THE 3-POINT SPECTRAL PICK INTERPOLATION PROBLEM AND AN APPLICATION TO HOLOMORPHIC CORRESPONDENCES

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Abstract. We provide a necessary condition for the existence of a 3-point holomorphic interpolant $F : \mathbb{D} \to \Omega_n$, $n \geq 2$. Our condition is inequivalent to the necessary conditions hitherto known for this problem. The condition generically involves a single inequality and is reminiscent of the Schwarz lemma. We combine some of the ideas and techniques used in our result on the $O(D, \Omega_n)$-interpolation problem to establish a Schwarz lemma—which may be of independent interest—for holomorphic correspondences from $D$ to a general planar domain $\Omega \subseteq \mathbb{C}$.

1. Introduction and statement of results

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$ centered at 0. Given $n \in \mathbb{Z}_{+}$ the $n^2$-dimensional spectral unit ball is the set $\Omega_n := \{ A \in M_n(\mathbb{C}) : \sigma(A) \subset \mathbb{D} \}$, where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices and $\sigma$ denotes the spectrum of a matrix. The interpolation problem referred to in the title of this article is the following problem:

$$(\ast)$$ Given $M$ distinct points $\zeta_1, \ldots, \zeta_M \in \mathbb{D}$ and matrices $W_1, \ldots, W_M \in \Omega_n$, $n \geq 2$, find conditions on the data $\{(\zeta_j, W_j) : 1 \leq j \leq M\}$ such that there exists a holomorphic map $F : \mathbb{D} \to \Omega_n$ satisfying $F(\zeta_j) = W_j$, $j = 1, \ldots, M$.

When such a function $F$ exists, we shall say that $F$ is an interpolant of the data.

One of the important steps towards understanding the problem $(\ast)$ was an operator-theoretic approach due to Bercovici, Foias and Tannenbaum. Using a spectral version of the commutant-lifting theorem, the authors in [4]—under the restriction that $\sup_{\zeta \in \mathbb{D}} \rho(F(\zeta)) < 1$, where $\rho$ denotes the spectral radius—provided a characterization for the existence of an interpolant. This characterization involves a search for $M$ appropriate matrices in $GL_n(\mathbb{C})$.

Another influential idea was introduced by Agler and Young in [1]. They observed that in the case $W_1, \ldots, W_M$ are all non-derogatory, then $(\ast)$ is equivalent to an interpolation problem from $\mathbb{D}$ to the $n$-dimensional symmetrized polydisc $G_n$, $n \geq 2$. This is a bounded domain in $\mathbb{C}^n$ (see [5] for the definition of $G_n$). Its relevance to $(\ast)$ is that, for “generic” matricial data $(W_1, \ldots, W_M)$, the problem $(\ast)$ descends to a region of much lower dimension with many pleasant properties. This idea has further been developed in [2], in the papers [7] and [8] by Costara, and in Ogle’s thesis [14]. A matrix $A \in M_n(\mathbb{C})$ is said to be non-derogatory if it admits a cyclic vector. It is a fact that $A$ being non-derogatory is equivalent to $A$ being similar to the companion matrix of its characteristic polynomial (see [11], p. 195, for instance). Recall: given a monic polynomial of degree $k$ of the form $p(t) = t^k + \sum_{j=1}^{k} a_j t^{k-j}$,
where \(a_j \in \mathbb{C}\), the companion matrix of \(p\) is the matrix \(C_p \in M_k(\mathbb{C})\) given by

\[
C_p := \begin{bmatrix}
0 & -a_k \\
1 & -a_{k-1} \\
\vdots & \vdots \\
0 & -a_1
\end{bmatrix}_{k \times k}.
\]

By way of the \(G_n\)-interpolation problem, Costara [8] and Ogle [14] arrived independently at a necessary condition for the existence of an interpolant for the problem \((\ast)\) when the data \((W_1, \ldots, W_M)\) are non-derogatory.

Bharali in [5] observed that when \(n \geq 3\), the necessary condition given in [8, 14] is not sufficient. He also established — for the case \(M = 2\) — a new necessary condition for the existence of an interpolant. Result 1.1 below is this necessary condition. It is reminiscent of the inequality in the classical Schwarz lemma; here \(M_\mathbb{D}(z_1, z_2)\) is the Möbius distance between \(z_1\) and \(z_2\), which is defined as:

\[
M_\mathbb{D}(z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \overline{z}_2 z_1} \right| \quad \forall z_1, z_2 \in \mathbb{D}.
\]

**Result 1.1** (Bharali, [5]). Let \(F \in \mathcal{O}(\mathbb{D}, \Omega_n), n \geq 2\), and let \(\zeta_1, \zeta_2 \in \mathbb{D}\). Write \(W_j = F(\zeta_j)\), and if \(\lambda \in \sigma(W_j)\), then let \(m(\lambda)\) denote the multiplicity of \(\lambda\) as a zero of the minimal polynomial of \(W_j\). Then:

\[
\max \left\{ \max_{\mu \in \sigma(W_2)} \prod_{\lambda \in \sigma(W_1)} \mathcal{M}_\mathbb{D}(\mu, \lambda)^{m(\mu)}, \max_{\mu \in \sigma(W_2)} \prod_{\lambda \in \sigma(W_1)} \mathcal{M}_\mathbb{D}(\lambda, \mu)^{m(\mu)} \right\} \leq \left| \frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta}_2 \zeta_1} \right|. \tag{1.1}
\]

The above theorem gives a necessary condition for the two-point interpolation problem without any restriction on the matrices, in contrast to the necessary condition in [8, 14]. In the same article, Bharali also shows that for each \(n \geq 3\), there exists a data-set for which (1.1) implies that it cannot admit an interpolant whereas the condition in [8, 14] is inconclusive.

The ideas behind Result 1.1 strongly influence a part of this work. One of the key tools introduced in [5] that lead to Result 1.1 are the following maps:

**Definition 1.2.** Given \(A \in M_n(\mathbb{C})\), let \(M_A\) denote its minimal polynomial and write:

\[
M_A(t) = \sum_{\lambda \in \sigma(A)} (t - \lambda)^{m(\lambda)}.
\]

The finite Blaschke product induced by \(M_A\) if \(A \in \Omega_n\):

\[
B_A(t) := \prod_{\lambda \in \sigma(A) \subseteq \mathbb{D}} \left( \frac{t - \lambda}{1 - \overline{\lambda} t} \right)^{m(\lambda)}. \tag{1.2}
\]

will be called the minimal Blaschke product corresponding to \(A\).

\(B_A\) induces, via the holomorphic functional calculus (which we will discuss in Section 2), a holomorphic self-map of \(\Omega_n\) that maps \(A\) to \(0 \in M_n(\mathbb{C})\). This sets up a form of the Schur algorithm on \(\Omega_n\), and yields an easy-to-check necessary condition for the existence of an interpolant for the data in \((\ast)\), for the case \(M = 3\). The existence of these maps \(B_A\) is extremely useful, since the automorphism group of \(\Omega_n\) does not act transitively on \(\Omega_n\) (see [15]), \(n \geq 2\) (whence the classical Schur algorithm is not even meaningful).

In [3], Baribeau and Kamara take a new look at the ideas in [5]. This they combine with an inequality — which may be viewed as a Schwarz lemma for algebroid multifications of the unit disc (see [13] for a definition) — due to Nokrane and Ransford [14, Theorem 1.1]. Before we present their result we need the following: given \(F \in \mathcal{O}(\mathbb{D}, \Omega_n)\) and \(\zeta_1 \in \mathbb{D}\), if we denote
by $B_1$ the minimal Blaschke product corresponding to $F(\zeta_1)$ then Theorem 1.3 in \cite{[3]} states, essentially, that for every $\zeta \in \mathbb{D}$ we have
\[
\sigma(B_1(F(\zeta))/\psi_1(\zeta)) = S \cup \sigma(F(\zeta)),
\]
where $S \subset \partial \mathbb{D}$ is a finite (possibly empty) set independent of $\zeta$, $F_1 \in \mathcal{O}(\mathbb{D}, \Omega_\nu)$, and
\[
\nu = \max_{\zeta \in \mathbb{D}} |\sigma(B_1(F(\zeta))/\psi_1(\zeta)) \cap \mathbb{D}|.
\]
Here and in what follows, for $\zeta_j \in \mathbb{D}$, $j = 1, 2, 3$, $\psi_j$ will denote the automorphism $\psi_j(\zeta) := (\zeta - \zeta_j)(1 - \overline{\zeta}_j)^{-1}$, $\zeta \in \mathbb{D}$, of $\mathbb{D}$. We are now in a position to state:

Result 1.3 (paraphrasing \cite{[3]} Corollary 3.1)). Let $\zeta_1, \zeta_2, \zeta_3$ be distinct points in $\mathbb{D}$. Let $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$, $n \geq 2$. Denote by $B_j$ the minimal Blaschke product corresponding to $F(\zeta_j)$, and suppose that $\sigma(B_1(F(\zeta))/\psi_1(\zeta)) \not\subset \partial \mathbb{D}$ for every $\zeta \in \mathbb{D}$. Let $\nu$ be the number given by (1.3). Then we have:
\[
\mathcal{H}^{M_B}(\sigma(B_1(W_2)/\psi_1(\zeta_2))) \cap \mathbb{D}, \sigma(B_1(W_3)/\psi_1(\zeta_3))) \cap \mathbb{D}) \leq \mathcal{H}_B(\zeta_2, \zeta_3)^{1/\nu} \tag{1.4}
\]
where $W_j = F(\zeta_j)$, $j = 2, 3$.

