On Symmetries of the WDVV Equations

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Abstract

For two solutions of the WDVV equations that are related by two types of symmetries of the equations given by Dubrovin, we show that the associated principal hierarchies of integrable systems are related by certain reciprocal transformation, and the tau functions of the hierarchies are either identical or related by a Legendre transformation. We also consider relationships between the Virasoro constraints and topological deformations of the principal hierarchies.

1 Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations, which arises in the study of 2d topological field theory in the beginning of 90’s of the last century, are given by the following system of PDEs for a function \( F = F(v^1, \ldots, v^n) \):

1. The variable \( v^1 \) is specified so that
   \[
   \eta_{\alpha\beta} := \frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta} = \text{constant}, \quad \det(\eta_{\alpha\beta}) \neq 0. \tag{1.1}
   \]

2. The functions \( c_{\alpha\beta\gamma} := \eta^{\nu\rho} c_{\nu\beta\gamma} \) with
   \[
   c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\nu \partial v^\beta \partial v^\gamma}, \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1} \tag{1.2}
   \]
   yield the structure constants of an associative algebra for given \( v = (v^1, \ldots, v^n) \), i.e., they satisfy
   \[
   c_{\alpha\beta\gamma} c_{\gamma\delta\lambda} = c_{\alpha\beta\delta} c_{\gamma\lambda\lambda} \quad \text{for any fixed } 1 \leq \alpha, \beta, \gamma, \nu \leq n. \tag{1.3}
   \]

Here and in what follows summation with respect to repeated upper and lower indices is assumed.

3. The function \( F \) satisfies the quasi-homogeneity condition
   \[
   \partial_E F = (3 - d) F + \frac{1}{2} A_{\alpha\beta} \, v^\alpha v^\beta + B_\alpha v^\alpha + C, \tag{1.4}
   \]
   here the Euler vector field has the form
   \[
   E = \sum_{\alpha=1}^{n} \left( 1 - \frac{d}{2} - \mu_\alpha \right) v^\alpha + r_\alpha \frac{\partial}{\partial v^\alpha}, \tag{1.5}
   \]
   and \( d, A_{\alpha\beta} = A_{\beta\alpha}, B_\alpha, C, \mu_\alpha, r_\alpha \) are some constants with \( \mu_1 = -\frac{d}{2} \).
These equations are satisfied by the primary free energy $F$ of the matter sector of a 2d topological field theory with $n$ primary fields as a function of the coupling constants [2, 3, 18]. In [5, 6] Dubrovin reformulated the WDVV equations in a coordinated free form by introducing the notion of Frobenius manifold structure on the space of the parameters $v^1, \ldots, v^n$, and revealed rich geometric structures of the WDVV equations which are important in the study of several different areas of mathematical research, such as the theory of Gromov - Witten invariants, singularity theory and nonlinear integrable systems [3, 5, 6, 10]. In particular, such geometrical structures enable one to associate a solution of the WDVV equations with a hierarchy of bihamiltonian integrable PDEs of hydrodynamic type which is called the principal hierarchy [10]. This hierarchy of integrable systems plays important role in the procedure of reconstructing a 2D topological field theory(TFT) from its primary free energy as a solution of the WDVV equations. In this construction, the tau function that corresponds to a particular solution of the principal hierarchy serves as the genus zero partition function, and the full genera partition function of the 2D TFT is a particular tau function of an integrable hierarchy of evolutionary PDEs of KdV type which is certain deformation of the principal hierarchy, such a deformation of the principal hierarchy is call the topological deformation [10].

In this paper we are to interpret certain symmetries of the WDVV equations in term of the associated principal hierarchy and its tau functions. The symmetries we consider here are given by Dubrovin in Appendix B of [6], where they are called symmetries of type-1 and type-2 respectively [6]. These symmetries are obtained from the Schlesinger transformations of the system of linear ODEs with rational coefficients which are associated to the Frobenius manifolds (see Remark 4.2 of [7] for details). It turns out that in terms of the principal hierarchies and their tau functions these symmetries have a simple and natural interpretation. On the principal hierarchies these symmetries act as reciprocal transformations, and on the associated tau functions these two types of symmetries either keep the tau functions unchanged or act as Legendre transformations. Recall that a symmetry of the WDVV equations consists of transformations

$$v^\alpha \mapsto \hat{v}^\alpha, \quad \eta_{\alpha\beta} \mapsto \hat{\eta}_{\alpha\beta}, \quad F \mapsto \hat{F}$$

(1.6)

that preserve the WDVV equations. The two types of symmetries given in [6] have the following form:

1. Type-1 symmetries: they are given by the transformations defined by

$$\hat{v}^\alpha = \eta^{\alpha\gamma} \frac{\partial^2 F(v)}{\partial \hat{v}^\gamma \partial v^\sigma}, \quad \hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}, \quad \frac{\partial^2 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} = \frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta}$$

(1.7)

for any given $1 \leq \kappa \leq n$ such that the matrix $(\eta^{\alpha\sigma}_{\kappa})$ is invertible. Note that in this case the transformed function $\hat{F}(\hat{v})$ satisfies the WDVV equations with $\hat{v}^\alpha$ as the specified variable, and the equations in (1.1) is replaced by

$$\hat{\eta}_{\alpha\beta} := \frac{\partial^3 \hat{F}}{\partial \hat{v}^\alpha \partial \hat{v}^\beta \partial \hat{v}^\kappa} = \text{constant}, \quad \det(\hat{\eta}_{\alpha\beta}) \neq 0.$$  

(1.8)

\[1\] In [1] the Lie algebra of general infinitesimal symmetries of the WDVV equations without the homogeneity condition is considered.
2. Type-2 symmetries: they are given by the transformation defined by
\[ \hat{v}^1 = -\frac{1}{2} \eta_{\sigma v^\gamma v^\gamma}, \quad \hat{v}^\alpha = \frac{v^\alpha}{v^n} \text{ for } \alpha \neq 1, n, \quad \hat{v}^n = \frac{1}{v^n}, \]
and in the expression of the Euler vector field \( \hat{E} \) given in (1.5) the constants \( r_{\alpha} = 0 \) whenever \( \mu_{\alpha} \neq 1 - \frac{4}{d} \). We also impose the additional conditions that in the cases when \( d = 1 \) and \( d = 2 \) the constants \( r_n \) and \( B_1 \) that appear in (1.9) and (1.10) vanishes respectively. Note that the transformation (1.9) is obtained from the one given in Append B of [6] by changing the signs of \( \hat{v}^1, \hat{v}^n \) and of \( \hat{F} \), we make this modification so that the above transformation is an involution.

We arrange the content of the paper as follows. We first recall in Sec. 2 the definition of the principal hierarchy and its tau function associated to a solution of the WDVV equations, or equivalently, to a Frobenius manifold. In Sec. 3 and Sec. 4 we show respectively that the actions of the type-1 and type-2 symmetries on the principal hierarchies are given by certain reciprocal transformations, and we give the transformation rule of the associated tau functions. In Sec. 5 we consider the transformation rule of the Virasoro constraints for the tau functions of the principal hierarchy. We conclude the paper with a discussion on actions of the symmetries of the WDVV equation on the topological deformations of the principal hierarchy.

