Eigenvalue decay of positive integral operators on compact two-point homogeneous spaces

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Abstract
We obtain decay rates for singular values and eigenvalues of integral operators generated by square integrable kernels on two-point homogeneous spaces in \( \mathbb{R}^{m+1}, m \geq 2 \), under assumptions on both, certain derivatives of the kernel and the integral operators generated by such derivatives. The assumptions on the kernel are all defined via the Laplace-Beltrami operator and the rates we present depend on both, the order of the Laplace-Beltrami operator used to define the smoothness conditions and the dimension \( m \).

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Abbreviated title: Eigenvalues of integral operators on homogeneous spaces.

1 Introduction
Let \( m \geq 1 \) be an integer and \( \mathbb{M} \) be a compact two-point homogeneous space of dimension \( m \). Such space is both a Riemannian \( m \)-manifold and a compact symmetric space of rank 1. A complete classification of this type of space is well-known. Summarizing, they are: the unit spheres \( \mathbb{S}^m, m = 1, 2, \ldots \); the real, complex, and quaternion projective spaces, \( \mathbb{P}^m(\mathbb{R}) (m = 2, 3, \ldots), \mathbb{P}^m(\mathbb{C}) (m = 4, 6, \ldots), \) and \( \mathbb{P}^m(\mathbb{H}) (m = 8, 12, 16, \ldots) \), respectively; and Cayley’s elliptic plane \( \mathbb{P}^{16} \) of dimension 16 (see more information in [1, 18, 21]).
In this paper, we will always consider $m \geq 2$. Let $dx$ be the usual volume element on $M$ and $L^2(M)$ the Hilbert space of all square-integrable complex functions on $M$ endowed with the inner product

$$\langle f, g \rangle_2 := \frac{1}{\sigma} \int_M f(x)\overline{g(x)} \, dx, \quad f, g \in L^2(M),$$

and the derived norm $|| \cdot ||_2$, the normalization constant being defined by $\sigma := \int_M dx$.

We will deal with integral operators defined by

$$K(f) = \int_M K(\cdot, y)f(y) \, dy,$$

in which the generating kernel $K: M \times M \to \mathbb{C}$ is an element of $L^2(M \times M)$. In this case, (1.1) defines a compact operator on $L^2(M)$.

If $K$ is positive definite in the sense that

$$\int_M \int_M K(x, y)f(x)\overline{f(y)} \, dx \, dy \geq 0, \quad f \in L^2(M),$$

then $K$ becomes a self-adjoint operator and the standard spectral theorem for compact and self-adjoint operators is applicable and we can write

$$K(f) = \sum_{n=0}^{\infty} \lambda_n(K) \langle f, f_n \rangle_2 f_n, \quad f \in L^2(M),$$

in which $\{\lambda_n(K)\}$ is a sequence of nonnegative reals (possibly finite) decreasing to 0 and $\{f_n\}$ is an $\langle \cdot, \cdot \rangle_2$-orthonormal basis of $L^2(M)$. The numbers $\lambda_n(K)$ are the eigenvalues of $K$ and the sequence $\{\lambda_n(K)\}$ takes into account possible repetitions implied by the algebraic multiplicity of each eigenvalue. The positive definiteness of $K$ means nothing but the positivity of the integral operator $K$. Since it relates to the inner product above, it is a common sense to call it $L^2$-positive definiteness.

We observe that the addition of continuity to $K$ implies that $K$ is also trace-class (nuclear) ([3, 4, 5]), that is,

$$\sum_{f \in B} \langle K^*K(f), f \rangle_2^{1/2} < \infty,$$

whenever $B$ is an orthonormal basis of $L^2(M)$. In particular,

$$\sum_{n=1}^{\infty} \lambda_n(K) = \int_M K(x, x) \, dx < \infty,$$

and we can extract the most elementary result on decay rates for the eigenvalues of such operators, namely,

$$\lambda_n(K) = o(n^{-1}).$$
If the integral operator \( K \) is compact but not self-adjoint then decay rates for the singular values of the operator becomes the focus. If \( T \) is a compact operator on \( L^2(\mathcal{M}) \), its eigenvalues can be ordered as \( |\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0, \) counting multiplicities ([8]). The singular values of \( T \) are, by definition, the eigenvalues of the compact, positive and self-adjoint operator \( |T| := (T^*T)^{1/2} \). The sequence \( \{s_n(T)\} \) of singular values of \( T \) can also be ordered in a decreasing manner, with repetitions being included according to their multiplicities as eigenvalues of \( |T| \). That being the case, the classical Weyl’s inequality ([5, p.52])

\[
\Pi_{j=1}^n |\lambda_j(T)| \leq \Pi_{j=1}^n s_j(T), \quad n = 1, 2, \ldots ,
\]

provides the convenient bridge between eigenvalues and singular values. We remark that the inequality characterizing the traceability of a compact non self-adjoint operator \( T \) on \( L^2(\mathcal{M}) \) reduces itself to

\[
\sum_{n=1}^{\infty} s_n(T) < \infty
\]

and the elementary decay presented before becomes \( s_n(K) = o(n^{-1}) \). Classical references on eigenvalues and singular values distribution of compact operators on Banach spaces are [8, 17].

