Face-on structure of barlenses and boxy bars: an insight from spectral dynamics

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ABSTRACT
Based on the spectral analysis of individual orbits of stars from different N-body models, we show that the face-on morphology of the so-called ‘face-on peanut’ bars (boxy bars) and barlenses is supported by different families of orbits. For ‘face-on peanut’ bars, the family of boxy orbits comes to the fore, and it is responsible for the unusual morphology of the bar in the central regions. In the models with compact bulges, the bars show a barlens morphology in their central parts. We found that the barlens supporting orbits come in two types, one of which give a square-like shape and the other have a rosette-like shape in the frame co-rotating with the bar. Such a shape is typical for orbits moving in simple spherical potentials. Both these types of orbits do not precess together with typical bar orbits. Square-shaped orbits were already known from some of the previous studies, while the second orbital type is revealed for the first time in the present work. Although quite simple, the family of rosette-like orbits is found to be the main building block of a barlens in our models. The detailed analysis of all bar orbits also allowed us to distinguish x\textsuperscript{2} orbital family and isolate the structure supported by its orbits. The x\textsuperscript{2} family is well-known, but, apparently, for the first time in studies of this kind, we have fully revealed its supported structure. We found that x\textsuperscript{2} family population increases with an increase in central matter concentration.

Key words: galaxies: bar – galaxies: kinematics and dynamics – galaxies: structure – galaxies: bulges

1 INTRODUCTION
Optical and near-infrared surveys indicate that from 45\% up to 80\% of disc galaxies in the local universe possess bars (e.g., Eskridge et al. 2000; Menéndez-Delmestre et al. 2007; Marinova & Jogee 2007; Barazza et al. 2008; Aguerri et al. 2009). The data scatter is explained by the fact that galaxies with bars are found more often in infrared surveys than in optical ones (Marinova & Jogee 2007), as well as whether statistics include only strong bars or weak ones too (Lüticcic et al. 2000; Eskridge et al. 2000). A prevalence of barred galaxies should not be surprising. Usually considered N-body models of stellar discs are almost always unstable with respect to the bar formation, and very special conditions are needed to suppress the instability. This can be done if a stellar disc is very hot dynamically (e.g., Athanassoula & Sellwood 1986), or a model has a very compact initial classical bulge (Fujii et al. 2018; Saha et al. 2018; Kataria & Das 2018), or the disc is embedded in a massive dark halo (e.g., Ostriker & Peebles 1973). Some other possibilities are discussed, for example, in Sellwood et al. (2019). Various numerical simulations have shown that after its formation, the bar grows in the vertical direction, thickens (Combes & Sanders 1981; Raha et al. 1991) and takes a ‘boxy’ or ‘peanut’-like (B/P) shape when viewed edge-on. The same B/P shape (B/P bulges) is also found in observations of edge-on disc galaxies (e.g., Lüticcic et al. 2000). Some of these bulges are accompanied by a pronounced X-structure (see, for example, SDSS image of NGC 128). The connection between B/P bulges in edge-on galaxies and bars is a subject of a long discussion although it is more or less established to the present day. It is believed that B/P bulges are the vertically thickened inner parts of bars. Most arguments follow from the kinematic studies of stars and ionized gas (Bertola & Capaccioli 1977; Kormendy & Illingworth 1982). These studies showed that B/P bulges rotate cylindrically and have a specific shape of the line-of-sight velocity distribution that arises due to the bar residing in the observed galaxies (Kuijken & Merrifield 1995; Bureau & Freeman 1999; Merrifield & Kuijken 1999; Veilleux et al. 1999; Chung & Bureau 2004). It is also important to note that at intermediate orientations and even in a face-on view, the B/P bulges can still have boxy shape and in this case their structure can be directly associated with the structure of the 3D bar (Erwin & Debattista 2013, 2017; Laurikainen & Salo 2017; Li et al. 2017). It is worth noting that the statistics of B/P
bulges (3D bars) and barred galaxies do not coincide. B/P-shaped bars are slightly smaller in number (Erwin & Debattista 2017; Li et al. 2017). At the same time, although barred galaxies are almost equally common among galaxies of different morphological types (Eskridge et al. 2000; Marinova & Jogee 2007), B/P-shaped bulges prefer to settle in early-type galaxies (Erwin & Debattista 2017; Li et al. 2017). A lot of them are found among Sa and Sb galaxies. Their fraction decreases significantly for later-type galaxies, they are almost never found in Scd galaxies and even later type galaxies (Erwin & Debattista 2017; Li et al. 2017).

Bars observed in nearly face-on galaxies do not necessarily show ‘boxy’ isophotes or a ‘peanut’-like shape, or X-shaped structures as, for example, IC 5240 (Laurikainen & Salo 2017). Often a face-on bar demonstrates a barlens, that is, it has a lens-like morphology in central regions with round or oval isophotes, and the lens itself is embedded in an elongated and narrow bar. Barlenses were massively introduced as a separate galactic structure only recently. That was done by Laurikainen et al. (2011) based on the analysis of $K_s$-band images of 206 early-type galaxies, although individual galaxies with barlenses have been studied for a long time (NGC 1097, NGC 4736, NGC 5728 Shaw et al. 1993; NGC 4442 Bettoni & Galletta 1994). There are also ‘face-on’ bars that do not demonstrate any special features but only elongated elliptical isophotes (see, for example, ESO 186-G062). Li et al. (2017) studied bars statistics for galaxies from the Carnegie-Irvine Galaxy Survey (CGS) and labelled such bars as “unn buckled bars” in contrast to B/P-shaped bars and bars with barlenses, which they call “buckled bars.” Thus, there are at least three types of face-on bars that differ morphologically. Apparently, buckled and unbuckled bar types tend to favour different physical conditions. Statistical studies showed that the buckled bar fraction increases up to 80% toward massive discs ($M_* > 10^{10.5} M_\odot$) residing in early-type galaxies and decreases with higher gas mass ratio (Li et al. 2017; Erwin & Debattista 2017). On opposite, unbuckled bars prefer later-type galaxies with a significant amount of gas.

Statistics of buckled and unbuckled bars, as well as of B/P-shaped bars and barlenses are available in only a few works, and they are not conventional and can be revised for larger samples. In addition, the fractions of different types of bars are highly dependent on the galaxy inclination. For example, in the CGS sample by Li et al. (2017), among 264 galaxies with bars identified in $I$-band images, the fractions of bars flagged as unbuckled and buckled are approximately 60% and 40%, respectively, regardless of the inclination. At the same time, the shares of barlenses and B/P-shaped bars among barred galaxies at low $0^\circ-40^\circ$ and moderate $40^\circ-55^\circ$ (in parentheses) inclinations are 4% (13%) for bars with boxy isophotes and X-structures and 36.5% (22%) for barlenses. The fraction of barlenses substantially diminishes at high inclinations $55^\circ-70^\circ$, while the fraction of galaxies with B/P/X-shaped features increases on the contrary. Laurikainen & Salo (2017) believe that such a complementarity between galaxies with B/P/X-shaped features and galaxies with barlenses indicates that these structures are the manifestation of the same phenomenon, i.e. B/P bulges and barlenses are the same structures but observed from a different viewing angle. At high galaxy inclinations, these structures are observed as B/P/X-shaped features, and viewed face-on, they are observed as barlenses. However, there exist galaxies at low inclination, whose bars look like ‘face-on peanut’ rather than a lens embedded in a bar (e.g. IC 4237, IC 5240, NFC 3227, NGC 4123, NGC 4725, Laurikainen & Salo 2017).

Thus, in early-type galaxies, there are at least two types of bars with different face-on morphology: ‘face-on peanuts’ with X-shaped features in central regions (X) and barlenses (BL).

Currently, there are works in which the difference in the morphology of the bars (in a face-on view) and B/P bulges (in edge-on view) is associated with the difference in the parameters of the underlying galaxy. First of all, $N$-body simulations give the relationship between the bar morphology and the parameters of the parent galaxy. For example, Athanassoula & Misiriotis (2002) showed that models with a small relative mass of the dark matter within four radial scales of the disc $M_b/M_d=0.75$ and without a classical bulge give rise to a weak bar that looks like an oval distortion while the models with $M_b/M_d=1.5$ give rise to a strong bar. Salo & Laurikainen (2017); Laurikainen & Salo (2017) concluded that the introduction of a moderate bulge in the model drastically changes the bar morphology. Models by Salo & Laurikainen (2017), which have a small classical bulge and a steep inner rotation curve slope, give BLs, whereas bulgeless models with a shallow rotation curve end up with an X-shaped feature visible even in a face-on view. We note that the bulge should not be too massive (concentrated). A massive and concentrated bulge completely damps the bar instability or leads to a strong delay in the bar formation (Saha & Elmegreen 2018). It was enough to have a bulge-to-disc (B/D) mass ratio about 0.08 at the beginning of the simulation, so that the growing B/P/X bulge takes a typical BL shape viewed face-on.

A BL morphology can be produced in $N$-body simulations, which include gas (SPH) and its physics (star formation, feedback and cooling), without the need to add any spheroidal bulge components in the initial models (Athanassoula et al. 2013, 2015). Apparently, the reason for BL morphology to appear in simulations by Athanassoula et al. (2013) is the increase in the gas concentration in the central area via the bar-induced inflow. However, it should be remembered that the well-developed barlenses are clearly more common in early-type S0s, which have consumed most of their gas but have classical bulges (Laurikainen & Salo 2017).

The accurate decomposition of galaxies with barlenses, taking into account the disc, bar, lens, and bulge, makes it possible to identify a bulge with $<B/T>-0.1$ (Laurikainen et al. 2014). Laurikainen et al. (2018) give even a smaller $B/T$-value ($-0.06$) in the B/D/bar/barlens-models for some of their galaxies. Nevertheless, only 21 galaxies among 46 barlens galaxies analysed by Laurikainen et al. (2018) demonstrate a bulge in the decomposition accounting for a separate bulge component. Thus, the presence of a small compact bulge is not the only way to obtain barlens morphology.