2 The principal hierarchy

Given a solution of the WDVV equations, the associated principal hierarchy consists of Hamiltonian systems of the following form:
\[ \frac{\partial \theta^\alpha}{\partial t^\beta} = \eta_{\gamma v^\gamma} \partial_x \left( \frac{\partial \theta_{\beta, q+1}^\alpha}{\partial v^\gamma} \right), \quad \alpha, \beta = 1, \ldots, n, \quad q \geq 0. \] (2.1)

Here the densities \( \theta_{\beta, q+1}^\alpha \) of the Hamiltonians \( H_{\beta, q} = \int \theta_{\beta, q+1}^\alpha(v(x))dx \) are given by the flat coordinates of the deformed flat connection of the corresponding Frobenius manifold [6]. Denote
\[ \theta_{\alpha}(z) = \sum_{p \geq 0} \theta_{\alpha, p}(v) z^p, \quad \alpha = 1, \ldots, n, \] (2.2)
then the functions \( \theta_{\alpha, p}(v) \) are determined by the equations
\[ \partial_x \partial_{\beta} \theta_{\alpha, p}(z) = z e^\gamma_{\alpha, \beta} \partial_x \theta_{\nu}(z), \quad \partial_x = \frac{\partial}{\partial v^\alpha}, \quad \alpha, \beta, \nu = 1, \ldots, n. \] (2.3)
\[ \partial_E \partial_{\beta} \theta_{\alpha, p}(v) = (p + \mu_{\alpha} + \mu_{\beta}) \partial_{\beta} \theta_{\alpha, p}(v) + \sum_{k=1}^{p} \partial_{\beta} \theta_{\alpha, p-k}(v) \left( R_k \right)_\alpha \] (2.4)
and are normalized by the conditions
\[ \theta_{\alpha,0} = \eta_{\alpha \gamma} v^\gamma := v_\alpha, \] (2.5)
\[ \partial_{\gamma} \theta_{\alpha}(z) \eta^{\gamma \nu} \partial_{\nu} \theta_{\beta}(-z) = \eta_{\alpha \beta}. \] (2.6)

Where the constant matrices \( R_1, R_2, \ldots \) have the properties

1. \( (R_k)^\gamma_{\alpha} \neq 0 \) only if \( \mu_\alpha - \mu_\beta = k \),
2. \( \eta_{\alpha \gamma} (R_k)^\gamma_{\beta} = (-1)^{k+1} \eta_{\beta \gamma} (R_k)^\gamma_{\alpha} \).

From the first property we see that we have only finitely many nonzero matrices \( R_1, \ldots, R_m \), the number \( m \) is determined by the particular solution of the WDVV equations. These matrices are defined up to conjugations
\[ R \mapsto GRG^{-1} \] (2.7)
given by nondegenerate constant matrices \( G \) that satisfy certain conditions, see [6, 7, 10] for details. These matrices form part of the monodromy data \((V, [R], \mu, <, >, e_1)\) of the Frobenius manifold at \( z = 0 \). Here \( V \) is the \( n \)-dimensional vector space spanned by \( e_1, \ldots, e_n \), \([R]\) is the equivalence class (w.r.t. the above conjugation) represented by the operator \( R \) that acts on \( V \) by \( Re_\alpha = R^\gamma_\alpha e_\gamma \), the action of the operator \( \mu \) on \( V \) is given by the diagonal matrix \( \mu = \text{diag}(\mu_1, \ldots, \mu_n) \), and the bilinear form is given by \( <e_\alpha, e_\beta> = \eta_{\alpha \beta} \). Note that the matrix \( \mu \) satisfies the anti-symmetry condition
\[ (\mu_\alpha + \mu_\beta) \eta_{\alpha \beta} = 0. \] (2.8)

The functions \( \theta_{\alpha, p}(v) \) satisfy the following tau-symmetry condition:
\[ \frac{\partial \theta_{\alpha, p}(v)}{\partial t^\alpha_{\beta, q}} = \frac{\partial \theta_{\beta, q}(v)}{\partial t^\alpha_{\beta, p}}, \quad \alpha, \beta = 1, \ldots, n. \] (2.9)

This property enables one to introduce the tau function \( \tau \) of the principal hierarchy (2.1). It is defined for any given solution \( v^\alpha = v^\alpha(t) \) of the hierarchy and required to satisfy the equations
\[ \frac{\partial^2 \log \tau}{\partial t^\alpha_{\beta, q}} = \theta_{\alpha, p}(v(t)), \quad \alpha = 1, \ldots, n, \quad p \geq 0. \] (2.10)

Note that the flow \( \frac{\partial}{\partial t} \) coincides with \( \frac{\partial}{\partial t^\alpha_{\beta, p}} \). In order to fix \( \tau \) up to a linear function of \( t^\alpha_{\beta, p} \), we are to use the functions \( \Omega_{\alpha, p; \beta, q}(v) \) defined by the following identities [6]:
\[ \sum_{\alpha, p; \beta, q} \Omega_{\alpha, p; \beta, q}(v) z^p w^q = \frac{\partial_{\gamma} \theta_{\alpha}(z) \eta^{\gamma \nu} \partial_{\nu} \theta_{\beta}(w) - \eta_{\alpha \beta}}{z + w}. \] (2.11)

Then for any solution \( v(t) = (v^1(t), \ldots, v^n(t)) \) of the principal hierarchy (2.1) we can fix, up to a factor of the form \( e^{\sum_{\alpha, p} t^\alpha_{\beta, p}} \), the tau function by the following relations:
\[ \frac{\partial^2 \log \tau}{\partial t^\alpha_{\beta, q}} = \Omega_{\alpha, p; \beta, q}(v(t)), \quad \alpha, \beta = 1, \ldots, n, \quad p, q \geq 0. \] (2.12)
3 Actions of the type-1 symmetries

The main results of this section were obtained in [10], we recollect them give the proofs here for the convenience of comparison with them the results that are related to the type-2 symmetries of the WDVV equations.

Let \( F = F(v^1, \ldots, v^n) \) be a solution of the WDVV equations with Euler vector field \( E \) of the form \( (1.4), (1.7) \). After the action of a symmetry of type-1, we obtain a new solution \( \hat{F} = \hat{F}(\hat{v}^1, \ldots, \hat{v}^n) \). Here we note that the unity vector field is \( \hat{\partial}_\alpha \) instead of \( \partial_\alpha \), and the first set of equations \( (1.5) \) of the WDVV equations is changed to \( (1.8). \)

We first note that in the new coordinates \( \hat{v}^1, \ldots, \hat{v}^n \) the Euler vector field \( \hat{E} \) given by \( (1.3) \) has the expression

\[
\hat{E} = \sum_{\alpha=1}^{n} \left( 1 - \frac{\hat{d}}{2} - \hat{\mu}_\alpha \right) \hat{v}^\alpha + \hat{\kappa}_\alpha \frac{\partial}{\partial \hat{v}^\alpha},
\]

where

\[
\hat{d} = -2\mu_\kappa, \quad \hat{\mu}_\alpha = \mu_\alpha, \quad \hat{\kappa}_\alpha = A_{\alpha \xi} \eta^{\xi \alpha}.
\]

Let us show that the Euler vector field \( \hat{E} \) of the new solution \( \hat{F} \) coincides with \( E \). In fact, by using \( (1.4), (1.7) \) we have

\[
\partial_E \hat{F}(\hat{v}) = \partial_E \hat{F}(v) = (1 + \mu_\alpha + \mu_\beta) \frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta} + A_{\alpha \beta}
\]

This yields the identity

\[
\partial_E \hat{F}(\hat{v}) = (3 - \hat{d})\hat{F}(\hat{v}) + \frac{1}{2} \hat{A}_{\alpha \beta} \hat{v}^\alpha \hat{v}^\beta + \hat{B}_\alpha \hat{v}^\alpha + \hat{C}
\]

for some constants \( \hat{A}_{\alpha \beta}, \hat{B}_\alpha, \hat{C} \).

