Charged particle trajectories in a toroidal magnetic and rotation-induced electric field around a black hole

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Abstract

Trajectories of charged particle in combined poloidal, toroidal magnetic field and rotation-induced unipolar electric field superposed in Schwarzschild background geometry have been investigated extensively in the context of accreting black holes. The main purpose of the paper is to obtain a reasonably well insight on the effect of spacetime curvature to the electromagnetic field surrounding black holes. The coupled equations of motion have been solved numerically and the results have been compared with that for flat spacetime. It is found that the toroidal magnetic field dominates the induced electric field in determining the motion of charged particles in curved spacetime. The combined electromagnetic field repels a charged particle from the vicinity of a compact massive object and deconfines the particle from its orbit. In the absence of toroidal magnetic field the particle is trapped in a closed orbit. The major role of gravitation is to reduce the radius of gyration significantly while the electric field provides an additional force perpendicular to the circular orbit. Although the effect of inertial frame dragging and the effect of magnetospheric plasma have been neglected, the results provide a reasonably well qualitative picture of the important role played by gravitation in modifying the electromagnetic field near accreting black holes and hence the results have potentially important implications on the dynamics of the fluid and the radiation spectrum associated with accreting black holes.

Key words: magnetic fields - relativity - quasars - stars: general
1. INTRODUCTION

From the nearest stellar object Sun to the furthest cosmological object quasar, the presence of the magnetic field effectively governs not only various physical phenomenon but also the evolution of many astrophysical objects.

The influence of the magnetic field on the dynamical flow of the infalling plasma around a compact object and on the emergent radiation spectrum have long been realized in the theoretical study of very high energy astrophysical objects like pulsars, quasars, Active Galactic Nuclei (AGN) and X-ray binaries. Unfortunately no convincing theory has been well established so far which can explain the radiation from these sources consistently and correctly. Though the definite mechanism has not been understood properly, it is generally believed that the radiation emission could be due to the plasma processes near compact objects like neutron stars and black holes. Thus it becomes necessary to consider plasma processes in intense-gravitational-field backgrounds, as such compact objects would be massive enough to produce significant spacetime curvature effects. Therefore one has to consider the dynamics of the fluid in curved spacetime which is quite an involved problem since one requires to solve the general relativistic magnetohydrodynamics equations.

It is well known that the best way to understand the structure of any field is to study the dynamics of test particles in that field. If a particle is charged then it deviates from its geodesic motion and the study of such trajectories would reveal informations about the influence of the interacting field on the spacetime geometry and vice-versa. The most important in the astrophysical context is the electromagnetic field. In an electromagnetic field a charged particle gyrates with a definite radius of gyration and that by and large determines the radiation spectrum of the object. The concept of radius of gyration in classical mechanics is modified when one considers curved spacetime\(^1\). Hence it is very important to investigate the trajectory of charged particles in curved spacetime with different types of
electromagnetic field configuration.

For this purpose one should look for solutions of Einstein-Maxwell equations which are asymptotically flat and have non-zero dipole magnetic field even in the absence of rotation. Although these systems of equations are formidable to solve in general, there are some solutions obtained by perturbation technique under the assumption that the electromagnetic field is weak compared to the gravitational field and thus it does not affect the basic geometry. This is achieved essentially by solving the Maxwell equations on a given curved spacetime background geometry. This is indeed a valid assumption for even the most intense magnetic field associated with pulsars carries an energy which is very small compared to the gravitational potential energy on the surface of a neutron star of one solar mass. Therefore it is considered that the magnetic field would not affect the spacetime curvature but the curvature could affect the magnetic field.

With such a consideration Ginzburg and Ozernoi\textsuperscript{2}, Petterson\textsuperscript{3}, Bicak and Dvorak\textsuperscript{4} have obtained solutions for dipole magnetic field on Schwarzschild background, whereas using a similar approach Chitre and Vishveshwara\textsuperscript{5}, Petterson\textsuperscript{6} have obtained solutions for stationary electromagnetic fields on Kerr background.

