Form factors with two-operator insertion and violation of transcendentality principles

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We present the first calculations of two-point two-loop form factors (FFs) with two identical operators insertion of supersymmetry protected half-BPS primary and unprotected Konishi in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills (MSYM) theory. To our surprise, the principle of maximal transcendentality which dictates the presence of identical highest weight terms in the scalar FFs of half-BPS and quark/gluon FFs in QCD is found to be violated. The conjecture that the FFs of half-BPS operator contain terms of uniform transcendentality weight is also found to break down. Moreover, the highest weight terms of half-BPS and Konishi FFs no longer match. The finite parts of FFs are also found not to exponentiate in contrast to the case of two-loop scalar FF of single half-BPS operator.

A generic quantum field theory is entirely specified by the knowledge of on-shell scattering amplitudes and off-shell correlation functions. There exists another class of fascinating objects, called form factors (FFs), which interpolate between amplitudes and correlators. This object is defined through the overlap between a state created by the action of a composite gauge invariant operator on the vacuum and a state consisting of only on-shell particles. The FFs in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills (MSYM) theory are expected to inherit much of the remarkable simplicity of the on-shell amplitudes, and at the same time to reflect some of the non-trivial behaviour of the off-shell correlators. In past few decades, FFs have been studied extensively starting from the seminal works in refs. [1–6]. Very recently, the first step is taken to go beyond the horizon of FFs with one-operator insertion and the scenario with two-operator insertion is addressed [7]. In this letter, we take this step forward by performing a state-of-the-art computation to explore the nature of two-loop two-point FFs with insertion of two identical operators. Consequently, for the first time, we examine the validation of several conjectures in view of generalised FFs.

In this work, we consider two local gauge invariant operators:

\[
O_{rt}^{\text{BPS}} = \phi_r^b \phi_t^b - \frac{1}{3} \delta_{rt} \phi_s^b \phi_s^b,
\]

\[
O^K = \phi_r^b \phi_t^b + \chi_r^b \chi_t^b,
\]

where \( O_{rt}^{\text{BPS}} \) and \( O^K \) are the SUSY protected half-BPS primary belonging to the stress-energy supermultiplet and unprotected Konishi operators, respectively. The scalar and pseudo-scalar fields are denoted by \( \phi_r^b \) and \( \chi_r^b \), respectively, where their number of generations is represented through \( r, s, t \in [1, n_g] \) with \( n_g = 3 \) in 4-dimensions. All the fields in MSYM theory transform under adjoint representation which is represented through the SU(N) colour index \( b \).

Understanding the analytical structures of on-shell amplitudes and FFs in MSYM has been an active area of investigation, not only to uncover the hidden structures of these quantities but also to establish the connections with other gauge theories, such as QCD. One of the most intriguing facts is the appearance of uniform transcendentality (UT) weight terms in certain class of quantities in MSYM. This is indeed an observational [4, 8–17], albeit unproven, fact. The two-point or Sudakov FFs of primary half-BPS operator belonging to the stress-energy supermultiplet is observed [1, 3, 11] to exhibit the UT property to three-loops, more specifically, they are composed of only highest transcendentality (HT) terms with weight \( 2L \) at loop order \( L \). This is a consequence of the existence of an integral representation of the FFs with every Feynman integral as UT [11]. Knowing the existence of such an integral basis has profound implication in choosing the basis of master integrals while evaluating Feynman integrals employing the method of differential equation [18]. The three-point FFs of half-BPS operator is also found to respect this wonderful UT property [12]. On the contrary, this property fails for the two- [19, 20] and three-point [16] FFs of the unprotected Konishi operator which are investigated up to three- and two-loops, respectively. Three-point FFs of a Konishi descendant operator is also found not to exhibit the UT property [21]. All the aforementioned results are in accordance with the general belief that the FFs of a supersymmetry (SUSY) protected operator, such as half-BPS primary, exhibit UT behaviour. Having seen the beautiful property of UT in FFs of one-operator insertion, the question arises whether it is respected for the two-point FFs with SUSY protected two-operator insertion and whether this property can be extrapolated to generalised FFs with \( n \)-number of operators insertion. In this

