Plancherel Theorem and the Left Ideals of the Group Algebra for the Jacobi Group.

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Abstract
Let $G = SL(2, \mathbb{R})$ be the $2 \times 2$ connected real semisimple Lie group and let $KAN$ be the Iwasawa decomposition of $SL(2, \mathbb{R})$. Let $J = H \rtimes SL(2, \mathbb{R})$ be the Jacobi group, which is the semidirect product of the two groups $H$ with $SL(2, \mathbb{R})$. It plays an important role in Quantum Mechanics. The purpose of this paper is to define the Fourier transform in order to obtain the Plancherel theorem for the group $J$. To this end a classification of all left ideals of the group algebra $L^1(H \rtimes AN)$.

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1 Introduction

1.1. The Jacobi group the semidirect product of the Heisenberg and the symplectic group $SL(2, \mathbb{R})$ plays an important role in quantum mechanics. In Quantum optics represent a physical realization of the coherent states associated to the Jacobi group. The Jacobi group is responsible for the squeezed states and has an important object in quantum mechanics, geometric quantization, optics. Abstract harmonic analysis is the field of the most modern branches of harmonic analysis, having its roots in the mid-twentieth century, is analysis on topological groups. If the group is neither abelian nor compact, no general satisfactory theory is currently known.
1.2. First In this paper I will define the Fourier transform in order to establish the Plancherel formula on the Jacobi group $H \rtimes SL(2, \mathbb{R})$, where $H$ is the 3-dimensional Heisenberg group and

$$SL(2, \mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det X = 1 \}$$  \hspace{1cm} (1)$$

Secondly, I will give classification for all left ideals of the group algebra $L^1(H \rtimes AN)$, where $AN$ is the solvable Lie group in the Iwasawa decomposition $KAN$ of $SL(2, \mathbb{R})$.

2 Fourier Transform and Plancherel Formula on $SL(2, \mathbb{R})$

2.1. In the following and far away from the representations theory of Lie groups we use the Iwasawa decomposition of $SL(2, \mathbb{R})$, to define the Fourier transform and to demonstrate Plancherel formula on the connected real semi-simple Lie group $SL(2, \mathbb{R})$. Therefore let $SL(2, \mathbb{R})$ be the real Lie group, which is

$$SL(2, \mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{R}^4 \text{ and } ad - bc = 1 \}$$  \hspace{1cm} (2)$$

and let $SL(2, \mathbb{R}) = KNA$ be the Iwasawa decomposition of $SL(2, \mathbb{R})$, where

$$K = \{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} = SO(2) : \phi \in \mathbb{R} \}$$

$$N = \{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \}$$

$$A = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^*_+ \}$$  \hspace{1cm} (3)$$

Hence every $g \in SL(2, \mathbb{R})$ can be written as $g = kan \in SL(2, \mathbb{R})$, where $k \in K$, $a \in A$, $n \in \mathbb{R}$.

2.2. We denote by $L^1(G)$ the Banach algebra that consists of all complex valued functions on the group $G$, which are integrable with respect to the Haar measure $dg$ of $G$ and multiplication is defined by convolution product on $G$, where $G = SL(2, \mathbb{R})$. And denote by $L^2(G)$ the Hilbert space of $G$. So we have for any $f \in L^1(G)$ and $\phi \in L^1(G)$

$$\phi \ast f(h) = \int_G f(g^{-1}h)\phi(g)dg$$  \hspace{1cm} (4)$$

The Haar measure $dg$ on a connected real semi-simple Lie group $G = SL(n, \mathbb{R})$, can be calculated from the Haar measures $dn$, $da$ and $dk$ on $N$; $A$ and $K$ respectively,
by the formula

\[ \int_G f(g) \, dg = \int_A \int_N \int_K f(ank) \, dadn \, dk \]  \hfill (5)

Keeping in mind that \( a^{-2\rho} \) is the modulus of the automorphism \( n \rightarrow ana^{-1} \) of \( N \) we get also the following representation of \( dg \)

\[ \int_G f(g) \, dg = \int_A \int_N \int_K f(ank) \, dadn \, dk = \int_N \int_A \int_K f(nak) a^{-2\rho} \, dndak \]  \hfill (6)

where

\[ \rho = 2^{-1} \sum_{\alpha \geq 0, \alpha \neq 0} m(\alpha) \alpha \]

and \( m(\alpha) \) denotes the multiplicity of the root \( \alpha \) see [17] or again \( \rho \) = the dimension of the nilpotent group \( N \). Furthermore, using the relation \( \int_G f(g) \, dg = \int_G f(g^{-1}) \, dg \), we receive

\[ \int_G f(g) \, dg = \int_K \int_A \int_N f(kan) a^{2\rho} \, dndak \]  \hfill (7)

2.3. Let \( \Gamma \) be a connected compact Lie group and let \( \mathfrak{g} \) be the Lie algebra of \( \Gamma \). Let \( (X_1, X_2, \ldots, X_m) \) a basis of \( \mathfrak{g} \), such that the both operators

\[ \Delta = \sum_{i=1}^{m} X_i^2 \]  \hfill (8)

\[ D_q = \sum_{0 \leq l \leq q} \left( -\sum_{i=1}^{m} X_i^2 \right)^l \]  \hfill (9)

are left and right invariant (bi-invariant) on \( \Gamma \), this basis exist see [2, p.564]. For \( l \in \mathbb{N} \), let \( D^l = (1 - \Delta)^l \), then the family of semi-norms \( \{\sigma_l, \, l \in \mathbb{N}\} \) such that

\[ \sigma_l(f) = \int_{\Gamma} |D^l f(y)|^2 \, dy \frac{1}{2} \]  \hfill (10)

define on \( C^\infty(\Gamma) \) the same topology of the Frechet topology defined by the semi-norms \( \|X^\alpha f\|_2 \) defined as

\[ \|X^\alpha f\|_2 = \int_{\Gamma} (|X^\alpha f(y)|^2 \, dy)^\frac{1}{2} \]  \hfill (11)

where \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \), see [2, p.565]

Let \( \hat{\Gamma} \) be the set of all equivalence classes of irreducible unitary representations of \( \Gamma \). If \( \gamma \in \hat{\Gamma} \), we denote by \( E_\gamma \) the space of representation \( \gamma \) and \( d_\gamma \) its dimension then we get
Definition 2.1. The Fourier transform of a function \( f \in C^\infty(\Gamma) \) is defined as
\[
Tf(\gamma) = \int_\Gamma f(x)\gamma(x^{-1})dx
\]  
(12)
where \( T \) is the Fourier transform on \( \Gamma \).

Theorem (A. Cerezo) 2.1. Let \( f \in C^\infty(\Gamma) \), then we have the inversion of the Fourier transform
\[
f(x) = \sum_{\gamma \in \hat{\Gamma}} d_\gamma tr[Tf(\gamma)]\gamma(x)
\]  
(13)
\[
f(I_\Gamma) = \sum_{\gamma \in \hat{\Gamma}} d_\gamma tr[Tf(\gamma)]
\]  
(14)
and the Plancherel formula
\[
\|f(x)\|_2^2 = \int_\Gamma |f(x)|^2 dx = \sum_{\gamma \in \hat{\Gamma}} d_\gamma \|Tf(\gamma)\|_{H.S}^2
\]  
(15)
for any \( f \in L^1(\Gamma) \), where \( I_\Gamma \) is the identity element of \( \Gamma \) and \( \|Tf(\gamma)\|_{H.S} \) is the Hilbert-Schmidt norm of the operator \( Tf(\gamma) \).

