ALMOST SOLITON DUALITY

GIDEON MASCHLER

Abstract. Gradient Ricci almost solitons were introduced by Pigola, Rigoli, Rimoldi and Setti [20]. They are defined as solitons except that the metric coefficient is allowed to be a smooth function rather than a constant. It is shown that any almost soliton is conformal to another almost soliton having a soliton function which is minus the original one. Uniqueness, and the case where both the source and target are solitons, are studied. Completeness of the target metric is also examined in the case where the source is Kähler and admits a special Kähler-Ricci potential in the sense of [10, 11].

1. Introduction

A gradient Ricci soliton on a manifold $M$ is a Riemannian metric $g$ satisfying

$$(1.1) \quad r + \nabla df = \lambda g,$$

where $r$ denotes the Ricci curvature of $g$, $\nabla df$ stands for the $g$-Hessian of a smooth function $f$ and $\lambda$ is a constant. We call $f$ the soliton function, $\lambda$ the metric coefficient and $(g, f)$ a soliton pair. If $f$ is constant the soliton is deemed trivial.

Ricci solitons have been intensively studied in recent years (cf. [12, 5, 18]), and their importance derives in part from the role they play in the study of the Ricci flow. More recently, the more general notion of a gradient Ricci almost soliton was introduced by Pigola, Rigoli, Rimoldi and Setti [20], and further studied in [2] and [1]. We will often employ the term “almost soliton” for brevity. Almost solitons and almost soliton pairs are also defined via (1.1), but with the metric coefficient $\lambda$ an arbitrary smooth function. In this case we call $(g, f)$ an almost soliton pair, yet $f$ is still called the soliton function. A more general notion considered in [17] is that of a pair $(g, f)$ as above satisfying a Ricci-Hessian equation

$$(1.2) \quad r + \alpha \nabla df = \lambda g,$$

in which the coefficients $\alpha$, $\lambda$ are smooth functions.

The classical problem of determining whether an Einstein manifold can be mapped conformally onto another Einstein manifold goes back to Brinkmann [3], and was addressed by many authors (see [16]). In work carried out very recently, the corresponding question for other metric types has been taken up by Jaregui and Wylie in [14]. In the context of generalized quasi-Einstein metrics, a category which includes almost solitons, it was found that under certain assumptions a strict classification holds, with metrics having such a “dual” metric being certain warped products with a one dimensional base. An important assumption that was made in this classification is that the conformal diffeomorphism preserves, in a certain sense, the generalized
quasi-Einstein structure. Remarkably, we show in this work that relaxing this condition just a little completely annuls the above-mentioned strictness. Namely, in the context of almost solitons and a single conformal class, we find a “dual” almost soliton conformal to any given one (and, in particular, to any given Ricci soliton). In particular, this gives many new examples of Ricci almost solitons.

More precisely, we have,

**Theorem A.** In the conformal class of an almost soliton $g$ on a manifold $M$ of dimension $n > 2$, there exists another almost soliton $\widehat{g}$, which is nonhomothetic to $g$ if $g$ is nontrivial. The metric $\widehat{g}$ is unique among almost solitons in the conformal class for which both their conformal factor to $g$ and their soliton function are smooth functions of the soliton function of $g$. If $g$ is complete and nontrivial, $g$ and $\widehat{g}$ cannot both be gradient Ricci solitons.

In terms of almost soliton pairs, the dual is given by

$$(g, f) \rightarrow (\widehat{g}, \widehat{f}) := (e^{\frac{-4f}{n-2}} g, -f).$$

This should be compared with

$$(g, f) \rightarrow (f^{-2} g, f^{-1}).$$

Both are involutions on the space of pairs, suitably defined. The latter was important in [17] and has an invariant subset consisting of pairs satisfying a Ricci-Hessian equation, while (1.3) possesses, by Theorem A, a smaller invariant set consisting of gradient Ricci almost soliton pairs. The relation between the two involutions can be seen by analogy. The equation describing the Ricci tensor of a metric conformal to a gradient Ricci almost soliton contains the Hessians of both the conformal factor and the soliton function of the latter metric. In this formula these Hessians have the following interchangeable role. Each of the conformal changes in (1.4) and (1.3) implies a particular functional relation between the conformal factor and the soliton function, which reduces the equation to a simpler one, namely a Ricci-Hessian equation in one case (with the Hessian of the conformal factor appearing in one of the terms), and an almost soliton equation in the other case (with one of the terms being the Hessian of the soliton function).

