Constructions and properties of a class of random scale-free networks

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Abstract

Complex networks have abundant and extensive applications in real life. Recently, researchers have proposed a number of complex networks, in which some are deterministic and others are random. Compared with deterministic networks, random network is not only interesting and typical but also practical to illustrate and study many real-world complex networks, especially for random scale-free networks. Here, we introduce three types of operations, i.e., type-A operation, type-B operation and type-C operation, for generating random scale-free networks \(N(p,q,r,t)\). On the basis of our operations, we put forward the concrete process of producing networks, which constitute the network space \(N(p,q,r,t)\), and then discuss their topological properties. Firstly, we calculate the range of the average degree of each member in our network space and discover that each member is a sparse network. Secondly, we prove that each member in our space obeys the power-law distribution with degree exponent \(\gamma = 1 + \frac{\ln(4-r)}{\ln 2}\), which implies that each member is scale-free. Next, we analyze the diameter, and find that the diameter may abruptly transform from small to large due to type-B operation. Afterwards, we study the clustering coefficient of network and discover that its value is only determined by type-C operation. Ultimately, we make an elaborate conclusion.

Keywords: Random network; degree distribution; diameter; clustering coefficient.

1 Introduction

Over the course of the recent decades, complex network is an interdisciplinary study that has attracted thousands of scholars from different fields, and is considered as a mathematical tool that connects the real world with theoretical research. Complex network is applied across a multitude of disciplines ranging from natural and physical sciences to social sciences and humanities. In general, a network can be viewed as a graph consisting of vertices (or nodes) connected by edges (or links). An extensive example of complex network contains the Internet, the World Wide Web (WWW), coauthorship network, citation network, annotated network, musical solos network, protein network, information network, peer-to-peer network, manage network\cite{1}-\cite{13}, and so forth.

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Erdős and Rényi, in 1960, defined a random network as a network with \( n \) vertices connected by \( m \) edges, which are randomly chosen from the \( n(n-1)/2 \) possible edges, it has been proven that the vertex degree distribution of random network is a Poisson distribution\[12\]. Yet, there are many real-life networks whose distribution is not Poisson distribution but heavily tail degree distribution, normally regarded as power-law distribution. Vertices in this type of distribution are dominated by several highly connected hubs – the vertices with the largest degree – implying that most of the vertices are connected to the hubs via one or very few steps.

Hereafter, the network can be considered as a scale-free one if degree distribution of the network is a power-law distribution. That is to say, its degree distribution obeys the following form

\[
P(k) \sim k^{-\gamma}, \quad 2 < \gamma \leq 3
\]

where \( P(k) \) represents a probability of randomly selecting a vertex of degree \( k \) from the entire network\[2\]. On the basis of the degree distribution formula \[1\], Dorogovtsev et al. \[21\] gave the definition of cumulative degree distribution and the computational formula to explain topological structure of deterministic scale-free networks as follows:

\[
P_{\text{cum}}(k) = \sum_{k' \geq k} \frac{N(k',t)}{n_v(t)} \sim k^{1-\gamma}
\]

where \( N(k',t) \) and \( n_v(t) \) stand for the number of vertices with degree \( k' \) and the network order \( N(t) \) at time step \( t \), respectively. Currently, many empirical studies have shown that scores of complex networks in society and nature have the scale-free property, namely, they follow either the degree distribution \[1\] or the cumulative degree distribution \[2\]. In this paper, we show that each member in our space obeys the power-law distribution with degree exponent \( \gamma = 1 + \frac{\ln(4-r)}{\ln 2} \), by calculating the cumulative degree distribution.

In other words, the degree distribution merely describe partial characteristics of network, it is not enough for ones who simply use the degree distribution to analyze complex networks. So, other statistical methods have been developed to depict the nature of network. Among of which, the prevalent statistical traits are diameter and clustering coefficient of network. If a network have smaller diameter and higher average clustering coefficient, it can be referred to as a small-world network which was discovered by Watts and Strogatz in 1998\[1\]. The well-known six degrees of separation theory reveals that the diameter is short in comparison with its size. Actually, in our real-life networks, the diameter of some networks generally increase exponentially with network order\[13\], while others usually grow logarithmically\[14\,15\,16\]. Here, we discover the fact that the diameters of these members in our space that do not contain type-B operation grow linearly with time \( t \), otherwise, the diameters increase exponentially.

The clustering coefficient of the network can be obtained by averaging the clustering coefficient of a vertex over all the vertices. The clustering coefficient of a vertex in network refers to the ratio of the number of actual existing edges between all vertices adjacent to it
and the number of all possible edges between them. A mass of social and natural networks have clustering characteristics, some networks have a low clustering coefficient\([17]\), while the remaining have a high clustering coefficient\([14, 15, 16]\). With the aim of providing a deeper understanding of the underlying mechanism that leads to different clustering coefficient, we need to probe the clustering coefficient of network in more detail. Then, we show that the clustering coefficient of these members in our network space that do not contain type-C operation is always zero, otherwise the clustering coefficient is not equal to zero. Combined with the diameter and clustering coefficient, we reveal that these members of our network space not contain type-B operation but contain type-C operation display better features. It may have smaller diameter and higher clustering coefficient, usually known as the small-world effect.

