Particles and Anti-Particles in a Relativistic Bose Condensate

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Abstract

We study the Bose-Einstein condensation (BEC) for a relativistic ideal gas of bosons. In the framework of canonical thermal field theory, we analyze the role of particles and anti-particles in the determination of BEC transition temperature. At the BEC transition point we obtain two universal curves, i.e. valid for any mass value: the scaled critical temperature as a function of the scaled charge density of the Bose system, and the density ratio of anti-particles versus the scaled critical temperature. Moreover, we numerically investigate charge densities and condensed fraction ranging from the non-relativistic to the ultra-relativistic temperature, where analytical results are obtained.
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I. INTRODUCTION

Nowadays more than twenty experimental groups have achieved Bose-Einstein condensation (BEC) in clouds of confined alkali-metal atoms.\(^1\) These studies have renewed the theoretical interest on non-relativistic\(^2\) but also relativistic BEC. In the past years, relativistic BEC has been analytically investigated by several authors using Euclidean-time functional integration\(^3–7\) but a quantitative numerical analysis with the temperature ranging from the non-relativistic to the ultra-relativistic limit has never been performed.

In this paper we consider BEC in the case of a relativistic non-interacting Bose gas described by a complex scalar field \(\phi(x)\). We derive the exact equation of motion of \(\phi(x)\) in the grand canonical ensemble of equilibrium statistical mechanics\(^8\) without invoking functional integration\(^9\) but using instead canonical field theory. By means of the Bogoliubov prescription\(^10\), we write down the equations of the condensate order parameter and of thermal particles and anti-particles. We numerically investigate the effect of particles and anti-particles in the determination of BEC transition temperature \(T_c\).

Moreover we study the fraction of anti-particles in the system as a function of temperature and derive ultra-relativistic formulas for \(T_c\) and the condensed fraction in a generic \(d\)-dimensional space.

II. SCALAR FIELD AND LEGENDRE ANTI-TRANSFORMATION

The Lagrangian density of a non-interacting complex scalar field \(\phi(x)\) is given by

\[
L = (\partial^\nu \phi)^+ (\partial_\nu \phi) - m^2 \phi^+ \phi ,
\]

where \(m\) is the mass of the identical bosons described by the scalar field. To study the finite-temperature properties of a field-theory one needs the Hamiltonian \(H\) of the system. In our case, the canonical conjugate momentum \(\Pi(x)\) of the scalar field \(\phi(x)\) is

\[
\Pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}^+ ,
\]

and the Hamiltonian density reads
\[ \mathcal{H} = \Pi^+ \Pi + \nabla \phi^+ \cdot \nabla \phi + m^2 \phi^+ \phi . \]  

(3)

Note that the invariance of the system under a global gauge U(1) field transformation implies the conservation of the electric current and the conserved charge density is

\[ Q = i \left( \phi^+ \dot{\phi} - \phi \dot{\phi}^+ \right) = i \left( \Pi^+ \phi^+ - \Pi \phi \right). \]  

(4)

Equilibrium statistical mechanics tells us that the grand canonical partition function \( Z \) of a quantum system of Hamiltonian \( H \) and conserved charge \( Q \) is given by

\[ Z = Tr \left[ e^{-\left( H - \mu Q \right)/T} \right], \]  

(5)

where \( T \) is the temperature of the thermal reservoir and \( \mu \) is the chemical potential.\(^8\) Thus the system in the grand canonical ensemble is described by an effective Hamiltonian density \( \mathcal{H}_\mu = H - \mu Q \), namely

\[ \mathcal{H}_\mu = \Pi^+ \Pi + \nabla \phi^+ \cdot \nabla \phi + V(\phi) - i\mu (\Pi^+ \phi^+ - \Pi \phi). \]  

(6)

To find the effective Lagrangian density we observe that the Hamilton equation for \( \dot{\phi} \) is given by

\[ \dot{\phi} = \frac{\partial \mathcal{H}_\mu}{\partial \Pi} = \Pi^+ + i\mu \phi. \]  

(7)

