Classification of the FRW universe with a cosmological constant and a perfect fluid of the equation of state \( p = w \rho \)

Te Ha, Yongqing Huang, Qianyu Ma, Kristen D. Pechan, Timothy J. Renner, Zhenbin Wu, and Anzhong Wang
Physics Department, Baylor University, Waco, TX 76798-7316
(Dated: May 4, 2009)

We systematically study the evolution of the Friedmann-Robertson-Walker (FRW) universe with a cosmological constant \( \Lambda \) and a perfect fluid that has the equation of state \( p = w \rho \), where \( p \) and \( \rho \) denote, respectively, the pressure and energy density of the fluid, and \( w \) is an arbitrary real constant. Depending on the specific values of \( w \), \( \Lambda \) and the curvature \( k \) of the 3-dimensional spatial space of the universe, we classify all the solutions into various cases. In each case the main properties of the evolution are studied in detail, including the periods of deceleration and/or acceleration, and the existence of big bang, big crunch, and big rip singularities. In some particular cases, the solutions reduce to those considered in some standard textbooks, where by some typos may be corrected. The methods used is simply the conservation law of kinetic and potential energies in classical mechanics, and undergraduate students can easily follow the analysis and apply it to the studies of other cosmological models of the universe.

PACS numbers: 98.80.-k; 98.80.Jk; 04.20.-q; 04.20.Jb

I. INTRODUCTION

Recent observations of supernova (SN) Ia reveal the striking discovery that our universe has lately been in its accelerated expansion phase \[1\]. Cross checks from the cosmic microwave background radiation and large scale structure all confirm this \[2\]. Such an expansion was predicted neither by the standard model of particle physics nor by the standard model of cosmology, and the underlying physics still remains a complete mystery \[3\]. Since the precise nature and origin of the acceleration have profound implications, understanding them is one of the major challenges of modern cosmology. As the Dark Energy Task Force (DETF) stated \[2\]: "Most experts believe that nothing short of a revolution in our understanding of fundamental physics will be required to achieve a full understanding of the cosmic acceleration."

Within the framework of general relativity (GR), to account for such an acceleration, it requires the introduction of either a tiny positive cosmological constant or an exotic component of matter that has a very large negative pressure and interacts with other components of matter weakly. This invisible component is usually dubbed as dark energy. For a perfect fluid with the equation of state, \( w = p/\rho \), this implies \( w < -1/3 \), where \( p \) and \( \rho \) denote, respectively, the pressure and energy density of the fluid. On the other hand, a tiny positive cosmological constant is well consistent with all observations carried out so far \[5\]. However, when we consider its physical origin, we run into other severe problems: (a) Its theoretical expectation values exceed observational limits by 120 orders of magnitude \[7\]. (b) Its corresponding energy density is comparable with that of matter only recently. Otherwise, galaxies would have not been formed. Considering the fact that the energy density of matter depends on time, one has to explain why only now the two are in the same order. (c) Once the cosmological constant dominates the evolution of the universe, it dominates forever. An eternally accelerating universe seems not consistent with string/M-Theory, because it is endowed with a cosmological event horizon that prevents the construction of a conventional S-matrix describing particle interaction \[8\]. Other problems with an asymptotical de Sitter universe in the future were explored in \[9\].

In view of all the above, dramatically different models have been proposed, including quintessence \[10\], DGP branes \[11\], and the \( f(R) \) models \[12\]. For details, see \[8\] and references therein. However, it is fair to say that so far no convincing model has been constructed.

To introduce such a fascinating subject to students, it is always challenging both physically and mathematically. In this paper, our purposes are two-fold: (i) By studying the concrete models of the FRW universe, we shall show that such a subject can be easily understood in terms of classical mechanics. The method is very basic, and undergraduate students with some knowledge of classical mechanics can easily follow. As a matter of fact, the only knowledge required is the conservation law of total energy of a classical particle with mass \( m \) moving under a potential \( V(x) \) \[13\],

\[
\frac{1}{2} m \dot{x}^2 + V(x) = E, \tag{1.1}
\]

where \( \dot{x} \equiv dx/dt \) and \( E \) is the total energy of the system, which is conserved without external force. Then, taking
derivative of the above equation with respect to \( t \), we find that
\[
\ddot{x} = -\frac{dV(x)}{dx}. \quad (1.2)
\]
Thus, once the potential \( V(x) \) is known in terms of \( x \), one can immediately tell if the particle is accelerating or decelerating, without integrating Eq.\( (1.1) \) explicitly. In addition, one can also easily determine the range of \( x \) that the motion allows. Therefore, if the problem of the evolution of the universe can be expressed in the above form, we can use the methods of classical mechanics to study its evolution, and classify all the possible models of the universe. (ii) The second purpose of this paper is to correct some typos appearing in some standard textbooks.

The rest of the paper is organized as follows: In Sec. II, we consider the Friedmann equation coupled with a cosmological constant and a perfect fluid with the equation of state \( p = w\rho \) for any given curvature \( k \). After writing it in the form of Eq.\( (1.1) \), we study the potential \( V(x) \) case by case, and deduce the main properties of each model of the universe. In Sec. III, we present our main conclusions.

It should be noted that classification of a (non-relativistic) matter coupled with a dark energy was considered recently in [14], and the corresponding Penrose diagrams were also presented.

In this paper, we shall use the notations and conventions defined in [15].

