An analytical study on fluid flow characteristics for flow within a microchannel of rectangular cross section

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Abstract. The present work considers fluid flow analysis within a microchannel of rectangular cross section with different exact and approximate analytical techniques. Thus, velocity slip at the solid wall of the channel is considered. The slip flow is transformed to a no-slip flow with some minor modifications by applying an average slip velocity. The flow is represented by the Navier-Stokes equation exposed to slip boundary conditions. The governing equation is solved considering separation of variables method (SOV) as an exact method and, Integral and Variational methods as approximate methods. The Poiseuille number and slip coefficient are also determined for each method. The present prediction agrees well with available literature. To predict accuracy level of all the techniques, the results are presented in a comparative scale along with numerical prediction. It is observed that the accuracy level for the velocity profile obtained in each technique depends on the aspect ratio whereas for the prediction of Poiseuille number and slip coefficient, all the technique show less dependency on the aspect ratio.

1. Introduction
In the recent years, fluid flow analysis within microchannel is an emerging area of research due to the growing demand of microfluidic systems and devices. It is used in various applications of medical and biomedical areas, computer chips, and chemical separations etc. In this regard, a new research area, namely Micro-electro-mechanical systems (MEMS) is established where non-continuum behaviour exists [1], and the components like micro-pumps, micro-valves, micro heat sinks and actuators are miniaturized, unified and assembled to develop a variety of micro systems and devices. However, a fundamental study on flow behaviours such as velocity distribution, Poiseuille number, slip coefficient etc. is vital for systematic process control as well as design of different microfluidic applications [2]. It is noticed that the flow in the micro level is connected with the insertion of slip velocity [3] and does not follow the classical continuum physics. Thus, the flow is coupled with a non-zero velocity at boundary walls. It occurs when the value of Knudsen number (Kn) ranges from 0.001 to 0.1 and corresponding flow is termed as slip-flow [3]. In such a flow, the momentum equation is coupled with the slip-flow boundary condition [4, 5]. In this connection, it is found that suitable analytical investigations for flow within the microchannel are less established.

The present work is focused accordingly to propose some suitable analytical methods for investigation of the flow behaviours within a microchannel. In this context, some related research works are reviewed for having concepts on the flow behaviours in microchannels. Kundu et al. [6] discussed different approximate analytical methods to determine the velocity distribution for laminar flow and
no-slip conditions at the channel boundary for channels of straight and rectangular cross section. They established various exact and in-exact analytical methods to determine velocity and temperature profiles. Chakraborty [7] considered flow problems within a microchannel of straight and arbitrary cross-section considering three solution techniques in general form, namely complex function analysis, membrane vibration analogy and variational method. Hooman [3] investigated forced convection based on superposition approach in microchannels of arbitrary cross section. It is found that incorporating a mean slip velocity and temperature jump, the no-slip/no-jump with some minor modifications is applicable. Application of this procedure to a triangular microchannel is validated with numerical analysis. Kuddusi [8] demonstrated slip flow in a rectangular microchannel of heated walls using integral transform methods. Morini [9] predicted heat transfer characteristics as function of aspect ratio considering a laminar fully developed flow of rectangular channels at constant wall temperature. Theofilis et al. [10] determined velocity distribution for fluid flow within a channel of rectangular cross section based on the Navier-Stoke equation. They considered a constant pressure gradient along length of the channel. Peng et al. [11] presented solution based on analytical methods for viscous fluid flow both for an equilateral triangular tube and irregular triangular tubes. The work is validated and compared with numerical results.

It is revealed from the existing literature that the analytical methods presented are complex, lengthy and laborious. Therefore, in the present work, establishment of three analytical techniques, namely Separation of variables (SOV), Integral and Variational methods on fluid flow within a straight rectangular microchannel is considered. The Navier–Stokes equation is solved assuming slip velocity at the solid boundary walls of the channel based on the analytical methods.

2. Description of the Physical Problem
The present study reflects a pressure driven fully developed laminar flow through a straight microchannel of rectangular cross section as shown in figure 1. The center of cross section of the channel is considered as origin of the Cartesian coordinate. Under steady state condition, the fluid flows only in the z-direction. A width of 2L parallel to the x-axis and a depth of 2l parallel to the y-axis are considered for the cross section.

