COHOMOLOGY OF RATIONAL FORMS AND
A VANISHING THEOREM ON TORIC VARIETIES

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ABSTRACT. We explicitly describe cohomology of the sheaf of differential forms with poles along a semiample divisor on a complete simplicial toric variety. As an application, we obtain a new vanishing theorem which is an analogue of the Bott-Steenbrink-Danilov vanishing theorem.

0. INTRODUCTION.

Phillip Griffiths in [G] calculated the cohomology of smooth hypersurfaces in a projective space. His method used the Gysin exact sequence, and the problem was reduced to finding the cohomology of the complement of the hypersurface in its ambient space. The latter cohomology was easily found due to the vanishing theorem of R. Bott in [Bot]:

\[ H^k(\mathbb{P}^m, \Omega^l_{\mathbb{P}^m}(X)) = 0 \] for an ample divisor \( X \) and \( k > 0 \).

This theorem was extended to an ample divisor on a complete toric variety (see [BFLM], [D] and [BC]).

As in the case of projective hypersurfaces, in order to compute the cohomology of \( X \) quasismooth hypersurfaces in complete simplicial toric varieties \( \mathbb{P}_\Sigma \), one needs to know the cohomology of the twisted sheaves

\[ H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}(X)). \]

An important case to consider is when the divisor \( X \) is semiample. For toric varieties this means that the corresponding line bundle is generated by global sections. The Bott vanishing theorem does not hold for semiample divisors. However, in this paper we explicitly calculated the cohomology of the twisted sheaves \( \Omega^l_{\mathbb{P}_\Sigma}(X) \) and, in particular, we discovered a new vanishing theorem for semiample divisors:

\[ H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}(X)) = 0 \] if \( k > l \) or \( l > k + i \),

where \( i \) is the Kodaira-Iitaka dimension of the divisor \( X \). When the divisor \( X \) is trivial this reduces to the well-known Danilov’s vanishing result for the cohomology of a complete simplicial toric variety:

\[ H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}) = 0 \] for \( k \neq l \).

Moreover, for \( k = l \), we recover Danilov-Jurkevich’s description of the cohomology of a complete simplicial toric variety.

The plan of the paper is as follows. In section 1, we compute cohomology of Ishida’s complexes that are necessary in studying cohomology of sheaves \( \Omega^l_{\mathbb{P}_\Sigma}(X) \). Then, in section 2, we calculated \( H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}(X)) \) for a semiample divisor \( X \) on a

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complete simplicial toric variety $P_\Sigma$. As a consequence of this result, we obtain the
vanishing theorem (1). We also find a dimension formula for this cohomology and
explicitly represent the generators of cohomology $H^k(P_\Sigma; \Omega^l_{P_\Sigma}(X))$ in terms of the
Čech cocycles.

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1. Cohomology of Ishida’s complexes.

Ishida’s complexes have appeared as a result of studying sheaves of differential
forms on toric varieties. In this section, we will compute the cohomology of Ishida’s
complexes of modules.

First, we fix some standard notation: $M$ is a lattice of rank $d$; $N = \text{Hom}(M, \mathbb{Z})$
the dual lattice; $\Sigma$ is a finite rational (usually simplicial) fan in the $\mathbb{R}$-scalar
extension $N_\mathbb{R}$; $\Sigma(k)$ is the set of all $k$-dimensional cones in $\Sigma$; $e_1, \ldots, e_n$
are the minimal integral (primitive) generators of the 1-dimensional cones $\rho_1, \ldots, \rho_n$ in
$\Sigma(1)$.

Definition 1.1. [O1] Let $\Sigma$ be a fan in $N_\mathbb{R}$ and $l = 0, \ldots, d$. Then Ishida’s $l$-th
complex of $\mathbb{Q}$-modules is denoted $C^l(\Sigma, l)$, where

$$ C^j(\Sigma, l) = \bigoplus_{\gamma \in \Sigma(j)} \bigwedge_{l-j} \gamma^\perp $$

(for simplicity, $\gamma^\perp := M_\mathbb{Q} \cap \gamma^\perp$) and the coboundary homomorphism $\delta$
are defined as the direct sum of $\delta_{\gamma, \tau} : \bigwedge_{l-j} \gamma^\perp \to \bigwedge_{l-j-1} \tau^\perp$, which are zero maps if $\gamma$ is not
a facet of $\tau$, while for $\gamma \prec \tau$, set $\delta_{\gamma, \tau}(w) = e_{\gamma, \tau} \cdot w$, where $e_{\gamma, \tau} \in N_\mathbb{Q}$ satisfies $\tau + (-\gamma) = \mathbb{R}_{\geq 0} e_{\gamma, \tau} + \mathbb{R} \gamma$.

Remark 1.2. Compared to the definition of Ishida’s complex in [O1, Section 3.2],
our definition allows the maps $\delta_{\gamma, \tau}$ to be defined up to a rational multiple.

Ishida’s complexes are also defined for star closed and star open subsets of a fan.

Definition 1.3. A subset $\Phi$ of a fan $\Sigma$ is called \textit{star closed} (\textit{star open}), if $\sigma \in \Phi$ and
$\sigma \prec \tau \in \Sigma$ imply $\tau \in \Phi$ (correspondingly, $\tau \in \Phi$ and $\sigma \prec \tau$ imply $\sigma \in \Phi$). For
such subsets, Ishida’s $l$-th complex of $\Phi$ is defined as

$$ C^l(\Phi, l) := \bigoplus_{\gamma \in \Phi \cap \Sigma(j)} \bigwedge_{l-j} \gamma^\perp $$

with the coboundary homomorphism $\delta$ as the sum of $\delta_{\gamma, \tau}$ for $\gamma, \tau \in \Phi$.

Example 1.4. Given a fan $\Sigma$ and a cone $\gamma \in \Sigma$, then the set

$$ \text{Star}_\gamma(\Sigma) := \{ \tau \in \Sigma | \tau \succ \gamma \} $$

is star closed in $\Sigma$. One can construct a fan $\widetilde{\text{Star}}_\gamma(\Sigma)$, called the \textit{star of} $\gamma$, which
consists of the images of the cones from $\text{Star}_\gamma(\Sigma)$ in the quotient space $N_\mathbb{R}/(N_\gamma)_\mathbb{R}$
with the corresponding quotient lattice. Note, then, that $l$-th Ishida’s complex $C^*(\text{Star}_\gamma(\Sigma), l)$ for the subset $\text{Star}_\gamma(\Sigma)$ is isomorphic (up to a shift) to Ishida’s
complex $C^{*-\dim \gamma}(\text{Star}_\gamma(\Sigma), l - \dim \gamma)$. The differential on the latter complex is
induced from the first one.
As in [Is, Proposition 1.8], we note the following statement.

