Abstract

I survey the physics of black holes in two and three spacetime dimensions, with special attention given to an understanding of their exterior and interior properties.

1 Introduction

The trickle of interest in lower-dimensional gravity that began a little more than a decade ago has turned in recent years into a virtual flood. There are, I think, two (related) reasons for this. First, the technical difficulties present in a wide range of problems in $(3+1)$ dimensional gravitation become significantly simpler in lower dimensions. The pedagogical value of this fact was noted some time ago by Collas [1]: one hopes that lower dimensional gravitational physics can provide insight into problems in $(3+1)$ dimensions by yielding a greater measure of computational simplicity without sacrificing too much of the conceptual complexity of the original problem. A number of physical problems have been approached in this manner, including the study of quantum gravitational effects in a mathematically tractable setting [2], and clarification of the conceptual issues associated with black hole physics [3, 4]. Second, there are some physical systems that are effectively confined to move in lower dimensions, such as cosmic strings and domain walls [5]. Indeed, the physics of strings and membranes necessitates the introduction of effective lower-dimensional geometries, and understanding of a number of problems in string theory (motivated by the
original work of Polyakov [6]) has been advanced by a study of lower-dimensional gravity.

It is not possible to give a comprehensive review of lower-dimensional black holes (let alone lower-dimensional gravity), in the space I have been allotted. Consequently this review will be somewhat idiosyncratic, expressing my own perspective on the subject and highlighting a number of issues and viewpoints on lower-dimensional black holes that I hope will provide a starting point for those interested in research in this field. I apologize in advance to those authors who might feel that their work is under-represented here.

I shall begin with a short review of the main features of lower-dimensional gravity. This will be followed by a discussion of the physics outside of lower-dimensional black holes, including gravitational collapse, mass/energy, thermodynamics, and quantum properties. I shall then turn to a consideration of the physics inside the event horizon, discussing recent research in this subject.

2 Ins and Outs of Lower-Dimensional Gravity

Perhaps the most direct way of approaching lower-dimensional gravity is to begin with a consideration of Einstein’s equations in $(D + 1)$ dimensions

$$G_{\mu\nu} = 8\pi G_{D+1}\text{T}_{\mu\nu} \quad (1)$$

where $G_{D+1}$ is Newton’s constant in $(D + 1)$-dimensions and $T_{\mu\nu}$ is the stress-energy tensor. For $D \geq 3$, a vanishing stress-energy implies a vanishing Einstein tensor $G_{\mu\nu}$, but not necessarily a vanishing Riemann tensor $R_{\mu\nu\rho\sigma}$: it is possible to have nonzero curvature in regions of spacetime where there is no stress-energy.

This feature is lost when $D \leq 2$. In $(2 + 1)$ dimensions it is possible to write the Riemann tensor completely in terms of the Einstein tensor

$$R_{\mu\nu\rho\sigma} \equiv \epsilon_{\mu\nu\gamma\epsilon_{\rho\sigma}} G_{\gamma\rho} \quad (2)$$

and so a vanishing of the latter necessarily implies a vanishing of the former. In $(1 + 1)$ dimensions the situation is even more extreme: the Einstein tensor vanishes for all metrics, and so adoption of the Einstein equations as the foundation of gravitational theory in $(1 + 1)$ dimensions necessarily implies the vanishing of the stress-energy tensor! One might therefore superficially conclude that there can be no interesting gravitational physics in lower dimensions. Fortunately the actual situation is considerably more interesting.

An early study of $(2 + 1)$ dimensional gravity [7] revealed that although spacetime was flat outside of matter, matter sources exert a gravitational influence that is topological in character: a point source will introduce a conical deficit into spacetime. Consequently vacuum gravity is only locally flat. There is no Newtonian limit to the field equations and a fluid in hydrostatic equilibrium will attain its maximal mass [8]. These features arise as a consequence
of the comparative mathematical simplicity relative to the \((3 + 1)\) dimensional case, and in recent years this simplicity has been exploited to develop a formulation of quantum gravity. This program has been quite successful; so successful that in fact there are six different formulations of quantum gravity in \((2 + 1)\) dimensions whose relationship remains an ongoing subject of research \cite{1}. One of the more recent developments in the subject has been the realization that black holes can exist if a negative cosmological constant is introduced \cite{10}, and that they can form as the endpoint of collapse of a disk of dust \cite{11}.

The triviality of the Einstein tensor in \((1 + 1)\) dimensions necessitates a more subtle approach to gravitation in this case. One such approach involves rewriting the Newtonian constant of gravity as

\[
G_{D+1} = (1 - D) \hat{G}_{D+1}/2, \quad \text{where} \quad \hat{G}_{2} \text{ is finite.}
\]

Upon rewriting the Einstein equations \((1)\) into their trace and trace-free parts, it is possible to show \cite{12} that as \(D \to 1\), the trace-free parts simply yield \(0 = 0\) whereas the trace part becomes

\[
R = 8\pi \hat{G}_{2} T
\]

where \(T = T_{\mu}^{\mu}\) is the trace of the conserved \((1 + 1)\)-dimensional stress energy. The above theory generalizes an early attempt at formulating a theory of gravity in two spacetime dimensions \cite{13} (in which the Ricci scalar was set equal to a constant) and was proposed several years ago \cite{14, 15} as a natural classical analogue of general relativity. The Newtonian limit exists \cite{16}, and a number of \((3 + 1)\) dimensional features of general relativity have analogues in the theory described by \cite{17}. Other approaches toward formulation of gravity in two spacetime dimensions have involved the consideration of non-local actions \cite{6}, higher-derivative theories \cite{18}, or setting the one-loop beta functions of the bosonic (or supersymmetric) non-linear sigma model to zero in a two-dimensional target space \cite{19}. In each approach the Ricci scalar becomes a non-vanishing function of the co-ordinates over some region of spacetime, permitting the spacetime to develop interesting features such as black-hole horizons and singularities. Considerable progress can be made in the quantization of such theories coupled to matter \cite{20, 21, 22, 23}, providing an attractive arena for testing ideas about quantum gravity \cite{24}.

The field equations of virtually all \((1 + 1)\)-dimensional theories of gravity can be derived from an action of the form \cite{21, 25}

\[
S = S_{G} + S_{M} = \int d^{2}x \sqrt{-g} \left( \frac{1}{8\pi \hat{G}_{2}} \left[ H(\psi) \frac{1}{2} g^{\mu\nu} \nabla_{\mu} \psi \nabla_{\nu} \psi + D(\psi) R \right] + L_{M}(\psi, \Phi) \right)
\]

where the first two terms form the gravitational part of the action and the remainder is the matter action. The scalar field \(\psi\) is called the dilaton, and \(\Phi\) denotes the presence of all other matter fields. The quantity

\[
J_{\mu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} g^{\nu\tau} \nabla_{\tau} F
\]
where

\[ \mathcal{F} = F_0 \int \psi ds D' e^{-\int \frac{dt}{\rho(t)}} \]  

(6)
is conserved \( (\nabla^\mu J_\mu = 0) \) regardless of whether or not the field equations are satisfied, provided \( \mathcal{L}_M \) has no metric dependence [25] \( (F_0 \) is a constant whose value is set by the boundary conditions on the spacetime). If a timelike Killing vector \( \xi^\mu \) exists, then one can define a conserved mass current density [25]:

\[ \mathcal{M} = \frac{1}{2} \left[ (\nabla D)^2 e^{-\int \frac{dt}{\rho(t)}} - F_0 \int dD V e^{-\int \frac{dt}{\rho(t)}} \right] \]  

(7)

whose space integral yields the total mass associated with a given (stationary) solution to the field equations.

By reparametrizing the metric and dilaton fields it is possible when \( \mathcal{L}_M = 0 \) to convert any given \( H(\psi) \) and \( D(\psi) \) to any other \( H(\psi) \) and \( D(\psi) \). Hence it is the matter action which introduces interesting physics, and a given choice of \( H(\psi) \) and \( D(\psi) \) describes which metric couples to matter in two spacetime dimensions. I shall employ several examples to illustrate this point. In all of these the metric is written in the coordinates:

\[ ds^2 = -\alpha(x) dt^2 + \frac{dx^2}{\alpha(x)} \]  

(8)

where \( \alpha \) shall be referred to as the metric function.

**R = T Theory**

This case corresponds to the choice \( H(\psi) = \frac{1}{2} \), \( D(\psi) = \psi \), and \( \mathcal{L}_M = \mathcal{L}_M(\Phi) \). In this case the matter action which is independent of the dilaton. This choice for \( H(\psi) \) and \( D(\psi) \) uniquely yields [3] \( (\text{with } 8\pi \hat{G}_2 \equiv \kappa) \), where the stress-energy tensor

\[ T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \]  

(9)
is conserved. There is also an auxiliary equation for the dilaton [3]. The full system of equations is such that the evolution of the gravity/matter system is (classically) unaffected by the evolution of the dilaton, although the converse is not true.