Smirnov & Sotnikova (2018) confirmed the results by Salo & Laurikainen (2017); Laurikainen & Salo (2017). The addition of a rather smooth but massive bulge in their high-resolution model led to a BL bar morphology. Other parameters (an initial disc thickness $Z_0$ and a Toomre parameter $Q$) were also varied in the simulations by Smirnov & Sotnikova (2018). In some cases, changes in these parameters also resulted in the transformation of X into BL without the need to include a classical bulge in the model.

The observational data indicate that barlenses come in different types themselves. For example, Laurikainen & Salo (2017) give examples of nearby face-on galaxies with barlenses and a weak X-shaped feature in the unsharp-masked images. These are intermediate cases between X and BL. Laurikainen & Salo (2017) also highlight an interesting type of galaxies where the barlens dominates the bar. Galaxies with such bar/barlens morphology are
very similar to models by Smirnov & Sotnikova (2018) with a large value of Toomre parameter $Q$.

This similarity suggests that the presence of a classical bulge cannot be considered as a unique reason for BL to appear. Unfortunately, it is very difficult to trace the relationship between $Z_g$ and $Q$ and the bar morphology in the observational data. At the same time, the connection between barlens morphology and the presence/absence of a bulge can be easily detected. That is why we will try to study the barlens structure using only models with an initial classical bulge in the present work.

An analysis of the $N$-body models of multi-component galaxies revealed a connection not only between the face-on morphology of bars and the parameters of the underlying galaxy, but also between the geometric parameters of the B/P bulge vertical structure, as well as their X-structures and the parameters of the parent galaxies (Smirnov & Sotnikova 2018, 2019). The existence of this connection is a direct consequence of the fact that in different potentials different families of orbits will be stable. Although the whole picture is also complicated by the fact that different stable orbital families have different populations. There is a large body of literature on orbits supporting a 3D bar. Different studies in the field of nonlinear dynamics (for example, Contopoulos & Papayannopoulos 1980; Athanassoula et al. 1983; Skokos et al. 2002a,b; Patsis et al. 2002; Patsis & Harsoula 2018; Patsis & Athanassoula 2019) have identified a lot of possible families of periodic and quasi-periodic orbits with a certain morphology in the bar potential while $N$-body simulations, especially those where the orbital frequencies of bodies are determined explicitly (for example, Voglis et al. 2007; Harsoula & Kalapotharakos 2009; Contopoulos & Harsoula 2013; Portail et al. 2015b; Gajda et al. 2016; Valluri et al. 2016; Chaves-Velasquez et al. 2017; Abbott et al. 2017; Lokas 2019), have revealed what is realised in conditions close to that in real galaxies. A comparative analysis of the orbital composition of various models, as in Parul et al. (2020), is a key point for understanding how the different types of bar morphology can be connected with the physical properties of real galaxies. And if the features of the vertical structure are determined by the difference in the distribution function of the orbits over the ratio of the vertical oscillations frequency to the in-plane frequency (Parul et al. 2020), then the features of the face-on morphology must be determined by the distribution function of the in-plane frequency ratios.

We analyse the orbital composition of four $N$-body models, two from Smirnov & Sotnikova (2018) and two new models. Using dominant frequencies we identified all orbital groups which do not enter to the outer disc. For all these orbits we analyse the distributions over the ratios of the in-plane frequencies and show how the dominance of one or another orbital group determine the face-on morphology of a bar and its features.

In Section 2 we present our numerical models and the overall picture of their evolution. Section 3 contains the description of the frequency analysis method. In Section 4 we give details of our algorithm for identifying a bar, an outer disc, and the ring separating a bar from a disc. In Section 5 we analyse the bar face-on morphology for all our models together with the distributions of frequency ratios for all orbits and identified all components in a bar, which are supported by different types of orbits. In Section 6 and Section 7, on the example of a particular numerical model we reveal the orbital anatomy of an elongated bar, a box-shaped bar and a lens embedded into the bar. In Section 8 we summarise the trends in the dominance of orbital families in all our models and show how the contribution of different orbital families to the overall bar changes from model to model. In Section 9, we review known orbits that support the morphological features of a bar and study typical orbits that can constitute a lens. In Section 10, we discuss the nature of the central lens from the point of view of the orbits, which are gradually involved in this structure, and give an interpretation of some of the observed features of barlenses. Finally, in Section 11 we summarise our results.

2 METHODS

2.1 Numerical models

To understand the physical structure of barlenses we first need to obtain appropriate models where barlenses arise. Here we rely on the results of the previous numerical and observational studies. According to Salo & Laurikainen (2017) one of the crucial parameters determining the appearance/absence of barlenses is the steepness of the rotations curve or the central matter concentration (CMC). Naturally, this parameter depends on the bulge contribution ($B/T$) as well as on how much its mass is actually compressed ($r_b/h$). The study of Salo & Laurikainen (2017) was focused on the connection between barlenses and classical bulges. Although, we should note that the CMC also depends on the properties of the dark matter halo, as well as on the physical processes associated with the gaseous component. Nevertheless, we follow Salo & Laurikainen (2017) in the present work and limit ourselves to the pure stellar models with dark halo profiles of NFW-type (Navarro et al. 1996).

In this way we try to ensure that barlenses we study have the same physical nature, the same formation mechanism and are supported by the the same types of orbits (if any) as in Salo & Laurikainen (2017). This is important because the models considered by Salo & Laurikainen (2017) gave the barlenses similar to those observed in real galaxies and were already compared with observational data (Salo & Laurikainen 2017; Laurikainen et al. 2018).

Figure 1. Rotation curves of the numerical models used in the present study. Different line styles are used to highlight the contribution of individual components. Models are arranged by the increase in the steepness of their rotation curves. Models notation (“X”, “Xb”, etc.) follows that of the main text.
Salò & Laurikainen (2017) chose the exact values of classical bulges parameters based on the results of the 2D decomposition of S4G galaxies from their previous works (Laurikainen et al. 2014; Salò et al. 2015). They considered two types of bulges, one with \( B/T = 0.08 \) and the other with \( B/T = 0.01 \) (both bulges have \( r_h/R_0 = 0.07 \)). These values seem rather low for a typical classical bulge (see, for example, Gao et al. 2020). However, Laurikainen et al. (2018) argued that \( B/T \) values strongly depend on the applied photometric model. More precisely, \( B/T \) strongly depend on whether or not barlens is included in the photometric model as a separate component. An overestimated \( B/T \) value can be two or three time greater (or even greater) than it actually is if the bar lens is not taken into account. Erwin et al. (2015) also obtained a similar result in case of composite bulges (when a galaxy posses a pseudo bulge and a classical bulge at the same time). Therefore, values of \( B/T \) in the range from about 0.01 and up to 0.1 seem reasonable from the perspective of the refined 2D decomposition.

In this work we consider four different numerical models, varying the parameters of classical bulges while all other components been the same. We found by trial and error that these four cases (which will be described in detail below) demonstrate fairly gradual transition of the bar morphology. This is important because it will allow us to obtain a general picture how and why bar lenses appear in various numerical models.

The details of our simulations are the following. We took as basis two models from our previous work (Smirnov & Sotnikova 2018). These models consisted of an exponential disc isothermal in the vertical direction,

\[
\rho_\delta(R, z) = \frac{M_\delta}{4\pi R_\delta^2 z_\delta^2} \exp(-R/R_\delta) \cdot \text{sech}^2(z/z_\delta),
\]

where \( M_\delta \) is the total mass of the disc and \( R_\delta \) and \( z_\delta \) are scale lengths in radial and vertical directions, respectively. The dark halo was modelled by a truncated sphere with the density profile close to the NFW profile (Navarro et al. 1996) but with a slightly steeper inner slope,

\[
\rho_h(r) = \frac{C_h}{(r/r_h)^\eta \left(1 + (r/r_h)^\eta\right)^{1/\gamma_h}},
\]

where \( r_h \) is the halo scale radius, \( \eta \) is the halo transition exponent, \( \gamma_h \) is the halo inner logarithmic density slope, \( C_h \) is the parameter defining the total mass of the halo \( M_h \). We adopted the following values: \( \eta = 4/9 \), \( \gamma_h = 7/9 \), \( \gamma_\delta = 31/9 \). \( C_h \) was chosen to produce a reasonable dark halo profile with \( M_h(r < 4 R_\delta)/M_\delta \approx 1.5 \) (see Fig. 1).

One model was pure bulgeless (hereinafter model X) while the second one possessed a classical bulge (model Xb) of a Hernquist (1990) profile,

\[
\rho_\delta(r) = \frac{M_b}{2\pi r (b_0 + r)^5},
\]

where \( b_0 \) is the scale parameter and \( M_b \) is the total bulge mass.

\( B/T \) value was rather large for this model, \( M_b/M_\delta = 0.2 \), compared to typical values from Laurikainen et al. (2018). But the bulge scale length was also rather large, \( r_b/R_\delta = 0.2 \), so the bulge mass was dispersed across rather large volume. The velocity curve is not so steep in this model (see Fig. 1).

Two new models we construct for this study were the same type but with more concentrated bulges. The first one has \( M_b/M_\delta = 0.1 \), \( r_b/R_\delta = 0.1 \) (BLX model) and the second one has \( M_b/M_\delta = 0.1 \), \( r_b/R_\delta = 0.05 \) (BL model), respectively. According to the CMC, the models can be arranged in the following way: BL > BLX > Xb > X (see Fig. 1).

We consider dynamically cold discs with Toomre parameter value \( Q = 1.2 \) at \( R = 2R_\circ \). The radial velocity dispersion profile was chosen to obey an exponential law:

\[
\sigma_r = \sigma_0 \exp(-R/2R_\circ),
\]

where \( \sigma_0 \) is the dispersion of radial velocities in the centre. Its value is derived from the condition on the Toomre parameter, \( Q(R = 2R_\circ) = 1.2 \).

