Modeling and analysis of fractal growth patterns from the material point

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Abstract. The growth of fractals from a material point is modeled using single-phase and multi-phase algorithms for linear fractals and polygonal fractal forms. The connection of fractal growth processes with the phenomena of formation in nature is considered. A method for analyzing growth processes and their characteristics is proposed. The growth laws of fractals metric characteristics and their transformation when changing the growth algorithm are revealed. The growth path of fractal points and the distribution of the slope angles of their elements are analyzed. The influence of stochastic fluctuations in the length of fractal elements on the process of its growth is modeled. It is shown that the algorithms and methods used for analyzing growth processes have a great potential for modernization and adaptation to a wide range of tasks.

1. Introduction
The range of scale value of the base variable is limited to the minimum and maximum values in the case of real natural fractals which are considered and constructed in various applications. They are determined by the conditions of fractal formation or by the tasks to be solved. In the modeling geometric fractals, for example, a unit-length segment is taken as the initial scale, which determines the maximum scale of such a problem [1]. The most significant feature of natural fractals and fractal processes is their growth from a material point. This process begins from the point of primitive form with the unbroken development of the fractal structure and the large-scale growth of all its elements. For example, it would be the growth of plants, individual fractal-like elements of living organisms (circulatory system, lungs, etc.), the development of turbulence in liquids and gases. This growth mechanism corresponds to a huge number of real natural processes, so it must be studied carefully for their numerical (computer) modeling.

In this paper, the processes of fractal growth from a material point are studied on simple geometric models with successive complication by the growth algorithm. Using the laws of composition of fractal algorithms [4], the growth of complex fractal forms is modeled, which resembles the processes of formation in nature. The regularities of the growth process are studied. The effect of stochasticity on the growth process is modeled.

2. Theoretical description of fractals in growth processes
Traditionally, the main fractal relations are written in universal form as the number of dimensions $C$ [2] and the length of the fractal line $L$ [3]. Variables contain indeterminate multipliers $C_1, C_2$, spatial scale
ε[2], and fractal dimension \( D[2] \). For the geometric fractals considered in this paper, it is convenient to define the spatial scale \( \varepsilon \) in relative form as:

\[
\varepsilon = \frac{a}{l}
\]

(1)

where \( a \) is the length of an elementary segment of the fractal, and \( l \) is the span of the fractal, i.e. the distance between its extreme points horizontally. Exactly, value \( l \) will increase during the growth of the fractal. Determining the scale in the relative format (1) makes it easy to generalize the quantitative relations for the fractals under consideration, using similar expressions for fractals obtained by splitting a single segment [5]. The standard algorithm for constructing fractals like the Koch Snowflake [6] consists of dividing elementary segments into \( m \) parts and replacing them with \( n \) segments.

For growth processes, the main relationships are \( l \) and \( L \), which follow the main patterns of growth of fractals. The span of the fractal \( l \) is determined by the number \( m \) and depends on it exponentially \( m^k \) with the exponent \( k \) equal to the depth of the fractal. The length of the fractal line \( L \) is determined only by the number \( n \) and depends on it exponentially \( n^k \) with the exponent \( k \) equal to the depth of the fractal. The ratio of the fractal line length and its span characterizes the development, complexity of the fractal line, and the density of its surface filling.

\[
\frac{L}{l} = \left( \frac{n}{m} \right)^k = m^{(D-1)k} = n^{(D-1)k}
\]

(2)

In this work, the growth of fractals like the «Koch Snowflake» [2] from a material point was performed with modification of the inclination angles of this geometric fractal main elements. However, the growth algorithms and their number may be different. Two growth algorithms have been developed, which are described below.

3. Single-phase growth algorithm

The main spatial scale is the size of the elementary segment \( a \) of the fractal in its completed form. Growth starts from a material point. When the segment grows to size \( a \), a primitive appears in the middle of it in the form of a polyline consisting of two symmetrical elements with dimensions \( ra \), and an angle of inclination to its branch \( \beta \) (figure 1). The coefficient \( r \) and angle \( \beta \) are model parameters and varied during the modeling.

![Figure 1. The main model geometric primitive with a bend, its coordinates and parameters.](image)

The growth rates of all four elements of this primitive are selected so that they simultaneously reach the size \( a \). Further, on each segment of length \( a \), the growth processes are completely repeated. This provides a characteristic feature of this algorithm: simultaneous formation of new primitives and their synchronous growth – all primitives are in the same growth phase. This is why the algorithm is called single-phase.

