GEOMETRY OF GENERATING FUNCTIONS AND LAGRANGIAN SPECTRAL INVARIANTS

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Abstract. Partially motivated by the study of topological Hamiltonian dynamics, we prove various $C^0$-aspects of the Lagrangian spectral invariants and the basic phase functions $f_H$, that is, a natural graph selector constructed by Lagrangian Floer homology of $H$ (relative to the zero section $o_N$). In particular, we prove that

$$\gamma^{log}(\phi_H(o_N)) := \rho^{log}(H; 1) - \rho^{log}(H; [pt]) = 0$$

as $\phi_H \to id$, provided $H$’s satisfy supp $X_H \subset DR(T^* N) \setminus o_B$ for some $R > 0$ and a closed subset $B \subset N$ with nonempty interior.

We also study the relationship between $f_H$ and $\rho^{log}(H; 1)$ and prove a structure theorem of the micro-support of the singular locus Sing$(\sigma_H)$ of the function $f_H$. Based on this structure theorem and a classification theorem of generic Lagrangian singularity in dim $N = 2$ obtained by Arnold’s school, we define the notion of cliff-wall surgery when dim $N = 2$: the surgery replaces a multi-valued Lagrangian graph $\phi_H(o_N)$ by a piecewise-smooth Lagrangian cycle that is canonically constructed out of the single valued branch $\Sigma_H := \text{Graph } d \phi_H \subset \phi_H(o_N)$ defined on an open dense subset of $N \setminus \text{Sing}(\sigma_H)$ of codimension 1.

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1. Introduction

We always assume that the ambient manifold $M$ or $N$ are connected throughout the entire paper.

1.1. Weak Hamiltonian topology of $Ham(M, \omega)$. In [OM], Müller and the author introduced the notion of Hamiltonian topology on the subset of the space $P(Homeo(M), id)$ of continuous paths on $Homeo(M)$ consisting of Hamiltonian paths $\lambda : [0,1] \to Symp(M, \omega)$ with $\lambda(t) = \phi_t^H$ for some time-dependent Hamiltonian $H$. We denote this subset by

$$P^h(\text{Symp}(M, \omega), id).$$

We would like to emphasize that we do not assume that $H$ is normalized unless otherwise said explicitly. This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proof of the various theorems of this paper.

In this subsection, we first recall the definition from [OM] of the Hamiltonian topology mostly restricted on the open manifold $T^*N$. While [OM] considers strong Hamiltonian topology, except Remark 3.27 therein, the more relevant topology in the present paper will be the weak Hamiltonian topology. We first recall its definition.

For a given continuous function $h : M \to \mathbb{R}$, we denote

$$\text{osc}(h) = \max h - \min h.$$ 

We define the $C^0$-distance $\overline{d}$ on $Homeo(M)$ by the symmetrized $C^0$-distance

$$\overline{d}(\phi, \psi) = d_{C^0}(\phi, \psi) + d_{C^0}(\phi^{-1}, \psi^{-1})$$

and the $C^0$-distance on $P(Homeo(M), id)$, again denoted by $\overline{d}$, by

$$\overline{d}(\lambda, \mu) = \max_{t \in [0,1]} \overline{d}(\lambda(t), \mu(t)).$$
This induces the corresponding \( C^0 \)-distance on \( \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega), id) \). The Hofer length of Hamiltonian path \( \lambda = \phi_H \) is defined by

\[
\text{leng}(\lambda) = \int_0^1 \text{osc}(H_t) \, dt = \| H \|
\]

Following the notations of [OM], we denote by \( \phi_H \) the Hamiltonian path \( \phi_H : t \mapsto \phi^t_H; [0,1] \to \text{Ham}(M,\omega) \).

**Definition 1.1.** Let \((M,\omega)\) be an open symplectic manifold. Let \( \lambda, \mu \) be smooth Hamiltonian paths with compact support in \( \text{Int} M \). The weak Hamiltonian topology is the metric topology induced by the metric

\[
d_{\text{ham}}(\lambda, \mu) := d(\lambda(1), \mu(1)) + \text{leng}(\lambda^{-1} \mu).
\]

(1.1)

1.2. **Hamiltonian \( C^0 \)-topology on \( \mathfrak{Ho}(o_N; T^*N) \).** Let \( N \) be a closed smooth manifold. We equip the cotangent bundle \( T^*N \) with the Liouville one-form \( \theta \) defined by

\[
\theta(x)(\xi) = \pi^*(d\pi(\xi)), \quad x = (q,p) \in T^*N.
\]

The canonical symplectic form \( \omega_0 \) on \( T^*N \) is defined by

\[
\omega_0 = -d\theta = \sum_{k=1}^n dq^k \wedge dp_k
\]

(1.2)

where \((q^1, \ldots, q^n, p_1, \ldots, p_n)\) is the canonical coordinates of \( T^*N \) associated to the coordinates \((q^1, \ldots, q^n)\) of \( N \).

Consider Hamiltonian \( H = H(t,x) \) such that \( H_t \) is asymptotically constant, i.e., the one whose Hamiltonian vector field \( X_H \) is compactly supported. We define

\[
\text{supp}_{\text{asc}} H = \text{supp} X_H := \bigcup_{t \in [0,1]} X_H.
\]

For each given \( K, R \in \mathbb{R}^+ \), we define

\[
\mathcal{PC}^\infty_{R,K} = \{ H \in C^\infty([0,1] \times T^*N, \mathbb{R}) \mid \text{supp}_{\text{asc}} H \subset D_R(T^*N), \| H \| \leq K \}
\]

(1.3)

which provides a natural filtration of the space \( C^\infty([0,1] \times T^*N, \mathbb{R}) \). We also denote

\[
\mathcal{PC}^\infty_R = \bigcup_{K \in \mathbb{R}^+} \mathcal{PC}^\infty_{R,K}, \quad \mathcal{PC}^\infty_{\text{asc}} = \bigcup_{R \geq 0} \mathcal{PC}^\infty_R.
\]

(1.4)

By definition, each element \( H_t \) is independent of \( x = (q,p) \) if \( |p| \) is sufficiently large and so carries a smooth function \( c_\infty : [0,1] \to \mathbb{R} \) defined by

\[
c_\infty(t) = H(t, \infty).
\]

Therefore we have the natural evaluation map

\[
\pi_\infty : \mathcal{PC}^\infty_{\text{asc}} \to C^\infty([0,1], \mathbb{R}).
\]

For each given smooth function \( c : [0,1] \to \mathbb{R} \), we denote

\[
\mathcal{PC}^\infty_{\text{asc}c} := \pi_*^{-1}(c).
\]

(1.5)

We then introduce the space of Hamiltonian deformations of the zero section and denote

\[
\mathfrak{Ho}(o_N; T^*N) = \{ \phi_H(o_N) \mid H \in \mathcal{PC}^\infty_{\text{asc}} \}. 
\]
following the terminology of \[W\], and
\[
\mathcal{J}(o_N;D^R(T^*N)) = \{\phi^1_H(o_N) \mid H \in \mathcal{P}C^\infty_{R,T}\}
\]
\[
\mathcal{J}K(o_N;D^R(T^*N)) = \{\phi^1_H(o_N) \mid H \in \mathcal{P}C^\infty_{R,K}\}. \tag{1.6}
\]

Now we equip a topology with \(\mathcal{J}(o_N;T^*N)\). One needs to pay some attention in finding the correct definition of the topology suitable for the study of Hamiltonian geometry of the set \(\mathcal{J}(o_N;T^*N)\). For this purpose, we introduce the following measurement of \(C^0\)-fluctuation of the Hamiltonian diffeomorphism of \(\phi^1_P\) along the zero section \(o_N \subset T^*N\),
\[
osc_{C^0}(\phi^1_P; o_N) := \max \left\{ \max_{x \in o_N} d(\phi^1_P(x), x), \max_{x \in o_N} d((\phi^1_P)^{-1}(x), x) \right\}.
\]
Using this we introduce the following restricted \(C^0\)-distance

**Definition 1.2.** Let \(L_0, L_1 \in \mathcal{J}(o_N; T^*N)\) with \(L_0 = \phi^1_P(0_N), L_1 = \phi^1_P(0_N)\). We define the following distance function
\[
d_{\mathcal{J}(o_N; T^*N)}(L_0, L_1) = \inf_{\{H: \phi^1_H(L_0) = L_1\}} \max \left\{ osc_{C^0}((\phi^1_P)^{-1}\phi^1_H\phi^1_P(o_N); o_N) \right\}, \tag{1.7}
\]
on \(\mathcal{J}(o_N; D^R(T^*N))\), which induces the metric topology thereon. We equip with \(\mathcal{J}(o_N; T^*N)\) the direct limit topology of \(\mathcal{J}(o_N; D^R(T^*N))\) as \(R, K \to \infty\) and call it the Hamiltonian \(C^0\)-topology of \(\mathcal{J}(o_N; T^*N)\).

For the main theorems proved in the present paper, we will also need to consider the following subset of Hamiltonian functions \(H\).

Let \(B \subset N\) be a given closed subset and \(o_B \subset o_N\) the corresponding subset of the zero section. Denote by \(T\) an open neighborhood of \(o_B\) in \(T^*N\). We define
\[
\mathcal{P}C^\infty_{asc,B} = \{H \in C^\infty([0, 1] \times T^*N, \mathbb{R}) \mid \text{supp} X_H \subset (T^*N \setminus B) \text{ is compact} \}. \tag{1.8}
\]
We have the filtration
\[
\mathcal{P}C^\infty_{asc,B} = \bigcup_{T \supset B} \bigcup_{R > 0} \mathcal{P}C^\infty_{R,T}
\]
over the set of open neighborhoods \(T\) of \(B\) and the positive numbers \(R > 0\) where
\[
\mathcal{P}C^\infty_T = \{H \in \mathcal{P}C^\infty_{asc,B} \mid \phi^1_H \equiv \text{id on } T\}. \tag{1.9}
\]
Here we would like to emphasize that the support condition on \(T \supset o_B\) is imposed only for the time-one map \(\phi^1_H\), but not for the whole path \(\phi_H\). This indicates relevance of the following discussion to the weak Hamiltonian topology described above.

Similarly as \(\mathcal{P}C^\infty_{R,K}\) above, we define \(\mathcal{P}C^\infty_{R,K,T}\). We define \(\mathcal{J}(o_N; T^*N)\) to be the subset
\[
\mathcal{J}(o_N; T^*N) = \{\phi^1_H(o_N) \mid H \in \mathcal{P}C^\infty_{asc,B}\}.
\]
This has the filtration
\[
\mathcal{J}(o_N; T^*N) = \bigcup_{K \geq 0} \bigcup_{T \supset B} \mathcal{J}(o_N; T^*N; T)
\]
where
\[
\mathcal{J}(o_N; D^R(T^*N)) = \{\phi^1_H(o_N) \mid H \in \mathcal{P}C^\infty_{R,K,T}\}.
\]
Definition 1.3. Equip with $\mathcal{J}_0^K_B(o_N; T^* N)$ the subspace topology of the Hamiltonian $C^0$-topology of $\mathcal{J}_0(o_N; T^* N)$. We then put on $\mathcal{J}_0^K_B(o_N; T^* N)$ the direct limit topology of $\mathcal{J}_0^K_B(o_N; T^* N)$ over $T \supset o_B$ and $K \geq 0$. We call this topology the Hamiltonian $C^0$-topology of $\mathcal{J}_0^K_B(o_N; T^* N)$.

Unravelling the definition, we can rephrase the meaning of the convergence $L_i \to L$ in $\mathcal{J}_0^K_B(o_N; T^* N)$ into the existence of $R, K > 0$, $T \supset o_B$ and a sequence $H_i$ such that $L_i = \phi^1_{H_i}(L)$ and

1. $\|H_i\| \leq K$ for all $i$,
2. $\text{supp} X_{H_i} \subset D^R(T^* N) \setminus o_B$ for all $i$,
3. $\phi^1_{H_i} \equiv \text{id}$ on $T$ for all $i$,
4. $d^\text{ham}_{C^0}(L_i, L) \to 0$ as $i \to \infty$.

Remark 1.4. (1) We refer to the proof of Lemma 7.5 and Remark 7.2 for the reason to take these particular support hypotheses (2), (3) imposed in our definition of Hamiltonian $C^0$-topology of $\mathcal{J}_0^K_B(o_N; T^* N)$. This topology may be regarded as the Lagrangian analog to the above mentioned weak Hamiltonian topology and seems to be the weakest possible topology with respect to which one can prove the $C^0$-continuity of spectral capacity $\gamma_\text{lag}$ which is stated in Theorem 1.1 below.

(2) The Lagrangianization Graph $\phi^1_F$ of Hamiltonian $F : [0, 1] \times M \to \mathbb{R}$ with $\text{supp} F \subset M \setminus B$, there exists an open neighborhood $U \supset B$ such that $\text{supp} F \subset M \setminus U$. Therefore provided the $C^0$-distance of $\mathcal{J}(\phi_F, \text{id}) =: \epsilon$ is so small that its graph is contained in a Weinstein neighborhood of the diagonal, such a graph will automatically satisfy

$$\phi^1_F(o_{\Delta B}) \subset T_\epsilon; \quad F(t, x) := F(t, x), \quad x = (x, y)$$

where $T_\epsilon$ is the $\epsilon$-neighborhood of $o_{\Delta B}$ in $T^* \Delta$ and hence is automatically contained in $\mathcal{J}_0^K_B(o_{\Delta \Delta}, T^* \Delta)$.

1.3. Lagrangian spectral invariants. For any given time-dependent Hamiltonian $H = H(t, x)$, the classical action functional on the space

$$\mathcal{P}(T^* N) := C^\infty([0, 1], T^* N)$$

is defined by

$$A^d_H(\gamma) = \int_0^1 \gamma^* \theta - \int_0^1 H(t, \gamma(t)) \, dt.$$ 

We define the subset $\mathcal{P}(T^* N; o_N)$ by

$$\mathcal{P}(T^* N; o_N) = \{ \gamma : [0, 1] \to T^* N \mid \gamma(0) \in o_N \}.$$ 

The assignment $\gamma \mapsto \pi(\gamma(1))$ defines a fibration

$$\mathcal{P}(T^* N; o_N) \to o_N \cong N$$

with fiber at $q \in N$ given by

$$\mathcal{P}(T^* N; o_N, T_q^* N) := \{ \gamma : [0, 1] \to T^* N \mid \gamma(0) \in o_N, \gamma(1) \in T_q^* N \}.$$ 

For given $x \in L_H$, we denote the Hamiltonian trajectory

$$z^H_x(t) = \phi^1_H((\phi^1_H)^{-1}(x))$$

which is a Hamiltonian trajectory such that, by definition,

$$z^H_x(0) = o_N, \quad z^H_x(1) = x.$$ (1.10)
We denote $L_H = \phi_1^H(o_N)$ and by $i_H : L_H \rightarrow T^*N$ the inclusion map.

Motivated by Weinstein’s observation that the action functional

$$\mathcal{A}^cl_H : \mathcal{P}(T^*N; o_N) \rightarrow \mathbb{R}$$

can be interpreted as the canonical generating function of $L_H$, the present author constructed a family of spectral invariants of $L_H$ by performing a mini-max theory via the chain level Floer homology theory in [Oh2, Oh3]. Indeed, the function defined by

$$h_H(x) = \mathcal{A}^cl_H(z^H_x)$$

is a canonical generating function of $L_H$ in that

$$i_H^* \theta = dh_H.$$ (1.12)

We call $h_H$ the basic generating function of $L_H$. As a function on $N$, not on $L_H$, it is a multi-valued function. Similarly, one may regard $N \rightarrow \phi_1^H(o_N)$ as a multi-valued section of $T^*N$.

