ON THE NON HOMOGENEOUS QUADRATIC BESSEL ZETA FUNCTION

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Abstract. We study the non homogeneous quadratic Bessel zeta function \( \zeta_{RB}(s, \nu, a) \) defined as the sum of the square of the positive zeros of the Bessel function \( J_\nu(z) \) plus a positive constant. In particular, we give explicit formulas for the main associated zeta invariants, namely poles and residua, \( \zeta_{RB}(0, \nu, a) \) and \( \zeta'_{RB}(0, \nu, a) \).

Keywords: zeta function, zeta invariant, regularized determinant

From the point of view of differential geometry and mathematical physics, the Riemann zeta function appears as the operator zeta function associated to the Laplacian operator on the line segment \([18][17][6][14]\). A natural generalisation of this setting, is to consider a Sturm Liouville operator instead, i.e. a singularity at one of the end points \([9][10][11][7][8][16]\). This leads again to a concrete zeta function, namely the Bessel zeta function, where the sum is extended on the positive zeros of the Bessel function \( J_\nu(z) \), and reduces for the opportune choice of \( \nu \) to the classical Riemann case. Such a function was first considered and studied by Stolarsky in \([21]\), where formulas for poles and residua are given, and more recently by other authors, who calculated the associated zeta invariants by different methods \([1][16]\). In these notes, we study the non homogeneous version of this function. We determine his poles and give formulas for the residua. In particular, we introduce two simple but quite general methods to calculate the value of the derivative at the origin, and therefore the regularized determinant of the associated Sturm-Liouville singular operator \([3][4][12][19]\).

Consider the constant singular Sturm Liouville operator

\[
L_\nu + q^2 = -\frac{d^2}{dx^2} + \frac{4\nu^2 - 1}{4x^2} + q^2,
\]
on the line interval \((0, l]\), with positive real \( \nu \) and \( q \) (the null cases can be easily obtained as limit cases). \( L_\nu \) has the discrete resolution \([13]\)

\[
\left\{ \lambda_{\nu,n}^2 + q^2 = \frac{j_{\nu,n}^2}{l^2} + q^2, \phi_{\nu,n}(x) = \frac{\sqrt{2x}J_\nu(\lambda_{\nu,n}x)}{lJ_{\nu+1}(j_{\nu,n})} \right\},
\]

where \( j_{\nu,n} \) are the positive zeros of the Bessel function \( J_\nu(z) \) ordered in increasing way \([22]\).
For analogy with the Riemann case, we consider the following non homogeneous quadratic Bessel zeta function defined by:

$$
\zeta_{RB}(s, \nu, a) = \sum_{n=1}^{\infty} (\pi n_{\nu,n}^2 + a^2)^{-s},
$$

for \( \text{Re}(s) > 1 \), where \( \pi n_{\nu,n} = j_{\nu,n} \) and \( a \) is real and positive, and we study its analytical extension. We can state our main result for the Bessel zeta function, that comes as a corollary of the more general result state in Proposition 3 below concerning the zeta function associated to the operator \( L_\nu + q^2 \).

**Proposition 1.** The function \( \zeta_{RB}(s, \nu, a) \) has an analytic extension to the complex \( s \)-plane, smooth in \( \nu \), up to a discrete set of simple poles at \( s = 1, -1, -3, \ldots \), whose residua can be computed using the known asymptotic expansions for the Bessel functions (more precisely, they are given by the residua of the function \( z(s, \nu, a, \pi) \) in Proposition 3 multiplied by 2); in particular

$$
\text{Res}_1 (\zeta_{RB}(s, \nu, a), s = 1) = 1,
$$

$$
\text{Res}_1 (\zeta_{RB}(s, \nu, a), s = -1) = -\frac{1}{4\pi^2} \left( \nu^2 - \frac{1}{4} - \pi^2 a^2 \right).
$$

This extension is regular at \( s = 0 \), and

$$
\text{Res}_0 (\zeta_{RB}(s, \nu, a), s = 0) = -\frac{1}{2} \left( \nu + \frac{1}{2} \right),
$$

$$
\text{Res}_0 (\zeta'_{RB}(s, \nu, a), s = 0) = -\frac{1}{2} \log 2\pi I_{\nu}(\pi a) a^{-\nu}.
$$

Notice that the values for the homogeneous case, follow immediately using the series expansion for the Bessel function \( I_{\nu}(z) \) for small \( z \), namely (see [10])

$$
\zeta'_{RB}(0, \nu, 0) = \frac{1}{2} \log 2^{\nu-1} \frac{\Gamma(\nu + 1)}{\pi^\nu}.
$$

We will study the more general setting, i.e. the function

$$
z(s, \nu, q, l) = \zeta(s; L_\nu + q^2) = \sum_{n=1}^{\infty} (\lambda_{\nu,n}^2 + q^2)^{-s}.
$$

Convergence of the series for \( \text{Re}(s) > \frac{1}{2} \) follows from classical estimates on the zeros of Bessel functions [22], and \( \zeta_{RB}(2s, \nu, a) = z(s, \nu, a, \pi) \).

