The Null Decomposition of Conformal Algebras

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Abstract

We analyze the decomposition of the enveloping algebra of the conformal algebra in arbitrary dimension with respect to the mass-squared operator. It emerges that the subalgebra that commutes with the mass-squared is generated by its Poincaré subalgebra together with a vector operator. The special cases of the conformal algebras of two and three dimensions are described in detail, including the construction of their Casimir operators.
1 Introduction

The study of symmetry groups has shown that invariants are of great interest in many areas in mathematics and physics, being used not only to label irreducible representations of Lie algebras, and decomposing generic representations into irreducible ones, but also in studying symmetry breaking [9]. (Symmetry breaking refers to systems which possess a certain symmetry and are perturbed in such a way that only part of the original symmetry still holds in the system. More often than not, the perturbed system presents more interest than the original one, since the full symmetry in general describes situations which lack reality.) In this process, one can still relate the invariants of the whole symmetry group with those of its subgroups [2].

The Lie algebra $\mathfrak{su}(2)$ played an important role in studying the mass spectrum of hadrons in particle physics. The central idea is that particles with the same spin, parity and close mass values should be related by some kind of symmetry [3]. In 1932, Heisenberg regarded the neutron and proton as forming basis states for the fundamental representations of $\mathfrak{su}(2)$ [5]. One of the quantum numbers used to label the particles is the isotopic spin (or isospin), which is conserved in reactions involving electromagnetic and strong interactions. This is not the only quantum number conserved in these reactions. In order to accommodate this extra quantity, in 1964 Gell-Mann and Ne’eman [6] grouped the hadrons into multiplets of $\mathfrak{su}(3)$. The basic observed particles fitted though into an eight-dimensional adjoint representation instead of the fundamental three-dimensional $\mathfrak{su}(3)$ representation.

A natural question to ask (from the point of view of supersymmetry) is whether there is some symmetry connecting the different octets in this eight-dimensional adjoint representation. In [1], Sparling suggested that such a symmetry arises naturally in the context of broken conformal invariance; there he addressed the question of what happens to a system transforming under the conformal group when the conformal invariance is broken by fixing the mass.

In the absence of gravitation, the physical space-time is the Minkowski space $\mathbb{M}$, which is a four-dimensional real vector space with a metric of signature $(3,1)$ (or $(1,3)$). As a manifold, this space is isomorphic to its group of translations. The transformations under the Lorentz group $SO(3,1)$ of rotations of $\mathbb{M}$ contain also important information on the physics of the system.
In order to include them in the formalism, $M$ has to be regarded as the homogeneous space of the Poincaré group, modulo the Lorentz group [3]. Many techniques are created for semisimple Lie groups, but unfortunately the Poincaré group is not semisimple. One solution that would allow using these techniques is to regard a suitable compactification of $M$ as a coset of the group of conformal transformations of $M$.

Physicists use the conformal group especially because it is simple. The downside is that its action on a quantum mechanical system changes its mass continuously. The Poincaré group, used extensively in particle physics and quantum field theory (QFT), is the part of the conformal group that commutes with the operator representing the square of the mass.

Quantum mechanics assumes that operators on a quantum system may be commuted with each other, and also multiplied together in an associative way. For a Lie algebra acting on a quantum system, this means that the action extends naturally to an action of the enveloping algebra of its Lie algebra. (The enveloping algebra of a given algebra $A$ is the smallest associative algebra that contains $A$ as a subspace and for which the commutator on this subspace reproduces the Lie bracket in $A$).

Therefore, from a QFT perspective, it is natural to study the subalgebra of the enveloping algebra of the conformal algebra that commutes with the mass. We find that this subalgebra, denoted $\mathcal{R}$, is strictly larger than the enveloping algebra of the Poincaré Lie algebra; the extra structure is encoded in a vector operator $R_a$ which is shown to commute with the mass of the system and with the translations $p_a$. When the conformal invariance is broken, provided the operator $R_a$ survives, no useful information is lost, only the dilation operator $D$ needs to be removed. The full conformal algebra is simply obtained by adding the generator $D$ with appropriate commutation relations to the $\mathcal{R}$-algebra.

We should remark here that the newly obtained $\mathcal{R}$-algebra, although finitely generated, is not a finite-dimensional Lie-algebra, as the commutator of the vector operator $R_a$ with itself is non-linear (actually cubic) in the generators.

For a finite-dimensional simple algebra $A$, the universal enveloping algebra $U$ contains a set of operators which can be used to label the representations of $A$. These operators, the Casimir invariants, generate the center of the associative algebra $U(A)$ and commute with any of its elements [3], [4].
In this work, we are interested in constructing the $\mathcal{R}$-algebras for spaces of various dimensions and their Casimir invariants. The four-dimensional case has been thoroughly studied in [1]; its origins lie in the study of the twistor model of hadrons which seeks to describe a hadron by means of three twistors [7], [8]. These types of representations (for $n = 4$) have also been used in the AdS/CFT correspondence [9], [10], [11].

The structure of the present work is as follows: in section two we generalize the four-dimensional theory to arbitrarily many dimensions and signatures. In section three we independently develop the three-dimensional theory and verify the consistency of these results with the generalized case. Section four treats the two-dimensional case, but calculations are only carried in the $(2, 0)$ signature.

There are many intermediate formulas we used to derive the final expressions included in this paper. Due to the fact that calculations are extremely lengthy, we listed most of the formulas used in an appendix.

Although this theory a priori only applies to the (non-compact) pseudo-orthogonal groups, many other (non-compact) Lie groups such as the pseudo-unitary groups are subgroups of the pseudo-orthogonal group in a natural way. As a consequence, this theory can be applied to these groups also, with appropriate modifications.

Finally note that the de Sitter algebra of $n$-dimensional de-Sitter space, $\mathfrak{SO}(1, n)$ and the anti-de-Sitter algebra of $n$-dimensional anti-de-Sitter space, $\mathfrak{SO}(2, n - 1)$ both fall within the scope of our theory [9].
2 The Conformal Algebra in \( n \)-dimensions

2.1 Basic Commutation Relations

We consider the conformal algebra of an \( n \)-dimensional affine space equipped with a metric of signature \((p, q)\), where \( p \) and \( q \) are non-negative integers such that \( p + q = n \). The algebra in question is the Lie algebra of \( \text{SO}(p+1, q+1) \).