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1 The transcendentality weight, \( \tau \), of a function, \( f \), is defined as the number of iterated integrals required to define it, e.g. \( \tau(\log) = 1 \), \( \tau(L_n) = n \), \( \tau(C_n) = n \) and moreover, we define \( \tau(f_1 f_2) = \tau(f_1) + \tau(f_2) \). Algebraic factors are assigned to weight zero and dimensional regularisation parameter \( \epsilon \) to -1.
letter, for the first time, we address this question and, to our surprise, find that the UT property breaks down at two-loop for the FF of half-BPS with two-operator insertion.

It is conjectured in ref. [22] that the HT weight parts of every two-point minimal FFs (presence of equal number of fields in the operator and external on-shell state) are identical and those are equal to that of half-BPS, $\mathcal{O}_{rt}^{BPS}$. This conjecture is verified to four-loops order in ref. [20] for the Sudakov FFs of operator $\mathcal{O}_K$. Naturally, it is curious to see if this conjecture holds true for the generalised two-point FFs with two-operator insertion. In particular, we address whether the HT parts of the two-point FFs with double $\mathcal{O}_{rt}^{BPS}$ and double $\mathcal{O}_K$ insertion match. It turns out this conjecture also fails for two-operator insertion.

The connection between quantities in MSYM and that of QCD is of fundamental importance. In addition to deepening our theoretical understanding, it is motivated from the fact that computing a quantity in QCD is much more difficult, and in the absence of our ability to calculate a quantity in QCD, if it is possible to obtain the result, at least partially, from that of simpler theory, such as MSYM. In refs. [9, 17, 23, 24], it is found that the anomalous dimensions of leading twist-two-operator in MSYM is identical to the HT counterparts in QCD [25], and consequently, the principle of maximal transcendentality (PMT) is conjectured. The PMT says that the algebraically most complex part of certain quantities in MSYM and QCD are identical. The conjecture is found to hold true for two-point FFs to three-loops level [11], more specifically, the HT pieces of quark and gluon FFs in QCD [26] are identical, up to a normalisation factor of $2^L$, to scalar FFs of the operator $\mathcal{O}_{rt}^{BPS}$ in MSYM upon changing the representation of fermions in QCD from fundamental to adjoint. The diagonal elements of the two-point pseudo-scalar [27] and tensorial FFs [28, 29] also obey the conjecture. The three-point scalar and pseudo-scalar FFs are also found to respect the PMT [12, 22, 30–36]. Employing this conjecture, the four-loop collinear anomalous dimension in the planar MSYM is computed [37]. In ref. [38], the asymptotic limit of energy-energy correlator and in ref. [17], the soft function are also observed to be consistent with PMT. The complete domain of validity of this principle is still unknown. For on-shell amplitudes, it fails even at one loop [39] with four or five external gluons. In this article, we investigate whether the wonderful conjecture of PMT holds true for two-point FFs of two-operator insertion with $\mathcal{O}_{rt}^{BPS}$. Surprisingly, the PMT also gets violated.

**COMPUTATION OF TWO-LOOP FORM FACTOR**

The Lagrangian density [40–43] encapsulating the dynamics of MSYM and describing the interactions with the gauge invariant local operators, $\mathcal{O}_{rt}^{BPS}$ and $\mathcal{O}_K$, is given by

$$\mathcal{L} = \mathcal{L}_{MSYM} + \mathcal{J}_{rt}^{BPS}\mathcal{O}_{rt}^{BPS} + \mathcal{J}_K\mathcal{O}_K.$$  \hspace{1cm} (2)

