Covariant model of a quarkonium with the funnel potential

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Abstract

The bound–state problem for the pion as a quarkonium with the funnel (Coulomb–plus–linear) interaction is solved in a framework that combines the bilocal approach to mesons with the covariant generalization of the instantaneous–potential model. The potential interaction leads to dynamical breaking of chiral symmetry. However, the Coulomb potential leads to ultraviolet divergences that must be subtracted. A careful choice of the renormalization prescription is needed in order to get the correct chiral limit. The mass, the lepton decay constant of the pion, as well as the pion decay width in two photons are calculated.
I. INTRODUCTION

The activity in the field of relativistic bound states has recently increased. References [1–5] are just some of examples. These papers were motivated, among other things, by the desire to formulate a covariant treatment for quarks interacting through a potential which would hopefully mimic nonperturbative QCD.

Our work is a continuation of the line of research where interquark interactions were modeled by the instantaneous potential. For the case of light quarks, Refs. [6–8] may serve as paradigmatic examples of such calculations. A related approach, using the Nambu–Jona–Lasinio model, can be exemplified by Refs. [9].

The successes of the potential model in describing heavy quarkonia are well known. For light quarks, however, besides many important qualitative successes, the potential model also exhibited weaknesses: first, the uncertainty concerning the question what potential can provide a realistic and yet reasonably tractable interaction without many free parameters; and second, the noncovariance of the instantaneous–potential approach to such highly relativistic constituents as $u, d, \text{and } s$ quarks.

As an example on the successful side, Le Yaouanc et al. [6] demonstrated the appearance of dynamical (spontaneous) chiral–symmetry breaking by generating the dynamical quark mass as well as the pion as a Goldstone boson in the chiral limit. They studied the power–law interaction ($V \propto r^\alpha$) between massless quarks and antiquarks most exhaustively in the simplest case of harmonic potential ($V \propto r^2$) where the gap equation (Schwinger–Dyson equations) reduced from an integral equation to a differential one. Adler and Davis [7] formulated the renormalization procedure for the Coulomb–like potential and performed a concrete numerical calculation for the linear confining quark–antiquark potential. In the latter case, only infrared divergences appeared. Trzupek [8] applied their renormalization procedure to the more realistic case of the funnel (linear–plus–Coulomb) potential where ultraviolet (UV) divergences were present. This situation was further complicated if finite quark masses were present [10,11]. Nevertheless, the aforementioned weaknesses led to un-
satisfactory quantitative results. For example, the investigations of Refs. [6–8] could not
yield the value for the pion decay constant better than four to five times smaller than the
experimental one, when the meson spectrum was fitted correctly. This was assumed to be the
consequence of noncovariance [3]. (Furthermore, it was expected that finite current quark
masses would improve the results). Our attention was thus attracted by the covariant general-
alization of the instantaneous–potential approach to bound states, formulated by Pervushin
and collaborators [12–15]. This approach was first applied to the harmonic potential as the
simplest case [16]; however the first concrete and correct numerical results were obtained
by our group [17]. Besides covariance, the effect of the finite current quark masses was also
included [17], while previous investigations in this line of research [6–8] had been concerned
with massless quarks only. Therefore, they had not been pertinent for studying the depen-
dence of the pion mass on the model parameters since in the chiral limit the pion mass is
vanishing. The mass of the pion from our Ref. [17] behaves as the square root of the current
quark mass, which is precisely the correct behavior of the (pseudo) Goldstone boson (see,
e.g., [18]). The pion decay constant, however, was found in Ref. [17] to be $F_\pi \approx 35 \text{ MeV}$
for the harmonic–potential strength which reproduced the experimental pion mass. The result
turned out to be practically the same as in Ref. [8], $F_\pi = \sqrt{2} f_\pi = \sqrt{2} 20 \text{ MeV} \approx 28 \text{ MeV}$.
The covariant approach removed certain ambiguities present in the definition of the pion
decay constant in Ref. [8], but did not solve the problem of its too small value. Obviously,
further studies of the form of the interaction potential were needed. In the present work we
therefore examine the funnel (Coulomb–plus–linear) potential

$$V(r) = V_C(r) + V_L(r) = \frac{4}{3} \left( \alpha_s \frac{r}{r} + \sigma r \right), \quad (1.1)$$

since it is known that, in QCD, the short–distance interactions are dominated by the
Coulomb interaction, while in the long–distance (small momentum, or $k \to 0$) regime,
$\alpha_s$ times gluon propagator seems to behave as $1/k^4$, corresponding to a linear confining
potential in the coordinate language. However, the ultraviolet divergences caused by the
Coulomb part pose some new difficulties. In fact, the issue of renormalization of the bound–
state equations for quarkonium with instantaneous interaction is the main point of this
work. After sketching in Sec. II how the representation of mesons by bilocal fields leads
to the Schwinger–Dyson equation (SDE) and the Bethe–Salpeter equation (BSE) in ladder
approximation, in Sec. III we formulate a renormalization scheme for the SDE with the
funnel potential. We discuss the limitations which such a scheme must suffer when various
approximations are introduced. We also compare our renormalization procedure with the
ones used so far in this context. In Sec. IV the Salpeter equation for the pion is solved, and
in Sec. V the pion decay constant is obtained. In Sec. VI we calculate the \( \pi^0 \rightarrow \gamma\gamma \) decay
width and conclude in Sec. VII.

II. MESONS AS BILOCAL FIELDS

When trying to model nonperturbative QCD, one may consider a very general interaction
kernel \( K \) entering in the effective action:

\[
W_{\text{eff}} = \int d^4x \left\{ \bar{q}(x) \left[ (i\not\partial - \hat{m}) - L(x) \right] q(x) \right. \\
+ \frac{1}{2} \int d^4y \bar{q}_{\alpha_2}(y) q_{\beta_2}(x) \left[ K(x - y) \right]_{\alpha_1,\beta_1;\alpha_2,\beta_2} q_{\beta_1}(y) \bar{q}_{\alpha_1}(x) \} .
\]

(2.1)

where \( \hat{m} \) is the current quark mass matrix, \( \hat{m} = \text{diag}(m_u, m_d, m_s) \). \( \alpha_i \) and \( \beta_i \) are spinor
indices, whereas color indices and flavor indices are suppressed. In (2.1) the summation over
repeated indices is understood. We assume that the interaction kernel \( K(x - y) \) can lead to
a bound \( q\bar{q} \) system. In (2.1) we have introduced \( L(x) \), an external operator coupled to the
quark current. For example, it can be the leptonic current \( l_\mu(x) \):

\[
L(x) = G_F \frac{\gamma^\mu}{\sqrt{2}} l_\mu \frac{1 - \gamma_5}{2} ,
\]

(2.2)

or a photon \( A_\mu(x) \):

\[
L(x) = e A_\mu(x) \gamma_\mu = e A(x) .
\]

