The Hele–Shaw flow as the sharp interface limit of the Cahn–Hilliard equation with disparate mobilities

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ABSTRACT

In this paper, we study the sharp interface limit for solutions of the Cahn–Hilliard equation with disparate mobilities. This means that the mobility function degenerates in one of the two energetically favorable configurations, suppressing the diffusion in that phase. First, we construct suitable weak solutions to this Cahn–Hilliard equation. Second, we prove precompactness of these solutions under natural assumptions on the initial data. Third, under an additional energy convergence assumption, we show that the sharp interface limit is a distributional solution to the Hele–Shaw flow with optimal energy-dissipation rate.

1. Introduction

The Hele–Shaw cell is made of two parallel horizontal sheets which are separated by a thin gap of width $b$. Between the two sheets, a viscous fluid fills an almost cylindrical domain. As the spacing $b$ between the plates vanishes, one considers the lower-dimensional cross-section $\Omega$ of the fluid. Formal arguments suggest that this limit is governed by the Hele–Shaw flow, see (2.2)–(2.3) below for the precise formulation. Otto [1] studied this reduced model for a ferrofluid in the presence of an external magnetic field to explain patterns observed in experiments [2]. There is yet another, less classical way in which the Hele–Shaw flow arises in a singular limit, namely as the sharp interface limit of a Cahn–Hilliard equation. This is suggested by formal matched asymptotic expansions [3]. The goal of this paper is to rigorously justify the connection between these two models.

The Cahn–Hilliard equation is a fundamental phase-field model describing the phase separation for preserved order parameters. We are interested in the case of disparate mobilities, i.e., the case when the mobility function vanishes in one of the two stable states but is non-degenerate in the other one. In that case, this degenerate parabolic PDE has a rich structure: it is the gradient flow of the Cahn–Hilliard energy in the
Wasserstein space of probability measures. Based on this gradient-flow structure, we first construct weak solutions to these degenerate Cahn–Hilliard equations and then analyze their convergence in the sharp-interface limit.

Elliott and Garcke [4] established the existence of solutions to such degenerate Cahn–Hilliard equations in a general setting; we also refer to Lisini et al. [5] for a similar result. We propose here an alternative construction based on the Wasserstein gradient flow structure, which is mostly soft. This is inspired by the seminal work of Jordan et al. [6]. Here, our energy is of higher order (as it depends on the gradient), and is therefore not geodesically convex as in the case of Jordan et al. [6]. For the most part of our proof, we do not rely on higher regularity as in Elliott and Garcke [4] and Lisini et al. [5], and we are confident that some of these ideas will be useful in other situations as well. One interesting result in its own right is the computation of the first variation of the Dirichlet energy in Wasserstein space relying only on the natural $H^1$ regularity, see Lemma 3.7. This is used to show that any limit of minimizing movements (or the JKO scheme) is a weak solution of our Cahn–Hilliard equation. In addition, we show that this weak solution saturates the optimal energy-dissipation rate. In the language of abstract gradient flows, this means that our solutions are also curves of maximal slope, which is probably the most natural weak solution concept for gradient flows. Another crucial observation is that the first variation of the Cahn–Hilliard energy in Wasserstein space is in divergence form. This is well-known for domain variations given by the transport equation $\partial_t u + \xi \cdot \nabla u = 0$ due to a result by Luckhaus and Modica [7]. In contrast, variations in Wasserstein space are given by conservation laws of the form $\partial_t u_n + \nabla \cdot (u_n \xi) = 0$. Our observation—which we already employ in the construction of our weak solution of the Cahn–Hilliard equation—is that also in this case, the first variation of the energy is in divergence form. This results in a stable notion of weak solutions and ultimately allows us to pass to the sharp-interface limit $\varepsilon \downarrow 0$ in our weak formulation and show that the limit is a weak solution of the Hele–Shaw flow under a typical assumption on the convergence of energies.

Similar sharp-interface limits have been studied in different settings, for example in the case of constant mobility [8, 9]. However, to the best of our knowledge, the present work is the first rigorous derivation of a sharp-interface limit for a Cahn–Hilliard equation with non-constant mobility function. Our approach differs from that of Alikakos et al. [8] and Chen [9] and is inspired by the work of Simon and one of the authors [10] who derive the sharp-interface limit of a system of Allen–Cahn equations, a second-order version of our problem here. On a conceptual level, our proof of the second main result is also similar to the work by Chambolle and one of the authors [11] who showed that the implicit time discretization of the Hele–Shaw flow produces varifold solutions which are slightly weaker than the solutions considered here. Jacobs et al. [12] introduced a thresholding-type scheme, similar to this implicit time discretization and proved its convergence to a weak solution under an energy convergence assumption.

We now state the setting in more detail and give an overview of our results. For a given (length) scale $\varepsilon > 0$ and a field $u : \mathbb{R}^d \to \mathbb{R}$ we consider the Cahn–Hilliard free energy
For initial data \( u_{e,0} \) we say that \( u_e \) solves the Cahn–Hilliard equation if

\[
\frac{\partial}{\partial t} u_e + \nabla \cdot \left( m(u_e) \left( -\nabla \frac{\delta E_e}{\delta u_e} \right) \right) = 0
\]

(1.2)

together with the initial condition \( u_e(\cdot,0) = u_{e,0} \). Here \( m : \mathbb{R} \to [0, \infty) \) is the mobility function and \( W : \mathbb{R} \to [0, \infty) \) is the standard double-well potential \( W(s) = \frac{1}{4} s^2 (s - 1)^2 \).

In this work, we want to consider a density-dependent mobility function which is degenerate in one of the two phases, say, \( u = 0 \). We focus on the prototypical example \( m(u) = u_+ = \max\{u, 0\} \). The crucial properties are that the mobility function is smooth and concave between the two energy wells 0 and 1, and degenerates in precisely one well, i.e., \( m(0) = 0 \) and \( m(1) > 0 \). Heuristically, it is clear that diffuse interfaces of thickness \( \sim \varepsilon \) will develop in this model, see Figure 1. The main goal of this work is to understand the behavior of the solutions \( u_e = u_e(x,t) \) in the singular limit \( \varepsilon \downarrow 0 \). It is well-known since the work of Modica and Mortola [13] and Modica [14] that the Cahn–Hilliard energy \( \Gamma \)-converges to a multiple of the perimeter functional. Therefore, our goal here can be formulated as extending this convergence to the corresponding gradient flows. On a conceptual level, the difficulties here are that, even for fixed \( \varepsilon \), the energy is not (geodesically) convex, and becomes singular in the limit \( \varepsilon \downarrow 0 \).

On the one hand, our result draws a connection between two well-known basic physical models. On the other hand, the Cahn–Hilliard equation can be used as a numerical scheme to approximate solutions to the Hele–Shaw flow [3]. In Glasner [3, Figure 2], a simulation shows the change of topology of a \textit{ferrofluid}, which is subject to a constant external magnetic field perpendicular to the plates. The long, narrow droplet breaks up multiple times, eventually leading to an array of circular droplets. Our diffuse interface model can be extended to describe this experiment by adding a term to the free energy.

\[
E_e(u) = \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \; dx.
\]

(1.1)
(1.1) describing the magnetic energy

\[ F_M(u) = 2\pi M^2 \int_{\mathbb{R}^d} uk_b * u \, dx, \]

where \( M \) is the magnetization and \( k_b \) is a convolution kernel depending on the plate spacing \( b \). For simplicity, we do not consider any additional terms in the energy in this work.

The remainder of this work is structured as follows. In Section 2, we give the heuristic idea behind our strategy, which is then formalized in the following sections. In Section 3 we prove existence of weak solutions to the Cahn–Hilliard equation (1.2). In Section 4 we investigate the sharp interface limit \( \varepsilon \downarrow 0 \). In Appendix A we construct well prepared initial data for a given initial configuration \( \Omega(0) = \Omega_0 \). Finally, in the other appendices, we gather some useful facts from the literature.

2. Heuristic idea behind our proof

In this work, we prove the following result; see Theorem 4.1 for the precise statement. We consider (weak) solutions \( u_\varepsilon \geq 0 \) of the Cahn–Hilliard equation, formally satisfying

\[
\begin{aligned}
\partial_t u_\varepsilon + \nabla \cdot j_\varepsilon &= 0, \\
j_\varepsilon &= u_\varepsilon \nabla \left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \right),
\end{aligned}
\]  

with given initial conditions \( u_\varepsilon(\cdot, 0) = u_{\varepsilon, 0} \). Here \( j_\varepsilon \) is the flux and the mobility \( m(u) = u \) degenerates in the phase \( \{ u = 0 \} \). We show that \( u_\varepsilon = u_\varepsilon(x, t) \) converges to a characteristic function \( \chi(x, t) = \chi_{\Omega(t)}(x) \) as the length scale \( \varepsilon \) vanishes and that, under an energy convergence assumption, there exists a flux field \( j \in L^2(\mathbb{R}^d \times (0, T); \mathbb{R}^d) \) such that the pair \( \{ \Omega(t) \}_{t \in [0, T], j} \) is a weak solution to the Hele–Shaw flow

\[
\begin{aligned}
\nabla \cdot j(\cdot, t) &= 0, & \text{in } \Omega(t), \\
V &= j(\cdot, t) \cdot \nu, & \text{on } \partial \Omega(t),
\end{aligned}
\]  

Figure 2. Illustration of interpolation functions. (a) A Hele–Shaw cell with spacing \( b \). (b) Dimension reduction leads to the Hele–Shaw flow.
\[
\begin{aligned}
\begin{cases}
    j(\cdot, t) = -\nabla p(\cdot, t), & \text{in } \Omega(t), \\
    \sigma H = p(\cdot, t), & \text{on } \partial\Omega(t).
\end{cases}
\end{aligned}
\tag{2.3}
\]

Here, \(\sigma\) denotes the surface tension, \(H\) denotes the mean curvature of the free boundary \(\partial\Omega(t)\) and \(\nu\) its exterior normal vector. In this sharp-interface model, the flux \(j\) can be viewed as a fluid velocity, and \(p\) plays the role of pressure. The first two equations (2.2) state that the flow is incompressible and that the free boundary is transported by the fluid velocity, a simple kinematic condition. The second two equations (2.3) are Darcy’s law, which governs the slow motion of fluids in porous media or in narrow regions, and the force balance along the free boundary between capillary forces and pressure.

In this section, we give the heuristic argument for this convergence. To this end, let us assume we have a smooth solution to (2.1). A direct computation then shows that the Cahn–Hilliard energy (1.1) is dissipated:

\[
\frac{d}{dt} E_e(u_e) = -\int_{\mathbb{R}^d} \frac{|j_e|^2}{u_e} \, dx \leq 0.
\]

In particular, this means that

\[
\sup_{t > 0} E_e(u_e) \leq E_e(u_e, 0) \quad \text{and} \quad \int_0^\infty \int_{\mathbb{R}^d} \frac{|j_e|^2}{u_e} \, dx \, dt \leq E_e(u_e, 0).
\]

The first estimate gives compactness in configuration space, while the second estimate gives us control in time. Indeed, by the Modica–Mortola/Bogomol’nyi trick [13, 15], i.e., a combination of the chain rule and Young’s inequality,

\[
\int_{\mathbb{R}^d} |\nabla (\phi \circ u_e)(x, t)| \, dx \leq E_e(u_e, \cdot, t)).
\]

Using \(v(\cdot, t') := \left(\frac{t}{t_0}\right)(\cdot, t'(t_2 - t_1) + t_1)\) in the Benamou–Brenier formula for optimal transport, which will also be rigorously justified later, and changing variables yields

\[
W_2^2(u_e(\cdot, t_2), u_e(\cdot, t_1)) \leq (t_2 - t_1) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|j_e|^2}{u_e} \, dx \, dt \leq E_e(u_e, 0).
\]

Hence, it is natural to expect some compactness by some variant of the Aubin–Lions lemma. More precisely, we will show in Section 4.2 that after passage to a subsequence

\[
u_e \to \chi \quad \text{and} \quad j_e \overset{*}{\to} j \quad \text{as } \varepsilon \downarrow 0
\]

for some characteristic function \(\chi\) and some flux \(j\), satisfying \(\text{ess sup}_{t > 0} \int |\nabla \chi| < \infty\) and \(\int_0^\infty \int_{\mathbb{R}^d} |\chi j|^2 \, dx \, dt < \infty\).

Now to verify that the limit \((\chi, j)\) satisfies the Hele–Shaw system (2.2)–(2.3) in a distributional sense, we observe that the Cahn–Hilliard equation (2.1) has a divergence structure. This is clear for the first equation, but it is not immediate for the second one relating the flux \(j_e\) to the first variation of the Cahn–Hilliard energy in Wasserstein space. However, since here we assume that \(u_e\) is smooth, some simple manipulations show that
\[ u_e \nabla \left( \varepsilon \Delta u_e - \frac{1}{\varepsilon} W'(u_e) \right) = \nabla \cdot T_e - \nabla \left( \left( \varepsilon |\nabla u_e|^2 + \frac{1}{\varepsilon} W'(u_e)u_e \right) - \nabla \cdot (\varepsilon u_e \nabla u_e) \right), \]

(2.4)

where \( T_e \) denotes the energy-stress tensor

\[ T_e = \left( \frac{\varepsilon}{2} |\nabla u_e|^2 + \frac{1}{\varepsilon} W(u_e) \right) I_d - \varepsilon \nabla u_e \otimes \nabla u_e, \]

which appears naturally when performing domain variations. Let us comment on the identity (2.4). First, it is not surprising that also in our case of variations in Wasserstein space, the energy-stress tensor appears. Second, when testing with a divergence-free vector field (in which case domain variations and variations in Wasserstein space are equivalent), the gradient-term vanishes and we recover the classical form known from domain variations. Third, the R.H.S. of equation (2.4) is in divergence form, which means that the outermost derivative can be put onto the test function. In addition, the last R.H.S. term contains yet another divergence, which can be put onto the test function, too. Then one ends up with only first-order operators on \( u_e \). This implies that the resulting weak formulation enjoys excellent compactness properties in the sense that given a sequence of weak solutions one only needs to show energy convergence (and does not need higher regularity) to prove that the limit is again a weak solution. Even better, (2.4) even allows us to pass to the sharp-interface limit \( \varepsilon \downarrow 0 \) : we only need to pass to the limit in these first-order terms, which can be done by an adaptation of the seminal work of Luckhaus and Modica [7], who in particular prove that if \( u_e \to \chi \) such that \( E_\varepsilon(u_e) \to \int |\nabla \chi| \), then \( T_e \rightharpoonup T = \sigma(I_d - \nu \otimes \nu)|\nabla \chi| \), where \( \nu \) is the measure theoretic outer unit normal given by the Radon–Nikodym derivative \(-\nabla \chi \big/ |\nabla \chi| \).

In order to turn this idea into a rigorous proof, we first construct weak solutions \( u_e \) in Section 3, which will already be based on the divergence structure (2.4). Then, in Section 4.2, we make rigorous the compactness, and in Section 4.3, we pass to the limit \( \varepsilon \downarrow 0 \) in our weak formulation.

### 3. Construction of weak solutions to the Cahn–Hilliard equation

The main result of this section is the following theorem on global-in-time existence of weak solutions to the degenerate Cahn–Hilliard equation (1.2).

**Theorem 3.1.** For each \( u_{e,0} \in \mathcal{A} \) and each \( \varepsilon > 0 \), there exists a weak solution to the Cahn–Hilliard equation in the sense of Definition 3.2 below.

Here \( \mathcal{A} \subset L^1(\mathbb{R}^d; [0, \infty)) \) denotes the set of nonnegative probability densities \( u \) with respect to the Lebesgue measure, which is denoted by \( \mathcal{L}^d \), with finite second moments, i.e.,

\[ \mathcal{A} := \left\{ u \in L^1(\mathbb{R}^d; [0, \infty)) : \int_{\mathbb{R}^d} u\,dx = 1, \, M_2(u) := \int_{\mathbb{R}^d} |x|^2 u(x)\,dx < \infty \right\}. \]

To define our weak solutions, we use the weak formulation of (2.1), which is formulated in terms of the field \( u_e \) and the flux \( j_e \).
**Definition 3.2** (Weak solution to Cahn–Hilliard). Let $\varepsilon > 0$ and let $u_{\varepsilon,0} \in A$. We say that $(u_{\varepsilon}, j_{\varepsilon})$ is a weak solution to the Cahn–Hilliard equation (2.1) if

(i) $u_{\varepsilon}$ and $j_{\varepsilon}$ are a distributional solution to (2.1), i.e., $u_{\varepsilon}(\cdot, 0) = u_{\varepsilon,0}$, and

\[
\int_{\mathbb{R}^d} u_{\varepsilon,0} \zeta(\cdot, 0) \, dx + \int_0^T \int_{\mathbb{R}^d} u_{\varepsilon} \partial_t \zeta + j_{\varepsilon} \cdot \nabla \zeta \, dx \, dt = 0
\]

(3.1)

holds for all $\zeta \in C^1_c([0,T])$, and

\[
\int_0^T \int_{\mathbb{R}^d} j_{\varepsilon} \cdot \nabla \zeta \, dx \, dt
\]

\[
= - \int_0^T \int_{\mathbb{R}^d} T_{\varepsilon} : \nabla \zeta \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left[ (\nabla \cdot \zeta) \left( \varepsilon |\nabla u_{\varepsilon}|^2 + \frac{1}{\varepsilon} W'(u_{\varepsilon})u_{\varepsilon} \right) + \varepsilon u_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla (\nabla \cdot \zeta) \right] \, dx \, dt
\]

(3.2)

for all $\zeta \in C^2_c([0,T]; \mathbb{R}^d)$, where $T_{\varepsilon}$ is the energy stress tensor

\[
T_{\varepsilon} := \left( \frac{\varepsilon}{2} |\nabla u_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(u_{\varepsilon}) \right) I_d - \varepsilon \nabla u_{\varepsilon} \otimes \nabla u_{\varepsilon}.
\]

(3.3)

Here $I_d$ denotes the identity matrix on $\mathbb{R}^d$.

(ii) The solution satisfies the optimal energy-dissipation rate

\[
\text{ess sup}_{T \in [0,T]} \left\{ \frac{E_{\varepsilon}(u_{\varepsilon}(\cdot, t'))}{T} + \int_0^T \int_{\mathbb{R}^d} |j_{\varepsilon}|^2 \, dx \, dt \right\} \leq E_{\varepsilon}(u_{\varepsilon,0}).
\]

(3.4)

Here, with a slight abuse of notation, we work with the convention $\frac{0}{0} = 0$. In particular we see that for a.e. $x$, $t$ if $u_{\varepsilon}(x,t) = 0$, then $j_{\varepsilon}(x,t) = 0$.

