Momentum and spin dynamics of Dirac particles at effective dimensional reduction

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Abstract. We consider the dynamics of Dirac particles moving in the curved spaces of variable dimension interpolating smoothly between 3- and 2-dimensional spaces and considered as a toy model for 2-dimensional structures in solid state physics. Performing the Foldy-Wouthuysen (FW) transformation of Dirac equation and passing to the classical limit, we derive the equations of motion of momentum and spin. The spin precesses with the variable angular velocity and may “flick” appearing in the remnant 2-dimensional space twice during the period.

1. Introduction
Modern solid state physics intensively investigates (almost) two-dimensional structures: graphene, fullerene, carbon nanotubes, carbon nanobuds, etc. Properties of particles placed into such structures are usually described by considering quantum theory at two dimensions. However, there is no doubt that real space remains 3-dimensional which may lead to qualitative difference in some observables. This especially concerns the particle (electron) spin properties which are crucially different at two and three dimensions (see, e.g., Ref. [1]). To investigate this problem, we consider the toy model [2] of the curved space of variable dimension smoothly changing from two and three. We use the Dirac equation for a correct description of spin-1/2 particle motion in the curved space rather than for taking into account relativistic effects (which are negligible in condensed matter physics) although the latter are included in our approach “for free” allowing its further applications to the processes at Large Hadron Collider in the case [3] of variable (momentum) space dimension. We use the earlier developed method of the FW transformation to obtain the equations of motion for particle momentum and spin. In the present work, we focus our attention just on spin properties.

2. Hamiltonian for the metric admitting the effective dimensional reduction
We consider an example of an effective (because we actually remain in the 3-dimensional space) dimensional reduction for a Dirac particle in a curved spacetime using the metric [2]:
\[ ds^2 = dt^2 - \rho_1^2(z)d\Phi_1^2 - \rho_2^2(z)d\Phi_2^2 - \rho_3^2(z)dz^2, \] (1)
where $\rho_3(z) = 1 + \phi'(z)^2 + (\rho_3(z))^2$. Hereinafter, primes denote derivatives with respect to $z$. The $(3+1)$-dimensional manifold defining this metric is a hypersurface in a flat pseudo-Euclidean $(5+1)$-dimensional space.

The transverse part of the metric (if $z$ is assumed to be a longitudinal coordinate) has a structure of the Clifford torus which is the product of two unit circles in the $4$-dimensional space from the construction of theory of quasicrystals \cite{8}.

We consider Clifford tori as a toy model of condensed media. Note that they are used for analyzing twisted materials \cite{4} and vesicles \cite{5, 6, 7}. There is also some qualitative similarity to projection of a tube in a $6$-dimensional space onto a $3$-dimensional space which was used for construction of theory of quasicrystals \cite{8}.

Taking the limit $\rho_1(z) \to 0$ or the limit $\rho_2(z) \to 0$ may lead to the reduction \cite{9} of dimension of the physical space from $d = 3$ to $d = 2$.

We apply the Dirac equation for curvilinear coordinates:

\begin{equation}
(i\hbar \gamma^a D_a - mc)\Psi = 0, \quad a = 0, 1, 2, 3,
\end{equation}

where the Dirac matrices $\gamma^a$ are defined in local Lorentz frame defined by tetrad $e^a_\mu$. The spinor covariant derivatives are given by

\begin{equation}
D_a = e^a_\mu D_\mu, \quad D_\mu = \partial_\mu + \frac{i}{4} \sigma^{ab} \Gamma_{\mu ab}.
\end{equation}

Here $\Gamma_{\mu ab} = -\Gamma_{\mu ba}$ are the Lorentz connection coefficients, $\Psi$ is the four-component wave function, and $\sigma^{ab} = \frac{i}{2} \left( \gamma^a \gamma^b - \gamma^b \gamma^a \right)$.

To find the Hamiltonian form of Dirac equation, one can use the general formulas obtained in Refs. \cite{10, 11} or substitute the given metric into the general equation for the Hermitian Dirac Hamiltonian (Eq. (2.21) in Ref. \cite{12}). The Hermitian Dirac Hamiltonian for given metric (1) was first derived in Ref. \cite{11} and can be presented in the form

\begin{equation}
\mathcal{H}_D = \beta m - \frac{i}{\rho_1} \alpha_1 \frac{\partial}{\partial \Phi_1} - \frac{i}{\rho_2} \alpha_2 \frac{\partial}{\partial \Phi_2} - \frac{i}{2} \alpha_3 \left( \frac{1}{\rho_3} \frac{\partial}{\partial z_1} \right),
\end{equation}

where $\{ \ldots, \ldots \}$ is an anticommutator.