Fourier did not actually assume any underlying group structure or representation theory but we typically associate his work with the case of the circle group in the following form using complex exponentials
\[
f(x) = \sum_{n=-\infty}^{\infty} Tm(m)e^{ixm} = \sum_{m=-\infty}^{\infty} c_me^{ixm}, \quad m \in \mathbb{Z}
\]  
(16)
where
\[
c_m = Tf(m) = \int_{SO(2)} f(x)e^{-ixm}dx
\]  
(17)

The group is \( SO(2) = S^1 = \mathbb{R}/\mathbb{Z} \) and the multiplicative characters are \( e^{inx} \), group homomorphisms from the circle \( K = SO(2) \) to the multiplicative group of non-zero complex numbers. Fourier actually preferred to express the coefficients using what is now known as the Plancherel formula
\[
\|f(x)\|_2^2 = \int_{SO(2)} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \sum_{n=-\infty}^{\infty} |Tm(m)|^2
\]  
(18)
where
\[
S^1 = SO(2) = \left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} : \phi \in \mathbb{R} \right\}
\]  
(19)
Definition 2.2. For any function \( f \in D(G) \), we can define a function \( \Upsilon(f) \) on \( G \times K = G \times SO(2) \) by

\[
\Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk_1) = f(kna_1)
\]

for \( g = kna \in G \), and \( k_1 \in K \). The restriction of \( \Upsilon(f) * \psi(g, k_1) \) on \( K(G) \) is \( \Upsilon(f) * \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) = f(g) \in D(G) \), and \( \Upsilon(f)(g, k_1) \downarrow_{K} = f(kna) \in D(G) \).

Remark 2.1. \( \Upsilon(f) \) is invariant in the following sense

\[
\Upsilon(f)(gh, h^{-1}k_1) = \Upsilon(f)(g, k_1)
\]

Definition 2.3. If \( f \) and \( \psi \) are two functions belong to \( D(G) \), then we can define the convolution of \( \Upsilon(f) \) and \( \psi \) on \( G \times K = G \times S^1 = G \times SO(2) \) as

\[
\Upsilon(f) * \psi(g, k_1) = \int_G \Upsilon(f)(gg_2^{-1}, k_1)\psi(g_2)dg_2
\]

\[
= \int_{SO(2)} \int_{N} \int_{A} \Upsilon(f)(kna_2^{-1}n_2^{-1}k^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2
\]

So we get

\[
\Upsilon(f) * \psi(g, k_1) \downarrow_{K(G)} = \Upsilon(f) * \psi(I_kna, k_1)
\]

\[
= \int_{SO(2)} \int_{N} \int_{A} f(na_2^{-1}n_2^{-1}k^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2
\]

\[
= \Upsilon(f) * \psi(na, k_1)
\]

where \( g_2 = k_2n_2a_2 \).

Definition 2.4. If \( f \in D(G) \) and let \( \Upsilon(f) \) be the associated function to \( f \), we define the Fourier transform of \( \Upsilon(f)(g, k_1) \) by

\[
\mathcal{F}\Upsilon(f)(I_{S^1}, \xi, \lambda, I_{S^1}) = \mathcal{F}\Upsilon(f)(I_{S^1}, \xi, \lambda, I_{S^1})
\]

\[
= \int_{S^1} \int_{N} \int_{A} \sum_{l=-\infty}^{\infty} \int_{S^1} T\Upsilon(f)(kna, k_1)e^{-ikdk}a^{-i\lambda}e^{-i(\xi, n)}e^{-imk_1}da_1dn_1dk_1
\]

\[
= \int_{S^1} \int_{N} \int_{A} [\Upsilon(f)(I_{S^1}na, k_1)]a^{-i\lambda}e^{-i(\xi, n)}e^{-imk_1}da_1dn_1dk_1
\]

(22)

where \( \mathcal{F} \) is the Fourier transform on \( AN \) and \( T \) is the Fourier transform on \( SO(2) \), and \( I_{S^1} \) is the identity element of \( S^1 = SO(2) \).

Plancherel’s Theorem on the Group \( G \) 2.2. For any function \( f \in L^1(G) \cap L^2(G) \), we get

\[
\int_{G} |f(g)|^2 dg = \int_{A} \int_{N} \int_{S^1} |f(kna)|^2 da_1dn_1dk_1 = \sum_{m=-\infty}^{\infty} \int_{R} \|T\mathcal{F}f(\lambda, \xi, m)\|_2^2 d\lambda d\xi
\]

(23)
\[ f(I_{ANI_{Sl}}) = \int_{N} \int_{A} \sum_{m = -\infty}^{\infty} \mathcal{F} f((\lambda, \xi, m)) d\lambda d\xi = \sum_{m = -\infty}^{\infty} \int_{R} \int_{R} \mathcal{T} f(\lambda, \xi, m) d\lambda d\xi \]  

(24)

where \( I_{A}, I_{N}, \) and \( I_{K} \) are the identity elements of \( A, N \) and \( K \) respectively, where \( \mathcal{F} \) is the Fourier transform on \( AN \) and \( \mathcal{T} \) is the Fourier transform on \( K \), and \( I_{K} \) is the identity element of \( K \).

**Proof:** First let \( \check{v} \) be the function defined by

\[ \check{v}(kna) = \overline{f((kna)^{-1})} = f(a^{-1}n^{-1}k^{-1}) \]  

(25)

Then we have

\[
\int_{G} |f(g)|^2 dg \\
= \mathcal{Y}(f) * \check{f}(I_{Sl}I_{N}I_{A}, I_{Sl}) \\
= \int_{G} \mathcal{Y}(f)(I_{Sl}I_{N}I_{A}(g_2^{-1}), I_{Sl}) \check{f}(g_2) dg_2 \\
= \int_{A} \int_{N} \int_{S_{1}} \mathcal{Y}(f)(a_2^{-1}n_2^{-1}k_2^{-1}, I_{Sl}) \check{f}(k_2n_2a_2) da_2 dn_2 dk_2 \\
= \int_{A} \int_{N} \int_{S_{1}} f(a_2^{-1}n_2^{-1}k_2^{-1})f((k_2n_2a_2)^{-1}) da_2 dn_2 dk_2 \\
= \int_{A} \int_{N} \int_{S_{1}} |f(a_2n_2k_2)|^2 da_2 dn_2 dk_2 
\]  

(26)
Secondly

\[
\mathcal{T}(f) \ast f(I_{S^1} I_{N A}, I_{S^1}) = \int_{S^1} \int_{\mathbb{R}^2} F(\mathcal{T}(f) \ast f)(I_{S^1}, \lambda, \xi, I_{S^1}) d\lambda d\xi
\]

\[
= \int_{S^1} \int_{\mathbb{R}^2} \sum_{m=-\infty}^{\infty} \int_{I_{S^1}} \mathcal{T}(f) \ast f(k_n a, k_1) e^{-i k_1 a} e^{-i \lambda a} e^{-i \xi a} e^{-i m k_1} d\lambda d\xi
\]

\[
= \sum_{m=-\infty}^{\infty} \int_{S^1} \int_{\mathbb{R}^2} \sum_{l=-\infty}^{\infty} \int_{I_{S^1}} \mathcal{T}(f) \ast f(I_{S^1} a, k_1) e^{-i l k_1 a} e^{-i \xi a} e^{-i m k_1} d\lambda d\xi
\]

\[
= \int_{S^1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{S^1} \mathcal{T}(f)(I_{S^1} a a^{-1} a^{-1} k_1) e^{-i \lambda a} e^{-i \xi a} e^{-i m k_1} d\lambda d\xi
\]

where

\[e^{-i \xi a} = e^{-i \xi n}(27)\]

