Entanglement and nonlocality versus spontaneous emission
in two – atom system

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Abstract: We study evolution of entanglement of two two-level atoms in the presence of dissipation caused by spontaneous emission. We find explicit formulas for the amount of entanglement as a function of time, in the case of destruction of the initial entanglement and possible creation of a transient entanglement between atoms. We also discuss how spontaneous emission influences nonlocality of states expressed by violation of Bell - CHSH inequality. It is shown that evolving system very quickly becomes local, even if entanglement is still present or produced.

1. Introduction

The process of spontaneous emission by a system of two - level atoms was extensively studied by several authors (see e.g. [1, 2, 3, 4]). In particular, in the case of spontaneous emission by two trapped atoms separated by a distance small compared to the radiation wavelength, where is a substantial probability that a photon emitted by one atom will be absorbed by the other, there are states of the system in which photon exchange can enhance or diminish spontaneous decay rates. The states with enhanced decay rate are called superradiant and analogously states with diminished decay rate are called subradiant [1]. It was also shown by Dicke, that the system of two coupled two-level atoms can be treated as a single four-level system with modified decay rates.

Another aspects of the model of the spontaneous emission are studied in the present paper. When the compound system of two atoms is in an entangled state, the irreversible process of radiative decay usually destroys correlations and the state becomes unentangled. In the model studied here, the photon exchange produces correlations between atoms which can partially overcome decoherence caused by spontaneous radiation. As a result, some amount of entanglement can survive, and moreover there is a possibility that this process can entangle separable states of two atoms. The idea that dissipation can create entanglement in physical systems, was recently developed in several papers [5, 6, 7, 8]. In particular, the effect of spontaneous emission on destruction and production of entanglement in two - atom system was discussed [9, 10, 11]. Possible production of robust entanglement for closely separated atoms was shown in Ref. [9], and the existence of transient entanglement induced by this process in a system of two atoms separated by an arbitrary distance was also studied [10, 11].

In this paper we also concentrate on arbitrarily separated atoms and consider the dynamics of entanglement. Similarly as in [11] we take some class of initial states, including interesting pure and mixed states, and discuss in details its time evolution as well as the evolution of its entanglement. Note that our initial states are different from that considered in Ref. [11]. We study also the interesting problem how dissipative process of spontaneous emission influences nonlocal properties of initial states. Nonlocality of quantum theory manifests by violation of Bell inequalities, and in the case of two two-level systems it can be quantified by some numerical parameter ranging from 0 for local states to 1 for states maximally violating some Bell inequality. Atomic dynamics studied in the paper enables also to consider time evolution of this parameter.
The model considered in the present paper consist of two two-level atoms coupled to a common thermostat at zero temperature and the reduced dynamics (in the Markovian approximation) is given by the semi-group \( \{ T_t \}_{t \geq 0} \) of completely positive linear mappings acting on density matrices (see e.g. [12]). The dynamics takes into account only spontaneous emission and possible photon exchange between atoms [13] [14], and the generator \( L_D \) of \( \{ T_t \}_{t \geq 0} \) is parametrized in terms of the spontaneous emission rate of the single atom \( \gamma_0 \) and the photon exchange rate \( \gamma \). In the case of atoms separated by an arbitrary distance \( R \), \( \gamma \) is strictly smaller then \( \gamma_0 \) and one can check that the relaxation process brings all initial states into the unique asymptotic state when both the atoms are in their ground states. In contrast to the small separation regime (\( \gamma = \gamma_0 \)) studied in Ref. [9], where the robust entanglement of non-trivial asymptotic states can be analysed, in the present case only the transient entanglement of some states can exist. To consider transient entanglement we need to know in details time evolution of initial states, not only its asymptotic behaviour, so the analysis of possible generation of entanglement is much more involved.

In this paper we calculate time evolution of an arbitrary initial density matrix. To obtain an analytic expression for entanglement as a function of time, we concentrate on the class of states which is left invariant during the evolution, and admits explicit formula for the measure of entanglement. Next we discuss in details how evolve pure initial states, both unentangled and entangled. We show that entanglement as a function of time can behave very differently depending on initial conditions: it may monotonically decrease to zero, increase to maximal value and then decrease to zero or even it can have local minimum and maximum. In particular, there are states for which induced transient entanglement is larger then initial entanglement. Our solution enables also to study nonlocality of the evolving system of two atoms. It turns out that the natural measure of nonlocality very quickly becomes equal to zero, even if entanglement is increasing in some time interval.
2. Entanglement and Nonlocality for a Pair of Two-Level Atoms

