Abstract

The problem of dimension reduction is of increasing importance in modern data analysis. In this paper, we consider modeling the collection of points in a high dimensional space as a union of low dimensional subspaces. In particular we propose a highly scalable sampling based algorithm that clusters the entire data via first spectral clustering of a small random sample followed by classifying or labeling the remaining out of sample points. The key idea is that this random subset borrows information across the entire data set and that the problem of clustering points can be replaced with the more efficient and robust problem of “clustering sub-clusters”. We provide theoretical guarantees for our procedure. The numerical results indicate we outperform other state-of-the-art subspace clustering algorithms with respect to accuracy and speed.

Keywords: dimension reduction, subspace clustering, sub-cluster, random sampling, scalability, handwritten digits, spectral clustering
1. Introduction

In data analysis, researchers are often given data sets with large volume and high dimensionality. To reduce the computational complexity arising in these settings, researchers resort to various dimension reduction techniques. To this end, traditional methods like PCA (Hotelling, 1933) use few principal components to represent the original data set; similarly, factor analysis (Cattell, 1952) seeks to get linear combinations of latent factors; subsequent works of PCA include kernel PCA (Schölkopf et al., 1998), generalized PCA (Vidal et al., 2005); manifold learning (Belkin and Niyogi, 2003) assumes data points collected from a high dimensional ambient space concentrate around a low dimensional manifold, and multi-manifold learning (Liu et al., 2011) considers the setting of a mixture of manifolds. In this paper, we focus on one of the simplest manifold, a subspace, and consider the subspace clustering problem. Specifically, we approximate the original data as an union of subspaces. Representing the data as a union of subspaces allows for more computationally efficient downstream analysis on a variety of problems such as motion segmentation (Elhamifar and Vidal, 2009), handwritten digits recognition (You et al., 2016a), and image compression (Hong et al., 2006).

1.1 Related Work

Many techniques have been developed for subspace clustering, see Vidal (2010) for a review. The mainstream methods usually include two phases: (1) calculating the affinity matrix; (2) applying spectral clustering (Ng et al., 2002) to the affinity matrix to compute a label for each data point. For phase (1), the property of self-representation is often used to calculate the affinity matrix: self-representation states that a point can be represented by a linear combination of other points in the same subspace. Specifically, Elhamifar and Vidal (2009) proposed the sparse subspace clustering (SSC) algorithm which solves the LASSO minimization problem $N$ times, where $N$ is the total number of data points. The drawback of SSC is its complexity of $O(N^2)$ in both time and space, which limits its application to large data sets. To address this limitation, a variety of methods have been proposed to replace LASSO in constructing the affinity matrix. Heckel and Bölcskei (2015) used inner products with thresholding (TSC) to calculate the affinity between each pair of points, Park et al. (2014) used a greedy algorithm to find for each point the linear space spanned by its neighbors, similarly Dyer et al. (2013) and You et al. (2016c) used orthogonal matching pursuit (OMP), You et al. (2016b) used elastic the net for subspace clustering (ENSC) and proposed an efficient solver by active set method. However, all these approaches require running spectral clustering on the full $N \times N$ affinity matrix. A Bayesian mixture model, was proposed for subspace clustering in Thomas et al. (2014), however MCMC based parameter inference is not computationally feasible to scale to large data. Zhou et al. (2018) used a deep learning based method which does not have theoretical guarantee.

Recently, there have been two methods that increase the scalability of sparse subspace clustering. In Peng et al. (2016) clusters a random subsample of the data and then uses this clustering to classify or label the out-of-sample data points. This method scales well when the random subsample is small, however a great deal of information is discarded as only the information in the subset is used. In You et al. (2016a) a divide and conquer strategy is used for SSC—the data set is split into several small subsets on which SSC is run, and
clustering results are merged. This method cannot reduce the computational complexity of the SSC by an order of magnitude so is limited in its ability to scale to large data.

1.2 Contribution

In this paper, we propose a novel, efficient sampling based algorithm with provable guarantees that extends the ideas in previous subsampling methods (Peng et al., 2016; You et al., 2016a). A key observation driving our algorithm is the observation that only a small fraction of the original data set is needed to recover the membership of each point, hence clustering a subset of the data should be adequate. In particular, for each point in the subsample we find the nearest neighbors in the complete data and use these points to construct a sub-cluster, these sub-clusters contain information from the entire data and not just the random sample. The affinity matrix is constructed from the sub-clusters rather than the random subset, the idea is that constructing the affinity matrix from the sub-clusters integrates information across the data and should be more robust.

We provide theoretical guarantees for our procedure in Section 3. The analysis reveals that under mild conditions, the subspaces can share arbitrarily many intersections as long as most of their principal angles are larger than a certain threshold. While our algorithm for finding neighborhood points is similar to that of Heckel and Bölcskei (2015), the assumptions underlying our theorems are more realistic. We take into account the fact that after normalization the noisy terms will no longer follow a multivariate normal distribution.

We apply our algorithm to synthetic and handwritten digit images data sets and demonstrate that our method is highly scalable and robust to noise with superior accuracy compared to other state-of-the-art methods.

1.3 Paper Organization

The rest of this paper is organized as follows: in Section 2, we describe the model setting and the algorithms used to implement our procedure and practical recommendations in choosing the parameters, in Section 3 we state theoretical guarantees for our procedure and explain in some detail the geometric and distributional intuitions underlying our procedure. The detailed proofs can be found in Appendix A. In Section 4, we present numerical experiments on three data sets and compare our method with state-of-the-art methods.

The rest of this paper is organized as follows: in Section 2, we describe the model setting and algorithms in our method and the practical recommendations in choosing the parameters, in Section 3 we list the theorems developed for our method and explain more about the intuitions behind our method, the detailed proofs can be found in Appendix A, in Section 4, we present our numerical experiments on three data sets and compare our method with state-of-the-art methods.

1.4 Notation

We are given a data set \( Y \) with \( N \) data points in \( \mathbb{R}^D \). If there is no noise, we assume each data point of \( Y \) lies in exactly one of \( K \) linear subspaces denoted by \( \{ \mathcal{S}_k \}_{k=1}^K \), otherwise we assume each point concentrates near one of the \( K \) subspaces. Here \( K \) is a known constant.
and $S_k$ is the $k$th linear subspace. The subspace clustering problem aims to assign to each point in $Y$ membership to a subspace (cluster) $S_k$.

Assuming subspace $S_k$ has dimension $d_k$ and we write $U_k \in \mathbb{R}^{D \times d_k}$ as the corresponding orthogonal base. The data set $Y$ contains $N$ points each of which is in $\mathbb{R}^D$. The number of points observed from cluster $S_k$ is $N_k$. We use $y_1^{(k)} \in \mathbb{R}^D$ to represent a single point concentrated around $S_k$, the set $\{y_1^{(k)}, ..., y_{N_k}^{(k)}\}$ contains all points that concentrate around $S_k$. Unless specified otherwise, lower bold letters are used to represent vectors, while capital bold letters are used to represent matrices. We use subscript with parenthesis to represent the order statistics of entries in a vector, for example $a_{(i)}$ is the $i$th smallest entry in vector $a$.

2. Sampling Based Subspace Clustering

In this section, we introduce our sampling based algorithm for subspace clustering (SBSC). In section 2.1 we present the detailed steps of this algorithm. In section 2.2 we discuss the issues regarding tuning parameters. In section 2.3 we provide comments on both the intuition underlying the procedure and the advantages of our procedure. Throughout this section, we assume the columns of $Y$ have unit $l_2$ norm.

2.1 The Algorithm for Sampling Based Subspace Clustering

Our main algorithm takes the raw data set $Y$ and several parameters as input and outputs the clustering assignment for each point in the data set, it proceeds in two stages (see Algorithm 1 for details):

1. Stage 1: Cluster a random sample:
   (a) Draw a subsample $\hat{Y}$ of $n \ll N$ points. Step 1 in Algorithm 1.
   (b) For each point $x_i \in \hat{Y}$ construct a sub-cluster $C_i$ that consists of the $d_{\max}$ nearest neighbors in $Y$ of $x_i$. Step 2 in Algorithm 1.
   (c) Compute the affinity matrix $D$ where each element $D_{ij}$ is the function of the distance between $C_i$ and $C_j$. Step 3 in Algorithm 1.
   (d) Sparsify the affinity matrix by removing possible spurious connections. Step 4 in Algorithm 1.
   (e) Cluster the points in $\hat{Y}$ based on spectral clustering of the sparsified adjacency matrix. Step 5 in Algorithm 1.

2. Stage 2: Label the points outside the random sample
   (a) Fit a ridge regression model to the subsample $\hat{Y}$ and corresponding labels $\ell$. Step in Algorithm 1.
   (b) Compute the residual error on each $Y \setminus \hat{Y}$ based on the ridge regression estimator and use the residual error to cluster the points $Y \setminus \hat{Y}$. See Algorithm 2 in Appendix B.

