The determination of residual stresses in a hardening elastoplastic sphere considering the temperature effects

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Abstract. Results of analytical and numerical simulation of cooling of the ideal elastoplastic and hardening sphere are giving in this article. Equations for calculation of the transient thermal fields which are common for both of models are described. Formulation of the mathematical models for solving the structural ideal elastoplastic and hardening problems are given. Analytical solution supplemented by numerical simulation with finite elements method in ANSYS Mechanical. During numerical simulation dependence of the radius of the elastoplastic boundary on the strain parameter $\chi$ and dependence of the magnitude of the strain parameter $\chi$ on the Poisson's ratio at which a repeated plastic flow of the material occurs were constructed.

There are a lot of works of different authors that devoted to the problem of determining the stress and strain state, as well as searching residual stress in a hollow sphere [1-4]. Accounting of temperature effects and the search of residual stresses are given in [5–7]. This paper solves the problem of determining residual stresses in the hardening elastoplastic sphere of radius $R$, that is under the influence of the temperature field dependent on the time $T(r,t)$. The case is considered when the sphere has a constant temperature $T_0$ and zero stresses at the initial moment, and the surface of that sphere cools down to zero temperature instantly and continues to maintain the same temperature. For example, this can be achieved by the immersion of the sphere in the fluid. The task is solved in spherical coordinates $(r, \phi, \theta)$. The following system of equations and relations is used in describing the symmetric deformation of the sphere:

- equilibrium equation

$$\frac{r}{2} \frac{d\sigma}{d\sigma} + \sigma_\phi - \sigma_\theta = 0$$ (1)

Here and below $\sigma_r, \sigma_\phi, \sigma_\theta$ are the components of the displacement tensor in the spherical coordinate system. In this context two of the three main stresses are equal to each other.

$$\sigma_\phi = \sigma_\theta$$

- Cauchy ratio for the components of total deformation
where $e_r, e_\phi$ are the components of the total strain tensor, $u$ is a component of the displacement vector in a spherical coordinate system.

- Hooke’s law for the components of the elastic deformation
  
  $$
  e_r^e = \frac{1}{E} (\sigma_r - \sigma_\phi) + \alpha (T - T_0),
  $$
  
  $$
  e_\phi^e = \frac{1}{2E} (\sigma_\phi - \sigma_r) + \alpha (T - T_0),
  $$
  
  where $E$ is the Young’s modulus, $\alpha$ is the coefficient of thermal expansion.

- Ishlinsky-Prager loading function [8], that has the form
  
  $$
  \left(\sigma_r - ce_r^p - \left(\sigma_\theta - ce_\theta^p\right)\right)^2 + \left(\sigma_\phi - ce_\phi^p - \left(\sigma_r - ce_r^p\right)\right)^2 + \left(\sigma_\theta - ce_\theta^p - \left(\sigma_\phi - ce_\phi^p\right)\right)^2 = 6k.
  $$
  
  where $e_r^p, e_\phi^p, e_\theta^p$ are the components of the plastic strain tensor, $c$ is the coefficient of hardening, $k$ is the yield stress.

- associated law of plastic flow
  
  $$
  de_r^p = d\lambda \left[4\left(\sigma_r - ce_r^p\right) - 2\left(\sigma_\theta - ce_\theta^p\right) - 2\left(\sigma_\phi - ce_\phi^p\right)\right],
  $$
  
  $$
  de_\phi^p = d\lambda \left[4\left(\sigma_\phi - ce_\phi^p\right) - 2\left(\sigma_r - ce_r^p\right) - 2\left(\sigma_\theta - ce_\theta^p\right)\right],
  $$
  
  $$
  de_\theta^p = d\lambda \left[4\left(\sigma_\theta - ce_\theta^p\right) - 2\left(\sigma_r - ce_r^p\right) - 2\left(\sigma_\phi - ce_\phi^p\right)\right],
  $$
  
  where $d\lambda$ is a scalar positive factor.

- the connection of complete deformations with plastic and elastic components
  
  $$
  e_r = e_r^p + e_r^e,
  $$
  
  $$
  e_\phi = e_\phi^p + e_\phi^e.
  $$

Temperature fields in this problem are determined by equations according to solution which are given in [5]

$$
T(r,t) = \frac{2RT_0}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left(\frac{n\pi r}{R}\right) e^{-\frac{a n^2 \pi^2 t}{R^2}},
$$

$$
\bar{T}(r,t) = 6T_0 \left(\frac{R}{\pi r}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2} \left(\cos \left(\frac{n\pi r}{R}\right) - \frac{R}{n\pi r} \sin \left(\frac{n\pi r}{R}\right)\right) e^{-\frac{a n^2 \pi^2 t}{R^2}},
$$

where $a$ is the thermal diffusivity coefficient.

