The field of knot theory has been a fundamental area of research for over 150 years. Framed knots are an extension that we can visualize as closed loops of knotted flat ribbons. Interest in framed knots follows from the critical role they play in low-dimensional topology. For example, a foundational result in the theory of 3-manifolds, the Lickorish–Wallace theorem [8], states that every closed, orientable, connected 3-manifold can be realized by performing a topological operation known as integer surgery on some framed link in the 3-dimensional sphere $S^3$. A diagrammatic process known as Kirby calculus [6] allows us to determine homeomorphic equivalence of 3-manifolds given by such a description with only a couple of simple moves on framed link diagrams, similar to the Reidemeister theorem for knots and links.

Framings also appear quite often when one is dealing with polynomial link invariants. Framed links can even be used to encode handlebody decompositions of 4-manifolds, though we will restrict ourselves to 3-dimensional topological considerations in what follows. The purpose of this article is to introduce the reader to various characterizations of framed knots and links, to show (or at least intuitively justify) their equivalence, and to discuss their application to 3-manifold topology. To that end, we assume that the reader has a basic understanding of algebraic topology. We will include proofs where appropriate and provide citations whenever the complexity of the details falls outside the scope of this paper. We will also provide auxiliary materials to help the reader visualize some of the more intuitive notions.

**Basic Definitions, Theorems, and Conventions**

A knot in the 3-sphere $S^3$ is a smooth one-to-one mapping $f : S^1 \to S^3$ (see Figure 1). Equivalently, a knot can be thought of as the set $f(S^1)$. We will work with these two definitions interchangeably. A link in $S^3$ is a finite collection of knots, called the components of the link, that do not intersect each other. Two links are considered equivalent if one can be deformed into the other without any of the knots intersecting itself or any other knots.¹

A link invariant is a quantity, defined for each link in $S^3$, that takes the same value for equivalent links.² Link invariants play a fundamental role in low-dimensional topology. In practice, we usually work with a link diagram of a link $L$, which is a projection of $L$ onto $S^2$ (or $\mathbb{R}^2$) that has a finite number of nontangential intersection points, called crossings. Each crossing corresponds to exactly two points of the link $L$. See Figure 1 for an example. To store the relative spatial information in the crossings, we usually draw a small break in the projection of the strand closest to the projection sphere (or plane in the case of $\mathbb{R}^2$) to indicate that it crosses under the other strand.

Framed knots $(K, V)$ in $S^3$ is a knot $K$ equipped with a continuous nonvanishing vector field $V$ normal to the knot, called a framing (see Figure 2). The magnitude of these vectors is largely irrelevant. Similarly, a framed link in $S^3$ is a link such that each of its component knots is equipped with a framing. A framed knot can be visualized as a tangled ribbon that has had its two ends glued after an even number of half-twists so as to yield an orientable surface. Note that this means that we exclude the cases in which the ribbon is glued together after an odd number of half-twists, e.g., a Möbius band (see Figure 3). To put it in more precise terms, the ribbon forms an embedded annulus, one of whose boundary components is identified with the specified knot $K$. For a given knot $K$, two framings on $K$ are considered to be equivalent if one can be transformed into the other by a smooth deformation.³ This is indeed an equivalence relation on the set of framings, and as such, the term “framing” will be used to refer to either an equivalence class or a representative vector field as context dictates.

Let us quickly demonstrate the equivalence between the definition of a framed knot and the conceptualization of the closed ribbon. Given a framed knot $(K, V)$, we can

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¹This is called ambient isotopy in the literature.

²The equivalence relation here is ambient isotopy.

³The precise notion of equivalence here is again ambient isotopy.
construct a ribbon by pushing the knot $K$ along the vector field $V$, sweeping out an area. Conversely, given a closed orientable ribbon in $S^3$, we can construct a framed knot by considering one of its boundary components to be the knot $K$ and choosing the vector field $V$ to lie in the ribbon, perpendicular to $K$ at every point of the knot. The magnitude of the vector field is unimportant, and it will be ignored for the remainder of this text (see Figure 4).

Given a knot, one can define infinitely many framings on it. Suppose that we are given a knot with a fixed framing. One may obtain a new framing from the existing one by cutting the ribbon and twisting it a nonzero integer multiple of $2\pi$ times around the knot and then reconnecting the edges. This operation leaves the knot itself fixed, and the reader should intuit that this is not a smooth deformation of the vector field. It is in fact impossible to have any smooth deformation between these two vector fields, but this is more easily shown using some of the characterizations that follow.

In the context of the previous operation, we see that the framing is associated with the number of “twists” the vector field performs around the knot, although it should not be immediately obvious how we can make such a definition precise. How does one count the number of twists a vector field makes around an object that is itself tangled up in the 3-sphere? What accounts for a clockwise rotation versus a counterclockwise one? As we will see, it is, in fact, possible to make such a definition, and knowing how many times the vector field is twisted around the knot allows one to determine the vector field completely up to a smooth deformation. The equivalence class of the framing is

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*Figure 1.* A knot is a smooth, linear, one-to-one mapping $f : S^1 \to S^3$. A knot diagram is obtained by projecting the image of the mapping onto a plane.

*Figure 2.* A framed trefoil.

*Figure 3.* A tangled ribbon.

*Figure 4.* Framed knots $\leftrightarrow$ Tangled ribbons.

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4See also a framed knots movie representation available at [http://www.youtube.com/watch?v=KoEBhD0C2Pw](http://www.youtube.com/watch?v=KoEBhD0C2Pw).
determined completely by this integer number of twists, called the framing integer. Our next goal is to show how the framing integer can be easily computed from a diagram using the linking number.

**Writhe, Linking and Self-Linking Numbers**

In practice, knots and links are frequently represented via diagrams. It would be useful, then, to have a combinatorial (diagrammatic) method for computing the framing integer. It turns out to be surprisingly easy to obtain such a method using the notion of the linking number. In what follows, an oriented knot is a knot that has been given an orientation. Similarly, an oriented link is a link each of whose components has been given an orientation.

**Definition 1.** (Lickorish [8]). Let \( J \) and \( K \) be two disjoint oriented knots represented by a link diagram \( D \). The linking number of \( J \) and \( K \), denoted by \( \text{lk}(J, K) \), is an integer, defined to be one-half the sum of the signs (see Figure 5) of the crossings between \( J \) and \( K \) in the diagram \( D \).

We reiterate that in this definition, self-crossings of the knots are not included in the summation.

It is easy to show that the set of linking numbers is actually an invariant of links. The proof is a direct application of Reidemeister’s theorem [13], which says that two diagrams of links represent the same link if and only if they are related through a finite sequence of three local moves called the Reidemeister moves (see Figure 6) and planar isotopy. We denote the three moves by \( \Omega_1 \), \( \Omega_2 \), and \( \Omega_3 \). Thus, to prove that the linking number is an invariant, one needs only to check that the linking number does not change under the three Reidemeister moves \( \Omega_1 \), \( \Omega_2 \), and \( \Omega_3 \).

For the invariance of the linking number under the first move, one can see immediately that \( \Omega_1 \) only adds or subtracts a self-crossing, and thus it leaves the summation unchanged. On the other hand, performing the move \( \Omega_2 \) will either introduce or remove two crossings with opposite signs. Either this new pair of crossings is two self-intersections of the same knot (which don’t appear in the summation), or both crossings occur between the two distinct knots, thus canceling each other in the summation. Finally, for the invariance of the linking number under the third move, \( \Omega_3 \), we note that the set of values being summed remains unchanged.\(^6\) Note also that for two knots \( K_1 \) and \( K_2 \), the definition implies that \( \text{lk}(K_1, K_2) = \text{lk}(K_2, K_1) \).

