On spontaneous breaking of continuous symmetry in 1+1–dimensional space–time

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Abstract

We analyse Coleman’s theorem asserting the absence of Goldstone bosons and spontaneously broken continuous symmetry in the quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time (Comm. Math. Phys. 31, 259 (1973)). We confirm that Coleman’s theorem reproduces the well–known statement by Wightman about the non–existence of a quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time in terms of Wightman’s observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$. Referring to our results (Eur. Phys. J. C 24, 653 (2002)) we argue that a formulation of a quantum field theory of a free massless (pseudo)scalar field in terms of Wightman’s observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ is motivated well by the possibility to remove the collective zero–mode of the “center of mass” motion of a free massless (pseudo)scalar field (Eur. Phys. J. C 24, 653 (2002)) responsible for infrared divergences of the Wightman functions. We show that in the quantum field theory of a free massless (pseudo)scalar field with Wightman’s observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ the continuous symmetry can be spontaneously broken. Coleman’s theorem reformulated for the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ does not refute this result. We construct the most general version of a quantum field theory of a self–coupled massless (pseudo)scalar field with a conserved current. We show that this theory satisfies Wightman’s axioms and Wightman’s positive definiteness condition with Wightman’s observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$ and possesses a spontaneously broken continuous symmetry. Nevertheless, in this theory the generating functional of Green functions exists only when the collective zero–mode is not excited by the external source.

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1 Introduction

In the literature the absence of a spontaneously broken continuous symmetry \[1\] and Goldstone bosons \[2\] in quantum field theories in two dimensional space–time is related \[3\]–\[6\] to the Mermin–Wagner–Hohenberg theorem \[1\], asserting the vanishing of spontaneous magnetization or long–range order in spin systems (Heisenberg models, the XY model and so on) at non–zero temperature, and Coleman’s proof of the non–existence of Goldstone bosons in the quantum field theory of a free massless (pseudo)scalar field \[2\]. Since we are interested in quantum field theories at zero–temperature we analyse below Coleman’s theorem \[2\] only.

The absence of Goldstone bosons has been related by Coleman \[2\] to the problem of the infrared divergences of the Wightman functions of a free massless (pseudo)scalar field which we call \(\vartheta\)

\[
D^{(+)}(x; \mu) = \langle \Psi_0 | \vartheta(x) \vartheta(0) | \Psi_0 \rangle = \int \frac{d^2k}{(2\pi)^2} F^{(+)}(k) e^{-ik \cdot x}
\]

\[
= \int \frac{d^2k}{2\pi} \theta(+k^0) \delta(k^2) e^{-ik \cdot x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} e^{-ik \cdot x} = -\frac{1}{4\pi} \ln [-\mu^2x^2 + i0 \cdot \varepsilon(x^0)],
\]

\[
D^{(-)}(x; \mu) = \langle \Psi_0 | \vartheta(0) \vartheta(x) | \Psi_0 \rangle = \int \frac{d^2k}{(2\pi)^2} F^{(-)}(k) e^{ik \cdot x}
\]

\[
= \int \frac{d^2k}{(2\pi)^2} \theta(-k^0) \delta(k^2) e^{ik \cdot x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} e^{ik \cdot x} = -\frac{1}{4\pi} \ln [-\mu^2x^2 - i0 \cdot \varepsilon(x^0)],
\]

(1.1)

where \(F^{(\pm)}(k) = 2\pi \theta(\pm k^0) \delta(k^2)\) are the Fourier transforms of the Wightman functions, \(\theta(k^0)\) is the Heaviside function, \(\delta(k^2)\) is the Dirac \(\delta\)–function of \(k^2 = (k^0)^2 – (k^1)^2\) in 2–dimensional momentum space, \(\varepsilon(x^0)\) is the sign function, \(x^2 = (x^0)^2 – (x^1)^2\), \(k \cdot x = k^0 x^0 – k^1 x^1\), \(k^0 = |k^1|\) is the energy of free massless (pseudo)scalar quantum with momentum \(k^1\) and \(\mu\) is the infrared cut–off reflecting the infrared divergences of the Wightman functions \[11\]. As has been stated by Klaiber in his seminal paper \[7\]: \textit{If one wants to solve the Thirring model, one has to overcome this problem.}\footnote{In this connection we would like to refer to the solution of the massless Thirring model suggested by Hagen \[8\], who has succeeded in solving the massless Thirring model avoiding the problem of infrared divergences of free massless (pseudo)scalar boson fields. The problem of infrared divergences does not appear within the path–integral solution of the massless Thirring model as well \[9\].}

Recently \[10\] we have shown that the fermion fields in the massless Thirring model evolve via a phase of spontaneously broken chiral \(U(1) \times U(1)\) symmetry. The wave function of the ground state of the massless Thirring model in the chirally broken phase coincides with the wave function of the superconducting phase of the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity. For the quantum field theory of a free massless (pseudo)scalar field we have shown \[10\]–\[18\] that this theory satisfies all requirements for a continuous symmetry to be spontaneously broken: (i) the ground state is not invariant under the continuous symmetry \[10\]–\[18\], (ii) the energy level of the ground state is infinitely degenerate \[10\]–\[15\] and (iii) Goldstone bosons appear \[10\]–\[14\] and they are the quanta of a free massless (pseudo)scalar field.
In this paper we would like to show that the results obtained in [10]–[18] do not contradict Coleman’s theorem [2] and Wightman’s axioms [19]–[21].

The quantum field theory of a free massless (pseudo)scalar field \( \vartheta(x) \) is described by the Lagrangian [2, 10, 11]

\[
\mathcal{L}(x) = \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x),
\]

where \( x = (x^0, x^1) \) is a 2–vector, which is invariant under field translations [2, 11]

\[
\vartheta(x) \to \vartheta'(x) = \vartheta(x) + \alpha,
\]

where \( \alpha \) is an arbitrary parameter \( \alpha \in \mathbb{R} \). The conserved current associated with these field translations is equal to

\[
j_\mu(x) = \partial_\mu \vartheta(x).
\]

The total charge is defined by the time–component of \( j_\mu(x) \) [10]

\[
Q(x^0) = \lim_{L \to \infty} Q_L(x^0) = \lim_{L \to \infty} \int_{-L/2}^{L/2} dx^1 j_0(x^0, x^1) = \lim_{L \to \infty} \int_{-L/2}^{L/2} dx^1 \frac{\partial}{\partial x^0} \vartheta(x^0, x^1),
\]

where \( L \) is the volume of the system. The time–component \( j_0(x^0, x^1) \) of the current \( j_\mu(x) \) coincides with the conjugate momentum of the \( \vartheta \)–field, \( j_0(x^0, x^1) = \vartheta(x^0, x^1) = \Pi(x^0, x^1) \). Due to this the operators \( j_0(x^0, x^1) \) and \( \vartheta(y^0, y^1) \) obey the equal–time commutation relation

\[
[j_0(x^0, x^1), \vartheta(y^0, y^1)] = [\Pi(x^0, x^1), \vartheta(y^0, y^1)] = -i \delta (x^1 - y^1).
\]

This is the canonical commutation relation which leads to

\[
i [Q(x^0), \vartheta(x)] = 1.
\]

As a result the total charge operator \( Q(x^0) \) given by [10] generates shifts of the \( \vartheta \)–field

\[
\vartheta'(x) = e^{+i \alpha Q(x^0)} \vartheta(x) e^{-i \alpha Q(x^0)} = \vartheta(x) + (i \alpha) [Q(x^0), \vartheta(x)] + \frac{1}{2!} (i \alpha)^2 [Q(x^0), [Q(x^0), \vartheta(x)]] + \ldots = \vartheta(x) + \alpha.
\]

For the further analysis it is convenient to denote

\[
\delta \vartheta(x) = \alpha i [Q(x^0), \vartheta(x)].
\]

According to Goldstone’s theorem [22, 23] the criterion for the existence of Goldstone bosons is the non–vanishing vacuum expectation value of \( \delta \vartheta(x) \). This reads

\[
\langle \Psi_0 | \delta \vartheta(x) | \Psi_0 \rangle = \lim_{L \to \infty} \alpha i \langle \Psi_0 | [Q_L(x^0), \vartheta(x)] | \Psi_0 \rangle \neq 0,
\]

\( ^2 \)As has been shown in [10] the parameter \( \alpha = -2\alpha_A \) is related to the chiral phase \( \alpha_A \) of global chiral rotations of Thirring fermion fields.
where $|\Psi_0\rangle$ is a vacuum wave function.

For the subsequent analysis of a free massless (pseudo)scalar field theory we use the expansion of the massless (pseudo)scalar field $\vartheta(x)$ and the conjugate momentum $\Pi(x)$ into plane waves

$$\vartheta(x) = \int_{-\infty}^{\infty} \frac{dk^1}{2\pi} 2^{1/2} \pi \frac{1}{2k^0} \left( a(k^1) e^{-ik\cdot x} + a^\dagger(k^1) e^{ik\cdot x} \right),$$

$$\Pi(x) = \int_{-\infty}^{\infty} \frac{dk^1}{2\pi} 2^{1/2} \pi \frac{1}{2i} \left( a(k^1) e^{-ik\cdot x} - a^\dagger(k^1) e^{ik\cdot x} \right),$$

(1.11)

where $a(k^1)$ and $a^\dagger(k^1)$ are annihilation and creation operators obeying the standard commutation relation

$$[a(k^1), a^\dagger(q^1)] = (2\pi) 2k^0 \delta(k^1 - q^1).$$

(1.12)

This gives the canonical commutation relation (1.6).

According to Coleman’s proof [2] the vacuum expectation value $\langle \Psi_0 | \delta \vartheta(x) | \Psi_0 \rangle$ should vanish, i.e.

$$\langle \Psi_0 | \delta \vartheta(x) | \Psi_0 \rangle = 0.$$  (1.13)

This has been interpreted [2]–[6] as the absence of Goldstone bosons and the proof of the impossibility for the continuous symmetry to be spontaneously broken in 1+1-dimensional space–time.

It is well–known that the spontaneous breaking of a continuous symmetry occurs when the ground state of the system is not invariant under the symmetry group [3, 22]. As has been shown in [10, 11, 14] the ground state of the system described by the Lagrangian (1.2) is not invariant under field translations (1.3). Therefore, the field–translation symmetry should be spontaneously broken and Goldstone bosons should appear [10]–[16]. According to Witten’s criterion for Goldstone bosons [24], Goldstone bosons should saturate low–energy theorems and Ward identities. As we have shown in [16], the quanta of the free massless (pseudo)scalar field $\vartheta(x)$, describing the bosonized massless Thirring model with fermion fields quantized in the chirally broken phase, saturate the low–energy theorems and axial–vector Ward identities. This means that they satisfy Witten’s criterion for Goldstone bosons [24]. The non–invariance of the ground state of the free massless (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian (1.2), can be demonstrated by acting with the operator $e^{i\alpha Q(0)}$ on the wave function of the ground state.

Recently [14, 15] we have shown that the BCS–type wave function of the ground state of the massless Thirring model in the chirally broken phase [10] bosonizes to the form

$$|\Omega_0\rangle = \exp \left\{ i \pi \frac{M^\alpha}{g} \int_{-\infty}^{\infty} dx^1 \sin \beta \vartheta(0, x^1) \right\} |\Psi_0\rangle,$$

(1.14)

where $M$ is a dynamical mass of Thirring fermion fields in the chirally broken phase, $g$ is the coupling constant of the Thirring model [10].

