A formula for the partition function of the $\beta\gamma$ system on the cone pure spinors

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Abstract

In this note, we propose a closed formula for the partition function $Z(t, q)$ of the $\beta\gamma$ system on the cone of pure spinors. We give the answer in terms of theta functions, $q$-Pochhammer symbols and Eisenstein series.

1 Introduction

The $\beta\gamma$ system on the cone of pure spinors $C$ is an integral part of the version of the string theory invented by N. Berkovits [5]. $C$ is an eleven-dimensional subvariety in 16-dimensional linear space with coordinates $\lambda, p_i, w_{ij}, 1 \leq i, j \leq 5, w_{ij} = -w_{ji}$ defined by the equations

$$
\begin{align*}
\lambda p_i - \text{Pf}_i(w) &= 0 \quad i = 1, \ldots, 5, \\
p w &= 0.
\end{align*}
$$

(1)

$\text{Pf}_i(w), 1 \leq i \leq 5$ are the principal Pfaffians of $w$. The action is common to all $\beta\gamma$ systems:

$$
S(\beta, \gamma) = \int_{\Sigma} \langle \bar{\partial} \beta, \gamma \rangle.
$$

The field $\beta$ is a smooth map $\beta: \Sigma \rightarrow C$, where $\Sigma$ is a Riemann surface, $\gamma$ is a smooth section of the pullback $\beta^* T^*_c \otimes T^*_\Sigma$. We advise the reader to consult [7] for the notation and discussion of issues related to definition of a $\beta\gamma$ systems on the nonsmooth $C$.

In [7], a geometric construction of the space of states $H^{i+\frac{\infty}{2}}, i = 0, \ldots, 3$ of this system was presented. It is a properly regularized space of the semi-infinite local cohomology of the space of polynomial maps $\text{Maps}(C^x, C)$. The support of the local cohomology lies at $\text{Maps}(C, C)$. The space $C$ is an affine cone over $\text{OGr}(5,10)$. $C^x \times \text{Spin}(10)$ is the groups of symmetries $C$, where $C^x$ acts by dilations. The group $C^x \times T \times \text{Spin}(10)$ acts by the symmetries of the pair $\text{Maps}(C, C) \subset \text{Maps}(C^x, C)$. The factor $T \cong C^x$ corresponds to loop rotations. The action of $C^x \times T \times \text{Spin}(10)$ survive the regularization and continue to act on $H^{i+\frac{\infty}{2}}$. It turns out (see [7]) that the formal character

$$
Z(t, q, z) = \sum_{i=0}^{3} (-1)^i \chi_{H^{i+\frac{\infty}{2}}} (t, q, z)
$$

(2)
is well defined as an element in \( Z((t, z_1, \ldots, z_5))(q) \cap \mathbb{Q}(t, z_1, \ldots, z_5))(q) \), where \( z_1, \ldots, z_5 \) are the coordinates on the Cartan subgroup \( T^5 \subset \text{Spin}(10) \). More precisely, \( Z(t, q, z) \) is a limit of coefficients of an infinite matrix product. The matrix product simplifies when \( z = 1 \). We use it to derive the formula for \( Z(t, q) := Z(t, q, 1) \).

It is necessary to state from the outset that the analysis presented in this note is based on experimentations with the formula for \( Z_N^N(t, q) \) using \textit{Mathematica} for finite \( N, N' \) and extrapolation of the found structures to infinite \( Ns \). Though somewhat loose in justification, the result looks convincing because it passes a number of consistency checks.

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## 2 The formula

As it is mentioned in the abstract, \( Z(t, q) \) will be expressed in terms of some standard special functions. We start with reviewing their definitions.

**The functions used in the formula** Recall that the \( q \)-Pochhammer symbol is an infinite product

\[
(t; q)_\infty := \prod_{n \geq 0} (1 - tq^n).
\]

It is used to write concisely the three-term identity

\[
\theta(t, q) = (1 - t^{-1})(q, q)_\infty (q/t, q)_\infty (qt, q)_\infty
\]

for the theta function

\[
\theta(q, t) := \sum_{n \in \mathbb{Z}} (-1)^n q^{-n(n+1)/2} t^n.
\]

Another ingredient of the formula are theta functions with characteristics:

\[
\epsilon_k(t, q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{7n(n+1)}{2} + kn} t^{7n+k} = t^k \left(1 - q^{-k} t^{-7}\right) \left(q^7, q^7\right)_\infty \left(q^{7-k} t^{-7}, q^7\right)_\infty \left(q^{k+7}, q^7\right)_\infty, k = 0, \ldots, 6.
\]

The proposed answer will also depend on the Eisenstein series

\[
E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3q^n}{1 - q^n}.
\]

