Average order in wreath products

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Abstract

We obtain an exact formula for the average order of elements of a wreath product of two finite groups. Then focusing our attention on $p$-groups for primes $p$, we give an estimate for the average order of a wreath product $A \wr B$ in terms of maximum order of elements of $A$ and average order of $B$ and an exact formula for the distribution of orders of elements of $A \wr B$. Finally, we show how wreath products can be used to find several rational numbers which are limits of average orders of a sequence of $p$-groups with cardinalities going to infinity.

Keywords: Group, average order, wreath product, maximum order, order distribution, semidirect product

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1 Introduction

For a finite group $G$, define the average order of elements to be $a(G) = \frac{\sum_{g \in G} \text{order}(g)}{|G|}$ and denote the maximum order of an element of $G$ by $m(G)$. It is well known that among all groups of a fixed cardinality, the cyclic group has the largest average order (see [1], [2]). Other authors have studied the average order function with a view to deriving characterization theorems for nilpotency and solvability (see [3], [4]).

Wreath products of groups often provide examples and counter-examples for various group theoretic questions. The famous Krasner-Kaloujnine embedding theorem shows that for any two groups $A$ and $H$, any extension of $A$ by $H$ is isomorphic to a subgroup of the wreath product $A \wr H$ (see [5]).

In this article, we obtain an exact formula for the average order of a wreath product of two finite groups. Following that, we focus our attention on $p$-groups for primes $p$. We give an estimate for the average order of a wreath product $A \wr B$ in terms of $m(A)$ and $a(B)$. In addition, we obtain an exact formula for the distribution of orders of elements of $A \wr B$, in terms of distributions of orders of elements of $A$ and $B$. Finally, we show how wreath products can be used to find several rational numbers which are limits of average orders of a sequence of $p$-groups with cardinalities going to infinity.
2 Notations and Conventions

1. Given two groups $A$ and $B$, let $K = \prod_{b \in B} A$. The group $B$ naturally acts on $K$ by $x \cdot (\alpha_b)_b = (\alpha_{x^{-1}b})_b$, for $x \in B$, $(\alpha_b)_b \in K$. The semidirect product $K \rtimes B$ is called the wreath product of $A$ by $B$ and is denoted by $A \wr B$.

2. $\mu : \mathbb{N} \to \mathbb{Z}$ denotes the Mobius function. Recall that if $n$ is square-free, $\mu(n) = (-1)^m$, where $m$ is the number of distinct prime divisors of $n$. If $n$ is not square-free, $\mu(n) = 0$.

3. For a natural number $n$, define $\tau(n) = \prod_{p|n, p \text{ prime}} (1 - \frac{1}{p})$. It is easy to see that $\tau(n) = \sum_{d|n} d \mu(d)$.

4. For real valued functions $f$ and $g$ we adopt the notation $g = O(f)$ to mean that there is a constant $M > 0$ such that $|g| \leq M |f|$ always.

5. Define $\psi(A, B) = a(A \wr B) \cdot a(B)^{m(A)}$, for $p$-groups $A$ and $B$.

6. Let $A$ be a $p$-group of cardinality $p^a$ and $m(A) = p^d$. For $k \in \mathbb{Z}$ define $r_{A,k} = \frac{1}{p^d} \times \text{Number of elements of order at most } p^{d-k}$.
   This essentially denotes the cumulative distribution function of the order distribution. So, $r_{A,k}$ is a non-increasing function of $k$, and $r_{A,k} = 1$ for $k \leq 0$, $r_{A,k} = 0$ for $k > d$.

7. Fix a prime $p$. Let us call a real number $\beta$ an “Average Order Limit” if there is a sequence of $p$-groups $G_n$ with $|G_n| \to \infty$ and $a(G_n) \to \beta$.

