Asymptotics of Young tableaux in the \((k, \ell)\) hook

A. Berele and A. Regev

July 23, 2010

Abstract. The asymptotics of the "\((k, \ell)\) hook" sums \(S_{k,\ell}^{(2z)}(n)\) (see (1)) were calculated in [1]. It was recently realized that in [1, Section 7] there are few misprints and certain confusion with regard to the notations, so that the precise asymptotics of \(S_{k,\ell}^{(2z)}(n)\) is not clear. Here we add more details and carefully repeat these calculations, which lead to explicit values for the asymptotics of \(S_{k,\ell}^{(2z)}(n)\).

Mathematics Subject Classification: 05A16, 34M30.

1 Introduction

Let \(\lambda\) be a partition and denote by \(f^{\lambda}\) the number of standard Young tableaux (SYT) of shape \(\lambda\). Let \(H(k, \ell; n)\) denote the partitions of \(n\) in the \((k, \ell)\) hook, namely \(H(k, \ell; n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda \vdash n \text{ and } \lambda_{k+1} \leq \ell\}\). In [1, Section 7] we computed the asymptotics, as \(n\) goes to infinity, of the sums

\[
S_{k,\ell}^{(2z)}(n) = \sum_{\lambda \in H(k,\ell; n)} (f^\lambda)^{2z}. \tag{1}
\]

That asymptotics has the form

\[
S_{k,\ell}^{(2z)}(n) \simeq a(k, \ell, 2z) \cdot \left(\frac{1}{n}\right)^{g(k,\ell,2z)} \cdot (k + \ell)^{2zn}
\]

for some functions \(a(k, \ell, 2z)\) and \(g(k, \ell, 2z)\).

It was recently realized that in [1, Section 7] there are few misprints and some confusion with the notations, so that the precise value of the constant term \(a(k, \ell, 2z)\) in that calculation is not clear. Here we add more details and carefully repeat these calculations, which lead to explicit values for the asymptotics of \(S_{k,\ell}^{(2z)}(n)\), namely the explicit expression for the functions \(a(k, \ell, 2z)\) and \(g(k, \ell, 2z)\). This is Theorem 4.1 below (see also Theorem 4.8).

Note that \(\Gamma\) here is the gamma function.
Theorem 1.1. As $n$ goes to infinity,

$$S_{k,\ell}^{(2z)}(n) \simeq a(k, \ell, 2z) \cdot \left(\frac{1}{n}\right)^{g(k,\ell,2z)} \cdot (k + \ell)^{2zn},$$

where

$$g(k, \ell, 2z) = \frac{1}{2} \cdot (z \cdot [k(k+1) + \ell(\ell+1) - 2] - (k + \ell - 1))$$

and

$$a(k, \ell, 2z) =$$

$$= \left[ \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{2}\right)^{k+\ell} \cdot \frac{(2\pi)^{k+\ell}}{k! \cdot \ell!} \cdot \frac{1}{(2\pi)^{k+\ell} \cdot (2z)^{-\frac{1}{2} \cdot (z \cdot [k(k-1) + \ell(\ell-1)] + k + \ell)}} \right]^{2z}$$

$$\cdot \left(\frac{1}{k + \ell}\right)^{z \cdot [k(k-1) + \ell(\ell-1)] + k + \ell} \cdot (\Gamma (1 + z))^{-k-\ell} \cdot \prod_{i=1}^{k} \Gamma (1 + zi) \cdot \prod_{j=1}^{\ell} \Gamma (1 + zj).$$

In the case $2z = 1$ we have (see Theorem 5.1 below)

Theorem 1.2.

$$S_{k,\ell}^{(1)}(n) \simeq a(k, \ell, 1) \cdot \left(\frac{1}{n}\right)^{\frac{1}{2} \cdot (k(k-1) + \ell(\ell-1))} \cdot (k + \ell)^{n}$$

where

$$a(k, \ell, 1) =$$

$$= \left(\frac{1}{2}\right)^{k\ell-k-\ell} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(2\pi)^{k+\ell}} \cdot \frac{1}{(k+\ell)^{\frac{1}{2} \cdot (k(k-1) + \ell(\ell-1))}} \cdot \prod_{i=1}^{k} \Gamma (1 + i/2) \cdot \prod_{j=1}^{\ell} \Gamma (1 + j/2).$$

For the evaluation of special cases note that

$$\Gamma \left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and

$$\Gamma (1 + x) = x\Gamma (x).$$

In several cases the sums $S_{k,\ell}^{(2z)}(n)$ can be evaluated and given by simple formulas which yield the corresponding asymptotics directly - independent of Theorem 1.1. These are the cases of $S_{1,1}^{(1)}(n)$ and $S_{1,1}^{(2)}(n)$, see Section 5, where we compare and verify that in these cases the direct asymptotics does agree with the asymptotics deduced from Theorem 1.1. It is also possible to use, say, ”Mathematica” to verify the validity of Theorem 1.1 in few special cases, see Section 5.1.2.
2 Preliminaries

2.1 Recalling the "strip" case \[5\]

Theorem 1.1 is a hook generalization of the following "strip" theorem 2.1, see \[5, Corollary 4.4\]. We remark that, even though Theorem 1.1 is proved under the assumption that both \(k, \ell \geq 1\), nevertheless it reduces to Theorem 2.1 when one substitutes \(\ell = 0\).

