A Global Bias-Correction DC Method for Biased Estimation under Memory Constraint

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Abstract

This paper introduces a global bias-correction divide-and-conquer (GBC-DC) method for biased estimation under the case of memory constraint. In order to introduce the new estimation, a closed representation of the local estimators obtained by the data in each batch is adopted to formulate a pro forma linear regression between the local estimators and the true parameter of interest. A least squares is used within this framework to composite a global estimator of the parameter. Thus, the main difference from the classical DC method is that the new GBC-DC method can absorb the information hidden in the statistical structure and the variables in each batch of data. Consequently, the resulting global estimator is strictly unbiased even if the local estimators have a non-negligible bias. Moreover, the global estimator is consistent under some mild conditions, and even can achieve root-$n$ consistency when the number of batches is large. The new method is simple and computationally efficient, without use of any iterative algorithm and local bias-correction. Moreover, the proposed GBC-DC method applies to various biased estimations such as shrinkage-type estimation and nonparametric regression estimation. Based on our comprehensive simulation studies, the proposed GBC-DC approach is significantly bias-corrected, and the behavior is comparable with that of the full data estimation.

Key words: Divide-and-conquer; memory constraint; bias-reduction; composition.

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1 Introduction

The divide-and-conquer (DC) is one of the most important algorithms in computer science to deal with large-scale datasets. When large-scale datasets cannot be fit into memory of a single computer, they are distributed in many machines over limited memory. Then, the local result, for example the estimator of a parameter, can be obtained by the batch of data in each machine, and finally, the global result can be achieved by aggregating these local results. See, e.g., Manku, Rajagopalan and Lindsay (1998); Greenwald and Khanna (2004); Zhang and Wang (2007); Guha and Mcgregor (2009) and the references therein. Up to now, there have been various types of aggregation methodologies for constructing the global estimator, for example, the naive average of the local estimators (e.g., Mcdonald et al. 2009; Zinkevich et al. 2010), and the relevant DC expressions (e.g., Chen, et al. 2006, and Lin and Xi, 2011) and representative approaches (e.g., Li and Yang, 2018, Wang, 2018). The related works include but are not limited to the DC expression for linear model of Chen et al. (2006), Lin and Xi (2011), and Schifano et al. (2016), the density estimation of Li, Lin and Li (2013), the parametric regression estimation of Chen and Xie (2014), and Zhang, Duchi and Wainwright (2015), the high-dimensional parametric regression estimation of Lee et al. (2017), semi-parametric regression estimation of Zhao, Cheng and Liu (2016), quantile regression processes of Volgushev, Chao and Cheng (2018), the M-estimator of Shi, Lu and Song (2017), and the distributed testing and estimation of Battey et al. (2018).

As shown by the existing literature (see, e.g., Li, Lin and Li, 2013; Zhang et al. 2013; Rosenblatt and Nadler 2016), for achieving the same asymptotic distribution for statistical inference as pooling all the data together, the number of batches is restricted. More specifically, a commonly used restriction is \( N = o(\sqrt{n}) \) (or equivalently \( n = o(m^2) \)), where \( n \) is the sample size, \( N \) is the number of batches and \( m = n/N \). Such a constraint on \( N \) cannot be satisfied in some applications such as sensor networks and streaming data because the number of batches can be large.
In order to relax the constraint, instead of one-shot aggregation via averaging, the aggregation with multiple rounds (e.g., iterative algorithm) was proposed recently by Jordan, Lee and Yang (2018) and Wang et al. (2017) for the case of differentiable loss function, and Chen, Liu and Zhang (2018) for quantile regression with non-differentiable loss. These methods are able to reduce both estimation bias and variance simultaneously and then obtain the standard result as pooling all the data together. It is known that bias reduction is more crucial than the variance reduction. Such a goal cannot be achieved by many classical inference methods that require to balance the variance and bias.

The estimation bias often appears in the procedure of statistical inference. The common examples are shrinkage-type estimations in linear and generalized linear models, and $M$- and $Z$-estimations in nonlinear model, and kernel estimation in nonparametric regression model. It is verified by our motivating examples in the next section that when the local estimators is biased (e.g., LASSO estimator), the global estimator by the naive average or the original DC expression cannot achieve $\sqrt{n}$-consistent and is even divergent for any choice of $N$. Thus, bias-correction is crucial in the procedure of aggregation. The existing methods use iterative algorithm (see, e.g., Wang et al., 2017) and local bias-correction (see, e.g., Lian, et al., 2018, and Keren and Yang, 2018) to reduce the bias of local estimators and then to control the bias of the global estimator. However, the iterative algorithm and the bias-correction for local estimators are computationally complex, and the bias-correction is not sufficient.

From a new perspective, we in this paper explore a global bias-correction divide-and-conquer (GBC-DC, for short) algorithm for the biased local estimations under the case of large sample size. The newly proposed GBC-DC methodology is motivated by a proven statistical technique, composition, which has received much attention in the literature. The early goal of the classical composition methods is only to reduce the estimation variance via optimizing the composite estimation
covariance; see Zou and Yuan (2008) for composite quantile linear regression estimation, see Kai, Li, and Zou (2010), and Sun, Gai, and Lin (2013) for composite nonparametric regression estimation, see Kai, Li, and Zou (2011) for composite semiparametric estimation, see Bradic, Fan, and Wang (2011) for composite variable selection of ultra-high-dimensional models. Recently, bias-reduction by composition has attained much attention as well in the literature. Based on the asymptotic or approximate representation of the initial estimator, Lin et al. (2019), Cheng et al. (2018) and Lin and Li (2008) introduced composite least squares to realize the targets of reducing estimation bias and optimizing estimation covariance, simultaneously. Moreover, the relevant composition methods were suggested by Wang and Lin (2015), and Tong and Wang (2005) for constructing the composite estimators of the derivative of nonparametric regression function and the residual variance in nonparametric regression, respectively.

It will be seen later that the main difference from the aggregation of DC in computer science is that the GBC-DC technique is able to sufficiently absorb the information of statistical structure and the variables in batches of data. To realize our goals aforementioned, we employ a closed representation of the local estimator computed on each batch of data to build a pro forma linear regression model in which the combination of the variables in each batch is regarded as the covariant and the local estimator is thought of as as response variable. Based on such a model and least squares, we composite a global estimator. It will be shown in the later development that this method has the following salient features.

1) **Bias-correction.** The new composition method sufficiently employs the information of the closed representation and the batches such that the resulting global estimator is strictly unbiased even if the local estimators have a non-negligible bias.

2) **Acceleration of convergence.** The convergence rate of the global estimator is accelerated such that the $\sqrt{n}$-consistency can be achieved under some mild
conditions.

3) Simplicity. Iterative algorithm and bias-correction for local estimators in the aggregation procedure are not needed. Furthermore, the structure of the resultant global estimator is simple, which is a least squares estimator and has a DC expression. Thus, the composition procedure is computationally simple and efficient. Because of the structure of least squares, we can construct its online updating version and make statistical inference in the case of data streams.

4) Generality. Although our method focuses mainly on linear model and related parameter estimation, the new technique is extended into other models such as nonlinear and nonparametric models.

All the features above are illustrated by our comprehensive simulation studies, which particularly show that the global estimator by GBC-DC is significantly bias-corrected, and the behavior is comparable with that of the full data estimation.