In order to state the next theorem, we need to define the notion of the connected sum \( K_1 \# K_2 \) of two knots \( K_1 \) and \( K_2 \). This sum is obtained by removing a single arc from each of the two knots, indicated by dotted lines in Figure 7. The two augmented knots are then joined by adding arcs in \( S^3 \setminus (K_1 \cup K_2) \), as indicated in the figure. The union of the two new arcs and the two deleted arcs must bound a topological disk that intersects the original knots only along the deleted arcs.

In the following, the knot \( K \) with reversed orientation will be denoted by \(-K\). We have the following theorem.

**Theorem 2.** Let \( K_1 \), \( K_2 \), and \( K_3 \) be three disjoint oriented knots in \( S^3 \). Then:

1. \( \text{lk}(K_1 \# K_2, K_3) = \text{lk}(K_1, K_3) + \text{lk}(K_2, K_3) \).
2. \( \text{lk}(K_1, -K_2) = \text{lk}(-K_1, K_2) = -\text{lk}(K_1, K_2) \).
3. Suppose that \( K_2 \) can be obtained from \( K_1 \) via a single crossing change. Then \( \text{lk}(K_1, K_3) = \text{lk}(K_2, K_3) \).

**Proof.** Item 1: Any new crossings that are introduced in the connected sum occur in canceling pairs. Thus additivity follows directly from the definition of the linking number.

Item 2 follows from the observation that by changing the orientation of only one of the pair, the sign of each crossing between the two knots changes.

Item 3 is immediate once we recall that self-crossings are not included in the calculation of linking number. \( \square \)

**Remark 3.** Item 3 of the previous theorem indicates that the linking number between two knots \( J \) and \( K \) is independent of the knot types of \( J \) and \( K \). By applying item 3 to certain self-crossings in the knots diagram of \( J \) and \( K \), one eventually obtains two trivial knots \( J' \) and \( K' \) that are linked such that \( \text{lk}(J, J') = \text{lk}(K, J') \). Figure 8 shows the two possibilities of \( K' \cup J' \).

A careful inspection of the difference between the two possibilities shown in Figure 8 hints at how we might define a notion of clockwise versus counterclockwise “twisting” of one knot about another. Note that item 2 of the previous theorem shows that any such definition can be made only relative to a choice of orientation on the knots. In fact, we shall see that the most natural definition of the framing integer arises from a choice of orientation on a Seifert surface of the knot (although this choice is equivalent to choosing one for the knot itself).

**Characterizations of the Linking Number**

In this section we give various geometric and combinatorial characterizations of the linking number and show the equivalence between them. The definitions of the linking number will be used in later sections.

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\(^6\)A linking number movie can be viewed at [https://www.youtube.com/watch?v=qOI5kCANE](https://www.youtube.com/watch?v=qOI5kCANE).
First, consider a knot \( K \) in \( S^3 \), and then its complement \( S^3 \setminus K \). A quick application of Alexander duality tells us that the first homology group satisfies

\[ H_1(S^3 \setminus K) \cong \mathbb{Z} \]

and is thus generated by a single element \( \gamma \) (see Figure 9).

Let \( J \) and \( K \) be two disjoint oriented knots in \( S^3 \). The curve \( J \) can be regarded as a loop in \( S^3 \setminus K \), so it represents an element of the first homology \( H_1(S^3 \setminus K) \). This group is generated by the curve \( \gamma \) (Figure 9), so we write \( J \) \( \in \) \( H_1(S^3 \setminus K) \) in terms of the generator \( \gamma \). Namely, \( J \) \( \cong \) \( s \gamma \) for some \( s \in \mathbb{Z} \). Theorem 5 below shows that this integer \( s \) is equal to \( \text{lk}(J, K) \) (see Figure 9).

We now give another characterization of the linking number. A Seifert surface of a knot is a compact, connected, orientable surface whose boundary is the knot. See Figure 10. When the knot \( K \) is oriented, we will always assume that the Seifert surface \( S \) of \( K \) is oriented such that \( \partial S = K \).

A Seifert surface movie can be viewed at http://www.youtube.com/watch?v=px3Gq_qvBQ.

For an orientable surface \( S \) with an oriented boundary we need to distinguish between the two sides of \( S \). We define the positive side to be the side whose oriented boundary runs counterclockwise as seen from that side of the surface. We denote this side by \( S^+ \). The side \( S^- \) is defined similarly (see Figure 11).

**Definition 4.** Let \( S \) be a Seifert surface for an oriented knot \( K \) in \( S^3 \). Let \( J \) be an oriented knot in \( S^3 \) that is disjoint from \( K \). A **positive** (respectively **negative**) **intersection** of \( S \) with \( J \) is a transverse intersection of \( S \) with \( J \) such that the
oriented curve $J$ passes from $S^-$ to $S^+$ (respectively $S^+$ to $S^-$). Assign weights $+1$ and $-1$ respectively to the positive and negative intersections of $S$ and $J$. The intersection number of $S$ and $J$, denoted by $S \cdot J$, is the sum of the weights of all transverse intersections.

The following theorem will prove that $S \cdot J$ is equal to $\text{lk}(J, K)$.

**Theorem 5.** Let $J$ and $K$ be disjoint oriented knots in $S^3$. Let $S$ and $S'$ be Seifert surfaces that bound $K$ and $J$ respectively. Then:

1. Suppose that $[\eta]$ generates $H_1(S^3 \setminus K) \cong \mathbb{Z}$, where $[\eta]$ is represented by a curve $\eta$ such that $\text{lk}(K, \eta) = 1$. Then if $[J] = s[\eta]$ for some $s \in \mathbb{Z}$, we have that $\text{lk}(J, K) = s$.
2. $\text{lk}(J, K) = J \cdot S = K \cdot S'$.

**Proof.** 1. Suppose that $[J] = s[\eta]$ for $s \in \mathbb{Z}$. Turn each positive crossing of $J$ under $K$ into an overcrossing by replacing $J$ with the connected sum $J \# (-\eta)$, keeping in mind that $-\eta$ denotes the curve $\eta$ with its orientation reversed. Turn each negative crossing of $J$ under $K$ into an overcrossing by replacing $J$ with the connected sum $J \# \eta$. Doing this for all undercrossings of $J$ with $K$ gives us two knots $K$ and $J \# (-\eta)$ that can be separated by a 2-sphere. We can therefore manipulate them via ambient isotopy in such a way that they share no crossings in a planar diagram, demonstrating that $\text{lk}(K, J \# (-\eta)) = 0$. Hence $\text{lk}(K, J \# (-\eta)) = \text{lk}(K, J) - s \text{lk}(K, \eta) = \text{lk}(K, J) - s = 0$, which yields the result. See Figure 12.

2. It is sufficient to show that $\text{lk}(K, J) = K \cdot S'$. The equality $\text{lk}(K, J) = J \cdot S$ follows by the symmetry of the linking number (statement 3 of Theorem 2). Consider the curve $J$ as an element in $H_1(S^3 \setminus K)$ that is generated by $[\eta]$. Write $[J]_{S^3 \setminus K} = s[\eta]$ for some $s \in \mathbb{Z}$. We know by statement 1 that $s = \text{lk}(J, K)$. The result follows if we show that $s = K \cdot S'$. Now let us set $n = K \cdot S'$.

