CHARACTERIZATION OF BALLS AS MINIMIZERS OF
AN ENDPOINT GAGLIARDO SEMINORM ON THE BOUNDARY

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Abstract. Given a bounded $C^2$ domain $\Omega \subset \mathbb{R}^d$ with $d \geq 3$, we prove a sharp inequality which relates the perimeter of $\partial \Omega$ to the endpoint Gagliardo seminorm in $W^{r,2}(\partial \Omega)$, corresponding to $r = 0$, of the normal vector field on $\partial \Omega$. The proof of the inequality relies on the use of Bessel potentials and a monotonicity formula; we also show that balls are the unique minimizers. For $1/2 < r < 1$, the Gagliardo seminorm of the normal vector field on $\partial \Omega$ is related to a fractional second fundamental form which arises in the study of nonlocal perimeters and nonlocal minimal surfaces.

1. Introduction

There are many results in the literature characterizing balls in terms of sharp inequalities of integral type. The classical isoperimetric inequality relating perimeter and volume is among the most famous ones. Other celebrated results are the Pólya-Szegö inequality on the Newtonian capacity [23], see also [22] for other Riesz capacities, and the Faber-Krahn inequality [13, 18] on the first eigenvalue of the Dirichlet Laplacian on a domain. Several results of this nature can be proved using rearrangement, a very powerful technique which, in may situations, is used both to show the inequality and to find the optimizers; it has important applications to Sobolev embeddings into Lebesgue spaces as well. One can also find in the literature sharp isoperimetric-type inequalities for fractional (or nonlocal) quantities which characterize balls as the unique minimizers, see [16] for example. Other interesting works are [24, 25] and [17] where, in the first ones, the balls are determined by the fact that the equilibrium distribution with respect to the Newtonian capacity is constant along the surface, and in the last one, the characterization is obtained in terms of the angle between the interior and exterior Hardy spaces.

In this article we characterize balls as minimizers of an endpoint Gagliardo seminorm on the boundary. More precisely, given $0 < r < 1$ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, the Sobolev-Slobodeckij trace space $W^{r,2}(\partial \Omega)$ is the space of functions $u \in L^2(\partial \Omega)$ such that $[u]_{r,\partial \Omega} < +\infty$, where the Gagliardo seminorm $[\cdot]_{r,\partial \Omega}$ is defined by

$$[u]_{r,\partial \Omega}^2 := \int_{\partial \Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{d-1+2r}} d\sigma(x) d\sigma(y).$$

Here $\sigma$ denotes the $(d-1)$-dimensional Hausdorff measure restricted to $\partial \Omega$ (the surface measure). The purpose of this work is to prove the following sharp inequality between $\sigma(\partial \Omega)$ and the seminorm $[\cdot]_{r,\partial \Omega}$ of the normal vector field on $\partial \Omega$ in the endpoint case $r = 0$, and to show that the equality is attained if and only if $\Omega$ is a ball. This is an scenario where, in principle, one cannot directly apply rearrangement arguments due to the lack of a volume constraint.

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Theorem 1.1. Let $d \geq 3$ be an integer, $\Omega \subset \mathbb{R}^d$ be a bounded $C^2$ domain and $\nu$ denote the outward unit normal vector field on $\partial \Omega$. Then,

$$
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \, d\sigma(x) \, d\sigma(y) \geq \sigma(\partial \Omega) \int_{S^{d-1}} |x - e|^{2 - d} \, d\sigma(x),
$$

where $S^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$ and $e \in S^{d-1}$ is any fixed unit vector. The equality in (1) is attained if and only if $\Omega$ is a ball.

This result can be rewritten in terms of the endpoint Gagliardo seminorm as follows: Given $d \geq 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded $C^2$ domain and $B \subset \mathbb{R}^d$ be a ball such that $\sigma(\partial \Omega) = \sigma(\partial B)$. Then, $|\nu|_{0,\partial \Omega} \geq |\nu|_{0,\partial B}$ and the equality holds if and only if $\Omega$ is a ball. Here, we used the symbols $\sigma$ and $\nu$ to denote the surface measure and the unit normal vector field on both $\partial \Omega$ and $\partial B$. We remark that the regularity assumptions on $\Omega$ in Theorem 1.1 are taken to avoid technicalities during the proofs in the article, and they can be relaxed substantially.

By a simple argument, in Section 4 we also prove that

$$
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\nu(x) - \nu(y)|}{|x - y|^{d-1}} \, d\sigma(x) \, d\sigma(y) \geq \sigma(\partial \Omega) \sigma(S^{d-1})
$$

and the equality in (2) is attained if and only if $\Omega$ is a ball. This is the result analogous to Theorem 1.1 when we replace $|\nu(x) - \nu(y)|^2$ by $|\nu(x) - \nu(y)|$. However, it is not clear how to get (1) from (2), since $|\nu(x) - \nu(y)|^2$ is smaller than $|\nu(x) - \nu(y)|$ for $x$ close to $y$.

Observe that $|\nu(x) - \nu(y)|^2 = 2 - 2 \nu(x) \cdot \nu(y) = 2(\nu(x) - \nu(y)) \cdot \nu(x)$ for all $x, y \in \partial \Omega$. Therefore, $|\nu|_{r,\partial \Omega}^2 = 2 \|c_{\partial \Omega,2r-1}\|_{L^2(\partial \Omega)}^2$ where, given $x \in \partial \Omega$,

$$
c_{\partial \Omega,s}(x) := \left( \text{P.V.} \int_{\partial \Omega} \frac{(\nu(x) - \nu(y)) \cdot \nu(x)}{|x - y|^{d+s}} \, d\sigma(y) \right)^{1/2}
$$

is the so-called $s$-fractional second fundamental form of $\partial \Omega$ at $x$. In particular, for smooth bounded domains $\Omega \subset \mathbb{R}^d$,

$$
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \, d\sigma(x) \, d\sigma(y) = \lim_{r \searrow 0} |\nu|_{r,\partial \Omega}^2 = \lim_{s \searrow -1} 2 \int_{\partial \Omega} c_{\partial \Omega,s}^2 \, d\sigma.
$$

For $0 < s < 1$, $c_{\partial \Omega,s}$ is an important object in the study of nonlocal minimal surfaces which arise as critical points of the $s$-fractional perimeter. Indeed, $c_{\partial \Omega,s}$ appears in the fractional Jacobi operator $J_{\partial \Omega,s}$ defined by

$$
J_{\partial \Omega,s} w(x) := \text{P.V.} \int_{\partial \Omega} \frac{w(y) - w(x)}{|x - y|^{d+s}} \, d\sigma(y) + c_{\partial \Omega,s}^2(x) w(x) \quad \text{for } x \in \partial \Omega,
$$

where $w : \partial \Omega \rightarrow \mathbb{R}$ is sufficiently smooth. The Jacobi operator $J_{\partial \Omega,s}$ was found in [11, 14] while computing the second variation of the $s$-fractional perimeter.

