Codes over subsets of algebras obtained by the Cayley-Dickson process

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Abstract. In this paper, we define binary block codes over subsets of real algebras obtained by the Cayley-Dickson process and we provide an algorithm to obtain codes with a better rate. This algorithm offers more flexibility than other methods known until now, similar to Lenstra’s algorithm on elliptic curves compared with \( p - 1 \) Pollard’s algorithm.

Keywords. Block codes, Cyclic codes, Integer codes, Codes over Gaussian Integers.

AMS Classification. 94B15, 94B05.

1. Introduction

Integer Codes are codes defined over finite rings of integers modulo \( m, m \in \mathbb{Z} \). Since these codes have a low encoding and decoding complexity, they had a significant development over the last years and are suitable for application in communication systems (see [Ko, Mo, Ii, Ha, Ma; 10]).

Some other codes similar to the Integer Codes, such as for example codes over Gaussian integers ([Hu; 94], [Gh, Fr; 10], [Ne, In, Fa, Pa; 01], [Ri; 95]) or codes over Eisenstein-Jacobi integers, have been intensively studied in recent years.

In this paper, we will extend the study of Integer Codes to codes over subsets of real algebras obtained by the Cayley-Dickson process. This idea comes in a natural way, starting from same ideas developed by Hubner in [Hu; 94], in which he regarded a finite field as a residue field of the Gaussian integer ring modulo a Gaussian prime, ideas extended to Hurwitz integers in [Gu; 13] and to a subset of the Octonions integers in [Fl; 15]. In this way, we regard a finite field as a residue field modulo a prime element from \( V \), where \( V \) is a subset of an algebra \( \mathbb{A}_d (\mathbb{R}) \), where \( \mathbb{A}_d (\mathbb{R}) \) is a real algebra obtained by the Cayley-Dickson process and \( V \) has a commutative and associative ring structure. In this way, we obtain an algorithm, called Main Algorithm, which allows us to find codes with a good rate. This algorithm offers more flexibility than other methods known until now, similar to Lenstra’s algorithm on elliptic curves compared with \( p - 1 \) Pollard’s algorithm. It is well known that for a prime \( p \), Lenstra’s algorithm replace the group \( \mathbb{Z}_p \) with the group of the rational points of an elliptic curve \( C_1 \) over \( \mathbb{Z}_p \) and, if this algorithm failed, the curve will be replaced with another curve \( C_2 \) over \( \mathbb{Z}_p \) and we can retake the algorithm (see [Si, Ta; 92]).
In the case of Main Algorithm, the algebra \( A_t(\mathbb{R}) \) and \( w \) offer this kind of flexibility since, for the same prime \( p \), these can be changed and the algorithm can be retake.

2. Preliminaries

In the following, we shortly recall the Cayley-Dickson process for the real algebras. Let \( A \) be a finite dimensional unitary algebra over the real field \( \mathbb{R} \) with a scalar involution \( \overline{\cdot} : A \to A, \ a \to \overline{a} \), i.e. a linear map satisfying the following relations: \( \overline{ab} = \overline{b}\overline{a}, \ \overline{a} = a, \) and \( a + \overline{a}, a\overline{a} \in \mathbb{R} \) for all \( a, b \in A \). The element \( \overline{a} \) is called the conjugate of the element \( a \), the linear form \( t : A \to \mathbb{R}, \ t(a) = a + \overline{a} \) and the quadratic form \( n : A \to \mathbb{R}, \ n(a) = a\overline{a} \) are called the trace and the norm of the element \( a \). Since \( n(a) = a\overline{a} = a(t(a) - a) \), it results that \( a^2 - t(a) a + n(a) = 0 \), for all elements \( a \in A \), therefore \( A \) is a quadratic algebra.

Let \( \gamma \in \mathbb{R} \) be a fixed non-zero element. On the vector space \( A \oplus A \), we define the following algebra multiplication:

\[
(a_1, a_2)(b_1, b_2) = (a_1b_1 + \gamma b_2\overline{a_2}, \overline{a_1}b_2 + b_1a_2).
\]

We obtain an algebra structure over \( A \oplus A \), denoted by \( (A, \gamma) \) and called the algebra obtained from \( A \) by the Cayley-Dickson process. We have \( \dim (A, \gamma) = 2 \dim A \).

Let \( x \in (A, \gamma) \), \( x = (a_1, a_2) \). The map

\[
\overline{\cdot} : (A, \gamma) \to (A, \gamma), \ x \to \overline{x} = (\overline{a_1}, -a_2),
\]

is a scalar involution of the algebra \( (A, \gamma) \), extending the involution \( \overline{\cdot} \) of the algebra \( A \).

If we take \( A = \mathbb{R} \) and we apply this process \( t \) times, \( t \geq 1 \), we obtain an algebra over \( \mathbb{R} \),

\[
A_t(\mathbb{R}) = \left( \bigoplus_{i=1}^{n-1} \mathbb{R} \right).
\]

In this algebra, the set \( \{e_0 = 1, e_1, ..., e_{n-1}\}, n = 2^t \), generates a basis with the properties:

\[
e_i^2 = \gamma_i 1, \ \gamma_i \in \mathbb{R}, \ \gamma_i \neq 0, \ i \in \{1, ..., n-1\}
\]

and

\[
e_i e_j = -e_j e_i = \beta_{ij} e_k, \ \beta_{ij} \in \mathbb{R}, \ \beta_{ij} \neq 0, \ i \neq j, \ i, j \in \{1, ..., n-1\},
\]

\( \beta_{ij} \) and \( e_k \) being uniquely determined by \( e_i \) and \( e_j \).

Algebras \( A_t(\mathbb{R}) \), obtained by the Cayley-Dickson process, are power-associative (i.e. the subalgebra \( < x > \) of \( A_t(\mathbb{R}) \), generated by any element \( x \in A_t(\mathbb{R}) \), is...
associative), flexible (i.e. \( x(yz) = (xy)z = xyz \), for all \( x, y, z \in A_4(\mathbb{R}) \)) and in general it is nonassociative.

For \( t = 2 \) and \( \gamma_1 = \gamma_2 = -1 \), we obtain the Quaternion division algebra, \( \mathbb{Q}(\mathbb{R}) \), for \( t = 3 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = -1 \), we obtain the Octonion division algebra, \( \mathbb{O}(\mathbb{R}) \), and for \( t = 4 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = -1 \), we obtain the Sedenion algebra, \( \mathbb{S}(\mathbb{R}) \). Due to the Hurwitz’s Theorem, for \( t \geq 4 \), all obtained algebras are not division algebras (i.e. we can find \( a, b \in A_t(\mathbb{R}) \), \( a \neq 0, b \neq 0 \), such that \( ab = 0 \)).

