QUADRATIC TORSION SUBGROUPS OF MODULAR JACOBIAN VARIETIES

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ABSTRACT

Let \( D \) be an odd square-free positive integer and \( C \) be a divisor of \( D \). For any quadratic character \( \chi \) modulo \( C \), we prove that a large portion of the \( \chi \)-part of the torsion subgroup of \( J_0(DC) \) coincides with that of its cuspidal subgroup.

1. Introduction

For a positive integer \( N \), let \( X_0(N) \) be the modular curve of level \( N \) over \( \mathbb{Q} \) and let \( J_0(N) \) be its Jacobian variety. The generalized Ogg’s conjecture asserts that \( \mathbb{Q} \)-rational torsion points on \( J_0(N) \) are all cuspidal. More precisely, let \( C_0(N) \) be the cuspidal subgroup of \( J_0(N) \) generated by those degree-zero divisor classes supported at the cusps of \( X_0(N) \). Denote by \( C_0(N)(\mathbb{Q}) \) the subgroup of \( \mathbb{Q} \)-rational points of \( C_0(N) \). Then the conjecture says that

\[
J_0(N)(\mathbb{Q})_{tor} = C_0(N)(\mathbb{Q}),
\]

where \( J_0(N)(\mathbb{Q})_{tor} \) is the torsion subgroup of the group of \( \mathbb{Q} \)-rational points on \( J_0(N) \). We know that

- \( J_0(p)(\mathbb{Q})_{tor} = C_0(p)(\mathbb{Q}) \) for any prime \( p \) (see [9]);
- \( J_0(p^r)(\mathbb{Q})_{tor} \otimes_{\mathbb{Z}} \mathbb{Z}[1/6p] = C_0(p^r)(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/6p] \) for any prime \( p \geq 5 \) and any integer \( r \geq 1 \) (see [8] or [5]).

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• $J_0(D)(\mathbb{Q})_{\text{tor}} \otimes \mathbb{Z} [1/6] = C_0(D) \otimes \mathbb{Z} [1/6]$ for any square-free positive integer $D$ (see [10]). Note that, when $D$ is square-free, all the cusps of $X_0(D)$ are $\mathbb{Q}$-rational and hence

$$C_0(D) = C_0(D)(\mathbb{Q}).$$

However, since cusps are not $\mathbb{Q}$-rational in general, it is natural to ask: What is the role played by the whole group $C_0(N)$ in $J_0(N)$? In this paper we take a first step in investigating this question. Let $N = DC$ with $D$ an odd square-free positive integer and let $C$ be a divisor of $D$. Then all cusps of $X_0(DC)$ are $\mathbb{Q}(\mu_C)$-rational (see §4.1). For any quadratic Dirichlet character $\chi$ modulo $C$ we define

$$J_0(DC)(\chi) := \{ P \in J_0(DC)(\overline{\mathbb{Q}})_{\text{tor}} | \sigma(P) = \chi(\sigma)P \text{ for any } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}.$$

Put

$$C_0(DC)(\chi) := C_0(DC) \cap J_0(DC)(\chi).$$

Then our main result is the following theorem.

**Theorem 1.1:** Let $D$ be an odd square-free positive integer and let $C$ be a divisor of $D$. Then, for any quadratic character $\chi$ modulo $C$, we have

$$J_0(DC)(\chi)[q^\infty] = C_0(DC)(\chi)[q^\infty]$$

for any prime $q$ not dividing $6D \varphi(D) \varpi(C/f_\chi)$, where $f_\chi$ is the conductor of $\chi$, $\varphi$ is the Euler function and

$$\varpi\left(\frac{C}{f_\chi}\right) = \prod_{p|C/f_\chi} (p^2 - 1).$$

Here is an outline of the proof. Let $\mathbb{T} = \mathbb{Z}[\{T_\ell\}]$ be the full Hecke algebra of level $DC$ and $I_\chi = (\{T_\ell - \chi(\ell) - \chi(\ell)\ell\}_{\ell|D})$ be an ideal in $\mathbb{T}$. Then the Eichler–Shimura relation implies that $J_0(DC)(\chi)[q^\infty]$ is a module over the ring $\mathbb{T}/I_\chi$. We are thus reduced to proving

$$J_0(DC)(\chi)[m^\infty] = C_0(DC)(\chi)[m^\infty]$$

for every maximal ideal $m$ of $\mathbb{T}$ containing $I_\chi$ and $q$. This consists of the following three ingredients:

1. For any such maximal ideal $m$, we associate an Eisenstein series $E$ which produces a subgroup $C(E)$ of $C_0(DC)(\chi)[m^\infty]$ by a method of Stevens.
(2) On the other hand, we obtain an Eisenstein ideal \( I(E) \) from the annihilators of \( E \) in the Hecke algebra, whose index in \( \mathbb{T} \) agrees with the order of \( C(E) \) (at least away from \( 6D \)).

(3) Finally, under the assumption \( (q, 6D \varphi(D) \varpi(C)) = 1 \), we will prove that \( J_0(DC)(\chi)[m^\infty] \) is a cyclic \( \mathbb{T}/I(E) \)-module, which implies

\[
|J_0(DC)(\chi)[m^\infty]| \leq |C(E)|
\]

and completes the proof. Note that we make the assumption about \( q \) to deal with the possible congruence between oldforms and newforms.

The paper is organized as follows. After recalling some preliminaries in §2, we construct in §3 an eigen-basis for the space of Eisenstein series of weight two and level \( DC \). While all these Eisenstein series are interesting, we will in this paper focus on those with quadratic characters. For any such an Eisenstein series, we calculate in §4 the order of its associated cuspidal subgroup. Then we give the proof of Theorem 1.1 as outlined above in the final section.

**Notations.** For any abelian group \( A \) and prime \( p \), let

\[
A_p := A \otimes \mathbb{Z} \mathbb{Z}_p
\]

with \( \mathbb{Z}_p \) the ring of \( p \)-adic integers.

For any positive integer \( N = \prod_{p \mid N} p^{\nu_p(N)} \), let

\[
\nu(N) := \sum_{p \mid N} v_p(N)m, \quad \psi(N) := \prod_{p \mid N} (p + 1) \quad \text{and} \quad \varpi(N) = \varphi(N) \cdot \psi(N).
\]

\( \mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) is the Poincaré upper half-plane. Let \( q: \mathcal{H} \to \mathbb{C}, z \mapsto e^{2\pi i z} \), be the function on \( \mathcal{H} \) which will be used in the Fourier expansions of modular forms.

For any function \( g \) on the upper half-plane and any \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{GL}_2^+(\mathbb{R}) \), we denote \( g|\gamma \) to be the function

\[
z \mapsto \det(\gamma)(cz + d)^{-2}g(\gamma z),
\]

where \( \gamma z = \frac{az + b}{cz + d} \).

If \( g \) is a modular form of some level and \( x \) is a cusp, then we denote \( a_0(g; [x]) \) to be the constant term of the Fourier expansion of \( g \) at \( x \).
2. Preliminaries

2.1. For any positive integer $N$, the modular curve $X_0(N)$ of level $N$ over $\mathbb{Q}$ is the coarse moduli space classifying all pairs $(E, H)$, where $E$ is a (generalized) elliptic curve over some $\mathbb{Q}$-scheme, and $H$ is a cyclic subgroup of $E$ of order $N$. This is a smooth projective curve over $\mathbb{Q}$ and we denote $J_0(N)$ to be its Jacobian variety. The Atkin–Lehner operator $w_N$ is an automorphism on $X_0(N)$ defined as

$$w_N(E, H) = (E/H, E[N]/H).$$

Then $w_N^2 = \text{id}_{X_0(N)}$. We will denote also by $w_N$ the induced automorphism on $J_0(N)$. We next briefly recall the definition of Hecke operators. For more details see §3 of [4]. For any prime $\ell$, let $X_0(N, \ell)$ be the coarse moduli space classifying all triples $(E, H, G)$, where $(E, H)$ is as above and $G$ is a cyclic subgroup of order $\ell$ such that $G \cap H = \{0\}$. Then there are two degeneracy morphisms

$$\begin{aligned}
\alpha_\ell & : X_0(N, \ell) \to X_0(N), \\
\beta_\ell & : X_0(N, \ell) \to X_0(N),
\end{aligned}$$

defined as

$$\alpha_\ell(E, H, G) := (E, H)$$

and

$$\beta_\ell(E, H, G) := (E/G, (H + G)/G).$$

Then, for any prime $\ell$, we define an endomorphism on $J_0(N)$ as

$$T_\ell^{(N)} := \beta_\ell \circ \alpha_\ell^*,$$

and define $\mathbb{T}(N)$ to be the sub-$\mathbb{Z}$-algebra of $\text{End}_\mathbb{Q}(J_0(N))$ generated by $T_\ell^{(N)}$ for all primes $\ell$. Unless necessary we denote $T_\ell^{(N)}$ simply as $T_\ell$.

2.2. Let $\mathcal{M}_2(\Gamma_0(N), \mathbb{C})$ be the space of weight-two modular forms of level $\Gamma_0(N)$. Let $S_2(\Gamma_0(N), \mathbb{C})$ (resp. $\mathcal{E}_2(\Gamma_0(N), \mathbb{C})$) be the subspace of cusp forms (resp. Eisenstein series) of $\mathcal{M}_2(\Gamma, \mathbb{C})$, so that we have

$$\mathcal{M}_2(\Gamma_0(N), \mathbb{C}) = S_2(\Gamma_0(N), \mathbb{C}) \oplus \mathcal{E}_2(\Gamma_0(N), \mathbb{C}).$$
For any prime $\ell$, we have a Hecke operator $T_{\ell}^{(N)}$ acting on $M_2(\Gamma_0(N), \mathbb{C})$ as
\[
T_{\ell}^{(N)}(g) = \begin{cases}
\sum_{k=0}^{\ell} g|_{(1 \ 1 \ \ell \ \ell)} + g|_{(0 \ 0 \ 1 \ 1)} & \text{if } \ell \nmid N, \\
\sum_{k=0}^{\ell} g|_{(1 \ 1 \ \ell \ \ell)} & \text{if } \ell \mid N.
\end{cases}
\]
Define $T^{(N)}$ to be the sub-$\mathbb{Z}$-algebra of $\text{End}_\mathbb{Q}(M_2(\Gamma_0(N), \mathbb{C}))$ generated by $T_{\ell}^{(N)}$ for all primes $\ell$. Unless necessary we denote $T_{\ell}^{(N)}$ simply as $T_{\ell}$. Since $T_{\ell}$ coincides with $T_{\ell}^{(N)}$ under the natural isomorphism $S_2(\Gamma_0(N), \mathbb{C}) \simeq H^0(J_0(N), \Omega_{J_0(N)}) \otimes \mathbb{Q} \mathbb{C}$ (see, for example, Corollary 3.17 of [4]), we can identify $T^{(N)}$ with the restriction of $T_{\ell}$ to the space of cusp forms.

2.3. Let $\text{cusp}(\Gamma_0(N))$ be the set of cusps of $X_0(N)$. Denote by $Y_0(N)$ the complement of $\text{cusp}(\Gamma_0(N))$ in $X_0(N)$. For any form $g$ in $M_2(\Gamma_0(N), \mathbb{C})$, let $\omega_g$ be the differential on $X_0(N)(\mathbb{C})$ whose pullback to $\mathcal{H}$ equals $g(z)dz$. If $E \in E_2(\Gamma_0(N), \mathbb{C})$, then $\omega_E$ is holomorphic on $Y_0(N)$ and hence there is an induced homomorphism
\[
\xi_E : H_1(Y_0(N)(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}, \quad [c] \mapsto \int_c \omega_E,
\]
where $[c]$ is a homology class represented by a 1-cycle $c$ on $Y_0(N)(\mathbb{C})$. Note that, for any small circle $c_x$ around a cusp $x$, we have
\[
\int_{c_x} \omega_E = 2\pi i \cdot \text{Res}_x(\omega_E) = e_x \cdot a_0(E; [x]),
\]
where $\text{Res}_x(\omega_E)$ is the residue of $\omega_E$ at $[x]$, $e_x$ is the ramification index of $X_0(N)$ over $X(1)$ at $x$, and $a_0(E; [x])$ is the constant term of the Fourier expansion of $E$ at the cusp $x$ (see P36 of [13]). Let
\[
\text{Div}^0(\text{cusp}(\Gamma_0(N)); \mathbb{C}) = \text{Div}^0(\text{cusp}(\Gamma_0(N)); \mathbb{Z}) \otimes \mathbb{Z} \mathbb{C}.
\]
Then we have the following $\mathbb{C}$-linear map (see [13], p. 35, or [14], §1):
\[
\delta_N : E_2(\Gamma_0(N), \mathbb{C}) \to \text{Div}^0(\text{cusp}(\Gamma_0(N)); \mathbb{C}),
\]
which is defined as
\[
E \mapsto 2\pi i \sum_{x \in \text{cusp}(\Gamma_0(N))} \text{Res}_x(\omega_E) \cdot [x].
\]
Definition 2.1: For any $E$ in $\mathcal{E}_2(\Gamma_0(N), \mathbb{C})$, let $\mathcal{R}_N(E)$ be the sub-$\mathbb{Z}$-module of $\mathbb{C}$ generated by the coefficients of $\delta_N(E)$. Let $\mathcal{R}_N(E)^\vee = \text{Hom}_\mathbb{Z}(\mathcal{R}_N(E), \mathbb{Z})$ be the dual $\mathbb{Z}$-module of $\mathcal{R}_N(E)$.