**Theorem 2.1.** Let

\[
S_k^{(2z)}(n) = \sum_{\lambda \in H(k,0;n)} (f^{\lambda})^{2z},
\]

then, as \(n\) goes to infinity,

\[
S_k^{(2z)}(n) \simeq a(k, 2z) \cdot \left(\frac{1}{n}\right)^{g(k,z)} \cdot k^{2zn}
\]

where

\[
g(k, z) = \frac{1}{2} \cdot (z(k^2 + k - 2) - (k - 1))
\]

and

\[
a(k, 2z) = \left[ \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot k^{k^2/2}\right]^{2z} \cdot \left(\frac{1}{k}\right)^{\left[k(k-1)+k\right]/2} \cdot \frac{1}{k!} \cdot \sqrt{\frac{z}{\pi}} \cdot (2\pi)^{k/2} \cdot (2z)^{-(z(k-1)+k)/2} \cdot \Gamma(1+z)^{-k} \cdot \prod_{j=1}^{k} \Gamma(1+zj).
\]

**Remark 2.2.** One of the main tools in proving Theorem 2.1 in \[5\] was the computation of the asymptotics of a single \(f^{\nu}\) where \(\nu \in H(k,0;n)\). Let \(\nu = (\nu_1, \ldots, \nu_k) \vdash n\). By \[5\ (F.1.1)\], when the \(a_i\)'s are bounded we have:

\[
f^{\nu} \simeq \gamma_k \cdot D_k(a_1, \ldots, a_k) \cdot e^{-\left(k/2\right)\left(\sum a_i^2\right)} \cdot \left(\frac{1}{n}\right)^{(k-1)(k+2)/4} \cdot k^n,
\]

where

\[
\gamma_k = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot k^{k^2/2} \quad \text{and} \quad D_k(a_1, \ldots, a_k) = \prod_{1 \leq i < j \leq k} (a_i - a_j).
\]

We apply (2) in what follows.
2.2 Preliminaries for the hook case

We assume that both $k, \ell \geq 1$ and we follow the notations of \cite{1} from Section 7.9 on. We assume that $\lambda \in H(k, \ell; n)$ and $\lambda_k \geq \ell$, namely $\lambda$ contains the $k \times \ell$ rectangle $R_{k, \ell}$. Thus $\lambda$ is made of the partitions $\nu, \mu'$ and of the rectangle $R_{k, \ell}$. We have $\lambda \vdash n$, $\nu \vdash n_k$, $\mu \vdash n_\ell$, $R_{k, \ell} \vdash k\ell$, $n = n_k + n_\ell + k\ell$, $\bar{n} = n - k\ell$.

By assumption $n_k \simeq n \cdot \frac{k}{k + \ell}$ and $n_\ell \simeq n \cdot \frac{\ell}{k + \ell}$. Now $\nu = (\nu_1, \ldots, \nu_k)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, and

$$\nu_i = \frac{n_k}{k} + a_i \sqrt{n_k}, \quad \mu_j = \frac{n_\ell}{\ell} + b_j \sqrt{n_\ell}.$$  \hfill (4)

Also

$$\nu_i = \frac{\bar{n}}{k + \ell} + \alpha_i \sqrt{\bar{n}}, \quad \mu_j = \frac{\bar{n}}{k + \ell} + \beta_j \sqrt{\bar{n}},$$  \hfill (5)

and we write $\alpha_1 + \cdots + \alpha_k = \alpha$ and $\beta_1 + \cdots + \beta_\ell = \beta$. It follows that $\beta = -\alpha$. The transition from the $a_i, b_j$ to the $\alpha_i, \beta_j$ is given by

$$n_k = \frac{\bar{n}k}{k + \ell} + \alpha \sqrt{\bar{n}}, \quad n_\ell = \frac{\ell \bar{n}}{k + \ell} - \alpha \sqrt{\bar{n}}, \quad \text{hence, since } \bar{n} \to \infty,$$

$$a_i = \left( \alpha_i - \frac{\alpha}{k} \right) \cdot \left( \frac{k}{k + \ell} + \frac{\alpha}{\sqrt{\bar{n}}} \right)^{-1/2} \simeq \left( \alpha_i - \frac{\alpha}{k} \right) \cdot \left( \frac{k}{k} \right)^{1/2}$$  \hfill (6)

and the difference l.h.s. $-$ r.h.s tends to zero as $n \to \infty$. Similarly

$$b_j = \left( \beta_j + \frac{\alpha}{\ell} \right) \cdot \left( \frac{\ell}{k + \ell} - \frac{\alpha}{\sqrt{\bar{n}}} \right)^{-1/2} \simeq \left( \beta_j + \frac{\alpha}{\ell} \right) \cdot \left( \frac{k + \ell}{\ell} \right)^{1/2}$$  \hfill (7)

and l.h.s. $-$ r.h.s $\to 0$.

3 Asymptotic of a single $f^\lambda$

We refer now to \cite{1} Section 7]. Up to Lemma 7.15 there, including that lemma, every detail was checked and verified, and we proceed from that point.