The idea of nuclearity can be extended as follows. For \( p > 0 \) we say that a compact operator \( T \) belongs to the Schatten \( p \)-class \( \mathcal{S}_p \) if

\[
\sum_{n=1}^{\infty} (s_n(T))^p < \infty.
\]

(1.2)

For \( p \geq 1 \), \( \mathcal{S}_p \) is a Banach space endowed with the norm

\[
\|T\|_p := \left( \sum_{n=1}^{\infty} (s_n(T))^p \right)^{1/p}.
\]

In particular, \( \mathcal{S}_2 \) coincides to the space of Hilbert-Schmidt operators. Of course, if \( T \in \mathcal{S}_p \) then its singular values satisfy \( s_n(K) = o(n^{-1/p}) \).

The object of study in this paper is the analysis of decay rates for the sequence \( \{\lambda_n(K)\} \) (depending on the case, the sequence \( \{s_n(K)\} \)) under additional assumptions on the kernel \( K \). Results of this very same nature can be found in [2] and references therein. In [2] we used the Laplace-Beltrami derivative to define a notion of smoothness to the kernel \( K \) and obtained some results about the asymptotic behavior of the eigenvalues or singular values of \( K \). The intention here is to invest in the very same question analyzed in [2]. However, in this more general setting we do not have the notion of devitative of the sphere. Then we use the Laplace-Beltrami operator to define the basic assumptions needed. The theory of Laplace-Beltrami derivative is fully discussed in the survey-like paper [15].

Two-point homogeneous spaces seem to be introduced in [21] in the early 1950’s. In 1990’s, russian mathematicians started studying approximation problems in these spaces.
and generalized some spherical problems ([13]). More recently, the interest in this field has increased and more papers about approximation theory in homogeneous spaces have appeared, for instance [11] [10] [11] [18] [19] [20].

The presentation of the paper is as follows. Section 2 contains basic material about harmonic analysis in two-point homogeneous spaces and the description of the main results of the paper. In Section 3, we state and prove some technical results to be used in Section 4, where we present the proofs for the main results along with other pertinent information.

2 Statement of the results

Each two-point homogeneous space $\mathbb{M}$ has an invariant Riemannian metric $d(\cdot, \cdot)$ and admits essentially one invariant second order differential operator, the Laplace-Beltrami operator $\Delta$.

The geometry of these spaces is similar in many aspects. For instance, all geodesics in one of them are closed and have the same length $2L$, in which $L = \max\{d(x, y) : x, y \in \mathbb{M}\}$ is the diameter of $\mathbb{G}/\mathbb{H}$. A function on $\mathbb{G}/\mathbb{H}$ is invariant under the left action of $\mathbb{H}$ on $\mathbb{G}/\mathbb{H}$ if and only if it depends only on the distance of its argument from the pole of $\mathbb{M}$. Let $\theta$ be the distance of a point from the pole. We can choose a geodesic polar coordinate system $(\theta, u)$, where $u$ is an angular parameter, in which the radial part of $\Delta$ can be written, up to a multiplicative constant, as

$$\Delta_\theta = \frac{1}{(\sin \lambda \theta)^\sigma (\sin 2\lambda \theta)^\rho} \frac{d}{d\theta} (\sin \lambda \theta)^\sigma (\sin 2\lambda \theta)^\rho \frac{d}{d\theta},$$

where

| Space       | $\sigma$ | $\rho$ | $\lambda$ | $m$     |
|-------------|----------|--------|-----------|---------|
| $S^m$       | 0        | $m - 1$| $\pi/2L$ | $m = 1, 2, 3, \ldots$ |
| $\mathbb{P}^n(\mathbb{R})$ | 0        | $m - 1$| $\pi/4L$ | $m = 2, 3, 4, \ldots$ |
| $\mathbb{P}^n(\mathbb{C})$ | $m - 2$ | 1      | $\pi/2L$ | $m = 4, 6, 8, \ldots$ |
| $\mathbb{P}^n(\mathbb{H})$ | $m - 4$ | 3      | $\pi/2L$ | $m = 8, 12, \ldots$ |
| $\mathbb{P}^{10}(\text{Cay})$ | 8        | 7      | $\pi/2L$ | $m = 16$ |

Furthermore, the change of variables $x = \cos 2\lambda \theta$ gives us

$$\Delta_x = (1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d}{dx} (1 - x)^{1+\alpha} (1 + x)^{1+\beta} \frac{d}{dx},$$

with $\alpha = (\sigma + \rho - 1)/2 = (d - 2)/2$ and $\beta = (\rho - 1)/2$.