1. The **cuspidal subgroup** $C_N(E)$ associated with $E$ is defined to be the subgroup of $J_0(N)$ generated by $\{w_N(\phi(\delta_N(E)))\}_{\phi \in \mathcal{R}_N(E)^\vee}$, where $\phi(\delta_N(E))$ is the divisor obtained by applying $\phi$ to the coefficients of $\delta_N(E)$.

2. The **Eisenstein ideal** $I_N(E)$ of $E$ is defined to be the image of $\text{Ann}_{\mathcal{T}(N)}(E)$ in $\mathcal{T}(N)$, where $\text{Ann}_{\mathcal{T}(N)}(E)$ is the annihilator of $E$ in $\mathcal{T}(N)$.

3. The **period** $P_N(E)$ of $E$ is defined to be the image of $\xi_E$.

Remark 2.2: The above definition of $C_N(E)$ is slightly different from that given in §1.8 of [13] and §1 of [14]. Here we add an action of $w_N$. This has the advantage of making $C_N(E)$ annihilated by $I_N(E)$. To see this, note that on the one hand we have $\delta_N(\mathcal{T}_\ell(E)) = {^t \mathcal{T}_\ell}(\delta_N(E))$ for any prime $\ell$, where $^t \mathcal{T}_\ell$ is the transpose of $\mathcal{T}_\ell$ (see p. 110 of [13]). On the other hand, we have $w_N \circ ^t \mathcal{T}_\ell = \mathcal{T}_\ell \circ w_N$ by (4) on p. 444 of [12], so the claim follows. However, since $w_N$ is an involution, this modification does not change the order of $C_N(E)$.

Remark 2.3: In the same way as above, we can define a $\mathbb{C}$-linear map

$$\delta_{\Gamma_1(N)} : \mathcal{E}_2(\Gamma_1(N), \mathbb{C}) \to \text{Div}^0(\text{cusp}(\Gamma_1(N)); \mathbb{C});$$

and, for any $E$ in $\mathcal{E}_2(\Gamma_1(N), \mathbb{C})$, we have similarly a homomorphism

$$\xi'_E : H_1(Y_1(N)(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}.$$ 

In particular, we can define $\mathcal{R}_{\Gamma_1(N)}(E)$ to be the module generated by the coefficients of $\delta_{\Gamma_1(N)}(E)$ and define

$$\mathcal{P}_{\Gamma_1(N)}(E) := \text{Im}(\xi'_E).$$

2.4. Take an $E$ in $\mathcal{E}_2(\Gamma_0(N), \mathbb{C})$. Then $E$ also belongs to $\mathcal{E}_2(\Gamma_1(N), \mathbb{C})$ as $\Gamma_0(N) \subseteq \Gamma_1(N)$. If we have the Fourier expansion $E = \sum_{n=0}^{\infty} a_n(E; [\infty]) \cdot q^n$, then for any Dirichlet character $\eta$, we define

$$L(E, \eta, s) := \sum_{n=1}^{\infty} \frac{a_n(E; [\infty]) \cdot \eta(n)}{n^s}.$$
Denote $S_N$ to be the set of all primes $p$ satisfying $p \equiv -1 \pmod{4N}$. Let $\mathfrak{X}_N$ be the set of all non-quadratic Dirichlet characters $\eta$ whose conductor is a prime in $S_N$, and let $\mathfrak{X}_N^\infty$ be the set of all non-quadratic Dirichlet characters $\eta$ whose conductor is a prime in $S_N$. For any $\eta \in \mathfrak{X}_N^\infty$ of conductor $p_\eta^d$, where $p_\eta \in S_N$ and $d$ is a positive integer, let

$$
\Lambda(E,\eta,1) := \frac{\tau(\overline{\eta}) \cdot L(E,\eta,1)}{2\pi i},
\Lambda_{\pm}(E,\eta,1) := \frac{1}{2}(\Lambda(E,\eta,1) \pm \Lambda(E,\eta \cdot (\overline{\eta^d}),1)).
$$

Here $(\overline{\eta^d})$ is the Legendre symbol associated to $p_\eta$. By Theorem 1.3 of [14], if $\mathcal{M}$ is a finitely generated sub-$\mathbb{Z}$-module of $\mathbb{C}$, then the following are equivalent:

\begin{align*}
\text{St1: } & \mathcal{P}_{\Gamma_1(N)}(E) \subseteq \mathcal{M}; \\
\text{St2: } & \mathcal{R}_{\Gamma_1(N)}(E) \subseteq \mathcal{M} \text{ and } \Lambda_{\pm}(E,\eta,1) \in \mathcal{M}[\eta, \frac{1}{p_\eta}] \text{ for any } \eta \in \mathfrak{X}_N; \\
\text{St3: } & \mathcal{R}_{\Gamma_1(N)}(E) \subseteq \mathcal{M} \text{ and } \Lambda_{\pm}(E,\eta,1) \in \mathcal{M}[\eta, \frac{1}{p_\eta}] \text{ for any } \eta \in \mathfrak{X}_N^\infty.
\end{align*}

Let $\pi_N^*: J_0(N) \to J_1(N)$ be the homomorphism induced by the natural projection $\pi_N: X_1(N) \to X_0(N)$. Let $\Sigma_N = \ker(\pi_N^*)$ be the Shimura subgroup of $J_0(N)$, which is finite and of multiplicative type as a group scheme over $\mathbb{Q}$. Put

$$
A^{(s)}_N(E) := (\mathcal{P}_{\Gamma_1(N)}(E) + \mathcal{R}_N(E))/\mathcal{R}_N(E).
$$

Then, by (4.3) of [14], there is an exact sequence

$$
0 \longrightarrow \Sigma_N \cap \mathcal{C}_N(E) \longrightarrow \mathcal{C}_N(E) \longrightarrow A^{(s)}_N(E) \longrightarrow 0,
$$

which enables us to determine the order of $\mathcal{C}_N(E)/(\Sigma_N \cap \mathcal{C}_N(E))$.

2.5. At the end of this section, we recall some basic properties of the collection of functions $\{\phi_{\underline{x}}\}_{\underline{x} \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}}$ due to Hecke (see [13], Chapter 2, §2.4) which will be needed later. For any $\underline{x} = (x_1, x_2) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$, the Fourier expansion of $\phi_{\underline{x}}$ at infinity is

$$
\phi_{\underline{x}}(z) + \delta(\underline{x}) \cdot \frac{i}{2\pi(z-\overline{z})} = \frac{1}{2}B_2(x_1) - P_{\underline{x}}(z) - P_{-\underline{x}}(z)
$$

for any $z \in \mathcal{H}$, where $B_2(t) = \langle t \rangle^2 - \langle t \rangle + \frac{1}{6}$ is the second Bernoulli polynomial and

$$
P_{\underline{x}}(z) = \sum_{k \in \mathbb{Q}_{>0}, k \equiv x_1(1)}^\infty \sum_{m=1}^\infty k \cdot e^{2\pi i m (kz+x_2)},
$$
and \( \delta(\underline{x}) \) is defined to be 1 or 0 according to whether \( \underline{x} = 0 \) or not. If \( \underline{x} \neq 0 \), then \( \phi_{\underline{x}} \) is a (holomorphic) Eisenstein series. Moreover, for any \( \underline{x} \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \) and \( \gamma \in \text{SL}_2(\mathbb{Z}) \), we have

\[
(2.3) \quad \phi_{\underline{x}}|\gamma = \phi_{\underline{x} \cdot \gamma},
\]

where \( \underline{x} \cdot \gamma \) is the natural right action of \( \gamma \) on the row vector of length two. The whole collection of functions satisfies the following distribution law:

\[
(2.4) \quad \phi_{\underline{x}} = \sum_{\underline{y}, \underline{x} \cdot \alpha = \underline{x}} \phi_{\underline{y}}|\alpha,
\]

where \( \alpha \) is any matrix in \( M_2(\mathbb{Z}) \) with positive determinant.

3. An eigen-basis for \( \mathcal{E}_2(\Gamma_0(DC), \mathbb{C}) \)

3.1. We first introduce some operators on the space \( C^\infty(\mathcal{H}, \mathbb{C}) \) of smooth \( \mathbb{C} \)-valued functions on \( \mathcal{H} \). Similar constructions have already been used in Definition 4.6 of [14]. When the character is trivial, these operators have been introduced by Yoo (see Definition 2.5 of [16]). For any prime \( p \), let

\[
\gamma_p : C^\infty(\mathcal{H}, \mathbb{C}) \to C^\infty(\mathcal{H}, \mathbb{C}), \quad g \mapsto g \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right).
\]

If \( \chi \) is a Dirichlet character of conductor \( f_\chi \), then, for any prime \( p \) not dividing \( f_\chi \), we define

\[
[p]_\chi^+ := 1 - \chi(p)\gamma_p,
\]
\[
[p]_\chi^- := 1 - p^{-1}\chi(p)^{-1}\gamma_p.
\]

If \( p_1 \) and \( p_2 \) are two primes not dividing \( f_\chi \), then \( [p_1]_\chi^+, [p_1]_\chi^-, [p_2]_\chi^+ \) and \( [p_2]_\chi^- \) commute with each other. So we can define, for any positive square-free integer \( M \) prime to \( f_\chi \), the following operators

\[
[M]_\chi^\pm := [p_1]_\chi^+ \circ [p_2]_\chi^+ \circ \cdots \circ [p_k]_\chi^+,
\]

with \( M = p_1 \cdot p_2 \cdots p_k \) in any order. When \( \chi = 1 \), we write \([M]_1^\pm\) as \([M]^\pm\) for simplicity.
LEMMA 3.1: Let $N$ be a positive integer. If $\chi$ is a character of conductor $f_\chi$ and $p$ a prime such that $(p, f_\chi) = 1$, then $[p]^{\pm}_\chi M_2(\Gamma_0(N), \mathbb{C}) \subseteq M_2(\Gamma_0(Np), \mathbb{C})$, $[p]^{\pm}_\chi (S_2(\Gamma_0(N), \mathbb{C})) \subseteq S_2(\Gamma_0(Np), \mathbb{C})$ and $[p]^{\pm}_\chi (E_2(\Gamma_0(N), \mathbb{C})) \subseteq E_2(\Gamma_0(Np), \mathbb{C})$.

Proof. It suffices to prove the same assertions for the operator $\gamma_p$. The first two assertions are clear. Since $\{\phi_{2, z \in (\mathbb{Q}/\mathbb{Z})^2 - 0}\}$ is a basis for the space of Eisenstein series of weight two, and $\gamma_p(\phi_{2, z}) = \phi_{2, |z|p}$ is still an Eisenstein series by (2.4), the third assertion follows.

LEMMA 3.2: Let $N$ be a positive integer. If $\chi$ is a character of conductor $f_\chi$ and $p$ a prime such that $(p, f_\chi) = 1$, then

1. $\mathcal{T}_\ell^{(Np)} \circ [p]^{\pm}_\chi = [p]^{\pm}_\chi \circ \mathcal{T}_\ell^{(N)}$ for any prime $\ell \neq p$;
2. if $p \nmid N$, then $\mathcal{T}_p^{(Np)} \circ [p]^{\pm}_\chi = \mathcal{T}_p^{(N)} - \gamma_p - p \cdot \chi(p)$ and $\mathcal{T}_p^{(Np)} \circ [p]^{-}_\chi = \mathcal{T}_p^{(N)} - \gamma_p - \chi(p)^{-1}$;
3. if $p \mid N$, then $\mathcal{T}_p^{(Np)} \circ [p]^{\pm}_\chi = \mathcal{T}_p^{(N)} - p \cdot \chi(p)$ and $\mathcal{T}_p^{(Np)} \circ [p]^{-}_\chi = \mathcal{T}_p^{(N)} - \chi(p)^{-1}$.

Proof. If $(p, \ell) = 1$, then $\gamma_p$ commutes with the $\ell$-th Hecke operators, so the first assertion follows. If $p \nmid N$, then we have

$$\mathcal{T}_p^{(Np)} \circ [p]^{\pm}_\chi(g) = g\left[1 - \chi(p) \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix}$$

$$= g\sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} - \chi(p) \cdot g\sum_{k=0}^{p-1} \begin{pmatrix} p & pk \\ 0 & p \end{pmatrix}$$

$$= \mathcal{T}_p^{(N)}(g) - \gamma_p(g) - p \cdot \chi(p) \cdot g,$$

for any $g \in M_2(\Gamma_0(N), \mathbb{C})$, and

$$\mathcal{T}_p^{(Np)} \circ [p]^{-}_\chi(g) = g\left[1 - p^{-1} \cdot \chi(p)^{-1} \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right] \sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix}$$

$$= g\sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} - p^{-1} \cdot \chi(p)^{-1} \cdot g\sum_{k=0}^{p-1} \begin{pmatrix} p & pk \\ 0 & p \end{pmatrix}$$

$$= \mathcal{T}_p^{(N)}(g) - \gamma_p(g) - \chi(p)^{-1} \cdot g,$$

which proves (2). The proof of (3) is similar so we omit it here.
3.2. From now on we fix an odd positive square-free integer $D$ and a positive divisor $C$ of $D$. The assumption that $D$ is odd will not be needed until the final section. Define $\mathcal{H}(DC)$ to be the set of all triples $(M, L, \chi)$ satisfying:

- $1 \leq M, L | D$ with $M \neq 1$;
- $D | ML | DC$;
- $\chi$ is a Dirichlet character modulo $(M, L)$.

