Exact solution of the EM radiation-reaction problem 
for classical finite-size and Lorentzian charged particles

Claudio Cremaschini
International School for Advanced Studies (SISSA), Trieste, Italy and
Consortium for Magnetofluid Dynamics, University of Trieste, Trieste, Italy

Massimo Tessarotto
Department of Mathematics and Informatics, University of Trieste, Trieste, Italy and
Consortium for Magnetofluid Dynamics, University of Trieste, Trieste, Italy
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An exact solution is given to the classical electromagnetic (EM) radiation-reaction (RR) problem, 
originally posed by Lorentz. This refers to the dynamics of classical non-rotating and quasi-rigid 
fine size particles subject to an external prescribed EM field. A variational formulation of 
the problem is presented. It is shown that a covariant representation for the EM potential of the self-field 
generated by the extended charge can be uniquely determined, consistent with the principles of 
classical electrodynamics and relativity. By construction, the retarded self 4-potential does not 
possess any divergence, contrary to the case of point charges. As a fundamental consequence, 
based on Hamilton variational principle, an exact representation is obtained for the relativistic 
equation describing the dynamics of a finite-size charged particle (RR equation), which is shown to 
be realized by a second-order delay-type ODE. Such equation is proved to apply also to the treatment 
of Lorentzian particles, i.e., point-masses with finite-size charge distributions, and to recover the 
usual LAD equation in a suitable asymptotic approximation. Remarkably, the RR equation admits 
both standard Lagrangian and conservative forms, expressed respectively in terms of a non-local effective Lagrangian and a stress-energy tensor. Finally, consistent with the Newton principle of 
determinacy, it is proved that the corresponding initial-value problem admits a local existence and 
uniqueness theorem, namely it defines a classical dynamical system.

I. INTRODUCTION

An unsolved theoretical problem is related to the description of the dynamics of classical charges with the inclusion 
of their electromagnetic (EM) self-fields, the so-called radiation-reaction (RR) problem ([1], [2], [3]). Despite efforts spent by the scientific community in more than one century of intensive theoretical research, 
an exact solution is still missing (see related discussion in Ref.[4]; for a review see Refs.[5–9]). In this regard, of 
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fundamental importance is the construction of the exact (i.e., non-asymptotic) relativistic equation of motion for a 
analytic (i.e., non-asymptotic) relativistic equation of motion for a classical charged particle in the presence of its EM self-field, also known as RR equation. This concerns, in particular, 
its treatment in the context of special relativity (SR) and classical electrodynamics (CE), namely imposing the following 
basic physical requirements, hereafter referred to as SR-CE Axioms:

1 Axiom #1: the Maxwell equations are fulfilled everywhere in the flat space-time $M^4 \subseteq \mathbb{R}^4$, with metric tensor $g_{\mu\nu}$. The Minkowski metric tensor is denoted as $\eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$. In particular the EM 4-potential $A^\mu$ is assumed of class $C^k(M^4)$, with $k \geq 2$;

2 Axiom #2: the Hamilton variational principle holds for a suitable functional class of variations $\{f\}$. In particular, 
the Hamilton principle must uniquely prescribe the particle world-line as a real function $r^\mu(s) \in C^k(\mathbb{R})$, with 
$k \geq 2$ for all $s \in \mathbb{R}$. The RR equation is then determined by the corresponding Euler-Lagrange (E-L) equations. 
Hence, $\{f\} \equiv \{f_i(s), i = 1, n\}$ is identified with the set of real functions of class $C^k(\mathbb{R})$, with $k \geq 2$:

$$\{f\} \equiv \left\{ f_i(s) : f_i(s) \in C^k(\mathbb{R}); \quad i = 1, n; \quad \text{and} \quad k \geq 2 \right\},$$

(1)

with functions $f_i(s)$ (for $i = 1, n$) to be properly defined. In particular, we shall require that the action functional 
is allowed to be of the general form

$$S_1(f, [f]) = \int_{-\infty}^{+\infty} ds L_1 \left( f(s), \frac{df(s)}{ds}, [f(s)], \left[ \frac{df(s)}{ds} \right] \right).$$

(2)

Here $L_1$ denotes a non-local variational particle Lagrangian, by assumption defined on a finite-dimensional phase-space, which depends at most on first-order derivatives $\frac{df(s)}{ds}$, with $f(s)$ belonging to the functional class
From the analysis of previous literature two important related problems arise: general relativity. Therefore a fully consistent relativistic theory should actually be formulated for to have the same support, as required, for example, by the energy-momentum conservation law in both special and for point mass particles having finite-size charge distributions was developed. However, charge and mass are expected first approach in this direction is provided by the paper by Nodvik (Nodvik, 1964) where a variational treatment possible alternative, in analogy with the case of the Debye screening problem in electrostatics. In this regard, a leading to a delay-type differential equation), the treatment of extended charge distributions emerges as the only time have been known for a long time (see for example the heuristic approach to the RR problem by Caldirola, 1956 the particle itself is actually finite-size that such a force can act on a charged particle only if aspects of the RR problem - and of the LAD and LL equations - are yet to find a satisfactory formulation/solution. This, in turn, implies that such methods allow one to determine - at most - only an asymptotic approximation for the correct RR equation.

For contemporary science the solution of the RR problem represents a fundamental prerequisite for the proper formulation of all relativistic theories, both classical and quantum ones, which are based on the description of relativistic dynamics for classical charged particles.

Since Lorentz famous paper several textbooks and research articles have appeared on the subject of RR. Many of them have criticized aspects of the RR theory, and in particular the LAD and LL equations (for a review see, where one can find the discussion of the related problems). However, despite contrary claims, rigorous results are scarce. In particular, most of previous investigations concern the treatment of point charges. These are usually based either on suitable asymptotic approximations or regularization schemes to deal with intrinsic divergences of the point-charge model. On the other hand, there is no obvious classical physical mechanism, consistent with the SR-CE axioms, which can explain the appearance of a finite EM self-force acting on a point charge. This should arise as a consequence of a finite delay time occurring between the particle position at the time of the generation of its EM self-field and the instantaneous particle position. It is well-known, as discovered by Lorentz himself (Lorentz, 1892 see also for example Landau and Lifschitz, 1951 that such a force can act on a charged particle only if the particle itself is actually finite-size. Therefore, although “ad hoc” models based on the adoption of a finite delay time have been known for a long time (see for example the heuristic approach to the RR problem by Caldirola, 1956 leading to a delay-type differential equation), the treatment of extended charge distributions emerges as the only possible alternative, in analogy with the case of the Debye screening problem in electrostatics. In this regard, a first approach in this direction is provided by the paper by Nodvik (Nodvik, 1964), where a variational treatment for point mass particles having finite-size charge distributions was developed. However, charge and mass are expected to have the same support, as required, for example, by the energy-momentum conservation law in both special and general relativity. Therefore a fully consistent relativistic theory should actually be formulated for finite-size particles. From the analysis of previous literature two important related problems arise:

- Issue #1 - Existence of an exact variational RR equation: this refers to the lack of an exact RR equation, based on Hamilton variational principle, even for classical point-particles (or point-masses). In fact, previous approaches have all been based on approximate (i.e., asymptotic) estimates. Example of this type leading to the well-known LAD equation (Lorentz, Abraham and Dirac) are those due to Nodvik and Medina. A critical aspect of the LAD equation, as well as of the related LL (Landau and Lifschitz, 1951 equation, is that it does not satisfy a variational principle in the customary sense, i.e., according to Axiom #2. Instead, the LL is non-variational, i.e., it does not admit a variational action at all. However, the problem arises whether, in the context of special relativity, an exact RR equation actually exists which holds for suitable
classical finite-size charged particles, and for Lorentzian particles as a limiting case, namely finite-size charges having point-mass distributions. Important related issues follow, such as the possibility for the resulting equation to admit a standard Lagrangian form in terms of a non-local effective Lagrangian function, and to be cast in an equivalent conservative form, as the divergence of an effective stress-energy tensor. Finally, the recovery of the customary LAD equation in a suitable approximation must be verified.

- **Issue #2 - Existence and uniqueness problem:** the second issue is related to the consistency of the variational RR equation with the SR-CE axioms and in particular with NDP. Therefore, the question arises whether an existence and uniqueness theorem for the corresponding initial value problem can be reached or not. Clearly the problem is relevant only for the exact RR equation (yet to be established).

Clearly, the possible solution of these problems has potential wide-ranging implications which are related to the description of relativistic dynamics of systems of classical finite-size particles both in special and general relativity.

**II. GOALS OF THE PAPER AND SCHEME OF THE PRESENTATION**

The aim of the research program, of which the first part is reported here, is to provide a consistent and exact theoretical formulation of the RR problem for classical charged particles with finite-size charge and mass distributions, addressing precisely issues #1 and #2. In this paper the case is considered of extended particles having mass and charge distributions localized on the same support, identified with a surface shell (see Section 3 for a complete rigorous definition). The result is obtained without introducing any perturbative or asymptotic expansion for the evaluation of EM self 4-potential and/or “ad hoc” regularization schemes for its point-particle limit. In particular, finite-size charge distributions are introduced in order to avoid intrinsic divergences (characteristic of the point-charge treatment) and achieve an analytical description of the RR phenomena which is consistent with the SR-CE axioms. A covariant representation for the EM self 4-potential is obtained, uniquely determined by the prescribed charge current density. This allows us to point out the characteristic non-local feature of the EM self-field, which is due to a causal retarded effect, produced by the finite spatial extension of the charge. Here we shall restrict the analysis to the treatment of charge and mass translational motion, leaving the inclusion of rotational dynamics to a subsequent study. Therefore, a suitable mathematical formulation of the problem is given, in which rotational degrees of freedom are effectively excluded from the present investigation. As a further result, it is proved that the exact RR equation here obtained also holds for classical non-rotating Lorentzian particles (Lorentz, 1892[10]), i.e., in the case in which the mass is regarded as point-wise localized and only the charge has a finite spatial extension. The approach here adopted is based on the variational formulation for finite-size charged particles earlier pointed out by Tessarotto et al. [22], in turn relying on the hybrid form of the synchronous variational principle [23, 24]. A key feature of this variational principle is the adoption of superabundant dynamic variables [25] (see also related discussion in Sections 5 and 7). Due to the arbitrariness of their definition, they can always be identified with the components of the particle position and velocity 4-vectors $r^\mu$ and $u^\mu$. This also implies that, by construction, the variational functional necessarily satisfies the property of covariance and MLC. Then, the corresponding E-L equations yield both the RR equation and also the required physical realizability constraints for $r^\mu$ and $u^\mu$, which allow one to identify them with physical observables.

