The Role Of Dimensional Analysis In Teaching Physics

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Abstract

Dimensional analysis is a simple qualitative method for determining essential connections between physical quantities. It is applicable to a multitude of physical problems, many of which can be introduced early on in a university physics curriculum. Despite the relative simplicity of the approach, it is rarely included in physics curricula. Here, we apply the dimensional analysis to the problem of radiation power of an electric dipole. We show a straightforward way to derive employing dimensional analysis without need for complex mathematical treatments and physical expressions.
I. INTRODUCTION

Dimensional analysis is a method that works with physical quantities and their physical dimensions. The approach can be used to estimate typical values of various quantities, but also to verify presumed relations between elements of a theory. Dimensional analysis is rarely a subject of secondary school or university physics curricula. This is rather unfortunate, as dimensional analysis has an important place in the development of qualitative thinking. Price in his essay\(^1\) describes three fundamental steps of dimensional analysis. The first step is to define the physical model – the identification of all dependent and independent variables. The second one, the mathematical step is to learn the consequences of the physical model and the general principle that complete equations are independent of the choice of units. The calculation that follows yields a dimensionless variable. The final step is to interpret the dimensionless basis set in the light of observations or existing theory. Then they describe it as a practical, systematic, objective, and quick mathematic method. It can be demonstrated on an example of e.g. a simple pendulum.

According to Price, one has to break down the problem in qualitative terms in order to understand its physical nature and to be able to correctly choose the physical quantities that play an important role in physical processes of the given problem. Bao\(^2\) suggests that rigorous training of physical knowledge in high schools has made significant impact on student’s ability in solving physics problems, while such training does not seem to have direct effects on their general ability in scientific reasoning. Marusic\(^3\) shows that traditionally formulated numerical exercises, which are frequently used in teaching physics, do not develop student’s abilities in physical thinking to a desirable extent, and thus affect negatively one of the most important aims of teaching physics. Blasial\(^4\) states that comprehension in physics is a complex and multi-stage process. It is relatively easy to describe a phenomenon, but more difficult to explain it, whereas predicting it, which in fact is the crowning achievement of adequate comprehension, seems to be the most difficult task of all.

Many studies indicate that the understanding of physical phenomena is deeper if the interpretation is firstly qualitative, and describes the physical properties that are most important in a phenomenon. It is useful if a quantitative description in terms of formulas follows after such introduction. Taber\(^5\) and Ploetzner\(^6\) have found that "quantitative information is likely to be more easily integrated into qualitative problem representations than
the other way around”. In our opinion, dimensional analysis is a suitable intermediate step in solving more complex problems. However, the question of complexity is age-dependent. Dimensional analysis is particularly suitable in cases where the mathematical complexity of the problems to be solved is too high for the given age group of pupils or students. Also in Millikan Lecture 2009 Eisenkraft emphasizes, that we should develop teaching strategies in such a way so the physics would become available to all, because everyone deserves an opportunity to reflect on the wondrous workings of our universe. Dimensional analysis could make physics available to students with lesser mathematical skills. Dimensional analysis is also suitable in cases where the mathematical complexity is not adequate in terms of benefit of calculations. For example, this facet of dimensional analysis had been used for derivation of the gravitational power radiated by a celestial body that moves on a circular orbit or the resistance force of the fluid that occurs when a body moves through it, and the speed of propagation of waves on water. Number of works in the recent literature attest to the extreme utility of this method in various disciplines of physics.

Dimensional analysis is one of the forms of well known Fermi problem solving. Mazur reflects on the complexity of physics problems in textbooks, and points out that problems too difficult may fail to help the development of physical thinking in students. Enrico Fermi had the ability to solve seemingly unsolvable and uncommon problems – all without additional information or complicated procedures and formulas. Mazur writes how a classic Fermi problem requires students to: 1. make assumptions, 2. make estimates, 3. develop a physical model, 4. work out that model. Numerical and algebraic answer is important and it should not be ignored, but it should not be the most important part of the solution. It should be noted that the creation of the physical model is the most important task. Taking this point of view, we consider dimensional analysis as one form of Fermi problem solving, and an extremely effective tool in teaching and research alike.

An interesting example of application of dimensional analysis is the derivation of power radiated by an electric dipole. The analogy between electric and magnetic dipole is straightforward, and one can repeat our steps to derive also the power radiated by a magnetic dipole. The formula for the power radiated by the dipole is derived in several publications in various ways. Jackson is a classic reference.

While evaluation of the radiative power generated by a dipole is one of the exactly solvable classical electromagnetic problem, the solution requires calculus. An alternative approach
is therefore desirable for both the non-calculus based courses, and for the exposition of the subject at the high-school level. It may seem first that taking the planetary model and applying dimensional analysis to calculate the radiation power of an electron encircling the nucleus of the atom on a circular trajectory may work. Unfortunately, the system of equations that this leads to is under-determined, and the dimensional analysis fails. In what follows, we avoid this difficulty by applying the dimensional analysis first to a more general problem without fixing any concrete geometry. Subsequent application to specific cases then only requires finding a factor of the order of one.

