Relativistic quantum dynamics of vector bosons in an Aharonov–Bohm potential

Luis B Castro and Edilberto O Silva

Departamento de Física, Universidade Federal do Maranhão, 65080–805, São Luís, Maranhão, Brazil

E-mail: lrb.castro@ufma.br (Luis B Castro) and edilbertoo@gmail.com (Edilberto O Silva)

Received 5 January 2017, revised 15 November 2017
Accepted for publication 22 November 2017
Published 13 December 2017

Abstract

The Aharonov–Bohm (AB) problem for vector bosons by the Duffin–Kemmer–Petiau (DKP) formalism is analyzed. Depending on the values of the spin projection, the relevant eigenvalue equation coming from the DKP formalism reveals an equivalence to the spin-1/2 AB problem. Using a boundary condition at the origin based on the self-adjoint extension method, we study the scattering and bound states problems. Expressions for $S$ matrix, phase shift and bound states are obtained.

Keywords: Duffin–Kemmer–Petiau equation, Aharonov–Bohm problem, self-adjoint extension, physical regularization

1. Introduction

The AB effect [1] has been the usual framework for investigating the arising of phases in the wave function of quantum particles in various physical models and has inspired a great deal of investigations in recent years. In the usual AB effect, the vector potential due to a solenoid gains an extraordinary physical meaning. It can affect the quantum behavior of a charged particle that never encounters an electromagnetic field. This phenomenon is intimately related to a non-local boundary condition which relates the change in the phase of an electron wave function to the amount of flux in the solenoid. The interest in this issue appears in the different contexts, such as solid state physics [2], cosmic strings [3–13] $\kappa$-Poincaré-Hopf algebra [14, 15], $\delta$-like singularities [16], supersymmetry [17], condensed matter [18], Lorentz symmetry violation [19–21], quantum chromodynamics [22], general relativity [23], nanophysics [24], quantum ring [25–28], black hole [29] and noncommutative theories [30–32].

In the AB problem of spin-1/2 particles a two-dimensional $\delta$ function appears as the mathematical description of the Zeeman interaction between the spin and the magnetic flux tube [16]. This interaction term is known to cause a splitting on the energy spectrum of atoms depending
on the spin state. In the AB problem of spin-1 particles [33, 34], this characteristic is also present. In [33], where the authors address the AB problem for spin-1 Yang–Mills particles, it was established that, for the case of spin-1/2, quasibound states exist for all noninteger flux parameter. The existence of these states is related to the penetration of the magnetic flux tube by the particle, which is sufficient to produce sensitivity to the sign of the flux. The difference for spin-1 Yang–Mills particles is that the quasibound states exist only for discrete values of the magnetic flux tube, so that penetration occurs only for flux values in a set of measure zero.

In this work, we solve the spin-1 AB problem for bound states in the context of the DKP formalism. In our approach, we consider the idealized picture of a magnetic flux tube of null radius which allows the particles to access the \( \rho = 0 \) region in a controlled way. Unlike the approach taken in [33], here, we modulate the problem with general boundary conditions. When the spin projection \( s^3 = 0 \), the radial operator can be expressed as a modified Bessel differential equation. In this case, the system does not admit bound-state solutions. On the other hand, when the spin projection \( s^{1,2} = 1, -1 \), as we mentioned above, we have the presence of a \( \delta \) function potential in the equation of motion. As it is well-known in quantum mechanics, the \( \delta \) function potential guarantees at least one bound state for the particle and this property is independent of its spin. For the system considered here, in first sight, the inclusion of the spin projection element \( s^{1,2} \) leads to an equation of motion equivalent to the equation for the spin-1/2 AB problem. Because of this, the problem can be addressed by the self-adjoint extension method [35, 36] with the application of appropriate boundary conditions at the origin. After imposing the boundary condition, we study the scattering and bound states problems. We determine expressions for the phase shift, \( S \) matrix and bound states.

2. A short review on Duffin–Kemmer–Petiau equation

The first-order DKP formalism [37–40] describes spin-0 and spin-1 particles and has been employed to study meson-nucleus interactions as an alternative to their conventional second-order Klein–Gordon (KG) and Proca counterparts. The equivalence between the DKP equation and the KG and Proca equations has been verified for the case of minimally coupled vector interactions [41–43], the algebraic structure of the DKP formalism permits more couplings than the KG and Proca theories [44, 45]. Indeed, the DKP formalism has been extensively used in the description of many processes in nuclear physics and elementary particles, yielding a better adjustment to the experimental data than the KG formalism in the analysis of \( K_0 \) decays, the decay-rate ratio \( \Gamma(\eta \rightarrow \gamma \gamma)/\Gamma(\pi^0 \rightarrow \gamma \gamma) \), level shifts and widths in pionic atoms [46–48]. Recently, the DKP formalism was used in other contexts, applications have been made in Bose–Einstein condensates [49, 50], in very special relativity (VSR) symmetries [51], in thermodynamics properties [52], in topological defects [53], in topological semimetals [54], in noninertial effect of rotating frames [55], in noncommutative phase space [56], among others.