**Lemma 3.3:** $\# \mathcal{H}(DC) = \dim_{\mathbb{C}} E_2(\Gamma_0(DC), \mathbb{C})$.

**Proof.** It is well known that the number of cusps of $X_0(DC)$ is equal to $\sum_{1 \leq d | DC} \varphi(d, DC/d)$ (see §2.1 of [5]), and hence

$$\dim_{\mathbb{C}} E_2(\Gamma_0(DC), \mathbb{C}) = \sum_{1 < d | DC} \varphi(d, DC/d).$$

Here $\varphi(d, DC/d)$ means applying Euler's $\varphi$-function to the greatest common divisor of $d$ and $DC/d$. Thus we only need to prove that

$$\# \mathcal{H}(DC) = \sum_{1 < d | DC} \varphi(d, DC/d).$$

We will first prove this when $C = D$. For any positive divisor $d$ of $D^2$, we can associate the following two positive integers:

$$M := \sqrt{d \cdot \left( \frac{D^2}{d} \right)}, \quad L := \sqrt{\frac{D^2}{d} \cdot \left( \frac{D^2}{d} \right)}$$

such that $1 \leq M, L | D$ and $D | ML | D^2$. Conversely, to any pair of integers $M$ and $L$ with $1 \leq M, L | D$ and $D | ML | D^2$, we can associate a positive divisor $d$ of $D$ as

$$d := \left[ \frac{M}{(M, L)} \right]^2 \cdot (M, L).$$

It is easy to see that the above establishes a bijection between $\{d : 1 \leq d | D^2\}$ and the set of all pairs of integers $M$ and $L$ with $1 \leq M, L | D$ and $D | ML | D^2$. Moreover, under this bijection, the divisor 1 of $D^2$ corresponds to the pair $M = 1$ and $L = D$, and we have $(d, D^2/d) = (M, L)$ if $d$ corresponds to $M$ and $L$. It follows that there is a bijection between

$$\{(d, \chi)| 1 < d | D^2, \chi : (\mathbb{Z}/(d, D^2/d) \cdot \mathbb{Z})^\times \to \mathbb{C}^\times \}$$

and

$$\mathcal{H}(D^2)$$

which proves the lemma in this situation.
In general, since $DC = \frac{D}{C} \cdot C^2$, any positive divisor $d$ of $DC$ can be uniquely decomposed as $d = d_0 \cdot d'$ with $1 \leq d_0 | \frac{D}{C}$ and $1 \leq d' | C^2$. If such a positive divisor $d'$ of $C^2$ corresponds to a pair of integers $m$ and $\ell$ with $1 \leq m, \ell | C$ and $C | m\ell | C^2$ as above, then we can associate with $d$ the pair of integers $M = d_0 \cdot m$ and $\frac{DC}{d_0} \cdot \ell$ which satisfies $1 \leq M, L | D$ and $D | ML | DC$. This establishes a bijection between \( \{d | 1 \leq d | DC\} \) and the set of all pairs of integers $M$ and $L$ with $1 \leq M, L | D$ and $D | ML | DC$. Moreover, we have $1 | D^2$ corresponds to the pair $M = 1$ and $L = D$, and $(d, \frac{DC}{d}) = (M, L)$ if $d$ corresponds to $M$ and $L$. It follows that there is a bijection between \( \{(d, \chi) | 1 < d | D^2, \chi : (\mathbb{Z}/(d, DC/d) \cdot \mathbb{Z})^\times \rightarrow \mathbb{C}\} \) and $H(DC)$ which completes the proof of the lemma. \[\blacksquare\]

**Definition 3.4:** For any Dirichlet character $\chi$ of conductor $f_\chi$, let

$$E_\chi := -\frac{1}{2g(\chi)} \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(a) \cdot \chi(b) \cdot \phi(\frac{a}{f_\chi}, \frac{b}{f_\chi})$$

where $g(\chi)$ is the Gauss sum of $\chi$. Then, for any $(M, L, \chi)$ in $H(DC)$, we define

$$E_{M, L, \chi} := \left[ \frac{L}{f_\chi} \right]_\chi \circ \left[ \frac{M}{f_\chi} \right]^+_\chi (E_\chi).$$

3.3. Let $\delta_\chi = 1$ or $0$ according to whether $\chi$ is trivial or not. It follows from (2.1) that

$$E_\chi = -\frac{\delta_\chi}{4\pi i(z - \bar{z})} - \frac{1}{g(\chi)} \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(a) \cdot \chi(b) \left( \frac{1}{4} B_2 \left( \frac{a}{f_\chi} \right) - P \left( \frac{a}{f_\chi}, \frac{b}{f_\chi} \right) \right).$$

Moreover, by (2.2) we have that

$$\sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(a) \cdot \chi(b) \cdot P \left( \frac{a}{f_\chi}, \frac{b}{f_\chi} \right)$$

$$= \sum_{k, m = 1}^{\infty} \frac{k\chi(k)}{f_\chi} \left( \sum_{y \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(y) e^{2\pi i \frac{my}{f_\chi} z} \right) e^{2\pi i \frac{mk}{f_\chi} z}$$

$$= \sum_{k, m = 1}^{\infty} \frac{k\chi(k)}{f_\chi} \left( \sum_{y \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(y) e^{2\pi i \frac{my}{f_\chi} z} \right) e^{2\pi i mk z}$$

$$= g(\chi) \sum_{k, m = 1}^{\infty} k \cdot \chi(k) \cdot \chi^{-1}(m) \cdot e^{2\pi i mk z},$$
where a Dirichlet character is extended to a function on \( \mathbb{Z} \) in the usual way. Therefore

\[
E_\chi = -\frac{\delta_\chi}{4\pi i(z - \bar{z})} + a_0(E_\chi; [\infty]) + \sum_{n=1}^{\infty} \sigma_\chi(n) \cdot q^n, \tag{3.1}
\]

where

\[
a_0(E_\chi; [\infty]) = \begin{cases}
-\frac{1}{24} & \text{if } \chi = 1, \\
0 & \text{otherwise,}
\end{cases} \tag{3.2}
\]

and

\[
\sigma_\chi(n) := \sum_{1 \leq d | n} d \cdot \chi(d) \cdot \chi^{-1}(n/d). \tag{3.3}
\]

**Lemma 3.5:** For any \((M,L,\chi)\) in \(H(\text{DC})\), \(E_{M,L,\chi}\) is a normalized form in \(E_2(\Gamma_0(\text{DC}), \mathbb{C})\).

**Proof.** If \(\chi\) is nontrivial so that \(f_\chi > 1\), then \(E_\chi\) belongs to \(E_2(\Gamma_0(f_\chi^2), \mathbb{C})\) by (2.3). Since \(ML\) divides \(DC\), it follows from Lemma 3.1 that

\[
E_{M,L,\chi} \in E_2(\Gamma_0(ML), \mathbb{C}) \subseteq E_2(\Gamma_0(\text{DC}), \mathbb{C}).
\]

On the other hand, if \((M,L,1)\) \(H(\text{DC})\), we take a prime divisor \(p\) of \(M\), which is possible because \(M > 1\). Since

\[
[p]^+(E_1) = -\frac{1}{2}(\phi_{(0,0)} - \gamma_p(\phi_{(0,0)}))
= -\frac{1}{2}\phi_{(0,0)} + \frac{1}{2} \sum_{b \in \mathbb{Z}/p\mathbb{Z}} \phi_{(0, b)}
= \frac{1}{2} \sum_{b \in (\mathbb{Z}/p\mathbb{Z})^\times} \phi_{(0, b)},
\]

it follows that \([p]^+(E_1)\) belongs to \(E_2(\Gamma_0(p), \mathbb{C})\), which implies that \(E_{M,L,1}\) belongs to \(E_2(\Gamma_0(\text{DC}), \mathbb{C})\) similarly as above. Finally, since both \([M/f_\chi]_\chi^+\) and \([L/f_\chi]_\chi^-\) preserve the first terms of Fourier expansions, we find that

\[
a_1(E_{M,L,\chi}; [\infty]) = a_1(E_\chi; [\infty]) = 1
\]

and hence complete the proof. \(\blacksquare\)
In particular, by the distribution law (2.4), we have

\[ T^{(f_x^2)}_\ell (E_\chi) = \begin{cases} (\chi(\ell)^{-1} + \ell \cdot \chi(\ell)) \cdot E_\chi & \text{if } \ell \nmid f_x, \\ 0 & \text{if } \ell \mid f_x. \end{cases} \]

**Proof.** Let \( \ell \) be a prime such that \((\ell, f_x) = 1\). By Proposition 2.4.7 of [13], for any integers \( x, y \) prime to \( f_x \), we have

\[ T^{(f_x^2)}_\ell (\phi_{(\frac{x}{f_x}, \frac{y}{f_x})}) = \phi_{(\frac{x}{f_x}, \frac{y}{f_x})} + \ell \cdot \phi_{(\frac{x'}{f_x}, \frac{y'}{f_x})}, \]

where \( \ell' \) is an integer such that \( \ell\ell' \equiv 1 \pmod{f_x} \). It follows that

\[ T^{(f_x^2)}_\ell (E_\chi) = (\chi(\ell)^{-1} + \ell \cdot \chi(\ell)) \cdot E_\chi. \]

On the other hand, by the distribution law (2.4), we have

\[ E_\chi = -\frac{1}{2g(\chi)} \sum_{x, y \in (\mathbb{Z}/f_x \mathbb{Z})^*} \chi(x) \cdot \chi(y) \cdot \phi_{(\frac{x}{f_x}, \frac{y}{f_x})} \left| f_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0, \]

so, if \( \ell \) is a prime divisor of \( f_x \), then

\[ T^{(f_x^2)}_\ell (E_\chi) \]

\[ = -\frac{1}{2g(\chi)} \sum_{x, y \in (\mathbb{Z}/f_x \mathbb{Z})^*} \chi(x) \cdot \chi(y) \cdot \phi_{(\frac{x}{f_x}, \frac{y}{f_x})} \left| f_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \]

\[ = -\frac{1}{2g(\chi)} \sum_{x, y \in (\mathbb{Z}/f_x \mathbb{Z})^*} \chi(x) \cdot \chi(y) \sum_{k=0}^{\ell-1} \phi_{(\frac{x}{f_x}, \frac{y}{f_x} + \frac{yk}{f_x})} \left| f_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0. \]

**Proposition 3.7:** For any \((M, L, \chi)\) in \( \mathcal{H}(DC) \), we have

\[ T^{(DC)}_\ell (E_{M,L,\chi}) = \begin{cases} (\chi(\ell)^{-1} + \ell \cdot \chi(\ell)) \cdot E_{M,L,\chi} & \text{if } \ell \nmid Dm \\ \chi(\ell)^{-1} \cdot E_{M,L,\chi} & \text{if } \ell \mid \frac{M}{(M, L)}, \\ \ell \cdot \chi(\ell) \cdot E_{M,L,\chi} & \text{if } \ell \mid \frac{L}{(M, L)}, \\ 0 & \text{if } \ell \mid (M, L). \end{cases} \]

In particular,

\[ E_2(\Gamma_0(DC), \mathbb{C}) = \bigoplus_{(M,L,\chi) \in \mathcal{H}(DC)} \mathbb{C} \cdot E_{M,L,\chi}. \]
Proof. It suffices to prove the first assertion. By Lemma 3.2(1) and Lemma 3.6, if \( \ell \) is a prime not dividing \( D \), then

\[
T^{(DC)}_{\ell}(E_{M,L,\chi}) = \left[ \frac{L}{f_{\chi}} \right]^{-} \circ \left[ \frac{M}{f_{\chi}} \right]^{+} \circ T^{(f_{\chi}^2)}_{\ell}(E_{\chi})
\]

\[
= (\chi(\ell)^{-1} + \ell \cdot \chi(\ell)) \cdot E_{M,L,\chi}.
\]

By Lemma 3.2(2), if \( \ell \) is a prime divisor of \( M/(M,L) \), then

\[
T^{(DC)}_{\ell}(E_{M,L,\chi}) = \left[ \frac{L}{f_{\chi}} \right]^{-} \circ \left[ \frac{M}{f_{\chi}} \right]^{+} \circ \chi(\ell)^{-1} \cdot E_{M,L,\chi}.
\]

The proofs for those prime divisors of \( L/(M,L) \) and \( (M,L)/(f_{\chi}) \) are similar to the above, so we omit them here. Finally, if \( \ell \) is a prime divisor of \( f_{\chi} \), then we find by Lemma 3.2(1) and Lemma 3.6 that

\[
T^{(DC)}_{\ell}(E_{M,L,\chi}) = \left[ \frac{L}{f_{\chi}} \right]^{-} \circ \left[ \frac{M}{f_{\chi}} \right]^{+} \circ T^{(f_{\chi}^2)}_{\ell}(E_{\chi}) = 0,
\]

which completes the proof of the proposition.

\[ \blacksquare \]

4. Orders of quadratic cuspidal subgroups

4.1. For any triple \( (M, L, \chi) \) in \( \mathcal{H}(DC) \), denote \( C^{(DC)}_{M,L,\chi} \) to be the cuspidal subgroup \( C_{DC}(E_{M,L,\chi}) \) associated with the Eisenstein series \( E_{M,L,\chi} \) as in Definition 2.1. When \( \chi \) is a quadratic character, we call \( C^{(DC)}_{M,L,\chi} \) a quadratic cuspidal subgroup.