Except for being independent of the dilaton, the choice of matter action is as arbitrary as in \( (3 + 1) \) dimensions, being constrained by conservation laws and positivity of energy. One can choose it to be a two-dimensional analogue of any desired corresponding \( (3 + 1) \) dimensional quantity; a number of implications of this have been explored in the literature [14, 17]. An interesting choice for black hole physics is [23]:

\[ S_M = \int d^2 x \sqrt{-g} [b(b(\nabla \phi)^2 + \Lambda e^{-2a\phi} - \gamma \phi R] \]  

(10)
which is the action for a Liouville field in curved spacetime. The field equations have the exact solution

\[
\begin{align*}
\alpha(x) &= 1 - \frac{\lambda^2}{\mu^2} e^{-2mx} \\
\phi(x) &= \frac{m}{a} x - \frac{1}{a} \log \frac{m}{\mu} \\
\psi(x) &= 2mx - 2 \log \frac{m}{\mu} + \kappa d \log \frac{m_0}{\lambda}
\end{align*}
\] (11)

where \( b = -\frac{2a^2}{\kappa} \), \( d = \gamma/a - 2/\kappa \), \( \lambda^2 = -\Lambda/2d \) and \( \mu \) and \( m_0 \) are positive constants. The signs of \( d \) and \( \Lambda \) must be chosen so that \( \lambda \) is real. Note that the choice of constants and origin of coordinates differs from that of ref. [23]. More general choices of the coupling constants \( a \) and \( b \) yield other exact solutions to the field equations [23].

2D String Theory

This case corresponds to the choice \( H(\psi) = 4 \exp[-2\psi] = 4D(\psi) \), and \( L_M = Q^2 \exp[-2\psi] \), and corresponds to the effective action of low energy bosonic string theory in two spacetime dimensions; the coupling constant \( Q \) may be given in terms of the central charge of the theory [19]. The field equations have the exact solution

\[
\begin{align*}
\alpha(x) &= 1 - 2m \frac{Q}{Q^2} e^{-Q(x-x_0)} \\
\psi(x) &= -\frac{Q}{2} (x-x_0)
\end{align*}
\] (12)

in the absence of any additional matter fields. These can be included in the theory if desired.

Generalized String-Inspired Theories

A natural generalization of the previous case involves taking \( H(\psi) = 4 \exp[-2\psi] = 4D(\psi) \), and

\[
S_M = \int \sqrt{-g} \left[ e^{-2\psi} \left( -\frac{1}{4} F_{\beta\sigma} F^{\beta\sigma} + Q^2 \right) + \sum_{n=2}^{k} a_n e^{2(n-1)\psi} - 8\pi \mathcal{L}_M \right],
\] (13)

where the \( \{a_n\} \) are dimensional coupling constants and \( F_{\mu\nu} \) is the electromagnetic field strength [26]. The field equations have the exact solution

\[
\begin{align*}
f &= F_{tx} = q e^{2\psi}, \\
\psi &= -\frac{Q}{2} (x-x_0)
\end{align*}
\] (14) (15)
\[ \alpha(x) = \frac{2}{Q} \left[ \mathcal{M}(x) - m \right] e^{2\psi}, \quad (16) \]

\[ \mathcal{M}(x) = \frac{Q}{2} e^{-2\psi(x)} \left[ 1 + \frac{q^2}{2Q^2} \alpha^4 \psi(x) - \frac{1}{Q^2} \sum_{n=2}^{k} a_n n e^{2n\psi(x)} \right]. \quad (17) \]

The spacetime may be said to have a black hole if there exists a region where \( \alpha < 0 \). In the coordinates of (8) the zeros of \( \alpha \) yield the locations of the event horizons; regions of positive \( \alpha \) are regions where \( \partial/\partial t \) is a timelike Killing vector.

Many other examples of two dimensional black holes exist whose properties have been explored to varying degrees in the literature [27, 28, 29, 30].

I would be remiss to close this section without remarking on the chief limitations of lower-dimensional theories of gravity: namely the absence of tidal forces and gravitational waves in the vacuum. Since \((1 + 1)\) dimensional gravity is typically formulated in terms of a dilaton field, the former limitation is perhaps not so serious, since all interesting solutions have a non-vanishing dilaton field. However helicity-2 gravitational waves propagating in regions of spacetime that are free of stress energy is one of the key features of general relativity, and is the last significant prediction of Einstein’s theory which remains to be explicitly verified (the binary pulsar PSR 1913+16 providing indirect verification of this phenomenon). The absence of this feature in lower dimensional theories of gravity has caused a number of authors to question their relevance for \((3 + 1)\) dimensional physics. However the last decade of research has time and again highlighted many features of lower dimensional gravitation and spacetime structure that are expected to survive in some form in the real \((3 + 1)\) dimensional world. Combined with the significant payoff in computational progress that results from the mathematical simplicity of such theories, this seems to me to more than warrant as thorough an investigation of this subject as is possible. Of course, as with any toy model of the real world, a healthy dose of caution is advised in extrapolating results.

3 Outside Looking In

In this section I shall review the main exterior properties of lower-dimensional black holes. These include their formation from gravitational collapse, their mass, energy and angular momentum, their thermodynamic properties and their quantum properties.

Gravitational Collapse

It is a well-known phenomenon in \((3 + 1)\) dimensions that under certain circumstances gravitational forces can overwhelm all other forces, causing complete gravitational collapse of a given system of matter [33], the end result of which is a black hole. The astrophysical implications of this process continue to be
an ongoing subject of research. That this phenomenon can also take place in lower dimensions indicates that lower dimensional black holes bear more than a superficial resemblance to their higher dimensional cousins.

Consider first the situation in \((2 + 1)\) dimensions. A circularly symmetric metric has the general form

\[
ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\theta^2
\]

and the Einstein equations \([1]\) with \(T_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu}\) have the form

\[
\begin{align*}
- \frac{1}{2}A' - \Lambda B &= 0 \\
- \frac{1}{2}AB' + \Lambda A &= 0 \\
- \frac{1}{2}r^2A'' - \Lambda r^2 &= 0
\end{align*}
\]

where the prime denotes a derivative with respect to \(r\) and \(\Lambda\) is the cosmological constant. The first two of these equations yield \(A(r) = 1/B(r) = C - \Lambda r^2\), where \(C\) is a constant of integration. The remaining equation yields no further constraints on \(C\) (as must be the case, due to the Bianchi identities). Hence the Einstein equations yield the exact solution

\[
ds^2 = -(C - \Lambda r^2)dt^2 + \frac{dr^2}{C - \Lambda r^2} + r^2d\theta^2
\]

which is the canonical form for a deSitter metric. Indeed, if \(\Lambda > 0\), then \((20)\) is the metric for deSitter spacetime, and \(C\) must be positive for \(\partial/\partial t\) to be a timelike direction. If \(C \neq 1\) then the metric has a conical singularity at the origin, indicative of the presence of a mass; in the \(\Lambda \to 0\) limit, we can identify \(C = 1 - 2GM\) where \(M\) is the mass associated with the conical singularity \([7]\).

If \(\Lambda < 0\) then the metric \((20)\) is that of anti-de Sitter spacetime. However here a novel feature emerges, since the sign of \(C\) can be either positive or negative. In the former case one simply has anti-de Sitter spacetime (with a conical singularity if \(C \neq 1\)). In the latter case the spacetime has an event horizon, signalling the presence of a black hole. The \(\Lambda > 0\), \(C < 0\) case cannot occur for spherically symmetric metrics in more than two spatial dimensions since the “angular” part of the Einstein equations force \(C = 1\) and the constant of integration is now \(m/r^{D-2}\) where \(D\) is the number of spatial dimensions – the metric is that of Schwarzschild anti de Sitter.

Given the simplicity of the above derivation it is somewhat remarkable that the existence of the \((2 + 1)\) dimensional black hole solution was uncovered by Banados \textit{et al.} \([10]\) nearly ten years after a significant research effort in \((2 + 1)\) dimensional gravity began \([7]\). Generalizing \((20)\) to include spin, it may be shown that the metric

\[
ds^2 = -N^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(V^2(r)dt + d\phi)^2
\]
where
\[ N^2(r) = f^2(r) = -m + \left( \frac{r}{\ell} \right)^2 + \left( \frac{j}{2r} \right)^2 \quad \text{and} \quad V^\phi(r) = -\frac{j}{2r^2} \]
is equivalent to anti-de Sitter spacetime with appropriate identifications \[34\].

