CONDITIONAL LOWER BOUND FOR THE \( k \)-TH PRIME IDEAL WITH GIVEN ARTIN SYMBOL

LOÏC GRENIÉ AND GIUSEPPE MOLTENI

Abstract. We prove an explicit upper bound for the \( k \)-th prime ideal with fixed Artin symbol, under the assumption of the validity of the Riemann hypothesis for the Dedekind zeta functions.

1. Introduction

For a number field \( K \), \( n_K \) denotes its dimension, \( \Delta_K \) the absolute value of its discriminant, \( r_1(K) \) the number of its real places, and \( r_2(K) \) the number of its imaginary places. Moreover, \( p \) denotes a nonzero prime ideal of the integer ring \( \mathcal{O}_K \) and \( N_p \) its absolute norm. The von Mangoldt function \( \Lambda_K \) is defined on the set of ideals of \( \mathcal{O}_K \) as \( \Lambda_K(I) := \log N_p \) if \( I = p^m \) for some \( p \) and \( m \geq 1 \), and is zero otherwise. Let \( K \subseteq L \) be a Galois extension of number fields with relative discriminant \( \Delta_{L/K} \). For \( \mathfrak{P} \) a prime ideal of \( L \) above a non-ramified \( p \), the Artin symbol \( \left[ \frac{L/K}{\mathfrak{P}} \right] \) denotes the Frobenius automorphism corresponding to \( \mathfrak{P}/p \). We further denote \( \left[ \frac{L/K}{p} \right] \) the conjugacy class of all the \( \left[ \frac{L/K}{\mathfrak{P}} \right] \). We then extend multiplicatively \( \left[ \frac{L/K}{p} \right] \) to the group of fractional ideals of \( K \) coprime to \( \Delta_{L/K} \).

Let \( C \) be any conjugacy class in \( G := \text{Gal}(L/K) \) and let \( \varepsilon_C \) be its characteristic function. Then the function \( \pi_C \) and the Chebyshev function \( \psi_C \) are defined as

\[
\pi_C(x) := \sharp \{ p: p \text{ non-ramified in } L/K, Np \leq x, \left[ \frac{L/K}{p} \right] = C \} = \sum_{p \text{ non-ram. } \text{Np} \leq x} \varepsilon_C \left( \left[ \frac{L/K}{p} \right] \right),
\]

\[
\psi_C(x) := \sum_{\mathfrak{J} \in \text{Conj}(\mathcal{O}_K) \text{ non-ram. } N\mathfrak{J} \leq x} \varepsilon_C \left( \left[ \frac{L/K}{\mathfrak{J}} \right] \right) \Lambda_K(\mathfrak{J}).
\]

In [5] we have proved the following explicit bound.

Theorem. Assume GRH holds. Let \( x \geq 1 \), then

\[
\left| \frac{|G|}{|C|} \psi_C(x) - x \right| \leq \sqrt{x} \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L.
\]

Date: June 6, 2019. File name: Conditional_lower_bound_for_the_k-th_prime_ideal_with_given_Artin_symbolXXVII.tex

2010 Mathematics Subject Classification. Primary 11R42, Secondary 11Y70.
This result concludes a quite long set of similar but partial computations, originated with Jeffrey Lagarias and Andrew Odlyzko’s paper [7] where this result is proved with undetermined constants, and which was followed by the result announced by Joseph Oesterlé [12] and the one of Bruno Winckler [15, Th. 8.1] (both with the same generality and explicit but larger constants), the one of Lowell Schoenfeld [14] (same bound but only for the case $L = K = \mathbb{Q}$), and our recent paper [4] (same conclusion, but only for the case $L = K$).

Bound (1.1) implies that for every class $C$ there is a prime ideal $\mathfrak{p}$ with

$$N\mathfrak{p} \leq \left(\frac{1}{2\pi \log \delta} + o(1)\right) \log \Delta_{L}(\log \log \Delta_{L})^{2}$$

which is not ramified and for which $[L/\mathbb{K}] = C$, where $\delta$ is any lower bound for the root discriminant of the family of fields for which we would like to apply the result: $\sqrt{3}$ is a possible value for all fields. This consequence of any bound similar to (1.1) is already discussed in Lagarias and Odlyzko’s paper, where in fact the existence of a bound of the form $c(\log \Delta_{L}(\log \log \Delta_{L})^{2})^{2}$ for some computable (but not explicit) constant $c$ is proved. In that paper, to remove the extra factor $(\log \log \Delta_{L})^{4}$, the authors also sketched a different approach using the smoothing kernel $(\frac{x^{s-1} - y^{s-1}}{s-1})^{2}$ instead of $\frac{x^{s}}{s}$, with a suitable choice of the parameters $x$ and $y$ in terms of $\log \Delta_{L}$.

