KÄHLER MANIFOLDS WITH REAL HOLOMORPHIC VECTOR FIELDS

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Abstract. For a Kähler manifold endowed with a weighted measure $e^{-f} \ dv$, the associated weighted Hodge Laplacian $\Delta f$ maps the space of $(p, q)$-forms to itself if and only if the $(1, 0)$-part of the gradient vector field $\nabla f$ is holomorphic. We use this fact to prove that for such $f$, a finite energy $f$-harmonic function must be pluriharmonic. Motivated by this result, we verify that the same also holds true for $f$-harmonic maps into a strongly negatively curved manifold. Furthermore, we demonstrate that such $f$-harmonic maps must be constant if $f$ has an isolated minimum point. In particular, this implies that for a compact Kähler manifold admitting such a function, there is no non-trivial homomorphism from its first fundamental group into that of a strongly negatively curved manifold.

In this paper, $(M, g)$ denotes a Kähler manifold of complex dimension $m$ with metric $g$ and complex structure $J$. For a smooth real valued function $f \in C^\infty (M)$, introduce a weighted measure of the form $dv_f := e^{-f} \ dv$, where $dv$ is the volume form induced from the metric $g$. With respect to the weighted volume form $dv_f$, the adjoint $d^* f$ of the exterior differential $d$ acting on $\Omega^p (M)$, the space of $p$ forms on $M$, is defined by

$$\int_M \langle d\omega, \theta \rangle e^{-f} = \int_M \langle \omega, d^* f \theta \rangle e^{-f},$$

for all $\omega \in \Omega^p (M)$ and $\theta \in \Omega^{p+1} (M)$. The weighted Hodge Laplacian $\Delta_f$ is then given by

$$\Delta_f := dd^* f + d^* f d.$$

Denote by $A^{p,q} (M)$ the space of $(p, q)$-forms on $(M, g)$. It is well known that the Hodge Laplacian $\Delta$ preserves the type of forms, i.e., $\Delta \omega \in A^{p,q} (M)$ for any $\omega \in A^{p,q} (M)$. This fact is important in the Hodge theory of Kähler manifolds. One may ask if the same holds true for the weighted Hodge Laplacian $\Delta_f$. Obviously, the case when both $p$ and $q$ are zero is trivially true. When $0 < p + q < 2m$ we note the following result.

Proposition 0.1. Let $(M, g)$ be a Kähler manifold and $f \in C^\infty (M)$. Then the weighted Hodge Laplacian $\Delta_f$ maps the space $A^{p,q} (M)$ of $(p, q)$-forms into itself if and only if $\nabla f$ is real-holomorphic.
Here, $\nabla f$ is said to be real holomorphic if its $(1, 0)$-part is a holomorphic vector field. In terms of local complex coordinates $\{z_1, z_2, \cdots, z_m\}$, that means that the complex vector field

$$X := g^{ik} \frac{\partial f}{\partial \bar{z}^k} \frac{\partial}{\partial z^i}$$

is holomorphic. This is equivalent to $f_{kl} = 0$ for all $k, l \in \{1, 2, \cdots, m\}$, in any local unitary frame $\{v_k\}_{k=1, \cdots, m}$. An alternative characterization is that $J(\nabla f)$ is a Killing vector field.

There are quite a few important classes of Kähler manifolds admitting a function with real holomorphic gradient vector field. One notable class is the gradient Kähler-Ricci solitons. Recall that a manifold $(M, g)$ is a gradient Ricci soliton if there exists a function $f \in C^\infty(M)$ such that its Ricci curvature and the Hessian of $f$ satisfy $\text{Ric} + \text{Hess}(f) = \lambda g$, for some $\lambda \in \mathbb{R}$. Since $(M, g)$ is Kähler, the equation can be expressed into $R_{ij} + f_{ij} = \lambda g_{ij}$ and $f_{ij} = 0$ under unitary frames. In particular, $\nabla f$ is a real holomorphic vector field. Another important class arises from Calabi’s extremal Kähler metric [1]. On a compact Kähler manifold $(N, g_0)$, one considers the following functional over the fixed Kähler class determined by $g_0$

$$F(g) = \int_N s^2(g) \, dv_g,$$

where $s(g)$ and $dv_g$ are the scalar curvature and the volume form of metric $g$, respectively. A critical point of this functional is called an extremal metric. It is shown by Calabi [1] that a metric $g$ is extremal if and only if $\nabla s(g)$ is real holomorphic. The last class we mention comes from eigenvalue estimates [19, 20]. For a compact Kähler manifold with Ricci curvature bounded below by a positive constant $k$, it says that the gradient vector field of the corresponding eigenfunction must be real holomorphic if the first nonzero eigenvalue achieves its optimal lower bound $2k$. In these examples, the existence of a real holomorphic vector field is required in the study of some important geometric questions. In [7], the existence of a function whose gradient is real holomorphic was also needed for obtaining the strong hypercontractivity of the weighted Laplacian. Inspired by this important work of Gross, a complete description of possible functions with this property on the complex hyperbolic space was obtained in [8].

We now briefly mention some previous results concerning manifolds admitting real holomorphic vector fields. In an influential paper [6], Frankel has shown that a one-parameter group of isometries acting on a Kähler manifold $M$ must be Hamiltonian, i.e., induced by a Killing vector field of the form $J(\nabla f)$ for some function $f$, if $M$ is simply connected or the action has nonempty fixed point set $Z$. Moreover, the Betti numbers of $M$ can be computed from those of the fixed point set $Z$. Later, in [5], it was shown that the Dolbeault cohomology $H^{p,q}(M) = 0$ for $|p - q| > \dim C Z$. More recently, our studies [13, 14] show that the existence of a smooth function $f$ such that $\nabla f$ is real holomorphic has important implications on the function theory of the manifold. In particular, it leads to various Liouville theorems for holomorphic or, more generally, harmonic functions on $M$. The interesting feature is that no curvature assumption is involved.

Here, we continue our investigation of manifolds with a real holomorphic vector field. First, we will use Proposition 0.1 to prove the following Liouville theorem.
Theorem 0.2. Let \((M, g)\) be complete Kähler manifold and suppose there exists \(f \in C^\infty (M)\) so that \(\nabla f\) is real holomorphic. Suppose that \(u\) is a \(f\)-harmonic function on \(M\) with finite total weighted energy \(\int_M |\nabla u|^2 e^{-f} < \infty\). Then \(u\) is pluriharmonic. If, in addition, \(f\) is proper, then \(u\) is constant on \(M\).

