ALL-DERIVABLE POINTS IN NEST ALGEBRAS

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ABSTRACT. Suppose that $\mathcal{A}$ is an operator algebra on a Hilbert space $H$. An element $V$ in $\mathcal{A}$ is called an all-derivable point of $\mathcal{A}$ for the strong operator topology if every strong operator topology continuous derivable mapping $\varphi$ at $V$ is a derivation. Let $\mathcal{N}$ be a complete nest on a complex and separable Hilbert space $H$. Suppose that $M$ belongs to $\mathcal{N}$ with $\{0\} \neq M \neq H$ and write $\tilde{M}$ for $M$ or $M^\perp$. Our main result is: for any $\Omega \in \text{alg}\mathcal{N}$ with $\Omega = P(\tilde{M})\Omega P(\tilde{M})$, if $\Omega_{\tilde{M}}$ is invertible in $\text{alg}\mathcal{N}_{\tilde{M}}$, then $\Omega$ is an all-derivable point in $\text{alg}\mathcal{N}$ for the strong operator topology.

1. Introduction

Let $K$ and $H$ be complex and separable Hilbert spaces of dimensions greater than one. $B(K,H)$ and $F(K,H)$ stand for the set of all bounded linear operators and the set of all finite rank operators from $K$ into $H$, respectively. When $H = K$, $B(K,H)$ and $F(K,H)$ are abbreviated to $B(H)$ and $F(H)$, respectively. The adjoint operator of $T$ is denoted by $T^*$. Suppose $x \in K$ and $y \in H$. The rank one operator $<\cdot , x> y$, from $K$ into $H$, is denoted by $y \otimes x$. If $\mathcal{N}$ is a complete nest on $H$, then the nest algebra $\text{alg}\mathcal{N}$ is the Banach algebra of all bounded linear operators which leave every member of $\mathcal{N}$ invariant. For $N \in \mathcal{N}$, $N_-$ stands for $\forall [M \in \mathcal{N}: M \subset N]$, and $\mathcal{N}_N$ stands for the nest $\{L \cap N : L \in \mathcal{N}\}$ in $N$. We write $P(N)$ for the orthogonal projection operator from $H$ onto $N$. The identity of $B(N)$ is denoted by $I_N$ and the restriction of an operator $T \in B(H)$ to the subspace $N$ is denoted by $T|_N$.

Suppose that $\mathcal{A}$ is a subalgebra of $B(H)$ and $V$ is an operator in $\mathcal{A}$. A linear mapping $\varphi$ from $\mathcal{A}$ into itself is called a derivable mapping at $V$ if $\varphi(ST) = \varphi(S)T + S \varphi(T)$ for any $S, T$ in $\mathcal{A}$ with $ST = V$. Operator $V$ is called an all-derivable point in $\mathcal{A}$ for the strong operator topology if every strong operator topology continuous derivable mapping $\varphi$ at $V$ is a derivation.

In recent years the study of all-derivable points in operator algebras has attracted many researchers’ attentions. Jing, Lu, and Li [4] proved that every derivable mapping $\varphi$ at 0 with $\varphi(I) = 0$ on nest algebras is a derivation. Li, Pan, and Xu [5] showed that every derivable mapping $\varphi$ at 0 with $\varphi(I) = 0$ on CSL algebras is a derivation. Zhu and Xiong proved the following results in \cite{6 7 8 9 10}: 1) every norm-continuous generalized derivable mapping at 0 on some CSL algebras is a generalized derivation; 2) every invertible operator in nest algebras is an all-derivable point for the strong operator topology; 3) $V$ is an all-derivable point in $\mathcal{M}_n$ if and only if $V \neq 0$, where $\mathcal{M}_n$ is the algebras of all $n \times n$ upper triangular matrices; and 4) every orthogonal projection operator $P(M)(\{0\} \neq M \in \mathcal{N})$

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is an all-derivable point in nest algebra \( \text{alg} \mathcal{N} \) for the strong operator topology.

The main purpose of this paper is to study the all-derivable points in nest algebras. Suppose that \( M \) belongs to \( \mathcal{N} \) with \( \{0\} \neq M \neq H \) and write \( \tilde{M} \) for \( M \) or \( M^\perp \). We shall prove: for any \( \Omega \in \text{alg} \mathcal{N} \) with \( \Omega = P(\tilde{M})\Omega P(\tilde{M}) \), if \( \Omega \mid \mathcal{M} \) is invertible in \( \text{alg} \mathcal{M}^\perp \), then \( \Omega \) is an all-derivable point in \( \text{alg} \mathcal{N} \) for the strong operator topology.

2. THREE LEMMAS

It is known that every operator \( S \) in \( B(H) \) can be uniquely expressed in the form of a \( 2 \times 2 \) operator matrix relative to the orthogonal decomposition \( H = M \oplus M^\perp \). Thus we immediately get the following proposition.

**Proposition 2.0** Let \( \mathcal{N} \) be a complete nest on a complex and separable Hilbert space \( H \). For an arbitrary \( M \) in \( \mathcal{N} \) with \( \{0\} \neq M \neq H \), we have

\[
\text{alg} \mathcal{N} = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} : X \in \text{alg} \mathcal{M}, Z \in \text{alg} \mathcal{M}^\perp, Y \in B(M^\perp, M) \right\}.
\]

The following three lemmas will be used to prove the main result of this paper in Section 3.