**Proposition 1.5.** Let $\Sigma'$ be a star closed subset of a fan $\Sigma$, equivalently $\Sigma'' = \Sigma \setminus \Sigma'$ be star open in $\Sigma$. Then there is a short exact sequence of Ishida’s complexes

$$0 \to C^*(\Sigma', l) \to C^*(\Sigma, l) \to C^*(\Sigma'', l) \to 0,$$

which gives a long exact sequence in cohomology:

$$0 \to H^0(\Sigma', l) \to H^0(\Sigma, l) \to H^0(\Sigma'', l) \to H^1(\Sigma', l) \to H^1(\Sigma, l) \to H^1(\Sigma'', l) \to \cdots.$$

**Proof.** Note that if $\Sigma'$ is star closed in $\Sigma$ and $\Sigma''$ is star open in $\Sigma$, then $C^j(\Sigma', l)$ and $C^j(\Sigma'', l)$ are $\mathbb{Q}$-submodules that decompose $C^j(\Sigma, l)$. Moreover, the differential $\delta$ for $C^*(\Sigma', l)$ is the same as $C^*(\Sigma, l)$, making it into a subcomplex. Similarly, there is a natural projection from $C^*(\Sigma, l)$ onto $C^*(\Sigma'', l)$ compatible with the differentials. \qed

We are interested to find cohomology of the Ishida complex in the case when the fan is a simplicial subdivision of a convex cone. First, consider the simplest case, when the fan consists of faces of a simplicial cone.

**Lemma 1.6.** Let $\tau$ be a simplicial cone in $\mathbb{N}_\mathbb{R}$. Then the higher dimensional cohomology of the complex

$$0 \to \bigwedge^l M_{\mathbb{Q}} \xrightarrow{\delta} \bigoplus_{\dim \rho = 1} \bigwedge^{l-1} \rho \xrightarrow{\delta} \bigoplus_{\dim \gamma = k} \bigwedge^{l-k} \gamma \xrightarrow{\delta} \cdots,$$

where the differential $\delta$ is defined as in Definition 1.1, vanishes and the zeroth cohomology of it is $\bigwedge^l \tau$.\[\]

**Proof.** Since the Koszul complex

$$\cdots \to \bigwedge^3 M_{\mathbb{Q}} \xrightarrow{e_\rho} \bigwedge^2 M_{\mathbb{Q}} \xrightarrow{e_\rho} \bigwedge^1 M_{\mathbb{Q}} \xrightarrow{e_\rho} \mathbb{Q} \to 0,$$

where $e_\rho$ is a $\mathbb{Q}$-generator of the 1-dimensional cone $\rho$, is known to be acyclic (see [D, Appendix 2]), we get that the zeroth cohomology of our complex is equal to

$$\bigcap_{\rho \subset \tau} \bigwedge^l \rho \cong \bigwedge^l \tau.\[\]

We use the induction on the dimension of the cone $\tau$ to prove the acyclicity of the complex. The statement holds for $\dim \tau = 1$, because the map

$$\bigwedge^l M_{\mathbb{Q}} \xrightarrow{e_\tau} \bigwedge^{l-1} \tau,$$

where $e_\tau$ is the minimal integral generator of $\tau$, is surjective.

For $\dim \tau > 1$, write $\tau = \tau' + \rho'$ where $\tau'$ and $\rho'$ are the facet and the edge of $\tau$, respectively. Let $\Sigma$ be the fan consisting of faces of $\tau$, and $\Sigma''$ be the fan consisting of faces of $\tau'$, then $\Sigma' = \Sigma \setminus \Sigma'' = \text{Star}_{\rho'}(\Sigma)$. So, by Proposition 1.5, we get a long exact sequence in cohomology:

$$\cdots \to H^k(\text{Star}_{\rho'}(\Sigma), l) \to H^k(\Sigma, l) \to H^k(\Sigma'', l) \to \cdots.$$

Note that $H^k(\Sigma'', l)$ vanishes for $k > 0$ by the induction assumption. On the other hand,

$$H^k(\text{Star}_{\rho'}(\Sigma), l) \cong H^{k-1}(\text{Star}_{\rho'}(\Sigma), l - 1).$$
Since $\text{Star}_{\gamma}(\Sigma)$ is a fan consisting of the faces of the simplicial cone $\bar{\tau}$, which is the image of $\tau$ in the quotient space $N_\mathbb{R}/(N_{\gamma})_{\mathbb{R}}$, the latter cohomology group vanishes for $k > 1$, again, by the induction. Hence, the middle term $H^k(\Sigma, l) = 0$ for $k > 1$ as well. To show that it also vanishes for $k = 1$, consider the first terms of the exact sequence:

$$0 \to H^0(\text{Star}_{\gamma}(\Sigma), l) \to H^0(\Sigma, l) \to H^0(\text{Star}_{\gamma}(\Sigma), l) \to H^1(\Sigma, l) \to 0.$$ 

Note that $H^0(\text{Star}_{\gamma}(\Sigma), l) = 0$, since there is no zeroth term in the corresponding complex. We already showed $H^0(\Sigma, l) \cong \bigwedge^{\dim \tau} l$ and $H^0(\Sigma', l) \cong \bigwedge^{\dim \tau'} l$. Also, $H^1(\text{Star}_{\gamma}(\Sigma), l) \cong H^0(\text{Star}_{\gamma}(\Sigma), l - 1) \cong \bigwedge^{\dim \tau - 1} l$. To show that $H^0(\Sigma', l) \to H^1(\text{Star}_{\gamma}(\Sigma), l)$ is onto, which will give the desired result, it suffices to show that

$$\dim H^0(\Sigma, l) - \dim H^0(\Sigma', l) + \dim H^1(\text{Star}_{\gamma}(\Sigma), l) = 0.$$ 

But the dimensions are the binomial coefficients $\binom{d - \dim \tau}{l}$, $\binom{d - \dim \tau + 1}{l}$ and $\binom{d - \dim \tau - 1}{l - 1}$, which satisfy the well known combinatorial identity. \quad \square

The next definition is motivated by the Chow ring of a complete simplicial toric variety (see [D]).