By considering the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases}
\frac{du}{d\tau} + J(\frac{du}{dt} - X^H_H(u)) = 0 \\
u(\tau, 0), u(\tau, 1) \in o_N,
\end{cases}$$

and applying a chain-level Floer mini-max theory, the author [Oh3] defined a homologically essential critical value, denoted by $\rho(H; a)$ associated to each cohomology class $a \in H^*(N)$. (A similar construction using the generating function method was earlier given by Viterbo [V1] and it is shown in [M, MO] that both invariants coincide modulo a normalization constant.) The number $\rho(H; a)$ depends on $H$, not just on $L_H = \phi_1^H(o_N)$.

1.4. **Statement of main results.** We will be particularly interested in the two spectral invariants $\rho^{\text{lag}}(F; 1)$, $\rho^{\text{lag}}(F; [pt]^\#)$ and their difference $\rho^{\text{lag}}(F; 1) - \rho^{\text{lag}}(F; [pt]^\#)$. This difference does not depend on the choice of normalization mentioned above. Therefore we can define a function

$$\gamma^{\text{lag}} : \mathfrak{so}(o_N; T^*N) \rightarrow \mathbb{R}$$

unambiguously by setting

$$\gamma^{\text{lag}}(L; o_N) := \rho^{\text{lag}}(F; 1) - \rho(F; [pt]^\#)$$

for $L = \phi_1^H(o_N)$. We call this function the spectral capacity of $L$ (relative to the zero section $o_N$). (See [V1], [Oh3].)

We denote by $\gamma^{\text{lag}}_B$ the restriction of $\gamma^{\text{lag}}$ to the subset $\mathfrak{so}_B(o_N; T^*N)$. The following Hamiltonian continuity result is the Lagrangian analog to Corollary 1.2 of [Sey1].

**Theorem 1.1** (Theorem 7.2). Let $N$ be a closed manifold. Then the function $\gamma^{\text{lag}}_B$ is continuous on $\mathfrak{so}_B(o_N; T^*N)$ with respect to the Hamiltonian $C^0$-topology defined above.

The following is a very interesting open question on the Hamiltonian $C^0$-topology.

**Question 1.5.** Is the full function $\gamma^{\text{lag}} : \mathfrak{so}(o_N; T^*N) \rightarrow \mathbb{R}$ continuous (without restricting to $\mathfrak{so}_B(o_N; T^*N)$ with $B$ having non-empty interior)?
The question seems to be an important matter to understand in $C^0$ symplectic topology. Indeed the affirmative answer to the question is a key ingredient in relation to Viterbo’s symplectic homogenization program \([V3]\). The question is sometimes called Viterbo’s conjecture. We refer to Theorem \([7.4]\) for the more precise statement on the relationship between the Hamiltonian $C^0$-distance $d_{C^0}$ and the spectral capacity $\gamma_B^{log}(\phi_F(o_N))$ and the support conditions (2), (3) of the Hamiltonian path $\phi_F$ given in Definition \([1.3]\).

To properly handle the individual number $\rho^\log(F;1)$, not just the difference of $\rho^\log(F;1)$ and $\rho^\log(F;[pt])$, and relate it to the Lagrangian submanifold $L_F = \phi_F^1(o_N)$ itself, not to the function $F$, we need to put an additional normalization condition relative to $L_F$. In this regard, it is useful to take the point of view $\{G1, A3, ZR\}$ for detailed study of the Maxwell set. (See \([W]\) for the more precise statement on the relationship between the Hamiltonian $C^0$-distance $d_{C^0}$ and the spectral capacity $\gamma_B^{log}(\phi_F(o_N))$ and the support conditions (2), (3) of the Hamiltonian path $\phi_F$ given in Definition \([1.3]\)).

The next result concerns an enhancement of the construction of basic phase function $f$ given in Definition \([1.1]\) and $L$ respectively.

\[ \int_M F(t,x)\omega^n = 0. \]

The next theorem concerns the structure of $\text{Sing}(\phi_F)$ in the weak Hamiltonian topology given in Definition \([1.3]\) and $L_i = \phi_F^{1}(o_N)$. Then $(\sigma_F, f_{F_i})$ converges uniformly in $J^1(N)$, whose limit defines a single-valued continuous section of $J^1(M)$ on $N \setminus \text{Sing}(\phi_F)$.

Here we define

\[ \text{Sing}(\phi_F) := \{ q \in N \mid f_F \text{ is not differentiable at } q \} \]

and call it the singular locus of $f_F$. It follows from definition that $\text{Sing}(\phi_F)$ is a subset of the so called Maxwell set of the Lagrangian projection $\phi_F^1(o_N) \to N$. (See \([G1, A3, ZR]\) for detailed study of the Maxwell set.)

We first note that for a generic choice of $F$, $\text{Sing}(\phi_F)$ is decomposed into the union of smooth manifolds

\[ \text{Sing}(\phi_F) = \bigcup_{k=1}^n S_k(\phi_F) \]

where $S_k(\phi_F)$ is the stratum of codimension $k$ in $N$. Along each connected component of the codimension one strata $S_1(\phi_F)$, $\Sigma_F$ has two branches. We denote by $f_F^\pm$ the restrictions of $f_F$ in a neighborhood of the component in each branch respectively.

The next theorem concerns the structure of $\text{Sing}(\phi_F)$ in the micro-local level.
Theorem 1.3 (Theorem 4.1). Let \( q \in S_1(F) \). Then
\[
df^-_F(q) - df^+_F(q) \in T^*_q N,
\]
which is contained in the conormal space \( \nu^*_q[S_1(\sigma_F); N] \subset T^*_q N \).

In dimension 2, a complete description of generic singularities of the Lagrangian projection is available (see \cite{G1, A3, ZR} for the precise statement). Based on this generic description of the singularities, we can precisely define the notion of cliff-wall surgery in dimension 2, which replaces the multi-valued graph \( \phi_1(F_{\partial N}) \) by a rectifiable Lagrangian cycle. A finer structure theorem is needed to perform similar surgery in higher dimension which will be studied elsewhere. It appears to the author that these results seem to carry some significance in relation to \( C^0 \)-symplectic topology and Hamiltonian dynamics, which may be worthwhile to pursue further in the future.

Finally we prove the following inequality between the basic phase function and the Lagrangian spectral invariants. The inequality stated in this theorem is closely related to Proposition 5.1 of \cite{V1}, whose statement and proof were formulated in terms of the generating function.

Theorem 1.4 (Theorem 6.1). For any Hamiltonian \( F = F(t, x) \), we have
\[
\rho^{\text{lag}}(F; [pt]^q) \leq \min f_F, \quad \max f_F \leq \rho^{\text{lag}}(F; 1).
\]

The proof of the second inequality uses a judicious usage of the triangle product in Lagrangian Floer homology \cite{Oh3, Se, FOOO} after a careful consideration of normalization problem in section 5.4. We would like to emphasize that the issue of normalization problem concerning \( \rho^{\text{lag}}(F; 1) \) is a delicate one when one would like to regard \( \rho^{\text{lag}}(F; 1) \) as an invariant attached to the Lagrangian submanifold itself, not just to the Hamiltonian \( F \). Once the second inequality is established, the first one easily follows from this and the behavior of spectral invariants \( \rho^{\text{lag}}(:, [q]) \) under the duality map \( F \mapsto F^*(t, x) = -F(t, r(x)) \) induced by the anti-symplectic reflection \( r : T^* N \to T^* N, r(q, p) = (q, -p) \) for \( x = (q, p) \) similarly as done in \cite{Oh3} for the duality map \( F \mapsto \tilde{F}(t, x) = -F(1-t, x) \). (We thank Seyfaddini for pointing out to us \cite{Sey2} that the first inequality should also hold in the presence of the second inequality in Theorem 6.1.) See also \cite{V1} for the similar consideration of this reflection map in the context of generating function techniques.

The research performed in this paper is partially motivated by the study of topological Hamiltonian dynamics and its applications to the problem of simplicity question on the area-preserving homeomorphism group of the 2-disc. We anticipate that these studies play some important role in the study of homotopy invariance of Hamiltonian spectral invariant function \( \phi_F \mapsto \rho(F; a) \) for a topological Hamiltonian path \( \phi_F \) in the sense of \cite{OM, Oh7} on any closed symplectic manifolds \((M, \omega)\). It should also be regarded as a natural continuation of the author’s study of Lagrangian spectral invariants performed in \cite{Oh2, Oh3}.

We thank F. Zapolsky for attracting our attention to the preprint \cite{MVZ} from which we have learned the Lagrangian version of the optimal triangle inequality, and S. Seyfaddini for sending us his very interesting preprint \cite{Sey1} before its publication, which greatly helps us in proving the Hamiltonian continuity of Lagrangian spectral capacity. We also thank A. Givental for many enlightening e-mail communications concerning the structure of Maxwell set, Proposition 4.2 and the cliff-wall surgery.
Notations and Conventions

We follow the conventions of [Oh2, Oh7] for the definition of Hamiltonian vector fields and action functional, and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants on general closed symplectic manifold. They are different from e.g., those used in [Po, EP] one way or the other, but coincide with those used in [Sey1].

1. We usually use the letter \( M \) to denote a symplectic manifold and \( N \) to denote a general smooth manifold.

2. The Hamiltonian vector field \( X_H \) is defined by \( dH = \omega(X_H, \cdot) \).

3. The flow of \( X_H \) is denoted by \( \phi_H : t \mapsto \phi_H^t \) and its time-one map by \( \phi_H^1 \in Ham(M, \omega) \).

4. We denote by \( z_H^q(t) = \phi_H^t(q) \) the Hamiltonian trajectory associated to the initial point \( q \).

5. We denote by \( z_H^x(t) = \phi_H^t((\phi_H^1)^{-1}(x)) \) the Hamiltonian trajectory associated to the final point \( x \).

6. \( \overline{\Pi}(t, x) = -H(t, \phi_H^t(x)) \) is the Hamiltonian generating the inverse path \((\phi_H^1)^{-1}\).

7. The canonical symplectic form on the cotangent bundle \( T^*N \) is denoted by \( \omega_0 = -d\theta \) where \( \theta \) is the Liouville one-form which is given by \( \theta = \sum_i p_i \, dq^i \) in the canonical coordinates \((q^1, \cdots, q^n, p_1, \cdots, p_n)\).

8. The classical Hamilton’s action functional on the space of paths in \( T^*N \) is given by

\[
A_H^d(\gamma) = \int \gamma^* \theta - \int_0^1 H(t, \gamma(t)) \, dt.
\]

9. We denote by \( o_N \) the zero section of \( T^*N \).

10. We denote \( \rho^{lag}(H; a) \) the Lagrangian spectral invariant on \( T^*N \) (relative to the zero section \( o_N \)) defined in [Oh2] for asymptotically constant Hamiltonian \( H \) on \( T^*N \).

11. We denote by \( f_H \) the basic phase function and its associated Lagrangian selector by \( \sigma_H : N \to T^*N \) given by \( \sigma_H(q) = df_H(q) \) at which \( df_H(q) \) exists.

2. Basic Generating Function \( h_H \) of Lagrangian Submanifold

In this section, we recall the definition of basic generating function. Let \( H = H(t, x) \) be a Hamiltonian on \( T^*N \) which is asymptotically constant i.e., one whose Hamiltonian vector field \( X_H \) is compactly supported. Denote by \( \mathcal{PC}^{\infty}_{as}(T^*N, \mathbb{R}) \) be the set of such a family of functions. We denote \( L_H = \phi_H(o_N) \) and denote by \( i_H : L_H \hookrightarrow T^*N \) the inclusion map.

Recall the classical action functional is defined as

\[
A_H^d(\gamma) = \int \gamma^* \theta - \int_0^1 H(t, \gamma(t)) \, dt
\]

on the space \( \mathcal{P}(T^*N) \) of paths \( \gamma : [0, 1] \to T^*N \), and its first variation formula is given by

\[
dA_H^d(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma} - X_H(t, \gamma(t)), \xi(t)) \, dt - \langle \theta(\gamma(0)), \xi(0) \rangle + \langle \theta(\gamma(1)), \xi(1) \rangle. \tag{2.1}
\]

For given \( q \in o_N \cong N \), we denote

\[
z_H^q(t) = \phi_H^t(q)
\]
which is a Hamiltonian trajectory such that
\begin{equation}
    z_H^q(0) = q \in o_N,
\end{equation}
which specifies the initial point \( q \in o_N \). (We remark that the notation here is slightly different from that of \([\text{Oh}2, \text{Oh}3]\) in that \( z_H^q \) therein denotes \( z_q^H \) in this paper. We adopt the current notation to be consistent with that of \([\text{Oh}8]\) and other recent papers of the author.)

We define the function \( h_H : [0,1] \times N \to \mathbb{R} \) by
\begin{equation}
    \tilde{h}_H(t,q) = \int (z_H^q|_{[0,t]})^* \theta - \int_0^t H(u, \phi_H^u(q)) \, du
\end{equation}
call it the space-time (or parametric) basic generating function in the fixed frame.

**Lemma 2.1.** The function \( \tilde{h}_H \) satisfies
\begin{align}
    d\tilde{h}_H(t,q) &= \left( (z_H^q)^* \theta(t) - H(t, z_H^q(t)) \right) dt + (\psi^t)^* \theta \\
    &= \psi^t_H \theta - H(t, z_H^q(t)) \, dt
\end{align}
where \( \psi : [0,1] \times N \to T^*N \) defined by \( \psi_H(t,q) = \phi_H^t|_{o_N} \) and \( \psi(t,q) = \psi(t,q) \).

It turns out that the following form of Hamiltonian trajectories
\begin{equation}
    z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))
\end{equation}
are also useful, which specifies the final point of the trajectory instead of the initial point as specified in the trajectory \( z_H^q \). Then we define
\begin{equation}
    h_H(t,x) = \tilde{h}_H(t,(\phi_H^1)^{-1}(x)), \quad x \in \phi_H^1(o_N)
\end{equation}
in the moving frame.

Now consider the Lagrangian submanifold \( \phi_H^1(o_N) \). We would like to point out that the function
\begin{align*}
    h_H(1,\cdot) : L_H &\to \mathbb{R} ; h_H(1,x) := \tilde{h}_H(1,(\phi_H)^{-1}(x))
\end{align*}
defines the natural generating function of \( L_H := \phi_H^1(o_N) \) in that \( d_xh_H = i_H^* \theta \) where \( i_H : L_H \to T^*N \) is the canonical inclusion map. The image of the map
\begin{equation}
    x \in L_H \mapsto (h_H(x),x)
\end{equation}
defines a canonical Legendrian lift of \( L_H \) in the one-jet bundle \( J^1(N) \cong \mathbb{R} \times T^*N \). We call \( h_H \) the basic generating function in the moving frame. We denote the corresponding Legendrian submanifold by \( R_H \). However, as a function on \( N \), \( h_H \) is multi-valued, while \( \tilde{h}_H \) is a well-defined single-valued function.