Theorem 0.2 was first established in [14] under a growth assumption on \(f\). It was used there to show that shrinking gradient Kähler-Ricci solitons must be connected at infinity. Our approach here enables us to remove this extra assumption.

In view of Theorem 0.2, it is natural to investigate the more general situation of harmonic maps between Kähler manifolds. We will show that the existence of a real holomorphic vector field on \(M\) implies analogous results for harmonic maps from \(M\) to another manifold \(N\) with negative curvature in a suitable sense. As is well known (see Schoen and Yau [16]), this leads to topological information of manifold \(M\). More precisely, we have the following result.

Theorem 0.3. Let \((M, g)\) be a complete Kähler manifold with a real holomorphic vector field \(\nabla f\) for some \(f \in C^\infty (M)\). Assume in addition that there exists an isolated minimum point \(x_0 \in M\) for \(f\). Then any \(f\)-harmonic map \(u : M \to N\) of finite total weighted energy into a Kähler manifold with strongly seminegative curvature must be constant.

We recall after [18] that the curvature \(K_{abcd}\) of a Kähler manifold is strongly seminegative if

\[
K_{abcd} \left( A^a B^b - C^a D^b \right) \left( A^d B^c - C^d D^c \right) \geq 0,
\]

for all complex numbers \(A^a, B^a, C^a, D^a\). We remark that no assumption on the curvature of \(M\) or the growth of \(f\) is involved in the theorem.

The assumption that \(f\) has an isolated minimum point is indeed necessary. To see this, consider a Kähler manifold \(N\) and let \(M = N \times \mathbb{C}\). The function \(f\) is taken to be constant on \(N\) and \(|z|^2\) on \(\mathbb{C}\), so \(\nabla f\) is clearly real holomorphic. Obviously, the projection map \(\pi : M \to N\) is a nonconstant weighted harmonic map from \(M\) to \(N\).

Examples of manifolds verifying the assumptions of Theorem 0.3 are abundant. They include steady Kähler Ricci solitons with positive Ricci curvature and scalar curvature going to zero at infinity as the potential function is strictly convex and attains its minimum value at its only critical point (see [3]). Such solitons have been constructed in [2]. They also include the complex projective spaces and complex hyperbolic spaces studied in [8]. For example, on the unit ball model of the complex hyperbolic space \(\mathbb{H}^n\) with Kähler form \(\omega = -\partial \bar{\partial} \log \left(1 - |z|^2\right)\), the weight \(f(z) = \frac{1}{1 - |z|^2}\) obviously has real holomorphic gradient and an isolated minimum at \(z = 0\).

As a consequence of Theorem 0.3 we get the following result concerning the fundamental group of such manifolds.

Corollary 0.4. Let \((M, g)\) be a compact Kähler manifold and assume there exists \(f\) which satisfies the assumptions in Theorem 0.3. Then there is no non-trivial homomorphism from \(\pi_1 (M)\) into that of a compact Kähler manifold with strongly seminegative curvature.

\(f\)-harmonic maps have been well studied in the literature, as they are natural objects in the presence of a smooth measure on a manifold. The interested reader may consult [11, 15] for some recent progress and a more extensive reference list.
Finally, in the last part of the paper, we prove a vanishing theorem for holomorphic forms. This result does not seem to follow from the previous work of [5, 6, 10, 21] even in the compact case as we impose no assumption on the size of the critical point set of \( f \).

**Theorem 0.5.** Let \((M, g)\) be a complete Kähler manifold with a bounded, real holomorphic vector field \( \nabla f \) for some \( f \in C^\infty (M) \). Assume in addition that there exists an isolated minimum point \( x_0 \in M \) for \( f \). Then, for any \( p \geq 0 \), all \( L^2 \)-holomorphic \((p, 0)\)-forms on \( M \) must be zero.

This theorem implies that a compact Kähler manifold admitting such a function has first Betti number equal to zero. It would be interesting to infer some information about higher Betti numbers, under the same assumptions.

1. **The weighted Laplacian and forms**

In this section, we prove Theorem 0.2. We begin by setting up the notations. First, to be consistent with our notation in previous works, given \( ds^2 := g_{k\bar{j}} dz^k d\bar{z}^j \) a Kähler metric on \( M \), the Riemannian metric that we consider is \( 4 \text{Re} (ds^2) \). So, with respect to this Riemannian metric, we have

\[
|\nabla u|^2 = g^{k\bar{j}} u_k u_{\bar{j}}, \quad \Delta u = g^{k\bar{j}} u_k u_{\bar{j}}.
\]

Any \( \omega \in A^{p,q} (M) \) will be written locally as

\[
\omega = \frac{1}{p!q!} \omega_{I\bar{J}} dz^I \wedge d\bar{z}^J,
\]

where \( |I| = p \) and \( |J| = q \). On \( A^{p,q} (M) \) we use the metric to define a Hermitian product by

\[
\langle \omega, \theta \rangle := \frac{1}{2^{p+q}p!q!} g^{I\bar{K}} g^{L\bar{J}} \omega_{I\bar{K}} \theta_{L\bar{J}}.
\]

The differential \( d : \Omega^p (M) \to \Omega^{p+1} (M) \) acting on \( p \) forms on \( M \), given by

\[
d\omega = dx^k \wedge \nabla_{\partial x^k} \omega,
\]

is decomposed as \( d = \partial + \bar{\partial} \), where \( \partial : A^{p,q} (M) \to A^{p+1,q} (M) \) and \( \bar{\partial} : A^{p,q} (M) \to A^{p,q+1} (M) \) are given by

\[
\partial \omega = dx^k \wedge \nabla_{\partial x^k} \omega, \quad \bar{\partial} \omega = dx^k \wedge \nabla_{\bar{\partial} x^k} \omega.
\]

We start to denote \( \partial_k := \partial_{\partial x^k} \) and \( \bar{\partial}_k := \partial_{\bar{\partial} x^k} \). These operators have adjoints \( d^*, \partial^* \) and \( \bar{\partial}^* \), respectively. We also have that \( d^* = \partial^* + \bar{\partial}^* \). We recall their well known formulas:

\[
d^* = -g^{\alpha\beta} i \left( \frac{\partial}{\partial x^\alpha} \right) \nabla_{\partial x^\beta}, \quad \partial^* = -\frac{1}{2} g^{k\bar{j}} i (\partial_k) \nabla_{\partial \bar{j}}, \quad \bar{\partial}^* = -\frac{1}{2} g^{k\bar{j}} i (\bar{\partial}_k) \nabla_{\bar{\partial} \bar{j}}.
\]

