On Baker type lower bounds for linear forms

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Abstract

We wish to give an axiomatic approach to (explicit) Baker type lower bounds for linear forms, over the ring $\mathbb{Z}_I$ of an imaginary quadratic field $I$, of given numbers $1, \Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$. In this work we are interested in simultaneous auxiliary functions case.

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1 Introduction

We wish to give an axiomatic approach to (explicit) Baker type lower bounds for linear forms of given numbers $\Theta_0, \ldots, \Theta_m \in \mathbb{C}^*$. Throughout this work, let $I$ denote an imaginary quadratic field with $\mathbb{Z}_I$ it’s ring of integers. By an explicit Baker type lower bound we mean any positive lower bound

$$|\beta_0\Theta_0 + \ldots + \beta_m\Theta_m| > F(H_0, \ldots, H_m, m)$$

valid for all $\beta = (\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}_I^{m+1} \setminus \{0\}$

with

$$\prod_{j=0}^m H_j \geq \hat{H} \geq 1, \quad H_j \geq h_j = \max\{1, |\beta_j|\},$$

where the dependence on each individual term $H_0, \ldots, H_m, m$ and numbers $\Theta_0, \ldots, \Theta_m$ is explicitly given in the functional dependence $F(H_0, \ldots, H_m, m)$ and the dependence on $\Theta_0, \ldots, \Theta_m, m$ is explicitly given in the constant $\hat{H} = \hat{H}(\Theta_0, \ldots, \Theta_m, m)$.

There seems to be demand to the above mentioned lower bounds (1) e.g. in the theory of uniformly distributed sequences, see [5]. Recently, we have seen a growing interest for such estimates also in the theory of MIMO-codes, see Lahtonen et al. [7] and Motahari et al. [10].

With the assumption that $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{Q}^*$ are distinct Baker [11] proved that there exist positive constants $\delta_1, \delta_2$ and $\delta_3$ such that

$$|\beta_0 e^{\gamma_0} + \ldots + \beta_m e^{\gamma_m}| > \frac{\delta_1 M^{1-\delta(M)}}{\prod_{j=0}^m h_j},$$

(2)

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for all\n$$\overline{\beta} = (\beta_0, ..., \beta_m)^T \in \mathbb{Z}^m \setminus \{0\}, \quad h_j = \max\{1, |\beta_j|\},$$
with\n$$\delta(M) \leq \frac{\delta_2}{\sqrt{\log \log M}}, \quad M = \max_{0 \leq j \leq m}\{|\beta_j|\} \geq \delta_3 > e. \quad (3)$$

Here we note that the constants $\delta_1, \delta_2, \delta_3$ in Baker’s work (2) are not explicitly given. Mahler [9] made Baker’s result completely explicit.

There are works, where the authors prove Baker type lower bounds for certain collection of numbers, say, values of functions belonging to class of $E$-, $G$- or $q$-hypergeometric functions evaluated at rational points. Usually the method is involved with a first or second kind Padé-approximation construction of certain auxiliary functions. The construction of a corresponding linear form (one auxiliary function) or simultaneous linear forms (several auxiliary functions) may be explicit or given by Siegel’s lemma. For definition of Siegel’s $E$- and $G$-functions we refer to [5]. The series

$$\sum_{n=0}^{\infty} q^{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{\prod_{j=1}^{n} (1 - q^j)}, \quad |q| < 1, \quad (4)$$

are a typical $q$-series. Väänänen and Zudilin [16] proved a first Baker type lower bound for certain class of $q$-series including the above series (4).

In this work we are interested in simultaneous auxiliary functions case. Fix $\Theta_1, ..., \Theta_m \in \mathbb{C}^*$ and put $\overline{n} = (n_1, ..., n_m)^T$, $N = N(\overline{n}) = n_1 + ... + n_m$. Assume that we have a sequence of simultaneous linear forms

$$L_{k,j}(\overline{n}) = A_{k,j}(\overline{n}) \Theta_j + A_{k,j}(\overline{n}), \quad k = 0, 1, ... m, \quad j = 1, ..., m, \quad \overline{n} \in \mathbb{Z}_d^m \geq 1,$$

where $A_{k,j} = A_{k,j}(\overline{n}) \in \mathbb{Z}_d$ satisfy certain determinant condition. Suppose also that

$$|A_{k,0}(\overline{n})| \leq e^{(aN + b \log N)g(N) + b_0 N(\log N)^1/2 + b_1 N + b_2 \log N + b_3}, \quad (5)$$

$$|L_{k,j}(\overline{n})| \leq e^{(dN - cn_j)g(N) + c_0 N(\log N)^1/2 + c_1 N + c_2 \log N + c_3}, \quad (6)$$

for $k, j = 0, 1, ..., m$, where $a, b, c, d, b_i, c_i$ are non-negative parameters satisfying $a, c - dm > 0$. Then, in the following special cases $g(N) \in \{1, \log N, N\}$, we shall prove that there exist explicit positive constants $F_i, G_i$ ($i \in \{1, 2, 3\}$), such that

$$|\beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m| > F_i \left( \prod_{j=1}^{m} (2mH_j) \right) - \frac{m}{e^{-dm} - \alpha(H)} \quad (7)$$

holds for all $\overline{\beta} = \{\beta_0, \beta_1, ..., \beta_m\} \in \mathbb{Z}_d^{m+1} \setminus \{0\}$ and

$$H = \prod_{j=1}^{m} (2mH_j) \geq G_i, \quad H_j \geq h_j = \max\{1, |\beta_j|\} \quad (8)$$

with an error term $\epsilon_i(H) \rightarrow 0.$
We shall study more closely the above three special cases where $F_l, G_l$ and the error term will be given explicitly in terms of the parameters $a, b, c, d, b_i, e_i$. Our result is axiomatic and we may expect the cases $g(N) \in \{1, \log N, N\}$ to be involved with $G_-, E$- and $q$-hypergeometric functions, respectively. It seems that our work is the first attempt to build an axiomatic theory for simultaneous linear forms case. Also, we note that our method is considerably different from the proofs presented so far, see e.g [1], [9], [11], [16] and [17].