The stress distribution throughout the elastic area is presented in the following form

$$
\sigma_r = \frac{2E\alpha}{3(1-\nu)} \left(c_2 - \frac{c_1}{r^2} - \bar{T}(r,t)\right),
$$

$$
\sigma_\phi = \frac{1}{3(1-\nu)} \left(2c_2 - \frac{c_1}{r^2} + \bar{T}(r,t) - 3\bar{T}(r,t)\right),
$$

(8)
where $\bar{T}(r,t)$ is the average temperature of a sphere of radius $r$, $c_2, c_3$ are the constants of integration, $\nu$ is a Poisson’s ratio (here and everywhere $\nu = 0.5$).

The sum of total deformations is determined by the following equation

$$e_r + 2e_{\phi} = 3\alpha(T - T_0)$$

If to substitute (2) into (9), it will have the following form

$$\frac{du}{dr} + 2\frac{u}{r} = 3\alpha(T - T_0)$$

The integration of this equation will have the following form, considering the condition $u(0, t) = 0$:

$$u(r, t) = \alpha r (\bar{T} - T_0)$$

The radial displacement $u(r, t)$, therefore, does not depend on the stress state and, consequently, on the occurrence of plastic zones. The found deformed state can be used to determine stress. The equation using the condition of symmetry (5) will have the form

$$d\sigma^p = d\lambda[4(\sigma_r - ce^p_{\phi}) - 4(\sigma_\phi - ce^p_{\phi})]$$

$$d\sigma^p = d\lambda[2(\sigma_r - ce^p_{\phi}) - 2(\sigma_\phi - ce^p_{\phi})]$$

$$d\sigma^p = d\lambda[2(\sigma_r - ce^p_{\phi}) - 2(\sigma_\phi - ce^p_{\phi})]$$

The solution to this equation will have the following form

$$e^p_{\phi} = -2e^p_{\phi}$$

If to substitute (12) into the plasticity condition (4), given the replacement $\sigma_s = 3^{1/2}k$, it will have the following form

$$|\sigma_\phi - \sigma_r - 3ce^p_{\phi}| = \sigma_s$$

The component of plastic deformations $e^p_{\phi}$ may be inferred from the equation of total deformations (6), Cauchy relations (2), and Hooke’s law (3)

$$e^p_{\phi} = -\frac{1}{6G}(\sigma_\phi - \sigma_\sigma),$$

where $G$ is the shear modulus.

If to substitute (14) into the equation (13), it will be obtained the equation of the stress difference in the plastic zone

$$\left(\sigma_\phi - \sigma_r\right) = \frac{2G}{c + 2G}\sigma_s$$

The equivalent stress $\bar{\sigma}$ is determined by the following equation, considering the hardening

$$\bar{\sigma} = \sigma_\phi - \sigma_r = \sigma_s + c(e^p_{\phi} - e^p_{\phi})$$

In an ideal plastic case, when $c = 0$, the equation (16) will have the following form. That ideal plastic case was described in [5].
The ratio that determines the equivalent stress can be obtained considering the ratio (8), (10), (15), from equation (16)

\[
\ddot{\sigma} = \left| \sigma_r - \sigma_p \right|
\]

The ratio (17) is valid as long as the equivalent stress increases or at least does not decrease at the considered point, that is when \( \frac{d}{dt}(\ddot{T} - T) \geq 0 \). If \( \frac{d}{dt}(\ddot{T} - T) < 0 \) the release comes and the stress changes follow Hooke’s law.

The point \( r = \rho \) will be indicated by the coordinate \( \rho(t) \) at which the plastic flow begins at time \( t \). The point \( r = \tau \) will be indicated by the coordinate \( \tau(t) \) at which the plastic flow begins at time \( t \). According to the equations \( \rho(t) \) and \( \tau(t) \) are determined by the relations

![Figure 1. The dependence of dimensionless \( \frac{\ddot{T} - T}{T_0} \) on the dimensionless radius \( \frac{at}{R^2} \).](image)
\[ T(\rho, t) - T(\rho, t) = \frac{\sigma_s}{3\alpha(2G+c)} \] (18)

and

\[ \frac{d}{dt}(\bar{T} - T) = 0 \] (19)

When the sphere cools, there will be three areas at the beginning: elastic area I, plastic area II, where the difference \( \bar{T} - T \) still continues to increase and finally, area III, where there is the release after the plastic flow that has taken place.

