Single-qubit optical quantum fingerprinting

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Introduction.– Quantum communication can significantly improve on the resource requirements compared to classical communication [1]. Fingerprinting, which enables an efficient way of inferring whether longer messages are identical or not, is a particularly striking example as quantum fingerprinting offers an exponential reduction of resources compared to classical fingerprinting [2]. In fact, even for single-qubit fingerprinting one can demonstrate an advantage of quantum protocols with respect to classical ones [3]. Here we establish the feasibility of single-qubit optical quantum fingerprinting, by theoretical analysis and also by experimentally generating and assessing the appropriate quantum optical states for encoding. In particular we (i) develop an optical protocol for single-qubit fingerprinting, (ii) show that two-photon coincidence measurements suffice as the approximation of the function

\[
\text{EQ}(x, y) = \begin{cases} 
0 & \text{if } x \neq y \\
1 & \text{if } x = y 
\end{cases}, \tag{1}
\]

and Roger is successful if \( z = \text{EQ}(x, y) \). Each message belongs to a set \( M = \{0, \ldots, m-1\} \) comprised of \( m \) different messages represented as bit strings of length \( n \equiv \lceil \log_2 m \rceil \) and each fingerprint to a set \( F = \{0, \ldots, f-1\} \) of \( f \) different fingerprints. Classically, \( g = \lceil \log_2 f \rceil \) and \( F = \{0, 1\}^g \) while in the quantum case \( F \subset \mathcal{H}_2^f \) for \( \mathcal{H}_2 = \text{span}\{\ket{0}, \ket{1}\} \). The protocol is evaluated according to the worst case scenario (WCS), in which Sapna always sends message pairs for which the probability for \( z \neq \text{EQ}(x, y) \) is maximized (i.e. performance in the WCS corresponds to the ‘guarantee’ on the protocol).

We consider two experimental scenarios: 1) Alice and Bob each simultaneously receive unentangled single photons [4] with polarization states expressed in the logical basis \( \ket{0} \) and \( \ket{1} \) and, 2) Alice and Bob share a source of entangled photon pairs in the singlet Bell state \( |\Psi^\prime\rangle \equiv (|0, 1\rangle - |1, 0\rangle)/\sqrt{2} \). In the first scenario we are able to show that a linear optical single-qubit quantum fingerprinting protocol outperforms classical fingerprinting without a shared resource. In the second scenario, Alice and Bob share entanglement, and we show that this protocol can yield perfect one-qubit fingerprinting for \( m = 4 \), outperforming one-bit fingerprinting for \( m = 4 \) with an arbitrary amount of shared randomness.

Encoding:– For any message \( w \in M \) that Alice or Bob receive, they transform their qubit to a unique \( |\Omega_w\rangle \) with \( |\Omega \equiv (\theta, \phi)\rangle \equiv \cos \frac{\phi}{2} |0\rangle + \exp(i\phi) \sin \frac{\phi}{2} |1\rangle \). The state can be understood geometrically by identifying \( \theta \) and \( \phi \) with azimuthal and polar angles of the (Bloch) sphere. We assume that Alice and Bob employ the same mapping: \( x = y \Leftrightarrow |\Omega_x\rangle = |\Omega_y\rangle \). Quantum fingerprinting allows \( m \) different qubit states so each message is distinctly encoded, but the distinguishability of these distinct states diminishes as \( m \) increases, with indistinguishability quantified by \( \delta(|\Omega', \Omega\rangle) \equiv |\langle \Omega' | \Omega \rangle|^2 = |\cos \frac{\phi'}{2} \cos \frac{\phi}{2} + \exp[i(\phi - \phi')] \sin \frac{\phi'}{2} \sin \frac{\phi}{2}|^2 \). Because of a nonzero overlap, Roger can misinterpret two different
messages as identical. In the WCS, the corresponding error rate depends on \( \delta_{\text{max}} \equiv \max_{(w \neq w')} \delta(\Omega_w, \Omega_w) \), and the strategy for qubit encoding should minimize \( \delta_{\text{max}} \).

Single-qubit fingerprinting is especially interesting because of its current feasibility. To demonstrate this, we analyze the case \( m = 4 \) \((n = 2)\). In this case \( \delta_{\text{max}} \) is minimized by the following set of four states,

\[
F = \{ \{ \Omega_w \}; \Omega_0 \equiv (\theta_0, \phi_0) = (0,0) \text{ or } \Omega_w = (2 \cos^{-1} \frac{1}{\sqrt{3}} \frac{2 \pi}{3} w) \text{ for } w = 1, 2, 3 \},
\]

and \( \delta = \frac{1}{3} \) for all pairs of different states \( \begin{array}{ll} \text{B} & \text{S} \end{array} \). We refer to the states \( \begin{array}{ll} \text{B} & \text{S} \end{array} \) as ‘tetrahedral states’ because the four states form the vertices of a tetrahedron on the Bloch sphere \( \mathbb{R}^3 \).