In general, the projection \( R \to \mathbb{R} \times N \) of any Legendrian submanifold \( R \subset J^1(N,\mathbb{R}) = \mathbb{R} \times T^*N \) is called the wave front \( [\text{El}] \) of the Legendrian submanifold \( R \). We denote by \( W_R \subset \mathbb{R} \times N \) by the front of \( R \). We also define the (Lagrangian) action spectrum of \( H \) on \( T^*N \) by
\begin{equation}
    \text{Spec}(H;N) = \{ A^H_x(z^H_x) \mid x \in L_H \cap o_N \}
\end{equation}
which also coincides with the set of critical values of \( h_H \). It follows that \( \text{Spec}(H;N) \) is a compact subset of \( \mathbb{R} \) of measure zero.
Remark 2.1. We would like to note that we have no a priori control of $C^0$ bound for the functions $h_H$ (or equivalently $\tilde{h}_H$), even when $H$ is bounded in $L^{(1,\infty)}$ norm. Getting this $C^0$-bound is equivalent to getting the bound for the actions of the relevant Hamiltonian chords. Indeed understanding the precise relationship between the action bound, the norm $\|H\|$ and the $C^0$-distance of the time-one map $\phi_H^1$ is a heart of the matter in $C^0$ symplectic topology.

In section 3, we recall construction of basic phase function $f_H$ from [Oh2] which is a particular single valued selection of the multivalued function $h_H$ on $N$ that has particularly nice properties in relation to the study of spectral invariants of the present paper. This function was constructed via the Floer mini-max arguments similarly as the spectral invariants $\rho_{ham}(H; a)$ is defined in [Oh2], and its $C^0$-norm is bounded by $\|H\|$.

3. Basic phase function and its associated Lagrangian selector

In this section, we first recall the definition of basic phase function constructed in [Oh2]. Then we introduce a crucial measurable map $\varphi^H : N \to N$, which is defined by a selection of of a single valued branch of the multivalued section $N \to L_H \subset T^*M$ followed by $(\phi_H^1)^{-1}$. We call this map the mass transfer map associated to the Hamiltonian $H$. It is interesting to note that such a selection process was studied e.g., in the theory of multi-valued functions, or $Q$-valued functions, in the sense of Almgren [Al] in geometric measure theory. In particular, in [DGT], existence of such a single valued branch is studied in the general abstract setting of metric spaces and a finite group action of isometries. It would be interesting to see whether there would be any other significant intrusion of the theory of multivalued functions into the study of symplectic topology.

3.1. Graph selector of wave fronts. The following theorem was proved in [Cha] and in [Oh2] by the generating function method and by the Floer theory respectively. (According to [PPS], the proof of this theorem was first outlined by Sikorav in Chaperon’s seminar.)

Theorem 3.1 (Sikorav, Chaperon [Cha], Oh [Oh2]). Let $L \subset T^*N$ be a Hamiltonian deformation of the zero section $o_N$. Then there exists a Lipschitz continuous function $f : N \to \RR$, which is smooth on an open subset $N_0 \subset N$ of full measure, such that $(q, df(q)) \in L$ for every $q \in N_0$. Moreover if $df(q) = 0$ for all $q \in N_0$, then $L$ coincides with the zero section $o_N$. The choice of $f$ is unique modulo the shift by a constant.

The details of the proof of Lipschitz continuity of $f$ is given in [PPS]. We denote by $\text{Sing} f$ the set of non-differentiable points of $f$. Then by definition

$$N_0 = \text{Reg} f := N \setminus \text{Sing} f$$

is a subset of full measure and $f$ is differentiable thereon.

We call such a function $f$ a graph selector in general following the terminology of [PPS] and denote the corresponding graph part of the front of the Legendrian submanifold $R$ by

$$G_f := \{(h_L(q, df(q)), q, df(q)) \mid q \in N \} \subset R.$$
By construction, the projection $\pi_R : G_f \to N$ restricts to a one-one correspondence and the function $f : \text{Reg} f \to \mathbb{R}$ continuously extends to $\text{Reg} f = N$.

By definition,
$$|df(q)| \leq \max_{x \in L} |p(x)| \quad (3.1)$$
for any $q \in N_0$, where $x = (q(x), p(x))$ and the norm $|p(x)|$ is measured by any given Riemannian metric on $N$. 

**Proposition 3.2.** As $d_H(L, o_N) \to 0$, $|df(q)| \to 0$ uniformly over $q \in N_0$.

In [Oh2], a canonical choice of $f$ is constructed via the chain level Floer theory, provided the generating Hamiltonian $H$ of $L$ is given. The author called the corresponding graph selector $f$ the basic phase function of $L = \phi_H^t(o_N)$ and denoted it by $f_H$. We give a quick outline of the construction referring the readers to [Oh2] for the full details of the construction.

### 3.2. The basic phase function $f_H$ and its Lagrangian selector

Another construction in [Oh2] is given by considering the Lagrangian pair
$$(o_N, T_q^*N), \ q \in N$$
and its associated Floer complex $CF(H; o_N, T_q^*N)$ generated by the Hamiltonian trajectory $z : [0, 1] \to T^*N$ satisfying
$$\dot{z} = X_H(t, z(t)), \ z(0) \in o_N, \ z(1) \in T_q^*N. \quad (3.2)$$

Denote by $\text{Chord}(H; o_N, T_q^*N)$ the set of solutions. The differential $\partial_{(H,J)}$ on $CF(H; o_N, T_q^*N)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation
$$\begin{cases}
\frac{\partial u}{\partial \tau} + J (\frac{\partial u}{\partial \tau} - X_H(u)) = 0 \\
u(\tau, 0) \in o_N, \ u(\tau, 1) \in T_q^*N.
\end{cases} \quad (3.3)$$

An element $\alpha \in CF(H; o_N, T_q^*N)$ is expressed as a finite sum
$$\alpha = \sum_{\text{Chord}(H; o_N, T_q^*N)} a_z[z], \ a_z \in \mathbb{Z}.$$ 

We denote the level of the chain $\alpha$ by
$$\lambda_H(\alpha) := \max_{z \in \text{supp} \alpha} \{A_H^d(z)\}.$$ 

The resulting invariant $\rho^{\alpha}(H; \{q\})$ is to be defined by the mini-max value
$$\rho^{\alpha}(H; \{q\}) = \inf_{\alpha \in [q]} \lambda_H(\alpha)$$
where $[q] \in H_0(\{q\}; \mathbb{Z})$ is a generator of the homology group $H_0(\{q\}; \mathbb{Z})$.

A priori, $\rho^{\alpha}(H; \{q\})$ is defined when $\phi_H^t(o_N)$ intersects $T_q^*N$ transversely but can be extended to non-transversal $q$’s by continuity. By varying $q \in N$, this defines a function $f_H : N \to \mathbb{R}$ which is precisely the one called the basic phase function in [Oh2]. (A similar construction of such a function using the generating function method was earlier given by Sikorav and Chaperon [Cha] ). We call the associated graph part $G_{f_H}$ the basic branch of the front $W_{R_H}$ of $R_H$.

**Theorem 3.3** ([Oh2] [Oh6]). There exists a solution $z : [0, 1] \to T^*N$ of $\dot{z} = X(t, z)$ such that $z(0) = q$, $z(1) \in o_N$ and $A_H^d(z) = \rho^{\alpha}(H; \{q\})$ whether or not $\phi_H^t(o_N)$ intersects $T_q^*N$ transversely.
We summarize the main properties of $f_H$ established in [Oh2].

**Theorem 3.4 (Oh2).** When the Hamiltonian $H = H(t, x)$ such that $L = \phi^1_H(o_N)$ is given, there is a canonical lift $f_H$ defined by $f_H(q) := \rho^{\alpha\beta}(H; \{q\})$ that satisfies

$$f_H \circ \pi(x) = h_H(x) = A_H^c(z^H_x)$$  \hspace{1cm} (3.4)

for some Hamiltonian chord $z^H_x$ ending at $x \in T^*_qN$. This $f_H$ satisfies the following property in addition

$$\|f_H - f_K\|_\infty \leq \|H - K\|.$$ \hspace{1cm} (3.5)

An immediate corollary of Theorem is

**Corollary 3.5.** If $H_i$ converges in $L^{(1, \infty)}$, then $f_{H_i}$ converges uniformly.

Based on this corollary, we will just denote the limit continuous function by

$$f_H := \lim_{i \to \infty} f_{H_i}$$  \hspace{1cm} (3.6)

when $H_i \to H$ in $L^{(1, \infty)}$-topology, and call it the basic phase function of the topological Hamiltonian $H$ or of the $C^0$-Lagrangian submanifold $L_H = \phi^1_H(o_N)$.

Note that $\pi_H = \pi|_{L_H} : L_H = \phi^1_H(o_N) \to N$ is surjective for all $H$ (see [LS] for its proof) and so $\pi_H^{-1}(\pi_H^{-1}(q)) \subset o_N$ is a non-empty compact subset of $o_N \cong N$. Therefore we can regard the ‘inverse’ $\pi_H^{-1} : N \to L_H \subset T^*N$ as a everywhere defined multivalued section of $\pi : T^*N \to N$.

We introduce the following general definition

**Definition 3.1.** Let $L \subset T^*N$ be a Lagrangian submanifold projecting surjectively to $N$. We call a single valued section $\sigma$ of $T^*N$ with values lying in $L$ a Lagrangian selector of $L$.

For any given Lagrangian selector $\sigma$ of $L = L_H = \phi^1_H(o_N)$, we define the map $\varphi^\sigma : N \to N$ to be

$$\varphi^\sigma(q) = (\phi^1_H)^{-1}(\sigma(q)).$$

Recall that the graph $G_{f_H}$ is a subset of the front $W_{R_H}$ of $R_H$ and for a generic choice of $H$ the set $\text{Sing} f_H \subset N$ consists of the crossing points of the two different branches and the cusp points of the front of $W_{R_H}$. Therefore it is a set of measure zero in $N$. (See [El], [PPS], for example.) Once the graph selector $f_H$ of $L_H$ is picked out, it provides a natural Lagrangian selector defined by

$$\sigma_H(q) := \text{Choice}\{x \in L_H \mid \pi(x) = q, A_H^c(z^H_x) = f_H(q)\}$$

via the axiom of choice where Choice is a choice function. It satisfies

$$\sigma_H(q) = df_H(q)$$  \hspace{1cm} (3.7)

whenever $df_H(q)$ is defined. We call this particular Lagrangian selector of $L_H$ the basic Lagrangian selector and the pair $(\sigma_H, f_H)$ the basic wave front of the Lagrangian submanifold $\phi^1_H(o_N)$.

The general structure theorem of the wave front (see [El], [PPS] for example) proves that the section $\sigma_H$ is a differentiable map on a set of full measure for a generic choice of $H$ which is, however, not necessarily continuous: This is because as long as $q \in N \setminus \text{Sing} f_H$, we can choose a small open neighborhood of $U \subset N \setminus \text{Sing} f_H$ of $q$ and $V \subset L_H = \phi^1_H(o_N)$ of $x \in V$ with $\pi(x) = q$ so that the projection $\pi|_V : V \to U$ is a diffeomorphism.
Then we define the mass transfer map \( \varphi^H : N \to N \) by
\[
\varphi^H(q) = (\phi^1_H)^{-1}(\sigma_H(q)). \tag{3.8}
\]
The map \( \varphi^H \) is measurable, but not necessarily continuous, which is however differentiable on a set of full measure for a generic choice of \( H \). And from its definition, it is surjective if and only if the Lagrangian submanifold \( \phi^1_H(o_N) \) is a graph of an exact one-form. On the other hand, the map \( \varphi^H \) may not be continuous along the subset \( \text{Sing} f_H \subset N \) which is a set of measure zero. By definition, we have
\[
f_H(q) = A^1_H \left( z_{\varphi^H}(q) \right) = \overline{h}_H(\varphi^H(q)). \tag{3.9}
\]
This relationship between \( f_H \) and \( \overline{h}_H \) is the reason why we introduce the transfer map \( \varphi^H \).

The following lemma is obvious from the definition of \( \varphi^H \). We note
\[
d_H(\phi^1_H(o_N), o_N) \leq \text{osc}_{C^0}(\phi^1_H; o_N)
\]
where \( d_H(\phi^1_H(o_N), o_N) \) is the Hausdorff distance.

**Lemma 3.6.** We have
\[
d(\varphi^H(x), x) \leq d_H(\phi^1_H(o_N), o_N) + \text{osc}_{C^0}(\phi^1_H; o_N) \leq 2\text{osc}_{C^0}(\phi^1_H; o_N)
\]
for all \( x \in N_0 \). In particular, if \( \text{osc}_{C^0}(\phi^1_H; o_N) \to 0 \), then \( \max_{x \in N_0} d(\varphi^H(x), x) \to 0 \) uniformly over \( x \in N_0 \).

4. **Singular locus of the basic phase function and cliff-wall surgery**

We first recall two important properties of the Liouville one-form \( \theta \):

1. \( \theta \) identically vanishes on any conormal variety. (See [Oh2, KO1] for the explanation on the importance of this fact in relation to the Lagrangian Floer theory on the cotangent bundle.)
2. For any one form \( \alpha \) on \( N \), we have \( \tilde{\alpha}^* \theta = \alpha \) where \( \tilde{\alpha} : N \to T^*N \) is the section map associated to the one-form \( \alpha \) as a section of \( T^*N \). In particular, we have
\[
\sigma_F^* \theta = df_F
\]
on \( N \setminus \text{Sing}(\sigma_F) \) and on each stratum of \( \text{Sing}(\sigma_F) \).

We note that the singular locus \( S(\sigma_F) \subset \Delta \) is a subset of the bifurcation diagram of the Lagrangian submanifold \( \phi^1_F(o_N) \): The bifurcation diagram is the union of the caustic and the Maxwell set where the latter is the set of points of which merge the different branches of the generating function \( h \). (See section 4 [G1] for the definition of bifurcation diagram of Lagrangian submanifold \( L \subset T^*N \) in general.)

For a generic \( F \), \( S(\sigma_F) \) is stratified into a finite union of smooth submanifolds
\[
\bigcup_{k=1}^n S_k(\sigma_F), \quad S_k(\sigma_F) = \text{Sing}_k(\sigma_F), \quad n = \dim N
\]
(see [Al] [Ell] [G1] e.g., for such a result) so that its conormal variety \( \nu^* S(\sigma_F) \) can be defined as a finite union of conormals of the corresponding strata. Each stratum \( \text{Sing}_k(\sigma_F) \) has codimension \( k \) in \( \Delta \). The stratum for some \( k \) could be empty. (See [KS]. See also [Kal, KO2, NZ, N] for the usages of such conormal varieties in relation to Lagrangian Floer theory.)

In \( \dim N = 2 \), there are two strata to consider, one \( S_1(\sigma_F) \) and the other \( S_2(\sigma_F) \).
For \( k = 1 \), each given point \( q \in S_1(\sigma_F) \) has a neighborhood \( A(q) \subset N \) such that \( A(q) \setminus S_1(\sigma_F) \) has two components. We also note that \( \Sigma_F \) carries a natural orientation induced from \( N \) by projection when \( N \) is orientable and so defines an integral current in the sense of geometric measure theory \[1\]. When \( N \) is oriented, \( S_1(F) \) is also orientable as a finite union of smooth hypersurfaces. We fix any orientation on \( S_1(F) \).

