Theory of Deep Q-Learning:  
A Dynamical Systems Perspective  

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Abstract  
Deep Q-Learning is an important algorithm, used to solve sequential decision making problems. It involves training a Deep Neural Network, called a Deep Q-Network (DQN), to approximate a function associated with optimal decision making, the Q-function. Although wildly successful in laboratory conditions, serious gaps between theory and practice prevent its use in the real-world. In this paper, we present a comprehensive analysis of the popular and practical version of the algorithm, under realistic verifiable assumptions. An important contribution is the characterization of its performance as a function of training. To do this, we view the algorithm as an evolving dynamical system. This facilitates associating a closely-related measure process with training. Then, the long-term behavior of Deep Q-Learning is determined by the limit of the aforementioned measure process. Empirical inferences, such as the qualitative advantage of using experience replay, and performance inconsistencies even after training, are explained using our analysis. Also, our theory is general and accommodates state Markov processes with multiple stationary distributions.

1 Introduction  
Automation and Artificial Intelligence are ubiquitous in our society, in electrical grids, commerce, transportation, etc. Reinforcement Learning (RL), a field that deals with the problem of ensuring that software agents take optimal decisions (actions) in an autonomous manner, is poised to play a pertinent role. The power of RL lies in solving sequential decision making problems in a model-free manner. Algorithms are called model-free when they learn to take optimal decisions merely by interacting with an unknown environment in which they operate, i.e., no model of the environment is assumed. This paradigm vastly differs from traditional decision making algorithms that greatly rely on accurate models of the environment [5]. While the model-free nature makes RL simple versatile and widely applicable, it requires enormous amounts of data and computational power to be successful.

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A popular variant of RL called Deep Reinforcement Learning (DeepRL) combines the fundamental principles of RL with the power of a Deep Neural Network (DNN). Recall that a DNN is a neural network with multiple layers in between the input and the output layers [11]. DeepRL has exhibited tremendous empirical success in recent years owing to availability of large amounts of data and colossal computational power, and due to innovations in DNN architectures. AlphaGo is a DeepRL algorithm that beat the best players in the board game Go [17]. In [14] the popular DeepRL algorithm Deep Q-Learning was introduced, and it achieved superhuman performance in playing ATARI video games. Recently, DeepRL algorithms for “autonomous continuous control” were developed, see for e.g., [13], where the control of self-driving cars is considered. The reader is referred to [2] for a brief survey of algorithms and associated results from DeepRL.

In this paper we focus on the Deep Q-Learning algorithm, since it is simple popular and performs well in a broad range of complex scenarios. Technically speaking, Deep Q-Learning seeks to find the optimal Q-function, which is a function used to pick optimal actions. Finding the optimal Q-function involves training a neural network, called the Deep Q-Network (DQN), to minimize the “squared Bellman loss (error)”. Training typically involves repeated interactions with a “simulator” or using “historical data”. Although popular and successful, Deep Q-Learning is always caveat with the lack of comprehensive theory and analyses. This is part of the reason why it is not ubiquitous in the real-world, although it performs very well in a laboratory setting. It is also observed that performance can sometimes be inconsistent, even when training is deemed sufficient. This could be due to several reasons, including but not limited to, inefficient training and the use of irrelevant training data. The unavailability of a comprehensive analysis of training, means that there is no scientific solution to fix these inconsistencies.

Recently, there has been concentrated effort towards developing concrete theories for Deep Q-Learning. Sufficient conditions are presented in [19] that guarantee convergence of Deep Q-Learning, provided the DQN only has Rectified Linear Units. The analysis requires strict conditions on the Bellman operator and the distribution of the state-action pairs. In [21] a non-asymptotic finite sample analysis of Deep Q-Learning with linear function approximation, instead of DNN approximation, is presented. While studies like the aforementioned focus on sufficient conditions for Deep Q-Learning convergence, [1] focuses on characterizing the conditions under which Deep Q-Learning is divergent. In [9] the topic of efficient exploration in policy optimization is explored from a theoretical perspective. These preliminary results are important and interesting. However, the simplifications made and assumptions imposed mean that they cannot not be used to analyze Deep Q-Learning as implemented in practice.  

1.1 Our Contributions
The performance of Deep Q-Learning greatly depends on the training procedure. It is empirically observed that performance is great in certain test-scenarios and poor in others. The hitherto available theory does not explain why this is so, and also does not explain several other empirical conclusions. The main contribution of this paper is a comprehensive analysis of Deep Q-Learning that explains, among others, the previously mentioned phenomenon. Further, the
assumptions involved are practical and verifiable.

We show that the squared Bellman loss is minimized over the set of state-action pairs, distributed in accordance to a measure obtained as a limit of a natural measure process associated with training. We also show that this limiting measure is stationary with respect to the state Markov process. Further, its empirical estimate can be used to retrain and boost performance. As stated earlier, the limiting measure is strongly shaped by the training process. It must be noted that, unlike previous literature, our analysis allows for multiple stationary distributions of the state Markov process.

The main idea behind experience replay is to relearn from past experiences. We show that experience replay affects the quality of performance by shaping the limiting distribution. Additionally, it may facilitate algorithm stability. Finally we note that, despite our focus on Deep Q-Learning, the theory is general, and can be used to analyze other Deep Learning algorithms.

1.2 Methodologies Used

For our analyses we utilize tools from the fields of Stochastic Approximation Algorithms (SA), Stochastic Processes, Measure Theory and Viability Theory [3, 6, 8, 10, 12]. The field of SA contains extensive theory around iterative algorithms that involve approximations and noisy observations. The o.d.e. method is a key technique from this theory, and we utilize it extensively in this paper. It involves transforming a discrete time algorithm into a continuous time trajectory such that the latter’s limit, as time tends to infinity, is identical to the limit of the algorithm. In other words, their asymptotic behaviors are identical. Further, the trajectory is a solution to a natural ordinary differential equation associated with the algorithm updates, see [8, 12]. As a solution to some o.d.e., the trajectory may be viewed as an evolving dynamical system with possible state constraints. This view facilitates further analysis using tools from Viability Theory [3].

1.3 Organization

Before we dive into the analysis, necessary preliminaries are presented in Section 2. Then, we present the Deep Q-Learning algorithm and the associated assumptions in Section 3. In Sections 4 and 5 we analyze the Deep Q-Learning algorithm and present our main result Theorem 1. The aforementioned analysis is presented while assuming that only squashing activations are used to construct the DQN, and that the training does not use experience replay. This is done to enhance clarity. Later in Section 6 we show that our analysis can be extended to DQN with general activations, and also to training with experience replay. We discuss an extension to accommodate general (non-squashing) activations in Section 6.1. The required modifications to account for experience replay is discussed in Section 6.2.

2 Preliminaries

In this section we present short backgrounds on Reinforcement Learning (RL), Deep Neural Network (DNN) and Deep Q-Network (DQN).
2.1 Reinforcement Learning

In RL an agent interacts with an environment over time, via actions. It takes the current (environment) state into consideration to pick an action, and receives a feedback in terms of a reward. The environment then moves to a new state. This is schematically represented in Figure 1. The goal in RL is to ensure that the agent takes a sequence of actions, such that the rewards accumulated over time are maximized.

![Figure 1: Snapshot of interaction at step n](image)

Formally speaking, the above stated interactions can be modelled as a Markov Decision Process (MDP). It is defined as a 5-tuple \((\mathcal{S}, \mathcal{A}, p, r, \alpha)\), where:

- **\(\mathcal{S}\)** is the state space. In typical applications \(\mathcal{S} \equiv \mathbb{R}^k, k > 0\).
- **\(\mathcal{A}\)** is the action space. In this paper, \(\mathcal{A}\) is a discrete finite set.
- **\(p\)** is the “controlled” transition kernel. We use \(p(\cdot | x, a)\) to represent the distribution of the next state given the current state and action.
- **\(r\)** is the reward function. In particular, \(r(x, a)\) denotes the reward associated with taking action \(a\) at state \(x\).
- **\(\alpha\)** is the discount factor with \(0 < \alpha \leq 1\). It is used to discount the relevance of future consequences of actions.

A policy \(\pi\) is defined as a function from \(\mathcal{S}\) to \(\mathcal{A}\). Given \(\pi\), we can associate a Value function \(V^\pi(x)\) with each \(x \in \mathcal{S}\), with \(V^\pi(x) := \mathbb{E} \left[ \sum_{n \geq 0} \alpha^n r(x_n, \pi(x_n)) \right] | x_0 = x\).

The goal in RL can be restated to find \(\pi^\star\) such that \(V^\pi^\star(x) = \max_{\pi^\star(x)}\) for all \(x \in \mathcal{S}\). In Dynamic Programming parlance \(\pi^\star\) is a solution to the infinite horizon discounted reward problem.

