ON $K$-STABILITY OF $\mathbb{P}^3$ BLOWN UP ALONG THE DISJOINT UNION OF A TWISTED CUBIC CURVE AND A LINE

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Abstract. We find all $K$-polystable smooth Fano threefolds that can be obtained as blowup of $\mathbb{P}^3$ along the disjoint union of a twisted cubic curve and a line.

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1. Introduction

Smooth Fano threefolds defined over the field $\mathbb{C}$ have been classified in [11, 12, 14, 15] into 105 families. The detailed description of these families can be found in [2]. In [2] the following problem was posed:

Calabi Problem. Find all $K$-polystable smooth Fano threefolds in each family.

This problem was solved for 72 families. A great contribution to solving this problem was made by the authors of [2]. After their work only 34 families were left. The same year I. Cheltsov and J. Park obtained the result for one more family [3]. Suppose $X$ is a general member of the family №$\mathcal{N}$, then [2 Main Theorem] tells us that $X$ is $K$-polystable $\iff N \notin \left\{ 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, 5.2 \right\}$. Suppose $X$ is a member of Family 3.12. Then we describe $X$ as the blowup $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ at a twisted cubic $C$ and line $L$ that is disjoint from $C$ (see Section 3 for explicit
description of all members of this family). These threefolds form a one-dimensional family. Groups of automorphisms of such threefolds are finite except for one threefold which has automorphism group $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. It was shown in [2, §5.18] that this threefold is $K$-polystable. Moreover, it was used to show that the general member of this family is $K$-polystable. Furthermore, in [2, §7.7] it was shown that there exists a non $K$-polystable member in this family and it was conjectured that all other smooth Fano threefolds in Family 3.12 are $K$-polystable. The goal of this work is to prove this conjecture and complete the description of all $K$-polystable smooth Fano threefolds of Picard rank 3 and degree 28 started in [2].

**Main Theorem.** *All the smooth threefolds except one in Family 3.12 are $K$-polystable.*

Hence, all smooth Fano threefolds in Family 2.12 except one described in [2, §7.7] admit a Kähler-Einstein metric.

1.1. **Plan of the paper.** In Section 2 we state the results which will use to prove Main Theorem. In Section 3 we will discuss the equivariant geometry of $\mathbb{P}^3$ which will help us to understand the equivariant geometry of $X$ in Family 3.12. We will focus our attention on the members in Family 3.12 for which the $K$-polystability has not been proved yet. In this section we show that $\text{Aut}(X) \cong \text{Aut}(\mathbb{P}^3, L+C)$ contains a subgroup $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We will show that there are no $G$-fixed points on $\mathbb{P}^3$, describe $G$-invariant quadrics containing $C$ on $\mathbb{P}^3$ and $G$-invariant lines on $\mathbb{P}^3$. At the end of this section we give description of the Mori cone and the cone of effective divisors on $X$. Finally, in Section 4 we prove our Main Theorem.

1.2. **Plan of the proof.** If $X$ is not $K$-polystable then it follows from [20, Corollary 4.14] that there exists a $G$-invariant prime divisor $F$ over $X$ such that $\beta(F) \leq 0$ where $\beta(F)$ was defined in [10], see also [2, Definition 1.2.1] and Section 2. Let $Z$ be the center of $F$ on $X$. Then $Z$ is not a point since $X$ has no $G$-fixed points, and $Z$ is not a surface by [9, Theorem 10.1], so that $Z$ is a $G$-invariant irreducible curve. Then we derive a contradiction as follows:

1. We use Abban-Zhuang theory (see [11]) and its corollary [2] Collorary 1.7.26] to exclude the case when $\pi(Z)$ is a line such that $\pi(Z) \neq L$ and $\pi(Z) \cap C = \emptyset$. This is done in Lemma 4.1.
2. We use Abban-Zhuang theory ([11] and its corollary [2] Collorary 1.7.26] to exclude the case when $\pi(Z) \subset L$. This is done in Lemma 4.2.
3. We use Abban-Zhuang theory (see [11]) and its corollary [2] Collorary 1.7.26] to show that $\pi(Z)$ is not contained in a $G$-invariant quadric passing through $C$. This is done in Lemma 4.3.
4. Using 1, 2, 3 we deduce that $\pi(Z)$ is not a line.
5. It follows from $\beta(F) \leq 0$ that $Z$ is contained in $\text{Nklt}(X, \lambda D)$ for some $G$-invariant effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ and $\lambda \in \mathbb{Q}$ such that $\lambda < \frac{3}{4}$. Moreover it follows from 2, 3 and description of the cone of effective divisors on $X$ that $Z$ is not contained in the surface in $\text{Nklt}(X, \lambda D)$. See Corollary 4.6.
6. Using 5 we derive that $\pi(Z) \not\subset C$. This is done in Corollary 4.7.
7. Finally, we use 6 to show that $\pi(Z)$ is a line in $\mathbb{P}^3$, which contradicts 4.
2. Preliminary results

Let $X$ be a Fano variety with Kawamata log terminal singularities, let $G$ be a reductive subgroup in $\text{Aut}(X)$, let $f : \tilde{X} \to X$ be a $G$-equivariant birational morphism, let $F$ be a $G$-invariant prime divisor in $\tilde{X}$, and let $n = \dim(X)$.

**Definition 2.1.** We say that $F$ is a $G$-invariant prime divisor over the Fano variety $X$. If $F$ is $f$-exceptional, we say that $F$ is an exceptional $G$-invariant prime divisor over $X$. We will denote the subvariety $f(F)$ by $C_X(F)$.

Let $S_X(F) = \frac{1}{(-K_X)^n} \int_0^\tau \text{vol}(f^*(-K_X) - xF)dx$, where $\tau = \tau(F)$ is the pseudo-effective threshold of $F$ with respect to $-K_X$, i.e. we have

$$\tau(F) = \sup \left\{ u \in \mathbb{Q}_{>0} \mid f^*(-K_X) - uF \text{ is big} \right\}.$$  

Let $\beta(F) = A_X(F) - S_X(F)$, where $A_X(F)$ is the log discrepancy of the divisor $F$.

**Theorem 2.2 ([9] Corollary 4.14]).** Suppose that $\beta(F) > 0$ for every $G$-invariant prime divisor $F$ over $X$. Then $X$ is $K$-polystable.

**Theorem 2.3 ([9] Theorem 10.1).** Let $X$ be any smooth Fano threefold that is not contained in the following 41 deformation families:

$$\#1.17, \#2.23, \#2.26, \#2.28, \#2.30, \#2.31, \#2.33, \#2.34, \#2.35, \#2.36, \#3.9, \#3.14, \#3.16, \#3.18, \#3.19, \#3.21, \#3.22, \#3.23, \#3.24, \#3.25, \#3.26, \#3.28, \#3.29, \#3.30, \#3.31, \#4.2, \#4.4, \#4.5, \#4.7, \#4.8, \#4.9, \#4.10, \#4.11, \#4.12, \#5.2, \#5.3, \#6.1, \#7.1, \#8.1, \#9.1, \#10.1.$$  

Then $S_X(Y) < 1$ for every irreducible surface $Y \subset X$, i.e. $X$ is divisorially stable.

**Theorem 2.4 ([2] Corollary 1.7.26]).** Let $X$ be a smooth Fano threefold, let $Y$ be an irreducible normal surface in the threefold $X$, let $Z$ be an irreducible curve in $Y$, and let $F$ be a prime divisor over the threefold $X$ such that $C_X(F) = Z$. Then

$$\frac{A_X(F)}{S_X(F)} \geq \min \left\{ \frac{1}{S_X(Y)}, \frac{1}{S(W^Y; Z)} \right\}.$$  

and

$$S(W^Y; Z) = \frac{3}{(-K_X)^3} \int_0^\tau (P(u))^2 \cdot \text{ord}_Z(N(u)|_Y)du + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_Y - vZ)dudv,$$

where $P(u)$ is the positive part of the Zariski decomposition of the divisor $-K_X - uY$, and $N(u)$ is its negative part.

**Lemma 2.5 ([2] Lemma 1.4.4]).** Let $X$ be a Fano variety with at most Kawamata log terminal singularities of dimension $n \geq 2$, $Z$ be a proper irreducible subvariety in $X$. Let $f : \tilde{X} \to X$ be an arbitrary $G$-equivariant birational morphism, let $F$ be a $G$-invariant prime divisor in $\tilde{X}$ such that $Z \subseteq f(F)$, and let $\tau(F)$ satisfy [2.0.1]. Suppose in addition that $X$ is smooth and $\dim(Z) \geq 1$. Then

$$\frac{A_X(F)}{S_X(F)} > \frac{n + 1}{n} \alpha_{G,Z}(X),$$
where
\[ \alpha_{G,Z}(X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the pair } (X, \lambda D) \text{ is log canonical at general point of } Z \text{ for any effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X \right\}. \]

Lemma 2.6 ([2, Corollary A.13]). Suppose \( X = \mathbb{P}^3 \) and \( B_X \sim_{\mathbb{Q}} -\lambda K_X \) for some rational number \( \lambda < \frac{3}{4} \). Let \( Z \) be the union of one-dimensional components of \( \text{Nklt}(X, B_X) \). Then \( \mathcal{O}_{\mathbb{P}^3}(1) \cdot Z \leq 1 \).

Lemma 2.7 ([2, Corollary A.15]). Suppose that \( X \) is a smooth Fano threefold, \( -K_X \) is nef and big, \( B_X \sim_{\mathbb{Q}} -\lambda K_X \) for some rational number \( \lambda < 1 \), and there exists a surjective morphism with connected fibers \( \phi : X \to \mathbb{P}^1 \). Set \( H = \phi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \). Let \( Z \) be the union of one-dimensional components of \( \text{Nklt}(X, \lambda B_X) \). Then \( H \cdot Z \leq 1 \).