(2.3)

Such external operators will make possible the weak and radiative decays, but the internal
structure of hadronic bound states will be dictated by the model kernel \( K \).
One can construct a theory of meson bound states by eliminating bilinear structures $q_\alpha(x)\bar{q}_\beta(y)$ in favor of bilocal fields $\chi_{\alpha,\beta}(x,y)$ \cite{19,23}, (introduced through the path integral in the generating functional) and then integrating out the remaining quark fields. In this way the action (2.1) becomes \cite{21}

$$W_{\text{eff}}[q,\bar{q}] \to W_{\text{eff}}[\chi]$$

$$= iN_c \text{Tr} \ln[i\not{\partial} - \hat{m} - L - \chi] + \frac{N_c}{2} (\chi, K^{-1}\chi)$$

$$= N_c \{ i\text{Tr} \ln(i\not{\partial} - \hat{m}) - i\text{Tr} \sum_{n=1}^\infty \frac{1}{n} [(i\not{\partial} - \hat{m})^{-1}(L + \chi)]^n + \frac{1}{2}(\chi, K^{-1}\chi) \},$$

(2.4)

where we have suppressed all indices and used shorthand:

$$(\chi, K^{-1}\chi) = \int d^4xd^4y \chi_{\beta_1\alpha_1}(x,y) \bar{K}_{\alpha_1\beta_1;\alpha_2\beta_2}(x,y) \chi_{\beta_2\alpha_2}(y,x),$$

(2.5)

and where $\text{Tr}$ (with the capital “T”) also includes the integration. (Below, “tr”, with small “t”, will denote a trace not including integration.) We can drop the external (weak or electromagnetic) operator $L$ while studying the bound–state equations determining the internal hadron structure. We shall reinstate $L$ later, while studying weak and electromagnetic decays.

We determine the classical solution $\chi_0$ conveniently written as $\chi_0(x - y)$

$$\equiv \sum(x,y) - \hat{m}\delta^4(x - y),$$

by varying $W_{\text{eff}}$ with respect to $\chi$:

$$\frac{\delta W_{\text{eff}}[\chi]}{\delta \chi} = 0 .$$

(2.6)

This yields the SDE for the quark self–mass operator $\Sigma$ in the ladder approximation:

$$\Sigma(x - y) = \hat{m}\delta^4(x - y) + iK(x - y)S(x - y),$$

(2.7)

where the “dressed” quark propagator is defined by

$$S^{-1}(x - y) = (i\not{\partial} - \hat{m})\delta^{(4)}(x - y) - \chi_0$$

$$= i\not{\partial}\delta^{(4)}(x - y) - \Sigma(x - y) .$$

(2.8)
Next, we expand the fields $\chi$ in the action around the minimum, $\chi(x,y) = \chi_0(x,y) + \mathcal{M}(x,y) = \sum (x-y) - \hat{m}\delta^{(4)}(x-y) + \mathcal{M}(x-y)$. As it will turn out that the fields $\mathcal{M}(x,y)$ represent mesons, we separate the part of the action containing $\mathcal{M}(x,y)$:

$$W_{\text{eff}}[\chi] = W_{\text{eff}}[\chi_0 + \mathcal{M}] = W_{\text{eff}}[\chi_0] + \tilde{W}_{\text{eff}}[\mathcal{M}] \ ,$$

where this part is given by

$$\tilde{W}_{\text{eff}}[\mathcal{M}] = \frac{N_c}{2}(\mathcal{M}, K^{-1}\mathcal{M}) - iN_c \sum_{n=2}^{\infty} \frac{1}{n} \text{Tr}\Phi^n$$

and where we have introduced

$$\Phi(x,y) \equiv \int d^4z S(x,z)\mathcal{M}(z,y)$$

and, as before, $\text{Tr}\Phi^n$ denotes

$$\text{Tr}\Phi^n \equiv \text{tr}\int d^4x_1 d^4x_2...d^4x_n \Phi(x_1,x_2)\Phi(x_2,x_3)...\Phi(x_n,x_1) \ .$$

Up to the terms of the order $O(M^2)$, varying the action $W_{\text{eff}}[\chi_0 + \mathcal{M}]$ or, equivalently, $\tilde{W}_{\text{eff}}[\mathcal{M}]$, with respect to the fluctuations $\mathcal{M}(x,y)$ gives the BSE in the ladder approximation for the bound state of a quark and an antiquark whose spectra and propagators are determined by Eq. (2.7) and whose flavors $a, b$ are now explicitly written out:

$$\mathcal{M}_{ab}(x,y) = iK(x-y) \int dx'dy' S_a(x-x')\mathcal{M}_{ab}(x', y')S_b(y' - y) \ .$$

Equation (2.13) is the BSE in the somewhat improved ladder approximation, since the quark propagators in it are not the free, bare ones, but $S$, containing the nontrivial self–energy function $\Sigma$.

In this paper we work in the isosymmetric limit, being concerned with pions only. Not having to distinguish quark masses for different flavors allows us to simplify the notation in the rest of this paper by dropping the indices $a$ and $b$. (However, in processes involving kaons, it will be necessary to keep track of quark flavors carefully.)
III. COVARIANT GENERALIZATION
OF THE POTENTIAL APPROACH AND THE RENORMALIZED
SCHWINGER–DYSON EQUATION FOR THE FUNNEL POTENTIAL

To solve the SDE and the BSE in practice, one must limit oneself to a tractable interaction kernel $K$, which is given by

$$K(k) = -iC_F g^2 \gamma^\mu \otimes \gamma^\nu D_{\mu\nu}(k).$$ \hspace{1cm} (3.1)

where $C_F$ is the second Casimir of the quark representation, here $4/3$ for the case of SU(3) triplet, $g$ is the strong coupling constant, and $D$ is the gluon propagator. An instantaneous approximation to the kernel $K$ leads to the potential model,

$$K(k) \approx i\gamma^0 \otimes \gamma^0 \bar{V}(k) - i\gamma^j \otimes \gamma^l \bar{V}_T(k) \left[ \delta^{jl} - \frac{k^j k^l}{|k|^2} \right].$$ \hspace{1cm} (3.2)

Further approximation consists in neglecting the transverse gluon exchange. Thus, the kernel $K$ becomes

$$K(k) \approx i\gamma^0 \otimes \gamma^0 \bar{V}(k).$$ \hspace{1cm} (3.3)

