A Note on T-duality of Strings in Plane-Wave Backgrounds

Shun’ya Mizoguchi†

High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan

Takeshi Mogami‡ and Yuji Satoh§

Institute of Physics, University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan

Abstract

We show, by direct computations of bosonic string spectra, the $O(d,d;\mathbb{Z})$ ($d = 1,2$) T-duality in the maximally supersymmetric IIB plane-wave background compactified on $S^1$ and $T^2$. Only half of the ordinary set of zero modes appear in the Hamiltonian. This “half” Narain lattice is proved to be stabilized by the T-duality group.
1 Introduction

Among string dualities, T-duality of flat toroidal compactifications is best understood. (See [1] for a review.) It typically contains an operation exchanging a compactification radius $R$ with $\alpha'/R$ [2], which is a special element of the full T-duality group $O(d, d; \mathbb{Z})$ [3] in the case of a $d$-torus.\footnote{We ignore the trivial $\mathbb{Z}_2$ factor of $O(d, d; \mathbb{Z})$.}

T-duality is associated with target-space isometries. The type IIB maximally supersymmetric plane-wave background [4, 5] has mutually commuting spacelike Killing vectors [4], and one naturally expects T-duality to hold also in this background. However, although one could still argue the classical equivalence, one needs some gauge fixing for quantization, and strictly speaking, in the presence of nontrivial Ramond-Ramond flux the direct comparison of the spectra has not been done in any covariant formulation. In light-cone gauge, the Green-Schwarz strings are known to be solvable [6], but then the world sheet theory becomes massive and that renders the known CFT proofs of T-duality unavailable. In this note, we show, by direct computations of string spectra in light-cone gauge, the $O(d, d; \mathbb{Z})$ ($d = 1, 2$) T-duality in the IIB maximally supersymmetric plane-wave background compactified on $S^1$ and $T^2$.

For the $S^1$ case, the mode expansions were already given in [7] and in its “IIA dual” in [8], though not in a form that can easily be compared with the IIB spectrum.\footnote{For a discussion of the strings on $S^1$ on the IIB side, see also [9].} We will display them in such a way that we may examine duality of the two theories. A curious feature of the zero modes will be found; on the IIB side there are no momentum modes but appear only winding modes, while on the IIA side both are present. We will show that the winding modes do not contribute to the IIA Hamiltonian, and they are really dual. Also, for the $T^2$ case, we will give a proof that the Hamiltonian is invariant under the full $O(2, 2; \mathbb{Z})$ action.

Since we are particularly interested in the zero modes, we restrict ourselves in this paper to the check of the bosonic spectra. For the fermionic sector, the T-duality transformation rule of Green-Schwarz strings was derived for $S^1$ compactification to quadratic order in $\theta$ [10] by using a generalized Buscher’s method [11]. In plane-wave backgrounds in general, the Green-Schwarz action has been shown [12] to truncate at quadratic order and take the form given in [10, 13]. The invariance of the fermionic spectrum may follow from the bosonic results and supersymmetry.

The notation that we use in this paper is as follows. The maximally supersymmetric plane-
wave background for type IIB strings is

\[ ds^2 = 2dx^+dx^- - \mu^2 x^i (dx^+)^2 + dx^i dx^i, \]
\[ F_5 + ijkl = 4\mu \epsilon_{ijkl}, \quad F_5 = 0 \quad \text{otherwise.} \]  

(1)

If \( i, j, \ldots = 1, \ldots, 8 \) are the indices for the transverse coordinates. \( \epsilon_{ijkl} \) is defined to be nonzero if \( \{i, j, k, l\} \) are the set \( \{1, 2, 3, 4\} \) or \( \{5, 6, 7, 8\} \), and \( \epsilon_{1234} = \epsilon_{5678} = +1 \).

It has 30 isometries \([4]\), and among them \( k_{S^+}^\pm (i \neq j) \) given by

\begin{align*}
  k_{S^+}^\pm &= k_{e_i} \pm k_{e_j}^*, \\
  k_{e_i} &= -\cos \mu x^+ \partial_i - \mu x^i \sin \mu x^+ \partial_-, \\
  k_{e_j}^* &= -\sin \mu x^+ \partial_i + \mu x^i \cos \mu x^+ \partial_-
\end{align*}

(2)

have a constant (unit) norm \([7]\).

