Fluctuations and non-Gaussianity in 3D+1 quantum nonlocal nonlinear waves: towards quantum gravity and quantum fluid technologies

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Nonlocal quantum fluids emerge as dark-matter models and tools for quantum simulations and technologies. However, strongly nonlinear regimes, like those involving multi-dimensional self-localized solitary waves (nonlocal solitons), are marginally explored for what concerns quantum features. We study the dynamics of 3D+1 solitons in the second-quantized nonlocal nonlinear Schrödinger equation. We theoretically investigate the quantum diffusion of the soliton center of mass and other parameters, varying the interaction length. 3D+1 simulations of the Ito partial differential equations arising from the positive P-representation of the density matrix validate the theoretical analysis. The numerical results unveil the onset of non-Gaussian statistics of the soliton, which may signal quantum-gravitational effects and be a resource for quantum computing. The non-Gaussianity arises from the interplay of the quantum diffusion of the soliton parameters and the stable invariant propagation. The fluctuations and the non-Gaussianity are universal effects expected for any nonlocality and dimensionality.

Three-dimensional (3D) self-localized nonlinear waves enter various fields of research [1, 2], but their quantum properties are unexplored. Classical three-dimensional solitary waves (in short, 3D solitons) need to be stabilized against catastrophic collapse. Nonlocality is a well known mechanism for the stabilization and nonlocal soliton are a fascinating research direction involving long-range Bose-Einstein condensates (BECs) [3-10], boson stars [6] and dark-matter models [11, 12]. However, a mean-field description that overlooks quantum effects provides limited information on the dynamics of self-trapped multidimensional waves. This limitation is specifically relevant as many recent investigations suggest the use of solitons as non-classical sources for quantum technologies and fundamental studies [13-18]. Results in 1D [19, 20] suggest that nonlocality frustrates fluctuations. However, despite ab-initio investigations on long-range interactions [21, 22], the quantum statistics of self-trapped 3D nonlocal solitons is an open issue.

In addition, recent work on gravitational interaction in BEC predicts non-Gaussian statistics [23]. Non-Gaussianity is a resource for continuous-variable quantum information science [24, 25] and its understanding in quantum fluids may enable new universal quantum processors. Also, emerging of non-Gaussian statistics in table-top experiments may open the way to study or simulate quantum gravity in the laboratory. Ref. [23] predicts that a BEC in a trap, once prepared in a squeezed state or a Schrödinger-cat state, triggers the non-Gaussian statistics measured by a signal-to-noise ratio (SNR) parameter, which reveals quantized-gravity. However - so far - no experiments or numerical simulations validate these theoretical predictions. Also, quantum fluctuations and non-Gaussianity in multidimensional self-trapped solitonic nonlocal condensates have never been considered before.

Here, we study theoretically and numerically the quantum dynamics of 3D+1 nonlocal solitons. We use a perturbative approach and analytically predict the quantum diffusion of the soliton position and other parameters. We validate our analytical results by an ab-initio numerical analysis based on the 3D+1 positive P-representation [21, 22, 26]. We compute the SNR parameter introduced in [23], which shows that non-Gaussianity arises in the quantum dynamics of 3D+1 nonlocal solitons, starting from a coherent state.

We consider the many-body Hamiltonian

\[ \hat{H} = \frac{\hbar^2}{2m} \int \nabla \hat{\psi}^\dagger \cdot \nabla \hat{\psi} \, d^3r + \int U(\mathbf{r} - \mathbf{r}') \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}) \, d^3r \, d^3r' , \]  

(1)

with \( m \) is the boson mass, and \( U \) is the interaction potential. We adopt the phase-space representation methods [21, 27] for studying the nonlocal interaction. The quantum field model is equivalent to a Fokker-Planck equation, which is mapped to Ito nonlinear partial dif-
ferential equations coupling two fields $\psi$, and $\psi^+$

\[ i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \psi U + \rho + \sqrt{\frac{\hbar}{m}} \xi_U \]

\[ -i\hbar \partial_t \psi^+ = -\frac{\hbar^2}{2m} \nabla^2 \psi^+ + \psi^+ U + \rho + t \sqrt{\frac{\hbar}{m}} \xi_U^+ \]  \hspace{1cm} (2)

where the asterisk denotes convolution integral. In $\rho = \psi^+ \psi$, $\xi_U$ and $\xi_U^+$ are independent noises such that

\[ \langle \xi_U(\mathbf{r}, t) \xi_U(\mathbf{r}', t') \rangle = \langle \xi_U^+(\mathbf{r}, t) \xi_U^+(\mathbf{r}', t') \rangle = U(\mathbf{r} - \mathbf{r}') \delta(t - t') \]  \hspace{1cm} (3)

