Nonstandard Cosmology With Constant and Variable Gravitational and Variable Cosmological “Constants” and Bulk Viscosity

A thesis submitted for the degree of Doctor of Philosophy (Physics)

by

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We have analyzed a nonsingular model with a variable cosmological term following the Carvalho et al. ansatz. The model was shown to approximate to the model of Freese et al. in one direction and to the Özer-Taha in the other. We have then included the effect of viscosity in this cosmology, as this effect has not been considered before. The analysis showed that this viscous effect could be important with a present contribution to the cosmic pressure, at most, of order of that of radiation. The model puts a stronger upper bound on the baryonic matter than that required by the standard model. A variable gravitational and cosmological constant were then introduced in a scenario which conserves the energy and momentum in the presence of bulk viscosity. The result of the analysis reveals that various models could be viscous. A noteworthy result is that some nonsingular closed models evolve asymptotically into a singular viscous one. The considered models solve for many of the standard model problems. Though the introduction of bulk viscosity results in the creation of particles, this scenario conserves energy and momentum. As in the standard model the entropy remains constant. We have not explained the generation of bulk viscosity but some workers attributes this to neutrinos. Though the role of viscosity today is minute it could, nevertheless, have had an important contribution at early times. We have shown that these models encompass many of the old and recently proposed models, in particular, Brans-Dicke, Dirac, Freese et al., Berman, Abdel Rahman and Kalligas et al. models. Hence we claim that the introduction of bulk viscosity enriches the adopted cosmology.
Chapter 1

Introduction

The standard model was successful in describing the Universe, since it predicts the existence of the observed cosmic microwave background radiation having a temperature of 2.75K. The other success was the formation of light elements during the first few minutes after the big bang. However, this model is fraught with some vexing problems. These problems remain unsolved within the context of the standard model. An extension to this model becomes inevitable, this extension may include an extension to the theory of general relativity.

Einstein introduced a cosmological term to the field equations to obtain a static Universe, but later, following the discovery of cosmic expansion, discarded this term. However, there are good reasons to believe that this term had an appreciable contribution at the beginning of the Universe. From a point of view of quantum field theory this term could correspond to a vacuum energy which was found to be very much bigger than the observed value. This discrepancy is known as the cosmological constant problem. In Particle Physics, the solution of this problem amounts to finding a mechanism that requires this term to vanish. An alternative to this view is to assume that as the Universe evolves, this term evolves and decreases to its present value. According to this view the cosmological term is small because the Universe is too old. Recently, cosmological models with this term decreasing with time were proposed (Özer & Taha, Freese et al., Gasperini, Chen & Wu, Abdel Rahman, Carvalho et al.). These models are known as vacuum decaying models.

In Chapter 2 and 3, we review the standard model concentrating on its successes and shortcomings. Chapter 4 deals with a general review to the nonstandard cosmology. A nonsingular model depending on the Carvalho et al. ansatz ($\Lambda = 3\beta H^2 + \frac{3\alpha}{R^2}$, where $\alpha, \beta$ are arbitrary dimensionless constants) is introduced in Chapter 5. In Chapter 6 we present a closed nonsingular viscous model with a cosmological constant of the Chen and Wu type and discuss its implications for the evolution of the Universe. Chapter 7 is devoted to a singular model with varying cosmological and gravitational constants containing bulk
viscosity varying as $\rho^n$, where $\rho$ is the density of matter and $n$ is a positive number. Various models are reproduced with appropriate values for $n$. A flat viscous model with $\beta = 0$ is presented in Chapter 8. The result of the analysis shows that $G$ increases with time in both radiation dominated and matter dominated phases of cosmic evolution. These results are in accordance with the previously found results of chapter 7. An appendix followed by some published papers is presented at the end of the thesis.

Finally we remark that in these cosmologies we have not discussed the issue of influence of bulk viscosity on the formation of galaxies. We will treat this in a future work.
Chapter 2

The Standard Model

2.1 Background

A cosmological model is a model of our Universe, taking into account and using all known physical laws, predicts (approximately) correctly the observed properties of the Universe, and in particular explains in detail the phenomena in the early Universe. In a more restricted sense cosmological models are solutions of Einstein’s field equations for a perfect fluid that reproduce the important features of our Universe. All cosmological models which differ only near the origin of the Universe must be accepted as equally valid. In fact a series of solutions are known which are initially inhomogeneous or anisotropic to a high degree, and which then increasingly come to approximate the observed Universe. All cosmological models which yield a red-shift($z$) and a cosmic background radiation can hardly be refuted. The possibility cannot be excluded that our Universe is neither homogeneous nor isotropic, but has those properties only approximately in our neighborhood. Every model is of course a great simplification of reality, and only by the study of many solutions can one establish which simplifications are allowed and which assumptions are essential.

2.2 Einstein’s field equations

These equations describe the gravitational field resulting from the distribution of matter in the Universe. They are nonlinear partial differential equations.

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \]  \hspace{1cm} (2.1)

where \( R_{\mu\nu} \) is the Ricci tensor and \( R \) is the curvature scalar, \( T_{\mu\nu} \) is the energy-momentum tensor of the source producing the gravitational field, and \( G \) is Newton’s gravitational constant.

The energy momentum tensor satisfies the following requirements.
(i) $T_{\mu\nu}$ is symmetric with respect to interchange of $\mu$ and $\nu$.

(ii) $T_{\mu\nu}$ is divergenceless, for energy and momentum to be conserved, (Bianchi identity):

$$T^{\mu\nu} ; \nu = 0$$  \hspace{1cm} (2.2)

The Ricci tensor is defined by

$$R_{\mu\nu} = \frac{\partial^2 \ln \sqrt{-g}}{\partial x^\mu \partial x^\nu} - \frac{\partial \Gamma_{\mu\ell}^{\ell}}{\partial x^\mu} + \Gamma_{\mu n}^{\nu n} \Gamma_{\nu m}^{\ell} - \Gamma_{\ell m}^{\nu} \frac{\partial \ln \sqrt{-g}}{\partial x^\ell},$$  \hspace{1cm} (2.3)

and

$$R = g^{\mu\nu} R_{\mu\nu},$$  \hspace{1cm} (2.4)

where $g_{\mu\nu}$ is the metric tensor, $g$ its determinant and $\Gamma_{\mu\nu}^{\ell}$ are the Christoffel symbols related to $g_{\mu\nu}$ by

$$\Gamma_{\mu\nu}^{\ell} = g^{\nu\alpha} (g_{\alpha\ell,j} + g_{\alpha k,j} - g_{jk,\alpha}).$$  \hspace{1cm} (2.5)

### 2.3 Robertson-Walker Metric

The distribution of matter and radiation in the observable Universe is homogeneous and isotropic. While this by no means guarantees that the entire Universe is smooth, it does imply that a region at least as large as our present observable Universe (Hubble’s volume) is smooth. So long as the Universe is spatially homogeneous and isotropic on a scale as large as Hubble’s volume, for purposes of our local Hubble’s volume we may assume that the entire Universe is homogeneous and isotropic. This is known as the Cosmological Principle. The metric for the space with homogeneous and isotropic sections which being maximally symmetric is the Robertson-Walker (RW),

$$ds^2 = dt^2 - R^2(t) \left( \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right)$$  \hspace{1cm} (2.6)

where $(t, r, \theta, \phi)$ are coordinates (referred to as comoving coordinates), $R(t)$ is the cosmic scale factor, and with appropriate scaling of the coordinates, $k$ can be chosen to be 1, -1, or 0 for spaces of constant positive (closed), negative (open), or spatial (flat) curvature, respectively. The time coordinate in the above equation is the proper (or clock) time- measured by an observer at rest in the comoving frame, i.e. $(r, \theta, \phi) = \text{constant}$

### 2.4 Expansion of the Universe

The RW line element contains the function $R(t)$, which can be any function of time, $t$. This line element necessarily requires that the Universe cannot be static. Because of the presence of the term
$R(t)$, an element of the spatial distance $dl$ changes with time, i.e. the distance between any pair of galaxies changes with time. Because of homogeneity, the cosmological fluid can not sustain any pressure gradient, the concomitant non-gravitational forces are absent. But when the only force present is gravitation, a static Universe is evidently not possible, it must either contract under gravity or expand against gravity. However, the analysis of the observed data from distant galaxies shows that the Universe is expanding. The recession velocity of a galaxy turns out to be related to its distance ($d$) by a simple law $v = Hd$ where the proportionality constant $H$ is the Hubble constant. Note that the recession has no effect whatsoever on individual bodies since a homogeneous medium generates no gravitational field inside a spherical cavity.

### 2.5 Distance measurements in RW metric

The determination of distance in astronomy is mostly done using the concepts and ideas of a three-dimensional Euclidean space. We therefore want to describe briefly how the laws of light propagation in RW metrics influence this distance. One possible way of determining the distance of an object is to compare its absolute brightness $L$, which is defined as the total radiated energy per unit time and is regarded as known, with the apparent brightness $I$ of the energy reaching the receiver per unit time per unit surface area. The Luminosity Distance $D_L$ is defined by

$$D_L = \sqrt{\frac{L}{4\pi I}}$$

so that in Euclidean space luminosity distance and geometrical distance coincide. eq.(2.6) can be written as

$$ds^2 = dt^2 - R^2[d\chi^2 + f(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)]$$

where

$$\chi = \frac{r}{\sqrt{1 - kr^2}}$$

In the RW space-time there exists a complicated relationship between the true distance $D$ and the brightness distance $D_L$. Due to the red-shift of the emitted light this becomes [5],

$$D_L = (1 + z)D \frac{f(\chi)}{\chi},$$

where

$$\begin{cases} f(\chi) = \sin \chi, & k = 1 \\ f(\chi) = \chi, & k = 0 \\ f(\chi) = \sinh \chi, & k = -1 \end{cases}$$

Since one observes stars with $z \gg 0$, $D$ and $D_L$ can differ from one another considerably.
A second possible way of determining distance is to compare the true diameter $\Delta$ of a cosmic source with the angle $\delta$ which it subtends at the Earth. In the RW metric, eq.(2.8) implies,

$$D_A \equiv \frac{\Delta}{\delta} = f(\chi)R(t_1) = \frac{D}{1+z} \frac{f(\chi)}{\chi}$$

(2.11)

These two examples of how to determine distance show clearly how the space curvature comes into astronomical considerations concerning the law of propagation of light. Of course optical methods can only be used to determine the distances of objects whose light reaches us.

### 2.6 The Friedmann equations

Having established the metric (RW), the solution of Einstein’s field equations requires a knowledge of the form of the energy-momentum tensor, $T_{\mu\nu}$. To be consistent with the symmetry of the metric we find that the simplest realization of such a tensor, $T_{\mu\nu}$, is that of a perfect fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $p(t)$. For a perfect fluid symmetry requirements dictate that the energy momentum tensor has the form:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},$$

(2.12)

where $u^\mu$ is the fluid four-velocity and for a comoving frame, $u^\mu = (1, 0, 0, 0)$ (i.e. a fluid at rest).

The equation of motion of a particle in a gravitational field is given by

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\Lambda\nu}u^\Lambda u^\nu = 0$$

(2.13)

Straightforward but tedious calculations show that the components of the Ricci tensor are [5]

$$\mathcal{R}_{00} = -3\ddot{R}/R,$$

(2.14)

$$\mathcal{R}_{ij} = -(\ddot{R}/R + 2\dot{R}^2 + 2k/R^2)g_{ij},$$

(2.15)

and the curvature scalar is

$$\mathcal{R} = -6(\ddot{R}/R + \dot{R}^2 + k/R^2).$$

(2.16)

The spatial 3-dimensional curvature is given by

$$3\mathcal{R} = \frac{6k}{R^2}$$

(2.17)

From eqs.(2.12) and (2.14-16), it follows that

$$\frac{\dddot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3} \rho,$$

(2.18)
and

\[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi Gp.\] (2.19)

The Einstein’s field equations (EFE) are related by the Bianchi identities and only two are independent. From eqs.(2.18) and (2.19) we obtain

\[\frac{\ddot{R}}{R} = -\frac{4\pi}{3} G(p + 3\rho)\] (2.20)

Today \(\dot{R} \geq 0\); if in the past \(p + 3\rho\) was always positive, then \(\dot{R}\) was always negative, and thus at finite time in the past \(R\) must have been equal to zero. This event, referred to as the big bang, is usually taken at time \(t = 0\).

At \(R = 0\), there is a singularity; extrapolating past the singularity is not possible in the framework of classical general relativity.

The constant defined by \(H = \frac{\dot{R}}{R}\) is not a genuine constant but in general varies with time as \(t^{-1}\) and hence \(H^{-1}\) sets a time scale for the expansion: \(R\) roughly doubles in a Hubble time.

Equations (2.18) and (2.19) are known as Friedmann’s equations. eq.(2.18) can be written as

\[\frac{k}{H^2 R^2} = \frac{\rho}{3H^2 / 8\pi G} - 1 \equiv \Omega - 1\] (2.21)

where \(\Omega = \rho/\rho_c\) and \(\rho_c = 3H^2 / 8\pi G\): \(\rho_c\) is known as the critical density of the Universe and \(\Omega\) as the density parameter. eqs.(2.18) and (2.20) give

\[\frac{k}{R^2} = (2q - 1)H^2\] (2.22)

where \(q = -\frac{\dot{R} \dddot{R}}{R^2}\), a dimensionless parameter, is known as the deceleration parameter which is a measure of slowing down the expansion of the Universe, is

\[q = \frac{\Omega}{2}(1 + 3p/\rho)\] (2.23)

Hence \(q_0 = \frac{\Omega_0}{2}(1 + 3p_0/\rho_0)\), where the subscript 0 denotes the present day quantities. The sign of \(k\) in eqs.(2.21) and (2.22) gives the sign of \(\Omega - 1\) and \(q\). At the moment observations only yield the bound \(-1 < q_0 < 2\).

The parameter \(k\) is normalized to the values

\[k = +1 \rightarrow \Omega > 1 \text{ closed}\]
\[k = 0 \rightarrow \Omega = 1 \text{ flat}\]
\[k = -1 \rightarrow \Omega < 1 \text{ open}\]

We will consider each of the above cases separately.
2.7 Successes and shortcomings of the Standard Model

The standard big-bang model of the Universe had three major successes:

(i) it predicts that something like Hubble’s law must hold for the Universe.

(ii) it predicts successfully the formation of light atomic nuclei from protons and neutrons a few minutes after the big-bang. This prediction gives the correct abundance ratios for $^3\text{He}$, $^4\text{He}$, and $^7\text{Li}$.

(iii) it predicts a relic of cosmic background radiation having a black-body spectrum with a temperature of 2.75K, today.

However, certain problems and puzzles remain in the standard model.

(i) the Universe displays a remarkable degree of large-scale homogeneity. This is evident in the cosmic microwave background radiation (CMBR) which is known to be uniform in temperature to about one part in $10^4$.

(ii) a certain amount of inhomogeneity must have existed in the primordial matter to account for the clumping of matter in galaxies and cluster of galaxies, etc., that we observe today. Any small inhomogeneity in the primordial matter rapidly grows into a large one with gravitational self-interaction. Thus one has to assume a considerable smoothness in the primordial matter to account for the inhomogeneity in the scale of galaxies at the present time. The problem becomes acute if one extrapolates to $10^{-43}$ seconds after the big-bang when one has to assume an unusual smoothness in the initial state of matter. This is known as the smoothness problem.

(iii) the present discrepancy between the observed density of matter and the required value. If $\Omega$ were initially equal to unity (flat Universe) it will stay equal to unity forever. On the other hand, if $\Omega$ were initially different from unity, its departure from unity will increase with time. The present value of $\Omega$ ranges between 0.1 and 2. For this to be the case the value of $\Omega$ would have had to be equal to 1 to one part in $10^{15}$ a second or so after the big-bang, which seems an unlikely situation. This is called the flatness problem.

To deal with these problems Alan Guth (1981), proposed a model of the Universe, known as the inflationary model, which does not differ from the standard model after a fraction of a second or so, but from about $10^{-45}$ to $10^{-30}$ seconds it has a period of extraordinary expansion, or inflation, during which a typical distance ($R$) increases by a factor of about $10^{50}$ more than the increase that would obtain in the standard model. Though inflationary models solve some of the problems of the standard model, they throw up problems of their own, which have not all been dealt with in a satisfactory manner [6].

The consideration of the Universe in the first second or so calls for a great deal of information from the theory of elementary particles, particularly in the inflationary models. This period is referred to as the
very early Universe.

### 2.8 Matter dominated (MD) Universe

1. **Flat model**

This is the simplest case \((k = 0)\) and is known as the Einstein de Sitter (ES) model. Equations (2.2) and (2.12) give

\[
\dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + p) = 0.
\]

(2.24)

The pressure \(p\) and the density \(\rho\) are related by the equation of state:

\[
p = p(\rho)
\]

(2.25)

which is taken as \(p = w\rho\) for a perfect fluid.

For dust \(w = 0\), for radiation \(w = 1/3\) and \(w = -1\) for a vacuum dominated Universe. Hence

\[
\rho \sim R^{-3(1+w)}
\]

(2.26)

For the matter dominated epoch \(p = 0\), i.e. \(w=0\), therefore eq.(2.24) gives

\[
R = R_0(t/t_0)^{2/3}
\]

(2.27)

where

\[
\rho_0 R_0^3 = \rho R^3.
\]

(2.28)

For this case \(\rho = \rho_c\), \(\Omega = 1\). The present value of \(\rho_c\) is \(2 \times 10^{-29}\) g cm\(^{-3}\). The present age of the Universe, \(t_0\) is given by

\[
t_0 = \frac{2}{3H_0}
\]

(2.29)

which has been found to be less than the range allowed by observations.

2. **closed model**

This is the case \(k = +1\). eqs.(2.20) and (2.18) give

\[
\frac{\dot{R}}{R} = -\frac{4\pi}{3} G \rho
\]

and

\[
\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} \rho - \frac{1}{R^2}
\]

The age of the Universe, \(t_0\), is given from the integral

\[
t = \int \frac{\sqrt{R} dR}{\sqrt{\alpha - R}}
\]

(2.30)
where \( \alpha = \frac{2q_0}{H_0(2q_0-1)^{3/2}} \). One has

\[
t_0 = \frac{q_0}{H_0(2q_0 - 1)^{3/2}}\left[\cos^{-1}\left(\frac{1 - q_0}{q_0}\right) - \sqrt{2q_0 - 1}\right].
\]  

(2.31)

Defining the red-shift \( z \) as \( (z + 1 = \frac{R}{R_0}) \), eq.(2.30) can be written as

\[
t = H_0^{-1} \int_{0}^{(1+z)^{-1}} \frac{dx}{[1 - \Omega_0 + \Omega_0 x^{-1}]^{1/2}}
\]  

(2.32)

From the above equation we see that the age of the Universe is a decreasing function of \( \Omega_0 \): large \( \Omega_0 \) implies faster deceleration, which in turns corresponds to a more rapidly expanding Universe in the past. The age of the Universe provides a very powerful constraint to the value of \( \Omega_0 \), and to the present energy density of the Universe. Independent measurements suggest that

\[
t_0 = 10 \text{ to } 20 \text{ Gyr}
\]

In terms of \( z \) and \( \Omega_0 \) one can write the age of the Universe as;

\[
t = \frac{2}{3} H_0^{-1} (1 + z)^{-3/2},
\]

for \( \Omega_0 = 1 \) and

\[
t = H_0^{-1} \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} \left[ - \cosh\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1 + z)}\right) + \frac{2(1 - \Omega_0)^{1/2}}{(\Omega_0 z + 1)^{1/2}\Omega_0(1 + z)}\right].
\]

for \( \Omega_0 < 1 \). A closed universe has a maximum radius of \( R_{\text{max}} = \frac{2q_0}{H_0(2q_0-1)^{3/2}} \) at \( t_{\text{max}} = \frac{\pi}{2H_0\sqrt{\Omega_0}} \).

3. Open model

This is the case \( k = -1 \). eq.(2.18) becomes

\[
\dot{R}^2 = (\frac{\beta}{R} + 1), \quad \beta = \frac{2q_0}{H_0(1 - 2q_0)^{3/2}}.
\]

The age of the Universe is

\[
t_0 = \frac{q_0}{H_0(1 - 2q_0)^{3/2}}\left[\sqrt{1 - 2q_0} - \ln\left(\frac{1 - q_0 + \sqrt{1 - 2q_0}}{q_0}\right)\right].
\]

Like in the Einstein-de Sitter model, the Universe in this model continues to expand forever.

### 2.9 Radiation dominated (RD) Universe

The solution of eq.(2.24) gives \( (p = \frac{1}{3}\rho) \)

\[
\rho \sim R^{-4}
\]  

(2.33)
and from eq.(2.18),

\[ R \sim t^{-1/2} \quad (2.34) \]

so that

\[ \rho \sim t^{-2} \quad (2.35) \]

In this case the curvature term is negligible in comparison with the second term in (2.18). Therefore, \( k = 0, 1, \) or \(-1\) does not change the physical results.

The radiation in this epoch is modeled by that of a black body. The density of the black body radiation is related to its temperature by

\[ \rho_r = aT_r^4 \quad (2.36) \]

where \( a = 8.418 \times 10^{-36} \text{gcm}^{-3} \text{K}^{-4} \).

According to eq.(2.33), the temperature of radiation is inversely proportional to the scale factor of the Universe: (This also follows from the fact that the black body radiation retains its spectrum during the expansion of the Universe)

\[ T_r \propto R^{-1}. \quad (2.37) \]

When \( R \) is very small, that is, in the early Universe, \( T_r \) can be very high. Equations (2.28) and (2.33) give

\[ \frac{\rho_r}{\rho_m} = \left( \frac{\rho_r}{\rho_m} \right)_0 \left( \frac{R_0}{R} \right). \quad (2.38) \]

This formula shows that the ratio of the radiation and matter densities is not invariant; rather, it decreases as the Universe expands. Although the value \((\rho_r/\rho_m)_0\) is very small, being only \(10^{-3}\); in the early Universe, i.e. that is when

\[ \frac{R_0}{R} \gg 1, \]

we had

\[ \frac{\rho_r}{\rho_m} \gg 1. \]

Namely, the radiation was the dominant component of the Universe and its temperature was

\[ T_r = (T_r)_0(0R_0/R) = T_{r0}(1 + z). \]

For \( T_{r0} = 2.7 \text{K} \) and \( z \sim 10^3 \), this corresponds to

\[ T_r > 2.7 \times 10^3 \sim 3000 \text{K}. \]

This phase of the Universe is called the radiation dominated (RD) Universe.