Closely related to the value function is the concept of Q-function, defined over state-action pairs \((x, a) \in \mathcal{S} \times \mathcal{A}\) by \(Q^\pi(x, a) := r(x, a) + \alpha \int V^\pi(x') p(dx' | x, a)\), where \(\pi\) is a fixed policy. The optimal Q-function is defined as:

\[
Q^\star(x, a) := r(x, a) + \alpha \int V^{\pi^\star}(x') p(dx' | x, a).
\]

Clearly, \(\max_{a \in \mathcal{A}} Q^\star(x, a) = V^{\pi^\star}(x)\) and \(\pi^\star(x) = \arg\max_{a \in \mathcal{A}} Q^\star(x, a)\) for all \(x \in \mathcal{S}\). Hence, in order to find \(\pi^\star\) it is sufficient to find \(Q^\star\). This is the idea behind
Q-Learning. Its variant, Deep Q-Learning, has shown tremendous promise in solving complex problems involving continuous state spaces, where Q-Learning typically fails. It involves parameterizing the optimal Q-function using a DNN, called the Deep Q-Network (DQN). The goal is to find the optimal set of parameters (DQN weights) $\theta^*$, by interacting with the environment, such that $Q(x,a;\theta^*) \approx Q^*(x,a)$ for all $(x,a) \in S \times A$. The DQN is trained to minimize the following squared Bellman loss over all state-action pairs $(x,a)$:

$$
[r(x,a) + \alpha \int_{a'\in A} \max_{a' \in A} Q(x',a';\theta) \ p(dx' | x,a) - Q(x,a;\theta)]^2
$$

### 2.2 Deep Q-Network (DQN)

Let us now delve into the architecture of DQN. As it is essentially an Artificial Neural Network, or simply a Neural Network (NN), we begin by describing one. In particular, we discuss the architecture of a **fully connected feedforward network** with real-valued vector inputs. **Activation functions** form the basic building blocks of a NN. The typical domain for an activation function $\sigma$ is $\mathbb{R}$ and its range $\mathbb{R}$ is usually a subset of $\mathbb{R}$, i.e., $\sigma : \mathbb{R} \rightarrow \mathbb{R} \subset \mathbb{R}$. Depending on whether the range of $\sigma$, $\mathbb{R}$, is compact or unbounded, it is said to be **squashing** or **non-squashing**, respectively. There are many activation functions, the following are a few examples considered in this paper: (a) Sigmoid $[1/(1+e^{-x})]$, (b) Hyperbolic Tangent $[(e^x-e^{-x})/(e^x+e^{-x})]$, (c) Gaussian Error Linear Unit $[x \int_{-\infty}^{\infty} e^{-y^2/2}/\sqrt{2\pi} \ dy]$, and (d) Sigmoid Linear Unit $[x/(1+e^{-x})]$. A NN is a collection of activations that are arranged in a sequence of layers, starting with an input layer, then followed by one or more hidden layers, and ending with the output layer. A NN with two or more hidden layers is called a Deep Neural Network (DNN). Figure 3 illustrates one such NN architecture. By convention, a NN is constructed from left to right starting with the input layer and ending with the output layer. Further, the layers are arranged in a feedforward architecture, in that any two successive layers constitute a **complete bipartite graph** with edges directed from the left layer into the right.

Let us now focus on a single activation $\sigma$ within some layer, illustrated in Figure 2. There are $k$ edges leading into and $m$ leading out of $\sigma$, where $m,k \geq 1$. When $\sigma$ is in the input layer, the in-edges connect the $k$ components of the input vector to it. As a part of other layers, the in-edges connect the $k$ activation-outputs from the previous layer to its input. Further, each in-edge is associated with a weight that equals the product of the corresponding previous layer activation output $\text{act}_i$ (or input component $x_i$) and network-weight $\theta_i$, $1 \leq i \leq k$. The input value to the activation is given by $\sum_{i=1}^{k} \text{act}_i \theta_i + b$ (or $\sum_{i=1}^{k} x_i \theta_i + b$), where $b$ is a tunable bias term.

Suppose $\sigma$ is part of an input or hidden layer, then the edges leading out of it, the out-edges, connect its output $\sigma(\sum_{i=1}^{k} \text{act}_i \theta_i + b)$ (or $\sigma(\sum_{i=1}^{k} x_i \theta_i + b)$) to the input of the $m$ activations in the following layer. Finally, if $\sigma$ is part
of the output layer, its output $\sigma \left( \sum_{i=1}^{k} \text{act}_i \theta_i + b \right)$ is combined with the output from other activations that also belong to the outer layer. In order to obtain the required final NN output. While our discussion is by no means complete we hope that it is sufficient for this paper. For more details the reader may refer to [11,20].

[Note on tunable biases] Moving forward, we assume that there are no tunable biases added to the activation inputs. In particular, we assume that the input is merely $\sum \theta_i \text{act}_i$, (or $\sum \theta_i x_i$ if the activation belongs to the input layer). We make this simplification for the sake of clarity in presentation. Our analysis will remain unaltered outside of minor bookkeeping to account for these biases.

![Figure 2: Single activation from some layer](image)

![Figure 3: Schematic Representation of a DQN](image)

We are now ready to discuss the DQN architecture. Its input is the state vector $x \in \mathbb{S}$, and its output is a vector of dimension $|\mathcal{A}|$. Hence the DQN output layer is a union of $|\mathcal{A}|$ separate (sub) output layers, one for each action. If $l(a)$ is the number of activations in the output layer associated with action $a$, then $Q(x, a; \theta) = \sum_{i=1}^{l(a)} \text{act}_a(i) \theta_a(i)$, where $\text{act}_a(i)$ is the activation-$i$ output and $\theta_a(i)$ is the associated network weight. Recall that we have ignored the
bias term in view of our previous “note on tunable biases”. Formally, the DQN is a parameterization of the vector \((Q(x,a))_{a \in A}\) where \(Q^*\) is the optimal Q-function. Please refer to Fig. 3 for an illustration of DQN. In practice, it has been found that limiting the number of hidden layers to two results in good empirical performance. Each edge in the DQN is associated with an edge-weight \(e\), which is the product of the output \(\text{act}\) from the previous activation and the network weight \(\theta_e\), see Fig. 3. In Deep Q-Learning one updates the DQN weights \(\theta := (\theta_e \mid e \text{ is an edge in the DQN})\) iteratively, in order to find \(\theta^*\) such that \(Q(x,a;\theta^*) \approx Q^*(x,a) \ \forall (x,a) \in \mathcal{S} \times \mathcal{A}\).

3 Deep Q-Learning and Assumptions

Deep Q-learning involves of the following iteration that updates the DQN weights to minimize the squared Bellman loss:

\[
\theta_{n+1} = \theta_n + \gamma(n) \nabla_{\theta} \ell(\theta_n, x_n, a_n), \quad \text{where} \quad (1)
\]

(i) \(\theta_n \in \mathbb{R}^d, x_n \in \mathcal{S} \text{ and } a_n \in \mathcal{A}, \text{ for } n \geq 0.\) The state space \(\mathcal{S}\) is assumed to be \(\mathbb{R}^n\) for some \(n \geq 1,\) and \(\mathcal{A}\) is a finite set of actions.

(ii) The loss gradient \(\nabla_{\theta} \ell(\theta_n, x_n, a_n) = \left[ (r(x_n, a_n) + \alpha \max_{a' \in \mathcal{A}} Q(x_{n+1}, a'; \theta_n) - Q(x_n, a_n; \theta_n)) \nabla_{\theta} Q(\theta_n, x_n, a_n) \right] \quad \text{where } \alpha \text{ is the discount factor. Since } a_n \text{ is the action taken at time } n, \nabla_{\theta} \ell(x_n, a_n; \theta_n) \text{ denotes the loss-gradient back-propagated via action } a_n.

(iii) \(\gamma(n)\) is the given step-size sequence satisfying the standard assumptions of non-summability and square summability.

Note that the loss gradient in (1) is calculated using the sample value \(\max_{a' \in \mathcal{A}} Q(x_{n+1}, a'; \theta_n)\) instead of the expected value \(\int \max_{a' \in \mathcal{A}} Q(x', a'; \theta_n)p(x' \mid x, a).\) This is because in real-world applications the transition kernel \(p\) is unknown. The algorithm observes the the next state \(x_{n+1}\) and the reward \(r(x_n, a_n),\) after applying \(a_n\) at state \(x_n\).

The state Markov process is determined by the transition kernel \(p(dy \mid x, a).\) In training, actions are picked through a policy that exploits the approximation capability of DQN, while simultaneously exploring new actions. In other words, the transition kernel is indirectly influenced by the network weights. Hence, we denote the controlled transition kernel by \(p(dy \mid x, a, \theta).\) For fixed weights \(\theta\) and a fixed stochastic policy \(\pi_\theta,\) the transition kernel is given by:

\[
\tilde{p}_\theta(dy \mid x) = \sum_{a \in \mathcal{A}} p(dy \mid x, a, \theta) \pi_\theta(x, da).
\]

The policy is subscripted with \(\theta\) to emphasize that it depends on the network weights (via exploitation). Let us suppose that \(\pi_\theta\) only exploits and does not explore, i.e., \(\pi_\theta(x) = \text{argmax}_{a \in \mathcal{A}} Q(x,a;\theta).\) Then the above stochastic policy is the Dirac measure given by \(\pi_\theta(x,a) = \delta_{\text{argmax}_{a \in \mathcal{A}} Q(x,a;\theta)},\) further the above kernel becomes:

\[
\tilde{p}_\theta(dy \mid x) = p \left( dy \mid x, \text{argmax}_{a \in \mathcal{A}} Q(x,a;\theta), \theta \right).
\]
3.1 Assumptions

Below, we list the assumptions required to analyze (1):

(A1) $\gamma(n) > 0$ for all $n \geq 0$, $\sum_{n \geq 0} \gamma(n) = \infty$ and $\sum_{n \geq 0} \gamma(n)^2 < \infty$. Further, the sequence is eventually monotonic.

(A2) (a) $\sup_{n \geq 0} \|\theta_n\|_2 < \infty$ a.s., (b) $\sup_{n \geq 0} \|x_n\|_2 < \infty$ a.s.

(A3) The state transition kernel $p(\cdot | x, a)$ is continuous in the $x$ coordinate.

(A4) The DQN is composed of activation functions that are squashing and twice continuously differentiable.

(A5) The reward function $r: \mathbb{S} \times \mathcal{A} \rightarrow \mathbb{R}$ is continuous.

The first assumption regarding the step-size sequence (learning rate) is standard to literature. Recall that the loss gradient in (1) is calculated using samples that are supposed to approximate expected values. The resulting sampling errors are controlled using step-sizes that are square summable. Assuming stability, i.e., (A2), is essential for analyzing the long-term behavior of (1). In this paper, we characterize the performance of DQN as a function of training, assuming its stability.

Consider two different environment states that are also “close neighbors”. Assumptions (A3) and (A4) state that the consequences (next states and rewards, respectively) of taking the same action in these states are similar. In addition to being natural, these assumptions ensure the performance of approximation based algorithms like Deep Q-learning. As long as the state-action pairs encountered during training are a rich enough representation of $\mathbb{S} \times \mathcal{A}$, (A3) and (A4) facilitate good approximation of the $Q$-function.