### 2.2 Simulations

Here we briefly describe various aspects of the simulations. The whole procedure is mostly repeating that from Smirnov & Sotnikova (2018) and we refer an interested reader to this work. An \( N \)-body representation of each model was prepared via mkgalaxy code of McMillan & Dehnen (2007). This code is a part of NEMO project (Teuben 1995) and free to use. We use 4k particles for the disc and 4.5k for the halo. Bulge particles have the same mass as disc particles. The total number of bulge particles is then determined according to \( M_b/M_\delta \) ratio. After self-consistent \( N \)-body representation of each model was obtained the evolution of the models was calculated via gryfalc08 code (Dehnen 2002). We use an adaptive time step with the maximal allowed value equal to 0.125 in simulation units which translates into \( \approx 1.65 \) Myr if we assume \( R_\circ = 3.5 \) kpc and \( M_\odot = 5 \cdot 10^9 M_\odot \). Hereinafter we measure time intervals in time units (t. u. for short) of our simulations, 1 t. u. = 14.4 Myr, and distances in units of \( R_\circ, 1 \) length unit = 3.5 kpc. Length unit is also denoted by l. u. for short. Here and below, if a variable goes without a unit of measurement, then its value is measured in the corresponding units of simulation. The softening length values for the disc, \( \epsilon_\delta \), and for the halo, \( \epsilon_h \), were scaled to our number of particles from the values given in McMillan & Dehnen (2007) for the same type of models. The resulted values were about \( 3.7 \cdot 10^{-3} R_\circ \) or \( \approx 13 \) pc for the disc and \( 12.9 \cdot 10^{-3} R_\circ \) or \( \approx 45 \) pc for the halo, respectively. We note that the choice of softening length is important for our problem because Salò & Laurikainen (2017) found that face-on bar morphology is actually dependent on it. They find that \( \epsilon_\delta = 0.02R_\circ \) do not lead to a bar lens while the smaller values indeed give it for their model. Our values of \( \epsilon \) are considerably smaller than 0.02 and therefore, there should be no problems with bar lens manifestation due to insufficient softening length. The evolution of the models was followed up to 8 Gyr. Each...
model is prone to a bar instability and a bar inevitably forms after several Gyr in each of them. It is important that each model leads to a different morphological type of a bar depending on the CMC (Fig 3). More concentrated models give rise to a bar with a barlens while those without classical bulges lead to a peanut-shaped bar (in a face-on view).

To study the orbital composition of the bar we apply the methods of spectral dynamics (Binney & Spergel 1982) in the following subsection. One of the main concerns of such an analysis is how reliable the frequencies we obtain in case of evolving bar/disc. The orbital frequencies of the bar particles are tied to the bar pattern speed. If the pattern speed vary then the frequencies vary too. The general strategy to deal with this problem is to choose some time interval where the bar pattern speed is more or less established. Fig 2 show how the pattern speed of the bar varies in our models. One can see that it is decreasing all the time which implies some frequency shift. However, in our previous work (Parul et al. 2020) we found that the actual frequency shift is small for most of the particles and its value is about the frequency measurement error (see figure 5 from Parul et al. 2020). In Parul et al. (2020) the frequency shifts were estimated for the time interval \( t = 400 – 500 \) t.u. For our convenience we select this time interval to apply the frequency analysis in the present work.

3 ANALYSIS OF DOMINANT FREQUENCIES

If we observe some structure stable enough to live over several Gyr then it should be supported by some kind of stable periodic orbits. Among them there are different families of orbits. A family of orbits is usually composed of orbits that share some morphological similarities (the general profile of the area they sweep or the number of turning points, for example). Strict definition of the family varies across different works (see, for example, Petersen et al. 2019) and depends on the method used to distinguish it. In the present work we distinguish orbital families by means of the dominant frequencies observed in the coordinate spectra of their members. Thus we follow classical work by (Binney & Spergel 1982) and more recent ones by Ceverino & Klypin (2007); Gajda et al. (2016); Portail et al. (2015a); Parul et al. (2020) where orbital families were studied in the context of the galaxy dynamics.

The analysis we carry out is almost the same as in Parul et al. (2020). The key features are the following:

(i) We work with self-consistent \( N \)-body snapshots (not the frozen potential).

(ii) All \( 4k \) particles composing the disc are processed, nothing excluded.

(iii) Each orbit is characterised by four time series of \( x, y, z \) and \( R \) (cylindrical radius) coordinates. Each orbit then can be characterised by the set of the dominant frequencies, \( f_x, f_y, f_z \) and \( f_R \), respectively.

(iv) We work in the reference frame co-rotating with the bar. It simplifies an analysis since we do not need to modify the obtained frequencies according to the changing pattern speed of the bar. Plus \( 2\pi f_c \approx \Omega \sim \Omega_c \) for the most of bar orbits (Gajda et al. 2016) which helps with the interpretation of the results.

The time series for each coordinate consisted of 801 data points, from 400 to 500 time units or from \( \approx 5.3 \) Gyr to \( \approx 6.6 \) Gyr. The orbit spectra were obtained by means of Fast Fourier Transform (FFT). The Nyquist frequency value \( \omega_n \) was about 1800 km/s/kpc while the frequency grid step \( \Delta \omega \) was about 4.5 km/s/kpc. The frequency resolution (due to discrete frequency grid) was improved using the procedure similar to the zero-padding but less time consuming (see Parul et al. 2020 for the details). The resulting frequency resolution \( \Delta \omega_{\text{imp}} \) was about 0.45 km/s/kpc. After we obtained the spectral characteristics of each orbit the next goal is to accurately clear bar particles from disc/ring debris. This will be done in the next section.

As we see below, for some important orbits we also need to distinguish the spectral line, which is second in amplitude. There is nothing strange for an orbit to have several spectral lines (see Binney & Spergel 1982) but this point is rather ignored in some recent studies, unfortunately. We extract the frequency of the second line in a simple way, subtracting the contribution of the dominant wave from the original spectrum and finding the maximum of the residue spectrum.

4 BAR, RING AND OUTER DISC

Before we start to analyse the orbits composing the 3D bar, we need to separate the bar from the disc. We will describe this procedure in the current section.

Portail et al. (2015b); Gajda et al. (2016) have identified the bar in their \( N \)-body simulations by the condition \( f_b/f_s = 2.0 \pm 0.1 \). Fig. 4 shows 2D distribution of frequencies \( f_x \) and \( f_y \) for our model BL. A bright straight line corresponding to the \( f_b/f_s = 2.0 \) is clearly visible. However, the situation with the identification of the bar, when it comes to all the particles in the model, is more complicated. First, particles in the region \( f_x = 0 \) and \( f_y = 0 \) do not belong to the bar, but to the outer regions of the disc. Secondly, above and below the resonance line \( f_b/f_s = 2.0 \), there is a significant number of particles that are located in the central areas of the bar, and they cannot be neglected while studying orbits that support the complex structure of the bar.

Thirdly, due to our choice of the reference frame co-rotating with the bar, the model has a specific radius, at which the angular frequency of the frame matches the angular frequency of the particles in the disc — the radius of co-rotation (CR hereinafter). This region is located near the ring that separates the bar from the outer disc (Fig 3). In the vicinity of CR, particles are unable to finish a turn around the centre during the considered time interval. Consequently, their orbital frequencies \( f_x \) or \( f_y \) tend to be smaller than the frequency resolution \( \Delta f = 1/\delta t \), where \( \delta t \) is the time interval on which we measure the frequencies. Using this feature, we can distinguish three major groups of orbits in all of our models. These groups are shown by different colours in Fig. 5, where we plot \( f_x \) against time-averaged orbital radius \( R_x \) for all \( 4k \) disc particles in the model BL. Given all of the above, the procedure for separating the bar from the outer areas of the disc includes several steps.

(i) The particles with \( f_x < \Delta f \) or \( f_y < \Delta f \) constitute a ring with two thickenings. The radius of the ring is roughly correspond to CR, and the thickenings are located at Lagrange points \( L_4 \) and \( L_5 \) arising in the analytical bar potentials (see Binney & Tremaine 2008). We identify the co-rotation radius \( R_c \) with the approximate “middle” of the ring in Fig. 5. A more precise definition is not required, as the groups are well separated by average orbital radii (see Fig. 5). Since the bars cannot extend beyond the co-rotation (Contopoulos 1980), this group has been removed from further analysis.

(ii) We associate the particles with \( f_x \geq \Delta f \) and \( f_y \geq \Delta f \) that are
further than CR with the outer disc. This group does not contain any bar orbits and is irrelevant for the current study, too.

(iii) The group with $f_x \geq \Delta f$ and $f_y \geq \Delta f$ located inside the co-rotation radius is the main contributor to the region that contains the bar. A detailed study of orbital families from this group is a primary goal of the current analysis and will be carried out in the following sections.

We attempted to refine the outlined bar selection scheme further. Fig. 6 shows the distribution of the orbits of the latter subset on $(R), -f_x/f_y$ plane. In this plot, most of the particles lie below the resonance line $f_x/f_y = 2$, which corresponds to ‘banana’-like orbits. The region above this line consists of two parts. The first one is a group of orbits with average radii smaller than about 0.8 length units. Since these orbits are located near the centre, we assume that they constitute a part of the bar and include them in further analysis. The second one is a “stripe” of orbits that are further from the centre and have higher $f_x/f_y$ than ‘banana’-like orbits. This group consists of orbits that inhabit the peripheral parts of the bar, where it connects to the ring. We decided to exclude all of them using the conditions $(R)_c > 0.8$ and $f_x/f_y > 2.25$ (see the shaded area in Fig. 6). The value of the frequency ratio was chosen so as not to lose ‘banana’-like orbits for sure. This is justified because this group consists of extended orbits, while we are mainly interested in the inner regions where the bar lens resides.

5 FACE-ON PORTRAIT OF A BAR: MORPHOLOGICAL AND ORBITAL ANALYSIS

Fig. 7 shows face-on portraits of our modelled bars, cleaned of the ring and outer disc as well as of the particles of the flat component inside the ring, which were cut out at the initial steps of the current analysis (see Section 4). The model without bulge (model X) demonstrates a peanut-shaped morphology in the face-on view with an X-shaped pattern in the center. Such a morphology is mainly supported by the so-called boxy orbits (see Section 9 and references therein). With an increase in B/D and a transition to a more concentrated bulge, i.e. with an increase of CMC, lens component makes its appearance (Fig. 7, from left to right, from top to down).

