leq i \leq h_F^+ \). We abbreviate \( \Gamma_1(\mathfrak{d}_F[t_i], n) \) to \( \Gamma \). We consider the Hilbert modular varieties \( Y(n) \) (defined by (1.11)) and \( Y_i \) (defined by (1.4)). Let \( C_i \) denote the set of all cusps of \( Y_i \).

We fix \( b_i \in S_i \). We consider the following subset \( H_{b_i} \) of \( \mathfrak{S}_{J,F}^* \):

\[ H_{b_i} := b_i + \sqrt{-1}(F \otimes \mathbb{R})_+^\infty \rightarrow \mathfrak{S}_{J,F}. \]

We define an action of \( \mathfrak{d}_{F,m_{\eta},+} \) on \( H_{b_i} \) by \( \varepsilon \ast (z_i)_{i \in J_F} = (\varepsilon^i z_i - (\varepsilon^i - 1)b_i)_{i \in J_F} \). Since \( (\varepsilon - 1)b_i \in \mathfrak{d}_{F}^{-1}[t_i]^{-1} \) for any \( \varepsilon \in \mathfrak{d}_{F,m_{\eta},+} \), we see that \( \varepsilon \ast (z_i)_{i \in J_F} \) is \( \Gamma \)-equivalent to \( (z_i)_{i \in J_F} \). Therefore we have \( H_{b_i}/\mathfrak{d}_{F,m_{\eta},+} \rightarrow Y_i \).

We extend it to a morphism on their compactifications as follows. Let \( (\mathfrak{S}_{J,F}^*)^{\text{BS}} \) denote the Borel–Serre compactification of \( \mathfrak{S}_{J,F}^* \), which is a locally compact manifold on which \( GL_2(F) \) acts (see, for example, [Ha §2.1], [Hida93 §1.8], [Hir §2.1]). We can describe the boundary of \( (\mathfrak{S}_{J,F}^*)^{\text{BS}} \) at the cusp \( \infty \) as follows. We put \( X = \{(y, x) \in (F \otimes \mathbb{R})_+^\infty \times (F \otimes \mathbb{R}) \mid \prod_{i \in J_F} y_i = 1 \} \). Then we have

\[ \mathfrak{S}_{J,F}^* \cong X \times (\mathbb{R}_+^\infty; (x_1 + \sqrt{-1}y_i)_{i \in J_F} \mapsto \left( \left( \prod_{i \in J_F} y_i \right)^{-\frac{1}{2}} y_i, x_i \right), \prod_{i \in J_F} y_i), \]

which is compatible with the action of \( \Gamma_\infty \). Here \( \Gamma_\infty \) denotes the stabilizer of \( \infty \) in \( \Gamma \), which acts trivially on the second factor of the right-hand side. The compactification of \( \mathfrak{S}_{J,F}^* \) at the cusp \( \infty \) is given by \( X \times (\mathbb{R}_+^\infty \cup \{\infty\}) \).

Since \( K_{\infty}w_JK_{\infty} = K_{\infty}w_J \), we have \( f_{J,i}(x) = f(xw_J) \). Then \( f_{J,i} \) has the Fourier expansion of the form

\[ f_{J,i}(z) = \sum_{\mu \in [t_i], \{\mu\} = J} a(\mu, f_{J,i})e_F(\mu z) \]
The action of $\Gamma_s$ image of manifold and its boundary at a cusp $s$ where $\Gamma_s \subset \text{SL}_2(F)$ such that $s = \alpha(\infty)$.

We define subsets $H_{b_i}^{bs}$, $\partial_{\infty}$, and $\partial_{b_i}$ of $X \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$ as follows. Let $X_{b_i}$ denote the image of $H_{b_i}$ in $X$ under the composition of the isomorphism $\tilde{\text{Y}}_{\text{UICHI}}$ HIRANO and the projection to $X$. We have $H_{b_i} \cong X_{b_i} \times \mathbb{R}_+$. We define $H_{b_i}^{bs}$, $\partial_{\infty}$, and $\partial_{b_i}$ by

$$H_{b_i}^{bs} = X_{b_i} \times (\mathbb{R}_{\geq 0} \cup \{\infty\}), \quad \partial_{\infty} = X_{b_i} \times \{\infty\}, \quad \partial_{b_i} = X_{b_i} \times \{0\}.$$

The action of $\mathfrak{o}_{F,m_{\eta},+}$ on $H_{b_i}$ extends canonically to an action on $H_{b_i}^{bs}$. We put $a_i = \begin{pmatrix} b_i & \sqrt{-1} \bar{y} \\ -1 & b_i \end{pmatrix}$. Note that $a_i(\infty) = b_i$. The embedding $H_{b_i} \hookrightarrow \mathfrak{Y}_{F,\mathfrak{m}_{\eta},+}(X \times \{\infty\})$ induces an $\mathfrak{o}_{F,m_{\eta},+}$-equivariant map

$$(3.6) \quad H_{b_i}^{bs} \cup \partial_{\infty} \to \mathfrak{Y}_{F,\mathfrak{m}_{\eta},+}(X \times \{\infty\})$$

because $a_i(\infty) = b_i + \sqrt{-1}y$. Therefore we have $H_{b_i}^{bs}/\mathfrak{o}_{F,m_{\eta},+} \to Y_{b_i}^{bs}$ and it induces

$$(3.7) \quad H_c^n(Y_{b_i}(A)) \to H_c^n(Y_{b_i}(A)) \simeq H^n(Y_{b_i}^{bs}, \partial(Y_{b_i}^{bs}); A)$$

$$\to H^n(H_{b_i}^{bs}/\mathfrak{o}_{F,m_{\eta},+}, \partial_{\infty}/\mathfrak{o}_{F,m_{\eta},+}, \partial_{b_i}/\mathfrak{o}_{F,m_{\eta},+}) \simeq H^n(H_{b_i}/\mathfrak{o}_{F,m_{\eta},+}, A)$$

for $A = \mathcal{O}$, $K$, or $\mathbb{C}$. Here $H_c^n(X, A)$ denotes the compactly supported cohomology of $X$ with coefficients in $A$, and $\partial(Y_{b_i}^{bs})$ denotes the boundary $\bigcup_{s \in \mathcal{C}} D_s$ of $Y_{b_i}^{bs}$.

We define the evaluation map

$$(3.8) \quad \text{ev}_{b_i,A} : \tilde{H}_c^n(Y_{b_i}(A)) \to A$$

by the composition of $(3.7)$ and the trace map $H_c^n(H_{b_i}/\mathfrak{o}_{F,m_{\eta},+}, A) \to A$, where

$$\tilde{H}_c^n(Y_{b_i}(A)) \simeq H_c^n(Y_{b_i}(A))/(A\text{-torsion})$$

Note that the definition of $\text{ev}_{b_i,A}$ depends only on $\bar{b}_i$ (because if $\bar{b}_i = \bar{b}_i'$, then there is $\gamma \in \Gamma$ such that $b = \gamma(b')$ and $\gamma(\infty) = \infty$ and hence it shall be denoted by $\text{ev}_{b_i,A}$.

Let us fix a Hilbert cusp form $f \in S_2(n, \mathcal{O})$. Let $[\omega_f]$ denote the associated cohomology class in $H^n(Y_{b_i}(\mathbb{C}, \mathcal{C})$. Let $[\omega_f]_c$ denote the compactly supported cohomology class in $H_c^n(Y_{b_i}(\mathbb{C})$ whose image in $H^n(Y_{b_i}(\mathbb{C})$ is $[\omega_f]$. By combining these observations and Proposition 3.3 we obtain the following cohomological description of the Dirichlet series defined by $\text{I.10}$.

**Proposition 3.5.** Let $f \in S_2(n, \mathcal{O})$ be a Hecke eigenform for all $T(\mathfrak{q})$ and $U(\mathfrak{q})$. Assume that $a[\omega_f]_c \in \tilde{H}_c^n(Y_{b_i}(A))$ for some $a \in A$. Then, under the same notation and assumptions as Proposition 3.5, we have

$$A(\eta) \equiv \sum_{b_i \in S_i} \sum_{\eta_i(b_i)} \eta_i(\bar{b}_i)^{-1} \text{ev}_{b_i,A}(a[\omega_f]_c) = a \tau(\eta^{-1}) \frac{D(1, f, \eta)}{(2\pi \sqrt{-1})^n}.$$

In the anti-holomorphic case, we also obtain the following cohomological description by the same argument as in the proof of Proposition 3.5 using Proposition 3.4 instead of Proposition 3.1.
Proposition 3.6. Under the same notation and assumptions as Proposition 3.2 and Proposition 3.3, we have

\[ A(\eta) \equiv \sum_{i=1}^{h_F} \sum_{b_i \in S_i} \eta_i(b_i)^{-1} e_{v_i} \left( \eta \left( a \right) \right)_{\mathbb{C}} \left( K_{\infty} w_j K_{\infty} \right) = a \eta_\infty (\nu_j) \tau(\eta)^{-1} \frac{D(1, f, \eta)}{(-2\pi \sqrt{-1})^n}. \]

3.3. Distribution property and interpolation property. In this subsection, we construct a \( \mathbb{C} \)-valued distribution attached to a Hilbert cusp form.