Using the fact that

\[
\int_{S^1} \int_{S^1} f(k_n a) d\lambda d\xi = \int_{S^1} \int_{S^1} f(ka) a^2 d\lambda d\xi
\]

and

\[
\int_{S^1} \int_{S^1} \int_{S^1} f(k_n a) e^{-i \xi a} d\lambda d\xi
\]

\[
= \int_{S^1} \int_{S^1} \int_{S^1} f(k_n a) e^{-i \xi a} a^2 d\lambda d\xi
\]

\[
= \int_{S^1} \int_{S^1} \int_{S^1} f(k_n a) e^{-i \xi a} a^2 d\lambda d\xi
\]

\[
= \int_{S^1} \int_{S^1} \int_{S^1} f(k_n a) e^{-i \xi a} d\lambda d\xi
\]
Then we get

\[
\mathcal{T}(f) * f(I_{S1}I_{N1}I_{A1}I_{S1})
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} f(na_{2}^{-1}n_{2}^{-1}k_{2}^{-1}, k_{1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(a_{2}^{-1}n_{2}^{-1}k_{2}^{-1}, k_{1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{2}^{-1}k_{1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{2}^{-1}k_{1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}e^{-imk_{2}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{1}^{-1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}e^{-imk_{2}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{1}^{-1})f(k_{2}n_{2}a_{2})e^{-imk_{1}}e^{-imk_{2}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{1}^{-1})f(a_{2}n_{2}k_{2})e^{-imk_{1}}e^{-imk_{2}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \int_{A} \sum_{m=-\infty}^{\infty} \int_{S1} \int_{S1} f(ak_{1}^{-1})f(a_{2}n_{2}k_{2})e^{-imk_{1}}e^{-imk_{2}}dk_{1}dk_{2} \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} T\mathcal{F}f(\lambda, \xi, m)T\mathcal{F}f(\lambda, \xi, m)d\lambda d\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_{m=-\infty}^{\infty} |T\mathcal{F}(f)(\lambda, \xi, m)|^2 d\lambda d\xi
3 Fourier Transform and Plancherel Formula $H$.

3.1. Let $H$ be the real Heisenberg group of dimension $2n + 1$ which consists of all matrices of the form

\[
\begin{pmatrix}
1 & x & z \\
0 & I & y \\
0 & 0 & 1
\end{pmatrix}
\]

(30)

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $I$ is the identity matrix of order $n$.

Let $H = \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$ be the group of the semi-direct product of the group $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, via the group homomorphism $\iota : \mathbb{R}^n \to Aut(\mathbb{R}^{n+1})$, which is defined by:

\[
\iota(x)(z,y) = (z + xy, y) = x(z,y)
\]

(31)

for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $xy = \sum_{i=1}^n x_i y_i$, where $Aut(\mathbb{R}^{n+1})$ is the group of all automorphism of $\mathbb{R}^{n+1}$.

3.2. Let $C^\infty(H)$, $\mathcal{D}(H)$, $\mathcal{D}'(H)$, $\mathcal{E}'(G)$ respectively the space of $C^\infty$-functions, $C^\infty$ with compact support, distribution and distribution with compact support on $G$. The Schwartz space $S(G)$ of $G$ can be considered as the Schwartz space $S(\mathbb{R}^{2n+1})$ of the vector group $\mathbb{R}^{2n+1}$. The action $\iota$ of the group $\mathbb{R}^n$ on $\mathbb{R}^{n+1}$ defines a natural action $\iota$ on the dual $(\mathbb{R}^n)^*$ of the group $\mathbb{R}^{n+1}$($((\mathbb{R}^{n+1})^* \simeq \mathbb{R}^{n+1}$) which is given by :

\[
x(\eta, \lambda) = (\eta, \eta x + \lambda)
\]

for any $\lambda \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, where :

\[
x(\eta, \lambda) = \iota(x)(\eta, \lambda)
\]

and

\[
\eta x = \sum_{i=1}^n \eta x_i
\]

Definition 3.1. For every $f \in S(G)$, one can define its Fourier transform $\mathcal{F}f$ by :

\[
\mathcal{F}f(\xi) = \int_G f(X) \ e^{-i\langle \xi, X \rangle} \ dX
\]

(32)

where $X = ((z, y); x) \in G$, $\xi = ((\eta, \lambda); \mu) \in G$, and $dX = dz \ dy \ dx$ the Lebesgue measure on $G$

\[
\langle \xi, X \rangle = z\eta + y\lambda + x\mu = z\eta + \sum_{i=1}^n \lambda_i y_i + \sum_{i=1}^n x_i \mu_i
\]
It is clear that the function $Ff \in S(G)$ and the mapping $f \mapsto Ff$ is a topological isomorphism vector space $S(G)$ onto itself.

**Theorem 3.1.** The Fourier transform $F$ satisfies:

$$\forall g \ast f(0) = \int_{G} Ff(\xi) \overline{Fg(\xi)} \, d\xi$$  \hspace{1cm} (33)

for every $f \in S(G)$ and $g \in S(G)$, where $\overline{g(X)} = g(X^{-1})$, $\xi = ((\eta, \lambda); \mu)$, $d\xi = d\eta d\lambda d\mu$, is the Lebesgue measure on $G = \mathbb{R}^{2n+1}$, and $\ast$ denotes the convolution product on $G$.

**Proof:** By the classical Fourier transform, we have:

$$\forall g \ast f(0) = \int_{G} F(g \ast f)(\xi) \, d\xi$$

$$= \int_{G} \int_{G} g \ast f(X) e^{-i\langle\xi,X\rangle} \, dX \, d\xi$$

$$= \int_{G} \int_{G} \int_{G} f(Y^{-1}X) \overline{g(Y^{-1})} e^{-i\langle\xi,X\rangle} \, dY \, dX \, d\xi$$

$$= \int_{G} \int_{G} \int_{G} f(YX) \overline{g(Y)} e^{-i\langle\xi,X\rangle} \, dY \, dX \, d\xi$$  \hspace{1cm} (34)

By change of variable $YX = X'$, with $X' = ((z, y); x)$ and $Y = ((z', y'); x')$ we get:

$$X = Y^{-1}X' = ((-x'(z', -y')) - x')(z, y); x)$$

$$= ((-x'(-z, z, y - y')); x - x')$$

this gives us:

$$e^{-i\langle\xi,X\rangle}$$

$$= e^{-i\langle\xi,Y^{-1}X'\rangle}$$

$$= e^{-i((-x'(\eta, \lambda); \mu); ((z-z', y-y'); x-x'))}$$

$$= e^{-i(((\eta - n x' + \lambda); \mu); ((z-z', y-y'); x-x'))}$$  \hspace{1cm} (35)

By the invariant of the Lebesgue measures $d\eta$, $d\lambda$, and $d\mu$ we obtain,
\[ g * f(0) = \int_G f(X) \overline{g(X)} \, dX \]
\[ = \int_G \int_G \int_G f(X) e^{-i\langle \xi, X \rangle} \overline{g(Y) e^{-i\langle \xi, Y \rangle}} \, dX \, dY \, d\xi \]
\[ = \int_G Ff(\xi) \overline{Fg(\xi)} \, d\xi \]

(36)

where 0 = ((0, 0); 0) is the identity of \( G \), whence the theorem.