For convenience I have set \( \Lambda = -1/\ell^2 \). As with the Kerr solution, the lapse function \( N(r) \) vanishes for two values of \( r \), namely \( r^+ \) and \( r^- \), where
\[ (r_{\pm})^2 = \frac{m\ell^2}{2} \pm \frac{\ell}{2} \sqrt{m^2\ell^2 - j^2} \]  

(22)
The larger of these, \( r^+ \), is specified as the black hole horizon and exists only for \( m > 0 \) and \( |j| \leq m\ell \); when \( |j| = m\ell \), \( r^+ = r^- \).

Recent research has indicated that there there are a wide class of \((2 + 1)\) dimensional black holes \[31\]. These arise as exact solutions to Einstein-Maxwell dilaton theory in \((2+1)\) dimensions. It is also possible to show that the black hole solution \[21\] is an exact solution to the Einstein equations with a topological matter source \[32\] – in this case \( \ell \) becomes a constant of integration whose appearance is contingent on the presence of the topological matter fields.

Turning now to gravitational collapse, consider a disk of collapsing dust surrounded by a vacuum region, with an exterior metric of the form
\[ ds^2 = -(R^2/\ell^2 - M)dt^2 + \frac{dR^2}{R^2/\ell^2 - M} + R^2d\theta^2. \]  

(23)
and an interior spacetime metric
\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2d\theta^2 \right) \]  

(24)
which describes the freely falling dust. In these coordinates \( T_{\mu\nu} = \rho u_\mu u_\nu \) is the stress-energy of the dust, where \( \rho(t) \) is the density of the dust and \( u_\mu = (1, 0, 0) \). Conservation of stress-energy \( T_{\mu\nu} = 0 \) then implies \( \rho a^2 = \rho_0 a_0^2 \), where \( \rho_0 \) is the initial density of the dust and \( a_0 \) is the initial scale factor. The field equations then have the solution
\[ a(t) = a_0 \cos(t/\ell) + \ell \dot{a}_0 \sin(t/\ell), \]  

(25)
where
\[ \dot{a}_0 = \sqrt{8\pi G \rho_0 a_0^2 - k} = a_0^2/\ell^2 \]  

(26)
yielding
\[ 8\pi G \rho_0 a_0^2 - k - a_0^2/\ell^2 \geq 0. \]  

(27)
since \( a(t) \) is real. This solution always collapses to \( a(t_c) = 0 \) in finite proper time; when \( a_0 = 0 \), this is \( t_c = \pi \ell/2 \).
Imposing appropriate matching conditions to make the dust edge a boundary surface yields [11]:

\[ M = \frac{a^2}{l^2} + k + \dot{a}^2 r_0^2 - 1 = 8\pi G\rho_0 a_0^2 r_0^2 - 1, \]  

(28)

where the dust edge is taken to be at \( r = r_0 \) in the interior coordinates, and at \( R = r_0 a(t) \) in the exterior coordinates. Collapse to a black hole occurs only for \( \rho_0 \) sufficiently large as \( M \) must be positive in this case; if \( \rho_0 \) is less than this critical value, the dust collapses to a conical singularity in anti de Sitter space.

In the case of collapse to a black hole, the curvature diverges at \( t = t_c \), and an event horizon (which is also a surface of infinite redshift) forms in a finite amount of exterior coordinate time [11].

If \( \Lambda \geq 0 \) then collapse to a black hole is not possible. If \( \Lambda = 0 \) then \( a(t) = a_0 + \dot{a}_0 t \), and collapse to a conical singularity occurs only if \( \dot{a}_0 < 0 \) [35]. If \( \Lambda > 0 \), then \( a(t) = a_0 \cosh(\sqrt{\Lambda}t) + \frac{\dot{a}_0}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}t) \) and collapse to a conical singularity is possible only if \( \dot{a}_0 < -a_0 \sqrt{\Lambda} \) [11].

These features are analogous to the (1+1)-dimensional case [16, 36]: a line of dust will collapse to a black hole only if the initial density is sufficiently large. The exterior spacetime may be taken to be that of two Rindler spacetimes with opposite acceleration [14, 37] or may be taken to be the full extension of Minkowski spacetime on either side of the line of dust [38]. In either case, additional boundary conditions need to be imposed once the endpoint of collapse has been reached.

Another form of collapse in (1 + 1) dimensions has been studied in the context of the string metric (12) [39]. In this case leftward-moving massless scalars divide spacetime into regions: a flat region (referred to as the linear dilaton vacuum) and a black hole region whose metric is that of (12), with the pulse of massless scalars at the boundary. The implications of this scenario for black hole radiation have been extensively studied in the last two years [40].

### Energy, Mass and Angular Momentum

The definition of mass-energy in gravitational theory is quite subtle, since spacetime curvature itself has stress-energy, making the localization of energy quite difficult. An early attempt to address this issue was developed for asymptotically flat spacetimes by Arnowitt, Deser and Misner [41]. However asymptotic flatness is not always an appropriate theoretical idealization and is never satisfied in reality. Furthermore, although the thermodynamic properties of black holes are expected to hold quite generally, much of the literature on black hole thermodynamics is restricted to the case of spacetimes that are asymptotically flat in spacelike directions.

In recent years there has been an effort by York and collaborators to develop a formalism for defining mass-energy that is independent of the assumptions of the asymptotic properties of spacetime curvature [42, 43]. This approach
has been referred to as the quasilocal formalism [44] and can be applied to gravitational and matter fields within a bounded, finite spatial region, so the asymptotic behavior of the gravitational field becomes irrelevant (a particularly important consideration for \((2 + 1)\) dimensional black holes). However even for asymptotically flat black hole spacetimes there are certain advantages to be gained by working in a spatially finite region: with the temperature fixed at a finite spatial boundary, the heat capacity is positive and there is no inconsistency in the black hole partition function[45]. However if the temperature is fixed at infinity, the heat capacity for a Schwarzschild black hole is negative and the formal expression for the partition function is not logically consistent [50].

I shall briefly review here the quasilocal formalism, highlighting those aspects which are pertinent for the lower dimensional cases. Further details may be found in ref. [47].

Consider a spacetime manifold \(M = \Sigma \times I\) of dimension \((D + 1)\) where \(\Sigma\) is a spacelike hypersurface whose boundary is \(\partial \Sigma = B\). The boundary of \(M\), \(\partial M\), consists of initial and final spacelike hypersurfaces \(t'\) and \(t''\), respectively (with induced metric denoted by \(h_{ij}\)), and a timelike hypersurface \(T = B \times I\) joining these (with induced metric \(\gamma_{ij}\)).

The gravitational action appropriate for fixation of the metric on \(\partial M\) is

\[
S_1 = \frac{1}{\kappa} \int_M d^{D+1}x \sqrt{-g} (R - 2\Lambda) + \frac{2}{\kappa} \int_{t''}^{t'} d^Dx \sqrt{h} K - \frac{2}{\kappa} \int_T d^Dx \sqrt{-\gamma} \Theta + S_M
\]

(29)

where \(\kappa = 8\pi G\) and \(\int_{t'}^{t''} d^Dx\) denotes the difference of integrals over the boundary elements \(t''\) and \(t'\). \(K\) is the trace of the extrinsic curvature \(K_{ij}\) for \(t'\) and \(t''\), defined with respect to the future pointing unit normal and \(\Theta\) is the trace of the extrinsic curvature \(\Theta_{ij}\) of the boundary element \(T\), defined with respect to the outward pointing unit normal.

Variation of (29) yields

\[
\delta S_1 = \text{(terms that vanish when the equations of motion hold)} + \int_{t'}^{t''} d^Dx P_{ij} \delta h_{ij} + \int_T d^Dx \pi^{ij} \delta \gamma_{ij}
\]

(30)

where by definition

\[
P_{ij} \equiv \frac{1}{\kappa} \sqrt{h} (K h_{ij} - K^{ij})
\]

(31)

is the gravitational momentum and

\[
\pi^{ij} \equiv -\frac{1}{\kappa} \sqrt{-\gamma} (\Theta \gamma^{ij} - \Theta^{ij}).
\]

(32)

Terms in \(\delta S_1\) that correspond to integrals over the corners \(t'' \cap T\) and \(t' \cap T\) will not be needed in the sequel. The functional \(S = S_1 - S^0\), where \(S^0\) is
a (background) functional of the metric on $\partial M$, yields the classical equations of motion when the metric is fixed on $\partial M$, since in that case $\delta S^0$ vanishes. Taking for simplicity, $S^0 = S^0[\gamma_{ij}]$ only entails the replacement of $\pi^{ij}$ by $\pi^{ij} - (\delta S^0/\delta \gamma_{ij})$.