First, we present an analysis for DQN with squashing activations, assuming (A4). Later, we forgo this assumption and present the steps involved in extending our analysis to general non-squashing activations.

4 Properties of the loss gradient

In this section, we present a thorough study of the loss gradient. The aim is to prove certain useful properties that facilitate an abstract view of the loss gradient, with lesser “moving parts”. In particular, we show that $\nabla \ell$ is (A) locally Lipschitz continuous in the $\theta$ coordinate, and (B) continuous in the $x$ and $a$ coordinates. Since $\mathcal{A}$ is finite, $\nabla \theta \ell$ is trivially continuous in the $a$ coordinate.

As for the rest, rather than showing them in one long-winded lemma, we shall break down the objective into several small auxiliary lemmata. In the end we put all these results together to obtain the required conclusions regarding $\nabla \theta \ell$.

Recall that every action is associated with a different output layer. Let $l(a)$ be the width of the output layer associated with action $a$, then $Q(x, a; \theta) = \sum_{i=1}^{l(a)} \text{act}_a(i) \theta_a(i)$.

**Lemma 1.** $\sup_{x \in \mathbb{S}, a \in \mathcal{A}} |Q(x, a; \theta)| \leq \hat{L} \|\theta\|_2$, for some $\hat{L} > 0$. 

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Proof. We begin by noting that activation functions considered herein are also squashing. Hence, absolute values of their outputs are bounded by some $0 < c < \infty$. Let us fix arbitrary $x \in \mathcal{S}$ and $a \in \mathcal{A}$, then

$$|Q(x, a; \theta)| \leq c \sum_{i=1}^{l(a)} |\theta_a(i)| = c \|\theta_a\|_1,$$

where $\|\cdot\|_1$ is the 1-norm. It now follows from $\|\theta_a\|_1 \leq l(a)\|\theta_a\|_2$, that $|Q(x, a; \theta)| \leq cl(a)\|\theta_a\|_2$. If we let $L := c l(a)$, then the statement of the lemma follows.

**Lemma 2.** $Q(x, a; \theta)$ is twice continuously differentiable in the $\theta$ coordinate for every $x \in \mathcal{S}$ and $a \in \mathcal{A}$.

**Proof.** Let us fix an arbitrary $\hat{x} \in \mathcal{S}$ and $\hat{a} \in \mathcal{A}$. We need to show the existence and continuity of $\partial^2 Q(\hat{x}, \hat{a}; \theta)/\partial \theta_a(i)^2$, where $\theta(i)$ is the $i$-th component of the DQN weight-vector $\theta$. Recall that each edge in the DQN is associated with an edge-weight $e := \text{act}\, \theta(i)$, where $\text{act}$ is the output of the activation located at the head of that edge, and $\theta(i)$ is the associated network weight.

To prove the lemma, we show that both $\partial^2 Q(\hat{x}, \hat{a}; \theta)/\partial \theta_a(i)^2$ and $\partial^2 Q(\hat{x}, \hat{a}; \theta)/\partial e^2$ are continuous. The proof involves inducting on the depth of the DNN, starting from the output layer. Recall that $Q(\hat{x}, \hat{a}; \theta) = \sum_{i=1}^{l(a)} e_a(i)$, where $e_a(i) := \text{act}_a(i) \theta_a(i)$. Then, $\partial^2 Q(\hat{x}, \hat{a}; \theta)/\partial \theta_a(i)^2 = \partial^2 Q(\hat{x}, \hat{a}; \theta)/\partial e_a^2 = 0$ for all $a \neq \hat{a}$. For $a = \hat{a}$, the required follows from the twice continuous differentiability of the activation units.

Now, we assume that the hypothesis is true for the $(l + 1)^{st}$ layer and prove for the $l^{th}$ layer. Let us focus on one edge from the $k^{th}$ layer and its associated edge-weight $e_i := \text{act}_i \theta_i$, see Figure 4 for an illustration. It follows directly from the back-propagation algorithm (chain rule) that:

$$\frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial e_i} = \frac{\partial \text{act}_a}{\partial e_i} \sum_{j=1}^{k} \left[ \frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial e_j} \theta_j \right],$$

$$\frac{\partial^2 Q(\hat{x}, \hat{a}; \theta)}{\partial e_i^2} = \left( \frac{\partial \text{act}_a}{\partial e_i} \right)^2 \sum_{j=1}^{k} \left[ \frac{\partial^2 Q(\hat{x}, \hat{a}; \theta)}{\partial e_j^2} \theta_j^2 \right] + \frac{\partial^2 \text{act}_a}{\partial e_i^2} \sum_{j=1}^{k} \left[ \frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial e_j} \theta_j \right].$$

From the induction hypothesis and the twice continuous differentiability of $\text{act}_a$, we get that $\frac{\partial^2 Q(\hat{x}, \hat{a}; \theta)}{\partial e_i^2}$ is continuous. Next, we observe:

$$\frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial \theta_i} = \frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial e_i} \frac{\partial e_i}{\partial \theta_i} = \text{act}_i \frac{\partial Q(\hat{x}, \hat{a}; \theta)}{\partial e_i},$$

Figure 4: Section of a DNN
Lemma 2. From this we conclude that there exists a neighborhood

\[ N(\theta) \]

of \( \theta \) such that (2) is satisfied for every \( \theta \) in this neighborhood. Finally, note that \( Q \) is twice continuously differentiable functions (activation units), we get that \( Q \) is locally Lipschitz continuous in that coordinate. Also, the Lipschitz constant may depend on \( \theta \). Let us fix arbitrary \( \hat{\theta} \in A \) and \( \theta \in \mathbb{R}^d \).

Since \( Q(x, a, \theta) \) is only via the action \( a \) and \( \hat{\theta} \), we conclude that:

\[
\max_{a \in A} Q(x, a; \theta_1) - \max_{a \in A} Q(x, a; \theta_2) \leq L(\theta, x) ||\theta_1 - \theta_2||_2.
\]

Hence, from Lemma 1 and the compactness of \( \mathcal{N}(\theta, \hat{x}) \) we conclude that:

\[
\sup_{\theta \in \mathcal{N}(\theta, \hat{x})} \sup_{a \in A} \left| Q(x, a; \theta) \right| < \infty.
\]

In particular, there exists a bounded measurable function \( \hat{F}_\theta : x \mapsto L(x, \theta) \) such that (2) is satisfied for every \( x \in S \), with \( \hat{F}_\theta(x) \) as the Lipschitz constant.

Hitherto presented arguments and observations yield:

\[
\int \max_{a \in A} Q(x, a; \theta_1) p(dx | \hat{x}, \hat{a}, \theta_1) - \int \max_{a \in A} Q(x, a; \theta_2) p(dx | \hat{x}, \hat{a}, \theta_2)
\leq ||\theta_1 - \theta_2||_2 \int L(\theta, x) p(dx | \hat{x}, \hat{a}) \leq L||\theta_1 - \theta_2||_2,
\]

where \( L = 2 \times \sup_{\theta \in \mathcal{N}(\theta, \hat{x})} \sup_{x \in S} \sup_{a \in A} |Q(x, a; \theta)| \).
We now show continuity in the \( x \) coordinate. Let us fix an arbitrary \( \hat{\theta} \in \mathbb{R}^d \). Define \( \hat{a}(x) := \arg\max_{a \in \mathcal{A}} Q(x, a; \hat{\theta}) \), then \( x_n \to x \) implies that \( \hat{a}(x_n) \to \hat{a}(x) \).

Define \( Q(x) := Q(x, \hat{a}(x), \hat{\theta}) \) for all \( x \in \mathbb{R}^d \), then we infer from Lemma 1 that \( Q \in \mathcal{C}_b(\mathcal{S}) \). Recall that we have assumed the transition kernel to be continuous in \( x \). Hence \( x_n \to x \) implies that \( p(\cdot \mid x_n, \hat{a}, \hat{\theta}) \overset{d}{\to} p(\cdot \mid x, \hat{a}, \hat{\theta}) \), i.e., the kernels converge in distribution. It now follows from the definition of “convergence in distribution” that \( \int Q(y)p(dy \mid x_n, \hat{a}, \hat{\theta}) \to \int Q(y)p(dy \mid x, \hat{a}, \hat{\theta}) \). In other words, we have the required, i.e., as \( n \to \infty \)

\[
x_n \to x \implies \int \max_{a \in \mathcal{A}} Q(y, a, \hat{\theta})p(dy \mid x_n, \hat{a}, \hat{\theta}) \to \int \max_{a \in \mathcal{A}} Q(y, a, \hat{\theta})p(dy \mid x, \hat{a}, \hat{\theta}).
\]

Finally, recall that \( \mathcal{A} \) is compact metrizable as it is a finite. Hence continuity in the “\( a \)” coordinate is trivial.

Let us define the following:

\[
\psi_n := \alpha \left[ \max_{a \in \mathcal{A}} Q(x_{n+1}, a; \theta_n) - \int \max_{a \in \mathcal{A}} Q(x, a; \theta_n)p(dx \mid x_n, a_n, \theta_n) \right] \nabla_\theta Q(x_n, a_n; \theta_n),
\]

and \( M_n := \sum_{m=0}^{n-1} \gamma(m)\psi_m, \ n \geq 0. \)

It can be shown that \( \{M_n\}_{n \geq 0} \) is a zero-mean Martingale with respect to the filtration \( \mathcal{F}_{n-1} := \sigma(x_m, a_m, \theta_m \mid m \leq n), \ n \geq 1 \). Recall that we assumed the stability of \( \mathcal{C} \) and the state sequence, i.e., sup \( \sup_{n \geq 0} \|\theta_n\| < \infty \) and sup \( \sup_{n \geq 0} \|x_n\| < \infty \) a.s. This, together with Lemma 2, lets us conclude that \( \sup_{n \geq 0} |Q(x_n, a_n; \theta_n)| < K_1 < \infty \), and that \( \|\nabla_\theta Q(x_n, a_n; \theta_n)\| < K_2 < \infty \), where \( K_1 \) and \( K_2 \) are possibly sample-path dependent. Hence sup \( \sup_{n \geq 0} \|\psi_n\| \leq K < \infty \), again \( K \) may be sample-path dependent. Finally, the square summability of the step-size sequence, assumption (A1), implies that \( \sum_n \gamma(m)^2 \|M_n\|^2 < \infty \) a.s. Convergence of the Martingale sequence \( \{M_n\}_{n \geq 0} \) follows from the Martingale Convergence Theorem, see [10].