Explicitly, we may write the following commutation relation:

\[
[M_{AB}, M_{CD}] = g_{AC}M_{BD} - g_{BC}M_{AD} - g_{AD}M_{BC} + g_{BD}M_{AC}. \tag{2.1}
\]

Here the upper case Latin indices run from 0 to \( n + 1 \). Also the Lie algebra generators \( M_{AB} \) are skew: \( M_{AB} = -M_{BA} \). The metric \( g_{AB} = g_{BA} \) is a real symmetric metric of signature \((p+1, q+1)\). We use a null decomposition of the metric \( g_{AB} \), describing it by the \((n + 2)\times(n + 2)\) symmetric matrix:

\[
g_{AB} = \begin{pmatrix}
0 & 0 & 1 \\
0 & g_{ab} & 0 \\
1 & 0 & 0 \\
\end{pmatrix}. \tag{2.2}
\]

So here the lower case Latin index runs from 1 to \( n \). Also we have the components \( g_{00} = g_{n+1n+1} = 0, g_{0a} = g_{a0} = g_{n+1a} = g_{an+1} = 0 \). The \( n \times n \)-symmetric matrix \( g_{ab} = g_{ba} \) has signature \((p, q)\). Now put

\[
M_{0a} = p_a, \ M_{n+1a} = q_a, \ M_{0n+1} = D, \text{and } M_{ab} = -M_{ba}, \tag{2.3}
\]

So the tensor \( M_{AB} = -M_{BA} \) can be described by an \((n + 2)\times(n + 2)\) skew matrix:

\[
M_{AB} = \begin{pmatrix}
0 & p_a & D \\
-p_b & M_{ab} & -q_b \\
-D & q_a & 0 \\
\end{pmatrix}. \tag{2.4}
\]

\( M_{ab} \) is itself an \( n \times n \) skew matrix and represents the analogue of the angular momentum in \( n \) dimensions, \( p_a \) represents the translations, \( q_a \) the special conformal transformations, and \( D \) is the dilation operator.

The generators of the full algebra are thus: \( p_a, q_a, M_{ab} \) and \( D \).
From equations (2.1)-(2.4), we obtain the following basic commutation relations:

\[
\begin{align*}
[p_a, p_b] &= 0, \quad [q_a, q_b] = 0, \quad [p_a, q_b] = M_{ab} + g_{ab}D, \\
[M_{ab}, p_c] &= g_{ac}p_b - g_{bc}p_a, \quad [M_{ab}, q_c] = g_{ac}q_b - g_{bc}q_a, \\
[D, p_a] &= -p_a, \quad [D, q_a] = q_a, \quad [D, M_{ab}] = 0.
\end{align*}
\] (2.5)

We will consider the following subalgebras, satisfying the commutation relations (2.5):

- the full algebra \( \mathcal{C} \), spanned by \( p_a, M_{ab}, D, \) and \( q_a \), of real dimension \( (n+1)(n+2)/2 \);
- the subalgebra \( \mathcal{D} \) of \( \mathcal{C} \), spanned by \( p_a, M_{ab}, \) and \( D \), of real dimension \( (n^2 + n + 2)/2 \); this is called the Weyl algebra;
- the subalgebra \( \mathcal{P} \) of \( \mathcal{C} \), spanned by \( p_a, \) and \( M_{ab} \), of real dimension \( n(n+1)/2 \); this is called the Poincaré algebra, or the Euclidean algebra, depending on the signature of the metric.

Following the notation introduced in [1], we denote by \( \mathcal{C}_e, \mathcal{D}_e, \mathcal{P}_e \) their corresponding enveloping algebras.

Throughout this paper we will use \( p^2 \) or \( p \cdot p \) to denote \( p_a p^a \), and also \( p^{-2} = (p^2)^{-1} = (p \cdot p)^{-1} = (p_a p^a)^{-1} \). Adjoin to each of the above algebras the element \( p^{-2} \) and its powers \( (p^{-2})^r \), where \( r \) is a positive integer, satisfying:

\[
p^{-2} p^2 = p^2 p^{-2} = 1,
\] (2.6)

and

\[
[p^{-2}, X] = -p^{-2} [p^2, X] p^{-2}, 
\] (2.7)

for any \( X \) in \( \mathcal{C}_e \). Note that \( p^{-2} \) is the inverse of \( p^2 \), where \( p^2 \) is the mass-squared operator of the system. The resulting algebras will be denoted by \( \mathcal{C}_e (p^2), \mathcal{D}_e (p^2), \mathcal{P}_e (p^2) \), so implicitly the representations that this work applies to are those of non-zero mass only. It is well known that \( p^2 \) is a Casimir invariant for \( \mathcal{P}_e \).
2.2 The $\mathcal{R}$-algebra

The first step in our analysis is the introduction of an operator constructed in $\mathcal{P}_e(p^2)$ that behaves like a position operator:

$$2p^2 x_a = M_{ab} p^b + p^b M_{ab} = 2x_a p^2,$$

satisfying the relation:

$$x \cdot p = \frac{n-1}{2}.$$  \hspace{1cm} (2.9)

Note that $x_a$ is not a full position operator since it obeys the constraint relation (2.9). Also note that we can write $x_a$ as:

$$p^2 x_a = M_{ab} p^b + k p_a,$$  \hspace{1cm} (2.10)

where we abbreviate the quantity $\frac{n-1}{2}$ by $k$: so we have $k = x \cdot p = \frac{n-1}{2}$. The operators $x_a$ and $p^2$ commute since $x_a$ lies in $\mathcal{P}_e(p^2)$.

Define also the analogue of the orbital angular momentum in $n$ dimensions:

$$L_{ab} = x_a p_b - x_b p_a$$  \hspace{1cm} (2.11)

and the intrinsic spin:

$$S_{ab} = M_{ab} - L_{ab} = -S_{ba}.$$  \hspace{1cm} (2.12)

It is now straightforward to verify that $S_{ab}$ is orthogonal to $p_a$:

$$S_{ab} p^b = p^b S_{ab} = 0.$$  \hspace{1cm} (2.13)

Finally define a projected metric tensor: $h_{ab} = g_{ab} - p^2 p_a p_b$. Note that $p^a h_{ab} = h_{ab} p^a = 0$ and $h_{a}^a = n - 1$. The commutation relations satisfied by these new operators can be found in appendix A.

The algebra $\mathcal{P}_e(p^2)$ can be written now in terms of the operators $x_a$, $p_a$, $S_{ab}$ (whose number of degrees of freedom are $n - 1$, $n$, and $(n - 1)(n - 2)/2$, respectively) as follows:

$$[p_a, p_b] = 0, \quad [x_a, p_b] = h_{ab},$$

$$p^2 [x_a, x_b] = -S_{ab} - x_a p_b + x_b p_a,$$  \hspace{1cm} (2.14)

$$[S_{ab}, p_c] = 0, \quad [p^2, S_{ab}] = 0,$$

$$p^2 [S_{ab}, x_c] = S_{ac} p_b - S_{bc} p_a,$$

$$[S_{ab}, S_{cd}] = h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}.$$
Next, we pass to the Weyl algebra $\mathcal{D}_e(p^2)$, which allows us to define a full position operator:

$$p^2 y_a = p^2 x_a - p_a(D - l) \quad (2.15)$$

such that $[y_a, p_b] = g_{ab}$. Here $l$ is a pure number to be determined later.

This new operator can be regarded as unconstrained if we think of $D$ as being defined in terms of $y_a$ by means of the relations $y \cdot p = 1 - D + k + l$ (see equation (2.20) below). Then $y_a$ obeys the commutation relations:

$$[p^2, y_a] = -2p_a, \quad p^2[y_a, y_b] = -S_{ab}. \quad (2.16)$$

Here as mentioned before, $D$ is the dilation operator satisfying:

$$[D, p_a] = -p_a, \quad [D, x_a] = x_a, \quad [D, M_{ab}] = 0. \quad (2.17)$$

(Additional commutation relations can be found in the appendix A.)