Here \( i (X) \omega \) denotes the interior product of \( \omega \) by \( X \), and \( \alpha, \beta \in \{1, \ldots, 2m\} \) are used to denote real coordinate indices. From now on, we use normal complex
Proposition 1.1. A forms \( f \) and sufficient condition on \( p < q < 2m \). Since this is clearly true for functions, from now on we let \( 0 \)

Proof. Note that \( \Delta := dd^* + d^*d \)

is positive and self adjoint. One can also define two other operators

\[
\Delta^\partial := \partial \partial^* + \partial^* \partial \quad \text{and} \quad \Delta^\bar{\partial} := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial},
\]

which map \( A^{p,q}(M) \) into itself. The fact that \((M,g)\) is Kähler implies

\[
\Delta = \Delta^\partial = \Delta^\bar{\partial}.
\]

In particular, \( \Delta \) preserves the space \( A^{p,q}(M) \).

Now let us assume we have a weight \( f \in C^\infty(M) \), which gives us a new volume form \( dv_f := e^{-f}dv \). We then have the corresponding adjoint operators \( d^*_f, \partial^*_f \) and \( \bar{\partial}^*_f \). For example,

\[
\int_M \langle d\omega, \theta \rangle e^{-f} = \int_M \langle \omega, d^*_f \theta \rangle e^{-f}.
\]

The corresponding formulas for these operators are easy to find:

\[
\begin{align*}
d^*_f &= d^* + i(\nabla f) \\
\partial^*_f &= \partial^* + i(\nabla^{1,0} f) \\
\bar{\partial}^*_f &= \bar{\partial}^* + i(\nabla^{0,1} f),
\end{align*}
\]

where

\[
\nabla^{1,0} f := \frac{1}{2}g^{jk} f_j \partial_j \quad \text{and} \quad \nabla^{0,1} f := \frac{1}{2}g^{jk} f_j \partial_k.
\]

Again, it holds that

\[
\nabla f = \nabla^{1,0} f + \nabla^{0,1} f \quad \text{and} \quad d^*_f = \partial^*_f + \bar{\partial}^*_f.
\]

From (1.1) it is easy to deduce the following formulas for the weighted Hodge Laplacian \( \Delta_f := dd^*_f + d^*_f d \), for \( \Delta^\partial_f := \partial \partial^*_f + \partial^*_f \partial \) and for \( \Delta^\bar{\partial}_f := \bar{\partial} \bar{\partial}^*_f + \bar{\partial}^*_f \bar{\partial} \).

\[
\begin{align*}
\Delta_f &= \Delta + \mathcal{L}_{\nabla f} \\
\Delta^\partial_f &= \Delta^\partial + \partial_i (\nabla^{1,0} f) + i(\nabla^{1,0} f) \partial \\
\Delta^\bar{\partial}_f &= \Delta^\bar{\partial} + \bar{\partial}_i (\nabla^{0,1} f) + i(\nabla^{0,1} f) \bar{\partial}.
\end{align*}
\]

We have denoted by \( \mathcal{L} \) the Lie derivative. Using (1.2), we can state the necessary and sufficient condition on \( f \) so that \( \Delta_f \) maps \((p,q)\) forms to \((p,q)\) forms, cf. [9]. Since this is clearly true for functions, from now on we let \( 0 < p + q < 2m \).

**Proposition 1.1.** The weighted Hodge Laplacian \( \Delta_f \) preserves the space of \((p,q)\) forms \( A^{p,q}(M) \) if and only if \( \nabla f \) is real holomorphic. In this case,

\[
\Delta_f = \Delta^\partial_f + \Delta^\bar{\partial}_f - \Delta.
\]

**Proof.** Note that

\[
\mathcal{L}_{\nabla f} = (d_i (\nabla^{1,0} f) + i(\nabla^{1,0} f) d_i) + (d_i (\nabla^{0,1} f) + i(\nabla^{0,1} f) d_i) \\
= (\partial_i (\nabla^{1,0} f) + i(\nabla^{1,0} f) \partial_i) + (\partial_i (\nabla^{0,1} f) + i(\nabla^{0,1} f) \bar{\partial}_i) \\
+ (\bar{\partial}_i (\nabla^{1,0} f) + i(\nabla^{1,0} f) \bar{\partial}_i) + (\partial_i (\nabla^{0,1} f) + i(\nabla^{0,1} f) \bar{\partial}_i) \\
= (\Delta^\partial_f - \Delta^\partial) + (\Delta^\bar{\partial}_f - \Delta^\bar{\partial}) + S \\
= \Delta^\partial_f + \Delta^\bar{\partial}_f - 2\Delta + S,
\]
where
\[ S = \left( \bar{\partial} (\nabla^{1,0} f) + i (\nabla^{1,0} f) \bar{\partial} \right) + \left( \partial (\nabla^{0,1} f) + i (\nabla^{0,1} f) \partial \right) = S_1 + S_2. \]

According to (1.2), we find that
\[ \Delta f = \Delta \partial f + \Delta \bar{\partial} f - \Delta + S. \]

Hence, we can finish the proof by computing \( S(\omega) \) for \( \omega \in A^{p,q}(M) \).

We fist note that in local coordinates
\[ S_1(\omega) = \bar{\partial} (\nabla^{1,0} f) \omega + i (\nabla^{1,0} f) \bar{\partial} \omega = \frac{1}{2} f_{k\bar{j}} dz^j \wedge i (\partial_k) \omega + \frac{1}{2} f_{k\bar{j}} (\bar{\partial} (\partial_k) + i (\partial_k) \bar{\partial}) \omega = \frac{1}{2} f_{k\bar{j}} dz^j \wedge i (\partial_k) \omega, \]
where we have used the fact that \( (\bar{\partial} (\partial_k) + i (\partial_k) \bar{\partial}) \omega = 0 \).