We expect that the observed transition in the face-on bar morphology is associated with some notable changes in the orbital composition of a bar. We understand the orbital composition as a set of orbital families that differ in the ratios of the in-plane frequencies $f_x$, $f_y$, and $f_R$. The ratio $f_R/f_y$ is usually employed to separate a bar according to the condition $f_R/f_y = 2$ (Portail et al. 2015b; Gajda et al. 2016). Nevertheless, Gajda et al. (2016)
showed that in their models, there are additional orbits in the region of the bar that can be identified only if the frequency $f_x$ is also involved. To accurately distinguish the orbital families that can be associated with a bar lens, we first would like to grasp the whole picture of what orbital families can actually constitute the bars in our models. To this aim, we want to distinguish all orbital families existing in the bar area for all of our models. We assume that each individual family is characterised by a unique set of frequency ratios $f_y/f_x$ and $f_z/f_x$. We begin with the description of a qualitative picture, that is, what type of orbital families on the plane $f_y/f_x-f_z/f_x$ can constitute a bar in each of the models and what new families start to emerge depending on the model. Although the ratios of dominant frequencies have long been used to classify different orbital families, we use the frequencies $f_y$ and $f_z$ together and for all orbits in our models, without exception. Thus, we obtain an extended classification of orbits in the bar area. Next, we will examine how quantitative changes in populations of different orbital families are associated with morphological changes, especially with lens morphology.

5.1 Main orbital families as building blocks of a bar

We start with a qualitative description of orbital families. Fig. 8 (left column) shows 2D maps of the ratios of dominant frequencies, $f_y/f_x$ and $f_z/f_x$. In such coordinates, the maps demonstrate a curiously regular pattern across different models. One can see a bright spot at $(f_y/f_x; f_z/f_x) = (0.5; 0.5)$ and at least four straight rays going from it. The spot and the rays can be associated with the corresponding orbital families. All these features are present on 2D maps for all models, regardless of their morphology.

For the spot and the vertical ray going upwards we have $f_x = 0.5f_y$. This equality was a criterion for the bar identification in Portail et al. (2015b) and Gajda et al. (2016). In our reference frame $2\pi f_x = \Omega - \Omega_y$, so the equality $f_x = 0.5f_y$ is practically equivalent to the “classic” condition for a bar $\Omega - \Omega_y = \pi/2$. This means that orbits falling into the spot and the vertical ray are mainly periodic orbits inside the bar plus quasi-periodic orbits around them. These orbits precess synchronously with a bar and can constitute a bar in its “classic” meaning (Portail et al. 2015b; Gajda et al. 2016). They are located at the inner Lindblad resonance (ILR) (Athanassoula 2003). The face-on projection of this family is presented in Section 6 (Fig. 9, “classic” bar). The plot is very similar to the ILR image from the work of Ceverino & Klypin (2007) (their figure 10), identified by the usual condition $\Omega - \Omega_y = \pi/2$. The use of the criterion to identify a bar based on Cartesian frequencies $f_y \approx 2f_x$ instead of the usual condition $\Omega - \Omega_y = \pi/2$ leads to the same bar morphology as if we used the latter criterion. Our bars look like a narrow and elongated structures as those typically distinguished by a “classic” condition in studies of this kind.

Thus, the bar orbits lie along a prominent strip, going vertically upward from a bright spot at the centre of a plot (Fig. 8, left). For the orbital family falling into the spot we have $f_y/f_x = 1.0 \pm 0.1$ and $f_z/f_x = 2.0 \pm 0.1$. We refer to this family as an x1-like family. The vertical strip is associated with another family with $f_y/f_x > 1.1$ and $f_z/f_x = 2.0 \pm 0.1$. It is constituted by the so-called box-shaped and boxlet orbits (Valluri et al. 2016; Abbott et al. 2017; Chaves-Velasquez et al. 2017; Gajda et al. 2016; see also Section 9.1).

Three more rays are noticeable on 2D frequency maps. Orbits falling into these rays do not contribute to the bar in its “classic” meaning because they do not precess synchronously with the bar and have either $f_y/f_x < 0.5$, or $f_z/f_x > 0.5$. At the same time, particles from these rays are definitely not from the outer disc, because here we consider particles only inside the ring, which separates the bar from the outer disc. The upper left ray begins near a central spot and has $f_y/f_x > 2.1$; $(f_x + f_y)/f_x = 1.0 \pm 0.1$. A bottom left ray have $f_y/f_x > 2.1$ and $f_z/f_x = 1.0 \pm 0.1$. There is a third ray going up to the right and having $f_y/f_x < 1.9$ with $f_z/f_x = 1.0 \pm 0.1$.

Orbits with $f_y/f_x > 2.1$ were mentioned by Harsoula & Kalapotharakos (2009). They were also found in the N-body simulations by Gajda et al. (2016). In Gajda et al. (2016), these orbits were found to satisfy an additional condition $(f_x + f_y)/f_y = 1$, that is, these are the orbits that fall into the upper left ray. There is another ray with $f_y/f_x > 2.1$ and an additional condition $f_z/f_x = 1.0$ (Fig. 8, left, line running from bottom to top, from left to right up to the point $f_y/f_x = 0.5; f_z/f_x = 0.5$). This orbital family was not explicitly mentioned in the literature.

Harsoula & Kalapotharakos (2009) did not give the morphology of the orbits with $f_y/f_x > 2.1$, but Gajda et al. (2016) noted that orbits similar to those of the upper left ray are not elongated along the bar and does not support it. As will be shown below, both left rays contribute to the lens.

Orbits from the third ray going up to the right were not explicitly described previously in the literature to our knowledge. They have $f_y/f_x < 1.9$ and $f_z/f_x = 1.0$. A possible complimentary branch along the line $(f_x + f_y)/f_y = 1.0$ is practically not populated in all our models. Thus, for $f_y/f_x < 1.9$ there is only one orbital family.

We studied the shape of the face-on isophotes for all particles of both branches with $f_y/f_x > 2.1$ and did not find a significant difference between the upper and lower rays, except that the orbits

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1 A more detailed discussion of the types of orbits is given in Section 9.

2 This is not pure x1 orbits, because they have a length in the vertical direction. Moreover, this family may contain orbits parented from the x2 planar family and other families of higher orders. For brevity, we call all of them “x1-like orbits”.
of the upper ray are assembled into a more compact structure than the orbits of the lower ray. For this reason, we view both of this families as sub-parts of the one building block of the entire bar morphology, that is, we will consider them only in conjunction with one another in further discussion. Here we should note that both of these branches are constituted by orbits with rather complex spectra. In some cases spectra show two waves of comparable amplitude and in some cases they do not. According to that, we believe that we need to perform a more rigorous analysis of orbits spectra to rightfully distinguish between these two orbital families. In the future, we intend to enter into more subtle details related to the difference in the spectra of these two families. But in this work we are mainly interested in the morphology of structures supported by orbits from different families, therefore, our perhaps not so rigorous treatment of these families is more than justified.

To simplify further discussion, we introduce the following notation for these three groups of orbits: “bl,” for orbits from the upper right ray and “bl,” for orbits from both bottom left and upper left rays. As we will see below, the orbits from these groups are the most important contributors to a barlens structure and hence the notation “bl” follows. Subscripts “u” and “o” reflect the position of these groups on 2D frequency map (Fig. 4) with respect to x-like and boxy orbits: “o” stands for ‘above’ (over) the bar ($f_R : f_x > 2 : 1$) while “u” stands for ‘beneath’ (under) the bar ($f_R : f_x < 2 : 1$). In practise we distinguish bl, and bl, groups by $f_R : f_x > 2.1$ and $f_R/f_x < 1.9$ conditions.

There are several other lines, but the number of particles along them is negligible compared the number of particles inside any previously mentioned ray or central spot and we exclude them from further discussion.

5.2 Distributions over the ratio of the in-plane frequencies

Fig. 8 (left column) demonstrates what orbital families contribute to the bar. At a qualitative level, all plots for all models look similar. Unlike left plots of Fig. 8, middle and right plots show 1D distributions over $f_y/f_x$ (middle column) and $f_R/f_x$ (right column) ratios for each of our models and they give an idea of

\[ \text{Figure 7. Face-on view of a bar for X and Xb (two upper plots) and BLX and BL (two bottom plots) models at } t = 450 \text{ t. u. (6 Gyr). All plots depict the same space area } (xy) = (-3, 3) \times (-3, 3). \]
Figure 8. Distribution of all orbits constituting the bar over the ratios of dominant frequencies for X, Xb, BLX and BL models (top down) calculated over the time interval $t = 400 - 500$ t. u. (5.3 – 6.6 Gyr). Left: 2D maps on the plane $f_R/f_x$–$f_y/f_x$. Middle and right: 1D distribution over the ratios $f_y/f_x$ and $f_R/f_x$. Bin width is equal to 0.01.
the quantitative population of the orbital families. It can be seen that these distributions have a slightly different shape depending on the particular model with the exception of the BL model which is quite peculiar. A comparative analysis of these distributions for different models will allow us to understand how the population of different orbital families changes quantitatively from one model to another and to identify what type of orbits give rise to a barlens in BL model.

First, we can see in Fig. 8 (middle) that the second wide peak in the distribution over \( f_x/f_z \) (between \( 1.0 < f_x/f_z < 2.0 \)) and the distribution as a whole shifts towards lower values of \( f_x/f_z \), with an increase in CMC (from top to bottom). All four distributions show the presence of orbits, which can be quasi-periodic and have a frequency ratio \( f_x/f_z \) expressed by a rational fraction (small peaks in the distributions in the region \( 1.0 < f_x/f_z < 2.0 \)). This ratio also becomes smaller when moving from the model X to the model BL: 5.3 for X model, 3.2 for Xb and BLX models and 4.3 for BL model. Besides these changes, the distribution over \( f_x/f_z \) becomes significantly less populated. We note that this part of the distribution is mainly consists of box-shaped orbits that form the vertical strips in Fig. 8, left (in \( f_x/f_h > 0.5 \) area, boxy bar). In Section 8 we will see that the length of a boxy bar formed by orbits with lower values of \( f_x/f_z \) is less than the length of a boxy bar consisting of orbits with larger frequency ratios. That is, the boxy bar becomes shorter and less bright with an increase in CMC, and this creates more favorable conditions for the manifestation of the central lens.

