On the way to understanding the electromagnetic phenomena

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Abstract

On the basis of the ordinary mathematical methods we discuss new classes of solutions of the Maxwell’s equations discovered in the papers by D. Ahluwalia, M. Evans and H. Múnera et al.

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Recently several authors found additional solutions to relativistic wave equations. Here they are listed:

- \( E = 0 \) solution of the Maxwell’s \( j = 1 \) equations [1] which was found on the basis of the consideration of the characteristic equation (in the momentum representation).

- \( B^{(3)} \) Evans-Vigier field [2], which was obtained as a cross-product of the transverse modes of electromagnetism: \( B^{(1)} \times B^{(2)} = iB^{(0)}B^{(3)*} \) and cyclic.

- Non plane-wave solutions of the Klein-Gordon equation [3a,b] by Múnera et al., which were obtained by using unconventional basis functions and “coupling anzatz”, see [3a, Eqs. (11,12)].

- Múnera and Guzmán generalized solution of Maxwell’s equations in terms of potentials [3c,d].

- Chubykalo and Smirnov-Rueda ‘method of separated potentials’, ref. [4], which permits us to consider a function with implicit dependence on time as full-value solution of the Maxwell’s (and/or D’Alembert) equations.

Why did so many new unexpected solutions appear at once? Let us look at this issue by using ordinary methods of solving the system of partial differential equations [5,6].

It is well known that the set of dynamical Maxwell’s equations are equivalent to the following set, e.g., [7, Eqs.(4.21,4.22)]:

\[
\begin{align*}
\nabla \times [E + iB] - i(\partial/\partial t)[E + iB] &= 0, \\
\nabla \times [E - iB] + i(\partial/\partial t)[E - iB] &= 0.
\end{align*}
\]

This is a system of partial differential equations. It is easy to see that the second equation is just the parity conjugate (\( x \rightarrow -x \)) of the first one if one uses ordinary interpretation of \( E \), a \textit{vector}, and \( B \), an \textit{axial vector}.

In the framework of this paper we shall look for solutions of (1a) in the generalized form

\[
A \equiv E + iB \sim a \exp(\lambda t + \kappa \cdot r),
\]

where \( \lambda \) and \( \kappa \) are some unknown parameters, which provide characteristic polynomo, and \( a = \text{column}(a_1 \quad a_2 \quad a_3) \) is some \textit{constant} vector, which is defined by the boundary and/or normalization conditions. Thus, at the moment we are not going to restrict our consideration by the plane waves. As a result of the use of the method of characteristic polynomo for the differential equation

\[
[(\partial/\partial t) + \mathbf{J} \cdot \mathbf{\nabla}]^j_i A^j = 0,
\]

with \((\mathbf{J}^i)_{jk} = -i\epsilon^{ijk}\), we obtain the algebraic equation for parameters \( \lambda \) and \( \kappa \):

\[
\text{Det}[\lambda + (\mathbf{J} \cdot \kappa)]_{ij} = 0.
\]

\[1\] Issues related with the source equations will be discussed in detail elsewhere.

\[2\] More rigorous consideration will be reported in the extended version.
It has solutions $\lambda = 0$ and $\lambda = \pm |\kappa|$. In fact, we repeated the procedure of ref. [1], but standing at the most general position we do not know yet, how $\lambda$ and $\kappa$ are connected with energy and momentum. Thus, the general solution of the first Maxwell equation (1a) may be presented, for instance, in the form:

$$E + iB = A_1 \exp[\alpha_1(|\kappa| t + \kappa \cdot r)] + A_2 \exp[\alpha_2(-|\kappa| t + \kappa \cdot r)] + A_3 \exp[\alpha_3(\kappa \cdot r)] ,$$

(4)

with the complex vectors $A_1$, $A_2$ and $A_3$ and the constants $\alpha_i$ to be defined from normalization and boundary conditions. We have several remarks: a) The plane waves are obtained only if associate $\lambda = \pm iE$ and $\kappa = \pm i\kappa$, what is not obligatory. It becomes clear that the Maxwell equations may describe physical states which are different from plane waves, so that the hypothesis on the quanta of light waves may be regarded as a particular case only, cf. [3a,4]; b) The solution with $\lambda = 0$ enters in the general solution of the system of differential equations. It may be removed only by means of the special choice of boundary conditions; c) In general, $\kappa$ can be substituted by $-\kappa$ (an analog of the space inversion transformation in the momentum representation), i.e. the solution can be written in several forms, which should be equivalent in the physical content; d) In the same way one can find the general solution of the second equation (1b).