**Definition 1.7.** Let $\Sigma$ be a fan in $N_\mathbb{R}$ with the integral generators of the 1-dimensional cones $e_1, \ldots, e_n$. Then define the Chow ring of $\Sigma$ as

$$A(\Sigma) := \mathbb{Q}[D_1, \ldots, D_n]/(P(\Sigma) + SR(\Sigma)),$$

where

$$P(\Sigma) = \left\langle \sum_{i=1}^{\dim \Sigma} (m, e_i)D_i : m \in M \right\rangle$$

and

$$SR(\Sigma) = \langle D_{i_1} \cdots D_{i_k} : \{e_{i_1}, \ldots, e_{i_k}\} \not\subset \sigma \text{ for all } \sigma \in \Sigma \rangle.$$

Note that $A(\Sigma)$ is a $\mathbb{Z}$-graded algebra assuming $\deg D_i = 1$.

**Theorem 1.8.** Let $\Sigma_\gamma$ be a simplicial subdivision of a convex cone $\tau$ in $N_\mathbb{R}$. Then cohomology of the corresponding $l$-th Ishida complex is

$$H^k(\Sigma_\gamma, l) \cong A(\Sigma_\gamma)_k \otimes \bigwedge^{l-k} \tau^\perp.$$ 

**Proof.** First, note that the $k$-th degree of the Chow ring $A(\Sigma_\gamma)$ is spanned by $D_{\gamma} := \prod_{e_\gamma' \in \gamma} D_i$, for $\gamma \in \Sigma_\gamma(k)$, because of the relations $P(\Sigma_\gamma)$ and $SR(\Sigma_\gamma)$. Also, the relations among $D_{\gamma}$ are

$$\sum_{\gamma' \succeq \gamma} \langle m, e_{\gamma' \gamma} \rangle D_{\gamma},$$

for all $\gamma' \in \Sigma_\gamma(k - 1)$, where $m \in M \cap \gamma'^\perp$ and $e_{\gamma' \gamma} \in N$ is a primitive generator of $\gamma$, but not of $\gamma'$. There is a natural homomorphism from $A(\Sigma_\gamma)_k \otimes \bigwedge^{l-k} \tau^\perp$ to the cohomology $H^k(\Sigma_\gamma, l)$, which sends $\sum_{\gamma \in \Sigma_\gamma(k)} D_{\gamma} \otimes a_{\gamma}$, for $a_{\gamma} \in \bigwedge^{l-k} \tau^\perp$, to

$$\oplus a_{\gamma} \in \ker(C^k(\Sigma_\gamma, l) \to C^{k+1}(\Sigma_\gamma, l)).$$
With this map the relations among $D_{\gamma}$ map to $\text{im}(C^{k-1}(\Sigma_{\tau}, l) \to C^{k}(\Sigma_{\tau}, l))$, so that the homomorphism $A(\Sigma_{\tau})_{k} \otimes \bigwedge^{l-k} \tau^\perp \to H^{k}(\Sigma_{\tau}, l)$ is well defined. We will prove by induction that this homomorphism is an isomorphism in a slightly more general situation: all maximal cones in the fan $\Sigma_{\tau}$ lie in $\tau$, have the same dimension equal to $\dim \tau$, and the support of the fan is topologically equivalent to a convex cone.

Suppose we are given a simplicial subdivision of $\tau$. Pick a cone of the dimension equal to $\dim \tau$ from this subdivision. Then for $\Sigma_{\tau}^{i}$ consisting of the faces of this cone, the statement easily follows from Lemma 1.6. Let us construct inductively, the following sequence of subfans $\Sigma_{\tau}^{i}$ of the subdivision of $\tau$. If $\Sigma_{\tau}^{i-1}$ is constructed, then $\Sigma_{\tau}^{i}$ is obtained by adding to $\Sigma_{\tau}^{i-1}$ a new $(\dim \tau)$-dimensional cone from the subdivision, which is adjacent along a facet to a cone from $\Sigma_{\tau}^{i-1}$, and, also, by adding all cones of the subdivision that have their edges among those of $\Sigma_{\tau}^{i-1}$ and the new edge $\rho$ of the new cone. So, the fan $\Sigma_{\tau}^{i}$ has one more 1-dimensional cone, than the fan $\Sigma_{\tau}^{i-1}$ does. It is clear that by doing this we get that the simplicial subdivision of $\tau$ coincides with $\Sigma_{\tau}^{i}$ for some $i$.

Now, note that $\Sigma_{\tau}^{i}$ is a disjoint union of the star open subset $\Sigma_{\tau}^{i-1}$ and the star closed subset $\text{Star}_{\rho}(\Sigma_{\tau}^{i})$. So, by Proposition 1.5 we have a long exact sequence:

\[ \cdots \to H^{k}(\text{Star}_{\rho}(\Sigma_{\tau}^{i}), l) \to H^{k}(\Sigma_{\tau}^{i}, l) \to H^{k}(\Sigma_{\tau}^{i-1}, l) \to \cdots. \]

By induction, we can assume that $A(\Sigma_{\tau}^{i-1})_{k} \otimes \bigwedge^{l-k} \tau^\perp \cong H^{k}(\Sigma_{\tau}^{i-1}, l)$. We also have $H^{k}(\text{Star}_{\rho}(\Sigma_{\tau}^{i}), l) \cong H^{k-1}(\text{Star}_{\rho}(\Sigma_{\tau}^{i}), l-1) \cong A(\text{Star}_{\rho}(\Sigma_{\tau}^{i}))_{k-1} \otimes \bigwedge^{l-k} \tau^\perp$, where the induction is applied to the fan $\text{Star}_{\rho}(\Sigma_{\tau}^{i})$ lying in the image $\tilde{\tau}$ of the cone $\tau$ in the quotient space $(N/N_{\rho})_{\mathbb{R}}$. Using the description of $A(\Sigma_{\tau}^{i})$ in terms of $D_{\gamma}$, one can easily show that there is a natural exact sequence:

\[ 0 \to A(\text{Star}_{\rho}(\Sigma_{\tau}^{i}))_{k-1} \to A(\Sigma_{\tau}^{i})_{k} \to A(\Sigma_{\tau}^{i-1})_{k} \to 0. \]

