Partial Information Stochastic Differential Games for Backward Stochastic Systems Driven by Lévy Processes

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1. Introduction

Consider a partial information two-person zero-sum stochastic differential game problem, where the system is governed by a backward stochastic differential equation driven by Teugels martingales and an independent Brownian motion. A sufficient condition and a necessary one for the existence of the saddle point for the game are proved. As an application, a linear quadratic stochastic differential game problem is discussed.

In the above, the processes \( u_1(\cdot) \) and \( u_2(\cdot) \) are open-loop control processes, which present the controls of the two players. Let \( U_1 \subset R^d_1 \) and \( U_2 \subset R^d_2 \) be two given nonempty convex sets. Under many situations, under which the full information \( \mathcal{F}_i \) is inaccessible for players, ones can only observe a partial information. For this, an admissible control process \( u_i(\cdot) \) for the player \( i \) is defined as a \( \mathcal{F}_i \)-predictable process with values in \( U_i \) s.t. \( E \int_0^T |u_i(t)|^2 \, dt < +\infty \), where \( i = 1, 2 \). Here, \( \mathcal{F}_i \subseteq \mathcal{F}_t \) for all \( t \in [0, T] \) is a given subfiltration representing the information available to the controller at time \( t \). For example, we could choose \( \mathcal{F}_t = \mathcal{F}_{(t-\delta)^+}, t \in [0, T] \), where \( \delta > 0 \) is a fixed delay of information.

The set of all admissible open-loop controls \( u_i(\cdot) \) for the player \( i \) is denoted by \( \mathcal{A}_i \), \( (i = 1, 2) \). \( \mathcal{A}_1 \times \mathcal{A}_2 \) is called the set of open-loop admissible controls for the players. We denote the strong solution of (1) by \( (y^{u_1,u_2}(\cdot), q^{u_1,u_2}(\cdot), z^{u_1,u_2}(\cdot)) \), or \( (y(\cdot), q(\cdot), z(\cdot)) \) if its dependence on admissible control \( (u_1(\cdot), u_2(\cdot)) \) is clear from context. Then, we call \( (y(\cdot), q(\cdot), z(\cdot)) \) the state process corresponding to the control process \( (u_1(\cdot), u_2(\cdot)) \) and call \( (u_1(\cdot), u_2(\cdot); y(\cdot), q(\cdot), z(\cdot)) \) the admissible quintuplet.

Roughly speaking, for the zero-sum differential game, Player I seeks control \( \pi_1(\cdot) \) to minimize (2), while Player II...
seeks control $u_2(\cdot)$ to maximize (2). Let $(\pi_1(\cdot), \pi_2(\cdot))$ be an optimal open-loop control satisfying
\[
J(\tilde{u}_1(\cdot), u_2(\cdot)) \leq J(\pi_1(\cdot), \pi_2(\cdot)) \leq J(u_1(\cdot), \pi_2(\cdot)),
\]
for all admissible open-loop controls $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We denote this partial stochastic differential game by Problem (P). We refer to $(\pi_1(\cdot), \pi_2(\cdot))$ as an open-loop saddle point of Problem (P). The corresponding strong solution $(\tilde{y}(\cdot), \tilde{\sigma}(\cdot), \tilde{\zeta}(\cdot))$ of (1) is called the saddle state process. Then, $(\pi_1(\cdot), \pi_2(\cdot); \tilde{y}(\cdot), \tilde{\sigma}(\cdot), \tilde{\zeta}(\cdot))$ is called a saddle quintuplet.

Game theory had been an active area of research and a useful tool in many applications, particularly in biology and economics. For the partial information two-person zero-sum stochastic differential games, the objective is to find a saddle point, for which the controller has less information than the player. It is natural to apply the theory to the stochastic optimal control problem.

In 2000, Nualart and Schoutens [4] got a martingale representation theorem for a type of Lévy processes through Øksendal [1] established a maximum principle for stochastic differential games by Problem (P). We refer to [2, 3] and the references therein, for more related results on the partial information stochastic differential games.