We present here two approaches to calculate the zeta invariants associated to \( z(s, \nu, q, l) \). In the first, we produce an analytic representation that can be effectively used to get all the invariants; this will be particularly useful to generalise the method for the calculation of the zeta invariants associated to a disk or to a cone [20]. In the second approach, we give a very general lemma (Lemma 4) to deal with regularized products, and we apply it to calculate \( z(0, \nu, q, l) \) and \( z'(0, \nu, q, l) \).
Let start with the first approach. Using Mellin transform \[13\] \[15\], we get the analytic representation

\[
z(s, \nu, q, l) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t, \nu, q, l) dt,
\]

where the trace of the heat operator is

\[
f(t, \nu, q, l) = \text{Tre}^{-tL^\nu} = \sum_{n=1}^\infty e^{-(\lambda^2_{\nu,n} + q^2)},
\]

and from this the complex representation

\[
f(t, \nu, q, l) = \frac{1}{2\pi i} \int_{\Lambda_c} e^{-\lambda t} R(\lambda, \nu, q, l) d\lambda,
\]

where the contour is \(\Lambda_c = \{ \lambda \in \mathbb{C} \mid \arg(\lambda - c) = \pi/4 \}\), for some \(0 < c < q^2\), and the trace of the resolvent is

\[
R(\lambda, \nu, q, l) = \sum_{n=1}^\infty \frac{1}{\lambda - (\lambda^2_{\nu,n} + q^2)}.
\]

We now observe that it is easy to express such function in terms of special functions. In fact, taking logarithmic derivative of the infinite product representation of the Bessel function \(I_{\nu}(z)\), we get \[22\]:

Lemma 1.

\[
R(\lambda, \nu, q, l) = \frac{\nu}{2z^2} - \frac{1}{2z} \frac{d}{dz} \log I_{\nu}(lz).
\]

Here, \(z = \sqrt{q^2 - \lambda}\), we set \(\arg(q^2 - \lambda) = 0\) on the line \((-\infty, q^2)\) and we fix the sector \(s_+ = \{ z \in \mathbb{C} \mid \arg z < \pi/2 \}\) for \(z\).

At this point it is worth observing that information about poles and residua of \(z(s, \nu, q, l)\) can be obtained using the representation introduced, asymptotics expansions for Bessel functions and classical arguments \[13\] \[15\]. This is an easy way for producing the results relative to the so called ‘constant case’ when studying regular singular operators \[9\] \[7\] \[8\]. More precisely, and for completeness, we can state the following results:

Lemma 2. For small \(t\),

\[
f(t, \nu, q, l) = \sum_{i=0}^l a_i(\nu, q, l) t^{(i-1)/2} + O(t^{l/2}) =
\]

\[
= \frac{l}{2\sqrt{\pi}} t^{-1/2} - \frac{1}{2} \left( \nu + \frac{1}{2} \right) + \frac{1}{2l\sqrt{\pi}} \left( \nu^2 - \frac{1}{4} - l^2 q^2 \right) t^{1/2} + O(t),
\]

where

\[
a_{2i}(\nu, q, l) = \frac{(-1)^i}{i!} \frac{q^{2i}}{2\sqrt{\pi}} \sum_{j=0}^\infty \frac{(\nu - j\nu + 1/2) \Gamma(j + \nu + \frac{1}{2})}{2^k j! k! \Gamma(\nu + k + \frac{1}{2})} \frac{q^{2j}}{l^{2k} k!^2} \Gamma(\nu - k + \frac{1}{2}) \frac{\Gamma(\nu + k + \frac{1}{2})}{\Gamma(\nu - k + \frac{1}{2})},
\]

\[
a_{2i+1}(\nu, q, l) = \frac{(-1)^{i+1}}{2i!} \frac{(\nu - \frac{1}{2}) q^{2i}}{2^k j! k! \Gamma(\nu - k + \frac{1}{2}) \Gamma(\nu + k + \frac{1}{2})} \sum_{j=0}^\infty \frac{(\nu - j\nu + 1/2) \Gamma(j + \nu + \frac{1}{2})}{2^k j! k! \Gamma(\nu + k + \frac{1}{2})} \frac{q^{2j}}{l^{2k} k!^2} \Gamma(\nu - k + \frac{1}{2}) \frac{\Gamma(\nu + k + \frac{1}{2})}{\Gamma(\nu - k + \frac{1}{2})}.
\]
Proposition 2. The function \( z(s, \nu, q, l) \) has an analytic extension to the whole complex \( s \)-plane, smooth in \( \nu \) and \( q \), up to a discrete set of simple poles at \( s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots \), with residua

\[
\text{Res}_1 \left( z(s, \nu, q, l), s = \frac{1}{2} - k \right) = \frac{a_{2k}(\nu, q, l)}{\Gamma\left(\frac{1}{2} - k\right)}, \quad k = 0, 1, 2, \ldots
\]