The remainder of this paper is organized in the following way. In Section 2, after the classical DC algorithm is briefly recalled, some motivating examples are investigated to motivate the methodological development. In Section 3, a unified framework for linear model is defined and the least squares global estimator is proposed via the newly defined model and least squares method, and the theoretical properties of the global estimator are investigated. The extensions of the new method to the cases of nonlinear and nonparametric models are discussed in Section 4. Simulation studies are provided in Section 5 to illustrate the new method. The proofs of theorems are relegated to Appendix.
2 Problem Formulation

2.1 Divide-and-conquer

We briefly recall general DC algorithm for statistical estimation. Let \( \{Z_1, \cdots, Z_n\} \) be the set of observation data, where the sample size \( n \) is extremely large. Our goal is to estimate a \( p \)-dimensional parameter \( \theta = (\theta^1, \cdots, \theta^p)^T \). We split the data index set \( \{1, \cdots, n\} \) into \( N \) subsets \( \mathcal{H}_1, \cdots, \mathcal{H}_N \), where the size of \( \mathcal{H}_j \) is \( m = |\mathcal{H}_j| \) satisfying \( n = Nm \). Correspondingly, the entire dataset \( \{Z_1, \cdots, Z_n\} \) is divided into \( N \) batches \( \mathcal{D}_1, \cdots, \mathcal{D}_N \) with \( \mathcal{D}_j = \{Z_i, i \in \mathcal{H}_j\} \). By swapping each batch of data \( \mathcal{D}_j \) into the memory, we can construct a local estimator of \( \theta \) as \( \hat{\theta}_j = g_j(\mathcal{D}_j) \) for \( \mathcal{D}_j \) with some function \( g_j(\cdot) \). The global estimator \( \hat{\theta} \) is then obtained by an aggregation of \( \hat{\theta}_j, j = 1, \cdots, N \), e.g., the naive average as \( \hat{\theta} = \frac{1}{N} \sum_{j=1}^N \hat{\theta}_j \) or the corresponding DC expression. Actually, the classical DC strategy typically requires a random data partition, that is, the batches of data stored in different computers are independent and have the same distribution. In our setting, however, the identical distribution assumption on \( \mathcal{D}_1, \cdots, \mathcal{D}_N \) is not necessary. We particularly consider the case of streaming data where the obtained data may not be identically distributed in different observation periods.

In this section, we mainly focuses on the following linear model:

\[
Y_i = X_i^T \beta + \varepsilon_i, \quad i = 1, \cdots, n,
\]

(2.1)

where \( \beta = (\beta^1, \cdots, \beta^p)^T \) is a \( p \)-dimensional vector of unknown parameters, and \( X_i = (X_i^1, \cdots, X_i^p)^T, \quad i = 1, \cdots, n \), are independent \( p \)-dimensional covariates, and the errors \( \varepsilon_i, i = 1, \cdots, n \), are independent and satisfy \( E[\varepsilon_i | X_i] = 0 \) and \( Var[\varepsilon_i | X_i] = \sigma^2 \).

For the regression model, the data batches are \( \mathcal{D}_j = \{(X_i, Y_i), i \in \mathcal{H}_j\}, j = 1, \cdots, N \).

2.2 Motivating examples and related issues

To proceed with the methodological development, we first look at the following shrinkage-type estimators.
Example 1. LASSO estimator. When the dimension $p$ is high in model (2.1), we use penalty-based methods to select variables and estimate parameters, simultaneously. Base on the subset $D_j$, the LASSO estimator (Tibshirani, 1996) of $\beta$ is given by

$$
\hat{\beta}_j = \arg \min_{\beta} \frac{1}{2m} \sum_{i \in D_j} (Y_i - X_i^T \beta)^2 + \lambda \|\beta\|_1,
$$

where $\lambda > 0$ is a regularization parameter. In the existing literature, $\lambda$ is supposed to satisfy $\lambda = O(m^{-\delta})$ for some constant $0 < \delta < 1$ (see, e.g., Knight and Fu, 2000).

In the remainder part of this subsection, however, we mainly consider the following special case:

$C0. \ 0 < \delta \leq 1/2.$

In the case of $C0$, the LASSO estimator has a non-ignorable bias because the convergence rate of the estimator is $m^{-\delta}$, instead of the standard one $m^{-1/2}$ (see, e.g., Knight and Fu, 2000). The constraint on $\lambda$ by $C0$ is sometimes used for selecting the variables and simplifying the model.

Suppose without loss of generality that $\beta^k \neq 0$ for $k = 1, \ldots, s$, and $\beta^k = 0$ for $k = s + 1, \ldots, p$. Denote by $\beta_S$ the significant subset of $\beta$, i.e., $\beta_S = (\beta^1, \ldots, \beta^s)^T$. Let $X_S = (x_1, \ldots, x_s)$ with $x_k = (X_{1k}, \ldots, X_{nk})^T$, and $X_{jS} = (x_{j1}, \ldots, x_{js})$ with $x_{jk} = (X_{ik} : l \in H_j)^T$. The existing literature (Wainwright, 2009; Huang et al, 2008) show that under some regularity conditions, the resultant estimator $\hat{\beta}_{jS}$ of the significant subset $\beta_S$ has the following closed representation:

$$
\hat{\beta}_{jS} = \beta_S - \left( \frac{1}{m} X_{jS}^T X_{jS} \right)^{-1} \lambda \text{sgn}(\beta_S) + \left( \frac{1}{m} X_{jS}^T X_{jS} \right)^{-1} \frac{1}{m} X_{jS}^T \varepsilon_j,
$$

where $\varepsilon_j = (\varepsilon_i : i \in H_j)^T$. From (2.2) we can see that the estimator is shrunken and has the estimation bias as $-E \left[ \left( \frac{1}{m} X_{jS}^T X_{jS} \right)^{-1} \right] \lambda \text{sgn}(\beta_S)$. Then, the naive average $\hat{\beta}_S = \frac{1}{N} \sum_{j=1}^N \hat{\beta}_{jS}$ has the bias as

$$
B(\hat{\beta}_S) = -\frac{1}{N} \sum_{j=1}^N E \left[ \left( \frac{1}{m} X_{jS}^T X_{jS} \right)^{-1} \right] \lambda \text{sgn}(\beta_S),
$$
which is of order $O(m^{-\delta})$. Similarly, the DC expression of LASSO estimator, which can be expressed as 
\[ \hat{\beta}_S = \left( \sum_{j=1}^{N} \frac{1}{m} X_{jS}^T X_{jS} \right)^{-1} \sum_{j=1}^{N} \frac{1}{m} X_{jS}^T X_{jS} \hat{\beta}_j S \] (see, e.g., Lin and Xi, 2011), has the bias of order $O(m^{-\delta})$ as well. Thus, under Condition $C0$, $\sqrt{n}\hat{\beta}_S$ has a bias of order $O(n^{1/2-\delta} N^{\delta})$, satisfying
\[
\sqrt{n}B(\hat{\beta}_S) = O(n^{1/2-\delta} N^{\delta}) \rightarrow \infty \text{ as } n, N \rightarrow \infty. \tag{2.3}
\]

This implies that the global estimator $\hat{\beta}_S$ cannot achieve $\sqrt{n}$-consistency for any choices of $N$ and $m$.

