Lie groups as four-dimensional special complex manifolds with Norden metric

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Abstract

An example of a four-dimensional special complex manifold with Norden metric of constant holomorphic sectional curvature is constructed via a two-parametric family of solvable Lie algebras. The curvature properties of the obtained manifold are studied. Necessary and sufficient conditions for the manifold to be isotropic Kählerian are given.

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1 Preliminaries

Let \((M, J, g)\) be a \(2n\)-dimensional almost complex manifold with Norden metric, i.e. \(J\) is an almost complex structure and \(g\) is a metric on \(M\) such that:

\[ J^2 x = -x, \quad g(Jx, Jy) = -g(x, y), \quad x, y \in X(M). \]  

The associated metric \(\tilde{g}\) of \(g\) on \(M\), given by \(\tilde{g}(x, y) = g(x, Jy)\), is a Norden metric, too. Both metrics are necessarily neutral, i.e. of signature \((n, n)\).

If \(\nabla\) is the Levi-Civita connection of \(g\), the tensor field \(F\) of type \((0, 3)\) is defined by

\[ F(x, y, z) = g((\nabla_x J)y, z) \]

and has the following symmetries

\[ F(x, y, z) = F(x, z, y) = F(x, Jy, Jz). \]

Let \(\{e_i\} (i = 1, 2, \ldots, 2n)\) be an arbitrary basis of \(T_pM\) at a point \(p\) of \(M\). The components of the inverse matrix of \(g\) are denoted by \(g^{ij}\) with respect to the basis \(\{e_i\}\). The Lie 1-forms \(\theta\) and \(\theta^*\) associated with \(F\) are defined by, respectively

\[ \theta(x) = g^{ij} F(e_i, e_j, x), \quad \theta^* = \theta \circ J. \]

The Nijenhuis tensor field \(N\) for \(J\) is given by

\[ N(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy]. \]

It is known [4] that the almost complex structure is complex iff it is integrable, i.e. \(N = 0\).
A classification of the almost complex manifolds with Norden metric is introduced in [2], where eight classes of these manifolds are characterized according to the properties of $F$. The three basic classes: $\mathcal{W}_1$, $\mathcal{W}_2$ of the special complex manifolds with Norden metric and $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric are given as follows:

\begin{equation}
\mathcal{W}_1 : F(x, y, z) = \frac{1}{2} [g(x, y)\theta(z) + g(x, z)\theta(y)] + g(x, Jy)\theta(Jz) + g(x, Jz)\theta(Jy)];
\end{equation}

\begin{equation}
\mathcal{W}_2 : F(x, y, Jz) + F(y, z, Jx) + F(z, x, Jy) = 0, \quad \theta = 0 \Leftrightarrow N = 0, \quad \theta = 0;
\end{equation}

\begin{equation}
\mathcal{W}_3 : F(x, y, z) + F(y, z, x) + F(z, x, y) = 0.
\end{equation}

The class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric is defined by $F = 0$ and is contained in each of the other classes.

Let $R$ be the curvature tensor of $\nabla$, i.e. $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$. The corresponding $(0,4)$-type tensor is defined by $R(x, y, z, u) = g(R(x, y)z, u)$. The Ricci tensor $\rho$ and the scalar curvatures $\tau$ and $\tau^*$ are given by:

\begin{equation}
\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \quad \tau = g^{ij}\rho(e_i, e_j), \quad \tau^* = g^{ij}\rho(e_i, Je_j).
\end{equation}

A tensor of type $(0,4)$ is said to be curvature-like if it has the properties of $R$. Let $S$ be a symmetric $(0,2)$-tensor. We consider the following curvature-like tensors:

\begin{equation}
\psi_1(S)(x, y, z, u) = g(y, z)S(x, u) - g(x, z)S(y, u) + g(x, u)S(y, z) - g(y, u)S(x, z),
\end{equation}

\begin{equation}
\pi_1 = \frac{1}{2}\psi_1(g), \quad \pi_2(x, y, z, u) = g(y, Jz)g(x, Ju) - g(x, Jz)g(y, Ju).
\end{equation}

It is known that on a pseudo-Riemannian manifold $M$ $(\dim M = 2n \geq 4)$ the conformal invariant Weyl tensor has the form

\begin{equation}
W(R) = R - \frac{1}{2(n-1)} \left\{ \psi_1(\rho) - \frac{\tau}{2n-1} \pi_1 \right\}.
\end{equation}

Let $\alpha = \{x, y\}$ be a non-degenerate 2-plane spanned by the vectors $x, y \in T_p M$, $p \in M$. The sectional curvature of $\alpha$ is given by

\begin{equation}
k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}.
\end{equation}

We consider the following basic sectional curvatures in $T_p M$ with respect to the structures $J$ and $g$: holomorphic sectional curvatures if $J\alpha = \alpha$ and totally real sectional curvatures if $J\alpha \perp \alpha$ with respect to $g$.