3. Geometry of Fano Threefolds in Family №3.12

3.1. Basic properties of Fano Threefolds in Family №3.12. Let \( C \) be the smooth twisted cubic curve in \( \mathbb{P}^3 \) that is the image of the map \( \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \) given by
\[ [x : y] \to [x^3 : x^2y : xy^2 : y^3] \]
let \( L \) be a line in \( \mathbb{P}^3 \) that is disjoint from \( C \), and let \( \pi : X \to \mathbb{P}^3 \) be the blow up of \( \mathbb{P}^3 \) along \( C \) and \( L \). Then \( X \) is a Fano threefold in family 3.12 and all threefolds in this family can be obtained this way. Note that there exists the following commutative diagram:

Where:
- \( \varphi \) is the blowup of a line \( L \),
- \( \theta \) is the blowup of a curve \( C \),
- \( \phi \) is the blowup of a curve \( \varphi^* L \),
- \( \theta \) is the blowup of a curve \( \varphi^* C \),
- the left dashed arrow is the linear projection from the line \( L \),
- the right dashed arrow is given by the linear system of quadrics that contain \( C \),
- \( \xi \) is a \( \mathbb{P}^1 \)-bundle,
- \( \nu \) is a \( \mathbb{P}^2 \)-bundle,
- \( \sigma \) is a non-standard conic bundle,
- \( \eta \) is a fibration into the del Pezzo surfaces of degree 6,
- \( \zeta \) is the contraction of the proper transforms of the quartic surface in \( \mathbb{P}^3 \) that is spanned by the secants of the curve \( C \) that intersect \( L \),
- \( \text{pr}_1 \) and \( \text{pr}_2 \) are projections to the first and the second factors, respectively.
Let $H$ be a plane in $\mathbb{P}^3$, $E_L$ be the exceptional surface of $\pi$ that is mapped to $L$, $E_C$ be the exceptional surface of $\pi$ that is mapped to $C$, $R$ be $\zeta$-exceptional surface. Then

$$R \sim Q \pi^* (4H) - 2E_C - E_L,$$

and

$$-K_X \sim Q \pi^* (4H) - E_C - E_L.$$

### 3.2. Construction of $R$

Consider the commutative diagram:

$$
\begin{array}{cccc}
X & \phi & \rightarrow & V \\
\downarrow & & \downarrow & \\
\mathbb{P}^3 & \pi & \rightarrow & \mathbb{P}^2
\end{array}
$$

Where $\xi$ is a $\mathbb{P}^1$-bundle given by the linear system $|\vartheta^*(2H) - E_C|$, $\vartheta$ is the blowup of $C$ and the dashed arrow is given by the linear system of quadrics containing $C$, $\phi$ is the blowup of $\vartheta^* L$. Denote $\tilde{L} = \vartheta^* L$. What is the image of $\tilde{L}$ in $\mathbb{P}^2$? We have that

$$\tilde{L} \cdot (\vartheta^*(2H) - E_C) = 2$$

which means that $\xi(\tilde{L})$ is a conic. The preimage of this conic are all the secants of $C$ which intersect $\tilde{L}$. Therefore, $\pi(R)$ is spanned by secants of $C$ that intersect $L$. Note that the class of the preimage is

$$\xi^*(\mathcal{O}_{\mathbb{P}^2}(2)) = 2(\vartheta^*(2H) - E_C) = \vartheta^*(4H) - 2E_C.$$

Note that moreover that $\xi(\tilde{L})$ is a smooth conic and $\xi$ is a $\mathbb{P}^1$-bundle thus the preimage of $\xi(\tilde{L})$ is a smooth surface so it is smooth along $\tilde{L}$ thus the class of $R$ in $\mathbb{P}^3$ is given by

$$R \sim Q \pi^* (4H) - 2E_C - E_L.$$

### 3.3. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{P}^3$ and Fano threefolds in Family 3.12

Note that $\text{Aut}(X) \cong \text{Aut}(X, C + L)$. On the other hand, we have

$$\text{Aut}(\mathbb{P}^3, C) = \text{PGL}_2(\mathbb{C}),$$

where $\text{Aut}(\mathbb{P}^3, C)$ is the group of automorphisms of $\mathbb{P}^3$ which fix $C$ as a set.

### 3.3.1. Types of threefolds in Family 3.12

We look at the projection from the line $L$ which is disjoint from $C$ to $\mathbb{P}^1$:

$$\phi_L : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1,$$

which gives a 3-cover of $\mathbb{P}^1$:

$$\phi_L|_C : C \rightarrow^{3:1} \mathbb{P}^1.$$
there is one ramification point of multiplicity 3 and two ramification points of multiplicity 2,

there are four ramification points of multiplicity 2.

We see that there are at least two ramification points on \( C \). By acting on \( C \) by the \( \text{PGL}(2, \mathbb{C}) \) we can make these points to be \( p_1 = [1 : 0] \), \( p_2 = [0 : 1] \) on \( C \). Now we look at the line \( L \). It is the intersection of 2 planes which are tangent to \( C \) at points \( p_1 \) and \( p_2 \) (note that these planes are different since the plane intersects the cubic \( C \) in three points, so the same plane cannot be tangent to \( C \) at two points) so it is given by the equations:

\[
L : \begin{cases}
x_0 = r_1 x_1, \\
x_3 = r_2 x_2.
\end{cases}
\]

Now we have 3 cases:

1. \( r_1 = r_2 = 0 \) so \( L \) is given by the equations:

\[
L : \begin{cases}
x_0 = 0, \\
x_3 = 0.
\end{cases}
\]

Here we have two ramification points of multiplicity 3. This case was described in [2]. The corresponding threefold \( X \) is \( K \)-polystable in this case.

2. \( r_1 = 0, r_2 \neq 0 \) (which is symmetric to the case \( r_1 \neq 0, r_2 = 0 \)) so \( L \) is given by the equations:

\[
L : \begin{cases}
x_0 = 0, \\
x_3 = r_2 x_2.
\end{cases}
\]

Using the action of \( \mathbb{C}^* \) by the matrix which fixes \( C \):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & r_2 & 0 & 0 \\
0 & 0 & r_2^2 & 0 \\
0 & 0 & 0 & r_2^3
\end{pmatrix}
\]

We can assume that \( L \) is given by

\[
L : \begin{cases}
x_0 = 0, \\
x_3 = x_2.
\end{cases}
\]

Here we have one ramification point of multiplicity 3 and two ramification points of multiplicity 2. This case was described in [2] where it was proved in that \( X \) is not \( K \)-polystable.

3. \( r_1 \neq 0, r_2 \neq 0 \) so \( L \) is given by the equations

\[
L : \begin{cases}
x_0 = r_1 x_1, \\
x_3 = r_2 x_2.
\end{cases}
\]
Using the action of \( \mathbb{C}^* \) by the matrix which fixes \( C \):
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3
\end{pmatrix},
\]
where \( \lambda \) satisfies \( \lambda^2 = \frac{r_1}{r} \). We can assume that \( L \) is given by
\[
L : \begin{cases}
x_0 = rx_1, \\ x_3 = rx_2.
\end{cases}
\]
Note that:
- \( r \neq 0 \) since otherwise we are in case (1),
- \( r \neq \pm 1 \) since otherwise \( L \) intersects \( C \) which is prohibited,
- \( r \neq \pm 3 \) since otherwise there exists a plane containing \( L \) which is tangent to \( C \) with multiplicity 3 (it is a plane given by \( x_3 + 3x_3 + 3x_1 + x_0 = 0 \) in case \( r = -3 \) and a plane \( -x_3 + 3x_3 - 3x_1 + x_0 = 0 \) in case \( r = 3 \)) so this case is projectively isomorphic to the case (2).

Now the involution on \( \mathbb{P}^3 \) given by \([x : y : z : w] \mapsto [w : z : y : x] \) fixes \( C \) an \( L \). We can do it for any pair of four ramification points on \( C \cong \mathbb{P}^1 \). This gives the action of \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). More precisely this group is generated by the involutions viewed on \( \mathbb{P}^1 \):
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & \frac{-r(r^2-5+(r^4-10r^2+9)^{1/2})}{2(r^2-3+(r^4-10r^2+9)^{1/2})} \\
\frac{r^2+3+(r^4-10r^2+9)^{1/2}}{4r} & -1
\end{pmatrix}
\]
The action on \( \mathbb{P}^3 \) is given by the map induced by \([x : y] \mapsto [x^3 : x^2y : xy^2 : y^3] \).

3.3.2. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-fixed points on \( X \). From now on we assume until the end of this section that we are in case (3) of the previous part and \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). In particular, \( \text{Aut}(X) \) is finite.

**Lemma 3.1.** There are no \( G \)-invariant planes in \( \mathbb{P}^3 \)

**Proof.** Note that \( G \hookrightarrow \text{Aut}(C) \) since \( C \) is a spatial curve. If there exists a \( G \)-invariant plane \( \Pi \) consider the intersection of \( \Pi \) with \( C \). There are three points in \( \Pi \cap C \) counted with multiplicities. Thus, since the order of \( G \) is 4 then there is a \( G \)-fixed point on \( C \cong \mathbb{P}^1 \), which is a contradiction. \( \square \)

**Corollary 3.2.** There are no \( G \)-fixed points in \( \mathbb{P}^3 \).

**Corollary 3.3.** The threefold \( X \) does not contain \( G \)-invariant points.

3.3.3. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-invariant Quadrics Containing \( C \). Let \( M \) be the linear system of quadrics in \( \mathbb{P}^3 \) that contain the curve \( C \).

**Lemma 3.4.** The linear system \( M \) is 3-dimensional, it contains exactly 3 \( G \)-invariant surfaces, and these surfaces are smooth.

**Proof.** Note that all this statement does not depend on the equivariant choice of coordinates. We know that groups isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) are conjugate in \( \text{PGL}_2(\mathbb{C}) \) so we can choose coordinates such that the generators of our group will look like:
\[
\tau_1 : [x : y] \mapsto [y : x],
\]
\[ \tau_2 : [x : y] \rightarrow [x : -y]. \]

Which gives us the action on \( \mathbb{P}^3 \) by:

\[
\tau_1 : [x_0 : x_1 : x_2 : x_3] \rightarrow [x_3 : x_2 : x_1 : x_0],
\]

\[
\tau_2 : [x_0 : x_1 : x_2 : x_3] \rightarrow [x_0 : -x_1 : x_2 : -x_3].
\]

The linear system \( \mathcal{M} \) is clearly 3-dimensional. We can provide the equations for 3 \( G \)-invariant quadrics containing \( C \):

\[
Q_1 : x_0x_3 = x_1x_2, \quad Q_2 : x_1^2 + x_2^2 = x_0x_2 + x_1x_3, \quad Q_3 : x_1^2 - x_2^2 = x_0x_2 - x_1x_3.
\]

Note that \((\tau_1, \tau_2)\) acts on the equation of:

- \( Q_1 \) by multiplying it by \((1, -1)\),
- \( Q_2 \) by multiplying it by \((1, 1)\),
- \( Q_3 \) by multiplying it by \((-1, 1)\).

Thus since all the pairs are different and \( \mathcal{M} \) is 3-dimensional there are exactly 3 \( G \)-invariant quadrics which we listed above. Note that these quadrics are smooth.

Now take a \( G \)-invariant quadric \( Q \in \mathcal{M} \) and look at the intersection of it with \( L \). Note that \( L \not\subset Q \) since \( L \) does not intersect \( C \). The intersection \( Q \cap L \) consists of two distinct points. This pair of points does not lie on curves of bidegree \((1, 0)\) or \((0, 1)\) (since these are the lines on \( Q \) and we know that \( L \not\subset Q \)). Now we see that the blowup \( \tilde{Q} \rightarrow Q \) at these points is a del Pezzo surface of degree 6.

