The Fine Structure of Preferential Attachment Graphs I: Somewhere-Denseness

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March 30, 2018

Abstract
Preferential attachment graphs are random graphs designed to mimic properties of typical real word networks. They are constructed by a random process that iteratively adds vertices and attaches them preferentially to vertices that already have high degree. We use improved concentration bounds for vertex degrees to show that preferential attachment graphs contain asymptotically almost surely (a.a.s.) a one-subdivided clique of size at least \((\log n)^{1/4}\). Therefore, preferential attachment graphs are a.a.s. somewhere-dense. This implies that algorithmic techniques developed for sparse graphs are not directly applicable to them. The concentration bounds state: Assuming that the exact degree \(d\) of a fixed vertex (or set of vertices) at some early time \(t\) of the random process is known, the probability distribution of \(d\) is sharply concentrated as the random process evolves if and only if \(d\) is large at time \(t\).

1 Introduction

Recently, there were more and more efforts to model real world networks using random graph models. One property that was observed for some real world networks is that they might follow a power law degree distributions and are scale-free, this has been studied in detail \[5, 9\]. One candidate to model these properties are the Barabási–Albert graphs or preferential attachment graphs, with a degree distribution that mimics the heavy-tailed distribution observed in many real-world networks \[1\].

These preferential attachment graphs are created by a random process that iteratively adds new vertices and randomly connects them to already existing ones.
The model has a free parameter $m$, that indicates the number of edges (or self-loops) attached to each newly created vertex. A random graph is started with one vertex that has exactly $m$ self-loops. Afterwards, more vertices are added iteratively. Every time a new vertex is added, $m$ random edges between this vertex and existing vertices are added. The probability that an edge from the new vertex to another vertex $v$ is added is proportional to the current degree of $v$ (see Section 2 for a more rigorous definition). This process creates a certain imbalance: The degree of low-degree vertices is unlikely to increase and the degree of high-degree vertices is likely to increase even further during the process. This effect is called “the rich get richer”-effect due to the tendency of high degree vertices to accumulate even more edges in the future. Vertices that have been added early in the random process have a higher expected degree than late vertices.

The preferential attachment model is particularly interesting from the point of mathematical analysis because of its simple formulation and interesting characteristics. The model has been widely studied in the literature [10, 20, 23]. Other random graph models that were created to simulate real world graphs include the Chung-Lu-
[7, 8], the Configuration-
[25] and the Kleinberg-model [21, 22].

Another often observed (and algorithmically exploitable) property of real networks is that they are sparse [12]. In a social network there may be some hubs connected to a lot of people, but the vast majority of members only has relatively few neighbors, especially compared to the set of all potential neighbors. Sparsity is a concept that has been deeply studied and has lead to interesting algorithmic applications. Many graph problems that are hard for general graphs become easier on sparse graph classes [3, 11, 14, 18, 19]. Before we can talk about algorithms, we have to discuss formally what it means for a graph to be sparse. One cannot simply say that a graph is sparse if it contains few edges. The one-subdivision of any graph has roughly two edges per vertex but has the same dense structure as the original graph and should therefore not be considered sparse. There are a number of measures for graph sparsity and one of the best known and most general, introduced by Nešetřil and Ossona de Mendez, is nowhere-denseness [27]. This concept generalizes many different sparse graph properties such as bounded degree, planarity, bounded treewidth, bounded genus or bounded expansion. Informally speaking, a graph class is nowhere-dense if one needs a subgraph with a large radius to construct a large clique-minor, where large means growing with $n$. A graph class that is not nowhere-dense is said to be somewhere-dense. For a rigorous definition of these two concepts see Section 2.2. While some graph properties like planarity are properties of a single graph, it should be noted that nowhere- and somewhere-denseness are not. They are properties of a graph class, just like having bounded degree. Let $C$ be a nowhere-dense graph class. It has been shown by Grohe, Kreutzer and Siebertz that for any fixed first-order formula it is possible
to decide whether it is satisfied in a graph from \( C \) in linear time (in the size of the graph) \([18]\). This leads to efficient algorithms on nowhere-dense graph classes for every problem that can be expressed as a short first-order formula, e.g., the parameterized clique or dominating set problem.

In this work we aim to transfer the deep structural and algorithmic results for sparse graph classes to random graph models. The concept of nowhere-denseness is defined for deterministic graph classes which means we have to find a definition that is suitable for random graph models. In this work (in line with earlier work \([12, 27]\)), we ask for a given random graph model whether there exists a nowhere-dense graph class \( C \) such that with probability 1 an instance with \( n \) vertices from this model belongs to \( C \) as \( n \) approaches infinity. If this is the case then the random graph model is asymptotically almost surely (a.a.s.) nowhere-dense. Similarly, if for all nowhere-dense graph classes \( C \) an instance from this model belongs to \( C \) with probability 0 as \( n \) approaches infinity, the random graph model is a.a.s. somewhere-dense. Again, for more details see Section 2.2.

We believe a.a.s. somewhere- and a.a.s. nowhere-denseness are important properties of random graph models, which reveal a lot of information about asymptotic structure and can help in the design of efficient algorithms. Efficient algorithms for random graph models which mimic real world networks may lead to efficient algorithms for real world networks as well.

If a random graph model is a.a.s. nowhere-dense it is not a.a.s. somewhere-dense, and vice versa. However, it is possible that a random graph model is neither a.a.s. somewhere- nor a.a.s. nowhere-dense, which can not happen for deterministic graph classes. To become more familiar with these concepts consider the following three simple random graph models: A graph with \( n \) vertices is

1. with probability \( 1/n \) complete and with probability \( 1 - 1/n \) empty,
2. with probability \( 1/2 \) complete and with probability \( 1/2 \) empty,
3. with probability \( 1 - 1/n \) complete and with probability \( 1/2 \) empty.

If we fix a clique-size \( k \), the probability that a graph from these models contains a \( k \)-clique as a subgraph converges to 0, \( 1/2 \) and 1, respectively as \( n \) approaches infinity. Therefore, the first model is a.a.s. nowhere-dense, the third is a.a.s. somewhere-dense and the second is neither a.a.s. somewhere- nor a.a.s. nowhere-dense.

So far it is only known that preferential attachment graphs are not a.a.s. nowhere-dense \([12]\). In this work we show that they are a.a.s. somewhere-dense, which has stronger implications. If a random graph model is a.a.s. nowhere-dense the well known model-checking algorithm for sparse graphs \([18]\) has with probability \( 1 - \varepsilon \) an efficient running time, where \( \varepsilon \) converges to zero with increasing graph size. If a random graph model is a.a.s. somewhere-dense then techniques developed for sparse graphs are not directly applicable. An efficient model-checking
algorithm for this graph model (if it exists) has to exploit additional information about this graph model. If, however, a random graph model is only known to be not a.a.s. nowhere-dense the picture is less clear. In this case we know that the probability of belonging to a fixed sparse graph class does not converge to zero. But it could be that for every $\varepsilon > 0$ with probability $1 - \varepsilon$ a graph from this model belongs to some sparse graph class that depends on $\varepsilon$. Then with probability $1 - \varepsilon$ the model-checking algorithm for nowhere-dense graphs runs in linear time. But this technique does not work for every random graph model that is not a.a.s. nowhere-dense. Note that both aforementioned algorithms do not necessarily have an efficient expected running time and in the long run we would be interested in algorithms that have expected linear running time.

**Somewhere-dense.** While many random graph models have already been classified (the Chung-Lu- and the configuration-model are a.a.s. nowhere-dense and the Kleinberg-Model is a.a.s. somewhere-dense [12]), it remained an open question whether preferential attachment graphs are a.a.s. somewhere-dense. It is known that they are not a.a.s. nowhere-dense, i.e., there is a non vanishing probability that there is a clique-minor of arbitrary size [12]. Our main result (Section 5) is that graphs constructed by the preferential attachment model are somewhere-dense in the limit, thus answering this question. We do so by showing that the probability that a preferential attachment graph of size $n$ contains a one-subdivided clique of size $\log(n)^{1/4}$ as a subgraph approaches one as $n$ approaches infinity. This means there exists a somewhere-dense graph class such that the probability that a preferential attachment graph $G_n^m$ of size $n$ and parameter $m$ belongs to this graph class approaches one as $n$ approaches infinity.

**Theorem 5.4.** Let $m \geq 2$. $G_n^m$ contains with a probability of at least $1 - \log(n)^{O(1)} e^{-c \log(n)^{1/4}}$ a one-subdivided clique of size $\lfloor \log(n)^{1/4} \rfloor$ for some positive constant $c$.

This asymptotic property might help to indicate which real-world networks can or cannot be modeled by a preferential attachment paradigm. A scale-free real-world network that follows the preferential attachment paradigm should also contain one-subdivided cliques of increasing size. Our results also imply that efficient model-checking algorithms for preferential attachment graphs need to be based on a deeper analysis of their structure. In the upcoming follow-up work we will show that the size of subdivided cliques in preferential attachment graphs does not exceed polylogarithmic size. This means that the density is unbounded, but only grows slowly. We will use this and further observations to construct efficient algorithms for preferential attachment graphs, which may lead to efficient algorithms for real networks in the future.
Tail Bounds. We also present a detailed analysis of the probability distribution of the degree of individual vertices during the preferential attachment process, including exponentially decreasing tail bounds. The results are crucial for showing that preferential attachment graphs are a.a.s. somewhere-dense but may also be interesting for other applications.

The preferential attachment process depends on a parameter $m \in \mathbb{N}$ which states how many edges are added per vertex. Let $t \in \mathbb{N}$ and $S$ be a set of vertices in a preferential attachment graph of size $t$. For $n \geq t$ we define $d^n_m(S)$ to be the sum over all degrees of vertices in $S$ in an preferential attachment graph of size $n$ with parameter $m$. The earlier a vertex had been added in the random process, the more time it had to accumulate neighbors. It can be shown that the expected degree of the $i$th vertex in a preferential attachment graph of size $n$, i.e., $E[d^n_m(v_i)]$ is approximately $m \sqrt{n/i}$ \[30\]. But the preferential attachment process is too unstable and chaotic to guarantee that the degree of a vertex is closely centered around its expected degree. In Figure 1 on the left the exact probability distribution of the first vertex in a preferential attachment graph of size 10000 is plotted in red. The probability that the $i$th vertex has only degree one at time $n$ is at least $1/n$, i.e., quite high. See Lemma 4.4 for stronger bounds for the first vertex.

If we, however, assume that the degree of a vertex already is relatively high without changing its expectation, things start to change: the exact probability distribution of the first vertex under the condition that it had roughly its expected value after 100 (1000) steps is plotted in blue (black). We observe that, since the expected value did not change much, the distribution became much more concentrated than in the unconditional distribution. On the right-hand side of
Figure 1: the distribution for the sum of the degrees of a vertex set of size 1, 20 and 50 is shown after 10000 steps. The absolute concentration of the probability distribution of the total degree does not change if we increase the size of the set, but since the curve is moved to the right the relative concentration is increasing. We show that if we assume $d_m^t(S)$ to be high, then the probability is also high, that for all $n > t$ the degree $d_m^n(S)$ is no more than a constant factor off from the expected degree $E[d_m^n(S) | d_m^t(S)]$. Also, the probability that $d_m^n(S)$ differs by more than a constant factor decreases exponentially for large $d_m^t(S)$. This is formalized by the following theorem. Note that the constants in this theorem and its proof are chosen to make the calculations easier and can most likely be improved.