Under the symmetry transformation induced by the operator $e^{i\alpha Q(0)}$ the wave function (1.14) transforms as follows [14, 15]

$$|\Omega_\alpha\rangle = e^{i\alpha Q(0)} |\Omega_0\rangle = \exp \left\{ i \pi \frac{M}{g} \int_{-\infty}^{\infty} dx^1 \sin \beta(\vartheta(0, x^1) - \alpha) \right\} |\Psi_0\rangle.$$  (1.15)
The wave functions $|\Omega_\alpha\rangle$ and $|\Omega_{\alpha'}\rangle$ are orthogonal and normalized to unity \[14\] \[15\]

$$\langle \Omega_{\alpha'} | \Omega_\alpha \rangle = \delta_{\alpha'\alpha}. \quad (1.16)$$

In \[11\] \[14\] we have shown that within the Schwinger formulation of the quantum field theory \[25\] the amplitude of the vacuum–to–vacuum transition $\langle \Omega_0^+ | \Omega_0^- \rangle_J$, where $J(x)$ is an external source of a free massless (pseudo)scalar field $\vartheta(x)$ and $|\Omega_0^-\rangle$ and $|\Omega_0^+\rangle$ are the vacuum states at $T = -\infty$ and $T = +\infty$, coincides with the generating functional of Green functions $Z[J]$ for the free massless (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian \[12\], defined by

$$Z[J] = \int \mathcal{D}\vartheta \exp \left\{ i \int d^2x \left[ \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) + \vartheta(x) J(x) \right] \right\}. \quad (1.17)$$

In order to get a non–vanishing value for $Z[J]$ we have to impose the constraint of the external source \[11\]

$$\int d^2x J(x) = 0, \quad (1.18)$$

which provides the suppression of the excitation of the collective zero–mode of the free massless (pseudo)scalar field $\vartheta(x)$ by the external source $J(x)$.

The collective zero–mode is responsible for the “center of mass” motion \[11\] and is the origin of the infrared divergences of the two–point Wightman functions in the quantum field theory of the free massless (pseudo)scalar field \[15\]. Hence, a removal of the “center of mass” motion allows to describe correlation functions in such a quantum field theory, defined by the generating functional of Green functions $Z[J]$ with the constraint \[1.18\], only in terms of the vibrational modes \[11\] \[14\] \[15\].

The time–dependent wave function \[1.14\] can be found in the usual way using the translation formula \[14\]

$$|\Omega_0^T \rangle = e^{+iHT} |\Omega_0\rangle = e^{+iHT} \exp \left\{ i \frac{\pi}{2} \frac{M}{g} \int_{-\infty}^{+\infty} dx^1 \sin \beta \vartheta(0, x^1) \right\} e^{-iHT} |\Psi_0\rangle =$$

$$= \exp \left\{ i \frac{\pi}{2} \frac{M}{g} \int_{-\infty}^{+\infty} dx^1 \sin \beta \vartheta(T, x^1) \right\} |\Psi_0\rangle, \quad (1.19)$$

where $H$ is the Hamilton operator of the free massless (pseudo)scalar field \[11\]

$$H = \frac{1}{2} \int_{-\infty}^{+\infty} dx^1 \left[ \left( \frac{\partial \vartheta(x)}{\partial x^0} \right)^2 + \left( \frac{\partial \vartheta(x)}{\partial x^1} \right)^2 \right]. \quad (1.20)$$

Then, we have taken into account that $H |\Psi_0\rangle = 0$ \[11\] \[15\].

The paper is organized as follows. In Section 2 we discuss a possible physical interpretation of the test functions for Wightman’s observables. We argue that in the case of a quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian \[12\], Wightman’s observables should be defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$. Such a reduction of the class of the test functions corresponds to the immeasurability of the collective zero–mode of the free massless (pseudo)scalar field describing the motion of the “center of mass”. As has been shown in \[11\] the collective zero–mode is
irrelevant to the dynamics of the free massless (pseudo)scalar field. The reduction of the test functions from \( \mathcal{S}(\mathbb{R}^2) \) to \( \mathcal{S}_0(\mathbb{R}^2) \) reconciles the problem of the correct formulation of a quantum field theory of a free massless (pseudo)scalar field within the path–integral approach in terms of the generating functional of Green functions (or the Schwinger external source approach) with Wightman’s axiomatic quantum field theory in terms of Wightman’s observables \[\text{[13]}\]. In Section 3 we show that the spontaneous breaking of continuous symmetry in the quantum field theory of a free massless (pseudo)scalar field with Wightman’s observables defined on the test functions from \( \mathcal{S}_0(\mathbb{R}^2) \) does not contradict Coleman’s theorem \[\text{[2]}\]. In Section 4 we construct a quantum field theory of a self–coupled massless (pseudo)scalar field with a conserved current using the K"allen–Lehmann representation for the description of two–point correlation functions. We show that Wightman’s observables in such a theory can be defined on the test functions from \( \mathcal{S}(\mathbb{R}^2) \). However, a non–vanishing value of the generating functional of Green functions can be gained only when the constraint on the external source \([\text{18}]\) is fulfilled. This agrees fully with Hasenfratz’s analysis of non–linear two–dimensional \(\sigma\)–models with \(O(N)\) symmetry \[\text{[26]}\]. In the Conclusion we discuss the obtained results.

2 Wightman’s axioms and quantum field theory of a free massless (pseudo)scalar field with spontaneously broken continuous symmetry

A quantum field theory in 1+1–dimensional space–time is well–defined within the axiomatic approach \[\text{[19, 20]}\] if it satisfies the following set of Wightman’s axioms \[\text{[20, 21]}\]:

- **W1** (Covariance). There is a continuous unitary representation of the inhomogeneous Lorentz group \( g \rightarrow U(g) \) on the Hilbert space \( \mathcal{H} \) of quantum theory states. The generators \( \mathcal{H} = (P^0, P^1) \) of the translation subgroup have spectrum in the forward cone \( (p^0)^2 - (p^1)^2 \geq 0, p^0 \geq 0 \). There is a vector \( |\Psi_0\rangle \in \mathcal{H} \) (the vacuum) invariant under the operators \( U(g) \).

- **W2** (Observables). There are field operators \( \{ \vartheta(h) : h(x) \in \mathcal{S}(\mathbb{R}^2) \} \) densely defined on \( \mathcal{H} \). The vector \( |\Psi_0\rangle \) is in the domain of any polynomial in the \( \vartheta(h) \)’s, and the subspace \( \mathcal{H}' \) spanned algebraically by the vectors \( \{ \vartheta(h_1) \ldots \vartheta(h_n)|\Psi_0\rangle ; n \geq 0, h_i \in \mathcal{S}(\mathbb{R}^2) \} \) is dense in \( \mathcal{H} \). The field \( \vartheta(h) \) is covariant under the action of the Lorentz group on \( \mathcal{H} \), and depends linearly on \( h \). In particular, \( U\dagger(g)\vartheta(h)U(g) = \vartheta(h_g) \).

- **W3** (Locality). If the supports of \( h(x) \) and \( h'(x) \) are space–like separated, then \([\vartheta(h), \vartheta(h')] = 0 \) on \( \mathcal{H}' \).

- **W4** (Vacuum). The vacuum vector \( |\Psi_0\rangle \) is the unique vector (up to scalar multiples) in \( \mathcal{H} \) which is invariant under time translations.

These axioms should be supplemented by Wightman’s positive definiteness condition \[\text{[20]}\] related to the positivity of the norm of a quantum state in the quantum field theory under consideration. Indeed, for quantum states \( |\Psi\rangle \) defined by

\[
|\Psi\rangle = \alpha_0|\Psi_0\rangle + \alpha_1 \int d^2x_1 h(x_1)\vartheta(x_1)|\Psi_0\rangle
\]
which are superpositions of $n$–particle quantum states $|\Psi_n\rangle$

$$|\Psi_n\rangle = \frac{1}{\sqrt{n!}} \int \ldots \int d^2x_1 \ldots d^2x_n h(x_1) \ldots h(x_n) \vartheta(x_1) \ldots \vartheta(x_n) |\Psi_0\rangle,$$

the vectors in the Hilbert space $\mathcal{H}$, should have a positive norm

$$\|\Psi\| = \|\alpha_0|\Psi_0\rangle + \alpha_1 \int d^2x_1 h(x_1) \vartheta(x_1)|\Psi_0\rangle + \frac{\alpha_2}{2!} \int \int d^2x_1 d^2x_2 h(x_1) h(x_2) \vartheta(x_1) \vartheta(x_2) |\Psi_0\rangle + \ldots \| \geq 0 \quad (2.3)$$

for all $\alpha_i \in \mathbb{R}^1 (i = 0, 1, \ldots)$ and test functions $h(x)$ from the Schwartz class $S(\mathbb{R}^2)$. In a quantum field theory of a free field the inequality (2.3) reduces to the constraint (2.4)

$$\int \int d^2x d^2y h^*(x) D^{(+)}(x - y; \mu) h(y) \geq 0,$$

which is the Wightman positive definiteness condition.

The problem of the correct formulation of a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ in agreement with Wightman’s axioms and Wightman’s positive definiteness condition spans many years and has a long history [19, 27]. The main problem concerns the observation that the two–point Wightman function $D^{(+)}(x; \mu)$ (1.1) does not satisfy Wightman’s positive definiteness condition on the test functions $h(x)$ from the Schwartz class $S(\mathbb{R}^2)$ [19].

As has been pointed out by Wightman [19] due to infrared divergences of the two–point Wightman function $D^{(+)}(x - y; \mu)$ one cannot formulate a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ on the class of test functions from $S(\mathbb{R}^2)$ consistent with Wightman’s positive definiteness condition. Therefore, in the sense of the non–existence of a quantum state $|\Psi\rangle$ defined by (2.1) with a positive norm, the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ with Wightman’s observables defined on the test functions $h(x)$ from the Schwartz class $S(\mathbb{R}^2)$ does not exist [19].

In order to avoid the problem of the violation of Wightman’s positive definiteness condition in a quantum field theory of a free massless (pseudo)scalar field Wightman has noticed that one can define Wightman’s observables on the test functions $h(x)$ from the Schwartz class $S_0(\mathbb{R}^2) = \{h(x) \in S(\mathbb{R}^2); h(0) = 0\} \subset S(\mathbb{R}^2)$ instead of $S(\mathbb{R}^2)$.

As has been shown in [11] the non–existence of a quantum field theory of a free massless (pseudo)scalar field in terms of the generating functional of Green functions $Z[J]$ or the amplitude of the vacuum–vacuum transitions $\langle \Omega^+_0 | \Omega^-_0 \rangle_j$ is related to the contribution of the collective zero–mode of the $\vartheta$–field describing the motion of the “center of mass”. Such a collective zero–mode can be removed from the system without influence on the evolution of vibrational modes [11]. A necessary and sufficient condition for the removal of the collective zero–mode of the $\vartheta$–field is the constraint (1.18).

In order to reconcile Schwinger’s formulation of a quantum field theory of a free massless (pseudo)scalar field in the form suggested in [11] and Wightman’s axiomatic approach
one has to understand how the constraint \([1.15]\) is related to the choice of the Schwartz class of test functions \(h(x)\).

