Nonperturbative Reduction of Yang-Mills Theory and Low Energy Effective Action

A. M. Khvedelidze $^a$ * and H.-P. Pavel $^b$

$^a$ Joint Institute for Nuclear Research, Dubna, Russia
$^b$ Fachbereich Physik der Universität Rostock, D-18051 Rostock, Germany

Abstract

The method of reduction of a non-Abelian gauge theory to the corresponding unconstrained system is exemplified for SU(2) Yang-Mills field theory. The reduced Hamiltonian which describes the dynamics of the gauge invariant variables is presented in the form of a strong coupling expansion. The physical variables are separated into fields, which are scalars under spatial rotations, and rotational degrees of freedom. It is shown how in the infrared limit an effective nonlinear sigma model type Lagrangian can be derived which out of the six physical fields involves only one of three scalar fields and two rotational fields summarized in a unit vector. Its possible relation to the effective Lagrangian proposed recently by Faddeev and Niemi is discussed.

1 Introduction

The perturbative reduction of non-Abelian gauge theories via gauge fixing, which ascribes the transverse components of the gauge field as the physical variables, is in accordance with asymptotic freedom seen in high energy scattering processes, but is not appropriate for the description of low

*Permanent address: Tbilisi Mathematical Institute, 380093, Tbilisi, Georgia
energy phenomena such as confinement. The alternative, nonperturbative, approaches to the reduction of gauge theories suggest different types of representations for physical variables [1]-[11] but none of them lead directly to the gauge-invariant formulation of the low energy problems and the question of variables relevant to the infrared behaviour of strong theory is still open.

The main task of the present report is to state the new unconstrained representation for SU(2) Yang-Mills system obtained recently [11] and to discuss the effective field theory which follows from this unconstrained formulation for description of infrared region. We shall give a Hamiltonian formulation of classical SU(2) Yang-Mills field theory entirely in terms of gauge invariant quantities, and separate six physical variables into scalars under ordinary space rotations and into “rotational” degrees of freedom. We shall obtain an effective low energy theory involving only two of the three rotational fields, summarized in a unit vector, and one of the three scalar fields, and shall discuss its possible relation to the effective soliton Lagrangian proposed recently in [16].

2 Elimination of the gauge degrees of freedom in an adapted canonical coordinate basis

The Hamiltonian dynamics of SU(2) gauge fields \( A_\mu^a(x) \) with the action

\[
S[A] := -\frac{1}{4} \int d^4x \, F_{\mu\nu}^a F^{a\mu\nu}, \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c, \tag{1}
\]

takes place on the definite domain of phase space spanned by the canonical variables \( (A_0^a, P^a := \partial L/\partial \partial_0 A_0^a) \), and \( (A_{ai}, E_{ai} := \partial L/\partial \partial_0 A_{ai}) \). This submanifold is defined by the three primary constraints

\[
P^a(x) = 0 \tag{2}
\]

and the non-Abelian Gauss law constraints

\[
\Phi_a := \partial_i E_{ai} + g\epsilon_{abc} A_{ci} E_{bi} = 0, \tag{3}
\]

which are first class

\[
\{\Phi_a(x), \Phi_b(y)\} = g\epsilon_{abc} \Phi_c \delta(x - y). \tag{4}
\]
According to the Dirac prescription \[12\] the evolution of the system is governed by the total Hamiltonian containing three arbitrary functions $\lambda_a(x)$

$$H_T := \int d^3 x \left[ \frac{1}{2} \left( E_{ai}^2 + B_{ai}^2(A) \right) - A_0^a \Phi_a + \lambda_a(x) P^a(x) \right], \quad (5)$$

where $B_{ai}(A) := \epsilon_{ijk} \left( \partial_j A_{ak} + \frac{1}{2} g \epsilon_{abc} A_{bj} A_{ck} \right)$ is the non-Abelian magnetic field. The presence of these arbitrary functions reflects the invariance of the Yang-Mills action (1) under the $SU(2)$ gauge transformations

$$A_\mu \rightarrow A'_\mu = U^{-1}(x) \left( A_\mu - \frac{1}{g} \partial_\mu \right) U(x), \quad (6)$$

and leads to the problem of isolation of the gauge-invariant functions, the observables, which are free of any constraints and have uniquely predictable dynamics. The reduction procedure consists of the elimination of the non-invariant, pure gauge degrees of freedom and the formulation of the corresponding unconstrained system equivalent to the initial one. Equivalence means that all observables of initial theory can be constructed in terms of unconstrained variables and have the same unique dynamics.