![Figure 1. The rectangular cross section of the microchannel](image)

3. Mathematical Modelling
The present analysis considers a pressure driven fully developed laminar flow within a straight rectangular microchannel. In the present work, the viscous incompressible flow is considered. Considering a hydro-dynamically developed flow, the conservation of momentum equation along the axial direction (z-axis) is given as [3, 6, 7, 8]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{dp}{\mu dz}
\]

Also \(dp/dz\) is the pressure gradient in the flow direction and assumed as a constant in the present study.
The following boundary conditions for Equation (1) are

\[ u = u_s \text{ at } x = \pm L \text{ and } u = u_s \text{ at } y = \pm l \]  
(2a)

\[ \partial u / \partial x = 0 \text{ at } x = 0 \text{ and } \partial u / \partial y = 0 \text{ at } y = 0 \]  
(2b)

where \( u_s \) is slip velocity at the solid walls of the channel.

The average velocity \( (u_m) \) of the flow is determined as

\[ u_m = \frac{1}{l} \iint_{0}^{l} u \, dx \, dy \]  
(3a)

The slip velocity \( (u_s) \) at the walls is presented [3, 8] as

\[ u_s = \left( (F - 2) / F \right) \beta \left( \partial u / \partial n \right)_{\text{wall}} \]  
(3b)

In equation (2a), the boundary conditions are not homogeneous. In order to convert the non-homogeneous boundary conditions to homogeneous, the governing equation (1) is represented as a non-dimensional equation by introducing following non-dimensional variables [3] as

\[ U = -\left( \mu / L^2 \right) \left( dp / dz \right) \left( u - u_s \right), \ X = x / L, \ Y = y / l \text{ and } A = l / L \]

Replacing these non-dimensional variables to equation (1), the governing momentum equation becomes

\[ \partial^2 U / \partial X^2 + \partial^2 U / \partial Y^2 + 1 = 0 \]  
(4)

The dimensionless form of the boundary conditions can be written as

\[ U = 0 \text{ at } X = \pm 1 \text{ and } Y = \pm 1 \]  
(5a)

\[ \partial U / \partial X = 0 \text{ and } \partial U / \partial Y = 0 \text{ at } X = 0, Y = 0 \]  
(5b)

The equation (4) is the dimensionless form of the Navier-Stokes equation for a fully developed laminar flow subjected to no-slip velocity boundary condition. Accordingly, the flow velocity \( U \) is regarded as the velocity for no-slip condition and stated as \( U_{ns} \). However, in the present microchannel flow, a non-zero velocity is present at the boundary walls. Assuming a constant slip velocity at the boundary, a normalized velocity \( (\bar{U}) \) of flow within the microchannel [3] is expressed as

\[ \bar{U}(X, Y) = BU_{ns} + 1 - B \]  
(6)

where \( B = 1 / \left( 1 + 2 - \frac{F}{U_{ns,m} \beta} \left( \frac{2A}{1 + A} \right)^2 \right) \) and the normalized no-slip velocity \( \bar{U}_{ns} = U_{ns} / U_{ns,m} \). The mean velocity for no-slip condition is determined as \( U_{ns,m} = \iint_{0}^{1} U_{ns} \, dx \, dy \). The equation (4) is solved to determine the velocity profile considering no-slip condition and hence the velocity profile for the slip flow is determined by the equation (6). In the present work, three analytical solution techniques, namely SOV, Integral and Variational methods are considered to derive the velocity distribution. For flow through microchannel, slip coefficient \( \beta_s \) which measures the velocity slip at the boundary is expressed as \( \beta_s = u_s / u_m \) and the Poiseuille number as \( Po = f \text{ Re} \).

Both the parameters are important in predicting fluid flow characteristics within the microchannel and expressed [3] as

\[ \beta_s = 1 - B \]  
(7a)

\[ Po = B(2 / U_{ns,m}) \left( 2A / (1 + A) \right)^2 \]  
(7b)

Finally, equations (1-7) are solved using the three solution techniques to predict the velocity distribution within the microchannel.

