CLASSIFICATION OF CONGRUENCES FOR MOCK Theta FUNCTIONS AND WEAKLY HOLOMORPHIC MODULAR FORMS

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Abstract. Let \( f(q) \) denote Ramanujan’s mock theta function
\[
f(q) = \sum_{n=0}^{\infty} a(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q^2)(1 + q^2)^2 \cdots (1 + q^n)^2}.
\]
It is known that there are many linear congruences for the coefficients of \( f(q) \) and other mock theta functions. We prove that if the linear congruence \( a(mn + t) \equiv 0 \pmod{\ell} \) holds for some prime \( \ell \geq 5 \), then \( \ell \mid m \) and \( (\frac{24t-1}{\ell}) \neq \left(\frac{-1}{\ell}\right) \). We prove analogous results for the mock theta function \( \omega(q) \) and for a large class of weakly holomorphic modular forms which includes \( \eta \)-quotients. This extends work of Radu [30] in which he proves a conjecture of Ahlgren and Ono for the partition function \( p(n) \).

1. Introduction and Statement of Results

In his famous last letter to Hardy, Ramanujan introduced 17 strange \( q \)-series which he called mock theta functions, many of whose coefficients encode interesting combinatorial information. A prototypical example is the function
\[
f(q) = \sum_{n=0}^{\infty} a(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q^2)(1 + q^2)^2 \cdots (1 + q^n)^2},
\]
whose coefficients are related to partition ranks (see [11]). The function \( f(q) \) resembles the generating function for partitions in the following way: by separating a partition into its Durfee square and two partitions into parts of size \( \leq n \), we have
\[
\sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 - q^2)(1 - q^2)^2 \cdots (1 - q^n)^2}.
\]
The latter generating function has many interesting arithmetic properties arising from its relation to the modular form
\[
\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi iz},
\]
via the well-known relation
\[
\frac{1}{\eta(z)} = \sum_{n=0}^{\infty} p(n)q^{n-1/24}.
\]
The arithmetic properties of \( p(n) \) have been the subject of a vast amount of research, much of which has focused on congruence properties satisfied by \( p(n) \) (for a few examples, see [2, 13, 14, 15, 21, 24]). We make particular mention of work of Ahlgren and Ono [11, 5, 27] which shows the existence of congruences
\[
p(mn + t) \equiv 0 \pmod{\ell^j}
\]
for all prime powers $\ell^j$ with $\ell \geq 5$. In their systematic theory, the progressions $\{mn + t\}$ are subprogressions of $\{\ell n + \beta\}$, where $\beta$ lies in one of $(\ell + 1)/2$ classes modulo $\ell$. In [5], they conjectured that congruences do not exist for $p(n)$ outside these progressions, and recently Radu [30] proved their conjecture. In particular, he showed that if

$$p(mn + t) \equiv 0 \pmod{\ell}$$

for some prime $\ell \geq 5$, then $\ell|m$ and $(\frac{24t-1}{\ell}) \neq (\frac{-1}{\ell})$.

Many authors have studied congruence properties of mock theta functions (see, for example, [6, 17, 18, 19, 37]). Recent work of Fuller, Friedlander, Goodson, and the author [7] shows the existence of linear congruences for all of Ramanujan’s mock theta functions. As an example, for any prime $\ell \geq 5$, the coefficients of $f(q)$ satisfy

$$a(mn + t) \equiv 0 \pmod{\ell}$$

for infinitely many progressions $\{mn + t\}$. As with $p(n)$, in each of these cases $\ell|m$ and $(\frac{24t-1}{\ell}) \neq (\frac{-1}{\ell})$. It is natural to ask whether the analogue of the conjecture in [4] holds in this case as well. Here we will prove

**Theorem 1.1.** Let $\ell \geq 5$ be prime. If for all $n$,

$$a(mn + t) \equiv 0 \pmod{\ell},$$

then $\ell|m$ and $(\frac{24t-1}{\ell}) \neq (\frac{-1}{\ell})$.

All known congruences for $f(q)$ are constructed by applying results from the theory of modular forms (see, for instance [35, 36]) to certain twists of $f(q)$ which are known to be weakly holomorphic modular forms. Theorem 1.1 implies that all linear congruences $a(mn + t) \equiv 0 \pmod{\ell}$ with $\ell \nmid (24t - 1)$ arise in this way.

**Corollary 1.2.** Suppose that $a(mn + t) \equiv 0 \pmod{\ell}$ for all $n$, and that $\ell \nmid (24t - 1)$. Then the function

$$q^{\frac{t-1}{24}\frac{1}{m}} \sum a(mn + t)q^n$$

is a weakly holomorphic modular form.

We will prove analogous statements for the mock theta function

$$\omega(q) = \sum_{n=0}^{\infty} c(n)q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{2n^2+2n}}{(1+q)^2(1+q^2)^2\cdots(1+q^{2n-1})^2}, \quad (1.2)$$

which appears naturally with $f(q)$ as a component of a vector-valued mock modular form (see, for example, [38]).

**Theorem 1.3.** Let $\ell \geq 5$ be prime. If for all $n$,

$$c(mn + t) \equiv 0 \pmod{\ell},$$

then $\ell|m$ and $(\frac{24t+2}{\ell}) \neq (\frac{-1}{\ell})$. Furthermore, if $\ell \nmid (3t + 2)$ then the function

$$q^{\frac{14+2t}{m}} \sum c(mn + t)q^n$$

is a weakly holomorphic modular form.
It is natural to ask whether these results extend to a larger class of modular forms. Treneer [35, 36] has extended the results of [4] to cover any weakly holomorphic modular form. In Section 5 we prove a general theorem which applies to the class of η-quotients, which appear as generating functions of many combinatorial objects. Using the standard notation

\[ f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}, \]

we define \( B := \sum \delta r_\delta \). If \( m \) is a positive integer, write \( m = 2^u 3^v m' \) and \( B = 2^s 3^t B' \) with \((m', 6) = (B', 6) = 1\) and define a divisor \( m_B \) of \( m \) by

\[ m_B := 2^{\min(r,u)} 3^{\min(s,v)} m'. \] (1.3)

The following theorem is an analogue of Theorems 1.1 and 1.3 for η-quotients.

**Theorem 1.4.** Let \( \ell \geq 5 \) be prime. Suppose that \( f(z) = q^{B/24} \sum a_f(n) q^n \) is an η-quotient as above with \( B < 0 \) and \( \ell \nmid BN \). If \( m \) and \( t \) are positive integers with \((m_B, N) = 1\) and

\[ a_f(mn + t) \equiv 0 \pmod{\ell} \]

for all \( n \), then \( \ell | m \) and \((\frac{24t + B}{\ell}) \neq (\frac{24t}{\ell})\).

We give some applications of Theorem 1.4.

**Example 1.** Let \( p_k(n) \) denote the number of partitions of \( n \) into \( k \) colors. Then we have the generating function

\[ \sum_{n=0}^{\infty} p_k(n) q^{n-k/24} = \eta^{-k}(z). \]

In [10], Andrews showed that if \( \ell \geq 5 \) is prime and \((\frac{8t + 1}{\ell}) \neq 1\) then

\[ p_{\ell-3}(\ell n + t) \equiv 0 \pmod{\ell}. \] (1.4)

Applying Theorem 1.4 with \( B = 3 - \ell \) we obtain the necessary condition \((\frac{24t + 3 - \ell}{\ell}) \neq (\frac{3 - \ell}{\ell})\), which is equivalent to \((\frac{2t + 1}{\ell}) \neq 1\). So we have

**Corollary 1.5.** Let \( \ell \geq 5 \) be prime. Then

\[ p_{\ell-3}(\ell n + t) \equiv 0 \pmod{\ell} \]

if and only if \((\frac{2t + 1}{\ell}) \neq 1\).

**Example 2.** Broken 1-diamond partitions are defined by the generating function

\[ \sum_{n=0}^{\infty} \Delta_1(n) q^{n-1/6} = \frac{\eta(2z)\eta(3z)}{\eta^3(z)\eta(6z)}. \]

In [25], Mortenson showed that if \( \ell \) is a prime satisfying

\[ \ell \equiv 1, 25, 37, 47, 59, \text{ or } 83 \pmod{84} \]

then there exist infinitely many progressions \( \{mn + t\} \subseteq \{\ell n + \beta\} \) for each \( \beta \) satisfying \((\frac{6\beta - 1}{\ell}) \neq (\frac{-1}{\ell})\) such that

\[ \Delta_1(mn + t) \equiv 0 \pmod{\ell}. \]

Theorem 1.4 shows that any such congruence with \( m \) odd must satisfy \( \{mn + t\} \subseteq \{\ell n + \beta\} \) for some \( \beta \) satisfying \((\frac{6\beta - 1}{\ell}) \neq (\frac{-1}{\ell})\).
Example 3. The Andrews-Stanley partition function \( t(n) \) (see [9, 32, 33]) is defined as the number of partitions \( \pi \) having the property that the number of odd parts of \( \pi \) is congruent to the number of odd parts of the conjugate partition \( \pi' \) modulo 4. The generating function for \( t(n) \) is

\[
\sum_{n=0}^{\infty} t(n) q^{n - 1/24} = \frac{\eta^2(2z) \eta^5(16z)}{\eta(z) \eta^5(4z) \eta^2(32z)}.
\]

In [34], Swisher proved that for each prime \( \ell \geq 5 \) there are infinitely many progressions \( \{mn + t\} \) for which \( t(n) \) and \( p(n) \) satisfy the simultaneous congruences

\[
t(mn + t) \equiv p(mn + t) \equiv 0 \pmod{\ell}.
\]

Theorem 1.4 shows that any linear congruence

\[
t(mn + t) \equiv 0 \pmod{\ell}
\]

must satisfy \( \ell | m \) and \( \left( \frac{2m - 1}{\ell} \right) \neq \left( \frac{1}{\ell} \right) \), mirroring the \( p(n) \) case.