Let \( B = \{1, e_2, ..., e_{2^t}\} \) be the basis in \( A_t(\mathbb{R}) \), where 1 is the unity. If \( x = x_1 + \sum_{i=2}^{2^t} x_ie_i \in A_t(\mathbb{R}) \), then its conjugate is the element \( \overline{x} = x_1 - \sum_{i=2}^{2^t} x_ie_i \) and the norm of the element \( x \) is \( n(x) = x\overline{x} = x_1^2 - \sum_{i=2}^{2^t} x_i^2 \). The norm \( n \), in general, is not multiplicative, i.e. for \( x, y \in A_t(\mathbb{R}) \), we have \( n(xy) \neq n(x)n(y) \). The real part of the element \( x \) is \( x_1 \) and its vector part is \( \sum_{i=2}^{2^t} x_ie_i \in A_t(\mathbb{R}) \).

The set \( \mathcal{A}_t(\mathbb{Z}) = \{z \in \mathcal{A}_t(\mathbb{R}) \mid z = x_1 + \sum_{i=2}^{2^t} x_ie_i, x_i \in \mathbb{Z}, i \in \{2, 3, ..., 2^t\}\} \) is called the integer elements of the real algebra \( \mathcal{A}_t(\mathbb{R}) \). This set has a ring structure. (see [Ma, Be, Ga; 09])

Let \( w = \alpha(1 + \sum_{i=2}^{2^t} e_i) \in \mathcal{A}_t(\mathbb{R}), \alpha \in \mathbb{R}, \), and let \( \mathcal{V} = \{a + bw \mid a, b \in \mathbb{Z}\} \). We note that \( t(x) = 2\alpha, n(x) = 2^t\alpha^2 \) and \( w^2 - 2\alpha w + 2^t\alpha^2 = 0 \). Since the algebra \( \mathcal{A}_t(\mathbb{R}) \) is a power associative algebra, it results that \( \mathcal{V} \) is an associative and a commutative ring.

**Remark 2.1.** For \( x \in \mathcal{V} \), we know that the following properties are equivalent:

i) \( x \) is an invertible element in the algebra \( \mathcal{V} \).
ii) \( n(x) = 1 \).
iii) \( x \in \{\pm 1\} \).

An element \( x \in \mathcal{V} \) is a prime element in \( \mathcal{V} \) if \( x \) is not an invertible element in \( \mathcal{V} \) and if \( x = ab \), it results that \( a \) or \( b \) is an invertible element in \( \mathcal{V} \).

**Proposition 2.2.** i) For \( x, y \in \mathcal{V} \), we have \( n(xy) = n(x)n(y) \).
ii) The ring \( \mathcal{V} \) is a division ring.

**Proof.** i) Denoting with \( q = 2^t - 1 \), let \( x = a + bw \) and \( y = c + dw \). We obtain 
\[
n(x)n(y) = \left[(a + b\alpha)^2 + b^2\alpha^2q\right]\left[(c + d\alpha)^2 + d^2\alpha^2q\right] = \\
(2aba + a^2 + b^2\alpha + b^2\alpha^2)(2ca\alpha + c^2 + d^2\alpha + d^2\alpha^2) = 2abc\alpha + 2a^2cd\alpha + 4abcd\alpha^2 + a^2c^2 + 2abcd\alpha^2 + b^2cd\alpha^3 + 2abd^2\alpha^3 + 2b^2\alpha^3 + 2bd^2\alpha^3 + 2c^2\alpha^2 + b^2d^2\alpha^2 + a^2d^2\alpha^2 + b^2c^2\alpha^2 + 2b^2d^2\alpha^2 + 2b^2d^2\alpha^2 + b^2d^2\alpha^2.
\]
Computing \( n(xy) \), we get...
\( n(xy) = [ac + (ad + bc)\alpha - \alpha^2bd(q + 1) + 2\alpha^2bd]^2 + \alpha^2[ad + bc + 2ab\alpha]^2 = 2abc\alpha + 2a^2cd\alpha + 4abcd\alpha^2 + a^2c^2 + 2abd^2\alpha^3 + 2b^2d^2\alpha^3 + 2abd^2\alpha^3 + 2b^2d^2\alpha^3 + a^2d^2\alpha^3 + b^2c^2\alpha^2 + b^2d^2\alpha + 2a^2d^2\alpha q^2 + b^2c^2\alpha q^2 + 2b^2d^2q^2\alpha^2 + b^2d^2q^2\alpha^2. \)

Therefore \( n(xy) = n(x)n(y). \)

ii) It results from i).

Remark 2.3. The above result is also true for all elements from the set 
\( V' = \{ a + bw \mid a, b \in \mathbb{R} \}. \)

In the following, we will consider \( \alpha = \frac{1}{x}, r \geq t - 1, t \geq 2. \)

Proposition 2.4. If \( x, y \in V, y \neq 0, \) with \( t \geq 2, \) then there are \( z, v \in V \) such that \( x = zy + v \) and \( n(v) < n(y). \)

Proof. Since \( y \neq 0, \) we have that \( y \) is an invertible element in \( \mathbb{A}_t(\mathbb{R}), \)

\[
\frac{1}{y} = a + bw, a, b \in \mathbb{R}. \]

Let \( m, n \in \mathbb{Z} \) such that \( |a - m| \leq \frac{1}{2} \) and \( |b - n| \leq \frac{1}{2}. \)

For \( z = m + nw, v = y[(a - m) + (b - n)w], \) it results that \( \tilde{y} = z + \tilde{\gamma}, \)

\[
\tilde{y} = z + \tilde{\gamma}, \text{ therefore } x = zy + v \text{ and } v = x - zy. \]

From here, we have that \( v \in \mathbb{V}_y. \)

We obtain that \( n(v) = n(y)n((a - m) + (b - n)w) = n(y)[(a - m) + \frac{1}{y}(b - n)]^2 + \frac{2^r + 1}{2^r} - (\frac{2^r + 1}{2^r} + \frac{2^r + 1}{2^r})n(y) = \frac{2^r + 1}{2^r} - n(y) \}

Definition 2.6. With the above notations, let \( \pi = x + yw \) be a prime integer in \( V \) and \( v_1, v_2 \) be two elements in \( V. \)

If there is \( v \in V \) such that \( v_1 - v_2 = v\pi, \)

then \( v_1, v_2 \) are called congruent modulo \( \pi \) and it is denoted by \( v_1 \equiv v_2 \mod \pi. \)

Proposition 2.7. The above relation is an equivalence relation on \( V \). The set of equivalence classes is denoted by \( \mathbb{V}_\pi \)

and is called the residue classes of \( V \) modulo \( \pi. \)

Proof. Denoting the elements from \( \mathbb{V}_\pi \) in bold, if \( v_1 \equiv v_2 \mod \pi \) and \( v_2 \equiv v_3 \mod \pi, \) then there are \( v, v' \in \mathbb{V} \) such that \( v_1 - v_2 = v\pi, \quad v_2 - v_3 = v'\pi. \)

It results that \( v_1 - v_3 = (v + v')\pi, \) therefore the transitivity holds.

