Simultaneous Type A $\mathcal{N}$-fold Supersymmetry with Two Different Values of $\mathcal{N}$

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Abstract

We investigate one-dimensional quantum mechanical systems which have type A $\mathcal{N}$-fold supersymmetry with two different values of $\mathcal{N}$ simultaneously. We find that there are essentially four inequivalent models possessing the property, one is conformal, two of them are hyperbolic (trigonometric) including Rosen–Morse type, and the other is elliptic.

Key words: quantum mechanics, quasi(-exact) solvability, $\mathcal{N}$-fold supersymmetry
PACS: 03.65.Ge, 11.30.Pb, 11.30.Na

1 Introduction

Discovery of quasi-exact solvability in quantum mechanics [1,2] has promoted the investigation of quantum systems which admit exact solutions. Recently, this concept was combined within the framework of $\mathcal{N}$-fold supersymmetry [3], which is a natural generalization [4] of ordinary supersymmetric quantum mechanics [5]. Furthermore, a systematic algorithm for constructing an $\mathcal{N}$-fold supersymmetric system was established based on the connection between (weak) quasi-solvability and $\mathcal{N}$-fold supersymmetry [6]. Up to now, three different families of $\mathcal{N}$-fold supersymmetric systems have been found for arbitrary finite integer $\mathcal{N}$, namely, type A [7,8], type B [9], and type C [6], which have correspondence with the classification of second-order linear differential operators preserving a monomial-type vector space [10].

One of the intriguing aspects of $\mathcal{N}$-fold supersymmetry is its dynamical breaking. We can actually observe the phenomenon through the nonperturbative

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effect due to the quantum tunneling [11,12]. In particular, there are several characteristic features which are different from the case of ordinary supersymmetric quantum mechanical systems [3]. In the case of $\mathcal{N}$-fold supersymmetry, semi-positive definiteness of the spectrum is not guaranteed, more than one state can be invariant with respect to one supercharge, and so on. Actually, it is shown that the latter fact can lead to a novel phenomenon, namely, partial breaking of $\mathcal{N}$-fold supersymmetry [6]. Hence, it is interesting to find out a realistic physical system with $\mathcal{N}$-fold supersymmetry.\(^1\)

On the other hand, ordinary supersymmetric quantum mechanical systems have been extensively studied more than two decades (for a review, see e.g. [14]). In particular, several realistic physical systems having ordinary supersymmetry have been found not only for nonrelativistic scalar systems but also for Pauli and Dirac systems [15,16,17,18,19,20,21,22,23]. Hence, it suggests that we shall begin with looking for an ordinary supersymmetric system which has $\mathcal{N}$-fold supersymmetry as well, as a starting point of the aforementioned purpose. This observation naturally leads us further to consider more general situation where a system has simultaneous $\mathcal{N}$-fold supersymmetry with two different values of $\mathcal{N}$. In this article, we show a family of that kind of quantum mechanical systems.

In the following, we first review the definition of $\mathcal{N}$-fold supersymmetry and extend it to the concept of simultaneous $\mathcal{N}$-fold supersymmetry with two different values of $\mathcal{N}$. In Section 3, we fully investigate the condition for simultaneous $\mathcal{N}$-fold supersymmetry with respect to two different type A $\mathcal{N}$-fold supercharges and classify the models possessing that property. Finally, we summarize the results and discuss some future problems.

## 2 $\mathcal{N}$-fold Supersymmetry and Quasi-solvability

First of all, we shall review the concept of $\mathcal{N}$-fold supersymmetry in one-dimensional quantum mechanics. Let $q$ denote a bosonic coordinate, and let $\psi$ and $\psi^\dagger$ be fermionic coordinates satisfying