We denote by \( A^\pm(q) \) the closure of each component of \( A(q) \setminus S_1(\sigma_F) \) in \( A(q) \) respectively. Here we denote by \( A^\pm(q) \) the component whose boundary orientation on \( \partial A^\pm(q) \) coincides with that of the given orientation on \( S_1(F) \) and by \( \partial A^-(q) \) the other one. Then each of \( A^\pm(q) \) is an open-closed domain with the same boundary

\[
\partial A^\pm(q) = A(q) \cap S_1(\sigma_F).
\]

Denote

\[
df^\pm_F(q) = \lim_{\mu \to q} df_F(p) = \lim_{\mu \to q} df_F(p) = \lim_{\mu \to q} df_F(p)
\]

obtained by taking the limit on \( A^\pm(q) \) respectively. The limits are well-defined from the definition of \( \sigma_F \) since \( \text{Im} \sigma_F = \text{Im} df_F \subset \phi^1_F(o_N) \) where \( \phi^1_F(o_N) \) is a smooth closed submanifold in \( T^*N \).

We now prove the following theorem. We refer to [Ch], [ZR] for a related statement.

**Theorem 4.1.** Let \( q \in S_1(F) \). Then

\[
df^-_F(q) - df^+_F(q) \in T^*_q N,
\]

which is contained in the conormal space \( T^*_q[S_1(\sigma_F); N] \subset T^*_q N \).

**Proof.** Let \( \vec{v} \in T_q S_1(\sigma_F) \) be any given tangent vector. Choose a smooth curve \( \gamma : (-\varepsilon, \varepsilon) \to S_1(\sigma_F) \) with \( \gamma(0) = q \). For any given sufficiently small \( \delta \geq 0 \), we define a family of \( \delta \)-shifted curves

\[
\gamma^\pm_\delta(t) = \exp_{\gamma(t)}(\pm \delta \vec{n}(t)),
\]

where \( \exp \) is the normal exponential map of \( S_1(\sigma_F) \) in \( N \) and \( \vec{n}(t) \) is the unit normal vector thereof at \( \gamma(t) \) towards the domain \( A^+(q) \). Then \( \gamma^+_\delta \) is mapped into \( \text{Int} A^+(q) \) and \( \gamma^-_\delta \) into \( \text{Int} A^-(q) \) for all sufficiently small \( \delta > 0 \). Note

\[
\gamma^+_0(t) = \gamma(t)
\]

for \( \delta = 0 \). Since \( f_F : N \to \mathbb{R} \) is a continuous function, we have the uniform convergence

\[
f_F(\gamma^+_\delta(t)) - f_F(\gamma^-_\delta(t)) \to 0
\]

as \( \delta \to 0 \) over \( t \in (-\varepsilon, \varepsilon) \). Furthermore since \( f_F \) is smooth up to the boundary on each of \( A^\pm(q) \) and \( df_F \) is uniformly differentiable up to the boundary of \( A^\pm(q) \) for either of \( \pm \),

\[
f_F(\gamma^+_\delta(t)) = f_F(\gamma^+_\delta(0)) + t df_F(\gamma^+_\delta(0))(\gamma^+_\delta)'(0) + O(|t|^2)
\]

\[
= f_F(\gamma^+_\delta(0)) + t df_F(\gamma^+_\delta(0)) + D \exp_{\gamma(0)}(\pm \delta \vec{n}(0))(\gamma'(0)) + O(|t|^2)
\]

where \( |O(|t|^2)| \leq C|t|^2 \) for a constant \( C > 0 \) uniformly over \( \delta \geq 0 \) and \( t \in (-\varepsilon, \varepsilon) \). Here \( D \exp_p(\vec{n})(\vec{v}) \) is the derivative

\[
D \exp_p(\vec{n})(\vec{v}) := \frac{d}{dt} \bigg|_{t=0} \exp_{\gamma(t)}(\vec{n}), \quad \vec{v} = \gamma'(0), \quad \gamma(0) = p,
\]
which is nothing but the covariant derivative of the Jacobi field along the geodesic $t \mapsto \exp_p(tv)$ with the initial vector $\vec{n}$ at $p$. (See [K] for an elegant exposition on the detailed study of exponential maps.) By letting $\delta \to 0$ and using the uniformity of the constant $C$ and the continuity of $f_F$, we obtain

$$f_F(\gamma(t)) = f_F(q) + \lim_{\delta \to 0} (t \lim_{\delta \to 0} df_F(\gamma^{\pm}_{\delta}(0))(\gamma^{\pm}_{\delta}(0))) + O(|t|^2)$$

Then by taking the difference of two equations for $\pm$ and dividing by $t$, utilizing the convergence $(\gamma^{\pm}_{\delta})'(0) \to \gamma'(0)$ as $\delta \to 0$ and then evaluating at $t = 0$, we obtain

$$0 = \lim_{\delta \to 0} \left( df_F^{\pm}(\gamma^{\pm}_{\delta}(0)) \right) (\gamma^{\pm}_{\delta}(0)) + df_F(\gamma^{\pm}_{\delta}(0)) - df_F(\gamma^{\pm}_{\delta}(0))$$

Recall that $\gamma(0) = p$ and $\gamma^{\pm}_{\delta}(0) \to p$, and $D \exp_p(\pm \delta \vec{n}(0))$ converges to $D \exp_p(\vec{0})$ as $\delta \to 0$, which is nothing but the identity map on $\nu_q S_1(\sigma_F)$ by the standard fact on the exponential map (see [K]). Therefore from this last equality, we derive

$$\left( df_F^{\pm}(q) - df_F^{-}(q) \right)(\vec{v}) = 0$$

by the definition of $df_F^{\pm}(q)$. Since this holds for all $\vec{v} \in T_q S_1(\sigma_F)$, the proposition for $k = 1$ is proved.

The boundary orientations of the two components arising from that of $\Sigma_F$, which in turn is induced from that of $N$ via $\pi_1$ have opposite orientations. We call the one whose projection to $S_1(\sigma_F)$ under $\pi_1$ coinciding with the given orientation the **upper branch** and the one with the opposite one the **lower branch** and denote them by

$$\partial^+ \Sigma_F, \partial^- \Sigma_F$$

respectively.

Now let $L_q$ be the line segment connecting the two vectors $df_F^{\pm}(q)$, i.e.,

$$L_q : u \in [0, 1] \mapsto \left. df_F^{\pm}(q) + u(df_F^{-}(q) - df_F^{+}(q)) \right| \subset T_q^* N. \quad (4.2)$$

This is an affine line that is parallel to the conormal space $\nu_q^* S_1(\sigma_F)$. Therefore the union

$$\Sigma_F[\pm] := \bigcup_{q \in S_1(\sigma_F)} L_q \quad (4.3)$$

is contained in the translated conormal

$$df_F^{\pm} + \nu^*[S_1(\sigma_F); N] \quad (4.4)$$

Here the bracket $[\pm]$ stands for the line segment $L_q$, and $\nu^*[S_1(\sigma_F); N]$ is the conormal bundle of $S_1(\sigma_F)$ in $N$. We would like to point out that since $df_F^{\pm}(q) - df_F^{-}(q) \in \nu^*[S_1(\sigma_F); N]$ we have the equality

$$df_F^{\pm}(q) + \nu_q^*[S_1(\sigma_F); N] = df_F^{-}(q) + \nu_q^*[S_1(\sigma_F); N]$$

for all $q \in S_1(\sigma_F)$. Therefore we can simply write (4.4) as

$$df_F + \nu^*[S_1(\sigma_F); N] \quad (4.5)$$

unambiguously.
Definition 4.1 (Basic Lagrangian selector chain). We denote by $\sigma_F$ the chain whose support is given by

$$\text{supp}(\sigma_F) := \Sigma_F$$

with the orientation given as above, and define its micro-support by

$$SS(\sigma_F) := df_F + \nu^* [S_1(\Sigma_F); N]$$

imitating the notation from [KS].

The two components of $\partial \sigma_F$ associated to each connected component of $S_1(\sigma_F)$ are the graphs of $df_F$ for the functions $f_F$ near $S_1(\sigma_F)$. Note that each connected component of $S_1(\sigma_F)$ gives rise to two components of $\partial \sigma_F$; $[-\nabla_+] \cap \sigma_F$. We can bridge the ‘cliff’ between the two branches of $\partial \sigma_F$ over each connected component of $S_1(\sigma_F)$ and

Definition 4.2 (Cliff wall chain). We define a ‘cliff wall’ chain $\sigma_F[-\nabla+]$ whose support is given by the union $\Sigma_F[-\nabla+] = \bigcup_{q \in S_1(\sigma_F)} L_q$

Then we define the chain $\sigma_F[-\nabla+]$ similarly as we define $\sigma_F$ by taking its closure in $T^*N$.

We emphasize that $\sigma_F[-\nabla+]$ lies outside the Lagrangian submanifold $\phi^1_F(oN)$. By definition, its tangent space at $x = (q, u)$ has natural identification with

$$T_x \Sigma_F[-\nabla+] \cong \nu^*_q S_1(\sigma_F) \oplus T_q S_1(\sigma_F).$$

Due to Theorem 4.1, it carries a natural direct sum orientation

$$o_{\Sigma_F[-\nabla+]}(q) = \{ df_F(q) - df_F(q) \} \oplus o_{S_1(\sigma_F)}(q).$$

Therefore $\Sigma_F[-\nabla+]$ carries a natural orientation and defines a current. Under the natural identification of $T_q N$ with $T^*_q N$ by the dual pairing, which induces an identification

$$\nu^*_q S_1(\sigma_F) \oplus T_q S_1(\sigma_F) \cong T_q S_1(\sigma_F) \oplus T_q S_1(\sigma_F)$$

as an oriented vector space. Then we have the relation

$$\partial \Sigma_F = -\partial \Sigma_F[-\nabla+]$$

along the intersection $\partial \Sigma_F \cap \partial \Sigma_F[-\nabla+]$. Remark 4.3. (1) We would like to note that the singular locus $S(\sigma_F) \subset \Delta$ is a subset of the bifurcation diagram of the Lagrangian submanifold $\phi^1_F(oN)$; The bifurcation diagram is the union of the caustic and the Maxwell set where the latter is the set of points of which merge the different branches of the generating function $h$. (See section 4 [G1] for the definition of bifurcation diagram of Lagrangian submanifold $L \subset T^*N$ in general.) But this detailed structure does not play any role in our proof except the one described.

(2) However we would like to note that each fiber of $SS(\sigma_F)$ is an affine space

$$df_F(q) + \nu^*_q [S_1(\Sigma_F); N]$$

at $q \in S_1(\Sigma_F)$, not a linear space. In fact, if we incorporate the orientation into consideration, one can refine this definition further to the ‘half space’ instead of the full affine space. We denote this refinement by $SS^+(\sigma_F)$. 

Then at a point \( q \) in the lower dimensional strata, it will be a ‘wedge domain’, i.e., the intersection of several space of this type. (See [KO1, KO2] for a usage of such domains in their quantization program of Eilenberg-Steenrod axiom.) We will come back to further discussion on the detailed structure of singularities elsewhere.

Next we consider the case of \( S_2(\sigma_F) \) and its relationship with \( \sigma_F \) and \( S_1(\sigma_F) \). Note that for a generic choice of \( F \), \( S_2(\sigma_F) \) consists of a finite number of points in \( N \) consisting of either a caustic point or a triple intersection point of the Maxwell set (see [A1], section 4 [G1] and 7.1 [ZR]).

The following proposition can be also derived from the general structure theorem of generic singularities of Lagrangian maps. We restrict the proposition to \( \dim N = 2 \) here postponing the precise statement for the high dimensional cases elsewhere.

**Proposition 4.2.** Assume \( \dim N = 2 \). For a generic choice of \( F \), the boundary of \( \sigma_F + \sigma_F[-\epsilon] \) is a finite union of triangles each of which is formed by the three line segments \( L_q \) given in (4.2) associated to a triple intersection point \( q \) of \( S(\sigma_F) \) contained in \( S_2(\sigma_F) \). Furthermore each triangle is the boundary of a 2-simplex contained in the fiber \( T_q^*N \).

**Proof.** This is an immediate consequence of the classification theorem of generic singularities in dimension 2 of Lagrangian maps originally proved by Arnold [A1]. (See also p. 55 and Figure 43 [A3], section 4 [G1] and section 7.1 [ZR].) \( \square \)

Now we define \( \sigma_F^{add} \) to be the union of these 2 simplices, and set

\[
\sigma_F^{add} = \sigma_F + \sigma_F[-\epsilon] + \sigma_F^{\Delta^2}.
\]

Then by construction, \( \sigma_F^{add} \) forms a mod-2 cycle.

This finishes the description of the basic Lagrangian cycle. A similar description can be given in the higher dimensional cases, which we will study elsewhere. This enables us to define the following important Lagrangian cycle.

**Definition 4.4** (Basic Lagrangian cycle and cliff-wall surgery). Let \( \dim N = 2 \). We call the cycle \( \sigma_F^{add} \) the basic Lagrangian cycle of \( \phi_F^1(o_N) \) (associated to the basic Lagrangian selector \( \sigma_F \)). We call the replacement of \( \phi_F^1(o_N) \) by the \( \Sigma_F^{add} \) the **cliff-wall surgery** of the \( \phi_F^1(o_N) \).

**Remark 4.5.**
(1) We also refer to [KO1, Ka, KO2] for a usage of the general conormal variety of an open-closed domain with boundary and corners, which also naturally occurs in micro-local analysis and in stratified Morse theory [KS].

(2) The basic Lagrangian cycle seems to be a good replacement of non-graph type Lagrangian submanifold \( \phi_F^1(o_N) \) in general for the study of various questions arising in Hamiltonian dynamics and symplectic topology. We hope to elaborate this point elsewhere.

**Remark 4.6.** We believe that this surgery will play an important role in the study of homotopy invariance of spectral invariants for the topological Hamiltonian paths \( \text{[Oh7]} \), which we hope to address elsewhere.
5. \textbf{Lagrangian Floer homology and spectral invariants}

In this section, we first briefly recall the construction of Lagrangian spectral invariants $\rho_{\text{lag}}(H; a)$ for $L_H = \phi_H^1(o_N)$ performed by the author in \cite{Oh3}. A priori, this invariant may depend on $H$, not just on $L_H$ itself. In \cite{Oh3}, we prove that

$$\rho_{\text{lag}}(H; a) = \rho_{\text{lag}}(F; a)$$

(5.1)

for all $a \in H^*(N; \mathbb{Z})$ if $L_H = L_F$, but modulo the addition of a constant and then somewhat ad-hoc normalization to remove this ambiguity of a constant.

5.1. \textbf{Definition of Lagrangian spectral invariants.} Consider the zero section $o_N$ and the space $P(o_N, o_N) = \{ \gamma : [0, 1] \to T^*N \mid \gamma(0), \gamma(1) \in o_N \}$.