In 2000, Nualart and Schoutens [4] got a martingale representation theorem for a type of Lévy processes through Teugels marginals, where Teugels marginals are a family of pairwise strongly orthonormal martingales associated with Lévy processes. Later, Nualart and Schoutens [5] proved the existence and uniqueness theory of BSDE driven by Teugels marginals. The above results are further extended to the case for one-dimensional BSDE driven by Teugels marginals and an independent multidimensional Brownian motion by Bahli et al. [6].

Since the theory of BSDE driven by Teugels marginals and an independent Brownian motion is established, it is natural to apply the theory to the stochastic optimal control problem. Now, the full information stochastic optimal control problem related to Teugels marginals has been in many literatures. For example, the stochastic linear quadratic problem with Lévy processes was studied by Mtsui and Tabata [7]. Motivated by [7], Meng and Tang [8] studied the general full information stochastic optimal control problem for the forward stochastic systems driven by Teugels marginals and an independent multidimensional Brownian motion and proved the corresponding stochastic maximum principle. Furthermore, Tang and Zhang [9] extended [8] to the Backward stochastic systems and obtained the corresponding stochastic maximum principle. For the case of the partial information, in 2012, Bahli et al. [10] studied the stochastic control problem for forward system and obtained the corresponding stochastic maximum principle. In the meantime, Meng et al. [11] extended [9] to the partial information stochastic optimal control problem of backward stochastic systems and obtained the corresponding optimality conditions. For the recent results about stochastic differential control problems or games, the readers are referred to [12–17] and the references therein.

However, to the best of our knowledge, there is little discussion on the partial information stochastic differential games for the system driven by Teugels marginals and an independent Brownian motion, which motivates us to write this paper. The main purpose of this paper is to establish partial information necessary and sufficient conditions for optimality for Problem (P) by using the results in [9]. The results obtained in this paper can be considered as a generalized form of stochastic optimal control problem to the two-person zero-sum case. As an application, a two-person zero-sum stochastic differential game of linear backward stochastic differential equations with a quadratic cost criterion under partial information is discussed and the optimal control is characterized explicitly by the adjoint processes.

The rest of this paper is organized as follows. We introduce useful notations and give needed assumptions in Section 2. Section 3 is devoted to present the sufficient condition for the existence of the optimal control problem. In Section 4, we establish the necessary condition of optimality. In Section 5, a linear quadratic stochastic differential game problem is solved by applying the theoretical results.

2. Preliminaries and Assumptions
Let $(\Omega, F, \mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a complete probability space. The filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right-continuous and generated by a $d$-dimensional standard Brownian motion $(W(t), 0 \leq t \leq T)$ and a one-dimensional Lévy process $((L(t), 0 \leq t \leq T))$. It is known that $L(t)$ has a characteristic function of the form: $\mathbb{E}e^{\theta t} = \exp[\alpha \theta t - 1/2 \sigma^2 \theta^2 t + t \int_0^\infty (e^{\theta x} - 1 - i \theta x 1_{\{x<1\}}) \nu(dx)]$, where $\alpha \in \mathbb{R}^1$, $\sigma > 0$ and $\nu$ is a measure on $\mathbb{R}^1$ satisfying the following: (i) $\int_0^\infty (1 + x^2) \nu(dx) < \infty$ and (ii) there exists $\epsilon > 0$ and $\lambda > 0$, s.t. $\int_0^\infty e^{\theta x} \nu(dx) < \infty$. These settings imply that the random variables $L(t)$ have moments of all orders. We denote by $\{H^i(t), 0 \leq t \leq T\}_{i=1}^\infty$ the Teugels marginals associated with the Lévy process $(L(t), 0 \leq t \leq T)$. Here, $H^i(t)$ is given by

\[
H^i(t) = c_{i,0} Y^{(0)}(t) + c_{i,1} Y^{(1)}(t) + \cdots + c_{i,n} Y^{(n)}(t),
\]

where $Y^{(i)}(t) = L^{(i)}(t) - E[L^{(i)}(t)]$ for all $i \geq 1$. $L^{(i)}(t)$ and $L^{(0)}(t)$ are so called power-jump processes with $L^{(i)}(t) = \int_0^t \Delta L(s)^i \, ds$ for $i \geq 2$, and the coefficients $c_{i,j}$ correspond to the orthonormalization of polynomials $1, x, x^2, \cdots$ w.r.t. the measure $\mu(dx) = x^2 \nu(dx) + \sigma^2 \delta_0(dx)$. The Teugels marginals $\{H^i(t)\}_{i=1}^\infty$ are pathwise strongly orthogonal and their predictable quadratic variation processes are given by

\[
\langle H^{(i)}(t), H^{(j)}(t) \rangle = \delta_{ij} t.
\]

