Regularized Solutions to Linear Rational Expectations Models

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Abstract

This paper proposes a computational method for obtaining regularized solutions to linear rational expectations models. The algorithm allows for regularization cross-sectionally as well as across frequencies. The algorithm is illustrated by a variety of examples.

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1 Introduction

The linear rational expectations model (LREM) occupies a fundamental place in theoretical and empirical macroeconomics. The model, 

\[ M_{-1}E_t X_{t+1} + M_0 X_t + M_1 X_{t-1} = \varepsilon_t, \quad t \in \mathbb{Z}, \]  

allows the present economic state \( X_t \) to depend not just on the past states \( X_{t-1} \) but also on expected states \( E_t X_{t+1} \) as well as exogenous economic forces \( \varepsilon_t \). It is therefore well suited for analysing the behaviour of a wide array of economic entities such as households, firms, and policy makers.

Recently, Al-Sadoon (2020) showed that solutions to (1) are not generally continuous with respect to the parameters, \( M_{-1}, M_0, \) and \( M_1 \), invalidating crucial assumptions for both frequentis and Bayesian estimation and inference methods. However, Al-Sadoon (2020) also showed that the solution that minimizes \( E ||X_0||^2 \) is unique, continuous, and even differentiable with respect to the parameters of (1) under certain regularity conditions. Theorem 5 of Al-Sadoon (2020) provides the following algorithm for computing the regularized solution if a solution to (1) exists. First, solve the auxiliary model

\[
M_{-1} M'_1 E_t Y_{t+2} + (M_{-1} M'_0 + M'_0 M'_1) E_t Y_{t+1} + M_1 M'_1 E_{t-1} Y_t \\
+ (M_{-1} M'_{-1} + M_0 M'_0) Y_t + (M_0 M'_{-1} + M_1 M'_0) Y_{t-1} + M_1 M'_{-1} Y_{t-2} = \varepsilon_t, \quad t \in \mathbb{Z}.
\]

The solution to this system exists and is unique. Second, compute

\[
X_{t}^{\text{reg}} = M'_1 E_t Y_{t+1} + M'_0 Y_t + M'_{-1} Y_{t-1}, \quad t \in \mathbb{Z}.
\]

This two-step algorithm computes the regularized solution when the elements of \( X \) are weighted equally; it is essentially a functional form of least squares regression. However, we may like to shrink the process by different amounts cross-sectionally as well as across frequencies. Thus, in this paper we opt for a simpler, yet more general, algorithm using the Sims (2002) framework.

This work is related to several more recent works. The main result of this paper builds on Lubik & Schorfheide (2003) and Al-Sadoon (2018). Farmer et al. (2015) and Bianchi & Nicolò (2019) provide alternative parametrizations of solutions to LREMs to Lubik & Schorfheide (2003). Funovits (2017) counts the dimension of the solution space to a given LREM. This paper can also be seen as part of the recent interest in frequency domain analysis of LREMs.
as exemplified by Onatski (2006), Tan & Walker (2015), and Tan (2019). Such methods have found important applications in addressing the identification problem for LREM as seen in Komunjer & Ng (2011), Qu & Tkachenko (2017), Kociecki & Kolasa (2018), and Al-Sadoon & Zwiernik (2019).

This paper is organized as follows. Section 2 reviews results of Sims (2002) and Lubik & Schorfheide (2003). Section 3 shows how regularization can be achieved and provides the main result of this paper. Section 4 provides illustrative examples of how regularization works. Section 5 concludes. The Matlab code for reproducing the computations presented in this paper can be found in the accompanying file, regular.zip.

2 Review

We begin by reviewing results developed by Sims (2002) and Lubik & Schorfheide (2003). We will adhere to their notation for ease of exposition. Because regularization is only defined in a stationary context, we will restrict attention to covariance stationary solutions.

Definition 1. Given \((\Gamma_0, \Gamma_1, \Psi, \Pi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times l} \times \mathbb{R}^{n \times k}\), an \(l\)-dimensional i.i.d. process \(z\) of mean zero and finite and positive definite variance matrix, and the formal LREM

\[
\Gamma_0 y(t) = \Gamma_1 y(t-1) + \Psi z(t) + \Pi \eta(t), \quad t \in \mathbb{Z},
\]

a solution to (2) is a pair \((y, \eta)\) such that:

(i) \(y\) is an \(n\)-dimensional process such that \(y(t)\) is measurable with respect to \(z(t), z(t-1), \ldots\) for all \(t \in \mathbb{Z}\).