The set of generators of $\text{CF}(H; o_N, o_N)$ is that of solutions $\dot{z} = X_H(t, z(t)), z(0), z(1) \in o_N$ and its Floer differential is defined by counting the number of solutions of

$$\begin{cases}
\frac{du}{d\tau} + J (\frac{du}{d\tau} - X_H(u)) = 0 \\
u(\tau, 0), \nu(\tau, 1) \in o_N.
\end{cases}$$

(5.2)

An element $\alpha \in \text{CF}(H; o_N, o_N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \text{Chord}(H; o_N, o_N)} a_z [z], \ a_z \in \mathbb{Z}.$$  

We define the level of the chain $\alpha$ by

$$\lambda_H(\alpha) := \max_{z \in \text{supp}\alpha} \{ A_H^\ell(z) \}. \quad (5.3)$$

For given non-zero cohomology class $a \in H^*(N, \mathbb{Z})$, we consider its Poincaré dual $[a]^\flat := PD(a) \in H_*(N, \mathbb{Z})$ and its image under the canonical isomorphism

$$\Phi : H_*(N, \mathbb{Z}) \to HF_*(H, J; o_N, o_N).$$

**Definition 5.1.** Let $(H, J)$ be a Floer regular pair relative to $(o_N, o_N)$ and let $(\text{CF}(H), \partial_{(H, J)})$ be its associated Floer complex. For any $0 \neq a \in H^*(N, \mathbb{Z})$, we define

$$\rho_{\text{lag}}(H; a) = \inf_{\alpha \in \Phi(a^\flat)} \{ \lambda_H(\alpha) \}. \quad (5.4)$$

One important result is the following basic property, called \textit{spectrality} in \cite{Oh6}, which is not explicitly stated in \cite{Oh2} but can be easily derived by a compactness argument. (See the proof in \cite{Oh6} given in the Hamiltonian context.)

**Proposition 5.1.** Let $H = H(t, x)$ be any, not necessarily nondegenerate, smooth Hamiltonian. Then for any $0 \neq a \in H^*(N, \mathbb{Z})$, there exists a point $x \in L_H \cap o_N$ such that

$$A_H^\ell(z_x^H) = \rho_{\text{lag}}(H; a).$$

In particular, $\rho_{\text{lag}}(H; a) \in \text{Spec}(H; N)$. 

5.2. Comparison of two Cauchy-Riemann equations. So far we have looked at the Hamiltonian-perturbed Cauchy-Riemann equation (5.2), which we call the dynamical version as in [Oh2].

On the other hand, one can also consider the genuine Cauchy-Riemann equation

$$\begin{cases} \frac{\partial v}{\partial t} + J^H \frac{\partial v}{\partial \tau} = 0 \\ v(\tau, 0) \in \phi_H(o_N), v(\tau, 1) \in o_N \end{cases} \tag{5.5}$$

for the path $u : \mathbb{R} \to \mathcal{P}(o_N, L)$ where $L = \phi_H(o_N)$ and

$$\mathcal{P}(o_N, L) = \{\gamma : [0, 1] \to T^*N | \gamma(0) \in L, \gamma(1) \in o_N\}$$

and $J^H = (\phi_H^{-1})^* J_L$. We call this version the geometric version.

We now describe the geometric version of the Floer homology in some more detail. We refer readers to [Oh2] for the discussion on the further comparison of the two versions in the point of moduli spaces and others. The upshot is that there is a filtration preserving isomorphisms between the dynamical version and the geometric version of the Lagrangian Floer theories.

We denote by $\tilde{\mathcal{M}}(L_H, o_N; J^H)$ the set of finite energy solutions and $\mathcal{M}(L_H, o_N; J^H)$ to be its quotient by $\mathbb{R}$-translations. This gives rise to the geometric version of the Floer homology $HF_*(o_N, o_H(o_N), J)$ of the type [Fl1, Oh3] whose generators are the intersection points of $o_N \cap \phi_H(o_N)$. An advantage of this version is that it depends only on the Lagrangian submanifold $L = \phi_H(o_N)$, only loosely on $H$. (The author proved in [Oh3] that $\rho(H; a)$ is the invariant of $L_H = \phi_H(o_N)$ up to this normalization by comparing these two versions of the Floer theory in [Oh2, Oh3].)

The following is a straightforward to check but is a crucial lemma.

**Lemma 5.2.** Let $L = \phi_H(o_N)$.

1. The map $\Phi_H : o_N \cap L \to \text{Chord}(H; o_N, o_N)$ defined by

$$x \mapsto z^H_x(t) = \phi_H((\phi_H^{-1}(x))$$

gives rise to the one-one correspondence between the set $o_N \cap L \subset \mathcal{P}(o_N, L)$ as constant paths and the set of solutions of Hamilton’s equation of $H$.

2. The map $a \mapsto \Phi_H(a)$ also defines a one-one correspondence from the set of solutions of (5.2) and that of

$$\begin{cases} \frac{\partial v}{\partial t} + J^H \frac{\partial v}{\partial \tau} = 0 \\ v(\tau, 0) \in \phi_H(o_N), v(\tau, 1) \in o_N \end{cases} \tag{5.6}$$

where $J^H = \{J^H_i\}, J^H_i := (\phi_H^{-1})^* J_L$. Furthermore, (5.6) is regular if and only if (5.2) is regular.

Once we have transformed (5.2) to (5.6), we can further deform $J^H$ to the constant family $J_0$ and consider

$$\begin{cases} \frac{\partial v}{\partial t} + J_0 \frac{\partial v}{\partial \tau} = 0 \\ v(\tau, 0) \in \phi_H(o_N), v(\tau, 1) \in o_N \end{cases} \tag{5.7}$$

This latter deformation preserves the filtration of the associated Floer complexes [Oh2]. A big advantage of considering this equation is that it enables us to study the behavior of spectral invariants for a sequence of $L_i$ converging to $o_N$ in weak Hamiltonian topology.
The following proposition provides the action functional associated to the equation (5.6), (5.7), which will give a natural filtration associated Floer homology $HF(L,o_N)$.

**Proposition 5.3.** Let $L$ and $h_L$ be as in Lemma 2.1. Let $\Omega(L,o_N;T^*N)$ be the space of paths $\gamma : [0,1] \to \mathbb{R}$ satisfying $\gamma(0) \in L,o_N$, $\gamma(1) \in o_N$. Consider the effective action functional

$$A^{\text{eff}}(\gamma) = \int \gamma^* \theta + h_H(\gamma(0)).$$

Then $dA^{\text{eff}}(\gamma)(\xi) = \int_0^1 \omega(\xi(t),\dot{\gamma}(t)) \, dt$. In particular,

$$A^{\text{eff}}(c_x) = h_H(x) = A^{\text{cl}}_H(z^H_x) \quad (5.8)$$

for the constant path $c_x \equiv x \in L \cap o_N$ i.e., for any critical path $c_x$ of $A^{\text{eff}}$.

We would like to highlight the presence of the ‘boundary contribution’ $h_H(\gamma(0))$ in the definition of the effective action functional above: This addition is needed to make the Cauchy-Riemann equation (5.5) or (5.7) into a gradient trajectory equation of the relevant action functional. We refer readers to section 2.4 [Oh2] and Definition 3.1 [KO1] and the discussion around it for the upshot of considering the effective action functional and its role in the study of Cauchy-Riemann equation.

**5.3. Triangle inequality for Lagrangian spectral invariants.** We recall from [Sc], [Oh6] that the triangle inequality of the Hamiltonian spectral invariants

$$\rho^{\text{ham}}(H \# F; a \cdot b) \leq \rho^{\text{ham}}(H; a) + \rho^{\text{ham}}(F; b)$$

for the product Hamiltonian $H \# F$ relies on the homotopy invariance property of spectral invariants which in turn relies on the existence of canonical normalization procedure of Hamiltonians on closed $(M,\omega)$ which is nothing but the mean normalization. On the other hand, one can directly prove

$$\rho^{\text{ham}}(H* F; a \cdot b) \leq \rho^{\text{ham}}(H; a) + \rho^{\text{ham}}(F; b)$$

more easily for the concatenated Hamiltonian. (See e.g., [FOOO4] for the proof.) Once we have the latter inequality, we can derive the former from the latter again by the homotopy invariance property of $\rho^{\text{ham}}(\cdot; a)$ for the mean-normalized Hamiltonians.

When one attempts to assign an invariant of Lagrangian submanifold $\phi^1_H(o_N)$ itself out of the spectral invariant $\rho^{\text{lag}}(H; a)$, one has to choose a normalization of the Hamiltonian relative to the Lagrangian submanifold. Since there is no canonical normalization unlike the Hamiltonian case, the invariance property of Lagrangian spectral invariants and so the triangle inequality is somewhat more nontrivial than the case of Hamiltonian spectral invariants. In this subsection, we clarify these issues of invariance property and of the triangle inequality.

The following parametrization independence follows immediately from the construction of Lagrangian spectral invariants and $L^{(1,\infty)}$-continuity of $H \mapsto \rho^{\text{lag}}(H; a)$.

**Lemma 5.4.** Let $H = H(t,x)$ be any, not necessarily nondegenerate, smooth Hamiltonian and let $\chi : [0,1] \to [0,1]$ a reparameterization function with $\chi(0) = 0$ and $\chi(1) = 1$. Then

$$\rho^{\text{lag}}(H; a) = \rho^{\text{lag}}(H^\chi; a)$$

where $H^\chi(t,x) = \chi'(t)H(\chi(t),x)$. 

---

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We first recall the following triangle inequality which was essentially proved in \cite{Oh3}. (See Theorem 6.4 and Lemma 6.5 \cite{Oh3}. In \cite{Oh3}, the cohomological version of the Floer complex was considered and hence the opposite inequality is stated. Other than this, the same proof can be applied here.)

**Proposition 5.5.** Let $H, F \in \mathcal{PC}_{asc}(T^*N; \mathbb{R})$, and assume $F$ is autonomous. Then we have

$$\rho^{lag}(H\#F; ab) \leq \rho^{lag}(H; a) + \rho^{lag}(F; b).$$

Moznzer, Vichery, and Zapolsky \cite{MVZ} proved the following form of the triangle inequality which uses the concatenated Hamiltonian $H \ast F$ instead of the product Hamiltonian $H\#F$.

**Proposition 5.6 (Proposition 2.4 \cite{MVZ}).** Let $H, K$ be compactly supported. Suppose $H(1, x) \equiv F(0, x)$ and $H \ast F$ be the concatenated Hamiltonian. Then

$$\rho^{lag}(H \ast F; ab) \leq \rho^{lag}(H; a) + \rho^{lag}(F; b)$$

for all $a, b \in H^*(N)$.

In particular, this proposition applies to all pairs $H, F$ which are compactly supported and boundary flat.

**Remark 5.2.** We suspect that \cite{MVZ} holds even for the non-autonomous $F$ as in the Hamiltonian case but we did not check this, since it is not needed in the present paper.

### 5.4. Assigning spectral invariants to Lagrangian submanifolds

In this subsection, we identify a class, denoted by $\mathcal{PC}_{ass,B}$, of Hamiltonians $H$ among those satisfying $\phi^1_H(o_N) = \phi^1_F(o_N)$, such that the equality

$$\rho^{lag}(H; a) = \rho^{lag}(F; a)$$

holds for all $H, F \in \mathcal{PC}_{ass,B}$. As the notation suggests, the class depends on the subset $B \subset N$.

We start with the following proposition. The proof closely follows that of Lemma 2.6 \cite{MVZ} which uses Proposition 5.6 in a significant way. We need to modify their proof to obtain a somewhat stronger statement, which replaces the condition “$\phi^1_H = \phi^1_F$” used in \cite{MVZ} by the conditions put in this proposition.

**Proposition 5.7 (Compare with Lemma 2.6 \cite{MVZ}).** Let $H, F \in \mathcal{PC}_{asc}(T^*N; \mathbb{R})$ be boundary-flat. Suppose in addition $H, F$ satisfy the following:

1. $\phi^1_H(o_N) = \phi^1_F(o_N)$,
2. $H \equiv c(t), F \equiv d(t)$ on a tubular neighborhood $T \supset B$ in $T^*N$ of a closed ball $B \subset o_N$ where $c(t), d(t)$ are independent of $x \in T$, and
3. they satisfy
   $$\int_0^1 c(t) \, dt = \int_0^1 d(t) \, dt.$$

Then $\rho^{lag}(H; a) = \rho^{lag}(F; a)$ holds for all $a \in H^*(N, \mathbb{Z})$ without ambiguity of constant.

**Proof.** We consider the Hamiltonian path $\phi_G : t \mapsto \phi^1_G$ with $G = \tilde{F} \ast H$ with $\tilde{F}(t, x) = -F(1 - t, x)$. This defines a loop of Lagrangian submanifold

$$t \mapsto \phi^1_G(o_N), \quad \phi^1_G(o_N) = o_N$$
and satisfies $\phi^G_t|_B \equiv id$ and

$$G(t, q) = \begin{cases} -c(1 - 2t) & 0 \leq t \leq 1/2 \\ d(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

for all $q \in B \subset T$ by definition $G = F^* H$.

We claim $\rho^{lag}(G; a) = 0$ for all $0 \neq a \in H^*(N)$. This will be an immediate consequence of the following lemma and the spectrality of numbers $\rho^{lag}(G; a)$.

**Lemma 5.8.** The value $A^G_C(z)$ does not depend on the Hamiltonian chord $z \in Chord(G; o_N, o_N)$. In particular, $A^G_C(z) = 0$.

**Proof.** Recall that any Hamiltonian chord in $Chord(G; o_N, o_N)$ has the form $z(t) = z^G_G(t)$ for some $q \in o_N$. Here we use the hypothesis $\phi^1_G(o_N) = o_N$. Consider any smooth path $\alpha : [0, 1] \rightarrow o_N$ with $\alpha(0) = q$, $\alpha(1) = q'$. Then

$$A^G_{c}(z^G_G(q')) - A^G_{c}(z^G_G(q)) = \int_0^1 \frac{d}{du} A^G_{c} \left( z^G_G^{\alpha(u)} \right) du.$$

But a straightforward computation using the first variation formula (2.1) implies

$$\frac{d}{du} A^G_{c} \left( z^G_G^{\alpha(u)} \right) = \left\{ \theta, \frac{\partial}{\partial u} (\phi_G(\alpha(u))) \right\} - \left\{ \theta, \frac{\partial}{\partial u} (\alpha(u)) \right\} = 0 - 0 = 0$$

since $\phi_G(\alpha(u)), \alpha(u) \in o_N$.

For the second statement, we have only to consider the constant path $z \equiv c_q \in B$ for which

$$A^G_{c}(c_q) = - \int_0^1 G(t, q) dt = \int_0^{1/2} c(1 - 2t) dt - \int_{1/2}^1 d(2t - 1) dt$$

$$= \int_0^1 c(t) dt - \int_0^1 d(t) dt = 0.$$

This proves the lemma. \qed

Once we have the lemma, we can apply the triangle inequality (5.10) $\rho^{lag}(H; a) \leq \rho^{lag}(F; a) + \rho^{lag}(G; 1) = \rho^{lag}(F; a)$ for any given $a \in H^*(N)$. By changing the role of $H$ and $F$ in the proof of the above lemma, we also obtain $\rho^{lag}(G; 1) = 0$ and then obtain $\rho^{lag}(F; a) \leq \rho^{lag}(H; a)$ by triangle inequality. This finishes the proof of the proposition. \qed

This proposition motivates us to introduce the following definitions

**Definition 5.3.** For each given $B \subset N$, we define

$$\mathcal{J}_B(o_N; T^* N) = \{ L \in \mathcal{J}_B(o_N; T^* N) | o_N \cap L \supset o_B \}.$$

When a function $c : [0, 1] \rightarrow \mathbb{R}$ is given in addition, we define

$$\mathcal{P}_c = \{ H \in \mathcal{P}_c : H_t \equiv c(t) \text{ on a neighborhood of } o_B \text{ in } T^* N \}

\text{ and } \int_0^1 c(t) dt = e \}.