Let \( p \) be a prime number such that \( (p, 6F_n(p)) = 1 \). We decompose \( p \) into prime ideals \( p_i \) of \( \mathfrak{p}_F \). Let \( f \in S_2(\mathfrak{n}, \mathcal{O}) \) be a normalized Hecke eigenform for all \( T(q) \) and \( U(q) \) with character \( \varepsilon \). We assume that \( f \) is \( p \)-ordinary, that is, for each \( j \) with \( 1 \leq j \leq r \), the \( T(p_j) \)-eigenvalue \( C(p_j, f) \) is prime to \( p \). Let \( \alpha_{p_j} \) be a unit root of the polynomial

\[ X^2 - C(p_j, f) X + \varepsilon(p_j) N(p_j) = 0. \]

Note that

\[ K_0(n) \left( \begin{array}{cc} \overline{w}_p j & 0 \\ 0 & 1 \end{array} \right) K_0(n) = \prod_{u \in \mathfrak{p}_F/p_j} K_0(n) \left( \begin{array}{cc} 1 & u \\ \overline{w}_p & 0 \end{array} \right) \prod K_0(n) \left( \begin{array}{cc} \overline{w}_p j & 0 \\ 0 & 1 \end{array} \right). \]

Then we have

\[ C(p_j, f) f(x) = \sum_{u \in \mathfrak{p}_F/p_j} f(x) \left( \overline{w}_p j \uplus u \\ 0 \uplus 1 \right) + \varepsilon(p_j) f(x) \left( \overline{w}_p j \uplus 0 \\ 0 \uplus 1 \right). \]

For \( \nu = (v_j)_{1 \leq j \leq r} \in (\mathbb{Z}_{\geq 0})^r \), we simply write \( p^\nu = \prod_{1 \leq j \leq r} p_j^{v_j} \) if there is no risk of confusion. For a non-zero ideal \( \mathfrak{a} \) of \( \mathfrak{p}_F \), let \( v(a) = (v(a)_j)_{1 \leq j \leq r} \in (\mathbb{Z}_{\geq 0})^r \) such that \( \mathfrak{a} p^{-v(a)} \subset \mathfrak{p}_F \) and \( \mathfrak{a} p^{-v(a)} \) is prime to \( p \). We use the convention \( \alpha^{-v(a)} = \prod_{1 \leq j \leq r} \alpha_{p_j}^{v_j} \).

For a non-zero ideal \( \mathfrak{m} \) of \( \mathfrak{p}_F \) and \( \mathfrak{a} = (a_i)_{1 \leq i \leq h_F^+} \in \bigoplus_{1 \leq i \leq h_F^+} (\mathfrak{m}^{-1} \mathfrak{p}_F^{-1} [t_i]^{-1} / \mathfrak{p}_F^{-1} [t_i]^{-1}) \times / \hat{\mathfrak{p}}_F^\times \), under the same notation in [3.2], we put

\[ H_\mathfrak{a} := \bigoplus_{1 \leq i \leq h_F^+} H_{a_i} \quad \text{and} \quad \text{ev}_{H_\mathfrak{a}} := \bigoplus_{1 \leq i \leq h_F^+} \text{ev}_{a_i}. \]

We note that the image of \( H_\mathfrak{a} \) under the right multiplication of \( \overline{w}_p j \uplus 0 \uplus 1 \) is expressed as \( H_{a(p_j)} \), where the element \( a(p_j) = (a(p_j)_j)_{1 \leq i \leq h_F^+} \in \bigoplus_{1 \leq i \leq h_F^+} \mathfrak{p}_j \mathfrak{m}^{-1} \mathfrak{p}_F^{-1} [t_i]^{-1} \) is explicitly given as follows: under the same notation in [3.2] and a fixed splitting of \( \mathfrak{p}_F \), by computing modulo the left action of \( G(F) \) and the right action of \( K_1(n) \), \( x_i \begin{pmatrix} y_{\infty} & a_i \\ 0 & 1 \end{pmatrix} \infty \equiv \begin{pmatrix} 1 & -a_i \\ 0 & 1 \end{pmatrix} \mathfrak{m}^{-1} \mathfrak{p}_F^{-1} [t_i]^{-1} \begin{pmatrix} y_{\infty} & 0 \\ 0 & 1 \end{pmatrix} \) and

\[ x_i \begin{pmatrix} y_{\infty} & a_i \\ 0 & 1 \end{pmatrix} \infty \equiv \begin{pmatrix} 1 & -a_i \\ 0 & 1 \end{pmatrix} \mathfrak{m}^{-1} \mathfrak{p}_F^{-1} [t_i]^{-1} \begin{pmatrix} y_{\infty} & 0 \\ 0 & 1 \end{pmatrix} \infty \equiv \begin{pmatrix} 1 & -(a')^{-1} a_i \\ 0 & 1 \end{pmatrix} \mathfrak{m}^{-1} \mathfrak{p}_F^{-1} [t_i]^{-1} \begin{pmatrix} y_{\infty} & 0 \\ 0 & 1 \end{pmatrix} \infty \equiv \begin{pmatrix} 1 & -(a')^{-1} a_i \\ 0 & 1 \end{pmatrix} x_k \begin{pmatrix} t' y_{\infty} & 0 \\ 0 & 1 \end{pmatrix} \infty, \]

where \( a' \in F^\times \) and \( t' \in (F \otimes \mathbb{R})^\times \) such that \( D^{-1} t_i^{-1} \overline{w}_p j \uplus 0 \equiv a' D^{-1} t_i^{-1} u t' \) for some \( u' \in \hat{\mathfrak{p}}_F^\times \).

Let \( R(p_j) \) be an operation such that \( R(p_j) \text{ev}_{H_\mathfrak{a}} = \text{ev}_{a(p_j)} \) associated to \( H_{a(p_j)} \). We note that the operations \( R(p_j) \) for \( 1 \leq j \leq r \) are compatible with each other.
For a non-zero ideal \( m \) of \( \mathfrak{o}_F \), we define a map

\[
\mu_{\ell, \alpha}(-, m) : \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \rightarrow \mathbb{C}
\]

by the following: for \( \mathfrak{a} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \),

\[
(3.11) \quad \mu_{\ell, \alpha}(\mathfrak{a}, m) = \alpha^{-v(m)} \left( \prod_{q | p} (1 - \alpha_q^{-1}\varepsilon(q) R(q)) \right) \text{ev}_\mathfrak{a}(C([\omega_F]_m)).
\]

**Proposition 3.7** (distribution property). Let \( \mathfrak{f} \in S_2(n, \mathcal{O}) \) be a normalized Hecke eigenform for all \( T(q) \) and \( U(q) \) with character \( \varepsilon \) and \( p \)-ordinary. Then, for a non-zero ideal \( m \) of \( \mathfrak{o}_F \) and \( \mathfrak{a} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \), we have

\[
\sum_{\substack{\mathfrak{b} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \\ \mathfrak{b} \equiv \mathfrak{a} \pmod{p}}} \mu_{\ell, \alpha}(\mathfrak{b}, mp) = \mu_{\ell, \alpha}(\mathfrak{a}, m).
\]

Here \( \mathfrak{b} \) runs over a complete set of representatives of \( \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \), whose image in \( \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \) under the canonical map is equal to \( \mathfrak{a} \).

**Proof.** We claim that, for every prime ideal \( p \) of \( \mathfrak{o}_F \) dividing \( p \), non-zero ideal \( m \) of \( \mathfrak{o}_F \), and \( \mathfrak{a} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \),

\[
(3.12) \quad \sum_{\substack{\mathfrak{b} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \\ \mathfrak{b} \equiv \mathfrak{a} \pmod{p}}} \mu_{\ell, \alpha}(\mathfrak{b}, mp) = \mu_{\ell, \alpha}(\mathfrak{a}, m).
\]

Here \( \mathfrak{b} \) runs over a complete set of representatives of \( \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \), whose image in \( \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \) under the canonical map is equal to \( \mathfrak{a} \).

For the moment, we admit the claim \( (3.12) \). Let \( T' \) be a complete set of representatives of \( \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \) in \( \bigoplus_{1 \leq i \leq h^+_p} m^{-1}\mathfrak{p}_1^{-1} \cdots \mathfrak{p}_{r-1}^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1} \). Then the left-hand side in the proposition is equal to

\[
\sum_{\substack{\mathfrak{b}' \in T' \\ \mathfrak{b}' \equiv \mathfrak{a} \pmod{\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}}} } \sum_{\substack{\mathfrak{b} \in \bigoplus_{1 \leq i \leq h^+_p} (m^{-1}\mathfrak{o}_F^{-1}[t_i]^{-1}/\mathfrak{o}_F^{-1}[t_i]^{-1})^\times /\mathfrak{o}_F^\times \\ \mathfrak{b} \equiv \mathfrak{b}' \pmod{\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}}}} \mu_{\ell, \alpha}(\mathfrak{b}, mp).
\]

Hence, by applying \( (3.12) \) to the triple \((\mathfrak{p}, mp_1 \cdots mp_{r-1}, \mathfrak{b}')\) instead of \((\mathfrak{p}, m, \mathfrak{a})\), the left-hand side in the proposition is equal to

\[
\sum_{\substack{\mathfrak{b}' \in T' \\ \mathfrak{b}' \equiv \mathfrak{a} \pmod{\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}}} } \mu_{\ell, \alpha}(\mathfrak{b}', mp_1 \cdots mp_{r-1}).
\]

Now, by the same argument as above, we can prove the proposition as desired.
It remains to prove the claim (3.12). The left-hand side of (3.12) is equal to

\[
\alpha^{-v(mp)} \sum_{b \equiv a \pmod{p}} \left( \prod_{q|p} \left( 1 - \alpha_q^{-1} \varepsilon(q) R(q) \right) \right) \text{ev}_{\mathbb{L},C}([\omega_f]_c)
\]

\[
= \alpha^{-v(mp)} \sum_{b \equiv a \pmod{p}} \left( \prod_{q|p, q \neq p} \left( 1 - \alpha_q^{-1} \varepsilon(q) R(q) \right) \right) \text{ev}_{\mathbb{L},C}([\omega_f]_c)
\]

\[- \alpha^{-v(mp^2)} \varepsilon(p) \sum_{b \equiv a \pmod{p}} \left( \prod_{q|p, q \neq p} \left( 1 - \alpha_q^{-1} \varepsilon(q) R(q) \right) \right) R(p) \text{ev}_{\mathbb{L},C}([\omega_f]_c).
\]

