The Gauged WZNW Model Perturbed By The Sigma Model Term

Oleg A. Soloviev

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, United Kingdom

Abstract

We discuss special perturbations of the gauged level $k$ WZNW model inspired by the $\sigma$-model perturbation of the nonunitary WZNW model. In the large $k$ limit there is a second conformal point in the vicinity of the ultraviolet fixed point. At the second critical point the conformal model has a rational Virasoro central charge which no longer corresponds to a coset construction. In spite of this fact the perturbative conformal model appears to be a unitary system as long as the underlying coset construction at the ultraviolet critical point is unitary.

March 1993
1 Introduction

Coset constructions [1] provide a fairly elegant description to a wide class of irreducible conformal representations. In contrast with many known algebraic conformal systems, coset constructions possess a proper Lagrangian formulation in terms of gauged Wess-Zumino-Novikov-Witten (GWZNW) models [2]. The significance of this can be appreciated, because many of GWZNW models admit quite an interesting space-time interpretations giving rise to exact string solutions [3]-[5]. This aspect of coset constructions is nowadays under extensive consideration.

Of no less importance is another aspect related to the Lagrangian formulation of coset constructions. Namely, this is the issue of understanding cosets as critical points of larger (massive) systems. This problem is in the intimate connection to possible perturbations of GWZNW models by relevant operators. Answering these questions might turn out to be helpful in understanding the configuration space of string solutions which is believed to form the coordinate space of string field theory.

In the present paper we are going to explore some special deformations of GWZNW models inspired by the \( \sigma \)-model perturbation of nonunitary conformal WZNW models [6],[7]. It was shown in [3],[7] that the conformal WZNW model with large negative level \( k \) allows a relevant perturbation by the \( \sigma \)-model term. The perturbed theory flows to the unitary conformal WZNW model with positive level \( |k| \). So that the two conformal points discovered by Witten [8] turn out to be the ultraviolet and infrared fixed points of the one renormalization group flow.

In the case of the GWZNW model, the conformal nonunitary WZNW model emerges within the path integral quantization of the gauge invariant system [2]. Therefore, there is an opportunity to apply the discovered perturbation of nonunitary conformal WZNW theories to unitary systems. It is quite plausible that this way might lead us to some unitary rational conformal models which are distinct from coset constructions.
2 The $\sigma$-model perturbation of the GWZNW

The crucial observation we are going to make use of is that at the classical level the GWZNW model can be described as a combination of usual conformal WZNW models

$$S_{GWZNW} = S_{WZNW}(hg\tilde{h};k) + S_{WZNW}(h\tilde{h};-k), \quad (2.1)$$

where $S_{WZNW}(f;l)$ is the level $l$ conformal WZNW model defined as follows

$$S_{WZNW}(f,l) = -\frac{l}{4\pi} \int [\text{Tr}|f^{-1}df|^2 + \frac{i}{3}d^{-1}\text{Tr}(f^{-1}df)^3]. \quad (2.2)$$

Here the matrix field $f$ is either $g$ taking its values on the Lie group $G$ or $h, \tilde{h}$ taking values on the subgroup $H$ of $G$. Respectively the level $l$ is either $k$ or $-k$.

The action of the GWZNW model in terms of the group element $g$ and the vector nonabelian fields $A_z$ and $\bar{A}_{\bar{z}}$ is obtained from eq. (2.1) upon the following definition

$$A_z = h^{-1}\partial h, \quad \bar{A}_{\bar{z}} = \bar{\partial}\tilde{h}\tilde{h}^{-1}. \quad (2.3)$$

Correspondingly the gauge symmetry in the variables $g, h, \tilde{h}$ is given by

$$g \to \Omega g\Omega^{-1}, \quad h \to h\Omega^{-1}, \quad \tilde{h} \to \Omega\tilde{h}, \quad (2.4)$$

where $\Omega$ is the parameter of the gauge group $H$. Obviously the product $h\tilde{h}$ is gauge invariant. As a matter of fact, in the GWZNW model presented in the variables $g, h, \tilde{h}$, the following operator

$$O = \text{const} : \text{Tr}[\partial(h\tilde{h})\bar{\partial}(h\tilde{h})^{-1}] : \quad (2.5)$$

has to be a gauge invariant operator, although the very action of the fields $h, \tilde{h}$ becomes changed.

