Symmetric Motion Planning

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Abstract. In this paper we study symmetric motion planning algorithms, i.e. such that the motion from one state $A$ to another $B$, prescribed by the algorithm, is the time reverse of the motion from $B$ to $A$. We experiment with several different notions of topological complexity of such algorithms and compare them with each other and with the usual (non-symmetric) concept of topological complexity. Using equivariant cohomology and the theory of Schwarz genus we obtain cohomological lower bounds for symmetric topological complexity. One of our main results states that in the case of aspherical manifolds the complexity of symmetric motion planning algorithms with fixed midpoint map exceeds twice the cup-length.

We introduce a new concept, the sectional category weight of a cohomology class, which generalises the notion of category weight developed earlier by E. Fadell and S. Husseini. We apply this notion to study the symmetric topological complexity of aspherical manifolds.

1. Introduction

The Motion Planning Problem is a central theme in Robotics, which invites applications of tools of algebraic topology. Any autonomous mechanical system or robot capable of performing tasks must first be told how to move between different states of the system. A Motion Planner in a given mechanical system is a rule which assigns to any pair of states $A$ and $B$ of the system a continuous motion from $A$ to $B$. Topology enters the picture by regarding the set of admissible states of a mechanical system as the points of a topological space, $X$, called the configuration space of the system. In practice this space is usually determined by several real parameters, and so has the natural topology induced by some Euclidean metric. The system becomes synonymous with the space $X$, and continuous motions of the system are represented by continuous paths in $X$ (continuous maps of the unit interval $I = [0, 1]$ to $X$).

Let $PX$ denote the space of all continuous paths in $X$, endowed with a suitable topology (such as the compact-open topology). The map $\pi: PX \to X \times X$ which takes a path in $X$ to its end-points, given by $\pi(\gamma) = (\gamma(0), \gamma(1))$, is a fibration in the sense of Serre. A Motion Planner in $X$ is then a function $s$ from the Cartesian product $X \times X$ to the path space $PX$, such that the composition $\pi \circ s$ is the

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identity function on \(X \times X\). Whilst such discontinuous functions always exist when \(X\) is path-connected, one may show that a continuous Motion Planner in \(X\) (which is therefore a continuous section of the fibration \(\pi\)) exists if and only if \(X\) is contractible \([8]\). Thus Motion Planners in \(X\) may have essential discontinuities, which reflect the homotopy properties of \(X\) and provide a measure of the complexity of the task of navigation in \(X\).

The problem has been studied extensively from this viewpoint by the first author in \([8, 9]\), and \([10]\), where a new homotopy invariant was explored. For any space \(X\), the Topological Complexity of \(X\) is a positive natural number \(\text{TC}(X)\), which may be defined in a number of equivalent ways. The definition we will give is based on the following concept due to A. S. Schwarz \([14]\).

**Definition 1.** Let \(p: E \to B\) be a continuous map and \(U \subseteq B\). Recall that a map \(s: U \to E\) is a section of \(p\) over \(U\) if \(p \circ s = 1_U\), the identity on \(U\). The Schwarz genus of \(p\), denoted \(g(p)\), is the minimum cardinality among coverings of \(B\) by open sets, over each of which \(p\) has a continuous section (in \([4]\) the term sectional category was used).

**Definition 2.** The Topological Complexity of a path-connected topological space \(X\), denoted \(\text{TC}(X)\), is the Schwarz genus of the path fibration \(\pi: PX \to X \times X\).

Thus \(\text{TC}(X) = g(\pi) \leq k\) iff there is an open cover \(\{U_1, \ldots, U_k\}\) of \(X \times X\) and continuous maps \(s_i: U_i \to PX\) such that \(\pi \circ s_i = 1_{U_i}\), for \(i = 1, \ldots, k\). The number \(\text{TC}(X)\) depends only on the homotopy type of \(X\) (see \([8]\)) and provides a measure of the intrinsic complexity of the Motion Planning problem in \(X\). This invariant is similar in spirit to the Lusternik-Schnirelmann category \(\text{cat}(X)\), and, whilst the two are independent, they satisfy the inequalities

\[
\text{cat}(X) \leq \text{TC}(X) \leq \text{cat}(X \times X).
\]

In addition, there are cohomological lower bounds for \(\text{TC}(X)\) based on ‘zero-divisors cup-length’ (see \([8]\) for details) and one may obtain upper bounds, for instance by constructing explicit Motion Planners in \(X\). With these considerations one may calculate the Topological Complexity of a large number of spaces, including spheres, simply connected symplectic manifolds (such as \(\mathbb{C}P^n\), \(n \geq 1\)) and tori. The paper \([10]\) contains a recent survey of such results, and along with \([14]\) is our basic reference.

In this paper we study several variations on the Motion Planning problem. We impose additional, quite natural, symmetry constraints on our Motion Planners, namely that the motion from \(A\) to \(A\) should be constant at \(A\), while the motion from \(B\) to \(A\) should be the motion from \(A\) to \(B\) traversed in the opposite direction. This is formalised as follows.

**Definition 3.** A Symmetric Motion Planner in \(X\) is a (possibly discontinuous) function \(s: X \times X \to PX\) such that \(\pi \circ s = 1_{X \times X}\) and the following conditions are satisfied for all \(t \in I:\)

\[
s(A, A)(t) = A, \quad s(B, A)(t) = s(A, B)(1 - t).
\]

Armed with this Definition, we will define in Section 2 a new invariant which measures the intrinsic complexity of Symmetric Motion Planning in a topological space. The Symmetric Topological Complexity of \(X\), denoted \(\text{TC}^S(X)\), is compared
with the usual $\text{TC}(X)$ and some examples are explored. Note that $\text{TC}^S(X)$, unlike $\text{TC}(X)$, is not homotopy invariant, and harder to deal with. The computation of this new invariant requires usable cohomological lower bounds like the one mentioned above for $\text{TC}(X)$; this is done in Section 3 of the present paper, using $\mathbb{Z}_2$-equivariant cohomology and results of N. E. Steenrod and A. Haefliger. We find that for a closed smooth manifold $X$,

$$\text{TC}^S(X) \geq \text{cup-length}(N) + 2,$$

where $N$ denotes the sub-ring of $H^*(X) \otimes H^*(X)$ spanned by the norm elements (elements $x \otimes y + y \otimes x$ with $x \neq y$).

In Section 5 we impose further constraints on our Symmetric Motion Planners, namely that the mid-point of a motion between $A$ and $B$ should depend continuously on $A$ and $B$. This has the effect of greatly increasing the complexity of navigation in certain cases. In particular we consider Symmetric Motion Planners such that the mid-point of each motion is a fixed base state $A_0 \in X$. One of our main results, Theorem 26 asserts that the Symmetric Topological Complexity with constant mid-point of a closed aspherical manifold $X$ exceeds twice the cup-length of $X$. We find that in the case of planar robot arm with $n$ revolving joints (having the torus $T^n$ as the configuration space) the symmetric topological complexity is $2n+1$ while $\text{TC}(X) = n+1$; thus in this instance the requirement that motions are symmetric and return to a fixed base state increases the complexity of navigation by $n$.

In Section 5 (which can be read independently of the rest of the paper) we employ a new tool for estimating the Schwarz genus of a fibration. We introduce the notion of sectional category weight of a cohomology class. It is a natural generalization of the usual category weight defined by Fadell and Husseini [7] and developed by Rudyak [15]. One may improve on the classical cohomological lower bound for the genus, given in [14], by finding cohomology classes of weight at least two.

In Section 6 we show that some cohomology classes which arise in the $\mathbb{Z}_2$-equivariant cohomology have sectional category weight $\geq 2$. This is the main ingredient of the proof of Theorem 26.

*Everywhere in this paper we denote by $G$ the cyclic group $\mathbb{Z}_2$ of order two. Cohomology is taken with coefficients in $G$ unless otherwise stated.*

### 2. Symmetric Topological Complexity

Let $X$ be a path-connected polyhedron. The path fibration

$$\pi : PX \to X \times X$$  \hspace{1cm} (2)

restricts to a fibration

$$\pi' : P'X \to F(X; 2),$$ \hspace{1cm} (3)

where $F(X; 2) = \{(x, y) \in X \times X \mid x \neq y\}$ is the space of ordered pairs of distinct points in $X$, and $P'X$ is the subspace $\{\gamma : I \to X \mid \gamma(0) \neq \gamma(1)\} \subseteq PX$ consisting of paths with distinct endpoints.