We note that the first term above is equal to

\[
(3.13) \quad \alpha^{-v(mp^2)} (C(p, f) - \varepsilon(p) R(p)) \left( \prod_{q|p, q \neq p} \left( 1 - \alpha_q^{-1} \varepsilon(q) R(q) \right) \right) \text{ev}_{\mathbb{L},C}([\omega_f]_c)
\]

and the second term above is equal to

\[
(3.14) \quad -\alpha^{-v(mp^2)} \varepsilon(p) N(p) \left( \prod_{q|p, q \neq p} \left( 1 - \alpha_q^{-1} \varepsilon(q) R(q) \right) \right) \text{ev}_{\mathbb{L},C}([\omega_f]_c).
\]

Here the former (3.13) follows from (3.10) and the following: under the same notation in (1.1) a fixed splitting of (3.1), and the identification \( \mathfrak{o}_F/p \cong \mathcal{O}_F/t_i^{-1}/\mathcal{O}_F[t_i]^{-1} \), by computing modulo the left action of \( G(F) \) and the right action of \( K_1(n) \),

\[
x_i \left( \begin{array}{ccc}
\infty & a_i & \varpi p - u \\
0 & 1 & 0
\end{array} \right) \equiv \left( \begin{array}{ccc}
1 & -a_i & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
D_{-1}^{-1} & 1 & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
y_0 & 0 & \varpi p - u \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
0 & 1 & 0 \\
0 & 1 & 0
\end{array} \right) \\
\left( D_{-1}^{-1} + u \varpi p - 1 \right)_{x_i} \left( \begin{array}{ccc}
1 & -a_i + u D_{-1}^{-1} & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
D_{-1}^{-1} & 1 & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
y_0 & 0 & \varpi p - u \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
0 & 1 & 0 \\
0 & 1 & 0
\end{array} \right)_x \\
\left( \begin{array}{ccc}
1 & -a'' \cdot (a_i + u D_{-1}^{-1}) & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
D_{-1}^{-1} & 1 & 0 \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
y_0 & 0 & \varpi p - u \\
0 & 1 & 0
\end{array} \right) \left( \begin{array}{ccc}
0 & 1 & 0 \\
0 & 1 & 0
\end{array} \right),
\]

where \( a'' \in F^{\times} \) and \( t'' \in (F \otimes \mathbb{R})^{\times} \) such that \( D_{-1}^{-1} \varpi_p = a'' D_{-1}^{-1} u'' t'' \) for some \( u'' \in \widehat{\mathcal{O}}_F^{\times} \). Here we note that the (1, 2) entry of the first matrix in the last line is a lift of \( a_i \). Furthermore the latter (3.14) follows from that a complete set of such representatives in (3.12) is given by \( \{(u + a_i)/\varpi_p | u \in \mathfrak{o}_F/p \cong \mathcal{O}_F/t_i^{-1}/\mathcal{O}_F[t_i]^{-1}\} \) and the map \( \text{ev}_{(u+a_i),c} \) depends only on \( (u+a_i) \). Since \( \varepsilon(p) N(p) = -\alpha_p^2 + C(p, f) \alpha_p \) by (3.3), therefore we obtain the claim (3.12) as desired. 

\[\square\]

**Proposition 3.8** (interpolation property). Let \( f \in S_2(n, \mathbb{C}) \) be a normalized Hecke eigenform with character \( \varepsilon \) and \( p \)-ordinary. Let \( \eta \) be a narrow ray class character of \( F \), whose conductor
is denoted by $m_{\eta}$, such that $m_{\eta}$ is prime to $O_{F}[t_i]$ for each $i$, and $np|m_{\eta}$. Then we have

$$\sum_{b \in \Theta_{i \leq h_{F}^{+}}(m_{\eta}^{-1}O_{F}[t_{i}])^{-1}/O_{F}[t_{i}]^{-1})^\times /O_{F}[t_{i}]^\times = \prod_{\eta \in \Theta_{i \leq h_{F}^{+}}(m_{\eta}^{-1}O_{F}[t_{i}])^{-1}/O_{F}[t_{i}]^{-1})^\times /O_{F}[t_{i}]^\times ,$$

where $b$ runs over a complete set of representatives of $\prod_{\iota=1}^{h_{F}^{+}}(m_{\eta}^{-1}O_{F}[t_{i}])^{-1}/O_{F}[t_{i}]^{-1})^\times /O_{F}[t_{i}]^\times$.

Proof. By the definition (3.11) and Proposition 3.5, it suffices to show that, for each $m$, $\sum_{b} \eta(b_{i})^{-1}R(p_{1})\cdots R(p_{m})ev_{b,\iota}(\omega_{F}[c]) = 0$.

Note that, by the definition of $R(p_{j})$, the value $R(p_{1})\cdots R(p_{m})ev_{b,\iota}(\omega_{F}[c])$ depends on $b$ modulo $\prod_{\iota=1}^{h_{F}^{+}}(m_{\eta}^{-1}O_{F}[t_{i}])^{-1}/O_{F}[t_{i}]^{-1})^\times /O_{F}[t_{i}]^\times$. Therefore, our assertion follows from that $\eta$ is a primitive character.

4. Integrality of $p$-adic $L$-functions

In this subsection, we prove the integrality of $p$-adic $L$-functions attached to Hilbert cusp forms divided by the canonical periods. In order to show it, we need the integrality of the relative cohomology class attached to a Hilbert cusp form.

4.1. Partial Eichler–Shimura–Harder isomorphism. In this subsection, we recall the Eichler–Shimura–Harder isomorphism (4.1), where we use the assumption $h_{F}^{+} = 1$ ([Hira §4.2]).

Let $K$ be a finite extension of the field $\Phi_{p}$ defined in §1.4 and $O$ the ring of integers of $K$. For $A = O$, $K$, or $\mathbb{C}$, let $H^{\rho}_{o}(Y(n), A)$ denote the compactly supported cohomology of $Y(n)$ with coefficients in $A$, and let $H^{n}_{\text{par}}(Y(n), A)$ denote the parabolic cohomology of $Y(n)$ with coefficients in $A$, that is, $H^{n}_{\text{par}}(Y(n), A) = \text{im}(H^{n}_{o}(Y(n), A) \rightarrow H^{n}(Y(n), A))$. Let $\tilde{H}^{n}_{\text{par}}(Y(n), A)$ denote the torsion-free part of $H^{n}_{\text{par}}(Y(n), A)$, that is, $\tilde{H}^{n}_{\text{par}}(Y(n), A) = H^{n}_{\text{par}}(Y(n), A)$ for $A = K$ or $\mathbb{C}$ and

$$\tilde{H}^{n}_{\text{par}}(Y(n), O) = \text{im}(H^{n}_{\text{par}}(Y(n), O) \rightarrow H^{n}_{\text{par}}(Y(n), K)).$$

As mentioned in [Hida88 §7], $H^{n}_{\text{par}}(Y(n), A)$ is a $W_{G}$-module and hence so is $\tilde{H}^{n}_{\text{par}}(Y(n), A)$, where $W_{G}$ is the Weyl group explained before Proposition 3.4. In the case where $n$ is even, if a character $\epsilon$ of $W_{G}$ satisfies $\{\iota \in J_{F} \mid \epsilon(-1) = -1\} \neq n/2$, then we have

$$H^{n}_{\text{par}}(Y(n), \mathbb{C})[\epsilon] \simeq S_{2}(n, \mathbb{C})$$

as Hecke modules, where we use the assumption that $h_{F}^{+} = 1$ ([Hira §4.2, (4.7)]). Here $W_{G}$ is identified with $\{\pm 1\}^{J_{F}}$ by the determinant map, and for a $W_{G}$-module $V$, $V[\epsilon]$ denotes the $\epsilon$-isotypic part $\{v \in V \mid w \cdot v = \epsilon(w)v \text{ for all } w \in W_{G}\}$. Thus the Hecke algebra $\mathcal{H}_{2}(n, O)$ is isomorphic to the $O$-subalgebra of $\text{End}_{O}(\tilde{H}^{n}_{\text{par}}(Y(n), O)[\epsilon])$.

4.2. Canonical periods. In this subsection, we recall the definition of the canonical periods (4.1) ([Hira §6.1]).

We keep the notation in §1.4. Let $f \in S_{2}(n, O)$ be a normalized Hecke eigenform for all $T(q)$ and $U(q)$ with character $\epsilon$. We put the character $\epsilon_{f} = 1$ or $\text{sgn}J_{F}$ of $W_{G}$. Let $p_{f}$ denote the prime ideal of the Hecke algebra $\mathcal{H}_{2}(n, O)$ generated by $T(q) - C(q, f)$ and $S(q) - \epsilon^{-1}(q)$ for all non-zero prime ideals $q$ of $\mathfrak{o}_{F}$ prime to $n$, and $U(q) - C(q, f)$ for all non-zero prime ideals $q$ of $\mathfrak{o}_{F}$ dividing $n$. The isomorphism (4.1) and the $q$-expansion principle over $\mathbb{C}$ imply
that \( \dim_{\mathbb{C}} \left( \tilde{H}^n_{\text{par}}(Y(n), \mathcal{O})[\epsilon_f, p_d] \right) = 1 \) and \( \text{rank}_{\mathbb{O}} \left( \tilde{H}^n_{\text{par}}(Y(n), \mathcal{O})[\epsilon_f, p_d] \right) = 1. \) We choose a generator \( [\delta]^{tr}_f \) of \( \tilde{H}^n_{\text{par}}(Y(n), \mathcal{O})[\epsilon_f, p_d] \). Let \( [\omega_f]^{tr}_f \) denote the projection of \([\omega]\) to the \( \epsilon_f \)-isotypic part \( H^*_n(\mathbb{C}) \). We define the canonical period \( \Omega_f^{tr} \in \mathbb{C}^X \) of \( f \) by

\[
[\omega_f]^{tr}_f = \Omega_f^{tr}[\delta_f]^{tr}_f.
\]