Charged particle dynamics in such electromagnetic fields have been extensively studied by Prasanna and Varma\textsuperscript{7} for the Schwarzschild background, by Prasanna and Vishveshwara\textsuperscript{8} and by Prasanna and Chakraborty\textsuperscript{9} for the Kerr background.

Prasanna and Sengupta\textsuperscript{10} have obtained trajectories of single charged particle in the presence of toroidal magnetic field superposed on the Schwarzschild background geometry. In that work it is shown that the toroidal component of the magnetic field repels the particles away and particularly the incoming ones get ejected in jet-like trajectories. Hence it has been pointed out that to produce jet like features from the accreting matter in the high energy astrophysical scenarios, it is very essential to have the toroidal component of the magnetic
field associated with the central gravitating body.

It is known that almost all of the celestial bodies do have a non-zero angular momentum and therefore one must consider the important role played by the angular velocity of the central object in determining not only the spacetime geometry exterior to it but also the electromagnetic field configurations. For this purpose Kerr metric has been used in the earlier works and the motion of the particle in the equatorial plane has been investigated. Since the exterior region in these studies is considered to be absolutely vacuum, the electric field induced by rotation of the object is obtained to be quadrupolar in nature. However this cannot be a realistic situation as it is well known that in the case of pulsar magnetosphere and in the accretion disc the region in which the motion of a charged particle is considered, are plasma filled. As a consequence Ohm’s law should be taken into consideration within the entire region while determining the electric field. Due to the Ohmic induction a rotation-induced unipolar electric field must be present in that region if one neglects the higher multipole by considering comparatively slow rotation.

In the present paper I have presented the trajectories of a charged particle under the combined poloidal, toroidal magnetic field and the rotation-induced unipolar electric field superposed in Schwarzschild background geometry. This is an extension of the work by Prasanna and Sengupta\textsuperscript{10}. The main purpose of the paper is to obtain a qualitative picture of the role played by toroidal magnetic field in curved spacetime in the vicinity black holes.

2. THE ELECTROMAGNETIC FIELD IN CURVED SPACETIME

2.1. The spacetime metric

Since we are dealing with the exterior geometry of black holes we need the vacuum solution to Einstein’s field equations as the self gravitational field of the surrounding plasma is negligible.

For a rotating compact object one may consider the Kerr metric which has the following
form

\[ ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} c^2 dt^2 - 2a^2 \frac{2mr \sin^2 \theta}{\rho^2} c dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2} \sin^2 \theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \]  

(1)

where

\[ \Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad a = J/m, \]

\( J \) being the total angular momentum and \( m = \frac{M G}{c^2} \), where \( M \) is the total gravitational mass of the object.

For the case of a black hole the effect due to the rapid rotation upon the spacetime curvature could be important and the Kerr metric may be more suitable. Nevertheless if one considers black holes with \( \frac{2m}{R} \gg \frac{a}{R} \) one can adopt the Schwarzschild metric, for small \( \frac{a}{R} \), the Kerr metric reduces to the Schwarzschild metric and the problem becomes much easier to tackle with under this approximation. In the present investigation we consider sufficiently slow rotation of the object so that the effect of rotation on the background geometry is insignificant. Hence we consider Schwarzschild background geometry. The investigation with Kerr background geometry is under progress.

The Schwarzschild metric is given by :

\[ ds^2 = -(1 - \frac{2m}{r})c^2 dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \]  

(2)

where \( m = \frac{M G}{c^2} \), \( M \) being the total gravitational mass of the object.