For Abelian constraints $\Psi_\alpha (\{ \Psi_\alpha, \Psi_\beta \} = 0)$ the reduction procedure can be achieved in the following two steps. One performs a canonical transformation to new variables such that part of the new momenta $P_\alpha$ coincide with the constraints $\Psi_\alpha$. After the projection onto the constraint shell, i.e. putting in all expressions $P_\alpha = 0$, the coordinates canonically conjugate to the $P_\alpha$ drop out from the physical quantities. The remaining canonical pairs are then gauge invariant and form the basis for the unconstrained system. For the case of non-Abelian constraints (4) it is clearly impossible to find such a canonical basis only via canonical transformation. However one can replace the set of non-Abelian constraints (4) by a new set of Abelian constraints which describe the same constraint surface in phase space and thus reduce the problem to the Abelian case (see e.g. \[13\], \[14\] and references therein). This problem of Abelianization of constraints is considerably simplified when studied in terms coordinates adapted to the action of the gauge group. The knowledge of the $SU(2)$ gauge transformations (5) which leaves the Yang-Mills action (1) invariant, directly prompts us with the choice of adapted coordinates by using the following point transformation to the new set of Lagrangian coordinates $Q_j$ ($j = 1, 2, 3$) and the six elements
\( Q^*_{ik} = Q^*_{ki} \) \((i, k = 1, 2, 3)\) of the positive definite symmetric \(3 \times 3\) matrix \( Q^* \)

\[
A_{ai} (\overline{Q}, Q^*) := O_{ak} (\overline{Q}) Q^*_{ki} - \frac{1}{2g} \epsilon_{abc} \left( O (\overline{Q}) \partial_i O^T (\overline{Q}) \right)_{bc}, \tag{7}
\]

where \( O(\overline{Q}) \) is an orthogonal \(3 \times 3\) matrix parametrised by the \( \overline{Q}_i \). \(^\dagger\) In the following we shall show that in terms of these variables the non-Abelian Gauss law constraints \((3)\) only depend on \( Q^* \) and its conjugated \( P^* \) and after Abelianization become \( \overline{P}_i = 0 \). The unconstrained variables \( Q^* \) and their conjugate \( P^* \) are gauge invariant, i.e. commute with the Gauss law and all observable quantities should depend only on \( Q^* \) and \( P^* \). The transformation \((8)\) induces a point canonical transformation linear in the new canonical momenta \( P^*_{ik} \) and \( \overline{P}_i \). Using the corresponding generating functional depending on the old momenta and the new coordinates,

\[
F_3 \left[ E; \overline{Q}, Q^* \right] := \int d^3 z \ E_{ai}(z) A_{ai} \left( \overline{Q}(z), Q^*(z) \right), \tag{8}
\]

one can obtain the transformation to new canonical momenta \( \overline{P}_i \) and \( P^*_{ik} \)

\[
\overline{P}_j(x) := \frac{\delta F_3}{\delta Q_j(x)} = -\frac{1}{g} \Omega_{jr} \left( D_i(Q^*) O^T E \right)_{ri}, \tag{9}
\]

\[
P^*_{ik}(x) := \frac{\delta F_3}{\delta Q^*_{ik}(x)} = \frac{1}{2} \left( E^T O + O^T E \right)_{ik}. \tag{10}
\]

Here

\[
\Omega_{ji}(\overline{Q}) := -\frac{i}{2} \text{Tr} \left( O^T (\overline{Q}) \frac{\partial O (\overline{Q})}{\partial \overline{Q}_j} J_i \right), \tag{11}
\]

with the \(3 \times 3\) matrix generators of \(SO(3)\), \((J_i)_{mn} := i \epsilon_{mn} \), and the corresponding covariant derivative \( D_i(Q^*) \) in the adjoint representation

\[
(D_i(Q^*))_{mn} := \delta_{mn} \partial_i - ig \left( J^k \right)_{mn} Q^*_{ki}. \tag{12}
\]

