Discrete Phase Transitions Associated to Topological Lattice Field Theories in Dimension $D \geq 2$

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ABSTRACT

We investigate the neighborhood of Topological Lattice Field Theories (TLFTs) in the parameter space of general lattice field theories in dimension $D \geq 2$, and discuss the phase structures associated to them. We first define a volume-dependent TLFT, and discuss its decomposition to a direct sum of irreducible TLFTs, which cannot be decomposed anymore. Using this decomposed form, we discuss phase structures and renormalization group flows of volume-dependent TLFTs. We find that TLFTs are on multiple first order phase transition points as well as on fixed points of the flow. The phase structures are controlled by the physical states on $(D - 1)$-sphere of TLFTs. The flow agrees with the Nienhuis-Nauenberg criterion. We also discuss the neighborhood of a TLFT in general directions by a perturbative method, so-called cluster expansion. We investigate especially the $Z_p$ analogue of the Turaev-Viro model, and find that the TLFT is in general on a higher order discrete phase transition point. The phase structures depend on the topology of the base manifold and are controlled by the physical states on topologically non-trivial surfaces. We also discuss the correlation lengths of local fluctuations, and find long-range modes propagating along topological defects. Thus various discrete phase transitions are associated to TLFTs.

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1. Introduction

The topological field theory (TFT)\[^{[1,2]}\] has been an important tool in theoretical physics and pure mathematics. In theoretical physics it has been appearing in the investigations of the non-perturbative aspects of string theories and quantum gravities. The topological phase of two-dimensional quantum gravity was discussed by Witten\[^{[3]}\]. In three dimensions, he pointed out that Chern-Simons theories with certain gauge groups are interesting models for quantum gravity\[^{[4]}\]. The three-dimensional Chern-Simons theories have been also useful in two-dimensional conformal field theories\[^{[5]}\] and integrable models\[^{[6]}\]. TFTs are also interesting from mathematical point of view concerning topological invariants of manifolds\[^{[7,8]}\].

Recently lattice constructions of TFTs have been being discussed. They are simpler than the usual continuum formulations of TFTs in the sense that they are free of infinitely many local spurious degrees of freedom and divergences. This simpleness is expected to facilitate the systematic constructions and classifications of TFTs.

In fact, in two dimensions, topological lattice field theories (TLFTs) have been constructed and classified extensively\[^{[9−16]}\]. It was shown that the class of TLFTs with degrees of freedom on links has one-to-one correspondence to semi-simple associative algebras\[^{[13−15]}\]. The relation between this class of TLFTs and known topological matter field theories in two dimensions\[^{[17,18]}\] was discussed\[^{[15]}\]. In general, it was shown\[^{[16]}\] that, starting with Atiyah’s axioms\[^{[2]}\], any unitary TFT can be decomposed as a direct sum of irreducible TFTs, which can be realized as TLFTs with degrees of freedom on vertices. Here the notion of irreducible TFTs was first introduced, and they are TFTs which can not be decomposed anymore. This notion will play an important role in the present paper.

In three dimensions, some systematic constructions of some series of TLFTs have been done, but any systematic classifications have not been done yet since they must be tightly related to the classification of three-dimensional manifolds. The first TLFT was constructed by Ponzano and Regge\[^{[19]}\], motivated by the three-
dimensional Regge calculus\textsuperscript{[20]}. The degrees of freedom of their TLFT are the labels of the irreducible representations of $SU(2)$. Their TLFT was generalized to a series of models with the quantum groups $SU_q(2)$ by Turaev and Viro\textsuperscript{[21]}. Dijkgraaf and Witten\textsuperscript{[22]} constructed models with discrete gauge groups. A relation between the Turaev-Viro model and 2D conformal field theories was pointed out, and some generalizations of the model were done\textsuperscript{[23,24]}. Boulatov\textsuperscript{[25]} constructed tensor-model versions of the Turaev-Viro model. (See also [26]-[31] for other discussions concerning Turaev-Viro model.) The relations to continuum formulations have not been studied enough. But it was shown by Turaev\textsuperscript{[32]} that the Turaev-Viro model is equivalent to the square of the Chern-Simons theory with the gauge group $SU(2)$. Ooguri and the present author\textsuperscript{[24,33]} discussed the equivalence between the Ponzano-Regge model and the Chern-Simons theory with gauge group $ISO(3)$. A new class of TLFTs was constructed by Chung, Fukuma and Shapere\textsuperscript{[34]}. This class of TLFTs is defined on arbitrary 3D cell complexes, not only on tetrahedral lattices. These TLFTs were shown to have one-to-one correspondence with a class of Hopf algebras.

In four dimensions, Ooguri\textsuperscript{[35]} generalized the Boulatov’s argument to four dimensions and obtained the 4D analogies of the Ponzano-Regge-Turaev-Viro model. He argued that the 4D analogue of the Ponzano-Regge model is the lattice realization of the $BF$-theory\textsuperscript{[36]}.

The recent developments on TLFTs have been concentrated only on constructions and classifications of TLFTs, and seem to uncover physical content. Since TLFTs are special cases of general lattice field theories, the investigation of the positions of TLFTs in the parameter space of general lattice field theories will clarify the specialties of TFTs and the background reason of their appearances in the investigations of quantum gravity and string theory. The present author investigated the phase structure around 2D TLFTs by a perturbative scheme and found that they are in general on multiple first order phase transition points\textsuperscript{[37]}. A usual continuum field theory describes the long-range modes near a fixed point of a renormalization group flow on a continuous phase transition surface. Thus,
complimentary to usual continuum field theories, TLFTs(TFTs) might be useful in describing the physics near a fixed point on a discrete phase transition surface.

This paper is an extension of the present author’s previous paper to dimension $D \geq 2$. The paper is organized as follows.

In section 2, we will define volume-dependent TLFTs in any dimension $D(\geq 2)$, and discuss their decompositions under certain conditions. We will review the physical structures of TLFTs$^{[21]}$. Then we will decompose a volume-dependent TLFT satisfying certain conditions into a direct sum of irreducible TLFTs with volume-dependent numerical factors. Using this decomposed form, we will show that a TLFT is in general on a multiple first order phase transition point, which is also a fixed point of the renormalization group flow in the parameter space of volume-dependent TLFTs. As an example, the procedure of the decomposition is shown in the two-dimensional lattice QCD$^{[17,38,10]}$.

In section 3, we will investigate the neighborhood of a TLFT by a perturbative method. We will discuss the method in detail, which is a kind of cluster expansion. Using this method, we will investigate the phase structure near a TLFT in general directions, and again will find the structure of first order phase transitions in the first order of the cluster expansion. The number of the different phases around it is shown to be equal to the number of the irreducible TLFTs of which the TLFT is a direct sum. The higher order contributions do not change this structure qualitatively, provided the perturbative treatment is valid around a TLFT and that the base manifold is topologically trivial. As a simple example, we will discuss the neighborhoods of the TLFTs in the parameter space of the 2D $Q$-state Potts model$^{[13]}$.

In section 4, we will investigate the phase structure near the discrete group $Z_p$ analogue$^{[22,25]}$ of the Turaev-Viro model on a topologically non-trivial manifold in three dimensions. This model is an irreducible TLFT, hence is not on a first order phase transition point. But we will find that the model is in general on a higher order discrete phase transition point, which is controlled by the physical states.
on topologically non-trivial surface. To show it, the higher orders of the cluster expansion are essential.

In section 5, we will give summary, comments and discussions.

2. Volume-dependent topological lattice field theories

For later discussions, it is more convenient to enlarge the parameter space of TLFTs. We will begin with defining volume-dependent TLFTs, which do not have any local degrees of freedom like TLFTs but are dependent on the volume of the manifold.

The rough description of the usual constructions of TLFTs is as follows. Topological field theories are field theories which are invariant under local deformations of background metric. Physical quantities of such theories depend only on the topology of the underlying manifold. In TLFTs, the independence of the background metric of TFTs corresponds to the independence of the lattice structure. Consider a simplicial decomposition of a given manifold. Usually lattice theories are defined by associating degrees of freedom to the simplices and assigning local weights. The partition function is defined as the summation of the products of the local weights over the degrees of freedom. The partition function of a TLFT must be invariant under the change of the simplicial decomposition of the manifold. The simplicial decompositions of a manifold can be generated by a certain set of local moves in any dimension\cite{39,40}. These local moves constrain the local weights. Solving these constraints, one obtains TLFTs.

In the present paper, we are not interested in concrete models of TLFTs, but in the general structures of TLFTs in an arbitrary dimension. Thus, for convenience, we use a more formal definition of TLFTs. Following Atiyah\cite{2}, we define a TLFT as a set of partition functions associated to simplicially decomposed manifolds possibly with boundaries. The partition functions should be independent of the simplicial decompositions of the insides of the manifolds. Hence these partition
functions must satisfy certain consistency conditions under gluing operations of the manifolds.

2.1 Definition of volume-dependent TLFTs

The lattices we consider in this paper are those obtained by simplicial decompositions of $D$-dimensional oriented connected manifolds possibly with boundaries. The lattices of boundaries can be colored by associating to each $l$-dimensional simplex ($l = 0, 1, \cdots, D - 1$) an element $x_l$ of a finite index set $X_l$.

The ingredients of a volume-dependent TLFT are complex-valued partition functions. A partition function is associated to a manifold $M$ which is decomposed with $D$-dimensional simplices and possibly has colored boundaries $\Sigma(c)$:

$$Z(M; A; \Sigma(c)). \quad (2.1)$$

Here $c$ denotes symbolically the coloring. The partition function is independent of the way how the inside of the manifold is decomposed with $D$-dimensional simplices but depends on the number $A$ and the topology of $M$ as well as the simplicial decomposition and the coloring of the boundaries.

The partition functions (2.1) must satisfy constraints under gluing operations of lattices. Let $(M; A; \Sigma)$ denote a lattice with uncolored boundaries. Consider a lattice $(M_3; A_3; \Sigma_3)$ obtained by identifying pairwise some of the $(D - 1)$-simplices of the boundaries of $(M_1; A_1; \Sigma_1)$ and $(M_2; A_2; \Sigma_2)$ (Fig.1). Let $b$ denote the sub-boundaries identified pairwise. Then $A_3 = A_1 + A_2$, $M_3 = M_1 \cup_b M_2$ and $\Sigma_3 = \Sigma_1 \cup_b \Sigma_2$, symbolically. We demand that the partition functions for the three lattices should satisfy the following constraint under the gluing operation:

$$Z(M_3; A_3; \Sigma_3(c_3)) = \sum_{c_{12}} Z(M_1; A_1; \Sigma_1(c_1))g_{b}^{c_1c_2}Z(M_2; A_2; \Sigma_2(c_2)). \quad (2.2)$$

Here $g_{b}^{c_1c_2}$ is the gluing tensor, and $c_{ib}(i = 1, 2)$ are the colorings associated to the
identified simplices of the sub-boundaries\(^*\). The summation is over the colorings associated to the simplices which become inner simplices after the gluing operation.

We demand also that the partition functions must satisfy the following constraint under a self-gluing operation:

\[
Z(M_2; A; \Sigma_2(c_2)) = \sum_{c_{inn}} g_b^{c_1 c_1'} Z(M_1; A; \Sigma_1(c_1)). \tag{2.3}
\]

Here \((M_2; A; \Sigma_2)\) is obtained by identifying some of the \((D - 1)\)-simplices of the boundaries \(\Sigma_1\) (Fig. 2).

2.2 Laplace decomposition

In this subsection, we will decompose a volume-dependent TLFT satisfying a certain condition into a direct sum of TLFTs by applying an analogue of the Laplace transformation to the volume-dependence of the partition functions. Then, in the following subsections, we will decompose each TLFT into a direct sum of irreducible TLFTs.

Let \(T^D\) denote a \(D\)-dimensional simplex, and \(\partial T^D\) its boundary lattice. We consider only a TLFT satisfying the following condition: A discrete analogue of the Laplace transformation is applicable to the volume dependence of the partition function for a \(D\)-dimensional ball \(B^D\) with \(\partial T^D\) as its boundary.

\[
Z(B^D; A; \partial T^D(c)) = \sum_i Z_i(B^D; \partial T^D(c)) \omega_i^A, \tag{2.4}
\]

where \(\omega_i\)’s are complex numbers and \(i\) is the label to distinguish different \(\omega_i\)’s.

First we will prove that the \(Z_i(B^D; \partial T^D(c))\)’s are orthogonal. Consider two lattices \((B^D; A_i; \partial T^D)\) \((i = 1, 2)\) and a lattice \((B^D; A_1 + A_2; \Sigma)\) obtained by gluing the two lattices identifying one of the \((D - 1)\)-simplices on \(\partial T^D\) of each lattice.