For the analysis of this relation we suggest to interpret the test functions \(h(x)\) as the \textit{apparatus functions} of the detector, which an observer uses for measurements of quanta of the \(\vartheta\)-field in terms of matrix elements of Wightman’s observables \(\vartheta(h)\) defined by

\[
\vartheta(h) = \int d^2 x \, h(x) \, \vartheta(x). \tag{2.5}
\]

For example, in terms of \(\vartheta(h)\) the matrix element \(\langle \Psi_0 | \vartheta(h) | k^1 \rangle\), where \(|k^1\rangle = a^\dagger(k^1)|\Psi_0\rangle\) is a one–particle state with momentum \(k^1\), should describe the amplitude of the registration of a massless quantum of the \(\vartheta\)-field with momentum \(k^1\) by the detector. The quantity \(P_{\text{det}}(k^1) = |\langle \Psi_0 | \vartheta(h) | k^1 \rangle|^2\) has the meaning of the probability to detect a massless quantum of the \(\vartheta\)-field with momentum \(k^1\).

In terms of the Fourier transform \(\tilde{h}(k^0, k^1)\) of the test function \(h(x)\) the probability \(P_{\text{det}}(k^1) = |\langle \Psi_0 | \vartheta(h) | k^1 \rangle|^2\) is equal to

\[
P_{\text{det}}(k^1) = |\tilde{h}(k^0, k^1)|^2 \neq 1. \tag{2.6}
\]

The probability \(P_{\text{free}}(k^1)\) of a massless quantum of the \(\vartheta\)-field with momentum \(k^1\) to be out the detector is equal to

\[
P_{\text{free}}(k^1) = 1 - P_{\text{det}}(k^1) = 1 - |\tilde{h}(k^0, k^1)|^2. \tag{2.7}
\]

Hence, if the test functions are equal to zero, \(h(x) = \tilde{h}(k) = 0\), this should mean that there are no devices which can detect massless quanta of the \(\vartheta\)-field, a massless quantum with a momentum \(k^1\) should be free with a probability \(P_{\text{free}}(k^1) = 1\).

This assertion can be fully confirmed by a direct calculation of \(P_{\text{free}}(k^1)\) in terms of the matrix element \(\langle \Psi_0 | \vartheta(x) | k^1 \rangle\), where the field \(\vartheta(x)\) is defined by the plane wave expansion \([1.11]\). One obtains

\[
P_{\text{free}}(k^1) = |\langle \Psi_0 | \vartheta(x) | k^1 \rangle|^2 = 1. \tag{2.8}
\]

Therefore, the function \(\tilde{h}(k^0, k^1)\) should be treated as a characteristic of the detector, the \textit{apparatus function} related to the \textit{resolving power} of the device.

One can notice that the probability \(P_{\text{det}}(k^1)\), defined by \([2.6]\), is a regular function of \(k^1\) in the limit \(k^1 \to 0\). Taking the limit \(k^1 \to 0\) in \([2.6]\) we get

\[
\lim_{k^1 \to 0} P_{\text{det}}(k^1) = |\tilde{h}(0, 0)|^2. \tag{2.9}
\]

Thus, the quantity \(|\tilde{h}(0, 0)|^2\) describes the probability of the detection of the infrared quanta \((k^0 = k^1 = 0)\). A more explicit meaning of the quantity \(|\tilde{h}(0, 0)|^2\) can be derived from the definition of Wightman’s observable \([2.5]\). Indeed, as has been stated in the Introduction the shift of the field \(\vartheta(x) \to \vartheta'(x) = \vartheta(x) + \alpha\) corresponds to the shift of the “center of mass” of the \(\vartheta\)-field. Under the shift \(\vartheta(x) \to \vartheta'(x) = \vartheta(x) + \alpha\) Wightman’s observable \([2.5]\) changes as follows

\[
\vartheta(h) \to \vartheta'(h) = \int d^2 x \, [\vartheta(x) + \alpha] \, h(x) = \int d^2 x \, \vartheta(x) \, h(x) + \alpha \int d^2 x \, h(x) = \int d^2 x \, \vartheta(x) \, h(x) + \alpha \tilde{h}(0, 0). \tag{2.10}
\]
It is seen that the test function \( \tilde{h}(0,0) \) feels the motion of the “center of mass” of the free massless (pseudo)scalar field \( \vartheta(x) \) defined by the collective zero–mode. Since the collective zero–mode can be deleted from the states defining correlation functions in terms of the generating functional of Green functions \( Z[J] \), one can use the detectors which are insensitive to the collective zero–mode. This can be gained by setting

\[
\tilde{h}(0,0) = 0. \tag{2.11}
\]

Therefore, in our interpretation the constraint \( \tilde{h}(0,0) = 0 \) means that the detector is insensitive to the collective zero–mode of the field \( \vartheta(x) \). Therefore, the requirement to define a quantum field theory of a free massless (pseudo)scalar field on the test functions from \( \mathcal{S}_0(\mathbb{R}^2) = \{ h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0,0) = 0 \} \) would correspond to the exclusion of the collective zero–mode from the observable states of the quantum field \( \vartheta(x) \) in terms of Wightman’s observables \( \vartheta(h) \).

Now let us show that a quantum field theory of a free massless (pseudo)scalar field \( \vartheta(x) \), described by the Lagrangian \( \mathcal{L}_2 \), satisfies all Wightman’s axioms and Wightman’s positive definiteness condition on the test functions from \( \mathcal{S}_0(\mathbb{R}^2) \) and unstable under spontaneous breaking of continuous symmetry \( \mathcal{L}_3 \).

As the validity of Wightman’s axioms \( \text{W1} \) (Covariance) and \( \text{W2} \) (Observables) is obviously fulfilled on the class of test functions \( h(x) \in \mathcal{S}_0(\mathbb{R}^2) \), and according to \( \text{W4} \) (Vacuum) the vacuum state is invariant under time translations, let us verify the fulfillment of Wightman’s axiom \( \text{W3} \) (Locality). For this aim we have to analyse the commutator

\[
[\vartheta(h), \vartheta(h')] = \int \int d^2x d^2y h(x) h'(y) [\vartheta(x), \vartheta(y)]. \tag{2.12}
\]

Since the commutator \( [\vartheta(x), \vartheta(y)] \) is equal to

\[
[\vartheta(x), \vartheta(y)] = D^+(x-y; \mu) - D^-(x-y; \mu) = -\frac{i}{2} \varepsilon(x^0 - y^0) \theta((x - y)^2), \tag{2.13}
\]

where \( \theta((x - y)^2) \) is the Heaviside function, the r.h.s. of (2.12) reads

\[
[\vartheta(h), \vartheta(h')] = \frac{i}{2} \int \int d^2x d^2y h(x) h'(y) \varepsilon(x^0 - y^0) \theta((x - y)^2). \tag{2.14}
\]

Due to the presence of the Heaviside function \( \theta((x - y)^2) \) it is obvious that the integrand vanishes if the supports of \( h(x) \) and \( h'(y) \) are space–like separated, i.e. \( (x - y)^2 < 0 \). It follows

\[
[\vartheta(h), \vartheta(h')] = 0. \tag{2.15}
\]

This corroborates the validity of Wightman’s axiom \( \text{W3} \) (Locality) within the quantum field theory of a free massless (pseudo)scalar field \( \vartheta(x) \) defined on the Schwartz class of test functions \( \mathcal{S}_0(\mathbb{R}^2) = \{ h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0,0) = 0 \} \).

Now we have to verify the fulfillment of Wightman’s positive definiteness condition \( \mathcal{L}_4 \). Using the Fourier transform \( F^+(k) = 2\pi \theta(k^0) \delta(k^2) \) of the Wightman function \( D^+(x-y; \mu) \) \( \mathcal{L}_1 \) and passing to the light–cone variables \( k_+ = k^0 + k^1, k_- = k^0 - k^1 \).
and $d^2k = \frac{1}{2} dk_+ dk_-$ we get
\[
\int \int d^2x d^2y h^*(x) D^{(+)}(x - y; \mu) h(y) = \int \frac{d^2k}{2\pi} |\tilde{h}(k)|^2 \theta(k^0) \delta(k^2) = \nonumber \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_+ dk_-}{4\pi} |\tilde{h}(k_+, k_-)|^2 \left[ \frac{\theta(k_+)}{k_+} \delta(k_-) + \frac{\theta(k_-)}{k_-} \delta(k_+) \right] \nonumber \\
= \frac{1}{2\pi} \int_0^{\infty} \frac{dk_+}{k_+} |\tilde{h}(k_+, 0)|^2. \tag{2.16}
\]
Since the test functions $h(x)$ belong to $S_0(\mathbb{R}^2) = \{ h(x) \in S(\mathbb{R}^2); \tilde{h}(0, 0) = 0 \}$, the integral over $k_+$ is positive defined and convergent.

Thus, a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ is well–defined on the class of test functions $h(x)$ belonging to $S_0(\mathbb{R}^2) = \{ h(x) \in S(\mathbb{R}^2); \tilde{h}(0, 0) = 0 \}$. All Wightman’s axioms including Wightman’s positive definiteness condition are fulfilled. The physical reason of the formulation of the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ on the test functions from $S_0(\mathbb{R}^2)$ is the insignificance of the collective zero–mode for the evolution of vibrational modes [1].

As we have shown in [1] such a quantum field theory of a free massless (pseudo)scalar field is unstable under spontaneous breaking of continuous symmetry (1.3). We would like to confirm this statement by calculating the spontaneous magnetization.

For this aim we suggest to consider the massless (pseudo)scalar field $\vartheta(x)$ coupled to an external “magnetic” field $h_\lambda(x)$ [28], where $h_\lambda(x)$ is a sequence of the Schwartz functions from $S_0(\mathbb{R}^2)$ with vanishing norm at $\lambda \rightarrow \infty$. The Lagrangian (1.2) should be changed as follows
\[
\mathcal{L}(x; h_\lambda) = \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) + h_\lambda(x) \vartheta(x). \tag{2.17}
\]
The Lagrangian (2.17) defines the action of a massless (pseudo)scalar field $\vartheta(x)$ coupled to the “magnetic”field $h_\lambda(x)$
\[
S[\vartheta, h_\lambda] = \int d^2x \mathcal{L}(x; h_\lambda) = \frac{1}{2} \int d^2x \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) + \int d^2x h_\lambda(x) \vartheta(x). \tag{2.18}
\]
Since the “magnetic” field $h_\lambda(x)$ belongs to the Schwartz class $S_0(\mathbb{R}^2)$ obeying the constraint
\[
\int d^2x h_\lambda(x) = \tilde{h}_\lambda(0) = 0, \tag{2.19}
\]
the action $S[\vartheta, h_\lambda]$ is invariant under the symmetry transformation (1.2).

