WEAK $Z$-STRUCTURES AND ONE-RELATOR GROUPS

M. CÁRDENAS, F. F. LASHERAS, AND A. QUINTERO

Abstract. Motivated by the notion of boundary for hyperbolic and $CAT(0)$ groups, Bestvina [2] introduced the notion of a (weak) $Z$-structure and (weak) $Z$-boundary for a group $G$ of type $F$ (i.e., having a finite $K(G,1)$ complex), with implications concerning the Novikov conjecture for $G$. Since then, some classes of groups have been shown to admit a weak $Z$-structure (see [15] for example), but the question whether or not every group of type $F$ admits such a structure remains open. In this paper, we show that every torsion free one-relator group admits a weak $Z$-structure, by showing that they are all properly aspherical at infinity; moreover, in the 1-ended case the corresponding weak $Z$-boundary has the shape of either a circle or a Hawaiian earring depending on whether the group is a virtually surface group or not. Finally, we extend this result to a wider class of groups still satisfying a Freiheitssatz property.

1. Introduction

We recall that a compact metrizable space $W$ is a compactification of a (path connected) locally compact metrizable space $X$ if it contains a homeomorphic copy $X \subset W$ as a dense open subset. Furthermore, we say that $W$ is a $Z$-compactification of $X$ if $Z = W - X$ is a $Z$-set in $W$, i.e., for every open set $U \subset W$ the inclusion $U - Z \hookrightarrow U$ is a homotopy equivalence; equivalently (if $X$ is an ANR), there is a homotopy $H : W \rightarrow W$ with $H_0 = id_W$ and $H_t(W) \subset X$ for all $t > 0$. And in this case, we say that $Z$ is a $Z$-boundary for $X$. The model example is that in which $W$ is a compact manifold and $Z \subseteq \partial W$ is a closed subset.

From now on all spaces will be ANRs; in fact, we will deal with (connected) locally finite CW-complexes.

In [2] Bestvina introduced the notion of $Z$-structure and $Z$-boundary for a group (of type $F$) as an attempt to generalize the already existing notion of boundary for hyperbolic and $CAT(0)$ groups; namely,

Definition 1.1. A $Z$-structure on a group $G$ is a pair $(W, Z)$ of spaces satisfying:

1. $W$ is a contractible ANR,
2. $Z$ is a $Z$-set in $W$,
3. $X = W - Z$ admits a proper, free and cocompact action by $G$, and
4. (nullity condition) For any open cover of $W$, and any compact subset $K \subseteq X$, all but finitely many translates of $K$ lie in some element of the cover.

Observe that it is not necessary that $W$ be finite-dimensional, by [26, 17]. If only conditions (1)-(3) are satisfied, then $(W, Z)$ is called a weak $Z$-structure on $G$, and we refer to $Z$ as a (weak) $Z$-boundary for $G$.
An additional condition can be added to the above:

(5) The action of $G$ on $X$ can be extended to $W$.

If conditions (1)-(5) are satisfied, then $(W, Z)$ is called an $EZ$-structure on $G$. It was shown in [11] that the Novikov conjecture holds for any (torsion free) group admitting an $EZ$-structure. Examples of groups admitting an $EZ$-structure are (torsion free) $\delta$-hyperbolic and $CAT(0)$ groups, see [3, 2].

Although not stated explicitly, it follows that such a group $G$ as in Definition 1.1 must be of type $F$ (see [15, Prop. 1.1]). The more conditions on a $Z$-structure on a group the better; nonetheless, a weak $Z$-boundary already carries significant information about the group and, when it exists, it is well-defined up to shape, and it is always a first step towards finding a stronger structure on it. The question whether or not every group of type $F$ admits a (weak) $Z$-structure still remains open. In this paper, we give a positive answer to this question for a class of groups containing all torsion free one-relator groups. For this, we first use some previous work from [4, 21] to show that they are all properly aspherical at infinity, and then combine this with some recent work in [5] to characterize the corresponding boundary in the 1-ended case. Our main results are Theorems 1.2 and 1.6 below.