2.1. Measure of entanglement. Consider two-level atom $A$ with ground state $|0\rangle$ and excited state $|1\rangle$. This quantum system can be described in terms of the Hilbert space $H_A = \mathbb{C}^2$ and the algebra $\mathfrak{A}_A$ of $2 \times 2$ complex matrices. If we identify $|1\rangle$ and $|0\rangle$ with vectors $(1 \ 0)$ and $(0 \ 1)$ respectively, then the raising and lowering operators $\sigma_+, \sigma_-$ defined by

$$\sigma_+ = |1\rangle\langle 0|, \quad \sigma_- = |0\rangle\langle 1|$$

can be expressed in terms of Pauli matrices $\sigma_1, \sigma_2$

$$\sigma_+ = \frac{1}{2} (\sigma_1 + i \sigma_2), \quad \sigma_- = \frac{1}{2} (\sigma_1 - i \sigma_2)$$

For a joint system $AB$ of two two-level atoms $A$ and $B$, the algebra $\mathfrak{A}_{AB}$ is equal to $4 \times 4$ complex matrices and the Hilbert space $H_{AB} = H_A \otimes H_B = \mathbb{C}^4$. Let $E_{AB}$ be the set of all states of the compound system i.e.

$$E_{AB} = \{ \rho \in \mathfrak{A}_{AB} : \rho \geq 0 \text{ and } \text{tr} \rho = 1 \}$$

The state $\rho \in E_{AB}$ is separable \cite{17], if it has the form

$$\rho = \sum_k \lambda_k P_k \otimes P_k', \quad \rho^A_k \in E_A, \rho^B_k \in E_B, \lambda_k \geq 0 \text{ and } \sum_k \lambda_k = 1$$

The set $E_{AB}^{\text{sep}}$ of all separable states forms a convex subset of $E_{AB}$. When $\rho$ is not separable, it is called inseparable or entangled. Thus

$$E_{AB}^{\text{ent}} = E_{AB} \setminus E_{AB}^{\text{sep}}$$

As a measure of the amount of entanglement a given state contains we take the entanglement of formation \cite{18}

$$E(\rho) = \min \sum \lambda_k E(P_k)$$

where the minimum is taken over all possible decompositions

$$\rho = \sum_k \lambda_k P_k$$

and

$$E(P) = -\text{tr}[(\text{tr}_A P) \log_2 (\text{tr}_A P)]$$

In the case of two two-level atoms, $E(\rho)$ is the function of another useful quantity $C(\rho)$ called concurrence, which also can be taken as a measure of entanglement \cite{15, 16}. Now we pass to the definition of $C(\rho)$. Let

$$\rho^\dagger = (\sigma_2 \otimes \sigma_2) \overline{\rho} (\sigma_2 \otimes \sigma_2)$$

where $\overline{\rho}$ is the complex conjugation of the matrix $\rho$. Define also

$$\hat{\rho} = (\rho^{1/2} \rho^{1/2})^{1/2}$$

Then the concurrence $C(\rho)$ is given by \cite{15, 16}

$$C(\rho) = \max (0, 2p_{\max}(\hat{\rho}) - \text{tr} \hat{\rho})$$

where $p_{\max}(\hat{\rho})$ denotes the maximal eigenvalue of $\hat{\rho}$. The value of the number $C(\rho)$ varies from 0 for separable states, to 1 for maximally entangled pure states.

Consider now the class of density matrices $\rho$

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & \rho_{24} \\ 0 & \rho_{32} & \rho_{33} & \rho_{34} \\ 0 & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}$$
where the matrix elements are taken with respect to the basis $|1\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$ and $|0\rangle \otimes |0\rangle$. One can check that for density matrices of the form (12)

$$C(\rho) = |\rho_{23}| - \sqrt{\rho_{22}\rho_{33}} - |\rho_{23}| - \sqrt{\rho_{22}\rho_{33}}$$

By positive-definiteness of $\rho$, $|\rho_{23}| \leq \sqrt{\rho_{22}\rho_{33}}$, and we have the result:

**Concurrence of a density matrix (12) is given by**

$$C(\rho) = 2|\rho_{23}|$$

As we will show, the class of density matrices given by (12) is invariant with respect to the time evolution considered in the paper, and formula (14) can be used to analyse the evolution of entanglement of states which initially have the form (12).

2.2. **Violation of Bell inequalities.** The contradiction between quantum theory and local realism expressed by the violation of Bell - CHSH inequality [19], can be studied in the case of two-qubit system using simple necessary and sufficient condition [20, 21]. Any state $\rho \in E_{AB}$ can be written as

$$\rho = \frac{1}{4} \left( I_2 \otimes I_2 + r \cdot \sigma \otimes I_2 + I_2 \otimes s \cdot \sigma + \sum_{n,m=1}^{3} t_{nm} \sigma_n \otimes \sigma_m \right)$$

where $I_2$ is the identity matrix in two dimensions, $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices, $r, s$ are vectors in $\mathbb{R}^3$ and $r \cdot \sigma = \sum_{j=1}^{3} r_j \sigma_j$. The coefficients

