Quantum algorithms for hedging and the Sparsitron

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Abstract

A paradigmatic algorithm for online learning is the Freund/Schapire Hedge algorithm with multiplicative weight updates. For multiple time steps, the algorithm constructs an allocation into different strategies or experts for which it is guaranteed that a certain regret is never much greater than the minimally achievable, even in an adversarial situation. This work presents quantum algorithms for such online learning in an oracular setting. For $T$ time steps and $N$ strategies, we exhibit run times of about $O\left(\text{poly}(T)\sqrt{N}\right)$ for passively estimating the hedging losses and for actively betting according to quantum sampling. In addition, we discuss a quantum analogue of a machine learning algorithm, the Sparsitron, which is based on the Hedge algorithm. The quantum algorithm inherits the provable learning guarantees from the classical algorithm and exhibits polynomial speedups. The speedups shown here may find relevance in both finance, for example for estimating hedging losses, and machine learning, for example for learning a generalized linear model or an Ising model.
I. INTRODUCTION

Optimization is a cornerstone of machine learning and artificial intelligence. A great deal of quantum algorithm developments have been focused on quantum speedups for optimization problems [1], and in particular for convex optimization problems. Generic convex optimization in the oracle model was discussed in [2, 3]. Linear programming (LP) involves optimizing a linear function of a vector subject to constraints and semidefinite programming (SDP) is optimizing a linear function of a matrix subject to positivity constraints. Several quantum algorithms have been focused on these two convex programs. For SDPs, the classical Arora-Kale framework [4] was first considered for quantum algorithms in [5]. Reference [6] gives an improved quantum algorithm. Linear programming is a special case of semidefinite programming using diagonal matrices. Linear programs can also be mapped to zero-sum games, for which a linear time classical solver is by Grigoriadis and Khachiyan [7]. For zero-sum games, quantum algorithms were obtained in [8, 9]. Beyond LPs and SDPs, quadratic constraints are discussed by Clarkson, Hazan and Woodruff [10], which includes for example the classification of data points with a margin, kernel-based classification, minimum enclosing ball and $\ell_2$-margin support vector machines. Reference [9] provides corresponding quantum algorithms.

These optimization algorithms generically assume a quantum oracle that can be queried in superposition. Such a setting goes back to the beginnings of quantum computing with algorithms such as Grover’s search algorithm or the Deutsch-Jozsa algorithm. In machine learning, data are usually generated from an external source, such as users providing ratings to movies or products. In this case, quantum random access memory (QRAM) is discussed as a way to make such data available to a quantum algorithm [11, 12]. As with large-scale quantum computers, substantial efforts are required to build large-scale quantum RAMs. The oracle framework used in this work encompasses such a QRAM data access and also the case of access to a computable function. For a quantum algorithm’s output, a possible way is to encode the output in a quantum state. A famous example is the work of Harrow, Hassidim and Lloyd (HHL) [13] for solving linear system of equations. The solution to a linear system $Ax = b$ is provided by a quantum state $|x\rangle$ upon which measurements can be performed to obtain classically relevant information. Under some well-discussed conditions [14], the HHL algorithm can achieve an exponential quantum speedup. In contrast, in the optimization works mentioned above and in the present work, the output of the algorithm is inherently classical. These algorithms are hybrid, that is partly of classical and partly of quantum nature, and designed in a modular way so that the quantum part of the algorithm can be treated as a separate building block. The algorithms make quantum improvements on parts of the (best) available classical algorithm while keeping its overall structure intact. In contrast to the HHL algorithm for example, here the quantum versus classical speedup is usually at most polynomial, in most cases at most quadratic in the domain size of the input function. The quantum algorithm can deliver some speedup in the dimension of the problem, while in other relevant parameters it might not necessarily achieve any speedup, sometimes it can even be worse than the best classical algorithm.

Consider a game with $T$ rounds and the chance to play a mixture of $N$ different strategies at each round, which can also be seen as an idealized version of sports betting or stock market trading. The Freund/Schapire Hedge algorithm with multiplicative weight updates [15] allows for a strategy such that the losses after $T$ rounds are not worse than $\sqrt{2T \log N + \log N}$ compared to the minimum achievable “offline” loss. The classical complexity for this strategy
is $O(TN)$. In most applications, $T$ is much smaller than $N$, e.g., $T = O(\log N)$. In this work, we provide various quantum algorithms in the online learning/hedging scenario. Assuming appropriate oracles for the online loss information, we exhibit quantum speedups for two settings, which can be considered the passive and active setting. In the passive setting, we are interested in estimating the total loss after $T$ rounds without ever writing down the full probability vector and without making active gambling decisions. We obtain an $\epsilon$-accurate estimate of the total loss with high probability and with a query complexity of $O\left(\frac{T^3}{\epsilon^2}N\right)$ and a gate complexity of $\tilde{O}\left(\frac{T^3}{\epsilon^2}\sqrt{N}\right)$ via amplitude estimation. We use the notation $\tilde{O}(\cdot)$ to hide poly-logarithmic factors in any of the variables. The improvement in $N$ and worsening in $T$ is acceptable in the common situation when $T$ is much smaller than $N$. Furthermore, we consider the active setting where at each round a gambling decision is made and a bet is placed. For this active setting we prepare the relevant quantum state with amplitude amplification, sample from it and bet on the outcome. In the case when every allocation of a particular strategy comes at a transaction cost, the sampling setting has the advantage of reducing such costs. We again obtain a quantum speedup in $N$ while the loss of such a strategy remains close to the minimum loss with high probability. We note prior work on quantum speedups for the Hedge algorithm and the related adaptive boosting technique in [16].

The main motivation for discussing the Hedge algorithm is its application in optimization and machine learning. In the second part of this work, we provide a quantum algorithm for the Sparsitron [17]. The Sparsitron is a supervised learning algorithm based on the multiplicative weights algorithm by Freund and Schapire. The algorithm can be used to learn a Generalized Linear Model (GLM) and Ising models from training examples. For each $N$-dimensional training example, a loss is computed which takes into account a non-linear, potentially non-convex, activation function and an inner product between the training example and a weight vector. For guaranteed learning from the data with a certain accuracy $\epsilon$ (to be defined,) the run time is about $\tilde{O}\left(\frac{N}{\epsilon^2}\right)$. We present a classical algorithm called SparsitronApprox, which estimates the inner products instead of computing them exactly (Theorem 7.) The runtime of this algorithm is $\tilde{O}\left(\frac{N}{\epsilon^2} + \frac{1}{\epsilon}\right)$, improving on the original algorithm if, say, $N > \frac{1}{\epsilon^5}$. Subsequently, we present a quantum algorithm called the Quantum Sparsitron (Theorem 8.) The quantum algorithm uses quantum inner product estimation and achieves a run time of about $\tilde{O}\left(\frac{\sqrt{N}}{\epsilon^7}\right)$, a polynomial quantum speedup compared to the classical algorithm with respect to the dimension of the data. As a corollary, we derive a polynomial quantum speedup for the learning of Ising models (Corollary 1.)

Regarding notation, we use $[N]$ to denote the set $\{1, \cdots, N\}$, where $N \in \mathbb{Z}_+$. We write a vector plainly as $x$ without any special furnishing, however we use $\vec{0}$ and $\vec{1}$ to denote the all 0s and all 1s vector, respectively. The $\ell_1$-norm of a vector $x \in \mathbb{R}^N$ is given by $\|x\|_1 := \sum_{j=1}^{N} |x_j|$. The maximum element of a vector $x \in \mathbb{R}^N$ is given by $\|x\|_{\text{max}} := \max_{j} |x_j|$, also sometimes denoted by $x_{\text{max}}$. As mentioned, we use the notation $\tilde{O}(\cdot)$ to hide poly-logarithmic factors in any of the variables.
II. CLASSICAL HEDGE ALGORITHMS

A. Original algorithm

We follow Ref. [15] for the discussion of the classical Hedge algorithm. We are given \( N \) strategies for a game that takes \( T \) rounds. Before each \( t \in [T] \), we choose an assignment (portfolio) of the \( N \) strategies. This assignment shall be given by the weights 
\[
\mathbf{w}(t) = (w_1^{(t)}, \ldots, w_N^{(t)})^\dagger \in [0, 1]^N,
\]
which form the probability vector
\[
\mathbf{p}(t) = (p_1^{(t)}, \ldots, p_N^{(t)})^\dagger = \frac{1}{\|\mathbf{w}(t)\|_1} (w_1^{(t)}, \ldots, w_N^{(t)})^\dagger.
\] (1)
The initial allocation is taken to be uniform, i.e., \( \mathbf{w}^{(1)} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)^\dagger \) and \( \mathbf{p}^{(1)} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)^\dagger \). The algorithm considers an online learning setting, where information arrives over time and the weights are updated accordingly. Specifically, at each time \( t \in [T] \), we observe the loss vector 
\[
\mathbf{l}(t) = (l_1^{(t)}, \ldots, l_N^{(t)})^\dagger \in [0, 1]^N.
\] To avoid further complexities, we assume that each loss \( l_j^{(t)} \) takes a constant number of bits to specify. The loss at time \( t \) is given by
\[
L(t) := \sum_{i=1}^{N} p_i^{(t)} l_i^{(t)} \equiv \mathbf{p}(t) \cdot \mathbf{l}(t) \in [0, 1].
\] (2)
A strategy to minimize losses was shown in Ref. [15]. Take \( \beta \in (0, 1) \). The strategy is based on multiplicative updates to the weights given the incoming loss information as 
\[
w_j^{(t)} = \beta^{t-1} l_j^{(t-1)} w_j^{(t-1)},
\]
which for the full path up to \( t \) is 
\[
w_j^{(t)} = \beta^{\sum_{t'=1}^{t-1} l_j^{(t')}} w_j^{(1)}.
\]
We also write \( \mathbf{w}(t) \odot \beta^{(t)} \), where \( \odot \) is the element-wise vector multiplication and \( \beta^{(t)} \) is understood element-wise. The accumulated loss of the algorithm (denoted by \( \mathcal{H} \) for hedge) over \( T \) rounds is
\[
L_\mathcal{H} := \sum_{t=1}^{T} \mathbf{p}(t) \cdot \mathbf{l}(t).
\] (3)
On the other hand, consider the “offline loss”, 
\[
L_{\text{min}}^{(T)} = \min_j \sum_{t=1}^{T} l_j^{(t)},
\]
which gives the minimum loss achievable when choosing the same single strategy for all rounds of the game. Ref. [15] shows a “regret” bound for the losses of the multiplicative update strategy as
\[
L_\mathcal{H} - L_{\text{min}}^{(T)} \leq \sqrt{2T \log N} + \log N,
\] (4)
much better than the naive bound \( L_\mathcal{H} - L_{\text{min}}^{(T)} \leq T \). The run time of the classical algorithm is given by \( \tilde{O}(TN) \). At every step, the algorithm updates \( N \) probabilities and there are \( T \) steps overall. The log factors in the run time are due to processing pointers of size \( \mathcal{O}(\log N) \). In summary the algorithm is given as follows.
Algorithm 1 The original Hedge algorithm by Freund and Schapire \cite{15}

**Input:** Number of strategies \(N\), number of rounds \(T\), parameter \(\beta \in (0, 1)\), \(w^{(1)} = \mathbf{1}/N \in \mathbb{R}^N\).

for \(t = 1\) to \(T\) do
  \(p^{(t)} \leftarrow w^{(t)}/\|w^{(t)}\|_1\).
  Receive loss vector \(l^{(t)}\).
  Suffer loss \(p^{(t)} \cdot l^{(t)}\).
  \(w^{(t+1)} \leftarrow w^{(t)} \odot \beta^{(t)}\).
end for

**Output:** \(p^{(T+1)} \leftarrow w^{(T+1)}/\|w^{(T+1)}\|_1\), \(L_H = \sum_{t=1}^{T} p^{(t)} \cdot l^{(t)}\).

B. Active gambling by sampling

In a simple extension to the previous algorithm, we model the cost of allocating individual bets in the portfolio. As before, we are using the multiplicative weight update algorithm \(\mathcal{H}\) for adapting the probability distribution as time goes on. The investment costs are taken to be as follows. Each allocation based on the vector \(p^{(t)}\), however small, will come with a unit cost \(C\). This cost can be thought of as a transaction cost, as, for example, buying even a single stock on the stock market comes with the cost of calling a broker and paying a broker fee. To minimize the cost, at each time \(t\), we sample an index \(j^{(t)}\) according to \(p^{(t)}\) and bet on that single outcome. In the quantum case we are presented with a quantum state \(|p^{(t)}\rangle\) from which samples are obtained.