For more details of Teugels marginals, we invite the reader to consult Nualart and Schoutens [4, 5]. Denote by $g$ the predictable sub-$\sigma$ field of $\mathcal{B}([0, T]) \times \mathcal{F}$; then, we introduce the following notation used throughout this paper.

In the following, we introduce some basic spaces:

(i) $H$: a Hilbert space with norm $\| \cdot \|_H$.
(ii) $\langle \alpha, \beta \rangle$: the inner product in $\mathbb{R}^n$, $\forall \alpha, \beta \in \mathbb{R}^n$.
(iii) $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm of $\mathbb{R}^n$, $\forall \alpha \in \mathbb{R}^n$. 

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Throughout this paper, we introduce the following basic assumptions on coefficients $(\xi, f, l, \phi)$.

**Assumption 1.** $\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n)$ and the random mapping $f$ is predictable w.r.t. $t$, Borel measurable w.r.t. other variables, and for almost all $(t, \omega) \in [0, T] \times \Omega$, $f(t, \omega, y, p, z, u_1, u_2)$ is Fréchet differentiable w.r.t. $(y, p, z, u_1, u_2)$ and the corresponding Fréchet derivatives $f_y, f_p, f_z, f_{u_1}, f_{u_2}$ are continuous and uniformly bounded.

**Assumption 2.** The random mapping $l$ is predictable w.r.t. $t$, Borel measurable w.r.t. other variables, and for almost all $(t, \omega) \in [0, T] \times \Omega$, $l$ is Fréchet differentiable w.r.t. $(y, p, z, u_1, u_2)$ with continuous Fréchet derivatives $l_y, l_p, l_z, l_{u_1}, l_{u_2}$. The random mapping $\phi$ is measurable, and for almost all $(t, \omega) \in [0, T] \times \Omega$, $\phi$ is Fréchet differentiable w.r.t. $y$ with continuous Fréchet derivative $\phi_y$.

Under Assumption 1, we can get from Lemma 2.3 in [9] that, for each $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$, system (1) admits a unique strong solution. Furthermore, by Assumption 2 and a priori estimate for BSDE driven by Teugels martingales (see Lemma 3.2 in [9]), it is easy to check that

$$||f(u_1(\cdot), u_2(\cdot))|| < \infty.$$  

Therefore, Problem $(P)$ is well defined.

### 3. A Partial Information Sufficient Maximum Principle

In this section, we want to study the sufficient maximum principle for Problem $(P)$.

In our setting, the Hamiltonian function $H : [0, T] \times R^n \times R^{n \times d} \times \hat{F} (R^n) \times U_1 \times U_2 \times R^d \longrightarrow R^l$ is of the following form:

$$H(t, y, q, z, u_1, u_2, k) = \langle k, f(t, y, q, z, u_1, u_2) \rangle + l(t, y, q, z, u_1, u_2).$$  

(17)
\[ \begin{aligned}
\frac{dk}{dt} &= -H_y(t, y, q, z, u_1, u_2, k)
- \sum_{i=1}^{d} H_q(t, y, q_i, z, u_1, u_2, k)dW_i(t) \\
&\quad - \int_{t}^{\infty} H_z(t, y, q, z, u_1, u_2, k)dH_i(t)
\end{aligned} \tag{18} \]

Under Assumptions 1 and 2, the forward stochastic differential equation (18) has a unique solution \( k(\cdot) \in \mathcal{S}_2^x (0, T; \mathbb{R}^n) \) by Lemma 2.1 in [9]. We now come to a verification theorem for Problem (P).