The proof of Theorem 1.1 relies in a nonlocal perimeter, in this case defined by

$$
\Lambda(\Omega, a) := \int_{\Omega} \int_{\Omega} G_a(x - y) \, dx \, dy,
$$

where $G_a$ is the fundamental solution of the Helmholtz operator $-\Delta + a^2$, namely,

$$
G_a(x) := \frac{a^{d/2-1}}{(2\pi)^{d/2}} |x|^{1-d/2} K_{d/2-1}(a|x|) \quad \text{for } x \in \mathbb{R}^d \setminus \{0\} \text{ and } a > 0.
$$

Here, $K_{d/2-1}$ denotes the modified Bessel function of the second kind and order $d/2 - 1$, see Section 2 for more details. The notion of nonlocal (or fractional) perimeter was introduced in the work of Caffarelli, Roquejoffre and Savin [6] regarding nonlocal minimal surfaces associated to the $s$-fractional perimeter, given by the Riesz kernel $|x|^{-d-s}$, and the fractional Laplacian.
Taking $r$ is known that the classical perimeter and the volume are obtained by taking the limit

$$\lim_{r \to 0} \text{perimeter} = \text{volume.}$$

In the case of the Riesz kernel $|x|^{-d-s}$, it is known that the classical perimeter and the volume are obtained by taking the limit $s \to 1$ and $s \to 0$, respectively, after a suitable rescaling; see [3, 20, 27] for the case of $s$-fractional curvatures. For other nonlocal perimeters, one can still recover the classical perimeter by a limiting argument based on rescaling, the reader may look at [9, 23], for example. Our proof of Theorem 1.1 is partially inspired in these ideas.

It is of interest to see if the methods presented in this work could be adapted to study $|u|_{r, \partial \Omega}$ in the general case $0 < r < 1$, where we cover the regime $0 < s < 1$ commented below (4) by taking $r = (s + 1)/2$. The expected inequality would be

$$\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d+1+2r}} \text{d}x \text{d}y \geq \sigma(\partial \Omega)^{-2} \int_{\partial \Omega} |x - e|^3 \text{d}x. \quad (5)$$

For $0 < r < 1$, the question of whether (5) holds or not requires further study.

Theorem 1.1 is a straightforward application of the following theorem. Its proof is based on a monotonicity formula involving Bessel potentials and the fundamental solution of the Helmholtz operator $-\Delta + a^2$. In more detail, we define

$$\Phi(\Omega, a) := \frac{1}{(2\pi)^{d/2}} \int_{\partial \Omega} \int_{\partial \Omega} \left( \int_{a|x|}^{+\infty} t^{d/2-1} K_{d/2-1}(t) dt \right) \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \text{d}x \text{d}y$$

and we show that $\Phi(\Omega, a)$ is monotone in $a \in (0, +\infty)$. To prove this monotonicity, we first find a sharp inequality between a solid integral and a boundary integral related to $G_a$, see Theorem 2.3 below. The proof of this inequality is mainly based on the Gauss-Green theorem and the Reflection Lemma which characterizes the balls of $\mathbb{R}^d$. This part of the article works for all integer $d \geq 2$ and is developed in Section 2. Using the sharp inequality, we prove that $\Phi(\Omega, a)$ is nonincreasing on $a \in (0, +\infty)$ and is constant if and only if $\Omega$ is a ball. Moreover, we can compute its limit when $a \to 0$ and $a \to +\infty$. In the former one we essentially get $|u|^2_{0, \partial \Omega}$ and in the latter one we obtain $\sigma(\partial \Omega)$ modulo some precise constants; the assumption $d \geq 3$ is only used to compute the limit when $a \to 0$, see Remark 3.2. The following theorem, which summarizes these conclusions, is the main result in this article; its proof is given in Section 3.

**Theorem 1.2.** Let $d \geq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded $C^2$ domain. Then,

$$\frac{\partial \Phi}{\partial a}(\Omega, a) \leq 0 \quad \text{for all } a > 0, \quad (7)$$

$$\lim_{a \to 0} \Phi(\Omega, a) = \kappa \int_{\partial \Omega} \int_{\partial \Omega} \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \text{d}x \text{d}y, \quad (8)$$

$$\lim_{a \to +\infty} \Phi(\Omega, a) = \kappa \sigma(\partial \Omega) \int_{\partial \Omega} |x - e|^{3-d} \text{d}x, \quad (9)$$

where $\kappa := (2\pi)^{-d/2} \int_0^{+\infty} t^{d/2-1} K_{d/2-1}(t) dt$ is a positive and finite constant.

The equality in (9) is attained for some (and thus for all) $a > 0$ if and only if $\Omega$ is a ball. This means that, as a function of $a \in (0, +\infty)$, if $\Omega$ is a ball then $\Phi(\Omega, a)$ is constant, and if $\Omega$ is not a ball then $\Phi(\Omega, a)$ is strictly decreasing.
A final comment is in order. The reader familiar with heat-flow monotonicity techniques will observe similarities with our approach. Several integral inequalities in euclidean analysis can be proved using adequate (sub/super)solutions of the heat equation \( \partial_t - \Delta \), for which certain monotone behavior holds in \( t > 0 \). Then, the evaluation at different times yields an inequality which, in many situations, generates sharp constants and identifies extremizers; see [2] for a survey on the subject. In certain cases the heat operator is replaced by other differential operators. In this work we use the Helmholtz operator to construct the flow.

Regarding the notation, given a bounded \( C^2 \) domain \( \Omega \subset \mathbb{R}^d \), throughout this work \( \sigma \) denotes the \((d-1)\)-dimensional Hausdorff measure restricted to \( \partial \Omega \) (the surface measure) and \( \nu \) the outward unit normal vector field on \( \partial \Omega \). We also denote by \(|\Omega|\) the Lebesgue measure of \( \Omega \) and, for simplicity of notation, we set \(|\partial \Omega| := \sigma(\partial \Omega)\).

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2. A Sharp Integral Inequality Involving Bessel Potentials

We begin this section by introducing the Bessel potential that will be used in the sequel, namely, a suitable fundamental solution of \( -\Delta + a^2 \) for \( a > 0 \). Given \( \alpha \geq 0 \), let \( K_\alpha \) denote the modified Bessel function of the second kind and order \( \alpha \), see [2] for the definition and properties. The Bessel function \( K_\alpha \) satisfies the differential equation

\[
t^2K''_\alpha(t) = (t^2 + \alpha^2)K_\alpha(t) - tK'_\alpha(t) \quad \text{for all } t > 0.
\]

