Inertial frame dragging and Mach’s principle in General Relativity

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March 24, 2022

Abstract

We define a new parameter ‘cumulative drag index’ for a particle in circular orbit in a stationary, axisymmetric gravitational field and study its behaviour in the two well known solutions of general relativity \(viz.,\) the Kerr spacetime and the Gödel spacetime, wherein the inertial frame dragging has an important role. As it shows similar behaviour for both co and counter rotating particles, it may indeed be an indication of the influence of the faraway universe on local physics and thus Machian.

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1 Introduction

It is indeed well known that ‘Mach’s principle’ which relates ‘inertia’ with the influence of distant sources in the universe in its original formulation has been a topic of interesting and intense discussions for a long time, particularly in the context of general relativity. For a recent survey of activities in this field, the best source is the proceeding of the Tübingen conference (1993) on this topic (Barbour and Pfister 1995 [1]), wherein as many as twenty different interpretations of the principle have been discussed. Subsequent to this meeting there have been some interesting arguments with conflicting conclusions concerning the celebrated Lense–Thirring effect with Rindler [2] claiming this to be anti–Machian, while Bondi and Samuel [3] claim it is Machian. These discussions do indeed pose a far more important question as to the nature of inertia and how one should define ‘inertial frames’ or inertial forces.

General relativity which is the most successful and complete theory of classical gravity, tried to do away with the concept of force by describing gravity as the curvature of spacetime geometry. However in certain contexts it may be useful to reintroduce the concept of force within the framework of general relativity and understand how the spacetime curvature influences the various parts of the acceleration acting on a test particle in a given spacetime. This was indeed made possible by a 3+1 conformal splitting of spacetime by Abramowicz et al. [4] and it has yielded several interesting new insights into the particle motion in curved spacetime. In the context of Machian definition of inertia one considers the induced effects of rotation in flatspace (Pfister and Braun 1985 [5]) and the entire definition depends only on the relative rotation, in the absence of which both Coriolis, and the centrifugal accelerations are zero. On the other hand, in a curved spacetime, the contribution of curvature in the definition of inertial forces would change the situation and could give a better analysis of the effects due to distant sources on local physics. In fact the work due to Abramowicz and
coworkers regarding the Newtonian forces in general relativity lead to some new discoveries like centrifugal reversal, reversal of Rayleigh criterion (Abramowicz and Prasanna 1990 [3]), and the explanation of the ellipticity maximum of a rotating, stationary configuration (Abramowicz and Miller [4]). There have been certain other discussions, wherein these new features are attributed to other reasons within the four–dim. formalism without using the concept of inertial forces (de Felice [5], Barbès et al. [9]). However in the context of Mach’s principle and general relativity, it is imperative to introduce the ‘inertial forces’ within the scope of general relativity and possibly separate the global and local effects.

Amongst the various effects of rotation, the most celebrated one is the Lense–Thirring effect which arises due to ‘Coriolis force’ coming from the relative rotation. It is amusing to note that this effect has been considered to be both anti–Machian and Machian as mentioned above thus leading to more confusion and leaving the conclusions to the different interpretation of the principle. As the effect arises due to ‘dragging of inertial frames’ it is best to define an index that incorporates the ratio of the relative acceleration to the total acceleration which could effectively subdue the contribution of the gravitational field and thereby allow one to get at the pure rotation effects.

In the following we attempt to consider this with the introduction of a new dimensionless parameter called the ‘cumulative drag index’ defined as the ratio of the difference between the Coriolis and gravitational force to the total force acting on a particle in circular orbit located at a distance where the centrifugal force acting on the particle is totally zero. Generally, it is assumed that the Coriolis and the centrifugal forces arise from the linear and the quadratic order of the angular velocity parameter and in flatspace thus either both are zero or both are present. On the other hand in general relativity the way we have introduced the ‘inertial forces’ using the optical reference geometry, it becomes apparent that there do exist orbits along which the centrifugal force vanishes but the Coriolis force can still be non zero. While in static spacetimes this orbit coincides with
that of the unstable photon orbit (Prasanna 1991 [10]), in stationary spacetime this orbit is different from the photon orbits.