**Lemma 2.1.** Let \( H \) be a complex and separable Hilbert space and let \( \mathcal{N} \) be a complete nest in \( H \). Suppose that \( \delta \) is a strong operator topology continuous linear mapping from \( \text{alg} \mathcal{N} \) into itself and \( \Gamma \) is an invertible operator in \( \text{alg} \mathcal{N} \). If the following equation

\[
(2.1) \quad \delta(\Gamma) = \delta(\Gamma S_1)S_2 + \Gamma S_1 \delta(S_2)
\]

holds for any \( S_1, S_2 \) in \( \text{alg} \mathcal{N} \) with \( S_1S_2 = I \), then \( \delta \) is an inner derivation.

**Proof.** Put \( S_1 = S_2 = I \) in Eq. (2.1), we have \( S_1S_2 = I \). It follows that \( \Gamma \delta(I) = 0 \). That is, \( \delta(I) = 0 \) since \( \Gamma \) is invertible in \( \text{alg} \mathcal{N} \). Put \( S_1 = I - aP \) and \( S_2 = I - bP \) in Eq. (2.1), where \( P \) is an idempotent in \( \text{alg} \mathcal{N} \) and \( a, b \) are two complex numbers such that \( a + b = ab = 1 \). Thus we get that \( S_1S_2 = I \). Furthermore, we have

\[
\begin{align*}
\delta(\Gamma) & = \delta(\Gamma - a\Gamma P)(I - bP) + (\Gamma - a\Gamma P)\delta(I - bP) \\
& = [\delta(\Gamma) - b\delta(\Gamma)P - a\delta(\Gamma)P + ab\delta(\Gamma)P] + [\Gamma\delta(I) - b\Gamma\delta(P) - a\Gamma P\delta(I) + ab\Gamma P\delta(P)].
\end{align*}
\]

It follows from \( \delta(I) = 0 \) that

\[
(2.2) \quad a\delta(\Gamma)P + b[\delta(\Gamma)P + \Gamma\delta(P)] = \delta(\Gamma)P + \Gamma P\delta(P).
\]

Interchanging the position of \( a \) and \( b \) in Eq. (2.2), we have

\[
(2.3) \quad b\delta(\Gamma)P + a[\delta(\Gamma)P + \Gamma\delta(P)] = \delta(\Gamma)P + \Gamma P\delta(P).
\]

It follows from Eq. (2.2) and Eq. (2.3) that \( \delta(\Gamma)P = \delta(\Gamma)P + \Gamma\delta(P) \). Notice that every rank-one operator in \( \text{alg} \mathcal{N} \) may be written as a linear combination of at most four idempotents in \( \text{alg} \mathcal{N} \) (see [3]) and
every finite rank operator in \( \text{alg}.\mathcal{N} \) may be represented as a sum of rank-one operators in \( \text{alg}.\mathcal{N} \) (see \cite{2}). Thus we obtain

\[
(2.4) \quad \delta(\Gamma F) = \delta(\Gamma)F + \Gamma\delta(F)
\]

for any \( F \) in \( F(H) \cap \text{alg}.\mathcal{N} \). By applying Erdős Density Theorem (see \cite{2}) to Eq. (2.4), we obtain that 

\[
\delta(\Gamma R) = \delta(\Gamma)R + \Gamma\delta(R) \quad \text{for any } R \text{ in } \text{alg}.\mathcal{N}.
\]

In particular,

\[
(2.5) \quad \delta(\Gamma S_1) = \delta(\Gamma)S_1 + \Gamma\delta(S_1).
\]

It follows from Eq. (2.1) and Eq. (2.5) that

\[
\delta(\Gamma)S_1 = \delta(\Gamma)S_1 + \Gamma\delta(S_1).
\]

We only prove (1). One can prove (2) similarly.

\textbf{Lemma 2.2.} Let \( K, H \) be two complex and separable Hilbert spaces with dimensions greater than one, and let \( \mathcal{N} \) and \( \mathcal{N}' \) be two complete nests in \( H \) and \( K \), respectively. Suppose that \( \varphi \) is a strong operator topology continuous linear mapping from \( B(K, H) \) into itself.

(1) If \( T\varphi(S) = 0 \) for any \( T \) in \( \text{alg}.\mathcal{N} \) and \( S \) in \( B(K, H) \) with \( TS = 0 \), then there exists an operator \( D \) in \( B(K) \) such that

\[
\varphi(S) = SD
\]

for any \( S \) in \( B(K, H) \).

(2) If \( \varphi(S)T' = 0 \) for any \( T' \) in \( \text{alg}.\mathcal{N}' \) and \( S \) in \( B(K, H) \) with \( ST' = 0 \), then there exists an operator \( D' \) in \( B(H) \) such that

\[
\varphi(S) = D'S
\]

for any \( S \) in \( B(K, H) \).

\textit{Proof.} We only prove (1). One can prove (2) similarly.