Suppose $K$ intersects $S'$ positively at a point $p$, as indicated in the left-hand image in Figure 13. Inspect the connected sum $J \# -\eta$ and notice that it cancels the intersection point between $S'$ and $K$ around $p$. Similarly, when $K$ intersects $S'$ negatively, the connected sum $J \# + \eta$ cancels one intersection between the surface $S'$ and $K$. Now let $-n\eta$ be the curve obtained from $n$ copies of $-\eta$, when $n > 0$, or $-n$ copies of $\eta$, when $n < 0$.

By our earlier observation, the homology element $[J \# -n\eta]_{S^3 \setminus K}$ bounds a surface that does not intersect the knot $K$. Hence the homology element $[J \# -n\eta]_{S^3 \setminus K}$ is equal to zero or $[J] = n[\eta]$. The result follows. □

If we orient the knot $K$, then for a framed knot $(K, \mathbf{V})$ we can define explicitly what we mean by the framing integer $n$ that describes the number of times the vector field twists around $K$.

**Definition 6.** Let $(K, \mathbf{V})$ be a framed knot. The self-linking number is given by $\text{lk}(K, K')$, where $K'$ is an oriented knot formed by a small shift of $K$ in the direction of the framing vector field and oriented parallel to the knot $K$ (see Figure 14).

We call the curve $K'$ given in Definition 6 the *pushoff* curve determined by the framed knot $(K, \mathbf{V})$. Note that the

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**Figure 10.** A Seifert surface for the figure-8 knot.

**Figure 11.** (a): An oriented knot $K$. (b) and (c): A Seifert surface of the oriented knot $K$. The Seifert surface determines two sides $S^+$ and $S^-$. 
self-linking number of a framed knot is independent of the orientation we choose for $K$, since at every crossing of $\gamma C_0 K$ the orientation of both arcs is reversed, leaving the sign unchanged. Note also that the self-linking number is the same if $K$ is shifted in the direction opposite to the framing.

The reason we have introduced this concept is that the self-linking number of a framed knot $(K, V)$ is equal to the framing integer that determines, or is determined by, $(K, V)$. This is evident by observing Figure 14 and noticing that locally, the vector field winds $\pm 1$ around the knot if and only if the pushoff $K'$ contributes $\pm 1$ to the self-linking number. Note that the definition of a framed knot $(K, V)$ is independent of the choice of orientation of the knot $K$. On the other hand, we have just shown that the self-linking number is independent of the orientation we choose for the knot $K$, so defining this number to be the framing integer agrees with our original definition of the framing.

Hence we will assume in what follows that these two concepts, the self-linking number and the framing integer, are the same, and we will use both terms interchangeably. The framing with self-linking number $n$ will be called the $n$-framing, and a knot with the $n$-framing will be referred to as $n$-framed. Hence we can define a framed knot in $S^3$ to be $(K, n)$, where $K$ is a knot in $S^3$ and $n$ is an integer. It will be useful in practice to have a standard way to choose a framing for a given a knot diagram.

**Definition 7.** The blackboard framing, defined for a plane knot projection, is given by a nonzero vector field that is everywhere parallel to the projection plane. See Figure 15.

The reason for calling this the blackboard framing is clear once we attempt to draw it: we simply choose a point on the knot, move transversely to the knot on one side (it doesn’t matter which!), and then follow the knot, staying on the same side of the arcs until the chalk returns to the

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**Figure 12.** Unlinking two knots $K$ and $J$ locally at a crossing corresponds to taking the connected sum of $J$ and the curve $\pm [\eta]$, where $[\eta]$ is a curve with $\text{lk}(K, \eta) = 1$.

**Figure 13.** The blue curve represents the knot $K$, the yellow curve represents the knot $J$, and the gray curve represents the curve $\eta$. The surface $S'$ is the surface colored in blue, and it is the Seifert surface of $J$. This figure shows that reducing the number of intersections between $S'$ and $K$ by 1 corresponds to taking the connected sum of $J$ and the curve $\pm [\eta]$.

**Figure 14.** The blue curve represents a knot $K$, and the green curve represents a pushoff curve $K'$. The yellow arrows represent a framing $V$ of the knot $K$. The pushoff curve $K'$ is determined by the knot $K$ and the framing $V$ and is obtained by “pushing” $K$ in the direction of $V$.

**Figure 15.** A trefoil diagram with its blackboard framing.
original pushoff. In this way, if we visualize the framed knot as a ribbon, it will lie flat on the blackboard.

The blackboard framing is also related to the notion of the writhe of a knot.

**Definition 8.** The *writhe* of a knot diagram is the sum of the signs of every crossing in the diagram.

Notice that since both possible choices of orientation give the same sign at each crossing, the writhe does not depend on the orientation of the knot. Note also that the writhe is invariant under $\Omega_2$ and $\Omega_3$ but not under the move $\Omega_1$. The notions of the writhe of a knot diagram and the self-linking of a framed knot given by a diagram with a blackboard framing are related, as we will show shortly.

The blackboard framing for a knot diagram $D$ of $K$ corresponds to one particular framing $n_0$ of $K$. This leads to a natural question: can we obtain a “framed knot diagram” corresponding to each of the possible framings for $K$? In other words, can we always represent a framed knot $(K, n)$ by a knot diagram with the blackboard framing? The answer is yes. In order to see this, we need to see the effect of the Reidemeister moves $\Omega_1$, $\Omega_2$, and $\Omega_3$ on the blackboard framing of a fixed knot diagram $K$. Notably, only $\Omega_1$ changes the blackboard framing, by exactly $\pm 1$. By applying an appropriate number of the moves $\Omega_1$, we can thus obtain a diagram of the knot with the desired blackboard framing.

What we have also discovered is that for framed knots (with blackboard framed diagrams), Reidemeister’s theorem does not hold immediately, because the move $\Omega_1$ changes the blackboard framing. Luckily, there is an analogous theorem, which will follow directly once we prove the following proposition.

**Proposition 9.** The self-linking number of a framed knot given by a diagram with blackboard framing is equal to the writhe of the diagram.

**Proof.** In the case of blackboard framing, the only crossings of $K$ with its pushoff $K'$ occur near the crossing points of $K$. The neighborhood of each crossing point looks like Figure 16.

There are two crossings of $K$ with $K'$, each with the same sign as the crossing of $K$. The claim follows directly from the definition of the linking number in $S^3$, and we now see some of the motivation for defining the writhe to be the total sum, while the linking number is one-half the sum of the crossings.

Figure 17 shows $F\Omega_1$, a modification of the Reidemeister move $\Omega_1$, which we will use in the next theorem.

**Theorem 10.** Two knot diagrams with blackboard framing $D_1$ and $D_2$ represent equivalent framed knots if and only if $D_1$ can be transformed into $D_2$ by a sequence of plane isotopies and local moves of the three types $F\Omega_1$, $\Omega_2$, and $\Omega_3$, where $F\Omega_1$ is given by Figure 17, and $\Omega_2$ and $\Omega_3$ are the usual Reidemeister moves.