The spaces $P'X$ and $F(X; 2)$ carry free $G$-actions, defined in the latter case by permutation of factors and in the former by sending a path $\gamma$ to its inverse $\overline{\gamma}$, given
by \( \tau(t) = \gamma(1 - t) \). Note that \( \pi' : P'X \to F(X; 2) \) is an equivariant map of free \( G \)-spaces. Hence the quotient map

\[
\pi_G := \pi' / G : P'X / G \to B(X; 2)
\]
is also a fibration, where \( B(X; 2) \) denotes the orbit space \( F(X; 2) / G \) of unordered pairs of distinct points in \( X \).

Comparing with Definition 3 we see that a Symmetric Motion Planner in \( X \) describes a function \( s : B(X; 2) \to P'X / G \) such that \( \pi_G \circ s \) is the identity map on \( B(X; 2) \). Conversely, such a function \( s \) completely describes a Symmetric Motion Planner, since the latter must map a point \((A, A)\) on the diagonal of \( X \times X \) to the constant path at \( A \). Of course a continuous map \( s \) of this kind may exist only in very few cases. We wish to measure the essential discontinuities of Symmetric Motion Planning in \( X \). Thus we are led to the following definition.

**Definition 4.** The *Symmetric Topological Complexity* of \( X \), denoted \( \text{TC}^S(X) \), is defined to be one plus the Schwarz genus of the fibration (4). In other words,

\[
\text{TC}^S(X) = 1 + g(\pi_G).
\]

We adopt the convention that the Schwarz genus of \( p : E \to B \) vanishes iff \( E = \emptyset = B \). Note that \( B(X; 2) \) is empty if and only if \( X \) is a single point; in this case \( \text{TC}^S(X) = 1 \). If \( X \) is not a single point then \( g(\pi_G) \geq 1 \) and therefore

\[
\text{TC}^S(X) \geq 2.
\]

**Example 5.** Let \( X \subseteq \mathbb{R}^n \) be a convex subset, not a single point. For \( A, B \in X \) define \( s(A, B)(t) = (1 - t)A + tB \) where \( t \in \mathbb{R} \). This defines a continuous equivariant section of (4) and hence \( \text{TC}^S(X) = 2 \).

**Example 6.** Let \( X \) be a finite tree. We may view \( X \) as a metric space by specifying length of every edge. Then for any pair of points \( A, B \) there is a unique constant speed curve of minimal length \( s(A, B) : [0, 1] \to X \) starting at \( A \) and ending at \( B \). We obtain a continuous equivariant section of (4) which implies \( \text{TC}^S(X) = 2 \).

**Example 7.** More generally, it is easy to see that for any contractible \( X \) which is not a single point one has \( \text{TC}^S(X) = 2 \). Indeed, since \( X \) is contractible there exists a continuous map \( x \mapsto \gamma_x \in PX \) such that \( \gamma_x(0) = x \) and \( \gamma_x(1) = x_0 \). Then setting \( s(A, B) \) to be equal the concatenation of \( \gamma_A \) and the inverse path to \( \gamma_B \) gives a symmetric equivariant section of (4).

One expects that the converse statement is true as well: any path-connected polyhedron \( X \) with \( \text{TC}^S(X) = 2 \) is contractible.

The following lemma will be used to justify Definition 4.

**Lemma 8.** Let \( U \) be an open subset \( U \subseteq B(X, 2) \). Denote by \( \hat{U} \) the preimage of \( U \) under the projection \( q : F(X, 2) \to B(X, 2) \). Any continuous section \( s : U \to P'X / G \) of fibration (4) over \( U \) determines a continuous equivariant section \( \hat{s} : \hat{U} \to P'X \) of fibration (4) over \( \hat{U} \).

**Proof.** Let \( (A, B) \in \hat{U} \). Then \( s(q(A, B)) \in P'X / G \) is an equivalence class containing two paths \( \gamma \) and \( \overline{\gamma} \), one of which goes from \( A \) to \( B \) and the other of which goes from \( B \) to \( A \). Without loss of generality we may assume that \( \gamma(0) = A \) and \( \gamma(1) = B \), and set \( \hat{s}(A, B) = \gamma \). In this way one describes a continuous equivariant section \( \hat{s} : \hat{U} \to P'X \).

\( \square \)
We now show that if $k = \text{TC}^S(X) - 1$ then we may partition $X \times X$ into $k$ disjoint subsets with a continuous symmetric motion planner on each. Given an open cover $\{U_1, \ldots, U_k\}$ of $B(X, 2)$ and a sequence of continuous sections $s_i : U_i \to P'X/G$, one uses Lemma 5 to obtain an open cover $\tilde{U}_1, \ldots, \tilde{U}_k$ of $F(X, 2)$ with equivariant continuous sections $\tilde{s}_i : \tilde{U}_i \to P'X$, where $i = 1, \ldots, k$. Consider a partition of unity $\{f_1, \ldots, f_k\}$ subordinate to the cover $\{U_1, \ldots, U_k\}$. Here $f_i : B(X, 2) \to \mathbb{R}_+$ is a continuous function with $\text{supp}(f_i) \subseteq U_i$ and $f_1 + \cdots + f_k = 1$. Composing with the projection $q : F(X, 2) \to B(X, 2)$ gives an equivariant partition $\tilde{f}_i = f_i \circ q$ of unity of $F(X, 2)$, where $i = 1, \ldots, k$, subordinate to $\{\tilde{U}_1, \ldots, \tilde{U}_k\}$. Define sets $V_i \subseteq F(X, 2)$ by

$$(A, B) \in V_i \iff \begin{cases} \tilde{f}_i(A, B) \geq 1/k, \\ \tilde{f}_j(A, B) < 1/k \quad \text{for all } j < i. \end{cases}$$

Then

(a) each $V_i$ is involution invariant and is contained in $\tilde{U}_i$;
(b) the sets $V_i$ are pairwise disjoint;
(c) $V_1 \cup \cdots \cup V_k = F(X, 2)$.

As a result one obtains a symmetric motion planning algorithm $s : X \times X \to PX$ by setting $s(A, B) = \tilde{s}_i(A, B)$ if $A \neq B$ and $(A, B) \in V_i$; we also set $s(A, A)(t) = A$.

**Corollary 9.** One has $\text{TC}(X) \leq \text{TC}^S(X)$.

**Proof.** In notations introduced in the paragraph preceding Corollary 9, the sets $\tilde{U}_1, \ldots, \tilde{U}_k$ constitute an open cover of $F(X, 2) = X \times X - \Delta$ where $\Delta$ is the diagonal, with continuous sections $\tilde{s}_i$ over each $\tilde{U}_i$. Hence it is enough to show that there is an open neighbourhood $\tilde{U}_0 \subseteq X \times X$ of $\Delta$ which supports a continuous equivariant section $\tilde{s}_0 : \tilde{U}_0 \to PX$ which takes each pair $(A, A)$ into the constant path at $A$, for $A \in X$.

Since $X$ is an ENR, we may find an embedding $e : X \to \mathbb{R}^n$ and an open neighbourhood $e(X) \subseteq N \subseteq \mathbb{R}^n$ which admits a retraction $r : N \to X$ onto $X$. Let $\tilde{U}_0$ be a neighbourhood of $\Delta$ in $X \times X$ which is involution invariant and such that for all $(A, B) \in \tilde{U}_0$ the straight line segment connecting $e(A)$ and $e(B)$ is contained in $N$. Now we define the desired section $\tilde{s}_0 : \tilde{U}_0 \to PX$ by $\tilde{s}_0(A, B)(t) = r((1 - t)e(A) + te(B))$. This completes the proof. \qed

Corollary 9 implies that all cohomological lower bounds for $\text{TC}(X)$ (see [8]) are valid also for $\text{TC}^S(X)$.

**Proposition 10.** For any $X$ we have $\text{TC}^S(X) \leq 2\text{dim}(X) + 2$ where $\text{dim}(X)$ denotes covering dimension of $X$. If $X$ is a closed smooth manifold then $\text{TC}^S(X) \leq 2\text{dim}(X) + 1$.

**Proof.** Since the genus of a fibration may not exceed the category of the base (see Theorem 5 of [14]), which in turn may not exceed the dimension of the base plus one, we obtain $\text{TC}^S(X) \leq \text{cat}(B(X; 2)) + 1 \leq \text{dim}(B(X; 2)) + 2 \leq 2\text{dim}(X) + 2$.