At the quantum level the action of the GWZNW in the variables $g, h, \tilde{h}$ is given by

$$S_{QGWZNW} = S_{WZNW}(hg\tilde{h}) + S_{WZNW}(h\tilde{h};-k-2c_V(H)) + S_{Gh}(b,c,\bar{b},\bar{c}). \quad (2.6)$$

Compared to the classical action in eq. (2.1), in the quantum action the second WZNW model of the product $h\tilde{h}$ has the level shifted by two times the eigenvalue of the quadratic
Casimir operator in the adjoint representation of the subalgebra $\mathcal{H}$. In addition, the QGWZNW model has the ghost-like contribution

$$S_{\text{Gh}} = \text{Tr} \int d^2 z (b \partial c + \bar{b} \partial \bar{c}). \quad (2.7)$$

The gauge symmetry (2.4) allows one to impose the following gauge condition

$$\tilde{h} = 1, \quad (2.8)$$

which does not entail any additional propagating Faddeev-Popov ghosts. In this gauge the operator $O$ in eq. (2.5) takes the form

$$O = \text{const} : \text{Tr}(\partial h \bar{\partial} h^{-1}) :, \quad (2.9)$$

which is nothing but the $\sigma$-model term.

A convenient choice for the constant in eq. (2.9) is

$$\text{const} = \frac{(k + 2 c_V(H))^2}{4}. \quad (2.10)$$

Then the operator $O$ can be written as a normal ordered product of the three operators

$$O = : \phi^{\alpha \bar{\alpha}} \cdot J_{\alpha} \cdot \bar{J}_{\bar{\alpha}} :, \quad (2.11)$$

where

$$\phi^{\alpha \bar{\alpha}} = \text{Tr}(h^{-1} t^\alpha h t^{\bar{\alpha}}),$$

$$J_{\alpha} = \frac{1}{2} \eta_{\alpha \beta} \text{Tr}[(k + 2 c_V(H)) h^{-1} \partial h t^\alpha], \quad (2.12)$$

$$\bar{J}_{\bar{\alpha}} = \frac{1}{2} \eta_{\bar{\alpha} \bar{\beta}} \text{Tr}[(k + 2 c_V(H)) \bar{\partial} h h^{-1} t^{\bar{\beta}}].$$

Here $t^\alpha$, $t^{\bar{\alpha}}$ are the generators of two copies of the Lie algebra $\mathcal{H}$.

The important point to be made is that normal ordering in the product (2.11) can be understood according to

$$O = \oint \frac{dw}{2\pi} \oint \frac{d\bar{w}}{2\pi} J_{\alpha}(w) \cdot \bar{J}_{\bar{\alpha}}(\bar{w}) \cdot \phi^{\alpha \bar{\alpha}}(z, \bar{z}) \left| z - w \right|^2, \quad (2.13)$$

where in the numerator the product is thought of as an operator product expansion (OPE) with respect to the conformal WZNW model $S_{WZNW}(h, -k - 2 c_V(H))$. Furthermore, one can compute the conformal dimensions of the operator $O$

$$\Delta = \bar{\Delta} = 1 - c_V(H)/(k + c_V(H)). \quad (2.14)$$
From the last formula it follows that in the large \( k \) limit, the given operator is to be classified as a quasimarginal relevant operator. Besides, the operator \( O \) preserves explicitly the global \( H \times H \) symmetry of the conformal theory \( S_{WZNW}(h, -k - 2c_V(H)) \). Because of this fact, the operator \( O \) has to obey the following fusion rule

\[
O \cdot O = \lfloor O \rfloor + \lfloor I \rfloor,
\]

(2.15)

where the square brackets denote the contributions of \( O \) and \( I \) and the corresponding descendants of \( O \) and \( I \). Here \( I \) is identity operator.

Thus, the given operator \( O \) has the properties which are appropriate for performing a renormalizable perturbation of the conformal WZNW model \( S_{WZNW}(h, -k - 2c_V(H)) \). Since the latter appears as an intrinsic part of the GWZNW model, the perturbation by the operator \( O \) results in the perturbation of the GWZNW theory. It is significant to point out that \( O \) does not belong to the subspace of operators of the \( G/H \) coset. Indeed, one can check that this operator \( O \) is not annihilated by the corresponding BRST operator of the GWZNW model \([11]\). At the same time, this operator \( O \) being an affine descendant of the affine-Virasoro primary field \( \phi^{a\bar{a}} \):

\[
O = J_{-1} \bar{J}_{-1} \phi,
\]

has positive norm (with respect to the \( SL(2, C) \) invariant vacuum)

\[
\|O\|^2 = \|J_{-1} \bar{J}_{-1} \phi > |^2 = (k + 2c_V(H))^2 \|\phi\|^2 > 0.
\]

* Note that the BRST operator selecting the states corresponding to the \( G/H \) coset is defined by the constraints which are nothing but conditions of independence of the partition function of the GWZNW model on the gauge fields \( A_z, \bar{A}_\bar{z} \)[1]. Whereas the gauge invariance requires a more weak restriction. Namely,

\[
\partial < \bar{J}^{tot} > - \bar{\partial} < J^{tot} > = 0,
\]

where

\[
< \bar{J}^{tot} > = \frac{\delta Z(A_z, \bar{A}_\bar{z})}{\delta A_z}|_{A_z, \bar{A}_\bar{z}=0} = 0, \quad < J^{tot} > = \frac{\delta Z(A_z, \bar{A}_\bar{z})}{\delta \bar{A}_\bar{z}}|_{A_z, \bar{A}_\bar{z}=0} = 0.
\]