II. CLASSIFICATION OF THE FRW UNIVERSE

The FRW universe is described by the metric \( \Sigma \),
\[
\begin{align*}
\quad d\tau^2 &= dT^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right), \quad (2.1)
\end{align*}
\]
in the spherically symmetric coordinates \( x^a = \{t, r, \theta, \phi\} \), \( (a = 0, 1, 2, 3) \), where \( k \) denotes the curvature of the three-dimensional spatial space of constant \( t \), and can be set \( k = 0, \pm 1 \), without loss of generality. \( a(t) \) is the expansion factor of the universe. It should be noted that Eq.\( (2.1) \) is invariant under the translation,
\[
\quad t' = t - t_a, \quad (2.2)
\]
where \( t_a \) is a constant. In the following we shall use this gauge freedom to fix the origin of the timelike coordinate \( t \). The expansion factor \( a(t) \) of the universe is determined through the Einstein field equations
\[
R_{ab} - \frac{1}{2} R g_{ab} = \kappa^2 T_{ab} + \Lambda g_{ab}, \quad (2.3)
\]
where \( \kappa^2 = 8\pi G/c^4 \) is the Einstein coupling constant, \( \Lambda \) denotes the cosmological constant, and \( T_{ab} \) the energy-momentum tensor of the matter field(s) filled in the universe. For a perfect fluid, we have
\[
T_{ab} = (\rho + p) u_a u_b - pg_{ab}, \quad (2.4)
\]
where \( u_a = \delta^t_a \) denotes the four-velocity of the fluid. It can be shown [15] that the Einstein field equations \( (2.3) \) for the metric \( (2.1) \) and energy-momentum tensor \( (2.4) \), have only two independent components, which can be cast in the form,
\[
\begin{align*}
H^2 &= \frac{8\pi G}{3} \rho + \frac{1}{3} \Lambda - \frac{k}{a^2}, \quad (2.5) \\
\frac{\dot{a}}{a} &= -\frac{4\pi G}{3} (\rho + 3p) + \frac{1}{3} \Lambda, \quad (2.6)
\end{align*}
\]
where \( H \equiv \dot{a}/a \). Note that in writing the above equation, we had chosen units such that the speed of light is one. On the other hand, the conservation law of matter fields, \( \nabla^a T_{ab} = 0 \), yields
\[
\dot{\rho} + 3H (\rho + p) = 0. \quad (2.7)
\]
It can be shown that this equation is not independent, and can be obtained from Eqs.\( (2.5) \) and \( (2.6) \).

Note that we have three unknowns, \( a, \rho \) and \( p \), but only two independent equations. Thus, to close the system, we need to have one more equation. Usually, this is given by the equation of state of the matter field. In this paper, we shall consider the case where
\[
p = w\rho, \quad (2.8)
\]
where \( w \) is an arbitrary real constant. Inserting Eq.\( (2.8) \) into Eq.\( (2.7) \), we find that it allows to integrate once and gives,
\[
\rho = \rho_0 \left( \frac{a_0}{a} \right)^{3(1+w)}, \quad (2.9)
\]
where \( \rho_0 \) and \( a_0 \) are the integration constants. Since \( \rho_0 \) represents the energy density when \( a = a_0 \), we shall assume that it is strictly positive \( \rho_0 > 0 \). Without loss of generality, we can always set \( a_0 = 1 \). Then, it can be shown that the Friedmann equation \( (2.8) \) can be cast in the form of Eq.\( (1.1) \) with \( m = 1, \ E = 0, \ x(t) = a(t) \), i.e.,
\[
\frac{1}{2} \dot{a}^2 + V(a) = 0, \quad (2.10)
\]
where
\[
V(a) = \frac{1}{2} k - \frac{1}{6} \Lambda a^2 - \frac{C}{a^{3+3w}}, \quad (2.11)
\]
with
\[
C \equiv \frac{4\pi G\rho_0}{3} > 0. \quad (2.12)
\]
When \( w = 0 \) the problem reduces to the one treated in [15]. To study the problem further, we consider the cases \( k = 0, \pm 1 \) separately.
A. \( k = 0 \)

When \( k = 0 \), Eq. (2.11) reduces to

\[
V(a) = - \frac{1}{6} \Lambda a^2 - \frac{C}{a^{1+3w}}. \tag{2.13}
\]

It is found convenient to consider the cases where \( \Lambda > 0 \), \( \Lambda = 0 \) and \( \Lambda < 0 \) separately.

1. \( \Lambda > 0 \)

When \( \Lambda > 0 \), Eq. (2.14) can be written as

\[
V(a) = - \frac{1}{6} \Lambda a^2 \left( 1 + \frac{\tilde{C}}{a^{1+3w}} \right), \tag{2.14}
\]

where \( \tilde{C} \equiv 6C/|\Lambda| > 0 \). It is found that, depending on the value of \( w \), the evolution of the universe can be significantly different. Thus, we shall further distinguish the following sub-cases:

\[
\begin{align*}
(i) \ w &> -\frac{1}{3}, \\
(ii) \ w &= -\frac{1}{3}, \\
(iii) \ -1 &< w < -\frac{1}{3}, \\
(iv) \ w &= -1, \quad (v) \ w < -1.
\end{align*} \tag{2.15}
\]

**Case A.1.1** \( w > -\frac{1}{3} \): Then, we find that \( V(a) \) is strictly negative, and \( V(a) \to -\infty \) for both \( a = 0 \) and \( a \to \infty \). It also has a maximum at \( a = a_m \equiv (3(1+3w)C/\Lambda)^{1/(3(1+w))} \), for which \( \tilde{a} < 0 \). As the universe expands to \( a = a_m \), it reaches the turning point, after which it starts to expand acceleratingly, i.e., \( \tilde{a}(a > a_m) > 0 \). The universe is asymptotically de Sitter, \( a(t) \propto e^{\sqrt{\Lambda/3}t} \).

**Case A.1.2** \( w = -\frac{1}{3} \): In this case, we find

\[
V(a) = - \frac{1}{6} \Lambda a^2 - C,
\]

\[
\tilde{a} = \begin{cases} 
< 0, & a < a_m, \\
= 0, & a = a_m, \\
> 0, & a > a_m.
\end{cases} \tag{2.16}
\]

\[
\dot{a} = - \frac{dV(a)}{da} = \frac{1}{3} \Lambda a + (3|w| - 1)C a^{3|w| - 2} \geq 0. \tag{2.17}
\]

The middle line of Fig. 1 is the potential for this case. Thus, in this case the universe is always accelerating except for the initial moment \( a = 0 \). On the other hand, from Eq. (2.19) we find that \( \rho \propto a^{-2} \to \infty \) as \( a \to 0 \), that is, a big bang singularity still occurs at \( a = 0 \).

**Case A.1.3** \( -1 < w < -\frac{1}{3} \): In this case, we find that the potential is that given by Fig. 1 and

\[
\dot{a} = - \frac{dV(a)}{da} = \frac{1}{3} \Lambda a + (3|w| - 1)C a^{3|w| - 2} \geq 0. \tag{2.18}
\]

where the equality holds only at \( a = 0 \) for \( w = -2/3 \). Thus, in this case the universe is always accelerating. Note that in the present case \( a = 0 \) still represents a big bang singularity, as can be seen from Eq. (2.19). It is also interesting to note that at \( a = 0 \) we have \( \dot{a} = \ddot{a} = 0 \), except for \( w = -2/3 \). Thus, when \( w \neq -2/3 \) the point \( a = 0 \) is a stationary point. However, it is not stable, and any perturbations will make the universe to expand. When \( w = -2/3 \) we have \( \dot{a}(a = 0) = 0 \) and \( \ddot{a}(a = 0) = (3|w| - 1)C > 0 \). Therefore, in the latter case \( a = 0 \) is not stationary, and the positive force will lead the universe automatically to expand.

