THE NUMBER OF HOMOMORPHISMS FROM THE HAWAIIAN EARRING GROUP

SAMUEL M. CORSON

Abstract. We show a dichotomy for groups of cardinality less than continuum. The number of homomorphisms from the Hawaiian earring group to such a group G is either the cardinality of G in case G is noncommutatively slender, or the number is $2^{2^{\aleph_0}}$ in case G is not noncommutatively slender. An example of a noncommutatively slender group with nontrivial divisible element is exhibited.

1. Introduction

The fundamental group of the Hawaiian earring, which we call the Hawaiian earring group and denote HEG, has become a subject of expanding use and interest in topology and group theory (see [MM], [Sm], [EK], [CC1], [CC2], [FZ], [ADTW]). Though not itself free, the structure of HEG has some analogies to that of free groups. The fundamental group of the infinitary torus $T^\infty$, which is isomorphic to the product $\prod_{\omega} \mathbb{Z}$, has comparable similarities to free abelian groups while not being itself free abelian. The group HEG is both residually and locally free, and the group $\prod_{\omega} \mathbb{Z}$ is residually and locally free abelian. There is an easy-to-define continuous map from the Hawaiian earring to $T^\infty$ which induces a surjection from HEG to $\prod_{\omega} \mathbb{Z}$. For these and other reasons, one can imagine HEG to be the unabelian version of $\prod_{\omega} \mathbb{Z}$.

These comparisons motivate an extension of certain abelian group notions to more general settings, where definitions involving $\prod_{\omega} \mathbb{Z}$ have HEG substituted. For example, an abelian group $A$ is said to be slender (see [F]) if for every homomorphism $\phi : \prod_{\omega} \mathbb{Z} \to A$ there is a natural number $n \in \omega$ such that $\phi \circ p_n = \phi$, where $p_n : \prod_{\omega} \mathbb{Z} \to \prod_{i=0}^{n-1} \mathbb{Z}$ is the retraction to the subgroup for which only the first $n$ coordinates are possibly nonzero. The Hawaiian earring group similarly has retraction maps $p_n$ to free subgroups $\text{HEG}_n$, which maps correspond to the topological retractions given by mapping all points not on the largest $n$ circles to the wedge point. Thus Eda defines a group $G$ to be noncommutatively slender (or n-slender for short) if for every homomorphism $\phi : \text{HEG} \to G$ there exists $n \in \omega$ such that $\phi \circ p_n = \phi$ (see [E1]). The abelian n-slender groups are precisely the slender abelian groups [E1, Theorem 3.3], so n-slenderness is conceptually an extension of slenderness.

Unsurprisingly, slenderness is better understood than n-slenderness. The slender groups are completely characterized in terms of subgroups: an abelian group is slender if and only if it is torsion-free and does not contain a subgroup isomorphic to $\mathbb{Q}$, $\prod_{\omega} \mathbb{Z}$ or the $p$-adic completion of the integers for any prime $p$ (see [Nu]). By

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contrast, no such straightforward characterization for n-slender groups is known. The problem of finding such a general characterization seems intractable. For example, if $G$ is the fundamental group of a path connected, locally path connected first countable Hausdorff space which lacks a universal cover then one gets a homomorphism (induced by a continuous map) from $\text{HEG}$ to $G$ witnessing that $G$ is not n-slender. Such fundamental groups seem varied and complicated.

Even for countable groups it is unknown as of this writing whether a group is n-slender if and only if it is torsion-free and does not contain $\mathbb{Q}$, though certainly it is necessary that n-slender groups not contain torsion or $\mathbb{Q}$. Many well known groups are known to be n-slender. For example, free groups, free abelian groups, torsion-free word hyperbolic groups, torsion-free one-relator groups, and Thompson’s group $F$ are n-slender (see [Hi], [D], [Co1], [Co2], [CoCo]). Despite our rather limited knowledge of small cardinality n-slender groups we get the following ($|X|$ is the cardinality of $X$ and $\text{Hom}(H,G)$ denotes the set of group homomorphisms from $H$ to $G$):

**Theorem A.** If $G$ is a group with $|G| < 2^{\aleph_0}$ then

$$|\text{Hom}(\text{HEG}, G)| = \begin{cases} |G| & \text{if } G \text{ is n-slender} \\ 2^{2^{\aleph_0}} & \text{if } G \text{ is not n-slender} \end{cases}$$

The difficulty in proving Theorem A lies in producing $2^{2^{\aleph_0}}$-many distinct homomorphisms from $\text{HEG}$ to $G$ given the existence of a map witnessing that $G$ is not n-slender. One cannot simply take such a map and precompose with sufficiently many endomorphisms of $\text{HEG}$, for $\text{HEG}$ has only $2^{\aleph_0}$-many endomorphisms (this follows immediately from [E2, Corollary 2.11]). Previously it was known that for any nontrivial finite group $G$ there exist $2^{2^{\aleph_0}}$-many surjections from $\text{HEG}$ to $G$ [CS]. More generally it is now known that given a compact Hausdorff topological group $G$ there are $|G|^{2^{\aleph_0}}$ homomorphisms from $\text{HEG}$ to $G$. This latter fact is seen by combining the main result of [T] with [Z, Theorem 0.1]. The dichotomy of Theorem A fails to hold when $G$ is of size $2^{\aleph_0}$ since $\text{HEG}$ is a group which is not n-slender and $\text{Hom}(\text{HEG}, G)$ is of cardinality $2^{\aleph_0}$. We give the abelian version of Theorem A as well (see Theorem 3.1), though it is a great deal easier to prove.

Theorem A is proved using the fundamental group of the harmonic archipelago [BS], which we denote HAG. The group HAG is a quotient of HEG but enjoys much greater freedom as to mappings, as is witnessed in the following theorem:

**Theorem B.** The group $\text{Aut}(\text{HAG})$ contains an isomorphic copy of the full symmetric group $S_{2^{\aleph_0}}$ on a set of cardinality continuum. Thus $\text{Aut}(\text{HAG})$ is of cardinality $2^{2^{\aleph_0}}$ and all groups of size at most $2^{\aleph_0}$ are subgroups.

The group $\text{Aut}(\text{HEG})$ is only of size $2^{\aleph_0}$ (using [E2, Corollary 2.11]).

Section 2 provides some necessary background definitions and results. In Section 3 we prove Theorems A and B. In Section 3 we turn our attention to homomorphisms from a group to an n-slender group and consider two subgroups which completely determine whether a group of small cardinality is n-slender. Using the machinery used to compare these subgroups we produce some more examples of n-slender groups, including an n-slender group with a nontrivial divisible element (see Example 3). We then prove a theorem characterizing the n-slender subgroups of HEG via subgroups and pose two questions related to this theorem.
2. Preparatory Results

We give some necessary background information, including a combinatorial characterization of the Hawaiian earring group and the harmonic archipelago group, as well as some relevant facts related to them. The Hawaiian earring group HEG can be described as a set of infinitary words on a countable alphabet. Let \( \{a_n\} \) be a countably infinite set where each element has a formal inverse. A word \( W \) is a function whose domain is a totally ordered set \( \mathbb{W} \), whose codomain is \( \{a_n\} \) such that for each \( n \in \omega \) the set \( \{i \in \mathbb{W} | W(i) \in \{a_n\}\} \) is finite. It follows that for any word \( W \) the domain \( \mathbb{W} \) is a countable total order.

We understand two words \( W_0 \) and \( W_1 \) to be the same, and write \( W_0 \equiv W_1 \), provided there exists an order isomorphism \( i : \mathbb{W}_0 \to \mathbb{W}_1 \) such that \( W_1(i(i)) = W_0(i) \). Let \( \mathbb{W} \) denote the set of \( \equiv \) equivalence classes. For each \( n \in \omega \) we define the function \( p_n : \mathbb{W} \to \mathbb{W} \) by setting \( p_n(W) = \{i \in \mathbb{W} | W(i) \in \{a_n\}\} \). Clearly \( p_m \circ p_n = p_m \) whenever \( m \geq n \). The word \( p_n(W) \) has finite domain, and we write \( W_0 \sim W_1 \) if for every \( n \in \omega \) we have \( p_n(W_0) \) equal to \( p_n(W_1) \) as elements in the free group over \( \{a_m\}_{m=0} \). Write \( [W] \) for the equivalence class of \( W \) under \( \sim \).