\(^*\) In most of concrete models of TLFTs, the gluing tensor is the product of the Kronecker’s deltas of the colors of the identified simplices with some numerical factors.
Since the partition functions must satisfy the constraint (2.2) for the gluing operation, one obtains

\[ Z(B^D; A_1 + A_2; \Sigma(c_3)) = \sum_{c_{inn}} Z(B^D; A_1; \partial T^D(c_1)) g^{c_1 c_2 b}_{T_{D-1}} Z(B^D; A_2; \partial T^D(c_2)). \] (2.5)

Using the assumption (2.4), the right-hand side is rewritten as

\[ \sum_{i,j} \sum_{c_{inn}} Z_i(B^D; \partial T^D(c_1)) g^{c_1 c_2 b}_{T_{D-1}} Z_j(B^D; \partial T^D(c_2)) \omega_i A_1 \omega_j A_2. \] (2.6)

Since the left-hand side of (2.5) is a function of \( A_1 + A_2 \), so must be (2.6). Thus the \( Z_i \)'s must be orthogonal, and we obtain the Laplace transformation for the left-hand side of (2.5) with

\[ \delta_{ij} Z_i(B^D; \Sigma(c_3)) = \sum_{c_{inn}} Z_i(B^D; \partial T^D(c_1)) g^{c_1 c_2 b}_{T_{D-1}} Z_j(B^D; \partial T^D(c_2)). \] (2.7)

In the self-gluing operation (2.3), if the Laplace transformation is applicable to the right-hand side, then so is the left-hand side. The \( Z_i \)'s in the both sides are related by

\[ Z_i(M_2; \Sigma_2(c_2)) = \sum_{c_{inn}} g^{c_1 c_2 b}_{b} Z_i(M_1; \Sigma_1(c_1)). \] (2.8)

Any partition function can be obtained by gluing operations of \( D \)-simplices and self-gluing operations. Thus repeating the same discussions as above, we obtain that the Laplace transformation using the same \( \omega_i \)'s is applicable to the volume-dependence of any partition function. Then the constraint (2.2) is rewritten using \( Z_i \)'s as

\[ \delta_{ij} Z_i(M_3; \Sigma_3(c_3)) = \sum_{c_{inn}} Z_i(M_1; \Sigma_1(c_1)) g^{c_1 c_2 b}_{b} Z_j(M_2; \Sigma_2(c_2)). \] (2.9)

The volume-dependence has disappeared in (2.8) and (2.9). Since \( Z_i \)'s are orthogonal for different \( i \)'s, the constraints (2.8) and (2.9) are the defining constraints for the partition functions of a TLFT for each \( i \), as will be explained in
the following subsection. Thus we have shown the following. A *volume-dependent TLFT with (2.4) is a direct sum of TLFTs with some volume-dependent numerical factors:*

\[
Z(M; A; \Sigma(c)) = \sum_{i=1}^{N} Z_i(M; \Sigma(c)) \omega_i^A,
\]

(2.10)

where \(N\) is a finite number*, and \(Z_i's\) are the partition functions of TLFTs.

2.3 Definitions concerning TLFTs

To prepare to decompose (2.10) further, we will discuss some elementary definitions concerning TLFTs in this subsection.

The ingredients of a TLFT are complex-valued partition functions. A partition function of a TLFT is associated to a manifold \(M\) which is simplicially decomposed with some \(D\)-simplices and possibly has colored boundaries:

\[
Z(M; \Sigma(c)).
\]

(2.11)

This partition function is independent of the way how the inside of \(M\) is decomposed and must satisfy the constraints (2.8) and (2.9).

The Hilbert space \(H_\Sigma\) associated to a simplicially decomposed boundary \(\Sigma\) is the module freely generated by all possible colorings of \(\Sigma\) over complex numbers. Thus a wave function in \(H_\Sigma\) is a complex-valued function of colorings on \(\Sigma; H_\Sigma = \{\Phi_\Sigma(c_\Sigma)\}\). The Hilbert space is finite dimensional since the index set \(X_i\) is finite.

Consider a \((D - 1)\)-dimensional closed manifold \(\Sigma\), and \(M = \Sigma \times [0, 1]\). Take simplicial decompositions \(\Sigma_1\) and \(\Sigma_2^*\) for the two boundaries of \(M\) so that one can take a simplicial decomposition of \(M\) consistent with \(\Sigma_1\) and \(\Sigma_2^*\). Here the symbol

* The finiteness can be shown as follows. In the following subsection, we will discuss the physical Hilbert spaces of TLFTs and volume-dependent TLFTs. It will be shown that the dimension of the physical Hilbert space on the \((D - 1)\)-sphere is non-zero and finite. Thus \(N\) must be finite.
* is for the reverse of orientation. We define a linear map which maps a state in $\mathcal{H}_{\Sigma_1}$ to a state in $\mathcal{H}_{\Sigma_2}$:

$$
\Phi_{\Sigma_2}(c_2) = P_{\Sigma_2\Sigma_1}(c_2, c_1)\Phi_{\Sigma_1}(c_1),
$$

$$
P_{\Sigma_2\Sigma_1}(c_2, c_1) = g^{c_2 c_2'}_{\Sigma_2} Z(\Sigma \times [0, 1]; \Sigma^*_{2'}(c_2'), \Sigma_1(c_1)),
$$

(2.12)

where the repeated indices for the colorings are summed over. From now on, it is supposed that the repeated indices for colorings are summed over unless otherwise stated.

Since the manifold $M = \Sigma \times [0, 1]$ can be obtained by gluing two $M$'s identifying the boundary $\Sigma$ of one $M$ and $\Sigma^*$ of another $M$, the linear maps satisfy the following equation:

$$
P_{\Sigma_3\Sigma_1}(c_3, c_1) = P_{\Sigma_3\Sigma_2}(c_3, c_2) P_{\Sigma_2\Sigma_1}(c_2, c_1).
$$

(2.13)

This equation (2.13) is derived from the definition (2.12) and the constraint (2.9) under the gluing operation above. Taking $\Sigma_3 = \Sigma_2 = \Sigma_1$, we see that $P_{\Sigma_1\Sigma_1} = P_{\Sigma_1\Sigma_1} P_{\Sigma_1\Sigma_1}$, *i.e.*, the map $P_{\Sigma_1\Sigma_1}$ is a projection map.

The physical Hilbert space $\mathcal{H}^{phys}_{\Sigma_1}$ is defined as the invariant subspace of $\mathcal{H}_{\Sigma_1}$ by the projection operator $P_{\Sigma_1\Sigma_1}$. The physical Hilbert space is finite dimensional since the Hilbert space is finite dimensional. Take a basis of the physical Hilbert space and its dual basis:

$$
\phi^i_{\Sigma_1}(c) \in \mathcal{H}^{phys}_{\Sigma_1},
$$

$$
\phi_i^{*\Sigma_1} \in \mathcal{H}^{phys*}_{\Sigma_1},
$$

$$
\phi_i^{*\Sigma_1}(c) \phi^j_{\Sigma_1}(c) = \delta_i^j,
$$

(2.14)

Taking $\Sigma_3 = \Sigma_1$ in (2.13), one finds $P_{\Sigma_1\Sigma_1} = P_{\Sigma_1\Sigma_2} P_{\Sigma_2\Sigma_1}$. Thus $P_{\Sigma_2\Sigma_1}$ is a one-to-one map from $\mathcal{H}^{phys}_{\Sigma_1}$ to $\mathcal{H}^{phys}_{\Sigma_2}$. Using the one-to-one maps $P_{\Sigma_1\Sigma_2}$ and $P_{\Sigma_2\Sigma_1}$, one
can obtain a natural basis and a natural dual basis for $\mathcal{H}_{\Sigma_2}^{\text{phys}}$.

\begin{equation}
\phi_{\Sigma_2}^i(c_2) = P_{\Sigma_2;\Sigma_1}(c_2, c_1)|\phi_{\Sigma_1}^{\cdot i}(c_1),
\end{equation}

\begin{equation}
\phi_{\Sigma_2}^{* i}(c_1) = \phi_{\Sigma_1}^{\cdot i}(c_1)P_{\Sigma_1;\Sigma_2}(c_1, c_2).
\end{equation}

From now on, the physical states labeled with the same index are supposed to be related in this way.

Consider a manifold $M$ with boundaries $\Sigma_1^*, \ldots, \Sigma_n^*, \Sigma_{n+1}, \ldots, \Sigma_m$ with certain simplicial decompositions. Following (2.12), define

\begin{equation}
P_{M;\Sigma_1^*;\ldots;\Sigma_n^*;\Sigma_{n+1},\ldots,\Sigma_m}(c_1, \ldots, c_n; c_n+1, \ldots, c_m)
= g_{c_1}^{c_n} \cdots g_{c_n}^{c_n} Z(M; \Sigma_1(c_1), \ldots, \Sigma_n(c_n), \Sigma_{n+1}(c_{n+1}), \ldots, \Sigma_m(c_m)),
\end{equation}

which can be regarded as a linear map from $\mathcal{H}_{\Sigma_{n+1}} \otimes \cdots \otimes \mathcal{H}_{\Sigma_m}$ to $\mathcal{H}_{\Sigma_1} \otimes \cdots \otimes \mathcal{H}_{\Sigma_n}$. Since gluing manifolds $\Sigma_i \times [0, 1]$ ($i = 1, \ldots, m$) to $M$ by identifying $\Sigma_i$’s and $\Sigma_i^*$’s does not change the manifold $M$, one can show, using the definitions (2.12) and (2.16) and the constraint (2.9) under the gluing operation, that

\begin{equation}
P_{M;\Sigma_1^*;\ldots;\Sigma_n^*;\Sigma_{n+1},\ldots,\Sigma_m}(c_1, \ldots, c_n; c_n+1, \ldots, c_m)
= P_{\Sigma_1;\Sigma_1}(c_1, c_1) \cdots P_{\Sigma_n;\Sigma_n}(c_n, c_n) P_{M;\Sigma_1^*;\ldots;\Sigma_n^*;\Sigma_{n+1},\ldots,\Sigma_m}(c_1, \ldots, c_n, c_n+1, \ldots, c_m)
\times P_{\Sigma_{n+1};\Sigma_{n+1}}(c_{n+1}, c_{n+1}) \cdots P_{\Sigma_m;\Sigma_m}(c_m, c_m),
\end{equation}

where $\Sigma_i$ and $\Sigma_i^*$ denote the same boundary possibly with different simplicial decompositions. The equation (2.17) is a generalization of (2.13). Thus, taking $\Sigma_i = \Sigma_i'$ in (2.17), the linear map (2.16) is in fact a map among the physical Hilbert spaces; $\mathcal{H}_{\Sigma_{n+1}}^{\text{phys}} \otimes \cdots \otimes \mathcal{H}_{\Sigma_m}^{\text{phys}}$ to $\mathcal{H}_{\Sigma_1}^{\text{phys}} \otimes \cdots \otimes \mathcal{H}_{\Sigma_n}^{\text{phys}}$. This fact leads us naturally to the following definition of a physical correlation function:

\begin{equation}
Z(M; (\Sigma_1)i_1, \ldots, (\Sigma_n)i_n, (\Sigma_{n+1})i_{n+1}, \ldots, (\Sigma_m)i_m)
\equiv \phi_{\Sigma_1}^{i_1}(c_1) \cdots \phi_{\Sigma_n}^{i_n}(c_n) P_{M;\Sigma_1^*;\ldots;\Sigma_n^*;\Sigma_{n+1},\ldots,\Sigma_m}(c_1, \ldots, c_n; c_n+1, \ldots, c_m)
\times \phi_{\Sigma_{n+1}}^{i_{n+1}}(c_{n+1}) \cdots \phi_{\Sigma_m}^{i_m}(c_m).
\end{equation}

The remarkable property of a physical correlation function is that it is independent of the simplicial decompositions of its boundaries. This fact can be proven by
using the definition of the basis of the physical Hilbert space (2.15) and (2.17). In particular, for any $\Sigma$,

$$Z(\Sigma \times [0,1]; (\Sigma)_i, (\Sigma)_j) = \delta^j_i,$$  \hspace{1cm} (2.19)

because the projection map (2.12) is the identity on $H_{\Sigma}^{\text{phys}}$.

### 2.4 Decomposition of a TLFT

Here we will discuss the decomposition of each TLFT in (2.10) into irreducible TLFTs. An irreducible TLFT is defined as a TLFT which cannot be decomposed further as a direct sum of TLFTs. This is an extension of the work by Durhuus and Jónsson\cite{16}, where they discussed the decomposition of a unitary two-dimensional TFT into a direct sum of irreducible TFTs.

The discussions will be roughly as follows. First we will discuss the correlation functions of $S^{D-1}$s, and will find a convenient basis of the physical Hilbert space on $S^{D-1}$, which we call the canonical basis. This basis was first introduced by Verlinde\cite{41} in the discussions of fusion rules of 2D conformal field theories, and was applied to 2D TLFTs in [13] - [15] and [16]. Then we will show that any partition(correlation) functions of a TLFT can be expressed as a sum of the partition(correlation) functions labeled by the labels of the canonical basis and that these labeled partition functions are orthogonal.

Here we begin with the notations to be used. $B_0$ denotes $S^{D-1}$ with a certain simplicial decomposition. Since, in the physical correlation functions, the simplicial decompositions of boundaries are irrelevant, we use $\phi^i$ for the physical state $\phi^i_{B_0}$ and $\phi_i$ for $\phi_{B_0 i}$. We also use $(\Sigma)^i(c)$ for $\phi^i_{\Sigma}(c)$ and $(\Sigma)_i(c)$ for $\phi^i_{\Sigma i}(c)$, for short.