Consider now the full algebra $\mathcal{C}_e(p^2)$. Recall from equation (2.5) that the vector operator $q_a$ generates special conformal transformations. Note that $p^2$ doesn’t commute with $q_a$:

$$[p^2, q_a] = p_a(2D - 1) - 2p^2 x_a \quad (2.18)$$

$$= p_a(2D - 1) - (M_{ab} p_b + p^b M_{ab}).$$

Our immediate goal is to construct an operator $Q_a$ from $\mathcal{D}_e(p^2)$ which obeys the same commutation relation with $p_b$ as does $q_a$, and therefore the difference $q_a - Q_a$ will then commute with $p_a$ and $p^2$.

Define:

$$Q_a = -y^b S_{ba} + \alpha(y \cdot y) p_a + \gamma(y \cdot p) y_a \quad (2.19)$$

made from $M_{ab}$, $D$ and $p_a$ only. When commuted with the $p$ operator, we obtain that:

$$[p_a, Q_b] = S_{ab} - 2\alpha y_a p_b - \gamma y_b p_a + \gamma g_{ab}(1 - y \cdot p).$$
Requiring that this commutation relation is the same as \([p_a, q_b]\), we obtain that
\[ \alpha = -\frac{1}{2}, \quad \gamma = 1, \quad l = -k, \quad \text{and} \quad y \cdot p = 1 - D. \quad (2.20) \]

Written in terms of the operator \(x_a\), we have:
\[ Q_a = x^b S_{ab} - \frac{1}{2} (x \cdot x) p_a - x_a D + \frac{1}{2} p^{-2} p_a \left( D^2 + D - k^2 \right), \quad (2.21) \]
satisfying:
\[ [p_a, Q_b] = M_{ab} + D g_{ab}, \]
\[ [p^2, Q_a] = p_a (2D - 1) - 2 p^2 x_a, \quad (2.22) \]
\[ 2p^2 [Q_a, Q_b] = -(n - 3)(n - 2) S_{ab} - 2 S_{b[d]} S^{cd} S_{a]c}. \]

Now we define:
\[ R_a = q_a - Q_a, \quad (2.23) \]
with the key commutation relations \([p_a, R_b] = 0\) and \([p^2, R_a] = 0\).

We obtain in this way the algebra which we will call the \(R\)-algebra. It is generated by \(R_a\), \(p_a\), and \(S_{ab}\) with the commutation relations:
\[ [p_a, p_b] = 0, \quad [R_a, p_b] = 0, \quad [S_{ab}, p_c] = 0, \]
\[ p^2 [S_{ab}, R_c] = p^2 h_{ac} R_b - p^2 h_{bc} R_a - g_{ac} (R \cdot p) p_b + g_{bc} (R \cdot p) p_a, \]
\[ [S_{ab}, S_{cd}] = h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}, \quad (2.24) \]
\[ 2p^2 [R_a, R_b] = ((n - 3)(n - 2) - 4 (R \cdot p)) S_{ab} + 2 S_{b[d]} S^{cd} S_{a]c} + (R^c S_{ac} + S_{ac} R^c) p_b - (R^c S_{bc} + S_{bc} R^c) p_a. \]

The last relation is derived by a very lengthy calculation.
The following theorem is the main result of this work and summarizes the results obtained in the style of [1].

**THEOREM 2.1:** Define in \( \mathcal{C}_e(p^2) \) the operator \( R_a \) by the formula:

\[
p^2 R_a = p^2 q_a - p^2 x^b S_{ab} + \frac{1}{2} p^2 (x \cdot x) p_a + p^2 x_a D - \frac{1}{2} p_a (D^2 + D - k^2).
\]

\( R_a \) is translationally invariant:

\[
[R_a, p_b] = 0.
\]

Also, \( p^4 (R_a - q_a) \in \mathcal{D}_e \), and \( R_a \) has the following commutation relations:

\[
[D, R_a] = R_a,
\]

\[
p^2 [S_{ab}, R_c] = p^2 h_{ac} R_b - p^2 h_{bc} R_a - g_{ac} (R \cdot p) p_b + g_{bc} (R \cdot p) p_a,
\]

\[
2p^2 [R_a, R_b] = ((n - 3)(n - 2) - 4 (R \cdot p)) S_{ab} + 2 S_{[b|d|} S_{c|d]} S_{a]c} + (R^c S_{ac} + S_{ac} R^c) p_b - (R^c S_{bc} + S_{bc} R^c) p_a.
\]

The freedom in choosing \( R_a \in \mathcal{C}_e(p^2) \), obeying

\[
[p^2, R_a] = 0, \; p^4 (R_a - q_a) \in \mathcal{D}_e,
\]

is \( R_a \rightarrow R_a + p^4 g_a \) where \( g_a \in \mathcal{P}_e \), and \( [D, g_a] = 5 g_a \). Also, \( \mathcal{C}_e(p^2) \) is generated by \( D \) and the \( \mathcal{R} \)-algebra.

### 2.3 Casimir Invariants

We have now an algebra generated by the operators \( R_a \), \( p_a \), and \( S_{ab} \) that commute with the mass. In order to simplify the expressions, we can introduce two new operators:

\[
S_a = R_b h^b_a,
\]

with \( h_{ab} = g_{ab} - p^{-2} p_a p_b \), and

\[
S = R \cdot p = R_a p^a.
\]

Note that \( p^a S_a = S_a p^a = 0 \) and \( p^a S_{ab} = S_{ab} p^a = 0 \).
The commutation relations become:

\[ [S_{ab}, S] = 0, \quad 2[S_a, S] = 2S^b S_{ab} - (n - 2) S_a, \]

\[ [S_{ab}, S_c] = h_{ac} S_b - h_{bc} S_a, \] \hspace{1cm} (2.27)

\[ [S_{ab}, S_{cd}] = h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}, \]

\[ 2p^2 [S_a, S_b] = ((n - 3) (n - 2) - 4S) S_{ab} + 2S_{[b|d|} S^{cd} S_{a|c].} \]

Note that by introducing these operators, \( S_a \) transforms like a vector with respect to the intrinsic spin \( S_{ab} \), and the commutation relations are significantly simpler. The generators of the \( \mathcal{R} \)-algebra are now the operators \( p_a, S_{ab}, S_a, \) and \( S \).