\textit{Case 1.} Suppose that \([0]_+ \neq [0] \). For any \( x \) in \( H \) and \( g \) in \( K \), it is clear that \( x \otimes g \) is in \( B(K, H) \). For an arbitrary vector \( z \in \{x\}^+(\subset H = [0]^+) \) and \( y \) in \([0]_+ \) with \( y \neq 0 \), \( y \otimes z \) in \( \text{alg}.\mathcal{N} \) and \( (y \otimes z)(x \otimes g) = 0 \). Under the hypothesis, we get that \( y \otimes z \varphi(x \otimes g) = 0 \). Thus there exists a vector \( \omega_{x,g} \) in \( K \) such that

\[
\varphi(x \otimes g) = x \otimes \omega_{x,g}.
\]

It follows that

\[
\varphi(x \otimes (f + g)) = x \otimes \omega_{x,f+g}.
\]

for any \( f, g \) in \( K \). Moreover,

\[
\varphi(x \otimes (f + g)) = \varphi(x \otimes f) + \varphi(x \otimes g) = x \otimes \omega_{x,f} + x \otimes \omega_{x,g}.
\]

Thus \( \omega_{x,f+g} = \omega_{x,f} + \omega_{x,g} \). Similarly, we obtain that \( \omega_{x,\text{alg}} = \lambda \omega_{x,g} \). Consequently, there exists a linear mapping \( L_x \) from \( K \) into \( K \) such that \( \varphi(x \otimes g) = x \otimes L_x g \). Furthermore, we have

\[
(x + v) \otimes L_{x+v}g = \varphi((x + v) \otimes g) = \varphi(x \otimes g) + \varphi(v \otimes g) = x \otimes L_x g + v \otimes L_v g
\]
for any \( v \) in \( H \). So \( x \otimes (L_{x+v} - L_x)g + v \otimes (L_{x+v} - L_x)g = 0 \), which implies that \( L_x = L_v \) when \( x \) and \( v \) are linearly independent. If \( x \) and \( v \) are linearly dependent, there exists some complex number \( t \) such that \( v = tx \). Since \( \dim H > 1 \), a vector \( u \) can be chosen from \( H \) such that \( u \) and \( x \) are linearly independent.

Thus \( L_x = L_u = L_v \) since \( u \) and \( v \) are linearly independent. This implies that \( L_x \) is independent of \( x \) for any \( x \in H \). If we write \( L = L_x \), then \( \varphi(x \otimes g) = x \otimes Lg \) for any \( x \) in \( H \) and \( g \) in \( K \). Next we shall prove that \( L \) is in \( B(K) \). In fact, for arbitrary sequence \((g_n)\) in \( K \) with \( g_n \rightarrow g \) and \( Lg_n \rightarrow h \), we have \( x \otimes Lg_n = \varphi(x \otimes g_n) \rightarrow \varphi(x \otimes g) = x \otimes Lg \). So \( x \otimes (Lg - h) = 0 \), namely \( Lg = h \). Therefore \( L \) is a closed operator. By the Closed Graph Theorem, we obtain that \( L \) is a bounded linear operator on \( K \).

Since \( L \in B(K) \), \( \varphi(x \otimes g) = x \otimes Lg = x \otimes gL^* \). We write \( D \) as \( L^* \). So \( \varphi(x \otimes g) = x \otimes gD \).

Furthermore, we have

\[ \varphi(FS) = FSD \]

for any \( S \) in \( B(K, H) \) and finite rank operator \( F \) in \( \text{alg} \mathcal{N} \). Since \( \varphi \) is a strong operator topology continuous linear mapping, it follows from Erdös Density Theorem that \( \varphi(S) = SD \) for any \( S \) in \( B(K, H) \).

**Case 2.** Suppose that \([0]_+ = [0] \). Then there exists a sequence \([N_n]\) in \( \mathcal{N} \) such that the following statements hold: 1) \( N_1 \supseteq N_2 \supseteq \cdots \supseteq N_j \supseteq N_{j+1} \supseteq \cdots \supseteq [0] \); 2) \( (P(N_n^\perp)) \) strongly converges to \( 0 \) as \( n \rightarrow +\infty \). It is obvious that

\[ N_1 \subseteq N_2 \subseteq \cdots \subseteq N_j \subseteq N_{j+1} \subseteq \cdots \subseteq H \]

and the sequence \( (P(N_n)) \) strongly converges to the unit operator \( I_H \) as \( n \rightarrow +\infty \). For an arbitrary integer \( n \) and \( x \) in \( N_n \), by imitating the proof of case 1, we can find a linear mapping \( D_{N_n} \) on \( K \) such that \( \varphi(x \otimes g) = x \otimes gD_{N_n} \) for any \( x \) in \( N_n \) and \( g \) in \( K \). Note that \( N_n \subseteq N_m \) \((m > n)\) and \( \varphi(x \otimes g) = x \otimes gD_{N_m} \) for any \( x \) in \( N_m \) and \( g \) in \( K \). So \( x \otimes gD_{N_n} = x \otimes gD_{N_m} \) for any \( x \in N_n \) and \( g \in K \). It follows that \( D_{N_n} = D_{N_m} \). Hence \( D_{N_n} \) is independent of \( N_n \). We write \( D \) as \( D_{N_n} \). Thus \( \varphi(x \otimes g) = x \otimes gD \) for any \( x \) in \( N_n \) and \( g \) in \( K \). For any \( x \) in \( H \), put \( x_0 = P(N_n)x \). Then we get that

\[ \varphi(x_0 \otimes g) = x_0 \otimes gD \]

That is,

\[ \varphi(P(N_n)x \otimes g) = P(N_n)x \otimes gD. \]

Since \( \varphi \) is a strong operator topology continuous linear mapping and \( P(N_n) \) strongly converges to \( I_H \) as \( n \rightarrow +\infty \), taking limit on both sides in the above equation, we obtain that \( \varphi(x \otimes g) = x \otimes gD \) for any \( x \) in \( H \) and \( g \) in \( K \). The rest of the proof is similar to case 1. The lemma is proved.