This extension is regular at \( s = 0 \), and

\[
\text{Res}_0 (z(s, \nu, q, l), s = 0) = -\frac{1}{2} \left( \nu + \frac{1}{2} \right).
\]

Notice that the contribution of the non homogeneity term \( q^2 \) is equally shared among all the poles; in other words, the homogeneous Bessel zeta function has the same poles, but with (possibly) different residua (see [21]).

The information available is not enough to deal with the derivative, that is an harder point; though, we introduce the following quite more general purpose result.

Lemma 3. Suppose the zeta function \( z(s, x) = \sum_{n=1}^{\infty} a_n(x)^{-s} \) has the following representation (everything smooth in \( x \)):

\[
z(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_x} e^{-\lambda t} R(\lambda, x)d\lambda dt,
\]

where the contour is as above, and the there is a primitive function \(-T(\lambda, x)\) for the function \( R(\lambda, x) = -\frac{A}{\lambda^2} T(\lambda, x)\), satisfying the following properties:

(a) \( T \) is analytic near \( \lambda = 0 \),

(b) for large \( \lambda \) and fixed \( x \), there is an asymptotic expansion

\[
T(\lambda, x) = \cdots + A(x) \log(-\lambda) + B(x) + \ldots
\]

Then, \( z(s, x) \) can be analytically extended at \( s = 0 \) and

\[
z(0, x) = -A(x), \\
z'(0, x) = -B(x) + T(0, x).
\]

Proof Integrating by part, first in \( \lambda \) and hence in \( t \), the given complex representation for \( z(s, x) \), we get

\[
z(s, x) = \frac{s}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_x} e^{-\lambda t} T(\lambda, x)d\lambda dt.
\]

At this point, it is convenient to isolate the constant part of \( T \) writing

\[
T(\lambda, x) = T_1(\lambda, x) + T_0(x);
\]

in fact, it is clear that \( T_0(x) \) gives no contribution, since

\[
\int_{\Lambda_x} e^{-\lambda t} T_0(x)d\lambda = 0.
\]

Next, since by definition the zeta function is well defined for large \( s \),

\[
z(s, x) = \frac{s^2}{\Gamma(s+1)} \int_0^{1} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_x} e^{-\lambda t} T_1(\lambda, x)d\lambda dt + s^2 f(s),
\]
where $f$ is regular near $s = 0$. Because of the pole at $\lambda = 0$, we have to split the complex integral as follows to use the expansion for large $\lambda$ (small $t$):

$$
\int_{\Lambda_c} = \int_{\Lambda_{-c}} - \int_{C_c} = \int_{\Lambda_{-c}} + T_1(0, x),
$$

where $C_c$ is a circle around the origin of radius $c$. Moreover, by assumption (b),

$$
\int_{\Lambda_{-c}} e^{-\lambda} T_1(\lambda, x) d\lambda dt = \int_{\Lambda_{-c}} e^{-\lambda} T_1(\lambda/t, \nu, q) d\lambda dt = \cdots + \gamma A(x) + A(x) \ln(-\lambda) + B(x) + \ldots,
$$

where we have explicit-ed only the relevant part. This means that we can write

$$z(s, x) = \frac{s}{\Gamma(s + 1)} \left[ \gamma A(x) - B(x) - \frac{1}{s} A(x) + T_1(0, x) \right] + s^2 g(s),$$

where again $g$ is regular near $s = 0$, and, from that, the thesis follows at once. \(\Box\)

Applying this argument to the function $z(s, \nu, q, l)$, we have

$$T(\lambda, \nu, q, l) = \nu \log lz - \log I_{\nu}(lz) - \log 2^{\nu} \Gamma(\nu + 1),$$

and hence we compute

$$A(\nu, q, l) = \frac{1}{2} \left( \nu + \frac{1}{2} \right), \quad B(\nu, q, l) = \frac{1}{2} \log 2\pi + \left( \nu + \frac{1}{2} \right) \log l - \log 2^{\nu} \Gamma(\nu + 1),$$

$$T(0, \nu, q, l) = \nu \log lq - \log I_{\nu}(lq) - \log 2^{\nu} \Gamma(\nu + 1).$$