A slight reformulation of this algebra can be achieved at the cost of introducing a scalar operator \( m \) which is the square root of \( p^2 \), together with its inverse \( m^{-1} \). Then we may put \( T_a = m^{-1} S_a \) and \( s_a = m^{-1} p_a \). The point here is that \( S, S_{ab}, T_a \) and \( s_a \) are dimensionless: \( [D, S] = 0, [D, S_{ab}] = 0, [D, T_a] = [D, s_a] = 0 \). The commutation relations become:

\[ [s_a, S] = 0, \quad [s_a, s_b] = 0, \quad [s_a, T_b] = 0, \quad [s_a, S_{bc}] = 0, \]

\[ [S_{ab}, S] = 0, \quad 2[T_a, S] = 2T^b S_{ab} - (n - 2) T_a, \]

\[ [S_{ab}, T_c] = h_{ac} T_b - h_{bc} T_a, \] \hspace{1cm} (2.28)

\[ [S_{ab}, S_{cd}] = h_{ac} S_{bd} - h_{bc} S_{ad} - h_{ad} S_{bc} + h_{bd} S_{ac}, \]

\[ 2[T_a, T_b] = ((n - 3) (n - 2) - 4S) S_{ab} + 2S_{[b|d|} S^{cd} S_{a|c]}, \]

\[ h_{ab} = g_{ab} - s_a s_b, \quad s_a s^a = 1, \quad h_{ab} s^b = 0, \quad S_{ab} s^b = 0, \quad T_a s^a = 0. \]

Note that the full conformal algebra is then recovered by adding in the two operators \( D \) and \( m \), which each commute with the operators \( S, s_a, T_a \) and \( S_{ab} \) and which obey the commutation relations:

\[ [D, m] = -m. \] \hspace{1cm} (2.29)

In particular the momentum operator is recovered as \( p_a = m s_a \).
Finally note that, at least formally, if we write \( m = e^x \) (which is intuitively reasonable when the mass is positive), then the commutation relation \([D, m] = -m\) is implied by the Heisenberg commutation relation:

\[ [x, D] = 1. \quad (2.30) \]

Thus we may consistently think of the enveloping algebra of the conformal algebra as a straight-forward product of the Heisenberg algebra with the (dimensionless) algebra generated by the operators \( S, s_a, T_a \) and \( S_{bc} \).

The following corollaries summarize our only general results on the Casimir invariants.

**Corollary 2.2:** The operator:

\[ C_1 = S + \frac{1}{4} S_{cd} S^{cd}, \quad (2.31) \]

is a Casimir invariant of the algebra \( C_e(p^2) \).

**Corollary 2.3:** \( C_1 \) and \( p^2 \) are Casimir invariants for the \( \mathcal{R} \)-algebra.

\( C_1 \) can be used to reduce the number of generators just to the operators \( S_a \) and \( S_{ab} \), together with \( C_1 \) itself.

It is known that \( SO(p + 1, q + 1) \) has rank \( 1 + \lfloor \frac{p}{2} \rfloor \), this being the number of Casimir invariants as well [3], [4]. In principle, these invariants are given by the scalar operators formed from products of \( M_{AB} \) with itself, but in practice this hasn’t been very helpful in finding the other Casimir invariants, due to the complexity of the expressions involved. Our attempts involved introducing yet another operator:

\[ U = S_a S^a. \quad (2.32) \]
The commutation relations involving the operator $U$ are as follows:

$$[U, S_{ab}] = 0, \quad [U, S] = 0,$$

(2.33)

$$p^2[U, S_a] = (\lambda \delta^b_a + \mu S^b_a + \nu S^c_a S^b_c + \rho S^d_a S^e_c S^b_e) S_b,$$

(2.34)

where

$$\lambda = (n - 2) (n - 1 + 2C_1 - \frac{1}{2} S_{ab} S^{ab}),$$

$$\mu = n (n - 1) + 4C_1 - S_{ab} S^{ab},$$

$$\nu = 3n - 4,$$

$$\rho = 2.$$

$U$ is a scalar with respect to $S_{ab}$ and $S$. It is expected that $U$ will be part of a Casimir invariant. To produce such an invariant, the building blocks must be of the general form:

$$S^a F_{ab}(S_{cd}, C_1) S^b + G (S_{cd}, C_1),$$

(2.36)

where $F_{ab}(S_{cd}, C_1)$ is a function of $S_{cd}$ and $C_1$ only, with $a$ and $b$ free indices, e.g. $f(C_1) S_{ad} S^d_c S^e_c S^b_e$, and $G (S_{cd}, C_1)$ is a scalar function of $S_{cd}$ and $C_1$, e.g. $g(C_1) S^a_c S^e_c S^b_e S^e_a$.

If necessary, one can assume that $F_{ab}(S_{cd}, C_1)$ is symmetric in the indices $a$ and $b$, since the skew part allows $S^a F S^b$ to be replaced by a commutator, which then can be assimilated into the $G$ term. However except in dimensions $n = p + q \leq 5$, we have not yet been able to construct the remaining Casimirs. The basic problem is the complexity of the relation (2.34).
3 The Conformal Algebra in Three Dimensions

Consider the infinitesimal generators of the conformal group of a three-dimensional space: the translations $p$, the special conformal transformations $q$, the angular momentum $J$, and the dilation operator $D$. These generators form a ten-dimensional algebra. We first show a direct construction of the $R$-algebra and then we show that this construction agrees with the specialization of the general case to $n = 3$.

3.1 Basic Commutation Relations

The basic commutation relations satisfied by these operators are:

$$
[D, p] = -p, \quad [D, J] = 0, \quad [D, q] = q, \\
[J \cdot a, p \cdot b] = -(a \times b) \cdot p, \\
[J \cdot a, q \cdot b] = -(a \times b) \cdot q, \\
[J \cdot a, J \cdot b] = -(a \times b) \cdot J, \\
[p \cdot a, p \cdot b] = 0, \quad [q \cdot a, q \cdot b] = 0, \\
[p \cdot a, q \cdot b] = (a \cdot b)D - (a \times b) \cdot J.
$$  (3.1)

In this section we will use the notation $J \cdot a$ instead of $J_a$ and $(a \times b) \cdot J$ instead of $\varepsilon_{abc}J^c$ in order to preserve some of the characteristics of a three-dimensional space.

These commutation relations ensure that the Jacobi identity is satisfied:

$$
0 = [p \cdot a, [q \cdot b, q \cdot c]] + [q \cdot c, [p \cdot a, q \cdot b]] + [q \cdot b, [q \cdot c, p \cdot a]].  \quad (3.2)
$$

Introduce the following notation used for the remaining of this section:

$$
[p, \times q] = p \times q + q \times p,  \quad (3.3)
$$
for any two vectors $p, q$. 

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3.2 Three-Dimensional Model

We can construct a model in $\mathbb{R}^3$ that will satisfy the commutation relation of the conformal algebra. This model has roots in quantum mechanics, where $p$ is the momentum operator, $x$ is the position operator, $J$ is the angular momentum and $D$ is the dilation operator:

\[ x_a = x_a, \quad p_a = \frac{\partial}{\partial x_a}, \quad J = x \times p, \quad D = x \cdot p, \tag{3.4} \]

with the well-known commutation relations:

\[ [a \cdot J, p \cdot b] = -(a \times b) \cdot p, \]
\[ [a \cdot J, x \cdot b] = -(a \times b) \cdot x, \tag{3.5} \]
\[ [p_a, x_b] = \delta_{ab}. \]

Define now an operator $q$ that will satisfy the key relation of the conformal algebra, equation (3.1vi):

\[ q = \frac{1}{2}(x \times J + xD) \tag{3.6} \]

Note that in this model we assumed that $J = x \times p, \quad D = x \cdot p$, which gives that $J \cdot p = p \cdot J = 0$, and this would not generally be true. Once we obtain an expression for the operator $q$ in terms of the generators of the algebra, we will drop all assumptions on any particular model.