Secondly, the height of the peak at \( f_h/f_0 = 2.0 \) (Fig. 8, right), which is populated by the orbits of the “classic” bar, decreases from the model X to the model BL. Moreover, the model without a classical bulge (model X) demonstrates one dominant family of orbits with \( f_h/f_0 = 2.0 \pm 0.1 \) and a very high peak at this value. At the same time, in the model BL, these orbits account only for 22% of all disc orbits versus 50% of the orbits involved in the whole bar. That is, an elongated bar becomes less populated.

Thirdly, orbits of novel types appear from model X to model BL, and they become quite numerous. They lie to the right and left of the peak at \( f_h/f_0 = 2.0 \) (Fig. 8, right). We previously distinguished these two groups of orbits as blue (\( f_h/f_0 < 1.9 \)) and blue (\( f_h/f_0 > 2.1 \)) using 2D frequency ratios maps. The orbital group bl_2 is present in all our models but it is not well populated in the model X. These orbits become more numerous in the BL model with barlens and they can contribute to the lens.

As to the bl_1 orbits, although models Xb and BLX have a bit more of such orbits than model X, there are still very few such orbits in these models. They manifest themselves quite prominently only in the model BL. We believe that this orbital family makes the main contribution to lens morphology.

The analysis performed above showed there is probably more than one group of orbits that can be associated with a lens (bl_1 and bl_2 orbits). That is, a lens has its own orbital composition consisting of at least two different groups of orbits. But before studying the orbital composition of the lens in detail, let us analyse in detail the structure of a “classic” bar usually extracted by the condition \( f_h/f_0 = 2.0 \pm 0.1 \).

6 THE ANATOMY OF A “CLASSIC” BAR IN BL MODEL

6.1 Two main families constituting a “classic” bar

We identify a “classic” bar according to the condition \( f_h/f_z = 2.0 \pm 0.1 \) as in Portail et al. (2015b); Gajda et al. (2016). For the model with a barlens (BL model) this component is presented in Fig. 9 (“classic” bar). This component looks like a very ordinary elongated bar, without any traces of the lens. In its centre, an X-shaped (or rather peanut-shaped) morphology is observed. There are also two weak vertical protrusions at \( z = 0 \).

Two different orbital families can be distinguished in this component. In Fig. 8, left they are grouped near a bright spot at \( f_h/f_0 = 0.5; f_x/f_h = 0.5 \) and along a vertical strip above this spot. These two group form two different orbital families: an x1-like bar and a boxy bar (see previous Section).

Fig. 9 demonstrates snapshots for these two families at \( t = 450 \), or \( \approx 6 \) Gyr. We have checked that the morphology of both configuration does not change over a time interval from \( t = 400 \) up to \( t = 500 \) (5.3 – 6.6 Gyr). Both families are equally populated (approximately 10-11% of all 4k particles are involved in each of these structures). But the structures formed by orbits from these families are strikingly different in morphology. An x1-like family gives a narrow and elongated bar with an inner bar-like structure superimposed on an outer bar and oriented perpendicular to it (Fig. 9, x1-like bar). Apparently, the inner bar is formed by the orbits parented from the x2 family (Contopoulos & Papayannopoulos 1980; Athanassoula et al. 1983). Box-shaped orbits constitute a face-on peanut (Fig. 9, boxy bar).

6.2 An x1-like bar

Two structures (a narrow and long outer bar and an inner perpendicular bar) in Fig. 9 (x1-like bar) can not be separated if we only use frequency ratios \( f_h/f_0 \) and \( f_x/f_z \). But if we look at 2D distribution of the ratios \( f_x/f_z \) and \( \langle |z|/|z| \rangle \), where \( \langle |z| \rangle \) and \( \langle |y| \rangle \) are the mean absolute values of \( x \) and \( y \) coordinates of orbits calculated over time interval \( t = 400 – 500 \), we will see two non-overlapping areas, where orbits contributing to the x1-like bar fall (Fig. 10). The boundary between two areas passes at \( \langle |y|/|x| \rangle = 1 \). The ratio \( \langle |y|/|x| \rangle \) characterises the flattening of the orbit along the major axis of the bar. An orbit is elongated along the major axis of the bar if this ratio is less than one. It is almost round if the ratio is equal to one, and elongated along the minor axis of the bar if the ratio is greater than one. Thus, in the area with \( \langle |y|/|x| \rangle < 1.0 \) lie orbits elongated along the bar major axis. At the same time, these orbits have \( f_x/f_z > 1.5 \). In the second area, orbits have \( \langle |y|/|x| \rangle > 1.1 \) and \( f_x/f_z < 1.5 \). Portail et al. (2015b) and Parul et al. (2020) argued that the smaller \( f_x/f_z \), the less elongated the orbits. And this is what we see in Fig. 10. It can be assumed that the first orbital group contribute to a narrow and extended bar and the second one constitutes the innermost bump in Fig. 9 (x1-like bar). Moreover, the first group shows an enhancement at \( f_x : f_z = 2 : 1 \). These are banana-like orbits that delineate the most remote parts of such a bar (Parul et al. 2020). There is a long tail of orbits with \( \langle |y|/|x| \rangle \geq 1 \), i.e. of orbits elongated along the bar minor axis. Apparently, these are orbits “genetically” related to the planar x2 family, but there are very few of them.

Snapshots for the individual orbital groups are presented in Fig. 9 (x1-p and x2-p)\(^3\). The right plot demonstrates a structure that is assembled from x2-p orbits. Although this orbital family has been known for a long time (Contopoulos & Papayannopoulos 1980; Athanassoula et al. 1983), it seems that for the first time we clearly distinguished this family as a whole structure in studies of

\(^3\) Such designations mean that these orbits are apparently parented from the planar families x1 and x2.
Figure 9. Face-on view of all bar building blocks for model BL at $t = 450$ t. u. (6 Gyr). All subplots depict the same space area $(xy) = (-2, 2) \times (-2, 2)$. 
In Fig. 9 we plot a snapshot depicting only bl\(_o\) orbits at \(t = 450\) (6 Gyr). We also note that the entire configuration does not change its shape over a period of time from \(t = 400\) to \(t = 500\). Rounded isophotes and a rather strong concentration of the matter towards the centre are striking.

There is another source from which orbits supporting the barlens morphology are drawn. These are the bl\(_o\) orbits with \(f_0/f_x > 2.1\) (Fig. 8, right plots, to the right of the peak at \(f_0/f_x = 2.0\)), or \(f_0/f_R < 0.5\) (Fig. 8, to the left of the vertical strip at \(f_0/f_R = 0.5\)). As we have mentioned, this group contains two branches: one with \((f_0 + f_y)/f_R = 1\) and the other one with \(f_y/f_x = 1\) (Fig. 8, left). The structures formed by the orbits of these two branches differ little morphologically. One difference is that the latter branch contains a larger number of elongated and flattened orbits. The distribution of all bl\(_o\) orbits over the ratio \((\langle y \rangle / \langle x \rangle)\) for BLX and BL models shows that a small number of orbits are very elongated along the major axis of the bar. In total, 17.42% of 4\(kk\) particles fall into the bl\(_o\) group. After removal\(^5\) of very elongated orbits, 13.65% remained. The distribution over the ratio \((\langle y \rangle / \langle x \rangle)\) for the remained orbits is shown in Fig. 11 (the dotted lines). The distribution has no prominent peak near \((\langle y \rangle / \langle x \rangle) \approx 1.1\). It rather wide but the structure associated with the bl\(_o\) orbits shows lens-like morphology (Fig. 9, bl\(_o\) subplot).

Without a doubt, the bl\(_o\) orbits definitely contribute to the barlens, but the morphology of the structure formed by these orbits is rather unusual. In contrast to the structure shown in Fig. 9 (bl\(_o\)) that has rounded isophotes, the isophotes of the structure formed by the bl\(_o\) orbits has a square-like shape (see Fig. 9, bl\(_o\) subplot). We will discuss orbits capable of supporting such an unusual morphology in Section 9.

Thus, it turns out that the barlens is a complex structure. The contributions of the rounded and square-shaped structures to the central barlens are comparable in the BL model (9.25% and 13.65%). The morphology of combined bl\(_o\) and bl\(_x\) parts depends on the distribution of matter in each structure, on how loose or concentrated it is. We will discuss this issue in the next Section 8. Concerning observational data we would like to note that although usually by a barlens we mean something that has circular isophotes, like those in NGC 1015, NGC 1398, NGC 4608, NGC 4643 galaxies (Laurikainen & Salo 2017, Fig. B.1), sometimes a barlens can have a square-shaped morphology. An example of a galaxy with such an unusual square-shaped barlens morphology is NGC 4314. The ‘square’ shape is especially noticeable in an unsharp-masked deprojected image of this galaxy (Laurikainen & Salo 2017, Fig. B.1).

7 THE ANATOMY OF A BARLENS

7.1 Main orbital blocks contributing to the barlens

We have identified three orbital groups that do not enter into x1-p, x2-p or boxy-shaped families constituting the bar. They form three rays in Fig. 8 (left) coming from the bright spot at the point (0.5,0.5) left-down, left-up and right-up. We previously introduced them as bl\(_x\) and bl\(_o\) orbit groups (see Section 5). The orbits that fall into these groups can contribute to the barlens. To support this hypothesis, we analyse the distribution of the average flattening of the orbits.

Fig. 11 shows the distribution over the ratio \((\langle y \rangle / \langle x \rangle)\). It can be seen that although there is a certain number of orbits with the ratio \((\langle y \rangle / \langle x \rangle) > 1\) in the model X, the number of these orbits is very small. There are more of them in models Xb and BLX, and this is due to the appearance of the bl\(_x\) orbits with \(f_R/f_x > 2.1\). And when we look at such a distribution for model BL, we observe the appearance of a large second peak in the distribution over \((\langle y \rangle / \langle x \rangle)\) near the ratio 1.05-1.10. There is only a hint of such a peak in BLX model (the model with traces of a barlens), but for model BL it becomes very noticeable\(^4\). And it is in this model that a new type of orbits with \(f_R/f_x < 1.9\) appears (bl\(_o\) orbits) and the peak is mainly constituted by these bl\(_o\) orbits (see the dashed lines in Fig. 11). The orbits of this type are expected to have a roundish shape and, apparently, make a decisive contribution to the lens.

