On the Geometry of Bäcklund Transformations

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Abstract

The geometry of an admissible Bäcklund transformation for an exterior differential system is described by an admissible Cartan connection for a geometric structure on a tower with infinite–dimensional skeleton which is the universal prolongation of a |1|–graded semi-simple Lie algebra.

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1 Towers on skeletons

In the following we consider infinite–dimensional objects in the p–category of objects obtained as projective limits of finite–dimensional ones [3].

Definition 1 An algebraic skeleton on a finite–dimensional vector space V is a triple (E, G, ρ), with G a p–Lie group, E = V ⊕ g, g the Lie algebra of G, and ρ a representation of G on E such that ρ(g)x = Ad(g)x, for g ∈ G, x ∈ g. An infinitesimal skeleton can be analogously defined via the representation of g on E.

Definition 2 Let (E, G, ρ) be a skeleton on V and Z a manifold of type V [3]. We say that a p–principal fibre bundle P(Z, G) provided with an absolute parallelism ω on P is a tower on Z with skeleton (E, G, ρ) if ω takes values in E and satisfies: \( R^*_g ω = ρ(g)^{-1} ω \), for g ∈ G; ω(\( \tilde{A} \)) = A, for A ∈ g; here \( R^*_g \) denotes the right translation and \( \tilde{A} \) the fundamental vector field induced on P from A.

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1.1 Cartan connections

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{k}$ a Lie subalgebra of $\mathfrak{g}$. Let $K$ be a Lie group with Lie algebra $\mathfrak{k}$ equipped with a representation $Ad : K \to GL(\mathfrak{g})$ such that its differential coincides with the adjoint representation of $\mathfrak{k}$ on $\mathfrak{g}$.

**Definition 3** Let $P(Z, K)$ be a principal fibre bundle over a manifold $Z$ with structure group $K$. A **Cartan connection** in $P$ of type $(\mathfrak{g}, K)$ is a $1$–form $\omega$ on $P$ with values in $\mathfrak{g}$ satisfying the following conditions: $\omega|_{T_\nu P} : T_\nu P \to \mathfrak{g}$ is an isomorphism $\forall \nu \in P$; $R^*_g \omega = Ad(g)^{-1} \omega$ for $g \in K$; $\omega(A) = 0$ for $A \in \mathfrak{k}$.

$(\mathfrak{g}, K, Ad)$ is a skeleton on $V$, with $\mathfrak{g} = \mathfrak{k} \oplus V$. Then it is clear that a Cartan connection $(P, Z, K, \omega)$ of type $(\mathfrak{g}, K)$ is a tower on $Z$.

In the following we assume the Lie algebra $\mathfrak{g}$ to be a generalized semi-simple $|1|$–graded Lie algebra i.e. $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

**Remark 1** According with the Lie algebra $|1|$–grading the Cartan connection form $\omega$ and its curvature $\kappa$ split as $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$ and $\kappa = \kappa_{-1} \oplus \kappa_0 \oplus \kappa_1$.

**Definition 4** Let $G$ be a semi-simple Lie group, with $|1|$–graded Lie algebra $\mathfrak{g}$ as above and $K$ the closed subgroup of $G$ corresponding to the Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. A **$K$–structure** on $Z$ is a principal fiber bundle $P \to Z$ with structure group $K$ equipped with a soldering form $\theta = \theta_{-1} \oplus \theta_0 \in \Omega^1(P, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ such that: $\theta_{-1}(\xi) = 0$, if and only if $\xi$ is a vertical vector; $\theta_0(X + Z) = Y$, $\forall Y \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$; $(R_b)^* \theta = Ad(b^{-1}) \theta$, $\forall b \in K$, where $Ad$ means the action on the vector space $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \simeq \mathfrak{g}/\mathfrak{g}_1$ induced by the adjoint action.

**Definition 5** Let $(P, \theta)$ be a $K$–structure on $Z$. A Cartan connection $\omega$ on $P$ is called **admissible** if and only if it is of the form $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$.

**Remark 2** If the $K$–structure has zero torsion, i.e. if it is a reduction of $L^2(Z) \to Z$ to $K$, then the curvature of the induced Cartan connection $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ is such that $\kappa_{-1} = 0$.

2 Bäcklund transformations and induced Cartan connections

Let $\pi : U \to X$, $\tau : Z \to X$, be two (vector) bundles with local fibered coordinates $(x^\alpha, u^A)$ and $(x^\alpha, z^i)$, respectively, where $\alpha = 1, \ldots, m = \text{dim}X$, $A = 1, \ldots, n = \text{dim}U - \text{dim}X$, $i = 1, \ldots, N = \text{dim}Z - \text{dim}X$. A system of non-linear field equations of order $k$ on $U$ is geometrically described as an exterior differential system $\nu$ on $J^kU$. The solutions of the field equations are (local) sections $\sigma$ of $U \to X$ such that $(j^k \sigma)^* \nu = 0$. We shall also denote by $J^\infty \nu$ (resp. $j^\infty \sigma$) the infinite order jet prolongation of $\nu$ (resp. $\sigma$).
2.1 Admissible Bäcklund transformations

Let $B$ be the infinite–order contact transformations group on $J^\infty U$.