The hook formula yields the factorisation $f^\lambda = A_1 \cdot A_2 \cdot A_3 \cdot A_4$ (see \cite{1} 7.14) where

$A_1 = n! / \bar{n}! \simeq n^{k\ell}$,

$A_2 = 1 / (\prod_R h_{ij}) \simeq ((k + \ell)/(2n))^{k\ell}$,

$A_3 = f^\nu \cdot f^\mu$, \quad and

$A_4 = \frac{\bar{n}!}{n_k! \cdot n_\ell!} \simeq \frac{1}{\sqrt{2\pi}} \cdot (k + \ell)^{-k\ell} \cdot \frac{k + \ell}{\sqrt{k\ell}} \cdot \frac{1}{\sqrt{n}} \cdot \frac{(k + \ell)^n}{k^{n_k} \cdot \ell^{n_\ell}} \cdot e^{-\alpha^2(k + \ell)^2/(2k\ell)}$. 
Note that $A_1 \cdot A_2 \simeq ((k + \ell)/2)^k$, so

$$f^\lambda = A_1 \cdot A_2 \cdot A_3 \cdot A_4 \simeq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2^{k\ell}} \cdot \frac{k + \ell}{\sqrt{k\ell}} \cdot \frac{1}{\sqrt{n}} \cdot \frac{(k + \ell)^n}{k^{n_k} \cdot \ell^{n_\ell}} \cdot e^{-\alpha^2(k + \ell)^2/(2k\ell)} \cdot f' \cdot f^\mu. \quad (8)$$

We analyze $f'$ and $f^\mu$. By (2) and by of [5, (F.1.1)],

$$f' \simeq \gamma_k \cdot D_k(a_1, \ldots, a_k) \cdot e^{-(k/2)(\sum a_i^2)} \cdot \left(\frac{1}{n_k}\right)^{(k-1)(k+2)/4} \cdot k^{n_k}, \quad (9)$$

where $\gamma_k = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot k^{k^2/2}$. We make the transition from the $a_i$ to the $\alpha_i$. By (6) $a_i - a_j \simeq (\alpha_i - \alpha_j) \cdot \sqrt{(k + \ell)/k}$, hence

$$D_k(a_1, \ldots, a_k) \simeq \left(\frac{k + \ell}{k}\right)^{k(k-1)/4} \cdot D_k(\alpha_1, \ldots, \alpha_k), \quad (10)$$

and similarly by (7)

$$D_\ell(b_1, \ldots, b_\ell) \simeq \left(\frac{k + \ell}{\ell}\right)^{\ell(\ell-1)/4} \cdot D_\ell(\beta_1, \ldots, \beta_\ell). \quad (11)$$

Claim:

$$e^{-\frac{1}{2} \sum a_i^2} \simeq e^{-\frac{1}{2} \sum \alpha_i^2} \cdot e^{\frac{k + \ell}{2k} \cdot \alpha^2} \quad (12)$$

Proof. By (6)

$$a_i^2 \simeq \frac{k + \ell}{k} \cdot \left(\frac{\alpha_i^2 - 2\alpha_i + \alpha_i^2}{k^2}\right) \quad \text{and l.h.s} - \text{r.h.s} \to 0 \text{ as } n \to \infty.$$

Since $\sum \alpha_i = \alpha$, we have

$$\sum a_i^2 \simeq \frac{k + \ell}{k} \cdot \sum \alpha_i^2 - \frac{k + \ell}{k^2} \cdot \alpha^2 \quad \text{and l.h.s} - \text{r.h.s} \to 0, \quad \text{so}$$

$$\frac{k}{2} \cdot \sum a_i^2 \simeq \frac{k + \ell}{2} \cdot \sum \alpha_i^2 - \frac{k + \ell}{2k} \cdot \alpha^2 \quad \text{and l.h.s} - \text{r.h.s} \to 0 \text{ as } n \to \infty,$$

and this implies (12).

Corollary 3.1. By (9), (10) and (12)

$$f' \simeq \gamma_k \cdot \left(\frac{k + \ell}{k}\right)^{k(k-1)/4} \cdot D_k(\alpha_1, \ldots, \alpha_k) \cdot e^{-\frac{1}{2} \sum \alpha_i^2} \cdot e^{\frac{k + \ell}{2k} \cdot \alpha^2} \cdot \left(\frac{1}{n_k}\right)^{(k-1)(k+2)/4} \cdot k^{n_k}, \quad (13)$$

and similarly (recall that $\beta = -\alpha$)

$$f^\mu \simeq \gamma_\ell \cdot \left(\frac{k + \ell}{\ell}\right)^{\ell(\ell-1)/4} \cdot D_\ell(\beta_1, \ldots, \beta_\ell) \cdot e^{-\frac{1}{2} \sum \beta_i^2} \cdot e^{\frac{k + \ell}{2\ell} \cdot \alpha^2} \cdot \left(\frac{1}{n_\ell}\right)^{(\ell-1)(\ell+2)/4} \cdot \ell^{n_\ell}. \quad (14)$$
Plugging (13) and (14) into (8) gives