Using the Legendre anti-transformation one can easily obtain the effective Lagrangian density \( \mathcal{L}_\mu = \dot{\phi}^+ \Pi^+ + \dot{\phi} \Pi - \mathcal{H}_\mu \), namely

\[ \mathcal{L}_\mu = \mathcal{L} + i\mu \left( \phi^+ \dot{\phi} - \phi \dot{\phi}^+ \right) + \mu^2 \phi^+ \phi. \]  

(8)

Therefore, the introduction of the chemical potential \( \mu \) is equivalent to the use of an effective Lagrangian \( \mathcal{L}_\mu \), that can be obtained with a shift

\[ \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + i\mu, \]  

(9)

in the time partial derivative of the bare Lagrangian \( \mathcal{L} \). It is important to observe that the same result could be obtained by means of the Euclidean-time functional integration\(^9\) and
that the shift found holds also in the fermionic case. Finally, the Euler-Lagrange equation of the effective Lagrangian density $\mathcal{L}_\mu$ reads

$$\left[\Box + 2i\mu \frac{\partial}{\partial t} - \mu^2 + m^2\right] \phi = 0 ,$$

(10)

where $\Box = \frac{\partial^2}{\partial x^2} - \nabla^2$ is the d’Alambert operator.

III. BOSE-EINSTEIN CONDENSATION AND BOGOLIUBOV PRESCRIPTION

In a Bosonic system one can separate Bose-condensed particles from non-condensed ones by means of the Bogoliubov prescription\textsuperscript{10} that is given by

$$\phi = \Phi + \eta ,$$

(11)

where

$$\Phi = \langle \phi \rangle = \frac{1}{Z} \text{Tr} \left[ \phi e^{\beta (H - \mu Q)} \right] ,$$

(12)

is the order parameter of the Bose condensate (a classical complex scalar field), namely the non-vanishing thermal average of the Bosonic field, and $\eta(x)$ is the operator of the non-condensed or thermal particles, such that $\langle \eta \rangle = 0$.

The exact equation of motion of the order parameter $\Phi(x)$ is obtained by calculating the thermal average over the equation of motion of the scalar field $\phi(x)$. If we have a static and homogeneous order parameter, then

$$\left[-\mu^2 + m^2\right] \Phi = 0 ,$$

(13)

from which it follows that there is macroscopic occupation of the lowest single-particle state ($\Phi \neq 0$) only if $|\mu| = m$.

The exact equation of motion of the fluctuation operator $\eta(x)$ is easily obtained by subtracting the exact equation of $\Phi(x)$ to the equation of $\phi(x)$. Note that, in the non-interacting case, the equations of $\phi(x)$, $\Phi(x)$ and $\eta(x)$ are formally identical but they have
different meanings. In addition, by using the Bogoliubov prescription and the effective Lagrangian, the grand canonical charge density of the system reads

\[ Q = Q_0 + \tilde{Q}, \quad (14) \]

where

\[ Q_0 = 2\mu|\Phi|^2 \quad (15) \]

is the contribution due to the Bose condensate and

\[ \tilde{Q} = i\left(\eta^+\dot{\eta} - \eta\dot{\eta}^+\right) \quad (16) \]

is the contribution due to thermal particles and anti-particles (see also Ref. 3).

**IV. BOSE CONDENSATE AND THERMAL PARTICLES**

The relativistic complex scalar field operator \( \eta(x) \) satisfies the equal-time commutation rule

\[ [\eta(x, t), \eta^+(y, t)] = \delta^3(x - y). \quad (17) \]

The operator \( \eta(x) \) can be Fourier decomposed into a single-particle basis of particles and anti-particles

\[ \eta(x) = \sum_k \left( \frac{e^{i(k\cdot x - \omega_k t)}}{\sqrt{2\omega_k}} a_k + \frac{e^{-i(k\cdot x - \bar{\omega}_k t)}}{\sqrt{2\bar{\omega}_k}} b_k^+ \right), \quad (18) \]

where we have used the symbols \( \omega_k \) and \( \bar{\omega}_k \) to indicate the value of energy for particles and anti-particles, respectively. The Bose operators for particles and anti-particles satisfy the canonical commutation relations

\[ [a_k, a_k^+] = [b_k, b_k^+] = \delta_{kk'}, \quad (19) \]

and all other commutators are zero. In addition, one imposes the following Bose-Einstein thermal averages
\[ \langle a_k^+ a_{k'} \rangle = \frac{1}{e^{\omega_k/T} - 1} \delta_{kk'} , \quad \langle b_k^+ b_{k'} \rangle = \frac{1}{e^{\bar{\omega}_k/T} - 1} \delta_{kk'} . \]  