**Example 2. Ridge estimator.** Under model (2.1), the Ridge estimator computed on subset $D_j$ is defined by

\[ \hat{\beta}_j = \arg \min_{\beta} \frac{1}{m} \sum_{i \in H_j} (Y_i - X_i^T \beta)^2 + \lambda \| \beta \|_2^2. \]

Let $X = (x_1, \ldots, x_n)$ with $x_k = (X_{1k}, \ldots, X_{nk})^T$ and $X_j = (x_{j1}, \ldots, x_{jn})$ with $x_{jk} = (X_{lk}^k : l \in H_j)^T$. It can be verified that the Ridge estimator has the following closed representation:

\[
\hat{\beta}_j = \beta - \left( \frac{1}{m} X_j^T X_j + \lambda I_p \right)^{-1} \lambda \beta + \left( \frac{1}{m} X_j^T X_j + \lambda I_p \right)^{-1} \frac{1}{m} X_j^T \epsilon_j, \tag{2.4}
\]

where $I_p$ is a $p \times p$ identity matrix. The estimator is shrunk and has the estimation bias of order $O(m^{-\delta})$. Then, the naive average $\hat{\beta} = \frac{1}{N} \sum_{j=1}^{N} \hat{\beta}_j$ has the estimation bias as

\[ B(\hat{\beta}) = -\frac{1}{N} \sum_{j=1}^{N} E \left[ \left( \frac{1}{m} X_j^T X_j + \lambda I_p \right)^{-1} \right] \lambda \beta, \]

which is of order $O(m^{-\delta})$. Similarly, the DC expression of Ridge estimator, which can be expressed as $\hat{\beta} = \left( \frac{1}{N} \sum_{j=1}^{N} \frac{1}{m} X_j^T X_j + \lambda I_p \right)^{-1} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{m} X_j^T X_j \hat{\beta}_j$ (see, e.g., Lin and Xi, 2011), has the bias of order $O(m^{-\delta})$ as well. Under Condition $C0$, $\sqrt{n}\hat{\beta}$ has a non-ignorable bias of order $O(n^{1/2-\delta} N^{\delta})$, specifically,

\[
\sqrt{n}B(\hat{\beta}) = O(n^{1/2-\delta} N^{\delta}) \rightarrow \infty \text{ as } n, N \rightarrow \infty. \tag{2.5}
\]
Therefore, the global estimator $\hat{\beta}$ cannot achieve $\sqrt{n}$-consistency for any choices of $N$ and $m$.

There are other examples of biased estimators (e.g., quantile estimator) satisfying that the resulting global estimators by naive average or the original DC expression have the non-ignorable bias as in (2.3) and (2.5). These examples indicate that the naive average and the original DC expression are invalid when the local estimators have a non-ignorable bias. The observation motivates us to develop new DC methodologies.

3 Global bias-correction estimate in linear model

3.1 Modeling

We use $\theta$ to denote the parameter vectors $\beta_S$ and $\beta$ respectively in Example 1 and Example 2, or a general parameter vector in a linear model. For convenience of modeling, suppose the dimension $p$ is fixed. The composite method proposed blow still applies to the case where $p$ depends on $n$. An interesting finding is that the closed representations (2.2) and (2.4) respectively for LASSO estimator and Ridge estimator can be expressed as the following unified form:

$$\hat{\theta}_j = \theta + V_m(D_j)\xi(\theta) + \epsilon_j, \; j = 1 \ldots, N. \tag{3.1}$$

In the above model, the matrices $V_m(D_j)$ depend on subsets $D_j$, the vector $\xi(\theta)$ is a function of $\theta$, and vectors $\epsilon_j$ have zero mean. In the following examples, the covariance matrix $Cov[\epsilon_j|D_j]$ is approximately equal to a positive definite matrix $\Sigma$. We then suppose $Cov[\epsilon_j|D_j] = \Sigma$, without loss of generality.

For the LASSO estimator in Example 1,

$$V_m(D_j) = - \left( \frac{1}{m} X_{js}^T X_{js} \right)^{-1}, \; \xi(\theta) = \lambda \text{sgn}(\theta), \; E[\epsilon_j|X_{js}] = 0, \; Cor[\epsilon_j|X_{js}] \approx \Sigma,$$
where $\Sigma = \sigma^2(E(XX^T))^{-1}$. Similarly, for the Ridge estimator in Example 2,

$$V_m(D_j) = -\left(\frac{1}{m}X_j^TX_j + \lambda I_p\right)^{-1}, \xi(\theta) = \lambda \theta, E[\epsilon_j|X_jS] = 0,$$

$$\text{Cov}[\epsilon_j|X_j] = \sigma^2\left(\frac{1}{m}X_j^TX_j + \lambda I_p\right)^{-1}\frac{1}{m}X_j^TX_j \left(\frac{1}{m}X_j^TX_j + \lambda I_p\right)^{-1} \approx \Sigma,$$

where $\Sigma = \sigma^2(E(XX^T))^{-1}$ as well.

Let $\hat{\theta}_j^k$ and $\epsilon_j^k$ be the $k$-th elements of $\hat{\theta}_j$ and $\epsilon_j$ respectively, and $v_m^k(D_j) = V_m^T(D_j)e_k$, where $e_k$ is a $p$-dimensional vector with the $k$-th element 1 and the others zero. By (3.1), we have

$$\hat{\theta}_j^k = \theta^k + \xi^T(\theta)v_m^k(D_j) + \epsilon_j^k, j = 1, \ldots, N. \quad (3.2)$$

Denote $V_k = (v_m^k(D_1), \ldots, v_m^k(D_N))^T$. According to the motivating examples aforementioned, we suppose the following conditions:

\begin{itemize}
  \item \textbf{C1.} $v_m^k(D_j) = O_p(1)$, $E[\epsilon_j^k|D_j] = 0$ and $\text{Var}[\epsilon_j^k|D_j] = \sigma^2$.
  \item \textbf{C2.} The inverse matrix $(V_k^TV_k)^{-1}$ exists uniformly for all $N$.
\end{itemize}

It can be seen that when $m$ is large enough, Condition \textbf{C1} is a direct result of the motivating examples. However, sometimes Condition \textbf{C2} is not satisfied. To verify the point of view, we look at the LASSO estimator, in which $V_m(D_j) = -\left(\frac{1}{m}X_j^TX_jS\right)^{-1}$. Under some regularity conditions, each component of $\frac{1}{m}X_j^TX_jS$ has the asymptotic normality with finite mean and variance of order $O(1/m)$. By the asymptotic normality of the function of multivariate normal variable (see, e.g., Theorem A in Chapter 3.3 of Serfling, 1980), we have that each component of $V_m(D_j)$ is asymptotically normally distributed with finite mean and variance of order $O(1/m)$. This shows that if $V_m(D_1), \ldots, V_m(D_N)$ are identically distributed, the matrix $V_k^TV_k$ is nearly degenerate, implying that Condition \textbf{C2} cannot be satisfied for identically distributed data.