Here, $\mathcal{H}_B$ denotes the Hausdorff distance induced by the Möbius distance (see \cite{[12]} p. 279) for the definition of Hausdorff distance) on the class of bounded subsets of $\mathbb{D}$.

Now we are ready to present the first result of this article (in what follows, $B_j$ will denote the minimal Blaschke product — as well as its extension to $\Omega_n$ — associated to the matrix $W_j$, $j = 1, 2, 3$):

**Theorem 1.4.** Let $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{D}$ be distinct points and let $W_1, W_2, W_3 \in \Omega_n$, $n \geq 2$. Let $m(j, \lambda)$ denote the multiplicity of $\lambda$ as a zero of the minimal polynomial of $W_j$, $j \in \{1, 2, 3\}$. Given $j, k \in \{1, 2, 3\}$ such that $j \neq k$, and $\nu \in \mathbb{D}$, we write:
\[
q(\nu, j, k) := \max \left\{ \left\lfloor \frac{m(j, \lambda)}{\text{ord}_B \lambda B_k} + 1 \right\rfloor : \lambda \in \sigma(W_j) \cap B_k^{-1}\{\nu}\right\}.
\]

Finally, for each $k \in \{1, 2, 3\}$ let
\[
G(k) := \max \left\{ \{1, 2, 3\} \setminus \{k\} \right\}, \text{ and } L(k) := \min \left\{ \{1, 2, 3\} \setminus \{k\} \right\}.
\]

If there exists a map $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $F(\zeta_j) = W_j$, $j \in \{1, 2, 3\}$, then for each $k \in \{1, 2, 3\}$, we have:

- either $\sigma(B_k(W_G(k))) \subset D(0, |\psi_k(\zeta_G(k))|), \sigma(B_k(W_L(k))) \subset D(0, |\psi_k(\zeta_L(k))|)$ and
\[
\max \left\{ \frac{\lambda B_k}{\psi_k(\zeta_G(k))}, \frac{\lambda B_k}{\psi_k(\zeta_L(k))} \right\} q(\nu, G(k), k)
\]
\[
\max \left\{ \frac{\lambda B_k}{\psi_k(\zeta_G(k))}, \frac{\lambda B_k}{\psi_k(\zeta_L(k))} \right\} q(\nu, L(k), k)
\]
\[
\mathcal{H}_B(\zeta_G(k), \zeta_L(k)),
\]
- or there exists a $\theta_0 \in \mathbb{R}$ such that
\[
B_k^{-1}\{e^{i\theta_0}\psi_k(\zeta_G(k))\} \subseteq \sigma(W_G(k)) \text{ and } B_k^{-1}\{e^{i\theta_0}\psi_k(\zeta_L(k))\} \subseteq \sigma(W_L(k)).
\]

Here, $[\cdot]$ denotes the greatest-integer function. Given $a \in \mathbb{C}$ and a function $g$ that is holomorphic in a neighbourhood of $a$, $\text{ord}_ag$ will denote the order of vanishing of $g$ at $a$ (with the understanding that $\text{ord}_ag = 0$ if $g$ does not vanish at $a$).
Remark 1.5. Theorem 1.4, unlike Result 1.3, incorporates information about the Jordan structure of the matricial data. Thus, Theorem 1.4 gives a more restrictive inequality than (1.4) if \( \nu = n \). Moreover, in Section 6 we will present a class of 3-point matricial data in \( \mathbb{D} \times \Omega_n \), \( n \geq 4 \), for which the condition (1.4) and that in [8, 14] provide no information while Theorem 1.4 implies that these data do not admit a \( \mathcal{O}(\mathbb{D}, \Omega_n) \)-interpolant.

The above discussion about the role of the Nokrane–Ransford result [13, Theorem 1.1] establishes how holomorphic correspondences are naturally related to the problem \((*)\). This is why we also consider holomorphic correspondences in this paper. Indeed, the method that we employ to provide the proof of Theorem 1.4 motivated our investigation into finding a Schwarz lemma for holomorphic correspondences, which are generalizations of algebroid multifunctions. Before we present our result, we need a few definitions:

Definition 1.6. Given domains \( D_i \subseteq \mathbb{C}^n \), \( i = 1, 2 \), a holomorphic correspondence from \( D_1 \) to \( D_2 \) is an analytic subvariety \( \Gamma \) of \( D_1 \times D_2 \) of dimension \( n \) such that \( \pi_1 | \Gamma \) is surjective (where \( \pi_1 \) denotes the projection onto \( D_1 \)).

A proper holomorphic correspondence \( \Gamma \) from \( D_1 \) to \( D_2 \) is a holomorphic correspondence (as defined above) such that \( \Gamma \cap (D_1 \times \partial D_2) = \emptyset \). We refer the reader to Section 5 for a discussion as to why holomorphic correspondences with the latter property are called proper holomorphic correspondences. A proper holomorphic correspondence \( \Gamma \) from \( D_1 \) to \( D_2 \) also induces the following set-valued map:

\[
F_\Gamma(z) := \{ w \in D_2 : (z, w) \in \Gamma \} \quad \forall z \in D_1.
\] (1.5)

The Carathéodory pseudo-distance, denoted by \( C_\Omega \), on a domain \( \Omega \) in \( \mathbb{C} \) is defined by:

\[
C_\Omega(p, q) := \sup\{ M_D(f(p), f(q)) : f \in \mathcal{O}(\Omega, \mathbb{D}) \}.
\] (1.6)

The reader will notice that we have defined \( C_\Omega \) in terms of the Möbius distance rather than the hyperbolic distance on \( \mathbb{D} \). This is done purposely because most conclusions in metric geometry that rely on \( C_\Omega \) are essentially unchanged if \( M_D \) is replaced by the hyperbolic distance on \( \mathbb{D} \) in (1.6), and because the Möbius distance arises naturally in the proof of our next theorem. We now present this theorem:

Theorem 1.7. Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) and let \( \Gamma \) be a proper holomorphic correspondence from \( \mathbb{D} \) to \( \Omega \). Then for every \( \zeta_1, \zeta_2 \in \mathbb{D} \) we have:

\[
\mathcal{H}^{C_\Omega}(F_\Gamma(\zeta_1), F_\Gamma(\zeta_2)) \leq M_D(\zeta_1, \zeta_2)^{1/n},
\]

where \( \mathcal{H}^{C_\Omega} \) denotes the Hausdorff distance induced by \( C_\Omega \), and \( n \) is the multiplicity of \( \Gamma \).

Here, the multiplicity \( n \) is as given by Lemma 5.2 (also see Remark 5.4) below. The inequality above reduces to the distance-decreasing property for the Carathéodory pseudo-distance if \( \Gamma \) is merely the graph of a holomorphic map \( F : \mathbb{D} \longrightarrow \Omega \). From this perspective, we can view Theorem 1.7 as a Schwarz lemma for proper holomorphic correspondences. It turns out that algebroid multifunctions are precisely the proper holomorphic correspondences from \( \mathbb{D} \) to itself (see Lemma 5.2). Hence, Theorem 1.7 generalizes [13, Theorem 1.1] by Nokrane–Ransford.

The above theorem is the consequence of a more precise inequality, which we present in Section 7 as Theorem 7.1. The proof of the latter theorem is closely related to the proof of Theorem 1.4. The proof of Theorem 1.4 is presented in Section 6, while the proof of Theorem 1.7 is presented in Section 7.
2. Some remarks on the holomorphic functional calculus

A very essential part of our proofs below is the ability, given a domain \( \Omega \subset \mathbb{C} \) and a matrix \( A \in M_n(\mathbb{C}) \), to define \( f(A) \) in a meaningful way for each \( f \in \mathcal{O}(\Omega) \), provided \( \sigma(A) \subset \Omega \). Most readers will be aware that this is what is known as the holomorphic functional calculus. We briefly recapitulate what the holomorphic functional calculus is so that we can make an observation about the boundary regularity of \( \Omega \) — where \( \Omega \) is as in the statement of Theorems 3.1 and 3.2 — which will be relevant to our proofs in Section 7.