In the formalism of DKP, the equation that describes a free charged boson is given by [40] (with units \( \hbar = c = 1 \))

\[
(i \beta^\mu \partial_\mu - M) \Psi = 0, \tag{1}
\]

where the matrices \( \beta^\mu \) satisfy the DKP algebra

\[
\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu \nu} \beta^\lambda + g^{\lambda \nu} \beta^\mu, \tag{2}
\]

and \( g^{\mu \nu} = \text{diag}(1, -1, -1, -1) \) is the metric tensor. The algebraic structure showed in (2) furnishes a set of 126 linearly independent matrices, which are part of three irreducible
representations: (i) a trivial one-dimensional representation (no physical meaning), (ii) a five-dimensional representation associated to scalar sector (spin-0 particles) and (iii) a ten-dimensional representation describing the vector sector (spin-1 particles). The DKP spinor contains superfluous components and the theory needs a constraint equation which allows to remove the excess of components. By multiplying the DKP equation by \(1 - \beta^0 \beta^0\) one can obtain the constraint equation, namely
\[
i \beta^j \beta^0 \beta^0 \partial_j \Psi = M \left(1 - \beta^0 \beta^0\right) \Psi, \tag{3}\]
where \(j\) runs from 1 to 3. Note that using the constraint equation, one can express components of the DKP spinor by their space derivatives in order to only remain the physical components of the DKP spinor. The second-order KG (Proca) equation is obtained when one selects the scalar (vector) sector of the DKP theory. A well-known conserved four-current is given by
\[
J^\mu = \frac{1}{2} \bar{\Psi} \beta^\mu \Psi, \tag{4}\]
where \(\Psi\) is the adjoint spinor and is given by \(\bar{\Psi} = \Psi^\dagger \eta^0\) with \(\eta^0 = 2 \beta^0 \beta^0 - 1\) in such a way that \(\eta^0 \beta^\mu \dagger = \eta^0 \beta^\mu\) (the matrices \(\beta^\mu\) are Hermitian with respect to \(\eta^0\)). In spite of the resemblance to the Dirac equation, the DKP equation employs singular matrices. Moreover, the time component of \(J^\mu\) in equation (4) is not positive definite and, consequently, the case of massless bosons can not be achieved through a limiting process [57]. Besides these characteristics, it is also known that the matrices \(\beta^\mu\) plus the unit operator create a ring consistent with integer-spin algebra and \(J^0\) may be interpreted as a charge density. As a final remark, since the factor \(1/2\) multiplying \(\bar{\Psi} \beta^\mu \Psi\) in equation (4) has no importance concerning the conservation law, it is only for obtain the charge density in agreed to the KG theory as well as its nonrelativistic limit [58].

### 3. Duffin–Kemmer–Petiau equation with external interactions

Taking into account interactions, the DKP equation takes the form
\[
(i \beta^\mu \partial_\mu - M - U) \Psi = 0, \tag{5}\]
where \(U\) is the more general potential matrix and it can be expressed in terms of 25 and 100 linearly independent matrices associate to five-dimensional (scalar sector) and ten-dimensional (vector sector) irreducible representations, respectively. Considering interactions, the four-current \(J^\mu\) obeys the following equation
\[
\partial_\nu J^\mu - \frac{i}{2} \bar{\Psi} \left(\eta^0 U^\dagger \eta^0 - U\right) \Psi = 0. \tag{6}\]
One can see that, if \(\eta^0 U^\dagger = \eta^0 U\) (Hermitian with respect to \(\eta^0\)), then one will have a conserved four-current. The general potential matrix \(U\) can be built on the basis of well-defined Lorentz structures. Considering the scalar sector (spin-0 particles) there are two scalar, two vector and two tensor terms [44], while in the vector sector (spin-1 particles) there are two scalar, two vector, a pseudoscalar, two pseudovector and eight tensor terms [45]. It should be mentioned here that the condition in equation (6) was employed to indicate a erroneous processing in the literature concerning analytical solutions for nonminimal vector interactions in the DKP theory [59–61].
### 3.1. Duffin–Kemmer–Petiau equation with minimal electromagnetic coupling

Let us consider the minimal vector interaction (electromagnetic interaction), so that the DKP equation can be expressed as

\[
(i\beta^\mu D_\mu - M) \Psi = 0,
\]

where \( D_\mu = \partial_\mu + ieA_\mu \) is the covariant derivative. The constraint equation (3) reads

\[
i\beta^k \beta^0 \beta^0 \partial_k \Psi - e\beta^k \beta^0 \beta^0 A_k \Psi = M \left(1 - \beta^0 \beta^0\right) \Psi,
\]

and the conserved four-current \( J^\mu \) remains with the same form as in equation (4).