It is interesting to note that the square root of the conformal factor in (1.3), namely $e^{\frac{-2f}{n-2}}$, has been employed in a number of works as an important conformal factor in its own right (aside from [14], see [6], [21], and also in the very recent [19]). Its effect on the first equation mentioned in the previous paragraph, is a simplification resulting in an equation involving, instead of a Hessian, the tensor product of a one form with itself, the one form being the differential of the soliton function.

The study of both involutions can be applied to the case where $g$ is Kähler. However, we will review the fact that any gradient Kähler-Ricci almost soliton is, in fact, a gradient Kähler-Ricci soliton. In the case of a Kähler metric conformal to a gradient Ricci almost soliton, a pertinent role is played by a potential for a Killing vector field. For the involution (1.4), this Killing potential is just the square root $\tau$ of the conformal factor, while for involution (1.3) it is the soliton function $f$. Both of these are the extreme special cases in the moduli space considered in [9], a work written in Polish. There, in the context of Kähler metrics conformal to gradient Ricci solitons, both $\tau$ and $f$ are regarded as essentially arbitrary functions of some Killing potential.
One consequence of Theorem 1 is that a complete gradient Kähler-Ricci soliton cannot have an image \( \hat{g} \) under the involution (1.3), which is Kähler with respect to some complex structure. This is in sharp contrast with the behaviour of the involution (1.4), as was shown in [17].

As mentioned above, the duality presented here furnishes many new examples of gradient Ricci almost solitons, for example the duals to known gradient Ricci solitons. After proving Theorem 1 via Propositions 2.1-2.3, we study in the final section the completeness question for the dual \( \hat{g} \), in the case where \( g \) is a gradient Kähler-Ricci soliton which is at the same time a metric with a special Kähler-Ricci potential, in the sense of [10]. These solitons belong to a type studied by Koiso [15], and also Cao [4]. In some cases \( \hat{g} \) is complete.

The author thanks the referee for various suggestions for improving the style of this work.

2. Duality and Ricci solitons

2.1. Conformal changes. Let \((M, g)\) be a Riemannian manifold of dimension \( n \), and \( \tau : M \to \mathbb{R} \) a \( C^\infty \) function. We write metrics conformally related to \( g \) in the form \( \hat{g} = g/\tau^2 \), and let \( \nabla, \hat{\nabla} \) denote the corresponding Levi-Civita connections as well as the metric gradient operators. The Koszul formula,

\[
2\hat{g}(\hat{\nabla}_u v, w) = d_w[\hat{g}(v, u)] + d_v[\hat{g}(w, u)] - d_u[\hat{g}(w, v)]
+ \hat{g}(v, [w, u]) + \hat{g}(w, [u, v]) - \hat{g}(w, [v, u])
\]

for smooth vector fields \( u, v, w \), with \( d_u \) etc. denoting directional derivatives, yields the following expression for the \( \hat{g} \)-Hessian of a function \( f : M \to \mathbb{R} \)

\[
\hat{\nabla} df = \nabla df + \tau^{-1}[2d\tau \odot df - g(\nabla \tau, \nabla f)g],
\]

where \( d\tau \odot df = (d\tau \odot df + df \odot d\tau)/2 \) with \( \odot \) denoting the tensor product. We will be concerned primarily with the case where \( df \wedge d\tau = 0 \), i.e., at points where \( df \neq 0 \), \( \tau \) is given locally as a composition \( \tau = H \circ f \) for some smooth function \( H : \mathbb{R} \to \mathbb{R} \).

In this case, (2.1) becomes

\[
\hat{\nabla} df = \nabla df + 2\tau^{-1}\tau' df \odot df - \tau^{-1}\tau' |\nabla f|^2 g,
\]

with ' denoting differentiation with respect to \( f \) and \( | \cdot | \) is the \( g \)-norm.