An advantage of our method is that one may easily see if the contribution to the lower bound is coming from the Diophantine method itself or from the auxiliary construction. In particular, when one has the above estimates (5)–(6) then the explicit depends on the parameters, in particular on $m$, is readily visible.

So far our axiomatic method is already applied in Leinonen’s work [8] and in a joint work [4] with A.-M. Ernvall-Hytönen and K. Leppälä. Leinonen [8] proves explicit Baker type results for a class of $q$-hypergeometric series (the case $g(N) = N$) improving and generalizing results of Väänänen and Zudilin [16]. The next application of our axiomatic method is our joint work [4] with A.-M. Ernvall-Hytönen and K. Leppälä concerning values of exponential function (the case $g(N) = \log N$). The work [4] gives a considerably improvement to the explicit versions, see Mahler [9] and Sankilampi [11], of Baker’s work [1] about exponential values at rational points. In particular, dependences on $m$ are improved.

2 Background from metrical theory

From the general metrical theory, see [2], [3], [5], [12], [13] we get the following well known results.

Let $1, \Theta_1, \ldots, \Theta_m \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then there exist infinitely many primitive vectors

$$(\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\}$$

with

$$h_j := \max\{1, |\beta_j|\}, \quad j = 1, \ldots, m,$$

satisfying

$$|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| < \frac{1}{\prod_{j=1}^m h_j}.$$  \hfill (9)

To the other direction we have. If there exist positive constants $c, \omega \in \mathbb{R}^+$ such that

$$|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| > \frac{c}{\left(\prod_{j=1}^m h_j\right)^\omega}$$ \hfill (10)

holds for all

$$(\beta_0, \ldots, \beta_m)^T \in \mathbb{Z}^{m+1} \setminus \{0\}$$

then

$$\omega \geq 1.$$ \hfill (11)
Sprindzhuk [15] proved the following. Let \( \epsilon > 0 \) be given. Then, in the sense of Lebesque measure, for almost all \( \Theta = t(\Theta_1, ..., \Theta_m) \in \mathbb{R}^m \setminus \{0\} \)
there exists a positive constant \( C = C(\Theta, \epsilon) \) such that
\[
|\beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m| > \frac{C}{(\log(1 + M))^{m+\epsilon} \prod_{j=1}^m h_j},
\]
for all \( \overline{\beta} = t(\beta_0, \beta_1, ..., \beta_m) \in \mathbb{Z}^{m+1} \setminus \{0\} \)
with \( h_j = \max\{1, |\beta_j|\}, \quad M = \max\{h_j\} \).

In the complex case Shidlovskii [13] studies linear forms over the ring of rational integers and gives the following

**Theorem 2.1.** [13] Let
\[
\Theta_0 = 1, \Theta_1, ..., \Theta_m \in \mathbb{C}
\]
and
\[
H \in \mathbb{Z}_{\geq 1}
\]
be given. Then there exists a non-zero rational integer vector
\[
(\beta_0, \beta_1, ..., \beta_m) \in \mathbb{Z}^{m+1} \setminus \{0\}, \quad |\beta_j| \leq H, \quad \forall j = 0, 1, ..., m,
\]
satisfying
\[
|\beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m| \leq \frac{c}{H^{(m-1)/2}}, \quad c = \sqrt{2} \sum_{j=0}^m |\Theta_j|.
\]

We are interested in linear forms over the ring of integers \( \mathbb{Z}_I \) in an imaginary quadratic field \( \mathbb{Q}(\sqrt{-D}) \), \( D \in \mathbb{Z}^+, \ D \not\equiv 0 \pmod{4} \). For that purpose we prove

**Theorem 2.2.** Let
\[
\Theta_1, ..., \Theta_m \in \mathbb{C}
\]
and
\[
H_1, ..., H_m \in \mathbb{Z}_{\geq 1}
\]
be given. Then there exists a non-zero integer vector
\[
(\beta_0, \beta_1, ..., \beta_m) \in \mathbb{Z}_I^{m+1} \setminus \{0\}, \quad |\beta_j| \leq H_j, \quad \forall j = 1, ..., m,
\]
satisfying
\[
|\beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m| \leq \left( \frac{2\tau D^{1/4}}{\sqrt{\pi}} \right)^{m+1} \frac{1}{H_1 \cdots H_m},
\]
where \( \tau = 1, \text{ if } D \equiv 1 \text{ or } 2 \pmod{4} \) and \( \tau = 1/2, \text{ if } D \equiv 3 \pmod{4} \).
3 Main theorem

Fix $\Theta_1, \ldots, \Theta_m \in \mathbb{C}^*$. Let $\mathbb{I}$ denote an imaginary quadratic field and $\mathbb{Z}_\mathbb{I}$ its ring of integers. Put

$$\pi = t(n_1, \ldots, n_m), \quad N = N(\pi) = n_1 + \ldots + n_m. \quad (17)$$

Assume that we have a sequence of simultaneous linear forms

$$L_{k,j}(\pi) = A_{k,0}(\pi)\Theta_j + A_{k,j}(\pi), \quad (18)$$

where

$$A_{k,j} = A_{k,j}(n) \in \mathbb{Z}_\mathbb{I}, \quad k, j = 0, 1, \ldots, m, \quad (19)$$

satisfy a determinant condition, say,

$$\Delta = \begin{vmatrix} A_{0,0} & A_{0,1} & \ldots & A_{0,m} \\ A_{1,0} & A_{1,1} & \ldots & A_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,0} & A_{m,1} & \ldots & A_{m,m} \end{vmatrix} \neq 0 \quad (20)$$

