Bose-Einstein condensation and superfluidity of a weakly-interacting photon gas in a nonlinear Fabry-Perot cavity

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Abstract

A field theoretical framework for the recently proposed photon condensation effect in a nonlinear Fabry-Perot cavity is discussed. The dynamics of the photon gas turns out to be described by an effective 2D Hamiltonian of a complex massive scalar field. Finite size effects turn out to be relevant for the existence of the photon condensate.
1 Introduction

This work originates from a very recent experimental proposal [1] in order to detect a Bose-Einstein condensation and a new superfluid state of light. The apparatus [1] consists basically of a planar Fabry-Perot cavity filled with a nonlinear polarizable medium responsible for an effective repulsive short-range four-photon interaction. The mirrors of the cavity have a low but finite transmittivity, allowing photons of an incident laser beam to enter and leave the cavity, so that a steady-state condition is achieved after many photon-photon collisions. Following [1], the weak interaction between the photons can be viewed as a four-photon term arising from a repulsive pairwise interaction provided by a self-defocusing Kerr nonlinear medium inside the Fabry-Perot cavity [2]. Moreover, due to the boundary conditions required by the Fabry-Perot, one is led with an effective two-dimensional nonrelativistic massive weakly-interacting photon gas confined in the cavity. The nonrelativistic regime is due to the paraxial propagation of the light inside the resonator [1, 2]. This effective 2D weakly-interacting photon gas displays a Bogoliubov type dispersion relation, suggesting the existence of a Bose-Einstein photon condensate and of a possible superfluid state of light. In particular, one of the tasks of the experiment is to investigate the existence of the sound-like waves corresponding to the collective phonon excitations in the photon superfluid, which, according to [1], should propagate with a velocity whose value $v_c$ is a few thousandths of the vacuum speed of light, $v_c = 4.2 \times 10^7 \text{ cm/s}$. A follow-up experiment is being planned [1] in order to demonstrate that the sound wave velocity $v_c$ is a critical velocity for the photon (super)fluid, corresponding to the existence of persistent currents.

It is useful now to spend a few words on the meaning of the Bose-Einstein condensation for this 2D photon gas, as is well known that there are quite severe restrictions on the existence of a Bose-Einstein condensation in momentum space for a weakly-interacting 2D gas in the thermodynamic limit for any nonvanishing temperature [3, 4]. We underline first of all that the system realized by the experimental setup of [1] is a zero-temperature Bose gas. We know that for an ideal Bose gas a macroscopic number $N_0$ of particles will condense occupying the zero-
momentum state; however the presence of interactions could strongly modify this picture, in such a way that the presence of the condensate is no longer so obvious \[6, 7\]. Nevertheless, in our case the interaction is very weak, and we may think that \(N_0\) remains a macroscopic number. Moreover, the presence of a nonvanishing interaction is crucial in order to imply a redefinition of the spectrum of the excitations which gives rise to the Bogoliubov dispersion relation.

Another remarkable issue is the fact that the photon gas ought to be considered here as a genuine finite-sized system, as the whole apparatus possesses a finite small volume (i.e. the cavity volume) and the average number \(N\) of photons inside the cavity is kept finite as well. Also, the Fabry-Perot boundary conditions play a crucial role in order to provide an effective mass for the photons \[1\], which turns out to be proportional to the inverse of the separation length \(L\) between the mirrors of the cavity. The situation looks very close to that of trapped Bose gases \[8, 9, 10\], for which the Bose-Einstein condensation has been observed even in the case of 1D dimensionally reduced systems \[13, 14\]. For these inhomogeneous finite-sized systems the existence of a Bose-Einstein condensation is not, strictly speaking, a phase transition \[11, 12, 4, 5\]. Rather, it is a direct consequence of the experimental evidence \[8, 9, 10, 11\] of a macroscopic occupation of the lowest state. It is worth remarking that the number \(N_0\) of photons occupying the lowest state has been estimated \[1\] to be of the order of \(N_0 \approx 8 \times 10^{11}\).

The aim of the present work is to propose and analyse a possible theoretical set up. We shall be able to show that the effective 2D weakly-interacting massive photon gas can be actually obtained by a four-photon \(QED\)-inspired Hamiltonian, once the gauge freedom and the Fabry-Perot boundary conditions have been properly taken into account.

