The hyperbolic volume of knots from quantum dilogarithm

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Abstract

The invariant of a link in three-sphere, associated with the cyclic quantum dilogarithm, depends on a natural number $N$. By the analysis of particular examples it is argued that for a hyperbolic knot (link) the absolute value of this invariant grows exponentially at large $N$, the hyperbolic volume of the knot (link) complement being the growth rate.
1 Introduction

Many known knot and link invariants, including Alexander [1] and Jones [2] polynomials, can be obtained from $R$-matrices, solutions to the Yang-Baxter equation (YBE) [3, 4]. Remarkably, many $R$-matrices in turn appear in quantum 3-dimensional Chern-Simons (CS) theory as partition functions of a 3-manifold with boundary, and with properly chosen Wilson lines [8]. The corresponding knot invariant acquires an interpretation in terms of a meanvalue of a Wilson loop. Note, that the quantum CS theory is an example of the topological quantum field theory (TQFT) defined axiomatically in [9].

Thurston in his theory of hyperbolic 3-manifolds [5] introduces the notion of a hyperbolic knot: a knot that has a complement that can be given a metric of negative constant curvature. The volume of the complement in this metric appears to be a topological invariant, called the hyperbolic volume of a knot [5, 6, 7]. In principle, there should exist a quantum generalization of this invariant for the following reason.

Consider Euclidean quantum 2+1 gravity with a negative cosmological constant, which is the CS theory with a non-compact gauge group [10], and calculate the partition function of a hyperbolic knot’s complement. The result would be a topological invariant, and the classical limit in the leading order would reproduce the hyperbolic volume of the complement. Unfortunately, quantum 2+1 gravity in its present formulation can not yet be used as a computational tool for this invariant.

The hyperbolic volume of ideal tetrahedra in three-dimensional hyperbolic space can be expressed in terms of Lobachevsky’s function, which is the imaginary part of Euler’s dilogarithm [11]. Therefore, it is natural to expect, that the quantum dilogarithm of [12, 13, 14] can lead to a generalized (deformed) notion of the hyperbolic volume.

In [15, 16] a link invariant, depending on a positive integer parameter $N$, has been defined via 3-dimensional interpretation of the cyclic quantum dilogarithm [12, 17], see also [18, 19]. The construction can be considered as an example of the simplicial (combinatorial) version of the 3-dimensional TQFT [20]. In this paper we argue that this invariant is in fact a quantum generalization of the hyperbolic volume invariant. Namely, let $(L)$ be the value of the invariant on a hyperbolic knot or link $L$ in three-sphere. We study the “classical” limit $N \to \infty$ of this invariant on particular examples of $L$, and show that for hyperbolic knots its absolute value grows exponentially, with the growth rate being given by the hyperbolic volume of the knot complement:

$$2\pi \log |(L)| \sim N V(L), \quad N \to \infty,$$

(1.1)

where $V(L)$ is the hyperbolic volume of $L$’s complement in $S^3$. Formula (1.1) is in agreement with the expected classical limit of Euclidean
quantum 2+1 gravity with a negative cosmological constant \[ V \]. It is thus possible that the simplicial TQFT, defined in terms of the cyclic quantum dilogarithm, can be associated with quantum 2+1 dimensional gravity.

2  The quantum invariant for three hyperbolic knots

Let \( \omega \) be a primitive \( N \)-th root of unity. Throughout this paper we will work with the following choice for this root:

\[
\omega = \exp(2\pi i/N). \tag{2.1}
\]

The result of calculation of the link invariant from papers [15, 16] for three simplest hyperbolic knots reads:

\[
\langle 4_1 \rangle = \sum_k |(\omega)_k|^2, \quad \text{("figure-eight" knot)}, \tag{2.2}
\]

\[
\langle 5_2 \rangle = \sum_{k \leq l} \frac{(\omega)_k^2}{(\omega)_l^2} \omega^{-k(l+1)}, \tag{2.3}
\]

\[
\langle 6_1 \rangle = \sum_{k+l \leq m} \frac{|(\omega)_m|^2}{(\omega)_k(\omega)_l} \omega^{(m-k-l)(m-k+1)}, \tag{2.4}
\]

where the summation variables run over \( \{0, \ldots, N-1\} \);

\[
(\omega)_k = \prod_{j=1}^k (1 - \omega^j), \quad k = 0, \ldots, N-1; \tag{2.5}
\]

and the asterisk means the complex conjugation. Strictly speaking only \( N \)-th powers of these quantities are invariants.

