Classes of Exactly Solvable Generalized Semi-Classical Rabi Systems

Roberto Grimaudo,* Antonio Sérgio Magalhães de Castro, Hiromichi Nakazato, and Antonino Messina

The exact quantum dynamics of a single spin-1/2 in a generic time-dependent classical magnetic field are investigated and compared with the quantum motion of a spin-1/2 studied by Rabi and Schwinger. The possibility regarding the scenario studied here as a generalization of that considered by Rabi and Schwinger is discussed and the notion of a time-dependent resonance condition is introduced and carefully legitimated and analyzed. Several examples help to disclose analogies and departures of the quantum motion induced in a generalized Rabi system with respect to that exhibited by the spin-1/2 in a magnetic field precessing around the z-axis. It is found that, under a generalized resonance condition, the time evolution of the transition probability $P_{\uparrow\downarrow}(t)$ between the two eigenstates of $\hat{S}^z$ may be dominated by a regime of distorted oscillations, or may even exhibit a monotonic behavior. At the same time, the authors succeed in predicting asymptotic behaviors with no oscillations in the time-dependence of $P_{\uparrow\downarrow}(t)$ under general conditions. New scenarios of experimental interest originating a Landau–Zener transition are brought to light.

1. Introduction

Spin language naturally arises in every physical scenario wherein, by definition, the states of the specific system under scrutiny live in a finite-dimensional Hilbert space. When the generally time-dependent Hamiltonian of the system may in particular be expressed as linear combination of the three generators of the SU(2) group, the corresponding quantum dynamical problem coincides with that of a spin $j$ in a time-dependent magnetic field. It is well known that the time evolution of the spin $j$ in this case is fully recoverable from that of a spin $j = 1/2$ subjected to the same magnetic field.[12] On the other hand, however, a closed general form of the $2 \times 2$ unitary evolution operator is still unavailable. As a consequence, to bring to light new dynamical scenarios of a spin-1/2 represents a target of basic and applicative importance in many contexts such as quantum optics,[13] quantum control,[14] quantum information, and quantum computing.[5,6]

Rabi[7,8] and Schwinger[9] exactly solved the quantum dynamics of a spin-1/2 in the now called Rabi scenario, that is, subjected to a static magnetic field $B_0$ along the z-axis and an r.f. magnetic field rotating in the $x$–$y$ plane with frequency $\omega_{xy}$, namely

$$B_z(t) = B_0 \left[ \cos(\omega_{xy} t) c_1 - \sin(\omega_{xy} t) c_2 \right] + B_0 c_3$$

$c_1$, $c_2$, and $c_3$ being fixed unit vectors in the laboratory frame. Their seminal papers show that the probability of transition between the two Zeeman states generated by $B_z = B_0 c_3$ is dominated by periodic oscillations reaching maximum amplitude under the so-called resonance condition $\Delta = \omega_{xy} - \omega_1 = 0$. Here, $\omega_1$ is the spin Larmor frequency. The exact treatment of this basic problem provides the robust platform for the NMR technology implementation.[18] The recently published special issue on semiclassical and quantum Rabi models[11] witnesses the evergreen attractiveness of this problem.

Searching new exactly solvable time-dependent scenarios of a single spin-1/2 could be very interesting and worth both from basic and applicative points of view, especially in the quantum control context. To pursue this target, over the last years, new methods have been developed to face the problem with an original strategy.[12–16] We stress that progresses along this direction might be of relevance even for treating time-dependent Hamiltonians describing interacting qubits or qudits.[17–19] In turn, exact treatments of such scenarios might stimulate the
interpretation of experimental results in fields from condensed matter physics\textsuperscript{[20,21]} to quantum information and quantum computing\textsuperscript{[22-25]}

Here, we construct the exact time evolution of a spin-1/2 subjected to time-dependent magnetic fields. Our investigation introduces, in a very natural way, three different classes of Generalized Rabi Systems (GRSs) wherein Rabi oscillations of maximum amplitude still survive. We succeed indeed in identifying generalized resonance conditions which are, as in the Rabi scenario, at the origin of the complete population transfer between the two Zeeman levels of the spin. We bring to light that, even at resonance, these oscillations might lose its periodic character, significantly differing, thus, from the sinusoidal behavior occurring in the Rabi scenario. Here, we have also considered time-dependent magnetic fields giving rise to exactly solvable models not satisfying the resonance condition. In this way, we are able to write down a special link among all the time-dependent parameters appearing in the Hamiltonian of the system even if the resonance condition is not met, for which, however, the dynamical problem is exactly solvable. The departure of the time evolution of the transition probability out of generalized resonance from the corresponding behavior in the Rabi scenario is illustrated with the help of two exemplary cases.

In Section 2, two sets of parametrized solutions of the dynamical problem of a single spin-1/2 in a time-dependent magnetic field are given. The generalized time-dependent resonance condition and the generalized out of resonance case are introduced and physically legitimated in Section 3. In the subsequent section (Section 4), several examples illustrate the occurrence of effects on the Rabi transition probability due to the magnetic field time-dependence. Finally, in Section 5, the results with possible applications and future outlooks are briefly discussed.