### 3.3.4. \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-invariant lines

Let us describe \( G \)-invariant lines in \( \mathbb{P}^3 \). As in proof of Lemma 3.4, we may assume that \( G \) is generated by

\[
\tau_1 : [x : y] \rightarrow [y : x],
\]

\[
\tau_2 : [x : y] \rightarrow [x : -y].
\]

Which gives us the action on \( \mathbb{P}^3 \) by:

\[
\tau_1 : [x_0 : x_1 : x_2 : x_3] \rightarrow [x_3 : x_2 : x_1 : x_0],
\]

\[
\tau_2 : [x_0 : x_1 : x_2 : x_3] \rightarrow [x_0 : -x_1 : x_2 : -x_3].
\]

In this case all \( G \)-invariant lines are of the form:

\[
\begin{cases}
\lambda x_0 + \mu x_2 = 0, \\
\lambda x_3 + \mu x_1 = 0,
\end{cases}
\]

where \([\lambda : \mu] \in \mathbb{P}^1\). All such lines do not intersect each other and lie on the quadric \( Q_4 \) given by \( x_1x_0 = x_2x_3 \). We see that \( \mathbb{P}^3 \) contains infinitely many \( G \)-invariant lines and all of them are contained in \( Q_4 \). Among them there are 3 lines that intersect \( C \). We can describe them explicitly. Let’s now look at the intersection of this quadric with \( C \). There
are exactly 6 such points:

\[ P_1 = [0 : 1] = [0 : 0 : 0 : 1] , \]
\[ P_2 = [1 : 0] = [1 : 0 : 0 : 0] , \]
\[ P_3 = [1 : 1] = [1 : 1 : 1 : 1] , \]
\[ P_4 = [1 : -1] = [1 : -1 : 1 : -1] , \]
\[ P_5 = [1 : i] = [1 : i : -1 : -i] , \]
\[ P_6 = [1 : -i] = [1 : -i : 1 : i] , \]

here in the third column are given the corresponding coordinates on \( C \subset \mathbb{P}^3 \). Note that \( \tau_1 \) exchanges \( P_1 \) and \( P_2 \), \( \tau_2 \) exchanges \( P_3 \) and \( P_4 \), \( \tau_2 \) exchanges \( P_5 \) and \( P_6 \) and we obtain three pairs of points which belong to the same line (different one for each pair). We denote these lines \( L_{12} , L_{34} , L_{56} \), where \( L_{ij} \) is the line connecting points \( P_i \) and \( P_j \).

**Lemma 3.5.** Suppose that \( Z \) is an irreducible curve on \( X \), \( \pi(Z) \) is its image on \( \mathbb{P}^3 \) and \( \pi(Z) \) is a line different from \( L \), \( L_{12} \), \( L_{34} \), \( L_{56} \) then \( Z \not\subset R \).

**Proof.** Suppose \( Z \) is contained in \( R \). Consider the following commutative diagram from section 3.2:

\[
\begin{array}{c}
R \\ \phi \downarrow \\
\pi \\
\pi(R) \subset \mathbb{P}^3 \rightarrow \mathbb{P}^2 \ni \xi \circ \phi(R)
\end{array}
\]

Where the bottom dashed arrow is given by the linear system of quadrics containing \( C \). Using the equations of quadrics which form the basis of the linear system \( \mathcal{M} \) defined in Section 3.3.3 we get the explicit map:

\[ \mathbb{P}^3 \rightarrow \mathbb{P}^2 \text{ where } [x_0 : x_1 : x_2 : x_3] \rightarrow [x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3]. \]

We know that \( \xi \circ \phi(R) \) is a conic. Let’s write its equation in \( \mathbb{P}^2 \) with coordinates \([x : y : z] : a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 = 0\).

We want to look at the preimage of this equation in \( \mathbb{P}^3 \) which will give the equation for \( \pi(R) \). Substituting \([x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3] \) into the defining equation of \( \xi \circ \phi(R) \) we get:

\[ \pi(R) : a_1x_3^2x_0^2 - a_2x_2x_3x_0^2 + a_4x_2^2x_0^2 + a_2x_3x_0x_1^2 - 2a_4x_2x_0x_1^2 + a_2x_2^2x_0x_1 + (-2a_1 + a_5)x_2x_3x_0x_1 - a_3x_3^2x_0x_1 + a_3x_3x_2^2x_0 - a_5x_3^3x_0 + a_4x_1^4 - a_2x_1^3x_2 - a_5x_3x_1^3 + (a_1 + a_5)x_2^2x_1^2 + a_3x_2x_3x_1^2 + a_6x_3^2x_1^2 - a_3x_3^2x_1 - 2a_6x_2^2x_3x_1 + a_6x_4^4 = 0. \]

Recall from section 3.3.4 that all \( G \)-invariant lines are of the form

\[
\begin{cases}
\lambda x_0 + \mu x_2 = 0, \\
\lambda x_3 + \mu x_1 = 0,
\end{cases}
\]
where $[\lambda : \mu] \in \mathbb{P}^1$. Now $L$ is given by

$$L = L_s : \begin{cases} x_0 + s x_2 = 0, \\ x_3 + s x_1 = 0, \end{cases}$$

for $s \in \mathbb{C}$. Note that $s \neq 0$ since otherwise $X$ would have an infinite group of automorphisms. Similarly $\pi(Z)$ is given by

$$\pi(Z) = L_t : \begin{cases} x_0 + t x_2 = 0, \\ x_3 + t x_1 = 0, \end{cases}$$

for $t \in \mathbb{C}$. By our assumption $L_s$ is contained in $\pi(R)$. This gives

$$\{a_1 = -1/s, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = (s^2 + 1)/s, a_6 = 1\}.$$  

So that $\pi(R)$ is given by

$$\pi(R) : x_2^2 x_0^2 s - x_3^2 x_0^2 - 2 x_2 x_0 x_1^2 s + (s^2 + 3) x_2 x_3 x_0 x_1 + (-s^2 - 1) x_3^2 x_0 + s x_1^4 +$$

$$+(-s^2 - 1) x_3 x_1^3 + s^2 x_1^2 x_2 + x_3 x_1^2 s - 2 x_2 x_3 x_1 s + s x_2^4 = 0.$$ 

Similarly since $L_t$ is contained in $\pi(R)$ we get

$$\begin{cases} -t^4 - 4 ts + (s^2 + 3)t^2 + s^2 = 0, \\ s + (-s^2 - 1)t + t^2 s = 0. \end{cases}$$

and the solution to this system is $s = t$ which means that $L_s = L$ and $L_t = \pi(Z)$ coincide contradicting the assumption on $Z$.  

\[\square\]

Remark 3.6. Let us use the assumptions and the notations from the proof of Lemma 3.5.

There is another way to show that the points in $\{b_1, b_2, b_3, b_4\}$ are in general position which as discussed above is equivalent to showing that $Z \not\subset R$. Suppose the opposite, i.e. that $Z \subset R$. Now as in the proof of lemma above we take suitable $t, s \in \mathbb{C}$ such that $L_s = L$ and $L_t = \pi(Z)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\pi} & X \\
\sigma \downarrow \quad & & \downarrow \\
\mathbb{P}^2 & \xrightarrow{\sigma} & \mathbb{P}^2 \\
\end{array}$$

Where the bottom dashed arrow is given by the linear system $\mathcal{M}$ of quadrics containing $C$ and a morphism $\sigma|_{\tilde{Q}_4}$ is given by the linear system $|\pi^*(2H) - E_C|$. Now we restrict this diagram to the quadric $Q_4$ which is the quadric which consists of $G$-invariant curves, $\tilde{Q}_4$ is the strict transform of $Q_4$ on $X$:

$$\begin{array}{ccc}
\tilde{Q}_4 & \xrightarrow{\pi|_{\tilde{Q}_4}} & \mathbb{P}^2 \\
\sigma|_{\tilde{Q}_4} \downarrow \quad & & \downarrow \\
Q_4 & \xrightarrow{\mathcal{M}|_{Q_4}} & \mathbb{P}^2 \\
\end{array}$$

The restriction $\pi|_{\tilde{Q}_4}$ is the blow up of the intersection points $C \cap Q_4 = \{P_1, \ldots, P_6\}$, which we described in 3.3.3. Observe that $\tilde{Q}_4$ is not a del Pezzo surface. Indeed, let $\tilde{L}_{ij}$ be in $\{\tilde{L}_{12}, \tilde{L}_{34}, \tilde{L}_{56}\}$ (here $L_{ij}$ is line connecting $P_i$ and $P_j$) then $\tilde{L}_{12}, \tilde{L}_{34}, \tilde{L}_{56}$ where $\tilde{L}_{ij}$ is the
strict transform of the line \( L_{ij} \) are \((-2)\)-curves in \( \bar{Q}_4 \) since the lines \( L_{12}, L_{34}, L_{56} \) lie in \( Q_4 \). So that they have trivial intersection with \(-K_{\bar{Q}_4}\). In fact, using the coordinates of the points \( P_1, ..., P_6 \) and the equation of \( Q_4 \) one can show that the lines \( L_{ij} \) are the only secants of the curve \( C \) that are contained in \( Q_4 \). On the other hand,

\[-K_{\bar{Q}_4} \sim (\pi^*(2H) - E_C)|_{\bar{Q}_4},\]

so that \(-K_{\bar{Q}_4}\) is nef and big, and the only curves in \( \bar{Q}_4 \) that has trivial intersection with \(-K_{\bar{Q}_4}\) are the three curves \( \bar{L}_{ij} \). Note that these curves do not intersect the curve \( Z \).

Taking the Stein factorization of the morphism \( \sigma|_{\bar{Q}_4} : \bar{Q}_4 \to \mathbb{P}^2 \), we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\bar{Q}_4 & \xrightarrow{\text{contraction of \((-2)\) curves}} & \bar{Q}_4 \\
\sigma|_{\bar{Q}_4} \downarrow & & \downarrow \beta \\
Q_4 & \xrightarrow{\pi|_{Q_4}} & \mathbb{P}^2 \\
\end{array}
\]

where \( Q_4 \) is a singular del Pezzo surface of degree 2 with three singular points of type \( A_1 \), \( \beta \) is the double cover given by \(|-K_{\bar{Q}_4}|\), and the dasharrow is the rational map given by the restriction of the linear system \( \mathcal{M} \) to \( Q_4 \). Suppose \( \bar{L}_s, \bar{L}_t \) are the images of \( \bar{L}_s, \bar{L}_t \) which are the strict transforms of \( L_s \) and \( L_t \) on \( Q_4 \). \( \bar{L}_s \) and \( \bar{L}_t \) do not pass through singular points, because \( L_s \) and \( L_t \) are disjoint from the lines \( L_{ij} \). Since both \( L_t, L_s \subset \pi(R) \) by assumption then by construction of \( R \) given in Section 3.2 we get that \( \beta(\bar{L}_s) = \beta(\bar{L}_t) \) is the same conic in \( \mathbb{P}^2 \). So we see that:

\[\bar{L}_s + \bar{L}_t = \beta^*(\mathcal{O}_{\mathbb{P}^2}(2)) = -2K_{\bar{Q}_4}.\]

By the adjunction formula we have:

\[K_{\bar{Q}_4} \cdot \bar{L}_s + \bar{L}_s^2 = -2 \Rightarrow -K_{\bar{Q}_4} \cdot \bar{L}_s = 2.\]

So we get

\[0 = \bar{L}_s \cdot \bar{L}_s + \bar{L}_t \cdot \bar{L}_s = -2K_{\bar{Q}_4} \cdot \bar{L}_s = 4.\]

Which gives us a contradiction. Thus \( Z \not\subset R. \)

3.4. Mori Cone. Let \( l_L, l_C, l_R \) be the general fibers of the natural projections \( E_L \to L, E_C \to C, R \to \sigma(R) \). Observe that we can contract any of two rays \( \mathbb{R}_{\geq 0}[l_L], \mathbb{R}_{\geq 0}[l_C], \mathbb{R}_{\geq 0}[l_R] \). Indeed \( l_C \) and \( l_L \) are contracted by \( \pi : X \to \mathbb{P}^3 \), \( l_R \) and \( l_C \) are contracted by \( \eta : X \to \mathbb{P}^1 \). Thus, these curves generate 3 extreme rays \( \mathbb{R}_{\geq 0}[l_L], \mathbb{R}_{\geq 0}[l_C], \mathbb{R}_{\geq 0}[l_R] \) of the Mori cone \( \text{NE}(X) \).