A pressure driven fully developed laminar flow through a straight microchannel of rectangular cross section as shown in figure 1. The center of cross section of the channel is considered as origin of the Cartesian coordinate. Under steady state condition, the fluid flows only in the \( z \)-direction. A width of \( 2L \) parallel to the \( x \)-axis and a depth of \( 2l \) parallel to the \( y \)-axis are considered for the cross section.
3.1. Separation of variable method (SOV)

In this section, an exact analytical method based on separation of variable method is presented. The velocity profile for no-slip condition as a solution of the equation (4) is considered as

\[ U_{ns}(X,Y) = \varphi(X,Y) + \psi(X) \]  

(8)

where \( \psi(X) \) refers to a particular solution of the Poisson equation (4) and \( \varphi(X,Y) \) represents solution of the corresponding homogeneous Laplace equation. The following equations are obtained as

\[ d^2\psi/dX^2 = -1 \]  

(9)

The boundary conditions identified for Equation (9) are given as

\[ dy/dX = 0 \text{ at } X = 0, \text{ and } y = 0 \text{ at } X = 1 \]

\[ \partial^2\psi/\partial X^2 + \partial^2\psi/A^2\partial Y^2 = 0 \]  

(10)

The necessary boundary conditions for the equation (10) are identified as

\[ dp/dX = 0 \text{ at } X = 0, \text{ and } \varphi = 0 \text{ at } Y = 1 \]

\[ dp/dY = 0 \text{ at } Y = 0, \text{ and } \varphi|_{Y=1} = -\psi(X) \]

Both the equations (9) and (10) are solved separately to obtain the \( \psi(X) \) and \( \varphi(X,Y) \). Hence, the normalized no-slip velocity, \( U_{ns}(X,Y) \), is obtained using the equation (8). Subsequently, the equation (6) is used to determine the velocity profile for slip flow within the microchannel as

\[ \mathcal{U} = \frac{B}{B_0} \left[ 1 - X^2 - \frac{32}{\pi^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} \cosh((2n+1)\pi X) \cos\left(\frac{(2n+1)\pi}{2} X\right) \right]^2 + 1 - B \]  

(11)

where \( B = \left[ 1 + \frac{2(1-F)}{FB_0} Kn \left( \frac{2A}{1+A} \right) \right]^{2^{-1}} \) and \( B_0 = \left[ 2 - \frac{128}{3 \pi^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^8} \tanh((2n+1)\pi A/2) \right] \).

The Poiseuille number (Po) is expressed as

\[ Po = 4 \left[ 2 - \frac{128}{3 \pi^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^8} \tanh((2n+1)\pi A/2) + \frac{2(1-F)}{F} Kn \left( \frac{2A}{1+A} \right)^2 \right] - 1 \]  

(12)

3.2. Solution method by Chakraborty [7]

In this section, equation (4) is solved in finite series form using the analytical technique proposed by Chakraborty [7] and finally the slip velocity is determined as

\[ \mathcal{U} = \frac{\pi B}{2 B_1} \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left[ 1 - \cos((2n+1)\pi A/2) \cos\left(\frac{(2n+1)\pi}{2} X\right) \right]^2 + 1 - B \right] \]  

(13)

where \( B = \left[ 1 + \frac{\pi^4 (1-F)}{32 FB_1} Kn \left( \frac{2A}{1+A} \right)^2 \right]^{-1} \) and \( B_1 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left[ 1 - \tan((2n+1)\pi A/2) \right] \).

The Poiseuille number (Po) is expressed as

\[ Po = \frac{\pi^4}{16} \left[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \left[ 1 - \tan((2n+1)\pi A/2) \right] + \frac{\pi^4 (1-F)}{32 F} Kn \left( \frac{2A}{1+A} \right)^2 \right] - 1 \]  

(14)