## 2 \( S^1 \) compactification revisited

We will first focus on the T-duality transformation along the isometry \( k_{S^+}^8 \). The coordinate system in which this isometry is manifest is \([7]\)

\begin{align*}
  x^+ &= X^+, \\
  x^- &= X^- - \mu X^7 X^8, \\
  x^I &= X^I \quad (I = 1, \ldots, 6), \\
  \begin{bmatrix} x^7 \\ x^8 \end{bmatrix} &= \begin{bmatrix} \cos \mu X^+ & \sin \mu X^+ \\ -\sin \mu X^+ & \cos \mu X^+ \end{bmatrix} \begin{bmatrix} X^7 \\ X^8 \end{bmatrix}.
\end{align*}

(3)

In these coordinates \( k_{S^+}^8 = -\partial/\partial X^8 \), and the metric becomes

\[ ds^2 = 2dX^+dX^- - \mu^2 X^I X^I (dX^+)^2 - 4\mu X^7 dX^8 dX^+ + dX^i dX^i. \]  

(4)

\( F_5 \) is unchanged.

### IIB spectrum

We will first examine the bosonic spectrum of a Green-Schwarz string in this background \([6, 7, 9]\). In light-cone gauge, the string action takes the form

\[ S_{IIB} = \int d\tau L_{IIB}, \]
\[ L_{IIB} = \int^{2\pi}_0 d\sigma \left[ p^+ \partial_\tau X^- + \frac{1}{4\pi \alpha'} \left( (\partial_\sigma X^i)^2 - (\partial_\sigma X^i)^2 - \bar{\mu}^2 (X^I)^2 - 4\bar{\mu} X^7 \partial_\tau X^8 \right) \right]. \]  

(5)
where we have set $X^+ = 2\pi \alpha' p^+ g^{-1}_{\sigma \sigma} \tau$ and $\bar{\mu} = 2\pi \alpha' p^+ \mu$. The Hamiltonian is

$$H_{IIB} = \frac{1}{4\pi \alpha'} \int_0^{2\pi} d\sigma \left[ \left(2\pi \alpha'\right)^2 \left(\Pi_I^2 + \Pi_7^2 + \Pi_8 + \frac{\bar{\mu}}{\pi \alpha'} X^7\right)^2 \right] + \left(\partial_\sigma X^i\right)^2 + \bar{\mu}^2 (X^I)^2 \right],$$

(6)

where

$$\Pi_I = \frac{1}{2\pi \alpha'} \partial_\tau X^I,$$
$$\Pi_7 = \frac{1}{2\pi \alpha'} \partial_\tau X^7,$$
$$\Pi_8 = \frac{1}{2\pi \alpha'} \left(\partial_\tau X^8 - 2\bar{\mu} X^7\right).$$

(7)

The mode expansions are straightforwardly obtained [7]. Equations of motion are

$$\ddot{X}^I - X^{I''} + \bar{\mu}^2 X^I = 0,$$
$$\ddot{X}^7 - X^{7''} + 2\bar{\mu} \dot{X}^8 = 0,$$
$$\ddot{X}^8 - X^{8''} - 2\bar{\mu} \dot{X}^7 = 0.$$

(8)

With the boundary condition

$$X^I(\tau, \sigma = 2\pi) = X^I(\tau, \sigma = 0) \quad (I = 1, \ldots, 6),$$

(9)

$X^I$'s are expanded as

$$X^I(\tau, \sigma) = x_0^I \cos \bar{\mu} \tau + \frac{\alpha'}{\bar{\mu}} p^I \sin \bar{\mu} \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(a_n^I f_n^+ + \bar{a}_n^I f_n^-\right),$$

(10)

where

$$f_n^\pm(\tau, \sigma) \equiv e^{-in(\rho_n \tau \pm \sigma)},$$
$$\rho_n \equiv \sqrt{1 + \frac{\bar{\mu}^2}{n^2}}.$$

The commutation relations are

$$[x_0^I, p^J] = i\delta^{IJ},$$
$$[a_n^I, a_m^J] = [\bar{a}_n^I, \bar{a}_m^J] = \frac{n}{\rho_n} \delta^{IJ} \delta_{n,-m}. $$

(12)

$X^7$ and $X^8$ have similar expansions to $X^I$ since $Y \equiv e^{\bar{\mu} \tau} (X^8 + i X^7)$ satisfies the same equation of motion as $X^I$. Their boundary conditions are

$$X^7(\tau, \sigma = 2\pi) = X^7(\tau, \sigma = 0), \quad X^8(\tau, \sigma = 2\pi) = X^8(\tau, \sigma = 0) + 2\pi R_{IIB} w^8 \quad (13)$$