$$t_{nm} = \text{tr}(\rho \sigma_n \otimes \sigma_m)$$

form a real matrix $T_{\rho}$. Define also real symmetric matrix

$$U_{\rho} = T_{\rho}^T T_{\rho}$$

where $T_{\rho}^T$ is the transposition of $T_{\rho}$. Consider now the family of Bell operators

$$B_{\text{CHSH}} = a \cdot \sigma \otimes (b + b') \cdot \sigma + a' \cdot \sigma \otimes (b - b') \cdot \sigma$$

where $a, a', b, b'$ are unit vectors in $\mathbb{R}^3$. Then CHSH inequality reads

$$|\text{tr}(\rho B_{\text{CHSH}})| \leq 2$$

Violation of inequality (19) by the density matrix (15) and some Bell operator (18) can be checked by the following criterion: Let

$$m(\rho) = \max_{j<k} (u_j + u_k)$$

and $u_j, j = 1, 2, 3$ are the eigenvalues of $U_{\rho}$. As was shown in [20, 21]

$$\max_{B_{\text{CHSH}}} \text{tr}(\rho B_{\text{CHSH}}) = 2 \sqrt{m(\rho)}$$

Thus (19) is violated by some choice of $a, a', b, b'$ iff $m(\rho) > 1$.

If we consider subclass of the class of states (12) consisting of density matrices of the form

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}$$

then we obtain the following expression for $m(\rho)$

$$m(\rho) = \max \left( 2C^2(\rho), (1 - 2\rho_{44})^2 + C^2(\rho) \right)$$

where $C(\rho) = 2|\rho_{23}|$ is the concurrence of the state $\rho$. Notice that the inequality

$$(1 - 2\rho_{44})^2 + C^2(\rho) > 1$$
is equivalent to
\[ |\rho_{23}|^2 > \rho_{44} (1 - \rho_{44}) \]

Let us introduce linear entropy of the state \( \rho \)
\[ S_L(\rho) = 1 - \text{tr} \, \rho^2 \]

For states (22)
\[ \text{tr} \, \rho^2 = \rho_{22}^2 + \rho_{33}^2 + \rho_{44}^2 + 2 |\rho_{23}|^2 \]
so using \( (\rho_{22} + \rho_{33} + \rho_{44})^2 = 1 \) we obtain
\[ S_L(\rho) = 2 (\rho_{22} \rho_{33} + \rho_{22} \rho_{44} + \rho_{33} \rho_{44} - |\rho_{23}|^2) \]

On the other hand
\[ |\rho_{23}|^2 - \rho_{44} (\rho_{22} + \rho_{33}) = |\rho_{23}|^2 - \rho_{44} (1 - \rho_{44}) > 0 \]
so
\[ \rho_{22} \rho_{33} - \frac{1}{2} S_L(\rho) = |\rho_{23}|^2 - \rho_{44} (\rho_{22} + \rho_{33}) > 0 \]
and we obtain the following result:

The states (22) violate some Bell-CHSH inequality if and only if
\[ |\rho_{23}| > \frac{1}{\sqrt{2}} \text{ or } \rho_{22} \rho_{33} > \frac{1}{2} S_L(\rho). \]
3. Spontaneous Emission and Evolution of Entanglement

We study time evolution of the system of two two-level atoms separated by a distance $R$ when we take into account only the dissipative process of spontaneous emission. The dynamics of such system is given by the master equation \[13, 14\]

\[
\frac{d\rho}{dt} = L_D\rho, \quad \rho \in \mathcal{E}_{AB}
\]

with the following generator $L_D$

\[
L_D\rho = \frac{1}{2} \sum_{k,l=A,B} \gamma_{kl} (2\sigma_-^k \rho \sigma_+^l - \sigma_+^k \sigma_-^l \rho - \rho \sigma_+^k \sigma_-^l)
\]

where

\[
\sigma_+ = \sigma_\pm \otimes I, \quad \sigma_- = I \otimes \sigma_\pm, \quad \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)
\]

and $\gamma_{AA} = \gamma_{BB} = \gamma_0$, $\gamma_{AB} = \gamma_{BA} = \gamma = g\gamma_0$. Here $\gamma_0$ is the single atom spontaneous emission rate, and $\gamma = g\gamma_0$ is a relaxation constant of photon exchange. In the model, $g$ is the function of the distance $R$ between atoms and $g \to 1$ when $R \to 0$. In this section we investigate the time evolution of the initial density matrix $\rho$ of the compound system, governed by the semi - group $\{T_t\}_{t \geq 0}$ generated by $L_D$. In particular, we will study the time development of entanglement of $\rho$, measured by concurrence. When $\gamma < \gamma_0$, the semi-group $\{T_t\}_{t \geq 0}$ is uniquely relaxing, with the asymptotic state $|0\rangle \otimes |0\rangle$. Thus, for any initial state $\rho$, the concurrence $C(\rho_t)$ approaches 0 when $t \to \infty$. But still there can be some transient entanglement between atoms \[9, 10, 11\]. In this section we study in details time evolution of a given initial state $\rho = (\rho_{jk})$. Direct calculations show that the state $\rho(t)$ at time $t$ has the following matrix elements with respect to the basis $|1\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |0\rangle \otimes |0\rangle$