Proposition 2.8. For each \( x, y \in V, \) there is \( \delta = (x, y), \) the greatest common divisor of \( x \) and \( y. \)

We also have that there are \( \gamma \) and \( \tau \in V, \)

such that \( \delta = \gamma x + \tau y, \) (the B"{e}zout’s Theorem).

Proof. We denote by \( J = \{ \gamma x + \tau y \mid \gamma, \tau \in \mathbb{V} \}. \)

We remark that if \( z = \gamma'x + \tau'y \in J \) and \( w \in \mathbb{V}, \)

we have \( wz = (w\gamma')x + (w\tau')y \in J. \)

We consider \( \delta_1 = \gamma_1x + \tau_1y \in J, \)

such that \( \delta_1 \) has the norm \( n(\delta_1) \) minimum in \( J. \)

We will prove that \( \delta = \delta_1. \)

From Proposition 2.4, we have that \( x = q_1\delta_1 + r_1, \)

with \( n(r_1) < n(\delta_1), q_1, r_1 \in \mathbb{V} \) and \( r_1 = x - q_1\delta_1 \in J. \)

Since \( n(r_1) < n(\delta_1) \) and \( \delta_1 \in J \)

has minimum norm in \( J, \)

it results that \( r_1 = 0, \) therefore \( \delta_1 \mid x. \)

In the same way, we will prove that \( \delta_1 \mid y. \)

Since \( \delta_1 = \gamma_1x + \tau_1y, \)

it results that each common divisor for \( x \) and \( y \) is a divisor for \( \delta_1, \)

therefore \( \delta \mid \delta_1 \) and finally \( \delta = \delta_1. \)

The above proposition generalized to elements in \( V \) Proposition 2.1.4. from [Da, Sa, Va:03].
Proposition 2.9. \( \mathbb{V}_\pi \) is a field isomorphic to \( \mathbb{Z}/p\mathbb{Z} \), \( p = n(\pi), p \) a prime number.

Proof.

For \( v_1, v_2 \in \mathbb{V}_\pi \), we define \( v_1 + v_2 = (v_1 + v_2) \mod \pi \) and \( v_1 \cdot v_2 = (v_1 v_2) \mod \pi \). These multiplications are well defined. Indeed, if \( v_1 \equiv v_1' \mod \pi \) and \( v_2 \equiv v_2' \mod \pi \), it results that \( v_1 - v_1' = u \pi, v_2 - v_2' = u' \pi, u, u' \in \mathbb{V} \), therefore \( (v_1 + v_2) - (v_1' + v_2') = (u + u') \pi \). Since \( v_1 = v_1' + u \pi \), \( v_2 = v_2' + u' \pi \), it results that \( v_1 v_2 = v_1' v_2' + M \pi \), with \( M \pi \) a multiple of \( \pi \).

Denoting in bold the equivalence classes from \( \mathbb{Z}_p \), let \( f \) be the map

\[
f : \mathbb{Z}_p \rightarrow \mathbb{V}_\pi, \quad f(m) = (m + \pi) \text{ mod } \pi, \text{ where } m \in \mathbb{m}.
\] (2.1)

Map \( f \) is well defined. Indeed, if \( m \equiv m' \mod p \) we have \( (m + \pi) - (m' + \pi) = m - m' = pq = \pi q \), \( q \in \mathbb{Z} \), therefore \( (m + \pi) \equiv (m' + \pi) \mod \pi \).

From Proposition 2.8, we have \( 1 = v_1 \pi + v_2 \pi \). If \( f(m) = v, v = (m + \pi) \mod \pi \in \mathbb{V} \), we define \( f^{-1}(v) = m = (v_1 \pi) + (v_2 \pi) = m \).

Map \( f \) is a ring morphism. Indeed, \( f(m) + f(m') = (m + \pi) \mod \pi + (m' + \pi) \mod \pi = (m + m' + \pi) \mod \pi = f(m + m') \) and \( f(m)f(m') = (m + \pi)(m' + \pi) \mod \pi = (mm' + \pi^2) \mod \pi = (mm' + \pi) \mod \pi \). We obtain that \( \mathbb{V}_\pi \) is isomorphic to \( \mathbb{Z}_p \). \( \square \)

Let \( x = a + bw \in \mathbb{V} \), therefore we have \( n(x) = (a + bw)^2 + q(ba)^2 \). For \( q = 2^t - 1 \) and for certain values of \( t \), we know the form of some prime numbers, as we can see in the proposition below.

Proposition 2.9. ([Co; 89], [Sa; 14])

Let \( p \in \mathbb{N} \) be a prime number.

1. There are integers \( a, b \) such that \( p = a^2 + 3b^2 \) if and only if \( p \equiv 1(\mod 3) \) or \( p = 3 \).
2. There are integers \( a, b \) such that \( p = a^2 + 7b^2 \) if and only if \( p \equiv 1, 2, 4(\mod 7) \) or \( p = 7 \).
3. There are integers \( a, b \) such that \( p = a^2 + 15b^2 \) if and only if \( p \equiv 1, 19, 31, 49(\mod 60) \).

\( \square \)

The label Algorithm for \( \mathbb{A}_t(\mathbb{R}) \).

1. We will fix \( t, \alpha \) and therefore \( w \).
2. We consider \( \pi \in \mathbb{V} \) a prime element, \( \pi = a + bw, a, b \in \mathbb{Z} \), such that \( n(\pi) = p = (a + bw)^2 + q(ba)^2 \), with \( p \) a prime positive number.
3. Let \( s \in \mathbb{Z} \) be the only solution to the equation \( a + bx = 0 \mod p \), \( x \in \{0, 1, 2, ..., p - 1\} \).
4. Let \( r = \left[ \frac{p - 1}{2} \right] \in \mathbb{N} \), where \( [ \cdot ] \) denotes the integer part.
5. Let \( k \in \mathbb{Z} \) and \( k \in \mathbb{Z}_p \) be its equivalence class modulo \( p \).
6. For all integers \( \sigma, \tau \in \{ -r - 1, ..., r \} \), let \( c = (s\tau + \sigma) \mod p \) and \( d = (\sigma + \tau\alpha)^2 + q(\tau\alpha)^2 \).
6. If \( d < p \) and \( c = k \), then we find the pairs \((\sigma, \tau)\) such that \( \mathbf{k} \) is the label of the element \( \sigma + \tau w \in \mathbb{V}_\pi \), that means \( \sigma + \tau s = k \mod p \) and \( n (\sigma + \tau w) \) is minimum. If we find more than two pairs satisfying the last condition, then we will choose that pair with the property that \(|\sigma| + |\tau| \leq |a| + |b| \). If there exist more than two pairs satisfying the last inequality, then we will choose one of them randomly.

3. Codes over \( \mathbb{V}_\pi \)

In the following, we will recall some definitions, which will be used in this section.