The same conclusion may be achieved also via different kernels. In particular, we have obtained (1.1) with the classical smoothing kernel $\frac{x^{s+1}}{s+1}$, and in this paper we combine some of the results we got there to prove the following claim.

**Theorem.** Assume GRH holds. Fix any class $C$ and any integer $k \geq 0$. Assume

$$\sqrt{x} \geq 1.075\log \Delta_{L} + \sqrt{2\left|\frac{G}{|C|}\right| k \log \left|\frac{G}{|C|}\right| k + 2\left|\frac{G}{|C|}\right| + 15},$$

where $k \log k$ is set to 0 for $k = 0$. Then $\pi_{C}(x) \geq k+1$.

The proof of this theorem shows that the constant 15 can be removed when the degree of the field is large enough. However, the main constant 1.075 is rooted in the method and can be improved only marginally. In particular it remains larger than 1. This implies that the case $k = 0$ of the theorem is weaker than the analogous conclusion of the paper by Eric Bach and Jonathan Sorenson [1, Th. 3.1], further improved for the case where $K = \mathbb{Q}$ and $L/\mathbb{Q}$ is abelian by Youness Lamzouri, Xiannan Li and Kannan Soundararajan [8, Th. 1.2] (see also [9]).

The claim giving at least two ideals (i.e., $k \geq 1$) cannot be reached with Lagarias–Odlyzko’s, Bach–Sorenson’s or Lamzouri–Li–Soundararajan’s approaches.

The case where $K = \mathbb{Q}$ and $C$ is the trivial class has been considered also in [3, Corollary 2.1], with similar conclusions, in particular with the same constant for $\log \Delta_{L}$ but a larger one for the $k \log k$ term.

For any field extension $L/K$ and any class $C$ fixed, the theorem says that we can find $k$ prime ideals in $C$ as soon as $x \gg \left|\frac{G}{|C|}\right| k \log k$, not uniformly in $L, K$ and $C$: this is the correct function of $k$. However, the implicit multiplicative constant is 2, while we know that the correct asymptotic value for this constant is 1. This overestimation represents the price we pay in order to get a uniform and totally explicit result.

**Acknowledgements.** The authors are members of the INdAM group GNSAGA.
2. Preliminary facts

We define two further functions which are closely related to $\pi_C$ and $\psi_C$ and easier to deal with. They are built using an arithmetical function which comes from the theory of Artin $L$-functions and extend $\varepsilon_C\left(\frac{L/K}{p}\right)$ to ramifying prime ideals. To wit, for any prime ideal $p \subseteq \mathcal{O}_K$ (possibly ramified) let $\mathfrak{P}$ be any prime ideal dividing $p\mathcal{O}_L$, let $I$ be the inertia group of $\mathfrak{P}$ and $\tau$ be one of the Frobenius automorphisms corresponding to $\mathfrak{P}/p$. Let

$$\theta(C; p^m) := \frac{1}{|I|} \sum_{a \in f} \varepsilon_C(\tau^m a).$$

Notice that $\theta(C; p^m) \in [0, 1]$, and that for non-ramified primes $\theta(C; p^m) = \varepsilon_C(p^m)$. We define

$$\psi(C; x) := \sum_{2 \leq \mathfrak{I} \subseteq \mathcal{O}_K} \sum_{\mathfrak{I} \subseteq \mathfrak{N} \leq x} \theta(C; \mathfrak{I}) \Lambda_K(\mathfrak{I}).$$

Observe that $\psi(C; x)$ and $\psi(C; x)$ agree except on ramified-prime-powers ideals. Let

$$\psi^{(1)}(C; x) := \int_0^x \psi(C; t) \, dt$$

and, for $s > 1$,

$$K(C; s) := \sum_{\mathfrak{I} \subseteq \mathcal{O}_K} \theta(C; \mathfrak{I}) \Lambda_K(\mathfrak{I})(\mathfrak{N}\mathfrak{I})^{-s}.$$