The paper is organized as follows. In Section 3 we present the derivation of the charge and mass current densities for the particle model adopted, while in Section 4 the exact solution for the EM self 4-potential generated by the non-rotating charge distribution is constructed (Lemma 1). On the basis of this result an explicit integral representation is obtained for the EM self-potential (Lemma 2). Subsequently, in Section 5 we proceed in detail to the construction of the variational functional. In particular, by making use of Lemma 3, the contributions from the EM-coupling with both the EM self-field (Subsection 5.1) and the external EM field (Subsection 5.2), as well as the inertial mass contribution (Subsection 5.4) are determined. Then, in Section 6 the resulting variational Lagrangian is derived. In Section 7 the variational formulation for finite-size particles is presented, based on a synchronous variational principle (THM.1). As a fundamental consequence, it is found that the RR equation is a covariant second-order delay differential equation which fulfills all SR-CE axioms and in particular ECP, GIP and MLC. The equation is proved to apply also to the particular case of Lorentzian particles. Then, in Section 8 the RR equation is shown to admit both standard Lagrangian and conservative forms (THM.2). Section 9 deals instead with the asymptotic behavior of the RR equation, showing that in the short delay-time approximation it recovers the customary LAD equation, while not admitting the point-charge limit (THM.3). Finally, in Section 10 it is proved that, under suitable physical assumptions, the RR equation here obtained fulfills also NDP and, consequently, admits a well-posed initial value problem (i.e., there is an existence and uniqueness theorem; see THM.4).
III. CHARGE AND MASS CURRENT DENSITIES

In this section we define the particle model, prescribing its mass and charge distributions, and determine the corresponding covariant expressions for the charge and mass current densities, both needed for the subsequent developments. Here we consider the treatment in the special relativity setting.

By definition, the particle is characterized by a positive constant rest mass \( m_0 \) and a non-vanishing constant charge \( q \), with surface mass and charge densities \( \rho_m \) and \( \rho_c \), respectively. We shall assume that the mass and charge distributions have supports \( \partial \Omega_m \) and \( \partial \Omega_c \). To define the particle mass and charge distributions on \( \partial \Omega_m \) and \( \partial \Omega_c \), let us assume initially that in a time interval \([-\infty, t_o]\) the particle is at rest with respect to an inertial frame (i.e., that external forces acting on the particle vanish identically). As a consequence, by assumption in the subset of the space-time \( M^4 \subseteq \mathbb{R}^4 \) in which \( t \in [-\infty, t_o] \), there is an inertial frame in which both the particle mass and charge distributions are at rest (particle rest-frame \( R_o \)). In this frame, we shall assume that there exists a point, hereafter referred to as center of symmetry (COS), whose position 4-vector \( r^\mu_{\text{COS}} \equiv (ct, r_o) \) spans the Minkowski space-time \( M^4 \subseteq \mathbb{R}^4 \) and with respect to which:

1) \( \partial \Omega_c \) and \( \partial \Omega_m \) are stationary spherical surfaces of radii \( \sigma > 0 \) and \( \sigma_m > 0 \) of equations \( (r - r_o)^2 = \sigma^2 \) and \( (r - r_o)^2 = \sigma_m^2 \);

2) the particle is quasi-rigid, i.e., the mass and charge distributions are stationary and spherically-symmetric respectively on \( \partial \Omega_m \) and \( \partial \Omega_c \);

3) in addition, consistent with the principle of energy-momentum conservation (see further discussion below), we shall assume the distributions of mass and charge densities to have the same support \( \partial \Omega_c \equiv \partial \Omega_m \), hence letting

\[
\sigma_m = \sigma. \tag{4}
\]

Finally, the case in which the mass is considered localized point-wise (Lorentzian particle) is recovered letting \( \sigma_m \neq \sigma \), with \( \sigma > 0 \) and \( \sigma_m = 0 \). In both cases the particle mass and charge distributions remain uniquely defined in any reference frame for arbitrary particle motion.

In this paper, we are concerned only with the investigation of the EM RR phenomenon on the translational dynamical motion of the charged particle. Hence, we require that the mass density (and, as a consequence, also the charge density) does not possess pure spatial rotation, nevertheless still allowing for space-time rotations (i.e., Thomas precession, see below). For definiteness, let us introduce here the Euler angles \( \alpha(s) \equiv \{\varphi(s), \theta(s), \psi(s)\} \) which define the orientation of the body-axis system \( K' \) with respect to the rest system \( K \) (according to the notations used by Nodvik [20]). Introducing the generalized velocities \( \frac{d\alpha(s)}{ds} \equiv \left\{ \frac{d\varphi}{ds}, \frac{d\theta}{ds}, \frac{d\psi}{ds} \right\} \), the condition of vanishing mass and charge spatial rotation in a time interval \( I \subseteq \mathbb{R} \) is thus prescribed imposing that the particular solution

\[
\alpha(s) = \alpha_o, \quad \frac{d\alpha(s)}{ds} = 0, \quad \tag{5}
\]

holds for all \( s \in I \). For a physical motivation for this assumption we refer to the discussion reported by Yaghjian [21].

Having specified the physical properties of the particle by means of the mass and charge distributions, we can now move on to obtaining the covariant expression for the corresponding charge and mass current densities. Since the charge and the mass have the same support, the mathematical derivation is formally the same for both of them. For convenience we start with the charge current \( j^\mu(r) \), introducing for it the representation used by Nodvik. For definiteness, let us denote \[20\]

\[
\begin{align*}
    s & \equiv \text{proper time of the COS}, \\
    r^\mu(s) & \equiv \text{COS 4-position}, \\
    \zeta^\mu & \equiv \text{charge element 4-position}.
\end{align*}
\]

Then, we define the displacement vector \( \xi^\mu \) as follows:

\[

\xi^\mu \equiv \zeta^\mu - r^\mu(s), \tag{6}
\]

from which we also have that \( \zeta^\mu = r^\mu(s) + \xi^\mu \). The physical meaning of the 4-vector \( \xi^\mu \) is that of a displacement between the particle COS and its boundary, where the charge is located. According to this representation, \( \xi^\mu \) is subject to the following two constraints \[20\]:

\[
\begin{align*}
    \xi^\mu \xi_\mu & = -\sigma^2, \\
    \xi_\mu u^\mu(s) & = 0.
\end{align*} \tag{7,8}
\]
where
\[ u^\mu(s) \equiv \frac{d}{ds} r^\mu(s) \] (9)
is the 4-velocity of the COS. The first equality (7) defines the boundary \( \partial \Omega_\sigma = \partial \Omega_m \). The second constraint (8) represents instead the constraint of rigidity for the particle. This implies that in the particle rest frame the 4-vector \( \xi^\mu \) has only spatial components. We can use the information from Eq. (7) to define the internal and the external domains with respect to the mass and charge distributions. In particular, if we define a generic displacement 4-vector \( X^\mu \in M^4 \) as
\[ X^\mu = r^\mu - r^\mu(s) \] (10)
which is subject to the constraint
\[ X^\mu u_\mu(s) = 0, \] (11)
then the following relations hold:
\[ X^\mu X_\mu \leq -\sigma^2 : \text{external domain}, \]
\[ X^\mu X_\mu > -\sigma^2 : \text{internal domain}, \]
\[ X^\mu X_\mu = \xi^\mu \xi_\mu = -\sigma^2 : \text{boundary location}. \]
To derive the current density 4-vector corresponding to the spherical charged shell we follow the presentation by Nodvik [20]. Consider first the charge-current density \( \Delta j^\mu(r) \) corresponding to a charge element \( \Delta q \) on the shell. This is expressed as follows:
\[ \Delta j^\mu(r) = c \Delta q \int_1^2 d\xi^\mu \delta^4 (r^\mu - \xi^\mu) = c \Delta q \int_{-\infty}^{+\infty} ds \left[ u^\mu + \frac{d\xi^\mu}{ds} \right] \delta^4 (x^\mu - \xi^\mu), \] (13)
where
\[ x^\mu = r^\mu - r^\mu(s). \] (14)
Note that, for the simplicity of the notation, here and in the rest of the paper the symbol \( r \) stands for the generic 4-vector \( r^\alpha \) when used as an argument of a function. Since the charge does not possess any pure spatial rotation, the relation
\[ \frac{d\xi^\mu}{ds} = \Gamma u^\mu \] (15)
holds, where \( \Gamma \equiv -\left( \frac{d\xi^\alpha}{ds} \xi^\alpha \right) \) carries the effect associated with the Thomas precession[20]. The expression for \( \Delta j^\mu(r) \) then becomes
\[ \Delta j^\mu(r) = c \Delta q \int_{-\infty}^{+\infty} ds u^\mu [1 + \Gamma] \delta^4 (x^\mu - \xi^\mu). \] (16)
To compute the total current of the charged shell we express the charge element \( \Delta q \) according to the constraint (8) as follows: \( \Delta q = q f(|\xi|) \delta(\xi^\alpha u_\alpha(s)) d^4 \xi \), where \( d^4 \xi \) is the 4-volume element in the \( \xi \)-space. Moreover, \( f(|\xi|) \) is referred to as the form factor, which describes the charge distribution of the moving body. In particular, for a spherically symmetric distribution this has the following representation:
\[ f(|\xi|) = \frac{1}{4\pi\sigma^2} \delta(|\xi| - \sigma), \] (17)
where \( |\xi| \equiv |\sqrt{\xi^\mu \xi_\mu}|. \) The total current density \( j^\mu(r) \) can therefore be obtained by integrating \( \Delta j^\mu(r) \) over \( d^4 \xi \). We get
\[ j^\mu(r) = q c \int_{-\infty}^{+\infty} ds u^\mu \int_1^2 d^4 \xi f(|\xi|) \delta(\xi^\alpha u_\alpha) [1 + \Gamma] \delta^4 (x^\mu - \xi^\mu) \]
\[ = q c \int_{-\infty}^{+\infty} ds u^\mu f(|x|) \delta(x^\alpha u_\alpha) [1 + \Gamma], \] (18)
where
\[ f(|x|) = \frac{1}{4\pi\sigma^2} \delta(|x| - \sigma) \] (19)
with \(|x| \equiv |\sqrt{x^\mu x_\mu}|\). Then we notice that
\[ \delta(x^\alpha u_\alpha(s)) = \frac{1}{|d(x^\alpha u_\alpha(s))/ds|} \delta(s - s_1) = \frac{1}{|1 + \Gamma|} \delta(s - s_1), \] (20)
where by definition \(s_1\) is the root of the algebraic equation
\[ u_\mu(s_1) [r_\mu - r_\mu(s_1)] = 0. \] (21)

Combining these relations, it follows that the integral covariant expression for the charge current density is given by
\[ j^\mu(r) = \frac{qc}{4\pi\sigma^2} \int_{-\infty}^{+\infty} ds u^\mu(s) \delta(|x| - \sigma) \delta(s - s_1). \] (22)

Finally, an analogous expression for the mass current density \(j^\mu_{mass}(r)\) can be easily obtained from \(j^\mu(r)\) by replacing the total charge \(q\) with the total mass \(m_o\), thus giving
\[ j^\mu_{mass}(r) = \frac{m_o c}{4\pi\sigma^2} \int_{-\infty}^{+\infty} ds u^\mu(s) \delta(|x| - \sigma) \delta(s - s_1). \] (23)
We remark that in both equations (22) and (23):
1) the dependence in terms of the 4-position \(r\) enters explicitly through \(|x| = |r^\mu - r^\mu(s)|\) in the form factor and implicitly through the root \(s_1\);
2) consistent with assumption (5), possible charge and mass spatial rotations have been set to be identically zero.