Given two words \( W_0 \) and \( W_1 \), we define their concatenation \( W_0W_1 \) to be the word whose domain is the disjoint union of \( \mathbb{W}_0 \) and \( \mathbb{W}_1 \) under the order extending that of the two subsets which places elements of \( \mathbb{W}_0 \) below those of \( \mathbb{W}_1 \), and such that

\[
W_0W_1(i) = \begin{cases} 
W_0(i) & \text{if } i \in \mathbb{W}_0 \\
W_1(i) & \text{if } i \in \mathbb{W}_1 
\end{cases}
\]

More generally suppose \( \{W_\lambda\}_{\lambda \in \Lambda} \) is a collection of words with \( \Lambda \) a totally ordered set and for each \( \lambda \in \Lambda \) the set \( \{\lambda \in \Lambda \mid (\exists i \in \mathbb{W}_\lambda)W_\lambda(i) \in \{a_n\}\} \) is finite. We obtain an infinite concatenation \( W = \prod_{\lambda \in \Lambda} W_\lambda \) by setting \( W \) to be the disjoint union \( \bigsqcup_{\lambda \in \Lambda} W_\lambda \) under the obvious order and setting \( W(i) = W_\lambda(i) \) where \( i \in \mathbb{W}_\lambda \). We'll use the notation \( \prod_{\lambda \in \Lambda} I_\lambda \) for the concatenation of ordered sets \( I_\lambda \) in the obvious way. Given \( W \in \mathbb{W} \) we let \( W^{-1} \) be the word whose domain is \( \mathbb{W} \) under the reverse order and such that \( W^{-1}(i) = (W(i))^{-1} \).

The set \( \mathbb{W}/\sim \) has a group structure defined by setting \( [W_0][W_1] = [W_0W_1] \) and \( [W]^{-1} = [W^{-1}] \). The identity element is the \( \sim \) class of the empty word \( E \). This group is isomorphic to the fundamental group of the Hawaiian earring and we denote it HEG. For each \( n \in \omega \) the word map \( p_n \) defines a retraction homomorphism, also denoted \( p_n \), which takes HEG to a subgroup which is isomorphic to the free group on \( \{a_m\}_{m=0} \), which we denote HEG\(_n\). Again, \( p_m \circ p_n = p_m \) whenever \( m \geq n \). The set of all elements of HEG which have a representative using no letters in \( \{a_m\}_{m=0} \) is also a retract subgroup, which we denote HEG\(_n^\ast \). There is a natural isomorphism HEG \( =\) HEG\(_n^\ast \) HEG\(_n^\ast^\).}

Definition 2.1. A group \( G \) is noncommutatively slender, or \( n \)-slender, if for every homomorphism \( \phi : \text{HEG} \to G \), there exists \( n \in \omega \) such that \( \phi \circ p_n = \phi \). Equivalently, \( G \) is \( n \)-slender if for every homomorphism \( \phi : \text{HEG} \to G \) there exists \( n \in \omega \) such that \( \text{HEG}^n \leq \ker(\phi) \). The \( n \)-slender groups do not contain torsion or \( \mathbb{Q} \) as a subgroup (E[1] Theorem 3.3, [Sa]). Free (abelian) groups, one-relator groups, and a host of other groups are \( n \)-slender (see [Co2], [CoCo] Theorems A, B).

As is the case with a free group, there exists a characterization of HEG which utilizes so-called reduced words. We say \( W \in \mathbb{W} \) is reduced if for every writing \( W = W_0W_1W_2 \) such that \( W_1 \sim F \) we have \( W_1 \sim E \). It is clear that if \( W \) is reduced then so is \( W^{-1} \), and if \( W \sim W_0W_1 \) is reduced then so are \( W_0 \) and \( W_1 \). We have the following (see [E1] Theorem 1.4, Corollary 1.7).
Lemma 2.2. Given $W \in \mathcal{W}$ there exists a reduced word $W_0 \in \mathcal{W}$ such that $[W] = [W_0]$, and this $W_0$ is unique up to $\equiv$. Moreover, letting $W_0$ and $W_1$ be reduced, there exist unique words $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ such that

1. $W_0 \equiv W_{0,0}W_{0,1}$
2. $W_1 \equiv W_{1,0}W_{1,1}$
3. $W_{0,1} \equiv W_{1,0}$
4. $W_{0,0}W_{1,1}$ is reduced

Thus one can consider HEG to be the set of all reduced words $\text{Red} \subseteq \mathcal{W}$ and define the binary operation via the concatenation (4).

Some endomorphisms of HEG can be defined by simply using mappings of the set $\{a_n\}_{n \in \omega}$. For this, suppose that $\{W_n\}_{n \in \omega}$ is a collection of words such that for every $m \in \omega$ the set $\{n \in \omega : (\exists i \in W_m)W_n(i) \in \{a_{n+1}^\pm\}\}$ is finite. Defining $f : \{a_{n+1}^\pm\}_{n \in \omega} \to \{W_n\}_{n \in \omega}$ by $f(a_{n+1}^\pm) = W_n^\pm$ one can extend $f$ to all of $\mathcal{W}$ by letting $f(W) = \prod_{n \in \mathcal{W}} f(W(i))$. This induces an endomorphism $\phi_f : \text{HEG} \to \text{HEG}$ by letting $\phi_f([W]) = [f(W)]$ (see [E2] Proposition 1.9). Surprisingly, every endomorphism is equal, up to conjugation, to a homomorphism defined in this way (this is essentially [E2] Corollary 2.11):

Lemma 2.3. If $\phi \in \text{Hom}(\text{HEG}, \text{HEG})$ there exists $h \in \text{HEG}$ and a mapping $f : a_n \to W_n$ such that $\phi(g) = h^{-1}\phi_f(g)h$.

An important quotient of HEG is the so called harmonic archipelago group HAG, which we define by HEG/\langle \langle \text{divides}.alt0 \rangle \rangle. This uncountable group is isomorphic to the fundamental group of the harmonic archipelago (see [CHM] or [Ho]). Let $\pi : \text{HEG} \to \text{HAG}$ denote the quotient map. We write $[[W_0]] = [[W_1]]$ if $\pi([W_0]) = \pi([W_1])$. Since $[[W]] = [[E]]$ for any finite $W \in \mathcal{W}$, it is clear that assuming $W \equiv W_0W_1W_2$ with $W_1$ finite, we have $[[W]] = [[W_0W_2]]$. In consequence, we obtain:

Lemma 2.4. For all $n \in \omega$, we have $\pi([\text{HEG}^n]) = \text{HAG}$. If HAG has a nontrivial homomorphic image in $G$ then $G$ is not n-slender.

Proof. The first claim follows from the fact that HEG $\simeq$ HEG$_n \ast$ HEG$^n$ and elements of HEG$_n$ have a representative which is a finite word. For the second claim, if $\phi(\text{HAG})$ has nontrivial image then $\phi \circ \pi$ witnesses that $G$ is not n-slender by the first claim. \hfill \Box

When $|G| < 2^{8\omega}$, the converse of the second claim in Lemma 2.4 holds as well. First we state a lemma (which is a special instance of [CC2] Theorem 4.4 (1)):

Lemma 2.5. If $|G| < 2^{8\omega}$ and $\phi : \text{HEG} \to G$ is a homomorphism then the sequence of images $\phi(\text{HEG}^n)$ eventually stabilizes.

The following is an argument of Conner [C]:

Lemma 2.6. Suppose $\phi : \text{HEG} \to G$ and $S \subseteq \bigcap_{n \in \omega} \phi(\text{HEG}^n)$ with $S$ countable. Then there exists a homomorphism $\psi : \text{HAG} \to G$ with $\psi(\text{HAG}) \supseteq S$.