Further, let

$$a, b, c, d, b_i, e_i \in \mathbb{R}_{\geq 0}, \quad a, c, c - d m > 0 \quad (21)$$

and suppose that

$$|A_{k,0}(\pi)| \leq Q(\pi) = e^{q(N)}, \quad |L_{k,j}(\pi)| \leq R_{j}(\pi) = e^{-r_{j}(\pi)}, \quad (22)$$

where

$$q(N) = (aN + b \log N)g(N) + b_0 N(\log N)^{1/2} + b_1 N + b_2 \log N + b_3, \quad (24)$$

$$- r_{j}(\pi) = (dN - c n_j)g(N) + e_0 N(\log N)^{1/2} + e_1 N + e_2 \log N + e_3, \quad (25)$$

for all $k, j = 0, 1, \ldots, m$.

Let the above assumptions be valid from $N \geq N_l$, in each our case $l = 1, 2, 3$, where the enumeration corresponds to the following three cases.

1. Typical examples are logarithm and Gauss hypergeometric functions.
2. Typical examples are exponential and Bessel functions.
3. Typical examples are $q$-series.

It has turned out that in applications we often meet the following situation in the above mentioned cases.

1. Here $g(N) = 1$ and

$$q(N) = a N + b \log N, \quad (26)$$

$$- r_{j}(\pi) = d N - c n_j + e_2 \log N. \quad (27)$$

2. Here $g(N) = \log N$ and $b = 0$. So

$$q(N) = a N \log N + b_0 N(\log N)^{1/2} + b_1 N + b_2 \log N + b_3, \quad (28)$$

$$- r_{j}(\pi) = (dN - c n_j) \log N + e_0 N(\log N)^{1/2} + e_1 N + e_2 \log N + e_3, \quad (29)$$

3. Here $g(N) = N$ and for simplicity we consider only the case $b = b_0 = b_2 = b_3 = e_0 = e_2 = e_3 = 0$. Thus

$$q(N) = a N^2 + b_1 N, \quad (30)$$

$$- r_{j}(\pi) = (dN - c n_j) N + e_1 N. \quad (31)$$
Theorem 3.1. Under the above assumptions (17)–(31) there exist explicit positive constants $F_l$ and $G_l$ not depending on $H$ such that

$$|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| > F_l \left( \prod_{j=1}^{m} (2mH_j) \right)^{-\frac{a}{c-dm} - \epsilon_l(H)}$$

(32)

for all

$$\overline{\beta} = (\beta_0, \beta_1, ..., \beta_m) \in \mathbb{Z}_l^{m+1} \setminus \{0\}$$

and

$$H = \prod_{j=1}^{m} (2mH_j) \geq G_l, \quad H_j \geq h_j = \max\{1, |\beta_j|\}$$

(33)

with an error term

$$\epsilon_l(H) \to 0.$$ 

Now we consider more closely our three cases, where the notation $f = \frac{2}{c-dm}$ will be used.

Theorem 3.2. Case 1. Denote

$$A_1 = \frac{a(dm^2 + e_2m)}{c-dm} + am + b + \log f, \quad B_1 = e_2m + b.$$ 

Then

$$F_1^{-1} = 2e^{A_1}, \quad \epsilon_1(H) = B_1 \frac{\log \log H}{\log H}.$$ 

for all

$$H \geq G_1 = \max\{m, N_1, e^{x_1/f}\}, \quad x_1 = \max\{S_1, 1\},$$

(34)

where $S_1$ is the largest solution of the equation

$$S = f(e_2m \log S + dm^2 + e_2m).$$

(35)

Before going to next cases we introduce a function $z : \mathbb{R} \to \mathbb{R}$, the inverse function of the function $y(z) = z \log z$ when $z \geq 1/e$ considered in [6].

Lemma 3.3. [6] The inverse function $z(y)$ of the function

$$y(z) = z \log z, \quad z \geq \frac{1}{e},$$

(36)

is strictly increasing. Define $z_0(y) = y$ and $z_n(y) = \frac{y}{\log z_{n-1}}$ for $n \in \mathbb{Z}^+$. Suppose $y > e$, then we have

$$z_1 < z_3 < \ldots < z < \ldots < z_2 < z_0.$$ 

Thus the inverse function may be given by the infinite nested logarithm fraction

$$z(y) = z_\infty = \frac{y}{\log \frac{y}{\log y} \ldots}, \quad y > e.$$

(37)
Theorem 3.4. Case 2. Denote now

$$A_2 = b_0 + \frac{ae_0m}{c - dm}, \quad B_2 = a + b_0 + b_1 + \frac{ae_1m}{c - dm},$$

$$C_2 = am + b_2 + \frac{a(dm^2 + e_2m)}{c - dm}, \quad D_2 = b_0m + \frac{ae_0m^2}{c - dm},$$

$$E_2 = (a + b_0 + b_1)m + b_2 + b_3 + \frac{a((2e_0 + e_1)m^2 + (e_2 + e_3)m)}{c - dm}.$$

Here

$$F_2^{-1} = 2e^{E_2}$$

and

$$\epsilon_2(H) = \xi(z, H) :=
A_2 \left( f \frac{z(f \log H)}{\log H} \right)^{1/2} + B_2 \frac{z(f \log H)}{\log H} + C_2 \frac{\log z(f \log H)}{\log H} + D_2 \frac{(\log z(f \log H))^{1/2}}{\log H}$$

for all

$$H \geq G_2 = \max\{m, N_2, e^{(x_2 \log x_2)/f}, e^{e/f}\}, \quad x_2 = \max\{S_2, 1\},$$

where $S_2$ is the largest solution of the equation

$$S \log S = f(e_0mS(\log S)^{1/2} + e_1mS + (dm^2 + e_2m) \log S + e_0m^2(\log S)^{1/2} \log S.\)$$