With these formulas, we obtain that the defining relation of the conformal algebra is satisfied:

\[ [p \cdot a, q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \tag{3.7} \]

and we also obtain a candidate for the operator $q$:

\[ q = \frac{1}{2(p \cdot p)} [(p \cdot J)J - p(J \cdot J) + pD(D + 3) + 2(p \times J)(D + 1)], \tag{3.8} \]

which can be seen not to commute with $p \cdot p$.

If we modify $q$ by a $p \cdot J$ term, we can define a new operator $Q$ by:
\[ 2(p \cdot p)Q = 2(p \cdot J)J - p(J \cdot J) + pD(D + 3) + 2(p \times J)(D + 1). \quad (3.9) \]

This choice of the operator \( Q \) still satisfies the conformal algebra:

\[ [p \cdot a, Q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \quad (3.10) \]

and

\[ [p \cdot p, Q] = p(2D + 1) + 2(p \times J) = [p \cdot p, q]. \quad (3.11) \]

Since neither \( q \) nor \( Q \) commute with \( p \cdot p \), this leads us to introducing a new operator, \( R \).

### 3.3 The \( \mathcal{R} \)-algebra

Construct the following operator: \( R = q - Q \), with the commutator:

\[ [p \cdot a, R \cdot b] = 0. \quad (3.12) \]

\( R \) is a vector operator, so we have the commutation relation:

\[ [J \cdot a, R \cdot b] = -(a \times b) \cdot R. \quad (3.13) \]

Moreover, \( R \) has the same dimension as does \( q \), so we have:

\[ [D, R] = R. \quad (3.14) \]

After lengthy calculations, we obtain:

\[ (p \cdot p)^2(Q \times Q) = 2(p \cdot J)p, \quad (3.15) \]

This result can be used to write:

\[ R \times R = (q - Q) \times (q - Q) = Q \times Q - (q \times Q + Q \times q), \quad (3.16) \]

where we took into account that \( q \times q = 0 \), and:

\[ (p \cdot p)^2(R \times R) = 2(p \cdot p)(p \cdot J)R - (p \cdot p)(p \times R) - 2(p \cdot J)p. \quad (3.17) \]
3.4 Casimir Operators

We have constructed the subalgebra with generators \(p\), \(J\), and \(R\), and obtained the complete commutation relations. This algebra, as desired, has the mass of the system as one of the Casimir operators.

For the remaining of this section, we are trying to construct the Casimir invariants of the algebra. The following commutation relations have been established:

\[
[R \cdot p, p] = 0, \quad [R \cdot p, J] = 0,
\]

\[
(p \cdot p)[R \cdot p, R] = -2(p \cdot J)(p \times R) + (R \cdot p)p - (p \cdot p)R,
\]

\[
[R \cdot R, p] = 0, \quad [R \cdot R, J] = 0,
\]

\[
(p \cdot p)^2[R \cdot R, R] = -2(p \cdot J)(p \times R) - (p \cdot p)R + (R \cdot p)p.
\]

Since not all these commutators vanish, we introduce a new operator:

\[
S = R - \lambda \frac{p}{p \cdot p}
\]

with

\[
S \cdot p = R \cdot p - \lambda,
\]

\[
S \cdot S = R \cdot R - 2\lambda(p \cdot p)^{-1}(R \cdot p) + (p \cdot p)^{-1}\lambda^2.
\]

By commuting it with the new generators of the algebra, namely \(R\), \(S\), and \(p\), we obtain that \(S \cdot S\) is a Casimir operator if \(\lambda = 1/2\). Note that the algebra is written now in terms of \(S\), \(p\), and \(J\).

With \(\lambda\) fixed at 1/2, we can show that the two Casimir operators of the algebra \(C_e\) have the simple expressions:

\[
C_1 = S \cdot S,
\]

\[
C_2 = S \cdot p + \frac{1}{pp}(p \cdot J)^2.
\]

The Casimir invariants of the \(R\)-algebra (more appropriately now called \(S\)-algebra) are \(C_1\), \(C_2\) and \(p \cdot p\).
3.5 Applying the $n$-d Theory to the 3-d Theory

As mentioned before, the two theories have been developed independently. In this subsection we show that the two approaches are consistent with each other. The main test is showing that the operators $Q$ and $R$ satisfy the same relations in both theories, and that the Casimir invariant in the $n$-dimensional theory when applied to $n = 3$ matches the one derived in the original three-dimensional theory.

We consider the following relation that will make the connection between the two cases:

$$M_{ab} = -\epsilon_{abc}J^c \quad \text{or} \quad J_c = -\frac{1}{2}\epsilon_{abc}M^{ab}. \quad (3.22)$$

One can show fairly easily that the commutation relations (3.1) are satisfied by these expressions.

We also have:

$$p^2x = -(p \times J) - p,$$

$$p^2L_{ab} = \epsilon_{abc}(p \cdot J)p^c - p^2\epsilon_{abc}J^c = (p \cdot J)(a \times b) \cdot p - (p \cdot p)(a \times b) \cdot J, \quad (3.23)$$

$$p^2S_{ab} = -\epsilon_{abc}(p \cdot J)p^c = -(p \cdot J)(a \times b) \cdot p,$$

which yields:

$$S_{ab}S^{ab} = 2p^{-2}(p \cdot J)^2. \quad (3.24)$$

Using these formulas in equation (2.19), we obtain the following expression for the $Q$ operator:

$$2(p \cdot p)Q = 2(p \cdot J)J - p(J \cdot J) + pD(D + 3) \quad (3.25)$$

$$+ 2(p \times J)(D + 1) + 2p - (p \cdot p)^{-1}(p \cdot J)^2p.$$

Note that the difference between equations (3.25) and (2.19) is given by the last two terms, $2p - (p \cdot p)^{-1}(p \cdot J)^2p$; as long as they commute with $p$ (and $p^2$), these terms will not change what we require from the $R$ operator (that is to commute with $p^2$). The choice of the $q$ and $Q$ operators (and, therefore, the Casimir operators), is not unique.
In the \( n \)-dimensional case, we obtained that one of the Casimir operators was:

\[
C = R \cdot p + \frac{1}{4} S_{ab} S^{ab}.
\]  
(3.26)

The corresponding Casimir operator in three-dimensions has been obtained from this general one and has the form:

\[
C = R \cdot p + \frac{1}{2} S_{ab} S^{ab}.
\]  
(3.27)

Although these two relations seem slightly different, they can be shown to be the same. If we prime the quantities arising from letting \( n = 3 \) in the general theory, we have that:

\[
2p^2 R' = 2p^2 R + p^{-2} (p \cdot J)^2 p - 2p,
\]  
(3.28)

and thus:

\[
2p^2 (R' \cdot p) = 2p^2 (R \cdot p) + (p \cdot J)^2 - 2p^2
\]  
(3.29)

or

\[
R' \cdot p = R \cdot p + \frac{1}{2} (p \cdot J)^2 - 1
\]  
(3.30)

\[
= R \cdot p + \frac{1}{4} S_{ab} S^{ab} - 1
\]

which means that

\[
R' \cdot p + \frac{1}{4} S_{ab} S^{ab} = R \cdot p + \frac{1}{2} S_{ab} S^{ab},
\]  
(3.31)

(where we ignored the constant \(-1\)).