**Lemma 2.3.** Let \( \mathcal{A} \) be an unital subalgebra of \( B(H) \), where \( H \) is a complex and separable Hilbert space. Suppose that \( \phi \) is a linear mapping from \( \mathcal{A} \) into itself. If \( \phi \) vanishes at every invertible operator in \( \mathcal{A} \), then \( \phi \) vanishes on \( \mathcal{A} \).

**Proof.** We only need to prove that \( \phi(T) = 0 \) for any operator \( T \) in \( \mathcal{A} \). Take a complex number \( \lambda \) with \( |\lambda| > ||T|| \). It follows that \( \lambda I - T \) is invertible in \( \mathcal{A} \). We thus see that \( \phi(\lambda I - T) = 0 \) by the hypothesis. Thus we have \( \phi(T) = \lambda \phi(I) \) by the linearity of \( \phi \). So \( \phi(T) = 0 \) for any \( T \) in \( \mathcal{A} \). 

\[ \square \]
3. All-derivable points in \( \text{alg}\mathcal{N} \)

In this section, we always assume that \( M \) belongs to \( \mathcal{N} \) with \( \{0\} \neq M \neq H \), and write \( \hat{M} \) for \( M \) or \( M^\perp \). Throughout the rest of this paper, every upper triangular \( 2 \times 2 \) operator matrix relative to the orthogonal decomposition \( H = M \oplus M^\perp \) always stands for the element of nest algebra \( \text{alg}\mathcal{N} \). The unit operator on \( M \) is denoted by \( I_M \). The following theorem is our main result.

**Theorem 3.1.** Let \( \mathcal{N} \) be a complete nest on a complex and separable Hilbert space \( H \). Suppose that \( M \) belongs to \( \mathcal{N} \) with \( \{0\} \neq M \neq H \) and write \( \hat{M} \) for \( M \) or \( M^\perp \). For any \( \Omega \in \text{alg}\mathcal{N} \) with \( \Omega = P(\hat{M})\Omega P(\hat{M}) \), if \( \Omega|_{\hat{M}} \) is invertible in \( \text{alg}\mathcal{N}_{\hat{M}} \), then \( \Omega \) is an all-derivable point in \( \text{alg}\mathcal{N} \) for the strong operator topology.

**Proof.** Let \( \varphi \) be a strong operator topology continuous derivable linear mapping at \( \Omega \) from \( \text{alg}\mathcal{N} \) into itself. We only need to show that \( \varphi \) is a derivation. For arbitrary \( X \) in \( \text{alg}\mathcal{N}_M \), \( Y \) in \( B(M^\perp, M) \) and \( Z \) in \( \text{alg}\mathcal{N}_{M^\perp} \), we write

\[
\left\{ \begin{array}{l}
\varphi\left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ 0 & A_{22}(X) \end{bmatrix}, \\
\varphi\left( \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ 0 & B_{22}(Y) \end{bmatrix}, \\
\varphi\left( \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix}.
\end{array} \right.
\]

Obviously, \( A_{ij}, B_{ij} \) and \( C_{ij}(i, j = 1, 2, i \leq j) \) are strong operator topology continuous linear mappings on \( \text{alg}\mathcal{N}_M, B(M^\perp, M), \) and \( \text{alg}\mathcal{N}_{M^\perp} \), respectively.

**Case 1.** Suppose that \( \hat{M} = M \). Then \( \Omega \) may be represented as the following matrix relative to the orthogonal decomposition \( H = M \oplus M^\perp \):

\[
\Omega = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix},
\]

where \( W \) is an invertible operator in \( \text{alg}\mathcal{N}_M \). The proof are divided into the following five steps:

**Step 1.** For arbitrary \( X_1, X_2 \) in \( \text{alg}\mathcal{N}_M \) with \( X_1X_2 = I_M \), taking \( S = \begin{bmatrix} WX_1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( T = \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix} \), then \( ST = \Omega \). Since \( \varphi \) is a derivable mapping at \( \Omega \) on \( \text{alg}\mathcal{N} \), we have

\[
\begin{bmatrix} A_{11}(W) & A_{12}(W) \\ 0 & A_{22}(W) \end{bmatrix} = \varphi(\Omega) = \varphi(S)T + S\varphi(T)
\]

\[
= \begin{bmatrix} A_{11}(WX_1) & A_{12}(WX_1) \\ 0 & A_{22}(WX_1) \end{bmatrix} \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} WX_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}(X_2) & A_{12}(X_2) \\ 0 & A_{22}(X_2) \end{bmatrix}
\]

\[
= \begin{bmatrix} A_{11}(WX_1)X_2 + WX_1A_{11}(X_2) & WX_1A_{12}(X_2) \\ 0 & 0 \end{bmatrix}.
\]
Furthermore,

\begin{align}
(3.1) \quad A_{11}(W) &= A_{11}(WX_1)X_2 + WX_1A_{11}(X_2), \\
(3.2) \quad A_{12}(W) &= WX_1A_{12}(X_2), \\
A_{22}(W) &= 0
\end{align}

for any $X_1, X_2$ in $\mathcal{N}_M$ with $X_1X_2 = I_M$. By Lemma 2.1 we get that $A_{11}$ is an inner derivation on $\mathcal{N}_M$. Then there exists an operator $A \in \mathcal{N}_M$ such that

\begin{equation}
(3.3) \quad A_{11}(X) = XA - AX
\end{equation}

for any $X$ in $\mathcal{N}_M$.

