Algorithms for geometrical operations with NURBS surfaces.

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Abstract

The aim of the paper is to show algorithms for geometrical manipulations on NURBS surfaces. These include generating NURBS surfaces that pass through given points, calculating the minimum distance to a point and include line to surface and surface to surface intersections.

Keywords: NURBS, Isogeometric analysis

1. Introduction

Nonuniform rational B-splines or NURBS have been used by the Computer Aided Design (CAD) community for decades. The reason for this is that they are very suitable for defining geometrical shapes and for geometrical operations. A great number of publications on NURBS exist, here we quote the NURBS book [8] and a paper [5] that give a good summary. With the publication of the book Isogeometric Analysis [4] it was suggested that NURBS would also benefit numerical simulation methods such as the Finite Element (FEM) and Boundary Element method (BEM). This was followed by a number of papers discussing the implementation, for example ([6, 1, 3, 10, 7, 9]). A book on the implementation of isogeometric methods into BEM simulation programs was published [2].

A proper implementation of NURBS into FEM and BEM programs, however required revisiting of some geometrical operations. While the CAD community is mainly interested in the graphical display of geometry, the simulation community is interested in generating an ‘analysis suitable’ geometry, i.e. one that allows a suitable volume or boundary discretisation. In this paper we concentrate of surfaces, as they are used in BEM, and present algorithms for some geometrical operations. It should be noted that CAD programs have very sophisticated algorithms for computing surface to surface intersections. However, these algorithms produce data that are not ‘analysis suitable’.

2. B-splines and NURBS

B-splines are an attractive alternative to Lagrange polynomials and Serendipity functions predominantly used in simulation. The basis for creating the functions is the knot vector. This is a vector containing a series of non-decreasing values of the local coordinate $\xi$:

$$\Xi = (\xi_0, \xi_1, \cdots, \xi_N)$$

A B-spline basis function of order $p = 0$ (constant) is given by:

$$B_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Higher order basis functions are defined by referencing lower order functions:

$$B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi_i}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi)$$

B-spines are associate with anchors. The location of the $i$-th anchor in the parameter space can be computed by:

$$\tilde{\xi}_i = \frac{\xi_{i+1} + \xi_{i+2} + \cdots + \xi_{i+p}}{p} \quad i = 0,1,\ldots,I-1.$$
Nonuniform rational B-splines or NURBS are based on B-splines but have improved properties for the definition of geometry. NURBS of order $p$ are defined as:

$$N_i^p(\xi) = \frac{B_{i,p}(\xi) w_i}{\sum_{j=0}^{I} B_{j,p}(\xi) w_j}$$

(5)

where $I + 1$ is the number of basis functions and $w_i$ are weights. This can be extended to two dimensions using a tensor product:

$$N_{ij}^{p,q}(\xi,\eta) = N_i^p(\xi) N_j^q(\eta)$$

(6)

where $N_j^q(\eta)$ is a NURBS of order $q$ in $\eta$ direction.

The coordinates of a point on a curve with the local coordinate $\xi$ can be computed by:

$$x(\xi) = \sum_{j=1}^{J} R_j(\xi)c_j$$

(7)

where $R_j(\xi)$ are NURBS basis functions defined in equation 5 except that numbering starts at 1 instead of 0. $c_j$ are control point coordinates. The vector tangential in direction $\xi$ can be computed by:

$$v_1 = \frac{\partial x(\xi)}{\partial \xi} = \sum_{j=1}^{J} \frac{\partial R_j(\xi)}{\partial \xi} c_j$$

(8)

The coordinates of a point on a surface with the local coordinate $\xi,\eta$ can be computed by:

$$x(\xi,\eta) = \sum_{j=1}^{J} R_j(\xi,\eta)c_j$$

(9)

where $R_j(\xi,\eta)$ are NURBS basis functions defined in equation 6 except that numbering starts at 1 instead of 0. The basis functions are numbered consecutively with a single index $j$ instead of two indices $i$ and $j$. For a surface the vector in direction $\eta$ can be additionally computed by:

$$v_2 = \frac{\partial x(\xi,\eta)}{\partial \eta} = \sum_{j=1}^{J} \frac{\partial R_j(\xi,\eta)}{\partial \eta} c_j$$

(10)

As an example we show in Figure 1 a surface created with the knot vectors:

$$\Xi = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(11)

$$H = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$

i.e. quadratic in $\xi$-direction and linear in $\eta$ direction. The control point coordinates and weights are shown in table 1.

| $i$ | $j$ | $x$ | $y$ | $z$ | $w$ |
|-----|-----|-----|-----|-----|-----|
| 0   | 0   | 1   | 0   | 1   | 1   |
| 1   | 0   | 1   | 1   | 0.707 |
| 2   | 0   | 0   | 1   | 1   |
| 0   | 1   | 1   | 1   | 0   |
| 1   | 1   | 1   | 1   | 0.707 |
| 2   | 1   | 1   | 0   | 1   |