Now we consider the relations of the densities \( \hat{\theta}_{\alpha,p}(\hat{v}) \) of the Hamiltonians of the principal hierarchy associated to \( \hat{F}(\hat{v}) \) with the ones that are associated to the original solution \( F(v) \) of the WDVV equations.

**Lemma 3.1**

i) The functions \( \hat{\theta}_{\alpha,p}(\hat{v}) \) for \( \hat{F}(\hat{v}) \) can be determined by the relations

\[
\frac{\partial \hat{\theta}_{\alpha,p}(\hat{v})}{\partial \hat{v}^\beta} = \frac{\partial \theta_{\alpha,p}(v)}{\partial v^\beta}, \quad \alpha, \beta = 1, \ldots, n, \quad p \geq 0
\]

and the normalization conditions

\[
\hat{\theta}_{\alpha,0}(\hat{v}) = \eta_{\alpha \gamma} \hat{v}^\gamma, \quad \alpha = 1, \ldots, n.
\]

ii) The monodromy data \( (\hat{V}, [\hat{R}], \hat{\mu}, <, >, \hat{\epsilon}_\alpha) \) at \( z = 0 \) of the Frobenius manifold associated to \( \hat{F}(\hat{v}) \) coincide with that of the Frobenius manifold associated to \( F(v) \). Here \( \hat{V} \) is the \( n \)-dimensional vector space spanned by \( \hat{\epsilon}_1, \ldots, \hat{\epsilon}_n \), \( [\hat{R}] \) is the equivalence class represented by the operator \( \hat{R} \) that acts on \( \hat{V} \) by \( \hat{R} \hat{v} = \hat{R}_\alpha \hat{v}^\alpha \), the action of the operator \( \hat{\mu} \) on \( \hat{V} \) is given by the diagonal matrix \( \hat{\mu} = \mu = \text{diag}(\mu_1, \ldots, \mu_n) \), and the bilinear form is given by \( <\hat{\epsilon}_\alpha, \hat{\epsilon}_\beta> = \eta_{\alpha \beta}. \)
Proof From the definition of the monodromy data \[6, 7\], it follows that in order to prove the lemma we only need to verify that the functions \( \hat{\theta}_{\alpha,p}(\hat{v}) \) determined by the conditions (3.5) and (3.6) satisfy (2.3)–(2.6). By using (1.7), (2.3) and (3.3) we have

\[
\frac{\partial^2 \hat{\theta}_i(z)}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} = \frac{\partial}{\partial \hat{v}^\alpha} \left( \frac{\partial \theta_i(z)}{\partial v^\beta} \right) = \frac{\partial^2 \theta_i(z)}{\partial v^\gamma \partial v^\delta} \frac{\partial v^\lambda}{\partial \hat{v}^\alpha} = z \hat{c}^\alpha_{\beta \gamma} \frac{\partial \theta_i(z)}{\partial v^\delta} \frac{\partial v^\lambda}{\partial \hat{v}^\alpha},
\]

so we prove that the functions \( \hat{\theta}_{\alpha,p}(\hat{v}) \) satisfy the recursion relation (2.3). Similarly, for the principal hierarchy (3.5) we have

\[
\frac{\partial}{\partial \hat{v}^\alpha} \left( \frac{\partial \hat{\theta}_a(x)}{\partial v^\beta} \right) = z \hat{c}^\alpha_{\beta \gamma} \frac{\partial \hat{\theta}_a(x)}{\partial v^\delta} \frac{\partial v^\lambda}{\partial \hat{v}^\alpha},
\]

which proves the validity of (2.4) for the functions \( \hat{\theta}_{\alpha,p}(\hat{v}) \). The relations (2.5) and (2.6) hold true obviously. The lemma is proved.

Let us proceed to consider the relation between the principal hierarchies associated to the solutions \( \hat{F}(v) \) and \( \hat{\hat{F}}(\hat{v}) \) of the WDVV equations. In the principal hierarchy (2.4) we have \( \frac{\partial \hat{\theta}_a}{\partial \hat{v}^\alpha} = \frac{\partial \hat{\theta}_a}{\partial v^\alpha} \), so we may identify the time variable \( \hat{t}^{1,0} \) with the spatial variable \( x \) and forget the flow \( \frac{\partial}{\partial \hat{v}^\alpha} \) in the hierarchy. Similarly, for the principal hierarchy

\[
\frac{\partial \hat{\theta}_a}{\partial \hat{v}^\alpha} = \hat{\eta}^\alpha, \frac{\partial}{\partial \hat{x}} \left( \frac{\partial \hat{\theta}_{a+1}}{\partial \hat{v}^\alpha} \right), \quad \alpha, \beta = 1, \ldots, n, q \geq 0
\]

we have \( \frac{\partial \hat{\theta}_a}{\partial \hat{v}^\alpha} = \frac{\partial \theta_a}{\partial x} \), so we may also identify \( \hat{t}^{\kappa,0} \) with the new spatial variable \( \hat{x} \) and forget the flow \( \frac{\partial}{\partial \hat{v}^\alpha} \) in the hierarchy. We will assume such an identification henceforth.

**Proposition 3.2** The principal hierarchy (3.2) associated to the solution \( \hat{\hat{F}}(\hat{v}) \) of the WDVV equation is obtained from the principal hierarchy (2.1) by the following reciprocal transformation

\[
\hat{x} = t^{\kappa,0}, \quad \hat{t}^{1,0} = x, \quad \hat{t}^{\alpha,p} = t^{\alpha,p}, \quad (\alpha, p) \neq (1, 0), (\kappa, 0).
\]

i.e. the principal hierarchy (3.2) is obtained from (2.1) simply by exchange the role of the spatial variable \( x \) and the time variable \( t^{\kappa,0} \). Moreover, any tau function \( \tau(t) \) of the principal hierarchy (2.1) yields a tau function \( \hat{\tau}(\hat{t}) \) of (3.2) by the formula

\[
\hat{\tau}(\hat{t}) = \tau(t)_{t^{\alpha,p} \rightarrow \hat{x}, x \rightarrow t^{\alpha,p}, \tilde{t}^{\kappa,0} \rightarrow \hat{t}^{\kappa,0}, (\alpha, p) \neq (1, 0), (\kappa, 0)}.
\]
Proof. Assume that \((\beta, q) \neq (1, 0), (\kappa, 0)\). Let \(v^1(t), \ldots, v^n(t)\) satisfy the principal hierarchy \((2.1)\), and \(\hat{\beta}^\alpha, \hat{\theta}_{\beta,q}(\hat{v})\) be defined as in \((1.7), (3.5), (3.6)\). Then after the reciprocal transformation \((3.9)\) we have

\[
\frac{\partial \hat{v}^\alpha}{\partial \hat{v}^{\beta,q}} = \frac{\partial \hat{v}^\alpha}{\partial \hat{v}^{\beta,q}} = \frac{\partial v^\lambda}{\partial \hat{v}^{\beta,q}} = \frac{\partial \theta_{\beta,q}}{\partial \hat{v}^{\beta,q}}v^\xi = c_{\alpha\lambda}^\xi \frac{\partial \theta_{\beta,q}}{\partial v^\xi}v^\xi
\]