3.5. Cone of Effective Divisors.  

Lemma 3.7. Suppose \( S \) is a surface in \( X \) then

\[S \sim a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C,\]

for \( a, b, c, e, f \in \mathbb{Z}_{\geq 0} \).

Proof. Suppose \( \pi(S) \subset \mathbb{P}^3 \) is the surface of degree \( d \) in \( \mathbb{P}^3 \). Then we have

\[S \sim d\pi^*(H) - m_LE_L - m_CE_C,\]

where \( m_L \) is the multiplicity of \( \pi(S) \) in \( L \), \( m_C \) is the multiplicity of \( \pi(S) \) in \( C \). Suppose that \( S \neq E_C, S \neq E_L \) and \( S \neq R \) for all \( n \). Now let’s intersect \( S \) with three extreme rays
\(l, l_C, l_R\) corresponding to \(L, C, R\):

- \(\pi^*(H) \cdot l_C = 0,\)
- \(\pi^*(H) \cdot l_L = 0,\)
- \(\pi^*(H) \cdot l_R = 1,\)
- \(E_L \cdot l_C = 0,\)
- \(E_L \cdot l_L = -1,\)
- \(E_L \cdot l_R = 1,\)
- \(E_C \cdot l_C = -1,\)
- \(E_C \cdot l_L = 0,\)
- \(E_C \cdot l_R = 2.\)

So we have that:

\[
S \cdot l_C = m_C \geq 0, \quad S \cdot l_L = m_L \geq 0, \quad S \cdot l_R = d - m_L - 2m_C \geq 0.
\]

Moreover if \(l_1\) is the general line intersecting \(L, l_2\) is the general secant of \(C\) then we get strict inequalities:

\[
S \cdot l_1 = d - m_L > 0, \quad S \cdot l_2 = d - 2m_C > 0
\]

Now we want to find the integer positive solutions for:

\[
d\pi^*(H) - m_L E_L - m_C E_C = a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C.
\]

Comparing the coefficients we get:

\[
d = a + 2b + 4c, \quad m_C = b + 2c - f, \quad m_L = a + c - e.
\]

The non-negative solution to this system can be given by

\[
\begin{cases}
  a = -2m_C + d, \\
  b = m_C, \\
  c = 0, \\
  e = -m_L - 2m_C + d, \\
  f = 0.
\end{cases}
\]

Thus, the cone of effective divisors over \(\mathbb{Z}\) is generated by \(\pi^*(H) - E_L, 2\pi^*(H) - E_C, R, E_L, E_C.\)

\[\square\]

**Corollary 3.8.** Cone of effective divisors of \(X\) is generated over \(\mathbb{Q}\) by \(\pi^*(H) - E_L, R, E_L, E_C.\) More precisely, suppose \(S\) is a surface in \(X\) then

\[
S \sim \mathbb{Q} a(\pi^*(H) - E_L) + cR + eE_L + fE_C,
\]

for unique \(a, c, e, f \in \mathbb{Q}_{\geq 0}\).

\[\square\]

### 4. Proof of the Main Theorem

Let \(C\) be a twisted cubic in \(\mathbb{P}^3\) that is the image of the map \(\mathbb{P}^1 \hookrightarrow \mathbb{P}^3\) given by

\[
x : y \rightarrow [x^3 : x^2y : xy^2 : y^3],
\]

and \(L\) be a line in \(\mathbb{P}^3\) that is disjoint from \(C\) given by

\[
L : \begin{cases}
x_0 = rx_1, \\
x_3 = rx_2
\end{cases}
\]

where \(r \neq 0, r \neq \pm 1, r \neq \pm 3\) in the coordinates presented in section 3.3.1. \(X\) is a smooth Fano 3-folds in Family 3.12 obtained by blowing up \(\mathbb{P}^3\) in \(C\) and \(L\), and \(G\) is the subgroup in \(\text{Aut}(X)\) such that \(G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) described in Section 3.1.

Suppose \(X\) is not \(K\)-polystable. By Theorem 2.2 there exists a \(G\)-invariant prime divisor \(F\) over \(X\) such that \(\beta(F) = A_X(F) - S_X(F) \leq 0\). Let us seek for a contradiction. Let \(Z = C_X(F)\). Then \(Z\) is not a point by Corollary 3.1, and \(Z\) is not a surface by Theorem 2.3, so that \(Z\) is a \(G\)-invariant irreducible curve.
Lemma 4.1. Suppose that \( \pi(Z) \neq L \) then \( \pi(Z) \) is not one of the \( G \)-invariant lines which does not intersect \( C \).

Proof. Let’s take a \( G \)-invariant line \( \pi(Z) \) that does not intersect \( C \) and consider a plane \( H \) which contains this line. It intersects a line \( L \) in one point and a twisted cubic \( C \) at three points. Let \( S \) be the proper transform of \( H \) on \( X \). In this case we have that the induced map \( \pi|_S : S \to H \) is the blowup of a plane \( H \) in 4 points \( b_1 = H \cap L, b_2, b_3, b_4 = H \cap C \).

We now need to check that these points are in general position to conclude that \( S \) is a del Pezzo surface of degree 5.

To prove that we need to show that the points in \( \{b_1, b_2, b_3, b_4\} \) are in general position which means that no three of them belong to the same line. Note that \( b_2, b_3, b_4 = H \cap C \) does not belong to the same line, because \( C \) is an intersection of quadrics. So the only option is that \( b_1 \) and two points from the set \( \{b_2, b_3, b_4\} \) belongs to the same line. Suppose \( H \) is a general plane and \( b_1 \) and 2 points among \( \{b_2, b_3, b_4\} \) are contained in one line \( \ell \).

From Section 3.2 we know that \( \pi(R) \) is spanned by secants of \( C \) that intersect \( L \), so \( H \) contains such secant \( \ell \). Moreover \( \pi(Z) \) intersects \( \ell \) so we see that \( \pi(Z) \) intersects a general secant of \( C \) that is contained in \( \pi(R) \). Then \( Z \subset R \) which contradicts Lemma 3.3. So we can choose the hyperplane \( H \) in such a way that the points in \( \{b_1, b_2, b_3, b_4\} \) are in general position.

Thus, \( S \) is a del Pezzo surface of degree 5 with the exceptional divisors \( E_1, E_2, E_3, E_4 \) corresponding to points \( b_1, b_2, b_3, b_4 \), \( L_{ij} \) are the preimages of lines connecting \( b_i \) and \( b_j \) for \( i \in \{1, \ldots, 4\} \). Recall that \( E_1, E_2, E_3, E_4 \) and \( L_{ij} \) generate the Mori Cone \( \overline{NE}(S) \). We have that

\[-K_X \sim \pi^*(4H) - E_C - E_L,\]

\[R \sim \pi^*(4H) - 2E_C - E_L,\]

and moreover

\[\pi^*(H)|_S \sim S|_S \sim Z, \quad E_L|_S \sim E_1, \quad E_C|_S \sim E_2 + E_3 + E_4.\]

By Theorem 2.3 we have \( S_X(S) < 1 \). Thus, we conclude that \( S(W^S_{\bullet\bullet}; Z) \geq 1 \) by Corollary 2.4. Let us compute \( S(W^S_{\bullet\bullet}; Z) \). Take \( u \in \mathbb{R}_{\geq 0} \). Observe that

\[-K_X - uS \sim (1 - u/3)R + u/3(H - E_L) + (1 - 2u/3)E_C,\]

which implies that \( -K_X - uS \) is pseudo-effective if and only if \( u \leq \frac{3}{2} \). Let \( P(u) = P(-K_X - uS) \) be a positive part of Zariski decomposition and \( N(u) = N(-K_X - uS) \) be a negative part of Zariski decomposition. Here we use the notations introduced in Theorem 2.4

\[P(u) = \begin{cases} -K_X - uS & \text{if } 0 \leq u \leq 1, \\ -K_X - uS - (u - 1)R & \text{if } 1 \leq u \leq \frac{3}{2}, \end{cases}\]

and

\[N(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u - 1)R & \text{if } 1 \leq u \leq \frac{3}{2}. \end{cases}\]

Then take any \( v \in \mathbb{R}_{\geq 0} \). Suppose \( P(u, v) \) is a positive part of the Zariski decomposition of \((-K_X - uS)|_S - vZ, N(u, v) \) is a negative part of the Zariski decomposition of \((-K_X -
To check when the divisor is nef we should choose the strongest inequality from the following system:

\[
P(u)|_S - vZ = (-K_X - uS)|_S - vZ = (4H - E_C - E_L - uS)|_S - vZ = 4Z - (E_2 + E_3 + E_4) - E_1 - uZ - vZ = (4 - u - v)Z - (E_1 + E_2 + E_3 + E_4).
\]

For \( u \in [0, 1] \) we find \( v \) such that the divisor \( P(u)|_S - vZ \) is nef. We have that:

- \( P(u, v) \cdot Z = 4 - u - v \),
- \( P(u, v) \cdot E_i = 1 \), for \( i \in \{1, \ldots, 4\} \),
- \( P(u, v) \cdot L_{ij} = 2 - u - v \), for \( i, j \in \{1, \ldots, 4\} \).

To check when the divisor is nef we should choose the strongest inequality from the following system:

\[
\begin{cases}
4 - u - v \geq 0, \\
2 - u - v \geq 0, \\
\end{cases} \Rightarrow v \leq 2 - u.
\]

Thus for \( v \leq 2 - u \) we have that the divisor \( P(u)|_S - vZ \) is nef. Note that for \( v = 2 - u \) we have that \( P(u, 2 - u)^2 = 0 \) so \( P(u)|_S - vZ \) is pseudo-effective until \( v \) satisfies \( v \leq 2 - u \).

We see that the Zariski decomposition for \( u \in [0, 1] \), \( v \leq 2 - u \) is given by

\[
P(u, v) = (4 - u - v)Z - (E_1 + E_2 + E_3 + E_4), \quad N(u, v) = 0.
\]

For \( u \in [1, 3/2] \) and \( v \in \mathbb{R}_{\geq 0} \) we have that:

\[
P(u)|_S - vZ = (-K_X - uS - (u - 1)R)|_S - vZ = (4H - E_C - E_L - uS - (u - 1)(4H - 2E_C - E_L))|_S - vZ = (8 - 5u - v)Z + (2u - 3)(E_2 + E_3 + E_4) + (u - 2)E_1.
\]

We have that:

- \( P(u, v) \cdot Z = 8 - 5u - v \),
- \( P(u, v) \cdot E_i = 2 - u \),
- \( P(u, v) \cdot E_i = 3 - 2u \) for \( i \in \{2, 3, 4\} \),
- \( P(u, v) \cdot L_{1i} = 3 - 2u - v \) for \( L_{1i} \in \{L_{12}, L_{13}, L_{14}\} \),
- \( P(u, v) \cdot L_{ij} = 3 - 2u - v \) for \( L_{ij} \in \{L_{23}, L_{34}, L_{24}\} \).