3.3. Integral method

In this segment, an approximate analytical result based on Integral method is presented. The governing equation (4) has been solved for Ritz profile and Kantorovich profile.
3.3.1. Ritz profile
Here, the equation (4) is solved using Integral method based on second-order approximation of the Ritz profile. Accordingly, a truncated series for the velocity profile is assumed as
\[ U_{ns} = (1 - X^2)(1 - Y^2)\left(a_0 + a_1 X^2\right) \]  
(15)

where \(a_0\) and \(a_1\) are two unknowns require two conditions as
\[
\int_0^1 \left( \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{A^2} \right) dX dY = 0
\]  
(16)

\[
\left( \frac{\partial^2 U}{\partial X^2} \right)_{X=0} + \left( \frac{\partial^2 U}{\partial A^2} \right)_{X=0} + 1 = 0
\]  
(17)

Both the equations (16) and (17) are solved to determine \(a_0\) and \(a_1\) and hence, equation (15) is used to obtain the normalized no-slip velocity \(U_{ns}(X, Y)\). Substituting \(U_{ns}(X, Y)\) in equation (6), the velocity profile is determined as
\[
\overline{U} = \frac{9B}{4} (1 - X^2)(1 - Y^2) \left( \frac{1 + A^2}{3 + 2A^2} \right) \left( \frac{2 + 25A^2}{1 + A^2} + 5X^2 \right) + 1 - B
\]  
(18)

where \(B = 1 + \frac{9(2 - F)}{F} \left( \frac{1 + A^2}{3 + 2A^2} \right) \left( \frac{(1 + A^2)}{2} - \frac{1}{1 + A} \right)^2 \) \(Kn\left( \frac{2}{1 + A} \right)^2 \) \(2\) \(7\) \(1\)

The Poiseuille number \(Po\) is determined as
\[
P_o = 18 \left( \frac{(3 + 2A^2)}{(1 + 10A^2)(1 + A^2)} + \left( \frac{9(2 - F)}{F} \right) \left( \frac{2}{1 + A} \right)^2 \right) \left( \frac{2}{1 + A} \right)^2 \]  
(19)

3.3.2. Kantorovich profile
The solution of equation (4) is determined by the Integral method using Kantorovich profile as
\[
U = (1 - Y^2) \left[ F_1(X) + F_2(X) \right]
\]  
(20)

Equations (16) and (17) are used to determine the unknowns \(F_1\) and \(F_2\), which are the functions of \(X\).

The successive equations are
\[
\left( A^4D^4 - 27A^2D^2 + 60 \right) F_1 = 30A^2
\]  
(21)

\[
\left( A^4D^4 - 27A^2D^2 + 60 \right) F_2 = 0
\]  
(22)

Hence, \(F_1\) and \(F_2\) are determined considering the boundary conditions \(F_1(1) = 0\) and \(F_2(1) = 0\).

Substituting the values of \(F_1\) and \(F_1\) in equation (20), the normalized no-slip velocity \(U_{ns}(X, Y)\), is obtained and finally, the slip velocity is by substituting \(U_{ns}(X, Y)\) in equation (6) found as
\[
\overline{U} = \frac{3B(1 - Y^2)}{2B_2} \left[ \left( \frac{2 - \beta}{1 - \alpha} \right) \cosh \sqrt{\beta} / A \right] \left[ \left( \frac{2 - \alpha}{1 - \beta} \right) \cosh \sqrt{\alpha} / A \right] \left( \frac{2}{1 + A} \right)^2 \]  
(23)

where \(B_2 = \left( \frac{2 - \alpha}{1 - \beta} \right) \left( \frac{2}{1 + A} \right)^2 \) \(Kn\left( \frac{2}{1 + A} \right)^2 \) \(2\) \(7\) \(1\)

The Poiseuille number \(Po\) is determined as
\[
P_o = 6 \left[ \left( \frac{2 - \alpha}{5(\beta - \alpha)} \right) \left( \frac{\tanh \sqrt{\alpha} / A}{\sqrt{\alpha} / A} \right) + \left( \frac{2 - \alpha}{5(\alpha - \beta)} \right) \left( \frac{\tanh \sqrt{\beta} / A}{\sqrt{\beta} / A} \right) \right] + 1 + \frac{3(2 - F)}{F} \left( \frac{2}{1 + A} \right)^2 \]  
(24)

where \(\alpha = \left( \frac{27/2 - \sqrt{489/2}}{2} \right)^2\) and \(\beta = \left( \frac{27/2 + \sqrt{489/2}}{2} \right)^2\)
3.4. Variational method
In this section, an approximate analytical solution of the flow problem is given by using Variational formulation. Both the Ritz and Kantorovich profiles are considered in the formulation. The necessary condition of this formulation is
\[ \delta I = 4 \iint_0^1 \left( \partial^2 U/\partial X^2 + \partial^2 U/\partial A^2 \partial Y^2 + 1 \right) \delta U \, dx \, dy = 0 \] (25)
where \( I \) represents a functional and \( \delta I \) refers a variation of the functional.