Define linear combinations of \( \epsilon_k(q, t) \):

\[
\theta_1(t, q) := -q^3 \epsilon_3(q, t) - q^4 \epsilon_4(q, t),
\]

\[
\theta_2(t, q) := q^2 \epsilon_2(q, t) + q^5 \epsilon_5(q, t), \quad \text{and}
\]

\[
\theta_3(t, q) := -q \epsilon_1(q, t) - q^6 \epsilon_6(q, t)
\]

and introduce abbreviation \( \theta_k(q) := \theta_k(1, q), k = 1, 2, 3. \)

The following conjecture contains the promised formula.
Conjecture 1 The partition function $Z(t, q)$ has the form

$$Z(t, q) = \frac{a(q)\theta_1(t, q) + b(q)\theta_2(t, q) + c(q)\theta_3(t, q)}{t^6\theta(t, q)^{11}} =: \Psi(t, q)$$

where the string functions $a, b, c$ are

$$a := \frac{1}{768(q, q)_{\infty}}(2744q^2E_2(q)\theta_3(q)\theta'_2(q) - 2744q^2E_2(q)\theta_2(q)\theta'_3(q) - 343qE_2(q)^2\theta_3(q)\theta'_2(q) + 3626qE_2(q)\theta_3(q)\theta''_2(q)$$

$$+ 343qE_2(q)^2\theta_2(q)\theta''_3(q) - 5194qE_2(q)\theta_2(q)\theta'_3(q) + 98E_2(q)^2\theta_2(q)\theta_3(q) - 476E_2(q)\theta_2(q)\theta_3(q)$$

$$+ 42E_4(q)\theta_3(q)\theta'_2(q) - 42E_4(q)\theta_2(q)\theta'_3(q) - 12E_4(q)\theta_2(q)\theta_3(q) - 6586q^3\theta''_2(q)\theta'_3(q)$$

$$+ 6586q^3\theta'_2(q)\theta''_3(q) - 29400q^2\theta_3(q)\theta''_2(q) + 37632q^2\theta'_2(q)\theta''_3(q) + 10584q^2\theta_2(q)\theta''_3(q)$$

$$- 25725q\theta_3(q)\theta'_2(q) + 18333q\theta_2(q)\theta'_3(q) + 1350\theta_2(q)\theta_3(q)),$$

$$b := -\frac{1}{768(q, q)_{\infty}}(2744q^2E_2(q)\theta_3(q)\theta''_2(q) - 2744q^2E_2(q)\theta_1(q)\theta''_3(q) - 343qE_2(q)^2\theta_3(q)\theta'_2(q) + 2842qE_2(q)\theta_3(q)\theta'_1(q)$$

$$+ 343qE_2(q)^2\theta_1(q)\theta'_3(q) - 5194qE_2(q)\theta_1(q)\theta''_3(q) + 147E_2(q)^2\theta_1(q)\theta_3(q) - 546E_2(q)\theta_1(q)\theta_3(q)$$

$$+ 42E_4(q)\theta_3(q)\theta'_2(q) - 42E_4(q)\theta_2(q)\theta'_3(q) - 18E_4(q)\theta_1(q)\theta_3(q) - 6586q^3\theta''_1(q)\theta'_3(q)$$

$$+ 6586q^3\theta'_1(q)\theta''_3(q) - 29400q^2\theta_3(q)\theta''_1(q) + 56448q^2\theta'_1(q)\theta''_3(q) + 1176q^2\theta_1(q)\theta''_3(q)$$

$$- 17325q\theta_3(q)\theta'_1(q) + 2205q\theta_1(q)\theta'_3(q) + 225\theta_1(q)\theta_3(q)),$$

$$c := \frac{1}{768(q, q)_{\infty}}(2744q^2E_2(q)\theta_2(q)\theta''_1(q) - 2744q^2E_2(q)\theta_1(q)\theta''_2(q) - 343qE_2(q)^2\theta_2(q)\theta'_1(q) + 2842qE_2(q)\theta_2(q)\theta'_2(q)$$

$$+ 343qE_2(q)^2\theta_1(q)\theta'_2(q) - 3626qE_2(q)\theta_1(q)\theta''_2(q) + 49E_2(q)^2\theta_1(q)\theta_2(q) - 70E_2(q)\theta_1(q)\theta_2(q)$$

$$+ 42E_4(q)\theta_2(q)\theta'_1(q) - 42E_4(q)\theta_1(q)\theta'_2(q) - 6E_4(q)\theta_1(q)\theta_2(q) - 6586q^3\theta''_1(q)\theta'_2(q)$$

$$+ 6586q^3\theta'_1(q)\theta''_2(q) - 10584q^2\theta_2(q)\theta''_1(q) + 18816q^2\theta'_1(q)\theta''_2(q) + 1176q^2\theta_1(q)\theta''_2(q)$$

$$- 9261q\theta_2(q)\theta'_1(q) + 1533q\theta_1(q)\theta'_2(q) + 27\theta_1(q)\theta_2(q)).$$