\(^\dagger\) In the strong coupling limit the representation \((3)\) reduces to the so-called polar representation for arbitrary quadratic matrices. In the general case we have the additional second term and \((7)\) has to be regarded as a set of partial differential equations for the \( \overline{Q}_i \) variables. The uniqueness and regularity of the suggested transformation \((7)\) depends on the boundary conditions imposed.
A straightforward calculation based on the linear relations (9) and (10) between the old and the new momenta leads to the following expression for the field strengths $E_{ai}$ in terms of the new canonical variables

$$E_{ai} = O_{ak}(Q) \left[ P^*_{ki} + \epsilon_{kis}^* D_{sl}^{-1}(Q^*) \left[ (\Omega^{-1}\mathcal{T})^e_l - \mathcal{S}_l \right] \right]. \tag{13}$$

Here $^*D^{-1}$ is the inverse of the matrix operator

$$^*D_{ik}(Q^*) := -i (J^m D_m(Q^*))_{ik}, \tag{14}$$

and

$$\mathcal{S}_k(x) := \epsilon_{klm} (P^*Q^*)_{lm} - \frac{1}{g} \partial_l P^*_{kl}. \tag{15}$$

Using the representations (13) and (13) one can easily convince oneself that the variables $Q^*$ and $P^*$ make no contribution to the Gauss law constraints (3)

$$\Phi_a := O_{as}(Q)\Omega_{sj}^{-1}\mathcal{T}_j = 0. \tag{16}$$

Here and in (13) we assume that the matrix $\Omega$ is invertible. The equivalent set of Abelian constraints is

$$\mathcal{T}_a = 0. \tag{17}$$

They are Abelian due to the canonical structure of the new variables.

After having rewritten the model in terms of the new canonical coordinates and after the Abelianization of the Gauss law constraints, the construction of the unconstrained Hamiltonian system is straightforward. In all expressions we can simply put $\mathcal{T}_i = 0$. In particular, the Hamiltonian in terms of the unconstrained canonical variables $Q^*$ and $P^*$ can be represented by the sum of three terms

$$H[Q^*, P^*] = \frac{1}{2} \int d^3x \left[ \text{Tr}(P^*)^2 + \text{Tr}(B^2(Q^*)) + \frac{1}{2} \hat{E}^2(Q^*, P^*) \right]. \tag{18}$$

The first term is the conventional quadratic “kinetic” part and the second the “magnetic potential” term which is the trace of the square of the non-Abelian magnetic field

$$B_{sk}(Q^*) := \epsilon_{klm} \left[ \partial_l Q^*_{sm} + \frac{g}{2} \epsilon_{sbc} Q^*_{ql} Q^*_{cm} \right]. \tag{19}$$
It is interesting that after the elimination of the pure gauge degrees of freedom the magnetic field strength tensor is the commutator of the covariant derivatives \( F_{ij} = [D_i(Q^*), D_j(Q^*)] \).

The third nonlocal term in the Hamiltonian \( H_3 \) is the square of the antisymmetric part of the electric field, \( E_s := (1/2)\epsilon_{sij}E_{ij} \), after projection onto the constraint surface. It is given as the solution of the system of differential equations\(^3\)

\[
* D_{ls}(Q^*)E_s = g\mathcal{S}_l ,
\]

with the derivative \(* D_{ls}(Q^*)\) defined in \( \text{(14)} \). Note that the vector \( \mathcal{S}_l(x) \), defined in \( \text{(13)} \), coincides up to divergence terms with the spin density part of the Noetherian angular momentum, \( \mathcal{S}_l(x) := \epsilon_{ijk}A^a_jE_{ak} \), after transformation to the new variables and projection onto the constraint shell. The solution \( \vec{E} \) of the differential equation \( \text{(20)} \) can be expanded in a \( 1/g \) series. The zeroth order term is

\[
E_s^{(0)} = \gamma_{sk}^{-1} \epsilon_{klm} (P^*Q^*)_{lm} ,
\]

with \( \gamma_{ik} := Q^*_{ik} - \delta_{ik}\text{Tr}(Q^*) \), and the first order term is determined as

\[
E_s^{(1)} := \frac{1}{g}\gamma_{sl}^{-1} [(\text{rot } \vec{E}^{(0)})_l - \partial_k P^*_k] \tag{22}
\]

from the zeroth order term. The higher terms are then obtained by the simple recurrence relations

\[
E_s^{(n+1)} := \frac{1}{g}\gamma_{sl}^{-1} (\text{rot } \vec{E}^{(n)})_s . \tag{23}
\]