IV. EM SELF 4-POTENTIAL - CASE OF NON-ROTATING CHARGE DISTRIBUTION

A prerequisite for the subsequent developments is the determination of the EM self-potential \((A_{\mu}^{(self)})\) produced by the spherical charged particle shell here introduced. In principle the problem could be formally treated by solving the Maxwell equations with the 4-potential written in terms of a suitable Green function according to standard methods. Remarkably, the solution can also be achieved in a more straightforward way based on the relativity principle and the covariance of Maxwell’s equations. This implies the possibility of obtaining a covariant representation of the EM 4-vector in a generic reference system once its definition is known in a particular reference frame. The approach is analogous to the derivation presented by Landau and Lifschitz [12] for the treatment of a point charge. The solution is provided by the following Lemmas.

Lemma 1 - Covariant representation for \(A_{\mu}^{(self)}(r)\)
Given validity of the assumptions on the particle structure introduced in the previous section and the results obtained for the current density, the following statements hold:

L11: Particle at rest in an inertial frame.
Let us assume that the particle is at rest in an inertial frame \(S_0\) and, according to (24), is non-rotating in this frame. By definition, in \(S_0\) the 4-vector potential of the self-field is written as \(A_{\mu}^{(self)}(r) = A_{S_0\mu}^{(self)}(r) \equiv \{ \Phi^{(self)}, 0 \}\), where
\[ \Phi^{(self)}(r, t) = \begin{cases} \frac{q}{R} (R \geq \sigma), \\ \frac{q}{R} \left( \frac{R}{\sigma} \right) (R < \sigma), \end{cases} \] (24)
(rest-frame representation) denote respectively the external and internal solutions with respect to the boundary of the shell. Here
\[ R \equiv |R|, \] (25)
\[ R = r - r(t'), \] (26)
with $r^\mu = (ct, \mathbf{r})$, $r'^\mu = (ct', \mathbf{r}' \equiv \mathbf{r}(t'))$, and $\mathbf{r}, \mathbf{r}' \equiv \mathbf{r}(t')$ being respectively a generic position 3-vector of $\mathbb{R}^3$ and the (stationary) position 3-vector of the particle COS. It follows that $\Phi^{(\text{self})}(r, t)$ can be equivalently represented as

$$
\Phi^{(\text{self})}(r, t) = \begin{cases} 
\frac{1}{2} (R \geq \sigma), \\
\frac{1}{2} (R < \sigma), 
\end{cases}
$$

(27)

where $t_{\text{rel}} \equiv t - t'$ is the following positive root

$$
t_{\text{rel}} \equiv t - t' = \begin{cases} 
t^{(\text{ext})}_{\text{rel}} \equiv \pm \frac{R}{c} & (R \geq \sigma), \\
t^{(\text{int})}_{\text{rel}} \equiv \pm \frac{\sigma}{c} & (R < \sigma). 
\end{cases}
$$

(28)

L12 : Particle with inertial motion in an arbitrary inertial frame.

Let us assume that when the particle is referred to an arbitrary inertial frame $S_I$ it has a constant 4-velocity $u^\alpha \equiv \frac{d\mathbf{r}'(t')}{ds'}$. Then, let us require that $t_{\text{rel}} \equiv t - t'$ is the positive root of the delay-time equation

$$
\tilde{R}^\alpha \tilde{R}_\alpha = \rho^2,
$$

(29)

with $\tilde{R}^\alpha$ being the bi-vector

$$
\tilde{R}^\alpha = r^\alpha - r'^\alpha(t')
$$

(30)

and

$$
\rho^2 = \begin{cases} 
0 & (X^\alpha X_\alpha \leq -\sigma^2), \\
\sigma^2 \left[ 1 + \frac{X^\alpha X_\alpha}{\sigma^2} \right] & (X^\alpha X_\alpha > -\sigma^2), 
\end{cases}
$$

(31)

where the displacement vector $X^\alpha$ is defined by Eqs. (30) and (7). For consistency, Eq. (31) provides the solution Eq. (28) when evaluated in the COS comoving frame. It follows that in the reference frame $S_I$ the EM self 4-potential have the internal and external solutions

$$
A^{(\text{self})}_\mu(r) = \begin{cases} 
q \frac{u^\alpha(r)}{R_{\alpha\mu}^{(\text{int})}} |_{t_{\text{rel}} = t_{\text{rel}}}^{t_{\text{rel}} = t_{\text{rel}}^{(\text{int})}} & (X^\alpha X_\alpha \leq -\sigma^2), \\
q \frac{u^\alpha(r)}{R_{\alpha\mu}^{(\text{ext})}} |_{t_{\text{rel}} = t_{\text{rel}}}^{t_{\text{rel}} = t_{\text{rel}}^{(\text{ext})}} & (X^\alpha X_\alpha > -\sigma^2),
\end{cases}
$$

(32)

where $R^{\alpha}$ is given by Eq. (37).

L13 : Particle with a non-inertial motion in an arbitrary frame.

Let us assume that the same particle is now referred to an arbitrary frame in which it has a time-dependent velocity $u_\mu(t')$. In this frame the EM self 4-potential $A^{(\text{self})}_\mu(r)$ takes the form:

$$
A^{(\text{self})}_\mu(r) = \begin{cases} 
q \frac{u^\alpha(t')}{R_{\alpha\mu}^{(\text{ext})}} |_{t_{\text{rel}} = t_{\text{rel}}^{(\text{ext})}}^{t_{\text{rel}} = t_{\text{rel}}} & (X^\alpha X_\alpha \leq -\sigma^2), \\
q \frac{u^\alpha(t')}{R_{\alpha\mu}^{(\text{int})}} |_{t_{\text{rel}} = t_{\text{rel}}}^{t_{\text{rel}} = t_{\text{rel}}^{(\text{int})}} & (X^\alpha X_\alpha > -\sigma^2),
\end{cases}
$$

(33)

where $u_\mu(t')$ is the 4-velocity of the COS with 4-position $r'^\alpha(t')$, i.e.,

$$
u_\mu(t') \equiv \frac{dr'^\alpha(t')}{ds'} = \gamma(t') \frac{dr'^\alpha(t')}{cdt'},
$$

(34)

and $t_{\text{rel}}^{(\text{ext})}$, $t_{\text{rel}}^{(\text{int})}$ are the positive roots of the delay-time equation (29).

Proof - L13) If the particle is at rest in an inertial frame $S_0$, from the form of the charge density (22) and the condition of non-rotation (5), the EM self 4-potential is stationary in $S_0$. Hence it takes necessarily the form $A^{(\text{self})}_\mu(r) = A^{(\text{self})}_\mu(0) \equiv \{ \Phi^{(\text{self})}, 0 \}$. Thus, denoting

$$
R \equiv |\mathbf{R}|,
$$

(35)

$$
\mathbf{R} = \mathbf{r} - \mathbf{r} \left( t - \frac{R - \mathbf{r}(t - \frac{R}{c})}{c} \right),
$$

(36)
with \( \mathbf{r} \) a generic position 3-vector of \( \mathbb{R}^3 \) and \( \mathbf{r}(t') = \mathbf{r}(t - \frac{\rho}{c}) \) the retarded-time position 3-vector, \( \Phi^{(self)} \) is written as

\[
\Phi^{(self)}(\mathbf{r}, t) = \begin{cases} 
\frac{2}{\beta} & (R \geq \sigma), \\
\frac{2}{\beta} & (R < \sigma).
\end{cases}
\] (37)

In other words, in the external/internal sub-domains (respectively defined by the inequalities \( R \geq \sigma \) and \( R < \sigma \)) the ES potential \( \Phi^{(self)} \) coincides with the ES potential of a point charge and a constant potential. In terms of the delay time \( t_{rel} = t - t' \) determined by Eq. (28) it is immediate to prove Eq. (27).

L1) Next, let us consider the same particle referred to an arbitrary inertial frame \( S_I \) in which the COS position vector \( r^\alpha(s') \) has a constant velocity

\[
u_\alpha \equiv u^\alpha(s') = \frac{d}{ds'} r^\alpha(s') = \text{const.}
\] (38)

Since by definition \( A^{(self)}_\mu(r) \) is a covariant 4-vector, its form in \( S_I \) is simply obtained by applying a Lorentz transformation \([12]\) according to Eq. (38). This requires

\[
A^{(self)}_\mu(r) = q \frac{u_\mu}{R^\alpha u_\alpha}
\] (39)

where \( \widehat{R}^\alpha = r^\alpha - r^\alpha(s') \). Denoting \( s' \equiv s'(t') \) and \( r^\alpha(s') \equiv (ct', \mathbf{r}(t')) \), let us now impose that \( t - t' \) is the positive root of the delay-time equation (29). The external and internal solutions in this case are given respectively by Eq. (32), as can be seen by noting that when \( u_\mu = (1, 0) \) the correct external and internal solutions (24) are recovered.

L1) The proof of the third statement is a basic consequence of the principle of relativity and of the covariance of the Maxwell equations. In fact we notice that both the solution (32) for the 4-vector potential and Eq. (29) for the delay time, which have been obtained for the specific case of an inertial frame, are already written in covariant form by means of the 4-vector notation. Hence, according to the principle of relativity, this solution is valid in any reference system related by a Lorentz transformation, and for a generic form of the 4-velocity \( u_\mu \) (cf Landau and Lifshitz \([12]\)).

Q.E.D.

We remark that Eq. (33) provides an exact representation (defined up to a gauge transformation) for the EM self 4-potential generated by the non-rotating finite-size charge considered here.

On the base of the conclusions of Lemma 1 it follows that \( A^{(self)}_\mu(r) \) can also be represented by means of an equivalent integral representation as proved by the following Lemma.

**Lemma 2 - Integral representation for** \( A^{(self)}_\mu(r) \)**

Given validity of Lemma 1, the EM self 4-potential Eq. (33) admits the equivalent integral representation

\[
A^{(self)}_\mu(r) = 2q \int_1^2 dr' \delta(\widehat{R}^\alpha \widehat{R}_\alpha - \rho^2),
\] (40)

with \( \rho^2 \) defined by Eq. (31) and \( r'_\mu = r_\mu(s') \).

**Proof** - In fact in the external and internal domains

\[
\delta(\widehat{R}^\alpha \widehat{R}_\alpha - \rho^2) = \begin{cases} 
\frac{\delta(s - s')}{2|R_\alpha \frac{ds}{ds'}|} & (X_\alpha X_\alpha \leq -\sigma^2), \\
\frac{\delta(s - s')}{2|R_\alpha \frac{ds}{ds'}|} & (X_\alpha X_\alpha > -\sigma^2),
\end{cases}
\] (41)

where \( \frac{ds}{ds'} = \frac{dX_\alpha X_\alpha}{ds} = 2X_\alpha u_\alpha(s') = 0 \) because of Eq. (11), while \( s' \) is determined by the delay-time equation (29). Hence, Eq. (40) manifestly implies Eq. (33).

Q.E.D.