Also recall the conformal change expression relating the Ricci tensors of \( g \) and \( \hat{g} \), with \( \Delta \) denoting the Laplace operator:

\[
\hat{r} = r + (n - 2)\tau^{-1}\nabla d\tau + [\tau^{-1}\Delta \tau - (n - 1)\tau^{-2}|\nabla \tau|^2] g.
\]

In the case where \( \tau \) depends locally on \( f \), we write the second term on the right in terms of \( f \), obtaining

\[
\hat{r} = r + (n - 2)\tau^{-1}\tau' \nabla df + (n - 2)\tau^{-1}\tau'' df \odot df + [\tau^{-1}\Delta \tau - (n - 1)\tau^{-2}|\nabla \tau|^2] g.
\]

2.2. Duality for almost solitons. With \( M, g, f \) and other notations as above, we consider the case where the pair \((g, f)\) forms an almost soliton pair.
Proposition 2.1. Let \( M \) be a manifold of dimension \( n > 2 \), and suppose \((g,f)\) denotes an almost soliton pair satisfying (1.1). Then \((\hat{g}, \hat{f}) := (g/\tau^2, -f)\), with \( \tau = e^{\frac{2f}{n-2}} \), is also an almost soliton pair. The latter pair satisfies \( \hat{\tau} + \hat{\nabla} d\hat{f} = \hat{\lambda} \hat{g} \), with

\[
\hat{\lambda} = \tau^2(\lambda + \beta + \delta),
\]

where \( \beta = \tau^{-1} \Delta \tau - (n - 1) \tau^{-2} |\nabla \tau|^2 \) and \( \delta = \tau^{-1} \tau'|\nabla f|^2 \) denote the coefficients of \( g \) in (2.3) and (2.2), respectively.

At times \( \hat{g} \) will be referred to as the almost soliton dual to \( g \), and [1.3] will then be called a duality.

\[\text{Proof.} \quad \text{Noting that } \hat{f} = -f, \text{ one has, by (2.4) and (2.2),}
\]

\[
\begin{align*}
\hat{\tau} + \hat{\nabla} d\hat{f} &= r + (n - 2) \tau^{-1} \tau' \nabla df + (n - 2) \tau^{-1} \tau'' df \otimes df + \beta g \\
&+ (-\nabla df' - 2\tau^{-1} \tau' df \otimes df + \delta g) \\
&= r + ((n - 2)\tau^{-1} \tau' - 1) \nabla df + ((n - 2)\tau^{-1} \tau'' - 2\tau^{-1} \tau') df \otimes df \\
&+ (\beta + \delta)\tau^2 \hat{g} \\
&= r + (2 - 1) \nabla df + \tau^{-1} \left( \frac{4}{n - 2} - \frac{4}{n - 2} \right) df \otimes df + (\beta + \delta)\tau^2 \hat{g} \\
&= r + \nabla df + (\beta + \delta)\tau^2 \hat{g} = (\lambda + \beta + \delta)\tau^2 \hat{g},
\end{align*}
\]

where in the penultimate equality we have used the expression for \( \tau \) as a function of \( f \).

\[\square\]

2.3. Uniqueness. We show here that among all choices of a conformal factor \( \tau \) and a soliton function \( \hat{f} \) in which both are functions of the initial soliton function \( f \), (essentially) only the combination of these two given in (1.3) gives rise to a dual almost soliton.

Proposition 2.2. On a manifold \( M \) of dimension \( n > 2 \), let \( g \) be a gradient Ricci almost soliton with soliton function \( f \). Suppose \( \tau(f), k(f) \) are smooth functions of a real variable, regarded as functions on \( M \) via composition with \( f \), and \( \tau(f) \) is non-constant. Assume \( (\tau(f))^{-2} g \) is a gradient Ricci almost soliton with soliton function \( \hat{f} = k(f) \). Then \( \tau(f) = e^{\frac{2f}{n-2}} \) and \( k(f) = -f \) up to an additive or, respectively, a multiplicative constant.

\[\text{Proof.} \quad \text{A computation similar to the one in Proposition 2.1 yields that for a dual almost soliton we have, for the coefficients of } \nabla df \text{ and } df \otimes df, \text{ the following two equations, respectively:}
\]

\[
(n - 2)\tau^{-1} \tau' + k' = 1, \quad (n - 2)\tau^{-1} \tau'' + k'' + 2 \tau^{-1} \tau' k' = 0.
\]

Subtracting the derivative of the first equation from the second equation yields, after dividing by the nonzero coefficient \( \tau' \) and rearranging, the equation

\[
\left( \log \tau \right)' = -\frac{2}{n - 2} k'.
\]

Substituting this in the first equation yields \( k' = -1 \), so that \( k(f) = -f \) up to an additive constant, and then from (2.6) we have \( \tau(f) = e^{\frac{2f}{n-2}} \), up to a multiplicative constant.

\[\square\]
2.4. The mapping problem for solitons. We now consider whether the image of a gradient Ricci soliton under the map (1.3) can itself be a soliton. It will turn out that this can happen only rarely. First, note that a gradient Ricci soliton satisfies
\[
\Delta f - |\nabla f|^2 = -2\lambda f + k
\]
for a constant $k$, with $f$ and $\lambda$ as in (1.1). This relation given by Hamilton (cf. [8]) is deduced by combining the trace of the soliton equation with a computation involving the Ricci identity.