The quantity $\mathcal{J}$ represents the off-shell state described by the corresponding operator. We are interested in investigating the two-point FFs with two-operator insertion of the following scattering processes

$$\phi(p_1) + \phi(p_2) \rightarrow \left\{ \begin{array}{l}
\mathcal{J}^{BPS}(p_3) + \mathcal{J}^{BPS}(p_4), \\
\mathcal{J}_K(p_3) + \mathcal{J}_K(p_4),
\end{array} \right.$$  \hspace{1cm} (3)

where $p_i$ are the corresponding four-momentum with $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m_{\lambda}^2 \neq 0$. The $m_{\lambda}^2$ is the invariant mass square of the colour singlet state described by the operators in (1) i.e. $\lambda \in \{BPS, K\}$. The underlying Mandelstam variables are defined as $s \equiv (p_1 + p_2)^2$, $t \equiv (p_1 - p_2)^2$ and $u \equiv (p_2 - p_3)^2$ satisfying $s + t + u = 2m_{\lambda}^2$. For convenience, we introduce the dimensionless variables $x, y, z$ through

$$s = m_{\lambda}^2 \frac{(1 + x)^2}{x}, \quad t = -m_{\lambda}^2 y, \quad u = -m_{\lambda}^2 z.$$  \hspace{1cm} (4)

In perturbation theory, the scattering amplitude of the processes (3) can be expanded in powers of the 't Hooft coupling constant, $a \equiv g^2 N(4\pi e^{-\gamma_E})^{-1}/(4\pi)^2$, as

$$|\mathcal{M}|_{\lambda} = \sum_{n=0}^{\infty} a^n |\mathcal{M}^{(n)}|_{\lambda},$$  \hspace{1cm} (5)

where the quantity $|\mathcal{M}^{(n)}|_{\lambda}$ represents the $n$-th loop amplitude of the process involving $\mathcal{J}_\lambda$. The quadratic Casimir in the adjoint representation of SU(N) group is given by $N$. The dimensional regulator, $\epsilon$, is defined through $d = 4 + \epsilon$ with the space-time dimension $d$. We regulate the theory by adopting a SUSY preserving modified dimensional reduction (DR) scheme [44, 45] which keeps the number of bosonic and fermionic degrees of freedom equal. This is achieved by changing the number of scalar and pseudo-scalar generations from $n_g = 3$ to $n_{g,\epsilon} = 3 - \epsilon/2$ in $d$-dimensions. The FFs are constructed out of the transition matrix elements through

$$\mathcal{F}_{\lambda} = 1 + \sum_{n=1}^{\infty} a^n \mathcal{F}_{\lambda}^{(n)} \equiv 1 + \sum_{n=1}^{\infty} a^n \left( \frac{|\mathcal{M}^{(0)}|}{|\mathcal{M}^{(0)}|} |\mathcal{M}^{(0)}|_{\lambda} \right).$$  \hspace{1cm} (6)

The primary objective of this letter is to compute the FFs to two-loops i.e. $\mathcal{F}_{\lambda}^{(1)}$ and $\mathcal{F}_{\lambda}^{(2)}$.

In contrast to the widely used method of unitarity to evaluate the on-shell amplitudes and FFs in MSYM, we employ the methodology based on Feynman diagrammatic approach. The relevant Feynman diagrams are generated using QGRAF [46]. Because of the presence of Majorana fermions in the theory, the generated diagrams are plagued with the wrong flow of fermionic currents which is rectified by an in-house algorithm based
on PYTHON. There are 440 and 606 number of Feynman diagrams at two-loop for the production of double $J_{tt}^{BPS}$ and $J^K$, respectively. The diagrams are passed through a series of in-house codes based on symbolic manipulating program FORM [47] in order to apply the Feynman rules, perform spinor, Lorentz and colour algebra. To ensure the inclusion of only physical degrees of freedom of gauge bosons, we include the ghosts in the loop. The resulting expressions of the matrix elements contain a large number of scalar Feynman integrals which are reduced to a much smaller set of master integrals (MIs) employing integration-by-parts (IBP) identities [48, 49] with the help of LiteRed [50, 51]. The integrals belong to the category of four-point families with two off-shell legs of same virtualities, which are computed in refs. [52, 53] as Laurent series expansion in dimensional regulator $\epsilon$. Employing the results of the MIs, we obtain the FFs (6) to two-loops.