### 3.2. Vector sector

Now, we focus attention on the vector sector (spin-1 particles) of the DKP theory. To select the physical components of the DKP spinor for the vector sector, we use the operator defined by [62]

\[
R^\mu = (\beta^1)^2 (\beta^2)^2 (\beta^3)^2 \left[ \beta^\mu \beta^0 - g^\mu_0 \right],
\]

which satisfies \( R^{\mu\nu} = R^\mu \beta^\nu \) and \( R^{\nu\mu} = -R^{\mu\nu} \). Moreover, as it is shown in [62], \( R^\mu \Psi \) and \( R^{\mu\nu} \Psi \) transform as a (pseudo)vector and (pseudo)tensor quantities under an infinitesimal Lorentz transformation, respectively. From the above definitions, the following property is obtained:

\[
R^{\mu\nu} \beta^\alpha = R^\mu g^{\nu\alpha} - R^\nu g^{\mu\alpha}.
\]

In this way, by applying the \( R^\mu \) and \( R^{\mu\nu} \) operators to the DKP equation (7), we obtain

\[
D_\mu (R^\mu \Psi) = -iM (R^\nu \Psi),
\]

\[
(R^{\mu\nu} \Psi) = -\frac{i}{M} U^{\mu\nu},
\]

\[
U^{\mu\nu} = D^\mu (R^\nu \Psi) - D^\nu (R^\mu \Psi),
\]

which leads to

\[
D_\mu U^{\mu\nu} + M^2 (R^\nu \Psi) = 0,
\]

\[
D_\mu (R^\mu \Psi) = \frac{ie}{2M^2} F_{\mu\nu} U^{\mu\nu},
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). From the results, we can conclude that all components of the column matrix \( R^\mu \Psi \) obey the Proca equation embedded in an electromagnetic field. So, this process selects the vector sector of DKP theory, making explicitly clear that it describes a spin-1 particle (vector sector) with an electromagnetic field.

According to [63], we can rewrite equation (14) in the form

\[
\left[D_\mu D^\mu + M^2 \right] R^\nu \Psi - D^\nu D_\mu R^\mu \Psi - \frac{ie}{2} R^\nu S^\nu_\alpha F_{\mu\alpha} \Psi = 0,
\]

where \( S^{\mu\nu} = [\beta^\mu, \beta^\nu] \). The term \( D^\nu D_\mu R^\mu \Psi \) is called the anomalous term because it has no equivalent in the spin-1/2 Dirac theory [40]. Nonetheless, it has been shown in [41, 42] that such an anomalous term disappears when the physical components of the DKP field are selected.
Following the same procedure of [63], equation (16) becomes
\[
[\mathcal{D}_\mu \mathcal{D}^\mu + M^2 - e (\mathbf{S} \cdot \mathbf{B})] \mathbf{R} \Psi = 0, \tag{17}
\]
where \( \mathbf{B} = \nabla \times \mathbf{A} \) and the spin operator \( \mathbf{S} = (S^1, S^2, S^3) \) is expressed by
\[
S^1 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{18}
\]
In this stage, it is worthwhile to mention that the last term in equation (17) is the called Zeeman interaction term, which is crucial to give meaning to the term that explicitly depends on the spin. Also, we can mention that the Zeeman term is only important for the vector sector (spin-1 particles) of the DKP theory, because that term is absent if we work with the scalar sector (spin-0 particles). For this reason, we only focus the vector sector of the DKP theory.

If the terms in the potential \( A^\mu = (A_0, \mathbf{A}) \) are time–independent one can write
\[
\Psi (\mathbf{r}, t) = \psi (\mathbf{r}) e^{-i E t}, \tag{19}
\]
where \( E \) is the energy of the vectorial boson, in such a way that the time–independent DKP equation for the vector sector becomes
\[
\left[ (p - eA)^2 + M^2 - (E - eA_0)^2 - e (\mathbf{S} \cdot \mathbf{B}) \right] \mathbf{R} \psi = 0. \tag{20}
\]
In the next section, we apply equation (20) to AB problem, giving a focus to spin effects through the \( e (\mathbf{S} \cdot \mathbf{B}) \) term. We shall see later that by carefully modulating the radial operator (20) with boundary conditions, it can provide both bound and scattering states.

4. The Aharonov–Bohm problem

Let us consider the particular case where the boson moves in the presence of the AB potential. The vector potential in the Coulomb gauge is given by \( (A_0 = 0) \)
\[
e A = -\frac{\phi}{\rho} \hat{\varphi}, \tag{21}
\]
where \( \phi \) is the flux parameter. The potential in (21) provides a magnetic field perpendicular to the plane \((\rho, \varphi)\), namely
\[
e B = -\phi \frac{\delta (\rho)}{\rho} \hat{z}, \tag{22}
\]
where \( \mathbf{B} \) is the magnetic field due to a solenoid.