As in [6, Ch. IV Sec. 4, p. 73] and [7, Sec. 5], we have the integral representation

$$\psi^{(1)}(C; x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(C; s) \frac{x^{s+1}}{s(s+1)} \, ds.$$

The function $\theta(C; \cdot)$ is a class function and therefore can be written as a linear combination of characters of irreducible representations of the group $G$. A clever trick (due to Deuring [2] and MacCluer [10], see also Lagarias and Odlyzko [7, Lemma 4. 1] and [5, p. 445–446]) allows to write this function as a linear combination of characters which are induced from characters of a certain cyclic subgroup of $G$ specified below. Explicitly,

$$K(C; s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L}(s, \chi, L/E),$$

where $g$ is any fixed element in $C$, $E := \mathbb{L}^H$ is the subfield of $\mathbb{L}$ fixed by $H := \langle g \rangle$ (which is the subgroup of $G$ which we alluded to), $L(s, \chi, \mathbb{L}/\mathbb{E})$ is the Artin $L$-function associated with the extension $\mathbb{L}/\mathbb{E}$ and the character $\chi$, and the sum runs on all irreducible characters $\chi$ of $H$. Since the extension is abelian, this coincides with a suitable Hecke $L$-function, by class field theory.

With (2.1), this equality produces the identity

$$\frac{|G|}{|C|} \psi^{(1)}(C; x) = -\sum_{\chi} \bar{\chi}(g) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(s, \chi, \mathbb{L}/\mathbb{E}) \frac{x^{s+1}}{s(s+1)} \, ds.$$

We introduce a special notation for the type of sum on characters as the one appearing in (2.3), and for any $f: \text{Gal}(\mathbb{L}/\mathbb{E}) \rightarrow \mathbb{C}$ we set

$$\mathcal{M}_C f := \sum_{\chi} \bar{\chi}(g)f(\chi).$$
With this language, Equality (2.3) reads
\[
(2.4) \quad \frac{|G|}{|C|} \psi^{(1)}(C;x) = \mathcal{M}_C I_\chi(x),
\]
where
\[
(2.5) \quad I_\chi(x) := -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{L'(s,\chi,L/E)}{L(s,\chi,L/E)} x^{s+1} \frac{ds}{s(s+1)}
\]

3. Some computations with Abelian Artin L-functions

Let \( E \subseteq \mathbb{L} \) be an abelian extension of fields and let \( \chi \) be any irreducible character of \( \text{Gal}(\mathbb{L}/E) \). We will use \( L(s,\chi) \) to denote \( L(s,\chi,L/E) \). Also, set \( \delta_\chi = 1 \) if \( \chi \) is the trivial character, and 0 otherwise.

We recall that for each \( \chi \) there exist non-negative integers \( a_\chi, b_\chi \) such that
\[
a_\chi + b_\chi = n_E
\]
and a positive integer \( Q(\chi) \) such that if we define
\[
(3.1) \quad \Gamma_\chi(s) := \left[ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right]^{a_\chi} \left[ \pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right) \right]^{b_\chi}
\]
and
\[
(3.2) \quad \xi(s,\chi) := [s(s-1)]^{\frac{1}{2}} Q(\chi)^{s/2} \Gamma_\chi(s) L(s,\chi),
\]
then \( \xi(s,\chi) \) satisfies the functional equation
\[
(3.3) \quad \xi(1-s,\bar{\chi}) = W(\chi) \xi(s,\chi),
\]
where \( W(\chi) \) is a certain constant of absolute value 1. Furthermore, \( \xi(s,\chi) \) is an entire function (by class field theory) of order 1 and does not vanish at \( s = 0 \), and hence by Hadamard’s product theorem we have
\[
(3.4) \quad \xi(s,\chi) = e^{A_\chi + B_\chi s} \prod_{\rho \in Z_\chi} \left( 1 - s / \rho \right) e^{s/\rho}
\]
for some constants \( A(\chi) \) and \( B(\chi) \), where \( Z_\chi \) is the set of zeros (multiplicity included) of \( \xi(s,\chi) \). They are precisely those zeros \( \rho = \beta + i\gamma \) of \( L(s,\chi) \) for which \( 0 < \beta < 1 \), the so-called “non-trivial zeros” of \( L(s,\chi) \). From now on \( \rho \) will denote a non-trivial zero of \( L(s,\chi) \).