Proof. Let $S = \{g_0, g_1, \ldots\}$ be an enumeration of $S$. For each $m \in \omega$ and $n \geq m$ select $[W_{(m,n)}] \in \text{HEG}^n$ such that $\phi([W_{(m,n)}]) = g_m$. Let $f : \omega \to ((m,n) \in \omega^2 : n \geq m)$ be a bijection. We get an endomorphism $\Xi : \text{HEG} \to \text{HEG}$ such that $\Xi([a_k]) = [W_{f(k)}]$, and letting $\phi_0 = \phi \circ \Xi$ we see that for each $m \in \omega$ there exist arbitrarily large $k \in \omega$ such that $\phi_0([a_k]) = g_m$. \hfill \Box
Now for each \( m \in \omega \) we let \( N_m = \{ k \in \omega : \phi_0([a_k]) = g_m \} \) and notice that \( \omega = \bigsqcup_{m\in\omega} N_m \) and each \( N_m \) is infinite. Enumerate each \( N_m \) in the standard way \( N_m = \{ k_{(m,0)}, k_{(m,1)}, \ldots \} \) so that \( k_{(m,i)} < k_{(m,i+1)} \). Let \( \gamma : \omega \to \omega^2 \) be a bijection. Define the endomorphism \( \tau : \text{HEG} \to \text{HEG} \) such that \( \tau([a_p]) = [a_{\gamma(p)}a_{\gamma(p)}^{-1}a_{\gamma(p)}a_{\gamma(p)}^{-1}a_{\gamma(p)}a_{\gamma(p)}^{-1}a_{\gamma(p)} \cdots] \) (here \( \text{proj}_i \) denotes projection to the \( i \) coordinate). Notice now that \( \phi \circ \tau : \text{HEG} \to \text{HEG} \) has \( \phi_1([a_j]) = 1_G \) for every \( j \in \omega \). Thus \( \phi_1 \) descends to a homomorphism \( \psi : \text{HAG} \to G \) with the same image as \( \phi_1 \). We shall be done if we show that \( S \subseteq \phi(\text{HEG}) = \psi(\text{HAG}) \).

Letting \( m \in \omega \) be given we notice that
\[
g_m = \phi_0([a_{k_{(m,0)}}]) = \phi_0([a_{k_{(m,0)}}a_{k_{(m,1)}}^{-1}a_{k_{(m,2)}}^{-1}a_{k_{(m,3)}}^{-1}a_{k_{(m,4)}}^{-1} \cdots]) = \phi_1 \circ \tau([a_{\gamma^{-1}(m,0)}a_{\gamma^{-1}(m,1)} \cdots]) = \psi([[a_{\gamma^{-1}(m,0)}a_{\gamma^{-1}(m,1)} \cdots]]) \in \psi(\text{HAG})
\]

\( \square \)

Combining Lemmas 2.7 and 2.8 one immediately obtains:

**Lemma 2.7.** If \( |G| < 2^{80} \) with \( G \) not n-slender, then there is a nontrivial homomorphism from \( \text{HAG} \) to \( G \).

Next we state a special case of [E3] Theorem 1.3:

**Lemma 2.8.** If \( \phi : \text{HEG} \to *_{j \in J} H_j \) is a homomorphism to a free product, then for some \( n \in \omega \) and \( j \in J \) the image \( \phi(\text{HEG}^n) \) lies in a conjugate of \( H_j \).

Consequently we obtain:

**Lemma 2.9.** If \( \phi : \text{HAG} \to *_{j \in J} H_j \) then for some \( j \in J \) the image \( \phi(\text{HAG}) \) is a subgroup of a conjugate of \( H_j \).

Finally, we note that extending the set \( \{ a_n^{+1} \}_{n \in \omega} \) to \( \{ a_n^{+1}, b_n^{+1}, c_n^{+1} \}_{n \in \omega} \), we analogously define the set \( \mathcal{W}_{a,b,c} \) of words, the set of reduced words \( \text{Red}_{a,b,c} \subseteq \mathcal{W}_{a,b,c} \) the Hawaiian earring group \( \text{HEG}_{a,b,c} \) and the harmonic archipelago group \( \text{HAG}_{a,b,c} \). Using the bijection \( \Gamma : \{ a_n^{+1} \}_{n \in \omega} \to \{ a_n^{+1}, b_n^{+1}, c_n^{+1} \}_{n \in \omega} \) given by
\[
\Gamma(a_n^{+1}) \mapsto \begin{cases} 
a_n^{+1} & \text{if } n = 3m \\
b_n^{+1} & \text{if } n = 3m + 1 \\
c_n^{+1} & \text{if } n = 3m + 2 
\end{cases}
\]

the elements of \( \mathcal{W} \) are placed in bijection with those of \( \mathcal{W}_{a,b,c} \) by letting \( \Gamma(W) = \prod_{i \in \mathcal{W}} \Gamma(W(i)) \). This bijection satisfies \( \Gamma(\text{Red}) = \text{Red}_{a,b,c} \) and induces an isomorphism \( \text{HEG} \cong \text{HEG}_{a,b,c} \) where the latter group is defined analogously to the former. This isomorphism also descends to an isomorphism \( \text{HAG} \cong \text{HAG}_{a,b,c} \). By deleting the elements of \( \{ b_n^{+1}, c_n^{+1} \}_{n \in \omega} \) we obtain retractions \( r_a : \mathcal{W}_{a,b,c} \to \mathcal{W}, r_b : \text{HEG}_{a,b,c} \to \text{HEG}, \) and \( r_c : \text{HAG}_{a,b,c} \to \text{HAG} \). We can similarly define the set of words \( \mathcal{W}_{b,c} \subseteq \mathcal{W}_{a,b,c} \) which have range disjoint from \( \{ a_n^{+1} \}_{n \in \omega} \), the set of reduced words \( \text{Red}_{b,c} = \mathcal{W}_{b,c} \cap \text{Red}_{a,b,c} \), an isomorph of the Hawaiian earring group \( \text{HEG}_{b,c} \cong \text{HEG}_{a,b,c} \) and an isomorph of the harmonic archipelago group \( \text{HAG}_{b,c} \cong \text{HAG}_{a,b,c} \).

The inclusion \( \text{HAG}_{b,c} \subseteq \ker(r_a) \) holds.
3. Theorems \( A \) and \( B \)

We begin by stating and proving the abelian version of Theorem \( A \).

**Theorem 3.1.** If \( A \) is an abelian group with \( |A| < 2^\omega \) then
\[
|\text{Hom}(\prod_\omega Z, A)| = \begin{cases} 
|A| & \text{if } A \text{ is slender} \\
2^{2^\omega} & \text{if } A \text{ is not slender}
\end{cases}
\]

**Proof.** Suppose that \( A \) is slender. If \( A \) is the trivial group then there is exactly one homomorphism from \( \prod_\omega Z \) to \( A \). If \( A \) is nontrivial then \( A \) is infinite, torsion-free. Since \( A \) is slender we see that \( \text{Hom}(\prod_\omega Z, A) = \bigcup_{n<\omega} \{ \phi : \prod_\omega Z \to A \mid \phi \circ p_n = \phi \} \). The set \( \{ \phi : \prod_\omega Z \to A \mid \phi \circ p_1 = \phi \} \) has cardinality exactly \( |A| \), and since \( A \) is infinite it is in fact true that for each \( n \in \omega \) we have \( \{ \phi : \prod_\omega Z \to A \mid \phi \circ p_n = \phi \} \) is of cardinality \( |A| \). Then \( \text{Hom}(\prod_\omega Z, A) \) has cardinality \( |A| \) in this case as well.

Suppose that \( A \) is not slender. By \( \text{Sø} \) we know that \( A \) must contain an isomorphic copy of \( \mathbb{Q} \) or of the cyclic group \( \mathbb{Z}/p \) for some prime \( p \). If \( \mathbb{Q} \leq A \), we take a subgroup \( F \leq \prod_\omega Z \) which is a free abelian group of rank \( 2^\omega \). The construction of such an \( F \) follows a straightforward induction. There are \( 2^{2^\omega} \) distinct homomorphisms from \( F \) to \( \mathbb{Q} \), and since \( \mathbb{Q} \) is an injective \( \mathbb{Z} \)-module, each of these homomorphisms may be extended to \( \prod_\omega Z \). Thus in this case there are at least \( 2^{2^\omega} \) homomorphisms from \( \prod_\omega Z \) to \( A \), and since \( |A| < 2^\omega \) we have precisely \( 2^{2^\omega} \) homomorphisms.

If \( \mathbb{Z}/p \leq A \) then we use the epimorphism \( \epsilon : \prod_\omega Z \to \prod_\omega (\mathbb{Z}/p) \) and notice that \( \prod_\omega (\mathbb{Z}/p) \) is a vector space over the field \( \mathbb{Z}/p \), so by selecting a basis we obtain a group isomorphism \( \prod_\omega (\mathbb{Z}/p) \cong \bigoplus_{2^\omega}(\mathbb{Z}/p) \). For each \( S \leq 2^\omega \) we get a homomorphism \( \epsilon_S : \bigoplus_{2^\omega}(\mathbb{Z}/p) \to \mathbb{Z}/p \) given by taking the sum of the \( S \) coordinates. Each such \( \epsilon_S \) is a distinct homomorphism and so each composition \( \epsilon_S \circ \epsilon : \prod_\omega Z \to \mathbb{Z}/p \) is distinct. Thus there exist at least \( 2^{2^\omega} \) homomorphisms from \( \prod_\omega Z \) to \( A \), and again by \( |A| < 2^\omega \) we see that there are exactly \( 2^{2^\omega} \) homomorphisms. \( \square \)

Next, we prove Theorem \( A \) modulo a proposition which proves the existence of many homomorphisms.