The on-shell amplitudes in MSYM are ultraviolet (UV) finite in 4-dimensions due to vanishing $\beta$-function. However, the FFs can exhibit UV divergences if the underlying operator is not SUSY protected which, in the present context, gets reflected by the presence of UV poles in the FFs, $F_K$, arising from the unprotected Konishi operator $O^K$. Being a property inherent to the operator, $O^K$ needs to be renormalised through multiplication of an operator renormalisation constant, $Z_K$, which reads

$$[O_K]_R = Z_K O_K,$$

where $[O_K]_R$ represents the corresponding renormalised operator. The $Z_K$ can be determined [9, 20, 54–57] by solving the underlying renormalisation group equation and analysing its Sudakov FFs. The result in terms of its anomalous dimensions, $\gamma_K$, is given by

$$Z_K = \exp \left( \sum_{n=1}^{\infty} \frac{a^n 2\gamma_K n}{n\epsilon} \right),$$

with $\gamma_{K,1} = -6$ and $\gamma_{K,2} = 24$. For the half-BPS operator, all the anomalous dimensions are identically zero, as guaranteed by the SUSY protection. The UV renormalised FFs are obtained through

$$[F_\lambda]_R = Z_\lambda^2 F_\lambda.$$

The UV finite FFs contain soft and collinear (IR) divergences resulting from the low momentum and vanishing angle configurations of the loop momentum. The IR divergences are universal [58–61] for an SU(N) gauge theory which can be expressed as exponentiation of a quantity containing universal light-like cusp and collinear anomalous dimensions. The renormalised FFs in (9) are found to exhibit the expected universal structure of the IR divergences which serves as the most stringent check of our calculation. We find that there is no additional divergence from the contact term of two-operator, unlike the di-Higgs production through gluon fusion in heavy quark effective theory [62, 63].

The BDS/ABDK ansatz [64, 65], which says the maximally helicity violating (MHV) amplitude in planar MSYM is exponentiated in terms of one-loop result along with the universal anomalous dimensions, gets violated for two-loop six-point amplitudes [66, 67]. In order to capture the deviation from the ansatz, a quantity called finite remainder is introduced [66, 67].

For the Sudakov FF of half-BPS operator at two-loop [1], both the IR divergence and finite part are found to be exponentiated, however, the finite part stops exhibiting this nature at three-loop [11]. In order to capture the deviation, following the line of thought for the MHV amplitudes, a finite remainder function (FR) for the FFs at two-loop is introduced in ref. [3] which reads

$$R^{(2)}_{\lambda} = F^{(2)}_{\lambda}(\epsilon) - \frac{1}{2} \left( F^{(1)}_{\lambda}(\epsilon) \right)^2 - f^{(2)}(\epsilon) F^{(1)}_{\lambda}(2\epsilon) - C^{(2)}$$

with $f^{(2)}(\epsilon) = -2\zeta_2 + \epsilon \zeta_3 - \frac{1}{5}\epsilon^2 \zeta_5^2$ and $C^{(2)} = \frac{8}{5}\zeta_3^2$. The quantities $f^{(2)}(\epsilon)$ and $C^{(2)}$ are independent of the number of operators and external states. Representing a two-loop FF in terms a quantity dictated by BDS/ABDK ansatz plus an extra part provides a nice way of representing the deviation from the exponentiation - the ansatz part captures the universal IR divergences that exponentiates whereas the extra part encapsulates the finite part in 4-dimensions. We compute the FR for both the FFs at two-loops and conclude that the finite parts of the two-point FFs with two-operator insertion do not exponentiate, unlike the case of single half-BPS operator insertion at two-loop [1]. The results of the form factors and finite remainders are provided as ancillary files with the arXiv submission.