To check when the divisor is nef we should choose the strongest inequality from the following system:

\[
\begin{cases}
8 - 5u - v \geq 0, \\
3 - 2u - v \geq 0, \\
2 - u - v \geq 0, \\
\end{cases} \Rightarrow v \leq 3 - 2u.
\]

Thus, for \( v \leq 3 - 2u \) we have that the divisor \( P(u)|_S - vZ \) is nef. We see that the Zariski decomposition for \( u \in [1, 3/2] \), \( v \leq 3 - 2u \) is given by

\[
P(u, v) = (8 - 5u - v)Z + (2u - 3)(E_2 + E_3 + E_4) + (u - 2)E_1, \quad N(u, v) = 0.
\]

Suppose \( v \geq 3 - 2u \). Let us find the Zariski decomposition of \( P(u)|_S - vZ \). \( P(u, v) \) is given by

\[
P(u, v) = (8 - 5u - v)Z + (2u - 3)(E_2 + E_3 + E_4) + (u - 2)E_1 - aL_{12} - bL_{13} - cL_{14},
\]
for some $a, b, c$. Note that we should have $P(u, v) \cdot L_{1i} = 0$ for $i \in \{2, 3, 4\}$ thus $a = b = c = -(3 - 2u - v)$. So we obtain that:

$$P(u, v) = (8 - 5u - v)Z + (2u - 3)(E_2 + E_3 + E_4) + (u - 2)E_1 + (3 - 2u - v)(L_{12} + L_{13} + L_{14}),$$

$$N(u, v) = -(3 - 2u - v)(L_{12} + L_{13} + L_{14}).$$

We have that:

- $P(u, v) \cdot Z = -11u - 4v + 17,$
- $P(u, v) \cdot E_i = -7u - 3v + 11,$
- $P(u, v) \cdot E_i = -v - 4u + 6$ for $i \in \{2, 3, 4\},$
- $P(u, v) \cdot L_{1i} = 0$ for $L_{1i} \in \{L_{12}, L_{13}, L_{14}\},$
- $P(u, v) \cdot L_{ij} = 5 - 3u - 2v$ for $L_{ij} \in \{L_{23}, L_{24}, L_{34}\}.$

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

$$\begin{align*}
-11u - 4v + 17 &\geq 0, \\
-7u - 3v + 11 &\geq 0, \\
-v - 4u + 6 &\geq 0, \\
5 - 3u - 2v &\geq 0,
\end{align*}$$

$$\Rightarrow \begin{align*}
v &\leq \frac{-11u + 17}{4}, \\
v &\leq \frac{-7u + 11}{3}, \\
v &\leq 6 - 4u, \\
v &\leq \frac{-3u + 5}{2}.
\end{align*}$$

So for $u \in [1, 7/5]$ we get $v \leq \frac{-3u + 5}{2}$ and for $u \in [7/5, 3/2]$ we get $v \leq 6 - 4u$. Note that

$$P(u, v)^2 = 2(3u - 5 + 2v)(4u - 6 + v).$$

Thus, for $v = \frac{-3u + 5}{2}$ or $v = 6 - 4u$ we have that $P(u, v)^2 = 0$. Note that for $v = 2 - u$ we have that $P(u, v)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective until $v$ is in these intervals.

The Corollary 24 gives us

$$S(W^S; Z) = \frac{3}{(-K^X)^3} \int_0^{3/2} (P(u)^2 \cdot S) \cdot \text{ord}_Z(N(u)|_S) du + \frac{3}{(-K^X)^3} \int_0^{3/2} \int_0^\infty \text{vol}(P(u)|_S - vZ) dv du.$$

Note that $\text{ord}_Z(N(u)|_S) = 0$ because $Z \not\subset R$. So we are only left with the second part of the integral which equals:

$$S(W^S; Z) = \frac{3}{28} \int_0^1 \int_0^{2u} ((4 - u - v)Z - (E_1 + E_2 + E_3 + E_4))^2 dv du +$$

$$+ \frac{3}{28} \int_1^{3/2} \int_0^{2u-3} ((8 - 5u - v)Z + (2u - 3)(E_2 + E_3 + E_4) + (u - 2)E_1)^2 dv du +$$

$$+ \frac{3}{28} \int_1^{7/5} \int_{3-2u}^{2u} ((8-5u-v)Z + (u-2)(E_2+E_3+E_4)+(u-2)E_1+(3-2u-v)(L_{12}+L_{13}+L_{14}))^2 dv du +$$

$$+ \frac{3}{28} \int_7^{7/5} \int_{3-2u}^{2u} ((8-5u-v)Z + (u-2)(E_2+E_3+E_4)+(u-2)E_1+(3-2u-v)(L_{12}+L_{13}+L_{14}))^2 dv du =$$

$$= \frac{3}{28} \int_0^1 \int_0^{2u} ((4-u-v)^2-4) dv du + \frac{3}{28} \int_1^{3/2} \int_0^{2u-3} (12u^2 + 10uv + v^2 - 40u - 16v + 33) dv du +$$

$$+ \frac{3}{28} \int_1^{7/5} \int_{3-2u}^{2u} (24u^2 + 22uv + 4v^2 - 76u - 34v + 60) dv du +$$

$$+ \frac{3}{28} \int_7^{7/5} \int_{3-2u}^{2u} (24u^2 + 22uv + 4v^2 - 76u - 34v + 60) dv du +$$
\[ + \frac{3}{28} \int_1^{\frac{7}{5}} \int_{\frac{3}{2} - 2u}^{\frac{6}{4} - 4u} (24u^2 + 22uv + 4v^2 - 76u - 34v + 60) dvdu = \frac{753}{1120} < 1. \]

The obtained contradiction completes the proof of the lemma. \[ \square \]

**Lemma 4.2.** One has \( Z \not\subset E_L \).

**Proof.** Suppose that \( Z \subset E_L \). Observe that \( E_L \cong \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( s \) be the section of the natural projection \( E_L \to L \) such that \( s^2 = 0 \), and \( l \) be a fiber of this projection. Then

\[
E_L|_{E_L} \sim -s + l, \quad \pi^*(H)|_{E_L} \sim l, \quad R|_{E_L} \sim s + 3l,
\]

and \( E_C \) and \( E_L \) are disjoint. By Theorem 2.3, we have \( S_X(E_L) < 1 \). Thus, we conclude that \( S(W^{E_L}; Z) \geq 1 \) by Corollary 2.4. Let us compute \( S(W^{E_L}; Z) \). Take \( u \in \mathbb{R}_{\geq 0} \). Observe that

\[
-K_X - uE_L \sim \mathcal{R} \left( \frac{1}{2} R + 2(\pi^*(H) - E_L) + \left( \frac{3}{2} - u \right) E_L, \right),
\]

which implies that \( -K_X - uE_L \) is pseudo-effective if and only if \( u \leq \frac{3}{2} \). Let \( P(u) = P(-K_X - uE_L) \) and \( N(u) = N(-K_X - uE_L) \). Then

\[
P(u) = \begin{cases} 
- K_X - uE_L \text{ for } 0 \leq u \leq 1, \\
(8 - 4u)\pi^*(H) - (3 - 2u)E_C - 2E_L \text{ for } 1 \leq u \leq \frac{3}{2}, 
\end{cases}
\]

and

\[
N(u) = \begin{cases} 
0 \text{ for } 0 \leq u \leq 1, \\
(u - 1)R \text{ for } 1 \leq u \leq \frac{3}{2}.
\end{cases}
\]

Suppose that \( Z \neq R|_{E_L} \) and \( Z \sim as + bl \). Note that \( a \geq 1 \) since \( \mathbb{P}^3 \) does not contain \( G \)-fixed points. Then using Corollary 2.4 we obtain

\[
S(W^{E_L}; Z) = \frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \operatorname{vol}(P(u)|_Q - v(as + bl)) dvdu \leq \frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \operatorname{vol}(P(u)|_Q - vs) dvdu.
\]

so it is enough to show that the last integral is less than 1 to deduce a contradiction. So suppose \( Z \sim s \). We have that:

\[
P(u)|_{E_L} - vs \sim_{\mathcal{R}} \begin{cases} 
(1 + u - v)s + (3 - u)l \text{ for } 0 \leq u \leq 1, \\
(2 - v)s + (6 - 4u)l \text{ for } 1 \leq u \leq 3/2.
\end{cases}
\]

Then Corollary 2.4 gives

\[
S(W^{E_L}; s) = \frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \operatorname{vol}(P(u)|_Q - vs) dvdu = \frac{3}{28} \int_0^1 \int_0^{1+u} 2(3 - u)(1 + u - v) dvdu + \frac{3}{28} \int_1^{3/2} \int_0^{2} 4(v - 2)(-3 + 2u) dvdu = \frac{13}{16} < 1.
\]

So we obtained a contradiction with Corollary 2.4.

Thus, for any \( G \)-invariant curve \( Z \subset E_L \) such that \( Z \neq R|_{E_L} \) we have \( S(W^{E_L}; Z) < 1 \)-contradiction with Corollary 2.4.
Suppose \( Z = R|_{E_L} \sim s + 3l \). Take any \( v \in \mathbb{R}_{\geq 0} \) then we have:

\[
P(u)|_{E_L} - vZ \sim_{\mathbb{R}} \begin{cases} 
(1 + u - v)s + (3 - u - 3v)1, & \text{for } 0 \leq u \leq 1, \\
(2 - v)s + (6 - 4u - 3v)1, & \text{for } 1 \leq u \leq 3/2.
\end{cases}
\]

Hence, if \( Z = R|_{E_L} \), then Corollary 2.4 gives

\[
S(W^*_{\bullet, \bullet}; Z) = \frac{3}{28} \int_1^2 (u - 1)E_L \cdot ((8 - 4u)\pi^*(H) - (3 - 2u)E_C - 2E_L)^2 du + \\
\frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \text{vol}(P(u)|_{E_L} - vZ)dvdu = \\
\frac{3}{28} \int_1^2 4(u - 1)(6 - 4u)du + \frac{3}{28} \int_0^1 \int_0^{\frac{3 - u}{2}} 2(1 + u - v)(3 - u - 3v)dvdu + \\
\frac{3}{28} \int_1^2 \int_0^{\frac{6 - 4u}{2}} 2(2 - v)(6 - 4u - 3v)dvdu = \frac{19}{56} < 1.
\]

The obtained contradiction with Corollary 2.4 and completes the proof of the lemma. \( \square \)

**Lemma 4.3.** Let \( Q \) be a \( G \)-invariant quadric surface in \( \mathbb{P}^3 \) that contains \( C \). Then \( Z \not\subset Q \).

**Proof.** Suppose that \( Z \subset Q \). Let us seek for a contradiction. Recall that \( \pi(Q) \) is a smooth quadric surface in \( \mathbb{P}^3 \) that contains the twisted cubic curve \( C \), and it does not contain line \( L \). Let us identify \( \pi(Q) = \mathbb{P}^1 \times \mathbb{P}^1 \) such that \( C \) is a curve in \( \pi(Q) \) of degree \((1,2)\). Then \( \pi \) induces a birational morphism \( \varphi: Q \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) that is a blow up of two intersection points \( \pi(Q) \cap L \), which are not contained in the curve \( C \). Moreover, the surface \( Q \) is a smooth del Pezzo surface of degree \( 6 \), because the points of the intersection \( \pi(Q) \cap L \) are not contained in one line in \( \pi(Q) \) since otherwise this line would be \( L \). But \( L \) is not contained in \( \pi(Q) \) which is a contradiction.