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1. \quad (1)$$

We define a super-Hamiltonian $H$ by

$$H = H^- \psi \psi^\dagger + H^+ \psi^\dagger \psi, \quad (2)$$

\(^1\) The models in Ref. [13] have unphysical magnetic moment.
where $H^\pm$ is a pair of ordinary scalar Hamiltonians:

$$H^\pm = -\frac{1}{2} \frac{d^2}{dq^2} + V^\pm_N(q). \quad (3)$$

With a monic $\mathcal{N}$th-order linear differential operator

$$P_N = \frac{d^N}{dq^N} + \sum_{k=0}^{N-1} w_k(q) \frac{d^k}{dq^k}, \quad (4)$$

$\mathcal{N}$-fold supercharges $Q^\pm_N$ are introduced by

$$Q^-_N = P^-_N \psi^t, \quad Q^+_N = P^+_N \psi, \quad (5)$$

where the operators $P^\pm_N$ are defined by

$$P^-_N = P_N, \quad P^+_N = P^t_N, \quad (6)$$

the superscript $t$ denoting the formal transposition. Then, the system $\mathbf{H}$ is said to have $\mathcal{N}$-fold supersymmetry if it satisfies

$$[Q^\pm_N, H] = 0. \quad (7)$$

One of the most important aspects of $\mathcal{N}$-fold supersymmetry is that the component Hamiltonians $H^-$ and $H^+$ are always weakly quasi-solvable with respect to the operators $P^-_N$ and $P^+_N$, respectively [3,8]. That is, $H^\pm$ leave the kernels of $P^\pm_N$ invariant:

$$H^\pm \mathcal{V}^\pm_N \subset \mathcal{V}^\pm_N, \quad \mathcal{V}^\pm_N = \ker P^\pm_N. \quad (8)$$

As a consequence, we can in principle diagonalize algebraically the Hamiltonians $H^\pm$ in the finite $\mathcal{N}$-dimensional vector spaces $\mathcal{V}^\pm_N$, which are thus called solvable sectors of $H^\pm$. If the space $\mathcal{V}^\pm_N^{(-)}$ is a subspace of a Hilbert space $L^2$ on which the Hamiltonian $H^{(+)}$ is defined, the elements of $\mathcal{V}^\pm_N^{(+)}$ provide a part of the exact eigenfunctions and thus $H^{(+)}$ is called quasi-exactly solvable.

Next, the system $\mathbf{H}$ is said to have $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetry if it commutes with two different $\mathcal{N}_i$-fold supercharges ($i = 1, 2$) simultaneously, namely,

$$[Q^{(i)}_{N_1}^\pm, \mathbf{H}] = [Q^{(i)}_{N_2}^\pm, \mathbf{H}] = 0. \quad (9)$$

Without loss of generality, we can assume $\mathcal{N}_1 \geq \mathcal{N}_2$. In this case, it is evident that the components $H^\pm$ of the system $\mathbf{H}$ preserve two vector spaces $\mathcal{V}^{(i)}_{N_1} = \ker P^{(i)}_{N_1} = \ker P^{(i)}_{N_2}$ separately, where $P^{(i)}_{N_i}$ are components of $Q^{(i)}_{N_i}$ ($i = 1, 2$), cf. Eq. (5). Hence, the solvable sectors $\mathcal{V}^{\pm}_{N_1,N_2}$ of $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetric Hamiltonians $H^\pm$ are generally given by

$$\mathcal{V}^\pm_{N_1,N_2} = \mathcal{V}^{(1)}_{N_1} \cup \mathcal{V}^{(2)}_{N_2}. \quad (10)$$
3 Type A \((\mathcal{N}_1, \mathcal{N}_2)\)-fold Supersymmetry

