First law and Smarr formula of black hole mechanics in nonlinear gauge theories

Yuan Zhang and Sijie Gao

Department of Physics, Beijing Normal University, Beijing 100875, People’s Republic of China
E-mail: zhangyuan@mail.bnu.edu.cn and sijie@bnu.edu.cn

Received 2 March 2018, revised 28 May 2018
Accepted for publication 4 June 2018
Published 21 June 2018

Abstract
Motivated by the fact that Bardeen black holes do not satisfy the usual first law and Smarr formula, we derive a generalized first law from the Lagrangian of the nonlinear gauge field coupled to gravity. In our treatment, the Lagrangian is a function of the electromagnetic invariant as well as some additional parameters. Consequently, we obtain new terms in the first law. With our formula, we find the correct forms of the first law for Bardeen black holes and BI black holes. By scaling arguments, we also derive a general Smarr formula from the first law. Our results apply to a wide class of black holes with nonlinear gauge fields.

Keywords: first law, regular black holes, BI theory, Smarr formula, nonlinear gauge theory

1. Introduction

It is well-known that the standard Einstein–Maxwell theory is described by the action

\[ S = \int \sqrt{-g} (R - F), \]  

(1)

where \( R \) is the scalar curvature of the spacetime and \( F = F_{ab}F^{ab} \) is the electromagnetic invariant. Nonlinear gauge (NLG)\(^1\) theories can be obtained by replacing \( F \) with a nonlinear function of \( F \)\(^2\). One particular NLG model was proposed in 1930s by Born and Infeld (BI) [1] as an attempt to construct charged particles with finite self-energy. The BI theory has been widely applied to quantum gravity and cosmology [2–5].

\(^1\) The terminology ‘nonlinear gauge’ we use in this paper has been referred to as ‘nonlinear electrodynamics’ by some other authors.

\(^2\) The Lagrangian can also depend on another invariant \( F_{ab} + F^{ab} \), where \( \ast F_{ab} \) is the Hodge dual of \( F_{ab} \). To make our demonstration simpler, we shall not discuss this case in the paper.
The Bardeen solution first appeared as an example of regular black holes [6]. Unlike other well-known black holes, Bardeen black holes do not possess central singularities. Bardeen’s solution was followed by various regular black hole solutions (see [7] for a review). However, the matter source of this solution remained unknown for many years. Ayón-Beato and García [8, 9](see [10] for a comment given by Bronnikov) successfully interpreted the source of Bardeen black hole as a magnetic monopole. Miskovic and Olea [11, 12] studied a general Lagrangian for NLG coupled to Einstein–Gauss–Bonnet gravity and found conserved charges. Recent developments on regular black holes can be found in [13–20].

Research on thermodynamic properties of NLG black holes has called attention in recent years. Azreg-Aïnou [21, 22] took into account the $p−V$ terms in NLG black hole thermodynamics. Breton studied the thermodynamical stability of the Bardeen black hole [23, 24]. Fan developed a general procedure to construct exact black hole solutions from NLG and studied the thermodynamics of these solutions [25, 26]. Ma and Zhao discussed the first law and thermodynamic stabilities of regular black holes by introducing a modified temperature [27, 28].

In fact, a general proof for the first law of black hole mechanics in the context of nonlinear gauge theory has been given by Rasheed [29]. By varying the Komar mass and the NLG Lagrangian, he found the following first law that applies to stationary black holes with NLG matter sources:

$$\delta M = \frac{\kappa}{8\pi} \delta \Lambda + \Omega_H \delta J + \Phi_H \delta Q + \Psi_H \delta P,$$

(2)

where $\Lambda, J, Q$ and $P$ are the area, angular momentum, electric charge and magnetic charge.

As demonstrated by Rasheed, the above first law holds for BI black holes. Since the Bardeen solution can be derived from the theory of nonlinear gauge field [9], Rasheed’s formula is expected to hold for Bardeen black holes. However, it is easy to check that Bardeen black holes do not satisfy equation (2).

Why does Rasheed’s formula break down for Bardeen black holes? By carefully examining Rasheed’s derivation, we notice that the Lagrangian used in [29] was assumed to be a pure function of the electromagnetic invariant $F$. But some other parameters, such as the mass and magnetic charge, also appear in the Lagrangian associated with Bardeen solutions [9]. These parameters are treated as constants when deducing the field equation from the Lagrangian. But to derive the first law, they must be treated as variables. By following Rasheed’s prescription, we derive a more general form of the first law and it is verified by the Bardeen solution.

A related issue is the Smarr formula which can be regarded as the integral form of the first law. By using the invariance of the Einstein-NLG theory under scale transformation, we prove a general Smarr formula. Compared to previous results, our formula includes extra terms which come from the additional parameters in the Lagrangian. By applying our result, we obtain the Smarr formula for Bardeen black holes.

The application of our formulae to BI black holes is more subtle. We have mentioned that Rasheed’s first law is satisfied by BI black holes. The BI Lagrangian depends on the vacuum polarization parameter $b$, which is a fundamental parameter and then need not be varied in deriving the field equation. However, as pointed by Rasheed, the first law he proposed does not correspond to a Smarr formula. By our argument, this is because the Lagrangian is not invariant under the scale transformation if $b$ is not allowed to change. This is similar to the case when the cosmological constant $\Lambda$ is present. To recover the first law, one needs to take $\Lambda$ as a variable, interpreted as the pressure [30, 31]. Following this idea, we treat $b$ as a variable and with the help of our formula, we derive the first law with the additional term proportional to $\delta b$. Then We show that the first law naturally gives rise to the correct Smarr formula.
This paper is organized as follows. In section 2, we briefly review the nonlinear gauge theory, deriving the equations of motion from the Einstein-NLG action. In section 3, we follow Rasheed’s treatment to derive the first law of a general NLG black hole. The major improvement in our derivation is assuming that the Lagrangian depends on some extra parameters. Consequently, we find new terms in our first law. In section 4, we derive the Smarr formula from the first law by using the scale-invariance argument. In section 5, by applying our general formulas, we deduce the first law and Smarr formula for Bardeen black holes. In section 6, we obtain the first law and the Smarr formula for BI black holes. Concluding remarks are given in section 7.