Recall that a gradient Ricci soliton is called steady if its metric coefficient function vanishes.

Proposition 2.3. Let $M$ be a manifold of dimension $n > 2$. If, on $M$, the image of a gradient Ricci soliton pair $(g, f)$ satisfying (1.1) under the map (1.3) is also a gradient Ricci soliton pair, then either $f$ is constant or both metrics are incomplete steady solitons and the constant $k$ of (2.7) is zero.

Proof. The proof resembles Proposition 6.1 in [14]. In fact, writing the metric coefficient $\hat{\lambda}$ of (2.5) explicitly, we have
\[
\hat{\lambda} = e^{\frac{4f}{n-2}} \left( \lambda + \frac{2}{n-2} \Delta f + \left( -4 \frac{n-1}{(n-2)^2} + \frac{2}{n-2} + \frac{4}{(n-2)^2} \right) |\nabla f|^2 \right),
\]
with $\tau = e^{\frac{2f}{n-2}}$. Using the latter relation along with $|\nabla \tau|^2 = (\tau')^2 |\nabla f|^2$ and $\Delta \tau = \tau' \Delta f + \tau'' |\nabla f|^2$ gives
\[
\hat{\lambda} = e^{\frac{4f}{n-2}} \left( \lambda + \frac{2}{n-2} (\Delta f - |\nabla f|^2) \right),
\]
and so
\[
(2.8) \quad \hat{\lambda} = e^{\frac{4f}{n-2}} \left( \lambda + \frac{2}{n-2} (\Delta f - |\nabla f|^2) \right).
\]

Equations (2.8) and (2.7) both hold exactly when
\[
(2.9) \quad \frac{n-2}{2} (\hat{\lambda} e^{\frac{4f}{n-2}} - \lambda) = -2\lambda f + k.
\]

Applying the operator $d$ to this equation, with $\lambda$ and $\hat{\lambda}$ constant, yields that solutions only occur for $n \neq 2$ if $f$ is constant or $\lambda = \hat{\lambda} = 0$. In the latter case, we proceed as in [14]. Namely, we have $\Delta f - |\nabla f|^2 = 0$ by (2.8). Thus, because the scalar curvature $R$ of a steady ($\lambda = 0$) soliton satisfies $R = -\Delta f$, we get $-R - |\nabla f|^2 = 0$. Now it is known [7] that for $g$ a complete steady soliton, $R \geq 0$. Consequently, in that case $|\nabla f|^2 = 0$, so that again $f$ must be constant. Thus if it is not constant, $g$ must be incomplete, and since $g$ and $\hat{g}$ are symmetric with respect to the involution (1.3), the metric $\hat{g}$ is also incomplete. Finally, from Equation (2.9), if $\lambda = \hat{\lambda} = 0$ then $k = 0$ as well. \qed

3. The Kähler case

3.1. Generalities. The Kähler condition forces an almost soliton to be a soliton.

Proposition 3.1. Let $M$ be a manifold of even dimension $n \geq 4$. A gradient Ricci almost soliton pair $(g, f)$ on $M$, for which $g$ is Kähler with respect to some complex structure $J$, is in fact a gradient Kähler-Ricci soliton.
This follows from Lemma 6.1(ii) of [10]. For completeness, we give the proof in a form that resembles Proposition 3.3 of [17]. For this, recall that a Killing potential $f$ on a Kähler manifold $(M, J, g)$ (with $J$ the almost complex structure), is a smooth function for which $u := J \nabla f$ is a Killing vector field, i.e. $L_u g = 0$ where $L_u$ is the Lie derivative.

Proof. From (1.1), since $g$ and the Ricci curvature $r$ are hermitian, so is $\nabla df$. By [10, Lemma 5.2], this implies that $f$ is a Killing potential. By [10, Lemma 5.5], this in turn implies that $\nabla df(J \cdot, \cdot) = d(\mathfrak{\mathcal{I}} \nabla f \omega) / 2$.