Example 4. The assumption \( \ell \nmid NB \) is necessary in general. Define a sequence \( a_\ell(n) \) of integers modulo \( \ell \) by the relation

\[
\sum_{n=0}^{\infty} a_\ell(n) q^{n - \ell / 24} := \eta^{-1}(\ell z) \equiv \eta^{-\ell}(z) \pmod{\ell}.
\]

Since \( \eta^{-1}(\ell z) \) and \( \eta^{-\ell}(z) \pmod{\ell} \) are supported only on exponents divisible by \( \ell \), the sequence \( a_\ell(n) \) satisfies \( a_\ell(\ell n + t) \equiv 0 \pmod{\ell} \) for all \( t \) coprime to \( \ell \).

The assumption \( (m_B, N) = 1 \) is also necessary. We give examples illustrating two cases when \( (m, N) > 1 \). Let \( \delta > 1 \) and define

\[
\sum_{n=0}^{\infty} b_3(n) q^{n - \delta / 24} := \eta^{-1}(\delta z).
\]

Then \( b_3(n) = 0 \) unless \( \delta | n \), so \( b_3(\delta n + t) = 0 \) for all \( t \) not divisible by \( \delta \). Here \( B = -\delta \) and \( N = \delta \), so \( (m_B, N) = (m, N) = \delta \). Now consider the sequence \( c_3(n) \) defined by

\[
\sum_{n=0}^{\infty} c_3(n) q^{n - 1/6} := \eta^{-1}(z) \eta^{-1}(3z).
\]

The sequence \( c_3(n) \) does not vanish trivially on any arithmetic progression as does \( b_3(n) \). Here \( B = -4 \) and \( N = 3 \), so \( (m_B, N) = 1 \). In this case \( m \) is allowed to be a multiple of 3.

2. Preliminaries

Ramanujan’s mock theta functions are examples of weight 1/2 mock modular forms, which are the holomorphic parts of harmonic Maass forms (see Sections 6 and 7 of [29] for definitions and details). Each harmonic Maass form \( F \) decomposes uniquely as \( F = f + NH \), where \( f \) is the holomorphic part (or mock modular form) and \( NH \) is the non-holomorphic part.

Define

\[
g_0(z) := \sum_{n \in \mathbb{Z}} (-1)^n (n + 1/3) q^{\frac{3}{2}(n + \frac{1}{3})^2},
\]

\[
g_1(z) := -\sum_{n \in \mathbb{Z}} (n + 1/6) q^{\frac{3}{2}(n + \frac{1}{3})^2}.
\]
From [23, Lemma 2(i)] we have (correcting a sign error),

\[
M(z) := q^{-1/24} f(q) \left( -2i \sqrt{3} \int_{-\tau}^{i\infty} \frac{g_1(\tau)}{(-i(z+\tau))^{1/2}} \, d\tau \right) \quad (2.1)
\]

and

\[
\Omega(z) := 2q^{2/3} \omega(q) \left( -2i \sqrt{3} \int_{-\tau}^{i\infty} \frac{g_0(\tau)}{(-i(2z+\tau))^{1/2}} \, d\tau \right) \quad (2.2)
\]

are the completed harmonic Maass forms associated to \( f(q) \) and \( \omega(q) \), respectively. These functions satisfy the following transformation laws (see [8, Theorems 2.1–2.4], [16, Proof of Corollary 2.3], [3, (3.13) and (4.2)]). For \( A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(2) \) with \( c > 0 \) we have

\[
M \left( \frac{az+b}{cz+d} \right) = w(A) (cz+d)^{1/2} M(z),
\]

(2.3)

where \( w(A) \) is the 24th root of unity given by

\[
w(A) := i^{-\frac{1}{2}} (-1) \frac{c+1+sd}{2} \exp \left( 2\pi i \left( \frac{1}{2} s(-d,c) \frac{a+d}{24} - \frac{a}{4} + \frac{3cd}{8} \right) \right).
\]

(2.4)

and \( s(d,c) \) is the Dedekind sum defined by

\[
s(d,c) := \sum_{r=1}^{c-1} \left( \frac{r}{c} - \left[ \frac{r}{c} \right] - \frac{1}{2} \right) \left( \frac{dr}{c} - \left[ \frac{dr}{c} \right] - \frac{1}{2} \right).
\]

(2.5)

For \( A = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \) with \( c > 0 \) we have

\[
\Omega(Az) = \begin{cases} w_1(A)(cz+d)^{1/2} \Omega(z) & \text{if } c \text{ is even}, \\ w_2(A)(cz+d)^{1/2} M(z/2) & \text{if } d \text{ is even}, \end{cases}
\]

(2.6)

where \( w_1(A) \) and \( w_2(A) \) are roots of unity defined by

\[
w_1(A) := i^{1/2} \exp \left( 2\pi i \left( \frac{a-1}{4} - \frac{s(-d,c/2)}{2} + \frac{3ab}{4} - \frac{a+d}{12c} \right) \right),
\]

\[
w_2(A) := (-i)^{1/2} \exp \left( 2\pi i \left( \frac{32a-d}{48c} - \frac{s(-d/2,c)}{2} - \frac{2a+b-3-3ab+3a/c}{4} \right) \right).
\]

(2.7)

The Dedekind sum \([2.5]\) satisfies the following transformation law which is an easy consequence of [\text{[23, Lemma 2]}].

**Lemma 2.1.** Let \( m, \lambda \) be positive integers with \( (m, 6) = 1 \). Then for every \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(m) \) with \( c > 0 \) and \( (a, 6) = 1 \) we have

\[
s(-d + \lambda c, mc) = s(-d, mc) + \lambda \frac{1-a^2}{12m} \text{ an even integer}.
\]

(2.8)

**Proof.** From [\text{[23, Lemma 2(i)]}] we have (correcting a sign error),

\[
s(d + c, mc) = s(d, mc) + \frac{1-a^2}{12m} \text{ an even integer}.
\]

Applying this iteratively to the matrices

\[
\left( \begin{array}{cc} -a & b - ja \\ c & -d + jc \end{array} \right) \in \Gamma_0(m), \quad 0 \leq j \leq \lambda - 1,
\]

we have

\[
s(-d + \lambda c, mc) = s(-d, mc) + \lambda \frac{1-a^2}{12m} \text{ an even integer}.
\]

(2.9)

**Proof.** From [\text{[23, Lemma 2(i)]}] we have (correcting a sign error),

\[
s(d + c, mc) = s(d, mc) + \frac{1-a^2}{12m} \text{ an even integer}.
\]

Applying this iteratively to the matrices

\[
\left( \begin{array}{cc} -a & b - ja \\ c & -d + jc \end{array} \right) \in \Gamma_0(m), \quad 0 \leq j \leq \lambda - 1,
\]

we have

\[
s(-d + \lambda c, mc) = s(-d, mc) + \lambda \frac{1-a^2}{12m} \text{ an even integer}.
\]
we obtain (2.8). □

We recall the transformation law for \( \eta(z) \) (see [22, Chapter 4, Theorem 2]). If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) with \( c > 0 \) we have

\[
\eta \left( \frac{az + b}{cz + d} \right) = \xi(A)(cz + d)^{1/2} \eta(z),
\]

where

\[
\xi(A) = \begin{cases} 
\left( \frac{c}{d} \right) \exp \left( \frac{2\pi i}{24} [(a + d)c - bd(c^2 - 1) + 3d - 3cd] \right) & \text{if } c \text{ is even}, \\
\left( \frac{d}{c} \right) \exp \left( \frac{2\pi i}{24} [(a + d)c - bd(c^2 - 1) - 3d] \right) & \text{if } c \text{ is odd}.
\end{cases}
\]

(2.10)

The proofs below adapt Radu’s methods in [30, 31] and require the following technical lemma which is a generalization of [31, Theorem 4.2]. Given an integer \( B = 2^r 3^s B' \) and a positive integer \( m = 2^u 3^v m' \) with \( (B', 6) = (m', 6) = 1 \), define \( Q_{m,B} := 2^a 3^b m' \), where

\[
\alpha := \begin{cases} 
0 & \text{if } r = 0, \\
\min(r, u) & \text{if } r = 1, 2, \\
u & \text{if } r \geq 3,
\end{cases}
\quad \beta := \begin{cases} 
0 & \text{if } s = 0, \\
v & \text{if } s \geq 1.
\end{cases}
\]

(2.11)

Note that the primes dividing \( Q_{m,B} \) are the same as those dividing \( m_B \) defined in (1.3).