We compute in a similar fashion and obtain
\[ S_2(\omega) = f_{k\bar{j}} dz^j \wedge i (\partial_k) \omega. \]

Hence, \( S(\omega) = 0 \) if and only if \( S_1(\omega) = S_2(\omega) = 0 \) for all \( \omega \in A^{p,q}(M) \). This happens if and only if \( f_{j\bar{k}} = f_{\bar{j}k} = 0 \), which is the same as \( \nabla f \) being real holomorphic. \( \square \)

We now use this result to demonstrate Theorem 1.2. In fact, we will prove a stronger statement. Let us denote \( B_{x_0}(R) \) the geodesic ball centered at point \( x_0 \) of radius \( R > 0 \).

**Theorem 1.2.** Let \((M, g)\) be complete Kähler manifold and \( f \in C^\infty(M) \) with \( \nabla f \) real holomorphic. Suppose that \( u \) is an \( f \)-harmonic function on \( M \) and that there exists a constant \( C > 0 \) so that
\[ \int_{B_{x_0}(R)} |\nabla u|^2 e^{-f} \leq CR^2, \]
for all \( R \geq R_0 \). Then \( u \) is pluriharmonic. If, in addition, \( f \) is proper, then \( u \) is constant on \( M \).

**Proof.** For an \( f \)-harmonic function \( u \), it can be checked that the 1–form \( \omega := du \) is also \( f \)-harmonic, \( \Delta_f \omega = 0 \). However, by splitting \( \omega = \partial u + \bar{\partial} u \) into \((1,0)\) and \((0,1)\) components, we find that \( \Delta_f \partial u = 0 \) as \( \Delta_f \) preserves the \((1,0)\) and \((0,1)\) forms by Proposition 1.1.

So the \((1,0)\) form \( \theta := \partial u \) verifies \( \Delta_f \theta = 0 \) and has growth rate
\[ \int_{B_{x_0}(R)} |\theta|^2 e^{-f} \leq CR^2. \]

Let \( \phi \) be the cut-off with support in \( B_{x_0}(2R) \) defined by
\[ \phi(x) = \begin{cases} 1 & \text{on } B_{x_0}(R) \\ \frac{1}{R} (2R - d(x_0, x)) & \text{on } B_{x_0}(2R) \setminus B_{x_0}(R) \end{cases} \]
By $\Delta_f \theta = 0$ we see that

\begin{equation}
(1.3) \quad 0 = \int_M \left( (dd^*_f + d^*_f d) \theta, \phi^2 \theta \right) e^{-f}
\end{equation}

\begin{align*}
&= \int_M \left( d^*_f \theta, d^*_f (\phi^2 \theta) \right) e^{-f} + \int_M \langle d \theta, d (\phi^2 \theta) \rangle e^{-f} \\
&\geq \int_M |d\theta|^2 \phi^2 e^{-f} + \int_M |d^*_f \theta|^2 \phi^2 e^{-f} - 2 \int_M |d^*_f \theta| |\theta| |\nabla \phi| e^{-f} \\
&- 2 \int_M |d\theta| |\theta| \phi |\nabla \phi| e^{-f},
\end{align*}

where in the last line we have used the Cauchy-Schwarz inequality and that

\begin{align*}
d (\phi^2 \theta) &= \phi^2 d\theta + d\phi^2 \wedge \theta \\
d^*_f (\phi^2 \theta) &= \phi^2 d^*_f \theta - i (\nabla \phi^2) \theta.
\end{align*}

It follows from (1.3) that

\begin{equation}
(1.4) \quad \int_M |d\theta|^2 \phi^2 e^{-f} + \int_M |d^*_f \theta|^2 \phi^2 e^{-f} \leq 8 \int_M |\theta|^2 |\nabla \phi|^2 e^{-f}.
\end{equation}

Since by the assumption $\int_M |\theta|^2 |\nabla \phi|^2 e^{-f} \leq 4C$, (1.4) implies that

\begin{equation}
(1.5) \quad \int_M |d\theta|^2 e^{-f} + \int_M |d^*_f \theta|^2 e^{-f} < \infty.
\end{equation}

Using (1.3) again, we obtain

\begin{align*}
&\int_M |d\theta|^2 \phi^2 e^{-f} + \int_M |d^*_f \theta|^2 \phi^2 e^{-f} \\
&\leq \frac{2}{R} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d^*_f \theta| |\theta| e^{-f} + \frac{2}{R} \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d\theta| |\theta| e^{-f} \right) \\
&\leq \frac{2}{R} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |\theta|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d^*_f \theta|^2 e^{-f} \right)^{\frac{1}{2}} \\
&\quad + \frac{2}{R} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |\theta|^2 e^{-f} \right)^{\frac{1}{2}} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d\theta|^2 e^{-f} \right)^{\frac{1}{2}} \\
&\leq 4\sqrt{C} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d^*_f \theta|^2 e^{-f} \right)^{\frac{1}{2}} + 4\sqrt{C} \left( \int_{B_{\rho}(2R) \setminus B_{\rho}(R)} |d\theta|^2 e^{-f} \right)^{\frac{1}{2}}.
\end{align*}

Together with (1.5), this implies that $d\theta = d^*_f \theta = 0$. Now that $u$ is pluriharmonic follows immediately from

\[ \overline{\partial} \partial u = \partial u = d \theta = 0. \]

The second conclusion that $u$ is constant follows as in [13]. Indeed, integrating by parts, we have

\begin{align*}
\int_{\{f \leq t\}} |\nabla u|^2 e^{-f} &= - \int_{\{f \leq t\}} u \Delta_f u e^{-f} + \int_{\{f = t\}} u \frac{\langle \nabla u, \nabla f \rangle}{|\nabla f|} e^{-f} \\
&= 0.
\end{align*}
The first integral above is zero because $\Delta f u = 0$, while the second term is zero because $u$ being pluriharmonic implies in particular that $\Delta u = 0$, hence $\langle \nabla u, \nabla f \rangle = 0$. □

In [12], a result similar to Theorem 1.2 was obtained for a harmonic function with its Dirichlet energy grows no faster than $o \left( R^2 \right)$ on a sequence of geodesic balls of radius $R_i$. Obviously, our result generalizes and strengthens this statement. The improvement to $O \left( R^2 \right)$ also enables us to conclude the following.

**Proposition 1.3.** Let $(M, g, f)$ be a complete Kähler shrinking Ricci soliton of complex dimension $m = 2$. Then any bounded harmonic function on $M$ must be constant.