Throughout this section we assume that \( d \geq 2 \) is an integer. Set

\[
G(x) := \frac{|x|^{1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(|x|), \quad H(x) := \frac{|x|^{1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(|x|)
\]

for \( x \in \mathbb{R}^d \setminus \{0\} \) and

\[
G_a(x) := a^{d-2}G(ax), \quad H_a(x) := a^{d-2}H(ax)
\]

for \( a > 0 \) and \( x \in \mathbb{R}^d \setminus \{0\} \). From [2] 9.6.24 or [23] (5) in page 181 we know that

\[
K_\alpha(t) = \int_0^{+\infty} e^{-t \cosh r} \cosh(\alpha r) \, dr,
\]

thus \( G_a(x) > 0 \) for all \( x \in \mathbb{R}^d \setminus \{0\} \). It is well known that \( G_a \in L^1(\mathbb{R}^d) \) and that, given \( f \in L^\infty(\mathbb{R}^d) \), the function

\[
\varphi(x) = (G_a * f)(x) = \int_{\mathbb{R}^d} G_a(x - y)f(y) \, dy
\]

belongs to \( L^1(\mathbb{R}^d) \) and satisfies \( -\Delta + a^2)\varphi = f \), see [23] Section 7.4 for example. Therefore, \( (-\Delta + a^2)G_a = \delta_0 \) in the sense of distributions, where \( \delta_0 \) denotes the Dirac measure centered at the origin. We refer to \( G_a \) as the Bessel potential. In particular, taking \( f = 1 \) it is clear that

\[
a^2 \int_{\mathbb{R}^d} G_a(x) \, dx = (-\Delta + a^2)(G_a * f) = 1.
\]

The next lemma contains some useful formulas involving \( G_a \) that will be used in the sequel.
Lemma 2.1. The following identities hold for all $a > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$:

\[
\frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = \frac{H_a(x)}{|x|} + \left( \frac{d}{2} - 1 \right) \frac{G_a(x)}{a|x|^2}, \tag{15}
\]

\[
\frac{\partial}{\partial a} \left( a^{d-3} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) \right) = a^{d-3} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right), \tag{16}
\]

Proof. Using (11) and (12) we compute

\[
\frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = \frac{\partial}{\partial a} \left( a^{d/2} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|) \right) = a^{d/2} \frac{|x|^{-d/2}}{(2\pi)^{d/2}} K'_{d/2-1}(a|x|) + \frac{d}{2} a^{d/2-1} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|)
\]

From here, (15) follows directly. Then, using (15) and (10),

\[
\frac{\partial^2}{\partial a^2} \left( \frac{G_a(x)}{|x|^2} \right) = \frac{\partial}{\partial a} \left( \frac{H_a(x)}{|x|} \right) + \left( \frac{d}{2} - 1 \right) \frac{G_a(x)}{a|x|^2} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right)
\]

\[
= \frac{\partial}{\partial a} \left( a^{d/2-1} \frac{|x|^{-d/2}}{(2\pi)^{d/2}} K'_{d/2-1}(a|x|) + \left( \frac{d}{2} - 1 \right) a^{d/2-2} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|) \right)
\]

\[
= (d-3)a^{d/2-2} \frac{|x|^{-d/2}}{(2\pi)^{d/2}} K'_{d/2-1}(a|x|) + a^{d/2-1} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|)
\]

\[
+ \frac{1}{2} (d-2)(d-3)a^{d/2-3} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|)
\]

\[
= \frac{d-3}{a^2} \left( \frac{H_a(x)}{|x|} \right) + \frac{(d-2)}{a} \frac{G_a(x)}{2|x|^2} + G_a(x) = \frac{d-3}{a} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) + G_a(x).
\]

Therefore,

\[
\left( \frac{\partial}{\partial a} - \frac{d-3}{a} \right) \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = G_a(x)
\]

and thus

\[
\frac{\partial}{\partial a} \left( a^{d-3} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) \right) = a^{d-3} \left( \frac{\partial}{\partial a} + 3 - \frac{d}{a} \right) \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = a^{d-3} G_a(x),
\]

which corresponds to (16). \(\square\)

Given a bounded $C^2$ domain $\Omega \subset \mathbb{R}^d$ and $a > 0$, we now focus on the nonlocal perimeter $\Lambda(\Omega, a)$ introduced in (4), whose kernel is the Bessel potential $G_a$; see [15, 21, 27] for other nonlocal perimeters. Since $G_a$ is nonnegative, from (14) we can trivially estimate $0 < a^2 \Lambda(\Omega, a) \leq |\Omega|$. Indeed, thanks to (14), we can write

\[
a^2 \Lambda(\Omega, a) = |\Omega| - \int_{\mathbb{R}^d} (a^2 G_a * \chi_\Omega) \chi_\Omega,
\]

where $\chi_\Omega$ denotes the characteristic function of $\Omega$.

Using the Gauss-Green theorem and that $(-\Delta + a^2)G_a = \delta_0$, in the following lemma we prove two identities which relate $\Lambda(\Omega, a)$ to certain double boundary integrals. These identities will be a key tool to prove the main theorem in this section, namely Theorem [2.3].
Lemma 2.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^2$ domain. Then, the following holds for all $\alpha > 0$:

\[
a^2 \Lambda(\Omega, \alpha) = \int_{\partial \Omega} \int_{\partial \Omega} G_a(x - y) \nu(x) \cdot \nu(y) \, d\sigma(x) \, d\sigma(y),
\]

\[
\int_{\partial \Omega} \int_{\partial \Omega} G_a(x - y) \left( \nu(x) \cdot \frac{x - y}{|x - y|^2} \right) \nu(y) \cdot \frac{x - y}{|x - y|^2} \, d\sigma(x) \, d\sigma(y)
\]

\[
= a^2 \Lambda(\Omega, \alpha) + a(d - 1) \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial a} \left( \frac{G_a(x - y)}{|x - y|^2} \right) \, dx \, dy.
\]

Proof. Recall that $(-\Delta + a^2)G_a = \delta_0$. Therefore, for $x, y \in \mathbb{R}^d$ with $x \neq y$ we have

\[
a^2 G_a(x - y) = (\Delta G_a)(x - y) = \text{div}_x ((\nabla G_a)(x - y)) = -\text{div}_x \nabla_y (G_a(x - y)),
\]

where $\text{div}_x$ and $\nabla_y$ mean the divergence and the gradient on the $x$ and $y$ variables, respectively. From (19) and the Gauss-Green theorem applied twice we easily get (17).