Let $X$ be a closed manifold with $\text{dim}(X) = n$. Let $Y \subseteq X \times X$ denote the space obtained by removing an open $G$-invariant tubular neighbourhood of the diagonal. Then $Y$ inherits a free $G$-action, and is $G$-equivariantly homotopy equivalent to
$F(X; 2)$. Its quotient $Y/G = Y'$ is therefore homotopy equivalent to $B(X; 2)$, and is a compact $2n$-manifold with boundary. A standard Morse theoretical argument gives that $Y'$ is homotopy equivalent to a complex $Y''$ of dimension $2n - 1$. Thus $\text{TC}^S(X) \leq \text{cat}(B(X; 2)) + 1 = \text{cat}(Y'') + 1 \leq \dim(Y'') + 2 = 2n + 1$. □

**Proposition 11.** $\text{TC}^S(S^n) \leq 3$ for any $n \geq 1$.

**Proof.** The inclusion $S^n \hookrightarrow F(S^n; 2)$ given by $A \mapsto (A, -A)$ is a $G$-equivariant homotopy equivalence, where $G$ acts on the sphere antipodally. We may describe the homotopy inverse $\phi: F(S^n; 2) \to S^n$ of this map as follows. Let $d$ denote the standard metric on $S^n$. Given any point $(A, B) \in F(S^n; 2)$, choose a geodesic great circle $\Gamma$ containing both $A$ and $B$ (note that if $A$ and $B$ are non-antipodal, so $A \neq -B$, then there is no choice to make). Then there is a unique pair of points $(A', B')$, both on $\Gamma$, which are antipodal ($B' = -A'$) and such that $d(A, A') = d(B, B') < d(A, B')$. We set $\phi(A, B) = A'$, and note that that this is independent of the choice of $\Gamma$ since if $A = -B$ then $A = A'$. Also note that $\phi$ is equivariant, since $\phi(B, A) = B' = -A'$.

We must show that $g(\pi_G) \leq 2$. Clearly, it suffices to cover $F(S^n; 2)$ by two $G$-invariant open sets, over each of which the restricted end-point map $\pi': P'S^n \to F(S^n; 2)$ has an equivariant section. Fix a point $A_0 \in S^n$, and let $U \subseteq S^n$ be a small open disk centred at $A_0$. Then $-U := \{-C \mid C \in U\}$ is a small open disk centred at $-A_0$. The sets

$$W_1 = \phi^{-1}(S^n - \{A_0, -A_0\}), \quad W_2 = \phi^{-1}U \cup \phi^{-1}(-U)$$

are clearly open and $G$-invariant, and cover $F(S^n; 2)$. We will describe equivariant sections $s_1, s_2$ of $\pi'$ over these sets.

Given $(A, B) \in W_1$, choose a geodesic great circle $\Gamma$ and pair of antipodal points $(A', B')$ on $\Gamma$, as in the construction of $\phi$. Let $\alpha$ (resp. $\beta$) denote the path along $\Gamma$ from $A$ to $A'$ (resp. from $B$ to $B'$). Since $\{A', B'\} \neq \{A_0, -A_0\}$ there is a unique path $\xi$ from $A'$ to $B'$ which travels along the geodesic great circle through $A_0$. We may then set $s_1(A, B) = \alpha \circ \xi \circ \beta$. This describes a continuous equivariant section $s_1: W_1 \to P'X$.

Given $(A, B) \in W_2$, the pair $(A', B')$ lies in either $U \times -U$ or $-U \times U$. In either case, there is a unique path $\eta$ from $A'$ to $B'$ of constant speed, which travels first along the geodesic arc from $A'$ to the centre of the disk containing $A'$, then along some fixed geodesic arc between $A_0$ and $-A_0$, then from the centre of the disk containing $B'$ to $B'$. It is not hard to see that setting $s_2(A, B) = \alpha \circ \eta \circ \beta$ describes a continuous equivariant section $s_2: W_2 \to P'X$ of $\pi'$ over $W_2$. □
In the next section we shall see that $\text{TC}^S(S^n) = 3$ for all $n \geq 1$. This should be compared with the corresponding result for $\text{TC}(S^n)$ which equals 2 for $n$ odd and 3 for $n$ even.

3. Cohomology of $B(X; 2)$

The main result of this Section, Theorem 12, gives a lower bound for $\text{TC}^S(X)$ in terms of the structure of the cohomology algebra of $X$ with $\mathbb{Z}_2$ coefficients, assuming that $X$ is a closed smooth manifold.

We use the following result of A. Schwarz (14, Theorem 4).

**Theorem 12.** Let $p: E \to B$ be a continuous map. Suppose there exist cohomology classes $\xi_1, \ldots, \xi_k \in H^*(B)$ with any coefficients, such that $p^*(\xi_1) = \ldots = p^*(\xi_k) = 0$ and the product $\xi_1 \cup \cdots \cup \xi_k$ is different from zero. Then $g(p) > k$.

We plan to apply this theorem to fibration (4). To make this work we need a good understanding of the cohomology algebra of $B(X; 2)$. A. Haefliger (11), who studied embeddings and immersions of manifolds, gave a very explicit description of $H^*(B(X, 2))$. In this Section we recall his results and apply them to our problem of estimating the Schwarz genus of (4).

For the remainder of this Section $X$ will denote a closed manifold and all cohomology will be taken with coefficients in the group $G = \mathbb{Z}_2$ which will be skipped from the notation. Let $\rho: EG \to BG$ be a universal principal $G$-bundle. Hence $EG$ is a contractible space with a free right $G$-action, and $\rho$ is the orbit map to $BG = EG/G$. Recall that the cohomology algebra of the group $G$ is $H^*(G) = H^*(BG) \cong \mathbb{Z}_2[\mu]$, the polynomial algebra over $G$ on one generator $\mu$ of degree 1.

The inclusion $F(X; 2) \hookrightarrow X \times X = X^2$ is $G$-equivariant, as are the fibrations $\pi$ and $\pi'$ (see (2) and (3)), allowing us to form the following diagram of homotopy orbit spaces.

$$
\begin{align*}
0 \xrightarrow{\pi_G} \xrightarrow{\pi} E \xrightarrow{\pi'} & \quad E \xrightarrow{\pi} \quad E \times_{G} P X \\
B(X; 2) \xrightarrow{\rho} \quad E \times_{G} F(X; 2) \xrightarrow{\rho} \quad E \times_{G} X^2
\end{align*}
$$

(7)

If $X$ and $Y$ are right and left $G$-spaces respectively, then $X \times_G Y$ denotes the orbit space of $X \times Y$ under the diagonal $G$-action. The homotopy equivalences on the left of the diagram result from the fact that $F(X; 2)$ and $P X$ are free $G$-spaces. The next Theorem describes the equivariant (Borel) cohomology ring $H^*(EG \times_G X^2)$ of the $G$-space $X^2$, where the $G$-action on $X^2$ permutes the factors.

Recall that all cohomology groups are with coefficients in $G = \mathbb{Z}_2$.

**Theorem 13.** Let $X$ be a finite polyhedron. Then there is an isomorphism of $H^*(G)$-algebras

$$
H^*(EG \times_G X^2) \cong H^*(G, H^*(X) \otimes H^*(X)).
$$

Here we view $H^*(X) \otimes H^*(X)$ as a $G$-module, where the non-trivial element of $G$ acts by $x \otimes y \mapsto y \otimes x$ (no signs are introduced here since we are working mod 2). Then the right hand side is cohomology of the group $G$ with coefficients in this $G$-module.
A. Haefliger credits this theorem to an unpublished work of N. E. Steenrod. One may interpret its statement as saying that the $G$-cohomology Cartan-Leray spectral sequence of the regular covering $EG \times X^2 \to EG \times_G X^2$ collapses at the $E_2$-term (see [3] or [12]). Below we describe the structure of this $E_2$-term $H^*(G, H^*(X) \otimes H^*(X))$ explicitly in terms of the cohomology ring $H^*(X)$.