\[
\rho_{11}(t) = e^{-2\gamma_0 t} \rho_{11}
\]

\[
\rho_{12}(t) = e^{-\frac{1}{2}\gamma_0 t} (\rho_{12} \cosh \frac{\gamma t}{2} - \rho_{13} \sinh \frac{\gamma t}{2})
\]

\[
\rho_{13}(t) = e^{-\frac{1}{2}\gamma_0 t} (\rho_{13} \cosh \frac{\gamma t}{2} - \rho_{12} \sinh \frac{\gamma t}{2})
\]

\[
\rho_{14}(t) = e^{-\gamma_0 t} \rho_{14}
\]

\[
\rho_{22}(t) = -e^{-\gamma_0 t} \left( \frac{\gamma_0^2 + \gamma_0^2}{\gamma_0^2 - \gamma^2} \rho_{11} + e^{-\gamma_0 t} \left[ \frac{1}{2} (\rho_{22} - \rho_{33}) + \frac{\gamma^2}{\gamma_0^2 - \gamma^2} \rho_{11} + \frac{1}{2} (\rho_{22} + \rho_{33}) \right] \cosh \gamma t \right) - \frac{2\gamma_0}{\gamma_0^2 - \gamma^2} \rho_{11} - \text{Re} \rho_{23} \sinh \gamma t \right]
\]

\[
\rho_{33}(t) = -e^{-\gamma_0 t} \left( \frac{\gamma_0^2 + \gamma_0^2}{\gamma_0^2 - \gamma^2} \rho_{11} + e^{-\gamma_0 t} \left[ \frac{1}{2} (\rho_{33} - \rho_{22}) + \frac{\gamma^2 + \gamma_0^2}{\gamma_0^2 - \gamma^2} \rho_{11} + \frac{1}{2} (\rho_{22} + \rho_{33}) \right] \cosh \gamma t \right) - \frac{2\gamma_0}{\gamma_0^2 - \gamma^2} \rho_{11} - \text{Re} \rho_{23} \sinh \gamma t \right]
\[ \rho_{23}(t) = -e^{-2\gamma_0 t} \frac{2\gamma_0}{\gamma_0^2 - \gamma^2} \rho_{11} + e^{-\gamma_0 t} \left[ -\frac{\gamma^2 + \gamma_0^2}{\gamma_0^2 - \gamma^2} \sinh \gamma t + \cosh \gamma t \rho_{11} + i \text{Im} \rho_{23} + \text{Re} \rho_{23} \cosh \gamma t \rho_{11} \right] - \frac{1}{2} (\rho_{22} + \rho_{33}) \sinh \gamma t \right] \\
\rho_{24}(t) = e^{-2\gamma_0 t} \left( \sinh \frac{\gamma}{2} t - \gamma \cosh \frac{\gamma}{2} t \right) + e^{-2\gamma_0 t} \left( \psi \cosh \frac{\gamma}{2} t - \sinh \frac{\gamma}{2} t \right) \rho_{12} + e^{-2\gamma_0 t} \left( \psi \sinh \frac{\gamma}{2} t - \cosh \frac{\gamma}{2} t \right) \rho_{13} + e^{-2\gamma_0 t} \left( \rho_{24} \cosh \frac{\gamma}{2} t - \rho_{34} \sinh \frac{\gamma}{2} t \right) \rho_{11} \\
\rho_{34}(t) = e^{-2\gamma_0 t} \left( \psi \sin \frac{\gamma}{2} t - \cosh \frac{\gamma}{2} t \right) + e^{-2\gamma_0 t} \left( \psi \cosh \frac{\gamma}{2} t - \sin \frac{\gamma}{2} t \right) \rho_{12} + e^{-2\gamma_0 t} \left( \psi \sin \frac{\gamma}{2} t - \cosh \frac{\gamma}{2} t \right) \rho_{13} + e^{-2\gamma_0 t} \left( \rho_{34} \cosh \frac{\gamma}{2} t - \rho_{24} \sin \frac{\gamma}{2} t \right) \rho_{11} \\
\rho_{44}(t) = 1 + e^{-2\gamma_0 t} \left( \frac{\gamma^2 + 2\gamma_0^2}{\gamma_0^2 - \gamma^2} \rho_{11} + e^{-\gamma_0 t} \left[ \frac{4\gamma \gamma_0}{\gamma_0^2 - \gamma^2} \sinh \gamma t - \frac{2(\gamma^2 + \gamma_0^2)}{\gamma_0^2 - \gamma^2} \cosh \gamma t \rho_{11} \right] - (\rho_{22} + \rho_{33}) \cosh \gamma t + 2 \text{Re} \rho_{23} \sinh \gamma t \right] \\
The remaining matrix elements can be obtained by the Hermiticity condition. One can simply check that the classes of states (12) and (22) are invariant with respect to the above time evolution. In particular, in both cases