We now focus on (18). A computation shows that

\[
\text{div}_x \left\{ G_a(x - y) \left( \nu(y) \cdot \frac{x - y}{|x - y|^2} \right) (x - y) \right\}
\]

\[
= \left( (x - y) \cdot (\nabla G_a)(x - y) + (d - 1)G_a(x - y) \right) \left( \nu(y) \cdot \frac{x - y}{|x - y|^2} \right),
\]

and that

\[
\text{div}_y \left\{ (x - y) \cdot (\nabla G_a)(x - y) + (d - 1)G_a(x - y) \right\} \frac{x - y}{|x - y|^2}
\]

\[
= -\frac{1}{|x - y|^2} \left\{ (d - 1)(d - 2)G_a(x - y) + 2(d - 1)(x - y) \cdot (\nabla G_a)(x - y) \right\}
\]

\[
+ (x - y)[D^2 G_a(x - y)](x - y)^t.
\]

By $(x - y)[D^2 G_a(x - y)](x - y)^t$ we mean $\sum_{i,j=1}^{d}(x_i - y_i)(x_j - y_j)\partial_i\partial_j G_a(x - y)$, where we also used the notation $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Therefore, by the Gauss-Green theorem, (20) and (21), we get

\[
\int_{\partial \Omega} \int_{\partial \Omega} G_a(x - y) \left( \nu(x) \cdot \frac{x - y}{|x - y|^2} \right) \nu(y) \cdot \frac{x - y}{|x - y|^2} \, d\sigma(x) \, d\sigma(y)
\]

\[
= \int_{\Omega} \int_{\Omega} \frac{1}{|x - y|^2} \left\{ (d - 1)(d - 2)G_a(x - y) + 2(d - 1)(x - y) \cdot (\nabla G_a)(x - y) \right\}
\]

\[
+ (x - y)[D^2 G_a(x - y)](x - y)^t \right\} \, dx \, dy.
\]

We can compute $\nabla G_a$ and $D^2 G_a$ on the right hand side of (22) using the definition of $G_a$ in terms of the Bessel function $K_{d/2-1}$. More precisely, by (12) and (11),

\[
\nabla G_a(x) = a^{d-1}(\nabla G)(ax) = \frac{a^{d/2-1}}{(2\pi)^{d/2}} \left( \left( 1 - \frac{d}{2} \right) K_{d/2-1}(a|x|) + a|x| K'_{d/2-1}(a|x|) \right) \frac{x}{|x|^{1+d/2}}
\]

and

\[
\partial_i \partial_j G_a(x) = \frac{a^{d/2-1}}{(2\pi)^{d/2}} \left\{ \delta_{i,j} \left( \left( 1 - \frac{d}{2} \right) K_{d/2-1}(a|x|) + a|x| K'_{d/2-1}(a|x|) \right) \right. 
\]

\[
+ \left. \frac{x_i x_j}{|x|^{1+d/2}} \left( \left( \frac{d^2}{4} - 1 \right) K_{d/2-1}(a|x|) + (1 - d)a|x| K'_{d/2-1}(a|x|) + a^2|x|^2 K''_{d/2-1}(a|x|) \right) \right\}.
\]
where \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) if \( i \neq j \). With this at hand, we obtain
\[
(d - 1)(d - 2)G_a(x - y) + 2(d - 1)(x - y) \cdot (\nabla G_a)(x - y) + (x - y)[D^2G_a(x - y)](x - y)^t
= \frac{a^{d/2 - 1}}{(2\pi)^{d/2}}|x - y|^{1-d/2} \left\{ \frac{1}{4} d(d - 2)K_{d/2-1}(a|x - y|) + \frac{1}{2} d a|x - y|K_{d/2-1}''(a|x - y|) \right\}. \tag{23}
\]
Using \( \ref{10} \) we see that \( \ref{24} \) can be rewritten as
\[
(d - 1)(d - 2)G_a(x - y) + 2(d - 1)(x - y) \cdot (\nabla G_a)(x - y) + (x - y)[D^2G_a(x - y)](x - y)^t
= \frac{a^{d/2 - 1}}{(2\pi)^{d/2}}|x - y|^{1-d/2} \left\{ (d - 1)a|x - y|K_{d/2-1}''(a|x - y|) \right\}. \tag{24}
\]
From \( \ref{24}, \ref{11} \) and \( \ref{12} \) we deduce that
\[
(d - 1)(d - 2)G_a(x - y) + 2(d - 1)(x - y) \cdot (\nabla G_a)(x - y) + (x - y)[D^2G_a(x - y)](x - y)^t
= (d - 1)a|x - y|H_a(x - y) + \left( a^2|x - y|^2 + \frac{1}{2} (d - 1)(d - 2) \right) G_a(x - y).
\]
Plugging this into \( \ref{22} \) we conclude that
\[
\int_{\partial \Omega \cap \Omega} \int_{\Omega} G_a(x - y) \left( \nu(x) \cdot \frac{x - y}{|x - y|} \right) \left( \nu(y) \cdot \frac{x - y}{|x - y|} \right) \, d\sigma(x) \, d\sigma(y)
= \int_{\Omega} \int_{\Omega} \left\{ (d - 1)a \frac{H_a(x - y)}{|x - y|} + a^2 \frac{1}{2} (d - 1)(d - 2) \right\} G_a(x - y) \, dx \, dy
= a^2 \Lambda(\Omega, a) + \int_{\Omega} \int_{\Omega} (d - 1) \left\{ a \frac{H_a(x - y)}{|x - y|} + \frac{1}{2} (d - 1) \right\} G_a(x - y) \, dx \, dy,
\]
which gives \( \ref{18} \) thanks to \( \ref{15} \).

The following is the main result in this section and provides a sharp inequality, which is only attained when \( \Omega \) is a ball, relating a solid and a boundary integral given in terms of the Bessel potential. From this sharp inequality we will extract the monotone behavior mentioned in the introduction which will lead to the proof of Theorem \( \ref{11} \) through Theorem \( \ref{12} \).

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain. Then,
\[
0 \leq 4 \int_{\Omega} \int_{\Omega} \frac{\partial}{\partial a} \left( a^2 \frac{\partial}{\partial a} \left( \frac{G_a(x - y)}{|x - y|^2} \right) \right) \, dx \, dy + \int_{\partial \Omega \cap \Omega} \int_{\Omega} G_a(x - y) |\nu(x) - \nu(y)|^2 \, d\sigma(x) \, d\sigma(y)
\]
for all \( a > 0 \). The equality is attained for some (and thus for all) \( a > 0 \) if and only if \( \Omega \) is a ball.

**Proof.** Using \( \ref{17} \), we can split
\[
a^2 \Lambda(\Omega, a) = \int_{\partial \Omega \cap \Omega} \int_{\Omega} G_a(x - y) \nu(x) \cdot \nu(y) \, d\sigma(x) \, d\sigma(y)
= \int_{\partial \Omega \cap \Omega} \int_{\Omega} G_a(x - y) \nu(x) \cdot \left( \nu(y) - 2 \frac{\nu(y) \cdot (x - y)}{|x - y|^2} (x - y) \right) \, d\sigma(x) \, d\sigma(y)
+ 2 \int_{\partial \Omega \cap \Omega} \int_{\Omega} G_a(x - y) \left( \nu(x) \cdot \frac{x - y}{|x - y|} \right) \left( \nu(y) \cdot \frac{x - y}{|x - y|} \right) \, d\sigma(x) \, d\sigma(y) =: I_1 + I_2. \tag{25}
\]
Furthermore, the equality in (26) is attained if and only if
\[ \Omega \]

\[ \text{hence the Cauchy-Schwarz inequality shows that} \]
\[
I_1 \leq \int_{\partial \Omega} \int_{\partial \Omega} G_a(x-y) \frac{1}{2} \left( |\nu(x)|^2 + |\nu(y) - 2 \frac{\nu(y) \cdot (x-y)}{|x-y|^2} (x-y)|^2 \right) d\sigma(x) d\sigma(y) 
= \int_{\partial \Omega} \int_{\partial \Omega} G_a(x-y) d\sigma(x) d\sigma(y). \tag{26}
\]