\(^4\) In the BL model the initially identified bl\(_x\) family contains 9.97% of orbits from 4\(kk\). We found that a small number of such orbits (0.75% from 4\(kk\)) are elongated and have \((\langle y \rangle / \langle x \rangle) < 0.5\). In the Appendix A, we describe an algorithm for getting rid of such orbits. As a result, we left in this family 9.22% of orbits from 4\(kk\).

\(^5\) See Appendix A
then in the model with the CMC (model BL) the contribution of both orbital families is the same (10.5% and 11%, respectively). Thus, judging by our models, the orbits forming a narrow and long bar become significant only in potentials with a central spherically symmetric component. Next, while passing to the model BL, boxy orbits become less elongated along the major axis of the bar. This fact is reflected in the distribution over the mean absolute value of the $x$ coordinate ($|x|$) of boxy bar orbits ($f_x/f_y > 1.0, f_y/f_x = 2.0 \pm 0.1$) calculated over the time interval $t = 400 – 500$ (Fig. 12). When passing from the model X to the model BL, orbits with large values of this parameter leave the distribution and the size of the box-shaped bar is almost halved.

The proportion of orbits that do not support the bar (bl$_u$ and bl$_o$ orbits) increases from the model X to the model BL. There is a significant difference in how these orbits manifest themselves in our models. The bl$_u$ orbits are present even in the X model (5%). It is not surprising that such orbits were found in studies of this kind even in models without CMC (for example, Gajda et al. 2016). The contribution of the bl$_o$ orbits becomes significant only for the model BL (9%). Their number is negligible even in the BLX model with 2500 t. u. (5.3 – 6.6 Gyr). Bin width is equal to 0.01. The dashed curves show the distribution of the same ratio but only for the bl$_o$ family with $f_y/f_x < 1.9$ while the dotted lines are for bl$_u$ orbits with $f_y/f_x > 2.1$.

Figure 11. 1D distribution of all bar particles over the ratio $(|y|)/(|x|)$ for X, Xb, BLX and BL models calculated for the time interval $t = 400 – 500$ (5.3 – 6.6 Gyr). Bin width is equal to 0.01. The dashed curves show the distribution of the same ratio but only for the bl$_o$ family with $f_y/f_x < 1.9$ while the dotted lines are for bl$_u$ orbits with $f_y/f_x > 2.1$.

Despite the different populations of the bl$_u$ and bl$_o$ groups in different models, orbits that do not support the bar remain approximately constant in size. The decrease in the size of the structure formed by box-shaped orbits, and the invariance of the length of the structure supported by the bl$_o$ and bl$_u$ orbits, leads to the following consequences. As we move to the model with a barlens (BL), the family of the box-shaped bar “sinks” in the remaining orbits, and the peanut-like structure ceases to be visible, giving way to a rounded shape clearly distinguishable even against the background of a bar.

Both found orbital groups with $f_y \neq 2f_x$ (bl$_u$ and bl$_o$) contribute to the morphology of a barlens, but their contribution is different. And this is due not only to the different populations of these groups. A good illustration of their different role in the lens building is the Fig. 13. In this figure, orbits that do not support the bar (bl$_u$, bl$_o$) are gradually excluded from BL and BLX models. In Fig. 13 one can see how each of the families affects morphology. For model BL, after exclusion of the bl$_o$ orbits (13.5%), the central lens is still visible, although its overall length is getting smaller. With the exception of only bl$_o$ orbits (9%), the innermost isophotes lose their rounded shape. In the centre there is even a hint of a peanut-like shape. However, outer isophotes still have a round shape, which is an indicator that there is a part of a barlens remaining. Only with the exception of both orbital groups, does the bar acquire a peanut-like shape. For BLX model, the exclusion of the bl$_o$ family (only 3.4%) immediately leads to a peanut-like shape, although some rounded outermost isophotes are still preserved. The exclusion of bl$_o$ orbits results in the following effect. Even the outermost isophotes begin to bend towards the centre along the line perpendicular to the bar. Thus, the family bl$_o$ is responsible for the roundness of the inner isophotes, and the family bl$_u$ gives elongation to the barlens in the direction perpendicular to the bar.

Figure 12. 1D distribution of the boxy bar particles over the value $(|y|)$ for BL, BLX, Xb and X models calculated over the time interval $t = 400 – 500$ t. u. (5.3 – 6.6 Gyr). Bin width is equal to 0.01.

Table 1. The percentage of orbits of each family in the models. The fraction is given relative to the total number of particles in the disc (4k$k)$.

| family | X | Xb | BLX | BL |
|--------|---|----|-----|----|
| bar:   | bl$_u$ | 5.36 | 7.82 | 9.98 | 13.49 |
| bl$_o$ | 0.96 | 1.05 | 3.42 | 9.22 |  |
| “classic” bar | 50.40 | 32.98 | 34.16 | 22.37 |  |

Table 1. The percentage of orbits of each family in the models. The fraction is given relative to the total number of particles in the disc (4k$k)$.

| x1-like bar | x1-p | 3.48 | 5.70 | 7.60 | 9.64 |
| x2-p | 0.30 | 0.83 | 1.91 | 1.71 |  |
Figure 13. Barlens “disassembling” into families. Top row — BL model, bottom row — BLX model. From left to right: all particles in the model, without bl\textsubscript{o}; without bl\textsubscript{i}; without bl\textsubscript{o} and bl\textsubscript{i}. All plots depict the same square $(xy) = (-2.2, -2.2) \times (2.2, 2.2)$ at $t = 450$ (6 Gyr).

9 TYPICAL ORBITS CONSTITUTING THE MAIN BLOCKS OF A COMBINED BAR

9.1 Bar supporting orbits

If we want to understand the physical reasons for the manifestation of different building blocks of a bar, the first thing to do is to examine what types of orbits constitute these blocks. Orbital analysis provides candidates for the backbone of the structure. We found such candidates by sorting all orbits based on frequency ratios $f_R/f_x$ and $f_z/f_x$. For further analysis, it is very important to indicate the stable periodic or quasi-periodic orbits which constitute the backbone of the structures. These orbits can trap regular orbits around them. The whole configuration supported by such orbits will retain its shape over many revolutions.

For the potential of a stationary triaxial ellipsoid the main types of orbits were distinguished by de Zeeuw (1985). They are “tubes” of different extensions, which rotate around the largest or smallest axes of an ellipsoid, and “boxes” that sweep the boxy-like area around the center. Numerous studies of analytical models of bars and bars in $N$-body models show that the main type of orbits supporting a bar in the disc plane are tube-shaped orbits rotating around the $z$-axis and elongated along the major axis of the bar (Athanassoula 2003). According to the nomenclature introduced in (Contopoulos & Papayannopoulos 1980; Athanassoula et al. 1983), it is customary to refer to such orbits as x1 family. The orbits from this family are in 2:1 resonance and they have two radial oscillations per one revolution around the centre. Another family of orbits in 2:1 resonance is the so-called x2 family (Contopoulos & Papayannopoulos 1980). Orbits of this family are elongated perpendicular to the bar.

In the 3D case, when we also have to take into account vertical perturbations, the situation becomes more complicated. New orbital families are generated by bifurcation from x1, x2 and planar families of higher order (Pfenniger 1984; Skokos et al. 2002a,b; Harsoula & Kalapotharakos 2009; Patsis & Katsanikas 2014a,b). In this case, we need to consider the orbits of $x_1 v_i$ ($i = 1, 2, 3, ...$) type bifurcated from the initially flat orbit at the vertical resonances $f_z/f_x = 2, 3, ...$ (Pfenniger & Friedli 1991; Skokos et al. 2002a). The whole sets of $x_1 v_i$ and $x_2 v_i$ families are usually referred as x1 and x2 orbital trees, respectively. However, we are not interested in the vertical structure of the B/P bulge in this work, therefore, we do not differentiate planar and non-planar orbits and refer to them as whole using “x1-p” and “x2-p” notations, respectively.

All these orbits are well known. They structure the phase space and create the backbone of the elongated bar and of the inner perpendicular bar (Fig. 9, x1-p and x2-p plots).

The structure like a face-on peanut (Fig. 9, a boxy bar plot) is built by orbits of another type. In addition to the “tubes”, our models include a significant number of box-like orbits. Valluri et al. (2016) examined a sample of 10,000 orbits in each of their two $N$-body models with bars and classified the orbits using a method based on frequency analysis. They considered only those orbits for which the radial oscillation frequency and the tangential frequency are in 2:1 resonance in a reference frame rotating with a bar pattern speed. Valluri et al. (2016) connected the majority of orbits with the quasi-periodic box-shaped orbit in a rotating triaxial potential. Like 3D orbits elongated along a bar, box-shaped orbits are parented from the x1 family. Based on their $N$-body simulations, Valluri et al. (2016); Abbott et al. (2017); Chaves-Velasquez et al. (2017) give a lot of examples of 3D non-periodic orbits in their $N$-body models, which have peanut-shaped projections both in their face-on and side-on views. Combined together these orbits can constitute the structure like that in Fig. 9 (a boxy bar plot). Valluri et al. (2016); Abbott et al. (2017); Gajda et al. (2016) also found the so-called resonant boxlet orbits (‘fish/prezels’, ‘brezels’). Such orbits can be a backbone of a face-on ‘peanut’ structure.