The reminder of this paper is organized by the following several sections. In Section 2, we introduce three types of operations in detail, namely, type-A operation, type-B operation and type-C operation, to successfully construct a network space \(\mathcal{N}(p, q, r, t)\) where probability parameters \(p, q, r\) hold on \(p + q + r = 1, 0 \leq p, q, r \leq 1\). After that, in Section 3, we show the elaborate process of construction and discuss some fundamental and typical topological structures of networks in our network space \(\mathcal{N}(p, q, r, t)\). Above all, we discover that each member in the network space has sparsity due to average degree approaches to a constant. Secondly, we prove that each member of our space is scale-free because it follows a power-law distribution with \(\gamma = 1 + \frac{\ln(4-r)}{\ln 2}\). Hereafter, we analyze the diameter may abruptly transform from small to large due to the type-B operation. Hence, we state that the diameters of these members in our space that do not contain type-B operation grow linearly with time \(t\), otherwise, the diameters increase exponentially. Afterwards, we study the clustering coefficient of the network and find type-C operation has a decisive influence on the clustering coefficient. Finally, we make a detailed conclusion.

2 Three Types of Operations

In this section, we will construct a category of random networks with tuning parameters which are generated from three different typical growth operations and then applied to span a network space, denoted as \(\mathcal{N}(p, q, r, t)\). Here, the probability parameters \(p, q, r\) hold \(p + q + r = 1\) with \(0 \leq p, q, r \leq 1\), and \(t\) stands for time step. In other words, for two arbitrary members \(N(p_i, q_i, r_i, t)\) and \(N(p_j, q_j, r_j, t)\) randomly selected from network space \(\mathcal{N}(p, q, r, t)\), they must have different “degree sequence”. To this end, we want to introduce three classical operations, referred in this paper type-A operation, type-B operation, type-C operation, which are explained in more detail, as follows

**Type-A operation** Add an edge \(xy\) on vertices \(x\) and \(y\) for a given edge \(uv\) with two vertices \(u\) and \(v\), as well join vertex \(u\) with \(x\) and \(v\) with \(y\) producing two new edges, which generates a cycle \(C_4\). Such a process is Type-A operation, which is also defined as
Fig. 1. The diagram of type-A operation, type-B operation as well as type-C operation.

A rectangle operation owing to each edge produces a rectangle [18], see Fig. 1(a).

**Type-B operation** Create two vertices $x$ and $y$ for a given active edge $uv$ with two vertices $u$ and $v$, connect vertex $x$ with two endpoints of edge $uv$ by two new edges, join vertex $y$ with vertex pair $u$ and $v$, and then remove the active edge $uv$, also obtain a cycle $C_4$. Such a process is Type-B operation, which is often defined as fractal operation because of each active edge applied fractal operation, or also known as a diamond operation due to each active edge forms a diamond [18], see Fig. 1(b).

**Type-C operation** Add a vertex $w$ for a given edge $uv$ with two vertices $u$ and $v$, and then connect vertex $w$ with $u$ and $v$, respectively. It produces a cycle $C_3$, also called a triangle $C_3$. Such a process is Type-C operation, which is commonly defined as triangle operation on account of each edge generates a triangle, see Fig. 1(c).

In fact, from a theoretical perspective, it is not necessary to distinguish two cycles $C_4$ obtained from type-A operation and type-B operation described above. Obviously, $C_4$ and $C_3$ completely differ in clustering coefficient according to the former is always equal to 0 while the latter is 1. Nevertheless, in this paper, we prefer to call the first cycle $C_4$ as a rectangle, the second cycle $C_4$ as a diamond, and the third cycle $C_3$ as a triangle. The reason is that it is convenient for us to describe and organize the content of our article. Care is necessary in such a situation, for instance, the isomorphism between two cycles $C_4$ does not guarantee the same topological structure, as we will describe the difference between two $C_4$ cycles later. Remember this in mind and stay tuned on following.

Generally, there exists two classical approaches to build the pre-existing network models. One is to first construct networks using some rules, in addition to degree sequence, and then study their topological structures, including the degree sequence. Such as ER-model [12], WS-model [11], BA-model [2], NW-model [19]. The other is to generate an available network consistency with the degree sequence. From the perspective of their appearance, the two processes mentioned above may be considered to be reversed. The latter is much more difficult theoretically than the former, it is likely for some networks to have identical degree sequence but to be in fact not isomorphic. In the previous work, some useful algorithms and methods have been provided to achieve this. According to
three types of operations we defined, our goal is to construct growing random networks \( \mathcal{N}(p, q, r, t) \), where \( p \) represents the probability of selecting rectangle operation, \( q \) stands for the probability of choosing fractal operation, \( r \) denotes the probability of selecting triangle operation. The degree of the vertex is the number of vertices of its neighbor set, represented by \( k \). Let the \( |X| \) be the cardinality of set \( X \).