3 The unconstrained Hamiltonian in terms of scalar and rotational degrees of freedom

Whereas the gauge fields transform as vectors under spatial rotations, the unconstrained fields \( Q^* \) and \( P^* \) transform as second rank tensors under spatial rotations.\(^2\) In order to separate the three fields which are invariant under the whole Poincaré group. We shall limit ourselves here to

\(^2\) We remark that for the solution of this equation we need to impose boundary conditions only on the physical variables \( Q^* \) in contrast to Eq. \( \text{(5)} \) for which boundary conditions only for the unphysical variables \( \overrightarrow{Q} \) are needed.

\(^3\) Note that for a complete analysis it is necessary to investigate the transformation properties of the field \( Q^* \) under the whole Poincaré group. We shall limit ourselves here to
spatial rotations from the three rotational degrees of freedom we perform the following main axis transformation of the original symmetric $3 \times 3$ matrix field $Q^*(x)$

$$Q^*(\chi, \phi) = R^T(\chi(x)) D(\phi(x)) R(\chi(x)), \quad (24)$$

with the orthogonal matrix $R(\chi)$ parametrized by the three Euler angles $\chi_i = (\phi, \theta, \psi)$ and the diagonal matrix $D(\phi) := \text{diag}(\phi_1, \phi_2, \phi_3)$. The main-axis transformation of the symmetric second rank tensor field $Q^*$ therefore induces a parametrization in terms of the three rotational degrees of freedom, the Euler angles $\chi_i$, which describe the orientation of the “intrinsic frame”, and the diagonal elements $\phi_i$ ($i = 1, 2, 3$) which are scalars under spatial rotations.

The momenta $\pi_i$ and $p_{\chi_i}$, canonical conjugate to the diagonal elements $\phi_i$ and the Euler angles $\chi_i$, can easily be found using the generating functional

$$F_3[\phi_i, \chi_i; \ P^*] := \int d^3x \ Tr \ (Q^* P^*) = \int d^3x \ Tr \ \left( R^T(\chi) D(\phi) R(\chi) P^* \right) \quad (25)$$

as

$$\pi_i(x) = \frac{\partial F_3}{\partial \phi_i(x)} = \text{Tr} \left( P^* R^T \alpha_i R \right),$$

$$p_{\chi_i}(x) = \frac{\partial F_3}{\partial \chi_i(x)} = \text{Tr} \left( \frac{\partial R^T}{\partial \chi_i} R \ [P^* Q^* - Q^* P^*] \right). \quad (26)$$

Here $\alpha_i$ are the diagonal matrices with the elements $(\alpha_i)_{lm} = \delta_{li} \delta_{mi}$. Together with the off-diagonal matrices $(\alpha_i)_{lm} = |\epsilon_{ilm}|$ they form an orthogonal basis for symmetric matrices. The original physical momenta $P_{ik}^*$ can then be expressed in terms of the new canonical variables as

$$P^*(x) = R^T(x) \left( \sum_{s=1}^{3} \pi_s(x) \alpha_s + \frac{1}{\sqrt{2}} \sum_{s=1}^{3} P_s(x) \alpha_s \right) R(x), \quad (27)$$

with

$$P_i(x) := \frac{\xi_i(x)}{\phi_j(x) - \phi_k(x)} \quad \text{(cyclic permutation } i \neq j \neq k), \quad (28)$$

the isolation of the scalars under spatial rotations and treat $Q^*$ in terms of “nonrelativistic spin 0 and spin 2 fields” in accordance with the conclusions obtained in the work [3].
and the $SO(3)$ left-invariant Killing vectors in terms of Euler angles $\chi_i = (\psi, \theta, \phi)$, 
\[ \xi_k(x) := \mathcal{M}(\theta, \psi)_{kl} p_{\chi l} , \]  
with the matrix 
\[ \mathcal{M}(\theta, \psi) := \begin{pmatrix} \sin \psi / \sin \theta, & \cos \psi, & -\sin \psi \cot \theta \\ -\cos \psi / \sin \theta, & \sin \psi, & \cos \psi \cot \theta \\ 0, & 0, & 1 \end{pmatrix} . \] 