Let \( C(\Gamma_1(0_F[t_1], n)) \) denote the set of all cusps of \( Y(n) \), and let \( C_{\infty} \) denote the subset of \( C(\Gamma_1(0_F[t_1], n)) \) consisting of cusps \( \Gamma_0(0_F[t_1], n) \)-equivalent to the cusp \( \infty \). Let \( D_{C_{\infty}}(n) \) denote the union of \( D_s \) for all \( s \in C_{\infty} \), where \( D_s \) is the boundary of \( Y(n) \) at a cusp \( s \) (cf. [Hi, §5.1], the Hecke correspondence \( U(q) \) preserves the component \( D_{C_{\infty}}(n) \). Let \( \mathbb{H}_2(n, \mathcal{O})' \) be the commutative \( \mathcal{O} \)-subalgebra of \( \text{End}\mathcal{O}(H^n(D_{C_{\infty}}(n), \mathcal{O})) \) generated by \( U(q) \) for all non-zero prime ideals \( q \) of \( \mathfrak{o}_F \) dividing \( n \), and \( m_f' \) the maximal ideal of \( \mathbb{H}_2(n, \mathcal{O})' \) generated by \( \varpi \) and \( U(q) - C(q, f) \) for all non-zero prime ideals \( q \) of \( \mathfrak{o}_F \) dividing \( n \).

As mentioned in [Hi, §2.4], by the relative Rham theory ([Bo, Theorem 5.2]), we can define the relative cohomology class \([\omega_f]_{\text{rel}} \in H^n(Y(n)_{\mathbb{B}}; D_{C_{\infty}}(n); \mathcal{O}) \) whose image in \( H^n(Y(n), \mathcal{O}) \) is \([\omega_f] \). Let \( [\delta_f]^{tr}_f \) denote the class \([\omega_f]_{\text{rel}}/\Omega_f^{tr} \) in \( H^n(Y(n)_{\mathbb{B}}; D_{C_{\infty}}(n); \mathcal{O}) \). We put

\[
\tilde{H}^n(Y(n)_{\mathbb{B}}; D_{C_{\infty}}(n); \mathcal{O}) = H^n(Y(n)_{\mathbb{B}}; D_{C_{\infty}}(n); \mathcal{O})/(\mathcal{O}-\text{torsion}),
\]

\[
H^n(Y(n), \mathcal{O}) = H^n(Y(n), \mathcal{O})/(\mathcal{O}-\text{torsion}).
\]

**Proposition 4.1.** Let \( f \in S_2(n, \mathcal{O}) \) be a normalized Hecke eigenform for all \( T(q) \) and \( U(q) \) with character \( \varepsilon \). Assume that \( H^n(\mathcal{D}_{C_{\infty}}(n), \mathcal{O})_{m_f'} \) is torsion-free, and \( C(q, f) \neq N(q) (\text{mod } \varpi) \) for some prime ideal \( q \) of \( \mathfrak{o}_F \) dividing \( n \). Then \([\delta_f]^{tr}_f \) is integral, that is,

\[
[\delta_f]^{tr}_f \in \tilde{H}^n(Y(n)_{\mathbb{B}}; D_{C_{\infty}}(n); \mathcal{O}).
\]

**Proof.** We abbreviate \( Y(n) \) to \( Y \) and \( D_{C_{\infty}}(n) \) to \( D_{C_{\infty}} \). We have an exact sequence

\[
H^{n-1}(D_{C_{\infty}}(n); \mathcal{O})_{m_f'} \rightarrow H^n(Y_{\mathbb{B}}; D_{C_{\infty}}; \mathcal{O})_{m_f'} \rightarrow H^n(Y; \mathcal{O})_{m_f'} \rightarrow H^n(D_{C_{\infty}}; \mathcal{O})_{m_f'}.
\]

By [Hi, Proposition 5.3], \( H^{n-1}(D_{C_{\infty}}; \mathcal{O})_{m_f'} \) is torsion. By the assumption, \( H^n(D_{C_{\infty}}; \mathcal{O})_{m_f'} \) is torsion-free. Therefore we obtain an exact sequence

\[
0 \rightarrow \tilde{H}^n(Y_{\mathbb{B}}; D_{C_{\infty}}; \mathcal{O})_{m_f'} \rightarrow \tilde{H}^n(Y; \mathcal{O})_{m_f'} \rightarrow H^n(D_{C_{\infty}}; \mathcal{O})_{m_f'}.
\]

Now the assertion follows from this exact sequence. \( \square \)

### 4.3. Integrality of \( p \)-adic \( L \)-functions for Hilbert modular forms.\(^{\spadesuit}\)

Let \( F_\infty \) denote the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \). We put \( \Gamma = \text{Gal}(F_\infty/F) \). Let \( \gamma \) be a topological generator of \( \Gamma \).

**Theorem 4.2.** Let \( f \in S_2(n, \mathcal{O}) \) be a normalized Hecke eigenform for all \( T(q) \) and \( U(q) \) with character \( \varepsilon \). Assume that \( f \) is \( p \)-ordinary. Let \( m \) be a non-zero ideal of \( \mathfrak{o}_F \) such that \( (m, p) = 1 \). Let \( g = (f \otimes 1_{\mathfrak{a}}) \otimes 1_m \in S_2(n', \mathcal{O}) \) (cf. [Sh, Proposition 4.4, 4.5]), where, for a non-zero ideal \( \mathfrak{a} \) of \( \mathfrak{o}_F \), \( 1_{\mathfrak{a}} \) is the trivial character modulo \( \mathfrak{a} \), and \( (n', p) = 1 \). Let \( \chi \) be a narrow ray class character of \( F \) whose conductor is \( n' \) such that \( \chi = \epsilon_f \) on \( W_G \). Assume that \( H^n(D_{C_{\infty}}(n'), \mathcal{O})_{m'_g} \) is torsion-free, where \( m'_g \) is the maximal ideal of \( \mathbb{H}_2(n', \mathcal{O})' \) defined before Proposition 4.1. If \( \chi \) is of type \( S \) (that is, \( F_\chi \cap F_\infty = F \), where \( F_\chi = \mathbb{Q}_{\ker(\chi)} \), then
there exists a $p$-adic $L$-function $L_p(f, \chi, T) \in \mathcal{O}(\chi)[[T]]$ satisfying the following interpolation property: for each finite order character $\rho$ of $\Gamma$ with conductor $\nu^\nu$,
\begin{equation}
L_p(f, \chi, \rho(\gamma) - 1) = \alpha^{-\nu^\nu} \tau(\chi^{-1} \rho^{-1}) \frac{D(1, f, \chi \rho)}{-2\pi i \Omega_{\rho}^{\nu} \Omega_{\rho}^{\nu}} \in \mathcal{O}(\chi, \rho).
\end{equation}
Here $\Omega_{\rho}^{\nu} \in \mathbb{C}^\times$ is the canonical period of $\rho \in S_2(n, \mathcal{O})$ defined by (4.2).

**Proof.** By Proposition 4.1, the class $[\delta_{\mu}]_{rel} = [\omega_{\delta}]_{rel} / \Omega_{\rho}^{\nu}$ is integral (because $C(q, g) = 0$ for every prime ideal $q$ dividing $n$). Hence Proposition 3.3 and 3.6 imply that the value in the right-hand side of (4.3) belongs to $\mathcal{O}(\chi, \rho)$ because $D(s, f, \chi \rho) = D(s, g, \chi \rho)$ and the fact as explained in [Hira, Proposition 2.5, 2.6] (that the evaluation maps factor through $H^n(Y(n')^{BS}, D_{C, \nu}(n'); \mathbb{C})$ as explained in [Hira, §2.4]), where we use the assumption that the conductor of $\chi$ is $n'$.

For a non-zero ideal $m$ of $\mathfrak{m}$ and $g \in (m^{-1} \Omega_{\rho}^{-1}[1]^{-1} \Omega_{\rho}^{-1}[1]^{-1})^{\times} / \Omega_{\rho}^{\nu}$, we define a map $\mu_{\nu}^{\nu}$ by
\begin{equation}
\mu_{\nu}^{\nu} = \nu(1 - \alpha_{\nu}^{-1} \varepsilon(q) R(q)) \varepsilon_{\mathfrak{m}, \mathfrak{m}, \mathfrak{m}}(\omega_{\mathfrak{m}}^{\nu})
\end{equation}

By Proposition 3.7, $\mu_{\nu}^{\nu}$ satisfies the distribution property. By Proposition 3.8, the distribution $\mu_{\nu}^{\nu} / \Omega_{\rho}^{\nu}$ interpolates the value in the right-hand side of (4.3). Now our assertion follows from the connection between $\mathcal{O}(\chi)$-valued measures on $\text{Gal}(F_\chi(\mu_{\nu}^{\nu}) / F)$ and elements of $\mathcal{O}(\chi)[[T]]$ (see, for example, [Del–Ri, R175, Sa]).

### 5. Equality between the Iwasawa $\lambda$-invariants

#### 5.1. The Iwasawa $\lambda$-invariants for Hilbert modular forms.

In this subsection, we define the analytic Iwasawa $\lambda$-invariant and state the main result of this paper.