2. Nonlinear gauge field coupled to gravity

As we have mentioned in the introduction, a general theory of nonlinear gauge field coupled to gravity can be described by the action [29]

\[ S = \int d^4x \sqrt{\text{det}g} [R - h(F)]. \]  

(3)

Denote the integrand of equation (3) by \( \mathcal{L} \), which is the Lagrangian density. \( \mathcal{L} \) can be viewed as a function of \( g_{ab} \) and \( A_a \), where \( A_a \) is the vector potential satisfying \( F_{ab} = \partial_a A_b - \partial_b A_a \). Let

\[ \mathcal{L} = \mathcal{L}_g - \mathcal{L}_{NL}, \]

(4)

where

\[ \mathcal{L}_g = \sqrt{-g} R, \]

(5)

\[ \mathcal{L}_{NL} = \sqrt{-g} h[F]. \]

(6)

We shall compute the variation of \( \mathcal{L} \) with respect to \( g_{ab} \) and \( A_a \). The standard calculation yields (see e.g. [32])

\[ \delta \mathcal{L}_g = (R_{ab} - \frac{1}{2} R g_{ab}) \delta g_{ab} + \text{boundary term}. \]  

(7)

The variation of \( \mathcal{L}_{NL} \) is

\[ \delta \mathcal{L}_{NL} = h(F) \delta \sqrt{-g} + \sqrt{-g} h'(F) \delta F \]

\[ = -\frac{1}{2} h(F) \sqrt{-g} g_{ab} \delta g_{ab} + \sqrt{-g} h'(F) \delta F. \]  

(8)

By substituting \( F = g^{ac} g^{bd} F_{ab} F_{cd} \), we have

\[ \delta \mathcal{L}_{NL} = -\frac{1}{2} h(F) \sqrt{-g} g_{ab} \delta g_{ab} + \sqrt{-g} h'(F) \delta (g^{ac} g^{bd} F_{ab} F_{cd}) \]

\[ = -\frac{1}{2} h(F) \sqrt{-g} g_{ab} \delta g_{ab} + 2 \sqrt{-g} h'(F) F_{ac} F_{bd} \delta g_{ab} \]

\[ + 2 h'(F) \sqrt{-g} F_{ab} \delta F_{ab} \]

\[ = -\frac{1}{2} h(F) \sqrt{-g} g_{ab} \delta g_{ab} + 2 \sqrt{-g} h'(F) F_{ac} F_{bd} \delta g_{ab} \]

\[ - \sqrt{-g} \partial_a [h'(F) F_{ab}] \delta A_b + \text{boundary term}. \]  

(9)

Combining equations (7) and (9) and discarding the boundary terms, one obtains
\[ \delta \mathcal{L} = \left( R_{ab} - \frac{1}{2} R g_{ab} - 8 \pi T_{ab} \right) \delta g^{ab} + 4 \sqrt{-g} \nabla_a G^{ab} \delta A_b, \]  
\tag{10} \]

where \( G^{ab} \) is defined by

\[ G^{ab} = h'(F) F^{ab} \]  
\tag{11} \]

and

\[ T_{ab} = \frac{1}{4 \pi} \left[ G_a^c F_b^c - \frac{1}{4} h(F) g_{ab} \right] \]  
\tag{12} \]

is the stress–energy tensor of the nonlinear gauge field.

Since the action \( S \) is a functional of \( A_a \) and \( g_{ab} \), \( \delta S = 0 \) yields the electromagnetic field equation

\[ \nabla_a G^{ab} = 0, \]  
\tag{13} \]

and Einstein’s equation

\[ R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi T_{ab}. \]  
\tag{14} \]

If the spacetime is stationary, possessing a timelike Killing vector field \( \xi^a \), the associated electric and magnetic field vectors are defined by

\[ E_a = F_{ab} \xi^b \]  
\tag{15} \]

\[ H_a = - * G_{ab} \xi^b, \]  
\tag{16} \]

where \[29\]

\[ * G_{ab} = \frac{1}{2} \epsilon_{abcd} G^{cd}. \]  
\tag{17} \]

Now we show that both \( E_a \) and \( H_a \) are closed forms. Since \( \xi^a \) is a Killing vector field, we have \( L_\xi F_{ab} = 0 \). Together with the fact that \( F_{ab} \) is a closed form, we can show that \( \nabla_a E_b = 0 \).

By using equations (B.7), (B.8) and (13), we can show that \( * G_{ab} \) is closed. Thus, \( H_a \) is closed.

Therefore, there exist an electric potential \( \Phi \) and a magnetic potential \( \Psi \) such that

\[ E_a = - \nabla_a \Phi, \]  
\tag{18} \]

\[ H_a = - \nabla_a \Psi. \]  
\tag{19} \]

The two scalar potentials can be determined uniquely by requiring them to vanish at infinity.

3. First law of NLG black hole mechanics

In this section, we shall derive the general form of the first law from the NLG action. The derivation follows closely the framework laid out by Rasheed \[29\]. However, we shall see the crucial difference in the final formula.

The key point of the derivation is to make a connection between the black horizon and infinity. Let \( \Sigma \) be a spacelike hypersurface starting from the horizon and extending to infinity (see figure 1). The 3-volume element of \( \Sigma \) is chosen as

\[ \epsilon_{abc} = \epsilon_{dabc} \delta^d, \]  
\tag{20} \]
where \( s^\alpha \) is the future-directed unit timelike vector field orthogonal to \( \Sigma \). \( \Sigma \) is bounded by two topological 2-spheres: \( S \) on the horizon and \( S_\infty \) at infinity. The volume element on \( S_\infty \) is specified as
\[
\epsilon_{ab} = \epsilon_{cab} n^c,
\]
where \( n^\alpha \) is the outward unit normal to \( S_\infty \). Let \( \xi^\alpha \) be a future-directed timelike Killing vector field. The Komar mass is defined by [29, 32]
\[
M = -\frac{1}{8\pi} \int_{S_\infty} \epsilon_{abcd} \nabla^c \xi^d,
\]
and the electric charge is given by
\[
Q = \frac{1}{8\pi} \int_{S_\infty} \epsilon_{abcd} G^{cd}.
\]
Note that in asymptotically spacetimes, the Komar mass agrees with the ADM mass [33]. Applying Stocks’ theorem on \( \Sigma \), equation (22) becomes
\[
M = -\frac{1}{8\pi} \int_S \epsilon_{abcd} \nabla^c \xi^d - \frac{1}{8\pi} \int_{\Sigma} d_\xi (\epsilon_{abcd} \nabla^c \xi^d).
\]
By standard calculation, the first integral in equation (24) yields [32]
\[
-\frac{1}{8\pi} \int_S \epsilon_{abcd} \nabla^c \xi^d = \frac{\kappa A}{4\pi},
\]
and the second integral yields
\[
-\frac{1}{8\pi} \int_{\Sigma} d_\xi (\epsilon_{abcd} \nabla^c \xi^d) = \frac{1}{4\pi} \int_{\Sigma} R_{ab} s^a \xi^b dV.
\]
Then using Einstein’s equation, equation (24) can be written as [32]
\[
M = \frac{\kappa A}{4\pi} + 2 \int_{\Sigma} \left( T_{ab} - \frac{1}{2} T g_{ab} \right) s^a \xi^b dV.
\]
So the variation of \( M \) is
\[
\delta M = \frac{1}{4\pi} (\kappa \delta A + A \delta \kappa) + 2 \delta \int_{\Sigma} \left( T_{ab} - \frac{1}{2} T g_{ab} \right) s^a \xi^b dV.
\]
Comparing it with equation (B.15) and using Einstein’s equation again, we obtain

\[ 2\delta M = \frac{1}{4\pi} \delta A + \int_{\Sigma} \epsilon_{abcd} \xi^d \delta T - \frac{1}{8\pi} \int_{\Sigma} \gamma^{ef} (8\pi T_{ef} - 4\pi g_{ef} T) \epsilon_{abcd} \xi^d \]
\[ + 2\delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abc} - \delta \int_{\Sigma} T_{sabcdc} \xi^d \epsilon_{abc}. \]  