Step (1a) computes a neighborhood of points around each sampled points by thresholding inner product similarities, the same method that was used in Heckel and Bölcskei.
The intuition behind this step is that for normalized data, two vectors are more likely to lie in the same linear subspace if the absolute magnitude of the inner product between the points is large. The idea of using the distance between the subclusters to define an affinity matrix in step (1c) relies on the self-representative property of linear subspaces — see Theorem 2 for technical details as well as some of the basic concepts underlying self-representation. There is both theoretical and empirical evidence that sparsification of an affinity matrix by setting smaller elements to zero improves clustering results (Belkin and Niyogi, 2003; Von Luxburg, 2007). For this reason in step (1d) we threshold the affinity matrix. Once the subsample is clustered the out-of-subsample points are labeled via a regression approach where a regression model is fit on the clustered data, specifically a ridge regression model. The residuals of the regression model on the out-of-subsample points is used to cluster these points, see Algorithm 2 in Appendix B.

| input | Data $Y$, number of subspaces $K$, sampling size $n$, neighbor threshold $d_{\text{max}}$, regularization parameters $\lambda_1$ and $\lambda_2$, residual minimization parameter $m$, affinity threshold $t_{\text{max}}$. |
|-------|-------------------------------------------------|
| output: | The label vector $\ell$ of all points in $Y$. |
| 1. | Uniformly subsample $n$ points $\hat{Y}$ from $Y$, the indices of these points are $\mathcal{I}$; |
| 2. | Construct subclusters for $q = 1$ to $n$ do |
| | $p = |\langle Y_q, Y_{\mathcal{I}_q} \rangle|$; |
| | $\mathcal{C}_q := \{ i : |\langle Y_q, Y_i \rangle| \geq p(N-d_{\text{max}}) \}$. |
| end | |
| 3. | Construct adjacency matrix $D_{ij} = e^{-d(Y_{C_i}, Y_{C_j})/2}$ for $i, j = 1, \ldots, n$ and |
| | $d(Y_{C_i}, Y_{C_j}) = ||Y_{C_i} - Y_{C_j}(Y_{C_j}^T Y_{C_j} + \lambda_1 I)^{-1} Y_{C_j}^T Y_{C_i})||_F$ |
| | $+||Y_{C_j} - Y_{C_i}(Y_{C_i}^T Y_{C_i} + \lambda_1 I)^{-1} Y_{C_i}^T Y_{C_j})||_F$. |
| 4. | Sparsify the adjacency matrix |
| for $i = 1$ to $n$ do |
| | $v := D_i$; |
| | for $j = 1$ to $n$ do |
| | if $D_{ij} \leq v(n-t_{\text{max}})$ then |
| | $D_{ij} := 0$ |
| end |
| end |
| 5. | Cluster $\hat{Y}$. First compute $\tilde{D} = D + D^T$ and label $\hat{Y}$ by via spectral clustering of $\tilde{D}$; |
| 6. | Label the remaining points: Fit a ridge regression model to $\hat{Y}$ and label the out-of-sample data $Y \setminus \hat{Y}$ based on the residual error; |

**Algorithm 1:** The steps in the Sampling Based Subspace Clustering (SBSC) algorithm.


2.2 Practical Recommendations for Parameter Setting

In Algorithm 1, we assume the number of clusters is known—there are a variety of methods for estimation of the number of clusters from data, see (Ng et al., 2002)). Intuitively, \( n \) should be large enough so that it can well represent the structure of whole data set while still be relatively small to reduce the computational complexity, in our numerical experiments, we choose \( n \) to be linear in \( K \log N \). See section 3 for theoretical considerations.

Ideally, each sub-cluster \( Y_{C_i} \) should well represent the subspace it belongs to, i.e. contains at least one basis of that subspace. Therefore we want \( d_{\text{max}} \) to be larger than the dimension of a sub-cluster which is unknown, for this reason we set \( d_{\text{max}} \) to be linear in \( D \).

In choosing \( \lambda \), we recommend using \( \lambda = \frac{1}{D} \max_{i=1,...,n} \left\{ \sum_{j=1}^{d} \frac{1}{a_{ij}} \right\} \) for the noisy case, and \( \lambda = \frac{1}{N} \max_{i=1,...,n} \left\{ \sum_{j=1}^{d} \frac{1}{a_{ij}} \right\} \) for the noiseless case, here \( a_{ij} \) correspond to the \( j \)th positive eigenvalue of \( Y_{C_i} Y_{C_i}^T \), see Appendix A for theoretical considerations.

2.2.1 Threshold Selection

The spectral clustering algorithm can deliver exact clustering result (Von Luxburg, 2007) if the graph induced by the affinity matrix \( D + D^T \) has no false connections; and has exactly \( K \) connected components. For a large threshold parameter \( t_{\text{max}} \) on the affinity matrix more entries in \( D \) will be kept and our algorithm is more likely to have false connections, while small \( t_{\text{max}} \) eliminates false connections but might incur non-connectivity.

Let us consider a heuristic situation: the subset we sampled contains exactly the same points (hence \( \frac{n}{K} \) points) for each cluster. Then if we choose the threshold index \( t_{\text{max}} \) to be \( \frac{n}{2K} \), the induced graph from our affinity matrix will have no false connection (given that points from same subspace have bigger similarities between each other) and the clusters themselves will be connected, therefore the spectral clustering algorithm will deliver the exact clustering result (Luxburg et al., 2005).

In reality clusters do not usually have same points in \( \hat{Y} \), hence we choose \( t_{\text{max}} \) to start from a relatively large number \( \frac{n}{0.5K} \) and gradually increase it. Based on different threshold values, we can generate different label vectors on the subset \( \hat{Y} \), intuitively label vectors that can deliver highly accurate results should be similar to each other or stable. Based on this intuition, we developed a simple adaptive algorithm for finding an “optimal” affinity threshold \( t_{\text{max}} \), see supplementary code for details.

2.2.2 Combining Runs of the Algorithm

In order to stabilize the results delivered by our algorithm, we designed an algorithm to combine the results from several runs of Algorithm 1. Please note that unlike the classification problem, we need to unify the label vectors before voting or in other words we need to deal with label switching, see the code for details on how label switching is addressed.
2.3 Comments on the Algorithm

In this section, we make some comments on our algorithm and try to explain the intuitions behind it.

2.3.1 Motivation of Sampling

A theoretical result was developed in Luxburg et al. (2005), where under certain assumptions, the spectral clustering results on subset $\tilde{Y}$ will converge to the results on whole data set $Y$. While the result is not directly applicable to our algorithm since it requires the distance function to be continuous and larger than a fixed constant, it gives us the insight that as the sample size $n$ increases properly with $N$, $\tilde{Y}$ is almost as informative as $Y$.

Another motivation of using sampling-based algorithm is the computational complexity. Traditional spectral clustering based algorithms need to build the “neighborhood” for each of the $N$ points (by lasso, OMP etc.), thus the complexity is usually at least $O(N^2)$ (both in time and space), while sampling based algorithms do this step only for the subset, using classification algorithms to label the out of sample points requires $O(N \log N)$ in time (given that $n, D$ are in order of $O(\log N)$) with much less memory.

2.3.2 Advantages over Existing Sampling Based Methods

While most sampling based algorithms use only the information in $\tilde{Y}$, our algorithm seeks to borrow information from $Y$ by finding closest points for each sampled point among the whole data set, which makes it possible to get a neighborhood with decent size and no false connections for each sampled point, note for methods that apply clustering algorithms purely on the subset, each sampled point only has few neighborhoods (in order to reduce false connection).

The affinity matrix we build on $\tilde{Y}$ is calculated from the subcluster-wise distance, under which the affinity between two points in $\tilde{Y}$ is measured by the affinity between the subclusters they belong to, hence the affinity matrix is robust to noises in the sense that points from same subspace are more likely to have big affinities.

3. Clustering Accuracy

In this section, we analyze several theoretical properties of Algorithm 1. Specifically, we proved that under mild conditions, our algorithm has subcluster preserving property (defined later) in stage 1 and can deliver exact out of sample classification in stage 2 with high probabilities. Throughout this section we conducted our analysis under the noisy case, also we assume all subspaces have same dimension $d$ for simplicity.

3.1 Model Specification for Provable Results

Note in Algorithm 1, we assume the data matrix $Y$ has unit column norm, this can always be achieved by normalizing each column of original data matrix. Specifically, we write the data generating equation for original data point as $\hat{y}_i^{(k)} = \zeta_i^{(k)} \mathbf{U}_k \mathbf{a}_i^{(k)} + e_i^{(k)}$, where $\mathbf{a}_i^{(k)} \in \mathbb{R}^d$ is sampled from an uniform distribution on the surface of $\mathbb{S}^{d-1}$, $\zeta_i^{(k)}$ is the random
scalar such that $\zeta_{i}^{(k)2} \sim \chi_{d}^{2}$. For the noisy case, we assume $\hat{e}_{i}^{(k)} \sim \mathcal{N}(0, d\sigma^{2}I_{D})$, and for the noiseless case we simply let $\hat{e}_{i}^{(k)} = 0$.

We write $y_{i}^{(k)}$ as the normalized version of $\hat{y}_{i}^{(k)}$, in this paper we only work with the normalized data points, where we have $y_{i}^{(k)} = U_{i}a_{i}^{(k)}$ for the noiseless case; and $y_{i}^{(k)} = \frac{u_{k}a_{i}^{(k)} + \sigma e_{i}^{(k)}}{||u_{k}a_{i}^{(k)} + \sigma e_{i}^{(k)}||_{2}}$ for the noisy case, here each entry in $e_{i}^{(k)}$ follows $t$-distribution with $d$ degrees of freedom, $||u_{k}a_{i}^{(k)} + \sigma e_{i}^{(k)}||_{2}$ is the normalization constant of point(vector) $y_{i}^{(k)}$. It is straightforward to show $\frac{||e_{i}^{(k)}||_{2}^{2}}{d} \sim F_{D,d}$.

Note that our distributional setting of the noises is more realistic than that of Heckel and Bőlcskei (2015), in which the authors assumed $e_{i}^{(k)}$ comes from multivariate normal distribution even after normalization.

We write $\lambda_{i}^{(ij)} \geq \lambda_{2}^{(ij)} \geq \ldots \geq \lambda_{d}^{(ij)}$ correspond to the cosine values of principal angles between $S_{i}$ and $S_{j}$, hence $\lambda_{1}^{(ij)} \leq 1$ and $\lambda_{d}^{(ij)} \geq 0$, also note $\lambda_{k}^{(ij)} = \lambda_{k}^{(ji)}$ for $1 \leq k \leq d$ and $1 \leq i < j \leq K$. For each subspace $S_{k}$, we define the maximal affinity vector to quantify its closeness to all other subspaces.

**Definition 1** For each subspace $S_{k}$, its uniformly maximal affinity vector with respect to other subspaces is $[\lambda_{1}^{(k)}, \ldots, \lambda_{d}^{(k)}]$ such that

$$\lambda_{i}^{(k)} = \max_{j \neq k} \lambda_{i}^{(kj)}.$$

Intuitively, when the maximal affinity vector of the $i$th subspace has small entries, we should be able to decrease the “false discovery” in $Y_{C_{i}}$s. Formally, we have the following definition.

**Definition 2** We say Algorithm 1 has subcluster preserving property if each of the $Y_{C_{i}}$‘s only contains points from same subspace.

