Infinitely many reducts of homogeneous structures

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joint work with Peter Cameron and Csaba Szabó

TU Dresden

Novi Sad, 17th June 2017
Basic concepts

Structure: $\mathfrak{A} = \langle A, C, F, R \rangle$, where
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- \( R \): set of relations \( \subset A^n \)
Basic concepts

Reducts

Reduct of a structure $\mathfrak{A}$: another structure on the same domain set; constants, functions and relations are definable in $\mathfrak{A}$. 
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Reducts

**Reduct** of a structure $\mathcal{A}$: another structure on the same domain set; constants, functions and relations are definable in $\mathcal{A}$.

Two reduct are called **interdefinable** iff they are reducts of one another.
Facts

Aut(\(A\)) : closed in Sym(\(A\)).

If \(B\) is a reduct of \(A\), then Aut(\(B\)) \(\supset\) Aut(\(A\)).

All structures: countable, \(\omega\)-categorical

Definition

\(A\) is \(\omega\)-categorical iff for all \(n\) Aut(\(A\)) has finitely many \(n\)-orbits.

Theorem (Ryll-Nardzewski, Engeler, Svenonius)

For \(\omega\)-categorcial structures \(B\) \(\mapsto\) Aut(\(B\)) is a bijection between the reducts of \(A\) (up to interdefinability) and the closed supergroups of Aut(\(A\)).
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Theorem (Ryll-Nardzewski, Engeler, Svenonius)

For $\omega$-categorical structures $B \mapsto \text{Aut}(B)$ is a bijection between the reducts of $\mathcal{A}$ (up to interdefinability) and the closed supergroups of $\text{Aut}(\mathcal{A})$. 

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Infinitely many reducts ...
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To classify all reducts of a structure \( \mathcal{A} \) (up to interdefinability).
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- random ordered graph (Bodirsky, Pinsker, Pongrácz, 2014)
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Conjecture (Thomas, 1991)

Every homogeneous, finite relational structure has finitely many reducts.
Reducts of the vector space

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\[ V = \mathbb{F}_2^\omega: \] countably infinite dimensional vector space over \( \mathbb{F}_2 \)

Difference: it is not homogeneous over a finite relational language.
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Proof:

Claim: \( (\mathcal{V}, R_1, R_2, \ldots, R_k) \) is not homogeneous.

Let \( n > \max(\ar(R_i), a_1, a_2, \ldots, a_n) \) be linearly independent.

Then \( a_i \mapsto a_i : i < n, a_n \mapsto a_1 + a_2 + \cdots + a_{n-1} \) does not extend to an automorphism.
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\[ \mathcal{V} = \mathbb{F}_2^{\omega} : \text{countably infinite dimensional vector space over } \mathbb{F}_2 \]

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Suppose \( \text{Aut}(\mathcal{V}) = \text{Aut}(\mathcal{V}, R_1, R_2, \ldots, R_k) \).

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**Theorem (B., Kalina, Szabó) (Bossière, Bodirsky)**

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### Theorem (B., Kalina, Szabó) (Bossière, Bodirsky)

\[ \mathcal{V} = \mathbb{F}_2^\omega \text{ has exactly 4 reducts.} \]

#### Model theoretical formulation

1. the vector space \( \mathcal{V} \) itself

#### Algebraic formulation

1. \( \text{Aut}(\mathcal{V}) \), the automorphism group of \( \mathcal{V} \)
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1. the vector space \( V \) itself
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3. the countably infinite dimensional affine space

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1. \( \text{Aut}(V) \), the automorphism group of \( V \)
2. \( \text{Sym}(V) \), the symmetric group
3. \( \text{Aff}(V) = \text{Aut}(V) \ltimes \text{Tr} \), the group of affine transformations on \( V \)
Reducts of the vector space

**Theorem (B., Kalina, Szabó) (Bossière, Bodirsky)**

\[ \mathcal{V} = F_2^\omega \] has exactly 4 reducts.

**Model theoretical formulation**

1. the vector space \( \mathcal{V} \) itself
2. the countably infinite set
3. the countably infinite dimensional affine space
4. the countably infinite set with a constant 0

**Algebraic formulation**

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3. \( \text{Aff}(\mathcal{V}) = \text{Aut}(\mathcal{V}) \rtimes \text{Tr} \), the group of affine transformations on \( \mathcal{V} \)
4. \( \text{Sym}(\mathcal{V})_0 \), the stabilizer of 0 in \( \text{Sym}(\mathcal{V}) \)
Reducts of the vector space

Theorem (B., Kalina, Szabó)
\[ \mathcal{V} = F_2^\omega \text{ has exactly 4 reducts}. \]

What if we add a constant?

Theorem (B., Cameron, Szabó)
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The construction

Algebraic description

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1. $V = W \oplus \langle c \rangle$
2. $W_n \leq W$, $\text{codim}(W_n) = n$
3. $h_n$: flipping along $W_n$ ($u \leftrightarrow u + c$ iff $u \in W_n \oplus \langle c \rangle$)

Observations:

$G_n$ only depends on $n$.

Why are they different?

Bertalan Bodor (TU Dresden)  Infinitely many reducts . . .  Novi Sad, 17th June 2017
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Definition

\[(x_1, \ldots, x_{2^n}) \in R_n \text{ iff}\]

Remark

This is not a first-order definition.

But! \(\text{Aut}(V, c)\) preserves \(R_n\).

Hence \(R_n\) is definable in \((V, c)\).

Result:

\(h_n\) preserves \(R_n + 1\).

In fact:

\(G_n = \text{Aut}(R_n + 1)\)

\(h_n + 1\) does not preserve \(R_n + 1\).

Consequence:

\(G_n \neq G_n + 1\).
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\[(x_1, \ldots, x_{2^n}) \in \mathcal{R}_n \text{ iff } \]

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\((x_1, \ldots, x_{2^n}) \in R_n\) iff

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Thank you for your attention!