Define the following symmetric tensors which reverse the orientations of the physical states on $S^{D-1}$:

$$\eta_{ij} = Z(T_{S^{D-1}}; \phi_i, \phi_j)$$

$$= (B_0)_i(c_1)g^{c_1c_2}_{B_0}(T_{S^{D-1}}; B^*_0(c_2), B^*_0(c_3))g^{c_3c_4}_{B_0}(B_0)_j(c_4),$$  \hspace{1cm} (2.20)

$$\eta^{ij} = Z(T_{S^{D-1}}; \phi^i, \phi^j).$$
where \( T_{S^{D-1}} = S^{D-1} \times [0, 1] \). Since the projection operator (2.12) is the identity on the physical Hilbert space, the following equation holds for any \( \Sigma \):

\[
\sum_i (\Sigma)^i(c_1)(\Sigma)_i(c_2) = P_{\Sigma \Sigma}(c_1, c_2),
\]

(2.21)

From now on it is supposed that contracted indices labeling the physical states are summed over unless otherwise stated. Using (2.21), (2.19) and (2.9) for the operation that two \( T_{S^{D-1}} \)s are glued to make one \( T_{S^{D-1}} \), we obtain

\[
\eta_{ij} \eta^{jk} = \delta^k_i,
\]

(2.22)

which shows that the tensors (2.20) are not degenerate. These tensors can be used to change the cases of the indices of \( \phi_i \) and \( \phi^i \). The proof is as follows:

\[
Z(M; \phi^i, \Sigma(c)) = Z(T_{S^{D-1}}; \phi^i, B_0(c_1))g_{B_0}^{c_1c_2}Z(M; B_0^*(c_2), \Sigma(c))
\]

\[
= Z(T_{S^{D-1}}; \phi^i, B_0(c_1))P_{B_0B_0}(c_1, c_2)g_{B_0}^{c_2c_3}Z(M; B_0^*(c_3), \Sigma(c))
\]

\[
= Z(T_{S^{D-1}}; \phi^i, B_0(c_1))\phi^j(c_1)\phi_j(c_2)g_{B_0}^{c_2c_3}Z(M; B_0^*(c_3), \Sigma(c))
\]

\[
= \eta^{ij}Z(M; \phi_j, \Sigma(c)),
\]

(2.23)

where we have used the invariance of the topology of \( M \) under gluing a \( T_{S^{D-1}} \) to \( M \) and the equations (2.9), (2.20) and (2.21).

Next we will obtain the fusion rule of the states on \( S^{D-1} \). Define the following rank-three symmetric tensor:

\[
N^{ijk} = Z(S^D; \phi^i, \phi^j, \phi^k),
\]

(2.24)

where the manifold truly treated can be uniquely obtained by taking away 3 \( D \)-simplices from \( S^D \). From now on, we will use such implications for simplicity. Using
this tensor (2.24), one can reduce the number of $S^{D-1}$s in correlation functions:

$$Z(M; \Sigma(c), \phi^i)N_i^{jk} = Z(M; \Sigma(c), B_0(c_1))(B_0)^i(c_1)(B_0)^j(c_2)g_{B_0}^{c_3}Z(S^D; B_0^*(c_3), \phi^j, \phi^k)$$

$$= Z(M; \Sigma(c), B_0(c_1))P_{B_0}B_0(c_1, c_2)g_{B_0}^{c_3}Z(S^D; B_0^*(c_3), \phi^j, \phi^k)$$

$$= Z(M; \Sigma(c), B_0(c_1))g_{B_0}^{c_3}Z(S^D; B_0^*(c_2), \phi^j, \phi^k)$$

$$= Z(M; \Sigma(c), \phi^j, \phi^k).$$

(2.25)

Here, from the third to the last line, we glued the manifold obtained by taking away one $D$-simplex from $S^D$ and the manifold obtained by taking away one $D$-simplex from $M$. Since the former manifold is topologically one $D$-simplex, the resultant manifold is simply $M$. In this derivation, we have used the constraint (2.9) for the gluing operation, (2.17), (2.18), (2.21) and (2.24). Taking particularly the three-sphere function as the partition function in the left-hand side of (2.25), one obtains

$$N_i^{jk}N_k^{lm} = N_i^{jlm} = N_i^{lk}N_k^{jm},$$

(2.26)

where $N_i^{jlm}$ is the four-sphere function defined as same as (2.24).

By the similar discussions as to derive (2.25), we obtain a formula

$$Z(M; \Sigma(c), \phi^i)Z(S^D; \phi_i) = Z(M; \Sigma(c)).$$

(2.27)

This formula implies also that, if the TLFT is not null, the dimension of the physical Hilbert space on $S^{D-1}$ must be non-zero. Substituting the first partition function of the left-hand side with a three-sphere function, we obtain

$$N^{ijk}Z(S^D; \phi_k) = Z(S^D; \phi^i, \phi^j)$$

$$= \eta^{ij},$$

(2.28)

where we have used the fact that $S^D$ with two holes is topologically equivalent to $T_{S^{D-1}}$ and (2.19).
From (2.25), (2.26) and (2.28), one can see that the physical states $\phi^i$'s form a commutative algebra with an identity. This algebra might be decomposed into mutually annihilating sub-algebras, and the decomposition will give the decomposition of the TLFT. One can discuss the decomposition of the algebra in general, but in the present paper we restrict our discussions only to the case that the algebra has no radical*. This is because, if a radical exists, one will have to discuss a wider class of TLFTs than those with (2.4). Let $\phi_R$ be a radical element, and consider a volume-dependent TLFT $Z_{\delta r}(M; A; \Sigma(c)) \equiv Z(M; \Sigma(c), \exp(A\delta r\phi_R))$. Since $\phi_R$ is a radical element and a physical state, the $Z_{\delta r}$ is a polynomial function of $A$. Thus, to discuss the neighborhood of a TLFT with radical, it is not sufficient to consider only the volume-dependence of (2.4).

Since an algebra without radical can be decomposed into a direct sum of simple sub-algebras, the commutative algebra can be simultaneously diagonalized. We call the following the canonical basis$^{[13-15,16]}$:

$$
\eta^{ij} = \delta^{ij}, \quad \eta_{ij} = \delta_{ij}, \quad N^{ijk} = \lambda^i \delta^{ij} \delta^{ik} \quad (\lambda^i \neq 0),
$$

(2.29)

which is supposed to be used from now on. From (2.25) and (2.29), we obtain the fusion rule:

$$
\phi^i \phi^j \sim \delta^{ij} \lambda^i \phi^i.
$$

(2.30)

One can see in (2.28) that the one-sphere function is the inverse of the coefficient of the three-sphere function:

$$
Z(S^D; \phi^i) = \frac{1}{\lambda^i}.
$$

(2.31)

From (2.27) and (2.31), we are lead to the fact that any correlation(partition)

* We are not sure whether a TLFT can have an algebra with a radical. In two dimensions, no such TLFTs are known$^{[9-15]}$. One can realize such cases only by taking a certain infinite limit$^{[15]}$. 
function can be expressed in a sum

\[ Z(M; \Sigma(c)) = \sum_{i=1}^{N} \frac{1}{\lambda_i} Z^i(M; \Sigma(c)), \]  

\( Z^i(M; \Sigma(c)) \equiv Z(M; \Sigma(c), \phi^i), \)  

where \( N \) is the dimension of the physical Hilbert space on \( S^{D-1} \). In other words, (2.32) is

\[ \sum_{i=1}^{N} \frac{1}{\lambda_i} \phi^i \sim 1. \]  

(2.33)

From now on, to avoid confusions, any repeated indices of physical states without the summation symbol are not supposed to be summed over.

To show that (2.32) is a direct sum, we will prove the orthogonality of \( Z^i \)'s. Suppose \( M_3 \) is obtained by gluing \( M_1 \) and \( M_2 \). Then, by using (2.9), (2.30) and (2.32),

\[ \sum_{c_{12}} \frac{1}{\lambda_i} Z^i(M_1; \Sigma_1(c_1)) g_{b}^{c_1c_2} \frac{1}{\lambda_j} Z^j(M_2; \Sigma_2(c_2)) = \frac{1}{\lambda_i \lambda_j} Z(M_3; \Sigma_3(c_3), \phi^i, \phi^j) \]

\[ = \delta^{ij} \frac{1}{\lambda_i} Z^i(M_3; \Sigma_3(c_3)). \]  

(2.34)

We will next discuss the decompositions of the physical Hilbert space and the physical correlation(partition) functions. The projection operator specifying the physical Hilbert space can be decomposed into

\[ P_{\Sigma \Sigma} = \sum_{i} P_{i \Sigma \Sigma} \]

\[ P_{i \Sigma \Sigma} = P_{j \Sigma \Sigma} \]

\[ P_{i \Sigma \Sigma}(c_2, c_1) \equiv \frac{1}{\lambda_i} g_{\Sigma^* c_2 c_1} Z^i(\Sigma \times [0, 1]; \Sigma^*(c_2), \Sigma(c_1)), \]  

where we have used (2.12), (2.32) and the orthogonality (2.34). Define the \( i \)-th physical Hilbert space \( \mathcal{H}_{\Sigma}^{phys,i} \) as the invariant subspace of the projection operator
$P^i_{\Sigma \Sigma}$. Then (2.35) implies that $\mathcal{H}^{\text{phys}}_{\Sigma} = \sum_{i} \oplus \mathcal{H}^{\text{phys},i}_{\Sigma}$. In particular, the dimension of each physical Hilbert space on $S^{D-1}$ is one: $\dim(\mathcal{H}^{\text{phys},i}_{S^{D-1}}) = 1$.

The physical states in the physical Hilbert spaces labeled by different $i$ and $j$, $\mathcal{H}^{\text{phys},i}_{\Sigma}$ and $\mathcal{H}^{\text{phys},j}_{\Sigma}$ ($i \neq j$), can not interact with each other. This is because the labeled partition(correlation) function is orthogonal to the physical Hilbert space with a different label:

$$Z^i(M; \Sigma_1(c_1), (\Sigma_2)^j) = Z^i(M; \Sigma_1(c_1), \Sigma_2(c_2)) P^j_{\Sigma_2 \Sigma_2}(c_2, c_3)(\Sigma_2)^j(c_3) = \delta^{ij} Z^i(M; \Sigma_1(c_1), (\Sigma_2)^j),$$  \hspace{1cm} (2.36)

where $(\Sigma_2)^j \in \mathcal{H}^{\text{phys},j}_{\Sigma_2}$, and, in this derivation, we have used (2.17), (2.35) and the orthogonality (2.34). The discussion is similar for the case of the dual states. Thus the linear map defined with the partition function $Z$ in (2.16) is a direct sum of the linear maps defined with $\frac{1}{N} Z^i$, which are the maps from $\mathcal{H}^{\text{phys},i}_{\Sigma_{m+1}} \otimes \cdots \otimes \mathcal{H}^{\text{phys},i}_{\Sigma_m}$ to $\mathcal{H}^{\text{phys},i}_{\Sigma_1} \otimes \cdots \otimes \mathcal{H}^{\text{phys},i}_{\Sigma_n}$, respectively. This concludes that the original TLFT is a direct sum of the theories whose partition functions are given by $\frac{1}{N} Z^i$.

The orthogonality (2.34) implies that $\frac{1}{N} Z^i$ of each $i$ satisfy the defining constraints of a TLFT (2.8) and (2.9). Thus $\frac{1}{N} Z^i(M; \Sigma(c))$ of each $i$ define a TLFT. Since the physical Hilbert space on $S^{D-1}$ of each TLFT is one-dimensional, each TLFT is an irreducible TLFT, which cannot be decomposed anymore into a direct sum of TLFTs.

In the present paper we do not discuss the general case of the algebra, but one will be able to extend the discussions so far to the general case. An irreducible TLFT will have the algebra which cannot be decomposed anymore into a direct sum of mutually annihilating sub-algebras.

Decomposing the TLFTs (without radical) in (2.10) into a direct sum of irreducible TLFTs, a partition function of a volume-dependent TLFT is a direct sum of those of irreducible TLFTs with volume-dependent numerical factors:

$$Z(M; A; \Sigma(c)) = \sum_{i=1}^{N} Z_i(M; \Sigma(c))(\omega_i)^A,$$  \hspace{1cm} (2.37)
where the $Z_i$ has only one physical state on $S^{D-1}$.

Since a volume-dependent TLFT with (2.4) is a direct sum of irreducible TLFTs, the natural definition of a physical Hilbert space of a volume-dependent TLFT is the direct sum of the physical Hilbert spaces of the irreducible TLFTs:

$$\mathcal{H}_\Sigma^{\text{phys}} = \sum_{i=1}^{N} \oplus \mathcal{H}_{\Sigma}^{\text{phys},i}.$$  

In fact, a partition function of a volume-dependent TLFT can be regarded as a multi-linear map among $\mathcal{H}_\Sigma^{\text{phys}}$ as same as the map (2.16) of a TLFT. The physical correlation function defined similarly to that of TLFTs is independent of the simplicial decomposition of $M$ and its boundary, but is dependent on the volume of the manifold $M$.

2.5 Renormalization group flow

Since a physical correlation function of a volume-dependent TLFT is dependent on the volume, there is a renormalization group flow. A renormalization group flow of a lattice theory is determined by how the change of the lattice structure is absorbed in the change of the parameters of the theory without changing the physical outcomes. Since a physical correlation function of a volume-dependent TLFT depends only on the volume $A$ of the lattice $M$, the invariance of the physical outcomes imposes only that $(\omega_i)^A = (\omega_i')^{A'}$ for each $i$ under the change of the volume from $A$ to $A'$. Taking the extensive variables $p_i = \ln(\omega_i)$ as the parameters of the theory, the flow is $p_i A = p_i' A'$. This implies the volume-dimension of the parameter $p_i$ is 1, and this agrees with the Nienhuis-Nauenberg criterion\[44\].

The fixed points of the renormalization group flow are the points with $\omega_i = 0, 1$ (or $p_i = -\infty, 0$) for all $i$. The point $p_i = -\infty$ is the infrared fixed point, and the point $p_i = 0$ is the ultraviolet fixed point. Since the volume-dependence vanishes on such a fixed point, the theory becomes a TLFT. On the other hand, one can make a volume-dependent TLFT from a TLFT by (2.37) with certain $\omega_i$s. The original TLFT corresponds to the point $\omega_i = 1$ for all $i$. Hence a TLFT can be regarded as being on a fixed point in the parameter space of a volume-dependent TLFT, and this fixed point has relevant operators with volume-dimension one. The number of such operators is equal to the dimension of the physical Hilbert space.
on $S^{D-1}$ of the TLFT.