BREAKDOWN OF UNIFORM TRANSCENDENTALITY

It is a general belief, albeit based on observations, that the FFs and FRs of a SUSY protected operator, such as half-BPS, exhibit the behaviour of UT i.e. they contain only HT weight terms. No deviation from this conjecture has ever been found. In this letter, for the first time, we report that the property of UT does not extrapolate to
the FFs of double insertion of SUSY protected operator. We find that though the FF of half-BPS primary is UT at one-loop, it no longer holds true at two-loop:

\[ \mathcal{F}_{\text{BPS}, \text{nHT}}^{(1)} = 0, \quad \mathcal{F}_{\text{BPS}, \text{nHT}}^{(2)} \neq 0, \quad (10) \]

where nHT represents the next-to-highest-transcendental terms. Remainder functions also obey same property. Therefore, the property of UT for SUSY protected operator can not be generalised to more general class of FFs with more than one-operator insertion. To be more specific, among the nHT terms at two-loop, only the transcendental 3 term is non-zero, the remaining lower ones identically vanish:

\[ \mathcal{F}_{\text{BPS}, \text{nHT}}^{(2)} = \mathcal{F}_{\text{BPS}, \tau(3)}, \quad \mathcal{F}_{\text{BPS}, \tau(<3)} = 0, \quad (11) \]

where the FFs are written as \( \mathcal{F}_{\tau}^{(n)} = \sum_{l=0}^{2n} \mathcal{F}_{\lambda}^{(n), \tau(l)} \). The \( \tau(l) \) represents the terms with transcendentality weight \( l \). Since the result is too big to be presented here, we provide a graphical presentation of the nHT terms of the FR, \( R_{\text{BPS}, \text{nHT}}^{(2)} \), through (A) in figure 2 to demonstrate the dependence on scaling variables \( x \) and \( y \). We plot the real (Re) and imaginary (Im) parts as a function of partonic invariant mass variable \( x \) for different choices of \( \cos(\theta) \), where \( \theta \) is the angle between one of the particles corresponding to half-BPS operator and one of the initial state scalars in their center of mass frame. It shows that as \( \cos(\theta) \) approaches 1 and \( x \) approaches 0 simultaneously, the non-zero nHT term gets closer to zero which implies the possible restoration of UT principle in this kinematic limit. The FR is also seen to be invariant under \( \cos(\theta) \leftrightarrow -\cos(\theta) \), as expected for a purely bosonic scattering. Since this symmetry is not used in the setup of the calculation, this serves as a strong check on the finite part of the results.

On the other hand, the UT is not a property for the two-point FFs with single insertion of unprotected Konishi operator which is verified to three-loops [19, 20]. The FFs with double insertion exhibit the behaviour consistent with this expectation:

\[ \mathcal{F}_{\text{K}, \text{nHT}}^{(1)} \neq 0, \quad \mathcal{F}_{\text{K}, \text{nHT}}^{(2)} \neq 0. \quad (12) \]

In ref. [22], in the context of FFs with one-operator insertion, it is conjectured that the HT weight parts of every two-point minimal FF, including that of Konishi, are identical to that of half-BPS. Through our computation, for the first time, we report the deviation from this conjecture, in particular, this property fails to hold true for the double insertion of operators. Our findings show that

\[ \mathcal{F}_{\text{K}, \text{HT}}^{(1)} \neq \mathcal{F}_{\text{BPS, HT}}^{(1)}, \quad \mathcal{F}_{\text{K}, \text{HT}}^{(2)} \neq \mathcal{F}_{\text{BPS, HT}}^{(2)}. \quad (13) \]

Hence, the conjecture fails to be extrapolated for the case involving two-operator insertion. This is demonstrated through (B) in figure 2 where we denote the difference between the HT terms in finite remainders of double half-BPS and Konishi as \( \delta_{\text{BPS,K}}^{(2), \text{HT}} \). It shows as \( \cos(\theta) \) increases, the difference between the HT terms decreases in regions with higher values of partonic centre of mass energy which indicates a possible restoration of the property in this kinematic limit.