By Theorem 2.3 we have \( S_X(Q) < 1 \). Then \( S(W^*_{\bullet, \bullet}; Z) \geq 1 \) by Corollary 2.4. Let us show that \( S(W^*_{\bullet, \bullet}; Z) < 1 \), which would give us the desired contradiction.

Take \( u \in \mathbb{R}_{\geq 0} \). Then

\[
-K_X - uQ \sim_{\mathbb{R}} 2\pi^*(H) - E_L + (1 - u)(2\pi^*(H) - E_C),
\]

which implies that \( -K_X - uQ \) is nef for every \( u \in [0,1] \). On the other hand, we have

\[
-K_X - uQ \sim_{\mathbb{R}} (4 - 2u)(\pi^*(H) - E_L) + (3 - 2u)E_L + (u - 1)E_C,
\]

so that the divisor \( -K_X - uS \) is pseudo-effective \( \iff u \in [0,\frac{3}{2}] \). Let \( P(u) = P(-K_X - uQ) \) and \( N(u) = N(-K_X - uQ) \). Then we have

\[
P(-K_X - uQ) = \begin{cases} 
-K_X - uQ & \text{for } 0 \leq u \leq 1, \\
(4 - 2u)\pi^*(H) - E_L & \text{for } 1 \leq u \leq \frac{3}{2},
\end{cases}
\]

and

\[
N(-K_X - uQ) = \begin{cases} 
0 & \text{for } 0 \leq u \leq 1, \\
(u - 1)E_C & \text{for } 1 \leq u \leq \frac{3}{2}.
\end{cases}
\]

Let us introduce some notation on \( Q \). Suppose \( \varphi: Q \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) is the blowup at points \( A_1, A_2 \). First, we denote by \( \ell_1 \) and \( \ell_2 \) the proper transforms on \( Q \) of general curves \( k_1 \) and
$k_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ of degrees $(1, 0)$ and $(0, 1)$, respectively. Second, we denote by $e_1$ and $e_2$ the exceptional curves of $\varphi$ which correspond to points $A_1, A_2$ respectively. Third, we let $F_{11}, F_{12}, F_{21}, F_{22}$ be the $(-1)$-curves on $Q$ such that

$$F_{11} \sim \ell_1 - e_1, \quad F_{12} \sim \ell_1 - e_2, \quad F_{21} \sim \ell_2 - e_1, \quad F_{22} \sim \ell_2 - e_2.$$  

Then

$$\pi^*(H)|_Q \sim \ell_1 + \ell_2, \quad E\ell_i|_Q \sim e_1 + e_2, \quad E|_Q \sim \ell_1 + 2\ell_2.$$  

Suppose $Z \neq E|_Q$, then $\varphi(Z)$ is a curve since $Z \neq e_1$ and $Z \neq e_2$, because neither $e_1$ nor $e_2$ is $G$-invariant. Now we have that $\varphi(Z) \sim ak_1 + bk_2$ and so

$$Z \sim a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2,$$

where $m_1$ is a multiplicity of $\varphi(Z)$ at point $A_1$, $m_2$ is a multiplicity of $\varphi(Z)$ at point $A_2$. Note that $G$ exchanges $A_1$ and $A_2$ and $Z$ is a $G$-invariant curve thus $m_1 = m_2 =: m$. We know that $Z \notin \{F_{11}, F_{12}, F_{21}, F_{22}\}$ since $F_{ij}$’s $(i, j \in \{1, 2\})$ are not $G$-invariant. Thus:

$$0 \leq Z \cdot F_{11} = (a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2)(l_1 - e_1) = b - m \Rightarrow b \geq m,$$

$$0 \leq Z \cdot F_{22} = (a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2)(l_2 - e_2) = a - m \Rightarrow a \geq m.$$  

Now we have that

$$Z \sim a\ell_1 + b\ell_2 - m(e_1 + e_2) =
\begin{cases}
\begin{array}{l}
(a - b)\ell_1 + m(\ell_1 + \ell_2 - e_1 - e_2) + (b - m)(\ell_1 + \ell_2), \quad a \geq b,
\end{array}
\end{cases}$$

$$\begin{cases}
\begin{array}{l}
(b - a)\ell_2 + m(\ell_1 + \ell_2 - e_1 - e_2) + (a - m)(\ell_1 + \ell_2), \quad b \geq a.
\end{array}
\end{cases}$$

So we can decompose each curve $Z$ as the sum of $\ell_1$, $\ell_2$, $\ell_1 + \ell_2 - e_1 - e_2$ with non-negative coefficients, i.e. $Z \sim c_1\ell_1 + c_2\ell_2 + c_3(\ell_1 + \ell_2 - e_1 - e_2)$ for some non-negative integers $c_1$, $c_2$, $c_3$. Note if for example $c_1 \geq 1$ then:

$$S(W^Q_{\bullet} ; Z) = \frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \text{vol}
\left((P(u)|_Q - v(c_1\ell_1 + c_2\ell_2 + c_3(\ell_1 + \ell_2 - e_1 - e_2)))\right)dvdu \leq \frac{3}{28} \int_0^{\frac{3}{2}} \int_0^\infty \text{vol}(P(u)|_Q - v\ell_1)dvdu.$$

and similarly for $c_2$ and $c_3$. So it is enough to get $S(W^Q_{\bullet} ; Z) < 1$ for $Z \sim \ell_1$, $Z \sim \ell_2$, $Z \sim \ell_1 + \ell_2 - e_1 - e_2$ to deduce a contradiction.

1. Suppose $Z \sim \ell_1$.

Take any $v \in \mathbb{R}_{\geq 0}$. Suppose $P(u, v)$ is a positive part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$, $N(u, v)$ is a negative part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$.

If $u \in [0, 1]$ then

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (3 - u - v)\ell_1 + 2\ell_2.$$  

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u, v) \cdot e_1 = 1,$
- $P(u, v) \cdot e_2 = 1,$
- $P(u, v) \cdot F_{11} = 1,$
- $P(u, v) \cdot F_{12} = 1,$
- $P(u, v) \cdot F_{21} = 2 - u - v,$
- $P(u, v) \cdot F_{22} = 2 - u - v.$
Thus, for $v \leq 2 - u$ we have that the divisor $P(u)|_Q - vZ$ is nef.
Suppose $v \geq 2 - u$. We find the Zariski decomposition of $P(u)|_Q - vZ$.

\[
P(u, v) = (-e_1 - e_2 + (3 - u - v)\ell_1 + 2\ell_2 + (2 - u - v)(F_{21} + F_{22})
\]

\[
= (-3 + u + v)e_1 + (-3 + u + v)e_2 + (3 - u - v)\ell_1 + (6 - 2u - 2v)\ell_2,
\]

\[
N(u, v) = -(2 - u - v)(F_{21} + F_{22}) = -(2 - u - v)(2\ell_2 - e_1 - e_2).
\]

We have that:

\begin{itemize}
  \item $P(u, v) \cdot e_1 = 3 - u - v$,
  \item $P(u, v) \cdot e_2 = 3 - u - v$,
  \item $P(u, v) \cdot F_{11} = 3 - u - v$,
  \item $P(u, v) \cdot F_{12} = 3 - u - v$,
  \item $P(u, v) \cdot F_{21} = 0$,
  \item $P(u, v) \cdot F_{22} = 0$.
\end{itemize}

We get $v \leq 3 - u$. Note that for $v = 3 - u$ we have that $P_2(u, 3 - u)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective if and only if $v$ satisfies $v \leq 3 - u$.

For $u \in [1, 3/2]$ and $v \in \mathbb{R}_{\geq 0}$ we have that:

\[
P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (4 - 2u - v)\ell_1 + (4 - 2u)\ell_2.
\]

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

\begin{itemize}
  \item $P(u, v) \cdot e_1 = 1$,
  \item $P(u, v) \cdot e_2 = 1$,
  \item $P(u, v) \cdot F_{11} = 3 - 2u$,
  \item $P(u, v) \cdot F_{12} = 3 - 2u$,
  \item $P(u, v) \cdot F_{21} = 3 - 2u - v$,
  \item $P(u, v) \cdot F_{22} = 3 - 2u - v$.
\end{itemize}

Thus for $v \leq 3 - 2u$ we have that the divisor $P(u)|_Q - vZ$ is nef.

Suppose $v \geq 3 - 2u$. We find the Zariski decomposition of $P(u)|_S - vZ$.

\[
P(u, v) = (-e_1 - e_2 + (4 - 2u - v)\ell_1 + (4 - 2u)\ell_2 + (3 - 2u - v)(F_{21} + F_{22}) =
\]

\[
= (-4 + 2u + v)e_1 + (-4 + 2u + v)e_2 + (4 - 2u - v)\ell_1 + (10 - 6u - 2v)\ell_2,
\]

\[
N(u, v) = -(3 - 2u - v)(F_{21} + F_{22}) = (3 - 2u - v)(2\ell_2 - e_1 - e_2).
\]

We have that:

\begin{itemize}
  \item $P(u, v) \cdot e_1 = 4 - 2u - v$,
  \item $P(u, v) \cdot e_2 = 4 - 2u - v$,
  \item $P(u, v) \cdot F_{11} = 6 - 4u - v$,
  \item $P(u, v) \cdot F_{12} = 6 - 4u - v$,
  \item $P(u, v) \cdot F_{21} = 0$,
  \item $P(u, v) \cdot F_{22} = 0$.
\end{itemize}

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

\[
\begin{align*}
v \leq 4 - 2u, \\
v \leq 6 - 4u,
\end{align*} \quad \Rightarrow v \leq 6 - 4u.
\]
We get \( v \leq 6 - 4u \). Note that for \( v = 6 - 4u \) we have that \( P_4(u, 6 - 4u)^2 = 0 \) so \( P(u)|_S - vZ \) is pseudo-effective if and only if \( v \leq 6 - 4u \). The Corollary [2,4] gives us:

\[
S(W^Q_u; Z) = \frac{3}{28} \int_0^1 \int_0^{\infty} \text{vol}\left(P(u)|_Q - v\ell_1\right)dvdu = \frac{3}{28} \int_0^1 \int_0^{2-u} \left(10 - 4u - 4v\right)dvdu + \\
+ \frac{3}{28} \int_0^1 \int_{2-u}^{3-u} 2(3 - u - v)^2dvdu + \frac{3}{28} \int_1^2 \int_0^{3-2u} \left(8u^2 + 4uv - 32u - 8v + 30\right)dvdu + \\
+ \frac{3}{28} \int_1^2 \int_{3-2u}^{6-4u} 2(4 - 2u - v)(6 - 4u - v)dvdu = \frac{109}{112}
\]

So we obtained a contradiction with Corollary [2,4] 2). Suppose \( Z \sim \ell_2 \).

Take any \( v \in \mathbb{R}_{>0} \). Abusing the notations suppose \( P(u, v) \) is a positive part of the Zariski decomposition of \((-K_X - uS)|_S - vZ\), \( N(u, v) \) is a negative part of the Zariski decomposition of \((-K_X - uS)|_S - vZ\).

