Solvability of the Dirac equation

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July 28, 2014

Abstract

In this paper, we study the Dirac equation by Hörmander’s $L^2$-method.
For Dirac bundles over 2-dimensional Riemannian manifolds, in compact case we give a sufficient condition for the solvability of the Dirac equation in terms of a curvature integral; in noncompact case, we prove the Dirac equation is always solvable in weighted $L^2$ space. On compact Riemannian manifolds, we give a new proof of Bär’s theorem comparing the first eigenvalue of the Dirac operator with that of the Yamabe type operator. On Riemannian manifolds with cylindrical ends, we obtain solvability in $L^2$ space with suitable exponential weights allowing mild negativity of the curvature. We also improve the above results when the Dirac bundle has a $\mathbb{Z}_2$-grading. Potential applications of our results are discussed.

1 Introduction

In many geometric problems, it is important to determine the solvability of the linear equation

$$Du = f$$

where $D$ is a Dirac operator on some Dirac bundle. For example, the classical Dirac operator on spin manifolds ([1]), the Dolbeault-Dirac operator in Kähler geometry, and the twisted Dirac operator in the normal bundle of instantons (associative submanifolds) in $G_2$ manifolds ([13]). In general, it is not easy to know when (1) is solvable. For the Dirac operator on spin manifolds, a sufficient condition was given by the positivity of the scalar curvature, dating back to a theorem of Lichnerowicz. However, the positive scalar curvature condition is not always necessary, as the Dirac operator on spin manifolds has the remarkable conformal covariance property ([14]), and a conformal change of metric could make the scalar curvature negative somewhere.

In this paper, starting with the Bochner formula, we establish weighted $L^2$-estimates and existence theorems for the Dirac equation, just as Hörmander’s $L^2$-method for the $\bar{\partial}$-equation ([12], [13]). In applications of the $L^2$-method, it is very important to construct good weight functions from geometric conditions ([3], [19]∼[21] and etc.). Sometimes one can gain “extra positivity” in suitable weighted $L^2$-spaces to establish vanishing theorems.
Let $\lambda_S$ be the function on $M$ defined in (10), which pointwisely is the first eigenvalue of some curvature operator. For Dirac equations on 2-dimensional Riemannian manifolds, we have

**Theorem 1.1** Let $\mathbb{S}$ be a Dirac bundle over a 2-dimensional Riemannian manifold $(M, g)$ and $D$ be the Dirac operator.

1. Suppose there exists a $C^2$ function $\varphi : M \to \mathbb{R}$ such that
   \begin{equation}
   \Delta \varphi + 2\lambda_S \geq 0 \quad \text{on } M. \tag{2}
   \end{equation}
   For any section $f$ of $\mathbb{S}$, if $\int_M \frac{|f|^2}{\Delta \varphi + 2\lambda_S} e^{-\varphi} < \infty$, then there exists a section $u$ of $\mathbb{S}$ such that
   \begin{equation}
   Du = f \quad \text{and} \quad \int_M |u|^2 e^{-\varphi} \leq \int_M \frac{|f|^2}{\Delta \varphi + 2\lambda_S} e^{-\varphi}. \tag{3}
   \end{equation}

2. If $M$ is noncompact, then for any $f \in L^2_{\text{loc}}(M, \mathbb{S})$, there exists a section $u \in L^2_{\text{loc}}(M, \mathbb{S})$ such that $Du = f$, where $L^2_{\text{loc}}(M, \mathbb{S})$ is the space of locally square integrable sections of $\mathbb{S}$.

By constructing the weight $\varphi$ from solutions of certain Dirichlet problem, we obtain the following Corollary, which essentially appeared in Theorem 2 of [2]:

**Corollary 1.2** If $M$ is a compact 2-dimensional Riemannian manifold without boundary, then
   \begin{equation}
   \lambda_{\text{min}}(D^2) \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_S, \tag{4}
   \end{equation}
   where $\lambda_{\text{min}}(D^2)$ is the first eigenvalue of $D^2$. Consequently, if
   \begin{equation}
   \int_M \lambda_S > 0, \tag{5}
   \end{equation}
   then for any $f \in L^2(M, \mathbb{S})$, there exists a section $u \in L^2(M, \mathbb{S})$ such that $Du = f$.

The corollary gives the solvability criteria for the Dirac equation on general Dirac bundles over two dimensional compact Riemannian manifolds. Its $\mathbb{Z}_2$-graded version (Corollary 7.3) has potential applications to the Fredholm regular criteria of $J$-holomorphic curves $\Sigma$ in symplectic manifolds $X$, as the linearization of the non-parameterized $J$-holomorphic curve equation, modulo zeroth order terms, is (half of) the Dirac equation on the Dolbeault complex $N_{\Sigma/X} \oplus N_{\Sigma/X} \otimes \Lambda^{0,1}(\Sigma)$, where $N_{\Sigma/X}$ is the normal bundle of $\Sigma$ in $X$, and the Fredholm regular property corresponds to the solvability of (half of) the Dirac equation. Especially when $X$ is a symplectic 4-manifold, condition (5) corresponds to the well-known Chern number condition of $N_{\Sigma/X}$, which gives the
“automatic transversality” for $J$-holomorphic curves $\Sigma$ in $X$ without any genericity assumption of $J$, and has nice applications to their moduli space theory (e.g. [11], [23]).

For spinor bundles over closed spin manifolds of dimension $\geq 3$, Hijazi obtained ([9]) a lower bound of the spectrum of the fundamental Dirac operator in terms of the first eigenvalue of the Yamabe operator, with the help of the conformal covariance of the Dirac operator and the transformation law of the scalar curvature under a conformal change of the given metric. By introducing a new connection, Bär ([2]) generalized this result, using a delicate technique of completing square, to any Dirac bundle over a closed Riemannian manifold.

As an application of Hörmander’s $L^2$-method, we construct weight functions by solving certain partial differential equations and give a new proof of Bär’s results.

**Theorem 1.3** (Theorem 3 [2]) Let $S$ be a Dirac bundle over a compact $n$-dimensional Riemannian manifold $(M, g)$ without boundary, and $D$ be the Dirac operator, $n \geq 2$. Then

$$\lambda_{\min}(D^2) \geq \frac{n}{n-1} \lambda_{\min}(L)$$

where $\lambda_{\min}(\cdot)$ means the first eigenvalue, $L = -\frac{n-1}{n-2}\Delta + \lambda_S$ if $n \geq 3$, and $L = -\frac{1}{2}\Delta + \lambda_S$ if $n = 2$.

Theorem 1.3 improves many vanishing theorems, which usually require pointwise non-negative curvature conditions. We relate the above theorem to the rigidity of instantons in $G_2$ manifolds, where the Dirac bundle is the normal bundle $N_{A/M}$ of a 3-dimensional instanton $A$ in a $G_2$ manifold $M$ (see Section 5 for their definitions).