Consider the ring $(H^*(X) \otimes H^*(X))^G$ of invariants of $H^*(X) \otimes H^*(X)$. It contains a subring consisting of diagonal elements, which are linear combinations of elements of the form $x \otimes x$ where $x \in H^*(X)$; denote this subring by $D$. It also contains a subring $N$ consisting of linear combinations of elements of the form $x \otimes y + y \otimes x$ where $x \neq y \in H^*(X)$, the so called norm elements. Then as $H^*(G)$-algebras, we have

$$H^*(G, H^*(X) \otimes H^*(X)) \cong H^*(G) \otimes D + N.$$  

This algebra is generated additively by elements of the form $\alpha = \mu^i \otimes x \otimes x$ where $x \in H^*(X)$ (the exponent $i \geq 0$ of $\mu$ is called the G-degree of $\alpha$), and the norm elements $x \otimes y + y \otimes x \in N$ (which have G-degree zero). It is generated as a $H^*(G)$-algebra by the sub-ring consisting of elements of G-degree 0, which is precisely the ring of invariants $(H^*(X) \otimes H^*(X))^G$. The action of $H^*(G)$ is free on $D$ and trivial on $N$.

One may now investigate the cohomology algebra $H^*(B(X; 2))$ with the aid of the exact sequence

$$\cdots \to H^*(EG \times_G X^2, EG \times_G F(X; 2)) \to H^*(EG \times_G X^2) \xrightarrow{j^*} H^*(B(X; 2)) \to \cdots$$

(Here $j: B(X; 2) \to EG \times_G X^2$ is the composition along the bottom of diagram (7)). Such an investigation leads to the main theorem of [11].

To describe the statement of the theorem, we must introduce several maps. We assume below that $X$ is a closed smooth $n$-dimensional manifold. Fixing a point $e \in EG$ we obtain inclusions

$$r: X^2 \hookrightarrow EG \times_G X^2, \quad r_0: X \to BG \times X = EG \times_G X,$$

given by $r[A, B] = [e, A, B]$ and $r_0[A] = ([e], A)$, which up to homotopy are independent of the choice of $e$. The induced maps $r^*: H^*(EG \times_G X^2) \to H^*(X^2)$ and $r_0^*: H^*(BG) \otimes H^*(X) \to H^*(X)$ in cohomology are given by

$$r^*(x \otimes y + y \otimes x) = x \otimes y + y \otimes x, \quad r^*(\mu^i \otimes x \otimes x) = \begin{cases} x \otimes x & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

$$r_0^*(\mu^i \otimes x) = \begin{cases} x & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

and can be described as ‘taking the G-degree 0 part’. We will also consider the generalised diagonal map

$$\Delta_G: BG \times X \to EG \times_G X^2, \quad \Delta_G([e], A) = [e, A, A].$$

The induced map

$$\Delta_G^*: H^*(EG \times G X^2) \to H^*(BG \times X)$$

plays an important rôle in the construction of the Steenrod square cohomology operations

$$Sq^i: H^*(X) \to H^{*+i}(X).$$
It may be completely described by noting that it is an $H^*(G)$-module homomorphism which vanishes on $N$, and for $x \in H^k(X)$ we have

$$\Delta^*_G(1 \otimes x \otimes x) = \sum_{i=0}^{k} \mu^{k-i} \otimes Sq^i(x) \in H^{2k}(BG \times X).$$

The diagonal embedding $\Delta: X \hookrightarrow X^2$ induces a Gysin or pushforward map

$$\Delta_!: H^{*-n}(X) \to H^*(X^2)$$

which is given by $\Delta_!(x) = (1 \times x) \cup \delta$, where $\delta \in H^n(X^2)$ is the diagonal class (for definitions and properties of this class see [13], Chapter 11).

Finally, there is the map

$$\varphi: H^{*-n}(BG \times X) \to H^*(BG \times X)$$

given by cup product with the element

$$\sum_{l=0}^{n} \mu^{n-l} \otimes w_l \in H^n(BG \times X),$$

where $w_l$ for $l = 0, \ldots, n$ is the $l$-th tangential Stiefel-Whitney class of $X$ [13]. Note that $\phi$ is injective since $w_0 = 1$.

**Theorem 14 (A. Haefliger [11]).** The following diagram commutes and has exact rows.

$\begin{array}{ccc}
0 & \to & H^{*-n}(X) \\
\uparrow & & \Delta_! \\
0 & \to & H^*(BG \times X)
\end{array}$

$\begin{array}{ccc}
\Delta & \to & H^*(X^2) \\
\uparrow r_0 & & \uparrow r^* \\
\varphi & \downarrow \Delta^*_G \\
& & H^*(BG \times X)
\end{array}$

An element $\alpha \in H^*(EG \times_G X^2)$ is in the kernel of $j^*$ if and only if: (a) There exists an element $\beta \in H^{*-n}(BG \times X)$ such that $\varphi(\beta) = \Delta^*_G(\alpha)$ (such an element $\beta$ is unique), and (b) $\Delta r_0^*(\beta) = r^*(\alpha)$.

This theorem facilitates explicit calculations of the algebra $H^*(B(X; 2))$ for certain manifolds $X$ for which the Stiefel-Whitney classes of $X$ and the action of the Steenrod algebra on $H^*(X)$ are known (the reader may enjoy checking for example that $H^*(B(S^n; 2)) \cong \mathbb{Z}_2[\mu]/(\mu^{n+1})$, which follows from the $G$-equivariant homotopy equivalence $S^n \to F(S^n; 2)$). A general description of $H^*(B(X; 2))$ in terms of these data, valid for all $X$, was obtained by Yo Ging-tzung [17]. Unfortunately the details are messy and not so instructive. Here we content ourselves with the following.

**Corollary 15.** $H^*(B(X; 2))$ contains a sub-ring $\tilde{N}$ isomorphic to $N$.

**Proof.** Consider a non-zero element $\alpha = x \otimes y + y \otimes x$ of the subring $N \subseteq H^*(EG \times_G X^2)$. Such an $\alpha$ satisfies condition (a) of Haefliger’s Theorem, with $\beta = 0$, since $\Delta^*_G(\alpha) = 0$. However it cannot satisfy condition (b), since $r^*$ is injective on $N$. Hence $[\alpha] := j^*(\alpha)$ is non-zero, and it follows that $j^*$ is injective.
on $N$, so $H^*(B(X; 2))$ contains a subring $\tilde{N} = \{ [\alpha] \mid \alpha \in N \}$ which is isomorphic with $N$. □

COROLLARY 16. The subring $\tilde{N}$ of $H^*(B(X; 2))$ is contained in $\ker(\pi^*_G)$ where $\pi^*_G: H^*(B(X; 2)) \to H^*(P'X/G)$ is the map induced by fibration $P'$.

PROOF. The generalised diagonal map factorises as

$$\Delta_G: BG \times X \xrightarrow{\phi} EG \times_G PX \xrightarrow{(1 \times_G \pi)^*} EG \times_G X^2,$$

where the map $\phi([e], A) = [e, \const(A)]$ is a homotopy equivalence; here $\const(A)$ denotes constant path at $A$. It follows for all $\alpha \in N$ that $(1 \times_G \pi)^* (\alpha) = 0$, since $\Delta_G^*(\alpha) = 0$. Applying cohomology to the diagram (7),

\[
\begin{array}{ccc}
H^*(P'X/G) & \xrightarrow{\pi^*_G} & H^*(EG \times_G PX) \\
\pi^*_G & & (1 \times_G \pi)^* \\
H^*(B(X; 2)) & \xrightarrow{j^*} & H^*(EG \times_G X^2)
\end{array}
\]

we see that $\pi^*_G [\alpha] = 0$ for all $[\alpha] \in \tilde{N}$. □

The following theorem is the main result of this Section:

THEOREM 17. Let $X$ be a closed smooth manifold. Then

$$\text{TC}^S(X) \geq \text{cup-length}(N) + 2. \tag{10}$$

PROOF. This follows by combining the previous Corollary with Theorem 12. □

COROLLARY 18. For any closed connected smooth manifold $X$ of dimension $\dim X > 0$ one has $\text{TC}^S(X) \geq 3$. In particular, $\text{TC}^S(S^n) = 3$ for $n \geq 1$.

PROOF. The fundamental class $a \in H^n(X)$ gives a non-zero element $a \otimes 1 + 1 \otimes a$ of $\ker(\pi^*_G)$, and so $\text{TC}^S(X) \geq 3$. The second statement now follows from Proposition 11. □

Comparing with the corresponding result for the usual topological complexity $\text{TC}(S^n) = 3$ for $n$ even while $\text{TC}(S^n) < \text{TC}^S(S^n)$ for $n \geq 1$ odd.