For an arbitrary invertible operator $X$ in $\mathcal{N}_M$, putting $X_2 = X, X_1 = X^{-1}$ in Eq. (3.2), then we get that $A_{12}(W) = WX^{-1}A_{12}(X)$, i.e., $W^{-1}A_{12}(W) = X^{-1}A_{12}(X)$. Taking $X = I_M$, we have $W^{-1}A_{12}(W) = A_{12}(I_M)$. Thus we get that

\[ A_{12}(X) = XA_{12}(I_M) \]

for any invertible operator $X$ in $\mathcal{N}_M$. It follows from Lemma 2.3 that $A_{12}(X) = XA_{12}(I_M)$ for any operator $X$ in $\mathcal{N}_M$. If we write $B$ for $A_{12}(I_M)$, then

\begin{equation}
(3.4) \quad A_{12}(X) = XB
\end{equation}

for any $X$ in $\mathcal{N}_M$.

\[ \text{Step 2. For arbitrary } Z_1, Z_2 \text{ in } \mathcal{N}_M^\perp \text{ with } Z_1Z_2 = 0 \text{ and } X_1, X_2 \text{ in } \mathcal{N}_M \text{ with } X_1X_2 = I_M, \text{ taking} \]

\[ S = \begin{bmatrix} WX_1 & 0 \\ 0 & Z_1 \end{bmatrix} \text{ and } T = \begin{bmatrix} X_2 & 0 \\ 0 & Z_2 \end{bmatrix}, \text{ then } ST = \Omega. \text{ Thus we have} \]

\[
\begin{bmatrix}
A_{11}(W) & A_{12}(W) \\
0 & A_{22}(W)
\end{bmatrix}
= \begin{bmatrix}
A_{11}(WX_1) + C_{11}(Z_1) & A_{12}(WX_1) + C_{12}(Z_1) \\
0 & A_{22}(WX_1) + C_{22}(Z_1)
\end{bmatrix}
\begin{bmatrix}
X_2 & 0 \\
0 & Z_2
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
WX_1 & 0 \\
0 & Z_1
\end{bmatrix}
\begin{bmatrix}
A_{11}(X_2) + C_{11}(Z_2) & A_{12}(X_2) + C_{12}(Z_2) \\
0 & A_{22}(X_2) + C_{22}(Z_2)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{11}(WX_1)X_2 + C_{11}(Z_1)X_2 & A_{12}(WX_1)Z_2 + C_{12}(Z_1)Z_2 \\
WX_1A_{11}(X_2) + WX_1C_{11}(Z_2) & +WX_1A_{12}(X_2) + WX_1C_{12}(Z_2)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & A_{22}(WX_1)Z_2 + C_{22}(Z_1)Z_2 \\
+Z_1A_{22}(X_2) + Z_1C_{22}(Z_2)
\end{bmatrix}.
\]
Furthermore,

\begin{align*}
(3.5) \quad A_{11}(W) &= A_{11}(WX_1)X_2 + C_{11}(Z_1)X_2 + WX_1A_{11}(X_2) + WX_1C_{11}(Z_2), \\
(3.6) \quad A_{12}(W) &= A_{12}(WX_1)Z_2 + C_{12}(Z_1)Z_2 + WX_1A_{12}(X_2) + WX_1C_{12}(Z_2), \\
(3.7) \quad A_{22}(W) &= A_{22}(WX_1)Z_2 + C_{22}(Z_1)Z_2 + Z_1A_{22}(X_2) + Z_1C_{22}(Z_2)
\end{align*}

for any \( Z_1, Z_2 \) in \( \text{alg.}\mathcal{M}^+ \) with \( Z_1Z_2 = 0 \) and \( X_1, X_2 \) in \( \text{alg.}\mathcal{M} \) with \( X_1X_2 = I_M \). Substituting the expression of \( A_{11}(W) \) in Eq. \((3.1)\) into Eq. \((3.5)\), and the expression of \( A_{12}(W) \) in Eq. \((3.4)\) into Eq. \((3.6)\), respectively, we have

\begin{align*}
(3.8) \quad 0 &= C_{11}(Z_1)X_2 + WX_1C_{11}(Z_2), \\
(3.9) \quad WB &= WX_1BZ_2 + C_{12}(Z_1)Z_2 + WB + WX_1C_{12}(Z_2)
\end{align*}

for any \( Z_1, Z_2 \) in \( \text{alg.}\mathcal{M}^+ \) with \( Z_1Z_2 = 0 \) and \( X_1, X_2 \) in \( \text{alg.}\mathcal{M} \) with \( X_1X_2 = I_M \). For an arbitrary \( Z \) in \( \text{alg.}\mathcal{M}^+ \), Putting \( Z_1 = 0 \) and \( Z_2 = Z \), it follows from Eq. \((3.8)\) and Eq. \((3.9)\) that

\[ C_{11}(Z) = 0 \]

and

\[ C_{12}(Z) = -BZ \]

for any \( Z \) in \( \text{alg.}\mathcal{M}^+ \). Taking \( Z_1 = I_M \) and \( Z_2 = 0 \) in Eq. \((3.7)\), we get that \( A_{22}(X_2) = A_{22}(W) \) for any \( X_2 \) in \( \text{alg.}\mathcal{M} \). Furthermore, \( A_{22}(X) = 0 \) for any invertible operator \( X \) in \( \text{alg.}\mathcal{M} \). It follows from Lemma \((2.2)\) that

\[ A_{22}(X) = 0 \]

for any \( X \) in \( \text{alg.}\mathcal{M} \). Using Eq. \((3.10)\) and Eq. \((3.7)\), we get that \( C_{22}(Z_1)Z_2 + Z_1C_{22}(Z_2) = 0 \), namely \( C_{22} \) is a derivable mapping at 0.