In gauge theory, it occurs often that $M$ are Riemannian manifolds with cylindrical ends (Definition 6.1), and exponential weights on the ends are frequently used to set up the moduli spaces (e.g. [3], [22]). On such manifolds we have the following existence theorem without assuming $\lambda_S$ is positive everywhere.

**Theorem 1.4** Suppose $S$ is a Dirac bundle over a Riemannian manifold $(M, g)$ with cylindrical ends, $\dim M \geq 3$. Suppose for some compact subset $K$, $M \setminus K$ is contained in the cylindrical ends, and there exist a constant $\alpha > 0$ such that

$$\lambda_S \geq \alpha > 0 \text{ on } M \setminus K. \quad (6)$$

Then there exists a constant $\beta > 0$ with the following significance: when

$$\lambda_S \geq -\beta \text{ on } K, \quad (7)$$

we have

1. for any section $f \in L^2(M, S)$, there exists a section $u \in W^{1,2}_S(M, S)$ such that

$$Du = f \text{ and } \|u\|_{W^{1,2}_S(M, S)} \leq C \|f\|_{L^2(M, S)}; \quad (8)$$
2. for any section $f \in L^p(M,S)$ ($p \geq 1$), there exists a section $u \in W^{1,p}(M,S)$ such that

$$Du = f$$

and

$$\|u\|_{W^{1,p}(M,S)} \leq C_p \|f\|_{L^p(M,S)}.$$  \hspace{1cm} (9)

Here $\delta_1, \cdots, \delta_m \geq 0$ are sufficiently small, $\delta = (\delta_1, \cdots, \delta_m)$ is the exponential weight (defined in (39)) for the cylindrical ends, and the constants $C$ and $C_p$ are independent on $f$.

Note that the $L^2$-solvability of the Dirac equation on a complete Riemannian manifold was established in Theorem 2.11 of [6], with the assumption that $\lambda_S \geq \alpha > 0$. Our theorem allows mild negativity of $\lambda_S$ on a compact domain $K \subset M$ (as in (7)), when $M$ is a Riemannian manifold with cylindrical ends.

Finally, for Dirac bundles with $\mathbb{Z}_2$-gradings, we establish similar results for the (half) Dirac operator $D^\pm$ in Section 7, as this is important for many geometric applications. Using the $\mathbb{Z}_2$-grading (which always exists on Dirac bundles over even dimensional manifolds) we improve Bär’s ([2]) first eigenvalue estimates of $D$ on even dimensional manifolds (see Corollary 7.4 and Corollary 7.6).

We conclude the introduction by some potential applications. In [14], we are going to generalize the integral condition (5) (or (52)) to give Fredholm regular criteria (“automatic transversality”) of $J$-holomorphic curves in general symplectic manifolds, as the linearized operator for $J$-holomorphic curves is similar to the (half) Dirac operator $D^\pm$ in Section 7. The criteria may be useful to study deformation of multiple covering $J$-holomorphic curves in Calabi-Yau manifolds. Since Hörmander’s $L^2$-method is not limited to Fredholm differential operators, it may be useful for the study of the symplectic Dirac operator (7), which is an analogue of the Riemannian Dirac operator on symplectic manifolds, while the symplectic spinor bundle is infinite dimensional and the operator is not necessarily Fredholm.

Acknowledgement. We would like to thank Professor Yum-Tong Siu and Professor Clifford Taubes for interest and support. We thank Harvard math department for the excellent research environment, and Hai Lin, Yi Xie and Jie Zhou for helpful discussions. The work of Q. Ji is partially supported by NSFC11171069 and NSFC11322103.

2 Dirac bundles and Dirac operators

In this section, we recall some basic facts of the Dirac operator and set up the notations. Let $M$ be a smooth $n$-dimensional Riemannian manifold with metric $g$. Let $Cl(M) \to M$ be the corresponding Clifford bundle, and let $S \to M$ be a bundle of left $Cl(M)$-modules with compatible metric and connection such that at any $x \in M$, for any unit vector $e \in T_xM$ and any $s, s' \in S_x$,

$$\langle e \cdot s, e \cdot s' \rangle = \langle s, s' \rangle$$  \hspace{1cm} (10)
where the $\cdot$ is the Clifford multiplication. Furthermore, for any smooth vector field $X$ on $M$ and smooth section $s$ of $S$,
\[\nabla (X \cdot s) = (\nabla X) \cdot s + X \cdot (\nabla s),\] (11)
where on the right hand side, $\nabla$ are the covariant derivatives of the Levi-Civita connection on $M$ and a connection on $S$ for the first and second terms respectively.

**Definition 2.1** A bundle $S$ of $\text{Cl}(M)$-modules satisfying \[10\text{ }\text{ }\] and \[11\text{ }\text{ }\] is called a Dirac bundle over $M$ (Definition 5.2 [15]) and has a canonically associated Dirac operator $D$ such that for any section $s$ of $S$,
\[Ds = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}s,\] (12)
where $\{e_i\}_{i=1}^{n}$ is any orthonormal basis of $T_x M$ for $x$ on $M$.

On the Dirac bundle $S$ we can define the canonical section $\mathfrak{R}$ of $\text{Hom}(S, S)$, such that for any smooth section $s$ of $S$,
\[\mathfrak{R} (s) = \frac{1}{2} \sum_{i,j=1}^{n} e_i \cdot e_j \cdot R_{e_i,e_j}(s),\] (13)
where $R_{V,W}$ is the curvature transform on $S$. Then we have the Bochner formula (c.f. Theorem 8.2 in Chapter II [15])
\[D^2 = \nabla^* \nabla + \mathfrak{R}.\] (14)
Especially, when $S$ is the spinor bundle over a spin manifold $M$, Lichnerowicz’s theorem says
\[\mathfrak{R} = \frac{1}{4} R \cdot \text{Id}_S,\] (15)
where $R$ is the scalar curvature of $(M, g)$.

For later applications, for any $x$ on $M$ and $\mathfrak{R}(x) \in \text{Hom}(S_x, S_x)$, we let the function
\[\lambda_S(x) = \text{the smallest eigenvalue of } \mathfrak{R}(x),\] (16)
where $\mathfrak{R}(x)$ is defined in \[13\text{ }\text{ }\]. Since $\mathfrak{R}(x)$ is differentiable with respect to $x$ on $M$, using the definition of eigenvalues by the Rayleigh quotient, it is not hard to see $\lambda_S(x)$ is a Lipschitz function on $M$.

## 3 The weighted $L^2$-estimates

Let $S$ be a Dirac bundle over a smooth Riemannian manifold $(M, g)$ and $D$ be the Dirac operator. For any smooth sections $s \in \Gamma(M, S)$ with compact support in, by \[14\text{ }\] and integration by parts, we have
\[
\int_M |Ds|^2 = \int_M |\nabla s|^2 + \langle s, \mathfrak{R}s \rangle.
\] (17)
We omit $dvol_g$ for integrals on $M$ in this paper).