Taking useful advantage of three types of operations mentioned above, let us turn our attention to constructing network space \( \mathcal{N}(p, q, r, t) \).

3 Construction and Properties of Random Scale-free Networks

Above all, the initial network \( \mathcal{N}(0) \) is a cycle \( C_4 \). Exploiting the three types of operations aforementioned, one can easily get \( \mathcal{N}(p, q, r, t) \) from \( \mathcal{N}(p, q, r, t-1) \) for any time step \( t \geq 1 \), i.e., utilizing type-A operation for each edge of network \( \mathcal{N}(p, q, r, t-1) \) with probability \( p \) or employing type-B operation for each edge of network \( \mathcal{N}(p, q, r, t-1) \) with probability \( q \) or applying type-C operation for each edge of network \( \mathcal{N}(p, q, r, t-1) \) with probability \( r \), shown in Fig.2. As discussed in the previous literatures, after \( t \) time steps, there are many possibilities for the member in our network space \( \mathcal{N}(p, q, r, t) \). To make this paper much self-contained, we simply find the upper and lower bounds of the topological structure of members in network space. In particular, our network space \( \mathcal{N}(p, q, r, t) \) consists of a subspace \( \mathcal{N}(p, q, 0, t) \) at \( r = 0 \) or a single deterministic member at \( r = 1 \); at \( r = 0 \), namely \( p + q = 1 \) is always true, it becomes a subspace of \( \mathcal{N}(p, q, r, t) \) whose certain topological properties have been thoroughly investigated in \[18\]; at \( r = 1 \), that is to say, \( p + q = 0 \), it becomes a deterministic network whose several topological properties have been exhaustively studied in \[20\]. At \( 0 < r < 1 \), it contains numerous random networks, to which we will pay more attention.

In order to better understand the properties of network \( \mathcal{N}(p, q, r, t) \), we discuss certain related quantities that determine topological structures. On the basis of above description, our goal is to research the topological structures of random networks with the tuning parameters, such as average degree, degree distribution, diameter, and clustering coefficient.

3.1 Average degree

Firstly, we calculate two basic quantities such as the number of all vertices and edges in \( \mathcal{N}(p, q, r, t) \), named network order and size, respectively. According to three types of operations we defined, the order and size have the relationship in the below formula.

\[
|V(p, q, r, t)| = 2(p + q)|E(p, q, r, t - 1)| + r|E(p, q, r, t - 1)| + |V(p, q, r, t - 1)| \\
|E(p, q, r, t)| = 4(p + q)|E(p, q, r, t - 1)| + 3r|E(p, q, r, t - 1)|
\]
Fig.2. The diagram of operation implemented on an edge. In the process of constructing $N(p,q,r,t)$ from $N(p,q,r,t-1)$ ($t \geq 1$), $p$ represents the probability of applying type-A operation to each edge of $N(p,q,r,t-1)$; $q$ represents the probability of applying type-B operation to each edge of $N(p,q,r,t-1)$; $r$ represents the probability of applying type-C operation to each edge of $N(p,q,r,t-1)$; and $p + q + r = 1$.

It is not difficult to obtain the order and size of $N(p,q,r,t)$ as follows

$$|V(p,q,r,t)| = 4 \times (2p + 2q + r)^{(4p + 4q + 3r)^t - 1}$$
$$|E(p,q,r,t)| = 4 \times (4p + 4q + 3r)^t = 4 \times (4 - r)^t$$

The order and size of member $N(p,q,r,t)$ in network space $N(p,q,r,t)$ is a tuning value by varying parameters $t$ and $r$. Together with the definition of average degree, the average degree $\langle k \rangle$ of random network is the average value of all such vertex degree $k_i$ over the entire network, and is calculated as below

$$\langle k \rangle = \frac{2|E(p,q,r,t)|}{|V(p,q,r,t)|} = \frac{2 \times 4 \times (4 - r)^t}{4 + 4 \times (2 - r)^{(4-r)^t - 1}}$$

One can easily discover that $\langle k \rangle$ is also associated with $t$ and $r$. In the limit of $t$,

$$\langle k \rangle = \frac{2 \times 4 \times (4 - r)^t}{4 + 4 \times (2 - r)^{(4-r)^t - 1}} \approx \frac{6 - 2r}{2 - r}$$

Obviously, it is not hard reveal that the $\langle k \rangle$ is independent of $p,q$, and completely determined by $r$. 
Fig. 3. The diagram of average degree with the probability parameter $r$, $0 \leq r \leq 1$, of network space $N(p, q, r, t)$. It is clear for the eye that the curve illustrates the value of average degree changing with probability parameter $r$.

