New model for rigorous analysis of LT-codes

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Abstract

We present a new model for LT codes which simplifies the analysis of the error probability of decoding by belief propagation. For any given degree distribution, we provide the first rigorous expression for the limiting error probability as the length of the code goes to infinity via recent results in random hypergraphs [2]. For a code of finite length, we provide an algorithm for computing the probability of error of the decoder. This algorithm improves the one of Karp, Luby, and Shokrollahi [5] by a linear factor.

1 Introduction

Fountain codes were originally introduced in [1] and were designed for robust and scalable transmission of data over lossy networks. Given a vector of input symbols \( (x_1, x_2, \ldots, x_k) \), a fountain code generates a stream of output symbols to be sent over the network. Each output symbol is generated independently by sampling from a fixed distribution on subsets of the input symbols and adding the symbols in the chosen subset. The sequence of output symbols, together with the positions of the input symbols whose sum they represent, is sent over a lossy network. The input word is decoded using the belief propagation algorithm which takes only linear time. The probability that the belief propagation decoder fails depends on the distribution from which output symbols were generated and on the number \( n \) of output symbols received.

Analysis of the error probability to date has been carried out under the assumption of a fixed number of received output symbols \( n \). Here we will change this assumption and say that the number of output symbols received is a random variable with mean \( n \). This assumption makes sense in applications and is not significantly different from the case of fixed number of output symbols, because the random variable is highly concentrated around \( n \). We will define the exact distribution in the following section. We refer to this as the Poisson model because the number of output symbols approaches the Poisson distribution as \( k \) goes to infinity. Intuitively, the Poisson model adds further independence between the random variables involved in the error-probability calculations, and thus significantly simplifies the analysis.

We will apply the new model to the analysis of a particular kind of fountain codes - the LT codes introduced by Michael Luby in [6]. The output symbols in LT codes are generated in the following way: \( d \) is chosen from a fixed probability distribution \( \Omega = (\Omega_1, \Omega_2, \ldots, \Omega_k) \) on the set \( 1, 2, \ldots, k \), after which the parity of \( d \) random input symbols is computed.

We are interested in two questions. Firstly, we look for an analytic expression for the limiting error-probability of belief propagation. The second question is that of designing an algorithm to compute the error-probability for finite-length codes.
The asymptotic analysis of LT codes to date has been based on a heuristic calculation, using the fact that in the limit of $k$ going to infinity the belief propagation iterations behave as if on a tree graph. With the new model we can apply recent results in the analysis of processes on random hypergraphs [2] to give an exact expression for the portion of symbols that can be decoded by belief propagation as $k$ goes to infinity.

For the finite-length analysis of LT codes, Karp, Luby, and Shokrollahi [5] proposed a dynamic programming algorithm. The size of the table is $O(n^3)$ and each entry is computed using $O(n^2)$ of the previous entries. Using generating polynomials representation and fast multi-point evaluation and interpolation of polynomials, the complexity of the algorithm is $O(n^3 \log^2 n)$. The Poisson model permits us to reduce the dynamic programming recursion to a table of size $O(k^2)$, and each entry is computed from $O(k)$ of the previous ones. Using generating polynomials representation, the complexity is reduced to $O(k^2 \log k)$.

In the next section we will review the factor graph representation of LT codes, the belief propagation algorithm for them, and we will define the new model precisely. Section 3 is dedicated to the asymptotic analysis, and Section 4 to the finite length analysis. We will conclude with a brief discussion of open problems.

2 Background and Definitions

It is convenient to think of the set of input and output symbols as the vertices of a bipartite graph. Every output symbol is connected by an edge to all input symbols in the set whose sum it represents as in figure 2.1.

2.1 Belief Propagation

In the setting of fountain codes the belief propagation algorithm is very simple. If there is an output symbol with a single undecoded neighbor, then the value of that input symbol can be computed. In this case, we say that a decodable input symbol becomes uncovered or decoded. Uncovering one symbol may result in other input symbols becoming decodable, and so on. The process stops when there are no decodable input symbols, or equivalently, there are no output symbols with a single
undecoded neighbor. We refer to the set of decodable input symbols as the *ripple* (note that in [5] the ripple is, instead, the set of output symbols that have only one undecoded symbol). At every step, one input symbol leaves the ripple and 0 or more input symbols join the ripple.

### 2.2 The Poisson model for LT codes

For a given set of input symbols of size $d$, let $p_d = n \Omega_d / \binom{k}{d}$ be the probability that an output symbol representing the parity of this set was received. Then by linearity of expectation the expected number of distinct output symbols is exactly $n$ and the expected number of output symbols from sets of size $d$ is $n \Omega_d$.