We consider the ring of Gaussian integers, \( \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\} \). We know that a prime integer \( p \) of the form \( p \equiv 1 \mod 4 \) can be written of the form \( p = \pi \bar{\pi} \), where \( \pi, \bar{\pi} \in \mathbb{Z}[i] \) and \( \bar{\pi} \) is the conjugate of \( \pi \). Let \( (\mathbb{Z}[i])_\pi \) be the set of the residue classes modulo \( \pi \). A block code \( C \) of length \( n \) over \( (\mathbb{Z}[i])_\pi \) is defined to be a set of codewords \( c = (c_1, ..., c_n) \), where \( c_i \in (\mathbb{Z}[i])_\pi \), \( i \in \{1, 2, ..., n\} \). For \( \alpha', \beta', \gamma' \in (\mathbb{Z}[i])_\pi \), with \( \gamma' = \alpha' - \beta' \mod \pi \), the Mannheim weight of \( \gamma' \), denoted by \( w_M(\gamma') \), is defined as

\[
 w_M(\gamma') = |Re(\gamma')| + |Im(\gamma')| ,
\]

where \( Re(\gamma') \) represents the real part of the element \( \gamma' \) and \( Im(\gamma') \) represents the imaginary part of the element \( \gamma' \). Using the Mannheim weight, we can define the Mannheim distance between \( \alpha' \) and \( \beta' \), denoted by \( d_M \), as follows

\[
d_M(\alpha', \beta') = w_M(\gamma') .
\]

For other details, the readers are referred to [Hu; 94].

Using ideas from the above definition and generalizing the Hurwitz weight from [Gu; 13] and Cayley-Dickson weight for the octonions, from [Fl; 15], in the same manner, we define the generalized Cayley-Dickson weight, for algebras obtained by the Cayley-Dickson process, denoted \( d_G \). We will fix \( t, \alpha, w \), therefore we will consider the elements in the algebra \( \mathbb{K}_t(\mathbb{R}) \). Let \( \pi \) be a prime in \( \mathbb{V}_\pi = a + bw \). Let \( x \in \mathbb{V}_\pi \), \( x = a_0 + b_0 w \). The generalized Cayley-Dickson weight of \( x \) is defined as \( w_G(x) = |a_0| + |b_0| \), where \( x = a_0 + b_0 w \mod \pi \), with \(|a_0| + |b_0| \) minimum.

The generalized Cayley-Dickson distance between \( x, y \in \mathbb{V}_\pi \) is defined as

\[
d_G(x, y) = w_G(x - y)
\]

and we will prove that \( d_G \) is a metric. Indeed, for \( x, y, z \in \mathbb{V}_\pi \), we have \( d_G(x, y) = w_G(\alpha_1) = |a_1| + |b_1| \), where \( \alpha_1 = x - y = a_1 + b_1 w \mod \pi \) is
an element in $\mathbb{V}_\pi$ and $|a_1| + |b_1|$ is minimum.

d_G (y, z) = w_G (\alpha_2) = |a_2| + |b_2|$, where $\alpha_2 = y - z = a_2 + b_2 \mod \pi$ is an element in $\mathbb{V}_\pi$ and $|a_2| + |b_2|$ is minimum.

d_G (x, z) = w_G (\alpha_3) = |a_3| + |b_3|$, where $\alpha_3 = x - z = a_3 + b_3 \mod \pi$ is an element in $\mathbb{V}_\pi$ and $|a_3| + |b_3|$ is minimum.

We obtain $x - z = \alpha_1 + \alpha_2 \mod \pi$ and it results that $w_G (\alpha_1 + \alpha_2) \geq w_G (\alpha_3)$, since $w_G (\alpha_3) = |a_3| + |b_3|$ is minimum, therefore $d_G (x, y) + d_G (y, z) \geq d_G (x, z)$.

In the following, we assume that $\pi$ is a prime in $\mathbb{V}$ with $n (\pi) = p$ a prime number of the form $n (\pi) = Mn + 1$, $M, n \in \mathbb{Z}, n \geq 0$, such that there are $\beta$ a primitive element (of order $p - 1$) in $\mathbb{V}_\pi$, with the properties $\beta^m = w$ or $\beta^{\frac{p-1}{M}} = -w$. We will consider codes of length $n = \frac{p-1}{M}$.

The below definitions and Theorems adapted and generalized to all algebras obtained by the Cayley-Dickson process some definitions from [Gu; 13], [Ne, In, Fa, Pa; 01], [Fl; 15] and Theorems 7,8,9,10,11,13,14,15 from [Ne, In, Fa, Pa; 01], Theorems 4,5,6,7 from [Gu; 13] and Theorems 2.3, 2.5, 2.7, 2.9 from [Fl; 15].

Let $C$ be a code defined by the parity-check matrix $H$,

$$
H = \begin{pmatrix}
1 & \beta & \beta^2 & \cdots & \beta^{n-1} \\
1 & \beta^{M+1} & \beta^2(M+1) & \cdots & \beta^{n-1}(M+1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \beta^{Mk+1} & \beta^2(Mk+1) & \cdots & \beta^{n-1}(Mk+1)
\end{pmatrix},
$$

(3.1)

with $k < n$. We know that $c$ is a codeword in $C$ if and only if $He^{t} = 0$. If we consider the associate code polynomial $c(x) = \sum_{i=0}^{n-1} c_i x^i$, we have that $c \left( \beta^{M+1} \right) = 0, l \in \{0, 1, ..., k\}$. For the polynomial $g(x) = (x - \beta) (x - \beta^{M+1}) \cdots (x - \beta^{M(k+1)})$, since the elements $\beta, \beta^{M+1}, \ldots, \beta^{M(k+1)}$ are distinct, from [10], Lemma 8.1.6, we obtain that $c(x)$ is divisible by $g(x)$, the generator polynomial of the code $C$. Since $g(x) / (x^n \pm w)$, it results that $C$ is a principal ideal in the ring $\mathbb{V}_\pi / (x^n \pm w)$.

If we suppose that a codeword polynomial $c(x)$ is sent over a channel and the error pattern $e(x)$ occurs, it results that the received polynomial is $r(x) = c(x) + e(x)$. The vector corresponding to the polynomial $r(x) = c(x) + e(x)$ is $r = c + e$ and the syndrome of $r$ is $S = Hr^t$, where $H$ is the above parity-check matrix.

**Theorem 3.1.** Let $C$ be the code defined on $\mathbb{V}_\pi$ by the parity check matrix

$$
H = \begin{pmatrix}
1 & \beta & \beta^2 & \cdots & \beta^{n-1}
\end{pmatrix}.
$$

(3.2)

It results that, the code $C$ is able to correct all error patterns of the form $e(x) = e_i x^i$, with $0 \leq w_C (e_i) \leq 1$.