3.4.1. Ritz profile
The velocity profile is considered by using Ritz profile as
\[ U_{ns} = (1 - X^2)(1 - Y^2) a_0 \] (26)
The constant \( a_0 \) is obtained by solving equation (25). Hence, the normalized velocity \( U_{ns}(X,Y) \) is determined using equation (26). The velocity profile is finally determined based on equation (6) as
\[ U = \left( 9B/4 \right)(1 - X^2)(1 - Y^2) + 1 - B \] (27)
where \( B = \left[ 1 + \left[ 18(2 - F)(1 + A^2)/5F \right] Kn[2/(1 + A)]^2 \right]^{-1} \).

The Poiseuille number \( (Po) \) is expressed as
\[ Po = \frac{36}{5} \frac{1}{(1 + A^2)} + \left[ \frac{18(2 - F)}{5F} Kn \left( \frac{2}{1 + A} \right)^2 \right]^{-1} \left( \frac{2}{1 + A} \right)^2 \] (28)

3.4.2. Kantorovich profile
The approximate solution of equation (4) is determined using Kantorovich profile as
\[ U_{ns} = (1 - Y^2) X_1(X) \] (29)
The equation (29) requires the condition stated in equation (25) to determine the unknown \( X_i(X) \). The corresponding differential equation for \( X_i \) is expressed as
\[ (4D^2/5 - 2A^2) X_1 + 1 = 0 \] (30)
The equation (30) is solved for \( X_1 \) as
\[ X_1 = C_1 \cosh(X \sqrt{5/2} / A) + C_2 \sinh(X \sqrt{5/2} / A) + A^2/2 \] (31)
The equation (31) is subjected to the boundary conditions \( X_i(0) = 0 \) and \( DX_i(1) = 0 \) to obtain the constants \( C_1 \) and \( C_2 \). Thereafter, the normalized no-slip velocity, \( U_{ns}(X,Y) \), is obtained by equation (29). Finally, the velocity profile for the given problem is found as
\[ U = \left[ 3B/2B_3 \right](1 - Y^2)(1 - \cosh(X \sqrt{5/2} / A) / \cosh(\sqrt{5/2} / A)) + 1 - B \] (32)
where \( B = \left[ 1 + \left[ 3(2 - F)/F B_3 \right] Kn \left( \frac{2}{1 + A} \right)^2 \right]^{-1} \) and \( B_3 = \left[ 1 - \tanh(X \sqrt{5/2} / A) \right] \).

The Poiseuille number \( (Po) \) is determined as
\[ Po = 6 \left[ 1 - \frac{\tanh(X \sqrt{5/2} / A)}{\sqrt{5/2} / A} + \left[ \frac{3(2 - F)}{F} Kn \left( \frac{2}{1 + A} \right)^2 \right]^{-1} \left( \frac{2}{1 + A} \right)^2 \right] \] (33)

3.5. Finite different method (FDM)
The Equation (4) is solved by employing the central difference scheme of finite difference method (FDM) to determine \( U_{ns}(X,Y) \) for no-slip conditions. Discretization of the equation (4) is carried out as
The normalized velocity $U_{ns}(X,Y)$, is obtained solving the equation (34) numerically. Finally, the velocity profile for the slip flow condition is determined as

$$U = B(U_{i,j} l/ \sum_j \sum U_{i,j} \Delta X \Delta Y) + 1 - B$$

(35)

where $B = \left[1 + \frac{(2 - F)}{F} \sum_j \sum U_{i,j} \Delta X \Delta Y} \right]^{-1} K, (\frac{2A}{1 + A})^2$, and the Poiseuille number ($Po$) is expressed as

$$Po = 2 \left[ \frac{\sum_j \sum U_{i,j} \Delta X \Delta Y}{\sum_j \sum \frac{(2 - F)}{F} \sum_j \sum U_{i,j} \Delta X \Delta Y} \right]^{-1} \frac{2A}{1 + A}$$