**Proof.** Suppose first that the diagrams represent equivalent framed knots. The associated knots $K_1$ and $K_2$ are isotopic, and thus the standard Reidemeister theorem tells us that the diagrams are related by a sequence of plane isotopies and the moves $\Omega_1$, $\Omega_2$, and $\Omega_3$. Note that by Proposition 9 above, $D_1$ and $D_2$ have the same writhe. We know that writhe is invariant under plane isotopies and the moves $\Omega_2$ and $\Omega_3$, and moreover, that every move $\Omega_1$ changes the writhe by exactly $\pm 1$, with the sign depending on the direction of the kink. Thus, there must be an even number of right kinks and left kinks in the sequence of moves connecting $D_1$ to $D_2$. By a sequence of plane isotopies and $\Omega_2$ and $\Omega_3$ moves, any kink can be moved anywhere along the knot. We can then pair them so that we get a set of moves of the form $F\Omega_1$, and this direction of the statement is proved. For the other direction, we need simply to note that the modified move $F\Omega_1$ is a combination of traditional Reidemeister moves and doesn’t change the writhe of a diagram. Hence two diagrams being related by a sequence of these moves means that the corresponding knots are isotopic, and they have the same framing.

The previous results can be summarized in the following statement. For every framed knot $(K, n)$ we can find a plane knot diagram that represents that framed knot. This plane diagram is unique up to modified Reidemeister moves and plane isotopy.

In the next section we introduce two important curves that are naturally related to a framed knot in $S^3$.
The Longitude and the Meridian

In this section, we provide various characterizations of two important curves that are related to the framing of a knot. These curves provide another homological characterization of the framing of a knot. Furthermore, we relate these curves to the self-linking number we introduced earlier. This definition of the framing plays an essential role in defining surgery on a 3-manifold.

Before we introduce these curves and their relationship to the framing of a knot, we need to discuss the homology and the homotopy groups of the torus.

Curves on the Torus

In this subsection we give a discussion of closed curves on the torus up to three equivalence relations: homology, homotopy, and ambient isotopy.

Let $S$ be an arbitrary surface. Let $f_i : [0, 1] \to S$ for $i = 1, 2$ be two loops ($f_i(0) = f_i(1)$) on the surface. It is easy to prove the following facts:

1. If $f_1$ is homotopic to $f_2$, then $f_1$ is homologous to $f_2$.
2. Suppose that $f_1$ and $f_2$ are embeddings. If $f_1$ is ambient isotopic to $f_2$, then $f_1$ is homotopic to $f_2$ in $S$.

For a generic surface $S$, the inverses of statements 1 and 2 are not true in general. On the torus, however, the inverse directions hold in special cases. We discuss this in the following.

Homology and Homotopy of the Torus

We give a brief discussion on the first homology and homotopy groups of the torus. See the first chapter of [14] for more details.

The fundamental group of the torus is $\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. In what follows, we will define a particular isomorphism between the fundamental group of $T^2$ and $\mathbb{Z} \oplus \mathbb{Z}$. To this end, we specify coordinates for the torus $T^2$ by $T^2 = S^1 \times S^1$, where we identify $S^1$ with the complex numbers of unit modulus. Then every point on $T^2$ has coordinates $(e^{2\pi i \theta}, e^{2\pi i \phi})$, where $0 \leq \theta, \phi \leq 1$. Furthermore, choose the counterclockwise orientation on $S^1$, and in this way, every map $f : S^1 \to T^2$ may be regarded as an element of $\pi_1(T^2)$. In particular, consider the maps $l : S^1 \to T^2$ and $m : S^1 \to T^2$ given by

$$m(e^{2\pi i \theta}) = (1, e^{2\pi i \theta}),$$
$$l(e^{2\pi i \theta}) = (e^{2\pi i \theta}, 1),$$

where $0 \leq \theta \leq 1$. These two maps represent the two generators of $\pi_1(T^2)$. See Figure 18.

We define an isomorphism between $\pi_1(T^2)$ and $\mathbb{Z} \oplus \mathbb{Z}$ by sending $m$ to $(0, 1)$ and $l$ to $(1, 0)$. Hence every class in $\pi_1(T^2)$ can be represented by $(n, m)$, where $n, m \in \mathbb{Z}$. We neglect the base point here, because $T^2$ is path connected.

Since $\pi_1(T^2)$ is abelian, we also know that the groups $\pi_1(T^2)$ and $H_1(T^2)$ are isomorphic. In other words, two closed curves in $T^2$ are homotopic if and only they are homologous.

The curves $m$ and $l$ are easily defined in the case that $T^2$ has the above parameterization. However, this definition is more involved when the torus $T^2$ is embedded in $S^3$. We study these curves below.

Knots on the Torus

Given a closed curve $C$ on the torus $T^2$ that represents a class in $\pi_1(T^2)$, we ask whether there exists a simple closed curve $C'$ in $T^2$ that is homotopic to $C$. In other words, can we represent a homotopy class in $\pi_1(T^2)$ by an embedding in $T^2$? The answer in general is no. For instance, we cannot find a simple closed curve that represents the homotopy class $(2, 0)$. On the other hand, one can find a simple closed curve that represents the class $(2, 3)$. The following two theorems answer this question.

**Theorem 11.** [14, p. 19] Let $c$ be a curve in $T^2$ with homotopy class $(a, b)$ in $\pi_1(T^2)$. The curve $c$ can be represented by an embedding $S^1 \to T^2$ if and only if the integers $a$, $b$ are coprime (that is, $\gcd(a, b) = 1$), one of them is zero and the other is $\pm 1$, or $a = b = 0$.

This theorem is useful when we want to know whether an embedded curve representation of a certain homotopy class exists on the torus. As Theorem 11 asserts, such an embedding exists if and only if the pair $(a, b)$, which completely characterizes the homotopy type of the class, satisfies the conditions mentioned in the theorem. The next theorem also relates the homotopy classes and isotopy classes of curves on $T^2$.

**Theorem 12.** [14] If two closed curves without self-intersection on the torus $T^2$ are homotopic, then they are ambient isotopic.

In summary, if we are given a homotopy class $(a, b)$ in $T^2$ such that $a = b = 0$ or one of them is zero and the other is $\pm 1$ or $\gcd(a, b) = 1$, then we can represent the class $(a, b)$ by a simple closed curve in $T^2$. Moreover, given two such representations of this homotopy class $(a, b)$, without self-intersection, one can find an ambient isotopy on $T^2$ that takes the first representation to the second one. The proofs of the previous two theorems are omitted, and the
The Longitude and the Meridian of an Embedded Torus in $S^3$

Let $K$ be an oriented knot in $S^3$. Let $N \subset S^3$ be a tubular neighborhood around $K$, i.e., a solid torus embedded in the 3-sphere whose core is the knot $K$. It is easiest to think of $N$ as just a thickening of $K$. Let $X$ denote the closure of $S^3 \setminus N$. We assume that $N$ is embedded in $S^3$, so that $X$ is a manifold. In this case, it is clear that $\partial X = \partial N = T^2$. The Mayer–Vietoris exact sequence for $S^3 = N \cup X$ with $N \cap X = T^2$ reads

$$H_2(S^3) \to H_1(X \cap N) \to H_1(X) \oplus H_1(N) \to H_1(S^3).$$

From basic homology theory we know that $H_1(S^3) = H_2(S^3) = 0$. Moreover, since $X \cap N$ is homeomorphic to a torus, we know from the previous section that $H_1(X \cap N) = Z \oplus Z$. Finally, since $N$ is homotopic to the knot $K$, it follows that $H_1(N)$ is isomorphic to $Z$; hence we can write equation (1) as follows:

$$0 \to Z \oplus Z \to H_1(X) \oplus H_1(N) \to 0.$$  

The sequence (2) is exact. Hence the middle map is an isomorphism, and thus $H_1(X)$ is isomorphic to $Z$. We will choose a specific isomorphism between $H_1(X)$ and $Z$ later in this section. By the Mayer–Vietoris theorem, the isomorphism

$$i_* \oplus j_* : H_1(X \cap N) \to H_1(X) \oplus H_1(N)$$

is given explicitly by $i_* \oplus j_*(\gamma) = (i_*\gamma, j_*\gamma)$, where $i : X \cap N \to X$ and $j : X \cap N \to N$ are the inclusion maps. Note that the map $i_*$ pushes curves on the surface $X \cap N$ into the knot’s exterior $X$, and similarly, the map $j_*$ pushes curves on $X \cap N$ into the solid torus $N$.