**Definition 3.1** (Weighted $L^2$-space) Let $\varphi : M \to \mathbb{R}$ be a $C^2$ function. For any sections $s$ and $s'$ of $\mathbb{S}$, let the weighted inner product of $s$ and $s'$ be

$$(s,s')_\varphi = \int_M \langle s, s' \rangle e^{-\varphi}.$$

Let $\|s\|_\varphi = \sqrt{(s,s)_\varphi}$ and denote by $L^2_\varphi (M, \mathbb{S})$ be the space of sections $s$ of $\mathbb{S}$ such that $\|s\|_\varphi < \infty$. We will drop the subscript $\varphi$ when $\varphi = 0$.

For the Dirac operator $D : L^2_\varphi (M, \mathbb{S}) \to L^2_\varphi (M, \mathbb{S})$, let $D_\varphi^*$ be its formal adjoint with respect to the measure $e^{-\varphi}dvol_g$. For $D_\varphi^*$, we have the following identity which is immediate from definitions.

**Lemma 3.2** For any smooth section $s$ of $\mathbb{S}$, we have

$$D_\varphi^* s = e^{\varphi} D (e^{-\varphi} s) = -\nabla \varphi \cdot s + Ds. \tag{18}$$

We will derive a weighted version of (17). Let $\Delta$ be the Laplace-Beltrami operator on $(M, g)$.

**Proposition 3.3** For any smooth section $s$ of $\mathbb{S}$ with compact support and any $C^2$ function $\varphi : M \to \mathbb{R}$, we have

$$\frac{n-1}{n} \int_M |D_\varphi^* s|^2 e^{-\varphi} + \frac{n-2}{n} \text{Re} \int_M \langle \nabla \varphi \cdot s, D_\varphi^* s \rangle e^{-\varphi} \geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 + \lambda \right] |s|^2 e^{-\varphi}. \tag{19}$$

Let $\varepsilon > 0$ be a constant, we have

$$\int_M |D_\varphi^* s|^2 e^{-\varphi} \geq C \int_M \left[ \Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda \right] |s|^2 e^{-\varphi}. \tag{20}$$

where $C = \frac{n}{2(n-1)+n-2\varepsilon}$. Especially, when $n = 2$ we have

$$\int_M |D_\varphi^* s|^2 e^{-\varphi} \geq \int_M (\Delta \varphi + 2\lambda) |s|^2 e^{-\varphi}. \tag{21}$$

**Proof.** Set $\sigma := e^{-\varphi/2} s$, then we know by (18)

$$\int_M |D_\varphi^* s|^2 e^{-\varphi} = \int_M \left[ e^{\varphi} D (e^{-\varphi} s) \right]^2 e^{-\varphi} \notag$$

$$= \int_M \left[ e^{\varphi} D (e^{-\varphi} \sigma) \right]^2 = \int_M \left| D\sigma - \frac{1}{2} \nabla \varphi \cdot \sigma \right|^2 \notag$$

$$= \int_M \left( |D\sigma|^2 + \frac{1}{4} |\nabla \varphi \cdot \sigma|^2 - \text{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle \right) \notag$$

$$= \int_M \left( |\nabla \sigma|^2 + \langle \sigma, R\sigma \rangle + \frac{1}{4} |\nabla \varphi|^2 |\sigma|^2 - \text{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle \right), \tag{22}$$
where in the last identity we have used (17).

We first rewrite \( \int_M |\nabla \sigma|^2 \) as follows

\[
\int_M |\nabla \sigma|^2 = \int_M \left| \nabla \left( e^{-\frac{\varphi}{2}} s \right) \right|^2 = \int_M \left| \nabla s - \frac{1}{2} d\varphi \otimes s \right|^2 e^{-\varphi} = \int_M \left( |\nabla s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} - \text{Re} \int_M \langle \nabla s, d\varphi \otimes s \rangle e^{-\varphi}.
\]

(23)

For the last term, we have

\[
\text{Re} \int_M \langle \nabla s, d\varphi \otimes s \rangle e^{-\varphi} = - \text{Re} \sum_{i=1}^n \int_M \langle \nabla e_i s, d\varphi (e_i) s \rangle e^{-\varphi} = - \text{Re} \int_M \langle \nabla \varphi s, s \rangle e^{-\varphi} = \frac{1}{2} \text{Re} \int_M \nabla \nabla (e^{-\varphi}) |s|^2 = \frac{1}{2} \int_M \left[ \text{div} \left( |s|^2 \nabla (e^{-\varphi}) \right) - |s|^2 \Delta (e^{-\varphi}) \right] = \frac{1}{2} \int_M \left[ \Delta \varphi - |\nabla \varphi|^2 \right] |s|^2 e^{-\varphi},
\]

(24)

where in the third line we have used \( Xf = \text{div} (fX) - f \text{div} X \) for the vector field \( X = \nabla (e^{-\varphi}) \) and function \( f = |s|^2 \).

From (23) and (24), we obtain

\[
\int_M |\nabla \sigma|^2 = \int_M \left( |\nabla s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} + \frac{1}{2} \int_M \left( \Delta \varphi - |\nabla \varphi|^2 \right) |s|^2 e^{-\varphi} = \int_M \left[ |\nabla s|^2 + \left( \frac{1}{2} \Delta \varphi - \frac{1}{4} |\nabla \varphi|^2 \right) |s|^2 \right] e^{-\varphi}.
\]

(25)

For the term \( \text{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle \), we have

\[
\text{Re} \langle \nabla \varphi \cdot \sigma, D\sigma \rangle = \text{Re} \left\langle e^{-\frac{\varphi}{2}} \nabla \varphi \cdot s, D \left( e^{-\frac{\varphi}{2}} s \right) \right\rangle = \text{Re} \left\langle \nabla \varphi \cdot s, e^\frac{\varphi}{2} D \left( e^{-\frac{\varphi}{2}} s \right) \right\rangle e^{-\varphi} = \text{Re} \left\langle \nabla \varphi \cdot s, D^\frac{\varphi}{2} (s) \right\rangle e^{-\varphi} = \text{Re} \left\langle \nabla \varphi \cdot s, D^\varphi (s) + \frac{1}{2} \nabla \varphi \cdot s \right\rangle e^{-\varphi},
\]

(26)

where in the fourth identity we have used Lemma 3.2.

By the Cauchy-Schwarz inequality, we get

\[
|\nabla s|^2 \geq \frac{1}{n} |Ds|^2.
\]

(27)
Combining (22), (25), (26), (27) and using (18), we have

\[
\int_M |D^*_s|^2 e^{-\varphi} \\
\geq \int_M \left[ \frac{1}{n} |D^*_s s + \nabla \varphi \cdot s|^2 + \left( \frac{1}{2} \Delta \varphi - \frac{1}{4} |\nabla \varphi|^2 \right) |s|^2 \right] e^{-\varphi} \\
+ \int_M \left( \lambda_\delta |s|^2 + \frac{1}{4} |\nabla \varphi|^2 |s|^2 \right) e^{-\varphi} - \text{Re} \int_M \langle \nabla \varphi \cdot s, D^*_s s + \frac{1}{2} \nabla \varphi \cdot s \rangle e^{-\varphi} \\
= \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 \right] |s|^2 e^{-\varphi} \\
\nonumber - 2 \left( \frac{1}{2} - \frac{1}{n} \right) \text{Re} \int_M \langle \nabla \varphi \cdot s, D^*_s s \rangle e^{-\varphi} + \frac{1}{n} \int_M |D^*_s|^2 e^{-\varphi} \\
\geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + \lambda_\delta \right] |s|^2 e^{-\varphi} \\
\nonumber - \left( \left( \frac{1}{2} - \frac{1}{n} \right) \varepsilon - \frac{1}{n} \right) \int_M |D^*_s|^2 e^{-\varphi},
\]

where \( \varepsilon \) is any positive constant. We have thus proved (20). The inequality (28) also gives the estimate (19).