Note that the simplest case \( N = 2 \) is related to a particular value of the Alexander polynomial \( \Delta_L(t) \):

\[
|\langle L \rangle| = \Delta_L(-1), \quad N = 2. \tag{2.6}
\]

In the next section we study another extreme case \( N \to \infty \).

3  The classical limit

Here we calculate explicitly the leading asymptotics at \( N \to \infty \) of the invariant for the hyperbolic knots from section 2 and justify formula (1.1).
First, for a positive real $\gamma$ and complex $p$ with $|\text{Re } p| < \pi + \gamma$, define two functions

$$f_{\gamma}(p) = S_{\gamma}(\gamma - \pi)/S_{\gamma}(p), \quad \overline{f}_{\gamma}(p) = S_{\gamma}(-p)/S_{\gamma}(\pi - \gamma),$$

where

$$S_{\gamma}(p) = \exp \frac{1}{4} \int_{-\infty}^{+\infty} \frac{e^{px}}{\sinh(\pi x) \sinh(\gamma x)} \frac{dx}{x},$$

the singularity of the integrand at $x = 0$ being put below the contour of integration. In [13] the function (3.3) is shown to be a particular solution to the functional equation:

$$(1 + e^{ip})S_{\gamma}(p + \gamma) = S_{\gamma}(p - \gamma).$$

Via this functional equation the definition of the function $S_{\gamma}(p)$ can be extended to the whole complex plane.

For fixed $p$ the leading asymptotics of $S_{\gamma}(p)$ at $\gamma \to 0$ is given by Euler’s dilogarithm:

$$S_{\gamma}(p) \sim \exp \frac{1}{2i\gamma} \text{Li}_2(-e^{ip}), \quad \gamma \to 0,$$

where

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du.$$  

For numerical calculations the following formula will be useful (see, for example, [21]):

$$\text{Im } \text{Li}_2(re^{i\theta}) = \varphi \log(r) + \Lambda(\varphi) + \Lambda(\theta) - \Lambda(\varphi + \theta),$$

where $0 < r \leq 1$,

$$\varphi = \varphi(r, \theta) = \arctan \left( \frac{r \sin \theta}{1 - r \cos \theta} \right),$$

and

$$\Lambda(\theta) = -\int_0^\theta \log |2 \sin \phi| \, d\phi,$$

is Lobachevsky’s function.

From (3.2) and (3.4) it is easy to see that $f_{\gamma}(p)$ and $\overline{f}_{\gamma}(p)$ are analytic continuations of the symbols $(\omega)_k$ and $(\omega)_k^*$ in the sense that

$$(\omega)_k = f_{\gamma}(-\pi + \gamma + 2k\gamma), \quad (\omega)_k^* = \overline{f}_{\gamma}(-\pi + \gamma + 2k\gamma),$$

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where
\[ \gamma = \pi/N. \] (3.11)

Formulae (3.10) enable us to rewrite the summations in (2.2), (2.3), (2.4) as contour integrals, one has just to replace symbols \((\omega)_k\) and \((\omega)_{\bar{k}}\) by their analytic continuations, and each summation, by a contour integral:
\[ \sum_k \rightarrow \frac{i}{4\gamma} \oint dp \tan \left( \frac{\pi^2 + \pi p}{2\gamma} \right) \] (3.12)
with an appropriately chosen contour.

In what follows \(\gamma\) will be assumed to be specified as in (3.11).