Now compose Equation (1.1) with $J$ and apply the differential operator $d$ to the result. Using the conclusion of the previous paragraph and the fact that the Kähler form $\omega$ and the Ricci form are closed, one arrives at $0 = d\lambda \wedge \omega$. But the operation $\wedge \omega$ is injective on 1-forms in dimensions four and above. Hence $d\lambda = 0$. $\square$

Note that for any gradient Ricci almost soliton it is known that the metric coefficient function is locally a function of the soliton function (see Remark 2.5 in [20]).

As mentioned in the introduction, unlike the situation for the involution (1.4), Propositions 2.1, 2.3 and 3.1 together imply that in the Kähler setting, the property of having a dual complex structure does not hold for the involution (1.3):

**Corollary 3.2.** Given a nontrivial complete gradient Kähler-Ricci soliton $g$ with soliton function $f$, the metric $e^{-4f}g$ is not Kähler with respect to any complex structure.

In fact, we know that the metric in question is an almost soliton. If it were Kähler, it would be a Kähler-Ricci soliton. Both this metric and $g$ would thus be gradient Ricci solitons, but in that case $g$ cannot be both complete and nontrivial.

The propositions above can be applied to the study of the setting analogous to the one considered in [17], namely that of a Kähler metric conformal to a gradient Ricci almost soliton. Note that, as the latter is the target metric in this setting, in the next proposition our notations for the metrics are reversed from those in the rest of this work.

**Proposition 3.3.** Suppose $g$ is a Kähler metric on a complex manifold $(M, J)$ of dimension $n \geq 4$, which is conformal to a gradient Ricci almost soliton $\hat{g} := g/\tau^2$ with soliton function $f$. If $f$ is a $g$-Killing potential and $\tau$ is locally a function of $f$ with $\tau(0) = 1$ and $\tau'(0) = -\frac{2}{n^2}$, then $g$ is a Kähler-Ricci soliton.

Proof. Since $g$ is conformal to an almost soliton, (2.1) and (2.3) yield

$$r + (n-2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1}d\tau \otimes df$$

$$= [\hat{\lambda}\tau^{-2} + (n-1)\tau^{-2}\nabla \tau^2 - \tau^{-1}\Delta \tau + \tau^{-1}g(\nabla \tau, \nabla f)]g,$$

with $\hat{\lambda}$ the almost soliton coefficient function of $(\hat{g}, f)$. If $d\tau \wedge df = 0$, we regard $\tau$ as a function of $f$, and this expression becomes, in similarity with Proposition 2.1

$$r + ((n-2)\tau^{-1}\tau' + 1) \nabla df + ((n-2)\tau^{-1}\tau'' + 2\tau^{-1}\tau') df \otimes df = (\hat{\lambda}\tau^{-2} - \beta + \delta) g,$$

with $\beta, \delta$ as in that proposition. As $f$ is a Killing potential for the Kähler metric $g$, the Hessian $\nabla df$ is hermitian with respect to the associated complex structure, and thus all the terms in the last equation are hermitian except for $df \otimes df$. Thus for the latter term, the coefficient $(n-2)\tau^{-1}\tau'' + 2\tau^{-1}\tau'$ must vanish, giving an ODE whose
solution with the given initial conditions is \( \tau = \exp(-2f/(n-2)) \). We can now apply Proposition 2.1 (with \( \hat{g} \) and \( g \) interchanged), to conclude that \( g \) is a gradient Ricci almost soliton. As \( g \) is Kähler, Proposition 3.1 yields that it is in fact a gradient Kähler-Ricci soliton. \( \Box \)

3.2. Example: Ricci solitons that admit a special Kähler-Ricci potential.

Metrics with special Kähler-Ricci potentials have been studied in [10, 11] and [17]. If a gradient Ricci soliton also admits such a potential, it is of a type first considered by Koiso [15] and also Cao [4]. We give here an independent viewpoint on such metrics and then consider their dual under the map induced by (1.3).

A Kähler metric \( g \) on a complex manifold \((M,J)\) admits a special Kähler-Ricci potential \( f \) if \( f \) is a Killing potential and, at each noncritical point of \( f \), all nonzero tangent vectors orthogonal to the complex span of \( \nabla f \) are eigenvectors of both the Ricci tensor and the Hessian of \( f \). This rather technical definition implies, by [10, Remark 7.4], the existence on an open set, of a Ricci-Hessian equation (1.2), which we reproduce here:

\[
(3.10) \quad r + \alpha \nabla df = \lambda g.
\]

The coefficients \( \alpha, \lambda \) are smooth functions, and in fact locally functions of \( f \). The metric \( g \) is called an SKR metric and \((g,f)\) is called an SKR pair.