**Theorem 1.2.** Every finitely generated, torsion free one-relator group $G$ admits a weak $Z$-structure. Moreover, if $G$ is 1-ended then the corresponding weak $Z$-boundary has the shape of either a circle or a Hawaiian earring depending on whether $G$ is a virtually surface group or not.

Bestvina [2] already pointed out that the Baumslag-Solitar group $\langle x, t; t^{-1}xt = x^2 \rangle$ admits a $Z$-boundary homeomorphic to the Cantor-Hawaiian earring, which is shape equivalent to the ordinary Hawaiian earring (see [24] as a general reference for shape theory). It is worth mentioning that for the particular class of Baumslag-Solitar groups a stronger $EZ$-structure has been recently described in [18].

More generally, we may construct the following class $C$ of finitely presented groups starting off from one-relator group presentations as follows. Let $G$ and $H$ be finitely generated one-relator groups, and assume $P = \langle X; r \rangle$ and $Q = \langle Y; s \rangle$ are presentations of $G$ and $H$ with a single relation, respectively. Let $V \subset X$ and $W \subset Y$, together with a bijection $\eta : V \rightarrow W$, be (possibly empty) subsets not containing all the generators involved in any of the relators of the corresponding presentation. We declare the corresponding amalgamated product $G \ast_F H$ associated with $\eta$ (over a free group of rank $\text{card}(V) = \text{card}(W)$) to be in our class $C$ together with the obvious presentation for it obtained from $P$ and $Q$. It seems natural to consider the class $C$ of all finitely generated one-relator groups together with those finitely presented groups which can be obtained by successive applications of the construction above, so that $C$ is closed under amalgamated products (over free subgroups) of the type just described. The group presentations obtained in this way still satisfy a Freiheitssatz property, see [21] for more details. Henceforth, we will refer to those groups as in $C$ as “generalized” one-relator groups.

**Remark 1.3.** The first interesting examples of groups in the class $C$ (other than one-relator groups and their free products) are those groups $G$ given by a presentation of the form $\langle X; r, s \rangle$, where $r, s$ are cyclically reduced words so that $r \in F(Y)$, $Y \subset X$, and $s \in F(X) - F(Y)$ misses at least one generator in $Y$ which occurs in $r$. Indeed, one can obtain $G$ as an amalgamated product $\langle Y; r \rangle \ast_F \langle (X - Y) \cup Z; s \rangle$.
over a free group of rank $\text{card}(Z)$, where $Z \subset Y$ is the subset consisting of those generators in $Y$ which occur in $s$.

**Remark 1.4.** From the above remark, one can see that Higman’s group $H$, with presentation $(a, b, c, d; a^{-2} b^{-1} a b, b^{-2} c^{-1} b c, c^{-2} d^{-1} c d, d^{-2} a^{-1} d a)$, is in $\mathcal{C}$. Indeed, $H$ can be expressed as an amalgamated product (of the type described above) of two copies of $(x, y, z; x^{-2} y^{-1} x y, y^{-2} z^{-1} y z)$ over a free subgroup of rank 2 (see [13]).

**Remark 1.5.** It was also shown in [21] that every finitely presented group $G$ given by a staggered presentation $P$ is in $\mathcal{C}$. We recall that $(X; R)$ is defined to be a staggered presentation if there are subsets $X_0 \subset X$ so that both $R$ and $X_0$ are linearly ordered in such a way that: (i) each relator $r \in R$ contains some $x \in X_0$; (ii) if $r, r'$ are relators with $r < r'$, then $r$ contains some $x \in X_0$ that precedes all elements of $X_0$ occurring in $r'$, and $r'$ contains some $y \in X_0$ that comes after all those occurring in $r$ (see [22]).

Theorem 1.2 above together with the results in [21] yield the following generalization.

**Theorem 1.6.** Every torsion free, 1-ended generalized one-relator group $G \in \mathcal{C}$ admits a weak $Z$-structure. Moreover, the corresponding weak $Z$-boundary has the shape of either a circle or a Hawaiian earring depending on whether $G$ is a virtually surface group or not.