**Theorem 3.8.** Let $0 < \varepsilon \leq 1/40$, $t, m \in \mathbb{N}$, $t > \frac{1}{\varepsilon}$ and $S \subseteq \{v_1, \ldots, v_t\}$. Then

$$\mathbb{P}\left[(1 - \varepsilon)\sqrt{\frac{n}{t}} d_m^t(S) < d_m^n(S) < (1 + \varepsilon)\sqrt{\frac{n}{t}} d_m^t(S) \text{ for all } n \geq t \bigg| d_m^t(S)\right] \geq 1 - \ln(15t)e^{-\varepsilon - O(1)d_m^t(S)}.$$

Informally speaking, we show that, while some members of a preferential attachment process behave chaotically, central hubs keep growing at a very predictable rate. The more important a member becomes during the preferential attachment process the higher and more predictable its growth rate becomes. One can say that not only “the rich get richer” but also “the rich are predictable.”

We use this result in our proof that preferential attachment graphs are a.a.s. somewhere-dense. We choose a set of so-called principal vertices with high degree, Theorem 3.8 then guarantees that the degrees of the principal vertices are at all times $n$ of the random process high, i.e., approximately $\sqrt{n}$. We then show that because their degree is high the principal vertices will become pairwise connected and therefore span a subdivided clique, which completes the proof.

Bollobás and Spencer have provided tail bounds on the number of vertices with degree $d$, where $d$ is small [5]. These tail bounds were obtained via martingales and the Azuma–Hoeffding inequality. This technique, however, breaks down if one tries to generalize the result to higher $d$, i.e., $d$ of order $\sqrt{n}$. Therefore, we present a novel approach to prove tail bounds. We use Chernoff bounds to obtain bounds that have high accuracy for a limited number of steps of the preferential attachment process and then use the union bound to combine these short-interval bounds to arbitrarily long intervals.

Our concentration bounds in Theorem 3.8 yield insights on how preferential attachment graphs form over time: Initially they are in a chaotic state but after a while central hubs will emerge. These hubs are then very likely to remain the central hubs at all times in the future. It is very unlikely that vertices which have been added after the early phase will outgrow the central hubs from the early phase.
Pólya Urns. In Section 3 we prove concentration bounds for degrees in preferential attachment graphs directly, using only Chernoff arguments. But one can also model the behavior of degrees over time using so called Pólya Urn processes. Pólya Urns can be used to model many interesting biological or physical processes such as the spreading of diseases or mixing behavior of particles [16]. There is a rich literature on probability distributions of Pólya Urns, both exact and in the limit [16, 24, 26]. In Section 4 we establish a tight connection between degrees in preferential attachment graphs and certain Pólya Urn processes and try to use established results from Pólya Urn theory to compute the probability distribution of degrees in preferential attachment graphs. Using Pólya Urns to simulate random graph models is an approach that is not yet well explored, but there are some results [2, 15].

At first, we observe in Lemma 4.1 that the degree of a vertex in a preferential attachment graph follows exactly the same probability distribution as a certain Pólya Urn process. We then reinterpret a result from Pólya Urn theory [16] as a closed formula for the degree distribution of the first vertex in a preferential attachment graph. We use this result to obtain tight bounds for the degree distribution of this vertex (see Proposition 4.2). These bounds appear to be previously unknown in the context of preferential attachment graphs. We believe that this strong connection between Pólya Urns and preferential attachment graphs could help to simplify and strengthen the bounds presented in Section 3 and may lead to further structural or algorithmic results for preferential attachment graphs. We finish Section 4 by giving some partial results for improving the bounds from Section 3.

2 Preliminaries

In this work we will denote probabilities by $P[*]$ and expectation by $E[*]$. We use common graph theory notation [13].

2.1 The Preferential Attachment Graph Model

We consider a random graph model to be a sequence of probability distributions. For every $n \in \mathbb{N}$ a random graph model describes a probability distribution on graphs with $n$ vertices. In this work we focus on the preferential attachment random graph model which we describe in this subsection. It has been ambiguously defined in the original article by Barabási and Albert [1]. The model generates random graphs by iteratively inserting new vertices and edges. It depends on a parameter, usually denoted by $m$, which indicates the number of edges attached to a newly created vertex. We follow the rigorous definition of Bollobás and Ricor-
For a fixed parameter $m$ the random process is defined by starting with a single vertex and iteratively adding vertices, thereby constructing a sequence of graphs $G^1_m, G^2_m, \ldots, G^t_m$, where $G^t_m$ has $t$ vertices and $mt$ edges. We define $d^t_m(v)$ to be the degree of vertex $v$ in the graph $G^t_m$. The random process for $m = 1$ works as follows. A random graph is started with one vertex $v_1$ that has exactly one self-loop. This graph is $G^1_1$. We then define the graph process inductively: Given $G^{t-1}_1$ with vertex set $\{v_1, \ldots, v_{t-1}\}$, we create $G^t_1$ by adding a new vertex $v_t$ together with a single edge from $v_t$ to $v_i$, where $i$ is chosen at random from $\{1, \ldots, t\}$ with

$$P[i = s] = \begin{cases} \frac{d^{t-1}_1(v_s)}{2t-1} & 1 \leq s \leq t \\ \frac{1}{2t-1} & s = t. \end{cases}$$

This means we add an edge to a random vertex with a probability proportional to its degree at the time.

For $m > 1$, the process can be defined by merging sets of $m$ consecutive vertices in $G^{mt}_m$ to single vertices in $G^m_m$. Let $v_1, \ldots, v_{mt}$ be the vertices of $G^t_1$ and $v'_1, \ldots, v'_t$ be the vertices of $G^t_m$. Then $v'_i = \{v_{im+1}, \ldots, v_{im+m}\}$. The graph $G^t_m$ is a multi graph. The number of edges between vertices $v'_i$ and $v'_j$ in $G^t_m$ equals the number of edges between the corresponding set of vertices in $G^{mt}_m$. Self-loops are allowed. We focus in most of our calculations on the case $m = 1$ and then reduce the case of arbitrary values for $m$ on this case.

In Section 3 we obtain concentration bounds for the total degree of a set of vertices $S \subseteq \{v_1, \ldots, v_t\}$ during the random process. We define the degree of a set $S$ at time $t > t_0$ as

$$d^t_m(S) = \sum_{v \in S} d^t_m(v).$$

For $m = 1$ we can explicitly state the probability distribution of $d^t_1(S)$, conditioned under $d^{t-1}_1(S)$ with $t > t_0$. We have

$$P[d^t_1(S) = x \mid d^{t-1}_1(S)] = \begin{cases} \frac{d^{t-1}_1(S)}{2t-1} & x = d^{t-1}_1(S) + 1 \\ 1 - \frac{d^{t-1}_1(S)}{2t-1} & x = d^{t-1}_1(S) \\ 0 & \text{otherwise.} \end{cases}$$

### 2.2 Sparsity

There are various ways to define the sparsity of a graph. We consider the concept of nowhere-denseness. In order to define it, we need the concept of shallow topological
Definition 2.1 (Shallow topological minor [28]). A graph $M$ is an $r$-shallow topological minor of $G$ if $M$ is isomorphic to a subgraph $G'$ of $G$ if we allow the edges of $M$ to be paths of length up to $2r + 1$ in $G'$. We call $G'$ a model of $M$ in $G$. For simplicity we assume by default that $V(M) \subseteq V(G')$ such that the isomorphism between $M$ and $G'$ is the identity when restricted to $V(M)$. The vertices $V(M)$ are called principal vertices and the vertices $V(G') \setminus V(M)$ subdivision vertices. The set of all $r$-shallow topological minors of a graph $G$ is denoted by $G \vartriangledown r$.

We define the clique size over all topological minors of $G$ as

$$\omega(G \vartriangledown r) = \max_{H \in G \vartriangledown r} \omega(H).$$

Definition 2.2 (Somewhere-dense [29]). A graph class $\mathcal{G}$ is somewhere-dense if for all functions $f$ there is an $r$ and a $G \in \mathcal{G}$, such that $\omega(G \vartriangledown r) > f(r)$.

Definition 2.3 (Nowhere-dense [29]). A graph class $\mathcal{G}$ is nowhere-dense if there exists a function $f$, such that for all $r$ and all $G \in \mathcal{G}$, $\omega(G \vartriangledown r) \leq f(r)$.

We note that $f$ can be an arbitrary function but is only allowed to depend on $r$. After defining what it means for a graph class to be nowhere- or somewhere-dense, we now transfer these concepts to random graph models.

Definition 2.4 (a.a.s. somewhere-dense). A random graph model $\mathcal{G}$ is a.a.s. somewhere-dense if for all functions $f$ there is an $r$, such that

$$\lim_{n \to \infty} P[\omega(G_n \vartriangledown r) > f(r)] = 1$$

where $G_n$ is a random variable modeling a graph of $n$ vertices randomly drawn from $\mathcal{G}$.

Definition 2.5 (a.a.s. nowhere-dense). A random graph model $\mathcal{G}$ is a.a.s. nowhere-dense if there exists a function $f$ such that for all $r$

$$\lim_{n \to \infty} P[\omega(G_n \vartriangledown r) \leq f(r)] = 1$$

where $G_n$ is a random variable modeling a graph of $n$ vertices randomly drawn from $\mathcal{G}$.

Observe that a graph class is somewhere-dense if and only if it is not nowhere-dense. The concepts are complementary. A random graph model, however, can both be not a.a.s. somewhere-dense and not a.a.s. nowhere-dense.
2.3 The Pólya Urn Process

We now describe a two-color Pólya Urn process. An exhaustive overview over Pólya Urns can be found in the book by Mahmoud [24]. The model works as follows: An urn contains balls of two colors $A$ and $B$. Initially at time $n = 0$ it contains $a_0$ balls of color $A$ and $b_0$ balls of color $B$. In the next step a ball is chosen uniformly at random from the urn and its color is observed. The ball remains in the urn in this step. Afterwards balls are added to (or removed from) the urn according to the replacement matrix

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{N}$. If the observed ball is of color $A$, then $\alpha$ new balls of type $A$ and $\beta$ new balls of type $B$ are placed into the urn; if the ball is of color $B$, then $\gamma$ new balls of type $A$ and $\delta$ new balls of type $B$ are placed into the urn. A negative value for $\alpha, \beta, \gamma, \delta$, denotes that balls have to be removed instead of added. In the next step another ball is chosen uniformly random from the urn and this procedure is repeated multiple times. One is interested in the distribution of colors in the urn after some number of $n$ steps.

**Definition 2.6.** We define the random variable $A(M, n, a_0, b_0)$ to be the number of balls of color $A$ after $n$ steps in the Pólya Urn process with replacement matrix $M$ and initial configuration $(a_0, b_0)$. If $M, a_0, b_0$ are clear from the context we also write $A_n$ as a shorthand.

We call an urn balanced with balance $\sigma$ if $\alpha + \beta = \gamma + \delta = \sigma$. Let $s_n$ be the number of balls in an urn at time $n$. For balanced urns we have $s_n = a_0 + b_0 + n\sigma = s_0 + n\sigma$. An urn is additive if $\alpha, \beta, \gamma, \delta \geq 0$. An urn is called triangular if $\gamma = 0$. For typographic reasons we write the replacement matrix as $[\alpha, \beta, \gamma, \delta]$ and an urn model as $(M, a_0, b_0)$, where $M$ is the replacement matrix and $(a_0, b_0)$ is the initial configuration.