Differentiating (3.2) and (3.3) logarithmically we obtain the identity
\[
(3.5) \quad \frac{L'(s,\chi)}{L(s,\chi)} = B_\chi + \sum_{\rho \in Z_\chi} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \frac{1}{2} \log Q(\chi) - \delta_\chi \left( \frac{1}{s} + \frac{1}{s-1} \right) - \Gamma'_\chi / \Gamma_\chi(s),
\]
valid identically in the complex variable \( s \).

Using (3.2), (3.3) and (3.5) one sees that
\[
(3.6) \quad \frac{L'(s,\chi)}{L(s,\chi)} = \frac{a_\chi - \delta_\chi}{s} + r_\chi + O(s) \quad \text{as } s \to 0,
\]
\[
(3.6) \quad \frac{L'(s,\chi)}{L(s,\chi)} = \frac{b_\chi}{s+1} + r'_\chi + O(s+1) \quad \text{as } s \to -1,
\]
where
\[
r_\chi = B_\chi + \delta_\chi - \frac{1}{2} \log \frac{Q(\chi)}{\pi^{a_\chi}} - \frac{a_\chi}{2} \Gamma(1) - \frac{b_\chi}{2} \Gamma\left(\frac{1}{2}\right).
\]
Comparing the previous formula for $r_\chi$ and (3.5) with $s = 2$, we further get

$$r_\chi = \frac{L'}{L}(2, \chi) - \sum_{\rho} \frac{2}{\rho(2-\rho)} + \frac{5}{2} \delta_\chi + b_\chi.$$  

Shifting the axis of integration in (2.5) arbitrarily far to the left, we collect the terms coming from the pole of $L$ at $s = 1$ (if any), the non-trivial zeros, the pole of the kernel (and of $L'/L$, if any) at $s = 0$, the pole of the kernel (and of $L'/L$, if any) at $s = -1$ and all the remaining terms coming from the trivial zeros of $L$. This procedure gives the identity

$$I_\chi(x) = \delta_\chi \frac{x^2}{2} - \sum_{\rho \in Z_\chi} \frac{x^{\rho+1}}{\rho(\rho+1)} - xr_\chi + r'_\chi + R_\chi(x) \quad \forall x > 1,$$

where $r_\chi$ and $r'_\chi$ are defined in (3.6) and $R_\chi(x)$ is the explicit function

$$f_1(x) := \sum_{r=1}^{\infty} \frac{x^{1-2r}}{(2r)(2r-1)}, \quad f_2(x) := \sum_{r=2}^{\infty} \frac{x^{2-2r}}{(2r-1)(2r-2)},$$

$$R_\chi(x) := -(a_\chi - \delta_\chi)(x \log x - x) + b_\chi(\log x + 1) - a_\chi f_1(x) - b_\chi f_2(x).$$

(with $x > 1$). The correctness of this procedure is proved in a way similar to [7, § 6], further simplified by the fact that the integral is absolutely convergent on vertical lines (see also [6, Ch. IV Sec. 4, p. 73]).

According to (2.4), in order to proceed we need to know the effect of the $M_C$ operator on each term in (3.9). To this effect, we recall a few lemmas that we will need in the following.

**Lemma 3.1 ([5, Lemma 1]).** Let

$$\mathbf{S} := \begin{cases} r_1(L) + r_2(L) & \text{if } g \text{ has order } 1, \\ r_2(L) - 2r_2(E) & \text{if } g \text{ has order } 2, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover let $\delta_C$ be defined to be 1 if $C$ is the trivial class and 0 otherwise. Then

$$M_C a_\chi = \sum_x \tilde{\chi}(g)a_\chi = \mathbf{S},$$

$$M_C b_\chi = \sum_x \tilde{\chi}(g)b_\chi = \delta_C n_{E} - \mathbf{S} = \delta_C n_{E} - \mathbf{S}.$$

From now on, we assume that $\mathbb{L}/E$ is cyclic, and let $Z$ be the multiset of zeros of the Dedekind zeta function $\zeta_L$. Thus $Z$ is the disjoint union of the sets $Z_\chi$ for $\chi \in \text{Gal}(\mathbb{L}/E)$.