**V. THE ACTION INTEGRAL**

In this section we derive the Hamilton action functional suitable for the variational treatment of finite-size charged particles introduced here and the investigation of their dynamics. As indicated in Section 3, the contributions due to
pure spatial charge and mass rotations will be ignored. In this case, the action integral is conveniently expressed in hybrid superabundant variables (see Tessarotto et al. [23]) as follows:

\[ S_1(r, u, \chi, [r]) = S_M(r, u) + S_C^{(self)}(r, [r]) + S_C^{(ext)}(r) + S_\chi(u, \chi), \]

where \( S_M, S_C^{(self)}, S_C^{(ext)} \) and \( S_\chi \) are respectively the inertial mass, the EM-coupling with the self and external fields, and the kinematic constraint contributions. For what concerns the notation, here \( r \) and \( u \) represent local dependencies with respect to the 4-vector position \( r^\mu \) and the 4-velocity \( u^\mu \), \([r]\) stands for non-local dependencies on the 4-vector position \( r^\mu \), while \( \chi \equiv \chi(s) \) is a Lagrange multiplier (see also below and the related discussion in THM.1 of Section 7).

Before addressing the explicit evaluation of \( S_1(r, u, \chi, [r]) \) we prove the following preliminary Lemma concerning the transformation properties of 4-volume elements under Lorentz transformations.

**Lemma 3 - Lorentz transformations and 4-volume elements**

Let us consider a Lorentz transformation (Lorentz boost) from an inertial reference frame \( S_I \) to a reference frame \( S_{NI} \) whose origin has 4-velocity \( u_\mu(s_2) \) with respect to \( S_I \), with \( s_2 \) being considered here an arbitrary proper time independent of \( r^\mu \in S_I \). By assumption \( u_\mu(s_2) \) is constant both with respect to the 4-positions \( r^\mu \in S_I \) and \( r'^\mu \in S_{NI} \) in the two reference frames. The relationship between the two 4-vectors \( r^\mu \in S_I \) and \( r'^\mu \in S_{NI} \) is expressed by the transformation law [26]

\[ r'^\mu = \Lambda^\mu_\nu (u_\mu(s_2)) r^\nu, \]

where \( \Lambda^\mu_\nu (u_\mu(s_2)) \) is the matrix of the Lorentz boost, which by definition depends only on the relative 4-velocity \( u_\mu(s_2) \) between \( S_I \) and \( S_{NI} \). Then it follows that the 4-volume element \( d\Omega \in S_I \) is invariant with respect to the Lorentz boost [43], in the sense:

\[ d\Omega = d\Omega', \]

with \( d\Omega' \in S_{NI} \) denoting the corresponding volume element in the transformed frame \( S_{NI} \).

**Proof** - The proof of this statement follows by considering the general transformation property of volume elements under arbitrary change of coordinates. Consider the invariant 4-volume element \( d\Omega \in S_I \) and assume a Minkowski metric tensor. By definition [12], for a generic change of reference frame the volume element transforms according to the law

\[ d\Omega = \frac{1}{J} d\Omega', \]

where

\[ J \equiv \left| \frac{\partial r^\mu}{\partial r'^\mu} \right| \]

is the Jacobian of the corresponding coordinate transformation. In the case considered here, the Lorentz boost [43] is described by the matrix \( \Lambda^\mu_\nu (u_\mu(s_2)) \) which depends only on the 4-velocity \( u_\mu(s_2) \), by assumption independent of the coordinates \( r^\nu \) and \( r'^\nu \). It follows that \( J \equiv 1 \), implying in turn Eq. (44).

Q.E.D.

We can now proceed to evaluate the various contributions to the action integral \( S_1(r, u, \chi, [r]) \) defined in Eq. (42).

### A. \( S_C^{(self)}(r, [r]) \): EM coupling with the self-field

The action integral \( S_C^{(self)}(r, [r]) \) containing the coupling between the EM self-field and the electric 4-current is of critical importance. For this reason and for the sake of clarity in this subsection the steps of its evaluation are reported in detail. According to the standard approach [12], \( S_C^{(self)} \) is defined as the 4-scalar

\[ S_C^{(self)}(r, [r]) = \int_{1}^{2} d\Omega \frac{1}{c^2} A^{(self)\mu}(r) j_\mu (r), \]
where \( A^{(\text{self})\mu}(r) \) is given by Eq. (10), \( j_\mu(r) \) by Eq. (22) and \( d\Omega \) is the invariant 4-volume element. In particular, in an inertial frame \( S_I \) with Minkowski metric tensor \( \eta_{\mu\nu} \), this can be represented as

\[
d\Omega = cdt dx dy dz,
\]

where \((x, y, z)\) are orthogonal Cartesian coordinates. The functional can be equivalently represented as

\[
S_C^{(\text{self})}(r, [r]) = \frac{q}{4\pi \sigma^2 c} \left( \frac{d\Omega}{d\Omega} \right) A^{(\text{self})\mu}(r) \int_{-\infty}^{+\infty} ds_1 \delta(s_2 - s_1) \times
\]

\[
\times \int_{-\infty}^{+\infty} dsu^\mu(s) \delta(|x'(s)| - \sigma) \delta(s - s_2),
\]

where \( s_1 \) is the root of the equation

\[
u_\mu(s_1) [r^\mu - r'^\mu(s_1)] = 0.
\]

Because of the principle of relativity, the integral (47) can be evaluated in an arbitrary reference frame. The explicit calculation of the integral (47) is then achieved, thanks to Lemma 3, by invoking the Lorentz boost (43) to the reference frame \( S_{NI} \) moving with 4-velocity \( \nu_\mu(s_2) \). In this frame, by construction \( d\Omega' = cdt'dx'dy'dz' = d\Omega \). In particular, introducing the spherical spatial coordinates \((ct', \rho', \phi', \vartheta')\) it follows that the transformed spatial volume element can also be written as \( cdt'dx'dy'dz' \equiv cdt' d\rho' d\phi' d\vartheta' \rho'^2 \sin \vartheta' \). In this frame the previous scalar equation becomes

\[
u_\mu(s_1) [r^\mu - r'^\mu(s_1)] = 0.
\]

On the other hand, performing the integration with respect to \( s_2 \) in Eq. (49), it follows that necessarily \( s_2 = s_1 \), so that from Eq. (51) \( s_1 \) is actually given by

\[
s_1 = ct = s_2.
\]

As a result, the integral \( S_C^{(\text{self})} \) reduces to

\[
S_C^{(\text{self})}(r', [r']) = \frac{q}{4\pi \sigma^2 c} \left( \frac{d\Omega}{d\Omega} \right) A^{(\text{self})\mu}(r') \int_{-\infty}^{+\infty} dsu^\mu(s) \delta(|x'(s)| - \sigma),
\]

with \( x'^\mu(s) = r'^\mu - r'^\mu(s) \). Moreover

\[
A^{(\text{self})\mu}(r') = 2q \int_{-\infty}^{+\infty} ds'' u_\mu(s'') \delta(\hat{R}^{\alpha\alpha} - \rho'^2),
\]

with \( \hat{R}^{\alpha\alpha} = r'^\alpha - r'^\alpha(s'') \) and, thanks to Lemma 1,

\[
\rho'^2 = \begin{cases}
0 & (X'^\alpha X'_\alpha \leq -\sigma^2), \\
\sigma^2 & (X'^\alpha X'_\alpha > -\sigma^2).
\end{cases}
\]

Notice here that in \( S_C^{(\text{self})}(r', [r']) \) the contributions of the external and internal domains for the self-field can be explicitly taken into account letting

\[
\delta(\hat{R}^{\alpha\alpha}) = \Theta(\sigma^2 + \xi^\alpha \xi_\alpha) \delta(\hat{R}^{\alpha\alpha} - \sigma^2 - X'^\alpha X'_\alpha) +
\]

\[
+\tilde{\Theta}(\sigma^2 + \xi^\alpha \xi_\alpha) \delta(\hat{R}^{\alpha\alpha} - \sigma^2),
\]

with \( \Theta(x) \) and \( \tilde{\Theta}(x) \) denoting respectively the strong and weak Heaviside step functions

\[
\tilde{\Theta}(x) = \begin{cases}
1 & x \geq 0 \\
0 & x < 0
\end{cases}
\]

\[
\Theta(x) = \begin{cases}
1 & x > 0 \\
0 & x \leq 0.
\end{cases}
\]
On the other hand, the only contribution to the integral (53) arises (because of the Dirac-delta in the current density) from the subdomain for which \(-\xi^\alpha \xi_\alpha - \sigma^2 = 0\). Hence, \(S_C^{(self)}\) simply reduces to the functional form:

\[
S_C^{(self)}(\alpha'[,\alpha']) = \frac{2q^2}{4\pi \sigma^2 c} \int_0^\pi d\theta' \sin \theta' \int_0^{2\pi} d\phi' \int_0^{+\infty} d\rho' \rho'^2 \times \nabla \delta(\hat{\alpha} \hat{\alpha}' R_\alpha R_\alpha) \int_{-\infty}^{+\infty} d\sigma \delta(|\alpha'(s)| - \sigma).
\]

(59)

The remaining spatial integration can now be performed letting

\[
\rho' \equiv |\alpha'(s)|
\]

and making use of the spherical symmetry of the charge distribution. The constraints placed by the two Dirac-delta functions \(\delta(\hat{\alpha} \hat{\alpha}' R_\alpha)\) and \(\delta(|\alpha'(s)| - \sigma)\) in the previous equation imply that both \(\hat{R}^{\alpha} \hat{R}_\alpha\) and \(|\alpha'(s)|\) are 4-scalars. Then, introducing the representation

\[
\hat{R}^{\alpha} \equiv \hat{r}^{\alpha} - r^{\alpha}(s'') = \hat{R}^{\alpha} + x^{\alpha}(s),
\]

(61)

with

\[
\hat{R}^{\alpha} \equiv r^{\alpha}(s) - r^{\alpha}(s'),
\]

\[
x^{\alpha}(s) \equiv r^{\alpha} - r^{\alpha}(s),
\]

(62)

(63)

it follows that

\[
\hat{R}^{\alpha} \hat{R}_\alpha = \hat{R}^{\alpha} \hat{R}_\alpha + x^{\alpha}(s) x^{\alpha}(s) + 2\hat{R}^{\alpha} x^{\alpha}(s)
\]

(64)

is necessarily a 4-scalar independent of the integration angles (\(\varphi', \theta'\)) when evaluated on the hypersurface \(\Sigma : \hat{R}^{\alpha} \hat{R}_\alpha = 0\). Similarly, the Dirac-delta \(\delta(|\alpha'(s)| - \sigma)\) warrants that \(x^{\alpha}(s) x^{\alpha}(s) = -\sigma^2\), which is manifestly a 4-scalar too. Let us now prove that necessarily

\[
\hat{R}^{\alpha} x^{\alpha}(s) \equiv 0.
\]

(65)

In fact, on \(\Sigma\) it must be

\[
\frac{d}{ds} \hat{R}^{\alpha} \hat{R}_\alpha = \frac{d}{ds'} \hat{R}^{\alpha} \hat{R}_\alpha = 0,
\]