We keep the notation in (4.3). Let $f \in S_2(n, \mathcal{O})$ be a normalized Hecke eigenform with character $\varepsilon$. We assume that $f$ is $p$-ordinary and $f$ satisfies the conditions (RR), (Parity), and $(\mu = 0)$ in (2.2). Let $n'$ be the least common multiple of $n^2$ and $m, m'$. We put $g = (f \otimes 1_n) \otimes 1_{m'} \in S_2(n', \mathcal{O})$ (cf. [Shi, Proposition 4.4, 4.5]). Let $H_2(n', \mathcal{O})$ be the commutative $\mathcal{O}$-subalgebra of $\text{End}_\mathbb{C}(H_2^n(Y(n'), \mathcal{O})) \oplus \text{End}_\mathbb{C}(H_2^n(Y(n'), \mathcal{O})) \oplus \text{End}_\mathbb{C}(H_2^{n+1}(Y(n'), \mathcal{O}))$ generated by $T(q), S(q)$ for all non-zero prime ideals $q$ of $\mathcal{O}_F$ prime to $n'$, and $U(q)$ for all non-zero prime ideals $q$ of $\mathcal{O}_F$ dividing $n'$, and $m$ the maximal ideal of $H_2(n', \mathcal{O})$ generated by $\mathfrak{m}$ and $T(q) - C(q, \mathfrak{m}), S(q) - \varepsilon(q) R(q)$ for all non-zero prime ideals $q$ of $\mathcal{O}_F$ prime to $n'$, and $U(q) - C(q, \mathfrak{m})$ for all non-zero prime ideals $q$ of $\mathcal{O}_F$ dividing $n'$.

Let $\chi$ be an $\mathcal{O}$-valued totally even narrow ray class character of $F$ whose conductor is $n'$. We define the analytic Iwasawa $\lambda$-invariant $\lambda_{\mathfrak{m}, \mathfrak{m}}^{\lambda}$ by
\begin{equation}
\lambda_{\mathfrak{m}, \mathfrak{m}}^{\lambda}(\mu_\mathfrak{m}) = \chi(L_\mathfrak{m}(f, \chi, T)) = \deg(P_{\chi}^{\lambda}(T)).
\end{equation}

Here $L_\mathfrak{m}(f, \chi, T) \in \mathcal{O}[[T]]$ is the $p$-adic $L$-function constructed in Theorem 4.2 and $P_{\chi}^{\lambda}(T)$ is the distinguished polynomial corresponding to $L_\mathfrak{m}(f, \chi, T)$ via the Weierstrass preparation theorem (see, for example, [Was, §7.1, Theorem 7.3]).

**Theorem 5.1.** Let $p$ be a prime number such that $p > n + 2$ and $p$ is prime to $n$ and $6\Delta_F$. Assume that $h_F^+ = 1$. Let $\chi$ be an $\mathcal{O}$-valued totally even narrow ray class character of $F$ satisfying the conditions at the beginning of (2.2). We assume the following two conditions:

(a) the local components $H_2^n(\partial(Y(n')^{BS}), \mathcal{O})_m$ and $H_2^{n+1}(Y(n'), \mathcal{O})_m$ are torsion-free;
(b) the local component \( H^n(D_{C_{\infty}}(n'), \mathcal{O}) \) is torsion-free, where \( m'_t \) is the maximal ideal of \( \mathbb{H}_2(n', \mathcal{O})' \) defined before Proposition 4.7.

Then we have

\[
\lambda^{\text{alg}}_{f \otimes \chi} = \lambda^{\text{an}}_{f \otimes \chi} = \lambda_{\varphi \chi, \Sigma_0} + \lambda_{\psi \chi, \Sigma_0}.
\]

5.2. **Proof of main result (Theorem 5.1)**. For the proof of Theorem 5.1 by the equality (2.8), it suffices to show that

\[
\lambda^{\text{an}}_{f \otimes \chi} = \lambda_{\varphi \chi, \Sigma_0} + \lambda_{\psi \chi, \Sigma_0}.
\]

It follows from a congruence between our \( p \)-adic \( L \)-function for \( g \) and the product of two Deligne–Ribet \( p \)-adic \( L \)-functions (Theorem 5.2) as explained below.

In order to state Theorem 5.2, we first define the \( p \)-adic \( L \)-functions \( \mathcal{L}_p(A_\varphi, \chi, T) \) and \( \mathcal{L}_p(A_\psi, \chi, T) \), and the non-primitive \( p \)-adic \( L \)-functions \( \mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, T) \) and \( \mathcal{L}_{p,0}^\Sigma_0(A_\psi, \chi, T) \) for the Galois representations \( A_\varphi \) and \( A_\psi \) appearing in (2.8) and (2.9). We put \( \Lambda = \mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]] ; \gamma \mapsto 1 + T \).

(i) The \( p \)-adic \( L \)-function \( \mathcal{L}_p(A_\varphi, \chi, T) \) is \( \Lambda \)-related to the Deligne–Ribet \( p \)-adic \( L \)-function \( L_p^{\text{DR}}(s, \chi \times \omega \psi^{-1}) \) by

\[
L_p^{\text{DR}}(s, \chi \times \omega \psi^{-1}) = \mathcal{L}_p(A_\varphi, \chi, \kappa(\gamma)^{-s} - 1)
\]

for any \( s \in \mathbb{Z}_p \). Here, \( \kappa(\gamma) \) is the element of \( 1 + p\mathbb{Z}_p \) which induces the action of \( \gamma \) on \( \mu_{p^\infty} \) under the identification \( \Gamma \simeq \text{Gal}(F(\mu_{p^\infty})/F(\mu_p)) \). The non-primitive \( p \)-adic \( L \)-function \( \mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, T) \) is \( \Lambda \)-related to the Deligne–Ribet \( p \)-adic \( L \)-function \( \mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, T) \) by

\[
\mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, \rho(\gamma) - 1) = L_p^{(s')}(0, \chi \times \omega \psi^{-1})
\]

for every non-trivial finite order character \( \rho \) of \( \Gamma \). Here \( L_p^{(s')}(s, *) \) denotes the complex \( L \)-functions formed from \( L_F(s, *) \) by omitting the Euler factors for non-zero prime ideals dividing \( s' \). Our assumption that \( \mu(\text{Sel}(F_{\infty}, A_{\varphi})^{\text{PD}}) = 0 \) and the Wiles theorem [W1990, Theorem 1.3, Theorem 1.4] (see also [Gre12, Proposition 9]) assert that \( \mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, T) \not\equiv \infty \Lambda \) and the \( \lambda \)-invariant of \( \mathcal{L}_{p,0}^\Sigma_0(A_\varphi, \chi, T) \) is equal to \( \lambda_{\chi \times \omega \psi^{-1}, \Sigma_0} = \lambda_{\varphi \chi, \Sigma_0} \).

(ii) The \( p \)-adic \( L \)-function \( \mathcal{L}_p(A_\psi, \chi, T) \) is \( \Lambda \)-related to the complex \( L \)-function \( L_p^{\text{DR}}(s, \chi \times \omega \psi^{-1}) \) by

\[
L_p^{\text{DR}}(s, \chi \times \omega \psi^{-1}) = \left( \frac{-1}{2^n \Delta_F^{1/2}} \right)^{-1} L_p(A_\psi, \chi, \kappa(\gamma)^s - 1)
\]

for every non-trivial finite order character \( \rho \) of \( \Gamma \). Here we remark that, by our assumption, \( \chi^{-1} \omega \psi^{-1} \) is non-trivial character. Then, \( \mathcal{L}_p(A_\psi, \chi, T) \) is related to the Deligne–Ribet \( p \)-adic \( L \)-function \( L_p^{\text{DR}}(s, \chi^{-1} \omega \psi^{-1}) \) by

\[
L_p^{\text{DR}}(s, \chi^{-1} \omega \psi^{-1}) = \left( \frac{-1}{2^n \Delta_F^{1/2}} \right)^{-1} \mathcal{L}_p(A_\psi, \chi, \kappa(\gamma)^s - 1)
\]
for any $s \in \mathbb{Z}_p$. The non-primitive $p$-adic $L$-function $\mathcal{L}_p^{\Sigma_0}(A_\psi, \chi, T) \in \Lambda$ is defined by the interpolation property
\[
\mathcal{L}_p^{\Sigma_0}(A_\psi, \chi, \rho(\gamma) - 1) = \tau(\chi^{-1} \psi^{-1} \rho^{-1}) \frac{L_F^{\{n\}}(1, \chi \psi \rho)}{(-2 \pi \sqrt{-1})^n}
\]
for every non-trivial finite order character $\rho$ of $\Gamma$. Again by the Wiles theorem, the $\mu$-invariant of $\mathcal{L}_p(A_\psi, \chi, T)$ is zero and its $\lambda$-invariant is equal to $\lambda_{\chi^{-1} \psi^{-1} \rho} = \lambda_{\psi \chi}$.

Next, we define the $p$-adic $L$-function $\mathcal{L}_p(G, \chi, T) \in \Lambda$ for the Eisenstein series $G \in M_2(n', \mathcal{O})$ with character $\varepsilon$ characterized by
\[
D(s, G) = L_F^{\{n'\}}(s, \psi)L_F^{\{n\}}(s - 1, \varepsilon \psi^{-1}).
\]
Note that Theorem 6.1 assures the existence of such $G$.