Since the Friedmann models are frequently used to interpret cosmological observations, we will now derive some of the observable quantities in these models:
2.10 The cosmological tests

2.10.1 The red-shift

Consider a galaxy $G_1$ at $(r, \theta, \phi)$ emitting light waves towards us. Let us denote by $t_0$ the present epoch of observation. At what time should a light wave leave $G_1$ in order to arrive at $r = 0$ at time $t = t_0$? Since a light signal moves in a null geodesic, $ds = 0$. Then the RW line element (eq.2.6) gives us

$$t = \pm \int \frac{R \, dr}{\sqrt{1 - kr^2}} .$$

Since $r$ decreases as $t$ increases along the null geodesic we should take the minus sign in the above equation.

Suppose that the light left $G_1$ at time $t_1$, hence from the above equation

$$\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} .$$

Thus if we know $R(t)$ and $k$ we know the answer to our question.

Assume the wave crest was emitted at $t_1$ and $t_1 + \Delta t_1$ and received at $t_0$ and $t_0 + \Delta t_0$, respectively. Then

$$\int_{t_1 + \Delta t_1}^{t_0 + \Delta t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} .$$

If $R$ is a slowly varying function of time, i.e. it effectively remains unchanged over the small intervals $\Delta t_1$ and $\Delta t_0$ we get, by subtracting eqs.(2.40) and (2.41)

$$\frac{\Delta t_0}{R(t_0)} = \frac{\Delta t_1}{R(t_1)} .$$

i.e.

$$\frac{\Delta t_0}{\Delta t_1} = \frac{R(t_0)}{R(t_1)} = 1 + z .$$

The quantity $z$ defines a red-shift $(\lambda_1 = c\delta t_1, \lambda_0 = c\delta t_0)$. The wave length of the light wave increases by a factor $z$ in transmission from $G_1$ to us, provided $R(t_0) > R(t_1)$. Thus Hubble’s observations of the red-shift is explained if $R(t)$ is an increasing function of time. This red-shift arises due to the passage of light through non-Euclidean space time. It does not arises from the Doppler effect, since in our coordinate frame all galaxies have constant $(r, \theta, \phi)$ coordinates. In non-Euclidean space-time it is not possible to attach an unambiguous meaning to the relative velocity of two objects separated by a great distance. Equation (2.43) may be compared with the gravitational red-shift which is characterized by the fact that if light from object B to A is red-shifted, the light from A to B is blue shifted. In the present case, if light from A to B is red-shifted, that from B to A will also be red-shifted provided $R(t)$ is an increasing function of time. We will therefore refer to the present red-shift as cosmological red-shift.
2.10.2 Luminosity distance ($D_L$)

This is defined by [2]

$$D_L = r_1 R_0 (1 + z). \quad (2.44)$$

1. Einstein-de Sitter model

eqs.(2.39) and (2.27) give

$$r_1 = \int_{t_1}^{t_0} \frac{dt}{R(t)} = \frac{1}{R_0} \int_{t_1}^{t_0} t_0^{2/3} t^{2/3} dt \quad (2.45)$$

$$r_1 = \frac{3t_0}{R_0} \left[ 1 - \left( \frac{t}{t_0} \right)^{\frac{1}{3}} \right] \quad (2.46)$$

Upon using eq.(2.29), this becomes

$$r_1 = \frac{2}{R_0 H_0} \left[ 1 - (1 + z)^{-1/2} \right]$$

The luminosity distance becomes

$$D_L = \frac{2}{H_0} \left[ 1 + z - (1 + z)^{1/2} \right]$$

2. Closed Model

$$D_L = \frac{1}{H_0 q_0} \left[ q_0 z + (q_0 - 1) (\sqrt{1 + 2q_0 z} - 1) \right] \quad (2.47)$$

3. Open Model

$$D_L = \frac{1}{H_0 q_0} \left[ q_0 z + (q_0 - 1) (\sqrt{1 + 2q_0 z} - 1) \right] \quad (2.48)$$

Note that the equations for $D_L$ for the closed and open models are the same.

2.10.3 The angular size

We will study how the apparent angular size varies with red-shift in different Friedmann models. We will assume that sources of a fixed size $d$ are observed at different red-shifts. Thus a source at $(r, \theta, \phi)$ with red-shift $z$ will subtend at the observer at $r = 0$, the angle

$$\Delta \theta_1 = \frac{d}{r_1 R(t_1)} = \frac{d(1 + z)^2}{D_L} \quad (2.49)$$

$\Delta \theta_1$ is defined in terms of $z$ and $q_0$. It is interesting to note that $\Delta \theta_1$ does not steadily decrease as $z$ increases. For $q_0 = 1/2$

$$\Delta \theta_1 = \frac{dH_0 (1 + z)^{3/2}}{(1 + z)^{1/2} - 1} \quad (2.50)$$

A minimum value for $\Delta \theta_1$ occurs at $z = 1.25$:

$$\Delta \theta_{1\text{ min}} = 6.75 H_0 d.$$

The cases ($q_0 > 1/2$) are more involved [2].
2.10.4 Source counts

The number of astronomical sources with red-shifts between \(z\) and \(z + dz\) is given by (applicable to all Friedmann models) \[2\]

\[
dN = 4\pi r^2 n(t) dr \sqrt{1 - kr^2} \tag{2.51}
\]

\[
dN = \frac{4\pi}{H_0^2} \frac{q_0 z + (q_0 - 1)(\sqrt{1 + 2q_0 z} - 1)^2}{q_0(1 + z)^3 \sqrt{1 + 2q_0 z}} \tilde{n} dz \tag{2.52}
\]

where \(\tilde{n} = \frac{n}{\pi r}\).

2.10.5 Particle horizon

The proper distance to the horizon in a RW space-time is given by

\[
d_H(t) = R(t) \int_0^t \frac{dt'}{R(t')} \tag{2.53}
\]

If \(R(t) \propto t^n\), then for \(n < 1\), \(d_H(t)\) is finite and is equal to \(\frac{t}{(1-n)}\), i.e. in spite the fact that all physical distances approach zero as \(R \rightarrow 0\), the expansion of the Universe precludes all but a tiny fraction of the volume of the Universe from being in casual contact. This is a vexing feature of the Standard Model. A more general expression for the \(d_H\) \[5\], is

\[
d_H(t) = \frac{1}{H_0(1 + z)} \int_0^{(1+z)^{-1}} dx \frac{dx}{[x^2(1 - \Omega_0) + \Omega_0 x^{(1-3w)}]^{1/2}} \tag{2.54}
\]

From this expression we see that if \(w < -1/3\), the integral will diverge and the horizon distance will be infinite. The present horizon distance is given by

\[
d_H = \frac{1}{H_0} \begin{cases} 
2 & k = 0, \quad q_0 = 1/2 \\
\frac{2}{\sqrt{2q_0 - 1}} \sin^{-1} \sqrt{\frac{2q_0 - 1}{2q_0}} & k = 1, \quad q_0 > 1/2 \\
\frac{2}{\sqrt{1 - 2q_0}} \sinh^{-1} \sqrt{\frac{1 - 2q_0}{2q_0}} & k = -1, \quad q_0 < 1/2
\end{cases}
\]

The existence of a finite \(d_H\) means that the Universe has a particle horizon.

2.10.6 Event horizon

A related question to that posed at the beginning of subsection 2.10.1 is whether a light signal sent out at the present time \((t_0)\) reaches all points of the Universe before its end at time \(t_E\). Since light travels a maximum coordinate distance

\[
d_{EH} = \int_{t_0}^{t_E} \frac{dt}{R(t)}
\]

there exists an event horizon if this is smaller than \(\pi\) or \(\infty\): we shall never learn any thing about events which at the present time are situated at a distance greater than the above distance \[5\]
2.11 References

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Chapter 3

Thermal History of the Universe

3.1 Thermodynamics of the Universe

3.1.1 Introduction

The first theoretical basis of the evolutionary view is thermodynamics. Applying the requirement of thermal equilibrium to the Universe, we have to say that the general tendency of the cosmic evolution is for the temperature to be the same all over. It was pointed out by Helmholtz in 1854 that the whole Universe will eventually be in a state of uniform temperature and will be falling in the state of thermal death (eternal rest). According to the thermodynamics any temperature difference between two systems will approach zero. Using thermodynamics, it was proved that for an expanding Universe, even if the initial temperature is the same, a temperature difference may still be generated. Consider a spherical region R and assume that the matter in R has already reached equilibrium at the beginning. Roughly speaking, there are two kinds of matter in the Universe, one is baryonic and the other is radiation. Let $\rho_m$ and $\rho_r$ be the mass densities of matter and radiation respectively. Their equations of state are

\[ p_r = \frac{1}{3} \rho_r \]  

\[ \rho_m \simeq m n + \frac{3}{2} n T \]  

where $n$ is the number density of the nonrelativistics particles and $m$ their rest mass. Here we have assumed that each particle has three degrees of freedom. The expansion of the Universe should be adiabatic. No exchange with the exterior for the exterior does not exist in the Universe. The expansion of the region R which is typical of the Universe, must also be adiabatic. Even though an exterior region for R exists, there is no difference between R and its exterior because the Universe is uniform throughout. The adiabatic expansion of a system satisfies

\[ dE = -pdV \]
where $E$ is the total energy of the system. This formula constitutes the basis for the thermodynamics of an expanding Universe. We now proceed to investigate the consequences of these equations.

3.1.2 Radiation under adiabatic expansion

The total energy density in the region $R$ is

$$E_r = V \rho_r, \quad V \sim R^3. \tag{3.4}$$

Substituting eq.(3.4) in (3.3) gives

$$d(R^3 \rho_r) = -p_r dR^3. \tag{3.5}$$

Then using eq.(3.1), this becomes

$$d(R^3 \rho_r) = -\frac{1}{3} \rho_r dR^3 \tag{3.6}$$

or

$$R^3 d\rho_r + \rho_r dR^3 = -\frac{1}{3} \rho_r dR^3 \tag{3.7}$$

i.e.

$$\frac{d\rho_r}{\rho_r} = -\frac{4}{3} \frac{dR^3}{R^3}. \tag{3.8}$$

The solution of this equation is

$$\rho_r \propto R^{-4}. \tag{3.9}$$

According to the thermodynamics of radiation the relation between $\rho_r$ and the temperature of radiation $T_r$ is

$$\rho_r \propto T^4. \tag{3.10}$$

From eqs.(3.9) and (3.10) we immediately get

$$T_r \propto R^{-1}.$$ 

This result shows that as the Universe expands the radiation temperature falls in inverse ratio to the scale factor $R$, provided only radiation exists.

3.1.3 Matter under adiabatic expansion

Assuming the presence of matter only we apply eq.(3.3) to obtain

$$d(R^3 \rho_m) = -p_m dR^3, \tag{3.11}$$
where \( p_m \) is the pressure of matter,
\[
p_m = nT_m.
\] (3.12)

Substituting eqs. (3.12) and (3.2) in (3.11), we then find
\[
d(R^3 m) + d(R^3 \frac{3}{2} nT_m) = -nT_m dR^3,
\] (3.13)

The total number of particles \( N = nV \) is conserved within \( V \), and so \( n \sim R^{-3} \). Thus, eq. (3.13) now becomes
\[
\frac{3}{2} d(R^3 nT_m) = -nT_m dR^3.
\] (3.14)

Using \( N = nV \), this becomes
\[
\frac{3}{2} \frac{dT_m}{T_m} = -\frac{dR^3}{R^3},
\] (3.15)

with the solution
\[
T_m \propto R^{-2}.
\] (3.16)

This result shows that, as the Universe expands, the matter temperature \( T_m \) also decreases but in a different manner to radiation.

### 3.1.4 Generation of temperature differences

\( T_r \) and \( T_m \) vary with \( R \) in different ways. Hence during the process of cosmic expansion it is impossible for \( T_r \) and \( T_m \) to be always equal. Even if initially \( T_r \) and \( T_m \) were equal, later on, with expansion, we will have \( T_r > T_m \). Therefore, the cosmic expansion saves the Universe from the final outcome of thermal death. According to thermodynamics, a system with thermal equilibrium has the same temperature for all the various components, but if radiation and matter always keep the same temperature, how can temperature difference ever appear?

In a system with complete thermal equilibrium all components must have the same temperature. However, a certain time is required to reach equilibrium, interaction between radiation and matter must be carried on for a length of time before the two can achieve the same temperature. If the time required to achieve uniform temperature is longer than the time scale of the cosmic expansion, then there will never be thermal equilibrium between radiation and matter. In this case, it is reasonable to separately solve the thermodynamics equations for the two components. Due to cosmic expansion, matter is not in a state of complete thermal equilibrium. Radiation and matter are separated from thermal equilibrium because there has not been sufficient time for the two components to achieve mutual equilibrium.
3.1.5 Equilibrium thermodynamics

Today the radiation in the Universe is comprised of the 2.75K microwave photons and the cosmic sea of 1.96K relic neutrinos. Because the early Universe was to a good approximation in thermal equilibrium, there should have been other relativistic particles present, with comparable abundance. The number density \( n \), energy density \( \rho \), and pressure of a dilute, weakly interacting gas of particles with \( g \) internal degrees of freedom, in thermal equilibrium at temperature \( T \), are given by [5]

\[
n = \frac{g}{(2\pi)^3} \int f(q) \, d^3 q, \tag{3.17}
\]

\[
\rho = \frac{g}{(2\pi)^3} \int E(q) f(q) \, d^3 q, \tag{3.18}
\]

and

\[
p = \frac{g}{(2\pi)} \int \frac{q^2}{3E} f(q) q^3 q \, d^3 q, \tag{3.19}
\]

where \( E^2 = q^2 + m^2 \) and \( f \) is the Fermi-Dirac (FD) or Bose-Einstein (BE) distribution for species in kinetic equilibrium.

\[
f(q) = (\exp \frac{E - \mu}{T} \pm 1)^{-1}, \tag{3.20}
\]

where \( \mu \) is the chemical potential of the species, +1 is chosen for FD species and −1 for BE species. It follows that

\[
\rho = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E^2 \, dE}{\exp(E - \mu/T) \pm 1}, \tag{3.21}
\]

\[
n = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2} E \, dE}{\exp(E - \mu/T) \pm 1}, \tag{3.22}
\]

\[
p = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{3/2} E \, dE}{\exp(E - \mu/T) \pm 1}, \tag{3.23}
\]

For \( T \gg m \) (relativistic limit) and \( T \gg \mu \)

\[
\rho = \left\{ \begin{array}{l}
\frac{\pi^2 gT^4}{30}, \quad \text{BE} \\
\frac{1}{8} \frac{\pi^2 gT^4}{30}, \quad \text{FD}
\end{array} \right\} \tag{3.24}
\]

and

\[
n = \left\{ \begin{array}{l}
\frac{\xi(3) T^3}{4\pi^2 gT^4}, \quad \text{BE} \\
\frac{3}{4} \frac{\xi(3) T^3}{gT^4}, \quad \text{FD}
\end{array} \right\} \tag{3.25}
\]

\( \xi(3) = 1.20206 \) is the Riemann zeta function. In the nonrelativistics limit

\[
n = g \left( \frac{mT}{2\pi} \right)^{3/2} \exp - \left( \frac{m - \mu}{T} \right), \tag{3.26}
\]

\[
\rho = mn, \tag{3.27}
\]
\[ p = nT \ll \rho \] \quad (3.28)

for BE and FD species. For relativistic particles, eqs. (3.21) and (3.23) give

\[ \rho_r = \frac{\pi^2}{30} g_* T^4 \] \quad (3.29)
\[ p_r = \frac{1}{3} \rho_r = \frac{\pi^2}{90} g_* T^4 \] \quad (3.30)

where \( g_* \), the total number of effectively massless degrees of freedom \((m_i \ll T)\), is given by

\[ g_* = \sum_{i=\text{boson}} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=\text{fermion}} g_i \left( \frac{T_i}{T} \right)^4 \] \quad (3.31)

and \( T_i \) is the temperature of species \( i \) that may not have the same temperature as photons. The factor \( \frac{7}{8} \) accounts for the difference in FD and BE statistics.

### 3.1.6 Entropy

Through most of the history of the Universe (in particular the early Universe) the reaction rates of particles in the thermal bath, \( \Gamma_{int} \), were much greater than the expansion rate, \( H \), and local thermal equilibrium should have been maintained. In this case the entropy per comoving volume element remains constant. The entropy in a comoving volume provides a very useful fiducial quantity during the expansion of the Universe. The second law of thermodynamics implies

\[ T dS \equiv d(\rho V) + pdV = d[(\rho + p)V] - V dp \] \quad (3.32)

The integrability condition \([5]\)

\[ \frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \] \quad (3.33)

relates the energy density and the pressure

\[ T \frac{dp}{dT} = \rho + p \] \quad (3.34)

or

\[ dp = \frac{\rho + p}{T} dT \] \quad (3.35)

Equation (3.32) becomes

\[ dS = \frac{1}{T} d[(\rho + p)V] - \frac{(\rho + p)V}{T^2} dT \] \quad (3.36)
\[ dS = d \left[ \frac{(\rho + p)V}{T} + \text{const.} \right] \] \quad (3.37)
The entropy per comoving volume is (up to an additive constant)

\[ S = \frac{R^3(\rho + p)}{T} \]  

(3.38)

The general relativistic field equations require

\[ dS = 0 \]  

(3.39)

This results implies that in thermal equilibrium the entropy, per comoving volume, is constant. It is useful to define the entropy density

\[ s \equiv \frac{S}{V} = \frac{\rho + p}{T}. \]  

(3.40)

At high temperature the entropy density is dominated by the contribution of the relativistic particles, so that to a very good approximation\(^4\)

\[ s = \frac{2\pi^2}{45} g_\ast T^3 \]  

(3.41)

\[ g_\ast = \sum_{i=\text{boson}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermion}} g_i \left(\frac{T_i}{T}\right)^3. \]  

(3.42)

During thermal equilibrium all particle species had a common temperature. Whenever a particle species becomes non-relativistic and decouples its entropy is transferred to the other relativistic particle species still present in the thermal plasma, causing \( T \) to decrease slightly less slowly. Massless particles that decoupled from the heat bath will not share in the entropy transfer as the temperature drops below the mass threshold of a species. Instead, the temperature of the decoupled massless particles scales as \( R^{-1} \).

3.2 The early radiation era

The Friedmann equation (2.18) for the radiation era is

\[ \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} \rho. \]  

(3.43)

During this epoch \( \rho \propto R^{-4} \), so that

\[ R \sim t^{1/2}. \]  

(3.44)

The energy density of the dominant black-body radiation in this era obeys the equation

\[ \rho = \frac{g}{2} a T^4, \quad a = \frac{\pi^2}{15}, \]  

(3.45)

\(^4s\) is also proportional to the density of relativistic particles and in particular

\( s = 1.8 g_\ast n_\gamma, \quad n_\gamma = \text{photon density} \)
where \( g = 2 \) for photons. Therefore, it follows that

\[
\frac{\dot{R}}{R} = -\frac{-T}{T}
\]

and that

\[
\left(\frac{\dot{T}}{T}\right)^2 = \frac{8\pi G g_* \pi^2}{30} T^4
\]

The solution of this latter equation is

\[
T = \left(\frac{3}{16\pi G g_* \alpha}\right)^{1/4} t^{-1/2} = \left(\frac{45}{16\pi^3 G g_*}\right)^{1/4} t^{-1/2},
\]

giving

\[
T(K) = \frac{1.805}{g_*^{1/4}} t^{-1/2}(sec)
\]

and

\[
\rho = \frac{3}{32\pi G} t^{-2}.
\]

To determine which types of particles will be in thermal equilibrium at a given temperature one proceeds as follows: we note that if the particle’s number density is \( n \), its velocity is \( v \) and its reaction cross section is \( \sigma \), then the average reaction rate is given by [6]

\[
\Gamma = \langle n \sigma v \rangle
\]

averaged over the thermal distribution. Some relevant reaction are: \( e^+ + e^- \rightarrow \mu + \bar{\mu}, e^+ + e^- \rightarrow \nu + \bar{\nu}, q + \bar{q} \rightarrow q + \bar{q}, \ell + \bar{\ell} \rightarrow q + \bar{q} \) where \( q, \ell \) stand for quarks and leptons respectively. A given species of particles will remain in thermal equilibrium so long as this reaction rate is sufficiently high compared to the age of the Universe, i.e. if

\[
\Gamma \gg H = \frac{1}{2t}
\]

Hence initially when \( n \) and \( v \) are very large we expect all particles to be in equilibrium, but once the temperature drops, and the energy of a particular particle falls below the relevant reaction threshold, \( \sigma \) will vanish and the particle will cease to be in equilibrium. Therefore, when \( \Gamma > H \) (\( \Gamma < H \)) the particle species couple (decouple) from the thermal bath.

For reactions mediated by massless gauge bosons [5] \( \Gamma \sim n \sigma |v| \sim \alpha^2 T \), during the RD epoch \( H \sim \frac{T^2}{m_{pl}} \) so that \( \frac{H}{\Gamma} \sim \frac{\alpha^2 m_{pl}}{T} \), (\( \sqrt{4\pi\alpha} = g \) =the gauge coupling strength).

Therefore, for \( T < \alpha^2 m_{pl} \sim 10^{16} \text{GeV} \), the reactions are occurring rapidly, while for \( T > \alpha^2 m_{pl} \sim 10^{16} \text{GeV} \), they are effectively “frozen out.”

\[ ^2 \text{e.g. } e^- + e^+ \rightarrow \gamma + \gamma, e^- + e^+ \rightarrow \mu^- + \mu^+ \]
For reactions mediated by massive gauge bosons:\[ \Gamma \sim n\sigma |v| \sim G_X^2 T^5 \text{ and } \Gamma \sim G_X m_{pl} T^3, \] where \( G_X \sim \frac{\alpha}{m_X} \) and \( m_X \) is the mass of the gauge bosons. Thus for \( m_X > T > G^{-2/3} m_{pl}^{-1/3} \sim (m_X/100\text{GeV})^{4/3} \text{MeV} \), the reactions are occurring rapidly, while for \( T < (m_X/100\text{GeV})^{4/3} \text{MeV} \), they are effectively frozen out.

### 3.3 Neutrino decoupling

For \( T \gg 1\text{MeV} \), neutrons would be in equilibrium with the rest of the Universe via the weak interaction processes (\( e^+ + e^- \rightarrow \nu + \bar{\nu}, \nu + e^- \rightarrow \nu + e^- , e^- + p \rightarrow \nu + n \), etc.). The weak interaction cross section \( \sigma \) is given by

\[
\sigma \simeq G_F^2 T^2
\]

where \( G_F \) is the Fermi’s constant. The number density of neutrinos is \( \propto T^3 \). The reaction rate for neutrinos (\( \Gamma_{wi} \)) therefore falls with decreasing temperature as \( T^5 \). Hence eq.(3.51) gives

\[
\frac{\Gamma_{wi}}{H} \propto T^3
\]

When eq.(3.52) is not fulfilled, neutrinos decouple (freeze out) from all interactions and begin a free expansion. The decoupling of \( \nu_\mu \) and \( \nu_\tau \) occurs at 3.5MeV whereas the \( \nu_e \) decouples at 2.3MeV. At decoupling, the neutrinos are still relativistic and thus their energy distribution is given by the Fermi distribution and their average temperature equals that of photons.