Finally, the calculation of the radiative power for the planetary model of atom in non-calculus based courses is not an art for arts sake. The problem is closely connected with the argumentation why the planetary model of the atom was rejected even before Rutherfords famous scattering experiment and the Bohrs model of atom. It is exciting for students to understand, also quantitatively, that classical electromagnetism predicts strongly radiating electrons in atoms what is in strong contradiction with the everyday observations.

II. RADIATION POWER OF AN ACCELERATING ELECTRIC CHARGE

A typical source of dipole radiation is a particle with electric charge undergoing simple harmonic motion. To derive its total radiation power $P$ in vacuum, we will assume a general case, namely a particle with electric charge $q$ and with an arbitrary acceleration $a$. Naturally, such an accelerated particle generates electromagnetic radiation. Since this radiation is an electromagnetic phenomenon, the radiation power $P$ will depend also on the electrical and magnetic characteristics of the vacuum: vacuum permittivity $\varepsilon_0$ and vacuum $\mu_0$. Because these two quantities determine the speed $c$ of electromagnetic waves in vacuum according to the relation $c = 1/\sqrt{\varepsilon_0\mu_0}$ we decide to replace the vacuum permeability $\mu_0$ with the speed of light in vacuum $c$, because the students are more familiar with the later quantity. Acceleration $a$ is a vector quantity, but we are interested in total power $P$ radiated by the electric charge of the particle in all directions which is a scalar. This leads us to assume that (due to the integration in all directions) there would only be dependence on the magnitude of acceleration $a$. There is no need to actually mention this reasoning to students, but it should be thought of in case questions arise about the vector nature of acceleration. We further assume that the radiated power does not depend on the mass of the moving charge,
because the source of electromagnetic field is the charge of the particle and not his mass. A particle with no electric charge and no magnetic dipole has no proper electric field and its acceleration does not generate electromagnetic waves (electromagnetic radiation). Therefore also the radiation is the same for protons and electrons.

Almost every physically important quantity may be expressed as a power series of other physical variables and fundamental constants. In this power series only one term has crucial contribution to the whole series - which one it is, may depend on the ranges of assumed variables. In many cases the power series consists from one term only.

In this spirit, mathematically, the formula for the power of radiation $P$ is given by the following expression

$$P = K q^\alpha c^\beta \varepsilon_0^\gamma a^\delta,$$

where $K$ is a dimensionless constant. For dimensions of power $P$, we write $[P] = ML^2T^{-3}$, dimension of speed of light $c$ is $[c] = LT^{-1}$, dimension of acceleration $a$ is $[a] = LT^{-2}$, dimension of charge $q$ is $[q] = IT$ and the dimension of permittivity of vacuum $\varepsilon_0$ is $[\varepsilon_0] = I^2T^4L^{-3}M^{-1}$. Here $I, T, L$ and $M$ are the dimensions of the basic physical quantities of current, time, length and mass, respectively. From the equality of right-hand and left-hand side we get

$$ML^2T^{-3} = (IT)^\alpha (LT^{-1})^\beta (I^2T^4L^{-3}M^{-1})^\gamma (LT^{-2})^\delta,$$

Simplifying the equation we obtain

$$ML^2T^{-3} = I^{\alpha + 2\gamma} T^{\alpha - \beta + 4\gamma - 2\delta} L^{\beta - 3\gamma + \delta} M^{-\gamma}.$$

From the equality of the right-hand and left-hand sides we obtain four equations, each expressing the equality for one of the exponents $\alpha, \beta, \gamma, \delta$ in individual dimensions

$$M : \quad 1 = -\gamma,$$
$$L : \quad 2 = \beta + \delta - 3\gamma,$$
$$T : \quad -3 = \alpha - \beta + 4\gamma - 2\delta,$$
$$I : \quad 0 = \alpha + 2\gamma.$$
After solving the equations we get $\alpha = 2$, $\beta = -3$, $\gamma = -1$ and $\delta = 2$. Thus, for the radiation power $P$ we can write

$$P = K \frac{q^2 a^2}{\varepsilon_0 c^3}. \quad (3)$$

### III. DISCUSSION

Dimensional analysis is a very suitable tool to learn for pupils in the lower classes where dimensional tests are already known and required as a part of certain problem. Students in the lower classes should also be able to solve a system of linear equations. There are many simple but interesting physical phenomena solvable by dimensional analysis: the period of a simple pendulum, the speed of sound in a string, in fluids and gases, the phase speed of waves on a surface of fluids and many other.