3 Main Results

Theorem 1 Let $A$, $B$ be finite groups with at least 2 elements. For $m \geq 1$, let $s_m$ be the number of elements of $A$ whose $m$-th power is 1. For $n|b$, let $d_n$ be the number of elements of $B$ of order $n$. Then

$$a(A \wr B) = \sum_{m \mid |A|, n \mid |B|} \frac{m s_m}{|A|^n d_n} \cdot \frac{|A|}{m} \tau \left( \frac{|A|}{m} \right).$$

Let $p$ be a prime. From now on, assume $A$, $B$ are $p$-groups, $|A| = p^a$, $|B| = p^b$, $a, b \geq 1$.

Theorem 2 With notations as in theorem 1,

$$a(A \wr B) = p^d a(B) - (p - 1) a(B) \sum_{n \leq b} k_n \cdot \left[ \sum_{m \leq p^{d-1}} p^m \left( \frac{v_m}{p^a} \right)^n \right],$$

where $k_n = \frac{p^{-n} d_{b-n} \cdot \psi(A, B)}{a(B)}$. Note that $k_n \geq 0 \forall n$, $\sum_{n \leq b} k_n = 1$.

Theorem 3 $a(B) \leq a(A \wr B) \leq p^d a(B)$. 

\[2\]
Note that Theorem 3 implies that $0 \leq \psi(A, B) \leq 1$.

**Theorem 4** With notations as in theorem 1,

$$a(A \wr B) = p^d a(B) - (p - 1)a(B) \sum_{n \leq b} k_n [p^{nd-1} \left( \frac{8p^{d-1}}{p^a} \right) p^n] + O(p^{d-1}a(B)).$$

Here, the implicit constant in $O(p^{d-1}a(B))$ is independent of $p$ as well as of the groups. Now, we explicitly write the order distribution of $A \wr B$ in terms of the order distributions of $A$ and $B$.

**Theorem 5** Let $m(A) = p^d$, $m(B) = p^e$. Then we have $m(A \wr B) = p^{d+e}$, and $r_{A,B,k} = \sum_{i=0}^{r} (r_{B,i} - r_{B,i+1}) r_{A,k-i}^1$.

**Corollary 1** $r_{A/Z/pZ,k} = (1 - p^{-1}) r_{A,k} + \frac{r_{A,k-1}}{p}$.

In view of theorem 4, it is natural to ask whether $a(A \wr B) = p^d a(B) + O(p^{d-1}a(B))$, i.e., whether “$\psi(A, B) = 1 + O(\frac{1}{p})$”. We shall show that this is not true in general. However, if we assume $A$ is abelian, then it is true.

**Corollary 2** Let $A$ be a $p$-group. Define $p$-groups $A_n$ recursively by $A_0 = A$, $A_n = A_{n-1} \wr \mathbb{Z}/p^k \mathbb{Z}$. Let $B$ be a $p$-group. Then $\psi(A_n, B) \to 0$ as $n \to \infty$.

Note that corollary 5.2 shows that “$\psi(A, B) = 1 + O(\frac{1}{p})$” cannot be true in general. Now we show how the situation differs if we assume $A$ to be abelian.

**Theorem 6** For abelian $p$-groups $A$, define $t(A)$ to be the unique positive integer such that $A \cong \mathbb{Z}/p^{d_1} \mathbb{Z} \oplus \mathbb{Z}/p^{d_2} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{d_k} \mathbb{Z} \oplus (\mathbb{Z}/p^d \mathbb{Z})^{t(A)}$, where $d_1 \leq d_2 \leq \ldots \leq d_k < d$. Then,

(i) For each noncyclic group $B$, $1 - p^{-t(A)p} \leq \psi(A, B) \leq 1$.

(ii) For cyclic groups $B$, $\psi(A, B) = 1 - p^{-t(A)} + O(p^{-t(A)-1})$.

In particular, $\psi(A, B) = 1 + O(\frac{1}{p})$ holds if $A$ is abelian.

Now we prove a result about limiting value of average orders for a sequence of groups whose sizes go to infinity.