3 Supporting evidences

The matrix product presentation for $Z(t, q)$ It was established in [7] that $Z(t, q)$ is the limit in the sense of formal power series convergence of a certain infinite matrix product. To state the result let us fix some additional notations:

$$B^3_0 := \frac{1 + 3t + t^2}{(1 - t)^3(1 - qt)^2},$$

$$A^3_0 := \frac{1 + 5t + 5t^2 + t^3}{(1 - t)^{11}},$$

$$K(t, q) := \begin{pmatrix}
\frac{t(t^2+3t+1)}{(t-1)^2(qt-1)} & \frac{(t^2+3t+1)(t^3+q^2)-5q(t+1)t^2}{q^2t(t-1)(qt-1)} \\
\frac{t(t+1)(t^2+4t+1)}{(t-1)^{10}} & \frac{(t^3+5t^2+5t+1)(t^3+q^2)-q(5t^2+14t+5)t^2}{q^2(t-1)^{10}}
\end{pmatrix}
$$
\[
\begin{pmatrix}
B_0^{r+1}
A_0^r
\end{pmatrix} := K(q^r t, q) \cdots K(q t, q) \begin{pmatrix}
B_0^1
A_0^0
\end{pmatrix},
\]
(10)

\[
A_N^{N'}(t, q) := A_0^{N'-N}(t q^N, q).
\]

It was verified in [7] that the limit of

\[
Z_N^{N'}(t, q) := A_N^{N'}(t, q) t^{4-4N} q^{-2+4N-2N^2}, N < 0
\]
(11)

\[
N \to -\infty, N' \to \infty
\]
coincides with \(Z(t, q)\) \(^2\).

**Poles of \(Z(t, q)\)** The rational function \(Z_N^{N'}(t, q)\) has a fairly complicated structure. Still, experiments with Mathematica show that \(Z_N^{N'}(t, q)\) has poles of multiplicity \(\dim \mathcal{C} = 11\) precisely at \(q^{-N}, \ldots, q^{-N'}\). It is natural to conjecture that in the limit \(N \to -\infty, N' \to \infty\) this pattern persists and \(Z\) is a meromorphic function for \(|q| < 1\) with poles at \(t = q^n, n \in \mathbb{Z}\) of multiplicity 11.

**\(Z(t, q)\) and a line bundle of degree 7** If this conjecture is true, then the product

\[
\Theta(t, q) := t^6 Z(t, q) \theta(t, q)^{11}
\]
(12)

is an analytic function for \(t \neq 0, |q| < 1\).

One of the results of [7] is that \(Z(t, q, g)\) as a formal power series in \(q\) satisfies

\[
\frac{Z(q t, q, g)}{Z(t, q, g)} = \frac{t^4}{q^2},
\]
(13)

\[
\frac{Z(1/t, q, g^{-1})}{Z(t, q, g)} = -t^8.
\]
(14)

It follows from the functional equations

\[
\theta(q t, q) = -\theta(t, q)/(qt), \quad \theta(1/t, q) = -t\theta(t, q)
\]
(15)

that \(\Theta(t, q)\) obeys

\[
\frac{\Theta(tq, q)}{\Theta(t, q)} = -\frac{1}{q^2 t^2},
\]
(16)

\[
\frac{\Theta(1/t, q)}{\Theta(t, q)} = t^7
\]
(17)

Stated differently, equation (15) says that \(\theta\) is a holomorphic section of a line bundle \(\mathcal{L}\) of degree one on the elliptic curve \(\mathbb{C}^\times/\{q^k\}\). To say that \(\Theta\) satisfy (16) is equivalent to saying that \(\Theta\) is a section of \(\mathcal{L}^\otimes 7\). The space of global sections of \(\mathcal{L}^\otimes 7\) has a basis \(\epsilon_k, k = 0, \ldots, 6\) \(^4\). Symmetry condition (17) determines a subspace in the span of \(\epsilon_k\) of dimension three with a basis \(\theta_i, i = 1, 2, 3\). As a consequence, we get

\[
\Theta(t, q) = a(q) \theta_1(t, q) + b(q) \theta_2(t, q) + c(q) \theta_3(t, q),
\]
(18)

which is equivalent to (5).
Equations on coefficients $a, b, c$ It remains to determine $a, b, c$. Denote the right-hand side in (5) by $\Psi(t, q)$. Identity
\[
\lim_{t \to 1} \partial_t^k Z(t, q)(1 - t)^{11} = \lim_{t \to 1} \partial_t^k \Psi(t, q)(1 - t)^{11}, \quad k = k_1, k_2, k_3
\] (19) produces three linear equations for $a(q), b(q), c(q)$. Denote by $x$ the vector $(a, b, c)$. We can write the system of equations on $x$ in the matrix form
\[
z' = Ax.
\] (20)
The coefficients of the matrix $A$ depend only on the functions $\theta_i$ and can be explicitly computed. The vector $z$ is more complicated because it depends on the unknown function $Z$. But if we manage to compute three Taylor coefficients of $Z(t, q)$ with respect to the variable $t$, we can easily find the functions $a, b, c$.