(36)

4. Results and discussion

The present study offers various analytical solutions for fluid flow through a rectangular microchannel. The Navier-Stokes equation with velocity slip boundary conditions has been solved using various analytical techniques to obtain the velocity profile, corresponding slip coefficient ($\beta_s$) and Poiseuille number ($Po$). The SOV method is used for an exact solution whereas the Integral and Variational methods are considered for the approximate solutions. The results obtained are compared with numerical solution and an existing work by Chakraborty [7] for the validation purpose.

In figure 2(a, b), velocity distribution obtained is presented along y-direction at $X = 0$ for $A = 1.0$ and $A = 0.5$, respectively, based on the separation of variables method, and compared with the solutions by numerical method and the existing method [7]. A good agreement of the present solution is observed with the numerical method and the existing method. Further, it is found in the literature that the Poiseuille number is an important parameter for the analysis of flow behaviour within a microchannel. The Poiseuille number is dependent on the Knudsen number ($Kn$) which normally ranges from 0.001 to 0.1 for the gas flow within microchannel. Hence, the Poiseuille number as a function of Knudsen number is evaluated and plotted in figure 3(a, b) for $A = 1.0$ and $A = 0.5$ based on the separation of variables method. The variation is compared with the numerical method and the existing method. It is noted that the present prediction matches exactly with the numerical one and the existing literature.
In this section, the SOV method is compared with other approximate techniques. Accordingly, the prediction of velocity filed and Poiseuille number is performed by the Integral Ritz and Integral Kantorovich methods. A comparison is made subsequently with the separation of variables method in figure 4(a, b) and figure 5(a, b) considering $A = 1.0$ and 0.5. It is observed in the figure 4 (a, b) that the velocity profiles obtained using the Integral Ritz and Integral Kantorovich methods follow same trend of the profile predicted by the separation of variables method with a slight deviation. In addition, the deviation is slightly higher in case of $A = 1.0$ than $A = 0.5$. It is also observed that the deviation becomes more in the flow towards the center of the channel.
In Fig. 5(a, b), the profile of Poiseuille number for the Integral Ritz and Integral Kantorovich methods is shown, which closely coincides with the profile of the separation of variables method. Hence, the Integral Ritz and Integral Kantorovich techniques also approximate the fluid field well through the microchannel with less error.

**Figure 4.** Comparison of the velocity profile \( (U) \) obtained by the Integral Ritz and Integral Kantorovich methods \( (F = 1, Kn = 0.01) \) with the profile by separation of variables method: (a) for \( A = 1 \) (b) for \( A = 0.5 \).

**Figure 5.** Variation of Poiseuille number \( (\dot{P}_o) \) with Knudsen number \( (Kn) \) as determined by separation of variables, Integral Ritz and Integral Kantorovich methods \( (F = 1) \): (a) for \( A = 1 \) (b) for \( A = 0.5 \).

Finally, in figure 6(a, b), the velocity profiles obtained by the Variational Ritz and Variational Kantorovich methods have been compared against the SOV method. It is found that the velocity profile in case of the Variational Kantorovich method agrees with that of the SOV method with a little deviation for both \( A = 1 \) and \( A = 0.5 \). However, in case of the Variational Ritz method, the velocity
profile is more acceptable for $A = 1$ compared to $A = 0.5$. In figure 7(a, b), it is noticed that the variation of Poiseuille number closely overlaps for both $A = 1$ and $A = 0.5$. Although, Variational Ritz shows more deviation for $A = 0.5$. Hence, Variational Katorovich method can be considered for the analysis with less error both for $A = 1$ and $A = 0.5$.

**Figure 6.** Comparison of the velocity profile ($U$) obtained by the Variational Ritz and Variational Kantorovich methods ($F = 1, Kn = 0.01$) with the profile by separation of variables method: (a) for $A = 1$ (b) for $A = 0.5$.