2. Resolution of $2 \times 2$ $\text{su}(2)$ Quantum Dynamical Problems

The problem of a single spin-1/2 subjected to a generic time-dependent magnetic field $B(t) = [B_x(t), B_y(t), B_z(t)]$ is investigated by assuming the $\text{su}(2)$ Hamiltonian model

$$H(t) = \begin{pmatrix} \Omega(t) & \omega(t) \\ -\omega(t) & -\Omega(t) \end{pmatrix}$$

with

$$\Omega(t) = \frac{\hbar \mu_B g}{2} B_z(t)$$

$$\omega(t) = \frac{\hbar \mu_B g}{2} [B_x(t) - i B_y(t)] \equiv \omega_x - i \omega_y \equiv |\omega(t)| e^{i \phi(t)}$$

Here, $\mu_B g$ is the magnetic moment associated with the spin-1/2, $g$ and $\mu_B$ being the appropriate Landé factor and the Bohr magneton, respectively. The entries $a(t) \equiv |a(t)| \exp(i \phi(t))$ and $b(t) \equiv |b(t)| \exp(i \phi(t))$ of the unitary time evolution operator

$$U(t) = \begin{pmatrix} a(t) & b(t) \\ -\overline{b^*}(t) & \overline{a^*}(t) \end{pmatrix}$$

generated by $H(t)$, must satisfy the Cauchy–Liouville problem

$$i \hbar U(t) = H(t) U(t), \quad U(0) = \mathbb{I},$$

which originates the following system of linear differential equations

$$\begin{cases} \dot{a}(t) = \frac{\Omega}{\hbar} a(t) - \frac{\omega}{\hbar} b^*(t) \\ \dot{b}(t) = \frac{\omega}{\hbar} a^*(t) + \frac{\Omega}{\hbar} b(t) \\ a(0) = 1, \quad b(0) = 0 \end{cases}$$

(6)

It is possible to demonstrate\textsuperscript{[15]} that if $\Theta(t)$ is a complex-valued $C^1$ (continuously differentiable) function of $t$ satisfying the nonlinear integral-differential Cauchy problem

$$\frac{1}{2} \dot{\Theta}(t) + \frac{|\omega(t)|}{\hbar} \sin \Theta(t) \cot \left[ \frac{2}{\hbar} \int_0^t |\omega(t')| \cos \Theta(t') dt' \right]$$

$$= \frac{\Omega(t)}{\hbar} + \frac{\phi(t)}{2}$$

(7)

$$\Theta(0) = 0$$

(8)

then the solutions of the Cauchy problem (6) can be represented as follows

$$a(t) = \cos \left[ \frac{1}{\hbar} \int_0^t |\omega(t')| \cos \Theta(t') dt' \right] \times \exp \left\{ i \left( \frac{\phi(t) - \phi(0)}{2} - \frac{\Theta(t)}{2} - \mathcal{R}(t) \right) \right\}$$

(9)

$$b(t) = \sin \left[ \frac{1}{\hbar} \int_0^t |\omega(t')| \cos \Theta(t') dt' \right] \times \exp \left\{ i \left( \frac{\phi(t) + \phi(0)}{2} - \frac{\Theta(t)}{2} + \mathcal{R}(t) - \frac{\pi}{2} \right) \right\}$$

(10)

with

$$\mathcal{R} = \int_0^t \sin \left[ \frac{2}{\hbar} \int_0^{t'} |\omega(t'')| \cos \Theta(t'') dt'' \right] dt'$$

(11)

Vice versa, if $a(t)$ and $b(t)$ are solutions of the Cauchy problem (6), then the representations given in Equations (9) and (10) are still valid and $\Theta(t)$ satisfies Equation (7).

Generally speaking, solving Equation (7) is a difficult task. This equation however may be exploited in a different way, giving rise to a strategy\textsuperscript{[15]} aimed at singling out exactly solvable dynamical problems represented by Equation (6). Fixing, indeed, at will the function $\Theta(t)$ in Equation (7), that is, $\Theta(t)$ regarded now as a parameter (function) rather than an unknown, determines a link between $\Omega(t)$ and $\omega(t)$ under which the corresponding dynamical problem may be exactly solved in view of Equations (9) and (10). It is important to underline, however, that, depending on the choice of the function $\Theta(t)$, it could be very difficult (if not impossible) sometimes to find analytical expressions of all quantities we need, in particular, for $\phi(t)$ and $\phi(t)$ because of the presence of the integral $\mathcal{R}$ in Equation (11). We emphasize that if we knew the solution of the Cauchy problem given in Equations (7) and (8), whatever $\Omega(t)$, $|\omega(t)|$, and $\phi(t)$ are, then we would be
in condition to solve in general the corresponding Cauchy dynamical problem expressed by Equation (6). It is worth noticing, however, that in the special physical scenario in which the driving term $\Omega(t) + h\phi_0(t)/2$ vanishes, the corresponding Cauchy problem admits the solution $\Theta(t) = 0$, by direct inspection. The physical implication of such a solution will be considered in detail in the next section.