We transform it to the FW representation by the method elaborated in Ref. \cite{13} which was earlier applied to non-Minkovskian spacetimes in Refs. \cite{12, 14, 15}. After the exact FW transformation, we get the result

\begin{equation}
\mathcal{H}_{FW} = \beta \sqrt{a + \hbar \Sigma} \cdot \mathbf{b},
\end{equation}

where

\begin{equation}
a = m^2 + \frac{\rho_1^2}{\rho_1^2} + \frac{\rho_2^2}{\rho_2^2} + \frac{1}{4} \left\{ \frac{1}{\rho_3}, p_3 \right\}^2, \quad \mathbf{b} = \frac{\rho_2^2}{\rho_2^2 \rho_3} p_2 e_1 - \frac{\rho_1^2}{\rho_1^2 \rho_3} p_1 e_2,
\end{equation}

and $(p_1, p_2, p_3) = \left( -\hbar \frac{\partial}{\partial \Phi_1}, -\hbar \frac{\partial}{\partial \Phi_2}, -\hbar \frac{\partial}{\partial \Phi_2} \right)$ is the generalized momentum operator. For the given time-independent metric, the operators $\mathcal{H}_{FW}, p_1,$ and $p_2$ are integrals of motion.

If terms of order of $\hbar^2$ and higher orders in $\hbar$ are neglected, Hamiltonian (6) takes the form

\begin{equation}
\mathcal{H}_{FW} = \beta \sqrt{a + \frac{\hbar \Pi}{4}} \cdot \left\{ \mathbf{b}, \frac{1}{\sqrt{a}} \right\},
\end{equation}

where $\Pi = \beta \Sigma$ is the polarization operator.

It can be shown that the classical limit of this Hamiltonian coincides with the purely classical Hamiltonian of a spinning particle obtained from general Eq. (3.17) in Ref. \cite{12} for the given metric.
Dynamics of momentum and spin

The effect of the spin on the momentum dynamics is defined by the second term in Eq. (8) which usually is much smaller than the first one. In the classical limit, neglecting this effect leads to the following equation of longitudinal motion of the particle:

\[ v_z \equiv \frac{dz}{dt} = \frac{P_3}{E\rho_3^2} \pm \sqrt{\frac{E^2 - m^2 - P_1^2}{E\rho_3(z)}} - \frac{P_2^2}{E\rho_3(z)^2}, \tag{9} \]

where \( E \) is the energy which is conserved. Different signs correspond to two different directions of the longitudinal particle motion.

When we consider the nontrivial case of \( P_1 \neq 0, P_2 \neq 0 \), there exists the \( z_f \) point in which \( v_z = 0 \). It can be shown that the particle acceleration at this point is nonzero, except for a rather specific relation between \( \rho_1(z) \) and \( \rho_2(z) \). Therefore, the \( z_f \) point is a turning point.

In the classical limit, the equation of (average) spin motion obtained with Hamiltonian (8) is given by

\[ \frac{ds}{dt} = \Omega \times s, \tag{10} \]

where

\[ \Omega = \left( \frac{\rho_2'(z)}{\rho_2^2(z)\rho_3(z)}E P_2, -\frac{\rho_1'(z)}{\rho_1^2(z)\rho_3(z)}E P_1, 0 \right). \tag{11} \]

Hereinafter, \( \Omega \) is the angular velocity of spin rotation, \( s \) is the unit spin vector, and \((P_1, P_2, P_3)\) is the classical generalized momentum \((P_1 = const, P_2 = const)\). Dynamics of spin depends only on the particle motion along the longitudinal axis, \( z \).

Because \( \frac{ds}{dt} = v_z(z)\frac{dz}{dt} \), Eqs. (10),(11) define a solvable system of first order homogeneous linear differential equations.

As a simplest example, we consider the case when \( \rho_1(z) = K/z, \rho_2(z) = const \). In this case the angular velocity

\[ \Omega = \left( 0, \frac{P_1}{KE\sqrt{1 + K^2/z^2}}, 0 \right) \tag{12} \]

is almost constant if \( z \gg \sqrt{K} \). As a result, the spin components change as

\[ s_1(t) = s_\perp \cos \left( \frac{P_1}{KE} t + \phi \right), \quad s_2(t) = s_\parallel, \quad s_3(t) = s_\perp \sin \left( \frac{P_1}{KE} t + \phi \right), \quad (s_\parallel)^2 + (s_\perp)^2 = 1, \tag{13} \]

where \( s_\perp \) and \( s_\parallel \) are the spin projections onto the \( e_2 \) axis and \( e_1 e_3 \) subspace.

When the particle spin crosses the above plane, it becomes observable in the (2+1)-dimensional spacetime. In this spacetime, spin "flickering" takes place so that the imaginable observer residing in the respective plane meets for a short time the pseudovector (violating therefore the parity invariance) whose origin is completely unexplainable in terms of 2-dimensional space.
4. Summary
We considered the Dirac fermion motion in the model curved space of variable dimension. Applying FW transformation to the Dirac equation we concentrated on the spin dynamics in the (semi)classical limit. In the simplest case under consideration the average spin precesses with the almost constant velocity. It spends most of the time out of the 2-dimensional space remaining after the effective dimensional reduction and appears in this space twice during each precession period, to the great surprise of the observers residing in this space.

We hope that the development of this model may provide the way to take into account the spin degrees of freedom of electrons confined to 2-dimensional surfaces in condensed matter physics.

The spin dynamics in spaces of variable dimension may be also important for the description of polarization effects at Large Hadron Collider in the presence of extra dimensions.

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