**Corollary 3.1.** In theorem 3.1, if we take \( g = \hat{f} \), we obtain the Plancherel formula on \( G \)

\[ \hat{f} * f(0) = \int_G |f(X)|^2 \, dX = \int_{\mathbb{R}^n} |\mathcal{F}f(\xi)|^2 \, d\xi \]

(37)

4 **Fourier Transform and Plancherel Formula on the Real Jacobi group** \( N \rtimes SL(2, \mathbb{R}) \)

4.1. Let \( H \) be the 3–dimensional Heisenberg group, with multiplication

\[(z_1, y_1, x_1)(z_2, y_2, x_2) = (z_1 + z_2 + x_1y_2 - x_2y_1, y_1 + x_1, x_2 + y_2)\]

(38)

The group \( H \) is isomorphic onto the following Heisenberg group of all matrices

\[ N = \{ \mathbb{R}^2 \rtimes_\sigma \mathbb{R} \simeq \begin{pmatrix} 1 & x & z \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : (z, y, x) \in \mathbb{R}^3 \} \]

(39)

where \( \sigma : \mathbb{R} \to Aut (\mathbb{R}^2) \) is the group homomorphism from the real group into the group \( Aut (\mathbb{R}^2) \) of all automorphisms of the vector group \( \mathbb{R}^2 \), defined as

\[ \sigma(x)(z, y) = (z + xy, y) \]

So the group \( H \) can be identified with the group \( N \), where the multiplication becomes as

\[(z_1, y_1, x_1)(z_2, y_2, x_2) = (z_1 + z_2 + x_1y_2, y_1 + x_1, x_2 + y_2)\]

(40)

Now we define the Jacobi group \( J \) as \( N \rtimes_\rho SL(2, \mathbb{R}) \) the semidirect of the Heisenberg group \( N \) and the real semisimple Lie group \( SL(2, \mathbb{R}) \), where \( \rho : \)
\( SL(2, \mathbb{R}) \rightarrow Aut(N) \) is the group homomorphism from the real group into the group \( Aut(N) \) of all automorphisms of the vector group \( N \), defined as

\[
\rho(M)(z, y, x) = (z, \begin{bmatrix} y & x \end{bmatrix} M) \\
= (z, \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \\
= (z, ya + xc, yb + xd)
\]

where \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). So, any element \( g \in J \) can be written in an unique way as \( g = (X, M) \) with \( M \in SL(2, \mathbb{R}) \) and \( X = (z, y, x) \in N \). Multiplication in \( J \) is then given as

\[
(X_1, M_1)(X_2, M_2) = (z_1, y_1, x_1, M_1)(z_2, y_2, x_2, M_2) \\
= (z_1 + z_2 + x_1 y_2 a_1 + x_1 x_2 c_1, y_1 + y_2 a_1 + x_2 c_1, x_1 + y_2 b_1 + x_2 d_1, M_1 M_2)
\]

where \( X_1 = (z_1, y_1, x_1) \in N, X_2 = (z_2, y_2, x_2) \in N, M_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in SL(2, \mathbb{R}) \)

From now on, our useful for the multiplication in \( J \) will be as

\[
(X_1, M_1)(X_2, M_2) = (X_1 \rho(M_1)(X_2), M_1 M_2)
\]

**Definition 4.1.** Let \( Q = H \times SL(2, \mathbb{R}) \rtimes \rho SL(2, \mathbb{R}) \) be the group with law:

\[
X \cdot Y = (X_1, M_1, M_2)(Y_1, N_1, N_2) \\
= (X_1 \rho(M_1)(Y_1), M_1 + N_1, M_2 + N_2)
\]

for all \( X = (X_1, M_1, M_2) \in Q \) and \( Y = (Y_1, N_1, N_2) \in Q \). From definition 4.1, the Jacobi group \( J \) can be identified with a subgroup \( N \times \{I_{SL(2, \mathbb{R})}\} \rtimes \rho SL(2, \mathbb{R}) \) of \( Q \). Let \( A = N \times SL(2, \mathbb{R}) \rtimes \rho \{I_{SL(2, \mathbb{R})}\} \) be the subgroup of \( Q \), which is the direct product of \( N \) with \( SL(2, \mathbb{R}) \)

**Definition 4.2.** For any function \( f \in \mathcal{D}(J) \), we can define a function \( \tilde{f} \) on \( Q \) by

\[
\tilde{f}(X, M_1, M_2) = f(MX, M_1 M_2)
\]

**Remark 4.1.** The function \( \tilde{f} \) is invariant in the following sense

\[
\tilde{f}(N^{-1}X, M_1, N^{-1}M_2) = \tilde{f}(N^{-1}X, M_1, N^{-1}M_2)
\]

**Theorem 4.1.** For any function \( \psi \in \mathcal{D}(J) \) and \( \tilde{f} \in \mathcal{D}(Q) \) invariant in sense (32), we get

\[
\psi \ast \tilde{f}(X, M_1, M_2) = \tilde{f} \ast_c \psi(X, M_1, M_2)
\]
where $*$ signifies the convolution product on $J$ with respect the variable $(X, M_2)$, and $\ast_c$ signifies the convolution product on $A$ with respect the variable $(X, M_1)$.

Proof: In fact we have

$$\psi \ast \tilde{f}(X, M_1, M_2)$$

$$= \int \int_{N \times S(2, \mathbb{R})} \tilde{f}(X, M_1, M_2) \psi(Y, M) dYdM$$

$$= \int \int_{N \times S(2, \mathbb{R})} f^{-1}(Y, M)(X, M_1, M_2) \psi(Y, M) dYdM$$

$$= \int \int_{N \times S(2, \mathbb{R})} f^{-1}(Y - X, M_1, M_1 M_2)(X, M_1, M_2) \psi(Y, M) dYdM$$

$$= \int \int_{N \times S(2, \mathbb{R})} f^{-1}(Y - X, M_1, M_1 M_2) \psi(Y', g') dYdM$$

$$= \int \int_{N \times S(2, \mathbb{R})} f(Y - X, M_1 M_2) \psi(Y, M) dYdM = \tilde{f} \ast_c \psi(X, M_1, M_2)$$

for any function $\psi \in \mathcal{D}(J)$ and $\tilde{f} \in \mathcal{D}(Q)$

**Definition 4.3.** For any $k_1 \in S^1$ let $\Gamma_k \Psi$ be the function defined by

$$\Gamma_k \Psi(v, g) = \Psi(X, gk_1)$$  \hspace{1cm} (48)

for any $v \in N$, $g \in \text{SL}(2, \mathbb{R})$ and $k_1 \in S^1$.