(40)

In this case the estimate (32) may be written as follows

$$|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \geq F_2 \left( z(f \log H) \right)^{-C_2} \frac{H^{\frac{n}{c - dm} - A_2 (f \frac{z(f \log H)}{\log H})^{1/2} - B_2 \frac{z(f \log H)}{\log H} - C_2 \frac{\log z(f \log H)}{\log H})^{1/2}}{\log H}.$$

(41)

Note, that

$$z(f \log H) < z_2(f \log H)$$

by (37) and thus

$$\epsilon_2(H) = \xi(z, H) < \xi(z_2, H).$$

(43)

Write now

$$\rho_2 = \rho_2(x_0) = \frac{\log x_0}{\log x_0 - \log \log x_0}.$$

Then (42) may further be estimated by using

$$z_2(f \log H) \leq \rho_2 f \left( 1 - \frac{\log f}{\log(f \log H)} \right) \frac{\log H}{\log \log H} <$$

$$\rho_2 f \frac{\log H}{\log \log H}$$

valid for all

$$f \log H \geq x_0 \geq e^e.$$

(46)

Hence by using the estimate (45) we get a more familiar looking lower bound in the next Corollary but weaker than (41) and that with (42) or (44).
Corollary 3.5. If
\[ f \log H \geq x_0 := \max\{f \log m, f \log N_2, x_2 \log x_2, e^x\}, \]  
then
\[ |\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \geq \frac{1}{2e^{E_2(f \rho)\rho}} \left( \frac{\log \log H}{\log H} \right)^{C_2} H^{-\frac{a}{c-dm}} - \frac{A_2 f \sqrt{\sigma}}{\log \log H} - \frac{B_2 f \sigma}{\log H} \sqrt{\log\left(\frac{f \log \log H}{\log \log H}\right)}, \]  
with
\[ \rho = \frac{\log x_0}{\log x_0 - \log \log x_0}. \]

The presentation (48) is a step towards what we may expect to see in the works considering lower bounds for linear forms of the values of \( E \)-functions. However, usually only the main error term i.e. the term corresponding to \( A_2 \) is given including implicitly the terms with \( B_2 \) and \( C_2 \). Hence in such situation explicit dependence on the parameters, say for example on \( m \), may become invisible. Next we like to mention that methods using Thue-Siegel’s Lemma seem to yield to the situation where \( A_2 \neq 0 \), which corresponds to main error term. However, if it were \( A_2 = 0 \), then the terms with \( B_2 \) and \( C_2 \) would come important. Consequently, it would be of great interest to find explicit Padé type approximations which would yield to the case \( A_2 = 0 \).

Theorem 3.6. Case 3. Now we have
\[ F_3^{-1} = 2e^{B_3}, \quad e_3(H) = A_3 \frac{1}{\sqrt{\log H}}, \]  
for all
\[ H \geq G_3 = \max\{m, N_3, e\}, \]  
where general \( A_3 \) and \( B_3 \) are given in the proof section. In a particular case, \( b_1 = e_1 = 0 \), they read
\[ A_3 = \frac{1}{\sqrt{c-dm}} \left( \frac{adm^2}{c-dm} + 2am \right), \]  
\[ B_3 = \frac{dm^2}{c-dm} \left( \frac{adm^2}{c-dm} + 2am \right) + am^2. \]

4 Proofs

4.1 Proof of Theorem [2.2]

For \( D \not\equiv 0 \pmod{4} \) the ring of integers may be given by
\[ \mathbb{Z}_1 = \mathbb{Z} + \mathbb{Z}(h + l \sqrt{-D}) \]  
with \( h = 0, l = 1 \), if \( D \equiv 1 \) or 2 \( \pmod{4} \) and \( h = l = 1/2 \), if \( D \equiv 3 \pmod{4} \).

We start with a simple principle. First we define a lattice
\[ \lambda = \mathbb{Z}(1, 0) + \mathbb{Z}(h, l \sqrt{D}), \quad \det \lambda = \frac{\sqrt{D}}{2h}, \]  
\[ 8 \]
and a complex disk

\[ \mathcal{D}_R = \{ x + y(h + l\sqrt{-D}) \in \mathbb{C} \mid x, y \in \mathbb{R}, |x + y(h + l\sqrt{-D})| \leq R \} \]

with a radius \( R > 0 \) and a corresponding real disk

\[ \mathcal{C}_R = \{ (v, w) \in \mathbb{R}^2 \mid v^2 + w^2 \leq R^2 \}, \quad \text{Vol } \mathcal{C}_R = \pi R^2. \]

Then

\[ x + y(h + l\sqrt{-D}) \in \mathcal{D}_R \cap \mathbb{Z}_l \iff (x + yh, yl\sqrt{D}) \in \mathcal{C}_R \cap \lambda. \tag{54} \]

Next we define a lattice

\[ \Lambda = (\mathbb{Z}(1, 0) + \mathbb{Z}(h, l\sqrt{D}))^{m+1}, \quad \det \Lambda = \left( \frac{\sqrt{D}}{2^{2h}} \right)^{m+1}. \tag{55} \]