(ii) \(\eta\) is a \(k\)-dimensional martingale difference sequence with respect to \(z\). That is, \(\eta(t)\) is measurable with respect to \(z(t), z(t-1), \ldots\) and \(E_t \eta(t+1) = 0\) almost surely for all \(t \in \mathbb{Z}\), where \(E_t(\cdot) = E(\cdot | z(t), z(t-1), \ldots)\).

(iii) The process \((z, y, \eta)\) is jointly covariance stationary.

(iv) The pair satisfies equation (2).

A solution \((y, \eta)\) is unique if for every other solution \((\tilde{y}, \tilde{\eta})\), \(y(t) = \tilde{y}(t)\) almost surely for all \(t \in \mathbb{Z}\). (For ease of exposition, we will drop the “almost surely” in the subsequent analysis).
Assuming, as Sims (2002) does, that $\det(\Gamma_0 + \Gamma_1 x)$ is not identically zero (i.e. it is impossible to cancel out any element of $y$ by elementary algebraic operations), then by Theorem VI.1.9 and Exercise VI.1.3 of Stewart & Sun (1990), there are orthogonal matrices $Q, Z \in \mathbb{R}^{n \times n}$ such that $Q\Gamma_0 Z$ and $Q\Gamma_1 Z$ are block upper triangular with either $1 \times 1$ or $2 \times 2$ blocks on the diagonal. Under the stronger assumption that $\det(\Gamma_0 + \Gamma_1 x) \neq 0$ for all $x \in \mathbb{C}$ with $|x| = 1$ (i.e. the aforementioned cancellation is impossible and there are no unit roots in the system), then these matrices can be partitioned conformably as

\[
Q\Gamma_0 Z = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix}, \quad Q\Gamma_1 Z = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \tag{3}
\]

where the polynomial $\det(\Lambda_{11} + \Omega_{11} x)$ has all its zeros outside the unit circle (this implies that $\Lambda_{11}$ is non-singular), and the polynomial $\det(\Lambda_{22} + \Omega_{22} x)$ has all its zeros inside the unit circle (this implies that $\Omega_{22}$ is non-singular). Note that Sims (2002) and Lubik & Schorfheide (2003) use the complex QZ decomposition but never explain how the final answer is real; using the real QZ decomposition obviates any need for such a discussion. As shown in the online appendix to Al-Sadoon (2018), this step is an implicit Wiener-Hopf factorization.

Now suppose $(y, \eta)$ is a solution to (2), define $w(t) = Z'y(t)$, and rewrite the system as

\[
\Lambda w(t) = \Omega w(t - 1) + Q\Psi z(t) + Q\Pi \eta(t), \quad t \in \mathbb{Z},
\]

If we partition

\[
w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}
\]

conformably with (3), then

\[
\Lambda_{22} w_2(t) = \Omega_{22} w_2(t - 1) + Q_2 \Psi z(t) + Q_2 \Pi \eta(t), \quad t \in \mathbb{Z},
\]

where

\[
Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\]

is partitioned conformably with (3). Applying the conditional expectation $E_{t-1}$ we obtain

\[
w_2(t - 1) = \Omega_{22}^{-1} \Lambda_{22} E_{t-1} w_2(t), \quad t \in \mathbb{Z},
\]
This implies that
\[ w_2(t) = (\Omega_{22}^{-1} \Lambda_{22})^{s-t} E_t w_2(s), \quad s \geq t. \]

Therefore,
\[ E\|w_2(t)\|^2 \leq \|(\Omega_{22}^{-1} \Lambda_{22})^{s-t}\|^2 E\|E_t w_2(s)\|^2 \]
\[ \leq \|(\Omega_{22}^{-1} \Lambda_{22})^{s-t}\|^2 E(\|E_t w_2(s)\|^2) \]
\[ = \|(\Omega_{22}^{-1} \Lambda_{22})^{s-t}\|^2 E\|w_2(s)\|^2, \quad s \geq t. \]

where the second inequality follows from Theorem 9.7 of Williams (1991). The covariance stationarity of \( y \) implies that \( E\|w_2(t)\|^2 = E\|w_2(s)\|^2 \). Since our choice of QZ decomposition ensures that the eigenvalues of \( \Omega_{22}^{-1} \Lambda_{22} \) are inside the unit circle, \( \|(\Omega_{22}^{-1} \Lambda_{22})^{s-t}\| < 1 \) for large enough \( s - t \) and then it must be the case that \( E\|w_2(t)\|^2 = E\|w_2(s)\|^2 = 0 \). Therefore,
\[ w_2(t) = 0, \quad t \in \mathbb{Z}. \]