**Theorem \( A \)** If \( G \) is a group with \( |G| < 2^\omega \) then
\[
|\text{Hom}(\text{HEG}, G)| = \begin{cases} 
|G| & \text{if } G \text{ is n-slender} \\
2^{2^\omega} & \text{if } G \text{ is not n-slender}
\end{cases}
\]

**Proof.** Suppose \( G \) is n-slender and \( |G| < 2^\omega \). If \( G \) is trivial then there is exactly one homomorphism from HEG to \( G \). If \( G \) is nontrivial then \( G \) is infinite and \( \text{Hom}(\text{HEG}, G) = \bigcup_{n<\omega} \{ \phi : \text{HEG} \to G \mid \phi \circ p_n = \phi \} \). Moreover since \( \text{HEG} \cong \text{HEG}_n \times \text{HEG}^n \) for all \( n \in \omega \) and \( \text{HEG}_n \) is a free group of rank \( n \), we get \( \text{Hom}(\text{HEG}, G) \) as a countable union of sets of cardinality \( |G| \). Thus \( |\text{Hom}(\text{HEG}, G)| = |G| \) in either case.

Suppose \( G \) is not n-slender and \( |G| < 2^\omega \). By Lemma \( 2.7 \) there exists a nontrivial homomorphism from HAG to \( G \). By Proposition \( 3.2 \) there are at least \( 2^{2^\omega} \) homomorphisms from HAG to \( G \) and precomposing these homomorphisms with the surjective map \( \pi : \text{HEG} \to \text{HAG} \) we obtain at least \( 2^{2^\omega} \) homomorphisms from HEG to \( G \). Since \( |\text{HEG}| = 2^\omega \) and \( |G| < 2^\omega \) we get \( |\text{Hom}(\text{HEG}, G)| = 2^{2^\omega} \). \( \square \)

**Proposition 3.2.** If \( |\text{Hom}(\text{HAG}, G)| > 1 \) then \( |\text{Hom}(\text{HAG}, G)| \geq 2^{2^\omega} \).
We prove Proposition 3.2 after a sequence of lemmas.

Lemma 3.3. If \( \phi_0 : \text{HAG} \to G \) is a nontrivial homomorphism there exists a homomorphism \( \phi : \text{HAG}_{a,b,c} \to G \) such that \( \phi([[a_0a_1a_2\cdots]]) \neq 1_G \) and \( \text{HAG}_{b,c} \leq \ker(\phi) \).

Proof. Select \( g \in \phi_0(\text{HAG}) \setminus \{1_G\} \). By Lemma 2.3 we select for each \( n \in \omega \) an element \( [W_n] \in \text{HEG}^n \) such that \( \phi_0 \circ \pi([W_n]) = g \). Let \( \sigma_0 : \text{HEG} \to \text{HEG} \) be the endomorphism determined by \( a_n \mapsto W_n \). Letting \( \phi_1 = \phi_0 \circ \sigma_0 : \text{HEG} \to G \), we have \( \phi_1([a_n]) = g \) for all \( n \in \omega \).

Now let \( \sigma_1 : \text{HEG} \to \text{HEG} \) be the endomorphism determined by \( a_n \mapsto a_na_{n-1}^{-1} \).

Notice that \( \phi_2 = \phi_1 \circ \sigma_1 \) satisfies \( \phi_2([a_n]) = 1_G \) for all \( n \in \omega \). Thus \( \phi_2 \) descends to a map \( \phi' : \text{HAG} \to G \) by letting \( \phi'([[W]]) = \phi_2([W]) \). Moreover we have \( \phi_2([[a_0a_1a_2\cdots]]) = g \) and so \( \phi'([[a_0a_1a_2\cdots]]) = g \). Letting \( r_a : \text{HAG}_{a,b,c} \to \text{HAG} \) be the retraction map, define \( \phi : \text{HAG}_{a,b,c} \to G \) by \( \phi = \phi' \circ r_a \). Thus \( \text{HAG}_{b,c} \leq \ker(\phi) \) and \( \phi([[a_0a_1a_2\cdots]]) \neq 1_G \).

Let \( \Sigma \) be a collection of subsets of \( \omega \) such that each \( S \in \Sigma \) is infinite, for distinct \( S_0, S_1 \in \Sigma \) the intersection \( S_0 \cap S_1 \) is finite, and \( |\Sigma| = 2^{2^{\alpha_0}} \). Such a construction is fairly straightforward (see [Ku] II.1.3)). For each \( S \in \Sigma \) define a word \( U_S \in \text{Red}_{b,c} \)

by \( \overline{U_S} = \omega \) and \( U_S(n) = \begin{cases} b_n & \text{if } n \in S \\ c_n & \text{if } n \notin S \end{cases} \). For each \( n \in \omega \) and \( S \in \Sigma \) let \( U_{S,n} = W \upharpoonright (\omega \setminus \{0, \ldots, n-1\}) \). Thus \( U_{S,0} = U_S \) and for each \( n \in \omega \) we have \( [U_{S,n}] \in \text{HEG}^n_{b,c} \).

Let \( T \) be a symbol such that \( T \notin \Sigma \) and define a word \( U_T \in \text{Red} \) by \( U_T \equiv a_0a_1a_2\cdots \) and let \( U_{T,n} \equiv a_na_{n+1}\cdots \).

Given a word \( W \in \text{Red}_{a,b,c} \) we say an interval \( I \subseteq \overline{W} \) participates in \( \Sigma \) for \( W \) if \( W \upharpoonright I \equiv U_{S,n} \) or \( W \upharpoonright I \equiv U_{S,n}^{-1} \) for some \( S \in \Sigma \) and \( n \in \omega \). Similarly, given a word \( W \in \text{Red}_{a,b,c} \) we say an interval \( I \subseteq \overline{W} \) is maximal in \( \Sigma \) for \( W \) if \( I \) participates in \( \Sigma \) and there does not exist a strictly larger interval \( \overline{W} \supset I \) which participates in \( \Sigma \).

Lemma 3.4. If \( W \in \text{Red}_{a,b,c} \) and \( I \subseteq \overline{W} \) is an interval which participates in \( \Sigma \) for \( W \) then \( I \) is contained in a unique interval \( I' \supset I \) which is maximal in \( \Sigma \) for \( W \).

Proof. Let \( I \subseteq \overline{W} \) satisfy the hypotheses. Suppose \( W \upharpoonright I \equiv U_{S,n} \). By the definition of the \( U_{S,n} \), it is clear that if \( m \in \omega \) and \( S' \in \Sigma \) also satisfy \( W \upharpoonright I \equiv U_{S',m} \) then \( S = S' \) and \( m = n \). Also, it cannot be that \( W \upharpoonright I \equiv U_{S,n}^{-1} \), since \( U_{S,n} \) has order type \( \omega \) and \( U_{S,n}^{-1} \) has order type \( -\omega \). If there does not exist an immediate predecessor \( i < \min(I) \) such that \( W \upharpoonright (I \cup \{i\}) \equiv U_{S,n-1} \) then \( I \) is maximal in \( \Sigma \) for \( W \). Otherwise we get \( W \upharpoonright (I \cup \{i\}) \equiv U_{S,n-1} \) and apply induction on \( n \).

The proof in case \( W \upharpoonright I \equiv U_{S,n}^{-1} \) is similar.

Lemma 3.5. Given \( W \in \text{Red}_{a,b,c} \), if subintervals \( I_0, I_1 \subseteq \overline{W} \) are both maximal in \( \Sigma \) for \( W \) then either \( I_0 \cap I_1 = \emptyset \) or \( I_0 = I_1 \).