**Proof.** Let $r(x) = c(x) + e(x)$ be the received polynomial, with $c(x)$ the codeword polynomial and $e(x) = e_i x^i$ denoting the error polynomial with $0 \leq w_C (e_i) \leq 1$. Since $\beta^n = w$ or $\beta^n = -w$, it results that $e_i = \beta^{ni}$. We have
the syndrome \( S = \beta^{i+nl} = \beta^L \), with \( i, L \in \mathbb{Z}, 0 \leq i, L \leq n - 1 \). If we reduce \( L \) modulo \( n \), we obtain \( i \), the location of the error, and from here, \( l = \frac{k-1}{n} \) and \( \beta^{nl} \), the value of the error. □

**Theorem 3.2.** Let \( C \) be a code defined by the parity-check matrix

\[
H = \begin{pmatrix}
1 & \beta & \beta^2 & \ldots & \beta^{n-1} \\
1 & \beta^{M+1} & \beta^{2(M+1)} & \ldots & \beta^{(n-1)(M+1)}
\end{pmatrix}.
\]  

(3.3)

Then \( C \) can correct error patterns of the form \( e(x) = e_i x^i, 0 \leq i \leq n - 1 \), with \( e_i \in \mathbb{V}_\pi \).

**Proof.** We consider the received polynomial, \( r(x) = c(x) + e(x) \) with \( c(x) \) the codeword polynomial and \( e(x) = e_i x^i \) the error polynomial with \( e_i \in \mathbb{V}_\pi \). It results that the corresponding vector of the polynomial \( r(x) \) is \( r = c + e \). We will compute the syndrome \( S \) of \( r \). We have \( e_i = \beta^j, 0 \leq j \leq Mn - 1 \). Therefore the syndrome is

\[
S = Hr^i = \begin{pmatrix}
s_1 = \beta^{i+j} = \beta^{M_1} \\
s_{M+1} = \beta^{(M+1)i+j} = \beta^{M_2}
\end{pmatrix}.
\]

We obtain \( \beta^{i+j-M_1} = 1 \), with \( i + j = M_1 \mod(p-1) \) and \( \beta^{(M+1)i+j-M_2} = 1 \), with \( (M+1)i + j = M_2 \mod(p-1) \). We get \( Mi = (M_2 - M_1) \mod(p-1) \), if there is, then the solution of the system is \( i = \frac{M_2 - M_1}{M} \mod n \) and \( j = (M_1 - i) \mod(p-1) \). From here, we can find the location and the value of the error. □

**Theorem 3.3.** Let \( C \) be a code defined by the parity-check matrix

\[
H = \begin{pmatrix}
1 & \beta & \beta^2 & \ldots & \beta^{n-1} \\
1 & \beta^{M+1} & \beta^{2(M+1)} & \ldots & \beta^{(n-1)(M+1)}
\end{pmatrix}.
\]  

(3.4)

Then \( C \) can find the location and can correct error patterns of the form \( e(x) = e_i x^i, 0 \leq i \leq n - 1 \), with \( e_i \in \mathbb{V}_\pi \), or can only correct error patterns of the above mentioned form.

**Proof.** Using notations from the above Theorem, we have \( e_i = \beta^j, 0 \leq j \leq Mn - 1 \). Therefore the syndrome is

\[
S = Hr^i = \begin{pmatrix}
s_1 = \beta^{i+q} = \beta^{M_1} \\
s_{M+1} = \beta^{(M+1)i+j} = \beta^{M_2} \\
s_{2M+1} = \beta^{(2M+1)i+j} = \beta^{M_3}
\end{pmatrix}.
\]

Since the matrix

\[
\begin{pmatrix}
1 & \beta & \beta^2 \\
1 & \beta^{M+1} & \beta^{2(M+1)} \\
1 & \beta^{2M+1} & \beta^{2(2M+1)}
\end{pmatrix}
\]

has its determinant equal to \( \beta^3 \beta^M (\beta^{2M} - 1)^3 \neq 0 \), it results that the rank of the matrix (3.4) is 3, then this system always has a solution. We obtain \( \beta^{i+j-M_1} = 1 \), with \( i + q = M_1 \mod(p-1) \), \( \beta^{(M+1)i+j-M_2} = 1 \), with \( (M+1)i + j = M_2 \mod(p-1) \), \( \beta^{(2M+1)i+j-M_3} = 1 \), with \( (2M+1)i + j = M_3 \mod(p-1) \)
1, with \((2M + 1) i + j = M_3 \mod (p - 1)\). We can find the location of the error if \(M i = (M_2 - M_1) \mod (p - 1)\) and \(Mi = (M_3 - M_2) \mod (p - 1)\) or, equivalently, \(i = \frac{M_3 - M_1}{M} \mod n\) and the value of the error \(e_i\) if \((M_1 - i) \mod (p - 1) = (M_2 - (M + 1) i) \mod (p - 1) = (M_3 - (2M + 1) i) \mod (p - 1) (= j)\).

**Theorem 3.9.** Let \(C\) be a code defined by the parity-check matrix

\[
H = \begin{pmatrix}
1 & \beta & \beta^2 & \ldots & \beta^{n-1} \\
0 & \beta M_{+1} & \beta^2(M+1) & \ldots & \beta^{n-1}(M+1) \\
0 & \beta 2M_{+1} & \beta^2(2M+1) & \ldots & \beta^{n-1}(2M+1) \\
0 & \beta 3M_{+1} & \beta^2(3M+1) & \ldots & \beta^{n-1}(3M+1)
\end{pmatrix}.
\]

Then \(C\) can correct error patterns of the form \(e(x) = e_i x^i + e_j x^j, 0 \leq i, j \leq n - 1, \) with \(e_i, e_j \in \mathbb{F}_p\).

**Proof.** We will prove this in the general case, when we have two errors. We have \(e_i = \beta^i \neq 0\) and \(e_j = \beta^j \neq 0, q', t' \in \mathbb{Z}\). We obtain the syndrome:

\[
S = H r^t = \begin{pmatrix}
s_1 = \alpha^{i+q'} + \alpha^{i+q} \\
s_{M+1} = \alpha^{M+1} q' + \alpha^{M+1} j + t' \\
s_{2M+1} = \alpha^{2M+1} q' + \alpha^{2M+1} j + t' \\
s_{3M+1} = \alpha^{3M+1} q' + \alpha^{3M+1} j + t'
\end{pmatrix}.
\]

We denote \(\beta^i = A\) and \(\beta^i = B\) and we get

\[
S = H r^t = \begin{pmatrix}
s_1 = A + B \\
s_{M+1} = \beta M_1 A + \beta M_1 B \\
s_{2M+1} = \beta 2M_1 A + \beta 2M_1 B \\
s_{3M+1} = \beta 3M_1 A + \beta 3M_1 B
\end{pmatrix}.
\]

If the system (3.6) admits only one solution, then code \(C\) can correct two errors. First, we will prove the following Lemma.

**Lemma.** With the above notations, if we have two errors, we obtain \(\beta^{M_i} \neq \beta^{M_j}, 0 \leq i, j \leq n - 1\) and \(s_1 s_{2M+1} \neq s_{M+1}^2\).