The total number of particles is $N_T = \int \psi^+ \psi \, dV$. In our numerical calculations below, we consider self-gravitating screened potential $U = -Gm^2 e^{-r/\Lambda}/r$, where $\Lambda$ is the interaction length. $G$ measures the coupling corresponding to the gravitational constant, but $U$ also models other long-range interactions as, e.g., thermal effects in photonic BEC [2]. The mean-field theory, Eq. (5) admits a stable solution in the absence of nonlinearity ($U = 0$), and $\psi^+ = \psi^*$. As detailed in the Supplementary Material (SM), we write the stochastic equations in dimensionless units as

\[ +i\partial_t \psi + \nabla^2 \psi - \psi U + \rho = s \]

\[ -i\partial_t \psi^+ + \nabla^2 \psi^+ + \psi^+ U + \rho = s^+ \]  \hspace{1cm} (5)

with $U = -e^{-r/\sigma}/r$, $s = \sqrt{\frac{1}{2} n_0 \xi_U \psi}$, and $s^+ = \sqrt{\frac{1}{2} n_0 \xi_U^+ \psi^*}$. $\sigma$ and $n_0$ are the dimensionless interaction length and particle number, respectively.

In the mean-field theory, Eq. (5) admits a stable radially-symmetric bound-state solution: a self-localized three-dimensional solitary wave [28]:

\[ \psi = u(x^a - X^a) \exp \left[ i \theta - iEt + \frac{i}{2} V^a(x_a - X_a) \right] \]  \hspace{1cm} (6)

$\psi^+ = \psi^*$, with $a = 1, 2, 3$, $x^a = x$, $x^2 = y$, $x^3 = z$, and omitting the sum symbol over repeated Latin indices. $u(x^a)$ is the real-valued soliton profile (Fig. 1), such that

\[ \Delta u - u^2 u = E u, \]  \hspace{1cm} (7)

The soliton energy $E$ is time-independent. For the position $X^a = X^a(t)$, we have (dot is time-derivative)

\[ \dot{X}^a = V^a \]

\[ \dot{V}^a = 0 \]  \hspace{1cm} (8)

We also have $\dot{\theta} = \frac{1}{2} V^2$, with $V^2 = \delta_{ab} V^a V^b$ and $\delta_{ab}$ the Kronecker symbol ($b = 1, 2, 3$).

Figure 1 shows the evolution of the time-invariant soliton profile obtained from Eq. (7), compared with the evolution in the absence of nonlinearity ($U = 0$). We also show the soliton compared with corresponding potential $U + \rho$. The field profile and the potential are computed numerically. We use a pseudo-spectral parallel relaxation procedure in a 3D Cartesian domain. Figure 1 shows the calculated classical bound state $u$. In the absence of interaction, the mass spreads upon evolution. In the presence of self-attraction, the soliton wave-packet is invariant upon propagation.

In the quantum regime, with $\xi_U \neq 0$ and $\xi_U^+ \neq 0$, the soliton, initially prepared in a coherent state, evolves with fluctuations depending on the particle number $n_0$ and the interaction length $\sigma$. The 3D+1 stochastic partial differential equations in [5] are solved by following Drummond and coworkers [21]. We adopt an iterative stochastic solver with pseudospectral discretization and parallelized with the FFTW [29] and the Message Passing Interface (MPI) protocol. Figure 2a shows the numerical solution of the stochastic equations (5), which unveils that the soliton undergoes a random-walk in position (Fig. 2c-e).