Now plugging this back into (4) we have that
\[ Q_2 \Psi z(t) + Q_2 \Pi \eta(t) = 0, \quad t \in \mathbb{Z}. \]

Multiplying on the right by \( z'(t) \), taking expectations, and utilizing the joint covariance stationarity of \( \eta \) and \( z \), we arrive at
\[ Q_2 \Psi E(z(0)z'(0)) + Q_2 \Pi E(\eta(0)z'(0)) = 0. \]

But since \( E(z(0)z'(0)) \) is invertible by assumption, a necessary condition for existence is
\[ \text{im}(Q_2 \Psi) \subseteq \text{im}(Q_2 \Pi). \quad (5) \]

It also follows that
\[ (Q_2 \Pi)^\dagger Q_2 \Psi z(t) + \eta(t) \in \ker(Q_2 \Pi), \quad t \in \mathbb{Z}, \]
where \((Q_2 \Pi)^\dagger\) is the Moore-Penrose generalized inverse of \( Q_2 \Pi \), and
\[ E_{t-1} \left( (Q_2 \Pi)^\dagger Q_2 \Psi z(t) + \eta(t) \right) = 0, \quad t \in \mathbb{Z}. \]

Thus, for a given matrix \( K \) whose columns form a basis for \( \ker(Q_2 \Pi) \) there is a martingale difference sequence with respect to \( z \), denoted by \( \nu \), such that
\[ K \nu(t) = (Q_2 \Pi)^\dagger Q_2 \Psi z(t) + \eta(t), \quad t \in \mathbb{Z}. \]
Every solution is therefore representable as
\[
\eta(t) = -(Q_2 \Pi)^\dagger Q_2 \Psi z(t) + K \nu(t)
\]
\[
w_1(t) = \sum_{s=0}^{\infty} (\Lambda_{11}^{-1} \Omega_{11})^s \Lambda_{11}^{-1} \left\{ \left( Q_1 \Psi - Q_1 \Pi (Q_2 \Pi)^\dagger Q_2 \Psi \right) z(t-s) + Q_1 \Pi K \nu(t-s) \right\}
\]
\[
w_2(t) = 0
\]
\[
y(t) = Z w(t),
\]
where \( \nu \) is a martingale difference sequence with respect to \( z \). Another representation is
\[
y(t) = \Theta_1 y(t-1) + \Theta_z z(t) + \Theta_\nu \nu(t), \quad t \in \mathbb{Z},
\]
with
\[
\Theta_1 = Z \begin{bmatrix} \Lambda_{11}^{-1} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} Z',
\]
\[
\Theta_z = Z \begin{bmatrix} \Lambda_{11}^{-1} (Q_2 \Psi - Q_2 \Pi (Q_2 \Pi)^\dagger Q_2 \Psi) & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
\Theta_\nu = Z \begin{bmatrix} \Lambda_{11}^{-1} Q_1 \Pi \\ 0 \end{bmatrix} K.
\]
Note that \( \nu \) is a martingale difference sequence with respect to \( z \). Another representation is
\[
y(t) = \Theta_1 y(t-1) + \Theta_z z(t) + \Theta_\nu \nu(t), \quad t \in \mathbb{Z},
\]
with
\[
\Theta_1 = Z \begin{bmatrix} \Lambda_{11}^{-1} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} Z',
\]
\[
\Theta_z = Z \begin{bmatrix} \Lambda_{11}^{-1} (Q_2 \Psi - Q_2 \Pi (Q_2 \Pi)^\dagger Q_2 \Psi) & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
\Theta_\nu = Z \begin{bmatrix} \Lambda_{11}^{-1} Q_1 \Pi \\ 0 \end{bmatrix} K.
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y(t) = \Theta_1 y(t-1) + \Theta_z z(t) + \Theta_\nu \nu(t), \quad t \in \mathbb{Z},
\]
with
\[
\Theta_1 = Z \begin{bmatrix} \Lambda_{11}^{-1} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} Z',
\]
\[
\Theta_z = Z \begin{bmatrix} \Lambda_{11}^{-1} (Q_2 \Psi - Q_2 \Pi (Q_2 \Pi)^\dagger Q_2 \Psi) & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
\Theta_\nu = Z \begin{bmatrix} \Lambda_{11}^{-1} Q_1 \Pi \\ 0 \end{bmatrix} K.
\]
Thus \( u \) is a covariance stationary process satisfying
\[
\Lambda u(t) = \Omega u(t - 1) + Q\Pi\psi(t), \quad t \in \mathbb{Z}.
\] (8)