**Corollary 3.2.**

\[
f^\lambda \simeq \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{2}\right)^{k\ell} \cdot \frac{k + \ell}{\sqrt{k\ell}} \cdot \frac{1}{\sqrt{n}} \cdot e^{-\frac{(k+\ell)^2}{2k\ell} \alpha^2} \cdot \frac{(k + \ell)^n}{k^{m_k^{1/n_k}}} \cdot f': f' \simeq \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{2}\right)^{k\ell} \cdot \frac{k + \ell}{\sqrt{k\ell}} \cdot \frac{1}{\sqrt{n}} \cdot e^{-\frac{(k+\ell)^2}{2k\ell} \alpha^2} \cdot \frac{(k + \ell)^n}{k^{m_k^{1/n_k}}}.
\]

Collecting terms in Corollary 3.2 we proved the following theorem (which is [1, Theorem 7.16]).

**Theorem 3.3.** With the notations of Section 2.2 we have

\[
f^\lambda \simeq c(k, \ell) \cdot D_k(\alpha_1, \ldots, \alpha_k) \cdot D_\ell(\beta_1, \ldots, \beta_\ell) \cdot e^{-\frac{k+\ell}{2} \sum \alpha_i^2} \cdot e^{+\frac{k+\ell}{2} \alpha^2} \cdot \left(\frac{1}{n_k}\right)^{(k-1)(k+2)/4} \cdot k^{n_k} \cdot \left(\frac{1}{n_\ell}\right)^{((\ell-1)(\ell+2)/4} \cdot \ell^{n_\ell},
\]

with

\[
c(k, \ell) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k+\ell-1} \cdot \left(\frac{1}{2}\right)^{k\ell} \cdot (k + \ell)^{(k^2+\ell^2)/2},
\]

and

\[
\theta(k, \ell) = \frac{1}{4} \cdot [k(k+1) + \ell(\ell + 1) - 2].
\]

**4 Asymptotics for the sums \( S_{k,\ell}^{(2z)}(n) \)**

By (1)

\[
S_{k,\ell}^{(2z)}(n) = \sum_{\lambda \in H(k,\ell,n)} (f^\lambda)^{2z}.
\]

As in [1, Theorem 7.18] (but with the additional factor \( e^{-\frac{(k+\ell)^2}{k\ell} u^2} \) in the integral), deduce

**Theorem 4.1.** With the notations of Theorem 3.3 as \( n \) goes to infinity we have

\[
S_{k,\ell}^{(2z)}(n) \simeq \left[ c(k, \ell) \cdot \left(\frac{1}{n}\right)^{\theta(k, \ell)} \cdot (k + \ell)^n \right]^{2z} \cdot (\sqrt{n})^{k+\ell-1} \cdot I(k, \ell, 2z),
\]
where $\sum x_i = u$, where
\[ I(k, \ell, 2z) = \int_{P(k, \ell)} \left[ D_k(x) \cdot D_\ell(y) \cdot e^{-\frac{k+\ell}{2} \left( \sum x_i^2 + \sum y_j^2 \right)} \right]^{2z} d^{(k+\ell-1)}(x, y), \]
and where $P(k, \ell) \subset \mathbb{R}^{k+\ell}$ is the domain
\[ P(k, \ell) = \{(x_1, \ldots, x_k; y_1, \ldots, y_\ell) \mid x_1 \geq \cdots \geq x_k; y_1 \geq \cdots \geq y_\ell; \sum x_i + \sum y_j = 0\}. \]
Note that $\sum y_j = -u$ since $\sum x_i = u$ and $\sum x_i + \sum y_j = 0$.

4.1 The evaluation of $I(k, \ell, 2z)$

Let
\[ A_{k, \ell}^{(2z)}(u) = \int_{\Omega(k, u)} \left[ D_k(x) \cdot e^{-\frac{k+\ell}{2} \sum x_i^2} \right]^{2z} d^{(k-1)}(x) \]
where $\Omega(k, u) \subset \mathbb{R}^k$, $\Omega(k, u) = \{x_1 \geq \cdots \geq x_k \mid \sum x_i = u\}$.

Similarly let
\[ B_{k, \ell}^{(2z)}(-u) = \int_{\Omega(\ell, -u)} \left[ D_\ell(y) \cdot e^{-\frac{k+\ell}{2} \sum y_j^2} \right]^{2z} d^{(\ell-1)}(y) \]
where $\Omega(\ell, -u) \subset \mathbb{R}^\ell$, $\Omega(\ell, -u) = \{y_1 \geq \cdots \geq y_\ell \mid \sum y_j = -u\}$.

We clearly have

**Lemma 4.2.** The integral \[15\] satisfies
\[ I(k, \ell, 2z) = \int_{-\infty}^{\infty} A_{k, \ell}^{(2z)}(u) \cdot B_{k, \ell}^{(2z)}(-u) \, du. \]

4.1.1 Evaluating $A_{k, \ell}^{(2z)}(u)$ and $B_{k, \ell}^{(2z)}(-u)$

Recall that $\sum x_i = u$ and make the substitution $x'_i = x_i - \frac{u}{k}$ (similarly $y'_j = y_j + \frac{u}{\ell}$), then $\sum x'_i = \sum y'_j = 0$; also $D_k(x') = D_k(x)$ and $D_\ell(y') = D_\ell(y)$. Also the Jacobians equal 1.