While one can analyze these issues further (and more rigorously) we stop here in order to be possible to publish an extended version elsewhere and because of volume restrictions of the journal. But, below we shall show that non-plane-wave solutions of the Maxwell’s equations, arise also from different viewpoint [2], they are not zero and that the field related with these unusual modes may be irrotational under certain conditions. Firstly, we write particular plane-wave solutions of the Maxwell’s equations in the form

$$A(r) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i(\omega t - \mathbf{k} \cdot r)} , \quad B(r) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} e^{-i(\omega t - \mathbf{k} \cdot r)} ,$$

(5)

with the objects $a = \text{column}(a_1 \quad a_2 \quad a_3)$ and $b = \text{column}(b_1 \quad b_2 \quad b_3)$ at the exponents being the constant vectors with respect to the space inversion operation. In order to form an axial vector one should add the space-inverted vectors to the defined ones. Thus, we obtain

$$C(r) = \frac{1}{2} \left( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) e^{i(\omega t - \mathbf{k} \cdot r)} - \left( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) e^{i(\omega t + \mathbf{k} \cdot r)} = \left( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \sin(\mathbf{k} \cdot \mathbf{r}) e^{i(\omega t - \pi/2)} ,$$

(6a)

$$D(r) = \frac{1}{2} \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) e^{-i(\omega t - \mathbf{k} \cdot r)} - \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) e^{-i(\omega t + \mathbf{k} \cdot r)} = \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \sin(\mathbf{k} \cdot \mathbf{r}) e^{-i(\omega t - \pi/2)} .$$

(6b)

We shall further prove the following theorems:

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3Here and below the notation may have nothing to do with the accustomed notation for the vectors of electric and magnetic fields.

4We still work in the coordinate representation and want to form an axial vector with respect $\mathbf{r} \rightarrow -\mathbf{r}$. We do not bother the properties of the vectors with respect to $\mathbf{k} \rightarrow -\mathbf{k}$.
Theorem 1. The quantity $F = C \times D$ conserves in time:
\[
\frac{\partial}{\partial t} F = 0.
\] (7)

Proof. By the straightforward calculation one can find the explicit form of the axial vector $F$. Here it is:
\[
F = \begin{pmatrix}
a_2 b_3 - a_3 b_2 \\
a_3 b_1 - a_1 b_3 \\
a_1 b_2 - a_2 b_1
\end{pmatrix} \sin^2(k \cdot r).
\] (8)

By definition the a and b are the constant vectors. Thus, Eq. (8) contains no dependence on the time $t$, so $\frac{\partial F}{\partial t} = 0$. Theorem is proven.

Theorem 2. If $A$ and $B$ chosen in the form (5) satisfy the Maxwell’s equations (1a,1b) respectively (or vice versa), the quantity $F = C \times D$ a) is irrotational; b) satisfies both equations (1a) and (1b); c) is zero in all space if and only if $A$ or $B$ is zero.

Proof. In order to prove a) and b) it is sufficiently to prove that $(J \cdot \nabla)^{ij} F^j = 0$ because of the operator identity $\nabla \times \equiv \text{curl}$, the definition of the $j = 1$ matrices and thanks to the proven Theorem 1. By direct calculations one comes to
\[
(J \cdot \nabla)^{ij} F^j = i \nabla \times F = i \sin 2(k \cdot r) \{k \times [a \times b]\} \equiv \\
\equiv i \sin 2(k \cdot r) \{a(k \cdot b) - b(k \cdot a)\} \equiv \\
\equiv i \sin 2(k \cdot r) \{a \times [k \times b] - b \times [k \times a]\}.
\] (9)

After using the Maxwell’s equations (1a,1b) one finds $k \times a = -i\omega a$ and $k \times b = +i\omega b$. Substituting these relations to (9) we are convinced that $F$ is irrotational and, thus, combining this statement with the previous one (conservation of $F$ in time) we prove that the quantity $F$ satisfies both Maxwell’s equations (1a) and (1b). Following the accustomed terminology it can be named as “longitudinal”.