(66)

\[
\frac{d}{ds} \hat{R}^{\alpha} x^{\alpha}(s) = u^{\alpha}(s) x^{\alpha}(s) - \hat{R}^{\alpha} u^{\alpha}(s) = -\hat{R}^{\alpha} u^{\alpha}(s) = -\frac{1}{2} \frac{d}{ds} [\hat{R}^{\alpha} \hat{R}_\alpha],
\]

(67)

\[
\frac{d}{ds'} \hat{R}^{\alpha} x^{\alpha}(s) = -u^{\alpha}(s') x^{\alpha}(s),
\]

(68)

\[
\frac{d}{ds'} \hat{R}^{\alpha} \hat{R}_\alpha = -2\hat{R}^{\alpha} u^{\alpha}(s').
\]

(69)

Therefore,

\[
\frac{d}{ds} \hat{R}^{\alpha} \hat{R}_\alpha = \frac{d}{ds} \hat{R}^{\alpha} \hat{R}_\alpha + 2\hat{R}^{\alpha} x^{\alpha}(s) = 0,
\]

(70)

\[
\frac{d}{ds'} \hat{R}^{\alpha} \hat{R}_\alpha = \frac{d}{ds'} \hat{R}^{\alpha} \hat{R}_\alpha + 2\hat{R}^{\alpha} x^{\alpha}(s) = -2\hat{R}^{\alpha} u^{\alpha}(s'') - 2u^{\alpha}(s') x^{\alpha}(s) = 0,
\]

(71)

from which it follows that, on \(\Sigma\), \(\hat{R}^{\alpha}\) is a 4-vector, since by definition both \(u^{\alpha}(s'')\) and \(x^{\alpha}(s)\) are 4-vectors too. Now we notice that

\[
\hat{R}^{\alpha} \hat{R}_\alpha = f(s, s'') = f(s'', s),
\]

(72)
with $f$ being a 4-scalar which is symmetric with respect to $s$ and $s''$, while by construction

$$\tilde{R}^\alpha x'_\alpha (s) = g (s, s'', \sigma) \neq g (s'', s, \sigma), \quad (73)$$

where $g$ is a non-symmetric 4-scalar with respect to the same parameters. On the other hand, Eq. (59) requires that $\tilde{R}^\alpha R'_\alpha$ must be symmetric in both $s$ and $s''$, so that, thanks to Eqs. (72) and (84), we can conclude that $g = g (\sigma)$ is a constant 4-scalar which can depend at most on $\sigma$. To determine the precise value of $g = \tilde{R}^\alpha x'_\alpha (s)$ we evaluate it in the COS comoving reference frame, where by definition $r_{\text{COS}} (s_0) = (s_0, 0)$ for all the COS proper times $s_0 \in [-\infty, +\infty]$. In this frame $\tilde{R}^\alpha = (s - s'', 0)$ has only time component and when $s_0 = s$ we get $g = \tilde{R}^\alpha x'_\alpha (s) = 0$ identically. On the other hand, since $g$ is a 4-scalar, it is independent of both $s$ and $s''$ and it is null when $s_0 = s$, we conclude that it must be null for all $s_0$ and in any reference frame, which proves Eq. (64).

Hence, as a result of the integration, the action integral $S_C^{(\text{self})}$ takes necessarily the expression

$$S_C^{(\text{self})} (r', [r']) = \frac{2g^2}{c} \int_1^2 dr'_\mu (s'') \int_1^2 dr'^\mu (s') \delta (\tilde{R}^\alpha R'_\alpha - \sigma^2). \quad (74)$$

Finally, since by construction $S_C^{(\text{self})}$ is a 4-scalar, it follows that the primes can be dropped thus yielding the following representation holding in a general reference frame:

$$S_C^{(\text{self})} (r, [r]) = \frac{2g^2}{c} \int_1^2 ds \frac{dr_\mu (s)}{ds} \int_1^2 ds' \frac{dr'^\mu (s')}{ds'} \delta (\tilde{R}^\alpha R'_\alpha - \sigma^2), \quad (75)$$

where for simplicity of notation $s''$ has been replaced with $s'$ and $\tilde{R}^\alpha$ now denotes

$$\tilde{R}^\alpha \equiv r^\alpha (s) - r^\alpha (s'). \quad (76)$$

It is worth pointing out the following basic properties of the functional $S_C^{(\text{self})}$:

1) it is a non-local functional in the sense that it contains a coupling between the past and the future of the dynamical system (see Eq. (3)). In fact it can be equivalently represented as

$$S_C^{(\text{self})} (r, [r]) = \frac{2g^2}{c} \int_{-\infty}^{+\infty} ds \frac{dr_\mu (s)}{ds} \int_{-\infty}^{+\infty} ds' \frac{dr'^\mu (s')}{ds'} \delta (\tilde{R}^\alpha R'_\alpha - \sigma^2); \quad (77)$$

2) furthermore, it is symmetric, namely it fulfills the property

$$S_C^{(\text{self})} (r_A, [r_B]) = S_C^{(\text{self})} (r_B, [r_A]), \quad (78)$$

where $r_A$ and $r_B$ are two arbitrary curves of the functional class $\{f\}$ (see Eq. (11)).

### B. $S_C^{(\text{ext})}$: EM coupling with the external field

The action integral $S_C^{(\text{ext})} (r)$ of the EM coupling with the external field is a 4-scalar defined as

$$S_C^{(\text{ext})} (r) = \int_1^2 d\Omega \frac{1}{c^2} A^{(\text{ext})}\mu (r) j_\mu (r), \quad (79)$$

where $A^{(\text{ext})}\mu (r)$ is the 4-vector potential of the external field, assumed to be assigned, and $j_\mu (r)$ is the current density given by Eq. (22). The evaluation of the action integral $S_C^{(\text{ext})}$ proceeds exactly in the same way as outlined for $S_C^{(\text{self})}$, with the introduction of the Lorentz boost (43), the spherical spatial coordinates and the use of the result from Lemma 3. The only difference now is that the vector potential $A^{(\text{ext})}\mu (r)$ does not possess spherical symmetry when evaluated in $S_{N\ell}$. As a result, spatial integration over the angle variables $\theta'$ and $\phi'$ cannot be computed explicitly. This leads to the introduction of the surface average EM external 4-potential $\overline{A}^{(\text{ext})}\mu$, which is defined in $S_{N\ell}$ as

$$\overline{A}^{(\text{ext})}\mu (r' (s), |x'|) \equiv \frac{1}{4\pi} \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' \left[A^{(\text{ext})}\mu (r' (s) + x'\mu)\right], \quad (80)$$
where we have used the relation \( \text{(14)} \). With this definition, the time and radial integrals can then be calculated using the Dirac-delta functions as outlined for the self-coupling action integral. After performing a final transformation to an arbitrary reference frame, this gives the following expression for \( S_C^{(ext)} \):

\[
S_C^{(ext)}(r) = \frac{q}{c} \int_1^2 A^{(ext)}(r, s, \sigma) \, dr, \tag{81}
\]

\[\text{C. } S_\chi(u, \chi): \text{kinematic constraint}
\]

The kinematic constraint concerns the normalization of the extremal 4-velocity of the COS. This is defined as

\[
S_\chi(u, \chi) = \int_{-\infty}^{+\infty} ds \chi(s) [u_\mu(s)u^\mu(s) - 1], \tag{82}
\]

where \( \chi(s) \) is a Lagrange multiplier.

\[\text{D. } S_M(r, u): \text{inertial mass functional}
\]

The action integral \( S_M \) of the inertial mass for the extended particle is here defined as the following 4-scalar:

\[
S_M(r, u) = \int_1^2 d\Omega \frac{1}{c} g_{\mu\nu} T_M^{\mu\nu}(r), \tag{83}
\]

where \( d\Omega \) denotes the invariant 4-volume element and \( T_M^{\mu\nu} \) the stress-energy tensor corresponding to the mass distribution of the finite-size charged particle. Notice that the choice of \( S_M \) is consistent with the customary definition of the stress-energy tensor \( T^{\mu\nu} \) (for a fluid or a field) in terms of \( T^{\mu\nu} = \frac{\delta L}{\delta g_{\mu\nu}} \), with \( L \) being a suitable Lagrangian function and \( \delta \) representing the variational derivative \( \text{[12]} \). Therefore, it is natural to identify \( S_M \) with the trace of the mass stress-energy tensor for the extended particle. In particular, the explicit representation of \( T_M^{\mu\nu} \) follows by projecting the mass current density \( j_{\text{mass}}^{\mu}(r) \) given in Eq. (23) along the velocity of a generic shell mass-element parameterized in terms of the proper time \( s \) of the COS. The procedure is completely analogous to that outlined in Section 3. Equivalently, \( T_M^{\mu\nu} \) can also be derived by considering the stress-energy tensor of a perfect fluid without pressure (since the mass is located on a shell by assumption) and imposing the rigidity constraints \( \text{(7)} \) and \( \text{(8)} \). Accordingly, one obtains the following expression:

\[
T_M^{\mu\nu}(r) = \frac{m_r c^2}{4\pi r^2} \int_{-\infty}^{+\infty} ds u^\mu(s) u^\nu(s) [1 + \Gamma] \delta(|x| - \sigma) \delta(s - s_1), \tag{84}
\]

with \( s_1 \) being the root of Eq. (21) and \( \Gamma \) the contribution of the Thomas precession. We notice that the stress-energy tensor thus defined is symmetric. With this definition, the action integral \( S_M \) becomes

\[
S_M(r, u) = \frac{m_r c^2}{4\pi r^2} \int_1^2 d\Omega \int_{-\infty}^{+\infty} ds u^\mu(s) u_\mu(s) [1 + \Gamma] \delta(|x| - \sigma) \delta(s - s_1). \tag{85}
\]

The integration over the 4-volume element can be performed explicitly in the same way as explained before (for the EM coupling action integral), by using the prescription of Lemma 3 and the transformation to local spatial spherical coordinates. In particular, here we notice that both \( u^\mu(s) \) and \( \frac{du^\mu}{ds} \) appearing in \( \Gamma \) are independent of the integration variables, while in the reference system \( S_{NI} \) introduced in Lemma 3 we have that \( \int_0^\pi d\phi \sin \theta \int_0^{2\pi} d\varphi \xi^\mu = 0 \), as it follows from the property of \( \xi^\mu \) to be a pure spatial vector in \( S_{NI} \). Thanks to this feature, the whole integral is straightforward, so that one obtains for \( S_M(r, u) \) the final expression:

\[
S_M(r, u) = \int_1^2 m_r c u_\mu dr^\mu \tag{86}
\]

holding in an arbitrary reference frame. Concerning the solution \( \text{(89)} \), a remark is in order. The choice of \( S_M \) given by Eq. (89) proves to be the correct one. In fact, as expected Eq. (89) is formally the same action integral of a point particle, with the difference that here \( u_\mu \) represents the 4-velocity of the COS rather than the one of a point mass.
VI. THE VARIATIONAL LAGRANGIAN

In this section we collect together all of the contributions to \( S_1 \) previously obtained. From the results of the previous section we can write the action integral \( S_1 \) as a line integral in terms of a variational Lagrangian \( L_1(\mathbf{r}, [\mathbf{r}], u, \chi) \) as follows [see Eq.(2)]:

\[
S_1 = \int_{-\infty}^{+\infty} ds L_1(\mathbf{r}, [\mathbf{r}], u, \chi).
\]  

(87)

More precisely, \( L_1(\mathbf{r}, [\mathbf{r}], u, \chi) \) is defined as:

\[
L_1(\mathbf{r}, [\mathbf{r}], u, \chi) = L_M(\mathbf{r}, u) + L_\chi(u, \chi) + L_C^{(\text{ext})}(\mathbf{r}) + L_C^{(\text{self})}(\mathbf{r}, [\mathbf{r}]),
\]

(88)

where

\[
L_M(\mathbf{r}, u) = m c u \frac{d\mathbf{r}^\mu}{ds},
\]

(89)

\[
L_\chi(u, \chi) = \chi(s) [u_\mu(s)u^\mu(s) - 1],
\]

(90)

\[
L_C^{(\text{ext})}(\mathbf{r}) = \frac{d\mathbf{r}^\mu}{ds} \frac{q}{c} \mathbf{F}_\mu^{(\text{ext})}(r(s), \sigma),
\]

(91)

denote the local contributions respectively from the inertial, the constraint and the external EM field coupling terms, while

\[
L_C^{(\text{self})}(\mathbf{r}, [\mathbf{r}]) = \frac{2q^2}{c} \frac{d\mathbf{r}^\mu}{ds} \int_1^2 d\mathbf{r}_s' \delta(\mathbf{R}_\mu - \mathbf{\sigma}^2)
\]

(92)

represents the non-local contribution arising from the EM self-field coupling.