We use the following methods to deal with the problem.
(i) Cases of fixed design and distribution heterogeneity. We first consider the case where the variables \(X_1, \ldots, X_n\) in model (2.1) are fixed designed. When the data sets \((X_i : i \in \mathcal{H}_1), \ldots, (X_i : i \in \mathcal{H}_N)\) are designed to be scattered, the matrix \(V_k^TV_k\) is invertible, i.e., Condition C2 is satisfied. On the other hand, consider the case where the sets \(D_1, \ldots, D_N\) are random, but not identically distributed. Such a distribution heterogeneity often appears under the situation of big data. A common example is streaming data, which may not be identically distributed in different observation periods. In this case, we can suppose that \(V_m(D_1), \ldots, V_m(D_N)\) are not identically distributed, and consequently, the matrix \(V_k^TV_k\) is invertible as well.

(ii) Case of distribution homogeneity. Consider the case where \((X_1, Y_1), \ldots, (X_n, Y_n)\) are identically distributed observations of \((X, Y)\). With the assumption, the regression coefficient of (2.1) is equal to

\[
\beta = (E[XX^T])^{-1} E[XY].
\]

Suppose \(E[X] = 0\), without loss of generality. Further suppose that we have known a subset \(\mathcal{H}^*\) of the index set \(\{1, \ldots, n\}\) such that \(E[X^kY] = 0, k \in \mathcal{H}^*\). This is a mild assumption for sparse high-dimensional model because we can easily estimate the marginal correlation \(E[X^kY]\). For non-sparse low-dimensional model, we can add some artificial variables into the model such that the added covariates are uncorrelated with \(Y\). Thus, the condition of \(E[X^kY] = 0 (k \in \mathcal{H}^*)\) is common. Suppose without loss of generality that \(\mathcal{H}^* = \{r, \ldots, p\}\). Construct an equivalent regression to (2.1) as

\[
Y = U^T \eta + \varepsilon, \quad (3.3)
\]

where \(U = AX + b\), matrix \(A = \text{diag}(1, \ldots, 1, a^r, \ldots, a^p)\) for some constants \(a^k \neq 0, k = r, \ldots, p\), and \(b\) is an eigenvector corresponding to a nonzero eigenvalue of the matrix \(E[XX^T] - AE[XX^T]A\). Under the case of distribution homogeneity, the matrix \(E[XX^T]\) can be consistently estimated using the data in a data set, for example \(D_1\). Then, when \(A\) is given, \(b\) can be consistently estimated using the
data in \( D_1 \). Particularly, when \( X \) can be standardized such that \( E[XX^T] = I_p \), the eigenvector can be chosen as
\[
b = (b^1, \ldots, b^r-1, 0, \ldots, 0)^T, \tag{3.4}
\]
where \( b^k, k = 1, \ldots, r - 1 \), are arbitrary constants satisfying \( \sum_{k=1}^{r-1} (b^k)^2 = 1 \). The proof for that \( b \) in (3.4) is an eigenvector of \( E[XX^T] - AE[XX^T]A \) is provided in Appendix. With the choices of \( A \) and \( b \), the models (2.1) and (3.3) are equivalent in the sense that
\[
\eta = (E[UU^T])^{-1}E[UY] = (E[XX^T])^{-1}E[XY] = \beta. \tag{3.5}
\]

The proof of (3.5) is provided in Appendix as well. Thus, estimating \( \beta \) in (2.1) is equivalent to estimating \( \eta \) in (3.5). Let \( U_i = A_iX_i + b, A_i = \text{diag}(1, \ldots, 1, a_i^1, \ldots, a_i^p) \) for \( i \in \mathcal{H}_j \). When \( (a_i^1, \ldots, a_i^p : i \in \mathcal{H}_j) \) have different values for different \( j \), \( (U_i : i \in \mathcal{H}_j), j = 1, \ldots, N \), are not identically distributed. Therefore, under the equivalent model (3.3), Condition C2 is satisfied.

Thus, both Condition C1 and Condition C2 can be easily satisfied. Under the two conditions, model (3.2) (or (3.1)) could be thought of as a linear regression model, in which \( \hat{\theta}_j^k \) (or vector \( \hat{\theta}_j \)) are the response variables (or response vectors), vector \( v^k_m(D_j) \) (or matrix \( V_m(D_j) \)) are the covariate vector (or covariate matrix), \( \xi(\theta) \) is the regression coefficient, \( \theta^k \) (or \( \theta \)) is the intercept, and \( \epsilon_j^k \) (or \( \epsilon_j \)) are the errors. Thus, the intercept \( \theta^k \) (or \( \theta \)) is the parameter of interest. Furthermore, models (3.1) and (3.2) are of DC expressions of regression. Such a structure is different from the composition methods in Lin et al. (2018), Cheng et al. (2018), Lin and Li (2008), Wang and Lin (2015), and Tong and Wang (2005). This is because these methods do not have DC structure and use a model-independent parameter (e.g., quantile and bandwidth) as an artificial covariate, which does not exist in the original model, but is identified from the estimation procedure.
3.2 Estimation

The above modeling procedures indicate that we can use the DC expressed model (3.1) or (3.2) to construct a global estimator. For simplicity, we mainly focus on model (3.2), which has univariate “response” \( \hat{\theta}_j \). Under the pro forma linear regression (3.2), the composite global estimator of \( \theta^k \) is naturally defined as the first component of the following least squares solution:

\[
(\tilde{\theta}^k, \tilde{\beta}^T)^T = \arg\min_{\theta, \beta} \sum_{j=1}^{N} (\hat{\theta}^k_j - \theta^k - \xi^T(\theta)v^k_m(D_j))^2.
\]  

(3.6)

It can be easily verified that the composite global estimator in (3.6) has the following simple expression:

\[
\tilde{\theta}^k = \overline{\theta}^k - \tilde{\xi}^T \overline{v}^k,
\]  

(3.7)

where \( \overline{\theta}^k = \sum_{j=1}^{N} \hat{\theta}^k_j \), \( \overline{v}^k = \sum_{j=1}^{N} v^k_m(D_j) \) and

\[
\tilde{\xi} = \left( \sum_{j=1}^{N} \left( v^k_m(D_j) - \overline{v}^k \right) \left( v^k_m(D_j) - \overline{v}^k \right)^T \right)^{-1} \sum_{j=1}^{N} \left( v^k_m(D_j) - \overline{v}^k \right) \hat{\theta}^k_j.
\]

The composite global estimator is a DC expression, without accessing the raw data. The global estimator is computational simple as it is computed directly on \( v^k_m(D_j) \) and \( \hat{\theta}^k_j \), without use of any iterative algorithm and local bias-correction, and has the form of least squares. Because of such a structure, we can construct its online updating version and make statistical inference in the case of streams (see, e.g., Schifano et al., 2016). Furthermore, the global estimator is unbiased (see Lemma 3.2 below), because such a DC expression sufficiently uses the structural information of regression (3.2) such that the unbiasedness can be achieved. We thus call it bias-corrected global estimator (BC-GE, for short). This is totally different from the original DC expressions (see the DC expressions of the LASSO and Ridge estimators given in Subsection 2.2).

