PERFECT COMMUTING-OPERATOR STRATEGIES FOR LINEAR SYSTEM GAMES

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Abstract. Linear system games are a generalization of Mermin’s magic square game introduced by Cleve and Mittal. They show that perfect strategies for linear system games in the tensor-product model of entanglement correspond to finite-dimensional operator solutions of a certain set of non-commutative equations. We investigate linear system games in the commuting-operator model of entanglement, where Alice and Bob’s measurement operators act on a joint Hilbert space, and Alice’s operators must commute with Bob’s operators. We show that perfect strategies in this model correspond to possibly-infinite-dimensional operator solutions of the non-commutative equations. The proof is based around a finitely-presented group associated to the linear system which arises from the non-commutative equations.

1. Introduction

Mermin [8] implicitly considers a non-local game that is sometimes called the magic square game (see also [11, 9, 1, 4]). This game is based around a system of linear equations over \(\mathbb{Z}_2\) with nine variables and six equations. In the game, Alice receives as input one of the six equations, and Bob receives as input one of the variables from the same equation. Without communicating with each other, Alice must output an assignment of the variables in her equation, and Bob must output an assignment of his variable. The players win if and only if Alice’s assignment satisfies her equation and their assignments are consistent in the common variable. Remarkably, Alice and Bob can always win Mermin’s game if they use entanglement; there is no way to achieve this without entanglement.

Cleve and Mittal [3] investigate the general case of games based on binary linear systems of the form \(Mx = b\), where \(M \in \mathbb{Z}_2^{m \times n}\) and \(b \in \mathbb{Z}_2^n\). A solution of such a system is a vector \(x \in \mathbb{Z}_2^n\) such that \(Mx = b\). It is convenient to

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†In fact, they consider a more general scenario called binary constraint system games, where each equation can be based on an arbitrary boolean function of inputs.
write these equations in multiplicative form, so a vector $x \in \{\pm 1\}^n$ satisfies equation $\ell$ if and only if
\[ x_1^{M_{\ell,1}}x_2^{M_{\ell,2}} \cdots x_n^{M_{\ell,n}} = (-1)^{b_\ell}. \]

An equivalent way of writing equation $\ell$ is
\[ x_{k_1}x_{k_2} \cdots x_{k_r} = (-1)^{b_\ell}, \]
where $V_\ell = \{k_1, k_2, \ldots, k_r\} = \{1 \leq k \leq n : M_{\ell,k} = 1\}$ is the set of indices of variables in equation $\ell$. The non-local game associated with a binary linear system $Mx = b$ is defined similarly to that of the magic square game. A classical strategy is one where Alice and Bob do not share entanglement. It can be shown that $Mx = b$ has a perfect classical strategy (i.e., a strategy with success probability 1) if and only if the system of equations has a solution.

An entangled quantum strategy is a strategy in which Alice and Bob share an entangled quantum state $|\psi\rangle$. In the tensor-product model, $|\psi\rangle$ is a bipartite state in a tensor product $H_A \otimes H_B$, and Alice and Bob’s measurements of this state are modeled as observables on $H_A$ and $H_B$ respectively. It is shown in [3] that a binary linear system game has a perfect entangled strategy in the tensor-product model if and only if the linear system has a finite-dimensional operator solution in the following sense:

**Definition 1** (Operator solution of binary linear system). An operator solution to a binary linear system $Mx = b$ is a sequence of bounded self-adjoint operators $A_1, \ldots, A_n$ on a Hilbert space $H$ such that:

(a) $A_i^2 = 1$ (that is, $A_i$ is a binary observable) for all $1 \leq i \leq n$.

(b) If $x_i$ and $x_j$ appear in the same equation (i.e., $i, j \in V_\ell$ for some $1 \leq \ell \leq m$) then $A_i$ and $A_j$ commute (we call this local compatibility).

(c) For each equation of the form $x_{k_1}x_{k_2} \cdots x_{k_r} = (-1)^{b_\ell}$, the observables satisfy
\[ A_{k_1}A_{k_2} \cdots A_{k_r} = (-1)^{b_\ell}1 \]
(we call this constraint satisfaction).