The next step is to foliate the boundary element $T$ into $(D-1)$--dimensional hypersurfaces $B$ with induced $(D-1)$--metrics $\sigma_{ab}$.(These boundary elements will be points if $D = 1$ – this case will be discussed later). The $(D)$--metric $\gamma_{ij}$ can be written as

$$\gamma_{ij} \, dx^i dx^j = -N^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt)$$

where $N$ is the lapse function and $V^a$ is the shift vector. This yields

$$\delta S|_T = \int_T d^{D-1}x \sqrt{\sigma} \left( -\varepsilon \delta N + j_a \delta V^a + (N/2) s^{ab} \delta \sigma_{ab} \right)$$

for the contribution to the variation of $S$ from the boundary element $T$. Here

$$\varepsilon = \frac{2}{N\sqrt{\sigma}} u_i \pi^{ij} u_j + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N} = \frac{2}{\kappa} k - \varepsilon_0$$

$$j_a = -\frac{2}{N\sqrt{\sigma}} \sigma_{ai} \pi^{ij} u_j - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V^a} \quad \text{and} \quad j_i = -\frac{2}{\sqrt{\hbar}} \sigma_{ij} P^{jk} n_k - (j_0)_i$$

$$s^{ab} = \frac{2}{N\sqrt{\sigma}} \sigma_{ai} \sigma^{ij} \sigma_{jk} - \frac{2}{N\sqrt{\sigma}} \frac{\delta S^0}{\delta \sigma_{ab}} = \frac{2}{\kappa} (k^{ab} + (n_\mu a^\mu - k) \sigma^{ab}) - (s_0)^{ab}$$

where $k_{ab}$ is the extrinsic curvature of $B$ considered as the boundary $B = \partial \Sigma$ of a spacelike hypersurface $\Sigma$ whose unit normal $u$ is orthogonal to $n$, $P^{ij}$ denotes the gravitational momentum for the hypersurfaces $\Sigma$ that are ‘orthogonal’ to $T$, and $a_\mu = u^\nu \nabla_\nu u_\mu$ denotes the acceleration of the unit normal $u_\mu$ for this family of hypersurfaces. Terms proportional to the functional derivatives of $S^0$ are respectively denoted as $\varepsilon_0$, $(j_0)_i$, and $(s_0)^{ab}$.

From its definition $-\sqrt{\sigma} \varepsilon$ is equal to the time rate of change of the action, where changes in time are controlled by the lapse function $N$ on $T$. Thus, $\varepsilon$ is identified as an energy surface density for the system and the total quasilocal energy is defined by integration over a $(D-1)$--surface $B$:

$$E = \int_B d^{D-1}x \sqrt{\sigma} \varepsilon$$

Similarly, $j_i$ is the momentum surface density and $s^{ab}$ is the spatial stress. When there is a Killing vector field $\xi$ on the boundary $T$, then

$$Q_\xi = \int_B d^{D-1}x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i$$

is conserved in the sense that it is independent of the particular surface $B$ (within $T$) that is chosen for its evaluation (provided there is no matter stress–energy in the neighbourhood of $T$). This property is not shared by the energy $E$. 

11
If the system contains a rotational symmetry given by a Killing vector field \( \xi^i = \zeta^i = (\partial / \partial \phi)^i \) then

\[
J = Q \zeta = \int_B d^{D-1}x \sqrt{\sigma} j_i \zeta^i
\]  

(38)

where the \((D-1)\)–surface \(B\) is chosen to contain the orbits of \(\zeta\). If the Killing vector field \(\xi\) is timelike, then the negative of the corresponding charge (2.18) defines a conserved mass for the system,

\[
M = -Q \xi = \int_B d^{D-1}x \sqrt{\sigma} N \varepsilon
\]

(39)

the latter equality holding if the Killing vector field is also surface forming, where \(B\) has a unit normal proportional to \(\xi\) and \(N\) is the lapse function defined by \(\xi = Nu\). If \(\xi\) does not have unit norm at \(B\), then the mass \(M\) will differ from the energy \(E\). In general the energy \(E\) evaluated on a given slice of \(T\) will not equal the conserved mass \(M\).

These distinctions between mass and energy are especially important for spacetimes that are not asymptotically flat. In the anti-de Sitter case the magnitude of the timelike Killing vector field diverges as it approaches infinity. It does not approach the unit normal to the (asymptotically) stationary time slices at spatial infinity, and the mass \(M\) and energy \(E\) do not coincide. Explicitly the metric in \((3 + 1)\) dimensions is

\[
ds^2 = -N^2(r) \, dt^2 + f^{-2}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)
\]

(40)

where \(N^2(r) = f^2(r) = 1 - 2m/r + r^2/\ell^2\). Choosing the boundary \(B\) to be a surface of constant \(r = R\), the extrinsic curvature is \(k = -2f(R)/R\) and \(\{36\}\) and \(\{39\}\) respectively give

\[
E = -\frac{16\pi}{\kappa} R \sqrt{1 - 2m/R + R^2/\ell^2} - 4\pi R^2 \varepsilon_0(R)
\]

(41)

\[
M = N(R)E
\]

(42)

Choosing \(\varepsilon_0(R) = -\frac{1}{2\pi R} \sqrt{1 + R^2/\ell^2}\) (i.e. the reference spacetime is anti-de Sitter space, so that \(E\) vanishes when \(m = 0\)) yields as \(R \to \infty\) a vanishing energy \(E \sim m\ell/R \to 0\) but finite mass \(M \to m\).

Turning now to the \((2 + 1)\) dimensional black hole, the metric is given by \(\{21\}\). Specifying \(B\) to be a surface of constant \(r = R\), it is straightforward to show \(\{47\}\) that \(k = -2f(R)/R\) and that \(\{36\}\) and \(\{39\}\) now yield

\[
E = -\frac{4\pi}{\kappa} \sqrt{-m + R^2/\ell^2 + j^2/(4R^2)} - 2\pi R \varepsilon_0(R)
\]

(43)

\[
M = N(R)E + \frac{j^2}{2R^2}
\]

(44)
and that

$$J = j$$  \hspace{1cm} (45)

from (44). The energy \(E\) and angular momentum \(J\) will vanish for the zero mass black hole (the metric (21) with \(m = 0, j = 0\)) if the choice \(\varepsilon_0(R) = -\frac{2}{\kappa}\) is made; as \(R \to \infty\) this implies \(E \sim m\ell/R \to 0\) and \(M \to m\).

These results are consistent with an analysis of the Hamiltonian for (2+1) gravity which shows that \(m\) and \(j\) are the ADM mass and angular momentum at infinity [34]. They are also consistent with the formulation of (2 + 1) gravity as a Chern-Simons gauge theory of the Poincare group [9]; in this case the parameters \(m\) and \(j\) may be interpreted in terms of Casimir invariants of mass and spin respectively [48].

The methods described here can also be applied straightforwardly to the (2 + 1) dimensional black hole solutions discussed in refs. [31, 32]. For the dilaton solutions one finds that there exists a well defined quasilocal mass at large \(R\) that is proportional to the expected constant of integration in the field equations [31]. However for the topological (2 + 1) black hole [32] one finds the rather surprising result that the mass is proportional to the parameter \(j\) and the angular momentum is proportional to the parameter \(m\)! The implications of this are still under investigation.

In (1+1) dimensions the development of the quasilocal formalism [19] begins with the action (4), modified to include boundary terms of the form

$$S_B = \int \left[ -2\varepsilon\sqrt{-g}\nabla_a (D(\psi) K n^a) \right]$$  \hspace{1cm} (46)

where \(n^a\) is a unit vector, \(n^a n_a = \varepsilon\) with \(\varepsilon = \pm 1\) for spacelike/timelike \(n^a\). The extrinsic curvature \(K_{ab}\) is defined as \(K_{ab} = -h^c_a\nabla_c n_b\), where \(h_{ab} = g_{ab} - \varepsilon n_a n_b\) and \(K = g^{ab}K_{ab} = h^{ab}K_{ab}\). An analysis similar to the one described above can be performed, the chief difference being that the \((D-1)\) dimensional boundary \(B\) now consists of either one or two points (depending on the choice of boundary). Although there is no formula analogous to (38) (since there is no angular momentum in (1+1) dimensions), the remaining formulae in the \(D = 1\) case are

$$E = -\frac{2}{\kappa} \left( n^a \nabla_a D(\psi) \right) - \mathcal{E}_0$$  \hspace{1cm} (47)

and

$$M = \frac{2}{\kappa} (u \cdot \xi) \left( n^a \nabla_a D(\psi) \right) - M_0$$  \hspace{1cm} (48)

for the quasilocal energy and mass respectively, where \(\xi\) is a timelike Killing vector. Applying these results to the (1+1) dimensional metrics discussed in the previous section yields for large \(x\)

$$E = M \to 2dm$$  \hspace{1cm} (49)
for the metric (11), and

\[ E = M \rightarrow 2m/\kappa \]  

(50)

for the metric (12). Note that if \( \gamma \neq 0 \) in (10), the formula in (48) must be modified so that \( D(\psi) \rightarrow D(\psi, \phi) \); provided no additional surface terms are added to the action, equation (50) yields the same result as (7) \[23, 49\]. Note that \( E = M \) at large \( x \) since these spacetimes are asymptotically flat.

