WEIGHTED LÉPINGLE INEQUALITY

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Abstract. We prove an estimate for weighted $p$-th moments of the pathwise $r$-variation of a martingale in terms of the $A_p$ characteristic of the weight. The novelty of the proof is that we avoid real interpolation techniques.

1. Introduction

Lépingle’s inequality [Lép76] is a moment estimate for the pathwise $r$-variation of martingales. Finite $r$-variation is a parametrization-invariant version of Hölder continuity of order $1/r$ and plays a central role in Lyons’s theory of rough paths [Lyo98].

Lépingle’s inequality also found applications in ergodic theory [Bou89] and harmonic analysis [NOT10], see [MSZ18] and [DOP17; DDU18] and references therein, respectively, for recent developments in these directions. Weighted inequalities in harmonic analysis go back to [Muc72], and weighted variational inequalities have been studied since [Cre+09]. A major motivation of the weighted theory is the Rubio de Francia extrapolation theorem that allows to obtain vector-valued $L^p$ inequalities for all $1 < p < \infty$ from scalar-valued weighted $L^p$ inequalities for a single $p$, see [Duo11, Section 3] for the most basic version of that result and [DPW17, Theorem 8.1] for a version applicable to martingales.

In this article, we prove a weighted version of Lépingle’s inequality for martingales with asymptotically sharp dependence on the $A_p$ characteristic of the weight. For dyadic martingales, weighted variational inequalities were first obtained in [DL12, Lemma 6.1] using the real interpolation approach as in [PX88; Bou89; JSW08; MSZ18]. The argument in the dyadic case relied on the so-called open property of $A_p$ classes, see e.g. [HPR12, Theorem 1.2], that is in general false for martingale $A_p$ classes, see the example in [BL79, §3] and [BB78]. Therefore, we use a new stopping time argument that is also simpler than the previous proofs of Lépingle’s inequality even in the classical, unweighted, case.

1.1. Notation. Let $(\Omega, (\mathcal{F}_n)_{n=0}^\infty, \mu)$ be a filtered probability space and $\mathcal{F}_\infty := \vee_{n=0}^\infty \mathcal{F}_n$. A weight is a positive $\mathcal{F}_\infty$-measurable function $w : \Omega \to (0, \infty)$. The corresponding weighted $L^p$ norm is given by $\|X\|_{L^p(\Omega,w)} := (\int_\Omega |X|^p w \, d\mu)^{1/p}$. For $1 < p < \infty$, the martingale $A_p$ characteristic of the weight $w$ is defined by

$$Q_p(w) := \sup_\tau \left\| \mathbb{E}(w \mid \mathcal{F}_\tau) \mathbb{E}(w^{-1/(p-1)} \mid \mathcal{F}_\tau)^{p-1} \right\|_{L^\infty(\Omega)},$$

where the supremum is taken over all adapted stopping times $\tau$. For comparison of our main result with the unweighted case, note that for $w \equiv 1$ we have $Q_p(w) = 1$ for all $1 < p < \infty$.

For $0 < r < \infty$, a sequence of random variables $X = (X_n)_n$, and $\omega \in \Omega$, the $r$-variation of $X$ at $\omega$ is defined by

$$V^r X(\omega) := V^r_n X_n(\omega) := \sup_{u_1 < u_2 < \cdots} \left( \sum_j |X_{u_{j+1}}(\omega) - X_{u_j}(\omega)|^r \right)^{1/r},$$

where the supremum is taken over arbitrary increasing sequences.

2010 Mathematics Subject Classification. 60G17, 60G42.

PZ was partially supported by the Hausdorff Center for Mathematics (DFG EXC 2047).
1.2. Main result. For an integrable \( \mathcal{F}_\infty \)-measurable function \( X : \Omega \to \mathbb{R} \), the associated martingale is defined by \( X_n := \mathbb{E}(X | \mathcal{F}_n) \). We have the following weighted moment estimate for the pathwise \( r \)-variation of this martingale.