2.2. The magnetic field

The vector potential for the poloidal magnetic field in Schwarzschild background geometry is given by Petterson\(^3\), Prasanna and Varma\(^7\), Wasserman and Shapiro\(^1^1\) as

\[ A_\phi = -\frac{3\mu \sin^2 \theta}{8m^3} [r^2 \ln(1 - \frac{2m}{r}) + 2mr + 2m^2]. \]  

(3)
Using the definition \( F_{ij} = (A_{ji} - A_{ij}) \) we get the components of the magnetic field in Schwarzschild background geometry as

\[
F_{\phi r} = \frac{3\mu \sin^2 \theta}{4m^2} \left[ \frac{r}{m} \ln(1 - \frac{2m}{r}) + (1 - \frac{2m}{r})^{-1} + 1 \right], 
\]

\[
F_{\phi \theta} = \frac{3\mu r^2 \sin \theta \cos \theta}{4m^3} \left[ \ln(1 - \frac{2m}{r}) + \frac{2m}{r}(1 + \frac{m}{r}) \right], 
\]

(4)

(5)

\( \mu \) being the dipole moment associated with the central source.

In magnetized accretion disks around a compact object the surrounding plasma is supposed to be tied to the magnetic field and its inertia would cause the magnetic field lines to be bent backward creating a toroidal component. The toroidal field component in Schwarzschild background geometry is given as\(^{10}\)

\[
F_{r\theta} = \frac{rB_\circ(r_\circ - 2m)}{r_\circ(r - 2m) \sin \theta}, 
\]

(6)

\( r_\circ \) and \( B_\circ \) being constants.

### 2.3. The induced electric field

We consider the electromagnetic field to be stationary. For this we require the magnetic dipole axis to be aligned with the rotational axis of the central object. In the present study we ignore any effects due to the surrounding plasma. Since the surrounding medium exterior to the object is considered to be plasma filled, there will be a rotation-induced unipolar electric field such that \( \mathbf{E} \cdot \mathbf{B} = 0 \) everywhere. The components of the induced electric field considered here are obtained by using the generalized Ohm’s law (assuming infinite conductivity of the medium)

\[
F_{\delta}^\gamma u^\delta = 0, 
\]

where \( u^\delta = (u^t, 0, 0, u^\phi) \) are the components of the four velocity vector of the corotating plasma.
In Schwarzschild background geometry the components of the induced unipolar electric field are given as\(^{12}\)

\[
F_{rt} = \frac{3\mu \omega \sin^2 \theta}{4m^2c} \left[ \frac{r}{m} \ln(1 - \frac{2m}{r}) + (1 - \frac{2m}{r})^{-1} + 1 \right],
\]

\[(7)\]

\[
F_{\theta t} = \frac{3\mu \omega r^2}{4m^3c} \sin \theta \cos \theta \left[ \ln(1 - \frac{2m}{r}) + \frac{2m}{r}(1 + \frac{m}{r}) \right],
\]

\[(8)\]

where \(\omega\) is the angular velocity of the central object as measured by an observer at infinity.

From equations (7) and (8) we obtain the potential for the electric field as

\[
A_t = \frac{3\mu \omega r^2 \sin^2 \theta}{8m^3c} \left[ \ln(1 - \frac{2m}{r}) + \frac{2m}{r}(1 + \frac{m}{r}) \right].
\]

\[(9)\]

The electric field arises due to the corotation of the plasma with the object and is contributed by the poloidal magnetic field. The toroidal magnetic field has no contribution to the electric field as can be seen from the generalized Ohm’s law.

Therefore in the present discussion we are considering a region with combined poloidal and toroidal magnetic field and rotation-induced unipolar electric field superposed on Schwarzschild background geometry.

3. THE EQUATIONS OF MOTION OF A CHARGED PARTICLE

3.1. In Schwarzschild background geometry

For the motion of a charged particle of charge \(e\) and rest mass \(m_\circ\) there are two constants of motion, the canonical angular momentum \(l\) and the energy \(E\) as given by

\[
v_\phi + eA_\phi = l,
\]

\[(10)\]

\[
v_t + eA_t = -E,
\]

\[(11)\]

wherein all quantities are normalized with respect to the particle rest mass \(m_\circ\). The velocity four vector of the charge particle, \(v^i\) is timelike, which for the metric of signature +2 is
expressed as

\[ g_{ij}v^iv^j = -1. \]