(29)

With some algebraic manipulations and the help of the formulas

\[ \epsilon_{abcd} \xi^d = s^d \xi_{dabc}, \quad \delta \epsilon_{abcd} = g_{ef} \epsilon_{abcd} \delta g^{ef}, \]

(30)

we have

\[ \delta M = \kappa \frac{1}{8\pi} \delta A - \frac{1}{2} \int_{\Sigma} \gamma^{ef} (T_{ef}) \epsilon_{abcd} \xi^d + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon^{(3)} \].  

(31)

To calculate the variation of the stress–energy tensor, we need to consider the variation of the Lagrangian \( \mathcal{L}_{NL} \). Rasheed assumed that the NLG is described by the function \( h(F) \), as shown in equation (3). Now we assume that \( h \) depends on some other parameters \( \beta_i \) as well. Note that these parameters are not universal constants. For example, in the Bardeen solution, \( \beta_i \) represent the mass and magnetic charge of the black hole. So their variations must be taken into account when formulating the first law. We shall see that this treatment is crucial to get the correct first law. Now equation (6) is written in the form

\[ \mathcal{L}_{NL} = \sqrt{-g} h(F, \beta_i). \]  

(32)

Previously, we have derived \( \delta \mathcal{L}_{NL} \) without varying \( \beta_i \) (see equation (9)). By adding the variations of \( \beta_i \), we find

\[ \delta \mathcal{L}_{NL} = 8\pi \sqrt{-g} T_{ab} \delta g^{ab} + 2\sqrt{-g} G^{ab} \delta F_{ab} + \sqrt{-g} \delta h, \]

(33)

where

\[ \delta h \equiv \sum_i \frac{\partial h}{\partial \beta_i} \delta \beta_i. \]

(34)

Substitution of equation (33) into the first integral in equation (31) yields

\[ -\frac{1}{2} \int_{\Sigma} \gamma^{ef} (T_{ef}) \epsilon_{abcd} \xi^d + \frac{1}{2} \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{2} \int_{\Sigma} \gamma^{ef} T_{ef} \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{16\pi} \int \left( \mathcal{L}_{NL} \right) \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{16\pi} \int \left( \mathcal{L}_{NL} \right) \epsilon_{abcd} \]
\[ = \frac{1}{16\pi} \int \left( h(F, \beta_i) \right) \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{16\pi} \int \left( h(F, \beta_i) \right) \epsilon_{abcd} \]
\[ = \frac{1}{16\pi} \int \left( h(F, \beta_i) \right) \epsilon_{abcd} \]
\[ = \frac{1}{8\pi} \int h(F, \beta_i) \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{8\pi} \int \left( G^{ab} \delta F_{ab} \xi^d \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \right) \]
\[ = \frac{1}{8\pi} \int \left( G^{ab} \delta F_{ab} \xi^d \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \right) \]
\[ = \frac{1}{8\pi} \int \left( G^{ab} \delta F_{ab} \xi^d \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \right) \]

(35)

where \( \epsilon_{abcd} = \frac{1}{\sqrt{-g}} \epsilon_{abcd} \) is the fixed volume element. So equation (31) becomes

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \frac{1}{4\pi} \int h(F, \beta_i) \xi^d \epsilon_{abcd} \]
\[ = \frac{1}{8\pi} \int \left( G^{ab} \delta F_{ab} \xi^d \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \right) \]
\[ = \frac{1}{8\pi} \int \left( G^{ab} \delta F_{ab} \xi^d \epsilon_{abcd} + \delta \int_{\Sigma} T_{cds} \xi^d \epsilon_{abcd} \right) \]

(36)

Substitution of equation (12) yields
\[ \delta M = \frac{\kappa}{8\pi} \delta A + \frac{1}{4\pi} \int \Gamma_{\xi} F_{\alpha \xi} \xi^{(3)} - \frac{1}{8\pi} \int G^{\alpha \beta} \delta F_{\alpha \beta} \xi^{(3)} + \frac{1}{16\pi} \int \delta h \xi^{(3)} \epsilon_{abcd}. \]  

Let

\[ I_1 = \frac{1}{4\pi} \int \Gamma_{\xi} F_{\alpha \xi} \xi^{(3)} \]  

\[ \delta I_2 = -\frac{1}{8\pi} \int G^{\alpha \beta} \delta F_{\alpha \beta} \xi^{(3)}. \]

So

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \delta I_1 + \delta I_2 - \sum_i \left( \frac{1}{16\pi} \int \frac{\partial h}{\partial \beta_i} \xi^{(3)} \epsilon_{abcd} \right) \delta \beta_i. \]

In the following calculation, we shall see that \( \delta I_1 \) and \( \delta I_2 \) are related to the variations of electric charge and magnetic charge.

**Calculating \( \delta I_1 \)**

Using

\[ E_a = F_{ab} \xi^{b} = -\nabla_a \Phi, \]

we may write

\[ I_1 = \frac{1}{4\pi} \int \Gamma_{\xi} \nabla_e \Phi \xi^{(3)} \]  

where \( \epsilon^{(3)} \) is the volume element on \( \Sigma \). From equation (13), we have

\[ I_1 = \frac{1}{4\pi} \int \nabla_e (\Gamma_{\xi} \Phi) \xi^{(3)} \]

\[ = \frac{1}{4\pi} \int \epsilon_{abcd} \nabla_e (G^{de} \Phi). \]  

Using equations (B.7) and (B.8) again, we find

\[ I_1 = \frac{1}{8\pi} \int S_{abcd}. \]  

where

\[ S_{ab} = \epsilon_{abcd} \Phi G^{cd}. \]  