The Meridian

Recall that $N$ is homeomorphic to a solid torus, and its boundary $X \cap N$ is homeomorphic to a torus. We then know that $H_1(X \cap N) \cong \pi_1(T^2)$ is generated by two curves, and by Theorem 25, each of them can be chosen to be simple and closed. One of these curves, denoted by $\eta$, can be chosen to encircle the knot $K$ and bound a disk in $N$. We can further choose the orientation on $\eta$ so that $lk(K, \eta) = 1$. Because $\eta$ bounds a disk in $N$, it is null-homologous in $N$, and hence $j_*[\eta] = 0$ in $H_1(N)$. The meridian $[\eta]$ represents a generator of $H_1(X \cap N)$, and thus every isomorphism $i_* \oplus j_*$ must map it to a generator in $H_1(X) \oplus H_1(N)$. Using our explicit definition of the map, we see that

$$i_* \oplus j_*([\eta])(i_*[\eta], j_*[\eta]) = (i_*[\eta], 0).$$

In other words, $i_*[\eta]$ generates the group $H_1(X)$. We use this generator to give a specific isomorphism $H_1(X) \to \mathbb{Z}$ defined by sending $i_*[\eta]$ to 1. We will denote the homology class $i_*[\eta]$ in $H_1(X)$ by $[\eta]$.

The Preferred Longitude

Note that the solid torus $N$ is homotopy equivalent to its core $K$, allowing us to represent the generator of $H_1(N)$ by the oriented knot itself. We then fix an isomorphism $H_1(N) \to \mathbb{Z}$ that maps $[K]$ to 1. In the previous section we defined the isomorphism $H_1(X) \to \mathbb{Z}$ that sends the homology class of the curve $\eta$ to 1 in $\mathbb{Z}$. Using these two isomorphisms, we can construct a specific isomorphism between $H_1(X) \oplus H_1(N)$ and $\mathbb{Z} \oplus \mathbb{Z}$ defined by $(0, [K]) \mapsto (0, 1)$ and $([\eta], 0) \mapsto (1, 0)$. We will assume this identification from now on.

Since $i_* \oplus j_*$ is an isomorphism, there exists a unique element $[\gamma]$ in $H_1(X \cap N)$ that maps to $(0, 1)$. Since the class $(0, 1)$ is a generator in $H_1(X) \oplus H_1(N)$, the image element $[\gamma]$ under the isomorphism $(i_* \oplus j_*)^{-1}$ must also be a generator in $H_1(X \cap N)$. Hence by Theorem 11, we can represent the class $[\gamma]$ by a simple closed curve (which will also be denoted by $\gamma$) on $T^2 \cong X \cap N$. We interpret $\gamma$ as follows:

$$i_* \oplus j_*([\gamma]) = (i_*[\gamma], j_*[\gamma]) = (0, 1) \in H_1(X) \oplus H_1(N),$$

means that $[\gamma]|_N$ is null-homologous in $H_1(X)$ and $[\gamma]|_N = [K]$ in $H_1(N)$.

Remark 13. If we consider a simple closed curve as a representative of the homology class $[\gamma]|_N$ and denote it by $\gamma$, then this curve can be seen to be obtained by an ambient isotopy of the knot $K$ inside $N$. We can choose the curves that connect the beginning of the ambient isotopy, namely $K$, to the end of it, $\gamma$, to be a collection of simple closed embedded curves in $N$, and hence these curves define a ribbon tangle or a framed knot with boundary that is the union of the knot $K$ and the curve $\gamma$.

We give some facts about the meridian and the preferred longitude in the following definition.

Definition 14. Let $K$ be an oriented knot in (oriented) $S^3$ with solid torus neighborhood $N$. A meridian $\eta$ of $K$ is a nonseparating simple curve in $\partial N$ that bounds a disk in $N$. A preferred longitude $\gamma$ of $K$ is a simple closed curve in $\partial N$ that is homologous to $K$ in $N$ and null-homologous in the exterior of $K$.

The previous discussion about meridian and longitude implies immediately the following theorem.

Theorem 15. Let $K$ be an oriented knot in (oriented) $S^3$ with solid torus neighborhood $N$. Then the following facts hold:

- The meridian $\eta$ is a simple closed curve that generates the kernel of the homomorphism $H_1(X \cap N) \to H_1(N)$.
- The preferred longitude $\gamma$ is a simple closed curve that generates the kernel of the homomorphism $H_1(X \cap N) \to H_1(X)$.
From this theorem we also obtain the following corollary.

Corollary 16. The median \( \eta \) is characterized by a simple closed curve on \( X \cap N \) that bounds a disk in \( N \). On the other hand, the preferred longitude \( \gamma \) is characterized by a simple closed curve on \( X \cap N \) that bounds a surface in \( X \).

It is important to notice that once we choose the natural orientation on the meridian and longitude, as in the construction above, these curves are unique on \( T^2 \) up to ambient isotopy.

Proposition 17. Let \( K \) be an oriented knot in \( S^3 \). Let \( N \) and \( X \) be as before. There exist two oriented curves \( \eta \) and \( \gamma \) unique up to ambient isotopy on \( T^2 = X \cap N \) that satisfy Definition 14.

Proof. The existence of the curves has already been established. For the uniqueness, suppose that \( \eta' \) is another curve on \( \cap N \) with the same properties of \( \eta \). Recall that the curve \( \eta \) was a representative of a certain homology class in \( H_1(T^2) \), and this homology class generates the kernel of the map \( H_1(X \cap N) \to H_1(N) \). This kernel is isomorphic to \( \mathbb{Z} \). Hence each of the curves \( \eta \) and \( \eta' \) must be a representative of a generator of the kernel, and hence \( [\eta] = \pm [\eta'] \). Now recall the construction of the meridian above and notice that we can choose the orientations of \( \eta \) and \( \eta' \) so that \( \text{lk}(K, \eta) = \text{lk}(K, \eta') = 1 \). With this choice we must now have \( [\eta] = [\eta'] \). Since \( \eta \) and \( \eta' \) are simple closed curves, again by construction, we conclude, by Theorem 12, that \( \eta \) and \( \eta' \) are ambient isotopic. Similarly, suppose that the curve \( \gamma' \) is a curve on \( \cap N \) with the same properties as those of \( \gamma \). These two curves are representatives of a generator of the kernel of the map \( H_1(X \cap N) \to H_1(X) \), and hence \( [\gamma] = \pm [\gamma'] \). The orientation of the two curves can be chosen so that they are both parallel to the oriented knot \( K \). We conclude that \( [\gamma] = [\gamma'] \), and thus, by Theorem 12, the curves \( \gamma' \) and \( \gamma' \) are ambient isotopic. \( \square \)

Remark 18. It is worth noting that while a meridian can be defined for a solid torus, a preferred longitude requires a specified embedding of the solid torus into \( S^3 \).