2.6 First order phase transitions and order parameters

We will consider the thermo-dynamical limit $A \to \infty$. From (2.37), a partition function of a volume-dependent TLFT is

$$Z(M; A) = \sum_{i=1}^{N} Z_i(M) \exp(p_i A),$$

where $M$ is without any boundaries and $N$ is the number of the irreducible TLFTs. The free energy per $D$-simplex in the thermo-dynamical limit is

$$f = -\lim_{A \to \infty} \frac{1}{A} \ln Z = -\max_i(p_i),$$

where $\max_i(p_i)$ is for taking the $p_i$ which has the largest real part among all the $p_i$s. Thus the point $p_i = 0$ for all $i$ is a multiple first order phase transition point with $N$ different phases around it*. In general, a TLFT is on a multiple first order phase transition point, and the number of the phases around it is equal to the number of the irreducible TLFTs of which the TLFT is the direct sum.

The order parameters are appropriately given by using the one-sphere functions. Define the one-sphere expectation values as follows:

$$\langle \phi^i \rangle^M = \lim_{A \to \infty} \frac{Z(M; A; \phi^i)}{Z(M; A)},$$

where $\phi^i$ is a physical state on $S^{D-1}$, and we take the canonical basis to label them. Using the orthogonality (2.36), we obtain

$$\frac{1}{\lambda^i} \langle \phi^i \rangle^M = \lim_{A \to \infty} \frac{1}{\lambda^i} \frac{Z_i(M; \phi^i) \exp(p_i A)}{\sum_{j=1}^{N} Z_j(M) \exp(p_j A)} = \delta^{im},$$

where we assumed the system is in the $m$-th phase, that is, $p_m$ has the greatest real part among $p_i (i = 1, \cdots, N)$. Thus the one-sphere expectation values give good order parameters.

* In general, $Z_i(M)$ might be zero, and then the discussions must be changed. Here we set aside such special cases.
Comments are in order. So far we have assumed implicitly that there are no constraints on the parameters of the theories and that we can take any values for the parameters $\omega_i$s or $p_i$s. But in some physical cases we cannot. An example is the 2-dimensional lattice QCD in the following subsection. This theory is a volume(area)-dependent TLFT, but cannot have any first order phase transition points if the positivity of the local weight is assumed.

2.7 An example — 2D lattice QCD

As a simple but physically interesting example of a volume(area)-dependent TLFT, we consider the 2-dimensional lattice QCD$^{[17,38,10,15]}$. As the index set $X_1$ for 1-simplex(edge) we take the Lie group $G = SU(n), U(n)$. The index set is infinite for this model, but nonetheless we can treat it straightforwardly in the same way as in the preceding subsections. The index set for 0-simplex(vertex) is not considered in this model. The partition function for a single 2-simplex(triangle) and the gluing tensor are defined as

$$
\begin{align*}
Z(\mathcal{T}^2; 1; \partial\mathcal{T}^2(g_1, g_2, g_3)) &= C_{g_1, g_2, g_3} = z(g_1 g_2 g_3), \\
z(ugu^{-1}) &= z(g), \\
g^{g_1 g_2} &= \delta(g_1, g_2^{-1}), 
\end{align*}
$$

(2.42)

where $u, g, g_i \in G$, and $\mathcal{T}^2$ is a 2-simplex(triangle). We take the local weight $z(g)$ as a positive real function of the group element of $G$, and it satisfies the invariance in the second line. The $g^{g_1 g_2}$ is the gluing tensor for a pair of 1-simplices, and is expressed by the delta function associated to the Haar measure of $G$, which is the measure used in summing over the internal colorings.

To show that the elementary data (2.42) define the partition functions of a volume-dependent TLFT, the following invariance of the product of two local
weights under the flip move (Fig. 4) is essential:

\[\int dg z(g_1 g_2 g) z(g_3 g_4) = \int d(g_2^{-1} g_4) z(g_1 g_2 (g_2^{-1} g_4)) z((g_2^{-1} g_4)^{-1} g_3 g_4)\]

\[= \int dg z(g_1 g g_4) z(g_1 g_2 g_3).\]  

(2.43)

It is known that all triangulations of a two-dimensional surface with the same number of triangles can be generated by the flip move\(^{[42]}\). A partition function \(Z(M; A; \Sigma(c))\) can be obtained from the elementary data (2.42) by using the rules (2.2) and (2.3). The invariance (2.43) guarantees that the obtained partition function is in fact independent of the triangulation of \(M\) and that the partition functions satisfy (2.2) and (2.3).

To perform the Laplace decomposition of the area-dependent TLFT, we will calculate \(Z(B^2; A; \partial \mathcal{T}^2(g_1, g_2, g_3))\), where \(B^2\) is a two-dimensional ball. The invariance in the second line of (2.42) enables one to expand the local weight \(z\) in the characters of \(G\):

\[z(g) = \sum_R \Lambda_R d_R \chi_R(g),\]  

(2.44)

where \(d_R\) is the dimension of the irreducible representation \(R\) of \(G\), and \(\Lambda_R\) is the coefficient of the expansion. The partition function \(Z(B^2; A; \partial \mathcal{T}^2(g_1, g_2, g_3))\) can be calculated easily using this expanded form (2.44) and the orthogonality of the characters:

\[Z(B^2; A; \partial \mathcal{T}^2(g_1, g_2, g_3)) = \sum_R (\Lambda_R)^A d_R \chi_R(g_1 g_2 g_3).\]  

(2.45)

This result (2.45) and the orthogonality of the characters imply that each \(d_R \chi_R\) gives the local weight of a TLFT. To see that each weight defines an irreducible TLFT, we will check \(\dim(\mathcal{H}^{phys \ R}_{S_1}) = 1\). We take a three-gon as the simplicially decomposed \(S^1\). The projection operator from a three-gon to another is calculated
to be
\[
P_{R}^{g_1,g_2,g_3}_{g'_1,g'_2,g'_3} = \int dg_4 dg_5 C_R^{g_1,g_2,g_3}_{g_4,g_5} C_R^{g_4,g_5}_{g'_1,g'_2,g'_3} \\
= \chi_R((g_1 g_2 g_3)^{-1}) \chi_R(g'_1 g'_2 g'_3), \\
C_R^{g_1,g_2,g_3} \equiv d_R \chi_R(g_1 g_2 g_3).
\]
This projection operator projects any wave-function \(\phi(g_1,g_2,g_3)\) to \(\chi_R((g_1 g_2 g_3)^{-1})\). Thus \(\mathcal{H}_{S}^{\text{phys}} R = \{\chi_R((g_1 g_2 g_3)^{-1})\}\) and its dimension is one.

Since the 2D lattice QCD is an area-dependent TLFT with the decomposition as a direct sum of an infinite number of TLFTs, one might expect that it has first order phase transition points. But, in fact, it is not valid because the coefficient \(\Lambda_0\) for the trivial representation in (2.44) has always the greatest absolute value. This inequality can be shown as the following. Since the local weight \(z\) is assumed to be positive real,

\[
|\Lambda_R| = \left| \frac{1}{d_R} \int dg \chi_R(g) z(g) \right| = \left| \frac{1}{d_R} \int dg \sum_{i=1}^{d_R} \rho_i(g) z(g) \right| < \int dg z(g) = \Lambda_0,
\]
where \(\rho_i\)s are the eigenvalues in the representation \(R\).

3. Neighborhood of topological lattice field theories

In the previous section we investigated volume-dependent TLFTs and showed that TLFTs are on fixed points of the renormalization group flow as well as on multiple first order phase transition points in the parameter space of volume-dependent TLFTs. In this section, we will consider general perturbative deformations of TLFTs. To treat such deformed systems, we will introduce a perturbative method, which is a kind of cluster expansion\[^{[43]}\]. We assume that such a perturbative method is available in the neighborhood of a TLFT.

Consider a manifold \(M\) simplicially decomposed with a certain number of \(D\)-simplices. The partition function of \(M\) of a TLFT is expressed as a product of
local partition functions of the TLFT:

\[
Z(M) = \sum_c \prod_q Z(T^D(q); c) \prod_b g^{cc'}_b,
\]
(3.1)

\[
Z(T^D(q); c) \equiv Z(T^D(q); \partial T^D(c)),
\]

where the summation is over the inner colorings, and the products are of the local partition functions over all the \( D \)-simplices and the gluing tensors.

We consider general changes of the local partition functions; \( Z(T^D(q); c) \rightarrow Z(T^D(q); c) + \delta Z(T^D(q); c) \). We assume that \( \delta Z \)s are small enough to justify the perturbative treatment in the following subsection. We will not consider the changes in the gluing tensors for simplicity, because they will not change the discussions essentially. Since we consider general changes, the deformed system is not a TLFT anymore. Thus the partition function depends on the simplicial decomposition of the manifold. Thus, from now on, we are supposed to consider certain simplicial decompositions and will take the thermo-dynamical limit that the total number of the \( D \)-simplices in \( M \) is taken to infinity.

3.1 Cluster expansion

Here we will introduce a systematic perturbative method, which approximates the partition function and the correlation functions of the deformed system. This method is a kind of cluster expansion\[^{[4]}\], but is unusual in the sense that it is a cluster expansion of an operator.

Here we begin with the partition function. The partition function of the deformed system is defined by

\[
Z(M; \delta Z) = \sum_c \prod_q (Z(T^D(q); c) + \delta Z(T^D(q); c)) \prod_b g^{cc'}_b,
\]
(3.2)

where \( q \) is the label for the \( D \)-simplices in \( M \). Expanding this partition function
(3.2) in $\delta Z$, we obtain

$$Z(M; \delta Z) = \sum_{n=0}^{A} \frac{1}{n!} \sum_{q_1, \cdots, q_n} u_n(q_1, \cdots, q_n),$$

$$u_0 = Z(M),$$

(3.3)

where $A$ is the total number of the $D$-simplices in $M$, and $Z(M)$ is the partition function of the original TLFT. The $u_n(q_1, \cdots, q_n)$ describes an $O((\delta Z)^n)$ contribution to the partition function, and the $q_i$s specify the $n$ $\delta Z$s coming from the $D$-simplices $q_1, \cdots, q_n$. The $u_n(q_1, \cdots, q_n)$ is supposed to be non-zero only when $q_1, \cdots, q_n$ are different from each other.

If we assume that the simplicially decomposed manifold $M$ has an enough ‘extension’, the lower order contributions can be calculated using the correlation functions of the original TLFT on $M$:

$$u_n(q_1, \cdots, q_n) = Z(M; U_n(q_1, \cdots, q_n)) + \theta(n - n(M))Z(M_1; U'_n(q_1, \cdots, q_n)) + \cdots,$$

(3.4)

where $n(M)$ is the number of $D$-simplices below which one does not have to consider correlation functions on manifolds different from $M$, and $\theta(n)$ is the discrete analogue of $\theta$-function: 1 for $n \geq 0$ and 0 for $n < 0$. An example of $M$ and $q_1, \cdots, q_n$ that one has to consider correlation functions on manifolds different from $M$ are given in Fig.5. $U_0$ is the identity and $U_n$ takes non-zero values only when the $q_i$s are different from each other. Since $n(M) = 1$ in $D = 1$, we will consider only the $D > 1$ cases. The $D = 1$ case can be treated very easily in another way. We assume that the thermo-dynamical limit is taken by refining the lattice structure of $M$ almost uniformly. Then $n(M) \to \infty$ in the thermo-dynamical limit, so that the right-hand side of (3.4) will be well approximated by the first term. Thus we will consider only the first term from now on.

The definition of the first term of (3.4) is as follows. Consider a lattice $M'$ obtained by taking away $n$ $D$-simplices $q_1, \cdots, q_n$ from $M$. Then $M'$ is a lattice
with some boundaries $\Sigma$, and a partition function $Z(M'; \Sigma(c))$ of the original TLFT can be associated to the lattice $M'$. Contracting the colorings of $Z(M'; \Sigma(c))$ with those of $\delta Z(T^D(q_i); c)s$, one obtains $Z(M; U_n(q_1, \cdots, q_n))$. Thus $U_n(q_1, \cdots, q_n)$ represents the operation to take away $n$ $D$-simplices $q_1, \cdots, q_n$ from $M$ and contract the colorings of the boundaries with those of $\delta Z(T^D(q_i); c)s$.

If the $n$ $D$-simplices $q_1, \cdots, q_n$ are separated, $Z(M; U_n(q_1, \cdots, q_n))$ is given by contracting the colorings of the $n$-sphere function of the original TLFT with those of $\delta Z(T^D(q_i); c)s$. Since the $n$-sphere function is independent of the positions $q_i$, the $q_i$ dependences of $Z(M; U_n(q_1, \cdots, q_n))$ come only from those of $\delta Z(T^D(q_i); c)s$. This property simplifies the systematic calculation of (3.3) drastically, because the leading contributions come from the cases that the $q_i$s are separated.

To consider sub-leading contributions, we introduce new operations $W_n(q_1, \cdots, q_n)$ satisfying

$$U_n(q_1, \cdots, q_n) = \sum_{k_1, \cdots, k_n=0}^{k_1, \cdots, k_n=n} \left( \prod_{i=1}^{n} \frac{1}{k_i!(q_i)!} \right) \sum_{\sigma_n} \overbrace{W_1(q_{\sigma_n(1)}) \cdots W_1(q_{\sigma_n(k_1)})}^{k_1 \text{ times}} \cdots \overbrace{W_n(\cdots, q_{\sigma_n(n)})}^{k_n \text{ times}},$$

(3.5)

where $\sigma_n$ denotes the permutation maps $\{1, \cdots, n\} \to \{1, \cdots, n\}$. The relation (3.5) defines $W_n$ iteratively with $U_n$ and $W_1 \sim W_{n-1}$. The relation between $W_n$s and $Z(M; \delta Z)$ is given by

$$Z(M; \delta Z) = Z(M; \exp(W)),$$

$$W = \sum_{n=1}^{\infty} \sum_{q_1, \cdots, q_n} W_n(q_1, \cdots, q_n),$$

(3.6)

which can be proven formally using the definitions (3.3), (3.4) and (3.5).