Recall the loss gradient from (1):

\[
\nabla_\theta \ell(\theta_n, x_n, a_n) = \left( r(x_n, a_n) + \alpha \max_{a' \in \mathcal{A}} Q(x_{n+1}, a'; \theta_n) - Q(x_n, a_n; \theta_n) \right) \nabla_\theta Q(x_n, a_n; \theta_n).
\]

Let us rewrite this using the definition of \( \psi_n \) as:

\[
\nabla_\theta \ell(\theta_n, x_n, a_n) = \left( r(x_n, a_n) + \alpha \int \max_{a' \in \mathcal{A}} Q(y, a'; \theta_n)p(dy \mid x_n, a_n, \theta_n) - Q(x_n, a_n; \theta_n) \right) \nabla_\theta Q(x_n, a_n; \theta_n) + \psi_n.
\]

Hence, (1) becomes:

\[
\theta_{n+1} = \theta_n + \gamma(n) \left[ \nabla_\theta \ell(\theta_n, x_n, a_n) + \psi_n \right], \quad \text{where} \quad (5)
\]
\[ \nabla_\theta \hat{\ell}(\theta_n, x_n, a_n) := \left( r(x_n, a_n) + \alpha \max_{a' \in \mathcal{A}} Q(y, a'; \theta_n) p(dy \mid x_n, a_n, \theta_n) - Q(x_n, a_n; \theta_n) \right) \nabla \theta Q(x_n, a_n; \theta_n). \]

Since the Martingale sequence \( \{M_n\}_{n \geq 0} \) converges a.s., the impact of \( \psi_n \) vanishes asymptotically. In other words, (1) and (5) is asymptotically identical to (has the same limiting set as):

\[
\theta_{n+1} = \theta_n + \gamma(n) \left[ \nabla_\theta \hat{\ell}(\theta_n, x_n, a_n) \right].
\]  

(6)

[Note on notation] Rather than keeping track of two versions of the loss gradients, \( \nabla_\theta \ell \) and \( \nabla_\theta \hat{\ell} \) from equations (1) and (6), respectively, we redefine \( \nabla_\theta \ell := \nabla_\theta \hat{\ell} \). With this slight abuse of notation, we hope to avoid unnecessary confusion. The reader does not need to track two different losses. In our analysis, from this point onward when we refer to (1), the associated loss gradient is:

\[ \nabla_\theta \ell(\theta_n, x_n, a_n) := \left( r(x_n, a_n) + \alpha \max_{a' \in \mathcal{A}} Q(y, a'; \theta_n) p(dy \mid x_n, a_n, \theta_n) - Q(x_n, a_n; \theta_n) \right) \nabla \theta Q(x_n, a_n; \theta_n). \]  

(7)

We are ready to state the main result of this section. As mentioned at the beginning of this section, its proof combines all of the above technical lemmata.

**Lemma 4.** \( \nabla_\theta \ell(\theta_n, x_n, a_n) \), redefined as (7), is continuous. Further, it is locally Lipschitz continuous in the \( \theta \) coordinate.

**Proof.** For the proof one can combine the consequences of (a) Lemmas 1, 2 and 3, (b) (A5), i.e., the continuity of the reward function \( r \), and (c) the fact that the sum and product of continuous and locally Lipschitz continuous functions are also continuous and locally Lipschitz continuous, respectively.

The Lipschitz constant from the above statement is local and changes with \( \theta \). However, as discussed before, following the proof of Lemma 2 it also depends on \( x \). If the domain of a locally Lipschitz continuous function is restricted to a compact subset, then the restricted function is Lipschitz continuous. Assumption (A2) states that \( \sup_{n \geq 0} \| \theta_n \|_2 < \infty \) and \( \sup_{n \geq 0} \| x_n \|_2 < \infty \) a.s. This can be used to conclude that \( \nabla_\theta \ell \) is Lipschitz continuous in the \( \theta \) coordinate, when restricted to an appropriate compact subset of \( \mathbb{R}^d \times \mathcal{S} \). It must be noted that the Lipschitz constant may be sample-path dependent. The reader may refer to the proof of Lemma 1 in [16] where something very similar is shown.

5 Convergence Analysis

To analyze the long-term behavior of (1), we first construct an associated continuous-time trajectory, such that they have identical limiting behaviors. Then, instead of (1) we may analyze the continuous time trajectory.

First we divide the time axis \([0, \infty)\) using the given step-size sequence as follows:

\[ t_n = 0 \text{ and } t_n = \sum_{m=0}^{n-1} \gamma(m) \text{ for } n \geq 1. \]
We now define the required trajectory $\overline{θ} \in C([0, ∞), \mathbb{R}^d)$ as follows:

(a) $\overline{θ}(t_n) = \overline{θ}_n$ for $n ≥ 0$.

(b) $\overline{θ}(t) = \overline{θ}(t_n) + \frac{t-t_n}{t_{n+1}-t_n} \left[ \overline{θ}(t_{n+1}) - \overline{θ}(t_n) \right]$ for $t \in (t_n, t_{n+1})$ and $n ≥ 0$.

As the sequence of actions taken are directly linked to the DQN-weights $θ$ via “exploitation”, we need to better understand them. To do this we define the following measure process:

$$\mu(t) = δ_{(x_n, a_n)}, \quad t \in [t_n, t_{n+1}),$$

where $δ_{(x,a)}$ is the Dirac measure that places mass 1 on the state-action pair $(x, a) \in S \times A$. Hence $µ : [0, ∞) → \mathcal{P}(S, A)$ defines a process of probability measures on $S \times A$. For our analysis, we need to define limits for the “left-shifted” measure process $\{µ([t_n, ∞))\}_{n≥0}$. To do this, we first define a metric space consisting of such measure processes below. The reader may note that the constructed metric space is similar to the one from [7].

We begin by observing that the action space $A$ is compact metrizable, since it is discrete and finite. Next we consider $S$. Since we have assumed that $S = \mathbb{R}^n$, it follows from the Alexandroff extension that $S$ is one-point compactifiable. In particular, the inverse stereographic projection $S^{-1} : S → S^n$ is such that $S^n \setminus S^{-1}(S) = (0, … , 1)$, where $S^n$ represents the $(n+1)$-dimensional Hausdorff compact sphere of radius 1 centered at the origin, and $(0, … , 1)$ is the “north pole”. In other words, the inverse stereographic projection is the required compactification embedding of $S$ into $S^n$, see [15].

Every measure $ν \in \mathcal{P}(S \times A)$ has a push forward counterpart in $\mathcal{P}(S^n \times A)$. It places mass 0 on $(0, … , 1) \times A$. Moving forward, note that we shall use the same symbol to represent, both the measure and its push forward counterpart. Finally, note that $S^n \times A$ is compact Hausdorff in the product topology.

Let us define $\mathcal{U}$ to be the space of all measurable functions $ν(·) = ν(·, dx, da)$ from $[0, ∞)$ to $\mathcal{P}(S^n \times A)$.

**Lemma 5.** $\mathcal{U}$ is compact metrizable. Further, this metric coincides with the coarsest topology that renders continuous the map

$$ν ↦ \int_0^T g(t) \int f dν(t) dt,$$

for all, $T > 0$, $f \in C(S^n \times A)$ and $g \in L^2([0, T], \mathbb{R})$.

**Proof.** By emulating the proof of Lemma 3 in [7] with “$S^n \times A$” replacing “$S$”, and making appropriate modifications, the required proof is obtained. We do not repeat it here, to avoid redundancies.

Define $\nabla^\ell(θ, ν) := \int \nabla_{θ}^\ell(θ, x, a) \ ν(dx, da)$, where $ν \in \mathcal{P}(S, A)$. Lemma 4 implies that $\nabla^\ell$ is continuous in both coordinates and locally Lipschitz continuous in the $θ$ coordinate. Further, $| |\nabla^\ell(θ, ν)|| ≤ K(1 + ||θ||)$, i.e., its growth is bounded as a function of $θ$ alone. Let us also define the following sequence of trajectories in $C([0, ∞), \mathbb{R}^d)$: $θ^n(t) = \overline{θ}(t_n) + \int_0^t \nabla^\ell(θ^n(s), μ^n(s)) \ ds$, where $μ^n(t) :=$
\( \mu(t_n + t), t \geq 0 \) and \( n \geq 0 \). In other words, we consider solutions to the set of non-autonomous ordinary differential equations: \( \{ \dot{\theta}^n(t) = \nabla \ell(\theta^n(t), \mu^n(t)) \} \). As stated earlier, to understand the long-term behavior of Deep Q-Learning, \( \{1\} \), one can study the behavior of the limit of sequence \( \{ \overline{\theta}(t_n, \infty) \} \).\( n \geq 0 \), in \( C([0, \infty), \mathbb{R}^d) \) as \( n \to \infty \). Suppose we are able to show:

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| \overline{\theta}(t_n + t) - \theta^n(t) \| = 0, \quad \text{for every } T > 0.
\]

Then instead of \( \{1\} \) or the associated trajectory \( \overline{\theta} \), we focus on the sequence of trajectories \( \{ \theta^n([0, \infty)) \} \). Now we may tap into the rich literature of tools and techniques available from viability theory \( [3, 4] \).

**Lemma 6.** \( \lim_{n \to \infty} \sup_{t \in [0, T]} \| \overline{\theta}(t_n + t) - \theta^n(t) \| = 0 \) for every \( T > 0 \).