We will write $\mathcal{B} = -\Delta_x$ and also call it Laplace-Beltrami operator on $\mathbb{M}$. Let denote $\mathcal{B}^r$ the $r$-th power of $\mathcal{B}$, $r = 0, 1, 2, \ldots$. The Sobolev space of order $r$ constructed from $\mathcal{B}$ is defined as in [12] p.37 and [18] by

$$W^r_2(\mathbb{M}) := \{ f \in L^2(\mathbb{M}) : \mathcal{B}^j f \in L^2(\mathbb{M}), j = 1, 2, \ldots, r \}.$$
The action of the Laplace-Beltrami operator on kernels is done separately: we keep one variable fixed and differentiate with respect to the other. The symbol $B^r_y K$ will indicate the $r$-th order of $B$ acting on the kernel $K$ with respect to the second variable $y$ (we will never differentiate with respect to the first variable $x$). For $r \in \mathbb{Z}_+$, we find convenient to introduce the following notation

$$K_{0,r}(x, y) := B^r_y K(x, y), \quad x, y \in M,$$

to abandon the operator symbol. The integral operator associated with $K_{0,r}$ will be written as $K_{0,r}$.

We are ready to describe the main results of the paper. We emphasize that all the results take for granted the ordering on either the eigenvalues or singular values mentioned before. At first, we will prove a theorem without the $L^2$-positive definiteness assumption on $K$ and obtain a decay rate for the sequence of singular values of $K$.

**Theorem 2.1** Let $r$ be a positive integer at least $(m + 1)/2$, $K \in L^2(M \times M)$ a kernel satisfying $K(x, \cdot) \in W^r_2(M)$, $x \in M$ a.e., and $p \in (m + 1, 2r + 1]$. If $K_{0,r}$ is bounded then

$$s_n(K) = o(n^{-1-(2r+1-p)/m}).$$

Clearly, the smaller the parameter $p$, the better the estimate. We observe that the fact that $B^r_y K(x, \cdot)$ exists for $x \in M$ a.e. does not imply that $K_{0,r}$ is a bounded operator. As so, the assumption on $K_{0,r}$ in Theorem 2.1 is reasonable. The next two results incorporate $L^2$-positive definiteness as an assumption. As so, they can describe decay rates for the eigenvalues of $K$ under certain hypotheses on either $K_{0,r}$ or $K_{0,r}$.

**Theorem 2.2** Let $K \in L^2(M \times M)$ be a $L^2$-positive definite kernel satisfying $K(x, \cdot) \in W^r_2(M)$, $x \in M$ a.e. If $K_{0,r}$ belongs to $L^2(M \times M)$ then

$$\lambda_n(K) = o(n^{-1/2-2r/m}).$$

If we replace the basic assumption in Theorem 2.2 by $K_{0,r} \in S_p$ then we can obtain an improvement on the previous decay rate.

**Theorem 2.3** Let $K \in L^2(M \times M)$ be a $L^2$-positive definite kernel satisfying $K(x, \cdot) \in W^r_2(M)$, $x \in M$ a.e. If $K_{0,r} \in S_p$ then

$$\lambda_n(K) = o(n^{(-1/p)-(2r/m)}).$$
3 Auxiliary results

The Hilbert space $L^2(M)$ can be decomposed as $L^2(M) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^m$, where $\mathcal{H}_n^m$ is the eigenspace of $\mathcal{B}$ with respect to the eigenvalue $\lambda_n(\mathcal{B}) := n(n+\alpha+\beta+1)$ ($\lambda_0(\mathcal{B}) = 1$). The elements of $\mathcal{H}_n^m$ are the well-known Jacobi polynomials $P_n^{(\alpha,\beta)}(x,y)$. Moreover, the Laplace-Beltrami operator and its powers satisfy

$$\langle \mathcal{B}^r f, g \rangle_2 = \langle f, \mathcal{B}^r g \rangle_2, \quad f, g \in W^r_2(M).$$

(3.3)

Each $\mathcal{H}_n^m$ is in $W^r_2(M)$ and so we have $W^r_2(M) = L^2(M)$. Thus, classical operator theory guarantees that there is a unique compact operator $J^r: L^2(M) \to L^2(M)$ satisfying

$$\mathcal{B}^r J^r f = J^r \mathcal{B}^r f = f, \quad f \in \bigcup_{n=1}^{\infty} \mathcal{H}_n^m,$$

(3.4)

with singular values given by

$$s_n(J^r) = \lambda_n(\mathcal{B}^r)^{-1} = n^{-r}(n+\alpha+\beta+1)^{-r}$$

(3.5)

and ordered in accordance with the spectral theorem for compact operators.