Let us now assume that $F = 0$ in all the space. If $a \neq 0$ and $b \neq 0$ this can occur only if $a \times b = 0$ for the propagating wave states. By definitions they are complex vectors. So, if denote $c = \Re e \, a$, $d = 3m \, a$, $e = \Re e \, b$ and $f = 3m \, b$ we can deduce that in order the searched cross product would be equal to zero it is necessary
\[
c \times e = +d \times f , \quad d \times e = -c \times f.
\] (10)

Let us firstly consider the case when $c$ and $e$ are not collinear, $d$ and $f$ are not collinear, i.e. the first relation is not equal to zero. It can be fulfilled if and only if the real vectors $c$, $d$, $e$ and $f$ are all coplanar. Thus, let us choose two vectors $c$ and $d$, which are implied to be linear independent, then other two can be expanded as follows
\[
e = a_{11} c + a_{12} d \quad , \quad f = a_{21} c + a_{22} d
\]
with real coefficients $a_{ij}$. Considering $c \times e$ and $d \times f$ we are convinced that the quantity $a_{12} = -a_{21}$. Considering the second equation in (10) we are convinced that $a_{11} = a_{22}$. Thus,
b = e + if = (a_{11} - ia_{12})(c + id) and, hence, b ∼ c_1e^{iβ}a. We have a contradiction with the statement that A and B, which are not phase free, satisfy different Maxwell's equations (1a) and (1b). Next, if d = λc from the set (10) we deduce that this can occur if and only if λ^2 = −1 what is again in contradiction with the fact that c, d, e and f are real vectors. Finally, if c = λ_1e and, then, d = λ_2f one deduces:

\[ d \times e = \lambda_2f \times e = -\lambda_1e \times f \]

and, therefore, λ_1 = λ_2 = λ. Again, b ∼ (1/λ)a and one has a contradiction with the conditions of the theorem. So, using the method of “from the inverse statement” we can say that a × b cannot be equal to zero and, hence, F ≠ 0. The end of the proof.

**Theorem 3.** If A and B are solutions of the same equations (1a) or (1b) and \( ω = ± |k| \), one can deduce the following relation for the axial vector F and the corresponding polar curl F:

\[ \text{curl} (\text{curl} F) + 4\nabla^2 F = 0 \]  \hspace{1cm} (11)

**Proof.** The theorem is proving by the direct calculations. One has

\[ \nabla \times F = ±4iω \cot(k \cdot r)F \]  \hspace{1cm} (12)

The signs depend on whether A and B satisfy simultaneously the first equation (1a), the sign is “−”, or the second one, the sign is “+”. Next,

\[ \nabla^2 F = 2k^2 \cos 2(k \cdot r) [a \times b] = 2k^2 \frac{\cos 2(k \cdot r)}{\sin^2(k \cdot r)}F \]  \hspace{1cm} (13)

and, if one takes into account (8,12),

\[ \nabla \times (\nabla \times F) = -8ω^2 \frac{\cos 2(k \cdot r)}{\sin^2(k \cdot r)}F \]  \hspace{1cm} (14)

Substituting these equations in (11) we are convinced in the validity of the theorem. It is necessary to stress that Eq. (11) is a relation, which was obtained after taking into account certain constraints between k, a, b and \( ω \). It cannot be considered as a dynamical equation. This is due to the operator identity curl curl ≡ grad div − \( \nabla^2 \). If we rewrite (11) with taking into account this identity we are convinced that the corresponding equation does not have solutions unless F = const, and/or \( k \cdot r = ±\frac{π}{4}, ±\frac{3π}{4}, \ldots \), or k ≡ 0.

The conclusion is: the Maxwell’s electromagnetic theory looked by a mathematician/theoretical physicist glance has richer structure comparing with views believed since the proposal of quantum field nature of the light. In the recent series of the papers (see for references [8]) we analyzed its shortcomings and advantages comparing with the more general Weinberg formalism [7]. The question, whether the former is equivalent to the latter, is still required further rigorous elaboration.

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