A note on the decay of noncommutative solitons

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Abstract

We propose an ansatz for the equations of motion of the noncommutative model of a tachyonic scalar field interacting with a gauge field, which allows one to find time-dependent solutions describing decaying solitons. These correspond to the collapse of lower dimensional branes obtained through tachyon condensation of unstable brane systems in string theory.

1 Introduction

Noncommutative field theory was shown to arise as the effective theory for the massless modes of string theory in the limit of a large B-field (see \cite{1}). The solitonic solutions found in noncommutative field theory in various limits (see \cite{1, 2, 3, 4} and later works) are believed to correspond to condensed branes \cite{6}, arising from unstable brane-antibrane configurations \cite{4}. In operator language, these solutions are given by finite rank projectors, whose Weyl–Moyal forms are localised functions.

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Although dynamical solutions involving noncommutative solitons were discussed [7] the aspects of the collapse process have still to be clarified.

The aim of this note is to study the dynamical process of solitonic collapse in noncommutative theory. We study the dynamics of rolling-down solutions in the model of a noncommutative tachyon field $\Phi$ interacting with a gauge field $A_\mu$. These solutions correspond to unstable noncommutative branes. They describe the classical decay of noncommutative solitons, in an infinite time interval.

The plan of the paper is as follows. First we consider the time-dependent ansatz which reduces the noncommutative field equations of motion to a chain of separate one-dimensional equations which can be implicitly solved. After that we discuss the implications of the solution, and present some conclusions.

## 2 Classical decay

The model is given by a $(2+1)$-dimensional action which, in operator form, reads

$$S = \int dt \text{Tr}_H \left( \frac{1}{2}(\nabla_0 \Phi)^2 + \frac{1}{2}[X_i, \Phi]^2 - V(\Phi) - \frac{1}{4g^2}[X_i, X_j]^2 + \frac{1}{2}(\nabla_0 X_i)^2 \right),$$

where $i = 1, 2$. The scalar field $\Phi$ and the gauge field $X_\mu$ are time-dependent operators acting on the Hilbert space $H$ on which the algebra

$$[x^1, x^2] = i\theta$$

is represented. $X_i$ is related to the conventional gauge field $A_i$ by $X_i = p_i + A_i$, where $p_i = \theta^{-1}\epsilon_{ij}x^j$.

The equations of motion for the field $\Phi$ are,

$$\nabla_0^2 \Phi + [X_i, [X_i, \Phi]] + V'(\Phi) = 0. \tag{3}$$

We consider the case in which the potential $\Phi$ is tachyonic-like, i.e. it has a local maximum at the origin and local minima at $\Phi = \lambda_i$.

The model admits static solutions (solitons),

$$\Phi_0 = \sum_i \lambda_i (1 - P_i), \quad [X, \Phi_0] = 0, \quad A_0 = 0, \tag{4}$$

\(^1\)After the initial version of this note was published a number of papers appeared discussing the falling-to-vacuum tachyon solutions [8].
where the operators $P_i, P_j$ for $i \neq j$, are projectors to finite dimensional orthogonal subspaces of the Hilbert space $\mathcal{H}$. In Weyl-Moyal language, these solutions are given by functions having the asymptotics $\Phi|_{x \to \infty} = \lambda_n$.

To show that the solution (4) is unstable, consider the following time-dependent ansatz. The gauge invariance of the model allows one to choose the operator $\Phi$ to be diagonal. This partially fixes the gauge. In the oscillator basis $\{|n\rangle\}$ given by

$$N |n\rangle = n |n\rangle, \quad N = aa, \quad a = \frac{1}{\sqrt{2\theta}}(x^1 + ix^2),$$  

the field $\Phi$ is described by an infinite set of one dimensional variables $\Phi_n(t) = \langle n | \Phi(t) | n \rangle, n = 0, 1, 2, \ldots$ In this basis, the above mentioned static soliton looks as follows:

$$\Phi_n = 0, \quad n \leq n_0; \quad \Phi_n = \lambda, \quad n > n_0.$$  

In what follows, we will allow dynamics for this finite number of $\Phi_n$’s. In this case, the second equation in the ansatz (4) is still valid for static $X_\mu$.

As a result, the field operator equations of motion split into a set of decoupled equations

$$\ddot{\Phi}_n + V'(\Phi_n) = 0, \quad 0 \leq n \leq n_0,$$

with initial conditions for $\Phi_n$ at $t = 0$ given by (3), supplemented with $\dot{\Phi}_n(t = 0) = 0$ for all $n$. These equations can be trivially integrated (see any textbook on classical mechanics), and the implicit form of the solution is

$$t(\Phi_n) = \int_0^{\Phi_n} d\Phi' \frac{1}{\sqrt{V(0) - V(\Phi')}}, \quad n \leq n_0,$$

and $\Phi_n = \lambda$, for $n > n_0$. If one integrates between a local maximum ($\Phi = 0$) and an adjacent minimum ($\Phi = \lambda$) of the potential, the above integral gives the time of fall, which diverges logarithmically as $\Phi$ approaches the origin.