Algorithm 2 Hedge algorithm with investment cost

**Input:** Number of strategies \(N\), number of rounds \(T\), parameter \(\beta \in (0, 1)\), investment cost \(C\), \(w^{(1)} = \mathbf{1}/N \in \mathbb{R}^N\).

for \(t = 1\) to \(T\) do
  \(p^{(t)} \leftarrow w^{(t)}/\|w^{(t)}\|_1\).
  Allocate portfolio at cost \(N \times C\).
  Receive loss vector \(l^{(t)}\).
  Suffer loss \(p^{(t)} \cdot l^{(t)}\).
  \(w^{(t+1)} \leftarrow w^{(t)} \odot \beta^{(t)}\).
end for

**Output:** \(L_H = \sum_{t=1}^{T} p^{(t)} \cdot l^{(t)}\).

First, we state a theorem for the efficient sampling from a probability vector.

**Theorem 1** (\(\ell_1\)-sampling \cite{18,19}). Given an \(N\)-dimensional probability vector \(p\). There exists a data structure to sample an index \(j \in [N]\) with probability \(p_j\) which can be constructed in time \(\tilde{O}(N)\). One sample can be obtained in time \(\tilde{O}(1)\).

A comment on this result. Refs. \cite{18} shows an algorithm with preparation in \(\mathcal{O}(N)\) and sampling in \(\mathcal{O}(1)\) assuming constant time operations for addition, comparison, and random number generation, among others. However, storing and processing the pointer \(j \in [N]\) takes \(\mathcal{O}(\log N)\) bits and operations, hence we take the slightly worse \(\mathcal{O}(N)\) and \(\mathcal{O}(1)\), respectively. The sampling Hedge algorithm is as follows.
Algorithm 3 Sampling Hedge algorithm with investment cost

Input: Number of strategies $N$, number of rounds $T$, parameter $\beta \in (0, 1)$, investment cost $C$, $w^{(1)} = \frac{1}{N} \in \mathbb{R}^N$.

for $t = 1$ to $T$ do
  $p^{(t)} \leftarrow \frac{w^{(t)}}{\|w^{(t)}\|_1}$.
  $j^{(t)} \leftarrow$ Sample from $p^{(t)}$.
  Invest in $j^{(t)}$-th strategy at cost $C$.
  Receive loss vector $l^{(t)}$.
  Suffer loss $l^{(t)}_{j^{(t)}}$.
  $w^{(t+1)} \leftarrow w^{(t)} \odot \beta l^{(t)}$.
end for

Output: $L_{\text{samp}} := \sum_{t=1}^T l^{(t)}_{j^{(t)}}$.

Theorem 2. The run times of the classical algorithms are

Algorithm 2 : $\tilde{O} (TN(1 + C))$, \hspace{1cm} (5)
Algorithm 3 : $\tilde{O} (T(N + C))$. \hspace{1cm} (6)

Proof. For algorithm 2 we have $T$ iterations, each of which takes $N$ steps to prepare the probability distribution and perform the multiplicative update. In addition, at each step the investment cost is $N \times C$. For algorithm 3 at each step we prepare the data structure according to Theorem 1, sample, and invest in the single sampled strategy. Hence the transaction cost part of the run time is $C$, independent of $N$. \hfill \qed

Theorem 3. Algorithm 3 outputs $L_{\text{samp}} := \sum_{t=1}^T l^{(t)}_{j^{(t)}}$ such that $E[L_{\text{samp}}] = L_\mathcal{H}$. For $s \in \mathbb{R}_+$, $L_{\text{samp}} - \min_j L_j \leq (s + 1)\sqrt{2T \log N + \log N}$ with probability at least $1 - e^{-2s^2}$.

Proof. The expectation value is $E[L_{\text{samp}}] = \sum_{t=1}^T E[l^{(t)}_{j^{(t)}}] = \sum_{t=1}^T \sum_{j=1}^N p^{(t)}_j l^{(t)}_{j} = \sum_{t=1}^T l^{(t)} = L_\mathcal{H}$. From Hoeffding’s inequality we have $P[|L_{\text{samp}} - L_\mathcal{H}| \geq s\sqrt{T}] \leq e^{-2s^2}$. Now, bound the difference to the minimum loss by $L_{\text{samp}} - \min_j L_j = L_{\text{samp}} - L_\mathcal{H} + L_\mathcal{H} - \min_j L_j \leq L_{\text{samp}} - L_\mathcal{H} + \sqrt{2T \log N + \log N} \leq s\sqrt{T} + \sqrt{2T \log N + \log N} \leq (s + 1)\sqrt{2T \log N + \log N}$, with probability at least $1 - e^{-2s^2}$. \hfill \qed

III. QUANTUM HEDGE ALGORITHMS

We now turn to quantum algorithms in the Hedge setting. We provide two simple algorithms, one for estimating the losses, one for the gambling setting, before discussing the Sparsitron. The algorithms are based on quantum minimum finding, see Lemma 7 in Appendix A, and amplitude amplification and estimation, see Lemma 8 Appendix A. We first discuss the rescaling and shifting of relevant quantities. We then discuss the quantum data input model and state preparation subroutines. In a passive setting, we use amplitude estimation to estimate the total loss of the Hedge algorithm given the data input. In an active setting, we discuss the allocation of a portfolio via amplitude amplification and sampling.
A. Bounds for estimated quantities

One of the quantities estimated via a quantum algorithm is the $\ell_1$-norm $\|w^{(t)}\|_1$ for all $t$. The algorithm starts with a uniform initial weight vector $w^{(1)} = 1/N$ and the minimum weight achievable after $T$ steps is $w^{(T)}_{\min} := \min_j w^{(T)}_j = \beta T/N$. In a situation where also the maximum weight $w^{(T)}_{\max} := \|w^{(T)}\|_{\max}$ is $O(\beta T/N)$, the $\ell_1$-norm is small, i.e., $\|w^{(T)}\|_1 \sim \beta T \sim 1/2^T$. To overcome this inconvenient lower bound, we rescale the weights in the estimation. For rescaling, we require lower bounds for the sum of losses. For each $1 \leq t \leq T$, define similar to before the minimum offline loss up to $t$, $L^{(t)}_{\min} := \min_j \sum_{t'=1}^t l_j^{(t')} \leq t$. Note that the maximum element of $w^{(t)}$ is

$$w^{(t)}_{\max} = \beta^{L^{(t-1)}_{\min}} / N. \quad (7)$$

We consider the rescaled weights $\frac{w^{(t)}_j}{w^{(t)}_{\max}} \leq 1$, which keep the expected loss $L^{(t)} \equiv \frac{w^{(t)}_j l^{(t)}_j}{\|w^{(t)}\|_1}$ the same because the $w^{(t)}_{\max}$ factor cancels out. However, we have the lower bound for the $\ell_1$-norm $\|\frac{w^{(t)}_j}{w^{(t)}_{\max}}\|_1 \geq 1$. In the quantum context, we find $L^{(t-1)}_{\min}$ via the minimum finding algorithm [20] in run time $\tilde{O}\left(\sqrt{N}\right)$ with high success probability. See Lemma 7 in Appendix A for the statement of the minimum finding algorithm. With Eq. (7) we then compute the maximum weight.

Another similar problem concerns the Hedge algorithm loss estimation. The (unnormalized) loss involves the sum $\sum_{j=1}^N l_j^{(t)} w_j^{(t)}$. If all $l_j^{(t)}$ are 0 then obviously the sum is 0. If we assume that at least one loss is non-zero, we can estimate $\sum_{j=1}^N l_j^{(t)} w_j^{(t)} / \max_j l_j^{(t)} w_j^{(t)}$, where we find $\max_j l_j^{(t)} w_j^{(t)}$ via the quantum maximum finding algorithm. However, to be more general and allow all-zero loss vectors, we shift the losses as $l_j^{(t)} \rightarrow 1 + l_j^{(t)}$ and estimate the quantity

$$\mathcal{L}^{(t)} := \frac{1}{\mathcal{L}^{(t)}_{\max}} \sum_{j=1}^N \left(1 + l_j^{(t)}\right) w_j^{(t)}. \quad (8)$$

The normalization $\mathcal{L}^{(t)}_{\max} := \max_j \left(1 + l_j^{(t)}\right) w_j^{(t)}$ is again found via the quantum maximum finding algorithm. Note that $\mathcal{L}^{(t)} \geq 1$. Using the shifted loss comes at the cost of translating multiplicative estimates into additive estimates, as will be shown below.

B. Quantum data input model

We translate the online learning setting into the quantum domain. The input data for the quantum algorithms are the losses experienced at every step $t$. First, we assume $T$ different oracles, where the sequential access to these oracles embodies the online setting.

**Data Input 1 (Loss oracles).** Assume $O(1)$ bits are sufficient to specify the losses $l_j^{(t)}$. For $t \in [T], j \in [N]$ and any in-range bit string $c$, assume unitaries $U_i^{(t)}$ such that $U_i^{(t)} |j\rangle |c\rangle = |j\rangle |c \oplus l_j^{(t)}\rangle$, operating on $O(\log N)$ quantum bits. Also trivially assume the initial loss
unitary $U_i^{(0)} = 1$ corresponding to $w^{(1)} = 1/N$. Denote by $\mathcal{I}_t = \{U_i^{(t')} : 0 \leq t' \leq t\}$ the set of unitaries up to a time $t$.

The oracles allow us to perform the following operations.

**Lemma 1.** Let $t \in [T]$, $\beta \in (0, 1)$, and $\eta \in (0, 1)$.

(i) Let the set of unitaries $\mathcal{I}_{t-1}$ as in Data Input (1) and knowledge of $w_{\text{max}}^{(t)}$ be given. There exists a quantum computation for the weights as $|j\rangle|\bar{0}\rangle \rightarrow |j\rangle\left|\frac{w_j^{(t)}}{w_{\text{max}}^{(t)}}\right\rangle$, where

$$\frac{w_j^{(t)}}{w_{\text{max}}^{(t)}} \equiv \frac{\beta^{t-t-1} l_j^{(t')}}{N w_{\text{max}}^{(t)}}$$

is computed to additive accuracy $\eta$.

(ii) Let the set of unitaries $\mathcal{I}_t$ as in Data Input (1) and knowledge of $L_j^{(t)} := \max_j (1 + l_j^{(t)}) w_j^{(t)}$ be given. There exists a quantum computation $|j\rangle|\bar{0}\rangle \rightarrow |j\rangle\left|1 + l_j^{(t)} w_j^{(t)} w_{\text{max}}^{(t)}\right\rangle$ with additive accuracy $\eta$.

Both operations take $O(T)$ queries to the data input and $O(T + \log N + \log(1/\eta))$ qubits and $O(T + \log(1/\eta))$ quantum gates.