**Theorem 7** Suppose $B_n$ is a sequence of $p$-groups with $a(B_n) \to \beta$. Then $a((\mathbb{Z}/p^k \mathbb{Z})^n \wr B_n) \to p \beta$. So, if $\beta$ is an Average Order Limit, then $p \beta$ is also. Hence $p \beta$ is an Average Order Limit for each nonnegative integer $r$.

Note that whatever the sizes of $B_n$ be, sizes of $(\mathbb{Z}/p^k \mathbb{Z})^n \wr B_n$ always go to infinity. So, taking $\{B_n\}$ to be a constant sequence $B$, we see that $p^r a(B)$ is an Average Order Limit, for any $r \geq 2$. This guarantees the existence of many non-integral Average Order Limits. For example, taking $B = (\mathbb{Z}/p^k \mathbb{Z})^b$ and $2 \leq r \leq b - 1$, we get $p^{r+b+1} \left( 1 - \frac{1}{p^r} \right)$ as an Average Order Limit.

**Corollary 3** Given any $p$-group $B$, and integer $r \geq 2$, there are sequences $G_n$, $H_n$ of $p$-groups with $|G_n| \to \infty$, $|H_n| \to \infty$ and $a(G_n \wr H_n) \to p^r a(B)$.
4 Proofs of the Theorems

We start with two preliminary lemmas.

**Lemma 1** For any semidirect product \( H \rtimes K \), we have \( a(H \rtimes K) \geq a(K) \). Also, for \((h, k) \in H \rtimes K\), we have \( \text{order}(k) | \text{order}(h) \).

**Proof.** For any \((h, k) \in H \rtimes K\), \((h, k)^m\) has \( k^m \) as 2nd coordinate. So, \((h, k)^m = 1 \implies k^m = 1\). So, \( \text{order}(k) | \text{order}(h) \), in particular \( \text{order}(h) \geq \text{order}(k) \), for all \( h \in H \). So, \( a(H \rtimes K) \geq \sum_{k \in K} \text{order}(k) = \frac{|H|}{|K|} \sum_{k \in K} \text{order}(k) = a(K) \).

**Lemma 2** Let \( p \) be a prime and \( b \geq 1 \) an integer. Then \( a(\mathbb{Z}/p^b\mathbb{Z}) = p^b + O(p^{b-1}) \).

**Proof.** In \( \mathbb{Z}/p^b\mathbb{Z} \), there are exactly \( \phi(p^n) = p^n(1 - p^{-1}) \) elements of order \( p^n \), for each \( 1 \leq n \leq b \). So,

\[
a(\mathbb{Z}/p^b\mathbb{Z}) = \frac{1 + \sum_{n=1}^{b} p^n \cdot p^n \cdot (1 - p^{-1})}{p^b} = \frac{1 + p(p - 1) \sum_{n=0}^{b-1} p^{2n}}{p^b} = 1 + \frac{(p^2 - p) \binom{2b-1}{p-1}}{p^b} = \frac{1 + \frac{p}{p+1}(p^2b - 1)}{p^b} = p^{-b} + \frac{p}{p+1}(p^b - p^b) = p^b + O(p^{b-1}).
\]

Let us recall some standard facts before starting the proof of theorem 1. For groups \( H, K \), and injective group homomorphism \( \psi : K \to Aut(H) \), we have an injective group homomorphism \( \psi : H \rtimes K \to Perm(H) \) defined by \( \psi((h, k)) = h \phi(k) \). So, \( H \rtimes K \) can be regarded as a subgroup of \( Perm(H) \), consisting of the transformations \( T_{h,k} \), defined by \( T_{h,k}(x) = h \phi(k)(x) \).

So, if \( A \) and \( B \) has at least 2 elements, \( A \) is the subgroup of \( Perm(\Pi_{b \in B} A) \), consisting of the permutations \( T_{a,x}, a \in \Pi_{b \in B} A, x \in B \), where \( T_{a,x}(b) = (\alpha_b \alpha_{x^{-1}} b, (a = (a_b)b)) \).