## 2 The Effective Hamiltonian

As the starting \(QED\)-inspired effective Hamiltonian describing a weak repulsive four-photon interaction we take the following gauge invariant expression

\[
H_{\text{eff}} = \int_V d^3x \left( \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + \frac{\lambda}{4} (A^T_{\mu} A^{T\mu})^2 \right),
\]

(2.1)
being the volume of the Fabry-Perot cavity. The coupling constant $\lambda$ is positive and $A^T_\mu$ stand for the transverse gauge invariant components of the gauge field, *i.e.*

$$A^T_\mu = (g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2})A^\nu = \frac{1}{\partial^2} \partial^\nu F_{\nu\mu},$$

where $g_{\mu\nu} = \text{diag}(+, - , - , - )$ is the flat Minkowski metric and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

In order to motivate the choice of the Hamiltonian (2.1) we underline that, according to [2], the field propagation inside the cavity in the paraxial approximation is described by a nonlinear Schrödinger equation. The effective Hamiltonian (2.1) can then be obtained by requiring that the Heisenberg equations of motion for the field reproduce the nonlinear Schrödinger equation in the semiclassical limit [2].

Although not needed, it is worth remarking that expression (2.1) can be immediately generalized in a gauge invariant way to a typical two-body interaction, namely

$$H_{\text{eff}} = \int_V d^3x \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 \right) + \frac{1}{4} \int_V d^3xd^3y \left( A^T A^T \right) (x) U(x - y) \left( A^T A^T \right) (y),$$

for some short-range repulsive potential $U(x - y)$.

Being interested in the analysis of the ground state of the Hamiltonian (2.1), we shall work in the static situation in which all fields are assumed to be time-independent. Accordingly, we shall make use of the so called temporal gauge

$$A_0 = 0,$$

$$\vec{A} = \vec{A}(\vec{x}),$$

which implies

\footnote{As we shall see in the next section, the boundary conditions required by the Fabry-Perot cavity do allow for usual integration by parts.}
\[ A_T^0 = 0 , \]  
\[ A_T^i = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) A_j = \frac{1}{\nabla^2} \partial_j F_{ji} , \quad i, j = 1, 2, 3, \]

where \( 1/\nabla^2 \) is the Green’s function of the three-dimensional laplacian,

\[ \frac{1}{\nabla^2} = -\frac{1}{4\pi |\vec{x} - \vec{y}|} . \]  
\[ (2.7) \]

For the Hamiltonian \((2.1)\) we get

\[ H_{\text{eff}} = \int d^3 x \left( \frac{1}{4} F_{ij} F^{ij} + \frac{\lambda}{4} \left( A_i^T A_i^T \right)^2 \right) . \]  
\[ (2.8) \]

Obviously, expression \((2.8)\) is left invariant by the time-independent gauge transformations, \( i.e. \)

\[ \delta A_i = \partial_i \eta(\vec{x}) . \]  
\[ (2.9) \]

This spatial type of gauge invariance can be fixed by imposing the axial gauge condition

\[ A_3(\vec{x}) = 0 , \]  
\[ (2.10) \]

naturally suggested by the geometry of the Fabry-Perot cavity. Here the \(z\)-axis is chosen to be coincident with the direction of propagation of the laser beam incident on the Fabry-Perot. The mirrors of the cavity lie in the transverse \(xy\) plane.

However, as is well known, condition \((2.10)\) does not fix completely the gauge freedom and allows for a further residual local invariance, corresponding to the gauge transformations on the plane \(xy\) orthogonal to the \(z\)-axis. In fact, owing to the equation \((2.10)\), for the field strength \(F_{ij}\) we get

\[ F_{a3} = -\partial_3 A_a , \]  
\[ F_{ab} = \partial_a A_b - \partial_b A_a , \quad a, b = 1, 2 . \]  
\[ (2.11) \]

It is apparent then that the components of \(F_{ij}\) are left invariant by the following \(z\)-independent transformations
\[ \delta A_a = \partial_a \eta(x_{\perp}) , \]  

(2.12)

where \( x_{\perp} = (x, y) \). This further residual gauge invariance (2.12) can be fixed by requiring the additional condition

\[ \partial_a A_a = 0 , \]  

(2.13)

from which it follows that the two gauge fields \( A_a \) can be identified with their transverse components. Finally, for the fully gauge fixed effective Hamiltonian we obtain

\[
H_{\text{eff}} = \int d^2 x_{\perp} dz \left( \frac{1}{2} F_{3a} F^{3a} + \frac{1}{4} F_{ab} F^{ab} + \frac{\lambda}{4} (A_a A_a)^2 \right) 
\]

(2.14)

\[
= \int d^2 x_{\perp} dz \left( \frac{1}{2} (\partial_3 A_a)(\partial_3 A_a) - \frac{1}{2} A_a \nabla^2_\perp A_a + \frac{\lambda}{4} (A_a A_a)^2 \right) ,
\]

where \( \nabla^2_\perp = \partial_a \partial_a \) is the two-dimensional laplacian. This Hamiltonian will be the starting point for the analysis of the spectrum of the excitations of the weakly coupled photon gas.