As the Universe cools and the energy approaches 0.8MeV, the reactions converting the protons into neutrons stop but the neutrons also decay into protons by the beta decay, \( n \rightarrow p + e + \bar{\nu} \), and therefore the ratio of protons to neutrons increases. The mean life of the neutron is 889s and in comparison with the age of the Universe which at this time is a few tens of seconds the neutrons are essentially stable. The electromagnetic cross sections are \( \propto T^{-2} \), and the reaction rate (\( \Gamma_{em} \)) is then \( \propto T \) so that

\[
\frac{\Gamma_{em}}{H} \propto \frac{1}{T}
\]

Equation (3.52) is satisfied for all temperatures, so, in contrast to the weak interactions, the electromagnetic interactions never freeze out. The reaction

\[
e^- + e^+ \rightarrow \gamma + \gamma
\]

creates new photons with energy 0.51MeV. This is higher than the ambient photon temperature at that time, so the photons population gets reheated. To see this consider eq.(3.41)

\[
s = \frac{2\pi^2}{45} g_{*s} T^3,
\]

\[3\text{e.g. } e^- + p \rightarrow n + \nu, e^- + e^+ \rightarrow \mu^- + \mu^+ , \ell + \bar{\ell} \rightarrow q + \bar{q} \]
Since entropy is conserved throughout, this is only possible if \( g_s s T^3 \) remains constant. By applying this argument to the situation when positrons and most of the electrons disappear by annihilation below 0.2 MeV. We denote quantities above this energy by +, and below it by −. Above this energy the particles in thermal equilibrium are \( \gamma, e^-, e^+ \). Then the entropy is
\[
s = \frac{11}{3} a T^3_+. \tag{3.58}
\]
Below this energy only photons contribute the factor \( g_s s = 2 \). Hence,
\[
T_- = \left( \frac{11}{4} \right)^{1/3} T_+ = 1.40 T_+ \tag{3.59}
\]
The number density of neutrinos is related to that of photons as
\[
n_\nu = \frac{3}{4} \frac{4}{11} n_\gamma \tag{3.60}
\]
When electrons become slow enough, they are captured into atomic orbits by protons, forming stable hydrogen atoms \((B.E = 13.6\text{eV})\). Actually the formation of hydrogen atom occurs at 0.3eV, because the released binding energy reheats the remaining electrons, and also because the large amount of entropy in the Universe favors free protons and electrons. When this recombination is completed, the photons find no more free electrons to scatter against, thus the photons decouple and the Universe becomes transparent to radiation \([3]\).

### 3.4 Nucleosynthesis

We left the story of the decoupling nucleons at the time when the weak interaction ceased and the conversion of protons to neutrons stopped because the energy in the thermal bath dropped below 0.8MeV. The neutrons and protons were then non-relativistic, so their number densities were each given by the Maxwell-Boltzmann distributions. For nuclear species \( A(Z) \) with mass number \( A \) this is given by
\[
n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} \exp(\frac{\mu_A - m_A}{T}) \tag{3.61}
\]
where \( \mu \) is the chemical potential of the species. At equilibrium this is related to the chemical potential of the proton \( (\mu_p) \) and neutron \( (\mu_n) \) by,
\[
\mu_A = Z\mu_p + (A - Z)\mu_n \tag{3.62}
\]
By eliminating the \( \exp(\mu_A/T) \) term, one can write \([5]\)
\[
n_A = g_A A^{3/2} 2^{-A} \left( \frac{2\pi}{m_N T} \right)^{3(A-1)/2} n_p^Z n_n^{A-Z} \exp(B_A/T) . \tag{3.63}
\]
\[ B_A \equiv Z m_p + (A - Z)m_n - m_A \quad , \] (3.64)

So the neutron to proton ratio is

\[
n_n/n_p = \exp(-Q/T) ,
\]

where \( Q = m_n - m_p \). At energies of the order of \( (m_n - m_p) = 1.293 \text{MeV} \) or less, this ratio is dominated by the exponential. Thus at \( T = 0.8 \text{MeV} \) one finds that the ratio has dropped from 1 to 1/5. Already at a few MeV, nuclear fusion reactions start to build up light elements. For a typical bound nucleus \( A \) with atomic mass \( A \) and number density \( n_A \) the mass fraction of the various nuclear species \( A(Z) \) is given by

\[
X_A \equiv \frac{n_A A}{n_N} \quad , \] (3.65)

where \( n_N \) is the total number density of nucleons. From eq.(3.63) it follows that

\[
X_A = g_A \left( \zeta (3)^{-1} n_p^{(1-A)/2} 2^{(3A-5)/2} A^{5/2} (T/m_N)^{3(A-1)/2} \right) \eta^{A-1} X_p X_n^{A-Z} \exp(B_A/T) \quad \] (3.66)

where \( \eta \equiv \frac{n_N}{n_p} = 2.68 \times 10^{-8} (\Omega_B h^2) \), \( \Omega_B \) being the baryonic fraction of the critical density. The equilibrium abundance of deuterons is given by :

\[
X_d = 16.3 \left( \frac{T}{m_N} \right)^{3/2} \eta X_p X_n \exp\left(\frac{B_d}{T}\right) . \quad (3.67)
\]

Since \( B_d \), the deuteron binding energy (= 2.22MeV), is low, \( X_d \) is not high enough to start fusion reactions leading to \(^2H, ^3H \) and \(^4He\) unless \( T \) drops to less than \( 10^9 \text{K} \). Although the deuterons are formed in very small quantities, they are of crucial importance to the final composition of matter. Photons of energy 2.22MeV or more photodisintegrate the deuterons into free protons and neutrons.

Consider the above process at equilibrium, and define the relative abundance by

\[
\frac{X_n X_p}{X_d} = \frac{4}{3} \frac{(2\pi T)^{3/2}}{n_B (2\pi)^3} \left( \frac{m_N^2}{m_d} \right)^{3/2} \exp\left(\frac{B_d}{T}\right) . \quad (3.68)
\]

where \( m_N = m_B \) is the nucleon mass.

The above equation tells us that as the Universe cools the equilibrium shifts in favor of \( d \) over \( p \) and \( n \) at \( T \approx 10^9 \text{K} \). We denote the temperature \( T_N \), when the above ratio equals to unity, as the nucleosynthesis temperature.

All evidence suggest that the number density of baryons (nucleons) is today very small. In particular, we are able to calculate it up to a multiplicative factor \( \Omega_B h^2 \), [3]

\[
n_B = \frac{\rho_B}{m_B} = \frac{\Omega_B \rho_c}{m_B} \approx 1.13 \times 10^{-5} \Omega_B h^2 \text{cm}^{-3} . \quad (3.69)
\]
The parameter $\Omega_B$ cannot be very much larger than unity, because that would close the Universe too fast.

Detailed calculations show that deuterons production becomes thermodynamically favorable only at energy of magnitude $0.07\text{MeV}$. Other nuclear reactions also commence at a few MeV. The $^3\text{He}$ is produced in the reactions

$$d + d \rightarrow ^3\text{He} + n,$$

$$p + d \rightarrow ^3\text{He} + \gamma,$$

with the binding energy $7.72\text{MeV}$. This reaction is hampered by the large entropy per nucleon, so it becomes thermodynamically favorable only at $0.11\text{MeV}$. The $^3\text{H}$ is produced in the fusion reactions

$$n + d \rightarrow ^3\text{H} + \gamma,$$

$$d + d \rightarrow ^3\text{H} + p,$$

$$n + ^3\text{He} \rightarrow ^3\text{H} + p,$$

with the binding energy $8.48\text{MeV}$. A very stable nucleus is the $^4\text{He}$ with a very large binding energy of $28.3\text{MeV}$. Once its production is favored by the entropy law, at about $0.28\text{MeV}$, there are no more $\gamma$-rays sufficiently energetic to photodisintegrate it. $^4\text{He}$ is mostly produced in the reaction

$$d + d \rightarrow ^4\text{He} + \gamma \quad (3.70)$$

However $^3\text{He}$ and $^3\text{H}$ production is preferred over deuteron fusion, so $^4\text{He}$ is only produced in a second step when these nuclei become abundant. The reactions are then

$$n + ^3\text{He} \rightarrow ^4\text{He} + \gamma,$$

$$d + ^3\text{He} \rightarrow ^4\text{He} + p,$$

$$p + ^3\text{H} \rightarrow ^4\text{He} + \gamma,$$

$$d + ^3\text{H} \rightarrow ^4\text{He} + n.$$

At energy of magnitude $0.1\text{MeV}$ when the temperature is $1.2 \times 10^9\text{K}$ and the time elapsed since the big bang is a little over 2 minutes, the beta decay of neutrons already noticeably converts neutrons to protons. At this point the $n_n/n_p$ ratio has reached its final value.

$$\frac{n_n}{n_p} \simeq \frac{1}{i} \quad (3.71)$$
These remaining neutrons have no time to decay before they fuse into deuterons and subsequently into \(^4\text{He}\). There they stay until today because bound neutrons don’t decay. The same number of protons as neutrons go into \(^4\text{He}\), and the remaining free protons are the nuclei of future hydrogen atoms. Thus the end result of nucleosynthesis taking place between 100 and 700 seconds after the big bang is a Universe composed almost entirely of hydrogen and helium nuclei.

Why not heavier nuclei? It is an odd circumstance of nature that although there exist stable nuclei composed of 1, 2, 3 and 4 nucleons, no nucleus of \(A = 5\) exits, and no stable one with \(A = 8\). In between these gaps, there exist the unstable nuclei \(^6\text{Li},^7\text{Be}\) and the stable \(^7\text{Li}\). Because of these gaps and because \(^4\text{He}\) is so strongly bound, nucleosynthesis essentially stops after \(^4\text{He}\) production. Only small minute quantities of stable nuclei \(^2\text{He},^3\text{He}\) and \(^7\text{Li}\) can be produced. The relic abundance of the light elements bears quite an important testimony of the Big Bang. The number of \(^4\text{He}\) is clearly half the number of protons as neutrons go into \(^4\text{He}\). Thus the excess number of protons going into the formation of hydrogen is \(n_p - n_n\). Usually one quotes the ratio of mass in \(^4\text{He}\) to total mass in \(^1\text{H}\) and \(^4\text{He}\), \[Y_4 = \frac{\text{mass in } ^4\text{He}}{\text{mass in } ^1\text{H} + ^4\text{He}} \simeq 0.25.\] (3.72)

At present, the best estimate of \(Y_4\) from observational data is in the range 0.22 – 0.24 with no directional variation. This is a strong support of the big bang hypothesis. The helium mass abundance \(Y_4\) depends sensitively on several parameters.

(i) The quantity \(\eta\) in eq.(3.69) \(\propto \Omega_B\). If the \(n_B\) increases \(\Omega_B\) and \(\eta\) also increase, and the entropy per baryon decreases. Remembering that the large entropy per baryons was the main obstacle to early deuteron and helium production, the consequence is that helium production can start earlier. But then the neutrons would have had less time to beta decay, so the \(n_n/n_p\) ratio would be larger than \(1/7\). It follows that more helium will be produced: \(Y_4\) increases.

(ii) An increase in the neutron mean life implies a decrease in the reaction rate \(\Gamma_{wi}\). Hence a longer mean life implies a higher decoupling temperature and an earlier decoupling time. As we have already seen, an earlier start of helium production leads to an increase in \(Y_4\).

(iii) The expansion rate \(H \propto \sqrt{g_*}\) which in turn depends on the number of neutrino families \(N_\nu\). If there were more than 3 neutrino families \(H\) would increase. Similarly if the number of neutrinos were very different from the number of antineutrinos, contrary to the assumption in standard model, \(H\) would also increase. From the nucleosynthesis value of \(\eta\) one can obtain an estimate of the baryon density parameter \(\Omega_B\). Combining eqs.(3.69) and (3.70) one gets \[\Omega_B = \frac{n_Bm_B}{\rho_c} = \frac{n_\gamma m_B}{\rho_c} \eta \simeq 3.65 \times 10^7 \eta h^{-2},\] (3.73)
The observational data [3] constrains \( \eta \) to

\[
2.8 < 10^{10} \eta < 4.2.
\]

This limit on \( \eta \) gives:

\[
0.010 < \Omega_B h^2 < 0.015.
\]

Note that \( h \) ranges from 0.5 to 0.85 [3] which implies

\[
0.01 < \Omega_B < 0.05.
\]

Thus we arrive at the very important conclusion that there is too little baryonic matter to close the universe. Either the Universe is then indeed open, or there must exist other non-baryonic matter.

### 3.5 The decoupling of photons

The relative abundance of free electrons of number density \( n_e \), free protons of number density \( n_p \) and free neutral \( H \)-atoms of number density \( n_H (= n_B - n_e) \) in thermal equilibrium at a given temperature is determined by

\[
\frac{n_e n_p}{n_H} = \left( \frac{m_e T}{2\pi} \right)^{3/2} \exp\left( -\frac{B_H}{T} \right)
\]

where \( B_H = 13.59 \text{ eV} \) is the hydrogen binding energy. The ionization fraction is given by

\[
x = \frac{n_e}{n_B},
\]

so that

\[
\frac{x^2}{1-x} = \frac{1}{n_B} \left( \frac{m_e T}{2\pi} \right)^{3/2} \exp\left( -\frac{B_H}{T} \right).
\]

For \( \Omega_0 h^2 = 0.1 \) [2], \( x = 0.003 \), at \( T = 3000 \text{K} \). Thus by this time most of the free electrons have been removed from the cosmological brew, and as a result the Universe becomes effectively transparent to radiation. This era signifies the beginning of a new phase when matter and radiation become decoupled. This phase continues to the present epoch. During this phase each photon frequency is red-shifted as \( R^{-1} \), the number density is \( \propto R^{-3} \) and the temperature is \( \propto R^{-1} \). A background radiation of temperature \( T \simeq 3 \text{K} \) therefore means that the red-shift at this epoch when matter decoupled from radiation was \( z \approx 10^3 \). The opaqueness of the Universe prevent us from “seeing” directly beyond the red-shift of \( z \approx 10^3 \). Thus any evidence of the big bang must come indirectly. The abundance of light nuclei and the MBR provide us with the only means of checking the very early history of the Universe.
3.6 References

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Chapter 4

Nonstandard Cosmological Models

4.1 Background

From eqs.(2.18) and (2.20) we see that if we want a static solution of Einstein’s equations, that is, one in which $\dot{R} = 0$, we must have $\rho + 3p = 0$, which is somewhat unphysical, because, if $\rho > 0$ then $p < 0$, and if $p = 0$ then $\rho = 0$. Therefore Einstein modified his equations by adding the so-called ‘cosmological term’ to the field equations, as follows

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$  \hspace{1cm} (4.1)

where $\Lambda$ is the cosmological constant. Hence eqs.(2.14-16) in (4.1) give

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$  \hspace{1cm} (4.2)

and

$$\frac{\dot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}.$$  \hspace{1cm} (4.3)

The energy conservation law (eq.(2.2)) takes the form

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = -\frac{\dot{\Lambda}}{8\pi G},$$  \hspace{1cm} (4.4)

so that the entropy $S$, defined in

$$dE + \rho dV = dQ \equiv TdS,$$  \hspace{1cm} (4.5)

is no longer constant. However, in the standard model the entropy is constant and that was considered to be one of the problems of the standard model. This is because the entropy of the Universe, today, is unexplainably very huge. Therefore, a variable cosmological term gives rise to an increasing entropy in conformity with the second law of thermodynamics. This is true if the cosmological constant decreases with expansion.
A more general relation between \( k \) and \( \Omega_0 \) is given by
\[
\Omega_0 (1 + 3w) = 2q_0 + \frac{2\Lambda}{3H_0^2},
\]
and
\[
\frac{3}{2} \Omega (1 + w) - q_0 - 1 = \frac{k}{R_0^2 H_0^2},
\]
where \( w \) is defined by the equation of the state \( p = w \rho \), and the subscript 0 denotes the present day quantities. Consider a Universe for which \( k \geq 0 \). It then follows for \( w = 0 \) that
\[
\Omega_0 \geq 1 - \frac{\Lambda_0}{3H_0^2}.
\]
Thus we see that the cosmological constant changes the simple relation between \( k \) and \( \Omega \), see eq.(2.21). For instance, a flat universe is no longer characterized by \( \Omega = 1 \). Note that although \( \Lambda \) is exceedingly small, the term \( \frac{\Lambda}{3H_0^2} \) may be of the order of unity. The value of \( \Omega = 1 \) is preferred on theoretical grounds, but the observational values are mostly much smaller than 1.

If we demand that \( R(t) = R_0 = \text{constant} \), and \( p = 0 \), we get
\[
\rho = \frac{\Lambda}{4\pi G}, \quad k = \frac{\Lambda}{R_0^2}.
\]
Thus \( \Lambda \) must be positive and therefore \( k = +1 \). This is the Einstein’s static model. Later on, Einstein regretted the addition of this term when he knew about the expansion of the Universe.

Many dynamical solutions with the cosmological term were studied by Lemaitre and they were accordingly known as Lemaitre models. In recent years other motivations have been found for introducing a cosmological term. Introducing this term is like introducing a fictitious ‘fluid’ with energy momentum tensor \( T'_{\mu\nu} \), given by
\[
T'_{\mu\nu} = (\rho' + p')u_\mu u_\nu - p'g_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu},
\]
so that the energy density and pressure of this fluid are given by \( \rho' = \frac{\Lambda}{8\pi G}, \quad p' = -\frac{\Lambda}{8\pi G} \). Hence eq.(4.1) can be written as
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G (T_{\mu\nu} + T'_{\mu\nu}).
\]

### 4.2 Limits on the cosmological constant

From eq.(4.4) one gets [32],
\[
q_0 = \frac{\Omega_0}{2} - \frac{\Lambda}{3H_0^2} \quad (4.12)
\]
\[
|q_0 - \Omega_0/2| = |\Lambda/3H_0^2|.
\]
Though $\Omega_0$ and $q_0$ are uncertain one can safely say that $-5 < q_0 < 5$ and $0 < \Omega_0 < 4$, hence $|\Lambda| = 21H_0^2$. By setting $H_0 = 100\text{kms}^{-1}\text{Mpc}^{-1}$, this leads to $\Lambda \sim 10^{-54}\text{cm}^{-2}$. Now in the Newtonian gravitational theory if $r$ is the distance of a point mass with respect to the centre of spherically symmetric distribution of matter, then the force on this unit mass is given by

$$F = -\frac{4\pi}{3}G\rho + \frac{\Lambda}{3}r$$

(4.13)

where $\Lambda$ is the Newtonian form of the cosmological constant:

(i) $\Lambda > 0$ implies a repulsive force and

(ii) $\Lambda < 0$ implies an attractive force.

The matter distribution ceases to be a bound system if $F$ is an outward force. This implies [32] that

$$\Lambda < 4\pi G\rho$$

(4.14)

which gives $\Lambda \sim 10^{-48}\text{cm}^{-2}$. From the above we see that $\Lambda$ acts at dimensions of order of galaxies and has no effect on the solar system.

### 4.3 De Sitter model

Consider a flat RW universe with $\rho(t) = \rho_0 =$ constant. Equation (4.2) becomes

$$\frac{\dot{R}}{R} = H = \text{const.}$$

(4.15)

Its solution is

$$R(t) \propto \exp HT$$

(4.16)

This is known as an inflationary solution. De Sitter obtained this solution for $p = \rho = 0$. However, such a solution is also possible even without $\Lambda$ but with $\rho = \text{const}$. The de Sitter universe is characterized by motion without matter in contrast to the Einstein's universe which is matter without motion.

One gets a similar form for $R(t)$ as in the steady-state theory of Bondi and Gold (1948) and Hoyle. However, unlike in the de Sitter model, which is empty, in the steady state theory there is continuous creation of matter due to the so-called c-field [33]. Although we started with $k = 0$ space-time; inflation is not restricted to $k = 0$. For a Universe with $k = +1$ one finds the solution [39]

$$R \propto H^{-1} \cosh HT ,$$

(4.17)

and for $k = -1$

$$R(t) \propto H^{-1} \sinh HT ,$$

(4.18)

which for large $t$ approaches the inflationary solution.
4.4 Importance of the cosmological constant

There is no convincing evidence available for a nonzero value to the cosmological constant. The present interest in the flat cosmological constant models has also appeared motivated by two reasons:

(i) a $\Lambda$ term helps to reconcile inflation with observations.

(ii) with a $\Lambda$ term it is possible to obtain, for flat universes, a theoretical age in the observed range, even for a high value of Hubble’s constant [34].

4.4.1 A varying cosmological term cosmology

The present estimates of the value of $\Lambda$ are very small. One way to resolve this dilemma with observations is to assume that $\Lambda$ is not a ‘pure’ constant, but rather decreases continuously with cosmic expansion. Hence, we can say $\Lambda$ is extremely small because the Universe is old. Cosmologists postulate a phenomenological law for the decay of this term [26,12].

Recently (Waga & Torres), considering the statistics of gravitational lensing, which is a powerful tool in constraining models of the Universe, especially those with a $\Lambda$-term, have shown that cosmologies with a varying cosmological term give a lower lensing rate. This is due to the fact that in a varying $\Lambda$ cosmology the distance to an object with red-shift $z$ is smaller than the distance to the same object in a constant $\Lambda$ model with the same $\Omega_0$. So, the probability that light coming from the object is affected by a foreground galaxy is reduced in a decaying $\Lambda$ cosmology [37].

Chen and Wu [36] advocated the possibility of particular $R$-dependence behaviour, i.e. $\Lambda \propto R^{-2}$. They argued in favor of this behaviour of $\Lambda$ from some very general arguments in line with quantum cosmology. From dimensional consideration and in the sprit of quantum cosmology, one can always write $\Lambda$ as $M_{\text{pl}}^4$ times a dimensionless product of quantities. For an ansatz for the evolitional behaviour of $\Lambda$, as in the common practice in quantum cosmology, it is more convenient to use the scale factor $R$ instead of the age of the Universe. Supposing that no other parameters are relevant here, the natural ansatz is that $\Lambda$ varies according to a power law in $R$. Theoretically, it can be obtained from some simple and general assumptions in line with quantum cosmology. Observationally, it is not in conflict with present data and may alleviate some problems in reconciling data with the inflationary scenario.

Therefore, one can write

$$\Lambda \propto M_{\text{pl}}^4 \left(\frac{r_{\text{pl}}}{R}\right)^n,$$

where $M_{\text{pl}}$ and $r_{\text{pl}}$ are the Planck mass and length respectively. With $n = 2$ in the above equation, both $\hbar$ and $G$ disappear, since general relativity is a classical theory, and therefore we have $\Lambda = \frac{\gamma}{R^2}$, where $\gamma$
is a dimensionless number of order unity. The case \( n = 3 \) or \( n = 1 \) would either lead to too big or too small value for the present cosmological constant \( \Lambda_0 \).