Another useful way of parameterizing the expressions of $a(t)$ and $b(t)$ is \cite{15}

$$a(t) = \left( \cos[\Phi(t)] - i \frac{\beta}{\sqrt{1 + \beta^2}} \sin[\Phi(t)] \right) \exp \left\{ i \frac{\phi_0(t)}{2} \right\}$$

(12)

$$b(t) = \frac{1}{\sqrt{1 + \beta^2}} \sin[\Phi(t)] \exp \left\{ i \left( \frac{\phi_0(t)}{2} - \frac{\pi}{2} \right) \right\}$$

(13)

with

$$\Phi(t) = \sqrt{1 + \beta^2} \int_0^t \frac{|\alpha(t')|}{\hbar} dt'$$

(14)

$\beta$ being an arbitrary real number and having put, without loss of generality, $\phi_0(0) = 0$. In this case, it is possible to check that they solve the system (6) if the following condition holds

$$\frac{\Omega(t)}{\hbar} + \frac{\phi_0(t)}{2} = \beta |\alpha(t)| \hbar$$

(15)

It is stressed that this last equation does only express the condition under which, whatever $\beta$ is, the representations (12) and (13) satisfy the Cauchy problem (6). This means that the real number $\beta$ plays in this case the role of parameter. When Equation (15) cannot be satisfied for any $\beta$, of course the solution of the dynamical problem exists but cannot be represented using Equations (12) and (13). In this case, there certainly exists a function $\Theta(t)$ enabling the representation of the solutions by using Equations (9) and (10). Finally, it is interesting to underline that Equation (7) turns into the simpler condition (15) on $B(t)$ under an appropriate choice of the parameter function $\Theta(t)$, \cite{15}

3. Generalized Resonance Condition and Out of Resonance Cases

The experimental setup considered by Rabi, as described in the introduction, leads to the Hamiltonian model (2) where

$$\Omega(t) = \frac{\hbar \mu g B_0}{2} \equiv \Omega_0$$

(16)

$$|\alpha(t)| = \frac{\hbar \mu g}{2} \sqrt{B_0^2(t) + B_z^2(t)} = \frac{\hbar \mu g}{2} B_z \equiv |\alpha_0|$$

(17)

$$\phi_0(t) = \nu_0 t \equiv \phi_0 t$$

(18)

Then, it is characterized by the three time-independent parameters: $\Omega_0$, $|\alpha_0|$, and $\phi_0$. Here, we generalize this Rabi scenario by making some out of or all these parameters time-dependent: $\Omega \to \Omega(t)$, $|\alpha| \to |\alpha(t)|$, and $\phi_0 t \to \phi_0(t)$.

First, we rewrite the general Hamiltonian (2) as follows

$$H = \Omega(t) \hat{\sigma}^z + \omega_x(t) \hat{\sigma}^x + \omega_y(t) \hat{\sigma}^y$$

(19)

with $\omega_{x/y}(t) = \hbar \mu g B_x/y/2$ and $\hat{\sigma}^{x/y/z}$ being Pauli matrices. Generalizing the approach in ref. \cite{8}, we pass from the laboratory frame to the time-dependent one tuned with $\phi_0(t)$, where the time-dependent Schrödinger equation for the transformed state

$$|\psi(t)\rangle = \exp[i \phi_0(t) \hat{\sigma}^z/2] |\tilde{\psi}(t)\rangle$$

(20)

is governed by the following effective time-dependent transformed Hamiltonian

$$\hat{H}_{\text{eff}}(t) = \left( \Omega(t) + \frac{\hbar}{2} \phi_0(t) \right) \hat{\sigma}^z + |\alpha(t)| \hat{\sigma}^x.$$  

(21)

It is worth noticing its strict similarity with the analogous one got in ref. \cite{8} where the unitary transformation is indeed a uniform rotation around the $z$-axis. In fact, it is enough to make $\Omega(t)$, $|\alpha(t)|$, and $\phi_0(t)$ time-independent in $\hat{H}_{\text{eff}}$ (GR stands for Generalized Rabi) to immediately recover the transformed Hamiltonian got by Rabi, \cite{16}. On the basis of this observation, it then appears natural to refer to the following condition

$$\frac{\Omega(t)}{\hbar} + \frac{\phi_0(t)}{2} = 0$$

(22)

as a generalized resonance condition, in accordance with the corresponding static resonance condition $\Omega_0 + \hbar \phi_0/2 = 0$ brought to light by Rabi in ref. \cite{7}. We underline that the generalized resonance condition does not lead to a time-independent transformed dynamical problem (as it happens in the Rabi scenario), but, whatever $H$ is, it easily enables the explicit construction of the time evolution operator describing the quantum motion of the spin in the laboratory frame. In view of Equation (15), the entries of such an operator are indeed exactly given by Equations (12) and (13) in the limit $\beta \to 0$, namely

$$a(t) = \cos \left[ \int_0^t \frac{|\alpha|}{\hbar} dt' \right] \exp \left\{ i \frac{\phi_0(t)}{2} \right\}$$

(23)

$$b(t) = \sin \left[ \int_0^t \frac{|\alpha|}{\hbar} dt' \right] \exp \left\{ i \frac{\phi_0(t)}{2} - i \frac{\pi}{2} \right\}$$

(24)

By definition, we say to be in generalized out of resonance when the left hand side of Equation (22) is non-vanishing, namely

$$\frac{\Omega(t)}{\hbar} + \frac{\phi_0(t)}{2} = \Delta(t) \neq 0$$

(25)

where $\Delta(t)$ is an arbitrary frequency-dimensioned well-behaved function of time. Let us observe that, on the basis of the structure of $\hat{H}_{\text{eff}}$ in Equation (21), when $\Delta(t)$ is proportional to $|\alpha(t)|$, the dynamical problem may be exactly solved. Indeed, this condition coincides with that expressed by Equation (15) which in turn enables one to write down exact solutions of the Cauchy problem (6) in the form given by Equations (12) and (13). Here, we report the exact solutions of special nontrivial out of resonance dynamical problems. Our aim is to illustrate the occurrence of analogies.
and differences in the time behavior on the Rabi transition probability
\[ P_\gamma^-(t) = |\langle - | U(t) |+\rangle|^2 = |b(t)|^2 \]  
(26)

\( (\hat{\sigma}^z|\pm\rangle = \pm|\pm\rangle) \), when the time evolution of the magnetic field acting upon the spin cannot be described as a perfect precession around the \( z \)-axis.