**Case A.1.4** \( w = -1 \): In this case, the matter acts as a vacuum energy, and the potential is given by

\[
V(a) = - \frac{1}{6} \Lambda_{eff} a^2, \tag{2.19}
\]

where \( \Lambda_{eff} \equiv \Lambda + 6C \). Therefore, in this case the universe is de Sitter, and

\[
a(t) = e^{\sqrt{\Lambda_{eff}/3}(t-t_0)}. \tag{2.20}
\]

Recall that the de Sitter space is free of any kind space-time singularities at \( a = 0 \) as well as at \( a = \infty \).

**Case A.1.5** \( w < -1 \): In this case, the behavior of the potential \( V(a) \) and \( a(t) \) are similar to the case \( -1 < w < -1/3 \), except for the fact that now \( \rho \propto a^{3|w| - 1} \) is not singular at \( a = 0 \), although it is at \( a = \infty \), which is usually called a big rip singularity. At \( a = 0 \) we have
\[ a(t) = \left\{ \begin{align*}
\left( \frac{6C}{\Lambda} \right)^{1/2} \sinh \left[ (1 + w) \sqrt{\frac{3\Lambda}{4}} (t - t_s) \right] \right. \\
\left. \frac{1}{1 + w} \sqrt{\frac{4}{3\Lambda}} \sinh^{-1} \left( \sqrt{\frac{\Lambda}{6C}} \right) \right. 
\end{align*} \right\}_{t = t_0}
\] (2.21)

for \( w \neq -1 \), where \( t_s \) is given by

\[ t_s = t_0 - \frac{1}{1 + w} \sqrt{\frac{4}{3\Lambda}} \sinh^{-1} \left( \sqrt{\frac{\Lambda}{6C}} \right), \] (2.22)

so that \( a(t = t_0) = 1 \). For \( w > -1 \), without loss of generality, we can always use the gauge freedom of Eq. (2.22) to set \( t_s = 0 \), so that the big bang singularity occurs at \( t = 0 \). This will be the case in the rest of this paper. When \( w = -1 \), the solution is that of de Sitter, given by Eq. (2.20), which is free of any kind of spacetime singularities, and the solution is valid for any \( t \), that is, \( t \in (-\infty, \infty) \). For the cases where \( w < -1 \), we shall not do such a translation, so that the big rip singularity happens exactly at \( t = t_s \). Note that when \( w < -1 \), from Eq. (2.21) we find that the solution is valid only when \( t \in (-\infty, t_s) \), for which we have

\[ a(t) = \left\{ \begin{align*}
\sqrt{\frac{6C}{\Lambda}} \sin \left[ \sqrt{\frac{3\Lambda}{4}} (|w| - 1) (t - t_s) \right] \\
\infty, \quad t = t_s, \\
1, \quad t = t_0, \quad (w < -1) \\
0, \quad t = -\infty, 
\end{align*} \right\}_{t = t_0}^{3|w| - 1} \] (2.23)

Since now we have \( \rho \propto a^{3|w| - 1} \) we find that the space-time is indeed not singular at \( a(t = -\infty) = 0 \), but singular at \( a(t = t_s) = \infty \).

In Fig. 2 we summarize the main properties of the solutions for \( k = 0 \) and \( \Lambda > 0 \) with different \( w \).

\[ V(a) = -\frac{C}{a^{1+3w}} \leq 0, \]
\[ \ddot{a} = -(3w + 1) \frac{C}{a^{2+3w}} \]
\[ a = \begin{cases} < 0, & w > -1/3, \\
= 0, & w = -1/3, \\
> 0, & w < -1/3. \end{cases} \] (2.24)

Fig. 3 shows the potential \( V(a) \). Similar to the last case, now we can also integrate the Friedmann equation (2.15) to obtain the explicit solutions of \( a(t) \) and \( \rho(t) \),

\[ a(t) = \begin{cases} 
3(1 + w) \sqrt{\frac{2}{3}} (t - t_s) \frac{1}{\sqrt{\frac{2}{3} |w| - 1}}, & w \neq -1, \\
\frac{\rho_0}{e^{\sqrt{\frac{2}{3} |w| - 1}}(t - t_0)}, & w = -1, \end{cases} \]
\[ \rho(t) = \begin{cases}
\frac{\rho_0}{e^{\sqrt{\frac{2}{3} |w| - 1}t}}, & w \neq -1, \\
\rho_0, & w = -1, \end{cases} \] (2.25)

where \( \rho_0 \equiv 2\rho_0/[9(1 + w)^2C] \), and

\[ t_s = t_0 - \frac{1}{1 + w} \sqrt{\frac{2}{9C}}. \] (2.26)

When \( w < -1 \), Eq. (2.25) shows that, to keep \( a(t) \) real and positive, we must require \( t \in (-\infty, t_s) \). As \( t \to -\infty \), both \( a(t) \) and \( \rho(t) \) vanish, while when \( t \to t_s \) all of them become unbounded, that is, a big rip singularity is developed there.

In Fig. 3 we summarize the main properties of the solutions for \( k = 0 \) and \( \Lambda = 0 \) with different values of \( w \).
The spacetime is de Sitter for $w > -1$. The energy density $\rho(t)$ for $k = 0$ and $\Lambda = 0$ is developed at $t = t_0$, at which we have $a(t) = \rho(t) = \infty$. During the whole process, the universe is always decelerating,

$$\ddot{a} = \frac{dV(a)}{da} < 0,$$  \hspace{1cm} (2.32)

as can be seen from Fig. 5.

**Case A.3.2** $w = -\frac{1}{3}$: In this case, we find

$$V(a) = \frac{1}{6} |\Lambda| a^2 - C = \begin{cases} -C, & a = 0, \\ -3C, & a \to \infty, \end{cases}$$

and the motion is also restricted to $a \leq a_m$. However, there is a fundamental difference between this case and the last two cases: The potential $V(a)$ has a minimum at $a = a_{m\min}$, at which its expanding velocity becomes zero, and afterwards it will start to collapse, until a big crunch singularity is developed at $a = 0$, as shown in Fig. 6.