2 Formalism

In an axially symmetric stationary gravitational field represented by the spacetime metric (with signature +2)

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2$$  \hspace{1cm} (1)

the four–acceleration $a^k$ acting on a particle in a circular orbit with constant angular velocity $\Omega$, may be decomposed covariantly as given by (Abramowicz et al. 1995)

$$a_k = \nabla_k \Phi + \gamma^2 v (n^i \nabla_i \tau_k + \tau^i \nabla_i n_k) + (\gamma v)^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{n}_k$$  \hspace{1cm} (2)

wherein the vector field $n^i$ corresponds to the locally non rotating observer (LNRF, Bardeen 1972 [11]) expressed in terms of the timelike Killing vector $\eta^i$ and the spacelike Killing vector $\xi^i$:

$$n^i = e^\Phi (\eta^i + \omega \xi^i), \hspace{0.5cm} \omega = -\langle \eta, \xi \rangle / \langle \xi, \xi \rangle.$$

Here $\Phi$ is the potential

$$\Phi = -\frac{1}{2} \ln[-\langle \eta, \eta \rangle - 2\omega \langle \xi, \eta \rangle - \omega^2 \langle \xi, \xi \rangle],$$  \hspace{1cm} (4)

$\tau^i$ is the unit vector orthogonal to $n^i$ along the circle, and $v$ the Lorentz speed related to the particle four–velocity $U^i$: $U^i = \gamma (n^i + v \tau^i)$, $\gamma$ being the Lorentz factor $= 1/\sqrt{1 - v^2}$. For the circular orbit $U^i$ may also be decomposed as $U^i = A (\eta^i + \Omega \xi^i)$ with $A$ being the redshift factor given by:

$$A^2 = -[\langle \eta, \eta \rangle + 2\Omega \langle \xi, \eta \rangle + \Omega^2 \langle \xi, \xi \rangle]^{-1}.$$  \hspace{1cm} (5)
In (2) the overhead tilde refers to the vectors defined in the conformally projected three space (ACL 1988) with the positive definite metric

\[ h_{ik} = g_{ik} + n_in_k, \quad \tilde{\tau}^i = e^\Phi \tau^i \]

and \( \tilde{\nabla}_i \) the covariant derivative with respect to the projected metric \( \tilde{h}_{ik} = e^{2\Phi}h_{ik} \).

The three terms on the right hand side of (2) are respectively identified as the gravitational, the Coriolis (Lense–Thirring), and the centrifugal acceleration (Abramowicz 1993). It is easy to see that the particle speed \( v \) is given by the relation

\[ v \tau^i = e^\Phi (\Omega - \omega) \xi^i, \]

using which one can write explicitly the three forces acting on a particle of rest mass \( m_0 \) (normalised) in the circular orbit with constant angular velocity \( \Omega \) as measured by the stationary observer at infinity, to be

**gravitational:**

\[ G_i = \nabla_i \Phi = -\frac{1}{2} \partial_i \left\{ \ln \left[ \frac{(g_{t\phi}^2 - g_{tt} g_{\phi\phi})}{g_{\phi\phi}} \right] \right\} \]  

**Coriolis:**

\[ (Co)_i = \gamma^2 v n^j (\nabla_j \tau_i - \nabla_i \tau_j) \]

\[ = A^2(\Omega - \omega) \sqrt{g_{\phi\phi}} \left\{ \partial_i \left( \frac{g_{t\phi}}{\sqrt{g_{\phi\phi}}} \right) + \omega \partial_i \sqrt{g_{\phi\phi}} \right\} \]

and **centrifugal:**

\[ (Cf)_i = -\frac{(\gamma v)^2}{2} \tilde{\nabla}_j \tilde{\tau}_i = -\frac{A^2(\Omega - \omega)^2}{2} g_{\phi\phi} \partial_i \left\{ \ln \left[ \frac{g_{\phi\phi}^2}{(g_{t\phi}^2 - g_{tt} g_{\phi\phi})} \right] \right\}. \]

As our interest lies in analysing the ‘dragging’ induced by the given spacetime, we shall consider only the orbit along which the centrifugal acceleration is zero. This as mentioned earlier is possible in the stationary case as these orbits do not coincide with unstable photon orbits as in the static case.
In such a situation one has for a particle in circular orbit on the equatorial plane \( \theta = \pi/2 \), the centrifugal acceleration along the radial direction to be zero if
\[
\left[ \frac{\partial_r g_{\varphi\varphi}}{g_{\varphi\varphi}} - \frac{1}{2} \frac{\partial_r (g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi})}{(g_{tt}^2 - g_{t\varphi} g_{\varphi\varphi})} \right]_{\theta=\pi/2} = 0 \tag{11}
\]
For specified \( g_{ij} \), this would give an algebraic equation, the real roots of which correspond to the location \( R \) of the orbits with the required condition. Evaluating the gravitational and Coriolis accelerations at this location \( (G)_{R,\pi/2} \) and \( (Co)_{R,\pi/2} \) from (8) and (9) one can get these two forces acting on the particle as functions of the angular velocity parameter \( \Omega \) and the rotation parameter associated with the background spacetime.