### 4. Examples of Generalized Rabi models

This section is aimed at showing that the Rabi transition probability \( P_\gamma^-(t) \) exhibits a remarkable sensitivity to possible different choices of the time-dependent magnetic fields under general conditions. The following examples are reported to illustrate such behavior.

#### 4.1. Examples of GRSs Dynamics under Generalized Resonance Condition

Let us consider, firstly, the generalized resonance condition in Equation (22). We know that, under such a condition, the time evolution operator is characterized by the time behavior of its two entries given in Equations (23) and (24), so that the transition probability reads

\[ P_\gamma^-(t) = \sin^2 \left( \int_0^t \frac{|\omega|}{\hbar} dt' \right) \]  
(27)

It is immediately evident that \( P_\gamma^-(t) \) exhibits different behaviors: it may be periodic or asymptotic. Indeed, for example, setting \( |\omega(t)| = |\omega_0| \text{sech}(|\omega_0| t/\hbar) \), obtainable by an \( x-y \) magnetic field varying over time as

\[ B_y = B_x(t) c_1 + B_z(t) c_2 \]

\[ = B_2 \text{sech}(|\omega_0| t/\hbar) \left[ \cos (\phi_0 t) c_1 - \sin (\phi_0 t) c_2 \right] \]  
(28)

we get

\[ P_\gamma^-(t) = \tan^2 \left( |\omega_0| t/\hbar \right) \]  
(29)

resulting in a Landau–Zener-like transition,\(^{[26]}\) that is an asymptotic aperiodic inversion of population. Figure 1a,b represent the transverse magnetic field in Equation (28) and the resulting transition probability in Equation (29), respectively, plotted against the dimensionless time \( \tau' = |\omega_0| t/\hbar \), with \( \phi_0/|\omega_0| = 10 \); c) the normalized magnetic field in Equation (31), parametrically represented in the \( x-y \) plane and the related d) transition probability in Equations (29) and (32) hold whatever the dimensionless parameter \( \gamma \tau' \) with \( |\omega_0|/\hbar \gamma = 9\pi/2 \) and \( \phi_0/\gamma = 10 \); e) the normalized transverse magnetic field in Equation (36), parametrically represented in the \( x-y \) plane in terms of \( \tau = \phi_0 t \) with \( A'/B_1 = 1 \) and \( \lambda = 10|\omega_0| \) and the related f) transition probability in Equation (37) for \( k = 1 \) and \( n = 10 \) and \( C = 1 \).

However, of course, it is easy to understand that it is possible to make choices either resulting in an oscillating but not periodic transition probability or exhibiting a periodic behavior, even if not coincident with that characterizing the Rabi scenario. If we consider, for example,

\[ |\omega(t)| = |\omega_0| e^{-\gamma t} \]  
(30)

reproducible by engineering the transverse magnetic field as

\[ B_y = B_x(t) c_1 + B_z(t) c_2 \]

\[ = B_z e^{-\gamma t} \left[ \cos (\phi_0 t) c_1 - \sin (\phi_0 t) c_2 \right] \]  
(31)

the resulting transition probability yields

\[ P_\gamma^-(t) = \sin^2 \left( \alpha (1-e^{-\gamma t}) \right) \]  
(32)

with \( \alpha = |\omega_0|/\hbar \gamma \). We point out that, for the sake of simplicity, in Equations (28) and (31), we have put \( \phi_0(t) = \phi_0 t \), even if, in general, the expression of the probability in Equations (29) and (32) hold whatever \( \phi_0(t) \) is, provided that Equation (22) is satisfied. Figure 1c shows the time behavior of the magnetic field in the \( x-y \) plane, against the dimensionless parameter \( \gamma t \), when \( \alpha = 9\pi/2 \) and \( \phi_0/\gamma = 10 \).

The time behavior of \( P_\gamma^-(t) \) as given in Equation (32) is reported in Figure 1d for \( \alpha = 9\pi/2 \). We recognize the existence of

---

*Figure 1.* a) The normalized magnetic field in Equation (28), parametrically represented in the \( x-y \) plane and the related b) transition probability in Equation (29) as a function of the dimensionless parameter \( \tau' = |\omega_0| t/\hbar \) with \( \phi_0/|\omega_0| = 10 \); c) the normalized magnetic field in Equation (31), parametrically represented in the \( x-y \) plane and the related d) transition probability in Equations (29) and (32) hold whatever the dimensionless parameter \( \gamma \tau' \) with \( |\omega_0|/\hbar \gamma = 9\pi/2 \) and \( \phi_0/\gamma = 10 \); e) the normalized transverse magnetic field in Equation (36), parametrically represented in the \( x-y \) plane in terms of \( \tau = \phi_0 t \) with \( A'/B_1 = 1 \) and \( \lambda = 10|\omega_0| \) and the related f) transition probability in Equation (37) for \( k = 1 \) and \( n = 10 \) and \( C = 1 \).
a transient wherein $P_\tau^+(t)$ exhibits aperiodic oscillations of maximum amplitude which, after a finite interval of time, turn into a monotonic increase that asymptotically approaches one. We emphasize that the number of complete oscillations, preceding the asymptotic behavior of $P_\tau^+(t)$ as well as $P_\tau^-(t)$ itself, is $\alpha$-dependent. Equation (32), indeed, predicts

$$P_\tau^+(\infty) = \sin^2(\alpha)$$  \hspace{1cm} (33)

which immediately leads to (with integer $n$)

$$P_\tau^+(\infty) = 0, \quad \alpha = n\pi$$
$$P_\tau^+(\infty) = 1, \quad \alpha = (2n+1)\pi$$  \hspace{1cm} (34)