\[
= c_{\alpha\lambda}^\xi \xi^\nu \frac{\partial \theta_{\beta,q}}{\partial v^\nu}v^\xi = c_{\alpha\lambda}^\xi \xi^\nu \frac{\partial \theta_{\beta,q}}{\partial v^\nu}v^\xi = c_{\alpha\lambda}^\xi \xi^\nu \frac{\partial \theta_{\beta,q}}{\partial v^\nu}v^\xi = c_{\alpha\lambda}^\xi \xi^\nu \frac{\partial \theta_{\beta,q}}{\partial v^\nu}v^\xi = \hat{\hat{q}}^{\alpha\nu} \frac{\partial \hat{\theta}_{\beta,q+1}}{\partial \hat{v}^{\nu,0}} \frac{\partial v^{\nu}}{\partial \hat{v}^{\nu,0}} = \hat{\hat{q}}^{\alpha\nu} \frac{\partial \hat{\theta}_{\beta,q+1}}{\partial \hat{v}^{\nu,0}} \frac{\partial v^{\nu}}{\partial \hat{v}^{\nu,0}}.
\]

(3.12)

In a similar way we can prove the validity of the above equation for \((\beta, q) = (1, 0), (\kappa, 0)\). So the reciprocal transformation \((3.9), (3.10)\) transforms the principal hierarchy \((2.1)\) to the principal hierarchy \((3.8)\).

The principal hierarchy \((2.1)\) possesses a bihamiltonian structure given by Frobenius manifolds. Certain bihamiltonian hierarchies so that the transformed ones are associated to Frobenius manifold, in such cases we can still perform the reciprocal transformation that exchanges the role of the spatial and time variables. It was shown in \([17]\) that such a transformation preserves the bihamiltonian property of the system, and the transformation rule of the bihamiltonian structure is similar to the one given above. Such transformations are applied in \([10]\) to certain bihamiltonian hierarchies so that the transformed ones are associated to Frobenius manifolds.

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Note that for a Hamiltonian system of hydrodynamic type the transformation rule of the Hamiltonian structure under linear reciprocal transformations was given by Pavlov in [15]. An interesting problem is whether we still have similar transformation rules when we apply linear reciprocal transformations to a Hamiltonian or bihamiltonian system which is certain deformation of a system of hydrodynamic type.

4 Actions of the type-2 symmetries

As we did in the last section, we denote by \( \hat{F}(\hat{v}^1, \ldots, \hat{v}^n) \) the solution of the WDVV equations that is obtained from a solution \( F(v) \) by the action of the type-2 symmetry (1.9). Note that in general the operator given by the gradient of the Euler vector field \( \hat{E} \) for \( \hat{F}(\hat{v}) \) is non-diagonalizable, for the convenience of the presentation of the results on the transformation rule of the principal hierarchies and their tau functions under the action of the type-2 symmetries, we assume that the in expression (1.5) of the Euler vector field \( E \) the constants \( r_\alpha, \alpha = 1, \ldots, n \) vanish. Under this assumption the Euler vector field for \( \hat{F}(\hat{v}) \) has the expression (see Lemma B.1 of [6])

\[
\hat{E} = \sum_{\alpha=1}^{n} (1 - \hat{d}/2 - \hat{d}_\alpha) \hat{v}^\alpha \frac{\partial}{\partial \hat{v}^\alpha},
\]

where

\[
\hat{d} = 2 - d, \quad \hat{d}_1 = \mu_n - 1, \quad \hat{d}_n = \mu_1 + 1, \quad \hat{d}_\alpha = \mu_{\alpha} \neq 1, n.
\]

In fact it coincides with the Euler vector field \( E \) for the function \( F(v) \). The function \( \hat{F}(\hat{v}) \) satisfies the following quasi-homogeneity condition:

\[
\partial_{\hat{E}} \hat{F} = (3 - \hat{d}) \hat{F} + \frac{1}{2} \hat{A}_{\alpha\beta} \hat{v}^\alpha \hat{v}^\beta + \hat{B}_\alpha \hat{v}^\alpha + \hat{C}.
\]

Here \( \hat{A}_{\alpha\beta}, \hat{B}_\alpha, \hat{C} \) are some constants.

It was shown in [6] that the functions

\[
\hat{\eta}_{\alpha\beta} = \frac{\partial^3 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta}, \quad \hat{c}_{\alpha\beta\gamma}(\hat{v}) = \frac{\partial^3 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta \partial \hat{v}^\gamma}
\]

have the following relations with the functions \( \eta_{\alpha\beta}, c_{\alpha\beta\gamma} \) defined in (1.1), (1.2):

\[
\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}, \quad \hat{c}_{\alpha\beta\gamma}(\hat{v}) = -\eta_{\lambda\mu} \frac{\partial v^\lambda}{\partial \hat{v}^\alpha} \frac{\partial v^\mu}{\partial \hat{v}^\beta} c_{\lambda\mu\gamma}(v).
\]

By taking \( \gamma = 1 \) in (4.5) we obtain

\[
\eta_{\alpha\beta} = (v^n)^{-2} \eta_{\lambda\mu} \frac{\partial v^\lambda}{\partial \hat{v}^\alpha} \frac{\partial v^\mu}{\partial \hat{v}^\beta}.
\]

We also have the following identities which will be used below:

\[
-v^n \delta_\alpha^\mu \frac{\partial v^\mu}{\partial \hat{v}^\beta} - v^n \delta_\beta^\mu \frac{\partial v^\mu}{\partial \hat{v}^\alpha} = \frac{\partial^2 v^\mu}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} + \eta_{\alpha\beta} \delta_\alpha^\mu v^n.
\]
Lemma 4.1 i) The functions $\hat{\theta}_{\alpha,p}(\hat{v})$ for the solution $\hat{F}(\hat{v})$ of the WDVV equations are given by the following formulae:

$$\hat{\theta}_{1,0}(\hat{v}) = \frac{1}{v^n}, \quad \hat{\theta}_{1,p}(\hat{v}) = (-1)^p \frac{\theta_n\hat{v}^{p-1}(v)}{v^n}, \quad p > 0,$$

$$\hat{\theta}_{\alpha,p}(\hat{v}) = (-1)^p \frac{\theta_{\alpha,p}(v)}{v^n}, \quad 2 \leq \alpha \leq n - 1, \quad p \geq 0,$$

$$\hat{\theta}_{n,p}(\hat{v}) = (-1)^{p+1} \frac{\theta_{n+1}(v)}{v^n}, \quad p \geq 0.$$

The validity of the normalization conditions (2.5), (2.6) is easy to see from the recursion relations (2.3) for $\hat{\theta}_{\alpha,p}(\hat{v})$.

Proof We need to verify that the functions $\hat{\theta}_{\alpha,p}(\hat{v})$ assume that (4.5)–(4.7) and (4.12).