REMARK 19. The subring $N \subseteq H^*(X) \otimes H^*(X)$ is contained in the kernel $I \subseteq H^*(X) \otimes H^*(X)$ of the homomorphism $\Delta^*: H^*(X \times X) \cong H^*(X) \otimes H^*(X) \to H^*(X)$ induced by the diagonal map, since we are working mod 2. This kernel $I$ is precisely the ring of zero-divisors, introduced in [8], whose cup-length plus one provides a lower bound for $\text{TC}(X)$, i.e.

$$\text{TC}(X) \geq \text{cup-length}(I) + 1. \tag{11}$$

Comparing (10) with (11) we see that cup-length$(I) \geq$ cup-length$(N)$ however in (11) there is an extra 1 which makes estimate (10) stronger in some cases.
4. Cohomology of $P'X$

In this Section we collect information about cohomology of the path spaces $P'X$. The results of this Section are not used in this paper and therefore we will state them without proofs.

Recall that $P'X$ is defined as the subspace of the full path space $PX$ consisting of all paths $\gamma: [0, 1] \rightarrow X$ with distinct end points $\gamma(0) \neq \gamma(1)$. In other words $P'X = PX - LX$ where $LX$ denotes the space of free loops in $X$, i.e. the space of all $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = \gamma(1)$. We consider $P'X$ and $LX$ with the topology induced from $PX$.

**Proposition 20.** Assume that $X$ is a closed connected orientable $n$-dimensional manifold. Let $\chi$ denote the Euler characteristic $\chi(X)$. Then

1. $H^i(P'X; \mathbb{Z}) \simeq H^i(X; \mathbb{Z})$ for $i < n - 1$;
2. If $\chi \neq 0$, there is an exact sequence

   $$0 \rightarrow H^{n-1}(X; \mathbb{Z}) \rightarrow H^{n-1}(P'X; \mathbb{Z}) \rightarrow \tilde{H}^0(LX; \mathbb{Z}) \rightarrow 0;$$

   In the case when $\chi = 0$ the group on the right should be replaced by the unreduced zero-dimensional cohomology $H^0(LX; \mathbb{Z})$;
3. There is an exact sequence

   $$0 \rightarrow \mathbb{Z}/\chi \mathbb{Z} \rightarrow H^n(P'X; \mathbb{Z}) \rightarrow H^1(LX; \mathbb{Z}) \rightarrow 0;$$
4. One has $H^i(P'X) \simeq H^{i-n+1}(LX)$ for $i > n$.

Below we briefly sketch the proof. One starts with the long exact sequence

$$\cdots \rightarrow H^i(PX) \rightarrow H^i(P'X) \xrightarrow{\delta} H^{i+1}(PX, P'X) \rightarrow \cdots$$

and observes that $H^i(PX) \simeq H^i(X)$ and $H^{i+1}(PX, P'X) \simeq H^{i+1-n}(LX)$. The second isomorphism is obtained as a composition of an excision and a Thom isomorphism. To justify it one notes that $LX$ has an open neighborhood in $PX$ which is homeomorphic to the total space of the rank $n$ vector bundle which is induced by the map $LX \rightarrow X$ (assigning to a loop its beginning) from the tangent bundle $TX \rightarrow X$ over $X$. For dimensional reasons the connecting homomorphism

$$\delta: H^i(LX) \rightarrow H^{i+n}(X)$$

can be nonzero only for $i = 0$; in that case it coincides with the restriction $H^0(LX) \rightarrow H^0(X)$ composed with multiplication by the Euler class of the tangent bundle.

Note that $H^0(LX; \mathbb{Z})$ can be identified with the group of all integer valued functions on the set of conjugacy classes of $\pi_1(X)$. The action of the natural involution $P'X \rightarrow P'X$ on the cohomology can be described using natural isomorphisms of Proposition 20. For instance we mention that the isomorphism of statement (4) of Proposition 20 expresses the action of the involution on $H^i(P'X)$ for $i > n$ through the cohomology of the free loop space $LX$ viewed together with the involution given by reversing directions of loops.

5. Mid-point maps

In this Section we study symmetric motion planning algorithms of a different type which leads to another notion of symmetric topological complexity of configuration spaces.
Recall that a Symmetric Motion Planner in $X$ is a function $s : X \times X \to PX$ which assigns to each pair $(A, B) \in X \times X$ a path $s(A, B)$ in $X$ from $A$ to $B$, satisfying $s(A, A)(t) = A$ and $s(B, A)(t) = s(A, B)(1 - t)$ for all $t \in I$. Considering just the mid-points of such paths, if we set

$$
\sigma(A, B) = s(A, B)(1/2)
$$

we obtain a function $\sigma : X \times X \to X$ which satisfies the conditions $\sigma(A, B) = \sigma(B, A)$ and $\sigma(A, A) = A$.

A continuous function $\sigma$ with these properties is called a “2-mean” on $X$. The question of existence of such means on spaces was considered by B. Eckmann, first in 1954 [6] and again 50 years later in relation to the problem of Social Choice in Economics [5]. For example, any CW-complex $X$ which admits a 2-mean must be an $H$-space, and either contractible or of infinite dimension. If we abandon the requirement $\sigma(A, A) = A$, however, then such maps are not so rare.

**Definition 21.** A midpoint map on a topological space $X$ is a continuous map

$$
\sigma : F(X; 2) \to X
$$

satisfying $\sigma(A, B) = \sigma(B, A)$ for all $(A, B) \in F(X; 2)$.

It is clear that such mid-point maps exist for any space $X$ (for example by setting $\sigma(A, B) = A_0$, where $A_0 \in X$ is a base point) and that they are classified by homotopy classes of continuous maps from $B(X; 2)$ to $X$.

**Proposition 22.** Any midpoint map on $S^n$ with $n$ even is homotopic to a constant map. For $n$ odd the midpoint maps on $S^n$ are classified by the integers (the so-called degree $d \in \mathbb{Z}$).

**Proof.** The inclusion $S^n \hookrightarrow F(S^n; 2)$ is $G$-homotopy equivalence, where $S^n$ is viewed as a $G$-space with antipodal action. Hence it induces a homotopy equivalence $\mathbb{RP}^n \to B(S^n; 2)$. The mid-point maps on $S^n$ are thus classified by homotopy classes of maps from $\mathbb{RP}^n$ to $S^n$, which (by Hopf’s Theorem) are in 1-1 correspondence with $H^n(\mathbb{RP}^n, \mathbb{Z})$. The latter group is trivial for $n$ even and $\mathbb{Z}$ for $n$ odd, which proves the claim. \hfill \Box

When $n$ is odd there exist mid-point maps on $S^n$ of arbitrary degree. This is illustrated in the case $n = 1$ by considering the map $\sigma : F(S^1; 2) \to S^1$ given by $\sigma(A, B) = (AB)^d$, where $d \in \mathbb{Z}$. This uses the structure of an Abelian group on $S^1$.

For any topological space $X$ and a midpoint map \(\sigma\) we now define a number $\text{TC}_{2}(X)$ which measures the essential discontinuities of Symmetric Motion Planners in $X$ whose mid-points are determined by $\sigma$. Consider the subspace

$$
E_\sigma' = \{ \gamma \in PX \mid \gamma(0) \neq \gamma(1), \gamma(1/2) = \sigma(\gamma(0), \gamma(1)) \} \subseteq PX
$$

consisting of paths with distinct endpoints and mid-point determined by $\sigma$. We denote by $\pi^\sigma : E_\sigma' \to F(X; 2)$ the fibration resulting from restricting the endpoint
map (2) to this subspace. This is a $G$-equivariant map of free $G$-spaces and so induces a fibration

\begin{equation}
\pi^G_\sigma := \pi^\sigma / G: E'_\sigma / G \to B(X; 2).
\end{equation}

**Definition 23.** The Symmetric Topological Complexity of $X$ with mid-point map $\sigma$ is the number $\text{TC}_\sigma^S(X)$ defined as one plus the Schwarz genus of the fibration $\pi^G_\sigma$.