As our third example, we consider the following modulation of $|\omega(t)|$

$$|\omega(t)| = |\omega_0| + A \cos(\lambda t), \quad 0 < A < |\omega_0|$$  \hspace{1cm} (35)

realizable by engineering the transverse magnetic field as

$$B_\perp(t) = [B_\perp + A' \cos(\lambda t)] \cos(\phi_0 t)$$
$$B_\parallel(t) = -[B_\parallel + A' \cos(\lambda t)] \sin(\phi_0 t)$$  \hspace{1cm} (36)

Here, $\lambda = n\phi_0$ with $n \in \mathbb{N}^+$, $A = \hbar\mu_B A'/2$, and then $0 < A' < B_\perp$, in view of Equations (35) and (17). The transverse field is represented in Figure 1f as a function of the dimensionless time parameter $\tau = \phi_0 t$, once more supposing for simplicity $\phi_\alpha(t) = \phi_0 t$.

In this case, the Rabi’s transition probability results

$$P_\tau^+(t) = \sin^2 \left[ C \left( \tau + \frac{k}{n} \sin(n\tau) \right) \right]$$  \hspace{1cm} (37)

with

$$C = \frac{|\omega_0|}{\hbar \phi_0}, \quad k = \frac{A'}{B_\perp}, \quad \tau = \phi_0 t, \quad n = \frac{\lambda}{\phi_0}$$  \hspace{1cm} (38)

The behavior of $P_\tau^+(t)$ in Equation (37) is shown in Figure 1f, having put $k = 1$, $n = 10$, and $C = 1$. Differently from the previous example, we see that, in this case, the characteristic sinusoidal behavior of the Rabi transition probability turns into a periodic population transfer, still of maximum amplitude, between the two energy levels of the spin. We emphasize that, in view of Equation (27), different time evolutions of $P_\tau^+(t)$ require different choices of $|\omega(t)|$ only, then regardless of $\Omega(t)$ and $\phi_\alpha(t)$ provided they are constrained by the generalized resonance condition (22). For this reason, in the plots in Figure 1, we have chosen the simplest case $\phi_\alpha(t) = \phi_0$ (implying $\Omega(t) = -\hbar\mu_B / 2$ by Equation (22)). Our choices for $|\omega(t)|$, indeed, are not merely mathematical choices, but they aim at furnishing physical scenarios in the grasp of the experimentalists. To this end, it is hence important to take care of $\Omega(t)$ and $\phi_\alpha(t)$. This consideration explains why we have chosen $\Omega(t) = -\hbar\mu_B / 2$ and $\phi_\alpha(t) = \phi_0 t$. It is worth noticing that we selected different, more complex, or also very exotic time-dependencies for such parameters, we would get just a mere mathematical speculation since no physical effects would be present in the physical quantity under scrutiny, $P_\tau^+(t)$.

for the latter, indeed, under the generalized resonance condition in Equation (22), $\Omega(t)$ and $\phi_\alpha(t)$ are not relevant physical parameters. We stress, however, that distinct realizations of the resonance condition, keeping the same $|\omega(t)|$, introduce significant changes in the dynamical behavior of the GRS with respect to the Rabi system. It is enough to consider, for example, that

$$(+|U(t)|^2 U(t)|+) = \mp 2\hbar\alpha(t)|b(t)| \cos(\phi_\alpha(t) + \phi_0)$$  \hspace{1cm} (39)

depend on both $\phi_\alpha(t)$ and $|\omega(t)|$, in view of Equations (23) and (24).

4.2. Examples of GRSs Dynamics in Generalized Out of Resonance Cases

In this subsection, we analyze the generalized out of resonance case, defined in Equation (25). Since it appears hopeless to have an exact closed treatment of the Cauchy problem in Equation (6) with an arbitrary $\Delta(t)$, we confine ourselves to the following specific forms

$$\hbar\Delta(t) = \begin{cases} \beta_0 |\omega(t)| \\ \beta(t)|\omega(t)| \end{cases}$$  \hspace{1cm} (40)

In the former case, the solutions $a(t)$ and $b(t)$ of the system in Equation (6) may be cast as reported in Equations (12) and (13) so that

$$P_\tau^+(t) = \frac{1}{1 + \beta_0^2} \sin^2 \left[ \sqrt{1 + \beta_0^2} \int_0^t |\omega(t')| \, dt' \right]$$  \hspace{1cm} (41)

In the limit $\beta_0 \to 0$, we recover Equation (27) from this equation. Thus, we may compare $P_\tau^+(t)$ in the resonant and this nonresonant cases when $|\omega(t)|$ is fixed in the same way. It is easy to convince oneself that the main effect of a positive value of the parameter $\beta_0$ on $P_\tau^+(t)$ is nothing but a scale effect determined by the ratio $1/(1 + \beta_0^2)$.

