NO UNCOUNTABLE POLISH GROUP CAN BE A RIGHT-ANGLED ARTIN GROUP

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Abstract. We prove that no uncountable Polish group can admit a system of generators whose associated length function satisfies the following conditions:

(i) if $0 < k < \omega$, then $\lg(x) \leq \lg(x^k)$;
(ii) if $\lg(y) < k < \omega$ and $x^k = y$, then $x = e$.

In particular, the automorphism group of a countable structure cannot be an uncountable right-angled Artin group. This generalizes results from [3] and [5], where this is proved for free and free abelian uncountable groups.

In a meeting in Durham in 1997, Evans asked if an uncountable free group can be realized as the group of automorphisms of a countable structure. This was settled in the negative by Shelah [3]. Independently, in the context of descriptive set theory, Becher and Kechris [1] asked if an uncountable Polish group can be free. This was also answered negatively by Shelah [4], generalizing the techniques of [3]. Inspired by the question of Becher and Kechris, Solecki [5] proved that no uncountable Polish group can be free abelian. In this paper we give a general framework for these results, proving that no uncountable Polish group can be a right-angled Artin group (see below for a definition). We actually prove more:

Theorem 1. Let $G = (G, d)$ be an uncountable Polish group and $A$ a group admitting a system of generators whose associated length function satisfies the following conditions:

(i) if $0 < k < \omega$, then $\lg(x) \leq \lg(x^k)$;
(ii) if $\lg(y) < k < \omega$ and $x^k = y$, then $x = e$.

Then $G$ is not isomorphic to $A$, in fact there exists a subgroup $G^*$ of $G$ of size $b$ (the bounding number) such that $G^*$ is not embeddable in $A$.

Proof. Let $\zeta = (\zeta_n)_{n < \omega} \in \mathbb{R}^\omega$ be such that $\zeta_n < 2^{-n}$, for every $n < \omega$, and $g = (g_n)_{n < \omega} \in G^\omega$ such that $g_n \neq e$ and $d(g_n, e) < \zeta_n$, for every $n < \omega$. Let $\Lambda$ be a set of power $b$ of increasing functions $\eta \in \omega^\omega$ which is unbounded with respect to the partial order of eventual domination. For transparency we also assume that for every $\eta \in \Lambda$ we have $\eta(0) > 0$. For $\eta \in \Lambda$, define the following set of equations:

$$\Gamma_\eta = \{x^{\eta(n)}_{n+1} = x_ng_n : n < \omega\}.$$

By [4], for every $\eta \in \Lambda$, $\Gamma_\eta$ is solvable in $G$. Let $b_\eta = (b_{\eta,n})_{n < \omega}$ witness it, i.e.:

$$b_\eta \in G^\omega \quad \text{and} \quad \bigwedge_{n < \omega} b^{\eta(n)}_{\eta,n+1} = b_{\eta,n}g_n.$$

Let $G^*$ be the subgroup of $G$ generated by $\{g_n : n < \omega\} \cup \{b_{\eta,n} : \eta \in \Lambda, n < \omega\}$. Towards contradiction, suppose that $\pi$ is an embedding of $G^*$ into $A$, and let $S$...
be a system of generators for $A$ whose associated length function $l_g = lg$ satisfies conditions (i) and (ii) of the statement of the theorem. For $\eta \in \Lambda$ and $n < \omega$, let:

\[
\pi(g_n) = g'_n, \quad \pi(b_{n,n}) = c_{n,n} \quad \text{and} \quad m_*(\eta) = lg(c_{n,0}).
\]

Now, $m_*$ is a function from $\Lambda$ to $\omega$ and so there exists unbounded $\Lambda_1 \subseteq \Lambda$ such that for every $\eta \in \Lambda_1$ the value $m_*(\eta)$ is a constant $m_*$. Fix such a $\Lambda_1$ and $m_*$, and let $f_1, f_2 \in \omega$ increasing satisfying the following:

1. $f_1(n) > lg(g'_n)$;
2. $f_2(n) = (m_* + 1) + \sum_{\ell < n} f_1(\ell)$.