**Thermodynamic Properties**

The methods of the previous section may now be employed to analyze the thermodynamic behaviour of lower dimensional black holes.

Consider first the temperature. For a spherically symmetric metric, it is given by \[42, 47\]

\[ T = \frac{1}{2\pi N(R)\kappa_H} \]  

(51)

where \( R \) is the boundary radius as described in the previous section and where

\[ \kappa_H^2 = \frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_H \]  

(52)

is the surface gravity at the event horizon; \( \xi^\mu \) is a timelike Killing vector outside the horizon. The factor \( N(R) \) is the redshift factor. It will approach unity in asymptotically flat spacetimes, but will diverge in anti-de Sitter spacetime.

For the (2 + 1) dimensional black hole the temperature is \[47\]

\[ T = \frac{1}{\sqrt{-m + R^2/\ell^2 + (J/2R)^2}} \left( \frac{r_+}{r_+} \right) \]  

(53)

where this result may also be derived by taking \( T \equiv \partial E/\partial S \), where \( S \) is the entropy of the black hole.

The entropy may be calculated using either the first law of thermodynamics or by computing the surface integral of the Noether charge associated with diffeomorphism invariance \[50\]. This latter method may be used to show that the entropy of the (2+1) black hole (4.1) is \( 4\pi r_+/\kappa \): twice the ‘area’ of its event horizon \[10\]. These results may further be employed to evaluate the heat capacity of the black hole \[17\]. At constant surface ‘area’ \( 2\pi R \) and constant angular momentum \( J \) this quantity is

\[ C_{R,J} \equiv \left( \frac{\partial E}{\partial T} \right) = \left( \frac{\partial E}{\partial r_+} \right) \left( \frac{\partial T}{\partial r_+} \right)^{-1} \]  

(54)

where the energy and temperature are expressed as functions of \( r_+ \), \( R \), and \( J \). It is straightforward to show that \( \partial T/\partial r_+ \) is positive, so that the temperature is a monotonically increasing function of \( r_+ \). This means that there is a unique black hole with a given temperature \( T(R) \) and a given angular momentum \( J \).
In \((1 + 1)\) dimensions for metrics of the form (8) the temperature is given by [14, 19]

\[
T = \frac{1}{4\pi} \alpha'(x_H) R(x)
\]  

(55)

and the entropy is

\[
S = \frac{4\pi}{\kappa} D(\psi(x_H))
\]  

(56)

where \(R(x)\) is a redshift factor whose value will depend upon the lapse function.

In the large \(x\) limit these quantities are respectively

\[
T = \frac{M}{4\pi d} \quad S = \frac{4\pi}{\kappa} d \ln(M/M_0)
\]  

(57)

for the metric (11) (where \(M_0 = 2dm_0\)) and

\[
T = \frac{Q}{4\pi} \quad S = \frac{2M}{Q}
\]  

(58)

for the metric (12).

Despite the similar functional form of the metrics (11) and (12), the thermodynamic properties of the two metrics are markedly different. For the Liouville black hole (11) the temperature is proportional to the mass and the entropy varies logarithmically with the mass, whereas in the string-inspired case (12) the temperature is constant and the entropy varies linearly with the mass.

I shall close this section by considering some of the implications of a black hole whose entropy varies logarithmically with the mass [51]. Some insight into the properties of such a black hole can be gained by considering its behaviour inside a box of radiation, a non-trivial problem in \((3 + 1)\) dimensions [52, 53]. In this case a black hole contained in a perfectly reflecting cavity filled with thermal energy will not be able to achieve thermal equilibrium if the cavity is sufficiently large.

For the analogous \((1 + 1)\) dimensional system, the metric is taken to be that of (11), with \(x \to |x|\): such a metric would describe the exterior of a collapsing line of matter in the background of a Liouville field – this metric can be extended everywhere that \(x \neq 0\), where there is a delta-function singularity in the curvature scalar. Consider a box of length \(L\) containing a black hole of mass \(M\) and thermal radiation at temperature \(T\), with \(mL \gg \ln(\lambda/\mu)\). The energy and entropy of this system are given by

\[
S = 2\pi \ln \left( \frac{M}{M_0} \right) + \frac{\pi}{3} TL \quad E = M + \frac{\pi}{6} T^2 L
\]  

(59)

where the latter term arises from the Stefan-Boltzmann law in two spacetime dimensions and \(2d\) has been set to unity for simplicity. Maximizing the entropy for a fixed energy \(E\) (the microcanonical ensemble) yields

\[
0 = \frac{\partial S}{\partial M} = \frac{2\pi}{M} - \frac{1}{T} \quad \Rightarrow \quad T = \frac{M}{2\pi}
\]  

(60)
providing an alternative derivation of (57). It is straightforward to show that \( \frac{\partial^2 S}{\partial M^2} < 0 \), guaranteeing a maximum.

Hence as long as the equilibrium condition (60) can be physically realized the entropy is always maximized. Clearly this cannot be trivial – indeed, insertion of (60) into (59) implies

\[
ML = -12\pi + 12\pi \sqrt{1 + \frac{EL}{6\pi}} \gg 2 \ln(m_0/\lambda) \quad .
\]

the latter inequality guaranteeing that the horizons are contained within the box, thereby constraining \( EL \) above a certain threshold.

Comparing the relative values of the entropies of a system of pure radiation and the combined black-hole/radiation system, it is possible to show that a phase transition can occur between the two systems [51]. For a black hole contained in a box of length \( L \) which respects the equilibrium condition (with \( ML \) is given by (61))

\[
S_{bh} = 2\pi \ln \left( \frac{ML}{M_0L} \right) + \frac{1}{6} ML
\]

whereas for pure radiation contained in a box of the same size

\[
S_{rad} = 2 \sqrt{\frac{1}{6} \pi EL} \quad .
\]

Fixing the value of \( M_0L \), it may be shown that the temperature initially increases with energy and the entropy is maximized by pure radiation. A phase transition is then reached, forming a black hole accompanied by a sharp drop in temperature [51].

It is furthermore possible to show that the functional form of the entropy

\[
S = 2\pi \ln \left( \frac{M}{M_0} \right)
\]

implies that it is thermodynamically favourable for lower-dimensional black holes to fragment [51, 54]. Consider a \((1+1)\)-dimensional system consisting of a black hole of mass \( M \) which has spawned \( n \) identical black holes of mass \( m \). The entropy is

\[
S_n = 2\pi \ln \left[ \frac{M - nm}{M_0} \left( \frac{m}{M_0} \right)^n \right]
\]

provided the subsystems are sufficiently separated to be regarded as independent. Eq. (65) becomes (64) for \( n = 0 \). It is straightforward to show that \( S_n \) is maximized if \( m = M/(n+1) \), implying

\[
S_n^{\text{max}} = 2\pi(n+1) \ln \left[ \frac{M}{(n+1)M_0} \right] \quad .
\]
When $n = 1$ the system of separated black holes has larger entropy than that of
a single black hole provided $M/M_0 > 4$; the situation is reversed if $M/M_0 > 4$.
For $n$ large enough to permit differentiation of $S_n^{\text{max}}$ with respect to $n$, one finds
(given $M$ and $M_0$) its maximal value ($\hat{S}_{n_0}$, say) to be at $n = n_0$, where

$$n_0 = \frac{M}{eM_0} - 1 \quad \text{and} \quad \hat{S}_{n_0} = 2\pi \frac{M}{eM_0} = 2\pi(n_0 + 1) \quad (67)$$

Strictly speaking $n_0$ must be an integer and $\hat{S}_{n_0}$ is a rising staircase curve as a
function of $n_0$.

To find the entropically best transition for a black hole of mass $M$ one must
ensure that $S_n > S_0$, or alternatively $M/M_0 > (n + 1)^{1+1/n}$. This yields the
number of equal black holes into which the original one will fragment. Setting
$S_0 = S_1$ determines the value of $M$ for which there is no gain in entropy for the
black hole system to either merge or fragment. This occurs for

$$\frac{M}{M_0} = 4 \quad \text{where} \quad S_0 = S_1 = 2\ln(2) = 1.386 \quad (68)$$

For no fragmentation $M/M_0 < 4$ and the fragmentation process always comes
to a stop, as one intuitively expects.

Lower dimensional black holes also give rise to several interesting points of
principle in thermodynamics which are discussed in ref. [54].