**Definition 4.4.** Let $f \in C^\infty_0(J)$, we define its Fourier transform by

$$\mathcal{F}_N \mathcal{F}_A \Psi(\eta, m, \xi, \lambda) = \int \int_{N \times A} \int_{S^1} \Psi(v, kNa) e^{-i \langle \eta, v \rangle} e^{-ikm} e^{-i\lambda n} dkdndv$$

where $\mathcal{F}_N$ is the Fourier transform on $N$, $kNa = g$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$, $v = (v_1, v_2, v_3) \in N$, and $dv = dv_1 dv_2 dv_3$ is the Lebesgue measure on $N$

$$(\eta, v) = \langle (\eta_1, \eta_2, \eta_3), (v_1, v_2, v_3) \rangle = \eta_1 v_1 + \eta_2 v_2 + \eta_3 v_3$$  \hspace{1cm} (49)

**Plancherel’s Theorem 4.2.** For any function $f \in L^1(J) \cap L^2(J)$, we get

$$\int_J |\Psi(v, g)|^2 dv dg = \int \int \sum_{m=-\infty}^{\infty} |\mathcal{F}_N \mathcal{F}_A \Psi(\eta, m, \xi, \lambda)| d\eta d\lambda d\xi$$  \hspace{1cm} (50)
Proof: For any function $\Psi \in L^1(J) \cap L^2(J)$, we get

$$\Gamma_{I_k} \Psi \ast \Psi(0, I_{SL(2,\mathbb{R})}, I_{SL(2,\mathbb{R})})$$

$$= \sum_{m=-\infty}^{\infty} [\Gamma_{k_1} \Psi \ast \Psi(0, I_{SL(2,\mathbb{R})}, I_{SL(2,\mathbb{R})})e^{-ikm}dk_1]$$

$$= \int_{N} \int_{SL(2,\mathbb{R})} \sum_{m=-\infty}^{\infty} \int_{S^1} \Psi((w, g)^{-1}(0, I_{SL(2,\mathbb{R})}, I_{SL(2,\mathbb{R})}))\Gamma_{k_1} \Psi(w, g)e^{-ikm}dk_1]dwdg$$

$$= \int_{N} \int_{SL(2,\mathbb{R})} \sum_{m=-\infty}^{\infty} \int_{S^1} \Psi(g^{-1}(0 - w), I_G, g^{-1}I_G)\Psi(w, k_1g)e^{-ikm}dk_1]dwdg$$

$$= \int_{N} \int_{SL(2,\mathbb{R})} \sum_{m=-\infty}^{\infty} \int_{S^1} \Psi(-w, I_Gg^{-1}, I_G)\Psi(w, k_1g)e^{-ikm}dk_1]dwdg$$

$$= \int_{N} \int_{SL(2,\mathbb{R})} \sum_{m=-\infty}^{\infty} \int_{S^1} \Psi(g^{-1}(-w), g^{-1})\Psi(w, k_1g)e^{-ikm}dk_1]dwdg$$

$$= \int_{N} \int_{SL(2,\mathbb{R})} \Psi(g^{-1}(-w), g^{-1})\Psi(w, I_{S^1}g)]dwdg$$

$$= \int_{H} \int_{SL(2,\mathbb{R})} \Psi(g^{-1}(-w), g^{-1})^{-1}\Psi(w, g)]dwdg$$

$$= \int_{H} \int_{SL(2,\mathbb{R})} \Psi(w, g)\Psi(w, g)]dwdg$$

$$= \int_{H} \int_{SL(2,\mathbb{R})} |\Psi(w, g)|^2 dwdg = \int_{J} |\Psi(w, g)|^2 dwdg$$

In other hand
$$\Gamma_k \Psi \ast \overline{\Psi}(0, I_G, I_G)$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}(\Gamma_k, \Psi \ast \overline{\Psi})(\eta, \xi, \lambda, m, I_G) e^{-ik_1 m} dk_1] d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\Gamma_k \Psi \ast \overline{\Psi}(v, lkna, I_G) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((w, g_2)^{-1} (v, lkna, I_G) \Gamma_k, \Psi(w, g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((g_2^{-1}(-w), g_2^{-1})(v, lkna, I_G) \Gamma_k, \Psi(w, g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((g_2^{-1} - w, g_2^{-1})(v, bkna, g_2^{-1} I_G) \Gamma_k, \Psi(w, g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((g_2^{-1} - w, I_kna, g_2^{-1} I_G) \Gamma_k, \Psi(w, g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((v - w, k_1 g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((v - w, akg_2^{-1}, I_G) \Psi(w, k_1 g_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((v - w, I_knaa^{-1} a^{-1} k_2^{-1}, I_G) \Psi(w, k_1 k_2 a_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$

$$= \int_{K} \int_{R} \int_{R} \sum_{m=-\infty}^{\infty} [F_{HTF}[\int_{K} \overline{\Psi}((v - w, I_knaa^{-1} a^{-1} k_2^{-1}, I_G) \Psi(w, k_1 k_2 a_2) \gamma(k_1^{-1}) dk_1] e^{-i(v, \eta)} e^{-i(n, \xi)} a^{-i\lambda} d\eta d\xi d\lambda$$
We continue our calculation.

\[
\Gamma_k \Psi(0, I_G, I_G)
= \int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi((v, an\gamma_{k}^{-1})I_G)\Psi(w, k_2 n_2 a_2) \\
\gamma(k^{-1}) \int dk_1 dk_2 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdn_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi((v, an\gamma_{k}^{-1}I_G)\Psi(w, k_2 n_2 a_2)\gamma(k^{-1})\gamma(k^{-1})) \\
dk_1 dk_2 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdn_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi((an\gamma_{k}I_G)\Psi(w, k_2 n_2 a_2)\gamma(k^{-1})\gamma(k^{-1})) \\
dk_1 dk_2 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdn_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi((an\gamma_{k}(v), an\gamma_{k})^{-1})\Psi(w, k_2 n_2 a_2)\gamma(k^{-1}) \\
\gamma(k^{-1}) \int dk_1 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdk_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi(-v, k^{-1}n^{-1}a^{-1})\Psi(w, k_2 n_2 a_2)\gamma(k^{-1})\gamma(k^{-1})) \\
dk_1 dk_2 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdn_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum_{m=-\infty}^{\infty} [F_{H}T_{F}] \int \int \Psi(w, k_2 n_2 a_2)\gamma^{*}(k^{-1})\gamma(k^{-1})) \\
dk_1 dk_2 e^{-i(v, n)} e^{-i(n, \xi)} a_{1}^{-i\xi} e^{-i(w, \eta)} e^{-i(n_2, \xi)} a_2^{-i\xi} dvdndwdn_2 da_2 d\eta d\xi d\lambda
\]

\[
\int \int \int \sum d_{\gamma} \|F_{R_{T}}T_{F}\Psi(\eta, \gamma, \xi, \lambda)\|_{H,S}^{2} d\eta d\xi d\lambda
\]

\[
\int \int \int \sum d_{\gamma} \|F_{R_{T}}T_{F}\Psi(\eta, \gamma, \xi, \lambda)\|_{H,S}^{2} d\eta d\xi d\lambda
\]