The relation (3.5) is of course meaningless before the products of $W_n$s are defined. From (3.5) one obtains $W_1(q) = U_1(q)$ without any such products. But to obtain $W_2$ we have to define the product $W_1(q_1)W_1(q_2)$. In fact there are various ways to define the product. If $q_1$ and $q_2$ are separated, a natural definition
of the product would be to identify $W_1(q_1)W_1(q_2)$ with $U_2(q_1, q_2)$, because the former can be regarded as the operation of taking away the $D$-simplices $q_1$ and $q_2$. But how about the cases that $q_1$ and $q_2$ coincide or are next to each other? We use ‘operator splitting regularization’ for the definitions of the products in general. The meaning is as follows. For example, $Z(M; W_1(q_1)W_1(q_2))$ is defined by contracting the colorings of $\delta Z(\mathcal{D}(q_i); c_i)$ ($i = 1, 2$) with those of the two-sphere function $Z(M; \partial \mathcal{D}(c_1), \partial \mathcal{D}(c_2))$ of the original TLFT. This definition coincides with $U_2(q_1, q_2)$ if $q_1$ and $q_2$ are separated. From now on, we assume that such ‘operator splitting regularization’ is used as the definitions of the products of $W_n$s.

Since $W_2(q_1, q_2) = U_2(q_1, q_2) - W_1(q_1)W_1(q_2)$ from (3.5), $W_2(q_1, q_2)$ takes non-zero values only when $q_1$ and $q_2$ coincide or next to each other. Such a cluster property of $W_n$s can be proven in general as follows. Consider two sets of positions: $Q = \{q_1, \ldots, q_m\}, Q' = \{q_{m+1}, \ldots, q_{m+m'}\}$. Suppose these two sets are separated from each other. We will prove by iteration that $W_{m+m'}(q_1, \ldots, q_{m+m'})$ is zero always for such separated $Q$ and $Q'$. We start with the following equation which is valid because the two sets are separated:

$$U_{m+m'}(q_1, \ldots, q_{m+m'}) = U_m(q_1, \ldots, q_m)U_{m'}(q_{m+1}, \ldots, q_{m+m'}), \quad (3.7)$$

where the definition of the product of $U_m$s follows that of $W_n$s. As was shown above, the cluster property is valid for $m = m' = 1$. Assume that $m + m' > 2$, and that $W_n (n < m + m')$ satisfies the cluster property. Then, from the relation (3.5),

$$U_{m+m'}(q_1, \ldots, q_{m+m'}) - W_{m+m'}(q_1, \ldots, q_{m+m'})$$

$$= \sum_{k_1, \ldots, k_{m+m'} = 1}^{m+m'} \left( \prod_{i=1}^{m+m'-1} \frac{1}{k_i!(i!)^{k_i}} \right) \sum_{\sigma_{m+m'}} W_1(q_{\sigma_{m+m'}(1)}) \cdots W_{m+m'-1}(\cdots, q_{\sigma_{m+m'}(m+m')})$$

$$= U_m(q_1, \ldots, q_m)U_{m'}(q_{m+1}, \ldots, q_{m+m'}), \quad (3.8)$$

where we used the cluster property of $W_n (n < m + m')$ from the second to the last line. Comparing with (3.7), we obtain $W_{m+m'}(q_1, \ldots, q_{m+m'}) = 0$. 

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Comments are in order. Because of the topological nature of the partition (correlation) functions of the TLFT, in the ‘operator splitting regularization’, one has to take care only of the topology of the neighborhoods of the operators. Thus it is rather simple in lower orders, but is complicated in higher orders since the topology of the neighborhood becomes complicated. But we are allowed to choose the neighborhood of an operator to make calculations as simple as possible. The difference of the choices will be absorbed in the operators with orders higher than the considering order. Consider, for example, a product $W_n(q_1, \cdots, q_n)W_1(q)$, where the $D$-simplices $q_1, \cdots, q_n$ surround an area of the lattice. Naively there are two possibilities of the splitting, say the inside or the outside of the area. We should define the inside of the area as a part of the neighborhood of $q_1, \cdots, q_n$, because then the splitting is always done to the outside and this facilitates the calculations.

The partition function $Z(M; \Sigma(c); \delta Z)$ with colored boundaries $\Sigma(c)$ can be also calculated by the cluster expansion. We start with the analogue of (3.3) and (3.4):

$$Z(M; \Sigma(c); \delta Z) = Z(M; U^{\Sigma(c)}),$$

$$U^{\Sigma(c)} = \sum_{n=0} U_n^{\Sigma(c)} \left( q_1, \cdots, q_n \right),$$

$$U_0^{\Sigma(c)} = \Sigma(c).$$

(3.9)

Special cares must be taken if some of the $q_1, \cdots, q_n$ are next to $\Sigma$. Define $W_n^{\Sigma}$ to single out such boundary effects:

$$U_n^{\Sigma(c)} \left( q_1, \cdots, q_n \right) = \sum_{m=0}^{n} \sum_{\sigma_n} \frac{1}{m!(n-m)!} W_m^{\Sigma(c)} \left( q_{\sigma_n(1)}, \cdots, q_{\sigma_n(m)} \right) U_{n-m} \left( q_{\sigma_n(m+1)}, \cdots, q_{\sigma_n(n)} \right),$$

(3.10)

where $U_n(q_1, \cdots, q_n)$ is defined by splitting operators from $\Sigma$. One can show that $W_n^{\Sigma(c)}$ satisfies a cluster property, that is, $W_n^{\Sigma(c)}(q_1, \cdots, q_n)$ takes zero if there exists a non-empty subset $Q$ of $\{q_1, \cdots, q_n\}$ separated from its compliment $\bar{Q}$ as
well as from $\Sigma$ (Fig.6). Using the $W_n^{\Sigma(c)}$, the partition function is expressed as

$$Z(M; \Sigma(c); \delta Z) = Z(M; W^{\Sigma(c)} \exp(W)),$$

$$W^{\Sigma(c)} = \sum_{n=0}^{\infty} \sum_{q_1, \cdots, q_n} \frac{1}{n!} W_n^{\Sigma(c)}(q_1, \cdots, q_n),$$

$$W_0^{\Sigma(c)} = \Sigma(c),$$

where the definition of $W$ is given by (3.5) and (3.6).

Consider the case that $\Sigma$ is composed of two boundaries $\Sigma_i$ ($i = 1, 2$) and they are separated by $d$ lattices. Consider a set $Q = \{q_1, \cdots, q_n\}$ for which $W_n(q_1, \cdots, q_n) \neq 0$. Then, if $n$ is smaller than the distance $d$, the set $Q$ is a sum of two sets $Q_1$ and $Q_2$, where $Q_1$ is in the neighborhood of $\Sigma_1$ but is separated from $Q_2$ and the boundary $\Sigma_2$, and vice versa for $Q_2$ (Fig.7). One can prove that, if $n < d$, $W_n^{\Sigma(c)}$ can be factorized:

$$W_n^{\Sigma_{12}(c)}(q_1, \cdots, q_n)$$

$$= \sum_{m=0}^{n} \sum_{\sigma_n} \frac{1}{m!(n-m)!} W_m^{\Sigma_1(c_1)}(q_{\sigma_n(1)}, \cdots, q_{\sigma_n(m)}) W_{n-m}^{\Sigma_2(c_2)}(q_{\sigma_n(m+1)}, \cdots, q_{\sigma_n(n)}),$$

(3.12)

where $W_n^{\Sigma_i}(q_1, \cdots, q_n)$ ($i = 1, 2$) has the cluster property that it is zero if there exists a subset of $\{q_1, \cdots, q_n\}$ separated from the compliment and the boundary $\Sigma_i$. An important point is that $W_n^{\Sigma_i}$ is independent of the other boundary. Substituting (3.11) with (3.12), we obtain

$$Z(M; \Sigma_{12}(c); \delta Z) = Z(M; (W^{\Sigma_1(c_1)}W^{\Sigma_2(c_2)} + W_d^{\Sigma_{12}(c)}) \exp(W)),$$

(3.13)

where $W_d^{\Sigma_{12}(c)}$ is the order of $(\delta Z)^d$.

3.2 First order phase transitions

In the previous section, we investigated volume-dependent TLFTs, and showed that a TLFT is on a multiple first order phase transition point and that the number of the phases around it is equal to the number of the irreducible TLFTs of which
the TLFT is the direct sum. This statement is exact because no approximation
is used, but is valid only in the parameter space of a volume-dependent TLFT. In
this section, we will show, in the approximation of the first order of the cluster
expansion, the same statement in the parameter space of the general deformations
of the local weights.

The operator $W_1(q)$ is to take away a $D$-simplex and contract the colorings
with those of $\delta Z(T^D(q); c)$. Thus the ‘operator splitting regularization’ implies

$$Z(M; \Sigma(c), \prod_{i=1}^{n} W_1(q_i)) = Z(M; \Sigma(c), B_0(c_1), \ldots, B_0(c_n)) \prod_{i=1}^{n} g_{B_0}^{c_i c_i'} \delta Z(T^D(q_i); c_i')$$

$$= Z(M; \Sigma(c), \phi^{i_1}, \ldots, \phi^{i_n}) \prod_{i=1}^{n} \phi_i(c_i) g_{B_0}^{c_i c_i'} \delta Z(T^D(q_i); c_i'),$$

(3.14)

where $B_0 = \partial T^D$ and $\phi_i$s are the physical states on $S^{D-1}$. Here we have used
(2.21). Thus, in the first order of the cluster expansion, $W$ is

$$W = \sum_q W_1(q) = A \delta Z \phi^i,$$

$$\delta Z_i \equiv \frac{1}{A} \sum_q \phi_i(c) g_{B_0}^{c c'} \delta Z(T^D(q); c'),$$

(3.15)

where $A$ is the total number of the $D$-simplices in $M$, and $\delta Z_i$s are the mean values
of the deformations of the local weights projected to the physical states on $S^{D-1}$.

Using (3.6), (2.30), (2.33) and (2.32), $Z(M; \delta Z)$ can be calculated easily in the
canonical basis:

$$Z(M; \delta Z) = Z(M; \exp(W))$$

$$= \sum_{i=1}^{N} \frac{1}{\lambda^i} Z(M; \phi^i \exp(W))$$

$$= \sum_{i=1}^{N} \frac{1}{\lambda^i} Z(M; \phi^i \sum_{n=0}^{N} \frac{1}{n!} (A \delta Z_i \phi^i)^i)$$

$$= \sum_{i=1}^{N} \frac{1}{\lambda^i} Z^i(M) \exp(A \lambda^i \delta Z_i),$$

(3.16)
where $N$ is the dimension of the physical Hilbert space on $S^{D-1}$, or the number of the irreducible TLFTs of which the TLFT is a direct sum. This is formally equivalent to (2.38) with $p_i = \lambda^i \delta Z_i$, and again the TLFT($\delta Z_i = 0$) is on a multiple first order phase transition point with $N$ different phases around it.

In the previous section, the order parameter is given by the one-sphere expectation value. Analogously, here we consider the following expectation value:

$$\langle 1 \frac{1}{\lambda^k} \phi^k \rangle_{M, \delta Z} \equiv \lim_{A \to \infty} \frac{1}{\lambda^k} \frac{Z(M; \delta Z; B_0(c))\phi^k(c)}{Z(M; \delta Z)}.$$  \hfill (3.17)

Using the discussions of the cluster expansion with boundaries in the previous subsection, we obtain, in the same way as (3.16),

$$Z(M; \delta Z; B_0(c))\phi^k(c) = \sum_{j=1}^N Z(M; (\phi^k + O(\delta Z)^j \phi^j) \exp(W))$$

$$= \sum_{j=1}^N (\delta^k_{j} + O(\delta Z)^k_{j})Z(M; \phi^j)\exp(A\lambda^j \delta Z_j),$$  \hfill (3.18)

where $O(\delta Z)^k_{j}$ comes from the correction to the boundary $B_0$ as in (3.11). Thus, substituting (3.17) with (3.16) and (3.18), we obtain

$$\langle \frac{1}{\lambda^k} \phi^k \rangle_{M, \delta Z} = \lim_{A \to \infty} \frac{1}{\lambda^k} \frac{\sum_{j=1}^N (\delta^k_{j} + O(\delta Z)^k_{j})Z(M; \phi^j)\exp(A\lambda^j \delta Z_j)}{\sum_{j=1}^N \frac{1}{\lambda^k}Z(M; \phi^j)\exp(A\lambda^j \delta Z_j)}$$

$$= \delta^k_m + O(\delta Z)^k_m,$$  \hfill (3.19)

where we assumed that the system is in the $m$-th phase, that is, $\lambda^m \delta Z_m$ has the greatest real part among $\lambda^i \delta Z_i$ ($i = 1, \cdots, N$).

3.3 Correlation lengths of fluctuations associated to trivial local operators

We here discuss the correlation lengths of fluctuations associated to trivial local operators in the cluster expansion and show that they vanish in the limit of TLFT $\delta Z \to 0$. Consider a closed lattice $M$ and make a new lattice $M'$ with two
boundaries $\Sigma_{1,2}$ by taking away some of the $D$-simplices in $M$. We assume that one of the boundaries $\Sigma_1$ is topologically trivial, that is, $\Sigma_1$ can be surrounded by a $S^{D-1}$ in $M'$ and separated from $\Sigma_2$ by the $S^{D-1}$. Thus what we consider here are the fluctuations associated to such a topologically trivial local boundary $\Sigma_1$.