If to add release low to (17), it will have the following form for any \( t \):

\[
\begin{align*}
\overline{\sigma} &= 6G\alpha \left( \bar{T}(r,t) - T(r,t) \right) \quad \text{in area I}, \\
\sigma &= \frac{2G}{c+2G}\sigma_s \\
\sigma &= \frac{2G}{c+2G}\sigma_s - 6G\alpha \left( \bar{T}(r,t) - T(r,t) - \bar{T}(r,t) + T(r,t) \right) \\
\end{align*}
\] (20)

Now there is no difficulty in stress calculation. If to integrate the equilibrium equation (1)

\[ \frac{d\sigma_r}{d\bar{\sigma}} = \frac{2}{r} \left( \sigma_\varphi - \sigma_r \right) = \frac{2}{r} \bar{\sigma} \]

and consider the boundary condition \( \sigma_r = 0 \) for \( r = R \), then it’s possible to get:

\[ \sigma_r = -2\int_{x=0}^{x=1} \overline{\sigma}(x,t) \, dx, \] (21)

where \( \overline{\sigma} \) is determined by ratio (20). In particular, the residual stresses \( T = 0 \) as \( t \to \infty \) will be determined by the following formulas:

\[
\begin{align*}
\sigma_r^0 &= 12G\alpha \int_{r_0}^{R} \left( \bar{T}[x, \tau(x)] - T[x, \tau(x)] \right) \frac{dx}{x} - \frac{4G}{c+2G}\sigma_s \ln \frac{R}{\rho_0} \quad \text{if } r \leq \rho_0, \\
\sigma_r^0 &= 12G\alpha \int_{r}^{R} \left( \bar{T}[x, \tau(x)] - T[x, \tau(x)] \right) \frac{dx}{x} - \frac{4G}{c+2G}\sigma_s \ln \frac{R}{r} \quad \text{if } r \geq \rho_0
\end{align*}
\] (22)

To find the circumferential residual stress, the ratio \( \sigma_\varphi = \sigma_r + \overline{\sigma} \), will be used

\[
\begin{align*}
\sigma_\varphi^0 &= \sigma_r^0 \quad \text{if } r \leq \rho_0, \\
\sigma_\varphi^0 &= \sigma_r^0 + \frac{2G}{c+2G}\sigma_s - 6G\alpha \left( \bar{T}[r, \tau(r)] - T[r, \tau(r)] \right) \quad \text{if } r \geq \rho_0
\end{align*}
\] (23)

Thus, the core of the sphere \( r \leq \rho_0 \) will be in a state of comprehensive and uniform stretching after cooling.

Thus, the relations are obtained for determining of the residual temperature stresses in a sphere of linear hardening material. On the basis of the above results, dependencies can be obtained to
determine the radius of the elastoplastic boundary. The obtained solution is a generality of the ideal plastic case presented in [5], to which it is reduced at the coefficient \( c = 0 \).

The solution of this problem for the case of ideal plasticity, considering the hardening, was carried out by using the engineering analysis package, which is ANSYS Mechanica. The solution was carried out in a 2D axisymmetric formulation using Transient Thermal and Transient Structural modules. Geometrically, this task is reduced to a quarter of a circle with given symmetry conditions relative to the X axis and axisymmetric conditions relative to the Y axis in ANSYS. It was constructed a structured grid (figure 2) that contains 12,000 Quad4 elements and 36391 nodes. The thickness of the mesh elements of the radial direction is reduced on the outer contour where the plastic zone begins.

![Figure 2. Grid for the calculated area.](image)

There were gradually conducted the following studies in modeling: the study of non-stationary thermal state of the sphere using the Transient Thermal module and the combined study of the stress-strain state under the action of the resulting non-stationary temperature field in the Transient Structural module.

The following boundary conditions are set during the modeling:

- a constant temperature 0 is set on the outer contour of the sphere;
- at the beginning, the temperature of the sphere is equal at all points and is determined in accordance with the loading parameter \( \chi \);
- a model of kinematically hardening elastoplastic material is given and it should correspond to the Ishlinsky-Prager model. When the hardening coefficient is 0, this model corresponds to the ideal plastic case.