From equation (10) we get

\[ v^\phi = \frac{d\phi}{ds} = \frac{l}{r^2 \sin^2 \theta} + \frac{3e\mu}{8m_o c^2 m^3} [\ln(1 - \frac{2m}{r}) + \frac{2m}{r}(1 + \frac{m}{r})]. \]  (12)

From equation (11) we get

\[ v^t = \frac{cdt}{ds} = (1 - \frac{2m}{r})^{-1} \{ E + \frac{3e\mu \omega r^2 \sin^2 \theta}{8m^3 c^3 m_o} [\ln(1 - \frac{2m}{r}) + \frac{2m}{r}(1 + \frac{m}{r})] \}. \]  (13)

From the covariant Lorentz equations

\[ v^i_jv^j = eF^i_jv^j, \]

where \( v^i = \frac{dx^i}{ds} \) is the four velocity of the particle and the semicolon denotes the covariant derivative with respect to the spacetime metric, we obtain

\[ \frac{d^2r}{ds^2} = \frac{m}{r^2}(1 - \frac{2m}{r})^{-1}(\frac{dr}{ds})^2 + r(1 - \frac{2m}{r}) \left( \left( \frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{ds} \right)^2 \right) - \]
\[ \frac{m}{r^2}(1 - \frac{2m}{r}) \left( \frac{cdt}{ds} \right)^2 + \frac{e}{m_o c^2} (1 - \frac{2m}{r}) \left\{ \frac{rB_o (r - 2m)}{r_o (r - 2m) \sin \theta} \frac{d\theta}{ds} - \right\} \]
\[ \frac{3\mu \sin^2 \theta}{2m^2} \left[ (1 - \frac{m}{r}) + \left( \frac{r}{2m} - 1 \right) \ln(1 - \frac{2m}{r}) \right] (1 - \frac{2m}{r})^{-1} \left( \frac{d\phi}{ds} - \omega \frac{dt}{ds} \right), \]  (14)

\[ \frac{d^2\theta}{ds^2} = -\frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} + \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 - \frac{e}{m_o c^2 r^2} \left[ \frac{rB_o (r - 2m)}{r_o (r - 2m) \sin \theta} \right] \frac{dr}{ds} - \]
\[ \frac{3\mu \sin \theta \cos \theta}{4m^3} \left\{ r^2 \ln(1 - \frac{2m}{r}) + 2m(r + m) \right\} \left( \omega \frac{dt}{ds} - \frac{d\phi}{ds} \right). \]  (15)

3.2. In flat spacetime

In order to investigate the effects of spacetime curvature we need to determine the trajectories of the charge particle in flat spacetime as well. These will provide the motion of a particle in the Newtonian mechanics under various electromagnetic conditions. However in the present discussion we neglect the toroidal magnetic field component for flat spacetime
since the poloidal magnetic and the induced electric fields are sufficient for a comparative study of the particle motion in curved and flat spacetimes.

Expanding in Taylor series and neglecting the higher order terms containing $m$ we obtain from equations (12) to (15) the equations of motion of a charge particle in electromagnetic field without gravitation as

$$\frac{d^2 r}{ds^2} = r\left\{ \left( \frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{ds} \right)^2 \right\} - \frac{e\mu \sin^2 \theta}{m_c^2 r^2} \left( \frac{d\phi}{ds} - \frac{\omega}{ds} \right),$$  

(16)

$$\frac{d^2 \theta}{ds^2} = -\frac{2}{r} \frac{d\rho}{ds} \frac{d\theta}{ds} + \sin \theta \cos \theta \left( \frac{d\phi}{ds} \right)^2 - \frac{2e\mu \sin \theta \cos \theta}{m_c^2 r^3} \left( \omega \frac{dt}{ds} - \frac{d\phi}{ds} \right),$$  

(17)

$$\frac{d\phi}{ds} = \frac{l}{r^2 \sin^2 \theta} - \frac{e\mu}{m_c^2 r^3},$$  

(18)

$$\frac{cdt}{ds} = E - \frac{e\mu \omega \sin^2 \theta}{m_c^2 r^3}.$$  

(19)