Proof. Suppose \( I_0 \cap I_1 \neq \emptyset \) and select \( i \in I_0 \cap I_1 \). If \( W(i) \) does not have superscript \(-1\) then by how we have defined the words \( U_{S,n} \) we see that \( I_0 \) and \( I_1 \) are both of order type \( \omega \). Since \( I_0 \cap I_1 \neq \emptyset \) and both \( I_0 \) and \( I_1 \) are intervals of order type \( \omega \), we get either \( I_0 \subseteq I_1 \) or \( I_1 \subseteq I_0 \) and since both are maximal in \( \Sigma \) for \( W \) we get \( I_0 = I_1 \) by Lemma 3.4. The case where \( W(i) \) has superscript \(-1\) is handled similarly.
Lemma 3.6. If \( W \in \text{Red}_{a,b,c} \) there is a unique decomposition \( \overline{W} = \prod_{\lambda} I_\lambda \) such that \( I_\lambda \) is either maximal in \( \Sigma \) for \( W \) or a maximal interval which does not intersect with any interval which is maximal in \( \Sigma \) for \( W \).

Proof. To begin we let \( \{ I_\lambda \}_{\lambda \in \Lambda} \) be the collection of intervals which are maximal in \( \Sigma \) for \( W \). Next, by Zorn’s Lemma we select all maximal intervals \( \{ I_\lambda \}_{\lambda \in \Lambda'} \) which are disjoint from the elements of \( \{ I_\lambda \}_{\lambda \in \Lambda''} \). Taking \( \Lambda = \Lambda' \cup \Lambda'' \) and endowing this set with the obvious ordering, we see that \( \overline{W} = \prod_{\lambda \in \Lambda} I_\lambda \). Uniqueness is clear. \( \square \)

Lemma 3.6 gives a word \( W \in \text{Red}_{a,b,c} \) a unique decomposition \( W = \prod_{\lambda} I_\lambda \). Now, given a function \( f : \Sigma \to \Sigma \cup \{ T \} \) we define a function \( F_f : \text{Red}_{a,b,c} \to \text{W}_a,b,c \) by letting

\[
F_f(W) = \prod_{\lambda} I_\lambda
\]

where

\[
W = \prod_{\lambda} W_\lambda
\]

is the aforementioned decomposition implied by Lemma 3.6 and

\[
W_\lambda = \begin{cases} W_\lambda \upharpoonright I_\lambda & \text{if } I_\lambda \text{ is not maximal in } \Sigma \text{ for } W \\ U_{f(S),n} & \text{if } W \upharpoonright I_\lambda \equiv U_{S,n} \text{ with } S \in \Sigma \\ U_{f(S),n}^{-1} & \text{if } W \upharpoonright I_\lambda \equiv U_{S,n}^{-1} \text{ with } S \in \Sigma \end{cases}
\]

The object \( \prod_{\lambda} W_\lambda \) is evidently a function whose domain is a totally ordered set which is order isomorphic to \( \overline{W} \). It has as codomain \( \{ a^{ n \pm 1 }, b^{ n \pm 1 }, c^{ n \pm 1 } \}_{n \in \omega} \). Moreover, for each \( n \in \omega \) the set of elements \( i \in \prod_{\lambda} W_\lambda \) for which the subscript of \( \prod_{\lambda} W_\lambda \) is \( \leq n \) has the same cardinality as the set of elements \( i \in \overline{W} \) such that \( W(i) \) has subscript \( \leq n \). Thus \( \prod_{\lambda} W_\lambda \in \text{W}_a,b,c \).

We check that the map \( \psi_f : \text{HEG} \to \text{HAG} \) given by \( \psi_f(W) = \lfloor [F_f(W)] \rfloor \) is a homomorphism. Towards this we give the following lemma.

Lemma 3.7. If \( W \in \text{Red}_{a,b,c} \) and \( W = W_0W_1 \) then \( \psi_f(W) = \psi_f(W_0)\psi_f(W_1) \).

Proof. If \( W = \prod_{\lambda} W_\lambda \) is the decomposition given by Lemma 3.6 then one of the following holds:

1. There exist \( \Lambda_0, \Lambda_1 \subseteq \Lambda \) such that all elements of \( \Lambda_0 \) are below those of \( \Lambda_1 \), \( W_0 = \prod_{\lambda \in \Lambda_0} W_\lambda \) and \( W_1 = \prod_{\lambda \in \Lambda_1} W_\lambda \).
2. There exists \( \zeta \in \Lambda \) such that \( W_\zeta = W_{0,\zeta}W_{1,\zeta} \) and \( W_0 = (\prod_{\lambda \prec \zeta} W_\lambda)W_{0,\zeta} \) and \( W_1 = W_{1,\zeta}(\prod_{\lambda \prec \zeta} W_\lambda) \).

In case (1) the decompositions of \( W_0 \) and \( W_1 \) given by Lemma 3.6 are respectively \( W_0 = \prod_{\lambda \in \Lambda_0} W_\lambda \) and \( W_1 = \prod_{\lambda \in \Lambda_1} W_\lambda \). Thus in this case we get \( F_f(W) \equiv F_f(W_0)F_f(W_1) \) and \( \psi_f(W) = \psi_f(W_0)\psi_f(W_1) \) is immediate.

In case (2) there are several subcases. We mention each of these subcases and state the Lemma 3.6 decomposition for \( W_0 \) and \( W_1 \).

\begin{enumerate}
  \item If \( \overline{W_\zeta} \) was maximal in \( \Sigma \) for \( W \) with \( W_\zeta = U_{S,n} \) then \( W_{1,\zeta} = U_{S,n'} \) for some \( n' > n \) and \( W_{0,\zeta} \) will be a finite word which is a prefix to \( U_{S,n} \).
  \item If in addition to 2.1 there is an immediate predecessor \( \zeta' < \zeta \in \Lambda \) and \( \overline{W_{\zeta'}} \) is not maximal in \( \Sigma \) for \( W \), then the decomposition of \( W_0 \) is \( W_0 = (\prod_{\lambda \prec \zeta'} W_\lambda)(W_{0,\zeta}W_{0,\zeta'}) \) and the decomposition of \( W_1 \) is \( W_1 = W_{1,\zeta}(\prod_{\lambda \prec \zeta} W_\lambda) \). By this we mean that the decomposition of word \( W_0 \) has index of order type \( \{ \lambda \in \Lambda \mid \lambda < \zeta' \} \) and the last word of the decomposition is \( W_{0,\zeta} \). The decomposition of \( W_1 \) has index of order type \( \{ \lambda \in \Lambda \mid \zeta \leq \lambda \} \) and the first word of the decomposition is \( W_{1,\zeta} \). Here we get
$F_f(W_0) \equiv (\Pi_{\lambda<\zeta} W'_\lambda)(W_{0,\zeta})$ and $F_f(W_1) \equiv W'_{1,\zeta} \Pi_{\zeta<\lambda} W'_\lambda$ and since the $[[\cdot]]$ class of a word is closed under modifying a finite subword we get

\[ \psi_f(W_0) \psi_f(W_1) = [[[\Pi_{\lambda<\zeta} W'_\lambda)(W_{0,\zeta})]][[W'_{1,\zeta} \Pi_{\zeta<\lambda} W'_\lambda]]] \]

2.2.1 If in addition to 2.1 there is an immediate successor $\zeta' < \zeta$ in $\Lambda$ and $W_{\zeta'}$ is maximal in $\Sigma$ for $W$ then the decomposition of $W_0$ is $W_0 = (\Pi_{\lambda<\zeta} W_{-1})W_{0,\zeta}$ and $W_1 = (\Pi_{\lambda<\zeta} W_{1,\zeta})W_{0,\zeta}$ are the decompositions. We get that

\[ \psi_f(W_0) \psi_f(W_1) = [[[\Pi_{\lambda<\zeta} W'_\lambda)(W_{0,\zeta})]][[W'_{1,\zeta} \Pi_{\zeta<\lambda} W'_\lambda]]] \]

2.2.2 If in addition to 2.1 there is an immediate successor $\zeta' < \zeta$ in $\Lambda$ and $W_{\zeta'}$ is maximal in $\Sigma$ for $W$ then the decomposition of $W_0$ is $W_0 = (\Pi_{\lambda<\zeta} W_{0,\zeta})W_{0,\zeta}$ and the decomposition of $W_1$ is $W_1 = (W_{1,\zeta})(W_{-1})_{\Lambda}$ $W_{\zeta}$ is maximal for $\Sigma$ in $\lambda$ and the decomposition of $W_1$ is $W_1 = (W_{1,\zeta})(W_{-1})_{\Lambda}$ $W_{\zeta}$ is the claim in this subcase follows along the same lines as 2.2.1.