**Case A.3.3** $-1 < w < -\frac{1}{3}$: In this case, from Fig. 5 we can see that $V(a = 0) = 0 = V(a_m)$, and the motion is initially accelerating. But, when it expands to $a = a_{m\min}$, it starts to decelerate until $a = a_m$, at which its expanding velocity becomes zero, and afterwards it will start to collapse, until a big crunch singularity is developed at $a = 0$, as shown in Fig. 6.

**Case A.3.4** $w = -1$: In this case, we have

$$V(a) = -\frac{1}{6} |\Lambda| a^2 \left( \frac{6C}{|\Lambda|} - 1 \right).$$  \hspace{1cm} (2.34)
Therefore, now there is solution only when $\Lambda_{eff} > 0$, for which the universe is de Sitter, and

$$a(t) = e^{\sqrt{\Lambda_{eff}/3}(t-t_0)}, \quad (2.35)$$

where $\Lambda_{eff} \equiv |\Lambda| - 6C$.

Case A.3.5) $w < -1$: In this case, there is a minimum $a_{min}$ for which $V(a < a_{min}) \geq 0$. Therefore, in contrast to the previous case, now the motion of the universe is restricted to $a \geq a_{min}$. The universe starts to expand from $a = a_{min}$ until $a(t_s) = \infty$ within a finite time, whereby a big rip singularity is formed, as shown in Fig. 6.

B. $k = 1$

In this case, the potential given by Eq. (2.11) can be written as

$$V(a) = \frac{1}{2} - \frac{1}{6} \Lambda \alpha^2 \left[ 1 + \left( \frac{\alpha}{\alpha} \right)^{3(1+w)} \right], \quad (2.36)$$

where $\alpha = (12C/\Lambda)^{1/(1+w)}$. Following the case of $k = 0$, we also distinguish the three cases, $\Lambda > 0$, $\Lambda = 0$ and $\Lambda < 0$.

1. $\Lambda > 0$

When $\Lambda > 0$, it is found convenient to further divide it into the five sub-cases defined as in Eq. (2.15).

Case B.1.1) $w > -\frac{1}{3}$: In this case, it can be shown that for any given $w$ and $\rho_0$ there always exists a critical value $\Lambda_c$ and radius $a_m$ that satisfy the conditions,

$$V(a_m, w, \rho_0, \Lambda_c) = 0, \quad V'(a_m, w, \rho_0, \Lambda_c) = 0, \quad (2.37)$$

where a prime denotes the ordinary differentiation with respect to $a$. It can be shown that the solutions of the above conditions are

$$a_m = \left[ 3 \left( 1 + w \right) C \right]^{1/w},$$

$$\Lambda_c = \left( \frac{1 + 3w}{1 + w} \right) \left[ 3 \left( 1 + w \right) C \right]^{1/3w} \quad (2.38)$$

As to be shown below, the solutions with $\Lambda > \Lambda_c$ have quite different properties from the ones with $\Lambda < \Lambda_c$. Therefore, in the following we shall further distinguish the three different cases, $\Lambda > \Lambda_c$, $\Lambda = \Lambda_c$ and $\Lambda < \Lambda_c$.

Case B.1.1.1) $w > -\frac{1}{3}, \Lambda > \Lambda_c$: In this case, the potential $V(a)$ is always negative for any given $a$, as shown in Fig. 7. Therefore, the corresponding solutions have no turning point. If the universe initially starts to expand from a big bang singularity at $a = 0$, it will expand forever, as shown by Fig. 8. However, the potential has a maximum at $a = a_m$, for which we have

$$\ddot{a} = \begin{cases} < 0, & a < a_m, \\ = 0, & a = a_m, \\ > 0, & a > a_m, \end{cases} \quad (2.39)$$

that is, the universe is initially decelerating. Once it expands to $a = a_m$, it starts to expand acceleratingly.

Case B.1.1.2) $w > -\frac{1}{3}, \Lambda = \Lambda_c$: In this case, there exists a static point $a_m$, at which we have $V(a_m) = V'(a_m) = 0$, as one can see from Fig. 7. Therefore, if the universe starts to expand from the big bang at $a = 0$, it will expand until $a = a_m$. The universe is decelerating during this period. Since at the point $a = a_m$, we have $\ddot{a} = 0 = \ddot{a}$, the universe will become static, once it reaches this point. However, it is not stable, and with small perturbations, the universe will either collapse until a singularity is developed at $a = 0$ or expand forever with

--

FIG. 6: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 0$ and $\Lambda < 0$. The spacetime is singular at $a = 0$ for all the cases with $w > -1$, (a big bang singularity). it is de Sitter for $w = -1$. When $w < -1$, a big rip singularity is developed at $t = t_s$, at which we have $a(t_s) = \rho(t_s) = \infty$.

FIG. 7: The potential given by Eq. (2.36) for $k = 1$, $w > -\frac{1}{3}$ and $\Lambda > 0$, where $\Lambda_c = \Lambda_c(w, \rho_0)$.

FIG. 8: The potential given by Eq. (2.36) for $k = 1$, $w > -\frac{1}{3}$ and $\Lambda > 0$, where $\Lambda_c = \Lambda_c(w, \rho_0)$. 

If the universe initially at $a = a_i > a_m$, from Fig. 7, we can see that it will expand forever. Since $V'(a)$ is always negative, so the universe in this region is always accelerating.

**Case B.1.1.3** $w > -\frac{1}{3}$, $0 < \Lambda < \Lambda_c$: In this case, $V(a) = 0$ has two real and positive roots, $a_1$ and $a_2$, as shown in Fig. 7. Without loss of generality, we assume that $a_2 > a_1$. Since $V(a) > 0$ for $a \in (a_1, a_2)$, the motion is forbidden in this region. Similar to the last case, depending on the initial conditions, the universe can have quite different evolutions. In particular, if it starts to expand from the big bang singularity at $a = 0$, it will expand until $a = a_1$, at which we have $\dot{a} = 0$ and $\ddot{a} < 0$. Since $\ddot{a} < 0$ at this point, the universe will start to collapse afterwards, until a big crunch singularity is developed at $a = 0$. If the universe starts to expand at $a_i \geq a_2$, it will expand forever. In the latter case, the universe is always accelerating, as can be seen from Fig. 7.

**Case B.1.2** $w = -\frac{1}{3}$: In this case, we have

$$V(a) = \frac{1}{2} (1 - C) - \frac{1}{6} \Lambda a^2.$$ (2.40)

Thus, depending on the value of $C(\rho_0)$, the motion of the universe will be different. In particular, when $C(\rho_0) < 1/2$, there exists a minimal $a_{min}$, for which $V(a < a_{min}) > 0$, that is, the motion in the region $0 < a < a_{min}$ is forbidden, as shown in Fig. 9. When $C(\rho_0) \geq 1/2$, the universe can start to expand from the big bang singularity at $a = 0$. In all the cases we have $V'(a) < 0$, so that the universe is always accelerating [cf. Fig. 10].

