Hawking radiation in an electro-magnetic wave-guide?

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It is demonstrated that the propagation of electro-magnetic waves in an appropriately designed wave-guide is (for large wave-lengths) analogous to that within a curved space-time – such as around a black hole. As electro-magnetic radiation (e.g., micro-waves) can be controlled, amplified, and detected (with present-day technology) much easier than sound, for example, we propose a set-up for the experimental verification of the Hawking effect. Apart from experimentally testing this striking prediction, this would facilitate the investigation of the trans-Planckian problem.

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Introduction

One of the major motivations behind the idea of black hole analogues ("dumb holes", see [1]) is the possibility of an experimental verification of the Hawking effect [2]. Apart from testing one of the most striking theoretical predictions of quantum field theory under the influence of external conditions, such an experiment would enable us to investigate the impact of ultra-high energy/momentum degrees of freedom (trans-Planckian problem) on the lowest-order Hawking effect and its higher-order corrections (with respect to the small ratio of Hawking temperature over Planck scale) by means of an analogue system. In view of the close relation between the Hawking effect and the concept of black hole entropy, these investigations are potentially relevant for the black hole information paradox etc.

The analogy between sound waves in moving fluids and scalar fields in curved space-times established in [1] can (in principle) be used to simulate a horizon in liquid Helium [3] or in Bose-Einstein condensates [4], for example (see also [5]). However, measuring the Hawking effect in those systems goes along with serious difficulties [6]. The main problem is the detection of sound waves corresponding to the realistically very low Hawking temperature.

On the other hand, electro-magnetic radiation is much easier to amplify and to detect with present-day technology and one might hope to exploit this advantage. Optical black hole analogues have been discussed for highly dispersive media which support the phenomenon of slow light [7] and for ordinary non-dispersive dielectrics [8]. However, it turns out that a horizon for slow light does not emit Hawking radiation [9] whereas an experimental realization by means of ordinary dielectrics is in principle possible but very challenging. In the following, we shall propose an alternative set-up in order to circumvent this difficulty.

Wave-guide

Let us consider the propagation of electro-magnetic waves in the wave-guide in Figure 1. For simplicity, and in order to avoid leakage and transverse modes etc., we assume the following hierarchy of dimensions of the wave-guide and the wave-length $\lambda$ of the propagating electro-magnetic waves

$$\delta z \ll \delta x \ll \Delta x, \Delta z \ll \Delta y \ll \lambda. \quad (1)$$

In this limit, the wave-guide possesses a large slow-down and we can omit Maxwell’s supplement $D$ in Oerstedt-Ampère’s law $\nabla \times H = J + \dot{D} \sim \oint \mathbf{dr} \cdot \mathbf{H} = I + \oint \oint dS \cdot \mathbf{D}$ in the upper region of the wave-guide (i.e., the surface integral over $S = \Delta x \times \Delta z$). The conditions (1) also ensure that the energy of the waves is basically confined to the wave-guide.

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FIG. 1. Circuit diagram and sketch of wave-guide.

The capacitances $C$ are realized by the parallel conducting plates at the bottom of each unit separated by insulating slabs with the dielectric permittivity $\varepsilon$ and thickness $\delta z$. The inductances $L$ are generated by the remaining space of the $\Delta x \times \Delta y \times \Delta z$ cells (magnetic permeability $\mu$) with the enclosing walls also being conductors.

Hence, the combination of Oerstedt-Ampère’s and Faraday’s law $\nabla \times E = -\dot{B}$ $\sim \oint \mathbf{dr} \cdot \mathbf{E} = -\frac{d}{dt} \oint dS \cdot \mathbf{B}$ gives the induction law $\Delta U = \frac{d}{dt} LI$ for the effective coils with a possibly time-dependent inductance $L(t)$ given by $L = \mu \Delta x \Delta z / \Delta y$ for a long coil $\Delta y \gg \Delta x, \Delta z$. Denoting the voltage impressed on the $n$-th capacitor by $U_n$, the above condition implies for the $n$-th coil $U_{n+1} - U_n = \frac{d}{dt} L_n I_n$. By means of Gauss’ law $\nabla \cdot \mathbf{D} = \rho \sim \oint dS \cdot \mathbf{D} = Q$ and the conditions (1), we ob-
tain for the $n$th capacitor $Q_n = C_n U_n$ with $Q_n$ denoting its charge and $C_n = \varepsilon_n \Delta x \Delta y / \delta z$ its possibly time-dependent capacitance. In the following [10] we shall assume that all the inductances are constant and equal ($L_n = L = \text{const}$) whereas the capacitances are generally time-dependent $C_n(t)$. In analogy to the electromagnetic field in vacuum, we may introduce effective potentials $A_n$ which automatically satisfy the induction law $U_{n+1} - U_n = \frac{d}{dt} L_n I_n$ via

$$U_n = L \frac{dA_n}{dt} \quad , \quad I_n = A_{n+1} - A_n \, . \quad (2)$$

Then Kirchoff’s law $\frac{d}{dt} Q_n = \frac{d}{dt} C_n U_n = I_n - I_{n-1}$ (i.e., charge conservation) implies the equation of motion

$$\frac{d}{dt} L C_n \frac{d}{dt} A_n = A_{n+1} - 2 A_n + A_{n-1} \, . \quad (3)$$