The discussion in this paragraph makes sense for any unital Banach algebra \( \mathcal{A} \), where we denote the norm on \( \mathcal{A} \) by \( \| \cdot \| \). Let \( a \in \mathcal{A} \) and write

\[
\text{Hol}(a) := \text{the set of all functions holomorphic in some neighbourhood of } \sigma(a).
\]

With the understanding that if \( f, g \in \text{Hol}(a) \), \( f + g \) and \( fg \) are defined and holomorphic on \( \text{dom}(f) \cap \text{dom}(g) \supset \sigma(a) \), which endows \( \text{Hol}(a) \) with the structure of a unital \( \mathbb{C} \)-algebra, the holomorphic functional calculus is an assignment \( \Theta_a : f \mapsto f(a) \) with the following properties:

1. \( \Theta_a : \text{Hol}(a) \rightarrow \mathcal{A} \) is a \( \mathbb{C} \)-algebra homomorphism.
2. \( \Theta_a(\text{id}_\mathbb{C}) = a \).
3. Let \( \{ f_\nu \} \subset \text{Hol}(a) \) and suppose there is an open set \( U \supset \sigma(a) \) such that \( U \subset \text{dom}(f_\nu) \)

for every \( \nu \in \mathbb{N} \). Suppose \( f \in \text{Hol}(a) \) is such that \( f_\nu \rightarrow f \) uniformly on compact subsets of \( U \). Then \( \| f_\nu(a) - f(a) \| \rightarrow 0 \) as \( \nu \rightarrow \infty \).

It is a basic result of the spectral theory of Banach algebras that an assignment \( \Theta_a : f \mapsto f(a) \) with the above properties exists.

We now specialize to the Banach algebra \( M_n(\mathbb{C}) \). Fix \( A \in M_n(\mathbb{C}) \). Then, it is well known that (see [9, Chapter 7, Section 1]) for any polynomial \( p(z) = \sum_{j=0}^m \alpha_j z^j \) such that

\[
p^{(j)}(\lambda) = f^{(j)}(\lambda) \quad \forall j : 0 \leq j \leq \nu(\lambda) - 1 \text{ and } \forall \lambda \in \sigma(A),
\]

where

\[
\nu(\lambda) := \min \left\{ k \in \mathbb{N} : \text{Ker}(\lambda I - A)^{k+1} = \text{Ker}(\lambda I - A)^k \right\}, \quad \lambda \in \sigma(A),
\]

the assignment

\[
\Theta_A(f) = f(A) := \sum_{j=0}^m \alpha_j A^j
\]

has the properties (i), (ii) and (iii) above. Note that, for \( \lambda \in \sigma(A) \), \( \nu(\lambda) \) is the exponent of \( (z - \lambda) \) in the minimal polynomial of \( A \). Now, given a non-empty open set \( \Omega \subset \mathbb{C} \) and \( A \in M_n(\mathbb{C}) \) such that \( \sigma(A) \subset \Omega \), one defines

\[
f(A) := \Theta_A(f) \quad \forall f \in \mathcal{O}(\Omega).
\]

By the foregoing discussion, we need to make no assumptions about the boundary of \( \Omega \) in defining \( f(A), f \in \mathcal{O}(\Omega) \), such that the assignment \( \mathcal{O}(\Omega) \ni f \mapsto f(A) \) (provided \( \sigma(A) \subset \Omega \)) behaves “naturally”. We consider this point relevant to make because in treatments of the assignment \( \mathcal{O}(\Omega) \ni f \mapsto f(a) \) in certain books, \( a \) belonging to a general unital Banach algebra \( \mathcal{A} \), this assignment is defined via a Cauchy integral and with certain conditions imposed on \( \partial \Omega \) when \( \Omega \not\subset \mathbb{C} \). A rephrasing of the above point in a manner that is more precise and relevant to the proofs in Section 7 is as follows.

Remark 2.1. Let \( \Omega \) be a non-empty open set in \( \mathbb{C} \) and let \( S_n(\Omega) := \{ A \in M_n(\mathbb{C}) : \sigma(A) \subset \Omega \} \), \( n \geq 2 \). Then for each \( f \in \mathcal{O}(\Omega) \) and \( A \in S_n(\Omega) \), we can define \( f(A) \) such that \( f(A) \) — fixing \( A \in S_n(\Omega) \) and writing \( f(A) := \Theta_A(f) \) — has the properties (i)–(iii) above (taking \( \mathcal{A} = M_n(\mathbb{C}) \), \( a = A \) and with \( \mathcal{O}(\Omega) \) and \( \Omega \) replacing \( \text{Hol}(a) \) and \( U \), respectively) \( \forall f \in \mathcal{O}(\Omega) \) without any conditions on \( \partial \Omega \) or on whether \( f \in \mathcal{O}(\Omega) \) extends to \( \partial \Omega \). With \( \Omega \) as above and
A ∈ S_n(Ω), the assignment O(Ω) ∋ f → f(A) will also be called the holomorphic functional calculus in our discussions below.

We end this section by stating the Spectral Mapping Theorem. When we invoke it in subsequent sections, it will be for the Banach algebra 𝒪 = M_n(ℂ).

**Result 2.2** (Spectral Mapping Theorem). Let 𝒪 be a unital Banach algebra. Then for every \( f ∈ \text{Hol}(a) \) and \( a ∈ 𝒪 \) we have

\[
\sigma(f(a)) = f(\sigma(a)).
\]

### 3. Minimal polynomials under the holomorphic functional calculus

In this section, we develop the key matricial tool needed in establishing Theorem 3.4 which is the computation of the minimal polynomial for \( f(A) \), given \( f ∈ O(Ω) \) and \( A ∈ Ω_n \), \( n ≥ 2 \). This is the content of Theorem 3.4. We begin with a few lemmas which will help us to establish Theorem 3.4. In what follows, given integers \( p < q, [p..q] \) will denote the set of integers \( \{p, p+1, ..., q\} \). Recall: \( [\cdot] \) denotes the greatest-integer function. Also, given \( A ∈ M_n(ℂ) \), we will denote its minimal polynomial by \( M_A \).

Let \( n ≥ 2 \). Given \( (α_1, α_2, ..., α_{n-1}) ∈ ℂ^{n-1} \), we define \( l(α_1, α_2, ..., α_{n-1}) ∈ [1..n] \) by

\[
l(α_1, α_2, ..., α_{n-1}) := \begin{cases} n, & \text{if } α_j = 0 \text{ for all } j ∈ [1..n-1], \\
\min\{j ∈ [1..n-1] : α_j ≠ 0\}, & \text{otherwise.}
\end{cases}
\]

**Lemma 3.1.** Let \( (α_0, α_1, ..., α_{n-1}) ∈ ℂ^n, n ≥ 2 \). Let \( A = \sum_{j=0}^{n-1} α_j N^j \), where \( N ∈ M_n(ℂ) \) is the nilpotent matrix of degree \( n \) given by \( (δ_{i+1,j})_{i,j=1}^n = δ_{μ,ν} \) being the Kronecker symbol. Then the minimal polynomial for \( A \) is given by:

\[
M_A(t) = (t - α_0)^{[(n-1)/l(α_1,α_2,...,α_{n-1})]+1},
\]

(3.1)

where \( l(α_1, α_2, ..., α_{n-1}) \) is as defined above.

**Proof.** The proof consists of two cases.

**Case 1.** \( α_j = 0 \) \( ∀ j ∈ [1..n-1] \).

This implies that \( l(α_1, α_2, ..., α_{n-1}) = n \), and hence \( [(n-1)/l(α_1, α_2, ..., α_{n-1})] = 0 \). The minimal polynomial in this case clearly is \( (t - α_0) \). This establishes (3.1) in this case.

**Case 2.** \( α_j ≠ 0 \) for some \( j ∈ [1..n-1] \).

We write \( l ≡ l(α_1, α_2, ..., α_{n-1}) \). Then \( A - α_0 I = α_l N^l + \cdots + α_{n-1} N^{n-1} \). Hence

\[
(A - α_0 I)^m = α_l^m N^{lm} + \text{(terms in } N^k \text{ with } k > lm).
\]

We observe thus that the power of \( (t - α_0) \) in \( M_A(t) \) must be the least integer \( m \) such that \( ml ≥ n \). It is elementary to see that that integer is \( [(n-1)/l]+1 \).

Let \( a ∈ ℂ \) and let \( g \) be a function that is holomorphic in a neighbourhood of \( a \). Then, \( \text{ord}_a g \) will denote the order of vanishing of \( g \) at \( a \). Recall: this means that \( \text{ord}_a g \) is the least non-negative integer \( j \) such that \( g^{(j)}(a) ≠ 0 \) (hence, \( \text{ord}_a g = 0 \) if \( g \) does not vanish at \( a \)).

**Lemma 3.2.** Let \( λ ∈ ℂ \) and let \( f ∈ O(Ω) \) be a non-constant function. Let \( J_n(λ) \) represent the \( n × n \) Jordan matrix associated to \( λ, n ≥ 2 \), and \( f(J_n(λ)) \) be the matrix given by the holomorphic functional calculus. Then the minimal polynomial of \( f(J_n(λ)) \) is given by

\[
M_{f(J_n(λ))}(t) = (t - f(λ))^{[n-1/\text{ord}_λ |f'|+1]}+1.
\]
Using the fact that $N_k$ gives an expression for $f(J_n(\lambda))$ in terms of the exponent of $(t - \lambda)$ in $M_{J_n(\lambda)}$, our task is to determine the analogous exponent in $M_{f(J_n(\lambda))}$, for which \eqref{eq:2.1} is not immediately helpful.