**Claim 1.** For every $\eta \in \Lambda_1$, $lg(c_{n,n}) < f_2(n)$.

**Proof.** By induction on $n < \omega$. The case $n = 0$ is clear by the choice of $f_1$ and $f_2$. Let $n = m + 1$. Because of assumption (i) on $A$, the choice of $\Lambda_1$ and the choice of $f_1$ and $f_2$, we have:

\[
lg(c_{n,n}) \leq lg(c_{n,n}^{(m)}) = lg(c_{n,m}g_m') \leq lg(c_{n,m}) + lg(g'_m) < f_2(m) + f_1(m) = f_2(n).
\]

Now, by the choice of $\Lambda_1$, we can find $\eta \in \Lambda_1$ and $n < \omega$ such that $\eta(n) > f_2(n + 2)$. Notice then that by the claim above and the choice of $f_1$ and $f_2$ we have:

1. $\eta(n) > f_2(n + 1) = f_2(n) + f_1(n) > lg(c_{n,n}) + lg(g'_n) \geq lg(c_{n,n}g'_n)$;
2. $\eta(n) > f_2(n + 2) \geq f_1(n + 1) > lg(g'_{n+1})$.

Thus, by (1) and the fact that $c_{n,n+1}^{(n)} = c_{n,n}g'_n$, using assumption (ii) we infer that $c_{n,n+1} = e$. Hence, $c_{n,n+2} = c_{n,n+1}g'_n = g'_n$.

Furthermore, if $\eta(n+1) > lg(g'_{n+1})$, then, again by assumption (ii), we have that $c_{n,n+2} = e$, and so $c_{n,n+2}^{(n+1)} = g'_n = e$, which contradicts the choice of $(g_n)_{n<\omega}$. Hence, $\eta(n) < \eta(n+1) \leq lg(g'_{n+1})$, contradicting (2). It follows that the embedding $\pi$ from $G^\eta$ into $A$ cannot exist. \hfill \blacksquare

**Definition 2.** Given a graph $\Gamma = (E, V)$, the right-angled Artin group $A(\Gamma)$ is the group with presentation $\langle V \mid ab = ba : aEb \rangle$.

Thus, for $\Gamma$ a graph with no edges (resp. a complete graph) $A(\Gamma)$ is a free group (resp. a free abelian group).

**Definition 3.** Let $A(\Gamma)$ be a right-angled Artin group and $lg$ its associated length function. We say that an element $g \in A(\Gamma)$ is cyclically reduced if it cannot be written as $g = hfh^{-1}$ with $lg(g) = lg(f) + 2$.

**Fact 4.** Let $A(\Gamma)$ be a right-angled Artin group, $lg$ its associated length function and $g \in A(\Gamma)$. Then:

1. $g$ can be written as $hfh^{-1}$ with $f$ cyclically reduced and $lg(g) = lg(f) + 2lg(h)$;
2. if $0 < k < \omega$ and $f$ is cyclically reduced, then $lg(f^k) = klg(f)$;
3. if $0 < k < \omega$ and $g = hfh^{-1}$ is as in (1), then $lg(hfh^{-1})^k = klg(f) + 2lg(h)$.

**Proof.** Item (1) is proved in [2, Proposition on pg.38]. The rest is folklore. \hfill \blacksquare
Corollary 5. No uncountable Polish group can be a right-angled Artin group.

Proof. By Theorem 1 it suffices to show that for every right-angled Artin group $A(\Gamma)$ the associated length function $lg$ satisfies conditions (i) and (ii) of the theorem, but by Fact 4 this is clear. □

As well known, the automorphism group of a countable structure is naturally endowed with a Polish topology which respects the group structure, hence:

Corollary 6. The automorphism group of a countable structure cannot be an uncountable right-angled Artin group.

The situation is different for right-angled Coxeter groups, in fact the structure $M$ with $\omega$ many disjoint unary predicates of size 2 is such that $Aut(M) = (\mathbb{Z}_2)^\omega$, i.e. $Aut(M)$ is the right-angled Coxeter group on $K_c$ (a complete graph on continuum many vertices). Notice that in this group for any $a \neq b \in K_c$ we have:

(i) $(ab)^2 = 1$;
(ii) $lg(ab) = 2 < 3$, $(ab)^3 = ab$ and $ab \neq e$.

We hope to investigate realizability of uncountable right-angled Coxeter groups as groups of automorphisms of countable structures in a future work.

References

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