**Proof.** The $j$-th multiplicative update at $t$ is

$$\frac{w_j^{(t)}}{w_{\text{max}}^{(t)}} \equiv \frac{\beta \sum_{t'=1}^{t-1} l_j^{(t')} - l_j^{(t-1)}}{N w_{\text{max}}^{(t)}} = 0.b_1 \ldots b_{\log(1/\eta)} + \eta_j \leq 1,$$ where $b_k$ are the bits for a binary approximation to the true value with additive error $\eta_j \leq \eta$. The computational register thus involves $O(T)$ ancilla qubits for the losses and $O(\log(1/\eta))$ ancilla qubits for storing the result. Use oracle queries and the basic quantum circuits for addition and exponentiation to perform the steps $|j\rangle|\bar{0}\rangle \rightarrow |j\rangle\left|l_j^{(1)}\right\rangle \ldots \left|l_j^{(t-1)}\right\rangle|\bar{0}\rangle \rightarrow |j\rangle\left|l_j^{(1)}\right\rangle \ldots \left|l_j^{(t-1)}\right\rangle\left|\frac{w_j^{(t)}}{w_{\text{max}}^{(t)}}\right\rangle$ in time linear in the size of the register. Uncomputing the loss registers leads to the result. The argument for the shifted loss is analogous, using the additional loss oracle $U_i^{(t)}$. \(\square\)

**C. Quantum algorithm to obtain the total loss of the Hedge algorithm**

Our first quantum algorithm is simple. At each time step, we receive a loss oracle according to Data Input (1). From this input, we estimate the total loss of the multiplicative weight update method. We never fully exhibit the full weight vector but rather only the total loss at each step given the weight vectors. The quantum algorithm is as follows.
Algorithm 4 Quantum estimation of the loss of the Freund/Schapire strategy

**Input:** Number of strategies \( N \), number of rounds \( T \), parameter \( \beta \in (0, 1) \), error \( \epsilon \in (0, T] \), success probability \( 1 - \delta \in (0, 1) \). Initial loss unitary \( U_0 = \mathbb{1} \) corresponding to \( w(1) = 1/N \).

for \( t = 1 \) to \( T \) do

\[ L_{\min}^{(t-1)} \leftarrow \text{Find min}_j \sum_{t'=1}^{t-1} j(t') \text{ using oracles } \{ U_{l_j}^{(t')}: t' \leq t-1 \} \text{ with success probability } 1 - \frac{\delta}{T}. \]

\[ w_{\max}^{(t)} \leftarrow \beta L_{\min}^{(t-1)}/N. \]

\[ \left\| w(t) \right\|_{w_{\max}^{(t)}} \left\| L_{\max}^{(t)} \right\|_1 \left\| w_{\max} \right\|_1 \]

Amplitude estimate \( \left\| w(t) \right\|_{w_{\max}^{(t)}} \) to relative accuracy \( \frac{\epsilon}{1/T} \) with success probability \( 1 - \frac{\delta}{T} \).

Receive loss oracle \( U_{l_j}^{(t)} \).

Find \( L_{\max}^{(t)} \) using oracles \( \{ U_{l_j}^{(t')}: t' \leq t \} \) with success probability \( 1 - \frac{\delta}{T} \).

\[ \widetilde{L}(t) \leftarrow \text{Amplitude estimate } L_{\max}^{(t)} \text{ to relative accuracy } \frac{\epsilon}{1/T} \text{ with success probability } 1 - \frac{\delta}{T}. \]

\[ \widetilde{L}(t) \leftarrow \frac{L_{\max}^{(t)}}{w_{\max}^{(t)} \left\| w(t) \right\|_{w_{\max}^{(t)}}} - 1. \]

(Recall Eq. (3))

end for

**Output:** \( \widetilde{L_H} = \sum_{t=1}^{T} \widetilde{L}(t) \).

See Section IIIA for the definition of the shifted loss. Recalling Eq. (8), note that the ratio of shifted loss and \( \ell_1 \)-norm is given by

\[
\frac{L(t)}{w_{\max}(t)} = w_{\max}(t) (1 + L(t)),
\]

hence contains the loss at time \( t \). Note that \( \frac{L(t)}{w_{\max}(t)} \leq 2 \). We perform an error analysis to determine the quality of the approximation and the resulting computational complexity. We follow Ref. [21] for some of the error analysis.

**Lemma 2.** Let \( \tilde{a} \) be an estimate of \( a > 0 \) such that \( |\tilde{a} - a| \leq \epsilon_a a \), with \( \epsilon_a \in (0, 1) \). Similarly, let \( \tilde{b} \) be an estimate of \( b > 0 \) and \( \epsilon_b \in (0, 1) \) such that \( |\tilde{b} - b| \leq \epsilon_b b \). Then the ratio \( a/b \) is estimated to relative error \( \frac{\tilde{a} - a}{a} \leq \left( \frac{\epsilon_a a + \epsilon_b a}{\beta(1 - \epsilon_b)} \right) \frac{a}{b}. \)

**Proof.** Note that \( b - \tilde{b} \leq |\tilde{b} - b| \leq \epsilon_b b \), from which we deduce \( \tilde{a} - a \leq \frac{\epsilon_a a + \epsilon_b a}{b} \leq \frac{\epsilon_a a + \epsilon_b a}{b(1 - \epsilon_b)}. \)

To tie it together, we have the following result.

**Lemma 3.** Let \( \epsilon_1, \epsilon_L \in (0, 1) \) and \( t \in [T] \). Let \( \left\| \frac{w(t)}{w_{\max}} \right\|_1 - \left\| \frac{w(t)}{w_{\max}} \right\|_1 \leq \epsilon_1 \left\| \frac{w(t)}{w_{\max}} \right\|_1 \), and

\[
\left| L(t) - \widetilde{L}(t) \right| \leq \epsilon_L L(t). \]

Then we obtain an estimate \( \widetilde{L}(t) \) of \( L(t) \) with additive error \( 2^{\epsilon_1 + \epsilon_L (1 - \epsilon_1)} \).

**Proof.** From Lemma 2, we have

\[
\left| \widetilde{L}(t) - L(t) \right| = \frac{L_{\max}(t)}{w_{\max}(t)} \left\| \frac{w(t)}{w_{\max}} \right\|_1 - \left\| \frac{w(t)}{w_{\max}} \right\|_1 \leq \frac{L_{\max}(t)}{w_{\max}(t)} \left\| \frac{w(t)}{w_{\max}} \right\|_1 \leq 2^{\epsilon_1 + \epsilon_L (1 - \epsilon_1)} (1 + L(t)).
\]
We can now obtain a statement on the accuracy of the loss, the run time, and the success probability of the quantum algorithm.

**Theorem 4.** Let \( \epsilon \in (0,T] \) and \( \delta \in (0,1) \). Algorithm 4 provides an estimate \( \widetilde{L}_{\mathcal{H}} \) of the total loss \( L_{\mathcal{H}} = \sum_{t=1}^{T} L^{(t)} \) such that \( |\widetilde{L}_{\mathcal{H}} - L_{\mathcal{H}}| \leq \epsilon \). This quantum algorithm requires \( \mathcal{O} \left( \frac{T^{3/2}}{\epsilon} \log \left( \frac{T}{\delta} \right) \right) \) queries to the oracles and \( \mathcal{O} \left( \frac{T^{3/2}}{\epsilon} \log \left( \frac{1}{\delta} \right) \right) \) gates and succeeds with probability at least \( 1 - \delta \).

**Proof.** For the correctness, from Lemma 3 we obtain \( |L^{(t)} - \widetilde{L}^{(t)}| \leq 2^{1+\epsilon/\epsilon_{L}} \), and thus, \( |\widetilde{L}_{\mathcal{H}} - L_{\mathcal{H}}| \leq 2T^{1+\epsilon/\epsilon_{L}} \). The algorithm sets \( \epsilon_{1} = \epsilon_{L} = \frac{\epsilon}{T} \), and thus obtains \( |L_{\mathcal{H}} - \widetilde{L}_{\mathcal{H}}| \leq \frac{4T \epsilon}{(6T - \epsilon)} \leq \epsilon \), since \( \epsilon \leq T \).

For the run time, the minimum findings [20] take a total run time of \( \mathcal{O} \left( T^{2} \sqrt{N} \log \left( \frac{T}{\delta} \right) \right) \), as sums of up to \( T \) terms have to be computed at each of the \( T \) iterations. The \( \frac{T}{\delta} \) factor arises because the success probability is taken to be at least \( 1 - \frac{\delta}{4T} \) for each minimum finding.

For the estimations to accuracy \( \frac{\epsilon}{T} \) for all \( t \in [T] \), use Lemma 5 with \( u_{j} = \frac{w_{(t)}^{(j)}}{w_{(t)}^{(\max)}} \) and again with \( u_{j} = \frac{1+\ell^{(t)}}{\ell_{\max}} w_{j}^{(t)} \). In both cases we know that \( \max_{j} u_{j} = 1 \). Choose the additive accuracy to represent \( u_{j} \) as \( \eta = \epsilon/(12TN) \). The run times for computing the entries \( u_{j} \) are hence given via Lemma 1 as \( \mathcal{O}(T) \) queries to the Data Input 1 and \( \mathcal{O}(T + \log(TN/\epsilon)) \) quantum gates. Also choose the success probability at least \( 1 - \frac{\delta}{4T} \). Thus, all the estimations of \( \left\| \frac{w^{(t)}}{w_{(t)}^{(\max)}} \right\|_{1} \) require \( \mathcal{O} \left( T \times T \times \frac{\sqrt{N}}{\epsilon_{L}} \log \left( \frac{T}{\delta} \right) \right) = \mathcal{O} \left( \frac{T^{3/2}}{\epsilon} \log \left( \frac{T}{\delta} \right) \right) \) queries to the oracles. The number of gates is \( \mathcal{O} \left( \frac{T^{3/2}}{\epsilon} \log \left( \frac{T}{\delta} \right) \log N + \log(TN/\epsilon) \right) = \tilde{\mathcal{O}} \left( \frac{T^{3/2}}{\epsilon} \log \left( \frac{1}{\delta} \right) \right) \). Analogous run times hold for the estimations of \( L^{(t)} \). Each single-step, single-estimate success probability is \( 1 - \frac{\delta}{4T} \). As we have four probabilistic steps per step \( t \), the overall success probability of the algorithm is \( (1 - \frac{\delta}{4T})^{4T} \geq 1 - \delta \).

**D. Active gambling with quantum sampling**

Next, we provide a quantum version of Algorithm 3. Instead of obtaining the sample classically, we prepare a corresponding quantum state and measure it. We approximately prepare the quantum states of square-root probabilities \( \ket{p^{(t)}} = \sum_{j=1}^{N} \sqrt{p_{j}^{(t)}} \ket{j} \), for every time \( t = 1, \ldots, T \), with the probabilities given in Eq. 1. For this preparation, we require again an estimate of the \( \ell_{1} \)-norm of the weights, which dominates the cost of the algorithm. Amplitude amplification produces an erroneous output state, see Lemma 8 with probabilities denoted by \( \tilde{p}_{j} \).