(20)

The energies \( \omega_k \) and \( \bar{\omega}_k \) are determined by inserting the decomposition of the field \( \eta(x) \) in its equation of motion, namely Eq. (10) with \( \eta(x) \) instead of \( \phi(x) \). In this way, one finds two decoupled algebraic equations:

\[ \omega_k^2 - 2\mu \omega_k + \mu^2 - m^2 - k^2 = 0 , \]  

(21)

\[ \bar{\omega}_k^2 + 2\mu \bar{\omega}_k + \mu^2 - m^2 - k^2 = 0 , \]  

(22)

which give the physical solutions

\[ \omega_k = \sqrt{k^2 + m^2} - \mu , \quad \bar{\omega}_k = \sqrt{k^2 + m^2} + \mu . \]  

(23)

The Fourier decomposition and the energies \( \omega_k \) and \( \bar{\omega}_k \) enable us to calculate the thermal average \( \tilde{\bar{q}} = \langle \bar{Q} \rangle \) of the non-condensed charged density, which is given by

\[ \tilde{\bar{q}} = \sum_{k \neq 0} \left[ \frac{1}{e^{(\sqrt{k^2+m^2} - \mu)/T} - 1} - \frac{1}{e^{(\sqrt{k^2+m^2} + \mu)/T} - 1} \right] . \]  

(24)

Thus, \( \tilde{\bar{q}} = n_1 - n_2 \), where \( n_1 = \sum_k \langle a_k^+ a_k \rangle \) is the average density of particles and \( n_2 = \sum_k \langle b_k^+ b_k \rangle \) is the average density of anti-particles. Note that \( n_2 \) is obtained from \( n_1 \) with the substitution \( \mu \rightarrow -\mu \). The chemical potential \( \mu \) describes both bosons and antibosons: the sign of \( \mu \) indicates whether particles outnumber antiparticles or vice versa. Moreover, because both \( n_1 \) and \( n_2 \) must be positive definite, it follows that \( |\mu| \leq m \).

Obviously, the total number of particles is not conserved because of the production of antiparticles, which becomes relevant when \( T \) is comparable with \( m \). The conserved quantity is the net charge density \( q = q_0 + \tilde{q} \), where \( q_0 = \bar{Q}_0 = 2\mu|\phi|^2 \) is the condensed charge density and \( \tilde{q} = \langle \bar{Q} \rangle = n_1 - n_2 \) is the the difference between the density of particles and the density of anti-particles. The condensed charge density \( q_0 \) is non-zero only below the BEC transition temperature \( T_c \). The condensed charge density corresponds to \( k = 0 \) in Eq. (24) and it is thus given by
\[ q_0 = \frac{1}{e^{(m-\mu)/T} - 1} - \frac{1}{e^{(m+\mu)/T} - 1} = n_1^{(0)} - n_2^{(0)}, \tag{25} \]

where \( n_1^{(0)} \) and \( n_2^{(0)} \) are the density of condensed particles and anti-particles, respectively. It is easy to show that

\[ \lim_{T \to 0} n_1^{(0)} (T) = q_0, \quad \lim_{T \to 0} n_2^{(0)} (T) = 0. \tag{26} \]

As expected, at \( T = 0 \) the particles are all in the condensate and there are no anti-particles. Moreover, in the limit \( T \to 0 \) the asymptotic behavior of the chemical potential \( \mu \) and of the density \( n_2^{(0)} \) of condensed anti-particles read

\[ \mu (T) \sim m - T \ln \left( \frac{q_0 + 1}{q_0} \right), \]

\[ n_2^{(0)} (T) \sim \frac{q_0 + 1}{q_0 \left( e^{2m/T} - 1 \right) - 1}. \tag{27} \]

To our knowledge, this is the first paper where these simple asymptotic relations have been explicitly written down.