By the field–shift (1.2) we get
\[
S[\vartheta, h_\lambda] \rightarrow S'[\vartheta, h_\lambda] = \frac{1}{2} \int d^2x \partial_\mu \vartheta'(x) \partial^\mu \vartheta'(x) + \int d^2x h_\lambda(x) \vartheta'(x) = \nonumber \\
= S[\vartheta, h_\lambda] + \alpha \int d^2x h_\lambda(x). \tag{2.20}
\]
Due to the constraint (2.19) the r.h.s. of (2.20) is equal to $S[\vartheta, h_\lambda]$. This confirms the invariance of the action under the symmetry transformations (1.2).
According to Itzykson and Drouffe [28] the magnetization $M(h_\lambda)$ can be defined by

$$M(h_\lambda) = \langle \Psi_0 | \cos \vartheta(h_\lambda) | \Psi_0 \rangle = \lim_{\mu \to 0} \exp \left\{ -\frac{1}{2} \int d^2 x \, d^2 y \, h_\lambda(x) \, D^{(+)}(x-y;\mu) \, h_\lambda(y) \right\} = \exp \left\{ -\frac{1}{4\pi} \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+,0)|^2 \right\}. \quad (2.21)$$

The momentum integral in the exponent of the r.h.s. of (2.21) is convergent. If we switch off the “magnetic” field taking the limit $h_\lambda \to 0$, this can be carried out adiabatically defining $h_\lambda(x) = e^{-\varepsilon \lambda} h(x)$ for $\lambda \to \infty$ with $\varepsilon$, a positive, infinitesimally small parameter, we get

$$M = \lim_{\lambda \to \infty} M(h_\lambda) = 1. \quad (2.22)$$

This agrees with our results obtained in [11]. The quantity $M$ is the spontaneous magnetization. Since the spontaneous magnetization does not vanish, $M = 1$, the continuous symmetry, caused by the field–shifts (1.2), is spontaneously broken. This confirms our statement concerning the existence of the chirally broken phase in the massless Thirring model [10].

Now let us analyse the properties of the field operator $\vartheta(h)$ under the symmetry transformations (1.3). The result obtained in (2.12) can be also proved by acting with the total charge operator $Q(x^0)$ defined by (1.3) or (1.16). We get

$$\vartheta'(h) = e^{+i\alpha Q(x^0)} \vartheta(h) e^{-i\alpha Q(x^0)} = \int d^2 y \, h(y) \, e^{+i\alpha Q(x^0)} \vartheta(y) e^{-i\alpha Q(x^0)} = \vartheta(h) + \alpha \int d^2 y \, h(y) \int_{-\infty}^\infty dx^1 \frac{\partial}{\partial x^0} [\vartheta(x), \vartheta(y)]. \quad (2.23)$$

Since the commutator $[\vartheta(x), \vartheta(y)]$ is defined by (2.13), the time derivative of this commutator reads

$$\frac{\partial}{\partial x^0} [\vartheta(x), \vartheta(y)] = -i |x^0 - y^0| \delta((x-y)^2) = \frac{1}{2i} \delta(x_+ - y_+) + \frac{1}{2i} \delta(x_- - y_-). \quad (2.24)$$

Substituting (2.24) in (2.23), integrating over $x^1$ and using the constraint (2.6) we obtain

$$\vartheta'(h) = e^{+i\alpha Q(x^0)} \vartheta(h) e^{-i\alpha Q(x^0)} = \vartheta(h) + \alpha \int d^2 y \, h(y) = \vartheta(h). \quad (2.25)$$

Thus, we have found that Wightman’s observables $\vartheta(h)$ are invariant under $\vartheta$–field shifts. In this sense the field operator $\vartheta(h)$ can be really treated as an observable to the same extent as electric $\vec{E}(t, \vec{r})$ and magnetic $\vec{B}(t, \vec{r})$ fields, invariant under gauge transformations and measurable, whereas the vector potential $A^\mu(t, \vec{r}) = (\varphi(t, \vec{r}), \vec{A}(t, \vec{r}))$, which is not invariant under gauge transformations and, correspondingly, an immeasurable quantity.

However, the invariance of the field operator $\vartheta(h)$ under symmetry transformations (1.3) tells nothing about the non–existence of Goldstone bosons in the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ described by the Lagrangian (1.2).
Indeed, the invariance of the field operator $\vartheta(h)$ provides the vanishing of the field variation $\delta \vartheta(h) = 0$ in the strong sense at the operator level. It is not related to the peculiar property of the vacuum wave function $|\Psi_0\rangle$ to be invariant under the symmetry transformations \( \{1,3\} \). Wightman’s axiom W4 (Vacuum), demanding the invariance of the vacuum wave function $|\Psi_0\rangle$ under time translations, does not require the invariance of $|\Psi_0\rangle$ under any internal symmetry group like that inducing the field shifts \( \{1,3\} \).

Hence, Wightman’s quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$, formulated on the test functions $h(x)$ from $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \hat{h}(0,0) = 0\}$, gives a nice possibility to deal with well-defined observable quantities but does not clarify the problem of the absence of Goldstone bosons and spontaneously broken symmetry in 1+1–dimensional space–time.

3 Coleman’s theorem

Coleman’s theorem [2], asserting the absence of Goldstone bosons and spontaneously broken continuous symmetry in 1+1–dimensional quantum field theories, is closely related to Wightman’s statement about the non–existence of a 1+1–dimensional quantum field theory of a free massless (pseudo)scalar field with Wightman’s observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$ \( \{19\} \).

The problem of the absence of Goldstone bosons and spontaneously broken continuous symmetry in 1+1–dimensional quantum field theories Coleman has investigated in terms of the Fourier transforms of the two–point correlation functions

$$ F^{(+)}(k) = \int d^2 x e^{i k \cdot x} \langle \Psi_0 | \vartheta(x) | \Psi_0 \rangle, $$
$$ F^{(+)}_{\mu}(k) = i \int d^2 x e^{i k \cdot x} \langle \Psi_0 | j_{\mu}(x) | \Psi_0 \rangle, $$
$$ F^{(+)}_{\mu\nu}(k) = \int d^2 x e^{i k \cdot x} \langle \Psi_0 | j_{\mu}(x) j_{\nu}(0) | \Psi_0 \rangle, $$

(3.1)

where $F^{(+)}(k)$ is the Fourier transform of the Wightman function. The Fourier transform $F^{(+)}_{\mu}(k)$ Coleman has determined by the expression \( \{2\} \)

$$ F^{(+)}_{\mu}(k) = \sigma k_{\mu} \delta(k^0), $$

(3.2)

where $\sigma$ is a parameter. Using the expression \( \{1,10\} \) the parameter $\sigma$ can be related to the vacuum expectation value $\langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle$. For this aim we suggest to consider the relation

$$ \int_{-\infty}^{\infty} dk^0 F^{(+)}_0(k^0,0) = i \frac{1}{2} \int_{-\infty}^{\infty} dx^1 \int_{-\infty}^{\infty} dx^0 \int_{-\infty}^{\infty} d\vartheta \ e^{i k^0 x^0} \langle \Psi_0 | [j_0(x^0, x^1), \vartheta(0)] | \Psi_0 \rangle. $$

(3.3)

The r.h.s. of \( \{3,3\} \) is obtained due to the fact that $F^{(+)}_{\mu}(k^0, k^1)$ is a real function of $k^0$ and $k^1$. Using the expression \( \{3,4\} \) and integrating over $k^0$ and $x^0$ we get

$$ \frac{1}{2} \sigma = i \pi \int_{-\infty}^{\infty} dx^1 \langle \Psi_0 | [j_0(0, x^1), \vartheta(0)] | \Psi_0 \rangle. $$

(3.4)
In terms of the total charge \( Q(0) \) related to the current \( j_0(0, x^1) \) by (3.5) we can transcribe it into the form

\[
i\langle \Psi_0 | Q(0), \vartheta(0) | \Psi_0 \rangle = \sigma \frac{\lambda}{2\pi}.
\]  

(3.5)

Due to equation (1.10) the vacuum expectation value \( \langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle \) can be expressed in terms of the parameter \( \sigma \) and reads

\[
\langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle = \sigma \frac{\lambda}{2\pi}.
\]

(3.6)

If \( \sigma = 0 \) this gives \( \langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle = 0 \) and according to the Goldstone theorem this should testify the absence of Goldstone bosons and spontaneously broken continuous symmetry.

For the proof of his theorem Coleman has considered the Cauchy–Schwarz inequality

\[
\left( \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F^{(\pm)}(k) \right) \left( \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F_0^{(\pm)}(k) \right) \geq \left[ \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F_0^{(+)}(k) \right]^2,
\]

(3.7)

where \( \tilde{h}_\lambda(k) \) is the Fourier transform of the sequence of test function \( h_\lambda(x) \) at \( \lambda \to \infty \) defined by

\[
h_\lambda(x) = h_\lambda(x_+, x_-) = \frac{1}{\lambda} f(x_+) g(x_-) + \frac{1}{\lambda} f(x_-) g(x_+),
\]

(3.8)

where \( k_- = k^0 - k^1, k_+ = k^0 + k^1 \) and \( x_+ = (x^0 + x^1)/2, x_- = (x^0 - x^1)/2 \) are the light–cone variables in momentum space and coordinate space–time.

According to Coleman the test functions (3.9) should belong to the Schwartz class \( S(\mathbb{R}^1) \otimes S_0(\mathbb{R}^1) \), where \( f(x_\pm) \in S(\mathbb{R}^1) \) with \( \tilde{f}(0) \neq 0 \) and \( g(x_\pm) \in S_0(\mathbb{R}^1) = \{g(x_\pm) \in S(\mathbb{R}^1); \tilde{g}(0) = 0 \} \).

For the analysis of the Cauchy–Schwarz inequality (3.7) Coleman has formulated a lemma

\[
\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} dk_- \tilde{f}(\lambda k_-) F(k_+, k_-) = c \delta(k_+),
\]

(3.10)

where \( c \) is a positive constant and \( F(k_+, k_-) \) is a positive Lorentz–invariant distribution.

Due to this lemma and the requirement for the test functions \( f(x_\pm) \) to belong to the Schwartz class \( S(\mathbb{R}^1) \) the massless mode with \( k^2 = k_+ k_- = 0 \) is excluded from the intermediate states defining the Fourier transform of the Wightman function \( F^{(+)}(k) \). Indeed, the massless mode contribution to \( F^{(+)}(k) \) should be proportional to \( \theta(k^0) \delta(k^2) = \theta(k_+ + k_-) \delta(k_+ k_-) \) (see (1.11)). The contribution of this term to (3.10) is given by

\[
\int_{-\infty}^{\infty} dk_- \tilde{f}(\lambda k_-) \theta(k_+ + k_-) \delta(k_- k_+) = \\
= \int_{-\infty}^{\infty} dk_- \tilde{f}(\lambda k_-) \theta(k_+ + k_-) \left( \frac{1}{|k_-|} \delta(k_+) + \frac{1}{|k_+|} \delta(k_-) \right) = \\
= \delta(k_+) \int_{-\infty}^{\infty} dk_- \tilde{f}(\lambda k_-) \frac{\theta(k_-)}{|k_-|} + \tilde{f}(0) \frac{\theta(k_+)}{k_+} = \\
= \delta(k_+) \int_{0}^{\infty} \frac{dk_-}{k_-} \tilde{f}(\lambda k_-) + \tilde{f}(0) \frac{\theta(k_+)}{k_+} = c(\lambda) \delta(k_+) + \tilde{f}(0) \frac{\theta(k_+)}{k_+}.
\]

(3.11)
The constant $c(\lambda)$ is defined by the integral
\[ c(\lambda) = \int_0^\infty \frac{dk_-}{k_-} \hat{f}(\lambda k_-) = \int_0^\infty \frac{dk_-}{k_-} \hat{f}(k_-), \]
where we have made a change of variables $\lambda k_- \to k_-$. The r.h.s. of (3.11) satisfies Coleman’s lemma (3.10) if and only if
\[ \hat{f}(0) = 0. \] (3.13)

However, the constraint (3.13) is fulfilled only for the test functions from $\mathcal{S}_0(\mathbb{R}^1)$. If the test functions $f(x_{\pm})$ belong to $\mathcal{S}(\mathbb{R}^1)$ with $\hat{f}(0) \neq 0$, as it has been assumed by Coleman, the coefficient $c(\lambda)$ given by (3.12) is divergent in the infrared region. This makes the relation (3.11) meaningless. Hence, due to Coleman’s lemma (3.10), the function $\vartheta(k^0)\delta(k^2) = \vartheta(k_+ + k_-)\delta(k_+k_-)$ is not a well-defined tempered distribution $[19]$.