A finite dimensional operator solution to a binary linear system $Mx = b$ is an operator solution in which the Hilbert space $H$ is finite dimensional.

The term local compatibility comes from quantum mechanics, where two observables commute if and only if they are compatible in the sense that they represent quantities which can be measured (or known) simultaneously. It is noteworthy that the result of [3] applies even when the Hilbert spaces $H_A$ and $H_B$ are allowed to be infinite dimensional; in this case, the operator solutions will still be finite dimensional.

In this paper we are interested in the commuting operator model for entanglement, in which $|\psi\rangle$ belongs to a joint Hilbert space $H$, and Alice and
Bob’s measurements are modeled as observables on $H$ with the property that Alice’s observables commute with Bob’s observables. This model—which clearly subsumes the tensor-product model—is used in algebraic quantum field theory. For any non-local game, a finite-dimensional strategy in the commuting-operator model can be converted into a strategy in the tensor-product model, but the precise relationship between the tensor-product model and the commuting-operator model is unknown in general. We refer to [13, 12, 7, 5] for more discussion.

The main result of our paper is that a binary linear system game has a perfect entangled strategy in the commuting operator model if and only if the linear system has a (possibly-infinite-dimensional) operator solution. As is typical with results of this type (compare for instance [10, Proposition 5.11]), the main difficulty arises in showing that an operator solution can be turned into a perfect strategy. In particular, an operator solution does not come with an entangled state. For this part of the proof, we make use of the fact that the relations for operator solutions in Definition 1 resemble (aside from the appearance of the scalar $(-1)$) the relations of a group presentation. If we represent $(-1)$ by a new variable $J$, we get a finitely-presented group $\Gamma$, which we call the solution group. We can then construct a tracial state on the group algebra of $\Gamma$ to use as our entangled state.

We do not know of any computational procedure that takes a description of a binary linear system $Mx = b$ as input and determines whether or not the game has a perfect entangled strategy. For tensor-product strategies, the characterization of perfect strategies in [3] can be used to certify the existence of a perfect tensor-product strategy, but cannot certify the non-existence of a perfect strategy. Interestingly, the situation seems to be reversed for commuting-operator strategies. We discuss this in some concluding remarks at the end of the paper.

All the results in this paper generalize to linear systems over $\mathbb{Z}_p$, $p$ a prime. For simplicity, we concentrate on the case of binary linear systems throughout. The generalization to arbitrary primes $p$ is briefly explained in the concluding remarks as well.

2. Main results

We now make some of the definitions from the introduction precise, starting with the definition of a linear system game.

Definition 2. Let $Mx = b$ be a binary linear system, so $M \in \mathbb{Z}_2^{m \times n}$ and $b \in \mathbb{Z}_2^m$. In the associated linear system game, Alice receives as input $s \in \{1, \ldots, m\}$, and Bob receives $t \in \{1, \ldots, n\}$, where $M_{s,t} = 1$. Alice outputs an assignment to the variables in equation $s$, and Bob outputs a bit. Alice and
Bob wins if Alice’s assignment satisfies equation $s$ and Alice’s assignment to variable $x_t$ is the same as Bob’s output bit.

We postpone the definition of commuting-operator strategies for linear system games to the following section. The next step is to define the solution group.

**Definition 3** (Solution group of a binary linear system). The solution group of a binary linear system $Mx = b$ is the group $\Gamma$ generated by $g_1, \ldots, g_n$ and $J$ satisfying the following relations (where $e$ is the group identity, and $[a, b] = aba^{-1}b^{-1}$ is the group commutator):

(a) $g_i^2 = e$ for all $1 \leq i \leq n$, and $J^2 = e$ (generators are involutions).

(b) $[g_i, J] = e$ for all $1 \leq i \leq n$ ($J$ commutes with each generator).