Let $R$ be such that $|\lambda| < R < 1$. Then, the power series expansion of $f$

$$f(z) := \sum_{k\in\mathbb{N}} \frac{f^{(k)}(0)}{k!} z^k$$

converges absolutely at each $z \in D(0; R)$.

Then by elementary properties (see Section 2) of the holomorphic functional calculus we get

$$f(J_n(\lambda)) = \sum_{k\in\mathbb{N}} \frac{f^{(k)}(0)}{k!} (J_n(\lambda))^k.$$  \hfill (3.2)

Note that $J_n(\lambda) = \lambda \mathbb{I} + N$, where $N$ is the nilpotent matrix as in Lemma 3.1. We can use the binomial expansion to get

$$(J_n(\lambda))^k = (\lambda \mathbb{I} + N)^k = \sum_{j=0}^{p(k)} \binom{k}{j} \lambda^{k-j} N^j,$$

where $p(k) := \min(k, n-1)$ and $k \in \mathbb{N}$. Hence (3.2) becomes

$$f(J_n(\lambda)) = \sum_{k\in\mathbb{N}} \frac{f^{(k)}(0)}{k!} \sum_{j=0}^{p(k)} \binom{k}{j} \lambda^{k-j} N^j.$$  \hfill (3.3)

The coefficient of $N^j$, $0 \leq j \leq n-1$, in (3.3) is

$$\sum_{k\geq j} \frac{f^{(k)}(0)}{k!} \binom{k}{j} \lambda^{k-j} = \sum_{k\geq j} \frac{f^{(k)}(0)}{(k-j)! j!} \lambda^{k-j} = \frac{f^{(j)}(\lambda)}{j!}, \quad j \in \mathbb{N}.$$  

Using the fact that $N^n = 0$, we get

$$f(J_n(\lambda)) = \sum_{j=0}^{n-1} \frac{f^{(j)}(\lambda)}{j!} N^j.$$  

From Lemma 3.1 we have $M_{f(J_n(\lambda))}(t) = (t - f(\lambda))^m$, where

$$m = \left\lceil \frac{n - 1}{l(f'(\lambda), f''(\lambda), \ldots, f^{(n-1)}(\lambda))} \right\rceil + 1.$$  

If $\text{ord}_\lambda f' \leq n - 2$, then $l(f'(\lambda), f''(\lambda), \ldots, f^{(n-1)}(\lambda)) = \text{ord}_\lambda f' + 1$, else $\text{ord}_\lambda f' + 1$ and $l(f'(\lambda), f''(\lambda), \ldots, f^{(n-1)}(\lambda)) > (n - 1)$. In both the cases we have:

$$\left\lceil \frac{n - 1}{l(f'(\lambda), f''(\lambda), \ldots, f^{(n-1)}(\lambda))} \right\rceil = \left\lceil \frac{n - 1}{\text{ord}_\lambda f' + 1} \right\rceil.$$  

From the last two expressions, the lemma follows. \hfill \Box

**Lemma 3.3.** Let $\lambda \in \mathbb{D}$ and $f \in \mathcal{O}(\mathbb{D})$, $f$ non-constant. Let $n_1 \leq n_2 \leq \cdots \leq n_q$ be a sequence of positive integers. Let $J = \oplus_{i=1}^{q} J_{n_i}(\lambda)$, where $J_{n_i}(\lambda)$ represents the $n_i \times n_i$ Jordan block associated to $\lambda$. Then the minimal polynomial for $f(J)$ is given by:

$$M_{f(J)}(t) = (t - f(\lambda))^{\left\lceil \frac{n_q - 1}{\text{ord}_\lambda f' + 1} \right\rceil + 1}.$$
Proof. Note that \( f(J) = \oplus_{i=1}^q f(J_{n_i}(\lambda)) \). If \( n_i = 1 \), for \( i = 1, \ldots, q \), then the following is obvious; else Lemma 3.2 gives us

\[
M_{f(J_{n_i}(\lambda))}(t) = (t - f(\lambda)) \left[ \frac{n_i - 1}{\text{ord}_\lambda f' + 1} \right] + 1 \quad \forall i \in [1..q].
\]

(3.4)

For each \( i \), we also have

\[
\left[ \frac{n_i - 1}{\text{ord}_\lambda f' + 1} \right] + 1 \leq \left[ \frac{n_q - 1}{\text{ord}_\lambda f' + 1} \right] + 1.
\]

(3.5)

Now the minimal polynomial for a matrix that is a finite direct sum of matrices is the least common multiple (in the ring \( \mathbb{C}[t] \)) of the minimal polynomials of the direct summands. This, together with (3.4) and (3.5), establishes the lemma.

\[\Box\]

Theorem 3.4. Let \( A \in \Omega_n, n \geq 2 \), and let \( f \in \mathcal{O}(\mathbb{D}) \) be a non-constant function. Suppose that the minimal polynomial for \( A \) is given by

\[
M_A(t) = \prod_{\lambda \in \sigma(A)} (t - \lambda)^{m(\lambda)}.
\]

Then the minimal polynomial for \( f(A) \) is given by

\[
M_{f(A)}(t) = \prod_{\nu \in f(\sigma(A))} (t - \nu)^{k(\nu)},
\]

where, \( k(\nu) = \max \left\{ \left[ \frac{m(\lambda) - 1}{\text{ord}_\lambda f' + 1} \right] + 1 : \lambda \in \sigma(A) \cap f^{-1}\{\nu\} \right\} \).

Proof. By the Spectral Mapping Theorem the minimal polynomial for \( f(A) \) is given by

\[
\prod_{\nu \in f(\sigma(A))} (t - \nu)^{k(\nu)}
\]

for some \( k(\nu) \in \mathbb{N} \). We must now show that \( k(\nu) \) are as stated above. Let \( \mathcal{G}(\nu) \), for each \( \nu \in f(\sigma(A)) \), denote the set

\[
\mathcal{G}(\nu) := \{ \lambda \in \sigma(A) : f(\lambda) = \nu \}.
\]

Then \( \{ \mathcal{G}(\nu) : \nu \in f(\sigma(A)) \} \) gives a partition of \( \sigma(A) \).

The Jordan canonical form tells us that \( \exists S \in GL_n(\mathbb{C}) \) such that

\[
A = S \left[ \oplus_{\nu \in f(\sigma(A))} \left[ \oplus_{\lambda \in \mathcal{G}(\nu)} \left[ \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right] \right] \right] S^{-1},
\]

(3.6)

where \( \{ J_{n_i^\lambda}(\lambda) : i \in [1..q(\lambda)] \} \) is the Jordan block-system associated to \( \lambda \) such that \( n_1^\lambda \leq n_2^\lambda \leq \cdots \leq n_{q(\lambda)}^\lambda \). Now from (3.6) and from the basic properties of the holomorphic functional calculus we get

\[
M_{f(A)} = M\left( \oplus_{\nu \in f(\sigma(A))} \left[ \oplus_{\lambda \in \mathcal{G}(\nu)} \left[ f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right] \right] \right),
\]

(3.7)

(we will sometimes write \( M_B \) as \( M(B) \) for convenience). Notice that if \( \nu_1 \neq \nu_2 \in f(\sigma(A)) \), the matrices \( \oplus_{\lambda \in \mathcal{G}(\nu)} f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right), j = 1, 2 \), have \( \{\nu_1\} \) and \( \{\nu_2\} \), respectively, as their spectra. Hence the minimal polynomials of these are relatively prime to each other. This implies that (from (3.7) above):

\[
M_{f(A)} = \prod_{\nu \in f(\sigma(A))} M\left( \oplus_{\lambda \in \mathcal{G}(\nu)} \left[ f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right] \right).
\]

(3.8)

The above is the consequence of the fact that the minimal polynomial of a direct sum of matrices is the least common multiple of the minimal polynomials of the individual matrices. This also implies that

\[
M\left( \oplus_{\lambda \in \mathcal{G}(\nu)} \left[ f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right] \right) = \text{lcm} \left\{ M \left( f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right) : \lambda \in \mathcal{G}(\nu) \right\}.
\]

(3.9)
For a fixed $\lambda \in \mathfrak{S}(\nu)$, recall that $n_1^\lambda \leq n_2^\lambda \leq \cdots \leq n_{q(\lambda)}^\lambda$. Furthermore, $n_q^\lambda = m(\lambda)$, $m(\lambda)$ being the multiplicity of $\lambda$ in $M_A$. Putting all of this together with Lemma 3.3 we have:

$$M\left( f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right)(t) = (t - \nu)^{\left\lfloor \frac{m(\lambda) - 1}{\ord f' + 1} \right\rfloor + 1} \forall \lambda \in \mathfrak{S}(\nu).$$  

(3.10)

Now, (3.10) and (3.9) together give us:

$$M\left( \left[ \oplus_{\lambda \in \mathfrak{S}(\nu)} f \left( \oplus_{i=1}^{q(\lambda)} J_{n_i^\lambda}(\lambda) \right) \right] \right) = (t - \nu)^{k(\nu)},$$

where $k(\nu)$ is as stated in our theorem. The above, in view of (3.8), gives the result. $\square$

4. TWO FUNDAMENTAL LEMMAS

In this section, we state two fundamental and closely related lemmas. Lemma 4.3 serves as the link between the two main results of this paper. Both lemmas are simple, once we appeal to Vesentini’s theorem. We begin by stating this result.