**Lemma 2.2.** Let \( m, t, B, N \in \mathbb{Z} \) with \( m, N > 0 \) and write \( m = 2^u 3^v m' \) with \( (m', 6) = 1 \). Let \( \alpha, \beta, \) and \( Q_{m,B} \) be as above. Suppose that \( \lambda \) is an integer with \( 0 \leq \lambda < m/Q_{m,B} \). Then there exists an integer \( a = a_\lambda \) with \( (a, 6mN) = 1 \) and

\[
t + \lambda Q_{m,B} \equiv ta^2 + B \frac{q^2 - 1}{24} \pmod{m}.
\]

(2.12)

**Proof.** Write \( Q = Q_{m,B} \) for convenience. We will construct integers \( b_0, \ldots, b_{u-\alpha} \) with \( (b_n, 6mN) = 1 \) and

\[
t + \lambda Q \equiv tb_n^2 + B \frac{b_n^2 - 1}{24} \pmod{2^n Q},
\]

and integers \( c_0, \ldots, c_{v-\beta} \) with \( (c_n, 6mN) = 1 \) and

\[
t + \lambda Q \equiv tc_n^2 + B \frac{c_n^2 - 1}{24} \pmod{2^n(a-3)n Q}.
\]

Then (2.12) holds with \( a = c_{v-\beta} \) since \( m = 2^{n-a-3} n \).

Let \( N' \) denote the largest divisor of \( N \) coprime to 6. We begin by constructing the integers \( b_n \). Let \( b_0 := 1 \). If \( \alpha = u \), we are done, so assume \( \alpha < u \). Then either \( \alpha = r = 0 \) or \( \alpha = \min(r, u) = r \). In either case, \( 2^r || B \). For \( n \geq 1 \), let \( x_{n-1} \) be the unique integer satisfying

\[
B \frac{b_n^2 - 1}{24} + t(b_{n-1}^2 - 1) - \lambda Q = x_{n-1} 2^{n-1} Q
\]

and define \( b_n := b_{n-1} + 2^{n+1-a} 3x_{n-1} Q N' \). Then since \( 2^n | Q \), we have

\[
B(b_n^2 - 1) \equiv B(b_{n-1}^2 - 1) + 2^{n+2-a} 3x_{n-1} Q B N' b_{n-1} \pmod{24 \cdot 2^n Q}.
\]
Since 2 \alpha BN'bn_1 is odd.

To construct the integers c_n, set \( c_0 := bu_\alpha \). If \( \beta = v \), we are done. If \( \beta < v \) then \( \beta = 0 \) and \( 3 \mid B \). In this case let \( y_{n-1} \) be the unique integer satisfying
\[
B \frac{c^2_{n-1} - 1}{24} + t(c_{n-1}^2 - 1) - \lambda Q = y_{n-1}2^{u_\alpha - \alpha}3^{n-1}Q
\]
and define \( c_n := c_{n-1}(2^{2+u_\alpha - \alpha}3^n y_{n-1}QN' - \epsilon) \), where \( \epsilon \in \{-1,1\} \) and \( \epsilon \equiv BN' \) (mod 3). Then
\[
B \frac{c^2_n - 1}{24} + t(c_n^2 - 1) - \lambda Q
\]
\[
\equiv B \frac{c^2_{n-1} - 1}{24} + t(c_{n-1}^2 - 1) - \lambda Q - \epsilon 2^{u_\alpha - \alpha}3^{n-1}y_{n-1}QBN'c_{n-1}^2 \quad \text{(mod } 2^{u_\alpha - \alpha}3^nQ)\]
\[
\equiv 2^{u_\alpha - \alpha}3^{n-1}y_{n-1}Q(1 - \epsilon BN'c_{n-1}^2) \quad \text{(mod } 2^{u_\alpha - \alpha}3^nQ)\]
\[
\equiv 0 \quad \text{(mod } 2^{u_\alpha - \alpha}3^nQ)\]
since \( \epsilon BN'c_{n-1}^2 \equiv \epsilon^2 \equiv 1 \) (mod 3).

We require two theorems of Deligne and Rapoport [20], which relate the expansion of a modular form at \( \infty \) with its expansions at other cusps. For a prime ideal \( \pi \) and a modular form \( f \) with \( \pi \)-integral coefficients, let \( v_{\pi}(f) \) denote the \( \pi \)-adic valuation of \( f \). Let \( \zeta_m := \exp(2\pi i/m) \).

**Theorem 2.3.** [20] VII, Cor 3.12 \( \text{Let } f \in M_k(\Gamma(N)) \cap \mathbb{Z}[\zeta_N][q], p \text{ a prime, and } \gamma \in \Gamma_0(p^m), \text{ where } p^m \mid |N|. \text{ Let } \pi \text{ be a prime of } \mathbb{Z}[\zeta_N] \text{ lying above } p. \text{ Then} \]
\[
v_{\pi}(f) = v_{\pi}(f|k\gamma).
\]

**Theorem 2.4.** [20] VII, Cor 3.13 \( \text{Let } f \in M_k(\Gamma(N)) \cap \mathbb{Z}[1/N, \zeta_N][q], \gamma \in \text{SL}_2(\mathbb{Z}). \text{ Then} \]
\[
f|k\gamma \in \mathbb{Z}[1/N, \zeta_N][q].
\]

3. **Proof of Theorem 1.1**

Define
\[
M_{m,t}(z) := \frac{1}{m} \sum_{\lambda=0}^{m-1} \zeta_m^{-\lambda(t-1/24)} M \left( \frac{z + \lambda}{m} \right) \quad (3.1)
\]
In order to work with the function \( M_{m,t}(z) \), we need to isolate progressions on which the non-holomorphic part of \( M_{m,t}(z) \) vanishes. We call a progression \( t \) (mod \( m \)) good if there exists a prime \( p|m \) with \( (\frac{-24t}{p}) = -1 \). By [3], if \( t \) (mod \( m \)) is good, then \( M_{m,t} \) is a weakly holomorphic modular form. Thus Corollary 1.2 follows immediately from Theorem 1.1 and (3.2) below.
Suppose \(a(mn + t) \equiv 0 \pmod{\ell}\) for some progression \(t \pmod{m}\) that is not good. Choose a prime \(p \geq 5\) with \(p \nmid \ell m\) and a quadratic non-residue \(x \pmod{p}\), and choose \(T\) satisfying
\[
T \equiv t \pmod{m},
\]
\[
T \equiv \frac{1-x}{24} \pmod{p}.
\]
Then the subprogression \(T \pmod{mp}\) is good. Note that the statements \(\ell | mp\) and \((\frac{24T-1}{\ell}) \neq (-\frac{1}{\ell})\) together imply that \(\ell | m\) and \((\frac{24t-1}{\ell}) \neq (-\frac{1}{\ell})\). Therefore to prove Theorem 1.1 we may assume that \(t \pmod{m}\) is a good progression. A calculation shows that if \(t \pmod{m}\) is good, then
\[
M_{m,t}(z) = q^{\frac{t-1}{24m}} \sum a(mn + t)q^n. \tag{3.2}
\]
We will need a proposition from [3] which describes the transformation of \(M_{m,t}\) under the action of \(\Gamma_0(N_m)\), where
\[
N_m := \begin{cases} 
2m & \text{if } (m, 6) = 1, \\
8m & \text{if } (m, 6) = 2, \\
6m & \text{if } (m, 6) = 3, \\
24m & \text{if } (m, 6) = 6.
\end{cases}
\]
If \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)\) has \(3 \nmid a\) then we define \(t_A\) to be any integer satisfying
\[
t_A \equiv ta^2 + \frac{1-a^2}{24} \pmod{m}. \tag{3.3}
\]

**Proposition 3.1.** For every \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)\) with \(3 \nmid a\) we have
\[
M_{24m, m,t} \mid_{12m} A = M_{24m, m,t_A}.
\]

The matrices in \(\Gamma_0(N_m)\) with \(3 \nmid a\) generate \(\Gamma_1(N_m)\). Therefore we have

**Proposition 3.2.** Suppose that \(t \pmod{m}\) is good. Then there exists \(s \in \mathbb{N}\) such that
\[
\Delta^s M_{m,t}^{24m} \in M_{12(m+s)}(\Gamma_1(N_m)). \tag{3.4}
\]

A useful consequence of these propositions is the following lemma.

**Lemma 3.3.** Suppose that \(t \pmod{m}\) is good. Let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_m)\) with \(3 \nmid a\) and let \(t_A\) be as in (3.3). If \(M_{m,t} \equiv 0 \pmod{\ell}\) then \(M_{m,t_A} \equiv 0 \pmod{\ell}\).

**Proof.** Let \(s\) be as in (3.4). Then by Proposition 3.1 we have
\[
\Delta^s M_{m,t}^{24m} \mid_{12(m+s)} A = \Delta^s M_{m,t_A}^{24m}.
\]
Applying Theorem 2.3 we conclude that
\[
M_{m,t_A} \equiv 0 \pmod{\ell}. \qedhere
\]

The following proposition proves the assertion in Theorem 1.1 that \(\ell | m\) and reduces us to the case \(m = Q\ell\), where \((Q, 6\ell) = 1\).