2.2.3 If in addition to 2.2 there is no immediate successor $\zeta' < \zeta$ in $\Lambda$ then $W_{\zeta}$ was not maximal for $\Sigma$ in $W$, then the decompositions of $W_0$ and $W_1$ are $W_0 = (\Pi_{\lambda<\zeta} W_{0,\zeta})W_{0,\zeta}$ and $W_1 = (W_{1,\zeta})(W_{-1})_{\Lambda}$ $W_{\zeta}$ is the claim in this subcase follows along the same lines as 2.2.1.

2.2.4 If in addition to 2.2 there is no immediate successor $\zeta' < \zeta$ in $\Lambda$ then $W_{\zeta}$ was not maximal for $\Sigma$ in $W$, then the decompositions of $W_0$ and $W_1$ are $W_0 = (\Pi_{\lambda<\zeta} W_{0,\zeta})W_{0,\zeta}$ and $W_1 = (W_{1,\zeta})(W_{-1})_{\Lambda}$ $W_{\zeta}$ is the claim in this subcase follows along the same lines as 2.2.1.

2.3.1 If in addition to 2.2 there is no immediate successor $\zeta' < \zeta$ in $\Lambda$ then $W_{\zeta}$ was not maximal for $\Sigma$ in $W$, then the decompositions of $W_0$ and $W_1$ are $W_0 = (\Pi_{\lambda<\zeta} W_{0,\zeta})W_{0,\zeta}$ and $W_1 = (W_{1,\zeta})(W_{-1})_{\Lambda}$ $W_{\zeta}$ is the claim in this subcase follows along the same lines as 2.2.1.
and the claim follows in all circumstances.

Lemma 3.8. The function $\psi_f : HEG_{a,b,c} \rightarrow HAG_{a,b,c}$ is a homomorphism which descends to a homomorphism $\phi_f : HAG_{a,b,c} \rightarrow HAG_{a,b,c}$.

Proof. Let $W_0, W_1 \in \text{Red}_{a,b,c}$. We let

\[
\begin{align*}
W_0 & \equiv W_{0,0}W_{0,1} \\
W_1 & \equiv W_{1,0}W_{1,1} \\
W_0^{-1} & \equiv W_{0,1}^{-1}W_{0,0}^{-1} \\
W_{0,0}W_{1,1} & \in \text{Red}_{a,b,c}
\end{align*}
\]

according the conclusion of Lemma 2.2. We have

\[
\begin{align*}
\psi_f(W_0)\psi_f(W_1) &= \psi_f(W_{0,0})\psi_f(W_{0,1})\psi_f(W_{1,0})\psi_f(W_{1,1}) \\
&= \psi_f(W_{0,0})\psi_f(W_{0,1})\psi_f(W_{0,1})\psi_f(W_{1,1}) \\
&= \psi_f(W_{0,0})\psi_f(W_{0,1})^{-1}\psi_f(W_{1,1}) \\
&= \psi_f(W_{0,0})\psi_f(W_{1,1}) \\
&= \psi_f(W_{0,0}W_{1,1})
\end{align*}
\]

where the first and last equality come from Lemma 3.7 and the third equality is clear since the map $\psi_f$ obviously satisfies $\psi_f(W^{-1}) = \psi_f(W)^{-1}$. Thus $\psi_f$ is a homomorphism and it is easy to check that $\bigcup_{n \in \mathbb{N}} HEG_{a,b,c} \subseteq \ker(\psi_f)$, and so $\psi_f$ induces a homomorphism $\phi_f : HAG_{a,b,c} \rightarrow HAG_{a,b,c}$.

Now we are ready to prove Proposition 3.2. Supposing there exists a nontrivial homomorphism from $HAG$ to $G$ we obtain by Lemma 3.3 a homomorphism $\phi : HAG_{a,b,c} \rightarrow G$ such that $\phi([U_T]) \neq 1_G$ and $HAG_{b,c} \subseteq \ker(\phi)$. Given a subset $S \subseteq \Sigma$ we define a function $f_S : \Sigma \rightarrow \Sigma \cup \{T\}$ by $f_S(S) = \begin{cases} S & \text{if } S \notin S \\ T & \text{if } S \in S \end{cases}$. The accompanying $\phi_{f_S} : HAG \rightarrow HAG$ satisfies

$$\phi_{f_S}([S]) = \begin{cases} [S] & \text{if } S \notin S \\ [T] & \text{if } S \in S \end{cases}$$

and so $[S] \in \ker(\phi \circ \phi_{f_S})$ if and only if $S \notin S$. Thus $|\text{Hom}(HAG,G)| \geq 2^{2^{\omega_0}}$.

We end this section by restating and proving Theorem B.

Theorem B. The group $\text{Aut}(HAG)$ contains an isomorphic copy of the full symmetric group $S_{2^{\omega_0}}$ on a set of cardinality continuum. Thus $\text{Aut}(HAG)$ is of cardinality $2^{2^{\omega_0}}$ and all groups of size at most $2^{\omega_0}$ are subgroups.

Proof. Supposing that $\sigma : \Sigma \rightarrow \Sigma$ is a bijection, we get an endomorphism $\phi_\sigma : HAG \rightarrow HAG$ and notice that $\phi_{\sigma^{-1}}\phi_\sigma = \phi_\sigma\phi_{\sigma^{-1}} = \text{Id}_{HAG}$. Thus the mapping $\phi_\sigma$ is an automorphism, and it is straightforward to check that $\phi_{\sigma_1\sigma_1} = \phi_{\sigma_1}, \phi_{\sigma_1}$, so $\sigma \mapsto \phi_\sigma$ is a homomorphism to $\text{Aut}(HAG)$. If $\sigma(S) = S' \neq S$ then $\phi_\sigma([S]) = [S'] \neq [S]$, and so the mapping $\sigma \mapsto \phi_\sigma$ has trivial kernel. Thus we see that $\text{Aut}(HAG)$ has a copy of $S_{2^{\omega_0}}$. From this, and since $HAG$ is of cardinality $2^{\omega_0}$, we see that $\text{Aut}(HAG)$ is of cardinality $2^{2^{\omega_0}}$. Since $S_{2^{\omega_0}} \leq \text{Aut}(HAG)$ every group of cardinality at most $2^{\omega_0}$ is also a subgroup by considering the left action of a group on itself.

\[\square\]
4. RESIDUAL SLENDERNESS AND SUBGROUPS

The techniques used so far suggest that residuality can be considered in determining the n-slenderness of small cardinality groups. Theorem 4.1 below makes this explicit. We explore two subgroups of a group $G$ whose triviality precisely determines n-slenderness in a group of small cardinality. Further, an example is shown in which these two subgroups are not equal. From this, a new family of groups is shown to be n-slender. We motivate and then present Theorem 4.8 which determines which subgroups of HEG are n-slender (they are precisely those which do not contain an isomorph of HEG as a subgroup). Finally, we leave the reader with two open questions.

**Theorem 4.1.** If $G$ is a group such that $|G| < 2^{80}$ then $G$ is n-slender if and only if $G$ is residually n-slender.

**Proof.** The direction ($\Rightarrow$) is obvious by using the identity homomorphism. For ($\Leftarrow$), we assume $G$ is residually n-slender and $|G| < 2^{80}$. If $G$ fails to be n-slender, we have by Lemma 2.7 a nontrivial homomorphism $\phi : \text{HAG} \rightarrow G$. Letting $g \in \phi(HAG) \setminus \{1_G\}$ we pick a homomorphism $\psi : G \rightarrow H$ such that $H$ is n-slender and $\psi(g) \neq 1_H$. Then $\psi \circ \phi$ is a nontrivial homomorphism from HAG to the slender group $H$, contradicting Lemma 2.4.

The statement of Theorem 4.1 can fail to hold if one drops the condition $|G| < 2^{80}$. For example, HEG is residually n-slender (since HEG is residually free) but the identity map witnesses that HEG is not n-slender. Theorem 4.1 motivates the following definition.