Hence, we get a commutative diagram with exact rows and with the right- and left-hand vertical maps being isomorphisms:

\[
\begin{array}{ccc}
0 & \to & A(\text{Star}_{\rho}(\Sigma_{\tau}^{i}))_{k-1} \otimes V \\
\downarrow & & \downarrow \\
0 & \to & H^{k}(\Sigma_{\tau}^{i}, l) \to H^{k}(\Sigma_{\tau}^{i-1}, l) \to 0,
\end{array}
\]

where $V = \bigwedge^{l-k} \tau^\perp = \bigwedge^{l-k} \tau^\perp$. By the 5-lemma (see [E]), the middle vertical map is also isomorphism.

**Corollary 1.9.** Let $\Sigma_{\tau}$ be a simplicial subdivision of a convex cone $\tau$ in $N_{\mathbb{R}}$. Then

\[
\dim H^{k}(\Sigma_{\tau}, l) = \left( d - \dim \tau \right) \sum_{j=0}^{k} \binom{d - \tau - j}{l - k} (-1)^{k-j} \cdot \#\Sigma_{\tau}(j).
\]

**Proof.** First, we guess the formula based on the answer for complete simplicial fans (see [O2, Corollary 4.2]). Then check that it also holds for a fan $\Sigma_{\tau}$ consisting of faces of a simplicial cone $\tau$ and prove it by the induction using the exact sequence from the proof of the above theorem. \(\square\)

**Remark 1.10.** One can consider Ishida’s complexes of $\mathbb{Z}$-modules. However, for a fan with singular cones, the corresponding cohomology of Ishida’s complexes may have torsion.
2. Cohomology of rational forms and a vanishing theorem.

Our goal in this section is to compute the cohomology $H^k(P_Σ, Ω^l_{P_Σ}(X))$ for all $k, l$ and a semiample divisor $X$ on a complete simplicial toric variety $P_Σ$. While not all of these spaces vanish for $k > 0$ as it was in the case of an ample divisor, we discover that some of them do vanish. The result for a trivial divisor gives the well-known description of the cohomology of a complete simplicial toric variety.

Before we can state our results, let us review some further notation. Let $P_Σ$ be a $d$-dimensional complete toric variety associated with the fan $Σ$ in $N_R$. We denote by $T_σ$ a torus corresponding to the cone $σ ∈ Σ$ and by $V(σ)$ the closure of $T_σ$ in $P_Σ$. Also, $D_1, \ldots, D_n$ are the torus invariant irreducible divisors in $P_Σ$, corresponding to the primitive generators $e_1, \ldots, e_n$ of the 1-dimensional cones. The polynomial ring $S = S(P_Σ) = \mathbb{C}[x_1, \ldots, x_n]$ is called the homogeneous coordinate ring of the toric variety $P_Σ$. A torus invariant Weil divisor $D = \sum_{i=1}^n a_i D_i$ on the complete toric variety gives rise to a convex polytope

$$Δ_D = \{m ∈ M_R : \langle m, e_i \rangle ≥ −a_i \text{ for all } i\} ⊂ M_R.$$

There is also a support function $ψ_D : N_R → \mathbb{R}$ which is linear on each cone $σ ∈ Σ$ and $ψ_D(e_i) = \langle m_σ, e_i \rangle = −a_i$ for all $e_i ∈ σ$ and some $m_σ ∈ M$.

We will work with semiample divisors which are conveniently classified by the following definition.

**Definition 2.1.** [M3] A semiample Cartier divisor $D$ (i.e., $O_{P_Σ}(D)$ is generated by global sections) on a complete toric variety $P_Σ$ is called $i$-semiample if the Kodaira-Iitaka dimension $κ(D) := \dim φ_D(P_Σ) = i$, where $φ_D : P_Σ → \mathbb{P}(H^0(P_Σ, O_{P_Σ}(D)))$ is the rational map defined by the sections of the line bundle $O_{P_Σ}(D)$.

These divisors satisfy the property:

**Theorem 2.2.** [M3] Let $[D] ∈ A_{d-1}(P_Σ)$ be an $i$-semiample divisor class on a complete toric variety $P_Σ$ of dimension $d$. Then, there exists a unique complete toric variety $P_{Σ_D}$ with a surjective morphism $π : P_Σ → P_{Σ_D}$, arising from a surjective homomorphism of lattices $π' : N → N_D$ which maps the fan $Σ$ into $Σ_D$, such that $π'[Y] = [D]$ for some ample divisor $Y$ on $P_{Σ_D}$. Moreover, $\dim P_{Σ_D} = i$, and, for a torus invariant $D$, the fan $Σ_D$ in $(N_D)_R := N_R/N'_R$, where $N'_R = \{v ∈ N : ψ_D(v) = −ψ_D(u)\}$ is a sublattice of $N$ and $ψ_D$ is the support function of $D$, is the normal fan of $Δ_D$, which lies in $(M_D)_R$, where $M_D := N'_R \cap M$.

To perform the calculation of the cohomology of the twisted sheaves we need to use Ishida’s complex of sheaves which is a resolution of $Ω^l_{P_Σ}$ (see [O1, Section 3.2]). Let us recall its construction. Define

$$Γ^k_{P_Σ} := \bigoplus_{γ \in Σ(k)} Ω^{l-k}_{V(γ)}(log D(γ)),$$

for $0 ≤ k ≤ l$, where $D(γ) := \sum_{γ'} V(γ')$ is the anticanonical divisor on $V(γ)$, and set $Γ^k_{P_Σ} := 0$ for $k < 0$ or $k > l$. Then, there are natural morphisms

$$Γ^k_{P_Σ} → Γ^{k+1,l}_{P_Σ},$$

induced by the Poincaré residue maps

$$R_{γ′, γ} : Ω^{l-k}_{V(γ)}(log D(γ)) → Ω^{l-k-1}_{V(γ′)}(log D(γ′)).$$
for the component $V(\gamma')$ of the divisor $D(\gamma)$ on $V(\gamma)$, if $\gamma \in \Sigma(k)$ is a face of $\gamma' \in \Sigma(k+1)$. If $\gamma \in \Sigma(k)$ is not a face of $\gamma' \in \Sigma(k+1)$, then the corresponding map $R_{\gamma', \gamma}$.