Each $\mathcal{H}_n^m$ has a finite dimension given by the formula

$$d_n^m = \frac{\Gamma(\beta+1)(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+1)}$$

(3.6)

for all $n \in \mathbb{N}$, if $M \in \{S^m, \mathbb{P}^m(\mathbb{C}), \mathbb{P}^m(\mathbb{H}), \mathbb{P}^{16}(\text{Cay})\}$. If $M = \mathbb{P}^m(\mathbb{R})$ then $\dim \mathcal{H}_n^m = d_n^m$ for $n \in \mathbb{N}$ even, and $\dim \mathcal{H}_n^m = 0$ for $n \in \mathbb{N}$ odd. Let denote $T_n^m = \bigoplus_{k=0}^{n} \mathcal{H}_k^m$ and $\tau_n^m = \dim T_n^m$. Precisely, for $S^m, \mathbb{P}^m(\mathbb{C}), \mathbb{P}^m(\mathbb{H}),$ and $\mathbb{P}^{16}(\text{Cay})$ we have the explicit expression

$$\tau_n^m = \frac{\Gamma(\beta+1)\Gamma(n+\alpha+\beta+2)\Gamma(n+\alpha+2)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha+2)\Gamma(n+\beta+1)\Gamma(n+1)}, \quad n = 0, 1, 2, \ldots.$$  

(3.7)

As for the case $M = \mathbb{P}^m(\mathbb{R})$ its not hard to see that $d_0^m = 1$ and

$$d_{2n}^m = \left( \begin{array}{c} m + 2n \\ m \end{array} \right) - \left( \begin{array}{c} m + 2(n-1) \\ m \end{array} \right), \quad n = 1, 2, \ldots,$$

(3.8)

and then

$$\tau_{2n}^m = \dim T_{2n}^m = \sum_{k=0}^{2n} d_k^m = \sum_{k=0}^{n} d_{2k}^m = \left( \begin{array}{c} m + 2n \\ m \end{array} \right), \quad n = 0, 1, 2, \ldots.$$  

(3.9)

It follows from (3.1) that

$$d_n^m = O(n^{m-1}), \quad \text{as } n \to \infty.$$  

(3.10)

$$\tau_n^m = O(n^m), \quad \text{as } n \to \infty.$$  

(3.11)
As so, if \( M \in \{ S^m, \mathbb{P}^m(\mathbb{C}), \mathbb{P}^m(\mathbb{H}), \mathbb{P}^{16}(\text{Cay}) \} \) we may think the sequence \( \{ s_n(J') \} \) is block ordered in such a way that the first block contains the singular value \( s_0(J') = 1 \) and the \((n + 1)\)-th block \((n \geq 1)\) contains \( d_n^m \) entries equal to \( n^{-\tau}(n + \alpha + \beta + 1)^{-\tau} \). For future reference, we notice that the first entry in the \((n + 1)\)-th block corresponds to the index

\[
d_0^m + d_1^m + \cdots + d_{n-1}^m + 1 = \tau_{n-1}^m + 1.
\]

As for the last one, it corresponds to

\[
d_0^m + d_1^m + \cdots + d_{n-1}^m + d_n^m = \tau_n^m.
\]

On the other hand, if \( M = \mathbb{P}^m(\mathbb{R}) \) then the sequence \( \{ s_n(J') \} \) is block ordered in such a way the first block contains the singular value \( s_0(J') = 1 \) and the \((n + 1)\)-th block \((n \geq 1)\) contains \( d_{2n}^m \) entries equal to \((2n)^{-\tau}(2n + \alpha + \beta + 1)^{-\tau} \). In this case, the first entry in the \((n + 1)\)-th block corresponds to the index

\[
d_0^m + d_2^m + \cdots + d_{2(n-1)}^m + 1 = \tau_{2(n-1)}^m + 1,
\]

and the last one corresponds to

\[
d_0^m + d_2^m + \cdots + d_{2(n-1)}^m + d_{2n}^m = \tau_{2n}^m.
\]

In the next three lemmas, we detach technical inequalities to be used in the proofs ahead. The first one shows the asymptotic behavior of the sequence \( \{ d_n^m \} \) while second and third ones include refinements to equation (3.11).

**Lemma 3.1** If \( M \in \{ S^m, \mathbb{P}^m(\mathbb{C}), \mathbb{P}^m(\mathbb{H}), \mathbb{P}^{16}(\text{Cay}) \} \) then there exists an integer \( \delta(m) \geq 1 \) such that

\[
\tau_n^m \leq 2n^m, \quad n \geq \delta(m).
\]

**Proof.** We keep equation (3.11) in mind and develop each case. For \( M = S^m \) we know \( \alpha = \beta = (m - 2)/2 \). Consequently there is \( \delta(S^m) \geq 0 \) such that

\[
\tau_n^m = \frac{2n^m}{m!} \left( 1 + \frac{c_1^{(1)}}{n} + \frac{c_2^{(1)}}{n^2} + \cdots + \frac{c_m^{(1)}}{n^m} \right) \leq 2n^m, \quad n \geq \delta(S^m), \tag{3.12}
\]

where \( c_1^{(1)}, \ldots, c_m^{(1)} \) do not depend upon \( n \).