Quantum Properties

The study of the quantum properties of lower dimensional black holes has been
the subject of intense research for the past two years. Much of the attention has
focussed on the model described by (12), since significant progress can be made
in solving the back-reaction problem associated with black hole evaporation. A
full discussion of this subject is beyond the scope of this paper and I refer the
reader to a recent review of the subject [10].

A recent study of Liouville black holes has shown that it is possible to make
significant progress on the back-reaction problem in this case as well [23]. These
metrics are solutions to the field equations of the $R = T$ theory, with matter
action [11]. A wide class of such solutions exists, one of which is the metric
(17). Coupling this system to a set of conformally invariant matter fields with
central charge $c_M$ and evaluating the path integral of to one-loop order yields
the result that the space of solutions maps into itself, so that a given classical
solution

$$ds^2 = -\alpha(x; G)dt^2 + \frac{dx^2}{\alpha(x; G)} \quad (69)$$

is mapped to

$$ds^2 = -\alpha(x; G_R)dt^2 + \frac{dx^2}{\alpha(x; G_R)} \quad (70)$$
where
\[ G_R = \frac{G}{1 - \frac{c M}{\bar{h} G}} \]  
(71)
is the renormalized gravitational constant. A more detailed study of this system is presently under investigation.

4 Inside Looking Out

The realization that a black hole can form from gravitationally collapsing matter naturally leads to the question of what the final fate of the collapsing matter is. In \((3 + 1)\) dimensions it has been demonstrated that the spacetime exterior to a collapsing body relaxes to that of a Kerr-Newman (KN) black hole, with radiative perturbations decaying as advanced time increases according to a power law, provided the (unproven) hypothesis of cosmic censorship is valid.

The interior situation is much less well understood. Infalling matter will either encounter a spacelike region of diverging curvature (at which point quantum gravitational effects presumably dominate) or alternatively will avoid the singularity and emerge into another universe via a ‘white hole’, the prototypical case being the KN geometry. However the stress-energy associated with massless test fields diverges at a null hypersurface inside the black hole called the Cauchy horizon, suggesting an instability in the interior geometry. Any object falling into a KN black hole must eventually cross the Cauchy horizon, and so an understanding of its stability is intimately connected to the question of the final fate of the infalling matter.

Further insight into the fate of the interior was made several years ago by Poisson and Israel, who demonstrated that the Cauchy horizon of the Reissner-Nordström solution forbids any evolution of spacetime beyond this horizon. The mass parameter becomes unbounded inside the black hole due to the presence of ingoing and backscattered outgoing radiation, and the Kretschmann scalar diverges. Ori later developed a simpler model of this phenomenon, and argued that the mass inflation singularity was too weak to forbid passage through the Cauchy horizon since its tidal forces do not necessarily destroy any physical objects. This extensibility problem remains a subject of some controversy.

Most recently mass inflation has also been shown to take place in lower-dimensional black holes. While several of the physical properties are quite analogous to the higher dimensional cases previously studied, some novel features emerge. I shall consider separately the \((1 + 1)\) and \((2 + 1)\) dimensional cases.
Inside a $(1 + 1)$ Dimensional Black Hole

Consider first the $(1 + 1)$ dimensional situation \cite{59}, where the action is given by \eqref{14}, with an additional term added to the matter action for a null fluid, whose stress-energy tensor is

\[
T_{\mu \nu} = \rho(v) l_\mu l_\nu .
\]

\text{(72)}

In this case the field equations have the exact solution \eqref{14}–\eqref{17}

\[
ds^2 = 2 \, dx \, dv - \alpha(x, v) \, dv^2 ,
\]

\text{(73)}

where $\alpha(x, v)$ is given by \eqref{16}, with $m \to m(v)$. The function $m(v)$ satisfies the differential equation

\[
dm{dv} = 8 \pi G \rho(v).
\]

\text{(74)}

For generic non-vanishing $a_n$ the spacetime described by the metric \eqref{73} has multiple horizons at $x_I$, where $\alpha(x_I) = 0$. The largest positive value of $x_I$ shall be called the outer horizon and the next largest value the Cauchy horizon.

Consider matching two patches of solution \eqref{14}–\eqref{17} along an outgoing null line. Denoting the chronological past of the union of the null ray and the Cauchy horizon by region I and its complement by region II, the mass functions in the respective regions are given by

\[
m = m_1(v_1) = m_0 - \delta m(v_1) \quad \text{and} \quad m = m_2(v_2)
\]

\text{(75)}

where $\delta m(v_1) = hv - p$ (in region I) models the radiative tail of the collapsing null matter, and has a power-law falloff in $(1 + 1)$ dimensions \cite{59}. The parameter $m_0$ corresponds to the final mass of the black hole.

The phenomenon of mass inflation in this context may be easily seen by matching the metric for an influx of radiation with different mass functions $m_1$ and $m_2$ along the outgoing null line. These requirements imply \cite{59}

\[
\alpha_1 \, dm_2 = \alpha_2 \, dm_1
\]

\text{(76)}

showing that near the Cauchy horizon, where $\alpha_1 \to 0$ as $v_1 \to \infty$, the inner mass function $m_2$ diverges, as all other quantities remain finite.

More explicitly, the coordinate system \eqref{73} implies that the outgoing null geodesic satisfies the equation

\[
2 \, \dot{x}(\lambda) = \alpha ( x(\lambda), v(\lambda) ) \, \dot{v}(\lambda) ,
\]

\text{(77)}

where $\lambda$ is an affine parameter and the dot denotes derivative with respect to $\lambda$. Without loss of generality, the parameter $\lambda$ may be taken to be zero at the Cauchy horizon and positive beyond that.
Consider first the case of only two horizons. The matching conditions imply that
\[ v_1(\lambda) \approx -\frac{1}{k_o} \int^\lambda e^{2\psi(X_c)} - 2\psi(X(\zeta)) \frac{d\zeta}{\zeta} \approx -\frac{1}{k_o} \int^\lambda \frac{d\zeta}{\zeta} = -\frac{1}{k_o} \ln |\lambda| \]  
(78)
in some negative neighborhood of \( \lambda = 0 \), and
\[ v_2(\lambda) \approx \int^\lambda e^{-2\psi(X_c)} / Z_2 \, d\zeta = e^{-2\psi(X_c)} \frac{\lambda}{Z_2}. \]  
(79)
which vanishes at the Cauchy horizon. Here \( X(\lambda) = x(\lambda) \) along the null ray and \( k_o = -\frac{1}{Q} M'(X_c) e^{2\psi(X_c)} > 0 \). The (\( \lambda \)-dependent) “mass” of the null particle can be defined as
\[ \Delta m(\lambda) := m_2(\lambda) - m_1(\lambda) = Q - Z_2 \dot{X}(\lambda). \]  
(80)
Near the Cauchy horizon, when \( \lambda \) is close to zero \( \Delta m \) is approximately
\[ \Delta m(v_1) \approx \frac{h Z_2}{k_o} e^{2\psi(X_c)} |v_1|^p e^{k_o v_1}. \]  
(81)
where \( Z_2 > 0 \) and \( m_2 \) is
\[ m_2(\lambda) = M - \delta m(\lambda) + \Delta m(\lambda) \]
\[ m_2(v_2) \approx M - h \left[ 1 + \frac{1}{k_o v_2} \right] |k_o| p |\ln |Z_2 e^{2\psi(X_c)} v_2| |^p, \]  
(82)
where (74) and (79) are used. This shows that in the case \( M'(X_c) \neq 0 \), the mass in region II becomes unbounded near the Cauchy horizon where \( v_2 \approx 0 \).

The presence of multiple horizons can change this picture [59]. Since our metric solutions are basically polynomials of a degree higher than 2, for certain values of the parameters \( a_n \) in (13) it is possible for \( k_o \) to vanish, i.e. for \( M'(X) \) to vanish at the (first) inner horizon. More generally, it is possible for the first \( N \) derivatives of \( M(X) \) to be zero at the Cauchy horizon, where \( N \) is some positive integer.

Since \( M(X) \) is a finite polynomial there will be some order of derivative of \( M(X) \) at \( X = X_c \) that will be non-zero. Denoting this by \( b \), such that \( M^{(i)}(X_c) = 0 \) but \( M^{(b+1)}(X_c) \neq 0 \) for integer \( b \geq 1 \) such that for every integer \( i \in [1, b] \), it may be shown that (78) is modified to
\[ v_1(\lambda) \approx \frac{1}{\hat{k} b} \lambda^{-b} \]  
(83)
where \( \hat{k} \) is a positive constant. Since \( v_1 \) no longer has logarithmic behaviour (81) becomes
\[ \Delta m(\lambda) \approx -h Z_2 e^{2\psi(X_c)} \hat{k}^{p-1} b^p \lambda^{p-b-1}, \]  
(84)
where $Z_2$ again must be positive in order to have a positive mass. Furthermore, as $X_c$ is the Cauchy horizon, we also have $\mathcal{M}(X_c) = M$.