In what follows, we shall investigate the \((\mathcal{N}_1, \mathcal{N}_2)\)-fold supersymmetric systems with respect to two type A \(\mathcal{N}_i\)-fold supercharges \((i = 1, 2)\), which we hereafter call type A \((\mathcal{N}_1, \mathcal{N}_2)\)-fold supersymmetric. It was briefly discussed in Ref. [8] in a different context, but here we will make a precise treatment. The component of the type A \(\mathcal{N}_i\)-fold supercharge is defined by

\[
P_{\mathcal{N}_i}^{(A)} = \left( \frac{d}{dq} + W_i(q) - \frac{\mathcal{N}_i - 1}{2} E_i(q) \right) \left( \frac{d}{dq} + W_i(q) - \frac{\mathcal{N}_i - 3}{2} E_i(q) \right) \times \cdots \]

\[
\cdots \times \left( \frac{d}{dq} + W_i(q) + \frac{\mathcal{N}_i - 3}{2} E_i(q) \right) \left( \frac{d}{dq} + W_i(q) + \frac{\mathcal{N}_i - 1}{2} E_i(q) \right). \tag{11}
\]

According to Ref. [8], the necessary and sufficient condition for type A \(\mathcal{N}_i\)-fold supersymmetry is the following:

\[
V_{\mathcal{N}_i}^{\pm}(q) = \frac{1}{2} W_i(q)^2 - \frac{\mathcal{N}_i^2 - 1}{24} \left( 2\dot{E}_i(q) - E_i(q)^2 \right) \pm \frac{\mathcal{N}_i}{2} \ddot{W}_i(q) - R_i, \tag{12}
\]

where the dot denotes derivative with respect to \(q\), \(R_i\) is a constant, and the functions \(W_i(q)\) and \(E_i(q)\) satisfy

\[
\left( \frac{d}{dq} - E_i(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E_i(q) \right) W_i(q) = 0 \text{ for } \mathcal{N} \geq 2, \tag{13}
\]

\[
\left( \frac{d}{dq} - 2E_i(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E_i(q) \right) E_i(q) = 0 \text{ for } \mathcal{N} \geq 3. \tag{14}
\]

In order that the pair of potentials \(V_{\mathcal{N}_1}^{\pm}\) be type A \((\mathcal{N}_1, \mathcal{N}_2)\)-fold supersymmetric, \(V_{\mathcal{N}_1}^{\pm}\) and \(V_{\mathcal{N}_2}^{\pm}\) must be identical up to an additive constant, namely, \(V_{\mathcal{N}_1}^{\pm} + R_3 = V_{\mathcal{N}_2}^{\pm} \equiv V_{\mathcal{N}_1, \mathcal{N}_2}^{\pm}\), where \(R_3\) is a constant. From Eq. (12) we immediately have

\[
\mathcal{N}_1 \dot{W}_1 = \mathcal{N}_2 \dot{W}_2, \tag{15}
\]

\[
W_1^2 - \frac{\mathcal{N}_1^2 - 1}{12} (2\dot{E}_1 - E_1^2) - 2R = W_2^2 - \frac{\mathcal{N}_2^2 - 1}{12} (2\dot{E}_2 - E_2^2), \tag{16}
\]

where \(R = R_1 + R_3 - R_2\). The first condition (15) is easily integrated as

\[
W_2 = \frac{\mathcal{N}_1}{\mathcal{N}_2} W_1 + C, \tag{17}
\]

where \(C\) is a constant. We note that we can assume

\[
2\dot{E}_i - E_i^2 \neq \text{const.} \iff \dot{E}_i - E_i \dot{E}_i \neq 0; \tag{18}
\]

otherwise, from Eqs. (16) and (17) we have \(W_i = \text{const.}\), which results in a trivial model \(V_{\mathcal{N}_1, \mathcal{N}_2}^{\pm} = \text{const}\). Noting further that \(W_i\) satisfy Eq. (13), we
conclude from the relation (17) that under the assumption (18)

$$2\dot{E}_1 - E_1^2 = 2\dot{E}_2 - E_2^2 (\neq \text{const.}),$$

(19)

and $C = 0$. To investigate the condition (19), it is more convenient to recall the fact that the type A Hamiltonians can be represented as [8]

$$H^\pm = e^{-\mathcal{W}_{R_i}^A} \left[ -A_i(z_i) \frac{d^2}{dz_i^2} + \left( \frac{N_i - 2}{2} A'_i(z_i) \pm Q_i(z_i) \right) \frac{d}{dz_i} \right. \\
\left. - \frac{(N_i - 1)(N_i - 2)}{12} A''_i(z_i) \pm \frac{N_i - 1}{2} Q'_i(z_i) - R_i \right] e^{\mathcal{W}_{R_i}^A},$$