Elliott and Garcke [4] have used a Galerkin approximation approach to construct weak solutions to Cahn–Hilliard equations with mobility functions $m \geq \delta$, and then take $\delta \downarrow 0$. To be self-contained, we give an alternative existence proof of weak solutions to (2.1). We utilize a minimizing movements scheme directly for the degenerate case $m(u) = u$ and exploit the connection to optimal transport as discovered by Jordan et al. [6]. Here our energy is of higher order, but non-negative.

Without loss of generality, by scaling, we assume $\varepsilon = 1$ and drop the index $\varepsilon$ in this section for notational simplicity.

### 3.1. The construction

Let $T \in (0, \infty)$. For $h > 0$ we consider the minimization problem

\[
\inf_{u \in A} \left\{ E(u) + \frac{1}{2h} d^2(u,u_0) \right\},
\]

(3.5)
where \( u_0 \in \mathcal{A} \) such that \( E(u_0) < \infty \). Here \( d(u, u_0) := W_2(u \mathcal{L}^d, u_0 \mathcal{L}^d) \) denotes the quadratic Wasserstein distance of densities, and the Wasserstein distance of measures \( \mu, \nu \) is given by (B.2). We refer to Appendix B for some basic facts and the standard notation we use here.

**Lemma 3.3 (Existence of minimizers).** Let \( u_0 \in \mathcal{A} \) such that \( E(u_0) < \infty \) and let \( h > 0 \). Then there exists \( u \in \mathcal{A} \) which minimizes (3.5).

We use this lemma inductively to construct a sequence \((u_n)_{n \in \mathbb{N}}\) such that

\[
u_n = \arg \min_u \left\{ E(u) + \frac{1}{2h} d^2(u, u_{n-1}) \right\}, \quad n = 1, 2, 3, \ldots \tag{3.6}\]

Then we define the approximation \( u_h : [0, \infty) \times \mathbb{R}^d \to \mathbb{R} \) by piecewise constant interpolation using \( u_h(x, t) = u_n(x) \) for \( t \in [hn, h(n + 1)) \). Finally, we show that \( u_n \) is precompact in \( L^1(0, T; L^1(\mathbb{R}^d)) \).

This variational algorithm is known as the minimizing movement/JKO scheme, and was first introduced in Jordan et al. [6] in the framework of Wasserstein gradient flows.

**Proof of Lemma 3.3.** We use the direct method to prove existence of a minimizer for (3.5).

Since \( E(u_0) + \frac{1}{2h} d^2(u_0, u_0) = E(u_0) < \infty \), the infimum (3.5) is bounded from above. It is also bounded from below, since \( E(u) + \frac{1}{2h} d^2(u, u_0) \) is nonnegative for all \( u \). It is well known both functionals \( u \mapsto E(u) \) and \( u \mapsto d^2(u, u_0) \) are lower semi-continuous w.r.t. \( L^1 \) convergence. For the first functional, this can simply be seen by writing \( W(u) = W(u)\chi_{[0,1]}(u) + W(u)\chi_{[0,1]}(u) \) so that \( E \) is the superposition of a convex l.s.c. functional and a continuous functional on \( L^1 \). For the latter, this follows e.g. from a direct argument on the level of the optimal transport plans, see Jordan et al. [6, eq. (21), (27)]. Hence we only need to show compactness.

Let \((u_l)_{l \geq 1} \subset \mathcal{A} \) be a minimizing sequence for \( E(\cdot) + \frac{1}{2h} d^2(\cdot, u_0) \); note that if \( u \not\in \mathcal{A} \), then \( d^2(u, u_0) = \infty \). For all sufficiently large \( l \) we then have

\[
E(u_l) + \frac{1}{2h} d^2(u_l, u_0) \leq E(u_0) + 1 < \infty,
\]

so we may assume w.l.o.g. that (3.7) holds for all \( l \). Now we observe that there exists \( M < \infty \) such that

\[
W(u) = \frac{1}{4} u^2(u - 1)^2 \geq \frac{1}{5} u^4 \quad \text{for all } |u| > M. \tag{3.8}
\]

Now let \( u \in \mathcal{A} \cap L^4(\mathbb{R}^d) \). Using (3.8) we have

\[
\int_{\mathbb{R}^d} u^4 \, dx = \int_{\{x : u(x) \leq M\}} u^4 \, dx + \int_{\{x : u(x) > M\}} u^4 \, dx
\leq M^3 \int_{\mathbb{R}^d} u \, dx + 5 \int_{\{x : u(x) > M\}} W(u) \, dx
\leq M^3 + 5E(u),
\]

and plugging in \( u = u_l \) we get by (3.7)
\[ \int_{\mathbb{R}^d} u_t^4 \, dx \leq M^3 + 5(E(u_0) + 1). \]

(3.9)

Thus \( u_t \) is uniformly bounded in \( L^4 \). Further we have by (3.7)

\[ \int_{\mathbb{R}^d} |\nabla u_t|^2 \, dx \leq 2(E(u_0) + 1), \]

hence \( \nabla u_t \) is also uniformly bounded in \( L^2 \). By Rellich’s Theorem [16, Chap. 5.7, Thm. 1] and a diagonal argument, there exists a subsequence \( u_{t_m} \) and \( u \in L^2_{loc} \) such that \( u_{t_m} \rightarrow u \) in \( L^2_{loc} \) as \( m \rightarrow \infty \).

To obtain \( L^1 \) convergence, it suffices to show that the second moments are uniformly bounded and \( L^1_{loc} \) convergence; indeed if the second moments are uniformly bounded from above, we have

\[ M_2(u_{t_m}) = \lim_{R \rightarrow \infty} \int_{B_R} |x|^2 u_{t_m}(x) \, dx \leq \lim_{R \rightarrow \infty} \limsup_{m \rightarrow \infty} \int_{B_R} |x|^2 u_{t_m}(x) \, dx \]

\[ \leq \limsup_{m \rightarrow \infty} \int_{\mathbb{R}^d} |x|^2 u_{t_m}(x) \, dx < \infty. \]

Then, for all \( R < \infty \) we have

\[ \int_{\mathbb{R}^d} |u_{t_m} - u| \, dx = \int_{B_R} |u_{t_m} - u| \, dx + \int_{\mathbb{R}^d \setminus B_R} |u_{t_m} - u| \, dx \]

\[ \leq \int_{B_R} |u_{t_m} - u| + \frac{1}{R^2} (M_2(u_{t_m}) + M_2(u)). \]

Taking the limit \( m \rightarrow \infty \) we get

\[ \limsup_{m \rightarrow \infty} \int_{\mathbb{R}^d} |u_{t_m} - u| \, dx \leq \frac{1}{R^2} \left( \sup_{m \in \mathbb{N}} M_2(u_{t_m}) + M_2(u) \right), \]

and taking \( R \rightarrow \infty \) on the R.H.S., we obtain

\[ \limsup_{m \rightarrow \infty} \int_{\mathbb{R}^d} |u_{t_m} - u| \, dx = 0. \]

Now we show that the second moments are uniformly bounded. To this end, let \( \gamma_l \) be the optimal plan in Kantorovich’s optimal transport problem (B.2) between \( u_t \) and \( u_0 \), i.e.,

\[ (\pi_x)_* \gamma_l = u_t \mathcal{L}^d, \quad (\pi_y)_* \gamma_l = u_0 \mathcal{L}^d, \quad d^2(u_t, u_0) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma_l(x, y). \]

We integrate the inequality

\[ |x|^2 = |x - y + y|^2 = |x - y|^2 + 2(x - y) \cdot y + |y|^2 \leq \left( 1 + \frac{1}{h} \right) |x - y|^2 + (1 + h) |y|^2 \]

against the optimal plan \( \gamma_l \) to get

\[ (3.10) \]
\[ M_2(u_l) = \int_{\mathbb{R}^d} |x|^2 u_l(x) \, dx \]
\[ \leq \left( 1 + \frac{1}{h} \right) d^2(u_l, u_{l-1}) + (1 + h) \int_{\mathbb{R}^d} |y|^2 u_0(y) \, dy \]
\[ \leq 2(1 + h)(E(u_0) - E(u_l)) + (1 + h)M_2(u_0) \]
\[ \leq 2(1 + h)E(u_0) + (1 + h)M_2(u_0). \]  

(3.11)

The R.H.S. is independent of \( l \), so we have a uniform bound on the second moments of \( u_l \).

Now we define \( u_n, \ n = 1, 2, 3, \ldots \) successively as a minimizer of

\[ \inf_{u \in A} E(u) + \frac{1}{2h} d^2(u, u_{n-1}). \]

The only assumption on \( u_0 \) in Lemma 3.3 was that \( u_0 \in \mathcal{A} \) with \( E(u_0) < \infty \), and any minimizer of \( E(\cdot) + \frac{1}{2h} d^2(\cdot, \cdot) \) is again in \( \mathcal{A} \). Thus it is guaranteed that a \( u_n \) exists for all \( n \in \mathbb{N} \). For a given time step size \( h > 0 \) define the piecewise constant time interpolation \( u_h : \mathbb{R}^d \times [0, \infty) \to \mathbb{R} \) by

\[ u_h(x, t) := u_n(x), \quad t \in [nh, (n + 1)h). \]  

(3.12)

Then we have a uniform bound on the energy of \( u_n \). The next lemma gives us a Gronwall-type estimate on the second moments of \( u_n \), which we need to prove compactness of \( u_h \).

**Lemma 3.4.** Let \( u_n \in \mathcal{A} \) be a sequence and \( h \in (0, 1) \) such that

\[ E(u_n) + \frac{1}{2h} d^2(u_n, u_{n-1}) \leq E(u_{n-1}) \quad \text{for all} \quad n = 1, 2, 3, \ldots \]

Then the second moments \( M_2(u_n) \) satisfy the estimate

\[ M_2(u_n) \leq Ce^{Ch}M_2(u_0) \quad \text{for all} \quad n = 1, 2, 3, \ldots \]

**Proof.** As in the proof of Lemma 3.3, we use (3.10) and (3.11), except instead of the optimal plan between \( u_n \) and \( u_0 \), we use the optimal plan between \( u_n \) and \( u_{n-1} \). Then

\[ M_2(u_n) \leq 2(1 + h)(E(u_{n-1}) - E(u_n)) + (1 + h)M_2(u_{n-1}). \]

This is equivalent to the inequality

\[ \frac{M_2(u_n) - M_2(u_{n-1})}{h} \leq 2(1 + h) \frac{E(u_{n-1}) - E(u_n)}{h} + M_2(u_{n-1}), \]

which we can rewrite as

\[ \frac{(M_2(u_n) + 2E(u_n)) - (M_2(u_{n-1}) + 2E(u_{n-1}))}{h} \leq M_2(u_{n-1}) + 2(E(u_{n-1}) - E(u_n)) \]

\[ \leq M_2(u_{n-1}) + 2(E(u_{n-1}) - E(u_n)). \]
Now Gronwall’s inequality \[17, \text{Prop. 3.1}\] yields
\[M_2(u_n) \leq Ce^{Ch}M_2(u_0)\]
for all \(h \in (0, 1)\).

Now we are ready to prove that the piecewise constant time interpolation is precompact in \(L^1\).

**Lemma 3.5** (Compactness). For all finite time horizons \(T < \infty\), there exists a subsequence \(h_l \to 0\) as \(l \to \infty\) such that \(u_{h_l} \to u\) for some \(u\) in \(L^1(0, T; L^p(\mathbb{R}^d))\) for all \(1 \leq p < 4\).

Of course, this lemma can be extended to \(L^1_{\text{loc}}([0, \infty); L^1(\mathbb{R}^d))\) convergence using a diagonal argument.

**Proof.** The proof is divided into two steps. First, we show convergence in \(L^1_{\text{loc}}\), and then post-process it to \(L^1\) convergence. Convergence in \(L^p\) for \(1 < p < 4\) then follows by interpolation.

**Step 1** (\(L^1_{\text{loc}}\) convergence). We have
\[
E(u_n) + \frac{1}{2h} d^2(u_n, u_{n-1}) \leq E(u_{n-1}),
\]

(3.13)
since \(u = u_{n-1}\) is admissible in the minimization problem for \(u_n\). Then by induction,
\[
E(u_n) + \frac{h}{2} \sum_{l=1}^n \left(\frac{d(u_l, u_{l-1})}{h}\right)^2 \leq E(u_0).
\]

(3.14)
Thus \(\nabla u_n\) is uniformly bounded in \(L^2\) and
\[
E(u_h(\cdot, T)) + \frac{1}{2} \int_0^{T-h} \left(\frac{d(u_h(\cdot, t), u_h(\cdot, t+h))}{h}\right)^2 \, dt \leq E(u_0).
\]

(3.15)
Further, as in (3.8)–(3.9), \(u_n\) is uniformly bounded in \(L^4\). Then there exists a constant \(C < \infty\) such that, for any \(T > 0\), we have
\[
\int_0^T \int_{\mathbb{R}^d} |u_h(x, t)|^4 \, dx \, dt \leq CTE(u_0),
\]
\[
\int_0^T \int_{\mathbb{R}^d} |\nabla u_h(x, t)|^2 \, dx \, dt \leq 2TE(u_0),
\]
\[
\frac{1}{2} \int_0^{T-h} \left(\frac{d(u_h(\cdot, t), u_h(\cdot, t+h))}{h}\right)^2 \, dt \leq E(u_0).
\]

Then \(u_h \in L^2(0, T; L^2(\mathbb{R}^d))\) and \(\nabla u_h \in L^2(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d))\) for all \(h\) with uniform bounds.

Now we want to apply the Aubin–Lions lemma D.3 to the sequence \(\{u_h\}\) with \(\mathcal{F} = E\) and \(g = d\). Then we get that there exists \(u : \mathbb{R}^d \times [0, T] \to \mathbb{R}\) such that \(u_h \to u\) in...
measure as $h \downarrow 0$, hence also in $L^1_{loc}(\mathbb{R}^d \times [0, T])$. Indeed, let $D := \{ u \in A : M_2(u) \leq C \}$ for some $C < \infty$ and let

$$
\mathcal{F}_D : L^1(\mathbb{R}^d) \to [0, \infty], \quad \mathcal{F}_D(u) = \begin{cases} E(u), & \text{if } u \in D, \\ \infty, & \text{else.} \end{cases}
$$

We need to show that $\mathcal{F}_D$ is a normal coercive integrand on $L^1(\mathbb{R}^d)$. To see this, observe first that since $A$ is closed and the constraint $M_2(u) \leq C$ is lower semi-continuous, $D$ is a closed subset of $L^1$. As in the proof of Lemma 3.3, $E$ is lower semi-continuous, and hence $\mathcal{F}_D$ is lower semi-continuous in $L^1$. To see that $\mathcal{F}_D$ is coercive, let $\{u_k\} \subset D$ such that $\mathcal{F}_D(u_k) \leq c < \infty$. By interpolation between $L^1$ and $L^4$ and the fact that $\|\nabla u_k\|_{L^2}^2 \leq 2E(u_k)$, we have that the sequence $u_k$ is uniformly bounded in $H^1(\mathbb{R}^d)$. As in the proof of Lemma 3.3, using Rellich’s theorem with a diagonal argument and the fact that $M_2(u_k) \leq C$ for all $k \geq 1$, there exists $u \in D$ such that $u_k \to u$ in $L^1(\mathbb{R}^d)$ as $k \uparrow \infty$. Using the lower semi-continuity of $\mathcal{F}_D$, we have $\mathcal{F}_D(u) \leq c$. Thus, recalling Remark D.4, $\mathcal{F}_D$ is indeed a normal coercive integrand on $L^1(\mathbb{R}^d)$. Condition (D.4) on $d$ is verified in Lemma 4.4, and so we can apply the Aubin–Lions lemma D.3. This concludes Step 1.

Step 2 ($L^1$ convergence). By Lemma 3.4 we know that

$$
M_2(u_n) \leq C e^{C_{coh}} M_2(u_0)
$$

(3.16) holds for all $h \in (0, 1)$. Now observe that for any $R < \infty$ we have

$$
\int_0^T \int_{\mathbb{R}^d} |u - u_{h_1}| \, dx \, dt \leq \int_0^T \int_{B_R} |u - u_{h_1}| \, dx \, dt + \frac{1}{R^2} \int_0^T \int_{\mathbb{R}^d} |x|^2 (|u| + |u_{h_1}|) \, dx \, dt
$$

$$
\leq \int_0^T \int_{B_R} |u - u_{h_1}| \, dx \, dt + \frac{CT}{R^2},
$$

where $C = \sup_{t \in [0, T]} \{M_2(u(\cdot, t)) + M_2(u_{h_1}(\cdot, t))\}$, which is finite by (3.16). Since we have $u_{h_1} \to u$ in $L^1_{loc}(\mathbb{R}^d \times [0, T])$ as $l \to \infty$, taking first $l \to \infty$ and then $R \to \infty$, we get strong convergence in $L^1(0, T; L^1(\mathbb{R}^d))$.

Since $u_{h_1}$ is uniformly bounded in $L^4$, we have $u \in L^4$. By interpolation between $L^p$ norms [16, Appendix B.2.h], we get $u_{h_1} \to u$ in $L^1(0, T; L^p(\mathbb{R}^d))$ as $l \to \infty$ for all $1 \leq p < 4$.

Remark 3.6. We remark that this proof also shows that $u_h$ is uniformly bounded in $L^2(0, T; H^1(\mathbb{R}^d))$ as $h \downarrow 0$ and that $u_h \rightharpoonup u$ weakly in $L^2(0, T; H^1(\mathbb{R}^d))$. Also the second moments of $u_n$ are uniformly bounded, i.e., there exists $C < \infty$ depending only on $E(u_0)$ and $M_2(u_0)$ such that for all $n$

$$
M_2(u_n) \leq C.
$$

(3.17)

The next step will be to compute the Euler–Lagrange equation. Before we do this, we compute the first variation of the Dirichlet energy w.r.t. variations Wasserstein space. For the remainder of this paper, $i_d$ denotes the identity map on $\mathbb{R}^d$. 