If \( u \in [0, 1] \) then

\[
P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (3 - u)\ell_1 + (2 - v)\ell_2.
\]

Now we find \( v \) such that the divisor \( P(u)|_Q - vZ \) is nef. We have that:

- \( P(u, v) \cdot e_1 = 1 \)
- \( P(u, v) \cdot e_2 = 1 \)
- \( P(u, v) \cdot F_{11} = 1 - v \)
- \( P(u, v) \cdot F_{12} = 1 - v \)
- \( P(u, v) \cdot F_{21} = 2 - u \)
- \( P(u, v) \cdot F_{22} = 2 - u \)

Thus for \( v \leq 1 \) we have that the divisor \( P(u)|_Q - vZ \) is nef.

Suppose \( v \geq 1 \). We find the Zariski decomposition of \( P(u)|_Q - vZ \).

\[
P(u, v) = (-e_1 - e_2 + (3 - u)\ell_1 + (2 - v)\ell_2) + (1 - v)(F_{11} + F_{12}) = \\
= (v - 2)e_1 + (v - 2)e_2 + (5 - u - 2v)\ell_1 + (2 - v)\ell_2,
\]

\[
N(u, v) = -(v - 1)(F_{11} + F_{12}) = -(v - 1)(2\ell_1 - e_1 - e_2).
\]

We have that:

- \( P(u, v) \cdot e_1 = 2 - v \)
- \( P(u, v) \cdot e_2 = 2 - v \)
- \( P(u, v) \cdot F_{11} = 0 \)
- \( P(u, v) \cdot F_{12} = 0 \)
- \( P(u, v) \cdot F_{21} = -v + 3 - u \)
- \( P(u, v) \cdot F_{22} = -v + 3 - u \)

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

\[
\begin{cases}
  v \leq 2, \\
  v \leq 3 - u,
\end{cases} \quad \Rightarrow \quad v \leq 2.
\]

We get \( v \leq 2 \). Note that for \( v = 2 \) we have that \( P_2(u, 2)^2 = 0 \) so \( P(u)|_S - vZ \) is pseudo-effective if and only if \( v \leq 2 \).

For \( u \in [1, 3/2] \) and \( v \in \mathbb{R}_{>0} \) we have that:

\[
P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (4 - 2u)\ell_1 + (4 - 2u - v)\ell_2.
\]

Now we find \( v \) such that the divisor \( P(u)|_Q - vZ \) is nef. We have that:
\[ P(u, v) \cdot e_1 = 1, \quad P(u, v) \cdot e_2 = 1, \quad P(u, v) \cdot F_{11} = 3 - 2u - v, \quad P(u, v) \cdot F_{12} = 3 - 2u - v, \quad P(u, v) \cdot F_{21} = 3 - 2u, \quad P(u, v) \cdot F_{22} = 3 - 2u. \]

Thus for \( v \leq 3 - 2u \) we have that the divisor \( P(u)|_Q - vZ \) is nef.

Suppose \( v \geq 3 - 2u \). We find the Zariski decomposition of \( P(u)|_S - vZ \).

\[
P(u, v) = (-e_1 - e_2 + (4 - 2u)\ell_1 + (4 - 2u - v)\ell_2) + (3 - 2u - v)(F_{11} + F_{12}) =
\]

\[
= (-4 + 2u + v)e_1 + (-4 + 2u + v)e_2 + (10 - 6u - 2v)\ell_1 + (4 - 2u - v)\ell_2,
\]

\[
N(u, v) = -(3 - 2u - v)(F_{11} + F_{12}) = (3 - 2u - v)(2\ell_1 - e_1 - e_2).
\]

We have that:

\[
\begin{align*}
\bullet & \quad P(u, v) \cdot e_1 = 4 - 2u - v, \quad \bullet & \quad P(u, v) \cdot F_{12} = 0, \\
\bullet & \quad P(u, v) \cdot e_2 = 4 - 2u - v, \quad \bullet & \quad P(u, v) \cdot F_{21} = 6 - 4u - v, \\
\bullet & \quad P(u, v) \cdot F_{11} = 0, \quad \bullet & \quad P(u, v) \cdot F_{22} = 6 - 4u - v.
\end{align*}
\]

We get \( v \leq 6 - 4u \). Note that for \( v = 6 - 4u \) we have that \( P_4(u, 6 - 4u)^2 = 0 \) so \( P(u)|_S - vZ \) is pseudo-effective if and only if \( v \leq 6 - 4u \). The Corollary 2.4 gives us:

\[
S(W^Q\bullet; Z) = \frac{3}{28} \int_0^{\frac{3}{4}} \int_0^1 \left( 2uv - 4u - 6v + 10 \right) dvdu + \frac{3}{28} \int_1^{\frac{3}{4}} \int_0^{\frac{3}{4}} \left( 8u^2 + 4uv - 32u - 8v + 30 \right) dvdu + \frac{3}{28} \int_1^{\frac{3}{4}} \int_{\frac{3}{4}}^{\frac{6}{4u}} 2(4 - 2u - v)(6 - 4u - v)dvdu = \frac{89}{112} < 1.
\]

So we obtained a contradiction with Corollary 2.4.

3). Suppose \( Z \sim \ell_1 + \ell_2 - e_1 - e_2 \).

Take any \( v \in \mathbb{R}_{\geq 0} \). Abusing the notations suppose \( P(u, v) \) is a positive part of the Zariski decomposition of \( (-K_X - uS)|_S - vZ \), \( N(u, v) \) is a negative part of the Zariski decomposition of \( (-K_X - uS)|_S - vZ \).

If \( u \in [0, 1] \) then:

\[
P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = (-1 + v)e_1 + (-1 + v)e_2 + (3 - u - v)\ell_1 + (2 - v)\ell_2.
\]

Now we find \( v \) such that the divisor \( P(u)|_Q - vZ \) is nef. We have that:

\[
\begin{align*}
\bullet & \quad P(u, v) \cdot e_1 = -v + 1, \quad \bullet & \quad P(u, v) \cdot F_{12} = 1, \\
\bullet & \quad P(u, v) \cdot e_2 = -v + 1, \quad \bullet & \quad P(u, v) \cdot F_{21} = 2 - u, \\
\bullet & \quad P(u, v) \cdot F_{11} = 1, \quad \bullet & \quad P(u, v) \cdot F_{22} = 2 - u.
\end{align*}
\]

Thus for \( v \leq 1 \) we have that the divisor \( P(u)|_Q - vZ \) is nef.

Suppose \( v \geq 1 \). We find the Zariski decomposition of \( P(u)|_Q - vZ \).

\[
P(u, v) = ((-1 + v)e_1 + (-1 + v)e_2 + (3 - u - v)\ell_1 + (2 - v)\ell_2) + (1 - v)(e_1 + e_2) =
\]

\[
= (3 - u - v)\ell_1 + (2 - v)\ell_2,
\]

\[
N(u, v) = -(v - 1)(e_1 + e_2).
\]

We have that:
\begin{itemize}
  \item \(P(u, v) \cdot e_1 = 0,\)
  \item \(P(u, v) \cdot e_2 = 0,\)
  \item \(P(u, v) \cdot F_{11} = 2 - v,\)
  \item \(P(u, v) \cdot F_{12} = 2 - v,\)
  \item \(P(u, v) \cdot F_{21} = -v + 3 - u,\)
  \item \(P(u, v) \cdot F_{22} = -v + 3 - u.\)
\end{itemize}

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

\[
\begin{aligned}
  &v \leq 2, \\
  &v \leq 3 - u, \\
\end{aligned}
\Rightarrow v \leq 2.
\]

We get \(v \leq 2.\) Note that for \(v = 2\) we have that \(P_2(u, 2)^2 = 0\) so \(P(u)|_S - vZ\) is pseudo-effective until \(v\) satisfies \(v \leq 2.\)

For \(u \in [1, 3/2]\) and \(v \in \mathbb{R}_{\geq 0}\) we have that:

\[
P(u)|_Q - vZ \sim (-K_X - uQ)|Q - vZ = (v-1)e_1 + (v-1)e_2 + (4 - 2u - v)\ell_1 + (4 - 2u - v)\ell_2.
\]

Now we find \(v\) such that the divisor \(P(u)|_Q - vZ\) is nef. We have that:

\[
\begin{itemize}
  \item \(P(u, v) \cdot e_1 = -v + 1,\)
  \item \(P(u, v) \cdot e_2 = -v + 1,\)
  \item \(P(u, v) \cdot F_{11} = 3 - 2u,\)
  \item \(P(u, v) \cdot F_{12} = 3 - 2u,\)
  \item \(P(u, v) \cdot F_{21} = 3 - 2u,\)
  \item \(P(u, v) \cdot F_{22} = 3 - 2u.\)
\end{itemize}
\]

Thus, for \(v \leq 1\) we have that the divisor \(P(u)|_Q - vZ\) is nef.

Suppose \(v \geq 3 - 2u.\) We find the Zariski decomposition of \(P(u)|_S - vZ.\)

\[
P(u, v) = ((v-1)e_1 + (v-1)e_2 + (4 - 2u - v)\ell_1 + (4 - 2u - v)\ell_2) + (1-v)(e_1 + e_2) =
\]

\[
(4 - 2u - v)\ell_1 + (4 - 2u - v)\ell_2,
\]

\[
N(u, v) = -(3 - 2u - v)(e_1 + e_2).
\]

We have that:

\[
\begin{itemize}
  \item \(P(u, v) \cdot e_1 = 0,\)
  \item \(P(u, v) \cdot e_2 = 0,\)
  \item \(P(u, v) \cdot F_{11} = 4 - 2u - v,\)
  \item \(P(u, v) \cdot F_{12} = 4 - 2u - v,\)
  \item \(P(u, v) \cdot F_{21} = 4 - 2u - v,\)
  \item \(P(u, v) \cdot F_{22} = 4 - 2u - v.\)
\end{itemize}
\]

We get \(v \leq 4 - 2u.\) Note that for \(v = 4 - 2u\) we have that \(P(u, 4 - 2u)^2 = 0\) so \(P(u)|_S - vZ\) is pseudo-effective until \(v\) satisfies \(v \leq 4 - 2u.\) The Corollary\textsuperscript{[2,4]} gives us:

\[
S(W^\otimes_{\bullet \bullet}; Z) = \frac{3}{28} \int_0^1 \int_0^\infty \text{vol}(P(u)|_Q - v\ell_2) dv du = \frac{3}{28} \int_0^1 \int_0^1 \left(2uv - 4u - 6v + 10\right) dv du +
\]

\[
+ \frac{3}{28} \int_0^1 \int_0^2 (2u - 2)(-3 + u + v) dv du + \frac{3}{28} \int_1^2 \int_0^1 \left(2(2u - 3)(2u + 2v - 5)\right) dv du +
\]

\[
+ \frac{3}{28} \int_1^2 \int_1^{4 - 2u} 2(-4 + 2u + v)^2 dv du = \frac{47}{56} < 1.
\]