We then define the ‘cumulative drag index’
\[
\mathcal{C} = [(Co)_R - (G)_R]/[(Co)_R + (G)_R] \tag{12}
\]
as the ratio of the relative drag acceleration to the total acceleration acting on the particle as measured by the locally non rotating observer.

### 3 Specific examples

#### 3.1 Kerr geometry

The spacetime metric outside a rotating black hole is, as well known, given by
\[
ds^2 = -(1 - \frac{2mr}{\Sigma})dt^2 - \frac{4mra}{\Sigma} \sin^2 \theta dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{B}{\Sigma} \sin^2 \theta d\varphi^2 \tag{13}
\]
with \( B = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \), \( \Delta = r^2 + a^2 - 2mr \), \( \Sigma = r^2 + a^2 \cos^2 \theta \).

Using these \( g_{ij} \) s in (8) to (11) the accelerations may be obtained explicitly on the equatorial plane \( \theta = \pi/2 \) as (see also Nayak and Vishveshwara [12]).
\[(G)_r = \frac{-m(a^2\Delta + r^4 + a^2r^2 - 2ma^2r)}{r\Delta(r^3 + a^2r + 2ma^2)} \quad (14)\]

\[(Co)_r = \frac{2ma(\Omega - \omega)(3r^2 + a^2)}{r[1 - \Omega^2(r^2 + a^2) - (2m/r)(1 - \Omega a)^2](r^3 + a^2r + 2ma^2)} \quad (15)\]

and

\[(Cf)_r = -\frac{(\Omega - \omega)^2[r^5 - 3mr^4 + a^2(r^3 - 3mr^2 + 6m^2r - 2ma^2)]}{r^2\Delta[1 - \Omega^2(r^2 + a^2) - (2m/r)(1 - \Omega a)^2]} \quad (16)\]

with \(\omega = 2ma/(r^3 + a^2r + 2ma^2)\).

It is well known that when \((\Omega - \omega)\) the angular velocity of the particle as measured by the Bardeen observer (LNRF) is zero, both Coriolis and centrifugal force vanish identically. On the other hand as seen from above, in this formulation of inertial forces, one can have the centrifugal force to be zero for \((\Omega - \omega) \neq 0\) if the algebraic expression

\[r^5 - 3mr^4 + a^2(r^3 - 3mr^2 + 6m^2r - 2ma^2) = 0 \quad (17)\]

(Iyer and Prasanna 1992 [13]) has real positive roots. It may be seen that for \(0 < a < 1\), there are at most three real roots of which at least one lies outside the event horizon. Fig. (1) shows the location of this root as a function of \(a\), the Kerr parameter. Denoting this root by \(R\), one can calculate \((G)_R\) and \((Co)_R\) from (14) and (15) and finally obtain the cumulative drag index to be

\[C = \frac{[2ma(\Omega - \omega)(3R^2 + a^2)]\Delta + mS(a^2\Delta + R^4 + a^2R^2 - 2ma^2R)}{[2ma(\Omega - \omega)(3R^2 + a^2)]\Delta - mS(a^2\Delta + R^4 + a^2R^2 - 2ma^2R)} \quad (18)\]

with \(S = 1 - \Omega^2(R^2 + a^2) - 2m(1 - \Omega a)^2/R\).

Fig. (2) gives the plots of \(C\) as a function of \(\Omega\) for different specific values of \(a\). As seen from the figures the function \(C\) has two zeros and two infinities as may be expected from the fact that both the numerator and the denominator are quadratic functions of \(\Omega\).
\( C \) is positive only for a very narrow range of values of \( |\Omega| \) (\( \ll 1 \)) whereas it is negative for all other values of \( \Omega \). Further, both the co rotating \((a\Omega > 0)\) and the counter rotating \((a\Omega < 0)\) ones have exactly the same range of values of \( \Omega \) for which \( C \) is positive, and this range increases with \( a \). (Table 1).