**Proof.** If \(\beta^{M_i} = \beta^{M_j}\), then \(\beta^{M(i-j)} = 1\) and \(Mn / (M(i-j))\), which is false. Supposing that \(s_1 s_{2M+1} - s_{M+1}^2 = 0\), we have \(s_1 s_{2M+1} = s_{M+1}^2\). For \(x = \beta^i + q'\), we can obtain \(\beta 2M_1 s_1 x + \beta 2M_1 s_{M+1} - \beta 2M_1 s_{M+1} = (\beta^{M_1} - \beta^{M_1})^2 x^2 + \beta 2M_1 s_1^2 + 2\beta^{M_1} x\). It results \(\beta^{M_1} x = \beta^{M_1} x^2 + 2\beta^{M_1} x\). From here, \(x = 0\) or \(x = \frac{2\beta^{M_1} x s_1 + \beta^{M_1} s_{M+1} + \beta^{M_1} s_{M+1} s_1}{(\beta^{M_1} - \beta^{M_1})^2} = s_1\). If we have \(x = \beta^i + q' = s_1\), this implies \(\beta^{i+q'} = 0\), which is false.

We now return to the proof of the Theorem and we know that the following conditions are fulfilled: \(\beta^{M_i} \neq \beta^{M_j}, 0 \leq i, j \leq n - 1\) and \(s_1 s_{2M+1} \neq s_{M+1}^2\). For \(B = s_1 - A\), it results that \(A (\beta^{M_i} - \beta^{M_j}) = s_{M+1} - s_{1} \beta^{M_j}\).
\[ A(\beta^{2M_i} - \beta^{2M_j}) = s_{2M+1} - s_1\beta^{2M_j} \]
\[ A(\beta^{3M_i} - \beta^{3M_j}) = s_{3M+1} - s_1\beta^{3M_j}. \]

We obtain \( s_{2M+1} - s_1\beta^{2M_j} = \left(s_{M+1} - s_1\beta^{M_j}\right)\left(\beta^{M_i} + \beta^{M_j}\right) \) and \( s_{3M+1} - s_1\beta^{3M_j} = \left(s_{M+1} - s_1\beta^{M_j}\right)\left(\beta^{2M_i} + \beta^{M_i}\beta^{M_j} + \beta^{2M_j}\right). \)

If we denote by \( s_M = \beta^{M_i} + \beta^{M_j} \) and \( p_M = \beta^{M_i}\beta^{M_j}, \) we have

\[ s_{2M+1} - s_{M+1}s_M + p_Ms_1 = 0 \]
and
\[ \left(s_{M+1} - s_1\beta^{M_j}\right)\left(s_M - p_M\right) = s_{3M+1} - s_1\beta^{3M_j}. \]

It results that
\[ p_M = \frac{s_{M+1}s_M - s_{2M+1}}{s_1} \]
and
\[ s_M(s_1s_{2M+1} - s_M^2) = s_1s_{3M+1} - s_{M+1}s_{2M+1}. \]

Therefore, we obtain
\[ s_M = \frac{s_1s_{3M+1} - s_{M+1}s_{2M+1}}{s_1s_{2M+1} - s_M^2} \]
\[ p_M = \frac{s_{M+1}s_{3M+1} - s_M^2}{s_1s_{2M+1} - s_M^2}. \]

From here, by solving the equation \( x^2 - s_Mx + p_M = 0, \) we find the locations and the values of the errors. \( \Box \)

4. Main algorithm and some examples

**Definition 4.1.** The rate of a block code is \( R = \frac{k}{n}, \) where \( k \) is the dimension of the block code and \( n \) is the length of the codewords.

**The Main Algorithm**

Let \( p \) be a prime number.

1. We find \( a, b, t \in \mathbb{N} \) such that we can write \( p \) under the form
\[ p = a^2 + (2^t - 1)b^2. \] (4.1.)

We remark that the values for \( a, b, t, \) if there exist, are not unique. Let \( \{a_l, b_l, t_l\}, l \in \{1, 2, ..., u\} \) all solutions of the equation (4.1).

2. Let \( p = n_jM_j + 1, \) with \( n_j, M_j \) not unique such that \( n_jM_j = p - 1, j \in \{1, 2, ..., v\}. \)

3. For \( l \in \{1, 2, ..., u\} \) and for \( j \in \{1, 2, ..., v\}, \) we find the algebra \( \mathbb{A}_{t_l}(\mathbb{R}) \), the element \( w = \frac{1}{r_l}(1 + \sum_{i=2}^{2^t} c_i) \in \mathbb{A}_{t_l}(\mathbb{R}), r_l \geq t_l - 1, \forall \mathbb{A}_{t_l}(\mathbb{R}), \) the element \( \pi \in \)
\[ \forall, \text{ such that } n(\pi) = p, \text{ we find } \forall_\pi \text{ such that } \forall_\pi \text{ is isomorphic to } \mathbb{Z}_p \text{ and we find } \beta \in \forall_\pi \text{ such that } \beta^{\rho_j} = w \text{ or } \beta^{\rho_j} = -w. \]

If the elements \( \{a_i, b_i, t_i\} \) don’t exist, then the algorithm stops.

If we have at least a solution for the equation (4.1) but we don’t find for \( j \in \{1, 2, ..., v\} \) the element \( \beta \in \forall_\pi \) such that \( \beta^{\rho_j} = w \) or \( \beta^{\rho_j} = -w \), then the algorithm stops. If we have solutions in both cases, then we go to the Step 4.

4. For each solution \( \{a_i, b_i, t_i\}, l \in \{1, 2, ..., u\} \), let \( J \subseteq \{1, 2, ..., v\} \). For each \( j \in J \), we have \( n_j \) such that \( \beta^{\rho_j} = w \) or \( \beta^{\rho_j} = -w \). We can change \( w \) by increasing the value of \( r_l \), if it is necessary, but working in the algebra \( \mathbb{A}_t(\mathbb{R}) \).

For each \( n_j \) we compute \( M_j \) and the rate of the obtained code, \( R_j = \frac{k_j}{n_j} \). Since we can suppose that the obtained codes have the same dimension \( k = k_j \), we will chose the indices \( l \in \{1, 2, ..., u\}, j \in J \), the pair \( \{a_i, b_i, t_i\} \) and the number \( n_j \) such that the rate \( R_j \) has the biggest value.

In the following, we will denote by Algorithm 1, the method described in [Gu; 13] and by Algorithm 2, the method described in [Fl; 15].

Remark 4.2. In the papers [Gu; 13] and [Fl; 15] were developed some algorithms which built binary block codes over subsets of integers in the real quaternion division algebra and in the real octonion division algebra. The above algorithm generalized these two algorithms to real algebras obtained by the Cayley-Dickson process. Moreover, the Main Algorithm can be generalized to almost all prime number, which in general the Algorithm 1 and the Algorithm 2 don’t make it. That means, in general, for a prime number \( p \), we can get the algebra \( \mathbb{A}_t(\mathbb{R}) \), the element \( w \in \mathbb{A}_t(\mathbb{R}) \), the subset \( \forall \subset \mathbb{A}_t(\mathbb{R}) \), \( \pi \in \forall \), such that \( n(\pi) = p \), we can find \( \forall_\pi \) with \( \forall_\pi \) isomorphic to \( \mathbb{Z}_p \), such that the obtained binary block code can have the highest rate.