We wish now to discuss some exactly solvable scenarios of generalized, out of resonance, Rabi problems wherein $\hbar\Delta(t) = \beta(t)|\omega(t)|$. The form of $\Delta(t)$ written before, naturally emerges when the independent variable $t$ in Equation (7) is substituted by

$$\tau(t) = \int_0^t |\omega(t')| \, dt'$$  \hspace{1cm} (42)

and a choice at will of $\Theta(\tau(t))$ is performed in accordance with ref. [15]. We stress however that the corresponding function $\beta(t)$ would be functionally dependent on $|\omega(t)|$, that is, we determine $\beta(\tau(t))$ once we have chosen $|\omega(t)|$. We emphasize that the fact that the function $\beta(t)$ is not independent of the function $|\omega(t)|$ does not spoil interest of such a particular procedure. In the following examples, we indeed report two applications of the general strategy here exposed.
4.2.1. Case 1

In this subsection and the following one, we will omit the $t$-dependence in $\tau(t)$ to save some writing. To illustrate the applicability of our parametrization given in Equations (9), (10), and (11), we cannot simply confine ourselves to assign at will $\Theta(\tau)$. Indeed, we must overcome the unique analytical difficulty related to the calculation of $\hat{\omega}(t)$ in Equation (11) as previously underlined. In practice, then, what is demanded is to search specific choices of $\Theta(\tau)$ such that the integral expressing $\hat{\omega}(t)$ becomes evaluable. The following two examples provide a successful application of such a strategy.

Assuming the solution of the Cauchy problem (7) and (8) as

$$\Theta(t) = 2 \tan^{-1}\left( \frac{2\tau}{\sqrt{2} + 4\tau^2} \right)$$

it is straightforward to show that

$$\int_0^t \frac{|\omega(t')|}{\hbar} \cos(\Theta(t')) dt' = \frac{1}{2} \tan^{-1}(2\tau)$$

Equation (7) immediately yields

$$\Delta(t) = \frac{4(1 + \tau^2)}{(1 + 4\tau^2)^{1/2} + 4\tau^2} |\omega(t)| = \beta(t)|\omega(t)|$$

Within such a scenario, the specialized expressions of Equations (9) and (10) result in

$$|a(t)| = \sqrt{\frac{1 + 4\tau^2 + 1}{2\sqrt{1 + 4\tau^2}}}$$

and

$$|b(t)| = \sqrt{\frac{1 + 4\tau^2 - 1}{2\sqrt{1 + 4\tau^2}}}$$

and

$$\phi_a(t) = \frac{\phi_a(t) - \phi_a(0)}{2} - \tan^{-1}\left( \frac{2\tau}{\sqrt{2} + 4\tau^2} \right) + \frac{i}{\sqrt{2}} \text{EllipticE}[i\sin^{-1}(2\tau), 1/2]$$

$$\phi_b(t) = \frac{\phi_b(t) + \phi_b(0)}{2} - \tan^{-1}\left( \frac{2\tau}{\sqrt{2} + 4\tau^2} \right) - \frac{i}{\sqrt{2}} \text{EllipticE}[i\sin^{-1}(2\tau), 1/2] - \frac{\pi}{2}$$

with $\text{EllipticE}(\phi, m) = \int_\phi^\phi [1 - m \sin^2(\theta)]^{1/2} d\theta$. It is interesting to consider a simple case in which $|\omega(t)| = \text{const.} = |\omega_0|$. In this instance, we have such a situation that $P^+_b(t) = |b(t)|^2$, $P^-_b(t) = |a(t)|^2$, which, notwithstanding its apparent similarity with the previous case given in Equation (43), leads, however, to a remarkable different temporal behavior of the correspondent generalized Rabi system. This time it results in

$$\int_0^t \frac{|\omega(t')|}{\hbar} \cos(\Theta(t')) dt' = \tan^{-1}(\tau)$$

so that the solutions of (6) read

$$|a(t)| = \frac{1}{\sqrt{1 + \tau^2}}$$

and

$$|b(t)| = \frac{\tau}{\sqrt{1 + \tau^2}} = |b(t)|\tau$$

and

$$\phi_a(t) = \frac{\phi_a(t) - \phi_a(0)}{2} - \tan^{-1}\left( \frac{\tau}{\sqrt{2} + \tau^2} \right) - \frac{1}{2} \left[ \frac{\tau \sqrt{2 + \tau^2}}{2} + \sin^{-1}\left( \frac{\tau}{\sqrt{2}} \right) \right]$$

$$\phi_b(t) = \frac{\phi_b(t) + \phi_b(0)}{2} - \tan^{-1}\left( \frac{\tau}{\sqrt{2} + \tau^2} \right) + \frac{1}{2} \left[ \frac{\tau \sqrt{2 + \tau^2}}{2} + \sin^{-1}\left( \frac{\tau}{\sqrt{2}} \right) \right] - \frac{\pi}{2}$$
Finally, the special form of $\Delta(t)$ underlying this specific scenario is

$$\Delta(t) = \frac{2 + (1 - \tau^2)(2 + \tau^2)}{2(1 + \tau^2)\sqrt{2 + \tau^2}} |\omega(t)| \frac{\hbar}{\omega} \tag{54}$$