4. NUMERICAL SOLUTIONS

As the orbit equations involve transcendental functions it is analytically impossible to make any general analysis. Hence we restore to numerical integration and get the qualitative picture of the nature of trajectories for different values of the physical parameters appearing in the equations. For convenience of calculations we shall consider the equations in dimensionless form by introducing the dimensionless quantities

$$\rho = \frac{r}{m}, \sigma = \frac{s}{m}, L = \frac{l}{m}, \tau = \frac{ct}{m}, W = \frac{m\omega}{c}, \lambda_T = \frac{eB_0}{m_c^2}, \lambda_P = \frac{e\mu}{m_c^2 m^2}.$$  

With this definition the equations of motion for curved spacetime read as

$$\frac{d^2 \rho}{d\sigma^2} = \frac{1}{\rho^2} \left( 1 - \frac{2}{\rho} \right)^{-1} \left( \frac{d\rho}{d\sigma} \right)^2 + \rho \left( 1 - \frac{2}{\rho} \right) \left\{ \left( \frac{d\theta}{d\sigma} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\sigma} \right)^2 \right\}$$

$$- \frac{1}{\rho^2} \left( 1 - \frac{2}{\rho} \right)^2 \left( \frac{d\tau}{d\sigma} \right)^2 + \frac{\Lambda \lambda_P (1 - \frac{2}{\rho_0})}{\sin \theta} \frac{d\theta}{d\sigma} - \frac{3\lambda_P}{2} \sin^2 \theta \left\{ (1 - \frac{1}{\rho}) + (\frac{\rho}{2} - 1) \ln (1 - \frac{2}{\rho}) \right\} \left( \frac{d\phi}{d\sigma} - W \frac{d\tau}{d\sigma} \right),$$  

(20)
\[
\frac{d^2 \theta}{d\sigma^2} = -\frac{2}{\rho} \frac{d\rho}{d\sigma} \frac{d\theta}{d\sigma} + \sin \theta \cos \theta \left( \frac{d\phi}{d\sigma} \right)^2 - \frac{\Lambda \lambda_P \left( 1 - \frac{2}{\rho} \right)}{\rho^2 \left( 1 - \frac{2}{\rho} \right) \sin \theta} \frac{d\rho}{d\sigma} \\
+ \frac{3}{4} \lambda_P \sin \theta \cos \theta \{ \ln \left( 1 - \frac{2}{\rho} \right) + \frac{2}{\rho} \left( 1 + \frac{1}{\rho} \right) \} \left( W \frac{d\tau}{d\sigma} - \frac{d\phi}{d\sigma} \right), \quad (21)
\]

\[
\frac{d\phi}{d\sigma} = \frac{L}{\rho^2 \sin^2 \theta} + \frac{3}{8} \lambda_P \left[ \ln \left( 1 - \frac{2}{\rho} \right) + \frac{2}{\rho} \left( 1 + \frac{1}{\rho} \right) \right], \quad (22)
\]

\[
\frac{d\tau}{d\sigma} = (1 - \frac{2}{\rho})^{-1} \left\{ E + \frac{3}{8} \lambda_P W \rho^2 \sin^2 \theta \left[ \ln \left( 1 - \frac{2}{\rho} \right) + \frac{2}{\rho} \left( 1 + \frac{1}{\rho} \right) \right] \right\}, \quad (23)
\]

where \( \Lambda = \frac{\lambda_T}{\lambda_P} \).