Remark 19. The preferred longitude \( \gamma \) is not determined completely by stating that it is a simple closed curve on \( \partial N \) that generates \( H_1(N) \). Indeed, there are infinitely many homology classes of curves on \( \partial N \) with this property. In fact, a curve that generates \( H_1(N) \) and is positively oriented with the knot is usually referred to as a longitudinal curve. Note that there are infinitely many curves on \( \partial N \), up to homotopy, that satisfy this condition. On the other hand, adding the condition that this curve is also trivial in \( H_1(X) \) determines that curve uniquely up to ambient isotopy on \( \partial N \), as we have shown in Proposition 17. This also explains the adjective “preferred” when we want to describe the preferred longitude to distinguish this curve among many other longitudinal curves on \( \partial N \).

Different Characterizations of the Meridian and the Preferred Longitude

It is useful to have many characterizations for the meridian and the longitude. The following theorem summarizes most of the characterizations of the meridian.

Theorem 20. Let \( K \) be an oriented knot in (oriented) \( S^3 \) and let \( X \) and \( N \) be defined as above. Suppose that \( \eta \) is essential in \( \partial N \). Then the following are equivalent:

1. \( \eta \) is homologically trivial in \( N \).
2. \( \eta \) is homotopically trivial in \( N \).
3. \( \eta \) bounds a disk in \( N \).

The choice of a meridian of a knot does not include any ambiguity. However, the choice of a preferred longitude needs more care. There is an easy characterization of the preferred framing given in terms of the linking number. This characterization is given in the following theorem.

Theorem 21. The preferred longitude \( \gamma \) of a knot \( K \) in \( S^3 \) is characterized by a simple closed curve on \( N \) such that \( \text{lk}(\gamma, K) = 0 \).

Proof. Viewing \( [\gamma]_X \) as an element of \( X \), we can write \( [\gamma]_X = n[\eta] \), where \( [\eta] \) is the generator of \( H_1(X) \), and \( n \) is some integer. By the definition of the linking number, the integer \( n \) is \( \text{lk}(\gamma, K) \). If \( [\eta] \) is a preferred longitude, then by definition, \( [\eta]_X = 0 \), and hence \( \text{lk}(\gamma, K) = 0 \). On the other hand, if \( \text{lk}(\gamma, K) = 0 \), then \( [\gamma]_X = 0 \), and the result follows. \( \square \)

The Relation Between the Longitudes and the Framings of a Knot

In this section we relate the notions of longitudes and framings of a knot. Let \( K \) be a knot in \( S^3 \). Every framing of \( K \) gives rise to a longitude of \( K \) on \( X \cap N \) and conversely. We first show that the zero framing corresponds to the preferred longitude.

Choose an orientation of the knot \( K \). We know that \( H_1(S^3 \setminus K) = H_1(X) = \mathbb{Z} \). We can choose the generator to be \( [\eta] \), the meridian of the tubular neighborhood around \( K \). Choose a framing \( V \) for \( K \). We know that this framing gives rise to another knot \( K' \) that is linked with \( K \), and the linking number between \( K \) and \( K' \) is precisely the framing integer determined by the framing \( V \). The curve \( K' \) represents an element of \( H_1(S^3 \setminus K) = H_1(X) \), and hence it can be written as \( m[\eta]_X \) for some integer \( m \). We conclude that every framing corresponds to some integer \( m \) in the homology of
the exterior of the knot $K$. In particular, the zero framing corresponds to the integer 0 and hence the linking number zero. Thus, by Theorem 21, the zero framing of a knot $K$ corresponds to the preferred longitude of the knot $K$. We have proved the following theorem.

**Theorem 22.** Let $K$ be a zero-framed knot in $S^3$. Suppose that $N$ is a tubular neighborhood of the knot $K$ that intersects the ribbon of $K$ in a simple closed curve $\gamma$. Then $\gamma$ is the preferred longitude of a tubular neighborhood of the knot $K$.

This theorem can be generalized to characterize any framing for a given knot. To see this, let $K$ be a framed knot and let $N$ be its tubular neighborhood and $X$ its exterior. Then $K$ intersects the torus $\partial N$ in a simple closed curve, say $d$, that winds $m'$ times around the meridian and one time around the longitude. Thus it can be represented by

$$d = m'[\eta] + [\gamma] \quad (4)$$

(see Figure 19). We want to show that $m'$ is precisely the framing integer of $K$. Recall that the framing number is the self-linking number of $K$, which is by definition $\text{lk}(d, K)$.

To this end, consider the image of the curve $d$ under the isomorphism $i_\ast \oplus j_\ast$. This can be seen to be

$$(i_\ast \oplus j_\ast)(d) = (i_\ast \oplus j_\ast)(m'[\eta] + [\gamma])$$

$$= m'(i_\ast \oplus j_\ast)([\eta]) + (i_\ast \oplus j_\ast)([\gamma])$$

$$= m'(i_\ast([\eta]), j_\ast([\eta])) + (i_\ast([\gamma]), j_\ast([\gamma]))$$

$$= m'(1, 0) + (0, 1)$$

$$(m', 1) = (m'i_\ast[\eta], j_\ast([\gamma])) \in H_1(X) \oplus H_1(N),$$

and thus $[d]_X = m'i_\ast[\eta] = m'[\eta]_X$.

Hence by the definition of the linking number, $m'$ must be $\text{lk}(d, K)$, and we are done.

There is a little more that we can say about the curve $d$ given in (4). Recall from Remark 19 that a longitudinal curve on $\partial N$ is a curve that generates $H_1(N)$ and is positively oriented with the knot $K$. We have proved that the curve $d$ defined in (4) actually satisfies this condition: it is obtained from the intersection of a framed knot with $\partial N$.

**Figure 19.** The curve $d$ is obtained by the intersection of the framed knot $K$ with the torus $\partial N$.

and it generates $H_1(N)$, since it is 1 in this group, as we have shown. Hence a longitudinal curve can be written in terms of the meridian and the preferred longitude as $m[\eta] + [\gamma]$ for some $m \in \mathbb{Z}$. In other words, the curve $d$ as given in (4) parameterizes all longitudinal curves on $\partial N$. We record this in the following theorem.

**Theorem 23.** Let $(K, V)$ be a framed knot in $S^3$. Suppose that $N$ is a tubular neighborhood of the knot $K$ that intersects the ribbon of $K$ in a simple closed curve $d$. Then $d$ is a longitudinal curve on $\partial N$. Furthermore, $d$ can be written as $d = m[\eta] + [\gamma]$, where $m \in \mathbb{Z}$ is the framing integer of $(K, V)$ and $\eta$ and $\gamma$ are the meridian and the preferred longitude respectively. Conversely, every longitudinal curve on $\partial N$ can be written as $m[\eta] + [\gamma]$ for some $m \in \mathbb{Z}$, and it corresponds to a framing $V$ of the knot $K$ whose framing integer equals $m$.

### Seifert Surfaces and Zero-Framed Knots

In this section we give the Seifert framing, which is a type of framing that can be associated with a knot $K$. We prove that this framing can be used to characterize the zero framing of a knot.