**Definition 4.2.** Given a group $G$ we define the slender kernel $\text{slk}(G)$ to be the intersection of all kernels of homomorphisms from $G$ to noncommutatively slender groups, that is

$$\text{slk}(G) = \bigcap \{\ker(\phi) \mid \phi : G \rightarrow H \text{ with } H \text{ n-slender}\}$$

This normal subgroup of $G$ records some of the obstruction which exists for the noncommutative slenderness of $G$. All torsion elements of $G$ are in $\text{slk}(G)$ and all subgroups isomorphic to $\mathbb{Q}$ are included therein. By Theorem 4.1 the slender kernel of $G$ is precisely the obstruction for noncommutative slenderness of $G$ when $|G| < 2^{80}$ that is, such a $G$ is noncommutatively slender if and only if $\text{slk}(G)$ is trivial. By Lemmas 2.7 and 2.8 we see that the existence of a nontrivial homomorphic image of HAG in $G$ is also a precise obstruction of n-slenderness for such groups $G$ of small cardinality. Let $\text{HAGim}(G)$ denote the subgroup

$$\text{HAGim} = \{\cup \{\phi(\text{HAG}) \mid \phi : \text{HAG} \rightarrow G \text{ is a homomorphism}\}\}$$

This subgroup is obviously normal. By the proof of Theorem 4.1 we have $\text{HAGim}(G) \leq \text{slk}(G)$ for any group $G$.

**Example 1.** We notice that $\text{HAGim}(G)$ need not be the union $\bigcup \{\phi(\text{HAG}) \mid \phi : \text{HAG} \rightarrow G \text{ is a homomorphism}\}$. To see this we let $G = (\mathbb{Z}/2) * (\mathbb{Z}/2)$. We know that $\mathbb{Z}/2$ is a homomorphic image of HAG (this group is not n-slender), and so in this case $\text{HAGim}(G) = G$. However, any homomorphism $\phi : \text{HAG} \rightarrow (\mathbb{Z}/2) * (\mathbb{Z}/2)$ has image which is contained in a conjugate of the first or the second copy of $\mathbb{Z}/2$ by applying Lemma 2.9. Thus if $\langle h_0 \rangle$ is the first copy of $\mathbb{Z}/2$ and $\langle h_1 \rangle$ is the second copy, the element $h_0h_1 \in G$ would never be in the image of a homomorphism from HAG.
Since \( \text{HAGim}(G) \leq \text{slk}(G) \) it seems natural to ask whether equality always holds. We give an example of a countable torsion-free group for which this fails, after first proving a lemma. Recall that a subgroup \( H \leq G \) is central provided \( H \) is a subgroup of the center of \( G \). Central subgroups are always normal.

**Lemma 4.3.** Suppose \( \{H_i\}_{i \in I} \) is a collection of groups and \( H \) is a group such that for each \( i \) we have a monomorphism \( \phi_i : H \to H_i \) with \( \phi_i(H) \) in the center of \( H_i \). Let \( *_{H}H_i \) denote the amalgamated free product obtained by identifying the copies of images of \( H \) via the maps \( \phi_i \). If \( \phi : \text{HEG} \to *_{H}H_i \), then for some \( n \in \mathbb{N} \), \( j \in I \) and \( g \in *_{H}H_i \) we have \( \phi(\text{HEG}^n) \leq g^{-1}H_jg \).

**Proof.** Since \( \phi_i(H) \) is central in \( H_i \) for every \( i \in I \), we have \( H \leq *_{H}H_i \) a central subgroup, and therefore normal. The isomorphism \( ( *_{H}H_i ) / H \cong *_{\ell_{I}}(H_i / H) \) is clear, and let \( \psi : *_{H}H_i \to *_{\ell_{I}}(H_i / H) \) be the quotient map. Given a map \( \phi : \text{HEG} \to *_{H}H_i \), we notice by Lemma 2.8 that for some \( n \in \mathbb{N} \), \( j \in I \) and \( h \in *_{H}H_i \), the inclusion \( \psi \circ \phi(\text{HEG}^n) \leq h^{-1}(H_j / H)h \). Selecting \( g \in *_{H}H_i \) satisfying \( \psi(g) = h \) it is easy to see that \( \phi(\text{HEG}^n) \leq g^{-1}H_jg \). \( \square \)

**Example 2.** Consider \( 2\mathbb{Z} \) both as a subgroup of \( \mathbb{Z} \) as well as a subgroup of \( \mathbb{Q} \). Let \( G \) be the amalgamated free product \( \mathbb{Z} *_{2\mathbb{Z}} \mathbb{Q} \) which identifies the copy of \( 2\mathbb{Z} \) in \( \mathbb{Z} \) with that in \( \mathbb{Q} \). As both \( \mathbb{Z} \) and \( \mathbb{Q} \) are torsion-free, the group \( G \) is torsion-free.

By Lemma 4.3 any homomorphism \( \phi : \text{HAG} \to G \) must either have \( \phi(\text{HAG}) \) as a subgroup of a conjugate of \( \mathbb{Z} \) or a conjugate of \( \mathbb{Q} \). As there is no nontrivial map from \( \text{HAG} \) to \( \mathbb{Z} \), we see that any nontrivial image of \( \text{HAG} \) must lie inside a conjugate of \( \mathbb{Q} \). Thus \( \text{HAGim}(G) \leq \langle \langle \mathbb{Q} \rangle \rangle \leq G \), and since each conjugate of \( \mathbb{Q} \) is in \( \text{HAGim}(G) \) we get \( \text{HAGim}(G) = \langle \langle \mathbb{Q} \rangle \rangle \).

Since \( \text{slk}(G) \geq \text{HAGim}(G) \) and \( \text{HAGim}(G) \) is of index 2 in \( G \), it is clear that \( \text{slk}(G) = G \) since \( n \)-slender groups are torsion-free.

**Theorem 4.4.** Assume the hypotheses of Lemma 4.3. If each of the groups \( H_i \) is \( n \)-slender, then so is \( *_{H}H_i \).

**Proof.** Assume the hypotheses and suppose \( \phi : \text{HEG} \to *_{H}H_i \) is a homomorphism. By Lemma 4.3 select \( n \in \mathbb{N} \), \( j \in I \) and \( g \in *_{H}H_i \) so that \( \phi(\text{HEG}^n) \leq g^{-1}H_jg \). Since \( H_j \) is slender there exists \( m \geq n \) so that \( \phi(\text{HEG}^m) \) is trivial. Thus \( *_{H}H_i \) is \( n \)-slender.

This theorem provides new examples of \( n \)-slender groups, as seen in the next example.

**Example 3.** Let \( \{s_n\}_{n \in \omega} \) be a sequence in \( \mathbb{Z} \setminus \{0\} \) such that \( s_0 = 1 \). Since \( \mathbb{Z} \) is \( n \)-slender, the amalgamated free product \( \{(a_n)_{n \in \omega} | (a_0 = a_0^+)_{n \in \omega}\} \) is \( n \)-slender by Theorem 4.3. If one lets \( s_n = n + 1 \), then the element \( a_0 \) has an \( n \)-th root for every \( n \in \omega \setminus \{0\} \). As far as the author is aware, this gives the first known example of an \( n \)-slender group which has a nontrivial divisible element. Notice that a nontrivial group in which every element is divisible cannot be \( n \)-slender. Such a group will either have torsion or it will be torsion-free, in which case one can easily piece together a subgroup which is isomorphic to \( \mathbb{Q} \).

We end by motivating a noncommutative version of a theorem regarding slender subgroups and posing two questions. The following is an immediate corollary to Theorem 4.4.
Corollary 4.5. Every subgroup of the Hawaiian earring group of cardinality \( < 2^{\aleph_0} \) is noncommutatively slender.

Although the group HEG is locally free, there exist countable subgroups which are not free (see the discussion following [Hi, Theorem 6]). Thus Corollary 4.5 gives examples of non-free n-slender groups. By contrast, the group \( \prod_\omega Z \) is \( \aleph_1 \)-free- that is, every countable subgroup is free abelian [Sp]. There is an analogous abelian version of Corollary 4.5 which follows immediately from the classification of slender groups of small cardinality in [Sa]:

Observation 4.6. Every subgroup of \( \prod_\omega Z \) of cardinality \( < 2^{\aleph_0} \) is slender.

In light of \( \aleph_1 \)-freeness and the fact that free abelian groups are slender, this observation does not furnish many new examples unless the continuum hypothesis fails. Moreover, in light of [Nu] one gets the stronger observation:

Observation 4.7. A subgroup of \( \prod_\omega Z \) fails to be slender if and only if it contains a subgroup isomorphic to \( \prod_\omega Z \).

We give the non-abelian analog to this stronger observation.