The BC-GE \( \tilde{\theta}^k \) in (3.8) is derived from the general model framework in (3.2). Particularly, for the LASSO estimator in Example 1, the local estimators of the
significant subset $\beta_S$ of $\beta$ may be different using different subsets $D_j$. We thus use
the majority voting methods proposed by Meinshausen and Buhlmann (2010), Shah
and Samworth (2013), and Chen and Xie (2014) to determine the significant subset $\beta_S$. After the significant subset $\beta_S$ is determined, the corresponding BC-GE of the
$k$-component of $\beta_S$ is
\[
\tilde{\beta}_S^k = \overline{\beta}_S^k - \tilde{\xi}^T \overline{v}_S^k,
\]
where $\overline{\beta}_S^k = \sum_{j=1}^N \tilde{\beta}_j^k$, $\overline{v}_S^k = \sum_{j=1}^N v_m^k(D_{jS})$, $v_m^k(D_{jS}) = \left( \frac{1}{m}X_j^T X_j + \lambda I_p \right)^{-1} e_k$, and
\[
\tilde{\xi} = \left( \sum_{j=1}^N \left( v_m^k(D_{jS}) - \overline{v}_S^k \right) \left( v_m^k(D_{jS}) - \overline{v}_S^k \right)^T \right)^{-1} \sum_{j=1}^N \left( v_m^k(D_{jS}) - \overline{v}_S^k \right) \tilde{\beta}_j^k.
\]
Similarly, for the Ridge estimator in Example 2, the corresponding BC-GE is
\[
\tilde{\beta}^k = \overline{\beta}^k - \tilde{\xi}^T \overline{v}^k,
\]
where $\overline{\beta}^k = \sum_{j=1}^N \tilde{\beta}_j^k$, $\overline{v}^k = \sum_{j=1}^N v^k(D_j)$, and
\[
v^k(D_j) = \left( \frac{1}{m}X_j^T X_j + \lambda I_p \right)^{-1} e_k,
\]
\[
\tilde{\xi} = \left( \sum_{j=1}^N \left( v^k(D_j) - \overline{v}^k \right) \left( v^k(D_j) - \overline{v}^k \right)^T \right)^{-1} \sum_{j=1}^N \left( v^k(D_j) - \overline{v}^k \right) \tilde{\beta}_j^k.
\]

3.3 Theoretical property

Actually, the BC-GE $\tilde{\theta}^k$ given in (3.8) is original least squares estimator under linear
regression model (3.2). Thus, its theoretical property is very simple. The following
lemma follows directly from the property of the least squares estimation.

**Lemma 3.1.** Under Conditions C1 and C2, the BC-GE $\tilde{\theta}^k$ given in (3.8) has mean
$\theta^k$ and variance
\[Var(\tilde{\theta}^k|V_k) = \sigma^2 e_1^T \left( (1, V_k)^T (1, V_k) \right)^{-1} e_1.\]
By the lemma and Conditions \( C1 \) and \( C1 \), we get the following lemma.

**Lemma 3.2.** Under Conditions \( C1 \) and \( C2 \), the BC-GE \( \tilde{\theta}^k \) given in (3.8) satisfies
\[
\tilde{\theta}^k = O_p(N^{-1}).
\]

Consequently, we get the following main results.

**Theorem 3.3.** Under Conditions \( C1 \) and \( C2 \), the BC-GE \( \tilde{\theta}^k \) is consistent in the sense that 1) it is always consistent for arbitrary choices of \( N \) and \( m \), and 2) when
\[
N = O(n),
\]

it is \( \sqrt{n} \)-consistent.

The theorem guarantees that the consistency of the BC-GE \( \tilde{\theta}^k \) for any choices of \( N \) and \( m \). Such a result cannot be attained by the existing methods. On the other hand, we need the condition (3.10) to achieve the \( \sqrt{n} \)-consistency of the BC-GE. The condition (3.10) means that \( N \) should be large enough. Such a constraint on \( N \) can be easily satisfied in some applications such as sensor networks and streaming data because the number of batches can be large.

Furthermore, in order to establish the asymptotic normality, we need the condition:

**C3.** The following limits exist:
\[
\frac{1}{N} \sum_{j=1}^{N} \mathbf{v}^k_m(D_j) \xrightarrow{p} E[\mathbf{v}^k], \quad \frac{1}{N} \sum_{j=1}^{N} \mathbf{v}^k_m(D_j)(\mathbf{v}^k_m(D_j))^T \xrightarrow{p} E[\mathbf{v}^k(\mathbf{v}^k)^T].
\]

In the above, the notation \( E[\mathbf{v}^k] \) stands for a fixed number, but is not always the expectation of a vector \( \mathbf{v}^k \), and the notation \( E[\mathbf{v}^k(\mathbf{v}^k)^T] \) denotes a fixed matrix, but is not always the expectation of a matrix \( \mathbf{v}^k(\mathbf{v}^k)^T \). This is because \( \mathbf{v}^k_m(D_j), j = 1, \cdots, N, \) may not be random, and even for the case of random variables, they may not be identically distributed. Obviously, the above condition is common. With this condition, the asymptotic normality holds; the following theorem states the details.
Theorem 3.3. Under Conditions C1-C3, the BC-GE $\tilde{\theta}^k$ has the asymptotic normality as

$$\sqrt{N} \left( \tilde{\theta}^k - \theta^k \right) \xrightarrow{d} N \left( 0, \sigma^2 \left( 1 + E[(v^k)^T] (Cov[v^k])^{-1} E[v^k] \right) \right)$$

for any choices of $N$ and $m$. Particularly, if

$$n/N \rightarrow r \text{ as } n \rightarrow \infty,$$

where $r$ is a positive constant, then,

$$\sqrt{n} \left( \tilde{\theta}^k - \theta^k \right) \xrightarrow{d} N \left( 0, r\sigma^2 \left( 1 + E[(v^k)^T] (Cov[v^k])^{-1} E[v^k] \right) \right).$$

The condition (3.11) is a concrete form of (3.10). By the theorem, we can compare the BC-GE with the full data estimator that is supposed to be computed on the entire data set. This theorem and the unbiasedness in Lemma 3.1 imply that the mean square error of the BC-GE is usually larger than that of the unbiased full data estimator (e.g., the full data least squares estimator under linear model). However, if the full data estimator is biased, the improvement of the BC-GE is significant. In the following, we use the LASSO estimator as an example to illustrate this point of view. Let $\beta_k$ be the $k$-th component of $\beta_S$ as in Example 1. Then, the full data LASSO estimator $\hat{\beta}_k$ has the mean square error as

$$MSE[\hat{\beta}_k] = \lambda^2 (\text{sgn}(\beta_S))^T E[v^k] (E[v^k])^T \text{sgn}(\beta_S) + \lambda^2 (\text{sgn}(\beta_S))^T Cov[v^k] \text{sgn}(\beta_S) + \frac{\sigma^2}{n}$$

$$= \lambda^2 (\text{sgn}(\beta_S))^T E[v^k (v^k)^T] \text{sgn}(\beta_S) + \frac{\sigma^2}{n},$$

where $\lambda = O(n^{-\delta})$ for some constant $0 < \delta < 1$. The proof of (3.12) is given in Appendix. When the full data LASSO estimator has a non-ignorable bias (i.e., $0 < \delta < 1/2$), the BC-GE $\tilde{\beta}^k$ is much better than the LASSO estimator $\hat{\beta}_k$ because...
\( \sqrt{n} \tilde{\theta}_k \) has a finite MSE, while the MSE of \( \sqrt{n} \hat{\beta}_k \) tends to infinity. When \( \delta = 1/2 \) (i.e., \( \lambda = cn^{-\delta} \) for a constant \( c > 0 \)), then

\[
MSE[\sqrt{n} \hat{\beta}_k] = c^2 (\text{sgn}(\beta_S))^T E[\mathbf{v}^k(\mathbf{v}^k)^T] \text{sgn}(\beta_S) + \sigma^2.
\]

It shows that when \( \mathbf{v}_m^k(D_j), j = 1, \cdot \cdot \cdot , N \), are very dispersed, \( MSE[\sqrt{n} \hat{\beta}_k] \) is larger than \( MSE[\sqrt{n} \tilde{\theta}_k] \). In this case, the BC-GE is better than the LASSO estimator as well.