For any positive integer \( N \), we have representatives for cusps on \( X_0(N) \) of the form \( \left[ \frac{xf_{\chi}}{N} \right] \), where \( d \mid N, d > 0 \) and \( (x,d) = 1 \) with \( x \) taken modulo \( (d,N/d) \). Moreover such a cusp \( \left[ \frac{xf_{\chi}}{N} \right] \) is defined over \( \mathbb{Q}(\mu_c) \) with \( c = (d,N/d) \) (see §2.1 of [5]). Put \( N = DC \). Then any positive divisor \( d \) of \( DC \) is of the form \( d = rs^2t \), where \( r \mid \frac{D}{C} \), \( s, t \mid C \) and \( (s,t) = 1 \). So we obtain a full set of representatives \( \{ [\frac{rs^2t_{\chi}}{DC}] \} \) for cusps on \( X_0(DC) \), with \( r,s,t \) as above and \( (x,rst) = 1 \) taken modulo \( t \).
Lemma 4.1: Let $p$ be a prime divisor of $D$ and $\left[\frac{rs^2tx}{DC}\right]$ a cusp of $X_0(DC)$. Then:

1. If $p \mid r$, then $\left[\frac{rs^2tx}{DC}\right] = \left[\frac{(r/p)s^2tx}{DC/p}\right]$ in $X_0(DC/p)$.
2. If $p \mid s$, then $\left[\frac{rs^2tx}{DC}\right] = \left[\frac{r(s/p)s^2tx}{DC/p^2}\right]$ in $X_0(DC/p^2)$.
3. If $p \mid t$, then $\left[\frac{rs^2tx}{DC}\right] = \left[\frac{rs^2(t/p)(px)}{DC/p^2}\right]$ in $X_0(DC/p^2)$.
4. If $p \mid \frac{D}{Cv}$, then $\left[\frac{rs^2tx}{DC}\right] = \left[\frac{rs^2t(px)}{DC/p}\right]$ in $X_0(DC/p)$.
5. If $p \mid \frac{C}{st}$, then $\left[\frac{rs^2tx}{DC}\right] = \left[\frac{rs^2t(p^2x)}{DC/p^2}\right]$ in $X_0(DC/p^2)$.

Proof. The first two assertions are obvious. Since the proofs of the last three assertions are similar, we will only give that of (3). We need to determine the unique cusp of the form $\left[\frac{r's^2t'x'}{DC/p^2}\right]$ on $X_0(DC/p^2)$ which equals $\left[\frac{rs^2tx}{DC}\right]$. Take

$$\gamma = \left( \begin{array}{cc} \frac{x}{DC} & u \\ \frac{rs^2t}{v} & v \end{array} \right) \in SL_2(\mathbb{Z})$$

such that $\gamma(\infty) = \frac{rs^2tx}{DC}$. Similarly take

$$\gamma' = \left( \begin{array}{cc} \frac{x'}{DC/p^2} & u' \\ \frac{r's^2t'}{v'} & v' \end{array} \right) \in SL_2(\mathbb{Z})$$

such that $\gamma'(\infty) = \frac{r's^2t'x'}{DC/p^2}$. Then there exists some $\left( \frac{\alpha}{p^2x^2}, \frac{\beta}{\delta}, \frac{\omega}{\gamma} \right)$ in $\Gamma_0(DC/p^2)$ such that

$$\left( \frac{\alpha}{DC}, \frac{\beta}{p^2x^2}, \frac{\omega}{\gamma} \right) \cdot \left( \begin{array}{cc} \frac{x}{DC} & u \\ \frac{rs^2t}{v} & v \end{array} \right)_{\infty} = \left( \begin{array}{cc} \frac{x'}{DC/p^2} & u' \\ \frac{r's^2t'}{v'} & v' \end{array} \right)_{\infty},$$

which implies that there is an integer $n$ such that

$$\left( \frac{\alpha}{DC}, \frac{\beta}{p^2x^2}, \frac{\omega}{\gamma} \right) \cdot \left( \begin{array}{cc} \frac{x}{DC} & u \\ \frac{rs^2t}{v} & v \end{array} \right) = \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right) \cdot \left( \begin{array}{cc} \frac{x'}{DC/p^2} & u' \\ \frac{r's^2t'}{v'} & v' \end{array} \right).$$

Then, by equating the left lower corner, we find that

$$\frac{DC/p^2}{v's^2t'} = \frac{DC}{p^2}x\delta + \frac{DC/p^2}{rs^2(t/p)}p\omega.$$ 

It follows that $rs^2(t/p) = r's^2t'$ and hence $r' = r$, $s' = s$ and $t' = t/p$. This in turn implies by the above equation that $1 = rs^2(t/p)x\delta + p\omega$ and, in particular, $\omega \equiv p^{-1} \pmod{t/p}$. It follows that $v' \equiv p^{-1}v \pmod{t/p}$ by equating the right
lower corner, which implies that
\[ x' \equiv v'^{-1} \equiv pv^{-1} \equiv px \pmod{t/p}. \]

Let \( K \) be a positive divisor of \( D \) and \( \alpha \) a positive divisor of \( K \). We leave to the reader, using Lemma 4.1, to verify the following equations: if \( (K, rst) = 1 \), then

\[ \left[ \frac{rs^2 t \alpha x}{DC} \right] = \left[ \frac{rs^2 t (\frac{K(K,C)}{\alpha} x)}{DC/K(K,C)} \right] \in X_0 \left( \frac{DC}{K(K,C)} \right); \tag{4.1} \]

and if \( K \mid t \), then

\[ \left[ \frac{rs^2 t \alpha x}{DC} \right] = \left[ \frac{rs^2 (\frac{t}{K}) (\frac{K}{\alpha} x)}{DC/K^2} \right] \in X_0 \left( \frac{DC}{K^2} \right). \tag{4.2} \]

4.2. For any cusp \( \left[ \frac{a}{c} \right] \) of \( X_0(N) \) with \((a, c) = 1\) choose \( (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z}) \) which maps \( \infty \) to \( \left[ \frac{a}{c} \right] \). Moreover, for any prime \( p \), we can and will always assume \( p \mid d \) when \( p \nmid c \), so that

\[
\begin{pmatrix}
  p & 0 \\
  0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} = \begin{cases}
  \begin{pmatrix}
    a & pb \\
    c/p & d
  \end{pmatrix} \begin{pmatrix}
    p & 0 \\
    0 & 1
  \end{pmatrix} & \text{if } p \mid c, \\
  \begin{pmatrix}
    ap & b \\
    c & d/p
  \end{pmatrix} \begin{pmatrix}
    1 & 0 \\
    0 & p
  \end{pmatrix} & \text{if } p \nmid c.
\end{cases}
\]

Take \( g \in \mathcal{M}_2(\Gamma_0(N), \mathbb{C}) \). If \( \chi \) is of conductor \( f_\chi \) and \( p \) is a prime not dividing \( f_\chi \), then

\[
a_0 \left( [p]_\chi^+ (g); \left[ \frac{a}{c} \right] \right) = \begin{cases}
  a_0(g; [\frac{a}{c}]) - p \cdot \chi(p) \cdot a_0(g; [\frac{ap}{c}]) & \text{if } p \mid c, \\
  a_0(g; [\frac{a}{c}]) - p^{-1} \cdot \chi(p) \cdot a_0(g; [\frac{ap}{c}]) & \text{if } p \nmid c,
\end{cases}
\]

and

\[
a_0 \left( [p]_\chi^- (g); \left[ \frac{a}{c} \right] \right) = \begin{cases}
  a_0(g; [\frac{a}{c}]) - \chi(p)^{-1} \cdot a_0(g; [\frac{ap}{c}]) & \text{if } p \mid c, \\
  a_0(g; [\frac{a}{c}]) - p^{-2} \cdot \chi(p)^{-1} \cdot a_0(g; [\frac{ap}{c}]) & \text{if } p \nmid c.
\end{cases}
\]
Thus, for any positive square-free integer $K$ prime to $f_\chi$, we find by induction that

$$a_0 \left( [K]^+_\chi (g); \left[ \frac{a}{c} \right] \right) = \begin{cases} \sum_{1 \leq \alpha \mid K} (-1)^{\nu(a)} \cdot \alpha \cdot \chi(\alpha) \cdot a_0 (g; \left[ \frac{a \alpha}{c} \right]) & \text{if } K \mid c, \\ \sum_{1 \leq \alpha \mid K} (-1)^{\nu(a)} \cdot \alpha^{-1} \cdot \chi(\alpha) \cdot a_0 (g; \left[ \frac{a \alpha}{c} \right]) & \text{if } (K, c) = 1, \end{cases}$$

and

$$a_0 \left( [K]^-_\chi (g); \left[ \frac{a}{c} \right] \right) = \begin{cases} \sum_{1 \leq \alpha \mid K} (-1)^{\nu(a)} \cdot \chi(\alpha)^{-1} \cdot a_0 (g; \left[ \frac{a \alpha}{c} \right]) & \text{if } K \mid c, \\ \sum_{1 \leq \alpha \mid K} (-1)^{\nu(a)} \cdot \alpha^{-2} \cdot \chi(\alpha) \cdot a_0 (g; \left[ \frac{a \alpha}{c} \right]) & \text{if } (K, c) = 1. \end{cases}$$

Now we come back to our situation. For any cusp $\left[ \frac{s^2tx}{f_\chi} \right] \in X_0 (f_\chi^2)$, choose some

$$\gamma = \begin{pmatrix} x \\ 0 \\ f_\chi^2 / s^2 t \\ u \\ 0 \\ v \end{pmatrix}$$

in $\text{SL}_2(\mathbb{Z})$ such that $\gamma[\infty] = [\frac{s^2tx}{f_\chi}]$. Then it follows from (2.1) and (2.4) that

$$a_0 \left( E_\chi ; \left[ \frac{s^2tx}{f_\chi} \right] \right) = a_0 (E_\chi | \gamma; [\infty])$$

$$= - \frac{1}{4g(\chi)} \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(a) \cdot \chi(b) \cdot B_2 \left( \frac{x a}{f_\chi} + \frac{b}{s^2 t} \right)$$

$$= - \frac{1}{4g(\chi)} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(b) \left( \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \chi(a) \cdot B_2 \left( \frac{x a}{f_\chi} + \frac{b}{s^2 t} \right) \right).$$

It is clear that the function in the above bracket depends only on $b$ modulo $s^2 t$. Since $\chi$ is primitive of conductor $f_\chi$, we find that $a_0 (E_\chi ; [\frac{s^2tx}{f_\chi}]) = 0$ unless $st = f_\chi$. Moreover, if $st = f_\chi$, then

$$a_0 \left( E_\chi ; \left[ \frac{s^2tx}{f_\chi} \right] \right)$$

$$= - \frac{1}{4g(\chi)} \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \sum_{b \in (\mathbb{Z}/f_\chi^2 \mathbb{Z})^\times} \chi(a) \cdot \chi(b) \cdot B_2 \left( \frac{x a}{f_\chi} + \frac{b}{s^2 t} \right)$$

$$= - \frac{\chi(x)^{-1}}{4g(\chi)} \sum_{a \in (\mathbb{Z}/f_\chi \mathbb{Z})^\times} \chi(a) \left( \sum_{b_0, k \in (\mathbb{Z}/f_\chi \mathbb{Z})} \chi(b_0) \cdot B_2 \left( \frac{as + b_0 + k f_\chi}{s f_\chi} \right) \right),$$
where the function in the bracket depends only on \( a \) modulo \( f_\chi/s \) and hence equals zero unless \( s = 1 \). It follows that

\[
(4.5) \quad a_0\left(E_\chi; \left[ \frac{s^2tx}{f_\chi^2} \right] \right) = \begin{cases} 
\chi(x)^{-1} \cdot n_\chi & \text{if } s = 1 \text{ and } t = f_\chi, \\
0 & \text{otherwise},
\end{cases}
\]

where

\[
n_\chi := - \frac{f_\chi}{4g(\chi)} \sum_{a,b \in \mathbb{Z}/f_\chi \mathbb{Z}} \chi(a) \cdot \chi(b) \cdot B_2\left(\frac{a+b}{f_\chi}\right).
\]

In particular, for any integer \( \alpha \) prime to \( f_\chi \), we have

\[
(4.6) \quad a_0\left(E_\chi; \left[ \frac{s^2t(\alpha x)}{f_\chi^2} \right] \right) = \chi(\alpha)^{-1} \cdot a_0\left(E_\chi; \left[ \frac{s^2tx}{f_\chi^2} \right] \right).
\]

**Lemma 4.2:** If \( \chi \) is a quadratic character modulo \( C \), then

\[
a_0\left(E_{D,f_\chi,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right) = \begin{cases} 
n_\chi \cdot \varphi\left(\frac{D}{f_\chi}\right) \cdot (-1)^{\nu\left(\frac{D}{f_\chi r^2 s}\right)} \chi\left(\frac{DC}{f_\chi rs^2tx}\right) \cdot (rs)^{-1} & \text{if } (s, f_\chi) = 1 \text{ and } f_\chi | t, \\
0 & \text{otherwise}
\end{cases}
\]

for any cusp \( \left[ \frac{rs^2tx}{DC} \right] \) in \( X_0(DC) \), where \( f_\chi \) is the conductor of \( \chi \). In particular, for any integer \( \alpha \) prime to \( D \), we have

\[
a_0\left(E_{D,f_\chi,\chi}; \left[ \frac{rs^2t(\alpha x)}{DC} \right] \right) = \chi(\alpha) \cdot a_0\left(E_{D,f_\chi,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right).
\]