Now $\sum x_i^2 = \frac{u^2}{k} + \sum x_i'^2$ hence
\[ e^{-\frac{k+\ell}{2} \sum x_i^2} = e^{-\frac{k+\ell}{2} \sum x_i'^2} \cdot e^{-\frac{k+\ell}{2k} u^2}. \]

It follows that
\[ A_{k, \ell}^{(2z)}(u) = e^{-\frac{(k+\ell)u}{k}} \cdot I_k^{(2z)} \]
where
\[ I_k^{(2z)} = \int_{\Omega'(k)} \left[ D_k(x') \cdot e^{-\frac{k+\ell}{2} \sum x_i'^2} \right]^{2z} d^{(k-1)}(x') \]
and where \( \Omega'(k) = \{ x'_1 \geq \cdots \geq x'_k \mid \sum x'_i = 0 \} \).

Similarly
\[
B^{(2z)}_{k, \ell}(-u) = e^{-\frac{(k+\ell)z^2-\pi^2}{k\ell}} \cdot I^{(2z)}_{\ell}
\]
where
\[
I^{(2z)}_{\ell} = \int_{\Omega(\ell)} \left[ D_{\ell}(y') \cdot e^{-\frac{k\ell}{\pi} \sum y_j^2} \right]^{2z} d^{(\ell-1)}(y')
\]
and where \( \Omega'(\ell) = \{ y'_1 \geq \cdots \geq y'_{\ell} \mid \sum y'_{j} = 0 \} \).

By (10) we have
\[
I(k, \ell, 2z) = I^{(2z)}_{k} \cdot I^{(2z)}_{\ell} \cdot \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)z^2-\pi^2}{k\ell}} \cdot e^{-\frac{(k+\ell)z^2-\pi^2}{k\ell}} du.
\]

Note that
\[
J := \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)z^2-\pi^2}{k\ell}} \cdot e^{-\frac{(k+\ell)z^2-\pi^2}{k\ell}} du = \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2}{2z}} du.
\]
Thus we proved

**Lemma 4.3.**
\[
I(k, \ell, 2z) = \int_{-\infty}^{\infty} A^{(2z)}(u) \cdot B^{(2z)}(-u) du = I^{(2z)}_{k} \cdot I^{(2z)}_{\ell} \cdot \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2}{2z}} du.
\]

**Calculate:**

Let \( J = \int_{-\infty}^{\infty} e^{-\frac{(k+\ell)^2}{2z}} du \) and denote \( r = \sqrt{\frac{z(k+\ell)^2}{k\ell}} \). Since \( \int_{-\infty}^{\infty} e^{-r^2} du = \sqrt{\pi}/r \), hence
\[
J = \sqrt{\frac{k \cdot \ell \cdot \pi}{z(k+\ell)^2}}.
\]

This implies

**Corollary 4.4.**
\[
I(k, \ell, 2z) = \sqrt{\frac{k \cdot \ell \cdot \pi}{z(k+\ell)^2}} \cdot I^{(2z)}_{k} \cdot I^{(2z)}_{\ell} = \\
= \sqrt{\frac{k \cdot \ell \cdot \pi}{z(k+\ell)^2}} \cdot \int_{\Omega(k)} \left[ D_k(x') \cdot e^{-\frac{k}{z} \sum x_i^2} \right]^{2z} d^{(k-1)}(x') \cdot \int_{\Omega(\ell)} \left[ D_{\ell}(y') \cdot e^{-\frac{k\ell}{\pi} \sum y_j^2} \right]^{2z} d^{(\ell-1)}(y')
\]
\[
= \sqrt{\frac{k \cdot \ell \cdot \pi}{z(k+\ell)^2}} \cdot \int_{\Omega(k)} \left[ D_k(x) \cdot e^{-\frac{k}{z} \sum x_i^2} \right]^{2z} d^{(k-1)}(x) \cdot \int_{\Omega(\ell)} \left[ D_{\ell}(y) \cdot e^{-\frac{k\ell}{\pi} \sum y_j^2} \right]^{2z} d^{(\ell-1)}(y)
\]

(in the last term we replaced \( x' \) by \( x \) and \( y' \) by \( y \)).
4.2 Evaluating $I_k^{(2z)}$ and $I_\ell^{(2z)}$

The key to the following evaluation is the celebrated Selberg integral \[2], \[9\]. Let

$$I(s, \beta) := \int_{\Omega(s)} \left[ |D_k(x)| \cdot e^{-\frac{1}{2} \sum x_i^2} \right]^\beta \, dx_1 \cdots dx_{s-1}$$

and let

$$\Psi_s^{(\beta)} = (\sqrt{2 \pi})^s \cdot \beta^{s/2 - \beta s(s-1)/4} \cdot \left[ \Gamma \left(1 + \frac{1}{2} \beta \right) \right]^{-s} \cdot \prod_{j=1}^s \Gamma \left(1 + \frac{1}{2} \beta j \right),$$

then it follows from the Selberg integral that

$$I(s, \beta) = \frac{1}{s!} \cdot \frac{1}{\sqrt{s}} \cdot \sqrt{\frac{\beta}{2 \pi}} \cdot \Psi_s^{(\beta)},$$

see \[5\] (F.4.1) and (F.4.3) . For elementary proofs of both the Mehta and Selberg integrals see \[3\].