(c) If $x_i$ and $x_j$ appear in the same equation (i.e., $i, j \in V_\ell$ for some $\ell$) then $[g_i, g_j] = e$ (local compatibility).

(d) $g_1^{M_{\ell_1}}g_2^{M_{\ell_2}} \cdots g_n^{M_{\ell_n}} = J^{b_\ell}$ for all $1 \leq \ell \leq m$ (constraint satisfaction).

As in the introduction, the last relation can be written as

$$\prod_{i \in V_\ell} g_i = g_{k_1} \cdots g_{k_r},$$

where $V_\ell = \{k_1, \ldots, k_r\}$ are the indices of variables in equation $\ell$.

We can now state our main result:

**Theorem 4.** Let $Mx = b$ be a binary linear system. Then the following statements are equivalent:

1. There is a perfect commuting-operator strategy for the non-local game associated to $Mx = b$.
2. There is an operator solution for $Mx = b$ (possibly on an infinite-dimensional Hilbert space).
3. The solution group for $Mx = b$ has the property that $J \neq e$.

The proof of Theorem 4 is given in the next section. For comparison, we note that the main result of [3] can also be phrased using the solution group.

**Theorem 5 ([3]).** Let $Mx = b$ be a binary linear system. Then the following statements are equivalent:

1. There is a perfect tensor-product strategy for the non-local game associated to $Mx = b$.
2. There is a finite-dimensional operator solution for $Mx = b$.
3. The solution group for $Mx = b$ has a finite-dimensional representation for which $J \neq e$.

Although the solution group is not mentioned explicitly in [3], the equivalence with condition (3) is straightforward. The requirement in [3] that the
Hilbert spaces \(H_A\) and \(H_B\) be separable can also be dropped, since every entangled state in \(H_A \otimes H_B\) can be written as
\[
\sum_{k=1}^{\infty} \alpha_k |u_k\rangle \otimes |v_k\rangle
\]
for some orthonormal sets \(\{|u_k\rangle : k \in \mathbb{N}\} \subset H_A\) and \(\{|v_k\rangle : k \in \mathbb{N}\} \subset H_B\). We thank Vern Paulsen for pointing this out.

3. Proofs

To prove Theorem 4, we start by looking at commuting-operator strategies for linear system games. It is straightforward (see for instance [3]) that Alice’s and Bob’s measurements in such a strategy can be represented by families of binary observables
\[
\{A_{i}^{(\ell)} : 1 \leq \ell \leq m, i \in V_\ell\} \text{ and } \{B_{j} : 1 \leq j \leq n\}
\]
respectively, where \(A_{i}^{(\ell)}\) is the observable for Alice’s assignment to variable \(x_i\) in equation \(\ell\), and \(B_{j}\) is the observable for Bob’s assignment to variable \(x_j\).

Thus we can formally define commuting-operator strategies as follows:

**Definition 6.** Let \(Mx = b\) be an \(m \times n\) binary linear system. A commuting-operator strategy for (the game associated to) \(Mx = b\) consists of a Hilbert space \(H\), a state \(|\psi\rangle \in H\), and two collections \(\{A_{i}^{(\ell)} : 1 \leq \ell \leq m, i \in V_\ell\}\) and \(\{B_{j} : 1 \leq j \leq n\}\) of self-adjoint operators on \(H\) such that:

(a) \((A_{i}^{(\ell)})^2 = B_{j}^2 = 1\) for all \(1 \leq \ell \leq m, i \in V_\ell\), and \(1 \leq j \leq n\). (\(A_{i}^{(\ell)}\) and \(B_{j}\) are binary observables).

(b) \(A_{i}^{(\ell)}B_{j} = B_{j}A_{i}^{(\ell)}\) for all \(1 \leq \ell \leq m, i \in V_\ell\), and \(1 \leq j \leq n\). (Alice’s operators commute with Bob’s operators).

(c) \(A_{i}^{(\ell)}A_{j}^{(\ell)} = A_{j}^{(\ell)}A_{i}^{(\ell)}\) for all \(1 \leq \ell \leq m\) and \(i, j \in V_\ell\). (Local compatibility).