2. Preliminaries

Given a non-compact (strongly) locally finite CW-complex $Y$, a *proper ray* in $Y$ is a proper map $\omega : [0, \infty) \to Y$. Recall that a proper map is a map with the property that the inverse image of every compact subset is compact. We say that two proper rays $\omega, \omega'$ define the same end if their restrictions to the natural numbers $\omega|\mathbb{N}, \omega'|\mathbb{N}$ are properly homotopic. This equivalence relation gives rise to the notion of *end determined by* $\omega$ as the corresponding equivalence class, as well as the space of ends $\mathcal{E}(Y)$ of $Y$ as a compact totally disconnected metrizable space (see [12] [13]). The CW-complex $Y$ is *semistable* at the end determined by $\omega$ if any other proper ray defining the same end is in fact properly homotopic to $\omega$; equivalently, if the fundamental pro-group $\pi_1(Y, \omega)$ is pro-isomorphic to a tower of groups with surjective bonding homomorphisms (see [13] Prop. 16.1.2). Recall that the homotopy pro-groups $\pi_n(Y, \omega)$ are represented by the inverse sequences (tower) of groups

$$\pi_n(Y, \omega(0)) \xrightarrow{\phi_1} \pi_n(Y - C_1, \omega(t_1)) \xrightarrow{\phi_2} \pi_n(Y - C_2, \omega(t_2)) \leftarrow \cdots$$

where $C_1 \subset C_2 \subset \cdots \subset Y$ is a filtration of $Y$ by compact subspaces, $\omega([t_i, \infty)) \subset Y - C_i$ and the bonding homomorphisms $\phi_i$ are induced by the inclusions and basepoint-change isomorphisms. One can show the independence with respect to the filtration. Also, properly homotopic base rays yield pro-isomorphic homotopy pro-groups $\pi_n$, for all $n$. If $Y$ is semistable at each end then we will simply say that $Y$ is semistable at infinity, and in this case two proper rays representing the same end yield the same (up to pro-isomorphism) homotopy pro-groups $\pi_n$. We refer to [13] [24] for more details.

Given a CW-complex $X$, with $\pi_1(X) \cong G$, we will denote by $\tilde{X}$ the universal cover of $X$, constructed as prescribed in ([13], §3.2), so that $G$ is acting freely on
the CW-complex $\tilde{X}$ via a cell-permuting left action with $G\backslash \tilde{X} = X$. The number of ends of an (infinite) finitely generated group $G$ represents the number of ends of the (strongly) locally finite CW-complex $\tilde{X}$, for some (equivalently any) CW-complex $X$ with $\pi_1(X) \cong G$ and with finite 1-skeleton, which is either 1, 2 or $\infty$ (finite groups have 0 ends [13, 27]). If $G$ is finitely presented, then $G$ is \textit{semistable at infinity} if the (strongly) locally finite CW-complex $\tilde{X}$ is so, for some (equivalently, any) CW-complex $X$ with $\pi_1(X) \cong G$ and with finite 2-skeleton. Observe that any finite-dimensional locally finite CW-complex is strongly locally finite, see [13].

The following result will be crucial for the proof of the main result in this paper.

\textbf{Proposition 2.1.} Let $\mathcal{P} = (X; r)$ be a finite presentation of a torsion free group with a single (cyclically reduced) relator $r \in F(X)$, and consider the associated 2-dimensional CW-complex $K_\mathcal{P}$. Then, the (contractible) universal cover $\tilde{K}_\mathcal{P}$ is properly aspherical at infinity, i.e., for any choice of base ray, the homotopy pro-groups $\text{pro} - \pi_n(\tilde{K}_\mathcal{P}) = 0$ are pro-trivial for $n \geq 2$. Furthermore, the fundamental pro-group $\text{pro} - \pi_1(\tilde{K}_\mathcal{P})$ is pro-(finitely generated free).

Observe that the universal cover $\tilde{K}_\mathcal{P}$ above is already known to be contractible (see [8]) and semistable at infinity (see [25]), and hence its homotopy pro-groups do not depend (up to pro-isomorphism) on the choice of the base ray.