Thus, if $\vartheta(x)$ is a free massless (pseudo)scalar field, described by the Lagrangian (1.1) with the Fourier transform $F^{(+)}(k) \propto \vartheta(k^0)\delta(k^2)$ of the two–point Wightman function, the existence of this field is prohibited by Coleman’s lemma (3.10). This agrees with Wightman’s statement if Wightman’s observables are defined on the test functions from $\mathcal{S}([\mathbb{R}^2])$.

However, as we have shown in Section 2 in the case of a quantum field theory of a free massless (pseudo)scalar field Wightman’s observables should be defined on the test functions from $\mathcal{S}_0([\mathbb{R}^1])$. In agreement with this reduction Coleman’s requirement $f(x_{\pm}) \in \mathcal{S}(\mathbb{R}^1)$ can be weakened and replaced by $f(x_{\pm}) \in \mathcal{S}_0(\mathbb{R}^1)$ with $\hat{f}(0) = 0$. As it is shown above on the test functions $f(x_{\pm}) \in \mathcal{S}_0([\mathbb{R}^1])$ the Fourier transform $F^{(+)}(k) = 2\pi \vartheta(k^0)\delta(k^2)$ of the Wightman function $D^{(+)}(x; \mu)$ (1.1) is well–defined tempered distribution $[19]$.

Let us show that Coleman’s constraint $\sigma = 0$ is not a solution of the Cauchy–Schwarz inequality (3.7) on the test functions (3.9) but the consequence of the lemma (3.10). The Fourier transforms (3.1) defined for a free massless (pseudo)scalar field $\vartheta(x)$ with $j_\mu(x) = \partial_\mu \vartheta(x)$ are equal to
\[
F^{(+)}(k) = \sigma \vartheta(k^0)\delta(k^2),
F^{(+)}_\mu(k) = \sigma k_\mu \vartheta(k^0)\delta(k^2),
F^{(+)}_{\mu\nu}(k) = \sigma k_\mu k_\nu \vartheta(k^0)\delta(k^2),
\] (3.14)

where the canonical value of the parameter $\sigma$ is $\sigma = 2\pi$. Nevertheless, following Coleman we keep it arbitrary and try to find the constraint from the solution of the Cauchy–Schwarz inequality defined on the test functions (3.8).

For convenience of the further analysis we suggest to rewrite the Cauchy–Schwarz inequality (3.7) as follows
\[ J(\lambda) J_{00}(\lambda) \geq J_0^2(\lambda), \] (3.15)

where $J(\lambda)$, $J_0(\lambda)$ and $J_{00}(\lambda)$ are momentum integrals of $F^{(+)}(k)$, $F^{(+)}_{00}(k)$ and $F^{(+)}_{00}(k)$ multiplied by $|\tilde{h}_\lambda(k)|^2/2\pi$, respectively.

\[ J(\lambda) = \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F^{(+)}(k) = \]
The same independence can be obtained for $J$ where we have used the condition $\tilde{g}(0) = 0$. Thus, due to the condition $\tilde{g}(0) = 0$ the momentum integral $J(\lambda)$ does not depend on $\lambda$

$$J(\lambda) = \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F(k) = \frac{\sigma}{2\pi} |\tilde{f}(0)|^2 \int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2,$$

(3.17)

The same independence can be obtained for $J_0(\lambda)$ and $J_{00}(\lambda)$

$$J_0(\lambda) = \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F_0^+(k) = \frac{\sigma}{4\pi} |\tilde{f}(0)|^2 \int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2,$$

$$J_{00}(\lambda) = \int \frac{d^2k}{2\pi} |\tilde{h}_\lambda(k)|^2 F_{00}^+(k) = \frac{\sigma}{8\pi} |\tilde{f}(0)|^2 \int_0^\infty \frac{dk_+k_+}{k_+} |\tilde{g}(k_+)|^2.$$

(3.18)

The Cauchy–Schwarz inequality reads

$$\frac{\sigma}{2\pi} |\tilde{f}(0)|^2 \int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2 \geq \left[ \frac{\sigma}{4\pi} |\tilde{f}(0)|^2 \int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2 \right]^2.$$

(3.19)

For $\tilde{f}(0) \neq 0$ and $\sigma \neq 0$ one can cancel the common factor $(\sigma |\tilde{f}(0)|^2/4\pi)^2$ and get

$$\int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2 \int_0^\infty \frac{dk_+k_+}{k_+} |\tilde{g}(k_+)|^2 \geq \left[ \int_0^\infty \frac{dk_+}{k_+} |\tilde{g}(k_+)|^2 \right]^2.$$

(3.20)

Thus, it is seen that (i) on the sequence of the test functions from the Schwartz class $\mathcal{S}(\mathbb{R}^1) \otimes \mathcal{S}_0(\mathbb{R}^1)$ the Coleman’s constraint is not a solution of the Cauchy–Schwarz inequality and (ii) setting $\sigma = 0$ the Cauchy–Schwarz inequality becomes a trivial identity $0 \equiv 0$ but tells nothing about the non–existence of Goldstone bosons.

This testifies that in Coleman’s treatment massless (pseudo)scalar quanta are excluded by the lemma formulated on the test functions from $\mathcal{S}(\mathbb{R}^1)$. On the test functions from $\mathcal{S}_0(\mathbb{R}^1)$ the Cauchy–Schwarz inequality becomes a trivial identity $0 \equiv 0$ due to $\tilde{f}(0) = 0$ for arbitrary $\sigma$. This confirms the existence of the quantum field theory of
a free massless (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian (1.2), with Wightman’s observables defined on the test functions from $S_0(\mathbb{R}^2) \supset S_0(\mathbb{R}^1) \otimes S_0(\mathbb{R}^1)$. Hence, no conclusion about the absence of spontaneously broken continuous symmetry can be derived from Coleman’s theorem with Coleman’s lemma formulated for the test functions $f(x_{\pm}) \in S_0(\mathbb{R}^1)$.

4 Quantum field theory of a massless (pseudo)scalar field in Källen–Lehmann representation for two–point Wightman functions

In this Section we construct a canonical quantum field theory of a massless self–coupled (pseudo)scalar field which satisfies Wightman’s axioms and Wightman’s positive definiteness condition with Wightman’s observables defined on the test functions from $S(\mathbb{R}^2)$. In such a theory the symmetry, related to the field–shifts (1.3), is spontaneously broken and Goldstone bosons are the quanta of the massless (pseudo)scalar field.

Let such a massless (pseudo)scalar field $\vartheta(x)$ be described by the Lagrangian

$$L(x) = L[\partial_\mu \vartheta(x)].$$

(4.1)

Due to the dependence on $\partial_\mu \vartheta(x)$ the Lagrangian (4.1) is invariant under the field–shifts (1.3). The current $j_\mu(x)$ related to the symmetry transformation (1.3) is defined by

$$j_\mu(x) = \frac{\delta L[\partial_\mu \vartheta(x)]}{\delta \partial_\mu \vartheta(x)}.$$ 

(4.2)

This current is conserved $\partial_\mu j^\mu(x) = 0$. The conjugate momentum $\Pi(x)$ of the $\vartheta$–field is equal to the time–component of the current

$$\Pi(x) = \frac{\delta L[\partial_\mu \vartheta(x)]}{\delta \dot{\vartheta}(x)} = j_0(x),$$

(4.3)

where $\dot{\vartheta}(x)$ is a time derivative. The canonical equal–time commutation relation (1.6) and the expression of the $\vartheta$–field variation (1.9) are retained for the quantum field theory of the massless $\vartheta$–field described by the Lagrangian (4.1).

The most useful tool for the analysis of the Fourier transforms (3.1) of the two–point correlation functions, defined in this theory, is the Källen–Lehmann representation [29]. Inserting a complete set of intermediate states we redefine the r.h.s. of the Fourier transforms (3.1) as follows

$$F^{(+)}_{\mu}(k) = \sum_n \int d^2 x e^{ik \cdot x} \langle \Psi_0 | j_\mu(x) | \Psi_0 \rangle = \sum_n \int d^2 x e^{ik \cdot x} \langle \Psi_0 | j_\mu(x) | \Psi_0 \rangle = \int d^2 x e^{ik \cdot x} \langle \Psi_0 | j_\mu(x) | \Psi_0 \rangle = \int d^2 x e^{ik \cdot x} \langle \Psi_0 | j_\mu(x) | \Psi_0 \rangle,$$

(4.4)
Due to the invariance of the vacuum state $|\Psi_0\rangle$ under space and time translations and Lorentz covariance $\langle \Psi_0|\vartheta(0)|\Psi_0\rangle = 0$, we have

\[
F^{(+)}(k) = |\langle \Psi_0|\vartheta(0)|\Psi_0\rangle|^2(2\pi)^2\delta^{(2)}(k) + \sum_{n \neq \Psi_0} \int d^2x \ e^{i k \cdot x} \langle \Psi_0|\vartheta(x)|n\rangle \langle n|\vartheta(0)|\Psi_0\rangle,
\]

\[
F^{(+)}_\mu(k) = i \sum_{n \neq \Psi_0} \int d^2x \ e^{i k \cdot x} \langle \Psi_0|j_\mu(x)|n\rangle \langle n|\vartheta(0)|\Psi_0\rangle,
\]

\[
F^{(+)}_{\mu\nu}(k) = \sum_{n \neq \Psi_0} \int d^2x \ e^{i k \cdot x} \langle \Psi_0|j_{\mu\nu}(x)|n\rangle \langle n|j_\nu(0)|\Psi_0\rangle. \tag{4.5}
\]

We would like to emphasize that the term proportional to $\delta^{(2)}(k)$ in the Fourier transform of the Wightman function appears only for a spontaneously broken symmetry and a non–invariant vacuum because $\langle \Psi_0|\vartheta(0)|\Psi_0\rangle \neq 0 \tag{24}$.