Pervushin and his group have found that the covariant generalization of the potential approach is possible, provided that the kernel $K$ is of a special form $K^n$:

$$K^n(k) = i\hbar \otimes \hat{V}(k_{\perp}),$$ \hspace{1cm} (3.4)

where $n$ is the timelike unit vector in the direction of the total momentum of the bound system, $n^\mu = P^\mu/\sqrt{P^2}$. For any vector $k^\mu = k^\mu_{\parallel} + k^\mu_{\perp}$, the components parallel and perpendicular to this axis are

$$k^\mu_{\parallel} = n^\mu k_P, \quad k_P = k \cdot n = k \cdot P/\sqrt{P^2},$$

$$k^\mu_{\perp} = k^\mu - k^\mu_{\parallel}, \quad k_{\perp} \cdot P = 0.$$ \hspace{1cm} (3.5)

The relativistic covariant formulation of potential models, given by the kernel (3.4), guarantees that the correct dispersion relation for the momentum and mass of the bound state, $P^2 = M^2$, is fulfilled.
For the interaction kernel of the form (3.4), the SDE (2.7) Fourier-transformed to momentum space is

$$S^{-1}(p) = \not{p} - m + i \int \frac{d^4k}{(2\pi)^4} \not{k} S(k) \not{\tilde{V}} (p_\perp - k_\perp) .$$  \hfill (3.6)

Equation (3.6) is valid for an arbitrary reference frame. For the quarkonium rest frame, \(n = (1, 0, 0, 0)\), and in the chiral limit, (3.6) reduces to the SDE studied by, e.g., Le Yaouanc et al. in their noncovariant approach with the power-law interaction, \(V(r) \propto r^\gamma\). In Ref. [17] we have already studied Eq. (3.6) for the particularly simple, harmonic case, with \(\gamma = 2\), where the integral SDE reduces to differential equations.

The UV divergences due to the Coulomb part of the potential require renormalization and introduction of counterterms. This will change Eq. (3.6) and its rest-frame version. Following Ref. [4] and its generalization to the massive case [10,11], we use the equations for renormalized vector, axial-vector and pseudoscalar vertices, and Ward identities, to set the renormalized SDE in the ladder approximation,

$$S^{-1}(p) = Z_2 \not{p} - Z_m m - ig^2 C_F \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \gamma^\nu D_\mu\nu (p - k) ,$$  \hfill (3.7)

where \(Z_2\) and \(Z_m\) are the wave function and mass renormalization constants defined by

$$S_0 = Z_2 S \quad \text{and} \quad m_0 = \frac{Z_m m}{Z_2} ,$$  \hfill (3.8)

where \(S_0\) and \(m_0\) are the bare quark propagator and the bare quark mass, respectively. Neglecting the retardation effects, Eq. (3.2), and the transverse gluon exchange, Eq. (3.3), in (3.7) we are provided with the renormalized version of Eq. (3.6):

$$S^{-1}(p) = Z_2 \not{p} - Z_m m + i \int \frac{d^4k}{(2\pi)^4} \gamma^0 S(k) \gamma^0 \tilde{V} (p - k) .$$  \hfill (3.9)

Since the perpendicular part of a four-vector reduces to the corresponding three-vector in the quarkonium rest frame, e.g., \(k_\perp \rightarrow k\), we will use the noncovariant notation to the end of this section. This will make easier the comparison of our results with the results of other authors in this line of research. Of course, all the expressions can be generalized back to those valid in moving frames, by substituting \(k \rightarrow k_\perp\), \(k^0 \rightarrow k_P\), and \(\gamma^0 \rightarrow \not{p}\).
Let us demonstrate the multiplicative renormalizability (MR) of this equation. The renormalization of the product $g^2 D$ is

$$g_0^2 D_0 = \left(\frac{Z_1}{Z_2}\right)^2 g^2 D ,$$

(3.10)

where the subscript 0 refers to the bare quantities and $Z_1$ is the vertex renormalization constant. The gauge invariance implies $Z_1 = Z_2$ and the renormalization–group (RG) invariance of $g^2 D$. However, the ladder approximation is consistent with $Z_1 = 1$, i.e., with no vertex renormalization (see, e.g., [25]). So, the renormalization of $g^2 D$, and hence of $\tilde{V}$ should be

$$g_0^2 D_0 = \left(\frac{1}{Z_2}\right)^2 g^2 D ,$$

(3.11)

$$\tilde{V}_0 = \left(\frac{1}{Z_2}\right)^2 \tilde{V} .$$

(3.12)

Now, suppose that $\{Z_2, Z_m\}$ and $\{Z'_2, Z'_m\}$ are two sets of renormalization constants. They may correspond to two different renormalization scales $\mu$ and $\mu'$. Since there is a definite relationship between the bare and renormalized quantities, we know the relationship between the quantities renormalized by primed and unprimed $Z$’s. Concretely, using (3.8) and (3.12), we transform the SDE, Eq. (3.9), and find that it does not change its form, so that the MR holds. Before further discussion of the MR, we shall rewrite the SD equation (3.7) using the conventional ansatz of the quark propagator $S$ through the functions $\omega$ and $\phi$:

$$S^{-1}(k) = k^0 \gamma^0 - \omega(k) \zeta^{-2}(k) ,$$

(3.13)

where the matrix $\zeta$ is defined as

$$\zeta(k) = \sin \frac{1}{2} \varphi(k) - \hat{k} \cdot \gamma \cos \frac{1}{2} \varphi(k) .$$

(3.14)

We can express the quark propagator as

$$S(k) = -\zeta(k) \left[ \frac{\frac{1}{2}(1 + \gamma^0)}{\omega(k) - k^0 - i\varepsilon} + \frac{\frac{1}{2}(1 - \gamma^0)}{\omega(k) + k^0 - i\varepsilon} \right] \zeta(k) .$$

(3.15)
Inserting (3.13) into the renormalized SD equation (3.7) yields the following integral equations for $\omega$ and $\varphi$:

\[
\begin{align*}
\omega(p) \sin \varphi(p) - Z_m m + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \sin \varphi(k) \bar{V}(p - k) &= 0, \\
\omega(p) \cos \varphi(p) - Z_2 |p| + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \cos \varphi(k)(\hat{k} \cdot \hat{p}) \bar{V}(p - k) &= 0.
\end{align*}
\] (3.16a, 3.16b)

Additionally, the equation $(Z_2 - 1)p^0 = 0$ arises, i.e., $Z_2 = 1$. However, the Eq. (3.16b) demands that $Z_2$ to be an infinite constant. Adler and Davis [7] have resolved that contradiction by splitting $Z_2$ into two parts,

\[
Z_2 \hat{p} \rightarrow Z_0 p^0 \gamma^0 - Z p \cdot \gamma,
\] (3.17)

and setting $Z_0 = 1$. However, we have to remind ourselves that $Z_2$ is the wave–function renormalization constant, which defines the renormalization of the quark propagator, $S_0 = Z_2 S$. We are faced with a dilemma, namely, whether $S$ should be renormalized with $Z_0$ or with $Z$. As one can expect, both possibilities change the form of the renormalized SD equation (3.9). Obviously, the consistency of MR is violated because one has to split $Z_2$ into $Z_0$ and $Z$. If unprimed and primed renormalization constants correspond to two different renormalization scales $\mu$ and $\mu'$, respectively, this inconsistency automatically shows that the invariance with respect to the changes of the renormalization scale $\mu$ is lost, and we cannot use the renormalization group (RG) equations to relate results for one arbitrary scale $\mu$ to that for some other scale $\mu'$. Nevertheless, as noted by Brown and Dorey [25], who explored the consistency of the MR of the SDE when various approximations are made, this does not mean that treatments that do fail such a consistency test cannot be useful, merely that is more difficult to relate their solutions to real physics. Of course, we must investigate the scale dependence of our results. We shall return to this point later in the text.