**Case B.1.3** $-1 < w < -\frac{1}{3}$: In this case, we find that $V'(a)$ is strictly negative for any $a \geq 0$ with $V(0) = 1/2$. Therefore, similar to the case $w = -1/3$ and $C < 1/2$, there exists a minimal $a_{min}$, for which $V(a < a_{min}) > 0$, and the motion in the region $0 < a < a_{min}$ is forbidden, as shown in Fig. 9. Thus, in the present case the universe starts to expand from a radius $a_i \geq a_{min}$ until $a = \infty$ without turning point. Again, because now $V'(a) < 0$ for any $a \geq a_{min}$, the universe is always accelerating. However, there is no any kind of singularities to be developed, either big bang, big crunch, or big rip, as shown by Fig. 10.

**Case B.1.4** $w = -1$: In this case, the potential is a simple parabola,

$$V(a) = \frac{1}{2} - \frac{1}{6} (\Lambda + 6C) a^2,$$ (2.41)

schematically shown by the top curve in Fig. 9. As a result, the motion is similar to the last case, except for the fact that now $\rho = \rho_0$.

**Case B.1.5** $w < -1$: In this case, we also have $V'(a) < 0$ and there exists a finite radius, $a_{min}$, such that when $a < a_{min}$ we have $V(a) > 0$, and when $a \geq a_{min}$ we have $V(a) \leq 0$. The only difference is that in the present case there is a big rip singularity that happens at $a = \infty$, as now we have $\rho \propto a^{2(|w|-1)}$, as shown in Fig. 10.
FIG. 11: The potential given by Eq. (2.42) for \( k = 1 \) and \( \Lambda = 0 \).

2. \( \Lambda = 0 \)

In this case, we find that

\[
V(a) = \frac{1}{2} - \frac{C}{a^{1+3w}}. \tag{2.42}
\]

Fig. 11 shows the potential for various values of \( w \), from which we can see that when \( w > -1/3 \), the motion of the universe is restricted to \( a \leq a_m \), where \( a_m \) is the solution of \( V(a) = 0 \). The universe starts to expand at the big bang singularity \( a = 0 \) until the turning point \( a = a_m \). Afterwards, it will start to collapse until a big crunch singularity is developed at \( a = 0 \), as shown in Fig. 12.

When \( w = -1/3 \), there is motion only for \( \mathcal{C} > 1/2 \), for which the universe expands linearly from a big bang singularity at \( a = 0 \) with \( \ddot{a} > 0 \).

When \( -1 \leq w < -1/3 \), the motion is possible only for \( a > a_{\text{min}} \), as shown in Figs. 11 and 12. The universe starts to expand from the initial point \( a_i \geq a_{\text{min}} \) with \( \ddot{a} > 0 \). No turning point exists, so the universe will expand forever. During the whole process, the matter density remains finite, so no singularity exists in this case.

When \( w < -1 \), it can be shown that the motion for \( a < a_{\text{min}} \) is also forbidden. As a result, no big bang singularity exists in the present case. But, a big rip singularity will be developed as \( a \to \infty \), as shown by Fig. 12. In the whole process, we have \( \ddot{a} > 0 \).

The potential for various values of \( w \) is shown in Fig. 11. The potential is non-positive only for \( a > a_{\text{min}} \), where \( a_{\text{min}} \) is the solution of \( V(a) = 0 \). The universe starts to expand at the big bang singularity \( a = 0 \) until the turning point \( a = a_m \). Afterwards, it will start to collapse until a big crunch singularity is developed at \( a = 0 \), as shown in Fig. 12.

When \( w = -1/3 \), there is motion only for \( \mathcal{C} > 1/2 \), for which the universe expands linearly from a big bang singularity at \( a = 0 \) with \( \ddot{a} > 0 \).

When \( -1 \leq w < -1/3 \), the motion is possible only for \( a > a_{\text{min}} \), as shown in Figs. 11 and 12. The universe starts to expand from the initial point \( a_i \geq a_{\text{min}} \) with \( \ddot{a} > 0 \). No turning point exists, so the universe will expand forever. During the whole process, the matter density remains finite, so no singularity exists in this case.

When \( w < -1 \), it can be shown that the motion for \( a < a_{\text{min}} \) is also forbidden. As a result, no big bang singularity exists in the present case. But, a big rip singularity will be developed as \( a \to \infty \), as shown by Fig. 12. In the whole process, we have \( \ddot{a} > 0 \).

In the case of \( \Lambda = 0 \), the potential is given by

\[
V(a) = \frac{1}{2} + \frac{1}{6} |a|^2 - \frac{\mathcal{C}}{a^{1+3w}}. \tag{2.43}
\]

3. \( \Lambda < 0 \)

In this case, we have

\[
V(a) = \frac{1}{2} + \frac{1}{6} |\Lambda| a^2 - \frac{\mathcal{C}}{a^{1+3w}}. \tag{2.43}
\]

This has the properties as shown by Fig. 13. In particular, when \( w > -1/3 \), we find that the universe starts to expand from a big bang singularity at \( a = 0 \) until a maximal radius \( a_m \) where \( V(a_m) = 0 \). Afterwards, the universe starts to collapse, and finally a big crunch is developed at \( t = 2t_m \), where \( t_m \) is determined by \( a_m = a(t_m) \). In the whole process, the universe is decelerating, as shown by Fig. 14.

When \( w = -1/3 \), the potential is non-negative for \( \mathcal{C} \leq 1/2 \), so the motion is forbidden. When \( \mathcal{C} > 1/2 \), we have \( V(a) < 0 \) for \( a < a_m \), where \( a_m \) is the root of \( V(a) = 0 \), as shown in Fig. 14. Thus, the motion now is possible in the region \( a < a_m \), for which the universe starts to expand from a big bang singularity at \( a = 0 \). Once it reaches its maximal at \( a_m \), it starts to collapse until a big crunch is developed at \( t = 2t_m \).