On the same dimensional grounds Carvalho et al. [42] assumed
\[
\Lambda \propto \frac{1}{l_p^2 (t_H)^2},
\]
where \( l_p, t_p \) are respectively the Planck length and time, \( n \) is an integer and \( t_H \approx H^{-1} \) is the Hubble time. Recalling the general relativity is a classical theory, in order to get rid of \( \hbar \) dependence of \( n \) one needs to put \( n = 2 \). Therefore, \( \Lambda \sim H^2 \).

On the other hand Özer and Taha [26] proposed a cosmological model with \( \Lambda = \frac{1}{R^2} \) (i.e. \( \gamma = 1 \)) based on the critical energy density assumption. Also a cosmological term varying as \( R^{-n}, 9/5 < n < 2 \), was first introduced by Gasperini [35].

Consider the Friedmann equation
\[
H^2 = \frac{8\pi G}{3} \rho - \frac{k}{R^2} + \frac{\Lambda}{3},
\]
For \( k = 0, \Lambda = 3\beta H^2, \beta = \text{const.} \), the above equation can be written as
\[
\beta = \frac{\rho_v}{\rho_v + \rho} = \text{const.},
\]
where \( \rho_v = \frac{\Lambda}{8\pi G} \) is the vacuum density. This equation is nothing but the Freese et al. ansatz. Hence the postulate \( \Lambda = 3\beta H^2 \) is equivalent to the Freese et al. one. One can also write the above Friedmann equation as
\[
\Omega + \frac{\Lambda}{3H^2} = 1,
\]
where \( \Omega \) is the density parameter. One can assume that the role of inflation can be described by specifying that it drives the Universe to a state of geometrical flatness, corresponding to \( \Omega + \frac{\Lambda}{3H^2} = 1 \). It is useful to regard the quantity on the left hand side of the latter equation as \( \Omega_{eff} = 1 \), with the term \( \frac{\Lambda}{3H^2} \) regarded as the vacuum energy contribution to the density parameter. Using this definition, we see that inflation always drives the Universe to \( \Omega_{eff} = 1 \).

In this work we will investigate some of these laws and discuss their implications on the evolution of the Universe.

### 4.5 Variable G models

#### 4.5.1 Theories in which G varies with time

Theories of this type were first proposed by Milne, Dirac, and Jordan. Later, Brans and Dicke (BD), Hoyle and Narlikar, and Dirac put forward more elaborate theories of this type. A variation of \( G \) with
time has a considerable effect on the evolution of the Earth and Sun, and on the orbits of the Moon and planets. If gravity has changed appreciably over the life time of the Earth, the radius of the Earth might have been affected. It has been suggested that the continents all fitted together at one time on a much smaller Earth. As the gravitational constant reduced, the Earth expanded to its present size and the continents were forced apart. Also a star, like the Sun, in its hydrogen burning phase [3] has a luminosity \( L \)

\[ L \propto G^{7.7}, \]  

(4.19)
as so would have been brighter in the past if \( G \) decreased with time \( t \). The effect of this on life on Earth would be enhanced by the fact that the Earth must be moving away from the Sun if \( G \) is declining. A too fast decline in \( G \) would lead to the Sun having already become a red giant. A varying \( G \) leads to variation in the Moon’s distance and period. The orbits of the planets are also modified, and this could show up in the radar time-delay experiments [24]. It should be emphasized that some theories in which \( G \) varies also predict other changes (to preserve energy conservation, e.g.) which can mask the above effects. A stronger limit on \( \dot{G} \) follows if \( ^4He \) and \( ^2H \) are believed to be synthesized in the big-bang hot phase. One then has

\[ \left| \frac{\dot{G}}{G} \right| \leq 10^{-12}\text{yr.}^{-1} \]  

(4.20)
The BD cosmology represents the simplest extension of general relativity (GR). In addition to the tensor gravitational fields represented by the metric tensor, there is a scalar field (related to the gravitational ‘constant’) which is a function of time. BD take \( \Lambda = 0 \) and seek to satisfy Mach’s principle, that local inertia properties should be determined by the gravitational field of the rest matter in the Universe, by taking

\[ G^{-1} \sim \sum_{\text{Universe}} \frac{m}{r} \]  

(4.21)
The models for \( k = 0 \) are particularly simple, since, there

\[ R \propto t^q, \quad G \propto t^r \]

where

\[ q = \frac{2(1 + w)}{4 + 3w}, \quad r = \frac{2}{4 + 3w} \]
w being the ‘coupling constant’ between the scalar field and geometry: \( w \rightarrow \infty \) gives the Einstein-de Sitter model. Note that for general \( w \), \( G\rho t^2 = \text{const.} \). Dirac’s 1937 model is obtained by setting \( w = -2/3 \). However, BD theory makes slightly different predictions from GR for the deflection of light by the Sun and for the advance of planets [25]. And if helium is synthesized in the big bang, there is
an even stronger limit, \(w > 100\) [24]. Also if \(G\) has had greater values in the past than it has now, one would expect a small mass density to have the same effects as a larger mass density has later. An increasing \(G\) would lead to a contraction of the earth.

### 4.6 A new formalism of variable \(G\) models

Ever since Dirac first considered the possibility of variable gravitational “constant”, \(G\) [4], there have been numerous modification of general relativity to allow for variable \(G\) [5]. These theories have not gained wide acceptance. However, recently [6-10] a modification has been proposed treating \(G\) and the cosmological ‘constant’ \(\Lambda\) as non constant coupling scalars. Einstein’s equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu},
\]

are considered, where \(G\) and \(\Lambda\) are coupling scalars and the other symbols have their usual meanings. The principle of equivalence then demands that only \(g_{\mu\nu}\) and not \(G\) and \(\Lambda\), must enter the equation of motions of particles and photons. This approach is appealing because it leaves Einstein's equation formally unchanged since variation in \(\Lambda\) is accompanied by a variation in \(G\) [6-10], in such a way the usual energy conservation law \(T_{\nu\mu} = 0\), holds. If we take the divergence of (5.4) and use the Bianchi identities, we get

\[
G_{\nu\tau} T^{\nu\mu} + \Lambda_{\nu\tau} g^{\nu\mu} = 0
\]

This approach, however, is non covariant and the field equations cannot be derived from a Hamiltonian. We note that the propagation equations for the scalars are not contained in eqs.(4.24). Despite this drawback, there are several advantages of the present approach. This approach could be a limit of a more viable fully covariant theory such as five dimensional Kaluza-Klein theory [11]. Various possibilities for variable \(G\) can be investigated. The problems of the standard model could be solved as in the inflationary scenario [12]. The variation of \(G\) with time is reasonable, since \(G\) couples geometry to matter, and in an expanding Universe we expect \(G = G(t)\). It is reasonable to assume that the Universe had always the Einstein-de Sitter critical density. For unless the Universe had this, it would have diverged, very rapidly from it. With the present Hubble constant \(H_p = 5 \times 10^{-11}\text{yr}^{-1}\), the present critical energy density \(\rho = \frac{3H_p^2}{8\pi G} \simeq 2 \times 10^{-47}\text{GeV}^4\). The current energy density of the Universe, on the other hand, is between \(10^{-47}\text{GeV}^4\) and \(10^{-48}\text{GeV}^4\)[26].

With these assumptions we will show that there are models in which \(G\) increases with time and other models in which it decreases.

Recently, the flat Friedmann Robertson Walker (FRW) models have been studied with the present
formalism. A number of solutions were presented including de Sitter-type ones relevant to inflation [30,31].

4.6.1 Bulk viscous solutions

The role of viscosity in cosmology has been studied by various authors [18,19,29]. It was initially hoped that neutrino viscosity could smooth all initial anisotropies and leads to the isotropic Universe that we observe today [13-15,22]. The bulk viscosity associated with the Grand Unified Theory (GUT) [6,16,17] phase transition can lead to an inflationary universe. It is also known that the introduction of bulk viscosity can avoid the big bang singularities [18,19,27]. When viscosity is introduced in a fluid the fluid becomes imperfect, i.e. the one in which the pressure, density and velocity vary appreciably over distances of the order of a mean free path, or over times of the order of mean free time, or both. For such fluids, the kinetic energy is dissipated as heat. For relativistic fluids, the dissipative effects play an important role in the history of the early Universe. The energy - momentum tensor of the imperfect fluid ($T'_{\alpha\beta}$) takes the form [23]

$$T'_{\alpha\beta} = T_{\alpha\beta} + \Delta T_{\alpha\beta} \quad (4.24)$$

where $(\Delta T_{\alpha\beta})$ is regarded as a correction term to the energy momentum-tensor of the perfect fluid $T_{\alpha\beta}$. In a comoving frame $\Delta T_{00} = 0$, and in a general frame it satisfies

$$u^\alpha u^\beta \Delta T_{\alpha\beta} = 0 \quad (4.25)$$

The most general form containing viscosity allowed by this condition and the second law of thermodynamics is (see Appendix A)

$$\Delta T_{\alpha\beta} = \zeta H^{\alpha\gamma} H^{\beta\delta} W_{\gamma\delta} + \eta H^{\alpha\beta} u^\gamma \quad (4.26)$$

where $W_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} - \frac{2}{3} g_{\alpha\beta} u^\gamma u_\gamma$ is the shear tensor and $H_{\alpha\beta} = g_{\alpha\beta} - u_\alpha u_\beta$ is the projection tensor on a hyperplane normal to $u^\alpha$. $\eta$ and $\zeta$ are the coefficient of bulk and shear viscosity, respectively. We will here consider the bulk viscosity only since the shear viscosity plays no role in a RW model. Therefore

$$\Delta T_{\alpha\beta} = \eta (g_{\alpha\beta} - u_\alpha u_\beta) u^\gamma_\gamma$$

Hence

$$T'_{\alpha\beta} = (\rho + p - \eta u^\gamma_\gamma) u_\alpha u_\beta - (p - \eta u^\gamma_\gamma) g_{\alpha\beta}$$

If we now replace $p - \eta u^\gamma_\gamma$ by $p^* [1]$, the above equation becomes

$$T'_{\alpha\beta} = (\rho + p^*) u_\alpha u_\beta - p^* g_{\alpha\beta}$$

---

[1] The total pressure $p^*$ accounts for the isotropic pressure $p$ plus viscous terms can be represented as a polynomial in $\theta$ ($\theta = u^\gamma_\gamma$): $p^* = p - \sum_{k=1}^{N} \alpha_k \theta^k$ where $\alpha_k$ are in general functions of $\rho$ [21,28,41].
Apart from the above replacement, this equation looks the same as that of a perfect fluid. Therefore, the introduction of the bulk viscosity does not alter the isotropy and homogeneity of the Universe.

The field equations with bulk viscosity can be obtained from the general relativistic field equations by replacing the pressure term, \( p \), by, \( p^* \), where

\[
p^* = p - \eta u^\gamma_{,\gamma}
\]

In a Robertson-Walker model, we have \( u^\gamma_{,\gamma} = 3H \), so that \( p^* = p - 3\eta H \).

In eq.(4.27) \( \eta \) is usually taken to have a power law form \[19\]

\[
\eta = \eta_0 \rho^n
\]

Where \( \eta_0 \geq 0 \), \( \rho \) is the energy density, \( n \) is a constant and \( H \) is the Hubble constant. In addition to the linear dependence of \( \eta \) upon \( H \), as in the above equation, some workers \[20,21,28\] have considered a quadratic dependence upon \( H \), i.e.

\[
p^* = p - 9\zeta H^2
\]

The \( \zeta = \text{const} \) models were analyzed by Romero \[20\] . The bulk viscous models considered so far are endowed with particle creation . In the Chapter 7 and 8 we will consider a model with variable \( G \) and \( \Lambda \) and bulk viscosity . This combination of \( G \), \( \Lambda \) and \( \eta \) has not been considered before. The model turns out to have many interesting features. Various models could be reproduced from this model by taking particular values of \( n \), where \( \eta = \eta_0 \rho^n \).

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Chapter 5

A Vacuum Decaying Cosmological Model

5.1 Introduction

The cosmological constant problem [1], that phase transitions in the early Universe would have left the cosmological constant larger than the observed upper bound, \( \Lambda \geq 10^{-84} \text{(GeV)}^2 \) [1, 2], by about 120 order of magnitude, is a major puzzle of cosmology and particle physics. Considerable efforts were made in seeking its solution [1]. One approach attempts to avoid this impasse by allowing \( \Lambda \) to vary smoothly with time so that models in which it was appreciable in the past could be constructed [1]. Examples are the models of Özer and Taha [1], Chen and Wu [4], as well as some of their later generalizations [5, 6], all of which require \( \Lambda \propto R^{-2} \), where \( R \) is the Robertson-Walker (RW) scale factor. Recently Carvalho, Lima and Waga [7] proposed the ansatz

\[
\Lambda = 3\beta H^2 + \frac{3\gamma}{R^2}
\]

(5.1)

where \( \beta \) and \( \gamma \) are dimensionless numbers (natural units being used) and \( H = \frac{\dot{R}}{R} \) is Hubble’s constant (an overdot denotes time differentiation). They suggested the \( \beta \)-term in Equation (5.1) on the basis of simple dimensional arguments consistent with quantum gravity. Special cases of eq.(5.1) when \( \beta=0 \) are: Chen and Wu [4] singular model (\( \gamma \) arbitrary), Özer - Taha (OT) [3] nonsingular cosmology (\( \gamma = 1 \)), and the singularity - free models of Ref. [6] (1/2 < \( \gamma \) < 1). Carvalho et al [7] have investigated the effect of \( \beta \)-term in eq.(5.1) on the singular model of Chen and Wu [4]. In the present work we study the implications of this term for nonsingular cosmologies of Özer - Taha [3] type.
5.2 Field equations

In a RW universe with a perfect fluid energy-momentum tensor, Einstein’s equations with a variable $\Lambda$ give ($\alpha \equiv 3/8\pi G$):

$$\alpha^{-1} \rho = \left[ \frac{\dot{R}}{R} \right]^2 + \frac{k}{R^2} - \frac{\Lambda(t)}{3},$$  \hspace{1cm}  (5.2)  \\
$$\frac{3}{2} \alpha^{-1} (\rho + p) = \left[ \frac{\dot{R}}{R} \right]^2 + \frac{k}{R^2} - \frac{\ddot{R}}{R}. \hspace{1cm} (5.3)$$

In these equations $\rho$ and $p$ are the cosmic energy density and pressure and $k$ the curvature index.

5.3 Nonsingular Model

The introduction of eq.(5.1) and the radiation equation of state $p = \frac{1}{3} \rho$ in eqs.(5.2) and (5.3) lead to ($\beta \neq \frac{1}{2}$):

$$\dot{R}^2 = \frac{(2\gamma - k)}{(1 - 2\beta)} + A_0 R^{-2+4\beta}, \hspace{1cm} (5.4)$$

$$\rho = \frac{\alpha(\gamma - \beta k)}{(1 - 2\beta)} R^{-2} + \alpha(1 - \beta) A_0 R^{-4+4\beta}, \hspace{1cm} (5.5)$$

$$\rho_v = \frac{\alpha(\gamma - \beta k)}{(1 - 2\beta)} R^{-2} + \alpha \beta A_0 R^{-4+4\beta}, \hspace{1cm} (5.6)$$

Where $A_0$ is a constant and $\rho_v = \frac{\alpha}{3} \Lambda$ is the vacuum energy density. Equations (5.4)-(5.6) were analyzed by Carvalho et al [7] for the case $A_0 > 0$ and $\beta < 1$ corresponding to a singular universe. As pointed by them [7] a ”natural extension (of their work ) would be to explore different scenarios obtained by making $A_0 < 0,”$ for which the cosmology is nonsingular. One possible line for such an investigation would be to consider a nonsingular cosmology of the OT type based on eq.(5.1). This is the theme of what follows.

5.4 Radiation Universe

Requiring $A_0 < 0$ in eq.(5.4) implies

$$\frac{2\gamma - k}{1 - 2\beta} > 0. \hspace{1cm} (5.7)$$

It is then possible for $R$ to have had an initial minimum nonvanishing value at $t=0$, say. The necessary condition for the existence of this minimum in an expanding Universe is $\dot{R} = 0$ at $t=0$. We explore this possibility. In order to reach $\dot{R} = 0$ at $t=0$ the exponent of $R$ in eq.(5.4) must be negative, implying that $\beta < 1/2$. Hence from eq.(5.7),

$$k < 2\gamma. \hspace{1cm} (5.8)$$
Let $R = R_0$ at $t=0$, when $\dot{R} = 0$. Clearly $R_0 \neq 0$ or else $\dot{R}^2(0) = -\infty$. Thus

$$\dot{R}^2 = \frac{(2\gamma - k)}{(1 - 2\beta)} \left[1 - \frac{R_0^{2-4\beta}}{R^{2-4\beta}}\right] < \frac{2\gamma - k}{1 - 2\beta}, \quad (5.9)$$

$$\rho = \frac{\alpha(\gamma - \beta k)}{(1 - 2\beta)R^2} \left[1 - \frac{(2\gamma - k)(1 - \beta)R_0^{2-4\beta}}{(\gamma - \beta k)R^{2-4\beta}}\right], \quad (5.10)$$

$$\rho_v = \frac{\alpha(\gamma - \beta k)}{(1 - 2\beta)R^2} \left[1 - \frac{\beta(2\gamma - k)R_0^{2-4\beta}}{(\gamma - \beta k)R^{2-4\beta}}\right], \quad (5.11)$$

From eq.(5.10),

$$\rho_0 = \frac{\alpha(k - \gamma)}{R_0^2} \quad (5.12)$$

which is the same as eq.(5.9) of Ref.[6]. Thus the physical condition $\rho_0 \geq 0$ requires.

$$k \geq \gamma. \quad (5.13)$$

Equations (5.8) and (5.12) lead to

$$1/2 < \gamma \leq k = 1, \quad (5.14)$$

implying a closed universe. The result (5.14) coincides with eq.(5.13) of Ref. [6]. Classical general relativity without or with constant $\Lambda$ does not explain the origin of cosmic entropy because classical Einstein’s equations are purely adiabatic and reversible. In the present cosmology, however, the change in the entropy $S$ is related to the temperature $T$ and the change in $\rho_v$, by [6]

$$T dS = -R^3 d\rho_v, \quad (5.15)$$

so that from eq.(5.11),

$$T \frac{dS}{dR} = \frac{2\alpha(\gamma - \beta)}{1 - 2\beta} \left[1 - \frac{2\beta(2\gamma - 1)(1 - \beta)R_0^{2-4\beta}}{(\gamma - \beta k)R^{2-4\beta}}\right], \quad (5.16)$$

in the radiation Universe. Therefore $\frac{dS}{dR} > 0$ provided the second term inside the bracket of eq.(5.16) is less than unity. Since $2\beta < 1$, $0 < (2\gamma - 1 \leq 1)$ but $\frac{1 - \beta}{2\gamma} \geq 1$, this would be guaranteed, for $R \geq R_0$, if we simply choose $\gamma = 1$. Henceforth we restrict ourselves to the model with $\gamma = 1, \beta \neq 0$. We will consider two cases separately

### 5.5 Nonsingular model: case 1.

Putting $\gamma = k = 1$ in eqs.(5.9)-(5.12) we obtain

$$\dot{R}^2 = \frac{1}{1 - 2\beta} \left[1 - \frac{R_0^{2-4\beta}}{R^{2-4\beta}}\right] < (1 - 2\beta)^{-1}, \quad (5.17)$$
\[ \rho = \alpha (1 - \beta) H^2, \tag{5.18} \]
\[ \rho_v = \frac{\alpha (1 - \beta)}{(1 - 2\beta) R^2} \left[ 1 - \frac{\beta R_0^{2-4\beta}}{(1 - \beta) R^{2-4\beta}} \right], \tag{5.19} \]
\[ \rho_0 = 0. \tag{5.20} \]

these equations extend the \( \gamma = 1, \beta = 0 \) OT model, albeit in a different direction than that of Ref.[6]. As in the OT model the initial Universe here also is empty (and cold) but the density parameter
\[ \Omega \equiv \frac{\rho}{\alpha H^2} = 1 - \beta < 1, \tag{5.21} \]
compared to \( \Omega \equiv 1 \) in the OT model. The density \( \rho \) attains a maximum
\[ \rho_{\text{max}} = \frac{\alpha}{2 R_0^2 [2(1 - \beta)]^{1-2\beta}} < \frac{\alpha}{2 R_0^2}, \tag{5.22} \]
at
\[ R = R_{\text{max}} = [2(1 - \beta)]^{\frac{1}{2-4\beta}} R_0 > R_0. \tag{5.23} \]

Note that in Ref.[6] \( \rho_0 < \alpha/2 R_0^2 \) and \( \rho_{\text{max}} \geq \alpha/4 R_0^2 \). An estimate of \( R_0 \) and hence an upper bound on \( \rho_{\text{max}} \) can be deduced as follows. From eq.(5.17),
\[ 0 < \ddot{R} \leq R_0^{-1}, \tag{5.24} \]
implying the existence of a natural cosmic acceleration limit in the radiation Universe. Such a maximal acceleration, of the order of Planck mass \( M_{pl} = G^{-1/2}, (h/2\pi = c = 1) \), has been discussed before [8]. Thus taking \( R_0^{-1} \sim M_{pl} \) yield \( R_0 \sim G^{1/2} \approx 8 \times 10^{-20} \text{(GeV)}^{-1} = 1.6 \times 10^{-33} \text{cm} \), so that \( \rho_{\text{max}} < 3 \times 10^9 \text{kgm}^{-3} \). Finally, for \( R \gg R_0, R \sim t \) (linear expansion) and \( \rho = \rho_v \sim t^{-2} \). The variation \( \rho_v \propto t^{-2} \)
is regarded by Berman [9] as more fundamental that expression of \( \rho_v \) in terms of \( R \), e.g. \( \rho_v \propto R^{-2} \) [3-6].