**Proof.** First we define the notation \( [t] \) for \( t \geq 0 \) as \( [t] := t_{\sup\{n | t_n \leq t\}} \). Next, we need to show that:

\[
\sup_{t \in [0, T]} \| \overline{\theta}(t_n + t) - \overline{\theta}([t_n + t]) \| \in \Theta(\gamma(n)).
\]

For this, we fix \( t \in [0, T] \), then \( [t_n + t] = t_{n+k} \) for some \( k \geq 0 \). Recall that

\[
\overline{\theta}(t_n + t) = \overline{\theta}(t_{n+k}) + \frac{t_n + t - t_{n+k}}{\gamma(n + k)} (\overline{\theta}(t_{n+k+1}) - \overline{\theta}(t_{n+k})).
\]

We use the following:

\[
| \overline{\theta}(t_{n+k+1}) - \overline{\theta}(t_{n+k}) | \leq \gamma(n+k) \| \nabla \ell(\overline{\theta}(t_{n+k}), x_{n+k}; a_{n+k}) \|; \quad \text{the stability of the algorithm, i.e., (A2); the monotonic property of the step-size sequence, i.e., (A1); and the boundedness of } \nabla \ell \text{ as a function of } \theta, \text{ to obtain } \| \overline{\theta}(t_n + t) - \overline{\theta}(t_{n+k}) \| \in \Theta(\gamma(n)). \]

Similarly, let us show that:

\[
\sup_{t \in [0, T]} \| \theta^n(t) - \theta^n([t_n + t] - t_n) \| \in \Theta(\gamma(n)).
\]

Again, \( [t_n + t] = t_{n+k} \) for some \( k \geq 0 \). We also have \( \| \theta^n(t) - \theta^n(t_{n+k} - t_n) \| = \int_{t_{n+k} - t_n}^{t_n} \nabla \ell(\theta(s), \mu^n(s)) \, ds \). Using arguments similar to the ones made before, the required statement directly follows. It follows from all of the above arguments that it is enough to show the following in order to prove the lemma:

\[
\sup_{t \in [0, T]} \| \overline{\theta}(t_n + t) - \theta^n([t_n + t] - t_n) \| \to 0.
\]

Once again we let \( [t_n + t] = t_{n+k} \) for some \( k \geq 0 \), and observe that

\[
\| \overline{\theta}([t_n + t]) - \theta^n([t_n + t] - t_n) \| \leq \sum_{m=n}^{n+k-1} \int_{t_m}^{t_{m+1}} \| \nabla \ell(\overline{\theta}(s), \mu^n(s - t_m)) - \nabla \ell(\theta^n(s - t_n), \mu^n(s - t_n)) \| \, ds,
\]

\[
\| \overline{\theta}(t_n + t) - \theta^n([t_n + t] - t_n) \| \leq \sum_{m=n}^{n+k-1} \int_{t_m}^{t_{m+1}} L \| \overline{\theta}(s) - \theta^n(s - t_n) \|. 
\]
Adding and subtracting $\theta^n([s] - t_n)$, the R.H.S. of above equation is less than or equal to

$$\sum_{m=n}^{n+k-1} \int_{t_m}^{t_{m+1}} \|\theta^n(s - t_n) - \theta^n([s] - t_n)\| + \sum_{m=n}^{n+k-1} \int_{t_m}^{t_{m+1}} \|\theta(s) - \theta^n([s] - t_n)\|.$$ 

Considering that $\|\theta(t_n + t) - \theta(t_{n+k})\|$ and $\|\theta^n(t) - \theta^n([t_n + t] - t_n)\| \in \Theta(\gamma(n))$, we get

$$\sum_{m=n}^{n+k-1} \int_{t_m}^{t_{m+1}} \|\theta^n(s - t_n) - \theta^n([s] - t_n)\| \leq \sum_{m=n}^{n+k-1} \Theta(\gamma(m)^2),$$

which goes to zero as $n \to \infty$. Now we use the discrete version of Gronwall’s inequality to get:

$$\|\theta([t_n + t]) - \theta^n([t_n + t] - t_n)\| \leq \left( L \sum_{m=n}^{n+k-1} \Theta(\gamma(m)^2) \right) \exp(LT).$$

\[\blacksquare\]

The family of trajectories $\{\theta^n([0, \infty))\}_{n \geq 0}$, in $C([0, \infty), \mathbb{R}^d)$, is equicontinuous and point-wise bounded. It follows from the Arzela-Ascoli theorem [6] that it is sequentially compact. Note that the topology of $C([0, \infty), \mathbb{R}^d)$ is the one induced by the topologies of $C([0, T], \mathbb{R}^d)$ for every $0 < T < \infty$. Now, let us consider the family $\{\mu^n\}_{n \geq 0} \subset \mathcal{U}$. As $\mathcal{U}$ is a compact metric space, $\{\mu^n\}_{n \geq 0}$ is sequentially compact. Hence, there is a common subsequence $\{m(n)\} \subset \{n\}$ such that $\mu^{m(n)} \to \mu^\infty$ in $\mathcal{U}$ and $\theta^{m(n)} \to \theta^\infty$ in $C([0, \infty), \mathbb{R}^d)$. With a slight abuse of notation, we have $\mu^n \to \mu^\infty$ in $\mathcal{U}$ and $\theta^n \to \theta^\infty$ in $C([0, \infty), \mathbb{R}^d)$. In other words, the sequences $\mu^n$ and $\theta^n$ are convergent in their respective spaces.

Below we state another important result. It states that convergence of the measure process in $\mathcal{U}$ implies that at every point in time, the corresponding measure sequence converges in distribution.

\textbf{Lemma 7.} If $\mu^n \to \mu^\infty$ in $\mathcal{U}$, then a.e. $\mu^n(t) \to \mu^\infty(t)$ in $\mathcal{P}(\mathbb{S} \times \mathcal{A})$ for $t \in [0, \infty)$.

\textbf{Proof.} We begin by recalling that the same notation is used to denote a measure on $\mathbb{S} \times \mathcal{A}$ and its push forward counterpart on $\mathcal{S}^\infty \times \mathcal{A}$. It follows from the definition of convergence in $\mathcal{U}$ that $\int_0^T g(s) \int f \mu^n(s, dx, da) \, ds \to \int_0^T g(s) \int f \mu(s, dx, da) \, ds$, as $n \to \infty$, for every $g \in L^2([0, T], \mathbb{R})$ and $f \in C(\mathcal{S}^\infty \times \mathcal{A})$. We claim that this implies, for every $f \in C(\mathcal{S}^\infty \times \mathcal{A})$, $\int f \mu^n(s, dx, da) \to \int f \mu(s, dx, da)$ a.e. for $s \in [0, \infty)$. Once this claim is proven, we can conclude that $s$-a.e. $\mu^n(s) \to \mu^\infty(s)$ in $\mathcal{P}(\mathcal{S}^\infty \times \mathcal{A})$, which finally yields the Lemma.

To prove the claim, we begin by assuming the contrary. In particular, we assume $\exists f \in C(\mathcal{S}^\infty \times \mathcal{A})$, $T > 0$, $\epsilon > 0$ and a non-zero Lebesgue measure set $A \in \mathcal{B}([0, T])$, such that at least one of the following holds:

(a) $\liminf_{n \to \infty} \int f \mu^n(s, dx, da) - \int f \mu(s, dx, da) > \epsilon$, $\forall \ s \in A$.

(b) $\liminf_{n \to \infty} \int f \mu^n(s, dx, da) - \int f \mu(s, dx, da) < -\epsilon$, $\forall \ s \in A$.

(c) $\limsup_{n \to \infty} \int f \mu^n(s, dx, da) - \int f \mu(s, dx, da) > \epsilon$, $\forall \ s \in A$. 

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(d) \( \limsup_{n \to \infty} \int f \mu^n(s, dx, da) - \int f \mu(s, dx, da) < -\epsilon, \forall s \in A. \)

We only present arguments for case (a), as the corresponding ones for the others are identical. Since \( f \) is bounded, we apply DCT to conclude that:

\[
\liminf_{n \to \infty} \int_0^T \mathbb{I}_A \left( \int f \mu^n(s, dx, da) - \int f \mu(s, dx, da) \right) ds > \epsilon \ l(A) > 0,
\]

where \( l(A) \) denotes the Lebesgue measure of \( A \). This directly contradicts the definition of convergence of measures in \( \mathcal{U} \).

It is left to show that \( \mu^n(t) \to \mu^\infty(t) \) in \( \mathcal{P}(\mathcal{S} \times \mathcal{A}) \) a.e. for \( t \in [0, \infty) \). To do this, we pick \( t \in [0, \infty) \) such that \( \mu^n(t) \to \mu^\infty(t) \) in \( \mathcal{P}(\mathcal{S} \times \mathcal{A}) \) and show that their pull back versions converge in \( \mathcal{P}(\mathcal{S} \times \mathcal{A}) \). This is done by showing that \( \limsup \mu^n(t, C) \leq \mu^\infty(t, C) \) for every closed set \( C \in \mathcal{B}(\mathcal{S} \times \mathcal{A}) \) (Portmanteau theorem [6]).

We first observe that the measures \( \{ \mu^n(t) \}_{0 \leq n \leq \infty} \) are tight as a consequence of (A2). Hence they place a mass of 0 on \( (0, \ldots, 1) \times \mathcal{A} \). If we restrict these measures to \( S^{-1}(\mathcal{S}) \times \mathcal{A} \), then \( \mu^n|_{S^{-1}(\mathcal{S}) \times \mathcal{A}} \Rightarrow \mu^\infty|_{S^{-1}(\mathcal{S}) \times \mathcal{A}} \).