Lemma 4.1 ii) Let $(\hat{V}, [\hat{R}], \hat{\mu}, <, \hat{e}_1)$ be the monodromy data at $z = 0$ of the Frobenius manifold associated to $\hat{F}(\hat{v})$ with $\hat{V}$ being the linear space spanned by $\hat{e}_1, \ldots, \hat{e}_n$. Then the operator $\hat{R}$ is given by the matrix elements

$$(\hat{R}_k)^\alpha\beta = (-1)^{k+n+1+\delta_\alpha^\beta} (\hat{R}_{k+\delta(\alpha)-\delta(\beta)})^\alpha(\alpha-1)^\delta(\alpha),$$

where we denote by $\delta(\alpha)$ the difference of two Kronecker delta functions

$$\delta(\alpha) := \delta_\alpha - \delta_\alpha^\varepsilon,$$

and we assume that $R_l = 0$ whenever $l \leq 0$. The operator $\hat{\mu}$ is given by (4.2) and the bilinear form is defined by $<\hat{e}_\alpha, \hat{e}_\beta> = \eta_{\alpha\beta}$.

Proof We need to verify that the functions $\hat{\theta}_{\alpha,p}(\hat{v})$ satisfy the equations (2.3). The validity of the normalization conditions (2.5), (2.6) is easy to see from the definition (1.9) of the new flat coordinates $\hat{v}^1, \ldots, \hat{v}^n$, the identity (1.6) and the relations

$$\hat{\theta}_{1,1}(\hat{v}) = \frac{\partial \hat{F}(\hat{v})}{\partial v^1} = \frac{1}{2} \eta_{\alpha\beta} v^\alpha v^\beta,$$

$$\frac{\partial \hat{\theta}_{\alpha,p}(\hat{v})}{\partial v^1} = \theta_{\alpha, p-1}(v) + \delta_{\alpha, 0} \delta_{p, 0} \quad \text{with} \quad \theta_{\alpha, -1}(v) = 0.$$

The recursion relations (2.3) for $\hat{\theta}_{\alpha,p}(\hat{v})$ can be verified by using the identities (1.6)–(1.7) and (4.12).

To prove the validity of the quasihomogeneity condition (2.1), let us first assume that $\alpha \neq 1, n$, then by using (4.3) and (1.9) we get

$$\frac{\partial}{\partial v^{n+1}} \hat{\theta}_{\alpha,p}(v) = E^\varepsilon \frac{\partial}{\partial v^\varepsilon} \left( \frac{(n+1-\beta) \partial \theta_{\alpha,p}(v)}{v^n} + \frac{(n+1-\beta) \partial \theta_{\alpha,p}(v)}{v^n} \right) \left( (-1)^p \theta_{\alpha,p}(v) \right).$$

$$=(-1)^{p+1} \frac{\partial \theta_{\alpha,p}(v)}{v^n} + \frac{(-1)^p \partial \theta_{\alpha,p}(v)}{v^n} + \sum_{k=1}^{p} \frac{\partial \theta_{\alpha,p-k}(v)}{v^n} (R_k)_{\alpha}^\varepsilon \left( p + \mu_\alpha + \mu_\beta \right) + \left( \frac{d}{2} - \hat{\mu}_{n+1-\beta} \right) \frac{\partial \theta_{\alpha,p}(v)}{v^n}.$$
The proof for the cases when $\alpha, \beta = 1, n$ are similar. The lemma is proved. \qed

As we explained in the last section, we identify the time variable $t^{1,0}$ of the principal hierarchy (24) with the spatial variable $x$. For the principal hierarchy that is associated to the solution $\hat{F}(\hat{v})$ of the WDVV equations (see (4.13) below) we also identify the time variable $t^{1,0}$ with the spatial variable $\hat{x}$.

**Proposition 4.2** The *principal hierarchy*

\begin{equation}
\frac{\partial \hat{v}^\alpha}{\partial \hat{v}^\beta} = \hat{q}^{\alpha \gamma} \frac{\partial}{\partial \hat{x}} \left( \frac{\partial \hat{v}^\beta_{,q+1}(\hat{v})}{\partial \hat{v}^\gamma} \right), \quad \alpha, \beta = 1, \ldots, n, \quad q \geq 0 \tag{4.13}
\end{equation}

associated to the solution $\hat{F} = \hat{F}(\hat{v})$ of the WDVV equation is related to the principal hierarchy (24) by the following reciprocal transformation:

\begin{equation}
dx = -v^\alpha dx = \sum_{(\alpha, p) \neq (1, 0)} \theta_{\alpha, p}(v) dt^{\alpha, p}, \tag{4.14}
\end{equation}

\begin{align*}
\hat{t}^{1,0} &= \hat{x}, & \hat{t}^{1, p} &= (-1)^p t^{\alpha, p-1}, & p \geq 1, \\
\hat{t}^{n, p} &= (-1)^{p+1} t^{1, p+1}, & \hat{t}^{\alpha, p} &= (-1)^p t^{\alpha, p}, & \alpha \neq 1, n, \quad p \geq 0. \tag{4.15}
\end{align*}

**Proof** From the definition of the reciprocal transformation we have

\begin{align*}
\frac{\partial}{\partial \hat{v}^{1, p}} &= (-1)^p \left( \frac{\partial}{\partial t^{1, p-1}} - \frac{\theta_{\alpha, p-1}(v)}{v^n} \frac{\partial}{\partial x} \right), \quad p \geq 1, \\
\frac{\partial}{\partial \hat{v}^{\alpha, p}} &= (-1)^p \left( \frac{\partial}{\partial t^{\alpha, p-1}} - \frac{\theta_{\alpha, p}(v)}{v^n} \frac{\partial}{\partial x} \right), \quad \alpha \neq 1, n, \quad p \geq 0, \\
\frac{\partial}{\partial \hat{v}^{n, p}} &= (-1)^{p+1} \left( \frac{\partial}{\partial t^{1, p+1}} - \frac{\theta_{1, p+1}(v)}{v^n} \frac{\partial}{\partial x} \right), \quad p \geq 0. \tag{4.16}
\end{align*}
Let $v^1(t), \ldots, v^n(t)$ be a solution of the principal hierarchy (2.1), and $\hat{\omega}^\alpha, \hat{\beta}_{\gamma,q}(\hat{v})$ be defined as in [15], [18]. Then for $\alpha, \beta \neq 1, n$ we have

$$
\frac{\partial \hat{\omega}^\alpha}{\partial \hat{\beta}_{\gamma,q}} = (-1)^q \left( \frac{\partial}{\partial \hat{\beta}_{\gamma,q}} \frac{\partial}{\partial \hat{v}^\alpha} \frac{\theta_{\beta,q}(v)}{v^n} \frac{\partial}{\partial x} + \left( \frac{\partial}{\partial \hat{v}^\alpha} \frac{\theta_{\beta,q}(v)}{v^n} \frac{\partial}{\partial x} \right) \right)
$$

In a similar way we can prove the validity of the above equation for other cases of $\alpha, \beta$. The proposition is proved.

From the definition (2.11) of the functions $\Omega_{\alpha,p,\beta,q}$ it follows that the functions $\hat{\Omega}_{\alpha,p,\beta,q} = \hat{\Omega}_{\alpha,p,\beta,q}(\hat{v})$ which are defined by the solution $\hat{F}(\hat{v})$ of the WDVV equations have the following expressions:

$$
\hat{\Omega}_{\alpha,p,\beta,q} = (-1)^{p+q+1} \hat{\omega}^\alpha \Omega_{\alpha+(n-1)\delta(\alpha),p-\delta(\alpha),\beta+(n-1)\delta(\beta),q-\delta(\beta)}(v) \hat{\beta}_{\gamma,q}(v)
$$

where $\delta(\alpha), \delta(\beta)$ are defined in (4.10).