### 5.6 Radiation and matter

Equation (5.2) and (5.3) may be combined to give
\[ \frac{dE}{dR} + 3 p R^2 = -\frac{\alpha}{3} R^3 \frac{d\Lambda}{dR}, \tag{5.25} \]
where \( E = \rho R^3 \) and \( \Lambda \) may be written as \( (\gamma = k = 1) \):
\[ \Lambda = \frac{3 \alpha^{-1} \beta \rho}{(1 - \beta)} + \frac{3}{R^2}. \tag{5.26} \]
These equations are assumed to hold throughout cosmic evolution. Following Özer and Taha \[3\] and Ref. \[6\] we assume the Universe to have evolved through: a very early era, \(R_0 \leq R \leq R_1\), say, of pure radiation (discussed in subsection ); a subsequent period of \(R_1 \leq R \leq R_2\) of matter generation; and, lastly, for \(R \geq R_2\), an era of radiation and conserved non relativistic matter reaching to the present. Except, perhaps, for the matter generation epoch the vacuum is assumed to decay into radiation only. Hence with the radiation and matter densities denoted by \(\rho_r\) and \(\rho_m\) respectively \((\rho_r + \rho_m = \rho)\), the energy, in volume \(R^3\), of non relativistic matter is \(E = \rho_m R^3 = \rho_m R_p^3 = E_{mp}\), where the subscript “p” designates present-day quantities. We also assume that non relativistic matter has zero pressure so that \(p = p_r = \frac{1}{3} \rho_r\). Chen and Wu \[4\] and also Carvalho \textit{et al} \[7\] assume that the vacuum decays into non relativistic matter in the matter-dominated (MD) universe. But Freese \textit{et al} \[10\] have demonstrated that such a scenario is not, apparently, favored by observations. Under our assumptions eqs.(5.25) and (5.26) yield, when \(R \geq R_2\),

\[
\frac{d\rho_r}{dR} + \frac{4(1 - \beta)}{R} \rho_r = \frac{3\beta \rho_m R^3_p}{R^3} + \frac{2\alpha(1 - \beta)}{R^3} .
\]

This has the solution

\[
\rho_r = \frac{3\beta \rho_m R^3_p}{(1 - 4\beta)R^3} + \frac{\alpha(1 - \beta)}{(1 - 2\beta)R^2} \left[1 + \frac{\omega R^2_{p-4\beta}}{R^{2-4\beta}}\right] ,
\]

where

\[
(1 + \omega) = \frac{(1 - 2\beta)R^2_p \rho_{mp} \left[\rho_{rp} - \frac{3\beta}{1 - 4\beta} \rho_{mp}\right]}{\alpha(1 - \beta)}
\]

\[
= (1 - 2\beta)[\rho_{rp} - \frac{3\beta}{1 - 4\beta} \rho_{mp}] [(1 - \beta) \rho_{vp} - \beta(\rho_{rp} + \rho_{mp})]^{-1}.
\]

Hence from eq.(5.26),

\[
\rho_v = \frac{\beta \rho_m R^3_p}{(1 - 4\beta)R^3} + \frac{\alpha(1 - \beta)}{(1 - 2\beta)R^2} \left[1 + \frac{\beta \omega R^2_{p-4\beta}}{(1 - \beta)R^{2-4\beta}}\right] ,
\]

and by eq.(5.2),

\[
\dot{R}^2 = \frac{\alpha^{-1} \rho}{(1 - \beta)R^2} = \frac{\alpha^{-1} \rho_m R^3_p}{(1 - 4\beta)R} + \frac{\omega R^2_{p-4\beta}}{(1 - 2\beta)R^{2-4\beta}} + (1 - 2\beta)^{-1}.
\]

From eq.(5.31),

\[
\Omega = 1 - \beta < 1 ,
\]

as before (see eq.(5.21)). Lastly, the present value of the deceleration parameter \(q = -R\ddot{R}/\dot{R}^2\) is

\[
q_p = [1/2 + \frac{\omega(1 - 4\beta)(1 - 2\beta)}{(1 + \omega)(1 - \beta)} \rho_{vp} - \frac{3\beta}{1 - 4\beta}] \times [1 + \frac{1 - 4\beta}{\beta} \rho_{vp} - \frac{3\beta}{1 - 4\beta}]^{-1}.
\]
For $\beta = 0$ eqs.(5.28)-(5.33) reduce to the corresponding OT results. Denote by $R_{eq}$ the value of $R$ at $t = t_{eq}$, the time when radiation and matter were balanced in equilibrium. Let $E_r = \rho_r R^3$, where $\rho_r$ is given by eq.(5.28), be the radiation energy. Then

$$E_r(R_{eq}) = E_{mp}$$  \hspace{1cm} (5.34)

$$E_r(R) > E_{mp} \text{ when } R < R_{eq}$$  \hspace{1cm} (5.35)

$$E_r(R) < E_{mp} \text{ when } R > R_{eq}$$

The condition that $E_r$ was decreasing as $R$ approached $R_{eq}$ would then imply that

$$(1 - 4\beta)\omega > \frac{R_{eq}^{2-4\beta}}{R_p^{2-4\beta}}.$$  \hspace{1cm} (5.36)

Hence either

$$\omega > 0 \text{ and } \beta < 1/4, \text{ or } \omega < 0 \text{ and } 1/4 < \beta < 1/2.$$  \hspace{1cm} (5.37)

We discuss each case separately.

1. $\omega > 0, \beta < 1/4$

From eq.(5.29) we have

$$\frac{3\beta}{1 - 4\beta} < \frac{\rho_r}{\rho_m} \equiv \delta_r.$$  \hspace{1cm} (5.38)

Hence

$$\beta < \delta_r.$$  \hspace{1cm} (5.39)

There are three subcases:

(a) $\beta = 0$: This corresponds to the OT \[3\] model.

(b) $\beta > 0$: Assuming $\delta_r \ll 1$, i.e. the present Universe is MD, we have $0 < \beta \ll 1$. Also from eq.(5.30) $d\rho_e/dR < 0$, so that entropy is continuously generated. From eq.(5.28),

$$\delta_r \equiv \frac{\rho_r}{\rho_m} = \frac{3\beta}{1 - 4\beta} + \frac{\alpha(1 - \beta)R}{(1 - 2\beta)\rho_{mp} R_p^3} \times [1 + \frac{\omega R_p^{2-4\beta}}{R^{2-4\beta}}].$$  \hspace{1cm} (5.40)

In the early radiation and matter universe $\delta_r \gg 1$. Using this condition in eq.(5.40) and noting from (5.38) that

$$3\beta(1 - 4\beta)^{-1} < \delta_r \ll 1$$ one has

$$1 + \frac{\omega R_p^{2-4\beta}}{R^{2-4\beta}} \gg \frac{(1 - 2\beta)\rho_{mp} R_p^3}{\alpha(1 - \beta)R}.$$ \hspace{1cm} (5.41)

But by condition (5.36)

$$\frac{\omega R_p^{2-4\beta}}{R^{2-4\beta}} > \frac{R_{eq}^{2-4\beta}}{(1 - 4\beta)R_p^{2-4\beta}} \gg 1.$$  \hspace{1cm} (5.42)
for \( R_{eq}/R \gg 1 \) and \( 0 < \beta < \delta_{rp} \ll 1 \). Hence from eqs.(5.31), (5.41) and (5.42),

\[
\dot{R}^2 \approx \frac{\omega R_p^{2-4\beta}}{(1 - 2\beta)R^2-4\beta} \approx \frac{\alpha^{-1}\rho_r R^2}{(1 - \beta)},
\]

(5.43)

when \( R_{eq}/R \gg 1 \). Consider now \( \rho_v \) of eq.(5.30). Assuming, plausibly, that for \( R \ll R_{eq} \) the vacuum energy was much more important than the energy of non relativistic matter, i.e. \( \rho_v/\rho_m = \delta_v \gg 1 \), we have

\[
1 + \frac{\beta \omega R_p^{2-4\beta}}{(1 - \beta)R^2-4\beta} \gg \frac{(1 - 2\beta)\rho_{mp} R_p^3}{\alpha(1 - \beta)R}.
\]

(5.44)

But by (5.36),

\[
\frac{\omega \beta R_p^{2-4\beta}}{(1 - \beta)R^2-4\beta} > \frac{\beta R_{eq}^{2-4\beta}}{(1 - (1/4\beta))R^2-4\beta}.
\]

(5.45)

For \( R \ll R_{eq} \), and provided \( \beta \) is not negligibly small (much smaller than \( \delta_{rp} \)), the RHS of (45) will be considerably larger than unity. (If \( \beta \ll \delta_{rp} \ll 1 \), the \( \beta \)-term in \( \Lambda \) may be dropped, which is subcase (a)). Then eqs.(5.30), (5.45), (5.44) and (5.43) imply

\[
\rho_v \approx \frac{\alpha \beta \omega R_p^{2-4\beta}}{(1 - 2\beta)R^4-4\beta} \approx \frac{\beta \rho_r}{(1 - \beta)}.
\]

(5.46)

The relation between \( \rho_v \) and \( \rho_r \) in this result is identical to eq.(5.7) in the paper of Freese et al. [10] where a parameter \( x = \rho_v/(\rho_r + \rho_v) \) replaces \( \beta \). But whereas letting \( x \to 0 \) in Ref. [10] produces standard cosmology, one is not entitled to the limit \( \beta \to 0 \) in eq.(5.46) here because the derivation of this equation assumed that \( \beta \) is not vanishingly small.

(Taking naively \( \beta \to 0 \) in eq.(5.46) one concludes erroneously that \( \rho_v = 0 \) in the OT model). Integration of eq.(5.43), taking approximately \( t = t_2 \approx 0 \) when \( R = R_2 \approx 0 \), gives

\[
R = \left[ \frac{2(1 - \beta)\omega^{1/2}R_p^{1-2\beta}}{(1 - 2\beta)} \right]^{1/(2(1 - \beta))} t_2^{1/(2(1 - \beta))}.
\]

(5.47)

Hence from eq.(5.46),

\[
\rho_v \approx \frac{\alpha \beta}{4(1 - \beta)^2} \approx \frac{\beta}{(1 - \beta)} \rho_r,
\]

(5.48)

which is the same as eq.(5.8) of Freese et al. [10]. As noted by them in eq.(5.48) has been suggested by various authors [11], with \( \beta \) presumably dependent on the particular model of vacuum decay. In the present model \( 0 < \beta < \delta_p \).

(c) \( \beta = -|\beta| < 0 \) : Then one has from condition (5.36),

\[
1 - \frac{|\beta| \omega R_p^{2+4|\beta|}}{(1 + |\beta|)R^2+|\beta|} \leq 1 - \frac{|\beta| R_{eq}^{2+4|\beta|}}{(1 + |\beta|)(1 + 4|\beta|)R^2+4|\beta|}.
\]

(5.49)
Thus provided $|\beta|$ is not vanishingly small both sides of this inequality will be negative when $R \sim R_2, R_{eq}/R \gg 1$. (If $|\beta|$ is vanishingly small subcase (a) is retrieved). It then follows from eq.(5.30) that $\rho_v < 0$ and $\frac{d\rho_v}{dR} > 0$ for $R \sim R_2$, implying a decreasing entropy in the early radiation and matter universe. Hence we exclude $\beta < 0$.

Observe that eq.(5.31) implies $\ddot{R} < 0$ if $\omega > 0$ and $\beta < 1/4$. On the other hand from eq.(5.24) $\ddot{R} > 0$ in the radiation universe. This reversal of the sign of $\ddot{R}$ is suggestive of an intermediate phase transition period separating the pure radiation and the radiation and matter eras.

2. $\omega = -|\omega| < 0$, $1/4 < \beta < 1/2$.

Consider eq.(5.31). Combining the second and third terms on its RHS and using condition (5.36) give

$$\frac{1}{(1-2\beta)}[1 - \frac{|\omega|}{R_{eq}^2}]$$ (5.50)

$$< \frac{1}{(1-2\beta)}[1 - \frac{R_{eq}^2}{(4\beta - 1)R^2}]$$.

For $R_2 \leq R \leq R_{eq}$ the expression to the right of the inequality, and so also that to its left, is negative. This leads to $\dot{R}^2 < 0$ and $\rho < 0$ in eqs.(5.31) and (5.31), which is physically inadmissible. Thus we conclude from this subsection that $\omega > 0$ and $0 < \beta < \delta_{rp}$, where $\delta_{rp}$ is present ratio of radiation-to-matter energy density.

### 5.7 Phase transition

From eqs.(5.25) and (5.26) one has

$$\frac{dE}{dR} = 2\alpha(1-\beta) + 3\beta R^2 \rho + -3(1-\beta)R^2 \rho$$, (5.51)

leading, on using $E_r = \rho_r R^3$ and eq.(28), to

$$3 \int_{R_0}^{R_2} R^2[(1-\beta)p - \beta \rho]dR = 2\alpha(1-\beta)(R_2 - R_0) - E_2$$

$$= -2\alpha(1-\beta)R_0 - \frac{(1-\beta)E_{mp}}{(1-4\beta)} + \frac{\alpha(1-\beta)(1-4\beta)R_2 F(\omega)}{(1-2\beta)}$$, (5.52)

where

$$F(\omega) \equiv 1 - \frac{\omega R_{eq}^2}{(1-4\beta)R_2^{2-4\beta}}$$ . (5.53)

But by condition (5.36),

$$F(\omega) < 1 - \frac{R_{eq}^{2-4\beta}}{(1-4\beta)^2 R_2^{2-4\beta}} < 0$$ (5.54)
Hence the integral in eq.(5.52) in negative implying that
\[ \frac{3p}{\rho} < 3\beta(1 - \beta)^{-1} < 1, \quad (\beta < 1/4), \]
or by condition (5.38),
\[ \frac{3p}{\rho} < 3\beta(1 - \beta)^{-1} < 3\beta(1 - 4\beta)^{-1} < \delta_{rp}, \quad (5.55) \]
during at least part of the period \((R_1, R_2)\). The condition (5.55) is clearly incompatible with the pure radiation equation of state. The condition (5.55) is incompatible with the equation \(\frac{3p}{\rho} = 1 - \frac{E_m}{E}\) of radiation and matter. For if one were to suppose that it is consistent with this equation one would have \(1 - \frac{E_m}{E} < \delta_{rp}\), which does not admit simultaneously the assumptions of a present MD universe \((\delta_{rp} \ll 1)\) and an early radiation-dominated universe \((E_m/E \ll 1)\).

Thus we interpret the era \((R_1, R_2)\) as a phase transition period associated with the creation of rest mass and the appearance of decelerated cosmic expansion. During this period the pressure may become negative. (If \(\beta = 0\) negative pressure must occur.)

### 5.8 Baryon-to-photon ratio

If the radiation produced by vacuum has a Planckian thermal distribution then its temperature \(T\) will be related to its density \(\rho_r\) by
\[ \rho_r = \frac{\pi^2}{30} g_{eff} T^4. \quad (5.56) \]
where \(g_{eff}\) is the effective number of spin degrees of freedom. Then in the early \((R \ll R_{eq})\) radiation and matter universe we have from eq.(5.56) and (5.48),
\[ T(t) \approx \left[ \frac{15\alpha}{2\pi^2(1 - \beta) g_{eff}} \right]^{1/4} t^{-1/2}, \quad (5.57) \]
which together with eq.(5.47) imply
\[ T \sim R^{\beta - 1}. \quad (5.58) \]
Equations (5.57) and (5.58) are identical to the corresponding results of Freese et al. [10]. If the photon Planckian spectrum is to be maintained the energy per photon \(\varepsilon_\gamma\) must red-shift like the temperature so that \(\varepsilon_\gamma \sim R^{\beta - 1}\). Then eqs.(5.48) and (5.47) imply that the photon number density
\[ n_\gamma \propto \rho_\gamma / \varepsilon_\gamma \sim R^{3(\beta - 1)}. \]
on the other hand the baryon number density \(n_B \sim R^{-3}\). Hence the baryon-to-photon ratio
\[ \eta \equiv \frac{n_B}{n_\gamma} \sim R^{-3\beta} \sim T^{3\beta/(1 - \beta)}, \quad R \ll R_{eq}. \quad (5.59) \]
This is eq.(12) of Freese et al.
5.9 Cosmic nucleosynthesis

Primordial synthesis of the light elements nuclei crucially depends on the temperature-time relation and the expansion rate of the early Universe. In the present model the temperature-time relation is given by eq.(5.57). The expansion rate, from eq.(5.43), is

\[ \frac{\dot{R}}{R} \approx \left( \frac{\alpha^{-1} \rho}{(1 - \beta)} \right)^{1/2}, \quad R \ll R_{\text{eq}} \quad (5.60) \]

coinciding with the rate deduced in Ref.[10]. Primordial nucleosynthesis depends also on \( \eta \) (eq.(5.59) here). Since eqs.(5.57)-(5.60) coincide with the corresponding relations of Freese et al. [10] they would lead to these authors nucleosynthesis constraints on \( \beta \), namely \( \beta \leq 0.1 \). But the present models require \( 0 < \beta \leq \delta_{rp} \). Hence it is consistent with the observed element abundance from primordial nucleosynthesis provided \( \delta_{rp} < 0.1 \) - a condition that readily obtained in a present MD universe.

5.10 Entropy generation

The decreases of \( \eta \) with temperature displayed by eq.(5.59) is a consequence of the production of the entropy by the vacuum. Requiring that \( \eta \) at nucleosynthesis falls in the range \( 10^{-10} \leq \eta \leq 10^{-9} \), and that at grand unification \( \eta_{GUT} < 10^{-4} \), Freese et al. [10] obtain \( \beta < 0.12 \), which is comparable to the constraint in the preceding section.

5.11 Deceleration parameter

Define

\[ \xi \equiv \delta_{rp} - \frac{3\beta}{1 - 4\beta} \quad (5.61) \]

By condition (5.38) \( \xi > 0 \). Hence from eq.(5.33) ,

\[ 0 < q_{p} < \frac{1}{2} + \frac{(1 - 2\beta)(1 - 4\xi)}{(1 - \beta)} [1 + \frac{(1 - 4\beta)\xi}{(1 - \beta)}]^{-1} < 1 - 2\beta. \quad (5.62) \]

More precisely since \( 0 < \beta < \delta_{rp} \ll 1 \), then \( \xi \ll 1 \) and \( 0 < q_{p} < 1/2 \).

5.12 Age of the Universe

By Equation (5.29) and (5.61)

\[ \frac{1}{(1 - 2\beta) R_{p}^{2}} = \frac{\alpha^{-1} \beta_{mp} \xi}{(1 - \beta)(1 + \omega)}, \quad (5.63) \]
and from eq.(5.31),
\[ H_p^2 = \frac{\alpha^{-1}\rho_{mp}(1 + \delta_p)}{(1 - \beta)}. \] (5.64)

Let \( R = uR_p \). For all \( u \geq u_2 = R_2/R_p \) eq. (5.31) can be written as
\[ u^2 = \frac{H_p^2}{(1 - \delta_p)}\left[\frac{\xi}{1 + \omega} + \frac{(1 - \beta)}{(1 - 4\beta)u} + \frac{\omega \xi}{(1 + \omega)u^{2 - 4\beta}}\right], \] (5.65)

Taking approximately \( t = t_2 \approx 0 \) when \( u = u_2 \approx 0 \) we may write for the age of the Universe
\[ t_p(\omega; \beta) = \frac{(1 + \delta_p)^{1/2}}{H_p} \int_0^1 u^{1 - 2\beta}\left[\xi u^{2 - 4\beta} + \frac{(1 - \beta)u^{1 - 4\beta}}{(1 - 4\beta)} + \frac{\omega \xi}{1 + \omega}\right]^{-1/2}du. \] (5.66)

It is readily verifiable that \( t_p(\omega; \beta), \beta < 1/4, \) is a decreasing function of \( \omega \) so that
\[ H_p t_p(\omega; \beta) < H_p t_p(0; \beta) = (1 + b)^{1/2}J, \] (5.67)

where
\[ J = \int_0^1 u^{1/2}[1 + bu]^{-1/2}du \]
\[ = \frac{(1 + b)^{1/2}}{b} + \frac{1}{2b\sqrt{b}} \ln\left[\frac{\sqrt{1 + b} - \sqrt{b}}{\sqrt{1 + b} + \sqrt{b}}\right] \]
and \( b = \frac{\xi(1 - 4\beta)}{(1 - \beta)} \ll 1. \) Thus \( J \to (1 + b)^{-1/2} \) or \( H_p t_p(0; \beta) \to 1 \) so that one has the upper bound \( t_p(\omega; \beta) < H_p^{-1} \approx 10^{42}(\text{GeV})^{-1} \approx 2 \times 10^{10}\text{yr} \)

### 5.13 Classical cosmological tests

Returning to eq.(5.31) using eqs.(5.63) to replace \( \rho_{mp} \) by \( \Delta \) and introducing the red-shift \( z(1 + z = R_p/R) \) we have
\[ \dot{R} = (1 - 2\beta)^{-1/2}[a(1 + z)[1 + \Delta^{-1}(1 - \beta)^{-1}(1 - 4\beta)(1 + z)^{1 - 4\beta}] + 1]^{1/2}, \] (5.68)

where by eq.(5.63), (5.64) and (5.61),
\[ a \equiv \omega(1 - \beta)(1 - 4\beta)^{-1}\Delta = \frac{(1 - \beta)(1 - 2\beta)R_p^2\gamma_p}{(1 - 4\beta)(1 + \delta_p)} \]
\[ = \frac{(1 - 2\beta)R_p^2\gamma_p}{(1 + b)} \] (5.69)

Now \( \Delta^{-1} = \frac{\omega \xi}{1 + \omega} < \xi \ll 1, \) so that for optical and radio cosmic source with \( z \leq 5 \) ( \( z \leq 1 \) if normal galaxies and not quasars are considered) \[ \Delta^{-1}(1 - \beta)^{-1}(1 - 4\beta)(1 + z)^{1 - 4\beta} \ll 1. \] Also \( a \equiv (1 + \omega)b^{-1} \gg 1 \), i.e. \( R_p^2\gamma_p \gg 1. \) This means that although \( \beta \ll 1, \) the \( \beta \)-term in \( \Lambda_p \) is still important, vis-à-vis the \( \gamma \)-term. Thus eq.(5.68) may be approximated simply by
\[ \dot{R} \approx \sqrt{a}(1 - 2\beta)^{-1/2}(1 + z)^{1/2}. \] (5.70)
The RW metric can be written as

\[ ds^2 = dt^2 - R^2[d\chi^2 + r^2(\chi)[d\theta^2 + \sin^2 \theta d\phi^2]] \] (5.71)

where \( r(\chi) = \sin \chi \) when \( k = 1 \). For light propagation along a radial (\( d\theta = d\phi = 0 \)) null geodesic from a distant cosmic source at \((\chi, \theta, \phi)\) to us at \( \chi = 0 \) eqs.(5.70) and (5.71) give

\[ d\chi = -\frac{dR}{RR} = \frac{(1 - 2\beta)^{1/2}dz}{\sqrt{a}(1 + z)^{3/2}}, \] (5.72)

which upon integration yields \((\chi(z = 0) = 0)\):

\[ \chi = \frac{2(1 - 2\beta)^{1/2}}{\sqrt{a}}[1 - (1 + z)^{-1/2}]. \] (5.73)

Because \( a \gg 1, \sin \chi \approx \chi \). Hence

\[ \xi \equiv R_p r(\chi) \approx \frac{2(1 + b)^{1/2}}{H_p} [1 - (1 + z)^{-1/2}] \]

\[ = (1 + b)^{1/2} \xi_{ES}; \] (5.74)

where \( \xi_{ES} \) denotes the product \( R_p r(\chi) \) in the \( k = 0 \) Einstein-de Sitter model. This result is important for the following cosmological tests:

### 5.13.1 Magnitude versus red-shift relation

The apparent and absolute bolometric magnitudes \( m \) and \( M \) of a galaxy of red-shift \( z \) are related by [16]

\[ m - M \equiv \mu = 5 \log_{10} D + 25, \] (5.75)

where \( \mu \) is the distance modulus and \( D = \xi(1 + z) \) is the luminosity distance of the galaxy, measured in \( Mpc \). From eq.(5.74),

\[ \mu \approx \mu_{ES} + \frac{5}{2} \log(1 + b). \] (5.76)

### 5.13.2 Galactic diameters

A galaxy of linear dimension \( dl \) has angular diameter

\[ d\theta = (1 + z)^{-1} dl \] (5.77)

Hence by eq.(5.74),

\[ d\theta \approx d\theta_{ES}(1 + b)^{-1/2}. \] (5.78)
5.13.3 Number counts of sources

The number of uniformly distributed optical and radio sources in the volume element \(dV\),

\[
dV = R^3 r^2(\chi) d\Omega d\chi, \quad d\Omega = \sin \theta d\theta d\phi,
\]

is [17]:

\[
dN = n R^3 r^2(\chi) d\Omega d\chi = n_p R_p^3 r^2(\chi) d\Omega d\chi
\]

where \(n(t) \propto \rho_m(t)\) is the number density and \(n R^3 \propto \rho_m R^3 = \rho_m R^3\) so that \(n R^3 = n_p R_p^3\). Noting that \(n_p = \rho_m^{ES} R_p^{ES} = (1 - \beta)(1 + \delta_p) = (1 - 4\beta)(1 + b)\),

\[
\frac{n_p}{n^{ES}_p} = \frac{\rho_m}{\rho_m^{ES}} = \frac{(1 - \beta)}{(1 + \delta_p)} = \frac{(1 - 4\beta)}{(1 + b)},
\]

and using eqs. (5.80), (5.74), (5.72) and (5.69) we obtain

\[
dN \approx \frac{(1 - 4\beta)n_p^{ES} \zeta^{2}_{ES}(1 + b)^{1/2} dz d\Omega}{H_p(1 + z)^{3/2}}.
\]

On the other hand in the MD universe of the Einstein-de Sitter model

\[
dN_{ES} = n_p^{ES} \zeta^{2}_{ES} R_p^{ES} d\Omega d\chi = \frac{n_p^{ES} \zeta^{2}_{ES} dz d\Omega}{H_p(1 + z)^{3/2}}.
\]

Hence

\[
dN \approx (1 - 4\beta)(1 + b)^{1/2} dN_{ES}
\]

Equation (5.76), (5.78) and (5.84) are the model’s predictions for the classical cosmological test. Because, \(b, \beta \ll 1\) these results are approximately the same as in the ES model.