Partitioning
\[
u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
\]
conformably with (3) and inspecting the second block of (8) first, we see immediately that
\[u_2(t) = 0, \quad t \in \mathbb{Z}.
\]

Plugging this back into the second block, we find that
\[Q_2\Pi\psi(t) = 0, \quad t \in \mathbb{Z}.
\]

It follows, as before, that for any matrix \( K \) whose columns are a basis for \( \text{ker}(Q_2\Pi) \), there is a martingale difference with respect to \( z \), call it \( \nu \), such that
\[
\psi(t) = K\nu(t), \quad t \in \mathbb{Z}.
\]

We therefore arrive at
\[
u(t) = \Lambda^{-1}_{11}\Omega_{11}u_1(t - 1) + \Lambda^{-1}_{11}Q_1\Pi K\nu(t), \quad t \in \mathbb{Z}.
\]

Now there are two cases to consider. If (7) holds, then the term containing \( \nu \) vanishes and the only covariance stationary process satisfying this equation is \( u_1 = 0 \). Thus, if (7) holds the solution is unique. On the other hand, if (7) does not hold, there exists a real vector \( v \neq 0 \) such that \( Q_1\Pi K v \neq 0 \). Now choose \( \nu(t) = vz_1(t) \) and construct \( u \) according to the equations above. Notice that \((z,u)\) is jointly covariance stationary and every element of \( u(t) \) is measurable with respect to \( z(t), z(t-1), \ldots \) for all \( t \in \mathbb{Z} \). Moreover, \( E\|u_1(t)\|^2 \neq 0 \) for all \( t \in \mathbb{Z} \). Then for any given solution \((y,\eta)\), there exists another solution \((\tilde{y},\tilde{\eta})\) with \( \tilde{y} = y + Zu \) and \( \tilde{\eta} = \eta + Kvz_1 \) with \( y(t) \neq \tilde{y}(t) \) for all \( t \in \mathbb{Z} \). That is, if (7) does not hold, uniqueness fails.

To summarize, we have proven the following.

**Theorem 1 (Sims (2002)).** If \( \det(\Gamma_0 + \Gamma_1x) \neq 0 \) for all \( x \in \mathbb{C} \) with \( |x| = 1 \), a solution to (2) exists if and only if (5) holds. A solution is unique if and only if (7) holds.
3 Regularization

Now that we have the general form of solutions (6), we can begin to discuss regularized solutions to (2).

We begin by noting that $\nu$ is a martingale difference sequence with respect to $z$ if and only if there exists another such martingale difference sequence, $\zeta$, such that

$$\nu(t) = B z(t) + \zeta(t), \quad E(z(t)\zeta(t)) = 0, \quad t \in \mathbb{Z}.$$ 

The process $\zeta$ is the residual from regressing $\nu(t)$ on $z(t)$.

Suppose a symmetric positive semi-definite matrix $W \in \mathbb{R}^{n \times n}$ is given and we are interested in selecting among all solutions to (2), a solution that minimizes

$$E\|W^{1/2}y(0)\|^2 = E(y'(0)W y(0)) = \text{tr} \left( W E(y(0)y'(0)) \right).$$