Thus, with $k = s$ and $\beta = 2z$ we deduce

**Lemma 4.5.**

$$I(k, 2z) = \frac{1}{k!} \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{\frac{2z}{2 \pi}} \cdot \Psi_k^{(2z)} =$$

$$= \frac{1}{k!} \cdot \frac{1}{\sqrt{k}} \cdot \sqrt{\frac{z}{\pi}} \cdot (\sqrt{2 \pi})^k \cdot (2z)^{-k/2 - 2zk(k-1)/4} \cdot \left[ \Gamma (1 + z) \right]^{-k} \cdot \prod_{j=1}^k \Gamma (1 + z j).$$

We rewrite \([17]\) as

$$I_k^{(2z)} = \int_{\Omega(k)} \left[ D_k(x_1, \ldots, x_k) \cdot e^{-\frac{k+\ell}{2} \sum x_i^2} \right]^{2z} \, d^{(k-1)}(x),$$

and make the transition from \([18]\) to \([21]\) as follows.

**Lemma 4.6.** With $I(k, 2z)$ given by \([18]\),

$$I_k^{(2z)} = \left( \frac{1}{k + \ell} \right)^{\frac{k(k-1)z + k-1}{2}} \cdot I(k, 2z)$$

**Proof.** In \([18]\) substitute $x_i = \sqrt{\frac{1}{k+\ell}} \cdot y_i$, then

$$D_k(x) = \left( \frac{1}{\sqrt{k+\ell}} \right)^{\frac{k(k-1)}{2}} D_k(y) = \left( \frac{1}{k + \ell} \right)^{\frac{k(k-1)}{2}} D_k(y)$$
and
\[ d^{(k-1)}(x) = \left( \frac{1}{\sqrt{k + \ell}} \right)^{k-1} d^{(k-1)}(y). \]

Thus
\[
I_k^{(2z)} = \left( \frac{1}{k + \ell} \right)^{\frac{k(k-1)x + k-1}{2}} \cdot \int_{\Omega(k)} D_k(y_1, \ldots, y_k) \cdot e^{-\frac{1}{2} \sum y_i^2} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^k \cdot (2z)^{-k/2-zk(k-1)/2} \cdot [\Gamma(1+z)]^{-k} \prod_{i=1}^{k} \Gamma(1+z_i) \cdot I(k, 2z).
\]

Together with Lemma 4.5 it implies

**Corollary 4.7.**

\[
I_k^{(2z)} = \left( \frac{1}{k + \ell} \right)^{\frac{\ell(\ell-1)x + \ell-1}{2}} \cdot \frac{1}{\ell!} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^\ell \cdot (2z)^{-\ell/2-\ell z(\ell-1)/2} \cdot [\Gamma(1+z)]^{-\ell} \cdot \prod_{j=1}^{\ell} \Gamma(1+z_j).
\]

Similarly
\[
I_\ell^{(2z)} = \left( \frac{1}{k + \ell} \right)^{\frac{\ell(\ell-1)x + \ell-1}{2}} \cdot \frac{1}{\ell!} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^\ell \cdot (2z)^{-\ell/2-\ell z(\ell-1)/2} \cdot [\Gamma(1+z)]^{-\ell} \cdot \prod_{j=1}^{\ell} \Gamma(1+z_j).
\]

By Corollary 4.3 we get that
\[
I_k^{(2z)} \cdot I_\ell^{(2z)} = \left( \frac{1}{k + \ell} \right)^{\frac{k(k-1)x + k-1}{2}} \cdot \frac{1}{k!} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^k \cdot (2z)^{-k/2-zk(k-1)/2} \cdot [\Gamma(1+z)]^{-k} \cdot \prod_{i=1}^{k} \Gamma(1+z_i) \cdot \left( \frac{1}{k + \ell} \right)^{\frac{\ell(\ell-1)x + \ell-1}{2}} \cdot \frac{1}{\ell!} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^\ell \cdot (2z)^{-\ell/2-\ell z(\ell-1)/2} \cdot [\Gamma(1+z)]^{-\ell} \cdot \prod_{j=1}^{\ell} \Gamma(1+z_j) = \left( \frac{1}{k + \ell} \right)^{\frac{k(k-1)x + k-1}{2} + \frac{\ell(\ell-1)x + \ell-1}{2}} \cdot \frac{1}{k!} \cdot \frac{1}{\ell!} \cdot \frac{z}{\pi} \left( \sqrt{2\pi} \right)^{k+\ell} \cdot (2z)^{-\frac{1}{2} \left[ (k(k-1) + \ell(\ell-1)) + k + \ell - 2 \right]} \cdot [\Gamma(1+z)]^{-k-\ell} \cdot \prod_{i=1}^{k} \Gamma(1+z_i) \cdot \prod_{j=1}^{\ell} \Gamma(1+z_j).\]
Therefore