Given the subcluster preserving property and assume $Y_{C_{i}}$ concentrates around $S_{k}$, we can write $Y_{C_{i}} = U_{i}B_{i} + E_{i}$. Here each column of $B_{i} \in \mathbb{R}^{d \times (d_{max}+1)}$ is a sample from uniform distribution on $S^{d-1}$ divided by its corresponding normalization constant, and each column of $E_{i} \in \mathbb{R}^{D \times (d_{max}+1)}$ is a noise vector divided by its corresponding normalization constant (see previous discussion), we write $B_{i,j}$ as the $j$th column of matrix $B_{i}$, and similarly for $E_{i,j}$, then the norm of $U_{k}B_{i,j} + E_{i,j}$ is 1. For convenience we also write $B_{i}$ as the “un-normalized” version of $B_{i}$, hence $B_{i}$ has unit column norm, similar notation is used for $E_{i}$.

In constructing the affinity matrix $D$, a desired property is: points that concentrate around same subspace have relatively bigger affinities hence smaller distances between each other, this property can be formally defined as:

**Definition 3** We say $Y_{C_{i}}$ has the correct neighborhood property with distance function $d(\cdot, \cdot)$ if

$$d(Y_{C_{i}}, Y_{C_{j}}) < d(Y_{C_{i}}, Y_{C_{k}})$$

for any $1 \leq j \neq k \leq n$ such that $Y_{C_{j}}$ concentrates around the same subspace with $Y_{C_{i}}$ and $Y_{C_{k}}$ concentrates around a different subspace.
3.2 Theoretical Properties of SBSC

In this section, we will discuss three theoretical properties regarding Algorithm 1. Detailed proof can be found in Appendix A.

3.2.1 Assumptions and Conditions

The assumptions and conditions needed to prove the results are summarized in this section.

A1. There exists a constant $c$ such that $5 \log N \leq d \leq c \min_{j=1,\ldots,K} \log N$.

A2. Both $D$ and $d_{\max}$ are in linear order of $d$.

A3. There exist positive constants $c_1, c_2, c_3 \in (1, +\infty)$, $\eta_1$, $g_1$ and $g_2$, such that if we write $T = \frac{4g_2 + 2g_2^2}{1 - g_2^2} + \frac{1 + g_2}{1 - g_2} g_1$, $r_{ji} = (g_1^2 - \lambda^2)^+_{ij}$, and $s_{ji} = (g_1^2 - \lambda^2)^-_{ij}$ (for any $j = 1, \ldots, K$), the following inequalities hold for any $j = 1, \ldots, K$: $2(1 - T^2)^{\frac{d - 1}{2}} \geq \frac{\sqrt{2\pi d d_{\max}}}{N_j^{\frac{d}{2}}} \left( \sum_{i=1}^{d} r_{ji} \right) \geq \frac{\sqrt{d} \sum_{i=1}^{d} \sqrt{r_{ji}^2}}{c_1}, \sum_{i=1}^{d} r_{ji} \geq \frac{d}{c_2}, \sum_{i=1}^{d} r_{ji}^2 > \sum_{i=1}^{d} s_{ji}^2,

A4. Rank of $B_3$ is $d$; if we write $r_{ji} = (\frac{\sqrt{d}}{g} - \lambda^2)^+_{ij}$, and $s_{ji} = (\frac{\sqrt{d}}{g} - \lambda^2)^-_{ij}$ (for any $j = 1, \ldots, K$, $i = 1, \ldots, d$), the following inequalities hold for any $j = 1, \ldots, K$: $\sum_{i=1}^{d} r_{ji} \geq \frac{\sqrt{d} \sum_{i=1}^{d} \sqrt{r_{ji}^2}}{c_1}, \sum_{i=1}^{d} r_{ji} \geq \frac{d}{c_2}, \sum_{i=1}^{d} r_{ji}^2 > \sum_{i=1}^{d} s_{ji}^2, \sum_{i=1}^{d} r_{ji} > c_3 \sum_{i=1}^{d} s_{ji}$.

A5. $\|\left(\hat{B}, \hat{B}_t\right)^{-1}\|_F < q_0 \sqrt{d}$ for some positive constant $q_0$.

A6. We let $\lambda = \frac{1}{3q_0 \sqrt{15d(d_{\max} + 1)}}$ and $g(d) = \frac{1}{3\sqrt{15d(d_{\max} + 1)}}$, and write $q = \frac{\lambda g(d)}{3D \sqrt{D}(d_{\max} + 1)}$, then $q > 1$ and $\frac{D(q - 1)}{2(\sqrt{D} + \sqrt{d})} \geq \sqrt{\frac{d(1 + \eta_2)}{5}}$ for some positive constant $\eta_2$.

A7. For any $j = 1, \ldots, K$, $R_j$ (see Algorithm 2) has dimension $d$, it only contains points from same subspace.

A8. $\cup_{i=1}^{\eta n} Y_{C_i}$ contains exactly $n(d_{\max} + 1)$ different points.

A9. We let $g(d) = \frac{1}{\sqrt{Dm}}$ and $\lambda = \frac{1}{q_0 \sqrt{dm}}$ (see Algorithm 2 for definition of $m$), and write $q = \frac{\lambda g(d)}{3D \sqrt{Dm}}$, then $q > 1$ and $\frac{D(q - 1)}{2(\sqrt{D} + \sqrt{d})} \geq \sqrt{\frac{d(1 + \eta_3)}{5}}$ for some positive constant $\eta_3$.

Assumption A1 says $d$ is in order of $\log N$, in modern big data setting, we often encounter data sets in which data grow exponentially in dimension.

Assumption A3 is the subspace separation assumption, it essentially says for each pair of subspaces, most of their principal angles should be larger (hence the corresponding cosine values are smaller) than some threshold. The condition $2(1 - T^2)^{\frac{d - 1}{2}} \geq \frac{\sqrt{2\pi d d_{\max}}}{N_j^{\frac{d}{2}}}$ seems
complicated, under high-dimension large sample size setting (given $A_1$ and $A_2$) it can be simplified as $T \in (0, \sqrt{1 - e^{-\frac{\sigma}{15}}})$ (the constant $c$ comes from assumption $A_1$). Similarly for $D(g_2^2D\sigma^2 - 1)\sqrt{D + g_2^2\sigma} \geq \sqrt{d/(1 + \eta)}$, we can write the LHS of it as $\sqrt{d/(1 + \eta)}(1 - 1 + g_2^2D\sigma^2)$, hence the inequality can be achieved once $\frac{g_2^2D\sigma^2}{D\gamma^2}$ is large enough, this requires $\sigma$ in the order of $O(\frac{1}{\sqrt{D}})$.

Assumption $A_4$ says each subcluster $Y_{C_i}$ contains at least one base for the subspace it belongs to; the second half of $A_4$ is very similar to $A_3$ (with $g_2^2$ replaced by $g_1^2$), it is generally weaker than $A_3$ since $A_3$ will require $T$ to be small (often smaller than $\sqrt{\frac{2}{5}}$) hence smaller $g_1$, while $A_4$ does not put any additional conditions on $T$.

Assumption $A_6$ aims to bound the error terms, it is fairly straightforward to see this requires $\sigma$ in the order of $O(d^{-4.5})$.

Assumption $A_7$ is similar to $A_4$, we want the “classifier” for each subspace to be informative.

3.2.2 Theorems of SBSC

Three theorems regarding Algorithm 1 are discussed in this section. For stage 1, we will show that subcluster preserving property and correct neighborhood property hold with large probabilities.

**Theorem 1 (Subcluster-Preserving)** Under assumptions $A_1$ to $A_3$, the probability that Algorithm 1 has subcluster preserving property is at least

$$1 - \sum_{j=1}^{K} \frac{n_j(N_j - d_{\text{max}})}{d_{\text{max}}(N_j + 1)(N_j^{1/3} + 1)^2} - 2(K - 1)ne^{-\epsilon^2} - \frac{2N}{N^{1+\eta} - 2},$$

where $\epsilon = \frac{(1 - \frac{1}{\sqrt{D}})\sqrt{d}}{2n1 + 4c_1^2 + 2c_2}$.

While the above theorem is for noisy case, we can easily modify it to the noiseless case by dropping $g_2$ and any assumption that contains $\sigma$, then the probability of achieving subcluster preserving property is at least (see Appendix A for details):

$$1 - \sum_{j=1}^{K} \frac{n_j(N_j - d_{\text{max}})}{d_{\text{max}}(N_j + 1)(N_j^{2/3} + 1)^2} - 2(K - 1)ne^{-\epsilon^2}.$$
here \( \lambda > 0 \) is the regularization parameter. For each \( Y_i \) that concentrates around \( S_k \), we write \( G_i = Y_i C_i - U_k B_i B_i^T U_k^T = \hat{E} B_i^T U_k^T + U_k B_i \hat{E}_i^T + \hat{E} \hat{E}_i^T \). Given that we have subcluster preserving property for all subclusters, and assume A4 to A6 are true. We pick \( \lambda = \frac{1}{3q_0 \sqrt{15d(d_{\max}+1)}} \), then all \( Y_i \)'s have the correct neighborhood property with the distance function above with probability at least

\[
1 - 4n(n - 1)e^{-\epsilon^2} - \frac{2N}{N^{1+\eta_2}} - \frac{2N}{N^{1+\eta_3}} - \frac{2N}{N^{1+\eta_3}} - \frac{2N}{N^{1+\eta_3}}.
\]

where \( \epsilon \) is the same as in Theorem 1.

For the noiseless case, we only need the first two terms in the above quantity.

In stage 2 of Algorithm 1, we want to develop the theoretical property of out of sample classifications. Note this is very similar to that of Theorem 2 since we essentially doing ridge regression for each out of sample point, and assigning the label corresponds to the cluster that delivers smallest residual.

**Theorem 3 (Out of Sample Classification)** Denote the points sampled in Algorithm 2 (see Appendix B) by \( R_j \in \mathbb{R}^{D \times m} \) \((j = 1,..,K)\), and assume A1 to A3, A6 to A9 are true. Then for \( \lambda = \frac{1}{q_0 \sqrt{dr}} \), Algorithm 2 delivers the exact out of sample classifications with probability at least

\[
1 - 2(K - 1)[N - n(d_{\max} + 1)]e^{-\epsilon^2} - \frac{2N}{N^{1+\eta_3}} - \frac{2N}{N^{1+\eta_3}} - \frac{2N}{N^{1+\eta_3}}.
\]

While the theorem above requires exact clustering in stage 1 (see Assumption A7), in numerical experiments we can allow certain degree of errors in stage 1. Note subcluster preserving property and correct neighborhood property together still cannot guarantee the no false discovery property in stage 1, unless the thresholding step can eliminate out all false connections in affinity matrix \( D \).