**Result 4.1** (Vesentini, [16]). Let $\mathcal{A}$ be a complex, unital Banach algebra and let $\rho(x)$ denote the spectral radius of any element $x \in \mathcal{A}$. Let $f \in \mathcal{O}(\mathbb{D}, \mathcal{A})$. Then, the function $\zeta \mapsto \rho(f(\zeta))$ is subharmonic on $\mathbb{D}$.

**Lemma 4.2.** Let $\Phi \in \mathcal{O}(\mathbb{D}, \overline{M_n})$ be such that there exists a $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \mathbb{D}$ satisfying $e^{i\theta_0} \in \sigma(\Phi(\zeta_0))$. Then $e^{i\theta_0} \in \sigma(\Phi(\zeta))$ for all $\zeta \in \mathbb{D}$.

**Proof.** Define $\tilde{\Phi} \in \mathcal{O}(\mathbb{D}, M_n(\mathbb{C}))$ by

$$\tilde{\Phi}(\zeta) = \Phi(\zeta) + e^{i\theta_0} \mathbb{I}, \forall \zeta \in \mathbb{D}.$$ 

By Result 4.1, $\rho \circ \tilde{\Phi}$ is subharmonic on $\mathbb{D}$. Notice that for each $\zeta \in \mathbb{D}$ we have $\rho \circ \tilde{\Phi}(\zeta) \leq 2$, and $\rho(\tilde{\Phi}(\zeta_0)) = 2$. By the maximum principle for subharmonic functions it follows that $\rho \circ \tilde{\Phi} \equiv 2$. As $\sigma(\tilde{\Phi}(\zeta)) = e^{i\theta_0} + \sigma(\Phi(\zeta))$ and $\sigma(\Phi(\zeta)) \subseteq \overline{\mathbb{D}}$, this implies that $e^{i\theta_0} \in \sigma(\Phi(\zeta)) \forall \zeta \in \mathbb{D}$. Hence the lemma. $\square$

The next lemma is, essentially, a fragment of a proof in [5, Section 3]. However, since it requires a non-trivial fact — i.e., the plurisubharmonicity of the spectral radius — we provide a proof.

**Lemma 4.3.** Let $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$ be such that $F(0) = 0$. Then, there exists $G \in \mathcal{O}(\mathbb{D}, \overline{\Omega_n})$ such that $F(\zeta) = \zeta G(\zeta)$ for all $\zeta \in \mathbb{D}$.

**Proof.** As $F(0) = 0$, there exists $G \in \mathcal{O}(\mathbb{D}, M_n(\mathbb{C}))$ such that $F(\zeta) = \zeta G(\zeta) \forall \zeta \in \mathbb{D}$. Fix a $\zeta \in \mathbb{D} \setminus \{0\}$ and let $R \in (0, 1)$ be such that $R > |\zeta|$. Then on the circle $|w| = R$ we have

$$\rho(F(w)) = R \rho(G(w))$$

$$\implies \rho(G(w)) = \frac{\rho(F(w))}{R} < \frac{1}{R} \forall w : |w| = R,$$

(4.1)

where $\rho$ denotes the spectral radius. We again appeal to Vesentini’s Theorem. Subharmonicity of $\rho \circ G$, the maximum principle and (4.1) give us:

$$\rho(G(\zeta)) < \frac{1}{R} \text{ (recall that } |\zeta| < R).$$

By taking $R \not> 1$, and from the fact that $\zeta \in \mathbb{D}$ was arbitrary, we get $\rho(G(\zeta)) \leq 1 \forall \zeta \in \mathbb{D}$. This is equivalent to $G \in \mathcal{O}(\mathbb{D}, \overline{\Omega_n})$. $\square$

5. Some notations and results in basic complex geometry

This section is devoted to a couple of results in the geometry and function theory in the holomorphic setting that are relevant to our proof of Theorem 1.7. Our first result pertains to the structure of a holomorphic correspondence $\Gamma$ from $\mathbb{D}$ to $\Omega$ with the properties as in Theorem 1.7. For this, we need the following standard result (see [6, Section 3.1], for instance).

**Result 5.1.** Let $\Omega_1 \subseteq X$ and $\Omega_2 \subseteq Y$ be open subsets in topological spaces $X$ and $Y$ respectively, with $\bar{\Omega}_2$ being compact. Let $A$ be a closed subset in $\Omega_1 \times \Omega_2$. Then the restriction to $A$ of the projection $(x, y) \mapsto x$ is proper if and only if $A \cap (\Omega_1 \times \partial \Omega_2) = \emptyset$.

Owing to the above result, any holomorphic correspondence $\Gamma$ from $D_1$ to $D_2$, which are domains in $\mathbb{C}^n$, such that $\overline{\Gamma} \cap (D_1 \times \partial D_2) = \emptyset$ is called a proper holomorphic correspondence. We can now state and prove the result that we need. This result is deducible, in essence, from [6, Section 4.2]. However, since that discussion pertains to a much more general setting, we provide a proof in the setup that we are interested in.

**Lemma 5.2.** Let $\Omega$ be a bounded domain in $\mathbb{C}$ and let $\Gamma$ be a proper holomorphic correspondence from $\mathbb{D}$ to $\Omega$. Then there exist an $n \in \mathbb{Z}_+$ and functions $a_1, \ldots, a_n \in \mathcal{O}(\mathbb{D})$ such that

$$
\Gamma = \left\{(z, w) \in \mathbb{D} \times \Omega : w^n + \sum_{j=1}^{n} (-1)^j a_j(z) w^{n-j} = 0 \right\}.
$$

**Proof.** Since $\pi_1 |_{\Gamma}$ is proper, it follows from the elementary theory of proper holomorphic maps that there exists a discrete set $A \subseteq \mathbb{D}$ and an $n \in \mathbb{Z}_+$ such that $(\Gamma \setminus \pi_1^{-1}(A), \mathbb{D} \setminus A, \pi_1 |_{\Gamma})$ is an $n$-fold analytic covering. Thus, given any $p \in \mathbb{D} \setminus A$, there exist an open neighborhood $V_p$ such that $p \in V_p \subseteq \mathbb{D} \setminus A$, and $n$ holomorphic inverse branches of $\pi_1 |_{\Gamma}$; $(\pi_1)^{-1}_{1,p}, \ldots, (\pi_1)^{-1}_{n,p} \in \mathcal{O}(V_p, \Gamma)$; such that the images of $(\pi_1)^{-1}_{j,p}$, $j = 1, \ldots, n$, are disjoint.

Let $\mathcal{J}_j$ denote the $j$-th elementary symmetric polynomial in $n$ symbols. Define:

$$
a_j(z) := \mathcal{J}_j((\pi_1)^{-1}_{1,z}(z), \ldots, (\pi_1)^{-1}_{n,z}(z)) \forall z \in \mathbb{D} \setminus A, \ 1 \leq j \leq n.
$$

Clearly, $a_j$ does not depend on the order in which $\{(\pi_1)^{-1}_{k,p}\}_{k \in [1 \ldots n]}$ are labeled, whence it is well-defined. Now, fix a $p \in \mathbb{D} \setminus A$. Provisionally, for $z \in V_p$, let us define:

$$(\pi_1)^{-1}_{j,z} := \text{the holomorphic inverse branch of } \pi_1 |_{\Gamma} \text{ that maps } z \text{ to } (\pi_1)^{-1}_{j,p}(z),$$

which is well-defined, because the images of $(\pi_1)^{-1}_{j,p}$, $j = 1, \ldots, n$, are disjoint. By the same reason $\{(\pi_1)^{-1}_{k,z}(z) : 1 \leq k \leq n\} = \{(\pi_1)^{-1}_{j,z}(z) : 1 \leq j \leq n\}.$ From the last two assertions, we have

$$a_j(z) = \mathcal{J}_j((\pi_1)^{-1}_{1,z}(z), \ldots, (\pi_1)^{-1}_{n,z}(z)) \forall z \in V_p.$$ 