**Proof.** For any cusp \( \left[ \frac{rs^2tx}{DC} \right] \), let

\[
\begin{align*}
K_r &:= (D/f_\chi, r) = r, \\
K_s &:= (D/f_\chi, s), \\
K_t &:= (D/f_\chi, t),
\end{align*}
\]
so that we have $D/f_x = K_rK_sK_tK$. Then it follows from (4.1), (4.2) and (4.3) that

$$a_0\left(E_{D,f_x,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right) = \sum_{1 \leq \alpha | K} (-1)^{\nu(\alpha)} \cdot \chi(\alpha) \cdot \alpha \cdot a_0\left(E_{D,f_x,\chi}; \left[ \frac{rs^2t(K(K,C)}{\alpha}x \right] \right)$$

$$= \sum_{1 \leq \alpha | K, 1 \leq \alpha_t | K_t} (-1)^{\nu(\alpha\alpha_t)} \cdot \chi(\alpha\alpha_t) \cdot \alpha\alpha_t$$

$$\times a_0\left(E_{\frac{D}{\alpha\alpha_t}f_x,\chi}; \left[ \frac{rs^2t(K(K,C)}{\alpha\alpha_t}x \right] \right),$$

and we find by (4.3) and Lemma 4.1(1),(2) that

$$a_0\left(E_{D,f_x,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right) = \sum (-1)^{\nu(\alpha_r,\alpha_s,\alpha_t)} \cdot \chi(\alpha_r\alpha_s\alpha_t) \cdot \frac{\alpha_t^{\alpha}}{\alpha_r^{\alpha_s}}$$

$$\times a_0\left(E_{\chi}; \left[ \frac{(\frac{s}{K_x})^2(\frac{t}{K_t})f_x^2}{K_x} \right] \right)$$

$$= \chi(K_tK(K,C)) \cdot \sum (-1)^{\nu(\alpha_r,\alpha_s,\alpha_t)} \cdot \frac{\alpha_t^{\alpha}}{\alpha_r^{\alpha_s}}$$

$$\times a_0\left(E_{\chi}; \left[ \frac{(\frac{s}{K_x})^2(\frac{t}{K_t})f_x^2}{K_x} \right] \right),$$

where $\alpha_r, \alpha_s, \alpha_t$ and $\alpha$ run through all positive divisors of $K_r, K_s, K_t$ and $K$ respectively. Thus, by (4.5) and (4.6), we have

$$a_0\left(E_{D,f_x,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right)$$

$$= \chi(K_tK(K,C)) \cdot \prod_{p | K_r K_s} (1-p^{-1}) \cdot \prod_{p | K_t K} (1-p) \cdot a_0\left(E_{\chi}; \left[ \frac{(\frac{s}{K_x})^2(\frac{t}{K_t})f_x^2}{K_x} \right] \right),$$

which is zero unless $s = K_s$ and $f_x K_t = t$, or equivalently, $(s,f_x) = 1$ and $f_x | t$. This proves the lemma because $K_r K_s = rs$, $K_t K = \frac{D}{rsf_x}$ and $(K,C) = \frac{C}{st}$ when these conditions are satisfied. 

**Lemma 4.3:** With notation as in the above lemma, then we have

$$a_0\left(E_{M,f_x,\chi}; \left[ \frac{rs^2tx}{DC} \right] \right)$$

$$= \begin{cases} n_x \varphi\left( \frac{D}{f_x} \right) \psi\left( \frac{D}{f_x} \right) \frac{M}{rsD} C \left[ \frac{rs^2tx}{DC} \right] & \text{if } (s, f_x) = 1, D = \frac{M}{st} | rs \text{ and } f_x | t, \\ 0 & \text{otherwise} \end{cases}$$
for any cusp \([rs^2tx/DC]\) in \(X_0(DC)\), where
\[
c_x \left[ \frac{rs^2tx}{DC} \right] := (-1)^{\nu(D/r)} \chi \left( \frac{DC}{r} \right).
\]
In particular, for any integer \(\alpha\) prime to \(D\), we have
\[
a_0 \left( E_{M, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC} \right] \right) = \chi(\alpha) \cdot a_0 \left( E_{M, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC} \right] \right).
\]
Proof. For any cusp \([rs^2tx/DC]\), let
\[
\begin{align*}
H_r &:= (D/M, r), \\
H_s &:= (D/M, s), \\
H_t &:= (D/M, t),
\end{align*}
\]
so that we have \(D/M = H_r H_s H_t H\). Then it follows from (4.1), (4.2) and (4.4) that
\[
a_0 \left( E_{M, f_x, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC} \right] \right) = \sum_{1 \leq \alpha \mid H} (-1)^{\nu(\alpha)} \cdot \chi(\alpha) \cdot a_0 \left( E_{M, f_x, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC/H(H, C)} \right] \right)
= \sum_{1 \leq \alpha \mid H, 1 \leq \alpha \mid H_t} (-1)^{\nu(\alpha, \alpha)} \cdot \chi(\alpha) \cdot a_0 \left( E_{M, f_x, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC/H^2(H, C)} \right] \right).
\]
So we find by (4.4), Lemma 4.1(1),(2) and Lemma 4.2 that
\[
a_0 \left( E_{M, f_x, \frac{D}{M}, \chi}, \left[ \frac{rs^2tx}{DC} \right] \right) = \sum (-1)^{\nu(\alpha, \alpha, \alpha, \alpha)} \cdot \chi(\alpha) \cdot \left( \alpha \cdot \alpha \right)^{-2}
\times a_0 \left( E_{M, f_x, \chi}, \left[ \frac{( \frac{r}{t} ) ( \frac{s}{t} )^2 ( \frac{r}{t} ) ( \frac{H(H, C)}{\alpha \omega} ) ( \alpha x )}{M \cdot (M, C)} \right] \right)
= \chi(HH_t(H, C)) \sum (-1)^{\nu(\alpha, \alpha, \alpha, \alpha)} \cdot \left( \alpha \cdot \alpha \right)^{-2}
\times a_0 \left( E_{M, f_x, \chi}, \left[ \frac{( \frac{r}{t} ) ( \frac{s}{t} )^2 ( \frac{r}{t} ) x}{M \cdot (M, C)} \right] \right),
\]
where \(\alpha, \alpha, \alpha, \alpha\) and \(\alpha\) run through all positive divisors of \(H_t, H_s, H_t\) and \(H\) respectively. The above sum is zero unless \(H_t = H = 1, (s, f_x) = 1\) and \(f_x \mid t\), or equivalently, \(\frac{D}{M} \mid rs, (s, f_x) = 1\) and \(f_x \mid t\). When these conditions are satisfied, then the assertion follows from the previous lemma.
PROPOSITION 4.4: For any \((M, L, \chi)\) in \(\mathcal{H}(DC)\) with \(\chi\) a quadratic character, we have

\[
a_0\left( E_{M, L, \chi}; \left[ \frac{rs^2tx}{DC} \right] \right)
= \begin{cases} 
  n_\chi \phi\left( \frac{D}{f_\chi} \right) \psi\left( \frac{L}{f_\chi} \right) \frac{\phi(s, M, L)}{rsL} c_{\chi} \cdot \left[ \frac{rs^2tx}{DC} \right] & \text{if } (s, f_\chi) = 1, (M, L) \mid st \text{ and } \frac{D}{M} \mid rs, \\
  0 & \text{otherwise}
\end{cases}
\]

for any cusp \([\frac{rs^2tx}{DC}]\) in \(X_0(DC)\), where \(f_\chi\) is the conductor of \(\chi\).

Proof. It remains to consider the case when \((M, L) \neq f_\chi\). For any cusp \([\frac{rs^2tx}{DC}]\), let

\[
\begin{align*}
W_s & := \left( \frac{ML}{f_\chi}, s \right), \\
W_t & := \left( \frac{ML}{f_\chi}, t \right),
\end{align*}
\]

so that \((M, L)/f_\chi = W_s W_t W\). Then it follows from (4.4) that

\[
a_0\left( E_{M, L, \chi}; \left[ \frac{rs^2tx}{DC} \right] \right)
= \sum (-1)^{\nu(\alpha)} \cdot \chi(\alpha) \cdot a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2t\alpha x}{DC} \right] \right)
= \sum (-1)^{\nu(\alpha t)} \cdot \chi(\alpha t) \cdot a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2t\alpha t x}{DC} \right] \right)
= \sum (-1)^{\nu(\alpha t \alpha_s)} \cdot \chi(\alpha t \alpha_s) \cdot \alpha_s^{-2} \cdot a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2t\alpha t \alpha_s x}{DC} \right] \right),
\]

where \(\alpha_s, \alpha_t\) and \(\alpha\) run over all positive divisors of \(W_s, W_t\) and \(W\) respectively. Since

\[
\left[ \frac{rs^2t\alpha t \alpha_s x}{DC} \right] = \left[ \frac{r(s\alpha t)^2(\frac{\alpha}{\alpha t})^2(\alpha_s x + \frac{DC}{\alpha^2})}{DC} \right]
\]

with \((\alpha_s x + \frac{DC}{\alpha^2}, D) = 1\) and \(\alpha_s x + \frac{DC}{\alpha^2} \equiv \alpha_s x \pmod{f_\chi}\), we find by Lemma 4.3 that

\[
a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2t\alpha t \alpha_s x}{DC} \right] \right) = (-1)^{\nu(\alpha t)} \cdot \chi(\alpha t \alpha_s) \cdot \alpha_t^{-1} \cdot a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2tx}{DC} \right] \right),
\]

and hence

\[
a_0\left( E_{M, L, \chi}; \left[ \frac{rs^2tx}{DC} \right] \right) = \sum (-1)^{\nu(\alpha \alpha_s)} \cdot \alpha_t^{-1} \cdot \alpha_s^{-2} \cdot a_0\left( E_{M, f_\chi, \frac{D}{\alpha^2}}; \left[ \frac{rs^2tx}{DC} \right] \right).
\]

The above sum is zero unless \(\frac{D}{M} \mid rs, (s, f_\chi) = 1, f_\chi \mid t\) and \(W = 1\), or equivalently, \(\frac{D}{M} \mid rs, (s, f_\chi) = 1\) and \((M, L) \mid st\). If these conditions are satisfied, then we derive the desired result from the previous lemma. \(\blacksquare\)
COROLLARY 4.5: If \((M, L, \chi) \in H(DC)\) with \(\chi\) a quadratic character of conductor \(f_X\), then
\[
R_{DC}(E_{M, L, \chi}) = n_{\chi} \frac{\varphi(D/f_X) \psi(L/f_X)(D/M, C)}{L/f_X} \mathbb{Z}.
\]

Proof. This follows immediately from the above result about constant terms, since the ramification index of \(X_0(DC)\) at the cusp \(\frac{r s^2 \mathcal{t}_X}{DC}\) is \(r s^2\). 

4.3. Now we turn to the calculation of the periods of the Eisenstein series \(E_{M, L, \chi}\) with \(\chi\) being a quadratic character.

Lemmas 4.6: For any quadratic character \(\chi\) with conductor \(f_X\) dividing \(C\), the Fourier expansion of \(E_{D, f_X, \chi}\) at \([\infty]\) is
\[
E_{D, f_X, \chi} = a_0(E_{D, f_X, \chi}; [\infty]) + \sum_{n=1}^{\infty} \sigma_{D/f_X}(n) \cdot \chi(n) \cdot q^n,
\]
where, for any positive integer \(n\), we have
\[
\sigma_{D/f_X}(n) := \sum_{1 \leq d | n, (d, D/f_X) = 1} d.
\]

Proof. We prove the statement by induction on \(\nu(D/f_X)\). If \(\nu(D/f_X) = 1\) and hence \(D = f_X\), then the assertion follows from (3.1) and (3.3) because \(\chi^2 = 1\). Now suppose \(\nu(D/f_X) > 1\). If \(p\) is a prime divisor of \(D/f_X\), then we find by induction that
\[
E_{D, f_X, \chi} = [p]_{X}^+(E_{D/p, f_X, \chi})
\]
\[
= \left( a_0(E_{D/p, f_X, \chi}) + \sum_{n=1}^{\infty} \sigma_{D/f_X}(n) \cdot \chi(n) \cdot q^n \right)
\]
\[
- p \cdot \chi(p) \cdot \left( a_0(E_{D/p, f_X, \chi}) + \sum_{n=1}^{\infty} \sigma_{D/f_X}(n) \cdot \chi(n) \cdot q^{pn} \right)
\]
\[
= a_0(E_{D, f_X, \chi}) + \sum_{n=1}^{\infty} (\sigma_{D/p f_X}(n) - p \cdot \sigma_{D/p f_X}(n/p)) \cdot \chi(n) \cdot q^n,
\]
where \(n/p\) is defined to be 0 when \(p \nmid n\). So the result follows because
\[
\sigma_{D/p f_X}(n) - p \cdot \sigma_{D/p f_X}(n/p) = \sigma_{D/f_X}(n)
\]
for any positive integer \(n\).
Lemma 4.7: For any \((M, L, \chi) \in \mathcal{H}(DC)\) with \(\chi\) a quadratic character of conductor \(f_X\), we have

\[
E_{M,L,\chi} = a_0(E_{M,L,\chi}) + \sum_{n=1}^{\infty} \sigma_{M,L}(n) \cdot \chi(n) \cdot q^n,
\]

where, for any positive integer \(n\), we have

\[
\sigma_{M,L}(n) := \begin{cases} 
(P_{\ell|D/M} \ell^{v_{\ell}(n)} \cdot \sigma_{D/f_X}(n) & \text{if } (n, (M, L)/f_X) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. First consider the case when \((M, L) = f_X\) and hence \(E_{M,L,\chi} = E_{M,f_X,\chi}\). We proceed by induction on \(\nu(D/M)\). When \(D/M = 1\) the result follows from Lemma 4.6. Now suppose \(D/M > 1\). If \(p\) is a prime divisor of \(D/M\), then we find by induction that

\[
E_{M,f_X,\chi} = [p]^{-1}_X(E_{M,f_X,\chi/p,\chi}) = a_0(E_{M,f_X,\chi/p,\chi};[\infty]) + \sum_{n=1}^{\infty} (\sigma_{M,f_X,\chi/p}(n) - \sigma_{M,f_X,\chi/p}(n/p)) \cdot \chi(n) \cdot q^n.
\]

Write \(n = m \cdot p^{v_p(n)}\) with \((m,p) = 1\), then

\[
\sigma_{M,f_X,\chi/p}(n) - \sigma_{M,f_X,\chi/p}(n/p) = (p^{v_p(n)} + \cdots + 1) \cdot \sigma_{M,f_X,\chi/p}(m) - (p^{v_p(n)-1} + \cdots + 1) \cdot \sigma_{M,f_X,\chi/p}(m) = p^{v_p(n)} \cdot \sigma_{M,f_X,\chi/p}(n),
\]

which proves the desired result.