Lemma 3.7. Let \( u_0 \in H^1(\mathbb{R}^d) \) and let \( \zeta \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d) \). Further, let \( \{t_t\}_{t \in \mathbb{R}} \) be the corresponding flux

\[
\begin{align*}
\partial_t t_t &= \zeta \circ t_t, \\
t_0 &= t_d,
\end{align*}
\]

and let \( u_t = (t_t)^* u_0 \) be the push-forward of \( u_0 \) under \( t_t \). Then

\[
\frac{d}{dt} \left| \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u_t|^2 \, dx \right| = \int_{\mathbb{R}^d} - \nabla u_0 \cdot \nabla \zeta \nabla u_0 - \frac{1}{2} |\nabla u_0|^2 \nabla \cdot \zeta - u_0 \nabla u_0 \cdot \nabla (\nabla \cdot \zeta) \, dx.
\]

\[
(3.18)
\]

Of course, formally, (3.18) is obvious using the continuity equation \( \partial_t u + \nabla \cdot (u \zeta) = 0 \). The main point of the lemma is that it applies in the optimal regularity setting \( u_0 \in H^1 \). Furthermore, the derivation of this formula is straightforward under the regularity condition \( u_0 \in H^2 \), see for example [5, Proposition 4.5]. Then one needs to prove right away that each solution \( u_n \) to the minimizing movements scheme (3.6) satisfies \( u_n \in H^2 \). A similar regularity result was also shown in Elliott and Garcke [4]. However, as we will see, \( H^1 \) is enough regularity to verify (3.18). After completion of this manuscript, it has come to our attention that a similar proof was given in Matthes et al. [18, Lemma 2.5]. However, the interchange of differentiation and integration is not justified in that paper, which is why we present the simple proof of the statement here. We believe that this lemma could be useful in other applications for which the regularity results from Lisini et al. [5] and Elliott and Garcke [4] are not applicable.

Proof. By definition of \( u_n \), we have

\[
\int_{\mathbb{R}^d} u_t \zeta \, dx = \int_{\mathbb{R}^d} u_0(x) \zeta(t_t(x)) \, dx \quad \text{for all } \zeta \in C^0_c(\mathbb{R}^d).
\]

Since \( t_t \) is invertible near \( t = 0 \), the above is equivalent to

\[
u_0 = \det(\nabla t_t) u_t \circ t_t,
\]

and using \( \partial_t t_t = \zeta \circ t_t \), we have

\[
\begin{align*}
\nabla t_t|_{t=0} &= I_d, \\
\partial_t \nabla t_t|_{t=0} &= \nabla \zeta, \\
\partial_t \det(\nabla t_t)|_{t=0} &= \text{tr} \nabla \zeta = \nabla \cdot \zeta.
\end{align*}
\]

First, we compute the gradient of (3.19)

\[
\nabla u_0 = u_t \circ t_t \nabla \det(\nabla t_t) + (\det(\nabla t_t)) (\nabla t_t)^T (\nabla u_t) \circ t_t.
\]

By the Jacobi formula for the gradient of the determinant, the first term reads

\[
\partial_k \det(\nabla t_t) = \det(\nabla t_t) \text{tr}( (\nabla t_t)^{-1} \partial_k \nabla t_t ).
\]
Therefore
\[ u_t \circ t_t \nabla \det \nabla t_t = u_t \circ t_t \det (\nabla t_t) (\text{tr}((\nabla t_t)^{-1} \partial_{x_i} \nabla t_t))^d \bigg|_{i=1} = u_0 \left( \text{tr}((\nabla t_t)^{-1} \partial_{x_i} \nabla t_t))^d \bigg|_{i=1} \right. \] (3.22)

Here \( v = (v_i)^d \) denotes the vector \( v \in \mathbb{R}^d \) with components \( v_i \). Rearranging terms in (3.21) and inserting (3.22) gives us
\[ (\nabla u_t) \circ t_t = \frac{1}{\det \nabla t_t} (\nabla t_t)^{-T} \left( \nabla u_0 - u_0 \left( \text{tr}((\nabla t_t)^{-1} \partial_{x_i} \nabla t_t))^d \bigg|_{i=1} \right) \].

Therefore
\[
\int_{\mathbb{R}^d} \frac{1}{2} |\nabla u_t|^2 \, dx = \int_{\mathbb{R}^d} \frac{1}{2} |(\nabla u_t) \circ t_t|^2 \det \nabla t_t \, dx \\
= \int_{\mathbb{R}^d} \frac{1}{2 \det \nabla t_t} |(\nabla t_t)^{-T} (\partial_{x_i} u_0 - u_0 \text{tr}((\nabla t_t)^{-1} \partial_{x_i} \nabla t_t))^d \bigg|_{i=1} |^2 \, dx.
\]

Now we can write the difference quotient as
\[
\delta_t \left( \frac{1}{2 \det \nabla t_t} \sum_{i,j,k} \left[ (\nabla t_t)^{-T} \right]_{ij} \left[ (\nabla t_t)^{-T} \right]_{jk} [\partial_{x_i} u_0 - u_0 v_j][\partial_{x_k} u_0 - u_0 v_k] \right) \]
\[
= \delta_t \left( \frac{1}{2 \det \nabla t_t} \sum_{i,j,k} \left[ (\nabla t_t)^{-T} \right]_{ij} \left[ (\nabla t_t)^{-T} \right]_{jk} \left( \partial_{x_i} u_0 (\partial_{x_k} u_0) - 2u_0 (\partial_{x_i} u_0) v_k + u_0^2 v_j v_k \right) \right).
\]

We compute the time derivatives for all terms:
\[
\partial_t (\nabla t_t)^{-T} \bigg|_{t=0} = -(\nabla t_t)^{-T} (\partial_t \nabla t_t)(\nabla t_t)^{-T} \bigg|_{t=0} = -\nabla \xi,
\]
\[
\partial_t v_k = \text{tr}((\nabla t_t)^{-1} \partial_{x_k} \nabla t_t) \bigg|_{t=0} = \text{tr}(\partial_{x_k} \nabla \xi) = \partial_k (\nabla \cdot \xi),
\]
\[
\partial_t \bigg|_{t=0} \frac{1}{\det \nabla t_t} = -\frac{1}{(\det \nabla t_t)^2} \partial_t \det \nabla t_t \\
= -\nabla \cdot \xi.
\]
Since $\xi$ has compact support, all terms are bounded in $L^\infty$. Now we split the difference quotient (3.24) into three terms:

$$\delta_t\left(\frac{1}{2\det\nabla t} \sum_{i,j,k} \left[ (\nabla t)^{-T} \right]_{ij} \left[ (\nabla t)^{-T} \right]_{ik} (\partial_x u_0)(\partial_x u_0) - 2u_0(\partial_x u_0)v_k + u_0^2v_k \right)$$

$$= \sum_{i,j,k} \delta_t\left(\frac{1}{2\det\nabla t} \left[ (\nabla t)^{-T} \right]_{ij} \left[ (\nabla t)^{-T} \right]_{ik} (\partial_x u_0)(\partial_x u_0) \right)$$

$$- 2\sum_{i,j,k} \delta_t\left(\frac{1}{2\det\nabla t} \left[ (\nabla t)^{-T} \right]_{ij} \left[ (\nabla t)^{-T} \right]_{ik} v_k \right) u_0(\partial_x u_0)$$

$$+ \sum_{i,j,k} \delta_t\left(\frac{1}{2\det\nabla t} \left[ (\nabla t)^{-T} \right]_{ij} \left[ (\nabla t)^{-T} \right]_{ik} v_k \right) u_0^2.$$ 

Since $u_0 \in H^1$, all three terms are a product of an $L^1$ function which is independent of $t$ and a function which converges uniformly in $L^\infty$ as $t \downarrow 0$. Therefore we can apply dominated convergence in (3.23) when passing to the limit $t \downarrow 0$ and obtain

$$\lim_{t \downarrow 0} \frac{1}{t} \left( \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u_t|^2 \, dx - \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u_0|^2 \, dx \right)$$

$$= \int_{\mathbb{R}^d} \frac{1}{2} \frac{\partial}{\partial t} \bigg|_{t=0} \left\{ \frac{1}{\det\nabla t} \left[ (\nabla t)^{-T} \right]_{ij} \left[ (\nabla t)^{-T} \right]_{ik} \right. \right. \left. \left. \partial_x u_0 - u_0 \text{tr}((\nabla t)^{-1}\partial_x \nabla t)_i \right\}^2 \bigg) \, dx$$

$$= \int_{\mathbb{R}^d} \frac{1}{2} \left\{ -|\nabla \cdot \xi| |\nabla u_0|^2 + 2\nabla u_0 \cdot \left( (\nabla \cdot \xi)^T \nabla u_0 - u_0 (\text{tr}(\partial t)_{t=0} \partial_x \nabla t)_i \right) \right\} \, dx$$

$$= \int_{\mathbb{R}^d} \frac{1}{2} \left\{ -|\nabla \cdot \xi| |\nabla u_0|^2 - 2\nabla u_0 \cdot \left( (\nabla \cdot \xi)^T \nabla u_0 + u_0 \nabla (\nabla \cdot \xi) \right) \right\} \, dx.$$

The previous lemma allows us now to compute the Euler–Lagrange equation for the minimizing movements scheme (3.5).

Let us first introduce some notation. For $n \in \mathbb{N}$ let $\gamma_n$ be the optimal plan for Kantorovich’s optimal transport problem (B.2) between $u_n$ and $u_{n-1}$ such that $(\pi_x)_y \gamma_n = u_{n-1} \mathcal{L}^d$ and $(\pi_y)_z \gamma_n = u_n \mathcal{L}^d$. It will also be convenient to define the transport map from $u_n \mathcal{L}^d$ to $u_{n-1} \mathcal{L}^d$ by $t_n$, so that $(t_n)_* u_n \mathcal{L}^d = u_{n-1} \mathcal{L}^d$. Using that $\gamma_n$ is absolutely continuous (cf. Prop. B.1 (i), (iii)), we define the flux

$$j_n(y) := \int_{[n \epsilon, (n+1) \epsilon]} \frac{1}{H} (y-x)^{d}\gamma_n(dx,y) = \int_{[n \epsilon, (n+1) \epsilon]} (y-t_n(y))u_n(y),$$

$$j_n(y,t) := j_n(y), \quad t \in [nh, (n+1)h)],$$

and the energy stress tensor

$$T_n := \frac{1}{2} |\nabla u_n|^2 + W(u_n) \right) I_d - \nabla u_n \otimes \nabla u_n,$$

$$T_h(\cdot,t) := T_n, \quad t \in [nh, (n+1)h).$$
Lemma 3.8. The Euler–Lagrange equation for (3.5) is given by
\[
\int_{\mathbb{R}^d} j_n \cdot \xi \, dx = -\int_{\mathbb{R}^d} T_n : \nabla \xi \, dx + \int_{\mathbb{R}^d} \left[ (\nabla \cdot \xi) \left( |\nabla u_n|^2 + W'(u_n)u_n + u_n \nabla u_n \cdot \nabla (\nabla \cdot \xi) \right) \right] \, dx
\]
for all \( \xi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) \).

Before proving the lemma let us briefly mention the following estimate on the flux.

Lemma 3.9. For any \( n \) and a.e. \( y \in \mathbb{R}^d \) such that \( u_n(y) \neq 0 \) we have
\[
\left| j_n(y) \right| \leq \int_{\mathbb{R}^d} \frac{1}{|y - x|^2} \gamma_n(dx, y).
\]

Further \( j_n(y) = 0 \) whenever \( u_n(y) = 0 \) a.e., and \( \|j_n\|_{L^1} \leq \frac{1}{h} E(u_0) \).

Proof. The inequality (3.27) follows directly from the definition of \( j_n \) and Jensen’s inequality. To obtain the \( L^1 \) bound on \( j_n \), observe that by Hölder’s inequality and (3.13)
\[
\int_{\mathbb{R}^d} |j_n| \, dx \leq \left( \int_{\mathbb{R}^d} \frac{|j_n|^2}{u_n} \, dx \right)^{1/2} \left( \int_{\mathbb{R}^d} u_n \, dx \right)^{1/2} \leq \frac{1}{h^2} d^2(u_n, u_{n-1}) \leq \frac{1}{h} E(u_0).
\]

Proof of Lemma 3.8. Let \( \xi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) \) and let \( \{t_t\}_{t \in \mathbb{R}} \) be the corresponding flux
\[
\partial_t t_t = \xi \circ t_t \quad \text{for all } t \in \mathbb{R} \text{ and } t_0 = i_d.
\]
Define \( \tilde{u}(\cdot, t) = (t_t)_*u_n \) as the push-forward of \( u_n \) under \( t_t \), i.e.,
\[
u_n = \det(\nabla t_t) \tilde{u}(\cdot, t) \circ t_t.
\]
By Villani [19, Thm. 5.34], \( \tilde{u} \) is the unique solution to the continuity equation
\[
\begin{align*}
\partial_t \tilde{u} + \nabla \cdot (\tilde{u} \xi) &= 0, \\
\tilde{u}(\cdot, t)\big|_{t=0} &= u_n.
\end{align*}
\]
To compute the first variation of the energy for \( \tilde{u} \), we have
\[
\frac{d}{dt} \bigg|_{t=0} E(\tilde{u}(\cdot, t)) = \frac{d}{dt} \bigg|_{t=0} \int_{\mathbb{R}^d} \frac{1}{2} |\nabla \tilde{u}(\cdot, t)|^2 \, dx + \int_{\mathbb{R}^d} W'(u_n)\partial_t \tilde{u}(\cdot, t)\big|_{t=0} \, dx.
\]

The first term \( \int_{\mathbb{R}^d} \frac{1}{2} |\nabla \tilde{u}|^2 \, dx \) is exactly the term we computed in Lemma 3.7. For the second term \( \int_{\mathbb{R}^d} W'(u_n)\partial_t \tilde{u}(\cdot, t)\big|_{t=0} \, dx \), the first variation of the potential energy, we plug in the continuity equation \( \partial_t \tilde{u} + \nabla \cdot (u_n \xi) = 0 \) and compute.
\[
\int_{\mathbb{R}^d} W'(u_n) \partial_t \tilde{u}(\cdot, t) \big|_{t=0} \, dx = \int_{\mathbb{R}^d} - W'(u_n) \nabla \cdot (u_n \xi) \, dx \\
= \int_{\mathbb{R}^d} - W'(u_n) \nabla u_n \cdot \xi - W'(u_n) u_n \nabla \cdot \xi \, dx \\
= \int_{\mathbb{R}^d} \nabla W(u_n) \cdot \xi - W'(u_n) u_n \nabla \cdot \xi \, dx \\
= \int_{\mathbb{R}^d} (W(u_n) - W'(u_n) u_n) \nabla \cdot \xi \, dx. 
\]

(3.29)

Furthermore, by Ambrosio et al. [20, Thm. 8.4.7], we have

\[
\frac{d}{dt} \left|_{t=0} \right. \frac{1}{2h} d^2(\tilde{u}(\cdot, t), u_{n-1}) = \frac{1}{h} \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - x) \cdot \tilde{\eta}^n(y) \, d\gamma_n(x, y) \\
= \int_{\mathbb{R}^d} j_n \cdot \xi \, dy.
\]

The Euler–Lagrange equation tells us that

\[
\frac{d}{dt} \left|_{t=0} \right. \left( E(\tilde{u}(\cdot, t)) + \frac{1}{2h} d^2(\tilde{u}(\cdot, t), u_{n-1}) \right) = 0. 
\]

(3.30)

Thus combining Lemma 3.7 with (3.29) and (3.30) gives (3.26).

Let us stress once more that our arguments up to this point did not rely on higher regularity of \( u_n \). The only additional ingredient to complete our proof is the following very natural energy convergence as \( h \downarrow 0 \). In our case, this follows immediately from results in the literature which we collect in Appendix C.

**Corollary 3.10** (Energy convergence). As \( l \uparrow \infty \), the energy converges:

\[
\lim_{l \uparrow \infty} \int_0^T E(u_{hl}(\cdot, t)) \, dt = \int_0^T E(u(\cdot, t)) \, dt.
\]

The following theorem concludes our construction of weak solutions to the Cahn–Hilliard equation. Together with Proposition 3.12 in the next section this implies Theorem 3.1.

**Theorem 3.11.** For any \( T < \infty \) and \( u_h \) given by (3.12) there exists a function \( u \in L^1(0, T; L^1(\mathbb{R}^d)) \) and a subsequence \( u_{hl} \) such that \( u_{hl} \to u \) in \( L^1(0, T; L^1(\mathbb{R}^d)) \) as \( h_l \to 0 \) and a vector field \( j \) such that \((u, j)\) is a weak solution to the Cahn–Hilliard equation (2.1) in the sense of Definition 3.2.

**Proof.** We proceed in two steps. First we show (3.1) and (3.2). In the second step we prove the energy dissipation inequality (3.4).

Step 1. Let \( \zeta \in C^2_c(\mathbb{R}^d \times (0, T)) \) and let \( \zeta_n(x) := \zeta(x, h_l n) \). Observe that, using an index shift, for sufficiently small \( h_l > 0 \) we have
\[ h_l \sum_{n=1}^{N} \int_{\mathbb{R}^d} \frac{u_n - u_{n-1}}{h_l} \zeta_n \, dx = -h_l \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \frac{\zeta_{n+1} - \zeta_n}{h_l} \, dx, \] (3.31)

where \( N = N(h_l) = [T/h_l] \). On the one hand, in the limit \( l \uparrow \infty \) the R.H.S. converges to

\[ -\int_0^T \int_{\mathbb{R}^d} u(x,t) \partial_t \zeta(x,t) \, dx \, dt, \]

because \( u_{h_l} \) is uniformly bounded in \( L^1 \), and the derivative \( \partial_t \zeta \) exists. On the other hand, by definition of \( \gamma_n \)

\[ \left| \int_{\mathbb{R}^d} \frac{u_n(x) - u_{n-1}(x)}{h_l} \zeta_n(x) \, dx - \int_{\mathbb{R}^d} \frac{y - x}{h_l} \cdot \nabla \zeta_n(y) \, d\gamma_n(x,y) \right| \]

\[ \leq \left( \sup_{\mathbb{R}^d} |\nabla^2 \zeta_n| \right) \frac{1}{2h_l} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma_n(x,y) \]

\[ = \|\nabla^2 \zeta_n\|_\infty \frac{1}{2h_l} d^2(u_n, u_{n-1}). \]

Then sum over \( n \) and insert the definition of \( j_n \) to obtain

\[ \left| h_l \sum_{n=1}^{N} \int_{\mathbb{R}^d} \frac{u_n(x) - u_{n-1}(x)}{h_l} \zeta_n(x) \, dx - \int_{\mathbb{R}^d} j_n \cdot \nabla \zeta_n \, dx \right| \]

\[ \leq \|\nabla^2 \zeta\|_\infty \frac{h_l}{2} \sum_{n=1}^{N} \left( \frac{d(u_n, u_{n-1})}{h_l} \right)^2 \]

\[ \leq \|\nabla^2 \zeta\|_\infty h_l E(u_0) \to 0. \]

Since \( j_{h_l} \) is uniformly bounded in \( L^1 \), by Evans and Gariepy [21, Thm. 1.41], there exists a Radon measure \( j \) such that

\[ j_{h_l} \rightharpoonup j, \quad \text{weakly} - * \text{ as Radon measures as } l \to \infty. \] (3.32)

Then the L.H.S. of (3.31) converges to

\[ -\int_0^T \int_{\mathbb{R}^d} j \cdot \nabla \zeta \, dx \, dt. \]

This gives us (3.1); of course with \( \varepsilon = 1 \).