This tells us that $a_j$ is $\mathbb{C}$-differentiable at each $p \in \mathbb{D} \setminus A$, whence $a_j \in \mathcal{O}(\mathbb{D} \setminus A)$. It is easy to see that, for each $q \in A$

$$\lim_{\mathbb{D} \setminus A \ni z \to q} a_j(z) = \sum_{(i_1, \ldots, i_j) \in \mathcal{J}_n(j)} \prod_{1 \leq k \leq j} w_{i_k},$$

where $\mathcal{J}_n(j)$ denotes the set of all increasing $j$-tuples of $[1 \ldots n]$, $j = 1, \ldots, n$, and $(\pi_1^{-1}\{q\} \cap \Gamma)^* :=$ an enumeration of the list of elements in $\pi_1^{-1}\{q\} \cap \Gamma$ repeated according to intersection multiplicity.
From (5.2) and Riemann’s removable singularities theorem, we conclude that \( a_j \) extends to a holomorphic function, \( j \in \{1..n\} \). Furthermore, by the definition of \( a_j|_{\Omega \setminus \mathcal{A}} \) and by (5.2), we conclude, from Vieta’s formulas that, fixing a \( z_0 \in \mathbb{D} \):

\[
\text{the list, repeated according to multiplicity, of the zeros of } w^n + \sum_{j=1}^{n} (-1)^j a_j(z_0)w^{n-j} = F_\Gamma(z_0)^* .
\]

As \( z_0 \in \mathbb{D} \) was arbitrary, the result follows. \( \square \)

**Remark 5.3.** We have used a notation in our proof of Lemma 5.2 — see (5.2) and the clarifications that follow — that will be useful in later discussions/calculations. Namely: if \( S \) is a non-empty set, and there is a multiplicity associated to each \( s \in S \), then we shall use the notation \( S^\ast \) to denote the list of elements of \( S \) repeated according to their multiplicity.

**Remark 5.4.** The positive integer \( n \) that appears in the above lemma is known as the multiplicity of \( \Gamma \). In general, when we have a proper holomorphic correspondence \( \Gamma \) from \( \Omega_1 \) to \( \Omega_2 \) with \( \dim(\Gamma) = \dim(\Omega_1) \), then there exists an analytic variety \( \mathcal{A} \subset \Omega_1 \) with \( \dim \mathcal{A} < \dim(\Gamma) \) such that the cardinality of \( \pi_1^{-1}\{z\} \cap \Gamma = k \) for all \( z \in \Omega \setminus \mathcal{A} \) (see [6, Section 3.7]). This generalizes the notion of multiplicity to higher dimensions.

We shall now look at an extremal problem associated to the Carathéodory pseudo-distance \( C_\Omega \) on the domain \( \Omega \) in \( \mathbb{C} \). Recall the discussion in Section 4 about the Carathéodory pseudo-distance, and the reasons that we prefer using the following definition:

\[
C_\Omega(p, q) := \sup \{ \mathcal{M}_D(f(p), f(q)) : f \in \mathcal{O}(\Omega, \mathbb{D}) \} = \sup \{ |f(q)| : f \in \mathcal{O}(\Omega, \mathbb{D}) : f(p) = 0 \} .
\]

(5.3)

The equality in (5.3) is due to the fact that the automorphism group of \( \mathbb{D} \) acts transitively on \( \mathbb{D} \) and the M"obius distance is invariant under its action. Applying Montel’s Theorem, it is easy to see that there exists a function \( g \in \mathcal{O}(\Omega, \mathbb{D}) \) such that \( g(p) = 0 \) and \( g(q) = C_\Omega(p, q) \). Such a function is called an extremal solution for the extremal problem determined by (5.3).

We will always consider domains \( \Omega \) in \( \mathbb{C} \) for which \( H^\infty(\Omega) \), the set of all bounded holomorphic functions in \( \Omega \), separates points in \( \Omega \). For such domains the Carathéodory pseudo-distance clearly is a distance. Moreover it turns out that for such domains there is a unique extremal solution (see the last two paragraphs in [10]). In Section 7 (since the domains considered there are bounded), we will always denote by \( G_\Omega(p, q; \cdot) \) the unique extremal solution determined by the extremal problem (5.3).

### 6. The proof of Theorem 1.4

This section is largely devoted to the proof of Theorem 1.4. Closely related to it is our example — referred to in Section 4 — that compares the necessary condition given by Theorem 1.4 with other necessary conditions for the existence of a 3-point interpolant from \( \mathbb{D} \) to \( \Omega_n \). This example is presented after our proof.

#### 6.1. The proof of Theorem 1.4

Let \( F \in \mathcal{O}(\mathbb{D}, \Omega_n) \) be such that \( F(\zeta_j) = W_j, j \in \{1, 2, 3\} \). Fix \( k \in \{1, 2, 3\} \) and write:

\[
\tilde{F}_k := B_k \circ F \circ \psi_k^{-1} ,
\]

where \( B_k, \psi_k \) are as described before the statement of Theorem 1.4. Notice that \( \tilde{F}_k \in \mathcal{O}(\mathbb{D}, \Omega_n) \) and satisfies \( \tilde{F}_k(\psi_k(\zeta_{L(k)})) = B_k(W_{L(k)}), \tilde{F}_k(\psi_k(\zeta_{G(k)})) = B_k(W_{G(k)}), \tilde{F}_k(0) = 0. \)
By Lemma 4.3 we get
\[ \tilde{F}_k(\zeta) = \zeta \tilde{G}_k(\zeta) \ \forall \zeta \in \mathbb{D}, \text{ for some } \tilde{G}_k \in \mathcal{O}(\mathbb{D}, \Omega_n). \] (6.1)

Two cases arise:

**Case 1.** \( \tilde{G}_k(\mathbb{D}) \subset \Omega_n. \)

In view of (6.1), we have
\[ \tilde{G}_k \left( \psi_k(\zeta(L_k)) \right) = W_{L(k), k}, \quad \tilde{G}_k \left( \psi_k(\zeta(G_k)) \right) = W_{G(k), k}, \] (6.2)

where \( W_{L(k), k} := B_k(W_{L(k)})/\psi_k(\zeta(L_k)) \) and \( W_{G(k), k} := B_k(W_{G(k)})/\psi_k(\zeta(G_k)) \). Now by Result 1.1 a necessary condition for (6.2) is
\[
\max \left\{ \max_{\eta \in \sigma(W_{L(k), k})} |b_{G(k), k}(\eta)|, \max_{\eta \in \sigma(W_{G(k), k})} |b_{L(k), k}(\eta)| \right\} \leq \mathcal{M}_\Delta \left( \zeta(L_k), \zeta(G_k) \right),
\] (6.3)

where \( b_{L(k), k} \) and \( b_{G(k), k} \) denote the minimal Blaschke product corresponding to the matrices \( W_{L(k), k} \) and \( W_{G(k), k} \). Given the definitions of the latter matrices, we need Theorem 3.4 to determine \( b_{L(k), k}, b_{G(k), k} \). By this theorem, we have
\[
b_{L(k), k}(t) = \prod_{\nu \in \sigma(B_k(W_{L(k)}))} \left( \frac{t - \nu/\psi_k(\zeta(L_k))}{1 - \nu/\psi_k(\zeta(L_k))} \right)^{q(\nu, L(k), k)} \] (6.4)
\[
b_{G(k), k}(t) = \prod_{\nu \in \sigma(B_k(W_{G(k)}))} \left( \frac{t - \nu/\psi_k(\zeta(G_k))}{1 - \nu/\psi_k(\zeta(G_k))} \right)^{q(\nu, G(k), k)} \] (6.5)

where \( q(\nu, L(k), k) \) and \( q(\nu, G(k), k) \) are as in the statement of Theorem 3.4. Now if \( \eta \in \sigma(W_{L(k), k}) \) or \( \eta \in \sigma(W_{G(k), k}) \), then \( \eta = \mu/\psi_k(\zeta(L_k)) \) for some \( \mu \in \sigma(B_k(W_{L(k)})) \) or \( \eta = \mu/\psi_k(\zeta(G_k)) \) for some \( \mu \in \sigma(B_k(W_{G(k)})) \), respectively, and conversely. This observation together with (6.5), (6.1) and (6.3) establishes the first part of our theorem.