9.2 Lens supporting orbits

9.2.1 The bl\textsubscript{o} orbits

Frequency analysis of $N$-body models revealed also non-bar orbital families in central parts of discs (Martinez-Valpuesta et al. 2006;
Face-on structure of barlenses and boxy bars: an insight from spectral dynamics

Voglis et al. 2007; Wozniak & Michel-Dansac 2009; Harsoula & Kalapotharakos 2009). A considerable number of particles having $(\Omega - \Omega_{\odot})/k < 0.5$ and $(\Omega - \Omega_{\odot})/k > 0.33$, that is, lying in between 2:1 and 3:1 planar resonances near $(\Omega - \Omega_{\odot})/k \approx 0.4 - 0.44$, were found by Martinez-Valpuesta et al. (2006) (figure 10, top left plot), Voglis et al. (2007) (figure 21, group A), Wozniak & Michel-Dansac (2009) (figure 8, top), Harsoula & Kalapotharakos (2009) (figure 6, right plots, groups A and B). All these orbital groups fall into group of orbits that we introduced as bl$_{o}$ in the present work. These orbits have frequency ratio $f_{x}/f_{y}$ in between 2.1 and at least 3.0. They produce square-like morphology when depicted all together (Fig. 9). Gajda et al. (2016) described an orbital family with a ‘square’ morphology with $f_{x}/f_{y} = 2.27$ and $(f_{x} + f_{y})/f_{k} = 1$. It is only a part of the upper left ray in Fig. 8, left. Thus, having analysed the frequencies for all, and not just for some pre-selected orbits, we determined all the orbits of this type, and not of individual representatives of this group falling into specific region of the mentioned ray.

Voglis et al. (2007); Harsoula & Kalapotharakos (2009) provide examples of regular orbits belonging to this family, including 5:2 resonant orbits. These orbits indeed have a ‘square’ morphology (figure 22, 5:2, group A, Voglis et al. 2007; figure 7, groups A B Harsoula & Kalapotharakos 2009). The authors have used time series for each orbit, the length of which corresponds to a time interval of hundreds of radial periods. In our models, the orbits of this family slowly precess at a speed lower than the angular speed of the bar and lag behind it, so $f_{x}/f_{y} > 2$. The orbits of these type make less than 20 revolutions, without having time to draw a square in our case. However, an ensemble of particles with arbitrary initial phases has no problem to do it, which we precisely observe in Fig. 9. Gajda et al. (2016) gave an example of a typical quasi-periodic orbit with $(f_{x} + f_{y})/f_{k} = 1$ in their figure 6, row (c) and concluded that this type of orbit does not seem to support the bar. Such an orbit also resembles the orbit in figure A6 in Patsis & Athanassoula (2019). The second bottom left ray of bl$_{o}$ group of orbits, that with $f_{x}/f_{y} = 1$ (Fig. 8, left), was not explicitly identified in other works, but the orbits from this branch produce a square-like structure which is quite similar to that of the first branch according to our analysis. They differ only in that the orbits with $f_{x}/f_{y} = 1$ are assembled into a more extended structure, while those with $(f_{x} + f_{y})/f_{k} = 1$ seem to be responsible for a more compact configuration. Perhaps it is precisely bl$_{o}$ orbits that create a backbone of an unusual square-like morphology as in Fig. 9, bl$_{o}$ subplot.

9.2.2 The bl$_{o}$ orbits

A noticeable group (group B) of orbits $\epsilon/(\Omega - \Omega_{\odot}) = 1.67$ is visible in figure 21 in Voglis et al. (2007). Figure 22 (Voglis et al. 2007) shows that such orbits are rosettes with many loops. In general, such orbits are undeservedly deprived of attention in the studies of the bar internal structure, and we believe that they can be identified with our bl$_{o}$ orbits. A possible lack of interest in such orbits is due to the fact that this family becomes noticeable only when a compact bulge is added to the N-body model, as is clearly visible in Fig. 8, left and in Table 1. The orbital composition of such models was practically not studied. Apparently, this is the first time we have identified all the orbits of this type in N-body simulations.

Unlike the orbits lying in the bar, the orbits of this family have a pronounced second peak in the periodograms $x(t)$ and $y(t)$. Moreover, for most of the orbits, in addition to the equality $f_{x}/f_{y} = 1$, the ratio $f_{y}/f_{x}$ is also equal to 1, where $f_{x}$ and $f_{y}$ are frequencies of the secondary peaks. We also found that orbits from these groups satisfy the following equalities: $(f_{x} + f_{y})/f_{k} = (f_{x} + f_{y})/f_{k} = 1$, $\varphi_{x} - \varphi_{y} = \pi/2$, $\varphi_{0}^{(2)} - \varphi_{0}^{(3)} = -\pi/2$ where $\varphi_{0}^{(2)}$, $\varphi_{0}^{(3)}$ are initial phases. In general we found that orbits of bl$_{o}$ group are fairly accurately described by sum of two oscillators along x-axis and sum of two oscillators along y-axis (see Eq. (5)), and such an orbit is a rosette:

$$x(t) = A_{1} \cos(2\pi f_{x} t + \varphi_{x}) + A_{1}^{(2)} \cos(2\pi f_{x}^{(2)} t + \varphi_{0}^{(2)}) + A_{1}^{(3)} \cos(2\pi f_{x}^{(3)} t + \varphi_{0}^{(3)}), \quad A_{1} > A_{1}^{(2)},$$

$$y(t) = A_{2} \cos(2\pi f_{y} t + \varphi_{y}) + A_{2}^{(4)} \cos(2\pi f_{y}^{(4)} t + \varphi_{0}^{(4)}), \quad A_{2} > A_{2}^{(4)},$$

(5)

where $A_{1}$, $A_{2}$, $A_{1}^{(2)}$ and $A_{2}^{(4)}$ are the corresponding amplitudes.

For most of the orbits, the second frequency is less than the first one $f_{y} < f_{x}$, and the orbit precesses in the same direction (prograde orbits). As a rule, they are rapidly precessing orbits. There are very few orbits with $f_{y} > f_{x}$. These are orbits that precess very slowly in the opposite direction (retrograde orbits). Fig. 14 shows different examples of bl$_{o}$ orbits which we have found in the BL model. Typical rosettes are shown in the upper two rows. The plots in the middle depict the density profile produced by each orbit over a long time interval ($\approx 1$ Gyr). These plots are obtained in the following way. We colour each point with coordinates $(x,y)$ according to how many times a particle passes through this point during the considered time interval. Such plots directly reflect the morphology produced by a large number of the given orbits. The plots on the right are spectra of $x(t)$, $y(t)$ and $R(t)$ oscillations.

Valluri & Merritt (1998) associate the appearance of such orbits in the potential of a triaxial ellipsoid with an increase in average stochasticity of orbits when moving to models with strong density cusps. They give an example of such an orbit under the guise of a stochastic tube orbit in their figure 7. It is believed that such an orbit is the result of the evolution of an unstable resonant tube orbit like a trefoil (their figure 6). Stable resonant orbits avoid the centre of the potential. On opposite, the orbits that can pass near the cuspy destabilizing centre typically become stochastic. Judging by the spectra, typical orbits from the bl$_{o}$ orbits are not chaotic. These are rather circular orbits in the axisymmetric potential of a concentrated bulge, which are strongly perturbed in the radial direction, but which can still be described by the model of two oscillators (Eq. (5)).

They appear to be regular orbits in the axisymmetric potential of a concentrated bulge. Some of them are slightly elongated along the minor axis of a bar. Among them, there are also $n$-foil type of orbits that have an appearance of stable periodic orbits (rows 3-5 in Fig. 14): trefoils with a frequency ratio $3:2$, quatrefoils with $4:3$ and cinquefoils with $5:4$. They are not elongated either along the major or the minor axis of the bar. It only natural that a mix of different types of such orbits plus the regular orbits around them that have rosette-like shape appears as a roundish structure which we observe as a barlens. Although, we should stress out again that these orbits constitute only a part of a barlens. There is also bl$_{o}$ orbits that also seem to contribute to it.

10 DISCUSSION

Kormendy (2013) argues that lenses in many galaxies are defunct bars and suggests that bars evolve away into lenses due to the secular evolution that increases the central mass concentration so much that the bar orbits can no longer precess synchronously (Kormendy 1979). Some theoretical aspects of bar-to-lens evolution were reviewed in Combes (2008, 2011a,b). In
these works, the appearance of the lens is associated with the gas inflow and gradual formation of the central mass concentrations (CMC). It is indeed well known that the addition of a CMCs like a central black hole leads to a bar weakening and even its possible destruction (Norman et al. 1996), Combes (2011a) also emphasises an additional important aspect of bar destruction. Besides the gas inflow there is also an exchange of momentum between gas and bar. Gas gradually loses its momentum and gives it to the bar. Inside the co-rotation, the bar is a pattern with a negative angular momentum and this exchange makes the central orbits rounder and the bar weaker. Simulations show that the gas infall of 1-2% of the disc mass is enough to transform a bar in a lens (e.g., Berentzen et al. 1998; Bournaud & Combes 2002; Bournaud et al. 2005). The formation of a central lens on the background of still existing bar (Athanassoula et al. 2015) is apparently associated with the same mechanism: partial destruction of the bar by growing CMC.

We specifically traced the evolution of the lens-like structure supported by bl orbits from the very beginning of the simulations. We rewound it in time to the very beginning of the model evolution and it turned out that the lens-like structure is formed in situ, and the influx of matter into this structure from other areas practically does not occur (see Fig. 15). Thus, we can state that bl orbits are indeed scattered by the axisymmetric compact bulge, with only a small contribution of non-axisymmetric bar. These orbits slip away from the bar in different directions, taking the shape of rosettes even in a reference frame rotating at the speed of the bar. We should note that such a behaviour of orbits is expected in a presence of a strong matter concentration to some degree and was already perceived by Kormendy (2013). He stated that “once an orbit escapes from its alignment with the bar, it phase-mixes azimuthally in a short time” which is exactly what we observe.

Disassemby of the barlens in our models into separate orbital families with its own special morphology allows us to offer an explanation for some observational features that are found in galaxies with barlenses.