Finally, (3.2) follows immediately from passing to the limit \( h \downarrow 0 \) in the Euler–Lagrange equation (3.26), since \( j_{h_l} \rightharpoonup j \) and \( u_{h_l} \to u \) in \( H^1 \) as \( h_l \downarrow 0 \).

Step 2 (Energy dissipation). The optimal energy dissipation follows from Proposition 3.12 below.
3.2. Optimal energy dissipation

In this subsection, we derive the optimal energy-dissipation inequality (3.33) below. We call this estimate optimal because it precisely captures the energy-dissipation rate $-\int \frac{|j|^2}{u} \, dy$. This is in contrast to the \textit{a priori} estimate

$$E(u(\cdot, T)) + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|j|^2}{u} \, dy \, dt \leq E(u_0),$$

cf. (3.13)–(3.15), which fails to be sharp by a factor $\frac{1}{2}$.

Proposition 3.12 (Optimal energy dissipation). The energy dissipation is optimal, i.e.,

$$E(u(\cdot, T)) + \int_0^T \int_{\mathbb{R}^d} \frac{|j|^2}{u} \, dy \, dt \leq E(u_0)$$

for a.e. $T < \infty$.

Before we prove Proposition 3.12, we introduce a few tools.

Fix $u \in \mathcal{A}$ such that $E(u) < \infty$. For $\tau > 0$ let

$$e_\tau(u) := \inf_{v \in \mathcal{A}} \left\{ E(v) + \frac{1}{2\tau} d^2(u, v) \right\}, \quad \mathcal{J}_\tau(u) := \arg \min_{v \in \mathcal{A}} \left\{ E(v) + \frac{1}{2\tau} d^2(u, v) \right\}.$$

By Lemma 3.3, $\mathcal{J}_\tau(u)$ is non-empty for all $\tau > 0$, and we define

$$z_\tau^+(u) := \sup_{u_t \in \mathcal{J}_\tau(u)} d(u, u_t), \quad z_\tau^-(u) := \inf_{u_t \in \mathcal{J}_\tau(u)} d(u, u_t).$$

Then $z_\tau^+(u) = z_\tau^-(u)$ for almost all $\tau$ and, by Ambrosio et al. [20, Thm. 3.1.4],

$$\frac{d}{d\tau} e_\tau(u) = -\frac{(z_\tau^-(u))^2}{2\tau^2}. \quad (3.34)$$

In particular

$$\frac{d^2(u_t, u)}{2\tau} + \int_0^\tau \frac{(z_\tau^-(u))^2}{2t^2} \, dt = E(u) - E(u_t) \quad \text{for all } u_t \in \mathcal{J}_\tau(u). \quad (3.35)$$

Proof of Proposition 3.12. For $h > 0$ and a solution $(u_n)_n$ to the minimizing movements scheme and corresponding flux $j_n$ as before, with piecewise constant time interpolation $u_n$ and $j_n$, we define a new interpolation as follows: Define $\bar{u}_n(t)$ by

$$\bar{u}_n(t) := \bar{u}_{n, \tau}, \quad t = nh + \tau, \quad (3.36)$$

where $0 < \tau < h$ and $\bar{u}_{n, \tau} \in \mathcal{J}_\tau(u_n)$, i.e., $\bar{u}_{n, \tau}$ minimizes $v \mapsto E(v) + \frac{1}{2\tau} d^2(u_n, v)$.

We can obtain a uniform bound on $d(u_n(t), \bar{u}_n(t))$: If $t = nh + \tau$ with $\tau < h$, then

$$d^2(u_n(u, t), \bar{u}_n(u, t)) = d^2(u_n, \bar{u}_{n, \tau}) \leq 2\tau E(u_n) \leq 2\tau E(u_0)$$

for all $n$ and all $0 < \tau < h$. Hence we have $d(u_n(u, t), \bar{u}_n(u, t)) \leq \sqrt{2E(u_0)} \sqrt{h}$. Since $u_n \rightarrow u$ in $L^1(0, T; L^1(\mathbb{R}^d))$ along a subsequence $h \downarrow 0$ (which we do not relabel in this proof), we have $u(u, t) \in \mathcal{A}$ for a.e. $t$ and $\int_0^T d^2(u_n(u, t), u(u, t)) \, dt \rightarrow 0$ as $h \downarrow 0$. Therefore
\[
\int_0^T d^2(\tilde{u}_h, u) \, dt \leq \int_0^T d^2(u_h, u) + d^2(\tilde{u}_h, u_h) \, dt \to 0,
\] (3.37)
as \(h \downarrow 0\) and since \(d\) metrizes weak-\(^*\) convergence on \(A\), we get that
\[
\tilde{u}_h \rightharpoonup^* u \quad \text{as} \quad h \downarrow 0.
\]

To prove the optimal dissipation inequality (3.33) we want to use (3.35). More precisely, we apply (3.35) to \(u = u^*_n\) and \(\tau = h\) and sum over \(n\):
\[
h \sum_{n=1}^N \frac{d^2(u_n, u_{n-1})}{2h} + h \sum_{n=0}^N \int T_0 \frac{h}{2\tau^2} \, dt \leq E(u_0) - E(u_h(Nh)).
\]

As with the piecewise constant time interpolation, we define
\[
\tilde{j}_{n, \tau}(y) := \frac{1}{\tau} \int_{\mathbb{R}^d} (y - x)\tilde{\gamma}_{n, \tau}(dx, y) = \frac{1}{\tau} (y - \tilde{\tau}_{n, \tau}(y))\tilde{u}_n(y),
\]
\[
\tilde{j}_h(y, t) := \tilde{j}_{n, \tau}, \quad t = nh + \tau,
\] (3.38)
where \(\tilde{\gamma}_{n, \tau}\) is the optimal measure in the definition of \(d(\tilde{u}_{n, \tau}, u_n)\) with \((\tau_n)_{\tau}^{\tilde{\gamma}_{n, \tau}} = u_n\mathcal{L}^d\).

Now we note that by Lemma 3.9 we have the lower bound
\[
\frac{1}{2} \int T_0 \int_{\mathbb{R}^d} \frac{|\tilde{j}_h|^2}{u_n} \, dx \, dt + \frac{1}{2} \int T_0 \int_{\mathbb{R}^d} \frac{|\tilde{\gamma}_h|^2}{\tilde{u}_n} \, dx \, dt
\]
\[
\leq h \sum_{n=1}^N \frac{d^2(u_n, u_{n-1})}{2h} + h \sum_{n=0}^N \int T_0 \frac{h}{2\tau^2} \, dt \, dx.
\]

Since \((u, j) \mapsto \frac{|j|^2}{u}\) is jointly convex, the functional \(\int \frac{|j|^2}{u} \, dx\) is lower semi-continuous with respect to weak convergence of \(j\) and strong convergence of \(u\) in \(L^2\). Hence our claim (3.33) follows once we have shown that \(\tilde{u}_h \rightharpoonup u\) in \(L^2(0, T; H^1(\mathbb{R}^d))\) and \(\tilde{j}_h \rightharpoonup j\) as \(h \downarrow 0\).

Indeed, by Lemma 3.9 and (3.35), \(\tilde{j}_h\) is uniformly bounded in \(L^1(0, T; L^1(\mathbb{R}^d; \mathbb{R}^d))\), hence \(\tilde{j}_h \rightharpoonup \tilde{j}\) for some \(\tilde{j}\) as Radon measures as \(h \downarrow 0\). Let \(\tilde{\xi} \in C_\infty(\mathbb{R}^d \times (0, T); \mathbb{R}^d)\) and let \(\xi_{n, \tau} := \tilde{\xi}(\cdot, nh + \tau)\). By the Euler–Lagrange equation (3.26) we have for all \(n\) and all \(0 < \tau < h\)
\[
\int T_0 \tilde{J}_{n, \tau} \cdot \tilde{\xi}_{n, \tau} \, dx = -\int T_0 \tilde{T}_{n, \tau} : \nabla \tilde{\xi}_{n, \tau} \, dx + \int T_0 \tilde{F}_{n, \tau} \, dx,
\]
where \(\tilde{T}_{n, \tau}\) denotes the energy stress tensor as defined in (3.25) for \(\tilde{u}_{n, \tau}\), instead of \(u_n\) and \(\tilde{F}_{n, \tau} = F(\tilde{u}_{n, \tau}, \nabla \tilde{u}_{n, \tau}, \nabla \tilde{\xi}, \nabla^2 \tilde{\xi})\) is the second R.H.S term in the Euler–Lagrange equation (3.26). Now let \(T = Nh\) for some \(N\) and sum over \(n\):
\[
\int T_0 \int_{\mathbb{R}^d} \tilde{j}_h \cdot \tilde{\xi} \, dx \, dt
\]
\[
= h \sum_{n=0}^N \int T_0 \int_{\mathbb{R}^d} \tilde{j}_{n, \tau} \cdot \tilde{\xi}_{n, \tau} \, dx \, dt
\]
\[
= h \sum_{n=0}^N \left( -\int T_0 \tilde{T}_{n, \tau} : \nabla \tilde{\xi}_{n, \tau} \, dx \, dt + \int T_0 \tilde{F}_{n, \tau} \, dx \, dt \right).\] (3.39)
Thus it suffices to show that $\tilde{u}_h \to u$ in $L^2(0, T; H^1(\mathbb{R}^d))$. Indeed, by the flow exchange Lemma C.3 applied to entropy functional $\mathcal{F} = \mathcal{U}$, we can apply Lemma C.1 also to $\tilde{u}_{n, \tau}$ (instead of $u_n$) as before to obtain that also $\tilde{u}_{n, \tau} \in H^2(\mathbb{R}^d)$ for all $n$ and all $0 < \tau < h$. By Remark 3.6, $\tilde{u}_h(nh) = u_n$ is uniformly bounded in $H^1$. By the estimate $E(\tilde{u}_{n, \tau}) \leq E(u_n) \leq E(u_0)$, we get that there exists $C < \infty$ depending only on $W$ such that

$$\int_0^T \|\tilde{u}_h\|_{H^1(\mathbb{R}^d)} \, dt \leq C E(u_0),$$

and as in (C.9),

$$\frac{\tau}{2} \int_{\mathbb{R}^d} (\Delta \tilde{u}_{n, \tau})^2 \leq \mathcal{U}(u_n) - \mathcal{U}(\tilde{u}_{n, \tau}) + C \tau E(\tilde{u}_{n, \tau}).$$

Hence $\tilde{u}_h$ is uniformly bounded in $L^2(0, T; H^2(\mathbb{R}^d))$, and $\tilde{u}_h \to u$ in $L^2(0, T; H^1(\mathbb{R}^d))$ as $h \downarrow 0$. Therefore the R.H.S. of (3.39) converges to the same limit as the one for $u_h$ instead of $\tilde{u}_h$, and hence $\tilde{u}_h \Rightarrow j$ as $h \downarrow 0$, where $j$ is the same as in Theorem 3.11. Now (3.33) follows from

$$\int_0^T \int_{\mathbb{R}^d} \frac{|j|^2}{u_\tau} \, dy \, dt \leq \liminf_{h \searrow 0} \left( \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|j_h|^2}{u_h} \, dx \, dt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{|j_h|^2}{\tilde{u}_h} \, dx \, dt \right)$$

$$\leq E(u_0) - E(u(\cdot, T)). \quad \blacksquare$$

### 4. The sharp interface limit

In Section 3 we have shown that for every $\varepsilon > 0$ with initial data $u_{\varepsilon, 0} \in \mathcal{A}$ such that $E_\varepsilon(u_{\varepsilon, 0}) < \infty$, there exists a weak solution $u_\varepsilon$ to the Cahn–Hilliard equation (2.1) in the sense of Definition 3.2. In the present section we show that there exists a subsequence $\varepsilon_l \to 0$ and a function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ such that $u_{\varepsilon_l} \to u$ in $L^1$ as $l \uparrow \infty$ under the following well preparedness condition on the initial data $u_{\varepsilon, 0}$:

$$u_{\varepsilon, 0} \to \chi_{\Omega_0} \quad \text{in } L^1,$$

$$E_\varepsilon(u_{\varepsilon, 0}) \to \sigma \mathcal{P}(\Omega_0),$$

$$\sup_{\varepsilon > 0} M_2(u_{\varepsilon, 0}) < \infty. \quad (4.1)$$

Here $\mathcal{P}(\Omega_0)$ denotes the perimeter of $\Omega_0$, and $\sigma = \int_0^1 \sqrt{2W(r)} \, dr$. Throughout this section we assume that the initial data are well prepared, and we set $E_0 := \sup_{\varepsilon > 0} E_\varepsilon(u_{\varepsilon, 0})$.

**Theorem 4.1.** Let $(u_{\varepsilon, j})$ be weak solutions to the Cahn–Hilliard equation (2.1) in the sense of Definition 3.2 with initial data $u_{\varepsilon, 0}$ satisfying (4.1).

Then there exists a subsequence $\varepsilon_l \to 0$ and a family of finite perimeter sets $(\Omega(t))_{t \in [0, T]}$ such that the following hold:

(i) For almost all $t \in [0, T]$ we have

$$u_{\varepsilon_l} \to \chi_\Omega \quad \text{in } L^1(0, T; L^1(\mathbb{R}^d)), \quad (4.2)$$
where \( \chi_\Omega(x,t) := \chi_{\Omega(t)}(x) \).

(i) There exists \( j \in L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d)) \) such that
\[
j_{\Omega_l} \rightharpoonup \chi_{\Omega(t)} j \quad \text{as Radon measures as } l \uparrow \infty. \tag{4.3}
\]

(ii) If in addition to (4.2) and (4.3),
\[
\limsup_{l \to \infty} \int_0^T E_{ij}(u_{ij}(.t)) \ dt \leq \int_0^T \sigma P(\Omega(t)) \ dt, \tag{4.4}
\]

then \((\Omega,j)\) is a weak solution to the Hele–Shaw flow in the sense of Definition 4.2 below.

Before we give the definition of weak solutions, we introduce some standard notation, see also [22, Chapter 12]. For an open set \( \Omega \subset \mathbb{R}^d \), \( BV(\Omega) \) denotes the space of functions \( u \in L^1(\Omega) \) with bounded variation in \( \Omega \). If \( \Omega \) is a set of finite perimeter, we denote by \( \partial^* \Omega \) its reduced boundary. The measure theoretic outward unit normal is denoted by \( \nu_\Omega : \partial^* \Omega \to S^{d-1} \). The Gauss–Green measure of the set of finite perimeter \( \Omega \) will be denoted by \( \mu_\Omega = -\nabla \chi_\Omega = \nu_\Omega \nabla \chi_\Omega \), and \( \mathcal{H}^{d-1} \) denotes the \((d-1)\)-dimensional Hausdorff measure.

**Definition 4.2** (Weak solution of the Hele–Shaw flow). Let \( d \geq 2 \) and \( T \in (0,\infty) \). Let \( \Omega := \{\Omega(t)\}_{t \in [0,T]} \) be a family of finite perimeter sets and let \( j \in L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d)) \). We say that the pair \((\Omega,j)\) is a weak solution to the Hele–Shaw flow if the following statements hold:

(i) For all \( \zeta \in C^1_c(\mathbb{R}^d \times [0,T)) \) we have
\[
\int_{\mathbb{R}^d} \chi_{\Omega(t)} \zeta(.,0) \ dx + \int_0^T \int_{\mathbb{R}^d} \chi_{\Omega(t)} \partial_t \zeta + \chi_{\Omega(t)} j \cdot \nabla \zeta \ dx \ dt = 0, \tag{4.5}
\]

where \( \chi_{\Omega(t)}(x,t) = \chi_{\Omega(t)}(x) \).

(ii) For all \( \xi \in C^1_c(\mathbb{R}^d \times (0,T);\mathbb{R}^d) \) with \( \nabla \cdot \xi = 0 \) we have
\[
\int_0^T \int_{\Omega(t)} \xi \cdot j(.,t) \ dx \ dt = -\sigma \int_0^T \int_{\partial^* \Omega(t)} (\nabla \cdot \xi - \nu_{\Omega(t)} \cdot \nabla \xi) \nu_{\Omega(t)} \ d\mathcal{H}^{d-1} \ dt. \tag{4.6}
\]

(iii) For a.e. \( T' \in [0,T] \) we have
\[
\sigma P(\Omega(T')) + \int_0^T \int_{\Omega(t)} |j|^2 \ dx \ dt \leq \sigma P(\Omega_0). \tag{4.7}
\]

This definition is motivated by certain properties of classical solutions of the Hele–Shaw flow. First of all, it is straight-forward to see that any classical solution to Hele–Shaw satisfies all properties stated in the definition. Second, if \( \Omega \) and \( j \) are smooth and a weak solution to the Hele–Shaw flow, then \((\Omega,j)\) should solve the Hele–Shaw flow in the classical sense, that is, \((\Omega,j)\) is a strong solution to (2.2)–(2.3). This is shown in Lemma 4.3 below.
Carefully note the slight difference in the space of admissible test vector fields in comparison to Definition 3.2. While there we allowed any vector field $\xi$, here we need to restrict to divergence-free vector fields. This is natural when recalling that $\xi$ represents a direction in which compute the first variation in Wasserstein space via the conservation law $\partial_t \mathcal{X}_s + \nabla \cdot (\mathcal{X}_s \xi) = 0$ for $s \neq 0$. Formally, it is clear that $\nabla \cdot \xi \neq 0$ will lead to the violation of the constraint $\chi \in \{0, 1\}$.

### 4.1. Consistency between weak and strong solutions

Now we show that if $\Omega := \bigcup_{t \in [0, T]} \Omega(t) \times \{t\}$ and $j$ are smooth and a weak solution to the Hele–Shaw flow, then $\Omega$ and $j$ solve the Hele–Shaw equations in the classical sense, that is, $\Omega$ and $j$ satisfy (2.2)–(2.3).

**Lemma 4.3.** Let $(\Omega, j)$ be a weak solution to the Hele–Shaw flow in the sense of Definition 4.2. If $j$ is smooth and $\Omega(t)$ evolves smoothly, then $(\Omega, j)$ is a classical solution to the Hele–Shaw flow (2.2)–(2.3).