This gives:

$$\text{Res}_0(z(s, \nu, q, l), s = 0) = -\frac{1}{2} \left( \nu + \frac{1}{2} \right),$$

and proves the following:

**Proposition 3.**

$$\text{Res}_0(z'(s, \nu, q, l), s = 0) = -\log \sqrt{2\pi l \frac{I_{\nu}(lq)}{q^\nu}}.$$  

Let’s turn to the second approach. First, we have the following:

**Lemma 4.** Suppose two sequences $a_n$ and $b_n$, $n = 1, 2, 3, \ldots$, of real positive numbers are given, and satisfy the following conditions:

(A) there are real $s_a$, $s_b$ such that the two series

$$\zeta_a(s) = \sum_{n=1}^{\infty} a_n^{-s}, \quad \zeta_b(s) = \sum_{n=1}^{\infty} b_n^{-s},$$

converge for $\text{Re}(s) > s_a, s_b$ respectively;

(B) the zeta function $\zeta_b(s)$ has an analytic extension at $s = 0$

(C) $|a_n - b_n| < K$ definitely for some constant $K$,

(D) $s_b < 1$. 

Then, the zeta function $\zeta_a(s)$ has an analytic extension at $s = 0$ with $\zeta_a(0) = \zeta_b(0)$, the infinite product

$$
\prod_{n=1}^{\infty} \frac{a_n}{b_n} = C
$$

converges absolutely, and

$$
\zeta'_a(0) = \zeta'_b(0) + \log C.
$$

Proof. Let $c_n = \frac{a_n}{b_n} - 1$. Then, $|c_n| < Kb_n^{-1}$ by (B) and (C) and hence $\sum_{n=1}^{\infty} c_n$ converges absolutely and so does the infinite product. We can assume $c_n < 1$, and hence

$$
\zeta_a(s) = \zeta_b(s) + \sum_{k=1}^{\infty} \left( \frac{-s}{k} \right) \sum_{n=1}^{\infty} b_n^{-s} c_n^k.
$$

Now, the series

$$
\sum_{n=1}^{\infty} b_n^{-s} c_n^k,
$$

converges in a neighbourhood of $s = 0$ for each $k$, by conditions (C) and (D). This means that $\zeta_a$ can be analytically extended to $s = 0$ by using the extension of $\zeta_b$. In particular, evaluating at $s = 0$ we get $\zeta_a(0) = \zeta_b(0)$ and

$$
\zeta'_a(0) = \zeta'_b(0) + \sum_{k=1}^{\infty} \frac{d}{ds} \left( \sum_{n=1}^{\infty} b_n^{-s} c_n^k \right)_{s=0} = \zeta'_b(0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} c_n^k = \zeta'_b(0) + \sum_{n=1}^{\infty} \log(1 + c_n) = \zeta'_b(0) + \log C.
$$

We can apply this lemma to the present case as follows. Let $a_n = \lambda_{\nu,n}^2 + q^2$, $b_n = \frac{\pi}{2} \left( n + \frac{1}{2} (\nu - \frac{1}{2}) \right)^2$; then, all the assumptions of Lemma 4 are satisfied, and $\zeta_b(s)$ is the Hurwitz zeta function $\zeta_H(2s, \frac{1}{2} (\nu + \frac{3}{2}))$. Thus,

$$
\zeta'_b(0) = 2 \log \Gamma \left( \frac{1}{2} \left( \nu + \frac{3}{2} \right) \right) - \log 2\pi,
$$

and we can compute (where $u = \frac{\nu}{2} (\nu - \frac{1}{2})$):

$$
\frac{1}{C} = \lim_{z \to 0^+} \prod_{n=1}^{\infty} \frac{1 + \frac{\pi^2 (z, u + q^2 (u + z^2))}{\pi^2 z^2 (u + z^2)^2}}{1 + \frac{\pi^2 (z, u + q^2 (u + z^2))}{\pi^2 z^2 (u + z^2)^2}} = \lim_{z \to 0^+} \prod_{n=1}^{\infty} \left( 1 + \frac{\pi^2 z^2 (u + z^2)^2}{\pi^2 z^2 (u + z^2)^2} \right) = \frac{z^\nu q^\nu I_\nu \left( i \sqrt{1 + q^2 z^2} \right)^2}{I_\nu (lq) \Gamma^2 (u + 1)} = \sqrt{2} (lq)^\nu \frac{I_\nu (lq) \Gamma^2 (u + 1)}{\pi^2 I_\nu (lq) \Gamma^2 (\frac{1}{2} (\nu + \frac{3}{2}))}.
$$
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References

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