To establish \( L^2 \)-existence theorem, we also need the following variant of Riesz representation Theorem:

**Lemma 3.4** (c.f. [12]) Let \( T : H_1 \rightarrow H_2 \) be a closed and densely defined operator between Hilbert spaces. For any \( f \in H_2 \) and any constant \( C > 0 \), the following conditions are equivalent

1. there exists some \( u \in \text{Dom}(T) \) such that \( Tu = f \) and \( \|u\|_{H_1} \leq C \).

2. \( |\langle f, s \rangle_{H_2}| \leq C \|T^* s\|_{H_1} \) holds for any \( s \in \text{Dom}(T^*) \).

We have the following proposition of the \( L^2 \)-existence result for the Dirac operator.

**Proposition 3.5** Let \( S \) be a Dirac bundle over a smooth Riemannian manifold \((M, g)\) and \( D \) be the Dirac operator. Let \( \varphi : M \rightarrow \mathbb{R} \) be a \( C^2 \) function and \( \varepsilon > 0 \) be a constant such that

\[
\Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda_\delta \geq 0 \text{ on } M.
\]

For any section \( f \in L^2_\varphi(M, S) \), if

\[
\int_M \frac{|f|^2}{\Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda_\delta} e^{-\varphi} < \infty,
\]

where...
then there exists a section \( u \in L^2_\varphi(M, \mathcal{S}) \) such that

\[
Du = f \quad \text{and} \quad \|u\|_\varphi^2 \leq \frac{1}{C} \int_M \Delta \varphi - \left(1 - \frac{2}{n}\right)(1 + \frac{1}{\varepsilon})|\nabla \varphi|^2 + 2\lambda_\mathcal{S} e^{-\varphi},
\]

where \( C = \frac{n}{2(n-1)+(n-2)\varepsilon} \).

**Proof.** Fix some open subset \( \Omega \) which is relatively compact in \( M \), we restrict \( \mathcal{S} \) to \( \Omega \) and introduce \( H_1 = H_2 = L^2_\varphi(\Omega, \mathcal{S}) \).

We let the closed, densely defined operator \( T : \text{Dom}(T) = \{u \in H_1 : Du \in H_1\} \rightarrow H_2 \) to be the maximal differential operator defined by the Dirac operator (acting on smooth sections) \( D : \Gamma(\Omega, \mathcal{S}) \rightarrow \Gamma(\Omega, \mathcal{S}) \).

From Proposition 3.3 we have for any compact supported section \( s \in \Gamma(\Omega, \mathcal{S}) \),

\[
\int_M |D^* s|^2 e^{-\varphi} \geq C \int_M \left(\Delta \varphi - \left(1 - \frac{2}{n}\right)(1 + \frac{1}{\varepsilon})|\nabla \varphi|^2 + 2\lambda_\mathcal{S}\right) |s|^2 e^{-\varphi}.
\]

Since \( T^* \) is given by the minimal differential operator defined by the formally adjoint operator \( D^* : \Gamma(\Omega, \mathcal{S}) \rightarrow \Gamma(\Omega, \mathcal{S}) \) (c.f. \[13\] for a more general result), we know that smooth sections of \( \mathcal{S} \) with compact support in \( \Omega \) are dense in \( \text{Dom}(T^*) \) with respect to the graph norm:

\[
s \mapsto \sqrt{\|s\|_H_2^2 + \|T^* s\|_{H_1}^2}.
\]

Because \( \varphi \) and \( \nabla \varphi \) are bounded on \( \Omega \), (31) holds for any \( s \in \text{Dom}(T^*) \).

Let \( f_\Omega \in L^2_\varphi(\Omega, \mathcal{S}) \). As we have proved that Proposition 3.3 is true for any \( s \in \text{Dom}(T^*) \), by Cauchy-Schwarz inequality, we get the following estimate for any \( s \in \text{Dom}(T^*) \)

\[
|(f_\Omega, s)_{H_2}|^2 \leq \frac{1}{C} \int_\Omega \Delta \varphi - \left(1 - \frac{2}{n}\right)(1 + \frac{1}{\varepsilon})|\nabla \varphi|^2 + 2\lambda_\mathcal{S} e^{-\varphi} \cdot \|T^* s\|_{H_1}^2
\]

Lemma 3.4 implies that there exists some \( u_\Omega \in \text{Dom}(T) \subseteq L^2_\varphi(\Omega, \mathcal{S}) \) such that

\[
Du_\Omega = f_\Omega \quad \text{and} \quad \int_\Omega |u_\Omega|^2 e^{-\varphi} \leq \frac{1}{C} \int_\Omega \Delta \varphi - \left(1 - \frac{2}{n}\right)(1 + \frac{1}{\varepsilon})|\nabla \varphi|^2 + 2\lambda_\mathcal{S} e^{-\varphi}.
\]

Given \( f \in L^2_\varphi(M, \mathcal{S}) \) with \( \int_M \Delta \varphi - \left(1 - \frac{2}{n}\right)(1 + \frac{1}{\varepsilon})|\nabla \varphi|^2 + 2\lambda_\mathcal{S} e^{-\varphi} < \infty \), we can finish the proof by applying (32) to an increasing sequence of open subsets...
\( \Omega_0 \subset \subset \Omega_1 \subset \subset \cdots \not\owns M \) to get a sequence of \( u_\nu \in L^2(\Omega_\nu, S) \) such that \( Du_\nu = f|_{\Omega_\nu} \) in the sense of distribution and

\[
\int_{\Omega_\nu} |u_\nu|^2 e^{-\varphi} \leq \frac{1}{C} \int_{\Omega_\nu} \Delta \varphi - \frac{1}{2} |f|^2 \left( 1 + \frac{1}{n} \right) |\nabla \varphi|^2 + 2\lambda S e^{-\varphi}, \nu = 0, 1, \ldots.
\]

The above uniform \( L^2 \)-estimate allows us obtain a desired solution of \( Du = f \) by taking a weak limit of \( \{u_\nu\} \). The proof is complete. \( \blacksquare \)

**Remark 3.6** Different from the Bochner formula of the \( \overline{\partial} \)-operator, for the Dirac operator \( D \), only \( |Ds|^2 \) is involved on the left side of (17). This simplifies the \( L^2 \)-estimates in our setting compared to [12], as we only need the minimal extension of the Dirac operator \( D \), for which the set of compactly supported smooth sections is dense in its domain with respect to the graph norm.