5 Left Ideals of the Group Algebra $L^1(N)$.

First, I will prove the solvability of any invariant differential operator on the connected solvable group $N = \mathbb{R}^{2} \ltimes_{\sigma} \mathbb{R}$.
larger group $E = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, with multiplication $(n, a, b)$ and $(m, x, y)$ as

$$ (n, a, b)(m, x, y) = (n + m + \sigma(b)x, a + x, b + y) \quad (51) $$

Let $F = \mathbb{R}^2 \times \mathbb{R}$ be the abelian group, which is the direct product of two real vector groups $\mathbb{R}^2$ and $\mathbb{R}$

**Definition 5.1.** For any function $f \in D(N)$, we can define a function $\tau f$ on $E$ by

$$ \tau f(n, a, b) = f(\sigma(a)n, ab) \quad (52) $$

**Remark 5.1.** The function $\tau f$ is invariant in the following sense

$$ \tau f(\sigma(x^{-1})n, xa, x^{-1}b) = \tau f(n, a, b) \quad (53) $$

Therefor denote by $\tau C^\infty(N)$ (resp. $\tau C^\infty(F)$) the image of $C^\infty(N)$ (resp. $C^\infty(F)$) then we have

$$ \tau C^\infty(N)|_N = C^\infty(N) $$

$$ \tau C^\infty(F)|_F = C^\infty(F) \quad (54) $$

**Definition 5.2.** Let be the mapping $\Lambda : \tau C^\infty(E)|_F \rightarrow \tau C^\infty(E)|_N$ defined by

$$ \Lambda(\tau f|_F)(z, y, 0) = \tau f|_N(z, 0, y) \quad (55) $$

is topological isomorphisms and its inverse is nothing but $\Gamma^{-1}$ defined by

$$ \Lambda^{-1}(\tau f|_N)(n, 0, a) = \tau f|_F(n, a, 0) \quad (56) $$

My main result is

**Theorem 5.1.** If $P_u$ any invariant differential operator on $N$ associated to the distribution $u \in \mathcal{U}$, then, we have

$$ P_u C^\infty(N) = C^\infty(N) \quad (57) $$

**Proof:** Let $Q_u$ be the invariant differential operator with constant coefficients on $K$ associated to $u$, then by the theory of differential operators with constant coefficients [20], we get

$$ Q_u \tau C^\infty(E)|_F = \tau C^\infty(E)|_F = C^\infty(F) \quad (58) $$

That means for any $\psi(n, a) \in C^\infty(F)$, there exist a function $\varphi(n, a, x) \in \tau C^\infty(E)|_F$, such that

$$ Q_u \varphi(n, a, 1) = u \ast_c \varphi(n, a, 0) = \psi(n, a) \quad (59) $$

The function $\psi(n, a)$ can be transformed as an invariant function $\psi \in \tau C^\infty(E)|_F$ as follows

$$ \psi(n, a) = \tau \psi(\rho(a^{-1})n, a, 0) \quad (60) $$

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In other side, we have

\[ \Lambda Q_u \varphi(n, a, 0) = Q_u \varphi(n, 0, a) = u * \varphi(n, 0, a) = u * \varphi(1, a) = P_u \varphi(n, 0, a) = \Lambda \tau \psi(\rho(a^{-1})n, a, 1) = \tau \psi(\sigma(a^{-1})n, 1, a) = \psi(n, a) \]  

(61)

So the proof of the solvability of any right invariant differential operator on \( N \).

If \( I \) is a subspace of \( L^1(N) \), we denote \( \tau I \) its image by the mapping \( \tau \), let \( J = \tau I|_F \). My main result is:

**Theorem 5.2.** Let \( I \) be a subspace of \( L^1(N) \), then the following conditions are equivalents.

(i) \( J = \tau I|_F \) is an ideal in the Banach algebra \( L^1(F) \).

(ii) \( I \) is a left ideal in the Banach algebra \( L^1(N) \).

**Proof:** (i) implies (ii) Let \( I \) be a subspace of the space \( L^1(N) \) and \( \tau I \) the image of \( I \) by \( \tau \) such that \( J = \tau I|_F \) is an ideal in \( L^1(F) \), then we have:

\[ u * _c \tau I|_F(n, a, 0) \subseteq \tau I|_F(n, a, 0) \]  

(62)

for any \( u \in L^1(F) \) and \( (n, a) \in F \), where

\[ u * _c \tau I|_F(n, a, 0) = \left\{ \int_F \tau f|_F \left[ n - m, a - b, 0 \right] u(m, b)dm \frac{db}{b}, \ f \in I \right\} \]  

(63)

It shows that

\[ u * _c \tau f|_F(n, a, 0) \in \tau I|_F(n, a, 0) \]  

(64)

for any \( \tau f \in \tau I \). Apply equation(32), we get

\[ \Gamma(u * _c \tau f|_F)(n, a, 0) = \int \tau f|_F \left[ n - m, a - b, 0 \right] u(m, b)dm \frac{db}{b}, \ f \in I \]  

(65)

(ii) implies (i), if \( I \) is an ideal in \( L^1(N) \), then we get

\[ u * \tau I|_N(n, 0, a) \subseteq \tau I|_N(n, 0, a) = I(n, a) \]  

(66)

where

\[ u * \tau I|_N(n, 1, a) = \left\{ \int_N \tau f|_N \left[ \sigma(-b)(n - m), 1, a - b \right] u(m, b)dm \frac{db}{b}, \ f \in I \right\} \]  

(67)
By equation (36), we obtain
\[ \chi^{-1}(u * \tilde{f} |_N)(n, 0, a) = u * c \tilde{f} |_F(n, a, 0) \in \chi^{-1}(u * \tilde{I} |_N)(n, 0, a) = u * \tilde{I} |_F(n, a, 0) \] (68)

**Corollary 5.1.** Let \( I \) be a subspace of the space \( L^1(N) \) and \( \tau I \) its image by the mapping \( \tau \) such that \( J = \tau I |_F \) is an ideal in \( L^1(F) \), then the following conditions are verified.

1. \( J \) is a closed ideal in the algebra \( L^1(F) \) if and only if \( I \) is a left closed ideal in the algebra \( L^1(N) \).
2. \( J \) is a prime ideal in the algebra \( L^1(F) \) if and only if \( I \) is a left prime ideal in the algebra \( L^1(N) \).
3. \( J \) is a maximal ideal in the algebra \( L^1(F) \) if and only if \( I \) is a left maximal ideal in the algebra \( L^1(N) \).
4. \( J \) is a dense ideal in the algebra \( L^1(F) \) if and only if \( I \) is a left dense ideal in the algebra \( L^1(N) \).

The proof of this corollary results immediately from theorem 5.2.

6 **Left Ideals of the Group Algebra** \( L^1(N \rtimes S) \).