**Proof.** Let $u$ be a bounded harmonic function. Then the $(1, 0)$ form $\theta = \partial u$ is harmonic. We claim that there exists $C > 0$ so that for all $R \geq R_0$,

$$\int_{B_{x_0}(R)} |\theta|^2 \leq CR^2. \tag{1.6}$$

Indeed, this follows from a reverse Poincaré inequality and the fact that $u$ is bounded. For a cut-off $\phi$ as in Theorem 1.2 we have

$$\int_M |\nabla u|^2 \phi^2 = -\int_M \langle \nabla u, \nabla \phi^2 \rangle u$$

$$\leq \frac{1}{2} \int_M |\nabla u|^2 \phi^2 + 2 \int_M u^2 |\nabla \phi|^2$$

$$\leq \frac{1}{2} \int_M |\nabla u|^2 \phi^2 + \frac{C}{R^2} \text{Vol} (B_{x_0} (2R))$$

$$\leq \frac{1}{2} \int_M |\nabla u|^2 \phi^2 + CR^2,$$

where in the last line we have used the fact that the volume growth of a shrinking Ricci soliton is at most Euclidean by [4].

This proves (1.6). Now Theorem 1.2 implies that $u$ is pluriharmonic. The conclusion that $u$ is constant follows from [13]. Indeed, we may lift $u$ to a holomorphic function on the universal covering $\tilde{M}$ of $M$, which we continue to denote by $u$. So we have a bounded holomorphic function on a complete Kähler shrinking Ricci soliton of complex dimension $m = 2$. According to [13], the space of holomorphic functions of any fixed polynomial growth order $d > 0$ is finite dimensional. This implies that the space of bounded holomorphic functions is trivial. The proposition is proved. □

## 2. Harmonic maps

In this section we prove Theorem 1.3. We let $(M, g)$ be a Kähler manifold of complex dimension $m$, admitting a function $f$ so that $\nabla f$ is real holomorphic. Consider another Kähler manifold $(N, h)$ of complex dimension $n$. A map $u : M \to N$ is called $f$–harmonic if $u$ is a critical point of the weighted energy

$$E_f (u) = \frac{1}{2} \int_M |du|^2 e^{-f}.$$
with respect to any compactly supported variation of $u$. We note that in local coordinates,

$$|du|^2 = 2 \left( |\partial u|^2 + |\bar{\partial} u|^2 \right) = 2 \left( h_{ab} g^{jk} u^a_j u^b_k + h_{ab} g^{jk} \bar{u}^a_j \bar{u}^b_k \right).$$

The Euler-Lagrange equation implies

$$\tau_f(u) := \tau(u) - i (\nabla f) du = 0,$$

where $\tau(u) = \text{div} (\nabla u)$ is the usual tension field of $u$. In local coordinates, this means that

$$\Delta_f u^a + g^{jk} \bar{\Gamma}^a_{bc} u^b_j u^c_k = 0,$$

where

$$\Delta_f u^a = g^{jk} \frac{\partial^2 u^a}{\partial z^j \partial \bar{z}^k} - \frac{1}{2} g^{jk} \left( u^a_j f_k + u^a_k f_j \right).$$

Here the indices $a, b = 1, 2, \ldots, n$ are used to indicate the local coordinates on $N$ and $\Gamma^a_{bc}$ are the Christoffel symbols on $N$. We now prove the following.

**Theorem 2.1.** Let $(M, g)$ be a complete Kähler manifold and suppose there exists $f \in C^\infty(M)$ so that $\nabla f$ is real holomorphic. Assume in addition that $f$ achieves its minimum at an isolated critical point $x_0 \in M$. Then any $f$–harmonic map $u : M \to N$ of finite total weighted energy into a Kähler manifold with strongly seminegative curvature must be constant.

We divide the proof of this theorem in two parts, each of independent interest. In the first lemma, we follow the ideas of Siu [18], with the necessary modifications in the weighted case inspired by our work in [14], to show that $u$ must be pluriharmonic. This, in particular, implies that $i (\nabla f) du = 0$.

**Lemma 2.2.** Let $(M, g)$ be a complete Kähler manifold and suppose there exists $f \in C^\infty(M)$ so that $\nabla f$ is real holomorphic. Then any $f$–harmonic map $u : M \to N$ of finite total weighted energy into a Kähler manifold $N$ of strongly seminegative curvature must be pluriharmonic. In particular, it is harmonic and $i (\nabla f) du = 0$.

**Proof.** By the hypothesis, $u : M \to N$ satisfies

$$\tau_f(u) = 0$$

$$\int_M |du|^2 e^{-f} < \infty.$$

Consider a cut-off function $\phi$ with support in $B_{x_0}(2R)$, $\phi = 1$ on $B_{x_0}(R)$ and $|\nabla \phi| \leq \frac{1}{R}$ on $M$. In the argument that follows, we write $du = \partial u + \bar{\partial} u$, where $\partial u$ and $\bar{\partial} u$ are given by

$$\partial u = \frac{\partial u^a}{\partial z^j} dz^j \otimes \frac{\partial}{\partial w^a}$$

and

$$\bar{\partial} u = \frac{\partial u^a}{\partial \bar{z}^j} d\bar{z}^j \otimes \frac{\partial}{\partial \bar{w}^a},$$

with $\{w^a\}_{a=1,\ldots,n}$ being the local complex coordinates on $N$. We further denote

$$D \bar{\partial} u = u^a_{jk} dz^j \otimes d\bar{z}^k \otimes \frac{\partial}{\partial w^a},$$

where

$$u^a_{jk} := \frac{\partial u^a_k}{\partial z^j} + \bar{\Gamma}^a_{bc} u^b_j u^c_k.$$
Integration by parts implies
\begin{equation}
\int_M |D\tilde{u}|^2 \phi^2 e^{-f} = \int_M \left| u^a_{jk}\right|^2 \phi^2 e^{-f} = \int_M u^a_{jk} \overline{u^b_{jk}} \phi^2 e^{-f} = - \int_M u^a_{jk} \overline{u^b_{jk}} \phi^2 e^{-f} + \int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} - \int_M u^a_{jk} \overline{u^b_{jk}} (\phi^2)_j e^{-f}. 
\end{equation}