\section*{V. NUMERICAL AND ANALYTICAL RESULTS}

The behavior of the Bose gas ranging from the non-relativistic to the ultra-relativistic regime can be numerically investigated by means of the non-condensed thermal charge density \( \bar{q} \) given by Eq. (24). In particular, we work in the thermodynamic limit substituting the sums in Eq. (24) with integrals and find

\[ \bar{q} = \frac{1}{2\pi^2} \int_{0}^{\infty} dk \left[ \frac{1}{e^{(\sqrt{k^2+m^2}-\mu)/T} - 1} - \frac{1}{e^{(\sqrt{k^2+m^2}+\mu)/T} - 1} \right] = \frac{1}{2\pi^2} \int_{0}^{\infty} dk \left[ n_1 (k) - n_2 (k) \right], \tag{28} \]

where \( n_1 (k) \) and \( n_2 (k) \) are the density profiles in momentum space for particles and antiparticles, respectively. It is important to stress that the previous formula has an useful scaling property: the chemical potential, the temperature and the momentum can be measured in units of \( m \) and the densities in units of \( m^3 \). This follows from the fact that, with \( \hbar = c = 1 \),
the mass, the chemical potential, the temperature and the momentum have the same unit: the energy, while the length is measured in units of the inverse of energy.

We first consider the case \( T > T_c \). Given the charge density and the temperature, the chemical potential is fixed by Eq. (28). In such a way one determines also the fraction of anti-particles in the system. As shown in Fig. 1, where we plot \( n_2/n_1 \) as a function of the scaled temperature \( T/m \) for different values of the scaled charge density \( q/m^3 \) of the Bose system, one can identify two regimes: the non-relativistic regime \( (T/m \ll 1) \) and the ultra-relativistic regime \( (T/m \gg 1) \). In the non-relativistic regime, the fraction of anti-particles of the system is negligible. In the ultra-relativistic regime the fraction of anti-particles becomes relevant. The curves with fixed charge density \( q = \tilde{q} (q_0 = 0) \) ends at the scaled critical temperature \( T_c/m \), where the Bose-Einstein condensate appears. The role of temperature in the formation of anti-particles is also shown in Fig. 2, where we plot the density profiles in momentum space of particles and anti-particles for three increasing values of the scaled temperature.

We observe that in the ultra-relativistic regime one can derive analytical results by performing a Taylor expansion of \( \tilde{q} \) at first order in \( \mu \). After straightforward but tedious calculations one finds

\[
\begin{align*}
n_1 &= \frac{\zeta(3)}{\pi^2} T^3 + \frac{\mu}{6} T^2, \\
n_2 &= \frac{\zeta(3)}{\pi^2} T^3 - \frac{\mu}{6} T^2, \\
\tilde{q} &= \frac{\mu}{3} T^2,
\end{align*}
\]

where \( \zeta(x) \) is the Riemann \( \zeta \)-function. These analytical results, confirmed by our numerical calculations, show that although \( n_2/n_1 \to 1 \) as \( T \to \infty \), the charge density \( \tilde{q} = n_1 - n_2 \) goes to infinity.

We have previously shown that the critical temperature \( T_c \) at which BEC occurs corresponds to \( |\mu| = m \). At the BEC transition temperature \( T_c \), the thermal charged density \( \tilde{q} \) can still be determined from Eq. (28). In fact, by inverting the function \( \tilde{q}(T_c, m = \mu) \) one finds the transition temperature. In Fig. 3 we plot two curves which do not depend on the value of the mass of Bosons in the gas (we call them universal curve). The first universal curve is the scaled critical temperature \( T_c/m \) as a function of the scaled charge density
q/m^2. The second universal curve is the ratio n_2/n_1 between anti-particles and particles as a function of the scaled critical temperature T_c/m. As expected, T_c grows with q and n_2/n_1. Moreover, at a fixed density ratio n_2/n_1, it is easier to get high-temperature BEC with heavy-mass particles. The two universal curves of Fig. 3 can be compared with ultra-relativistic analytical results. From (29) one immediately finds that the critical temperature (µ = m) is given by

\[ T_c = \left( \frac{3q}{m} \right)^{1/2}, \]

and the density ratio reads

\[ \frac{n_2}{n_1} = \frac{\zeta(3) T_c^3 - 6T_c^2 \mu}{\zeta(3) T_c^3 + 6T_c^2 \mu}. \]  