Lemma 4.3 The tau function of the principal hierarchy (2.1) defined by (2.12) satisfies the following equations:

$$
\frac{\partial}{\partial \tau^p} \frac{\partial}{\partial x} \log \tau = 0, \quad \forall (\alpha, p) \neq (1, 0).
$$

Proof We first consider the case when $\alpha = 1, p \geq 1$. By using the relations (4.10) we have

$$
\frac{\partial}{\partial \tau^p} \frac{\partial}{\partial x} \log \tau = (-1)^p (-1)^{p} \frac{\partial}{\partial \tau^p} \frac{\partial}{\partial x} \log \tau = \frac{\theta_{n,p-1}}{v^n} \frac{\partial^2}{\partial x^2} \log \tau
$$

Here we used the relation (2.12) and the fact that $\Omega_{\alpha,p,1,0} = \theta_{\alpha,p}$ which follows from (2.11). For the cases when $\alpha \neq 1, p \geq 1$ the proof of the equation (4.18) is similar. When $(\alpha, p) = (1, 0)$ we have

$$
\frac{\partial}{\partial x} \frac{\partial}{\partial x} \log \tau = - \frac{1}{v^n} \frac{\partial^2}{\partial x^2} \log \tau = - v^n = -1.
$$
The lemma is proved. □

It follows from the above lemma that up to the addition of a constant we have
\[ \hat{x} = -\frac{\partial}{\partial x} \log \tau. \] (4.20)
The constant can be absorbed by a translation of \( \hat{x} \) in the definition of the reciprocal transformation (4.14), (4.15), so we will assume from now on the validity of (4.20).

**Proposition 4.4** Any tau function \( \tau(t) \) of the principal hierarchy (2.1) defined by (2.12) yields a tau function \( \hat{\tau}(\hat{t}) \) of the transformed principal hierarchy (4.13) by the following Legendre transformation:
\[ \log \hat{\tau} = x \frac{\partial \log \tau}{\partial x} - \log \tau. \] (4.21)
together with the change of independent variables (4.15) and (4.20).

**Proof** We only need to prove the validity of the equation (2.12) for \( \hat{\tau} \) w.r.t. its independent variables \( \hat{t}^{\alpha,p} \). For \( \alpha, \beta \neq 1, n \), by using (4.8), (4.16) and (4.17) we have
\[ \frac{\partial^2 \log \hat{\tau}}{\partial t^{\alpha,p} \partial t^{\beta,q}} = \left( -1 \right)^{p+q+1} \left( \frac{\partial}{\partial \hat{t}^{\alpha,p}} - \frac{\theta_{\alpha,p}}{v^n} \frac{\partial}{\partial \hat{x}} \right) \left( \partial \log \tau + x \frac{\partial \log \tau}{\partial x} \right) = \left( -1 \right)^{p+q+1} (\bar{\Omega}_{\alpha,p;\beta,q} - \frac{\theta_{\alpha,p} \theta_{\beta,q}}{v^n}) = \hat{\Omega}_{\alpha,p;\beta,q}. \]

For other values of \( \alpha, \beta \) we can verify the validity of the same equation, so the proposition is proved. □

Note that the transformation (4.15), (4.20), (4.21) is an involution, its inverse is given by (4.15) and
\[ x = -\frac{\partial}{\partial x} \log \hat{\tau}, \quad \log \tau = \hat{x} \frac{\partial \log \hat{\tau}}{\partial x} - \log \hat{\tau}. \] (4.22)

The flat metric \( \hat{\eta} \) and the intersection form \( \hat{g} \) of the Frobenius manifold associated to \( \hat{F}(\hat{v}) \) give a bihamiltonian structure for the principal hierarchy (4.13). In the flat coordinates \( \hat{v}^1, \ldots, \hat{v}^n \), due to the identities (4.5), (4.6) and (3.15), the compatible Hamiltonian operators
\[ \hat{P}^{\alpha\beta}_1 = \hat{\eta}^{\alpha\beta} \partial_{\hat{x}}, \quad \hat{P}^{\alpha\beta}_2 = \hat{g}^{\alpha\beta}(\hat{v}) \partial_{\hat{x}} + \hat{\Gamma}^{\alpha\beta}_{\gamma}(\hat{v}) \hat{v}^\gamma \] (4.23)
have the following relation with the Hamiltonian operators given in (3.14):
\[ \hat{\eta}_{\alpha\beta} d\hat{v}^\alpha d\hat{v}^\beta = (v^n)^{-2} \eta_{\alpha\beta} dv^\alpha dv^\beta, \]
\[ \hat{g}_{\alpha\beta}(\hat{v}) dv^\alpha d\hat{v}^\beta = -(v^n)^{-2} g_{\alpha\beta}(v) dv^\alpha dv^\beta. \]

Here \( (g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}, (\hat{g}_{\alpha\beta}) = (\hat{g}^{\alpha\beta})^{-1} \). Such transformation rule for Hamiltonian structures of hydrodynamic type is given in [12], and for more general type of Hamiltonian structures is recently given in [13].
5 Virasoro constraints of the tau functions

In this section, we consider the problem of how the actions of the symmetries of the WDVV equations change the Virasoro constraints of the tau functions.

It was shown in [11] that the principal hierarchy (2.1) possesses an infinite number of Virasoro symmetries. In terms of its tau function these symmetries can be represented in the form

\[
\frac{\partial \log \tau}{\partial s_m} = \sum a_m^{\alpha,p,\beta,q} \frac{\partial \log \tau}{\partial t^{\alpha,p}} \frac{\partial \log \tau}{\partial t^{\beta,q}} + \sum b_m^{\beta,q} \frac{\partial \log \tau}{\partial t^{\beta,q}} + \sum c_m^{\alpha,p,\beta,q} \frac{\partial \log \tau}{\partial t^{\beta,q}}, \quad m \geq -1. \tag{5.1}
\]

The coefficients that appear in the above expressions are some constants, they define a set of linear differential operators

\[
L_m = \sum a_m^{\alpha,p,\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + \sum b_m^{\beta,q} t^{\alpha,p} t^{\beta,q} + \sum c_m^{\alpha,p,\beta,q} t^{\alpha,p} t^{\beta,q} + \delta_{m,0} c \tag{5.2}
\]

which give a representation of half of the Virasoro algebra

\[
[L_i, L_j] = (i - j)L_{i+j} + n \frac{\delta^3 - i \cdot j}{12} \delta_{i+j,0}, \quad i, j \geq -1. \tag{5.3}
\]

Here \( c \) is a constant.