By [O1, Theorem 3.6], the following sequence is exact
\[ 0 \to \Omega_{P_\Sigma} \to I_{P_\Sigma}^{0,l} \to I_{P_\Sigma}^{1,l} \to \cdots \to I_{P_\Sigma}^{l,l} \to 0, \]

since $P_\Sigma$ is simplicial. Twisting this sequence by $O_{P_\Sigma}(X)$ gives another exact sequence
\[ 0 \to \Omega_{P_\Sigma}(X) \to I_{P_\Sigma}^{0,l}(X) \to I_{P_\Sigma}^{1,l}(X) \to \cdots \to I_{P_\Sigma}^{l,l}(X) \to 0, \tag{3} \]

which is a resolution of the twisted sheaf. Here, $I_{P_\Sigma}^{k,l}(X) := I_{P_\Sigma}^{k,l} \otimes O_{P_\Sigma}(X)$. We will now use this resolution to compute the cohomology of the twisted sheaves.

**Theorem 2.3.** Let $\beta = [X] \in A_{d-1}(P_\Sigma)$ be a semiample divisor class on a complete simplicial toric variety $P_\Sigma$, and let $\pi : P_\Sigma \to P_{\Sigma,X}$ be the associated canonical contraction. Then
\[ H^k(P_\Sigma, \Omega_{P_\Sigma}(X)) \cong \bigoplus_{\sigma \in \Sigma_X} (S/(x_j : \pi(\rho_j) \subset \sigma))_{\beta - \beta_0 + \beta_1} \otimes A^\sigma(\Sigma) \otimes \bigwedge^{l-k}(M_X \cap \sigma) \]

where $\beta_0 = \deg(\prod_{j=1}^n x_j)$, $\beta_1 = \deg(\prod_{j=1}^n \pi(\rho_j) \subset \sigma)$, and
\[ A^\sigma(\Sigma) := \mathbb{Q}[D_1, \ldots, D_n]/(P(\Sigma) + SR(\Sigma) + \langle D_j : \pi(\rho_j) \not\subset \sigma \rangle) \]

is defined similar to Definition 1.7.

**Proof.** Since
\[ \Omega_{V(\gamma)}(\log D(\gamma)) \cong O_{V(\gamma)} \otimes \bigwedge^{l-k}(M \cap \gamma) \tag{4} \]

(see [O1, Corollary 3.2]), the sheaf $I_{P_\Sigma}^{k,l}(X)$ is a direct sum of the semiample sheaves $O_{V(\gamma)}(X)$. By the vanishing of the higher dimensional cohomology of a semiample sheaf (see [D, Corollary 7.3]), it follows that the resolution (3) is acyclic, whence the cohomology of the twisted sheaf $\Omega_{P_\Sigma}(X)$ can be computed as the cohomology of the complex of global sections (see [I, Proposition 4.3]):
\[ 0 \to H^0(P_\Sigma, I_{P_\Sigma}^{0,l}(X)) \to H^0(P_\Sigma, I_{P_\Sigma}^{1,l}(X)) \to \cdots \to H^0(P_\Sigma, I_{P_\Sigma}^{l,l}(X)) \to 0. \]

By the identifications (4) and the isomorphisms (see [C])
\[ H^0(V(\gamma), O_{V(\gamma)}(X)) \cong S(V(\gamma))_{\beta_\gamma}, \]

where $\sigma \in \Sigma_X$ is the smallest cone containing the image of $\gamma$, this complex can be rewritten as
\[ 0 \to S_\beta \otimes \bigwedge M \to \bigoplus_{\rho \in \Sigma(1)} S(V(\rho))_{\beta_\rho} \otimes \bigwedge^{l-1}(M \cap \rho) \to \]

\[ \cdots \to \bigoplus_{\gamma \in \Sigma(k)} S(V(\gamma))_{\beta_\gamma} \otimes \bigwedge^{l-k}(M \cap \gamma) \to \cdots \to \bigoplus_{\gamma \in \Sigma(l)} S(V(\gamma))_{\beta_\gamma} \to 0, \tag{5} \]

where the map $\delta$ sends $g \otimes w \in S(V(\gamma))_{\beta_\gamma} \otimes \bigwedge^{l-k}(M \cap \gamma)$, for $\gamma \in \Sigma(k)$, to the direct sum of $g_{\gamma'} \otimes (w, x_\gamma') \in S(V(\gamma'))_{\beta_{\gamma'}} \otimes \bigwedge^{l-k-1}(M \cap \gamma')$, for $\gamma' \subset \gamma \in \Sigma(k+1)$, such that $g_{\gamma'}$ is the image of $g$ induced by the restriction $H^0(V(\gamma), O_{V(\gamma)}(X)) \to H^0(V(\gamma'), O_{V(\gamma')}(X))$, and $e_{\gamma', \gamma}$ is the minimal integral generator of the cone $\rho \in \Sigma(1)$ contained in $\gamma'$ but not in $\gamma$. 


Next, notice that the monomials in $S_{\beta}$ naturally correspond to the lattice points of the polytope $\Delta_X$, by $[C]$. With respect to this identification, the monomials in $S(V(\gamma))_{\beta^\gamma} \cong (S/(x_j : \pi(\rho_j) \subset \sigma))_{\beta}$ (see $[M3]$) are the lattice points in the face of $\Delta_X$ corresponding to the minimal cone $\sigma \in \Sigma_X$ containing the image of $\gamma$. The natural grading of $S(V(\gamma))_{\beta^\gamma}$, for $\gamma \in \Sigma$, by the lattice points of $\Delta_X$ induces a grading on the sequence (5), and it is not difficult to see that the maps $\delta$ in (5) respect this grading. A monomial in $S_{\beta}$ can be uniquely written as $\prod_{\pi(\rho_j) \subset \sigma} x_j$ times a monomial in $(S/(x_j : \pi(\rho_j) \subset \sigma))_{\beta-\beta_0+\beta_1^m}$, where $\sigma \in \Sigma_X$ corresponds to the minimal face of $\Delta_X$ containing the lattice point associated to the monomial. From here, it follows that the $k$-th cohomology of (5) is isomorphic to the direct sum, by $\sigma \in \Sigma_X$, of the tensor products of the complex spaces $(S/(x_j : \pi(\rho_j) \subset \sigma))_{\beta-\beta_0+\beta_1^m}$ and the $k$-th cohomology of the complex

$$0 \to \bigwedge^l M \xrightarrow{\delta} \bigoplus_{\rho \in \Sigma(1)} \bigwedge^{l-1} (M \cap \rho^\perp) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \bigoplus_{\gamma \in \Sigma(k)} \bigwedge^{l-k} (M \cap \gamma^\perp) \xrightarrow{\delta} \cdots.$$ 