Using again the invariance of the vacuum state $|\Psi_0\rangle$ under space and time translations we obtain \tag{29}

\[
F^{(+)}(k) = |\langle \Psi_0|\vartheta(0)|\Psi_0\rangle|^2(2\pi)^2\delta^{(2)}(k) + \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_S(m^2) dm^2,
\]

\[
F^{(+)}_\mu(k) = -\varepsilon^\mu_{\nu} k^\nu \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_V(m^2) dm^2,
\]

\[
F^{(+)}_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu}) \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_T(m^2) dm^2, \tag{4.6}
\]

where $\rho_i(m^2)$ ($i = S, V, T$) are the Källen–Lehmann spectral functions defined by \tag{29}

\[
(2\pi)^2 \sum_{n \neq \Psi_0} \delta^{(2)}(k - p_n) |\langle n|\vartheta(0)|\Psi_0\rangle|^2 = \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_S(m^2) dm^2,
\]

\[
(2\pi)^2 \sum_{n \neq \Psi_0} \delta^{(2)}(k - p_n) \langle \Psi_0|j_\mu(0)|n\rangle \langle n|\vartheta(0)|\Psi_0\rangle =
\]

\[
i \varepsilon^\mu_{\nu} k^\nu \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_V(m^2) dm^2,
\]

\[
(2\pi)^2 \sum_{n \neq \Psi_0} \delta^{(2)}(k - p_n) \langle \Psi_0|j_{\mu\nu}(0)|n\rangle \langle n|j_\nu(0)|\Psi_0\rangle =
\]

\[
= (k_\mu k_\nu - k^2 g_{\mu\nu}) \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_T(m^2) dm^2. \tag{4.7}
\]

We notice that for the massless state $-\varepsilon^\mu_{\nu} k^\nu \varepsilon(k^1) = k_\mu$.

Thus, in the Källen–Lehmann representation the Fourier transforms $F^{(+)}(k)$, $F^{(+)}_\mu(k)$ and $F^{(+)}_{\mu\nu}(k)$ are defined by

\[
F^{(+)}(k) = |\langle \Psi_0|\vartheta(0)|\Psi_0\rangle|^2(2\pi)^2\delta^{(2)}(k) + \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_S(m^2) dm^2,
\]

\[
F^{(+)}_\mu(k) = -\varepsilon^\mu_{\nu} k^\nu \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_V(m^2) dm^2,
\]

\[
F^{(+)}_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu}) \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_T(m^2) dm^2. \tag{4.8}
\]
These are the most general forms of distributions in the quantum field theory of a massless self–coupled (pseudo)scalar field $\vartheta(x)$ in 1+1–dimensional space–time satisfying Wightman’s axioms $W1 - W4$ and current conservation $\partial^\mu j_\mu(x) = 0$.

Now let us analyse Wightman’s positive definiteness condition on the test functions $h(x)$ from the Schwartz class $S(\mathbb{R}^2)$. We get

$$\langle h, h \rangle = \iiint d^2 x d^2 y h^*(x) D^{(+)}(x-y) h(y) = \int \frac{d^2 k}{(2\pi)^2} |\tilde{h}(k)|^2 F^{(+)}(k) =$$

$$= |\langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle|^2 |\tilde{h}(0)|^2 + \int_0^\infty dm^2 \rho_S(m^2) \int \frac{d^2 k}{(2\pi)^2} |\tilde{h}(k)|^2 \theta(k^0) \delta(k^2 - m^2) =$$

$$= |\langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle|^2 |\tilde{h}(0)|^2 + \int_0^\infty dm^2 \rho_S(m^2) \int_0^\infty \frac{dk_+ dk_-}{8\pi^2} |\tilde{h}(k_+, k_-)|^2 \theta(k_+) \theta(k_-) \delta(k_+ k_- - m^2) =$$

$$= \langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle^2 |\tilde{h}(0)|^2 + \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_S(m^2) \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2 \geq 0. \tag{4.9}$$

It is convenient to rewrite the second term as follows

$$\int_0^\infty \frac{dm^2}{8\pi^2} \rho_S(m^2) \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2 = \rho_S(0) \int_0^\infty \frac{dk_+}{k_+} \int_0^\infty \frac{dm^2}{8\pi^2} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2$$

$$+ \int_0^\infty \frac{dm^2}{8\pi^2} [\rho_S(m^2) - \rho_S(0)] \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2. \tag{4.10}$$

In the first integral we suggest to make a change of variables $m^2/k_+ = k_-$. This gives

$$\int_0^\infty \frac{dm^2}{8\pi^2} \rho_S(m^2) \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2 = \rho_S(0) \int_0^\infty \frac{dk_+}{k_+} \int_0^\infty \frac{dk_-}{8\pi^2} |\tilde{h}(k_+, k_-)|^2$$

$$+ \int_0^\infty \frac{dm^2}{8\pi^2} [\rho_S(m^2) - \rho_S(0)] \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2. \tag{4.11}$$

Substituting (4.11) in (4.9) we obtain

$$\langle h, h \rangle = \iiint d^2 x d^2 y h^*(x) D^{(+)}(x-y) h(y) =$$

$$= |\langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle|^2 |\tilde{h}(0)|^2 + \rho_S(0) \int_0^\infty \frac{dk_+}{k_+} \int_0^\infty \frac{dk_-}{8\pi^2} |\tilde{h}(k_+, k_-)|^2$$

$$+ \int_0^\infty \frac{dm^2}{8\pi^2} [\rho_S(m^2) - \rho_S(0)] \int_0^\infty \frac{dk_+}{k_+} |\tilde{h}(k_+, \frac{m^2}{k_+})|^2 \geq 0. \tag{4.12}$$

This testifies the fulfillment of the positive definiteness of the scalar product $\langle h, h \rangle \geq 0$ on the test functions $h(x) \in S(\mathbb{R}^2)$ in the quantum field theory of a massless self–coupled (pseudo)scalar field $\vartheta(x)$ with the Wightman functions defined by (4.8).

Our change of variable can be illustrated by an example.

$$\int_0^\infty dm^2 \int_0^\infty \frac{dk_+}{k_+} \left| \tilde{h}(k_+, \frac{m^2}{k_+}) \right|^2 = \int_0^\infty dk_+ \int_0^\infty \frac{dm^2}{k_+} \left| \tilde{h}(k_+, \frac{m^2}{k_+}) \right|^2 =$$

$$= \int_0^\infty dm^2 \int_0^\infty \frac{dk_+}{k_+} \frac{1}{4} \exp \left\{ - \frac{1}{2} \left( k_+ + \frac{m^2}{k_+} \right) \right\} = \int_0^\infty dm \, m K_0(m) = 1. \tag{4.13}$$
After the change of variable \( m^2/k_+ = k_- \) we get

\[
\int_0^\infty dm^2 \int_0^\infty \frac{dk_+}{k_+} \frac{1}{4} \exp \left\{ -\frac{1}{2} \left( k_+ + \frac{m^2}{k_+} \right) \right\} = \int_0^\infty dk_+ \int_0^\infty \frac{dk_-}{k_-} \exp \left\{ -\frac{1}{2} \left( k_+ + k_- \right) \right\} = 1. \tag{4.14}
\]

It is easy to show that the Wightman function \( D^{(+)}(x) \) as well as the Fourier transform \( F^{(+)}(k) \) is a tempered distribution defined on the test functions \( h(x) \in \mathcal{S}(\mathbb{R}^2) \). For this aim we have to calculate the functional \((h, D^{(+)})\) given by

\[
(h, D^{(+)}) = \int d^2x h^*(x) D^{(+)}(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{h}^*(k) F^{(+)}(k) = |\langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle|^2 \tilde{h}^*(0) + \int_0^\infty dm^2 \rho_S(m^2) \int_0^\infty \frac{dk_+}{k_+} \tilde{h}^*(k_+, m^2) = \langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle^2 \tilde{h}^*(0) + \rho_S(0) \int_0^\infty dm^2 \left[ \rho_S(m^2) - \rho_S(0) \right] \int_0^\infty \frac{dk_+}{k_+} \tilde{h}^*(k_+, m^2)
\]

\[
+ \frac{1}{8\pi^2} \int_0^\infty dm^2 \left[ \rho_S(m^2) - \rho_S(0) \right] \int_0^\infty \frac{dk_+dk_-}{k_+k_-} \tilde{h}^*(k_+, k_-) \tag{4.15}
\]

The r.h.s. of (4.13) contains only convergent integrals. This testifies that the Wightman function \( D^{(+)}(x) \) and the Fourier transform \( F^{(+)}(k) \) are tempered distributions for Schwartz’s test functions \( h(x) \in \mathcal{S}(\mathbb{R}^2) \) satisfying Wightman’s positive definiteness condition.

Due to the necessity to fulfill Wightman’s positive definiteness condition, imposing to keep \( \rho_S(0) \) finite or zero, we suggest that the contribution of the state with \( m^2 = 0 \) is screened in the Fourier transform \( F^{(+)}(k) \). This is, of course, a dynamical effect caused by the self-coupling of the \( \vartheta \)-field leading to the influence of all intermediate states \( |n\rangle \). Therefore, distinct contributions of the state with \( m^2 = 0 \) can be only to the Fourier transforms \( F^{(+)}_\mu(k) \) and \( F^{(+)}_{\mu
u}(k) \). Isolating the contributions of the state with \( m^2 = 0 \) in the spectral functions \( \rho_\nu(m^2) \) and \( \rho_T(m^2) \) and setting

\[
\rho_\nu(m^2) = \sigma \delta(m^2) + \rho'_\nu(m^2), \\
\rho_T(m^2) = \sigma' \delta(m^2) + \rho'_T(m^2), \tag{4.16}
\]

we obtain the Fourier transforms \( F^{(+)}(k) \), \( F^{(+)}_\mu(k) \) and \( F^{(+)}_{\mu
u}(k) \) in the following form

\[
F^{(+)}(k) = |\langle \Psi_0 | \vartheta(0) | \Psi_0 \rangle|^2 (2\pi)^2 \delta^{(2)}(k) + \theta(k^0) \int_0^\infty \delta(k^2 - m^2) \rho_S(m^2) dm^2,
\]

\[
F^{(+)}_\mu(k) = \sigma k_\mu \theta(k^0) \delta(k^2) - \varepsilon_{\mu\nu} k^\nu \varepsilon(k^1) \theta(k^0) \int_{M^2}^\infty \delta(k^2 - m^2) \rho'_\nu(m^2) dm^2,
\]

\[
F^{(+)}_{\mu\nu}(k) = \sigma' k_\mu k_\nu \theta(k^0) \delta(k^2) + (k_\mu k_\nu - k^2 g_{\mu\nu}) \theta(k^0) \int_{M^2}^\infty \delta(k^2 - m^2) \rho'_T(m^2) dm^2, \tag{4.17}
\]

where the spectral functions \( \rho'_\nu(m^2) \) and \( \rho'_T(m^2) \) contain only the contributions of the states with \( m^2 > 0 \) and the scale \( M^2 \) isolates the state with \( m^2 = 0 \) from the states with \( m^2 > 0 \).
The origins of the Fourier transforms given by (4.8) are defined by

\[
D^{(+)}(x) = \langle \Psi_0 | \partial(0) | \Psi_0 \rangle^2 + \frac{1}{4\pi^2} \int_0^\infty dm^2 \rho_s(m^2) K_0(m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)}),
\]

\[
iD^{(+)}_\mu(x) = -i \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \int_{-\varphi_0}^{\varphi_0} d\varphi \left( e^{-m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)} \cosh \varphi} - 1 \right),
\]

\[
D^{(+)}_{\mu\nu}(x) = (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) \frac{1}{4\pi^2} \int_0^\infty dm^2 \rho_F(m^2) K_0(m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)}),
\]

where \( K_0(m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)}) \) is the McDonald function and \( \varphi_0 \) is defined by

\[
\varphi_0 = \frac{1}{2} \ln \left( \frac{x^0 + x^1 - i0}{x^0 - x^1 - i0} \right).
\]