\textbf{Remark 2.2.} Recall that the (finite) 2-dimensional CW-complex $K_\mathcal{P}$ associated to $\mathcal{P}$ is constructed as follows. The 0-skeleton consists of a single vertex and the 1-skeleton $K_1^\mathcal{P}$ consists of a bouquet of circles, one for each element of the basis $x_i \in X$, all of them sharing the single vertex in $K_\mathcal{P}$. Finally $K_\mathcal{P}$ is obtained from $K_1^\mathcal{P}$ by attaching a 2-cell $d$ via a PL map $S^1 \to K_1^\mathcal{P}$ which spells out the single relator $r$. Note that every lift in the universal cover $\tilde{d} \subset \tilde{K}_\mathcal{P}$ of the 2-cell $d \subset K_\mathcal{P}$ is a disk as $r$ is a cyclically reduced word. Moreover, by the Magnus’ Freiheitssatz (see [22, 23]) every subcomplex of the 1-skeleton $K_1^\mathcal{P}_\mathcal{P}$ not containing all the 1-cells involved in the relator $r$ lifts in the universal cover $\tilde{K}_\mathcal{P}$ to a disjoint union of trees.

Proposition [24] follows immediately from the following lemma, which is an enhancement of [3, Prop. 2.7].

\textbf{Lemma 2.3.} Let $\mathcal{P} = (X; r)$ be a finite, torsion free group presentation with a single (cyclically reduced) relator $r \in F(X)$, and consider the associated 2-dimensional CW-complex $K_\mathcal{P}$. Then, the universal cover $\tilde{K}_\mathcal{P}$ is proper homotopy equivalent to another 2-dimensional CW-complex $\tilde{K}_\mathcal{P}$ which has a filtration $\tilde{C}_1 \subset \tilde{C}_2 \subset \cdots \subset \tilde{K}_\mathcal{P}$ by finite contractible subcomplexes satisfying (for any choice of base ray):

(a) The tower $\{1\} \leftarrow \pi_1(\tilde{K}_\mathcal{P} - \tilde{C}_1) \leftarrow \pi_1(\tilde{K}_\mathcal{P} - \tilde{C}_2) \leftarrow \cdots$ consists of finitely generated free groups of increasing rank, with the bonding maps being the obvious projections, and

(b) The tower $\{1\} \leftarrow \pi_n(\tilde{K}_\mathcal{P} - \tilde{C}_1) \leftarrow \pi_n(\tilde{K}_\mathcal{P} - \tilde{C}_2) \leftarrow \cdots$ is the trivial tower, $n \geq 2$.

\textbf{Remark 2.4.} In fact, the proper homotopy equivalence in the statement of Lemma [24] can be replaced by a “strong” proper homotopy equivalence, i.e., a (possibly infinite) sequence of internal collapses and/or expansions, carried out in a proper fashion. See [4] for more details.
Proof. Indeed, the proof of this lemma is that of [4, Prop.2.7], only that now we extend it, by taking a closer look, so that it covers part (b) here. The proof there goes by induction on the length of the relator $r \in F(X)$ in such a presentation $\mathcal{P} = \langle X : r \rangle$. It consists of a simultaneous double induction argument keeping track of two possible cases, depending on whether there is a generator in $X$ whose exponent sum in $r$ is zero or not, see §3 and §4 in [4] respectively.

In the first case (§3 in [4]), one shows that the induction lies on the fact that $K_{\mathcal{P}}$, an intermediate cover of the CW-complex $K_{\mathcal{P}}$, is made out, up to homotopy, of blocks $K_{\mathcal{P}'}$, where $\mathcal{P}'$ satisfies the inductive hypothesis. In fact, its universal cover $\tilde{K}_{\mathcal{P}'}$ is being slightly altered (within their proper homotopy type) to a CW-complexes $\tilde{K}_{\mathcal{P}'}$ so that their copies can be assembled together resulting into a new CW-complex $\tilde{K}_{\mathcal{P}}$ strongly proper homotopy equivalent to the universal cover of $K_{\mathcal{P}}$. This new CW-complex $\tilde{K}_{\mathcal{P}}$ consists of copies of the various CW-complexes $\tilde{K}_{\mathcal{P}'}$ above, glued together along trees (which were already present in the universal cover of $K_{\mathcal{P}}$, that correspond to the intersections of the different copies of $K_{\mathcal{P}'}$ and whose existence is a consequence of the Magnus’ Freiheitssatz, see Remark 2.2.