### 4. Experimental Results

In this section, we test the scalability and tolerance to noise of our algorithm on synthetic data set and compare the performance of our algorithm with other state-of-the-art algorithms on clustering images of handwritten digits. We use clustering accuracy (see You et al., 2016c), nmi (see Zhou et al., 2018) and running time as the metrics for performance evaluation. All the reported results of our algorithm are averaged over 10 trials.

For randomized algorithms (sampling based algorithms etc.), we also report the standard deviations of accuracy and nmi. The parameters setup for other algorithms can be found directly in the supplementary codes.

#### 4.1 Results on Synthetic Data Set

The data generation mechanism of our synthetic data was based on section 3.1. Specifically, we have \( K = 15 \) linear subspaces, each has dimension \( d_i = 5 \) and \( N_i = 10000 \) points (so
Figure 1: **Tolerance to Noise:** For each noise level, we generate 10 data sets and our algorithm was applied on each of them. The final result was average over 10 different data sets for each noise level.

From Figure 1, we can see that our algorithm can deliver exact clustering in the noiseless case. While as noise level increases both accuracy and nmi decrease correspondingly, they are still acceptable.

4.1.1 Tolerance to Noise

In this section, we test the tolerance to noise of our algorithm. The noise level $\sigma$ ranges from 0 (the noiseless case) to 0.25. For each noise level, we simulated 10 data sets and apply our algorithm on each of them, the final result was averaged over these 10 independent simulations, we also report the standard deviations of accuracy and nmi with respect to each noise level.

4.1.2 Scalability

In this section, we test the scalability of our algorithm. Specifically, we gradually increase $N_i$ from 10000 to 8000, so the corresponding $N$ increases from 150000 to 1200000. Throughout this section we fix $\sigma = 0.2$.

As shown from Figure 2, the relationship between running time and $N$ seems to be almost linear, which agrees with our complexity analysis on Algorithm 1.
Figure 2: **Scalability:** We change the number of points $N$ from 150000 to 1200000, for each $N$ we generate 10 data sets and the final result was average over 10 independent runs.

### 4.2 Results on Handwritten Digits

In this section we apply our algorithm to the Pen-Based Handwritten Digits (Pendigits) data set and MNIST data set. While they are both handwritten digits data sets, the Pendigits data set is medium sized with low dimension, MNIST is relatively large with high dimension.

The results are compared to other state-of-the-art algorithms, include scalable representation-based algorithms (Peng et al., 2016), SSC and its varies (Elhamifar and Vidal (2009), You et al. (2016a), You et al. (2016c), You et al. (2016b)), TSC (Heckel and Bölcskei, 2015). We attended to replicate their results on our machine to make direct comparisons, some of the results were copied from the original papers due to the unavailability of codes.

#### 4.2.1 Pen-Based Handwritten Digits Data

The Pendigits data set contains $N = 10992$ samples with $K = 10$ clusters, the dimension of ambient is $D = 16$. In implementing our algorithm on Pendigits data set, we set $n = 300$, $d_{\text{max}} = 4$, $m = 10$. To show the impacts of bagging number, we report the results based on three different bagging numbers 6, 8 and 10.

In Table 1, we can see TSC performs well with the highest accuracy, nmi and decent speed. Here SBSC performs decently, note the performance of SBSC gets better as we increase the number of bagging, which is consistent with the intuition.
### Table 1: Results on Pendigits

The first three methods are Algorithm 1 based on three different bagging numbers. The accuracy and nmi for sampling-based algorithms were averaged over 10 independent runs, here we also report the standard deviations of accuracy and nmi.

| Method    | Accuracy (%) | NMI (%) | Runtime (sec.) |
|-----------|--------------|---------|----------------|
| SBSC(6)   | 81.375 (1.0338) | 74.26 (1.091) | 7.6 |
| SBSC(8)   | 81.746 (0.7875) | 74.908 (1.1471) | 10.5 |
| SBSC(10)  | 82.213 (0.9179) | 75.55 (1.267) | 13.4 |
| SSSC      | 71.99 (2.06) | 69.19 (0.91) | 18.3 |
| SLSR      | 73.95 (1.3) | 66.72 (0.93) | 12.1 |
| SLRR      | 78.58 (2.57) | 71.77 (1.8) | 4.5 |
| SSC       | 73.8 (2.57) | 73.11 (1.8) | 5590 |
| TSC       | 87.36 | 83.50 | 11.5 |
| SSC-OMP   | 11.1 | 2.37 | 5 |
| ENSC      | 82.82 | 76.07 | 30.9 |

Table 1: **Results on Pendigits**: The first three methods are Algorithm 1 based on three different bagging numbers. The accuracy and nmi for sampling-based algorithms were averaged over 10 independent runs, here we also report the standard deviations of accuracy and nmi.

#### 4.2.2 The MNIST Database of Handwritten Digits

The MNIST database (MNIST) contains $N = 70000$ data points, each point represents an image of handwritten digit. The original data was transferred into $\mathbb{R}^{500}$ by convolutional neural network and PCA (You et al., 2016c). Again the number of clusters $K$ is 10.

Throughout this section, we choose $n = 500$, $d_{\text{max}} = 29$, $m = 100$, number of bagging equals to 6. Although not reported here, Algorithm 1 is robust to the changes of parameters. From Table 2 we can see that SBSC delivers the highest accuracy with minimal running time. The high efficiency of SBSC makes it possible to use 6 baggings without taking too much time.

#### 5. Conclusions

To the best of our knowledge, our algorithm is the first scalable subspace clustering algorithm with theoretical performance guarantee. Empirically, it can deliver accurate clustering result with high efficiency.

While the idea of subsampling was discussed by other researchers before (Peng et al., 2016), the highlights of this paper are finding neighborhood points among the whole data set and using cluster-wise distance to cluster points in the subsample, in turn this is robust to sampling bias noises.

In calculating cluster-wise distances and classifying out of sample points, ridge regression seems to be the most direct method, please note the algorithm itself is highly flexible, readers
| Method      | Accuracy (%) | NMI (%)     | Runtime (sec.) |
|------------|--------------|-------------|----------------|
| SBSC (6)   | 97.0699 (0.1883) | 92.4602 (0.3508) | 381            |
| SSC (DC1)* | 96.55        | NA          | 5254           |
| SSC (DC2)* | 96.1         | NA          | 4390           |
| SSC (DC5)* | 94.9         | NA          | 1596           |
| SCCC       | 81.72 (0.9)  | 84.41 (1.03) | 829            |
| SLSR       | 76.45 (3.72) | 77.92 (2.37) | 530            |
| SLRR       | 77.47 (4.42) | 81.46 (1.8)  | 719            |
| TSC        | 85.04        | 89.73       | 1276           |
| SSC-OMP    | 81.51        | 84.45       | 1036           |
| ENSC       | 93.79        | 88.8        | 1214           |

Table 2: **Results on MNIST**: The results for randomized algorithms are average over 10 independent runs. The results of methods with star marks are copied from original paper. NA means not available.

are encouraged to try different distance functions, classification methods and even metrics in finding neighborhood points.
Appendix A. Proofs of Theorems

In this section, we will prove the theorems from Section 3. The following Lemmas are used to prove Theorem 1.

**Lemma 1** Let \( b \in \mathbb{R}^d \) sampled uniformly from \( S^{d-1} \), and \( \lambda_k(k = 1, ..., d) \) be constants such that \( 1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0 \). For constant \( g_1 \in (\lambda_d, \lambda_1) \), we write \( r_i = (g_1^2 - \lambda_i^2)_+ \) and \( s_i = (g_1^2 - \lambda_i^2)_- \). Assuming that \( \sum_{i=1}^{d} r_i > \sum_{i=1}^{d} s_i \), then

\[
\sum_{i=1}^{d} |\lambda_i b_i|^2 < g_1^2
\]

with probability at least \( 1 - 2e^{-r^2} \), where

\[
\epsilon = \frac{\sum_{i=1}^{d} (r_i - s_i)}{\sqrt{\sum_{i=1}^{d} r_i^2} + \sqrt{\sum_{i=1}^{d} s_i^2}}.
\]

**Proof** We write \( b_i = \frac{z_i}{\sqrt{\sum_{i=1}^{d} z_i^2}} \) as the \( i \)th entry of \( b \), where \( z_i \)'s are i.i.d. \( N(0, 1) \) random variables. The goal is to bound

\[
\mathbb{P} \left[ \sum_{i=1}^{d} (g_1^2 - \lambda_i^2)_- \cdot z_i^2 \geq \sum_{i=1}^{d} (g_1^2 - \lambda_i^2)_+ \cdot z_i^2 \right] = \mathbb{P} \left[ \sum_{i=1}^{d} s_i \cdot z_i^2 \geq \sum_{i=1}^{d} r_i \cdot z_i^2 \right].
\]

Note \( g_1 \in (\lambda_d, \lambda_1) \), hence both \( \sum_{i=1}^{d} r_i \) and \( \sum_{i=1}^{d} s_i \) are strictly positive.