All the theoretical properties aforementioned will be illustrated by the simulation studies given in Section 5.

4 Extensions

We extend the method proposed above into the cases of nonlinear and nonparametric models.

4.1 Global bias-correction estimate in nonlinear model

Consider the following nonlinear model:

\[
Y_i = q(\theta, X_i) + \varepsilon_i, i = 1, \cdot \cdot \cdot , n,
\]

where \( q(\cdot, \cdot) \) is a given function, and the error term satisfies \( E(\varepsilon|X) = 0 \) and \( Var(\varepsilon|X) = \sigma^2 \). The parameter \( \theta \) can be estimated by least squares method. More generally, we consider the following \( M \)- and \( Z \)-estimators of \( \theta \). For the case of \( M \)-estimator, the local estimator \( \hat{\theta}_j = g_j(D_j) \) is defined as the minimizer of the following objective function:

\[
M_j(\theta) = \frac{1}{m} \sum_{i \in H_j} m(\theta, Z_i),
\]

where \( m(\theta, z) \) is a given function. A common choice of \( m(\theta, z) \) is \( (y - q(\theta, x))^2 \). For the case of \( Z \)-estimator, the local estimator \( \hat{\theta}_j = g_j(D_j) \) is defined as the solution of
the following equation:
\[
\Psi_j(\theta) = \frac{1}{m} \sum_{i \in H_j} \psi(\theta, Z_i) = 0,
\]
where the estimating function \( \psi(\theta, z) = (\psi_1(\theta, z), \cdots, \psi_p(\theta, z))^T \) is a known \( p \)-dimensional vector-valued function satisfying \( E[\psi(\theta, Z)] = 0 \). For example, \( \psi(\theta, z) \) can be chosen as the derivative of \( m(\theta, z) \) with respect to \( \theta \) if it exists. Under some regularity conditions (see, e.g., van der Vaart, 1998; Jurečková, 1985; Jurečková and Sen, 1987), we have the following asymptotic representation:
\[
\hat{\theta}_j = \theta - \frac{1}{\sqrt{m}} D^{-1}(\theta) \frac{1}{\sqrt{m}} \sum_{i \in H_j} \psi(\theta, Z_i) + O_p\left( \frac{1}{m^{1/2}} \right), \quad j = 1, \cdots, N,
\]
(4.2)
where \( D(\theta) \) is the derivative matrix of \( E[\psi(\theta, Z)] \) with respective to \( \theta \) if it exists, and \( \gamma \) is a constant satisfying \( 1/2 < \gamma \leq 1 \). It is known that \( \gamma = 1 \) if \( \psi(\theta, z) \) is twice differentiable with respect to \( \theta \), but \( \gamma = 3/4 \) if \( \psi(\theta, z) \) has jump discontinuities; see, for example, Jurečková (1985), Jurečková and Sen (1987), and He and Shao (1996).

By the two methods, the local estimator is biased usually.

By (4.2) and the same argument as used in (3.2), we get the following pro forma linear model:
\[
\hat{\theta}^k_j = \theta^k + \xi^T(\theta) \psi(D_j) + \epsilon_j, \quad j = 1, \cdots, N,
\]
(4.3)
where \( \xi^T(\theta) = -\frac{1}{\sqrt{m}} e_k^T M^{-1}(\theta) \) and \( \psi(D_j) = \frac{1}{\sqrt{m}} \sum_{i \in H_j} \psi(\hat{\theta}_j, Z_i) \). The main difference from model (3.2) is that here the error \( \epsilon_j \) is not unbiased for zero. Actually, it is an infinitesimal of higher order than \( \xi^T(\theta) \psi(D_j) \). Then, by the above model and the same argument as used in (3.8), we get the BC-GE of \( \theta^k \) as
\[
\tilde{\theta}^k = \bar{\theta}^k - \tilde{\xi}^T \overline{\psi},
\]
(4.4)
where \( \bar{\theta}^k = \sum_{j=1}^N \hat{\theta}^k_j, \overline{\psi} = \sum_{j=1}^N \psi(D_j) \) and
\[
\tilde{\xi} = \left( \sum_{j=1}^N \left( \psi(D_j) - \overline{\psi} \right) \left( \psi(D_j) - \overline{\psi} \right)^T \right)^{-1} \sum_{j=1}^N \left( \psi(D_j) - \overline{\psi} \right) \hat{\theta}^k_j.
\]
The key for a valid estimator is that the matrix \( \sum_{j=1}^{N} (\psi(D_j) - \bar{\psi}) (\psi(D_j) - \bar{\psi})^T \)
is invertible. We thus need the condition: the model is fixed design, or the data sets \( D_j, j = 1, \cdots, N \), are not identically distributed, or the data sets \( \psi(D_j), j = 1, \cdots, N \), are transformed such that the resulting data sets are not identically distributed; for more details see the related discussions in Subsection 3.1. On the other hand, because the expectation of \( \epsilon_j \) is not zero, the theoretical property of the BC-GE \( \tilde{\theta}^k \) in (4.4) is different from and more complex than those in linear model. It will be investigated in the future.

4.2 Global bias-correction estimate in nonparametric model

Consider the following nonparametric regression:

\[
Y_i = r(X_i) + \varepsilon_i, i = 1, \cdots, n,
\]

where \( r(x) \) is a smooth nonparametric regression function for \( x \in [0, 1] \), and the error term satisfies \( E(\varepsilon|X) = 0 \) and \( Var(\varepsilon|X) = \sigma_\varepsilon^2 \). Under certain regularity conditions (see, e.g., Bhattacharya and Gangopadhyay, 1990; Chaudhuri, 1991; Hong, 2003), a commonly used kernel estimator \( \hat{r}_j(x) \) (e.g., N-W estimator) computed on \( D_j \) has following Bahadur representation:

\[
\hat{r}_j(x) = r(x) + \frac{1}{\sqrt{hm}} v_h^{-1}(x) \sqrt{\frac{h}{m} \sum_{i \in H_j} K_h(X_i - x)(Y_i - r(x))} + O_p \left( \frac{1}{m^{3(1-\varsigma)/4}} \right) \tag{4.5}
\]

for \( x \in [0, 1] \) and \( j = 1, \cdots, N \), where \( K_h(x) = h^{-1}K(x/h) \), \( K(\cdot) \) is a kernel function, \( h \) is bandwidth satisfying \( h = O(m^{-\varsigma}) \) for some constant \( 0 < \varsigma < 1 \), and \( v_h(x) = E[K_h(X-x)] \). Suppose \( m = n^\tau \) for some constant \( \tau \) satisfying \( \varsigma < 1 - 2\tau/3 \). Then, the error term \( O_p \left( 1/m^{3(1-\varsigma)/4} \right) \) is an infinitesimal of higher order than the second term on the right hand side of (4.5). In this case the local estimators is always biased.