4.1. Aharonov–Bohm problem for the spin-1 sector

Now, we consider the effect of AB flux field on vector bosons. Substituting equation (21) in equation (20), we obtain
\[
\left[ \left( \frac{1}{i} \nabla + \frac{\phi}{\rho} \hat{\varphi} \right)^2 + \phi S \frac{\delta (\rho)}{\rho} \right] \mathbf{R} \psi = (E^2 - M^2) \mathbf{R} \psi, \tag{23}
\]
where \( S \) is the matrix
\[
\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\] (24)

and
\[
\nabla = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z},
\] (25)
is the gradient operator in cylindrical coordinates. At this stage, we can use the invariance under boosts along the \(z\)-direction and adopt the usual decomposition
\[
R \psi = R^i \psi (\rho, \varphi, z) = f^{(i)}_m (\rho) e^{im\phi} e^{ipz};
\] (26)
with \(m \in \mathbb{Z}\). Inserting equation (26) into equation (23), we get
\[
H f^{(i)}_m (\rho) = k^2 f^{(i)}_m (\rho),
\] (27)
with \(k = \sqrt{E^2 - M^2 - p_z^2}\), where
\[
\mathcal{H} = \mathcal{H}_0 + \phi s \delta (\rho) \rho,
\] (28)
\[
\mathcal{H}_0 = - \frac{d^2}{d\rho^2} - \frac{1}{\rho} \frac{d}{d\rho} + \frac{(m + \phi)^2}{\rho^2},
\] (29)
and \(\hat{s} = (1, -1, 0)\) represents the eigenvalues of the operator \(S\) acting on the DKP spinor \(f^{(i)}_m (\rho)\). Equation (27) describes the quantum dynamics of vector bosons in the presence of the AB potential. From equation (27) we can see that scattering states occur only if \(k \in \mathbb{R}\), whereas bound states occur only if \(k = i|k|\).

At this level, it is worthwhile to note that the solution for this problem can be separated in two cases. The first case is when the spin projection \(s^3 = 0\). In this case, the Pauli term is absent and the radial operator \(\mathcal{H}\) becomes \(\mathcal{H}_0\) and \(f^{(3)}_m\) can be expressed as a solution of the modified Bessel differential equation. We can see that the system does not admit bound-state solutions. On the other hand, the second case is when \(s^1, s^2 = 1, -1\). In this case the radial operator \(\mathcal{H}\) is equivalent to equation (29) of [64] (see also [16, 65, 66]) which governs the quantum dynamics of the usual spin-1/2 AB problem, namely
\[
- \frac{d^2}{d\rho^2} - \frac{1}{\rho} \frac{d}{d\rho} + \frac{(m + \phi)^2}{\rho^2} + \phi s \delta (\rho) \rho,
\] (30)
where \(s\) is twice the spin value, with \(s = +1\) for spin ‘up’ and \(s = -1\) for spin ‘down’, and \(\phi\) is the flux parameter.

In order to study the dynamics of the system for \(s^{1,2} = 1, -1\), it is necessary to solve equation (27). However, as for the case studied in [64], it also involves a singularity in the \(\rho = 0\) region.

By exploiting the nature of the \(\delta\) function in equation (28), we can see that the system admits at least a bound state. On the other hand, if we taking into account such a function, \(\mathcal{H}\) is not essentially self-adjoint. In this case, we must find the self-adjoint extensions of the operator \(\mathcal{H}_0\) corresponding to different types of boundary conditions. Such self-adjoint extensions are based in boundary conditions at the origin and infinity [67–69].

Let us investigate the Hamiltonian \(\mathcal{H}_0\). By using the unitary operator \((uf) (\rho) = \sqrt{\rho} f (\rho)\) we write \(\mathcal{H}_0\) as

\[
L B Castro and E O Silva
J. Phys. A: Math. Theor. 51 (2018) 035201
\[ \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0 = -\frac{d^2}{d\rho^2} - \frac{1}{\rho^2} \left[(m + \phi)^2 - \frac{1}{4}\right]. \quad (31) \]

which is essentially self-adjoint if
\[ |m + \phi| \geq 1, \quad (32) \]
while for
\[ |m + \phi| < 1, \quad (33) \]
it admits an one-parameter family of self-adjoint extensions [35], \( \mathcal{H}_{0,\zeta_m} \), where \( \zeta_m \) is the self-adjoint extension parameter. In order to characterize \( \mathcal{H}_{0,\zeta_m} \), we use the approach of [70], which is based in a boundary conditions at the origin. In this approach, the boundary condition is a mathematical limit allowing divergent solutions of the Hamiltonian (29) at isolated points, provided they remain square integrable. Before applying the method allows us to clarify some questions. First, we can not impose any boundary condition (e.g. \( \psi = 0 \) at \( \rho = 0 \)) without discovering which boundary conditions are allowed by the operator \( \mathcal{H}_0 \). Second, the self-adjoint extension provides an infinity of possible boundary conditions, so that it can not give us the true physics of the problem. Nevertheless, once the physics at \( \rho = 0 \) is known [71, 72], it is possible to determine any arbitrary parameter coming from the self-adjoint extension, and then we have a complete description of the problem. Since we have a singular point, described by the \( \delta \) function, we must guarantee that the Hamiltonian is self-adjoint in the region of motion. In other words, we must ensure that the Hamiltonian operator \( \mathcal{H}_0 \), with domain \( \mathcal{D}(\mathcal{H}_0) \), must satisfy both requirements that \( \mathcal{D}(\mathcal{H}_0^\dagger) = \mathcal{D}(\mathcal{H}_0) \) and \( \mathcal{H}_0^\dagger = \mathcal{H}_0 \), respectively [35]. Then, according to [70], all self-adjoint extensions \( \mathcal{H}_{0,\zeta_m} \) of \( \mathcal{H}_0 \) are parametrized by the boundary condition at the origin,
\[ \zeta_m f(0) = f(1), \quad (34) \]
with
\[ f(0) = \lim_{\rho \to 0^+} \rho^{m+\phi} f_m(\rho), \quad (35) \]
\[ f(1) = \lim_{\rho \to 0^+} \frac{1}{\rho^{m+\phi}} \left[ f_m(\rho) - f(0) \frac{1}{\rho^{m+\phi}} \right]. \quad (36) \]
As pointed out in [36], the self-adjoint extension parameter \( \zeta_m \) have a physical interpretation; it represents the scattering length of \( \mathcal{H}_{0,\zeta_m} \). In particular, for \( \zeta_m = 0 \), we have the free Hamiltonian (without the \( \delta \) function) with regular wave functions at the origin and, for \( \zeta_m \neq 0 \), the boundary condition in equation (34) permit a \( \rho^{-|m+\phi|} \) singularity in the wave functions at the origin.