Let us note that
\[ u^a_{jkj} = \frac{\partial u^a_{jk}}{\partial \overline{z^j}} - \Gamma^h_{jk} u^a_{jh} + \Gamma^h_{kj} u^a_{j}e. \]

We now investigate each term in (2.1). First, a well known computation in [18] yields
\begin{equation}
u^a_{jkj} = u^a_{k} + K^a_{bcd} u^b_k \overline{u^c_{j}} - K^a_{bcd} u^b_j \overline{u^c_{k}}, \end{equation}
where
\[ K^a_{bcd} = \frac{\partial \Gamma^a_{bc}}{\partial \overline{u^d}} \]
is the curvature tensor on $N$. The hypothesis that the curvature of $N$ is strongly seminegative implies that
\[ K^a_{b\overline{c}d} \left( u^b_k \overline{u^c_j} - u^b_j \overline{u^c_k} \right) = \frac{1}{2} K^a_{b\overline{c}d} \left( u^b_k \overline{u^c_j} - u^b_j \overline{u^c_k} \right) \leq 0. \]

Therefore, from this computation and (2.1) we conclude the following
\begin{equation}
\int_M |D\tilde{u}|^2 \phi^2 e^{-f} \leq - \int_M \tau^a_{\overline{e}} (u^a_k) \overline{u^c_{k}} \phi^2 e^{-f} + \int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} - \int_M u^a_{jk} \overline{u^b_{jk}} (\phi^2)_j e^{-f}. \end{equation}

In a similar fashion, we get
\begin{equation}
\int_M |D\tilde{u}|^2 \phi^2 e^{-f} \leq - \int_M \tau^a_{\overline{e}} (u^a_k) \overline{u^c_{k}} \phi^2 e^{-f} + \int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} - \int_M u^a_{jk} \overline{u^b_{jk}} (\phi^2)_j e^{-f}. \end{equation}

Adding (2.3) and (2.4) and integrating by parts, we obtain that
\begin{equation}
\int_M |D\tilde{u}|^2 \phi^2 e^{-f} \leq \int_M \tau^a_{\overline{e}} (u^a_k) \tau^b_{\overline{e}} (u^b_k) \phi^2 e^{-f} + \frac{1}{2} \int_M \tau^a_{\overline{e}} (u^a_k) \left( \overline{u^c_{k}} \phi^2 \right)_k + \overline{u^c_{k}} \left( \phi^2 \right)_k e^{-f} + \frac{1}{2} \int_M u^a_{jk} \left( \overline{u^c_{k}} f^e + \overline{u^b_{jk}} f^e \right) \phi^2 e^{-f} - \frac{1}{2} \int_M u^a_{jk} \left( \overline{u^c_{k}} \phi^2 \right)_j + \overline{u^c_{k}} \left( \phi^2 \right)_j e^{-f}. \end{equation}

Note the first term in the right side of (2.5) is zero as $u$ is $f$—harmonic. Furthermore, integration by parts implies
\begin{equation}
\int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} = - \int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} + \int_M u^a_{jk} \overline{u^b_{jk}} f^e \phi^2 e^{-f} - \int_M u^a_{jk} \overline{u^b_{jk}} (\phi^2)_j e^{-f}, \end{equation}
where we have used the fact that \( f_{jk} = f_{kj} = 0. \)

One obtains a similar formula for \( \int_M u_j^a u_k^a f_k \phi^2 e^{-f}. \) Putting together, we see that the third term in the right side of (2.10) becomes

\[
(2.6) \quad \frac{1}{2} \int_M u_j^a u_k^a (\overline{\nabla} u_j f_k + \overline{u_j f_k}) \phi^2 e^{-f} = -\frac{1}{2} \int_M (u_j^a f_j + u_k^a f_k) \overline{\nabla}(u) \phi^2 e^{-f} + \int_M \text{Re} (u_j^a u_k^B f_k) \phi^2 e^{-f} - \frac{1}{2} \int_M u_j^a u_k^a (\phi^2)_j e^{-f}.
\]

Plugging into (2.5), we conclude

\[
(2.7) \quad \int_M |\overline{\partial} u|^2 \phi^2 e^{-f} \leq -\frac{1}{4} \int_M |u_j^a f_j + u_k^a f_k|^2 \phi^2 e^{-f} + \int_M \text{Re} (u_j^a u_k^B f_k) \phi^2 e^{-f} + \int_M \overline{\nabla}(u) \phi^2 e^{-f} - \frac{1}{2} \int_M u_j^a u_k^a (\phi^2)_j e^{-f}.
\]

Note that

\[
\frac{1}{4} |u_j^a f_j + u_k^a f_k|^2 - \text{Re} (u_j^a u_k^B f_k) = \frac{1}{4} |u_j^a f_j - u_k^a f_k|^2 \geq 0.
\]

Also, the last term in (2.7) can be estimated as

\[
\int_M |u_j^a u_k^a (\phi^2)_j|^2 e^{-f} \leq 2 \int_M |D \overline{\partial} u| |du| |\nabla| \phi |e^{-f}.
\]

Hence, by the Cauchy-Schwarz inequality, (2.7) becomes

\[
(2.8) \quad \frac{1}{2} \int_M |\overline{\partial} u|^2 \phi^2 e^{-f} \leq -\frac{1}{4} \int_M |u_j^a f_j - u_k^a f_k|^2 \phi^2 e^{-f} + 2 \int_M |du|^2 |\nabla| \phi |^2 e^{-f} + \int_M \text{Re} (u_j^a u_k^B f_k) \phi^2 e^{-f} + \frac{1}{2} \int_M u_j^a u_k^a (\phi^2)_j e^{-f}.
\]

We now deal with the other terms as follows. We have that

\[
(2.9) \quad \int_M \overline{\nabla}(u) i (\overline{\nabla| \phi|}^2) du^a e^{-f} - \frac{1}{2} \int_M u_j^a u_k^a f_j (\phi^2)_j e^{-f} - \frac{1}{2} \int_M u_j^a u_k^a (\phi^2)_j e^{-f} = \frac{1}{4} \int_M (u_j^a f_j - u_k^a f_k) (\overline{u_j^a \phi_j^a} - \overline{u_k^a \phi_k^a}) e^{-f}
\]

\[
\leq \frac{1}{4} \int_M |u_j^a f_j - u_k^a f_k|^2 \phi^2 e^{-f} + \int_M |du|^2 |\nabla| \phi |^2 e^{-f}.
\]