The desired filtration for $\tilde{K}_{\mathcal{P}}$ is then the result of assembling the filtrations we encounter on the various complexes $\tilde{K}_{\mathcal{P}'}$, which already have one by induction, as we grow towards infinity. This can be carefully done in such a way that if two of these CW-complexes $\tilde{K}_{\mathcal{P}'}$ meet along a tree inside $\tilde{K}_{\mathcal{P}}$ then each of the members of the corresponding filtration for each of them intersects that tree in a connected subtree.

Finally, given a compact subset $\tilde{C}_n \subset \tilde{K}_{\mathcal{P}}$ from this resulting filtration, the generalized van-Kampen theorem yields that the fundamental group $\pi_1(\tilde{K}_{\mathcal{P}} - \tilde{C}_n)$ is the free product of a free group together with the various $\pi_1(\tilde{K}_{\mathcal{P}'} - \tilde{C}_n')$ (finitely generated free by induction), where $\tilde{C}_n = \tilde{K}_{\mathcal{P}'} \cap \tilde{C}_n \neq \emptyset$.

The novelty here consists of adding part (b) of the statement to the induction hypothesis, and observing that each neighborhood of infinity of the form $U = \tilde{K}_{\mathcal{P}} - \tilde{C}_n$ is an assembly of the various neighborhoods of infinity $U' = \tilde{K}_{\mathcal{P}'} - \tilde{C}_n'$ (with $\tilde{C}_n = \tilde{K}_{\mathcal{P}'} \cap \tilde{C}_n \neq \emptyset$) together with all those (contractible) copies $\tilde{K}_{\mathcal{P}'} \subset \tilde{K}_{\mathcal{P}}$ which do not intersect $\tilde{C}_n$. Moreover, if two of the neighborhoods of infinity $U'$ (corresponding to two different copies of $\tilde{K}_{\mathcal{P}'}$) intersect inside $\tilde{K}_{\mathcal{P}}$, then they do it along the various components of $T - \tilde{C}_n$, where $T \subset \tilde{K}_{\mathcal{P}}$ is the corresponding tree along which those copies of $\tilde{K}_{\mathcal{P}'}$ are glued together inside $\tilde{K}_{\mathcal{P}}$. This way, the universal cover $\tilde{U}$ of $U = \tilde{K}_{\mathcal{P}} - \tilde{C}_n$ is the result of putting together the universal covers $\tilde{U}'$ of the various neighborhoods of infinity $U' = \tilde{K}_{\mathcal{P}'} - \tilde{C}_n'$ glued along connected subtrees, together with all those copies $\tilde{K}_{\mathcal{P}'} \subset \tilde{K}_{\mathcal{P}}$ which do not intersect $\tilde{C}_n$, each one glued to the rest along a copy of the corresponding tree from the construction indicated above. Thus, the induction hypothesis guarantees that each $\tilde{U}'$ is a contractible CW-complex and hence part (b) follows for $\tilde{K}_{\mathcal{P}}$.