4 2-dimensional manifolds

We give the proof of Theorem 1.1. \( \square \)

**Proof.** The first statement follows directly from the \( n = 2 \) case of Proposition 3.3.

For the second statement, by the first statement it suffices to prove that there is a \( \varphi \in C^2(M) \) such that

\[
\Delta \varphi + 2\lambda S \geq 0 \text{ on } M \text{ and } \int_M \frac{|f|^2}{\Delta \varphi + 2\lambda S} e^{-\varphi} < \infty. \tag{33}
\]

We first construct some nonnegative proper exhaustion function \( \psi \in C^2(M) \) such that \( \Delta \psi + 2\lambda S \geq 1 \) on \( M \). Since \( M \) is a noncompact 2-dimensional Riemannian manifold, there always exists a nonnegative exhaustion function \( \phi \in C^\infty(M) \) which is strictly subharmonic. Then, we choose a nonnegative function \( \kappa \in C^\infty[0, +\infty) \) such that

\[
\kappa'(t) > 0, \kappa''(t) \geq 0 \text{ for } t \geq 0, \kappa'(\nu) \geq \sup_{\Omega_{\nu+1} \setminus \Omega_\nu} \frac{1 - 2\lambda S}{\Delta \phi} \text{ for } \nu = 0, 1, 2, \ldots, \tag{34}
\]

where \( \Omega_\nu := \{x \in M \mid \phi(x) < \nu\} (\nu = 0, 1, 2, \ldots) \). Set \( \psi = \kappa \circ \phi \), then

\[
\Delta \psi = \kappa' \circ \phi \cdot \Delta \phi + \kappa'' \circ \phi \cdot |\nabla \phi|^2 \geq \kappa' \circ \phi \cdot \Delta \phi.
\]

Consequently, by the monotonicity of \( \kappa' \) and (34), we obtain \( \Delta \psi + 2\lambda S \geq 1 \) on \( M \). Since \( \kappa(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty \), \( \psi = \kappa \circ \phi \) is also an exhaustion function.
Now we construct the desired function $\varphi$ satisfying (33). If we set $\Omega_\nu = \psi^{-1}(-\infty, \nu), \nu = 0, 1, 2, \cdots$, then $\emptyset = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \supset M$. Let $\chi \in C^\infty[0, +\infty)$ such that

$$\chi(\nu) \geq \nu + \log \int_{\Omega_{\nu+1}} |f|^2 \ (\nu = 0, 1, 2, \cdots), \ \chi' \geq 1, \ \chi'' \geq 0.$$

Define $\varphi = \chi \circ \psi$, then we have

$$\Delta \varphi = \chi' \circ \psi \cdot \Delta \psi + \chi'' \circ \psi \cdot |\nabla \psi|^2 \geq \Delta \psi \geq 1 - 2\lambda_S,$$

and

$$\int_M \frac{|f|^2}{\Delta \varphi + 2\lambda_S} e^{-\varphi} = \sum_{\nu \geq 0} \int_{\Omega_{\nu+1} \setminus \Omega_{\nu}} \frac{|f|^2}{\Delta \varphi + 2\lambda_S} e^{-\varphi} \leq \sum_{\nu \geq 0} e^{-\chi(\nu)} \int_{\Omega_{\nu+1} \setminus \Omega_{\nu}} |f|^2 \leq \sum_{\nu \geq 0} e^{-\nu} < \infty.$$

Hence, we have constructed the desired weight function $\varphi$ satisfying (33) and the proof is therefore complete. 

Then we give the proof of Corollary 1.2.

**Proof.** Since $\lambda_{\min}(D^2) \geq 0$, the inequality (11) is trivial if $\int_M \lambda_S \leq 0$. Now let us assume $\int_M \lambda_S > 0$. The equation $\Delta \varphi + 2\lambda_S = \frac{2}{\operatorname{Vol}(M)} \int_M \lambda_S$ always has a solution $\varphi$, as the integral of $\lambda_S - \frac{1}{\operatorname{Vol}(M)} \int_M \lambda_S$ over $M$ is zero. Take such $\varphi$ as the weight function in (3). For any $f \in L^2(M, S)$, by our condition

$$\int_M \frac{|f|^2}{\Delta \varphi + 2\lambda_S} e^{-\varphi} = \frac{\operatorname{Vol}(M)}{2 \int_M \lambda_S} \int_M |f|^2 < \infty.$$

So by Theorem 1.1 there is a $L^2$ section $u$ such that $Du = f$. From (3) we have

$$\frac{\int_M |Du|^2 e^{-\varphi}}{\int_M |u|^2 e^{-\varphi}} \geq \frac{2}{\operatorname{Vol}(M)} \int_M \lambda_S.$$

By the Rayleigh quotient, the desired estimate $\lambda^2 \geq \frac{2}{\operatorname{Vol}(M)} \int_M \lambda_S$ follows, where $\lambda$ is a square root of $\lambda_{\min}(D^2)$. (To get rid of the weight $e^{-\varphi}$ in (35), we consider the operator $\tilde{D} = e^{-\frac{\varphi}{2}} \circ D \circ e^{\frac{\varphi}{2}}$ and section $\tilde{u} = e^{-\frac{\varphi}{2}} u$, and notice $D^2$ and $\tilde{D}^2$ have the same eigenvalues). 

**Remark 4.1** Corollary 1.2 also follows from (21), by taking the same $\varphi$ as above and the nontrivial section $s$ such that $D(e^{-\varphi}s) = \lambda e^{-\varphi}s$. 

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For any two dimensional Riemannian manifold \((M, g)\), the vector bundle 
\[ \Lambda^* M := \bigoplus_{\ell=0}^2 \Lambda^\ell (T^* M) \]
has a natural structure of a Clifford bundle over \((M, g)\) and the associated Dirac operator is given by 
\[ D = d + \delta \]
where \(\delta\) is the codifferential operator. Since \(D^2 = (d + \delta)^2 = \Delta_d\) (the Hodge Laplacian operator), the Poisson equation \(\Delta_d u = f\) is always solvable on noncompact two dimensional Riemannian manifolds (we do not require \(M\) is orientable).

5 Compact manifolds

Using Hörmander’s \(L^2\)-method we give a simple proof of Bär’s Theorem \[3\].

**Proof.** (of Theorem \[3\]) For any \(\varphi \in C^\infty(M)\), we can choose a non-trivial section \(s \in \Gamma(M, S)\) such that 
\[ D(e^{-a\varphi} s) = \lambda e^{-a\varphi} s \]
where \(\lambda\) is a square root of \(\lambda_{\text{min}}(D^2)\) and \(a = 0\) if \(n = 2\), \(a = \frac{n}{2(n-1)}\) if \(n \geq 3\).