**Proof of Theorem 1.** Note that lemma 1 yields \( \text{order}(x) | \text{order}(T_{a,x}) \) for all \((a, x) \in A \ast B\). Fix \( x \in B \). Let \( d = \text{order}(x) \). For \( m \geq 1 \), we have \( T_{a,x}^{dm - 1}(a) = (\alpha_b \alpha_{x^{-1}} b, (a = (a_b)b)) \).

So,

\[
T_{a,x}^{dm - 1} = 1 \iff \alpha_b \alpha_{x^{-1} b} \alpha_{x^{-1} b}^{m - 1} b = 1 \quad \forall \quad b \in B
\]

Name this condition (1).

Multiplication by \( x \) divides \( B \) into \( \frac{|B|}{d} \) orbits of size \( d \), and (1) is equivalent to saying that product of \( \alpha_b \)'s, for \( b \) running in each orbit in cyclic order, has \( m \)'th power = 1.
Number of $\alpha \in \prod_{b \in B} A$ satisfying (1) is exactly $\binom{|A|^{d-1}s_m}{B}$. The reason is as follows. Fix a point $p$ in a orbit. For each $b \neq p$ in that orbit, $\alpha_b$ can be chosen to be anything in $A$. After that $\alpha_p$ has exactly $s_m$ choices. Altogether we get $\alpha_b$’s for $b$ running over a fixed orbit has exactly $|A|^{d-1}s_m$ choices. There are $\binom{|B|}{d}$ orbits. So in total we have $\binom{|A|^{d-1}s_m}{B}$ choices for $\alpha$.

From (1), it is clear that $T^{d|A|} = 1$. So, $\text{order}(T_{\alpha_{x^{-1}}}) = |A|$

For $m \geq 1$, let $g_m = \text{number of } \alpha \in \prod_{b \in B} A \text{ with order}(T_{\alpha_{x^{-1}}}) = dm$. We have just shown that unless $m \mid |A|$, we have $g_m = 0$. Also, for all $k \geq 1$,

$$\sum_{m \mid k} g_m = \text{number of } \alpha \text{ with } (T_{\alpha_{x^{-1}}})^{dk} = 1 = \binom{|A|^{d-1}s_m}{B}$$

By Mobius inversion, $g_m = |A|^{B(d-1)} \sum_{n \mid m} \mu(m) s_n^m$ for all $m \geq 1$. So,

$$\sum_{\alpha \in \prod_{b \in B} A} \text{order}(T_{\alpha_{x^{-1}}}) = \sum_{m} \binom{A}{B} m \mu(m) s_n^m$$

This is true for any $x \in B$ or order $d$. There are $d_n$ elements of order $n$ in $B$, for each $n \mid |B|$. So,

$$\sum_{x \in A \setminus B \text{ order } x} = \sum_{n \mid |B|} d_n \binom{A}{B} m \mu(m) s_n^m$$

Proof of Theorem 2. Theorem 1, together with the observation that $s_p^m = p^a$ for $m \geq d$ yields