(iii) The $p$-adic $L$-function $\mathcal{L}_p(G, \chi, T) \in \Lambda$ is defined by the interpolation property
\[
\mathcal{L}_p(G, \chi, \rho(\gamma) - 1) = \tau(\chi^{-1} \psi^{-1} \rho^{-1}) \frac{D(1, G, \chi \rho)}{(-2 \pi \sqrt{-1})^n}
= L_F^{\{n'\}}(0, \chi \varepsilon \psi^{-1} \rho) \tau(\chi^{-1} \psi^{-1} \rho^{-1}) \frac{L_F^{\{n\}}(1, \chi \psi \rho)}{(-2 \pi \sqrt{-1})^n}
\]
for every non-trivial finite order character $\rho$ of $\Gamma$. Then clearly we have
\[
\mathcal{L}_p(G, \chi, T) = \mathcal{L}_p^{\Sigma_0}(A_\psi, \chi, T)\mathcal{L}_p^{\Sigma_0}(A_\psi, \chi, T).
\]
Therefore, the $\mu$-invariant of $\mathcal{L}_p(G, \chi, T)$ is zero and the $\lambda$-invariant of $\mathcal{L}_p(G, \chi, T)$ is equal to $\lambda_{\chi \psi \rho}$. Now Theorem 5.1 follows from the following:

**Theorem 5.2.** Under the same assumptions as Theorem 5.1, we have
\[
\mathcal{L}_p(f, \chi, T) \equiv U(T)\mathcal{L}_p(G, \chi, T) \pmod{\varpi \Lambda}.
\]
Here $U(T)$ is a unit in $\Lambda^\times$ characterized by
\[
U(\rho(\gamma) - 1) = u' \alpha^{-\nu_\rho} \frac{\tau(\chi^{-1} \rho^{-1})}{\tau(\chi^{-1} \psi^{-1} \rho^{-1})}
\]
for every non-trivial finite order character $\rho$ of $\Gamma$ with conductor $p^{\nu_\rho}$, and some $p$-adic unit $u' \in \mathcal{O}^\times$. In particular, we have
\[
\lambda_{f \otimes \chi}^{an} = \lambda_{\chi \psi \rho} + \lambda_{\chi \psi \rho}.\varepsilon.
\]
**Proof.** By our assumption, in order to apply [Hira, Theorem 6.1] to the quadruplet $(g, G, n', \chi \rho)$ instead of $(f, E, n, \eta)$, it suffices to check the congruence $g \equiv G \pmod{\varpi}$, that is,
\[
C(q, g) \equiv C(q, G) \pmod{\varpi}
\]
for every prime ideal $q$ of $\mathcal{O}_F$. For every prime ideal $q$ of $\mathcal{O}_F$ such that $q \nmid n'p$, we have
\[
C(q, g) = C(q, f) = \text{Tr}(\rho_T(\text{Frob}_q))
\]
\[
\equiv \psi(\text{Frob}_q) + \varphi(\text{Frob}_q)
\equiv \psi(\text{Frob}_q) + \det(\rho_T)\psi^{-1}(\text{Frob}_q)
= \psi(\text{Frob}_q) + \varepsilon \chi_{\text{cycl}} \psi^{-1}(\text{Frob}_q)
= \psi(q) + \varepsilon(q)\psi^{-1}(q)N(q)
= C(q, G) \pmod{\varpi}.
\]
For every prime ideal \( \mathfrak{p} \) of \( \mathcal{O}_F \) lying above \( p \), by (2.1), (RR), and (Parity), we obtain
\[
C(p, g) = C(p, f) \equiv \psi(\text{Frob}_p) \equiv C(p, G) \pmod{\varpi}.
\]
Moreover, by the definitions of \( g \) and \( G \), for every prime ideal \( \mathfrak{q} \) of \( \mathcal{O}_F \) dividing \( n' \), we have
\[
C(q, g) = 0, \quad C(q, G) = 0.
\]
Thus, by applying [Hira, Theorem 6.1], we get the congruence
\[
\tau(\chi^{-1} \rho^{-1}) \frac{D(1, g, \chi \rho)}{(-2\pi \sqrt{-1})^n \Omega_g^1} \equiv u' \tau(\chi^{-1} \rho^{-1}) \frac{D(1, G, \chi \rho)}{(-2\pi \sqrt{-1})^n} \pmod{\varpi}
\]
for every non-trivial finite order character \( \rho \) of \( \Gamma \) with conductor \( p^{n_r} \), where \( u' \) is a unit in \( \mathcal{O}^\times \). Therefore we obtain
\[
\mathcal{L}_p(f, \chi, \rho(\gamma) - 1) = \alpha^{-\nu_r} \tau(\chi^{-1} \rho^{-1}) \frac{D(1, g, \chi \rho)}{(-2\pi \sqrt{-1})^n \Omega_g^1}
\equiv u' \alpha^{-\nu_r} \tau(\chi^{-1} \rho^{-1}) \frac{D(1, G, \chi \rho)}{(-2\pi \sqrt{-1})^n}
= u' \alpha^{-\nu_r} \frac{\tau(\chi^{-1} \rho^{-1})}{\tau(\chi^{-1} \rho^{-1})} \tau(\chi^{-1} \rho^{-1}) \frac{D(1, G, \chi \rho)}{(-2\pi \sqrt{-1})^n}
= U(\rho(\gamma) - 1) \mathcal{L}_p(G, \chi, \rho(\gamma) - 1) \pmod{\varpi}
\]
for every non-trivial finite order character \( \rho \) of \( \Gamma \) with conductor \( p^{n_r} \). This proves the congruence (5.2). Now the equality (5.3) follows from (5.2) and \( \mu(\mathcal{L}_p(G, \chi, T)) = 0 \). \( \square \)

6. Modularity of residually reducible representations

6.1. Modularity of residually reducible representations. In this subsection, we prove a modularity theorem of residually reducible representations.

In order to do it, we recall the definition and properties of Hilbert Eisenstein series of parallel weight 2 (see, for more detail, [Shi §3], [Da–Da–Po §2]):

**Theorem 6.1.** Let \( \psi_1 \) be \( \mathcal{O} \)-valued narrow ray class characters of \( F \), whose conductor are denoted by \( m_{\psi_1} \), with sign \( \tau_i \in (\mathbb{Z}/2\mathbb{Z})^r \) for \( i = 1, 2 \). We put \( n = m_{\psi_1}m_{\psi_2} \) and \( \varepsilon = \psi_1\psi_2 \). If \( \varepsilon \) is totally even, then there exists \( E_2(\psi_1, \psi_2) = (E_2(\psi_1, \psi_2), 1)_{1 \leq i \leq h_F^+} \in M_2(n, \mathcal{O}) \) with character \( \varepsilon \), called a Hilbert Eisenstein series, such that
\[
D(s, E_2(\psi_1, \psi_2)) = L_F(s, \psi_1)L_F(s - 1, \psi_2),
\]
\[
C(a, E_2(\psi_1, \psi_2)) = \sum_{c | a} \psi_1 \left( \frac{a}{\chi} \right) \psi_2(c) N(c) \quad \text{for each non-zero ideal} \, a \, \text{of} \, \mathcal{O}_F.
\]

Hereafter we assume that \( h_F^+ = 1 \). Let \( n \) be a non-zero ideal of \( \mathcal{O}_F \) prime to \( 6p\Delta_F \) and \( \mathfrak{a}_F[t_1] \). We fix narrow ray class characters \( \varphi \) and \( \psi \) of \( F \) satisfying the conditions (\( \mu = 0 \)), \( m_{\varphi}m_{\psi} = n \), and

(Eis) \( \varphi \) and \( \psi \) are \( \mathcal{O} \)-valued totally even (resp. totally odd) such that \( \varphi \) is non-trivial.

Then \( \varphi \) and \( \psi \) satisfy the condition [Hira §5, (Eis condition)].

Let \( E \) denote the Hilbert Eisenstein series \( E_2(\varphi, \psi) \in M_2(n, \mathcal{O}) \) attached to \( \varphi \) and \( \psi \) as Theorem 6.1. We define the character \( \varepsilon_E \) of the Weyl group \( W_G \) by \( \varepsilon_E = \text{sgn}^i \tau \) (resp. \( \varepsilon_E = 1 \)) if both \( \varphi \) and \( \psi \) are totally even (resp. totally odd). We put \( \varepsilon = \varphi \psi \).
For each cusp $s \in C(\Gamma_1(\mathfrak{d}_F[t_1], \mathfrak{n}))$, let $a_s(0, E_1)$ denote the constant term of $E_1$ at $s$ (cf. [13]). Let $s_0 \in C(\Gamma_1(\mathfrak{d}_F[t_1], \mathfrak{n}))$ such that $v_p(a_{s_0}(0, E_1)) \leq v_p(a_s(0, E_1))$ for each $s \in C(\Gamma_1(\mathfrak{d}_F[t_1], \mathfrak{n}))$, where $v_p$ denotes the $p$-adic valuation. We put
\[ C = a_{s_0}(0, E_1). \]

**Theorem 6.2.** Let $p$ be a prime number $> 3$ such that $p$ is prime to $\mathfrak{n}$ and $\Delta_F$. We assume the following conditions and the following three conditions:

(a) $H^n(\partial (Y(\mathfrak{n})^{BS}), \mathcal{O})_m$, $H^{n+1}(Y(\mathfrak{n}), \mathcal{O})_m$, and $H^n(D_{C, \omega}(\mathfrak{n}), \mathcal{O})_{m_E}$ are torsion-free, where $m$ (resp. $m'_E$) is the Eisenstein maximal ideal of the Hecke algebra $\mathcal{H}_2(\mathfrak{n}, \mathcal{O})$ (resp. $\mathcal{H}_2(n, \mathcal{O}')$) defined before [Hira, Theorem 5.5]; (resp. [Hira, Proposition 5.3]);

(b) $C(q, \mathbf{E}) \equiv N(q) (mod \wp)$ for some prime ideal $q$ dividing $\mathfrak{n}$;

(c) the ideal $(C) \neq 0, \mathcal{O}$.