Denote

\[ a + b(h + l\sqrt{-D}) = -(z_1 \Theta_1 + \ldots + z_m \Theta_m), \quad z_k = x_k + y_k(h + l\sqrt{-D}), \]

\[ v_k = x_k + y_kh, \quad w_k = y_kl\sqrt{D}, \quad x_k, y_k \in \mathbb{R}, \quad k = 0, 1, \ldots, m, \]

and define the following sets

\[ \mathcal{D} = \{ (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1} \mid |z_0 - (a + b(h + l\sqrt{-D}))| \leq R_0; \quad |z_k| \leq H_k, \quad k = 1, \ldots, m \}, \]

and

\[ \mathcal{C} = \{ (v_0, w_0, v_1, \ldots, v_m, w_m) \in \mathbb{R}^{2m+2} \mid (v_0 - (a + lh))^2 + (w_0 - bl\sqrt{D})^2 \leq R_0^2, \quad v_k^2 + w_k^2 \leq H_k^2, \quad k = 1, \ldots, m \}, \]

where

\[ R_0 := \left( \frac{2\tau D^{1/4}}{\sqrt{\pi}} \right)^{m+1} \frac{1}{H_1 \cdots H_m}. \tag{56} \]

First we note that \( \mathcal{C} \) is a symmetric convex body. For the volume of \( \mathcal{C} \) we get

\[ \text{Vol } \mathcal{C} = \int \ldots \int \left( \int \int \ldots \int \frac{dv_0 dw_0}{(v_0 - (a + bh))^2 + (w_0 - bl\sqrt{D})^2 \leq R_0^2} \right) dv_1 dw_1 \ldots dv_m dw_m = \]

\[ \pi R_0^2 \int \ldots \int \left( \int \int \ldots \int \frac{dv_1 dw_1}{v_1^2 + w_1^2 \leq H_1^2} \right) dv_2 dw_2 \ldots dv_m dw_m = \]

\[ \ldots = \pi^{m+1} H_1^2 \cdots H_m^2 R_0^2 = \pi^{m+1} H_1^2 \cdots H_m^2 \left( \frac{2\tau D^{1/4}}{\pi} \right)^{m+1} \frac{1}{H_1^2 \cdots H_m^2} = 2^{2m+2} \left( \frac{\sqrt{D}}{2^{2h}} \right)^{m+1} = 2^{2m+2} \det \Lambda. \tag{57} \]
Thus by Minkowski’s convex body theorem, see [12], there exists a non-zero lattice vector
\((x_0 + y_0 h, y_0 l \sqrt{D}, \ldots, x_m + y_m h, y_m l \sqrt{D}) \in C \cap \Lambda \setminus \{0\}\). \hfill (58)

Consequently, by the above principle \(54\), we get a non-zero integer vector
\((\beta_0, \beta_1, \ldots, \beta_m) = (x_0 + y_0 (h + l \sqrt{-D}), \ldots, x_m + y_m (h + l \sqrt{-D})) \in D \cap \mathbb{Z}_{m+1} \setminus \{0\}\)

with
\(|\beta_k| \leq H_k, \quad \forall k = 1, \ldots, m,\) \hfill (59)

and
\(|\beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m| \leq \left(\frac{2^\tau D^{1/4}}{\sqrt{\pi}}\right)^{m+1} \frac{1}{H_1 \cdots H_m}.\) \hfill (60)

### 4.2 Proof of Theorems 3.1–3.6

We start our proof with a classical way and then we will shortly describe a basic idea needed for Baker type estimates.

### 4.3 A classical part

We use the notation
\(\Lambda := \beta_0 + \beta_1 \Theta_1 + \ldots + \beta_m \Theta_m, \quad \beta_j \in \mathbb{Z}_d\)

for the linear form to be estimated. Using our simultaneous linear forms
\(L_{k,j}(\overline{\pi}) = A_{k,0}(\overline{\pi}) \Theta_j + A_{k,j}(\overline{\pi}),\)

see [13], we get
\(A_{k,0} \Lambda = G_k + \beta_1 L_{k,1}(\overline{\pi}) + \ldots + \beta_m L_{k,m}(\overline{\pi}),\) \hfill (61)

where
\(G_k = G_k(\overline{\pi}) = A_{k,0}(\overline{\pi}) \beta_0 - \beta_1 A_{k,1}(\overline{\pi}) - \ldots - \beta_m A_{k,m}(\overline{\pi}) \in \mathbb{Z}_d.\) \hfill (62)

If now \(G_k \neq 0\), then by \(61\) and \(62\) we get
\[1 \leq |G_k| = |A_{k,0} \Lambda - (\beta_1 L_{k,1} + \ldots + \beta_m L_{k,m})| \leq \]
\[|A_{k,0}| |\Lambda| + \sum_{j=1}^{m} |\beta_j| |L_{k,j}| \leq Q(\overline{\pi}) |\Lambda| + \sum_{j=1}^{m} H_j R_j(\overline{\pi}).\] \hfill (63)

Here we should have, say
\[\sum_{j=1}^{m} H_j R_j(\overline{\pi}) \leq \frac{1}{2},\] \hfill (64)

in order to get a lower bound
\[1 \leq 2 |\Lambda| Q(\overline{\pi})\] \hfill (65)

for our linear form \(\Lambda\).
4.4 Basic idea roughly

Here we outline a rough version of the proof by studying the case $b = b_1 = b_2 = b_3 = e_0 = e_1 = e_2 = e_3 = 0$, for simplicity. It starts by fixing the remainders and heights:

$$H_j R_j(N) = \frac{1}{2m} \iff 2mH_j = e^{r_j(\pi)} = e^{(-dN + cn_j)g(N)} \Rightarrow \quad (66)$$

$$e^{-dmN + \sum_{j=1}^m n_j)g(N)} = e^{(c-dm)N g(N)} = \prod_{j=1}^m (2mH_j) \Rightarrow$$

$$Q(N) = e^{aNg(N)} = \left( \prod_{j=1}^m (2mH_j) \right)^{-\frac{1}{2d}} \Rightarrow$$

$$1 \leq 2|\Lambda|Q(N) = 2|\Lambda| \left( \prod_{j=1}^m (2mH_j) \right)^{-\frac{1}{2d}}.$$