Furthermore, the equality in (26) is attained if and only if
\[
\nu(x) = \nu(y) - 2 \frac{\nu(y) \cdot (x-y)}{|x-y|^2} (x-y) \quad \text{for all } x, y \in \partial \Omega. \tag{27}
\]

But, since \( \Omega \) is bounded, the Reflection Lemma shows that (27) holds if and only if \( \Omega \) is a ball, see [7, Lemma 5.3 on page 45]. Therefore, combining (25), (26) and (27), we get that
\[
0 \leq \int_{\partial \Omega} \int_{\partial \Omega} G_a(x-y) d\sigma(x) d\sigma(y) + a^2 \Lambda(\Omega, a) + 2a(d-1) \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) dx dy, \tag{28}
\]
and the equality is attained if and only if \( \Omega \) is a ball. Thanks to (17), we can rewrite (28) as
\[
-2a(d-1) \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) dx dy \leq \int_{\partial \Omega} \int_{\partial \Omega} G_a(x-y) \left(1 + \nu(x) \cdot \nu(y)\right) d\sigma(x) d\sigma(y). \tag{29}
\]
Subtracting \( 2a^2 \Lambda(\Omega, a) \) on both sides of (29) and using (16) and (17), we arrive at
\[
-2 \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial}{\partial a} \left( a^2 \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) \right) dx dy = -2 \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial}{\partial a} \left( a^{d-1} a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) \right) dx dy 
= -2(a^{d-1}) \int_{\partial \Omega} \int_{\partial \Omega} \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) dx dy 
\]
which proves the inequality in the statement of the theorem. As before, the equality is attained if and only if \( \Omega \) is a ball.
3. A monotonicity formula related to the perimeter

In this section we deal with the function $\Phi(\Omega, a)$ introduced in \((1)\). We prove that it is monotone on $a$ thanks to Theorem 2.3. Furthermore, $\Phi(\Omega, a)$ is constant if and only if $\Omega$ is a ball, and it is strictly decreasing otherwise. As we explained in the introduction, we compute its limit when $a \to 0$ and $a \to +\infty$, obtaining $|\nu|^2_{0, \partial \Omega}$ and $\sigma(\partial \Omega)$ modulo some precise constants, respectively.

Throughout this section, we assume that $d \geq 3$ is an integer, see Remark 3.2 in what concerns the case $d = 2$. In order to study the asymptotic behavior of $\Phi(\Omega, a)$ with respect to $a$, we introduce two auxiliary functions related to the Bessel potential. Set

$$W(x) := \frac{1}{(2\pi)^{d/2}} \int_{|x|}^{+\infty} t^{d/2-1}K_{d/2-1}(t) \, dt, \quad F(x) := \left(1 - \frac{d}{2}\right)\frac{G(x)}{|x|} - H(x)$$

(30)

for $x \in \mathbb{R}^d \setminus \{0\}$ and

$$W_a(x) := W(ax), \quad F_a(x) := a^d F(ax)$$

(31)

for $a > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$, where $G$ and $H$ are given in \((11)\). The following lemma states the relation between $W, F$ and $G$, as well as some properties that will be useful for computing the above-mentioned limits with respect to $a$.

**Lemma 3.1.** The following identities hold for all $a > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$:

$$\int_a^{+\infty} G_s(x) \, ds = W_a(x) |x|^{d-1},$$

(32)

$$-a^2 \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = F_a(x) \frac{|x|^2}{|x|}.$$

(33)

Furthermore,

- (i) $\lim_{a \to +\infty} W_a(x) = 0$ and $\lim_{a \to 0} W_a(x) = \kappa$ for all $x \in \mathbb{R}^d \setminus \{0\}$, where $0 < \kappa < +\infty$ is the constant given in Theorem 1.2, namely,

$$\kappa := \frac{1}{(2\pi)^{d/2}} \int_0^{+\infty} t^{d/2-1}K_{d/2-1}(t) \, dt = \frac{2^{1-d/2}}{|S^{d-1}|} \int_0^{+\infty} \frac{\cosh \left((d/2 - 1)t\right)}{(\cosh t)^{d/2}} \, dt,$$

(34)

(ii) $F(x) > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\int_{\mathbb{R}^d}(1 + |x|^{-1})F(x) \, dx < +\infty$.

**Proof.** Using \((12)\), \((14)\) and a change of variables, we obtain

$$\int_a^{+\infty} G_s(x) \, ds = \int_a^{+\infty} s^{d-2}|s|x|^{1-d/2} \frac{(2\pi)^{d/2}}{|S^{d-1}|} K_{d/2-1}(s|x|) \, ds$$

$$= \frac{|x|^{1-d}}{(2\pi)^{d/2}} \int_a^{+\infty} t^{d/2-1}K_{d/2-1}(t) \, dt = |x|^{1-d}W_a(x),$$

which is \((32)\). Regarding \((33)\), by \((15)\), \((12)\), \((30)\) and \((31)\),

$$\frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = \frac{H_a(x)}{|x|} + \left( \frac{d}{2} - 1 \right) \frac{G_a(x)}{|x|^2} = \frac{a^d-2}{a^2|x|} \left( H(ax) + \left( \frac{d}{2} - 1 \right) \frac{G(ax)}{|ax|} \right) = -\frac{F_a(x)}{|x|^2}. $$

We now adress to (i) and (ii) in the lemma. Regarding (i), it is clear from \((13)\) that $\kappa > 0$. Moreover, by \((13)\), Fubini’s theorem and a change of variables we see that, for every $\alpha \geq 0$,

$$\int_{0}^{+\infty} t^{\alpha} K_a(t) \, dt = \int_{0}^{+\infty} \int_{0}^{t} t^{a-1} e^{-t \cosh r} \cosh(\alpha r) \, dr \, dt$$

$$= \int_{0}^{+\infty} \cosh(\alpha r) \frac{1}{(\cosh r)^{\alpha+1}} \int_{0}^{+\infty} s^\alpha e^{-s} \, ds \, dr = \Gamma(\alpha + 1) \int_{0}^{+\infty} \frac{\cosh(\alpha r)}{(\cosh r)^{\alpha+1}} \, dr,$$

(35)
where \( \Gamma \) denotes the Gamma function. Since \( \Gamma(d/2)|S^{d-1}| = 2\pi^{d/2} \), see [15 Proposition 0.7], (35) proves (34). Observe also that, for \( r > 0 \),
\[
0 \leq \frac{\cosh(\alpha r)}{(\cosh r)^{\alpha+1}} = 2^\alpha \frac{e^{\alpha r} + e^{-\alpha r}}{(e^r + e^{-r})^{\alpha+1}} \leq 2^\alpha \frac{2e^{\alpha r}}{e^{(\alpha+1)r}} = 2^{\alpha+1}e^{-r},
\]
which is integrable in \((0, +\infty)\). Therefore, (35) and (36) show that \( \kappa < +\infty \). With this at hand, that \( \lim_{a \to +\infty} W_a(x) = 0 \) and \( \lim_{a \to 0} W_a(x) = \kappa \) follow by dominated convergence. The proof of (i) is complete.