The local compatibility requirement comes from the fact that Alice must measure the observables \(A_{i}^{(\ell)}\), \(i \in V_\ell\), simultaneously. Using this definition, we can identify perfect strategies as follows:

**Proposition 7.** A commuting-operator strategy \((H, |\psi\rangle, \{A_{i}^{(\ell)}\}, \{B_{j}\})\) is perfect if and only if

(1) \(A_{i}^{(\ell)}|\psi\rangle = B_{j}|\psi\rangle\) for all \(1 \leq \ell \leq m\) and \(i \in V_\ell\) (consistency between Alice and Bob), and

(2) \(\prod_{i \in V_\ell} A_{i}^{(\ell)}|\psi\rangle = (-1)^{b_{\ell}}|\psi\rangle\) for all \(1 \leq \ell \leq m\) (constraint satisfaction).

**Proof.** Alice’s output is always consistent with Bob’s if and only if
\[
\langle \psi | A_{i}^{(\ell)}B_{j}|\psi\rangle = 1
\]
Similarly, if $i$ with operators, and hence is unitary. Since $|\psi\rangle$ is a unit vector, the above equation holds if and only if

$$A_i^{(\ell)} B_i |\psi\rangle = |\psi\rangle.$$  

Since $A_i^{(\ell)}$ is an involution, this equation is equivalent to the identity in part (1) of the proposition.

Similarly, Alice’s assignment for equation $\ell$ is always a satisfying assignment if and only if

$$\langle \psi \rangle (-1)^{by} \prod_{i \in V_{\ell}} A_i^{(\ell)} |\psi\rangle = 1.$$  

Again, $(-1)^{by} \prod_{i \in V_{\ell}} A_i^{(\ell)}$ is unitary, so the above equation is equivalent to the identity in part (2) of the proposition. \hfill \Box

Using Proposition 7, we can prove the first part of Theorem 4.

**Lemma 8.** Let $\left(\mathcal{H}, |\psi\rangle, \{A_i^{(\ell)} \}, \{B_j\} \right)$ be a perfect commuting-operator strategy for $Mx = b$, and let $\mathcal{H}_0 = \mathcal{A}|\psi\rangle$, where $\mathcal{A}$ is the unital algebra generated by $\{A_i^{(\ell)} \}$, and $\mathcal{A}|\psi\rangle = \{A|\psi\rangle : A \in \mathcal{A} \}$. Finally, let $Q_i := A_i^{(\ell)}|_{\mathcal{H}_0}$ for some $\ell$ with $i \in V_{\ell}$. Then $Q_1, \ldots, Q_n$ is an operator solution for $Mx = b$.

**Proof.** Let $\mathcal{B}$ be the unital algebra generated by $\{B_j\}$. By Proposition 7, we know that $A_i^{(\ell)}|\psi\rangle = B_i|\psi\rangle$ for all $i \in V_{\ell}$. Since $\mathcal{A}$ and $\mathcal{B}$ commute, if follows immediately that for every $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $A|\psi\rangle = B|\psi\rangle$. In particular, this tells us that $\mathcal{A}|\psi\rangle = \mathcal{B}|\psi\rangle$, and consequently that $\mathcal{H}_0 = \mathcal{B}|\psi\rangle$.

Now suppose we have $A, A' \in \mathcal{A}$ such that $A|\psi\rangle = A'|\psi\rangle$. Then

$$AB|\psi\rangle = BA|\psi\rangle = BA'|\psi\rangle = A'B|\psi\rangle$$

for all $B \in \mathcal{B}$. By continuity, we conclude that $A|_{\mathcal{H}_0} = A'|_{\mathcal{H}_0}$. Suppose that variable $x_i$ belongs to equations $\ell$ and $\ell'$, or in other words that $i \in V_{\ell} \cap V_{\ell'}$. Then

$$A_i^{(\ell)}|\psi\rangle = B_i|\psi\rangle = A_i^{(\ell')}|\psi\rangle$$

by Proposition 7 again. We conclude that $A_i^{(\ell)}|_{\mathcal{H}_0} = A_i^{(\ell')}|_{\mathcal{H}_0}$, and thus $Q_i = A_i^{(\ell)}|_{\mathcal{H}_0}$ is independent of the choice of $\ell$.