3 Degree Bounds in Preferential Attachment Graphs

The main contribution of this section is to show that under certain conditions the degree of vertices is closely centered around their expected value. Assume a subset $S$ of the first $t$ vertices with a total degree of $d^t_1(S)$ at time $t$. If $d^t_1(S)$ is high then the total degree of $S$ at time $n$ is close to its expected value, which is approximately $\sqrt{n/t} \cdot d^t_1(S)$ [30]. The higher $d^t_1(S)$ is, the closer the degree is
centered at time $t$. This is formalized in Theorem 3.8, which is proven at the end of this section. This section consists of multiple lemmas that together prove Theorem 3.8. We separately show upper and lower bounds and then join these bounds together. These upper and lower bounds are proven by first giving bounds which hold for a short time interval of time (Section 3.1) and then extending these bounds for longer intervals (Section 3.2).

Due to the technical nature of this section, we consider the set $S$ to be fixed and write $D(t) := D([t])$ as shorthand for $d_1^1(S)$ to avoid having large formulas as a superscript. We also define $D(t) := D(\lfloor t \rfloor)$ for $t \in \mathbb{R}$.

### 3.1 Short-Term Bounds

In the following we show that for small $\delta$ from time-step $t$ to $(1 + \delta)t$ it is very likely that we increase the degree of the set $S$ by a factor of $1 + \delta/2 + O(\delta^2)$. A lower bound is established in Lemma 3.1 and an upper bound follows from Lemma 3.2 and 3.3.

**Lemma 3.1.** Let $0 < \delta < 1$ and $t \geq \frac{2}{\delta^2}$. Then

$$
P\left[D((1 + \delta)t) \leq (1 + \frac{\delta}{2} - 2\delta^2)D(t) \mid D(t)\right] \leq e^{-\frac{1}{16}\delta^3D(t)}.
$$

**Proof.** For every $t' \in \mathbb{R}$ $D(t') = D([t'])$. For every $t' \in \mathbb{N}$ either $D(t') = D(t' - 1)$ or $D(t') = D(t' - 1) + 1$. Let $N$ be the number of integers between $t$ and $(1 + \delta)t$. Let $\Delta_i$ with $1 \leq i \leq N$ be the Bernoulli variable indicating that $D([t] + i) = D([t] + i - 1) + 1$ and $\Delta = \Delta_1 + \cdots + \Delta_N$. Then $D(t) + \Delta = D((1 + \delta)t)$. Furthermore

$$
P[\Delta_i = 1 \mid \Delta_1, \ldots, \Delta_{i-1}, D(t)] = \frac{D([t] + i - 1)}{2([t] + i) - 1} \geq \frac{D(t)}{2(1 + \delta)t}.
$$

Let $X = X_1 + \cdots + X_N$ be the sum of identically distributed Bernoulli variables with

$$
P[X_i = 1] = \frac{D(t)}{2(1 + \delta)t}. \quad (1)
$$

We consider two experiments: The first game is $N$ tosses of a fair coin. The second one is $N$ tosses of a biased coin, where the probability that the $i$th coin comes up head depends on the outcome of the previous coins but always is at least 1/2. Obviously, the probability of at least $s$ heads in the second experiment is at least as high as the probability of at least $s$ heads in the first experiment. The same argument implies

$$
P[\Delta \leq s \mid D(t)] \leq P[X \leq s \mid D(t)]. \quad (2)
$$
With \( t \geq \frac{2}{\delta^2} \) and (11) we get

\[
N \geq \delta t - 1 \geq (\delta - \frac{1}{2}\delta^2)t
\]

and

\[
E[X \mid D(t)] = NP[X_i = 1 \mid D(t)] \geq \frac{(\delta - \frac{1}{2}\delta^2/2)D(t)}{2(1 + \delta)}.
\] (3)

In contrast to \( \Delta \), we can directly apply Chernoff bounds to \( X \):

\[
P[X \leq (1 - \delta)E[X \mid D(t)] \mid D(t)] \leq e^{-\frac{1}{4}\delta^2E[X \mid D(t)]}.
\] (4)

Combining the above inequality with (3), (2) and (1) yields

\[
P[\Delta \leq (1 - \delta)E[X \mid D(t)] \mid D(t)] \leq P[\Delta \leq (1 - \delta)E[X \mid D(t)] \mid D(t)] \leq e^{-\frac{1}{4}\delta^2E[X \mid D(t)]} \leq e^{-\frac{\delta^3\delta^4/4}{2(1 + \delta)^2}D(t)}
\] (5)

\[
\leq e^{-\frac{\delta^3\delta^4/4}{2(1 + \delta)^2}D(t)}.
\] (6)

The left and right hand side of the following inequality are identical for \( \delta = 0 \) and their difference is increasing for \( 0 \leq \delta \leq 1 \). Hence for \( 0 \leq \delta \leq 1 \)

\[
\frac{(1 - \delta)(\delta - \delta^2/2)}{2(1 + \delta)} \geq \frac{\delta}{2} - 2\delta^2.
\] (7)

Combining (5), (7), and \( D((1 + \delta)t) = \Delta + D(t) \) yields

\[
P[\Delta \leq (\delta/2 - 2\delta^2)D(t) \mid D(t)] = P[D((1 + \delta)t) \leq (1 + \delta/2 - 2\delta^2)D(t) \mid D(t)] \leq e^{-\frac{\delta^3\delta^4}{2(1 + \delta)^2}D(t)}.
\]

Lemma 3.2. Let \( 0 < \delta < 1, t \geq \frac{2}{\delta^2}, k > 0 \) and \( \Delta = D((1 + \delta)t) - D(t) \). Then

\[
P[\Delta \geq (\delta/2 + \delta^2/2)(D(t) + k) \mid D(t), \Delta \leq k] \leq e^{-e^{k}\delta^4D(t)}.
\]

Proof. This proof is similar to the one of Lemma 3.1. Let \( N \) be the number of integers between \( t \) and \( (1 + \delta)t \). Let \( \Delta_i \) with \( 1 \leq i \leq N \) be the Bernoulli variable indicating that \( D([t] + i) = D([t] + i - 1) + 1 \). Then \( \Delta = \Delta_1 + \cdots + \Delta_N \). Furthermore

\[
P[\Delta_i = 1 \mid \Delta_1, \ldots, \Delta_{i-1}, \Delta \leq k, D(t)] = \frac{D([t] + i + 1)}{2([t] + i - 1)} \leq \frac{D(t) + k}{2t}.
\]
Let \( X = X_1 + \cdots + X_N \) be the sum of identically distributed Bernoulli variables with
\[
P[X_i = 1] = \frac{D(t) + k}{2t}.
\]
(8)

With a similar argument as in the proof of Lemma 3.1 we have
\[
P[\Delta \geq s \mid D(t)] \leq P[X \geq s \mid D(t)].
\]
(9)

With \( t \geq \frac{2}{\delta^2} \) and (8) we get
\[
N \leq \delta t + 1 \geq (\delta + \frac{1}{2}\delta^2)t
\]
and
\[
E[X \mid D(t)] = NP[X_i = 1 \mid D(t)] \geq \frac{(\delta + \delta^2/2)(D(t) + k)}{2}.
\]
(10)

Chernoff bounds applied on \( X \) yield:
\[
P \left[ X \geq (1 + \delta)E[X \mid D(t)] \right] \leq e^{-\frac{1}{2}\delta^2 E[X \mid D(t)]}.
\]
(11)

For \( 0 \leq \delta \leq 1 \) we have
\[
\frac{(1 + \delta)(\delta + \delta^2/2)}{2} \leq \frac{\delta}{2} + \frac{\delta^2}{2}.
\]
(12)

Combining (12), (10), (9) and (11) yields
\[
P \left[ \Delta \geq (\delta/2 + \delta/2)(D(t) + k) \mid D(t) \right] \leq P \left[ \Delta \geq (1 + \delta)E[X \mid D(t)] \mid D(t) \right] \leq e^{-\frac{1}{4}\delta^2 E[X \mid D(t)]} \leq e^{-\frac{1}{4}\delta^3 D(t)}.
\]

Lemma 3.3. Let \( 0 < \delta \leq \frac{1}{\sqrt{2}} \) and \( t \geq \frac{2}{\delta^2} \). Then
\[
P \left[ D((1 + \delta)t) \geq (1 + \delta/2 + 2\delta^2)D(t) \mid D(t) \right] \leq \ln(2t)e^{-\frac{1}{4}\delta^3 D(t)}
\]

Proof. By setting \( \Delta = D((1 + \delta)t) - D(t) \), \( \beta = (\delta + \delta^2)/2 \), \( p = e^{\frac{1}{4}\delta^3 D(t)} \) and using Lemma 3.2 we get
\[
P \left[ \Delta \geq \beta D(t) + \beta k \mid D(t), \Delta \leq k \right] \leq p.
\]
(13)
We define bounds \( k_i \) recursively by \( k_0 = \delta t \) and \( k_{i+1} = \beta D(t) + \beta k_i \) for \( i \geq 0 \). Let us proof by induction on \( i \) that

\[
P[\Delta \geq k_i \mid D(t)] \leq ip. \tag{14}
\]

The base case \( i = 0 \) follows from \( \Delta \leq k_0 \) being always true. Let us now assume that \( i \geq 0 \) and split the left-hand side of (14) into

\[
P[\Delta \geq k_{i+1} \mid D(t)] = P[\Delta \geq k_{i+1} \mid D(t), \Delta \geq k_i] P[\Delta \geq k_i \mid D(t)]
+ P[\Delta \geq k_{i+1} \mid D(t), \Delta < k_i] P[\Delta < k_i \mid D(t)].
\]

By induction we know that \( P[\Delta \geq k_i \mid D(t)] \leq ip \) and by (13) we know that \( P[\Delta \geq k_{i+1} \mid D(t), \Delta < k_i] \leq \bar{p} \). Together this is at most \((i + 1)p\). This concludes the proof of (14). Solving the linear recurrence relation \( k_i \) yields

\[
k_i = \beta D(t) + \beta^2 D(t) + \cdots + \beta^i D(t) + \beta^i k_0 \leq \frac{\beta}{1 - \beta} D(t) + \beta^i k_0.
\]

With \( 0 < \delta < 1/e^2 \) we have

\[
\frac{\beta}{1 - \beta} \leq \frac{\delta}{2} + \delta^2 3/2
\]

and

\[
\beta^{\ln(e^2 t)} k_0 \leq \left(\frac{\delta + \delta^2}{2}\right)^{\ln(e^2 t)} k_0 \leq \delta^{\ln(e^2 t)} \delta t \leq \delta^2 \delta^{\ln(2t)} t \leq \delta^2 (2t)^{\ln(\delta)} t \leq \delta^2 (2t)^{-2} t \leq \frac{\delta^2}{2}
\]

Therefore

\[
k_{\ln(e^2 t)} \leq \frac{\beta}{1 - \beta} D(t) + k_0^{\ln(e^2 t)} \leq (\delta/2 + 2\delta^2) D(t). \tag{15}
\]

Combining (14) and (15) gives us with \( i = \ln(e^2 t) \) and \( D((1 + \delta) t) = \Delta + D(t) \)

\[
P[D((1 + \delta) t) \geq (1 + \delta/2 + 2\delta^2) D(t) \mid D(t)] = P[\Delta + D(t) \geq (1 + \delta/2 + 2\delta^2) D(t) \mid D(t)] = P[\Delta \geq \delta/2 + 2\delta^2 D(t) \mid D(t)] \leq P[\Delta \geq k_{\ln(e^2 t)} \mid D(t)] \leq \ln(e^2 t) e^{1/2 \delta^2 D(t)}.
\]

\[\Box\]
3.2 Long-Term Bounds

In the previous subsection we established bounds for a small interval from step $t$ to step $(1 + \delta)t$ with an error of order $\delta^2$. In this subsection we combine these bounds into long-term bounds. We get these bounds by defining positions $t_0 = t$ and $t_{k+1} = (1 + \delta_k)t_k$ with $k \in \mathbb{N}$ and using the union bound to guarantee that for each interval from time $t_k$ to $t_{k+1}$ the short-term bounds hold. The choice of $\delta_k$ is of high importance for the success of this strategy. It turns out that we need the product $\prod_{i=1}^{\infty} (1 + \delta_i)$ to diverge, but the error $\prod_{i=1}^{\infty} (1 + \delta^2_i)$ to converge. Eventually, we settled for $\delta_k = \varepsilon/k^{2/3}$, which satisfies both conditions.