**Theorem 1** (partial information sufficient maximum principle). Let Assumptions 1 and 2 hold. Let \((\Pi_1(\cdot), \Pi_2(\cdot), \bar{y}(\cdot), \bar{q}(\cdot), \bar{z}(\cdot))\) be an admissible quintuplet and \(\bar{K}(\cdot)\) the unique strong solution of the corresponding adjoint equation (18). Suppose that the Hamiltonian function \(H\) satisfies the following conditional maximum principle:

\[ \inf_{u_1 \in U_1} \mathbb{E} \left[ H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), u_1, \Pi_1(t), \bar{K}(t)) \right] \geq 0 \tag{19} \]

(i) Suppose that, for all \( t \in [0, T] \), \( \phi(y) \) is convex in \( y \), and

\( (y, q, z, u_1) \mapsto H(t, y, q, z, u_1, \Pi_1(t), \bar{K}(t)) \)

is convex. Then, for all \( u_2(\cdot) \in \mathcal{A}_2 \),

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) \leq J(u_1(\cdot), \Pi_2(\cdot)) \tag{21} \]

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), \Pi_2(\cdot)) \tag{22} \]

(ii) Suppose that, for all \( t \in [0, T] \), \( \phi(y) \) is concave in \( y \), and

\( (y, q, z, u_2) \mapsto H(t, y, q, z, \Pi_1(t), u_2, \bar{K}(t)) \)

is concave. Then, for all \( u_1(\cdot) \in \mathcal{A}_1 \),

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) \geq J(\Pi_1(\cdot), u_2(\cdot)) \tag{24} \]

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2} J(\Pi_1(\cdot), u_2(\cdot)) \tag{25} \]

(iii) If both cases (i) and (ii) hold (which implies, in particular, that \( \phi(y) \) is an affine function), then \((\Pi_1(\cdot), \Pi_2(\cdot))\) is an open-loop saddle point and

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2} \left( \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), u_2(\cdot)) \right) \]

\[ = \inf_{u_1(\cdot) \in \mathcal{A}_1} \left( \sup_{u_2(\cdot) \in \mathcal{A}_2} J(u_1(\cdot), u_2(\cdot)) \right) \tag{26} \]

**Proof**

(i) In the following, we consider a stochastic optimal control problem over \( \mathcal{S}_1^y \), where the system is

\[ y(t) = \xi + \int_{t}^{T} f(s, y(s), q(s), z(s), u_1(t), \Pi_2(t))ds \]

\[ - \sum_{i=1}^{d} \int_{t}^{T} q_i'(s)dW_i(s) - \int_{t}^{T} z'(s)dH_i(s), \]

with the cost functional

\[ J(u_1(\cdot), \Pi_2(\cdot)) = \mathbb{E}[\phi(y(0)) + \int_{0}^{T} J(t, y(t), q(t), z(t), u_1(t), \Pi_2(t))dt]. \tag{27} \]

Our optimal control problem is to minimize \( J(u_1(\cdot), \Pi_2(\cdot)) \) over \( u_1(\cdot) \in \mathcal{A}_1 \), i.e., find \( \Pi_1(\cdot) \in \mathcal{A}_1 \) such that

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), \Pi_2(\cdot)). \tag{28} \]

Then, for this case, it is easy to check that the Hamilton is \( H(t, y, q, z, u_1, \Pi_2(t), k) \), and for the admissible control \( \Pi_1(\cdot) \in \mathcal{A}_1 \), the corresponding state process and the adjoint process is still \( (\bar{y}(t), \bar{q}(t), \bar{z}(t)) \) and \( \bar{K}(t) \), respectively. And the optimality condition is

\[ \inf_{u_1(\cdot) \in \mathcal{A}_1} \mathbb{E} \left[ H(t, \bar{y}(t), \bar{q}(t), \bar{z}(t), u_1, \Pi_1(t), \bar{K}(t)) \right] \geq 0 \tag{29} \]

Thus, from the partial information sufficient maximum principle for optimal control (see Theorem 1 in [9]), we conclude that \( \Pi_1(\cdot) \) is the optimal control of the optimal control problem, i.e.,

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), \Pi_2(\cdot)). \tag{30} \]

The proof of (i) is complete.

(ii) This statement can be proved in a similar way as (i).