The conclusion is remarkable. Indeed, although the extended particle can be regarded as a continuous system carrying mass and charge current densities, the variational functional here defined is similar to that of a point particle subject to appropriate interactions. In fact, because of the rigidity constraint and the spherical symmetry imposed on the charge and mass distributions, the variational action \( S_1 \) is actually reduced from a volume integral to a line integral over the proper time of the COS. This is realized by means of the volume integration performed in the reference frame \( S_{NI} \) and thanks to Lemma 3.

The procedure introduces the surface-average operator acting both on the external and the self EM coupling terms. As a result, the Lagrangian \( \mathcal{L}_1 \) must be interpreted as prescribing the dynamics for the COS of the charged particle in terms of averaged EM fields, integrating all the force contributions to the translational motion on the shell. Furthermore, we recall once again the formal analogy between the Lagrangian \( L_M(\mathbf{r}, u) \) and the one of a point particle, when \( u_\mu \) is interpreted as the 4-velocity of the point mass rather than that of the COS of the shell. This means that the dynamics of the finite-size particle is effectively described in terms of a point particle with a finite-size charge distribution. Hence, the mathematical problem is formally the same of that for a Lorentzian particle. Therefore, this proves that the particular case of a Lorentzian particle is formally included in the present description, in the limit in which the radius of the mass distribution \( \sigma_m \) is sent to zero while keeping the charge spatial extension fixed (\( \sigma > 0 \)). The conclusion manifestly follows within the framework of special relativity, in which any possible curvature effects due to the EM field and the mass of the particle itself are neglected.

VII. THE VARIATIONAL PRINCIPLE AND THE RR EQUATION

In this section we shall determine the explicit form of the relativistic RR equation for the non-rotating charged particle. As pointed out earlier [22], this goal can be uniquely attained by means of a synchronous variational principle, in analogy with the approach originally developed for point particles by Nodvik in terms of an asynchronous principle (Nodvik, 1964 [20]). In particular, we intend to prove that, in the present case, the exact RR equation can be uniquely and explicitly obtained by using the hybrid synchronous Hamilton variational principle defined in the previous section and given by Eq.(42). In this case the action functional is expressed by means of superabundant hybrid (i.e., non-Lagrangian) variables and the variations are considered as synchronous, i.e., they are performed by keeping constant the particle COS proper time. Taking into account the results presented in the previous sections, the appropriate form of the Hamilton variational principle is given by the following theorem:

**THM.1 - Hybrid synchronous Hamilton variational principle**

*In validity of the SR-CE axioms, let us assume that:*
1. the Hamilton action $S_1(r,u,\chi,[r])$ is defined by Eq. (42), with $A_{\mu}^{(self)}$ given by Eq. (40) and $\chi(s)$ being a suitable Lagrange multiplier;

2. the real functions $f(s)$ in the functional class $\{f\}$ [see Eq. (1)] are identified with

$$f(s) \equiv [r^\mu(s),u_\mu(s),\chi(s)],$$

with synchronous variations $\delta f(s) \equiv f(s) - f_1(s)$ belonging to

$$\{\delta f\} \equiv \{\delta f_i(s) : \delta f_i(s) = f_i(s) - f_1(s) ; i = 1,n \text{ and } \forall f(s), f_1(s) \in \{f\},$$

here referred to as the functional class of synchronous variations;

3. the extremal curve $f \in \{f\}$ of $S_1$, which is the solution of the equation

$$\delta S_1(r,u,\chi,[r]) = 0,$$

exists for arbitrary variations $\delta f(s)$ (hybrid synchronous Hamilton variational principle);

4. if $r^\mu(s)$ is extremal, the line element $ds$ satisfies the constraint $ds^2 = n_{\mu\nu}dr^\mu(s)dr^\nu(s)$;

5. the 4-vector field $A_{\mu}^{(ext)}(r)$ is suitably smooth in the whole Minkowski space-time $M^4$;

6. the E-L equation for the extremal curve $r^\mu(s)$ is determined subject to the constraint that the delay-time $s_{ret}$ (namely the root of the delay-time equation (103) below) must be chosen consistently with ECP.

It then follows that:

TI1) If all the synchronous variations $\delta f_i(s)$ ($i=1,n$) are considered as being independent, the E-L equations for $\chi(s)$ and $u_\mu$ following from the synchronous hybrid Hamilton variational principle (95) give respectively

$$\frac{\delta S_1}{\delta \chi(s)} = u_\mu u^\mu - 1 = 0,$$

$$\frac{\delta S_1}{\delta u_\mu} = m c dr^\nu + 2 \chi u^\nu ds = 0.$$

Instead, the E-L equation for $r_\mu$

$$\frac{\delta S_1}{\delta r^\mu(s)} = 0$$

yields the following covariant (and hence also MLC) 4-vector, second-order delay-type ODE:

$$m c \frac{du_\mu(s)}{ds} = \frac{q}{c} \frac{F_{\mu\nu}^{(ext)}(r(s))}{ds} dr^\nu(s) + \frac{q}{c} \frac{F_{\mu\nu}^{(self)}(r(s),r(s'))}{ds} dr^k(s),$$

which is identified with the RR equation of motion for the COS of a spherical shell non-rotating charge particle. Here

$$F_{\mu\nu}^{(ext)} = \partial_\mu A_\nu^{(ext)} - \partial_\nu A_\mu^{(ext)}$$

denotes the surface-average [defined according to Eq. (5)] of the Faraday tensor carried by the externally-generated EM 4-vector and evaluated at the particle 4-position $r^\mu(s)$. In addition, $F_{\mu\nu}^{(self)}$ - in MLC 4-vector representation - is the surface-averaged Faraday tensor of the corresponding EM self-field, given by

$$F_{\mu\nu}^{(self)} = -\frac{2q}{R^\alpha u_\alpha(s')} \frac{d}{ds'} \left\{ \frac{u_\mu(s')\tilde{R}_k - u_k(s')\tilde{R}_\mu}{R^\alpha u_\alpha(s')} \right\}_{s'=s-s_{ret}}. $$

Imposing the constraint $ds' = \gamma(t') c dt'$, this implies also

$$F_{\mu\nu}^{(self)} = -\frac{2q}{c} \left[ (t - t') - \frac{1}{c^2} \frac{dr(t')}{dt} \cdot (r - r(t')) \right] \frac{d}{dt'} \left\{ \frac{v_\mu(t')\tilde{R}_k - v_k(t')\tilde{R}_\mu}{c^2 \left[ (t - t') - \frac{1}{c^2} \frac{dr(t')}{dt} \cdot (r - r(t')) \right]_{t'=t}} \right\}_{t'=t-t_{ret}}.$$
Here \( u^\nu = \frac{dt^\mu}{ds} \) denotes the COS 4-velocity and \( v^\mu(t) = \frac{du^\mu}{dt} \), while \( s_{ret} = s - s' \) is the positive root of the delay-time equation

\[
R^\alpha R_\alpha - \sigma^2 = 0. \tag{103}
\]

**T12)** The E-L equations (96), (97) and (98) imply that the extremal functional takes the form

\[
S(r, t, u, \chi) = S_1(r, t, u, \chi) = \left( -\frac{m_0 c}{2} \right).
\tag{104}
\]

**T13)** If \( F^\mu_{\nu}(r) \equiv 0 \) for all \( s \leq s_1 \in \mathbb{R} \), a particular solution of Eq. (99), holding for all \( s \leq s_1 \) is provided by the inertial motion, i.e.,

\[
\frac{dr^\mu}{ds} = u^\mu = \text{const.},
\]
\[
\frac{du^\mu}{ds} = 0,
\]

in agreement with the Galilei principle of inertia.

**T14)** The RR equation Eq. (97) also holds for a Lorentzian particle having the same charge distribution of the finite-size particle \((\sigma > 0)\) and carrying a point-mass with position and velocity 4-vectors \( r^\mu(s) \), \( w^\mu(s) \).

**Proof - T11** and T12) The proof proceeds as follows. Since \( \frac{d}{ds} \delta \left( R^\alpha R_\alpha - \sigma^2 \right) = \frac{d}{ds} \delta \left( R^\alpha R_\alpha - \sigma^2 \right) = 0 \), the variations with respect to \( \chi(s) \) and \( u_0 \) deliver respectively the two E-L equations (96) and (97). Hence, the Lagrange multiplier \( \chi \) must be for consistency

\[
2\chi = -m_0 c,
\tag{107}
\]

so that, ignoring gauge contributions with respect to \( \chi \), the extremal functional \( S_1(r, t, u, \chi, r) \) takes the form [104] [statement T12]. To prove also Eq. (98), we notice that the synchronous variation of \( S_{C}^{(self)} \) has the form

\[
\delta S_{C}^{(self)} = \delta A + \delta B,
\tag{108}
\]

where

\[
\delta A = \frac{-4\pi^2}{c^2} \eta_{\nu \mu} \int_1^2 \nu \beta \left[ \int_1^2 \nu \beta \delta \left( R^\alpha R_\alpha - \sigma^2 \right) \right],
\]
\[
\delta B = \frac{2q}{c^2} \eta_{\nu \mu} \int_1^2 \nu \beta \left[ \int_1^2 \nu \beta \delta \left( \nu \beta \sigma - \sigma^2 \right) \right],
\tag{109}
\]

and \( \nu' \equiv \nu'(s') \) and \( \nu'' \equiv \nu''(s) \). Then we can write

\[
\delta A = \frac{2q}{c^2} \eta_{\nu \mu} \int_1^2 \nu \beta \beta \left[ B_k^\nu \right]_{\nu = t - t_{ret}},
\]
\[
\delta B = \frac{2q}{c^2} \eta_{\nu \mu} \int_1^2 \nu \beta \beta \left[ B_k^\nu \right]_{\nu = t - t_{ret}},
\tag{110}
\]

where \( B_k^\nu \) is

\[
B_k^\nu = -\frac{q}{c} \left[ \frac{1}{c^2} \frac{d\nu'}{dt'} \cdot \left( r' - r \right) \right] \frac{d}{dt'} \left( \frac{v''(t') R_k}{(t' - t) - \frac{1}{c^2} \frac{d\nu'}{dt'} \cdot \left( r' - r \right)} \right)
\tag{111}
\]

(the details of the derivation of these identities are provided in Appendix A). Finally, from the results given in Appendix A, the variation with respect to \( r^\mu \) yields

\[
\frac{\delta S_1}{\delta r^\mu} = -m_0 c u_\mu(s) + \frac{q}{c} d_{\nu k} \left[ F^\mu_{\nu k}^{(self)} \right] + \frac{q}{c} \left[ \partial_{\mu} A^\nu - \partial_{\nu} A_{\mu} \right] \left( \nu s \right) \right] dr^\nu,
\tag{112}
\]

where

\[
F^\mu_{\nu k}^{(self)} = 2(B_{k \mu} - B_{\mu k}),
\tag{113}
\]
from which Eqs. (103)-(106) follow. This yields the RR equation being sought, i.e., the exact relativistic equation of motion for the translational dynamics of the COS of a finite-size spherical shell charge particle subject to the simultaneous action of a prescribed external EM field and of its EM self-field.