The input data for the simulation and material parameters are presented in table 1.

| Parameter                              | Value           |
|----------------------------------------|-----------------|
| Radius of the sphere, mm               | 20              |
| Simulation time, s                     | 25              |
| Young’s modulus, MPa                   | 200000          |
| Poisson ratio                          | 0.5             |
| Linear expansion coefficient, 1/K      | \( 1.2 \cdot 10^{-5} \) |
| Conductivity coefficient, W/(m·K)     | 43.255          |
| Heat capacity, J/(kg·K)                | 434             |
| Yield stress, MPa                      | 250             |
| Hardening factor, MPa                  | 12500           |
| The initial temperature of the sphere, K | \( \chi=0.1 \), \( T_0=520.83 \) |
|                                        | \( \chi=0.3 \), \( T_0=173.61 \) |
|                                        | \( \chi=0.5 \), \( T_0=104.17 \) |
|                                        | \( \chi=0.7 \), \( T_0=74.41 \) |
According to the calculation results, the dependence of the residual stress \( \frac{\sigma_r}{K}, \frac{\sigma_\theta}{K}, \frac{\sigma_\phi}{K} \) (figures 3–5) on the dimensionless radius of the sphere is obtained for various values of the strain parameter for the case of an ideal plastic material.

**Figure 3.** Dependence of \( \frac{\sigma_r}{K} \) on the radius at various values of \( \chi \) for the case of ideal plasticity.

**Figure 4.** Dependence of \( \frac{\sigma_\phi}{K} \) on the radius at various values of \( \chi \) for the case of ideal plasticity.
Figure 5. Dependence of $\frac{\sigma_{eq}}{K}$ on the radius at various values of $\chi$ for the case of ideal plasticity.

The obtained solution is in line with the results obtained by the analytical solution. It confirms the absence of re-plastic deformation if $1 > \chi > 0.5$. If the value of the strain parameter is $\chi < 0.5$, then repeated plastic deformation occurs, which is not considered in the solution presented in [5]. The area of constant equivalent stress at figure 5 is clearly highlighted on the outer part of the sphere at $\chi = 0.1$ (from radius 0.7 to 1) and $\chi = 0.3$ (from radius 0.96 to 1) and this corresponds to the repeated plastic flow of the material.

The dependence of the radius of the elastoplastic boundary on the strain parameter with a completely cooled sphere was also constructed. This graph is shown in figure 6.

Figure 6. Dependence of the radius of the elastoplastic boundary on the strain parameter $\chi$. 
Further modeling was carried out considering the hardening of the material. According to the calculation results, the dependence of the residual stresses \( \frac{\sigma_r}{K}, \frac{\sigma_\varphi}{K}, \frac{\sigma_\theta}{K} \) (figures 7-9) on the dimensionless radius of the sphere for different values of the strain parameter for kinematically hardening material was obtained. It should be noted that in accordance with table 1, the hardening coefficient is set as \( c = 12,500 \text{ MPa} \). This coefficient \( c \) gives the material a pronounced hardening, which is not typical for structural steels. However, such a coefficient allows to show the difference with the ideal plastic case.

**Figure 7.** Dependence of \( \frac{\sigma_r}{K} \) on the radius at various values of \( \chi \) for a hardening material.

**Figure 8.** Dependence of \( \frac{\sigma_\varphi}{K} \) on the radius at various values of \( \chi \) for a hardening material.
Figure 9. Dependence of $\frac{\sigma_{eq}}{K}$ on the radius at various values of $\chi$ for hardening material.

The equivalent residual stresses are less than in the case of ideal plasticity for a hardening material. It should be noted that there also were different studies on the effect of Poisson’s ratio on the stress-strain state of the sphere. The consideration of the compressibility of the material led to a decrease of the value of $\chi$ at which the plastic flow of the material occurred. The dependence of the strain parameter on the Poisson's ratio is shown in figure 10.

Figure 10. Dependence of the magnitude of the strain parameter $\chi$ on the Poisson's ratio at which a repeated plastic flow of the material occurs.
Thus, an analytical solution is obtained for the problem of cooling ideally elastoplastic and hardening spheres in this work. A comparison of analytical solutions and solutions based on the finite element method was held and it showed strong topology of the obtained solutions. In addition, the dependences of the radius of the elastoplastic boundary on the value of the strain parameter were obtained. The influence of the Poisson coefficient on the stress-strain state of the sphere was also investigated.

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