This shows that the two Casimirs operators are the same, hence the two theories agree.
4 The Conformal Algebra in Two Dimensions

Consider the case, \( n = p + q = 2 \). This situation is relevant for the Lorentz group \( SO(1,3) \) (or \( SO(3,1) \)) and for the ultra-hyperbolic group \( SO(2,2) \) which is used in studying solitons and integrable systems [12], [13]. The results are obtained by directly applying the general theory to \( n = 2 \); as a consequence, we only describe how the quantities considered there change and their new commutation relations.

In two dimensions, any skew quantity with two indices must be a multiple of the completely antisymmetric tensor \( \varepsilon_{ab} \). We have thus:

\[
M_{ab} = J \varepsilon_{ab}.
\] (4.1)

The \( \varepsilon_{ab} \) tensor satisfies the following identity:

\[
\varepsilon_{ab} \varepsilon^{cd} = \sigma \left( \delta^c_a \delta^d_b - \delta^d_a \delta^c_b \right),
\]

with \( \sigma = -1 \) for \( (p,q) = (1,1) \), and \( \sigma = 1 \) for \( (p,q) = (0,2) \) or \( (2,0) \). In this section we consider \( \sigma = 1 \).

We have the following basic commutation relations for the \( C_e \) algebra:

\[
[p_a, p_b] = 0, \quad [q_a, q_b] = 0,
\]

\[
[J, p_a] = \varepsilon^b_a p_b, \quad [J, q_a] = \varepsilon^b_a q_b,
\]

\[
[D, p_a] = -p_a, \quad [D, q_a] = q_a,
\]

\[
[J, J] = 0, \quad [D, D] = 0, \quad [D, J] = 0,
\]

\[
[p_a, q_b] = \varepsilon_{ab} J + g_{ab} D.
\] (4.2)

The new operators corresponding to (2.9)-(2.11) are now:

\[
p^2 x_a = \varepsilon_{ab} J p^b + \frac{1}{2} p_a,
\]

\[
L_{ab} = J \varepsilon_{ab} = M_{ab},
\]

\[
S_{ab} = 0.
\] (4.3)

The operator \( Q_a \) defined in (2.19) becomes in this case:

\[
2p^2 Q_a = -J^2 p_a + p_a D^2 - 2 \varepsilon_{ab} J p^b D,
\] (4.4)
which can be shown to satisfy:

\[ [p_a, Q_b] = \varepsilon_{ab} J + g_{ab} D, \]  

(4.5)
as desired.

This new operator also satisfies:

\[ [Q_a, Q_b] = 0, \]  

(4.6)
which is consistent with equation (2.22iii) since \( S_{ab} = 0 \).

As in the previous cases, let \( R_a = q_a - Q_a \). The \( R \)-algebra will be now generated by the operators \( p_a, R_a \) and \( J \) satisfying the following commutation relations:

\[ [R_a, p_b] = 0, \]
\[ [R_a, R_b] = 0, \]
\[ [R_a, J] = 0, \]
\[ [p_a, p_b] = 0, \]
\[ [J, p_a] = \varepsilon^b_a p_b. \]  

(4.7)

It is easy to show that in this case the two Casimir invariants of the \( C_e(p^2) \) algebra are:

\[ C_1 = R \cdot p, \]
\[ C_2 = R \cdot R, \]  

(4.8)
and the Casimir invariants of the \( R \)-algebra are \( C_1, C_2 \), and \( p^2 \).
5 Conclusions

In this work we have given a complete description of the $\mathcal{R}$-algebras in $n$-dimensions and the lower-dimensional cases: $n = 2$, and $n = 3$.

In reference [1], the $\mathcal{R}$-algebra, for the case $n = 4$, was used to construct the family of unitary irreducible representations that constitute the discrete series of the conformal group. We expect that in general the $\mathcal{R}$-algebras constructed here can be used in the same way.

Finding the Casimir operators for these theories was not a trivial task, but it proved to be extremely difficult in the general case, preventing us from obtaining a complete result.

We have studied the 5-dimensional case as well; these results will be presented in a separate paper, as the approach is quite different, requiring spinor techniques to make the calculations tractable. Such techniques can also be used to simplify the original 4-dimensional case.

We are currently investigating the possibility of extending these results to super-conformal algebras as well [9], [10].
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Appendix A: $n$-dimensional case

$$[p_a, p_b] = 0, \quad [q_a, q_b] = 0,$$

(A.1)

$$[M_{ab}, p_c] = g_{ac}p_b - g_{bc}p_a,$$

(A.2)

$$[M_{ab}, q_c] = g_{ac}q_b - g_{bc}q_a,$$

(A.3)

$$[M_{ab}, M_{cd}] = g_{ac}M_{bd} - g_{bc}M_{ad} - g_{ad}M_{bc} + g_{bd}M_{ac},$$

(A.4)

$$[D, p_a] = -p_a, \quad [D, q_a] = q_a,$$

(A.5)

$$[D, D] = 0, \quad [D, M_{ab}] = 0,$$

(A.6)

$$[p_a, q_b] = M_{ab} + g_{ab}D,$$

(A.7)

$$p^2 x_a = \frac{1}{2} (M_{ab} p^b + p^b M_{ab}) = M_{ab} p^b + k p_a,$$

(A.8)

$$p^4 (x \cdot x) = (M_{ab} p^b)(M^c_p p^c) - \frac{(n - 1)^2}{4} p^2,$$

(A.9)

$$[p^2, q_a] = p_a (2D - 1) - 2p^2 x_a,$$

(A.10)

$$[M_{ab}, x_c] = g_{ac} x_b - g_{bc} x_a,$$

(A.11)

$$[p^2, x_a] = 0,$$

(A.12)

$$[x_a, x_b] = -M_{ab},$$

(A.13)

$$p^4 (x \cdot x) = (M_{ab} p^b)(M^c_p p^c) - \frac{(n - 1)^2}{4} p^2,$$

(A.14)

$$[x \cdot x, p_a] = 2x_a M_{ab},$$

(A.15)

$$[p^2, x_a] = 0,$$

(A.16)

$$[x_a, p_b] = g_{ab} - p^{-2} p_a p_b := h_{ab},$$

(A.17)

$$x \cdot p = -p \cdot x := k = \frac{n - 1}{2},$$

(A.18)

$$L_{ab} = x_a p_b - x_b p_a, \quad L_{ab} p^a p^b = 0,$$

(A.19)

$$[M_{ab}, L_{cd}] = g_{ac} L_{bd} - g_{bc} L_{ad} - g_{ad} L_{bc} + g_{bd} L_{ac},$$