From expression (84), it is clear that two distinct possible scenarios can occur. If $p < 1 + 1/b$, $\Delta m$ will be unbounded as $\lambda \to 0$, and the inner mass parameter $m_2$ will also inflate because $m_2 = m_1 + \Delta m$, just as in the previous situation. However if $p > 1 + 1/b$, there will not be any mass inflation at all because the exponent of $\lambda$ in (84) is positive. When $\lambda \to 0$, the mass of the particle tends to zero when $p$ is large enough; the boundary particle ‘deflates’ as it approaches the Cauchy horizon.

An analogue of this situation in $(3+1)$ dimensions would require the existence of a black hole solution with vanishing surface gravity at the inner horizon. Whether or not such a solution can be obtained under physically reasonable circumstances is not yet clear.

4.1 Inside a (2 + 1) dimensional Black Hole

It is also possible to show that mass inflation occurs for the (2 + 1) dimensional black hole \cite{62, 63}, explicitly including effects due to rotation.

Consider modifying the Einstein equations (1) in (2 + 1) dimensions by including an electromagnetic stress-energy tensor, and the stress-energy tensor of a rotating null fluid with energy density $\hat{\rho}$ and angular momentum density $\hat{\omega}$, that is

$$[\mathcal{T}_{\mu\nu}] = \begin{bmatrix} \hat{\rho}(v, r) & 0 & -\hat{\omega}(v, r) \\ 0 & 0 & 0 \\ -\hat{\omega}(v, r) & 0 & 0 \end{bmatrix},$$

in addition to including a cosmological constant. The field equations then have the exact solution \cite{62}

$$ds^2 = \left[ r^2/\ell^2 + m(v) + 4 \pi G q^2 \ln(r/r_o) \right] dv^2 + 2 dv dr - j(v) dv d\theta + r^2 d\theta^2,$$

(85)

where $m(v)$ and $j(v)$ satisfy the differential equations

$$\frac{dm(v)}{dv} = 16 \pi G \rho(v) \quad \text{and} \quad \frac{dj(v)}{dv} = 16 \pi G \omega(v)$$

and $\hat{\rho}(v, r) = \rho(v)/r + j(v) \omega(v)/(2 r^3)$ and $\hat{\omega}(v, r) = \omega(v)/r$, as dictated by the conservation laws, with $\rho$ and $\omega$ arbitrary functions of $v$.

Consider a pulse, $S$, of outgoing null radiation between the Cauchy and outer horizons in the background spacetime \cite{62}. We can model this by matching two patches of solution \cite{62} with different $m$ and $j$ along $S$. The region enclosed by the ring and its complement will be characterized by their distinct values $m_a(v_a)$ and $j_a(v_a)$ where $a$ has value of 2 (1) for the enclosed (non-enclosed)
region. The two regions have different masses and the Cauchy horizon cannot coincide with the inner horizon. If \( j_1 = j_2 \), the null ring will rotate at the same pace as the spacetime in both regions. However, when \( j_1 \neq j_2 \), \( S \) will carry intrinsic spin.

As before, continuity of inflow along a null curve with tangent vector

\[
l^\mu = \left( \frac{2}{N^2}, 1, \frac{j}{r^2 N^2} \right)
\]

yields

\[
\frac{d m_1 - d(j_1^2)/(4 R^2)}{N_1^2} = \frac{d m_2 - d(j_2^2)/(4 R^2)}{N_2^2}.
\]  
(86)

as the \((2 + 1)\) dimensional analogue of (76), where \( R(\lambda) \) is such that \( 2 \pi R \) is the perimeter of \( S \). Taking the affine parameter \( \lambda \) to be zero at the Cauchy horizon and positive behind that, and in addition taking \( M \) and \( J \) to be the respective asymptotic values of \( m_1 \) and \( j_1 \) we can again see that inflationary behaviour will occur: when the ring is close to the Cauchy horizon \( R = r_c \), \( v_1 \) approaches infinity and the right hand side of (86) diverges.

Note that the inclusion of angular momentum implies that it is the quantity \( E \equiv m(v) - j^2(v)/(4r^2) \) which inflates. This quantity is proportional to the total energy of spacetime at large \( r \), neglecting electromagnetic contributions as discussed above. A more detailed analysis shows that \[62, 63\]

\[
v_1(\lambda) \approx - \frac{1}{\hat{k}_o} \ln |\lambda| \quad v_2(\lambda) \approx \frac{2}{Z_2} \lambda.
\]  
(87)

where \( \hat{k}_o \) is a positive constant and that

\[
E_{\text{ring}}(\lambda) \approx - \frac{Z_2}{2 \hat{k}_o \lambda} \delta E(\lambda),
\]

where \( Z_2 \) must be positive so that \( E_{\text{ring}} > 0 \). If we assume that \( \delta E(\lambda) \) decays to zero via a power law \( hv_1^p \) \[71, 72, 74\], we obtain

\[
E_2(v_2) \approx M - \frac{J^2}{4r_c^2} - \frac{h}{v_2} \frac{\hat{k}_o}{p-1} \ln \frac{Z_2 v_2}{2} \ln^{-p}.
\]  
(88)

As a result, \( E_2(v_2) \) goes to infinity while \( S \) approaches the Cauchy horizon because \( v_2 \) tends to zero from below at that instant.

We close by considering tidal distortions at the horizon. In the triad frame, the relevant components of the Riemann tensor look like the following:

\[
R^1_{001} = \frac{\partial_{\alpha} \partial_{\alpha} - \partial_{r} \partial_{\alpha}}{2r} \left[ \frac{\partial_{\alpha} (j^2)}{8r^3} - \frac{\partial_{\alpha} j}{8r} \frac{\partial_{\alpha} \alpha}{r} \right],
\]

\[
R^1_{002} = R^2_{001} = - \frac{j}{4} \partial_r \left( \frac{\partial_{\alpha} \alpha}{r} \right) + \frac{\partial_{\mu} j}{2r^2},
\]

\[
R^2_{002} = - \frac{1}{2} \partial_{r} \alpha
\]
where $\alpha = -g_{vv}$. It is clear that the most divergent component is $R^1_{001}$. The tidal distortion is finite since one can approximate the distortion by integrating the above components twice with respect to $v$ and obtain a finite result. Furthermore the Kretschmann scalar of the BTZ solution is given by

$$R_{abcd} R^{abcd} = \frac{2}{r^2} \left[ \partial_r \alpha(v, r) \right]^2 + \left[ \partial_r r \alpha(v, r) \right]^2$$

which is obviously bounded at Cauchy horizon. Hence there is no reason to terminate the classical extension of the spacetime beyond the Cauchy horizon.

Some of the qualitative features of this phenomenon carry over to $(3 + 1)$ dimensions. It is possible to show that a black string will also undergo mass inflation in a manner similar to that described above.

5 Outlook

Although research in lower dimensional black holes has mushroomed since the advent of the subject five years ago, much remains to be done. I shall close by suggesting two interesting lines of research in the subject which might be pursued.

- **Black Hole Radiation**

  To a large extent, the surge of interest in lower dimensional black holes was caused by the realization that such models offer the possibility of explicitly incorporating back reaction effects due to their mathematical simplicity. However to date the physics of this problem has been studied virtually exclusively in the context of the metric (12). This is a 2-dimensional black hole whose temperature is constant and whose entropy is proportional to the mass. As such it is a very special model. It would be of interest to see to what extent the physics of black hole radiation (including the back reaction) is dependent upon these properties. The wide variety of black hole metrics available in two dimensions permit such an investigation, and this remains largely unexplored territory. The Liouville black hole metric (11) provides a most interesting contrast: as it evaporates, its temperature and entropy both decrease, and it can reach a state of vanishing entropy at finite mass in a finite amount of time, reminiscent of a remnant. Evolution of the spacetime beyond this point is at present unclear.

- **Black Hole Interiors**

  Lower dimensional black holes afford a much more detailed investigation of the physics of black hole interiors. It is reasonable to expect that much more progress could be made on incorporating quantum effects due to the mathematical simplicity of the problem. Some work has been done
in this area [67, 68], but there is still much to be done. The possibility of preventing mass inflation in lower dimensions [59], raises the intriguing question as to the circumstances under which this could occur in higher dimensions. Presumably some form of dilatonic gravity will be required.