By Stokes’s theorem and the boundary condition \( \Phi \to 0 \) at infinity, we have

\[ I_1 = \frac{1}{8\pi} \int S_{ab} = \frac{1}{8\pi} \int \epsilon_{abcd} \Phi G^{cd} = \Phi_H Q. \]

where \( \Phi_H \equiv \Phi |_{H} \) is the value of \( \Phi \) on the horizon and in the last step, we have used the result that \( \Phi \) is constant on the horizon [32]. Hence

\[ \delta I_1 = \Phi_H \delta Q + Q \delta \Phi_H. \]  

**Calculating \( \delta I_2 \)**

Consider

\[ Y^a = \epsilon^{abcd} (\ast G_{cd} \delta E_b - H_b \delta F_{cd}). \]
Denote the two terms on the right-hand side of equation (48) by \( t_1^r \) and \( t_2^r \), respectively. Then

\[
\begin{align*}
t_1^r &= \epsilon^{abcd} G_{cd} \delta E_b \\
&= \frac{1}{2} \epsilon^{abcd} c_{def} G^{ef} \delta E_b \\
&= \frac{1}{2} \epsilon^{cdab} c_{def} G^{ef} \delta E_b \\
&= -\frac{1}{2} \frac{1}{2} \epsilon^{abef} [\epsilon_{a'b'} c_{def} G^{ef}] \delta E_b \\
&= -2 G^{ab} \xi^e \delta F_{be}
\end{align*}
\]

where we have used that \( \xi^a \) is a fixed vector field, i.e. its variation is zero.

\[
\begin{align*}
t_2^r &= \epsilon^{abcd} G_{be} \xi^e \delta F_{cd} \\
&= -\frac{1}{2} \epsilon^{bacd} \epsilon_{beij} G_{ij} \xi^e \delta F_{cd} \\
&= \frac{1}{2} \frac{1}{3} \epsilon^{a[b} c_{d]} \delta G^{ef} \xi^e \delta F_{cd} \\
&= 3 \epsilon^a c^{G^{df} \xi^e \delta F_{cd}} \\
&= 3 \frac{2}{3} \epsilon^a c^{G^{df} \xi^e \delta F_{cd}}
\end{align*}
\]

Adding equations (49) and (50), we have

\[
\begin{align*}
Y^a &= \epsilon^a G^{cd} \delta F_{cd}
\end{align*}
\]

So equation (39) can be written as

\[
\delta I_2 = -\frac{1}{8\pi} \int \epsilon^{abcd} Y^d Y^{(3)}
\]

\[
= -\frac{1}{8\pi} \int \epsilon^{abcd} G^{ef} \delta E_b - H_k \delta F_{ef}
\]

\[
= -\frac{1}{8\pi} \int \epsilon^{abcd} \ast G_{ef} \delta E_b + \frac{1}{8\pi} \int \epsilon^{abcd} H_k \delta F_{ef}
\]

\[
= -\frac{1}{4\pi} \int \epsilon^{abcd} G_{ef} \delta \nabla^e \Phi - \frac{1}{8\pi} \int \epsilon^{abcd} \ast (\nabla_k \Psi) \delta F_{ef}
\]

\[
= -\frac{1}{4\pi} \int \epsilon^{abcd} \nabla^e (G^{ef} \delta \Phi) - \frac{1}{8\pi} \int \epsilon^{abcd} \ast (\nabla_k \Psi) \delta F_{ef}
\]

\[
= -\frac{1}{8\pi} \int \epsilon^{abcd} \ast \Psi \nabla_k \delta F_{ef}.
\]

The last term vanishes because \( \nabla^k [F_{ef}] = 0 \). So

\[
\delta I_2 = -\frac{1}{4\pi} \int \epsilon^{abcd} \nabla^e (G^{ef} \delta \Phi) + \frac{1}{8\pi} \int \epsilon^{abcd} \ast (\nabla_k \Psi) \delta F_{ef}.
\]

By applying equations (B.7) and (B.8) and Stokes’ theorem, we obtain
\[ \delta I_2 = -\frac{1}{8\pi} \int_S \epsilon_{abcd} G^{cd} \delta \Phi_H - \frac{1}{16\pi} \int_S \epsilon_{abcd} \epsilon^{ef} \Psi \delta F_{ef} \]

\[ = -Q \delta \Phi_H + \frac{\Psi_H}{4\pi} \int S \delta F_{cd} \]  

(54)

Since

\[ P = -\frac{1}{8\pi} \int_S \epsilon_{abcd} * F^{cd} = \frac{1}{4\pi} \int S F_{ab} \]  

(55)

is the magnetic charge of the black hole, equation (54) becomes

\[ \delta I_2 = -Q \delta \Phi_H + \Psi_H \delta P. \]  

(56)

By substituting equations (47) and (56) into equation (40), we obtain the final form of the first law

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \Phi_H \delta Q + \Psi_H \delta P + \sum_i K_i \delta \beta_i \]  

(57)

where

\[ K_i = \frac{1}{16\pi} \int_S \frac{\partial h}{\partial \beta_i} \xi^a dabc. \]  

(58)

Differing from Rasheed’s result, our expression contains the variations with respect to \( \beta_i \). We shall see, in the following sections, that the extra terms are crucial to get the correct first law and Smarr formula.

4. Smarr formula for NLG black holes

In this section, we shall use the scaling arguments proposed by Wald [34] to derive the Smarr formula from equation (57). Suppose we make the transformation \( A_a \rightarrow \alpha A_a \) with \( \alpha \) being a constant. In Einstein–Maxwell theory, one may choose \( g_{ab} \rightarrow \alpha^2 g_{ab} \) such that the theory is invariant [34]. Since most NLG theories can reduce to Einstein–Maxwell theory in some limits, we make the same choice for the metric. Consequently, we have

\[ \sqrt{-g} \rightarrow \alpha^4 \sqrt{-g}, \quad R \rightarrow \alpha^{-2} R, \quad F \rightarrow \alpha^{-2} F. \]  

(59)

To make the theory invariant, \( h \) must change as (see equation (4))

\[ h(F, \beta_i) \rightarrow \alpha^{-2} h(F, \beta_i). \]  

(60)

The Killing vector field \( \xi^a \) should change as

\[ \xi^a \rightarrow \alpha^{-1} \xi^a. \]  

(61)

Furthermore, we have

\[ M \rightarrow \alpha M, \quad \kappa \rightarrow \alpha^{-1} \kappa, \quad A \rightarrow \alpha^2 A, \quad \Phi_H \rightarrow \Phi_H, \]  

(62)

\[ Q \rightarrow \alpha Q, \quad \Psi_H \rightarrow \Psi_H, \quad P \rightarrow \alpha P. \]  

(63)

\[ ^3 \text{To make the presentation simpler, we have not taken the angular momentum into account in the derivation above. So the } \Omega_H dJ \text{ term is absent. However, there is no problem to add this term to our formula if the spacetime possesses a nonvanishing angular momentum.} \]
We assume
\[ \beta_i \rightarrow \alpha^b \beta_i. \]  
(64)

The value of \( b_i \) depends on the specific form of \( h(F, \beta_i) \) such that equation (60) holds. Consequently,
\[ K_i \rightarrow \alpha^{1-b_i}. \]  
(65)

Since the theory is invariant under the above transformation, the quantities after the transformation should also satisfy the first law (57). Then by substituting these quantities into equation (57), we obtain
\[ M = \frac{k}{4\pi} A + \Phi_H Q + \Psi_H P + \sum_i b_i K_i \beta_i. \]  
(66)

This is the Smarr formula for NLG black holes.