**Figure 7.** Variation of Poiseuille number ($Po$) with Knudsen number ($Kn$) as determined by separation of variables, Variational Ritz and Variational Kantorovich methods ($F = 1$): (a) for $A = 1$ (b) for $A = 0.5$.

It is already stated that the slip coefficient which measures the velocity slip at the solid boundary is of particular importance for flow within microchannel. The work accordingly predicts the slip coefficient evaluated at different Knudsen numbers for different analytical methods. The respective values of slip coefficient are summarized in table 1 for $A = 1.0$ and in table 2 for $A = 0.5$. 
### Table 1. A comparison of the slip coefficients for different exact and approximate analytical methods at $A = 1.0$

| Solution methods | $Kn = 1.00E-03$ | $Kn = 0.003162$ | $Kn = 1.00E-02$ | $Kn = 0.031623$ | $Kn = 1.00E-01$ |
|------------------|----------------|----------------|----------------|----------------|----------------|
| FDM              | 0.017241       | 0.052562       | 0.149254       | 0.356824       | 0.636943       |
| Chakraborty[7]   | 0.017241       | 0.052562       | 0.149254       | 0.356824       | 0.636943       |
| SOV              | 0.017241       | 0.052562       | 0.149254       | 0.356824       | 0.636943       |
| Integral Ritz    | 0.016393       | 0.050066       | 0.142857       | 0.345141       | 0.625000       |
| Integral Kantorovich | 0.016949     | 0.051703       | 0.147059       | 0.352843       | 0.632911       |
| Variational Ritz | 0.017544       | 0.053451       | 0.151515       | 0.360897       | 0.641026       |
| Variational Kantorovich | 0.017241     | 0.052562       | 0.149254       | 0.356824       | 0.636943       |

### Table 2. A comparison of the slip coefficient for different exact and approximate analytical methods at $A = 0.5$

| Solution methods | $Kn = 1.00E-03$ | $Kn = 0.003162$ | $Kn = 1.00E-02$ | $Kn = 0.031623$ | $Kn = 1.00E-01$ |
|------------------|----------------|----------------|----------------|----------------|----------------|
| FDM              | 0.007737       | 0.024064       | 0.072333       | 0.1978         | 0.438116       |
| Chakraborty[7]   | 0.007737       | 0.024064       | 0.072333       | 0.1978         | 0.438116       |
| SOV              | 0.007737       | 0.024064       | 0.072333       | 0.1978         | 0.438116       |
| Integral Ritz    | 0.007353       | 0.022888       | 0.068966       | 0.189787       | 0.425532       |
| Integral Kantorovich | 0.007605     | 0.023659       | 0.071174       | 0.195054       | 0.433839       |
| Variational Ritz | 0.007874       | 0.024483       | 0.073529       | 0.200623       | 0.442478       |
| Variational Kantorovich | 0.007737     | 0.024064       | 0.072333       | 0.1978         | 0.438116       |

### 5. Conclusions
The present study offers various analytical solutions for fluid flow through a rectangular microchannel to obtain the exact and approximate results. The reduced form of the Navier-Stokes equation with velocity slip boundary conditions is solved using the analytical methods to determine velocity profile, Poiseuille number and slip coefficient. For exact analysis, the SOV method is used whereas for the approximate analysis, the Integral Ritz, Integral Kantorovich, Variational Ritz and Variational Kantorovich methods are considered. It is seen that the momentum equation has been solved with the approximate methods with less effort and error. The predictions obtained by the offered techniques are compared with the numerical method and existing work in literature for validation purpose for $A = 1.0$ and $A = 0.5$. Subsequently, Poiseuille number and slip coefficient are determined at different Knudsen numbers for the various analytical methods. For velocity distribution, the Integral Ritz and Integral Kantorovich methods provide closer solution compared to the Variational Ritz and Variational Kantorovich methods. For prediction of Poiseuille number, almost all the methods predict results with less deviation. Therefore, the proposed approximate methods can be used for analysis of fluid flow within microchannels of rectangular cross section under laminar fully developed conditions.
6. References

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