(iii) If both (i) and (ii) hold, then

\[ J(\Pi_1(\cdot), u_2(\cdot)) \leq J(\Pi_1(\cdot), \Pi_2(\cdot)) \leq J(u_1(\cdot), \Pi_2(\cdot)), \tag{32} \]

for any \((u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2 \), i.e.,

\[ J(\Pi_1(\cdot), \Pi_2(\cdot)) \leq \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), \Pi_2(\cdot)) \]

\[ \leq \sup_{u_2(\cdot) \in \mathcal{A}_2} \left( \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), u_2(\cdot)) \right). \tag{33} \]

On the contrary,
Corollary 1. Suppose that $\mathcal{F}_t = \mathcal{F}_r$. Moreover, suppose that, for all $t \in [0, T]$, the following maximum principle holds:

$$J(\pi_1(\cdot), \pi_2(\cdot)) \geq \inf_{u_1(\cdot) \in \mathcal{A}_t} \left( \sup_{u_2(\cdot) \in \mathcal{S}_t} J(u_1(\cdot), u_2(\cdot)) \right).$$

(34) Now, due to the inequality,

$$\inf_{u_1(\cdot) \in \mathcal{A}_t} \left( \sup_{u_2(\cdot) \in \mathcal{S}_t} J(u_1(\cdot), u_2(\cdot)) \right) \geq \sup_{u_2(\cdot) \in \mathcal{S}_t} \left( \inf_{u_1(\cdot) \in \mathcal{A}_t} J(u_1(\cdot), u_2(\cdot)) \right),$$

(35) we have

$$J(\pi_1(\cdot), \pi_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_t} \left( \sup_{u_2(\cdot) \in \mathcal{S}_t} J(u_1(\cdot), u_2(\cdot)) \right) = \sup_{u_2(\cdot) \in \mathcal{S}_t} \left( \inf_{u_1(\cdot) \in \mathcal{A}_t} J(u_1(\cdot), u_2(\cdot)) \right).$$

(36) The proof of the theorem is completed. If the control process $(u_1(\cdot), u_2(\cdot))$ is admissible adopted to the filtration $\mathcal{F}_r$, we have the following full information sufficient maximum principle.

4. Partial Information Necessary Maximum Principle

In this section, we give a necessary maximum principle for Problem (P).

Theorem 2 (a partial information necessary maximum principle). Under Assumptions 1 and 2, let $(\pi_1(\cdot), \pi_2(\cdot))$ be an optimal control of Problem (P). Suppose that $(\overline{\gamma}(\cdot), \overline{\xi}(\cdot), \overline{\zeta}(\cdot))$ is the state process of system (i) corresponding to the admissible control $(\pi_1(\cdot), \pi_2(\cdot))$. Let $(\mathcal{K}(\cdot))$ be the unique solution of the adjoint equation (18) corresponding $(\pi_1(\cdot), \pi_2(\cdot); \overline{\gamma}(\cdot), \overline{\xi}(\cdot), \overline{\zeta}(\cdot))$. Then, for $i = 1, 2$, we have, for all $u_1 \in \mathcal{A}_t$

$$\langle E \left[ H_n(t) \right| \mathcal{F}_t \right|, u_i - \pi_i(t) \rangle \geq 0, \text{ a.s. a.e.,}$$

(45) $H_n(t) = H_n(t, \overline{\gamma}(t), \overline{\xi}(t), \overline{\zeta}(t), \pi_1(t), \pi_2(t), \mathcal{K}(t))$. (46)

Proof. Since $(\overline{\pi}_1(\cdot), \overline{\pi}_2(\cdot))$ is a saddle open-loop control, then $(\pi_1(\cdot), \pi_2(\cdot))$ is an open-loop saddle point, i.e.,

$$J(\pi_1(\cdot), u_2(\cdot)) \leq J(\pi_1(\cdot), \pi_2(\cdot)) \leq J(u_1(\cdot), \pi_2(\cdot)).$$

(47) So, we have

$$J_1(\pi_1(\cdot), \pi_2(\cdot)) = \min_{u_1(\cdot) \in \mathcal{A}_t} J_1(u_1(\cdot), \pi_2(\cdot)),$$