TI.3) The proof of Eqs. (105)-(106) is straightforward. In fact, let us assume that in the interval \([-∞, s_1]\) the motion is inertial, namely that \(\frac{d}{ds}u_\mu \equiv 0\), \(\forall s \in [-∞, s_1]\). This implies that in \([-∞, s_1]\) it must be \(u_\mu \equiv u_{0\mu}\), with \(u_{0\mu}\) denoting a constant 4-vector velocity. It follows that \(\forall s, s' \in [-∞, s_1]\), \(r_\mu(s) = r_\mu(s') + u_{0\mu}(s-s')\) and \(R_\mu = u_{0\mu}(s-s')\). Hence, by direct substitution in Eq. (102) we get that \(v_\mu(t')\overline{R}_k - v_k(t')\overline{R}_\mu = 0\), which by consequence implies also that \(dv_\mu H_{\mu k} \equiv 0\) identically in this case.

TI.4) The proof follows immediately from the definition of Lorentzian particle given above by noting that in the context of SR the variational particle Lagrangian \(L_1(r, [r], u, \chi)\) [see Eq. (88)] formally coincides with that of a Lorentzian particle characterized by a finite charge distribution [i.e., with \(\sigma > 0\)], subject to the simultaneous action of the averaged external and EM self-fields \(\overline{T}^{(ext)}_{\mu\nu}\) and \(\overline{T}^{(self)}_{\mu\nu}\).

Q.E.D.

We notice that, by assumption, the varied functions \(f(s) \equiv [r_\mu(s), u_\mu(s), \chi(s)]\) are unconstrained, namely they are solely subject to the requirement that end points and boundary values are kept fixed. This implies that all of the 9 components of the variations \(\delta f(s)\), namely \(\delta r_\mu(s), \delta u_\mu(s), \delta \chi(s)\), must be considered independent. On the other hand, the extremal curves \(f(s)\) of \(S_1([r, u, \chi, [r]])\), the solution of the hybrid Hamilton variational principle, satisfy all of the required physical constraints, so that only 6 of them are actually independent. In fact, the resulting E-L equations determine, besides the RR equation (99), also the relationship between \(r_\mu(s)\) and \(u_\mu(s)\), namely

\[
u_\mu(s) = \frac{dr_\mu(s)}{ds},
\]

as well as the physical constraint

\[
u_\mu(s)u_\mu(s) = 1.
\]

As a consequence, \(r_\mu(s)\) and \(u_\mu(s)\) coincide respectively with the physical 4-position and 4-velocity of the COS mass particle. Therefore only 3 components of the 4-velocity are actually independent, while the first component of the 4-position \(ct\) can always be represented in terms of the proper length \(s\) (so that only the spatial part of the position 4-vector actually defines a set of independent Lagrangian coordinates).

A further basic feature of the RR equation concerns the validity of GIP and its meaning in this context. In fact, let us assume that the external EM field is non-vanishing in the time interval \(I_{12} \equiv [s_1, s_2]\), while it vanishes identically in \(I_2 \equiv [s_2, +∞]\). Then, the inertial solution (105) and (106) does not hold, by definition, in \(I_{12}\) and is only achieved in an asymptotic sense in \(I_2\), i.e., in the limit \(s \to +∞\). In fact, the non-local feature of the RR effect prevents the particle from reaching the inertial state in a finite time interval. It is concluded, therefore, that GIP must be intended as holding in the past, namely in the time interval \(s \leq s_1\in \mathbb{R}\), where by assumption no external EM field is acting on the particle.

VIII. STANDARD LAGRANGIAN AND CONSERVATIVE FORMS OF THE RR EQUATION

In this section we discuss some developments about the physical properties of the non-local RR equation obtained, which exactly describes the translational dynamics of the COS of a spherical-shell non-rotating charged particle. Remarkably, the variational principle (THM.1) implies that the E-L equations (106)- (109) can be cast in an equivalent way either:

1) in a standard Lagrangian form, namely expressed in the form of Lagrange equations defined in terms of a suitable non-local effective Lagrangian \(L_{eff}\);

2) in a conservative form, as the divergence of a suitable effective stress-energy tensor.

The result is provided by the following theorem.

THM.2 - RR equation in standard Lagrangian and conservative forms

Given validity of THM.1, it follows that:

T21) Introducing the non-local real function

\[
L_{eff} \equiv L_M(r, u) + L_\chi(u, \chi) + L^{(ext)}_{C}(r) + 2L^{(self)}_{C}(r, [r]),
\]

(116)
here referred to as non-local effective Lagrangian, the E-L equations (96), (97) and (112) take respectively the form

$$\frac{\partial L_{\text{eff}}}{\partial \chi(s)} = 0,$$

$$\frac{\partial L_{\text{eff}}}{\partial \mu(s)} = 0,$$

$$\frac{d}{ds} \frac{\partial L_{\text{eff}}}{\partial \nu(s)} - \frac{\partial L_{\text{eff}}}{\partial \nu^\prime(s)} = 0.$$

(117) (118) (119)

These will be referred to as E-L equations in standard Lagrangian form.

The stress-energy tensor of the system $T_{\mu\nu}$ is uniquely determined in terms of $L_{\text{eff}}$. As a consequence, the RR equation (99) can also be written in conservative form as

$$T_{\mu\nu,\nu} = 0,$$

(120)

where $T_{\mu\nu} = T^{(M)}_{\mu\nu} + T^{(EM)}_{\mu\nu}$ is the surface-averaged total stress energy tensor, obtained as the sum of the corresponding tensors for the mass distribution and the EM field which characterize the system.

Proof - T21) The proof follows immediately by noting that the Hamiltonian action (42) defines a symmetric functional with respect to local and non-local dependencies, i.e., such that

$$S_1(r_A, r_B, u, \chi) = S_1(r_B, r_A, u, \chi).$$

(121)

Because the E-L equations (117)-(119) are written in terms of local partial derivative differential operators, the effective Lagrangian $L_{\text{eff}}$ must be therefore distinguished from the corresponding variational Lagrangian function $L_1$ which enters the Hamilton action and which contains non-local contributions. These features imply the definition (116), which manifestly satisfies the E-L equations in standard form (117)-(119).

T22) The proof of this statement is straightforward, by first recalling that the Lagrangian of the distributed mass is analogous to that of a point mass particle. Moreover, the stress-energy tensor of the total EM field $T^{(EM)}_{\mu\nu}$, to be defined in terms of $L_{\text{eff}}$ according to the standard definition (see for example Landau and Lifshitz [12]) becomes

$$T^{(EM)}_{\mu\nu} = T^{(EM-ext)}_{\mu\nu} + T^{(EM-self)}_{\mu\nu}.$$

(122)

Then, given validity to the Maxwell equations, it follows that

$$T^{(EM)}_{\mu\nu,\nu} = F_{\mu\nu} j^\nu = \left[ F^{(ext)}_{\mu\nu} + F^{(self)}_{\mu\nu} \right] j^\nu.$$

(123)

Gathering the mass and the field contributions, substituting the expressions for $F^{(ext)}_{\mu\nu}$ and $F^{(self)}_{\mu\nu}$ obtained in THM.1, and performing the integration over the 4-volume element finally proves that the equation (120) actually coincides with the extremal RR equation (99).

Q.E.D.

The expression (120) represents the conservative form of Eq. (99), and hence - consistent with the surface integration procedure here adopted - it holds for the surface-averaged EM external and self-fields $T^{(ext)}_{\mu\nu}$ and $T^{(self)}_{\mu\nu}$, defined respectively by Eqs.(100) and (101). It is important to remark that the result holds both for finite-size and Lorentzian particles. On the other hand, a local form of the conservative equation - analogous to Eq. (120) - and holding for the local EM fields is in principle achievable too. However, this last conclusion generally applies only to finite-size particles with the same support for the mass and charge distributions, i.e., for which Eq. (4) holds.

IX. SHORT DELAY-TIME ASYMPTOTIC APPROXIMATION

In this section we consider the asymptotic properties of the RR equation, considering the customary approximation in the treatment of the problem, which leads to the LAD equation (Dirac, 1938 [3]). This is the power-series expansion of the retarded EM self-potential in terms of the dimensionless parameter $\epsilon \equiv \frac{(s-s')}{s}$, to be assumed as
infinitesimal (short delay-time ordering), \( s - s' \) denoting the proper-time difference between observation (s) and emission (s'). The same approach was also adopted by Nodvik [20] in the case of flat space-time and by DeWitt and Brehme [27] and Crowley and Nodvik [28] in their covariant generalizations of the LAD equation valid in curved space-time. It is immediate to show that the following result holds:

**THM.3 - First-order, short delay-time asymptotic approximation**

Let us introduce the 4-vector \( G_\mu \) defined as

\[
ds G_\mu = q \frac{F^{(self)}_{\mu k} \ dr^k}{c}, \tag{124}
\]

and invoke the asymptotic ordering

\[
0 < \epsilon \ll 1. \tag{125}
\]

Then:

\( T3_1 \) Neglecting corrections of order \( \epsilon^N \), with \( N \geq 1 \) (first-order approximation), the following asymptotic approximation holds for \( G_\mu \)

\[
G_\mu \equiv \left\{ -m_{\sigma EM} \frac{d}{ds} u_\mu + g_\mu \right\} \left[ 1 + O(\epsilon) \right], \tag{126}
\]

where \( g_\mu \) denotes the 4-vector

\[
g_\mu = \frac{2q^2}{3c} \left[ \frac{d^2}{ds^2} u_\mu - u_\mu(s) u^k(s) \frac{d^2}{ds^2} u_k \right], \tag{127}
\]

with

\[
m_{\sigma EM} \equiv \frac{q^2}{c^2 \sigma} \frac{1}{1 + \frac{v(t(s))}{c^2}} \tag{128}
\]

being the EM mass and \( \gamma(t(s)) \equiv 1/\sqrt{1 - v^2(t(s))/c^2} \).