We turn next to the local classification of metrics with a special Kähler-Ricci potential. We consider only the non-trivial case, of metrics that are not local products of Kähler metrics. In this case the metrics come in the following families involving one free function, described using a variant of the Calabi construction.

Let \( \pi : (L, \langle \cdot, \cdot \rangle) \to (N, h) \) be a Hermitian holomorphic line bundle over a Kähler-Einstein manifold of complex dimension \( m - 1 \). Assume that the curvature of \( \langle \cdot, \cdot \rangle \) is a multiple of the Kähler form of \( h \). Consider, on \( L \setminus N \) (the total space of \( L \) excluding the zero section), the metric \( g \) given by

\[
(3.11) \quad g|_H = 2|f_c|^2 \pi^* h, \quad g|_V = \frac{Q(f)}{(p \cdot \ell)^2} \text{Re} \langle \cdot, \cdot \rangle,
\]

where

- \( V, H \) are the vertical/horizontal distributions of \( L \), respectively, the latter determined via the Chern connection of \( \langle \cdot, \cdot \rangle \),
- \( \ell \) is the norm induced by \( \langle \cdot, \cdot \rangle \),
- \( f \) is a function on \( L \setminus N \) obtained as follows: one fixes an open interval \( I \) and a positive \( C^\infty \) function \( Q(f) \) on \( I \), solves the differential equation \((a/Q) df = d(\log \ell)\) to obtain a diffeomorphism \( \ell(f) : I \to (0, \infty) \), and defines \( f(\ell) \) as the inverse of this diffeomorphism (composed on the norm \( \ell \)),
- \( c \) and \( p \neq 0 \) are constants and \( f_c = f - c \).

For \( m \geq 2 \), the pair \((g,f)\) is a nontrivial SKR pair. Conversely, for any nontrivial SKR metric \((M,J,g,f)\) with \( m > 2 \), any point that is not a critical point for \( f \) has a neighborhood biholomorphically isometric to an open set in some triple \((L,g,f(\ell))\) as above (this is a special case of [10, Theorem 18.1]).

The function \( Q \) of a nontrivial SKR metric as above is given by \( Q(f) = 2f_c \phi(f) \), where \( \phi(f) \) is a solution to the ordinary differential equation derived from (3.10).
Namely,
\[(f_c)^2 \phi'' + (f_c)[m - (f_c)\alpha] \phi' - m\phi = -\text{sgn}(\phi)\kappa/2\]
holds at points which are not critical for \(f\) and for which \(\phi'(f)\) is nonzero, with \(\kappa\) the Einstein constant of the metric \(h\) [17, Proposition 4.1].

We now suppose that the Ricci-Hessian equation of an SKR metric also defines a gradient Ricci-soliton, in other words \(\alpha = 1\) and \(\lambda\) is constant in (3.10). Substituting \(\alpha = 1\) in (3.12), the resulting differential equation has solutions \(\phi(f)\), which we take for positive \(\phi\), given by
\[(3.13) \quad \phi(f) = \frac{1}{f_c m} \left( A \sum_{k=0}^{m} \frac{f_c^k}{k!} + B e^{f_c} \right) - \frac{\kappa}{2m}\]
for constants \(A, B\). One can verify that for these \(\phi\) the function \(\lambda\) is indeed constant, as it must be by Proposition 3.1 from the formula
\[
\lambda = \alpha \phi + (\alpha f_c - (m + 1)) \phi' - f_c \phi'',
\]
valid for any such SKR metric (see [17, Section 4.2]).

The almost soliton dual under (1.3) to an SKR metric of this type is thus given by
\[(3.14) \quad \tilde{g}\vert_H = 2|f_c| e^{-\frac{4f}{m}} \pi^* h, \quad \tilde{g}\vert_\nu = \frac{Q(f)}{(p \cdot f)^2} e^{\frac{4f}{m}} \Re \langle \cdot, \cdot \rangle.
\]
We now examine conditions under which \(\tilde{g}\) is complete. Assuming the manifold \(N\) is compact, it is enough to check completeness for the \(\tilde{g}\)-geodesics normal to, say, the zero section of \(L\), an \(f\)-critical manifold. Since the conformal factor is a function of \(f\), it follows that these geodesics coincide, as unparametrized curves, with the \(g\)-geodesics normal to the zero section.