Because $r$ has a value range from 0 to 1, put it another way, $r \in [0, 1]$. At $r = 0$, it means that $N(p, q, r, t)$ only contains type-A operation ($p = 1$) or merely contains type-B operation ($q = 1$), or contains both of them ($p + q = 1, 0 < p, q < 1$). In this special case, the average degree $\langle k \rangle$ of $N(p, q, r, t)$, where $\langle k \rangle \approx \frac{6 - 2r}{2 - r} = \frac{6 - 2 \times 0}{2 - 0}$, is small and approximately equal to 3. It should be mentioned that the network with same average degree have been discussed in [18]. We can note that when $t$ is large enough, the resulting networks $N(p, q, r, t)$ are sparse networks whose vertices have many fewer connections than is possible.

At $r = 1$, it indicates that $N(p, q, r, t)$ contains neither type-A operation nor type-B operation. In this particular case, the average degree $\langle k \rangle$ of $N(p, q, r, t)$, where $\langle k \rangle \approx \frac{6 - 2r}{2 - r} = \frac{6 - 2 \times 1}{2 - 1}$, is close to 4. It is interesting to note that the identical average degree have been observed analytically in [13], Pseudofractal graphs [21], recursive graphs with $q = 1$ [22], and Apollonian networks with $d = 1$ [23]. One can point out that when $t$ tends to infinite, the resulting network $N(p, q, r, t)$ is a sparse network with fewer edges as possible.

Based on our above analysis and Fig.3, we have the following proposition.

**Proposition 1** For any member $N(p, q, r, t)$ of network space $\mathcal{N}(p, q, r, t)$, the average degree must hold the following inequality

\[
3 \leq \langle k \rangle = \frac{6 - 2r}{2 - r} \leq 4.
\]  

(3)

From Eq.(3), it is worth noting that each $r$ corresponds to a unique $\langle k \rangle$. The Eq.(3),
also, indicates that the value of average degree of the network with mix growth modes will fall into a unique interval, where the lower and upper bounds of the interval are the minimum and maximum values of its constituent ingredients, respectively. Besides, the average degree of each member in network space is independent of the network size. It implies that the network order and network size both increases linearly with time \( t \). In other words, in the limits of \( t \), each member of network space \( \mathcal{N}(p, q, r, t) \) is a sparse network independent of \( p, q \), where \( r \) determines the sparsity.

### 3.2 Degree distribution

Degree distribution is one of most fundamental and important topological features of a network. It is a standard for judging whether the network is scale-free or not. Actually, our networks have scale-free property. The following analysis hold on this viewpoint.

Now we analyze the degree distribution about random networks \( \mathcal{N}(p, q, r, t) \), when a new vertex \( i \) is joined to the networks at a certain step \( t_i (t_i \geq 1) \), according to three types of operations give in Fig.1, there are three cases for newly added vertex \( i \). It has been found that the vertex \( i \) has a degree of 2 no matter which operation is taken. We denote by \( k_i(t) \) the degree of vertex \( i \) at time \( t \). From the construction process of network, the degree \( k_i(t) \) evolves with time as \( k_i(t) = 2k_i(t-1) = 2^{t+1-t_i} \), namely, the degree of vertex \( i \) is increased by a factor 2 at each time step.

Considering the process of constructing network, we can see the degree spectrum of network in our space is a series of discrete real values. In order to relate the degree distribution parameter of our network to the power-law exponent of a continuous degree distribution for random scale-free networks, we take full advantage of the method proposed by Dorogovtsev[21], as follows

\[
    P_{\text{cum}}(k) = \frac{|V(p, q, r, t_i)|}{|V(p, q, r, t)|} = \frac{4 + 4 \times (2 - r)(4 - r)^{t_i-1}}{4 + 4 \times (2 - r)(4 - r)^{t-1}}
\]

where \( P_{\text{cum}}(k) \) is the probability that the degree of a vertex in the network is greater than \( k_i(t) \).

So for large \( t \), we omit the constant and get the formula as below.

\[
    P_{\text{cum}}(k) = \frac{4 + 4 \times (2 - r)(4 - r)^{t_i-1}}{4 + 4 \times (2 - r)(4 - r)^{t-1}} \approx (4 - r)^{t_i-t}
\]

Due to \( k_i(t) = 2^{t-t_i} \), thus, substituting for \( t_i \) using \( t_i = t + 1 - \frac{\ln k}{\ln 2} \), one can obtain

\[
    P_{\text{cum}}(k) \sim k^{1-\gamma}
\]

where \( \gamma = 1 + \frac{\ln(4-r)}{\ln 2} \).
Fig. 4. The diagram of degree exponent with probability parameter, $0 \leq r \leq 1$, of network space $\mathcal{N}(p, q, r, t)$. Obviously, the curve indicates the degree exponent varies with the probability parameter $r$.

Therefore, the cumulative degree distribution $P_{\text{cum}}(k)$ of the random network follows a power-law form $P_{\text{cum}}(k) \sim k^{1-\gamma}$ with the degree exponent $\gamma = 1 + \frac{\ln(4-r)}{\ln 2}$. It is evident that $\gamma$ is independent of $p, q$, as well associated with $r$.