Nayak and Vishveshwara (1996) have calculated the gyroscopic precession rate \( \tau_1 \) in the Kerr geometry. Using their expression for \( \theta = \pi/2 \) plane, the precession rate is given by

\[
\tau_1 = \frac{\Omega r^3 - 3m\Omega(1 - \Omega a)r^2 + ma(1 - \Omega a)^2}{r^2[-\Omega^2 r^3 + (1 - \Omega^2 a^2)r - 2m(1 - \Omega a)^2]} \tag{19}
\]

which when \( a = 0 \), is identically zero at \( r = 3m \), where the centrifugal force also vanishes in Schwarzschild spacetime. On the other hand in the present case for \( r = R \), where \( Cf = 0 \), we have (after rescaling with \( m \))

\[
\tau_1 = \frac{(\alpha^3 + 3\alpha R^2)\Omega^2 + (R^3 - 3R^2 - 2\alpha^2)\Omega + \alpha}{-R^2(R^3 + R\alpha^2 + 2\alpha^2)\Omega^2 + 4\alpha R^2\Omega + (R^3 - 2R^2)} \tag{20}
\]

As may be seen easily this would have real zeros if and only if \( R \), which is also a function of \( \alpha = a/m \) (eq. 17) satisfies the relation \( R(R - 3)^2 > 4\alpha^2 \). On the other hand \( \tau_1 \) is infinite for

\[
\Omega_\pm = \frac{(2\alpha \pm R\Delta^{1/2})}{(R^3 + R\alpha^2 + 2\alpha^2)}
\]

outside the event horizon \( (\Delta = 0) \) and further is positive in the entire range

\[
[(2\alpha - R\Delta^{1/2})/(R^3 + R\alpha^2 + 2\alpha^2)] < \Omega < [(2\alpha + R\Delta^{1/2})/(R^3 + R\alpha^2 + 2\alpha^2)]
\]

and negative outside this range (fig. 3). It is interesting to note (fig. 4) that for values of \( \Omega \) where the index \( C \) is zero, \( \tau_1 \) is negative for co rotating particles and positive for counter rotating particles.
3.2 Gödel spacetime

One of the simplest homogeneous cosmological model is Gödel’s solution of Einstein’s equations which represents a pressure free, dusty universe having a non zero cosmological constant as given by the spacetime metric (Hawking and Ellis 1972 [14])

\[ ds^2 = 2a^2 \left( -dt^2 + dr^2 + dz^2 + f(r) \, d\varphi^2 + g(r) \, d\varphi \, dt \right) \] (21)

with \( a^2 = 1/8\pi \rho \), \( f = \sinh^2 r - \sinh^4 r \), \( g = 2\sqrt{2} \sinh^2 r \). \( \rho \) being the matter density which should be positive everywhere. Using these metric functions in the expressions (8) to (10) for the accelerations of a test particle in circular orbit one finds:

\[ (G)_r = -g(2fg' - f'g)/2f(f + g^2) \] (22)

\[ (Co)_r = -2A^2(\Omega - \omega)a^2(gf' - g'f)/f \] (23)

\[ (Cf)_r = -A^2(\Omega - \omega)^22a^2[(2g^2 + f)f' - 2gg'f]/(g^2 + f)f \] (24)

with \( \omega = -g/f \), \( A^2 = [2a^2(1 - 2\Omega g - \Omega^2 f)]^{-1} \). If we now consider the particular orbit on which the centrifugal force is zero we get

\[ [(2g^2 + f)f' - 2gg'f]_R = 0 \] (25)

and using this in the other two we get

\[ (G)_R = -f'/2f \, , \, \, (Co)_R = 4a^2A^2(\Omega - \omega)f'/2g \, . \] (26)
With these, the index $C$ turns out to be

$$C = \frac{(f\Omega + 2g)(1 - g\Omega)}{\Omega(f + 2g^2 + \Omega fg)}$$  \hspace{1cm} (27)$$

which clearly shows the two zeros and the two infinities of $C$. Fig. (6) shows $C$ as a function of $\Omega$ and again like in the earlier case, $C$ is positive only for a very small range of values viz., $-(f + 2g^2)/fg < \Omega < -2g/f$ and $0 < \Omega < 1/g$, which again is the same for both co and counter rotating particles.

4 Discussion

It is indeed very significant that the ‘cumulative drag index’ has the same sign for both co and counter rotating particles in any given range of values of the angular velocity parameter $\Omega$, irrespective of the spacetime geometry. As the two examples sighted above are very typical for the discussion of ‘inertial frame dragging’, the index could characterise the asymptotic effects if there are any. In fact the Gödel universe though is pathological (being achronal) is really well suited to study pure geometric effects as it is homogeneous and pressure free. The Kerr solution on the other hand being the spacetime devoid of any matter distribution, is again free from pressure gradient or electromagnetic effects. Thus for a particle in circular orbit in these spacetimes, the only forces acting on it are the gravitational, Coriolis, and centrifugal. By considering the orbit where the centrifugal force is zero, we have restricted further the influence arising from the spacetime surrounding the particle. If the positivity of the cumulative drag index is to show the influence of the distant universe on the particle, then the fact that the influence on both co and counter rotating particles exist to the same extent as measured by the locally non rotating observer is of special interest.