One drawback of the dimensional analysis is that we cannot determine the value of dimensionless constant $K$ and, therefore, our relation is only qualitative. In the case of basic physical phenomena such constants can be determined from experiment as, for example, for the period of a mathematical pendulum. In a general case one can assume, that $K$ expresses some geometric properties of the problem and therefore its magnitude is of the order of one – this is an acceptable approximation if there is no possibility to obtain a more precise estimation.

To confirm the correctness of our argumentation, it may be interesting to obtain constant $K$ by comparing results from the literature. Our derivation says nothing about the nature of acceleration $a$, which affects the value of the so-far undetermined constant $K$. The two simplest-to-analyze cases are: 1. a particle in a uniform circular motion, and 2. a particle undergoing simple harmonic motion. Another possible assumption could be that of a constant acceleration ($a = const$), which holds for Bremsstrahlung, and collisions of two particles (in the simplest approximation).

In the case of simple harmonic motion in one direction, we can determine the constant $K$ from Eq. (4) which was derived by Jefimenko\textsuperscript{20} (see 16-8, page 561)

$$P = \frac{p_0^2 \omega^4}{12\pi \varepsilon_0 c^3} = \frac{1}{4\pi \varepsilon_0} \frac{p_0^2 \omega^4}{3c^3}, \quad (4)$$

where $p_0$ is the dipole moment ($p_0 = qr_0$, where $r_0$ is the amplitude of oscillation) and $\omega$ is the angular velocity. Alternatively, one can utilize the following relation which was derived
by Sedlak\(^2\) (see 5.3.3 page 341)

\[ P = \frac{ck^4p_0^2}{12\pi\varepsilon_0} = \frac{1}{4\pi\varepsilon_0} \frac{ck^4p_0^2}{3}, \]  

(5)

where \(k\) is the wavenumber, \(c\) is the speed of light, and \(p_0\) is the dipole moment. Both of these relations, can be brought to the same form of

\[ P = \frac{q^2a^2}{12\pi\varepsilon_0c^3} = \frac{1}{4\pi\varepsilon_0} \frac{q^2a^2}{3c^3}, \]  

(6)

From Eq. (6) we can determine constant \(K_1\), which has the value of \(K_1 = \frac{1}{12\pi} \approx 0.027\).

In the case of a uniform circular motion of particle in one plane (with acceleration \(a = r\omega^2\)), we determine constant \(K\) from the relation derived by Jackson\(^1\) (see 14.2 page 665).

In Gauss unit system it reads

\[ P = \frac{2q^2a^2}{3c^3}, \]  

(7)

and after conversion to SI system it has the form of

\[ P = \frac{2}{4\pi\varepsilon_0} \frac{q^2a^2}{3c^3}. \]  

(8)

And from Eq. (8) we can determine constant \(K_2\) which has the value of \(K_2 = \frac{1}{6\pi} \approx 0.053\).

One can appreciate that in SI units (used here) \(\varepsilon_0\) is mainly accompanied by the geometric factor 4\(\pi\)(this is the consequence of the form of the Maxwell’s equation \(\nabla \cdot \mathbf{E} = \rho/\varepsilon_0\) – in vacuum). Taking into consideration this fact one obtain the formula for the power of radiation Eq. (3) in the form

\[ P = K \frac{q^2a^2}{4\pi\varepsilon_0c^3}. \]  

(9)

Now, we determine \(K_1 = \frac{1}{3}\) and \(K_2 = \frac{3}{2}\), and it is in perfect accordance with our statement about the magnitude of the constant \(K\).

IV. CONCLUSION

The aim of this article is to point out to the utility of a simple qualitative method that is dimensional analysis. We have calculated the radiation power of a general, accelerated charged particle, and the particular case of the electric dipole radiation is also discussed.
This derivation is very simple, and it does not require high level of mathematical skills. The only requirement with respect to the student’s prior knowledge is the solution of a simple system of linear equations. The derivation shown above has been used with success by the authors in the introductory courses to Quantum physics and Nuclear physics, in particular to argue quantum properties of atoms; By using Eq. (8) one can calculate that a classical approach gives the radiation power of electron in the hydrogen atom as $P \approx 10^{-8}$ W. Should this be the actual radiated power, a human body with its $10^{28}$ electrons would emit power comparable to that at the peak output of a nuclear bomb, what is clearly in contradiction to the everyday experience. The dimensional analysis is especially useful method for students with weaker mathematical skills. One can say that dimensional analysis is one of the forms of the Fermi problem solving strategy, because it naturally leads the student to think about the physical nature of the problem. As in the Fermi approach, the dimensional method breaks the problem into a sequence of steps that are easier for students to follow, namely definition of the physical model; assumptions - determining variables and parameters; using simple mathematics to compile equations and solving them, which finally leads to an approximate solution.

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