Thanks to $r \in [0, 1]$, at $r = 0$, it shows that $\mathcal{N}(p, q, r, t)$ not contain type-C operation. In this special circumstance, we can find that the network $\mathcal{N}(p, q, r, t)$ follows a scale-free rule $P_{\text{cum}}(k) \sim k^{1-\gamma}$ with the degree index $\gamma = 1 + \frac{\ln(4-r)}{\ln 2} = 3$. Note that the same degree index has been obtained in the deterministic networks \cite{13}, classical BA networks \cite{24} and typical random Sierpinski networks \cite{25}. However, our network is different from BA networks and random Sierpinski networks. It has been observed that our network is deterministic while BA network and random Sierpinski network are random.

At $r = 1$, it displays that $\mathcal{N}(p, q, r, t)$ merely do type-C operation. In this particular case, one can discover that the degree exponent $\gamma$ of the network $\mathcal{N}(p, q, r, t)$ follows a power-law rule $P_{\text{cum}}(k) \sim k^{1-\gamma}$, where $\gamma = 1 + \frac{\ln(4-1)}{\ln 2} = 3$. Apparently, $2 < \gamma = 1 + \frac{\ln 3}{\ln 2} \leq 3$, it indicates that the network $\mathcal{N}(p, q, r, t)$ is scale-free. It is notice that the same degree exponent have been obtained in Pseudofractal graphs \cite{21}, recursive graphs with $q = 2$ \cite{22}, Apollonian networks with $d = 2$ \cite{23}, and deterministic scale-free networks\cite{26}, etc.

Combining with previous analysis and Fig.4, it is easily to obtain the following proposition.

**Proposition 2** For any member $\mathcal{N}(p, q, r, t)$ of network space $\mathcal{N}(p, q, r, t)$, the power-
law exponent of $N(p, q, r, t)$ must satisfy the following inequality

$$2 < 1 + \frac{\ln 3}{\ln 2} \leq \gamma = 1 + \frac{\ln(4 - r)}{\ln 2} \leq 3. \tag{4}$$

From Eq.(4), we note that, $\gamma = 1 + \frac{\ln(4 - r)}{\ln 2}$, each $r$ corresponds to a unique $\gamma$. According to the expression of $\gamma$, the degree exponent is independent of $p, q$, and only affected by $r$, i.e., it illustrates that the value of $\gamma$ varies with $r$. This result shows that the value of power-law exponent of the network with mix growth patterns will fall into a restricted scope, where the lower bound and the upper bound are the minimum and maximum values of its constituents, respectively. In addition, it is apparent that each member of network space $\mathcal{N}(p, q, r, t)$ is a scale-free network due to degree distribution is power-law distribution. It means that these members in network space most vertices have very low degrees and yet there are a few vertices having very high degrees. Consequently, the number of vertices that satisfy the condition $k \gg \langle k \rangle$ is rare. This kind of complex networks is referred to as heterogeneous networks, where the high degree vertices are also called hubs. It has been found that many real-world networks, including such typical one as the Internet, WWW, metabolic networks, social networks, coauthorship networks etc., are power-law distribution with $2 < \gamma \leq 3$. It is also worth remarking that the present framework can be included to networks with $2 < \gamma \leq 3$, recovering qualitatively the same results. In short, as $t$ tends to infinite, all member of network space $\mathcal{N}(p, q, r, t)$ are scale-free networks independent of $p, q$, where $r$ determine the degree exponent.

### 3.3 Diameter

The small-world behavior describes the fact that there is a relatively short distance between most pairs of vertices in most real-life networks. The distance between two vertices is the least number of edges to get from one vertex to other. The longest shortest path between all pairs of vertices is called diameter, which is one of the major evaluation indexes. Diameter is itself a feature of graph structure and can be applied to characterize communication delay over a network. Typically, for coauthorship networks, diameter means that a social communication efficiency of author over the entire network. In general, the larger diameter is, the lower communication efficiency is. Hence, computing the precise diameter can be done analytically and giving the solution process as below. The diameter of the network denoted by $D(p, q, r, t)$, is defined to be the largest of all distances in the network. Here, we will introduce the main idea of analysis.

Before continuing, let us concentrate on three types of operations again, which help us to deduce iterative expression for the diameter of the network $N(p, q, r, t)$.

Clearly, according to the evolution of constructing the network, it is noticeable that the diameter $D(p, q, r, t)$ of network $N(p, q, r, t)$ must be based on the diameter $D(p, q, r, t-1)$ of network $N(p, q, r, t-1)$. In the light of the definition of diameter $D(p, q, r, t-1)$, assume that there exists a path with length just equal to $u_1 u_2, ..., u_D(p, q, r, t-1)+1$ in network
of one of all operations. Now we discuss the calculation process of diameter in three cases.