**Proposition 3.4.** Suppose that \(t \pmod{m}\) is good and write \(m = 2^u3^w\ell Q\) with \((Q, 6\ell) = 1\). If \(M_{m,t} \equiv 0 \pmod{\ell}\) then \(j > 0\). If additionally \(\ell \nmid (24t - 1)\) then \(M_{Q\ell,t} \equiv 0 \pmod{\ell}\).
Proposition 3.5. We begin by proving that $M_{Q^3, t} \equiv 0 \pmod{\ell}$. Applying Lemma 2.2 with $B = -1$ and $N = 1$ we see that as $a$ ranges over integers with $(a, 6m) = 1$, the quantity $t_a$ covers each of the progressions $t + \lambda Q^3 \pmod{m}$ for $0 \leq \lambda < 2^a 3^v$. This fact and Lemma 3.3 together imply that

$$M_{m, t + \lambda Q^3} \equiv 0 \pmod{\ell}$$

for $0 \leq \lambda < 2^a 3^v$. We have

$$\bigcup_{\lambda=0}^{2^a 3^v-1} \{mn + (t + \lambda Q^3)\} = \bigcup_{\lambda=0}^{2^a 3^v-1} \{Q^3(2^a 3^v n + \lambda) + t\} = \{Q^3 n + t\},$$

since every integer can be written as $2^a 3^v n + \lambda$ for some $\lambda$ with $0 \leq \lambda < 2^a 3^v$. Therefore $M_{Q^3, t} \equiv 0 \pmod{\ell}$.

We will now show that $j > 0$. Suppose, by way of contradiction, that $j = 0$, so that $M_{Q^3, t} = M_{Q, t}$. For a sufficiently large $s \in \mathbb{N}$, define

$$g := \Delta^s M_{Q, t}^2 \in M_{12(Q + s)}(\Gamma_1(2Q)).$$

Then by Proposition 9, the leading term in the $q$-expansion of $g$ at the cusp $1/2$ is given by

$$g|_{12(Q + s)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = Q^{-12Q} q^{-2^a 3^v + \ldots}.$$

By assumption, $M_{Q, t} \equiv 0 \pmod{\ell}$, so the modular form $\ell^{-24Q} g$ has integral coefficients. But by Theorem 2.41 the coefficients of

$$\ell^{-24Q} g|_{12(Q + s)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

lie in $\mathbb{Z}[1/2Q, \zeta_{2Q}]$, which contradicts $\ell \nmid Q$.

We now show that $M_{Q, t} \equiv 0 \pmod{\ell}$ under the assumption $\ell \nmid (24t - 1)$. By Lemma 4.6, for each $r$ with $0 \leq r < \ell - 1$ there exists an integer $a_r$ with $(a_r, 6Q\ell) = 1$ such that

$$a_r^2 (24t - 1) \equiv 24(t + Q\ell r) - 1 \pmod{Q^3}.$$

Therefore $t + Q\ell r = t_a$ for some $A \in \Gamma_0(2Q\ell)$ as in (3.3), so by Lemma 3.3 we have $M_{Q^3, t + Q\ell} \equiv 0 \pmod{\ell}$. Since

$$\bigcup_{r=0}^{\ell - 1} \{Q^3 n + (t + Q\ell r)\} = \bigcup_{r=0}^{\ell - 1} \{Q^3 \ell n + r + t\} = \{Q^3 n + t\},$$

we conclude that $M_{Q^3, t} \equiv 0 \pmod{\ell}$.

We will use the following proposition to prove that $\left(\frac{24t - 1}{\ell}\right) \neq \left(\frac{-1}{\ell}\right)$.

**Proposition 3.5.** If $(Q, 6\ell) = 1$ and $t \equiv (1 - Q^2)/24 \pmod{\ell}$ then

$$M_{Q^3, t}^{48Q^3} \Delta^s|_{24Q^3 + 12s} \begin{pmatrix} 1 & 0 \\ 2\ell & 1 \end{pmatrix} = Q^{-24Q\ell^2} q^{-2Q^2 \ell + s + \ldots}.$$ (3.5)

Assume for the moment that the proposition is true and suppose, by way of contradiction, that $\left(\frac{24t - 1}{\ell}\right) = \left(\frac{-1}{\ell}\right)$. We will construct an integer $t'$ satisfying both $t' = t_a$ for some $A \in \Gamma_0(2Q\ell)$ and $t' \equiv (1 - Q^2)/24 \pmod{\ell}$. Let $\alpha$ be an integer satisfying

$$(24t - 1)\alpha^2 \equiv -1 \pmod{\ell}.$$
Since \( \ell \nmid \alpha Q \), there exists an integer \( a \) with \( (a, 6Q\ell) = 1 \) such that 
\[
a \equiv Q\alpha \pmod{\ell}.
\]
Let \( t' \) be any integer satisfying
\[
a^2(24t - 1) \equiv (24t' - 1) \pmod{Q\ell}.
\]
Then \( t' = t_A \) for some \( A \in \Gamma_0(2Q\ell) \) as in (3.3) and
\[
t' \equiv \frac{1 - Q^2}{24} \pmod{\ell}.
\] (3.6)
By Proposition 3.4 and Lemma 3.3 we have
\[
M_{Q\ell,t'} \equiv 0 \pmod{\ell}.
\] (3.7)
For \( s \) sufficiently large, define
\[
g' := M_{Q\ell,t'}^{48Q\ell^2} \Delta^s \in M_{24Q\ell^2+12s}(\Gamma_1(2Q\ell)).
\]
We have \( g' \equiv 0 \pmod{\ell} \), so by Theorem 2.3 we have
\[
g'|_{24Q\ell^2+12s} \left( \frac{1}{2\ell} \begin{array}{c} 0 \\ 1 \end{array} \right) \equiv 0 \pmod{\ell},
\]
but this contradicts Proposition 3.5.

Proof of Proposition 3.5. We compute the leading coefficient of \( M_{Q\ell,t} \) at the cusp \( 1/2\ell \). By (3.1) we have
\[
M_{Q\ell,t} \left( \frac{1}{2\ell} \begin{array}{c} 0 \\ 1 \end{array} z \right) = \frac{1}{Q\ell} \sum_{\lambda=0}^{Q\ell-1} \zeta_{Q\ell}^{-\lambda(t-1/24)} M \left( \frac{1}{\lambda} \begin{array}{c} 1 \\ Q\ell \end{array} \left( \frac{1}{2\ell} \begin{array}{c} 0 \\ 1 \end{array} z \right) \right)
\]
\[
= \frac{1}{Q\ell} \sum_{\lambda=0}^{Q\ell-1} \zeta_{Q\ell}^{-\lambda(t-1/24)} M \left( A_{\lambda} \left( \frac{d_{\lambda}}{0} \begin{array}{c} \lambda' \\ Q\ell/d_{\lambda} \end{array} z \right) \right)
\]
where \( d_{\lambda} := (1 + 2\ell\lambda, Q) \), the integer \( \lambda' \) satisfies the congruence
\[
\frac{1 + 2\ell\lambda}{d_{\lambda}} \lambda' \equiv \lambda \pmod{Q\ell/d_{\lambda}},
\]
and \( A_{\lambda} \) is the matrix
\[
A_{\lambda} := \left( \begin{array}{cc} 1 + 2\ell\lambda/d_{\lambda} & -1+2\ell\lambda/Q\ell/d_{\lambda} \\ 2\ell^2Q/d_{\lambda} & -2\ell\lambda' + d_{\lambda} \end{array} \right) \in \Gamma_0(2).
\]
Recall the transformation law for \( M(z) \) in (2.3). Since \( M(z) = q^{-1/24} + \ldots \), the leading term of
\[
(2\ell z + 1)^{-1/2} M_{Q\ell,t} \left( \frac{1}{2\ell} \begin{array}{c} 0 \\ 1 \end{array} z \right)
\]
arises from those \( \lambda \) for which \( d_{\lambda} = Q \). For these \( \lambda \) we can take \( \lambda' = Q\lambda \). The leading coefficient is
\[
K = \frac{1}{Q\ell} \sum_{d_{\lambda}=Q} \zeta_{Q\ell}^{-\lambda(t-1/24)} \omega(A_{\lambda}) Q^{1/2} e^{-2\pi i Q\lambda/24\ell}.
\]
We must show that $K^{48Q^{\ell^2}} = Q^{-24Q^{\ell^2}}$, so it is enough to compute $K$ up to a $48Q^{\ell^2}$-th root of unity. To this end, we will factor out any terms in the exponent of $\omega(A_\lambda)$ which are independent of $\lambda$ and collect them in roots of unity denoted $\omega_1, \omega_2, \text{etc.}$ If $A_\lambda = (a \ b \ c \ d)$, note that

$$\frac{c + 1 + ad}{2} = \ell^2 + 1 - 2\ell^2 \lambda^2 \equiv 0 \pmod{2}$$

and

$$- \frac{a + d}{24c} - \frac{a}{4} + \frac{3dc}{8} - \frac{Q\lambda}{24\ell} = \frac{Q(36\ell^4 - 1)}{48\ell^2} - \frac{(1 + 2\ell\lambda)(12\ell^2 + 1)}{48\ell^2} \frac{3}{2}\ell^3 Q\lambda$$

$$= -\lambda\frac{(12\ell^2 + 1)}{24Q\ell} - \frac{\lambda}{2} + \frac{3Q^2\ell^2 - 1}{4Q} \frac{Q^2 - 1}{48Q\ell^2} + \lambda\left(\frac{1 - 3Q\ell^3}{2}\right) \text{ independent of } \lambda \text{ an integer}.$$ 

Choose $\alpha \in \mathbb{Z}$ so that $\alpha Q \equiv 1 \pmod{2\ell}$ and $3 \nmid \alpha$. Then, applying Lemma 2.1 to the matrix $(\alpha \ 0 \ \delta \ \lambda) \in \Gamma_0(\ell)$, we have

$$s(-Q + 2\ell\lambda, 2\ell^2) = s(-Q, 2\ell^2) + \ell\frac{1 - \alpha^2}{12\ell} + \text{an even integer.}$$