**Theorem 1.1.** For every \( 1 < p < \infty \), there exists a constant \( C_p < \infty \) such that, for every \( r > 2 \), every filtered probability space \( \Omega \), every weight \( w \) on \( \Omega \), and every integrable function \( X : \Omega \to \mathbb{R} \), we have

\[
\|V^r X\|_{L^p(\Omega, w)} \leq C_p \sqrt{\frac{r}{r-2}} Q_p(w)^{\max(1,1/(p-1))} \|X\|_{L^p(\Omega, w)}. \tag{1.2}
\]

**Remark 1.2.** By the monotone convergence theorem, Theorem 1.1 extends to càdlàg martingales.

**Remark 1.3.** The example in [Qia98, Theorem 2.1] shows that, for \( p = 2 \), the constant in (1.2) must diverge at least as

\[
\sqrt{\log \frac{r}{r-2}} \quad \text{when} \quad r \to 2. \tag{1.3}
\]

Indeed, it is proved there that, if \( (X_n)_{n=0}^N \) is a martingale with i.i.d. increments that are Gaussian random variables with zero expectation and unit variance, then \( (V^2 X)^2 \geq cN \log \log N \) with probability converging to 1 as \( N \to \infty \) for every \( c < 1/12 \). In this case, choosing \( r \) such that \( r-2 = 1/ \log N \), by Hölder’s inequality, we obtain

\[
V^2 X \leq N^{1/2-1/r} V^r X \leq CV^r X.
\]

This would lead to a contradiction if the constant in (1.2) diverges slower than stated in (1.3). The growth rate of the constant in (1.2) as \( r \to 2 \) is important e.g. in Bourgain’s multi-frequency lemma, as explained in [Zor15, §3.2].

**Remark 1.4.** The growth rate of the constant in (1.2) as \( r \to 2 \) is also related to endpoint estimates, in which the \( \ell^p \) norm in (1.1) is replaced by an Orlicz space norm. The results of [Tay72] for the Brownian motion suggest that it might be possible to use a Young function that decays as \( \log \log x \) when \( x \to 0 \). Such an estimate would imply an estimate of the form (1.3) for the constant in (1.2), and it would have useful consequences for rough differential equations, see [Dav08, Remark 5]. Our method allows to use Young functions that decay as \( x^2/(\log x^{-1})^{1+\epsilon} \) when \( x \to 0 \).

**Remark 1.5.** A Fefferman–Stein type weighted estimate that substitutes (1.2) in the case \( p = 1 \) can be deduced from Corollary 2.4 and [Osè17, Theorem 1.1].

2. Stopping times and a pathwise \( r \)-variation bound

In this section, we estimate the \( r \)-variation of an arbitrary adapted process pathwise by a linear combination of square functions. We consider an adapted process \( (X_n)_n \) with values in an arbitrary metric space \( (X, d) \) and extend the definition of \( r \)-variation (1.1) by replacing the absolute value of the difference by the distance. We have the following metric spaces \( X \) in mind.

1. In Theorem 1.1, we will use \( X = \mathbb{R} \) (and \( \rho = 2 \) below).
2. In applications to the theory of rough paths, one takes \( X \) to be a free nilpotent group, see [FV10, §9].
3. When \( X \) is a Banach space with martingale cotype \( \rho \in [2, \infty) \), Corollary 2.4 can be used to recover [PX88, Theorem 4.2].

**Definition 2.1.** Let \( M_t := \sup_{0 \leq \tau' \leq t} d(X_{\tau'}, X_{\tau'}) \). For each \( m \in \mathbb{N} \), define an increasing sequence of stopping times by

\[
\tau_0^{(m)}(\omega) := 0, \quad \tau_{j+1}^{(m)}(\omega) := \inf\{t \geq \tau_j^{(m)}(\omega) \mid d(X_t(\omega), X_{\tau_j^{(m)}(\omega)}(\omega)) \geq 2^{-m} M_t(\omega)\}. \tag{2.1}
\]
Lemma 2.2. Let $0 \leq t' < t < \infty$ and $m \geq 2$. Suppose that
\begin{equation}
2 < d(X_{t'}(\omega), X_t(\omega))/(2^{-m}M_t(\omega)) \leq 4. \tag{2.2}
\end{equation}
Then there exists $j$ with $t' < \tau_{j}^{(m)}(\omega) \leq t$ and
\begin{equation}
d(X_{t'}(\omega), X_t(\omega)) \leq 8d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega)(\omega)). \tag{2.3}
\end{equation}

Proof. We fix $\omega$ and omit it from the notation. Let $j$ be the largest integer with $\tau' := \tau_j^{(m)} \leq t$. We claim that $\tau' > t'$. Suppose for a contradiction that $\tau' < t'$ (the case $\tau' = t'$ is similar but easier). By the hypothesis (2.2) and the assumption that $t, t'$ are not stopping times, we obtain
\begin{equation}
2 \cdot 2^{-m}M_t < d(X_{t'}, X_t) \leq d(X_{t'}, X_{t'}) + d(X_{t'}, X_t) < 2^{-m}M_{t'} + 2^{-m}M_t \leq 2 \cdot 2^{-m}M_t,
\end{equation}
a contradiction. This shows $\tau' > t'$.