In order to prove (ii) we need to use the asymptotic behavior of \( K_\alpha(t) \) and \( K'_\alpha(t) \) as \( t \to +\infty \) when \( \alpha \geq 0 \). By [28 page 206], we know that
\[
K_\alpha(t) = \left( \frac{\pi}{2t} \right)^{1/2} \frac{e^{-t}}{\Gamma(\alpha + 1/2)} \int_{0}^{+\infty} e^{-r}r^{\alpha-1/2} \left( 1 + \frac{r}{2t} \right)^{\alpha-1/2} dr.
\]
Therefore, for \( t > 0 \) big enough,
\[
K_\alpha(t) \leq \left( \frac{\pi}{2t} \right)^{1/2} \frac{2e^{-t}}{\Gamma(\alpha + 1/2)} \left\{ \int_{0}^{1} \frac{dr}{\sqrt{r}} + \int_{1}^{+\infty} e^{-r}r^{2\alpha} dr \right\}.
\]
From (37) we deduce that there exists \( C_\alpha > 0 \) only depending on \( \alpha \) such that
\[
K_\alpha(t) \leq C_\alpha t^{-1/2}e^{-t} \quad \text{for} \quad t \to +\infty.
\]
(38)

Concerning \( K'_\alpha \), note that
\[
\cosh r \cosh(\alpha r) = \frac{1}{4}(e^r + e^{-r})(e^{\alpha r} + e^{-\alpha r}) = \frac{1}{2}(\cosh((\alpha + 1)r) + \cosh(|\alpha - 1|r)),
\]
thus using (12) we see that
\[
K'_\alpha(t) = -\int_{0}^{+\infty} e^{-t} \cosh r \cosh(\alpha r) dr = -\frac{1}{2}(K_{\alpha+1}(t) + K_{|\alpha-1|}(t))
\]
for all \( t > 0 \). Then, (39) and (38) prove that
\[
|K'_\alpha(t)| \leq C_\alpha t^{-1/2}e^{-t} \quad \text{for} \quad t \to +\infty.
\]
(40)

With these estimates at hand, we are ready to deal with the first statement in (ii). Fix \( x \in \mathbb{R}^d \setminus \{0\} \). From (16) and using that \( K_{d/2-1} \) is a positive function, we know that
\[
\frac{\partial}{\partial a} \left( a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) \right) = a^{3-d}G_a(x) > 0
\]
for all \( a > 0 \). Additionally, using (15), (12) and (11) we see that
\[
a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = a \frac{H(ax)}{|x|} + \left( \frac{d}{2} - 1 \right) \frac{G(ax)}{|x|^2}
\]
\[
= a^{2-d/2} \frac{|x|^{-d/2}}{(2\pi)^{d/2}} K'_{d/2-1}(a|x|) + \left( \frac{d}{2} - 1 \right) a^{1-d/2} \frac{|x|^{-1-d/2}}{(2\pi)^{d/2}} K_{d/2-1}(a|x|),
\]
and therefore
\[
\lim_{a \to +\infty} a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) = 0
\]
by (38) and (10). In conclusion, (11) and (12) prove that \( a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) \), as a function of \( a > 0 \), is strictly increasing and converges to 0 at infinity, thus
\[
a^{3-d} \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) < 0
\]
for all $a > 0$. Then, applying (15) to (43) and taking $a = 1$, we obtain that

$$0 > \frac{\partial}{\partial a} \left( \frac{G_a(x)}{|x|^2} \right) \bigg|_{a=1} = \frac{H(x)}{|x|} + \left( \frac{d}{2} - 1 \right) \frac{G(x)}{|x|^2} = -\frac{F(x)}{|x|},$$

where we used (30) in the last equality above. Therefore, $F(x) > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

Finally, let us address the second statement in (ii). For this purpose, we need to study the asymptotic behavior of $tK'_a(t) + \alpha K_a(t)$ as $t \to 0$ when $\alpha \geq 1/2$. We are going to consider two different cases: $\alpha > 1/2$ and $\alpha = 1/2$, which correspond to $d > 3$ and $d = 3$, respectively, since we are denoting $\alpha = d/2 - 1$. Assume first that $\alpha > 1/2$. Using (2) 9.6.25 we can write

$$K_\alpha(t) = \frac{2\pi \Gamma(\alpha + 1/2)}{(\pi t)^{\alpha + 1/2}} \int_0^{+\infty} \frac{\cos(t)}{(r^2 + 1)^{\alpha + 1/2}} dr$$

for all $t > 0$, thus

$$|tK'_a(t) + \alpha K_a(t)| = \frac{2\pi \Gamma(\alpha + 1/2)}{(\pi t)^{\alpha + 1/2}} \int_0^{+\infty} \frac{r \sin(t)}{(r^2 + 1)^{\alpha + 1/2}} dr.$$  \hspace{1cm} (45)

These computations are justified because the integrals appearing in (44) and (45) converge absolutely, since we are assuming that $\alpha > 1/2$. By a change of variables, if $t > 0$ is small enough,

$$\int_0^{+\infty} \frac{r \sin(t)}{(r^2 + 1)^{\alpha + 1/2}} dr = t^{2\alpha - 1} \left| \int_0^{+\infty} \frac{s \sin(s)}{(s^2 + t^2)^{\alpha + 1/2}} ds \right| \leq t^{2\alpha - 1} \left\{ \int_0^1 \frac{s ds}{(s^2 + t^2)^{\alpha + 1/2}} + \int_1^{+\infty} s^{-2\alpha} ds \right\} \leq C. \hspace{1cm} (46)$$

Therefore, (45) and (46) yield that $|tK'_a(t) + \alpha K_a(t)| \leq C$ if $t > 0$ is small enough. Combining this with (45) and (40) we finally deduce that, for $\alpha > 1/2$,  

$$\begin{cases} |tK'_a(t) + \alpha K_a(t)| \leq O(e^{-t/2}) & \text{for } t \to +\infty, \\ |tK'_a(t) + \alpha K_a(t)| \leq O(1) & \text{for } t \to 0. \end{cases} \hspace{1cm} (47)$$

Assume now that $\alpha = 1/2$. In this case $K_\alpha$ has a simple representation (see [2] 10.2.17 for example), that is, $K_{1/2}(t) = \sqrt{\frac{\pi}{t}} t^{-1/2} e^{-t}$ for $t > 0$. Then,

$$|tK'_{1/2}(t) + \frac{1}{2} K_{1/2}(t)| = \sqrt{\frac{\pi}{2}} t^{1/2} e^{-t}.$$  \hspace{1cm} (48)