Effective geometry In the continuum limit (i.e., one wave-length involves many units $\lambda \gg \Delta x$), the above equation of motion approaches the wave equation

$$\left( \frac{\partial}{\partial t} c^2 \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) A = 0 \, , \quad (4)$$

with the space-time dependent velocity of propagation

$$c = \frac{\Delta x}{\sqrt{LC}} = \sqrt{\frac{\delta z}{\epsilon \mu \Delta y}} \ll c_0 \, . \quad (5)$$

If we arrange $c^2(t, x)$ according to $c^2(t, x) = c^2(x + vt)$ with a constant velocity $v$ and transform into the co-moving frame $(x \rightarrow x + vt)$, the wave equation becomes

$$\left[ \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) c^2 \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) - \frac{\partial^2}{\partial x^2} \right] A = 0 \, . \quad (6)$$

Unfortunately, in 1+1 dimensions, the Maxwell equations are trivial and the scalar field is conformally invariant – which prevents the introduction of an effective geometry at this stage (i.e., in 1+1 dimensions). On the other hand, the wave-guide is not really 1+1 dimensional, the $y$-dimension just does not contribute due to $\lambda \gg \Delta y$, see [11]. Taking into account this "silent" $y$-dimension, the above wave equation allows for the identification of an effective metric via $\Box_{\text{eff}} A = \frac{\partial}{\partial x}(\sqrt{g_{\text{eff}}} \frac{\partial}{\partial x} A) = 0$ in the 2+1 dimensional Painlevé-Gullstrand-Lemaître form [12]

$$g_{\text{eff}}^{\mu \nu} = \begin{pmatrix} 1 & v & 0 \\ v & v^2 - c^2 & 0 \\ 0 & 0 & -c^2 \end{pmatrix} \, . \quad (7)$$

Hence, the propagation of electro-magnetic waves within the wave-guide is equivalent to that in a curved space-time as described by the above metric. Note that the scenario under consideration is similar to the super-sonic domain wall discussed in [13].

As one would expect, the metric describes a horizon at $v^2 = c^2$ with a surface gravity corresponding to the gradient of the propagation velocity $c$ (since $v = \text{const}$) and, therefore, a Hawking temperature of

$$T_{\text{Hawking}} = \frac{\hbar}{2\pi k_b} \left| \frac{\partial c}{\partial x} \right|_{v^2 = c^2} \, . \quad (8)$$

As demonstrated in [9], the (classical) wave equation is not enough for the prediction of Hawking radiation – the (quantum) commutation relations have to match as well. If we start from the effective action of the wave-guide

$$\mathcal{A}_{\text{eff}} = \int dt \sum_n \frac{1}{2} \left( C_n U_n^2 - L_n I_n^2 \right)$$

$$\rightarrow \frac{L \Delta x}{2} \int dt dx \left( \frac{1}{c^2} \left[ \frac{\partial A}{\partial t} \right]^2 - \left[ \frac{\partial A}{\partial x} \right]^2 \right) \, , \quad (9)$$

and perform the usual canonical quantization procedure, we indeed obtain the correct commutation relations for the Hawking effect – i.e., the conversion of the zero-point oscillations (with energy $\hbar \omega / 2$) to real photons by the external conditions $C_n(t)$.

Switching In the following we present a microscopic model for the controlled change of the capacitances $C_n(t)$. Let us consider an insulating material (e.g., a semiconductor at very low temperatures) where the electrons are localized and their dynamics (under the influence of an electro-magnetic field) can be described by the usual non-relativistic first-order Lagrangian density

$$\mathcal{L} = \Psi^\dagger \left( i \partial_t + \frac{\nabla^2}{2m} - V + ie \frac{\mathbf{A} \cdot \nabla}{m} - e \Phi \right) \Psi \, . \quad (10)$$