5.14 Nonsingular model: case 2.

Here we will take the case for which \(\gamma = 1 - \beta\) and \(k = 1\), so that \(\rho_0 \neq 0\). Equations (5.9), (5.10) and (5.11) become

\[
\dot{R}^2 = 1 - \left(\frac{R_0}{R}\right)^{2-4\beta},
\]

\[
\rho = \frac{\alpha}{R^2} [1 - (1 - \beta)\left(\frac{R_0}{R}\right)^{2-4\beta}],
\]

\[
\rho_v = \frac{\alpha}{R^2} [1 - \beta(\frac{R_0}{R})],
\]

\[A_0 = R_0^{-2+4\beta}, \quad \frac{\alpha^2}{R_0^4} > 0.\]

And for physical reason we will take \(\beta > 0\). The rate of change of entropy at temperature \(T\) is given by

\[
T \frac{dS}{dR} = -R^3 \frac{d\rho_v}{dR}.
\]
The equation shows that \( \frac{dS}{dR} > 0 \) for all \( R \geq R_0 \) and thereby solving the entropy problem of the standard model.

The density parameter \( \Omega \) defined by

\[
\Omega = \frac{\rho}{\rho_c}, \quad \rho_c = \alpha H^2
\]

becomes

\[
\Omega = 1 - \beta + \beta \frac{\dot{R}}{R^2} \quad (5.90)
\]

The maximum of \( \rho \) is attained at

\[
R = R_{mx} = [2(1 - \beta)^{1/2(1-2\beta)}] \quad (5.91)
\]

\[
\rho_{mx} = \rho(R_{mx}) = 1 - 2\beta + \frac{\alpha}{2(1-\beta)} \frac{\dot{R}}{R_{mx}^2} \quad (5.92)
\]

Using eq.(5.9) one can estimate the value of \( R_0 \) since \( \ddot{R} = \frac{1-2\beta}{R_0} < R_0^{-1} \) and that signal a limit for a cosmic acceleration in the early Universe. Such as limit is of the order of Planck mass \((M_{pl} = G^{-1/2})\).

This Universe accelerates less rapidly that the one with \( \gamma = 1 \). Thus starting with some non zero initial density allows the Universe to be less accelerating than with zero one; and with a less maximum density than the one with \( \gamma = 1 \).

### 5.15 Radiation and matter

In the wake of pure radiation era, the rest mass is generated during the period, say, \( R_1 \leq R \leq R_2 \) [6].

For \( R > R_2 \), after creation of the rest mass, the matter energy \((E_m)\), \( E_m = \rho_m R^3 \) stayed constant so that \( E_m = \rho_m R^3 = E_{mp} = \rho_{mp} R_p^3 \), where “p” denotes present-day quantities. Thus for the present case with \( \Lambda \) given by eq.(5.1) and \( \gamma = 1 - \beta, \ k = 1 \) one gets

\[
\rho_r = \frac{3\beta E_{mp}}{1-4\beta} R^{-3} + \alpha R^{-2}[1 + \omega(R_p/R)^{2-4\beta}] , \quad (5.93)
\]

where \( \omega + 1 = \alpha^{-1} \rho_{mp} R_p^2 (\delta_p - \frac{3\beta}{1-4\beta}) \) and \( \delta_p = \rho_{rp}/\rho_{mp} \).

The radiation energy \( E_r, \rho_r = \rho, R^3 \) is given by

\[
E_r = \frac{3\beta E_{mp}}{1-4\beta} R^{-3} + \alpha R^{-2}[1 + \omega(R_p/R)^{2-4\beta}] , \quad (5.94)
\]

\[
\rho_r = \frac{\beta E_{mp}}{1-4\beta} R^{-3} + \alpha R^{-2}[1 + \frac{\beta\omega}{1-\beta}(R_p/R)^{2-4\beta}] , \quad (5.95)
\]

and

\[
\dot{R}^2 = \frac{\alpha^{-1}}{1-\beta} R^2 = \frac{\alpha^{-1} E_{mp}}{1-4\beta} R^{-1} + \frac{\omega}{1-\beta}(R_p/R)^{2-4\beta} . \quad (5.96)
\]
The acceleration parameter, \( q \),

\[
q = -\frac{R \ddot{R}}{R^2}
\]  

(5.97)

The present value is

\[
q_p = \frac{1/2 + [\omega \alpha (1-4\beta)/(1-\beta)]\rho_{mp}R_p^2}{1 + [(1-\beta + \omega)(1-4\beta)/(1-\beta)]\rho_{mp}R_p^2} \approx 1/2 .
\]  

(5.98)

Denote by \( R_{eq} \) the value of \( R \) at \( t = t_{eq} \), the time when radiation and matter were balanced in equilibrium

\[ E_r(R_{eq}) = E_{mp} . \]

The condition that \( E_r \) was decreasing as \( R \) approached \( R_{eq} \) would then imply that

\[(1-4\beta)\omega > \left(\frac{R_p}{R}\right)^2-4\beta .\]  

(5.99)

Thus either \( \omega > 0 \) and \( \beta < 1/4 \) as \( \omega < 0 \) and \( 1/4 < \beta < 1/2 \).

However, the later case implies that \( \dot{R}^2 < 0, \rho < 0 \) for \( R_2 < R < R_{eq} \). In Ref.[3] one approximately has \( \rho_r R^4 \) constant or \( \rho_t \) constant for \( R < R_{eq} \) in the radiation and matter universe. In the present case for \( R \ll R_{eq} \) the first two terms in eq.(5.95) contribute negligibly to \( \rho_r \) and therefore

\[
\rho_r \approx \frac{\alpha \omega}{R} \left(\frac{R_p}{R}\right)^2-4\beta ,
\]  

(5.100)

\[
\rho_v = \frac{\beta}{1-\beta} \left(\frac{R_p}{R}\right)^2-4\beta ,
\]  

(5.101)

and

\[
\dot{R}^2 = \frac{\omega}{1-\beta} \left(\frac{R_p}{R}\right)^2-4\beta .
\]  

(5.102)

This leads to

\[
\rho_v = \frac{\beta}{1-\beta} \rho_r, \quad R \ll R_{eq} .
\]  

(5.103)

This is the same relation as postulated by Freese et al. [10] which suggests that in the early phase of radiation and matter \( \rho_r \) and \( \rho_v \) red-shift at the same rate. The solution of eqs.(5.102) and (5.103) is

\[ R \sim t^{1/2(1-\beta)} ,
\]  

(5.104)

and

\[
\rho_v = \frac{\beta}{1-\beta} \rho_r \approx \frac{\alpha \beta}{4(1-\beta)^2} t^2 .
\]  

(5.105)

valid in the early radiation and matter (i.e. \( R \ll R_{eq} \)) epochs.

This behaviour of \( \rho_v \sim t^{-2} \) has been noted by many workers. Though the model of \( \gamma = 1, k = 1 \) and \( \gamma = 1 - \beta \) has different features at the early Universe, they give the same results in the wake of early radiation and matter phases of the Universe that are identical to Freese et al.
5.16 Phase transition

We consider, finally, the matter generation period $R_1 \leq R \leq R_2$. For $R \geq R_2$ eq. (5.95) implies $\ddot{R} > 0$ if $\omega > 0$ and $\beta < 1/4$ in contrast with $\ddot{R} > 0$ for $R < R_1$ as previously noted. Thus the appearance of rest-mass ushers in decelerated expansion of the Universe. Equations (5.25) and (5.26) with $\gamma = 1 - \beta$, $k = 1$

$$\frac{dE}{dR} = 2\alpha(1 - 2\beta) + 3\beta R^2 \rho - 3(1 - \beta)pR^2$$

(5.106)

and using eq.(5.94) one gets

$$3 \int_{R_0}^{R_2} R^2[(1 - \beta)p - \beta \rho]dR = 2\alpha(1 - 2\beta)(R_2 - R_0) + E_0 - E_2 ,$$

(5.107)

$$E_2 = E_m(R_p) + E_r(R_2)$$

$$= 2\alpha(1 - 2\beta)(R_2 - R_0) + \beta \alpha R_0 - \rho_{mp}R_p^3 - \frac{3\beta}{1 - 4\beta}\rho_{mp} R_p^3 - \alpha R_2[1 + \omega(R/R)^{2 - 4\beta}] ,$$

(5.108)

$$= -\alpha(2 - 5\beta) R_0 - \frac{(1 - \beta)}{1 - 4\beta}\rho_{mp} R_p^3 + \alpha (1 - 4\beta) F(\omega) ,$$

(5.109)

where

$$F(\omega) = 1 - \frac{\omega}{1 - 4\beta} \left(\frac{R_p}{R}\right)^{2 - 4\beta} < 0$$

as long as $\beta < 1/4$ the above integral is negative, implying that $3p/\rho \ll 1$ for some values of $R$ in the interval $(R_1, R_2)$. During this period the Universe undergone a phase transition, an era separating the pure radiation and the radiation and matter epochs. It was noted by Freese et al. that the constraint that an early radiation epoch be followed by a matter dominated era requires that $x = \beta < 1/4$ in both matter and radiation epochs. Nucleosynthesis in this model proceeds the same as in the previous one.

5.17 Discussion and concluding remarks

Our aim has been to discuss a nonsingular Robertson-Walker cosmological model with a varying cosmological constant $\Lambda = 3\gamma R^{-2} + 3\beta H^2$. This form of $\Lambda$ is due to Carvalho et al. [7]. In the considered model the Universe was initially (t=0) empty and had a minimum scale factor $R_0 > 0$. Subsequently it evolved through consecutive phase of pure radiation, matter generation and radiation and matter. The absence of the initial singularity requires $2\beta < 1$ and $\frac{1}{2} < \gamma \leq k = 1$. In the very early pure radiation era $\ddot{R} > 0$. For $R \gg R_0$ in this era $R \sim t$, compared to $R \sim t^{1/2}$ in the standard model. On the other hand generation of entropy throughout the pure radiation era requires [see eq.(16)] $\frac{2\beta(2\gamma - 1)(1 - \beta)}{(\gamma - \beta)} < 1$. To satisfy this condition we have chosen, for simplicity, $\gamma = 1$ and have, henceforth, confined our selves
to this case. The model is thus an extension of the Özer-Taha model \((\beta = 0, \gamma = 1)\) albeit in a different direction than that followed by Ref. [6]. General physical consideration and an increasing entropy in the radiation and matter period place a stronger restriction on \(\beta\), viz. \(0 < \beta \leq \delta_{rp}\), where \(\delta_{rp}\) is the present ratio of radiation to matter energy densities. This constraint is quite stringent if one assumes, as we have done here, that the present Universe is matter-dominated so that \(\delta_{rp} \ll 1\). Under these conditions the pure radiation and the radiation matter eras are separated by a phase transition period that reverses the sign of \(\ddot{R}\) from \(\ddot{R} \geq 0\) to \(\ddot{R} < 0\), ushering in decelerated expansion. But the precise phase transition equation of state is not known. In the early radiation and matter universe the model is virtually indistinguishable from the flat-space decaying-vacuum singular cosmology of Freese et al. [10]. The condition \(0 < \beta < \delta_{rp}\) means that the nucleosynthesis constraint of Freese et al., \(\beta \leq 0.1\), is easily satisfied for \(\delta_{rp} \ll 1\). This provides an interesting connection between the present and early Universe. The Özer-Taha model and the model of Freese et al. are very dissimilar, as already noted by the latter authors [10]. It is therefore remarkable that an extension of the one converges to the other. Freese et al. make the basic postulate that the vacuum and radiation energy densities red-shift at the same rate, for large \(R\). This feature emerges here in the very early radiation and matter era, as a mere by-product of the approach. In the cosmology of Carvalho et al. the Freese et al. scenario corresponds to \(\gamma = k = 0, \beta \neq 0\). Here also \(\beta \neq 0\), but \(\gamma = k = 1\). As to the age of the Universe and the deceleration parameter, our model predicts \(t_p < H_p^{-1}\) and \(0 < q_p < 1/2\). On the other hand, in the singular cosmology of Carvalho et al. \(\beta\) is a free parameter that can be adjusted to produce \(t_p > H_p^{-1}\) and \(q_p < 0\). We have also examined the implications of the model for the classical cosmological tests. The results approach Einstein-de Sitter model predictions. In the present work the density parameter \(\Omega = 1 - \beta = \text{constant}\), before and after the phase transition. Hence \(1 - \delta_{rp} < \Omega < 1\) although this implies the absence of flatness fine-tuning problem it would apparently require the existence of dark matter. Finally, the proposed cosmology predicts a continuously expanding Universe.

### 5.18 References

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Chapter 6

A Nonsingular Viscous Cosmological Model

6.1 Introduction

In one approach for solving the cosmological constant problem [1, 2] cosmologies with decaying vacuum energy are introduced ([3,4] and references therein). But in most of the proposed models dissipative effects that were probably very important in the early Universe are not considered. On the other hand the influence of bulk viscosity on cosmic evolution in standard Friedmann models has been discussed in many works (for an excellent review see [5]). In these works the coefficient of bulk viscosity $\eta$, say, is usually taken to have a power law dependence on the cosmic energy density $\rho$, viz. $\eta \propto \rho^n$, $n$ constant. Recently Calvao, de Oliveira, Pavón and Salim [6] generalized the model of Chen and Wu [7], in which the vacuum energy decreases with cosmic expansion as the inverse square of the Robertson-Walker (RW) scale factor, by appending to the energy-momentum tensor a bulk viscosity term with $\eta \propto \rho$. The resulting field equations in this case are found to have no analytical solutions in general and methods of qualitative analysis of differential equations are used to deduce the cosmological implications of the model.

The present paper also attempts to incorporate bulk viscosity in a decaying vacuum cosmology. But unlike Ref. [6] we take $n = 1/2$ in the dependence of $\eta$ on $\rho$, i.e. $\eta \propto \rho^{1/2}$. For this case the solutions of the field equations for flat cosmological models with vanishing cosmological constant $\Lambda$ appears to have a special status:

They are the only solutions with structural stability [8]. In addition, Beesham [9] has shown that the $n = 1/2$ models are equivalent to the flat variable-$\Lambda$ cosmologies of Berman [10]. Because of these peculiarities of the models with $\eta \propto \rho^{1/2}$ we limit our self to this case.

We also assume a Universe with exactly the critical density. Consequently the vacuum energy decays
as in the Özer - Taha model [11]. The critical density assumption, which is indicated by inflation, is theoretically appealing because the apparent closeness of the present and critical densities would else be difficult to understand on other than anthropic grounds.

Our model which is nonsingular and closed is introduced in Sec.5.2. Several implications of this model turn out to be equivalent to the corresponding consequences in previously proposed non viscous cosmologies [12 – 14].

6.2 The Model

In a homogeneous isotropic universe with a perfect fluid energy-momentum tensor, Einstein’s equations with a variable cosmological ‘constant’ Λ give (κ ≡ 3/8πG)[11]:

\[ \frac{\dot{R}^2}{R^2} = \kappa^{-1} \rho + \frac{\Lambda}{3} - \frac{k}{R^2}, \]

\[ \frac{d}{dR} (\rho R^3) + 3pR^2 + \frac{\kappa}{3} R^3 \frac{d\Lambda}{dR} = 0, \]

where \( \rho \) and \( p \) are the cosmic energy density and pressure, respectively, and \( k \) the curvature index. To obtain the field equations with bulk viscosity we replace \( p \) by the effective pressure [15]

\[ p^* = p - 3\eta H, \]

where \( H = \frac{\dot{R}}{R} \) is the Hubble’s constant and \( \eta \) is the coefficient of bulk viscosity.

As remarked in the introduction we set \( \rho \) equal to the critical density.

\[ \rho = \rho_c = \kappa H^2, \]

throughout cosmic evolution. Consequently eq.(6.10) gives the Özer-Taha [11] result

\[ \Lambda = \frac{3k}{R^2} \]

The form \( \Lambda \propto R^{-2} \) has also been suggested by Chen and Wu [7] on the basis of dimensional arguments consistent with quantum gravity. Pavón [16], on the other hand, has used the Landau-Lifshitz fluctuation theory to study the physical consistency of several \( \Lambda \) variations and has concluded that \( \Lambda \) should vary as \( R^{-2} \). The detailed observational consequences of this behaviour have been examined in Refs. [7, 11, 13, 17, 18]. As mentioned in the introduction we choose \( \eta \propto \rho^{1/2} \). Specifically we write

\[ \eta = \frac{\eta_0}{2(6\pi)^{1/2}} \rho^{1/2}, \quad \eta_0 \geq 0, \text{ const..} \]
Together with eq.(6.13) and the usual equation of state
\[ p = (\gamma - 1)\rho, \quad 0 \leq \gamma \leq 2 \]
this leads to
\[ p^* = \left[ \gamma - 1 - \frac{\eta_0}{M_{Pl}} \right] \rho, \quad (6.7) \]
where \( M_{Pl} = G^{-1/2} \) is the Planck mass. Equation (6.11), with \( p \rightarrow p^* \), and eqs.(6.16) and (6.14) give
\[ \frac{d\rho}{dR} + \frac{3}{R} \left[ \gamma - \frac{\eta_0}{M_{Pl}} \right] \rho = \frac{2k}{R^3}, \quad (6.8) \]
which has the solution
\[ \kappa^{-1} \rho R^2 = \dot{R}^2 = \frac{2k}{3\gamma - 2 - 4\beta} + AR^{-3\gamma+2+4\beta}, \quad (6.9) \]
where \( \beta = \frac{3\eta_0}{4M_{Pl}} \) and \( A \) is a constant. In particular for the radiation- \( (\gamma = 4/3) \) dominated (RD) phase of the Universe
\[ \dot{R}^2 = \frac{k}{1-2\beta} + AR^{-2+4\beta}. \quad (6.10) \]
Carvalho, Lima and Waga [12] have proposed a model Universe based on the cosmological constant ansatz
\[ \Lambda = 3\beta H^2 + \frac{3\alpha}{R^2}, \quad (6.11) \]
where \( \alpha \) and \( \beta \) are dimensionless constants of order unity. The detailed cosmological implications of this postulate have been explored in [12,3,14]. In particular a nonsingular scenario based on eq.(6.19) was studied in [14].

It follows from eq.(6.20) that in the RD universe [12]
\[ \dot{R}^2 = \frac{2\alpha - k}{1-2\beta} + A_0 R^{-2+4\beta}, \quad (6.12) \]
and
\[ \rho = \frac{\kappa (\alpha - \beta k)}{1-2\beta} R^{-2} + \kappa (1-\beta) A_0 R^{-4+4\beta}. \quad (6.13) \]
For \( \alpha = k \) eqs.(6.21) and (6.19) are formally equivalent. It is therefore interesting to ask to what extent their equivalence is reflected in the corresponding cosmologies. To answer this question we consider a nonsingular scenario as in Ref. [14].

A Universe with a non vanishing minimum scale factor \( R_0 \) at \( t = 0 \) arises from eq.(6.19) if \( A < 0 \), \( \beta < 1/2 \) and \( k = 1 \) (closed universe). Then
\[ \rho = \frac{\kappa \dot{R}^2}{R^2} = \frac{\kappa}{R^2(1-2\beta)} \left[ 1 - \frac{R_0^{2-4\beta}}{R^{2-4\beta}} \right], \quad (6.14) \]
with \( \rho_0 = 0 \) and \( \rho \) maximum \( (= \frac{\kappa}{2(1-\beta)R_{max}^{2-4\beta}}) \) at
\[ R = R_{max} = \left[ 2(1-\beta) \right]^{1/2} R_0. \]
An estimate of $R_0$ may be obtained as follows. From eq.(6.19) $0 < \dot{R} \leq R_0^{-1}$ so that $R_0^{-1}$ represents a cosmic acceleration limit. Such a limit of the order of $M_{pl}$ has been suggested before [19]. Taking $R_0^{-1} \sim M_{pl}$ yields $R_0 \sim 10^{-33}$ cm.

We next investigate the implications of the model in the cosmic phase of noninteracting radiation of density $\rho_r$ and pressureless non relativistic matter of density $\rho_m$ ($\rho_r + \rho_m = \rho$).