The other gauge fixing conditions we have adopted are $\det g_{\mu \nu} = -1, \partial_\sigma g_{\sigma \sigma} = 0$ and $g_{\tau \sigma}(\sigma = 0) = 0$. Then $g_{\sigma \sigma}$ is constant everywhere, and is set to be unity.
where \( w^8 \in \mathbb{Z} \) is the winding number. Writing \( X \equiv X^8 + iX^7, \bar{X} \equiv X^8 - iX^7 \), they are

\[
\begin{align*}
X(\tau, \sigma) &= x^8_0 + ix^7_0 + R_{IIB}w^8\sigma + a_0 \sqrt{\frac{\alpha'}{\mu}} e^{-2i\bar{\mu}\tau} + e^{-i\bar{\mu}\tau}\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( a_n f^+_n + \bar{a}_n f^-_n \right),
\end{align*}
\]

\[
\begin{align*}
\bar{X}(\tau, \sigma) &= x^8_0 - ix^7_0 + R_{IIB}w^8\sigma + \bar{a}_0 \sqrt{\frac{\alpha'}{\mu}} e^{2i\bar{\mu}\tau} + e^{i\bar{\mu}\tau}\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \bar{a}_n f^+_n + a_n f^-_n \right)
\end{align*}
\]

(14)

with

\[
[x^7_0, x^8_0] = \frac{\alpha'i}{2\bar{\mu}}
\]

(15)

and

\[
\begin{align*}
[a_0, \bar{a}_0] &= 1, \\
[a_n, \bar{a}_m] &= [\bar{a}_n, a_m] = \frac{2n}{\rho_n} \delta_{n,-m} \quad (n, m \neq 0),
\end{align*}
\]

(16)

where the zero-mode part is different from [7] but is equivalent. Note that, unlike the ordinary toroidal case, the compact boson \( X^8 \) does not have quantized momenta owing to “Coriolis’s force” in the 7-8 plane.\(^4\) Therefore the “Narain lattice” is a one-dimensional lattice here.

**IIA spectrum**

We next consider the IIA background obtained by the T-duality transformation along \( \partial/\partial X^8 \) [8], which can be read off from the reduction to nine dimensions. The metric and non-vanishing components of other fields are

\[
\begin{align*}
\mathrm{ds}_{IIA}^2 &= 2dX^+dX^- - \mu^2 ((X^I)^2 + 4(X^7)^2)(dX^+)^2 + dX^i dX^i, \\
F_{4 + 567} &= 4\mu, \quad H_{+79} = - 2\mu
\end{align*}
\]

(17)

with \( i = 1, \ldots, 7, 9 \) and \( I = 1, \ldots, 6 \), where \( X^9 \) denotes the coordinate of the dual circle. All other components vanish. The bosonic light-cone string action reads

\[
\begin{align*}
S_{IIA} &= \int d\tau L_{IIA}, \\
L_{IIA} &= \int_0^{2\pi} d\sigma \left[ p^+ \partial_\tau X^- + \frac{1}{4\pi\alpha'} ((\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2 \right. \\
&\left. - \bar{\mu}^2 ((X^I)^2 + 4(X^7)^2) - 4\bar{\mu}X^7 \partial_\sigma X^9 \right].
\end{align*}
\]

(18)

The canonical momenta \( \Pi_I \) and \( \Pi_7 \) are the same as IIB, and

\[
\Pi_9 = \frac{1}{2\pi\alpha'} \partial_\tau X^9.
\]

\(^4\)This is a realization of a noncommutative cylinder without (tangent) constant \( B \) field (see (15)).
The Hamiltonian is
\[
H_{IIA} = \frac{1}{4\pi\alpha'} \int_0^{2\pi} d\sigma \left[ (2\pi\alpha')^2 \Pi_i^2 + (\partial_\sigma X^i)^2 + \bar{\mu}^2 ((X^I)^2 + 4(X^7)^2) + 4\bar{\mu} X^7 \partial_\sigma X^9 \right].
\] (20)

Since the \(X^I\) equations of motion are identical to those before taking T-dual, their mode expansions remain the same. The \(X^7\) and \(X^9\) equations are
\[
\ddot{X}^7 - X^7'' + 4\bar{\mu}^2 X^7 + 2\bar{\mu} X^9' = 0,
\]
\[
\ddot{X}^9 - X^9'' - 2\bar{\mu} X^7' = 0
\] (21)
and their boundary conditions are
\[
X^7(\tau, \sigma = 2\pi) = X^7(\tau, \sigma = 0), \quad X^9(\tau, \sigma = 2\pi) = X^9(\tau, \sigma = 0) + 2\pi R_{IIA} w^9.
\] (22)

\(w^9 \in \mathbb{Z}\) is the winding number. If \((X^7, X^9)\) is a solution with \(w^9 = 0\), so is \((X^7 - \frac{R_{IIA}}{2\mu} X^9, X^9 + R_{IIA} w^9 \sigma)\) with winding number \(w^9\). All the winding solutions are obtained in this way. Therefore it is enough to consider solutions with \(w^9 = 0\).