(A.20)

$$[L_{ab}, p_c] = g_{ac} p_b - g_{bc} p_a,$$

(A.21)

$$S_{ab} = M_{ab} - L_{ab},$$

(A.22)

$$[M_{ab}, S_{cd}] = g_{ac} S_{bd} - g_{bc} S_{ad} - g_{ad} S_{bc} + g_{bd} S_{ac},$$

(A.23)

$$[L_{ab}, x_c] = g_{ac} x_b - g_{bc} x_a - p^{-2} (S_{ac} p_b - S_{bc} p_a),$$

(A.24)
\[ [S_{ab}, p_c] = 0, \quad S_{ab}p^b = 0, \quad [p^2, S_{ab}] = 0, \quad (A.25) \]
\[ p^2[S_{ab}, x_c] = S_{ac}p_b - S_{bc}p_a, \quad (A.26) \]
\[ [S_{ab}, S_{cd}] = h_{ac}S_{bd} - h_{bc}S_{ad} - h_{ad}S_{bc} + h_{bd}S_{ac}, \quad (A.27) \]
\[ [S_{ab}, x^b] = 0, \quad (A.28) \]
\[ p^2 y_a = p^2 x_a - p_a D, \quad (A.29) \]
\[ [p^2, y_a] = -2p_a, \quad p^2[y_a, y_b] = -S_{ab}, \quad (A.30) \]
\[ y_a p_b - y_b p_a = L_{ab}, \quad (A.31) \]
\[ [D, M_{ab}] = 0 = [D, S_{ab}] = [D, L_{ab}], \quad (A.32) \]
\[ [D, p^2] = -2p^2, \quad [D, p^{-2}] = 2p^{-2}, \quad (A.33) \]
\[ [D^2, p^2] = 4p^2(1 - D), \quad (A.34) \]
\[ [D, p^{-2}p_a] = p^{-2}p_a, \quad (A.35) \]
\[ [D^2, p_a] = p_a(-2D + 1), \quad (A.36) \]
\[ [D, y_a] = y_a, \quad [D, x_a] = x_a, \quad [D, Q_a] = Q_a, \quad (A.37) \]
\[ [D, x \cdot x] = 2(x \cdot x), \quad (A.38) \]
\[ f(D)x_a = x_a f(D + 1), \quad f(D)p_a = p_a f(D - 1), \quad (A.39) \]

for any polynomial function \( f \).

\[ Q_a = x^b S_{ab} - \frac{1}{2} (x \cdot x) p_a - x_a D + \frac{1}{2} p^{-2}p_a (D^2 + D - k^2), \quad (A.40) \]
\[ [p^2, Q_a] = [p^2, q_a], \quad [p_a, Q_a] = [p_a, q_a], \quad (A.41) \]
\[ [R_a, p_b] = 0, \quad (A.42) \]
\[ [D, R_a] = 0, \quad (A.43) \]
\[ [R_a, p^2] = 0, \quad (A.44) \]
\[ [M_{ab}, R_c] = g_{ac}R_b - g_{bc}R_a, \quad (A.45) \]
\[ p^2[x_a, R_b] = g_{ab}(R \cdot p) - R_a p_b, \quad (A.46) \]
\[ p^2[S_{ab}, R_c] = p^2 h_{ac}R_b - p^2 h_{bc}R_a - g_{ac}(R \cdot p)p_b + g_{bc}(R \cdot p)p_a, \quad (A.47) \]
\[2p^2[R_a, R_b] = ((n - 3)(n - 2) - 4R \cdot p)S_{ab} + 2S_{[b|d]S^{cd}S_{a|c}}
\]
\[+ (R^c S_{ac} + S_{ac}R^c) p_b - (R^c S_{bc} + S_{bc}R^c) p_a, \quad (A.48)\]
\[[S_a^c, S_{bc}] = (n - 3)S_{ab}, \quad (A.49)\]
\[[S_{ab}, x \cdot x] = 2x^c p_b S_{ac} - 2x^c p_a S_{bc} - 2S_{ab}, \quad (A.50)\]
\[R \cdot x = x \cdot R - p^{-2}(n - 1)(R \cdot p), \quad (A.51)\]
\[[x_a, R \cdot p] = 0, \quad (A.52)\]
\[[x \cdot R, p_a] = R_a - p^{-2}p_a(R \cdot p), \quad (A.53)\]
\[[S_{ac}, R^c] = (n - 2)[p^{-2}(R \cdot p)p_a - R_a], \quad (A.54)\]
\[[R \cdot p, R_a] = -\frac{1}{2}(R^b S_{ab} + S_{ab} R^b), \quad (A.55)\]
\[[S_{ab}, R \cdot p] = 0, \quad (A.56)\]
\[[S_{ac}S^{ac}, R_b] = 2(S_{bc}R^c + R^c S_{bc}), \quad (A.57)\]
\[[S_{ab}S^{ab}, S_{cd}] = 0, \quad (A.58)\]
\[S_a = R_b h_a^b, \quad S_a p^a = 0, \quad (A.59)\]
\[S = R \cdot p, \quad [S, S] = 0, \quad (A.60)\]
\[[S_{ab}, S] = 0, \quad [S_a, S] = S^b S_{ab} - \frac{n - 2}{2} S_a, \quad (A.61)\]
\[[S_{ab}, S_c] = h_{ac} S_b - h_{bc} S_a, \quad (A.62)\]
\[p^2[S_a, S_b] = \frac{(n - 3)(n - 2)}{2} S_{ab} + S_{[b|d]S^{cd}S_{a|c}} - 2SS_{ab}, \quad (A.63)\]
\[U = S_a S^a, \quad (A.64)\]
\[[U, S_{ab}] = 0 = [U, S] = 0, \quad (A.65)\]
\[p^2[U, S_a] = 2S_a^d S_d^e S_c^b S_b + (n(n - 1) + 4C_1 - S_{cd} S^{cd}) S_a^b S_b \]
\[+ (3n - 4)S_a^e S_c^b S_b + (n - 2) \left( n - 1 + 2C_1 - \frac{1}{2} S_{cd} S^{cd} \right) S_a \quad (A.66)\]
Appendix B: 3-dimensional case

\[ [D,p] = -p, \quad [D,J] = 0, \quad [D,q] = q, \] (B.1)

\[ [J \cdot a, p \cdot b] = -(a \times b) \cdot p, \] (B.2)

\[ [J \cdot a, q \cdot b] = -(a \times b) \cdot q, \] (B.3)

\[ [J \cdot a, J \cdot b] = -(a \times b) \cdot J, \] (B.4)

\[ [p \cdot a, p \cdot b] = 0, \quad [q \cdot a, q \cdot b] = 0, \] (B.5)

\[ [p \cdot a, q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \] (B.6)

\[ [p, \times q] := p \times q + q \times p, \] (B.7)

\[ (a \times b) \cdot (J \times p + p \times J) = -2(a \times b) \cdot p, \] (B.8)

\[ [J \times p] = -2p, \quad [J \times q] = -2q, \] (B.9)

\[ J \times J = -J, \] (B.10)

\[ [J, p \cdot p] = 0, \] (B.11)

\[ [J, J \cdot J] = 0, \] (B.12)