Finally, we complete the proof by induction on \(\nu(\frac{(M,L)}{f_X})\). If \(p\) is a prime divisor of \((M, L)/f_X\), then we find by induction that

\[
E_{M,L,\chi} = [p]^{-1}_X(E_{M,L/p,\chi}) = a_0(E_{M,L/p,\chi};[\infty]) + \sum_{n=1}^{\infty} (\sigma_{M,L/p}(n) - \sigma_{M,L/p}(n/p)) \cdot \chi(p) \cdot q^n,
\]

which proves the lemma because \(\sigma_{M,L/p}(n) - \sigma_{M,L/p}(n/p) = 0\) if \(p \mid n\).

Proposition 4.8: For any \((M, L, \chi) \in \mathcal{H}(DC)\) with \(\chi^2 = 1\), we have

\[
\mathcal{P}_{\Gamma_1(DC)}(E_{M,L,\chi}) = \frac{g(\chi)}{L} \mathbb{Z} + \mathcal{R}_{\Gamma_1(DC)}(E_{M,L,\chi}).
\]
Proof. Denote $f_\chi$ to be the conductor of $\chi$. For any character $\eta$ of conductor $f_\eta$, prime to $D$, it follows from Lemma 4.7 that

$$L(E_{M,L,\chi}, \eta, s) = \prod_{p | M/f_\chi} (1 - \chi_\eta(p) \cdot p^{1-s}) \cdot \prod_{p | L/f_\chi} (1 - \chi_\eta(p) \cdot p^{-s}) \cdot L(\chi_\eta, s-1) \cdot L(\chi_\eta, s).$$

In particular, if $\chi_\eta(-1) = 1$, then $\Lambda(E_{M,L,\chi}, \eta, 1) = 0$; on the other hand, if $\chi_\eta(-1) = -1$, then

$$\Lambda(E_{M,L,\chi}, \eta, 1) = -\frac{\eta(-f_\chi)\chi(f_\eta)g(\chi)}{2f_\chi} \cdot \prod_{p | M/f_\chi} (1 - \chi_\eta(p)) \cdot \prod_{p | L/f_\chi} (1 - \frac{\chi_\eta(p)}{p}) \cdot B_{1, \chi_\eta} \cdot B_{1, \chi^\eta}.$$

Therefore $\frac{g(\chi)}{\Gamma_1(D_C)(E_{M,L,\chi})}$ satisfies the condition (St3) by Theorem 4.2(b) of [14], so that $\mathcal{P}_{\Gamma_1(D_C)}(E_{M,L})$ is contained in $\frac{g(\chi)}{\Gamma_1(D_C)(E_{M,L,\chi})}$. To complete the proof, it is thus sufficient to show that $\mathcal{P}_{\Gamma_1(D_C)}(E_{M,L,\chi})$ contains $\frac{g(\chi)}{\Gamma_1(D_C)(E_{M,L,\chi})}$. Let $q$ be an arbitrary prime. Let $p' \in S_{DC}$ be a prime different from $q$. Then, for all but finitely many $\eta \in \mathfrak{X}_{DC}$ whose conductor is a power of $p'$, both

$$\prod_{p | M/f_\chi} (\chi(p) - \eta(p)) \quad \text{and} \quad \prod_{p | L/f_\chi} (\chi(p) \cdot p - \eta(p))$$

are $q$-adic units. So it follows from [14, Theorem 4.2(c)] that $\frac{L}{\Gamma_1(D_C)(E_{M,L,\chi})}$ is a $q$-adic unit for infinitely many $\eta \in \mathfrak{X}_{DC}$ and hence completes the proof. 

Theorem 4.9: For any $(M, L, \chi) \in \mathcal{H}(DC)$ with $\chi$ a quadratic character of conductor $f_\chi$, we have

$$\mathcal{C}^{(DC)}_{M,L,\chi} \otimes \mathbb{Z}[1/6] \simeq \frac{\frac{g(\chi)}{f_\chi^{\bullet}} \mathbb{Z} + \varphi\left(\frac{f_\chi^{\bullet}}{f_\chi}\right)\psi\left(\frac{f_\chi}{f_\chi}\right)\left(\frac{D}{M}, C\right)\mathbb{Z}}{\varphi\left(\frac{f_\chi^{\bullet}}{f_\chi}\right)\psi\left(\frac{f_\chi}{f_\chi}\right)\left(\frac{D}{M}, C\right)\mathbb{Z}} \otimes \mathbb{Z}[1/6].$$

Proof. Since $\mathcal{R}_{DC}(E_{M,L,\chi})$ contains $\mathcal{R}_{\Gamma_1(DC)}(E_{M,L,\chi})$, we find by Corollary 4.5 and Proposition 4.8 that

$$A^{(s)}(E_{M,L,\chi}) = \frac{\mathcal{P}_{\Gamma_1(DC)}(E_{M,L,\chi}) + \mathcal{R}_{DC}(E_{M,L,\chi})}{\mathcal{R}_{DC}(E_{M,L,\chi})} \simeq \frac{\frac{g(\chi)}{f_\chi^{\bullet}} \mathbb{Z} + \varphi\left(D/f_\chi\right)\psi\left(L/f_\chi\right)\left(\frac{D}{M}, C\right)\mathbb{Z}}{\varphi\left(D/f_\chi\right)\psi\left(L/f_\chi\right)\left(\frac{D}{M}, C\right)\mathbb{Z}}.$$
Thus, to prove the theorem, it suffices to show that $C^{(DC)}_{M,L,\chi} \cap \sum_{DC}$ is annihilated by 6.

- If $\chi = 1$, then $C^{(DC)}_{M,L,\chi} \cap \sum_{DC}$ is $\mathbb{Q}$-rational and of multiplicative type. So it must be contained in $\mu_2$ and hence annihilated by 2.

- If $\chi$ is nontrivial so that $f_\chi > 1$ is odd, then by Proposition 3.7, $T_p$ annihilates $C^{(DC)}_{M,L,\chi}$ for each prime divisor $p$ of $f_\chi$. On the other hand, by [6], any such $T_p$ acts on $\sum_{DC}$ as multiplication by $p$. So we find that $C^{(DC)}_{M,L,\chi} \cap \sum_{DC} \subseteq \mu_{f_\chi}$. However, since $\chi$ can be cyclotomic only if $f_\chi = 3$, we find that $\sum_{DC} \cap C^{(DC)}_{M,L,\chi}$ is zero away from 3 and hence completes the proof. ■

Remark 4.10: For any character $\chi$ of conductor $f_\chi$, let

$$d_\chi := \sum_{a,b \in \mathbb{Z}/f_\chi \mathbb{Z}} \chi(a)\chi(b)B_2\left(\frac{a + b}{f_\chi}\right),$$

which is clearly $6f_\chi$-integral. Since $g(\chi)^2 = \pm f_\chi$, it follows that

$$\frac{g(\chi)}{f_\chi n_\chi} = -4g(\chi)^2 \cdot \frac{1}{f_\chi^2} \cdot \frac{1}{d_\chi} = \pm \frac{4}{f_\chi d_\chi},$$

and we find from Theorem 4.9 that

(4.7) $C^{(DC)}_{M,L,\chi} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{6f_\chi}] \simeq \mathbb{Z}[\frac{1}{6f_\chi}] / \varphi(\frac{P_\chi}{\psi(\frac{P}{\chi}(\frac{P}{\chi}, C))d_\chi} \mathbb{Z}[\frac{1}{6f_\chi}].$

5. Proof of Theorem 1.1

5.1. We first recall some notations from the algebraic theory of modular forms. For more details we refer the reader to §1 of [10]. For any positive integer $N$, let $M^B_2(\Gamma_0(N), \mathbb{Z})$ be the sub-$\mathbb{Z}$-module of $M_2(\Gamma_0(N), \mathbb{C})$ consisted of those forms whose Fourier expansions at infinity have coefficients in $\mathbb{Z}$. Put

$$S^B_2(\Gamma_0(N), \mathbb{Z}) = S^B_2(\Gamma_0(N), \mathbb{C}) \cap M^B_2(\Gamma_0(N), \mathbb{Z}).$$

Then, for an arbitrary commutative ring $R$, we define

$$\begin{cases} M^B_2(\Gamma_0(N), R) := M^B_2(\Gamma_0(N), \mathbb{Z}) \otimes_{\mathbb{Z}} R, \\ S^B_2(\Gamma_0(N), R) := S^B_2(\Gamma_0(N), \mathbb{Z}) \otimes_{\mathbb{Z}} R. \end{cases}$$

When $N$ is invertible in $R$, then, by a moduli theoretic method, we can define $R$-modules

$$S^B_2(\Gamma_0(N), R) \subseteq M^B_2(\Gamma_0(N), R).$$
Furthermore, if $R$ is flat over $\mathbb{Z}[1/N]$, then by (1.3.4) of [10] we have
\[ \begin{cases} M_2^B(\Gamma_0(N), R) = M_2^A(\Gamma_0(N), R), \\ S_2^B(\Gamma_0(N), R) = S_2^A(\Gamma_0(N), R). \end{cases} \]

Note that, for any $g \in C^\infty(\mathcal{H}, \mathbb{C})$ and $z \in \mathcal{H}$, we have
\[ \begin{cases} [p]_\chi^+(g)(z) = g(z) - p \cdot \chi(p) \cdot g(pz), \\ [p]_\chi^-(g)(z) = g(z) - \chi(p)^{-1} \cdot g(pz), \end{cases} \]
where $\chi$ is a Dirichlet character and $p$ a prime not dividing the conductor of $\chi$. So it follows from (3.1) and (3.2) that $E_{M,L,\chi}$ belongs to $M_2^B(\Gamma_0(\mathcal{D}C), \mathbb{Z}[1/6])$.

5.2. For any $(M, L, \chi) \in \mathcal{H}(\mathcal{D}C)$, denote by $I^{(\mathcal{D}C)}_{M,L,\chi}$ the Eisenstein ideal $I^{(\mathcal{D}C)}(E_{M,L,\chi})$ as in Definition 2.1. When $\chi$ is a quadratic character, we call $I^{(\mathcal{D}C)}_{M,L,\chi}$ a quadratic Eisenstein ideal. Then, since $\chi^2 = 1$, all values of $\chi$ are either 1 or $-1$, and it follows from Proposition 3.7 that $I^{(\mathcal{D}C)}_{M,L,\chi}$ is generated by the following elements of $\mathbb{T}(\mathcal{D}C)$:
- $T_\ell - \chi(\ell) \cdot \ell$ for primes $\ell \nmid D$;
- $T_p - \chi(p)$ for primes $p \mid \frac{M}{(M,L)}$;
- $T_p - \chi(p) \cdot p$ for primes $p \mid \frac{L}{(M,L)}$;
- $T_p$ for primes $p \mid (M,L)$.

**Lemma 5.1:** Notations are as above. Then:

1. $\mathbb{T}(\mathcal{D}C)/I^{(\mathcal{D}C)}_{M,L,\chi}$ is a finite cyclic group;
2. there is a surjection $\mathbb{T}(\mathcal{D}C)/I^{(\mathcal{D}C)}_{M,L,\chi} \twoheadrightarrow \mathbb{C}(\mathcal{D}C)$.

**Proof.** Since $T_\ell$ is congruent to an integer modulo $I^{(\mathcal{D}C)}_{M,L,\chi}$ for any prime $\ell$, there is a surjection $\phi : \mathbb{Z} \twoheadrightarrow \mathbb{T}(\mathcal{D}C)/I^{(\mathcal{D}C)}_{M,L,\chi}$. Thus, to prove (1), we only need to show $\ker(\phi)$ is nonzero. Suppose to the contrary that $\ker(\phi) = 0$ so that $\phi$ is...
an isomorphism, then we obtain a normalized eigenform
\[ f = \sum_{n=1}^{\infty} a_n(f)q^n \]
with \(a_\ell(f) = \chi(\ell)(1 + \ell)\) for any prime \(\ell \nmid D\). But this contradicts the Ramanujan bound, so \(\ker(\phi)\) must be nonzero.

By Theorem 3.2.4 of [13], \(C_{M,L,\chi}^{(DC)}\) is a cyclic \(T(DC)\)-module. Since \(I_{M,L,\chi}^{(DC)}\) annihilates \(C_{M,L,\chi}^{(DC)}\) (see Remark 2.2), the action of \(T(DC)\) on \(C_{M,L,\chi}^{(DC)}\) induces the desired surjection.