Recently, a similar Smarr formula was obtained directly from NLG Lagrangians [35]. Now we show that our formula (66) is equivalent to that in [35]. According to our analysis,
\[ h(\alpha^{-2} F, \alpha^b \beta_i) = \alpha^{-2} h(F, \beta_i) \]  
(67)

holds for any \( \alpha \). Thus, differentiating both sides of equation (67) with respect to \( \alpha \) and then taking \( \alpha = 1 \) yield
\[ \sum_i b_i \frac{\partial h}{\partial \beta_i} \beta_i = -2 \left( h - \frac{\partial h}{\partial F} \right) = -8\pi T, \]  
(68)

where equation (12) has been used in the last step. Equation (68) shows clearly that the last term in (66) is the integral of the trace of the stress–energy tensor, which is in agreement with the result in [35] (Balart and Fernando [36] derived a similar formula for spherically symmetric NLG black holes). The authors of [35] also considered to express the integral as a product of conjugate pair. However, the Lagrangian in [35] has been confined to the form \( h(\beta, F) = \beta^{-1} \tilde{h}(\beta F) \), where \( \tilde{h} \) is a real differentiable function. The Lagrangians we have considered can have more than one dependent variables and then our Smarr formula has wider applications, such as in the Bardeen case below.

In the following sections, we will apply our results to two well-known NLG solutions.

5. Application to Bardeen black holes

The Bardeen model is described by line element [9]
\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \]  
(69)

where
\[ f(r) = 1 - \frac{2M r^2}{(r^2 + q^2)^{3/2}}. \]  
(70)

The horizon is located at \( f(r = r_h) = 0 \), which gives the relation
\[ M = \frac{(r_h^2 + q^2)^{3/2}}{2r_h^2}, \]  
(71)

where \( r_h \) is the horizon radius. The surface gravity is given by
\[ \kappa = \frac{1}{2} f'(r_h) = \frac{Mr_h(-2q^2 + r_h^2)}{(q^2 + r_h^2)^{3/2}}. \] (72)

The volume element is chosen as
\[ \epsilon_{abcd} = r^2 \sin^2 \theta \, dr_a \wedge dr_b \wedge d\theta_c \wedge d\phi_d. \] (73)

Then by our convention, the induced volume-elements on the \( t = \text{constant} \) hypersurface and the two-sphere are
\[ \epsilon_{abc} = \frac{1}{\sqrt{f(r)}} \left( \frac{\partial}{\partial t} \right) \epsilon_{dabc} = \frac{1}{f(r)} r^2 \sin \theta \, dr_a \wedge d\theta_b \wedge d\phi_c. \] (74)

and
\[ \epsilon_{ab} = \sqrt{f(r)} \left( \frac{\partial}{\partial r} \right)^c \epsilon_{cab} = r^2 \sin \theta \, d\theta_a \wedge d\phi_b. \] (75)

Ayón-Beato and García first found the Bardeen solution can be derived from the following NLG Lagrangian [9]4
\[ h(F, M, q) = \frac{12M}{q^3} \left( \frac{\sqrt{2}q^2 F}{1 + \sqrt{2}q^2 F} \right)^{5/2}. \] (76)

Here \( M \) and \( q \) correspond to the extra parameters \( \beta_i \). Without loss of generality, we shall assume \( q > 0 \). As derived in [9], the field-strength is given by
\[ F_{ab} = q \sin \theta (d\theta_a d\phi_b - d\phi_a d\theta_b). \] (77)

So
\[ F = \frac{q^2}{2q^2}. \] (78)

One can check that \( M \) is just the Komar mass or ADM mass of the spacetime.

Performing the integration (55) on any two-sphere on \( \Sigma \), we find
\[ P = \int F_{\theta a} d\theta d\phi = q. \] (79)

Thus, \( q \) is the magnetic charge of the black hole.

From equation (16), we find
\[ H_a = \frac{15Mq r^4}{2(q^2 + r^2)^{3/2}} dr_a. \] (80)

Then, the magnetic potential is
\[ \Psi(r) = \frac{3M}{2q} \left( 1 - \frac{r^3}{(q^2 + r^2)^{3/2}} \right). \] (81)

where the integration constant has been chosen such that \( \Psi \to 0 \) as \( r \to \infty \). The magnetic potential on the horizon can be obtained immediately by

4 To be consistent with our convention, \( h \) differs from its original expression in [9] by a factor of 4.
\[ \Psi_H = \Psi(r_h). \]  

Equation (58) now corresponds to the following two quantities

\[
K_q = \frac{1}{16\pi} \int_0^\infty \frac{\partial h}{\partial q} \, \kappa_{abc} \, d\epsilon = \frac{1}{4} \int_0^\infty \frac{\partial h}{\partial r} \, r^2 \, dr = \frac{3M}{2q} \left[ (2q^2 r_h^3 + r_h^2) (q^2 + r_h^2)^{-5/2} - 1 \right].
\]

\[
K_M = \frac{1}{16\pi} \int_0^\infty \frac{\partial h}{\partial M} \, \kappa_{abc} \, d\epsilon = \frac{1}{4} \int_0^\infty \frac{\partial h}{\partial r} \, r^2 \, dr = \frac{3}{4} \int_0^\infty \left( 1 - \frac{r_h}{r_h} (q^2 + r_h^2)^{-3/2} \right).
\]

Therefore, the first law (57) can be expressed as

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \Psi_H \delta q + K_q \delta q + K_M \delta M. \] (84)

One can verify equation (84) by varying \( M \) in equation (71). Thus, we have found the correct first law for Bardeen black holes from our general formula.