Let $\mathcal{O}_{1,2}$ be the operations to make the colored boundaries $\Sigma_{1,2}(c_{1,2})$, respectively. As was discussed previously, these operators are modified in the cluster expansion. Assume that each operator is modified to $\mathcal{O}'_{1,2}$ respectively. Then the composite operator $\mathcal{O}_1\mathcal{O}_2$ is modified to $\mathcal{O}'_1\mathcal{O}'_2 + \mathcal{O}_3$, where the $\mathcal{O}_3$ is of order $(\delta Z)^d$ when the two boundaries are separated by $d$ lattices. Then the expectation value of the correlation function of the two operators is calculated as

$$
\langle \mathcal{O}_1\mathcal{O}_2 \rangle^{M,\delta Z} \equiv \lim_{A \to \infty} \frac{Z(M; \mathcal{O}_1, \mathcal{O}_2; \delta Z)}{Z(M; \delta Z)} = \lim_{A \to \infty} \left( \frac{Z(M; \mathcal{O}'_1\mathcal{O}'_2 \exp(W))}{Z(M; \exp(W))} + \frac{Z(M; \mathcal{O}_3 \exp(W))}{Z(M; \exp(W))} \right),
$$

where $W$ is that in (3.6). Since the operator $\mathcal{O}_1$ is a trivial operator, the first term can be factorized:

$$
\lim_{A \to \infty} \frac{Z(M; \mathcal{O}'_1\mathcal{O}'_2 \exp(W))}{Z(M; \exp(W))} = \lim_{A \to \infty} \sum_{k=1}^{N} \frac{Z(S^D; \phi_k O'_1)Z(M; \mathcal{O}'_2\phi_k \exp(W))}{\sum_{k=1}^{N} \frac{1}{\lambda} Z(M; \exp(W)\phi^k)}
\quad = \sum_{k=1}^{N} \frac{Z(S^D; \phi_m O'_1)Z(M; \mathcal{O}'_2\phi^m \exp(W))}{\lambda Z(M; \exp(W)\phi^m)},
$$

where we cut $M'$ at the $S^{D-1}$ and used the constraint (2.9), (2.21) with $\Sigma = S^{D-1}$ and (2.33) in the first line, and we assumed that the system is in the $m$-th phase. On the other hand, the expectation values of the one-point function of each operator are

$$
\langle \mathcal{O}_1 \rangle^{M,\delta Z} = \lim_{A \to \infty} \frac{Z(M; \exp(W)\phi^m)Z(S^D; \phi^m O'_1)}{\lambda Z(M; \exp(W)\phi^m)} = \lim_{A \to \infty} \lambda^m Z(S^D; \phi^m O'_1), \quad (3.22)
$$

$$
\langle \mathcal{O}_2 \rangle^{M,\delta Z} = \lim_{A \to \infty} \frac{Z(M; \mathcal{O}'_2 \exp(W)\phi^m)}{Z(M; \exp(W)\phi^m)},
$$

where we have used the same cut as above in the first line. Thus, combining (3.20),

(3.21)
(3.21) and (3.22), we obtain the connected correlation function as
\[
\langle O_1 O_2 \rangle_{c}^{M,\delta Z} = \lim_{A \to \infty} \frac{Z(M; O_3 \exp(W))}{Z(M; \exp(W))} \sim O((\delta Z)^d). 
\]
(3.23)

This concludes that the correlation length is of order \(-1/\ln(\delta Z)\), which vanishes in the limit of TLFT \(\delta Z \to 0\).

3.4 An example — 2D Q-state Potts model

The previous example of 2D lattice QCD cannot be on a multiple first order phase transition point, if we respect the physical constraints. But the 2D Q-state Potts model has a multiple first order phase transition point of a TLFT in the physical parameter space\([13]\).

The Potts spins are located on the 1-simplices, and they interact among nearest neighbors and with external magnetic fields. This model is defined by the following local partition function and gluing tensor:

\[
g^{ij} = \delta^{ij}, \quad C_{ijk} = \kappa W(i, j)W(j, k)W(k, i), \\
W(i, j) = \exp(\beta(\delta_{ij} - 1) + h_i + h_j),
\]

where \(\beta\) is the inverse temperature, \(h_i\)s the external magnetic fields, and the Roman indices run from 1 to \(Q\). As was discussed in the previous example, this model will be a volume-dependent TLFT if the ‘flip invariance’ is satisfied. One can easily show that the necessary and sufficient condition of the invariance is \(\beta = 0\) or \(\infty\)\([13]\).

For \(\beta = \infty\), \(W(i, j) = \delta_{ij} \exp(2h_i)\). To perform the Laplace decomposition, we calculate \(Z(B^2; A; B_0(i, j, k))\), where \(B^2\) denotes a disk and \(B_0(i, j, k)\) a three-gon with coloring \(i, j, k\):

\[
Z(B^2; A; B_0(i, j, k)) = \delta_{ij} \delta_{ik} (\kappa \exp(6h_i)) A = \sum_{\alpha=1}^{Q} \delta^\alpha_i \delta^\alpha_j \delta^\alpha_k (\kappa \exp(6h_\alpha)) A. 
\]

(3.25)
The local weight \(\delta^\alpha_i \delta^\alpha_j \delta^\alpha_k\) trivially defines an irreducible TLFT for each \(\alpha\). Thus the point \(\kappa \exp(6h_\alpha) = 1\) for all \(\alpha\) must be a multiple first order phase transition
point with \( Q \) different phases around it. On the other hand, since, after redefining \( \kappa \) and the magnetic fields properly, this point corresponds to the point of zero temperature and zero magnetic fields of the \( Q \)-state Potts model, it is a multiple first order phase transition point with \( Q \) different phases labeled by the directions of the spin. Thus the two results coincide. The cluster expansion of the partition function near the TLFT has a direct relation to the low temperature expansion of the Potts model. This suggests that the cluster expansion near a TLFT is at best an asymptotic expansion in general.

For \( \beta = 0 \), \( W(i, j) = \exp(h_i + h_j) \). One obtains easily

\[
Z(B^2; A; B_0(i, j, k)) = \kappa(\sum_{l=1}^{Q} \exp(4h_l))^{3/2} (\sum_{l=1}^{Q} \exp(4h_l))^{-3/2} \exp(2(h_i + h_j + h_k)).
\]

(3.26)

Here one can see that the local weight \((\sum_{l=1}^{Q} \exp(4h_l))^{-3/2} \exp(2(h_i + h_j + h_k))\) is of the form \( \varphi_i \varphi_j \varphi_k \) with \( \varphi_i = (\sum_{l=1}^{Q} \exp(4h_l))^{-1/2} \exp(2h_i) \). Hence the local weight can be transformed to the form \( \delta_i \delta_j \delta_k \), and it defines an irreducible TLFT. This concludes that the TLFT at \( \beta = 0 \) and \( \kappa(\sum_{l=1}^{Q} \exp(4h_l))^{3/2} = 1 \) is an irreducible TLFT and is not on a first order phase transition point. On the other hand, this point corresponds to the infinite temperature limit of the Potts model, and hence is not on a first order phase transition point. The cluster expansion near the TLFT corresponds to the high temperature expansion of the Potts model.

4. Neighborhood of three-dimensional \( Z_p \) model

So far we have discussed the physical states on \( S^{D-1} \) of a TLFT and their relations to the phase structure of first order phase transitions. We gave some examples in two dimensions. Since, in two dimensions, a boundary has always the topology of \( S^1 \), all the physical states are those on \( S^1 \). Thus only the structure of first order phase transitions is associated to a two-dimensional TLFT. But how about the physical states on non-trivial boundaries in higher dimensions? We will
not give here any general statements about it. We will discuss it in a simple threedimensional model on a manifold with a certain topology. The method is again the cluster expansion. But here higher orders will play essential roles.

4.1 Definition and physical states

The model we discuss here is a simpler analogue of the model by Turaev and Viro\cite{Tur21}. We consider the $Z_p$ gauge group in place of the quantum groups. (See \cite{Tur22} and \cite{Tur25}.)

Consider a simplicially decomposed three-dimensional manifold $M$ possibly with some boundaries. The degrees of freedom are on its 1-simplices, and they are integers running from 0 to $p - 1$. The coloring of the lattice is performed by the assignment of these integers to all the 1-simplices. The weight for each inner 0-simplex is $1/p$, and that for each 0-simplex on boundaries is $1/\sqrt{p}$. The weight for each 3-simplex is given by the $Z_p$ analogue of $6j$-symbol:

$$\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} = \delta^{(p)}_{j_1+j_2+j_3,0}\delta^{(p)}_{j_4+j_5-j_6,0}\delta^{(p)}_{j_2-j_4+j_6,0}\delta^{(p)}_{j_1+j_4-j_5,0}, \quad (4.1)$$

where $\delta^{(p)}_{ij}$ takes 1 for $i = j \mod p$, but 0 otherwise. The weight (4.1) is a flatness condition for the gauge group $Z_p$, that is, the summation of $j_i$s over the 1-simplices surrounding each 2-simplex should vanish with mod $p$. The partition function is defined as

$$Z(M; \Sigma) = p^{-(v_i+v_b/2)} \sum \prod T \begin{vmatrix} j_1^T & j_2^T & j_3^T \\ j_4^T & j_5^T & j_6^T \end{vmatrix}, \quad (4.2)$$

where $v_i$ and $v_b$ are the total numbers of the inner 0-simplices and those on the boundaries, respectively. The summation is over all the colorings, that is, all the assignments of the $p$ integers to the 1-simplices. $j_i^T$ represents the six integers associated to each 3-simplex and the product is taken over all the 3-simplices in $M$.

All the triangulations of the inside of a three-dimensional manifold can be related by the two local moves (1,4) and (2,3) and their inverse moves\cite{Tur40} (Fig.8).
The partition functions (4.2) are invariant under these moves, and hence they define a TLFT.

The boundary of a 3-manifold is a closed 2-manifold. Let \( B_g \) denote a triangulated surface with genus \( g \). Since (4.2) consists of the products of the flatness conditions, one can show easily that the basis of the physical Hilbert space on \( B_g \) can be labeled by the currents on \( \alpha \) and \( \beta \) cycles of \( B_g \): \( \mathcal{H}_g = \{ \{ j_{\alpha_i} j_{\beta_i} \mid i = 1, \ldots, g \} \mid j_{\alpha_i}, j_{\beta_i} = 0, \ldots, p - 1 \} \). Hence the dimension of the physical Hilbert space is given by \( \dim(\mathcal{H}_g) = p^{2g} \). Especially, the dimension of the physical Hilbert space on the sphere is one, so this system is not on a first order phase transition point.

4.2 Evaluation of the partition function

We will calculate by the cluster expansion the partition function of the system obtained by shifting the local weights by little amounts from those of the \( Z_p \) model. Since the deformed system is not a TLFT anymore, we have to fix the lattice. The lattice we consider is a simplicially decomposed manifold \( M_g \) with the topology of \( S^1 \times B_g \). Since we finally consider the thermo-dynamical limit, we can consider a surface composed of 2-simplices in \( M_g \) which cuts \( M_g \) almost perpendicularly to the direction along \( S^1 \) at any place \( t(\in S^1) \). We assume the followings for the triangulated surface \( t \times B_g \) (Fig.9):

1. The holes are regularly ordered and the length of the circumference of each hole is about \( l_1 \) lattices.
2. The length of the cycle winding through two neighboring holes is about \( l_2 \) lattices.
3. \( l_1 < l_2 \).

In the thermo-dynamical limit, the length of \( S^1 \) and the extent of \( B_g \) will become so large that the partition function of the deformed system can be well approximated by a correlation function of the original TLFT on \( M_g \): \( Z(M_g; \delta Z) = Z(M_g; \exp(W)) \).

The first order of the cluster expansion of \( W \) is the operator taking away one
3-simplex. Thus the ‘operator splitting regularization’ implies

\[ W = Af_0\phi_0, \quad (4.3) \]

where \( \phi_0 \) is the physical state on \( S^2 \), and \( f_0 \) is the order of \( \delta Z \). Let us consider the \( p \)-th order (\( p > 1 \)). The \( p \)-th order of \( W \) comes from the \( p \) 3-simplices forming a cluster. If \( p < l_1 \), the \( p \) 3-simplices can be always surrounded by a \( S^2 \) in \( M_g \). Taking such a \( S^2 \) of a small size, one can define the neighborhood of the cluster as the neighborhood of the ball whose boundary is the \( S^2 \). Then the ‘operator splitting regularization’ projects the operation of taking away the \( p \) 3-simplices to the physical state \( \phi_0 \). Thus the cluster expansions below the \( l_1 \)-th order can be included in the form (4.3).