In the original \( iD^{(+)}_\mu(x) \) of the Fourier transform \( F^{(+)}_\mu(k) \) we suggest to isolate the contribution of the state with \( m^2 = 0 \) from the contributions of the states with \( m^2 > 0 \). For this aim we transcribe the r.h.s. of \( iD^{(+)}_\mu(x) \) as follows

\[
iD^{(+)}_\mu(x) = -i \varepsilon_{\mu\nu} \frac{\partial \varphi_0}{\partial x^\nu} \left[ \frac{1}{4\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \right]
\]

\[
- \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \int_{-\varphi_0}^{\varphi_0} d\varphi \left( e^{-m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)} \cosh \varphi} - 1 \right) =
\]

\[
= \left[ \frac{1}{2\pi} \int_0^\infty dm^2 \rho_V(m^2) \right] \frac{i}{2\pi} \frac{x^\mu}{-x^2 + i0 \cdot \varepsilon(x^0)}
\]

\[
- \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \int_{-\varphi_0}^{\varphi_0} d\varphi \left( e^{-m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)} \cosh \varphi} - 1 \right) =
\]

\[
= \int_0^\infty dm^2 \rho_V(m^2) \int \frac{d^2 q}{(2\pi)^2} q_\mu \theta(q^0) \delta(q^2) e^{-iq \cdot x}
\]

\[
- \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \int_{-\varphi_0}^{\varphi_0} d\varphi \left( e^{-m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)} \cosh \varphi} - 1 \right). \quad (4.20)
\]

This defines \( iD^{(+)}_\mu(x) \) in the following general form

\[
iD^{(+)}_\mu(x) = \int_0^\infty dm^2 \rho_V(m^2) \int \frac{d^2 q}{(2\pi)^2} q_\mu \theta(q^0) \delta(q^2) e^{-iq \cdot x}
\]

\[
- \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{1}{8\pi^2} \int_0^\infty dm^2 \rho_V(m^2) \int_{-\varphi_0}^{\varphi_0} d\varphi \left( e^{-m \sqrt{-x^2 + i0 \cdot \varepsilon(x^0)} \cosh \varphi} - 1 \right). \quad (4.21)
\]

The first term describes the contribution of the state with \( m^2 = 0 \), whereas the second one contains the contributions of all states with \( m^2 > 0 \). Since the contribution of the state with \( m^2 = 0 \) is defined by the expression \( F^{(+)}_\mu(k; m^2 = 0) = \sigma k_\mu \theta(k^0) \delta(k^2) \) we get the sum rules for the spectral function \( \rho_V(m^2) \), which read

\[
\int_0^\infty dm^2 \rho_V(m^2) = \sigma. \quad (4.22)
\]
Now let us consider the vacuum expectation value \( \langle \Psi_0 | [j_\mu(x), \vartheta(0)] | \Psi_0 \rangle \). Following the standard procedure expounded above we get

\[
\langle \Psi_0 | [j_\mu(x), \vartheta(0)] | \Psi_0 \rangle = \left. i \int^\infty_0 d^2m^2 \rho_V(m^2) \int \frac{d^2k}{(2\pi)^2} \varepsilon_{\mu\nu} k^\nu \varepsilon(k^1) \theta(k^0) \delta(k^2 - m^2) \left( e^{-ik \cdot x} + e^{+ik \cdot x} \right) \right].
\] (4.23)

The vacuum expectation value of the equal–time commutation relation for the time–component of the current \( j^0(0, x^1) \) and the field \( \vartheta(0) \) reads

\[
\langle \Psi_0 | [j^0(0, x^1), \vartheta(0)] | \Psi_0 \rangle = -\frac{1}{2\pi} \int^\infty_0 d^2m^2 \rho_V(m^2) \left( \frac{k^1}{\sqrt{(k^1)^2 + m^2}} - 1 \right) \cos(k^1 x^1).
\] (4.24)

It is seen that the state with \( m^2 = 0 \) does not contribute to the second term. Therefore, the second term in the r.h.s. of (4.24) is defined by only the spectral function \( \rho'_V(m^2) \). This gives

\[
\langle \Psi_0 | [j^0(0, x^1), \vartheta(0)] | \Psi_0 \rangle = -\frac{1}{2\pi} \int^\infty_{M^2} d^2m^2 \rho'_V(m^2) \left( \frac{k^1}{\sqrt{(k^1)^2 + m^2}} - 1 \right) \cos(k^1 x^1).
\] (4.25)

Since time–component of the current \( j_\mu(x) \) coincides with the conjugate momentum, i.e. \( \Pi(x) = j^0(x) \), the l.h.s. of (4.25) is equal to \( -i \delta(x^1) \). Using the canonical equal–time commutation relation (1.6) for the l.h.s. of (4.24) we derive the sum rules

\[
\int^\infty_0 d^2m^2 \rho_V(m^2) = 2\pi.
\] (4.26)

Comparing (4.26) with (4.22) we get

\[
\sigma = 2\pi.
\] (4.27)

This is a model–independent result which rules out Coleman’s result asserting \( \sigma = 0 \).

As the second term in the r.h.s. of (4.25) can be never proportional to \( \delta(x^1) \) it should be identically zero. This yields

\[
\rho'_V(m^2) \equiv 0.
\] (4.28)

Hence, the spectral function \( \rho_V(m^2) \) is equal to

\[
\rho_V(m^2) = \sigma \delta(m^2) = 2\pi \delta(m^2).
\] (4.29)

This means that in the case of current conservation \( \partial^\mu j_\mu(x) = 0 \) the Fourier transform \( F^{(+)}_\mu(k) \) is defined by only the contribution of the state with \( m^2 = 0 \). This confirms that the expression (2.4) postulated by Coleman is general for the canonical quantum field
theories with conserved current $\partial^\mu j_{\mu}(x) = 0$ in 1+1-dimensional quantum field theories of a massless (pseudo)scalar field $\vartheta(x)$, but this rules out Coleman’s constraint $\sigma = 0$.

We would like to emphasize that in the non-canonical quantum field theory, whether such a theory could exist, for which $j_0(x) \neq \Pi(x)$, the parameter $\sigma$ can be arbitrary but the spectral function $\rho'_\nu(m^2)$ does not vanish. This means that if Coleman would have considered a non-canonical quantum field theory, the equation (13) of Ref. [2] for the Fourier transform $F^{(+)}_\mu(k)$ should contain this term. Recall that the second term in (13) of Ref. [2] is caused by parity violation and is not related to the contribution of $\rho'_\nu(m^2)$, which conserves parity. Dropping the term caused by $\rho'_\nu(m^2)$ Coleman has asserted implicitly that he is in the framework of a canonical quantum field theory.

Multiplying (4.24) by $i\alpha$ and integrating over $x^1$ we obtain $\langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle$, which reads

$$\langle \Psi_0 | \delta \vartheta(0) | \Psi_0 \rangle = i \alpha \int_{-\infty}^{\infty} dx^1 \langle \Psi_0 | [j_{\mu}(x), \vartheta(0)] | \Psi_0 \rangle = \frac{\alpha}{2\pi} \int_{0}^{\infty} dm^2 \rho_V(m^2) = \alpha,$$  \hspace{1cm} (4.30)

where we have taken into account the discussion above and the expression for the spectral function $\rho_V(m^2)$ given by (4.22). In agreement with the Goldstone theorem [22] the non-vanishing value of the $\vartheta$-field variation (4.30) testifies the existence of Goldstone bosons and spontaneously broken continuous symmetry (1.3).

In order to analyse the properties of the spectral function $\rho_T(m^2)$ we suggest to calculate the vacuum expectation value of the commutator $[j_{\mu}(x), j_{\nu}(0)]$. In terms of the spectral function $\rho_T(m^2)$ the result reads

$$\langle \Psi_0 | [j_{\mu}(x), j_{\nu}(0)] | \Psi_0 \rangle = \int_0^{\infty} dm^2 \rho_T(m^2) \int \frac{d^2k}{(2\pi)^2} (k_{\mu}k_{\nu} - k^2 g_{\mu\nu}) \theta(k^0) \delta(k^2 - m^2) (e^{-ik \cdot x} - e^{+ik \cdot x}).$$ \hspace{1cm} (4.31)

Considering the component $\langle \Psi_0 | [j_0(x^1), j_1(0)] | \Psi_0 \rangle$ at $x^0 = 0$ we get

$$\langle \Psi_0 | [j_0(x^1), j_1(0)] | \Psi_0 \rangle = \int_0^{\infty} dm^2 \rho_T(m^2) \int \frac{d^2k}{(2\pi)^2} k_0 k_1 \theta(k^0) \delta(k^2 - m^2) (e^{+ik^1 x^1} - e^{-ik^1 x^1}) = \frac{1}{2\pi} \int_0^{\infty} dm^2 \rho_T(m^2) i \frac{\partial}{\partial x^1} \delta(x^1).$$ \hspace{1cm} (4.32)

Since due to Schwinger [31] the commutator $[j_0(0, x^1), j_1(0)]$ is defined by

$$[j_0(0, x^1), j_1(0)] = iS \frac{\partial}{\partial x^1} \delta(x^1),$$ \hspace{1cm} (4.33)

where $S$ is a Schwinger term, we get the sum rules

$$\int_0^{\infty} dm^2 \rho_T(m^2) = 2\pi S.$$ \hspace{1cm} (4.34)

This testifies that the integral over $m^2$ of the spectral function $\rho_T(m^2)$ is convergent.

The analysis of Wightman’s observables $\vartheta(h)$ on the test functions $h(x)$ from $S(\mathbb{R}^2)$ and $S_0(\mathbb{R}^2)$, which we have carried out in Section 2, is applicable to the canonical quantum
field theory of a massless self–coupled (pseudo)scalar field \( \vartheta(x) \) formulated in this Section. Indeed, for the general case \( h(x) \in S(\mathbb{R}^2) \) the Wightman observable \( \vartheta(h) \) is not invariant under the field–shifts (1.3). We get

\[
\vartheta'(h) = e^{-iQ(x^0)\vartheta(h)} e^{iQ(x^0)} = \vartheta(h) + \alpha \int d^2 x \, h(x).
\]

This yields the variation of Wightman’s observable

\[
\delta \vartheta(h) = \alpha \int d^2 x \, h(x).
\]

It is important to emphasize that \( \delta \vartheta(h) \) given by (4.36) is not an operator–valued quantity. Due to this the vacuum expectation value coincides with the quantity

\[
\langle \Psi_0 | \delta \vartheta(h) | \Psi_0 \rangle = \delta \vartheta(h) = \alpha \int d^2 x \, h(x).
\]

Hence, the variation of Wightman observable \( \delta \vartheta(h) \) can say nothing about spontaneous breaking of continuous symmetry.

Then, narrowing the class of the test functions from \( S(\mathbb{R}^2) \) to \( S_0(\mathbb{R}^2) \) one gets the Wightman observable \( \vartheta(h) \) invariant under shifts (1.3) and the variation of Wightman’s observable \( \delta \vartheta(h) \) identically zero, \( \delta \vartheta(h) = 0 \). However, this does not give a new information about the Goldstone bosons and a spontaneously broken continuous symmetry in addition to that we have got on the class of the test functions from \( S(\mathbb{R}^2) \).