(20)

where the prime denotes derivative with respect to $z_i$, $A_i(z_i)$ and $Q_i(z_i)$ are polynomials of at most fourth- and second-degree, respectively, and related to $E_i(q)$ and $W_i(q)$ by

$$A_i(z_i) = \frac{1}{2} (\dot{z}_i)^2 = a_4^{(i)} z_i^4 + a_3^{(i)} z_i^3 + a_2^{(i)} z_i^2 + a_1^{(i)} z_i + a_0^{(i)},$$

(21)

$$A'_i(z_i) = \ddot{z}_i = E_i \dot{z}_i,$$  

(22)

$$Q_i(z_i) = -W_i \dot{z}_i = b_2^{(i)} z_i^2 + b_1^{(i)} z_i + b_0^{(i)}.$$  

(23)

The gauge potentials $\mathcal{W}_{R_i}^A$ in Eq. (20) are given by

$$\mathcal{W}_{R_i}^A = \frac{N_i - 1}{2} \int dq E_i \mp \int dq W_i.$$  

(24)

From the relation (22), we obtain

$$2\dot{E}_i - E_i^2 = \frac{4H[A_i]}{(\dot{z}_i)^2} = \frac{1}{(\dot{z}_i)^2} \left[ 4A_i A''_i - 3(A'_i)^2 \right],$$

(25)

$$\dot{E}_i - E_i \dot{E}_i = -\frac{24 T[A_i]}{(\dot{z}_i)^3} = \frac{1}{(\dot{z}_i)^3} \left[ 4A_i^2 A''_i - 6A_i A'_i A''_i + 3(A'_i)^3 \right],$$

(26)

where $H[A_i]$ and $T[A_i]$ are algebraic covariants called the Hessian of $A_i$ and the Jacobian of $A_i$ and $H[A_i]$, respectively [24,25]. Thus, we see from Eqs. (21) and (25) that the condition (19) is equivalent to $H[A_1]/A_1 = H[A_2]/A_2 (\neq \text{const.})$. The latter equation is satisfied if and only if $A_1(z_1) = A_2(z_2)$ and $(\dot{z}_1)^2 = (\dot{z}_2)^2$. Hence, we have

$$z_1 = z_2 \equiv z, \quad A_1(z) = A_2(z) \equiv A(z).$$

(27)

\footnote{Here we fix the irrelevant multiplicative constants for $H[A_i]$ and $T[A_i]$ after Refs. [8,26], which are different from those in Refs. [24,25].}

\footnote{More precisely, we have $z_1 = \pm z_2 + z_0$ where $z_0$ is a constant. But we can always redefine the variables with the aid of the $GL(2,K)$ transformation (30) and (31), under which the Hamiltonians (20) are invariant, so that $z_1 = z_2$.}
In this case, Eq. (22) implies

\[ E_1 = E_2 = E = \ddot{z} \dot{z}. \]  \hspace{1cm} (28)

From this relation together with Eq. (17) \((C = 0)\) two supercharges coincide when \(\mathcal{N}_1 = \mathcal{N}_2\). Thus, we hereafter assume \(\mathcal{N}_1 > \mathcal{N}_2\). With the use of Eqs. (17) with \(C = 0\), (19), (21), (23), and (25)–(28), the condition (16) and the assumption (18) are rewritten as

\[ Q^2_1 + \frac{\mathcal{N}_2^2}{3} H[A] + \frac{4\mathcal{N}_2^2 R}{\mathcal{N}_1^2 - \mathcal{N}_2^2} A = 0, \quad T[A] \neq 0. \]  \hspace{1cm} (29)