When \( -1 < w < -1/3 \), similar to the last case, the potential is negative only for \( a < a_m \), as shown in Fig. 14. In particular, a big rip (crunch) singularity happens at \( t = 0 \) \( (t = 2t_m) \). The difference is that now there exists a time \( t_{\text{max}} \) so that for \( 0 < t < t_{\text{max}} \) or \( 2t_m - t_{\text{max}} < t < 2t_m \) the universe is accelerating, while during the time \( t_m - t_{\text{max}} < t < 2t_m - t_{\text{max}} \) it is decelerating, where \( t_{\text{max}} \) is the root \( V(t_{\text{max}}) = 0 \).

When \( w = -1 \), the potential is non-positive only for \( \mathcal{C} > |\Lambda|/6 \) and \( a \geq a_{\text{min}} \), where \( a_{\text{min}} \) is the root of \( V(a) = 0 \), as shown in Fig. 13. Therefore, in this case the universe starts to expand always at an initial radius \( a_i \geq a_{\text{min}} \). The universe will expand forever with \( \ddot{a} > 0 \). However, the spacetime is not singular even when \( a = \infty \).

When \( w < -1 \), the potential is non-positive only for \( a \geq a_{\text{min}} \), where again \( a_{\text{min}} \) is the root of \( V(a) = 0 \), as shown in Fig. 13. The evolution of the universe in this case is similar to the last one, except for that now a big rip singularity will be developed at \( a = \infty \).
V(a)

FIG. 13: The potential given by Eq. (2.43) for \( k = 1 \) and \( \Lambda = 0 \): (a) for \( w > -1/3 \); (b) for \( w = -1/3 \) and \( C > 1/2 \); (c) for \(-1 < w < -1/3\); (d) for \( w = -1 \) and \( C > |\Lambda|/6 \); and (e) for \( w < -1 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
w & 1/2 & 0 & 1/2 & \infty \\
\hline
V(a) & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
\end{array}
\]

FIG. 14: The expansion factor \( a(t) \), the acceleration \( \ddot{a}(t) \), and the energy density \( \rho(t) \) for \( k = 1 \) and \( \Lambda < 0 \). There are both big bang and big crunch singularities in the case with \( w > -1 \), while only a big bang singularity occurs in the case \( w = -1 \), and no singularities for the case \( w = -1 \), while there is a big rip singularity at \( a = \infty \) for \( w < -1 \).

C. \( k = -1 \)

When \( k = -1 \), the potential is given by

\[
V(a) = -\frac{1}{2} - \frac{1}{6} \Lambda a^2 - \frac{C}{a^{1+3w}} \quad (2.44)
\]

To study the motion of the universe in this case, it is also found convenient to distinguish the three cases \( \Lambda > 0 \), \( \Lambda = 0 \) and \( \Lambda < 0 \), and in each case there are five sub-cases with different choices of \( w \).

1. \( \Lambda > 0 \)

In this case, we find that \( V(a) \to -\infty \) as \( a \to \infty \), and

\[
V(a) = \begin{cases} 
-1/2, & w < -1/3, \\
-(1/2 + C), & w = -1/3, \\
-\infty, & w > -1/3, 
\end{cases} \quad (2.45)
\]

when \( a \to 0 \), as shown by Fig. 15. Thus, when \( w < -1/3 \), the potential has a maximum at \( a_m \) where \( V(a_m) = 0 \). The universe starts to expand from a big bang at \( a = 0 \). Initially, it is decelerating, \( \ddot{a} < 0 \). However, when it expands to \( a_m \), it turns to expand acceleratingly, \( \ddot{a} > 0 \), as shown in Fig. 16. When \(-1 < w \leq -1/3 \), the universe expands from a big bang at \( a = 0 \) until \( a = \infty \), and there is no turning point. It expands with \( \ddot{a} > 0 \) in the whole process. The case of \( w = -1 \) is similar to the case of \(-1 < w \leq -1/3 \), except that the spacetime is not singular either at \( a = 0 \) or at \( a = \infty \), as shown in Fig. 16. When \( w < -1 \), one can see that the universe starts to expand from \( a = 0 \) with \( \ddot{a} > 0 \) for any given \( a \). There is no big bang singular at \( a = 0 \), but there is a big rip singularity at \( a = \infty \).

2. \( \Lambda = 0 \)

In this case, we have

\[
V(a) = -1/2 - \frac{C}{a^{1+3w}} < 0, \\
\ddot{a} = 0, \quad w = -1, (2.46)
\]

when \( a \to 0 \).
FIG. 16: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$ and $\Lambda > 0$. There are a big bang singularity for $w > -1$, no singularity for $w = -1$, and a big rip singularity at $a = \infty$ for $w < -1$.

When $a \in [0, \infty)$, as shown by Fig. 17, we also have

$$\rho(a) = \begin{cases} 
\infty, & w > -1/3, \\
\rho_0, & w = -1/3, \\
0, & w < -1/3,
\end{cases} \quad (2.47)$$

as $a \to 0$, and

$$\rho(a) = \begin{cases} 
0, & w > -1/3, \\
\rho_0, & w = -1/3, \\
\infty, & w < -1/3,
\end{cases} \quad (2.48)$$

as $a \to \infty$. Fig. 18 shows the motion of the universe for each given $w$.

3. $\Lambda < 0$

In this case, we have

$$V(a) = -\frac{1}{2} + \frac{1}{6}|\Lambda|a^2 - \frac{C}{a^{1+3w}}. \quad (2.49)$$

Depending on the values $\Lambda$, $C$, and $w$, the potential will have quite different properties. In the following we shall study all of them case by case.

**Case C.3.1** $w > -1/3$: In this case, the potential is shown schematically in Fig. 19, from which we can see that it is non-positive only for $a \leq a_m$, where $a_m$ is the positive root of $V(a) = 0$. Clearly, in this case there is a big bang singularity at $a = 0$, from which the universe starts to expand until $a = a_m$. Afterwards, it will collapse so that finally a big crunch singularity is developed at $t = 2t_m$, at which we have $a(2t_m) = 0$, as shown by Fig. 20.

**Case C.3.2** $w = -1/3$: The potential in this case is similar to the last case, except for the fact that now $V(0) = -1/2 - C$, as shown in Fig. 19. The motion of the universe is qualitatively the same as that in the last case, as shown by Fig. 20.

**Case C.3.3** $-1 < w < -1/3$: In this case the potential has a minimum at $a = a_{m_{\text{min}}}$, as shown in Fig. 19, for which we find that $\ddot{a} < 0$ for $a < a_{m_{\text{min}}}$, and $\ddot{a} > 0$ for $a > a_{m_{\text{min}}}$, as shown by Fig. 20.