By Lemma \[2\] we know 
\[ D^*_s s = D^* e^{a\varphi} s + (a - 1) \nabla \varphi \cdot s = \lambda s + (a - 1) \nabla \varphi \cdot s. \]

Substituting the above identity into \(19\) and noticing \(\text{Re} \langle s, \nabla \varphi \cdot s \rangle = 0\), we have 
\[ \lambda^2 \int_M |s|^2 e^{-\varphi} \geq \int_M \left( \Delta \varphi - |\nabla \varphi|^2 + 2\lambda \right) |s|^2 e^{-\varphi}, \text{ if } n = 2, \quad (36) \]
and 
\[ \lambda^2 \int_M |s|^2 e^{-\varphi} \geq \frac{n-1}{n-1} \int_M \left( \frac{1}{2} \Delta \varphi - \frac{n-2}{4(n-1)} |\nabla \varphi|^2 + \lambda \right) |s|^2 e^{-\varphi}, \text{ if } n \geq 3. \quad (37) \]

Let \(\psi \in C^\infty(M)\) be an eigenfunction 
\[ L \psi = \lambda_{\text{min}}(L) \psi. \]
Without loss of generality, we may assume \(\psi > 0\) on \(M\). Set 
\[ \varphi = -\log \psi \text{ if } n = 2, \text{ and } \varphi = -\frac{2(n-1)}{n-2} \log \psi \text{ if } n \geq 3, \]
then Theorem \[3\] follows from \(36\) and \(37\). \(\blacksquare\)

An example of a Dirac bundle over a 3-manifold is the normal bundle of an instanton in a \(G_2\) manifold. In Physics, \(G_2\)-manifolds are internal spaces for compactification in M-theory in eleven dimensional spacetimes, similar to the role of Calabi-Yau threefolds in string theory. Counting instantons in \(G_2\) manifolds is similar to counting holomorphic curves in Calabi-Yau threefolds.
Definition 5.1 A $G_2$ manifold is a 7-dimensional Riemannian manifolds $(M, g)$ equipped with a parallel cross product $\times$. An instanton (or associative submanifold) $A$ is a 3-dimensional submanifold whose tangent spaces are closed under the cross product.

Let $N_{A/M}$ be the normal bundle of $A$ in $M$. Regarding $N_{A/M}$ as a left Clifford module over $A$ with the $G_2$ cross product as the Clifford multiplication, it is a twisted spinor bundle over $A$, with the normal connection $\nabla^\perp$ inherited from the Levi-Civita connection $\nabla$ on $M$ (Section 5 [18]). All 3-manifolds nearby $A$ can be parameterized by sections $V$ of $N_{A/M}$.

Given the cross product $\times$, one can define a $TM$-valued 3-form $\tau$ on $M$ as

$$\tau(u,v,w) = -u \times (v \times w) - g(u,v)w + g(u,w)v,$$

for $u, v$ and $w \in TM$. Then $A$ is an instanton if and only if $\tau|_A = 0$ (c.f. [8]). Using $\tau$ McLean (Section 5 [18]) defined a nonlinear function

$$F : C^{1,\alpha}(A, N_{A/M}) \to C^\alpha(A, N_{A/M}) \quad (0 < \alpha < 1)$$

such that instantons nearby $A$ correspond to the zeros of $F$. He computed

$$\left. \frac{d}{dt} \right|_{t=0} F(tV) = DV,$$

where $V \in A, N_{A/M}$, and $D$ is the twisted Dirac operator on $N_{A/M}$ over $A$. Then he proved

Theorem 5.2 (Theorem 5-2 [18]) Infinitesimal deformations of instantons at $A$ are parametrized by the space of harmonic twisted spinors on $A$, i.e. the kernel of $D$.

More detailed exposition of McLean’s proof can be found in Theorem 9 and Section 4.2 of [17].

We relate Theorem 1.3 to rigidity of instantons in $G_2$ manifolds, i.e. situation that the moduli space of instantons near $A$ is a zero dimensional smooth manifold.

Corollary 5.3 If an instanton $A$ is compact and $\lambda_{\min}(L) > 0$, then $A$ is rigid. Here $L = -2\Delta_{(A,g)} + \lambda N_{A/M}$, and $\Delta_{(A,g)}$ is the Laplace-Beltrami operator on $A$ with the induced metric from $(M, g)$.

Proof. Let $S$ be $N_{A/M}$ and $D$ be the twisted Dirac operator in Theorem 1.3. By our condition the kernel and cokernel of $D : W^{1,2}(A, N_{A/M}) \to L^2(A, N_{A/M})$ vanish (note $D$ is self-adjoint). A standard interpolation argument implies that $D : W^{1,p}(A, N_{A/M}) \to L^p(A, N_{A/M}) (p > 3)$ is surjective (e.g. estimates between (36)$\sim$(37) in [17]), which in turn implies that $D : C^{1,\alpha}(A, N_{A/M}) \to$
$C^\alpha (A, N_{A/M})$ is surjective by the Schauder estimate of $D$ and Sobolev embedding $W^{1,p} \hookrightarrow C^0$. By the implicit function theorem for the nonlinear function $F$, the moduli space $F^{-1}(0)$ near $A$ is a zero dimensional smooth manifold, and $A$ is rigid.

Clearly if $\lambda_{N_{A/M}} > 0$ everywhere on $M$, then $\lambda_{\min}(L) > 0$, but not vice versa. So Corollary 5.3 provides a potentially weaker condition to guarantee the rigidity of $A$. The computation of $\mathcal{R}_{N_{A/M}}$ in terms of the curvature of $M$ and the second fundamental form of $A$ can be found in Section 5.3 of [5].

6 Manifolds with cylindrical ends

Between the compact and noncompact cases, there is an important case of manifolds with cylindrical ends. There are many works of differential operators on such manifolds, going back to the work of Lockhart and McOwen ([16]), and occurring often in gauge theory and low dimensional topology (e.g. [4], [22]).

Definition 6.1 Let $(M, g)$ be a $n$-dimensional Riemannian manifold. We say it is a manifold with cylindrical ends if outside a compact subset, $M$ consists of product Riemannian manifolds $E_\nu \simeq [0, +\infty) \times B_\nu$ ($\nu = 1, 2, \cdots, m$), where each $B_\nu$ is a $(n-1)$-dimensional compact Riemannian manifold. Each $E_\nu$ is called a cylindrical end, and the $[0, +\infty)$ direction is called the cylindrical direction.

For analysis on cylindrical manifolds, one often needs the Sobolev space with exponential weights. In other words, one chooses a smooth weight function $\varphi$ on $M$ such that

$$\varphi|_{E_\nu} = -\delta_\nu \tau_\nu$$

for some constant $\delta_\nu \geq 0$, and large $\tau_\nu$ \hspace{1cm} (39)

$(\nu = 1, 2, \cdots, m)$, and then defines the weighted Sobolev norm

$$\|f\|_{W^{k,p}_\delta} := \left\|e^{-\frac{\varphi}{p}}f\right\|_{W^{k,p}}$$

where $\delta = (\delta_1, \cdots, \delta_m)$. Different choices of $\varphi$ satisfying (39) only result in equivalent Banach spaces.