**Lemma 3.2 ([5, Lemma 2]).** Let $f$ be any complex function with $\sum_{\rho \in Z} |f(\rho)| < \infty$. Then

$$M_C \sum_{\rho \in Z_\chi} f(\rho) = \sum_{\rho \in Z} \epsilon(\rho)f(\rho)$$

where, for any $\rho \in Z$, $|\epsilon(\rho)| = 1$ and $\epsilon(7) = \epsilon(\overline{7})$.

The following lemma comes from (3.8) and Lemmas 3.1 and 3.2.
Lemma 3.3 ([5, Lemma 3]).
\[ \mathcal{M}_C r_\chi = 2 \sum_{\rho \in Z} \frac{\epsilon(\rho)}{\rho(2-\rho)} \frac{n_E}{n_E|C|} \sum_{\mathfrak{P} \in \mathcal{D}_C} \theta(C; \mathfrak{P}) \Lambda_K(\mathfrak{P}) (N\mathfrak{P})^2 + n_E \delta_C - S + \frac{5}{2}. \]

Lemma 3.4 ([5, Lemma 5]). Define for any \( x > 1 \), \( R_C(x) := \mathcal{M}_C R_\chi(x) \). Then
\[ R_C(x) = \int_0^x \log u \, du - S \int_1^{x+1} \log u \, du + \frac{n_L}{2} \left[ \log(x^2-1) + x \log \left( \frac{x+1}{x-1} \right) \right]. \]

Lemma 3.5 ([5, Lemma 10]). Assume GRH. Then
\[ \sum_{\rho \in Z} \frac{1}{|\rho(\rho+1)|} \leq 0.5375 \log \Delta_L - 1.0355 n_L + 5.3879 - 0.2635 r_1(L). \]

We finally prove three technical lemmas.

Lemma 3.6. Assume GRH. Then
\[ \mathcal{M}_C r_\chi \leq 1.075 \log \Delta_L - 1.571 n_L + 13.276. \]

Proof. By Lemma 3.3, we have
\[ \mathcal{M}_C r_\chi \leq 2 \sum_{\rho \in Z} \frac{1}{|\rho(2-\rho)|} + n_E \delta_C - S + \frac{5}{2}. \]

A brief check shows that \( n_E \delta_C - S \leq r_2(L) \leq \frac{1}{2} n_L \). Moreover, \( |\rho(2-\rho)| = |\rho(\rho+1)| \), thus Lemma 3.5 applies here and the result follows. \( \square \)

Lemma 3.7. We have
\[ -\mathcal{M}_C r'_\chi \leq \log \Delta_L. \]

Proof. As a consequence of (3.7), we have
\[ \mathcal{M}_C r'_\chi = -\mathcal{M}_C \frac{L'}{L}(2, \bar{\chi}) - \mathcal{M}_C \log Q(\chi) + n_E \left( \log \pi - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{3}{2} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} (1) \right) \mathcal{M}_C 1. \]

Letting \( C_1 \) to be the class of \( g^{-1} \), we see from (2.2) that
\[ -\mathcal{M}_C \frac{L'}{L}(2, \bar{\chi}) = \frac{|G|}{|C|} K(C_1; 2) \]
which, by definition, is a positive real. Moreover,
\[ |\mathcal{M}_C \log Q(\chi)| = \left| \sum_{\chi} \bar{\chi}(g) \log Q(\chi) \right| \leq \sum_{\chi} \log Q(\chi) = \log \Delta_L, \]
by the product formula for conductors. The result follows because \( n_E \mathcal{M}_C 1 = n_L \delta_C \geq 0 \) and \( \log \pi - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{3}{2} \right) - \frac{1}{2} \frac{\Gamma'}{\Gamma} (1) = 1.41 \ldots \) is positive. \( \square \)

Lemma 3.8. If \( \mathbb{L} \neq \mathbb{Q} \), for any \( x > 1 \),
\[ -R_C(x) \leq (n_L - 1) \int_1^{x+1} \log u \, du. \]
Proof. Consider the formula for $R_C(x)$ given in Lemma 3.4 When $r_2(L) \geq 1$ we have $S \leq r_1(L) + r_2(L) = n_L - r_2(L) \leq n_L - 1$ producing

$$-R_C(x) \leq S \int_1^{x+1} \log u \, du \leq (n_L - 1) \int_1^{x+1} \log u \, du.$$  