Acknowledgements

I am grateful to Jolien Creighton and Jim Chan for their comments to me during the preparation of this manuscript. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

References

[1] P. Collas, Am. J. Phys. 45 (1977) 833
[2] S.D. Odintsov and I.L. Shapiro, Mod. Phys. Lett. A7 (1992) 437; E. Eliadze and S.D. Odintsov, Nucl. Phys. B399 (1993) 581.
[3] R.B. Mann, Gen. Rel. Grav. 24 (1992) 433.
[4] J. Harvey and A. Strominger, ‘Quantum Aspects of Black Holes’ hep-th-9209055 (EFI-92-41).
[5] see for example, A. Vilenkin and E.P.S. Shellard, Cosmic Strings and Other Topological Defects, (Cambridge University Press, 1995).
[6] A.M. Polyakov, Phys. Lett B103 (1981) 207.
[7] S. Deser, R. Jackiw & G. ’tHooft, Ann. Phys. 52, 220(1984); S. Deser and R. Jackiw, Ann. Phy. 153 (1984) 405.
[8] N. Cornish and Frenkel, Phys. Rev. D43 (1991) 2555; Phys. Rev. D47 (1993) 714.
[9] S. Carlip in Proc. of the 5th Canadian Conference on General Relativity and Relativistic Astrophysics, eds. R.B. Mann and R.G. McLenaghan (World Scientific, 1994).
[10] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992).
[11] R.B. Mann & S.F. Ross, Phys. Rev. D 47, 3319 (1993).
[12] R.B. Mann and S.F. Ross, Class. Quant. Grav. 10 (1993) 1405.
[13] R. Jackiw, Nucl. Phys. B252 (1985) 343; C. Teitelboim, Phys. Lett. B126 (1983) 41, 46.
[14] R.B. Mann, A. Shiekh, and L. Tarasov, Nucl. Phys. B\textbf{341} (1990) 134.

[15] R.B. Mann, Found. Phys. Lett. \textbf{4} (1991) 425.

[16] A.E. Sikkema and R.B. Mann, Class. Quantum Grav. \textbf{8} (1991) 219.

[17] R.B. Mann, S.M. Morsink, A.E. Sikkema and T.G. Steele, Phys. Rev. D\textbf{43} (1991) 3948; S.M. Morsink and R.B. Mann, Class. Quant. Grav. \textbf{8} (1991) 2257; J.D. Christensen and R.B. Mann, Class. Quant. Grav. \textbf{9} (1992) 1769; K.C.K. Chan and R.B. Mann, Class. Quantum Grav. \textbf{10} (1993) 913.

[18] H.-J. Schmidt, J. Math. Phys. \textbf{32} (1991) 1562.

[19] G. Mandal, A.M. Sengupta, and S.R. Wadia, Mod. Phys. Lett. \textbf{6} (1991) 1685; E. Witten, Phys. Rev. D\textbf{44} (1991) 314.

[20] F. David, Mod. Phys. Lett. A\textbf{3} (1988) 1651; J. Distler and H. Kawai, Nucl. Phys. B\textbf{321} (1989) 509.

[21] T. Banks and M. O’Loughlin, Nucl. Phys. B\textbf{362}, (1991) 649 .

[22] R.B. Mann, Phys. Lett. B\textbf{294} (1992) 310.

[23] R.B. Mann, Nucl. Phys. B\textbf{418} (1994) 231.

[24] J. Gegenberg and G. Kunstatter, gr-qc/9501017.

[25] R.B. Mann, Phys. Rev. D\textbf{47} (1993) 4438.

[26] O. Lechtenfeld and C. Nappi Phys. Lett. B\textbf{288} (1992) 72.

[27] J.P.S. Lemos & P.M. Sa, Class. Quant. Grav. \textbf{11} (1994) L11; Phys. Rev. D\textbf{49} (1994) 2897.

[28] M. Trodden, V. Mukhanov and R. Brandenberger, Phys. Lett. B\textbf{316} 483.

[29] M.J. Perry and E. Teo Phys. Rev. Lett. \textbf{70} (1993) 2669.

[30] K.C.K Chan and R.B. Mann, WATPHYS-TH94/10 gr-qc/9501028 (1995).

[31] K.C.K. Chan and R.B. Mann, Phys. Rev. D\textbf{50} (1994) 6385.

[32] S. Carlip, J. Gegenberg and R.B. Mann, gr-qc/9410021 (1994).

[33] S.W. Hawking and G.F.R. Ellis \textit{The large scale structure of space-time}, New York: Cambridge University Press (1973).

[34] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D\textbf{48}, 1506 (1993).

[35] S. Giddings, J. Abbott and K. Kuchař, Gen. Rel. Grav. \textbf{16} (1984) 751.
[36] R.B. Mann and S.F. Ross, Class. Quant. Grav. 9 (1992) 2335; R.B. Mann, M.S. Morris and S.F. Ross, Class. Quant. Grav. 10 (1993) 1477.

[37] J.D. Brown, M. Henneaux and C. Teitelboim, Phys. Rev. D33 (1986) 319; J.D. Brown, Lower Dimensional Gravity, (World Scientific, 1988).

[38] M. Kriele, Class. Quant. Grav. 9 (1992) 1863.

[39] C. G. Callan, S. B. Giddings, J. A. Harvey, A. Strominger, Phys. Rev. D 45, (1992) 1005; S. Hawking, Phys. Rev. Lett. 69 (1992) 406.

[40] A. Strominger, Lectures on Black Hole Physics [hep-th 9501071] (1995).

[41] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (Wiley, New York, 1962).

[42] J. W. York, Phys. Rev. D. 33 2092 (1986); B. F. Whiting and J. W. York, Phys. Rev. Lett. 61 1336 (1988); J. D. Brown, G. L. Comer, E. A. Martinez, J. Melmed, B. F. Whiting, and J. W. York, Class. Quantum Grav. 7 1433 (1990); J. D. Brown, E. A. Martinez, and J. W. York, Phys. Rev. Lett. 66 2281 (1991).

[43] O. B. Zaslavskii, Phys. Lett. A. 152 463 (1991); O. B. Zaslavskii, Class. Quantum Grav. 8 L103 (1991).

[44] J. D. Brown and J. W. York, Phys. Rev. D. 47 1407 (1993); J. D. Brown and J. W. York, Phys. Rev. D. 47 1420 (1993).

[45] S. W. Hawking in General Relativity, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[46] Wald R. M. Wald, in Black Hole Physics, edited by V. DeSabbata and Z. Zhang (Kluwer Academic Publishers, Dordrecht, 1992).

[47] J.D. Brown, J. Creighton and R.B. Mann, Physical Review D50 (1994) 6394.

[48] D. Cangemi, M. Leblanc and R.B. Mann, Phys. Rev. D. 48, 3606 (1993).

[49] J. D. Brown, J. Creighton and R.B. Mann, paper in preparation.

[50] R. M. Wald, Phys. Rev. D 48 3427 (1993); R. C. Myers, Phys. Rev. D50 (1994) 6412.

[51] R.B. Mann and T.G. Steele, Class. Quant. Grav. 9 (1992) 475.

[52] P.C.W. Davies, Rep. Prop. Phys. 41 (1978) 1313.

[53] G.W. Gibbons, M.J. Perry, Proc. R. Soc. Lond. A358 (1978) 1313.
[54] P.T. Landsberg and R.B. Mann, Class. Quant. Grav. 10 (1993) 2373.

[55] R. Penrose in Battlle Rencontres, edited by C.M. De Witt and J.A. Wheeler, New York: Benjamin (1968).

[56] E. Poisson and W. Israel, Phys. Rev. Lett. 63, 16, 1663 (1989).

[57] A. Ori, Phys. Rev. Lett. 67, 7, 789 (1991).

[58] A. Bonanno, S. Droz, W. Israel and S. Morsink, Physical Review D50 (1994) 7372.

[59] J.S.F. Chan and R.B. Mann, Physical Review D50 (1994) 7376.

[60] S. Droz, Phys. Lett. A 191, 211 (1994).

[61] V. Husain, Phys. Rev. D 50, 2361 (1994).

[62] J.S.F. Chan, K.C.K. Chan and R.B. Mann, preprint gr-qc/9406049, WATPHYS-TH94/06.

[63] J.S.F. Chan, and R.B. Mann, preprint gr-qc/9411064, WATPHYS-TH94/08.

[64] R.H. Price, Phys. Rev. D 5, 10, 2419 (1972).

[65] R. Balbinot, P.R. Brady, W. Israel and E. Poisson, Phys. Lett. A 161, 3, 223 (1991).

[66] J.S.F. Chan and R.B. Mann, preprint gr-qc/9409034, WATPHYS TH94/07.

[67] W. Israel in Directions in General Relativity: Proceedings of the 1993 International Symposium, Vol. I, edited by B.L. Hu, M.P. Ryan Jr. and C.V. Vishveshwara, New York: Cambridge University Press (1993).

[68] R. Balbinot and P.R. Brady, Class. Quantum Grav. 11, 1763 (1994).