By (4.5), we get the following pro forma linear model:

\[
\hat{r}_j(x) = r(x) + \alpha(x) \phi_h(x, D_j) + \epsilon_j, j = 1, \cdots, N, \tag{4.6}
\]
where $\alpha(x) = \frac{1}{\sqrt{hm}} v_h^{-1}(x)$ and $\phi_h(x, D_j) = \sqrt{\frac{k}{m}} \sum_{i \in H_j} K_h(X_i - x)(Y_i - \hat{r}_j(x))$. Then, by the above model and the same argument as used previously, we get the BC-GE of $r(x)$ as

$$\tilde{r}(x) = \overline{r}(x) - \tilde{\alpha}_h(x) \overline{\phi}_h(x),$$

(4.7)

where $\overline{r}(x) = \sum_{j=1}^{N} \tilde{r}_j(x)$, $\overline{\phi}_h(x) = \sum_{j=1}^{N} \phi_h(x, D_j)$ and

$$\tilde{\alpha}_h(x) = \left( \sum_{j=1}^{N} \left( \phi_h(x, D_j) - \overline{\phi}_h(x) \right) (\phi_h(x, D_j) - \overline{\phi}_h(x))^T \right)^{-1} \times \sum_{j=1}^{N} \left( \phi_h(x, D_j) - \overline{\phi}_h(x) \right) \tilde{r}_j(x).$$

Because of the nonzero expectation of $\epsilon_j$ and the dependence between the estimator and the choice of $h$, the theoretical property of the BC-GE $\tilde{r}(x)$ in (4.7) is more complex than those aforementioned. It will be investigated in the future as well.

5 Simulation Studies

The goal of this section is to comprehensively evaluate the performance of the proposed method by a series of simulations. To this end, the newly proposed BR-GE for biased LASSO, Ridge and N-W estimators is compared respectively with the naive averaging estimators and DC-expression estimators from LASSO and Ridge estimators in linear model, and the naive averaging estimators and DC-expression estimators from N-W estimator in nonparametric model. Various experiment conditions such as the correlation among data, and heterogeneity or homogeneity of the distributions of data are overall considered in the procedure simulation studies. As an object of reference, the full data estimator that is computed on the entire dataset is considered as well. The mean squared error (for the parametric model) and the mean integrated squared error (for the nonparametric model) are used to measure the performance of the involved estimators. The simulation results of the estimation bias are also reported for checking the bias-correction of the new method.
5.1 Linear model with heterogeneously distributed data

Experiment 1. LASSO-based estimators. Here we investigate the performance of the BR-GE for biased LASSO estimator. The dataset with size $n = 10000$ are generated from the linear model

$$Y = X^T \beta + \varepsilon,$$

(5.1)

where $\beta = (3, 1, -1, -2, 0, 0, \cdots, 0)^T$, a 20-dimensional vector, and $\varepsilon$ follows the standard normal distribution $N(0, 1)$. In the procedure of simulation, the heterogeneously distributed data $X_i$ are generated from $N_p(\mu_i, \Sigma)$, where $\Sigma = (\sigma_{ij})_{p \times p}$ with $\sigma_{ij} = 0.5^{|i-j|}$, and $\mu_i$ are generated from $N_p(0, I)$. For the LASSO-based estimators, the penalty parameter $\lambda$ is selected by 5-fold CV cross-validation criterion. The number of batches $N$ takes the values 10, 20, 50, 100, 200 and 400, respectively.

For the linear model above, we mainly focus on the significant subset $\beta_S$ of $\beta$, i.e., $\beta_S = (\beta^{k_1}, \cdots, \beta^{k_s})^T$ with $\beta^{k_l} \neq 0$ for $l = 1, \cdots, s$. As shown in Subsection 3.2, the local estimators of $\beta_S$ may be different across different subsets $D_j$, thus, the majority voting method is employed to determine the significant subset $\beta_S$. For the details see Meinshausen and Buhlmann (2010), Shah and Samworth (2013), and Chen and Xie (2014).

Figure 1 shows the estimation bias of all the estimators considered, and Figure 2 presents the mean square error of the involved estimators. We have the following findings:

1) The newly proposed BR-GE performs comparably well with the full data estimator. Actually, the difference between the BR-GE and full data estimator is negligible, and the bias and mean square error of both estimators are nearly zero for any choices of $N$.

2) Under criteria of estimation bias and mean square error, the BR-GE is much better than the naive averaging estimator and the DC-expression estimator.
uniformly for any choices of $N$. Furthermore, the bias and mean square error of the naive averaging estimator and DC-expression estimator are increasing with the number $N$, and both estimators are almost collapsed when $N$ is large.

3) The naive averaging estimator is the worst one among the estimators considered for any choices of $N$.

(Figure 1 and Figure 2 about here)

Experiment 2. Ridge-based estimators. Here we examine the behavior of the BR-GE estimator for the Ridge estimation. For the linear regression model, the regression coefficient is chosen as $\beta = (2, 0.5, -1, -2)^T$, a 4-dimensional vector, and the covariance matrix of the covariate vector $X$ is chosen as $\Sigma = (\sigma_{ij})_{p \times p}$ with $\sigma_{ij} = 0.95^{|i-j|}$. The other experiment conditions are designed as the same as those in Experiment 1. Because this is non-sparse and low dimensional regression, and the correlation among the components of $X$ is relatively strong, we can use Ridge estimation method to estimate $\beta$.

Figure 3 and Figure 4 report the bias and mean square error of all the estimators. It can be seen that the simulation results in Figure 3 and Figure 4 are the almost same as those in Figure 1 and Figure 2 of Experiment 1. In simple terms, the BR-GE is the best one, the naive averaging estimator is the worst one among all the estimators for any choices of $N$, and particularly, when $N$ is large, the BR-GE is significantly better than the naive averaging estimator and the DC-expression estimator.

(Figure 3 and Figure 4 about here)

5.2 Linear model with identically distributed data

5.3 Nonparametric model
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Figure 1: Bias of the estimators in Experiment 1
Figure 2: Mean square error of the estimators in Experiment 1
Figure 3: Bias of the estimators in Experiment 2
Figure 4: Mean square error of the estimators in Experiment 2
Appendix: Proofs

Proof of (3.4). When \([XX^T] = I_p\), we have

\[ E[XX^T] - AE[XX^T]A = \text{diag} \left( 1, \cdots, 1, 1 - (a^r)^2, \cdots, 1 - (a^p)^2 \right). \]

Since \(b = (b^1, \cdots, b^p)^T\) is an eigenvector of the matrix \(E[XX^T] - AE[XX^T]A\), we have the following equation:

\[ \text{diag} \left( 1, \cdots, 1 - (a^r)^2, \cdots, 1 - (a^p)^2 \right) \left( b^1, \cdots, b^p \right)^T = \sum_{l=1}^{p} b_l^2 \left( b^1, \cdots, b^p \right)^T. \]

Then,

\[ b^k = b^k \sum_{l=1}^{p} b_l^2 \text{ for } k = 1, \cdots, r - 1, \text{ and } (1 - (a^k)^2) b^k = b^k \sum_{l=1}^{p} b_l^2 \text{ for } k = r, \cdots, p. \]

It shows that \(b = (b^1, \cdots, b^{r-1}, 0, \cdots, 0)^T\) with \(\sum_{k=1}^{p} b_k^2 = 1\) is a solution to the above equation. The proof is completed. □