New operators arise when we consider the \( l_1 \)-th order. There appear the cases that \( l_1 \) 3-simplices form the non-trivial loop surrounding one of the \( g \) holes in \( M_g \). Then the 3-simplices can not be surrounded by a \( S^2 \) in \( M_g \). The simplest definition of the neighborhood of such 3-simplices is to choose a small torus which surrounds the 3-simplices around the hole. Then such operations are projected to the physical states on the torus by the ‘operator splitting regularization’. Thus we obtain

\[ W = Af_0\phi_0 + T \sum_{k=1}^{T} \sum_{i,j=0}^{p-1} f_{ij}^k \phi_{ij}^k, \quad (4.4) \]

where \( T \) is the number of the lattices along \( S^1 \). The \( \phi_{ij}^k \) denotes the physical states on the torus around the \( k \)-th hole in \( M_g \), and is labeled by the two integers along the \( \alpha \) and \( \beta \) cycles of the torus. The \( f_{ij}^k \)'s are the mean values of the coefficients coming from the projections of the operations to \( \phi_{ij}^k \): \( f_{ij}^k = \frac{1}{T} \sum_t f_{ij}^k(t) \). Thus they are the order of \( (\delta Z)^l_1 \). Since, from (2.33), the first term of (4.4) is merely the all over factor of the partition function, we will consider only the second term. To calculate \( Z(M_g; \exp(W)) \), we begin with determining the ‘matrix elements’ of \( \phi_{ij}^k \). Consider a manifold \( T_{B_g} = [0, 1] \times B_g \) with an insertion of the operator \( \phi_{ij}^k \). Then,
using similar discussions to obtain the physical Hilbert space before, the ‘matrix elements’ of \( \Phi_{ij} \) are obtained from the flatness condition as

\[
Z(T_{g^*}; \psi_g^{\{i_j\}}, \Phi_k, \psi_g^{\{i_{j'}\}}) = \delta_{i,i_k} \prod_{l=1}^{g} \delta_{j_l,j_l'} \delta_{h_{l},j'}^{(p)}.
\]

(4.5)

where \( \psi_g^{\{i_j\}} \) and \( \psi_g^{\{i_{j'}\}} \) are the physical states on the boundaries \( B_{g^*} \) and \( B_g \), respectively. Here we took the normalization of the inserted operator \( \Phi_{ij} \) such that there appear no numerical factors in the right-hand side of (4.5). The case that there are two insertions can be considered by taking the square of the matrix (4.5), and we obtain the fusion rule

\[
\Phi_{ij} \Phi_{i'j'} \sim \delta_{i,i'}^{(p)} \Phi_k^{i,j + j'}.
\]

(4.6)

For simplicity, we will use \( \varphi_{ij}^{i_j} \equiv (1/p) \sum_{l=0}^{p-1} \exp(i2\pi j_l/p) \Phi_{ij}^{i_j} \) in place of \( \Phi_{ij} \), where the \( i \) in \( i2\pi \) denotes the imaginary unit, and do not confuse it with the label for the states. Then the fusion rule and \( W \) become

\[
\varphi_{ij}^{i_j} \varphi_k \sim \delta_{i,i'}^{(p)} \delta_{j,j'}^{(p)} \Phi_k^{ij},
\]

\[
W = \sum_{k=1}^{g} \sum_{i,j=0}^{p-1} h_{ij}^{k} \varphi_{ij}^{i_j},
\]

(4.7)

where \( h_{ij}^{k} \)s are the order of \( (\delta Z)^{l_i} \). Then, using the fusion rule (4.7) and \( Z(M_g; \varphi_{i_1j_1}^{i_{j_1}}, \cdots, \varphi_{i_gj_g}^{i_{j_g}}) = 1 \) obtained by taking the trace of (4.5), we obtain

\[
Z(M_g; \exp(W)) = Z(M_g; \prod_{k=1}^{g} \sum_{i,j=0}^{p-1} \varphi_{ij}^{i_j} \exp(Th_{ij}^{k})) = \prod_{k=1}^{g} \sum_{i,j=0}^{p-1} \exp(Th_{ij}^{k}).
\]

(4.8)

Thus the free energy of the system is

\[
F(M_g; \delta Z) = -\sum_{k=1}^{g} \ln(\sum_{i,j=0}^{p-1} \exp(Th_{ij}^{k})).
\]

(4.9)

4.3 Phase structure
Since we assume $\delta Z \ll 1$, if one takes $l_1 \to \infty$ in the thermo-dynamical limit, the system will become uninteresting. Thus we fix $l_1$ and $l_2$ in the thermo-dynamical limit. Since then the number of holes $g$ is proportional to the extent of $B_g$, it is natural to consider the free energy per hole and per lattice along $S^1$:

$$f(\delta Z) = \lim_{T,g \to \infty} \frac{1}{gT} F(M_g; \delta Z) = -\max_{ij}(h_{ij}),$$  

(4.10)

where $\max_{ij}(h_{ij})$ is to take the $h_{ij}$ which has the greatest real part among all the $h_{ij}$ $(i,j = 0, \cdots, p-1)$, and we assumed for simplicity that $h_{ij}^k$ does not depend on the hole: $h_{ij}^k = h_{ij}$. From (4.10), we conclude that the TLFT is on a $l_1$-th order discrete phase transition point with $p^2$ different phases around it.

Comments are in order. Firstly, the number of the different phases around the TLFT would depend on the physical constraints of the system. Secondly, the order of the phase transition $l_1$ is not universal, since $l_1$ depends heavily on the local lattice structures around holes. Thirdly, if $l_1$ is large, the phase transition is very weak and hence the phase structure might become different because of possible large fluctuations. Finally, the order of the phase transitions is $l_1$ only on the point of the TLFT ($\delta Z = 0$) in general. An example is as follows. Suppose the free energy is given by $-\max(x^q y^{l_1-q}, x^{q'} y^{l_1-q'})$ ($q \neq q'$). Then a phase transition line is at $x = y$. But the line is a first order phase transition line except the point $x = y = 0$, where the order is $l_1$.

4.4 Correlation length and order parameter

In the preceding section, we showed that the correlation lengths of the fluctuations associated to a local trivial operator vanish in the limit of TLFT. In this section, we investigate the correlation lengths of the fluctuations associated to local non-trivial operators in the perturbed $Z_p$ model. We will show that the correlation lengths of the fluctuations associated to the operators which wind through two neighboring holes diverge in the limit of TLFT.

We first discuss the correlation lengths of the fluctuations associated to the torus around a hole. Consider a triangulated torus $B_1$ around the $k$-th hole and
an arbitrary boundary \( \Sigma \) obtained by taking away some 3-simplices from \( M_g \). We assume they are separated by \( d \) lattices. We start with evaluating the following partition function in the cluster expansion of the \( l_1 \)-th order:

\[
\lim_{T \to \infty} \sum_{c'} Z(M_g; \Sigma(c), B_1(c'); \delta Z) \varphi^{ij}(c') = \lim_{T \to \infty} \sum_{i',j'=0}^{p-1} Z(M_g; (\Sigma(c) D^{ij}_{k} \varphi^{ij}_{k} + \mathcal{O}_d) \exp(W)) \\
= \lim_{T \to \infty} Z(M_g; (\Sigma(c) D^{ij}_{k} \varphi^{mn}_{k} + \mathcal{O}_d) \exp(W)) \\
= \lim_{T \to \infty} (D^{ij}_{k} \varphi^{mn}_{k} Z(M_g; \delta Z) + Z(M_g; \mathcal{O}_d \exp(W))).
\]

(4.11)

Here the modifications of the operators by the cluster expansion are considered, and \( D^{ij}_{k} \) are the coefficients obtained by projecting the modified \( B_1(c) \) to the \( \varphi^{ij}_{k} \)s. Hence \( D^{ij}_{k} = \delta_{im} \delta_{jn} + O(\delta Z) \). We assumed that, in the limit \( T \to \infty \), \( h^{k}_{mn} \) dominates among \( h^{k}_{ij} \) \((i, j = 0, \cdots, p-1)\), and used \( \exp(W) = \prod_{k=1}^{p} \sum_{i,j=0}^{p-1} \varphi^{ij}_{k} \exp(T h^{k}_{ij}) \) and the fusion rule (4.7) in the last line. The operator \( \mathcal{O}_d \) is the modification of the composite operator like \( W^{\Sigma_{\Sigma_{\Sigma 12}}(c)}_{d} \) in (3.13), and is of order \( (\delta Z)^d \). Substituting (4.11) with \( \Sigma = \emptyset \), we obtain another relation

\[
\lim_{T \to \infty} \sum_{c'} Z(M_g; B_1(c'); \delta Z) \varphi^{ij}(c') = \lim_{T \to \infty} D^{ij}_{k} \varphi^{mn}_{k} Z(M_g; \delta Z),
\]

(4.12)

From (4.11) and (4.12), we obtain

\[
\langle \Sigma(c) \varphi^{ij}_{k} \rangle_{M_g, \delta Z} = \lim_{T \to \infty} \frac{Z(M_g; \Sigma(c), B_1(c'); \delta Z) \varphi^{ij}_{k}(c')}{Z(M_g; \delta Z)}
\]

(4.13)

Hence the correlation lengths of the fluctuations associated to \( B_1 \) are the order of \(-1/\ln \delta Z\), which vanish in the limit of the TLFT \((\delta Z \to 0)\).

From (4.12), one can see that the order parameters are given by the following one-\( \varphi \) expectation values:

\[
\langle \varphi^{ij}_{k} \rangle_{M_g, \delta Z} = D^{ij}_{k} = \delta_{im} \delta_{jn} + O(\delta Z).
\]

(4.14)

Next we consider the fluctuations associated to the torus winding through
two neighboring holes. We denote the physical states associated to the torus by \( \Gamma_{ij}^{kl} \), where \( kl \) labels the two neighboring holes and \( ij \) the physical states (Fig.10). Using the similar discussions as before, the ‘matrix elements’ of these operators are obtained from the flatness condition:

\[
Z(T_{B_g}; \psi_g^{(i_m,j_m)}_{\ast}, \Gamma_{ij}^{kl}, \psi_g^{(i_m,j_m)}) = \delta^{(p)}(j_j - j'_j) \prod_{m=1}^{g} \delta^{(p)}(i_{m,i_{m}+\delta_{km}i_{m} - \delta_{lm}i_{m}}^j, j_{m,j_{m}}^j), \quad (4.15)
\]

where we normalized the \( \Gamma \)s such that there are no numerical factors in the right-hand side of (4.15). For convenience, we will use the operators \( \gamma_{ij}^{kl} = \sum_q \Gamma_{ij}^{kl} \exp(i2\pi qj/p) \) in place of \( \Gamma \).

One has to take into account the deformations of the operators in the cluster expansion. But, since the main contribution comes from the original operators, we will neglect the deformations. This simplification does not change the conclusions concerning the correlation lengths.

By using the fusion rule (4.7) and taking the trace of the product of the matrices (4.5) and (4.15), the one-\( \gamma \) function is evaluated as

\[
Z(M_{g}; \gamma_{ij}^{kl}, \delta Z) = Z(M_{g}; \gamma_{ij}^{kl} \exp(W)) = \sum_{\{i_m,j_m\}} Z(T_{B_g}; \psi_g^{(i_m,j_m)}_{\ast}, \gamma_{ij}^{kl} \exp(W), \psi_g^{(i_m,j_m)}) \\
= \delta^{(p)}(j_j^0) Z(M_{g}; \delta Z). \quad (4.16)
\]

Thus the one-\( \gamma \) expectation value is

\[
\langle \gamma_{ij}^{kl} \rangle_{M_{g},\delta Z} = \delta^{(p)}(j_j^0). \quad (4.17)
\]

We next consider the two-\( \gamma \) function. We insert two \( \gamma \)s separated by \( t \) lattices in the direction along \( S^1 \). As same as above, the two-\( \gamma \) function is calculated
straightforwardly as

\[
Z(M_g; \gamma_{kl}^i(t), \gamma_{kl'}^j(0); \delta Z) = \sum_{\{i_m, j_m\}\{i'_m, j'_m\}} Z(T_{B_g}; \psi_g^{\{i_m, j_m\}*}, \exp(W(T - t))\gamma_{kl}^i, \psi_g^{\{i'_m, j'_m\}}) \\
\times Z(T_{B_g}; \psi_g^{\{i'_m, j'_m\}*}, \exp(W(t))\gamma_{kl'}^j, \psi_g^{\{i_m, j_m\}}) \\
p^{-2g} \sum_{\{i_m, j_m\}\{i'_m, j'_m\}} \exp(i2\pi (j'_{k'} - j'_{l'}) + j(j_k - j_l)/p)) \\
\times \prod_{m=1}^g \delta^{(p)}_{i_m', i_m + \delta_{m_k'}i' - \delta_{m_l'}i'} \delta^{(p)}_{i_m, i_m' + \delta_{m_k}i - \delta_{m_l}i} H(t; m; i_m, j_m - j_m').
\]

where \(W(T - t)\) and \(W(t)\) are the short expressions for the \(W_s\) for the intervals \([t, T]\) and \([0, t]\), respectively. We assumed for simplicity that the \(h_{ij}^k\)s at \([0, t]\) and \([t, T]\) are equal, and the \(H\) is defined as \(H(t; m; i, j) \equiv \sum_{k=0}^{p-1} \exp(th_{ik} + i2\pi kj/p)\).

To evaluate (4.18) further, we will consider two cases separately. First consider the case \(k = k', l = l'\). Then the Kronecker’s deltas in (4.18) requires \(i = -i'\), \(j = -j'\), and we obtain

\[
Z(M_g; \gamma_{kl}^i(t), \gamma_{kl'}^j(0); \delta Z) = Z(M_g; \delta Z)\delta^{(p)}_{i+i', 0} \delta^{(p)}_{j+j', 0} \left( \sum_{r,s=0}^{p-1} \exp(T h_{rs}^k) \right)^{-1} \left( \sum_{r,s=0}^{p-1} \exp(T h_{rs}^l) \right)^{-1} \\
\times \left( \sum_{r,s=0}^{p-1} \exp(th_{r-s,0}^k + (T - t)h_{r-s,j}^k) \right) \left( \sum_{r,s=0}^{p-1} \exp(th_{r+s,0}^l + (T - t)h_{r+s,j}^l) \right).
\]

In the thermo-dynamical limit \(T \to \infty\), we fix the separation \(t\). Suppose \(h_{mk,nk}^k\) and \(h_{ml,nl}^l\) have the largest real values among \(h_{rs}^k\) and \(h_{rs}^l\) \((r, s = 0, \ldots, p - 1)\), respectively. Then we obtain in the thermo-dynamical limit

\[
\langle \gamma_{kl}^i(t) \gamma_{kl'}^j(0) \rangle = \delta^{(p)}_{i+i', 0} \delta^{(p)}_{j+j', 0} \exp(-t(h_{mk,nk}^k + h_{ml,nl}^l - h_{mk-i,nk+j}^k - h_{ml+i,nl-j}^l)).
\]

Combining (4.20) and (4.17), one can see that the correlation lengths associated
to the $\gamma_{kl}^{ij} ((i, j) \neq (0, 0))$ are of order $(\delta Z)^{-l_1}$. Then, in the limit of TLFT, these correlation lengths diverge.