In fact, unparametrized \(g\)-geodesics form integral curves of \(v := \nabla f\) (see [11, Section 8]), and thus \(\nabla_v v = \alpha v\) for some function \(\alpha\). Now for vector fields \(x, y\) on \(M\), we have the conformal change formula \(\tilde{\nabla}_x y = \nabla_x y - (d_x f) y - (d_y f) x + g(x, y) \nabla f\).
Thus \(\tilde{\nabla}_v v = (\alpha - 2d_x f + |\nabla f|^2) v := \beta v\), and therefore the \(\tilde{g}\)-geodesics also coincide with integral curves of \(\nabla f\).

To obtain the arc length formula, we first reparametrize an integral curve \(\tilde{x}(t)\) of \(\nabla f\) so that \(f\) itself is the new parameter, giving the form \(x(f)\). Then the velocity is given by \(x'(f) = \psi'(f) \tilde{x}'(\psi(f)) = \psi'(f) \nabla f \vert_{\tilde{x}(\psi(f))}\), where \(t = \psi(f)\) is the reparametrization map and the prime denotes differentiation. To compute \(\psi'(f)\) we apply \(g(\cdot, \nabla f)\) to this equation, giving
\[
1 = \frac{d}{df} f = d_x f = g(x', \nabla f) = \psi'(f) g(\nabla f, \nabla f) := \psi'(f) Q.
\]
Thus the velocity \(x'\) is given by \(\frac{1}{Q} \nabla f\). (The notation \(Q\) corresponds to \(Q(f)\), as in fact the expression \(g(\nabla f, \nabla f)\) is locally a function of \(f\) (see [10, Lemmas 7.5 and 11.1]), namely \(Q(f)\) of Equation (3.12)). Hence the \(\tilde{g}\)-arc length \(s\) of the curve \(x(f)\) is characterized by
\[
\frac{ds}{df} = \sqrt{\tilde{g}(x'(f), x'(f))} = \sqrt{e^{-\frac{4f}{m}} g(x'(f), x'(f))} = e^{-\frac{2f}{m}} \sqrt{\frac{1}{Q^2} Q} = e^{-\frac{2f}{m}} \frac{1}{\sqrt{Q}}.
\]
Completeness thus depends on the non-integrability of \( \int_{\inf f}^{\sup f} e^{-\frac{2f}{\sqrt{Q}}} \frac{1}{\sqrt{Q}} df \), where the infimum and supremum are with respect to the range of values of \( f \).

We distinguish a few cases. If the range of \( f \) is a finite interval, we focus, say, on the endpoint \( a := \inf f \). An infinite contribution to the integral near \( a \) can clearly only occur if \( a \) is a zero of \( Q \). Assuming we have such a zero, if \( Q'(a) \neq 0 \) then \( 1/\sqrt{Q} \) is asymptotic to \( (f-a)^{-1/2} \) near \( a \), so that the integral is finite. However, under these conditions \( g \) extends smoothly to the level set \( \{ f = a \} \) (see Remark 4.3 and Lemma 4.4 of [11]), hence \( \tilde{g} \) extends as well and the finite length curve \( x(f) \) has a limit at \( a \), so that, assuming the same behaviour at \( \sup f \), we see that \( \tilde{g} \) is complete (in fact the manifold is then compact).

Next, if \( Q(a) = Q'(a) = 0 \), then \( 1/\sqrt{Q} \) is asymptotic to \( (f-a)^p \) with \( p \leq -1 \) so that the integral diverges and \( x(t) \) has infinite length to the critical set. If this occurs also at \( \sup f \) then \( \tilde{g} \) is complete.