The square norm of $\nabla J$ is defined by $\|\nabla J\|^2 = g^{ij}g^{kl}g((\nabla e_i J)e_k, (\nabla e_j J)e_l)$. Then, by (1.2) we get

\begin{equation}
\|\nabla J\|^2 = g^{ij}g^{kl}g^{pq}F_{ikp}F_{jtl},
\end{equation}

where $F_{ikp} = F(e_i, e_k, e_p)$.

An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called an isotropic Kähler manifold with Norden metric [3].
2 Almost complex manifolds with Norden metric of constant holomorphic sectional curvature

In this section we obtain a relation between the vanishing of the holomorphic sectional curvature and the vanishing of $\|\nabla J\|^2$ on $W_2$-manifolds and $W_3$-manifolds with Norden metric.

In [1] it is proved the following

**Theorem A. ([1])** An almost complex manifold with Norden metric is of pointwise constant holomorphic sectional curvature if and only if

\[
3\{R(x,y,z,u) + R(x,y,Jz,Ju) + R(Jx,Jy,z,u) + R(Jx,Jy,Jz,Ju) \\
- R(Jy,z,x,u) + R(Jx,Jz,y,u) - R(y,z,Jx,Ju) + R(x,z,Jy,Ju) \\
- R(Jx,z,y,Ju) + R(Jy,z,x,Ju) - R(x,Jz,Jy,u) + R(y,Jz,Jx,u) \\
- R(Jx,z,Jy,u) + R(Jy,z,Jx,u)
\} = 8H\{\pi_1 + \pi_2\}
\]

for some $H \in FM$ and all $x,y,z,u \in \mathcal{X}(M)$. In this case $H(p)$ is the holomorphic sectional curvature of all holomorphic non-degenerate 2-planes in $T_pM$, $p \in M$. ■

Taking into account (1.7) and (1.8), the total trace of (2.1) implies

\[
H(p) = \frac{1}{4n^2}(\tau + \tau^{**}),
\]

where $\tau^{**} = g^{ij}g^{jk}R(e_i,e_j,e_k,e_l)$.

In [5] we have proved that on a $W_2$-manifold it is valid

\[
\|\nabla J\|^2 = 2(\tau + \tau^{**}),
\]

and in [3] it is proved that on a $W_3$-manifold

\[
\|\nabla J\|^2 = -2(\tau + \tau^{**}).
\]

Then, by Theorem A, (2.2), (2.3) and (2.4) we obtain

**Theorem 2.1.** Let $(M,J,g)$ be an almost complex manifold with Norden metric of pointwise constant holomorphic sectional curvature $H(p)$, $p \in M$. Then

(i) $\|\nabla J\|^2 = 8n^2H(p)$ if $(M,J,g) \in W_2$;

(ii) $\|\nabla J\|^2 = -8n^2H(p)$ if $(M,J,g) \in W_3$. ■

Theorem 2.1 implies

**Corollary 2.2.** Let $(M,J,g)$ be a $W_2$-manifold or $W_3$-manifold of pointwise constant holomorphic sectional curvature $H(p)$, $p \in M$. Then, $(M,J,g)$ is isotropic Kählerian iff $H(p) = 0$.

In the next section we construct an example of a $W_2$-manifold of constant holomorphic sectional curvature.
3 Lie groups as four-dimensional $\mathcal{W}_2$-manifolds

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra corresponding to a real connected Lie group $G$. If $\{X_1, X_2, X_3, X_4\}$ is a basis of left invariant vector fields on $G$ and $[X_i, X_j] = C^k_{ij} X_k$ ($i, j, k = 1, 2, 3, 4$) then the structural constants $C^k_{ij}$ satisfy the anti-commutativity condition $C^k_{ij} = -C^k_{ji}$ and the Jacobi identity $C^k_{ij} C^l_{ks} + C^k_{js} C^l_{ki} + C^k_{si} C^l_{kj} = 0$.