Lemma 3.4 and Lemma 3.5 bridge the gap between the bounds for the recursively defined small intervals and the expected growth of the degree of a vertex over a longer period of time. These lemmas state that if the degree differs by a factor of $(1 \pm \varepsilon)$ from its expected value then there has been one short interval where the allowed error of order $\delta^2$ has been exceeded. The lemmas do not directly mention degrees of vertices but a monotone increasing function $f(t)$. This function will later be replaced with the degree of a vertex. Lemma 3.6 then uses the short-term bounds from Lemma 3.4 and Lemma 3.5 to give bounds for each interval from time $t$ to $t_{k+1}$. Finally, Lemma 3.7 combines Lemma 3.6 and Lemma 3.6 into bounds for preferential attachment graphs with parameter $m = 1$ and Theorem 3.8 generalizes this result for arbitrary $m$.

**Lemma 3.4.** Let $0 < \varepsilon \leq 1/8$, $t > 0$, and $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. For every $k \in \mathbb{N}$ let $\delta_k = \frac{k}{k^{2/3}}$, $h_k = \prod_{i=1}^{k} (1 + \delta_i)$, and $c_k = \prod_{i=1}^{k-1} (1 + \frac{1}{2} \delta_i - 2 \delta^2_i)$.

If there is an $n \in \mathbb{N}$, such that $t < n$ and $f(n) < (1 - \varepsilon)\sqrt{\frac{n}{t}}f(t)$, then there is a $k \in \mathbb{N}$ such that $f((1 + \delta_k)h_k t) < (1 + \frac{3}{2} \delta_k - 2 \delta^2_k)f(h_k t)$ and $f(h_k t) \geq c_k f(t)$.

**Proof.** Consider any $n \in \mathbb{N}$, $n \geq t$. Let $k(n) \in \mathbb{N}$ be the maximal value such that $h_{k(n)}t \leq n$. Then $\frac{n}{1 + \delta_{k(n)}} \leq h_{k(n)}t$, because of the maximality of $k(n)$. Notice that

$$(1 - \varepsilon)\sqrt{\frac{n}{t}} \leq e^{-\frac{\varepsilon}{2}}e^{-\frac{\varepsilon}{2}}\sqrt{\frac{n}{t}} = e^{-\frac{\varepsilon}{2}}\sqrt{\frac{n}{te^\varepsilon}} \leq e^{-\frac{\varepsilon}{2}}\sqrt{\frac{n}{t(1 + \delta_{k(n)})}} \leq e^{-\frac{\varepsilon}{2}}\sqrt{h_{k(n)}}$$

and for all $k \in \mathbb{N}$

$$c_k = \prod_{i=1}^{k-1} (1 + \frac{1}{2} \delta_i - 2 \delta^2_i) \geq \prod_{i=1}^{k-1} e^{\frac{1}{2} \delta_i - 3 \delta^2_i} \geq \left(\prod_{i=1}^{k-1} e^{\delta_i} \right)^{\frac{1}{2}} \prod_{i=1}^{\infty} e^{-3 \delta^2_i} \geq \left(\prod_{i=1}^{k-1} (1 + \delta_i) \right)^{\frac{1}{2}} e^{-4 \varepsilon^2} \geq e^{-\frac{\varepsilon}{2}}\sqrt{h_k}.$$
Combining the upper two inequalities gives us \((1 - \varepsilon)\sqrt{n/t} \leq c_k\). We assumed 
\(f(n) < (1 - \varepsilon)\sqrt{\frac{n}{t}} f(t)\). Monotonicity of \(f\) yields 
\[f(h_k(n)t) \leq f(n) < (1 - \varepsilon)\sqrt{\frac{n}{t}} f(t) \leq c_k(n)f(t).\]

Let \(J = \{ j \geq 0 \mid f(h_{j+1}t) < c_{j+1}f(t) \}\). The set \(J\) is not empty because \(k(n) - 1 \in J\) by the equation above. Furthermore, \(0 \notin J\) because \(h_1 = c_1 = 1\) and therefore \(f(h_1t) = f(t) = c_1f(t)\).

Let now \(k\) be the minimal value in \(J\). Then \(k > 0\), \(f(h_k t) \geq c_k f(t)\), and \(f(h_{k+1} t) < c_{k+1} f(t)\). At last, we have 
\[f((1 + \delta_k)h_k t) = f(h_{k+1} t) < c_{k+1} f(t) = (1 + \frac{1}{2}\delta_k - 2\delta_k^2) c_k f(t) \leq (1 + \frac{1}{2}\delta_k - 2\delta_k^2) f(h_k t).\]

\[\Box\]

**Lemma 3.5.** Let \(0 < \varepsilon \leq 1/40\), \(t > 0\), and \(f: \mathbb{R} \to \mathbb{R}\) be an increasing function. For every \(k \in \mathbb{N}\) let \(\delta_k = \frac{\varepsilon}{2\sqrt{t}}\), \(h_k = \prod_{i=1}^{k-1}(1 + \delta_i)\), and \(c_k = \prod_{i=1}^{k-1}(1 + \frac{1}{2}\delta_i + 2\delta_i^2)\).

If there is an \(n \in \mathbb{N}\), such that \(t < n\) and \(f(n) > (1 + \varepsilon)\sqrt{\frac{n}{t}} f(t)\), then there is a \(k \in \mathbb{N}\) such that \(f((1 + \delta_k)h_k t) > (1 + \frac{1}{2}\delta_k + 2\delta_k^2) f(h_k t)\) and \(f(h_k t) < c_k f(t)\).

**Proof.** This proof is similar to the one of Lemma 3.4. Consider any \(n \in \mathbb{N}\), \(n \geq t\). Let \(k(n) \in \mathbb{N}\) be the minimal value such that \(n \leq h_k(n) t\). Then \(h_k(n) t \leq n(1 + \delta_k(n))\), because of the maximality of \(k(n)\). Notice that 
\[(1 + \varepsilon)\sqrt{\frac{n}{t}} \geq e^{\frac{1}{4}c\varepsilon} e^{\frac{1}{2}\varepsilon} \sqrt{\frac{n}{t}} = e^{\frac{1}{2}\varepsilon} \sqrt{\frac{n(1 + \delta_k(n))}{t}} \geq e^{\frac{1}{2}\varepsilon} \sqrt{h_k(n)}\]

and for all \(k \in \mathbb{N}\)
\[c_k = \prod_{i=1}^{k-1}(1 + \frac{1}{2}\delta_i + 2\delta_i^2) \leq \prod_{i=1}^{k-1} e^{\frac{1}{2}(\delta_i - \delta_i^2)} = \prod_{i=1}^{k-1} e^{\frac{1}{2}(\delta_i - \delta_i^2)} \leq \prod_{i=1}^{k-1} e^{\frac{1}{2}\varepsilon} \leq e^{\frac{1}{2}\varepsilon} \sqrt{h_k}.\]

Combining the upper two inequalities gives us \((1 + \varepsilon)\sqrt{n/t} \geq c_k\). We assumed 
\(f(n) > (1 + \varepsilon)\sqrt{\frac{n}{t}} f(t)\). Monotonicity of \(f\) yields 
\[f(h_k(n) t) \geq f(n) > (1 + \varepsilon)\sqrt{\frac{n}{t}} f(t) \geq c_k(n) f(t).\]
Let $J = \{ j \geq 0 \mid f(h_{j+1}) > c_{j+1}f(t) \}$. The set $J$ is not empty because $k(n) - 1 \in J$ by the equation above. Furthermore, $0 \notin J$ because $h_1 = c_1 = 1$ and therefore $f(h_1t) = f(t) = c_1f(t)$. Let now $k$ be the minimal value in $J$. Then $k > 0$, $f(h_kt) \leq c_kf(t)$, and $f(h_{k+1}t) > c_{k+1}f(t)$. At last, we have

$$f((1 + \delta_k)h_kt) = f(h_{k+1}t) > c_{k+1}f(t)$$

$$= (1 + \frac{1}{2}\delta_k + 2\delta_k^2)c_kf(t) \geq (1 + \frac{1}{2}\delta_k + 2\delta_k^2)f(h_kt).$$

Lemma 3.6. Let $0 < \varepsilon \leq 1/40$, $t > \frac{1}{e\varepsilon}$. For every $k \in \mathbb{N}$ let $\delta_k = \frac{\varepsilon}{k^{3/4}}$, $h_k = \prod_{i=1}^{k-1}(1 + \delta_i)$, $c_k^+ = \prod_{i=1}^{k-1}(1 + \frac{1}{4}\delta_i + 2\delta_i^2)$ and $c_k^- = \prod_{i=1}^{k-1}(1 + \frac{1}{4}\delta_i - 2\delta_i^2)$. Then

$$P\left[D((1 + \delta_k)h_kt) < (1 + \frac{1}{2}\delta_k - 2\delta_k^2)D(h_kt), D(h_kt) \geq c_k^-D(t) \mid D(t)\right]$$

$$\leq e^{-\frac{1}{40}\delta_k^2c_k^-D(t)},$$

$$P\left[D((1 + \delta_k)h_kt) > (1 + \frac{1}{2}\delta_k + 2\delta_k^2)D(h_kt), D(h_kt) \leq c_k^+D(t) \mid D(t)\right]$$

$$\leq \ln(2t)e^{-\frac{1}{40}\delta_k^2c_k^+D(t)}.$$

Proof. At first we focus on the first bound. By the law of total probability

$$P\left[D((1 + \delta_k)h_kt) < (1 + \frac{1}{2}\delta_k - 2\delta_k^2)D(h_kt), D(h_kt) \geq c_k^-D(t) \mid D(t)\right]$$

$$\leq P\left[D((1 + \delta_k)h_kt) < (1 + \frac{1}{2}\delta_k - 2\delta_k^2)D(h_kt) \mid D(h_kt) \geq c_k^-D(t)\right].$$

The second line of this equation states the probability that the degree of a vertex is in the future below a certain threshold under the condition that it is currently above a certain threshold. We can bound this probability if we assume that it currently is not above, but exactly at the threshold:

$$P\left[D((1 + \delta_k)h_kt) < (1 + \frac{1}{2}\delta_k - 2\delta_k^2)D(h_kt) \mid D(h_kt) \geq c_k^-D(t)\right]$$

$$\leq P\left[D((1 + \delta_k)h_kt) < (1 + \frac{1}{2}\delta_k - 2\delta_k^2)D(h_kt) \mid D(h_kt) = c_k^-D(t)\right].$$

Similarly, the probability that the degree of a vertex is in the future above a certain threshold under the condition that it is currently below a certain threshold can be
bounded by assuming that it is currently exactly at the threshold. It is therefore sufficient to prove the following two bounds

\[ P\left( (1 + \delta_k) h_k t \leq (1 + \frac{1}{2} \delta_k - 2 \delta_k^2) D(h_k t) \mid D(h_k t) = c_k^+ D(t) \right) \leq e^{-\frac{1}{2} \delta_k^2 c_k^+ D(t)}, \]

\[ P\left( (1 + \delta_k) h_k t > (1 + \frac{1}{2} \delta_k + 2 \delta_k^2) D(h_k t) \mid D(h_k t) = c_k^+ D(t) \right) \leq \ln(e2t) e^{-\frac{1}{2} \delta_k^2 c_k^+ D(t)}. \]

Lemma 3.1 and 3.3 state that if \( 0 \leq \delta_k = e/k^{3/2} \leq 1/e^2 \) and \( h_k t \geq 2/\delta_k^2 \) for every \( k \) then these bounds are true. We observe that for \( 0 \leq \varepsilon \leq 1/8 \) the first precondition is always satisfied. We will finish the proof by showing that \( h_k t \geq 2/\delta_k^2 \) for every \( k \).