Since

$$E(y(0)y'(0)) = \sum_{j=0}^{\infty} \Theta_j^\prime \left( \Theta z \Sigma zz \Theta' + \Theta z \Sigma zz B' \Theta' + \Theta \nu B \Sigma zz \Theta' + \Theta \nu \nu B \Sigma zz B' \Theta' + \Theta \nu \nu CC' \Theta' \right) \Theta_j,'$$

where $CC' = E(\zeta(0)\zeta(0))$, finding a regularized solution is equivalent to minimizing

$$\mathcal{L} = \frac{1}{2} \text{tr} \left( W \sum_{j=0}^{\infty} \Theta_j^\prime \left( \Theta z \Sigma zz \Theta' + \Theta z \Sigma zz B' \Theta' + \Theta \nu B \Sigma zz \Theta' + \Theta \nu \nu B \Sigma zz B' \Theta' + \Theta \nu \nu CC' \Theta' \right) \Theta_j,' \right)$$

with respect to $B$ and $C$. Using the properties of the trace of a product of matrices,

$$\mathcal{L} = \frac{1}{2} \text{tr} \left( \left( \Theta z \Sigma zz \Theta' + \Theta z \Sigma zz B' \Theta' + \Theta \nu B \Sigma zz \Theta' + \Theta \nu \nu B \Sigma zz B' \Theta' + \Theta \nu \nu CC' \Theta' \right) \Xi \right),$$

where

$$\Xi = \sum_{j=0}^{\infty} \Theta_j^\prime W \Theta_j.'$$

Note that $\Xi$ is the unique solution to the Lyapunov equation

$$\Xi = \Theta_1^\prime \Xi \Theta_1 + W.$$

See Section B.1.8 of Lindquist & Picci (2015). Taking the gradient of $\mathcal{L}$, we obtain the following first order conditions for $B^*$ and $C^*$

$$\Theta'_\nu \Xi (\Theta_z + \Theta_\nu B^*) = 0$$

$$\Theta'_\nu \Xi \Theta_\nu C^* = 0.$$
If $\Theta' \Xi \Theta$ is invertible, there exists a unique solution to the first order conditions given by

$$B^* = -(\Theta'_v \Xi \Theta_v)^{-1} \Theta'_v \Xi z, \quad C^* = 0.$$  

If $\Theta'_v \Xi \Theta_v$ is not invertible, then the set of all solutions to the first order conditions is given by

$$B^* = -(\Theta'_v \Xi \Theta_v)^{\dagger} \Theta'_v \Xi z + X, \quad C^* = Y,$$

for arbitrary $X$ and $Y$ of the appropriate sizes such that $\text{im}(X), \text{im}(Y) \subseteq \ker(\Theta'_v \Xi \Theta_v)$. We shall have no use for these expressions as the whole point of regularization is to eliminate indeterminacy.

Thus, we have proven that a regularized solution to (2) exists if and only if solutions to (2) exist and the regularized solution is unique if and only if $\Theta'_v \Xi \Theta_v$ is invertible, in which case it has the representation

$$y_{\text{reg}}(t) = \Theta_1 y(t - 1) + \Theta_{\text{reg}} z(t), \quad t \in \mathbb{Z}, \quad (9)$$

where

$$\Theta_{\text{reg}} = (I - \Theta_v \Xi \Theta_v)^{-1} \Theta'_v \Xi \Theta_v.$$  

The intuition of this result is quite simple. Write

$$\Theta'_v \Xi \Theta_v = \begin{bmatrix} \Theta'_v W^{1/2} & \Theta'_v \Theta'_1 W^{1/2} & \Theta'_v \Theta'_2 W^{1/2} & \cdots \\ W^{1/2} \Theta_1 & \Theta_1 W^{1/2} & \Theta_1 W^{1/2} & \cdots \\ W^{1/2} \Theta_2 & \Theta_2 W^{1/2} & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$  

Now if $W^{1/2} \Theta_v$ is of full column rank, then the regularized solution is unique. That is, if $W$ attaches non-trivial weight to every contemporaneous instances of indeterminacy, then regularization eliminates indeterminacy. More generally, we have proven that regularization eliminates indeterminacy if and only if $W$ attaches non-trivial weight to every instance of indeterminacy whether contemporaneous or lagged. From a linear systems point of view, regularization leads to uniqueness if and only if the triple $(\Theta_1, \Theta_v, W^{1/2})$ is input observable (Sain & Massey, 1969), which is to say, again, that the weight matrix detects all of the indeterminacy in the system.
The above analysis, suggests a further generalization. We have constructed an algorithm for minimizing