Here $m$ denotes the mass and $e$ the charge of the electrons, $V(r)$ is the static one-particle potential, and $\mathbf{A}$ and $\Phi$ are the vector and scalar potentials of the electromagnetic field, respectively. The scalar potential $\Phi$ describes a slow and small electric test field $\mathcal{E}(t)$ and, in addition, the vector potential $\mathbf{A}$ contains a fast and strong external laser field inducing electronic transitions. Performing an expansion into eigenfunctions of the undisturbed operator $V - \nabla^2 / (2m)$, we shall assume that only the three lowest-lying eigenstates with the eigenfunctions $f_a(r)$, $f_b(r)$, and $f_c(r)$ and the eigenvalues $\omega_a < \omega_b < \omega_c$, respectively, are relevant $\Psi(t) = \psi_a(t) f_a(r) e^{-i \omega_a t} + \psi_b(t) f_b(r) e^{-i \omega_b t} + \psi_c(t) f_c(r) e^{-i \omega_c t}$. Note that we have used $\omega_b$ instead of $\omega_c$ in the last term, which will be more convenient later on.

The frequency of the laser field is chosen such that it matches the transition from the ground state $a$ to the first excited state $b$ – but not from $b$ to $c$ or from $a$ to $c$. Furthermore, we assume that the ground state $a$ is strongly localized and hence does not couple significantly to the small and slow electric test field $\mathcal{E}(t)$ – whereas the states $b$ and $c$ are more spread out and thus do couple. Under
these circumstances, the insertion of the above expansion into the original Lagrangian in Eq. (10) and the application of the rotating wave approximation together with the dipole approximation yields the reduced Lagrangian

\[ \mathcal{L}_{\text{red}} = i\dot{\psi}_a^* \psi_a + i\dot{\psi}_b^* \psi_b + i\dot{\psi}_c^* \psi_c - \Delta \omega \psi_c^* \psi_c + + (\kappa \mathcal{E}(t) \psi_b^* \psi_c + \Omega(t) \psi_a^* \psi_b + \text{h.c.}) . \]  

(11)

Here \( \Delta \omega = \omega_c - \omega_b > 0 \) denotes the energy difference between the states \( a \) and \( b \), the dipole moment \( \kappa \) describes the coupling to the small and slow electric test field \( \mathcal{E}(t) \), and \( \Omega(t) \) is the Rabi frequency of the laser. Hence, the (approximated) equations of motion read \( i\dot{\psi}_a + \Omega(t) \psi_b = 0 \), \( i\dot{\psi}_b + \kappa \mathcal{E}(t) \psi_c + \Omega(t) \psi_a = 0 \), and \( i\dot{\psi}_c - \Delta \omega \psi_c + \kappa \mathcal{E}(t) \psi_b = 0 \). Since the test field \( \mathcal{E}(t) \) and thus the induced disturbance \( \psi_c \) are small (linear response), we may neglect the term \( \kappa \mathcal{E}(t) \psi_c \), such that the dynamics for \( \psi_a \) and \( \psi_b \) decouple and depend on \( \Omega(t) \) only. In the adiabatic régime, where the change of \( \mathcal{E}(t) \) is much slower than \( \Delta \omega \), we can solve the remaining equation for \( \psi_c \) via \( \psi_c(t) = \kappa \mathcal{E}(t) \psi_b(t)/\Delta \omega \), which is equivalent to a mixing of the states \( \psi_b \) and \( \psi_c \) due to the disturbance \( \mathcal{E}(t) \) as in stationary perturbation theory. In the adiabatic limit, we may integrate out (i.e., average over) the microscopic degrees of freedom \( \psi_{a,b,c} \) in the reduced Lagrangian in Eq. (11) leading to the contribution to the effective Lagrangian for the test field \( \mathcal{E}(t) \)

\[ \mathcal{L}_{\text{eff}} = |\psi_b(t)|^2 \frac{\kappa^2}{\Delta \omega} \mathcal{E}^2(t) . \]  

(12)

Obviously this corresponds to a varying dielectric \([14]\) permittivity \( \varepsilon(t) \) and thus capacitance \( C_n(t) \) which can be controlled (non-linear optics) by the laser field, i.e., its Rabi frequency \( \Omega(t) \). Note that, in contrast to resonant phenomena such as electro-magnetically induced transparency, this effective Lagrangian is valid for all adiabatic frequencies (i.e., far below \( \Delta \omega \)).

Discussion  As we have observed in the previous considerations, it is possible (under the assumptions and approximations made) to increase the capacitances in a controlled way by shining a laser beam on the material without necessarily generating dissipation and noise etc. Note that a decoherence-free modulation of \( \varepsilon(t) \) requires the dipole moment for the transition \( a \leftrightarrow b \) to be small and the laser beam to contain a large number of photons – such that it can be described classically as an external field, i.e., stimulated emission/absorption only. In contrast, the spontaneous decay back to ground state \( a \) is potentially associated with dissipation, noise and decoherence – and hence the life-time of the excited state \( b \) should be much longer than all other time-scales relevant for the effective geometry.