To derive the corresponding Smarr formula, we need to determine \( \kappa_i \) which was introduced in equation (66). Note that \( K_q \) and \( K_M \) correspond to \( K_i \) in equation (66). Since \( M \rightarrow \alpha M \), one sees immediately that the transformation

\[ q \rightarrow \alpha q \] (85)

just leads to the transformation (60). Therefore, application of equation (66) yields the Smarr formula

\[ M = \frac{\kappa}{4\pi} A + \Psi_H q + K_q \delta q + K_M \delta M. \] (86)

In fact, by rearranging the coefficients, equation (84) can be written in a simpler form

\[ y \delta M = \frac{\kappa'}{8\pi} \delta A + \psi_H' \delta q, \] (87)

where

\[
\kappa' = \left( A^2 - 32\pi^2 q^2 \right) \sqrt{A^2 + 16\pi^2 q^2}/A^3.
\]

\[
\Psi_H' = \frac{6\pi q \sqrt{A^2 + 16\pi^2 q^2}}{A^2}.
\]

It is easy to verify that

\[ M = \frac{1}{4\pi} \kappa' A + \Psi_H q, \] (90)

which can be regarded as a simplified version of the Smarr formula.

6. First law and Smarr formula in BI theory

As we have mentioned in the Introduction, one important example of NLG is BI theory. The Lagrangian describing BI theory is [29]

\[
h(F, b) = \frac{4}{b^2} \left( 1 - \sqrt{1 + \frac{1}{2} b^2 F} \right).
\] (91)

where \( b \) is a constant called the BI vacuum polarization [31]. According to equations (11) and (12), \( G^{ab} \) and \( T_{ab} \) are given by [29]
\[ G^{ab} = \frac{F^{ab}}{\sqrt{1 + \frac{1}{2} b^2 F}}. \]  
\[ (92) \]

\[ T_{ab} = \frac{b^2 F_a^c F_{bc} + \left( \sqrt{1 + \frac{1}{2} b^2 F} - 1 - \frac{1}{2} b^2 F \right) g_{ab}}{b^2 \sqrt{1 + \frac{1}{2} b^2 F}}. \]  
\[ (93) \]

The BI solution is a spherically symmetric solution associated with the Lagrangian (91). The metric takes the form

\[ ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right), \]  
\[ (94) \]

where the function \( m(r) \) satisfies

\[ m'(r) = \frac{1}{b^2} \left( \sqrt{r^4 + b^2 Q^2} - r^2 \right), \]  
\[ (95) \]

and the corresponding field strength tensor is

\[ F_{ab} = \frac{Q}{\sqrt{r^4 + b^2 Q^2}}. \]  
\[ (96) \]

This solution satisfies Einstein’s equation. One can verify that \( Q \) is the electric charge and \( M = \lim_{r \to \infty} m(r) \) is the ADM mass. By integration, we find

\[ m(r) = M - \frac{1}{b^2} \int_r^\infty dx \left( \sqrt{x^4 + b^2 Q^2} - x^2 \right) \]
\[ = M - \frac{r^4 - r^2 \sqrt{b^2 Q^2 + r^4} + 2b^2 Q^2 F_1 \left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{b^2 Q^2}{r^4}\right)}{3b^2 r}, \]  
\[ (98) \]

where \( _2F_1 \) is the hypergeometric function. Since on the horizon

\[ m(r_h) = \frac{r_h}{2}, \]  
\[ (99) \]

the mass in equation (98) can be written as

\[ M = \frac{r_h}{2} + \frac{r^4_h - r^2_h \sqrt{b^2 Q^2 + r^4_h} + 2b^2 Q^2 F_1 \left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{b^2 Q^2}{r^4_h}\right)}{3b^2 r_h}. \]  
\[ (100) \]

It is straightforward to calculate the surface gravity \( \kappa \) and the electric potential on the horizon \( \Phi_H \):

\[ \kappa = \frac{1}{2r_h} - \frac{1}{b^2 r_h} \left( \sqrt{r^4_h + b^2 Q^2} - r^2_h \right), \]  
\[ (101) \]

\[ \Phi_H = \int_{r_h}^\infty \frac{dr}{\sqrt{r^4 + b^2 Q^2}} = \frac{Q}{r_h} _2F_1 \left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{b^2 Q^2}{r^4_h}\right). \]  
\[ (102) \]

Now we verify the first law. From equation (100), it is easy to get
\[ \delta M = \frac{b^2 + 2r_h^4 - 2\sqrt{r_h^2 + b^2Q^2}}{2b^2} \delta r_h + \frac{Q}{r_h^2} F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4} - \frac{b^2Q^2}{r_h^2} \right) \delta Q. \]  

(103)

By comparing with equations (101) and (102), we see that equation (103) is just the first law

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \Phi_H \delta Q. \]  

(104)

Rasheed noticed that equation (104) does not correspond to an integral form, i.e. the Smarr formula. This failure can be explained clearly by our argument in section 4. Equation (91) suggests that \( b \) must change with \( F \) to make the theory invariant. This is critical to get the Smarr formula from the first law. If we treat \( b \) as a variable, the last term in our formula (57) becomes

\[ K \equiv -\frac{1}{16\pi} \int_{r_h}^{\infty} 4\pi \frac{\partial h}{\partial b} r^2 dr. \]  

(105)

From equation (91), we have

\[ \frac{\partial h}{\partial b} = \frac{2(4\sqrt{2} + \sqrt{2b^2F - 4\sqrt{2 + b^2F}})}{b^2(\sqrt{2} + \sqrt{b^2F})}. \]  

(106)

Now substitute

\[ F = -\frac{2Q^2}{b^2Q^2 + r^4} \]  

(107)

into equation (106), we obtain

\[ r^2 \frac{\partial h}{\partial b} = \frac{4}{b^3(\sqrt{2} + \sqrt{b^2F})} \left( b^2Q^2 + 2r_h^4 - 2r_h^2 \sqrt{b^2Q^2 + r^4} \right). \]  

(108)

Then by performing the integral (105), we obtain

\[ K = -\frac{1}{4} \int_{r_h}^{\infty} \frac{\partial h}{\partial b} r^2 dr \]

\[ = -\frac{1}{3b^3r_h} \left[ 2r_h^4 - 2r_h^2 \sqrt{b^2Q^2 + r_h^4} + b^2Q^2 F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4} - \frac{b^2Q^2}{r_h^2} \right) \right]. \]  

(109)

Thus, by applying our formula (57), we finally have

\[ \delta M = \frac{\kappa}{8\pi} \delta A + \Phi_H \delta Q + K \delta b. \]  

(110)

From equation (100) one can verify \( K = \frac{\partial M}{\partial b} \). Equation (110) can be viewed as an extended version of the first law.