**Proof.**

Step 1 ($(\Omega, j)$ solves (2.2)). By (4.6), we have

$$
\int_0^T \int_{\mathbb{R}^d} \mathcal{X}_{\Omega(t)} \partial_t \eta + \mathcal{X}_{\Omega(t)} j \cdot \nabla \eta \, dx \, dt = 0
$$

for all $\eta \in C^1_c((0, T))$. Now let $t_0 \in (0, T)$ and $x_0 \in \text{Int}(\Omega(t_0))$, the interior of $\Omega(t_0)$. Since $\Omega(t)$ evolves smoothly, there exists $\epsilon > 0$ such that $B_\epsilon(x_0) \subset \Omega(t)$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Let $\zeta \in C^1_c(B_\epsilon(x_0) \times (t_0 - \epsilon, t_0 + \epsilon))$ and let $\phi(t) := \int_{B_\epsilon(x_0)} \zeta(x, t) \, dx$. Then $\phi(t_0 - \epsilon) = \phi(t_0 + \epsilon) = 0$. Using differentiation under the integral, we have

$$
\int_{\Omega(t)} \partial_t \zeta(x, t) \, dx = \int_{B_\epsilon(x_0)} \partial_t \zeta(x, t) \, dx = \frac{d}{dt} \int_{B_\epsilon(x_0)} \zeta(x, t) \, dx.
$$

The R.H.S. vanishes after integration over $(t_0 - \epsilon, t_0 + \epsilon)$. By (4.8) we have

$$
0 = \int_0^T \int_{\mathbb{R}^d} \mathcal{X}_{\Omega(t)} \partial_t \zeta + \mathcal{X}_{\Omega(t)} j \cdot \nabla \zeta \, dx \, dt
$$

$$
= \int_{t_0 - \epsilon}^{t_0 + \epsilon} \frac{d}{dt} \phi(t) \, dt + \int_{t_0 - \epsilon}^{t_0 + \epsilon} j \cdot \nabla \zeta \, dx \, dt
$$

$$
= \int_{t_0 - \epsilon}^{t_0 + \epsilon} j \cdot \nabla \zeta \, dx \, dt.
$$

Therefore

$$
\nabla \cdot j(x_0, t_0) = 0 \quad \text{for all } (x_0, t_0) \in \Omega.
$$

It remains to show that $V = j \cdot \nu$. 
Let \( V(t) : \partial \Omega(t) \to \mathbb{R} \) be the normal velocity of the boundary \( \partial \Omega(t) \) at time \( t \) and let \( \zeta \in C^1_c(\mathbb{R}^d \times (0, T)) \). Then, by Evans \cite[Appendix C4, Thm. 6]{16},

\[
0 = \int_0^T \int_{\partial \Omega(t)} \partial_t \zeta \, dx \, dt + \int_0^T \int_{\partial \Omega(t)} \zeta V \, d\mathcal{H}^{d-1} \, dt,
\]

and by (4.8) we know that

\[
0 = \int_0^T \int_{\partial \Omega(t)} (-\partial_t \zeta - j \cdot \nabla \zeta) \, dx \, dt.
\]

Adding these two identities, using \( \nabla \cdot j = 0 \) in \( \Omega \) and the divergence theorem, we get

\[
0 = \int_0^T \int_{\partial \Omega(t)} \zeta V \, d\mathcal{H}^{d-1} \, dt - \int_0^T \int_{\Omega(t)} \nabla \cdot (\zeta j) \, dx \, dt
\]

\[
= \int_0^T \int_{\partial \Omega(t)} \zeta V \, d\mathcal{H}^{d-1} \, dt - \int_0^T \int_{\partial \Omega(t)} \zeta j \cdot \nu \, d\mathcal{H}^{d-1} \, dt
\]

\[
= \int_0^T \int_{\partial \Omega(t)} (V - j \cdot \nu) \, d\mathcal{H}^{d-1} \, dt. \tag{4.9}
\]

By the fundamental lemma of the calculus of variations, we have \( V - j \cdot \nu = 0 \) on \( \partial \Omega(t) \) for all \( t \), since (4.9) holds for all \( \zeta \in C^1_c(\mathbb{R}^d \times (0, T)) \). Hence \( j \) solves (2.2).

**Step 2 ((\( \Omega, j \)) solves (2.3)).** We want to show that there exists a function \( p : \Omega \to \mathbb{R} \) such that

\[
\begin{cases}
  j(\cdot, t) = -\nabla p(\cdot, t), & \text{in } \Omega(t), \\
  p(\cdot, t) = \sigma H, & \text{on } \partial \Omega(t). 
\end{cases}
\]

Fix \( t \in [0, T] \) and let first \( \xi \in C^\infty_c(\Omega(t); \mathbb{R}^d) \) such that \( \nabla \cdot \xi = 0 \) in \( \Omega(t) \). By (4.6)

\[
\int_{\Omega(t)} \xi \cdot j(\cdot, t) \, dx = -\sigma \int_{\partial \Omega(t)} (\nabla \cdot \xi - \nu \cdot \nabla \xi \nu) \, d\mathcal{H}^{d-1}. \tag{4.10}
\]

Since \( \xi \) has compact support in \( \Omega \), the R.H.S. is zero. Thus, for all \( t \), we have

\[
j(\cdot, t) \perp_{L^2} \{\nabla \cdot \xi = 0, \xi \cdot \nu = 0\},
\]

that is, \( j(\cdot, t) \) is perpendicular to the set of divergence-free vector fields on \( \Omega(t) \) w.r.t. the \( L^2 \)-inner product. By the Helmholtz–Weyl decomposition \cite[Thm. III.1.1, Thm. III.2.3]{23} there exists \( p \in H^1_{\text{loc}}(\Omega) \) such that

\[
j = -\nabla p \quad \text{in } \Omega(t) \text{ for all } t. \tag{4.11}
\]

Now, let \( \xi \in C^1_c(\mathbb{R}^d; \mathbb{R}^d) \) with \( \nabla \cdot \xi = 0 \) and plug (4.11) into (4.10) to obtain

\[
-\int_{\Omega(t)} \xi \cdot \nabla p(\cdot, t) \, dx = -\sigma \int_{\partial \Omega(t)} (\nabla \cdot \xi - \nu \cdot \nabla \xi \nu) \, d\mathcal{H}^{d-1}
\]

\[
= -\sigma \int_{\partial \Omega(t)} \xi \cdot H \nu \, d\mathcal{H}^{d-1}.
\]

Using the divergence theorem and the fact that \( \nabla \cdot \xi = 0 \), we also have

\[
\int_{\Omega(t)} \xi \cdot \nabla p(\cdot, t) \, dx = \int_{\Omega(t)} \nabla \cdot (p(\cdot, t) \xi) \, dx = \int_{\partial \Omega(t)} p(\cdot, t) \xi \cdot \nu \, d\mathcal{H}^{d-1}.
\]
Therefore
\[ \int_{\partial \Omega(t)} (p(\cdot, t) - \sigma H) \zeta \cdot \nu \, d\mathcal{H}^{d-1} = 0. \]

Again, by the fundamental lemma of the calculus of variations, \( p(\cdot, t) - \sigma H = 0 \) on \( \partial \Omega(t) \) for all \( t \).

\[ \square \]

### 4.2. Compactness

In this section we prove (i) and (ii) of Theorem 4.1.

To get precompactness of \( u_\varepsilon \) in \( L^1 \), an important tool will be a variant of Aubin–Lions lemma, see Theorem D.3 below. Before we go into the proof, we perform some preliminary computations, which will be needed to complete the proof.

To define \( F \), we let
\[ \phi : \mathbb{R} \to \mathbb{R}, \quad \phi(s) = \int_0^s \sqrt{2W(r)} \, dr, \quad (4.12) \]

and
\[ F : L^1(\mathbb{R}^d) \to [0, \infty], \quad F(u) = \begin{cases} \int_{\mathbb{R}^d} |\nabla (\phi \circ u)|, & \text{if } \phi \circ u \in BV(\mathbb{R}^d) \\ \infty, & \text{else}. \end{cases} \]

We start with an upper bound on \( F \) in terms of the energy \( E_\varepsilon \), as well as a bound on the energy \( E_\varepsilon \), which we need later.

Let \( u \in A \). By using Young’s inequality \( 2ab \leq a^2 + b^2 \) with \( a = \frac{1}{\sqrt{\varepsilon}} \sqrt{2W(u)} \) and \( b = \sqrt{\varepsilon}|\nabla u| \), we get
\[ |\nabla (\phi \circ u)| = |\phi'(u)||\nabla u| \leq \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u). \quad (4.13) \]

Then we have
\[ F(u) = \int_{\mathbb{R}^d} |\nabla (\phi \circ u)| \, dx \leq E_\varepsilon(u). \quad (4.14) \]

It follows that
\[ \int_{\mathbb{R}^d} |(\phi \circ u)(x + z) - (\phi \circ u)(x)| \, dx \leq |z|E_\varepsilon(u). \quad (4.15) \]

In particular if \( u = u_\varepsilon(\cdot, t) \) is a weak solution to the Cahn–Hilliard equation, we can use (3.4) and integrate (4.14) over \((0, T)\) to obtain the estimate
\[ \sup_{\varepsilon > 0} \int_0^T F(u_\varepsilon(\cdot, t)) \, dt \leq \sup_{\varepsilon > 0} \int_0^T E_\varepsilon(u_\varepsilon(\cdot, t)) \, dt \leq T \sup_{\varepsilon > 0} E(u_\varepsilon, 0) \leq TE_0. \quad (4.16) \]

Hence if \( U = \{u_\varepsilon|_{[0, T]}\}_{\varepsilon > 0}, \) then the tightness condition (D.3) holds for all \( T > 0 \). It remains to check (D.4) and finding a normal coercive integrand.

To check condition (D.4), we use the following lemma.
Lemma 4.4 (A Hölder bound on g). There exists a constant \( C > 0 \) depending only on \( T \) and \( E_c(u_c, 0) \), such that for any \( \varepsilon > 0 \) and \( 0 \leq t_0 < t_1 \leq T \), it holds

\[
g(u_c(\cdot, t_1), u_c(\cdot, t_0)) \leq C|t_1 - t_0|^{1/2}.
\]

Then (D.4) follows immediately, since

\[
\lim_{h \to 0} \sup_{\varepsilon > 0} \int_{t_0}^{T-h} g(u_c(\cdot, t+h), u_c(\cdot, t)) \, dt \leq \lim_{h \to 0} C \int_{t_0}^{T-h} h^{1/2} \, dt = 0.
\]

Proof. By the Benamou–Brenier formula [19, Thm. 8.1], we have

\[
g^2(u_c(\cdot, t_0), u_c(\cdot, t_1)) = \inf_{\rho, v} \left\{ \int_0^1 \int_{\mathbb{R}^d} \rho(x, t)|v(x, t)|^2 \, dx \, dt \right\},
\]

where we minimize over all \( \rho, v \) such that \( \partial_t \rho + \nabla \cdot (\rho v) = 0 \) and
\[
\rho(\cdot, 0) = u_c(\cdot, t_0), \quad \rho(\cdot, 1) = u_c(\cdot, t_1).
\]

Recalling from Lemma 3.9 that \( j_c(y) = 0 \) if \( u_c(y) = 0 \), using our usual convention \( 0_0 = 0 \), we are allowed to define

\[
\rho_c(x, t) := u_c(x, t(t_1 - t_0) + t_0), \quad v_c(x, t) := (t_1 - t_0) \frac{j_c(x, t(t_1 - t_0) + t_0)}{u_c(x, t(t_1 - t_0) + t_0)},
\]

which are admissible for the infimum, since by (2.1)

\[
\partial_t \rho_c + \nabla \cdot (\rho_c v_c) = 0
\]

in the sense of distributions. Plugging this competitor into (4.17), using the change of variables formula and then (3.4), we have

\[
g^2(u_c(\cdot, t_0), u_c(\cdot, t_1)) \leq \int_0^1 \int_{\mathbb{R}^d} \frac{|j_c|^2}{g^2} \, dx \, dt
\]

\[
= (t_1 - t_0) \int_0^1 \int_{\mathbb{R}^d} \frac{|j_c|^2}{u_c} \, dx \, dt
\]

\[
= (t_1 - t_0) E_c(u_c(\cdot, 0))
\]

\[
\leq (t_1 - t_0) E_0.
\]

To apply Theorem D.3, we slightly adjust the integrand \( F \). For \( D \subset L^1(\mathbb{R}^d) \), define

\[
F_D(u) := \begin{cases}
\int_{\mathbb{R}^d} |\nabla (\phi \circ u)|, & \text{if } u \in D, \ \phi \circ u \in BV(\mathbb{R}^d), \\
\infty, & \text{else}.
\end{cases}
\]

Then, with suitable assumptions on \( D \), \( F_D \) is a normal coercive integrand on \( L^1(\mathbb{R}^d) \), as the following lemma shows.

Lemma 4.5. Let \( D \subset L^1(\mathbb{R}^d) \) be closed such that \( \int_{\mathbb{R}^d} u \, dx = 1 \) and \( u \geq 0 \) for all \( u \in D \), and

\[
\sup_{u \in D} \|u\|_{L^1(\mathbb{R}^d)} < \infty, \quad \sup_{u \in D} M_2(u) < \infty.
\]

Then \( F_D \) is a normal coercive integrand on \( L^1(\mathbb{R}^d) \).
Note that for our purposes it suffices to choose
\[ D = \left\{ u \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} u \, dx = 1, \|u\|_{L^4} \leq C, \, M_2(u) \leq C', \, \phi \circ u \in BV(\mathbb{R}^d) \right\}, \]
for some \( C, C' < \infty \). Indeed, \( D \) is closed, since the constraints \( \int u \, dx = 1, \|u\|_{L^4} \leq C \) and \( M_2(u) \leq C' \) as well as the total variation are lower semi-continuous with respect to \( L^1 \) convergence.

**Proof.** By Remark D.4, we only need to show that \( F_D \) is lower semi-continuous and coercive.

Step 1 (coercivity). Let \( c > 0 \) and let \( (u_k)_{k \geq 1} \subset D \) be a sequence with \( F_D(u_k) \leq c \), and define
\[ w_k := \phi \circ u_k : \mathbb{R}^d \to \mathbb{R}. \]
We need to show that there exists a subsequence \( u_{k_l} \) and \( u : \mathbb{R}^d \to \mathbb{R} \) such that
\[ u_{k_l} \to u \text{ in } L^1(\mathbb{R}^d) \text{ as } l \uparrow \infty, \]
and \( F_D(u) \leq c \). The latter will follow from lower semi-continuity of \( F_D \) (see Step 2).

First we show that there exists a subsequence \( w_{k_l} \) as \( l \uparrow \infty \) such that \( w_{k_l} \to w \) pointwise a.e. in \( \mathbb{R}^d \) for some \( w \). By assumption we have
\[ \int_{\mathbb{R}^d} |\nabla w_k| = F_D(u_k) \leq c. \]
In particular we have \( u_k \in D \) for all \( k \). Further there exist \( M, C < \infty \) such that
\[
\int_{\mathbb{R}^d} |w_k| \, dx = \int_{\mathbb{R}^d} |\phi(u_k)| \, dx \\
\leq \int_{\{|u_k| > M\}} \frac{1}{6} |u_k|^3 + O(u^2) \, dx + \int_{\{|u_k| \leq M\}} |\phi(u_k)| \, dx \\
\leq \int_{\{|u_k| > M\}} C|u_k|^3 \, dx + (\text{Lip}\phi)_{[0,M]} \int_{\mathbb{R}^d} |u_k| \, dx < \infty.
\]
Thus \( w_k \in BV(\mathbb{R}^d) \) for all \( k \) and the sequence \( w_k \) is uniformly bounded in \( BV(\mathbb{R}^d) \), because \( u_k \) is uniformly bounded in \( L^3 \) (by interpolation between \( L^1 \) and \( L^4 \)). By Rellich’s Theorem [16, Chap. 5.7, Thm. 1] and a diagonal argument, there exists a subsequence converging in \( L^1_{loc}(\mathbb{R}^d) \) to some \( w \). Passing to a further subsequence if necessary, we may assume that \( w_{k_l} \to w \) a.e. in \( \mathbb{R}^d \) as \( l \uparrow \infty \). Now define \( u := \phi^{-1} \circ w \). Then
\[ u_{k_l}(x) = \phi^{-1}(w_{k_l}(x)) \to \phi^{-1}(w(x)) = u(x) \quad \text{for a.e.x. } \in \mathbb{R}^d \text{ as } l \uparrow \infty. \]
To get \( L^1 \) convergence of \( u_{k_l} \), after relabeling, it suffices to show that [24, Thm. 4.16]
\[
\sup_k \int_{\mathbb{R}^d} |u_k(x - z) - u_k(x)| \, dx \xrightarrow{z \to 0} 0, \quad (4.18)
\]
\[
\sup_k \int_{\mathbb{R}^d \setminus B_R(0)} |u_k| \, dx \to 0. \tag{4.19}
\]

To verify (4.19), we use the fact that the second moments are uniformly bounded to obtain
\[
\sup_k \int_{\mathbb{R}^d \setminus B_R} u_k(x) \, dx \leq \frac{1}{R^2} \sup_k \int_{\mathbb{R}^d \setminus B_R} |x|^2 u_k(x) \, dx \\
\leq \frac{1}{R^2} \sup_k M_2(u_k) \\
\leq \frac{C}{R^2} R^{d\infty} 0.
\]

To verify (4.18), let \( \varepsilon > 0 \) and let \( M = 2 \). For \( R < \infty \) consider the decomposition
\[
u_k = (u_k \wedge M + (u_k \vee M - M)) \chi_{B_R} + u_k \chi_{\mathbb{R}^d \setminus B_R}. \tag{4.21}
\]

We treat those three terms separately. Let us start with the first one. The term \( u_k \wedge M \) converges to \( u \wedge M \) pointwise a.e., hence by the dominated convergence theorem we have \( u_k \wedge M \to u \wedge M \) in \( L^p_{\text{loc}} \) for all \( p < \infty \), and we can choose \( \delta_1 > 0 \) small enough such that for all \( |z| < \delta_1 \) and all \( k \)
\[
\int_{\mathbb{R}^d} \left| \left( (u_k \wedge M) \chi_{B_R}(x-z) - ((u_k \wedge M) \chi_{B_R}(x) \right| \, dx < \varepsilon/4.
\]

The last term can be made small by choosing \( R < \infty \) sufficiently large. Precisely, we can choose a fixed \( R < \infty \) such that for all \( |z| < 1 \) and all \( k \) we have
\[
\int_{\mathbb{R}^d} \left| \left( (u_k \wedge M) \chi_{\mathbb{R}^d \setminus B_R}(x-z) - ((u_k \wedge M) \chi_{\mathbb{R}^d \setminus B_R}(x) \right| \, dx \leq \frac{2}{R^2} \sup_k M_2(u_k) < \varepsilon/4.
\]