Putting (2.9) into (2.8), we get

\[
(2.10) \quad \frac{1}{2} \int_M |\overline{\partial} u|^2 \phi^2 e^{-f} \leq 4 \int_M |du|^2 |\nabla| \phi |^2 e^{-f}.
\]

Since \( \int_M |du|^2 e^{-f} < \infty, \) it is easy to see that as \( R \to \infty, \)

\[
\int_M |du|^2 |\nabla| \phi |^2 e^{-f} \to 0.
\]

Therefore, by letting \( R \to \infty \) in (2.10), we conclude that \( \overline{\partial} u = 0 \) or \( u \) is pluriharmonic. In particular, this implies that \( u \) is a harmonic map. Hence, \( \tau(u) = 0 \) and \( i (\nabla f) du = 0. \) This proves the lemma. \( \square \)
We point out that Lemma 2.2 also holds under energy growth assumption on $u$ that
\[ \int_{B_{x_0}(R)} |du|^2 e^{-f} \leq CR^2, \quad \text{for all } R \geq R_0. \]
The argument for this improvement is similar to that of Theorem 1.2.

We next present a local result that holds for harmonic maps between any two Riemannian manifolds. We show that $u$ must be constant if $u$ is harmonic and $i(\nabla f) du = 0$. 

**Lemma 2.3.** Let $(M, g)$ be a complete Riemannian manifold and $u : \Omega \to N$ a harmonic map from a domain $\Omega \subset M$ into a Riemannian manifold $N$. If $i(\nabla f) du = 0$ on $\Omega$ for a smooth function $f$ and $f$ has unique minimum point $x_0 \in \Omega$, then $u$ must be constant.

**Proof.** Since this lemma is stated in a Riemannian setting, we will denote here
\[ du = u^a_k dx^k \otimes \frac{\partial}{\partial y^a}, \]
where $\{x^k\}_{k=1,\ldots,2m}$ and $\{y^a\}_{a=1,\ldots,2n}$ are real coordinates on $M$ and $N$. The fact that $u$ is harmonic and $i(\nabla f) du = 0$ means that
\begin{align}
\Delta u^a + g^{kj} \Gamma^a_{bc} u^b u^c_j &= 0, \\
g^{kj} u^a_k f_j &= 0.
\end{align}

Let $\delta > 0$ be sufficiently small so that $u(B_{x_0}(\delta)) \subset B_{y_0}(\rho)$, where $y_0 = u(x_0)$, and the exponential map $\exp_{y_0} : B_0(\rho) \subset T_{y_0}N \to B_{y_0}(\rho)$ is a diffeomorphism. Under the induced normal coordinates, we have that for $y \in B_{y_0}(\eta)$,
\[ |h_{ab}(y) - \delta_{ab}| \leq C \eta \quad \text{and} \quad \left| \frac{\partial h_{ab}}{\partial y^c} (y) \right| \leq C \eta \]
for all $\eta \leq \rho$, where $C$ is a constant independent of $\eta$.

We normalize $f$ so that $f(x_0) = 0$. Since $x_0$ is an isolated critical point, there exists $\varepsilon > 0$ small enough so that the level set $\{f = \varepsilon\}$ has a connected component completely contained in $B_{x_0}(\delta)$. Denote by
\[ D(\varepsilon) := \{f \leq \varepsilon\} \cap B_{x_0}(\delta) \]
and note that $\partial D(\varepsilon)$ has unit normal vector $\nu := \frac{\nabla f}{|\nabla f|}$. Clearly, $u(D(\varepsilon)) \subset B_{y_0}(\eta)$ with $\eta \to 0$ as $\varepsilon \to 0$.

Integrating by parts,
\begin{align}
\int_{D(\varepsilon)} g^{kj} h_{ab} u^a_k u^b_j &= - \int_{\partial D(\varepsilon)} h_{ab} (\Delta u^a) u^b - \int_{D(\varepsilon)} \langle dh_{ab}, du^a \rangle u^b \\
&\quad + \int_{\partial D(\varepsilon)} \frac{1}{|\nabla f|} g^{kj} h_{ab} u^a_k u^b f_j \\
&\leq C \eta \int_{D(\varepsilon)} |du|^2,
\end{align}
where we have used (2.11) so that the boundary term is zero, and that
\[ |\Delta u^a| = |g^{kj} \Gamma^a_{bc} u^b_k u^c_j | \leq C \eta |du|^2, \]
as well as
\[ \langle dh_{ab}, du^a \rangle u^b = g^{kj} \frac{\partial h_{ab}}{\partial x^k} u^a_j u^b_k \leq C\eta |du|^2. \]

By choosing \( \varepsilon > 0 \) to be sufficiently small, this implies that \( |du| = 0 \) on \( D(\varepsilon) \). Therefore, \( u \) is constant on \( D(\varepsilon) \). By the unique continuation property, \( u \) must be constant on \( \Omega \).

We can now prove Theorem 2.1. Using Lemma 2.2, we see that \( u : M \to N \) must be harmonic and \( i(\nabla f) du = 0 \) on \( M \). Now Lemma 2.3 says that \( u \) must be constant on \( M \). This proves Theorem 2.1.

It turns out under the hypothesis in Theorem 2.1, we also have Liouville property for harmonic maps, not just for weighted ones. The idea is to show that \( i(\nabla f) du = 0 \) again and then appeal to Lemma 2.3.

**Theorem 2.4.** Let \( (M, g) \) be a compact Kähler manifold and suppose there exists \( f \in C^\infty(M) \) so that \( \nabla f \) is real holomorphic. Assume in addition that \( f \) achieves its minimum at an isolated critical point. Then any harmonic map \( u : M \to N \) into a Kähler manifold of strongly seminegative curvature must be constant.

**Proof.** Since \( u : M \to N \) is harmonic and \( N \) has strongly seminegative curvature, Siu’s theorem in [18] implies that \( u \) is pluriharmonic, or
\[ u^a_{jk} = 0. \]  

We now define the flow induced by the vector field \( J(\nabla f) \).
\[
\frac{d\phi_t}{dt} = J(\nabla f)(\phi_t) \quad \phi_0 = \text{Id}.
\]

Since \( \nabla f \) is real holomorphic, \( J(\nabla f) \) is a Killing vector field. So \( \phi_t \) is a one parameter group of isometries of \( M \). In particular,
\[ u_t := u \circ \phi_t \]
is a continuous family of harmonic maps from \( M \) to \( N \). Since \( N \) has strongly seminegative curvature, it has nonpositive sectional curvature as well. We now use the uniqueness theorem for harmonic maps in [17] to show that \( u_t = u \) for all \( t \geq 0 \).