**Definition 24.** Given a Seifert surface for a knot, the associated **Seifert framing** is obtained by taking a vector field perpendicular to the knot and inward tangent to the Seifert surface.

The Seifert framing provides a useful characterization for the zero-framing of a knot.

**Theorem 25.** The self-linking number obtained from a Seifert framing is always zero.

**Proof.** Suppose that $N$ is a tubular neighborhood of a knot $K$, and $X$ its exterior. Let $S$ be the Seifert surface of $K$, and let $K'$ be the intersection curve $\partial N \cap S$. It is clear that $K'$ is a simple closed curve on $\partial N$. The curve $K'$ bounds the Seifert surface $S$ in $X$, and hence it is trivial in $H_1(X)$. Thus, $K'$ is precisely the preferred longitude, and by Theorem 22, we conclude that $\text{lk}(K', K) = 0$. Hence the framing obtained from the Seifert surface is zero.

Alternatively, Theorem 25 can be seen to be true by considering a different definition of the linking number. Namely, let $K$ and $K'$ be as stated in Theorem 25. Recall that $\text{lk}(K, K') = S \cdot K'$, where $S \cdot K'$ is the intersection number between the surface $S$ and the knot $K'$. From the way we construct $K'$, we see that the intersection number between $S$ and $K'$ is zero. It is worth mentioning here that even though it looks as if there were infinitely many intersections between $S$ and $K'$, those intersections are not transverse intersections, and hence they do not contribute to the
number $S \cdot K'$. In other words, one needs to push the surface $S$ a little bit away from the knot $K'$ so that it does not intersect $K'$.

**Framing Characterization Using the Fundamental Group of SO(2)**

Another characterization of the framing of a knot is obtained by considering the fundamental group $\pi_1(\text{SO}(2))$ of the special orthogonal group SO(2). While less intuitive, this description will hint at how exactly we can make precise this notion of what it means to count “twists” around a knot.

Recall that SO(2) is the group of all $2 \times 2$ real matrices with determinant equal to 1 and that geometrically, these linear maps form the set of rotations in the plane about the origin. As such, the group is topologically a circle and can be parameterized by an angle $\theta$ corresponding to the angle of the rotation.

Suppose that we are given a knot $K$ and a vector field $V$ representing the zero-framing of $K$, and we choose an arbitrary orientation on the knot. We can consider this as a sort of reference framing that is used to create a well-defined map from the set of framings of $K$ into $\pi_1(\text{SO}(2))$. The knot $K$ is an embedded circle, so we can identify points of the knot with their preimages in this embedding to get a parameterization $t \mapsto e^{it_\theta} \cdot K(t)$ in terms of the standard unit circle $S^1$, where $t \in [0, 1]$. With this notation, the vector field $V$ at the point $K(t)$ will be denoted by $V(t)$. For every point $K(t)$ on $K$, construct a vector field $N$ such that the ordered set $(K(t), V(t), N(t))$ forms a right-hand basis of $\mathbb{R}^3$ for every $t \in [0, 1]$. Here the vector $K(t)$ is a unit vector that has the direction of the orientation of the knot $K$ and forms a right-hand basis along with the vectors $V(t)$ and $N(t)$. In other words, the vector field $K$ is a nonzero vector tangential to the knot, with the direction dictated by the knot’s orientation. As such, it suffices to choose $N = K \times V$ to get the desired right-hand basis. At every point $K(t)$, the associated vectors $V(t)$ and $N(t)$ lie in a plane perpendicular to the knot (see Figure 20).

Let $(K, W)$ be any choice of framing on $K$. Using the same process as before, we associate to every point $K(t)$ a pair of nonzero orthogonal vectors $(W(t), M(t))$ that span the plane normal to the knot. Choose an element $0 \leq \theta(t) < 2\pi$ of SO(2) that represents the rotation of the basis $V(0), N(0)$ around the axis given by $K(0)$ to obtain $(W(0), M(0))$. Proceeding backward along the knot, we can create a smooth map $\phi_W : t \mapsto \theta(t)$ that encodes the rotation of the framing $W$ with respect to $V$. Note that $\phi(1) = 2\pi m$ must be a multiple of $2\pi$, because $V(0) = V(1)$ and $W(0) = W(1)$. This multiple is exactly the framing integer.

**Remark 26.** The decision to construct $\phi_W$ by proceeding backward along the knot is made to ensure consistency with other characterizations of the framing integer given in this text, such as the self-linking number. We could alternatively have defined the map $\phi_W$ by proceeding forward along the knot and negating the value $m$. Or we could have chosen to reverse the parameterization of SO(2). It is worth taking a minute to consider what other choices were made in this construction that could have the effect of changing the framing integer.

**Remark 27.** The reason for presenting this characterization of a framing as an element of the fundamental group of SO(2) (i.e., the circle) rather than simply stopping at the framing integer is that this interpretation allows us to see easily that a framing $W$ constructed from $W$ by cutting the ribbon and giving it a full twist before reconnecting the ends of the ribbon is indeed distinct from $W$. The effect of this operation would add/subtract $2\pi$ to the value of $\phi_W(1)$, thus creating a distinct element of the fundamental group of SO(2). This is exactly analogous to the traditional calculation of the fundamental group of the circle using covering maps. Any ambient isotopy that could map $W$ to $W'$ would induce a homotopy between $\phi_W$ and $\phi_W'$; since we know that they are not homotopic, no such isotopy can exist.

On the other hand, a loop $f$ in SO(2) gives rise to a continuous family of elements in SO(2) that can be used to construct a smooth vector field on $K$. Perturbing the curve $f$ inside $S^1$ in a way that respects its homotopy type will change the vector field $V$ only up to some ambient isotopy. This yields a bijection between elements of $\pi_1(\text{SO}(2))$ and the set of equivalence classes of framings of $K$.

**Framed Knots and 3-Manifolds**

In the introduction of this article we mentioned a theorem of Lickorish–Wallace (Theorem 28 below). In this section we will give the ingredients necessary to state the theorem and illustrate how framed knots are used in lower-dimensional topology.

Lickorish was trying to answer a question posed by Bing [2], who gave a partial solution to the Poincaré conjecture. Bing’s question states, “Which compact connected 3-manifolds can be obtained from the 3-sphere using the following process: deleting disjoint polyhedral tori and sewing them back in a different way.” As we will see in this section, all closed and orientable 3-manifolds can be realized in this way, and importantly for us, framed knots play a central role in this.

**Theorem 28.** Every closed, orientable, connected 3-manifold can be realized as integer surgery on some framed link in the 3-dimensional sphere $S^3$.

Intuitively, Theorem 28 can be used in the fundamental quest for 3-manifold topology to obtain a complete classification of all compact orientable 3-manifolds. The problem is that this list might have redundancies: two different framed links may correspond to the same 3-manifold. To determine when two links give rise to the same 3-manifold, Kirby [6] studied the necessary moves on framed links, similar to Reidemeister moves, now called Kirby moves. More precisely, Kirby calculus [6] states that two framed links produce the same 3-manifold if and only if the links
are related by a sequence of moves called Kirby moves. Thus Kirby calculus in conjunction with Theorem 28 can be used in the fundamental quest for 3-manifold topology to obtain a complete classification of all compact orientable 3-manifolds.