We define an almost complex structure $J$ and a compatible metric $g$ on $G$ by the conditions, respectively:

\begin{equation}
J X_1 = X_3, \quad J X_2 = X_4, \quad J X_3 = -X_1, \quad J X_4 = -X_2,
\end{equation}

\begin{equation}
g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \quad g(X_i, X_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4.
\end{equation}

Because of (1.1), (3.1) and (3.2) $g$ is a Norden metric. Thus, $(G, J, g)$ is a 4-dimensional almost complex manifold with Norden metric.

From (3.2) it follows that the well-known Levi-Civita identity for $g$ takes the form

\begin{equation}
2g(\nabla X_j, X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_i, X_j], X_k).
\end{equation}

Let us denote $F_{ijk} = F(X_i, X_j, X_k)$. Then, by (1.2) and (3.3) we have

\begin{equation}
2F_{ijk} = g([X_i, J X_j] - J [X_i, X_j], X_k) + g([J X_k, X_i] - [J X_i, X_j], X_k) + g([X_k, J X_j] - [J X_k, X_j], X_i).
\end{equation}

According to (1.6) to construct an example of a $\mathcal{W}_2$-manifold we need to find sufficient conditions for the Nijenhuis tensor $N$ and the Lie 1-form $\theta$ to vanish on $\mathfrak{g}$.

By (1.2), (1.5), (3.2) and (3.4) we compute the essential components $N^i_{jk} (N(X_i, X_j) = N^i_{jk} X_k)$ of $N$ and $\theta_i = \theta(X_i)$ of $\theta$, respectively, as follows:

\begin{equation}
N^1_{12} = C^1_{12} - C^1_{21} - C^3_{23} + C^3_{14}, \quad \theta_1 = 2C^1_{13} - C^1_{12} + C^2_{14} + C^2_{23} - C^3_{34},
\end{equation}

\begin{equation}
N^2_{12} = C^2_{12} - C^2_{21} - C^3_{13} + C^3_{14}, \quad \theta_2 = 2C^2_{12} + C^1_{13} + C^2_{14} + C^3_{23} + C^3_{34},
\end{equation}

\begin{equation}
N^3_{12} = C^3_{12} - C^3_{21} + C^3_{23} - C^3_{14}, \quad \theta_3 = 2C^3_{13} + C^1_{14} + C^2_{34} + C^3_{23} + C^3_{34},
\end{equation}

\begin{equation}
N^4_{12} = C^4_{12} - C^4_{21} + C^3_{14} - C^3_{14}, \quad \theta_4 = 2C^4_{12} - C^1_{14} + C^3_{14} + C^3_{23} - C^3_{34}.
\end{equation}

Then, (1.6) and (3.5) imply

**Theorem 3.1.** Let $(G, J, g)$ be a 4-dimensional almost complex manifold with Norden metric defined by (3.1) and (3.2). Then, $(G, J, g)$ is a $\mathcal{W}_2$-manifold iff for the Lie algebra $\mathfrak{g}$ of $G$ are valid the conditions:

\begin{equation}
C^1_{13} = C^1_{12} - C^3_{23} = C^3_{34} - C^3_{14}, \quad C^3_{13} = - (C^3_{12} + C^3_{24}) = - (C^3_{14} + C^3_{34}),
\end{equation}

\begin{equation}
C^2_{14} = C^1_{12} - C^3_{14} = C^3_{34}, \quad C^3_{24} = - (C^3_{12} + C^3_{14}) = - (C^3_{34}).
\end{equation}

where $C^k_{ij}$ ($i, j, k = 1, 2, 3, 4$) satisfy the Jacobi identity. $\blacksquare$
One solution to (3.6) and the Jacobi identity is the 2-parametric family of solvable Lie algebras \( \mathfrak{g} \) given by

\[
[X_1, X_2] = \lambda X_1 - \lambda X_2, \quad [X_2, X_3] = \mu X_1 + \lambda X_4,
\]

(3.7)

\[
\mathfrak{g} : \quad [X_1, X_3] = \mu X_2 + \lambda X_4, \quad [X_2, X_4] = \mu X_1 + \lambda X_3,
\]

\[
[X_1, X_4] = \mu X_2 + \lambda X_3, \quad [X_3, X_4] = -\mu X_3 + \mu X_1, \quad \lambda, \mu \in \mathbb{R}.
\]

Let us study the curvature properties of the \( \mathcal{W}_2 \)-manifold \((G, J, g)\), where the Lie algebra \( \mathfrak{g} \) of \( G \) is defined by (3.7).