$$a(A \setminus B) = \sum_{n \leq b} p^a \sum_{p^a} p^n d_{p^b-n}$$

$$- (p-1) \sum_{m \leq a-1, n \leq b} p^m \sum_{p^m} p^n d_{p^b-n}$$

$$= p^a \sum_{n \leq b} p^n d_{p^b-n}$$

$$- (p-1) \sum_{d \leq m \leq a-1, n \leq b} p^m \sum_{p^m} p^n d_{p^b-n}$$

$$- (p-1) \sum_{m \leq d-1, n \leq b} p^m \sum_{p^m} p^n d_{p^b-n}$$

$$= p^a a(B) - (p-1)p^d \sum_{m \leq a-d-1} p^m \sum_{n \leq b} p^n d_{p^b-n}$$
\[(p - 1) \sum_{m \leq d - 1, n \leq b} \frac{\mu_m}{p^m} (\frac{s_m}{p^n})^n d_{p^n} = p^d a(B) - (p^d - 1) p^d a(B) \]
\[(p - 1) \sum_{m \leq d - 1, n \leq b} \frac{\mu_m}{p^m} (\frac{s_m}{p^n})^n d_{p^n} = p^d a(B) - (p - 1) \sum_{m \leq d - 1, n \leq b} p^m (\frac{s_m}{p^n})^n d_{p^n} \]
\[(p - 1) \sum_{m \leq d - 1, n \leq b} \frac{\mu_m}{p^m} (\frac{s_m}{p^n})^n d_{p^n} = p^d a(B) - (p - 1) \sum_{n \leq k} p^n d_{p^n} - \sum_{m \leq d - 1} p^m (\frac{s_m}{p^n})^n \]
\[(p - 1) a(B) - (p - 1) a(B) \sum_{n \leq b} k_n \| \sum_{m \leq d - 1} p^m (\frac{s_m}{p^n})^n \].

**Proof of theorem 3.** Theorem 2 proves the second inequality, and the first inequality follows from lemma 1.

**Proof of theorem 4.** Let us look at theorem 2 more closely. \( \frac{s_m}{p^n} < 1 \) \( \forall m \leq d - 1 \). So, \( \sum_{m \leq d - 2} p^m (\frac{s_m}{p^n})^n \leq \sum_{m \leq d - 2} p^m = \frac{d^2 - 1}{d - 1} \), for each \( n \leq b \).
We also have \( k_n \geq 0 \) \( \forall n \), \( \sum_{n \leq b} k_n = 1 \). Hence,
\[(p - 1) a(B) \sum_{n \leq b} k_n \| \sum_{m \leq d - 2} p^m (\frac{s_m}{p^n})^n \] = \( O(p^d a(B)) \).

**Proof of theorem 5.** Let \( x \in B \), order \( (x) = p^{e_1} \), \( \alpha \in \prod_{b \in B} A \). Multiplication by \( x \) divides \( B \) into \( p^{d - e_1} \) orbits of size \( p^{e_1} \); let \( b_1, b_2, ..., b_{p^{e_1}} \) be representatives of distinct orbits. By (1) of theorem 1, the order of \( T_{\alpha, x} \) equals \( m(\alpha) = \sum_{i=1}^{d} \alpha_i \).

If \( x \in B \) is of order \( p^e \), and \( y \in A \) is of order \( p^e \), then the order of \( T_{\alpha, x} \) equals \( p^d + e \), where \( \alpha \) is defined by \( \alpha_1 = y \), \( \alpha_2 = 1 \forall b \neq 1 \). So, \( m(\alpha) = p^d + e \).

Now we prove the second statement.

Again, let \( x \in B \), order \( (x) = p^{e_1} \). Let \( k \leq d + e - e_1 \). Note that
\[
T_{\alpha, x} = 1 \iff (\alpha_0, \alpha_{b_1}, ..., \alpha_{p^{e_1} - 1}) = 1 \forall i.
\]

**Lemma 3** For all \( k \), \( \lim_{n \to \infty} r_n k = 1 \).

**Proof.** We proceed by induction on \( k \). For \( k \leq 0 \), we have \( r_n k = 1 \) \( \forall n \), and nothing to show. For the induction step, \( k \geq 1 \) and assume \( \lim_{n \to \infty} r_n k-1 = 1 \). By corollary 5.1., \( r_{n+1,k} = (1 - p^{-1}) r_n k + \frac{r_{n,k-1}}{p} \). So,
\[ \lim\inf_{n \to \infty} r_{n+1,k} = (1-p^{-1}) \lim\inf_{n \to \infty} r_{n,k} + p^{-1}, \] as induction hypothesis implies \( \lim_{n \to \infty} r_{n,k-1} = 1. \]