\( T3_2 \) The point-charge limit of the RR equation (99) does not exist.

Proof - \( T3_1 \) The proof is straightforward and follows by performing explicitly the perturbative expansion with respect to \( \epsilon \). By dropping the terms which vanish in the limit \( \epsilon \to 0 \), this yields Eq.(126). The proof of \( T3_2 \), instead, follows by noting that the limit obtained by letting \( \sigma \to 0^+ \)

(point-charge limit) is not defined, since

\[
\lim_{\sigma \to 0^+} m_{\sigma EM} = \infty. \tag{130}
\]

Q.E.D.

As basic consequences, in the first-order approximation the RR equation (99) recovers the LAD equation. Moreover, in a similar way, by introducing a suitable approximate reduction scheme, also the LL equation (Landau and Lifschitz, 1951 [12]) can be immediately obtained.

**X. THE FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM**

THMs.1 and 2 show that in the presence of RR the non-local Lagrangian system \( \{ x, L \} \) admits E-L equations [Eq.(99)] which are of delay differential type. This feature is not completely unexpected, since model equations of this type have been proposed before for the RR problem (see for example [18]). In general, for a delay-type differential equation there is nothing similar to the existence and uniqueness theorem holding for an initial condition of the type

\[
x(s_o) = x_o. \tag{131}
\]
In fact, no finite set of initial data is generally enough to determine a unique solution. The possibility of having, under suitable physical assumptions, an existence and uniqueness theorem therefore plays a crucial role in the proper formulation of the RR problem. In fact, for consistency with the SR-CE axioms, and in particular with NPD, the existence of a classical dynamical system \[ \text{[13]} \] must be warranted. The result can be obtained by requiring that there exists an initial time \( s_o \) before which for all \( s < s_o \), the particle motion is inertial (see also the related discussion in Ref. \[ \text{[15]} \]). The assumption has also been invoked to define the particle mass and charge distributions (see Section 3).

In view of THM.1 this happens if the external EM force vanishes identically for all \( s < s_o \) and is (smoothly) “turned on” at \( s = s_o \). In this regard, we here point out the following theorem:

**THM.4 - The fundamental theorem for the RR equation**

Given validity of THM.1, let us assume that:

1. **REQUIREMENT #1:** at time \( t_o \) the initial condition \[ \text{[13]} \] holds;

2. **REQUIREMENT #2:** the external force \( F^{(\text{ext})}_{\mu\nu}(r,s) \) is of the form \( F^{(\text{ext})}_{\mu\nu}(r,s) = \Theta(s-s_o)F^{(\text{ext})}_{1\mu\nu}(r) \), i.e., \( F^{(\text{ext})}_{\mu\nu} \) is “turned on” at the proper time \( s = s_o \). In particular we shall take \( F^{(\text{ext})}_{\mu\nu}(r,s) \) to be a smooth function of \( s \), of class \( C^k (M^4 \times I) \), with \( k \geq 1 \);

3. **REQUIREMENT #3:** more generally, let us require that for an arbitrary initial state \( x(s_1) = x_1 \in \Gamma \) there always exists \( \{ x(s_o) = x_o, s_o \} \in \Gamma \times I \), with \( s_o = s_1 - s_{\text{rel}} \), such that at time \( s_o \), \( x(s_o) \) is inertial, i.e., before \( s_o \) the external force \( F^{(\text{ext})}_{\mu\nu} \) vanishes identically, so that the dynamics is of the form provided by Eqs. \[ \text{[103]} - \text{[106]} \].

It then follows that the solution of the initial-value problem \[ \text{[99]} - \text{[131]} \], subject to REQUIREMENTS #1-#3, exists at least locally in a subset \( I \equiv [-\infty, s_0] \cup [s_0, s_n] \subseteq \mathbb{R} \) with \( [s_0, s_n] \) a bounded interval, and is unique (fundamental theorem).

**Proof** - Eq. \[ \text{[99]} \] can be cast in the form of a delay-differential equation, i.e.,

\[
\frac{d x(s)}{d s} = X(x(s), x(s - s_{\text{rel}}), s),
\]

subject to the initial condition

\[
x(s_o) = x_o.
\]

Here \( x(s) \) and \( x(s - s_{\text{rel}}) \) denote respectively the “instantaneous” and “retarded” states \( x(s) \) and \( x(s - s_{\text{rel}}) \), while \( X(x(s), x(s - s_{\text{rel}}), s) \) is a suitable \( C^2 \) real vector field depending smoothly on both of them. The proof of local existence and uniqueness for Eq. \[ \text{[132]} \], with the initial conditions \[ \text{[133]} \] and the Requirements #1-#3, requires a generalization of the fundamental theorem holding for ordinary differential equations (in which the vector field \( X \) depends only on the local state \( x(s) \)).

Let us first consider the case in which the solution \( x(s) \) of the initial-value problem \[ \text{[132]} \] and \[ \text{[133]} \] is defined in the half-axis \( [-\infty, s_o] \) : by assumption this solution exists, is unique and is that of inertial motion [see Eqs. \[ \text{[103]} - \text{[106]} \]].

Next, let us consider the proper time interval \( I_{o,1} \equiv [s_o, s_1] \equiv s_o + s_{\text{rel}} \). Thanks to the Requirement #3, by assumption in \( I_{o,1} \) the particle is subject only to the action of the external force (produced by \( A^{(\text{ext})}_{\mu}(x) \)), since \( F^{(\text{ext})}_{\mu\nu} \) vanishes by definition if \( s < s_o + s_{\text{rel}} \). Hence, in the same time interval the solution exists and is unique because the differential equation \[ \text{[132]} \] is of the form

\[
\frac{d x(s)}{d s} = X^{\text{ext}}(x(s), s),
\]

with \( X^{\text{ext}}(x(s), s) \) being, by assumption, a smooth vector field (see THM.1). Eq. \[ \text{[134]} \] is manifestly a local ODE for which the fundamental theorem (for local ODEs) holds. Hence, existence and uniqueness is warranted also in \( I_{o,1} \).

Finally, let us consider the sequence of proper time intervals \( I_{k,k+1} \equiv [s_k, s_{k+1} = s_k + s_{\text{rel}}] \), for the integer \( k = 1, 2, 3,...n \), where \( n \geq 2 \). In this case, for any proper time \( s \in I_{k,k+1} \), the advanced-time solution \( x(s - s_{\text{rel}}) \) appearing in the vector field \( X \equiv X(x(s), x(s - s_{\text{rel}}), s) \) can be considered as a prescribed function of \( s \), determined in the previous time interval \( I_{k,k-1} \). Therefore, \( X \) is necessarily of the form \( X \equiv X(x(s), s) \), so that for \( s > s_1 \), Eq. \[ \text{[132]} \] can be viewed again as a local ODE. We conclude that, thanks to the fundamental theorem holding for local ODEs, the local existence (in a suitable bounded proper time interval \( I \equiv [s_1, s_n] \)) and uniqueness of solutions of the problem \[ \text{[132]} - \text{[133]} \] is assured under the Requirements #1-#3. This proves the statement.

Q.E.D.
XI. CONCLUSIONS

In this paper we have shown that the RR problem originally posed by Lorentz for classical non-rotating finite-size and Lorentzian particles can exactly be solved analytically within the SR setting.

For these particles, the resulting relativistic dynamics in the presence of the RR force, i.e., the classical RR equation, has been found analytically by taking into account the exact covariant form of the EM self 4-potential. In particular, this has been uniquely determined consistently with the basic principles of classical electrodynamics and special relativity. In addition, the RR equation has been proved to be variational in the functional class of synchronous variations with respect to the Hamilton variational principle, defined in terms of a non-local variational Lagrangian function. The same equation has been shown: 1) to admit the standard Lagrangian form in terms of the non-local effective Lagrangian \( L_{\text{eff}} \); 2) to admit a conservative form; 3) to recover the usual asymptotic LAD and LL equations in the first-order short delay-time approximation; 4) not to admit the point-charge limit. From the mathematical point of view, the RR equation is a delay-type second order ODE, which fulfills GIP in the sense of THM.1, relativistic covariance and MLC. As a consequence, provided suitable physical requirements are imposed, the initial-value problem for the RR equation is well-posed, defining the classical dynamical system required by NDP.

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XII. APPENDIX A: VARIATIONAL CALCULATIONS

Here we report the proof of identities \([103]-[111]\) in THM.1. Let us first notice that

\[
d \left[ \int_1^2 dr'^\nu \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2) \right] = dr^k \int_{-\infty}^\infty ds'^\nu (s') \frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2). \tag{135}\]

Hence the variations \( \delta A \) and \( \delta B \) given in Eqs.\([110]\) are respectively

\[
\delta A = -\frac{4q^2}{c} \eta_{\alpha\beta} \int_1^2 dr'^\nu \delta \rho \int_{-\infty}^\infty ds'^\nu (s') \frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2), \tag{136}\]

while

\[
\delta B = \frac{4q^2}{c} \eta_{\alpha\beta} \int_1^2 dr'^\nu \delta \rho \int_{-\infty}^\infty ds'^\nu (s') \frac{\partial}{\partial r^k} \delta(\tilde{R}^k \tilde{R}_k - \sigma^2). \tag{137}\]

Let us now evaluate the partial derivative \( \frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2) \). Invoking the chain rule, this becomes

\[
\frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2) = \frac{\partial(\tilde{R}^\alpha \tilde{R}_\alpha)}{\partial r^k} \frac{d\delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2)}{ds'} = \frac{d\delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2)}{ds'} \frac{2\tilde{R}_k}{d(\tilde{R}^\alpha \tilde{R}_\alpha)}, \tag{138}\]

and so

\[
\frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2) = -\frac{\tilde{R}_k}{\tilde{R}^\alpha \tilde{R}_\alpha (s')} \frac{d}{ds'} \left\{ \frac{\delta(s - s' - s_{ret})}{2 |\tilde{R}^\alpha \tilde{R}_\alpha (s')|} \right\}. \tag{139}\]

It follows that

\[
\frac{\partial}{\partial r^k} \delta(\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2) = -\frac{\tilde{R}_k}{c^2 \left[ (t - t') - \frac{1}{c^2} \frac{dr'}{dt'} \cdot (r - r') \right]} \times
\times \frac{d}{ds'} \left\{ \frac{\delta(t - t' - t_{ret})}{2c^2 \gamma (t')} \left[ (t - t') - \frac{1}{c^2} \frac{dr(t')}{dt'} \cdot (r - r') \right] \right\}, \tag{140}\]
where \( \mathbf{r}' = r(t') \), \( t = t(s) \) and \( t' = t(s') \). Substituting Eq. (140) into Eqs. (136) and (137), and then directly integrating, it follows immediately that \( \delta A \) and \( \delta B \) have the form (110).