Quantum Finance represents the synthesis of the techniques of quantum theory (quantum mechanics and quantum field theory) to theoretical and applied finance. After a brief overview of the connection between these fields, we illustrate some of the methods of lattice simulations of path integrals for the pricing of options. The ideas are sketched out for simple models, such as the Black-Scholes model, where analytical and numerical results are compared. Application of the method to nonlinear systems is also briefly overviewed. More general models, for exotic or path-dependent options are discussed.

1. Introduction

a Financial markets have undergone a tremendous growth in the last decades and in order to meet the need of customers new and complex financial instruments have been developed. Risk assessment models and the quantification of returns, given the huge amount of trading involved worldwide, requires more sophisticated approaches than in the past. Non-

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linearities play a key role in all of this, and, from this side, the field is probably largely unexplored.

In general, intensive numerical simulations and fast algorithms are needed to obtain useful results. Analytical results are limited, except for simple models such as the Black-Sholes model and other similar models.

Use of a path integral formulation has some advantages. First, it is in close relation to the lagrangean description of diffusion processes, second, it opens the way to the use of quantum mechanical methods.

We will introduce the field at a non-expert level through a crash course (next few sections). Then we will come to briefly illustrate where nonlinearities appear and review some of the simplest equations one can write down, generalizing the Black-Sholes model. We will then proceed to discuss the path integral formulation of the Black-Sholes and the model of barrier options, resorting to a lagrangian path integral formulation. Strategies to solve the path integral are then briefly presented, together with some results. Detailed simulations, algorithms of analysis and related applications will be presented elsewhere \(^1\). The review sections are standard knowledge in the field, and are based on \(^2\).

2. A view on Theoretical Finance

2.1. Simulating the Complexity of Finance

The simulation of financial markets can be modeled, from a theoretical viewpoint, according to two separate approaches: a bottom up approach and (or) a top down approach.

For instance, the modeling of financial markets starting from diffusion equations and adding a noise term to the evolution of a function of a stochastic variable is a top down approach. This type of description is, effectively, a statistical one.

A bottom up approach, instead, is the modeling of artificial markets using complex data structures (agent based simulations) using general updating rules to describe the collective state of the market. The number of procedures implemented in the simulations can be quite large, although the computational cost of the simulation becomes forbidding as the size of each agent increases. Readers familiar with Sugarscape Models and the computational strategies based on Growing of Artificial Societies \(^4\) have probably an idea of the enormous potentialities of the field. However, one would expect that the bottom up description should become comparable to the top down description for a very large number of simulated agents.

The bottom up approach should also provide a better description of ex-
treme events, such as crashes, collectively conditioned behaviour and market incompleteness, this approach being of purely algorithmic nature. A top-down approach is, therefore, a model of reduced complexity and follows a statistical description of the dynamics of complex systems (for an introduction see ⁵).

2.2. Forward, Futures Contracts and Options

Let the price at time t of a security be $S(t)$. A specific good can be traded at time t at the price $S(t)$ between a buyer and a seller. The seller (short position) agrees to sell the goods to the buyer (long position) at some time T in the future at a price $F(t, T)$ (the contract price). Notice that contract prices have a 2-time dependence (actual time t and maturity time T). Their difference $\tau = T - t$ is usually called time to maturity. Equivalently, the actual price of the contract is determined by the prevailing actual prices and interest rates and by the time to maturity.

Entering into a forward contract requires no money, and the value of the contract for long position holders and strong position holders at maturity T will be

$$(-1)^p (S(T) - F(t, T))$$

where $p = 0$ for long positions and $p = 1$ for short positions. Futures Contracts are similar, except that after the contract is entered, any changes in the market value of the contract are settled by the parties. Hence, the cashflows occur all the way to expiry unlike in the case of the forward where only one cashflow occurs. They are also highly regulated and involve a third party (a clearing house). Forward, futures contracts and, as we will see, options go under the name of derivative products, since their contract price $F(t, T)$ depend on the value of the underlying security $S(T)$.

To complete this crash course on financial instruments we need to define options. Options are derivatives that can be written on any security and have a more complicated payoff function than the futures or forwards. For example, a call option gives the buyer (long position) the right (but not the obligation) to buy or sell the security at some predetermined strike-price at maturity. A payoff function is the precise form of the price. Path dependent options are derivative products whose value depends on the actual path followed by the underlying security up to maturity. In the case of path-dependent options, since the payoff may not be directly linked to an explicit right, they must be settled by cash. This is sometimes true for futures and plain options as well as this is more efficient.
3. Langevin Evolution

In the top down description of theoretical finance, a security $S(t)$ follows a random walk described by an Ito-Weiner process (or Langevin equation) as

$$\frac{dS(t)}{S(t)} = \phi dt + \sigma R(t) dt,$$  \hspace{1cm} (2)

where $R(t)$ is a Gaussian white noise with zero mean and uncorrelated values at time $t$ and $t'$ ($R(t)R(t') = \delta(t-t')$. $\phi$ is the drift term or expected return, while $\sigma$ is a constant factor multiplying the random source $R(t)$, termed volatility.