But \( \lim\inf_{n \to \infty} r_{n+1,k} = \lim\inf_{n \to \infty} r_{n,k}. \) Hence, \( \lim\inf_{n \to \infty} r_{n,k} = (1-p^{-1}) \lim\inf_{n \to \infty} r_{n,k} + p^{-1}; \) that is, \( \lim\inf_{n \to \infty} r_{n,k} = 1. \) Since we always have \( 0 \leq r_{n,k} \leq 1, \) we get \( \lim_{n \to \infty} r_{n,k} = 1. \) Induction completes the proof.

**Proof of corollary 5.2.** Let \( m(A) = p^d, |B| = p^b. \) By theorem 5, \( m(A_n) = p^{d+n} \) for all \( n. \) By theorem 2,

\[
\psi(A_n, B) = 1 - \frac{p^{-1}}{p^d+n} \sum_{r \leq b} k_r \left( \sum_{m=0}^{d+n-1} p^m p^{r-m} \right) \\
= 1 - \left( 1 - p^{-1} \right) \sum_{r \leq b} k_r \left( \sum_{m=0}^{d+n-1} p^{-(d+n-1-m)} r_{n,(d+n-1-m)+1} \right) \\
= 1 - \left( 1 - p^{-1} \right) \sum_{r \leq b} k_r \left( \sum_{m=0}^{d+n-1} p^{-m} r_{n,m+1} \right) \quad \text{(replace } m \text{ by } d+n-1-m). \]

Fix \( M \in \mathbb{N}. \) For all sufficiently large \( n, \) we have \( d+n-1 > M, \) hence \( \psi(A_n, B) \leq 1 - (1-p^{-1}) \sum_{r \leq b} k_r \left( \sum_{m=0}^{M} p^{-m} r_{n,m+1} \right). \) Since \( \lim\inf_{n \to \infty} r_{n,m+1} = 1 \) by lemma 3, we get

\[
\limsup_{n \to \infty} \psi(A_n, B) \leq 1 - (1-p^{-1}) \sum_{r \leq b} k_r \left( \sum_{m=0}^{M} p^{-m} r_{n,m+1} \right) \\
= 1 - (1-p^{-1}) \sum_{m=0}^{M} p^{-m} \quad \text{(as } \sum_{r \leq b} k_r = 1 \text{ for all } M \in \mathbb{N}). \]

Taking \( M \to \infty, \) we get \( \limsup_{n \to \infty} \psi(A_n, B) \leq 0, \) that is, \( \lim_{n \to \infty} \psi(A_n, B) = 0. \)

**Proof of theorem 6.** We use the notation of theorem 2, and write \( t \) instead of \( t(A). \) Projection gives a surjective group homomorphism \( \phi: A \to (\mathbb{Z}/p^d\mathbb{Z})^t. \)

Note that the subgroup of \( p^{d-1} \)-torsion elements of \( A \) is \( \phi^{-1}(p\mathbb{Z}/p^d\mathbb{Z}) \), and for \( 0 \leq m \leq d-2 \), the subgroup of \( p^{d-m} \)-torsion elements of \( A \) is contained in \( \phi^{-1}(p^m\mathbb{Z}/p^d\mathbb{Z}) \). So, \( \frac{p^{d-1}}{p^m} = p^{-1}, \) and for each \( 0 \leq m \leq d-2, \) we have \( \frac{s_m}{p^m} \leq p^{-mt} \leq p^{-2t}. \) By theorem 2,

\[
p^d a(B) - a(A \langle B \rangle) = (p-1) a(B) k_0 \cdot \left[ \sum_{m=0}^{d-1} p^m \frac{s_m p^m}{p^d} \right] \\
= (p-1) a(B) \sum_{1 \leq n \leq b} k_n \cdot \left[ \sum_{m=0}^{d-1} p^m \left( \frac{s_m}{p^d} \right) p^n \right] \\
\leq (p-1) a(B) \sum_{1 \leq n \leq b} k_n \sum_{m=0}^{d-1} \frac{p^m}{p^d p^n} \\
= (p-1) a(B) \left( \sum_{1 \leq n \leq b} k_n p^{-tp^n} \right) \cdot \frac{p^d-1}{p-1} \\
\leq a(B) \sum_{1 \leq n \leq b} p^{d-tp^n} k_n \\
\leq a(B) p^{d-tp} \sum_{1 \leq n \leq b} k_n \leq a(B) p^{d-tp}. \]