\[ [p, J \cdot J] = p \times J - J \times p = 2p \times J + 2p, \] (B.13)

\[ [q, J \cdot J] = q \times J - J \times q = 2q \times J + 2q, \] (B.14)

\[ [J, J \cdot J] = 0, \] (B.15)

\[ 2(p \cdot p)Q = 2(p \cdot J)J - p(J \cdot J) + pD(D + 3) + 2(p \times J)(D + 1), \] (B.16)

\[ [p \cdot a, Q \cdot b] = (a \cdot b)D - (a \times b) \cdot J, \] (B.17)

\[ [p \cdot p, Q] = p(2D + 1) + 2(p \times J) = [p \cdot p, q], \] (B.18)

\[ J \cdot p = p \cdot J, \quad J \cdot q = q \cdot J, \] (B.19)

\[ [J \cdot p, p] = 0, [J \cdot p, J] = 0, [J \cdot p, q] = J(D - 1) - q \times p, \] (B.20)

\[ [q, \times p] = [p, \times q] = -2J, \] (B.21)

\[ (p \times J) \times J = -p \times J + (p \cdot J)J - p(J \cdot J), \] (B.22)

\[ J \times (p \times J) = -p \times J - (p \cdot J)J + p(J \cdot J), \] (B.23)

\[ p \times (p \times J) = (p \cdot J)p - (p \cdot p)J, \] (B.24)

\[ (p \times J) \times p = -(p \cdot J)p + (p \cdot p)J, \] (B.25)
\[ [p \cdot J, J \cdot J] = 0, \]  
\[ p \cdot (p \times J) = 0, \]  
\[ (p \times J) \cdot p = -2p^2, \]  
\[ (p \times J) \cdot J = -p \cdot J, \]  
\[ J \cdot (p \times J) = -p \cdot J, \]  
\[ (p \times J) \times (p \times J) = (p \cdot p)J, \]  
\[ (p \times J) \cdot (p \times J) = (p \cdot p)(J \cdot J) - (p \cdot J)^2, \]  
\[ [p \cdot p, J \cdot J] = 0, \]  
\[ x_a p_b - x_b p_a = (a \times b) \cdot (x \times p), \]  
\[ [p_a, (p \times J)_b] = (a \cdot b)(p \cdot p) - p_a p_b, \]  
\[ [J \cdot q, p] = -(p \times q) - J(D + 1), \]  
\[ (p \times J) \times q = (p \cdot q)J - (q \cdot J)p - 2(p \times q) - J(D + 1), \]  
\[ p \times (q \times J) = (p \cdot J)q - (p \times q) - (p \cdot q)J, \]  
\[ (p \times q) \times J = (p \cdot J)q - (q \cdot J)p - 2(p \times q) - J(D + 1), \]  
\[ [J \cdot q, p] = -(p \times q) - J(D + 1), \]  
\[ (p \cdot p)^2(Q \times Q) = 2(p \cdot J)p, \]  
\[ R = q - Q, \]  
\[ [p \cdot p, R] = 0, \]  
\[ [p \cdot a, R \cdot b] = 0, \]  
\[ [D, R] = R, \]  
\[ [J \cdot a, R \cdot b] = -(a \times b) \cdot R, \]  
\[ (p \cdot p)^2(R \times R) = 2(p \cdot p)(p \cdot J)R - (p \cdot p)(p \times R) - 2(p \cdot p)(p \cdot J)p, \]  
\[ (p \times R) \times p = (p \cdot p)R - (p \cdot R)p, \]  
\[ p \times (p \times R) = (p \cdot R)p - (p \cdot p)R, \]  
\[ [p \cdot J, R] = p \times R, \]  
\[ [(p \cdot J)^2, R] = 2(p \cdot J)(p \times R) + (p \cdot p)R - (R \cdot p)p, \]  
\[ (p \times R) \times R = R(p \cdot R) - (R \cdot R)p, \]  
\[ R \times (p \times R) = (R \cdot R)p - (p \cdot R)R, \]  
\[ 28 \]
\[ [R \cdot p, p] = 0, \quad [R \cdot p, J] = 0, \]  
\[ (p \cdot p)[R \cdot p, R] = -2(p \cdot J)(p \times R) + (R \cdot p)p - (p \cdot p)R, \]  
\[ [R \cdot R, p] = 0, \quad [R \cdot R, J] = 0, \]  
\[ S = R - \frac{1}{2} \frac{p}{p \cdot p}, \]  
\[ S \cdot p = R \cdot p - \frac{1}{2}, \]  
\[ (p \cdot p)(S \cdot S) = (p \cdot p)(R \cdot R) - (R \cdot p) + \frac{1}{4}. \]  

(B.53)  
(B.54)  
(B.55)  
(B.56)  
(B.57)  
(B.58)
Appendix C: 2-dimensional case

\[
M_{ab} = J \varepsilon_{ab}, \quad (C.1)
\]

\[
[p_a, p_b] = 0, \quad [q_a, q_b] = 0, \quad (C.2)
\]

\[
[J, p_c] = \varepsilon_c^b p_b, \quad [J, q_c] = \varepsilon_c^b q_b, \quad (C.3)
\]

\[
[D, p_a] = -p_a, \quad [D, q_a] = q_a, \quad (C.4)
\]

\[
[J, J] = 0 = [D, D] = [D, J], \quad (C.5)
\]

\[
[p_a, q_b] = \varepsilon_{ab} J + g_{ab} D, \quad (C.6)
\]

\[
[M_{ab}, M_{cd}] = 0, \quad (C.7)
\]

\[
p^2 x_a = \varepsilon_{ab} J p^b + \frac{1}{2} p_a, \quad (C.8)
\]

\[
x \cdot p = -p \cdot x = \frac{1}{2}, \quad (C.9)
\]

\[
p^2 (x \cdot x) = J^2 - \frac{1}{4}, \quad (C.10)
\]

\[
[J^2, p_a] = -p_a + 2 \varepsilon_a^b p_b J, \quad (C.11)
\]

\[
L_{ab} = M_{ab}, \quad S_{ab} = 0, \quad (C.12)
\]

\[
p^2 Q_a = -\frac{1}{2} J^2 p_a + \frac{1}{2} p_a D^2 - \varepsilon_{ab} J p^b D, \quad (C.13)
\]

\[
[p_a, Q_b] = \varepsilon_{ab} J + g_{ab} D = [p_a, q_b], \quad (C.14)
\]

\[
p^4 [Q_a, Q_b] = 0, \quad (C.15)
\]

\[
R_a = q_a - Q_a, \quad (C.16)
\]

\[
[R_a, p_b] = 0, \quad [R_a, R_b] = 0, \quad [R_a, J] = 0, \quad (C.17)
\]

\[
[D, R_a] = R_a, \quad [p^2, R_a] = 0, \quad (C.18)
\]

\[
[R \cdot p, p_a] = 0 = [R \cdot p, J] = [R \cdot p, R_a], \quad (C.19)
\]

\[
[R \cdot R, p_a] = 0 = [R \cdot R, J] = [R \cdot R, R_a] = 0, \quad (C.20)
\]