Recognize that this is the Ishida complex of $\mathbb{Z}$-modules for a simplicial subdivision of the convex cone $\tilde{\pi}^{-1}(\sigma)$ induced by the fan $\Sigma$. Since we tensor the cohomology groups of this complex with complex spaces, we can discard torsion and use Theorem 1.8. After noting $M \cap (\tilde{\pi}^{-1}(\sigma))^\perp = M_X \cap \sigma^\perp$ the result easily follows. \hfill $\square$

As a consequence of the above theorem, we get the following vanishing result, which is an analogue of the Bott-Steenbrink-Danilov vanishing theorem (see $[BFLM]$).

**Theorem 2.4.** Let $X$ be an $i$-semiample divisor on a complete simplicial toric variety $P_\Sigma$. Then

$$H^k(P_\Sigma, \Omega^l_{P_\Sigma}(X)) = 0$$

if $k > l$ or $l > k + i$.

**Proof.** If $k > l$, then $\bigwedge^{l-k}(M_X \cap \sigma^\perp) = 0$ in Theorem 2.3. One can also obtain $H^k(P_\Sigma, \Omega^l_{P_\Sigma}(X)) = 0$ in this case directly from the complex of global sections of sheaves arising from the exact sequence (3), since that complex does not have nonzero terms after $l$.

If $l > k + i$ then $l-k > i$, while the rank of $M_X \cap \sigma^\perp$ is no more than $i$. So, $\bigwedge^{l-k}(M_X \cap \sigma^\perp)$ vanishes again for all $\sigma \in \Sigma_X$. \hfill $\square$

**Remark 2.5.** When $X$ is a trivial divisor (i.e., $i = 0$), this theorem gives the vanishing part of the cohomology of the complete simplicial toric variety $P_\Sigma$: $H^k(P_\Sigma, \Omega^l_{P_\Sigma}) = 0$ for $k \neq l$. Moreover, by Theorem 2.3, we get $H^k(P_\Sigma, \Omega^l_{P_\Sigma}) \cong \mathbb{C} \otimes A(\Sigma)_k$, which is the Danilov-Jurkevich description of the cohomology of a complete simplicial toric variety with coefficients in $\mathbb{C}$ (see $[D]$). We should also remark that Theorem 2.4 can not be extended to arbitrary complete toric varieties, because in the case when $X$ is trivial we get $H^k(P_\Sigma, \Omega^l_{P_\Sigma})$, which may be nontrivial for $k < l$ (see $[ADu]$). However, for $k > l$ and a complete toric variety $P_\Sigma$, cohomology $H^k(P_\Sigma, \Omega^l_{P_\Sigma}) = 0$ by $[D]$. It will be interesting to see if the vanishing result of Theorem 2.4 holds for complete toric varieties when $k > l$.

We can also deduce a generalization of the Kodaira vanishing theorem for toric varieties, which was proved by a different method in $[Mu]$ and $[BBo]$ (see also $[CDi]$).
**Theorem 2.6.** Let $X$ be an $i$-semiample divisor on a $d$-dimensional complete simplicial toric variety $P_\Sigma$. Then $H^k(P_\Sigma, \Omega^0_{P_\Sigma}(X)) = 0$ for $k \neq d - i$.

**Proof.** If $l = d$ in Theorem 2.3, then $\bigwedge^{l-k}(M_X^\perp \cap \sigma^\perp) \neq 0$ for $d - k \leq i - \dim \sigma$. For such $\sigma$, we have $\dim \pi^{-1}(\sigma) = \dim \sigma + d - i \leq k$. By the relations in $A^\sigma(\Sigma)_k$, it is clear that this $\mathbb{Q}$-module vanishes for $\dim \pi^{-1}(\sigma) < k$. Let us show that in the case of $\dim \pi^{-1}(\sigma) = k$, we also have $A^\sigma(\Sigma)_k = 0$. Indeed, the maximal cones $\gamma$ of dimension $k$ that subdivide the convex cone $\dim \pi^{-1}(\sigma)$ correspond to the generators $D_\gamma$ of $A^\sigma(\Sigma)_k$. By the relations (2) that come from facets of the maximal cones, all $D_\gamma$ are multiples of each other. Moreover, they are all zero, if $\sigma \neq \emptyset$, since we also have the relation coming from the facet of a maximal cone, which lies on the boundary of $\pi^{-1}(\sigma)$. But $\sigma \neq \emptyset$ because $\dim \pi^{-1}(\sigma) = k \neq d - i$ while $\dim \pi^{-1}(\emptyset) = d - i$. \hfill $\Box$

**Corollary 2.7.** Let $X$ be an $i$-semiample torus invariant divisor on a $d$-dimensional complete simplicial toric variety $P_\Sigma$ as in Theorem 2.3. Then

$$\dim H^k(P_\Sigma, \Omega^k_{P_\Sigma}(X)) = \sum_{\Gamma \prec \Delta_X} \ell^\ast(\Gamma) \binom{\dim \Gamma}{l-k} \sum_{j=0}^{k} \binom{d - \dim \Gamma - j}{k-j} (-1)^{k-j} \# \Sigma_{\sigma}(j),$$

where the sum is by faces of the polytope $\Delta_X$, $\sigma \in \Sigma_X$ corresponds to $\Gamma$, $\Sigma_{\sigma} = \{ \tau \in \Sigma | \pi(\tau) \subset \sigma \}$ is a subfan of $\Sigma$, and $\ell^\ast(\Gamma)$ denotes the number of interior lattice points inside $\Gamma$.