Table 1: Control point coordinates and weights for definition of surface.
2.1. Trimming

We can cut off a portion of the surface by a process called trimming. We first define the limits of the trimmed space using basis functions \( \hat{R}_{i,p}(\xi) \) and \( \hat{R}_{j,q}(\eta) \) and control points \( \hat{c}_{ij} \). We then map from a \( \xi, \eta \) space to a local \( \hat{\xi}, \hat{\eta} \) space:

\[
\left\{ \hat{\xi}, \hat{\eta} \right\} = \sum_{j=0}^{J} \hat{R}_j(\xi, \eta) \hat{c}_{ij} \tag{12}
\]

Finally we map form the \( \hat{\xi}, \hat{\eta} \) coordinate system to the \( x,y,z \) coordinate system using equation (9). In Figure 2 we show an example on how the surface shown in Figure 1 can be trimmed.
3. Generating NURBS curves and surfaces through a set of points

We can determine the NURBS parameters \( c_j \) in such a way that the defined curve or surface either exactly goes through a number of points with the coordinates \( x(\xi_i, \eta_i) \) or is a least square approximation.

First we gather all coordinates of the given points in vector \( \{x\} \) and of the unknown control point coordinates in vector \( \{c\} \):

\[
\{x\} = \begin{bmatrix} x(\xi_1, \eta_1) \\ x(\xi_2, \eta_2) \\ \vdots \end{bmatrix}, \quad \{c\} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \end{bmatrix}
\]  

(13)

Using (7) or (9) we arrive at the following matrix equation:

\[
\{x\} = [A]\{c\}
\]  

(14)

For a curve the matrix \([A]\) is given for example by:

\[
[A] = \begin{pmatrix}
R_1(\xi_1) & 0 & 0 & R_2(\xi_1) & 0 & 0 & \cdots \\
0 & R_1(\xi_1) & 0 & 0 & R_2(\xi_1) & 0 & \cdots \\
0 & 0 & R_1(\xi_1) & 0 & 0 & R_2(\xi_1) & \cdots \\
R_1(\xi_2) & 0 & 0 & R_2(\xi_2) & 0 & 0 & \cdots \\
0 & R_1(\xi_2) & 0 & 0 & R_2(\xi_2) & 0 & \cdots \\
0 & 0 & R_1(\xi_2) & 0 & 0 & R_2(\xi_2) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]  

(15)

We can solve for the control point coordinates by:

\[
\{c\} = [A]\{x\}
\]  

(16)

where \( \\backslash \) denotes either a solution by inverting \([A]\) or a least square approximation of \( \{x\} \) depending on the number of points. If the number of points is equal to the number of control points then the
curve/surface will go exactly through the specified points.

**Algorithm 1** Algorithm for computing control point coordinates through a set of points

**Require:** NURBS functions $R_j$ and local coordinates $\xi_i, (\eta_i)$, Vector of point coordinates $\{x\}$

1: assemble matrix $[A]$
2: solve for control point coordinates $\{c\} = [A]\{x\}$
3: return control point coordinates $\{c\}$

Algorithm 1 summarises the steps. An example of curve fitting is shown in Figure 3 of surface fitting in Figure 4.

4. Calculating the closest distance between a point and a surface

To calculate the closest distance of a point to a surface we use an iterative technique. We start with a point on the surface and compute the vector to the specified point. Using dot products with vectors $v_1$ and $v_2$ we estimate increments in the local coordinates. We iterate until the distance from the specified point to the surface does not change. For surfaces which are very curved we need to limit the increments to a maximum value using a damping coefficient (about 0.5 to 0.75) otherwise the iteration may diverge.

**Algorithm 2** Algorithm for computing minimum distance to a surface

**Require:** Coordinates of point $x_P$, NURBS of surface, Max. number of iterations $I$, damping $damp$, Tolerance.

1: Set max. increments: $max_\xi = 1, max_\eta = 1$
2: Select starting point coordinates $\xi_0, \eta_0$ and compute point on surface $x_{S0}$
3: Compute $R_0 = \|x_P - x_{S0}\|
4: for $i = 0$ to $I$ do
5: Compute unit vectors $v_1$ and $v_2$ in $\xi$- and $\eta$-directions.
6: $L_\xi = |v_1|, L_\eta = |v_2|$
7: $\Delta \xi = v_1 \cdot (x_P - x_{S_i})/L_\xi$
8: $\Delta \eta = v_2 \cdot (x_P - x_{S_i})/L_\eta$
9: $\xi_{i+1} = \xi_i + \Delta \xi$
10: if $\xi_{i+1} > max_\xi$ then
11: $\xi_{i+1} = max_\xi$
12: else if $\xi_{i+1} < 0$ then
13: $\xi_{i+1} = 0$
14: end if
15: $\eta_{i+1} = \eta_i + \Delta \eta$
16: if $\eta_{i+1} > max_\eta$ then
17: $\eta_{i+1} = max_\eta$
18: else if $\eta_{i+1} < 0$ then
19: $\eta_{i+1} = 0$
20: end if
21: Compute new point $x_{S_{i+1}}(\xi_{i+1}, \eta_{i+1})$ on surface.
22: Compute distance $R_{i+1} = \|x_P - x_{S_{i+1}}\|$
23: if $R_{i+1} - R_i < $Tolerance then
24: Exit loop
25: end if
26: Set $max_\xi = max_\xi \ast damp$
27: Set $max_\eta = max_\eta \ast damp$
28: end for
29: return Minimum distance $R$, local and global coordinates of point on surface.