In this case, it is easy to see that if $|\omega| = \text{const.} = |\omega_0|$, $P_{\omega}^- (t) = |b(t)|^2 \quad (P_{\omega}^+ (t) = |a(t)|^2)$ goes from 0 (1) to 1 (0) asymptotically. These behaviors, reproducing the transition probabilities in the Landau–Zener scenario, are illustrated by full blue and dashed red lines, respectively, in Figure 2d. In this case, the time behavior of the “detuning” $h\Delta(t)/|\omega_0|$ is characterized by an asymptotic linear dependence on $t$, as shown in Figure 2c. As in the resonant scenario, even here different time-dependencies of the magnetic field may give rise to qualitatively different time evolutions of $P_{\omega}^\pm (t)$ with respect to the Rabi scenario. We emphasize that the scenarios and the formulas reported for the two examples are valid whatever the time-dependence of $|\omega(t)|$ is. Of course, depending on the choice of $|\omega(t)|$ the expressions for $\beta(t)$ and for $a(t)$ and $b(t)$ could become complicated functions of time, but the related time-dependent scenario keeps the property to be an exactly solvable case for the spin-1/2 dynamical problem.

As a final remark, we want to emphasize that if it was very hard to get analytical expressions for $a(t)$ and $b(t)$, in Equations (9) and (10), respectively, depending on the choice of $\Theta(t)$ and two of the three Hamiltonian parameters. Nevertheless, such a bottleneck does not influence our capability to predict the Rabi transition probability and the expression of $\Delta(t)$ in order to know how to engineer the magnetic fields to get the desired time evolution. Indeed, we would be always able to find accordingly the expressions of $|a(t)|$ and $|b(t)|$. Thus, as a consequence, given $|\omega(t)|$, when $\Omega(t)$ and $\phi(t)$ are chosen in such a way to generate the same detuning $\Delta(t)$, the related different physical scenarios share the same analytical expressions of $|a(t)|$ and $|b(t)|$. Then, all physical observables depending only on these quantities share the same expressions as well, for example, $P_{\omega}^- (t)$ or

$$\langle \pm |U(t)^\dagger U(t)|\pm \rangle = \pm \hbar (|a(t)|^2 - |b(t)|^2) \tag{55}$$

5. Conclusions

The Rabi scenario consists of a spin-1/2 subjected to a time-dependent magnetic field precessing around the quantization axis ($\mathcal{Z}$) and is characterized by three time-independent parameters: $\Omega_0$, $|\omega_0|$, and $\phi_0$. Rabi shows that when $\Omega_0 + \hbar \omega_0 / 2 = \Delta = 0$, the transverse magnetic field acts as a probe of the energy separation $2\Delta$ due to the longitudinal field alone. The measurable physical quantity revealing $\Omega_0$ is the transition probability $P_{\omega}^- (t) = \langle - |U(t)|+ \rangle$, which, at resonance, oscillates between 0 and 1 with frequency now referred to as Rabi frequency.

In this work, we generalize this Rabi scenario by assuming an $SU(2)$ general time-dependent Hamiltonian model where then $\Omega_0$, $|\omega_0|$, and $\phi_0$ are replaced with time-dependent counterparts. Along the lines of the Rabi approach, we first show that, in the rotating frame with the time-dependent angular frequency $\phi(t)$, the condition $\Omega(t) + \hbar \omega(t)/2 = \Delta(t) = 0$ plays the same role of the Rabi resonance condition in the Rabi scenario.

Such an occurrence makes of basic interest a direct comparison between the Rabi scenario and its generalized version on both time-dependent resonance and out of resonance ($\Delta(t) \neq 0$) cases. To bring to light the occurrence of analogies and differences, we have focussed our attention on the study of the transition probability $P_{\omega}^- (t)$ between the two eigenstates of $\Sigma$.

We show that, on resonance, $P_{\omega}^- (t)$ depends only on the integral of $|\omega(t)|$. Our examples illustrate that this circumstance determines a transition probability characterized by three possible different regimes: oscillatory (the only one dominating the Rabi scenario), monotonic, and mixed which means an initial oscillatory transient followed by an asymptotic monotonic behavior.

To capture significant dynamical consequences stemming from the detuning time dependence, we have constructed exactly solvable problems and analyzed the corresponding quantum dynamics of the spin-1/2. We have thus highlighted that when $\Delta(t)$ is proportional to $|\omega(t)|$, the main effect emerging in the time behavior of $P_{\omega}^- (t)$ is a scale effect both in amplitude and in frequency (like in the Rabi scenario).

We have further investigated two specific exactly solvable scenarios of experimental interest for which $\Delta(t)/|\omega(t)|$ varies over time. One of them predicts a Landau–Zener transition, while the other an equal weighted coherent superposition of the two states of the system. It is important to underline that our examples illustrate exactly solvable cases where all the dynamical aspects and features of the spin-1/2 system under scrutiny may be brought to light. We highlighted, however, that when one is interested in the Rabi transition probability $P_{\omega}^- (t)$ only, the knowledge of $|a(t)|$ and $|b(t)|$ is enough. We point out that this circumstance leads us to wider and richer classes of physical scenarios, since we need not worry about possible analytical difficulties stemming from Equation (11).