A convenient renormalization prescription that determines $Z$ and $Z_m$ uniquely is given, e.g., in Refs. [26,27]. The authors specify that the quark propagator $S(p)$, for a given spacelike $p^2 = -\mu^2$, agrees with free-field theory. We adopt this choice, adjusted for a special form of the kernel (3.4) and the propagator (3.13),
$$S^{-1}(p)|_{p} = \mu = \hat{p} - m.$$  \hspace{1cm} (3.18)

Imposing (3.18) on the SD equation (3.7) yields the renormalization constants

\begin{align*}
Z &= 1 + \frac{1}{2\mu} \int \frac{d^3k}{(2\pi)^3} \cos \varphi(k)(\hat{\mu} \cdot \hat{k})\bar{V}(\mu - k), \\
Z_m &= 1 + \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k)\bar{V}(\mu - k). \hspace{1cm} (3.19a)
\end{align*}

Using (3.19), the SD equation (3.16) becomes

\begin{align*}
\omega(p) \sin \varphi(p) &= m \\
&+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k)[\bar{V}(p - k) - \bar{V}(\mu - k)] = 0, \\
\omega(p) \cos \varphi(p) &= |p| \\
&+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \cos \varphi(k)[(\hat{p} \cdot \hat{k})\bar{V}(p - k) - |\vec{p}|(\hat{\mu} \cdot \hat{k})\bar{V}(\mu - k)] = 0. \hspace{1cm} (3.20a)
\end{align*}

For $m = 0$, there is no mass renormalization, and the functions $\varphi_D$ and $\omega_D$ of the dynamical quark propagator $S_D$ have to satisfy the equations

\begin{align*}
\omega_D(p) \sin \varphi_D(p) + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi_D(k)\bar{V}(p - k) &= 0, \\
\omega_D(p) \cos \varphi_D(p) &= |p| \\
&+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \cos \varphi_D(k)[(\hat{p} \cdot \hat{k})\bar{V}(p - k) - |\vec{p}|(\hat{\mu} \cdot \hat{k})\bar{V}(\mu - k)] = 0. \hspace{1cm} (3.21a)
\end{align*}

Now, it is obvious that the SD equation (3.20) does not reduce to Eq. (3.21) in the limit $m \to 0$, because $\lim_{m \to 0} Z_m m \neq 0$. Moreover, we see that $\varphi(p) = \varphi_D(p)$ and $\omega(p) = \omega_D(p)$ is a solution to the SD equation (3.20) for $m = m' \equiv \omega_D(\mu) \sin \varphi_D(\mu)$. So, the chiral limit is reached in the SD equation (3.20) for $m \to m'$ and not for $m \to 0$ as it should be. This is an artifact of the renormalization prescription (3.18). Pagels [28] showed that the normalization condition (3.18) precluded the presence of a dynamically generated term in the quark propagator $S(p)$, and he argued in favor of Weinberg’s zero–mass renormalization scheme. However, it is possible to recover the proper chiral–limit behavior by redefinition of the mass renormalization constant, Eq. (3.19b):
\[ Z_m = 1 + \frac{1}{m} \omega_D(\mu) \sin \varphi_D(\mu) + \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k) \tilde{V}(\mu - k) . \quad (3.22) \]

This corresponds to the redefinition of the normalization condition \([3.18]\):

\[ S^{-1}(p)|\mu| = \mu = [\hat{p} - m - \omega_D(p) \sin \varphi_D(p)]|\mu| = \mu . \quad (3.23) \]

Equation \(3.20a\) becomes

\[ \omega(p) \sin \varphi(p) - \omega_D(\mu) \sin \varphi_D(\mu) - m \]
\[ + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k) [\tilde{V}(p - k) - \tilde{V}(\mu - k)] = 0 , \quad (3.24) \]

whereas Eq. \(3.20b\) remains unchanged.

The renormalization constants \(Z\), Eq. \(3.19a\), and \(Z_m\), Eq. \(3.22\), are defined in terms of the potential \(\tilde{V}\). We are considering the case of funnel potential, which is a sum of the Coulomb–like potential \(\tilde{V}_C\) and the linear potential \(\tilde{V}_L\). The integrals involving the linear potential are finite, so we can drop \(\tilde{V}_L\) from the definition of the renormalization constants, \textit{i.e.}, we can define

\[ Z = 1 + \frac{1}{2|\mu|} \int \frac{d^3k}{(2\pi)^3} \cos \varphi(k) (\hat{\mu} \cdot \hat{k}) \tilde{V}_C(\mu - k) , \quad (3.25a) \]
\[ Z_m = 1 - \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3} \sin \varphi_D(k) \tilde{V}_C(\mu - k) + \frac{1}{2m} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k) \tilde{V}_C(\mu - k) . \quad (3.25b) \]

The first integral in the \(Z_m\) definition, Eq. \(3.25a\), multiplied by \(m\), now plays the role of \(m'\). Omitting this term will cause the incorrect chiral limit behavior we have met using the renormalization constants \([3.19]\). The renormalization constants \([3.25]\) are those we have actually used in our numerical calculation. The corresponding SDE is

\[ \omega(p) \sin \varphi(p) - [m - \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi_D(k) \tilde{V}_C(p - k)] \]
\[ + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k) [\tilde{V}_C(p - k) - \tilde{V}_C(\mu - k)] \]
\[ + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sin \varphi(k) \tilde{V}_L(p - k) = 0 , \quad (3.26a) \]

\[ \omega(p) \cos \varphi(p) - |p| \]
Let us now make a comparison with the renormalization prescription of Refs. [8,10,29]. The renormalization constant $Z$ used by them is

$$Z = 1 + \frac{1}{2|p|} \int \frac{d^3k}{(2\pi)^3} (\hat{p} \cdot \hat{k}) \tilde{V}_C(p - k).$$

(3.27)

One way in which Eq. (3.27) differs from our $Z$, Eq. (3.25b), is the absence of the factor $\cos \varphi(k)$. This is not a critical difference, as the modification of Eq. (3.25) by making the substitution $\cos \varphi(k) \to 1$ would change $Z$ only by a finite value. As noted by Adler and Davis [7], the crucial point is that Eq. (3.27) corresponded to a momentum-dependent subtraction of UV divergences when the coupling constant $\alpha_s$ was running, i.e., when $\alpha_s$ was momentum dependent. (For that reason, they rejected the SDE with the Coulomb-like interaction of Finger–Mandula [30].) For $\alpha_s = \text{const}$, however, they noted that the infinite part of (3.27) defined a momentum-independent UV subtraction. Nevertheless, this counterterm still has a momentum-dependent finite part. This is conceptually objectionable even if the UV infinity is successfully subtracted.