Here \( A_z, \bar{A}_\bar{z} \) are background gauge fields associated with the quantum \( A_z, \bar{A}_\bar{z} \).
Furthermore, the Virasoro central charge of the nonunitary WZNW model $S_{WZNW}(h, -k - 2c_V(H))$ is positive. Indeed,

$$c_{WZNW}(-k - 2c_V(H)) = \frac{(k + 2c_V(H)) \dim H}{k + c_V(H)} > \dim H.$$ 

Thus, the conclusion is: since the operator $O$ has the positive conformal dimensions given by eq. (2.14), this operator provides a unitary representation of the Virasoro algebra.

From now on by the perturbed GWZNW model we will understand the following theory

$$S_{PGWZNW} = S_{GWZNW} - \epsilon \int d^2z \ O(z, \bar{z}).$$

(2.16)

3 Perturbed conformal point

We go on to compute the renormalization beta function corresponding to the coupling $\epsilon$. Away of criticality, where $\epsilon \neq 0$, the beta function is defined according to (see e.g. [9])

$$\beta = [2 - (\Delta + \bar{\Delta})] \epsilon - \pi C \ \epsilon + \mathcal{O}(\epsilon^3),$$

(3.17)

where $\Delta$, $\bar{\Delta}$ are given by eq. (2.14). The constant $C$ is the coefficient of the three point function

$$< O(z_1)O(z_2)O(z_3) > = C \ \prod_{i<j}^3 \frac{1}{|z_{ij}|^{\Delta + \bar{\Delta}}}$$

(3.18)

when the two point functions are normalized to unity.

The coefficient $C$ in the large $k$ limit is given by [7]

$$C = c_V(H) + \mathcal{O}(1/k).$$

(3.19)

With the given $C$ one can easily solve equation (3.17) to find the perturbative fixed point

$$\epsilon_2 = \frac{-2}{\pi k}.$$ 

(3.20)

Note that the value of the perturbative conformal point in eq. (3.20) can be corrected by higher orders in $1/k$. It is quite remarkable that one can obtain the exact value of $\epsilon_2$. Indeed, the second conformal point $\epsilon_2$ has to coincide with the second critical point of the WZNW model. This is given as follows

$$\epsilon_2 = \frac{-2}{\pi (k + 2c_V(H))}.$$ 

(3.21)
Obviously in the large $k$ limit this solution goes to the perturbative expression in eq. (3.20).

Once we know the perturbative critical point exactly, we can compute the exact Virasoro central charge at this point $\epsilon_2$. We find

$$c(\epsilon_2) = c(G/H) + c_{WZNW}(k + 2c_V(H)) - c_{WZNW}(-k - 2c_V(H))$$

$$= c(G/H) - \frac{2(k + 2c_V(H))c_V(H) \dim H}{k^2 + 4kc_V(H) + 3c_V(H)^2}. \tag{3.22}$$

Here $c_{WZNW}(l)$ denotes the Virasoro central charge of the level $l$ WZNW model. Thus we come to conclusion that the perturbative Virasoro central charge $c(\epsilon_2)$ is less than $c(G/H)$. According to Zamolodchikov’s $c$-theorem \cite{Zamolodchikov} the magnitude of the Virasoro central charge at the infrared critical point, that is $\epsilon_2$, must be less than at the ultraviolet critical point, $\epsilon = 0$, if the flow between these points is unitary. Since the flow in the case under consideration behaves in the full agreement with the $c$-theorem, it is suggestive that the conformal system by the perturbation of the GWZNW model gives rise to the unitary theory as long as the GWZNW model describes a unitary coset. Note that it is not obvious from the expression for the central charge given by eq. (2.22) that the underlying conformal model corresponds to another unitary coset construction. Thus it may turn out that the BRST projection of GWZNW models is not only way of deriving unitary Virasoro representations.

4 Conclusion

We have considered the perturbation of the $G/H$ GWZNW model by the gauge invariant operator coinciding with the $\sigma$-model term of the WZNW model on the subgroup $H$. We exhibited that the perturbed GWZNW model arrives at the perturbative (infrared) critical point which has a rational Virasoro central charge. Surprisingly this central charge does not appear to be a combination of the Virasoro central charges of some unitary cosets, but we were not able to prove it precisely. We found that the flow agrees with the $c$-theorem for unitary systems. Based on it we conjecture that the obtained conformal system is
unitary provided the $G/H$ coset projection of the GWZNW model is unitary.

Acknowledgement: I would like to thank J. M. Figueroa-O’Farrill, C. M. Hull and I. Vaysburd for useful discussions. I would also like to thank the SERC for financial support.

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