Table 2 gives the locations of the zeros and infinities of the general function $\tau_1(r)$ for different values of $a$ for $\Omega = 0.1$. The precession rate $\tau_1$ has no
zeros for $a$ higher than a critical value for every $\Omega$, as is clear from fig. (5). As observed by Nayak and Vishveshwara $\tau_1$ is zero at a different location than the zeros of the centrifugal force and thus there always exists a non zero precession even when the cumulative drag index is zero. It is interesting to note that this precession corresponding to $\mathcal{C} = 0$ is negative for co rotating particles and positive for counter rotating particles.

In the above we have restricted the analysis to the circular motion of the particle in the equatorial plane. However, one might get more information if one considers the behaviour of the drag index for other values of $\theta$, particularly to compare with the precession rate at poles and the equator, of the gravitating sources. In fact the Lense–Thirring dragging effect is also considered as a kind of gravimagnetic effect (Will 1995) \cite{15} by comparing the dragging of inertial frames to the influence of the magnetic field of a rotating electrical conductor. It would indeed be useful to consider the discussion of the ‘forces’ to including the electromagnetic fields and then look for an index when both centrifugal and Coriolis forces are absent, thus having only gravitational and electromagnetic effects in static geometry. Comparing the behaviour of such an index with $\mathcal{C}$, one can perhaps get a better feeling about the local and global effects associated with rotation.

**Acknowledgement**

It is a great pleasure to thank Urs M. Schaudt for his help in getting the plots as well as in preparing the ms in the \LaTeX{} mode. Discussions with Urs M. Schaudt, Herbert Pfister, Jiri Bicak, Jurgen Ehlers and Wolfgang Rindler, who went through the manuscript, helped in clarifying many points and in the final presentation. The work was done while the author was visiting the Institut für Theor. Phys., Universität Tübingen, Germany, during the spring of 1996. He is grateful to the Alexander von Humboldt Stiftung, Bonn, for the financial grant.
which made this visit possible.

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Table 1: Gives the $\Omega$ values for co (+) and counter (-) rotating particles, for which $C$ is zero (1) and infinity (2).

| $a$ | $R$    | $\Omega_{1+}$ | $\Omega_{2+}$ | $\Omega_{1-}$ | $\Omega_{2-}$ | $(\Omega_{1} - \Omega_{2})$ |
|-----|--------|----------------|----------------|----------------|----------------|-----------------------------|
| 0.1 | 2.9978 | 0.211269       | 0.189022       | -0.1742        | -0.1964        | 0.02224                     |
| 0.5 | 2.9445 | 0.29437        | 0.180094       | -0.10537       | -0.21965       | 0.11428                     |
| 0.9 | 2.8226 | 0.39668        | 0.17724        | -0.04084       | -0.26027       | 0.21944                     |
| 1.0 | 2.7830 | 0.42640        | 0.17722        | -0.02535       | -0.27452       | 0.24917                     |

Table 2: Gives the location of the event horizon ($r_e$) and the location of the infinities and zero of $\tau_1(r)$.

| $a$ | $r_e$   | $r_{1\infty}$ | $\tau_1$ | $r_{2\infty}$ |
|-----|---------|---------------|----------|----------------|
| 0.1 | 1.99499 | 2.046606      | 2.84927  | 8.81822        |
| 0.5 | 1.86603 | 1.87568       | –        | 8.91667        |
| 0.9 | 1.43581 | 1.72113       | –        | 8.98669        |
| 1.0 | 1.0     | 1.68466       | –        | 9.0            |
Figure 1: Location of $R$ (↑) as a function of $a$ (→).
Figure 2: Plots of $C (\uparrow)$ as a function of $\Omega (\rightarrow)$ for different values of $a$. 
Figure 3: Gyroscopic precession rate $\tau_1(R)$ as a function of $\Omega$ ($\rightarrow$).
Figure 4: $C \ (\cdots)$ and $\tau_1(R) \ (\rightarrow)$ as a function of $\Omega \ (\rightarrow)$. 
Figure 5: \( \tau_1(r) \) (\( \uparrow \)) as a function of \( r \) (\( \rightarrow \)) for \( \Omega = 0.1 \) and different values of \( a \) (0.1 to 0.4).

Figure 6: \( C(\uparrow) \) for Gödel universe, as a function of \( \Omega \) (\( \rightarrow \)).