**Step 3.** For arbitrary \( Y \) in \( B(M^+, M) \) and \( X_1, X_2 \) in \( \text{alg.}\mathcal{M} \) with \( X_1X_2 = I_M \), taking \( S = \begin{bmatrix} WX_1 & Y \\ 0 & 0 \end{bmatrix} \) and \( T = \begin{bmatrix} X_2 & 0 \\ 0 & 0 \end{bmatrix} \), then \( ST = \Omega \). Thus we have

\[
\begin{bmatrix}
A_{11}(W) & A_{12}(W) \\
0 & 0
\end{bmatrix} = \varphi(\Omega) = \varphi(S)T + S\varphi(T)
\]

\[
= \begin{bmatrix}
A_{11}(WX_1) + B_{11}(Y) & A_{12}(WX_1) + B_{12}(Y) \\
0 & B_{22}(Y)
\end{bmatrix}
\begin{bmatrix}
X_2 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
WX_1 & Y \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A_{11}(X_2) & A_{12}(X_2) \\
0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{11}(WX_1)X_2 + B_{11}(Y)X_2 + WX_1A_{11}(X_2) + WX_1A_{12}(X_2) \\
0 & 0
\end{bmatrix}.
\]
Furthermore,

\[ A_{11}(W) = A_{11}(WX_1)X_2 + B_{11}(Y)X_2 + WX_1A_{11}(X_2). \]

Since \( A_{11} \) is an inner derivation and \( X_2 \) is an invertible operator in \( \text{alg.} \mathcal{N}_M \), we have

\[ B_{11}(Y) = 0 \]

for any \( Y \) in \( B(M^+, M) \).

**Step 4.** For arbitrary \( Y \) in \( B(M^+, M) \), \( Z \) in \( \text{alg.} \mathcal{N}_M \), taking \( S = \begin{bmatrix} I_M & Y \\ 0 & 0 \end{bmatrix} \) and \( T = \begin{bmatrix} W & -YZ \\ 0 & Z \end{bmatrix} \), then \( ST = \Omega \). Thus we have

\[
\begin{pmatrix}
A_{11}(W) & A_{12}(W) \\
0 & 0
\end{pmatrix}
= \varphi(\Omega) = \varphi(S)T + S\varphi(T)
\]

\[
= \begin{pmatrix}
0 & A_{12}(I_M) + B_{12}(Y) \\
0 & B_{22}(Y)
\end{pmatrix}
\begin{pmatrix}
W & -YZ \\
0 & Z
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
I_M & Y \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
A_{11}(W) & A_{12}(W) - B_{12}(YZ) + C_{12}(Z) \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
W & -YZ \\
0 & Z
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
A_{12}(I_M) + B_{12}(Y) \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
A_{11}(W) \\
0
\end{pmatrix}
\begin{pmatrix}
W & -YZ \\
0 & Z
\end{pmatrix}
\]

Furthermore,

\[ A_{12}(W) = (A_{12}(I_M) + B_{12}(Y))Z + A_{12}(W) \]

\[ -B_{12}(YZ) + C_{12}(Z) + Y(C_{22}(Z) - B_{22}(YZ)), \]  \tag{3.11}

\[ 0 = B_{22}(Y)Z \]  \tag{3.12}

for any \( Y \) in \( B(M^+, M) \) and \( Z \) in \( \text{alg.} \mathcal{N}_M \). Putting \( Z = I_M \) in Eq. (3.12), we have

\[ B_{22}(Y) = 0 \]

for any \( Y \) in \( B(M^+, M) \). Taking \( Z = I_M \) in Eq. (3.11), we get \( A_{12}(I_M) + C_{12}(I_M) + YC_{22}(I_M) = 0 \) for any \( Y \in B(M^+, M) \). So \( C_{22}(I_M) = 0 \). Since \( C_{22} \) is a derivable mapping at 0, \( C_{22} \) is a derivation on \( \text{alg.} \mathcal{N}_M \) (see [11]). Thus \( C_{22} \) is inner, and so there is an operator \( C \in \text{alg.} \mathcal{N}_M \) such that

\[ C_{22}(Z) = ZC - CZ \]

for any \( Z \) in \( \text{alg.} \mathcal{N}_M \).

**Step 5.** For arbitrary idempotent \( Q \) in \( \text{alg.} \mathcal{N}_M \) and \( Y \) in \( B(M^+, M) \), we write \( Q\lambda \) for \( Q + \lambda I_M \). Obviously there exist two complex numbers \( \lambda_1, \lambda_2 \) such that \( \lambda_1 + \lambda_2 = -\lambda_1\lambda_2 = -1 \). So \( Q_{\lambda_1}Q_{\lambda_2} = \)
$Q_{d_2}Q_{d_1} = I_M$ and $Q_{d_1} + Q_{d_2} = 2Q - I_M$. Taking $S = \begin{bmatrix} WQ_{d_1} & -WQ_{d_1}Y \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} Q_{d_2} & Y \\ 0 & I_{M^+} \end{bmatrix}$, then $ST = \Omega$. Thus we have

$$
\begin{bmatrix} A_{11}(W) & A_{12}(W) \\ 0 & 0 \end{bmatrix} = \varphi(\Omega) = \varphi(S)T + S\varphi(T)
$$