Coefficients of the matrix $A$ Let us find the matrix $A$ first. The convenient choice for $k_i$ in (19) is $0, 2, 4$. The function $\theta(t, q)$ has a zero of order one at $t = 1$ so that $\frac{\theta(t, q)}{t - 1}$ is regular. Denote
\[
\theta_0(q) := \lim_{t \to 1} \frac{\theta(t, q)}{1 - t}.
\] The right-hand-side of fist equation Eq\(_0\) (19), the $k = 0$ case, becomes
\[
\lim_{t \to 1} \Psi(t, q)(1 - t)^{11} = \frac{a \theta_0 + b \theta_2 + c \theta_3}{\theta_0^3}.
\] (21) The second equation Eq\(_2\), corresponding to $k = 2\begin{cases} 1 \end{cases}$, contain $t$-partial derivatives of functions at $t = 1$. Note that $\theta$ and $\epsilon_k$ satisfy the heat equations:
\[
t^2 \partial_t^2 \theta + t \partial_t \theta = 2q \partial_q \theta, \quad t^2 \partial_t^2 \epsilon_k + 8t \partial_t \epsilon_k - k(k + 7)\epsilon_k = 14q \partial_q \epsilon_k.
\] In addition, linear combinations $\theta_i$ of $\epsilon_k$ satisfy (17). These equations allow to express $t$-derivatives of $\theta_i$ at $t = 1$ in terms of $q$-derivatives of $\theta_i$. Thus equations [3] and the results of [1] imply that
\[
\theta_0(q) = -(q, q)^3, \quad \lim_{t \to 1} \partial_t^3 \frac{\theta(t, q)}{1 - t} = -\theta_0(q), \quad \lim_{t \to 1} \partial_t^2 \frac{\theta(t, q)}{1 - t} = \theta_0 \times \left(\frac{23}{12} + \frac{E_2}{12}\right),
\]
\[
\lim_{t \to 1} \partial_t^3 \frac{\theta(t, q)}{1 - t} = \theta_0 \times \left(\frac{11}{2} - \frac{1}{2} E_2\right),
\]
\[
\lim_{t \to 1} \partial_t^4 \frac{\theta(t, q)}{1 - t} = \theta_0 \times \left(\frac{1689}{80} + \frac{23}{8} E_2 + \frac{1}{48} E_2^2 - \frac{1}{120} E_4\right),
\]
\[
\lim_{t \to 1} \partial_t^5 \frac{\theta(t, q)}{1 - t} = \theta_0 \times \left(-\frac{1627}{16} - \frac{145}{8} E_2 - \frac{5}{16} E_2^2 + \frac{1}{8} E_4\right).
\] Similarly
\[
\theta_1(q) := \theta_1(1, q) = 2(q^3, q^7) \infty(q^4, q^7) \infty(q^7, q^7) \infty, \quad \partial_t \theta_1(1, q) = -7/2 \theta_1(q), \quad \partial_t^2 \theta_1(1, q) = 16 \theta_1(q) + 14 q \partial_q \theta_1(q),
\]

\[1\text{Eq}_1\] is proportional to Eq\(_0\)
\[ \partial^3_t \theta_1(1, q) = -90\theta_1 - 189q\partial_q \theta_1, \quad \partial^4_t \theta_1(1, q) = 600\theta_1 + 2268q\partial_q \theta_1 + 196q^2 \partial^2_q \theta_1, \]

\[ \theta_2(q) := \theta_2(1, q) = -2(q^2, q^7)_\infty(q^5, q^7)_\infty(q^7, q^7)_\infty, \quad \partial_t \theta_2(1, q) = -7/2\theta_2, \quad \partial^2_t \theta_2(1, q) = 18\theta_2 + 14q\partial_q \theta_2, \]

\[ \partial^3_t \theta_2(1, q) = -1170\theta_2 - 189q\partial_q \theta_2, \quad \partial^4_t \theta_2(1, q) = 900\theta_2 + 2324q\partial_q \theta_2 + 196q^2 \partial^2_q \theta_2, \]

\[ \theta_3(q) := \theta_3(1, q) = 2(q^2, q^7)_\infty(q^6, q^7)_\infty(q^7, q^7)_\infty, \quad \partial_t \theta_3(1, q) = -7/2\theta_3, \quad \partial^2_t \theta_3(1, q) = 22\theta_3 + 14q\partial_q \theta_3, \]

\[ \partial^3_t \theta_3(1, q) = -1710\theta_3 - 189q\partial_q \theta_3, \quad \partial^4_t \theta_3(1, q) = 1524\theta_3 + 2436q\partial_q \theta_3 + 196q^2 \partial^2_q \theta_3. \]