The other possibility considered by Adler and Davis [7] for the Coulomb-like counterterm, again in connection with the unrenormalized equations of Amer et al. [31,32]. This alternative $Z$ is given by Eq. (2.17) of Adler and Davis [7] and is obviously a completely momentum-independent choice. It can be shown that it is given by (3.27) when $|p|$ tends to zero:

$$Z = 1 + \lim_{|p| \to 0} \frac{1}{2|p|} \int \frac{d^3k}{(2\pi)^3} (\hat{p} \cdot \hat{k}) \tilde{V}_C(p - k).$$

(3.28)

It is evident that (3.28) is momentum independent simply because $|p|$ is fixed to a specific value – zero. However, there is a subtlety that seems to have been unnoticed so far. After performing angular integration in Eq. (3.28), we obtain

$$Z = 1 - \frac{4\alpha_s}{3\pi} \int_0^\infty \frac{dk}{k}.$$

(3.29)
This means that, for \( k \to 0 \), the UV renormalization constant (3.28) introduces a new IR divergence, which has not been present in the original SDE! If \( \alpha_s \) is not constant but runs with \( k \), then \( \alpha_s \) have to be under the integral; however, this does not change the situation. One possibility for avoiding troubles for \( p \to 0 \) is to introduce an IR cutoff, but this requires a new unphysical parameter. Alkofer and Amundsen \[33\] used the renormalization constant (3.28) but no IR cutoff, which makes their results suspicious \[1\]. On the other hand, our renormalization constant (3.25) also reduces to (3.28) as \( |\mu| \to 0 \) apart from the factor \( \cos \varphi(k) \) in the integrand. For \( |k| \to 0 \), \( \varphi(k) \to \pi/2 \) and \( \cos \varphi(k) \to 0 \), which makes the integration in (3.25) IR regular.

To summarize, our \( Z \) differs from that used (so far) most frequently in this line of research \[7,10,8,29\], in that the variable momentum is replaced by the fixed renormalization scale \( \mu \). At the same time it encompasses the alternative possibility \[7,33\] for \( Z \) in the limit \( \mu \to 0 \). The finite part is chosen so that no new IR divergence arises in this limit.

Our solutions are given in Fig. 1 while numerical methods are detailed in the Appendix.

The linear potential \( \tilde{V}_L \) makes the integrals in the SDE (3.26) IR divergent. The formulae (3.26) can be rewritten such that the equations for \( \varphi \) and \( \omega \) are separated. The equation for \( \varphi \) needs no IR regularization \[7\]. The other equation expresses \( \omega \) as a functional of \( \varphi \) and requires IR regularization. We adopt the procedure used by Le Yaouanc et al. \[4\], where

\[
V_L(r) = \frac{4}{3} \sigma r = \frac{4}{3} \sigma \lim_{\xi \to 0} \frac{2}{\xi^2} e^{-\xi r} - 1 + \xi r,
\]

which gives the IR–finite quark energy \( \omega(k) \). For light quarks, \( \omega(k) \) is negative for small momenta, as expected from earlier works, e.g., \[3,10\]. This is in fact a signature of dynamical chiral–symmetry breaking.

To conclude this section, we have successfully solved the SDE for massive quarks with the funnel–potential interaction, treating IR and UV divergences carefully. With the solution for

\[1\] Alkofer and Amundsen studied the temperature– and momentum–dependent \( \alpha_s \), but the claim of the singularity in \( Z \) for \( p \to 0 \) still holds.
\( \varphi(p) \), we can proceed to solving and examining the Salpeter equation for a light pseudoscalar quark–antiquark bound state.

**IV. THE SALPETER EQUATION FOR THE PION**

Solving the SDE (3.26) for \( \varphi(k) \) provided us with the necessary input for solving the BSE (2.13). By Fourier transforming to momentum space, this equation becomes

\[
\Gamma(q|P) = i \int \frac{d^4q'}{(2\pi)^4} K(q - q') S(q' + \frac{P}{2}) \Gamma(q'|P) S(q' - \frac{P}{2}) ,
\]

(4.1)

where \( 2q = p_a - p_b \) and \( P = p_a + p_b \) are the relative and the total momentum of the quark–antiquark pair, respectively, and \( \Gamma(q|P) \) is the momentum vertex function of their bound state. Using the instantaneous potential for the interaction kernel reduces the BSE to the Salpeter equation. This equation is still manifestly Lorentz covariant for the special form (3.4) of the instantaneous interaction, as shown by Refs. \[13\]–\[15\]. It is convenient to write this equation in terms of the quarkonium wave function \( \Psi(q_{\perp}) \) or its transformed mate \( \psi^P \):

\[
\Psi^P(q_{\perp}) = i \int \frac{dq_P}{(2\pi)^2} [S(q + \frac{P}{2}) \Gamma(q_{\perp}|P) S(q - \frac{P}{2})] \equiv \zeta(q_{\perp}) \psi^P(q_{\perp}) \zeta(q_{\perp}).
\]

(4.2)

In this work we are interested only in pseudoscalar mesons. It can be shown that \( P \psi^P = -\psi^P P \), so that the decomposition of \( \psi^P \) for pseudoscalar mesons in the Dirac matrices is simply

\[
\psi^P(q_{\perp}) = \gamma_5 [L_1(q_{\perp}) + \frac{P}{\sqrt{P^2}} L_2(q_{\perp})] .
\]

(4.3)

The BSE (1.1) written in terms of (1.2) and (1.3) and for the form (3.4) is boost invariant. To solve it for \( L_1 \) and \( L_2 \), we are free to choose the rest frame of the bound system \( q_{\perp} = (0, q) \), \( P = (M_\pi, 0) \). The mass \( M_\pi \) of the pseudoscalar bound state, the pion, is the eigenvalue of the BSE. (The treatment of the kaon is in principle identical to that of the pion, except that one of the quarks would have to be the significantly heavier strange quark).
Inserting (4.2), (4.3), and (3.4) in (4.1) yields the coupled system of integral equations, which at first sight seems to be IR infinite,

\[ M_\pi L_2(p_\perp) + 2\omega(p_\perp)L_1(p_\perp) + \int \frac{d^3k_\perp}{(2\pi)^3} \tilde{V}(p_\perp - k_\perp)L_1(k_\perp) = 0, \quad (4.4a) \]

\[ M_\pi L_1(p_\perp) + 2\omega(p_\perp)L_2(p_\perp) + \int \frac{d^3k_\perp}{(2\pi)^3} \tilde{V}(p_\perp - k_\perp)[\sin \varphi(p_\perp) \sin \varphi(k_\perp) \]
\[ - (\hat{p}_\perp \cdot \hat{k}_\perp) \cos \varphi(p_\perp) \cos \varphi(k_\perp)]L_2(k_\perp) = 0. \quad (4.4b) \]

However, as in the SDE, the IR infinity of the unregularized quark energy \( \omega(k) \) cancels the IR infinity of the integrals with kernels (see [7]).