Algorithm 2 summarises the steps. Figure 5 illustrates the procedure.
5. Intersection between a line and a surface

To calculate the intersection of a line with a surface we also use an iterative process. We define the line with a one dimensional NURBS and compute a vector $\mathbf{V}_L$ representing the line. We define $\xi$ as the local coordinate along the line. Starting with $\xi = 1$ which corresponds to the end of $\mathbf{V}_L$ (global coordinate $x_0$) we compute the closest point to the surface as $x_1$. We then compute and increment of $d\xi$ by projecting this point normal to the line, arriving at a new point on the line closer to the surface. We repeat this process until $d\xi$ is very small, i.e. the point on the line corresponds to a point on the surface. We explain the procedure in Figure 6.

![Image of NURBS surface with a line and iterative procedure](image)

Algorithm 3 outlines the steps.
**Algorithm 3** Algorithm for computing the intersection of a line with a surface

**Require:** NURBS of surface, NURBS of line, number of iterations \( n_{iter} \), tolerance for convergence \( tol \)

1: compute vector of line \( V_L \), line length \( L \),
2: Set \( \xi = 1 \) (= end of line).
3: for \( i = 1 \) to \( n_{iter} \) do
4:   compute \( x_{i-1}(\xi) \) on line
5:   compute \( x_i \) as the point on the surface closest to \( x_{i-1} \)
6:   compute \( d\xi_i = \frac{1}{L} (x_i - x_{i-1}) \cdot V_L \)
7:   compute local coordinate of point on line \( \xi = \xi - d\xi_i \)
8:   if \( d\xi < tol \) then
9:     Exit
10: end if
11: end for
12: return Coordinates of intersection point \( x_i \), local coordinates of point on line \( \xi \)

![Figure 7](image-url)

**Figure 7:** Example of 2 intersecting surfaces.

6. Intersection between a surface and a surface

As explained earlier CAD programs have sophisticated and fast intersection algorithms. However, they don’t lead to analysis suitable data. Here we present a simple algorithm that can be used to generate a suitable Boundary element mesh. We explain the determination of the intersection between 2 surfaces on an example involving the surface defined in section 1 and a vertical cylinder depicted in Figure 7.

We specify the cylinder (surface 1) as the **intersecting** surface and the other surface as the **intersected** surface (surface 2). Next we subdivide surface 1 in \( \xi \) direction and generate vertical lines as shown in Figure 8 left. The number of lines will determine the quality of the intersection curve. We use algorithm 3 to compute the intersection points with surface 2. Next we use algorithm 1 to compute the parameters of a trimming curve that passes through the calculated points. We then trim surface 1 as shown in Figure 8 right.

**Remark:** These trimming curve parameters can also be taken from data produced by CAD programs.

To produce analysis suitable data the next steps are more involved. We perform these steps in the local \( \xi, \eta \) coordinate system of surface 2. First we divide surface 2 into 4 subsurfaces as shown in Figure 9 left. For each subsurface we determine trimming curves that go through the computed intersection points and then trim the subsurfaces (Figure 9 middle). We finally map to global coordinates (Figure 9 right). Algorithm 4 summarises the steps.
Figure 8: Steps 1 and 2 of the process, showing lines and the computed intersection points. Shown on the right is the trimmed intersecting surface.

Figure 9: Step 3 to 5 of intersection process showing the intersection points in the $\xi, \eta$ coordinate system of the intersected surface. The original surface is split into 4 subsurfaces which are trimmed and then mapped to the global system. The final outcome consisting of 6 surfaces is shown on the right.

Algorithm 4 Algorithm for computing the intersection of a surface with a surface

Require: NURBS of surface 1 and surface 2, number of intersection lines \textit{nlines}

1: Subdivide surface 1 in $\xi$ direction into \textit{nlines} points.
2: Generate \textit{nlines} lines in $\eta$ direction.
3: Intersect these lines with surface 2 to generate intersection points
4: Use the local $\xi, \eta$ coordinates of intersection points to compute trimming curve for surface 1
5: Trim surface 1
6: Subdivide surface 2 into four subsurfaces in the local coordinate system $\xi, \eta$.
7: Determine trimming curves that go through the intersection points.
8: Trim the subsurfaces
9: Map to global coordinates
10: return Analysis suitable data
7. Summary

In this short paper some algorithms were presented that the author has used to implement isogeometric methods into a BEM simulation program. It is hoped that it will be of benefit to developers of simulation software.

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