We can now check that $Q_1, \ldots, Q_n$ is an operator solution. Since $\mathcal{H}_0$ is $\mathcal{A}$-invariant,

$$Q_i^2 = (A_i^{(\ell)})^2 |_{\mathcal{H}_0} = 1_{\mathcal{H}_0}.$$  

Similarly, if $i$ and $j$ both belong to $V_{\ell}$, then

$$Q_i Q_j = A_i^{(\ell)} A_j^{(\ell)} |_{\mathcal{H}_0} = A_j^{(\ell)} A_i^{(\ell)} |_{\mathcal{H}_0} = Q_j Q_i.$$
Finally,
\[
\prod_{i \in V_\ell} A_i^{(\ell)} |\psi\rangle = (-1)^{b_\ell} |\psi\rangle
\]
by Proposition 7 and hence
\[
\prod_{i \in V_\ell} Q_i = \prod_{i \in V_\ell} A_i^{(\ell)} |\psi_0\rangle = (-1)^{b_\ell} \mathbb{1}_{\mathcal{H}_0}
\]
for all \(1 \leq i \leq \ell\).

The second part of Theorem 4 is easy to prove.

Lemma 9. If \(Mx = b\) has an operator solution then \(J \neq e\) in the solution group \(\Gamma\) of \(Mx = b\).

Proof. Suppose \(A_1, \ldots, A_n\) is an operator solution for \(Mx = b\). By Definitions 4 and 5, the map sending
\[
g_i \mapsto A_i, \; 1 \leq i \leq n \quad \text{and} \quad J \mapsto -\mathbb{1}
\]
is a representation of \(\Gamma\) with \(J \neq 1\). It follows that \(J \neq e\) in \(\Gamma\).

Proof of Theorem 4. We have shown in Lemmas 8 and 9 that (1) implies (2) and (2) implies (3). It remains to show that (3) implies (1). Suppose that \(J \neq e\) in the solution group \(\Gamma\). We need to construct a perfect commuting-operator strategy for \(Mx = b\).

Let \(\mathcal{H} = \left\{ \sum_{g \in \Gamma} \alpha_g |g\rangle : \alpha_g \in \mathbb{C} \text{ such that } \sum_{g \in \Gamma} |\alpha_g|^2 < \infty \right\}\)
be the completion of the group algebra of \(\Gamma\). Given \(g \in \Gamma\), let \(L_g\) and \(R_g\) be the left and right multiplication operators for \(g\) on \(\mathcal{H}\), so
\[
L_g |h\rangle = |gh\rangle \quad \text{and} \quad R_g |h\rangle = |hg\rangle.
\]
Clearly, \(L_g\) and \(R_g\) are unitary. Furthermore,
\[
L_g R_h = R_h L_g, \quad L_g L_h = L_{gh}, \quad \text{and} \quad R_g R_h = R_{hg}
\]
for all \(h, g \in \Gamma\). We set
\[
A_i^{(\ell)} := L_{g_i} \text{ for all } 1 \leq \ell \leq m, \; i \in V_\ell,
\]
and
\[
B_j := R_{g_i} \text{ for all } 1 \leq j \leq n.
\]
Finally we set
\[
|\psi\rangle := \frac{|e\rangle - |J\rangle}{\sqrt{2}}.
\]
Since \(J \neq e\) in \(\Gamma\), \(|\psi\rangle\) is a well-defined unit vector in \(\mathcal{H}\). Since
\[
L_{g_i}^2 = L_{g_i} = L_e = 1 = R_{g_i}^2, \text{ for all } 1 \leq i \leq n
\]
and