**Definition 6** The group of (infinitesimal) Bäcklund transformations for the system $\nu$ is the closed subgroup $K$ of $B$ which leaves invariant solution submanifolds of $J^\infty \nu$. The group of (infinitesimal) generalized Bäcklund transformations for the system $\nu$ is the closed subgroup $K$ of $B$ which leaves invariant $J^\infty \nu$.

Let $\pi : U \to X$, $\tau : Z \to X$, be vector bundles as the above and $\pi^1 : J^1U \to X$, $\tau^1 : J^1Z \to X$, the first order jet prolongations bundles, with local fibered coordinates $(x^\alpha, u^A, u^{A^i}_i)$, $(x^\alpha, z^i, z^{\alpha i}_i)$, respectively. Furthermore, let $(\partial_\beta, \partial_A, \partial^\beta_A), (\partial_\beta, \partial_i, \partial^\beta_i)$ and $(dx^\beta, du^A, du^{A^i}_i)$, $(dx^\beta, dz^i, dz^{\alpha i}_i)$ be local bases of tangent vector fields and 1–forms on $J^1U$ and $J^1Z$, respectively.

**Definition 7** We define a Bäcklund map to be the fibered morphism over $Z$: $\phi : J^1U \times_X Z \to J^1Z : (x^\alpha, u^A, u^{A^i}_i, z^i) \mapsto (x^\alpha, z^i, z^{\alpha i}_i)$, with $z^{\alpha i}_i = \phi^1_\alpha (x^\beta, u^A, u^{A^i}_i; z^i)$.

The fibered morphism $\phi$ is said to be an admissible Bäcklund transformation for the differential system $\nu$ if $\phi^1_\alpha = D_\alpha \phi^\beta$ and the integrability conditions coincide with the exterior differential system $\nu$.

**Remark 3** By pull–back of the contact structure on $J^1Z$, the Bäcklund morphism induces an horizontal distribution, the induced Bäcklund connection, on the bundle $(J^1U \times_X Z, J^1U, \pi^1_0(\eta))$.

**Theorem 1** The following statements are equivalent:

1. $\phi$ is an admissible Bäcklund transformation for the differential system $\nu$.
2. The induced Bäcklund connection is $K$–invariant, where $K$ is a normal subgroup $K \subset (\tilde{K} \cap K) \subset B$ leaving invariant (the infinite order prolongation of) $\nu$ and its solutions.

Let now $Z$ be a vector bundle over the basis $X$, the fibers of which are modelled over the homogeneous space $\tilde{K}/\hat{K}$ such that fibers are vector spaces of type $\tilde{\mathfrak{r}}_- = \bigoplus_{p<0} \tilde{\mathfrak{r}}_p$, with $\tilde{\mathfrak{r}}_-$ a graded abelian Lie algebra. Let $P$ be a tower on $Z$ with algebraic skeleton $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{r}}, \text{Ad})$, where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{K}$. Suppose $U$ be a vector bundle (over the same basis $X$) with a left action $\lambda$ of $\tilde{K}$ on $U$ (as a manifold). For each tower $(P, Z, \tilde{K}, \omega)$ we have a vector bundle $U_\lambda(Z) = P \times_{\tilde{K}} U$ over $X$ and vice versa.

Assume $\tilde{\mathfrak{g}}$ to be the universal prolongation of a $|1|$–graded semi-simple Lie algebra $\mathfrak{g}$ such that $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{r}}_-$ are (the prolongation of) $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, respectively.

**Theorem 2** A Bäcklund transformation admissible for an exterior differential system induces the tower $(P, Z, \tilde{K}, \omega)$, where $\omega$ is an admissible Cartan connection for a $\tilde{K}$-structure over $Z$.  

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Proof. It follows from Remark 2 and Theorem 1. In fact, a Bäcklund morphism can be seen as a $\bar{K}$–equivariant section of the bundle $U_\lambda(Z) \to Z$; it induces a reduction of $L^2(Z) \to Z$ defining a $\bar{K}$–structure on $Z$ with zero torsion. The admissible Cartan connection is the tower $(P, \bar{K}, \theta)$ induced from a $\mathfrak{g}_0$–principal connection on the underlying first order structure.

As a consequence one can build a cohomological theory of complete integrability for (nonlinear) exterior differential systems. Cohomological conditions are in fact given for a graded simple Lie algebra to be a universal prolongation [7]. This topic will be developed elsewhere.

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