Using also (38) and (40), we conclude that

$$\begin{cases} |tK'_{1/2}(t) + \frac{1}{2} K_{1/2}(t)| \leq O(e^{-t/2}) & \text{for } t \to +\infty, \\ tK'_{1/2}(t) + \frac{1}{2} K_{1/2}(t) \leq O(t^{1/2}) & \text{for } t \to 0. \end{cases} \hspace{1cm} (49)$$

We are ready to prove the second statement in (ii). By (39), (41) and a change of variables to polar coordinates, we have

$$\int_{\mathbb{R}^d} \left( 1 + \frac{1}{|x|} \right) F(x) \, dx = \left[ \frac{d-1}{2(\pi)^d/2} \right] \int_0^{+\infty} r^{d/2-2} (r + 1) \left\{ \left( 1 - \frac{d}{2} \right) K_{d/2-1}(r) - r K'_{d/2-1}(r) \right\} dr. \hspace{1cm} (49)$$

Then, that \(\int_{\mathbb{R}^d} (1 + |x|^{-1}) F(x) \, dx < +\infty\) follows by (47) if $d > 3$ and by (48) if $d = 3$. \hfill \Box
Remark 3.2. Assume that \( d = 2 \). Then, the estimates in the proof of Lemma 3.1(ii) to bound the integral \( \int_{\mathbb{R}^d} (1 + |x|^{-1}) F(x) \, dx \) fail. Indeed, by (30), (11) and (29), we have
\[
F(x) = -H(x) = -\frac{1}{2\pi} K'_0(|x|) = \frac{1}{2\pi} K_1(|x|). 
\]
(50)
It is known that \( K_1(t) \sim \Gamma(1)t^{-1} \) for \( t \to 0 \), see [2] 9.6.9]. But then, arguing as in (49) and using (50), we see that
\[
\int_{\mathbb{R}^2} |x|^{-1} F(x) \, dx = - \int_{0}^{+\infty} K'_0(r) \, dr = \int_{0}^{+\infty} K_1(r) \, dr = +\infty,
\]
thus the second statement in Lemma 3.1(ii) does not hold when \( d = 2 \). We must stress that this is the unique point where we require that \( d \geq 3 \); the finiteness of \( \int_{\mathbb{R}^d} |x|^{-1} F(x) \, dx \) is used in (50) below. The rest of the arguments in the article work for all integer \( d \geq 2 \).

Let \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain and \( a > 0 \). By (51), (30), and (11), we see that \( \Phi(\Omega, a) \) defined in (60) rewrites as
\[
\Phi(\Omega, a) = \int_{\partial\Omega} \int_{\partial\Omega} W_\alpha(x-y) \frac{\nu(x)-\nu(y)}{|x-y|^{d-1}} \, d\sigma(x) \, d\sigma(y) + 4 \int_{\Omega} \int_{\Omega} \frac{F_a(x-y)}{|x-y|} \, dx \, dy.
\]
For simplicity of notation, we also introduce the constant
\[
\tilde{\kappa} := \int_{S^{d-1}} |x-e|^{3-d} \, d\sigma(x) < +\infty,
\]
where \( e \in S^{d-1} \) is any unit vector. For example, when \( d = 3 \) we trivially get \( \tilde{\kappa} = |S^2| = 4\pi \).

Proof of Theorem 1.2 Thanks to (32) and (33), we see that
\[
\frac{\partial \Phi}{\partial a}(\Omega, a) = - \int_{\partial\Omega} \int_{\partial\Omega} G_a(x-y) \nu(x) - \nu(y))^2 \, d\sigma(x) \, d\sigma(y)
- 4 \int_{\Omega} \int_{\Omega} a^2 \frac{\partial}{\partial a} \left( \frac{G_a(x-y)}{|x-y|^2} \right) \, dx \, dy.
\]
Then (7) follows directly from Theorem 2.3 which also shows that the equality in (7) is attained for some (and thus for all) \( a > 0 \) if and only if \( \Omega \subset \mathbb{R}^d \) is a ball.

We now focus on (9). Given \( R > 0 \), set \( B_R := \{ x \in \mathbb{R}^d : |x| < R \} \). Take \( R \) big enough so that \( \overline{\Omega} \subset B_{R/2} \). Then, we can split
\[
\int_{\Omega} \int_{\Omega} F_a(x-y) \, dx \, dy = \int_{B_R} \int_{B_R} F_a(x-y) \, |x-y|^{-1} \chi \Omega(x) \chi \Omega(y) \, dx \, dy + \int_{B_R} \int_{\Omega \setminus B_R} F_a(x-y) \, |x-y|^{-1} \, dx \, dy.
\]
(52)
In order to deal with the two terms on the right hand side of (52), recall that \( F_a(x) = a^d F(ax) \) for \( x \in \mathbb{R}^d \setminus \{0\} \) and that \( F \) is a positive and radial function such that \( 0 < \| F \|_{L^1(\mathbb{R}^d)} < +\infty \), see Lemma 3.1(ii) and (31). In particular, a change of variables gives \( \int_{\mathbb{R}^d} F_a(x) \, dx = \| F \|_{L^1(\mathbb{R}^d)} \) and, for every \( \epsilon > 0 \),
\[
\lim_{a \to +\infty} \int_{|x| > \epsilon a} F_a(x) \, dx = \lim_{a \to +\infty} \int_{|x| > \epsilon a} F(x) \, dx = 0.
\]
(53)
Therefore, \( F_a/\| F \|_{L^1(\mathbb{R}^d)} \) is a positive and radial approximation of the identity as \( a \to +\infty \).

Concerning the first term on the right hand side of (52), since \( F_a \) is radial, an application of Fubini’s theorem gives that
\[
\int_{B_R} \int_{B_R} F_a(x-y) \, |x-y|^{-1} \chi \Omega(x) \chi \Omega(y) \, dx \, dy = \int_{B_R} \int_{B_R} F_a(x-y) \chi \Omega(x) - \chi \Omega(y) \, |x-y|^{-1} \, dx \, dy.
\]
Therefore, \[9, Theorem 1\] shows that
\[
\lim_{a \to +\infty} \int_{B_R} \int_{B_R} \frac{F_a(x - y)}{|x - y|} \chi_\Omega(x) \chi_{\Omega^c}(y) \, dx \, dy = C_0 |\chi_\Omega|_{BV(B_R)},
\]
where \(C_0 > 0\) is some constant only depending on \(d\) and
\[
|\chi_\Omega|_{BV(B_R)} := \sup \left\{ \int_{B_R} \chi \, \text{div} \varphi : \varphi \in C^1_c(B_R; \mathbb{R}^d), |\varphi| \leq 1 \text{ in } B_R \right\}.
\]
It is well known that \(|\chi_\Omega|_{BV(B_R)} = C_1 |\partial \Omega|\) whenever \(\overline{\Omega} \subset B_R\) (see [19], for example), where \(C_1 > 0\) is some constant only depending on \(d\). Thus (54) yields
\[
\lim_{a \to +\infty} \int_{B_R} \int_{B_R} \frac{F_a(x - y)}{|x - y|} \chi_\Omega(x) \chi_{\Omega^c}(y) \, dx \, dy = C_2 |\partial \Omega|
\]
for some constant \(C_2 > 0\) only depending on \(d\).