Indeed, lifting \( u_t \) to the universal coverings \( \tilde{M} \) and \( \tilde{N} \), we get a family of harmonic maps \( \tilde{u}_t : \tilde{M} \to \tilde{N} \). Using the fact that \( u_t \) is homotopic to \( u \) and \( N \) has nonpositive curvature, a standard computation shows that \( \tilde{r}^2(\tilde{u}_t, \tilde{u}_0) \) descends to \( M \) and is subharmonic, where \( \tilde{r} \) is the distance function on \( \tilde{N} \). Therefore, for each fixed \( t \), \( \tilde{r}^2(\tilde{u}_t, \tilde{u}_0) \) is a constant function on \( M \) as \( M \) is compact. However, at the minimum point \( x_0 \) of \( f \), \( \phi_t(x_0) = x_0 \). This means that \( u_t(x_0) = u_0(x_0) \) for all \( t \geq 0 \). In turn, it shows that \( \tilde{r}^2(\tilde{u}_0(\tilde{x}_0), \tilde{u}_t(\tilde{x}_0)) = 0 \). Hence, \( \tilde{r}^2(\tilde{u}_0, \tilde{u}_t) = 0 \) and \( u_0 = u_t \) for all \( t \geq 0 \).

We now differentiate the equation \( u_0 = u_t \) in \( t \) and get that
\[
0 = \frac{d}{dt} u_t = i(J(\nabla f)) du.
\]
This means, in complex coordinates, that
\[(2.15)\]
\[u_k^a f_k = u_k^a f_k.\]
Using (2.14), we see that
\[(2.16)\]
\[(u_k^a f_k)_{\bar{j}} = u_k^a f_k = 0\]
\[(u_k^a f_k)_{\bar{j}} = u_k^a f_k = 0,\]
where we have also made use of \(f_{kj} = f_{\bar{k}\bar{j}} = 0\) as \(\nabla f\) is real holomorphic. By (2.15) and (2.16), we conclude that
\[(u_k^a f_k)_{\bar{j}} = (u_k^a f_k)_{\bar{j}} = 0.\]
This forces the function \(u_k^a f_k\) to be constant on \(M\). Since \(f\) has a critical point on \(M\), \(u_k^a f_k = 0\) on \(M\). This proves that \(i(\nabla f) d\mu = 0\).

As mentioned earlier, the existence of a function \(f\) with \(\nabla f\) being real holomorphic on \(M\) has strong implications on the topology of \(M\). An early result in this direction was proved by Frankel [6], which states that all odd Betti numbers of a compact Kähler manifold must be zero if it has a Killing vector field whose zero set is non-empty and discrete. Howard [10] proved a result of similar nature. In particular, it says that a projective manifold \(M\) admitting a holomorphic vector field with nonempty and discrete zero set has only trivial holomorphic \((p,0)\)-forms.

We establish below a vanishing result for holomorphic forms under an assumption that is more in the spirit of the preceding Liouville type results.

**Theorem 2.5.** Let \((M,g)\) be a complete Kähler manifold. Suppose there exists \(f \in C^\infty(M)\) such that \(\nabla f\) is real holomorphic and bounded on \(M\). Assume in addition that \(f\) has an isolated minimum point in \(M\). Then any \(L^2\) holomorphic \((p,0)\)-form on \(M\) must be zero for all \(p \geq 0\).

**Proof.** We proceed by induction on \(p\). For \(p = 0\), this is certainly true as any \(L^2\) holomorphic function must be constant. Let us assume that the result holds for \((p-1)\). We now prove it for \(p\)-forms. Consider
\[\omega = \frac{1}{p!} \omega_{i_1,\ldots,i_p} dz^{i_1} \wedge \ldots \wedge dz^{i_p},\]
an \(L^2\) holomorphic \(p\)-form. Now the \((p-1)\) form
\[\theta = \omega (\cdot,\ldots,\cdot, \nabla f) = \frac{1}{(p-1)!} \left( \omega_{i_1,\ldots,i_p, f_{i_p}} \right) dz^{i_1} \wedge \ldots \wedge dz^{i_{p-1}}\]
is holomorphic as \(\nabla f\) is real holomorphic and \(\omega\) is holomorphic. It is also in \(L^2\) as \(\nabla f\) is bounded on \(M\). By the induction hypothesis, \(\theta = 0\). Hence,
\[(2.17)\]
\[\omega_{i_1,\ldots,i_p, f_{i_p}} = 0.\]
The rest of the argument is local and around an isolated minimum point \(x_0\) of \(f\). Note that since \(\omega\) is holomorphic and in \(L^2\), it is also harmonic, closed, and co-closed. Thus, in a fixed complex local coordinate chart \(U\) at \(x_0\), we know that \(\omega\) is exact, i.e., \(\omega = \partial \eta\) for a \((p-1,0)\) form \(\eta\) defined on \(U\).
We normalize $f$ so that $f(x_0) = 0$. Since $x_0$ is an isolated critical point, there exists $\varepsilon > 0$ small enough so that the level set $\{ f = \varepsilon \}$ has a connected component completely contained in $U$. Let

$$D(\varepsilon) := \{ f \leq \varepsilon \} \cap U$$

and note that $\partial D(\varepsilon)$ has normal vector $\nu := \frac{\nabla f}{|\nabla f|}$. On $D(\varepsilon)$ we have that $\omega = \partial \eta$. Therefore,

$$\int_{D(\varepsilon)} |\omega|^2 = \int_{D(\varepsilon)} \langle \omega, \partial \eta \rangle = -\int_{D(\varepsilon)} \langle \partial^* \omega, \eta \rangle + \int_{\partial D(\varepsilon)} \langle \omega, df \wedge \eta \rangle \frac{1}{|\nabla f|} = 0$$

as $\omega$ is co-closed and $\omega (\cdot, \ldots, \cdot, \nabla f) = 0$. This proves that $\omega = 0$ on $D(\varepsilon)$. Thus, $\omega = 0$ on $M$ by the unique continuation property. This proves the theorem. \qed

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