The antisymmetric part $\vec{E}$ of the electric field appearing in the unconstrained Hamiltonian (18) is given by the following expansion in a $1/g$ series, analogous to (21) - (23), 
\[ E_i = R^T_{is} \sum_{n=0}^{\infty} \mathcal{E}_s^{(n)} , \] 
with the zeroth order term 
\[ \mathcal{E}_s^{(0)} := -\frac{\xi_i}{\phi_j + \phi_k} \ (\text{cycl. perm. } i \neq j \neq k) , \] 
the first order term given from $\mathcal{E}^{(0)}$ via 
\[ \mathcal{E}_s^{(1)} := \frac{1}{g} \frac{1}{\phi_j + \phi_k} \left[ \left( (\nabla_{X_j} \vec{E}^{(0)})_k - (\nabla_{X_k} \vec{E}^{(0)})_j \right) - \Xi_i \right] , \] 
with cyclic permutations of $i \neq j \neq k$, and the higher order terms of the expansion determined via the recurrence relations 
\[ \mathcal{E}_s^{(n+1)} := \frac{1}{g} \frac{1}{\phi_j + \phi_k} \left( (\nabla_{X_j} \vec{E}^{(n)})_k - (\nabla_{X_k} \vec{E}^{(n)})_j \right) . \] 

Here the components of the covariant derivatives $\nabla_{X_i}$ in the direction of the vector field $X_i(x) := R_{ik} \theta_k$, 
\[ (\nabla_{X_i} \vec{E})_b := X_i \mathcal{E}_b + \Gamma^d_{ib} \mathcal{E}_d , \] 
are determined by the connection depending only on the Euler angles 
\[ \Gamma^b_{ia} := (RX_i R^T)_{ab} . \]
It is easy to check that the connection $\Gamma^b_{ia}$ can be written in the form

$$\Gamma^b_{ia} = i(J^*)_{ab} (\mathcal{M}^{-1})_{sk} X_i \chi_k,$$

(37)

using the matrix $\mathcal{M}$ given in terms of the Euler angles $\chi_i = (\psi, \theta, \phi)$ in (30), which expresses the dual nature of the Killing vectors $\xi_i$ in (30), and the Maurer-Cartan one-forms $\omega^i$ defined by

$$RdR^T =: \omega_i J^i, \quad \omega_i = (\mathcal{M}^{-1})^i_k d\chi_k.$$  

(38)

The source terms $\Xi_k$ in (33), finally, are given as

$$\Xi_1 = \Gamma^1_{22}(\pi_1 - \pi_2) + \frac{1}{2} X_1 \pi_1 - \Gamma^2_{23} \mathcal{P}_2 - \Gamma^1_{23} \mathcal{P}_1 - 2\Gamma^1_{12} \mathcal{P}_3 + X_2 \mathcal{P}_3 + (2 \leftrightarrow 3),$$

(39)

and its cyclic permutations $\Xi_2$ and $\Xi_3$.

The unconstrained Hamiltonian therefore takes the form

$$H = \frac{1}{2} \int d^3 x \left( \sum_{i=1}^{3} \pi_i^2 + \frac{1}{2} \sum_{cyc.} (\phi_j - \phi_k)^2 + \frac{1}{2} \vec{E}^2 + V \right),$$

(40)

where the potential term $V$

$$V[\phi, \chi] = \sum_{i=1}^{3} V_i[\phi, \chi]$$

(41)

is the sum of

$$V_1[\phi, \chi] = \left( \Gamma^1_{12}(\phi_2 - \phi_1) - X_2 \phi_1 \right)^2 + \left( \Gamma^1_{13}(\phi_3 - \phi_1) - X_3 \phi_1 \right)^2 + \left( \Gamma^1_{23}\phi_3 + \Gamma^1_{32}\phi_2 - g\phi_2\phi_3 \right)^2,$$

(42)

and its cyclic permutations. We see that, via the main-axis-transformation of the symmetric second rank tensor field $Q^*$, the rotational degrees of freedom, the Euler angles $\chi$ and their canonical conjugate momenta $p_{\chi}$, have been isolated from the scalars under spatial rotations, and appear in the unconstrained Hamiltonian only via the three Killing vector fields $\xi_k$, the connections $\Gamma$, and the derivative vectors $X_k$. 