4.5 Tuning

Now a direct generalization of (66) would be

$$r_j(\pi) = \log(2mH_j), \quad (67)$$

where

$$r_j(\pi) = (-dN + cn_j)g(N) - e_0 N (\log N)^{1/2} - e_1 N - e_2 \log N - e_3.$$  

However, (67) will be too rough and thus we tune it into right frequency by defining

$$B_j = \log(2me^{dmg(S)+e_0 m((\log S)^{1/2} + 2) + e_1 m + e_2 H_j}) \quad (68)$$

and stating a new system of equations

$$\sum_{j=1}^m s_j = S, \quad (69)$$

$$r_j(\pi) = B_j, \quad j = 1, ..., m. \quad (70)$$

Here (70) reads

$$(-dS + cs_j)g(S) - e_0 S (\log S)^{1/2} - e_1 S - e_2 \log S - e_3 =$$

$$\log(2mH_j) + dmg(S) + e_0 m((\log S)^{1/2} + 2) + e_1 m + e_2$$

which by (69) gives

$$(c - dm)S g(S) - e_0 m S (\log S)^{1/2} - e_1 m S - e_2 m \log S - e_3 m =$$

$$\log \prod_{j=1}^m (2mH_j) + dm^2 g(S) + e_0 m^2 ((\log S)^{1/2} + 2) + e_1 m^2 + e_2 m. \quad (71)$$
Hence $S$ is a solution of the equation
\[(c - dm)S - dm^2 g(S) - e_0 m S (\log S)^{1/2} - e_1 m S - e_2 m \log S - e_3 m\]
and consequently
\[s_j = \frac{B_j/g(S) + dS}{c}.
\]
Put $\sigma_j = \lfloor s_j \rfloor$ and write $\bar{\sigma} = \ell(\sigma_1, ..., \sigma_m)$, $\bar{1} = \ell(1, ..., 1)$, then
\[\bar{\sigma} \leq \sigma < \bar{\sigma} + 1.
\]
First we note that
\[N(\bar{\sigma} + 1) = N(\bar{\sigma}) + m \leq S + m.
\]
Write also
\[M_j(\bar{n}) = -dN(\bar{n}) + cn_j, \quad j = 1, ..., m,
\]
so that
\[r_j(\bar{n}) = M_j(\bar{n}) g(N(\bar{n})) - e_0 N(\bar{n})(\log N(\bar{n}))^{1/2} - e_1 N(\bar{n}) - e_2 \log N(\bar{n}) - e_3,
\]
Then we have
\[M_j(\bar{\sigma} + 1) = M_j(\bar{\sigma}) + l(c - dm)
\]
which shows, that $M_j(\bar{\sigma} + 1)$ is increasing with $l \in \mathbb{N}$. In particular, we have
\[M_j(\bar{\sigma}) < M_j(\bar{\sigma} + \bar{1}).
\]
But, note that we have only
\[M_j(\bar{\sigma}) < M_j(\bar{\sigma} + \bar{1}) + dm
\]
For simplicity from now on we suppose that
\[S \geq m.
\]
Further, because $g(x)$ is increasing, we get
\[r_j(\bar{n}) = M_j(\bar{n}) g(S) - e_0 S (\log S)^{1/2} - e_1 S - e_2 \log S - e_3 <
\]
\[(M_j(\bar{\sigma} + \bar{1}) + dm) g(S) - e_0 S (\log S)^{1/2} - e_1 S - e_2 \log S - e_3 \leq
\]
\[M_j(\bar{\sigma} + \bar{1}) g(N(\bar{\sigma} + \bar{1})) + dm g(S) - e_0 S (\log S)^{1/2} - e_1 S - e_2 \log S - e_3 =
\]
\[r_j(\bar{\sigma} + \bar{1}) + dm g(S) + e_0 N(\bar{\sigma} + \bar{1}) (\log N(\bar{\sigma} + \bar{1}))^{1/2} +
\]
\[e_1 N(\bar{\sigma} + \bar{1}) + e_2 \log N(\bar{\sigma} + \bar{1}) - e_0 S (\log S)^{1/2} - e_1 S - e_2 \log S.
\]
Hence
\[r_j(\bar{n}) < r_j(\bar{\sigma} + \bar{1}) + dm g(S) + e_0 m ((\log S)^{1/2} + 2) + e_1 m + e_2,
\]which is the reason to define (68).
According to the non-vanishing of the determinant (20) and the assumption 

$$\beta = t(\beta_0, \beta_1, ..., \beta_m) \neq 0$$

it follows that

$$G_k(\sigma + 1) \in \mathbb{Z} \setminus \{0\}$$

(81)

with some integer $k \in [0, m]$. Now we may prove the following essential estimate

$$m \sum_{j=1}^{m} H_j R_j(\sigma + 1) = m \sum_{j=1}^{m} H_j e^{-r_j(\sigma + 1)} <$$

$$\sum_{j=1}^{m} H_j e^{-B_j + dm(S) + e_0 m((\log S)^{1/2} + 2) + e_1 m + e_2} = \frac{1}{2}. $$

(82)

Hence by (63) we get

$$1 < 2|\Lambda| Q(\sigma + 1) = 2|\Lambda| e^{q(N(\sigma + 1))} \leq 2|\Lambda| e^{q(S + m)},$$

(83)

where

$$q(S + m) = (a(S + m) + b \log(S + m))g(S + m) + \frac{b_0(S + m)(\log(S + m))^{1/2} + b_1(S + m) + b_2 \log(S + m) + b_3.}$$