On the other hand, if $r_2(L) = 0$ then when $\delta_C = 0$ we have $S = 0$ and $R_C(x) > 0$, while if $\delta_C = 1$ we have $S = n_L$, or $\delta_C > 1$ because $L \not\subseteq \mathbb{Q}$, and

$$-R_C(x) \leq n_L \int_1^{x+1} \log u \, du - \int_0^x \log u \, du - \left[ \log(x^2 - 1) + x \log \left( \frac{x+1}{x-1} \right) \right]$$

$$= (n_L - 1) \int_1^{x+1} \log u \, du - \log(x-1) - x \log \left( \frac{x}{x-1} \right) \leq (n_L - 1) \int_1^{x+1} \log u \, du. \quad \square$$

4. PROOF OF THE THEOREM

When $L = \mathbb{Q}$ the claim follows easily by Chebyshev’s bound $\pi(x) \geq \frac{x}{2 \log x}$, For the next computations we assume $L \not\subseteq \mathbb{Q}$.

**Lemma 4.1.** Let $x \geq 400$ and $y > 0$, then

$$(x-y) \log y \leq x(\log x - \log(2 \log x)).$$

Proof. Let $f_x(y) := (x-y) \log y$. Its maximum is attained at a unique point $y_0(x) \in (1, x)$, with $y_0(\log y_0 + 1) = x$. The formula shows that $y_0$ grows as a function of $x$. A simple computation shows that

$$\frac{f_x(y_0)}{x} \log x + \log \log x = g(1 + \log y_0),$$

where

$$g(z) := \log \left( 1 + \frac{\log z - 1}{z} \right) + \frac{1}{z} - 1.$$  

This function decreases for $z \geq e$ and is lower than $-\log 2$ when $z \geq 5.3193$. Since $5.3193^{4.3193} = 399.67 \ldots$, the claim is proved. \square

Let $a_C(n) := \sharp\{\mathfrak{p} : \mathfrak{p} \text{ unramified , } \frac{L/\mathbb{Q}}{\mathfrak{p}} = C, \ N\mathfrak{p} = n\}$ and let also

$$\vartheta_C(x) := \sum_{\mathfrak{p} \text{ non-ram. } N\mathfrak{p} \leq x} \varepsilon_C \left( \frac{L/\mathbb{Q}}{\mathfrak{p}} \right) \log N\mathfrak{p} = \sum_{n \leq x} a_C(n) \log n, \quad \vartheta_C^{(1)}(x) := \int_0^x \vartheta_C(t) \, dt.$$  

Then, by Lemma 4.1 for $x \geq 400$,

$$\vartheta_C^{(1)}(x) = \sum_{n \leq x} a_C(n)(x-n) \log n \leq x(\log x - \log(2 \log x)) \sum_{n \leq x} a_C(n)$$

(4.1)

$$= \pi_C(x)(x(\log x - \log(2 \log x))).$$

Now we produce a lower bound for $\vartheta_C^{(1)}(x)$ out of a lower bound for $\psi^{(1)}(C; x)$.

To ease the notation we set $g_C := |G|/|C|$ and observe that this is a positive integer. By (2.3), (3.1) and Lemma 3.2 we get

$$g_C \psi^{(1)}(C; x) = M_CI(x) = \frac{x^2}{2} \sum_{\rho \in \mathbb{Z}} e(\rho) \frac{x^{\rho+1}}{\rho(\rho+1)} xMCr_C + MCR_C + R_C(x)$$
which with the GRH assumption yields
\[ \frac{x^2}{2} - g_c \psi^{(1)}(C; x) \leq x^{3/2} \sum_{\rho \in \mathbb{H}} \frac{1}{|\rho(\rho+1)|} + xM_C r_{\chi} - M_C r'_{\chi} - R_C(x). \]

With Lemmas 13, 14, 15, this gives
\[ \frac{x^2}{2} - g_c \psi^{(1)}(C; x) \leq (0.5375(x^{3/2}+2x)+1)\log \Delta_L + x^{3/2}(5.41-1.0355n_L) \]
\[ + \left( \int_{1}^{x+1} \log u \, du - 1.571x \right) n_L + 13.276x - \int_{1}^{x+1} \log u \, du. \]