**Protocol.**—Alice and Bob map their two-bit messages to the tetrahedral states, and Roger’s task is to assess EQ\((x,y)\) by measuring and inferring whether \( |\Omega_2\rangle \equiv |\Omega_y\rangle \). The original proposals \( \begin{array}{ll} \text{2} & \text{4} \end{array} \) provided Roger with a controlled swap gate and an ancilla qubit (Fig. 1 (a)). The ancilla is prepared as \( (|0\rangle + |1\rangle)/\sqrt{2} \) and entangled with the fingerprint states as follows: the two fingerprint states are not swapped if the ancilla is in the state \( |0\rangle \) and swapped otherwise. The ancilla then passes through a Hadamard gate and is measured in the logical basis with outcome \( r \in \{0,1\} \) corresponding to the ancilla being in state \( |r\rangle \). This strategy yields a one-sided error protocol because Roger’s error rate when Sapna sends \( x = y \) is \( p_{\text{err}}^{\text{same}} = 1 - \frac{1}{2}[1 + \delta] = 0 \). In the WCS, Sapna always sends different states so that, when Roger obtains \( r = 0 \), he infers \( z \) with error rate \( p_{\text{err}} = p_{\text{err}}^{\text{WCS}} = 1 - \frac{1}{2}[1 - \delta] \). For \( m = 4 \) and tetrahedral encoding, we obtain \( p_{\text{err}}^{\text{WCS}} = \frac{2}{3} \).

A controlled swap gate is not available in a linear optical system, but we show that it is not required. If Alice and Bob each send a single photonic qubit encoded in polarization to Roger, then Roger only needs to measure whether the photons are in the same polarization. This measurement can be accomplished with the use of a Bell state discriminator that can distinguish between \( |\Psi^-\rangle \) and the other three Bell states. Optically this discrimination is achieved by directing each of Alice’s and Bob’s photons into separate input ports of a symmetric beam splitter and observing photon count events from two photodetectors placed at the output ports. A coincidence detection implies the state of the photon pair before the beam splitter was not orthogonal to \( |\Psi^-\rangle \) because the other three Bell states result in two photons leaving the beam splitter through the same port \( \text{X} \). These states exhibit a Hong-Ou-Mandel (HOM) dip in the coincidence rate \( \text{X} \) as the delay of the incidence photons is varied.

For \( m = 4 \), Alice and Bob each receive two-bit messages from Sapna, which are used to encode their photonic qubit into one of the tetrahedral states. Their photons are transmitted to Roger who infers using a symmetric beam splitter whether the messages were the same or different. This protocol is depicted in Fig. 1(b). Ideally Alice and Bob would have separate single-photon-on-demand sources, but practically they will be supplied with correlated, unentangled photons from a down-conversion source. Later we consider the case that Alice and Bob share entangled photons.

Following the same notation as for the controlled swap case, Roger assigns \( r := 0 \) for a no-coincidence and \( r := 1 \) for a coincidence event, then employs (as before) the pure strategy \( z = 1 - r \). The result \( r = 1 \) guarantees the messages are unequal but \( r = 0 \) only indicates that the messages were possibly the same. In fact this HOM dip protocol is equivalent to the controlled swap version of single qubit quantum fingerprinting because if Sapna sends \( x \neq y \), the probability that Alice’s and Bob’s photons do not trigger a coincidence detection is identically \( p_{\text{err}}^{\text{diff}} = p_{\text{err}} = 1 - \frac{1}{2}[1 - \delta] \). Thus Sapna always sends different messages in the WCS. For \( m = 4 \), \( p_{\text{err}}^{\text{WCS}} = \frac{2}{3} \).