For \( M = \mathbb{P}^m(\mathbb{C}) \) we know \( \alpha = (m - 2)/2 \) and \( \beta = 0 \). As so, there is \( \delta(\mathbb{P}^m(\mathbb{C})) \geq 0 \) such that

\[
\tau_n^m = \left[ \frac{n^{m/2}}{(m/2)!} \left( 1 + \frac{c_1^{(2)}}{n} + \cdots + \frac{c_m^{(2)}}{n^{m/2}} \right) \right]^2 \leq 2n^m, \quad n \geq \delta(\mathbb{P}^m(\mathbb{C})), \tag{3.13}
\]

in which \( c_1^{(2)}, \ldots, c_m^{(2)} \) do not depend upon \( n \).
If $M = \mathbb{P}^m(\mathbb{H})$ then $\alpha = (m - 2)/2$ and $\beta = 1$. Thus, there is $\delta(\mathbb{P}^m(\mathbb{H})) > 0$ such that
\[
\tau_n^m = \frac{(n + 1 + m/2)}{(n + 1)(1 + m/2)} \left( \frac{(n + m/2)!}{n!(m/2)!} \right)^2 \leq 2n^m, \quad n \geq \delta(\mathbb{P}^m(\mathbb{H})). \quad (3.14)
\]
If $M = \mathbb{P}^{16}(Cay)$ then $\alpha = (m - 2)/2$, $\beta = 3$, and there is $\delta(\mathbb{P}^{16}(Cay)) > 0$ such that
\[
\tau_n^{16} = \frac{(n + 12)(n + 11)(n + 10)(n + 9)}{1980(n + 4)(n + 3)(n + 2)(n + 1)} \left( \frac{(n + 8)!}{(n)!8!} \right)^2 \leq 2n^{16}, \quad (3.15)
\]
since $n \geq \delta(\mathbb{P}^{16}(Cay))$.

To conclude, we define $\delta(m) = \max\{\delta(\mathbb{S}^m), \delta(\mathbb{P}^m(\mathbb{C})), \delta(\mathbb{P}^m(\mathbb{H})), \delta(\mathbb{P}^{16}(Cay))\}$.

**Lemma 3.2** If $M = \mathbb{P}^m(\mathbb{R})$ then there exists an integer $\delta(m) \geq 1$ such that
\[
2\tau_{2n}^m + 1 \leq (3n)^m, \quad n \geq \delta(m).
\]

**Proof.** We start using (3.9) to write
\[
\tau_{2n}^m = \frac{(2n)^m}{m!} \left( 1 + \frac{c_1^{(3)}}{n} + \frac{c_2^{(3)}}{n^2} + \cdots + \frac{c_m^{(3)}}{n^m} \right), \quad n = 1, 2, \ldots, \quad (3.16)
\]
where $c_1^{(3)}, \ldots, c_m^{(3)}$ do not depend upon $n$. Then, we take an integer $\delta(m) > 1$ and obtain
\[
\tau_{2n}^m \leq 2n^m \left( 1 + \frac{1}{2} \right) = 3n^m, \quad n \geq \delta(m). \quad (3.17)
\]
Now, it is not hard to see that
\[
2\tau_{2n}^m + 1 \leq 6n^m + 1 \leq (3n)^m, \quad n \geq \delta(m), \quad (3.18)
\]
and the proof is completed.

**Lemma 3.3** If $m$ is an integer at least 2 then
\[
(n + 1)^m - (n^m + 1) + 1 \leq m2^{m-1}n^{m-1}, \quad n \geq 1.
\]

**Proof.** It suffices to apply the mean value theorem to the function $x^m$ on the interval $[n, n + 1]$ and estimate the resulting formula conveniently.
4 Proofs of the main results

This section contains proofs for Theorems 2.1-2.3. They depend upon some general properties of compact operators and their singular values which we now describe in a form adapted to our needs. They can be found in standard references on operator theory such as [4, 5, 8, 17] and depend on the ordering of eigenvalues and singular values as previously mentioned.

**Lemma 4.1** Let $T$ be a compact operator on $L^2(\mathbb{M})$. The following assertions hold:

(i) If $T$ is self-adjoint then

$$s_n(T) = |\lambda_n(T)|, \quad n = 1, 2, \ldots;$$

(ii) If $A$ is a bounded operator on $L^2(\mathbb{M})$ then both, $AT$ and $TA$, are compact. In addition,

$$\max\{s_n(AT), s_n(TA)\} \leq \|A\| s_n(T), \quad n = 1, 2, \ldots;$$

(iii) If $A$ is a linear operator on $L^2(\mathbb{M})$ of rank at most $l$, then

$$s_{n+l}(T) \leq s_n(T + A), \quad n = 1, 2, \ldots;$$

(iv) If $A$ is a compact operator on $L^2(\mathbb{M})$ then

$$s_{n+k-1}(AT) \leq s_n(A)s_k(T), \quad n, k = 1, 2, \ldots.$$

The following additional lemma regarding the singular values of an integral operator generated by a square-integrable kernel is proved in [8, p.40].

**Lemma 4.2** If $K \in L^2(\mathbb{M} \times \mathbb{M})$ then

$$\sum_{n=1}^{\infty} s_n^2(K) = \|K\|_2^2.$$

The key idea behind the proof of the main results previously stated resides in the following estimation for the singular values of $K$, which holds when $K$ is smooth enough.