We note that the condition (29) is expressed as algebraically covariant form under the projective transformations \(GL(2, K)\) \((K = \mathbb{R} \text{ or } \mathbb{C})\) on \(A(z)\) and \(Q_1(z)\) (cf. Refs. \([8,26]\)):

\[ A(z) \mapsto \hat{A}(z) = \Delta^{-2}(\gamma z + \delta)^4 A \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right), \]  \hspace{1cm} (30)

\[ Q_1(z) \mapsto \hat{Q}_1(z) = \Delta^{-1}(\gamma z + \delta)^2 Q_1 \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right), \]  \hspace{1cm} (31)

where \(\alpha, \beta, \gamma, \delta \in K\) and \(\Delta = \alpha \delta - \beta \gamma \neq 0\). We can easily show that the polynomial \(A(z)\) of at most fourth-degree is transformed to one of the five canonical forms listed in Table 1 by the \(GL(2, \mathbb{C})\) transformation. Hence, it is sufficient for us to examine each of the five cases separately.

| Case | Canonical Form | \(H[A]\) | \(T[A]\) |
|------|---------------|----------|---------|
| I 1/2 | 0             | 0        |
| II 2z  | -3            | nonzero  |
| III \(2\nu z^2\) | \(-4\nu^2 z^2\) | 0        |
| IV \(2\nu(z^2 - 1)\) | \(-4\nu^2(z^2 + 2)\) | nonzero  |
| V \(2z^3 - g_2 z/2 - g_3/2\) | \(-3z^3 - 3g_2 z^2/2 - 6g_3 z - 3g_3^2/16\) | nonzero  |

Table 1

Canonical forms of \(A(z)\) and their corresponding \(H[A]\) and \(T[A]\). The parameters \(\nu, g_2, g_3 \in \mathbb{C}\) satisfy \(\nu \neq 0\) and \(g_3^2 - 27g_2^3 \neq 0\).

From the last column of Table 1, we see that for Cases I and III \(T[A] = 0\) and thus we obtain trivial models. Therefore, we need to investigate the condition (29) only for Cases II, IV, and V.

Before proceeding to the case-by-case study, we shall investigate the solvable sectors \(\mathcal{V}^{(A)}_{\mathcal{N}_1, \mathcal{N}_2}^{(A)}\) of the type A \((\mathcal{N}_1, \mathcal{N}_2)\)-fold supersymmetric Hamiltonians \(H^{(A)}_{\mathcal{N}_1, \mathcal{N}_2}\), namely, the vector spaces preserved by \(H^{(A)}_{\mathcal{N}_1, \mathcal{N}_2}\). Since type A \(\mathcal{N}_i\)-fold
supersymmetric Hamiltonian preserves the kernel of type A $N$-fold supercharge, we have

$$V_{N_1,N_2}^{(A)\pm} = V_{N_1}^{(A)\pm} \cup V_{N_2}^{(A)\pm},$$

(32)

where $V_{N_i}^{(A)\pm}$ are given by

$$V_{N_i}^{(A)\pm} = \ker D_{N_i}^{(A)\pm} = e^{-W_{N_i}^{\pm}} \langle 1, z, \ldots, z^{N_i-1} \rangle.$$  

(33)

Substituting Eqs. (17), (24) and (28) into the latter equation, we finally obtain

$$V_{N_1}^{(A)\pm} = (z) -\frac{N_1-1}{2} \exp \left( \pm \int dq W_1 \right) \langle 1, z, \ldots, z^{N_1-1} \rangle,$$

(34)

$$V_{N_2}^{(A)\pm} = (z) -\frac{N_2-1}{2} \exp \left( \pm \frac{N_1}{N_2} \int dq W_1 \right) \langle 1, z, \ldots, z^{N_2-1} \rangle.$$  

(35)

4 Classification of the Models

In this section, we shall present a detailed classification of the three non-trivial cases characterized by nonzero $T[A]$ in Table 1.