Similarly, the equations of motion for flat spacetime read as (without the toroidal field components)

\[
\frac{d^2 \rho}{d\sigma^2} = \rho \left\{ \left( \frac{d\theta}{d\sigma} \right)^2 + \sin^2 \theta \left( \frac{d\phi}{d\sigma} \right)^2 \right\} - \frac{\lambda_P}{\rho^2} \sin^2 \theta \left( \frac{d\phi}{d\sigma} \right)^2 - \frac{W}{\rho} \frac{d\tau}{d\sigma}, \quad (24)
\]

\[
\frac{d^2 \theta}{d\sigma^2} = -\frac{2}{\rho} \frac{d\rho}{d\sigma} \frac{d\theta}{d\sigma} + \sin \theta \cos \theta \left( \frac{d\phi}{d\sigma} \right)^2 - \frac{2}{\rho^3} \sin \theta \cos \theta \left( W \frac{d\tau}{d\sigma} - \frac{d\phi}{d\sigma} \right), \quad (25)
\]

\[
\frac{d\phi}{d\sigma} = \frac{L}{\rho^2 \sin^2 \theta} - \frac{\lambda_P}{\rho^3}, \quad (26)
\]

\[
\frac{d\tau}{d\sigma} = E - \frac{\lambda_P W}{\rho} \sin^2 \theta. \quad (27)
\]

In order to solve the equations of motion we need to prescribe the initial conditions.

We assume initially \( \rho = \rho_o, \theta = \theta_o, \phi = \phi_o \) and \( \tau = \tau_o \). Also at \( \rho_o, \frac{d\theta}{d\sigma} = \left( \frac{d\theta}{d\sigma} \right)_o, \frac{d\phi}{d\sigma} = \left( \frac{d\phi}{d\sigma} \right)_o \).

Throughout our discussion we take \( \left( \frac{d\phi}{d\sigma} \right)_o = 0 \) and \( \tau_o = 0 \). Therefore, from equation (22) we get

\[
L = -\frac{3}{8} \lambda_P \rho_o^2 \sin^2(\theta_o) \left[ \ln \left( 1 - \frac{2}{\rho_o} \right) + \frac{2}{\rho_o} \left( 1 + \frac{1}{\rho_o} \right) \right], \quad (28)
\]

From equation (23) we get

\[
\left( \frac{d\tau}{d\sigma} \right)_o = (1 - \frac{2}{\rho_o})^{-1} \left\{ E + \frac{3}{8} \lambda_P W \rho_o^2 \sin^2(\theta_o) \left[ \ln \left( 1 - \frac{2}{\rho_o} \right) + \frac{2}{\rho_o} \left( 1 + \frac{1}{\rho_o} \right) \right] \right\}. \quad (29)
\]
From the spacetime metric (2) we get

$$\left(\frac{d\rho}{d\sigma}\right) = \pm(1 - \frac{2}{\rho_{\circ}})\{(1 - \frac{2}{\rho_{\circ}})(d\tau_{\circ})^2 - \rho_{\circ}^2(d\theta_{\circ})^2 + 1\}^{1/2}. \quad (30)$$

Equations (28) - (30) provide the initial conditions for the motion of the particle in curved spacetime.

Similarly, from equation (26) we obtain

$$L = \frac{\lambda p}{\rho_{\circ}} \sin^2(\theta_{\circ}). \quad (31)$$

From equation (27) we get

$$\left(\frac{d\tau}{d\sigma}\right) = E - \frac{\lambda p W}{\rho_{\circ}} \sin^2(\theta_{\circ}). \quad (32)$$

From the Minkowskian metric

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

we get

$$\left(\frac{d\rho}{d\sigma}\right) = \pm[1 + (\frac{d\tau}{d\sigma})^2 - \rho_{\circ}^2(\frac{d\theta}{d\sigma})^2]^{1/2}. \quad (33)$$

Equations (31) - (33) give the initial conditions for the motion of the particle in flat spacetime.

The equations are numerically solved by using the 4th order adaptive step size Runge-Kutta method. The integrations are performed between finite limits with an accuracy of $10^{-7} - 10^{-8}$, while changing the step size appropriately through self inspection depending upon the initial conditions. The trajectories are obtained in spherical polar co-ordinates which are then converted into Cartesian co-ordinates by using the relations