This error rate appears relatively high, yet it is superior to classical one-bit fingerprinting with one-sided error, in which failure is guaranteed for at least one pair of messages, resulting in a 100% WCS error rate \( \text{X} \). Of course 100% failure rate for the classical case can be improved by allowing Roger a random strategy, but then the quantum protocol can be improved in the same way, always maintaining its superiority over the classical case \( \text{X} \).
### TABLE I: Experimental visibilities of the Hong-Ou-Mandel dip for each pair of tetrahedral states

|       | Bob |
|-------|-----|
| Alice |     |
| 0     | 0.88| 0.31| 0.24| 0.26 |
| 1     | 0.30| 0.88| 0.25| 0.40 |
| 2     | 0.44| 0.30| 0.89| 0.25 |
| 3     | 0.20| 0.30| 0.35| 0.89 |

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**Experiment.**—The feasibility of this protocol has been illustrated by creating simultaneous pairs of tetrahedral states and analyzing the dip achieved by Roger’s set-up in Fig.1(b). To create correlated photons, a Ti:Sapphire laser tuned to a wavelength of 790 nm emitted 170 fs pulses that were frequency doubled and then down-converted in a type I configuration via a 2-mm beta-barium borate crystal. Output photons were spectrally filtered with a 2-nm interference filter and transmitted through λ/2 and λ/4 waveplates which were rotated to convert the polarization state in each channel into one of the tetrahedral states. The two photons were then overlapped in free space on a symmetric beam splitter and subjected to measurements with single-photon counting modules; the experimental results are presented in Fig. 1(c) where state $|\Omega_1\rangle$ is mixed with itself and each of the other three states. The largest dip in Fig. 1(c) corresponds to the traditional HOM dip with two identical states mixing at the beam splitter, and the degree of distinguishability is varied by controlling the relative delay between the two photons. The experimental coincidence rates as a fraction of the maximum coincidence rate $R/R_{\text{max}}$ for all 16 possible fingerprint pairs is given in Table I and is consistent with Clarke et al.’s experimental results for tetrahedral states.

Due to birefringence in the beam splitter and limitations on constructing the unitary transformation required to create perfect tetrahedral states, the visibilities vary, but the dip depth $d$ for mixing identical states is consistently at 88% or higher, as shown in the diagonal elements of Table I. Ideally Table I would have unity for all diagonal elements (ie. $d = 1$) and 1/3 for all off-diagonal elements. We use Bob’s state $|\Omega_1\rangle$ as the reference state for assessing feasibility, which consistently produces visibilities of approximately 30% when mixed with Alice’s other three states. Note that in the non-ideal case, $d < 1$, and the error probabilities change. That is, $p_{\text{err}}^{\text{diff}} = \frac{1}{2}(1 + d \delta_{\text{diff}})$ and $p_{\text{err}}^{\text{same}} = 1 - \frac{1}{4}(1 + d \delta_{\text{same}})$, where for the tetrahedral states $\delta_{\text{same}} = 1$ and $\delta_{\text{diff}} = 1/3$.

This experimental scheme is not directly applicable to fingerprinting because Alice and Bob are not aware when a photon pair has been produced, so the amount of information the parties send to Roger cannot be traced. This problem can be resolved by using either deterministic single-photon sources or heralded single-photon sources based on two down-converters. Another issue is that a large fraction of information is lost due to poor single-photon detection efficiency. This can be overcome by Roger using number-resolving detectors, post-selecting the data on the registration of two photons, and requesting Alice and Bob to repeat their messages if photons were lost.

**Two-sided errors.**—The one-sided error protocol is predicated on unitary diagonal elements of Table I as this is impossible, an experimental protocol must be assessed for two-sided errors because, even if Sapna sends the same messages to Alice and Bob, Roger is no longer guaranteed to obtain the measurement outcome $r = 1$. Allowing Roger to err on both inferences lowers the classical WCS error probability bound from 1 to $p_{\text{err}}^{\text{WCS}} \geq 0.5$. Thus, quantum fingerprinting is advantageous provided that Roger’s strategy yields $p_{\text{err}}^{\text{WCS}} < 0.5$ when permitting a two-sided error protocol.

Whereas Roger followed a pure strategy for protocols with one-sided error this restriction is unnecessary for a two-sided error protocol. As such we introduce a procedure for producing a successful two-sided error protocol where Roger incorporates randomness and follows a mixed strategy instead.

The mixed strategy is as follows. Roger makes an initial inference $z^* = 1 - r$ as before. If $z^* = 1$ Roger infers $z = 0$ with probability $\pi_0$ and if $z^* = 0$ Roger infers $z = 1$ with probability $\pi_1$.

The success rate is

$$\pi_0 p_{\text{err}}^{\text{diff}} + (1 - \pi_1)(1 - p_{\text{err}}^{\text{diff}}),$$

$$\pi_1 p_{\text{err}}^{\text{same}} + (1 - \pi_0)(1 - p_{\text{err}}^{\text{same}}),$$

for Sapna supplying $x \neq y$ and $x \neq y$ respectively. Roger then chooses values of $\pi_0$ and $\pi_1$ such that his success rate is identical for both cases making Sapna’s choice of messages irrelevant: all cases correspond to a WCS. We solve both success equations based on the values in Table I where $d = 0.88$, $p_{\text{err}}^{\text{same}} \sim 0.06$, and $p_{\text{err}}^{\text{diff}} \sim 0.65$ and find that the success rate can achieve 0.59 which is above the classical threshold of 0.5. This optimal case is achieved by setting $\pi_0 = 0.37$ and $\pi_1 = 0$, which means that Roger’s best strategy is to treat the protocol as if it were one-sided, thereby invoking randomness only on the side with error.