Let \( S = SL(2, \mathbb{R})/SO(2) \) the symmetric space of the real semi simple Lie group \( SL(2, \mathbb{R}) \), which is diffeomorphism on the group
\[ S = SL(2, \mathbb{R})/SO(2) = \{ \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix} , a \in \mathbb{R}^*_+ \} \] (69)

The group \( S \) is isomorphic onto the group \( \mathbb{R} \times \mathbb{R}^*_+ \) semidirect product of the two group \( \mathbb{R} \) and \( \mathbb{R}^*_+ \) where \( \varphi : \mathbb{R}^*_+ \rightarrow Aut(\mathbb{R}) \) is the group homomorphism from the real group into the group \( Aut(\mathbb{R}) \) of all automorphisms of the vector group \( \mathbb{R} \), defined as
\[ \varphi(x)(n) = xn \]

First, I will prove the solvability of any invariant differential operator on the connected solvable group \( S \). Therefore denote by \( W = \mathbb{R} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \), with the following law defined as
\[ (n, x, y)(m, a, b) = (n + \varphi(y)m, xa, yb) = (n + m + y^2a, xa, yb) \] (70)

for any \((n, a) \in S \), and \((m, b) \in S \), here \( \varphi(a)m = a^2m \). Let \( K \) be the group \( \mathbb{R} \times \mathbb{R}^*_+ \), which is the direct product of the group \( \mathbb{R} \) with the group \( \mathbb{R}^*_+ \). So the group \( S \) can be identified with the subgroup \( \mathbb{R} \times \mathbb{R}^*_+ \times \{1\} \) of \( W \) and \( K \) can be identified with the subgroup \( \mathbb{R} \times \mathbb{R}^*_+ \times \{1\} \) of \( W \).
Definition 6.1. For any function $f \in D(S)$, we can define a function $\tau f$ on $W$ by

$$
\tau f(n,a,b) = f(\varrho(a)n,ab)
$$

(71)

Remark 6.1. The function $\tau f$ is invariant in the following sense

$$
\tau f(\varrho(x^{-1})n,xa,x^{-1}b) = \tau f(n,a,b)
$$

(72)

Now denote by $\tau (C^\infty(S))$ (resp. $\tau (C^\infty(K))$) the image of $C^\infty(S)$ (resp.$C^\infty(K)$) by the transformation $\tau$, then we have

$$
\tau(C^\infty(S))|_{S} = C^\infty(S)
$$

$$
\tau(C^\infty(K))|_{K} = C^\infty(K)
$$

(73)

Definition 6.2. Let be the mapping $\chi : \tau(C^\infty(K))|_{K} \rightarrow \tau(C^\infty(S))|_{S}$ defined by

$$
\tau f|_{K} (z,y,1) \rightarrow \tau f|_{N}(z,1,y)
$$

(74)

$$
\tau f|_{K} (n,a,1) \rightarrow \tau f|_{S}(n,1,a)
$$

(75)

is topological isomorphisms and its inverse is nothing but $\chi^{-1}$ defined by

$$
\tau f|_{S} (n,1,a) \rightarrow \tau f|_{K}(n,a,1)
$$

(76)

My main result is

Theorem 6.1. If $P_u$ any invariant differential operator on $S$ associated to the distribution $u \in \mathcal{U}$, then, we have

$$
P_u C^\infty(S) = C^\infty(S)
$$

(77)

Proof: Let $Q_u$ be the invariant differential operator with constant coefficients on $K$ associated to $u$, then by the theory of differential operators with constant coefficients [20], we get

$$
Q_u \tau(C^\infty(K))|_{K} = \tau(C^\infty(K))|_{K} = C^\infty(K)
$$

(78)

That means for any $\psi(n,a) \in C^\infty(K)$, there exist a function $\varphi(n,a,x) \in \tau(C^\infty(K))|_{K}$, such that

$$
Q_u \varphi(n,a,1) = u \ast_c \varphi(n,a,1) = \psi(n,a)
$$

(79)

The function $\psi(n,a)$ can be transformed as an invariant function $\psi \in \tau(C^\infty(K))|_{K}$ as follows

$$
\psi(n,a) = \tau \varphi(\varrho(a^{-1})n,a,1)
$$

(80)
In other side, we have
\[
\chi Q_u \varphi(n, a, 1) = Q_u \varphi(n, 1, a) = u \ast_\varepsilon \varphi(n, 1, a) = u \ast \varphi(n, 1, a) = \chi \tau \psi(g(a^{-1})n, a, 1) = \tau \psi(g(a^{-1})n, 1, a) = \psi(n, a)
\] (81)

So the proof of the solvability of any right invariant differential operator on \( S \).

If \( I \) is a subspace of \( L^1(S) \), we denote by \( \tau I \) its image by the mapping \( \tau \), let \( \omega = \tau I|_K \). My main result is:

**Theorem 6.2.** Let \( I \) be a subspace of \( L^1(S) \), then the following conditions are equivalents.

(i) \( \omega = \tau I|_K \) is an ideal in the Banach algebra \( L^1(K) \).

(ii) \( I \) is a left ideal in the Banach algebra \( L^1(S) \).

**Proof:** (i) implies (ii) Let \( I \) be a subspace of the space \( L^1(S) \) such that \( \omega = \tau I|_K \) is an ideal in \( L^1(K) \), then we have:
\[
\chi \tau \psi(g(a^{-1})n, a, 1) \subseteq \tau I|_K(n, 1, a) = I(n, a)
\] (82)

for any \( \chi \tau \psi(g(a^{-1})n, a, 1) \in \tau I|_K(n, a, 1) \) for any \( \tau f \in \tau I \). According to equation (82), we get
\[
\chi(\chi \tau \psi(g(a^{-1})n, a, 1) = \chi(n, a, 1) = \tau I|_K(n, 1, a) = I(n, a)
\] (85)

(ii) implies (i), if \( I \) is an ideal in \( L^1(S) \), then we get
\[
\chi(n, a, 1) = \chi(n, 1, a) \subseteq \tau I|_S(n, 1, a) = I(n, a)
\] (86)

where

\[
u \ast \tau I|_S(n, 1, a) = \int_S \tau f|_S \rho(-b)\rho(n - m), 1, a - b) u(m, b) dm \frac{db}{b}, \ \ f \in I
\] (87)
By equation (36), we obtain

\[
\begin{align*}
\chi^{-1}(u \ast \tau f_S)(n,1,a) \\
&= u \ast \tau f_S|_K(n,a,1) \in \chi^{-1}(u \ast \tau I|_S)(n,a,1) \\
&= u \ast \tau I|_K(n,a,1)
\end{align*}
\]

(88)

**Corollary 6.1.** Let \( I \) be a subspace of the space \( L^1(S) \) and \( \tau I \) its image by the mapping \( \tau \) such that \( \omega = \tau I|_K \) is an ideal in \( L^1(K) \), then the following conditions are verified.

1. \( \omega \) is a closed ideal in the algebra \( L^1(K) \) if and only if \( I \) is a left closed ideal in the algebra \( L^1(S) \).
2. \( \omega \) is a prime ideal in the algebra \( L^1(K) \) if and only if \( I \) is a left prime ideal in the algebra \( L^1(S) \).
3. \( \omega \) is a maximal ideal in the algebra \( L^1(K) \) if and only if \( I \) is a left maximal ideal in the algebra \( L^1(S) \).
4. \( \omega \) is a dense ideal in the algebra \( L^1(K) \) if and only if \( I \) is a left dense ideal in the algebra \( L^1(S) \).

The proof of this corollary results immediately from theorem 6.2.