As detailed in the SM, to study the quantum regime, we derive equations for the soliton parameters by (5)

\[ M \dot{X}^a = MV^a + F_X^a(X, t) \]

\[ M \dot{V}^a = F_V^a(X, t) \]  \hspace{1cm} (9)

with $X = (X^1, X^2, X^3)$ the soliton position, $F_X^a$ and $F_V^a$ stochastic terms, and $M = \int u^2 \, d^3r$ the dimensionless soliton mass. We have in (9)

\[ F_X^a = \frac{1}{\sqrt{n_0}} \int \rho(x - X)(x^a - X^a) \xi_+(x, t) \, d^3r \]  \hspace{1cm} (11)

with $\xi_+ = (\xi_U + \xi_U^+)/\sqrt{2}$ being $\langle \xi_+(x', t') \xi_+(x, t) \rangle = U(x - x') \delta(t - t')$.

The stochastic terms in Eqs. (9) are such that

\[ \langle F_X^a(X, t) F_X^b(X, t') \rangle = \frac{1}{n_0} Q_X^a \delta_{ab} \delta(t - t') \]  \hspace{1cm} (12)

with the correlation coefficient $\langle U_{12} = U(r_1 - r_2) \rangle$.

\[ Q_X^a = \int \int x_1^a x_2^a \rho(r_1) \rho(r_2) U_{12} \, d^3r_1 \, d^3r_2 \]  \hspace{1cm} (13)

Results for $F_V^a$, $\theta$ and $E$ are given in the SM. Eqs. (9) give, at the lowest order in $t$

\[ \langle |X^a(t)|^2 \rangle = D_X^a t \]  \hspace{1cm} (14)

with the diffusion coefficient $D_X^a = Q_X^a / n_0 M^2$.

For the radially symmetric soliton, we show in Fig. 2a the diffusion coefficient $D_X^r = D_x^r = D_x^3 = D_X$, as obtained by the numerical profile $u$ computed with the screened gravitational potential $U$. One finds that for a growing $\sigma$ the quantum diffusion is frustrated, as it happens in 1D [19, 20]. This can be deduced from Eq. (13), indeed, as $\sigma \rightarrow \infty$, one has $U_{12} \approx$ constant, and $Q_X^a \rightarrow 0$, as for the soliton profile $\rho(x) = \rho(-x)$.

We compare Eq. (14) with the full 3D+1 stochastic simulations and we find excellent agreement, as shown in
FIG. 1. Self-gravitating solitonic core. (a) Comparison of time dynamics with and without interaction \((G = 0)\); 3D isodensity surfaces at different instants for freely evolving fields (top panel) and in the presence of the nonlinearity (bottom panel) with the time-invariant self-trapped wave-packet. (b) Two dimensional projection (average in the \(z\)-direction) of the density profile (blue) and resulting long-range potential (yellow)

Fig. 2c-e where we report the dynamics of solitary waves with \(n_0M \approx 10^6\) atoms.

The diffusion constant \(D = hD_X/2m\) in the original physical units reads \(\rho_{1,2} = \rho(r_{1,2})\)

\[
D = \frac{Gm^2}{\hbar} \int \int \frac{x_1 \rho_1 x_2 \rho_2 \ e^{-|r_1 - r_2|/\Lambda}}{\rho d^3r_1 \rho d^3r_2 \ d^3r_1 d^3r_2}.
\]

Eq. (15) shows the interplay of quantum and gravitational effects through the ratio \(Gm^2/\hbar\).

In our stochastic simulations, the initial state is a coherent state, whose statistical properties change upon evolution. Here we follow [23] to determine if deviations from Gaussianity arise. We report in Fig. 3a the evolution of the statistical distribution of the density \(\rho(r = 0)\) as computed by Eqs. (5) at the center of the classical solitonic core. The initial state is coherent, and the histogram is localized in the initial value of the peak. Upon evolution, the distribution spreads and manifestly displays a bell-shaped non-Gaussian profile. Similar behavior is also obtained for the quadratures of the field (not reported).