With these definitions, the proposition enables us to unambiguously define the following spectral invariant attached to $L$. 
Theorem 6.1. Suppose $L \in \mathcal{S}^\infty(o_N; T^*N)$ and let $e \in \mathbb{R}$ be given. For each given such $e$, we define a spectral invariant of $L \in \mathcal{S}^\infty(B_e)(o_N; T^*N)$ by

$$
\rho^{(B,e)}(L; a) := \rho^\text{log}(H; a), \quad L = \phi_H^1(o_N)
$$

for a (and so any) $H \in \mathcal{PC}^\infty_{(B,e)}$.

With this definition, we have the following obvious lemma

**Lemma 5.9.** Let $H \in \mathcal{PC}^\infty_{(B,e)}$, then $\tilde{H}, \tilde{H} \in \mathcal{PC}^\infty_{(B,-e)}$.

Then we prove the following duality statement of $\rho^{(B,e)}$.

**Proposition 5.10.** Let $H \in \mathcal{PC}^\infty_{(B,e)}$ and $L = \phi_H^1(o_N)$. We denote $\tilde{L} = \phi_{\tilde{H}}^1(o_N) = \phi_{\tilde{H}}^1(o_N)$. Then

$$
\rho^{(B,-e)}(\tilde{L}; 1) = -\rho^{(B,e)}(L; [pt]^#).
$$

(5.11)

**Proof.** By the above lemma, $\tilde{H} \in \mathcal{PC}^\infty_{(B,-e)}$ and so $\rho^{(B,-e)}(\tilde{L}; 1)$ is given by

$$
\rho^{(B,-e)}(\tilde{L}; 1) = \rho^\text{log}(H; 1)
$$

by definition. But it was proven in [V1; Oh2; Oh3] that

$$
\rho^\text{log}(\tilde{H}; 1) = -\rho^\text{log}(H; [pt]^#)
$$

(5.12)

which follows from the Poincaré duality argument, by studying the time-reversal flow of the Floer equation $\tilde{u}$ defined by $\tilde{u}(\tau, t) = u(-\tau, 1 - t)$. The map $\tilde{u}$ satisfies the equation

$$
\begin{align*}
\frac{\partial \tilde{u}}{\partial \tau} + \tilde{J} \left( \frac{\partial \tilde{u}}{\partial \tau} - X_{\tilde{H}}(\tilde{u}) \right) &= 0 \\
\tilde{u}(\tau, 0), \quad \tilde{u}(\tau, 1) &\in o_N.
\end{align*}
$$

Furthermore this equation is compatible with the involution of the path space

$$
\iota : \Omega(o_N, o_N) \rightarrow \Omega(o_N, o_N)
$$

defined by $\iota(\gamma)(t) = \tilde{\gamma}(t)$ with $\tilde{\gamma}(t) = \gamma(1 - t)$ and the action functional identity

$$
\mathcal{A}^\text{cl}_H(\tilde{\gamma}) = -\mathcal{A}^\text{cl}_H(\gamma).
$$

We refer to [Oh3] for the details of the duality argument in the Floer theory used in the derivation of (5.12).

On the other hand, by definition,

$$
\rho^{(B,e)}(H; [pt]^#) = \rho^{(B,e)}(L; [pt]^#)
$$

since $H \in \mathcal{PC}^\infty_{(B,e)}$. This finishes the proof. \hfill \Box

6. Comparison theorem of $f_H$ and $\rho^\text{log}(H; 1)$

We first remark that both $\rho^\text{log}(H; 1)$ and $f_H$ remain unchanged under the change of $H$ outside a neighborhood of $\bigcup_{t \in [0,1]} \phi_t^1(o_N)$.

The main theorem we prove in this section is the following which is closely related to Proposition 5.1 [V1].

**Theorem 6.1.** For any Hamiltonian $H \in \mathcal{PC}^\infty_{\text{asc}}$, 

$$
\rho^\text{log}(H; [pt]^#) \leq \min f_H, \quad \max f_H \leq \rho^\text{log}(H; 1).
$$
For the purpose of studying comparison result given in the next section, we start with this section by adding the following additional symmetry property of $f_H$ and $\rho^ag$ under the reflection $r : T^*N \to T^*N$ defined by $r(q,p) = (q, -p)$. Such a reflection argument was used by Viterbo [V1] in the proof of similar identities in the context of generating function method.

### 6.1. Anti-symplectic reflection and basic phase function.

**Proposition 6.2.** Consider the canonical reflection map $r : T^*N \to T^*N$ given by $r(q,p) = (q, -p)$ and define the Hamiltonian $H^r$ to be $H^r(t,x) = -H(t,r(x))$ for $x = (q,p)$. Then

$$f_{H^r} = -f_H, \quad \rho^ag(H^r; 1) = -\rho^ag(H; [pt]^\#)$$

**Proof.** We observe that the map satisfies $r^*\theta = -\theta$ and in particular is anti-symplectic. It also preserves the zero section and each individual fibers of $T^*N$ and so induces the corresponding reflection map on the path space

$$r : \Omega(L, T^*_qN) \to \Omega(L^r, T^*_qN); z = (q,p) \mapsto r(z) = (q, -p)$$

for each given base point $q \in N$, where $L^r := r(L) = \phi^1_{H^r}(o_N)$. A straightforward computation also shows

$$A_{H^r}(r(z)) = -A_H(z). \quad (6.1)$$

We then consider $J$ satisfying $r^*J = -J$. For example, the standard Sasakian almost complex structure $J_g$ associated any Riemannian metric $g$ on $N$ [Fl3] is such an almost complex structure. Therefore the set of such $J$’s is non-empty. It is also not difficult show that the set is a contractible infinite dimensional manifold. (See Lemma 4.1 [FOOO3] for its proof.)

Then a straightforward computation shows that this reflection map induces one-one correspondence

$$u \mapsto u'; \quad u'(\tau,t) := r(u(-\tau,t))$$

between the set of solutions of the Floer equation [Fl3] and those associated to

$$\begin{cases} \frac{du'}{d\tau} + J(\frac{du'}{d\tau} - X_{H^r}(u')) = 0 \\ u'(\tau,0) \in o_N, u'(\tau,1) \in T^*_qN. \end{cases}$$

Furthermore all the generic transversality statements are equivalent for $u$ and $u'$ for $J$’s satisfying $r^*J = -J$ via the transformation of the Hamiltonian $H \mapsto H^r$. Therefore $r$ induces canonical isomorphism

$$r_* : HF_*(H; o_N, T^*_qN) \to HF_*(H^r; o_N, T^*_qN).$$

We also recall the canonical isomorphism established for arbitrary generic $H$ in [Oh2]

$$HF_*(H; o_N, T^*_qN) \cong H_*(\{pt\}) \cong \mathbb{Z}$$

which has rank 1. Therefore $(r)_*(|pt|_H) = \pm|pt|_{H^r}$. The first equality then follows from these observations and [6.1] by the general construction of spectral invariants $\rho^ag(H; \{q\})$ given in section 5 especially (the Lagrangian version of) Conformality Axiom [Oh6].

A similar consideration based on [6.1] with the boundary condition

$$u'(\tau,0) \in o_N, u'(\tau,1) \in o_N.$$
Figure 1. Multi-section Lagrangian $L$

gives rise to the second identity by the same kind of duality argument as done to prove (5.12) in [Oh3]. We omit the details by referring readers thereto for the details. This finishes the proof.

6.2. Analysis of Example 9.4 [Oh2]. Before giving the proof of Theorem 6.1, we illustrate the inequalities by a concrete example, which is a continuation of Example 9.4 [Oh2].

Example 6.1. Consider the Lagrangian submanifold $L$ in $T^*S^1$ pictured as in Figure 1 whose coordinates we denote by $(q,p)$. One can check that the wave front projection of $L$, i.e., the graph of the multi-valued function $h_H$ of the associated Hamiltonian $H$ such that $L = \phi^1_H(o_{S^1})$ can be drawn as in Figure 2 in $S^1 \times \mathbb{R}$ whose coordinates we denote by $(q,a)$.

Here we denote by $z_i = (q_i, 0)$ below for $i = 0, \ldots, 3$ the intersections of $L$ with the zero section, and by $x_i$ $i = 1, 2$ the caustics and by $y$ the point at which the two regions between the graph and the dotted line have the same area in Figure 1. Note that the points $z_i$'s are the critical points of the multi-valued generating function $h_H$ (or correspond to critical points of the action functional), $x_i$'s to the cusp points of the wave front and $y$ is the crossing point of two different branches of the wave front projection.

Using the continuity of the basic phase function $f_H$ where $L = \phi^1_H(o_N)$, one can easily see that the graph of $f_H$ is the one bold-lined in Figure 2. We would like to note that the value $\min_{q \in \mathbb{N}} f_H(q)$ is not a critical value of $A_H^a$, and the branch of the wave front containing the point $(q_1, a_1)$ associated to the critical point $z_1$ of $h_H$ is eliminated from the graph of the basic phase function $f_H$.

We note that the Floer complex $CF(L, o_{S^1}) \cong \bigoplus_{i=0}^3 \mathbb{Z}\{z_i\}$ and its boundary map is given by

$$\partial(z_0) = z_1 - z_3, \partial(z_1) = 0 = \partial(z_2), \partial(z_3) = z_3 - z_1.$$  
(Here we take $\mathbb{Z}$-coefficients to avoid precise checking of the signs which is irrelevant for the study of this example.) From this we derive

$$\ker \partial = \mathbb{Z}\{z_1, z_3, z_0 + z_2\}, \quad \text{Im} \partial = \mathbb{Z}\{z_1 - z_3\}.$$ 

Therefore the class 1 is realized by the Floer cycle $z_0 + z_2$ (or any other class of the form $z_0 + z_2 + \partial(\alpha)$) and the class $[pt]^\#$ is realized by the Floer cycle of the form $z_1$. 

(or any other class of the form $z_1 + \partial(\beta)$. A simple examination of Figure 1 and 2 and comparison of the action values associated to the intersection points $z_1$ and $z_3$ shows that the infimum of the level $\lambda_H(z_1 + \partial(\beta))$, which is nothing by $\rho(H; [pt]^{#})$ by definition, is realized by the Floer cycle represented by the intersection point $z_1$.

Therefore we obtain

$$
\rho(H; [pt]^{#}) = A_H(z_1^H)
$$

which is denoted as $a_1$ in Figure 2, where $z_1^H$ is the Hamiltonian path given by $z_1^H(t) := \phi_t^H(\phi_1^H)^{-1}(z_1)$.

Combining the above discussion on $\rho^{lag}$ and comparing them with the values of $f_H$, we can easily obtain from Figure 2 that

$$
\rho(H; [pt]^{#}) < \min f_H < \max f_H = \rho(H; 1)
$$

It is interesting to observe two peculiar phenomena in this example:

1. the minimum of $f_H$ is realized at a non-smooth point $y \in N$ of the function $f_H$, and
2. the value $\rho(H; [pt]^{#})$ is realized by the ‘local maximum’ of the branch of $h_H$ containing the point $(q_1, a_1) \in S^1 \times \mathbb{R}$ where $q_1 = \pi(z_1)$ and $a_1 = A_H(z_1^H)$.

6.3. Proof of comparison result on $\rho^{lag}(H; 1)$ and $f_H$. We now go back to the proof of Theorem 6.1. We first remark that the second inequality in Theorem 6.1 immediately follows by applying the first inequality to the Hamiltonian $H^r$ and combining Proposition 6.2.

Therefore it remains to prove the inequality $\max f_H \leq \rho(H; 1)$, which will occupy the rest of this section.

We first recall the definition of the triangle product described in [Oh3, FO] and put it into a more modern context in the general Lagrangian Floer theory such as in [FOOO1] and in other more recent literatures.

Let $q \in N$ be given. Consider the Hamiltonians $H : [0, 1] \times T^*N \to \mathbb{R}$ such that $L_H$ intersects transversely both $o_N$ and $T_q^*N$. We consider the Floer complexes

$$
CF(L_H, o_N), \quad CF(o_N, T_q^*N), \quad CF(L_H, T_q^*N)
$$
each of which carries filtration induced from the effective action function given in Proposition 5.3. We denote by \( v(\alpha) \) the level of the chain \( \alpha \) in any of these complexes.

More precisely, \( CF(L_H, o_N) \) is filtered by the effective functional
\[
A^{(1)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0)),
\]
\( CF(o_N, T_q^* N) \) by
\[
A^{(2)}(\gamma) := \int \gamma^* \theta,
\]
and \( CF(L_H, T_q^* N) \) by
\[
A^{(0)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0))
\]
respectively. We recall the readers that \( h_H \) is the potential of \( L_H \) and the zero function the potentials of \( o_N, T_q^* N \).

We now consider the triangle product in the chain level, which we denote by \( m_2 : CF(L_H, o_N) \otimes CF(o_N, T_q^* N) \to CF(L_H, T_q^* N) \) following the general notation from [FOOO1], [Se]. This product is defined by considering all triples \( x_1 \in L_H \cap o_N, x_2 \in o_N \cap T_q^* N, x_0 \in L_H \cap T_q^* N \) with the polygonal Maslov index \( \mu(x_1, x_2; x_0) \) whose associated analytical index, or the virtual dimension of the moduli space
\[
M_3(D^2; x_1, x_2; x_0) := \tilde{M}_3(D^2; x_1, x_2; x_0)/PSL(2, \mathbb{R})
\]
of \( J \)-holomorphic triangles, becomes zero and counting the number of elements thereof. The precise formula of the index is irrelevant to our discussion which, however, can be found in [Se], [FOOO2].

**Definition 6.2.** Let \( J = J(z) \) be a domain-dependent family of compatible almost complex structures with \( z \in D^2 \). We define the space \( \tilde{M}_3(D^2; x_1, x_2; x_0) \) by the pairs \( (w, (z_0, z_1, z_2)) \) that satisfy the following:

1. \( w : D^2 \to T_q^* N \) is a continuous map satisfying \( \overline{\partial}_J w = 0 \ D^2 \setminus \{z_0, z_1, z_2\} \),
2. the marked points \( \{z_0, z_1, z_2\} \subset \partial D^2 \) with counter-clockwise cyclic order,
3. \( w(z_1) = x_1, w(z_2) = x_2 \) and \( w(z_0) = x_0 \),
4. the map \( w \) satisfies the Lagrangian boundary condition
   \[
w(\partial_1 D^2) \subset L_H, \ w(\partial_2 D^2) \subset o_N, \ w(\partial_3 D^2) \subset T_q^* N
   \]
   where \( \partial_i D^2 \subset \partial D^2 \) is the are segment in between \( x_i \) and \( x_{i+1} \) (\( i \) mod 3).