4.1 Case II: $A(z) = 2z$, $z(q) = q^2$

In this case, the condition (29) is satisfied only if

$$b_2^{(1)} = b_1^{(1)} = 0, \quad R = 0, \quad b_0^{(1)2} = N_2^2.$$  

(36)

We set $b_0^{(1)} = N_2$ without loss of generality. The functions $E$ and $W_1$ in the supercharge, potentials, and solvable sectors are given by the followings.

Supercharge:

$$E(q) = \frac{1}{q}, \quad W_1(q) = -\frac{N_2}{2q}.$$  

(37)

Potentials:

$$V_{N_1,N_2}^{\pm} (q) = \frac{(N_1 \pm N_2 - 1)(N_1 \pm N_2 + 1)}{8q^2}.$$  

(38)

Solvable sectors:

$$V_{N_i}^{(A)\pm} = q^{\frac{N_i+N_{i+1}-1}{2}} \langle 1, q^2, \ldots, q^{2(N_i-1)} \rangle \quad (i = 1, 2).$$  

(39)
4.2 Case IV: $A(z) = 2\nu(z^2 - 1)$, $z(q) = \cosh 2\sqrt{\nu}q$

In this case, the condition (29) is satisfied only if

$$b_2^{(1)} = 0, \quad b_1^{(1)2} - \frac{4N_2^2\nu^2}{3} + \frac{8N_2^2R\nu}{N_1^2 - N_2^2} = 0, \quad b_1^{(1)}b_0^{(1)} = 0, \quad b_0^{(1)2} - \frac{8N_2^2\nu^2}{3} - \frac{8N_2^2R\nu}{N_1^2 - N_2^2} = 0. \quad (40)$$

Hence, we shall consider in what follows the two cases separately, namely, $b_0^{(1)} = 0$ first and $b_1^{(1)} = 0$ next. When $b_0^{(1)} = 0$, the condition (40) is equivalent to

$$b_0^{(1)} = 0, \quad b_1^{(1)2} = 4N_2^2\nu^2, \quad R = \frac{N_1^2 - N_2^2}{3}\nu. \quad (41)$$

We set $b_1^{(1)} = 2N_2\nu$ without loss of generality. The functions $E$ and $W_1$ in the supercharge, potentials, and solvable sectors are given by the followings.

**Supercharge:**

$$E(q) = \frac{2\sqrt{\nu}\cosh 2\sqrt{\nu}q}{\sinh 2\sqrt{\nu}q}, \quad W_1(q) = -\frac{N_2\sqrt{\nu}\cosh 2\sqrt{\nu}q}{\sinh 2\sqrt{\nu}q}. \quad (42)$$

**Potentials:**

$$V_{N_1,N_2}^{\pm} = \frac{(N_1 \pm N_2 - 1)(N_1 \pm N_2 + 1)\nu}{2\sinh^2 2\sqrt{\nu}q} + \frac{3N_1^2 + N_2^2 - 1}{6}\nu. \quad (43)$$

**Solvable sectors:**

$$V_{N_1}^{(4)\pm} = (\sinh 2\sqrt{\nu}q)\frac{N_1 \pm N_1 - 1}{2}\quad (i = 1, 2). \quad (44)$$

Next, when $b_0^{(0)} = 0$, the condition (40) is equivalent to

$$b_1^{(0)} = 0, \quad b_0^{(0)2} = 4N_2^2\nu^2, \quad R = \frac{N_1^2 - N_2^2}{6}\nu. \quad (45)$$

We set $b_0^{(1)} = 2N_2\nu$ without loss of generality. The functions $E$ and $W_1$ in the supercharge, potentials, and solvable sectors are given by the followings.