The numerics for solving (4.4) is discussed in the Appendix along with the numerics for the SDE. The solutions \( L_1 \) and \( L_2 \) are displayed in Fig. 1.

Having solved the massless version of the SDE, Eq. (3.26), we have found that \( \omega_D(\mu) \sin \varphi_D(\mu) \to 0 \) as \( \mu \to \infty \). So, in the limit of infinitely large renormalization point \( \mu \), the normalization condition (3.18) approaches the normalization condition (3.23). The improper behavior of the theory, artificially generated by the normalization condition (3.18), will disappear for \( \mu \to \infty \). Fig. 2 shows the pion mass \( M_\pi \) as a function of the renormalized quark mass \( m \) for three different renormalization prescriptions. The dotted lines relate to the normalization condition (3.18), slightly modified by leaving out the linear potential from definition of the renormalization constants, and \( \mu = 1 \, GeV \). In this case, \( M_\pi \to 0 \) for \( m \to m'(\mu) = \omega_D(\mu) \sin \varphi_D(\mu) \) (\( = 0.70 \, MeV \) for \( \alpha_s = 0.4 \) and 4.2 \( MeV \) for \( \alpha_s = 0.8 \)). The dashed lines relate to the same normalization condition (3.18), but for \( \mu = 5 \, GeV \). Now, \( \omega_D(\mu) \sin \varphi_D(\mu) \sim 0.1 \, MeV \) and the improper behavior of \( M_\pi(m) \) has almost disappeared. However, if we use the normalization condition (3.23), we obtain \( \lim_{m \to 0} M_\pi(m) = 0 \) for arbitrary choice of \( \mu \) (solid lines in Fig. 2). This particular scheme is therefore preferred, and we will discuss our results for the pion decay constant \( F_\pi \) and the \( \pi^0 \to \gamma\gamma \) decay width using this scheme.

Finally, let us remark that from the low–momentum behavior of the SD solution \( \varphi \) one can read off the constituent quark mass \( m^* \) as defined by Adler and Davis [7]. Reference [7] implies \( m^* = -1/\varphi'(0) \). \( m^* \) is a linear function of \( m \) and is depicted in Fig. 4. These results
are consistent with the constituent mass of Ref. [7] which is \( m^* = 70 \text{ MeV} \) for the massless case and the pure linear potential with \( \sigma = (350 \text{ MeV})^2 \). As a function of \( \sigma \), \( m^* \) grows as square root (Fig. 5).

V. THE PION DECAY CONSTANT \( F_\pi \)

So far we have been concerned with describing the hadronic structure which is determined in this context by the quark–quark interaction \( K \). Therefore, for simplicity, we have so far omitted the external local operator \( L(x) \), which is used in the present approach to describe weak and radiative decays. However, rederiving the bilocal action \( \tilde{W}_{eff} \) in the presence of \( L(x) \) shows that we can consistently reinstate \( L(x) \) by the substitution

\[
\mathcal{M}(x, y) \rightarrow \mathcal{M}(x, y) + L(x)\delta^{(4)}(x - y) .
\] (5.1)

The matrix element for the leptonic decay of pseudoscalar mesons is then [17]

\[
< l^\pm \nu\mid \tilde{W}_{eff} \mid \pi^\pm > = < l^\pm \nu\mid \frac{i}{2}N_c \text{Tr}(S(M + L))^2 \mid \pi^\pm >
\] (5.2)

\[
= < l^\pm \nu\mid iN_c \text{Tr}(SMSL) \mid \pi^\pm > .
\] (5.3)

It is expressed through the axial–current matrix element which is conveniently parametrized by the pion decay constant \( F_\pi \). Evaluating (5.3) thus yields [13] for equal \( u \) and \( d \) quark masses

\[
F_\pi = \frac{4N_c}{M_\pi} \int \frac{d^3q_\bot}{(2\pi)^3} L_2(q_\bot) \sin \varphi(q_\bot) .
\] (5.4)

In Ref. [17] we obtained the first correct numerical results for the decay constants in the bilocal approach with the harmonic potential not only for the pion, but also for the kaon and their radial excitations. (Of course, (5.4) was generalized for different quark masses because of the kaon.)

The variation of \( F_\pi \) with \( \sigma \) (for \( m = 7 \text{ MeV} \)) is given in Fig. 6. \( F_\pi \) grows monotonically with \( \sigma \) roughly like a square root, Fig. 7.
At $\mu = 1 \text{ GeV}$ and for fixed $\sigma = (350 \text{ MeV})^2$, our $M_\pi$ is fitted to the experimental value for $m = 1.92 \text{ MeV}$ when $\alpha_s = 0.4$, and for $m = 2.80 \text{ MeV}$ when $\alpha_s = 0.8$. Unfortunately, Fig. 6 shows that $F_\pi$ is then too small, being typically about $20 - 30 \text{ MeV}$. We remark that in the earlier treatments of the linear or funnel potential $F_\pi$ has been even smaller, e.g., $F_\pi = \sqrt{2} f_\pi = 16 \text{ MeV}$ in Ref. [7], and $F_\pi = (16 - 34) \text{ MeV}$ in Ref. [8]. However, very close to the massless regime, at $m = 0.22 \text{ MeV}$ for $\alpha_s = 0.4$ and at $m = 0.41 \text{ MeV}$ for $\alpha_s = 0.8$, the correct experimental ratio $M_\pi/F_\pi$ is obtained. This means that we can fit both $M_\pi$ and $F_\pi$ provided we rescale all dimensional quantities, including $\mu$. Therefore, if we rescale by a factor of 7.9, we get $M_\pi = 140 \text{ MeV}$ and $F_\pi = 132 \text{ MeV}$ at $m = 1.71 \text{ MeV}$, $\sigma = (7.9 \times 350 \text{ MeV})^2$ at the renormalization scale $\mu = 7.9 \text{ GeV}$. Similarly, for $\alpha_s = 0.8$ we again practically reproduce the experimental values if we increase the scale by a factor of 6: at $\mu = 6.0 \text{ GeV}$, $m = 2.47 \text{ MeV}$, and $\sigma = (6.0 \times 350 \text{ MeV})^2$, we get $M_\pi = 140 \text{ MeV}$ and $F_\pi = 132 \text{ MeV}$. Clearly, continually varying $\alpha_s$ and $\sigma$ would give experimental values of $M_\pi$ and $F_\pi$ for continuum of different quark masses $m$ but also different scales $\mu$.