When \( x \geq 400 \) the term in \( n_L \) appearing in the last line is bounded by \( n_L(x\log x-2.55) \) and the sum of the last two terms by \( 8.3x \). Thus we have
\[ \frac{x^2}{2} - g_c \psi^{(1)}(C; x) \leq (0.5375(x^{3/2}+2x)+1)\log \Delta_L + x^{3/2}(5.41-1.0355n_L) \]
\[ + n_L x \log x - 2.4n_L x + 8x. \]

Now we remove the contribution to \( \psi^{(1)}(C; x) \) of the prime powers \( p^n \) with \( m \geq 2 \). Let
\[ \theta(C; x) := \sum_{p \leq x} \theta(C; p) \log(Np), \quad \theta^{(1)}(C; x) := \int_{0}^{x} \theta(C; t) \, dt. \]
The estimation in [13, Th. 13] gives \( 0 \leq \psi^{(1)}(C; x) - \theta^{(1)}(C; x) \leq 1.432^{2}x^{3/2}n_{K}. \) Thus
\[ \frac{x^2}{2} - g_c \theta^{(1)}(C; x) \leq (0.5375x^{3/2}+1.075x+1)\log \Delta_L \]
\[ + x^{3/2}(5.4-0.082n_L) + n_L x \log x - 2.4n_L x + 8x \]
which simplifies to
\[ (4.2) \hspace{1cm} \frac{x^2}{2} - g_c \theta^{(1)}(C; x) \leq (0.5375x^{3/2}+1.075x+1)\log \Delta_L + 2n_L x + 5.4x^{3/2} + 8x. \]

The quantities \( \theta(C; x) \) and \( \theta_C(x) \) differ only by the contribution of the ramified prime ideals to \( \theta(C; x) \). In fact,
\[ 0 \leq \theta(C; x) - \theta_C(x) \leq \sum_{p \leq x} \log Np \leq \sum_{p \text{ ram.}} \log Np \leq \log(N\Delta_{L/K}) \leq \log \Delta_L. \]

Hence,
\[ 0 \leq \theta^{(1)}(C; x) - \theta^{(1)}_C(x) \leq (x-1)\log \Delta_L, \]
which with (4.2) gives
\[ (4.3) \hspace{1cm} \frac{x^2}{2} - g_c \theta^{(1)}_{C}(x) \leq (0.5375x^{3/2}+g_c x+1.075x)\log \Delta_L + 2n_L x + 5.4x^{3/2} + 8x. \]

By (4.1) and (4.3), in order to have \( \pi_C(x) > k \) it is sufficient to have
\[ \frac{x^2}{2} > (0.5375x^{3/2}+g_c x+1.075x)\log \Delta_L + 2n_L x + 5.4x^{3/2} + 8x + k g_c x (\log x - \log(2\log x)), \]
i.e.,

\[
\sqrt{x} > \left(1.075 + \frac{2g_c + 2.15}{\sqrt{x}}\right) \log \Delta_L + \frac{n_L}{\sqrt{x}} + 10.8 + \frac{16}{\sqrt{x}} + 2k_g \frac{\log x - \log(2 \log x)}{\sqrt{x}}
\]

which is true when

\[
\sqrt{x} = 1.075 \log \Delta_L + \sqrt{2g_c k \log(g_c k) + 2g_c + 15}.
\]

**Proof.** Let

\[
A := 1.075 \log \Delta_L + 2g_c + 15
\]

and

\[
B := \left(1.075 + \frac{2g_c + 2.15}{\sqrt{x}}\right) \log \Delta_L + \frac{n_L}{\sqrt{x}} + 10.8 + \frac{16}{\sqrt{x}}.
\]

To show that (4.4) holds with the indicated value of \(x\), it is sufficient to prove

\[
A + \sqrt{2g_c k \log(g_c k)} > B + 2k_g \frac{\log x - \log(2 \log x)}{\sqrt{x}}.
\]