The advantage of this algorithm is its simplicity. The «rigid» symmetry of the main geometric primitive ensures high accuracy of growth: after each stage of reaching the growing elements of size \( a \),
a classic version of the «Koch Snowflake» type fractal is obtained. An example of modeling fractal growth in accordance with this algorithm is shown in figure 2. It is seen that the primitives of the new fractal elements are generated at the same time and grow simultaneously. The growth is proportional in all directions, but the «rigidity» of the algorithm is a problem of reliability or realism of the model. Synchronous growth of all fractal elements does not correspond well to the growth processes of real fractal-like structures.

4. Multi-phase algorithm growth

Generalization of the previous algorithm was performed by getting rid of «rigid synchronization» in the growth of fractal elements. The growth of each of its elementary segments was performed independently with a certain step $\Delta a$ in each cycle. Since the initial lengths of segments have different values, their growth phases also differ. This is why this growth algorithm is called multi-phase. In one version of the algorithm, the growth of all fractal elements stopped simultaneously. This led to the fact that there were sprouts that did not reach size $a$, i.e. the growth process was incomplete, the growth goal was not achieved. The elimination of this disadvantage was achieved by independently increasing each element to the limit value of $a$. An example of growth modeling using a multi-phase algorithm is shown in figure 3. Of particular interest is the disproportionate, mostly longitudinal growth of the fractal at the first stages. The number of sprouts grows very quickly, they are in different growth phases. This indicates that in the process of growth, the model is not a fractal in the strict sense of the word. At the final stages of growth, all elements become size $a$, the model takes a complete form and becomes a fractal of a certain order. The symmetry and synchronicity of the growth of the left and right branches of the fractal were observed.
This algorithm has a great potential for its modernization by introducing the dependence of the step \( \Delta a \) on the depth of the fractal and the growth phase. This will increase its adaptability for modeling the growth processes of real fractal-like objects and the ability to generate more diverse fractals.

Obviously, for single-phase and multi-phase algorithms, the result is the same fractal. This indicates that the fractalization of the growth process creates a "distributed memory effect" of the overall shape of the growing object in the details of its structure. In other words, combining the process of large-scale growth with fractal structure decomposition allows you to grow objects with a regular internal structure and a complex global shape.

5. Analysis of the growth process
Based on the modeling of the growth of geometric fractals performed in this paper, we can offer methods for analyzing these processes.

First of all, it is necessary to analyze the metric characteristics of growing fractals. The span and length of the fractal line described by the \( l \) and \( L \) ratios for the modeled fractals is shown in figure 4 as a function of the number of growth cycles. The difference in the growth dynamics of these parameters for single-phase and multi-phase growth models is very significant, which was noted during the analysis of figure 2 and figure 3. The obtained regularities characterize two classical growth laws: exponential (exponential) and logistic. The length and span of a fractal growing using a single-phase algorithm are described by a polyline (figure 4, line a). Its vertices (figure 4: f1, f2) are well approximated by the exponential theoretical dependencies \( l \) and \( L \). Linear sections between vertexes correspond to a large-scale growth of fractal segments of a constant number, and new segments are born at nodal points and the slope of linear sections increases. For a multi-phase algorithm, the data is approximated by a logistic curve, which in General has the form:

\[
N(t) = \frac{N_m N_0}{N_0 + (N_m - N_0) e^{-\varepsilon t}}
\]

Its parameters for this task are replaced with the following. The role of time \( t \) is played by the number of fractal construction cycles. The initial number of segments is \( N_0 = 1 \). The optimal number of fractal elements is \( N_m = 4^k \) according to the ratio \( N = n^k \). The relative growth rate \( \varepsilon \) is defined as the approximation parameter for each curve. Thus, for our case, the logistic law takes the form (in parentheses of the second term, we neglect the unit, since \( 4^k >> 1 \)):
\[ N(t) = \frac{4^k}{1+4^k e^{-\alpha t}} \] (4)

For length and span the corresponding growth laws according to \( L \) and \( D \) take the form:

\[ L(t) = \frac{4^k a}{1+4^k e^{-\alpha t}} \] (5)

\[ l(t) = \frac{(2+2\cos \beta)^k a}{1+4^k e^{-\alpha t}} \] (6)

These functions are well approximated [7] by the corresponding curves in figure 4.