**Proof.** Cones $\sigma$ of $\Sigma_X$ correspond to the faces $\Gamma$ of the polytope $\Delta_X$ and monomials in $(S/(x_j : \pi(\rho_j) \subset \sigma))_{\beta - \beta_0 + \beta_j}$ correspond to monomials in $S_\beta$ that are divisible by $\prod_{\pi(\rho_j) \subset \sigma} x_j$ and not divisible by $x_j$ for $\pi(\rho_j) \subset \sigma$. Such monomials are in one-to-one correspondence with the interior lattice points of $\Gamma$. Next, note that $A^\sigma(\Sigma) \simeq A(\Sigma_{\pi^{-1}(\sigma)})$, where $\Sigma_{\pi^{-1}(\sigma)}$ is the simplicial subdivision of $\pi^{-1}(\sigma)$ induced by $\Sigma$. Since $M_X^\perp \cap \sigma^\perp = M \cap \pi^{-1}(\sigma)^\perp$, Corollary 1.9 and Theorem 2.3 give us the formula. \hfill $\Box$

**Example 2.8.** E. Materov in [Ma] calculated the Bott formula for ample divisors on complete simplicial toric varieties. Let $X$ be ample on $P_\Sigma$, then $X$ is $d$-semiample and $\Sigma_X = \Sigma$ consists of simplicial cones. For $k = 0$ in Corollary 2.7, we get

$$\dim H^0(P_\Sigma, \Omega^0_{P_\Sigma}(X)) = \sum_{\Gamma \prec \Delta_X} \ell^\ast(\Gamma) \binom{\dim \Gamma}{l}.$$

For $k > 0$, the combinatorial identity

$$\sum_{j=0}^{k} \binom{m-j}{k-j} (-1)^{k-j} \binom{m}{j} = 0,$$

which follows from

$$\sum_{j=0}^{k} \binom{m}{j} t^j (1-t)^{m-j} = (t+1-t)^m = 1,$$

implies $\dim H^k(P_\Sigma, \Omega^k_{P_\Sigma}(X)) = 0$. 

If $X$ is trivial on $P_{\Sigma}$, then $\Sigma_X$ and $\Delta_X$ are just points. We get

$$\dim H^k(P_{\Sigma}, \Omega^1_{P_{\Sigma}}) = \begin{cases} \sum_{j=0}^{k-1} (d-j) \cdot (k-j) \cdot (-1)^{k-j} \cdot \#(\Sigma), & k = l, \\ 0, & k \neq l. \end{cases}$$

These are the formulas in Theorems 2.14 and 3.6 in [Ma].

The next thing we will do is to describe $H^k(P_{\Sigma}, \Omega^1_{P_{\Sigma}}(X))$ via Čech cohomology. The toric variety $P_{\Sigma}$ has an affine open cover $U = \{ U_\tau \}$, where

$$U_\tau = \{ x \in P_{\Sigma} \mid \prod_{i \in \tau} x_i \neq 0 \}$$

and $\tau \in \Sigma(d)$ have the maximal dimension. This cover induces an affine open cover on all subvarieties $V(\gamma) \subset P_{\Sigma}$ as well. Using the notation in [BC], let $m \in M$ correspond to the differential form $\omega \in \Omega^1_{P_{\Sigma}}(X)$ via Čech cohomology.

**Proposition 2.9.** Let $X$ be a semiample divisor on a complete simplicial toric variety $P_{\Sigma}$ defined by $f = 0$ for $f \in S_\beta$. Then, under the isomorphism of Theorem 2.3 and the natural isomorphism $H^k(U, \Omega^1_{P_{\Sigma}}(X)) \cong H^k(P_{\Sigma}, \Omega^1_{P_{\Sigma}}(X))$, we have that $A \otimes D_{i_1} \cdots D_{i_k} \otimes \omega$, where $A \in S_{\beta - \beta_0 + \delta \tau}$, $\gamma = \{e_{i_1}, \ldots, e_{i_k}\} \in \Sigma$, $\pi(\gamma) \subset \Sigma \subset \Sigma_X$ and $\omega \in \wedge^{l-k}(M_X \cap \sigma^k)$, correspond to the Čech cocycle

$$\left\{ \frac{A}{f} \prod_{\tilde{\pi}(e_i) \notin \sigma} x_i (m_i^{1_{i_1}} - m_i^{1_{i_2}}) \wedge (m_i^{2_{i_2}} - m_i^{2_{i_3}}) \wedge \cdots \wedge (m_i^{k_{i_{k-1}}} - m_i^{k_{i_{k-1}}}) \wedge \omega \right\}_{\tau_0 \cdots \tau_k}$$

with $m_i^j = 0$ if $e_i \notin \tau$, and, for $e_i \in \tau$, $m_i^j$ is determined by $\langle m_i^j, e_i \rangle = 1$ and $\langle m_i^j, e_j \rangle = 0 = 0$ for $e_j \in \tau, j \neq i$.

**Proof.** We have the resolution (3) of the sheaf of rational forms $\Omega^1_{P_{\Sigma}}(X)$. When dealing with an acyclic resolution of sheaves, one can apply the following standard trick. Introduce auxiliary sheaves $K_j$ as kernels of the morphisms $T^{j+1}_P(X) \to T^j_P(X)$. Then, by the exactness of (3), we get short exact sequences of sheaves

$$0 \to K_j \to T^j_P(X) \to K_{j+1},$$

which give rise to the long exact sequences in cohomology:

$$\cdots \to H^{k-j}(P_{\Sigma}, T^j_P(X)) \to H^{k-j}(P_{\Sigma}, K_j) \to H^{k-j+1}(P_{\Sigma}, K_{j-1}) \to H^{k-j+1}(P_{\Sigma}, T^{j+1}_P(X)) \to \cdots.$$ 

Since higher dimensional cohomology of $T^j_P(X)$ vanishes, the connecting homomorphisms are isomorphisms for $k-j>0$ and epimorphisms for $k-j=0$. Thus, we get isomorphisms:

$$H^0(P_{\Sigma}, K_k)/\text{im} H^0(P_{\Sigma}, T^{k-1}_P(X)) \cong H^1(P_{\Sigma}, K_{k-1}) \cong H^2(P_{\Sigma}, K_{k-2}) \cong \cdots \cong H^{k-1}(P_{\Sigma}, K_1) \cong H^k(P_{\Sigma}, \Omega^1_{P_{\Sigma}}(X)).$$