For nonzero modes, we make an ansatz:
\[
X^7(\tau, \sigma) = A^7 e^{i(\omega_n \tau + n\sigma)}, \quad X^9(\tau, \sigma) = A^9 e^{i(\omega_n \tau + n\sigma)}.
\] (23)
\(A^7, A^9\) and \(\omega_n\) are constants \((n \neq 0)\). Plugging them into the equations of motion, one finds that the solutions satisfy either
\[
\omega_n = \bar{\mu} \pm n\rho_n, \quad \frac{A^9}{A^7} = \frac{-\bar{\mu} \mp n\rho_n}{in},
\] (24)
or
\[
\omega_n = -\bar{\mu} \pm n\rho_n, \quad \frac{A^9}{A^7} = \frac{-\bar{\mu} \mp n\rho_n}{in}.
\] (25)

On the other hand, the modes independent of \(\sigma\) satisfy
\[
\ddot{X}^7 + 4\bar{\mu}^2 X^7 = 0, \quad \ddot{X}^9 = 0.
\] (26)
Taking into account the winding modes together, we obtain the expansions
\[
X^7_{IIA} = -\frac{R_{IIA}}{2\mu} w^9 + \frac{1}{2i} \sqrt{\frac{\alpha'}{\bar{\mu}}} (a_0 e^{-2\bar{\mu}i\tau} - \bar{a}_0 e^{2\bar{\mu}i\tau})
\]
\[
+ \frac{1}{2} \sqrt{\frac{\alpha'}{2}} e^{-i\bar{\mu} \tau} \sum_{n \neq 0} \frac{1}{n} \left( a_n f_+ + \bar{a}_n f_+ \right) - \frac{1}{2} \sqrt{\frac{\alpha'}{2}} e^{i\bar{\mu} \tau} \sum_{n \neq 0} \frac{1}{n} \left( a_n f_- + \bar{a}_n f_- \right),
\] (27)
\[
X^9 = R_{IIA} w^9 \sigma + x_0^9 + \alpha' p^9 \tau
\]
\[
+ \frac{1}{2} \sqrt{\frac{\alpha'}{2}} e^{-i\bar{\mu} \tau} \sum_{n \neq 0} \frac{\bar{\mu} - n\rho_n}{in^2} \left( a_n f_+ - \bar{a}_n f_- \right) - \frac{1}{2} \sqrt{\frac{\alpha'}{2}} e^{i\bar{\mu} \tau} \sum_{n \neq 0} \frac{\bar{\mu} + n\rho_n}{in^2} \left( a_n f_+ - \bar{a}_n f_- \right)
\] (28)
with \([x_0^9, p^9] = i\) and the same relations as (16).

**Comparison of the spectra**

We have put the subscript “IIA” on \(X^7\) in order to distinguish from the IIB expansion, though we have used the same oscillators; this is justified since \(X^7_{IIA}\) (27) coincides with \(X^7 = (2i)^{-1}(X - \bar{X})\) (14) up to constant modes, of which the Hamiltonians are independent.\(^5\) Therefore, just like the ordinary toroidal compactification, no distinction is necessary between \(X^7\) and \(X^7_{IIA}\).

On the other hand, unlike \(X^8\), the “dual” boson \(X^9\) has not only a winding mode but also a momentum mode, since the equation of motion for the \(X^9\) zero modes is free and the same as the flat case. One can argue the quantization of this momentum by noting that \(p^9\) is a constant part of the canonical momentum \(2\pi\Pi_9\) (or by requiring the solution to the Klein-Gordon equation \(e^{ip^9X^9}\) to be single-valued), to get

\[
p^9 = \frac{n^9}{R_{IIA}} \quad (n^9 \in \mathbb{Z}).
\]

(29)

Therefore, the Narain lattice on the “dual” background appears to be spanned by \(n^9\) and \(w^9\), and hence does not look like dual to the one-dimensional lattice for \(X^8\). However, an explicit computation shows that the \(w^9\) dependences cancel in the Hamiltonian, and the lattice becomes again one-dimensional. More precisely, one may confirm that the following relations hold:

\[
2\pi\alpha'\Pi_8 \ (IIB) = \partial_\sigma X^9 \ (IIA),
\]

\[
2\pi\alpha'\Pi_9 \ (IIA) = \partial_\sigma X^8 \ (IIB),
\]

(30)

provided that\(^6\)

\[
w^8 = n^9, \quad R_{IIB} = \frac{\alpha'}{R_{IIA}},
\]

(31)

and then

\[
H_{IIB} = H_{IIA}.
\]

(32)

We note that, on the IIA side, there is an infinite degeneracy of the spectrum in the Hilbert space associated with the redundancy of \(w^9\). The corresponding degeneracy on the IIB side

\(^5\)Due to the quantization of \(x_0^7\) discussed below, the constant modes also agree if the parameters are appropriately identified.