With the above algorithm, we have a much higher flexibility, similar to the Lenstra algorithm for elliptic curves compared with \( p - 1 \) Pollard algorithm. It is well known that for a prime \( p \), Lenstra’s algorithm replace the group \( \mathbb{Z}_p^* \) with the group of the rational points of an elliptic curve \( C_1 \) over \( \mathbb{Z}_p \) and if this algorithm failed, the curve will be replaced with another curve \( C_2 \) over \( \mathbb{Z}_p \) and we can retake the algorithm (see [Si, Ta; 92]).

In the case of Main Algorithm, the algebra \( \mathbb{A}_t(\mathbb{R}) \) and \( w \) offer this kind of flexibility since, for the same prime \( p \), these can be changed and the algorithm can be retake, with better chances of success.

We will explain this in the following examples.

Example 4.3. Let \( p = 29 \). We have \( a = 1, b = 2 \) and \( t = 3 \), therefore \( p = 1 + 7 \cdot 4 \) with unique decomposition. It results that we can use the real Octonion algebra. If we apply Algorithm 2, we have \( w = \frac{1}{4} \left(1 + \sum_{i=2}^{8} e_i\right) \), \( \pi = -1 \cdot 4 \cdot w, p = 29, n = 4, s = 22, \beta = 1 - w, \beta^4 = -w \mod \pi \), therefore we can define codes.

If we apply the Main Algorithm for \( w = \frac{1}{4} \left(1 + \sum_{i=2}^{8} e_i\right) \), we have \( \pi = -1 + 8w, n = 4, s = 11 \) which is the label for the element \( w \in \forall_\pi \). We remark that
we can’t find $\beta \in \mathcal{V}_\pi$ such that $\beta^4 = w$, as we can see from the MAPLE’s procedures below.

```plaintext
for i from -15 to 14 do for j from -15 to 14 do
c := (11*j+i)mod 29; d := ((7/4)*j)^2+(i+(1/4)*j)^2;
if d < 29 and c = 11 then print(i, j);fi;od;od;

A := 8^{-1} mod 29; for a to 29 do
b := a^4 mod 29; if b = 11 then print(a);fi;od;

But, if we increase $\alpha$ we still work on octonions and we take $w = 1 + 8 \sum_{i=2}^{16} e_i$, with the label $s = 24$. We obtain $\beta = -1 - w$ with the label 4 such that $\beta^4 = w$, therefore we can define codes. In this situation, both algorithms can be applied with success.

Example 4.4. Let $p = 71 = 64 + 7 \cdot 1$, with unique decomposition. Therefore $a = 8, b = 1, t = 3$. Then we work on real Octonion algebra. If we apply the Algorithm 2, we have $w = \frac{1}{4} \left(1 + \sum_{i=2}^{16} e_i \right)$, $\pi = 7 + 2w, p = 71, n = 10, s = 32, \beta = 2 - 2w, \beta^{10} = w \mod \pi$(see [Fl; 15])

If we apply the Main Algorithm and first we take $w = \frac{1}{4} \left(1 + \sum_{i=2}^{16} e_i \right)$, we have $\pi = 7 + 4w, p = 71, n = 10, s = 16$ which is the label for the element $w \in \mathcal{V}_\pi$. We remark that we can’t find $\beta \in \mathcal{V}_\pi$ such that $\beta^{10} = w$ (even if we increase the value of $r$, as in Example 4.3), as we can see in the procedure below.

```plaintext
A := -7*4^{-1}mod71; for a from 1 to 71 do b := a^{10} mod 71;
if b = 16 then print(a);fi;od:

Therefore, the Algorithm 2 is better than the Main Algorithm.

Example 4.5. For $p = 31 = 6 \cdot 5 + 1$, we have $p = 4 + 3 \cdot 9 = 16 + 15 \cdot 1$, therefore $t \in \{4, 16\}$ and we can use the real Quaternion algebra or the real Sedenion algebra. If we apply the Main Algorithm for sedenions, we have $w = \frac{1}{8} \left(1 + \sum_{i=2}^{16} e_i \right)$. We get $\pi = 3 + 8w, p = 31, s = 19$ and we remark that we can’t use it for the sedenions since we can’t find $\beta \in \mathcal{V}_\pi$ such that $\beta^5 = w$. Therefore, we will use the Main Algorithm only for Quaternion algebra, which can be applied in this case.
Example 4.6. Let $p = 61$. We have that $p = 4 \cdot 3 \cdot 5 + 1 = 1 + 60 = 1 + 15 \cdot 4 = 49 + 3 \cdot 4$, therefore $t \in \{4, 16\}$ and we can use the real Quaternion algebra or the real Sedenion algebra.

If we take $p$ under the form $p = 61 = 7^2 + 3 \cdot 2^2$, we use the real Quaternion algebra. For $w = \frac{1}{2} \left( 1 + \sum_{i=2}^{4} e_i \right)$, we get $\pi = 5 + 4 w$, the label for $w$ is $s = 14, n = 10(p = 6 \cdot 10 + 1)$ and we have $\beta = -4 + w, \beta_{10} = w$, as we can see from in below procedures:

\begin{verbatim}
A := -5*4^{-1} mod 61; for a to 61 do
   b := a^10 mod 61; if b = 14 then print(a); fi; od;
   14 10 17 26 29 30 30 31 32 35 44 51
for i from -31 to 30 do for j from -31 to 30 do
   c :=(14*j+i) mod 61 d := (3/4)*j^2+(i+(1/2)*j)^2;
   if d < 61 and c = 10 then print(i, j) fi; od; od;
   -4, 1
   1, 5
   5, -4
\end{verbatim}

In this case, the rate code is $R_1 = \frac{6k}{p-1} = \frac{k}{10}$, where $k$ is the dimension of the code, since we can’t find $\beta$ such that $\beta^6 = w$ or $\beta^{M_j} = w$, for $M_j \mid p-1, j \in \{1, 2, ..., v\}$.