The quantum algorithm is as follows.
Algorithm 5 Gambling algorithm with quantum sampling

**Input:** Number of strategies $N$, number of rounds $T$, parameter $\beta \in (0, 1)$, investment cost $C$, desired loss sampling bias $\epsilon \in (0, T]$, success probability $1 - \delta \in (0, 1)$. Initial loss unitary $U_l^{(0)} = 1$ corresponding to $w^{(i)} = \overline{1}/N$.

for $t = 1$ to $T$

- $L_{\min}^{(t-1)} \leftarrow \text{Find } \min_j \sum_{t'=1}^{t-1} l_j^{(t')} \text{ using oracles } \{U_l^{(t')} : t' \leq t-1\} \text{ with success probability } 1 - \frac{\delta}{4T}$.
- $w^{(t)}_{\max} \leftarrow \beta L_{\min}^{(t-1)} / N$.
- $\| \frac{w^{(t)}}{w^{(t)}_{\max}} \|_1 \leftarrow \text{Amplitude estimate } \| \frac{w^{(t)}}{w^{(t)}_{\max}} \|_1$ to relative accuracy $\frac{\delta}{4T}$ with success probability $1 - \frac{\delta}{4T}$.
- Prepare quantum state $|\tilde{p}^{(t)}\rangle$ using amplitude amplification.
- $j^{(t)} \leftarrow \text{Sample from } |\tilde{p}^{(t)}\rangle$ by measuring in the computational basis.
- Allocate portfolio in $j^{(t)}$-th strategy at cost $C$.
- Receive loss oracle $U_l^{(t)}$.
- $l_j^{(t)} \leftarrow \text{Receive loss by measuring second register of } U_l^{(t)} |j^{(t)}\rangle \langle 0|$.
- Suffer loss $l_j^{(t)}$.

end for

**Output:** $L_{\text{samp}}^Q := \sum_{t=1}^{T} l_j^{(t)}$.

Theorem 5. Let $\epsilon \in (0, T]$ and $\delta \in (0, 1)$. Algorithm 5 outputs $L_{\text{samp}}^Q := \sum_{t=1}^{T} l_j^{(t)}$ with a bias $|E[L_{\text{samp}}^Q] - L_{\tilde{H}}| \leq \epsilon$ with success probability at least $1 - \delta$. For $s \in \mathbb{R}_+$, it holds that $L_{\text{samp}}^Q - \min_j L_j \leq \epsilon + (s+1)\sqrt{2T \log N} + \log N$ with success probability at least $(1 - \delta)(1 - e^{-2s^2})$. This quantum algorithm requires $O\left(\frac{T^3\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right)\right)$ queries to the oracles and $O\left(\frac{T^3\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right)\right)$ gates. The total cost is $\tilde{O} \left( T \times C + \frac{T^3\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right) \right)$.

Proof. The expectation value is $E[p|L_{\text{samp}}^Q] = \sum_{t=1}^{T} E[p_{j^{(t)}}^{(t)}] = \sum_{t=1}^{T} \sum_{j=1}^{N} \tilde{p}_j^{(t)} l_j^{(t)} =: \tilde{L}_{\tilde{H}}$. We estimate $\| \frac{w^{(t)}}{w^{(t)}_{\max}} \|$ to relative accuracy $\frac{\delta}{4T}$, hence from Lemma 3 (iii) we have $\|\tilde{p}^{(t)} - \bar{p}^{(t)}\|_1 \leq \frac{\delta}{4T}$. Thus, $\sum_j l_j^{(t)} \left( \bar{p}_j^{(t)} - \tilde{p}_j^{(t)} \right) \leq \|\bar{p}^{(t)} - p^{(t)}\|_1 \|l^{(t)}\|_\infty \leq \frac{\delta}{4T}$. Thus, we have the bias $|\tilde{L}_{\tilde{H}} - L_{\tilde{H}}| \leq \epsilon$.

The success probability for obtaining this bias is $(1 - \frac{\delta}{4T})^{2T} \geq (1 - \delta)$.

For the difference to the minimum loss strategy, we find $L_{\text{samp}}^Q - \min_j L_j \leq |L_{\text{samp}}^Q - \tilde{L}_{\tilde{H}}| + |\tilde{L}_{\tilde{H}} - L_{\tilde{H}}| + L_{\tilde{H}} - \min_j L_j \leq |L_{\text{samp}}^Q - \tilde{L}_{\tilde{H}}| + \epsilon + \sqrt{2T \log N} + \log N \leq \epsilon + (s+1)\sqrt{2T \log N} + \log N$.

Here, we used that $|L_{\text{samp}}^Q - \tilde{L}_{\tilde{H}}| \leq s\sqrt{2T \log N}$ with probability $1 - e^{-2s^2}$ as in the classical algorithm, see Theorem 2. Hence, we have a total success probability for this regret bound of $(1 - \delta)(1 - e^{-2s^2})$.

The minimum findings take a total run time of $O\left(T^2 \sqrt{N} \log \left(\frac{T}{\delta}\right)\right)$ [20]. Per Lemma 3 with $u_j = w_j^{(t)}/w^{(t)}_{\max}$, amplitude estimation dominates the algorithm and takes $O\left(\frac{T^3\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right)\right)$ queries to the oracles and $O\left(\frac{T^3\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right)\right)$ gates as in Theorem 4.
IV. SPARSITRON

At a high level, statistical classification in machine learning is performed with two main approaches. Let $X$ denote the observable variables (features) and $Y$ the target variables (labels.) The first approach is using a discriminative model. In this approach, one models the conditional distribution $P(Y|X=x)$, i.e., the probability of the labels given an input example $x$. On the other hand, a generative model constructs a joint distribution $P(X,Y)$ of variables and labels. In this more general approach, inferences in both directions features $\rightarrow$ label and label $\rightarrow$ features can be made.

Undirected graphical models, or Markov random fields, are a powerful, modern statistical tool for modeling high-dimensional probability distributions [17, 22]. The joint probability distribution of such a model depends on an underlying graph, where the presence of an edge gives conditional dependence and the absence of an edge gives conditional independence. The Ising model is a special type of Markov random field which uses binary variables and pairwise interactions. Consider here $N$ binary variables $Z_j \in \{-1, 1\}$ for $j \in [N]$. Associated with these variables is an undirected dependency graph. The graph enters a probability distribution

$$P[Z = z] \propto \exp\left( \sum_{i,j:i\neq j} A_{ij} z_i z_j + \sum_i \theta_i z_i \right),$$

where $A \in \mathbb{R}^{N \times N}$ is the graph adjacency matrix and $\theta \in \mathbb{R}^N$ describes bias terms. An important task in a machine learning context is the following unsupervised learning task: Given samples from the distribution Eq. (10) on $\{-1, 1\}^N$, learn the matrix $A$.

Focussing on a particular variable $Z_j$, and setting the other variables to $x \in \{-1, 1\}^{[N] \setminus \{j\}}$, we have for the conditional distribution $P[Z_j = -1|Z_{\neq j} = x] = \frac{P[Z_j = -1, Z_{\neq j} = x]}{P[Z_{\neq j} = x]} = \frac{1}{1 + \exp(\sum_{k \neq j} A_{jk} z_k - \theta_j)} = \sigma(w \cdot x + \theta_j)$. Here, we have used the sigmoid function $\sigma(z) = 1/(1 + e^{-z})$ and the vector $w \in \mathbb{R}^{[N] \setminus \{j\}}$ with $w_k = -2A_{jk}$.

This single-variable conditional probability suggests a way to turn the original unsupervised learning problem into a supervised learning problem. Setting $X = (Z_k, k \neq j)$ and $Y = (1 - Z_j)/2$ turns all variables except the $j$-th one into features and the $j$-th variable into a label. Note that $\mathbb{E}[Y|X = x] = P[Z_j = -1|Z_{\neq j} = x] = \sigma(w \cdot x + \theta_j)$. We have the following problem statement for learning such a generalized linear model (GLM.) Given samples $(X,Y)$ with the conditional mean function $\mathbb{E}[Y|X = x] = \sigma(w \cdot x + \theta)$, learn $w$ and $\theta$.

Reference [17] developed the Sparsitron, an efficient classical method to solve GLMs based on the Freund/Schapire Hedge algorithm. One assumption is that a true $w$ with $||w||_1 \leq \lambda$ exists where $\lambda \geq 0$ is known. Without loss of generality one can take $w \geq 0$ and that $||w||_1 = \lambda$, see [17]. The classical algorithm assumes query access to a training and a test set. We first show the original classical algorithm, then an approximate classical algorithm, then the quantum algorithm. The classical algorithm for the Sparsitron is given by the following.
Algorithm 6 Sparsitron \[17\]

Input: Parameter $\beta \in (0, 1)$, norm $\lambda \geq 0$, training set $(x^{(t)}, y^{(t)}) \in [-1, 1]^N \times [0, 1]$ for $t \in [T]$, test set $(a^{(m)}, b^{(m)}) \in [-1, 1]^N \times [0, 1]$ for $m \in [M]$, $w^{(1)} = \mathbf{I}/N \in \mathbb{R}^N$.

for $t = 1$ to $T$ do
  $p^{(t)} \leftarrow \frac{w^{(t)}}{\|w^{(t)}\|}$,
  $l^{(t)} \leftarrow \frac{1}{2} \left( \mathbf{I} + (\sigma (\lambda p^{(t)} \cdot x^{(t)}) - y^{(t)}) x^{(t)} \right)$,
  $w^{(t+1)} \leftarrow w^{(t)} \odot \beta^{(t)}$
end for

for $t = 1$ to $T$ do
  $\varepsilon^{(t)} \leftarrow \frac{1}{M} \sum_{m=1}^{M} (\sigma (\lambda p^{(t)} \cdot a^{(m)}) - b^{(m)})^2$
end for

Output: $v = \lambda p^{(t')}$ for $t' = \arg \min_{t \in [T]} \varepsilon^{(t)}$.

The algorithm consists of two loops. The first loop goes over all the training examples and is equivalent to the Hedge algorithm. For each training example $x^{(t)}$, a prediction vector $\lambda p^{(t)}$ is constructed. From this vector one can compute the label prediction for the training example by using the activation function as $\sigma (\lambda p^{(t)} \cdot x^{(t)})$. Deviations from the prediction are minimized across the loop by computing a loss $l^{(t)} = \frac{1}{2} \left( \mathbf{I} + (\sigma (\lambda p^{(t)} \cdot x^{(t)}) - y^{(t)}) x^{(t)} \right) \in [0, 1]^N$, which takes into account the difference of the prediction and the actual label $y^{(t)}$. The second loop uses the test set and computes the so-called “empirical risk” $\varepsilon^{(t)}$ to test the quality of each prediction vector $p^{(t)}$. The empirical risk is an approximation to the true risk $\varepsilon^{(t)} := E_{(X,Y) \sim D} [ (\sigma (\lambda p^{(t)} \cdot X) - \sigma (w \cdot X))^2 ]$, as only $M$ test examples are used to compute the expectation value. The algorithm finally returns the vector $v = \lambda p^{(t')}$ which performs best on the test set. The provable learning guarantee and the run time is summarized in the following theorem. Plugging in $T$ and $M$, the run time can be expressed as $\tilde{O} \left( \frac{N \lambda^2}{\epsilon^2} \log^2 \frac{1}{\delta^2} \right)$.

Theorem 6 (Sparsitron \[17\]). Let $D$ be a distribution on $[-1, 1]^N \times \{0, 1\}$ where for $(X, Y) \sim D$, $E[Y|X=x] = \sigma(w \cdot x)$ for a non-decreasing 1-Lipschitz function $\sigma: \mathbb{R} \rightarrow [0, 1]$. Suppose that $\|w\|_1 \leq \lambda$ for a known $\lambda \geq 0$. Then, there exists an algorithm that for all $\epsilon, \delta \in (0, 1)$ given $T = \mathcal{O} (\lambda^2 (\log (N/\delta \epsilon))/\epsilon^2)$ independent examples from $D$, produces a vector $v \in \mathbb{R}^N$ such that with probability at least $1 - \delta$,

$$E_{(X,Y) \sim D} [ (\sigma (v \cdot X) - \sigma (w \cdot X))^2 ] \leq \epsilon. \quad (11)$$

The run time of the algorithm is $\mathcal{O} (N \times T \times M)$, where $M = \mathcal{O} (\log (T/\delta)/\epsilon^2)$. Moreover, the algorithm can be run in an online manner.

We now discuss our approximate Sparsitron. We have discussed in Theorem \[1\] the construction of a data structure to sample from a probability vector $p$. Next, we show a method to sample related inner products efficiently. The number of samples scales with $1/\epsilon^2$, in contrast to using quantum estimation which scales with $1/\epsilon$. The proof is standard and adapted from \[23\] which shows the $\ell_2$-sampling case.

Lemma 4. Let $\epsilon, \delta \in (0, 1)$. Given query access to $x \in [-1, 1]^N$ and $\ell_1$-sampling access to an $N$-dimensional probability vector $p$. We can determine $p \cdot x$ to additive error $\epsilon$ with success
probability at least $1 - \delta$ with $O\left(\frac{\|x\|_2^2 \log \frac{1}{\delta}}{\epsilon^2}\right)$ queries and samples, and $\tilde{O}\left(\frac{\|x\|_2^2 \log \frac{1}{\delta}}{\epsilon^2}\right)$ time complexity.