The general construction is by now well-known and e.g., given in [FOOO1]. In the current context of exact Lagrangian submanifolds, the detailed construction is also given in [Oh3] and [Se]. One important ingredient in relation to the study of the effect on the level of Floer chains under the product is the following (topological) energy identity where the choice of the effective action functional plays a crucial role. For readers’ convenience, we give its proof here.
Proposition 6.3. Suppose \( w : D^2 \to T^*N \) be any smooth map with finite energy that satisfy all the conditions given in (6.2) but not necessarily \( J \)-holomorphic. We denote by \( c_x : [0,1] \to T^*N \) the constant path with its value \( x \in T^*N \). Then we have

\[
\int w^* \omega_0 = \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}). \tag{6.3}
\]

Proof. Recall \( \omega_0 = -d\theta \) and \( i^*\theta = dh_H \) on \( L_H \) and \( i^*\theta = 0 \) on \( \alpha_N \) and \( T^*_qN \) where \( i \)'s are the associated inclusion maps of \( L_H, \alpha_N, T^*_qN \subset T^*N \) respectively. Therefore

\[
\int_{D^2} w^* \omega_0 = -\int_{\partial D^2} w^* \theta = -\int_{\partial_1 D^2} w^* \theta - \int_{\partial_2 D^2} w^* \theta - \int_{\partial_3 D^2} w^* \theta
\]

\[
= -\int_{\partial_1 D^2} w^* dh_H - 0 - 0 = h_H(w(z_1)) - h_H(w(z_2))
\]

\[
= h_H(x_1) - h_H(x_0) = \mathcal{A}^{(1)}(c_{x_1}) - \mathcal{A}^{(0)}(c_{x_0})
\]

\[
= \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}).
\]

Here the last equality comes since \( \mathcal{A}^{(2)}(c_{x_2}) = \int c_{x_2}^* \theta = 0 \). This finishes the proof. \( \square \)

An immediate corollary of this proposition from the definition of \( m_2 \) is that the map \( (6.2) \) restricts to

\[
m_2 : CF^\lambda(L_H, \alpha_N) \otimes CF^\mu(\alpha_N, T^*_qN) \to CF^{\lambda+\mu}(L_H, T^*_qN).
\]

It is straightforward to check that this map satisfies

\[
\partial(m_2(x,y)) = m_2(\partial(x),y) \pm m_2(x,\partial(y))
\]

and in turn induces the product map

\[
\ast_F : HF^\lambda(L_H, \alpha_N) \otimes HF^\mu(\alpha_N, T^*_qN) \to HF^{\lambda+\mu}(L_H, T^*_qN) \tag{6.4}
\]

in homology. This is because if \( w \) is \( J \)-holomorphic \( \int w^* \omega \geq 0 \). (We refer to [Oh3] and [FO] for the general construction of product map \( m_2 \) and to [Oh3], [MVZ] for the study of filtration. Similar study of filtration is also performed in [Sc], [Oh6] in the Hamiltonian Floer homology setting.)

With these preparations, we are ready to wrap-up the proof of Theorem 6.1.

Proof of Theorem 6.1 We consider a Floer cycle \( \alpha \) representing the fundamental class \( 1^i = [M] \in HF(L_H, \alpha_N) \) and \( \beta = \{q\} \) representing the unique generator of \( HF(\alpha_N, T^*_qN) \cong \mathbb{Z} \). Then by definition

\[
v(\alpha) \geq \rho^{lag}(H;1), \quad v(\beta) = \rho^{lag}(0;[q]) = 0.
\]

Then its product cycle \( m_2(\alpha,\beta) \in CF(L_H, T^*_qN) \) represents the homology class \( [q] \in CF(L_H, T^*_qN) \cong \mathbb{Z} \) and so \( v(m_2(\alpha,\beta)) \geq \rho^{lag}(H;[q]) = f_H(q) \) by definition of the latter. Applying the triangle inequality, we obtain

\[
v(\alpha) + 0 = v(\alpha) + v(\beta) \geq v(m_2(\alpha,\beta)) \geq \rho^{lag}(H;[q]) = f_H(q).
\]

Therefore we have derived

\[
v(\alpha) \geq f_H(q)
\]

for all cycle \( \alpha \in CF(L_H, \alpha_N) \) representing \( [M] \). By definition of \( \rho^{lag}(H;1) \), this proves

\[
\rho^{lag}(H;1) \geq f_H(q).
\]
Since this holds for any point \( q \in N \), we have proved \( \rho^{\text{log}}(H; 1) \geq \max f_H. \)

7. A Hamiltonian \( C^0 \) continuity of spectral Lagrangian capacity

We first recall the definition of the function \( \gamma_{\text{log}}^L : \mathcal{B}_B(0_N; T^*N) \to \mathbb{R} \) defined by

\[
\gamma_{\text{log}}^L(L) = \rho_{\text{log}}^L(H; 1) - \rho_{\text{log}}^L(L_H; [pt]#)
\]

for \( L = \phi_H^1(0_N) \) with \( H \in \mathcal{P}_{\text{asc}}^N \).

In this section, we prove the following Hamiltonian \( C^0 \)-continuity result of the function which is the Lagrangian analog to Theorem 1 [Sey1].

**Theorem 7.1.** The function \( \gamma_{\text{log}}^L : \mathcal{B}_B(0_N; T^*N) \to \mathbb{R} \) is continuous with respect to the Hamiltonian \( C^0 \)-topology in the sense of Definition [12].

The triangle inequality of \( \gamma_{\text{log}}^L \) stated in section 5.3 implies the inequalities

\[
\left| \gamma_{\text{log}}^L(L_1) - \gamma_{\text{log}}^L(L_2) \right| \leq \max \left\{ \gamma_{\text{log}}(\phi^{-1}_H(\phi_H^1(0_N)); 0_N), \gamma_{\text{log}}(\phi^{-1}_H(\phi_H^1(0_N)); 0_N) \right\}.
\]

We also note that for \( L_k = \phi_H^1(0_N) \in \mathcal{B}_B(0_N; T^*N) \) for \( k = 1, 2 \)

\[
\max \{ d_{C^0}(\phi^{-1}_H(\phi_H^1(0_N)); 0_N), d_{C^0}(\phi^{-1}_H(\phi_H^1(0_N)); 0_N) \} \to 0
\]

if and only if

\[
\max \{ d_{C^0}(\phi^{-1}_H(\phi_H^1(0_N)), \phi_H^1(0_N)), d_{C^0}(\phi^{-1}_H(\phi_H^1(0_N)), (\phi_H^1)^{-1}(0_N)) \} \to 0
\]

provided we assume \( \text{supp} \phi_H^k \) is compact and so \( \text{supp} \phi_H^k \subset D^B(T^*N) \setminus T, k = 1, 2 \), for some \( R > 0 \) and \( T \supset B \). The latter assumption is already embedded in the definition of Hamiltonian topology given in Definition 1.2.

Therefore to prove the above theorem, it is enough to prove the continuity of \( \gamma_{\text{log}}^L \) at the zero section \( 0_N \) in \( \mathcal{B}_B(0_N; T^*N) \).

By unravelling the definition of Hamiltonian \( C^0 \)-topology on \( \mathcal{B}_B(0_N; T^*N) \) given in Definition 1.2, we now rephrase the continuity statement at the zero section \( 0_N \) more explicitly. For this purpose, we introduce the notation

\[
\text{osc}_{C^0}(\phi^1_H; 0_N) := \max \left\{ \max_{x \in 0_N} d(\phi^1_H(x), x), \max_{x \in 0_N} d((\phi^1_H)^{-1}(x), x) \right\}.
\]

Then it is easy to see that this continuity at \( 0_N \) is equivalent to the following

**Theorem 7.2.** Let \( \lambda_i = \phi_{H_i} \), where \( H_i \in \mathcal{P}_{\text{asc}}^N \) is a sequence such that

1. \( H_i \in \mathcal{P}_{\text{asc}}^N \) for some \( R, K > 0 \) for all \( i \) and \( s \in [0, 1] \),
2. There exists a closed ball \( B \subset N \) such that \( \phi_{H_i}^1 \equiv \text{id} \) on \( B \) for all \( t \in [0, 1] \) for all \( i \),
3. There exists a uniform neighborhood \( T \supset o_B \) in \( T^*N \) such that \( \phi_{H_i}^1 \equiv \text{id} \) on \( T \) for all \( i \),
4. \( \lim_{i \to \infty} \text{osc}_{C^0}(\phi^1_H; 0_N) = 0 \).

Then

\[
\lim_{i \to \infty} (\rho_{\text{log}}^L(H_i; 1) - \rho_{\text{log}}^L(L_{H_i}; [pt]#)) = 0.
\]

The proof of this theorem is an adaptation to the Lagrangian context of the one used by Seyfaddini in his proof of Theorem 1 (or rather Corollary 1.3) [Sey1]. The proof is also a variation of Ostrover’s scheme used in [Os] and is an adaptation thereof. In our proof, we however use the Lagrangian analog to the notion of ‘\( \varepsilon \)-shiftability’ introduced by Seyfaddini [Sey1] instead of ‘displaceability’ used in [Os].
and in other literature such as [EP], [U]. In the Lagrangian context here, the ε-shiftable domain is realized as the graph of \( df \) of a function \( f \) having no critical points on the corresponding domain. In this regard, it appears to the author that the notion of ε-shiftability becomes more geometric and intuitive in the Lagrangian context than in the Hamiltonian context.

7.1. ε-shifting of the zero section by the differential of function. Fix a Riemannian metric \( g \) and the Levi-Civita connection on \( N \). They naturally induces a metric on \( T^*N \). Denote the latter metric on \( T^*N \) by \( \tilde{g} \) and the corresponding distance function by \( \tilde{d}(x,y) \) for \( x, y \in T^*N \). We denote by \( D'(T^*N) \) the disc bundle of \( T^*N \) of radius \( r \).

The following is the well-known fact on this metric \( \tilde{g} \), which can be easily checked.

**Lemma 7.3.** The metric \( \tilde{g} \) carries following properties:

1. \( \tilde{g} \) is invariant under the reflection \( (q,p) \mapsto (q,-p) \) and in particular \( o_N \) is totally geodesic.
2. There exists a sufficiently small \( r = r(N, g) > 0 \) depending only on \((N, g)\) such that
   (a) for all \( d(q,q') < r \) \( \tilde{d}(o_N, o_N) = d(q,q') \),
   (b) for all \( x \in D'(T^*N) \), which we denote \( x = (q(x), p(x)) \),
   \[
   d(o_{q(x)}, x) \geq \max\{|p(x)|, d(q, q(x))\} \geq |p(x)|
   \] (7.1)
   where \( |p(x)| \) is the norm on \( T_{q(x)}N \).

From now on, we will drop ‘tilde’ from \( \tilde{d} \) and just denote by \( d \) even for the distance function of \( \tilde{g} \) on \( T^*N \) which should not confuse readers.

Consider the subset
\[
C^\infty_{\text{crit}}(N; B) = \{ f \in C^\infty(N) \mid \text{Crit } f \subset \text{Int } B \}.
\]
The set \( C^\infty_{\text{crit}}(N; B) \subset C^\infty(N) \) has the filtration
\[
C^\infty_{\text{crit}}(N; B) = \bigcup_T C^\infty_{\text{crit}}(N; B, T)
\]
where \( C^\infty_{\text{crit}}(N; B, T) \) is the subset of \( C^\infty_{\text{crit}}(N; B) \) that consists of \( f \)'s satisfying
\[
\text{Graph}(df|_B) \subset T.
\] (7.2)

It is easy to check that \( C^\infty_{\text{crit}}(N; B, T) \neq \emptyset \) for any such \( T \supset o_B \) by considering the \( \lambda f \) for a sufficiently small \( \lambda > 0 \) for any given Morse function \( f \) with \( \text{Crit } f \subset \text{Int } B \).

We now introduce the collection, denoted by \( T(B,r) \), of the pairs \( (T, f) \) consisting of a tubular neighborhood \( T \supset o_B \) in \( T^*N \) and a Morse function \( f \in C^\infty_{\text{crit}}(N; B, T) \) such that
\[
\text{Graph } df \subset D'(T^*N)
\] (7.3)
for the constant \( r = r(N, g) \) given in Lemma [7.3],

By the choice of the pair \( (T, f) \in T(B,r) \), we have
\[
\min \left\{ \min_{p \in N \setminus B} |df(p)|, d_H(N \setminus B, \text{Crit } f) \right\} > 0.
\]
where \( d_H(N \setminus B, \text{Crit } f) \) is the Hausdorff distance.
Definition 7.1. We define a positive constant
\[
C_{(f;B,T)} := \min \left\{ \min_{p \in N \setminus B} |df(p)|, d_H(N \setminus B, \text{Crit } f) \right\}
\] (7.4)
By definition of \(C_{(f;B,T)}\), if \(q \in N \setminus B\), we have
\[
|df(q)|, d(q, \text{Crit } f) \geq C_{(f;B,T)} > 0.
\] (7.5)
Lemma 7.4. For any \(f \in C_{\text{crit}}^\infty(N;B,T)\),
\[
C_{(\delta f;B,T)} = \min_{p \in N \setminus B} |d(\delta f)(p)|
\]
whenever \(\delta > 0\) is so small that
\[
\min_{p \in N \setminus B} |d(\delta f)(p)| < d_H(N \setminus T, B).
\]
In particular, for such \(\delta > 0\),
\[
\lambda C_{(\delta f;B,T)} = C_{(\lambda \delta f;B,T)}
\] (7.6)
for any \(\lambda \leq 1\).

Proof. First note that the distance \(d_H(N \setminus B, \text{Crit } (\delta f))\) does not depend on \(\lambda\) and that
\[
\min_{p \in N \setminus B} |\delta df(p)| = \delta \min_{p \in N \setminus B} |df(p)| \to 0
\]
as \(\delta \to 0\). Therefore the minimum in the definition
\[
C_{(\delta f;B,T)} = \min \left\{ \min_{p \in N \setminus B} |d(\delta f)(p)|, d_H(N \setminus B, \text{Crit } (\delta f)) \right\}
\]
is realized by \(\min_{p \in N \setminus B} |d(\delta f)(p)|\) for all sufficiently small \(\delta > 0\). Then the lemma follows.

Now we consider the Hamiltonians \(H\) adapted to the triple \((f;B,T)\) as in the definition of Hamiltonain \(C^\infty_{\text{asc}}\)-topology of \(\mathfrak{so}_B(o_N;T^*N)\).

Lemma 7.5. Let \(T \supset o_B\) in \(T^*N\) and \(H \in \mathcal{P}C_{\text{asc}}^{\infty}B\) satisfy
\[
\phi^1_H \equiv id
\] (7.7)
on \(T\). Then we have
\[
L_f \cap o_N = \phi^1_H(L_f) \cap o_N
\]
whenever \(H\) satisfies
\[
osc_{C^\infty_{\text{asc}}}(\phi^1_H; o_N) < C_{(f;B,T)}.
\] (7.8)
In particular all the Hamiltonian trajectories of \(H\#(f \circ \pi)\), are constant equal to \(o_p\) for some point \(p \in \text{Crit } f\) for such Hamiltonian \(H\).

Proof. In the proof, we will denote \(p \in N\) and the corresponding point in the zero section of \(T^*N\) by \(o_p\) for the notational consistency.

Obviously we have \(\text{Crit } f = L_f \cap o_B \subset \phi^1_H(L_f) \cap o_N\) since we assume \(\phi^1_H \equiv id\) on a neighborhood, \(T\), of \(o_B \supset \text{Crit } f\).

We will now prove the opposite inclusion \(\phi^1_H(L_f) \cap o_N \subset L_f \cap o_B\). Suppose \(o_p \in \phi^1_H(L_f) \cap o_N\). Then we have \((\phi^1_H)^{-1}(o_p) \in L_f\).