9
4 The infrared limit of unconstrained $SU(2)$ gluodynamics

From the expression (40) for the unconstrained Hamiltonian one can analyse the classical system in the strong coupling limit up to order $O(1/g)$. Using the leading order (33) of the $\vec{E}$ we obtain the Hamiltonian

$$H_S = \frac{1}{2} \int d^3x \left( \sum_{i=1}^{3} \pi_i^2 + \sum_{\text{cycl.}} \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + V[\phi, \chi] \right).$$

(43)

For the further investigation of the low energy properties of $SU(2)$ field theory a thorough understanding of the properties of the term in (41) containing no derivatives,

$$V_{\text{hom}}[\phi_i] = g^2[\phi_1^2 \phi_2^2 + \phi_2^2 \phi_3^2 + \phi_3^2 \phi_1^2],$$

(44)

is crucial. The classical absolute minima of energy correspond to vanishing of the positive definite kinetic term in the Hamiltonian (43). The stationary points of the potential term (44) are

$$\phi_1 = \phi_2 = 0, \quad \phi_3 \text{ arbitrary},$$

(45)

and its cyclic permutations. Analysing the second order derivatives of the potential at the stationary points one can conclude that they form a continuous line of degenerate absolute minima at zero energy. In other words the potential has a “valley” of zero energy minima along the line $\phi_1 = \phi_2 = 0$. They are the unconstrained analogs of the toron solutions [17] representing constant Abelian field configurations with vanishing magnetic field in the strong coupling limit. The special point $\phi_1 = \phi_2 = \phi_3 = 0$ corresponds to the ordinary perturbative minimum.

For the investigation of configurations of higher energy it is necessary to include the part of the kinetic term in (43) containing the angular momentum variables $\xi_i$. Since the singular points of this term just correspond to the absolute minima of the potential there will a competition between an attractive and a repulsive force. At the balance point we shall have a local minimum corresponding to a classical configuration with higher energy.

We now would like to find the effective classical field theory to which the unconstrained theory reduces in the limit of infinite coupling constant $g$, if we
assume that the classical system spontaneously chooses one of the classical zero energy minima of the leading order $g^2$ part \( (44) \) of the potential. As discussed above these classical minima include apart from the perturbative vacuum, where all fields vanish, also field configurations with one scalar field attaining arbitrary values. Let us therefore put without loss of generality (explicitly breaking the cyclic symmetry)

$$\phi_1 = \phi_2 = 0, \quad \phi_3 \text{ -- arbitrary},$$

(46)
such that the potential \( (44) \) vanishes. In this case the part of the potential \( (11) \) containing derivatives takes the form

$$V_{\text{inh}} = \phi_3(x)^2[(\Gamma^2_{13}(x))^2 + (\Gamma^2_{23}(x))^2 + (\Gamma^2_{33}(x))^2 + (\Gamma^3_{11}(x))^2 + (\Gamma^3_{21}(x))^2 + (\Gamma^3_{31}(x))^2] + [(X_1\phi_3)^2 + (X_2\phi_3)^2] + 2\phi_3(x)[\Gamma^3_{31}(x)X_1\phi_3 + \Gamma^3_{32}(x)X_2\phi_3]$$

(47)

Introducing the unit vector

$$n_i(\phi, \theta) := R_3(\phi, \theta),$$

(48)
pointing along the 3-axis of the “intrinsic frame”, one can write

$$V_{\text{inh}} = \phi_3(x)^2(\partial_i\vec{n})^2 + (\partial_i\phi_3)^2 - (n_i\partial_i\phi_3)^2 - (n_i\partial_i n_j)\partial_j(\phi_3^2).$$

(49)

Concerning the contribution from the nonlocal term in this phase, we obtain for the leading part of the electric fields

$$\mathcal{E}^{(0)}_1 = -\xi_1/\phi_3, \quad \mathcal{E}^{(0)}_2 = -\xi_2/\phi_3.$$ 