Proof of (3.5). By model (2.1), \(E[X] = 0\) and \(U = AX + b\), we have \(E[Y] = 0\) and

\[ \eta = (E[UU^T])^{-1} E[UY] = (AE[XX^T]A + bb^T)^{-1} AE[XY]. \]

Note that \(E[X^kY] = 0, k = r, \cdots, p\) and \(A = \text{diag}(1, \cdots, 1, a^r, \cdots, a^p)\) with arbitrary constants \(a^k \neq 0, k = r, \cdots, p\). We have \(AE[XY] = E[XY]\). Thus, to prove (3.5), we only need to solve the following equation:

\[ AE[XX^T]A + bb^T = E[XX^T], \]

which can be rewritten as

\[ bb^T = E[XX^T] - AE[XX^T]A. \]

We then have the following equation:

\[ \|b\|^2 b = (E[XX^T] - AE[XX^T]A)b. \]
This implies $b$ is an eigenvector of the matrix $E[XX^T] - AE[XX^T]A$. With the choice of $b$, we have

$$\eta = (E[UU^T])^{-1}E[UY] = (E[XX^T])^{-1}E[XY] = \beta.$$ 

The proof is completed. \(\square\)

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**Proof of Lemma 3.1.** It is the direct result of the properties of expectation and variance of original least squares estimation. \(\square\)

**Proof of Lemma 3.2.** By the formula for the block matrix inversion, we have

$$\text{Var}(\tilde{\theta}^k|V_k) = (1^T 1)^{-1} + (1^T 1)^{-1}1^T V_k D^{-1} V_k^T 1 (1^T 1)^{-1},$$

where $D = V_k^T V_k - V_k^T 1 (1^T 1)^{-1} 1^T V_k$. It follows from the definition of $V_k$ that

$$V_k^T V_k = \sum_{j=1}^{N} v_m^k(D_j)(v_m^k(D_j))^T, V_k^T 1 = \sum_{j=1}^{N} v_m^k(D_j),$$

$$V_k^T 1 (1^T 1)^{-1} 1^T V_k = \frac{1}{N} \sum_{j=1}^{N} v_m^k(D_j) \sum_{j=1}^{N} (v_m^k(D_j))^T,$$

$$V_k^T V_k - V_k^T 1 (1^T 1)^{-1} 1^T V_k = \sum_{j=1}^{N} v_m^k(D_j)(v_m^k(D_j))^T - \frac{1}{N} \sum_{j=1}^{N} v_m^k(D_j) \sum_{j=1}^{N} (v_m^k(D_j))^T.$$

The results above, Lemma 3.1 and Condition C1 together imply that $(1^T 1)^{-1} = N^{-1}$, $V_k^T V_k = o_p(N)$ and $V_k^T 1 = o_p(N)$. These result in $V_k^T 1 (1^T 1)^{-1} 1^T V_k = o_p(N)$, $D = O(N^{-1}) + o_p(N) = o_p(N)$ and $V_k D^{-1} V_k = o_p(N^{-1})$. Consequently, $1^T V_k D^{-1} V_k 1 = o_p(N)$ and $(1^T 1)^{-1} 1^T V_k D^{-1} V_k^T 1 (1^T 1)^{-1} = o_p(N^{-1})$. Therefore, we have $\text{Var}(\tilde{\theta}^k|V_k) = o_p(N^{-1})$. The proof is completed. \(\square\)

**Proof of Theorem 3.3.** It is a direct result of Lemma 3.2. \(\square\)

**Proof of Theorem 3.4.** By the definition of the estimator, we have

$$(\tilde{\theta}^k, \tilde{\xi}^T)^T = (\theta^k, \xi^T)^T + (1, V_k)^T (1, V_k)^{-1} (1, V_k)^T \epsilon^k,$$
where \( \epsilon^k = (\epsilon_1^k, \ldots, \epsilon_N^k)^T \). This shows that \((\tilde{\theta}^k, \tilde{\xi}^T)^T - (\theta^k, \xi^T)^T\) has mean zero and covariance \(\sigma^2 ((1, V_k^T(1, V_k))^{-1}\), and is normally distributed, asymptotically. Thus, we only need to calculate the asymptotic variance of \(\tilde{\theta}^k\).

The proof of Lemma 3.2 and Condition C3 indicate that

\[
D = \frac{1}{N} \sum_{j=1}^{N} v_m^k(D_j)(v_m^k(D_j))^T - \frac{1}{N} \sum_{j=1}^{N} v_m^k(D_j) \frac{1}{N} \sum_{j=1}^{N} (v_m^k(D_j))^T
\]

\[
\overset{p}{\rightarrow} E[v^k(v^k)^T] - E[v^k]E[(v^k)^T].
\]

and moreover,

\[
\text{Var}(\sqrt{N} \tilde{\theta}^k|V_k) = \sigma^2 \left( 1 + \frac{1}{N} \sum_{j=1}^{N} (v_m^k(D_j))^T D^{-1} \frac{1}{N} \sum_{j=1}^{N} v_m^k(D_j) \right)
\]

\[
\overset{p}{\rightarrow} \sigma^2 \left( 1 + E[(v^k)^T] (E[v^k(v^k)^T] - E[v^k]E[(v^k)^T])^{-1} E[v^k] \right).
\]

Particularly, under condition (3.11),

\[
\text{Var}(\sqrt{n} \tilde{\theta}^k|V_k) \overset{p}{\rightarrow} r \sigma^2 \left( 1 + E[(v^k)^T] (E[v^k(v^k)^T] - E[v^k]E[(v^k)^T])^{-1} E[v^k] \right).
\]

The proof is completed. \(\Box\)

**Proof of (3.11).** Similar (2.2), the LASSO estimator of \(\beta^k\) has the following representation:

\[
\hat{\beta}^k = \beta^k - \lambda(v^k)^T \text{sgn}(\beta_S) + (v^k)^T \frac{1}{n} X_S^T \varepsilon,
\]

where \(v^k = e^T (\frac{1}{n} X_S^T X_S)^{-1} \). Thus, its bias is \(-\lambda E[v^k] \text{sgn}(\beta_S)\) and variance is

\[
\text{Var}(\hat{\beta}^k) = \lambda^2 (\text{sgn}(\beta_S))^T \text{Cov}(v^k) \text{sgn}(\beta_S) + \frac{\sigma^2}{n} E \left[ (v^k)^T \frac{1}{n} X_S^T X_S v^k \right]
\]

\[
= \lambda^2 (\text{sgn}(\beta_S))^T \text{Cov}(v^k) \text{sgn}(\beta_S) + \frac{\sigma^2}{n}.
\]

Then, the mean square error of \(\hat{\beta}^k\) is

\[
\text{MSE}[\hat{\beta}^k] = \lambda^2 (\text{sgn}(\beta_S))^T E[v^k](E[v^k])^T \text{sgn}(\beta_S) + \lambda^2 (\text{sgn}(\beta_S))^T \text{Cov}(v^k) \text{sgn}(\beta_S) + \frac{\sigma^2}{n}
\]

\[
= \lambda^2 (\text{sgn}(\beta_S))^T E[v^k(v^k)^T] \text{sgn}(\beta_S) + \frac{\sigma^2}{n}.
\]

The proof is completed. \(\Box\)