If we consider $p$ under the form $p = 1 + 15 \cdot 4$, we use the real Sedenion algebra, we get $n = 4$ and for $w = \frac{1}{8} \left( 1 + \sum_{i=2}^{16} e_i \right)$, we have $\pi = -1 + 16 w$, the label for $w$ is $s = 42$ and $\beta = 2 + 2 w$. In this case, the rate of the code is $R_2 = \frac{15k}{p-1} = \frac{k}{4}$ and it is greater than $R_1$. We remark that we can use both algebras to define codes, but in the second case, we have chance to obtain a better rate. (The dimension $k$ is considered the same, in both situations).

\begin{verbatim}
A :=16^{-1}mod 61; for a to 61 do b := a^10 mod 61;
   if b = 42 then print(a); fi; od;
   42 25 30 31 36
for i from -31 to 30 do for j from -31 to 30 do
   c :=42*j+i mod 61 d := (15/64)*j^2+(i+(1/8)*j)^2;
   if d < 61 and c = 25 then print(i, j) fi; od; od;
   -6, 8
   -5, -8
   -2, 5
   -1, -11
\end{verbatim}
Example 4.7. Let $p = 151 = 4 + 3 \cdot 49 = 16 + 15 \cdot 9 = 6 \cdot 25 + 1$

We have $t \in \{2, 4\}$ and will use the real Quaternion algebra or real Sedenion algebra. For $w = \frac{1}{2} \left( 1 + \sum_{i=2}^{4} e_i \right)$, we have $\pi = -3 + 14w, n = 25$ and $s = 140$, the label for $w$. In this case, we can't find an element $\beta$, such that $\beta^{25} = w, \beta^6 = w, \beta^{15} = w$, etc, as we can see in the procedure below.

```
A:=3*14^(-1) mod 151; for a to 151 do b:=a^25 mod 151; if b = 140 then print(a);fi;od:
```

```
140
```

But, as we remarked, the number $p$ can be written under the form $p = 16 + 15 \cdot 9 = 25 \cdot 6 + 1$, then if we take $t = 4$, we can use the real Sedenion algebra. We consider $w = \frac{1}{8} \left( 1 + \sum_{i=2}^{16} e_i \right)$. We obtain $\pi = 1 + 24w, n = 6$ and $s = 44$, the label for $w$. We can find $\beta$, such that $\beta^6 = w \text{ mod } \pi$ and $\beta = 3 - 3w$, with the label $s = 22$.

```
A:=-24^(-1)mod 151; for a to 151 do b:=a^6 mod 151; if b = 44 then print(a);fi;do;
```

```
44 22 51 100 122 129
```

```
for i from -76 to 75 do for j from -76 to 75 do
c := 44*j+i mod 151; d:= (15/64)*j^2+(i+(1/8)*j)^2;
if d < 151 and c = 22 then print(i, j);fi;od;od;
```

```
-9, 11
-4, -20
-3, 4
3, -3
4, 21
9, -10
```

Example 4.8. Let $p = 149 = 25 + 31 \cdot 4 = 121 + 7 \cdot 4$. In this situation, $t \in \{3, 5\}$ and we can use the real Octonion algebra or a real Cayley-Dickson algebra of dimension 32.

We can't use the Algorithm 2 for octonions, since we can't obtain the element $\beta$ and $p$ is not under the form $7k+1$. For $w = \frac{1}{4} \left( 1 + \sum_{i=2}^{8} e_i \right)$, we have $\pi = 9 + 8w$.

We consider $p = 1 + 4 \cdot 37$ and we can't find an element $\beta$, even if we take $p = 2k + 1$ or $4k + 1$ or $37k + 1$. 
A := -9*8^{-1} mod 149; for a to 149 do
b := a^2 mod 149; if b = 92 then print(a);fi;od;
92

A := -9*8^{-1} mod 149; for a to 149 do
b := a^4 mod 149; if b = 92 then print(a);fi;od;
92

A := -9*8^{-1} mod 149; for a to 149 do
b := a^37 mod 149 if b = 92 then print(a);fi;od;
92

But we can choose another \( \alpha \). For example, for \( w = 1 + 8 \sum \) if \( i = 2 \), we have
\[
\pi = 9 + 16w,
\]
and \( s_1 = 46 \), the label for \( w \). If we consider \( p = 74n + 1 \), \( n = 2 \), we get \( \beta = -2 + 4w \), with label \( s_2 = 33 \). In this case, the rate of the code is
\[
R_1 = \frac{24k}{p+1} = \frac{k}{2}.
\]
For \( p = 37n + 1 \), \( n = 4 \), the label of \( \beta = 4w \) is \( s_3 = 35 \). In this case the rate of the code is
\[
R_2 = \frac{37k}{p+1} = \frac{k}{4}.
\]
We have \( R_2 < R_1 \). Therefore the code in the first case is better, since it can have a greater rate as in the second case. For \( p = 2k + 1 \) or \( 4k + 1 \), we can’t find \( \beta \).

A := -9*16^{-1} mod 149; for a to 149 do
b := a^2 mod 149; if b = 46 then print(a);fi;od;
46 33 116

for i from -75 to 74 do for j from -75 to 74 do
c := (46*j+i) mod 149; d := (31/64)*j^2+(i+(1/8)*j)^2;
if d < 149 and c = 33 then print(i, j);fi;do;do;
-4, 17
-2, 4
0, -9
9, 7
11, -6

A := -9*16^{-1} mod 149; for a to 149 do
b := a^4 mod 149; if b = 46 then print(a);fi;do;
46 35 50 99 114

for i from -75 to 74 do for j from -75 to 74 do
c := (46*j+i) mod 149; d := (31/64)*j^2+(i+(1/8)*j)^2;
if d < 149 and c = 35 then print(i, j);fi;do;do;
-11, 1
-2, 17
A := -9*16^{-1} mod 149; for a to 149 do
  b := a^37 mod 149; if b = 46 then print(a); fi; od;

If we work on a real algebra of dimension 32, let $w = \frac{1}{16} \left( 1 + \sum_{i=2}^{32} e_i \right)$. We have $\pi = 3 + 32w$, $s = 107$, the label for $w$, $\beta = 4$, with the label $s = 4$, $n = 4$, $p = 37 \cdot 4 + 1$, as we can see in the procedures below.

A := -3*32^{-1} mod 149; for a to 149 do
  b := a^4 mod 149; if b = 107 then print(a); fi; od;

107 4 27 122 145

for i from -75 to 74 do for j from -75 to 74 do
  c := (107*j+i) mod 149; d := (31/256)*j^2+(i+(1/16)*j)^2;
  if d < 149 and c = 4 then print(i, j); fi; od; od;

-8, 21
-7, -18
-4, 14
-3, -25
0, 7
1, -32
4, 0
8, -7
12, -14

We can work on both algebras to obtain codes with good rate.

Conclusions. Regarding a finite field as a residue field modulo a prime element from $V$, where $V$ is a subset of a real algebra obtained by the Cayley-Dickson process with a commutative ring structure, in this paper, we obtain an algorithm, called Main Algorithm, which allows us to find codes with a good rate. This algorithm offers more flexibility than other methods known until now, similar to Lenstra’s algorithm on elliptic curves compared with $p – 1$ Pollard’s algorithm.

As a further research, we intend to improve this algorithm. This thing can be done if first we can find answers at the following questions:

i) When the equation (4.1) has solutions?

ii) When we have more than one solution for the equation (4.1)?

iii) If we have solution(s) for equation (4.1), when we can find the element $\beta$?
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