Thus classically the soliton will decay into an infinite amount of time. This is the standard situation for unstable equilibrium points in classical mechanics. However, quantum fluctuations will kick the soliton out of the origin in a finite time.

In particular, if one limits oneself to the lowest terms of the tachyonic potential, $V(\Phi) = V(0) - \frac{1}{2}m^2\Phi^2 + g\Phi^4$, the classical solution is

$$\Phi_n(t) = \frac{m}{\sqrt{g}} \frac{2e^{-mt}}{1 + e^{-2mt}}, \quad n \leq n_0,$$
which satisfies $\Phi(t = -\infty) = 0$ and $\Phi(t = 0) = \frac{m}{\sqrt{g}} = \lambda$ for $n \leq n_0$. In operator form this solution looks like

$$\Phi(t) = \frac{m}{\sqrt{g}} \frac{2e^{-mt}}{1 + e^{-2mt}} P_{n_0},$$

(10)

where $P_{n_0}$ is the projector operator to the first $n_0+1$ states $P_{n_0} = \sum_{n=0}^{n_0} |n\rangle \langle n|$. We also note the existence of Euclidean-time (instantonic) solutions of the equations of motion, which interpolate between two degenerate vacua:

$$\Phi(\tau) : \quad \tau = \int_{\lambda_i}^{\Phi} \frac{d\tau}{\sqrt{V(\Phi)}}, \quad \tau = it.$$  

(11)

In the real time approach they correspond to tunnelling between the vacua.

Let us make a remark. We considered time-dependent solutions with static gauge field configuration, leaving aside the possible evolution of the gauge fields. There are indications however that those may decay as well. Indeed, the most general gauge field background satisfying (4) is given by the block operator,

$$X_\mu = \begin{pmatrix} X_\mu^{(n)} & 0 \\ 0 & X_\mu^{(\infty)} \end{pmatrix},$$

(12)

where $X_\mu^{(n)}$ are $n \times n$ Hermitian matrices, while $X_\mu^{(\infty)}$ are (infinite-dimensional) Hermitian operators. Both of them satisfy

$$\dot{X}_\mu - \frac{1}{g} [X_\nu, [X_\nu, X_\mu]] = 0,$$

where $X$ stands for either $X^{(n)}$ or $X^{(\infty)}$. In particular one can choose a solution where $X^{(n)} = 0$ and $X^{(\infty)}$ are some new $p'_\mu$ satisfying the commutation relation on the infinite dimensional subspace $[p'_\mu, p'_\nu] = i\theta_{\mu\nu}^{-1}$.

The finite dimensional part of the gauge field given by $X^{(n)}$ possesses only static solutions satisfying $[X_\mu^{(n)}, X_\nu^{(n)}] = 0$, which are stable since they have zero energy (density). So the only trouble may come from the infinite-dimensional sector $X^{(\infty)}_\mu$ which, beyond the commuting solutions of the above type, may also have solutions with c-number commutator. These gauge field configurations possess finite energy density proportional to the square of the inverse noncommutativity parameter $(\theta_{\mu\nu}^{-1})^2$, and therefore potentially may decay to a commutative solution with $\theta_{\mu\nu}^{-1} = 0$, or equivalently

$$[X_\mu^{(\infty)}, X_\nu^{(\infty)}] = 0,$$

(13)
which has zero energy density. For a description of the noncommutative model around the background\[13\] we refer the reader to the paper\[9\].

The solution found in this paper gives one channel of decay. As it often happens with unstable systems, there could be other channels on which the system classically slides down. In the quantum theory these channels would be assigned different probabilities. Such an analysis, however, goes beyond the scope of the present work. We just mention that, from another point of view, there are indications of instabilities of purely bosonic noncommutative Yang–Mills models at one-loop level\[10\].

3 Radiation

Finally, we discuss the possible radiation during the decaying process. We consider for that matter the general tachyonic action (1) where \(\nabla_0 = \partial_0 + [A_0,\cdot]\), and all fields are Hilbert space operators.