**Supercharge:**

$$E(q) = \frac{2\sqrt{\nu}\cosh 2\sqrt{\nu}q}{\sinh 2\sqrt{\nu}q}, \quad W_1(q) = -\frac{N_2\sqrt{\nu}}{\sinh 2\sqrt{\nu}q}. \quad (46)$$
Potentials:

\[ V_{N_1,N_2}^\pm = \frac{(N_1 \mp N_2 - 1)(N_1 \mp N_2 + 1)}{2 \sinh^2 2\sqrt{\nu q}} \nu \pm \frac{N_1 N_2 \nu}{2 \sinh^2 \nu q} + \frac{N_2^2 - 1}{6} \nu. \]  

(47)

Solvable sectors:

\[ V_{N_1}^{(4)\pm} = (\sinh 2\sqrt{\nu q})^{\frac{N_{1}^2 - 1}{2}} (\tanh \sqrt{\nu q})^{\frac{N_{2}^2 - 1}{2}} \times \langle 1, \cosh 2\sqrt{\nu q}, \ldots, (\cosh 2\sqrt{\nu q})^{N_1 - 1} \rangle \ (i = 1, 2). \]

(48)

4.3 Case V: \( A(z) = 2z^3 - g_2 z/2 - g_3/2, z(q) = \wp(q) \)

In this case, the condition (29) is satisfied only if

\[ \begin{align*}
& b_2^{(1)^2} = N_{2}^{2}, \quad 2b_2^{(1)}b_1^{(1)} + \frac{8N_{2}^2 R}{N_{1}^2 - N_{2}^2} = 0, \quad 2b_2^{(1)}b_0^{(1)} + b_1^{(1)^2} - \frac{N_{2}^2 g_2}{2} = 0, \\
& 2b_1^{(1)}b_0^{(1)} - 2N_{2}^2 g_3 - \frac{2N_{2}^2 R g_2}{N_{1}^2 - N_{2}^2} = 0, \quad b_0^{(1)^2} - \frac{N_{2}^2 g_2^2}{16} - \frac{2N_{2}^2 R g_3}{N_{1}^2 - N_{2}^2} = 0.
\end{align*} \]

(49)

We set \( b_2^{(1)} = N_{2} \) without loss of generality. Then, the latter set of conditions is equivalent to

\[ \begin{align*}
& b_2^{(1)} = N_{2}, \quad b_1^{(1)} = -\frac{4N_{2}}{N_{1}^2 - N_{2}^2} R, \quad b_0^{(1)} = -\frac{N_{2}}{4} g_2 - \frac{N_{2}(N_{1}^2 - N_{2}^2)}{4R} g_3, \\
& \frac{32R^3}{(N_{1}^2 - N_{2}^2)^3} - \frac{2R}{N_{1}^2 - N_{2}^2} g_2 - g_3 = 0.
\end{align*} \]

(50)

(51)

If we introduce the values of the Weierstrass function \( \wp(q) \) at the half of the fundamental periods 2\( \omega_l \)

\[ e_l = \wp(\omega_l) \ (l = 1, 2, 3), \]

(52)

which all satisfy the algebraic equation of third-degree \( 4e_l^3 - e_l g_2 - g_3 = 0 \), a solution of Eq. (51) is represented as

\[ R = \frac{N_{1}^2 - N_{2}^2}{2} e_k, \]

(53)

where \( k = 1, 2, \) or 3. Substituting the latter expression for \( R \) into Eq. (50), we obtain

\[ \begin{align*}
& b_2^{(1)} = -2N_{2}e_k, \quad b_1^{(1)} = -N_{2}(H_k^2 - e_k^2), \quad b_0^{(1)} = -N_{2}(H_k^2 - e_k^2),
\end{align*} \]

(54)

where \( H_k^2 \) is defined by

\[ H_k^2 = 3e_l^2 - \frac{g_2}{4} = (e_l - e_m)(e_l - e_n) \ (l = 1, 2, 3; \ l \neq m \neq n \neq l). \]

(55)
Finally, the functions $E$ and $W_1$ in the supercharge, potentials, and solvable sectors are given by the followings.