(27) \[ \rho_{23}(t) = e^{-\gamma_0 t} \left[ \text{Re} \rho_{23} \cosh \gamma t + i \text{Im} \rho_{23} - \frac{1}{2} (\rho_{22} + \rho_{33}) \sinh \gamma t \right] \]

So we obtain the result:

For initial states (12) or (22) the concurrence at time \( t \) is given by

(28) \[ C(\rho(t)) = 2 e^{-\gamma_0 t} \left| \left( \text{Re} \rho_{23} \cosh \gamma t + i \text{Im} \rho_{23} - \frac{1}{2} (\rho_{22} + \rho_{33}) \sinh \gamma t \right) \right| \]

where \( \rho_{jk} \) are matrix elements of the initial state.

Let us consider pure initial states \( \Psi \in \mathbb{C}^4 \) belonging to the class (12). The most general pure state of this type can be written as

(29) \[ \Psi = \cos \phi \cos \psi |1 \rangle \otimes |0 \rangle + \sin \phi \cos \psi e^{i \Theta} |0 \rangle \otimes |1 \rangle + \sin \psi e^{i \Xi} |0 \rangle \otimes |0 \rangle \]

with \( \phi, \psi \in [0, \frac{\pi}{2}], \Theta, \Xi \in [0, 2\pi] \). Using (14) we see that concurrence of (29) is given by

(30) \[ C(\Psi) = \cos^2 \psi \sin 2\phi \]

By (28), the time evolution of this initial concurrence is described by the following function

(31) \[ C(\rho(t)) = e^{-\gamma_0 t} \cos^2 \psi \left| \sin 2\phi \cos \Theta \cosh \gamma t - \sin \gamma t - i \sin 2\phi \sin \Theta \right| \]

The function (31) is simple to analyse when \( \phi = 0 \) or \( \frac{\pi}{2} \). In this case

(32) \[ \Psi = e^{i \Theta} \cos \psi |0 \rangle \otimes |1 \rangle + e^{i \Xi} \sin \psi |0 \rangle \otimes |0 \rangle \]

and

(33) \[ C(\rho(t)) = e^{-\gamma_0 t} \cos^2 \psi \sinh \gamma t \]
From the formula (33) we see that initial concurrence equal to zero, increases in the time interval 
\([0, t_{\text{max}}]\), where

\begin{equation}
\label{eq:tm}
t_{\text{max}} = \frac{1}{2\gamma} \ln \frac{\gamma_0 + \gamma}{\gamma_0 - \gamma}
\end{equation}

to the maximal value

\begin{equation}
\label{eq:Cmax}
C_{\text{max}} = \cos^2 \psi \frac{\gamma}{\gamma_0 - \gamma} \left( \frac{\gamma_0 + \gamma}{\gamma_0 - \gamma} \right)^{-\frac{\gamma_0 + \gamma}{2\gamma}}
\end{equation}

and then asymptotically goes to zero (see Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{\(C(\rho(t))\) for initial states (32) with \(\psi = 0\) and different values of \(\gamma/\gamma_0\)}
\end{figure}

If \(\phi\) is arbitrary, we can put for simplicity \(\psi = 0\). Then

\begin{equation}
\label{eq:psi1}
\Psi = \cos \phi |1\rangle \otimes |0\rangle + \sin \phi e^{i\Theta} |0\rangle \otimes |1\rangle
\end{equation}

and \(C(\Psi) = \sin 2\phi\). The evolution of this initial concurrence is given by

\begin{equation}
\label{eq:evolution}
C(\rho(t)) = e^{-\gamma_0 t} \left| \sin 2\phi \cos \Theta \cosh \gamma t - \sinh \gamma t - i \sin 2\phi \sin \Theta \right|
\end{equation}

Depending on values of \(\phi\) and \(\Theta\), the function (37) can be strictly decreasing to zero, or can have one maximal value for some \(t\), or even can have one minimal and one maximal value. Let us discuss all these possibilities by choosing some special initial states from the class (36).