The term “surgery” mentioned in Theorem 28 refers to the idea of performing “surgery” on a 3-manifold. Intuitively, a surgical operation on a 3-manifold typically involves removing a manifold with boundary from a 3-manifold and then gluing it back to a 3-manifold via a homeomorphism. There are multiple types of surgeries on 3-manifolds such as integer surgery and rational surgery. In the remaining part of the paper we will talk briefly about integer and rational surgeries on the 3-sphere.

In order to gain intuition, we begin with a few simple examples to show how one can obtain 3-manifolds by gluing “simpler” 3-manifolds together.

Our first example is the 3-sphere. It is intuitively clear that one can obtain the 3-sphere by gluing two 3-disks along their boundaries. This intuition can actually be generalized to the 3-sphere, and the reader may convince herself that gluing two 3-disks along their boundaries gives the 3-sphere. See Figure 21. A key fact here is that there is only one way to glue D^3 to itself. Roughly speaking, there is essentially only one homeomorphism between S^3 and itself. This exhausts the list of 3-manifolds that can be obtained by gluing D^3 to itself. We shall not prove this result here. The next natural choice of simple 3-manifolds is the solid torus. What are the different manifolds that one can obtain by gluing two solid tori along their boundaries? It turns out that in this case, we can obtain infinitely many manifolds! We describe this next.

Let ST_1 and ST_2 be two solid tori. Let f : ∂ST_1 → ∂ST_2 be a homeomorphism between their boundaries that sends the meridian of ∂ST_1 to the longitude of ∂ST_2. It turns out that the manifold obtained by this gluing is again S^3. To see this, denote by D^2 the meridional disk of ST_1. We thicken the disk D^2 a little bit to obtain D^2/ε and cut this part out of ST_1. We obtain in this way another piece that is homeomorphic to D^3. See Figure 22 (a). If we now glue D^2/ε to the solid torus ST_2 along the longitude, as indicated in Figure 22 (b), we obtain a space that is homeomorphic to D^3 as well. To finish the gluing process of ST_1 and ST_2, we need to glue the remaining boundaries together. However, the resulting two manifolds are exactly two 3-disks, and by our earlier discussion, there is only one way to glue such manifolds together along their boundaries.
boundaries. Hence the resulting manifold is again $S^3$. See Figure 22 (c).

The first thing of which the reader should be aware from the previous example is that the manifold obtained from gluing $ST_1$ to $ST_2$ by sending the meridian of the first one to the longitude of the second one was completely determined by where we sent the meridian. It turns out that the resulting manifold is always completely determined by the image of the meridian under the gluing homeomorphism. But what are the the other 3-manifolds that one could obtain if we chose to map the meridian of $o ST_1$ to the another closed and simple (that is, without self-intersection) curve on $o ST_2$?

Recalling Theorem 11, we can characterize simple closed curves on $o ST_2$ by two coprime integers $p$ and $q$, where $p$ is the number of times the curve winds around the meridian, and $q$ is the number of times the curve winds around the longitude. If we choose to map the meridian of $o ST_1$ to a curve $(p, q)$ in $o ST_2$, where $p$ and $q$ are coprime, then the resulting 3-manifold is called a lens space, and it is denoted by $L(p, q)$. See Figure 23.

Finally, when we map the meridian of $ST_1$ to the meridian of $ST_2$, then the resulting manifold is homeomorphic to $S^2 \times S^1$. We shall not prove this fact here. The reader is referred to [1, 14] for more details.

We are now ready to see how framed knots can be used to obtain 3-manifolds. Consider a knot $K$ in the 3-sphere and let $N(K)$ be a tubular neighborhood of $K$ as before. By cutting along the torus boundary $\partial N(K)$ of $N(K)$, we obtain the complement $S^3 \setminus \text{Int}(N(K))$, which has a boundary that is homeomorphic to a torus. Let $b$ be a homeomorphism between the boundaries of $D^2 \times S^1$ and $S^3 \setminus \text{Int}(N(K))$. Consider the 3-manifold obtained by gluing $D^2 \times S^1$ to $S^3 \setminus \text{Int}(N(K))$ via the homeomorphism $b$. Just as before, the final 3-manifold is completely determined by where we send the meridian of $D^2 \times S^1$.

The 3-manifold obtained is closed and orientable, and we say that it is obtained from the 3-sphere by surgery along the knot $K$. This manifold, as we saw before, is completely determined by the image of the meridian. Up to isotopy, we can assume that the meridian goes to a $(p, q)$-
curve on the torus boundary of the knot, where \( p \) and \( q \) are coprime. Moreover, it can be shown that the surgery that glues the meridian to the \((p, q)\)-curve is the same as the surgery that sends the meridian to the \((-p, -q)\)-curve. Thus this surgery of the 3-sphere is completely determined by the fraction \( p/q \), which is called the surgery index. We call the above operation on \( S^3 \) a rational surgery with rational index \( r = p/q \).

Our earlier discussion about lens spaces \( L(p, q) \) implies that these spaces can be obtained by performing a rational surgery on the unknot. We now show how framed links are naturally related to the notion of surgery.

**Integer Surgery and Framed Links**

In this final part we briefly introduce integer surgery and then show its relationship to framed links. First we state the definition of integer surgery.

**Definition 29.** If the integer \( q \) is equal to \( \pm 1 \), then we say that we have integer surgery on \( S^3 \).

We now explain the relationship between integer surgery and framed knots, recalling that a framed knot \( K \) determines a longitudinal curve on \( \partial N \), where \( N \) is the solid torus neighborhood of \( K \). Moreover, as we illustrated earlier, this curve can be written as a \((p, 1)\)-curve on the torus, where \( p \) is the framing integer. Hence the information given by a framed knot, the knot and its framing integer, is precisely the same information one needs to perform integer surgery on \( S^3 \). Thus by Theorem 28, every compact orientable 3-manifold can be represented by a link diagram with an integer on each link component. We have turned all of 3-manifold theory into a version of knot theory!

As an example of a 3-manifold obtained from framed knots, recall from our earlier discussion that \( S^2 \times S^1 \) can be obtained by gluing two solid tori along their boundaries by sending the meridian of the boundary of the first solid torus to the meridian of the boundary of the second solid torus. Another equivalent way to obtain the same 3-manifold is to perform an integer surgery on \( S^3 \) along the zero-framed unknot. Notice here that when \( K \) is the zero-framed unknot, the manifold \( S^3 \setminus \text{Int}(N(K)) \) is homeomorphic to the solid torus. This solid torus is “flipped inside out.” In particular, the longitudinal curve in \( S^3 \setminus \text{Int}(N(K)) \) corresponds to a meridian curve in the standardly embedded solid torus.

Working with framed link diagrams in the context of 3-manifolds is advantageous, because one can use a link diagram with its blackboard framing as a well-defined method to denote the 3-manifold obtained by performing the surgery on \( S^3 \) along that link. The blackboard framing of the link, along with the link diagram, completely determines the surgery and hence the manifold itself. Hence framed link diagrams can be used to define 3-manifold invariants. Indeed, every quantity defined on framed link diagrams that is invariant under \( \Omega_q \) and \( \Omega_r \) as well as under Kirby moves can be considered an invariant of 3-manifolds. An interesting family of knots and 3-manifold invariants called the quantum invariants has been at the center of interest in low-dimensional topology for decades, and framed knots play an important role in these invariants. The Jones polynomial \([4]\) was the first invariant discovered from this family, and then later, Kauffman \([5]\) showed that this invariant can be defined via framed links. For more details about this subject, see \([11]\).

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