Observe that for \( 0 \leq k \leq 1 \) we have \( h_k t \geq 2/\varepsilon^2 \geq 2/\delta_k^2 \). We can therefore assume \( k \geq 2 \). First, we need a lower bound for \( h_k \)

\[
h_k = \prod_{i=1}^{k-1} \left( 1 + \frac{\varepsilon}{i^{2/3}} \right) \geq \prod_{i=1}^{k-1} e^{\varepsilon i^{2/3}} \geq e^{3 \varepsilon (k-1)^{1/3}} \geq e^{2 \varepsilon k^{1/3}}.
\]

One can show that \( e^{x}/x^4 \geq e^{4}/256 \) for \( x > 0 \). We therefore get for \( x = 2 \varepsilon k^{1/3} \)

\[
h_k \geq e^{2 \varepsilon k^{1/3}} = \frac{e^{2 \varepsilon k^{1/3}}}{(2 \varepsilon k^{1/3})^4} \frac{16 \varepsilon^6}{\delta_k^2} \geq \frac{16 \varepsilon^6}{\delta_k^2} \cdot \frac{16 \varepsilon^6}{256} \geq \frac{\varepsilon^6}{\delta_k^2}.
\]

Since \( t \geq \frac{1}{e^2} \) it follows that \( h_k t \geq \frac{2}{\delta_k^2} \). \( \square \)

**Lemma 3.7.** For \( 0 < \varepsilon \leq 1/40 \) and \( \frac{1}{e^2} < t < N \) we have

\[ P\left[ (1 - \varepsilon) \sqrt{\frac{N}{t} D(t)} < D(n) < (1 + \varepsilon) \sqrt{\frac{N}{t} D(t)} \ \text{for all} \ n \geq t \mid D(t) \right. \]

\[ \geq 1 - 2 \ln(e2t) \varepsilon^{-6} \exp\left(-\varepsilon^{15} 10^{-24} D(t)\right) \]

**Proof.** Observe that

\[ P\left[ (1 - \varepsilon) \sqrt{\frac{N}{t} D(t)} < D(n) < (1 + \varepsilon) \sqrt{\frac{N}{t} D(t)} \ \text{for all} \ n \geq t \mid D(t) \right. \]

\[ \geq 1 - (p^+ + p^-) \]

with

\[ p^- := P\left[ D(n) < (1 - \varepsilon) \sqrt{\frac{N}{t} D(t)} \ \text{for some} \ n \geq t \mid D(t) \right] \]

\[ p^+ := P\left[ D(n) > (1 + \varepsilon) \sqrt{\frac{N}{t} D(t)} \ \text{for some} \ n \geq t \mid D(t) \right]. \]
We proceed by finding upper bounds for \( p^+ \) and \( p^- \). For \( k \in \mathbb{N} \) let \( \delta_k = \frac{\varepsilon_k}{k^{1/3}} \), 
\[ h_k = \prod_{i=1}^{k-1} (1 - \delta_i), \quad c_k^- = \prod_{i=1}^{k-1} (1 - \frac{1}{2} \delta_i - 2\delta_i^2) \text{ and } c_k^+ = \prod_{i=1}^{k-1} (1 - \frac{1}{2} \delta_i + 2\delta_i^2). \]
Every function \( f(t) : \mathbb{R} \to \mathbb{R} \) which is a realization of the random variables \( D(t) \) is monotone increasing. It follows using Lemma 3.4, Lemma 3.5, the union bound over all possible choices of \( k \), and Lemma 3.6 that

\[
p^- \leq \sum_{k=0}^{\infty} P\left[ D((1 + \delta_k)h_k t) < (1 + \frac{1}{2} \delta_k - 2\delta_k^2) D(h_k t), D(h_k t) \geq c_k^- D(t) \mid D(t) \right] 
\leq \sum_{k=0}^{\infty} e^{-\frac{1}{16} \delta_k^3 c_k^-} D(t),
\]

\[
p^+ \leq \sum_{k=0}^{\infty} P\left[ D((1 + \delta_k)h_k t) > (1 + \frac{1}{2} \delta_k + 2\delta_k^2) D(h_k t), D(h_k t) \leq c_k^+ D(t) \mid D(t) \right] 
\leq \sum_{k=0}^{\infty} \ln(2t) e^{-\frac{1}{16} \delta_k^3 c_k^+} D(t).
\]

We finish this proof with a longer calculation.

\[
p^+ + p^- \leq (\ln(2t) + 1) \sum_{k=0}^{\infty} e^{-\frac{1}{16} \delta_k^3 c_k^-} D(t)
= \ln(15t) \sum_{k=0}^{\infty} \exp\left(-\frac{\varepsilon^3}{16k^2} \prod_{i=1}^{k-1} (1 + \frac{\varepsilon}{4i^{2/3}} - \frac{\varepsilon^2}{16i^{4/3}}) D(t)\right)
= \ln(15t) \sum_{k=0}^{\infty} \exp\left(-\frac{\varepsilon^3}{16k^2} e^{\frac{k^{1/3}}{3}} D(t)\right)
\leq \ln(15t) \left( \sum_{k=0}^{\varepsilon^{-6}} \exp\left(-\frac{\varepsilon^{15}}{16} D(t)\right) + \sum_{k=\varepsilon^{-6}+1}^{\infty} \exp\left(-\frac{\varepsilon^3}{16k^2} e^{\frac{k^{1/6}}{3}} D(t)\right) \right)
\leq \ln(15t) \left( \varepsilon^{-6} \exp\left(-\frac{\varepsilon^{15}}{16} D(t)\right) + \sum_{k=\varepsilon^{-6}+1}^{\infty} \exp\left(-\varepsilon^3 k 10^{-23} D(t)\right) \right)
\leq \ln(15t) \varepsilon^{-6} \exp\left(-\varepsilon^{15} 10^{-24} D(t)\right)
\]

\[ \square \]

The last step in this section is to generalize the previous lemma for different parameters \( m \) of the preferential attachment process.
Theorem 3.8. Let $0 < \varepsilon \leq 1/40$, $t, m \in \mathbb{N}$, $t > \frac{1}{\varepsilon}$ and $S \subseteq \{v_1, \ldots, v_t\}$. Then

$$P\left[ (1 - \varepsilon) \sqrt{\frac{n}{t}} d_m^t(S) < d_m^n(S) < (1 + \varepsilon) \sqrt{\frac{n}{t}} d_m^t(S) \right] \geq 1 - \ln(15t) e^{-\varepsilon O(1)} d_m^t(S).$$

Proof. As stated in the introduction, we can simulate $G_m^n$ via $G_{mn}^1$, by merging every $m$ consecutive vertices into a single one. Let $G_m^n$ be a graph with vertices $V = \{v_1, \ldots, v_n\}$. We can assume that this graph has been constructed from a graph $G_{mn}^1$ with vertex set $V' = \{v'_1, \ldots, v'_{mn}\}$ by merging the vertices $v'_{i+1}, \ldots, v'_{i+m}$ into vertex $v_i$ for $1 \leq i \leq n$. Let $S' \subseteq V'$ be the set of vertices in $G_{mn}^1$ which are merged into $S$. Since the graph allows multi-edges it is easy to see that $d_m^n(S) = d_{mn}^1(S')$. This means that $d_m^n(S)$ and $d_{mn}^1(S')$ have the same probability distribution. Lemma 3.7 gives with $d_{mn}^1(S') = D(mn)$ the following statement

$$P\left[ (1 - \varepsilon) \sqrt{\frac{n}{t}} d_{mn}^1(S') < d_{mn}^1(S') < (1 + \varepsilon) \sqrt{\frac{n}{t}} d_{mn}^1(S') \right] \geq 1 - \ln(15t) e^{-\varepsilon O(1)} d_m^t(S).$$

which proves the claim with $d_m^n(S) = d_{mn}^1(S')$. \qed

4 Preferential Attachment Graphs and Pólya Urns

In this section we consider a two-color Pólya Urn process and its relation to the probability distribution of degrees in preferential attachment graphs. At first, we observe in Lemma 4.1 that the degree of vertices in a preferential attachment graph follows exactly the same probability distribution as the number of balls of one color in a certain Pólya Urn process. This means that concentration results for distributions in Pólya Urns can be easily restated as concentration results for degrees of vertices. Our concentration bounds presented in Section 3 were obtained using no other methods than Chernoff bounds. By using the rich set of existing results on Pólya Urns it might be possible to dramatically improve these results. This section builds on results by Flajolet, Dumas and Puyhaubert [16].

The three authors give a closed expression for the color distribution for a specific urn process with fixed initial configuration after $n$ steps (see Proposition 4.2). We use Lemma 4.1 to rephrase this result as a closed expression for the exact probability distribution of the degree of the first vertex in a preferential attachment graph of size $n$. Knowing the exact distribution, we then give asymptotically tight
concentration bounds for the degree of the first vertex in Lemma 4.3. We observe that $P[d_1^n(v_1) > c \sqrt{n}] \leq e^{-c^2/4}$ for any $c, n \in \mathbb{N}$. Our results in Section 3 did not yield unconditional concentration bounds for the degree of the first vertex.

Unfortunately, no one seems to know a closed expression for the degree of arbitrary vertices under the condition that they have a certain degree at some earlier time, i.e., probabilities of the form $P[d_1^n(S) = k \mid d_1^t(S) = l]$. The results by Flajolet, Dumas and Puyhaubert [16] can be restated as expressions for such probabilities. These are, however, not closed expressions as they contain an alternating sum. The summands of this alternating sum are very large but alternate in sign such that they almost completely cancel each other out. This means it is very hard to get an approximate closed form for this sum. Good approximations for each summand may still lead to bad approximations for the whole sum because of cancellation. We were unable to use these results to get an approximation for probabilities of the form $P[d_1^n(S) = k \mid d_1^t(S) = l]$. However, we were able to replace the alternating sum with a non-alternating sum. For each probability of the form $P[d_1^n(S) = k \mid d_1^t(S) = l]$ we discovered an equivalent sum where each summand is non-negative. This result is stated in Lemma 4.9. This sum might now be easier to approximate because no summands cancel each other out and a good approximation for each summand could lead to a good approximation for the whole sum. We believe that this result can help improving the bounds presented in Section 3. We also observed, using computer algebra tools, that probabilities of the form $P[d_1^n(S) = k \mid d_1^t(S) = l]$ can be expressed in terms of generalized hypergeometric functions. There exists a rich set of tools for analyzing generalized hypergeometric functions [17], which also might lead to improved bounds.