Now we write \( X = \sum_{i=1}^{d} s_i \cdot z_i^2 \) and \( Y = \sum_{i=1}^{d} r_i \cdot z_i^2 \), applying Lemma 1 in Laurent and Massart (2000) we have for positive constants \( \epsilon_1 \) and \( \epsilon_2 \)

\[
\mathbb{P} \left[ X \geq \sum_{i=1}^{d} s_i + 2 \sqrt{\sum_{i=1}^{d} s_i^2 \epsilon_1 + 2 s_1 \epsilon_1^2} \right] \leq e^{-\epsilon_1^2},
\]

\[
\mathbb{P} \left[ Y \leq \sum_{i=1}^{d} r_i - 2 \sqrt{\sum_{i=1}^{d} r_i^2 \epsilon_2} \right] \leq e^{-\epsilon_2^2}.
\]

We set \( \epsilon_1 = \epsilon_2 \) and

\[
\sum_{i=1}^{d} s_i + 2 \sqrt{\sum_{i=1}^{d} s_i^2 \epsilon_1 + 2 s_1 \epsilon_1^2} = \sum_{i=1}^{d} r_i - 2 \sqrt{\sum_{i=1}^{d} r_i^2 \epsilon_2}.
\]

Solving the above quadratic equation we have

\[
\epsilon_1 = \epsilon_2 = \frac{\sum_{i=1}^{d} (r_i - s_i)}{\sqrt{\sum_{i=1}^{d} r_i^2} + \sqrt{\sum_{i=1}^{d} s_i^2}}.
\]
Consequently
\[
\mathbb{P}[X \geq Y] \leq \mathbb{P}[X \geq \sum_{i=1}^{d} s_i + 2\sqrt{\sum_{i=1}^{d} s_i^2 \epsilon_1 + 2s_1 \epsilon_1^2}] + \mathbb{P}[Y \leq \sum_{i=1}^{d} r_i - 2\sqrt{\sum_{i=1}^{d} r_i^2 \epsilon_2}]
\]
\[
\leq e^{-\epsilon_1^2} + e^{-\epsilon_2^2}.
\]
Substituting \( \epsilon_1 \) and \( \epsilon_2 \) into the inequality above yields the result. \( \blacksquare \)

Based on Lemma 1, we can build a concentration inequality for F-distributed random variables.

**Corollary 1** Let \( X \sim F(m, n) \) and \( m, n \geq 2 \), then for constant \( q > 1 \), we have
\[
\mathbb{P}[X \geq q] \leq 2e^{-\epsilon^2},
\]
where \( \epsilon = \frac{1}{2}[-(\sqrt{m} + \frac{am}{\sqrt{n}}) + \sqrt{(\sqrt{m} + \frac{am}{\sqrt{n}})^2 + 2m(q-1)}] \).

**Proof** We write \( b_i = \frac{z_i}{\sum_{i=1}^{m+n} z_i^2} \), and \( X = \frac{\left(\sum_{i=1}^{m} z_i^2\right)/m}{\left(\sum_{i=m+1}^{m+n} z_i^2\right)/n} \) where \( z_i \sim i.i.d. N(0, 1), i = 1, ..., m+n \). It follows
\[
\mathbb{P}[X \geq q] = \mathbb{P}\left[\sum_{i=1}^{m} \frac{1}{mq} \cdot z_i^2 \geq \sum_{i=m+1}^{m+n} \frac{1}{n} \cdot z_i^2\right].
\]
The corollary follows by selecting \( \lambda_i^2 = \frac{1}{2} + \frac{1}{mq} \) for \( i = 1, ..., m \), \( \lambda_i^2 = \frac{1}{2} - \frac{1}{n} \) for \( i = m+1, ..., m+n \), and \( g_i^2 = \frac{1}{2} \) in Lemma 1. \( \blacksquare \)

Lemma 2 provides an inequality for quantiles of the beta distribution.

**Lemma 2** For any \( j = 1, ..., K \) and \( \rho = \frac{9}{10} \), we have
\[
Q_{1 - \frac{d_{\max}}{N_j}} \geq T^2,
\]
where \( Q_{1 - \frac{d_{\max}}{N_j}} \) is the \((1 - \frac{d_{\max}}{N_j})\) quantile of \( \beta(\frac{1}{2}, \frac{d-1}{2}) \), and \( T \) is given in assumption A3.

**Proof** It suffices to show
\[
F_{\left(\frac{1}{2}, \frac{d-1}{2}\right)}(T^2) \leq 1 - \frac{d_{\max}}{N_j},
\]
where \( F_{\left(\frac{1}{2}, \frac{d-1}{2}\right)}(\cdot) \) is the CDF of beta distribution with parameters \( \left(\frac{1}{2}, \frac{d-1}{2}\right) \). Let \( B(a, b) \) be beta function with parameters \((a, b)\). The above inequality can be rewritten as
\[
\int_{T^2}^{1} t^{-\frac{1}{2}}(1-t)^{\frac{d-3}{2}} dt \geq B\left(\frac{1}{2}, \frac{d-1}{2}\right) \cdot \frac{d_{\max}}{N_j}.
\] (1)
For (1) to hold we need
\[
\int_{T^2}^{1} (1-t)^{\frac{d-3}{2}} \, dt \geq \frac{\Gamma(1/2) \Gamma(\frac{d-1}{2})}{\Gamma(d/2)} \frac{d_{\text{max}}}{N_j^\rho},
\]
based on the inequality on gamma functions (see Jameson (2013)) it suffices to have
\[
2(1 - T^2) \frac{d_{\text{max}}}{2} \geq \frac{\sqrt{2 \pi d} d_{\text{max}}}{N_j^\rho}.
\]
This is Assumption A3.

Lemma 3 states a bound on the order statistics of beta distributed random variables.

Lemma 3 For any \( j = 1, \ldots, K \), we let \( B_{(1)}, B_{(2)}, \ldots, B_{(N_j-1)} \) be the order statistics from a sample of \( N_j - 1 \) i.i.d \( \beta(\frac{1}{2}, \frac{d_{\text{max}}}{2}) \) random variables, we have
\[
\mathbb{P} \left[ B_{(N_j-d_{\text{max}})} \leq T^2 \right] \leq \frac{(N_j - d_{\text{max}})}{d_{\text{max}}(N_j + 1)(N_j^{1/10} - 1)^2}.
\]

Proof Let \( U(i) = F_{(\frac{1}{2}, \frac{d_{\text{max}}}{2})}(B(i)) \), where \( F_{(\frac{1}{2}, \frac{d_{\text{max}}}{2})} \) is the CDF of the beta distribution with parameters \( (\frac{1}{2}, \frac{d_{\text{max}}}{2}) \). Note that \( U(i) \)'s are the order statistics of the uniform distribution.

From Lemma 2 we know
\[
\mathbb{P} \left[ B_{(N_j-d_{\text{max}})} \leq T^2 \right] \leq 1 - \frac{d_{\text{max}}}{N_j^{9/10}}.
\]
Therefore by (2), Chebyshev’s inequality and basic properties of uniform order statistics
\[
\mathbb{P} \left[ B_{(N_j-d_{\text{max}})} \leq T^2 \right] \leq \mathbb{P} \left[ U_{(N_j-d_{\text{max}})} \leq 1 - \frac{d_{\text{max}}}{N_j^{9/10}} \right]
\]
\[
\leq \frac{\text{Var} \left[ U_{(N_j-d_{\text{max}})} \right]}{(\frac{d_{\text{max}}}{N_j} - \frac{d_{\text{max}}}{N_j^{9/10}})^2} = \frac{(N_j - d_{\text{max}})}{d_{\text{max}}(N_j + 1)(N_j^{1/10} - 1)^2}.
\]

Proof of Theorem 1. Throughout this proof we consider the noisy case. Let the event \( \mathcal{E}_{1i} \) be “\( Y_{C_i} \) only contains points in same subspace”, then \( \mathcal{E}_1 = \bigcap_{i=1}^n \mathcal{E}_{1i} \) is the event that Algorithm 1 has subcluster preserving property. Let the event \( \mathcal{E}_2 = \{ \sigma ||e_i^{(k)}||_2 < g_2, \forall i, k \} \), where \( g_2 \) is from assumption A3, \( \mathcal{E}_2 \) says the norms of errors are bounded in a range.

Using the fact \( \mathbb{P}[\mathcal{E}_1] \geq \mathbb{P}[\mathcal{E}_1|\mathcal{E}_2] + \mathbb{P}[\mathcal{E}_2] - 1 \geq 1 - \sum_{i=1}^n \mathbb{P}[\mathcal{E}_{1i}|\mathcal{E}_2] + \mathbb{P}[\mathcal{E}_2] - 1 = \mathbb{P}[\mathcal{E}_2] - \sum_{i=1}^n \mathbb{P}[\mathcal{E}_{1i}|\mathcal{E}_2] \). We will build a bound on \( \mathbb{P}[\mathcal{E}_{11} | \mathcal{E}_2] \) first, and use the above relation combine with union bound inequality to finish the proof.
Assuming that \( y_1^{(1)} \) is one of the sampled points and \( Y_{C_1} \) is the subcluster associate with it. Recall that to construct \( Y_{C_1} \), we calculate the absolute inner products between \( y_1^{(1)} \) and all other points in \( Y \) and select the points that correspond to the largest \( d_{\text{max}} \) absolute inner products among the \((N - 1)\) absolute inner products we calculated (see step 3 to 5 in Algorithm 1).

Now we write \( \hat{A}_i^k = |\langle y_1^{(1)}, y_i^{(k)} \rangle| \). To achieve subspace preserving property in \( Y_{C_1} \), we need the largest \((d_{\text{max}} + 1)\) values among \( \hat{A}_i^k \)s are all from \( \hat{A}_i^k \)s, mathematically this means \( \mathcal{E}_{11} \) is

\[
\hat{A}_{(N - d_{\text{max}})}^{(1)} \leq \max_{k \neq 1} \max_{i = 1, \ldots, N_k} \hat{A}_i^k,
\]

here \( \hat{A}_{(N - d_{\text{max}})}^{(1)} \) is the \((N - d_{\text{max}})\)th order statistics among \( \hat{A}_i^k \)s.