\[ E\|W^{1/2}y(0)\|^2 = \text{tr} \left( \frac{1}{2\pi} \int W f(\omega)d\omega \right), \]

where \( f \) is the spectral density of \( y \). The above expression allows us to choose different weights along the cross-section of \( y \). More generally, we may consider choosing weights on frequencies of oscillation of \( y \). In particular, we may consider minimizing

\[ \mathcal{L} = \frac{1}{2} \text{tr} \left( \frac{1}{2\pi} \int W(\omega)f(\omega)d\omega \right), \]

where \( W \) is a bounded measurable function, with \( W(\omega) \) Hermitian positive semi definite and \( W(\omega)^* = W(-\omega)' \) for all \( \omega \in (-\pi, \pi] \). If, for example, we like to impose that the solution should display the frequency characteristics of the business cycle, we could choose

\[ W(\omega) = \begin{cases} 
0, & 2\pi/32 \leq |\omega| \leq 2\pi/4, \\
I, & \text{otherwise},
\end{cases} \]

which penalizes oscillations of period smaller than a year and greater than eight years in quarterly data.

To that end, we first note that

\[ f(\omega) = (I - \Theta_1 e^{-i\omega})^{-1} (\Theta_z \Sigma z z \Theta'_z + \Theta_z \Sigma z z B'\Theta'_\nu + \Theta_\nu B \Sigma z z \Theta'_z \\
+ \Theta_\nu B \Sigma z z B'\Theta'_\nu + \Theta_\nu CC'\Theta'_\nu) (I - \Theta'_1 e^{i\omega})^{-1}. \]

This implies that

\[ \mathcal{L} = \frac{1}{2} \text{tr} \left( (\Theta_z \Sigma z z \Theta'_z + \Theta_z \Sigma z z B'\Theta'_\nu + \Theta_\nu B \Sigma z z \Theta'_z + \Theta_\nu B \Sigma z z B'\Theta'_\nu + \Theta_\nu CC'\Theta'_\nu) \Xi \right), \]

where

\[ \Xi = \frac{1}{2\pi} \int (I - \Theta'_1 e^{i\omega})^{-1} W(\omega)(I - \Theta_1 e^{-i\omega})^{-1} d\omega. \]

It is easily checked that \( \Xi \) is a real symmetric positive semi definite matrix and that it reduces to our previous expression when \( W(\omega) \) is constant. Following the same line of argument as above, we arrive finally at the main result of the paper.

**Theorem 2.** A solution to (2) that minimizes (10) exists if and only a solution to (2) exists. The regularized solution (9) is unique if and only if \( \Theta'_z \Xi \Theta_\nu \) is invertible.
4 Examples

Next we illustrate the methodology with simple examples. The computations can be found in the Matlab code accompanying this paper.

4.1 The Cagan Model

Consider first, the Cagan model with mean zero independent and identically distributed shocks

\[ X_t = 2E_t X_{t+1} + \varepsilon_t, \quad t \in \mathbb{Z}. \]

There are infinitely many solutions to this system. To compute the regularized solution minimizing \( EX_t^2 \), we reformulate this model as

\[
\begin{align*}
y(t) &= \begin{bmatrix} X_t \\ E_t X_{t+1} \end{bmatrix}, \quad z(t) = \varepsilon_t, \quad \eta(t) = X_t - E_{t-1} X_t, \quad t \in \mathbb{Z}
\end{align*}
\]

with

\[
\Gamma_0 = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

The weight matrix for this problem is

\[
W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

This implies that

\[
\Theta_1 = \begin{bmatrix} 0 & 1.0000 \\ 0 & 0.5000 \end{bmatrix}, \quad \Theta_{\text{reg}} = \begin{bmatrix} 0.2500 \\ -0.3750 \end{bmatrix}.
\]

This is indeed the correct answer as solving for the first element we obtain \( 0.5 \left( \frac{0.5 - t}{1 - 0.5t} \right) \varepsilon_t \), which was obtained analytically in Al-Sadoon (2020). This regularized solution is actually a white noise process and therefore has a flat spectral density. We may instead impose that the solution avoid empirically unlikely frequencies. If we use the weight matrix

\[
W(\omega) = \begin{cases} 
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & 2\pi/32 \leq |\omega| \leq 2\pi/4, \\
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \text{otherwise}, 
\end{cases}
\]

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we obtain a different regularized solution with the spectral density plotted in the Figure 1.