5. Scattering and bound states analysis

In the scattering analysis it is more convenient to write the solution for equation (27) (in the \( \rho \neq 0 \) region) in terms of Bessel functions
\[ f_m^{(i)}(\rho) = a_m J_{|m+\phi|}(k\rho) + b_m Y_{|m+\phi|}(k\rho), \quad (37) \]
with \( a_m \) and \( b_m \) being constants.
Now, we must replace the solution (37) in the boundary condition (34). Since \( \lim_{\rho \to 0^+} \rho^{2-2|m+\phi|} \) is divergent if \( |m + \phi| > 1 \), then \( b_m \) must be zero. On the other hand, \( \lim_{\rho \to 0^+} \rho^{2-2|m+\phi|} \) is finite for \( |m + \phi| < 1 \), so that there arises the contribution of the irregular solution \( Y_{m+\phi}(k \rho) \). The reason for the presence of this irregular solution contributing to \( f_m^{(i)}(\rho) \) stems from the fact the Hamiltonian (28) is not a self-adjoint operator when \( |m + \phi| < 1 \). Because of this characteristic the solution \( Y_{m+\phi}(k \rho) \) must be associated with a self-adjoint extension of the operator (29) [73, 74]. After we take into account these considerations, we get (for \( |m + \phi| < 1 \))

\[
\frac{b_m}{a_m} = \frac{-\zeta_m k^{2|m+\phi|} \Gamma (1 - |m + \phi|) \sin (|m + \phi| \pi)}{4 |m+\phi| \Gamma (1 + |m + \phi|) + \zeta_m k^{2|m+\phi|} \Gamma (1 - |m + \phi|) \cos (|m + \phi| \pi)}.
\]

(38)

Since \( \delta \) is a short range potential, it follows that the behavior of \( f_m(\rho) \) for \( \rho \to \infty \) is given by [75]

\[
f_m(\rho) \approx \sqrt{\frac{2}{\pi k \rho}} \cos \left( k \rho - \frac{|m|}{2} \pi - \frac{\pi}{4} + \delta_m^\omega \right),
\]

(39)

where \( \delta_m^\omega \) is the scattering phase shift, which represents a measure of the argument difference to the asymptotic behavior of the solution \( J_{|m|}(k \rho) \) of the radial free equation. Using equation (38) and the asymptotic behavior of \( J_{|m|}(z) \) and \( Y_m(z) \) [76],

\[
J_{|m|}(k \rho) \approx \sqrt{\frac{2}{k \pi \rho}} \cos \left( k \rho - \frac{\nu \pi}{2} - \frac{\pi}{4} \right),
\]

(40)

\[
Y_m(k \rho) \approx \sqrt{\frac{2}{k \pi \rho}} \sin \left( k \rho - \frac{\nu \pi}{2} - \frac{\pi}{4} \right),
\]

(41)

into equation (37), we obtain

\[
f_m(\rho) \sim a_m \sqrt{\frac{2}{\pi k \rho}} \times \left[ \cos \left( k \rho - |m + \phi| \frac{\pi}{2} - \frac{\pi}{4} \right) - \sin \left( k \rho - |m + \phi| \frac{\pi}{2} - \frac{\pi}{4} \right) \Omega_m^\omega \right],
\]

(42)

where

\[
\Omega_m^\omega = \frac{\zeta_m k^{2|m+\phi|} \Gamma (1 - |m + \phi|) \sin (|m + \phi| \pi)}{4 |m+\phi| \Gamma (1 + |m + \phi|) + \zeta_m k^{2|m+\phi|} \Gamma (1 - |m + \phi|) \cos (|m + \phi| \pi)}.
\]

(43)