Let $D$ be the largest degree with positive probability. Let $N_d$ for $d = 1, \ldots, D$ be a random variable denoting the total number of output symbols of degree $d$, and $N = \sum_{d=1}^{D} N_d$ is the total number of output symbols. The distribution of $N_d$ is binomial $B\left(\binom{k}{d}, p_d\right)$. By concentration inequalities in [4]:

\begin{align*}
\Pr[N \geq n + \Delta] & \leq \exp\left(-\frac{\Delta^2}{2(n + \Delta/3)}\right), \quad (1) \\
\Pr[N \leq n - \Delta] & \leq \exp\left(-\frac{\Delta^2}{2n}\right). \quad (2)
\end{align*}

### 3 Asymptotic Analysis of LT Codes

The above random model is almost identical to the Poisson random hypergraph model of Darling and Norris [2]. The process that they study, called the *hypergraph collapse process*, is identical to the uncovering of input symbols in the belief propagation algorithm. In order to restate their result in our setting, we need some notation. Let $n = (1 + \delta)k$ for some constant $\delta \geq 0$. Let $\beta_1 = -\ln(1 - (1 + \delta)\Omega_1)$ and $\beta_d = (1 + \delta)\Omega_d$, for $d = 2, \ldots, k$. From these we define the power series

\[ \beta(t) = \sum_{d \geq 0} \beta_d t^d \]

and its derivative:

\[ \beta'(t) = \sum_{d \geq 1} d\beta_d t^{d-1}. \]

The statement of the theorem is in terms of the roots of the function $\beta'(t) + \log(1 - t)$. Let

\[ z^* = \inf\{t \in [0, 1) : \beta'(t) + \log(1 - t) < 0\} \land 1 \]

and suppose there are no roots of $\beta'(t) + \log(1 - t)$ in $[0, z^*)$. Notice that in particular if $\beta$ is a polynomial (as is the case in the LT-codes setting) then $z^* < 1$.

**Theorem 1** [2] Assuming $z^* < 1$ and there are no roots of $\beta'(t) + \log(1 - t)$ in $[0, z^*)$, then as $k$ goes to infinity the fraction of recoverable input symbols goes to $z^*$ in probability.

Therefore as a first test for the quality of a particular degree distribution, one can compute the roots of $\beta'(t) + \log(1 - t)$. In fact, $(1 - t)(\beta'(t) + \log(1 - t))$ is the expected fraction of output symbols which have a unique undecoded neighbor, when fraction $t$ of the input symbols have been decoded. This is equivalent to the expression obtained from the tree analysis.
4 Finite-length Analysis of LT Codes

For codes of finite length, we are interested in calculating the probability that all input symbols can be recovered. In this section we give an algorithm for computing this probability for a given degree distribution.

4.1 Recursion of probability distributions

Let $X_u$, for $u = 1, \ldots, k$, be random variables that denote the size of the ripple when $u$ symbols are undecoded, or equivalently, we will sometimes say at step $u$. In particular, $X_k$ is the number of input symbols for which a degree-1 output symbol was generated. This number has a binomial distribution $B(k, p_1)$. The decoding process stops when $X_u = 0$. The distribution of the size of the ripple at step $u$ depends only on the size of the ripple at step $u$. If $X_u = 0$ then the process stops and $X_{u-1} = 0$. If $X_u > 0$, then one of the symbols in the ripple is decoded. This results in $Y_u$ input symbols joining the ripple. $Y_u$ is distributed as the binomial distribution $B(u - X_u, q_u)$, where $q_u$ is the probability that a symbol joins the ripple at step $u$ (when the $(k-u+1)$-st symbol is decoded). An input symbol $a$ joins the ripple at this time if and only if there is an output symbol with neighbors: the symbol $a$, the last decoded symbol, and any set of symbols among the other $k-u$ decoded symbols. Therefore,

$$q_u = 1 - \prod_{d=2}^{\min\{D, k-u+2\}} (1 - p_d)^{\binom{k-u}{d-2}}.$$

Finally, $X_{u-1} = X_u - 1 + Y_u$. Therefore, for every $1 \leq r \leq u - 1$ and $1 \leq s \leq r + 1$,

$$\Pr[X_{u-1} = r \mid X_u = s] = \Pr[Y_u = r - s + 1] = \binom{u-r}{r-s+1} q_u^{r-s+1} (1 - q_u)^{u-r-1},$$

This gives us an expression for the distribution of $X_{u-1}$ in terms of the distribution of $X_u$:

$$\Pr[X_{u-1} = r] = \sum_{s=1}^{r+1} \Pr[X_u = s] \Pr[X_{u-1} = r \mid X_u = s].$$

The probability that belief propagation cannot complete the decoding is exactly the probability that $X_1 = 0$. This can be computed by dynamic programming. Let $Q(u, r)$ be $\Pr[X_u = r]$, for every $u = 0, \ldots, k$, and $r = 0, \ldots, u$. Then

$$Q(k, r) = \binom{k}{r} p_1^r (1 - p_1)^{k-r}, \text{ for } r = 0, \ldots, k$$

$$Q(u-1, r) = \sum_{s=1}^{r+1} Q(u, s) \binom{u-s}{r-s+1} q_u^{r-s+1} (1 - q_u)^{u-r-1}, \text{ for } r = 1, \ldots, u - 1$$

$$Q(u-1, 0) = Q(u, 0) + Q(u, 1) (1 - q_u)^{u-1}.$$

Finally, the probability of error of the decoder is $Q(1, 0)$. 
4.2 Complexity of the algorithm