**Lemma 4.3** Let $K$ be an element of $W^r_2(\mathbb{M})$. If $K_{0,r}$ is bounded then

$$s_{n+1}(K) \leq s_n(K_{0,r}J^r), \quad n = 1, 2, \ldots.$$

**Proof.** Consider the orthogonal projection $Q$ of $L^2(\mathbb{M})$ onto $\oplus_{\ell=1}^{\infty} \mathcal{H}_{\ell}^{m+1}$. Since $I - Q$ is a projection onto the orthogonal complement of $\oplus_{\ell=1}^{\infty} \mathcal{H}_{\ell}^{m+1}$ then $K - KQ$ is an operator on $L^2(\mathbb{M})$ of rank at most 1. Using Lemma 4.1(iii), we may deduce that

$$s_{n+1}(K) \leq s_n(K - K(I - Q)) = s_n(KQ), \quad n = 1, 2, \ldots \quad (4.19)$$
To proceed, we need a convenient decomposition for $KQ$. Looking at the action of $KQ$ on a generic element $f$ from $L^2(M)$ and using (3.4) we see that

$$KQ(f) = \int_M K(\cdot, y)Qf(y) \, d\sigma_m(y) = \int_M K(\cdot, y)D^r J^r Qf(y) \, d\sigma_m(y).$$

Since $K \in W_2^r(M)$, we employ (3.3) to obtain

$$KQ(f) = \int_M K_0(\cdot, y)J^r(Qf)(y) \, d\sigma_m(y) = K_0 J^r Q(f),$$

that is, $KQ = K_0 J^r Q$. Now, assuming $K_0$ is bounded, we can apply (4.19) and Lemma 4.1-(ii) to see that

$$s_{n+1}(K) \leq s_n(KQ) \leq \|Q\| s_n(K_0 J^r), \quad n = 1, 2, \ldots.$$

The proof is complete.

The following technical result is borrowed from [9].

**Lemma 4.4** Let $\{a_n\}$ be a decreasing sequence of positive real numbers. If the series

$$\sum_{n=1}^{\infty} n^\alpha a_n^\beta$$

is convergent for some positive constants $\alpha$ and $\beta$ then $a_n = o(n^{-(\alpha+1)/\beta}).$

We now proceed to the proofs of the main results in the paper.

**Proof.** [Proof of Theorem 2.1] We assume $K_0$ is bounded and show that

$$\sum_{n=1}^{\infty} n^{(2r+1-p)/m} s_n(K) < \infty. \quad (4.20)$$

Lemma 4.4 takes care of the rest. In the first half of the proof we intend to derive the convergence of the series

$$\sum_{n=1}^{\infty} n^{2r+m-p} s_n m(K). \quad (4.21)$$

An application of Lemma 4.3-(ii) in the inequality provided by Lemma 4.3 leads to

$$s_{n+1}(K) \leq \|K_0\| s_n(J^r), \quad n = 1, 2, \ldots.$$ 

Keeping in mind the information provided at the end of the previous section, we can deduce that

$$n^r(n + \alpha + \beta + 1)^r \sum_{k=\tau_{n-1}+1}^{\tau_n} s_{k+1}(K) \leq \|K_0\| d^m_n, \quad n = 1, 2, \ldots.$$
Using the fact that the sequence \( \{s_n(K)\} \) is decreasing, we can estimate in the previous inequality to reach

\[
n^r(n + \alpha + \beta + 1)r s_{\tau_n+1}(K) \leq \|K_{0,r}\|, \quad n = 1, 2, \ldots.
\] (4.22)

Invoking Lemma 3.1 to select \( \delta = \delta(m) \geq 1 \) so that

\[
\tau_n^m + 1 \leq 2n^m + 1 \leq (2n)^m, \quad n \geq \delta,
\]

the previous inequality can be reduced to

\[
n^r(n + \alpha + \beta + 1)r s_{(2n)^m}(K) \leq \|K_{0,r}\|, \quad n \geq \delta.
\] (4.23)

Clearly \( n^{2r} \leq n^r(n + \alpha + \beta + 1)^r \) and we can write

\[
n^{2r} s_{(2n)^m}(K) \leq c\|K_{0,r}\|, \quad n \geq \delta.
\]

It is now clear that

\[
\sum_{n \geq \delta} n^{2r+m-p} s_{(2n)^m}(K) \leq c\|K_{0,r}\| \sum_{n \geq \delta} n^{m-p} < \infty,
\]
due to the fact that \( p - m > 1 \). Consequently,

\[
\sum_{n \geq \delta} (2n)^{2r+m-p} s_{(2n)^m}(K) \leq 2^{2r+m} \sum_{n \geq \delta} n^{2r+m-p} s_{(2n)^m}(K) < \infty
\]

and

\[
\sum_{n \geq \delta} (2n + 1)^{2r+m-p} s_{(2n+1)^m}(K) \leq 4^{2r+m} \sum_{n \geq \delta} n^{2r+m-p} s_{(2n)^m}(K) < \infty.
\]