There remains the case where, say, \( \sup f = \infty \). Making the substitution \( u = 1/f_c \) gives, using the explicit expression (3.13) for \( \phi(f) \) in \( Q(f) = 2f_c \phi(f) \):

\[
\int_1^\infty e^{-\frac{2f}{\sqrt{Q}}} \frac{1}{\sqrt{Q}} df = -\int_0^1 e^{-\frac{1}{m-1}u^{-1}} e^{-\frac{m-1}{m-2} u^{-2}} \left( A\sum_{k=0}^m \frac{u^{-k}}{k!} + Be^{u^{-1}} \right)^{-\frac{3}{2}} u^{-2} du.
\]

Assuming \( B \neq 0 \) (the analysis of the case \( B = 0 \) is similar), neglecting the polynomial in \( u^{-1} \), the integrand of the last expression is asymptotic to

\[
\exp \left( -\left( -\frac{1}{m-1} - \frac{1}{2} \right) u^{-1} \right) u^{-\left( \frac{m-1}{2} + 2 \right)}
\]

as \( u \to 0^+ \). Over, say, the interval \([0, 1]\), the integral of this quantity converges for \( m > 1 \), so that \( \tilde{g} \) is incomplete.

References

[1] Barros, A., Batista, R., Ribeiro, E., Jr: Compact almost Ricci solitons with constant scalar curvature are gradient. [arXiv:1209.2720]
[2] Barros, A., Ribeiro, E., Jr.: Some characterizations for compact almost Ricci solitons. Proc. Amer. Math. Soc. 140 (2012), 1033-1040.
[3] Brinkmann, H. W.: Einstein spaces which are mapped conformally on each other. Math. Ann. 94 (1925), 119-145.
[4] Cao, H.-D.: Existence of gradient Kähler-Ricci solitons. In: Elliptic and parabolic methods in geometry, Minneapolis MN 1994, 1–16 A. K. Peters, Wellesley MA (1996).
[5] Cao, H.-D.: Recent progress on Ricci solitons. Recent advances in geometric analysis, 1–38, Adv. Lect. Math. (ALM), 11, Int. Press, Somerville, MA, 2010.
[6] Catino, G.: Generalized quasi-Einstein manifolds with harmonic Weyl tensor. Math. Z. 271 (2012), 751–756.
[7] Chen, B.-L., Strong uniqueness of the Ricci flow. J. Differential Geom. 82 (2009), 363–382.
[8] Chow, B., Chu, S.-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci flow: techniques and applications. Part I, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007.
[9] Derdzinski, A.: Solitony Ricciiego, Wiadomosci Matematyczne, 48 (2012), 1–32.
[10] Derdzinski, A., Maschler, G.: Local classification of conformally-Einstein Kähler metrics in higher dimensions. Proc. London Math. Soc. 87 (2003), 779–819.
[11] Derdzinski, A., Maschler, G.: Special Kähler-Ricci potentials on compact Kähler manifolds. J. reine angew. Math. 593 (2006), 73–116.
[12] Eminenti, M., La Nave, G.; Mantegazza, C.: Ricci solitons: the equation point of view. Manuscripta Math. 127 (2008), 345–367.
[13] Hamilton, R.S.: The Ricci flow on surfaces. In: Mathematics and general relativity, Santa Cruz CA, 1986. Contemp. Math., vol. 71, pp. 237–262. AMS, Providence RI (1988).
[14] Jauregui, J., Wylie, W.: Conformal diffeomorphisms of gradient Ricci solitons and generalized quasi-Einstein manifolds. arXiv:1209.1118v1.
[15] Koiso N.: On rotationally symmetric Hamilton’s equation for Kähler-Einstein metrics. In: Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18-I, pp. 327–337. Academic Press, Boston MA (1990).
[16] Kühnel, Wolfgang: Conformal transformations between Einstein spaces. In: Conformal geometry (Bonn, 1985/1986), 105146. Aspects Math., E12, Vieweg, Braunschweig, 1988.
[17] Maschler, G: Special Kähler-Ricci potentials and Ricci solitons. Ann. Global Anal. Geom. 34 (2008), 367–380.
[18] Petersen, P., Wylie, W. On the classification of gradient Ricci solitons. Geom. Topol. 14 (2010), 2277–2300.
[19] Phong D. H., Song J., Sturm, J.: Degeneration of Kähler-Ricci solitons on Fano manifolds. arXiv:1211.5849
[20] Pigola, S., Rigoli, M., Rimoldi, M., Setti, A. Ricci almost solitons. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 10 (2011), 757–799.
[21] Zhang, Z.: Degeneration of shrinking Ricci solitons. Int. Math. Res. Not. IMRN (2010), No. 21, 4137–4158.

Department of Mathematics and Computer Science, Clark University, Worcester, Massachusetts 01610, U.S.A.
E-mail address: gmaschler@clarku.edu