Now, $\delta(\prod_{\tilde{\pi}(e_i) \notin \sigma} x_i) \omega/f$ is a global section of $\Omega^{l-k}_{V(\gamma)}(\log D(\gamma))$, which is a subsheaf of $T^{l-k}_P(X)$. Moreover, one can easily check that its restriction by the homomorphism

$$H^0(P_{\Sigma}, T^{l-k}_P(X)) \to H^0(P_{\Sigma}, T^{k+1-l}_P(X))$$

...
vanishes, whence \( A(\prod_{\mathfrak{p}(e_i) \not\in \sigma} x_i) \omega/f \in H^0(\mathbf{P}_\Sigma, K_k) \). In Čech cohomology this section is represented by the cocycle
\[
\left\{ A \left( \prod_{\mathfrak{p}(e_i) \not\in \sigma} x_i \right) \omega/f \right\}_\tau \in \check{H}^0(\mathcal{U}_{\gamma}, \Omega^{l-k}_{\gamma}(\log D(\gamma))). \tag{6}
\]
To find its image by the connecting homomorphism \( \check{H}^0(\mathbf{P}_\Sigma, K_k) \to \check{H}^0(\mathbf{P}_\Sigma, K_{k-1}) \), we can use the commutative diagram:
\[
\begin{array}{ccc}
0 & \to & C^1(\mathcal{U}, K_{k-1}) \to C^1(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{k-1,l}(X)) \to C^1(\mathcal{U}, K_k) \to 0 \\
& \uparrow & \uparrow & \uparrow \\
0 & \to & C^0(\mathcal{U}, K_{k-1}) \to C^0(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{k-1,l}(X)) \to C^0(\mathcal{U}, K_k) \to 0.
\end{array}
\]
The homomorphism \( C^0(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{k-1,l}(X)) \to C^0(\mathcal{U}, K_k) \) is induced by the Poincaré residue maps
\[ R_{\gamma, \gamma'} : C^0(\mathcal{U}_{\gamma'}, \Omega^{l-k+1}_{\gamma'}(\log D(\gamma'))) \to C^0(\mathcal{U}_{\gamma}, \Omega^{l-k}_{\gamma}(\log D(\gamma))), \]
where \( \gamma' \) is a facet of \( \gamma \in \Sigma(k) \). The cocycle (6) has a lift by this homomorphism to the cochain
\[
\left\{ \frac{A}{f} \prod_{\mathfrak{p}(e_i) \not\in \sigma} x_i m^{l_k}_i \wedge \omega \right\}_\tau \in C^0(\mathcal{U}_{\gamma}, \Omega^{l-k+1}_{\gamma}(\log D(\gamma'))),
\]
where \( \gamma' = (e_i, \ldots, e_{i-1}) \in \Sigma(k-1) \), which can also be thought as the cochain in \( C^0(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{k-1,l}(X)) \). Applying the Čech coboundary to this cochain, we get the Čech cocycle
\[
\left\{ \frac{A}{f} \prod_{\mathfrak{p}(e_i) \not\in \sigma} x_i (m^{l_k}_i - m^{l_k}_{i_0}) \wedge \omega \right\}_{\tau_0 \ldots \tau_k}.
\]
Repeating the above procedure and using the commutative diagrams
\[
\begin{array}{ccc}
0 & \to & C^{k-j+1}(\mathcal{U}, K_{j-1}) \to C^{k-j+1}(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{j-1,l}(X)) \to C^{k-j+1}(\mathcal{U}, K_k) \to 0 \\
& \uparrow & \uparrow & \uparrow \\
0 & \to & C^{k-j}(\mathcal{U}, K_{j-1}) \to C^{k-j}(\mathcal{U}, \mathcal{I}_{\mathbf{P}_\Sigma}^{j-1,l}(X)) \to C^{k-j}(\mathcal{U}, K_j) \to 0,
\end{array}
\]
one can check that the image of the cocycle (6) by the sequence of homomorphisms
\[
\check{H}^{k-j}(\mathbf{P}_\Sigma, K_j) \to \check{H}^{k-j+1}(\mathbf{P}_\Sigma, K_{j-1})
\]
is precisely the cocycle
\[
\left\{ \frac{A}{f} \prod_{\mathfrak{p}(e_i) \not\in \sigma} x_i (m^{l_k}_{i_1} - m^{l_k}_{i_0}) \wedge (m^{l_k}_{i_2} - m^{l_k}_{i_1}) \wedge \cdots \wedge (m^{l_k}_{i_k} - m^{l_k}_{i_{k-1}}) \wedge \omega \right\}_{\tau_0 \ldots \tau_k}
\]
from \( \check{H}^k(\mathcal{U}, \Omega^{l}_{\mathbf{P}_\Sigma}(X)) \).

\begin{example}
Let us apply Proposition 2.9 in the case when \( \mathbf{P}_\Sigma \) is the projective space \( \mathbb{P}^n \), and \( X \) is a trivial divisor. If \( x_1, \ldots, x_{n+1} \) are the homogeneous coordinates on \( \mathbb{P}^n \), then the open cover \( \mathcal{U} \) is given by the open sets \( U_\tau = \{ x \in \mathbb{P}^n | x_j = 0 \} \), where \( \tau = (e_1, \ldots, e_j, \ldots, e_{n+1}) \), and \( m^l_j = \frac{dx_j}{x_j} - \frac{dx_{j+1}}{x_j} \). Hence, the following cocycles represent \( H^k(\mathbb{P}_\Sigma, \Omega^l_{\mathbb{P}_\Sigma}) \cong \mathbb{C} \):
\[
\left\{ \left( \frac{dx_{j_1}}{x_{j_1}} - \frac{dx_{j_2}}{x_{j_2}} \right) \wedge \left( \frac{dx_{j_2}}{x_{j_2}} - \frac{dx_{j_3}}{x_{j_3}} \right) \wedge \cdots \wedge \left( \frac{dx_{j_k}}{x_{j_k}} - \frac{dx_{j_{k+1}}}{x_{j_{k+1}}} \right) \right\}_{j_1 \ldots j_k}.
\]
\end{example}
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