The importance of this formula is that it corresponds to the Smarr formula. By the analysis in section 4, we see immediately from equation (91) that the transformation \( b \to \alpha b \) preserves the action. Therefore, equation (66) yields

\[ M = \frac{\kappa A}{4\pi} + \Phi_H Q + Kb. \]  

(111)

This is the desired Smarr formula for BI black holes. In fact, the same extended first law and Smarr formula were already given in [31]. However, as mentioned in [31], the first law remained to be proved from a general perturbation theory techniques. Our formulae (57) and (66) show explicitly how to read off the first law and Smarr formula from a NLG Lagrangian.
7. Conclusions

We have derived a generalized first law and Smarr formula for black holes in nonlinear gauge theories. From scaling arguments, we also derived the Smarr relation corresponding to the first law. In our prescription, it is crucial to consider extra parameters in the Lagrangian, which lead to additional terms in the first law and the Smarr relation. We showed that these terms in the Smarr relation can be written as an integral of the trace of the stress–energy tensor, in agreement with the result in [35] and [36]. Our formulas hold for Bardeen and BI black holes, for which the usual first law and Smarr formula break down. Although a similar Smarr formula for NLG theories has been found by Gulin and Smolić [35], our Smarr formula is derived directly from the first law and thus can be expressed as a sum of conjugate pairs. Moreover, we have considered a more general class of Lagrangians which depend on multiple extra variables, such as in the Bardeen case.

Our work suggests that there are two kinds of variables in Lagrangians: the dynamical variables, such as the electromagnetic field, and nondynamical variables, such as $b$ in the BI theory. When deriving the equations of motion of the theory, only dynamical fields should be varied and nondynamical variables are held fixed. When deriving the first law and Smarr formula, all variables should be varied. It is not difficult to generalize this argument to theories beyond nonlinear gauge field.

Acknowledgments

This research was supported by NSFC Grants No. 11775022 and 11375026. We thank anonymous referees for helpful comments.

Appendix A. Calculating $\delta \kappa$

In this section, we derive the variation of the surface gravity of a static black hole. The derivation follows closely the treatment in [37] with more details. The surface gravity is defined on the horizon $H$ by

$$\kappa = n^a \xi^b \nabla_a \xi_b,$$

(A.1)

where $\xi^a$ is the Killing vector field normal to $H$ and $n^a$ is a null vector field on $H$ satisfying $n^a n_a = -1$. Note that in the static spacetime$^5$,

$$\delta \xi^a = 0.$$  

(A.2)

Using the diffeomorphism freedom, the horizon can remain unchanged after perturbation. This means that in the perturbed spacetime $\xi_a$ is still the normal to the horizon, i.e.

$$\delta \xi_a = f \xi_a,$$

(A.3)

where $f$ is a function. Consequently,

$$\delta n^a = n^a - n^a = gn^a,$$

(A.4)

where $g$ is another function. Since

$$\xi'_a n'^a = -1,$$

(A.5)

$^5$For rotating black holes, $\xi^a$ usually takes the form $\xi^a = k^a + \Omega_H \phi^a$, where $k^a$ and $\phi^a$ represent the timelike and axial Killing vectors, and $\Omega_H$ is the horizon angular velocity. In this case, $\delta \xi_a = \Omega_H \phi^a$. 

Y Zhang and S Gao
Class. Quantum Grav. 35 (2018) 145007
and \( f \) and \( g \) are small quantities, we have
\[
f + g = 0. \tag{A.6}
\]
Thus,
\[
\delta(n^a \xi_b) = \xi_b \delta n^a + n^a \delta \xi_b = g \xi_b n^a + fn^a \xi_b = 0 \tag{A.7}
\]
i.e.
\[
n^a \delta \xi_b + \xi_b \delta n^a = 0. \tag{A.8}
\]
From the fact that \( \xi^a \) is a Killing vector field for both the unperturbed and perturbed spacetime, we obtain
\[
\mathcal{L}_\xi \delta \xi_a = \xi^b \nabla_b \delta \xi_a + \delta \xi_b \nabla_a \xi^b = 0. \tag{A.9}
\]
Now we calculate \( \delta \kappa \). First, we write
\[
k = n^a \xi^b \nabla_a \xi_b = \frac{1}{2} n^a \xi^b \nabla_a \xi_b - \frac{1}{2} n^a \xi^b \nabla_b \xi_a. \tag{A.10}
\]
Then the variation can be written as
\[
\delta \kappa = \frac{1}{2} n^a \xi^b \nabla_a \delta \xi_b - \frac{1}{2} n^a \xi^b \nabla_b \delta \xi_a + \delta n^a \xi^b \nabla_a \xi_b
\[
= \frac{1}{2} n^a \xi^b \nabla_a \delta \xi_b - \frac{1}{2} n^a \xi^b \nabla_b \delta \xi_a - n^a \delta \xi_b \nabla_a \xi^b
\[
= \frac{1}{2} n^a \xi^b \nabla_a \delta \xi_b - \frac{1}{2} n^a \xi^b \nabla_b \delta \xi_a + n^a \xi^b \nabla_b \delta \xi_a
\[
= \frac{1}{2} (n^a \xi^b + n^b \xi^a) \nabla_a \delta \xi_b. \tag{A.11}
\]
where equation (A.8) has been used in the second step and equation (A.9) has been used in the third step. Since \( \delta \xi_b = f \xi_b \), we have
\[
\delta \kappa = \frac{1}{2} (n^a \xi^b + n^b \xi^a) \xi_b \nabla_a f \tag{A.12}
\]
\[
= \frac{1}{2} n^b \xi_b \xi^a \nabla_a f = -\frac{1}{2} \nabla^a (f \xi_a) = -\frac{1}{2} \nabla^a \delta \xi_a. \tag{A.13}
\]
Now that \( \delta \xi_a = \xi^b \delta g_{ab} \), we find [37]
\[
\delta \kappa = -\frac{1}{2} \nabla^a (\xi^b \delta g_{ab}) = -\frac{1}{2} \xi^b \nabla^a \delta g_{ab}, \tag{A.14}
\]
where we have used the fact that \( \nabla^a \xi^b \) is antisymmetric in the last step.

**Appendix B. Calculating \( \delta M \)**

We shall derive a useful formula containing the mass variation. Note that on the horizon, the induced volume element can be specified as [32]
\[
\epsilon_{ab} = \epsilon_{abcd} n^c \xi^d, \tag{B.1}
\]
where \( \xi^a \) is the Killing vector field and null normal to the horizon and \( n^a \) is the inward null vector field satisfying \( n^a \xi^a = -1 \). Equation (B.1) is equivalent to
\[ \epsilon_{abcd} = \epsilon_{ab} \wedge \xi_c \wedge \eta_d. \]  

So on the horizon \( S \)

\[
\int_S \epsilon_{abcd} w^{cd} = \int_S \epsilon_{ab} \wedge \xi_c \wedge \eta_d w^{cd} \\
= \int_S \epsilon_{ab}(\xi_c \eta_d - \eta_c \xi_d) w^{cd}.
\]  

(B.3)

Taking

\[ w^{cd} = \xi^d (\nabla e \gamma^{ce} - \nabla^c \gamma), \]  

(B.4)

where

\[ \gamma_{ab} = \delta_{gab} \]  

(B.5)

and indices are raised by \( g^{ab} \). Then equation (B.3) gives

\[
\int_S \epsilon_{abcd} w^{cd} = \int_S \epsilon_{ab}(\xi_c \eta_d - \eta_c \xi_d) \xi^d (\nabla e \gamma^{ce} - \nabla^c \gamma) \\
= -\int_S \epsilon_{ab} \xi^c (\nabla e \gamma^{ce} - \nabla^c \gamma) \\
= -\int_S \epsilon_{ab} \xi^c \nabla^c \gamma \\
= 2\Lambda \delta \kappa
\]  

(B.6)

where equation (A.14) has been used.