$$
= \begin{bmatrix} A_{11}(WQ_{d_1}) & A_{12}(WQ_{d_1}) - B_{12}(WQ_{d_1}Y) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{d_2} & Y \\ 0 & I_{M^+} \end{bmatrix}
$$

$$
+ \begin{bmatrix} WQ_{d_1} & -WQ_{d_1}Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}(Q_{d_2}) & A_{12}(Q_{d_2}) + B_{12}(Y) + C_{12}(I_{M^+}) \\ 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{11}(WQ_{d_1})Q_{d_2} + WQ_{d_1}A_{11}(Q_{d_2}) & A_{11}(WQ_{d_1})Y + A_{12}(WQ_{d_1}) - B_{12}(WQ_{d_1}Y) + WQ_{d_1}A_{12}(Q_{d_2}) + WQ_{d_1}B_{12}(Y) + WQ_{d_1}C_{12}(I_{M^+}) \\ 0 & 0 \end{bmatrix}
$$

Furthermore,

$$
A_{12}(W) = A_{11}(WQ_{d_1})Y + A_{12}(WQ_{d_1}) - B_{12}(WQ_{d_1}Y) + WQ_{d_1}A_{12}(Q_{d_2}) + WQ_{d_1}B_{12}(Y) + WQ_{d_1}C_{12}(I_{M^+}).
$$

(3.13)

Interchanging the position of $d_1$ and $d_2$ in Eq. (3.13), we have

$$
A_{12}(W) = A_{11}(WQ_{d_2})Y + A_{12}(WQ_{d_2}) - B_{12}(WQ_{d_2}Y) + WQ_{d_2}A_{12}(Q_{d_1}) + WQ_{d_2}B_{12}(Y) + WQ_{d_2}C_{12}(I_{M^+}).
$$

(3.14)

Subtracting Eq. (3.14) from Eq. (3.13), we have

$$
A_{11}(W)Y - B_{12}(WY) + WB_{12}(Y) = 0.
$$

Adding Eq. (3.13) to Eq. (3.14), we have

$$
2[A_{11}(WQ)Y - B_{12}(WQY) + WQB_{12}(Y)] - [A_{11}(W)Y - B_{12}(WY) + WB_{12}(Y)] = 0.
$$

It follows that

$$
A_{11}(WQ)Y - B_{12}(WQY) + WQB_{12}(Y) = 0.
$$

Since every rank one operator in $\text{alg}\mathcal{N}_M$ can be represented as a linear combination of at most four idempotents in $\text{alg}\mathcal{N}_M$(see [3]), we get that the above equation is valid for each rank-one operator in $\text{alg}\mathcal{N}_M$. Furthermore, it is valid for every finite rank operator in $\text{alg}\mathcal{N}_M$(see [2]). Therefore, by the Erdős Density Theorem(see [2]), we have

$$
A_{11}(WX)Y - B_{12}(WXY) + WXB_{12}(Y) = 0
$$

for any $X$ in $\text{alg}\mathcal{N}_M$ and $Y$ in $B(M^+, M)$. If we take $X$ in $\text{alg}\mathcal{N}_M$ and $Y$ in $B(M^+, M)$ in the above equation such that $XY = 0$, from Eq. (3.3) we can get

$$
(WXA - AWX)Y + WXB_{12}(Y) = 0.
$$
That is, $X(AY + B_{12}(Y)) = 0$. By Lemma 2.2 (1), we can pick an operator $G$ from $B(M^\perp)$ such that $AY + B_{12}(Y) = YG$, i.e., $B_{12}(Y) = YG - AY$ for any $Y$ in $B(M^\perp, M)$. Substituting the expressions of $A_{12}, B_{12}, C_{12}, B_{22}$ and $C_{22}$ into Eq. (3.11), we can obtain that $Y[(G - C)Z - Z(G - C)] = 0$ for any $Y$ in $B(M^\perp, M)$ and $Z$ in $\Alg{\mathcal{N}}_M$. Thus $G - C$ in $(\Alg{\mathcal{N}}_M, Y)$. Thus there exists a complex number $\lambda$ such that $G - C = -\lambda I_{M^\perp}$ (The commutant of nest algebra is trivial.). Finally, we can obtain that $B_{12}(Y) = Y(C - \lambda I_{M^\perp}) - AY$. That is,

$$B_{12}(Y) = YC - AY - \lambda Y$$

for any $Y$ in $B(M^\perp, M)$.

In summary, we get that

$$\varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ 0 & A_{22}(X) \end{bmatrix} = \begin{bmatrix} XA - AX & XB \\ 0 & 0 \end{bmatrix}$$

$$\varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ 0 & B_{22}(Y) \end{bmatrix} = \begin{bmatrix} 0 & YC - AY - \lambda Y \\ 0 & 0 \end{bmatrix}$$

$$\varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ 0 & C_{22}(Z) \end{bmatrix} = \begin{bmatrix} 0 & -BZ \\ 0 & ZC - CZ \end{bmatrix}$$

for any $X$ in $\Alg{\mathcal{N}}_M$, $Y$ in $B(M^\perp, M)$ and $Z$ in $\Alg{\mathcal{N}}_M$. Hence we obtain that

$$\varphi\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} - \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} - \lambda \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\begin{bmatrix} A + \frac{1}{2}\lambda I_{M^\perp} & B \\ 0 & C - \frac{1}{2}\lambda I_{M^\perp} \end{bmatrix} - \begin{bmatrix} A + \frac{1}{2}\lambda I_{M^\perp} & B \\ 0 & C - \frac{1}{2}\lambda I_{M^\perp} \end{bmatrix}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}.$$

Thus $\varphi$ is an inner derivation.