By (3.2), (3.3) and (3.7) we obtain the components of the Levi-Civita connection:

\[
\nabla_{X_1} X_2 = \lambda X_1 + \mu(X_3 + X_4), \quad \nabla_{X_2} X_3 = -\lambda(X_1 + X_2) - \mu X_4,
\]

(3.8)

\[
\nabla_{X_3} X_4 = -\lambda X_1, \quad \nabla_{X_4} X_3 = \mu X_1,
\]

\[
\nabla_{X_5} X_1 = \nabla_{X_1} X_5 = \nabla_{X_2} X_4 = -\lambda X_4, \quad \nabla_{X_3} X_2 = -\lambda X_3.
\]

Taking into account (3.4) and (3.7) we compute the essential non-zero components of \( F \):

\[
F_{144} = -F_{214} = F_{312} = \frac{1}{2} F_{322} = \frac{1}{2} F_{411} = F_{412} = -\lambda,
\]

(3.9)

\[
F_{112} = \frac{1}{2} F_{122} = \frac{1}{2} F_{211} = F_{212} = -F_{314} = F_{414} = \mu.
\]

The other non-zero components of \( F \) are obtained from (1.3).

By (1.11) and (3.9) for the square norm of \( \nabla J \) we get

\[
\|\nabla J\|^2 = -32(\lambda^2 - \mu^2).
\]

Further, we obtain the essential non-zero components \( R_{ijks} = R(X_i, X_j, X_k, X_s) \) of the curvature tensor \( R \) as follows:

\[
-\frac{1}{2} R_{1221} = -R_{1341} = -R_{2342} = R_{3123} = \frac{1}{2} R_{3443} = R_{4124} = \lambda^2 + \mu^2,
\]

(3.11)

\[
R_{1331} = R_{1441} = R_{2332} = R_{2442} = -R_{1324} = -R_{1423} = \lambda^2 - \mu^2,
\]

\[
R_{1231} = R_{1241} = R_{2132} = R_{2142} = -R_{3113} = -R_{3213} = -R_{4114} = -R_{4214} = 2\lambda \mu.
\]

Then, by (1.7) and (3.11) we get the components \( \rho_{ij} = \rho(X_i, X_j) \) of the Ricci tensor and the values of the scalar curvatures \( \tau \) and \( \tau^* \):

\[
\rho_{11} = \rho_{22} = -4\lambda^2, \quad \rho_{33} = \rho_{44} = -4\mu^2,
\]

(3.12)

\[
\rho_{12} = \rho_{34} = -2(\lambda^2 + \mu^2), \quad \rho_{13} = \rho_{14} = \rho_{23} = \rho_{24} = 4\lambda \mu,
\]

\[
\tau = -8(\lambda^2 - \mu^2), \quad \tau^* = 16\lambda \mu.
\]

Let us consider the characteristic 2-planes \( \alpha_{ij} \) spanned by the basic vectors \( \{X_i, X_j\} \): totally real 2-planes - \( \alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34} \) and holomorphic 2-planes - \( \alpha_{13}, \alpha_{24} \). By (1.10) and (3.11) for the sectional curvatures of the holomorphic 2-planes we obtain

\[
k(\alpha_{13}) = k(\alpha_{24}) = -(\lambda^2 - \mu^2).
\]
Then it is valid

**Theorem 3.2.** The manifold \((G, J, g)\) is of constant holomorphic sectional curvature.

Using (1.9), (3.11) and (3.12) for the essential non-zero components 
\(W_{ijks} = W(X_i, X_j, X_k, X_s)\) of the Weyl tensor \(W\) we get:

\[
\begin{align*}
\frac{1}{2}W_{1221} &= W_{1331} = W_{1441} = W_{2332} = W_{2442} = \frac{1}{2}W_{3443} \\
&= -\frac{1}{3}W_{1324} = -\frac{1}{3}W_{1423} = \frac{1}{3}(\lambda^2 - \mu^2).
\end{align*}
\]

Finally, by (1.9), (3.10), (3.12), (3.13) and (3.14) we establish the truthfulness of

**Theorem 3.3.** The following conditions are equivalent:

(i) \((G, J, g)\) is isotropic Kählerian;
(ii) \(|\lambda| = |\mu|\);
(iii) \(\tau = 0\);
(iv) \((G, J, g)\) is of zero holomorphic sectional curvature;
(v) the Weyl tensor vanishes.
(vi) \(R = \frac{1}{4}\psi_1(\rho)\).

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