Proof. (of Theorem 1.4). Without loss of generality we assume $M \setminus K = \cup_{\nu=1}^m E_\nu$, where $E_\nu$ are the cylindrical ends of $M$, (otherwise we can always enlarge $K$). We choose a smooth cut-off function $\rho : M \rightarrow [0, 1]$ such that $\rho \equiv 0$ on $K$, and on $E_\nu$, $\rho$ is a function of the variable $\tau_\nu$ in the cylindrical direction satisfying

$$\rho(\tau_k) = \begin{cases} 0, & \text{if } 0 \leq \tau_\nu \leq 1 \\ 1, & \text{if } \tau_\nu \geq 2 \end{cases}$$

and $0 \leq \rho'(\tau_\nu), \ |\rho''(\tau_\nu)| \leq 2$. 

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Let $\mu > 0$ be the first eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta \eta = \mu \eta \text{ on } M \setminus \bigcup_{\nu=1}^{m} (3, \infty) \times B_{\nu}. \quad (41)$$

By the nodal domain theorem, we can find an eigenfunction function $\eta > 0$ on $M \setminus \bigcup_{\nu=1}^{m} (3, \infty) \times B_{\nu}$. Setting $A = \frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} > 0$, from (41) it follows that for sufficiently large $\varepsilon > 0$

$$\left( -\frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} \Delta + 2\lambda_{\text{S}} \right) \eta^{\gamma} = \eta^{\gamma-2} \left( -A \gamma \eta \Delta \eta - A \gamma (\gamma - 1) |\nabla \eta|^{2} + 2\lambda_{\text{S}} \eta^{2} \right) \geq \eta^{\gamma-1} \left( -A \gamma \Delta \eta + 2\lambda_{\text{S}} \eta \right) \geq \eta^{\gamma} \left( A \gamma \mu - 2\beta \right) > 0 \quad (42)$$
on $M \setminus \bigcup_{\nu=1}^{m} [2, \infty) \times B_{\nu}$, provided that $\beta$ satisfies the condition

$$0 < \beta < \frac{\gamma \mu}{2 - \frac{2}{n}}$$

where $\gamma \in (0, 1)$ is a constant to be determined. Let

$$\phi := -\frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} \log \eta,$$

then on $M \setminus \bigcup_{\nu=1}^{m} [2, \infty) \times B_{\nu}$, by (42) we have

$$\Delta (\gamma \phi) - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla (\gamma \phi)|^{2} + 2\lambda_{\text{S}}$$

$$= \eta^{\gamma} \left[ -\frac{1}{(1-\frac{2}{n})(1+\frac{1}{\varepsilon})} \Delta \eta^{\gamma} + 2\lambda_{\text{S}} \eta^{\gamma} \right] > 0. \quad (43)$$

We define a function $h : \bigcup_{\nu=1}^{m} E_{\nu} \to \mathbb{R}$, such that

$$h (\tau_{\nu}, b_{\nu}) = -\delta_{\nu} \tau_{\nu}, \quad (44)$$

where the constants $\delta_{\nu} > 0$ are to be determined. Then we define the weight function $\varphi : M \to \mathbb{R}$ as

$$\varphi = \gamma (1 - \rho) \phi + \rho h. \quad (45)$$

It is easy to check that $\varphi$ is smooth and is globally defined on $M$.

By (43) $\sim$ (45), for sufficiently small $\delta_{1}, \cdots, \delta_{m} \geq 0$ and sufficiently large $\varepsilon > 0$, we have

$$\Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^{2} + 2\lambda_{\text{S}}$$

$$\begin{cases} > 0 & \text{on } K \cup \bigcup_{\nu=1}^{m} [0, 1] \times B_{\nu}, \\ = -\left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) \delta_{\nu}^{2} + 2\lambda_{\text{S}} > 0 & \text{on } \bigcup_{\nu=1}^{m} [2, \infty) \times B_{\nu}. \end{cases}$$

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On each $[1,2] \times B_\nu$, if $\gamma$ and $\delta_1, \cdots, \delta_m$ are sufficiently small, then
\[
\Delta \varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right)|\nabla \varphi|^2 + 2\lambda_\delta \geq -C_1(\gamma + |h| + |\nabla h|) + 2\lambda_\delta \geq \alpha,
\]
where $C_1 > 0$ is some constant depending on $\phi$ and $\rho$. So we have
\[
\Delta \varphi - \left(1 - \frac{2}{n}\right) \left(1 + \frac{1}{\varepsilon}\right)|\nabla \varphi|^2 + 2\lambda_\delta \geq \alpha_1 \text{ on } M
\]
for some constant $\alpha_1 > 0$.

Therefore by Proposition 3.5 and (47), for any $f \in L^2(\varphi, M, S)$, there exists
\[
u \in L^2(\varphi, M, S) \text{ such that } Du = f \text{ and } \|u\|^2_\varphi + \|\nabla u\|^2_\varphi \leq C_2 \int_M |f|^2 e^{-\varphi}
\]
for some constants $C_2$ and $C$. By elliptic regularity of $D$ and the cylindrical structure on $M$, we have $\|u\|^2_\varphi + \|\nabla u\|^2_\varphi \leq C \|f\|^2_\varphi$, so (8) is proved.

With the $L^2$-estimate (8) ($\delta = 0$ case) on the cylindrical manifold $M$, it is standard to derive the $L^p$ estimate (9) (c.f. Section 3.4 [4]). The theorem is proved.

7 $\mathbb{Z}_2$-graded Dirac operators

In several important applications, one needs to consider the $\mathbb{Z}_2$-graded Dirac bundle, i.e. $\mathcal{S}$ has a parallel decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ so that $Cl^i(M) \cdot \mathcal{S}^j \subseteq \mathcal{S}^{ij}$ for all $i,j \in \mathbb{Z}_2 \cong \{+, -\}$, where $ij$ is the sign of the product of the signs $i$ and $j$, and $Cl^i(M) (i = +, -)$ are the even and odd parts of $Cl(M)$. The Dirac operator $D$ splits accordingly
\[
D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix},
\]
where $D^\pm : \Gamma(\mathcal{S}^\pm) \rightarrow \Gamma(\mathcal{S}^\mp)$, and $D^+$ and $D^-$ are adjoint of each other. Note that for Dirac bundles over even dimensional Riemannian manifolds, there always exists a natural $\mathbb{Z}_2$-grading (c.f. Section 6 [15]).

The solvability of the “half” Dirac equation $D^\pm u = f$ is of interest for several reasons: e.g. in quaternion-valued analysis, the correct generalization of analytical functions on $\mathbb{C}$ are solutions of $D^+ u = 0$ on $\mathbb{H}$, as requiring $u$ to be quaternion differentiable only yields linear functions; $D^\pm$ may have nonzero Fredholm index to produce nontrivial invariants; $D^\pm$ arises as the linearized operator (modulo zeroth order terms) in many moduli problems, like those of $J$-holomorphic curves and solutions of the Seiberg-Witten equation. As we will
see, consideration of the $\mathbb{Z}_2$-grading also improves eigenvalue estimates of the Dirac operator $D$ on even dimensional manifolds.