For the second term, let \( \omega_R(z) := \mathcal{L}^d(B_{R+|z|} \setminus B_R) \). Then, by the triangle inequality and since \( \phi \) is monotonically increasing on \([1, \infty)\), we have
\[
\int_{\mathbb{R}^d} \left| \left( (w_k \vee \phi(M)) \chi_{B_R}(x-z) - ((w_k \vee \phi(M)) \chi_{B_R}(x) \right| \, dx \\
\leq M \omega_R(z) + \int_{\mathbb{R}^d} \left| \left( (w_k \vee \phi(M)) \chi_{B_R}(x-z) - ((w_k \vee \phi(M)) \chi_{B_R}(x) \right| \, dx \\
\leq M \omega_R(z) + \left( \text{Lip } \phi^{-1} \right) \int_{\mathbb{R}^d} \left| \left( (w_k \vee \phi(M)) \chi_{B_R}(x-z) - ((w_k \vee \phi(M)) \chi_{B_R}(x) \right| \, dx.
\]

Note that since \( M = 2 > 1 \), we have \( \text{Lip } \phi^{-1} < \infty \). Moreover we have
\[
\int_{\mathbb{R}^d} \left| \left( (w_k \vee \phi(M)) \chi_{B_R}(x-z) - ((w_k \vee \phi(M)) \chi_{B_R}(x) \right| \, dx \\
\leq \|w_k \vee \phi(M) \chi_{B_{2R}}\|_{L^2(\mathbb{R}^d)} (\omega_R(z))^{1/2} + \int_{B_{R+|z|}} \left| (w_k \vee \phi(M))(x-z) - (w_k \vee \phi(M))(x) \right| \, dx.
\]

The second term on the R.H.S. converges to zero uniformly in \( k \) as \(|z| \to 0\), because \( w_k \) converges in \( L^1_{\text{loc}} \), and \( s \mapsto s \vee \phi(M) \) is Lipschitz. Thus we can choose \( \delta_2 > 0 \) such that
for all $|z| < \delta_2$ and all $k$

\[
\left( \frac{\text{Lip } \varphi^{-1}}{|M, \infty|} \right) \int_{B_{R_k(z)}} |(w_k \vee \varphi(M))(x - z) - (w_k \vee \varphi(M))(x)| \, dx < \varepsilon/4.
\]

Finally, the term $\|w_k \vee \varphi(M)\|_{L^2}$ is bounded by a uniform constant and $\omega_R$ is continuous with $\omega_R(z) \to 0$ as $|z| \to 0$, thus we may choose $\delta_3 > 0$ such that for all $|z| < \delta_3$

\[
2\omega_R(z) + \left( \frac{\text{Lip } \varphi^{-1}}{|M, \infty|} \right) \|w_k \vee \varphi(M)\|_{L^2(\mathbb{R}^d)}(\omega_R(z))^{1/2} < \varepsilon/4.
\]

Therefore we can conclude that for all $|z| < \delta := \min(\delta_1, \delta_2, \delta_3)$ by (4.21) and the triangle inequality,

\[
\sup_k \int_{\mathbb{R}^d} |u_k(x - z) - u_k(x)| \, dx < \varepsilon.
\]

Step 2 (Lower semi-continuity). Let $u_k \to u$ in $L^1$ with $u_k \in D$. Since $D$ is closed, we have $u \in D$. By interpolation between $L^1$ and $L^4$, we get $u_k \to u$ in $L^3$ for a subsequence, and

\[
w_k := \varphi \circ u_k \to \varphi \circ u =: w \text{ in } L^1 \text{ as } k \uparrow \infty.
\]

Moreover $F_D$ can be represented as

\[
F_D(w) = \int_{\mathbb{R}^d} |\nabla w| = \sup \left\{ \int_{\mathbb{R}^d} (\nabla \cdot \xi) w \, dx : \xi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d), \|\xi\|_\infty \leq 1 \right\},
\]

i.e., $F_D$ can be written as the supremum of continuous linear functionals on $L^1(\mathbb{R}^d)$. Hence $F_D$ is lower semi-continuous on $L^1(\mathbb{R}^d)$.

Proof of Theorem 4.1 (i) and (ii). We are now in the position to prove the first two items of Theorem 4.1. The proof of the last item is given in the next subsection.

Step 1 (Proof of (i)). We show that there exists a family of finite perimeter sets $(\Omega(t))_{t \in [0, T]}$ such that $u_e(\cdot, t) \to \chi_{\Omega(t)}$ in $L^1$ for a.e. $t \in [0, T]$.

By Theorem D.3, there exists a subsequence $u_{e_l}$ and $u$ such that $u_{e_l} \to u$ in $M(0, T; L^1(\mathbb{R}^d))$. Then for a further subsequence we have for a.e. $t$

\[
\lim_{l \to \infty} \|u_{e_l}(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R}^d)} = 0.
\]

Then $u_{e_l} \to u$ in $L^1(0, T; L^1(\mathbb{R}^d))$ as $l \to \infty$ by dominated convergence.

Now we show that $u(x, t) \in \{0, 1\}$ a.e. in $\mathbb{R}^d$ for a.e. $t \in [0, T]$. By assumption, the energies are uniformly bounded by

\[
E_{e_l}(u_{e_l}(\cdot, t)) = \int_{\mathbb{R}^d} \frac{e_l}{2} |\nabla u_{e_l}(\cdot, t)|^2 + \frac{1}{e_l} W(u_{e_l}(\cdot, t)) \, dx \leq E_0 < \infty.
\]
Therefore, by Fatou’s lemma

\[
\int_{\mathbb{R}^d} W(u(x, t)) \, dx = \int_{\mathbb{R}^d} \liminf_{l \to \infty} W(u_{l_i}(x, t)) \, dx \\
\leq \liminf_{l \to \infty} \int_{\mathbb{R}^d} W(u_{l_i}(x, t)) \, dx = 0.
\]

Recall that \( W(s) = \frac{1}{4} s^2 (s - 1)^2 \), hence for a.e. \((x, t) \in \mathbb{R}^d \times [0, T]\) we have \( u(x, t) \in \{ s \in \mathbb{R} : W(s) = 0 \} = \{0, 1\} \). Let \( \Omega(t) := \{ x \in \mathbb{R}^d : \lim_{l \to 0} u_{l_i}(x, t) = 1 \} \).

It remains to show that \( \Omega(t) \) has finite perimeter. By Fatou’s lemma we have for all \( \zeta \in C_0^1(\Omega(t); \mathbb{R}^d) \) with \( |\zeta| \leq 1 \),

\[
\int_{\mathbb{R}^d} \sigma \chi_{\Omega(t)} \text{div} \zeta \, dx \leq \liminf_{l \to \infty} \int_{\mathbb{R}^d} (\phi \circ u_{l_i})(x, t) \text{div} \zeta \, dx \\
\leq \liminf_{l \to \infty} \int_{\mathbb{R}^d} |\nabla (\phi \circ u_{l_i})|(x, t) \, dx,
\]

where \( \sigma = \phi(1) \). Now take the supremum over \( \zeta \) and use (4.14) to get

\[
\sigma P(\Omega(t)) = \sup \left\{ \int_{\mathbb{R}^d} \sigma \chi_{\Omega(t)} \text{div} \zeta \, dx : \zeta \in C_0^1(\Omega(t); \mathbb{R}^d), \ |\zeta| \leq 1 \right\}
\leq \liminf_{l \to \infty} \int_{\mathbb{R}^d} |\nabla (\phi \circ u_{l_i})|(x, t) \, dx \\
\leq \liminf_{l \to \infty} E_{l_i}(u_{l_i}(\cdot, t)) < \infty.
\]

Step 2 (Proof of (ii)). Using Cauchy–Schwarz and (3.4), we get a uniform bound on \( j_{l_i} \) in \( L^1 \):

\[
\int_0^T \int_{\mathbb{R}^d} |j_{l_i}| \, dx \, dt \leq \left( \int_0^T \int_{\mathbb{R}^d} u_{l_i} \, dx \, dt \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^d} |j_{l_i}|^2 \, dx \, dt \right)^{1/2} \leq \sqrt{T} \left( \int_0^T \int_{\mathbb{R}^d} \frac{|j_{l_i}|^2}{u_{l_i}} \, dx \, dt \right)^{1/2} \leq \sqrt{T} (2E_0)^{1/2} = \sqrt{2T} E_0.
\]

Then, by Evans and Gariepy [21, Thm. 1.41], there exists a subsequence \( j_{l_i} \) and a Radon measure \( j \) such that

\[
j_{l_i} \overset{*}{\rightharpoonup} j \quad \text{weakly} - * \quad \text{as Radon measures as} \quad l \uparrow \infty.
\]

Now let \( \Omega := \bigcup_{t \in [0, T]} \Omega(t) \times \{ t \} \subset \mathbb{R}^{d+1} \). We show that \( \text{supp} \, j \subset \Omega \). To this end, let \( 0 < t_0 < t_1 < T \), and let \( U \subset \mathbb{R}^d \) be open. We localize the estimate (4.23) in time and space to obtain
\[
\int_{t_0}^{t_1} \int_U |j_{\xi}| \, dx \, dt \leq \left( \int_{t_0}^{t_1} \int_U u_{\xi} \, dx \, dt \right)^{1/2} \left( \int_{t_0}^{t_1} \int_U \frac{|j_{\xi}|^2}{u_{\xi}} \, dx \, dt \right)^{1/2} \\
\leq \left( \int_{t_0}^{t_1} \int_U u_{\xi} \, dx \, dt \right)^{1/2} \left( \int_{t_0}^{T} \int_{\mathbb{R}^d} \frac{|j_{\xi}|^2}{u_{\xi}} \, dx \, dt \right)^{1/2} \\
\leq \left( \int_{t_0}^{t_1} \int_U u_{\xi} \, dx \, dt \right)^{1/2} \left( 2E_0 \right)^{1/2}. \tag{4.24}
\]

Now passage to the limit $\varepsilon \downarrow 0$ gives
\[
|j|(U \times (t_0, t_1)) \leq \liminf_{l \to \infty} \int_{t_0}^{t_1} \int_U |j_{\xi}| \, dx \, dt \leq \sqrt{2E_0} \left( \int_{t_0}^{t_1} \int_U j_{\Omega(t)} \, dx \, dt \right)^{1/2}.
\]

If we choose $U$ such that $\mathcal{L}^{d+1}((U \times (t_0, t_1)) \cap \Omega) = 0$, then the R.H.S. is zero. Now, let $x \in \mathbb{R}^d$ and let $t \in (0, T)$. If $(x, t) \in \mathbb{R}^{d+1} \setminus \Omega$, there exist $0 < t_0 < t_1 < T$ and $U \subset \mathbb{R}^d$ such that $U \times (t_0, t_1) \subset \mathbb{R}^{d+1} \setminus \Omega$. Then $|j|(U \times (t_0, t_1)) = 0$, and therefore $\text{supp } j \subset \Omega$.

Again using joint convexity, by weak convergence of $j_{\xi}$ we have
\[
\int_{\Omega(t)} \int_0^T |j|^2 \, dx \, dt \leq \liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} \frac{|j_{\xi}|^2}{u_{\xi}} \, dx \, dt. \tag{4.25}
\]

In particular $j \in L^2(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d))$.

4.3. Convergence

**Proof of Theorem 4.1** (iii). The proof of item (iii) is divided into two steps. First, we verify (4.5) and (4.6). In the second step, we prove the optimal dissipation inequality (4.7).

Step 1. To show that (4.5) holds, we pass to the limit in the corresponding equation for $u_{\xi}$. Let $\xi \in C_1^1(\mathbb{R}^d \times [0, T])$,
\[
\int_{\mathbb{R}^d} u_{\xi, \partial_0^s \xi} \, dx + \int_0^T \int_{\mathbb{R}^d} u_{\xi} \partial_t \xi + j_{\xi} \cdot \nabla \xi \, dx \, dt = 0.
\]

Now the first term converges by (4.1). The second and third term converge by (i) and (ii) of Theorem 4.1.

It remains to show (4.6), i.e., for all $\xi \in C_1^1(\mathbb{R}^d \times (0, T); \mathbb{R}^d)$ with $\nabla \cdot \xi = 0$ we have
\[
\int_0^T \int_{\Omega(t)} j \cdot \xi \, dx \, dt = -\int_0^T \int_{\mathbb{R}^d} \nabla \cdot \xi - \nu_{\Omega} \cdot \nabla \xi \nu_{\Omega} \, d|\mu_{\Omega}| \, dt \tag{4.26}.
\]

Here, $\nu_{\Omega}(\cdot, t) : \partial^*\Omega(t) \to S^{d-1}$ is the measure theoretic outward unit normal and $\mu_{\Omega}(\cdot, t) = \mu_{\Omega(t)}$ is the Gauss–Green measure at time $t$. 

First, we combine (4.4) and (4.22) to get
\[
\lim_{l \to \infty} \int_0^T E_{\varepsilon l}(u_{\varepsilon l}(\cdot, t)) \, dt = \sigma \int_0^T P(\Omega(t)) \, dt. \tag{4.27}
\]
Let \( \tilde{\zeta} \in C^2(\mathbb{R}^d \times (0, T); \mathbb{R}^d) \) such that \( \nabla \cdot \tilde{\zeta} = 0 \), and let \( T_{\varepsilon} \) as in (3.3), i.e.,
\[
T_{\varepsilon} := \left( \frac{\varepsilon}{2} |\nabla u_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(u_{\varepsilon}) \right) I_d - \varepsilon \nabla u_{\varepsilon} \otimes \nabla u_{\varepsilon}.
\]
We want to derive (4.26) by passing to the limit in (3.2), i.e.,
\[
\int_0^T \int_{\mathbb{R}^d} j_{\varepsilon l} \cdot \zeta \, dx \, dt = - \int_0^T \int_{\mathbb{R}^d} T_{\varepsilon} : \nabla \zeta \, dx \, dt.
\]
Since \( j_{\varepsilon l} \to \chi_\Omega^l \) as \( l \to \infty \), it suffices to show that
\[
T_{\varepsilon} \, dx \, dt \rightharpoonup \sigma(I_d - \nu_\Omega \otimes \nu_\Omega) \, d|\mu_\Omega| \, dt \quad \text{weakly} \quad -^* \quad \text{as Radon measures.} \tag{4.28}
\]
For the first term, we note that it is sufficient to test \( \frac{\varepsilon}{2} |\nabla u_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(u_{\varepsilon}) \) with \( \zeta \in C^0(\mathbb{R}^d \times (0, T)) \) such that \( 0 \leq \zeta \leq 1 \). We have, by (4.13),
\[
\int_0^T \int_{\mathbb{R}^d} \zeta \left( \frac{\varepsilon}{2} |\nabla u_{\varepsilon}|^2 + \frac{1}{\varepsilon} W(u_{\varepsilon}) \right) \, dx \, dt \geq \int_0^T \int_{\mathbb{R}^d} \zeta |\nabla (\phi \circ u_{\varepsilon})| \, dx \, dt.
\]
Hence we have
\[
\liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} \zeta \left( \frac{\varepsilon_l}{2} |\nabla u_{\varepsilon_l}|^2 + \frac{1}{\varepsilon_l} W(u_{\varepsilon_l}) \right) \, dx \, dt \geq \sigma \int_0^T \int_{\mathbb{R}^d} \zeta \, d|\mu_\Omega| \, dt.
\]
Now we test with \( \eta = 1 - \zeta \) to obtain
\[
\liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} (1 - \zeta) \left( \frac{\varepsilon_l}{2} |\nabla u_{\varepsilon_l}|^2 + \frac{1}{\varepsilon_l} W(u_{\varepsilon_l}) \right) \, dx \, dt \geq \sigma \int_0^T \int_{\mathbb{R}^d} (1 - \zeta) \, d|\mu_\Omega| \, dt.
\]
Combining these two inequalities with (4.27), we get
\[
\lim_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} \zeta \left( \frac{\varepsilon_l}{2} |\nabla u_{\varepsilon_l}|^2 + \frac{1}{\varepsilon_l} W(u_{\varepsilon_l}) \right) \, dx \, dt = \sigma \int_0^T \int_{\mathbb{R}^d} \zeta \, d|\mu_\Omega| \, dt.
\]
This gives us
\[
\left( \frac{\varepsilon_l}{2} |\nabla u_{\varepsilon_l}|^2 + \frac{1}{\varepsilon_l} W(u_{\varepsilon_l}) \right) \, dx \, dt \rightharpoonup \sigma |\mu_\Omega| \, dt,
\]
and
\[
|\nabla (\phi \circ u_{\varepsilon_l})| \, dx \, dt \rightharpoonup \sigma |\mu_\Omega| \, dt \quad \text{as} \quad l \to \infty. \tag{4.29}
\]
For the second term, we define
\[
\nu_{\varepsilon l} = - \frac{\nabla (\phi \circ u_{\varepsilon_l})}{|\nabla (\phi \circ u_{\varepsilon_l})|},
\]
with the convention \( \nu_{\varepsilon_l} = e_1 \) if \( \nabla (\phi \circ u_{\varepsilon_l}) = 0 \). Let \( \nu^* \in C^1_c(\mathbb{R}^d \times (0, T); \mathbb{R}^d) \) and let
Lemma 4.6. \( E_{t_i}(u_{t_i}; \nu^*) := \int_{0}^{T} \int_{\mathbb{R}^d} \left| \frac{\nabla u_{t_i}}{|\nabla u_{t_i}|} \right| \, dx \, dt, \)

\( E(\Omega; \nu^*) := \sigma \int_{0}^{T} \int_{\mathbb{R}^d} |\nu_{t_i} - \nu^*|^2 \, d|\mu_{t_i}| \, dt. \) (4.30)