VI. THE $\pi^0 \rightarrow \gamma \gamma$ DECAY WIDTH

If the external operator is taken to be $L(x) = Q A_\mu(x) \gamma^\mu$, where $Q = e \text{ diag}(Q_u, Q_d, Q_s) = e \text{ diag}(2/3, -1/3, -1/3)$, it will make possible the radiative processes as the $\pi^0 \rightarrow \gamma \gamma$ decay computed in [17] for the harmonic interaction. We consider the present case of the funnel potential more realistic but still extremely oversimplified, so that we do not intend that our calculation should compete with the standard description via the Adler–Bell–Jackiw anomaly and PCAC, which yields an almost experimental decay width. We calculate this decay more as a further test of the quality of the funnel interquark interaction. We should however stress that our approach is in a way more ambitious than most of the other approaches, including the standard anomaly calculation: these, namely, always contain the step when one actually parametrizes the unknown hadronic structure with the pion decay constant $F_\pi$. On the contrary, the present calculation is not parametrizing but trying to
describe the pion structure and in this respect it is more microscopic.

The $\pi^0 \to \gamma\gamma$ transition is caused by the cubic term from $\tilde{W}_{\text{eff}}$ because it contains subterms with one meson bilocal $M$ and two photon fields $A_\mu$. The transition matrix element is thus

$$ A_{\pi^0\gamma\gamma} = <\gamma(k,\sigma)\gamma(k',\sigma')|iN_c\text{Tr}[MSQ^2ASQ^2]|\pi^0(P)> , \quad (6.1) $$

where the symbol “Tr” also includes the integrations over coordinates. Equation (6.1) in fact corresponds to the $\gamma_5$ triangle graph except that the propagator lines emanate out of a pseudoscalar bilocal bound–state vertex and that these propagators are not free but dressed ones. Transforming to momentum space,

$$ A_{\pi^0\gamma\gamma} = \frac{(2\pi)^4\delta^{(4)}(P+k+k')}{\sqrt{(2\pi)^32^3P_0k_0k'_0}} T_{\pi^0\gamma\gamma} , \quad (6.2) $$

where

$$ T_{\pi^0\gamma\gamma} \equiv 2iN_c\epsilon^2 \frac{Q_u^2 - Q_d^2}{\sqrt{2}} \epsilon_\mu(k,\sigma)\epsilon_\nu(k',\sigma') I^{\mu\nu} , \quad (6.3) $$

Inserting $\Gamma$ and $S$, rearranging, and integrating over the parallel component $q_P$ and performing the spinor trace,

$$ I^{\mu\nu} = 4\epsilon^\alpha\beta^{\mu\nu} \frac{P_\alpha}{M_\pi} I_\beta , \quad (6.5) $$

$$ \Re I_\beta = \int \frac{d^3q_1}{(2\pi)^3} \mathcal{J}_\beta(q_1, k'_1, \varphi) \frac{L_2(q_1)[E(q_1) + E((q + k')_\perp)] - L_1(q_1)M_\pi}{[E(q_1) + E((q + k')_\perp)]^2 - \frac{M_\pi^2}{4}} , \quad (6.6) $$

$$ \Im I_\beta = -\frac{\pi}{2} \int \frac{d^3q_1}{(2\pi)^3} \mathcal{J}_\beta(q_1, k'_1, \varphi) \{\delta[E(q_1) + E((q + k')_\perp) + \frac{M_\pi}{2}] \times [L_1(q_1) + L_2(q_1)] - \delta[E(q_1) + E((q + k')_\perp) - \frac{M_\pi}{2}]\}[L_1(q_1) - L_2(q_1)] , \quad (6.7) $$
\[ J_\beta(q_\perp, k'_\perp, \varphi) = - \frac{[q + k'_\perp]}{|(q + k'_\perp)|} \sin \varphi(q_\perp) \cos \varphi((q + k'_\perp) \rangle \]
\[ + \frac{(q_\perp)_\beta}{|q_\perp|} \sin \varphi((q + k'_\perp) \rangle \cos \varphi(q_\perp) . \]  

(6.8)

Since \( I_\beta \) is a function of \( M_\pi \) and of only one four vector, \( (k'_\perp) \),
\[ I_\beta = (k'_\perp)_\beta C[E, L_1, L_2, \varphi, M_\pi] , \]  

(6.9)

where \( C \) is a dimensionless Lorentz scalar functional of \( E, L_1, L_2, \) and \( \varphi \) and a function of \( M_\pi \) and \( k'_\perp \). We can thus extract it numerically by evaluating (6.3) in, say, the rest frame as the easiest one. Noting that

\[ \epsilon^{\alpha \beta \mu \nu} P_\alpha(k'_\perp)_\beta = - \epsilon^{\alpha \beta \mu \nu} k_\alpha k'_\beta, \]

summing \( |A_\pi^{\sigma \gamma \gamma}|^2 \) over the polarizations \( \sigma, \sigma' \), and integrating over the phase space of the outgoing photons yields

\[ \Gamma(\pi^0 \rightarrow \gamma \gamma) = \alpha^2 M_\pi 8\pi |C|^2 . \]  

(6.10)

The dependence of \( \Gamma \) on \( m \) (for fixed \( \mu = 1 \ GeV \) and \( \sigma = (350 \ MeV)^2 \)) is depicted in Fig. 8. The experimental value \( \Gamma_{exp} = (7.7 \pm 0.5) \ eV \) is reached for \( \alpha_s = 0.4 \) at \( m = 11.7 \ MeV \) and for \( \alpha_s = 0.8 \) at \( m = 12.1 \ MeV \). For such quark masses, \( M_\pi \) is already too large, while \( F_\pi \) is still too small. Since \( \Gamma \) falls rather quickly with \( m \), in a good approximation proportional to \( m^{3/2} \), it is far too small in the range of \( m' \)s for which we could fit both \( M_\pi \) and \( F_\pi \) using rescaling, as shown in the preceding section.

On the other hand, for \( \mu = 6.7 \ GeV, \ m = 25.6 \ MeV, \) and \( \alpha_s = 0.4 \) (or \( \mu = 5.0 \ GeV, \ m = 23.3 \ MeV, \) and \( \alpha_s = 0.8 \)) we can also fit \( F_\pi \) and \( \Gamma \) to experiment using rescaling, but then \( M_\pi \) becomes too large by a factor of 7 to 10. The variation of \( \sigma \) with fixed \( m = 7 \ MeV \) does not yield better results either. This is reminiscent of the situation in the harmonic-oscillator case \[17\] where it was impossible to fit \( M_\pi, F_\pi \) and \( \Gamma \) simultaneously. In this case, however, we have not explored the whole parameter space and it is still possible that appropriate choice of \( \alpha_s, m, \sigma, \) and \( \mu \) would fit all three quantities simultaneously.