### 3 The spectrum of the excitations

In order to analyse the spectrum of the Hamiltonian (2.14), we have first to properly take into account the boundary conditions of the problem. These require the vanishing of the fields at the reflecting surfaces of the mirrors of the Fabry-Perot cavity, i.e.

\[ A_a(x_{\perp}, z) = \frac{1}{\sqrt{L}} \tilde{A}_a(x_{\perp}) \sin \left( \frac{\pi}{L} n_0 z \right) , \]  

(3.15)

where \( L \) is the distance between the mirrors and where the fixed integer \( n_0 \) is related to the frequency \( \omega \) of the laser beam incident on the cavity through \( n_0 \pi / L = \omega \). It should be pointed out that the form of the field (3.15) requires the assumption that the spacing between the modes of the cavity is so large that only one longitudinal mode is excited by the laser beam [2]. Concerning now the \( xy \) plane, periodic boundary conditions will be assumed.
Inserting equation (3.13) in the expression (2.14) and performing the integration over the $z$-axis, we easily get the following 2D dimensionally reduced effective Hamiltonian

$$H_{\text{eff}} = \int d^2 \mathbf{x}_\perp \left( -\frac{1}{4} \tilde{A}_a \nabla^2 _\perp \tilde{A}_a + \frac{m^2}{4} \tilde{A}_a \tilde{A}_a + \frac{3}{32L} \lambda (\tilde{A}_a \tilde{A}_a)^2 \right), \quad (3.16)$$

where $m = n_0 \pi / L = \omega$ is the effective mass of the photon gas confined in the Fabry-Perot cavity. It is worth mentioning that the paraxial approximation guarantees that the photons have a finite effective mass also when tunneling effects due to the low but finite transmittivity of the mirrors are taken into account [1]. Setting

$$\tilde{A}_1 = (\varphi + \varphi^\dagger), \quad (3.17)$$

$$\tilde{A}_2 = -i(\varphi - \varphi^\dagger),$$

we obtain the final form of the four-photon Hamiltonian

$$H_{\text{eff}} = \int d^2 \mathbf{x}_\perp \left( -\varphi^\dagger \nabla^2 _\perp \varphi + m^2 \varphi^\dagger \varphi + \frac{3}{2L} \lambda (\varphi^\dagger \varphi)^2 \right), \quad (3.18)$$

describing an effective weakly-interacting massive 2D photon gas. This Hamiltonian displays a $U(1)$ global phase symmetry, which follows from the $O(2)$ rotational invariance in the $xy$ plane of the dimensionally reduced 2D effective Hamiltonian (3.16). Moreover, for a paraxial propagation of the light inside the cavity [1], we have $p_\perp = \sqrt{p_1^2 + p_2^2} \ll m$, so that the photon gas is in fact nonrelativistic. In order to obtain the spectrum of the Hamiltonian we expand the fields $\varphi, \varphi^\dagger$ in Fourier modes

$$\varphi = \frac{1}{\sqrt{A}} \sum_p a_p e^{i \mathbf{p}_\perp \cdot \mathbf{x}_\perp}, \quad (3.19)$$

$$\varphi^\dagger = \frac{1}{\sqrt{A}} \sum_p a_p^\dagger e^{-i \mathbf{p}_\perp \cdot \mathbf{x}_\perp},$$

where $A$ is the available area of the cavity in the transverse plane $xy$. Thus
\[ H_{\text{eff}} = \sum_p \epsilon_p a_p^\dagger a_p + \frac{3\lambda}{2} V \sum_{p_1+p_2=p_3+p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} , \quad (3.20) \]

where \( V = AL \) is the volume of the cavity and

\[ \epsilon_p = \sqrt{\vec{p}^2 + m^2} . \quad (3.21) \]