Using the fact that \(-\frac{\nabla u_{t_i}}{|\nabla u_{t_i}|} = \nu_{t_i}\) a.e., we compute

\[
\left| \int_{0}^{T} \int_{\mathbb{R}^d} \nu_{t_i} \cdot \nabla \bar{\xi} \nu_{t_i} |\nabla (\phi \circ u_{t_i})| \, dx \, dt - \sigma \int_{0}^{T} \int_{\mathbb{R}^d} \nu_{t_i} \cdot \nabla \bar{\xi} \nu_{t_i} \, d|\mu_{t_i}| \, dt \right|
\leq \left| \int_{0}^{T} \int_{\mathbb{R}^d} \nu^* \cdot \nabla \bar{\xi} \nu_{t_i} |\nabla (\phi \circ u_{t_i})| \, dx \, dt - \sigma \int_{0}^{T} \int_{\mathbb{R}^d} \nu^* \cdot \nabla \bar{\xi} \nu_{t_i} \, d|\mu_{t_i}| \, dt \right|
+ \left| \int_{0}^{T} \int_{\mathbb{R}^d} (\nu_{t_i} - \nu^*) \cdot \nabla \bar{\xi} \nu_{t_i} |\nabla (\phi \circ u_{t_i})| \, dx \, dt \right|
+ \left| \int_{0}^{T} \int_{\mathbb{R}^d} (\nu_{t_i} - \nu^*) \cdot \nabla \bar{\xi} \nu_{t_i} \, d|\mu_{t_i}| \, dt \right|
\leq \left| \int_{0}^{T} \int_{\mathbb{R}^d} \nu^* \cdot \nabla \bar{\xi} \cdot |\nabla (\phi \circ u_{t_i})| \, dx \, dt - \sigma \int_{0}^{T} \int_{\mathbb{R}^d} \nu^* \cdot \nabla \bar{\xi} \nu_{t_i} \, d|\mu_{t_i}| \, dt \right|
+ \left( \int_{0}^{T} \int_{\mathbb{R}^d} \left| \nabla \bar{\xi} \right|^2 |\nabla (\phi \circ u_{t_i})| \, dx \, dt \right)^{1/2} E_{t_i}^{1/2}(u_{t_i}; \nu^*)
+ \left( \int_{0}^{T} \int_{\mathbb{R}^d} \left| \nabla \bar{\xi} \right|^2 \, d|\mu_{t_i}| \, dt \right)^{1/2} E^{1/2}(\Omega; \nu^*).
\]

By Lemma 4.6 below,
\( E_{t_i}(u_{t_i}; \nu^*) \to E(\Omega; \nu^*) \) as \( l \uparrow \infty. \)

Note further that \( \nu^* \cdot \nabla \bar{\xi} \) is an admissible test function in the weak convergence of \( \nu_{t_i} \, dx \, dt \to \nu_{t_i} |\mu_{t_i}| \, dt. \) Therefore, first letting \( l \uparrow \infty \) and then \( \nu^* \to \nu_{t_i} \) we obtain
\[
\lim_{l \to \infty} \left| \int_{0}^{T} \int_{\mathbb{R}^d} \nu_{t_i} \cdot \nabla \bar{\xi} \nu_{t_i} |\nabla (\phi \circ u_{t_i})| \, dx \, dt - \sigma \int_{0}^{T} \int_{\mathbb{R}^d} \nu_{t_i} \cdot \nabla \bar{\xi} \nu_{t_i} \, d|\mu_{t_i}| \, dt \right| = 0.
\]

Step 2 (Optimal energy dissipation). Using that \( u_{t_i} \to \chi_{t_i} \) in \( L^2(0, T; L^1(\mathbb{R}^d)) \) and (4.25), we have for a.e. \( T' \in [0, T] \)

\[
\sigma P(\Omega(T')) + \int_{0}^{T} \int_{\Omega(t)} |j|^2 \, dx \, dt \leq \liminf_{l \to \infty} \left( E_{t_i}(u_{t_i}, T') + \int_{0}^{T} \int_{\mathbb{R}^d} \frac{|j_{t_i}|^2}{u_{t_i}} \, dx \, dt \right)
\leq \liminf_{l \to \infty} E_{t_i}(u_{t_i}, 0)
= \sigma P(\Omega_0).
\]

Lemma 4.6. Assume \( u_{t_i} \to \chi_{t_i} \) in \( L^1 \) and \( \int_{0}^{T} E_{t_i}(u_{t_i}, t) \, dt \to \sigma \int_{0}^{T} P(\Omega(t)) \, dt \) as \( l \uparrow \infty. \)
For \( \nu^* \in C^1(\mathbb{R}^d \times (0, T); \mathbb{R}^d) \) let \( E_{t_i}(u_{t_i}; \nu^*) \) and \( E(\Omega; \nu^*) \) as in (4.30). Then
\[ \lim_{l \to \infty} \mathcal{E}_{\varepsilon_l}(u_{\varepsilon_l}; \nu^*) = \mathcal{E}(\Omega; \nu^*). \]

Proof. We expand the square \(|\nu_{\varepsilon_l} - \nu^*|^2 = 1 + |\nu^*|^2 - 2\nu^* \cdot \nu_{\varepsilon_l}\) and integrate the last term by parts

\[ \mathcal{E}_{\varepsilon_l}(u_{\varepsilon_l}; \nu^*) = \int_0^T \int_{\mathbb{R}^d} (1 + |\nu^*|^2) |\nabla (\phi \circ u_{\varepsilon_l})| \, dx \, dt - 2 \int_0^T \int_{\mathbb{R}^d} (\phi \circ u_{\varepsilon_l}) \nabla \cdot \nu^* \, dx \, dt, \]

\[ \mathcal{E}(\Omega; \nu^*) = \sigma \int_0^T \int_{\mathbb{R}^d} (1 + |\nu^*|^2) \, d|\mu_\Omega| \, dt - 2\sigma \int_0^T \int_{\mathbb{R}^d} \nu^* \cdot \nu^* \, d|\mu_\Omega| \, dt. \]

The first terms converge by (4.29). The last term in the second line reads, by definition of the Gauss-Green measure \(\mu_\Omega\),

\[ 2\sigma \int_0^T \int_{\mathbb{R}^d} \nu^* \cdot \nu^* \, d|\mu_\Omega| \, dt = 2\sigma \int_0^T \int_{\mathbb{R}^d} \chi_\Omega \nabla \cdot \nu^* \, dx \, dt. \]

Now by the \(L^1\) convergence \(\phi \circ u_{\varepsilon_l} \to \sigma \chi_\Omega\) as \(l \uparrow \infty\), the claim follows. \(\square\)

The next lemma shows that the energy contribution of both summands in the energy density is essentially the same as \(\varepsilon \downarrow 0\). This was first shown by Luckhaus and Modica [7]. For the convenience of the reader we recall the proof here.

Lemma 4.7. ([7, Lemma 1]). Let \(a_{\varepsilon_l} = (\varepsilon_l/2)^{1/2} |\nabla u_{\varepsilon_l}|\) and \(b_{\varepsilon_l} = \varepsilon_l^{-1/2} W^{1/2}(u_{\varepsilon_l})\). Further let \(\{\Omega(t)\}_{t \in [0, T]}\) be a family of finite perimeter sets such that (4.2) holds. If

\[ \int_0^T E_{\varepsilon_l}(u_{\varepsilon_l}, t) \, dt \to \sigma \int_0^T P(\Omega(t)) \, dt \tag{4.31} \]

as \(l \uparrow \infty\), then

\[ \liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} (a_{\varepsilon_l} - b_{\varepsilon_l})^2 \, dx \, dt = \liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} |a_{\varepsilon_l}^2 - a_{\varepsilon_l} b_{\varepsilon_l}| \, dx \, dt \]

\[ \quad = \liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} |a_{\varepsilon_l}^2 - b_{\varepsilon_l}^2| \, dx \, dt = 0. \tag{4.32} \]

Proof. On the one hand, by (4.31),

\[ \lim_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} a_{\varepsilon_l}^2 + b_{\varepsilon_l}^2 \, dx \, dt = \sigma \int_0^T P(\Omega(t)) \, dt. \]

On the other hand, by (4.29) and lower semi-continuity,

\[ \liminf_{l \to \infty} \int_0^T 2 a_{\varepsilon_l} b_{\varepsilon_l} \, dx \, dt = \liminf_{l \to \infty} \int_0^T \int_{\mathbb{R}^d} |\nabla (\phi \circ u_{\varepsilon_l})| \, dx \, dt \geq \sigma \int_0^T P(\Omega(t)) \, dt. \]

Therefore

\[ \liminf_{l \to \infty} \int_0^T (a_{\varepsilon_l} - b_{\varepsilon_l})^2 \, dx \, dt = 0. \]
Moreover, by the Hölder inequality and the uniform bound on \( a_{ij} \) in \( L^2 \), we have

\[
\int_0^T \int_{\mathbb{R}^d} \left| a_{ij}^2 - a_{ij} b_{ij} \right| \, dx \, dt \leq C \left( \int_0^T \int_{\mathbb{R}^d} |a_{ij}|^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^d} |a_{ij} - b_{ij}|^2 \, dx \, dt \right)^{1/2},
\]

The same argument applies for \( |b_{ij}^2 - a_{ij} b_{ij}| \). Finally, observe that

\[
\int_0^T \int_{\mathbb{R}^d} \left| a_{ij}^2 - b_{ij}^2 \right| \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^d} |a_{ij}^2 - a_{ij} b_{ij}| \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} |b_{ij}^2 - a_{ij} b_{ij}| \, dx \, dt.
\]

This concludes the proof.

\[\square\]

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Appendix A. Construction of well-prepared initial data

In this section we construct initial data which satisfies the well-preparedness condition (4.1).

The construction is based on a famous result, which was first published by Modica and Mortola [13] in 1977, to construct suitable initial conditions in order to recover a solution to the Hele–Shaw flow for a given initial configuration $\Omega_0$.

Luciano Modica and Stefano Mortola have shown that the functionals defined on $L^2(\mathbb{R}^d)$ by

$$E_\varepsilon(u) = \int_{\mathbb{R}^d} \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx$$

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\[ E_0(u) := \begin{cases} \sigma P(\Omega; \mathbb{R}^d), & \text{if } u = \chi_{\Omega}, \\ \infty, & \text{else.} \end{cases} \]

The following lemma is a slight variant of their recovery sequence and provides well-prepared initial conditions.

**Lemma A.1.** Let \( \Omega_0 \subset \mathbb{R}^d \) be open, bounded, with \( C^2 \)-boundary and such that \( \mathcal{L}^d(\Omega_0) = 1 \). Then there exists a sequence \( (\bar{u}_{e,0})_{e>0} \) such that \( \bar{u}_{e,0} \in A \) with \( \sup_e M_2(\bar{u}_{e,0}) < \infty \) and

\[
\limsup_{e \downarrow 0} E_e(\bar{u}_{e,0}) \leq E_0(\chi_{\Omega_0}).
\]

**Proof.** The idea is simply to scale the recovery sequence of Modica–Mortola to ensure \( \int u_{e,0} \, dx = 1 \). Then we only need to check that the second moments are uniformly bounded.

Step 1 (Optimal profile and Modica-Mortola). Let \( \bar{W}(t) = \frac{1}{2}(t^2 - 1)^2 \), and let \( \tilde{q} : \mathbb{R} \to \mathbb{R} \) be the 1-d optimal profile, that is,

\[
\begin{cases}
\tilde{q}'' = \bar{W}(\tilde{q}), \\
\tilde{q}(0) = 0, \\
\lim_{z \to -\infty} \tilde{q}(z) = -1, \\
\lim_{z \to +\infty} \tilde{q}(z) = 1.
\end{cases}
\]

Then \( \tilde{q}(z) = \tanh(z) \). Let \( q(z) = \frac{1}{2} \tanh(z) + \frac{1}{2} \). Then

\[
q(z) \leq Ce^{-|z|/C}, \quad \text{as } z \to -\infty.
\]

Further let \( s_{\Omega_0}(x) = \text{dist}(x, \Omega_0) - \text{dist}(x, \mathbb{R}^d \setminus \Omega_0) \) be the signed distance function w.r.t. \( \Omega_0 \), and consider the one-parameter family of functions

\[
u_{e,0}^a := q\left(-\frac{s_{\Omega_0}(ax)}{e}\right), \quad a \in (0, \infty).
\]

For \( a = 1 \) we obtain the standard recovery sequence for \( \chi_{\Omega_0} \) in the \( \Gamma \)-convergence of \( E_e \) to \( E_0 \) [14, Thm. I]. In particular, they showed

\[
\limsup_{e \downarrow 0} E_e(u_{e,0}^1) \leq \sigma P(\Omega_0).
\]

Step 2 (Rescaling and volume constraint). We show that there exist \( a_e \in (0, \infty) \) such that the functions \( \nu_{e,0}^a \) satisfy (4.1), \( \nu_{e,0}^a \in A \) and \( \nu_{e,0}^a \to \chi_{\Omega_0} \) in \( L^1 \) as \( e \downarrow 0 \).

The one-parameter family \( (\nu_{e,0}^a)_{a \in (0, \infty)} \) satisfies

\[
\int_{\mathbb{R}^d} \nu_{e,0}^a(x) \, dx = \int_{\mathbb{R}^d} \nu_{e,0}^1(ax) \, dx = \frac{1}{a^d} \int_{\mathbb{R}^d} \nu_{e,0}^1(x) \, dx.
\]

Hence there exists \( a_e \in (0, \infty) \) such that \( \|\nu_{e,0}^a\|_{L^1} = a_e^{-d}\|\nu_{e,0}^1\|_{L^1} = 1 \).

Let \( \bar{u}_{e,0} := \nu_{e,0}^{a_e} \). Since \( \nu_{e,0}^1 \to \chi_{\Omega_0} \) in \( L^1 \) and \( \mathcal{L}^d(\Omega_0) = 1 \), we have

\[
a_e \downarrow 1.
\]

Let \( \chi_{\Omega_0,e}(x) := \chi_{\Omega_0}(a_e x) \) and consider

\[
\int_{\mathbb{R}^d} |\bar{u}_{e,0} - \chi_{\Omega_0}| \, dx \leq \int_{\mathbb{R}^d} |\bar{u}_{e,0} - \chi_{\Omega_0,e}| \, dx + \int_{\mathbb{R}^d} |\chi_{\Omega_0,e} - \chi_{\Omega_0}| \, dx.
\]
For the first term we have
\[
\int_{\mathbb{R}^d} |\bar{u}_{e,0} - \chi_{\Omega_0}| \, dx = \int_{\mathbb{R}^d} |u_{e,0}^1(a, x) - \chi_{\Omega_0}(a,x)| \, dx
\]
\[
= \frac{1}{a_\varepsilon^2} \int_{\mathbb{R}^d} |u_{e,0}^1(x) - \chi_{\Omega_0}(x)| \, dx.
\]
Here the R.H.S. converges to zero as \( \varepsilon \downarrow 0 \). For the second term, we have
\[
\chi_{\Omega_{e,\varepsilon}^0} \rightarrow \chi_{\Omega_0} \quad \text{in} \quad L^1
\]
because \( a_\varepsilon \rightarrow 1 \). Therefore \( \bar{u}_{e,0} \rightarrow \chi_{\Omega_0} \) in \( L^1 \).

Step 3 (\( \Gamma \)-limsup inequality and moment bound). It remains to show that the second moment of \( \bar{u}_{e,0} \) is uniformly bounded in \( \varepsilon \) and that \( \limsup_{\varepsilon \downarrow 0} E_c(\bar{u}_{e,0}) \leq \sigma E_0(\chi_{\Omega_0}) \). Since we have \( a_\varepsilon \rightarrow 1 \), \( q(z) \leq C e^{-|z|^2/C} \) as \( z \rightarrow -\infty \), and \( \Omega_0 \) is bounded, we can choose \( R < \infty \) sufficiently large, and \( \delta > 0 \) small enough such that \( |a_\varepsilon - 1| < 1/2 \) for all \( \varepsilon < \delta \) and
\[
\bar{u}_{e,0}(x) \leq C e^{-|x|^2/2C} \quad \text{for all} \quad x \in \mathbb{R}^d \setminus B_R(0).
\]
Then for any \( p \geq 1 \) and \( \varepsilon \leq \delta \)
\[
M_p(\bar{u}_{e,0}) = \int_{\mathbb{R}^d} |x|^p \bar{u}_{e,0}(x) \, dx \leq R^p L^d(B_R(0)) + C \int_{|x| > R} |x|^p e^{-|x|^2/2C} \, dx.
\]
The R.H.S. is uniformly bounded in \( \varepsilon \), hence \( \sup_{\varepsilon \leq \delta} M_2(\bar{u}_{e,0}) < \infty \).

For the \( \Gamma \)-limsup inequality, observe that
\[
E_c(\bar{u}_{e,0}) = \int_{\mathbb{R}^d} \frac{\varepsilon}{2} \left| \nabla \bar{u}_{e,0} \right|^2 \, dx + \int_{\mathbb{R}^d} \frac{1}{2} \mathcal{W}(\bar{u}_{e,0}) \, dx
\]
\[
= \int_{\mathbb{R}^d} \frac{\varepsilon}{2} \left| \nabla (u_{e,0}^1(a,x)) \right|^2 \, dx + \int_{\mathbb{R}^d} \frac{1}{2} \mathcal{W}(u_{e,0}^1(a,x)) \, dx
\]
\[
= \frac{1}{a_\varepsilon^{d-2}} \int_{\mathbb{R}^d} \frac{\varepsilon}{2} \left| \nabla u_{e,0}^1(x) \right|^2 \, dx + \frac{1}{a_\varepsilon^{d-1}} \int_{\mathbb{R}^d} \frac{1}{2} \mathcal{W}(u_{e,0}^1(x)) \, dx.
\]
Therefore, since \( a_\varepsilon \rightarrow 1 \) as \( \varepsilon \downarrow 0 \), for any \( \delta > 0 \), it holds
\[
\limsup_{\varepsilon \downarrow 0} E_c(\bar{u}_{e,0}) \leq \limsup_{\varepsilon \downarrow 0}(1 + \delta)E_c(u_{e,0}^1)
\]
\[
\leq (1 + \delta)\sigma P(\Omega_0; \mathbb{R}^d);
\]
hence \( \limsup_{\varepsilon \downarrow 0} E_c(u_{e,0}^1) \leq \sigma P(\Omega_0; \mathbb{R}^d) \).

Appendix B. Recap on optimal transport

We recall the quadratic optimal transport problem in the Euclidean setting, see Villani [19] and Ambrosio et al. [20, Part II]. Let \( \mathcal{P}(\mathbb{R}^d) \) denote the space of probability measures on \( \mathbb{R}^d \). Given \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \), Monge’s formulation asks for an optimal transport map \( t \) which minimizes
\[
\inf \left\{ \int_{\mathbb{R}^d} |x - t(x)|^2 \, d\mu(x) : t_\#\mu = \nu \right\}, \tag{B.1}
\]
where \( \int_{\mathbb{R}^d} f(y) \, d(t_\#\mu)(y) = \int_{\mathbb{R}^d} f(t(x)) \, d\mu(x) \) for all \( f \in C_b(\mathbb{R}^d) \). On the other hand, Kantorovich’s formulation looks at all measures \( \gamma \) with marginals \( \mu \) and \( \nu \) and asks for a minimizer to
\[
\inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x,y) : \gamma \in \Gamma(\mu, \nu) \right\}, \tag{B.2}
\]
where
\[ \Gamma(\mu, \nu) := \left\{ \gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : (\pi_x)_\# \gamma = \mu, (\pi_y)_\# \gamma = \nu \right\}, \]
and \( \pi_x, \pi_y : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) denote the projections onto the first and second factor, respectively. We observe that Kantorovich’s formulation (B.2) is exactly the definition of the squared Wasserstein distance \( W_2^2(\mu, \nu) \). Moreover, it is well known that the squared Wasserstein distance satisfies
\[ W_2^2(\mu, \nu) = \inf \left\{ \int_{\mathbb{R}^d} |x - t(x)|^2 \, d\mu(x) : t#\mu = \nu \right\}, \quad \text{(B.3)} \]
whenever the R.H.S. is well-posed, which is not always the case, e.g. when \( \mu \) is a Dirac mass but \( \nu \) is not.