\[I(\ell, 2z) = \frac{1}{k + \ell} \cdot \sqrt{\frac{k \log \pi}{z}} \cdot \left( \frac{1}{k + \ell} \right)^{\frac{1}{2} \left[ z - (k(l-1) + m(l-1)) + k + \ell - 2 \right]} \cdot \frac{1}{k! \cdot \ell!} \cdot \frac{z}{\sqrt{2\pi}} \cdot \left( \sqrt{\frac{z}{2\pi}} \right)^{k + \ell} \cdot (2z)^{-\frac{1}{4} \left[ z - (k(l-1) + m(l-1)) + k + \ell \right]} \cdot \left( \Gamma \left( 1 + z \right) \right)^{-k - \ell} \cdot \prod_{i=1}^{k} \Gamma (1 + zi) \cdot \prod_{j=1}^{\ell} \Gamma (1 + zj) \]

so, after cancellations,

\[I(\ell, 2z) = \left( \frac{1}{k + \ell} \right)^{\frac{1}{2} \left[ z - (k(l-1) + m(l-1)) + k + \ell \right]} \cdot \frac{1}{k! \cdot \ell!} \cdot \frac{z}{\sqrt{2\pi}} \cdot \left( \sqrt{\frac{z}{2\pi}} \right)^{k + \ell} \cdot (2z)^{-\frac{1}{4} \left[ z - (k(l-1) + m(l-1)) + k + \ell \right]} \cdot \left( \Gamma \left( 1 + z \right) \right)^{-k - \ell} \cdot \prod_{i=1}^{k} \Gamma (1 + zi) \cdot \prod_{j=1}^{\ell} \Gamma (1 + zj) .\]

Combined with Theorem 4.1, we have proved

**Theorem 4.8.** *(This is also Theorem 1.1)*

\[S_{k, \ell}^{(2z)} \approx a(k, \ell, 2z) \cdot \left( \frac{1}{n} \right)^{g(k, \ell, 2z)} \cdot (k + \ell)^{2n}, \]

where

\[g(k, \ell, 2z) = \frac{1}{2} \cdot (z \cdot [k(1 + \ell + 1) - 2] - (k + \ell - 1)) \]

and

\[a(k, \ell, 2z) = [c(k, \ell)]^{2z} \cdot I(\ell, 2z) = \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{k + \ell - 1} \cdot \frac{1}{2} \cdot (k + \ell)^{(k^2 + \ell^2)/2} \right]^{2z} \cdot \left( \frac{1}{k + \ell} \right)^{\frac{1}{2} \left[ z - (k(l-1) + m(l-1)) + k + \ell \right]} \cdot \frac{1}{k! \cdot \ell!} \cdot \sqrt{\frac{z}{\sqrt{2\pi}}} \cdot \left( \sqrt{\frac{z}{2\pi}} \right)^{k + \ell} \cdot (2z)^{-\frac{1}{4} \left[ z - (k(l-1) + m(l-1)) + k + \ell \right]} \cdot \left( \Gamma \left( 1 + z \right) \right)^{-k - \ell} \cdot \prod_{i=1}^{k} \Gamma (1 + zi) \cdot \prod_{j=1}^{\ell} \Gamma (1 + zj) .\]
5 Some special cases

5.1 The case $2z = 1$

Here

$$g(k, \ell, 1) = \frac{1}{4} \cdot (k(k - 1) + \ell(\ell - 1)).$$

We calculate $a(k, \ell, 1)$. Recall that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(x+1) = x\Gamma(x)$ so $\Gamma(1+1/2) = \sqrt{\pi}/2$. Thus, with $x = 1/2$,

$$(\Gamma(1 + 1/2))^{-k-\ell} = \left(\frac{2}{\sqrt{\pi}}\right)^{k+\ell},$$

so

$$a(k, \ell, 1) =$$

$$= \left[ \left(\frac{1}{\sqrt{2\pi}}\right)^{k+\ell-1} \cdot \left(\frac{1}{2}\right)^{k\ell} \cdot (k + \ell)^{(k^2 + \ell^2)/2} \right] \cdot \left(\frac{1}{k + \ell}\right) \cdot \frac{1}{k! \cdot \ell!} \cdot \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{2}{\sqrt{\pi}}\right)^{k+\ell} \cdot \left(\frac{1}{\sqrt{2\pi}}\right)^{k+\ell} \cdot \prod_{i=1}^{k} \Gamma(1 + i/2) \cdot \prod_{j=1}^{\ell} \Gamma(1 + j/2) =$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{2k\ell - 2k - 2\ell - 1} \cdot \left(\frac{1}{\sqrt{\pi}}\right)^{k+\ell-1} \cdot (k + \ell)^{\frac{1}{2}[k(k-1)+\ell(\ell-1)]} \cdot \frac{1}{k! \cdot \ell!} \cdot \frac{1}{\sqrt{2\pi}} \cdot \prod_{i=1}^{k} \Gamma(1 + i/2) \cdot \prod_{j=1}^{\ell} \Gamma(1 + j/2).$$

The factor $1/\sqrt{2\pi}$ cancels and we have

Theorem 5.1.