Next consider the case $(k, l) \neq (k', l')$. Since then the Kronecker’s deltas in (4.18) restrict $i$ and $i'$ to be zero, we obtain

$$Z(M_g; \gamma_{kl}^{ij}(t), \gamma_{k'l'}^{i'j'}(0); \delta Z) = \delta_{i,0}^{(p)} \delta_{j,0}^{(p)} \delta_{i',0}^{(p)} \delta_{j',0}^{(p)} Z(M_g; \delta Z). \quad (4.21)$$

Hence,

$$\langle \gamma_{kl}^{ij} \gamma_{k'l'}^{i'j'} \rangle_{M_g, \delta Z} = \langle \gamma_{kl}^{ij} \rangle_{M_g, \delta Z} \langle \gamma_{k'l'}^{i'j'} \rangle_{M_g, \delta Z}. \quad (4.22)$$

This implies that the non-zero contributions to the connected two $\gamma$ functions of $(k, l) \neq (k', l')$ will come only from the cluster deformations of the composite operators, and hence are of order $(\delta Z)^d$ if the two $\gamma$s are separated by $d$ lattices in the direction perpendicular to the $S^1$. Thus the correlation lengths to that direction are of order $-1/\ln \delta Z$, which vanish in the limit of the TLFT.

We have obtained that, in the limit of the TLFT, the fluctuations associated to the operator $\gamma$ have the infinite correlation lengths in the direction along $S^1$, but have zero correlation lengths in the directions perpendicular to $S^1$. The fluctuations propagate only in the one-dimensional direction along the holes. We could not find any local fluctuations which propagate in all directions in $M_g$ in the perturbed $Z_p$ model.
5. Summary, comments and discussions

We investigated the thermo-dynamical natures near TLFTs, and found that they are in general on discrete phase transition points, and that they are on fixed points of renormalization group transformations at least in a restricted sense.

First we discussed the decomposition of a volume-dependent TLFT satisfying a certain condition and showed that it is a direct sum of irreducible TLFTs with volume-dependent numerical factors. In the parameter space of the volume-dependent TLFTs we can discuss the phase structure and the renormalization group flow exactly, and showed that a TLFT is on a fixed point of the flow and in general on a multiple first order phase transition point. The number of the different phases around a TLFT is equal to the number of the irreducible TLFTs of which the TLFT is a direct sum.

To generalize the discussion out of the parameter space of the volume-dependent TLFTs, we introduced a kind of cluster expansion. By this perturbative scheme we found the same phase structure around a TLFT in the first order of the cluster expansion. Higher orders do not change the result qualitatively provided the base manifold is trivial.

To specify the roles of the non-trivial topology of the base manifold and the physical states on non-trivial $(D - 1)$-dimensional boundaries, we investigated the neighborhood of the $Z_p$ analogue of Turaev-Viro model in three dimensions. Then we found another phase structure, that is, the TLFT is on a higher order discrete phase transition point controlled by the physical states on boundaries with non-trivial topologies.

We also studied the correlation lengths of fluctuations associated to various boundaries in the cluster expansion near TLFTs. The correlation lengths of the fluctuations associated to trivial local operators were shown to be zero in the limit of TLFT. On the other hand, in the $Z_p$ model, we found the fluctuations whose correlation lengths diverge in the limit of TLFT. But these fluctuations propagate
only along the topological defects. In this sense, a TLFT is on a point where only the fluctuations propagating along topological defects remain and have infinite correlation lengths.

Comments are in order. We restricted our discussions in the cases that the numbers of the elements of the index sets $X_i$ are finite. This restriction makes the numerical factors of the three-sphere functions $\lambda_i$ to be non-zero, and then we could obtain the decomposition theorem of a TLFT. But some continuum formulations of TFTs might allow $\lambda$ to be zero. The twisted $N = 2$ Landau-Ginzburg model \cite{45} with a certain super-potential is such a case. The TLFT corresponding to the model is discussed in the paper by Fukuma, Hosono and Kawai \cite{15}, where a certain infinite limit must be taken to show the correspondence. Moreover it was pointed out \cite{25,34} that the zero-coupling limit of 3D lattice QCD corresponds to the Ponzano-Regge model. Can we apply our arguments around the Ponzano-Regge model? It would be too stupid to argue that there are no local long-range fluctuations just around the zero-coupling limit of the 3D lattice QCD. Thus we insist that our discussions are applicable only to TLFTs with finite degrees of freedom, and new discussions are needed for the other types of TLFTs and TFTs.

As we have shown, the physical states on $S^{D-1}$ of a TLFT label the phases around a multiple first order phase transition point. The continuum correspondences of the physical states on $S^{D-1}$ would be scalar observables of TFTs. Thus one could expect that such scalar observables would have special roles in the dynamics of TFTs such as ‘spontaneous symmetry breaking’. Such discussions were done by Kawamoto and Watabiki \cite{46} in their generalized Chern-Simons theory, which contains fields of forms of all orders. Here we cannot comment anything about it, since the applicability of our results is obscure as we mentioned in the last paragraph. Relating TLFTs to TFTs and developing analogous discussions for TFTs remain as future problems.

The thermo-dynamical natures near a fixed point of a renormalization group transformation on a continuous phase transition surface can be well analyzed by
the renormalization group technique and renormalizable field theories. This is because of the universality, or because some finite number of interacting long-range modes with some finite number of couplings control the system essentially. How about discrete phase transitions? In general we cannot expect universalities around such phase transition surfaces, so predictions are difficult. But when we analyzed the thermo-dynamical natures near TLFTs by the cluster expansion, the deviations could be projected to the physical states of TLFTs, and we could obtain some qualitative natures near TLFTs. In this sense, the thermo-dynamical natures near TLFTs are described by the physical observables of TLFTs. If there were a map to transform any point near a discrete phase transition surface to the neighborhood of a TLFT, one could talk about general discrete phase transitions. But the existence of such a map is obscure. Some difficulties of the renormalization group technique near a first order phase transition surface were pointed out\cite{17}. We can define a renormalization group flow in the parameter space of the volume-dependent TLFTs, but will have to overcome difficulties in the outside. TLFTs can generate various systems with discrete phase transitions, but whether they are useful in the investigations of discrete phase transitions is not clear.

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REFERENCES

1. E. Witten, *Commun. Math. Phys.* **117** (1988), 353.

2. M. Atiyah, *Publ. Math. IHES* **68** (1989), 175.

3. E. Witten, *Nucl. Phys.* **B340** (1990), 281.

4. E. Witten, *Nucl. Phys.* **B311** (1988), 46.

5. G. Moore and N. Seiberg, *Phys. Lett.* **B220** (1989), 422.

6. E. Witten, *Nucl. Phys.* **B322** (1989), 629; E. Witten, *Nucl. Phys.* **B330** (1990), 285.

7. E. Witten, *Commun. Math. Phys.* **121** (1989), 351.

8. E. Witten, *Commun. Math. Phys.* **117** (1988), 353.

9. J.F. Wheater, *Phys. Lett.* **B223** (1989), 451.

10. J.F. Wheater, *Phys. Lett.* **B264** (1991), 161.

11. T. Jönsson, *Phys. Lett.* **B265** (1991), 141.

12. T. Jönsson, *Nucl. Phys. (Proc. Suppl.)* **B 25A** (1992), 176.

13. C. Bachas and P.M.S. Petropoulos, *Commun. Math. Phys.* **152** (1993), 191.

14. M. Bordemann, T. Filk and C. Nowak, “Topological actions on 2-dimensional graphs and (graded) metrised algebras”, preprint THEP 92/18.

15. M. Fukuma, S. Hosono and H. Kawai, “Lattice topological field theory in two-dimensions”, preprint CLNS 92/1173.

16. B. Durhuus and T. Jönsson, “Classification and construction of unitary topological field theories in two dimensions”, preprint COPENHAGEN-MI-14-1993.

17. A. Migdal, *Sov. Phys. J.E.T.P.* **42** (1975), 413.

18. T. Eguchi and S.K. Yang, *Mod. Phys. Lett.* **A5** (1991), 1693.
19. G. Ponzano and T. Regge, in *Spectroscopic and Group Theoretical Methods in Physics*, ed. F. Bloch (North-Holland, Amsterdam, 1968).

20. T. Regge, *Nuovo Cimento* 19 (1961), 558.

21. V.G. Turaev and O.Y. Viro, *Topology* 31 (1992), 865.

22. R. Dijkgraaf and E. Witten, *Commun. Math. Phys.* 129 (1990), 393.

23. B. Durhuus, H. Jacobsen and R. Nest, *Nucl. Phys.(Proc. Suppl.)* 25A (1992), 109; B. Durhuus, H. Jacobsen and R. Nest, *Rev. Math. Phys.* 5 (1993), 1.

24. H. Ooguri and N. Sasakura, *Mod. Phys. Lett* A6 (1991), 3591.

25. D.V. Boulatov, *Mod. Phys. Lett* A7 (1992), 1629.

26. F. Archer and R.M. Williams, *Phys. Lett.* B273 (1991), 438.

27. S. Mizoguchi and T. Tada, *Phys. Rev. Lett.* 68 (1992), 1795.

28. F.J. Archer, *Phys. Lett.* B295 (1992), 199.

29. G. Felder and O. Grandjean, “On combinatorial three-manifold invariants”, ETH preprint (1992).

30. S. Mizoguchi, *Int. J. Mod. Phys.* A8 (1993), 3909.

31. H. Ooguri, *Prog. Theor. Phys.* 89 (1993), 1.

32. V.G. Turaev, *C.R. Acad. Sci. Paris* 313 (1991), 395; V.G. Turaev, *J. Diff. Geom.* 36 (1992), 35.

33. H. Ooguri, *Nucl. Phys.* B382 (1992), 276.

34. S. Chung, M. Fukuma and A. Shapere, “Structure of topological lattice field theories in three dimensions”, preprint CLNS 93/1200.

35. H. Ooguri, *Mod. Phys. Lett* A7 (1992), 2799.

36. A.S. Schwarz, *Lett. Math. Phys.* 2 (1978), 247; G.T. Horowitz, *Commum. Math. Phys.* 125 (1989), 417.

37. N. Sasakura, *Phys. Lett.* B316 (1993), 329.
38. B.Ye. Rusakov, *Mod. Phys. Lett* **A5** (1990), 693.

39. J.W. Alexander, *Ann. Math.* **31** (1930), 292.

40. M. Gross and S. Varsted, *Nucl. Phys.* **B378** (1992), 367.

41. E. Verlinde, *Nucl. Phys.* **B300** (1988), 360.

42. D.V. Boulatonov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Nucl. Phys.* **B275** (1986), 641.

43. See, for example, C. Itzykson and J.M. Drouffe, *Statistical field theory*, Cambridge (1989), vol.2, chapter 7.

44. B. Nienhuis and M. Nauenberg, *Phys. Rev. Lett.* **35** (1975), 477.

45. For review see, R. Dijkgraaf, H. Verlinde and E. Verlinde, “Notes on Topological String Theory and 2D Quantum Gravity”, Trieste Spring School 1990: 91-156;
R. Dijkgraaf, “Intersection Theory, Integrable Hierarchies and Topological Field Theory”, NATO ASI: Cargese 1991: 95-158.

46. N. Kawamoto and Y. Watabiki, *Phys. Rev.* **D45** (1992), 605; N. Kawamoto and Y. Watabiki, *Nucl. Phys.* **B396** (1993), 326.

47. See, for example, A.C.D. van Enter, R. Fernandez and A.D. Sokal, *Phys. Rev. Lett.* **66** (1991), 3253, and references therein.
FIGURE CAPTIONS

1) Two lattices $M_1$ and $M_2$ with boundaries are glued at sub-boundaries $b$.

2) A lattice $M_2$ is obtained by identifying sub-boundaries $b$ of $M_1$.

3) Two lattices with $\partial T^D$ as their simplicially decomposed boundaries are glued at one of their $T^{D-1}$s of each boundary. This figure is for $D = 3$.

4) The 2D lattice QCD is invariant under the flip move, and hence is an area-dependent TLFT.

5) If the $\delta Z$s come from the $D$-simplices shown in the figure, to estimate the contributions to the partition function, one has to use correlation functions of the TLFT on manifolds different from $M$.

6) If the set $Q$ is separated from the boundaries and the compliment $\bar{Q}$ near the boundaries, then $W_n^\Sigma$ is zero.

7) If the two boundaries are separated, the lower order terms of $W_n^{\Sigma_{12}}$ factorize.

8) The (1,4) and (2,3) moves of the simplicial decomposition and their inverses. In three dimensions, all the simplicial decompositions of a manifold can be generated by these moves.

9) The geometrical condition of $t \times B_g$ ($t \in S^1$).

10) $\Gamma_{kl}^{ij}$ is a physical state on the torus winding through the two neighboring holes $k$ and $l$. 
This figure "fig1-1.png" is available in "png" format from:

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