Proof. Define a random variable $Z$ with outcome $x_j$ with probability $p_j$. Note that $E[Z] = \sum_j p_j x_j = p \cdot x$. Also, $\mathbb{V}[Z] \leq \sum_j x_j^2 p_j \leq \|x\|_2^2$. Take the median of $6 \log 1/\delta$ evaluations of the mean of $9/(2\epsilon^2)$ samples of $Z$ to be within $\epsilon \sqrt{\mathbb{V}[Z]} \leq \epsilon \|x\|_2$ of $p \cdot x$ with probability at least $1 - \delta$ in $O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$ queries.

Since here $\|x\|_2 \leq 1$ we have $O\left(\frac{\|x\|_2^2 \log \frac{1}{\delta}}{\epsilon^2}\right) = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$.

We now show the approximate Sparsitron algorithm. The only difference to Algorithm \[ \] is the inner product estimation. Since that estimation introduces errors that propagate across the iterations, we furnish the weight and probability vectors with a tilde to denote the difference to the original vectors. In addition, the estimated quantities are also furnished with a tilde.

**Algorithm 7 SparsitronApprox**

**Input:** Error $\epsilon \in (0, 1)$, probability $\delta \in (0, 1)$, parameter $\beta \in (0, 1)$, norm $\lambda > 0$, training set $(x^{(t)}, y^{(t)}) \in [-1,1]^N \times \{0,1\}$ for $t \in [T]$, test set $(a^{(m)}, b^{(m)}) \in [-1,1]^N \times \{0,1\}$ for $m \in [M]$, $\tilde{w}^{(1)} = I/N \in \mathbb{R}^N$.

for $t = 1$ to $T$ do

\[ \tilde{p}^{(t)} \leftarrow \frac{\tilde{w}^{(t)}}{\|\tilde{w}^{(t)}\|_1} \]

\[ \tilde{p}^{(t)} \cdot x^{(t)} \leftarrow \text{Estimate } \tilde{p}^{(t)} \cdot x^{(t)} \text{ to accuracy } \frac{\epsilon}{\lambda M} \text{ with success probability } 1 - \frac{\epsilon}{2T}. \]

\[ \tilde{p}^{(t)} \leftarrow \frac{1}{\lambda} \left( I + \sigma \left( \lambda \tilde{p}^{(t)} \cdot x^{(t)} - y^{(t)} \right) x^{(t)} \right). \]

\[ \tilde{w}^{(t+1)} \leftarrow \tilde{w}^{(t)} \odot \beta^{(t)} \]

end for

for $t = 1$ to $T$ do

for $m \equiv 1$ to $M$ do

\[ \tilde{p}^{(t)} \cdot a^{(m)} \leftarrow \text{Estimate } \tilde{p}^{(t)} \cdot a^{(m)} \text{ to accuracy } \frac{\epsilon}{16\lambda} \text{ with success probability } 1 - \frac{\delta}{2MT}. \]

end for

\[ \tilde{\varepsilon}^{(t)} \leftarrow \frac{1}{M} \sum_{m=1}^{M} \sigma \left( \lambda \tilde{p}^{(t)} \cdot a^{(m)} - b^{(m)} \right)^2 \]

end for

**Output:** $\tilde{v} = \lambda \tilde{p}^{(t')}$ for $t' = \arg\min_{t \in [T]} \tilde{\varepsilon}^{(t)}$.

The provable learning guarantee for this algorithm follows the original work but relies on three additional ideas. First, it is not important that the probabilities $\tilde{p}^{(t)}$ follow exactly the original probabilities $p^{(t)}$. It is only important that we have a final guarantee from the Hedge algorithm. If the inner product estimations are not too inaccurate we only obtain a small additional error to the Hedge error bound. Second, as mentioned before the empirical risk is an approximation to the true risk as only $M$ test examples are used. The inner product estimation leads to an approximate empirical risk $\tilde{\varepsilon}$ which is used to bound the true risk via two applications of triangle inequalities. Third, each inner product estimation is probabilistic, hence we bound the overall success probability of the algorithm with a union bound together with the probabilistic behavior of the original algorithm.
As we show below, the run time of the approximate classical algorithm is about \( \tilde{O}(T(N + \frac{MA}{\epsilon^2} \log \frac{1}{\delta})) \). The multiplicative updates and maintaining of a sampling data structure still cost \( \tilde{O}(N) \). Plugging in \( T \) and \( M \), the run time can be expressed as \( \tilde{O}(N^2 \log \frac{1}{\delta} + \frac{\lambda^6}{\epsilon^2} \log \frac{1}{\delta}) \). In some ranges of parameters, this run time can be considered an improvement over the original Sparsitron which has a run time of \( \tilde{O}(N^2 \log^2 \frac{1}{\delta}) \).

**Theorem 7 (Approximate Sparsitron).** Let \( \mathcal{D} \) be a distribution on \([-1, 1]^N \times \{0, 1\}\) where for \((X, Y) \sim \mathcal{D}, E[Y|X = x] = \sigma(w \cdot x)\) for a non-decreasing 1-Lipschitz function \( \sigma : \mathbb{R} \rightarrow [0, 1] \). Suppose that \( \|w\|_1 \leq \lambda \) for a known \( \lambda \geq 0 \) and for \( \epsilon, \delta \in (0, 1) \) given \( T = \mathcal{O}(\lambda^2 (\log (N/\delta \epsilon))/\epsilon^2) \) independent examples from \( \mathcal{D} \). Algorithm \( \mathcal{A} \) produces a vector \( v \in \mathbb{R}^n \) such that with probability at least \( 1 - 3\delta \),

\[
E_{X,Y \sim D} [(\sigma(v \cdot X) - \sigma(w \cdot X))^2] \leq \epsilon.
\]

The run time of the algorithm is \( \tilde{O}(T \left( N + \frac{MA}{\epsilon^2} \log \frac{1}{\delta}\right)) \), where \( M = \mathcal{O}(\log(T/\delta)/\epsilon^2) \).

Again, the algorithm can be run in an online manner.

**Proof.** We perform a similar analysis as in [17]. From the original Hedge algorithm we know that

\[
\sum_{t=1}^{T} \tilde{p}^{(t)} \cdot \tilde{l}^{(t)} \leq \min_{j} \tilde{L}_j + \sqrt{2T \log N + \log N},
\]

where \( \tilde{L}_j = \sum_{t=1}^{T} \tilde{l}^{(t)} \). Let \( \tilde{p}^{(t)} \cdot x^{(t)} \) be an estimate of \( p^{(t)} \cdot x^{(t)} \) with error \( |p^{(t)} \cdot x^{(t)} - \tilde{p}^{(t)} \cdot x^{(t)}| \leq \epsilon_{px} \). From the 1-Lipschitz property, it is easy to see that

\[
|\sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)})| \leq \lambda \epsilon_{px}.
\]

Define the random variable \( Q^{(t)} = \tilde{p}^{(t)} \cdot \tilde{l}^{(t)} - w \cdot \tilde{l}^{(t)}/\lambda \).

We have

\[
\mathbb{E}_{(x^{(t)}, y^{(t)})} \left[ Q^{(t)} \right| (x^{(t)}, y^{(t)}), \ldots, (x^{(t-1)}, y^{(t-1)}) = \mathbb{E}_{(x^{(t)}, y^{(t)})} \left[ (\tilde{p}^{(t)} - w/\lambda) \cdot \tilde{l}^{(t)} \right]
\]

\[
= \frac{1}{2} \mathbb{E}_{(x^{(t)}, y^{(t)})} \left[ (\tilde{p}^{(t)} - w/\lambda) \cdot \left( \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - y^{(t)} \right) x^{(t)} \right]
\]

\[
= \frac{1}{2} \mathbb{E}_{(x^{(t)}, y^{(t)})} \left[ (\tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)}/\lambda) \left( \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - \sigma(w \cdot x^{(t)}) \right) \right].
\]

From the inequality above we obtain (i) \( \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) \geq \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - \lambda \epsilon_{px} \) and (ii) \( \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) \leq \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) + \lambda \epsilon_{px} \). For any \( \zeta \in \mathbb{R}_+ \), we have from (i)

\[
\zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) \geq \zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - \zeta \lambda \epsilon_{px} = \zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - |\zeta| \lambda \epsilon_{px}.
\]

For any \(-\zeta \in \mathbb{R}_+ \), we have from (ii)

\[
\zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) \geq \zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) + \zeta \lambda \epsilon_{px} = \zeta \sigma(\lambda \tilde{p}^{(t)} \cdot x^{(t)}) - |\zeta| \lambda \epsilon_{px}.
\]
Noting that here $\zeta = \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)}/\lambda$, continue with the lower bound for the expectation value

$$\cdots \geq \frac{1}{2\lambda} E_{(x^{(t)}, y^{(t)})} \left[ \left( \lambda \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)} \right)^2 \right] - \frac{\epsilon_{px}}{2} E_{(x^{(t)}, y^{(t)})} \left[ \left( \lambda \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)} \right) \right]$$

$$\geq \frac{1}{2\lambda} E_{(x^{(t)}, y^{(t)})} \left[ \left( \lambda \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)} \right)^2 \right] - \frac{\epsilon_{px}}{2} E_{(x^{(t)}, y^{(t)})} \left[ \left( \lambda \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)} \right) \right].$$

The first term is the risk $\frac{1}{2\lambda} \varepsilon(\lambda \tilde{p}^{(t)})$. The error (second) term is, using $\|x^{(t)}\|_\infty \leq 1$,

$$E_{(x^{(t)}, y^{(t)})} \left[ \lambda \tilde{p}^{(t)} \cdot x^{(t)} - w \cdot x^{(t)} \right] \leq \lambda E_{(x^{(t)}, y^{(t)})} \left[ \|\tilde{p}^{(t)} - w/\lambda\|_1 \right] \leq 2\lambda.$$

Hence, for the risk we have

$$\frac{1}{2\lambda} \varepsilon(\lambda \tilde{p}^{(t)}) \leq E_{(x^{(t)}, y^{(t)})} \left[ Q^{(t)} \right] \left( x^{(1)}, y^{(1)}), \ldots, (x^{(t-1)}, y^{(t-1)}) \right] + \lambda \epsilon_{px}.$$

Combining with the martingale bounds for the expectation value, we obtain with probability at least $1 - \delta$ that

$$\sum_{t=1}^{T} E_{(x^{(t)}, y^{(t)})} \left[ Q^{(t)} \right] \left( x^{(1)}, y^{(1)}), \ldots, (x^{(t-1)}, y^{(t-1)}) \right] \leq \sum_{t=1}^{T} Q^{(t)} + O \left( \sqrt{T \log(1/\delta)} \right).$$

Hence, we obtain with probability at least $1 - \delta$ that

$$\frac{1}{2\lambda} \sum_{t=1}^{T} \varepsilon(\lambda \tilde{p}^{(t)}) \leq \sum_{t=1}^{T} Q^{(t)} + O \left( \sqrt{T \log(1/\delta)} \right) + \lambda \epsilon_{px} T \leq \min_j \bar{L}_j - \sum_{t=1}^{T} w \cdot \tilde{l}^{(t)}/\lambda + \sqrt{2T \log N} + \log N + O \left( \sqrt{T \log 1/\delta} \right) + \lambda \epsilon_{px} T \leq \sqrt{2T \log N} + \log N + O \left( \sqrt{T \log 1/\delta} \right) + \lambda \epsilon_{px} T,$$

because $\min_j \bar{L}_j - \sum_{t=1}^{T} w \cdot \tilde{l}^{(t)}/\lambda \leq 0$ since $\lambda = \|w\|_1$. From the upper bound for the sum, we obtain an upper bound for the minimum element as $\min_{t \in [T]} \varepsilon(\lambda \tilde{p}^{(t)}) \leq \frac{1}{T} \sum_{t=1}^{T} \varepsilon(\lambda \tilde{p}^{(t)})$. Similar to the original work, setting $T > C' \lambda^2 \log(N/\delta)/\epsilon^2$ with a constant $C''$ and $\epsilon_{px} = \frac{\epsilon}{4\lambda}$, we obtain

$$\min_{t} \varepsilon(\lambda \tilde{p}^{(t)}) \leq O \left( \frac{\sqrt{2T \log N} + \log N + \sqrt{T \log 1/\delta}}{T} + 2\lambda^2 \epsilon_{px} \right) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