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

The results are presented graphically.
5. RESULTS AND DISCUSSIONS

As obtained by Prasanna and Varma\textsuperscript{7}, when the magnetic field is large, the motion of a charged particle looks similar to that in the flat geometry wherein the particle gyrates in a given tube of lines reflecting between two mirror points located symmetrically with respect to the equatorial plane (see Figure 1a). On the other hand, if the magnetic field is lower, then the particle oscillations up and down the equatorial plane damp continuously as \( r \) decreases and eventually the particle gets sucked in by the central object. The inclusion of the induced unipolar electric field changes the nature of the orbit by introducing a \( \mathbf{E} \times \mathbf{B} / B^2 \) drift. The particle still remains trapped in a mirror configuration reflecting between two points on either side of the equatorial plane (see Figure 1b) just like the case without the electric field. Figure 1c shows the trajectory of the particle in a combined poloidal and unipolar electric field superposed in flat spacetime. A comparison between Figure 1b and 1c reveals that the gravitational field mainly contributes in decreasing the radius of gyration of the particle significantly i.e., in curved spacetime the particle gyrates in much tighter orbit than it does in flat spacetime.

From equation (12) to equation (15) we notice that unlike the case of pure poloidal field, a particle on the plane \( \theta = \pi/2 \) having zero initial velocity in the meridional direction is not confined to the equatorial plane as the acceleration along the \( \theta \) direction is still non-zero in the presence of toroidal magnetic field. The inclusion of rotational motion of the object modifies the initial conditions. From the initial conditions we notice that the initial velocity of the particle is well defined even if \( E = 0 \) since the particle acquires its initial velocity due to its interaction with the electric field. The situation is similar to the case for the Kerr background geometry\textsuperscript{8}. But here we have considered a rigid rotation neglecting the dragging of inertial frame effect. Since it is difficult to obtain the trajectory of a charged particle in the off equatorial plane with the Kerr geometry due to the presence of non-zero off diagonal terms.
in the metric, the particle trajectory has not been investigated in that situation although the situation is more realistic in that case for astrophysical interest. The present study can very well provide a nearly similar qualitative picture with two differences i.e., the absence of the dragging of inertial frame effect and the rotation-induced quadrupolar electric field. As mentioned in the introduction, if we consider comparatively slow rotation, the inertial frame dragging becomes negligible and the consideration of rotation-induced unipolar electric field is much more realistic in the astrophysical situation than that of quadrupolar electric field.

Prasanna and Sengupta\textsuperscript{10} investigated the trajectories of a charged particle in combined poloidal and toroidal magnetic field superposed in Schwarzschild background geometry. The effect of rotation and hence the electric field has not been considered in that study. In that work it is found that when the toroidal field is extremely small compared to the poloidal field the particle trajectory appears somewhat similar to that in the case of pure poloidal field till the particle reaches a distance two Schwarzschild radii away from the central object where gravity seems to dominate and pull the particle in. As the magnitude of the toroidal field is increased, the nature of the trajectory alters and when the magnitude of the toroidal field becomes significant compared to that of the poloidal field the particle gets deconfined even very near to the central gravitating object and gets pushed away to infinity. The most interesting feature is that if the initial velocity of the particle is towards the central object then in the presence of the toroidal field the particle comes very near to the object and then gets ejected away to infinity.

As the earlier investigation has not included the electric field induced by rotation one may expect significant changes in the trajectories in more realistic case. But here we show that the above situation remains the same even if one includes the electric field. Figure 2 and Figure 3a show the trajectory of a charged particle in combined toroidal, poloidal magnetic field and induced unipolar electric field superposed in Schwarzschild background
geometry under different initial conditions. In these situations the particle starting at any
given position and having its radial velocity inwards comes towards the central object slowly
and as it approaches its minimum value of r (depending on \( r_0 \)) and of \( \theta \) (depending upon
the angular momentum) the particle gets bounced off in a straight line trajectory. This
phenomenon could be a potential mechanism for the acceleration of high energy cosmic ray
particles. In case the particle’s radial velocity is directly away from the center, the particle
spirals around the central body as depicted in Figure 3a. It is very interesting to note that
near the object the nature of the particle trajectory remains the same (as shown in Figure
2) whether one includes the electric field or not. However, at a very large distance from
the central object the presence of the electric field becomes visible as the two trajectories
differ significantly. In the case for the closed orbit the trajectory remains almost the same
with and without the presence of the electric field as observed from Figure 3a and Figure
3b. Thus one is lead to a very interesting conclusion that the presence of toroidal magnetic
field totally dominates the role of the electric field in determining the motion of a charged
particle in the presence of a strong gravitational field.