**Case C.3.4** $w = -1$: In this case, depending on the ratio $6C/|\Lambda|$, there are three distinguished sub-cases. When $6C/|\Lambda| < 1$, the potential is non-positive only when $a \leq a_m$ where $a_m = [3/(|\Lambda| - 6C)]^{1/2}$, as shown by Fig. 21. Then, the universe starts to expand from $a = 0$ until $a = a_m$. Afterwards, it will start to collapse until $a = 0$ again. But in the whole process, no spacetime singularity is developed. So, the universe is oscillating between $a = 0$ and $a = a_m$, as shown in Fig. 22.

When $6C/|\Lambda| = 1$, we find that $V(a) = -1/2$, and the universe expands linearly starting from $a = 0$. There is no turning point, and no spacetime singularity, as shown by Figs. 21 and 22.

When $6C/|\Lambda| > 1$, we find that $V(a) < -1/2$ for any given $a$. Then, starting from $a = 0$, the universe expands always acceleratingly ($\ddot{a} > 0$) until $a = \infty$, as shown by Figs. 21 and 22. No spacetime singularity is developed during the whole process.

**Case C.3.5** $w < -1$: In this case, it can be shown that for any given $\rho_0$ there always exists a critical value $\Lambda_c$ so that $V(a) = 0$ has two root positive roots when $|\Lambda| > |\Lambda_c|$; one positive root when $|\Lambda| = |\Lambda_c|$; and no
are the two positive roots of $V$ is positive in the region $a > 0$. Therefore, the motion of the universe now is restricted to the regions $0 < a < a_1$ and $a > a_2$, depending on its initial condition. If the universe starts to expand at $a = 0$, it will expand until its maximal radius $a = a_1$, and then collapse until $a = 0$. In the whole process, we have $\ddot{a} < 0$. Since at $a = 0$ the spacetime is not singular, so the universe will start to expand again. This process will be repeating endlessly, as shown in Fig. 23. However, if it starts to expand at a radius $a_i > a_2$, the universe will expand forever and never stops, as now $\ddot{a} > 0$ for any given $a \geq a_2$. A big rip singularity will be finally developed at $a = \infty$, since now we have $\rho \to \infty$, as $a \to \infty$.

Case C.3.5.1 $|\Lambda| > |\Lambda_c|$: In this case, the potential is positive in the region $a_1 < a < a_2$, where $a_1$ and $a_2$ are the two positive roots of $V(a) = 0$ with $a_2 > a_1$. Therefore, the motion of the universe now is restricted to the regions $0 \leq a \leq a_1$ and $a \geq a_2$, depending on its initial condition. If the universe starts to expand at $a = 0$, it will expand until its maximal radius $a = a_1$, and then collapse until $a = 0$. In the whole process, we have $\ddot{a} < 0$. Since at $a = 0$ the spacetime is not singular, so the universe will start to expand again. This process will be repeating endlessly, as shown in Fig. 23. However, if it starts to expand at a radius $a_i \geq a_2$, the universe will expand forever and never stops, as now $\ddot{a} > 0$ for any given $a \geq a_2$. A big rip singularity will be finally developed at $a = \infty$, since now we have $\rho \to \infty$, as $a \to \infty$.

Case C.3.5.2 $|\Lambda| = |\Lambda_c|$: In this case, there exists a static point $a_m$, at which we have $V(a_m) = V'(a_m) = 0$, as one can see from Fig. 23 where $a_m = [3 (|w| - 1) |\Lambda|]^{-1/3(|w| - 1)}$. Therefore, if the universe starts to expand from $a = 0$, it will expand until $a = a_m$ with $\ddot{a} < 0$. Since at the point $a = a_m$, we have $\ddot{a} = 0 = \dddot{a}$, the universe will become static at this point. However, it is not stable, and with small perturbations, the universe will either collapse until $a = 0$ or expand forever with $\ddot{a} > 0$. It should be noted that the spacetime is not singular at $a = 0$. So, if it collapses, it will start to expand again when the point $a = 0$ is reached. If the universe initially at $a = a_i > a_m$, from Fig. 23 we can see that it will expand forever. Since $V'(a)$ is always positive root when $|\Lambda| < |\Lambda_c|$, as shown by Fig. 19 where $\Lambda_c$ is the solution of the equations $V'(a_m, \Lambda_c) = 0$, and $V'(a_m, \Lambda_c) = 0$. It can be shown that it is given by,

$$\Lambda_c = \left(\frac{3 |w| - 1}{|w| - 1}\right) \left[3 (|w| - 1) |\Lambda|\right]^{1/3(|w| - 1)}.$$  (2.50)

FIG. 19: The potential given by Eq.(2.49) for $k = -1$, $\Lambda < 0$ and $w > -1$.

FIG. 20: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$, $\Lambda < 0$ and $w > -1$. There are both big bang and big crunch singularities for all the cases with $w > -1$.

FIG. 21: The potential given by Eq.(2.49) for $k = -1$, $\Lambda < 0$ and $w = -1$.

FIG. 22: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$, $\Lambda < 0$ and $w = -1$. The spacetime is not singular in any of these cases.
ing. A big rip singularity will be also finally developed at all the cases, except for the first sub-cases of $\Lambda < 0$, where $\Lambda_0$ is given by Eq.(2.50). There are big rip singularities in all cases, except for the first sub-cases of $|\Lambda| > |\Lambda_0|$; (b) is for $|\Lambda| = |\Lambda_0|$; and (c) is for $|\Lambda| < |\Lambda_0|$.

![Graph (a)](image)

**FIG. 23:** The potential given by Eq.(2.49) for $k = -1$, $w < -1$ and $\Lambda < 0$, where $\Lambda_0$ is given by Eq.(2.50). There are big rip singularities in all cases, except for the first sub-cases of $|\Lambda| > |\Lambda_0|$; (b) is for $|\Lambda| = |\Lambda_0|$; and (c) is for $|\Lambda| < |\Lambda_0|$.

![Graph (b)](image)

![Graph (c)](image)

**TABLE:**

| $|\Lambda| > |\Lambda_0|$ | $|\Lambda| = |\Lambda_0|$ | $|\Lambda| < |\Lambda_0|$ |
|------------------------|------------------------|------------------------|
| $a(t)$                 | $0$, $a = a_m$         | $0$, $a = a_m$         |
| $\rho(t)$              | $0$, $a = a_m$         | $0$, $a = a_m$         |

FIG. 24: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$, $w < -1$ and $\Lambda < 0$, where $\Lambda_0$ is given by Eq.(2.50). There are big rip singularities in all the cases, except for the first sub-cases of $|\Lambda| > |\Lambda_0|$ and $|\Lambda| = |\Lambda_0|$.

negative, so the universe in this case is always accelerating. A big rip singularity will be also finally developed at $a = \infty$.