To compute the values for \( (1-q_u) \) for \( u = 1, \ldots, k \), we proceed again by dynamic programming. We store the values for all of the factors \( f(u,d) = (1-p_d)^{\binom{u-2}{d-2}(u-2)} \) for every \( d = 2, 3, \ldots, \min\{D, k-u+2\} \). There are \( O(Dk) \) entries and \( f(u-1,d) = f(u,d) \cdot \binom{k-u+1}{d-1} \), which takes \( O(\log k) \) operations to compute. Therefore precomputing \( q_u \) takes \( O(Dk \log k) \) operations.

The recursion for \( Q(u,r) \) can be computed more efficiently if we represent it by generating polynomials. We will proceed in a manner similar to [5]. Let \( Q_u(x) = \sum_{r=1}^u Q(u,r)x^{r-1} \). Then the recursion can be written as:

\[
Q_k(x) = \frac{1}{x}((p_1 x + (1-p_1))^k - (1-p_1)^k),
\]

\[
Q_{u-1}(x) = \frac{(q_u x + (1-q_u))^{u-1}}{x} Q_u\left(\frac{x}{q_u x + (1-q_u)}\right) - \frac{(1-q_u)^{u-1}}{x} Q_u(0).
\]

Finally, the probability of success is \( Q_1(x) \) (which is a constant). We compute the sequence of polynomials \( Q_u(x) \) for \( u = k, \ldots, 1 \), in the following way: Suppose we have computed the coefficients of \( Q_u \). We choose \( k \) non-zero points \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k \), and compute \( \hat{y}_i = \frac{\hat{x}_i}{q_u \hat{x}_i + (1-q_u)} \) for \( i = 1, \ldots, k \). We evaluate \( Q_u \) at the points \( \hat{y}_i \) using the multipoint evaluation algorithm, which takes \( O(\log k) \) operations per point. Given these values, evaluating \( Q_{u-1}(\hat{x}_i) \) takes another \( O(\log k) \) operations. We can then interpolate the coefficients of \( Q_u \) by fast polynomial interpolation, which takes \( O(k \log k) \) operations. Therefore, the time complexity of our algorithm is \( O(k^2 \log k) \).

4.3 Implications for the case with fixed number of output symbols

The algorithm above outputs the probability that belief propagation fails, given that the number of output symbols is a random variable with expected value \( n \), as described. Let’s denote this probability by \( P_p(n) \). We can use this algorithm to get bounds for the probability that belief propagation fails, when there is a fixed number of output symbols, which we denote by \( P_f(n) \). We use the fact that the probability that the decoder fails is monotone decreasing in the number of output symbols, and

\[
P_p(n) = \sum_{\hat{n}=0}^{2^k-1} \Pr[N = \hat{n}] \times P_f(\hat{n}).
\]

Let \( n_1 \leq n \leq n_2 \). Using the concentration inequalities (11) and (21) we get the bounds:

\[
P_p(n_1) - \exp\left(\frac{3(n-n_1)^2}{2(2n_1 + n)}\right) \leq P_p(n) \leq \frac{P_p(n_2)}{1 - \exp\left(\frac{(n_2-n)^2}{2n_2}\right)}.
\]

5 Discussion

Our approach presented here is applicable to general fountain codes, as well as some classes of LDPC codes. Di et al. [3] gave algorithms for the finite-length analysis of regular LDPC codes (i.e. left and right degrees are constant). Our method is applicable to codes with Poissonian degree distribution on the right.
References

[1] J. Byers, M. Luby, M. Mitzenmacher, and A. Rege, *A digital fountain approach to reliable distribution of bulk data*, in Proceedings of ACM SIGCOMM, 1998, pp. 56-67.

[2] R.W.R. Darling and J.R. Norris, *Structure of large random hypergraphs*, Ann. Appl. Probab. 15, no. 1A (2005), pp. 125-152.

[3] C. Di, D. Proietti, E. Telatar, T. Richardson, and R. Urbanke, *Finite-length analysis of low-density parity-check codes on the binary erasure channel*, IEEE Trans. Inform. Theory, vol.48 (2002), pp. 1570-1579.

[4] S. Janson, *On Concentration of Probability*, Contemporary Combinatorics (Proceedings, Workshop on Probabilistic Combinatorics at the Paul Erds Summer Research Center, Budapest 1998), pp. 289-301.

[5] R. Karp, M. Luby, A. Shokrollahi, *Finite Length Analysis of LT Codes*, in Proceedings of ISIT 2004, p.37.

[6] M. Luby, *LT-codes*, in Proceedings of FOCS, 2002, pp. 271-280.