The convergence of the series in (4.21) follows. To close the proof, we will use this convergence to show that

\[
\sum_{n=1}^{\infty} n^{(2r+1-p)/m} s_n(K) = \sum_{n=1}^{\infty} \sum_{k=0}^{(n+1)^m-(n^m+1)} (n^m + k)^{(2r+1-p)/m} s_{n^m+k}(K)
\]

converges. Call the inner sum in the double sum above \( S(n) \) and observe that

\[
S(n) \leq [(2n)^m]^{(2r+1-p)/m} \sum_{k=0}^{(n+1)^m-(n^m+1)} s_{n^m+k}(K).
\] (4.24)

The sequence \( \{s_n(K)\} \) being decreasing, (4.24) can be reduced to

\[
S(n) \leq (2n)^{2r+1-p} s_{n^m}(K)[(n + 1)^m - (n^m + 1) + 1].
\]
Invoking Lemma 3.3, we now see that
\[ S(n) \leq 4^r m 2^{m-1} n^{2r+m-p} s_{nm}(K). \]

It follows that
\[ \sum_{n=1}^{\infty} S(n) \leq 4^r m 2^{m-1} \sum_{n=1}^{\infty} n^{2r+m-p} s_{nm}(K) < \infty. \]

The proof is complete. \(\blacksquare\)

**Proof.** [Proof of Theorem 2.2] We proceed as in the proof of Theorem 2.1. We assume \( K_{0,r} \in L^2(M \times M) \) and show that
\[ \sum_{n=1}^{\infty} n^{4r/m} \lambda_n^2(K) < \infty. \]

Combining Lemma 4.3 with Lemma 4.1-(iv) we can deduce the inequalities
\[ s_{n+k}(K) \leq s_{n+k-1}(K_{0,r} J^n) \leq s_k(K_{0,r}) s_n(J^n), \quad n, k = 1, 2, \ldots, \]
while the setting in Theorem 2.2 allows us to write
\[ \lambda_{n+k}(K) \leq s_k(K_{0,r}) s_n(J^n), \quad n, k = 1, 2, \ldots. \]

Next, we square both sides of (4.25) and sum in \( k \), letting \( k \) run inside the \((n+1)\)-th block of the sequence of the singular values of \( J^n \):
\[ n^{2r}(n + \alpha + \beta + 1)^{2r} \sum_{k=n_{n-1}+1}^{\tau_n^m} \lambda_{t_n+k}^2(K) \leq \sum_{k=n_{n-1}+1}^{\tau_n^m} s_k^2(K_{0,r}). \]

Estimating on the left-hand side leads to
\[ n^{4r} \sum_{k=n_{n-1}+1}^{\tau_n^m} \lambda_{t_n+k}^2(K) \leq \sum_{k=n_{n-1}+1}^{\tau_n^m} s_k^2(K_{0,r}), \quad n = 1, 2, \ldots. \]

Due to Lemma 4.2 it is now clear that
\[ \sum_{n=1}^{\infty} n^{4r} \sum_{k=n_{n-1}+1}^{\tau_n^m} \lambda_{t_n+k}^2(K) \leq \|K_{0,r}\|^2_2 < \infty. \]

To proceed, we apply Lemma 3.1 to select a constant \( \delta = \delta(m) \geq 1 \) so that
\[ 2 \tau_{n_k}^m \leq 2^2 n^m \leq (2n)^m, \quad n \geq \delta. \]
Since equation 3.10 gives us $C > 0$ such that
\[ n^{m-1} \leq C d_n^m, \quad n \geq \beta = \beta(m), \] (4.28)
and $\{\lambda_n(K)\}$ decreases, choosing $\gamma = \max\{\delta(m), \beta(m)\}$, we now see that
\[ \sum_{n=\gamma}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2 (K) \leq C \sum_{n=\gamma}^{\infty} (2n)^{4r} d_n^m \lambda_{(2n)^m}^2 (K) \]
\[ \leq C 2^{4r} \sum_{n=1}^{\infty} n^{4r} \sum_{k=r_{n-1}+1}^{r_{n}} \lambda_{n_{n+k}}^2 (K). \]
Hence,
\[ \sum_{n=\gamma}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2 (K) < \infty, \]
due to (4.26). On the other hand,
\[ \sum_{n=\gamma}^{\infty} (2n+1)^{4r+m-1} \lambda_{(2n+1)^m}^2 (K) \leq 2^{4r+m} \sum_{n=\gamma}^{\infty} (2n)^{4r+m-1} \lambda_{(2n+1)^m}^2 (K) \]
\[ \leq 2^{4r+m} \sum_{n=\gamma}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2 (K) < \infty, \]
due to the previous calculations. Hence, we may infer that
\[ \sum_{n=1}^{\infty} n^{4r+m-1} \lambda_{n^m}^2 (K) < \infty. \]
Repeating the same trick used in the second half of the proof of Theorem 2.1 leads to
\[ \sum_{n=1}^{\infty} \sum_{k=0}^{(n+1)^m-(n^m+1)} (n^m + k)^{4r/m} \lambda_{n_{n+k}}^2 (K) < \infty. \]
Thus,
\[ \sum_{n=1}^{\infty} n^{4r/m} \lambda_{n}^2 (K) < \infty, \]
and an application of Lemma 4.4 closes the proof. \[ \blacksquare \]
Proof. [Proof of Theorem 2.3] We assume $K_{0,r}$ belongs to $S_p$ and show first that