We assume that the vacuum couples to radiation only [11, 13, 14, 18]. Hence $p = \rho_r/3$ and the matter energy $E_m = \rho_m R^3 = \rho_{mp} R^3_p = E_{mp}$, where subscript “p” denotes present-day quantities. Then eqs.(6.11), (6.14) and (6.16) give (k=1)

$$\frac{d\rho_r}{dR} + \frac{4(1 - \beta)}{R} \rho_r = \frac{2\kappa}{R^3} + \frac{4\beta}{R^4} E_{mp} \tag{6.15}$$

which has the solution

$$\rho_r = \frac{4\beta E_{mp}}{(1 - 4\beta)R^3} + \frac{\kappa}{(1 - 2\beta)R^2} \left[ 1 + \frac{\omega R_p^{2-4\beta}}{R^{2-4\beta}} \right], \tag{6.16}$$

where

$$(1 + \omega) = (1 - 2\beta)\kappa^{-1}\rho_{mp} R_p^2 (\delta_p - \frac{4\beta}{1 - 4\beta}), \tag{6.17}$$

$\delta_p = \frac{\rho_{r_{eq}}}{\rho_{mp}}$ being the present ratio of radiation-to-matter energy densities. It follows from eq.(6.24) and conservation of non relativistic matter that

$$\dot{R}^2 = \kappa^{-1} \rho R^2 = \frac{\kappa^{-1}\rho_{mp} R_p^3}{(1 - 4\beta)R} + \frac{\omega R_p^{2-4\beta}}{(1 - 2\beta)R^{2-4\beta}} + (1 - 2\beta)^{-1}. \tag{6.18}$$

Denote $R$ at $t = t_{eq}$, the time when radiation and matter were equal, by $R_{eq}$, i.e. $E_r(R_{eq}) = E_m(R_{eq}) = E_{mp}$, where $E_r(R) = \rho_r R^3$ is the radiation energy. Then the condition that $E_r$ was decreasing as $R$ approached $R_{eq}$ leads to

$$(1 - 4\beta)\omega > \frac{R_{eq}^{2-4\beta}}{R_p^{2-4\beta}}. \tag{6.19}$$

Implying that either $\omega > 0$ and $\beta < 1/4$ or $\omega < 0$ and $\beta > 1/4$. But the latter case would, by virtue of eq.(6.26), imply that $\dot{R}^2 < 0$ for $R \leq R_{eq}$ and is therefore excluded.

Equation (6.27) is formally identical with eq.(6.24) in Ref. [14]. But whereas $\beta$ there determines the coefficient of $H^2$ term in the cosmological constant ansatz, here it is a measure of bulk viscosity.

The condition $\beta < 1/4$ implies, from eq.(6.26), that $\dot{R} < 0$ compared to $\ddot{R} > 0$ in the pure radiation era. Assuming that a period of rest-mass generation $R_1 \leq R \leq R_2$ , say, separated the pure radiation and the matter and radiation epochs one can readily show, in exactly the same way as in [14], that this period is a phase transition era during which the pressure becomes small or negative.

The conditions $\omega > 0$, $\beta < 1/4$ and eq.(6.25) lead to $4\beta < \delta_p$. Hence if the present Universe is matter-dominated (MD) $\frac{\omega}{M_{pl}} < 4\delta_p/3 \ll 1$ , or, from eq.(6.16), that the present bulk viscosity contribution to
the effective pressure $p^*$ is at most of the order of the radiation pressure. To examine nucleosynthesis in the present cosmology we proceed as follows. Noting that $\rho_{mp}R_p^3 = \rho_{r,eq}R_{eq}^3$ where $\rho_{r,eq}$ is the radiation density at equilibrium we obtain from eqs.(6.26), (6.24) and (6.27) the approximate relation

$$\dot{R}^2 \approx \frac{\omega R_p^{2-4\beta}}{(1-2\beta)R^{2-4\beta}} \approx k^{-1}\rho_rR^2, \quad (6.20)$$

valid for $R \ll R_{eq}, \beta \ll 1$ and $\rho_m \ll \rho_r$.

Equation (6.28) is approximately the standard ($k = 0$) model’s cosmic expansion rate. Counting three neutrino species one therefore has for the neutron-to-proton “freeze-out” temperature $T_{F10}(\equiv T_F/10^{10}K) \approx 1.11$, the same as in standard cosmology. Also from eq.(6.28) $\dot{\rho}_r/\rho_r \approx -4(1-\beta)\dot{R}/R \approx -4\dot{R}/R$ leading to the standard temperature-time relation.

From eqs.(6.28) and (6.25), using $\rho_r = \frac{\pi^2}{30} g_{eff}T^4$, one has

$$R^{-1}R_p = \left[ \frac{1 + \omega}{\omega \Delta} \right]^{1/4(1-\beta)} \left[ \frac{T}{T_p} \right]^{1/(1-\beta)} \approx \frac{1 + \omega}{\omega \Delta} \frac{T}{T_p}, \quad (6.21)$$

where

$$\Delta \approx 1 - 4\beta \delta_p^{-1}. \quad (6.22)$$

Denoting the baryon number density by $n_B$ we then have

$$n_B \approx n_{Bp}\left[ \frac{1 + \omega}{\omega \Delta} \right]^{3/4} \left[ \frac{T}{T_p} \right]^3 \quad (6.23)$$

for $T \leq T_N$, the nucleosynthesis temperature, and where $n_{Bp} = \kappa m_N^{-1} \Omega_B H_p^2$, with $m_N$ being the nucleon mass and $\Omega_B$ the present baryonic fraction of the critical density. The equation determining $T_N$(inGeV) therefore becomes

$$\frac{B_d}{T_N} + \frac{3}{2} \ln T_N + \ln(\Omega_B h^2) - 14.47 = 0, \quad (6.24)$$

where $B_d(=2.23\text{MeV})$ is the deuteron binding energy, $h$ is the normalized Hubble constant ($0.4 \leq h \leq 0.8$ [20]) and

$$\Omega_B = \left[ \frac{1 + \omega}{\omega \Delta} \right]^{3/4} \Omega_B \geq \Omega_B, \quad (6.25)$$

with the equality sign holding for the non viscous ($\Delta \to 1$) standard ($\omega \to \infty$) case.

Apart from the replacement $\Omega_B \to \bar{\Omega}_B$, eq.(6.32) coincides with the corresponding standard model relation [21]. Requiring $\bar{\Omega}_B$ to satisfy the standard model constraints on $\Omega_B$, i.e. [22]

$$0.015 \leq \bar{\Omega}_B \leq 0.070 \quad (6.26)$$
gives $T_N$(standard). With $T_F$ and $T_N$ as in standard cosmology and the nuclear physics aspects of nucleosynthesis unaltered the standard nucleosynthesis scenario is reproduced. The condition (6.34) implies that

$$\frac{\Omega_B}{0.070} \leq \Delta^{3/4} \leq 1,$$

(6.27)

irrespective of the size of the contribution of vacuum energy. In the presence of viscosity ($\Delta < 1$) the constraint (6.35) implies a stronger upper bound on the baryonic density than required by the standard model.

### 6.3 Concluding remarks

We have presented a homogeneous isotropic cosmological model with decaying vacuum energy and bulk viscosity. Our aim has been to generalize an earlier work by Calvao et al. [6] in which the bulk viscosity coefficient depends linearly on the cosmic energy density $\rho$. In the present model it varies as $\rho^{1/2}$.

Our model is nonsingular, closed and, in several aspects, equivalent to previously proposed non viscous variable-$\Lambda$ models [12 – 14]. This type of equivalence of viscous and non viscous variable-$\Lambda$ cosmologies has been noted and discussed by Beesham [9].

A noteworthy feature of the present work is that primordial nucleosynthesis proceeds as in standard cosmology provided the Universe today is matter dominated. This condition implies that the present bulk viscosity contribution to the cosmic pressure is, at most, of the order of the radiation pressure. Another consequence of nucleosynthesis is that the upper limit on the baryon density is lower than the standard value. This strengthens the case for nonbaryonic dark matter.

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Chapter 7

A Viscous Universe with Variable $G$ and $\Lambda$

7.1 Introduction

The role of viscosity in cosmology has been studied by several authors [1-4]. The bulk viscosity associated with grand unified theory phase transition can lead to inflationary universe scenario. It was well known that in an early stage of the Universe when neutrino decoupling occurred, the matter behaves like viscous fluid [22]. The coefficient of viscosity is known to decrease as the Universe expands. Beesham [20] studied a universe consisting of a cosmological constant ($\Lambda \sim t^{-2}$) and bulk viscosity. He showed that the Berman model could be viscous model for $n = 1/2$.

More recently Abdel Rahman considered a model in which the gravitational constant, $G$, varies with time but energy was conserved [11]. Other models have been considered in the literature by Sistero and Kalligas et al. In the present work we will investigate the effect of viscosity in a universe where $G$ and $\Lambda$ vary in such a way that energy is conserved.

7.2 The Model

In a Robertson Walker universe

$$d\tau^2 = dt^2 - R^2(t)[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)],$$

(7.1)

where $k$ is the curvature index.

Einstein’s field equations with time dependent cosmological and gravitational “constants”

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} + \Lambda g_{\mu\nu},$$

(7.2)
and the perfect fluid energy momentum tensor

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \]  \hfill (7.3)

yield the two independent equations

\[ 3\frac{\ddot{R}}{R} = -4\pi G(3p + \rho - \frac{\Lambda}{4\pi G}), \]  \hfill (7.4)
\[ 3\frac{\dot{R}^2}{R^2} = 8\pi G(\rho + \frac{\Lambda}{8\pi G}) - \frac{3k}{R^2}. \]  \hfill (7.5)

Elimination of \( \ddot{R} \) gives

\[ 3(p + \rho)\dot{R} = -(\frac{\dot{G}}{G}\rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G})R. \]  \hfill (7.6)

The conservation of energy and momentum yields

\[ 3(p + \rho) = -R \frac{d\rho}{dR}. \]  \hfill (7.7)

The effect of bulk viscosity in the field equation is to replace \( p \) by \( p - 3\eta H \), where \( \eta \) is the viscosity coefficient. It follows immediately that

\[ 9\eta H \dot{R} = (\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G})R, \]  \hfill (7.8)
and

\[ \dot{\rho} + 3H(\rho + p) = 0. \]  \hfill (7.9)

eq.(8) can be written as

\[ 9\eta H \frac{H}{R} = \frac{G'}{G}\rho + \frac{\Lambda'}{8\pi G}, \]  \hfill (7.10)

where prime denotes derivative w.r.t. scale factor \( R \) while dot is the derivative w.r.t to cosmic time \( t \).

In what follows we will consider a flat universe, \( k = 0 \)

Equation (7.5) and (7.17) lead to

\[ 8\pi G\rho = 3(1 - \beta)H^2, \]  \hfill (7.11)
and the equation of state

\[ p = (\gamma - 1)\rho \]  \hfill (7.12)
in eq.(7.8) and (7.9) lead to

\[ \rho = AR^{-3\gamma}, \]  \hfill (7.13)

where \( A \) is a constant.

\[ 9\eta H \frac{H}{R} = 2\frac{H'}{H}\rho - \rho' + 2\beta\frac{H'}{H}\rho \]  \hfill (7.14)
or
\[
\frac{H'}{H^2} + \frac{3\gamma}{2(1+\beta)R} = \frac{9\eta_0 A^{n-1}}{2(1+\beta)} R^{-3\gamma n + 3\gamma - 1},
\] (7.15)
where we have taken the viscosity coefficient to have the power law
\[
\eta = \eta_0 \rho^n, \quad \eta_0 \geq 0, \quad n \text{ const.}
\] (7.16)
and the ansatz [30]
\[
\Lambda = 3\beta H^2, \quad \beta \text{ const.}
\] (7.17)
The solution of eq.(7.15) is obtained as follows
Let \( y = \frac{1}{H} \) and \( a = \frac{3\gamma(1-\beta)}{2} \). Therefore
\[
\frac{d}{dR} y R^{-a} = -\frac{9(1-\beta)\eta_0 A^{n-1}}{2} R^{-3\gamma n + 3\gamma - a - 1},
\] (7.18)
\[
y R^{-a} = \frac{9(1-\beta)\eta_0 A^{n-1}}{2(3\gamma n - 3\gamma + a)} R^{-3\gamma n + 3\gamma - a},
\] (7.19)
\[
y = \frac{9(1-\beta)\eta_0 A^{n-1}}{2(3\gamma n - 3\gamma + a)} R^{-3\gamma n + 3\gamma},
\] (7.20)
and finally
\[
H = \frac{2(3\gamma n - 3\gamma + a)}{9(1-\beta)\eta_0 A^{n-1}} R^{3\gamma n - 3\gamma}.
\] (7.21)
eq(7.11) and (7.21) give
\[
G = \frac{3D^2(1-\beta)}{8\pi A} R^{3\gamma(2n-1)},
\] (7.22)
where
\[
D = \frac{2(2n-1-\beta)}{9(1-\beta)\eta_0 A^{n-1}},
\]
eq(7.21) gives
\[
R(t) = \left[3D\gamma(1-n)\right]^\frac{1}{3\gamma(1-n)} t^\frac{1}{3\gamma(1-n)}.
\] (7.23)
Hence eq.(7.13) and (7.22) become
\[
\rho(t) = A't^\frac{1}{1-n},
\] (7.24)
\[
G(t) = B't^\frac{2n-1}{1-n},
\] (7.25)
and
\[
\eta(t) = A_0 t^\frac{n}{1-n},
\] (7.26)
where
\[
A' = A[3m\gamma(1-n)]^\frac{1}{1-n},
\]
\[B' = \frac{3m^2}{8\pi A}(3m\gamma(1-n))^{\frac{2n-1}{1-n}}, \text{ and } A_0 = A^n \eta_0.\]

The Hubble parameter is

\[H(t) = \frac{1}{3\gamma(1-n)} \frac{1}{t} \quad (7.27)\]

where \(0 \leq n \leq 1\). This condition on \(n\) rules out some models with \(n > 1\) [3]. The cosmological constant becomes

\[\Lambda = \frac{\beta}{3\gamma^2(1-n)^2 t^2} \quad (7.28)\]

This law of variation of \(\Lambda\) is thought to be fundamental [20]. The vacuum energy density \(\rho_v\) is given by

\[\rho_v = \frac{\Lambda}{8\pi G}, \quad (7.29)\]

and from eq.(7.11) and (7.17) we obtain

\[\rho_v = \frac{\beta}{1-\beta} \rho. \quad (7.30)\]

For an expanding Universe, i.e. \(H > 0\), we must have \(D > 0\). This implies that

\[2n - \beta - 1 > 0,\]

or

\[\beta < 2n - 1. \quad (7.31)\]

Whether \(G\) increases or decreases depends on the value of \(n\). For \(n > 1/2\) \(G\) increases with time and for \(n < 1/2\) \(G\) decreases with time and for \(n = 1/2\) \(G\) remains constant. The condition (7.31) now gives \(\Lambda > 0\), \(G\) increases with time

and \(\Lambda < 0\), \(G\) decreases with time or remains constant.

To solve the persisting age problem of the ES model we must have \(t H > 2/3\). This result require that \(G\) is an increasing function of time. Recently, Massa (1995) proposed a model in which \(G\) increases with time. An increasing \(G\) would cause the Planck length \(l_P = \sqrt{G}\) to be an increasing function of time, and the quantum fluctuations on the metric would be vanishingly small in the very early Universe. A fully classical description of the Universe for all \(t > 0\) would be possible. This is perhaps one of the reasons to consider the increasing \(G\) model [6].

Following Freese et al., one can put a stringent constrain on the value of \(\beta\). The parameter \(x\) of Freese et al. is equivalent to \(\beta\) and since \(x \leq 0.1\) this implies \(\beta \leq 0.1\) for the nucleosynthesis constraints to hold. The deceleration parameter is given by

\[q = -\frac{\dddot{R}}{\dot{R}^2},\]
This shows that the deceleration parameter is constant. The constant deceleration models have considered by Berman and Som [9,15]. Equation (7.27) can be written as

$$H = \frac{1}{(1 + q)} \frac{1}{t},$$

(7.33)

and for the present phase “p”

$$t_p = \frac{1}{(1 + q_p)} \frac{1}{H_p}.$$  

(7.34)

It is evident that negative $q_p$ would increase of the present age of the Universe. From eq.(7.25) we obtain

$$\frac{\dot{G}}{G} = \frac{2n - 1}{1 - n} \frac{1}{t},$$

(7.35)

and the present value is

$$\left(\frac{\dot{G}}{G}\right)_p = \frac{2n - 1}{1 - n} \frac{1}{t_p} = \frac{2n - 1}{1 - n} (1 + q_p)H_p.$$  

(7.36)

A power law dependence of $G$ was obtained by Kalligas et al.[19], it has been shown to lead naturally to $\Lambda \sim t^{-2}$. Unlike the model of Abdel Rahman and Beesham, this model shows a constant $G$ does not imply constant $\Lambda$. We see that the quantity $G\rho$ satisfies the condition for a Machian cosmological solution, i.e. $G\rho \sim H^2$, (see [25]). This also follows from the model of Kalligas et al.

The relationship between our model and that due to Kalligas et al. is manifested in the following replacement

$$n = \frac{1 + n_K}{2 + n_K}$$

and

$$\beta = \frac{n_K}{2 + n_K},$$

where

$n_K$ : $n$ due to Kalligas et al. This furnishes the resemblance. Hence Kalligas et al. model is equivalent to a viscous model.

### 7.3 The horizon problem

The horizon distance, i.e. the size of the causally connected region, is given by

$$d_H = R(t) \int_{t_0}^{t} \frac{dt'}{R(t')}.$$
\[ d_H(t, t_0) = \frac{3\gamma - 3\gamma n}{3\gamma - 3\gamma n - 1}(t, t_0). \]

We would like to have \(3\gamma - 3\gamma n = 1\), so

\[ n = \frac{3\gamma - 1}{3\gamma}. \]

Note that \(3\gamma - 3\gamma n > 0\) implies \(n < 1\).

In what follows we will discuss some classes of models.

### 7.4 Model with \(n=1\)

Equation (7.7) becomes

\[ \frac{d}{dR}yR^{-a} = -\frac{9(1 - \beta)\eta_0 A^{a-1}}{2}R^{1-a}, \quad (7.37) \]

\[ yR^{-a} = \frac{9(1 - \beta)\eta_0 A^{a-1}}{a}R^{-a}, \quad (7.38) \]

\[ y = \frac{9(1 - \beta)\eta_0 A^{a-1}}{a} = \text{const}, \quad (7.39) \]

or

\[ H = \frac{a}{9(1 - \beta)\eta_0 A^{a-1}} = \frac{\gamma}{3\eta_0} \equiv H_0. \quad (7.40) \]

Hence

\[ R(t) = F \exp H_0 t \quad , \quad F \quad \text{const.}, \]

which is an inflationary solution.

Such a solution has been obtained by several authors [2,5,13]. Here the density is not constant but has the following variation

\[ \rho = AF^{-3\gamma} \exp -3\gamma H_0 t. \]

Such a solution was obtained by Berman and Som for the Brans-Dicke theory for the scalar field \(\phi\) where \(\phi = 1/G\) [26].

In the present case, however, \(G\) is not constant during this epoch, viz.

\[ G(t) = M \exp 3\gamma H_0 t, \]

where

\[ M = \frac{3H_0^2 F^{3\gamma}(1 - \beta)}{8\pi A}. \]
7.5 Model with $n = 1/2$, $\gamma = 1$

Equations (7.23)–(7.25) become

$$ R(t) = Ft^{\frac{2}{3}}, \quad F \text{ const.} $$

$$ \rho = A_0 t^{-2}, \quad A_0 \text{ const.} $$

and

$$ G = \text{const.} $$

The Hubble parameter is $H(t) = \frac{2}{3}t^{-1}$. This is the flat FRW universe result. The deceleration parameter is

$$ q = -3n + 2 = 1/2 $$

Since several authors claim that the age of the Universe computed from the FRW flat model tends to be smaller than the range given by observation, $0.6 < H_p t_p < 1.4$, our model could give a better value for any departure from $n = 1/2$.

However, it was found that only $n = 1/2$ solution are structurally stable [21]. It was shown by Beesham that Berman solution (a power law for $R$) is a viscous solution with $n = 1/2$. The relationship between our model and Berman’s [9] is

$$ m = 3\gamma(1 - n) $$

The value of $m$ in our case is not put by hand, but emerges naturally from the dependence of the viscosity on the energy density ($\eta \sim \rho^n$) in a given epoch. This solution seems more elegant.

7.6 Model with $n = 0$, $\gamma = 1$

Equations (7.23)–(7.25) give

$$ R(t) = F't^{1/3}, $$

$$ \rho = A't^{-1}, $$

and

$$ G(t) = Bt^{-1}, $$

where $F'$, $A'$ and $B$ are constants. This is a model of constant bulk viscosity. It resembles the Brans-Dicke model [16]. We see that

$$ \frac{\dot{G}}{G} = -\frac{1}{t} = -3H^{-1}, \quad (7.41) $$
\[
\frac{\dot{G}}{G}_p = -3H_p^{-1}.
\] (7.42)

This solution was obtained by Berman [15] for the Bertolami equation for the present phase.

Note that in GR, \( k = 0 \)

\[
\rho = \frac{1}{6\pi G t^2}.
\] (7.43)

Whether our result is acceptable or not depends upon the value we measure for \( \left( \frac{\dot{\rho}}{\rho} \right)_p \) for the present phase.

This also resembles the Dirac no creation model. For this class of solutions \( q = 2 \).

### 7.7 Model with \( n=2/3, \gamma = 1 \)

The scale factor is given by

\[
R(t) = R_0 t, \quad R_0 \text{ const.},
\]

and

\[
\rho = A't^{-3},
\]

\[
G = Bt,
\]

where \( A' \) and \( B \) are constants.

This linear variation of \( G \) has been found by Berman [15] for the Bertolami solution for the Brans-Dicke theory (BD) with a time varying cosmological constant for the present phase. For this model \( q = 0 \).

### 7.8 Model with \( n=3/4, \gamma = 4/3 \)

The scale factor is given by

\[
R(t) = Ft,
\]

and

\[
\rho = A't^{-4},
\]

\[
G = Bt^2,
\]

where \( F, A' \) and \( B \) are constants.

This solution was obtained by Berman [15] for the Bertolami theory for the radiation era. He also found that \( T \propto R^{-1} \), preserving Stefan’s law. It was also found by Abdel Rahman that a variable \( G \) and \( \Lambda \)
model lead to a similar result for the radiation universe [11]. In his model he considered $\Lambda \sim R^{-2}$. For this class of models $q = 0$.

More recently, (1995) Massa has considered a model which support an increasing $G$ constant. In his work his considered a “maximal power hypothesis (MPH) in the Einstein-Cartan theory of gravity. Equivalence of his work and ours requires $n > 1/2$.

7.9 Model with $n=1/2$, $\gamma = 4/3$

For this model

$$R(t) = F t^{1/2},$$

$$\rho = A' t^{-2},$$

and

$$G = B = \text{const.},$$

where $F$, $A'$ and $B$ are constants.