(a) If we choose the type-A operation, the diameter \( D(p, q, r, t) \) is obtained by \( D(p, q, r, t-1) \), the path of length of \( D(p, q, r, t) \) is equal to \( D(p, q, r, t-1)+2 \), namely, it is available for all paths of various diameter in network \( N(p, q, r, t-1) \). Consequently, we may find an recursive expression between \( D(p, q, r, t) \) and \( D(p, q, r, t-1) \), that is, \( D(p, q, r, t) = D(p, q, r, t-1) + 2 \). Together with known term \( D(p, q, r, 0) = 3 \), we shortly obtain a closed-form formula of diameter \( D(p, q, r, t) \), i.e., \( D(p, q, r, t) = 2(t + 1) \) for all \( t \geq 2 \). From another point of view, there is a fact that \( \ln|V(p, q, r, t)| \sim \ln 4^t = (t + 1)\ln 4 \), we discover a connection between diameter \( D(p, q, r, t) \) and order \( |V(p, q, r, t)| \) under the approximate relationship \( D(p, q, r, t) \sim \ln|V(p, q, r, t)| \). Hence, more usually, as \( t \) tends to infinite, the diameter \( D(p, q, r, t) \) has at most a logarithmic diameter with network order.

(b) If we select the type-B operation, the diameter \( D(p, q, r, t) \) is derived from \( D(p, q, r, t-1) \), it is different from type-A operation, we can find an iterative relationship between diameter \( D(p, q, r, t) \) and \( D(p, q, r, t-1) \), which was regarded as \( D(p, q, r, t) = 2D(p, q, r, t-1) \). Compared with type-A operation, the diameter of type-B operation is to substitute each edge \( uv \) for a length \( 2 \) path which does contain the factor \( 2 \). Combining with this initial condition \( D(p, q, r, 0) = 3 \) and inferring a solution of diameter \( D(p, q, r, t) \), \( D(p, q, r, t) = 2^{t+1} \) for all \( t \geq 2 \). It is unlikely that doing type-A operation on network \( N(p, q, r, t-1) \), diameter \( D(p, q, r, t) \) is not approximately equal to \( \ln|V(p, q, r, t)| \) but for a squared value of order of network \( N(p, q, r, t) \), directly showing \( N(p, q, r, t) \) is large-scale. Obviously, when the order of network is large, the diameter is an exponential increasing.

(c) If we take the type-C operation, one can deduce the diameter \( D(p, q, r, t) \) by \( D(p, q, r, t-1) \), similar to type-A operation, we reveal an recursive connection between \( D(p, q, r, t) \) and \( D(p, q, r, t-1) \), namely, \( D(p, q, r, t) = D(p, q, r, t-1)+2 \). Combined with initial term \( D(p, q, r, 0) = 3 \), we easily get a closed-form formula of diameter \( D(p, q, r, t) \), i.e., \( D(p, q, r, t) = 2(t + 1) \) for any \( t \geq 2 \). From another angle, there exists an result that \( \ln|V(p, q, r, t)| \sim \ln 4^{t+1} = (t + 1)\ln 4 \), we show an relationship between diameter and order, namely, \( D(p, q, r, t) \sim \ln|V(p, q, r, t)| \). So, when \( t \) approaches to infinite, the diameter \( D(p, q, r, t) \) grows logarithmically with increasing network order.

Thanks to the three cases of above analysis, we in theory compute the expected value of \( D(p, q, r, t) \) of network \( N(p, q, r, t) \) on the basis of three cases. Because the probability of selecting each operation is \( p, q, r \), respectively, it is easily to find that

\[
D(p, q, r, t) = p \times (D(p, q, r, t-1) + 2) + q \times 2D(p, q, r, t-1) + r \times (D(p, q, r, t-1) + 2) = (1 + q) \times D(p, q, r, t-1) + 2 - 2q
\]

Together with initial condition, then, we have

\[
D(p, q, r, t) = (1 + q)^t \times D(p, q, r, 0) + (2 - 2q)t
\]

Therefore, it is clear that \( D(p, q, r, t) \) is independent of \( p, r \) and relevant with \( q \) and \( t \).
Fig. 5. The diagram of diameter with constraints, $0 \leq t \leq 1000$ and $0 \leq q \leq 1$, of network space $\mathcal{N}(p, q, r, t)$. It also demonstrates that the curve represents the value of diameter mainly relies on time $t$ and probability parameter $q$.

Because of $q \in [0, 1]$, at $q = 0$, it means that $\mathcal{N}(p, q, r, t)$ merely contains type-A operation, or just contains type-C operation, or contains both of them. In this case, we can compute the diameter of networks $\mathcal{N}(p, q, r, t)$. $D(p, q, r, t) = (1+0)^t \times D(p, q, r, 0) + 2t$, then we get $D(p, q, r, t) = 2(t + 1)$ which increase linearly with time for all $t \geq 1$. Note that the same diameter has been obtained in the deterministic network [23]. It should be mentioned that the diameters $D(p, q, r, t)$ in our space that do not contain type-B operation grow logarithmically with network order. This phenomena can be easily found in a large quantity of networks, which indicates that network $\mathcal{N}(p, q, r, t)$ is like such networks with small diameter.