After simplifications Eq2 and Eq4 become

\[ \lim_{t \to 1} \partial^2_t \Psi(t, q)(1 - t)^{11} = \frac{1}{12} \theta_1^1 \times \]

\[ \times (11E_2 \theta_1 - 168q\theta'_1 - 23\theta_1) \alpha + (11E_2 \theta_2 - 168q\theta'_2 - 47\theta_2) \beta + (11E_2 \theta_3 - 168q\theta'_3 - 95\theta_3) \gamma, \]

\[ \lim_{t \to 1} \partial^4_t \Psi(t, q)(1 - t)^{11} = \frac{1}{2400^11} \times \]

\[ \times ((-18480qE_2 \theta'_1 + 605E_2^2 \theta_1 - 990E_2 \theta_1 + 22E_4 \theta_1 + 47040q^2 \theta''_1 + 58800q\theta'_1 + 363\theta_1) \alpha \]

\[ (-18480qE_2 \theta'_2 + 605E_2^2 \theta_2 - 3630E_2 \theta_2 + 22E_4 \theta_2 + 47040q^2 \theta''_2 + 72240q\theta'_2 + 3003\theta_2) \beta \]

\[ (-18480qE_2 \theta'_3 + 605E_2^2 \theta_3 - 8910E_2 \theta_3 + 22E_4 \theta_3 + 47040q^2 \theta''_3 + 99120q\theta'_3 + 14043\theta_3) \gamma. \]

The matrix A consists of coefficients of a, b, c from equations (21)2223).

The vector z components Computation of the vector z from (20) relies on the extrapolation of the results obtained with Mathematica. We use that Z(t, q) has presentation as a limit of (11), which makes it possible computation of the q-series limt→1 \( \partial^k_t Z(t, q)(1 - t)^{11} \) with arbitrary precision. Due to integrality of the coefficients, limt→1 \( \partial^k_t Z_N(t, q)(1 - t)^{11} \mod q^r \) stabilizes for sufficiently large N and N'. The first computation shows that

\[ \frac{1}{12} \lim_{t \to 1} Z(t, q)(1 - t)^{11} = \]

\[ = 1 + 22q + 275q^2 + 2530q^3 + 18975q^4 + 124230q^5 + 702328q^6 + 3661900q^7 + 17627775q^8 + \ldots O(q^{20}). \]

The database [http://oeis.org](http://oeis.org) hints that these series is the generating function for the sequence [A023020](http://oeis.org/A023020). \( a_n \) is the number of partitions of n into parts of 22 kinds. The generating function coincides with the Taylor expansion of \( \frac{1}{q^2 q^7}. \)

Another computation with Mathematica shows that

\[ \frac{1}{98} (q, q)^{22} \lim_{t \to 1} \partial^2_t Z(t, q)(1 - t)^{11} = 1/6 + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \ldots O(q^{20}). \]

If we drop the term 1/6, the sequence \( a_n \) of Taylor coefficients of the remaining series is [A000203](http://oeis.org/A000203). \( a(n) = \sigma(n) \), the sum of the divisors of n. The generating function for \( a_n \) is \( (1 - E_2(q))/24. \) The same way we find that

\[ (q, q)^{22} \lim_{t \to 1} \partial^4_t Z(t, q)(1 - t)^{11} = \frac{42}{5} - 12E_2(q) + 4E_2^2(q) - \frac{2}{3} E_4 + O(q^{20}). \]
To summarize, the vector $z$ in (20) is conjecturally equal
\[
\left( \frac{12}{(q,q)_{\infty}^{22}}, \frac{20 - 4E_2(q)}{(q,q)_{\infty}^{22}}, \frac{42}{(q,q)_{\infty}^{22}} - 12E_2(q) + 4E_2^2(q) - \frac{2}{3}E_4 \right)
\]