As for the second case (§4 in [4]), in which there is no generator in $X$ whose exponent sum in $r$ is zero, the proof goes somehow the other way around. An auxiliary CW-complex $K_{\mathcal{P}}$ is built. For such $K_{\mathcal{P}}$, the induction hypothesis applies since it has a generator whose exponent sum is zero in the presentation $\mathcal{P}'$, which lies under the inductive hypothesis for the previous case (§3 in [4]). As above, its universal cover can be slightly altered (within its proper homotopy type) to a new CW-complex $\tilde{K}_{\mathcal{P}'}$ which is made out of blocks, corresponding to copies of
our candidates for the CW-complex $\hat{K}_P$ in question, glued together along copies of the real line. Given an appropriate filtration $C_n' \subset K_{P'}$ by compact subsets (provided by the induction hypothesis) satisfying the required properties for $K_{P'}$, one can get the desired filtration on each copy $\hat{K}_P$ inside $\hat{K}_{P'}$ simply by considering the intersections $C_n = \hat{K}_P \cap C_n'$. Observe that this procedure may yield different choices for the desired filtration on each of those copies of $\hat{K}_P$, but they all satisfy the required properties (a)-(b). Indeed, by induction, each neighborhood of infinity in $\hat{K}_{P'}$ of the form $U' = \hat{K}_{P'} - C_n'$ has finitely generated free fundamental group and trivial higher homotopy groups. From here, the argument is similar to the one given above, concluding that the corresponding neighborhoods of infinity $U = \hat{K}_P - C_n$ in each copy $\hat{K}_P$ inside $\hat{K}_{P'}$ behave in the same way (as each $\pi_1(U)$ is now a free factor of $\pi_1(U')$).

A tower of groups $F \equiv \{1 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots\}$ consisting of finitely generated free groups of non-decreasing rank and the obvious projections as bonding maps will be said to be “telescopic” (or of telescopic type). One can always associate to any given telescopic tower a 1-ended locally-finite (simply connected) 2-dimensional CW-complex $Y_m$, $0 \leq m \leq \infty$, whose fundamental pro-group realizes that telescopic tower as follows. Set $Y_0 = \{\ast\} \times [0,\infty)$ (a copy of $\mathbb{R}_+$). Assume $Y_n$ constructed, $n \in \mathbb{N} \cup \{0\}$. Then, $Y_{n+1}$ consists of the proper wedge of $Y_n$ and a copy $S^1 \times [n,\infty) \cup D^2 \times \{n\}$ of $\mathbb{R}^2$ attached along $Y_0$. Finally, we set $Y_\infty = \cup_{n \geq 0} Y_n$. Indeed, one can easily check that for some $0 \leq m \leq \infty$ and some filtration $\{J_n\}_{n \geq 1}$ of $Y_m$, there is a pro-isomorphism $\psi = \{\psi_n\}_{n \geq 1} : \text{pro} - \pi_1(Y_m) \rightarrow F$, where each $\psi_n : \pi_1(Y_m - J_n) \rightarrow F_n$ is an isomorphism between finitely generated free groups. Observe that the proper homotopy type of $Y_m$ can be represented by a subpolyhedron of $\mathbb{R}^3$, see the figure below.

![Figure 1](image_url)

**Corollary 2.5.** With the above notation, in the 1-ended case the universal cover $\hat{K}_P$ is proper homotopy equivalent to either $Y_1(=\mathbb{R}^2)$ or $Y_\infty$.

**Proof.** According to the above, by Lemma 2.3 (a), there is some $0 \leq m \leq \infty$ and a pro-isomorphism $\psi = \{\psi_n\}_{n \geq 1} : \text{pro} - \pi_1(Y_m) \rightarrow \text{pro} - \pi_1(\hat{K}_P)$, with each $\psi_n : \pi_1(Y_m - J_n) \rightarrow \pi_1(\hat{K}_P - C_n)$ being an isomorphism between finitely generated free groups. Moreover, by Lemma 2.3 (b) and [6] Prop. 3.3, there is a proper map $f : Y_m \rightarrow \hat{K}_P$ inducing the pro-isomorphism $\psi$; in fact, $f$ is a weak proper homotopy equivalence, as $Y_m$ is clearly properly aspherical at infinity as well, and hence $f$ induces pro-isomorphisms between all the homotopy pro-groups. Therefore, by the corresponding proper Whitehead theorem (see [10] Thm. 5.5.3] or [11 § 8], for instance) $f$ is in fact a proper homotopy equivalence.
It remains to show that \( m = 1 \) or \( \infty \). For this, observe that \( m > 0 \) since otherwise \( \hat{K}_P \) (and hence \( \tilde{K}_P \)) would be proper homotopy equivalent to a 3-manifold with a single plane on its boundary (as \( Y_0 = \mathbb{R}_+ \) thickens to a 3-dimensional half-space), which is not possible by [5, Cor. 5.14]. Furthermore, \( Y_m \) (and hence \( \tilde{K}_P \)) must be proper homotopy equivalent to a 3-manifold with boundary (by means of a regular neighborhood of the subpolyhedron of \( \mathbb{R}^3 \) in the figure above) which can only have either two or infinitely many plane boundary components, by [5, Cor. 5.11, 5.14]. The rest of the proof follows from this and the fact that the first option only occurs in the case of a virtually surface group, see [5, Thm. 5.17]. □