One can show that if

\[ S_{ab} = \epsilon_{abcd} w^{cd}, \]  

(B.7)

then

\[ dS_{ab} = 2\epsilon_{abcd} \nabla e w^{[cd]}. \]  

(B.8)

With the help of equation (B.8), we can apply the Stocks’s theorem to \( \Sigma \)

\[
\int\Sigma 2\epsilon_{abcd} \nabla e w^{[cd]} \\
= -\int_S \epsilon_{abcd} w^{cd} + \int_{\infty} \epsilon_{abcd} w^{cd} \\
= -2\Lambda \delta \kappa + \int_{\infty} \epsilon_{abcd} \xi^c (\nabla e \gamma^d_e - \nabla^d \gamma). 
\]  

(B.9)

The integral at infinity gives \(-8\pi \delta M\) [32], where \( M \) is the Komar mass. Thus

\[
\int\Sigma 2\epsilon_{abcd} \nabla e w^{[cd]} = -2\Lambda \delta \kappa - 8\pi \delta M.
\]  

(B.10)

Let

\[ v^d = \nabla e \gamma^d_e - \nabla^d \gamma, \]  

(B.11)
Then
\[
\int \Sigma 2 \varepsilon_{abcd} \nabla^{[a} v^{b]}
\]
\[
= 2 \int \Sigma \varepsilon_{abcd} \nabla^{[a} \xi^{b]} v^{c]
\]
\[
= \int \Sigma \varepsilon_{abcd} \xi^{d} \nabla a v^{c}.
\]
(B.12)

where \( \mathcal{L}_\xi v^a = 0 \) has been used.

Standard calculation yields [32]
\[
\delta R = \nabla^a v_a + R_{ab} \delta g^{ab} = \nabla^a v_a - R^{ab} \delta g_{ab}.
\]
(B.13)

Thus, equation (B.12) becomes
\[
\int \Sigma 2 \varepsilon_{abcd} \nabla^{[a} v^{b]} = \int \Sigma \varepsilon_{abcd} \xi^{d} \left( \delta R + R^{ef} \delta g_{ef} \right).
\]
(B.14)

Substituting equations (B.14) into (B.10), we finally obtain
\[
\int \Sigma \varepsilon_{abcd} \xi^{d} \left( \delta R + \gamma^{ef} R_{ef} \right) = -2 \Lambda \delta \kappa - 8 \pi \delta M.
\]
(B.15)

ORCID iDs

Sijie Gao [https://orcid.org/0000-0002-8179-8514]

References

[1] Born M and Infeld L 1934 Foundations of the new field theory Proc. R. Soc. A 144 425
[2] Fradkin E and Tseytlin A 1985 Phys. Lett. B 163 123
[3] Bergshoeff E, Sezgin E, Pope C and Townsend P 1987 Phys. Lett. B 188 70
[4] Leigh R 1989 Mod. Phys. Lett. A 4 2767
[5] Bäñados M and Ferreira P G 2010 Phys. Rev. Lett. 105 011101
[6] Bäñados M and Ferreira P G 2014 Phys. Rev. Lett. 113 119901 (erratum)
[7] Bardeen J 1968 Proc. GR5 (Tbilisi, USSR)
[8] Lemos J P S and Zanchin V T 2011 Phys. Rev. D 83 124005
[9] Ayón-Beato E and García A 1998 Phys. Rev. Lett. 80 5056
[10] Ayón-Beato E and García A 2000 Phys. Lett. B 493 149
[11] Bronnikov K A 2000 Phys. Rev. Lett. 85 4641
[12] Miskovic O and Olea R 2011 Phys. Rev. D 83 024011
[13] Miskovic O and Olea R 2011 Phys. Rev. D 83 064017
[14] Neves J C S and Saa A 2014 Phys. Lett. B 734 44
[15] Neves J C S 2017 Int. J. Mod. Phys. A 32 1750112
[16] Manko V S and Ruiz E 2016 Phys. Lett. B 760 759
[17] Abdujabbarov A et al Phys. Rev. D 93 104004
[18] Stuchlík Z and Schee J 2014 Int. J. Mod. Phys. D 23 1550020
[19] Schee J and Stuchlík Z 2015 J. Cosmol. Astropart. Phys. JCAP06(2015)048
[20] Ghosh S G and Amir M 2015 Eur. Phys. J. C 75 553
[21] Macedo C F B, Oliveira E S and Crispino L C B 2015 Phys. Rev. D 92 024012
[22] Azreg-Aïnou M 2015 Phys. Rev. D 91 064049
[23] Azreg-Aïnou M 2015 Eur. Phys. J. C 75 34
[24] Breton N 2015 Ann. Phys. 354 440
[25] Breton N 2005 Gen. Relativ. Gravit. 37 643
[25] Fan Z Y 2017 Eur. Phys. J. C. 77 266
[26] Fan Z Y and Wang X 2016 Phys. Rev. D 94 124027
[27] Ma M-S and Zhao R 2014 Class. Quantum Grav. 31 245014
[28] Ma M-S 2015 Ann. Phys. 362 529
[29] Rasheed D A 1997 arXiv:hep-th/9702087
[30] Kastor D, Ray S and Traschen J 2009 Class. Quantum Grav. 26 195011
[31] Gunasekaran S, Kubiznak D and Mann R B 2012 J. High Energ. Phys. JHEP11(2012)110
[32] Wald R M 1984 General Relativity (Chicago, IL: The University of Chicago Press)
[33] Ashtekar A and Ashtekar A M 1979 On conserved quantities in general relativity J. Math. Phys. 20 793
[34] Sudarsky D and Wald R M 1993 Phys. Rev. D 47 R5209
[35] Gulin L and Smolík I 2018 Class. Quantum Grav. 35 025015
[36] Balart L and Fernando S 2017 Mod. Phys. Lett. A 32 1750219
[37] Bardeen J M, Carter B and Hawking S W 1973 Commun. Math. Phys. 31 161