$$S_{k,\ell}^{(1)}(n) \simeq a(k, \ell, 1) \cdot \left(\frac{1}{n}\right)^{\frac{1}{2}(k(k-1)+\ell(\ell-1))} \cdot (k + \ell)^{n}$$

where

$$a(k, \ell, 1) = \left(\frac{1}{2}\right)^{k\ell - k - \ell} \cdot \left(\frac{1}{\sqrt{\pi}}\right)^{k+\ell} \cdot (k + \ell)^{\frac{1}{2}[k(k-1)+\ell(\ell-1)]} \cdot \frac{1}{k! \cdot \ell!} \cdot \prod_{i=1}^{k} \Gamma(1 + i/2) \cdot \prod_{j=1}^{\ell} \Gamma(1 + j/2).$$

5.1.1 A case with $2z = 1$

1. $k = \ell = 1$. Then by Theorem 5.1 $a(1, 1, 1) = 1/2$ and $g(1, 1, 1) = 0$, so

$$S_{1,1}^{(1)}(n) \simeq \frac{1}{2} \cdot 2^n.$$
On the other hand we know that

\[ S_{1,1}^{(1)}(n) = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}, \]

which verifies Theorem 5.1 (or Theorem 1.1) in that case.

5.1.2 Using ”Mathematica”

For small \( k \) and \( \ell \) it is possible to write an explicit formula for, say, \( S_{k,\ell}^{(1)}(n) \). By Theorem 1.1

\[ S_{k,\ell}^{(1)}(n) \approx A(k, \ell, n). \]

Now form the ratio \( S_{k,\ell}^{(1)}(n)/A(k, \ell, n) \). Using, say, ”Mathematica”, calculate that ratio for increasing values of \( n \), verifying that these values become closer and closer to 1 as \( n \) increases. This indicates the validity of Theorem 1.1. We demonstrate this in the case \( k = 2 \) and \( \ell = 1 \).

Here \( \frac{1}{4} \cdot (k(k-1) + \ell(\ell-1)) = 1/2 \) and \( a(2, 1, 1) = \frac{1}{4} \cdot \sqrt{\frac{3}{\pi}} \). Thus by Theorem 1.1 (or 5.1)

\[ S_{2,1}^{(1)} \approx \frac{1}{4} \cdot \sqrt{\frac{3}{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 3^n. \]

Next we deduce a relatively simple formula for \( S_{2,1}^{(1)}(n) \). By [6]

\[ S_{2,1}^{(1)}(n) = S(2, 1; n) = \]

\[ = \frac{1}{4} \left( \sum_{r=0}^{n-1} \binom{n-r}{\frac{n-r}{2}} \left( \sum_{r=0}^{n-1} \binom{n-r}{j} \right) \right) + \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k) \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} + 1. \]

Also for \( n \geq 2 \) it can be proved by the WZ method [3, 10] that

\[ 2 \sum_{j=1}^{n} \binom{n}{j} \binom{n-j}{j} = \]

\[ = \sum_{r=0}^{n-1} \binom{n-r}{\frac{n-r}{2}} \left( \sum_{r=0}^{n-1} \binom{n-r}{j} \right) + \sum_{k=1}^{\frac{n}{2}} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)} \]

(22)

(for an elementary proof of Equation (22) (due to I. Gessel), see [8]). Hence

\[ S_{2,1}^{(1)} \approx \frac{1}{2} \cdot \sum_{j=1}^{n} \binom{n}{j} \binom{n-j}{j} \]

If indeed

\[ S_{2,1}^{(1)} \approx \frac{1}{4} \cdot \sqrt{\frac{3}{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 3^n \]
then

\[ \frac{2 \cdot \sqrt{n}}{3n} \cdot \sum_{j \geq 1} \binom{n}{j} \left(\frac{n-j}{j}\right) \approx \sqrt{\frac{3}{\pi}} = 0.977205. \] (23)

Indeed, "Matematica" gives the following values \( \text{lhs}(n) \) for the left hand side of (23):

\[
\text{lhs}(10) = 0.958821, \quad \text{lhs}(100) = 0.975373, \quad \text{lhs}(1000) = 0.977022,
\]

\[
\text{lhs}(2000) = 0.977113, \quad \text{lhs}(3000) = 0.977144 \quad \text{etc.}
\]

This, in a sense, verifies Theorem 1.1 in this case.

5.1.3 A special case with \( z = 1 \)

Let \( k = \ell = z = 1 \). By Theorem 4.8

\[ S_{1,1}^{(2)}(n) \approx a \cdot \left(\frac{1}{n}\right)^g \cdot 2^{2n} \quad \text{where} \quad a = \frac{1}{4\sqrt{\pi}} \quad \text{and} \quad g = \frac{1}{2}. \] (24)

Now

\[ S_{1,1}^{(2)}(n) = \sum_{j \geq 0} \binom{n-1}{j}^2 = \binom{2(n-1)}{n-1} \]

and by Stirling’s formula

\[ \binom{2(n-1)}{n-1} \approx \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \cdot 4^{n-1}, \]

agreeing with (24), hence again, Theorem 1.4 is verified in this case.

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A. Berele, Department of Mathematics, DePaul University, Chicago, Ill 60614
demail: aberele at condor.depaul.edu

A. Regev, Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: amitai.regev at weizmann.ac.il