To quantify the deviation from Gaussianity, we consider the SNR introduced in [23]

\[
\text{SNR} = \frac{|\kappa_4|}{\sqrt{\text{var} k_4}}
\]

here \(\kappa_4\) is the fourth cumulant of the statistical distribution \(\text{var} k_4\) is its uncertainty (details in the SM). For a Gaussian statistics, all the cumulants higher than second order vanish, hence SNR measures deviation from non-Gaussianity including the uncertainty \(\text{var} k_4\) due to a finite number of samples. We compute SNR for the density and the field quadratures with similar results.

At variance with [23], we account for the heterogeneous features of SNR, i.e., we measure SNR in different spatial locations. Figure 3b shows the 3D isosurface of the SNR at different instants. The statistical distributions at different positions becomes non-Gaussian with time. Figure 3c shows the spatially averaged value of the SNR, which demonstrates that a self-trapped solitonic wave-packet develops a non-Gaussian statistic upon evolution when starting from a coherent state. Results in Fig. 3 refer to a representative case with \(n_0M \approx 10^4\) atoms; we found these dynamics for different interaction lengths and number of atoms.

To understand the physical origin of the non-Gaussianity, we observe that - at the lowest order in \(t\)
- the soliton parameters $X^a, V^a, \theta,$ and $E$, are the time-integral of white noise terms (i.e., Wiener processes). Thus they are the sum of many independent variables and hence obey Gaussian statistics. Non-Gaussianity arises from the fact that the soliton profile is a nonlinear function of these parameters, and any observable depends on the soliton profile. In general terms, the statistical distribution of a nonlinear function of a Gaussian variable is expected to be non-Gaussian. Thus, as far as the soliton is stable with respect to fluctuations, non-Gaussianity arises. On the contrary, an unstable soliton would break into linear waves and generate Gaussian states. Nonlocal solitons are stable self-trapped nonlinear waves, and their robustness against quantum fluctuations induce non-Gaussianity.

In conclusion, we studied theoretically and by first-principle numerical simulations the 3D+1 dynamics of non-local self-gravitating boson fluids. The quantum
noise induces diffusion in the self-localized wave-packet position determined by the degree of nonlocality and the particle number. The theoretical results agree with ab-initio 3D+1 simulations with no fitting parameters. The quantum diffusion varies with the ratio \( \frac{mG}{\hbar} \) and is due to the interplay of the quantum fluctuations and the long-range self-interaction. This interplay causes non-Gaussian statistics that spread in the solitonic core upon evolution. We remark that this is a universal phenomenon that is not dependent on the specific interaction potential \( U \) but arises from the general stability properties of solitons.

The results open the way to using non-Gaussian multidimensional solitary waves as non-classical reservoirs for continuous-variable quantum information and as quantum simulators for quantum gravity models. Notably enough, the numerical simulations suggest that signatures of a quantized gravity may arise even without careful preparation of the initial state as a Schrödinger cat (or a squeezed state), but starting from a coherent solitonic state. Also, the results show the relevance of quantum fluctuations in cold dark-matter models, which can potentially impact the investigation of self-gravitating BEC and enable tests within astrophysical observations.