**Theorem 5.2:** There is an isomorphism
\[ T(DC)/I_{M,L,\chi}^{(DC)} \otimes_{\mathbb{Z}} \mathbb{Z}[1/6D] \simeq C_{M,L,\chi}^{(DC)} \otimes_{\mathbb{Z}} \mathbb{Z}[1/6D]. \]

**Proof.** We need to show, for any prime \(p\) not dividing \(6D\), that the \(p\)-part of the index agrees with that of the order of \(C_{M,L,\chi}^{(DC)}\) given as in (4.7). Fix such a prime \(p\). By Lemma 5.1(1), we have \(T(DC)/I_{M,L,\chi}^{(DC)} \simeq \mathbb{Z}/p^m\mathbb{Z}\) for some integer \(m \geq 0\). By Theorem 2.2 of [11] there is a perfect pairing of \(\mathbb{Z}\)-modules
\[ (T(DC)/I_{M,L,\chi}^{(DC)})_p \times S^B_2(\Gamma_0(DC), \mathbb{Z}/p^m\mathbb{Z})[I_{M,L,\chi}^{(DC)}] \rightarrow \mathbb{Z}/p^m\mathbb{Z}, \]
and hence
\[ S^B_2(\Gamma_0(DC), \mathbb{Z}/p^m\mathbb{Z})[I_{M,L,\chi}^{(DC)}] = (\mathbb{Z}/p^m\mathbb{Z}) \cdot \theta \]
for some normalized eigenform \(\theta\). On the other hand, since \(E_{M,L,\chi} \pmod{p^m}\mathbb{Z}\) belongs to \(M^B_2(\Gamma_0(DC), \mathbb{Z}/p^m\mathbb{Z})\) which is normalized and has the same Hecke eigenvalues as those of \(\theta\), we obtain the following constant form:
\[ A := E_{M,L,\chi} \pmod{p^m}\mathbb{Z} - \theta \]
\[ = a_0(E_{M,L,\chi}; [\infty]) \pmod{p^m}. \]

Below we distinguish two situations:

- **If** \(L \neq 1\) **so that** \(a_0(E_{M,L,\chi}; [\infty]) = 0\) **by Proposition 4.4**, then \(A = 0\) and hence \(E_{M,L,\chi} \pmod{p^m}\mathbb{Z} = \theta\) **by the q-expansion principle** (see Proposition 1.2.10 of [10]). Thus we find that \(E_{M,L,\chi} \pmod{p^m}\mathbb{Z}\) belongs
to \( S_2^B(\Gamma_0(DC), \mathbb{Z}/p^m\mathbb{Z}) \). Since \( p \) is prime to \( D \), it follows from Lemma (1.3.5) of [10] that \( E_{M,L,\chi} \mod p^m\mathbb{Z} \) belongs to \( S_2^A(\Gamma_0(DC), \mathbb{Z}/p^m\mathbb{Z}) \), and hence vanishes at all cusps. By Proposition 4.4, the constant term at any cusp with \( rs = \frac{D}{M}, (s, f_\chi) = 1 \) and \((M, L) | t\) (for example at the cusp \( \frac{(M, L)}{M(M, C)} \)) is of the form

\[
    u \cdot \varphi\left(\frac{D}{f_\chi}\right) \cdot \psi\left(\frac{L}{f_\chi}\right) \cdot d_\chi,
\]

where \( u \) is a \( p \)-adic unit. It follows that \( p^m \) divides \( \varphi(D) \cdot \psi(L) \cdot d_\chi \), which proves the desired assertion in view of (4.7).

- If \( L = 1 \), then \( E_{M,L,\chi} = E_{D,1} \) whose constant term at infinity is \( \pm \frac{1}{24} \varphi(D) \). Let \( q \) be an auxiliary prime with \( q \nmid D \) and \( q \neq \pm 1 \mod p \). Let \( B(q) \) be the operator introduced on p. 289 of [10], which equals \( \frac{1}{q} \gamma_q \) when base change to \( \mathbb{C} \). Then

\[
    0 = (1 - q \cdot B(q))(A)
    = E_{D,q} \mod p^m\mathbb{Z} - (1 - q \cdot B(q))(\theta),
\]

which implies that \( E_{D,q} \mod p^m\mathbb{Z} \) is a cuspform. In particular, \( a_0(E_{D,q};[1]) \) is congruent to zero modulo \( p^m \), which implies by Proposition 4.4 that \( p^m \) divides \( \varphi(Dq) \cdot \psi(q) = \varphi(D) \cdot (q^2 - 1) \). It follows that \( p^m \) divides \( \varphi(D) \) as desired in view of (4.7) and hence completes the proof.

5.3. For any prime divisor \( p \) of \( D \), there are two degeneracy morphisms

\[
    \pi_1^{(p)}, \pi_p^{(p)} : X_0(DC) \to X_0(DC/p),
\]

which are defined as

\[
    \pi_1^{(p)}(E, H) = (E, H[DC/p]) \quad \text{and} \quad \pi_p^{(p)}(E, H) = (E/H[p], H/H[p]).
\]

Thus, from the Picard functoriality, we obtain a homomorphism

\[
    \iota_p = \pi_1^{(p)*} + \pi_p^{(p)*} : J_0(DC/p)^2 \to J_0(DC)
\]

between abelian varieties over \( \mathbb{Q} \). Define

\[
    \begin{cases}
        J_0(DC)_{p\text{-old}} := \text{Im}(\iota_p), \\
        J_0(DC)_{p\text{-new}} := J_0(DC)/J_0(DC)_{p\text{-old}}.
    \end{cases}
\]
It is not difficult to see that both \( J_0(DC)_p\text{-old} \) and \( J_0(DC)_p\text{-new} \) are stable under the action of \( T(DC) \). Define

\[
\begin{aligned}
T(DC)^{p\text{-old}} &:= \text{Im}(T(DC) \to \text{End}_\mathbb{Q}(J_0(DC)_p\text{-old})), \\
T(DC)^{p\text{-new}} &:= \text{Im}(T(DC) \to \text{End}_\mathbb{Q}(J_0(DC)_p\text{-new})).
\end{aligned}
\]

For simplicity we use the same symbols for the Hecke operators in \( T(DC) \) and their images in \( T(DC)^{p\text{-old}} \) and \( T(DC)^{p\text{-new}} \). There are thus two surjective \( \mathbb{Z} \)-algebra homomorphisms,

\[
\begin{aligned}
T(DC) &\twoheadrightarrow T(DC)^{p\text{-old}}, \\
T(DC) &\twoheadrightarrow T(DC)^{p\text{-new}},
\end{aligned}
\]

which combine to give the following homomorphism with nilpotent kernel:

\[
T(DC) \to T(DC)^{p\text{-old}} \times T(DC)^{p\text{-new}}.
\]

Let \( R \) be the sub-\( \mathbb{Z} \)-algebra of \( T(DC/p) \) generated by \( T^{(DC/p)}_\ell \) for all those primes \( \ell \neq p \). Since, for any prime \( \ell \neq p \) and \( i = 1 \) or \( p \), we have

\[
T^{(DC)}_\ell \circ \pi_i^{(p)*} = \pi_i^{(p)*} \circ T^{(DC/p)}_\ell,
\]

the diagonal action of \( R \) on \( J(DC/p)^2 \) induces an inclusion of \( R \) in \( \text{End}_\mathbb{Q}(J_0(DC)_p\text{-old}) \). Now:

- If \( p \) is a prime divisor of \( D/C \) so that \( p \) exactly divides \( DC \), then by the lemma on p. 491 of [15] we have

\[
R \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] = T(DC/p) \otimes_{\mathbb{Z}} \mathbb{Z}[1/2].
\]

Moreover, if \( p \) is odd, then \( R = T(DC/p) \) (loc. cit.) and there is an isomorphism

\[
T(DC/p)[x]/(x^2 - T_p^{(DC/p)}x + p) \simeq T(DC)^{p\text{-old}},
\]

which maps \( T^{(DC/p)}_\ell \) to \( T^{(DC)}_\ell \) for any prime \( \ell \neq p \), and maps \( x \) to \( T_p^{(DC)} \) (see §7 of [12]).

- If \( p \) is a prime divisor of \( C \) so that \( p^2 \) divides \( DC \), then straightforward verification shows that

\[
\begin{aligned}
T_p^{(DC)} &= \pi_p^{(p)*} \circ \pi_1^{(p)}, \\
T_p^{(DC/p)} &= \pi_1^{(p)*} \circ \pi_p^{(p)*},
\end{aligned}
\]
which implies

\[
\begin{aligned}
T_p^{(DC)} & \circ \pi_p^{(p)*} = \pi_p^{(p)*} \circ T_p^{(DC/p)}, \\
T_p^{(DC)} & \circ \pi_1^{(p)*} = p \cdot \pi_p^{(p)*}.
\end{aligned}
\] (5.5)

In particular, the subvariety \(\pi_p^{(p)*}(J_0(\mathcal{D}/p))\) of \(J_0(\mathcal{D})_{p\text{-old}}\) is stable under the action of \(\mathbb{T}(\mathcal{D})_{p\text{-old}}\), and we obtain an induced surjection

\(\varphi_p' : \mathbb{T}(\mathcal{D})_{p\text{-old}} \to \mathbb{T}(\mathcal{D}/p)\),

which maps \(T_\ell^{(DC)}\) to \(T_\ell^{(DC/p)}\) for any prime \(\ell\). Furthermore, let

\[
J_0^{\text{new}}_0(\mathcal{D}) := J_0(\mathcal{D})_{p\text{-old}}/\pi_p^{(p)*}(J_0(\mathcal{D}/p)),
\]

which is also stable under \(\mathbb{T}(\mathcal{D})_{p\text{-old}}\), and define

\[
\mathbb{T}(\mathcal{D})_{(1)}^{\text{new}} := \text{Im}(\mathbb{T}(\mathcal{D})_{p\text{-old}} \to \text{End}_\mathbb{Q}(J_0^{\text{new}}_0(\mathcal{D})_{(1)})).
\]

Then we obtain a surjection

\[
\varphi_p'' : \mathbb{T}(\mathcal{D})_{\text{b-old}} \to \mathbb{T}(\mathcal{D})_{(1)}^{\text{new}},
\]

which maps \(T_p^{(DC)}\) to zero by (5.5). Finally, the two surjections \(\varphi_p'\) and \(\varphi_p''\) combine to give the following homomorphism with nilpotent kernel:

\[
(5.6) \quad \varphi_p : \mathbb{T}(\mathcal{D})_{p\text{-old}} \to \mathbb{T}(\mathcal{D}/p) \times \mathbb{T}(\mathcal{D})_{(1)}^{\text{new}}.
\]

5.4. Recall that \(D\) is an odd square-free positive integer and \(C\) is a positive divisor of \(D\). Fix a quadratic character \(\chi\) with conductor \(f_\chi\) dividing \(C\). Define an ideal in \(\mathbb{T}(\mathcal{D})\) as follows:

\[
I_\chi := (T_\ell^{(DC)} - \chi(\ell) - \chi(\ell) \cdot \ell)_{\ell \nmid D}.
\]

**Lemma 5.3:** \(\mathbb{T}(\mathcal{D})/I_\chi\) is a finite ring.

**Proof.** Since \(\mathbb{T}(\mathcal{D})\) is finite over \(\mathbb{Z}\), \(\mathbb{T}(\mathcal{D})/I_\chi\) is finitely generated over \(\mathbb{Z}\). Thus, to prove the lemma, it suffices to show that \((\mathbb{T}(\mathcal{D})/I_\chi) \otimes_{\mathbb{Z}} \mathbb{C} = 0\).

Suppose to the contrary that \((\mathbb{T}(\mathcal{D})/I_\chi) \otimes_{\mathbb{Z}} \mathbb{C}\) is nonzero. Therefore, by the Hilbert Nullstellensatz, there exists a nonzero \(\mathbb{Z}\)-algebra homomorphism from \(\mathbb{T}(\mathcal{D})/I_\chi\) to \(\mathbb{C}\). Then, due to the duality between the Hecke algebra and the weight-two cusp forms, we obtain a normalized cuspidal eigenform whose \(\ell\)-th Hecke eigenvalue is \(\chi(\ell) + \chi(\ell) \ell\) for any prime \(\ell \nmid D\). But this contradicts the Ramanujan bound, so \((\mathbb{T}(\mathcal{D})/I_\chi) \otimes_{\mathbb{Z}} \mathbb{C}\) must be zero. 

\[\blacksquare\]
In particular, $\mathbb{T}(DC)/I_\chi$ is an artinian ring. Therefore, for any prime $q$, we have

\begin{equation}
\mathbb{T}(DC)_q/I_\chi \simeq \prod_m \mathbb{T}(DC)_m/I_\chi,
\end{equation}

where $m$ runs over all the maximal ideals containing the ideal $(q, I_\chi)$.

**Proposition 5.4:** Let $q$ be a prime with $(q, 2D) = 1$. Then, for any maximal ideal $m$ in $\mathbb{T}(DC)$ containing $(q, I_\chi)$, we have

\begin{equation}
T_{\ell}^{(DC)} \equiv \begin{cases} 
\chi(\ell) + \chi(\ell)\ell & \text{if } \ell \nmid D, \\
\chi(\ell) \text{ or } \chi(\ell)\ell & \text{if } \ell \mid D, \\
0, \chi(\ell) \text{ or } \chi(\ell)\ell & \text{if } \ell \mid C.
\end{cases}
\end{equation}

Moreover, if $p$ is a prime divisor of $f_\chi$, then

\[ T_p^{(DC)} \equiv 0 \pmod{m} \quad \text{and} \quad \mathbb{T}(DC)^{p\text{-old}} = 0. \]

**Proof.** For simplicity we will also denote by $m$ for its images in various old- or new-quotients of $\mathbb{T}(DC)$. For any prime $\ell$ not dividing $D$, the assertion follows directly from the definition of $I_\chi$. Let $\rho_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{T}(DC)/m)$ be the unique semi-simple Galois representation associated to $m$, which is unramified outside $qN$ and the matrix $\rho_m(\text{Frob}_\ell)$ has characteristic polynomial

\[ X^2 - T_{\ell}^{(DC)} X + \ell \]

for any prime $\ell$ not dividing $qN$ (see §5 of [12]). Thus, by the Chebotarev density theorem and the theorem of Brauer–Nesbitt, we have $\rho_m \simeq \chi \oplus \chi_{\epsilon_q}$ with $\epsilon_q$ the modulo $q$ cyclotomic character.