The fact that \( \Gamma \) depends on the current quark mass as \( m^{3/2} \), is another manifestation of consistency with PCAC, as discussed in Sec. V of Ref. \[17\].
VII. CONCLUSION

We have studied the pion as the quarkonium bound by the funnel potential. We did it using the covariant generalization of the instantaneous–potential model proposed in the framework of the effective bilocal Lagrangian \([13–16]\). Provided that the covariant BSE solutions thus obtained describe bound states sufficiently well, they can be useful for understanding what happens in the experiments probing the quark substructure. In these experiments both nonperturbative bound–state effects and relativistic recoil effects can be important, as in the program of CEBAF, for example.

Our work has extended the line of research which employs instantaneous interquark potentials (e.g., Refs. \([6–8,10,11]\) and \([33]\)) by using the boost–invariant potential ansatz, by introducing and analyzing the effects of the finite quark mass, and by improving and generalizing the subtraction procedure for UV divergences caused by the Coulomb part of the interaction. The initial choice for the renormalization prescription, essentially following from the standard prescription where infinities are subtracted at a spacelike point \([26]\), has been shown to preclude the correct chiral limit. However, by appropriately modifying the renormalization prescription, we have been able to recover the correct chiral limit. With this (naturally preferred) scheme, our pion exhibits the qualitatively correct (pseudo)–Goldstone behavior. We have also shown how the MP breaking occurs in this context and explored the dependence of the results on the subtraction point \(\mu\). In particular, we have found that, as \(\mu\) grows, the results in the first renormalization scheme gradually approach the results in the second, preferred scheme.

As far as the quantitative results are concerned, the description of the pion is still not satisfactory, as the usage of the funnel potential has not (yet) resulted in the improvement of \(M_\pi\), \(F_\pi\), and \(\Gamma(\pi^0 \rightarrow \gamma\gamma)\) with respect to the values we obtained with the harmonic interaction \([17]\).

These results are not definitive as we have not yet systematically explored the parameter space. This may be one task for the future work, but further search for an improved form
of the potential certainly remains as the other, and even more important task.

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APPENDIX:

The SD equation (3.26) and the BS equation (4.4) are nonlinear integral equations with singular kernels. In order to solve these equations numerically, we discretize them, i.e., we approximate integral equations with a finite set of coupled nonlinear equations. This is realized by discretization of the momentum variable,

\[ k_i = k_{\text{max}} \left( \frac{i - 1}{N} \right)^n \quad (i = 1, \ldots, N + 1). \quad (A1) \]

\( k_{\text{max}} \) is the “numerical” cutoff; it should be chosen large enough to eliminate the effects of the boundary condition at “infinity”. \( N + 1 \) is the number of points. For \( n = 1 \) Eq. (A1) corresponds to an equidistant mesh, i.e., uniformly distributed points, while for \( n > 1 \) the points are denser near \( k = 0 \). A finite set of variables is defined, e.g., for the SDE we define \( \varphi_i = \varphi(k_i), \ i = 1, \ldots, N + 1 \). The values \( \varphi_1 \) and \( \varphi_{N+1} \) are fixed by the boundary conditions \( \varphi(0) = \pi/2 \) and \( \lim_{k \to +\infty} \varphi(k) = 0 \) (see, e.g., [6]). Equation (3.26) must be satisfied for every \( k_i, \ i = 2, \ldots, N \); thus we have obtained a system of \( N - 1 \) nonlinear coupled equations for \( \varphi_1, \ldots, \varphi_{N-1} \).

A problem arises from the singular nature of the integration kernel. The integrals have to be calculated as principal values. So, we define the continuous function \( \varphi(k) \) as a cubic spline of \( \varphi_1, \ldots, \varphi_{N+1} \) and perform integration using Gaussian quadrature, adapted for principal–value calculation. The system of \( N - 1 \) coupled nonlinear equations is solved using a modification of Brent’s methods. A similar procedure was applied to the BSE (4.4), treating \( M_\pi \) as an additional variable.
For $k_{\text{max}}$ larger than a few $MeV$, the solutions has been found to be independent of $k_{\text{max}}$. For an equidistant mesh, $n = 1$, the number of points have to be $N \geq 100$ to obtain a solution independent of $N$. For a mesh with $n = 3$, this is $N \approx 25$. The meshes with larger $N$ have an inadequate distribution of points in respect to the solution $\varphi(k)$. This optimization is important because the computer time consumed behaves as $\propto N^2$. 
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FIGURES

FIG. 1. Solutions $\varphi$ of the SDE, Eq. (3.28), and $L_1$ and $L_2$ of the BSE, Eqs. (4.3), are plotted versus $q$ for $m = 7 \, \text{MeV}$ and $\sigma = (350 \, \text{MeV})^2$. Solid lines pertain to the case $\alpha_s = 0.4$, and dashed lines to $\alpha_s = 0.8$. Starting values of the solutions are $\varphi(0) = \pi/2$, $L_1(0) = 1$, and $L_2(0) < 0$.

FIG. 2. Pion mass $M_\pi$ vs. quark mass $m$ for three different values of $\alpha_s$ (0.0, 0.4, 0.8) and $\sigma = (350 \, \text{MeV})^2$. The dotted lines essentially correspond to the normalization condition (3.18) and $\mu = 1 \, \text{GeV}$. The dashed lines relate to the same scheme, but for $\mu = 5 \, \text{GeV}$. The solid lines are obtained by using the normalization condition (3.23), slightly modified, as explained in Sec. IV. The renormalization point is $\mu = 1 \, \text{GeV}$.

FIG. 3. Pion mass $M_\pi$ vs. string tension $\sigma$. As in Fig. 2, the solid lines relate to the normalization condition (3.23).

FIG. 4. Constituent mass $m^*$ vs. quark mass $m$. The curves are as in Fig. 2.

FIG. 5. Constituent mass $m^*$ vs. string tension $\sigma$. The curves are as in Fig. 4.

FIG. 6. Pion decay constant $F_\pi$ vs. quark mass $m$. The curves are as in Fig. 3.

FIG. 7. Pion decay constant $F_\pi$ vs. string tension $\sigma$. The curves are as in Fig. 5.

FIG. 8. Decay width for $\pi^0 \rightarrow \gamma\gamma$ vs. quark mass $m$. The curves are as in Fig. 4.

FIG. 9. Decay width for $\pi^0 \rightarrow \gamma\gamma$ vs. string tension $\sigma$. The curves are as in Fig. 6.
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Fig. 3
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Fig. 7
Fig. 9