The matrix $A$ has the determinant
\[
\Delta = \frac{8\tilde{\Delta}}{\theta_0^{33}}
\]
\[
\tilde{\Delta} = (-343q^3\theta_3\theta_1\theta_0' + 343q^3\theta_2\theta_0''\theta_3' + 343q^3\theta_3\theta_0'\theta_0'' - 343q^3\theta_2\theta_1\theta_0'\theta_3' + 343q^3\theta_1\theta_0'\theta_0'' + 98q^2\theta_2\theta_3\theta_0'' + 98q^2\theta_2\theta_1\theta_0'\theta_3' + 147q^2\theta_1\theta_0'\theta_0'' + 196q^2\theta_1\theta_2\theta_0'') + 49q^2\theta_1\theta_2\theta_0'' + 42q_2\theta_3\theta_0' + 126q_1\theta_3\theta_0' + 84q_1\theta_2\theta_0' + 6q_1\theta_2\theta_0'').
\]

It is not hard to check with Mathematica that
\[
\tilde{\Delta} = -48(q,q)_{\infty}^{15} + O(q^{300}).
\]

We conjecture that this is an exact equality. We use it and the Kramer’s rule to derive from (20) the formulas (6, 7, 8).

## 4 Some consistency checks

Note that by construction the function $\Psi$ (5) satisfies equations (13, 14). $Z(t, q)$ is the solution of the same set of equations. The $q$-expansion of $\Psi$ is

\[
\Psi(t, q) = -\frac{t^3 + 5t^2 + 5t + 1}{(t - 1)^{11}} - \frac{q(46t^3 + 86t^2 + 86t + 46)}{(t - 1)^{11}} + \frac{q^2(t^{11} - 11t^{10} + 55t^9 - 181t^8 - 567t^7 - 947t^6 - 947t^5 - 567t^4 - 181t^3 + 55t^2 - 11t + 1)}{(t - 1)^{11}t^4}
\]
\[
+ \frac{2q^3(8t^{13} - 65t^{12} + 195t^{11} - 143t^{10} - 1011t^9 - 2657t^8 - 3917t^7 - 3917t^6 - 2657t^5 - 1011t^4 - 143t^3 + 195t^2 - 65t + 1)}{(t - 1)^{11}t^5}
\]
\[
- \frac{q^4(-126t^{15} + 794t^{14} - 1491t^{13} - 559t^{12} + 3597t^{11} + 18745t^{10} + 38767t^9 + 54123t^8 + 54123t^7 + 38767t^6 + 18745t^5 + 336t^{17} + 1662t^{16} + 1810t^{15} + 2173t^{14} + 1337t^{13} - 21131t^{12} - 65159t^{11} - 122387t^{10} - 162607t^9 - 162607t^8 - 122387t^7 - 162607t^6 - 162607t^5 + 122387t^4 + 162607t^3 + 162607t^2 + 122387t - 1)}{(t - 1)^{11}t^7} + O(q^6)
\]

It agrees with the expansion from (2).

Another interesting consistency check gives comparison of the functions $Z(t, q)$ and $\Psi(t, q)$ at $t = -1$. Literal comparison is not very fruitful because by virtue of (17). $Z(-1, q) = \Psi(-1, q) = 0$. To get a nonzero
result, we used Mathematica to compute the derivatives $\partial_t Z(-1, q)$ and $\partial_t \Psi(-1, q)$. Two values agree by giving

$$-1024(q, q)^{22}_{\infty} \partial_t Z(-1, q) = -1024(q, q)^{22}_{\infty} \partial_t \Psi(-1, q) = 1 - 48q + 1104q^2 - 16192q^3 + 170064q^4 - 1362336q^5 + 8662720q^6 - 44981376q^7 + 195082320q^8 + O(q^9).$$

(24)

The coefficients $a_n$ (up to a sign) coincide with the sequence \texttt{A000156}. $a_n$, according to the database, is the number of ways of writing $n$ as a sum of 24 squares. This is why it is very plausible that the series (24) is the expansion of

$$\left( \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \right)^{24} = (1 - q)^{24}(q, q^2)^{24}(q^2, q^3)^{24}(q^3, q^4)^{24}.$$ 

5 Functions $\theta_k$

$\theta_k, k = 1, 2, 3$ are closely related to Rogers-Selberg functions (see e.g. \cite{8}, \cite{3}). They satisfy

$$A(q) := \sum_{n \geq 0} \frac{q^{2n^2}}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2n})(1 + q)(1 + q^2) \cdots (1 + q^{2n})} = \frac{(q^3, q^7)_{\infty}(q^4, q^7)_{\infty}(q^7, q^7)_{\infty}}{(q^2, q^2)_{\infty}} = \frac{\theta_1}{2(q^2, q^2)_{\infty}},$$

$$B(q) := \sum_{n \geq 0} \frac{q^{2n^2 + 2n}}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2n})(1 + q)(1 + q^2) \cdots (1 + q^{2n})} = \frac{(q^2, q^7)_{\infty}(q^5, q^7)_{\infty}(q^7, q^7)_{\infty}}{(q^2, q^2)_{\infty}} = -\frac{\theta_2}{2(q^2, q^2)_{\infty}},$$

$$C(q) := \sum_{n \geq 0} \frac{q^{2n^2 + 2n}}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2n})(1 + q)(1 + q^2) \cdots (1 + q^{2n+1})} = \frac{(q^4, q^7)_{\infty}(q^6, q^7)_{\infty}(q^7, q^7)_{\infty}}{(q^2, q^2)_{\infty}} = \frac{\theta_3}{2(q^2, q^2)_{\infty}}.$$ 

For a list known identities these functions obey see \cite{6}.