**Case 2.** \( \tilde{G}_k(\mathbb{D}) \cap \partial \Omega_n \neq \emptyset \).

Let \( \zeta_0 \in \mathbb{D} \) such that \( e^{i\theta_0} \in \sigma(\tilde{G}_k(\zeta_0)) \) for some \( \theta_0 \in \mathbb{R} \). By Lemma 4.2 we have \( e^{i\theta_0} \in \sigma(G_k(\zeta)) \) for every \( \zeta \in \mathbb{D} \). By (6.1), \( e^{i\theta_0} \zeta \in \sigma(F_k(\zeta)) \). Let \( \Phi = F \circ \psi_k^{-1} \). Then \( \Phi \in \mathcal{O}(\mathbb{D}, \Omega_n) \) and we have:
\[
e^{i\theta_0} \zeta \in \sigma(B_k \circ \Phi(\zeta)) = B_k \{ \sigma(\Phi(\zeta)) \} \ \forall \zeta \in \mathbb{D},
\]
where the last equality is an application of the Spectral Mapping Theorem. For each \( \zeta \in \mathbb{D} \), let \( \omega_\zeta \in \sigma(\Phi(\zeta)) \) be such that \( B_k(\omega_\zeta) = e^{i\theta_0} \zeta \). Notice that if \( \zeta_1 \neq \zeta_2 \) then \( \omega_{\zeta_1} \neq \omega_{\zeta_2} \), whence \( E := \{ \omega_\zeta : \zeta \in \mathbb{D} \} \) is an uncountable set. Notice that \( \omega_\zeta \) satisfies:
\[
B_k(\omega_\zeta) = e^{i\theta_0} \zeta \text{ and } \det(\omega_\zeta I - \Phi(\zeta)) = 0 \ \forall \zeta \in \mathbb{D}.
\]

This implies \( \det(\omega_\zeta I - \Phi(\zeta)) = 0 \) for every \( \omega_\zeta \in E \). As \( E \) is uncountable, it follows that \( \det(\omega_\zeta I - \Phi(\zeta)) = 0 \) for every \( \omega_\zeta \in \mathbb{D} \). Thus, as \( B_k \) maps \( \mathbb{D} \) onto itself, it follows that
\[
B_k^{-1 \{ e^{i\theta_0} \zeta \}} \subset \sigma(\Phi(\zeta)) = \sigma(F \circ \psi_k^{-1}(\zeta)) \ \forall \zeta \in \mathbb{D}.
\] (6.6)

Putting \( \zeta = \psi_k(\zeta(L_k)) \) and \( \zeta = \psi_k(\zeta(G_k)) \) respectively in (6.6) we get
\[ B_k^{-1 \{ e^{i\theta_0} \psi_k(\zeta(L_k)) \}} \subset \sigma(F(\zeta(L_k))) = \sigma(W_{L(k)}) \text{ and } B_k^{-1 \{ e^{i\theta_0} \psi_k(\zeta(G_k)) \}} \subset \sigma(F(\zeta(G_k))) = \sigma(W_{G(k)}). \]

We now present our example that compares the condition given by Theorem 1.4 with that of Costara and Ogle and Baribeau–Kamara as alluded to in Remark 1.5.

**Example 6.1.** For each \( n \geq 4 \), there is a class of 3-point data-sets for which Theorem 1.4
implies that it cannot admit any $O(D, \Omega_n)$-interpolant, whereas the conditions given by \[8, 14\] and by Result \[13\] provide no information.

Let $0, a$ and $b$ be distinct points in $D$. For each $n \geq 4$, we will construct a class of 3-point matricial data—of the form $\{(0, A, B)\}$ such that $0, A, B \in \Omega_n$, where $A$ and $B$ will depend on $n, a, b$—for which the aforementioned statement holds true. To this end, consider the matrices:

$$A := \sum_{j=0}^{n-1} \alpha_j N^j,$$
where $\alpha_j \in \mathbb{C}$ is such that $\alpha_j = 0 \ \forall j \in [0 \ldots n-3]$ and $\alpha_{n-2} \neq 0$,

$$B := \text{diag}[^{\beta_1, \ldots, \beta_n}]: = \text{the diagonal matrix with distinct entries } \beta_i \text{ such that } \beta_1 = 0 \text{ and such that}
\begin{align*}
&\bullet \beta_i^2 \neq \beta_j^2 \ \forall i \neq j, \\
&\bullet |\beta_i| < |b| \ \forall i \in [1 \ldots n], \text{ and} \\
&\bullet |\beta_i|^2 < M_D(a, b) \ \forall i \in [1 \ldots n].
\end{align*}

We shall see the relevance of these conditions presently. Here, $N$ is the nilpotent matrix introduced in Lemma \[3.1\].

Notice that by Lemma \[3.1\] the minimal polynomial for $A$ is given by $M_A(t) = t^2$ while its characteristic polynomial is $t^n$. As $n \geq 4$, $A$ is not a non-derogatory matrix. Thus, for the data $\{(0,0), (a, A), (b, B)\}$ the result given by Costara and Ogle cannot be applied in this setting, and hence yields no information.

Let us compute the form that the condition \[1.4\] takes for these data by setting $(\zeta_1, W_1) = (0,0)$, $(\zeta_2, W_2) = (a, A)$ and $(\zeta_3, W_3) = (b, B)$. Observe: $B_1(t) = t$ for every $t \in D$. The Spectral Mapping Theorem then implies that $\sigma(B_1(W_1)) = B_1(\sigma(W_1)) = \sigma(W_1)$ for $j = 2, 3$. Hence we have:

$$\mathcal{H}^{M_D}(\sigma(B_1(W_2)) \cap D, \sigma(B_1(W_3)) \cap D) = \max_{1 \leq i \leq n} |\beta_i| ^{1/n}.$$

Since $\beta_i$’s are all distinct, $\nu = n$. So the condition \[1.4\] turns out to be:

$$\max_{1 \leq i \leq n} |\beta_i| ^{1/n} \leq M_D(a, b)^{1/n}. \quad (6.7)$$

Permuting the roles of $(\zeta_j, W_j)$, $j = 1, 2, 3$, in Result \[13\] provides two other conditions. In the case when $(\zeta_1, W_1) = (b, B)$ the condition \[1.4\] holds trivially, because its left-hand side reduces to 0. On the other hand if $(\zeta_1, W_1) = (a, A)$, then under the restriction $\beta_i^2 \neq \beta_j^2$, for $i \neq j$, and that $|\beta_i|^2 < M_D(a, b)$ for every $i \in [1 \ldots n]$, we leave it to the reader to check that this condition turns out to be:

$$\max_{1 \leq i \leq n} |\beta_i|^2 \psi_i(b) \leq |b|^{1/n}. \quad (6.8)$$

Now let us compute the form that the necessary condition provided by Theorem \[1.4\] takes for the above data in the case $k = 1$, $L(k) = 2$, $G(k) = 3$. To this end, we observe first:

$$\max_{\mu \in \sigma(B_1(W_2))} \prod_{\nu \in \sigma(B_1(W_3))} M_D\left(\frac{\mu}{\psi_1(\zeta_2)}, \frac{\nu}{\psi_1(\zeta_3)}\right)^{q(\nu, 3, 1)} = \prod_{i=1}^{n} |\beta_i|^2 \psi_i^3(b)^{q(\beta_i, 3, 1)} ,$$

which is equal to zero since $q(\beta_i, 3, 1) = 1$ for every $i \in [1 \ldots n]$ and $\beta_1 = 0$. On the other hand:

$$\max_{\mu \in \sigma(B_1(W_3))} \prod_{\nu \in \sigma(B_1(W_2))} M_D\left(\frac{\mu}{\psi_1(\zeta_3)}, \frac{\nu}{\psi_1(\zeta_2)}\right)^{q(\nu, 2, 1)} = \max \left\{ \left| \frac{\beta_i}{b} \right|^{q(\beta_i, 2, 1)} : i \in [1 \ldots n]\right\}.$$
Notice $m(2, 0)$—the multiplicity of 0 as a zero of $M_A$—is 2 and $\text{ord}_0 B'_1 = 0$ whence $q(0, 2, 1) = 2$. Hence, one of the conditions given by Theorem 1.4 is:

$$\max_{1 \leq i \leq n} \left| \frac{\beta_i}{b} \right|^2 \leq M_D(a, b). \quad (6.9)$$

Notice that

$$|b| M_D(a, b)^{1/2} < \min \left\{ |b| M_D(a, b)^{1/n}, |b|^{1/2n} M_D(a, b)^{1/2} \right\}, \quad \forall n \geq 3. \quad (6.10)$$

Now, let us choose $\{\beta_1, \ldots, \beta_n\} \subset \mathbb{D}$ such that, in addition to the conditions listed above,

$$|\beta_i| \leq \min \left\{ |b| M_D(a, b)^{1/n}, |b|^{1/2n} M_D(a, b)^{1/2} \right\}, \quad \forall i \in [2..n],$$

and such that for some $i_0 \in [2..n]$,

$$|\beta_{i_0}| > |b| M_D(a, b)^{1/2}.$$

This is possible owing to (6.10). We now see that all forms of the condition arising from Result 1.3 are satisfied by the given data-set, while (6.9) does not hold true. Hence, Theorem 1.4 implies that there does not exist an $F \in O(D, \Omega_n)$ such that $F(0) = 0$, $F(a) = A$ and $F(b) = B$ while Result 1.3 provides no information.