Recall we can write \( y_i^{(k)} = \frac{u_k a_i^{(k)} + \sigma e_i^{(k)}}{||u_k a_i^{(k)} + \sigma e_i^{(k)}||_2} \), from triangle inequality we know that

\[
||u_k a_i^{(k)}||_2 - ||\sigma e_i^{(k)}||_2 \leq ||u_k a_i^{(k)} + \sigma e_i^{(k)}||_2 \leq ||u_k a_i^{(k)}||_2 + ||\sigma e_i||_2,
\]

hence conditioning on \( E_2 \) we know the normalization constants are bounded in \([1 - g_2, 1 + g_2]\). For fixed \( y_1^{(1)} \) we write \( A_i^k = ||y_1^{(1)}||_2 \cdot ||y_i^{(k)}||_2 \cdot \hat{A}_i^k \), it is fairly straightforward to get the following relation

\[
P \left[ A_{(N - d_{\text{max}})}^{(1)} \leq \frac{1 + g_2}{1 - g_2} \max_{k \neq 1} \max_{1 \leq i \leq N_k} A_i^k \mid E_2 \right] \geq P \left[ \mathcal{E}_{11} \mid E_2 \right], \tag{3}
\]

Now we are going to bound \( P[\mathcal{E}_{11} \mid E_2] \) by bounding the LHS of inequality (3). Conditioning on \( E_2 \) and write \( B_i = ||\langle a_i^{(1)}, a_i^{(1)} \rangle||^2, i = 2, \ldots, N_1 - 1 \), we have the following inequalities

\[
A_{(N - d_{\text{max}})}^{(1)} = \sqrt{B_{(N - d_{\text{max}})}} + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle
\]

\[
\geq \sqrt{B_{(N - d_{\text{max}})}} - \sigma ||e_i^{(1)}||_2 - \sigma ||e_i^{(1)}||_2 - \sigma ||e_i^{(1)}||_2 \max_{i \neq 1} ||e_i^{(1)}||_2
\]

\[
\geq \sqrt{B_{(N - d_{\text{max}})}} - 2g_2 - g_2^2.
\]

Similarly we have

\[
\max_{k \neq 1} \max_{1 \leq i \leq N_k} A_i^k = \max_{k \neq 1} \max_{1 \leq i \leq N_k} |\langle U_1^{(1)}, U_k a_i^{(k)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_k^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_1^{(1)}, e_i^{(1)} \rangle + \sigma \langle U_k^{(1)}, e_i^{(1)} \rangle|
\]

\[
\leq \max_{k \neq 1} \max_{1 \leq i \leq N_k} \left| \langle U_1^{(1)}, U_k a_i^{(k)} \rangle \right| + \sigma \max_{k \neq 1} \max_{1 \leq i \leq N_k} \|e_i^{(k)}\|_2
\]

\[
\quad + \sigma ||e_i^{(1)}||_2 + \sigma ||e_i^{(1)}||_2 \max_{k \neq 1} \max_{1 \leq i \leq N_k} ||e_i^{(k)}||_2
\]

\[
\leq \max_{k \neq 1} \max_{1 \leq i \leq N_k} \left| \langle U_1^{(1)}, U_k a_i^{(k)} \rangle \right| + 2g_2 - g_2^2.
\]

Pick \( T \) from assumption A3, we have the LHS of (3) has the following upper bound

\[
P \left[ T \leq Q \mid E_2 \right] + P \left[ B_{(N - d_{\text{max}})} \leq T^2 \mid E_2 \right], \tag{4}
\]

where

\[
Q = (1 + \frac{1 + g_2}{1 - g_2} (2g_2 + g_2^2) + \frac{1 + g_2}{1 - g_2} \max_{k \neq 1} \max_{1 \leq i \leq N_k} \left| \langle U_1^{(1)}, U_k a_i^{(k)} \rangle \right|.
\]

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Note that under the noiseless case we have $g_2 = 0$, hence $Q = \max_{k \neq 1} \max_{1 \leq i \leq N_k} |\langle U_1 a_1^{(1)}, U_k a_i^{(k)} \rangle|$, this quantity intuitively measures the closeness between subspaces as it will be large if $U_1$ and $U_k$ have small principal angles. Larger $g_2$ hence larger magnitude of errors will have “inflation” effect on $T$ (simply look at $\frac{1 + g_2}{1 - g_2}$), this means for larger magnitude of errors we need the subspaces to be well separated to achieve subspace preserving property.

For the first term in (4) we have

$$\mathbb{P} [T \leq Q \mid \mathcal{E}_2] = \mathbb{P} \left[ g_1 \leq \max_{k \neq 1} \max_{1 \leq i \leq N_k} |\langle U_1 a_1^{(1)}, U_k a_i^{(k)} \rangle| \right].$$

For fixed $k$ we can use svd to write

$$\langle U_1 a_1^{(1)}, U_k a_i^{(k)} \rangle = a_1^{(1)T} W_{1k} \Lambda_{1k} V_{1k}^T a_i^{(k)} := b_k^T \Lambda_{1k} V_{1k}^T a_i^{(k)},$$

here both $b_k$ and $V_{1k}^T a_i^{(k)}$'s are sampled uniformly from $S^{d-1}$(orthogonal transformation), therefore

$$\mathbb{P} \left[ g_1^2 \leq \max_{k \neq 1} \max_{1 \leq i \leq N_k} |b_k^T \Lambda_{1k} V_{1k}^T a_i^{(k)}|^2 \right] \leq \sum_{k \neq 1} \mathbb{P} \left[ g_1^2 \leq \max_{1 \leq i \leq N_k} |b_k^T \Lambda_{1k} V_{1k}^T a_i^{(k)}|^2 \right] \leq \sum_{k \neq 1} \mathbb{P} \left[ g_1^2 \leq \sum_{i=1}^d |\lambda_i^{(1)} b_{ki}|^2 \right] \leq \sum_{k \neq 1} \mathbb{P} \left[ g_1^2 \leq \sum_{i=1}^d |\lambda_i^{(1)} b_{ki}|^2 \right]$$

where inequality (5) uses the union bound inequality and (6) comes from Cauchy-Schwarz inequality, (7) uses Definition 1. Since $b_k$ is uniformly distributed on $S^{d-1}$, we can further write (7) as

$$(K - 1) \mathbb{P}[g_1^2 \leq \sum_{i=1}^d |\lambda_i^{(1)} b_i|^2],$$

where $b$ is uniformly distributed on $S^{d-1}$. Lemma 1 is applicable directly to the quantity above: $\mathbb{P}[g_1^2 \leq \sum_{i=1}^d |\lambda_i^{(1)} b_i|^2] \leq 2e^{-\epsilon'^2}$ where

$$\epsilon' = \frac{\sum_{i=1}^d (r_i - s_i)}{\left( \sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2} \right) + \sqrt{\left( \sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2} \right)^2 + 2s_1 \sum_{i=1}^d (r_i - s_i)}}$$

$$(\sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2}) + \sqrt{\left( \sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2} \right)^2 + 2s_1 \sum_{i=1}^d (r_i - s_i)}$$

$$= \frac{-\left( \sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2} \right) + \sqrt{\left( \sqrt{\sum_{i=1}^d r_i^2} + \sqrt{\sum_{i=1}^d s_i^2} \right)^2 + 2s_1 \sum_{i=1}^d (r_i - s_i)}}{2s_1}$$

$$20$$
where \( t \) can be found as in Corollary 1. After some manipulations and using Assumption A3 we have

\[
t > \frac{D(\frac{g_2^2}{D\sigma^2} - 1)}{2(\sqrt{D} + \frac{g_2^2}{\sigma^2\sqrt{d}} + \sqrt{d})} \geq \sqrt{\frac{d(1 + \eta)}{5}}.
\]
Therefore we have $\mathbb{P}[g_2 \leq \sigma ||e||_2] \leq 2e^{-t^2} \leq \frac{2}{N^{1+\eta}}$.

Now we note that

$$\mathbb{P}\left[g_2 \leq \sigma \max_{k=1, \ldots, K} \max_{1 \leq i \leq N_k} ||e_i^{(k)}||_2\right] = 1 - \mathbb{P}[g_2 > \sigma \max_{k=1, \ldots, K} \max_{1 \leq i \leq N_k} ||e_i^{(k)}||_2]$$

$$= 1 - \prod_{k=1}^N (1 - \mathbb{P}[g_2 \leq \sigma ||e_i^{(k)}||_2])$$

$$\leq 1 - (1 - 2e^{-t^2})^N$$

$$\leq \frac{2N}{N^{1+\eta} - 2},$$

where the last inequality comes from the Taylor expansion of $e^{-2N}/N^{1+\eta}$. Therefore $\mathbb{P}[\mathcal{E}_2]$ is lower bounded by $1 - \frac{2N}{N^{1+\eta} - 2}$.

Finally, the above arguments hold for any $y_i^{(k)}$, putting everything together and applying the union bound inequality yield the result:

$$\mathbb{P}[^{\mathcal{E}_2}] \geq 1 - \sum_{j=1}^K \frac{n_j(N_j - d_{\max})}{d_{\max}(N_j + 1)(N_j^{1/2} - 1)^2} - 2(K-1)n\epsilon e^{-2} \leq \frac{2N}{N^{1+\eta} - 2}. \quad (9)$$

For the noiseless case we have $\sigma = 0$, we no longer need to consider the normalization constants and we can let $g_1 = T$ directly, finally we can drop the last term at the RHS of (9). \hfill \blacksquare

To prove Theorem 2, we will use the following equation

$$(W^TW + \lambda I_{d_2})^{-1}W^T = W^T(WW^T + \lambda I_{d_1})^{-1}, \quad (10)$$

here $W \in \mathbb{R}^{d_1 \times d_2}$ and $\lambda$ is a positive constant. Please see Chapter 4 in Murphy (2012) for detailed proof. Throughout the proof of Theorem 2, the subscript of identity matrix $I$ will be omitted as the dimension of it should be fairly clear from the content.

**Proof of Theorem 2.** Define $\mathcal{I} = \{(i, j) : 1 \leq i < j \leq n$ and $Y_{C_i}, Y_{C_j}$ concentrate around different subspace}, and $\mathcal{J} = \{(i, j) : 1 \leq i < j \leq n$ and $Y_{C_i}, Y_{C_j}$ concentrate around the same subspace}. Similar to the proof of Theorem 1, we use $\mathcal{E}_1$ to denote the event that correct neighborhood property holds for all $Y_{C_i} s$, and use $\mathcal{E}_2$ to denote the event that

$$\max_{k=1, \ldots, K} \max_{1 \leq i \leq N_k} \sigma ||e_i^{(k)}||_2 < l_1,$$

$l_1$ is a positive constant that we will specify later.

Again we use the relation $\mathbb{P}[\mathcal{E}_1] \geq \mathbb{P}[\mathcal{E}_1|\mathcal{E}_2] + \mathbb{P}[\mathcal{E}_2] - 1$. Specifically, we will show that conditioning on $\mathcal{E}_2$, there is a deterministic bound $l_2$ on $d(Y_{C_i}, Y_{C_j})_{(i, j) \in \mathcal{J}}$, therefore we have $\mathbb{P}[\mathcal{E}_1|\mathcal{E}_2] \geq \mathbb{P}[d(Y_{C_i}, Y_{C_j})_{(i, j) \in \mathcal{I}} > l_2[\mathcal{E}_2] \geq 1 - \sum_{(i,j) \in \mathcal{I}} \mathbb{P}[d(Y_{C_i}, Y_{C_j}) \leq l_2[\mathcal{E}_2]]$.