It remains to verify (2.3). Assume that $M_{t'} < M_t/2$. Then, for some $t' < t'' \leq t$, we have $d(X_{t'}, X_{t''}) \geq M_t/2 \geq 2^{-m}M_{t''}$, contradicting maximality of $\tau'$. It follows that
\begin{equation}
d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega)) \geq 2^{-m}M_{t'} \geq 2^{-m}M_t/2 \geq d(X_{t'}, X_t)/8. \tag{2.4}
\end{equation}

Lemma 2.3. For every $0 < \rho < \rho < \infty$, we have the pathwise inequality
\begin{equation}
V_t^r(X_t(\omega))^r \leq 8^\rho \sum_{m=2}^{\infty} (2^{-(m-2)}M_\infty(\omega))^{r-\rho} \sum_{j=1}^{\infty} d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega))^\rho. \tag{2.4}
\end{equation}

Proof. We fix $\omega$ and omit it from the notation. Let $(u_l)$ be any increasing sequence. For each $l$ with $d(X_{u_l}, X_{u_{l+1}}) \neq 0$, let $m = m(l) \geq 2$ be such that
\begin{equation}
2 < d(X_{u_l}, X_{u_{l+1}})/(2^{-m}M_{u_{l+1}}) \leq 4.
\end{equation}
Such $m$ exists because the distance is bounded by $M_{u_{l+1}}$.

Let $j$ be given by Lemma 2.2 with $t' = u_l$ and $t = u_{l+1}$. Then
\begin{equation}
d(X_{u_l}, X_{u_{l+1}})^r \leq 8^\rho d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega))^\rho \cdot (2^{-m}M_{u_{l+1}})^{r-\rho}.
\end{equation}
Since each pair $(m, j)$ occurs for at most one $l$, this implies
\begin{equation}
\sum_l d(X_{u_l}, X_{u_{l+1}})^r \leq 8^\rho \sum_{m,j} d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega))^\rho \cdot (2^{-(m-2)}M_\infty)^{r-\rho}.
\end{equation}
Taking the supremum over all increasing sequences $(u_l)$, we obtain (2.4). \hfill \Box

Corollary 2.4. For every $0 < \rho < \rho < \infty$, we have the pathwise inequality
\begin{equation}
V_t^r(X_t(\omega))^\rho \leq 8^\rho \sum_{m=2}^{\infty} 2^{-(m-2)(r-\rho)} \sum_{j=1}^{\infty} d(X_{\tau_{j}^{(m)}-1}(\omega), X_{\tau_{j}^{(m)}}(\omega))^\rho. \tag{2.5}
\end{equation}

Proof. By the monotone convergence theorem, we may assume that $X_n$ becomes independent of $n$ for sufficiently large $n$. In this case,
\begin{equation}
M_\infty(\omega) \leq V_t^r(X_t(\omega)) < \infty.
\end{equation}
Substituting this inequality in (2.4) and canceling $V_t^r(X_t(\omega))^{r-2}$ on both sides, the claim follows. \hfill \Box

3. Proof of the weighted Lépingle inequality

Estimates in weighted spaces $L^p(\Omega, w)$ for differentially subordinate martingales with sharp dependence on the characteristic $Q_\rho(w)$ were obtained in [TTV15] in the discrete case (a simpler alternative proof is in [Lac17]) and [DP19] in the continuous case (a simpler alternative proof is in [DP16]). By Khintchine's inequality, these results imply the following weighted estimate for the martingale square function.
Theorem 3.1 (cf. [DP16]). Let \((X_j)_{j=0}^\infty\) be a martingale on a probability space \(\Omega\). Then, for every \(1 < p < \infty\), we have
\[
\left\| \sum_{j=1}^{\infty} |X_j - X_{j-1}|^2 \right\|_{L^p(\Omega, w)}^{1/2} \leq C_p Q_p(w)^{\max(1,1/(p-1))} \|X\|_{L^p(\Omega, w)},
\]
(3.1)
where the constant \(C_p < \infty\) depends only on \(p\), but not on the martingale \(X\) or the weight \(w\).