**Proposition 24.** One has $\text{TC}_\sigma^S(X) \leq \text{TC}_\sigma^S(X)$.

**Proof.** The fibrations $\pi^G_\sigma$ and $\pi^G_\sigma$ are related as shown in the diagram

\[
\begin{array}{ccc}
E'_\sigma / G & \xrightarrow{\varphi} & P'X / G \\
\varpi & \downarrow \pi^G_\sigma & \downarrow \pi^G_\sigma \\
B(X; 2) & \xrightarrow{\varphi} & B(X; 2)
\end{array}
\]

where the map $\varphi$ is induced by the inclusion $E'_\sigma \subseteq P'X$. Any section $s: U \to E'_\sigma / G$ of $\pi^G_\sigma$ on an open set $U \subseteq B(X; 2)$ therefore gives rise to a section $\varphi \circ s$ of $\pi^G_\sigma$ on $U$, and the conclusion follows. \(\square\)

The proof of Proposition 10 works with $\text{TC}_\sigma^S(X)$ replacing $\text{TC}_\sigma^S(X)$ and gives:

**Proposition 25.** For any $X$ we have $\text{TC}_\sigma^S(X) \leq 2\dim(X) + 2$. If $X$ is a closed smooth manifold then $\text{TC}_\sigma^S(X) \leq 2\dim(X) + 1$.

The main result of this paper concerning $\text{TC}_\sigma^S(X)$ can be stated as follows:

**Theorem 26.** Let $X$ be a smooth closed aspherical manifold. Then

\begin{equation}
\text{TC}_\sigma^S(X) \geq 2\text{cl}(X) + 1
\end{equation}

where $\sigma: F(X; 2) \to X$ is the constant mid-point map and $\text{cl}(X)$ denotes the largest integer $k$ such that there exist $k$ cohomology classes $u_1, \ldots, u_k \in H^*(X; \mathbb{Z}_2)$ of positive degree whose cup-product is non-zero, $u_1 \cdots u_k \neq 0$.

Recall that a path-connected topological space $X$ is said to be aspherical if $\pi_i(X) = 0$ for all $i > 1$.

Theorem 26 will be proven in the final Section 7. We first need to sharpen our tools by introducing a new variant on category weight. This will be done in Section 6.

We conclude this Section by two examples:

**Example 27.** Consider the closed orientable surface $\Sigma_g$ of genus $g \geq 1$. This has cup-length $\text{cl}(\Sigma_g) = 2$, and so $\text{TC}_\sigma^S(\Sigma_g) \geq 5$ by Theorem 26. Proposition 25 gives $\text{TC}_\sigma^S(X) \leq 2n + 1$ whenever $X$ is a closed $n$-manifold. Hence $\text{TC}_\sigma^S(\Sigma_g) = 5$. This agrees with the usual Topological Complexity except in the case of the torus $T^2 = \Sigma_1$ which has $\text{TC}(T^2) = 3$, see [8].

**Example 28.** Consider the $n$-dimensional torus $T^n$, the Cartesian product of $n$ copies of $S^1$. This models the configuration space of a planar robot arm with $n$ revolving joints. It has $\text{cl}(T^n) = n$, and Theorem 26 implies that $\text{TC}_\sigma^S(T^n) = 2n + 1$. The usual topological complexity of the torus is $\text{TC}(T^n) = n + 1$.

The last example shows that $\text{TC}_\sigma^S(X)$ can be much larger than $\text{TC}(X)$. 
6. Sectional category weight

The usual cohomological lower bound for the Lusternik-Schnirelmann category of a space $X$ states that if $u_1, \ldots, u_k \in H^*(X)$ are non-zero cohomology classes of positive degree such that their cup product $u = u_1 \cdots u_k \neq 0$ is nonzero then $\text{cat}(X) > k$. E. Fadell and S. Husseini [7] improved this estimate by assigning to each cohomology class $u$ an integer weight, denoted $\text{cwgt}(u)$ and called its category weight, such that if $0 \neq u = u_1 \cdots u_k$ then

$$\text{cat}(X) > \text{cwgt}(u) = \text{cwgt}(u_1 \cdots u_k) \geq \sum_i \text{cwgt}(u_i).$$

Note that to improve on the cup-length estimate one must find indecomposable classes of category weight at least 2, which Fadell and Husseini did using Steenrod operations [7]. This notion was developed further by Y. B. Rudyak, who gave a homotopy invariant version (known as strict category weight) and proved that non-trivial Massey products also have weight at least 2 [15].

Our aim here is to improve on Schwarz’s original cohomological lower bound for the genus of a fibration $p: E \to B$ (given in Theorem 12) by extending the ideas of Fadell and Husseini mentioned above. To do this we note that each cohomology class $\xi \in H^*(B)$ can be assigned an integer weight with respect to $p$, called its sectional category weight, and defined as follows.

**Definition 29.** Let $\xi \in H^*(B)$ be a cohomology class. We define the sectional category weight of $\xi$ with respect to $p$, denoted $\text{wgt}_p(\xi)$, to be the largest integer $k$ such that for any continuous map $f: X \to B$ with $g(f^*p) \leq k$ we have $f^*(\xi) = 0$. Note that $\text{wgt}_p(\xi) \geq 0$, since any fibration over a non-empty base has genus at least 1, and so the condition $f^*(\xi) = 0$ whenever $g(f^*p) \leq 0$ is vacuously satisfied. We set $\text{wgt}_p(\xi) = \infty$ when $\xi = 0$.

**Proposition 30.** A class $\xi \in H^*(B)$ has $\text{wgt}_p(\xi) \geq 1$ if and only if $p^*(\xi) = 0 \in H^*(E)$.

**Proof.** First suppose $p^*(\xi) = 0$, and let $f: X \to B$ be a map such that $g(f^*p) \leq 1$, so that $f^*p$ has a section. From the pullback diagram

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E \\
\downarrow{f^*p} & & \downarrow{p} \\
X & \xrightarrow{f} & B
\end{array}
$$

we see that $(f^*p)^*f^*(\xi) = \overline{f^*p}^*(\xi) = 0$, which implies $f^*(\xi) = 0$ since $(f^*p)^*$ is injective. Hence $\text{wgt}_p(\xi) \geq 1$.

Conversely suppose that $\text{wgt}_p(\xi) \geq 1$, and consider the pull-back diagram

$$
\begin{array}{ccc}
p^*(E) & \xrightarrow{p^*} & E \\
p^*p & \downarrow{p} & \downarrow{p} \\
E & \xrightarrow{p} & B.
\end{array}
$$

Clearly the fibration $p^*p$ has a section given by the diagonal map, and so $p^*(\xi) = 0$ by the definition of sectional category weight. \qed
Proposition 31. For any non-zero class \( \xi \in H^*(B) \),

\[
\text{wgt}_p(\xi) < g(p).
\]

Proof. Suppose the converse is true, and we have \( \xi \) such that \( \text{wgt}_p(\xi) \geq g(p) = k \), say. The identity map \( 1 : B \to B \) has \( g(1^*p) = g(p) = k \), and so \( 1^*\xi = \xi \) must be zero.

Proposition 32. For the cup product of classes \( \xi_1, \ldots, \xi_l \in H^*(B) \) we have

\[
\text{wgt}_p(\xi_1 \cdots \xi_l) \geq \sum_{i=1}^l \text{wgt}_p(\xi_i).
\]

Proof. Of main interest is the case when \( \xi = \xi_1 \cdots \xi_l \neq 0 \) (the other case is true by convention). Letting \( k_i = \text{wgt}_p(\xi_i) \), we must show that \( \text{wgt}_p(\xi) \geq k = \sum k_i \).

So suppose that \( f : X \to B \) is a continuous map with \( g(f^*p) \leq k \). We may find a covering \( \Omega = \{U_1, \ldots, U_k\} \) of \( X \) by open sets, above each of which the fibration \( f^*p \) has a section. Partition the cover \( \Omega \) into \( l \) families \( \Omega_1, \ldots, \Omega_l \) such that \( \Omega_i \) consists of \( k_i \) open sets. We now define

\[
A_i = \bigcup_{U_i \in \Omega_i} U_j.
\]

Note that \( X = \bigcup_i A_i \), and \( f^*(\xi_i)|_{A_i} = 0 \) for \( i = 1, \ldots, n \) since \( \text{wgt}_p(\xi_i) = k_i \). A standard argument now gives that \( f^*(\xi) = f^*(\xi_1) \cdots f^*(\xi_l) = 0 \), as required.

The last three Propositions together imply the following sharpened version of Theorem 12.

Theorem 33. Let \( p : E \to B \) be a fibration. If \( \xi_1, \ldots, \xi_l \in H^*(B) \) are positive dimensional cohomology classes whose product is non-zero, then

\[
g(p) > \sum_{i=1}^l \text{wgt}_p(\xi_i).
\]

Hence we may improve on the lower bound given by Theorem 12 by finding indecomposable elements of sectional category weight at least 2. Finding such elements is greatly facilitated by the next Proposition regarding fibrewise joins.