6 $Z(t, q, g)$ in the general case

Some of the above arguments extend to $Z(t, q, g)$. In particular the divisor of poles of $Z(t, q, g)$ satisfies equations

$$1 - q^{4i} tv_\alpha(z) = 0$$

where $v_\alpha(z)$ are the weights of the spinor representation. In order to be more explicit recall (see see e.g. \cite{7}) that coordinates $\lambda^\alpha$ on the spinor representation of Spin(10) can be labelled by elements of the set

$$\alpha \in \mathbb{E} := \{(0), (ij), (k)| 1 \leq i < j \leq 5, 1 \leq k \leq 5\}$$

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Let $\widetilde{T}^5$ be the two-sheeted cover of the maximal torus $T^5 \subset \text{SO}(10)$ and $z = (z_1, \ldots, z_5)$ be the image of $g$ under projection $T^5 \to T^5$. The action $\Pi$ of $g \in \widetilde{T}^5$ on $\lambda^8$ is given by the formula

$$\Pi(g)\lambda^8 v_{\alpha}(z)\lambda^8,$$

or in more details:

$$\Pi(g)\lambda^{(0)} = \det^{-\frac{1}{2}}(z)\lambda^{(0)},$$
$$\Pi(g)\lambda^{(ij)} = \det^{-\frac{1}{2}}(z)z_i z_j \lambda^{(ij)},$$
$$\Pi(g)\lambda^{(k)} = \det^{\frac{1}{2}}(z)z_k^{-1} \lambda^{(k)},$$
$$\det^{\frac{1}{2}}(z) = \sqrt{z_1 \cdots z_5}.$$

(25)

Introduce the product

$$\theta_{E}(t, q, z) := \prod_{\alpha \in E} \theta(tv_{\alpha}(z), q)$$

If the conjecture about the structure of the poles of $Z(t, q, z)$ is correct, then

$$\Xi(t, q, z) := t^6 Z(t, q, z)\theta_{E}(t, q, z)$$

is an analytic function for $|q| < 1$ and $t, z \in \mathbb{C} \times \widetilde{T}^5$. Equations (13) (15) imply that $\Theta(t, q, z)$ satisfies

$$\Xi(qt, q, z) = \frac{1}{t^{12}q^{12}}\Xi(t, q, z)$$

That is $\Xi(t, q, z)$ is a section of $L^\otimes 12$. Let us fix a basis

$$\eta_k(t, q) = \sum_{n \in \mathbb{Z}} q^{\frac{12n(n+1)}{2}+kn} t^{12n+k}, k = 0, \ldots, 11$$

in the space of global sections of $L^\otimes 12$. Functions $\eta_k$ satisfy

$$\eta_k(qt, q) = 1/(tq)^{12}\eta_k(t, q), \quad \eta_k(1/t, q) = q^{12-2k}t^{12}\eta_{12-k}(t, q), \quad \eta_k(t, q) = q^{k+12}\eta_{k+12}(t, q)$$

The function $\Xi(t, q, z)$ is a linear combination

$$\Xi(t, q, z) = \sum_{k=0}^{11} c_k(q, z)\eta_k(t, q).$$

(17) implies that

$$c_k(q, z) = -q^{12-2k}c_{12-k}(q, z^{-1}), k \neq 0, 6,$$
$$c_0(q, z) = -c_6(q, z^{-1}), \quad c_6(q, z) = -c_0(q, z^{-1})$$

Determination of the coefficients $c_i(q, z)$ is more difficult than in the case $z = 1$ and will be postponed for the future publications.

Note that after specialization $g = 1 \Xi(t, q, 1) = \Theta(t, q)\theta(t, q)^5$ and $\theta_{E}(t, q, 1) = \theta(t, q)^{16}$ giving as a fraction the function $Z(t, q)$. 