Since $\lambda\alpha \equiv \lambda \pmod{\ell}$ and $1 - Q^2\alpha^2 \equiv 0 \pmod{24\ell}$, we have

$$\lambda\frac{1 - \alpha^2}{24\ell} = \lambda\frac{1 - \alpha^2Q^2}{24\ell} - \ell\frac{\lambda\alpha^2}{\ell} \left(\frac{1 - Q^2}{24}\right)$$

$$= -\frac{\lambda\alpha}{\ell} \left(\frac{1 - Q^2}{24}\right) + \text{an integer} \ .$$

Since $6c s(-d, c) \in \mathbb{Z}$ (see [12, Theorem 3.8]), the term $e^{\pi is(-Q, 2\ell^2)}$ is a $24\ell^2$-th root of unity. Therefore

$$K = \frac{\omega_1}{Q^{1/2\ell}} \sum_{d_\lambda = Q} \exp \left(-2\pi i \left(\frac{\lambda(t - 1/24)}{Q\ell} - \frac{\lambda\alpha}{\ell} \left(\frac{1 - Q^2}{24}\right) + \frac{\lambda}{2} + \lambda\left(\frac{12\ell^2 + 1}{24Q\ell^2}\right)\right)\right).$$

Let $\lambda_0$ be the smallest $\lambda$ for which $d_\lambda = Q$. Then each $\lambda$ is of the form $\lambda = \lambda_0 + \delta\ell$ for some $1 \leq \delta \leq \ell - 1$. Factoring out those terms which do not depend on $\delta$ we obtain

$$K = \frac{\omega_2}{Q^{1/2\ell}} \sum_{\delta = 0}^{\ell - 1} \exp \left(-2\pi i \delta \left(\frac{t}{\ell} - \frac{Q\alpha}{\ell} \left(\frac{1 - Q^2}{24}\right) + \frac{Q + \ell}{2}\right)\right).$$

Note that $(Q + \ell)/2$ is an integer. Since $Q\alpha \equiv 1 \pmod{\ell}$ and $(1 - Q^2)/24$ is an integer, we obtain

$$K = \frac{\omega_2}{Q^{1/2\ell}} \sum_{\delta = 0}^{\ell - 1} \exp \left(-2\pi i \delta \left(\frac{t}{\ell} - \frac{1 - Q^2}{24}\right)\right).$$

By assumption, $t \equiv (1 - Q^2)/24 \pmod{\ell}$ so each term in the sum is 1. Therefore

$$K^{48Q^{\ell^2}} = Q^{-24Q^{\ell^2}},$$

as desired. \qed
4. Proof of Theorem 1.3

Define

$$
\Omega_{m,t}(z) := \frac{1}{m} \sum_{\lambda=0}^{m-1} \zeta_m^{-\lambda(t+2/3)} \Omega \left( \frac{z + \lambda}{m} \right).
$$

(4.1)

In this case we call a progression \( t \mod m \) good if for some \( p|m \) we have \( \left( \frac{-3t-2}{p} \right) = -1 \). By [3], if \( t \mod m \) is good, \( \Omega_{m,t} \) is weakly holomorphic. If \( t \mod m \) is not good, we can reduce to the case of a good progression \( T \mod mp \) as before, so to prove Theorem 1.3 we may assume that \( t \mod m \) is a good progression. A calculation shows that if \( t \mod m \) is good then

$$
\Omega_{m,t} = \frac{q^{t+2/3}}{m} \sum c(mn + t)q^n.
$$

Thus, as in the \( f(q) \) case, the second statement in Theorem 1.3 follows from the first statement.

We will need results from [3] which describe the transformation of \( \Omega_{m,t} \) under the action of \( \Gamma_0(2N_m) \). If \( A = (a \ b \ c \ d) \in \Gamma_0(2N_m) \) has \( 3 \nmid a \) then we define \( t_A \) to be any integer satisfying

$$
t_A \equiv ta^2 + \frac{2}{3}(a^2 - 1) \pmod m.
$$

(4.2)

Proposition 4.1. For every \( A = (a \ b \ c \ d) \in \Gamma_0(2N_m) \) with \( 3 \nmid a \) we have

$$
\Omega_{m,t}^{24m} \mid_{12m} A = \Omega_{m,t_A}^{24m}.
$$

Proposition 4.2. Suppose that \( t \mod m \) is good. Then there exists \( s \in \mathbb{N} \) such that

$$
\Delta^s \Omega_{m,t}^{24m} \in M_{12(m+s)}(\Gamma_1(2N_m)).
$$

We will need an analogue of Lemma 3.3 which can be proved similarly.

Lemma 4.3. Suppose that \( t \mod m \) is good. Let \( A = (a \ b \ c \ d) \in \Gamma_0(2N_m) \) with \( 3 \nmid a \), and let \( t_A \) be as in (4.2). If \( \Omega_{m,t} \equiv 0 \pmod m \) then \( \Omega_{m,t_A} \equiv 0 \pmod \ell \).

We now show that \( \ell|m \) and reduce to the case \( m = Q\ell \) with \( (Q, 3\ell) = 1 \).

Proposition 4.4. Suppose that \( t \mod m \) is good and write \( m = 3^s\ell^jQ \) with \( (Q, 3\ell) = 1 \). If \( \Omega_{m,t} \equiv 0 \pmod \ell \) then \( j > 0 \). If additionally \( \ell \nmid (3t + 2) \) then \( \Omega_{Q\ell,t} \equiv 0 \pmod \ell \).

Proof. Applying Lemma 2.2 with \( B = 16 \) and \( N = 1 \) we see that as \( a \) ranges over integers with \( (a, 6m) = 1 \) the quantity \( t_A \) covers each of the progressions \( t + \lambda Q\ell^j \pmod m \), for \( 0 \leq \lambda < 3^s \). So, as in the proof of Proposition 3.2, we can conclude that \( \Omega_{Q\ell,t} \equiv 0 \pmod \ell \).

To show that \( j > 0 \), suppose by way of contradiction that \( j = 0 \), and define

$$
h := \Delta^s \Omega_{Q\ell,t}^{24Q} \in M_{12(Q+s)}(\Gamma_1(2Q)).
$$

Then by [3] Proposition 12, we have

$$
h \mid_{12(Q+s)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (-1)^Q(2Q)^{-12Q}q^{-Q^2/2+s} + \ldots.
$$

By assumption, \( \Omega_{Q\ell,t} \equiv 0 \pmod \ell \), so the modular form \( \ell^{-24Q}h \) has integral coefficients. By Theorem 2.4, the coefficients of

$$
\ell^{-24Q}h \mid_{12(Q+s)} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
$$
lie in $\mathbb{Z}[1/4Q, \zeta_{4Q}]$, which contradicts $\ell \mid Q$.

We show that $\Omega_{Q\ell,t} \equiv 0 \pmod{\ell}$. By the argument in \cite[Lemma 4.6]{30}, for each $r$ with $0 \leq r < \ell^{-1}$ there exists an integer $a_r$ with $(a_r, 6Q\ell) = 1$ such that

$$a_r^2(3t + 2) \equiv 3(t + Q\ell r) + 2 \pmod{Q\ell^2}.$$ 

So $t + Q\ell r = t_A$ for some $A \in \Gamma_0(16Q\ell^2)$ as in (4.2). Therefore, as in the proof of Proposition 3.4 we can conclude that $\Omega_{Q\ell,t} \equiv 0 \pmod{\ell}$. □

We will use the following proposition to prove that $(\frac{3t+2}{\ell}) \neq \left( \frac{-1}{\ell} \right)$.

**Proposition 4.5.** If $4 \mid Q$, $(Q, 3\ell) = 1$ and $t \equiv -(Q^2 + 32)/48 \pmod{\ell}$ then

$$\Omega_{Q\ell,t}^{48Q\ell^2} \Delta^s|_{24Q\ell^2+12s} \left( \frac{1}{\ell} 0 \right) = (2Q)^{-24Q\ell^2} q^{-Q^2\ell^2+s} \cdots .$$ (4.3)

Assume for the moment that the proposition is true and assume, without loss of generality, that $4 \mid Q$. Suppose, by way of contradiction, that $(\frac{3t+2}{\ell}) = \left( \frac{-1}{\ell} \right)$. We will construct an integer $t'$ that satisfies both $t = t_A$ for some $A \in \Gamma_0(16Q\ell^2)$ and $t \equiv -(Q^2 + 32)/48 \pmod{\ell}$. If $(\frac{3t+2}{\ell}) = \left( \frac{-1}{\ell} \right)$, then there exists an $\alpha \in \mathbb{Z}$ with

$$(3t + 2)\alpha^2 \equiv -1 \pmod{\ell}.$$ 

Since $\ell \nmid \alpha$ and $(Q, 3\ell) = 1$, there exists $a \in \mathbb{Z}$ with $(a, 3Q\ell) = 1$ such that

$$4a \equiv Q\alpha \pmod{\ell}.$$ 

Let $t'$ be any integer satisfying

$$a^2(3t + 2) \equiv (3t' + 2) \pmod{Q\ell}.$$ 

Then $t' = t_A$ for some $A \in \Gamma_0(16Q\ell^2)$ as in (4.2) and

$$t' \equiv \frac{-(Q/4)^2 + 2}{3} \equiv -\frac{Q^2 + 32}{48} \pmod{\ell}.$$ (4.4)

So by Lemma 4.3 we have $\Omega_{Q\ell,t'} \equiv 0 \pmod{\ell}$. Define

$$h' := \Omega_{Q\ell,t'}^{48Q\ell^2} \Delta^s \in M_{24Q\ell^2+12s}(\Gamma_1(4Q\ell)),$$

We have $h' \equiv 0 \pmod{\ell}$, so by Theorem 2.3 we have

$$h'|_{24Q\ell^2+12s} \left( \frac{1}{\ell} 0 \right) \equiv 0 \pmod{\ell},$$

but this contradicts Proposition 4.5.