(50)

Since the third component \( \mathcal{E}^{(0)}_3 \) and \( \mathcal{P}_3 \) are singular in the limit \( \phi_1, \phi_2 \to 0 \), it is necessary to have \( \xi_3 \to 0 \). The assumption of a definite value of \( \xi_3 \) is in accordance with the fact that the potential is symmetric around the 3-axis for small \( \phi_1 \) and \( \phi_2 \), such that the intrinsic angular momentum \( \xi_3 \) is conserved in the neighbourhood of this configuration. Hence we obtain the following effective Hamiltonian up to order \( O(1/g) \)

$$H_{\text{eff}} = \frac{1}{2} \int d^3x \left[ \frac{\pi_3^2}{\phi_3^2} + \frac{1}{\phi_3^2}(\xi_1^2 + \xi_2^2) + (\partial_i\phi_3)^2 + \phi_3^2(\partial_i\vec{n})^2 \right. \\
\left. - (n_i\partial_i\phi_3)^2 - (n_i\partial_i n_j)\partial_j(\phi_3^2) \right].$$

(51)
After the inverse Lagrangian transformation we obtain the corresponding nonlinear sigma model type effective Lagrangian for the unit vector $\vec{n}(t, \vec{x})$ coupled to the scalar field $\phi_3(t, \vec{x})$

$$L_{\text{eff}}[\phi_3, \vec{n}] = \frac{1}{2} \int d^3x \left[ (\partial_\mu \phi_3^2)^2 + \phi_3^2 (\partial_\mu \vec{n})^2 + (n_i \partial_i \phi_3)^2 + n_i (\partial_i n_j) \partial_j (\phi_3^2) \right].$$

(52)

In the limit of infinite coupling the unconstrained field theory in terms of six physical fields equivalent to the original $SU(2)$ Yang-Mills theory in terms of the gauge fields $A_\mu^a$ reduces therefore to an effective classical field theory involving only one of the three scalar fields and two of the three rotational fields summarized in the unit vector $\vec{n}$. Note that this nonlinear sigma model type Lagrangian admits singular hedgehog configurations of the unit vector field $\vec{n}$. Due to the absence of a scale at the classical level, however, these are unstable. Consider for example the case of one static monopole placed at the origin,

$$n_i := x_i / r, \quad \phi_3 = \phi_3(r), \quad r := \sqrt{x_1^2 + x_2^2 + x_3^2}. \quad (53)$$

Minimizing its total energy $E$

$$E[\phi_3] = 4\pi \int dr \phi_3^2(r) \quad (54)$$

with respect to $\phi_3(r)$ we find the classical solution $\phi_3(r) \equiv 0$. There is no scale in the classical theory. Only in a quantum investigation a mass scale such as a nonvanishing value for the condensate $<0|\phi_3^2|0>$ may appear, which might be related to the string tension of flux tubes directed along the unit-vector field $\vec{n}(t, \vec{x})$. The singular hedgehog configurations of such string-like directed flux tubes might then be associated with the glueballs. The pure quantum object $<0|\phi_3^2|0>$ might be realized as a squeezed gluon condensate $|18\rangle$. Note that for the case of a spatially constant condensate,

$$<0|\phi_3^2|0> = 2m^2 = \text{const.}, \quad (55)$$

the quantum effective action corresponding to (52) should reduce to the lowest order term of the effective soliton Lagangian discussed very recently by Faddeev and Niemi $|10\rangle$

$$L_{\text{eff}}[\vec{n}] = m^2 \int d^3x (\partial_\mu \vec{n})^2. \quad (56)$$
As discussed in [16], for the stability of these knots furthermore a higher order Skyrmion-like term in the derivative expansion of the unit-vector field $\vec{n}(t, \vec{x})$ is necessary. To obtain it from the corresponding higher order terms in the strong coupling expansion of the unconstrained Hamiltonian (40) is under present investigation.