\(^6\)Here we have used the scheme in which the metric is kept fixed but the radius \(R\) describes the change of the physical circumference. In the next section we fix the radii but get the metric transformed.
comes from the noncommutative coordinate $x_0^7$, which is the constant piece of $2\pi \Pi_8$ and, hence, quantized as $-\alpha' n^8/(2\tilde{\mu} R_{IIB})$.\footnote{We thank Y. Imamura and T. Yoneya for comments on this point. See also [9].}

To compare the physical spectra, we have to consider the constraints, details of which for plane-wave backgrounds are found in [6]. The physical spectrum then needs to satisfy the “level-matching” condition:

$$P \equiv \int_0^{2\pi} d\sigma \Pi_i \partial_\sigma X^i = 0. \quad (33)$$

The oscillator part of $P$ is computed straightforwardly, while the quantization conditions for the zero modes give $P^\text{zeromodes}_{IIB} = n^8 w^8$, $P^\text{zeromodes}_{IIA} = n^9 w^9$. If $w^8 = n^9$ and $n^8 = w^9$,

$$P_{IIB} = P_{IIA} \quad (34)$$

holds, and the physical states also correspond one to one. This establishes that this T-dual pair is “really dual”. Note that, when $w^8 \neq 0$ ($n^9 \neq 0$), the physical-state condition resolves the infinite degeneracy in the Hilbert space and selects a unique $n^8(w^9)$ for the physical states.

It would be worth noting that the relations (30) directly follow from the usual duality transformation [11] (introducing $X^9$ as a Lagrange multiplier and integrating out $X^8$) in the string action (5), and the classical coincidence of the Hamiltonians follows. Our direct check shows that it is also consistent quantum mechanically, mode by mode, in particular that the Narain lattices agree despite the apparent asymmetry of zero modes.

$T^d$ compactifications ($d = 1, \ldots, 4$)

The coordinate system (3) can be trivially generalized so that up to four commuting Killing vectors can be manifestly seen. For instance, the Killing vectors $k_{s_2}^+, k_{s_4}^+, k_{s_6}^+$ and $k_{s_8}^+$ are manifest in the coordinate system

$$x^+ = X^+, \quad x^- = X^- - \mu(1^{1}X^2 + \cdots + X^7X^8), \quad \begin{bmatrix} x_{2j-1}^2 \\ x_{2j}^2 \end{bmatrix} = \begin{bmatrix} \cos \mu X^+ & \sin \mu X^+ \\ -\sin \mu X^+ & \cos \mu X^+ \end{bmatrix} \begin{bmatrix} X_{2j-1}^2 \\ X_{2j}^2 \end{bmatrix} \quad (j = 1, \ldots, 4). \quad (35)$$

The metric then reads

$$ds^2 = 2dX^+dX^- - 4\mu(1^{1}dX^2 + \cdots + X^7dX^8)dX^+ + dX^idX^i \quad (i = 1, \ldots, 8). \quad (36)$$

With this choice of a set of Killing vectors, $F_5$ is again unchanged.
In this case, the transverse boson mode expansions are nothing but four copies of \((X^7, X^8)\) (14) in the \(S^1\) case, and hence contain a four-dimensional winding-number lattice but no momentum lattice. Then a simultaneous T-duality flip along \(X^2, X^4, X^6\) and \(X^8\) directions converts it to a four-dimensional momentum lattice.

3 \(O(2, 2; \mathbb{Z})\) symmetry

In \(T^d\) compactifications for \((d \geq 2)\), the simultaneous T-duality flip we have considered so far is a special element of \(O(d, d; \mathbb{Z})\) transformation, realized by conjugating the \(O(d, d)\) metric \(L\) to the scalar matrix. We will now examine the full \(O(2, 2; \mathbb{Z})\) duality in the plane-wave background compactified on \(T^2\).

The metric that we use for a \(T^2\) compactification is

\[
\text{ds}^2 = 2dX^+dX^- - \mu^2X^I X^I (dX^+)^2 - 4\mu(X^5dX^6 + X^7dX^8) dX^+ + dX^i dX^i \tag{37}
\]

\((i = 1, \ldots, 8; I = 1, \ldots, 4)\). The dilaton and \(B\) fields are zero. We have taken \(k^I_{65}^+\) and \(k^I_{87}^+\) as the manifest Killing vectors. The change of coordinates is an obvious generalization of (35) [7]. \(F_5\) is the same as before but anyway irrelevant for the bosonic spectrum.