The Heisenberg group \( N \) is the semi-direct product of the two vector Lie group \( \mathbb{R}^2 \times_a \mathbb{R} \). I extend the group \( M = N \times S \) by considering the new group \( V = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times S \) with the following law

\[
X \cdot Y = (n_3, n_2, n_1, n_4, a_1, a_2, a_3)(m_3, m_2, m_1, m_4, b_1, b_2, b_3)
\]

\[
= (n_3 + m_3 + \sigma(n_4)(m_3, m_2), n_2 + m_2, n_1 + m_1, n_4 + m_4, (a_1b_1, a_2b_2, a_3b_3))
\]

\[
= (n_3 + m_3 + n_4m_2, n_2 + m_2, n_1 + m_1, n_4 + m_4, (a_1b_1, a_2b_2, a_3b_3))
\]

(89)

Denote by \( B = \mathbb{R}^2 \times \mathbb{R} \times S \) the commutative Lie group of the direct product of three Lie groups \( \mathbb{R}^2, \mathbb{R} \), and \( S \). In this case the group \( M = N \times S \) can be identified with the sub-group \( \mathbb{R}^2 \times \{0\} \times \mathbb{R} \times S \) and the group \( B = \mathbb{R}^2 \times \mathbb{R} \times S \) can be identified with the sub-group \( \mathbb{R}^2 \times \mathbb{R} \times \{0\} \times S \).

**Definition 6.3.** Any function \( \psi \in C^\infty(M) \) can be extended to a unique function \( \Xi \psi \) belongs to \( C^\infty(V) \), as follows

\[
\Xi \psi((n_3, n_2, n_1, n_4), s)
\]

\[
= \psi((\sigma(n_1)(n_3, n_2), n_1 + n_4), s)
\]

\[
= \psi((n_1(n_3, n_2), n_1 + n_4), s)
\]

\[
= \psi((n_3 + n_1n_2, n_2, n_1 + n_4), s)
\]

(90)

for any \((n_3, n_2, n_1, n_4) \in N \times \mathbb{R}, s \in S, n_1(n_3, n_2) = (n_3 + n_1n_2, n_2) = \sigma(n_1)(n_3, n_2) \) if \( I \) is a subspace of \( L^1(M) \), we denote \( \Xi I \) its image by the mapping \( \Xi \). Let \( J = \Xi I \mid_B \).

My main result is:

**Theorem 6.3.** Let \( I \) be a subspace of \( L^1(K) \), then the following conditions are equivalents.

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\( i \) \( J = \overline{I} |_B \) is an ideal in the Banach algebra \( L^1(B) \).
\( ii \) \( I \) is a left ideal in the Banach algebra \( L^1(M) \).

For the proof of this theorem, I refer to my book \[9, ChapI, theorem 3.1.\]

**Corollary 6.2.** Let \( \Xi \) be a subspace of the space \( L^1(M) \) and \( \Xi I \) its image by the mapping \( \Xi \) such that \( J = \Xi I |_B \) is an ideal in \( L^1(B) \), then the following conditions are verified.

1. \( J \) is an ideal in the algebra \( L^1(B) \) if and only if \( I \) is a closed ideal in the algebra \( L^1(M) \) if and only if \( I \) is a closed left ideal in the algebra \( L^1(N \rtimes S) \).
2. \( J \) is a prime ideal in the algebra \( L^1(B) \) if and only if \( I \) is a prime ideal in the algebra \( L^1(M) \) if and only if \( I \) is a prime left ideal in the algebra \( L^1(N \rtimes S) \).
3. \( J \) is a maximal ideal in the algebra \( L^1(B) \) if and only if \( I \) is a maximal ideal in the algebra \( L^1(M) \) if and only if \( I \) is a left maximal ideal in the algebra \( L^1(N \rtimes S) \).
4. \( J \) is a dense ideal in the algebra \( L^1(B) \) if and only if \( I \) is a dense ideal in the algebra \( L^1(M) \) if and only if \( I \) is a left dense ideal in the algebra \( L^1(N \rtimes S) \).

For the proof of this theorem, I refer to **Theorem 6.2.** and **Corollary 6.1.**

**References**

[1] M. F. Atiyah, Resolution of Singularities and Division of Distributions, Comm. on Pure and App. Math, vol, 23, pp.145-150, 1970.
[2] A. Cerezo and F. Rouviere, "Solution elemetaire d’un operator differentielle lineare invariant agauch sur un group de Lie reel compact" Annales Scientiques de E.N.S. 4 serie, tome 2, n°4,p 561-581, 1969.
[3] Chirikjian, G. S., and A. Kyatkin, A., (2000), Engineering Applications in Non-commutative Harmonic Analysis, Johns Hopkins University, Baltimore, Maryland, CRC Press.
[4] K. El- Hussein., (1989), Operateurs Differentiels Invariants sur les Groupes de Deplacements, Bull. Sc. Math. 2e series 113., p. 89-117.
[5] K. El- Hussein., (2009), Eigendistributions for the Invariant Differential operators on the Affine Group. Int. Journal of Math. Analysis, Vol. 3, no. 9, 419-429.
[6] K. El- Hussein., (2010), Fourier transform and invariant differential operators on the solvable Lie group G4, in Int. J. Contemp. Maths Sci. 5. No. 5-8, 403-417.
[7] K. El- Hussein., (2011), On the left ideals of group algebra on the affine group, Int. Math Forum, Int, Math. Forum 6, No. 1-4, 193-202.
[8] K. El- Hussein., (2013), Non Commutative Fourier Transform on Some Lie Groups and Its Application to Harmonic Analysis, International Journal of Engineering Research & Technology (IJERT) Vol. 2 Issue 10, 2429- 2442.
1. K. El- Hussein, (2015), Abstract Harmonic Analysis on Poincare Space-Time, Book, LAP Lambert Academic Publishing (May 21, 2015).

2. Harish-Chandra; (1952), Plancherel formula for $2\times2$ real unimodular group, Proc. nat. Acad. Sci. U.S.A., vol. 38, pp. 337-342.

3. Harish-Chandra; (1952), The Plancherel formula for complex semi-simple Lie group, Trans. Amer. Mth. Soc., vol. 76, pp. 485-528.

4. S. Helgason., (2005), The Abel, Fourier and Radon Transforms on Symmetric Spaces. Indagationes Mathematicae. 16, 531-551.

5. L. Hormander, 1983, The analysis of Linear Partial Differential Operator I”, Springer-Verlag, 1983.

6. Kirillov, A. A., ed, (1994), Representation Theory and Noncommutative Harmonic Analysis I, Springer-Verlag, Berlin.

7. H. Lewy, An Example of a Smooth Linear Partial Differential Operator without Solution, Annals of Mathematics, Vol. 66, No. 2, 1957, pp. 155-158.

8. D. M"uller, and M. Peloso Non-Solvability for a Class of Left-Invariant Second-Order Differential Operators on the Heisenberg Group, Transaction of the American Mathematical Society Volume 355, Number 5, Pages 2047-2064 S 0002-9947(02)03232-4 Article electronically published on December 18, 2002.

9. B. Malgrange, Existence and Approximation des Solutions des équations aux Derivées Partielles et des équations de Convolutions, Ann. Inst. Fourier Grenoble, 6, 271, 1955.

10. Nicolas Lerner, A Tribute to Lars Hormander, Matapli100, 25/4/2013.

11. W. Rudin., (1962), Fourier Analysis on Groups, Interscience Publishers, New York, NY.

12. F. Treves, Linear Partial Differential Equations with Constant Coefficients, Garden and Breach, 1966.

13. Vaen Deal., A., (2007), The Fourier transform in quantum group theory, preprint [math.RA/0609502 at http://lanl.arXiv.org].

14. G. Warner., (1970), Harmonic Analysis on Semi-Simple Lie Groups, Springer-verlag Berlin heidelberg New york.