4.1 Relation Between Pólya Urns and Degrees in Preferential Attachment Graphs

We now observe that the degree of a vertex of a preferential attachment graph follows the same probability distribution as the color $A$ in the balanced triangular urn of the form

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$ 

We fix a vertex $v$ of the preferential attachment process. The degree of $v$ is represented by the number of balls of color $A$ and the degree of all other vertices is represented by the balls of color $B$. In each step the total degree of the graph grows by two. Either the degree of $v$ grows by one and the degree of the vertex attaching to $v$ grows by one (first line of $M$) or the degree of $v$ stays the same and the degree of two other vertices grows by one (second line of $M$). This relationship is formalized in the following lemma.
Lemma 4.1. Let $M = [1, 1, 0, 2]$. Let $n \geq t$. For every $d, k \in \mathbb{N}$
\[
P\left[d_1^n(v_t) = k\right] = P\left[A(M, n - t + 1, 1, 2t - 2) = k\right]
\]
\[
P\left[d_1^n(S) = k \mid d_1^n(S) = d\right] = P\left[A(M, n - t, d, 2t - 1 - d) = k\right].
\]

Proof. We start with the first equality. The random variables $d_1^n(v_t)$ and $A(M, n - t + 1, 1, 2t - 2)$ both describe how a quantity (the degree of a vertex or the number of balls of color $A$) behaves over time. Initially, this quantity is one and then there are $n - t + 1$ rounds in which it may be increased by one. In each round either two edges or two balls are added. Therefore in both random processes the probability that the quantity increases in the $i$th round is $x/(2t - 1 + 2i)$, assuming it currently is $x$. It follows that both random variables follow the same distribution. Intuitively, one can visualize this equality as follows. The color $A$ refers to the degree of the first vertex, color $B$ to the sum of the other degrees and $s_n$ to the sum of all degrees at time $n$. We can think of it as at time 0 there is a single vertex with degree one, which represents a stub and at the next time step this gets extended to a self-loop and a new vertex with a stub is inserted. At the next step this stub is either extended to an edge to $v_1$ or to a self-loop to $v_2$.

Now consider the second equality. If we assume that $d_1^n(S) = d$ then there are only $n - t$ rounds after the $t$th round in which the degree of the set $S$ may increase. The probability that the degree or the number of balls of color $A$ increases in the $i$th round is again $x/(2t - 1 + 2i)$, assuming it currently is $x$. It follows that $d_1^n(S)$ under the condition $d_1^n(S) = d$ and $A(M, n - t, d, 2t - 1 - d)$ follow the same distribution. \qed

4.2 Tight Bounds for the First Vertex in Preferential Attachment Graphs

Due to Flajolet, Dumas and Puyhaubert [16], we get the following closed form expression by solving the urn model ($[1, 1, 0, 2], 1, 0$).

Proposition 4.2 ([16]). Given the urn model ($[1, 1, 0, 2], 1, 0$), we get
\[
P[A_n = k] = \frac{k - 1}{n} 2^{k-1} \binom{2n-k}{n-1}. \binom{2n}{n}^{-1}
\]

By Lemma 4.1, this yields a formula for $P[d_1^n(v_1) = k]$. We can directly derive asymptotically tight bounds for the degree of the first vertex.

Lemma 4.3. Let $n, c \in \mathbb{N}$. Then $P[d_1^n(v_1) > c\sqrt{n}] \leq e^{-\frac{c^2}{2}}$. This bound is asymptotically tight in $n$. 22
Proof. Let \( M = [1, 1, 0, 2] \), \( a_0 = 1 \) and \( b_0 = 0 \). By Lemma 4.1 we have

\[
P[d_1^n(v_1) > c\sqrt{n}] = \sum_{i = \lceil c\sqrt{n} \rceil}^{\infty} P[A_n = i].
\]

One can use Proposition 4.2 and computer algebra tools to verify that

\[
\lim_{n \to \infty} \sum_{i = \lceil c\sqrt{n} \rceil}^{\infty} P[A_n = i] = e^{-c^2/4}
\]

and that the limit is approached from below.

From Lemma 4.3 we can follow a special case.

Lemma 4.4. For every \( \varepsilon > 0 \) there exists an \( N_\varepsilon \) such that for all \( N_\varepsilon < n \in \mathbb{N} \)

\[
P[d_1^n(v_1) \leq \varepsilon \sqrt{n}] \geq \frac{1}{n}.
\]

Proof. We use Lemma 4.3

\[
P[d_1^n(v_1) \leq c\sqrt{n}] = 1 - P[d_1^n(v_1) > c\sqrt{n}] > 1 - e^{-c^2/4}.
\]

We set \( c = 2\sqrt{\log(\frac{n}{n-1})} \). For large enough \( n \) we have \( c \leq \varepsilon \) for every \( \varepsilon > 0 \). Therefore for every \( \varepsilon > 0 \) there exists an \( N_\varepsilon \) such that for all \( n > N_\varepsilon \)

\[
P[d_1^n(v_1) \leq \varepsilon \sqrt{n}] \geq P \left[ d_1^n(v_1) \leq 2\sqrt{\log(\frac{n}{n-1})\sqrt{n}} \right] > 1 - e^{-\log(\frac{n}{n-1})} = \frac{1}{n}.
\]

\[\square\]

4.3 Partial Results for Improved Degree Bounds in Preferential Attachment Graphs

According to Lemma 4.1 a simple formula for the probability distribution in the urn \( [1, 1, 0, 2] \) for arbitrary \( a_0 \) and \( b_0 \) leads to a simple formula for the probability distribution of degrees in preferential attachment graphs. There is no closed formula for the \( [1, 1, 0, 2] \) matrix and arbitrary \( a_0 \) and \( b_0 \) in the aforementioned work. With Proposition 4.5 by Flajolet, Dumas and Puyhaubert we have an exact formula for the probability distribution of any balanced triangular urn and arbitrary values \( a_0 \) and \( b_0 \).
Proposition 4.5 ([16]). For \((M, a_0, b_0)\) with \(M = \begin{pmatrix} \alpha & \sigma - \alpha \\ 0 & \sigma \end{pmatrix}\), we get

\[
P[A_n = a_0 + k\alpha] = 
\frac{\Gamma(n + 1)\Gamma(n + \frac{a_0}{\alpha})}{\Gamma(n + \frac{a_0}{\alpha} + 1)} \left( k + \frac{a_0}{\alpha} - 1 \right) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left( n + \frac{b_0 - ai}{\sigma} - 1 \right).
\]

Unfortunately, this result contains a difficult alternating sum. In the following lemma, we observe that the alternating sum collapses for the special case \(\sigma = 2, \alpha = 1\) and \(b_0 = 0\). This yields a simpler closed expression. This result can be seen as a generalization of Proposition 4.2 to arbitrary values for \(a_0\).

Lemma 4.6. For \([1, 1, 0, 2], a_0, 0\), we get

\[
P[A_n = a_0] = 
\frac{\Gamma(n + 1)\Gamma(\frac{1}{2} + n)}{\Gamma(\frac{1}{2} + n)\Gamma(\frac{1}{2})} \left( k + a_0 - 1 \right) \frac{k}{n} \frac{2k}{2^n} \frac{\binom{2n-k-1}{n-1}}{2^n}.
\]

Proof. The sum in Proposition 4.5 is hard to simplify but by setting \([1, 1, 0, 2], a_0, 0\) and using the fact that the two values of Proposition 4.2 and Proposition 4.5 have to be equal to \(P[A_n = 1 + k]\), we get

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \frac{n - j}{2} - 1 \right) = \frac{k}{n} \frac{2k}{2^n} \frac{\binom{2n-k-1}{n-1}}{2^n}.
\]

Now we can get a closed formula for the sum:

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} \left( \frac{n - i}{2} - 1 \right) = \frac{k}{n} \frac{2k}{2^n} \frac{\binom{2n-k-1}{n-1}}{2^n} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(n + 1)\Gamma(\frac{1}{2})}.
\]

Since the sum does not depend on \(a_0\) we directly get a closed expression for arbitrary \(a_0\) and \(b_0 = 0\) by inserting that value and \(M\) into Proposition 4.5. \(\square\)

For the rest of this section we want to find a simpler formula for the probability distribution of the \([1, 1, 0, 2]\) urn for arbitrary values \(a_0, b_0\). We want to avoid an alternating sum as in Proposition 4.5. For this, we first consider a simpler urn model with replacement matrix \([2, 0, 0, 2]\). This is a scaled version of the original Pólya Urn with matrix \([1, 0, 0, 1]\).

Proposition 4.7 ([16]). Let \(I = [2, 0, 0, 2]\) and \(a_0, b_0\) arbitrary,

\[
P[A_n = a_0 + 2k] = 
\frac{\binom{a_0/2+k-1}{a_0/2-1} \binom{n+b_0/2-k-1}{b_0/2-1} \binom{(a_0+b_0)/2+n-1}{(a_0+b_0)/2-1}}{\binom{a_0/2+k-1}{a_0/2-1} \binom{n+b_0/2-k-1}{b_0/2-1} \binom{(a_0+b_0)/2+n-1}{(a_0+b_0)/2-1}}.
\]

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Now we can rewrite the probability distribution of the $[1, 1, 0, 2]$ urn for arbitrary $a_0, b_0$ as a sum involving the distribution on the $[2, 0, 0, 2]$ urn and the $[1, 1, 0, 2]$ urn restricted to $b_0 = 0$. For these two settings closed formulas are given by Lemma 4.6 and Proposition 4.7. Since the sum is a sum of probabilities, every summand is non-negative.

**Lemma 4.8.** Let $M = [1, 1, 0, 2]$ and $I = [2, 0, 0, 2]$.

$$P[A(M, n, a_0, b_0) = a_0 + k] = \sum_{i=0}^{\infty} P[A(M, i, a_0, 0) = a_0 + k] P[A(I, n, a_0, b_0) = a_0 + 2i]$$

**Proof.** When we look at the urn $([1, 1, 0, 2], a_0, b_0)$ after $n$ steps, we distinguish between three kinds of balls: Balls of type $A$, balls of type $B$ which were added after drawing a ball of type $A$ and all other balls of type $B$. Let us call the last two subtypes $B_1$ and $B_2$, respectively. We define the type $A'$ to be the union of type $A$ and $B_1$. Let $A_n'$ be the number of balls of this type in the urn after $n$ steps.

The law of total probability states

$$P[A(M, n, a_0, b_0) = a_0 + k] = \sum_{i=0}^{\infty} P[A(M, n, a_0, b_0) = a_0 + k | A_n' = a_0 + 2i] P[A_n' = a_0 + 2i].$$

Initially there are $a_0$ balls of type $A'$ and $b_0$ balls of type $B_2$. If a ball of type $A'$ was drawn then two more balls of type $A'$ are added and if a ball of type $B_2$ was drawn then two more balls of type $B_2$ are added. Therefore

$$P[A_n' = a_0 + 2i] = P[A(I, n, a_0, b_0) = a_0 + 2i].$$

Observe that $P[A(I, n, a_0, b_0) = a_0 + 2i] = 0$ for $i > n$. Now we will consider $P[A(M, n, a_0, b_0) = a_0 + k | A_n' = a_0 + 2i]$, i.e., the probability that $k$ balls of color $A$ were added under the condition that $2i$ balls of color $A'$ were added. Initially there are $a_0$ balls of color $A$ and zero balls of color $B_1$. In each round the number of balls of color $A$ increases by one if and only if the number of balls of color $A'$ increases by two. There are two possibilities that the number of balls of color $A'$ may increase: Either a ball of color $A$ was drawn and a ball of color $A$ and $B_1$ was added or a ball of color $B_1$ was drawn and two balls of color $B_1$ were added. Therefore the distribution of balls of color $A$ and $B_1$ among color $A'$ can be modeled by

$$P[A(M, n, a_0, b_0) = a_0 + k | A_n' = a_0 + 2i] = P[A(M, i, a_0, 0) = a_0 + k].$$

$\square$
If we replace the probabilities in the sum of the previous lemma with the closed formulas from Lemma 4.6 and Proposition 4.7 we get the following formula with positive summands.