We compactify the manifest coordinates \(X^6, X^8\) as

\[
X^6 \sim X^6 + 2\pi R^6, \quad X^8 \sim X^8 + 2\pi R^8, \tag{38}
\]

then the \(O(2, 2; \mathbb{Z})\) transformation \(g\) acting on the scalar matrix \(\mathcal{M}\) as \(g\mathcal{M}g^T\) is given by

\[
g = Q^{-1}\mathbf{g}_\mathbb{Z} Q, \quad Q = \begin{bmatrix} R^6/\alpha' \\ R^8/\alpha' \\ 1/R^6 \\ 1/R^8 \end{bmatrix}, \tag{39}
\]

where \(\mathbf{g}_\mathbb{Z}\) is an \(O(2, 2; \mathbb{R})\) matrix with integer entries. Any \(\mathbf{g}_\mathbb{Z}\) can be written [1] as \(\mathbf{g}_\mathbb{Z} = w^s g_1 g_2\), where

\[
w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad s = 0 \text{ or } 1, \tag{40}
\]

and \(g_1 \in G_1, \ g_2 \in G_2\) with \(G_1, G_2 = SL(2, \mathbb{Z})\) respectively represented by the matrices of
the form

\[ g_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad g_2 = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}, \quad g_2 = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}. \] (41)

\( a, b, c, d \) are integers satisfying \( ad - bc = 1 \). We have taken the \( O(d, d) \) metric \( L \) as

\[ L = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \] (42)

so that \( gLg^T = L \).

How the \( O(2, 2; \mathbb{Z}) \) acts on a supergravity background may be read off from the dimensionally reduced NS-NS sigma model [14]. It is easy to see that \( G_1 \), one \( SL(2, \mathbb{Z}) \) factor of \( O(2, 2; \mathbb{Z}) \), causes no change in the spectrum, since it can be absorbed into the target space modular transformation of compactified \( (X^6, X^8) \). \( w \) is a T-duality flip along a single circle, and the invariance of the Hamiltonian was confirmed in the previous section. Therefore, we have only to prove the invariance under the action of \( G_2 \) and \( wG_1 \) (= \( G_2w \)). First we check \( G_2 \).

Writing

\[ \tilde{b} = \frac{R^6 R^8}{\alpha'^2} b, \quad \tilde{c} = \frac{\alpha'}{R^6 R^8} c, \] (43)

we obtain the new metric in the form

\[ ds^2_{new} = 2dX^+dX^- - \mu^2 \left( X^I X^I + \frac{4\tilde{c}^2}{\tilde{c}^2 + \tilde{d}^2} X^\hat{m} X^\hat{m} \right) (dX^\pm)^2 \] (44)

\[ - \frac{4d\mu}{\tilde{c}^2 + \tilde{d}^2} (X^5 dX^6 + X^7 dX^8) dX^+ dX^- + dX^I dX^I + dX^\hat{m} dX^\hat{m} + \frac{1}{\tilde{c}^2 + \tilde{d}^2} dX^m dX^m, \]

and nonvanishing components of the \( B \) field

\[ B_{+6}^{new} = \frac{2\tilde{c}^2}{\tilde{c}^2 + \tilde{d}^2} X^7, \quad B_{+8}^{new} = -\frac{2\tilde{c}^2}{\tilde{c}^2 + \tilde{d}^2} X^5, \quad B_{68}^{new} = \frac{\alpha \tilde{c} + \tilde{b} \tilde{d}}{\tilde{c}^2 + \tilde{d}^2}. \] (45)

The bosonic string action reads

\[ S_{new} = \int d\tau L_{new}, \] (46)

\[ L_{new} = \int_0^{2\pi} d\sigma \left[ p^+ \partial^- X^- + \frac{1}{4\pi \alpha'} \left( (\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2 - \tilde{\mu}^2 \left( X^I \right)^2 + \frac{4\tilde{c}^2}{\tilde{c}^2 + \tilde{d}^2} (X^\hat{m})^2 \right) \right. \]

\[ + \left. (\partial_\tau X^\hat{m})^2 - (\partial_\sigma X^\hat{m})^2 + \frac{1}{\tilde{c}^2 + \tilde{d}^2} ((\partial_\tau X^m)^2 - (\partial_\sigma X^m)^2) \right] \]
Using if modes are found to be \( SL \) Therefore, the set of nonzero modes is

\[
\hat{\alpha} X^6 \hat{\partial}_\sigma X^8 - \hat{\partial}_\sigma X^8 \hat{\partial}_\sigma X^6 \]

\[
\frac{4\hat{\mu}}{c^2 + d^2} (d(\hat{X}_5 \hat{\partial}_\tau \hat{X}_6 + \hat{X}_7 \hat{\partial}_\tau \hat{X}_8) + \hat{c}(\hat{X}_7 \hat{\partial}_\sigma \hat{X}_6 - \hat{X}_5 \hat{\partial}_\sigma \hat{X}_8)) \]