**Proof of Proposition 4.5.** By (4.1) we have

$$\Omega_{Q\ell,t} \left( \left( \frac{1}{\ell} 0 \right) z \right) = \frac{1}{Q\ell} \sum_{\lambda=0}^{Q\ell-1} \zeta_{Q\ell}^{-\lambda(t+2/3)} \Omega \left( \left( \frac{1}{\ell} \lambda \right) \left( \frac{1}{\ell} 0 \right) z \right)$$ 

$$= \frac{1}{Q\ell} \sum_{\lambda=0}^{Q\ell-1} \zeta_{Q\ell}^{-\lambda(t+2/3)} \Omega \left( B_{\lambda} \left( \frac{d_{\lambda}}{Q\ell} \frac{\Lambda'}{Q\ell/d_{\lambda}} \right) z \right),$$
where \( d_\lambda := (1 + \ell \lambda, Q) \), the integer \( \lambda' \) is chosen to satisfy \( \frac{1 + \ell \lambda}{d_\lambda} \lambda' \equiv \lambda \pmod{Q/\ell d_\lambda} \), and

\[
B_\lambda := \begin{pmatrix}
\frac{1 + \ell \lambda}{d_\lambda} & -\frac{1 + \ell \lambda}{d_\lambda} \lambda' + \lambda \\
Q/\ell d_\lambda & \frac{Q/\ell d_\lambda}{d_\lambda - \ell \lambda'}
\end{pmatrix} \in \text{SL}_2(\mathbb{Z}).
\]

Recall that the transformation law (2.6) for \( \Omega(z) \) depends on the parity of the lower entries of \( B_\lambda \). Note that if \( Q/d_\lambda \) is odd, then \( d_\lambda \) is even so that \( 6 \mid d_\lambda \). If \( 4 \mid Q \), then \( \ell \lambda' := \left( 1 + \frac{1 + \ell \lambda}{d_\lambda} \lambda' \right) \equiv \lambda \pmod{Q/\ell d_\lambda} \). Applying Lemma 2.1 to the matrix \( B_\lambda \) of \( Q/\ell d_\lambda \), we may take \( \lambda' = Q\lambda \) so that

\[
\Omega_{Q,\ell,t} \left( \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} z \right) = \frac{1}{Q/\ell d_\lambda} \sum_{Q/d_\lambda \text{ even}} \zeta_{Q/\ell}^{-\lambda(2t+3)} \omega_1(B_\lambda) \left( \frac{d_\lambda z + \lambda'}{Q/\ell d_\lambda} \right)^{1/2} \Omega \left( \frac{d_\lambda z + \lambda'}{Q/\ell d_\lambda} \right)
\]

for which \( \Omega(z) \) arises from those \( \lambda \) for which \( d_\lambda = Q \). For these \( \lambda \) we may take \( \lambda' = \lambda Q \) so that

\[
B_\lambda = \begin{pmatrix}
\frac{1 + \ell \lambda}{Q/\ell} & -\lambda^2 \\
Q(1 - \ell \lambda) & Q
\end{pmatrix}.
\]

Therefore

\[
(\ell z + 1)^{-1/2} \Omega_{Q,\ell,t} \left( \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix} z \right) = K q^{-Q/48\ell} + \cdots,
\]

where

\[
K = \frac{1}{Q/\ell} \sum_{d_\lambda = Q} \zeta_{Q/\ell}^{-\lambda(2t+3)} \left( \frac{Q/\ell}{2} \right)^{1/2} \omega_2(B_\lambda) e^{-\frac{2\pi i \lambda Q}{48\ell}}.
\]

We will compute \( K \) up to \( 48Q\ell^2 \)-th roots of unity denoted \( \omega_1, \omega_2, \) etc. Choose \( \alpha \in \mathbb{Z} \) such that \( \alpha Q \equiv 2 \pmod{\ell} \) and \( 3 \nmid \alpha \). Applying Lemma 2.1 to the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), we obtain

\[
s(Q\ell\lambda/2 - Q/2, \ell^2) = s(-Q/2, \ell^2) + \frac{Q\lambda}{2} \frac{1 - \alpha^2}{12\ell} + \text{an even integer}.
\]

Since \( 6\ell^2 s(-Q/2, \ell^2) \in \mathbb{Z} \), the term \( e^{-\pi i s(-Q/2, \ell^2)} \) is a \( 12\ell^2 \)-th root of unity which is independent of \( \lambda \). If \( B_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then by (4.5) we obtain

\[
\frac{32a - d}{48c} = \sum_{d_\lambda = Q} \lambda(Q^2 + 32) - 3\lambda \frac{Q^2}{4Q\ell} + \lambda^2 \frac{Q^2}{4Q} + \frac{1}{2Q} + \frac{3}{4} - \frac{Q^2 + 4}{48Q\ell^2}.
\]

So we have

\[
K = \frac{\omega_1}{\ell \sqrt{2Q}} \sum_{d_\lambda = Q} e^{2\pi i R_\lambda},
\]
where

\[ R_\lambda = -\frac{\lambda(t + 2/3)}{Q\ell} + \frac{\lambda(Q^2 + 32)}{48Q\ell} - \frac{Q\lambda(1 - \alpha^2)}{48\ell} - \frac{3\lambda}{4} + \frac{\lambda^2}{4} - \frac{\lambda(3\lambda + 2\ell + 3\lambda^2\ell)}{4Q} - \frac{Q\lambda}{48\ell} \]

\[ = -\frac{t\lambda}{Q\ell} - \frac{\lambda(Q^2 + 32)}{48Q\ell} + \frac{\lambda(\alpha^2Q^2 - 4 + 12\ell(Q\lambda - 3\lambda - 2\ell - 3\lambda^2))}{48Q\ell} \]

The \( \lambda \) for which \( d_\lambda = Q \) are of the form \( \lambda = \lambda_0 + \delta \) for \( 0 \leq \delta < \ell \). Replacing \( \lambda \) by \( \lambda_0 + \delta \) and recalling that \( 4 \mid Q \), then ignoring the terms which are integers, we obtain

\[ K = \frac{\omega_2}{\ell \sqrt{2Q}} \sum_{\delta = 0}^{\ell - 1} e^{2\pi i R'_\delta} \]

where

\[ R'_\delta = -\frac{t\delta}{\ell} - \frac{\delta(Q^2 + 32)}{48\ell} + \frac{\delta}{\ell} \cdot \frac{\alpha^2Q^2 - 4 - 12\ell(9\ell\lambda_0^2 + 6\lambda_0 + 2\ell)}{48} \]  \hspace{1cm} (4.6)

Since \( d_{\lambda_0} = (1 + \ell\lambda_0, Q) = Q \) and \( 4 \mid Q \), we have \( 1 + \ell\lambda_0 \equiv 0 \pmod{4} \). Thus

\[ 9\ell\lambda_0^2 + 6\lambda_0 + 2\ell \equiv \ell \pmod{4} \]

which implies that

\[ R'_\delta = -\frac{t\delta}{\ell} - \frac{\delta(Q^2 + 32)}{48\ell} + \frac{\delta}{\ell} \cdot \alpha^2Q^2 - 4 - 12\ell^2 \]

\hspace{1cm} \text{an integer.} \hspace{1cm} (4.7)

Recall that we chose \( \alpha \) so that \( \alpha^2 \equiv 1 \pmod{3} \) and \( \alpha Q \equiv 2 \pmod{\ell} \). These facts, together with \( 4 \mid Q \), imply that

\[ \alpha^2Q^2 - 4 - 12\ell^2 \equiv 0 \pmod{48\ell} \]

so the third term in (4.7) is also an integer. We assumed that \( t \equiv -(Q^2 + 32)/48 \pmod{\ell} \), so in fact \( R'_\delta \in \mathbb{Z} \). Therefore

\[ K = \frac{\omega_2}{\ell \sqrt{2Q}} \sum_{\delta = 0}^{\ell - 1} 1 = \frac{\omega_2}{\sqrt{2Q}} \]

\[ \square \]

5. **Weakly holomorphic modular forms**

In this section we state and prove a general theorem about a large class of weakly holomorphic modular forms which contains \( \eta \)-quotients. For an integer or half-integer \( k \), integers \( B \) and \( N \) (with \( 4 \mid N \) if \( k \not\in \mathbb{Z} \)), and a Dirichlet character \( \chi \) modulo \( N \), we define

\[ S(B, k, N, \chi) := \{ \eta^B(z)F(z) : F(z) \in M_k^!(\Gamma_0(N), \chi) \} \]

Here \( M_k^!(\Gamma_0(N), \chi) \) denotes the space of meromorphic modular forms of weight \( k \) with Nebentypus \( \chi \) on \( \Gamma_0(N) \) whose poles, if any, are supported at the cusps. A form \( F \in M_k^!(\Gamma_0(N), \chi) \) transforms under \( \gamma = (a \ b \ c \ d) \in \Gamma_0(N) \) as

\[ F|_k \gamma = \rho(\gamma)\chi(d)F, \]

where

\[ \rho(\gamma) := \begin{cases} 1 & \text{if } k \in \mathbb{Z}, \\ \epsilon_d^{-2k}(\frac{a}{d})^{2k} & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases} \]
and
\[ \epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4} \end{cases} \]
(see [28, Chapter 1] for details). The following theorem generalizes [30, Theorem 1.2] for any \( f \in \mathcal{S}(B, k, N, \chi) \).