By comparing the above expression with equation (39), we can extract the phase shift:

\[
\delta_m^\omega = \Delta_m + \theta_m^\omega,
\]

(44)

where

\[
\Delta_m = \frac{\pi}{2} \left( |m| - |m + \phi| \right),
\]

(45)

is the usual phase shift of the scattering process and

\[
\theta_m^\omega = \arctan (\Omega_m^\omega).
\]

(46)
Therefore, the scattering operator $S_m^\zeta$ ($S$ matrix) for the self-adjoint extension is

$$S_m^\zeta = e^{2i\Delta_m} = e^{2i\Delta_m} e^{2i\Delta_m}.$$

(47)

Using (46), equation (47) can be rewritten as

$$S_m^\zeta = e^{2i\Delta_m} \left[ \frac{\zeta^k 2|m+\phi| \Gamma(1 - |m + \phi|) e^{i\pi|m+\phi|} + 4|m+\phi| \Gamma(1 + |m + \phi|)}{\zeta^k 2|m+\phi| \Gamma(1 - |m + \phi|) e^{-i\pi|m+\phi|} + 4|m+\phi| \Gamma(1 + |m + \phi|)} \right].$$

(48)

Hence, for any value of the self-adjoint extension parameter $\zeta_m$, there is an additional scattering. If $\zeta_m = 0$, we have a scattering process with Dirichlet boundary condition, whose result is known to be

$$S_m^0 = e^{2i\Delta_m}.$$

(49)

If we assume that $\zeta_m = \infty$, we get

$$S_m^\infty = e^{2i\Delta_m + 2i\pi|m+\phi|}.$$

(50)

In accordance with the general theory of scattering, the poles of the $S$ matrix in the upper half of the complex plane [77] determine the positions of the relativistic bound states in the energy scale. These poles occur when the denominator of equation (48) is equal to zero with the replacement $k \to ik$, with $\kappa = \sqrt{-\left[(E_m^1)^2 - M^2 - p_z^2 \right]}$. So, we have

$$\zeta_m (ik)^2|\phi| e^{-i\pi|m+\phi|} \Gamma(1 - |m + \phi|) + 4|m+\phi| \Gamma(1 + |m + \phi|) = 0.$$  

(51)

Solving this equation for $E$, we obtain

$$(E_m^1)^2 = M^2 - 4 \left[ -\frac{1}{\zeta_m} \frac{\Gamma(1 + |m + \phi|)}{\Gamma(1 - |m + \phi|)} \right] + p_z^2.$$  

(52)

By analysing equation (52), we see that the poles of the operator $S_m^\zeta$ only occur for negative values of $\zeta_m$. It is also possible to express the operator $S_m^\zeta$ in terms of the bound state energy as

$$S_m^\zeta = e^{2i\Delta_m} \left[ \frac{e^{2i\pi|m+\phi|} - \left(\kappa/k\right)^{2|m+\phi|}}{1 - \left(\kappa/k\right)^{2|m+\phi|}} \right].$$

(53)

Note that the above results dependent of the self-adjoint extension parameter $\zeta_m$. However, in [64], an alternative method has been used to address the spin-$1/2$ AB problem using boundary conditions at the origin in order to obtain bound states. In this approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for equation (27) and the solutions for the problem $H_0$ plus self-adjoint extensions. As a result, we find an expression which is just the energy (52) with the self-adjoint extension parameter given explicitly as

$$(E_m^1)^2 = M^2 - 4 \left[ \frac{\left(\phi_k + |m + \phi|\right)}{\phi_k - |m + \phi|} \right] \Gamma(1 + |m + \phi|) \Gamma(1 - |m + \phi|) + p_z^2,$$  

(54)

where $\rho_0$ is a finite very small radius. This radius may be understood as a kind of physical regularization for the $\delta$ function in equation (28).

As for the spin-$1/2$ AB problem, we can get explicitly an expression for the parameter $\zeta_m$ by direct comparison between equations (52) and (54) as
\[ \zeta_m = -\frac{2^{m+\phi}|m+\phi|}{\rho_0 \left( \frac{\partial s^4 - |m+\phi|}{\partial s^4 + |m+\phi|} \right)}. \] (55)

The component of the DKP spinor for bound-state solutions for the spin projection \( s^{1,2} = 1, -1 \) is given by

\[ R_i \psi(\rho, \phi, z) = C_m K_{|m+\phi|} \left( |k^{(i)}| \rho \right) e^{i m \phi} e^{ipz}, \] (56)

where \( C_m \) is a normalization constant, \( K_{|m+\phi|} \) are the modified Bessel functions of second kind. The important result in equation (55) was found for the first time in [66], where was proposed a general regularization procedure to obtain the self-adjoint extension parameter for both state bound and scattering problem for the spin-1/2 AB problem in conical space in \((1+2)\) dimensions. To ensure that equation (52) is a real number, the self-adjoint extension parameter must be negative, i.e. \( \zeta_m < 0 \). This condition ensures that the system admits relativistic bound states.