9
7 Concluding remarks

Partition function $Z_{X}$ for $X = Q$ a smooth affine quadric of dimension $n - 1$ is known [1]:

$$Z_{Q}(t, q) = \frac{1 - t^2 (qt^2, q)_{\infty}(qt^{-2}, q)_{\infty}}{(1 - t)^n (qt, q)_{\infty}^n(qt^{-1}, q)_{\infty}^n}$$

It satisfies

$$\frac{Z_{Q}(qt, q)}{Z_{Q}(t, q)} = (-1)^n t^{n-4} q^{-1},$$

$$\frac{Z_{Q}(t^{-1}, q)}{Z_{Q}(t, q)} = -(-t)^{n-2}. \quad (26)$$

Functions $Z_{C}(t, q), Z_{Q}(t, q)$ have some common features. In both cases $\lim_{t \to 1} Z_{X}(t, q)(1-t)^{\dim X} = c_{X} \frac{1}{(q, q)^{\dim X}},$

where $c_{X}$ is some constant. Functions $Z_{X}(t, q)$ have poles of multiplicity $\dim X$ at points $\{q^{k}\}$.

To further extend the analogy we need to digress. The spaces of polynomial maps $\mathbb{C} \to X$ of degree $N$ is a cone over the space of Drinfeld’s quasimaps $QMaps_{N}(X)$ to projectivization of $X$. The space $QMaps_{N}(X)$ is not smooth but still has a well defined line bundle of algebraic volume forms $K$. $K^{*} = O(a(X) + Nb(X))$ in for some constants $a(X), b(X)$. As usual $O(n)$ is the power of the tautological line bundle. Exponents of $t$ in (13) and (26) coincide with $b(C)$ and $b(Q)$. Thus functions $\Theta_{C}(t, q)$ and $\Theta_{Q}(t, q)$ are sections of $L^{\dim C - a(C)} = L^{7}$ and $L^{\dim Q - a(Q)} = L^{3}$ respectively. Finally $P_{X}(t) = \lim_{q \to 0} Z_{X}(t, q)$ is the classical Poincaré series of the algebra of homogeneous functions on $X$.

It is tempting to say that this is a common features of an elliptic generalization of Poincaré series of an algebra of functions $X$ that should exist for a class of conical varieties whose members are $C$ and $Q$. This class contains in the class of local conical Calabi-Yao varieties $X$, whose base $B(X)$ is Fano of sufficiently large index. In this generalization

$$Z_{X}(t, q) = \frac{\Theta_{X}(t, q)}{t^{l(X)\theta}(t, q)^{\dim X}}.$$

$l(X) \in \mathbb{Z}^{>0}$, $\Theta_{X}$ is a section of $L^{\dim X - a(X)}$. The denominator in this formula is similar to the denominator in the Kac formula for the character of an integrable representations of an affine Lie algebra $\hat{g}$ at a positive level.

It appears that the first nontrivial Laurent coefficients of $Z_{Q_{n}}(t, q)$ for a quadric at $t = 1$ can be expressed through algebraic combinations of $(q, q)_{\infty}, E_{2}, E_{4}$ and probably $E_{6}$. The formulas are similar to the ones that have already appeared in Section [3]

$$\lim_{t \to 1} Z_{Q_{n}}(t, q)(1-t)^{n-1} = 2(q, q)^{2-2n}$$

$$\lim_{t \to 1} \partial_{t}(Z_{Q_{n}}(t, q)(1-t)^{n-1}) = (q, q)^{2-2n}$$

$$\lim_{t \to 1} \partial_{t}^{2}(Z_{Q_{n}}(t, q)(1-t)^{n-1}) = (q, q)^{2-2n} \frac{(n-4)(1-E_{2})}{6}$$

$$\lim_{t \to 1} \partial_{t}^{3}(Z_{Q_{n}}(t, q)(1-t)^{n-1}) = (q, q)^{2-2n} \frac{(4-n)(1-E_{2})}{4}$$

$$\lim_{t \to 1} \partial_{t}^{4}(Z_{Q_{n}}(t, q)(1-t)^{n-1}) = (q, q)^{2-2n} \left( \frac{(n-4)^{2}E_{2}^{2}}{24} - \frac{(n-4)(6+E_{2})}{12} + \frac{(n-16)E_{4}}{60} + \frac{5n^{2} + 58n - 288}{120} \right)$$
Introduce an increasing multiplicative filtration on the algebra of function generated by $E_2, E_4, E_6$. The $n$-th derivative of $Z_{Q_n}(t,q)(1-t)^{n-1}$ at $t = 1$ is a multiple of $(q,q)^{-2\dim Q}_\infty$. The factor belongs to $n$ filtration space of the algebra. This parallels between $Z_{Q_n}$ and $Z_C$ suggests that this also holds for a more general $\mathcal{X}$. It raises a question whether coefficients of the $E_2, E_4, E_6$-monomials in the above formulas can be computed in terms of the characteristic classes of some bundles on the base of the cone $\mathcal{X}$.

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