Recall that the fibrewise join of two fibrations \( p_1 : E_1 \to B \) and \( p_2 : E_2 \to B \) over the same base is a certain fibration \( p_1 \ast p_2 : E_1 \ast_B E_2 \to B \), whose fibre has the homotopy type of the join \( F_1 \ast F_2 \) (see 14 or 11, for example). The total space \( E_1 \ast_B E_2 \) may be described as the subspace

\[
\{(e_1, e_2, t) \in E_1 \times E_2 \times I \mid p_1(e_1) = p_2(e_2)\} \subseteq E_1 \times E_2 \times I
\]

modulo the relations \((e_1, e_2, 0) \sim (e'_1, e_2, 0)\) and \((e_1, e_2, 1) \sim (e_1, e'_2, 1)\) for all \( e_1, e'_1 \in E_1 \) and \( e_2, e'_2 \in E_2 \). The projection \( p_1 \ast p_2 \) of this fibration is given by \( p_1 \ast p_2((e_1, e_2, t)) = p_1(e_1) = p_2(e_2) \).

One may also define the fibrewise join of an arbitrary number of fibrations. For a fibration \( p : E \to B \), denote by \( p(k) : E(k) \to B \) the \( k \)-fold fibrewise join of \( p : E \to B \) with itself. A. Schwarz (14, Theorem 3) proved that \( g(p) \leq k \) if and only if \( p(k) \) has a continuous section.

Proposition 34. If \( p(k)^*(\xi) = 0 \) then \( \text{wgt}_p(\xi) \geq k \).
Proof. Let \( f : X \to B \) be such that \( f^*p : f^*(E) \to X \) has genus not more than \( k \). The fibrations \( f^*p(k) \) and \((f^*p)(k)\) are homeomorphic. Hence we have the following diagram,

\[
\begin{array}{ccc}
 f^*(E)(k) & \longrightarrow & E(k) \\
 (f^*p)(k) & \downarrow f & p(k) \\
 X & \longrightarrow & B
\end{array}
\]

in which the map \((f^*p)(k)\) admits a section and hence induces a monomorphism in cohomology. If \( p(k)^*(\xi) = 0 \) then \((f^*p)(k)^*f^*(\xi) = 0\) and hence \( f^*(\xi) = 0 \). \( \Box \)

To conclude this Section we describe the homotopy invariance of sectional category weight.

Lemma 35. Suppose \( p_1 : E_1 \to B \) and \( p_2 : E_2 \to B \) are fibre homotopy equivalent fibrations, and \( \xi \in H^*(B) \). Then \( \text{wgt}_{p_1}(\xi) = \text{wgt}_{p_2}(\xi) \).

Proof. Given any continuous map \( f : X \to B \), the pullback fibrations \( f^*p_1 \) and \( f^*p_2 \) are fibre homotopy equivalent. Since fibre homotopy equivalent fibrations have the same genus (this follows immediately from Proposition 6 of Schwarz [14]), this means that \( g(f^*p_1) = g(f^*p_2) \) for all \( f \), and the conclusion follows. \( \Box \)

Proposition 36. Let \( p : E \to B \) be a fibration, \( \xi \in H^*(B) \) a cohomology class and \( g : A \to B \) a continuous map. Then, \( \text{wgt}_{g^*p}g^*(\xi) \geq \text{wgt}_p(\xi) \). If \( g \) is a homotopy equivalence, then \( \text{wgt}_{g^*p}g^*(\xi) = \text{wgt}_p(\xi) \).

Proof. The first statement is immediate from the definition. Suppose \( k : B \to A \) is a homotopy inverse for \( g \). Then \( g \circ k \simeq 1 \) implies \( k^*g^*(\xi) = \xi \) and \( k^*g^*p \) is fibre homotopy equivalent to \( p \), and so by Lemma 35

\[
\text{wgt}_p(\xi) = \text{wgt}_{k^*g^*p}k^*g^*(\xi) \geq \text{wgt}_{g^*p}g^*(\xi) \geq \text{wgt}_p(\xi).
\]

\( \Box \)

7. Proof of Theorem 26

Recall that the number \( \text{TC}_G^2(X) \) is defined to be one plus the genus of the fibration \( E_\sigma \). The space

\[
E_\sigma = \{ \gamma : I \to X \mid \gamma(1/2) = A_0 \} \subseteq PX
\]

of paths in \( X \) with mid-point \( A_0 \) admits an involution \( \gamma \mapsto \overline{\gamma} \) where \( \overline{\gamma}(t) = \gamma(1-t) \), and it contains \( E_\sigma^F \) (given by \( \ref{23} \)) as a free \( G \)-invariant subspace (the subspace consisting of such paths with distinct endpoints). The endpoint map \( q : E_\sigma \to X^2 \) is a \( G \)-invariant fibration, of which \( \pi^\sigma : E_\sigma^F \to F(X; 2) \) is a restriction. We obtain a diagram of homotopy orbit spaces

\[
\begin{array}{ccc}
E_\sigma/G & \simeq & EG \times_G E_\sigma^F \\
\downarrow \pi_G^\sigma & & \downarrow 1 \times_G \pi^\sigma \\
B(X; 2) & \simeq & EG \times_G F(X; 2)
\end{array}
\]

\[
\begin{array}{ccc}
& \longmapsto & \\
\downarrow & \longmapsto & \\
1 \times_G q & = & p
\end{array}
\]

\[\text{wgt}_p(\xi) = \text{wgt}_{k^*g^*p}k^*g^*(\xi) \geq \text{wgt}_{g^*p}g^*(\xi) \geq \text{wgt}_p(\xi).\]
By Theorem 13 and the subsequent paragraphs, every cohomology class $u \in H^k(X)$ determines cohomology classes

(21)  $\alpha_u = 1 \otimes u \otimes u \in H^{2k}(EG \times_G X^2)$,  $\beta_u = j^*(\alpha_u) \in H^{2k}(B(X, 2))$.

The main ingredient of the proof of Theorem 26 consists of the following statement:

**Theorem 37.** For any $u \in H^k(X)$ with $k > 0$, the cohomology class $\alpha_u \in H^{2k}(EG \times_G X^2)$ has sectional category weight at least 2 with respect to the fibration $p: EG \times_G E_\sigma \rightarrow EG \times_G X^2$. Hence, the cohomology class $\beta_u \in H^{2k}(B(X, 2))$ has sectional category weight at least 2 with respect to the fibration $\pi^*_G: E_G/G \rightarrow B(X, 2)$.

The proof of Theorem 37 rests on the following two lemmas, the first of which concerns the fibrewise join

$q * q: E_\sigma \ast_{X^2} E_\sigma \rightarrow X^2$

of two copies of the fibration $q: E_\sigma \rightarrow X^2$. Let $\Omega X$ denote the space of loops in $X$ based at the base-point $A_0 \in X$. We think of points of $\Omega X$ as maps $\omega: I \rightarrow X$ such that $\omega(0) = \omega(1) = A_0$. Let $S$ denote the unreduced suspension functor. There is a map

$h: S(\Omega X \times \Omega X) \rightarrow X^2$

defined by setting $h[\omega_1, \omega_2, t] = (\omega_1(1/2), \omega_2(1/2))$.

**Lemma 38.** There is a homotopy equivalence $\tau: E_\sigma \ast_{X^2} E_\sigma \rightarrow S(\Omega X \times \Omega X)$ which satisfies $h \circ \tau = q * q$. Furthermore, $\tau$ is $G$-equivariant with respect to the diagonal action of $G$ on $E_\sigma \ast_{X^2} E_\sigma$ and the action on $S(\Omega X \times \Omega X)$ which swaps the loop factors.

**Proof of Lemma 38.** (Compare Theorem 21 of [14] and figure below.) Points of $E_\sigma \ast_{X^2} E_\sigma$ are equivalence classes $[\gamma_1, \gamma_2, t]$ where $\gamma_1, \gamma_2$ are paths in $X$ with the same initial and final points and mid-point $A_0$ (see figure below). We set $\tau[\gamma_1, \gamma_2, t] = [\omega_1, \omega_2, t]$, where

$$h_1(s) = \begin{cases} \gamma_2(1/2 - s) & \text{if } s \in [0, 1/2), \\ \gamma_1(s - 1/2) & \text{if } s \in [1/2, 1], \end{cases} \quad \omega_2(s) = \begin{cases} \gamma_2(1/2 + s) & \text{if } s \in [0, 1/2), \\ \gamma_1(3/2 - s) & \text{if } s \in [1/2, 1]. \end{cases}$$

We then have $\tau[\gamma_1, \gamma_2, t] = [\omega_2, \omega_1, t]$ and $h \circ \tau = q * q$ by design.