6. Conclusions

In this work, we have addressed the spin-1 AB problem in the context of the DKP formalism. We have assumed vector bosons incidents on a flux tube, characterized by a \( \delta \) function. We found that our problem can be separated in two cases depending on the spin projection. For \( s^3 = 0 \), the term that depend explicitly of the spin is absent (\( \delta \) function is absent). For this case, we can see that the system does not admit bound-state solutions. Otherwise, for \( s^{1,2} = 1, -1 \), the radial operator is equivalent to usual spin-1/2 AB problem and consequently, after we apply the physical regularization procedure, it is equivalent to the problem of a particle in the presence of a \( \delta \) function potential in one dimension in quantum mechanics. In this sense, the self-adjoint extension approach was used to study the case \( s^{1,2} \) of the spin-1 AB scattering problem. The results were obtained by imposing the boundary condition in equation (34), which comes from the von Neumann theory of the self-adjoint extensions. This boundary condition permit a singularity in the wave functions at the origin and also the possibility of studying the problem for bound states. We have showed that the vector sector (spin-1 sector), which gives rise to a point interaction, changes the scattering phase shift and consequently the \( S \) matrix. Through the study of the poles of the \( S \) matrix, we derive an expression for the relativistic bound states. Finally, by direct comparison with results in the literature, where it was used an alternate procedure to determine bound states to the spin-1/2 AB problem, we derived an expression for the self-adjoint extension parameter \( \zeta_m \), which is given in terms of the physical parameters of the problem, as expected.

Acknowledgments

This work was partially supported by the CNPq, Brazil, Grants nos 455719/2014-4 (Universal), 427214-2016/5 (Universal), 304105/2014-7 (PQ) and 303774/2016-9 (PQ), FAPEMA, Brazil, Grants nos 01852/14 (PRONEM) and 01202/16 (Universal).

ORCID iDs

Luis B Castro \( \odot \) https://orcid.org/0000-0002-7017-1791
Edilberto O Silva \( \odot \) https://orcid.org/0000-0002-0297-5747
References