### 3.1 Knot 4_1 (figure-eight knot)

The exact formula (2.2) can be rewritten as a contour integral:
\[ \langle 4_1 \rangle = \frac{i}{4\gamma} \oint_C dp \tan \left( \frac{\pi^2 + \pi p}{2\gamma} \right) f_\gamma(p) \overline{f}_\gamma(p), \] (3.13)

Where contour \(C\) encircles \(N\) points:
\[ -\pi + \gamma + 2k\gamma \quad 0 \leq k < N, \] (3.14)
in the counterclockwise direction. In the large \(N\) (small \(\gamma\)) limit integral (3.13), by using (3.5), asymptotically can be approximated by
\[ \langle 4_1 \rangle \sim \int dz \exp \frac{i}{2\gamma} [\text{Li}_2(z) - \text{Li}_2(1/z)]. \] (3.15)

The saddle point approximation of the last integral gives the result:
\[ \langle 4_1 \rangle \sim \exp \frac{V(4_1)}{2\gamma}, \quad \gamma = \pi/N \rightarrow 0, \] (3.16)
where
\[ V(4_1) = 4\Lambda(\pi/6) = 2.02988321 \ldots \] (3.17)

Formula (3.17) is in agreement with the known hyperbolic volume of the figure-eight knot complement \([6]\).

### 3.2 Knot 5_2

The sum (2.3) at large \(N\) is approximated by the double contour integral
\[ \langle 5_2 \rangle \sim \int dzdu \exp \frac{i}{2\gamma} [2\text{Li}_2(z) + \text{Li}_2(1/u) + \alpha(z,u) - \pi^2/2], \] (3.18)
where
\[ \alpha(z, u) = \log(z) \log(u). \] (3.19)
The stationary points are described by the algebraic equations:
\[ u + z = uz, \quad u = (1 - z)^2. \] (3.20)
The maximal contribution to the integral comes from the solution \((z_0, u_0)\) to (3.20) with the property:
\[ \text{Im } z_0 < 0 < \text{Im } u_0. \] (3.21)
Thus, we obtain for asymptotics of the absolute value of the invariant:
\[ |\langle 5_2 \rangle| \sim \exp \frac{V(5_2)}{2\gamma}, \quad \gamma = \pi/N \to 0, \] (3.22)
where
\[ V(5_2) = -\text{Im } (2\text{Li}_2(z_0) + \text{Li}_2(1/u_0) + \alpha(z_0, u_0)) = 2.82812208... \] (3.23)
in agreement with [6].

### 3.3 Knot 6_1

The sum (2.4) at large \(N\) is approximated by the triple integral
\[ \langle 6_1 \rangle \sim \int dzdudv \exp \frac{i}{2\gamma} \left[ \text{Li}_2(z) - \text{Li}_2(1/z) - \text{Li}_2(u) + \text{Li}_2(1/v) + \alpha(uv/z, z/u) + 2\pi i \log(u/z) \right], \] (3.24)
where \(\alpha(z, u)\) is defined in (3.19). The stationary points are solutions to the algebraic system of equations:
\[ z(1 - z)^2 = -u^2v, \quad z^2(1 - u) = u^2v, \quad z(1 - v) = -uv. \] (3.25)
The maximal contribution to the integral comes from the solution \((z_0, u_0, v_0)\) with
\[ \text{Im } z_0 < 0 < \text{Im } (u_0v_0). \] (3.26)
Thus, the asymptotics of the absolute value of the integral reads
\[ |\langle 6_1 \rangle| \sim \exp \frac{V(6_1)}{2\gamma}, \quad \gamma = \pi/N \to 0, \] (3.27)
where
\[ V(6_1) = -\text{Im } (\text{Li}_2(z_0) - \text{Li}_2(1/z_0) - \text{Li}_2(u_0) + \text{Li}_2(1/v_0) + \alpha(u_0v_0/z_0, z_0/u_0) + 2\pi i \log(u_0/z_0)) = 3.16396322... \] (3.28)
again in agreement with [6].
4 Summary

We have demonstrated on three examples of hyperbolic knots, that the link invariant, defined in [15, 16] via cyclic quantum dilogarithm, in the asymptotic limit \( N \to \infty \) reproduces the hyperbolic volume of a knot, see formula (1.1). This result implies a possible relation of the corresponding combinatorial TQFT to quantum 2+1-dimensional gravity.

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