Next, we consider an arbitrary closed subset \( C \in \mathcal{B}(\mathcal{S} \times \mathcal{A}) \). Since the stereographic projection is bicontinuous, \( \hat{C} := \{(S^{-1}(x), a) \mid (x, a) \in C \} \) is closed in \( S^{-1}(\mathcal{S}) \times \mathcal{A}, \) equipped with subspace topology (with respect to \( \mathcal{S} \times \mathcal{A} \)). Clearly, \( \limsup \mu^n(t, C) \leq \mu^\infty(t, C) \). Now, as \( \mu^n(t, C) \) is the push forward measure of \( \mu^n(t, C) \) for all \( 0 \leq n \leq \infty \), we get the required.

We can use one of the many available measurable selection theorems [18] to drop the a.e. clause in the statement of Lemma 8. Hence, we have hitherto shown that \( \theta^n \to \theta^\infty \) in \( C([0, \infty), \mathbb{R}^d) \) and \( \mu^n(s) \Rightarrow \mu^\infty(s) \) for all \( s \in [0, \infty) \).

We now need to show that \( \hat{\theta}^\infty \) is a solution of \( \hat{\theta}(t) = \nabla \ell(\theta(t), \mu^\infty(t)) \). Then, one can study the limiting behavior of a solution to the o.d.e. \( \hat{\theta}(t) = \nabla \ell(\theta(t), \mu^\infty(t)) \), to understand the long-term behavior of the Deep Q-Learning algorithm given by (1).

**Lemma 8.** \( \theta^\infty \) is a solution to \( \hat{\theta}(t) = \nabla \ell(\theta(t), \mu^\infty(t)). \)

**Proof.** Fix an arbitrary \( T > 0 \). We need to show that

\[
\sup_{t \in [0, T]} \left\| \theta^n(t) - \theta^\infty(0) - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) ds \right\| \to 0.
\]

Let us first consider the following:

\[
\left\| \theta^n(0) + \int_0^t \nabla \ell(\theta^n(s), \mu^n(s)) ds - \theta^\infty(0) - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) ds \right\|,
\]

(8)

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\[ \|\theta^n(0) - \theta^\infty(0)\| + \left\| \int_0^t \nabla \ell(\theta^n(s), \mu^n(s)) \, ds - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds \right\| \\
+ \left\| \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds \right\|. \] (9)

Next, we note the following:

(A) From Lemma 7, we have \( \mu^n(s) \xrightarrow{\text{d}} \mu^\infty(s) \) (converges in distribution on \( S \times A \) for all \( s \in [0, T] \).

(B) From (A2), i.e., the stability of the algorithm, and the boundedness of \( \nabla \ell \) as a function of \( \theta \), we get \( \nabla \ell(\theta^\infty(s), \cdot) \in C_b(S \times A) \). Hence, as a consequence of note (A), \( \int \nabla \ell(\theta^\infty(s), x, a) \mu^n(s) \to \int \nabla \ell(\theta^\infty(s), x, a) \mu^\infty(s) \) for all \( s \in [0, T] \).

Using the Dominated Convergence Theorem (DCT) \[10\] we get:

\[ \left\| \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds \right\| \to 0. \] (10)

Further, it follows from the Arzela-Ascoli theorem that the convergence in (10) is uniform over \([0, T]\).

Since \( \nabla \ell \) is locally Lipschitz continuous in \( \theta \), we get:

\[ \left\| \int_0^t \nabla \ell(\theta^n(s), \mu^n(s)) \, ds - \int_0^t \nabla \ell(\theta^\infty(s), \mu^\infty(s)) \, ds \right\| \leq L \int_0^t \|\theta^n(s) - \theta^\infty(s)\| \, ds. \] (11)

As \( \theta^n \to \theta^\infty \) uniformly over \([0, T]\), we get that the L.H.S. of (11) \( \to 0 \) uniformly over \([0, T]\). The discussion surrounding (10) and (11) implies that (9) \( \to 0 \) and hence (8) \( \to 0 \), uniformly over \([0, T]\). As \( T \) is arbitrary, the Lemma follows. \( \square \)

To develop a better understanding of Deep Q-Learning, we need to study \( \mu^\infty \), the limiting distribution over the state-action pairs. In the following lemma we show that \( \mu^\infty(t, dx \times A) \) is stationary with respect to the state Markov process, \( \forall \ t \geq 0 \). Recall that \( p(\cdot | x, a, \theta) \) is the controlled transition kernel of the state Markov process. We use \( p(\cdot | x, A, \theta) \) to denote the probability associated with transitioning out of state \( x \) (when some action is picked). We use \( p(dy | x, A, \theta) \mu(dx \times A) \) to denote \( \int_y p(dy | x, a, \theta) \mu(dx, da) \). In words, it represents the probability to transition from state \( x \) to state \( y \), given that \((x, a) \sim \mu \).

**Lemma 9.** \( \mu^\infty(t, dy \times A) = \int_y p(dy | x, A, \theta^\infty(t)) \mu^\infty(t, dx \times A) \) for all \( t \in [0, \infty) \). In other words, the limiting marginal constitutes a stationary distribution over the state Markov process.
Further, the limit in (16) equals
\[\xi_n := \sum_{m=0}^{n-1} \gamma(m) \left[ f(x_{m+1}) - \int_{S} f(y)p(dy \mid x_m, a_m, \theta_m) \right]. \tag{12}\]

Since \(f\) is bounded and \(\sum_{n \geq 0} \gamma(n)^2 < \infty\), the quadratic variation process associated with the above Martingale is convergent. It follows from the Martingale Convergence Theorem \cite{10} that \(\xi_n\) converges almost surely. Hence for \(t > 0\),
\[\sum_{m=n}^{\tau(n,t)} \gamma(m) \left[ f(x_{m+1}) - \int_{S} f(y)p(dy \mid x_m, a_m, \theta_m) \right] \to 0 \text{ a.s.}, \tag{13}\]
where \(\tau(n,t) := \min\{m \geq n \mid t_m \geq t_n + t\}\). Since the steps-sizes are eventually decreasing, hence \(\sum_{m=n}^{\tau(n,t)} [\gamma(m) - \gamma(m+1)] f(x_{m+1}) \to 0 \text{ a.s.}\). Then (13) becomes:
\[\sum_{m=n}^{\tau(n,t)} \gamma(m) \left[ f(x_{m}) - \int_{S} f(y)p(dy \mid x_m, a_m, \theta_m) \right] \to 0 \text{ a.s.} \tag{14}\]

Using the definition of \(\mu\), we rewrite (14) as:
\[
\int_{t_n}^{t_{n+1}} \int_{S \times \mathcal{A}} \left[ f(x) - \int_{S} f(y)p(dy \mid x, a, \bar{\theta}(s)) \right] \mu(s, dx, da)ds \to 0 \text{ a.s.} \tag{15}\]

Let us define a new function \(\hat{f}(x, a) := f(x)\) for all \((x, a) \in S \times \mathcal{A}\), then \(\hat{f} \in C_b(S \times \mathcal{A})\). Since \(\mu(t_n) \to \mu^\infty(\cdot)\) in \(\mathcal{M}\), it follows that as \(n \to \infty:\)
\[
\int_{t_n}^{t_{n+1}} \int_{S \times \mathcal{A}} \hat{f}(x, a)\mu(s, dx, da)ds \to \int_{0}^{t} \int_{S \times \mathcal{A}} \hat{f}(x, a)\mu^\infty(s, dx, da)ds. \tag{16}\]

Further, the limit in (16) equals \(\int_{0}^{t} \int_{S \times \mathcal{A}} \mu^\infty(s, dx \times \mathcal{A})ds\).

Recall that \((x, a, \theta) \mapsto p(\cdot \mid x, a, \bar{\theta})\) is a continuous map. Since \(f\) is a convergence determining function in \(\mathcal{P}(S)\), it follows that \(\int_{S} f(y)p(dy \mid x, a, \bar{\theta}(s)) \to \int_{S} f(y)p(dy \mid x, a, \theta^\infty(s))\) for all \(s \in [0, t]\). Define \(h_n(s, x, a) := \int_{S} f(y)p(dy \mid x, a, \bar{\theta}(t_n + s))\) and \(h_\infty(s, x, a) := \int_{S} f(y)p(dy \mid x, a, \theta^\infty(s))\). For a fixed \(s \in [0, t]\), \(h_n(s, \cdot)\), \(n \geq 0\), and \(h_\infty(s, \cdot)\) belong to \(C_b(S \times \mathcal{A})\). Hence,
\[
\int_{S \times \mathcal{A}} h_n(s, x, a)\mu(t_n + s, dx, da) \to \int_{S \times \mathcal{A}} h_\infty(s, x, a)\mu^\infty(s, dx, da).
\]

It then follows from Dominated Convergence Theorem (DCT) \cite{10} that:
\[
\int_{t_n}^{t_{n+1}} \int_{S \times \mathcal{A}} h_n(s, x, a)\mu(s, dx, da)ds \to \int_{0}^{t} \int_{S \times \mathcal{A}} h_\infty(s, x, a)\mu^\infty(s, dx, da)ds.
\]

In other words, we have
\[
\int_{t_n}^{t_{n+1}} \int_{S \times \mathcal{A}} f(y)p(dy \mid x, a, \bar{\theta}(s))\mu(s, dx, da)ds \to \int_{0}^{t} \int_{S \times \mathcal{A}} f(y)p(dy \mid x, a, \theta^\infty(s))\mu^\infty(s, dx, da)ds. \tag{17}\]
From [15], [16] and [17] we get:
\[
\int_0^t \int_{S \times \mathcal{A}} f(x) \mu^\infty(s, dx, da) ds = \int_0^t \int_{S \times \mathcal{A}} \int_S f(y)p(dy \mid x, a, \theta^\infty(s)) \mu^\infty(s, dx, da) ds.
\]  
(18)