Then there exist a finite extension $K'$ of $K$ with the ring of integers $\mathcal{O} \to \mathcal{O}'$ and a uniformizer $\wp'$ such that $(\wp') \cap \mathcal{O} = (\wp)$, and a Hecke eigenform $f \in S_2(\mathfrak{n}, \mathcal{O}')$ for all $T(q)$ and $U(q)$ with character $\varepsilon$ such that $f \equiv \mathbf{E} (mod \wp')$, that is, for every non-zero ideal $\mathfrak{a}$ of $\mathfrak{o}_F$,
\[ C(\mathfrak{a}, f) \equiv C(\mathfrak{a}, \mathbf{E}) (mod \wp'). \]

**Proof.** We use the same notation as in the proof of [Hira, Theorem 5.5]. By the proof of [Hira, Theorem 5.5, (5.8)], the assumption [14] implies that there exists a non-zero class $e_0 \in H^n_{par}(Y(\mathfrak{n}), \mathcal{O})[\varepsilon]), \omega \cap H^n_{par}(Y(\mathfrak{n}), \mathcal{O})[\varepsilon])$ such that $e_0$ is cohomologous to $-\omega_E$ modulo $\wp$, and the Hecke eigenvalues of $e_0$ are the same as those of $-\omega_E$ modulo $\wp$ for all $t \in \mathcal{H}_2(\mathfrak{n}, \mathcal{O})$. Now the Deligne–Serre lifting lemma ([Del–Ser, Lemma 6.11]) in the case where $R = \mathcal{O}$, $M = H^n_{par}(Y(\mathfrak{n}), \mathcal{O})[\varepsilon])$, and $T = \mathcal{H}_2(\mathfrak{n}, \mathcal{O})$ says that there exist a finite extension $K'$ of $K$ with the ring of integers $\mathcal{O} \to \mathcal{O}'$ and a uniformizer $\wp'$ such that $(\wp') \cap \mathcal{O} = (\wp)$, and a non-zero eigenvector $e \in H^n_{par}(Y(\mathfrak{n}), \mathcal{O})[\varepsilon]) \otimes \mathcal{O}'$ for all $t \in T$ with eigenvalues $\lambda(V(q))$ such that
\[ \lambda(V(q)) \equiv C(q, \mathbf{E}) (mod \wp'). \]

for all non-zero prime ideals $q$ of $\mathfrak{o}_F$ prime to $\mathfrak{n}$ (resp. dividing $\mathfrak{n}$) and $V(q) = T(q)$ (resp. $U(q)$). By the isomorphism [14, (1.1)], we obtain a Hecke eigenform $f \in S_2(\mathfrak{n}, \mathcal{C})$ for all $T(q)$ and $U(q)$ such that $e = [\omega_E]$. By using the relation between Hecke eigenvalues and Fourier coefficients, we may assume that $f \in S_2(\mathfrak{n}, \mathcal{O}')$ with character $\varepsilon$. Therefore we obtain the desired congruence $f \equiv \mathbf{E} (mod \wp')$. \hfill $\Box$

Let $\kappa'$ denote the residue field of $\mathcal{O}'$.

**Corollary 6.3.** Under the same notation and assumptions as Theorem 6.2, the residual Galois representation $\overline{\rho}_f = \rho_f (mod \wp') : G_F \to GL_2(\kappa')$ associated to $f$ is reducible and of the form
\[ \overline{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi_{\chi_{cyc}} \end{pmatrix} (mod \wp'). \]

**Proof.** By Theorem 6.2 the Hecke eigenform $f \in S_2(\mathfrak{n}, \mathcal{O}')$ satisfies the congruences
\[ C(\mathfrak{a}, f) \equiv \sum_{c|a} \varphi \left( \frac{a}{c} \right) \psi(c) \chi_{cyc}(c) (mod \wp') \]
for every non-zero ideal $\mathfrak{a}$ of $\mathfrak{o}_F$. Let $\overline{T}$ denote a 2-dimensional $\kappa'$-vector space with the action of $G_F$ via $\overline{\rho}_f$. For the proof, it suffices to show that all of the constituents of $\overline{T}$ are
isomorphic to \( \kappa'(\varphi) \) or \( \kappa'(\psi \omega) \). We write \( \rho \) for \( \mathcal{P}_F \) to simplify the notation. Let
\[
W = \mathcal{T} \oplus \mathcal{T}^* ,
\]
where \( \mathcal{T}^* = \text{Hom}(\mathcal{T}, \kappa'(\varepsilon \omega)) \). We fix a prime ideal \( q \) of \( \mathcal{O}_F \) prime to \( np \). We consider the characteristic polynomial of \( \text{Frob}_q \) acting on \( W \). We note that \( \rho \) satisfies the relation
\[
\rho(\text{Frob}_q)^2 - C(q, f)\rho(\text{Frob}_q) + \varepsilon(q)\chi_{\text{cyc}}(q) = 0 .
\]
Let \( \alpha(q) \) and \( \beta(q) \) denote the solutions of \( X^2 - C(q, f)X + \varepsilon(q)\chi_{\text{cyc}}(q) = 0 \). Let \( G \) denote a finite quotient of \( G_F \) through which the action acting on \( W \) factors. Let \( N_{\alpha(q)} \) and \( N_{\beta(q)} \) denote the generalized eigenspaces of \( \rho(\text{Frob}_q) \) with respect to \( \alpha(q) \) and \( \beta(q) \), respectively. Then we have \( \mathcal{T} = N_{\alpha(q)} \oplus N_{\beta(q)} \). Since the operation \( \text{Hom}(*, \kappa'(\varepsilon \omega)) \) interchanges the eigenvalues of the action of \( \text{Frob}_q \), the characteristic polynomial of \( \text{Frob}_q \) acting on \( W \) is \( (X - \alpha(q))^2(X - \beta(q))^2 \). On the other hand, the congruence \( (6.1) \) implies that the characteristic polynomial of \( \text{Frob}_q \) acting on \( \kappa'(\varphi)^{\mathfrak{O}_2} \oplus \kappa'(\psi \omega)^{\mathfrak{O}_2} \), which is regarded as a \( G \)-module, is also \( (X - \alpha(q))^2(X - \beta(q))^2 \). By the Chebotarev density theoremCany element of \( G \) is the image of some \( \text{Frob}_q \) such that \( q \) is prime to \( np \). Therefore, by the Brauer–Nesbitt theorem, we obtain
\[
W^{\text{ss}} \simeq \kappa'(\varphi)^{\mathfrak{O}_2} \oplus \kappa'(\psi \omega)^{\mathfrak{O}_2} .
\]
Here \( W^{\text{ss}} \) denotes the semisimplification of \( W \). Hence there exists a filtration
\[
0 = T_0 \subseteq T_1 \subseteq T_2 = \mathcal{T}
\]
of \( \mathcal{T} \) such that, for each \( i \) with \( 1 \leq i \leq 2 \),
\[
T_i/T_{i-1} \simeq \kappa'(\alpha_i) ,
\]
where \( \alpha_i = \varphi \) or \( \psi \omega \). Since \( \rho \) is totally odd, we have \( \alpha_1 \neq \alpha_2 \). This proves the corollary. \( \square \)

6.2. Real quadratic field case. In this subsection, we give examples satisfying all the assumption of the Main Theorem \( 0.1 \) by using Corollary \( 6.3 \). In order to do it, we prove \( [\mathfrak{w}] \) of Theorem \( 6.2 \) in certain case (Proposition \( 6.4 \) and \( 6.5 \)) and give a Hilbert Eisenstein series satisfying \( [\mathfrak{l}] \) and \( [\mathfrak{x}] \) of Theorem \( 6.2 \) (Example \( 6.6 \)).

In this subsection, we assume that \( F \) is a real quadratic field with \( h^+_F = 1 \). The following proposition is obtained by [Hira] Proposition 5.8:

Proposition 6.4. Assume that \( n \) is prime to \( 6\Delta_F \). If \( p \) is prime to \( 6n \) and \( \mathfrak{p}(\mathfrak{o}_{F,+})/\mathfrak{o}_{F,n}^{\times 2} \), then \( H^2(F(Y(n), \mathcal{O}) \) is torsion-free.

Let \( u \) denote the fundamental unit of \( F \). We put \( u_+ = u \) (resp. \( u_+^2 \)) if \( N(u) = 1 \) (resp. \( N(u) = -1 \)). Let \( \langle n \rangle \) denote the index \( [\Gamma_1(\mathfrak{o}_F[t_1], \mathfrak{o}_F) : \Gamma_1(\mathfrak{o}_F[t_1], \mathfrak{n})] \) of the subgroup \( \Gamma_1(\mathfrak{o}_F[t_1], \mathfrak{n}) \) in \( \Gamma_1(\mathfrak{o}_F[t_1], \mathfrak{o}_F) \), where the bar means image in \( \text{GL}_2(F) / (\text{GL}_2(F) \cap F^\times) \).

Proposition 6.5. Assume that \( n \) is prime to \( 6\Delta_F \). If \( p \) is prime to \( \iota(n) \) and \( u_+^{\langle n \rangle} - 1 \), then \( H^2(\partial (Y(n)_{\mathfrak{BS}}), \mathcal{O}) \) is torsion-free.