6. CONCLUSIONS

In order to understand the effect of curved spacetime on the electromagnetic field sur-
rounding compact objects the trajectories of charged particles have been investigated in the
context of accretion onto black holes. For this purpose the trajectories of a charged parti-
cle in combined poloidal, toroidal magnetic field and rotation-induced unipolar electric field
superposed in Schwarzschild background geometry have been obtained. It is found that the
toroidal component of the magnetic field dominates the role of unipolar electric field in de-
termining the motion of a charged particle in the presence of strong gravitational field. In
the presence of a non-zero toroidal magnetic field an incoming particle approaches very near
to the central object and then gets ejected away to infinity. This phenomenon could be a
plausible mechanism for the production of high energy cosmic ray. A rigorous magnetohydro-
dynamical investigation including the toroidal component of the magnetic field superposed 
in curved spacetime could give rise to the production of jet like phenomenon observed in 
quasars and AGN. In the absence of toroidal field the particle is trapped into a closed orbit 
in the same way it behaves in the presence of only poloidal magnetic field except a regular 
oscillation perpendicular to the orbital motion is produced by the presence of the electric 
field.

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Figure Captions

**Fig. 1a.** – Projection of the trajectory of a positively charged particle (in purely poloidal magnetic field superposed on curved spacetime) in XY plane. The various parameters are $E = 2.0$, $L = 91.77$, $\lambda_P = 80$, $\rho_o = 2.5$, $(\frac{d\rho}{d\sigma})_o = -1.92$, $\theta_o = \pi/2$, $(\frac{d\theta}{d\sigma})_o = 0.3$, $\phi_o = 0^\circ$. In all figures a positive value of $(\frac{d\rho}{d\sigma})_o$ implies that the initial motion of the particle to be away from the center whereas a negative value to be towards the center.

**Fig. 1b.** – Projection of the trajectory of a positively charged particle in poloidal magnetic and unipolar electric field superposed in curved spacetime. The parameters are the same as used for Figure 1a except $(\frac{d\rho}{d\sigma})_o = -1.122$ and $W = 0.01$.

**Fig. 1c.** – Same as Figure 1b but without gravitational field. The parameters are the same except $(\frac{d\rho}{d\sigma})_o = -1.8$ and $L = 32.0$.

**Fig. 2.** – Projection of the trajectory of a charged particle on XY, XZ and YZ planes. The solid line represents the trajectory in the combined poloidal, toroidal magnetic and unipolar electric field superposed in curved spacetime while the dashed line represents that without the electric field. The parameters are $E = 5.0$, $L = 17.9$, $\lambda_P = 80.0$, $\lambda_T = 4.0$, $\rho_o = 6.0$, $\theta_o = \pi/2$, $(\frac{d\theta}{d\sigma})_o = 0.0$, $W = 0.01$ and $\phi_o = 0.0$. With the electric field $(\frac{d\rho}{d\sigma})_o = -4.89$, without the electric field $(\frac{d\rho}{d\sigma})_o = -4.93$.

**Fig. 3a.** – Projection of the trajectory of a charged particle in XY plane in the combined poloidal, toroidal magnetic and unipolar electric field superposed in curved spacetime. The parameters are the same as given for Figure 2 but $\rho_o = 3.5$, $(\frac{d\rho}{d\sigma})_o = +4.63$, $L = 41.382$, and $\lambda_T = 40$.

**Fig. 3b.** – Projection of the trajectory of a charged particle in XY plane in the combined poloidal, toroidal magnetic but without unipolar electric field. The parameters are the same as given for Figure 3a except $(\frac{d\rho}{d\sigma})_o = +4.96$. 