where \( \hat{m} = 5, 7 \) and \( m = 6, 8 \). The \( X^I \) modes are inert under the \( SL(2, \mathbb{Z}) \), while the \( X^m \) and \( X^m \) equations of motion are

\[
\ddot{X}^5 - X^{5\prime} + \frac{2\hat{\mu}}{c^2 + d^2} (d \dot{X}^6 - \hat{c} X^{6\prime} + 2\hat{\mu} c^2 X^5) = 0,
\]

\[
\ddot{X}^7 - X^{7\prime} + \frac{2\hat{\mu}}{c^2 + d^2} (d \dot{X}^8 + \hat{c} X^{8\prime} + 2\hat{\mu} c^2 X^7) = 0,
\]

\[
\ddot{X}^6 - X^{6\prime} - 2\hat{\mu}(d \dot{X}^5 + \hat{c} X^7) = 0,
\]

\[
\ddot{X}^8 - X^{8\prime} - 2\hat{\mu}(d \dot{X}^7 - \hat{c} X^5) = 0.
\]

(47)

It can be verified that, for the solutions of the form \( e^{i(\omega_n \tau + n \sigma)} \), \( \omega_n \) does not depend on \( c \) and \( d \), but is always given by

\[
\omega_n = \pm \hat{\mu} \pm \sqrt{\hat{\mu}^2 + n^2}.
\]

(48)

Therefore, the set of nonzero modes is \( SL(2, \mathbb{Z}) \) independent. On the other hand, the zero modes are found to be

\[
X^5_{\text{zeromodes}} = \frac{1}{2\hat{\mu}} (-d \alpha^\prime p^6 + \hat{c} R^8 w^8),
\]

\[
X^6_{\text{zeromodes}} = x^6_0 + \alpha^\prime p^6 \tau + R^6 w^6 \sigma,
\]

\[
X^7_{\text{zeromodes}} = \frac{1}{2\hat{\mu} c^2} (-d \alpha^\prime p^8 - \hat{c} R^6 w^6),
\]

\[
X^8_{\text{zeromodes}} = x^8_0 + \alpha^\prime p^8 \tau + R^8 w^8 \sigma
\]

(49)

if \( c \neq 0 \). They depend both on the momenta \( p^m \) and on the winding numbers \( w^m \in \mathbb{Z} \). Using \( f_n^\pm(\tau, \sigma) \) (11), we obtain the mode expansions

\[
X^m(\tau, \sigma) = x^m_0 + y^m + \frac{1}{2i} \sqrt{\frac{\alpha^\prime}{\hat{\mu}}} (a^m_0 e^{-2i\hat{\mu} \tau} - \bar{a}^m_0 e^{2i\hat{\mu} \tau}) \]

\[
+ \frac{1}{2} \sqrt{\frac{\alpha^\prime}{2}} e^{-i\hat{\mu} \tau} \sum_{n \neq 0} \frac{1}{n} (a^m_n f^m_n + \bar{a}^m_n f^m_{-n}) - \frac{1}{2} \sqrt{\frac{\alpha^\prime}{2}} e^{i\hat{\mu} \tau} \sum_{n \neq 0} \frac{1}{n} (a^\bar{m}_n f^m_{+n} + \bar{a}^\bar{m}_n f^m_{-n}),
\]

\[
y^m = \begin{cases} 
+ \frac{2w^8}{\hat{\mu} c^2} (\hat{m} = 5), \\
- \frac{2w^8}{\hat{\mu} c^2} (\hat{m} = 7),
\end{cases}
\]

\[
X^6(\tau, \sigma) = x^6_0 + \alpha^\prime p^6 \tau + R^6 w^6 \sigma + \frac{1}{2} \sqrt{\frac{\alpha^\prime}{\hat{\mu}}} (a^5_0 e^{-2i\hat{\mu} \tau} + \bar{a}^5_0 e^{2i\hat{\mu} \tau})
\]

(51)
respectively, which imply the quantization conditions

\[ H_{\text{zeromodes}} = \alpha' \left( (p^6)^2 + (p^8)^2 \right). \]  

At first sight it appears to depend on \( c \). However, the constant pieces of \( 2\pi \Pi_m \) (\( m = 6, 8 \)) are

\[ \frac{p^6}{c^2} = \frac{a}{c} \frac{R^8}{\alpha'} w^8, \quad \frac{p^8}{c^2} = \frac{a}{c} \frac{R^6}{\alpha'} w^6, \]  

respectively, which imply the quantization conditions

\[ \frac{p^6}{c} = \frac{R^8}{\alpha'} w^8, \quad \frac{p^8}{c} = -\frac{R^6}{\alpha'} w^6, \]  

where

\[ \begin{bmatrix} w^8 \\ n^6 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} w^8 \\ n^6 \end{bmatrix}, \quad \begin{bmatrix} w^6 \\ n^8 \end{bmatrix} = \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \begin{bmatrix} w^6 \\ n^8 \end{bmatrix} \]
with \( n^6, n^8 \in \mathbb{Z} \). Since \( w^m, n^m \ (m = 6, 8) \) run over \( \mathbb{Z} \), \( H_{\text{zeromodes}} \) is invariant under the \( SL(2, \mathbb{Z}) \). Similarly, we find that

\[
P_{\text{zeromodes}} = n^6 w^6 + n^8 w^8,
\]  

(58)

and \( P_{\text{zeromodes}} \) is also invariant.