**Shared entanglement.**—Thus far Alice and Bob have been denied any communication, but experimentally it is straightforward to provide Alice and Bob with an entangled pair of photons. We show that shared entanglement allows perfect single-qubit quantum fingerprinting for $m = 4$ and, furthermore, exceeds the classical limit. The classical analog to this case corresponds to the performance in the WCS for Alice and Bob sharing random bits that are secret from Sapna.

We allow Alice and Bob to share the Bell singlet state $|\Psi^-\rangle$. Alice and Bob each receive a two-bit message from
Sapna and apply one of the four Pauli operations according to which message has been sent. The result is that the state sent to Roger is one of the four Bell states. If Alice and Bob perform the same Pauli operation, $|\Psi^-\rangle$ is invariant (up to a global phase); if Alice and Bob apply different transformations, $|\Psi^-\rangle$ maps to a different Bell state. Thus, for Roger to infer whether the messages are the same or different, he needs only to detect whether he has received the state $|\Psi^-\rangle$ or not. The Bell state discriminator, in the form of a HOM dip apparatus discussed earlier suffices as a discriminator between the Bell state $|\Psi^-\rangle$ and the other three Bell states $|\Phi^\pm\rangle$ and $|\Psi^\pm\rangle$. For a perfectly efficient setup, a coincidence is guaranteed for an input Bell state $|\Psi^-\rangle$, and no coincidence occurs for the other Bell states. Therefore, the protocol can achieve $p_{\text{err}} = 0$ and by consuming one ebit for each pair of two-bit messages delivered Sapna.

The physics underlying this fingerprinting scheme resembles that employed in quantum dense coding [9], but the purposes that these two communication protocols serve are quite different. Whereas, in the latter case, a shared ebit is used to communicate a classical two-bit message from Alice to Bob, the former allows a third party (Roger) to compare two two-bit messages. If Alice and Bob share one random bit (in the case of a shared ebit, Alice and Bob could convert their ebit to a shared random bit if they wish), Roger’s success rate for classical one-bit fingerprinting regardless of how many random bits Alice and Bob share. If Alice and Bob share one random bit (in the case of a shared ebit, Alice and Bob could convert their ebit to a shared random bit if they wish), Roger’s success rate for classical one-bit fingerprinting rises from zero to $\frac{1}{2}$ when Roger follows a pure strategy. If Alice and Bob share an arbitrarily large number of random bits, Roger’s success rate improves but cannot exceed $\frac{1}{2}$ for any fixed number of random bits [11]. Of course limited detector efficiency for the entangled protocol will diminish the success rate, but any success rate beyond $\frac{1}{2}$ is superior to the classical case.

Conclusions.— We have proposed an optical protocol for single-qubit fingerprinting, experimentally demonstrated its functionality for the case $m = 4$, and shown that tetrahedral states can be produced that meet the requirements for beating the classical one-bit fingerprinting protocol for $m = 4$. We have also proven that single-qubit quantum fingerprinting with shared entanglement can succeed with a zero error rate, which beats the classical fingerprinting protocol with an arbitrary amount of shared randomness between Alice and Bob. The experimental results show that, in reality, two-sided errors must be accounted for, but we have shown that Roger’s best strategy is to randomly vary his inference of whether the states are the same but not change his guesses as to whether they are different, and this approach yields a performance, given experimentally obtained parameters, that exceeds the classical error bound. Quantum fingerprinting is an excellent example of the new field of quantum communication complexity [1], and our results here open this field to experiments. Further work is now underway on quantum fingerprinting with two qubits and beyond, which will allow scaling and complexity issues to be fully investigated.

Note:— Optical quantum fingerprinting was considered by Massar [13], but his protocol is very different: Alice and Bob share a single photon, and the protocol uses two-slit interference as an alternative approach to Roger’s strategy. In our protocol Alice and Bob each have independent photons, with or without shared entanglement. Despite the related names, the two protocols are entirely different and with different aims: our goal is to ensure that quantum fingerprinting operates within the strict confines of the simultaneous message passing model, inspired by de Beaudrap’s analysis of single-qubit fingerprinting [2].

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