Using Lebesgue’s theorem we get that a.e. on \([0,t]::
\[
\int_{S \times \mathcal{A}} f(x) \mu^\infty(s, dx, da) = \int_S \int_{\mathcal{A}} f(y)p(dy \mid x, a, \theta^\infty(s)) \mu^\infty(s, dx, da).
\]
Applying Fubini’s theorem [10] to swap the double integral on the R.H.S. of the above equation, gives us:
\[
\int_S f(x) \mu^\infty(s, dx, \mathcal{A}) = \int_S \int_{\mathcal{A}} p(dy \mid x, \mathcal{A}, \theta^\infty(s)) \mu^\infty(s, dx, \mathcal{A}).
\]
Since \(f\) is a convergence determining function, we get that \(\mu^\infty(s, dy, \mathcal{A}) = \int_S \int_{\mathcal{A}} p(dy \mid x, \mathcal{A}, \theta^\infty(s)) \mu^\infty(s, dx, \mathcal{A}).\)
 Hence, we have shown that the limiting distribution over the state-action pairs \(\mu^\infty\) is such that, almost everywhere on \([0,\infty),\) its marginal over the state space constitutes a stationary distribution over the state Markov process with transition kernel \(p(\cdot \mid x, \mathcal{A}, \theta).\)

Let us quickly show that the family of measures \(\{\mu^\infty(t, dx, da)\}_{t \geq 0}\) is tight.
From previous discussions and observations, given \(t \geq 0\), we can find \(\{n(m)\}_{m \geq 0} \subset \{n\}_{n \geq 0}\) such that
\[
\lim_{n(m) \to \infty} \mu(t_{n(m)}, dx, da) \xrightarrow{d} \mu^\infty(t, dx, da).
\]
Using the Portmanteau Theorem [6] we get \(\mu^\infty(t, \mathcal{H} \times \mathcal{A}') \geq \limsup_{n(m) \to \infty} \mu(t_{n(m)}, \mathcal{H} \times \mathcal{A}')\), where \(\mathcal{H} \subset S\) is compact and \(\mathcal{A}' \subset \mathcal{A}\). Given \(\epsilon > 0\) there exists \(\mathcal{H}(\epsilon) \subset S\), compact, such that \(\inf_{m \geq 0} \mu(t_{n(m)}, \mathcal{H}(\epsilon) \times \mathcal{A}') \geq 1 - \epsilon\) for any \(\mathcal{A}' \subset \mathcal{A}\), as \(\mu(t_{n(m)})_{m \geq 0}\) is tight. Hence \(\mu^\infty(t, \mathcal{H}(\epsilon) \times \mathcal{A}') \geq 1 - \epsilon\). Since \(t\) was arbitrary, we get that \(\{\mu^\infty(t, dx, da)\}_{t \geq 0}\) is tight.

Tightness implies that it is relative compact in the Prokhorov metric. This combined with the stability of [1] yields \(\{n(k)\}_{k \geq 0} \subset \{n\}_{n \geq 0}\) such that both
\[
\lim_{n(k) \to \infty} \mathcal{H}(t_{n(k)}) \quad \text{and} \quad \lim_{n(k) \to \infty} \mu(t_{n(k)}, dx, da)
\]
respectively. The properties of these limits, let call them \(\mathcal{H}^\infty\) and \(\mu^\infty\), determine the long-term behavior of [1]. Lemmata 1 to 3 were stated and proved to build up to the most important result of this paper, which concerns the limiting behavior of [1]. We state and prove this result below, following which we discuss its implications.

**Theorem 1.** Assuming (A1)-(A5), the limit \(\mathcal{H}^\infty\) of the deep Q-learning algorithm, [1], is such that \(\nabla \ell(\mathcal{H}^\infty, \mu^\infty) = 0\) and \(\mathcal{H}^\infty(dx \times \mathcal{A})\) is a stationary distribution of the state Markov process \(x\).

**Proof.** From previous lemmata we know that [1] tracks \(\theta\), a solution to the non-autonomous o.d.e. \(\dot{\theta}(t) = \nabla \ell(\theta(t), \mu^\infty(t))\). Further, there is a sample path dependent compact subset of \(\mathbb{R}^d \times \mathcal{H}\), such that \(\theta\) remains inside of it. This
is because the algorithm is assumed to be stable, i.e., \( \theta_n \in K \forall n \geq 0 \). To determine the limit of the algorithm, \( \theta_{\infty} \), we need \( \lim_{t \to \infty} \theta(t) \).

To analyze \( \dot{\theta}(t) = \tilde{\nabla} \ell(\theta(t), \mu_{\infty}(t)) \), we transform it into an autonomous o.d.e. through the standard change of variables trick. For this, we define \( s(t) := t^{1+\frac{1}{1-s(t)}} \), then \( \dot{s}(t) = (1-s(t))^2 \) and \( t = s(t)^{1/s(t)} \). We get the following transformed autonomous o.d.e.:

\[
\dot{\theta}(t) = \tilde{\nabla} \ell \left( \theta(t), \mu_{\infty} \left( \frac{s(t)}{1-s(t)} \right) \right), \quad (1-s(t))^2.
\] \( (19) \)

Before proceeding we state the following useful theorem, paraphrased to suit us:

**[Theorem 2, Chapter 6 of [3]]** Let \( F \) be a continuous map from a closed subset \( \hat{K} \subset X \) to \( X \). Let \( x(\cdot) \) be a solution trajectory of \( \dot{x}(t) = F(x(t)) \) such that it is inside \( \hat{K} \). Then the solution converges to \( x^* \), an equilibrium of \( F \).

To utilize the theorem, we define the following:

\[
X := \mathbb{R}^d \times [0,1]; \quad \hat{K} := K \times [0,1]; \quad F : \hat{K} \to X \text{ such that } F(\theta,s) := \left( \tilde{\nabla} \ell \left( \theta, \mu_{\infty} \left( \frac{s(t)}{1-s(t)} \right) \right), (1-s(t))^2 \right).
\]

It now follows from the above theorem that the transformed o.d.e \((19)\) converges to \((\theta_{\infty},1)\), an equilibrium of \( F \). Further, 1 is the unique equilibrium point of \((1-s)^2\), and \( \theta_{\infty} \) is a equilibrium of \( \tilde{\nabla} \ell(\theta_{\infty}, \pi_{\infty}) \), where \( \lim_{t \to \infty} \mu_{\infty}(t) \overset{d}{\Rightarrow} \pi_{\infty} \).

We discussed the existence of the limit \( \pi_{\infty} \) in the paragraph before stating this theorem.

We showed in Lemma 9 that \( \mu_{\infty}(t) \) is a stationary distribution of the state Markov process \( x \) for all \( t \geq 0 \), i.e.,

\[
\mu_{\infty}(t, dx \times A) = \int_S p(dy | x, A, \theta_{\infty}(t)) \mu_{\infty}(t, dx \times A).
\]

Letting \( t \to \infty \) on both sides of the above equation yields,

\[
\pi_{\infty}(dy \times A) = \int_S p(dy | x, A, \theta_{\infty}) \pi_{\infty}(dx \times A).
\]

In other words, the marginal over the states, \( \pi_{\infty}(dx \times A) \), is stationary with respect to the state process.

5.1 On practical implications of the theory

The primary goal of Deep Q-Learning is to find the optimal DQN-weights \( \theta^* \) such that \( \arg\max_{a \in A} Q(x,a; \theta^*) = \arg\max_{a \in A} Q^*(x,a) \), where \( Q^* \) is the optimal Q-function. This is achieved by minimizing the squared Bellman loss, see \( (1) \).

Theorem 1 states that the Deep Q-Learning algorithm given by \( (1) \) converges to \( \theta_{\infty} \), a local minimizer of the average squared Bellman loss. The averaging over state-action pairs is induced by the limiting measure \( \pi_{\infty} \in \mathcal{P}(S \times A) \). In particular, we have:

\[
\int \nabla_{\theta} \ell(\theta_{\infty}, x, a) \pi_{\infty}(dx, da) = 0.
\] \( (20) \)
Lemma 9 states that the limiting marginal distribution $\pi^\infty(dx \times A)$, over the state space $S$, is stationary. Deep Q-Learning is typically employed in complex environments with multiple stationary distributions. Since $\pi^\infty$ captures the long-term behavior of the training process, it directly depends on the distribution of the data encountered during training. As the squared Bellman loss is minimized on an average in accordance to $\pi^\infty$, the quality of learning is entirely captured by $\pi^\infty$. In particular, the trained DQN approximates the optimal Q-factors accurately for state-action pairs that are distributed in accordance to $\pi^\infty$. Performance is therefore good when encountering states arising from the “limiting marginal”.

Fix $a \in A$. Let $S(a)$ be a measurable subset of $S$ such that $a$ is the optimal action associated with every $x \in S(a)$. For the sake of illustration, we consider a scenario wherein $\pi^\infty(S(a) \times A) > 0$ and $\pi^\infty(S(a) \times a) = 0$. Colloquially speaking, the set of state-action pairs given by $\{(x, a) | x \in S\}$ were not encountered during training. This could happen, for e.g., due to poor exploration-exploitation trade-offs, or due to improper initialization of the DQN weights. The Q-factors may hence be poorly approximated on $S(a) \times a$, and the trained DQN-agent cannot be expected to take optimal actions in these states. This explains the observation that Deep Q-Learning, in practice, is sometimes inconsistent. Hitherto present literature, see for e.g., [19, 21] do not explain such behaviors. Since DQN is usually trained using a simulator, it may be possible to empirically estimate $\pi^\infty$. This knowledge may help identify scenarios wherein DQN is undertrained. Thus avoiding circumstances, such as the above illustrated one.

The theory presented herein is comprehensive as it completely characterizes Deep Q-Learning performance as a function of the training process. The assumptions involved are practical and easy to verify. Further, we believe that the analysis is general and may be applied to understand other Deep Learning algorithms.

6 Extending our analysis

Thus far, we have considered neural network architectures with squashing activation functions. Also, we have not accounted for the use of experience replay. Recall that experience replay is a concept which allows RL agents to reuse and relearn from past experiences. In this section we briefly discuss extensions of our analysis to account for (a) general (non-squashing) activations, and (b) experience replay.