**VIOLATION OF THE PRINCIPLE OF MAXIMAL TRANSCENDENTALITY**

The conjecture of PMT establishes a bridge between MSYM and QCD. It states that the HT terms of certain quantities in MSYM and QCD are identical upon converting the fermions in QCD from fundamental to adjoint representation through \( C_A = C_F = 2n_f T_F = N \), where \( C_A \) and \( C_F \) are the Casimirs in adjoint and fundamental representations, respectively, \( n_f \) is the number of light quark flavours and \( T_F \) is the normalisation factor in fundamental representation. The conjecture is found to hold true to three-loops for two-point FFs with one-operator
insertion while comparing the quark/gluon FFs in QCD and that of half-BPS primary. The question arises if the PMT carries over to more general class of FFs and correlation functions. Through our computations, for the first time, we find that the PMT gets violated for the FFs with two-operator insertion. The HT terms of two-point FFs of double half-BPS operator do not match with that of the di-Higgs produced either through gluon fusion or through bottom quark annihilation:

\[ F_{gg \rightarrow HH,HT}^{(n)} \neq F_{BPS,HT}^{(n)}, \quad F_{bb \rightarrow HH,HT}^{(n)} \neq F_{BPS,HT}^{(n)}. \]

To demonstrate the non-matching, we present its behaviour through (C) in figure 2, capturing the difference between the HT terms of di-Higgs production through gluon fusion and that of half-BPS at one-loop which we denote by \( \delta_{gg}^{(1)} \). Hence, the conjecture of PMT can not be extrapolated to general class of FFs and correlation functions. Unlike the UT property, the violation remains intact throughout the kinematic regions.

### CONCLUSIONS

In this letter, for the first time, we present the form factors with insertion of two identical local gauge invariant operators to two-loops in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory by performing a state-of-the-art computation. In particular, we compute the scalar FFs with double insertion of half-BPS primary and Konishi operators employing the Feynman diagrammatic approach under modified dimensional reduction scheme. Through this calculation, we take the step forward to go beyond the FFs of one-operator insertion and enter into the domain of more general class of FFs. To validate our computations, we check the infrared poles which agree with the predictions. Moreover, the appearance of expected kinematic symmetry inherent to the bosonic FFs provides a strong check on the finite parts of the FFs.

The findings enable us to reach a number of remarkable conclusions. For the first time, the conjecture that the FFs of SUSY protected operators are always UT is found to breakdown at two-loop for the case of two-operator insertion. In particular, though the FF of double half-BPS primary is UT at one-loop, it fails to exhibit this property at two-loop. In accordance with our expectation, we find the FFs of SUSY unprotected operator Konishi to be not UT.

The conjecture that the HT weight terms of every two-point minimal FF are identical to that of half-BPS is found to get violated for two-operator insertion. The HT weight terms of unprotected Konishi are not identical to that of half-BPS both at one- and two-loop.

The conjecture of PMT, which says HT weight terms of quark/gluon FFs in QCD and that of half-BPS primary is found to be violated for two-operator insertion. The HT weight terms of double half-BPS FFs do not match with that of di-Higgs production through gluon fusion or bottom quark annihilation.

By computing the finite remainder function at two-loop, we confirm that the finite parts of the FFs of double half-BPS do not exponentiate, in contrast to the corresponding FFs of single BPS at two-loop. In addition to providing a better understanding of the nature of generalised FFs, our work opens the door for further analytic calculations of general class of FFs in \( \mathcal{N} = 4 \) SYM.

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