Without loss of generality we assume that $Y_{C_1}$ and $Y_{C_2}$ concentrate around $S_1$ and $Y_{C_3}$ concentrates around $S_2$, we first find the bound $l_2$, and then upper bound $\mathbb{P}[d(Y_{C_1}, Y_{C_2}) \leq l_2[\mathcal{E}_2]]$ (note $(1, 3) \in \mathcal{I}$), finally the results will be proved by using union bound inequality.

We can explicitly write $d(Y_{C_1}, Y_{C_2})$ as

$$||Y_{C_1} - Y_{C_2}(Y_{C_2}^TY_{C_2} + \lambda I)^{-1}Y_{C_2}^TY_{C_1}||_F + ||Y_{C_2} - Y_{C_1}(Y_{C_1}^TY_{C_1} + \lambda I)^{-1}Y_{C_1}^TY_{C_2}||_F, \quad (11)$$
and $Y_C = U_1 \hat{B}_1 + \hat{E}_1$ (similar forms for $Y_C$ and $Y_C$). For convenience we also write $E_{ji}$ as the $i$th column of matrix $E_j$, here $E_j$ is the un-normalized error matrix corresponds to subcluster $Y_C$. Using equation (10) and after manipulations, the first term in (11) can be rewritten as

$$
\|Y_C - Y_C (Y_C^T Y_C + \lambda I)^{-1} Y_C^T Y_C \|_F
= \|Y_C - (Y_C^T Y_C + \lambda I)(Y_C^T Y_C + \lambda I)^{-1} Y_C \|_F
= \lambda \| (Y_C^T Y_C + \lambda I)^{-1} Y_C \|_F
< \lambda \| (Y_C^T Y_C + \lambda I)^{-1} - (U_1 B_2 \tilde{B}_2^T U_1^T + \lambda I)^{-1} \|_F \| Y_C \|_F
+ \lambda \| (U_1 B_2 \tilde{B}_2^T U_1^T + \lambda I)^{-1} U_1 B_i \|_F
+ \lambda \| (U_1 B_2 \tilde{B}_2^T U_1^T + \lambda I)^{-1} \|_F \| \tilde{E}_1 \|_F.
$$

(12)

Now we are going to find the bounds of the three terms in (12). For convenience we write $H = U_1 B_2 \tilde{B}_2^T U_1^T + \lambda I$, note the first term in (12) can be rewritten as

$$
\lambda \| (G_2 + H)^{-1} - H^{-1} \|_F (\sqrt{d_{max}} + 1).
$$

The bound $l_1$ guarantees the norm of error terms are small compared to $\lambda$, so the difference $\| (G_2 + H)^{-1} - H^{-1} \|_F$ is small.

We pick $g(d) = \frac{1}{3\sqrt{15 D(d_{max}+1)}}$, $l_1 = \frac{\lambda g(d)}{3\sqrt{D(d_{max}+1)}} < \frac{1}{d}$ (recall $\lambda = \frac{1}{3\sqrt{15 D(d_{max}+1)}}$).

Conditioning on $E_2$ we have

$$
\|E_j\|_F \leq \frac{\lambda g(d)}{3\sqrt{D(d_{max}+1)}} \quad \text{for } j=1,\ldots,n,
$$

(13)

and the normalization constants of $\tilde{E}_{j,i}$'s are bounded in $[1 - \frac{1}{d}, 1 + \frac{1}{d}]$. We then have

$$
\|G_2\|_F = \| \tilde{E}_2 \tilde{B}_2^T U_1^T + U_1 \hat{B}_2 \hat{E}_2^T + \hat{E}_2 \hat{E}_2^T \|_F
\leq \| \tilde{E}_2\|_F \| \tilde{B}_2 \hat{E}_2^T \|_F + \| U_1 \hat{B}_2 + \hat{E}_2 \|_F \| \hat{E}_2^T \|_F
\leq (1 + \frac{1}{1 - \frac{1}{d}}) \sqrt{d_{max} + 1} \|E_2\|_F
< \frac{\lambda}{\sqrt{D}} g(d),
$$

the above analysis used triangle inequalities and the bounds of normalization constants.

Using the fact $\|H^{-1}\|_F < \frac{\sqrt{D}}{\lambda}$ together with inequality above lead to $\|H^{-1} G_2\|_F \leq \|H^{-1}\|_F \|G_2\|_F < g(d) < 1$, hence $\lim_{m \to \infty} (H^{-1} G_2)^m = 0$. From Theorem 4.29 in Schott (2016) we know $(I + H^{-1} G_2)^{-1} = \sum_{j=0}^{\infty} (H^{-1} G_2)^j$ and

$$
\| (G_2 + H)^{-1} - H^{-1} \|_F = \| H^{-1} G (I + H^{-1} G_2)^{-1} H^{-1} \|_F
$$

(14)
\[ \lambda \Vert (G_2 + H)^{-1} - H^{-1} \Vert_F \lesssim \frac{g(d) \sqrt{D}}{1 - g(d)} \lambda \sqrt{d_{\max} + 1} \cdot \sqrt{d_{\max} + 1} \]

Plugging back we then have for the first term in (12)

\[ \lambda \Vert (G_2 + H)^{-1} - H^{-1} \Vert_F \lesssim \frac{g(d) \sqrt{D}}{1 - g(d)} \lambda \sqrt{d_{\max} + 1} \cdot \sqrt{d_{\max} + 1} \]

For convenience we write \( \hat{B}_i \hat{B}_i^T = V_i^T A_i V_i \) as the svd of \( \hat{B}_i \hat{B}_i^T \). Note the fact that \( \hat{B}_i \hat{B}_i^T \) shares the same positive eigenvalues with \( U_j \hat{B}_i \hat{B}_i^T U_j^T \), we use \( a_{ij} \) to denote the \( j \)th diagonal element of \( A_i \), hence \( a_{ij} > 0 \) since we assume the rank of \( \hat{B}_i \)s is \( d \). Using (10) and after some manipulations, for the second term in (12) we have

\[ \lambda \Vert (U_1 \hat{B}_2 \hat{B}_2^T U_1^T + \lambda I)^{-1} U_1 \hat{B}_1 \Vert_F \]

\[ = \Vert \hat{B}_1 - \hat{B}_2 \hat{B}_2^T (\hat{B}_2 \hat{B}_2^T + \lambda I)^{-1} \hat{B}_1 \Vert_F \]

\[ = \lambda \Vert (I - V_2^T \begin{bmatrix} \frac{a_{21}}{a_{21} + \lambda} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{a_{2d}}{a_{2d} + \lambda} \end{bmatrix} V_2) \hat{B}_1 \Vert_F \]

\[ \leq \lambda \cdot \begin{bmatrix} \frac{a_{21}}{a_{21} + \lambda} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{a_{2d}}{a_{2d} + \lambda} \end{bmatrix} \cdot \Vert F \cdot \Vert \hat{B}_1 \Vert_F \]

\[ < \lambda \cdot \begin{bmatrix} \frac{a_{21}}{a_{21} + \lambda} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{a_{2d}}{a_{2d} + \lambda} \end{bmatrix} \cdot \Vert F \cdot \frac{\sqrt{d_{\max} + 1}}{1 - \frac{1}{d}} \]

\[ \leq \frac{\lambda g_0 \sqrt{d(d_{\max} + 1)}}{1 - \frac{1}{d}} \]

from (13) we also have for the third term in (12)

\[ \lambda \Vert (U_1 \hat{B}_2 \hat{B}_2^T U_1^T + \lambda I)^{-1} \Vert_F \Vert \hat{E}_1 \Vert_F \]

\[ < \frac{\lambda g(d)}{3 \sqrt{(d_{\max} + 1)(1 - \frac{1}{d})}} \]

\[ < \frac{\lambda g_0 \sqrt{d(d_{\max} + 1)}}{1 - \frac{1}{d}} \]

Hence conditioning on \( E_2 \), (12) can be upper bounded by \( \frac{3 \lambda g_0 \sqrt{d(d_{\max} + 1)}}{1 - \frac{1}{d}} \). Note this quantity is deterministic and does not depend on the choices of \( B_s \) and \( \hat{U}_s \), also the manipulation

\[ 24 \]
on RHS of (11) is symmetric, therefore we pick \( l_2 = \frac{6\lambda q_0 \sqrt{d(d_{\text{max}} + 1)}}{1 - \frac{1}{d}} \) is the hard bound on \( d(Y_{C_1}, Y_{C_3}) \).

Now we consider \( \mathbb{P}[d(Y_{C_1}, Y_{C_3}) \leq l_2 | \mathcal{E}_2] \). We can explicitly write \( d(Y_{C_1}, Y_{C_3}) \) as

\[
\|Y_{C_1} - Y_{C_3}(Y_{C_3}^T Y_{C_3} + \lambda I)^{-1} Y_{C_3}^T Y_{C_1}\|_F + \|Y_{C_3} - Y_{C_1}(Y_{C_1}^T Y_{C_1} + \lambda I)^{-1} Y_{C_1}^T Y_{C_3}\|_F. \tag{14}
\]

Note the following relation

\[
\mathbb{P}[d(Y_{C_1}, Y_{C_3}) \leq l_2 | \mathcal{E}_2] \leq \mathbb{P} [\|Y_{C_1} - Y_{C_3}(Y_{C_3}^T Y_{C_3} + \lambda I)^{-1} Y_{C_3}^T Y_{C_1}\|_F \leq l_2/2 | \mathcal{E}_2]
+ \mathbb{P} [\|Y_{C_3} - Y_{C_1}(Y_{C_1}^T Y_{C_1} + \lambda I)^{-1} Y_{C_1}^T Y_{C_3}\|_F \leq l_2/2 | \mathcal{E}_2]. \tag{15}
\]

To bound the first term in (14), we only need to use reverse triangle inequalities and follow the same procedure as before to lower bound them, specifically

\[
\lambda\|(U_2 \hat{B}_3 \hat{B}_3^T U_2^T + \lambda I)^{-1} U_1 \hat{B}_1\|_F
= \|U_1 \hat{B}_1 - U_2(I - \lambda V_3^T \left[ \begin{array}{cccc} \frac{1}{\alpha_{31} + \lambda} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\alpha_{3d} + \lambda} \end{array} \right] V_3) U_2^T U_1 \hat{B}_1\|_F
\geq \|U_1 \hat{B}_1 - U_2 U_2^T U_1 \hat{B}_1\|_F - \lambda\|U_2 V_3^T \left[ \begin{array}{cccc} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\alpha_{3d} + \lambda} \end{array} \right] V_3 U_2^T U_1 \hat{B}_1\|_F
\geq \|U_1 \hat{B}_1 - U_2 U_2^T U_1 \hat{B}_1\|_F - \frac{\lambda q_0 \sqrt{d(d_{\text{max}} + 1)}}{1 - \frac{1}{d}}, \tag{17}
\]

inequality (17) comes from the bound we found before.