[1] Aharonov Y and Bohm D 1959 Phys. Rev. 115 485
[2] Kronig R L and Penney W G 1931 Proc. R. Soc. A 130 499
[3] Nouri-Zonoz M and Parvizi A 2013 Phys. Rev. D 88 023004
[4] Vilenkin A and Shellard E P S 2000 Cosmic Strings and Other Topological Defects (Cambridge: Cambridge University Press)
[5] Chu Y Z, Mathur H and Vachaspati T 2010 Phys. Rev. D 82 063515
[6] Sitenko Y A and Vlasiud N D 2010 J. Phys. A: Math. Theor. 43 354014
[7] Yu A Sitenko A M 1995 J. Exp. Theor. Phys. 81 831
[8] de Sousa Gerbert P 1989 Phys. Rev. D 40 1346
[9] Hohensee M A, Estey B, Hamilton P, Zeilinger A and Müller H 2012 Phys. Rev. Lett. 108 230404
[10] Bázerra V B 1997 J. Math. Phys. 38 2553–64
[11] Bakke K and Furtado C 2010 Ann. Phys. 322 447–55
[12] Furtado C, Bázerra V B and Moraes F 2000 Mod. Phys. Lett. A 15 253
[13] Aliev A and Gal’tsov D 1989 Ann. Phys. 193 142–65
[14] Roy P and Roychoudhury R 1995 Phys. Lett. B 359 339
[15] Andrade F M and Silva E O 2013 Phys. Lett. B 719 467–71
[16] Hagen C R 1990 Phys. Rev. Lett. 64 503
[17] Jakubsky V, Nieto L M and Plyushchay M S 2010 Phys. Lett. B 692 51–6
[18] Slobodeniuk A O, Sharapov S G and Loktev V M 2011 Phys. Rev. B 84 125306
[19] Belich H and Bakke K 2014 Phys. Rev. D 90 025026
[20] Belich H, Silva E O, Ferreira M M Jr and Orlando M T D 2011 Phys. Rev. D 83 125025
[21] Bakke K, Belich H and Silva E O 2011 J. Math. Phys. 52 063505–7
[22] Hashimoto K and Iizuka N 2010 Phys. Rev. D 82 105023
[23] Dolan S R, Oliveira E S and Crispino L C 2011 Phys. Lett. B 701 485–9
[24] Dmitriev A P, Gornyi I V, Kachorovskii V Y and Polyakov D G 2010 Phys. Rev. Lett. 105 036402
[25] Schelter J, Trauzettel B and Recher P 2012 Phys. Rev. Lett. 108 106603
[26] Tanaka A and Cheon T 2010 Phys. Rev. A 82 022104
[27] Baltateanu D 2011 Phys. Lett. A 375 2952–7
[28] Bakke K and Furtado C 2012 J. Math. Phys. 53 023514
[29] Anacleto M A, Brito F A and Passos E 2013 Phys. Rev. D 87 125015
[30] Das A, Falomir H, Nieto L M and Penney W G 2011 Phys. Rev. D 84 045002
[31] Chaichian M, Prenajder P, Sheikh-Jabbari M and Tureanu A 2002 Phys. Lett. B 527 149–54
[32] Li K and Dull S 2006 Eur. Phys. J. C 46 825–8
[33] Horner M L and Goldhaber A S 1997 Phys. Rev. D 55 5951–6
[34] Hagen C R and Ramaswamy S 1990 Phys. Rev. D 42 3524–33
[35] Reed M and Simon B 1975 Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. (New York: Academic)
[36] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 2004 Solvable Models in Quantum Mechanics 2nd edn (Providence, RI: American Mathematical Society)
[37] Petiau G 1936 Publ. Acad. R. Belg., Cl. Sci., Mem. (Collect. 8) 16 2
[38] Kemmer N 1938 Proc. R. Soc. A 166 127
[39] Duffin R J 1938 Phys. Rev. 54 1114
[40] Kemmer N 1939 Proc. R. Soc. A 173 91
[41] Nowakowski M 1998 Phys. Lett. A 244 329
[42] Lunardi J T, Pimentel B M, Teixeira R G and Valverde J S 2000 Phys. Lett. A 268 165
[43] Castro L B and de Castro A S 2014 Phys. Rev. A 90 022101
[44] Guertin R and Wilson T L 1977 Phys. Rev. D 15 1518
[45] Vojta A, Pfeiffer L and Mathews P M 1979 J. Phys. A: Math. Gen. 12 665
[46] Fischbach E, Iachello F, Lande A, Nieto L M and Scott C K 1971 Phys. Rev. Lett. 26 1200
[47] Fischbach E, Nieto L M and Scott C K 1974 Prog. Theor. Phys. 51 1585
[48] Friedman E, Källermann G and Batty C J 1986 Phys. Rev. C 34 2244
[49] Casana R, Fainberg V Y, Pimentel B M and Valverde J S 2003 Phys. Lett. A 316 33
[50] Abreu L M, Gadelha A L, Pimentel B M and Santos E S 2015 Physica A 419 612
[51] Cavalcanti R M T, Hoff da Silva J M and da Rocha R A 2014 Eur. Phys. J. Plus 129 246
[52] Wang Z, Long Z W, Long C Y and Zhang W 2015 AdHEP 2015 901675
[53] Castro L B 2015 *Eur. Phys. J.* C **75** 287
[54] Palumbo G and Meichanetzidis K 2015 *Phys. Rev.* B **92** 235106
[55] Castro L B 2016 *Eur. Phys. J.* C **76** 61
[56] Hassanabadi H, Molaee Z and Zarrinkamar S 2012 *Eur. Phys. J.* C **72** 2217
[57] Krajcik R A and Nieto M M 1974 *Phys. Rev.* D **10** 4049–63
[58] Cardoso T R, Castro L B and de Castro A S 2010 *J. Phys. A: Math. Theor.* **43** 055306
[59] Cardoso T R, Castro L B and de Castro A S 2009 *Can. J. Phys.* **87** 857
[60] Cardoso T R, Castro L B and de Castro A S 2012 *J. Phys. A: Math. Theor.* **45** 075302
[61] Castro L B and de Oliveira L P 2014 AdHEP **2014** 784072
[62] Umezawa H 1956 *Quantum Field Theory* (Amsterdam: North-Holland)
[63] Abreu L M, Santos E S and Viana J D M 2010 *J. Phys. A: Math. Theor.* **43** 495402
[64] Andrade F M, Silva E O and Pereira M 2013 *Ann. Phys., NY* **339** 510–30
[65] Park D K and Oh J G 1994 *Phys. Rev.* D **50** 7715
[66] Andrade F M, Silva E O and Pereira M 2012 *Phys. Rev.* D **85** 041701
[67] Gesztesy F, Albeverio S, Hoegh-Krohn R and Holden H 1987 *J. Reine Angew. Math.* **380** 87
[68] Dabrowski L and Stovicek P 1998 *J. Math. Phys.* **39** 47–62
[69] Adami R and Teta A 1998 *Lett. Math. Phys.* **43** 43–54
[70] Bulla W and Gesztesy F 1985 *J. Math. Phys.* **26** 2520
[71] Filgueiras C, Silva E O, Oliveira W and Moraes F 2010 *Ann. Phys., NY* **325** 2529
[72] Silva E O, Andrade F M, Filgueiras C and Belich H 2013 *Eur. Phys. J.* C **73** 2402
[73] Audretsch J, Jasper U and Skarzhinsky V D 1995 *J. Phys. A: Math. Gen.* **28** 2359
[74] Coutinho F A B, Nogami Y and Fernando Perez J 1992 *Phys. Rev.* A **46** 6052
[75] de Oliveira C R and Pereira M 2010 *J. Phys. A: Math. Theor.* **43** 354011
[76] Abramowitz M and Stegun I A (ed) 1972 *Handbook of Mathematical Functions* (New York: Dover)
[77] Bennaceur K, Dobaczewski J and Ploszajczak M 1999 *Phys. Rev.* C **60** 034308