Figure 1: Regularized Solutions to the Cagan Model.

4.2 A Non-generic System

Consider now the system

\[
E_t X_{1t+2} = \varepsilon_{1t}, \quad t \in \mathbb{Z},
\]
\[
\theta E_t X_{1t+1} + X_{2t} = \varepsilon_{2t},
\]

The shocks are again zero mean independent and identically distributed. This system also has infinitely many solutions. Its solutions can exhibit quite nasty discontinuities as demonstrated in Al-Sadoon (2020).

In order to modify the system to make it more suitable for solving via our algorithm, suppose we use the second equation to obtain

\[
\theta E_t X_{1t+2} + E_t X_{2t+1} = 0, \quad t \in \mathbb{Z},
\]

and then combine this equation with the first equation of the original system to obtain

\[
E_t X_{2t+1} = -\theta \varepsilon_{1t}, \quad t \in \mathbb{Z},
\]
\[
\theta E_t X_{1t+1} + X_{2t} = \varepsilon_{2t},
\]
This system is equivalent to the original one, provided $\theta \neq 0$. We can now set

$$y(t) = \begin{bmatrix} X_{1t} \\ X_{2t} \\ E_t X_{1t+1} \\ E_t X_{2t+1} \end{bmatrix}, \quad z(t) = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \quad \eta(t) = \begin{bmatrix} X_{1t} - E_{t-1}X_{1t} \\ X_{2t} - E_{t-1}X_{2t} \end{bmatrix}, \quad t \in \mathbb{Z}$$

with

$$\Gamma_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & \theta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} -\theta & 0 \\ 0 & 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The weight matrix is

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $\theta = 10^{-6}$, the first three impulse responses of the non-regularized solution are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 10^6 \\ -10^{-6} & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, these are quite far from the impulse responses of the $\theta = 0$ model, which ought to be

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, the first three impulse responses of the regularized solution are

$$\begin{bmatrix} 0 & -0.0002 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 10^{-6} \\ -10^{-6} & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The continuity of regularized solutions is proven in Theorem 6 of Al-Sadoon (2020).

### 4.3 A New Keynesian Model

Consider the New Keynesian model of Lubik & Schorfheide (2004).
\[ \Gamma_0 = \begin{bmatrix} -1 & -\tau & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 \\ -(1 - \rho_R)\psi_2 & -(1 - \rho_R)\psi_1 & 1 & 0 & (1 - \rho_R)\psi_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} -1 & 0 & -\tau & 1 & 0 \\ \kappa & -1 & 0 & 0 & -\kappa \\ 0 & 0 & \rho_R & 0 & 0 \\ 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & \rho_z \end{bmatrix}, \]

\[ \Psi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -1 & 0 \\ 0 & -\beta \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

The model is calibrated using Lubik & Schorfheide’s estimates reported in their Table 3 in the column titled “Pre-Volcker (Prior 1)”. Figure 2 plots the impulse responses of the first three variables to the three shocks. The impulse responses are generated from the non-regularized solution, the solution regularized with constant weight matrix with equal weights on the first three variables, and the solution regularized with a variable weight matrix emphasizing business cycle frequencies in the first three variables.

Clearly, regularization produces more stable dynamics. Unlike the case in Figure 1, however, regularizing by constant and variable weight matrices did not produce dramatically different results.

5 Conclusion

This paper has provided necessary and sufficient conditions for existence and uniqueness of regularized solution along with an algorithm for computing regularized solutions to LREMs. This work suggests at least three venues for further investigation.

Al-Sadoon & Zwiernik (2019) studied the identification problem for uniquely solvable models of the form (1). It is not clear how their results might extend to the larger parameter space where regularized solutions exist. Resolving this issue is a necessary first step before taking regularized solutions to the data.

Various numerical methods for obtaining Wiener-Hopf factorizations exist in the mathe-
Figure 2: Regularized Solutions to the New Keynesian Model.

The results presented in this work are applicable to the setting where the information set includes all exogenous variables. Recent work has sought to relax this assumption (e.g. Huo & Takayama (2015), Rondina & Walker (2017), Angeletos & Huo (2018), and Han et al. (2019)). This would certainly be a very fruitful venue for follow up work.

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