Fortunately, one can simulate a black hole horizon (for \( c \) decreases \( \leftrightarrow \varepsilon \) increases) by illuminating the material in its ground state \( a \). The white hole horizon occurs at the transition back to the ground state \( a \) and one has to make sure that the two horizons are far apart such that the noise generated by the white hole horizon does not propagate to the black hole horizon and prevents the detection of the Hawking radiation. The dispersion relation following from Eq. (3) reads (for \( k_y = 0 \))

\[ \omega^2 = \frac{4}{LC} \sin^2 \left( \frac{k\Delta x}{2} \right) \approx c^2 k^2 - \frac{\Delta \omega^4}{12LC} k^4 , \]  

(13)

and thus the group and phase velocities are (for moderate frequencies \([15]\)) less or equal to \( c^3 = \Delta \omega^2/(LC) \), i.e., we have a sub-luminal dispersion relation – which is typical for a lattice. As a result, disturbances from the white hole horizon (with moderate frequencies) cannot propagate to the black hole horizon.

In order to answer the main question of whether it will be possible to actually measure the Hawking effect one has to estimate the Hawking temperature. According to Eq. (8) the Hawking temperature is basically determined by the characteristic time scale on which \( c \) changes. For our microscopic model, this switching time is related to the Rabi frequency \( \Omega \), which must much smaller than the optical frequencies \( \omega_{a,b,c} \) for the rotating wave approximation to apply. Nevertheless, with strong laser pulses, it is possible to pump a large number of electrons in a semiconductor from the ground state into an excited state in 10-100 picoseconds – whereas the (spontaneous) decay back to the ground state can be much slower \([16]\). (In a semi-conductor, the effective one-particle potential \( V \) and hence the dipole moment \( \kappa \) and energy gap \( \Delta \omega \) can be very different from those of atomic transitions.) In this case, the order of magnitude of the Hawking temperature could be \( 10 - 100 \) mK, which is a really promising value since there already exist amplifiers and detectors (for micro-waves) with a noise temperature of order 10 mK, see e.g., \([17]\). Of course, one should cool down the apparatus below that temperature.

One advantage of the present proposal is that it allows for large velocities, say \( c_0/10-c_0/100 \) (in contrast to Bose-Einstein condensates \([4]\), for example, with a sound speed of order mm/s). Furthermore, there are no walls in relative motion with respect to the medium, which could lead to the Miles \([18]\) instability (via momentum transfer). With the above values, the thermal wave-length of the Hawking radiation would be of order millimeter – hence the minimum size \( \Delta x \) representing the analogue of the Planck length (knee wave-length) should be smaller than that. Consequently, it would be sufficient to illuminate a slab with a thickness way below one millimeter by the laser.

The power of the radiation can be inferred from the \((1+1 \text{ dimensional})\) energy-momentum tensor (in the comoving frame)

\[ \frac{dE}{dt} = T_0^1 = \frac{\pi}{12h} (k_B T_{\text{Hawking}})^2 , \]  

(14)
where we can get an order of magnitude between $10^{-14}$ and $10^{-16}$ Watt, which vastly exceeds the threshold in [17]. Owing to the Doppler effect (slow-down) the created power in the laboratory frame is even larger. The 1+1 dimensional character of the above expression reflects the fact that we have excluded transverse modes via assumption (1). With appropriate generalizations (e.g., $\lambda \gg \Delta y$), we could also reproduce higher (e.g., 2+1) dimensional behavior [11].

There are basically two major possibilities for the geometry of the set-up – a line or a circle. In both cases one may accumulate energy over some time by building a very long line (e.g., as a spiral) or using many revolutions in the circle. In the latter scenario, however, the problem of how to deal with the white hole horizon has to be solved.

The control of the space-time dependence of $c(t, x)$ can be achieved actively or passively: An active control could be realized via sweeping the laser beam externally (e.g., shining through holes in the lower capacitor plate), for example. Alternatively (passive control), one could arrange optical fibers filling the capacitors – either wound up around the capacitor plates or aligned along the wave-guide – in a way such that the linear or non-linear (e.g., self-focusing) optical pulse propagation exactly generates the desired behavior $c(t, x)$.

The Hawking effect could be measured by connecting an amplifying circuit (e.g., field effect transistors) to the end of the wave-guide (an alternative method would be a bolometer) subject to impedance matching – which can be achieved by manipulating $\mu/\varepsilon$, for example. In case the measurement is not fast enough, that amplifier should be disconnected (switched off) before the arrival of the horizon – inducing additional noise etc. The thermal spectrum of the Hawking radiation could be determined with successive band-passes.

In summary, the present proposal for an experimental verification of the Hawking effect (in black hole analogues) appears to be just at the edge of the present experimental capabilities – but not far beyond them (as is some other scenarios).

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