**Theorem 5.1.** Let \( \ell \geq 5 \) be prime. Suppose that
\[ f(z) = q^{B/24} \sum_{n \geq n_0} a_f(n)q^n \in \mathcal{S}(B, k, N, \chi) \]
has a pole at \( \infty \), leading coefficient equal to 1, and rational \( \ell \)-integral coefficients. Suppose that \( \ell \nmid N(24n_0 + B) \) and \((m_B, N) = 1 \) (with \( m_B \) defined in (1.3)). If
\[ a_f(mn + t) \equiv 0 \pmod{\ell}, \]
then \( \ell | m \) and \( \left( \frac{24t + B}{\ell} \right) \neq \left( \frac{24n_0 + B}{\ell} \right) \).

Before proving Theorem 5.1 we show that Theorem 1.4 is a corollary.

**Proof of Theorem 1.4.** Suppose that \( f(z) \) is the \( \eta \)-quotient
\[ f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta} = q^{B/24} \sum_{n=0}^{\infty} a_f(n)q^n, \]
and that \( a_f(mn + t) \equiv 0 \pmod{\ell} \). Recall that
\[ B = \sum_{\delta \mid N} \delta r_\delta, \quad (5.1) \]
and write \( f(z) = \eta^B(z)F(z) \) with
\[ F(z) = \frac{f(z)}{\eta^B(z)} = \eta^{-B}(z) \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}. \]
To apply Theorem 5.1 we use a standard criterion [26] to show that \( F(z) \in M^!_k(\Gamma_0(N^4), \chi) \) for some character \( \chi \). In light of (5.1), the condition
\[ \sum_{\delta \mid N} \delta r_\delta - B \equiv 0 \pmod{24} \quad (5.2) \]
is satisfied trivially. The condition
\[ N^4 \left( \sum_{\delta \mid N} \frac{r_\delta}{\delta} - B \right) \equiv 0 \pmod{24} \quad (5.3) \]
is also satisfied (to see this, consider cases depending on the value of \((N, 6)\)). We apply Theorem 5.1 with \( n_0 = 0 \) to obtain \( \ell | m \) and \( \left( \frac{24t + B}{\ell} \right) \neq \left( \frac{B}{\ell} \right) \). \( \square \)
Proof of Theorem 5.1. Suppose that
\[ f(z) = q^{B/24} \sum_{n=-n_0}^{\infty} a_f(n)q^n = q^{n_0+B/24} + \cdots \in S(B,k,N,\chi) \]
has a pole at \( \infty \) and rational \( \ell \)-integral coefficients, and that \( \ell \nmid N(24n_0+B) \). Given \( m \) and \( t \) with \( (m_B,N) = 1 \), define
\[ f_{m,t} := \frac{1}{m} \sum_{\lambda=0}^{m-1} \zeta_m^{-\lambda(t+B/24)} f \left( \frac{z+\lambda}{m} \right) \]
\[ = q^{t\ell B/24} \sum_{n} a_f(mn+t)q^n. \]

We require a transformation law for \( f_{m,t} \). Define
\[ N_m = \begin{cases} m & \text{if } (m,6) = 1, \\ 8m & \text{if } (m,6) = 2, \\ 3m & \text{if } (m,6) = 3, \\ 24m & \text{if } (m,6) = 6. \end{cases} \]
Note that this differs slightly from the previous definition of \( N_m \). If \( A = (a \ b \ c \ d) \in \Gamma_0(N_m) \) has \( (a,6) = 1 \) then we define \( t_A \) to be any integer satisfying
\[ t_A \equiv ta^2 + B\frac{a^2-1}{24} \pmod{m}. \] (5.5)

The following is proved in [3, Section 5].

Proposition 5.2. Suppose \( f \) is as above, and define \( \kappa = 24mN(k+B/2). \) For every \( A = (a \ b \ c \ d) \in \Gamma_0(NN_m) \) with \( (a,6) = 1 \) we have
\[ f_{24mN}^{24mN} \mid_A \kappa A = f_{24mN}^{24mN}. \]
In particular, there exists \( s \in \mathbb{N} \) such that
\[ \Delta^s f_{24mN}^{24mN} \in M_{\kappa+12s}(\Gamma_1(NN_m)). \]

We need the following analogue of Lemma 3.3. The proof is similar so we omit it here.

Lemma 5.3. Suppose that \( f \) is as above and that \( A = (a \ b \ c \ d) \in \Gamma_0(NN_m) \) has \( (a,6) = 1 \), and define \( t_A \) as in (5.5). If \( f_{m,t} \equiv 0 \pmod{\ell} \) then \( f_{m,t_A} \equiv 0 \pmod{\ell} \).

For the remainder of this section, assume that \( f_{m,t} \equiv 0 \pmod{\ell} \). Proposition 5.4 will show that \( \ell | m \) and reduce us to a simpler case. We fix some notation. As in Section 2 write \( B = 2^r3^rB' \) and \( m = 2^s3^sm' \) with \( (B',6) = (m',6) = 1 \), and write \( Q_{m,B} = 2^s3^{3r}m' = Q\ell \) with \( (Q,\ell) = 1 \). Recall that \( \alpha \) and \( \beta \) are defined in (2.11) by
\[ \alpha := \begin{cases} 0 & \text{if } r = 0, \\ \min(r,u) & \text{if } r = 1,2, \\ u & \text{if } r \geq 3, \end{cases} \]
\[ \beta := \begin{cases} 0 & \text{if } s = 0, \\ v & \text{if } s \geq 1. \end{cases} \] (5.6)

Since \( \ell \nmid N \) and the primes dividing \( Q_{m,B} \) are the same as those dividing \( m_B, \) we have \( (Q,N) = 1 \). There are two main cases to consider depending on the value of \( r. \) To facilitate this, define parameters \( M \in \mathbb{N} \) and \( \varepsilon \in \{0,1,2\} \) as follows. If \( r = 1,2, \) define \( M := \frac{24N}{(B,3)} \) and \( \varepsilon := \alpha = \min(r,u). \) If \( r \neq 1,2, \) define \( M := \frac{24N}{(B,24)} \) and \( \varepsilon := 0. \)
Proposition 5.4. Assume the notation above. If \( f_{m,t} \equiv 0 \pmod{\ell} \), then \( j > 0 \). If additionally \( \ell \nmid (24t + B) \) then \( f_{Q\ell,t} \equiv 0 \pmod{\ell} \).

Proof. Applying Lemma 2.2 we see that for each \( \lambda \) with \( 0 \leq \lambda < m/Q\ell \) there exists \( a \in \mathbb{Z} \) with \( (a, 6mN) = 1 \) such that

\[
t + \lambda Q\ell^j \equiv ta^2 + B\frac{a^2 - 1}{24} \pmod{m}.
\]

As in the proof of Proposition 3.4, we conclude by Lemma 5.3 that \( f_{Q\ell,t} \equiv 0 \pmod{\ell} \).

Suppose now that \( j = 0 \). From [3, Section 5], the leading coefficient in the expansion of \( f_{Q,t} \) at the cusp \( 1/N \) is

\[
\xi Q^B/2 + k - 1
\]

where \( \xi \) is a \( 24QN \)th root of unity. Using Theorem 2.4 as before with the modular form \( \ell - 24QN \Delta_s f_{24QN}^2 \), which has integral coefficients, we obtain a contradiction. Therefore \( j > 0 \).

Lastly, we use [30, Lemma 4.6] as in the proof of Proposition 3.4 to conclude that \( f_{Q\ell,t} \equiv 0 \pmod{\ell} \) if \( \ell \nmid (24t + B) \). \( \square \)

We will use the following to prove that \( (24t + B) \ell \neq (24n_0 + B) \ell \).

Proposition 5.5. Assume the notation above. Let \( \kappa = 24Q\ell N(k + B/2) \) and define \( R := Q/2^s \). If

\[
t \equiv R^2n_0 + B\frac{R^2 - 1}{24} \pmod{2^s\ell},
\]

then

\[
\Delta^s f_{24Q\ell N} \mid_{\kappa+12s} \left( \frac{1}{M\ell} \ 0 \\ 1 \right) = R^{k - 24Q\ell N}q^{QRN\ell + 24RQn_0 + s} + \ldots .
\]

Assume for the moment that the proposition is true. Suppose, by way of contradiction, that \( (24t + B) \ell = (24n_0 + B) \ell \). We will construct an integer \( t' \) satisfying both \( t' = t_A \) for some \( A \in \Gamma_0(NN\ell) \) and \( t' \equiv R^2n_0 + B(R^2 - 1)/24 \pmod{2^s\ell} \). There exists an \( a' \in \mathbb{Z} \) with

\[
(24t + B)a'^2 \equiv 24n_0 + B \pmod{\ell},
\]

and since \( 2^s|(B, 24) \), we have

\[
(24t + B)a'^2 \equiv 24n_0 + B \pmod{2^s\ell}.
\]

Let \( a \) be an integer with \( (a, 6Q\ell) = 1 \) such that \( a \equiv Ra' \pmod{\ell} \) and let \( t' \) be any integer satisfying

\[
a'^2(24t + B) \equiv (24t' + B) \pmod{Q\ell}.
\]

Then

\[
t' \equiv R^2n_0 + B\frac{R^2 - 1}{24} \pmod{2^s\ell}
\]

and by Lemma 5.3 we have \( f_{Q\ell,t'} \equiv 0 \pmod{\ell} \), but this is not compatible with Proposition 5.5.