We underline that the knowledge of the exactly solvable problems reported here provide stimulating ideas for technological applications with single qubit devices. In addition, it furnishes ready-to-use tools for interacting qudits systems being of relevance in several fields, from condensed matter physics to quantum information and quantum computing. In this connection, we wish to remark the relevance of the analytical results reported here to the real problems in the control of several qubits. Hamiltonian models describing a system of $N$ interacting qubits subjected to controlled external magnetic fields have been recently reported and exactly treated on the basis of the strategy underlying the methods here reported. The general idea is to investigate models of experimental interest characterized by symmetry properties allowing the reduction of the quantum dynamics of the multi-qubit system to that of a single qubit. The ability of controlling $N$ coupled spins controlling the time evolution of only one spin is a protocol of interest in quantum information technology. Concerning such a last comment, it is possible to see that, under specific conditions brought to light in ref. [27], the time-dependent scenarios analyzed in Sections 4.2.1 and 4.2.2 may be of applicable interest to generate GHZ states of the $N$ spins in the system or a complete inversion of all the spins from their up (down) state to the down (up) one. The interesting physical aspect consists of the fact that such time-dependent magnetic fields are applied upon only one of the spins (ancilla) and the type of high non-local...
coupling allows to reverberate the physical effect on all the other spins.

Acknowledgements
A.S.M.C. acknowledges the Brazilian agency CNPq’s (BR) financial support (Grant No. 453835/2014-7). R.G. acknowledges the economical support by research funds difc 3100050001d08+ (University of Palermo) in memory of Francesca Palumbo.

Conflict of Interest
The authors declare no conflict of interest.

Keywords
exact single-qubit dynamics, exactly solvable time-dependent models, semiclassical Rabi model

Received: June 13, 2018
Revised: October 18, 2018
Published online: November 19, 2018

[1] F. T. Hioe, J. Opt. Soc. Am. B 1987, 4, 1327.
[2] S. Haroche, J. M. Raimond, Exploring the Quantum: Atoms Cavities And Photons, Oxford University Press, Oxford, UK 2006.
[3] D. Daems, A. Ruschhaupt, D. Sugny, S. Guerin, Phys. Rev. Lett. 2013, 111, 050404.
[4] A. Greilich, S. E. Economou, S. Spatzek, D. R. Yakovlev, D. Reuter, A. D. Wieck, T. L. Reinecke, M. Bayer, Nat. Phys. 2009, 5, 262.
[5] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, UK 1990.
[6] I. S. Oliveira, T. J. Bonagamba, R. S. Sarthour, J. C. C. Freitas, E. R. deAzevedo, NMR Quantum Information Processing, Elsevier, Amsterdam/Oxford 2007.
[7] I. I. Rabi, Phys. Rev. 1937, 51, 652.
[8] I. I. Rabi, N. F. Ramsey, J. Schwinger, Rev. Mod. Phys. 1954, 26, 167.
[9] J. Schwinger, Phys. Rev. 1937, 51, 648.
[10] L. M. K. Vandersypen, I. L. Chuang, Rev. Mod. Phys. 2004, 76, 1037.
[11] D. Braak, Q.-H. Chen, M. T. Batchelor, E. Solano, J. Phys. A: Math. Theor. 2016, 49, 300301.
[12] V. G. Bagrov, D. M. Gitman, M. C. Baldiotti, A. D. Levin, Ann. Phys. (Berlin, Ger.) 2005, 14, 764.
[13] M. Kuna, J. Naudts, Rep. Math. Phys. 2010, 65, 77.
[14] E. Barnes, S. Das Sarma, Phys. Rev. Lett. 2012, 109, 060401.
[15] A. Messina, H. Nakazato, J. Phys. A: Math. Theor. 2014, 47, 445302.
[16] L. A. Markovich, R. Grimaudo, A. Messina, H. Nakazato, Ann. Phys. (Amsterdam, Neth.) 2017, 385, 522.
[17] R. Grimaudo, A. Messina, H. Nakazato, Phys. Rev. A 2016, 94, 022108.
[18] R. Grimaudo, A. Messina, P. A. Ivanov, N. V. Vitanov, J. Phys. A 2017, 50, 175301.
[19] R. Grimaudo, Y. Belousov, H. Nakazato, A. Messina, Ann. Phys. (Amsterdam, Neth.) 2018, 392, 242.
[20] R. Calvo, J. E. Abud, R. P. Sartoris, R. C. Santana, Phys. Rev. B 2011, 84, 104433.
[21] Y. B. Borozdina, E. Mostovich, V. Enkelmann, B. Wolf, P. T. Cong, U. Tutsch, M. Lang, M. Baumgarten, J. Mater. Chem. C 2014, 2, 6618.
[22] J. R. Petta, Science 2005, 309, 2180.
[23] M. Anderlini, P. J. Lee, B. L. Brown, J. Sebby-Strabley, W. D. Phillips, J. V. Porto, Nature 2007, 448, 452.
[24] X. Wang, L. S. Bishop, J. P. Kestner, E. Barnes, K. Sun, S. D. Sarma, Nat. Comm. 2012, 3, 997.
[25] H. Bluhm, S. Foletti, I. Neder, M. Rudner, D. Mahalu, V. Umansky, A. Yacoby, Nat. Phys. 2011, 7, 109.
[26] a) L. Landau, Phys. Z. Sowjetunion 1932, 2, 46; b) C. Zener, Proc. R. Soc. A 1932, 137, 696.
[27] R. Grimaudo, L. Lamata, E. Solano, A. Messina, Phys. Rev. A 2018, 98, 042330.