Laurikainen & Salo (2017) give examples of six galaxies with ‘bl’ (barlens) in their classification and a weak X-shaped feature in the unsharp-masked images. The most impressive examples are IC 1067 and NGC 4902 galaxies (Laurikainen & Salo 2017, their figure B.4, p. 54). The face-on views of models with the gas from Athanassoula et al. (2015); Laurikainen et al. (2014) demonstrate a barlens but almost all their models have traces of an X-like morphology. The same can be noted about the model by Salo & Laurikainen (2017) with a low-mass bulge of the classical type ($B/T = 0.01$) and the model by Smirnov & Sotnikova (2018) with a massive bulge ($B/T = 0.2$, figure 9). But Shen & Sellwood (2004) noted that for a given central mass, dense objects cause the greatest destructive effect on the bar, while significantly more diffuse objects have a lesser effect. That is why the model by Salo & Laurikainen (2017) with $B/T = 0.08$, but with the same effective radius as the model with $B/T = 0.01$, leads to the formation of a strong lens without traces of the face-on peanut. The same is true for our BLX and BL models.

Since the manifestation of a particular morphology is determined by the delicate balance between the populations of different orbital families, we can assume that in galaxies with barlenses and traces of X-shaped structures, viewed face-on, the central mass is not too compact and phase mixing has not yet occurred. The box-shaped part of the bar is squeezed with an increase in central density concentration, drowns in the orbital family forming the lens (first of all, bl), but still continues to shine through it.

Another tiny feature of the barlens structure that almost no one paid attention to in the literature is as follows. The external isophotes of some barlenses show rather square outlines with slightly rounded corners. The clear example is NGC 4314 galaxy from the sample by Laurikainen & Salo (2017). The authors depict elliptical isophotes for it (figure C.11, p. 83) but the image itself shows a lens of a box-shaped morphology (figure B.1, p. 34). NGC 4314 was considered in Athanassoula et al. (2015) (figure 1) but in context of an ordinary rounded lens embedded in a thin and much longer component. We can only assume that the bl, orbital family with ‘square’ morphology comes to the fore in such galaxies, and the orbital family bl is lost against its background or is confined in central regions.

Table 1 shows that in our models there are orbits, which we designated as x2-p. The population of this group of orbits increases with the transition to models with a central concentration. Apparently, this is a mixture of tube-shaped orbits of the x2-tree and x4-tree. The main difference between x2 and x4 orbits is the sign of the angular momentum around the $z$ axis: the former rotate in the same direction as the bar in the inertial reference frame,

![Figure 14](image-url)
and the latter rotate retrograde (Contopoulos & Papayannopoulos 1980). It is interesting to note that in the model X we found no x2 orbit, but there is a small number of x4 retrograde orbits. The same result was obtained in Valluri et al. (2016); Voglis et al. (2007), who used models with initial conditions that did not include a classical bulge. Moreover, in all considered models with a bulge, the inverse relation is observed, and x2 orbits are much more populated than x4 orbits. In the BL model, the structure supported by these orbits has a large extent in the vertical direction. There is a great temptation to identify the structure supported by these orbits with an inner bar perpendicular to the main bar because galaxies with bulges tend to have double bars de Lorenzo-Cáceres et al. (2019). However, even in the BL model, this structure has a large extent in the vertical direction and, apparently, makes an additional contribution to the lens.

Identification of the lens as a separate orbital structure helps to understand how it can contribute to the overall photometric profile. Unlike conventional photometric decomposition, for our models we can build a radial profile of the lens as it is, without the need to account for other components. Fig. 16 shows the surface density profile along the major axis of the bar for the BL model. The plot depicts an overall surface density profile, including a bulge, and an individual profile of the lens, which is the sum of $b_1$ and $b_2$ orbit groups. An outer exponential disc is clearly visible on the overall profile. A little hump near $r_0 = 3$ coincides with the ring location, while the elongated bar gives a characteristic shoulder at $r_0 = 1$. As for the lens, its surface brightness profile is very close to exponential. This is in good agreement with the results of the B/D/barlens/bar photometric decomposition of a number of galaxies (Laurikainen et al. 2018), as well as numerical models with bar lenses (Salo & Laurikainen 2017): the surface brightness profiles of bar lenses are nearly exponential.

11 CONCLUSIONS

We analysed the orbital composition of the stellar disc for four N-body models with bars that differ in mass and scales of the initial classical bulge. We focused on the morphology of the orbits in the disc plane. In full agreement with the results described in Salo & Laurikainen (2017), we found that as the central concentration of the bulge increases, the morphology of the face-on bar changes. A smooth transition occurs from the peanut-shaped bar to the bar with a bar lens.

We translated the results of our N-body simulations into the “language” of orbits and suggested a solution to a number of dynamic phenomena. To investigate the orbital structure of the studied models, we used the frequency analysis method pioneered by Binney & Spergel (1982). We applied FFT analysis to all particle orbits ($N = 4kk$) of the stellar discs of our models. Since the orbits that determine the long-lived structure in the stellar disc should be quasi-periodic, their spectra must consist of a discrete set of peaks. We calculated two main frequencies from the time series of Cartesian coordinates and a cylindrical radius for all of the orbits in the disc. Work of such a volume of studying the morphology of orbits in a bar before this was carried out only in Parul et al. (2020). Based on the obtained frequencies and their ratios, we have identified all particles that contribute to the entire bar in our models. Further, a classification scheme for orbital families based on two in-plane frequency ratios ($f_1/f_2$ and $f_3/f_4$) was extended and applied to all particles in the area of the bar. The analysis of ratios of different frequencies allowed us to distinguish all orbits that structure the phase space and are responsible for the morphological features of the so-called B/P bulges in our models. These orbits include those that are not elongated along the bar and constitute the lens.

A comparison of the populations of different families in different models showed that the peanut-shaped morphology of the bar is created mainly by box-shaped orbits located at the inner Lindblad resonance. This confirms the results of previous studies, based both on the analysis of a large number of orbits in “frozen” potentials and on the analysis of a sample of orbits in N-body potentials.

Using the “orbital language”, we have proposed an explanation of the difference between the two types of bars that stand out in a face-on view, bar lenses and peanut-shaped bars. Although our models demonstrate both types of bars and each of the bars contain many possible families of regular orbits, there is always a dominant one that is responsible for particular morphological features.

One of the most important results of our research is that the dominance of one or another orbital family is determined, first of all, by the potential of the galaxy. This initial idea turned out to be very fruitful and led us to the following conclusions.

(i) In galaxies with classical bulges, albeit of small mass, the family of box-shaped orbits decreases in number, and the structure formed by these orbits becomes shorter. At the same time, the x1-like orbital family, which support a long and narrow bar, is growing in number.

(ii) In potentials with a high matter concentration, two ‘non classic’ types of orbits come to the fore. These orbits do not support the bar, but apparently branch off from the well-known x1 family.

(iii) One of these orbital group ($b_1$, in our notation) maintains the rounded shape in the central area of the bar, turning the part of it into a lens. If before that there were only qualitative discussions about the possible types of orbits that inhabit the bar lens, now both the structure itself and the families of orbits supporting it are directly highlighted in the N-body models.

(iv) The motion of stars belonging to this orbital family is most affected by the bulge potential. It is the compact bulge that scatters the orbits in the central regions and prevents the capture of fast-precessing orbits in a bar. We believe that this orbital group is a key component of a bar lens, without which it is impossible to obtain its characteristic roundish isophotes, which are observed in many galaxies with a bar lens. In terms of orbits shape, such orbits are very simple, but they have not explicitly stood out in such studies.

(v) The orbit language, which made it possible to distinguish the lens as a separate structure, can be useful in constructing photometric models of galaxies. A modelled density profile could be inserted into packages for the photometric decomposition of galaxies. Dividing the bar into separate families will help to further study the difference in the vertical structure of the lens and the rest of the B/PS bulge. And finally, the orbital composition of the lens opens up great opportunities for creating observational kinematic tests and further study of galaxies with bar lenses.

(vi) We also proposed an explanation of the unusual morphology of some of the galaxies with bar lenses. For example, traces of X-shaped structures observed in galaxies with bar lenses in unsharp-masked images can indicate that the structure of a bar lens is determined by the delicate balance between the populations of different orbital families, in particular, by the role of the box-shaped part of the bar. This is also evidenced by...
the existence of galaxies with barlenses, which are not so much rounded as ‘square’-like isophotes. Perhaps an ‘extended’ family bl, is responsible for such isophotes.

(vii) Finally, although the existence of x2-tree orbits in the central areas of the bar has been known for a long time, we were able to completely isolate the structure supported by these orbits. The length of this structure in the direction of the bar minor axis is quite large, and it seems to form two protrusions against the background of an elongated bar. We also showed that the population of this family in our models increases with the transition to galaxies with a classical bulge.

DATA AVAILABILITY
The data underlying this article will be shared on reasonable request to the corresponding author.

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Figure 15. Time evolution of a structure assembled from particles that make up bl, group captured at several moments from 0 to 450 t. u. (≈6 Gyr). Face-on views are displayed in the square (xy) = (−2, 2) × (−2, 2).

Figure 16. Surface density radial profiles of the total model and lens along the bar major axis. The width of a slit is 0.12 length units.
APPENDIX A: EXTRACTING ELONGATED ORBITS FROM THE FAMILIES

When analysing the orbits supporting the barlens, we were faced with a situation where a small number of orbits elongated along the major axis of the bar were mixed with the main ensemble of orbits, which manifested itself as a roundish structure. Apparently, these elongated orbits are the orbits that have not yet had time to scatter into the bulge potential. They cannot be considered part of the barlens and thus should be excluded from blu scatter into the bulge potential. They cannot be considered part of these elongated orbits are the orbits that have not yet had time to or, as manifested itself as a roundish structure. Apparently, with a situation where a small number of orbits elongated along the barlens, we were faced from the families.

Figure A1. 2D distribution of orbits in blu (left panel) and blo (right panel) families on $\langle Ry \rangle / \langle |x| \rangle$ plane for model BL calculated for the time interval $t = 400 - 500$.

corresponding to the groups described above. The leftmost island is constituted by orbits that are close to the centre and contribute to the lens, while the rightmost one corresponds to the elongated orbits described above. For blu orbits, the groups are well separated for all models, and finding an appropriate delimiting line is not a problem. However, for blu, these two parts are connected by an "isthmus". In this case, we draw the border approximately along the middle of the isthmus. In all cases, we exclude the area below and to the right of the boundary, described by the linear equation $\langle |y| \rangle / \langle |x| \rangle = \alpha \langle R \rangle + \beta$. 

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