It remains to check that $\tau$ is a homotopy equivalence. Denote by $W_0$ (respectively $W_1$) the closed subspace of $E_\sigma \ast_{X^2} E_\sigma$ consisting of points of the form $[\gamma_1, \gamma_2, 0]$ (resp. $[\gamma_1, \gamma_2, 1]$). Both of these subspaces are homeomorphic to $E_\sigma$, hence are contractible within themselves. It is not difficult to construct a deformation of
the space $E_\sigma \star X \times X E_\sigma$ which contracts each to a point within itself. The proof is completed by noting that, up to homeomorphism, $\tau$ is the quotient map collapsing $W_0$ and $W_1$ to the vertices of the suspension, and so is a homotopy equivalence by Lemma 7.1.5 of Spanier [16].

Our next result generalises to the equivariant setting Proposition 1 of Schwarz [14] concerning the join of two fibrations associated with a principle fibration.

Let $E$ be a free $G$-space and let $q_i : Y_i \to X$, $i = 1, 2$ be two equivariant maps of $G$-spaces which are fibrations over the common base space $X$. The group $G$ acts diagonally on the total space $Y_1 \star X Y_2$ of the fibrewise join $q_1 \ast q_2$.

**Lemma 39.** The following two fibrations over $E \times_G X$ are homeomorphic:

$1 \times_G (q_1 \ast q_2) : E \times_G (Y_1 \star X Y_2) \to E \times_G X$,

and

$(1 \times_G q_1) \ast (1 \times_G q_2) : (E \times_G Y_1) \ast_{E \times_G X} (E \times_G Y_2) \to E \times_G X$.

**Proof of Lemma 39.** We define a map

$\mu : E \times_G (Y_1 \star X Y_2) \to (E \times_G Y_1) \ast_{E \times_G X} (E \times_G Y_2)$

and show that it is a homeomorphism. A point in the domain of $(22)$ is an equivalence class $[e, [y_1, y_2, t]]$, where $e \in E$, $y_i \in Y_i$ and $t \in I$ such that $q_1(y_1) = q_2(y_2) \in X$. For any $g \in G$ we have $[e, [y_1, y_2, t]] = [ge, [gy_1, gy_2, t]]$. A point in the range is a class $[e_1, [y_1, y_2], t]$ where $e_1, e_2 \in E$, $y_i \in Y_i$ and $t \in I$ such that $[e_1, q_1(y_1)] = [e_2, q_2(y_2)] \in E \times_G X$. We set

$\mu[e, [y_1, y_2, t]] = [[e, y_1], [e, y_2], t]$.

This map clearly is well-defined and is continuous. We will show that $\mu$ is onto; the proof that it’s 1-1 runs similarly.

A point $x = [e_1, [y_1, y_2], t]$ in the range has $[e_1, q_1(y_1)] = [e_2, q_2(y_2)] \in E \times_G X$. Hence there is a unique $g \in G$ with $e_1 = ge_2$ and $q_1(y_1) = gq_2(y_2)$. Therefore,

$x = [e_1, [y_1, [e_1, gy_1], t]] = \mu[e_1, [y_1, gy_2, t]].$

(Note that in this way one may describe a continuous inverse for $\mu$).

Hence $\mu$ is a homeomorphism. Verifying that $\mu$ is fibre preserving is trivial. □

**Proof of Theorem 37.** We will show that $(p \ast p)^* (\alpha_u) = 0$. The first statement of Theorem 37 then follows from Proposition 34 and the second statement follows from the first using Proposition 36 since $\pi^*_G$ is the pullback fibration $j^* p$ where $j : B(X, 2) \to EG \times_G X^2$ is the inclusion. Applying Lemma 39 we find that $(p \ast p)^* (\alpha_u) = 0$ if and only if $(1 \times_G (q \ast q))^* (\alpha_u) = 0$.

Lemma 38 gives a homotopy equivalence

$\tau : E_\sigma \star X^2 E_\sigma \simeq S(\Omega X \times \Omega X)$.

Since $X$ is an aspherical manifold, it is a $K(\pi, 1)$ with discrete fundamental group. Thus $\Omega X \times \Omega X$ has the homotopy type of a discrete set of points, and $S(\Omega X \times \Omega X)$ of a wedge of circles. Therefore the map $(q \ast q)^* : H^{2k}(X^2) \to H^{2k}(E_\sigma \star X^2 E_\sigma)$ takes values in a zero group when $k > 0$; in particular, $(q \ast q)^*(u \otimes u) = 0$.
The map $q \ast q \colon E_\sigma \times_X E_\sigma \to X^2$ is $G$-equivariant, and so gives a morphism of the associated Cartan-Leray spectral sequences (see [2], Chapter IV, Section 3),

$$H^p(G, H^q(E_\sigma \times_X E_\sigma)) \Rightarrow H^{p+q}(E_G \times_G (E_\sigma \times_X E_\sigma)) \quad \text{(20)}$$

and

$$H^p(G, H^q(X^2)) \Rightarrow H^{p+q}(E_G \times_G X^2).$$

The lower spectral sequence collapses at the $E_2$ term, by Theorem 13. Hence we may view $\alpha_u = 1 \otimes u \otimes u \in H^{2k}(E_G \times_G X^2)$ as an element of $H^0(G, H^{2k}(X^2))$, which maps to zero under the map on coefficients induced by $q \ast q$. The claim follows by naturality.

Proof of Theorem 26. Suppose $\text{cl}(X) = m$, and let $u_1, \ldots, u_m \in H^*(X)$ be positive dimensional classes with non-zero product. Let $\alpha_i$ and $\beta_i$ denote the elements $\alpha_{u_i}$ and $\beta_{u_i}$. Note that whilst the product $\alpha_1 \cdots \alpha_m \in H^*(E_G \times_G X^2)$ is non-zero, the product

$$\beta_1 \cdots \beta_m = j^*(\alpha_1 \cdots \alpha_m) \in H^*(B(X; 2))$$

may vanish. However, letting $\gamma_m = [u_m \otimes 1 + 1 \otimes u_m] \in H^*(B(X; 2))$ we find that the product

$$\beta_1 \cdots \beta_{m-1} \gamma_m = [u_1 \cdots u_m \otimes u_1 \cdots u_{m-1} + u_1 \cdots u_{m-1} \otimes u_1 \cdots u_m]$$

is non-zero, by Corollary 15.

We now observe that $\text{wgt}_{\pi_G^*} \gamma_m \geq 1$. Indeed, since $E_\sigma \times E_\sigma$ is a contractible space with free $G$-action, there is a homotopy equivalence $E_G \times_G E_\sigma \simeq B\mathsf{G}$. Now applying cohomology to the diagram (20) we obtain

$$
\begin{array}{ccc}
H^*(E_\sigma^G) & \xrightarrow{(\pi_G^*)^*} & H^*(B\mathsf{G}) \\
\downarrow & & \downarrow \Theta^* \\
H^*(B(X; 2)) & \xrightarrow{j^*} & H^*(E_G \times_G X^2),
\end{array}
$$

where $\Theta \colon B\mathsf{G} \to E_\sigma \times E_\sigma X^2$ is the map $[e] \mapsto [e, A_0, A_0]$. Now $\Theta^*(u_m \otimes 1 + 1 \otimes u_m) = 0$, which implies $(\pi_G^*)^*(\gamma_m) = 0$. This proves the claim in view of Proposition 30.

The proof of Theorem 26 is completed by applying Theorem 33 to the product $\beta_1 \cdots \beta_{m-1} \gamma_m$, since

$$g(\pi_G^* \beta_i) > \sum_{i=1}^{m-1} \text{wgt}_{\pi_G^*} \beta_i + \text{wgt}_{\pi_G^*} \gamma_m \geq 2(m-1) + 1 = 2m - 1.$$ 

Hence $g(\pi_G^* \beta) \geq 2m$ and so $\text{TC}_{\sigma}^\pi(X) \geq 2m + 1$, as stated.

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\text{References}
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