**Lemma 4.9.** Given \([1, 1, 0, 2], a_0, b_0\),

\[
P[A_n = a_0 + k] = \frac{\Gamma\left(\frac{a_0}{2}\right)\Gamma_{(k)}^\left(\frac{k+a_0-1}{k}\right) 2^k k}{\Gamma\left(\frac{1}{2}\right)\Gamma_{(t-3/2)}^\left(n_{=0}^{\infty} \frac{n^{(2i-k-1)}(i-1/2)(n-1-i)}{i^{(2i)}\Gamma}\right)}
\]

**Proof.** Combining Lemma 4.6, Proposition 4.7 and Lemma 4.8. \(\square\)

**Theorem 4.10.**

\[
P\left[d_1^n(v_t) = k\right] = \frac{2^k k^{(n-t+1)}(n-1/2)}{\Gamma\left(\frac{1}{2}\right)\Gamma_{(t-3/2)}^\left(n_{=0}^{\infty} \frac{n^{(2i-k-1)}(i-1/2)(n-1-i)}{i^{(2i)}\Gamma}\right)}
\]

\[
P\left[d_1^n(S) = k \mid d_1^n(S) = d\right] = \frac{\Gamma\left(d_2^n\right)\Gamma_{(k)}^\left(\frac{k+d-1}{k}\right) 2^k k}{\Gamma\left(\frac{1}{2}\right)\Gamma_{(t-3/2)}^\left(n_{=0}^{\infty} \frac{n^{(2i-k-1)}(i-1/2)(n-1-i)}{i^{(2i)}\Gamma}\right)}
\]

**Proof.** Inserting the values of Lemma 4.1 into Lemma 4.9. \(\square\)

By employing algebraic techniques we were also able to transform the formula in Lemma 4.9 into a generalized hypergeometric function, for which further methods exist [17].

**Proposition 4.11.** Given \([1, 1, 0, 2], a_0, b_0\),

\[
P[A_n = a_0 + k] = \frac{\Gamma\left(\frac{a_0}{2}\right)\Gamma_{(k)}^\left(\frac{k+a_0-1}{k}\right) 2^k k}{\Gamma\left(\frac{1}{2}\right)\Gamma_{(t-3/2)}^\left(n_{=0}^{\infty} \frac{n^{(2i-k-1)}(i-1/2)(n-1-i)}{i^{(2i)}\Gamma}\right)}
\]

where \(g(n, k, a_0, b_0) = F\left(\frac{n}{2}, \frac{1}{2} - \frac{k}{2}, -\frac{k}{2}, -n; 1, 1 - k, -\frac{k}{2} - n + 1; 1\right)\) and \(F\) is the generalized hypergeometric function.

It remains an open question whether the formulas in Lemma 4.9 or Proposition 4.11 can be further simplified or approximated to improve the bounds from Section 3.
4.4 Remark: Pólya Urns and Graphs with More Edges

The previous subsections gave results for the preferential attachment process with parameter $m = 1$. One can use the reduction in Section 2.1 to generalize these results to arbitrary $m$. However, Farczadi and Wormald \[15\] defined $G_m$ directly via a balanced urn model and we briefly state their formulation for the distribution of $d_n(v_i)$. They set

$$a_0 = m$$
$$b_0 = 2i$$
$$s_0 = a_0 + b_0 = 2i,$$

use the replacement matrix

$$M = \begin{pmatrix} m & m \\ 0 & 2m \end{pmatrix},$$

and do $n' = n - i$ trials. Using Flajolet et al.’s methods they obtain

$$E[d_n(v_i)] = \frac{m \Gamma(i) \Gamma(n + \frac{i}{2})}{\Gamma(i + \frac{1}{2}) \Gamma(n)}$$

$$E[d_n(v_i)^2] = \frac{m(m+1)n}{i} + 2E[d_n(v_i)].$$

They show how urn models can help (re-)proving statements about preferential attachment graphs.

5 Preferential Attachment Graphs are Somewhere-Dense

In this section we show that preferential attachment graphs are a.a.s. somewhere-dense. We do so by analyzing the probability that a preferential attachment graph of size $n$ contains a one-subdivided clique of size $k := \log(n)^{1/4}$ as a subgraph. Let this probability be $p_n$. We show that $\lim_{n \to \infty} p_n = 1$. The proof works as follows: We start with a small preferential attachment graph and pick a set of $k$ vertices with high degree. These are the principal vertices. We then add vertices to the graph according to the preferential attachment process. A one-subdivided clique of size $k$ arises, if for every pair of principal vertices $v$ and $w$, a new vertex $u$ is added that is adjacent to both $v$ and $w$. The core of this proof is to show that after $n = 2^{k^4}$ vertices have been inserted, with high probability there is at least one connecting vertex for every pair of principal vertices.

We now describe an urn experiment that illustrates for a pair of principal vertices $v, w$ the probability that a new vertex $u$ is connected to both $u$ and $v$. 

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This experiment has no connection to Pólya Urns and is solely used for illustrative purposes. The experiment consists of multiple rounds.

In the $i$th round (for simplicity we assume $i \geq 10$), we define an urn containing $i$ balls, where $\sqrt{i}$ balls are red, $\sqrt{i}$ balls are blue, and the rest is black. In each round we draw two balls uniformly at random from the urn. The experiment succeeds if we draw a red and a blue ball in the same round. It is easy to see that the probability of success in the $i$th round equals $2(\sqrt{i}/i)^2$. We observe that eventually the experiment succeeds because

$$1 - \prod_{i=10}^{\infty} \left(1 - 2\left(\frac{\sqrt{i}}{i}\right)^2\right) = 1.$$ 

This experiment behaves similarly to the process of connecting two principal vertices. Two principal vertices are connected in round $i$ if the vertex $v_i$ is connected to both principal vertices. The expected degree of the first vertex in a preferential attachment graph of size $i$ is proportional to $\sqrt{i}$. If we (naively) assume that the degrees of $u$ and $v$ at time $i$ are at all times exactly $\sqrt{i}$ then a new vertex throws an edge to $v$ or $w$ with probability proportional to $\sqrt{i}/i$ and is connected to both with probability roughly $2(\sqrt{i}/i)^2$. Therefore, the probability that in the $i$th step a new vertex $u$ connects both $v$ and $w$ is proportional to the probability that in the $i$th round of the urn experiment a red and a blue ball is drawn. Using similar arguments we show in this section that the success probability of building a one-subdivided clique also is high.

If we, however, alter the urn experiment and assume that in the $i$th round there are only $\sqrt{i}/\log(i)$ red or blue balls we cannot guarantee success because

$$1 - \prod_{i=10}^{\infty} \left(1 - 2\left(\frac{\sqrt{i}}{i}\right)^2\right) \neq 1.$$ 

This means if the expected value of the degrees were just a logarithmic factor smaller then our proof would not work. This suggests that preferential attachment graphs are “just barely” a.a.s. somewhere-dense\(^1\). This also means we need lower bounds which guarantee that the degrees of our principal vertices are not much smaller than their expected value, e.g., at most a constant factor off. Bounds which guarantee a factor of $1/\log(i)$ would not be sufficient.

Unfortunately the probability distribution of the degree of a vertex is only well centered around its expected value if its initial degree is already large (see right side of Figure 1). We use Lemma 5.1 to find in a graph of size $k^4$ with high probability

\(^1\) In a follow up work we will show that after removing a small number of vertices from a preferential attachment graph, the probability is very high that the graph belongs to a nowhere dense graph class.
principal vertices which have a degree of roughly \( k \). These vertices will be our principal vertices. Their degree is centered closely around their expected value and therefore close enough to \( \sqrt{i} \) for our proof to work.

Our main proof is done in Theorem 3.8. There we argue that with high probability for every pair of principal vertices there will eventually be a vertex which is connected to both. At first we tried to prove this by showing that with high probability the degrees of the principal vertices are always high (roughly \( \sqrt{7} \) in the \( i \)th round for every \( i \)) and then showing that the probability that the principal vertices will be connected under the condition that the degrees of the principal vertices are always high. This method did not work out because of the subtle dependencies between the event that the \( i \)th vertex connects two principal vertices and the event that the principal vertices have a certain degree in round \( j \geq i \).

One can, however, easily compute the probability that a new edge of the \( i \)th vertex is connected to a principal vertex under the (weaker) condition that the principal vertex has at least a certain degree at time \( i - 1 \). Let \( B_i \) be the event that the degree of a vertex is at least half the expected degree at time \( i \). The event \( B_i \) on its own is very likely and gives us a good bound for the probability that \( v_{i+1} \) is connected to the principal vertex. Our calculations work in a way where whenever we assume a new event \( B_i \) the probability \( P(\overline{B_i}) \) is added to our failure probability (see Lemma 5.3). So if we were to assume all events \( B_i, B_{i+1}, B_{i+2}, \ldots \) our bound quickly becomes meaningless as the sum \( P(\overline{B_i}) + P(\overline{B_{i+1}}) + P(\overline{B_{i+2}}) + \ldots \) becomes larger than one. But if we assume exponentially spaced events \( B_i, B_{2i}, B_{4i}, B_{8i}, \ldots \) our bound on the failure probability stays small enough and new vertices are still likely to be connected to our principal vertices, allowing us to show in Theorem 5.4 that \( G^m_n \) contains a large clique.

We now proceed to prove the results of this section. The first Lemma shows that there are some vertices which have a reasonably high degree after a short number of steps.

**Lemma 5.1.** Let \( 64000 \leq k \in \mathbb{N} \). With a probability of at least \( 1 - 2ke^{-ck} \) (for some positive constant factor \( c \)) there exists a set of vertices \( X \subseteq \{v_1, \ldots, v_{k^2}\} \), \( |X| = k \) such that \( d^k_m(x) \geq \frac{1}{2}mk \) for all \( x \in X \).

**Proof.** We partition the first \( k^2 \) vertices into \( k \) sets of \( k \) vertices. Let \( S \) be one of these sets. We know that \( d^k_m(S) \geq mk \). Theorem 3.8 therefore implies with \( t = k^2 \), \( n = k^4 \) and \( \varepsilon = 1/40 \)

\[
P\left[d^k_m(S) \leq \frac{1}{2}mk^2 \right] \leq P\left[d^k_m(S) \leq (1 - 0.1)\sqrt{\frac{k^4}{k^2}mk} \right] \leq 2e^{-ck}
\]

where \( c \) is the positive constant factor of Theorem 3.8. This theorem additionally requires \( k^2 = t \geq 4/\varepsilon^3 \) which is satisfied for \( k \geq 64 \). With a probability of at least
Each of the $k$ sets have at time $k^4$ a total degree of at least $\frac{1}{2}mk^2$ by the union bound. Let now $x_i$ be the vertex in the $i$th set that has the highest degree after $k^4$ steps and let $X = \{x_1, \ldots, x_k\}$. Since every set contains at most $k$ vertices, the vertex with the highest degree has a degree of at least $\frac{1}{2}mk$.

We now bound the probability that two principal vertices $v_a, v_b$ become connected under the condition that they have high degree.

**Lemma 5.2.** We consider the preferential attachment process with $m \geq 2$. Let $k \in \mathbb{N}$ and $v_a, v_b$ be any vertices. Let $B_i$ be the event that $d^i_m(v_a), d^i_m(v_b) \geq m\sqrt{i}/4k$. Let $A_{j,i}$ with $j \leq i$ be the event that the first two edges of at least one of the vertices $v_j, \ldots, v_i$ are adjacent to $v_a$ and $v_b$, respectively. Then $P(\bar{A}_{i+1,2i} \mid \bar{A}_{j,i}, B_i) \leq e^{-\frac{1}{256}k^2}$ for $k^4 \leq i$ and $j \leq i$.