\[ L_{g_i} L_{g_j} = L_{g_j} L_{g_i} = L_{g_i} L_{g_j} \]

for all \( 1 \leq \ell \leq m \) and \( i, j \in V_\ell \),

it is clear that \( \{ A_i^{(\ell)} \}, \{ B_i \}, \) and \( |\psi\rangle \) form a valid commuting-operator strategy for \( Mx = b \). To show that they form a perfect strategy, observe that

\[ A_i^{(\ell)} |\psi\rangle = \frac{|g_i\rangle - |g_i J\rangle}{\sqrt{2}} = \frac{|g_i\rangle - |J g_i\rangle}{\sqrt{2}} = B_i |\psi\rangle \]

for all \( 1 \leq \ell \leq m \) and \( i \in V_\ell \), and that

\[ \prod_{i \in V_\ell} A_i |\psi\rangle = \prod_{i \in V_\ell} L_{g_i} |\psi\rangle = L_{J^\ell} |\psi\rangle, \]

for all \( 1 \leq \ell \leq m \). If \( b_\ell = 0 \), then

\[ L_{J^\ell} |\psi\rangle = L_{e} |\psi\rangle = |\psi\rangle, \]

while if \( b_\ell = 1 \) then

\[ L_{J^\ell} |\psi\rangle = L_{J} |\psi\rangle = \frac{|J\rangle - |e\rangle}{\sqrt{2}} = -|\psi\rangle. \]

Therefore, \( \prod_{i \in V_\ell} A_i |\psi\rangle = (-1)^{b_\ell} |\psi\rangle \) for all \( 1 \leq \ell \leq m \). Thus the strategy we have constructed is perfect by Proposition 7.

\[ \Box \]

4. Concluding remarks

As mentioned in the introduction, we do not know of any computational procedure which can determine if a binary linear system has a perfect entangled strategy. Arkhipov showed that, in the special case where each variable appears in exactly two constraints, there is a polynomial-time algorithm to determine if a perfect entangled strategy exists [2] (in this case, a game has a perfect commuting-operator strategy if and only if it has a perfect tensor-product strategy). For the general case, we can attempt to use the characterization of perfect strategies in [3] by searching for operator solutions over \( \mathbb{C}^d \), \( d \in \mathbb{N} \). It is decidable to determine if there is an operator solution over \( \mathbb{C}^d \) for fixed \( d \), and thus this naive procedure is guaranteed to find a perfect strategy if one exists. However, if a perfect strategy does not exist, then the naive procedure does not halt. We note that, for arbitrarily large \( d \), Ji gives examples of binary linear systems which have finite-dimensional operator solutions, but for which the solutions require dimension at least \( d \) [6].

In contrast, there is no apparent way to search through operator solutions over infinite-dimensional Hilbert spaces. What we can do instead is try to show that \( J = e \) in the group \( \Gamma \) by searching through products of the defining relations. Using our characterization, we see that this procedure will halt if and only if the linear system game does not have a perfect strategy in
the commuting-operator model. Thus this problem would be decidable if the
tensor-product model and commuting-operator model were equivalent. Determin-
ing whether or not these two models are equivalent is a well-known open
problem due to Tsirelson [13].

As also mentioned in the introduction, the results in this paper generalize to
linear systems over $\mathbb{Z}_p$. The non-local game associated to a system over $\mathbb{Z}_p$ is
defined in exactly the same way, although Alice and Bob output assignments
from $\mathbb{Z}_p$ rather than $\mathbb{Z}_2$. Similarly, commuting-operator strategies are modelled
using measurements based on unitary operations $U$ with $U^p = \mathbb{1}$, rather than
$U^2 = \mathbb{1}$. Likewise, the definition of the solution group must be changed so that
$g_i^p = e$ and $J^p = e$. Finally, the state $|\psi\rangle$ in the proof of Theorem 4 becomes

$$\frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} \zeta^{-i} |J^i\rangle,$$

where $\zeta$ is a primitive $p$th root of unity. Otherwise all definitions, propositions,
and proofs are the same.

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