Because $g(x)$ is increasing we get

$$g(S + m) = g(S) + mV(S), \quad V(S) = \max_{S \leq x \leq S + m} \{g'(x)\}. $$

(84)

Or, remembering the assumption $m \leq S$, we may use the following estimates

$$\left\{ \begin{array}{l}
\log(S + m) \leq \log S + 1; \\
(\log(S + m))^{1/2} \leq (\log S)^{1/2} + 1.
\end{array} \right. $$

(85)

Consequently

$$q(S + m) \leq a S g(S) + Y(S), $$

(86)

where

$$Y(S) = am g(S) + am SV(S) + am^2 V(S) + bg(S + m) \log(S + m) + \frac{b_0(S + m)(\log(S + m))^{1/2} + b_1(S + m) + b_2 \log(S + m) + b_3.}$$

From (72) we get

$$S g(S) = \frac{\log H}{c - dm} + \frac{X(S)}{c - dm}, $$

(87)

where

$$X(S) = dm^2 g(S) + e_0 m S(\log S)^{1/2} + e_1 m S + e_2 m \log S + e_0 m^2 ((\log S)^{1/2} + 2) + e_1 m^2 + e_2 m + e_3 m. $$

Hence

$$Q(N) \leq H \frac{m}{c - dm} + Z(S), $$

(88)
where

\[ Z(S) = \frac{1}{\log H} \left( \frac{a}{c - dm} X(S) + Y(S) \right) \]

Case 1. We have

\[ Z(S) = \frac{1}{\log H} \left( \frac{a}{c - dm} (dm^2 + e_2 m \log S + e_2 m) + am + b \log(S + m) \right) \leq \]

\[ \frac{1}{\log H} \left( \frac{a(dm^2 + e_2 m)}{c - dm} + am + b \right) + \frac{e_2 m + b}{\log H} \log(S). \]

Here (72) reads

\[ (c - dm)S - dm^2 - e_2 m \log S - e_2 m = \log H. \] (89)

Let \( S_1 \) be the largest solution of the equation

\[ (c - dm)S - dm^2 - e_2 m \log S - e_2 m = \frac{1}{2}(c - dm)S. \] (90)

Then

\[ S \leq f \log H, \quad f = \frac{2}{c - dm}, \quad \forall S \geq x_1 = \max\{S_1, 1\}. \] (91)

Thus

\[ Z(S) \leq A \frac{1}{\log H} + B \frac{\log \log H}{\log H}, \] (92)

\[ A = \frac{a(dm^2 + e_2 m)}{c - dm} + am + b + \log f, \quad B = e_2 m + b \] (93)

and

\[ 1 < 2|\Lambda|Q(N) \leq |\Lambda|2e^A H^{\frac{a}{c - dm} + \frac{B \log \log H}{\log H}}, \] (94)

where

\[ \Lambda = \beta_0 + \beta_1 \Theta_1 + ... + \beta_m \Theta_m \]

is our linear form. This proves Case 1.

Case 2. Here

\[ q(S + m) \leq a(S + m) \log(S + m) + b_0(S + m) (\log(S + m))^{1/2} + \]

\[ b_1(S + m) + b_2 \log(S + m) + b_3 \leq aS \log(S) + Y(S), \]

\[ Y(S) = b_0S(\log S)^{1/2} + (a + b_0 + b_1)S + (am + b_2) \log S \]

\[ b_0m(\log S)^{1/2} + (a + b_0 + b_1)m + b_2 + b_3. \]

From (72) we get

\[ S \log S = \frac{\log H}{c - dm} + \frac{X(S)}{c - dm}, \] (95)

where

\[ X(S) = e_0mS(\log S)^{1/2} + e_1mS + (dm^2 + e_2 m) \log S + \]

\[ e_0m^2(\log S)^{1/2} + (2e_0 + e_1)m^2 + e_2 m + e_3 m. \]
Here (72) has the form
\[
((c - dm)S - dm^2) \log S - e_0 m S (\log S)^{1/2} - e_1 m S - e_2 m \log S - e_0 m^2 ((\log S)^{1/2} + 2) - e_1 m^2 - e_2 m - e_3 m = \log H.
\] (100)

Let \( S_2 \) be the largest solution of the equation
\[
((c - dm)S - dm^2) \log S - e_0 m S (\log S)^{1/2} - e_1 m S - e_2 m \log S - e_0 m^2 ((\log S)^{1/2} + 2) - e_1 m^2 - e_2 m - e_3 m = c - dm \frac{2}{S \log S}.
\] (101)

then
\[
S \log S \leq f \log H, \quad f = \frac{2}{c - dm}, \quad \forall \ S \geq x = \max\{1, S_2\},
\] (102)

and by (37) we get
\[
S \leq z(f \log H) \leq z_2(f \log H) = \frac{f \log H}{\log \frac{f \log H}{\log f \log H}}
\] (103)
valid for
\[
f \log H > e.
\] (104)

Note the estimate
\[
\frac{S(\log S)^{1/2}}{\log H} = \frac{S^{1/2}(\log S)^{1/2}}{\log H} \leq \left( f \frac{z(f \log H)}{\log H} \right)^{1/2} \leq \left( f \frac{z_2(f \log H)}{\log H} \right)^{1/2},
\] (105)
too. By using the notation
\[
\xi(z, H) = A \left( f \frac{z(f \log H)}{\log H} \right)^{1/2} + B \frac{z(f \log H)}{\log H} + C \frac{\log z(f \log H)}{\log H} + D \frac{(\log z(f \log H))^{1/2}}{\log H},
\]

we have
\[
Q(N) \leq e^{E H \frac{a}{2 \pi i} + \xi(z, H)},
\] (106)
where the error term satisfies
\[
\xi(z, H) \leq \xi(z_2, H).
\] (107)
Note that
\[ D \frac{(\log z(f \log H))^{1/2}}{\log H} = o \left( C \frac{\log z(f \log H)}{\log H} \right), \]
\[ C \frac{\log z(f \log H)}{\log H} = o \left( B \frac{z(f \log H)}{\log H} \right) \]
and
\[ B \frac{z(f \log H)}{\log H} = o \left( A \left( \frac{z(f \log H)}{\log H} \right)^{1/2} \right) \]
for any \( H \) big enough. Thus
\[ A \left( \frac{z(f \log H)}{\log H} \right)^{1/2} \]
will be the main error term, if \( A \neq 0 \).