a. Let \(\Theta = 0\). Then

\begin{equation}
\label{eq:psi2}
\Psi = \cos \phi |1\rangle \otimes |0\rangle + \sin \phi |0\rangle \otimes |1\rangle
\end{equation}

and

\begin{equation}
\label{eq:Cpsi2}
C(\rho(t)) = e^{-\gamma_0 t} \left| \sin 2\phi \cosh \gamma t - \sin \gamma t \right|
\end{equation}
The function (39) is decreasing to zero in the interval \([0, t_{\text{min}}]\) where

\[(40)\]
\[t_{\text{min}} = \frac{1}{2\gamma} \ln \frac{1 + \sin 2\phi}{1 - \sin 2\phi}\]

Then in the interval \([t_{\text{min}}, t_{\text{max}}]\), with

\[(41)\]
\[t_{\text{max}} = \frac{1}{2\gamma} \ln \frac{(1 + \sin 2\phi)(\gamma_0 + \gamma)}{(1 - \sin 2\phi)(\gamma_0 - \gamma)}\]

(39) increases to the maximal value

\[(42)\]
\[C_{\text{max}} = \gamma \frac{|\cos 2\phi|}{\sqrt{\gamma_0^2 - \gamma^2}} \left( \frac{(1 + \sin 2\phi)(\gamma_0 + \gamma)}{(1 - \sin 2\phi)(\gamma_0 - \gamma)} \right)^{-\frac{2\phi}{\pi}}\]

For \(t > t_{\text{max}}\), (39) goes asymptotically to 0. For sufficiently small initial concurrence \(C_{\text{max}} > C(\Psi)\) but for larger entanglement of the initial state, the maximal entanglement produced during the evolution is smaller then \(C(\Psi)\) (see Fig. 2. below).

![Fig. 2. \(C(\rho(t))\) for initial states (38), with \(\phi = \pi/40, \pi/20\) and \(\gamma/\gamma_0 = 0.75\)](image)

b. Let \(\Theta = \pi\). Then

\[(43)\]
\[\Psi = \cos \phi \ket{1} \otimes \ket{0} - \sin \phi \ket{0} \otimes \ket{1}\]

and

\[(44)\]
\[C(\rho(t)) = e^{-\gamma_0 t} \left| \sin 2\phi \cosh \gamma t + \sinh \gamma t \right|\]

If \(\sin 2\phi \geq \frac{\gamma}{\gamma_0}\) then function (44) is monotonically decreasing to 0. On the other hand, if \(\sin 2\phi < \frac{\gamma}{\gamma_0}\) then at time

\[(45)\]
\[t_{\text{max}} = \frac{1}{2\gamma} \ln \frac{(1 - \sin 2\phi)(\gamma_0 + \gamma)}{(1 + \sin 2\phi)(\gamma_0 - \gamma)}\]
(44) attains local maximum

\[ C_{\text{max}} = \frac{\gamma |\cos 2\phi|}{\sqrt{\gamma_0^2 - \gamma^2}} \left( \frac{(1 - \sin 2\phi)(\gamma_0 + \gamma)}{(1 + \sin 2\phi)(\gamma_0 - \gamma)} \right)^{-\frac{\gamma_0}{2\gamma}} \]

\( C_{\text{max}} \) is always greater than initial concurrence \( C(\Psi) \) and

\[ C_{\text{max}} \to \frac{1 + \sin 2\phi}{2} \quad \text{when} \quad \gamma \to \gamma_0 \]

and

\[ C_{\text{max}} \to \frac{\gamma}{\gamma_0} \quad \text{when} \quad \sin 2\phi \to \frac{\gamma}{\gamma_0} \]

Thus, for entangled pure initial states (43) the dissipative process of spontaneous emission increases entanglement, provided that the initial entanglement was smaller than \( \frac{\gamma}{\gamma_0} \) (see Fig. 3).

\[ \text{Fig. 3. } C(\rho(t)) \text{ for initial states (43) with } C(\Psi) = 0.1, 0.4, 0.7, 1 \text{ and } \gamma/\gamma_0 = 0.75 \]

c. Let \( \Theta = \pi/2 \). Then

\[ \Psi = \cos \phi |1\rangle \otimes |0\rangle + i \sin \phi |0\rangle \otimes |1\rangle \]

and

\[ C(\rho(t)) = e^{-\gamma_0 t} \left| \sinh \gamma t + i \sin 2\phi \right| \]