Theorem 4.8. A subgroup of HEG fails to be n-slender if and only if it contains a subgroup isomorphic to HEG.

For this we first prove a straightforward lemma regarding free groups.

Lemma 4.9. Let \( F(X) \) be the free group on generators \( X \), let \( Y \subseteq X \) and \( w \in F(X) \) be a reduced word that utilizes an element of \( X \setminus Y \). Letting \( t \) be a symbol such that \( t \notin X \), the mapping \( f : \{t\} \cup Y \to \{w\} \cup Y \) given by \( f(t) = w \) and \( f(y) = y \) for \( y \in Y \) extends to an isomorphism \( \phi : F(\{t\} \cup Y) \to (\{w\} \cup Y) \).

Proof. By freeness of \( F(\{t\} \cup Y) \) we get an extension \( \phi : F(\{t\} \cup Y) \to (\{w\} \cup Y) \), and we need only check that this is injective. We consider elements of free groups as reduced words in the appointed generators. Write \( w \equiv w_0 w_1 w_2 w_3^{-1} w_4 \), where \( w_0 \) is the maximal prefix of \( w \) which uses only letters in \( Y^{\pm 1} \), \( w_3 \) is the maximal suffix of \( w \) which uses only letters in \( Y^{\pm 1} \), and \( w_2 \) is the cyclic reduction of the remaining middle word \( w_1 w_2 w_3 w_4^{-1} \). Since \( w \) uses a letter not in \( Y \), the remaining middle word \( w_1 w_2 w_3^{-1} \) is not the empty word, and therefore \( w_2 \) is also nonempty.

Supposing we have a multiplication \( wvw \) with \( v \) nontrivial \( F(Y) \), we consider the maximum extent of letter cancellation. We have

\[
wwv \equiv w_0 w_1 w_2^{-1} w_3 w_4 w_5 w_6 w_7 w_8^{-1} w_9
\]

and the greatest extent of letter cancellation has \( w_3 w_4 w_5 \) cancelling entirely, so that \( w_1^{-1} w_2 w_3 w_4 w_5 \) cancels entirely, but the cancellation may go no further since \( w_2 \) was cyclically reduced. Supposing we have multiplication \( wvw^{-1} \) with \( v \) nontrivial in \( F(Y) \), we consider the maximum extent of cancellation within

\[
wwv^{-1} \equiv w_0 w_1 w_2^{-1} w_3 w_4^{-1} w_5 w_6^{-1} w_7 w_8 w_9^{-1}
\]

Since \( v \) is nontrivial we see that \( w_3 w_4 w_5^{-1} \) is not trivial. Thus in both cases, the nontrivial word \( w_2^{-1} \) remains untouched after a maximal cancellation, and the multiplication \( w^{-1} w^{-1} \) yields the same conclusion, as do products \( ww \) and \( w^{-1} w^{-1} \). This is sufficient for showing that a nontrivial reduced word in \( F(\{t\} \cup Y) \) maps nontrivially.

\[
\square
\]
Proof. (of Theorem 4.8) Only the ($\Rightarrow$) direction is nontrivial. Suppose $G \leq \text{HEG}$ is not $n$-slender and let $\phi_0 : \text{HEG} \to G$ witness this. By Lemma 2.3 we can conjugate both $G$ and the homomorphism and obtain a new homomorphism $\phi_1$ determined by a mapping $a_n \mapsto W_n$ where $[W_n] \in \text{HEG}^n$ and $j_n \not\sim \infty$ which witnesses that the conjugate of $G$ is not $n$-slender. It will be sufficient to find a subgroup of this conjugate which contains HEG as a subgroup, so without loss of generality we replace $G$ with this conjugate.

Since $\phi_1$ witnesses the negation of $n$-slenderness, we can select a sequence $V_n$ of reduced words such that $[V_n] \in \text{HEG}^n$ and $\phi_1(V_n) \in \text{HEG}^n \setminus \{1\}$ and we let $U_n$ be the reduced word such that $U_n \in \phi(V_n)$. The mapping $f : a_n \mapsto U_n$ induces an endomorphism $\phi_2 : \text{HEG} \to \text{HEG}$ such that $\phi_2(\text{HEG}) \leq G$ and $\phi_2([a_n]) \in \text{HEG}^n \setminus \{1\}$ for all $n \in \omega$. Now, for each $n \in \omega$ there exists $m_n \in \omega$ such that $p_{m_n}((a_n)) = 1$. Thus by using a subsequence we produce an endomorphism $\phi$ such that $\phi(\text{HEG}) \leq G$, $p_{m_n}([a_n]) = 1$ and $\phi([a_{n+1}]) \in \text{HEG}^{m_n}$. This last condition guarantees that $p_{m_{n-1}} \circ \phi \circ p_n = p_{m_{n-1}} \circ \phi$. It also guarantees that given $n < j$ there exists a letter utilized in $p_{m_{j-1}} \circ \phi([a_n])$ which is not one of the generators of the free group $\text{HEG}_{m_{n-1}} \cap \text{HEG}^{m_j}$.

We claim that for each $n \in \omega$ the restriction $p_{m_{n-1}} \circ \phi \upharpoonright \text{HEG}_n$ is an injection. Fix $n$. We know $p_{m_{n-1}} \circ \phi([a_0])$ utilizes a letter that is not a generator in $\text{HEG}_{m_{n-1}} \cap \text{HEG}^{m_n}$, so by Lemma 4.9 we get the isomorphism

$$\langle t_0 \rangle * (\text{HEG}_{m_{n-1}} \cap \text{HEG}^{m_n}) \cong \{ p_{m_{n-1}} \circ \phi([a_0]) \} \cup (\text{HEG}_{m_{n-1}} \cap \text{HEG}^{m_n})$$

Since the subgroup $\{ p_{m_{n-1}} \circ \phi([a_1]), \ldots, p_{m_{n-1}} \circ \phi([a_{n-1}]) \}$ is included in the group $\text{HEG}_{m_{n-1}} \cap \text{HEG}^{m_n}$, we get $\text{a fortiori}$ that

$$\langle t_0 \rangle * (p_{m_{n-1}} \circ \phi([a_1]), \ldots, p_{m_{n-1}} \circ \phi([a_{n-1}]) \rangle = p_{m_{n-1}} \circ \phi([a_0]), \ldots, p_{m_{n-1}} \circ \phi([a_{n-1}])$$

Continuing to argue in this manner we get that $\{ p_{m_{n-1}} \circ \phi([a_0]), \ldots, p_{m_{n-1}} \circ \phi([a_{n-1}]) \}$ is a free group in its listed generators, so $p_{m_{n-1}} \circ \phi \upharpoonright \text{HEG}_n$ is injective.

Now, given $[W] \in \text{HEG} \setminus \{1\}$ we select $n \in \omega$ such that $p_n([W]) \neq 1$. Then $p_{m_{n-1}} \circ \phi \circ p_n([W]) \neq 1$ since $p_{m_{n-1}} \circ \phi \upharpoonright \text{HEG}_n$ is injective. Thus

$$1 \neq p_{m_{n-1}} \circ \phi \circ p_n([W]) = p_{m_{n-1}} \circ \phi([W])$$

so that $\phi([W]) \neq 1$. Then $\phi$ is a monomorphism and we are done. $\square$

The classification in Theorem 4.8 cannot be strengthened by more generally considering subgroups of residually free groups, since $\prod_{n} \mathbb{Z}$ is residually free, fails to be $n$-slender, and does not contain HEG as a subgroup.

In light of the result of \cite{N} we ask whether the analogous situation holds in the non-abelian case:

**Question 4.10.** Does there exist a countable set of groups $\{ G_n \}_{n \in \omega}$ such that a group fails to be $n$-slender if and only if it includes one of the $G_n$ as a subgroup?

As a weakening of Question 4.10 we ask whether the main result of \cite{S} holds in the non-abelian case:

**Question 4.11.** Does there exist a countable set of groups $\{ G_n \}_{n \in \omega}$ such that a countable group fails to be $n$-slender if and only if it includes one of the $G_n$ as a subgroup?

As has already been mentioned, it is even unknown whether there exists a countable group not containing $\mathbb{Q}$ or torsion which fails to be $n$-slender.
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IKERBASQUE- BASQUE FOUNDATION FOR SCIENCE AND MATHEMATICA SAILA, UPV/EHU, SARRIENA S/N, 48940, LEIOA - BIZKAIA, SPAIN
E-mail address: sammyc973@gmail.com