If \( B \) is noncyclic, \( k_0 = 0; \) so \( p^d a(B) - a(A \langle B \rangle) \leq a(B) p^{d-tp}. \) Dividing by \( p^d a(B), \) we get \( 1 - p^{-tp} \leq \psi(A, B). \) If \( B \) is the cyclic group \( \mathbb{Z}/p^d\mathbb{Z}, \) we have
\[ k_0 = \frac{d(p^h)}{n(B)} = \frac{1}{p} - \frac{p^h}{n(B)} = (1 + O(\frac{1}{p}))(1 + O(\frac{1}{t})) = 1 + O(\frac{1}{p}). \]

Here, we used lemma 2, and the observation that \((1 + O(p^{-1}))^{-1} = 1 + O(p^{-1})\). So, \(\sum_{m \leq d-1} p^{m_s_m} = p^{d-1} + O(\sum_{m \leq d-2} p^{m-2t}) = p^{d-1} + O\left(\frac{p^{d-1} - 1}{p-1}\right) \cdot p^{d-2t} = p^{d-1} + O(\frac{p^{d-2-2t}}{p^t})\).

Hence, \(\frac{(p-1)a(B)k_0 \cdot \sum_{m \leq d-1} p^{m_s_m}}{p^d a(B)} = \frac{(p-1)(1 + O(p^{-1}))(p^{d-1} + O(p^{d-2-2t}))}{p^t} = \frac{p^{d-t} + O(p^{d-1-t})} = p^{-t} + O(p^{-1}).\)

We have shown
\[ 0 \leq p^t a(B) - a(A \uplus B) - (p - 1)a(B)k_0 \cdot \sum_{m \leq d-1} p^{m_s_m} \leq a(B)p^{d-t}. \]

So, dividing by \(p^t a(B)\) we get
\[ \psi(A, B) = 1 - \frac{(p-1)a(B)k_0 \cdot \sum_{m \leq d-1} p^{m_s_m}}{p^d a(B)} + O(p^{-t}) \]
\[ = 1 + p^{-t} + O(p^{-t+1}) + O(p^{-1}) = 1 + p^{-t} + O(p^{-1}). \]

Proof of theorem 7. Let \(|B_n| = p^{bn}.\) By theorem 2,
\[ a((Z/pZ)^n \uplus B_n) = p \cdot a(B_n) - (p - 1)a(B_n) \sum_{m \leq b_n} k_{m,n}(\frac{1}{p^n})p^n, \]
for some \(0 \leq k_{m,n} \leq 1,\) with \(\sum_{m \leq b_n} k_{m,n} = 1\) for each \(n.\)

So, for all sufficiently large \(n,\) so that \(a(B_n) \leq \beta + 1;\) we have
\[ |a((Z/pZ)^n \uplus B_n) - p\beta| \leq p|a(B_n) - \beta| + (p - 1)(\beta + 1)p^{-n} \to 0. \]

Proof of corollary 7.1. Let \(G_n = (Z/pZ)^n, H_n = (Z/pZ)^n \uplus ((Z/pZ)^n \uplus (\ldots((Z/pZ)^n \uplus B)),\) there are \(r - 1\) \((Z/pZ)^n)’s here. Now corollary 7.1 follows by repeated application of theorem 7.

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6 Declaration of interest

No potential conflict of interest was reported by the author.

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