The closeness to the empirical risk is achieved as in the original work by choosing $M = C'' \log(T/\delta)/\epsilon^2$, with a constant $C'''$, such that for all $t \in [T]$ with probability $1 - \delta$

$$|\varepsilon(\lambda \tilde{p}^{(t)}) - \bar{\varepsilon}(\lambda \tilde{p}^{(t)})| \leq \frac{\epsilon}{8}.$$

Note that the algorithm selects $\bar{v} = \lambda \tilde{p}^{(t')}$ for $t' = \arg \min_{t \in [T]} \bar{\varepsilon}(\lambda \tilde{p}^{(t)})$, where $\bar{\varepsilon}$ is the empirical risk estimated from the imprecise inner products. The error for these inner products
used for estimating the empirical risk is set to $\frac{\epsilon}{10\lambda}$, hence $|\hat{\varepsilon}(t) - \varepsilon(t)| \leq \frac{\epsilon}{8}$ from Lemma 5 below. Therefore, the true risk can be bounded as

$$\varepsilon(\tilde{v}) \leq \frac{\epsilon}{8} + \hat{\varepsilon}(\tilde{v}) \leq \frac{\epsilon}{4} + \hat{\varepsilon}(\tilde{v}) = \frac{\epsilon}{4} + \min_{t \in [T]} \hat{\varepsilon}(\lambda \tilde{p}^{(t)})$$

(27)

$$= \frac{\epsilon}{4} + \min_{t \in [T]} (\hat{\varepsilon}(\lambda \tilde{p}^{(t)}) - \hat{\varepsilon}(\lambda \tilde{p}^{(t)}) + \hat{\varepsilon}(\lambda \tilde{p}^{(t)})) \leq \frac{3\epsilon}{8} + \min_{t \in [T]} \hat{\varepsilon}(\lambda \tilde{p}^{(t)})$$

(28)

$$\leq \frac{3\epsilon}{8} + \min_{t \in [T]} (\hat{\varepsilon}(\lambda \tilde{p}^{(t)}) - \varepsilon(\lambda \tilde{p}^{(t)}) + \varepsilon(\lambda \tilde{p}^{(t)})) \leq \frac{\epsilon}{2} + \min_{t \in [T]} \varepsilon(\lambda \tilde{p}^{(t)}) \leq \epsilon.$$  

(29)

Next, we discuss the run time. Computing the loss vector and maintaining the $\tilde{p}^{(t)}$ sampling data structure costs $\tilde{O}(N)$. Computing the multiplicative update costs $O(N)$. Estimating the inner product at every step to accuracy $\frac{\epsilon}{8\lambda^2}$ with success probability $1 - \frac{\delta}{2T}$ costs $\tilde{O}\left(\frac{MT}{\epsilon^2}\right)$. Estimating the $M$ inner products for risk estimation at every step to accuracy $\epsilon/(16\lambda)$ with success probability $1 - \frac{\delta}{2MT}$ costs $\tilde{O}\left(\frac{\lambda^2}{\epsilon^2} \log \frac{MT}{\delta}\right)$.

Hence the total run time is $\tilde{O}\left(T \left(N + \frac{\lambda^2}{\epsilon^2} \log \frac{T}{\delta} + M \frac{\lambda^2}{\epsilon^2} \log \frac{MT}{\delta}\right)\right) = \tilde{O}\left(T \left(N + \frac{M \lambda^4}{\epsilon^2} \log \frac{1}{\delta}\right)\right)$. The success probability of the inner loop is $(1 - \frac{\delta}{2MT})^M \geq 1 - \frac{\delta}{2T}$. The success probability of all the estimations of the algorithm is $(1 - \frac{\delta}{2T})^{2T} \geq 1 - 3\delta$. Together with the success probability of the martingale estimation and the risk estimation this gives a total success probability of the algorithm of at least $1 - 3\delta$.

As mentioned, the inner product estimation affects also the estimation of the empirical risk.

**Lemma 5.** Let $\sigma(x)$ be 1-Lipschitz and $\varepsilon_{pa} \in (0, 1)$. For all $m \in [M]$, let $\tilde{p}^{(t)} \cdot a^{(m)}$ be an estimate of $\tilde{p}^{(t)} \cdot a^{(m)}$ with error $|\tilde{p}^{(t)} \cdot a^{(m)} - \tilde{p}^{(t)} \cdot a^{(m)}| \leq \varepsilon_{pa}$. Then the estimator $\tilde{\varepsilon}$ of the empirical risk $\varepsilon$ arising from the imprecise inner products is accurate as $|\tilde{\varepsilon}(t) - \varepsilon(t)| \leq 2\lambda \varepsilon_{pa}$.

**Proof.** Note that $b^{(m)} \in [0, 1]$ and the Lipschitz constant of $x^2$ on $[-1, 1]$ is 2. Then,

$$|\tilde{\varepsilon}(t) - \varepsilon(t)| = \frac{1}{M} \sum_{m=1}^{M} \left(\sigma\left(\lambda \tilde{p}^{(t)} \cdot a^{(m)}\right) - b^{(m)}\right)^2 - \sum_{m=1}^{M} \left(\sigma\left(\lambda \tilde{p}^{(t)} \cdot a^{(m)}\right) - b^{(m)}\right)^2 \leq \frac{2\lambda}{M} \sum_{m=1}^{M} \left|\tilde{p}^{(t)} \cdot a^{(m)} - \tilde{p}^{(t)} \cdot a^{(m)}\right| \leq 2\lambda \varepsilon_{pa}.$$  

□

**V. QUANTUM SPARSITRON**

In this section, we construct a quantum algorithm for the Sparsitron. The algorithm is again based on quantum minimum finding, see Lemma 7 in Appendix A and amplitude amplification and estimation, see Lemma 8 Appendix A. Similar to the quantum Hedge algorithms above, the core idea is to never explicitly store the weight vector $w^{(t)}$. Rather, norms and inner products are estimated and stored. Access to the current values of these quantities and quantum access to a new training datum allow to prepare a new loss oracle and a new weight quantum state. This state preparation can then in turn be used to compute the new norm and inner products. We can expect to obtain a quantum speedup in
the dimension \( N \). On the other hand, we do not expect a quantum speedup in the number of samples \( T \) as the provable learning guarantees are classical and invoke the Hedge algorithm. In fact, we obtain again a worsening of the performance in \( T \). If \( \lambda \) and \( 1/\epsilon \) are \( \text{poly} \log N \), this worsening is however tolerable as \( T \) is then also \( \text{poly} \log N \).

We first specify the input model. Here, we assume quantum access to the training and test data. The access model can be turned into an online setting by providing sequential access to the unitaries.

**Data Input 2** (Training and test set). Assume \( O(1) \) bits are sufficient to store \( x_j^{(t)} \) and \( a_{j}^{(m)} \). Assume to be given access to \( T \) unitaries \( U_{\text{train}} \) and \( M \) unitaries \( U_{\text{test}} \) on \( O(\log N) \) qubits that perform the operations \( |j\rangle|0\rangle \rightarrow |j\rangle|c \oplus x_j^{(t)}\rangle \) and \( |j\rangle|0\rangle \rightarrow |j\rangle|c \oplus a_j^{(m)}\rangle \) for \( j \in [N] \), \( t \in [T] \), and \( m \in [M] \), respectively, with an arbitrary in-range bit string \( c \).

The data access allows to arithmetically compute the desired losses in quantum superposition.

**Lemma 6** (Loss quantum circuits). Let \( \eta > 0 \). Given Input 2 and classical access to the numbers \( \lambda \geq 0 \), \( h \in [-1, 1] \) and \( y \in [0, 1] \). For \( t \in [T] \), the quantum operation \( |j\rangle|0\rangle \rightarrow |j\rangle|l_j^{(t)}\rangle \equiv |j\rangle\left|\frac{1}{2} \left(1 + (\sigma (\lambda h) - y) x_j^{(t)}\right)\right| \) for \( j \in [N] \) can be constructed on \( O(\log N + \log 1/\eta) \) qubits, where \( l_j^{(t)} \) is encoded to additive accuracy \( \eta \). The run time is \( O(\log N + \log 1/\eta) \). We denote these quantum circuits by \( U_j^{(t)} \).

**Proof.** Use quantum access to \( x_j^{(t)} \) and the well-known quantum circuits for basic arithmetic operations.

Hence, we have constructed the loss units, which in the previous quantum Hedge algorithm were assumed to be given in Input 1. With these loss units, we can compute the weights via Lemma 1 and perform minimum finding and \( \ell_1 \)-norm estimation, as before. Next, we turn to estimating inner products, the core step in the Sparsitron. Consider the inner product \( h^{(t)} := \sum_j \frac{w_j^{(t)}}{\|w^{(t)}\|} x_j^{(t)} \) and the corresponding unnormalized inner product \( \sum_j w_j^{(t)} x_j^{(t)} \). Instead of this inner product, we estimate a shifted inner product for two reasons: (i) if all \( x_j^{(t)} \) are zero the same argument as in Section III A for the shifted loss applies, (ii) even if that is not the case, since \( x_j^{(t)} \) are \([-1, 1]\) there can be cancelation effects which make the inner product zero or very close to zero.

We proceed with the algorithm for the Quantum Sparsitron.
Algorithm 8 Quantum Sparsitron

Input: Error $\epsilon \in (0, 1)$, probability $\delta \in (0, 1)$, parameter $\beta \in (0, 1)$, norm $\lambda \geq 0$, quantum query access to training set $(x^{(t)}, y^{(t)}) \in [-1, 1]^N \times [0, 1]$ for $t \in [T]$ and test set $(a^{(m)}, b^{(m)}) \in [-1, 1]^N \times [0, 1]$ for $m \in [M]$. Initial loss unitary $U^{(0)}_t = 1$ corresponding to $w^{(1)} = 1/N$.

for $t = 1$ to $T$ do

\[
\begin{align*}
\tilde{L}^{(t-1)}_{\text{min}} & \leftarrow \text{Find } \min_{j} \sum_{t'=1}^{t-1} \tilde{l}^{(t')}_{j} \text{ using oracles } \{U^{(t')}_j : t' \leq t - 1\} \text{ with success probability } 1 - \frac{\delta}{5T}.
\end{align*}
\]

\[
\tilde{w}^{(t)}_{\text{max}} \leftarrow \beta^{\tilde{L}^{(t-1)}_{\text{min}}} / N.
\]

\[
\left\| \frac{\tilde{w}^{(t)}}{\tilde{w}^{(t)}_{\text{max}}} \right\|_1 \leftarrow \text{Quantum estimate } \left\| \frac{\tilde{w}^{(t)}}{\tilde{w}^{(t)}_{\text{max}}} \right\|_1 \text{ to relative accuracy } \frac{1}{10} \min \{ \epsilon, \frac{\epsilon}{16\lambda} \} \text{ with success probability } 1 - \frac{\delta}{5T}.
\]

\[
\tilde{h}^{(t)}_{\text{max}} \leftarrow \text{Find } \max_{j} \tilde{w}^{(t)}_{j} (x^{(t)}_{j} + 3) \text{ using } \{U^{(t')}_{j} : t' \leq t - 1\} \text{ with success probability } 1 - \frac{\delta}{5T}.
\]

\[
\tilde{h}^{(t)}_{\text{tmp}} \leftarrow \text{Quantum estimate } \sum_{j} \frac{\tilde{w}^{(t)}_{j} (x^{(t)}_{j} + 3)}{\tilde{h}^{(t)}_{\text{max}}} \text{ to relative accuracy } \frac{1}{10} \frac{\epsilon}{16\lambda} \text{ with success probability } 1 - \frac{\delta}{5T}.
\]

\[
\tilde{h}^{(t)} \leftarrow \frac{\tilde{h}^{(t)}_{\text{tmp}}}{\tilde{h}^{(t)}_{\text{max}}} \left\| \frac{\tilde{w}^{(t)}}{\tilde{w}^{(t)}_{\text{max}}} \right\|_1 - 3 \text{ (to additive accuracy } \frac{\epsilon}{16\lambda} ).
\]