5 Concluding remarks

Following the Dirac formalism for constrained Hamiltonian systems we have formulated classical $SU(2)$ Yang-Mills gauge theory entirely in terms of unconstrained gauge invariant local fields. All transformations which have been used, canonical transformations and the Abelianization of the constraints, maintain the canonical structures of the generalized Hamiltonian dynamics. We identify the unconstrained fields with symmetric positive definite second rank tensor fields $Q^*$ and $P^*$ under spatial rotations. The three scalar fields are separated from the three rotational degrees of freedom via the main-axis transformation of the field $Q^*$. Our unconstrained representation of the Hamiltonian furthermore allows us to derive an effective low energy Lagrangian for the rotational degrees of freedom coupled to one of the scalar fields suggested by the form of the classical potential in the strong coupling limit. The dynamics of the rotational variables in this limit is summarized by the unit vector describing the orientation of the intrinsic frame. Due to the absence of a scale in the classical theory the singular hedgehog configurations of the unit vector field is found to be unstable classically. Only in a quantum treatment, which is under present investigation, a nonvanishing value for the vacuum expectation value for one of the three scalar field operators, and hence a mass scale, can occur. For the case of a spatially constant scalar quantum condensate we expect to obtain the first term of a derivative expansion proposed recently by Faddeev and Niemi [16]. As shown in their work such a soliton Lagrangian allows for stable massive knotlike configurations which might be related to glueballs. For the stability of the knots higher order terms in the derivative expansion, such as the Skyrme type fourth order term in [16], are necessary. Their derivation in the framework of the unconstrained theory, proposed in this paper, is also under investigation.

The work of A.M.K. was partially supported by the Russian Foundation for Basic Research under grant No. 98-01-00101. H.-P. P. acknowledges sup-
port by the Deutsche Forschungsgemeinschaft under grant No. Ro 905/11-2 and by the Heisenberg-Landau program for providing a grant.

References

[1] J. Goldstone, R. Jackiw, Phys. Lett. B 74, 81 (1978).
[2] V. Baluni, B. Grossman, Phys. Lett B 78, 226 (1978).
[3] A.G. Izergin, V.F. Korepin, M.E. Semenov - Tyan - Shanskii, L.D. Faddeev, Teor. Mat. Fiz. 38 , 3 (1979).
[4] A. Das, M. Kaku, P.K. Townsend, Nucl.Phys B 149, 109 (1979).
[5] M.Creutz, I.J.Muzinich, and T.N.Tudron, Phys. Rev. 19 , 531 (1979).
[6] N.H. Christ, T.D. Lee, Phys. Rev. 22 , 939 (1980).
[7] Yu. Simonov, Sov. J. Nucl. Phys. 41, 835 (1985).
[8] V.V. Vlasov, V.A. Matveev, A.N. Tavkhelidze, S.Yu. Khlebnikov, M.E. Shaposhnikov, Phys. of Elem. Part. Nucl. 18, 5 (1987).
[9] K. Haller, Phys. Rev. D 36, 1839 (1987).
[10] M. Bauer, D. Z. Freedman and P. E. Haagensen, Nucl. Phys. B 428, 147 (1994).
[11] A.M. Khvedelidze and H.-P. Pavel, Hamiltonian Reduction of SU(2) Yang-Mills Field Theory, hep-th/9808082, MPG-VT-UR 157/98; On the Infrared Limit of Unconstrained SU(2) Yang-Mills Theory, hep-th/9808089, MPG-VT-UR 158/98; Unconstrained Hamiltonian Formulation of SU(2) Gluodynamics, hep-th/9808102, MPG-VT-UR 159/98.
[12] P.A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, (Yeshive University Press, New York, 1964).
[13] M. Henneaux, C. Teitelboim, Quantization of Gauge Systems , (Princeton University Press, Princeton, NJ, 1992).
[14] S.A. Gogilidze, A.M. Khvedelidze, V.N. Pervushin, J. Math. Phys. 37, 1760 (1996).

[15] S.A. Gogilidze, A.M. Khvedelidze, D. M. Mladenov and H.-P. Pavel, Phys. Rev. D 57, 7488 (1998).

[16] L. Faddeev and A.J. Niemi, “Partially Dual Variables in $SU(2)$ Yang-Mills Theory”, Electronic Archive hep-th/9807069 (1998).

[17] M. Lüscher, Nucl. Phys. B 219, 233 (1983).

[18] D. Blaschke, H.-P. Pavel, V.N. Pervushin, G. Röpke and M.K. Volkov, Phys. Lett. B 397 (1997) 129.