Remark 2.6. In terms of [5], every 1-ended, torsion free one-relator group is proper \( 2 \)-equivalent to either \( \mathbb{Z} \times \mathbb{Z} \) or \( \mathbb{F}_2 \times \mathbb{Z} \) by [5, Thm 5.1], as one relator groups are properly 3-realizable, see [4, 21]; in fact, given a presentation \( \mathcal{P} \) as above, the universal cover of \( K_P \) itself is proper homotopy equivalent to a 3-manifold (by considering a regular neighborhood of the above subpolyhedron in \( \mathbb{R}^3 \)) with no need to take wedge with a single 2-sphere, thus answering in the affirmative a conjecture posed in [4] (in the torsion free case). Observe that the third option \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) from [5, Thm 5.1] is ruled out by [5, Cor. 5.14], and the first option \( \mathbb{Z} \times \mathbb{Z} \) only occurs in the case of a virtually surface group, by [5, Thm. 5.17].

3. Proof of the main results

The purpose of this section is to prove Theorems 1.2 and 1.6. For this, we need the following previous result, which is a combination of other well known results.

Lemma 3.1. Let \( X \) be a locally finite \( n \)-dimensional (PL)CW-complex. If the following two conditions hold:

(a) \( X \) is inward tame, and
(b) For any choice of base ray, the fundamental pro-group \( \text{pro} \ - \pi_1(X) \) is pro-finitely generated free

then the product \( X \times I^{2n+5} \) admits a \( \mathbb{Z} \)-compactification, with \( I = [0,1] \).

Proof. Let \( I^\infty \) denote the Hilbert cube. It is well known that the product \( Y = X \times I^\infty \) is a Hilbert cube manifold (see [29, 9]) which satisfies again properties (a) and (b) from the statement, as \( I^\infty \) is compact and contractible and so \( X \) and \( Y \) are proper homotopy equivalent. In particular, \( Y \) is inward tame. Moreover, the Chapman-Siebenmann obstructions for a Hilbert cube manifold admitting a \( \mathbb{Z} \)-compactification ([7, Thms 3, 4], see also [16, §3.8.2]) vanish for \( Y \) since \( \text{pro} \ - \pi_1(Y) \)
can be represented by an inverse sequence
\[ \pi_1(Y) \leftarrow \pi_1(N_1) \leftarrow \pi_1(N_2) \leftarrow \cdots \]
where \( \{N_i\}_i \) is a nested cofinal sequence of neighborhoods of infinity in \( Y \) with \( \pi_1(N_i) \) a finitely generated free group, \( i \geq 1 \); in fact, each \( N_i \) can be taken as a product \( N_i = M_i \times I^\infty \), where \( M_i \) is a neighborhood of infinity in \( X \). Thus \( Y = X \times I^\infty \) admits a \( \mathbb{Z} \)-compactification. Finally, the results in [12] show that \( X \times I^{2n+5} \) admits a \( \mathbb{Z} \)-compactification as well.

We now proceed with the proof of the main results.