Thus, changing the growth algorithm from single-phase to multi-phase with a limited growth depth leads to a significant transformation of the growth law of the metric characteristics of the fractal. The logistic law is one of the most universal growth processes characteristics of various nature.

Figure 5. Successive stages of growth of a polygonal (triangle) fractal shape from a material point located in the center, and the growth trajectory of the fractal nodal points. The density of points on the trajectories reflects the growth rate.

Important information about the growth process is provided by the growth trajectories of the characteristic points of the fractal. Figure 5 shows several intermediate shapes as the polygonal fractal shape grows (on the left) and the corresponding trajectories of their characteristic points (on the right). As you increase the speed of growth of the trajectory from a solid line turns into a point. The distance between points is proportional to the growth rate. Depending on the complexity of the fractal boundaries, the paths for different points may intersect or overlap. If necessary, such data can be used to construct vector fields of displacements and growth rates.

An essential structural and topological characteristic of a growing fractal is the set and dynamics of the appearance of different orientation angles of fractal segments. Theoretically, the set of angles for fractals of the type used can take values such as:

\[ \phi_k = \pm k \beta, \quad k = 0,1,2,3, \ldots \] (7)

Where different signs correspond to different symmetric branches of the fractal. Thus, for an ideal fractal, the number of different angles increases in proportion to the depth of the fractal \( k \) with a step equal to the base angle of the fractal, all angles \( \phi_k \) are multiples of the base angle. Visually, the angular structure of a fractal is represented by a diagram (figure 6) in polar coordinates.

\[ \rho_k = 1+0.2k \quad \phi_k = \pm k \beta, \quad k = 0,1,2,3, \ldots \] (8)
A coefficient of 0.2 in the radial variable is selected to stretch the chart in this direction to prevent overlapping angles and thus display information about all segments of the fractal.

The General considerations and the type of angle diagrams allow us to draw the following conclusions about the angular structure of fractals in the process of their growth: the lower the value of the base angle, the denser the set of angles in the fractal, but the speed of filling the corner space decreases (figure 6, left); if the base angle $\beta$ is a multiple of the full angle 360°, then the set of angles is finite (figure 6, in the center). And the more proportional the angles $\beta$ and 360°, the smaller the set of angles in the fractal; if the angles $\beta$ and 360° are not proportional, the set of angles in the fractal is limited only by its depth (figure 6, on the right); for complex compositions of fractal algorithms[4], the angular structure of the fractal can be made arbitrarily diverse.

6. Effect of stochasticity on growth processes
Real growth processes are always subject to a wide range of stochastic perturbations. In addition, growth is a cumulative, integral process, so any disturbance in its early stages leads to consequences in the following stages. Therefore, in growth processes, stochastic effects can be much more significant than in stationary States. In this paper, the effect of stochasticity on the growth of fractals was modeled by random variations in the length of fractal segments with an amplitude of $\Delta a_i$. Three fragments of such modeling for a multi-phase growth algorithm are presented in figure 7.

The intensity of stochastic fluctuations of the entire fractal structure is highest in the areas of the highest growth rate of the fractal, which is well described by the logistic laws (5, 6). The growth results are radically different from those presented in figure 3. First of all, there is a significant asymmetry of all elements of the fractal, distortion of the structure of a significant part of the details. The cumulative effect noted above is well manifested. As the amplitude of stochastic oscillations $\Delta a_i$ increases, reversible self-intersections first appear in the structure of the growing fractal, i.e. they disappear with further growth. In the end, there are also irreversible self-intersections, which can be considered unacceptable growth defects.
7. Conclusion
Modeling the growth of fractals from a material point reveals geometric, phase, and statistical patterns common to a wide range of growth processes. The conducted modeling and analysis of model experiments indicate that many growth processes are not a trivial increase in size, but are a complex scale-fractal process. Large-scale growth provides an increase in size, volume, and mass. Fractalization of the growth process provides the internal structure and global shape of the growing object. This is due to the fact that the global or macroscopic form of a fractal is determined by the laws of the internal microscopic structure. This conditionality of the macroscopic structure by the microscopic law (algorithm) is the reason for the significant role of fractals in nature, the processes of formation and growth. The use of more complex combinational laws of fractal algorithms, discussed in [4], will allow us to model the most arbitrary fractal forms with variation both in the depth of the fractal boundary and along its perimeter.

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