The necessity to use the test functions from \( S_0(\mathbb{R}^2) \) for the definition of Wightman’s observables in the quantum field theory of a self–coupled massless (pseudo)scalar field, described by the Lagrangian (4.1), can be demonstrated by example of the definition of the generating functional of Green functions. Denoting the generating functional of Green functions as \( Z[J] \) we define

\[
Z[J] = \langle \Psi_0 | T \left( e^{i \int d^2 x \vartheta(x) J(x)} \right) | \Psi_0 \rangle = \\
= \int D\vartheta e^{i \int d^2 x \left\{ \mathcal{L}[\partial_\mu \vartheta(x)] + \vartheta(x) J(x) \right\}} | \Psi_0 \rangle = \\
= e^{i \int d^2 x \mathcal{L}' \left( -i \partial_\mu \delta J(x) \right)} Z[J],
\]

where \( Z[J] \) is determined by (1.17) and \( \mathcal{L}' \) is the Lagrangian of the self–interaction of the massless (pseudo)scalar field \( \vartheta(x) \)

\[
\mathcal{L}'[\partial_\mu \vartheta(x)] = \mathcal{L}[\partial_\mu \vartheta(x)] - \frac{1}{2} \partial_\mu \vartheta(x) \partial^\mu \vartheta(x).
\]

Since in order to get a non–vanishing value for \( Z[J] \) one needs the constraint (1.18), the collective zero–mode is deleted from the intermediate states of correlation functions. This result agrees well with Hasenfratz [26] who showed that the removal of the collective zero–mode motion of the self–coupled massless scalar fields in lattice \( \sigma \)–models with \( O(N) \) symmetry in one and two dimensional volumes does not affect the evolution of the system but allows to construct a self–consistent perturbation theory for the calculation of correlation functions.

This means that in the quantum field theory of a self–coupled massless (pseudo)scalar field \( \vartheta(x) \), described by the Lagrangian (4.1), Wightman’s observables should be defined on the test functions \( h(x) \) from \( S_0(\mathbb{R}^2) \).
5 Conclusion

We have shown that Coleman’s constraint $\sigma = 0$, interpreted as a proof for the absence of Goldstone bosons and spontaneously broken continuous symmetry in 1+1–dimensional quantum field theories, is not a consequence of the Cauchy–Schwarz inequality, but follows from Coleman’s lemma. Formulating the lemma on the test functions from $\mathcal{S}(\mathbb{R}^1) \otimes \mathcal{S}_0(\mathbb{R}^1)$ Coleman has followed Wightman’s axioms demanding the definition of Wightman’s observables on the test functions from $\mathcal{S}(\mathbb{R}^2) \supset \mathcal{S}(\mathbb{R}^1) \otimes \mathcal{S}_0(\mathbb{R}^1)$. Due to his lemma Coleman removed the massless state from the Fourier transform of the Wightman function, which has entailed the exclusion of this state from all two–point correlation functions. This agrees with Wightman’s assertion about the non–existence of the quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time, when Wightman’s observables are defined on the test functions $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, $h(x) \in \mathcal{S}(\mathbb{R}^2)$.

However, we have motivated the reduction of the test functions $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}_0(\mathbb{R}^2) = \{ h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0 \}$. The definition of Wightman’s observables on the test functions from $\mathcal{S}_0(\mathbb{R}^2) = \{ h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0 \}$ instead of $\mathcal{S}(\mathbb{R}^2)$ is justified by the removal of the collective zero–mode from the massless (pseudo)scalar field $\vartheta(x)$. The collective zero–mode does not affect the evolution of a free massless (pseudo)scalar field.

Wightman’s observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ do not measure the collective zero–mode due to the constraint $\tilde{h}(0) = 0$. This makes a quantum field theory of a free massless (pseudo)scalar field well–defined within Wightman’s axiomatic approach.

The removal of the collective zero–mode agrees well with Hasenfratz’s approach to one and two dimensional quantum field theories [26], who showed that a self–consistent perturbation theory for the calculation of correlation functions in $\sigma$–models for self–coupled massless scalar fields with $O(N)$ symmetry can be developed only removing the collective zero–mode.

The reformulation of Coleman’s lemma for the test functions from $\mathcal{S}_0(\mathbb{R}^1)$ reduces the Cauchy–Schwarz inequality to the identity $0 \equiv 0$ for an arbitrary parameter $\sigma$. This certifies that Coleman’s analysis of 1+1–dimensional quantum field theories does not contradict the existence of Goldstone bosons and spontaneous breaking of continuous symmetry in the quantum field theory of the free and the self–coupled massless (pseudo)scalar field for Wightman’s observables defined on the test functions from the Schwartz class $\mathcal{S}_0(\mathbb{R}^2)$.

The physical states of quanta of vibrational modes of a free massless (pseudo)scalar field in a quantum field theory with Wightman’s observables, defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$, are determined in a positive–definite Hilbert space [19]. According to Nakanishi [33] the use of test functions from the Schwartz class $\mathcal{S}_0(\mathbb{R}^2)$ leads to a violation of localizability of the Schwartz distributions. This entails the impossibility to define consistently the support of the Schwartz distributions [33, 34]. Due to this the formulation of a quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time with Wightman’s observables, defined on the test functions from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, and physical states, determined in an indefinite–metric Hilbert space, is more advantageous [33].

An analysis of Coleman’s theorem for a two–dimensional quantum field theory of a free massless (pseudo)scalar field with Wightman’s observables, defined on the test functions
from the Schwartz class $S(\mathbb{R}^2)$, and physical states, determined in an indefinite–metric Hilbert space, we are planning to perform in a forthcoming publication.

An interesting analysis of Coleman’s theorem [2], applied to the 1+1–dimensional quantum field theory of self–coupled “charged” bosons with an internal non–Abelian continuous symmetry, has been recently suggested by Chigak Itoi [35].

References

[1] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966); P. C. Hohenberg, Phys. Rev. 158, 383 (1967); N. D. Mermin, J. Math. Phys. 8, 1061 (1967).
[2] S. Coleman, Comm. Math. Phys. 31, 259 (1973).
[3] C. Itzykson and J.–B. Zuber, in QUANTUM FIELD THEORY, McGraw–Hill Book Company, New York, 1980.
[4] C. Itzykson and J.–M. Drouffe, in STATISTICAL FIELD THEORY, From Brownian motion to renormalization and lattice gauge theory, Vol. I, Cambridge University Press, Cambridge, 1989, pp.219–224.
[5] J. Zinn–Justin, in QUANTUM FIELD THEORY AND CRITICAL PHENOMENA, Clarendon Press • Oxford, 1993, pp.549–551.
[6] K. Huang, in QUANTUM FIELD THEORY, From Operators to Path Integrals, John Willey & Sons, Inc., New York, 1998, pp.363–367.
[7] B. Klaiber, in LECTURES IN THEORETICAL PHYSICS, Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut and W. Brittin, Gordon and Breach, New York, 1968, Vol. X, part A, pp.141–176.
[8] C. R. Hagen, Nuovo Cim. B 51, 169 (1967); M. Faber, C. R. Hagen, and A. N. Ivanov, Correlation functions of left–right fermion densities in Hagen’s approach to the massless Thirring model, 2003 (unpublished).
[9] K. Furuya, Re. E. Gamboa Saravi and F. A. Schaposnik, Nucl. Phys. B 208, 159 (1982); R. Banerjee, Z. Phys. C 25, 251 (1984); C. M. Naón, Phys. Rev. D 31, 2035 (1985).
[10] M. Faber and A. N. Ivanov, Eur. Phys. J. C 20, 723 (2001), hep–th/0105057.
[11] M. Faber and A. N. Ivanov, Eur. Phys. J. C 24, 653 (2002), hep–th/0112184.
[12] M. Faber and A. N. Ivanov, Is the energy of the ground state of the sine–Gordon model unbounded from below for $\beta^2 > 8\pi$ ?, hep–th/0205249, J. of Phys. A 36, issue 28 (2003).
[13] M. Faber and A. N. Ivanov, Quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time as a test for the massless Thirring model, hep–th/0206244.
[14] M. Faber and A. N. Ivanov, Phys. Lett. B 563, 231 (2003).

[15] M. Faber and A. N. Ivanov, On the ground state of a free massless (pseudo)scalar field in two dimensions, hep-th/0212226.

[16] M. Faber and A. N. Ivanov, Goldstone bosons in the massless Thirring model. Witten’s criterion, hep-th/0305174.

[17] M. Faber and A. N. Ivanov, Dynamical breaking of conformal symmetry in the massless Thirring model, hep-th/0305203.

[18] M. Faber and A. N. Ivanov, On the vacua in the massless Thirring model, hep-th/0305229.

[19] A. S. Wightman, Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields, in HIGH ENERGY ELECTROMAGNETIC INTERACTIONS AND FIELD THEORY, Cargèse Lectures in Theoretical Physics, edited by M. Levy, 1964, Gordon and Breach, 1967, pp.171–291.

[20] R. F. Streater and A. S. Wightman, in PCT, SPIN AND STATISTICS, AND ALL THAT, Princeton University Press, Princeton and Oxford, Third Edition, 1980.

[21] J. Glimm and A. Jaffe, in QUANTUM PHYSICS, A Functional Integral Point of View, Springer–Verlag, New York, 1981.

[22] J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[23] K. Yoshida, Nucl. Phys. B 105, 272 (1976).

[24] E. Witten, Nucl. Phys. B 145, 110 (1978).

[25] J. Schwinger in PARTICLES AND SOURCES, Gordon and Breach, New York 1969 and PARTICLES, SOURCES AND FIELDS, Addison–Wesley Publishing Co., Massachusetts 1970.

[26] P. Hasenfratz, Phys. Lett. B 141, 385 (1984).

[27] B. Schroer, Fortschr. der Physik 11, 1 (1963); R. F. Streater and I. F. Wilde, Nucl. Phys. B 24, 561 (1970); N. Nakanishi, Progr. Theor. Phys. 57, 269 (1977); Z. Phys. C 4, 17 (1980). L. K. Hadjijivanov, Free massless scalar fields in two dimensions, Preprint of JINR, E2–80–445, 1980; G. Morchio, D. Pierotti, and F. Strocchi, J. Math. Phys. 31, 1467 (1990).

[28] (see [2] pp.219–224).

[29] H. Lehmann, Nuovo Cim. 11, 342 (1954); G. Källen in GUANTUMELEKTRODYNAMIK, Handbuch der Physik, Springer, 1958; G. Källen in ELEKTRON-TARTEILECHENPHYSIK, Bibliographisches Institut, Mannheim, 1964 and references therein; (see [2] pp.203–204).
[30] M. Faber and A. N. Ivanov, On the solution of the massless Thirring model with fermion fields quantized in the chiral symmetric phase, hep-th/0112183.

[31] J. Schwinger, Phys. Rev. Lett. 3, 296 (1959).

[32] N. N. Bogoliubov and D. V. Schirkov, in INTRODUCTION TO THE THEORY OF QUANTIZED FIELDS, Interscience, New York, 1959.

[33] N. Nakanishi (private communication).

[34] L. Schwartz, in THÉRIE DES DISTRIBUTIONS, Vol. 1 & 2, Hermann, Paris, 1957, pp.26–28.

[35] Chigak Itoi, Coleman’s theorem on physical assumptions for no Goldstone bosons in two dimensions, hep-th/0303118.