So we obtained a contradiction with Corollary\textsuperscript{[2,4]}

Thus, for any \(G\)-invariant curve \(Z \subset Q\) such that \(Z \neq E_C|Q\) we have \(S(W^\otimes_{\bullet \bullet}; Z) < 1\) which is impossible by Corollary\textsuperscript{[2,4]}
Suppose \( Z = E_C|_Q \), then

\[
S(W^Q \cdot Z) = \frac{3}{28} \int_0^\frac{3}{4} \left( P(u) \cdot P(u) \cdot Q \right) \text{ord}_Z \left( N(u)|_Q \right) du + \frac{3}{28} \int_0^\frac{3}{4} \int_0^\infty \text{vol}(P(u)|_Q - vZ) dvdu = 
\]

\[
= \frac{3}{28} \int_1^\frac{3}{4} (u-1)((4-2u)^2 - E_L) \cdot (2\pi^*(H) - E_C) du + \frac{3}{28} \int_0^\frac{3}{4} \int_0^\infty \text{vol}(P(u)|_Q - vZ) dvdu = 
\]

\[
= \frac{3}{28} \int_1^\frac{3}{4} (u-1)(2(4 - 2u)^2 - 2) du + \frac{3}{28} \int_0^\frac{3}{4} \int_0^\infty \text{vol} \left( P(u)|_Q - v\ell_1 + 2\ell_2 \right) dvdu \leq 
\]

\[
\leq \frac{5}{224} + \frac{3}{28} \int_0^\frac{3}{4} \int_0^\infty \text{vol} \left( P(u)|_Q - v\ell_1 \right) dvdu = \frac{5}{224} + \frac{109}{112} = \frac{223}{224} < 1.
\]

So we obtained a contradiction with Corollary \([2.4]\). This completes the proof of the lemma. \( \square \)

**Corollary 4.4.** The curve \( \pi(Z) \) is not a line that intersect \( C \).

**Proof.** Recall from Section \([3.3.4]\) that there are exactly 3 \( G \)-invariant lines in \( \mathbb{P}^3 \) that intersect the curve \( C \). These are the lines \( L_{12}, L_{34}, L_{56} \). We have that \( L_{12} \subset Q_2 \cap Q_3, \ L_{34} \subset Q_1 \cap Q_2, \ L_{34} \subset Q_1 \cap Q_3 \). Thus, the lemma above gives us the result. \( \square \)

By Lemma \([2.5]\), one has \( a_{G,Z}(X) < \frac{3}{4} \). Thus, by lemma \([2.1]\) Lemma 1.4.1] and it’s proof, there is a \( G \)-invariant effective \( \mathbb{Q} \)-divisor \( D \) on the threefold \( X \) such that \( D \sim Q - K_X \) and \( Z \subset \text{Nklt}(X, \lambda D) \) for some positive rational number \( \lambda < \frac{3}{4} \).

**Lemma 4.5.** Let \( S \) be an irreducible surface in \( X \). Suppose that \( S \subset \text{Nklt}(X, \lambda D) \). Then either \( S \in |\pi^*(2H) - E_C| \) and \( S \) is \( G \)-invariant or \( S = E_L \).

**Proof.** We have \( D \sim Q 4\pi^*(H) - E_C - E_L \) and \( \lambda < \frac{3}{4} \). By assumption we have \( D = aS + \Delta \) where \( a \in \mathbb{Q} \) such that \( a \geq \frac{1}{\lambda} > \frac{4}{3} \) and \( \Delta \) is an effective \( \mathbb{Q} \) divisor on \( X \) whose support does not contain \( S \).

Assume \( S = E_C \). Then we get

\[
\pi^*(4H) - E_C - E_L \sim Q aE_C + \Delta \Rightarrow \Delta \sim Q \pi^*(4H) - (1 + a)E_C - E_L = R - (a - 1)E_C
\]

-contradiction.

Assume \( S \neq E_L, S \neq E_C \) then \( \pi(S) \subset \mathbb{P}^3 \) is the surface of some degree \( d \). We have that:

\[
4H \sim Q a\pi(S) + \pi(\Delta) \Rightarrow 4 \geq ad \Rightarrow d = 1 \text{ or } d = 2.
\]

The latter holds since \( a > \frac{4}{3} \). Then \( S \) is given by

\[
S \sim Q d\pi^*(H) - mLE_L - mCE_C.
\]

By Corollary \([3.8]\) we know that the cone of effective divisors is generated by \( E_L, E_C, R, \ H - E_L \) so we have that:

\[
\Delta \sim Q \pi^*(4H) - E_C - E_L - a(\pi^*(4H) - (1 + a)E_C - E_L = R - (a - 1)E_C) \sim
\]

\[
\sim a_1E_L + a_2E_C + a_3(\pi^*(4H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L)
\]

\[23\]
for $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$, $a_4 \geq 0$, which gives us a system of equations:

\[
\begin{aligned}
-4a_3 - a_4 - d + 4 &= 0, \\
-1 + m_L - a_1 + a_3 + a_4 &= 0, \\
-1 + m_C - a_2 + 2a_3 &= 0, \\
a_1 \geq 0, &\quad a_2 \geq 0, &\quad a_3 \geq 0, &\quad a_4 \geq 0, &\quad a > 4/3
\end{aligned}
\]

(4.0.1)

Thus, if $d = 2$ then $(m_L, m_C) = (0, 1)$ or $(m_L, m_C) = (1, 1)$ so we have the following options:

- $m_C = 1$ and $m_L = 1$. This gives us the linear system $|S| = |2\pi^*(H) - m_L E_L - m_C E_C|$ which on $\mathbb{P}^3$ corresponds to the linear system of quadrics which contain a line $L$ and a cubic $C$. But this is impossible since by assumption $L$ and $C$ so not intersect. Thus, this linear system does not contain effective divisors.

- $m_C = 1$, $m_L = 0$. Suppose $S \in |2\pi^*(H) - E_C|$ and $S$ is not $G$-invariant. We have that $S \subset \text{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum a_i D_i$ where $D_i$-s are irreducible components of $D$, $a_i \in \mathbb{Q}_{>0}$ and we have $S = \tilde{D}_i$ for some $i$. We assumed that $S$ is not $G$-invariant thus if we take a non-trivial element $g \in G$ we will have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j$. Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:

\[
\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a(2\pi^*(H) - E_C) + \Delta.
\]

Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:

\[
4(1-a)\pi^*(H) + (-1 + 2a)E_C - E_L \sim_{\mathbb{Q}} \Delta.
\]

By Corollary 3.8 we know that the cone of effective divisors is generated by $E_L$, $E_C$, $R$, $H - E_L$ so we can write $\Delta$ as:

\[
\Delta = a_1 E_L + a_2 E_C + a_3 (4\pi^*(H) - 2E_C - E_L) + a_4 (H - E_L).
\]

For $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$, $a_4 \geq 0$. Solving the system of equations on coefficients we get that it has no solutions. Suppose $S \in |2\pi^*(H) - E_C|$ and $S$ is $G$-invariant. This case is possible.

Similarly, if $d = 1$ then $(m_L, m_C) = (0, 0)$ or $(m_L, m_C) = (1, 0)$ so we have the following options:

- $m_C = 0$, $m_L = 1$. Suppose $S \subset |\pi^*(H) - E_L|$. We have that $S \subset \text{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum a_i D_i$ where $D_i$-s are irreducible components of $D$, $a_i \in \mathbb{Q}_{>0}$ and we have $S = \tilde{D}_i$ for some $i$. Note that $S \subset |\pi^*(H) - E_L|$ where $|\pi^*(H) - E_L|$ does not have $G$-invariant elements. Take a non-trivial element $g \in G$. We have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j$. Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:

\[
\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a(\pi^*(H) - E_L) + \Delta.
\]

Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:

\[
(4 - 2a)\pi^*(H) - E_C - (1 - 2a)E_L \sim_{\mathbb{Q}} \Delta.
\]
By Corollary 4.8 we know that the cone of effective divisors is generated by $E_L,$ $E_C,$ $R,$ $H - E_L$ so we can write $\Delta$ as:
$$\Delta = a_1E_L + a_2E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L).$$

For $a_1 \geq 0,$ $a_2 \geq 0,$ $a_3 \geq 0,$ $a_4 \geq 0.$ Solving the system of equations on coefficients we get that it has no solutions.

- $m_C = 0,$ $m_L = 0.$ Suppose $S \in |\pi^*(H)|.$ We have that $S \subseteq \text{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum_i a_iD_i$ where $D_i$-s are irreducible components of $D,$ $a_i \in \mathbb{Q}_{>0}$ and we have $S = D_i$ for some $i.$ Note that $S \in |\pi^*(H)|$ where $|\pi^*(H)|$ does not have $G$-invariant elements. Take a non-trivial element $g \in G.$ We have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j.$ Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:
$$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a\pi^*(H) + \Delta.$$ 

Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:
$$(4 - 2a)\pi^*(H) - E_C - E_L \sim_{\mathbb{Q}} \Delta.$$ 

By Corollary 4.8 we know that the cone of effective divisors is generated by $E_L,$ $E_C,$ $R,$ $H - E_L$ so we can write $\Delta$ as:
$$\Delta = a_1E_L + a_2E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(H - E_L).$$

For $a_1 \geq 0,$ $a_2 \geq 0,$ $a_3 \geq 0,$ $a_4 \geq 0.$ Solving the system of equations on coefficients we get that it has no solutions.

We see that we excluded all options except $S \in |\pi^*(2H) - E_C|$ and $S$ is $G$-invariant or $S = E_L.$

\textbf{Corollary 4.6.} $\pi(Z)$ is not a surface in $\text{Nklt}(X, \lambda D).$

\textbf{Corollary 4.7.} One has $Z \not\subset E_C.$

\textit{Proof.} Suppose that $Z \subset E_C.$ Observe that $\pi(Z)$ is not a point, since $\mathbb{P}^3$ does not have $G$-fixed points by Lemma 3.1. Hence, we see that $\pi(Z)$ is the twisted cubic $C.$

Let $S$ be a general fiber of $\eta.$ Then $S \cdot Z \geq 3,$ which contradicts Lemma 2.7. 

\textbf{Lemma 4.8.} The curve $\pi(Z)$ is the line.

\textit{Proof.} Let $\overline{D} = \pi(D),$ $\overline{Z} = \pi(Z).$ We see that $\overline{Z}$ is a $G$-invariant curve in $\mathbb{P}^3$ such that such that $Z$ is not contained in a $G$-invariant surface in $|2\pi^*(H) - E_C|$ (by Lemma 4.3), $Z \not\subset E_L$ (by Lemma 4.2) and $Z \not\subset E_C$ (by Lemma 4.7). Then $\overline{Z} \subset (\mathbb{P}^3, \lambda \overline{D})$ and $\overline{Z}$ is not contained in any surface contained in $\text{Nklt}(\mathbb{P}^3, \lambda \overline{D})$ by Lemma 4.5. Now we apply Lemma 2.6 and get that $\mathcal{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} \leq 1.$ Thus $\mathcal{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} = 1$ so $\pi(Z)$ is a line.

\textbf{Corollary 4.9.} Such irreducible curve $Z$ does not exist.

\textit{Proof.} By Lemma 4.8 we know that $\pi(Z)$ is a line. We have that $\pi(Z) \neq L$ (by Lemma 4.2), $\pi(Z)$ is not one of the $G$-invariant lines which does not intersect $C$ and $\pi(Z) \neq L$ (by Lemma 4.1) and $\pi(Z)$ is not one of the $G$-invariant lines which intersect $C$ (by Corollary 4.4). So such irreducible curve $Z$ does not exist.

This completes the proof of Main Theorem.
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