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[1] Y. Kivshar and G. P. Agrawal, *Optical solitons* (Academic Press, New York, 2003).
[2] B. Malomed, Multidimensional self-trapping in linear and nonlinear potentials, (2021), arXiv:2111.00547
[3] S. K. Turitsyn, Spatial dispersion of nonlinearity and stability of multidimensional solitons, Teor. Mat. Fiz. 64, 226 (1985).
[4] V. M. Pérez-García, V. V. Konotop, and J. J. García-Ripoll, Dynamics of quasicollapse in nonlinear schrödinger systems with nonlocal interactions, Phys. Rev. E 62, 4300 (2000)
[5] O. Bang, W. Krolikowski, J. Wyller, and J. J. Rasmussen, Collapse arrest and soliton stabilization in non-local nonlinear media, Phys Rev E 66, 046619 (2002).
[6] D. O’Dell, S. Giovanazzi, G. Kurizki, and V. M. Akulin, Bose-einstein condensates with \( 1/r \) inter-atomic attraction: Electromagnetically induced “gravity”, Phys. Rev. Lett. 84, 5687 (2000)
[7] J. Klaers, J. Schmitt, F. Vewinger, and M. Weitz, Bose–Einstein condensation of photons in an optical microcavity, Nature 468, 545 (2010)
[8] I. Carusotto and C. Ciuti, Quantum fluids of light, Rev. Mod. Phys. 85, 299–366 (2013)
[9] M. Calvanese Strinati and C. Conti, Bose-einstein condensation of photons with nonlocal nonlinearity in a dye-doped graded-index microcavity, Phys. Rev. A 90, 043853 (2014)
[10] N. Defenu, T. Donner, T. Macrì, G. Pagano, S. Ruffo, and A. Trombettoni, Long-range interacting quantum systems (2021), arXiv:2109.01063
[11] A. Paredes and H. Michinel, Interference of dark matter solitons and galactic offsets, Phys. Dark Universe 12, 50 (2016)
[12] J. Garnier, K. Baudin, A. Fusaro, and A. Picozzi, Incoherent localized structures and hidden coherent solitons from the gravitational instability of the schrödinger-poisson equation, (2021), arXiv:2108.13250
[13] C. Conti, Quantum gravity simulation by nonparaxial nonlinear optics, Phys. Rev. A 89, 061801 (2014)
[14] Q.-Y. Liang, A. V. Venkastramani, S. H. Cantu, T. L. Nicholson, M. J. Gullans, A. V. Gershkov, J. D. Thompson, C. Chin, M. D. Lukin, and V. Vuletić, Observation of three-photon bound states in a quantum nonlinear medium, Science 359, 783 (2018)
[15] L. D. M. Villari, D. Faccio, F. Biancalana, and C. Conti, Quantum soliton evaporation, Phys. Rev. A 98, 043859 (2018)
[16] O. V. Marchukov, B. A. Malomed, V. Dunjko, J. Ruhl, M. Olshanii, R. G. Hulet, and V. A. Yurovsky, Quantum fluctuations of the center of mass and relative parameters of nonlinear schrödinger breathers, Phys Rev Lett 125, 050405 (2020)
[17] C. Conti, Boson sampling discrete solitons by quantum
[18] A. Alodjants, D. Tsarev, T. V. Ngo, and R.-K. Lee, Enhanced nonlinear quantum metrology with weakly coupled solitons in the presence of particle losses, Physical Review A 105, 012606 (2022).

[19] V. Folli and C. Conti, Frustrated Brownian Motion of Nonlocal Solitary Waves, Phys. Rev. Lett. 104, 193901 (2010).

[20] S. Batz and U. Peschel, Frustrated quantum phase diffusion and increased coherence of solitons due to nonlocality, Phys. Rev. A 83, 033826 (2011).

[21] P. D Drummond and S. Chaturvedi, Quantum simulations in phase-space: from quantum optics to ultra-cold physics, Phys Scripta 91, 073007 (2016).

[22] S. Wüster, J. F. Corney, J. M. Rost, and P. Deuar, Quantum dynamics of long-range interacting systems using the positive- \( p \) and gauge- \( p \) representations, Phys Rev E 96, 013309 (2017).

[23] R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, and A. Iyer, Non-gaussianity as a signature of a quantum theory of gravity, PRX Quantum 2, 010325 (2021).

[24] C. Hughes, M. G. Genoni, T. Tufarelli, M. G. A. Paris, and M. S. Kim, Quantum non-gaussianity witnesses in phase space, Phys. Rev. A 90, 013810 (2014).

[25] Q. Zhuang, P. W. Shor, and J. H. Shapiro, Resource theory of non-gaussian operations, Phys. Rev. A 97, 052317 (2018).

[26] P. D. Drummond, P. Deuar, and K. V. Kheruntsyan, Canonical bose gas simulations with stochastic gauges, Phys. Rev. Lett. 92, 040405 (2004).

[27] C. W. Gardiner and P. Zoller, Quantum Noise, 3rd ed. (Springer-Verlag, Berlin, 2004).

[28] I. M. Moroz, R. Penrose, and P. Tod, Spherically-symmetric solutions of the schrödinger-newton equations, Classical Quant Grav 15, 2733 (1998).

[29] M. Frigo and S. G. Johnson, The design and implementation of FFTW3, Proceedings of the IEEE 93, 216 (2005), special issue on “Program Generation, Optimization, and Platform Adaptation”.