**Proof of Theorem 1.2.** Suppose a given torsion free finitely presented group \( G \) admits a finite presentation \( \mathcal{P} = \langle X; r \rangle \) with a single (cyclically reduced) relator \( r \in F(X) \). If \( G \) is 2-ended then \( G \) must be the group of integers \( \mathbb{Z} \) (see [27, Thm. 5.12]) which easily admits a weak \( \mathbb{Z} \)-structure just by adding two points as its boundary. Assume now \( G \) is 1-ended. Then, by Corollary 2.5, the universal cover \( \widetilde{K}_\mathcal{P} \) is proper homotopy equivalent to either the plane \( \mathbb{R}^2 \) or the locally finite subpolyhedron of \( \mathbb{R}^3 \) shown in figure 1, which are both easily shown to be inward tame, and hence so is \( K_\mathcal{P} \). On the other hand, Proposition 2.1 ensures condition (b) in Lemma 3.1 above. Therefore, the (contractible) CW-complex \( \widetilde{K}_\mathcal{P} \times I^9 \) admits a \( \mathbb{Z} \)-compactification. Observe that the proper, free and cocompact \( G \) action on \( \widetilde{K}_\mathcal{P} \) yields a proper, free and cocompact \( G \) action on \( \widetilde{K}_\mathcal{P} \times I^9 \) in the obvious way, thus providing a weak \( \mathbb{Z} \)-structure on \( G \) whose associated weak \( \mathbb{Z} \)-boundary has the shape of the \( \mathbb{Z} \)-boundary of a \( \mathbb{Z} \)-compactification of either the plane or the subpolyhedron shown in figure 1, see [16, Cor. 3.8.15]. In the case of the plane this \( \mathbb{Z} \)-boundary has the shape of a circle, and in the second case one can easily show that the corresponding \( \mathbb{Z} \)-boundary has the shape of a Hawaiian earring, as claimed.

Finally, if \( G \) is infinite ended then \( G \) decomposes as a free product of groups (as \( G \) is torsion free) by the Stallings’s structure theorem (see [13,27]). Moreover, being \( G \) a one-relator group, it follows from Grushko’s theorem that \( G \) is a free product of a free group and a one-relator group with at most one end. See [22, Prop. II.5.13] for details. Both factors admit a weak \( \mathbb{Z} \)-structure and hence so does their free product, by the proof of [28, Thm. 2.9].

Just as we did in section §2 with respect to the work in [3], a closer look at the proofs of [21 Thm. 1.13] and [21 Prop. 1.18] yields the following generalization of Proposition 2.1 and Lemma 2.3 (in the 1-ended case).

**Proposition 3.2.** Let \( \mathcal{P} = \langle X; R \rangle \) be a finite aspherical presentation of a torsion free, 1-ended generalized one-relator group \( G \in \mathcal{C} \), with each \( r \in R \) being a cyclically reduced word in \( F(X) \), and consider the associated 2-dimensional CW-complex \( \widetilde{K}_\mathcal{P} \). Then, the (contractible) universal cover \( \widetilde{K}_\mathcal{P} \) is properly aspherical at infinity, i.e., for any choice of base ray, the homotopy pro-groups \( \pi_n(\widetilde{K}_\mathcal{P}) = 0 \) are pro-trivial for \( n \geq 2 \), and the fundamental pro-group \( \pi_1(\widetilde{K}_\mathcal{P}) \) is pro-isomorphic to a telescopic tower

Thus, the proof of Theorem 1.6 is the same as that of Theorem 1.2 in the 1-ended case.
Remark 3.3. It is worth pointing out that sometimes the strategy followed to prove that some classes of 1-ended groups admit a weak $\mathcal{Z}$-structure includes showing that the fundamental pro-group is pro-(finitely generated free). Under semistability at infinity, this property about the fundamental pro-group amounts to saying that the groups under study are \textit{properly 3-realizable}, i.e., they can be realized by a finite 2-dimensional CW-complex whose universal cover is proper homotopy equivalent to a 3-manifold. See [20, Thm. 1.2] and [5, Thm. 5.22]. The above is the case of this and other papers, see [15] for instance. At the time of writing it is unknown whether there is a relation between proper 3-realizability and the existence of a weak $\mathcal{Z}$-structure.

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