For the first term in inequality (17) we have

\[
\|U_1 \hat{B}_1 - U_2 U_2^T U_1 \hat{B}_1\|_F = \sqrt{\text{Tr}[(\hat{B}_1^T \hat{B}_1 - \hat{B}_1^T \hat{B}_1) U_2^T U_2^T U_1 \hat{B}_1]}
= \|\sqrt{I - \Lambda_{12}^2 \hat{B}_1 W}\|_F \geq \frac{\|\sqrt{I - \Lambda_{12}^2 \hat{B}_1}\|_F}{1 + \frac{1}{d}},
\]

where \( W \) is the diagonal matrix with its diagonal equals to the reciprocal of normalization constants of each column of \( \hat{B}_1 \) (simply note \( \hat{B}_1 = B_1 W \)), and \( \hat{B}_1 = MB_1 \) is a orthogonal transformation of \( B_1 \) (here \( M \) is the right orthogonal matrix in the SVD of \( U_2 U_1 \)), \( \Lambda_{12} \)
is the diagonal matrix that takes $\lambda_i^{12}$ ($i = 1, \ldots, d$) as its $i$th diagonal entry. Therefore, eventually the first term at the RHS of (15) can be upper bounded by

$$
P \left[ || \sqrt{I - A_{12}^2 \tilde{B}_1} ||_F \leq \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} \cdot 6 \lambda q \sqrt{d(d_{\max} + 1)} \right],$$

using the fact $d \geq 5$ we can get $P[|| \sqrt{I - A_{12}^2 \tilde{B}_1} ||_F \leq \sqrt{\frac{3}{5}}]$ is an upper bound of the quantity above. Here we only focus on the first column of $\sqrt{I - A_{12}^2 \tilde{B}_1}$, specifically

$$
P \left[ || \sqrt{I - A_{12}^2 \tilde{B}_1} ||_F^2 \leq \frac{3}{5} \right] \leq P \left[ \sum_{i=1}^{d} (1 - \lambda_i^{(1)}) b_i^2 \leq \frac{3}{5} \right],$$

(18)

here $b$ follows an uniform distribution on $\mathbb{S}^{d-1}$, similarly as in the proof of Theorem 1 we can write $b_i = \frac{z_i^2}{\sum_{i=1}^{d} z_i^2}$, where $z_i$ is $i.i.d. \sim N(0, 1)$. For convenience we write $\lambda_i = \lambda_i^{(1)}$, and $T^2 = 1 - \frac{3}{5} = \frac{2}{5}$. The RHS of (18) is

$$
P \left[ \sum_{i=1}^{d} (T^2 - \lambda_i^2) \cdot z_i^2 \geq \sum_{i=1}^{d} (T^2 - \lambda_i^2) \cdot z_i^2 \right].$$

From assumption A4 and Lemma 1 we have

$$
P \left[ \sum_{i=1}^{d} (T^2 - \lambda_i^2) \cdot z_i^2 \geq \sum_{i=1}^{d} (T^2 - \lambda_i^2) \cdot z_i^2 \right] \leq 2e^{-\epsilon^2},$$

here $\epsilon$ is the same as in Theorem 1. Using analogous manipulations we can have similar results for the second term in (15). Therefore $P[d(Y_{C_1}, Y_{C_1}) \leq l_2|E_2] \leq 4e^{-\epsilon^2}$.

Finally we are going to lower bound $P[E_2]$. Following similar procedure in the proof of Theorem 1 we have

$$
P \left[ \sigma e_i^{(k)} \geq l_1 \right] \leq 2e^{-t^2},$$

here $t = \frac{D(d-1)}{2(\sqrt{D} + \frac{\lambda_0 d}{\sqrt{d_0}} + \sqrt{d})}$ and $q = \frac{\lambda_0 d}{3D\sqrt{D(d_{\max} + 1)}}$, from assumption A6 and union bound inequality we know

$$
P [E_2] \geq 1 - \frac{2N}{N^{1+\eta_2} - 2}.$$

From union bound inequality we know the correct neighborhood property is achieved at all sub-clusters with probability at least

$$1 - \sum_{\forall (i, j) \in I} P[d(Y_{C_i}, Y_{C_j}) \leq l_2|E_2] + P[E_2] - 1 \geq 1 - 4n(n - 1)e^{-\epsilon^2} - \frac{2N}{N^{1+\eta_2} - 2}. \quad (19)$$

For the noiseless case, $E_i$s are just $0$, so we do not need to consider the norm of them, the last term at the RHS of (19) can be dropped. Note that for the noiseless case, $Y_{C_1} Y_{C_1}^T$ and $B_i B_i^T$ share the same positive eigenvalues, and these values are actually known to us (can be
directly calculated from our data set), hence we may simply choose \( \lambda = \frac{1}{D} \max_{i=1,...,n} \sqrt{\frac{1}{a_{ij}^2}} \), it is fairly easy to modify the above discussions to see this \( \lambda \) will guarantee correct neighborhood property with probability goes to 1 as we increase \( N \).

**Proof of Theorem 3.** The proof is similar to that of Theorem 2. Again we use \( E_1 \) to denote the event that all out of sample points are correctly classified, and \( E_2 \) to denote the event that the norm of error vectors are bounded in a range, specifically we let \( E_2 \) to be \( \{ \max_{k=1,...,K} \max_{i=1,...,N_k} \sigma \| e_i^{(k)} \| < l_1 \} \), here \( l_1 \) is a positive constant that we will specify later, later we will show this \( l_1 \) is different from the \( l_1 \) in the proof of Theorem 2.

We first consider the probability that one point get incorrectly classified, and then use union bound inequality to finish the proof.

Assume \( y = U_1 \hat{a} + \hat{e} \) is a out of sample point that concentrates around \( S_1 \). Then \( y \) would be incorrectly classified if for some \( k \neq 1 \)

\[
\| y - R_1 (R_1^T R_1 + \lambda I)^{-1} R_1^T y \| _2 \geq \| y - R_k (R_k^T R_k + \lambda I)^{-1} R_k^T y \| _2. \tag{20}
\]

Now we pick \( g(d) = \frac{1}{\sqrt{Dm}} \) and recall \( \lambda = \frac{1}{q_0 \sqrt{dm}} \), and let \( l_1 = \frac{3 \lambda g(d)}{3 \sqrt{Dm}} \). Conditioning on \( E_2 \) and following similar procedure in the proof of Theorem 2, we can find \( \frac{3 \lambda g(d)}{1 - \frac{d}{3}} \) is a deterministic upper bound on LHS of (20).

For fixed \( k = 2 \), the manipulation of RHS of (20) is similar as before. In short we want to upper bound

\[
P \left[ \| \sqrt{I - \Lambda_{12}^2} b \| _2 \leq \frac{1 + \frac{1}{2} \cdot 6 \lambda q_0 \sqrt{d}}{1 - \frac{d}{3}} \right],
\]

here \( b \) is a uniformly distributed r.v. from unit sphere. The quantity above is bounded by

\[
P[\| \sqrt{I - \Lambda_{12}^2} b \| _2 \leq \sqrt{\frac{3}{5}}],
\]

which is further upper bounded by \( 2e^{-\epsilon^2} \), where \( \epsilon \) is the same as in proof of Theorem 1. Finally, similarly as before we have \( P[E_2] \geq 1 - \frac{2N}{N^{1+\eta_3}} \).

Therefore, we have exact out of sample classification with probability at least

\[
1 - 2(K - 1)[N - n(d_{\max} + 1)]e^{-\epsilon^2} - \frac{2N}{N^{1+\eta_3}} - \frac{2}{N^{1+\eta_3}}.
\]
Appendix B.

In this appendix we provide the regression method to classify the out-of-subsample points $Y \setminus \hat{Y}$.

\begin{algorithm}
\textbf{input} : Data $Y$ to be classified, $R$ and $\ell$ are the training data and labels, $m$ is the residual minimization parameter $m$, $\lambda$ is the regularization parameter
\textbf{output}: The label vector $o$ of all points in $Y$

1. Generate subsamples of training data
   \begin{algorithmic}
   \FOR{$k = 1$ \TO $K$}
   \STATE Denote as $R_k$ as $m$ training points uniformly sampled from those with label $k$;
   \STATE Similarly denote as $Y_k$ the vector label values $k$;
   \ENDFOR
   \end{algorithmic}

2. Compute the projection matrix for each cluster label
   \begin{algorithmic}
   \FOR{$k = 1$ \TO $K$}
   \STATE $P_k := R_k (R_k^T R_k + \lambda \mathbf{I})^{-1} R_k^T$
   \ENDFOR
   \end{algorithmic}

3. Compute residuals for all points in $Y$
   \begin{algorithmic}
   \FOR{$i = 1$ \TO $N$}
   \FOR{$k = 1$ \TO $K$}
   \STATE $r_k(i) := (I - P_k) Y_i$;
   \ENDFOR
   \ENDFOR
   \end{algorithmic}

4. Assign cluster value as minimum residual
   \begin{algorithmic}
   \FOR{$i = 1$ \TO $N$}
   \STATE $o_i = \arg \min_k \{r_k\}$;
   \ENDFOR
   \end{algorithmic}

\textbf{Algorithm 2:} The steps in the Residual Minimization by Ridge Regression algorithm.
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