Construct unitary $U^{(t)}_t$ that prepares $|j\rangle |\tilde{l}^{(t)}_{j}\rangle$ with $\tilde{l}^{(t)}_{j} = \frac{1}{2} \left( \tilde{I} + \left( \sigma \left( \lambda\tilde{h}^{(t)}_{j} \right) - y^{(t)} \right) x^{(t)}_{j} \right)$.

for $m = 1$ to $M$ do

\[
\tilde{z}^{(t,m)}_{\text{tmp}} \leftarrow \text{Quantum estimate } \sum_{j} \tilde{w}^{(t)}_{j} (a^{(m)}_{j} + 3) \text{ using } \{U^{(t')}_{j} : t' \leq t - 1\} \text{ with success probability } 1 - \frac{\delta}{10MT}.
\]

\[
\tilde{z}^{(t,m)} \leftarrow \frac{\tilde{z}^{(t,m)}_{\text{tmp}}}{\tilde{z}^{(t,m)}_{\text{max}}} \left\| \frac{\tilde{w}^{(t)}}{\tilde{w}^{(t)}_{\text{max}}} \right\|_1 - 3 \text{ (to additive accuracy } \frac{\epsilon}{16\lambda} ).
\]

end for

\[
\tilde{z}^{(t)} \leftarrow \frac{1}{M} \sum_{m=1}^{M} \left( \sigma \left( \lambda\tilde{z}^{(t,m)}_{j} \right) - b^{(m)} \right)^2
\]

end for

Output: $\left( t', \tilde{h}^{(1)}, \ldots, \tilde{h}^{(t')} \right)$, $\left\| \frac{\tilde{w}^{(t')}}{\tilde{w}^{(t')}_{\text{max}}} \right\|_1$, $\tilde{w}^{(t')}_{\text{max}}$ for $t' = \arg \min_{t \in [T]} \tilde{z}^{(t)}$.

Note that the output of the algorithm are inner product estimates and a norm estimate. The output is not a full classical vector, which would take $O(N)$ to write down. The inner products estimates also allow also to prepare a quantum state proportional to the desired vector $q$ from which one can take samples or compute inner products with other quantum states. Using $T$ and $M$, the run time is given by $\tilde{O} \left( \frac{\lambda^2}{\epsilon^2} \log^4 \left( \frac{1}{\delta} \right) \right)$, compared to the run time of $\tilde{O} \left( N \frac{\lambda^2}{\epsilon^2} \log^4 \left( \frac{1}{\delta} \right) \right)$ of the approximate Sparsitron and the run time of $\tilde{O} \left( N \frac{\lambda^2}{\epsilon^2} \log^2 \lambda \right)$ of the original Sparsitron. The statement is as follows.

**Theorem 8** (Quantum Sparsitron). Let $\mathcal{D}$ be a distribution on $[-1, 1]^N \times \{0, 1\}$ where for $(X, Y) \sim \mathcal{D}, E[Y|X = x] = \sigma(w \cdot x)$ for a non-decreasing 1-Lipschitz function $\sigma$ :
\( \mathbb{R} \to [0, 1] \). Suppose that \( \|w\|_1 \leq \lambda \) for a known \( \lambda \geq 0 \). Let \( \epsilon, \delta \in (0, 1) \) and given \( T = O(\lambda^2 (\log(N/\delta\epsilon))/\epsilon^2) \) independent examples from \( \mathcal{D} \) accessed via Data Input \( \mathcal{D} \). Algorithm \( \mathcal{D} \) returns \( (t', \hat{h}^{(1)} \cdots \hat{h}^{(t')}, \sqrt{\hat{w}^{(t')}_{\text{max}}} , \sqrt{\hat{w}^{(t')}_{\text{max}}} ) \), i.e., some \( t' \in [T] \), inner product estimates, a norm estimate, and a maximum weight. Given this output, there exists a vector \( q \in \mathbb{R}^N \) such that its coordinates \( q_j \) can be constructed separately in time \( \tilde{O}(T) \) and \( q \) satisfies with probability at least 1 \(- 3\delta \) that

\[
E_{(X,Y)\sim \mathcal{D}}[(\sigma(q \cdot X) - \sigma(w \cdot X))^2] \leq \epsilon.
\]  

The run time of the algorithm to obtain the output is \( \tilde{O} \left( \frac{T^2 T\sqrt{N}}{\epsilon} \log \left( \frac{1}{\delta} \right) \right) \), where \( M = O(\log(T/\delta)/\epsilon^2) \). Again, the algorithm can be run in an online manner. In addition, an \( \epsilon \)-approximation to the quantum state \( |q\rangle = \sum_{j=1}^{N} \sqrt{q_j/\|q\|_1} |j\rangle \) can be prepared in time \( \tilde{O} \left( \frac{\sqrt{N}}{\epsilon} \log \left( \frac{1}{\delta} \right) \right) \) with success probability at least \( 1 - \delta \).

Proof. We first discuss the inner product estimation. Fix \( 1 \leq t \leq T \). Given the Input \( \mathcal{D} \) and the Sparsitron inner product estimates up to time \( t \), \( \hat{h}^{(1)} \cdots \hat{h}^{(t-1)} \), and the corresponding unitaries \( U^{(1)}_t, \ldots, U^{(t-1)}_t \). Together with the weight computation Lemma \( \mathcal{D} \) and quantum access to the training data, construct the operation that computes \( |j\rangle \langle j|\hat{u}^{(t)}_j \left( x_j^{(t)} + 3 \right) \rangle \) by basic arithmetic operations and uncomputing unnecessary registers.

We obtain \( \hat{w}^{(t)}_{\text{max}} = \max_j \hat{w}^{(t)}_j \) and \( h^{(t)}_{\text{max}} = \max_j \hat{w}^{(t)}_j \left( x_j^{(t)} + 3 \right) \) by quantum maximum finding. Each maximum finding requires \( O \left( T\sqrt{N} \log \frac{T}{\delta} \right) \) queries and \( \tilde{O} \left( T\sqrt{N} \log \left( \frac{T}{\delta} \right) \right) \) quantum gates to success probability at least \( 1 - \frac{\delta}{5T} \).

We obtain an estimate \( \tilde{h}^{(t)} \) of the inner product \( h^{(t)} = \sum_j \frac{\hat{w}^{(t)}_j}{\|u^{(t)}_j\|_1} x_j^{(t)} \) to additive accuracy \( \frac{\epsilon}{8\lambda^2} \) with run time \( \tilde{O} \left( \frac{T^2 T\sqrt{N}}{\epsilon} \log \left( \frac{1}{\delta} \right) \right) \) quantum gates as follows. Apply Lemma \( \mathcal{D} \) with \( u_j = \frac{\hat{w}^{(t)}_j (x_j^{(t)} + 3)}{h^{(t)}_{\text{max}}} \) to obtain an estimate \( \tilde{h}^{(t)}_{\text{tmp}} = \|\hat{u}\|_1 \) to relative accuracy \( \epsilon_u = \frac{\epsilon}{16 \lambda^2} \) with success probability \( 1 - \frac{\delta}{5T} \), noting that \( u_{\text{max}} = 1 \). Similarly, apply Lemma \( \mathcal{D} \) with \( \frac{\hat{w}^{(t)}_j}{\|\hat{w}^{(t)}_{\text{max}}\|_1} \) to obtain the respective \( \ell_1 \)-norm to relative accuracy \( \epsilon_1 = \frac{1}{16} \min \left\{ \frac{\epsilon}{8\lambda^2}, \frac{\epsilon}{10\lambda} \right\} \) with success probability \( 1 - \frac{\delta}{5T} \). Both estimations take a run time of \( \tilde{O} \left( \frac{T^2 T\sqrt{N}}{\epsilon} \log \left( \frac{T}{\delta} \right) \right) \). Translate between the outcome of the amplitude estimations and the desired inner product via \( \tilde{h}^{(t)} = \frac{h^{(t)}_{\text{max}}}{\|\hat{w}^{(t)}_{\text{max}}\|_1} \tilde{h}^{(t)}_{\text{tmp}} \) - 3. From Lemma \( \mathcal{D} \) we have \( \|\tilde{h}^{(t)} - h^{(t)}\|_1 \leq \frac{h^{(t)}_{\text{tmp}}}{\|\hat{w}^{(t)}_{\text{max}}\|_1} \|\tilde{h}^{(t)}_{\text{tmp}} - h^{(t)}_{\text{tmp}}\|_1 \leq \frac{h^{(t)}_{\text{tmp}}}{\|\hat{w}^{(t)}_{\text{max}}\|_1} \|\tilde{h}^{(t)}_{\text{tmp}} - h^{(t)}_{\text{tmp}}\|_1 \leq \frac{\epsilon}{8\lambda^2} \).
\(z_{t,m}^{(t,m)}\) to relative accuracy \(\frac{1}{16}\epsilon\) with success probability \(1 - \frac{\delta}{10MT}\) at cost \(\tilde{O}\left(\frac{\sqrt{N}}{\epsilon} \log \left(\frac{MT}{\delta}\right)\right)\).

The norm was already estimated to relative accuracy \(\leq \frac{1}{16}\epsilon\), hence the inner product \(z_{t,m}^{(t,m)}\) is estimated to additive accuracy \(\epsilon, \delta\) model. Given product estimates and norms such that every element of a matrix \(A\) from the Ising distribution, there exists a quantum algorithm that produces classical inner products with time \(\tilde{O}\left(\frac{\sqrt{N}}{\epsilon} \log \left(\frac{4}{\delta}\right)\right)\). The success probability of the inner loop is \((1 - \frac{\delta}{10MT})^{2M} \geq 1 - \frac{\delta}{5T}\). The success probability of all the probabilistic steps in the algorithm is \((1 - \frac{\delta}{5T})^{5T} \geq 1 - \delta\).

From the output of the algorithm, the construction of a single element \(q_j\) of a classical vector takes time \(\tilde{O}(T)\). That is because

\[
q_j = \lambda \beta^{\sum_{t=1}^{T} \frac{1}{2} \left(1 + \left(\sigma(h(t)) - y(t)\right)\sigma^{(t)}_j\right)} \cdot \frac{1}{u_{\max}} \frac{\|u_j^{(t-1)}\|}{\left\|u_{\max}^{(t-1)}\right\|_1}, \tag{31}
\]

where the computation of the sum takes in time \(\tilde{O}(T)\). For the quantum state preparation, use Lemma \(\Box\) with \(u_j = q_j/q_{\max}\), where \(q_{\max} = \max_j q_j\), and the efficient computability of \(q_j\) in \(\tilde{O}(T)\). The maximum finding can be done as before. By Lemma \(\Box\) estimating \(\left\|\frac{q_j}{q_{\max}}\right\|_1\) and preparing \(|q\rangle = \sum_{j=1}^N \sqrt{q_j} |j\rangle\) takes a run time of \(\tilde{O}\left(\frac{T\sqrt{N}}{\epsilon} \log \left(\frac{1}{\delta}\right)\right)\).

The guarantee of the algorithm and the success probability is the same as in Theorem \(\Box\), as the inner products were estimated to the same accuracy and the individual success probabilities lead to a total success probability of at least \(1 - 3\delta\).