Further, we note that the estimate (106) may be written as follows
\[ Q(N) \leq e^E (z(f \log H))^C H^{a/\log H} + A \left( \frac{z(f \log H)}{\log H} \right)^{1/2} + B \frac{z(f \log H)}{\log H} + D \frac{(\log z(f \log H))^{1/2}}{\log H} \] (108)
which by (107) implies
\[ Q(N) \leq e^E (z_2(f \log H))^C H^{a/\log H} + A \left( \frac{z_2(f \log H)}{\log H} \right)^{1/2} + B \frac{z_2(f \log H)}{\log H} + D \frac{(\log z_2(f \log H))^{1/2}}{\log H} . \] (109)

Next we shall prove the estimates (114), (115) valid with (116). For that purpose we recall that
\[ \rho_2(x) = \frac{\log x}{\log x - \log \log x} \]
and note that
\[ z_2(x) \leq \rho_2(x_0) \frac{x}{\log x}, \quad \forall x \geq x_0 \geq e^e. \]

Further we have
\[ z_2(fy) \leq \rho_2(x_0) f \frac{y}{\log fy} = \rho_2(x_0) f \left( 1 - \frac{\log f}{\log fy} \right) \frac{y}{\log y} \]
In particular we have
\[ z_2(f \log H) \leq \rho_2(x_0) f \left( 1 - \frac{\log f}{\log(f \log H)} \right) \frac{\log H}{\log \log H} < \rho_2(x_0) f \frac{\log H}{\log \log H} \] (110)
for all
\[ f \log H \geq x_0 \geq e^e. \]
Hence
\[ Q(N) \leq e^{E_2} \left( f \rho \frac{\log H}{\log \log H} \right)^{C_2} H^{a/\log H} + A_2 \frac{f \log H}{\log \log H} + B_2 \frac{f \log H}{\log \log H} + D_2 \frac{(\log f \log H)^{1/2}}{\log \log H}, \quad \rho \geq \rho_2(x_0). \] (111)
by (107), (110). Now substitute (106), (108) and (111), respectively, into
\[ 1 < 2|\Lambda|Q(N) \] (112)
proving (38), (41) and (48). This ends the proof of Case 2.

Case 3. Here (72) reads

\[ ((c - dm)S - dm^2)S - e_1 mS - e_1 m^2 = \log H \]

which implies

\[ S = \frac{dm^2 + e_1 m + \sqrt{(dm^2 + e_1 m)^2 + 4(e_1 m^2 + \log H)(c - dm)}}{2(c - dm)}. \]

For convenience, we will use the following estimate

\[ 1 = S_3 \leq S \leq v_1 + v_2 \sqrt{\log H}, \quad v_2 = \frac{1}{\sqrt{c - dm}}. \]

\[ v_1 = \frac{dm^2 + e_1 m + \sqrt{(dm^2 + e_1 m)^2 + 4e_1 m^2(c - dm)}}{2(c - dm)}. \]

Now

\[ Z(S) = (w_1 S + w_2) \frac{1}{\log H} \]

\[ w_1 = \frac{a(dm^2 + e_1 m)}{c - dm} + (2am + b_1), \quad w_2 = am^2 + bm + e_1 m^2. \]

So

\[ Z(S) \leq \frac{B}{\log H} + \frac{A}{\sqrt{\log H}}, \quad B = v_1 w_1 + w_2, \quad A = v_2 w_1. \]

Hence

\[ Q(N) \leq H \frac{a}{c - dm} \frac{b}{\log H} + \frac{A}{\sqrt{\log H}} = e^B H \frac{a}{c - dm} + \frac{A}{\sqrt{\log H}}. \]

In particular, if \( b_1 = e_1 = 0 \), then

\[ A = \frac{1}{\sqrt{c - dm}} \left( \frac{adm^2}{c - dm} + 2am \right), \]
\[ B = \frac{dm^2}{c - dm} \left( \frac{adm^2}{c - dm} + 2am \right) + am^2. \]

This proves Case 3.

Yet we need to determine lower bound \( G_t \). In each case, there are some assumptions imposed for \( H \). The determinant condition (20) and the conditions (74), (79), (91) and (102) should be satisfied. So, we put

\[ f_1 = x_1/f, \quad f_2 = (x_2 \log x_2)/f, \quad f_3 = S_3 \]

and

\[ G_t = \max\{m, N_l, e^f_t\}. \]

If

\[ H \geq G_t, \]

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then Theorem (3.1) is proved. Finally we note, that in the Case 2 the condition (104) is needed in Theorem (3.4) and in Corollary (3.5) we need the assumption (4.5), too. The condition (114) applied in the Case 2 shows, in particular, that
\[ f \log H \geq f \log G_2 \geq x_2 \log x_2 \] (116)
and thus in (110) we may choose
\[ \rho = \frac{\log(x_0)}{\log(x_0) - \log \log(x_0)}, \quad x_0 = \max\{f \log m, f \log N_2, x_2 \log x_2, e^e\}. \] (117)

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