The solution of the general equations of motion,

\[
\nabla_0^2 \Phi + [X_i, [X_i, \Phi]] + V'(\Phi) = 0, \tag{14}
\]

\[
\nabla_0^2 X_i + \frac{1}{g^2} [X_k, [X_k, X_i]] + [\Phi, [X_i, \Phi]] = 0, \tag{15}
\]

\[
[X_i, \dot{X}_i] + [\Phi, \nabla_0 \Phi] = 0, \tag{16}
\]

is given by the ansatz (in diagonal gauge),

\[
\ddot{\phi} + V'(\phi) = 0, \quad [Y_i, \phi] = [A_0, \phi] = 0. \tag{17}
\]

Let us label the components of the background solution as follows \(\phi_a = \phi(t) \neq \lambda, \ a = 1, \ldots, N\) and \(\phi_n = \lambda, \ n = 0, 1, \ldots\) where \(\lambda\) is the vacuum of the tachyonic potential. In above index \(a\) spans the solitonic subspace of the Hilbert space while \(n\) spans its infinite dimensional completion.

According to these notations the gauge field is split in two blocks corresponding to respectively tachyonic subspace and its completion, each of these blocks satisfying separately,

\[
[Y_k^{(A)}, [Y_k^{(A)}, Y_i^{(A)}]] = 0, \tag{18}
\]

where capital \(A\) denotes respectively blocks corresponding either to solitonic space or to its completion.

Fields in the solution are assumed to have the following block structure,

\[
\phi = \begin{pmatrix} \phi & 0 \\ 0 & \lambda \end{pmatrix}, \quad Y_i = \begin{pmatrix} Y_i^{(0)} & 0 \\ 0 & Y_i' \end{pmatrix}. \tag{19}
\]
From (18), the solitonic block should be diagonal (in the basis in which the tachyon is diagonal), while the infinite dimensional block in the completion space could be chosen e.g. to satisfy

$$[Y'_i, Y'_j] = B_{ij}, \quad \det B \neq 0,$$

(20)
giving rise to a noncommutative space by itself.

We now show that the decaying tachyonic soliton is a source for various particles living in the space generated by $Y'$ satisfying (20). To see this, consider the fluctuation fields around the solution,

$$\Phi = \phi + u,$$

(21)

$$X_i = Y_i + a_i,$$

(22)

$$A_0 = a_0.$$  

(23)

Inserting the above into the action yields after some algebra, up to second order in fluctuations,

$$S[u, a] = S[\phi, Y] + \frac{1}{2} (\nabla_0 u)^2 - \frac{1}{2} V''(\phi) u^2 - \frac{1}{2} [Y_i, u]^2 + \frac{1}{2g^2} (\nabla_0 a_\mu)^2 - \frac{1}{4g^2} [Y_i, a_\mu]^2 + \frac{1}{2} [a_\mu, \phi]^2 + \dot{\phi}[a_0, u] - [Y_i, u][a_i, \phi].$$

(24)

The lines two and three in the above equation give the free propagators of respectively $u$ and $a_\mu$ fields while the last line provides the interaction vertices. We also used the gauge fixing condition $\nabla_0 a_0 - [Y_i, a_i] = 0$.

If we split the fields $u$ and $a_\mu$ into blocks according to the block structure (19),

$$u = \begin{pmatrix} c & v \\ \bar{v} & u' \end{pmatrix}, \quad a_\mu = \begin{pmatrix} c_\mu & b_\mu \\ \bar{b}_\mu & a'_\mu \end{pmatrix},$$

(25)

the decaying soliton field is a source for the particles $b$, $\bar{b}$, $v$ and $\bar{v}$, as follows,

1. $\frac{1}{2} [a_\mu, \phi]^2 \rightarrow [b_\mu, \phi][\bar{b}_\mu, \phi]$ — a pair of charged photons $b_\mu \bar{b}_\mu$.

2. $\dot{\phi}[a_0, u] + [Y_i, u][a_i, \phi] \rightarrow \dot{\phi}[b_\nu, \phi] + [Y_i, v][\bar{b}_i, \phi] + \text{h.c.}$ — a charged photon + the scalar with opposite charge, where h.c. stands for Hermitian conjugate.

We see that at tree level the decay process produces only particles in the fundamental representation of the unbroken gauge group. Because of gauge invariance emission is possible for only totally neutral pairs.

\footnote{More precisely: in the bi-fundamental representation of each factor.}
4 Conclusion

Let us conclude with the following:

- In string theory framework, noncommutative solitons correspond to the \( D(p-2) \)-branes which result from condensation of unstable \( D(p) \)-brane systems. As our analysis shows, the \( D(p-2) \)-brane appears to be unstable as well, decaying into nothing.

- In their classical rolling down, solitons will be reflected back and reach their initial values (in infinite time). As we have shown, however, the decaying brane can emit definite types of charged particles and antiparticles. In this case, the field will not reach its initial value, but will oscillate, with decreasing amplitude, around the vacuum. Then noncommutative solitons could have cosmological applications, playing the role of the inflaton field.

- For a general noncommutative potential the solitons are unstable iff there are states in the Hilbert space for which the noncommutative field \( \Phi \) takes values corresponding to unstable points of the potential (as a commutative function). In the case of solitons living at local minima, they are unstable due to tunnelling to the global minimum.

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