Fig. 3 shows that while the formula of the density ratio n_2/n_1 is valid only in the ultra-relativistic region (for T_c → 0 it predicts the wrong limit n_2/n_2 → −1), the formula of the critical temperature is quite accurate also at low temperatures. Note that the formula of the critical temperature has been first obtained by Kapusta. We now extend it to the case of a ultra-relativistic gas in d-dimensional space.

The charge density our system of non-interacting bosons can be re-written as

\[ \tilde{q} = \int_0^\infty d\epsilon \rho(\epsilon) \left[ \frac{1}{e^{(\epsilon-\mu)/T} - 1} - \frac{1}{e^{(\epsilon+\mu)/T} - 1} \right], \]

where \( \rho(\epsilon) \) is the density of states. It can be obtained from the formula

\[ \rho(\epsilon) = \int \frac{d^dk}{(2\pi)^d} \delta(\epsilon - H(k)), \]

where \( H(k) \) is the classical single-particle Hamiltonian of the system in a d-dimensional space. The classical single-particle Hamiltonian of a relativistic ideal gas is \( H = \sqrt{k^2 + m^2} \) and the density of states reads

\[ \rho(\epsilon) = \frac{2\pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \epsilon^{(d-2)/2}. \]

In the ultra-relativistic limit the density of states is simply \( \rho(\epsilon) = (2\pi^{d/2})/((2\pi)^d \Gamma(d/2)) \epsilon^{(d-1)}. \) In this case, by using again the Taylor expansion of q at first order in \( \mu \) with \( T = T_c \) one finally obtains
\[ T_c = \left( \frac{(2\pi)^d \Gamma(d/2) q}{4\pi^{d/2} \Gamma(d) \zeta(d-1) m} \right)^{1/(d-1)}, \tag{35} \]

where \(\Gamma(x)\) is the factorial function. Because \(\zeta(1) = \infty\), it follows that for a homogeneous relativistic gas there is BEC only for \(d > 2\), as in the case of a non-relativistic homogeneous gas (see also Ref. 12, Ref. 13).

Below \(T_c\), a macroscopic number of particle occupies the single-particle ground-state of the system \((q_0 \neq 0)\). The Eq. (28) gives the charge density \(\tilde{q} = q - q_0\) of non-condensed particles. In this way, form Eq. (15), one determines the order parameter \(\Phi\), that is such that

\[ |\Phi|^2 = \frac{\tilde{q}(T) - \tilde{q}(T_c)}{2m}. \tag{36} \]

Thus, the condensed fraction \(q_0/q\) can be numerically calculated as \(1 - \tilde{q}(T)/\tilde{q}(T_c)\). In Fig. 4 we show the condensed fraction as a function of the temperature for different values of the scaled charge density \(q/m^3\) of the Bosonic particles. In the ultra-relativistic regime, from (28) and (29) one finds

\[ \frac{q_0}{q} = 1 - \left( \frac{T}{T_c} \right)^2, \tag{37} \]

namely the condensed fraction has an inverted-parabola shape. It is important to stress that, although we are able to determine the charge density \(q_0\) of the Bose condensate, our formalism cannot tell us the fraction of anti-particles into the condensate.