6.1 Two times continuously differentiable non-squashing activations

The hitherto presented analysis accounts for DQN architectures with differentiable squashing activations. In this section, we discuss modifications to our analysis that allow for general activations as well. In particular, the modifications account for activations such as Sigmoid Linear Unit (SiLU), Gaussian Error Linear Unit (GELU), etc.

Let us begin by understanding the role of squashing activations in our analysis. In Lemma 1, the squashing property is used to find a $x$-independent $\hat{L}$ such that $|Q(x, a; \theta)| \leq \hat{L}||\theta||_2$. Note that Lemma 1 is true even when the activations
are non-squashing, provided \(S\) is a compact metric space. Since (A2) states that \(\sup_{n \geq 0} \|x_n\|_2 < \infty\) a.s., there is a sample path dependent compact set \(S_c \subset S\) such that \(x_n \in S_c \forall n \geq 0\). Using this information, we may modify the statement of Lemma 4 as follows:

**Lemma 10.** \(\forall \theta \in \mathbb{R}^d \sup_{a \in A} |Q(x, a; \theta)| \leq \tilde{L}\|\theta\|_2\), and \(\tilde{L} > 0\) is dependent on \(x\). Further, there is a sample path dependent \(\tilde{L}\), independent of \(x\), such that \(\sup_{x \in S, a \in A} |Q(x, a; \theta)| \leq \tilde{L}\|\theta\|_2\), where \(S_c\) is as defined above.

Parts of the analysis using Lemma 4 must now be modified to use Lemma 10. Other Lemmata, for e.g., Lemma 3 do not change when using Lemma 10 instead of Lemma 4.

### 6.2 Experience replay

Now, we extend our analysis to account for experience replay, an idea that allows the RL agent to relearn from past experiences. Specifically, at time \(T\), the agent has ready access to \(\{x_k, a_k, r(x_k, a_k), x_{k+1}\}_{T-H+1 \leq k \leq T}\), the history of states encountered, actions taken, rewards received and transitions made. The optimal size of the experience replay \(H\) is problem dependent, and tunable. At time \(T\), to update the NN weights \(\theta\), the agent first samples a mini-batch of size \(\hat{H} < H\) from the experience replay and calculates the following average loss gradient:

\[
\frac{1}{\hat{H}} \sum_{i=1}^{\hat{H}} \nabla_{\theta} \ell \left( \theta_T, x_{k(T,i)}, a_{k(T,i)} \right), \text{ where } T - H + 1 \leq k(T, i) \leq T.
\]

The DQN weights are updated as follows:

\[
\theta_{n+1} = \theta_n + \gamma(n) \left[ \frac{1}{\hat{H}} \sum_{i=1}^{\hat{H}} \nabla_{\theta} \ell \left( \theta_n, x_{k(n,i)}, a_{k(n,i)} \right) \right]. \tag{21}
\]

To analyze (21), we must redefine \(\mu\). For \(t \in [t_n, t_{n+1})\), redefine \(\mu(t)\) to be the probability measure (on \(S \times A\)) that places a mass of \(\frac{1}{\hat{H}}\) on \((x_{k(n,i)}, a_{k(n,i)})\) for \(1 \leq i \leq \hat{H}\). With the new definition of \(\mu\), for \(t = t_n\) we get:

\[
\nabla \ell(\bar{\theta}(t), \mu(t)) = \int \nabla_{\theta} \ell(\bar{\theta}(t), x, a) \mu(t) = \frac{1}{\hat{H}} \sum_{i=1}^{\hat{H}} \nabla_{\theta} \ell \left( \theta_{n}, x_{k(n,i)}, a_{k(n,i)} \right).
\]

Emulating the proofs of the Lemmata up to Lemma 9 for the new \(\mu\), shows that (21) tracks a solution to the non-autonomous o.d.e. \(\dot{\theta}(t) = \nabla \ell(\theta(t), \mu^\infty(t))\). Again, \(\mu^\infty\) is a limit of the redefined measure process sequence \(\{\mu([t, \infty))\}_{t \geq 0}\) in \(\mathcal{U}\).

**Lemma 9** states the the limiting marginal measure process \(\mu^\infty(t, dx \times A)\) is stationary with respect to the state Markov process for every \(t \geq 0\). For it to hold in the presence of experience replay we redefine \(\xi_n\) and \(\mathcal{F}_n\) as follows:

\[
\xi_n := \sum_{m=0}^{n-1} \frac{1}{\hat{H}} \sum_{i=1}^{\hat{H}} \left( f(x_{k(m,i)+1}) - \int f(y) p(dy \mid x_k(m,i), a_{k(m,i)}, \theta_{k(m,i)}) \right).
\]
\( \mathcal{F}_{n-1} = \sigma(\{x_m, a_m, \theta_m, \Xi_m \mid m \leq n\}) \) for \( n \geq 1 \), where \( \{\Xi_n\}_{n \geq 0} \) is the random process associated with mini-batch sampling. Typically the mini-batches are all sampled independently over time, hence \( \{\Xi_n\}_{n \geq 0} \) constitutes an independent sequence of random variables. With these modifications the rest the steps involved in the proof of Lemma 9 may be readily emulated. This would directly lead to the statement of the main result, Theorem 1. In conclusion, Deep Q-Learning with experience replay, (21), converges to \( \hat{\theta}_\infty \) such that \( \nabla_\theta \ell(\hat{\theta}_\infty, \hat{\mu}_\infty) = 0 \), where \( \hat{\mu}_\infty \) is a limit of \( \{\hat{\mu}_\infty(t)\}_{t \geq 0} \) as \( t \to \infty \), and \( \hat{\mu}_\infty \) is the limiting measure process of the redefined \( \mu \)-process. Again, \( \hat{\mu}_\infty(dx \times \mathcal{A}) \) is stationary with respect to the state Markov process.

It is a common belief among deep learning practitioners that experience replay plays an important role in stabilizing the DQN training. In regards to the long-term behavior, we show that the use of experience replay has a qualitative effect on learning. This is because the limiting measure \( \hat{\mu}_\infty \) is shaped by the mini-batches sampled from experience replay during training, and it is richer than the one resulting from no experience replay.

7 Conclusion

In this paper, we presented a comprehensive analysis of Deep Q-Learning under practical and verifiable assumptions. An important contribution of this paper is the complete characterization of the DQN performance as a function of training. We obtained this by analyzing the limit of a closely associated measure process (on the state-action pairs). We were able to explain empirical inferences regarding Deep Q-Learning, in particular, with regards to inconsistent behavior, and qualitative advantage of using experience replay.

Moving forward, we are interested to explore sufficient conditions for Deep Q-Learning stability. The algorithm considered herein may only be used in scenarios with discrete action spaces. However, there are actor-critic DeepRL algorithms that are capable of solving continuous action space problems. We would like to extend our analysis to include such algorithms as well.

References

[1] Joshua Achiam, Ethan Knight, and Pieter Abbeel. Towards characterizing divergence in deep q-learning. *arXiv preprint arXiv:1903.08894*, 2019.

[2] Kai Arulkumaran, Marc Peter Deisenroth, Miles Brundage, and Anil Anthony Bharath. Deep reinforcement learning: A brief survey. *IEEE Signal Processing Magazine*, 34(6):26–38, 2017.

[3] J-P Aubin and Arrigo Cellina. *Differential inclusions: set-valued maps and viability theory*, volume 264. Springer Science & Business Media, 2012.

[4] Jean-Pierre Aubin, Alexandre M Bayen, and Patrick Saint-Pierre. *Viability theory: new directions*. Springer Science & Business Media, 2011.

[5] Dimitri P Bertsekas. *Reinforcement learning and optimal control*. Athena Scientific Belmont, MA, 2019.
[6] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[7] Vivek S. Borkar. Stochastic approximation with ‘controlled Markov’ noise. *Systems & control letters*, 55(2):139–145, 2006.

[8] Vivek S Borkar. *Stochastic approximation: a dynamical systems viewpoint*, volume 48. Springer, 2009.

[9] Qi Cai, Zhuoran Yang, Chi Jin, and Zhaoran Wang. Provably efficient exploration in policy optimization. *arXiv preprint arXiv:1912.05830*, 2019.

[10] Rick Durrett. *Probability: Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[11] Daniel Graupe. *Principles of artificial neural networks*, volume 7. World Scientific, 2013.

[12] Harold Kushner and G George Yin. *Stochastic approximation and recursive algorithms and applications*, volume 35. Springer Science & Business Media, 2003.

[13] Timothy P Lillicrap, Jonathan J Hunt, Alexander Pritzel, Nicolas Heess, Tom Erez, Yuval Tassa, David Silver, and Daan Wierstra. Continuous control with deep reinforcement learning. In *ICLR (Poster)*, 2016.

[14] Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *nature*, 518(7540):529–533, 2015.

[15] James R. Munkres. *Topology. 2nd ed.* Upper Saddle River, NJ: Prentice Hall, 2nd ed. edition, 2000.

[16] Arunselvan Ramaswany, Adrian Redder, and Daniel E Quevedo. Optimization over time-varying networks with unbounded delays. *arXiv preprint arXiv:1912.07055*, 2019.

[17] David Silver, Julian Schrittwieser, Karen Simonyan, Ioannis Antonoglou, Aja Huang, Arthur Guez, Thomas Hubert, Lucas Baker, Matthew Lai, Adrian Bolton, et al. Mastering the game of go without human knowledge. *nature*, 550(7676):354–359, 2017.

[18] Daniel H Wagner. Survey of measurable selection theorems. *SIAM Journal on Control and Optimization*, 15(5):859–903, 1977.

[19] Zhuoran Yang, Yuchen Xie, and Zhaoran Wang. A theoretical analysis of deep q-learning. *arXiv preprint arXiv:1901.00137*, 2019.

[20] Bayya Yegnanarayana. *Artificial neural networks*. PHI Learning Pvt. Ltd., 2009.

[21] Shaofeng Zou, Tengyu Xu, and Yingbin Liang. Finite-sample analysis for sarsa with linear function approximation. In *Advances in Neural Information Processing Systems*, pages 8668–8678, 2019.