As a consequence of Ito calculus, differentials of functions of random variables, say $f(S,t)$, do not satisfy Leibniz’s rule, and for an Ito-Weiner process with drift (2) one easily obtains for the time derivative of $f(S,t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \phi S \frac{\partial f}{\partial S} + \sigma S \frac{\partial f}{\partial S} R.$$  \hspace{1cm} (3)

The Black-Scholes model is obtained by removing the randomness of the stochastic process shown above by introducing a random process correlated to (3). This operation, termed hedging, allows to remove the dependence on the white noise function $R(t)$, by constructing a portfolio $\Pi$, whose evolution is given by the short-term risk free interest rate $r$

$$\frac{d\Pi}{dt} = r\Pi.$$  \hspace{1cm} (4)

A possibility is to choose $\Pi = f - \frac{\partial f}{\partial S} S$. This is a portfolio in which the investor holds an option $f$ and short sells an amount of the underlying security $S$ proportional to $\frac{\partial f}{\partial S}$. A combination of (3) and (4) yields the Black-Scholes equation

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} = rf.$$  \hspace{1cm} (5)

There are some assumptions underlying this result. We have assumed absence of arbitrage, constant spot rate $r$, continuous balance of the portfolio, no transaction costs and infinite divisibility of the stock.

The quantum mechanical version of this equation is obtained by a change of variable $S = e^x$, with $x$ a real variable. This yields

$$\frac{\partial f}{\partial t} = H_{BS} f$$  \hspace{1cm} (6)

\*short selling of the stock should be possible
with an Hamiltonian $H_{BS}$ given by

$$H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} \sigma^2 - r \right) \frac{\partial}{\partial x} + r. \quad (7)$$

Notice that one can introduce a quantum mechanical formalism and interpret the option price as a ket $|f\rangle$ in the basis of $|x\rangle$, the underlying security price. Using Dirac notation, we can formally reinterpret $f(x,t) = \langle x|f(t)\rangle$, as a projection of an abstract quantum state $|f(t)\rangle$ on the chosen basis.

In this notation, the evolution of the option price can be formally written as $|f,t\rangle = e^{iH}|f,0\rangle$, for an appropriate Hamiltonian $H$.

In the presence of a stochastic volatility, the description is more involved, but also more interesting.

In general, the description of these processes is driven by two correlated white noise functions $R_1$ and $R_2$

$$\frac{dV}{dt} = \lambda + \mu V + \zeta V^\alpha R_1$$
$$\frac{dV}{dt} = \phi S + \sigma \sqrt{V} + \mu V + \zeta V^\alpha R_2 \quad (8)$$

with $V = \sqrt{\sigma}$ and $\langle R_1(t)R_2(t') \rangle = 1/\rho \delta(t-t')$, $\rho$ being the correlation parameter. However, since volatility is not traded in the market (the market is said to be incomplete), perfect hedging is not possible, and an additional term, the market price of volatility risk $\beta(S,V,t,r)$, is in this case introduced. $\beta$ can be modeled appropriately. In some models, a redefinition of the drift term $\mu$ in (8) in the evolution of the volatility is sufficient to hedge such more complex portfolios, which amounts to an implicit choice of $\beta(S,V,t,r)$. We just quote the result for the evolution of an option price in the presence of stochastic volatility, which, in the Hamiltonian formulation are given by

$$\frac{\partial f}{\partial t} = H_{MG} f \quad (9)$$

with

$$H_{MG} = -\left( r - \frac{e^y}{2} \right) \frac{\partial}{\partial x} - \left( \lambda e^{-y} + \mu - \frac{\zeta^2}{2} e^{2y(\alpha-1)} \right) \frac{\partial}{\partial y} - \frac{e^y}{2} \frac{\partial^2}{\partial x^2}$$

$$-\rho \zeta e^{y(\alpha-1/2)} \frac{\partial^2}{\partial x \partial y} - \zeta^2 \frac{e^{2y(\alpha-1)}}{2} \frac{\partial^2}{\partial y^2} + r. \quad (10)$$

which is nonlinear in the variables $x = \log(S)$ and $y = \log(V)$. For general values of the parameters, the best way to obtain the pricing of the options in this model is by a simulation of the path integral.
4. Monte Carlo Simulations of option pricing

Simulations of the functional integral are rather straightforward and we should omit any detail about them since they have been known ever since by the high energy physics community. However, to reach out to a less specialized audience, we will provide a simple illustration of the method.