For an ideal gas of charged massless Bosons \((m = 0)\) it follows, form Eq. (30) and Eq. (35), that \(T_c = \infty\) and \(q_0 = q\): at any temperature, all net charge resides in the Bose condensate. Nevertheless, if the thermal average of the charge is not conserved \((\mu = 0)\), i.e. a gas of photons, then BEC does not take place. Finally, by using the previously discussed procedure, one finds that the Bose condensed fraction for a ultra-relativistic gas in a d-dimensional space is given by

\[ \frac{q_0}{q} = 1 - \left( \frac{T}{T_c} \right)^{d-1}, \tag{38} \]

remembering that \(T_c \to \infty\) as \(d \to 2\).
VI. CONCLUSIONS

We have studied thermal properties of a non-interacting relativistic Bose gas by analyzing in detail the fraction of anti-particles in the system. By using a finite-temperature operator formalism, we have obtained the equation of the Bose condensate, described by a complex classical order parameter, and the equation of non-condensed particles and anti-particles. At zero temperature the particles are all in the condensate and there are no anti-particles. In the limit of zero temperature we have determined the asymptotic behavior of the density of condensed anti-particles. The charge density and the density of particles and anti-particles have been analyzed as a function of temperature raging from the non-relativistic to the ultra-relativistic regime. We have determined two universal curves at the BEC transition point: the scaled critical temperature $T_c/m$ as a function of the scaled charge density $q/m^3$ of the Bose gas, and the ratio between anti-particles and particles as a function of the scaled critical temperature $T_c/m$. Moreover, we have investigated the condensed fraction as a function of the scaled temperature for increasing values of the scaled charge density of the gas. Finally, analytical results have been found in the ultra-relativistic region. Our analytical formulas for a ultra-relativistic Bose gas in d-dimensional space generalize previous findings with $d = 3$.

In conclusion, we observe that detailed analytical and numerical investigations can be also performed in the case of an interacting relativistic Bose gas, at least in the Bogoliubov-Popov mean-field approximation. This is one of our future projects.
REFERENCES

1. M.H. Anderson, et al., Science 269, 189 (1995); K.B. Davis, et al. Phys. Rev. Lett. 75, 3969 (1995); C.C. Bradley, et al., Phys. Rev. Lett. 75, 1687 (1995).

2. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

3. J.I. Kapusta, Phys. Rev. D 24, 426 (1981).

4. H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 23, 1497 (1981); H.E. Haber and W.A. Weldon, Phys. Rev. D 25, 502 (1982)

5. J. Bernstein and S. Dodelson Phys. Rev. Lett. 66, 683 (1991).

6. D.J. Toms, Phys. Rev. Lett. 69, 1152 (1992); D.J. Toms, Phys. Rev. D 50, 6457 (1994).

7. K. Shiokowa and B.L. Hu, Phys. Rev. D 60, 105016 (1999).

8. K. Huang, Statistical Mechanics (John Wiley, New York, 1987).

9. J.I. Kapusta, Finite Temperature Field Theory (Cambridge Univ. Press, Cambridge, 1989); M. Le Bellac, Thermal Field Theory (Cambridge Univ. Press, Cambridge, 1996).

10. N.N. Bogoliubov, J. Phys. U.S.S.R. 11, 23 (1941); S.T. Beliaev, Sov. Phys. JEPT 7, 289 (1958).

11. M. Modugno, Rivista del Nuovo Cimento 23, N.5, 1 (2000).

12. L. Salasnich, Int. J. Mod. Phys. B 14, 405 (2000).

13. L. Salasnich, J. Math. Phys. 41, 8016 (2000).
FIG. 1. Density ratio $n_2/n_1$ between particles and anti-particles vs scaled temperature $T/m$. Curves for different values of the scaled charge density $q/m^3$ of the Bose gas above the critical temperature $T_c$, where each curve ends.
FIG. 2. Density profile in momentum space for particles (full line) and anti-particles (dashed line). Scaled charge density of the Bose gas: $q/m^3 = 0.1$. From left to right: $T/m = 1$, $T/m = 1.5$, $T/m = 2$. 
FIG. 3. Universal curves at the BEC transition point. Scaled critical temperature $T_c/m$ vs scaled charge density $q/m^3$ (above). Density ratio $n_2/n_1$ vs scaled critical temperature $T_c/m$ (below). Full lines are numerical results and dashed lines are analytical results in the ultra-relativistic limit.
FIG. 4. Condensed fraction $q_0/q$ vs scaled temperature $T/m$. Curves for different values of the scaled charge density $q/m^3$ of the Bose gas below the critical temperature.