For a generic solution of the principal hierarchy its tau function satisfies the Virasoro constraints [10]

\[
A_m(t; \tau) = 0, \quad m \geq -1. \tag{5.4}
\]

Here we denote the r.h.s. of (5.4) by \( A(t; \tau) \), and the shifted variable \( t \) in the above expression is defined by

\[
i^{\alpha,p} = t^{\alpha,p} - c^{\alpha,p} \tag{5.5}
\]

for some constants \( c_{\alpha,p} \). In particular in 2d topological field theory the partition functions are given by the tau functions which are specified by the Virasoro constraints with \( c^{\alpha,p} = \delta^f_1 \delta^f_1 \) [11] [14], and the first Virasoro constraint is the string equation

\[
\sum_{p \geq 1} t^{\alpha,p} \frac{\partial \log \tau}{\partial t^{\alpha,p-1}} + \frac{1}{2} t^{\alpha,0} t^{\beta,0} = \frac{\partial \log \tau}{\partial \bar{x}}. \tag{5.6}
\]

Let \( F(v) \) and \( \hat{F}(\hat{v}) \) be solutions of the WDVV equations that are related by a type-1 or type-2 symmetry. We denote by

\[
\dot{A}(\dot{t}; \dot{\tau}) = \sum a_m^{\alpha,p,\beta,q} \frac{\partial \log \dot{\tau}}{\partial t^{\alpha,p}} \frac{\partial \log \dot{\tau}}{\partial t^{\beta,q}} + \sum b_m^{\beta,q} \frac{\partial \log \dot{\tau}}{\partial t^{\beta,q}} + \sum c_m^{\alpha,p,\beta,q} \frac{\partial \log \dot{\tau}}{\partial t^{\beta,q}}, \quad m \geq -1 \tag{5.7}
\]

the r.h.s. of the Virasoro symmetries of the principal hierarchy (3.8) or (4.13) associated to \( \hat{F}(\hat{v}) \).
Proposition 5.1 Let $\tau(t)$ be the tau function of the principal hierarchy (2.1) associated to the solution $F(v)$ of the WDVV equations. Then for the tau function $\hat{\tau}(t)$ obtained from $\tau(t)$ by applying the type-1 symmetry we have

$$\hat{A}_m(\hat{\tau}) = A_m(t; \tau), \quad m \geq -1,$$

(5.8)

and for the tau function $\tilde{\tau}(t)$ obtained from $\tau(t)$ by applying the type-2 symmetry we have

$$\tilde{A}_m(\tilde{\tau}) = (-1)^{m+1} A_m(t; \tau), \quad m \geq -1.$$

(5.9)

Proof The validity of (5.8) follows from Proposition 3.2 obviously. To verify the validity of (5.9), we note that the relations (4.16) and (4.21) yield

$$\frac{\partial \log \hat{\tau}}{\partial \tau^1, p} = -t^{1,0}, \quad \frac{\partial \log \tilde{\tau}}{\partial \tau^1, p} = (-1)^{p+1} \frac{\partial \log \tau}{\partial \tau^{n,p-1}}, \quad p \geq 1,$$

$$\frac{\partial \log \tilde{\tau}}{\partial \tau^{n,p}} = (-1)^{p} \frac{\partial \log \tau}{\partial \tau^{1,p+1}}, \quad p \geq 0.$$

From the above equations we obtain

$$\hat{A}_{-1}(\hat{\tau}) = \sum_{p \geq 1} \hat{\tau}^{n,p} \frac{\partial \log \hat{\tau}}{\partial \tau^{n,p-1}} + \frac{1}{2} \eta_{\alpha \beta} \hat{\tau}^{\alpha,0} \hat{\tau}^{\beta,0} = A_{-1}(t; \tau).$$

The proof of the relation (5.9) for $m \geq 0$ is similar, so we omit it here. The proposition is proved.

From the above proposition we see that after the action of the type-1 and type-2 symmetries of the WDVV equations, the topological solution of the principal hierarchy (2.1) that is specified by the Virasoro constraints (5.4), (5.5) with
\[ \epsilon^{\alpha,p} = \delta^{\alpha}_1 \delta^{p}_1 \] is transformed to a tau function of the principal hierarchies (3.8) and (4.13) respectively, they satisfy the Virasoro constraints

\[ \hat{A}_m(\hat{t}; \hat{\tau}) = 0, \quad \tilde{\epsilon}^{\alpha,p} = \tilde{\epsilon}^{\alpha,p}_1. \] (5.10)

with

\[ \hat{c}^{\alpha,p} = \begin{cases} \delta^{\alpha}_1 \delta^{p}_1, & \text{for the type-1 symmetry,} \\ -\delta^{\alpha}_n \delta^{p}_0, & \text{for the type-2 symmetry.} \end{cases} \] (5.11)

Note that the tau function \( \hat{\tau}(\hat{t}) \) for the topological solution of the principal hierarchy (3.8) satisfies the Virasoro constraints (5.10) with \( \hat{c}^{\alpha,p} = \delta^{\alpha}_1 \delta^{p}_1 \).

6 Conclusion

For two solutions of the WDVV equations related by the type-1 or type-2 symmetries, we have shown that the associated principal hierarchies are related by certain reciprocal transformation, and their tau functions are either identical or related by a Legendre transformation. We also considered the relation of the Virasoro constraints for their tau functions.

It was shown in [10] that the principal hierarchy associated to a semisimple Frobenius manifold has a unique deformation of the form

\[ \frac{\partial w^{\alpha}}{\partial \varepsilon^{2g+1}} = \eta^{\alpha\gamma} \partial_x \left( \frac{\partial \theta_{\beta,g+1}(w)}{\partial w^{\gamma}} \right) + \sum_{g \geq 1} \varepsilon^{2g} K^{g}_{\beta,g} (w; w_x, \ldots, w^{(2g+1)}). \] (6.1)

Here the \( K^{g}_{\beta,g} \) are polynomials of \( w_x^\gamma, \ldots, \partial^{2g+1} w^{\gamma} \) with coefficients depending smoothly on \( w^1, \ldots, w^n \). Such a deformation is called the topological deformation of the principal hierarchy. It preserves the tau structure of the principal hierarchy and has an infinite number of Virasoro symmetries. Moreover, in terms of the tau function the Virasoro symmetries are required to be linearized, i.e. they can be represented by

\[ \frac{\partial \tau(t; \varepsilon)}{\partial \varepsilon^m} = \varepsilon^2 L_m |_{t^0 \rightarrow t^{\alpha}}, \quad m \geq -1. \]

Here and in what follows we use \( \tau(t; \varepsilon) \) to denote the tau function of the topological deformation of the principal hierarchy, and we redenote by

\[ \tau^{[0]}(x, t) = e^{\mathcal{F}_0(x,t)} \]

the tau function of the principal hierarchy. For a semisimple Frobenius manifold that is associated to a 2d topological field theory, the topological deformation of the principal hierarchy is supposed to determine the partition function of the model via its tau function specified by the Virasoro constraints

\[ L_m |_{t^0 \rightarrow t^{\alpha}}, \quad \delta^{\alpha}_1 \delta^{p}_1 \tau(t; \varepsilon) = 0, \quad m \geq -1. \] (6.2)

When \( m = -1 \) the above constraint is just the string equation (5.6).

We then have the following natural question: For any two semisimple Frobenius manifolds related by the type-1 or type-2 symmetries of the WDVV equations, what is the relationship between the topological deformations of their principal hierarchies?
From the construction of the topological deformation of the principal hierarchy given in [10] we know that the deformed hierarchy (6.1) is related to the principal hierarchy (2.1) via a so called quasi-Miura transformation of the form

$$w^{\alpha} = v^{\alpha} + \eta^{\alpha\gamma} \frac{\partial^2}{\partial x \partial y_{\gamma,0}} \sum_{y \geq 1} \varepsilon^{2y} F_y(v; v_x, \ldots, \partial_x^{3y-2} v), \, \alpha = 1, \ldots, n. \quad (6.3)$$

Here the functions $F_y$ are determined by the loop equation associated to the semisimple Frobenius manifold [10]. In particular, we have

$$F_1(v, v_x) = \frac{1}{24} \det (c_{\alpha\beta}(v) v_{\alpha}^2) + G(v), \quad (6.4)$$

where $G(v)$ is the $G$-function of the Frobenius manifold [9]. The tau function $\tau(t; \varepsilon)$ of the deformed hierarchy is related to a solution of the deformed hierarchy by the formula

$$w^{\alpha}(t) = \varepsilon^2 \eta^{\alpha\gamma} \frac{\partial^2 \log \tau(t; \varepsilon)}{\partial x \partial y_{\gamma,0}},$$

and the tau function has the genus expansion

$$\tau(t; \varepsilon) = e^{\sum_{g \geq 0} \varepsilon^{2g-2} F_g(t)}. \quad (6.5)$$

Here $F_g(t) = F_g(v(x, t), \ldots, \partial_x^{3g-2} v(x, t))$.