Case 2. $\widehat{M} = M^\perp$. Then $\Omega$ may be represented as the following operator matrices relative to the orthogonal decomposition $H = M \oplus M^\perp$:

$$\Omega = \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix},$$
where $W$ is an invertible operator in $\alg \mathcal{M}^\perp$. Since the proof is similar to case 1, the sketch of the proof is given below. The proof is divided into the following six steps:

**Step 1.** For arbitrary $Z_1, Z_2$ in $\alg \mathcal{M}^\perp$ with $Z_1Z_2 = I_{\mathcal{M}^\perp}$, taking $S = \begin{bmatrix} 0 & 0 \\ 0 & WZ_1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 \\ 0 & Z_2 \end{bmatrix}$, then $ST = \Omega$. Since $\varphi$ is derivable at $\Omega$, by imitating the proof of Case 1, we get that $C_{12}(Z) = B'Z$ for any $Z$ in $\alg \mathcal{M}^\perp$, where $B' = C_{12}(I_{\mathcal{M}^\perp})$. It follows from Lemma 2.1 that there exists an operator $C'$ in $B(\mathcal{M}^\perp)$ such that $C_{22}(Z) = ZC' - C'Z$ for any $Z$ in $\alg \mathcal{M}^\perp$.

**Step 2.** For arbitrary $Z_1, Z_2$ in $\alg \mathcal{M}^\perp$ with $Z_1Z_2 = I_{\mathcal{M}^\perp}$ and $X_1, X_2$ in $\alg \mathcal{M}$ with $X_1X_2 = 0$, taking $S = \begin{bmatrix} X_1 & 0 \\ 0 & WZ_1 \end{bmatrix}$ and $T = \begin{bmatrix} X_2 & 0 \\ 0 & Z_2 \end{bmatrix}$, then $ST = \Omega$. By Lemma 2.3 and imitating the proof of case 1, we may get that $C_{11}(Z) = 0$ for any $Z$ in $\alg \mathcal{M}^\perp$. Since $C_{11}$ vanishes on $\alg \mathcal{M}^\perp$, we obtain that $A_{11}$ is derivable at 0. It follows from the expression of $C_{12}$ that $A_{12}(X) = -XB'$ for any $X$ in $\alg \mathcal{M}$. We also get that $A_{22}(Z) = 0$ for any $X$ in $\alg \mathcal{M}$.

**Step 3.** For arbitrary $Z_1, Z_2$ in $\alg \mathcal{M}^\perp$ with $Z_1Z_2 = I_{\mathcal{M}^\perp}$ and $Y$ in $B(\mathcal{M}^\perp, M)$, taking $S = \begin{bmatrix} 0 & 0 \\ 0 & WZ_1 \end{bmatrix}$ and $T = \begin{bmatrix} Y \\ 0 \end{bmatrix}$, then $ST = \Omega$. Furthermore, we get that $B_{22}(Y) = 0$ for any $Y$ in $B(\mathcal{M}^\perp, M)$.

**Step 4.** For an arbitrary $Y$ in $B(\mathcal{M}^\perp, M)$, taking $S = \begin{bmatrix} I_{\mathcal{M}^\perp} & -YW^{-1} \\ 0 & I_{\mathcal{M}^\perp} \end{bmatrix}$ and $T = \begin{bmatrix} Y \\ 0 \end{bmatrix}$, then $ST = \Omega$.

**Step 5.** For arbitrary idempotent $Q'$ in $\alg \mathcal{M}^\perp$ and $Y$ in $B(\mathcal{M}^\perp, M)$, we write $Q'_1$ for $Q' + I_{\mathcal{M}^\perp}$. Then there exist two complex numbers $\lambda_1, \lambda_2$ such that $\lambda_1 + \lambda_2 = -\lambda_1\lambda_2 = -1$. So $Q'_{\lambda_1}Q'_{\lambda_2} = Q'_{\lambda_2}Q'_{\lambda_1} = I_{\mathcal{M}^\perp}$ and $Q'_{\lambda_1} + Q'_{\lambda_2} = 2Q' - I_{\mathcal{M}^\perp}$. Taking $S = \begin{bmatrix} I_{\mathcal{M}^\perp} & Y \\ 0 & WQ'_{\lambda_1} \end{bmatrix}$ and $T = \begin{bmatrix} -YQ'_{\lambda_2} \\ 0 \end{bmatrix}$, then $ST = \Omega$.

**Step 6.** For arbitrary $X$ in $\alg \mathcal{M}$ and $Y$ in $B(\mathcal{M}^\perp, M)$, take $S = \begin{bmatrix} X & -XY \\ 0 & W \end{bmatrix}$ and $T = \begin{bmatrix} Y \\ 0 \end{bmatrix}$, then $ST = \Omega$. It follows from $B' = C_{12}(I_{\mathcal{M}^\perp})$ and the expression of $A_{12}$ that $A_{11}(X)Y - B_{12}(XY) + XB_{12}(Y) = 0$ and $A' + D'$ in $(\alg \mathcal{M})'$ (see [11]). Hence there exists a complex number $\lambda'$ such that
Thus $\varphi$ is an inner derivation. This completes the proof. $\square$

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