**Proof.** Let $u > 0$. $P(A_{i+u,i+u} \mid B_i)$ is the probability that vertex $v_{i+u}$ is adjacent to both $v_a$ and $v_b$ under the condition that $v_a$ and $v_b$ have degree at least $m\sqrt{i}/4k$ at some earlier time $i$. When vertex $v_{i+u}$ is inserted, the random process draws $m \geq 2$ edges from $v_{i+u}$ to earlier vertices. The probability that some vertex is chosen equals its degree divided by the total number of edges in the graph at this time. The degree of $v_a$ and $v_b$ is at least $m\sqrt{i}/4k$ at this point in time. Also there is a total of at most $2(i+u)m$ edges in the graph. We can therefore bound

$$P(A_{i+u,i+u} \mid B_i) \geq \left(\frac{\sqrt{i}}{8(i+u)k}\right)^2.$$

The same argument holds if we additionally assume some of the earlier vertices not to be adjacent to both $v_a$ and $v_b$. Let $j < i$. Then

$$P(A_{i+u,i+u} \mid \bar{A}_{j,i+u-1}, B_i) \geq \left(\frac{\sqrt{i}}{8(i+u)k}\right)^2.$$

We now consider the probability that no vertex in a sequence of vertices is adjacent to both $v_a$ and $v_b$. The chain rule yields

$$P(\bar{A}_{i+1,2i} \mid \bar{A}_{j,i}, B_i) = \prod_{u=1}^{i} P(\bar{A}_{i+u,i+u} \mid \bar{A}_{j,i+u-1}, B_i)$$

$$\leq \prod_{u=1}^{i} \left(1 - \left(\frac{\sqrt{i}}{8(i+u)k}\right)^2\right)$$

$$\leq \left(1 - \left(\frac{\sqrt{i}}{16ik}\right)^2\right)^i \leq \left(1 - \frac{1}{256ik^2}\right)^i \leq e^{-\frac{1}{256}k^2}.$$

□
Imagine a sequence of events $A_0, \ldots, A_{l-1}$ such that a preferential attachment graph contains a large subdivided clique if any one of these events is false. This means it is sufficient to show that the probability $P[A_0 \cap \cdots \cap A_{l-1}]$ is small. Assume we can only bound the probability of event $A_i$ under the condition $B_i$. The following lemma gives a good approximation of $P[A_0 \cap \cdots \cap A_{l-1}]$ if the events $B_i$ have a high probability.

**Lemma 5.3.** Let $A_0, \ldots, A_{l-1}, B_0, \ldots, B_{l-1}$ be events. Then

$$P[\bar{A}_0 \cap \cdots \cap \bar{A}_{l-1}] \leq \sum_{i=1}^{l-1} P[\bar{B}_i] + \prod_{i=1}^{l-1} P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1} \cap B_i].$$

**Proof.** As a first step we apply the chain rule and the law of total probability. Let $i < l$. Then

$$P[\bar{A}_0 \cap \cdots \cap \bar{A}_i] = P[\bar{A}_0 \cap \cdots \cap \bar{A}_{i-1}] P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1}]$$

$$= P[\bar{A}_0 \cap \cdots \cap \bar{A}_{i-1}] P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1} \cap B_i] P[B_i]$$

$$+ P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1} \cap \bar{B}_i] P[\bar{B}_i]$$

$$\leq P[\bar{A}_0 \cap \cdots \cap \bar{A}_{i-1}] P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1} \cap B_i] + P[\bar{B}_i].$$

We can now recursively apply this inequality, and use an upper bound of 1 for all factors in front of $P[B_j]$ when expanding the product, to get

$$P[\bar{A}_0 \cap \cdots \cap \bar{A}_{l-1}] \leq \sum_{i=1}^{l-1} P[\bar{B}_i] + \prod_{i=1}^{l-1} P[\bar{A}_i | \bar{A}_0 \cap \cdots \cap \bar{A}_{i-1} \cap B_i],$$

which proves the claim. 

We now use Lemma 5.1 and Lemma 5.2 to prove the main result of this section.

**Theorem 5.4.** Let $m \geq 2$. $G^m_n$ contains with a probability of at least $1 - e^{-c \log(n)^{1/4}}$ a one-subdivided clique of size $\lfloor \log(n)^{1/4} \rfloor$ for some positive constant $c$.

**Proof.** Let $k \in \mathbb{N}$. We will prove this theorem by showing that $k$ vertices in $G^m_n$ are with high probability pairwise connected by a path of length two and thereby span a one-subdivided clique. Later, the value for $n$ will be chosen as $k^{\frac{4}{92}k^\frac{1}{3}}$ which implies $k \geq \log(n)^{\frac{1}{4}}$ for $k \geq 2$.

We know for vertices with high degree that their degree will be centered closely around their expected value in the future. Let us therefore assume that there are vertices $v_1, \ldots, v_k$ such that $d_m^k(v_i) \geq \frac{1}{2} mk$ for $1 \leq i \leq k$. We call these vertices principal vertices. Lemma 5.1 states that these principal vertices exist with a probability of at least $1 - 2ke^{-ck}$ for some $c > 0$. Let us fix a pair of principal
vertices \( v_a, v_b \) and show that with high probability there is a vertex that is adjacent to both of these principal vertices. The higher the degree of \( v_a \) and \( v_b \) the higher the probability that a new vertex is adjacent to both \( v_a \) and \( v_b \). For \( i \geq k^4 \) we define \( B_i \) to be the event that \( d'_m(v_a), d'_m(v_b) \geq m\sqrt{i}/4k \). Since we assume \( d_m^{k^4}(v_a) \geq \frac{1}{2}mk \) Theorem 3.8 states with \( t = k^4 \) and \( \varepsilon = \frac{1}{2} \), that

\[
P(\bar{B}_i) = P\left[d'_m(v_a) < \frac{m\sqrt{i}}{4k} = \frac{1}{2} \sqrt{k^4 \frac{1}{2} mk} \right] \leq 2e^{-ck}.
\]

We define \( A_{j,i} \) with \( j \leq i \) to be the event that at least one of the vertices \( v_j, \ldots, v_i \) is adjacent to both \( v_a \) and \( v_b \). We will prove the claim by showing that \( P(A_{k^4+1,k^4+2^i}) \) converges to zero for an appropriate choice of \( l \). We divide our vertices into windows which double in size. We set \( \tilde{A}_i = \tilde{A}_{k^4+1,k^4+2^i} \) to be the event that none of the vertices \( v_{i+1}, \ldots, v_{2^i} \) is adjacent to \( v_a \) and \( v_b \) for \( i' = k^4 \). It holds that \( \tilde{A}_{k^4+1,k^4} = A_0 \cap \cdots \cap A_{k^4-1} \). Furthermore, by setting \( B_i = B_{k^4} \), Lemma 5.2 states that \( P(\bar{A}_i \mid A_0 \cap \cdots \cap A_{i-1} \cap B_i) \leq e^{-\frac{i}{2k^4e^2}} \). By Lemma 5.3

\[
P(\bar{A}_{k^4+1,k^4+2^l}) = P(\bar{A}_0 \cap \cdots \cap \bar{A}_{k^4-1})
\leq \sum_{i=0}^{l-1} P(B_i) + \prod_{i=0}^{l-1} P(\bar{A}_i \mid A_0 \cap \cdots \cap \bar{A}_{i-1} \cap B_i)
\leq \sum_{i=0}^{l-1} 2e^{-ck} + e^{-\frac{i}{2k^4e^2}} = 2lke^{-ck} + e^{-\frac{l}{2k^4e^2}}.
\]

We define \( l = k^3, n = k^42^l \) and

\[
p = k^3 2ke^{-ck} + e^{-\frac{k^3}{2k^4e^2}} \geq P(\bar{A}_{k^4+1,n}).
\]

This means that in \( G_m^n \) the probability that there exists a vertex which connects the principal vertices \( v_a \) and \( v_b \) is at least \( 1 - p \). According to the union bound, the probability that for all \( \binom{k}{2} \) pairs of principal vertices there exists a vertex which connects them is bounded by \( 1 - \binom{k}{2} p \). In Lemma 5.2 only the first two edges of the connecting vertex are considered. Therefore a connecting vertex may only connect a single pair of principal vertices. This means every pair of principal vertices has a unique connecting vertex, i.e., the principal vertices span a one-subdivided clique.

So far, all our calculations were based on the condition that there are \( k \) principal vertices with reasonably high degree in the beginning. According to Lemma 5.1 the probability \( k \) such vertices do not exist is at most \( 2ke^{-ck} \) for some \( c > 0 \). So by law of total probability, we can add this error probability to the conditional bound to get an unconditional bound. This means that \( G_m^n \) contains no one-subdivided \( k \) clique with a probability of at most

\[
\binom{k}{2} p + 2ke^{-ck} \leq \text{poly}(k)e^{-c'k} = \text{polylog}(n)e^{-c'\log(n)^{1/2}} \leq e^{-c''\log(n)^{1/2}}
\]

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for some positive constant $c''$.

The main result of this section follows directly from the previous theorem and Definition 2.4.

**Corollary 5.5.** $G^n_m$ is a.a.s. somewhere-dense for $m \geq 2$.

## 6 Conclusion

Our analysis of preferential attachment graphs resulted in two main results: (1) a tail bound result stating the degree of vertices, under the condition that the degree had a certain value $D(t)$ at an earlier point in time $t$, is a small factor away from its expectation is exponentially small in $D(t)$ and (2) that preferential attachment graphs are a.a.s. somewhere-dense. For the first result we used Chernoff bounds to obtain bounds which are reasonably tight for small time-frames and then used the union bound to extend these bounds to large time-frames. The second result was obtained by using the fact that even though for a single vertex the probability distribution of its degree is not concentrated the probability distribution of sets of vertices is indeed concentrated. Among such a set we choose the principal vertices for a large clique-minor.

Recently a more general preferential attachment model has been introduced that has an additional parameter $\delta$ \[30\]. This new parameter captures how much the degree of a vertex changes the probability that another vertex connects to it, where $\delta = 0$ is the standard model considered in this paper and $\delta = \infty$ corresponds to uniform attachment where the degrees do not matter. Another notable case is $\delta \leq -1$, where self-loops no longer occur. It would be interesting to see for what values of $\delta$ this model is a.a.s. somewhere- or nowhere-dense and if our techniques can still be applied.

The way we (sequentially) constructed the graph follows the definition by Bollobás and Spencer \[5\] but since the original definition by Barabási and Albert \[1\] was ambiguous there are two other natural ways to define the preferential attachment step \[2\]: In the **Independent Model** a new vertex $v_t$ chooses its $m$ neighbors $w_1, \ldots, w_m$ independently at the same time with repetitions and the **Conditional Model** is the same but the $w_1, \ldots, w_m$ have to be different. The model we use is known as the **Sequential Model**, where the neighbors are chosen after one another (which means the probabilities are updated in-between). It has the nice property that it can also be easily modeled as a Pólya Urn, which is not the case for the other interpretations. It would be interesting to generalize our results in Section 3 and 5 to the Independent and the Conditional Model.

In Section 4 we tried to use Pólya Urns to find simple formulas for the probability distribution of vertex degrees. It is open if this approach can be used to
improve the bounds presented in Section 3. On the other hand it might be interesting to see if the bounds from Section 3 can be used to improve certain bounds for Pólya Urns. Nice bounds for the original Pólya Urn with replacement matrix $[1, 0, 0, 1]$ were obtained using martingale techniques. Since martingales were not applicable our setting we used Chernoff bounds on multiple intervals. This technique might be flexible enough to provide bounds for other replacement matrices of Pólya Urns as well.

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