This special value for $n$ gives a constant $G$ in both radiation and matter epochs. This is equivalent to a FRW flat universe. For this class $q = 1$.

7.10 Model with $\eta = \eta_0 H$

Using eq.(7.13) in eq.(7.14) we obtain

$$\frac{H'}{H^3} + \frac{3\gamma (1 - \beta)}{2R} \frac{1}{H^2} = \frac{9(1 - \beta)\eta_0}{2A} R^{3\gamma - 1}$$

(7.44)

Let $y = 1/H^2$. This becomes

$$\frac{dy}{dR} - \frac{3\gamma (1 - \beta)}{R} y = \frac{-9(1 - \beta)\eta_0}{A} R^{3\gamma - 1},$$

(7.45)

$$\frac{d}{dR} y R^{-2a} = \frac{-9(1 - \beta)\eta_0}{A} R^{3\gamma - 1 - 2a},$$

(7.46)

$$y = \frac{9(1 - \beta)\eta_0}{A(2a - 3\gamma)} R^{3\gamma},$$

(7.47)

and hence

$$H^2 = NR^{-3\gamma}, \quad N = \frac{A(2a - 3\gamma)}{9(1 - \beta)\eta_0}.$$  

(7.48)

Substituting this in eq.(7.11) and using eq.(7.13) yields

$$G = \frac{3N}{8\pi A} = \text{const.}$$
Hence
\[ R(t) = N^{\frac{1}{3\gamma}} t^{\frac{1}{3\gamma}}. \]

This reduces to the flat FRW model with constant \( G \). This is equivalent to the solution with \( n = 1/2 \).

Therefore the assumption \( \eta \sim H \) is equivalent to \( \eta \sim \rho^{1/2} \) [12].

### 7.11 Model with \( n = 2/3 \), \( \gamma = 2 \)

The scale factor is given by
\[ R(t) = Ft^{1/2}, \]
and
\[ \rho(t) = A't^{-3}, \]
\[ G(t) = Bt, \]
\[ \eta(t) = \eta_0 t^{-2}, \]
where \( F, A' \) and \( B \) are constants.

This result is obtainable from Berman [2] if we let \( A = \frac{1}{16\pi}, B = \frac{1}{4} \) and \( m = 2 \). For this model \( q = 1 \).

### 7.12 Model with \( n = 1/2 \), \( \gamma = 2 \)

The scale factor is given by
\[ R = Ft^{1/3}, \]
and
\[ \rho = A't^{-2}, \]
\[ G = B = \text{const.}, \]
where \( F \) and \( A' \) are constants.

This is the solution for the BD theory for the present phase, as shown by Berman and Som (1990). For this model \( q = 2 \). This solution also found by Beesham for Bianchi type I models for \( n = 0 \) (where \( \gamma = n + 2 \), i.e., \( \rho \sim t^{-\gamma} \)). Barrow showed that \( \rho \sim t^{-\gamma} \) dominates the viscous term for all fluids with \( 1 \leq \gamma \leq 2 \) [3].
7.13 Model with $n = 1/2, \ \gamma = 2/3$

The scale factor is given by

$$R = Ft,$$

and

$$\rho = A't^{-2},$$

$$G = \text{const.},$$

where $F$ and $A'$ are constants.

These are the solutions obtained by Pimentel [14] for the scalar field of the second-self creation theory proposed by Barber, assuming a power law of the scalar field and the expansion factor. The resemblance is evident if we put

$$n = \frac{n_P + 3(\gamma - 1)}{6\gamma}$$

$n_P : n$ due to Pimentel. When $n = 1/2$, the present case, $n_P = 3$. For this class of model $q = 0$. There is no horizon problem associated with this solution.

7.14 Model with $n=1/2 \quad \gamma = 1/3$

The scale factor is given by

$$R = Ft^2,$$

and

$$\rho = A't^{-2},$$

$$G = \text{const.},$$

where $F$ and $A'$ are constants.

This is the wall-like matter. For this solution $q = -1/2$. This solution has been obtained by Berman for the radiation universe, i.e. a wall-like matter behaves the same as radiation in a viscous universe.

7.15 Model with $n = 0, \ \gamma = 1/3$

The scale factor is given by

$$R = Ft,$$
and

\[ \rho = A't^{-1}, \]
\[ G = Bt^{-1}, \]

where \( F, A' \) and \( B \) are constants.

This solution which solves for the power law is also Machian i.e. \( G\rho \sim H^2 \), (see [25]). Such solution has been noted by Berman and Som for the constant deceleration type with \( m \neq 0 \) [25]. This solution corresponds to the case \( m = 1 \). In this case we see that the viscosity is constant, i.e. \( \eta = \eta_0 \). This solution is a wall-like matter solution. This model is free of the horizon problem. This solution has been obtained by Pimentel [24] for the solution of the Brans-Dicke theory with a constant bulk viscosity for \( k \neq 0 \) solution. He has shown that these solutions satisfy the Machian condition and the second Dirac hypothesis. Singh and Devi [23] studied cosmological solutions in Brans-Dicke theory involving particle creation and obtained a similar solution for \( k = 0 \). Some other solutions are

### 7.16 The Pimentel solution for the scalar tetradic theory

**A**  

**Case I**  

This solution of Pimentel [4] is equivalent to our solution provided

\[ n_P = (\beta_P - 2 + 3\gamma) \]

and \( \beta_P = 2 - 6\gamma(1 - n) \), where the subscript “P” is the Pimentel value. Therefore this solution is a viscous solution. The viscosity (\( \eta \)) varies as \( t^{-\frac{(2n_P - \beta_P + 2)}{(2 - \beta_P)}} \). The condition \( \beta_P \neq 2 \) is equivalent to \( n \neq 1 \).  

Note that

\[ \gamma = 2 \left( 1 - \frac{1}{\omega n_P} \right) \]

or

\[ \omega = \frac{2}{3\gamma(2n - 1)(\gamma - 2)} \]

**Case II**  

This solution is equivalent to our solution provided we make the following substitution

\[ 3\gamma(1 - n) = 1 \]

or

\[ n_P = \frac{2n - 1}{1 - n} = 3\gamma - 2 \]
and therefore

\[ \omega = \frac{2(n - 1)}{(2n - 1)(\gamma - 2)}. \]

The viscosity coefficient in this case varies as

\[ \sim t^{-(1 + n_P)}(\beta_P = 0). \]

The viscosity term in this case \( \sim \exp\left(-\frac{(n_P + 1)}{n_P + 2}\right)t. \)

### 7.17 Berman solution

Berman studied a constant deceleration model [2]. eqs. (7.14) and (7.15) of Berman are equivalent to eq.(7.24) provided

\[ \beta = 2n - 1 \leq 0, \]

\[ A = \frac{1}{12\pi\gamma^2(1 - n)}, \]

and

\[ B = \frac{2n - 1}{3\gamma^2(1 - n)^2}. \]

We conclude that Berman solution is equivalent to a bulk viscous model with variable \( G \) and \( \Lambda \). The viscosity term here varies as \( \eta \sim t^{-\left(1 + \frac{n_P}{1 - n}\right)} \).

More recently Johri and Desikan (1994) have considered cosmological models in Brans-Dicke theory with constant deceleration parameter. Their solution for a flat universe, [their eqs.(65)–(67)] are equivalent to our solution, i.e. eqs.(7.23)–(7.25) for the replacement of

\[ \beta = 3\gamma - 3\gamma n - 1 \quad \text{and} \quad \alpha = 3\gamma(1 - 2n). \]

### 7.18 Cosmological expansion in the presence of quadratic bulk viscosity (\( \zeta \))

This term appears as \( 3\zeta H^2 \) in the pressure term.

Let us consider \( \zeta = \text{const.} \) case. It follows that

\[ (9\eta H + 3\zeta H^2)\frac{\dot{R}}{R} = \frac{\dot{G}}{G\rho} + \frac{\dot{\Lambda}}{8\pi G} \]

Using eq.(7.4) we obtain

\[ H' + \frac{3\gamma(1 - \beta)}{2R}H - \frac{9(1 - \beta)\eta_0}{2}R^{3\gamma - 3\gamma n - 1}H^2 - \frac{3\zeta(1 - \beta)}{2}R^{3\gamma - 1}H^3 = 0 \]
This equation admits a power law solution of the form

\[ H = \alpha R^m, \quad \alpha \text{ const.} \]

Substituting this in the above equation,
we get \( m = -\frac{3\gamma}{2} \) and \( n = 1/2 \)
Hence
\[ H = \alpha R^{-\frac{3\gamma}{2}}, \]
or
\[ R = \left( \frac{3\gamma \alpha}{2} \right)^{\frac{2}{3}} t^{\frac{2}{3}} \]
This is the familiar FRW flat universe solution. If we take a general power law for \( \zeta \), i.e. \( \zeta \sim \rho^r \) for some \( r \), it follows that only \( r = 1 \) is possible. This case has been studied by Wolf [5]. He showed that a constant \( \rho \) leads to the inflationary solution. This model is similar to the one considered before \( (\eta = \eta_0 \rho) \). We see for all these models one has the Machian solution \( G\rho \sim H^2 \).

### 7.19 Brans-Dicke solution

In the Brans-Dicke theory [29] the scalar field is related to the gravitational constant \( G \) as \( \phi \propto 1/G \).
This theory is equivalent to a bulk viscous solution with
\[ n = \frac{3\omega + 2}{6(1 + \omega)}, \quad (7.49) \]
in which case the viscosity coefficient varies as
\[ \eta \sim t^{-\frac{3\omega + 2}{6(1 + \omega)}}, \quad (7.50) \]
As in the Brans-Dicke theory when \( \omega \to \infty \) the theory approximates to Einstein de Sitter, in the present case we have \( n \to 1/2 \).

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Chapter 8

A flat Viscous Universe with Increasing G

8.1 Introduction

In most variable $G$ models [7,8] $G$ is a decreasing function of time. But the possibility of increasing $G$ has only been considered recently [2]. Massa [5] proposed a model in scale of increasing $G$ depending on a “maximal power hypothesis (MPH).” Recently [1], we have considered a cosmological model with variable $G$ and $\Lambda$ and bulk viscosity. Various solutions are listed and all of them satisfy Mach’s condition [4].

8.2 The Model

In a Robertson Walker universe

$$d\tau^2 = dt^2 - R^2(t)[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] \quad (8.1)$$

where $k$ is the curvature index.

Einstein’s field equations with time dependent cosmological and gravitational “constants”

$$R_{\mu \nu} - \frac{1}{2}g_{\mu \nu}R = 8\pi G T_{\mu \nu} + \Lambda g_{\mu \nu} \quad (8.2)$$

and the perfect fluid energy momentum tensor

$$T_{\mu \nu} = (\rho + p)u_\mu u_\nu - pg_{\mu \nu} \quad (8.3)$$

yield the two independent equations

$$\frac{\ddot{R}}{R} = -4\pi G(3p + \rho - \frac{\Lambda}{4\pi G}), \quad (8.4)$$
\[
3 \frac{\ddot{R}^2}{R^2} = 8\pi G (\rho + \frac{\Lambda}{8\pi G}) - \frac{3k}{R^2}
\]  
(8.5)

Elimination of \( \ddot{R} \) gives
\[
3(p + \rho) \dot{R} = -\left( \frac{\dot{G}}{G} \rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G} \right) R.
\]  
(8.6)

The conservation of energy and momentum yields
\[
3(p + \rho) = -R \frac{d\rho}{dR}.
\]  
(8.7)

The effect of bulk viscosity in the field equation is to replace \( p \) by \( p - 3\eta H \), where \( \eta \) is the viscosity coefficient. It follows immediately that
\[
9 \eta H \dot{R} = \left( \frac{\dot{G}}{G} \rho + \frac{\dot{\Lambda}}{8\pi G} \right) R
\]  
(8.8)

and
\[
\dot{\rho} + 3H (\rho + p) = 0.
\]  
(8.9)

Equation (8.8) can be written as
\[
9 \eta \frac{H}{R} = \frac{G'}{G} \rho + \frac{\Lambda'}{8\pi G},
\]  
(8.10)

where prime denotes derivative w.r.t. scale factor \( R \) and dot is the derivative w.r.t. cosmic time \( t \). In what follow we will consider the flat universe, \( k = 0 \). Equation (8.5) becomes
\[
3H^2 = 8\pi G \rho + \Lambda.
\]  
(8.11)

We take the Chen and Wu ansatz for \( \Lambda \) [3]
\[
\Lambda = \frac{3\alpha}{R^2}, \quad \alpha \text{ const.},
\]  
(8.12)

and the viscosity to have the form
\[
\eta = \eta_0 \rho^n, \quad \eta_0 \geq 0, n \text{ const.}
\]  
(8.13)

Using the equation of the state \( p = (\gamma - 1)\rho \) in eq.(8.9) we obtain
\[
\rho = AR^{-3\gamma}, \quad A \text{ const.}
\]  
(8.14)

Substituting eqs.(8.12), (8.13) and (8.14) in (8.10) yields
\[
G' - 9\eta_0 A^{n-1} R^{-3\gamma n + 3\gamma - 1} HG - \frac{3\alpha}{4\pi A} R^{2\gamma - 3} = 0.
\]  
(8.15)

This equation admit a power law solution of the form
\[
G = FR^m, \quad F, m \text{ const.}
\]  
(8.16)
Inserting this in the above equation we get
\[ H = C \, R^{3\gamma(n-1)} \] (8.17)
and the condition,
\[ m = 3\gamma - 2, \] (8.18)
where \( C = \left( m - \frac{3\eta}{4\pi F A} \right) / (9\eta_0 A^{n-1}) \). From eqs.(8.16) and (8.18) we see that in the radiation epoch \( (\gamma = 4/3) \, G \propto R^2 \) and in the matter epoch \( G \propto R \). As long as \( \gamma > 2/3 \), \( G \) continues to increase.

This result agrees with that obtained by Abdel Rahman [2] for the critical density model. Hence his model is equivalent to a viscous model. Now eq.(8.17) gives
\[ R = \left[ 3\gamma(1-n)C \right]^{[1/3\gamma(1-n)]} t^{[1/3\gamma(1-n)]} \] (8.19)
Hence eqs.(8.16), (8.14) and (8.12) become
\[ G = G_0 t^{[(3\gamma-2)/3\gamma(1-n)]}, \] (8.20)
\[ \rho = \rho_0 t^{[-1/(1-n)]}, \] (8.21)
\[ \Lambda = \Lambda_0 t^{[-2/3\gamma(1-n)]}, \] (8.22)
and
\[ H = \frac{1}{3\gamma(1-n)t}, \] (8.23)
where \( G_0, \rho_0 \) and \( \Lambda_0 \) are constants. For an expanding Universe, i.e. \( H > 0 \), we must have \( C > 0 \). This implies \( \alpha < \frac{4}{3}\pi FA(3\gamma - 2) \), and for \( \alpha > 0, \gamma > 2/3 \). Since some authors think that the variation of \( \Lambda \propto t^{-2} \) is essential, we impose the condition
\[ 3\gamma(1-n) = 1. \] (8.24)
Hence eqs.(8.19) and (8.20) become
\[ R = R_0 t, \quad R_0 \text{ const.} \] (8.25)
\[ G = G_0 t^{[(2n-1)/(1-n)]}, \quad G_0 \text{ const.} \] (8.26)
This solution has been obtained by Ref.1 for a different ansatz of \( \Lambda \). In the next section we will consider two cases separately.
8.3 I. Radiation dominated (RD) universe

This is characterized by $p = \frac{1}{3} \rho$. The condition (8.24) gives $n = 3/4$. Hence, eqs.(8.21), (8.20) and (8.13) become

$$\rho \propto t^{-4},$$

(8.27)

$$G \propto t^2,$$

(8.28)

and

$$\eta \propto t^{-3}.$$  

(8.29)

We see that $G$ increases with time. In Ref.2, for $t \gg R_0$ the gravitational constant $G \propto t^2$. Though the model of Ref.2 is nonsingular and closed yet it evolves towards this singular viscous model. One also notices that $T \propto R^{-1}$ a result that is expected to hold in this era. An increasing $G$ is recently supported by Massa [6].

8.4 II. Matter dominated (MD) universe

This is a dust filled universe ($\gamma = 1$). The condition (8.24) gives $n = 2/3$. Hence, eqs.(8.21), (8.20) and (8.13) become

$$\rho \propto t^{-3},$$

(8.30)

$$G \propto t,$$

(8.31)

and

$$\eta \propto t^{-2}.$$  

(8.32)

We see that $G$ increases linearly with time. This solution has been found by Berman [3] for the Bertolami solution for the Brans-Dicke theory with a time dependent cosmological constant. Moreover, Ref.2 predicts that $G \propto R$ for $R \to \infty$ (asymptotically).

8.5 An inflationary solution

This corresponds to the case $n = 1$. Hence, eq.(8.17) gives

$$H = C = \text{const.}.$$  

(8.33)

Therefore

$$R = \text{const.} \exp(Ct)$$  

(8.34)
This solution has been obtained by [9,10]. Hence eqs.(8.14) and (8.15) become

\[ \rho = N \exp(-3\gamma H_0 t) \quad , \quad N \text{ const} \quad (8.35) \]

and

\[ G = M \exp[(3\gamma - 2)H_0 t] \quad , \quad M \text{ const.} \quad (8.36) \]

### 8.6 Concluding remarks

We have analyzed a flat viscous cosmological model with varying $G$ and $\Lambda$. The gravitational constant is shown to increase quadratically with time in the pure radiation era and linearly in the matter dominated era. We have also shown that Abdel Rahman model approaches this model asymptotically, i.e. for $t \gg R_0$. The model assumes to solve the horizon and monopole problems of the standard model. In this model we relax the assumption of the critical density. The cosmological constant retains its evolution with time, $\Lambda \propto t^{-2}$, since this is assumed to be fundamental. We see that the viscosity becomes more important in the matter than in the radiation epoch. This model corresponds to one of the models we have considered recently.
8.7 References

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8.8 Appendix A

In order to obtain the equations describing the motion of a viscous fluid, we have to include some additional terms in the equation of motion of an ideal fluid. The equation of motion of a viscous fluid may therefore be obtained by adding to the ideal momentum flux a term $\sigma'_{ik}$ which gives the irreversible viscous transfer of momentum in the fluid. Thus we write the momentum flux density in a viscous fluid in the form

$$T^{\prime}_{ik} = \rho v_i v_k - p \delta_{ik} + \sigma'_{ik}.$$ 

The general form of the tensor $\sigma'_{ik}$ can be found as follows. Processes of internal friction occur in a fluid only when different fluid particles move with different velocities, so that there is a relative motion between various parts of the fluid. Hence $\sigma'_{ik}$ must depend on the space derivatives of the velocity. If the velocity gradients are small we may suppose that the momentum transfer due to viscosity depends only on the first derivatives of the velocity. To the same approximation, $\sigma'_{ik}$ may be supposed a linear function $\frac{\partial v_i}{\partial x_k}$. There can be no term in $\sigma'_{ik}$ independent of $\frac{\partial v_i}{\partial x_k}$, since $\sigma'_{ik}$ must vanish for $v = \text{constant}$. Next, we notice that $\sigma'_{ik}$ must also vanish when the whole fluid is in uniform rotation, since it is clear that in such a motion no internal friction occurs in the fluid. The sum

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}$$

are linear combination of the derivatives $\frac{\partial v_i}{\partial x_k}$, and vanish when $v = \Omega \times r$, where $\Omega$ is the angular velocity. Hence $\sigma'_{ik}$ must contain just these symmetrical combinations of the derivatives $\frac{\partial v_i}{\partial x_k}$. The most general tensor of rank two satisfying the above condition is

$$\sigma'_{ik} = \zeta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_\ell}{\partial x_\ell} \right) - \eta \delta_{ik} \frac{\partial v_\ell}{\partial x_\ell}$$

with coefficients $\zeta$ and $\eta$ independent of the velocity. In making this statement we use the fact that the fluid is isotropic, as a result of which its properties must be described by scalar quantities only (in this case, $\zeta, \eta$). The constants $\zeta$ and $\eta$ are called coefficients of viscosity. Note that $\zeta$ and $\eta$ are functions of temperature and pressure and therefore are not constants throughout the fluid.

8.9 Appendix B

The apparent luminosity is related to the absolute one as by

$$l = \frac{LR^2(t_1)}{4\pi R^4(t_0)r_1^2}$$

(8.37)

\^1See e.g. L.D.Landau and E.M. Lifshitz, Fluid Mechanics (Pergamon Books Ltd. 1987)
where \( r_1 \) is the source coordinate, and \( t_1, t_0 \) are the time of emission and reception respectively. Hence

\[
d_L = \frac{R^2(t_0)}{R^2(t_1)} r_1. \tag{8.38}
\]

But

\[
\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \equiv f(r_1). \tag{8.39}
\]

The scale factor can be expanded in power of \((t_0 - t)\) as

\[
R(t) = R(t - (t_0 - t)) = R(t_0)\left[1 - (t_0 - t)H_0 - 1/2(t_0 - t)^2q_0H_0^2 + ...\right] \tag{8.40}
\]

where \( f(r_1) = \sin^{-1} r_1, r_1, \sin r_1 \), according to whether \( k = 1, 0, -1 \).

The red-shift \( z \) is

\[
z = (t_0 - t_1)H_0 + (t_0 - t_1)^2(q_0/2 + 1)H_0^2 + ... \tag{8.41}
\]

Inverting this, the look-back time is

\[
t_0 - t_1 = H_0^{-1}z - H_0^{-1}(1 + q_0/2)z^2 + ... \tag{8.42}
\]

Therefore

\[
r_1 + O(r^3) = R_0^{-1}[t_0 - t_1 + 1/2H_0(t_0 - t_1)^2 + ...
\]

\[
= (R_0H_0)^{-1}[z - 1/2(1 + q_0)z^2 + ...]
\]

\[
d_L = H_0^{-1}[z + 1/2(1 - q_0)z^2 + ...] \tag{8.43}
\]

Hence

\[
\ell = \frac{L}{4\pi d_L^2} = \frac{LH_0^2}{4\pi z^2}[1 + (q_0 - 1)z + ...]
\]

\( \ell \) is usually expressed in terms of apparent bolometric magnitude \( m \). This is defined by

\[
l = 10^{-2m/5} \times 2.52 \times 10^{-5} \text{ erg/cm}^2/\text{s}.
\]

The absolute luminosity is defined as the apparent magnitude of the source would have at a distance of 10pc.

\[
L = 10^{-2M/5} \times 3.02 \times 10^{35} \text{ erg s}^{-1}.
\]

The distance modulus can be found as follows

\[
d_L = 10^{1+(m-M)/5} \text{ pc}
\]

and hence

\[
m - M = 25 - 5\log H_0 + 1.086(1 - q_0)z + ...
\]

or

\[
m - M = 5\log d_L - 5
\]