**Case C.3.5.3** $|\Lambda| < |\Lambda_0|$: In this case, the potential $V(a)$ is always negative for any given $a$, as shown in Fig. 23. Therefore, the corresponding solutions have no turning point. If the universe initially starts to expand from $a = 0$, it will expand forever. However, the potential has a maximum at $a = a_m$, for which we have

$$\ddot{a} = \begin{cases} < 0, & a < a_m, \\ 0, & a = a_m, \\ > 0, & a > a_m, \end{cases}$$

(2.51)

that is, the universe is initially decelerating. Once it expands to $a = a_m$, it starts to expand acceleratively. Similar to the last two cases, a big rip singularity will also finally developed at $a = \infty$.

**III. CONCLUSIONS**

In this paper, we have systematically studied the solutions of the Friedmann-Robertson-Walker (FRW) universe with a cosmological constant and a perfect fluid that has the equation of state $p = w\rho$, where $p$ and $\rho$ denote, respectively, the pressure and energy density of the fluid, and $w$ is an arbitrary real constant. Writing the motion of the universe in the form of Eq.(1.1), we have been able to classify all the solutions according to the different values of $k$, $\rho_0$, $w$ and $\Lambda$, by simply using the knowledge of one-dimensional motion in classical mechanics [13], where $k = 0, \pm 1$ denotes the curvature of the FRW 3-dimensional spatial space, $\rho_0$ the energy density of the matter field when $a = 1$, and $\Lambda$ the cosmological constant. All these solutions are classified and presented in Figs. 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22 and 24. Some particular cases were already discussed in various standard textbooks, whereby some typos may be corrected.

In this paper, we have studied the FRW universe in the Einstein theory of gravity. Clearly, such studies can be easily applied to all the models where the evolution of the universe can be cast in the form of Eq.(1.1), including brane worlds in string/M theory [16], asymmetric branes [17], and cosmological models in the Horava-Lifshitz gravity [18].

[1] A.G. Riess et al., Astron. J. 116, 1009 (1998); S. Perlmutter et al., Astrophys. J. 517, 565 (1999).
[2] A.G. Riess et al., Astrophys. J. 607, 665 (2004); P. Astier et al., Astron. and Astrophys. 447, 31 (2006); D.N. Spergel et al., astro-ph/0603440; W.M. Wood-Vasey et al., astro-ph/0701041; T.M. Davis et al., astro-ph/0701510.
[3] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000); P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003); T. Padmanabhan, Phys. Rep. 380, 235 (2003); V. Sahni, "Dark Matter and Dark Energy," arXiv:astro-ph/0403324 (2004); The Physics of the Early Universe, edited by E. Papantonopoulos (Springer, New York 2005), P. 141; T. Padmanabhan, Proc. of the 29th Int. Cosmic Ray Conf. 10, 47 (2005); E.J. Copeland, M. Sami, and S. Tsujikawa, "Dynamics of dark energy," arXiv:hep-th/0603057 (2006); E.W. Kolb, "Cosmology and the Unexpected," arXiv:0709.3102; E. Linder, "Mapping the cosmological expansion," arXiv:0801.296; J.A. Frieman, M.S. Turner, and D. Huterer, "Dark Energy and the Accelerating Universe," arXiv:0803.0982.
[4] A. K. Albrecht et al., Report of the dark energy task force, arXiv:astro-ph/0606991.
[5] S. Sullivan, A. Cooray, and D.E. Holz, arXiv:0706.3730; A. Mantz, et al., arXiv:0709.4294.
[6] A.D. Chernin, et al., arXiv:0704.2753, arXiv:0706.4171.
[7] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989); S.M. Carroll, arXiv:astro-ph/0004075; T. Padmanabhan, Phys. Rept. 380, 235 (2003); S. Nobbenhuis, arXiv:gr-qc/0411093; J. Polchinski, arXiv:hep-th/0603249; J.M. Cline, arXiv:hep-th/0612129

[8] W. Fischler, et al., JHEP, 07, 003 (2001); J.M. Cline, ibid., 08, 035 (2001); E. Halyo, ibid., 10, 025 (2001); S. Hellerman, ibid., 06, 003 (2003).

[9] L.M. Krauss and R.J. Scherrer, arXiv:0704.0221 and references therein.

[10] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998); A. R. Liddle and R. J. Scherrer, Phys. Rev. D59, 023509 (1999); P. J. Steinhardt, L. M. Wang and I. Zlatev, ibid., 59, 123504 (1999).

[11] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B484, 112 (2000); C. Deffayet, ibid., 502, 199 (2001); V. Sahni and Y. Shtanov, JCAP, 0311, 014 (2003).

[12] S. Capozziello, S. Carloni, and A. Troisi, arXiv:astro-ph/0303041; S.M. Carroll, et al, Phys. Rev. D70, 043528 (2003); S. Nojiri and S.D. Odintsov, ibid., 68, 123512 (2003).

[13] H. Goldstein, C. Poole, and J. Safko, Classical Mechanics, Third edition (Addison Wesley, New York, 2002).

[14] T. Chiba, R. Takahashi, and N. Sugiyama, Class. Quantum Grav. 22, 3745 (2005).

[15] R. d’Inverno, Introducing Einstein’s Relativity (Clarendon Press, Oxford, 2003).

[16] Y.-G. Gong, A. Wang, and Q. Wu, Phys. Lett. B663, 147 (2008) [arXiv:0711.1597]; and A. Wang and N.O. Santos, Phys. Lett. B 669, 127 (2008) [arXiv:0712.3938].

[17] A. Padilla, Class. Quantum Grav. 22, 681 (2005); 22, 1087 (2005); K. Koyama and K. Koyama, Phys. Rev. D72, 043511 (2005); C. Charmousis, R. Gregory, and A. Padilla, arXiv:0706.0857; Y. Shtanov, et al, arXiv:0901.3074 and references therein.

[18] P. Horava, JHEP, 03, 020 (2009); Phys. Rev. D79, 084008 (2009); Phys. Rev. Lett. 102, 161301 (2009); E. Kiritsis and G. Kofinas, arXiv:0904.1334; H. Lü, J. Mei, and C.N. Pope, arXiv:0904.1595; and references therein.