One can show that if \( |\sin 4\phi| < \gamma/\gamma_0 \) then (50) achieves local minimum at

\[ t_{\text{min}} = \frac{1}{2\gamma} \ln \frac{\gamma_0 \cos 4\phi - \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{\gamma_0 - \gamma} \]

and local maximum at

\[ t_{\text{max}} = \frac{1}{2\gamma} \ln \frac{\gamma_0 \cos 4\phi + \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{\gamma_0 - \gamma} \]
with the corresponding values of concurrence

\begin{equation}
C_{\text{min}} = \left( \frac{\gamma_0 \cos 4\phi - \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{\gamma_0 - \gamma} \right)^{-\frac{20}{\pi}} \sqrt{\frac{\gamma^2 \cos 4\phi - \gamma \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{2(\gamma_0^2 - \gamma^2)}}
\end{equation}

and

\begin{equation}
C_{\text{max}} = \left( \frac{\gamma_0 \cos 4\phi + \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{\gamma_0 - \gamma} \right)^{-\frac{20}{\pi}} \sqrt{\frac{\gamma^2 \cos 4\phi + \gamma \sqrt{\gamma^2 - \gamma_0^2 \sin^2 4\phi}}{2(\gamma_0^2 - \gamma^2)}}
\end{equation}

For other cases, the function (50) is monotonically decreasing to 0 (see Fig. 4. below).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{$C(\rho(t))$ for initial states (49) with $\phi = \pi/20$, $\pi/40$ and $\gamma/\gamma_0 = 0.75$}
\end{figure}
4. Evolution of Nonlocality

Nonlocality of quantum theory manifesting by violation of Bell inequalities is strictly connected with the existence of entangled states. It is known that every pure entangled state violates some Bell inequality (see e.g. [22]). But for mixed entangled states it is no longer true [17]. So it is interesting to discuss how dissipative process of spontaneous emission influences nonlocal properties of initial states. For simplicity we restrict the class of states considered below to density matrices of the form (22). For that class we can apply the results of Sect. 2.2.

As the initial states we take the states (43). Note that at time \( t \), the state \( \rho(t) \) will have the form (22). The initial entanglement is non-zero, so violation of some Bell - CHSH inequality occurs at time \( t = 0 \). What happens during the evolution? Consider the inequality

\[
\rho_{22}(t)\rho_{33}(t) > \frac{1}{2} S_L(\rho(t))
\]

which is sufficient to nonlocality of the state \( \rho(t) \). Observe that

\[
S_L(\rho(0)) = 0 \quad \text{and} \quad S_L(\rho(t)) \quad \text{is increasing in some time - interval.}
\]

On the other hand,

\[
\rho_{22}(0)\rho_{33}(0) \geq |\rho_{23}(0)|^2 > 0
\]

and \( \rho_{22}(t)\rho_{33}(t) \) asymptotically goes to 0, so there is some non - empty interval \( 0 \leq t < t_1 \) for which the inequality (55) is satisfied. Thus for \( 0 \leq t < t_1 \), all states \( \rho(t) \) will still have nonlocal properties. We may also introduce the time \( t_n \) after which nonlocality is lost. To this end, besides \( t_1 \) consider \( t_2 \) such that

\[
|\rho_{23}(t_2)| = \frac{1}{2\sqrt{2}}
\]

Then

\[
t_n = \max(t_1, t_2)
\]

By the results of Sect. 2.2, all states \( \rho(t) \) for \( t \geq t_n \) will admit local hidden variable model. To illustrate this concept, consider as initial states \( \Psi^+ \) and \( \Psi^- \) (symmetric and antisymmetric states)

\[
\Psi^\pm = \frac{1}{\sqrt{2}} \left( |1\rangle \otimes |0\rangle \pm |0\rangle \otimes |1\rangle \right)
\]

Let \( \rho^\pm(t) \) denote corresponding density matrices at time \( t \). Then

\[
|\rho_{23}^\pm(t)| = \frac{1}{2} e^{-\gamma_0 (\gamma_0 \pm \gamma) t}
\]

and

\[
\rho_{22}^\pm(t)\rho_{33}^\pm(t) = \frac{1}{4} e^{-2(\gamma_0 \pm \gamma) t}, \quad S_L(\rho^\pm(t)) = 2 \left( e^{-\gamma_0 (\gamma_0 \pm \gamma) t} - e^{-2(\gamma_0 \pm \gamma) t} \right)
\]

We see that

\[
t_1 = \frac{1}{\gamma_0 + \gamma} \ln \frac{5}{4}, \quad t_2 = \frac{1}{\gamma_0 + \gamma} \ln \frac{2}{2}
\]

Thus

\[
t_n = \frac{1}{\gamma_0 + \gamma} \ln \frac{2}{2}
\]

Note that for antisymmetric state \( \Psi^- \) and \( \gamma \) close to \( \gamma_0 \), \( t_n \) goes to infinity.