Once the model is given, one determines the underlying lagrangean. We assume a discretization of the time to maturity $\tau$ in intervals $\epsilon = \tau/N$, with $N$ an arbitrary (large) integer.

For instance, for the Black Scholes model one gets the action

$$S_{BS} = \epsilon \sum_{i=1}^{N} L_{BS}(i)$$

(11)

with

$$L_{BS}(i) = -\frac{1}{2\sigma^2} \left( \frac{x_i - x_{i-1}}{\epsilon} + r - \frac{\sigma^2}{2} \right)^2$$

(12)

where we have introduced discretized positions $(x_i)$ for the variable $x = \log(S)$ which identifies the quantum mechanical state of the system. We will refer to it as to the stock price. The propagator for the stock price is given by the pricing kernel

$$p_{BS}(x, x', \tau) = \int DX_{BS} e^{S_{BS}}$$

$$= \langle x | e^{-\tau H_{BS}} | x' \rangle$$

(13)

with

$$\int DX_{BS} = \Pi_{t=0}^{\tau} \int_{-\infty}^{\infty} dx(t).$$

(14)

For barrier options there is an analogous procedure, except that now we need to introduce a generic potential $V(x)$ in the corresponding Hamiltonian

$$H_V = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} \sigma^2 - V(x) \right) \frac{\partial}{\partial x} + V(x).$$

(15)

The pricing kernel is the fundamental quantity to compute using the functional integral. Related attempts can be found in the literature.

For this purposes, we have used a standard Metropolis algorithm. If thermalization is slow, it is possible to resort to use sequentially Metropolis updates and cluster updates. The latter is an update for the embedded Ising dynamics in the lattice variables $x_i/|x_i|$ (Swendsen-Wang, Wolff), and is
included in for a faster generation of the thermalized paths of the stock price \( x(t) \).

For processes involving a stochastic volatility \( (y = \log(V)) \) the expression of the path integral is more complicated and can be found in \(^6\). From now on we will just consider the case of a constant volatility.

If we denote by \( g(x, K) \) the payoff function, with a strike price \( K \), in this case the value of the option (its price) is given by the Feynman-Kac formula

\[
 f(t, x) = \int_{-\infty}^{\infty} dx' (x|x'\rangle e^{-\frac{(T-t)H_{BS}\rho}}|x'\rangle g(x', K).
\] (16)

In actual simulations, it is convenient to compute directly the option price rather than the propagator itself. The simulation is done by taking the initial point \( x \) fixed, and letting the final point evolve according to its quantum dynamics. In this way a path \( (x, x') \) is generated. After the first thermalization, \( x' \) is allowed to undergo quantum fluctuations, at fixed \( x \). Each \( x' \) is then convoluted with the payoff function and an average is performed. Finally, this procedure is repeated for several \( x \) values, so to obtain the option price at time to maturity \( \tau \).

Figs. 1, 2 and 3, illustrate some simple results obtained by the monte carlo method. For illustrative purposes, we show the behaviour of the Black-Scholes model. Fig. 1 shows a typical thermalized path, generated from a given initial value \( x \) (at current time \( t = \tau \)) assuming a maturity of 300 days, while in Fig. 2 we have plotted several path for different starting values \( x \) of the stock at current time \( \tau \). We have chosen an interest rate \( r = 0.05 \) and a 12 percent volatility \( \sigma \). Finally, in Fig. 3 we compare the analytical and the numerical evaluation of the Black-Scholes option price with a low resolution for (16), in order to separate the two curves, which otherwise would overlap completely, in order to illustrate the convergence of the Metropolis algorithm.

Barrier options can be analyzed similarly, equivalently, by this method or by the Langevin method. We show in Fig. 4 the evaluation of the price of the option using the Langevin method in the presence of a step potential sitting at a value of the stock price given by \( x_0 = \log(S_0) \), with \( S_0 = 100 \). Compared to the Black-Scholes now the price has been discounted.

Applications of the method to the determination of various pricing kernels is underway. More details will be given elsewhere \(^1\).
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Figure 2. Several thermalized paths for (Black-Scholes) with $r=0.05$ and $\sigma = 0.12$.

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Figure 3. Call option price for strike price $3$ versus the logarithm of the initial value of the stock $x_0 = \log(S_0)$. The parameters are fixed as in figs 1. Shown is the analytical result vs the monte carlo result, with a low resolution of 10,000 configurations.

Result for Potential -0.95 for $x>x_0$ and 0.05 for $x<x_0$ ($S_0=100$)

$t=1$ year, volatility $= 0.25$/year, 50,000 configurations, 128 time steps

Figure 4. Plot of the option price versus the stock price obtained by a Langevin simulation of the path integral, with a potential step.