$$\sum_{n=1}^{\infty} n^{2rp + m - 1} (\lambda_n m(K))^p < \infty,$$

and then we prove

$$\sum_{n=1}^{\infty} n^{2rp/m} (\lambda_n(K))^p < \infty.$$

The proof is identical to the proof of Theorem 2.2 until (4.25). From there, we can write

$$\lambda_{\tau n + k}(K) \leq s_k(K_{0,r}) s_{\tau n}(J^r) = s_k(K_{0,r}) n^{-r} (n + \alpha + \beta + 1)^{-r}, \quad n, k = 1, 2, \ldots.$$ 

Thus

$$n^{2r} \lambda_{\tau n + k}(K) \leq n^r (n + \alpha + \beta + 1)^r \lambda_{\tau n + k}(K) \leq s_k(K_{0,r}), \quad n, k = 1, 2, \ldots,$$

which implies that

$$n^{2rp}(\lambda_{\tau n + k}(K))^p \leq (s_k(K_{0,r}))^p, \quad n, k = 1, 2, \ldots.$$ 

Since $K_{0,r} \in S_p$, by adding on $k$ and $n$ leads to

$$\sum_{n=1}^{\infty} n^{2rp} \sum_{k=\tau_{n-1}+1}^{\tau_n} (\lambda_{\tau n + k}(K))^p \leq \sum_{n=1}^{\infty} \sum_{k=\tau_{n-1}+1}^{\tau_n} (s_k(K_{0,r}))^p \leq \sum_{n=1}^{\infty} (s_n(K_{0,r}))^p < \infty.$$ 

Proceeding as in the previous proof, since $2\tau_n \leq 2^2 n^m \leq (2n)^m$, for $n \geq \gamma$, there is a positive constant $c = c(r, p, m)$ such that

$$\sum_{n \geq \gamma} (2n)^{2rp + m - 1} (\lambda_{(2n)m}(K))^p \leq c \sum_{n \geq \gamma} n^{2rp} d_n^m (\lambda_{(2n)m}(K))^p \leq c \sum_{n \geq \gamma} n^{2rp} \sum_{k=\tau_{n-1}+1}^{\tau_n} (\lambda_{(2n)m}(K))^p \leq c \sum_{n \geq \gamma} n^{2rp} \sum_{k=\tau_{n-1}+1}^{\tau_n} (\lambda_{2\tau n}(K))^p \leq c \sum_{n \geq \gamma} n^{2rp} \sum_{k=\tau_{n-1}+1}^{\tau_n} (\lambda_{\tau n + k}(K))^p < \infty.$$ 

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Moreover, we show in the same way there is $c_1 = c_1(r, p, m) > 0$ such that

$$\sum_{n \geq \gamma} (2n + 1)^{2rp + m - 1} (\lambda_{(2n+1)^m}(K))^p \leq c_1 \sum_{n \geq \gamma} n^{2rp} \sum_{k=r_{n-1}+1}^{r_n} (\lambda_{r_n+k}(K))^p < \infty.$$  

Hence,

$$\sum_{n=1}^{\infty} n^{2rp + m - 1} (\lambda_n(K))^p < \infty.$$  

Finally, there is a positive constant $c_2(r, p, m)$ such that

$$\sum_{n=1}^{\infty} n^{2rp} (\lambda_n(K))^p = \sum_{n=1}^{\infty} \sum_{k=0}^{(n+1)^m - n^m + 1} (n^m + k)^{2rp} (\lambda_{n^m+k}(K))^p  
\leq \sum_{n=1}^{\infty} \sum_{k=0}^{(n+1)^m - n^m + 1} [(2n)^m \lambda_{n^m+k}(K))^p  
= 2^{2rp} \sum_{n=1}^{\infty} n^{2rp} \sum_{k=0}^{(n+1)^m - n^m + 1} (\lambda_{n^m+k}(K))^p  
\leq 2^{2rp} \sum_{n=1}^{\infty} n^{2rp} (\lambda_{n^m}(K))^p \sum_{k=0}^{(n+1)^m - n^m + 1} 1  
= 2^{2rp} \sum_{n=1}^{\infty} n^{2rp} (\lambda_{n^m}(K))^p [(n+1)^m - n^m - 1]  
\leq c_2(r, p, m) \sum_{n=1}^{\infty} n^{2rp} (\lambda_{n^m}(K))^p (n)^{m-1}  
\leq c_2(r, p, m) \sum_{n=1}^{\infty} n^{2rp + m - 1} (\lambda_{n^m}(K))^p < \infty.$$  

Using Lemma 4.4 follows that

$$\lim_{n \to \infty} n^{\frac{1}{2r} + \frac{2r}{m}} \lambda_n(K) = 0,$$

and the proof is complete.
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