An alternative proof that deals directly with the square function (3.1) appears in [BO18], but it is carried out only for continuous time martingales with continuous paths.

Proof of Theorem 1.1. By extrapolation, see [DPW17, Theorem 8.1], it suffices to consider \(p = 2\). We will in fact give a direct proof for \(2 \leq p < \infty\). A similar argument also works for \(1 < p < 2\), but gives a poorer dependence on \(r\) than claimed in (1.2).

Let \(\tau_j^{(m)}\) be the stopping times constructed in (2.1), and let
\[
S_{(m)}(\omega) := \left( \sum_{j=1}^{\infty} |X_{\tau_j^{(m)}(\omega)} - X_{\tau_{j-1}^{(m)}(\omega)}|^2 \right)^{1/2}
\]
denote the square function of the sampled martingale \((X_{\tau_j^{(m)}})_{j}\). Then Corollary 2.4 with \(\mathcal{X} = \mathbb{R}\) and \(\rho = 2\) gives
\[
V^r X \leq 8 \left( \sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} S_{(m)}^2 \right)^{1/2}.
\]

Since \(2 \leq p < \infty\), by Minkowski’s inequality, this implies
\[
\|V^r X\|_{L^p(\Omega, w)} \leq 8 \left( \sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} \|S_{(m)}\|_{L^p(\Omega, w)}^2 \right)^{1/2}.
\]

Inserting the square function estimates (3.1) for the sampled martingales \((X_{\tau_j^{(m)}})_{j}\) on the right-hand side above, we obtain
\[
\|V^r X\|_{L^p(\Omega, w)} \leq 8C_p Q_p(w) \|X\|_{L^p(\Omega, w)} \left( \sum_{m=2}^{\infty} 2^{-(m-2)(r-2)} \right)^{1/2}
\]
\[= 8C_p \left( 1 - 2^{-(r-2)} \right)^{-1/2} Q_p(w) \|X\|_{L^p(\Omega, w)}.\]
This implies (1.2).

\[\square\]

Remark 3.2. One can also directly apply Theorem 3.1 for \(1 < p < 2\), without passing through the extrapolation theorem. But this seems to lead to a faster growth rate of the constant in (1.2) as \(r \to 2\).

Remark 3.3. The unweighted Lépingle inequality (Theorem 1.1 with \(w \equiv 1\)) follows from Corollary 2.4 and the usual Burkholder–Davis–Gundy (BDG) inequality.

Remark 3.4. Corollary 2.4 can be used to recover the \(p\)-variation rough path BDG inequality [CF19, Theorem 4.7]. For convex moderate functions \(F(x) = x^p\) with \(1 < p < \infty\), the required estimate for the square function appearing in (2.5) can be deduced from the usual BDG inequality and [KZ19, Proposition 3.1]. The latter result can be extended to arbitrary convex moderate functions \(F\) using the Davis martingale decomposition.

Remark 3.5. Let \(\rho \in [2, \infty)\), and let \(\mathcal{X}\) be a Banach space with martingale cotype \(\rho\). Using Corollary 2.4 and the \(p\)-function bounds for \(\mathcal{X}\)-valued martingales in [Pis16, Theorem 10.59], we see that, for every \(1 < p < \infty\) and \(r > \rho\), every filtered probability space \(\Omega\), and every integrable function \(X : \Omega \to \mathcal{X}\), we have
\[
\|V^r X\|_{L^p(\Omega)} \leq C_{X,p} \frac{r}{r - \rho} \|X\|_{L^p(\Omega)}.
\]
(3.2)
In fact, it is possible to obtain a slightly better dependence on $r$, which we omit for simplicity. There is also an endpoint version of (3.2) at $p = 1$, in which $X$ is replaced by the martingale maximal function on the right-hand side.

The vector-valued estimate (3.2) was first proved in [PX88, Theorem 4.2], with an unspecified dependence on $r$. The dependence on $r$ stated in (3.2) can also be obtained using Theorem 1.3 and Lemma 2.17 in [MSZ18], as well as real interpolation, but this method does not work at the endpoint $p = 1$.

Acknowledgment. This work was partially supported by the Hausdorff Center for Mathematics (DFG EXC 2047).

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