Consider first the case \(p \in B\). In this case since we assume \(\phi^1_H = id\) on a neighborhood of \(o_B\), it in particular implies \(o_p = (\phi^1_H)^{-1}(o_p)\) for all \(i\) and hence \(o_p \in o_B \cap L_f \equiv \text{Crit } f\).
Now we will show that \( p \) cannot lie in \( N \setminus B \). Suppose \( p \in N \setminus B \) to the contrary and write
\[
(\phi^1_H)^{-1}(o_p) = df(p')
\]
for some \( p' \in N \). Therefore
\[
d(o_p, df(p')) = d(o_p, (\phi^1_H)^{-1}(o_p)) \leq \text{osc}_{C^\alpha}(\phi^1_H; o_N).
\]
Furthermore we also have \( |df(p')| \leq d(o_p, df(p')) \) by Lemma \( 7.3 \) since Graph \( df \subseteq D'(T^* N) \). Therefore we have shown
\[
|df(p')| \leq \text{osc}_{C^\alpha}(\phi^1_H; o_N) < C_{(f,B,T)}.
\]
(7.9)
This in particular implies \( (\phi^1_H)^{-1}(o_p) = df(p') \) must lie in Graph \( df|_B \subseteq T \) for otherwise \( |df(p')| \geq C_{(f,B,T)} \) by definition of \( C_{(f,B,T)} \) which would contradict to (7.3).

This in turn implies \( (\phi^1_H)^{-1}(o_p) \in T \). But \( \phi^1_H \) is assumed to be the identity map on \( T \) and hence follows
\[
o_p = (\phi^1_H)^{-1}(o_p) = df(p').
\]
In particular \( df(p') \in o_N \) and so \( p' \in \text{Crit } f \) and hence \( o_{p'} = df(p') \). This implies \( p = p' \) and so \( d(p, \text{Crit } f) = 0 \), i.e., \( p \in \text{Crit } f \subseteq B \), a contradiction to the hypothesis \( p \in N \setminus B \). Therefore \( p \) cannot lie in \( N \setminus B \) and hence proves \( o_p \in o_B \cap L_f \equiv \text{Crit } f \) for any \( o_p \in \phi^1_H(L_f) \cap o_N \). This then finishes the proof of the first statement
\[
L_f \cap o_N = \phi^1_H(L_f) \cap o_N.
\]
(7.10)
To prove the second statement, the first statement of the lemma implies that all the Hamiltonian trajectories of \( H \# f \circ \pi \) ending at a point \( \phi^1_H(L_f) \cap o_N \) have the form
\[
z_p^{H \# f \circ \pi}(t) = \phi^1_H(t) \circ \pi((\phi^1_H)^{-1}(o_p))
\]
for some intersection point \( o_p \in \phi^1_H(L_f) \cap o_N = L_f \cap o_N \). By definition, we have
\[
z_p^{H \# f \circ \pi}(1) = o_p.
\]
But we also have \( df(p) = 0 \) and \( (\phi^1_H)^{-1}(o_p) = o_p \) since
\[
o_p \in \phi^1_H(L_f) \cap o_N = L_f \cap o_N \subset o_B \cap \text{Crit } f
\]
and \( \phi^1_H \equiv \text{id} \) near \( p \). Therefore
\[
(\phi^1_H)^{-1}(o_p) = (\phi^0)\mathcal{f}(o_p) = o_p.
\]
Therefore
\[
z_p^{H \# f \circ \pi}(t) = \phi^1_H(t) \circ \pi((\phi^1_H)^{-1}(o_p)) = \phi^1_H(t) \circ \pi(o_p) = \phi^1_H(t) \circ \pi(o_p) \]
since \( df(p) = 0 \) and \( \phi^1_H(o_p) = o_p \) for all \( t \in [0, 1] \). The last statement follows since we assume \( \text{supp } \phi_H \cap o_B = \emptyset \). By compactness of \( \text{supp } \phi_H \) and the closeness of \( B \), \( \text{supp } \phi_H \cap o_B = \emptyset \) implies \( \phi^1_H \equiv \text{id} \) for all \( t \in [0, 1] \) on a neighborhood \( T' \supset o_B \) in \( T^* N \).

This finishes the proof. \( \square \)

**Remark 7.2.** We would like to mention that in the above proof, the choice of the neighborhood \( T' \supset B \) is allowed to vary depending on \( H \)'s. This is because our Hamiltonian \( C^\alpha \)-topology requires only \( \text{supp } \phi^1_H \cap o_B = \emptyset \) for \( t \in [0, 1] \), not the existence of uniform neighborhood \( T \supset o_B \) independent of \( H \). It only requires existence of such uniform neighborhood for the time-one map \( \phi^1_H \).
Remark 7.3. In fact all the discussion in this subsection can be generalized by replacing the differential $df$ by any closed one form $\alpha$ and $\text{Crit } f$ by the zero set of $\alpha$. But we restrict to the exact case since the discussion in the next subsection seems to require the exactness of the form.

7.2. Lagrangian capacity versus Hamiltonian $C^0$-fluctuation. In fact, Theorem 7.2 is an immediate consequence of the following comparison result between the Lagrangian capacity $\gamma_B^\text{lag}(L) = \rho^\text{lag}(H; 1) - \rho^\text{lag}(H; [pt])$ and the Hamiltonian $C^0$-fluctuation $\text{osc}_{C^0}(\phi^1_H; \alpha_N)$ for $L = \phi^1_H(\alpha_N)$ for $H \in \mathcal{P}^\text{asc}_B$, which itself has some independent interest in its own right.

Theorem 7.6. Let $B \subset N$ be a closed ball and $(T, f) \in \mathcal{T}(B, T)$. Consider the set of Hamiltonians $H$ satisfying $\text{supp } \phi_H \cap o_B = \emptyset$ and assume

$$\text{osc}_{C^0}(\phi^1_H; \alpha_N) < C(f; B, T)$$

Then we have

$$\frac{\gamma_B^\text{lag}(L)}{\text{osc}_{C^0}(\phi^1_H; \alpha_N)} \leq \frac{2 \text{osc}_f}{C(f; B, T)}$$

(7.11)

for $L = \phi^1_H(\alpha_N)$.

We would like to mention that the right hand side of (7.11) does not depend on the scale change of $f$ to $\delta t$ for $\delta > 0$.

The following question seems to be an interesting question to ask in regard to the precise estimate of the upper bound in this theorem and Question 1.5.

Question 7.4. For given $H$ satisfying the condition in Theorem 7.6 what is an optimal estimate of the constant $\frac{2 \text{osc}_f}{C(f; B, T)}$ in terms of $B$, $T$ and $H$? For example, can we obtain an upper bound independent of $B$ or $T$?

The rest of the section is occupied by the proof of Theorem 7.6. The following proposition is a crucial ingredient of the proof, which is a variation of Proposition 2.6 [Os], Proposition 3.3 [EP], Proposition 3.1 [U] and Proposition 2.3 [Sey1].

Proposition 7.7. Let $H \in \mathcal{P}^\text{asc}_B$ in $T^*N$ such that

$$\text{supp } \phi_H \cap o_B = \emptyset.$$ 

(7.12)

Take any $f \in C^\infty_c(N; B)$ such that (7.3) holds. Then

$$\rho^\text{lag}(H; 1) - \rho^\text{lag}(H; [pt]) \leq 2 \text{osc}_f.$$ 

(7.13)

Proof. Denote $L_f := \text{Graph } df$, $L_t = \phi^1_H(L_f) = \phi^1_H(\text{Graph } df)$. Note that the condition (7.12) implies

$$H_t|_B \equiv c_B(t)$$

(7.14)

for a function $c_B = c_B(t)$ depending only on $t$ but not on $x \in B$.

The following lemma is the analogue of Lemma 5.1 [Os].

Lemma 7.8.

$$\rho^\text{lag}(H\# f; 1) - \rho^\text{lag}(H\# f; [pt]) \leq \text{osc}_f.$$ 

(7.15)

Proof. By the spectrality of $\rho^\text{lag}(\cdot; 1)$ in general, we have

$$\rho^\text{lag}(H\# f \circ \pi; 1) = A^d_{(H\# f \circ \pi; 1)}(\gamma_{H\# f \circ \pi}^\text{lag});$$

$$\rho^\text{lag}(H\# f \circ \pi; [pt]) = A^d_{(H\# f \circ \pi; [pt])}(\gamma_{H\# f \circ \pi}^\text{lag});$$

where

$$A^d_{(H\# f \circ \pi)}(\gamma_{H\# f \circ \pi}^\text{lag});$$

$$A^d_{(H\# f \circ \pi; [pt])}(\gamma_{H\# f \circ \pi}^\text{lag});$$

and

$$\gamma_{H\# f \circ \pi}^\text{lag};$$

are the asymptotic Lagrangian capacities associated to $H\# f \circ \pi$ and $H\# f \circ \pi; [pt]$, respectively.
for some \( p_\pm \in L_f \cap o_N \). Using the second statement of Lemma \[\text{(7.3)}\] we compute
\[
A^d_{(H \# f \circ \pi)} \left( z^{H \# f \circ \pi} \right)_{p_+} - A^d_{(H \# f \circ \pi)} \left( z^{H \# f \circ \pi} \right)_{p_-} \\
= - \int_0^1 (H \# f \circ \pi)(t, p_+) \, dt + \int_0^1 (H \# f \circ \pi)(t, p_-) \, dt \\
= - \int_0^1 c_B(t) \, dt - f(p_+) + \int_0^1 c_B(t) \, dt + f(p_-) \\
= - f(p_+) + f(p_-) \leq \max f - \min f = \text{osc} \, f.
\]
Here for the equality in the line next to the last, we use the identity
\[
(H \# f \circ \pi)(t, p_\pm) = H(t, p_\pm) + f(\phi_H^t(p_\pm)) = c_B(t) + f(p_\pm).
\]
This finishes the proof. \qed

On the other hand, we have
\[
\phi_H^1(L_f) = \phi_H^1(\phi_{f \circ \pi}(o_N)) = \phi_H^1(o_N)
\]
and so by the triangle inequality, Proposition \[\text{(5.6)}\]
\[
\rho^{\log}(H \#(f \circ \pi); 1) \geq \rho^{\log}(H; 1) - \rho^{\log}(- f \circ \pi; 1) \\
\rho^{\log}(H \#(f \circ \pi); [pt]#) \leq \rho^{\log}(H; [pt]#) + \rho^{\log}(f \circ \pi; 1).
\]
(One can also use Proposition \[\text{(5.6)}\] using the concatenation \( H \ast (f \circ \pi) \) instead. Here \( f \circ \pi \) is not boundary flat, which is required in Proposition \[\text{(5.6)}\] but one can always reparameterize the flow \( t \mapsto \phi_f^t \) by multiplying \( \chi(t) \) to \( f \circ \pi \) so that the perturbation is as small as we want in \( L^{(1,\infty)} \) topology which in turn perturbs \( \rho \) slightly. See Lemma 5.2 \[\text{[Oh1]}\], Remark 2.5 \[\text{[MVZ]}\] for the precise statement on this approximation procedure. This enables us to apply the triangle inequality in Proposition \[\text{(5.6)}\] in the current context.)

Therefore subtracting the second inequality from the first and using the identity
\[
\rho^{\log}(- f \circ \pi; 1) = \max f, \quad \rho^{\log}(f \circ \pi; 1) = - \min f
\]
(see \[\text{[Oh3]}\] for its proof), we obtain
\[
\rho^{\log}(H \#(f \circ \pi); 1) - \rho^{\log}(H \#(f \circ \pi); [pt]#) \geq \rho^{\log}(H; 1) - \rho^{\log}(H; [pt]#) - (\max f - \min f)
\]
which in turn gives rise to
\[
\rho^{\log}(H; 1) - \rho^{\log}(H; [pt]#) \leq \rho^{\log}(H \#(f \circ \pi); 1) - \rho^{\log}(H \#(f \circ \pi); [pt]#) + (\max f - \min f) \leq 2 \text{osc} \, f.
\]
We have finished the proof of the proposition. \qed

We now go back to the proof of Theorem \[\text{(7.6)}\].

Let \( H \in \mathcal{P}^{\text{asc}; B} \) and \( T \supset o_B \) such that \( \phi_H^1 = \text{id} \) on \( T \) and assume \[\text{(7.8)}\].

If \( \text{osc}_{C^0}(\phi_H^1; o_N) = 0 \), we have \( \phi_H^1(o_N) = o_N \) and so \( \rho^{\log}(H; 1) - \rho^{\log}(H; [pt]#) = 0 \) for which \[\text{(7.13)}\] obviously holds. Therefore we assume \( \text{osc}_{C^0}(\phi_H^1; o_N) \neq 0 \).

Recall from Lemma \[\text{(7.5)}\] that the choice of \( f \) depends only on the ball \( B \) and the neighborhood \( T \supset o_B \) in \( T^* N \). Then we choose \( \lambda > 0 \) such that
\[
\text{osc}_{C^0}(\phi_H^1; o_N) = \lambda C_{(f; B, T)}
\]
\[ \lambda = \frac{\text{osc}_{C^0}(\phi_H^1; o_N)}{C_{(f;B,T)}}. \]

Obviously we have
\[ \text{osc}_{C^0}(\phi_H^1; o_N) < (\lambda + \varepsilon) C_{(f;B,T)} \]
for all \( \varepsilon > 0 \). We note that both \( d_H(N \setminus B, \text{Crit}(\delta f)) \) and the ratio \( \frac{\text{osc}_{f}}{C_{(f;B,T)}} \) do not depend on the choice of \( \delta > 0 \).

Therefore we can replace \( f \) by \( \delta f \) for a sufficiently small \( \delta > 0 \), if necessary, so that
\[ \min_{p \in N \setminus B} |d(\lambda f)(p)| < d_H(N \setminus B, \text{Crit}(\delta f)) \tag{7.16} \]
which in turn implies
\[ \lambda C_{(f;B,T)} = C_{(\lambda \delta f;B,T)} \]
by Lemma 7.4. From now on, we assume
\[ \min_{p \in N \setminus B} |d(\lambda f)(p)| < d_H(N \setminus B, \text{Crit}(f)) \tag{7.17} \]
without loss of any generality.

Lemma 7.4 also implies
\[ (\lambda + \varepsilon) C_{(f;B,T)} = C_{(\lambda + \varepsilon) f;B,T} \]
for all small \( \varepsilon > 0 \) such that
\[ \min_{p \in N \setminus B} |(\lambda + \varepsilon) df(p)| < d(N \setminus B, \text{Crit}(f)). \]

For example, we can choose any \( \varepsilon > 0 \) so that
\[ 0 < \varepsilon < \frac{d(N \setminus B, \text{Crit}(f))}{\min_{p \in N \setminus B} |df(p)|}. \tag{7.18} \]

Since (7.13) holds for any pair \( H, f \) that satisfy (7.12) and (7.8), applying it to the pair \( (H, \lambda + \varepsilon f) \) for \( T \supset B \) chosen above independently of \( i \)'s, we derive
\[
\rho^{\lambda \varepsilon g}(H;1) - \rho^{\lambda \varepsilon g}(H;[pt]^\#) \leq 2\text{osc}((\lambda + \varepsilon) f) = 2(\lambda + \varepsilon)\text{osc}_f
\]
\[
= 2 \left( \frac{\text{osc}_{C^0}(\phi_H^1; o_N)}{C_{(f;B,T)}} + \varepsilon \right) \text{osc}_f.
\]

Since this holds for all \( \varepsilon > 0 \) satisfying (7.18), it follows
\[ 0 \leq \rho^{\lambda \varepsilon g}(H;1) - \rho^{\lambda \varepsilon g}(H;[pt]^\#) \leq 2 \left( \frac{\text{osc}_{f}}{C_{(f;B,T)}} \right) \text{osc}_{C^0}(\phi_H^1; o_N) \tag{7.19} \]

letting \( \varepsilon \to 0 \). This finishes the proof of Theorem 7.6. \( \square \)

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