Proof of Proposition 5.5. By (5.4) we have

\[
f_{Q\ell,t} \left( \begin{pmatrix} 1 & 0 \\ M\ell & 1 \end{pmatrix} z \right) = \frac{1}{Q\ell} \sum_{\lambda=0}^{Q\ell-1} \xi_{Q\ell}^{-\lambda(t+B/24)} f \left( C_\lambda \begin{pmatrix} d_\lambda & \lambda' \\ 0 & Q\ell/d_\lambda \end{pmatrix} z \right),
\]
where \(d_\lambda = (1 + \lambda \ell M, Q)\), the integer \(\lambda'\) satisfies
\[
\frac{1 + \lambda \ell M}{d_\lambda} \lambda' \equiv \lambda \pmod{Q\ell/d_\lambda},
\]
and \(C_\lambda\) is the matrix
\[
\begin{pmatrix}
\frac{1 + \lambda \ell M}{d_\lambda} & -\frac{1 + \lambda \ell M}{d_\lambda} \\
M\ell\lambda + 1 & q
\end{pmatrix} \in \Gamma_0(N).
\]
By assumption, \(f\) has a pole at \(\infty\), so the leading term of
\[
(M\ell z + 1)^{-k-B/2} f_{Q,t}\left(\begin{pmatrix} 1 & 0 \\ M\ell & 1 \end{pmatrix} z\right)
\]
arises from those \(\lambda\) for which \(d_\lambda\) is maximized. Since \((Q, N) = 1\) and since \(3|Q\) only if \(3|B\), we see that \((Q, M) = 2^\varepsilon\). Thus, the largest value of \(d_\lambda\) is \(R\). For these \(\lambda\) we can take \(\lambda' = R\lambda\), so that
\[
C_\lambda = \begin{pmatrix}
\frac{1 + \lambda \ell M}{R} & -\frac{M\lambda^2}{2} \\
2\ell \lambda^2 R & R(1 - M\ell \lambda)
\end{pmatrix}.
\]
We compute the terms of (5.8) for which \(d_\lambda = R\). Recall that \(f(z) = \eta^B(z)F(z)\). We consider first the transformation of \(F(z)\). We have
\[
(M\ell z + 1)^{-k} F\left(C_\lambda\begin{pmatrix} R & \lambda R \\ 0 & 2^\varepsilon \ell \end{pmatrix} z\right) = \rho(C_\lambda) R^k \chi(R) \chi(1 - M\ell \lambda) F\left(\begin{pmatrix} R & \lambda R \\ 0 & 2^\varepsilon \ell \end{pmatrix} z\right), \tag{5.9}
\]
where
\[
\rho(C_\lambda) = \begin{cases} 
1 & \text{if } k \in \mathbb{Z}, \\
\left((-1)^{2k} \frac{2^\varepsilon \ell M^2 R}{M\ell \lambda R - R}\right)^{\varepsilon + \mu} & \text{if } k \in \frac{1}{2} + \mathbb{Z}.
\end{cases}
\]
Note that since \(N|M\), we have \(\chi(1 - M\ell \lambda) = 1\). We show that \(\rho(C_\lambda)\) is independent of \(\lambda\). Suppose that \(k \in \frac{1}{2} + \mathbb{Z}\). Then \(4|N\), so \(R(1 - M\ell \lambda) \equiv R \pmod{4}\), thus \(\varepsilon_{R - M\ell \lambda R} = \varepsilon_R\). Write \(M = 2^\mu M'\) with \(M'\) odd and \(\mu \geq 2\). We have
\[
\frac{2^\varepsilon \ell M^2 R}{M\ell \lambda R - R} = \left(\frac{2^\varepsilon M}{R}\right) \left(\frac{2}{M\ell \lambda - 1}\right)^{\varepsilon + \mu} \left(\frac{M'}{M\ell \lambda - 1}\right) = \left(\frac{2^\varepsilon M}{R}\right),
\]
since \(\varepsilon = 0\) when \(\mu = 2\). Therefore \(\rho(C_\lambda) = \rho\) is independent of \(\lambda\).

We now consider the transformation of \(\eta^B(z)\). We have
\[
(M\ell z + 1)^{-B/2} \eta^B\left(C_\lambda\begin{pmatrix} R & \lambda R \\ 0 & 2^\varepsilon \ell \end{pmatrix} z\right) = \xi(C_\lambda)^B R^{B/2} \eta^B\left(\begin{pmatrix} R & \lambda R \\ 0 & 2^\varepsilon \ell \end{pmatrix} z\right), \tag{5.10}
\]
with \(\xi(C_\lambda)\) defined in (2.10). Write \(C_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\). If \(B\) is even then the factor \((\frac{c}{d})^B\) or \((\frac{d}{c})^B\) is equal to 1. Suppose that \(B\) is odd. Then \(\varepsilon = 0, 8|M\) and \(c\) is even. In this case we have
\[
\left(\frac{c}{d}\right) = \left(\frac{M^2}{M\ell \lambda R - R}\right) = \left(\frac{M'}{R}\right) \left(\frac{2}{M\ell \lambda - 1}\right)^{\mu} \left(\frac{M'}{M\ell \lambda - 1}\right) = \left(\frac{M'}{R}\right),
\]
by a similar argument as before. Note that \(24|Bc\) by the definition of \(M\). If \(2^\varepsilon M\) is even, we have
\[
\exp\left(\frac{2^\varepsilon B}{24}[(a + d)c - bd(c^2 - 1) + 3d - 3 - 3cd]\right) = \exp\left(\frac{2^\varepsilon B}{24}[Bbd + 3Bd - 3B]\right) = \xi_{s^{(R-1)}}^B \exp\left(\frac{2^\varepsilon}{24} Bbd\right)
\]
since $Bd \equiv BR \pmod{24}$. If $2^e M$ is odd, we have
\[
\exp\left(\frac{2\pi i}{24} B[(a + d)c - bd(c^2 - 1) - 3c]\right) = \exp\left(\frac{2\pi i}{24} Bbd\right).
\]
It remains to show that $24 \mid Bbd$. We have
\[
Bbd = -\frac{BM}{2^e} \lambda^2 R(1 - M\ell\lambda) \equiv 0 \pmod{24}
\]
since $2^e \cdot 24 \mid BM$. So, in each case $\xi(C_\lambda) = \xi^B$ is independent of $\lambda$.

From (5.8), (5.9), and (5.10) we obtain
\[
f_{Q\ell,t}|_{k+B/2} \left(\begin{array}{c} 1 \\ M\ell \\ 0 \\ 1 \end{array}\right) = \omega_1 \frac{R^{k+B/2}}{Q\ell} \sum_{d_\lambda = R} \zeta_{Q\ell}^{-\lambda(t+B/24)} f\left(\left(\begin{array}{c} R \\ \lambda R \end{array}\right) z\right)
\]
where $\omega_1 = \rho \chi(R)\xi^B$ is a 24N-th root of unity. Since $f(z) = q^{B/24} + \cdots$ and $B/24 + n_0 < 0$, the leading coefficient $K$ of (5.11) is given by
\[
K = \omega_1 \frac{R^{k+B/2-1}}{2^e \ell} \sum_{d_\lambda = R} \exp\left(2\pi i \left(-\frac{\lambda t}{Q\ell} - \frac{\lambda B}{24Q\ell} + \frac{\lambda BR}{24 \cdot 2^e \ell} + \frac{n_0 \lambda R}{2^e \ell}\right)\right).
\]
Let $\lambda_0$ be the smallest positive integer for which $d_\lambda = R$. Then the $\lambda$ for which $d_\lambda = R$ are of the form $\lambda_0 + \delta R$ for $0 \leq \delta \leq 2^e \ell - 1$. Write $\lambda = \lambda_0 + \delta R$. Then
\[
-\frac{\lambda t}{Q\ell} - \frac{\lambda B}{24Q\ell} + \frac{\lambda BR}{24 \cdot 2^e \ell} + \frac{n_0 \lambda R}{2^e \ell}
= \frac{\delta}{2^e \ell} \left(\frac{R^2 - 1}{24} + n_0 R^2 - t\right) + \frac{\lambda_0}{2^e \ell} \left(\frac{R^2 - 1}{24R} + n_0 R - t\right) \quad \text{independent of } \delta.
\]
So, for some 24Q\ell N-th root of unity $\omega_2$, we have
\[
K = \omega_2 \frac{R^{k+B/2-1} \cdot 2^e \ell - 1}{2^e \ell} \sum_{\delta = 0} \exp\left(2\pi i \frac{\delta}{2^e \ell} \left(-t + \frac{R^2}{24} + n_0 R^2\right)\right).
\]
By assumption,
\[
t \equiv \frac{R^2 - 1}{24} + n_0 R^2 \pmod{2^e \ell},
\]
so $K = \omega_2 R^{k+B/2-1}$. \hfill \Box

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