(48) $$J_2(\overline{\pi}_1(\cdot), \overline{\pi}_2(\cdot)) = \max_{u_1(\cdot) \in \mathcal{A}_t} J_2(\overline{\pi}_1(\cdot), u_2(\cdot)).$$

(49)

By (48), $\pi_1(\cdot)$ can be regarded as an optimal control of the optimal control problem, where the controlled system is (27) and the cost functional is (28). Then, for this case, it is easy to check that the Hamilton is $H(t, y, q, z, u_1, n_2(t), k)$, and for the optimal control $\overline{\pi}_1(\cdot) \in \mathcal{F}_t$, the corresponding optimal state process and the adjoint process is still $(\overline{\gamma}(t), \overline{\xi}(t), \overline{\zeta}(t))$ and $\mathcal{K}(t)$, respectively. Thus, applying the partial necessary stochastic maximum principle for optimal control problems (see Theorem 2 in [9]), we can obtain (45) for $i = 1$. Similarly, from (49), we can obtain (45) for $i = 2$. The proof is complete.

5. Example: Linear Quadratic Problem

In this section, we will apply our stochastic maximum principles to a linear quadratic problem under partial information, i.e., consider the game problem to the following quadratic cost functional over $(u_1, u_2)$ valued in $R^{m_1} \times R^{m_2}$:
where the state process \((y(\cdot), q(\cdot), z(\cdot))\) is the solution to the controlled linear backward stochastic system below:

\[
\begin{align*}
J(u_1(\cdot), u_2(\cdot)) &= E\langle M, y(0) \rangle + E \int_0^T \langle E(s), y(s) \rangle \, ds \\
&\quad + \sum_{i=1}^d \langle F^i(s), q^i(s) \rangle \\
&\quad + \sum_{i=1}^\infty \langle G^i(s), z^i(s) \rangle \\
&\quad + \langle N_1(s)u_1(s), u_1(s) \rangle \\
&\quad - \langle N_2(s)u_2(s), u_2(s) \rangle \, ds,
\end{align*}
\]

\[(50)\]

\[
\begin{align*}
dy(t) &= -\mathcal{A}(t)y(t) + \sum_{i=1}^d B^i(t)q^i(t) + \sum_{i=1}^\infty C^i(t)z^i(t) + D_1(t)u_1(t) + D_2(t)u_2(t) \, dt \\
&\quad + \sum_{i=1}^d q^i(t)dW^i(t) + \sum_{i=1}^\infty z^i(t)dW^i(t),
\end{align*}
\]

\[(51)\]

This problem is denoted by Problem (LQ). To study this problem, we need the assumptions on the coefficients as follows.

**Assumption 3.** The matrix-valued functions \(\mathcal{A} : [0, T] \rightarrow \mathbb{R}^{n \times n}; B^i : [0, T] \rightarrow \mathbb{R}^{n \times n}, i = 1, 2, \ldots, d; C^i : [0, T] \rightarrow \mathbb{R}^{m \times n}, i = 1, 2; E : [0, T] \rightarrow \mathbb{R}^{m \times m}; F^i : [0, T] \rightarrow \mathbb{R}^{m \times n}, i = 1, 2, \ldots, d, G^i : [0, T] \rightarrow \mathbb{R}^{m \times m}, i = 1, 2\) and the matrix \(M \in \mathbb{R}^n\) are uniformly bounded. Moreover, \(N_i\) is uniformly positive, i.e., \(N_i \geq \delta I\), \((i = 1, 2)\) for some positive constant \(\delta\).

**Assumption 4.** There is no further constraint imposed on the control processes; the set all admissible control processes is

\[
\mathcal{A}_1 \times \mathcal{A}_2 = \left\{ (u_1(\cdot), u_2(\cdot)) : (u_1(\cdot), u_2(\cdot)) \text{ is} \right\}
\]

\[
G_5 - \text{predictable process with values in} \quad \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \text{ and} \quad E \int_0^T |u(t)|^2 \, dt < \infty.
\]

\[(52)\]

In what follows, we will utilize the stochastic maximum principle to study the dual representation of the game Problem (LQ).