**Supercharge:**

$$E(q) = \frac{12\varphi(q)^2 - g^2}{2\varphi'(q)}, \quad W_1(q) = -N_2\frac{\varphi(q)^2 - 2e_k\varphi(q) - H_k^2 + e_k^2}{\varphi'(q)}. \quad (56)$$

**Potentials:**

$$V_{N_1,N_2}^\pm = \frac{(N_1 + N_2 - 1)(N_1 + N_2 + 1)}{8} \left( \varphi(q) + \sum_{l=1}^{3} \frac{H_l^4 - 2e_l^2H_l^2 + 4e_l^4}{H_l^2(\varphi(q) - e_l)} \right)$$

$$\mp \frac{N_1N_2}{2} \sum_{l=1}^{3} \frac{e_k^2H_l^2 + (e_k + 2e_l)e_l(5H_l^2 - 12e_l^2)}{H_l^2(\varphi(q) - e_l)} - \frac{N_1(N_1 \pm N_2)}{2} e_k. \quad (57)$$

**Solvable sectors:**

$$V_{N_i}^{(A)\pm} = \prod_{l=1}^{3} \left( \varphi(q) - e_l \right)^{-N_i - 1} \left( \varphi(q) - e_l \right)^{-N_i - 1}$$

$$\times \langle 1, \varphi(q), \ldots, \varphi(q)^{N_i - 1} \rangle \quad (i = 1, 2). \quad (58)$$

5 Concluding Remarks

In this article, we have solved the condition for simultaneous $\mathcal{N}$-fold supersymmetry with respect to two type A $\mathcal{N}$-fold supercharges and classified all the possible models. It turns out that there are essentially four inequivalent models possessing the property, the first is conformal, the second is Rosen–Morse type, the third is another hyperbolic ($\nu > 0$) or trigonometric ($\nu < 0$) type, and the last is elliptic. However, it does not mean that we have exhausted the investigation for $(N_1, N_2)$-fold supersymmetric models. Indeed, it is interesting to study possibility for another family of simultaneous $\mathcal{N}$-fold supersymmetry, namely, with respect to two type B, two type C, or two different types of $\mathcal{N}$-fold supercharges.

One may be curious about the relation between type A $(N_1, N_2)$-fold supersymmetry investigated here and type C $\mathcal{N}$-fold supersymmetry with $\mathcal{N} = N_1 + N_2$ in Ref. [6] since they are similar in the sense that both of them preserve two type A monomial spaces of dimension $N_1$ and $N_2$. In this respect, we would like to stress at least three significant differences between them. First, a pair of Hamiltonians of the former is related by $N_1$th- and $N_2$th-order differential operators while that of the latter is by a differential operator of order $N_1 + N_2$. Second, the gauge potentials which connect the spaces acted by the
Hamiltonians and the monomial spaces are different between the former and the latter systems. Third, solvable sectors of type C $\mathcal{N}$-fold supersymmetry always decomposes as a direct sum of two spaces while those of type A ($\mathcal{N}_1, \mathcal{N}_2$)-fold supersymmetry may not, that is, the two spaces which together constitute it through Eq. (32) can have nontrivial intersection, $\mathcal{V}_{\mathcal{A}_1}^{(A)\pm} \cap \mathcal{V}_{\mathcal{A}_2}^{(A)\pm} \neq \emptyset$.

In this article, we have not touched upon underlying mathematical details such as those investigated in Ref. [27] since our present motivation rather comes from physical applications. Thus, it is interesting to see, for instance, what kind of supercharges we will obtain for our case after the process of the optimization of supercharges in Ref. [27].

Finally, we note that we have not restricted the potentials to be real so that we could classify the models into less number of inequivalent classes. Hence, some of the models presented here may not describe a realistic physical system, which is one of the reason why we have avoided to discuss normalizability of the solvable sectors. In this respect, we have found that a certain subclass of the present models would actually fit in describing a physical system and would lead to dynamical (non-)breaking of $\mathcal{N}$-fold supersymmetry, which we would like to report in a subsequent publication [28].

Acknowledgements

This work was partially supported by the National Science Council of the Republic of China under the Grant No. NSC-93-2112-M-032-009.

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11
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