Regarding the second term on the right hand side of (52), using that \(\overline{\Omega} \subset B_{R/2}\), Fubini’s theorem and a change of variable, we can easily estimate
\[
\int_{B_R} \int_{\Omega} \frac{F_a(x - y)}{|x - y|} \, dx \, dy \leq \int_{B_R} \int_{B_{R/2}} \frac{F_a(x - y)}{|x - y|} \, dx \, dy
\]
\[
\leq \frac{2}{R} \int_{B_{R/2}} \int_{|y| > R/2} F_a(x - y) \, dy \, dx = \frac{2}{R} |B_{R/2}| \int_{|y| > R/2} F_a(y) \, dy,
\]
thus (53) yields
\[
\lim_{a \to +\infty} \int_{B_R} \int_{\Omega} \frac{F_a(x - y)}{|x - y|} \, dx \, dy = 0.
\]

By (34), (50) and (51), we have \(0 < W_a(x) < \kappa\) for all \(a > 0\) and \(x \in \mathbb{R}^d \setminus \{0\}\) because \(K_{d/2-1}\) is a positive function. Also, by the regularity of \(\Omega\), there exists \(M > 0\) such that \(|\nu(x) - \nu(y)| \leq M|x - y|\). Hence we can estimate
\[
0 \leq W_a(x - y) \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \leq \kappa M^2 |x - y|^{3-d},
\]
which is absolutely integrable in \(\partial \Omega \times \partial \Omega\). Therefore, by dominated convergence and Lemma 3.1(i) we get
\[
\lim_{a \to +\infty} \int_{\partial \Omega} \int_{\partial \Omega} W_a(x - y) \frac{|\nu(x) - \nu(y)|^2}{|x - y|^{d-1}} \, d\sigma(x) \, d\sigma(y) = 0.
\]
Finally, a combination of (52), (50), (56) and (57) shows that
\[
\lim_{a \to +\infty} \Phi(\Omega, a) = C_3 |\partial \Omega|.
\]
for some constant \(C_3 > 0\) only depending on \(d\). The precise value of \(C_3\) can be tracked from [9] and computing \(\|F\|_{L^1(\mathbb{R}^d)}\). However, later on we will easily deduce that \(C_3 = \kappa \bar{\kappa}\) with \(\kappa\) as in Theorem 1.2 and \(\bar{\kappa}\) as in (51) by looking at the case of balls. This will yield (9). Once this is known, the fact that \(\Phi(\Omega, a) = \kappa \bar{\kappa} |\partial \Omega|\) for all \(a > 0\) if \(\Omega\) is a ball and that \(\Phi(\Omega, a)\) is strictly decreasing in \(a \in (0, +\infty)\) and converges to \(\kappa \bar{\kappa} |\partial \Omega|\) when \(a \to +\infty\) if \(\Omega\) is not a ball follows by (7) and (9).

Let us now deal with (5). A change of variables and Lemma 3.1(ii) show that
\[
\frac{1}{a} \int_{\mathbb{R}^d} \frac{F_a(x)}{|x|} \, dx = a^d \int_{\mathbb{R}^d} \frac{F(ax)}{|ax|} \, dx = \int_{\mathbb{R}^d} \frac{F(y)}{|y|} \, dy < +\infty.
\]
Hence,
\[
0 \leq \lim_{a \to 0} \int_{\Omega} \int_{\Omega} F_a(x-y) \frac{dx \, dy}{|x-y|} \leq \lim_{a \to 0} \int_{\Omega} \int_{\mathbb{R}^d} F_a(x-y) \frac{dy \, dx}{|x-y|} \leq |\Omega| \int_{\mathbb{R}^d} \frac{F(y)}{|y|} \, dy \lim_{a \to 0} a = 0. \tag{59}
\]

Additionally, by monotone convergence and Lemma 3.11(i),
\[
\lim_{a \to 0} \int_{\partial \Omega} \int_{\partial \Omega} W_a(x-y) \frac{\nu(x) - \nu(y)^2}{|x-y|^{d-1}} \, d\sigma(x) \, d\sigma(y) = \kappa \int_{\partial \Omega} \int_{\partial \Omega} \frac{\nu(x) - \nu(y)^2}{|x-y|^{d-1}} \, d\sigma(x) \, d\sigma(y). \tag{60}
\]
Then (58) is a consequence of (59) and (60).

Finally, assume that \( \Omega \) is a ball of radius \( R > 0 \). Then \( |\partial \Omega| = |S^{d-1}|R^{d-1} \), thus by (58), (5), and the equality in (7) we see that
\[
C_3 |\partial \Omega| = \kappa \int_{\partial \Omega} \int_{\partial \Omega} \frac{\nu(x) - \nu(y)^2}{|x-y|^{d-1}} \, d\sigma(x) \, d\sigma(y) = \kappa R^{d-1} |S^{d-1}| \int_{S^{d-1}} |x - e|^{3-d} \, d\sigma(x) = \kappa |\partial \Omega| \int_{S^{d-1}} |x - e|^{3-d} \, d\sigma(x), \tag{61}
\]
where \( e \in S^{d-1} \) is any unit vector. We used the invariance of \( S^{d-1} \) under rotations in the second equality above. Then, (61) leads to \( C_3 = \kappa \kappa \). In particular, (58) gives (9). This finishes the proof of the theorem. \( \square \)

4. Proof of (2)

By [15, Proposition 3.19] we know that
\[
- \int_{\partial \Omega} \frac{(x-y) \cdot \nu(y)}{|x-y|^d} \, d\sigma(y) = \frac{1}{2} |S^{d-1}| \tag{62}
\]
for all \( x \in \partial \Omega \). If we integrate this equality over all \( x \in \partial \Omega \), we symmetrize the resulting integral, and we apply Cauchy-Schwarz inequality on the integrand, we get
\[
\frac{1}{2} |S^{d-1}| |\partial \Omega| = - \int_{\partial \Omega} \int_{\partial \Omega} \frac{(x-y) \cdot \nu(y)}{|x-y|^d} \, d\sigma(y) \, d\sigma(x)
\leq \frac{1}{2} \int_{\partial \Omega} \int_{\partial \Omega} \frac{\nu(x) - \nu(y)}{|x-y|^{d-1}} \, d\sigma(y) \, d\sigma(x),
\]
which is (2). Furthermore, the equality in (62) is attained if and only if \( \nu(x) - \nu(y) = \lambda(x-y) \) for some constant \( \lambda > 0 \) and all \( x, y \in \partial \Omega \), and this holds if and only if \( \Omega \) is a ball of radius \( 1/\lambda \).

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