Of course, expression (3.20) is nothing but the starting Hamiltonian of [1]. Now we can proceed with the standard analysis of the weakly-interacting Bose gas within the Bogoliubov approximation [1, 12]. Assuming that the number \( N_0 \) of excitations occupying the zero-momentum state is macroscopic, \( \text{i.e.} \ N_0 \gg 1 \), and neglecting higher order interaction terms above the condensate, the Hamiltonian (3.20) is diagonalized by means of a Bogoliubov transformation

\[
\begin{align*}
a_p &= u_p \alpha_p + v_p \alpha_{-p}^\dagger , \\
a_p^\dagger &= u_p \alpha_p^\dagger + v_p \alpha_{-p} ,
\end{align*}
\tag{3.22}
\]

with

\[
u_p^2 - v_p^2 = 1 .
\]

The resulting spectrum is easily worked out and turns out to be given by

\[ H_{\text{eff}} = \sum_{p \neq 0} \tilde{\epsilon}_p \alpha_p^\dagger \alpha_p + E_0 , \quad (3.23) \]

with

\[
\tilde{\epsilon}_p = \left[ \left( \epsilon_p - m + 3\lambda \frac{N}{V} \right)^2 - \left( 3\lambda \frac{N}{V} \right)^2 \right]^{1/2} , \quad (3.24)
\]

\( N \) being the total average number of photons in the cavity. As is well known, expression (3.24) provides a nonvanishing critical velocity for the phonon sound-like waves

\[ \text{We recall here that } N_0 \text{ is of the order of } 8 \times 10^{11}, \text{ so that the condition } N_0 \gg 1 \text{ is in fact verified.} \]
\[ v_c = \lim_{p \to 0} \frac{\tilde{\epsilon}_p}{p} = \sqrt{\frac{3\lambda N}{mV}}. \]  
\[ (3.25) \]

Let us finally display the time evolution of the fields in the Bogoliubov approximation (3.23). Making use of the relations

\[ u_p^2 + v_p^2 = \frac{\left(\epsilon_p - m + 3\lambda \frac{N}{V}\right)}{\tilde{\epsilon}_p}, \]  
\[ 2uv_p = -3\frac{N}{V} \lambda \tilde{\epsilon}_p, \]  

for the time evolution of the mode \( a_p \) we obtain

\[ i\frac{\partial a_p(t)}{\partial t} = (\epsilon_p - m) a_p(t) + 3\lambda \frac{N}{V} \left(a_p(t) + a_{-p}^\dagger(t)\right), \]  
\[ a_p(t) = e^{iH_{\text{eff}}t} a_p e^{-iH_{\text{eff}}t}, \]  

in agreement with the classical nonlinear argument of [1].

### 4 Conclusion

In this work we have derived a possible theoretical set up for the photon condensation effect proposed in [1], starting from a nonlinear QED–inspired Hamiltonian. In particular, we have shown that the dynamics of the photon gas inside the Fabry-Perot cavity filled with a nonlinear polarizable medium is described in terms of a 2D dimensionally reduced effective Hamiltonian of a massive complex scalar field. This implies that the transverse dimensions of the cavity should play a crucial role in the photon condensation. In fact, according to our field theory description, if one takes as possible thermodynamic limit \( N, A \to \infty, N/A = \text{const.} \), the lowest lying phonon of the Bogoliubov spectrum would play the role of the Goldstone boson corresponding to the spontaneous breaking of the \( U(1) \) symmetry of the Hamiltonian.

\[ ^{\text{We remind that } A \text{ is the available area of the cavity in the transverse plane } xy.} \]
However, it is well known that spontaneous symmetry breaking cannot take place in two-dimensions, due to the infrared divergencies associated to massless scalar fields \[14\].

Therefore, as in the case of trapped Bose gases, the photon gas has to be considered as a genuine finite-sized system, the transverse dimensions of the cavity providing a natural infrared cut-off\[4\].

Thus, we expect that these finite size conditions should be actually realized in the experimental framework in order to observe the photon condensate. The role played by the finiteness of the system is under investigation. In this context, it would be also very interesting to study the possibility of the existence of a Kosterlitz-Thouless phase transition for the 2D photon (super)fluid.

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\[4\]Notice in fact that finite size effects are rather relevant in the condensation of 2D dimensionally reduced atomic gases, for very similar reasons: long wavelenght phonons destabilize the long range order of the condensate \[5\].
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