At $q = 1$, it implies that $\mathcal{N}(p, q, r, t)$ contain neither type-A operation nor type-C operation. We obtain that $D(p, q, r, t) = (1+1)^t \times D(p, q, r, t - 1)$, one can easily find $D(p, q, r, t) = 2^{t+1}$ for all $t \geq 2$. It is noticeable that the diameter $D(p, q, r, t)$ is increasing with exponentially. It is different from case $q = 0$, diameter $D(p, q, r, t) = 2^{t+1}$ which is not linear with network order but for a squared value of network order, directly indicating $\mathcal{N}(p, q, r, t)$ is large-scale. Apparently, when the network order is large, the diameter increase exponentially.

On account of aforementioned analysis and Fig. 5, one can get the following proposition.

**Proposition 3** For any member $\mathcal{N}(p, q, r, t)$ of network space $\mathcal{N}(p, q, r, t)$, the diameter of $\mathcal{N}(p, q, r, t)$ must meet the following inequality

$$2(t + 1) \leq D(p, q, r, t) \leq 2^{t+1}. \quad (5)$$
From Eq. (5), it is worth noticing that the diameter is mainly affected by $q$. This result shows that the value of diameter of the network with mix growth modes will fall into a restricted range, where the lower bound and the upper bound are the minimum and maximum values of its constituent ingredients, respectively. For the member of network space, the diameter may abruptly transform a small value to a high value due to the type-B operation. Together with our analysis, we discover that the diameters of these members in our space that do not contain type-B operation grow linearly with time $t$, otherwise, the diameters increase exponentially. From another perspective, we can also say that the diameters of these members in our space that do not contain type-B operation grow logarithmically with network order, otherwise, the diameters increase exponentially. It should be mentioned that diameter can be applied in many real-life networks for characterizing the maximum communication delay over a network.

3.4 Clustering coefficient

Clustering is another vital property of a network, which provides measure of local structure within the network. The most immediate measure of clustering is the clustering coefficient $C_i$ for every vertex $i$. By definition, clustering coefficient of a vertex $i$ is the ratio of the total number $E_i$ of edges that actually exist between all $k_i$ its nearest neighbors and the number of $k_i(k_i - 1)/2$ of all possible edges between them, i.e., $C_i = 2E_i/[k_i(k_i - 1)]$. The clustering coefficient $\langle C \rangle$ of the entire network is average of all vertex $C_i$. Now we will compute the clustering coefficient of every vertex and their average value. As shown in previous researches, most networks are highly transitive or clustered, i.e., a friend may be two friends individually, who may then become acquainted with one another through their common friend, and so end up friends themselves. “Transitivity” or “clustering” has a different physical meanings in different networks, such as coauthorship networks and WWW. To better simulate the actual network, a subspace with non-clustering have been investigated in our another paper[18]. Here, our goal is to research networks with tuning clustering coefficient by varying the value of $r$ in the following.

Due to type-A operation and type-B operation always forms a cycle $C_4$, while type-C operation continuously generates a cycle $C_3$, it is evident that the clustering coefficient of $C_4$ is always equal to 0 while the clustering coefficient of $C_3$ is 1. It shows that merely type-C operation increase the clustering coefficient of the entire network. Put it another way, $r$ has a decisive influence on the clustering coefficient of the whole network. Intentionally, there are two factors that have an impact on the average clustering coefficient of network $N(p, q, r, t)$, the first is the value of $r$, and the second is that the total number of times of performing type-C operation at time $t$. Now we introduce the main ideas and the process in detail.

Because of the three case of aforementioned description, we in theory compute the average degree of $\langle c \rangle$ of network $N(p, q, r, t)$. We divide into 2 steps.

Firstly, we find the clustering coefficient for each vertex in $N(p, q, r, t)$, we can derive
a closed formula for the clustering coefficient $C$ and list in the below table

Table-I the clustering coefficient of $c_i$ of vertices degree $k_i$

| $k_i$ | 2   | $2^2$ | $2^3$ | $\cdots$ | $2^{t-1}$ | $2^t$ | $2^{t+1}$ |
|------|-----|-------|-------|-----------|----------|-------|----------|
| $c_i$ | 1   | $\frac{1}{3}$ | $\frac{3}{14}$ | $\cdots$ | $\frac{(2^t-2)}{2^{t-(2^t-1)}}$ | $\frac{2^t-2}{2^{t-(2^t-1)}}$ | $\frac{2^t+1-2}{2^{t-(2^t+1-1)}}$ |

Secondly, we calculate the proportion of the vertices with a clustering coefficient of $c_i$ in network $N(p,q,r,t)$. In order to obtain the clustering coefficient $\langle c \rangle$ of the entire network $N(p,q,r,t)$, it is necessary for us to give the degree distribution spectrum, namely, the probability $p_i(p_i = \frac{n_{k_i(t)}}{|V(p,q,r,t)|})$ of vertices degree $k_i$ is shown in the following table