7. A Schwarz lemma for holomorphic correspondences

This section is dedicated to the proof of Theorem 1.7. However, as hinted at in Section 1, there is a more precise inequality, from which Theorem 1.7 follows. We begin, therefore, with the following:

**Theorem 7.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}$ and let $\Gamma$ be a proper holomorphic correspondence from $D$ to $\Omega$. Then, for every $\zeta_1, \zeta_2 \in \mathbb{D}$ we have

$$\max \left\{ \max_{\mu \in F_\Gamma(\zeta_2)} \prod_{\nu \in F_\Gamma(\zeta_1)} C_\Omega(\nu, \mu), \max_{\mu \in F_\Gamma(\zeta_1)} \prod_{\nu \in F_\Gamma(\zeta_2)} C_\Omega(\nu, \mu) \right\} \leq M_D(\zeta_1, \zeta_2),$$

where $F_\Gamma(\cdot)^*$ is as in Section 4, read together with Remark 5.3.

We remind the reader that $C_\Omega$ here is as defined by (5.3). The proof of the above theorem is an easy consequence of the following lemma.

**Lemma 7.2.** Let $\Gamma$ be as in the statement of Theorem 1.7. Then

$$\max_{\mu \in F_\Gamma(\zeta)} \prod_{p \in F_\Gamma(0)^*} C_\Omega(p, \mu) \leq |\zeta|, \quad \forall \zeta \in \mathbb{D}.$$ 

**Proof.** By Lemma 5.2, there exists a positive integer $n$ and functions $a_j \in O(\mathbb{D})$, $j = 1, \ldots, n$, such that

$$\Gamma = \left\{ (z, w) \in \mathbb{D} \times \Omega : w^n + \sum_{j=1}^{n} (-1)^j a_j(z) w^{n-j} = 0 \right\}.$$ 

Note that by our definition of $F_\Gamma(\cdot)^*$

$$F_\Gamma(\cdot)^* = \text{the list, repeated according to multiplicity, of the zeros of}$$

$$w^n + \sum_{j=1}^{n} (-1)^j a_j(\cdot) w^{n-j}.$$
Let us now define an open set in $M_n(\mathbb{C}) : S_n(\Omega) := \{ A \in M_n(\mathbb{C}) : \sigma(A) \subset \Omega \}$. Let $\Phi \in \mathcal{O}(\mathbb{D}, M_n(\mathbb{C}))$ that is defined by

$$\Phi(z) := \text{companion matrix corresponding to the polynomial } w^n + \sum_{j=1}^{n} (-1)^j a_j(z) w^{n-j}.$$ 

In the notation introduced by Remark 5.3, $\sigma(A)^\bullet$ will denote the list of eigenvalues of $A$ repeated according to their multiplicity. In this notation, we have $\sigma(\Phi(z))^\bullet = F_\Gamma(z)^\bullet \subset \Omega$. Hence $\Phi(\mathbb{D}) \subset S_n(\Omega)$. Now we choose an arbitrary $z \in \Omega$ and fix it. Consider $B \in \mathcal{O}(\Omega)$ defined by

$$B := \prod_{p \in F_\Gamma(0)\,^\bullet} G_\Omega(p, z; \cdot),$$

where $G_\Omega(p, z; \cdot)$ denotes the Carathéodory extremal for points $p, z \in \Omega$, whose existence was discussed in Section 5. As $B$ is holomorphic in $\Omega$, it induces — via the holomorphic functional calculus — a map (which we continue to denote by $B$) from $S_n(\Omega)$ to $M_n(\mathbb{C})$. The Spectral Mapping Theorem tells us that $\sigma(B(A)) = B(\sigma(A)) \subset \mathbb{D}$ for every $A \in S_n(\Omega)$. Hence $B(A) \subset \Omega_n$ for every $A \in S_n(\Omega)$.

**Claim.** $B(\Phi(0)) = 0$.

To see this we write:

$$B(w) = \left( \prod_{p \in F_\Gamma(0)\,^\bullet} (w - p) \right) g(w), \quad w \in \Omega,$$

where $g \in \mathcal{O}(\Omega)$. Hence, since — by the holomorphic functional calculus — the assignment $f \mapsto f(\Phi(0))$, $f \in \mathcal{O}(\Omega)$, is multiplicative, as discussed in Section 2 (also see Remark 2.1), we get

$$B(\Phi(0)) = \left( \prod_{p \in F_\Gamma(0)\,^\bullet} (\Phi(0) - p \mathbb{1}) \right) g(\Phi(0)).$$

As $F_\Gamma(0)\,^\bullet = \sigma(\Phi(0))\,^\bullet$, Cayley–Hamilton Theorem implies that the product term in the right hand side of the above equation is zero, whence the claim.

Consider the map $\Psi$ defined by:

$$\Psi(\zeta) := B \circ \Phi(\zeta), \quad \zeta \in \mathbb{D}.$$ 

It is a fact that $\Psi \in \mathcal{O}(\mathbb{D})$. Moreover from the above claim and the discussion just before it, we have $\Psi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ with $\Psi(0) = 0$. By Lemma 4.3 there exists $\tilde{\Psi} \in \mathcal{O}(\mathbb{D}, \overline{\Omega}_n)$ such that

$$\Psi(\zeta) = \zeta \tilde{\Psi}(\zeta) \quad \forall \zeta \in \mathbb{D}. \tag{7.1}$$

From the definition of $\Psi$, (7.1), and from the Spectral Mapping Theorem, we get

$$B(\sigma(\Phi(\zeta))\,^\bullet) = \zeta \sigma(\tilde{\Psi}(\zeta))\,^\bullet \implies |B(\mu)| \leq |\zeta| \quad \forall \mu \in \sigma(\Phi(\zeta))\,^\bullet = F_\Gamma(\zeta)^\bullet, \quad \forall \zeta \in \mathbb{D}.$$ 

This in turn implies that

$$\prod_{p \in F_\Gamma(0)\,^\bullet} |G_\Omega(p, z; \mu)| \leq |\zeta| \quad \forall \zeta \in \mathbb{D}, \forall \mu \in F_\Gamma(\zeta)^\bullet.$$ 

Since $z$ is arbitrary we can take $z = \mu$ for some $\mu \in F_\Gamma(\zeta)^\bullet$. This with the observation that $G_\Omega(p, \mu; \mu) = C_\Omega(p, \mu)$ establishes the lemma. \(\square\)

We are now ready to give:
The proof of the Theorem 7.1. Fix ζ₁, ζ₂ ∈ ℍ and consider the correspondence ̂Γ such that

\[ F_{̂Γ}(\zeta)^* = F_{̂Γ}(ϕ_{ζ₁}^{-1}(ζ))^* . \]

It is easy to see that ̂Γ is a proper holomorphic correspondence from ℍ to Ω. So from the lemma above and with the observation that \( F_{̂Γ}(0)^* = F_{̂Γ}(ζ₁)^* \) we get

\[ \max_{μ∈F_{̂Γ}(ζ)} \prod_{p∈F_{̂Γ}(ζ)} C_Ω(p, μ) \leq |ζ| \quad ∀ζ ∈ ℍ. \tag{7.2} \]

Putting ζ = ϕ_{ζ₁}(ζ₂) in (7.2) gives us

\[ \max_{μ∈F_{̂Γ}(ζ₂)} \prod_{p∈F_{̂Γ}(ζ₁)} C_Ω(p, μ) \leq M_Ω(ζ₁, ζ₂). \tag{7.3} \]

Interchanging the role of ζ₁ and ζ₂ in the above discussion, we get

\[ \max_{μ∈F_{̂Γ}(ζ₁)} \prod_{p∈F_{̂Γ}(ζ₂)} C_Ω(p, μ) \leq M_Ω(ζ₁, ζ₂). \tag{7.4} \]

From (7.4) and (7.3) the result follows.

Theorem 1.7 is a corollary of Theorem 7.1. This is almost immediate; we probably just require a few words about the Hausdorff distance induced by \( C_Ω(μ, Ω) \). We refer the reader to [12] p. 279 for the definition of the Hausdorff distance—which is a distance on the set of non-empty closed, bounded subsets of a distance space \((X, d)\). In our case \((X, d) = (Ω, C_Ω)\) and it is easy to check that

\[ H_{CΩ}(F_{̂Γ}(ζ₁), F_{̂Γ}(ζ₂)) = \max \left\{ \max_{w∈F_{̂Γ}(ζ₁)} \text{dist}_{CΩ}(w, F_{̂Γ}(ζ₂)), \max_{w∈F_{̂Γ}(ζ₂)} \text{dist}_{CΩ}(w, F_{̂Γ}(ζ₁)) \right\}, \]

given ζ₁, ζ₂ ∈ ℍ, and where, given \( p ∈ Ω, 0 \neq S ⊂ Ω, \text{dist}_{CΩ}(p, S) := \inf_{q∈S} C_Ω(p, q) \). Clearly, there exists \( j, k ∈ \{1, 2\} \) with \( j \neq k \) such that

\[ H_{CΩ}(F_{̂Γ}(ζ₁), F_{̂Γ}(ζ₂))^n \leq \max_{μ∈F_{̂Γ}(ζ)} \prod_{ν∈F_{̂Γ}(ζ)'ν} C_Ω(ν, μ), \]

where \( n \) is the multiplicity of Γ. Combining the above inequality with the inequality in Theorem 7.1, we establish Theorem 1.7.

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