Finally, if \( c = 0 \), the zero modes are simply given by

\[
\begin{align*}
X^5_{\text{zeromodes}} &= x^5_0, & X^6_{\text{zeromodes}} &= x^6_0 + R^6 w^6 \sigma, \\
X^7_{\text{zeromodes}} &= x^7_0, & X^8_{\text{zeromodes}} &= x^8_0 + R^8 w^8 \sigma,
\end{align*}
\]

(59)

where the commutation relations of \((x^m_0, x^m)\) are the same as in (53). The spectrum then amounts to two sets of the IIB spectrum found in the previous section, with constant shifts of the canonical momenta. The invariance of the spectrum is easily confirmed. Thus we have shown the \( G_2 \) invariance.

One can similarly check the \( wG_1 = G_2 w \) invariance. In this case much labor is saved because the system after the transformation is reduced to that of the IIB bosons \( X^7, X^8 \) and the IIA bosons \( X^7_{\text{IA}}, X^9 \) in the \( S^1 \) compactification, with a constant compact \( B \) field. Then the check of the invariance for the nonzero-mode sector is trivial, and for the zero-mode sector is also straightforward. This completes our proof of the \( O(2, 2; \mathbb{Z}) \) T-duality in the \( T^2 \) compactified IIB maximally supersymmetric plane wave.

\section{Conclusions and discussion}

We have shown, by direct computations of bosonic string spectra, the \( O(d, d; \mathbb{Z}) \) \((d = 1, 2)\) T-duality in the maximally supersymmetric IIB plane-wave background compactified on \( S^1 \) and \( T^2 \). Only half of the ordinary set of zero modes appear in the Hamiltonian. The “half” Narain lattice is stabilized by the T-duality group.

A natural question may be raised here about modular invariance. Let us focus on the case of the action of \( G_2 \). Other cases are similar. First, the partition function is given by

\[
Z \sim \text{Tr} e^{-2\pi \tau_2 (\alpha' p^+ p^- + H) + 2\pi i \tau_1 P},
\]

(60)

where \( \tau_1 + i \tau_2 \) is the modular parameter. Since the zero-mode piece of \( P \) is given by (58) for both \( c = 0 \) and \( c \neq 0 \) and the Hamiltonian is independent of \((n^6, n^8)\), the summation over these integers in the trace gives \( \delta_{w^6, 0} \delta_{w^8, 0} \). (Precisely, this holds for irrational \( \tau_1 \). The
case of rational $\tau_1$ may be safely handled, since its measure is zero.) This in turn makes the summation over $(w^6, w^8)$ trivial. Thus, modular invariance does not require pairs of integers in the Hamiltonian, which is different from the flat case. In this way, mechanism of T-duality works in an interesting manner, not to make the “half” Narain lattice conflict with modular invariance. (For a related discussion in a somewhat different context, see [15].) For the uncompactified plane-wave background (1), the modular invariance of the one-loop vacuum amplitude has been discussed in [16]. It would be an interesting open problem to show the modular invariance also in the compactified cases.

The background we have considered has more commuting isometries. For example,

$$\{k_{S_{21}^+}, k_{S_{43}^+}, k_{S_{65}^+}, k_{S_{25}^+}, k_{S_{45}^+}, k_{S_{67}^+}, k_{S_{81}^+}\}$$

are a (non-maximal) set of mutually commuting Killing vectors with unit norm. The first and last four’s are mutually orthogonal, but some two of the whole set are not. Therefore, while it is natural to expect that the T-duality group extends at least up to $O(4, 4; \mathbb{Z})$, it would be interesting to see whether and how it goes beyond that.

Finally, one of the underlying motivations for the present work is to investigate whether $E_{10}$ [17] is the duality of the maximally supersymmetric plane wave. Being a one-dimensional system with a null Killing vector (in the Rosen coordinates), it is a solution to the null reduction to one dimension [18]. Since both are, in a sense, something ultimate, it is tempting to speculate that the maximally supersymmetric plane wave might realize $E_{10}$ as its U-duality.

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