We have

\[
(A - B - 1) \sqrt{x} = (2g_c + 3.2) \sqrt{x} - ((2g_c + 2.15) \log \Delta_L + 4n_L + 16) \geq 1.4 \log \Delta_L - 4n_L + 72
\]

which is positive, according to entry \(b = 4\) in [11, Table 3]. Since \(A - B > 1\), our claim will hold if

\[
\sqrt{2g_c k \log(g_c k)} + 1 \geq 2g_c \frac{\log x - \log(2 \log x)}{\sqrt{x}},
\]

i.e. \(k = 0\) or

\[
\frac{\sqrt{\log y}}{\sqrt{2y}} + \frac{1}{2y} \geq \frac{\log x - \log(2 \log x)}{\sqrt{x}},
\]

where \(y := g_c k \geq 1\). The right-hand side decreases if \(x \geq 30\) hence is at most 0.2 and the left-hand side is larger than 0.2 for \(1 \leq y \leq 120\). We thus assume \(y \geq 120\), and in that case \(x \geq 2y \log y \geq 30\), hence (4.6) holds if

\[
\frac{\sqrt{\log y}}{\sqrt{2y}} \geq \frac{\log(2y \log y) - \log(2 \log(2y \log y))}{\sqrt{2y \log y}}
\]

i.e.

\[
\log y \geq \log(2y \log y) - \log(2 \log(2y \log y))
\]

which is obviously true in this range. \(\square\)

This proves the claim under the assumption that \(x \geq 400\). The exceptions to this condition are the cases where

\[
1.075 \log \Delta_L + \sqrt{2g_c k \log(g_c k)} + 2g_c + 15 < 20
\]

and this happens only when \(g_c = 1\), \(\Delta_L \leq 16\) and \(k \leq 2\).

For these remaining cases we check directly the existence of the corresponding ideals. We observe that \(g_c = |G|/|C| = 1\) if and only if \(|G| = 1\) and hence \(L = K\). Moreover, \(\Delta_L \leq 16\) implies \(n_L \leq 2\). Hence it is sufficient to check that, in quadratic fields, there are at least three ideals of norm at most \(\lfloor (1.075 \log 3 + 17)^2 \rfloor = 330\). They exist because the primes above 2, 3 and 5 have norm at most 25.
References

1. E. Bach and J. Sorenson, *Explicit bounds for primes in residue classes*, Math. Comp. 65 (1996), no. 216, 1717–1735.
2. M. Deuring, *Über den Tschebotareffschen Dichtigkeitsatz*, Math. Ann. 110 (1935), no. 1, 414–415.
3. L. Grenié and G. Molteni, *Explicit smoothed prime ideals theorems under GRH*, Math. Comp. 85 (2016), no. 300, 1875–1899.
4. L. Grenié and G. Molteni, *Explicit versions of the prime ideal theorem for Dedekind zeta functions under GRH*, Math. Comp. 85 (2016), no. 298, 889–906.
5. L. Grenié and G. Molteni, *An effective Chebotarev density theorem under GRH*, J. Number Theory 200 (2019), 441–485.
6. A. E. Ingham, *The distribution of prime numbers*, Cambridge University Press, Cambridge, 1990.
7. J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem*, Algebraic number fields: $L$-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, pp. 409–464.
8. Y. Lamzouri, X. Li, and K. Soundararajan, *Conditional bounds for the least quadratic non-residue and related problems*, Math. Comp. 84 (2015), no. 295, 2391–2412.
9. Y. Lamzouri, X. Li, and K. Soundararajan, *Corrigendum to “Conditional bounds for the least quadratic non-residue and related problems”*, Math. Comp. 86 (2017), no. 307, 2551–2554.
10. C. R. MacCluer, *A reduction of the Čebotarev density theorem to the cyclic case*, Acta Arith. 15 (1968), 45–47.
11. A. M. Odlyzko, *Discriminant bounds*, [http://www.dtc.umn.edu/~odlyzko/unpublished/index.html](http://www.dtc.umn.edu/~odlyzko/unpublished/index.html) 1976.
12. J. Oesterlé, *Versions effectives du théorème de Chebotarev sous l’hypothèse de Riemann généralisée*, Astérisque 61 (1979), 165–167.
13. J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.
14. L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II*, Math. Comp. 30 (1976), no. 134, 337–360, *Corrigendum in Math. Comp. 30* (1976), no. 136, 900.
15. B. Winckler, *Théorème de Chebotarev effectif*, [arXiv:1311.5715](http://arxiv.org/abs/1311.5715) 2013.

(L. Grenié) Dipartimento di Ingegneria Gestionale, dell’Informazione e della Produzione, Università di Bergamo, viale Marconi 5, 24044 Dalmine Italy
E-mail address: loic.grenie@gmail.com

(G. Molteni) Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italy
E-mail address: giuseppe.molteni1@unimi.it