We first define the Hamiltonian function \(H : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{m \times d} \times \mathbb{F}^T(\mathbb{R}^n) \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^n \rightarrow \mathbb{R}^1\) by

\[
H(t, y, q, z, u_1, u_2, k) = -\langle k, \mathcal{A}(t)y(t) + \sum_{i=1}^d B^i(t)q^i(t) + \sum_{i=1}^\infty C^i(t)z^i(t) + D_1(t)u_1(t) + D_2(t)u_2(t) \rangle \\
+ \sum_{i=1}^\infty \langle F^i(t), q^i(t) \rangle + \langle N_1(t)u_1(t), u_1(t) \rangle \\
+ \sum_{i=1}^\infty \langle G^i(t), z^i(t) \rangle - \langle N_2(t)u_1(t), u_1(t) \rangle.
\]

\[(53)\]

Then, the adjoint equation corresponding to an admissible quintuplet \((u_1(\cdot), u_2(\cdot); y(\cdot), q(\cdot), z(\cdot))\) can be rewritten as

\[
dk(t) = (\mathcal{A}^\top k - E)dt - \sum_{i=1}^d \left((B^i)^\top k - E^i\right)dW^i - \sum_{i=1}^\infty \langle (C^i)^\top k - G^i \rangle dH^i,
\]

\[
k(0) = -M.
\]

\[(54)\]

Under Assumption 3, for any admissible quintuplet \((u_1(\cdot), u_2(\cdot); y(\cdot), q(\cdot), z(\cdot))\), the adjoint equation (54) has a unique solution \(k(\cdot)\) in view of Lemma 2.1 in [9].

It is time to give the dual characterization of the optimal control.

**Theorem 3.** Let Assumptions 3 and 4 be satisfied. Then, a necessary and sufficient condition for an admissible quintuplet \((u_1(\cdot), u_2(\cdot); y(\cdot), q(\cdot), z(\cdot))\) to be a saddle quintuplet
of Problem (LQ) is that the control \((u_1(\cdot), u_2(\cdot))\) has the representation

\[
\begin{align*}
    u_1(t) &= \frac{1}{2} N_1^{-1}(t) D_1^*(t) E[k(t) | \mathcal{G}_t], \\
    u_2(t) &= \frac{1}{2} N_2^{-1}(t) D_2^*(t) E[k(t) | \mathcal{G}_t], \tag{55}
\end{align*}
\]

where \(k(\cdot)\) is the unique solution of the adjoint equation (54) corresponding to the admissible quintuplet \((u_1(\cdot), u_2(\cdot) ; y(\cdot), q(\cdot), z(\cdot))\).

Proof. For the necessary part, let \((u_1(\cdot), u_2(\cdot) ; y(\cdot), q(\cdot), z(\cdot))\) be an admissible quintuplet satisfying (55). By the classical representation (55), we have, a.e., a.s.,

\[
H_{u_1}(t, y(t), q(t), z(t), u_1(t), u_2(t), k(t)) = 0. \tag{56}
\]

Noticing the definition of \(H\) in (53), we obtain

\[
2N_1(t)u_1(t) + D_1^*(t) E[k(t) | \mathcal{G}_t] = 0, \text{ a.e. a.s.,} \tag{57}
\]

\[
-2N_2(t)u_2(t) + D_2^*(t) E[k(t) | \mathcal{G}_t] = 0. \text{ a.e. a.s.} \tag{58}
\]

So, the saddle point \((u_1(\cdot), u_2(\cdot))\) has the dual presentation (55).

For the sufficient part, let \((u_1(\cdot), u_2(\cdot) ; y(\cdot), q(\cdot), z(\cdot))\) be an admissible quintuplet satisfying (55). By the classical technique of completing squares, from (55), we can claim that \((u_1(\cdot), u_2(\cdot) ; y(\cdot), q(\cdot), z(\cdot))\) satisfies the optimality condition (19) in Theorem 1. Moreover, from Assumptions 3 and 4, it is easy to check that all other conditions in Theorem 1 are satisfied. Hence, \((u_1(\cdot), u_2(\cdot) ; y(\cdot), q(\cdot), z(\cdot))\) is a saddle quintuplet by Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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