By (13) we see the curvature operator $\mathfrak{R}(\cdot)$ splits as well:

$$\mathfrak{R}(\cdot) = \begin{bmatrix} \mathfrak{R}^+(\cdot) & 0 \\ 0 & \mathfrak{R}^-(\cdot) \end{bmatrix},$$

where $\mathfrak{R}^+ (\cdot) \in \text{Hom}(S^+, S^+)$, and $\mathfrak{R}^- (\cdot) \in \text{Hom}(S^-, S^-)$.

Our main technical tool, Proposition 3.3, extends immediately to $D^\pm$:

**Proposition 7.1** For any smooth section $s$ of $S^\mp$ with compact support and any $C^2$ function $\varphi : M \to \mathbb{R}$, we have

$$\int_M \frac{n-1}{n} |(D^\pm)^* s|^2 e^{-\varphi} + \frac{n-2}{n} \text{Re} \int_M \langle \nabla \varphi \cdot s, (D^\pm)^* s \rangle e^{-\varphi} \geq \int_M \left[ \frac{1}{2} \Delta \varphi - \left( \frac{1}{2} - \frac{1}{n} \right) |\nabla \varphi|^2 + \lambda_{S^\mp} \right] |s|^2 e^{-\varphi}.$$

and

$$\int_M |(D^\pm)^* s|^2 e^{-\varphi} \geq C \int_M \left[ \frac{1}{2} \Delta \varphi - \left( 1 - \frac{2}{n} \right) \left( 1 + \frac{1}{\varepsilon} \right) |\nabla \varphi|^2 + 2\lambda_{S^\mp} \right] |s|^2 e^{-\varphi},$$

where $C$ is the constant in Proposition 3.3.

**Proof.** Consider the Dirac operator $D^+ : L^2_\varphi (M, S^+) \to L^2_\varphi (M, S^-)$ (the $D^-$ case is similar). Let $(D^+)^*_\varphi$ be its formal adjoint with respect to the measure $e^{-\varphi} dvol_g$. Then it is easy to observe

$$(D^+)^* s = e^\varphi D^- (e^{-\varphi} s) = -\nabla \varphi \cdot s + D^- s$$

for smooth sections $s$ of $S^-$. When we restrict the sections from $\Gamma (S)$ to $\Gamma (S^-)$, by (48) and (49), $D^2$ becomes $D^+ D^-$, and $\mathfrak{R}$ becomes $\mathfrak{R}^-$ in the Bochner formula (14), i.e.

$$D^+ D^- = \nabla^* \nabla + \mathfrak{R}^-.$$

The remaining part of the proof is the same as Proposition 3.3, except that $\lambda_S$ is replaced by $\lambda_{S^-}$, where

$$\lambda_{S^{\pm}} (x) := \text{the smallest eigenvalue of } \mathfrak{R}^\pm (x)$$

at any $x \in M$. The proposition follows. $
$

**Remark 7.2** From Proposition 7.1 we know that Proposition 3.5 still holds if we replace simultaneously $D$ by $D^\pm$ and $\lambda_S$ by $\lambda_{S^\mp}$.

From Proposition 7.1 and Remark 7.2 similarly we obtain the results of $D^\pm$ parallel to Theorem 1.1, Corollary 1.2 and Theorem 1.4, by replacing $D$ by $D^\pm$, and $\lambda_S$ by $\lambda_{S^\mp}$ in the corresponding statements. For example, we can refine Corollary 1.2 to the following
Corollary 7.3 Let $S$ be a $\mathbb{Z}_2$-graded Dirac bundle over a compact 2-dimensional Riemannian manifold $M$ without boundary, and $D^\pm$ be the Dirac operators, then

$$\lambda_{\min}(D^\pm D^\mp) \geq \frac{2}{\text{Vol}(M)} \int_M \lambda_{S^\mp}, \quad (51)$$

where $\lambda_{\min}(D^\pm D^\mp)$ is the first eigenvalue of $D^\pm D^\mp$. Consequently, if

$$\int_M \lambda_{S^\mp} > 0, \quad (52)$$

then for any $f \in L^2(M, S^\mp)$, there exists a section $u \in L^2(M, S^\pm)$ such that $D^\pm u = f$.

As an immediate corollary, we improve the estimate for $\lambda_{\min}(D^2)$ in Corollary [1.2] as follows.

Corollary 7.4 Under the hypothesis in Corollary 7.3, it holds that

$$\lambda_{\min}(D^2) \geq \frac{2}{\text{Vol}(M)} \min \left\{ \int_M \lambda_{S^+}, \int_M \lambda_{S^-} \right\}. \quad (53)$$

To extend Theorem 1.3 we need to estimate the first eigenvalue of the self-adjoint operators $D^- D^+$ and $D^+ D^-$. We have a partial result in this direction. Setting

$$v = e^{-\left(1 - \frac{2}{n}\right)\phi} > 0 \quad (n \geq 3)$$

we have

$$\Delta \phi - \left(1 - \frac{2}{n}\right)|\nabla \phi|^2 + 2\lambda_{S^\pm} = v^{-1} L^\pm v, \quad (54)$$

where the operator

$$L^\pm : = -\frac{n}{n-2} \Delta + 2\lambda_{S^\pm}. \quad (55)$$

By the argument in the proof of Proposition 3.3 using Remark 7.2, we have

Corollary 7.5 If there exists a positive function $v$ such that $L^\mp v > 0$ on a compact Riemannian manifold $M$, then $D^\pm u = f$ is always solvable in $L^2(M, S^\mp)$.

From Proposition 7.1 we can also improve Theorem 1.3 of Bär. As in the proof of Theorem 1.3, let $s \in \Gamma(M, S)$ be a non-trivial section such that $D(e^{-a\phi}s) = \lambda e^{-a\phi}s$, which implies

$$(D^\pm)_{a\phi}s^\mp = \lambda s^\pm.$$

By applying Proposition 7.4 to $s^\mp$ in the same way as in the proof of Theorem 1.3, we have the following
Corollary 7.6 Let $S$ be a $\mathbb{Z}_2$-graded Dirac bundle over a compact $n$-dimensional Riemannian manifold $(M,g)$ without boundary, and $D$ be the Dirac operator, $n \geq 2$. Then

\[ \lambda_{\text{min}}(D^2) \geq \frac{n}{n-1} \min \{ \lambda_{\text{min}}(L^+), \lambda_{\text{min}}(L^-) \} \]

where $L^\pm = -\frac{n-1}{n-2} \Delta + \lambda_{S}^\pm$ if $n \geq 3$, and $L^\pm = -\frac{n}{2} \Delta + \lambda_{S}^\pm$ if $n = 2$.

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