Proof. We write \( \Gamma \) for \( \Gamma_1(\mathfrak{o}_F[t_1], \mathfrak{n}) \) to simplify the notation. We may assume \( \Gamma = \Gamma_1(\mathfrak{o}_F, \mathfrak{n}) \) by taking conjugation. For the proof, it suffices to show that \( H^2(\Gamma_1, \mathcal{O}) = H^2(\alpha^{-1}\Gamma_1 \cap B_\infty, \mathcal{O}) \) is torsion-free for each cusp \( s \in C(\Gamma) \), where \( \alpha \in \text{SL}_2(\mathfrak{o}_F) \) such that \( \alpha(\infty) = s \), and \( B_\infty \) denotes the standard Borel subgroup of upper triangular matrices. We simply write
\[
G = \alpha^{-1}\Gamma_1(\mathfrak{o}_F, \mathfrak{n}) \alpha \cap B_\infty , \quad N = \alpha^{-1}\Gamma_1 \cap B_\infty .
\]
We note that \( \alpha^{-1}\Gamma_1(\mathfrak{o}_F, \mathfrak{n}) \alpha = \text{GL}_2(\mathfrak{o}_F) \). As mentioned in [Gha, p.260], for the proof, it suffices to show that \( H^1(\mathfrak{N}, K/\mathcal{O}) \) is divisible. By (the proof of) [Gha] §3.4.2, Proposition
where \( \overline{N}^{ab} \) and \( \overline{G}^{ab} \) denote the maximal abelian quotients of \( \overline{N} \) and \( \overline{G} \), respectively. Let \( \psi : \overline{N}^{ab} \to \overline{G}^{ab} \) denote the canonical morphism induced by \( \overline{N} \hookrightarrow \overline{G} \). We have a diagram

\[
\begin{array}{ccc}
0 & \to & \overline{N}^{ab} \\
\downarrow & & \downarrow \\
\ker(\psi) & \to & \overline{N}^{ab} \xrightarrow{\psi} \overline{G}^{ab} \xrightarrow{\cok(\psi)} 0.
\end{array}
\]

Then the assumption on \( \iota(n) \) implies that \( \cok(\psi) \) has finite order prime to \( p \). Hence \( \psi \) induces an exact sequence

\[
0 \to \Hom(\overline{G}^{ab}, K/O) \to \Hom(\overline{N}^{ab}, K/O) \to \Hom(\ker(\psi), K/O) \to 0.
\]

Thus it is enough to show that

\[(6.2) \quad \Hom(\ker(\psi), K/O) = 0.\]

We simply write

\[
\overline{G}_{T_\infty} = \overline{G}/(\overline{G} \cap U_\infty), \quad \overline{N}_{T_\infty} = \overline{N}/(\overline{N} \cap U_\infty),
\]

where \( U_\infty \) (resp. \( T_\infty \)) denotes the unipotent radical (resp. the standard torus) of \( B_\infty \). We have the exact sequence

\[(6.3) \quad 0 \to \overline{N} \cap U_\infty \to \overline{N} \to \overline{N}_{T_\infty} \to 0.\]

We note that \( \overline{N}_{T_\infty} \) acts on \( \overline{N} \cap U_\infty \) via the exact sequence \( (6.3) \). Let \( \varphi_N : \overline{N}^{ab} \to \overline{N}_{T_\infty} \) denote the morphism induced by \( \overline{N} \to \overline{N}_{T_\infty} \), and let \( N' \) denote the image of the morphism \( (\overline{N} \cap U_\infty)_{T_\infty} \to \overline{N}^{ab} \) induced by \( \overline{N} \cap U_\infty \to \overline{N} \to \overline{N}^{ab} \). We note that \( \ker(\varphi_N) \subset N' \). Since the morphism \( \overline{N}_{T_\infty} \to \overline{G}_{T_\infty} \) is injective, we have \( \ker(\psi) \subset \ker(\varphi_N) \) and hence \( \ker(\psi) \subset N' \).

Thus we obtain

\[(6.4) \quad \Hom(\ker(\psi), K/O) \hookrightarrow \Hom(N', K/O) \hookrightarrow \Hom((\overline{N} \cap U_\infty)_{\overline{N}_{T_\infty}}, K/O).\]

Now, for the proof of \( (6.2) \), it suffices to show that the last term

\[
\Hom((\overline{N} \cap U_\infty)_{\overline{N}_{T_\infty}}, K/O) = 0.
\]

Since \( \overline{G}_{T_\infty} \simeq \sigma_{F,+}(\text{by } \text{Gha} \ \S 3.4.2)) \), we have \( (\sigma_{F,+})^{\iota(n)}(\overline{N}_{T_\infty}) \hookrightarrow \overline{G}_{T_\infty} \hookrightarrow \sigma_{F,+}. \) It induces \( (\overline{N} \cap U_\infty)_{(\sigma_{F,+})^{\iota(n)}} \hookrightarrow (\overline{N} \cap U_\infty)_{\overline{N}_{T_\infty}} \), which implies

\[
\Hom((\overline{N} \cap U_\infty)_{\overline{N}_{T_\infty}}, K/O) \hookrightarrow \Hom((\overline{N} \cap U_\infty)_{(\sigma_{F,+})^{\iota(n)}}, K/O).
\]

Since \( \overline{G} \cap U_\infty \simeq \sigma_F \) (by \text{Gha} \ \S 3.4.2), we have \( \iota(n)\sigma_F \hookrightarrow \overline{N} \cap U_\infty \hookrightarrow \sigma_F. \) Hence the assumption on \( \iota(n) \) and the snake lemma for

\[
\begin{array}{ccccccccc}
0 & \to & \overline{N} \cap U_\infty & \xrightarrow{\iota(n)} & \sigma_F & \xrightarrow{\sigma_F/(\overline{N} \cap U_\infty)} & 0 \\
\downarrow & & \downarrow \times(u_{\epsilon}^{\iota(n)} - 1) & & \downarrow \times(u_{\epsilon}^{\iota(n)} - 1) \quad \downarrow \times(u_{\epsilon}^{\iota(n)} - 1) \\
0 & \to & \overline{N} \cap U_\infty & \xrightarrow{\iota(n)} & \sigma_F & \xrightarrow{\sigma_F/(\overline{N} \cap U_\infty)} & 0.
\end{array}
\]
implies that
\[ \text{Hom}( (\mathcal{N} \cap U_\infty)_{(\sigma_F^\vee)^\vee}, K/\mathcal{O}) \simeq \text{Hom}(\sigma_F/(u_+^{(\mathfrak{n})} - 1), K/\mathcal{O}). \]

Now, by combining them, the assumption on \( u_+^{(\mathfrak{n})} - 1 \) implies the claim \( (6.2) \) as desired. \( \square \)

**Example 6.6.** We give examples satisfying all the assumptions of the Main Theorem \( 0.1 \) in the case where \( F = \mathbb{Q}(\sqrt{2}) \). We have \( \sigma_F = \mathbb{Z}[\sqrt{2}], h_F^+ = 1, \Delta_F = 8, u = 1 + \sqrt{2}, \) and \( u_+ = 3 + 2\sqrt{2} \). We put \( G = \text{Gal}(K/\mathbb{Q}) \) and \( H = \text{Gal}(K/F) \), where \( K = \mathbb{Q}(\sqrt{2}, \sqrt{20149}) \).

Let \( \varepsilon : H \to \{ \pm 1 \} \) be the non-trivial character whose conductor is a prime ideal \( (20149) \) of \( \sigma_F \). Let \( \varepsilon_1 : \text{Gal}(\mathbb{Q}(\sqrt{20149})/\mathbb{Q}) \to \{ \pm 1 \} \) be the non-trivial character whose conductor is a prime ideal \( (20149) \) of \( \mathbb{Z} \), and \( \varepsilon_2 : \text{Gal}(\mathbb{Q}(\sqrt{2 \cdot 20149})/\mathbb{Q}) \to \{ \pm 1 \} \) the non-trivial character whose conductor is an ideal \( (8 \cdot 20149) \) of \( \mathbb{Z} \). Then we have \( \text{Ind}_H^G \varepsilon_1 = \varepsilon_1 \oplus \varepsilon_2 \) as \( G \)-modules and hence \( L_\mathbb{Q}(s, \text{Ind}_H^G \varepsilon_1) = L_\mathbb{Q}(s, \varepsilon_1) \cdot L_\mathbb{Q}(s, \varepsilon_2) \). Since \( L_F(s, \varepsilon) = L_\mathbb{Q}(s, \text{Ind}_H^G \varepsilon) \), thus we obtain
\[ L_F(-1, \varepsilon) = 2^2 \cdot 5 \cdot 281 \cdot 4951 \cdot 13417. \]

Let \( p \) be a prime number \( \in \{ 281, 4951, 13417 \} \). Let \( E \) denote the Eisenstein series \( \in M_2((20149)^2, \mathcal{O}) \) characterized by
\[ D(s, E) = L_F^{(20149)}(s, \varepsilon)L_F^{(20149)}(s - 1, 1). \]

In order to apply Theorem 6.2 (and Corollary 6.3) to the pair \( (p, E) \), it suffices to check the assumptions of Proposition 6.3 and 6.5. The former follows from that \( p \) is prime to \( 2(\sigma_F/(20149)^2) \) and the latter follows from that \( p \) is prime to \( 4 \cdot (20149)^2 \) and \( u_+^{(20149)^2} - 1 \). Here, for a non-zero ideal \( \mathfrak{a} \) of \( \sigma_F \), \( \iota^1(\mathfrak{a}) \) denotes the index \( [\Gamma_1^1(\sigma_F, \sigma_F) : \Gamma_1^1(\sigma_F, \mathfrak{a})] \), which is explicitly given by
\[ \iota^1(\mathfrak{a}) = \frac{1}{2^2}(\sigma_F/\mathfrak{a})^\vee N(\mathfrak{a}) \prod_{q | \mathfrak{a}} \left( 1 + \frac{1}{N(q)} \right) \text{ if } \mathfrak{a} \neq (2) \]
(cf. [117, Theorem 4.2.5]). Now, by applying Corollary 6.3, there exists \( \mathfrak{f}_0 \in S_2((20149)^2, \mathcal{O}') \) such that \( p \mathfrak{f}_0 \simeq \chi_1(20149) \chi_{20149}(\mod \varpi^2) \), where \( 1_{(20149)^2} \) denotes the trivial character modulo \( (20149) \). Let \( \theta \) be the non-trivial character of \( \text{Gal}(F(\sqrt{-19})/F) \) whose conductor is a prime ideal \( (19) \) of \( \sigma_F \). We put \( \mathfrak{n} = (20149)^2(19)^2 \) and \( \mathfrak{f} = \mathfrak{f}_0 \otimes \theta \in S_2(\mathfrak{n}, \mathcal{O}') \). Thus, by the same way as above, \( \mathfrak{f} \) and \( \mathfrak{n} \) satisfy all the assumptions of the Main Theorem 0.1.

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