We now show the application of the Quantum Sparsitron to Ising models as a corollary. Please refer to the beginning of Section [IV] for a brief introduction to the problem. The width of an Ising model is defined as \(\lambda(A, \theta) = \max_i (\sum_j |A_{ij}| + |\theta_i|)\), see Eq. (10) for the definition of \(A\) and \(\theta\).

**Corollary 1 (Quantum Learning of Ising models).** *Given an \(N\)-variable Ising model with width \(\leq \lambda\) for \(\lambda \geq 0\). Given quantum query access to the entries of the samples from the Ising model. Given \(\epsilon, \delta \in (0,1)\), and \(T = \tilde{O} \left(\lambda \exp(\tilde{O}(\lambda))/\epsilon^4 \right) \log(N/\delta\epsilon)\) independent samples from the Ising distribution, there exists a quantum algorithm that produces classical inner product estimates and norms such that every element of a matrix \(A^*\) can be computed in time \(\tilde{O}(T)\). For the matrix \(A^*\) it holds that \(\|A - A^*\|_\infty \leq \epsilon\) with probability at least \(1 - \delta\). The run time of the algorithm is \(\tilde{O}\left(\frac{\sqrt{N}}{\epsilon} \log \left(\frac{1}{\delta}\right)\right)\). Quantum states of the columns/rows of the matrix \(A^*\) with \(\epsilon\) distance can be prepared in time \(\tilde{O}\left(\frac{T\sqrt{N}}{\epsilon}\right)\). Again, the algorithm can be run in an online manner.*

**Proof.** The algorithm applies the Quantum Sparsitron \(\tilde{O}(N)\) times hence the run time is \(\tilde{O}\left(\frac{\sqrt{N}}{\epsilon} \log \left(\frac{T}{\delta}\right)\right)\). As the Quantum Sparsitron has the same guarantee as the classical Sparsitron, we obtain the same guarantee for the Ising model learning. \(\Box\)
VI. DISCUSSION AND CONCLUSION

In the main part of this work, we have presented a quantum machine learning algorithm with both provable learning guarantee and provable quantum speedup over the best known classical algorithm. The starting point is a classical algorithm called the Sparsitron, which is a dimensionally sample-optimal algorithm for generalized linear models under modest assumptions. Generalized linear models have a large range of applications, including logistic and Poisson regression, and they appear also in a variety of problems such as the learning of Ising models and Markov Random Fields (MRFs.)

The run time of our quantum algorithm, the Quantum Sparsitron, shows a speedup polynomial in the dimension of the problem and a slowdown in the error dependency, while the sample complexity remains the same as for the classical algorithm. The setting here is the standard quantum gate model, i.e., many logical quantum bits with the physical errors being kept under control via error correction. In addition, we assume the availability of oracles which provide access to the training examples. The training examples can be given via efficiently computable circuits or via classical data collected from sampling the true distribution and quantum RAM access to these data. The main quantum subroutines are the well-known amplitude amplification and estimation algorithms, which are here applied in a way that the provable learning guarantee and success probability of the classical algorithm are preserved. Due to the use of amplitude amplification and oracles, the algorithm can be considered far-term in nature, requiring significantly more resources than the presently available 50-100 noisy qubits.

The optimization problem here is non-convex, as the Lipschitz function defining the generalized linear model can be a non-convex function (such as the sigmoid function.) In the sense that the classical Sparsitron solves this non-convex problem, the quantum algorithm solves the same non-convex problem. While many of the recent quantum algorithms are for convex problems, such as LPs and SDPs [5, 6], this work can be seen as an interesting extension of the same underlying quantum techniques to non-convex problems.

On the classical side, we have shown that the Sparsitron run time (but not the sample complexity) can be sped up via inner product estimation techniques. Randomized linear algebra is a well-studied area which has recently also found application in the discussion of dimension-efficient classical algorithms for various problems considered for quantum machine learning [23–28]. Our work relates to these results in the sense that we have started at a near-optimal classical machine learning algorithm. In analogy to the present work, it will be interesting to take [23–28] as starting points and to exhibit provable quantum speedups for the various problems.

VII. ACKNOWLEDGEMENTS

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Note added: We note very recent work [29], using quantum techniques for speeding up
the Hedge algorithm in the adaptive boosting context.

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Appendix A: Quantum subroutines

Lemma 7 ([20]). Given quantum access to a vector $u \in [0, 1]^N$ via the operation $|j\rangle |\bar{0}\rangle \rightarrow |j\rangle |u_j\rangle$ on $O \left( \log N + \log \frac{1}{\eta} \right)$ qubits, where $u_j$ is encoded to additive accuracy $\eta > 0$. Then,
we can find the minimum \( u_{\min} = \min_j u_j \) with success probability \( 1 - \delta \) with \( \mathcal{O}\left(\sqrt{N \log \frac{1}{\delta}}\right) \) queries and \( \mathcal{O}\left(\sqrt{N \log \left(\frac{1}{\eta}\right)} \log \left(\frac{1}{\eta}\right)\right) \) quantum gates.

The minimum finding can be turned straightforwardly into a maximum finding algorithm. Next, we state the results for estimating the \( \ell_1 \)-norm of a vector and preparing states encoding the square root of the vector elements. We follow Ref. [21] regarding parts of the error analysis.

**Lemma 8.** Given a non-zero vector \( u \in [0,1]^N \), with \( \max_j u_j = 1 \). Given quantum access to \( u \) via the operation \( |j\rangle |0\rangle \rightarrow |j\rangle |u_j\rangle \) on \( \mathcal{O}(\log N + \log 1/\eta) \) qubits, where \( u_j \) is encoded to additive accuracy \( \eta > 0 \). Then:

(i) There exists a unitary operator that prepares the state \( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle (\sqrt{u_j} |0\rangle + \sqrt{1-u_j} |1\rangle) \) with a single query and number of gates \( \mathcal{O}(\log N + \log 1/\eta) \). Denote this unitary by \( U_X \).

(ii) Let \( \epsilon_1 > 0 \) and the additive accuracy be \( \eta \leq \epsilon_1/(2N) \). There exists a quantum algorithm that provides an estimate \( \|u\|_1 \) of the \( \ell_1 \)-norm \( \|u\|_1 \) such that \( \|u\|_1 - \|\tilde{u}\|_1 \leq \epsilon_1 \|u\|_1 \), with probability at least \( 1 - \delta \). The algorithm requires \( \mathcal{O}\left(\frac{\sqrt{N}}{\epsilon_1} \log(1/\delta)\right) \) queries to the oracles and \( \mathcal{O}\left(\sqrt{N} \log(1/\delta)\right) \) gates.

(iii) Let \( \epsilon \in (0,1] \) and \( \eta \leq \epsilon/4N \). Also let \( \|\tilde{u}\|_1 \geq 1 \) be an estimate of \( \|u\|_1 \) to relative accuracy \( \epsilon/4 \). An approximation \( |\tilde{p}\rangle = \sum_{j=1}^{N} \sqrt{p_j} |j\rangle \) to the state \( |u\rangle := \sum_{j=1}^{N} \sqrt{u_j/\|u\|_1} |j\rangle \) can be prepared, using \( \mathcal{O}(\sqrt{N}) \) calls to the unitary of (i) and \( \mathcal{O}\left(\sqrt{N}\right) \) additional gates. The approximation in \( \ell_1 \)-norm of the probabilities is \( \|\tilde{p} - \frac{u}{\|u\|_1}|\|_1 \leq \epsilon \).

**Proof.** For (i), prepare a uniform superposition of all \( |j\rangle \) with \( \mathcal{O}(\log N) \) Hadamard gates. With the quantum query access, perform \( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle |0\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle |u_j\rangle |0\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle |u_j\rangle (\sqrt{u_j} |0\rangle + \sqrt{1-u_j} |1\rangle) \). The steps consist of an oracle query and a controlled rotation. The rotation is well-defined as \( u_j \leq 1 \) and costs \( \mathcal{O}(\log 1/\eta) \) gates. Then uncompute the data register \( |u_j\rangle \) with another oracle query.

For (ii), define a unitary \( U = (U_X)^{1/2} (1 - 2 |0\rangle \langle 0|) U_X \), with \( U_X \) from (i). Define another unitary by \( V = 1 - 1 \otimes |0\rangle \langle 0| \). Using \( K \) applications of \( U \) and \( V \), Amplitude Estimation [30] allows to estimate the quantity \( a = \frac{\|u\|_1}{N} \) to accuracy \( \tilde{a} - a \leq 2\pi \frac{\sqrt{a(1-a)}}{N} + \frac{\pi^2}{K} \). Taking \( K > \frac{6\pi}{\epsilon_1} \sqrt{N} \), one obtains \( \tilde{a} - a \leq \frac{\pi}{K} \left(2\sqrt{a} + \frac{\pi}{a}\right) < \epsilon_1 \frac{6}{\sqrt{N}} \left(2\sqrt{a} + \frac{\pi^2}{12}\right) \frac{1}{N} \leq \epsilon_1 \frac{6}{\sqrt{N}} \left(3\sqrt{a}\right) = \frac{\epsilon_1 \sqrt{\|u\|_1}}{2N} \). Since \( \|u\|_1 \geq 1 \) by assumption, we have \( \tilde{a} - a \leq \frac{\epsilon_1 \sqrt{\|u\|_1}}{2N} \). Also, there is an inaccuracy arising from the additive error \( \eta \) of each \( u_j \). As it was assumed that \( \eta \leq \epsilon_1/(2N) \), the overall multiplicative error \( \epsilon_1 \) is obtained for the estimation. For performing a single run of amplitude estimation with \( K \) steps, we require \( \mathcal{O}(K) = \mathcal{O}\left(\frac{\sqrt{N}}{\epsilon_1}\right) \) queries to the oracles and \( \mathcal{O}\left(\frac{\sqrt{N}}{\epsilon_1} (\log N + \log(N/\epsilon_1))\right) \) gates.
For (iii), rewrite the state from (i) as $\sqrt{\frac{\|u\|_1}{N}} \sum_{j=1}^{N} \sqrt{\frac{1-u_j}{N-\|u\|_1}} |j\rangle |0\rangle + \sqrt{1 - \frac{\|u\|_1}{N}} \sum_{j=1}^{N} \sqrt{\frac{1-u_j}{N-\|u\|_1}} |j\rangle |1\rangle$. Now amplify the $|0\rangle$ part using Amplitude Amplification [30], to prepare $\sum_{j=1}^{N} |j\rangle \sqrt{\frac{u_j}{\|u\|_1}}$. The amplification requires $O\left(\sqrt{\frac{N}{\|u\|_1}}\right) = O\left(\sqrt{N}\right)$ calls to the unitary of (i), as $\|u\|_1 \geq 1$. Each step requires $O(\log N)$ additional gates. Taking into account that only the $\eta$-additive approximation to $u_j$, denoted by $\tilde{u}_j$, is available, evaluate the $\ell_1$-distance. One obtains $\left\| \tilde{p} - \frac{u}{\|u\|_1} \right\|_1 = \left\| \frac{\tilde{u}}{\|u\|_1} - \frac{u}{\|u\|_1} \right\|_1 \leq \sum_j \left|\frac{\tilde{u}_j}{\|u\|_1} - \frac{u_j}{\|u\|_1}\right| + \sum_j \left|\frac{u_j}{\|u\|_1} - \frac{u_j}{\|u\|_1}\right| \leq \frac{N\eta}{\|u\|_1} + \frac{\epsilon\|u\|_1}{4\|u\|_1}$. For the second term, $\|u\|_1 - \|\tilde{u}\|_1 \leq \frac{\epsilon}{4}\|u\|_1$ was used, which also obtains $\frac{1}{\|u\|_1} \leq \frac{1}{\|u\|_1(1-\epsilon/4)} \leq \frac{2}{\|u\|_1}$ for $\epsilon \leq 1$. Since $\eta \leq \epsilon/(4N)$, the distance is $\left\| \tilde{p} - \frac{u}{\|u\|_1} \right\|_1 \leq \epsilon$ as desired. \(\square\)