The following proposition gives a characterization of optimal transport plans for absolutely continuous measures w.r.t. the Lebesgue measure. For \( 1 \leq p < \infty \) let
\[ \mathcal{P}_p := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \, d\mu(x) < \infty \right\}, \quad \mathcal{P}_p^a := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu = u\mathcal{L}^d \right\}. \]

**Proposition B.1.** (Existence of optimal transport maps [25, Thm. 2.3]). For any \( \mu, \nu \in \mathcal{P}_2^a(\mathbb{R}^d) \), Kantorovich’s optimal transport problem has a unique solution \( \gamma \). Moreover:

(i) The optimal plan \( \gamma \) is induced by a transport map \( t \), i.e., \( \gamma = (i_d, t)_\# \mu \), where \( i_d \) is the identity map on \( \mathbb{R}^d \). In particular, \( t \) is the unique solution of Monge’s optimal transport problem (B.1).

(ii) The map \( t \) coincides \( \mu \)-a.e. with the gradient of a convex function \( \varphi : \mathbb{R}^d \to (-\infty, \infty] \), whose finiteness domain \( D(\varphi) \) has non-empty interior and satisfies
\[ \mu(\mathbb{R}^d \setminus D(\varphi)) = \mu(\mathbb{R}^d \setminus D(\nabla \varphi)) = 0. \]

(iii) If \( s \) is the optimal transport map between \( \nu \) and \( \mu \), then
\[ s \circ t = i_d \quad \mu \text{-a.e. in } \mathbb{R}^d \quad \text{and} \quad t \circ s = i_d \quad \nu \text{-a.e. in } \mathbb{R}^d. \]

**Proposition B.2.** ([20, Thm. 7.2.2]). Let \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \) and let \( \gamma \in \Gamma(\mu, \nu) \) be optimal. Then the map
\[ t \mapsto \mu_t := ((1 - t)\pi_x + t\pi_y)_\# \gamma \]
is a constant speed geodesic. Conversely, any constant speed geodesic \( \mu_t : [0, 1] \to \mathcal{P}_2(\mathbb{R}^d) \) joining \( \mu \) and \( \nu \) has this representation for a suitable \( \gamma \in \Gamma(\mu, \nu) \).

Here we say that \( \mu_t \) is a constant speed geodesic if
\[ W_2(\mu_s, \mu_t) = (t - s)W_2(\mu_0, \mu_1) \]
whenever \( 0 \leq s \leq t \leq 1 \).

**Proposition B.3.** ([20, Thm. 8.3.1]). Let \( \mu_t : [0, 1] \to \mathcal{P}_p^a(\mathbb{R}^d) \) be a constant speed geodesic. Then there exists a Borel vector field \( \nu : (x, t) \mapsto \nu_t(x) \) such that
\[ \nu_t \in L^0(\mu_t, \mathbb{R}^d; \mathbb{R}^d) \quad \text{for } L^1 \text{-a.e. } t \in [0, 1], \]
and the continuity equation
\[ \partial_t \mu_t + \nabla \cdot (\nu_t \mu_t) = 0 \]
holds in the sense of distributions, i.e.,
\[
\int_0^1 \int_{\mathbb{R}^d} \partial_t \zeta(x, t) + v_t(x) \cdot \nabla \zeta(x, t) \, d\mu_t(x) \, dt \quad \text{for all } \zeta \in C^1_c(\mathbb{R}^d \times (0, 1)). \tag{B.4}
\]

Here

\[L^p(\mu, \mathbb{R}^d; \mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{R}^d : \int_{\mathbb{R}^d} |u|^p \, d\mu < \infty \right\}.\]

Appendix C. Higher regularity and flow exchange lemma

We recall here some useful facts from the theory of Wasserstein gradient flows, which will allow us to prove the convergence of energies.

**Lemma C.1.** ([5, Lemma 4.3]). Let \( \tilde{v} \in H^1(\mathbb{R}^d) \) such that \( E(\tilde{v}) < \infty \). Suppose \( v_t : [0, \infty) \to H^1(\mathbb{R}^d) \) is a solution to the heat flow

\[
\begin{align*}
\partial_t v_t &= \Delta v_t, \quad \text{on } (0, \infty) \times \mathbb{R}^d, \\
v_0 &= \tilde{v}, \quad \text{on } \mathbb{R}^d,
\end{align*}
\]

and

\[
\lim \inf_{t \downarrow 0} \frac{1}{t} (E(v_t) - E(v_0)) > -\infty. \tag{C.2}
\]

Then \( \tilde{v} \in H^2(\mathbb{R}^d) \), and

\[
- \lim \inf_{t \downarrow 0} \frac{1}{t} (E(v_t) - E(v_0)) \geq \int_{\mathbb{R}^d} (\Delta v_0)^2 \, dx - CE(v_0), \tag{C.3}
\]

where the constant \( C \) depends only on \( W \).

**Proof.** By parabolic smoothing we have \( v \in C^\infty(\mathbb{R}^d \times (0, \infty)) \). Thus, for \( t > 0 \), we can compute the derivative \( \frac{d}{dt} E(v_t) \)

\[
\frac{d}{dt} E(v_t) = \int_{\mathbb{R}^d} \nabla v_t \cdot \nabla \partial_t v_t + W'(v_t) \partial_t v_t \, dx
\]

\[= - \int_{\mathbb{R}^d} (\Delta v_t)^2 + W''(v_t) |\nabla v_t|^2 \, dx
\]

\[\leq - \int_{\mathbb{R}^d} (\Delta v_t)^2 \, dx + C \int_{\mathbb{R}^d} |\nabla v_t|^2 \, dx,
\]

where \( C \) is a constant with \(-C \leq W''\). Observing that

\[
\int_{\mathbb{R}^d} |\nabla v_t|^2 \, dx \leq \int_{\mathbb{R}^d} |\nabla v_0|^2 \, dx,
\]

and using that the map \( t \mapsto v_t \) is continuous in \( H^1(\mathbb{R}^d) \), we get that \( t \mapsto E(v_t) \) is continuous at \( t = 0 \), and by (C.2) there exists a constant \( C < \infty \) such that for all \( t \leq t_0 \) sufficiently small

\[-C < \frac{1}{t} (E(v_t) - E(v_0)) \leq - \frac{1}{t} \int_0^t \int_{\mathbb{R}^d} (\Delta v_s)^2 \, dx \, ds + C \|\nabla v_0\|_{L^2(\mathbb{R}^d)}^2.
\]

Thus the family \( \{v_t\}_{t \leq t_0} \) is weakly precompact in \( L^2(\mathbb{R}^d) \). Since \( v_t \to \tilde{v} \) strongly in \( H^1(\mathbb{R}^d) \) as \( t \downarrow 0 \), we get \( \tilde{v} \in H^2(\mathbb{R}^d) \). \( \square \)

We want to apply Lemma C.1 with initial data \( u_m \), the solution to the minimizing movement scheme (3.6). To verify (C.2), we need the flow exchange lemma below.
**Definition C.2.** Let \( F : A \to (-\infty, \infty] \) be a proper, lower semi-continuous functional and let \( \lambda > 0 \). Let \( \text{Dom} \, F := \{ u \in A : F(u) < \infty \} \) denote the domain of \( F \). A continuous semi-group \( S_t : \text{Dom} \, F \to \text{Dom} \, F, \, t \geq 0, \) is a \( \lambda \)-flow for \( F \) if it satisfies the following Evolution Variational Inequality (EVI)

\[
\frac{1}{2} \limsup_{t \to 0} \frac{d^2(S_t(u), v) - d^2(u, v)}{t} + \lambda \frac{d^2(u, v) + F(u)}{2} \leq F(v)
\]  

(C.4)

for all densities \( u, v \in \text{Dom} \, F \) with \( d(u, v) < \infty \).

**Lemma C.3.** (Flow exchange lemma [18, Thm. 3.2]). Let \( S_t \) be a \( \lambda \)-flow for a proper, lower semi-continuous functional \( F \) in \( A \) and let \( (u_n)_{n \geq 0} \) be a solution to the minimizing movements scheme (3.6) with time-step size \( h > 0 \). If \( u_n \in \text{Dom} \, F \), then

\[
F(u_n) - F(u_{n-1}) \leq h \left( \liminf_{t \to 0} \frac{E(S_t(u_n)) - E(u_n)}{t} - \frac{\lambda}{2} d^2(u_n, u_{n-1}) \right).
\]  

(C.5)

We want to apply the flow exchange lemma C.3 to the entropy functional

\[
\mathcal{U}(u) := \int_{\mathbb{R}^d} u \log u \, dx.
\]  

(C.6)

By Ambrosio et al. [20, Prop. 9.3.9] \( \mathcal{U} \) is geodesically convex in \( A \) with respect to the Wasserstein distance in the sense that the map \( t \mapsto \mathcal{U}(u_t) \) is convex for every geodesic \( u_t \) in \( A \).

**Lemma C.4.** The semi-group \( S_t \) induced by solutions to the heat equation (C.1) on \( A \cap C^\infty(\mathbb{R}^d) \) extends to a \( 0 \)-flow for \( \mathcal{U} \).

**Proof.** We recall the well known fact that the heat equation (C.1) is the \( W_2 \) gradient flow of the entropy functional \( \mathcal{U} \), for a reference see e.g. [6, Thm. 5.1]. Further, since \( \mathcal{U} \) is geodesically convex in \( A \) w.r.t. the Wasserstein distance \( d \), \( \mathcal{U} \) satisfies the Evolution Variational Inequality (C.4) with \( \lambda = 0 \), for a reference see for example [20, Thm. 11.1.4].

**Proposition C.5.** ([5, Prop. 4.1]). Let \( (u_n)_{n \in \mathbb{N}} \) be a solution to the minimizing movement scheme (3.6) with time-step size \( h > 0 \). Then (up to a subsequence)

(i) \( u_n \in H^2(\mathbb{R}^d) \) for all \( n \) with a uniform bound,

(ii) \( u_n \rightharpoonup u \) strongly in \( L^2(0, T; H^1(\mathbb{R}^d)) \) as \( h \downarrow 0 \) for all \( T > 0 \),

(iii) \( u_n \rightharpoonup u \) weakly in \( L^2(0, T; H^2(\mathbb{R}^d)) \) as \( h \downarrow 0 \) for all \( T > 0 \).

To prove Proposition C.5, we need the following lemma.

**Lemma C.6.** There exists \( \alpha < 1 \) and a constant \( C < \infty \) depending only on \( d \) such that for all \( u \in A \) we have

\[
\int_{\mathbb{R}^d} u \log u \, dx \geq -C(M_2(u) + 1)^\alpha.
\]  

(C.7)

Furthermore if \( (u_n)_{n \in \mathbb{N}} \) is a solution to the minimizing movement scheme (3.6), then there exists a constant \( C < \infty \) such that for all \( n \) we have

\[
||\mathcal{U}(u_n)|| < C.
\]  

(C.8)

**Proof.** Equation (C.7) is shown in the proof of Jordan et al. [6, Prop. 4.1]. Now let \( (u_n)_n \) be a solution to (3.6). Then, by Remark 3.6 and interpolation between \( L^1 \) and \( L^4 \), there exists \( C < \infty \).
such that for all $n$

$$M_2(u_n) \leq C, \quad \int_{\mathbb{R}^d} u_n^2 \, dx \leq C.$$ 

Noting that

$$\mathcal{U}(u) \leq \int_{\mathbb{R}^d} u^2 \, dx$$

for any $u \in \text{Dom } \mathcal{U}$, this concludes the proof of (C.8).

**Proof of Proposition C.5.** We apply the flow exchange Lemma C.3 with $\mathcal{F} = \mathcal{U}$. Then (C.2) holds, and by Lemma C.1 applied to $\tilde{v} = u_n$ we have $u_n \in H^2(\mathbb{R}^d)$. Then, by (C.3) and (C.5), we get

$$\frac{h}{2} \int_{\mathbb{R}^d} (\Delta u_n)^2 \, dx \leq \mathcal{U}(u_{n-1}) - \mathcal{U}(u_n) + C\text{E}(u_n).$$

(C.9)

Now observe that, by interpolation between $L^1$ and $L^4$, $\mathcal{U}(u_0) \leq \|u_0\|_{L^2}^2 < \infty$ and recall that $E(u_n) \leq E(u_0)$. Let $N := \lceil T/h \rceil$ and sum over $n$

$$\|\Delta u_n\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \leq \sum_{n=1}^N \int_{\mathbb{R}^d} (\Delta u_n)^2 \, dx \leq \mathcal{U}(u_0) - \mathcal{U}(u_N) + C\text{E}(u_0).$$

By Lemma C.6, the R.H.S. is uniformly bounded as $N \to \infty$. By the uniform bound on $u_h$ in $L^2(0,T;H^2(\mathbb{R}^d))$, see Remark 3.6, we get that $u_h$ is uniformly bounded in $L^2(0,T;H^2(\mathbb{R}^d))$ as $h \downarrow 0$ for all $T < \infty$:

$$\int_0^T \|u_h(\cdot,t)\|^2_{H^2(\mathbb{R}^d)} \, dt \leq C\text{E}(u_0) < \infty.$$ 

(C.10)

Therefore $u_h$ converges, up to a subsequence, weakly in $L^2(0,T;H^2(\mathbb{R}^d))$ as $h \downarrow 0$. We recall that there exists $u$ such that $u_h \to u$ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$ and weakly in $L^2(0,T;H^1(\mathbb{R}^d))$. By interpolation with the uniform bound (C.10) we get $u_h \to u$ strongly in $L^2(0,T;H^1(\mathbb{R}^d))$ as $h \downarrow 0$.}

**Appendix D. A version of the Aubin–Lions lemma**

In this appendix, we recall a version of Aubin–Lions lemma in the setting of Banach spaces from Rossi and Savaré [26]. In particular we can apply this result with time compactness w.r.t. the Wasserstein distance.

Let us first recall the notion of convergence in measure:

**Definition D.1.** Let $B$ be a separable Banach space and let $\mathcal{M}([0,T];B)$ denote the space of measurable $B$-valued functions on $[0,T]$. A sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}([0,T];B)$ converges in measure to $u \in \mathcal{M}([0,T];B)$ if

$$\lim_{n \to \infty} \left| \{ t \in (0,T) : \| u_n(t) - u(t) \|_B \geq \sigma \} \right| = 0 \quad \text{for all } \sigma > 0.$$ 

(D.1)

Aubin–Lions lemma relies on the existence of a normal coercive integrand $\mathcal{F}$, so we briefly recall the definition.

**Definition D.2.** Let $B$ be a separable Banach space and let $T > 0$. A functional $\mathcal{F} : (0,T) \times B \to [0,\infty]$ is coercive if $\{u \in B : \mathcal{F}(u) \leq c\}$ is compact for all $c < \infty$ and a.e. $t \in (0,T)$. Further, $\mathcal{F}$ is a normal integrand if

1. $\mathcal{F}$ is $\mathcal{L} \otimes \mathcal{B}(B)$-measurable, and
(2) the maps $u \mapsto F_t(u)$ are lower semi-continuous for a.e. $t \in (0, T)$.

(3) If in addition, $g : B \times B \to [0, \infty]$ is a lower semi-continuous map, we say that $g$ is compatible with $\mathcal{F}$ if the following holds for a.e. $t \in (0, T)$: If $u, v \in B$ such that $\mathcal{F}(t, u), \mathcal{F}(t, v) < \infty$, then

$$u = v \quad \text{whenever} \quad g(u, v) = 0.$$  \hfill (D.2)

**Theorem D.3.** ([26, Thm. 2]). Let $B$ be a separable Banach space, let $T > 0$, and let $\mathcal{U}$ be a family of measurable $B$-valued functions on $(0, T)$. If there exists a normal coercive integrand $\mathcal{F} : (0, T) \times B \to [0, \infty]$ and a l.s.c. map $g : B \times B \to [0, \infty]$ compatible with $\mathcal{F}$ such that

$$\mathcal{U} \text{ is tight w.r.t. } \mathcal{F}, \text{ i.e., } S := \sup_{u \in \mathcal{U}} \int_0^T \mathcal{F}(t, u(t))dt < \infty, \quad \text{(D.3)}$$

and

$$\limsup_{h \to 0} \sup_{u \in \mathcal{U}} \int_0^{T-h} g(u(t+h), u(t))dt = 0, \quad \text{(D.4)}$$

then $\mathcal{U}$ is precompact in $\mathcal{M}([0, T]; B)$.

For us, $B$ will be $L^1(\mathbb{R}^d)$, and $g$ will be the Wasserstein distance: Let

$$g : L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to [0, \infty], \quad \begin{cases} g(u, v) = W_2(u\mathcal{L}^d, v\mathcal{L}^d), & \text{if } u, v \in \mathcal{A}, \\ \infty, & \text{else}, \end{cases}$$

where $W_2 : \mathcal{P}_2 \times \mathcal{P}_2 \to [0, \infty]$ is the Wasserstein-distance. Then $g$ is a metric on $L^1(\mathbb{R}^d) \cap \mathcal{A}$, so the compatibility condition (D.2) holds.

**Remark D.4.** We will only be interested in the case that $\mathcal{F}$ is independent of the time variable $t$, in which case we do not have to worry about measurability thanks to the following basic measure-theory statement. If $\mathcal{F} : B \to [0, \infty]$ is lower semi-continuous, then $\mathcal{F}$ is measurable. Indeed, it is enough to show that $\mathcal{F}^{-1}([0, a]) \subset B$ is measurable for any $0 \leq a \leq \infty$ since we can write any half-open interval $(a, b]$ with $0 \leq a < b \leq \infty$ as $(a, b] = [0, b] \setminus [0, a]$. By assumption these sets $\mathcal{F}^{-1}([0, a])$ are closed, hence measurable.