For the type-2 symmetry of the WDVV equations, we see from the above mentioned construction that the topological deformation of the principal hierarchy associated to $\hat{F}(\hat{v})$ is obtained from that of the principal hierarchy (2.1) by using $t^n, 0$ as the new spatial variable $\hat{x}$. The time variables are given by $\hat{t}^{\alpha,p} = t^{\alpha,p}$ for $(\alpha, p) \neq (k, 0)$ and $\hat{t}^{k,0} = \hat{x}$, and the tau functions of these topological deformations of the principal hierarchies are related by $\hat{\tau}(\hat{x}, \hat{t}) = \tau(x, t)$.

Based on the results of Propositions 4.2, 4.4 we have the following conjecture for the type-2 symmetry of the WDVV equations.

**Conjecture 6.1** For the type-2 symmetry of the WDVV equations, the topological deformation of the principal hierarchy (4.13) associated to $\hat{F}(\hat{v})$ is obtained, up to a Miura type transformation, from that of the principal hierarchy (2.1) by the following Legendre transformation of the tau function:

$$\log \hat{\tau}(\hat{t}; \varepsilon) = \log \tau(t; i \varepsilon) - x \frac{\partial \log \tau(t; i \varepsilon)}{\partial x}$$

$$\hat{x} = \varepsilon^2 \frac{\partial \log \tau(t; i \varepsilon)}{\partial x}, \quad \hat{t}^{1,p} = (-1)^p t^{1,p}, \quad p \geq 1$$

$$\hat{t}^{n,p} = (-1)^{p+1} t^{n+1,p+1}, \quad \hat{t}^{\alpha,p} = (-1)^p t^{\alpha,p}, \quad \alpha \neq 1, n, \quad p \geq 0. \quad (6.6)$$

Let us explain the validity of this conjecture at the approximation up to $\varepsilon^2$. To this end we perform a genus expansion of $\hat{\tau}(\hat{t}; \varepsilon)$ as follows:

$$\hat{\tau}(\hat{t}; \varepsilon) = e^{\sum_{g \geq 0} \varepsilon^{2g-2} F_g(t)}. \quad (6.7)$$

Then by using the genus expansion (6.5) of the tau function $\tau(t; \varepsilon)$ we can rewrite the first equation of (6.6) in the form

$$\varepsilon^{-2} \left( \hat{F}_0(\hat{t}) + \varepsilon^2 \frac{\partial \hat{F}_0(\hat{t})}{\partial \hat{x}} \right) + \hat{F}_1(\hat{t}) + O(\varepsilon^2)$$

$$= -\varepsilon^2 F_0(t) + F_1(t) - x \left( -\varepsilon^{-2} \frac{\partial F_0(t)}{\partial x} + \frac{\partial F_1(t)}{\partial x} \right) + O(\varepsilon^2). \quad (6.8)$$
Here we expand $\hat{t}^{\alpha,p} = \hat{t}^{\alpha,p}(t; \varepsilon)$ that are defined in (6.6) in the form

$$\hat{t}^{1,0} = \hat{x} = \hat{x}_0 + \varepsilon^2 \hat{x}_1 + \mathcal{O}(\varepsilon^4) = -\frac{\partial F_0(t)}{\partial x} + \varepsilon^2 \frac{\partial F_1(t)}{\partial x} + \mathcal{O}(\varepsilon^4),$$

and $\hat{t}^{\alpha,p} = \hat{t}_0^{\alpha,p}$ for $(\alpha, p) \neq (1, 0)$. By comparing the coefficients of $\varepsilon^{-2}$ of the left and right sides of (6.8) we get

$$\hat{F}_0(\hat{t}_0) = -F_0(t) + \frac{\partial F_0(t)}{\partial x}.$$

From this it follows that

$$x = -\frac{\partial \hat{F}_0(\hat{t}_0)}{\partial \hat{x}_0}.$$

Then the coefficients of $\varepsilon^0$ of the equation (6.8) yields

$$\hat{F}_1(\hat{t}_0) = F_1(t).$$

(6.10)

The formula (6.9) coincides with the Legendre transformation (4.21) between the tau functions of the principal hierarchies. From the formula (6.10) we get

$$\hat{F}_1(\hat{t}_0) = F_1(v, v_x)$$

(6.11)

On the other hand, from (4.5) it follows that

$$\frac{1}{24} \log \det(\hat{c}_{\gamma\beta\gamma}(\hat{v})\hat{v}_x^n) = \frac{1}{24} \log \det(c_{\gamma\beta\gamma}(v)\nu_x^n) - \frac{n}{24} \log v^n.$$

(6.12)

And by using the result of [16] the G-function of the Frobenius manifold associated to $\hat{F}(\hat{v})$ is given by

$$\hat{G}(\hat{v}) = G(v) + (\frac{n}{24} - \frac{1}{2}) \log v^n.$$

(6.13)

Thus we arrive at

$$\hat{F}_1(\hat{t}_0) = \left(\frac{1}{24} \log \det(\hat{c}_{\gamma\beta\gamma}(\hat{v})\nu_x^n) + \hat{G}(\hat{v}) + \frac{1}{2} \log v^n\right)_{v=v(t)}$$

$$= \left(\frac{1}{24} \log \det(c_{\gamma\beta\gamma}(v)\nu_x^n) + \hat{G}(\hat{v}) - \frac{1}{2} \log \nu^n\right)_{\hat{v}=\hat{v}(t_0)}.$$  

(6.14)

Then from the formula (6.4) for the topological deformation of the principal hierarchy and the equations

$$\frac{\partial \hat{v}_n}{\partial \hat{t}_0} = \eta_{\gamma\sigma} \hat{v}_x^n, \quad \alpha = 1, \ldots, n$$

(6.15)

we see that the hierarchy of equations satisfied by

$$\hat{w}_\alpha = \varepsilon^2 \eta_{\alpha\gamma} \frac{\partial^2 \log \hat{F}(\hat{t}; \varepsilon)}{\partial \hat{x}_0 \partial \hat{t}_0}, \quad \alpha = 1, \ldots, n$$

(6.16)

is related, at the approximation up to $\varepsilon^2$, to the topological deformation of the principal hierarchy associated to $\hat{F}(\hat{v})$ by the following Miura type transformation

$$\hat{w}_\alpha \rightarrow \hat{w}_\alpha + \varepsilon^2 \frac{\hat{w}_\alpha \hat{w}_n - \hat{w}_\alpha \hat{w}_n}{2(\hat{w}_n)^2} + \mathcal{O}(\varepsilon^4), \quad \alpha = 1, \ldots, n.$$  

(6.17)
We will return to the extension of the Legendre transformation to the topological
deformation of a principal hierarchy in separate publications.

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