It is also interesting to study in more details time evolution of the measure of nonlocality which may be defined as follows

\[
n(\rho) = \max(0, m(\rho) - 1)
\]

As is well known, \( m(\rho) \leq 2 \), so \( 0 \leq n(\rho) \leq 1 \) and larger value of \( n(\rho) \) greater than 1 means violation of CHSH inequality to a larger extent. Since \( m(\rho) = 2 \) for maximally entangled pure states which maximally violate CHSH inequalities, for them \( n(\rho) = 1 \). To obtain analytic expression for \( n(\rho(t)) \) we can utilize formula (23). Formula (23) is further simplified if we take such initial states that

\[
|\rho_{23}(t)| < \frac{1}{2\sqrt{2}}
\]
for all $t$. This condition can be achieved if

$$\frac{\gamma}{\gamma_0} < \frac{1}{\sqrt{2}}$$

Then inequality (58) is satisfied and $m(\rho(t)) > 1$ if and only if

$$\text{(59)} \quad (1 - 2\rho_{44}(t))^2 + C^2(\rho(t)) > 1$$

and to study the time evolution of nonlocality, we only need to know time-dependence of the left hand side of (59). If we take initial state with $\sin 2\phi < \frac{\gamma}{\gamma_0} < \frac{1}{\sqrt{2}}$, then during the time evolution $C(\rho(t))$ increases to the maximal value $C_{\text{max}} < \frac{1}{\sqrt{2}}$ in the interval $[0, t_{\text{max}}]$. At the same time,

$$\text{(60)} \quad (1 - 2\rho_{44})^2 = \left(2e^{-\gamma_0 t} (\cosh \gamma t + \sin 2\phi \sinh \gamma t) - 1\right)^2$$

decreases so fast that left hand side of (59) is a decreasing function of $t$ (see Fig. 5, 6 below). Thus we obtain the result:

*For initial states (43) and $\sin 2\phi < \frac{\gamma}{\gamma_0} < \frac{1}{\sqrt{2}}$, nonlocality of $\rho(t)$ given by $n(\rho(t))$ decreases during the time evolution even if the entanglement increases.*

![Fig. 5](image_url)  
*Fig. 5. $C(\rho(t))$ (dotted line) and $(1 - 2\rho_{44}(t))^2$ (solid line) for initial state (43) with $C(\Psi) = 0.3$ and $\gamma/\gamma_0 = 0.7$*
Fig. 6. \( n(\rho(t)) \) for initial state (43) with \( C(\Psi) = 0.3 \) and \( \gamma/\gamma_0 = 0.7 \)

Numerical analysis indicates that for other initial states (43) for which \( m(\rho(t)) \) is defined by the whole expression (23), the nonlocality \( n(\rho(t)) \) also monotonically goes to 0 irrespective of the evolution of entanglement (see Fig. 7 below).

Fig. 7. \( n(\rho(t)) \) for initial states (43) with \( C(\Psi) = 0.8, 0.9, 1.0 \) and \( \gamma/\gamma_0 = 0.7 \)
References

[1] R.H. Dicke, Phys. Rev. 93, 99(1954).
[2] M. Dillard, H.R. Robl, Phys. Rev. 184, 312(1969).
[3] R.H. Lehmberg, Phys. Rev. 181, 32(1969).
[4] R.H. Lehmberg, Phys. Rev. A 2, 883(1970); A 2, 889(1970).
[5] M.B. Plenio, S.F. Huelga, Phys. Rev. Lett. 88, 197901(2002).
[6] M.S. Kim, J. Lee, D. Ahn, P.L. Knight, Phys. Rev. A65, 040101(R)(2002).
[7] S. Schneider, G.J. Milburn, Phys. Rev. A65, 042107(2002).
[8] M.B. Plenio, S.F. Huelga, A. Beige, P.L. Knight, Phys. Rev. A59, 2468(1999).
[9] L. Jakóbczyk, J. Phys. A 35, 6383(2002), J. Phys. A 36, 1537(2003) (Corrigendum).
[10] Z. Ficek, R. Tanasi, quant-ph/0302124
[11] Z. Ficek, R. Tanasi, quant-ph/0307045
[12] R. Alicki, K. Lendi, Quantum Dynamical Semigroups and Applications, Lecture Notes in Phys. Vol 286, Springer, Berlin, 1987.
[13] G.S. Agarwal, Quantum Statistical Theories of Spontaneous Emission and their Relation to Other Approaches, Springer, Berlin, 1974.
[14] Z. Ficek, R. Tanasi, Phys. Rep. 372, 369(2002)
[15] S. Hill, W.K. Wootters, Phys. Rev. Lett. 78, 5022(1997).
[16] W.K. Wootters, Phys. Rev. Lett. 80, 2254(1998).
[17] R.F. Werner, Phys. Rev. A40, 4277(1989).
[18] Ch. Bennett, P.D. DiVincenzo, J. Smolin, W.K. Wootters, Phys. Rev. A 54, 3824(1996).
[19] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys. Rev. Lett. 23, 880(1969).
[20] R. Horodecki, P. Horodecki, M. Horodecki, Phys. Lett. A 200, 340(1995).
[21] R. Horodecki, Phys. Lett. A 210, 223(1996).
[22] N. Gisin, Phys. Lett. A 154, 201(1991).