Table-II the degree spectrum of degree $k_i$

| $k_i$ | 2   | $2^2$ | $2^3$ | $\cdots$ | $2^{t-1}$ | $2^t$ | $2^{t+1}$ |
|------|-----|-------|-------|-----------|----------|-------|----------|
| $p_i$ | $\frac{3-r}{4-r}$ | $\frac{3-r}{(4-r)^2}$ | $\frac{3-r}{(4-r)^3}$ | $\cdots$ | $\frac{3-r}{(4-r)^{t-1}}$ | $\frac{3-r}{(4-r)^t}$ | $\frac{3-r}{(4-r)^{t+1}}$ |

On the basis of above detailed discussion, it is easily to obtain the clustering coefficient $\langle c \rangle$ of the entire network-model is

$$\langle c \rangle = r \sum p_i c_i$$

Therefore, it seems that $\langle c \rangle$ is independent of $p,q$ and have a relationship with $r$. Owing to $r \in [0,1]$, for first particular case with $r = 0$, this limiting case of network has a self-similar structure that allows one to calculate the exact value $C$ analytically. Since the network $N(p,q,r,t)$ not contain type-C operation, there no exists triangle, and then $N(p,q,r,t)$ have no clustering. So the clustering coefficient of every vertex in $N(p,q,r,t)$ is zero. In this situation, the average value of clustering coefficient in $N(p,q,r,t)$ is always equal to 0. The clustering coefficient of whole network is always zero.

For second special case with $r = 1$, this limiting case of network $N(p,q,r,t)$ only contains type-C operation. Substituting $r$ into the Eq.(6), it is noticeable that the clustering coefficient is approximately to 0.7566. Apparently, when the parameters $r$ is larger, the clustering coefficient is higher.

As mentioned previously, it can be get the following proposition.

**Proposition 4** For any member $N(p,q,r,t)$ of network space $\mathcal{N}(p,q,r,t)$, the clustering coefficient must hold the following inequality

$$0 \leq \langle c \rangle = r \sum p_i c_i \leq 0.7566$$

From Eq.(7), consider the general case with $0 \leq r \leq 1$, it is worthwhile to study the processes taking place upon the network to discover the different effects on dynamic processes. As we all known, if we give a fixed value of $r$, there exists a unique clustering coefficient $\langle c \rangle$ corresponding to $r$. According to our operations mentioned above, it shows that type-C operation always have an effect on clustering coefficient of network.
This result shows that the value of clustering coefficient of the network with mix growth modes will fall within a restricted range, where the lower bound and the upper bound are the minimum and maximum values of its constituents, respectively. For member of network space $N(p, q, r, t)$, the clustering coefficient range 0 from 0.7566.

4 Conclusion

We have introduced three types of operations to produce our network $N(p, q, r, t)$. In terms of our operations, we have shown the detailed process of generating our network, which constitute the network space $N(p, q, r, t)$, and then discussed their topological properties such as, average degree, degree distribution, diameter, and clustering coefficient. Above all, we have computed the average degree of each member in our network space and discovered that all member are sparse network. Secondly, we have proven that each member in our space follows the power-law distribution with degree exponent $\gamma = 1 + \frac{\ln(4-r)}{\ln 2}$, which means that all networks are always scale-free except that the degree exponent is different.

Hereafter, we have analyzed the diameters of several members in our network space, where the diameters of some members may suddenly transform from a small value to a large value. The reason is that the type-B operation is a fractal operation, if each type-B operation is performed, the value of the diameter increase exponentially. In this circumstances, the diameter is so large that the communication efficiency is low. Put it another way, by setting the parameters $q = 0$, one can make the diameter of certain networks in our space become relatively small.

Eventually, we have studied the clustering coefficient in our network space, and discovered that the clustering coefficient of some member may abruptly leap to a high value from 0, while others may gradually increase from 0. Because the type-C operation is a triangle operation, each time type-C operation is performed, the value of the clustering coefficient will increase, that is to say, by effectively varying the parameters $r$, one can make the clustering coefficient of certain networks in our space become relatively higher. Therefore, small diameter network can be obtained with $q = 0$, while high clustering coefficient network can be obtained with larger $r$. Combined with small-world effect and our analysis, for the $q = 0$ and high $r$ case, we have found that these network satisfying the above case have small diameter and high clustering coefficient, that is, it indicates that those networks in our network space are small-world networks.

It is clear that the network in our network space serves as a better feature by changing the probability parameters, for example, we can obtain a small-world network with small diameter and high clustering coefficient by setting $q = 0$ and higher $r$. Especially, the clustering coefficient may abruptly transform from zero to high value. Accomplishment notwithstanding, research on complex networks is far from enough and requires long-term long-sustainable endeavor. New discoveries, developments, enhancements, and
improvements are still needed. In the future, we will devote more efforts to investigate complex networks in order to better help people understand and apply it to explain some phenomena in real life.

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