- Let $p$ be a prime divisor of $D/C$. If $\mathbb{T}(DC)^{p\text{-new}}_m$ is nonzero and hence $\mathbb{T}(DC)/m \simeq \mathbb{T}(DC)^{p\text{-new}}/m$, then $\rho_m$ comes from a $p$-new form. Then it follows from Theorem 3.1,(e) of [3] that $\rho_m \simeq \eta \oplus \eta_{\epsilon_q}$ when restricted to $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\eta(\text{Frob}_p) = T_p^{(DC)} \pmod{m}$, which implies that

\[ T_p^{(DC)} \equiv \chi(p) \text{ or } \chi(p)p \pmod{m}. \]

- Still assume that $p$ divides $D/C$. If $\mathbb{T}(DC)^{p\text{-new}}_m = 0$, then

\[ \mathbb{T}(DC)/m \simeq \mathbb{T}(DC)^{p\text{-old}}/m. \]

By (5.3) and (5.4), the oddness of $q$ implies that

\[ \mathbb{T}(DC/p)_q[x]/(x^2 - T_p^{(DC/p)} x + p) \simeq \mathbb{T}(DC)^{p\text{-old}}_q \]
In particular, \( T(DC/p) \) is a subring of \( T(DC)_{q}^{p\text{-old}} \). Let \( n \) be the inverse image of \( m \) in \( T(DC/p) \). Then \( n \) contains the element

\[
T_{\ell}(DC/p) - \chi(\ell) - \ell\chi(\ell)
\]

for any prime \( \ell \nmid D \). In particular, \( \rho_{n} \simeq \rho_{m} \) by the Chebotarev density theorem and the theorem of Brauer–Nesbitt. Now since \( q \neq p \) and \( (p, DC/p) = 1 \), we have \( T(DC/p) \equiv \chi(p) + \chi(p)p \pmod m \) by Proposition 5.1 of [12]. Therefore \( T_{p}(DC) \equiv \chi(p) \) or \( \chi(p)p \pmod m \) as desired.

- Let \( p \) be a prime divisor of \( C \). If \( T(DC)_{m}^{p\text{-new}} \) is nonzero and hence \( T(DC)/m \simeq T(DC)^{p\text{-new}}/m \), then \( T_{p}(DC) \equiv 0 \pmod m \) by the newform theory as \( p^{2} \) divides \( DC \).

- Still assume that \( p \) is a prime divisor of \( C \). If \( T(DC)_{m}^{p\text{-new}} = 0 \), then \( T(DC)/m \simeq T(DC)^{p\text{-old}}/m \), and hence one of \( \varphi'_{p}(m) \) and \( \varphi''_{p}(m) \) must be maximal. In the first case there is an induced isomorphism

\[
T(DC)^{p\text{-old}}/m \simeq T(DC/p)/m,
\]

so we are done by the above, already proved, result. In the second case, we have \( T(DC)^{p\text{-old}}/m \simeq T(DC)_{(1)}^{p\text{-new}}/m \), which implies that

\[
T_{p}(DC) \equiv 0 \pmod m
\]

because \( \varphi''_{p}(T_{p}(DC)) = 0 \).

- Finally, let \( p \) be a prime divisor of \( f_{\chi} \). Suppose \( T(DC)^{p\text{-old}}/m \) is nonzero; then we find as above that one of \( \varphi'_{p}(m) \) and \( \varphi''_{p}(m) \) must be maximal. Since \( \varphi'_{p}(m) \subseteq T(DC/p) \), \( \varphi''_{p}(m) \subseteq T(DC)_{(1)}^{p\text{-new}} \), and \( T(DC)_{(1)}^{p\text{-new}} \) is a quotient ring of \( T(DC/p) \) as we remarked before, we will always obtain some maximal ideal \( n \) in \( T(DC/p) \) which contains \( T_{\ell}(DC/p) - \chi(\ell) - \ell\chi(\ell) \) for all those primes \( \ell \) not dividing \( D \). In particular \( \rho_{n} \simeq \chi \oplus \chi\epsilon_{q} \), which is ramified at \( p \).

However, since \( p \) exactly divides \( (DC/p) \), the restriction of \( \rho_{n} \) to \( \text{Gal}(\overline{Q}_{p}/Q_{p}) \) has an unramified quotient by Theorem 3.1.(e) of [3]. This is a contradiction. So we have \( T(DC)_{m}^{p\text{-old}} = 0 \) and hence

\[
T(DC)/m \simeq T(DC)^{p\text{-new}}/m,
\]

which implies \( T_{p}(DC) \equiv 0 \pmod m \) by the newform theory. \( \blacksquare \)
5.5. Let $q$ be a prime with $(q, 2D) = 1$. Fix a maximal ideal $\mathfrak{m}$ in $\mathbb{T}(DC)$ containing $(q, I_\chi)$. Let

$$\mathcal{P}_1(\mathfrak{m}) = \{ p \text{ a prime} : p \mid D, \quad T_p^{(DC)} \equiv 0, \chi(p) \pmod{\mathfrak{m}} \},$$

$$\mathcal{P}_2(\mathfrak{m}) = \{ p \text{ a prime} : p \mid D, \quad T_p^{(DC)} \equiv 0, \chi(p)p \pmod{\mathfrak{m}}, \text{ and } p \not\equiv 1 \pmod{q} \},$$

and we define

$$\begin{cases} M = \prod_{p \in \mathcal{P}_1(\mathfrak{m})} p, \\ L = \prod_{p \in \mathcal{P}_2(\mathfrak{m})} p. \end{cases}$$

**Lemma 5.5:** With notation as above, then $(M, L, \chi)$ belongs to $\mathcal{H}(DC)$.

**Proof.** It is clear that $\mathfrak{m} = (q, I^{(DC)}_{M, L, \chi})$. To prove $(M, L, \chi) \in \mathcal{H}(DC)$ we only need to show that $M > 1$. Suppose to the contrary that we have $M = 1$. Then $\chi$ is the trivial character and $\mathfrak{m}$ is a so-called rational Eisenstein prime in the sense of [17]. Moreover, we have now $T_p^{(DC)} \equiv p \pmod{q}$ for any prime divisor $p$ of $D$, which contradicts Theorem 1.3 of [17]. So we must have $M > 1$ and hence completes the proof. 

**Remark 5.6:** By the above lemma, to any maximal ideal $\mathfrak{m}$, we have an associated Eisenstein series $E_{M, L, \chi}$ such that $\mathfrak{m} = (q, I^{(DC)}_{M, L, \chi})$.

**Proof of Theorem 1.1.** Let $q$ be a prime such that $(q, 6D \varphi(D) \varpi(\frac{D}{N})) = 1$. By the Eichler–Shimura relation, $J_0(DC)(\chi)[q^\infty]$, as an étale group scheme over $\mathbb{Z}[1/D]$, is annihilated by $I_\chi$. Therefore $J_0(DC)(\chi)[q^\infty]$ is a $\mathbb{T}(DC)_q/I_\chi$-module, and we have by (5.7) the following decompositions:

$$\begin{cases} J_0(DC)(\chi)[q^\infty] = \bigoplus_{\mathfrak{m}} J_0(DC)(\chi)[\mathfrak{m}^\infty], \\ C_0(DC)(\chi)[q^\infty] = \bigoplus_{\mathfrak{m}} C_0(DC)(\chi)[\mathfrak{m}^\infty], \end{cases}$$

where $\mathfrak{m}$ runs over all maximal ideals containing $(q, I_\chi)$. Fix such a maximal ideal $\mathfrak{m}$. Let $(M, L, \chi)$ be the associated triple as in Lemma 5.5. We claim that $J_0(DC)(\chi)[\mathfrak{m}^\infty]$ is annihilated by $I^{(DC)}_{M, L, \chi}$.

Let $p$ be a prime divisor of $D$. Then there is an exact sequence of $\mathbb{T}(DC)$-modules

$$0 \longrightarrow J_0(DC)_{p\text{-old}}(\chi)[\mathfrak{m}^\infty] \longrightarrow J_0(DC)(\chi)[\mathfrak{m}^\infty] \longrightarrow J_0(DC)_{p\text{-new}}(\chi)[\mathfrak{m}^\infty].$$
• If $p$ divides $f_\chi$, then $J_0(DC)_{p,\text{old}}(\chi)[m^\infty] = 0$ by Proposition 5.4. It follows that $J_0(DC)(\chi)[m^\infty]$ is contained in $J_0(DC)_{p,\text{new}}(\chi)[m^\infty]$, and is therefore annihilated by $T^{(DC)}_p$.

• We next prove $T(DC)_{p,\text{new}} = 0$ if $p$ divides $D/f_\chi$. For otherwise there would exist a $p$-newform with an associated $q$-adic representation $\rho$ such that $\bar{\rho} = \rho_m \simeq \chi \oplus \chi e_q$, where $\bar{\rho}$ is the semisimplification of the reduction of $\rho$ modulo $q$. So we are in the so-called degenerate case (see [2] and [7]). However, when $p$ divides $C$, this degeneration is impossible since $(q, p^2 - 1) = 1$ (see Proposition 1.1 of [1]). When $p$ divides $D/C$, then the reduction of $\bar{\rho}$ to $I_p$ is trivial as $(q, p - 1) = 1$, and hence $\text{codim}(\bar{\rho}|_{I_p}) = 0$. Since the Swan conductor does not change under reduction (see Proposition 1.1 of [7]), the degeneration of $\rho$ at $p$ would imply that $\text{codim}(\rho|_{I_p}) = 1$, so we find that

$$\rho|_{I_p} \simeq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

where $b$ is a non-trivial homomorphism from $I_p$ to a pro-$q$ group. But this is impossible again because $(q, p - 1) = 1$ and hence the assertion follows. In particular, for any prime divisor $p$ of $D/f_\chi$, we have $J_0(DC)(\chi)[m^\infty] = J_0(DC)_{p,\text{old}}(\chi)[m^\infty]$.

• Now, if $p$ divides $D/C$, then $p$ is either a divisor of $M/(M, L)$ or a divisor of $L/(M, L)$. Since $J_0(DC)_{p,\text{old}}$ is a quotient of $J_0(DC/p)^2$, we find by the Eichler–Shimura relation again that $T^{(DC/p)}_p$ acts on $J_0(DC)(\chi)[m^\infty]$ as multiplication by $\chi(p) + \chi(p)p$. Then it follows from (5.4) that $(T^{(DC)}_p - \chi(p))(T^{(DC)}_p - \chi(p)p)$ annihilates $J_0(DC)(\chi)[m^\infty]$. Since $p \neq 1 \pmod{q}$, $J_0(DC)(\chi)[m^\infty]$ is either annihilated by $T^{(DC)}_p - \chi(p)$ when $p$ divides $M/(M, L)$, or annihilated by $T^{(DC)}_p - \chi(p)p$ when $p$ divides $L/(M, L)$.

• Finally, if $p$ divides $C/f_\chi$, then $p$ divides $(M, L)/f_\chi$. Recall that there is a ring homomorphism $T(DC)_{p,\text{old}} \to T(DC)_{p,\text{new}}^{(1)} \times T(DC/p)$ with nilpotent kernel. If $T(DC)_{p,\text{old}}/m$ is isomorphic to $T(DC/p)/m$, then we obtain a maximal Eisenstein ideal $m$ in $T(DC/p)$ such that $T^{(DC/p)}_p \equiv 0 \pmod{m}$, which contradicts Proposition 5.4. Therefore $T(DC/p)/m = 0$ and $J_0(DC)_{p,\text{old}}(\chi)[m^\infty]$ is contained in $J_0(DC)_{p,\text{new}}^{(1)}(\chi)[m^\infty]$, which implies that $J_0(DC)(\chi)[m^\infty]$ is annihilated by $T^{(DC)}_p$ as $T^{(DC)}_p$ acts as zero on $J_0(DC)_{p,\text{new}}^{(1)}$. We have thus completed the proof of the claim.
In particular, $J_0(DC)(\chi)[m^\infty]$ is a $\mathbb{T}(DC)_q/I_{M,L,\chi}^{(DC)}$-module. Consider $J_0(DC)[m]$ as a finite flat group scheme over $\mathbb{Z}_q$. Note that $J_0(DC)(\chi)[m]$ is contained in $J_0(DC)[m]^{et}$. By composing the injection $J_0(DC)(\chi) \hookrightarrow J_0(DC)(\overline{\mathbb{F}}_q)$ with $J_0(DC)(\overline{\mathbb{F}}_q) \hookrightarrow H^0(X_0(DC)/\overline{\mathbb{F}}_q, \Omega)$ (see Proposition 14.7 of [9]), we obtain an injection $J_0(DC)(\chi)[m] \hookrightarrow H^0(X_0(DC)/\overline{\mathbb{F}}_q, \Omega)[m]$. Since $H^0(X_0(DC)/\overline{\mathbb{F}}_q, \Omega)[m]$ is isomorphic to $S^B(\Gamma_0(DC), \overline{\mathbb{F}}_q)[m]$, it follows from the $q$-expansion principle that $\dim_{\overline{\mathbb{F}}_q}(J_0(DC)(\chi)[m]) \leq 1$ and so $J_0(DC)(\chi)[m^\infty]$ is a cyclic group. Therefore

$$|J_0(DC)(\chi)[m^\infty]| \leq |\mathbb{T}(DC)_q/I_{M,L,\chi}^{(DC)}|$$

$$= |C^{(DC)}_{M,L,\chi}|,$$

with the second equality holding by Proposition 5.2. Since $C^{(DC)}_{M,L,\chi}$ is clearly contained in $J_0(DC)(\chi)[m^\infty]$, the above inequality implies that

$$J_0(DC)(\chi)[m^\infty] = C^{(DC)}_{M,L,\chi}[q^\infty] \subseteq C_0(DC)(\chi)[m^\infty],$$

which completes the proof. 

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