TOPOSES FROM FORCING FOR INTUITIONISTIC ZF WITH
ATOMS

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Abstract. We introduce the forcing model of IZFA (Intuitionistic Zermelo-
Fraenkel set theory with Atoms) for every Grothendieck topology and prove
that the topos of sheaves on every site is equivalent to the category of ‘sets in
this forcing model’.

1. Introduction

For a complete Heyting algebra $H$, the Heyting-valued model $V^{(H)}$ of Intuition-
istic Zermelo-Fraenkel set theory (IZF) is obtained by carrying out the definition
of the Boolean-valued model $V^{(B)}$ of ZFC with $H$ in place of a complete Boolean
algebra $B$. Then it can be shown [2, pp. 179–181] that the topos $\text{Sh}(H)$ of sheaves
on $H$ is equivalent to the category $\text{Set}^{(H)}$ of ‘sets in $V^{(H)}$’, which is defined more
precisely as follows:

- we identify elements $u, v$ of $V^{(H)}$ when the truth value $\|u = v\|_{V^{(H)}} \in H$
is equal to 1,
- the objects of $\text{Set}^{(H)}$ are the (identified) elements of $V^{(H)}$,
- the arrows of $\text{Set}^{(H)}$ are those (identified) elements $f$ of $V^{(H)}$ for which
  $\|f\text{ is a function}\|_{V^{(H)}} = 1$.

In this paper, for every Grothendieck topology $J$ on every small category $C$, we
introduce the forcing model of IZFA (Intuitionistic Zermelo-Fraenkel set theory
with Atoms) as an extended version of Heyting-valued models of IZF and prove
that the topos $\text{Sh}(C, J)$ of sheaves on $(C, J)$ is equivalent to the category $\text{Set}^{(C, J)}$
of ‘sets in this forcing model’.

This forcing for IZFA is a modification of forcing for IZF in [7]. The points of
modification are as follows:

1. the universe of the forcing model of IZFA includes the arrows of $C$ as atoms
   while that of [7] is without atoms,
2. the forcing model of IZFA is defined without using toposes directly since
   this formulation is more convenient for the author and for other set theorists
   than that of [7].

The point (1) is necessary to prove that the categories $\text{Sh}(C, J)$ and $\text{Set}^{(C, J)}$
are equivalent, which is the main theorem (Theorem 3.8). The point (2) will enable
category theorists and set theorists to communicate more with each other.

As a related work, it is shown in [1] that every Grothendieck topos has an equiva-
 lent topos which is the universe of some model of IZFA. Our result is stronger

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than it since our forcing model has only set many atoms while the model in [1] has proper class many atoms.

In section 2, we define forcing for IZFA and present some propositions on it. In section 3, we define the category \( \text{Set}^{(C,J)} \) for each site \((C,J)\) and prove the main theorem.

Notation and terminology:
- On Grothendieck topologies or sheaves, we adopt the terminology of Chapter III.
- \( \text{Ob}(C) \) is the class of all objects of a category \( C \).
- \( \text{Arr}(C) \) is the class of all arrows of a category \( C \).
- \( \text{Hom}_C(\text{any}, B) := \bigcup_{A \in \text{Ob}(C)} \text{Hom}_C(A, B) \).
- \( \text{Hom}_C(\text{any}, \text{any}) := \bigcup_{B \in \text{Ob}(C)} \text{Hom}_C(A, B) \).
- \( \mathcal{L}_e \) is the first-order language with two binary predicate symbols \( = \) (equality), \( \in \) (membership).
- \( \mathcal{L}_{\text{atom}} \) is the first-order language obtained by adding two unary predicate symbols \( *: \text{atom}, *: \text{set to } \mathcal{L}_e \).

2. FORCING FOR IZFA

Let \((C,J)\) be a site.

In this section, we introduce the forcing model \( (W^{(C,J)}, \mathcal{P}^{(C,J)}) \), which consists of the class-valued presheaf \( W^{(C,J)} \) and the forcing relation \( \mathcal{P}^{(C,J)} \). The definition of this forcing is a modification of forcing for IZF in [7]. After giving the definition, we present some propositions on it, which are used in the next section. Most proofs of these propositions are omitted in this paper since we can prove them almost by arguments similar to that of forcing for ZFC with posets familiar to set theorists.

2.1. Definition of forcing. We fix two injective class functions \( x \mapsto x^{(\text{atom})} \) and \( x \mapsto x^{(\text{set})} \) on \( V \) whose ranges \( \{ x^{(\text{atom})} \mid x \in V \} \) and \( \{ x^{(\text{set})} \mid x \in V \} \) are disjoint. For example, \( x^{(\text{atom})} := (x,0) \) and \( x^{(\text{set})} := (x,1) \).

Definition 2.1. We define a presheaf \( W^{(C,J)}_{\alpha} : \mathcal{C}^{\text{op}} \to \text{Set} \) for each ordinal \( \alpha \) by transfinite recursion as follows:
- Case: \( \alpha = 0 \) For \( A \in \text{Ob}(C) \),
  \[ W^0_{(C,J)}(A) := \left\{ k^{(\text{atom})} \mid k \in \text{Hom}_C(A, \text{any}) \right\}. \]

For \( f \in \text{Hom}_C(A, B) \), we define a function \( W^0_{(C,J)}(f) : W^0_{(C,J)}(B) \to W^0_{(C,J)}(A) \) by
  \[ W^0_{(C,J)}(f) \left( k^{(\text{atom})} \right) := (k \circ f)^{\text{atom}}. \]

- Case: successor ordinal \( \alpha + 1 \) For \( A \in \text{Ob}(C) \), we define \( W_{\alpha+1}^{(C,J)}(A) \) to be the set of all \( a^{(\text{set})} \) satisfying two conditions (1) and (2):
  1. \( a \subseteq \bigcup_{f \in \text{Hom}_C(\text{any}, A)} W^0_{(C,J)}(\text{dom } f) \times \{ f \}, \)
  2. \( (W^0_{\alpha}^{(C,J)}(g)(b), f \circ g) \in a \) for all \( (b,f) \in a \) and all \( g \in \text{Hom}_C(\text{any}, \text{dom } f) \).

Let \( W_{\alpha+1}^{(C,J)}(A) := W_{\alpha+1}^{(C,J)}(A) \cup \bigcup_{f} W_{\alpha}^{(C,J)}(B) \). For \( f \in \text{Hom}_C(A, B) \), we define a function \( W^0_{\alpha+1}^{(C,J)}(f) : W_{\alpha+1}^{(C,J)}(B) \to W_{\alpha+1}^{(C,J)}(A) \) by:
  \( W_{\alpha+1}^{(C,J)}(f) \left( x^{(\text{set})} \right) := \left\{ (y,g) \mid g \in \text{Hom}_C(\text{any}, A), (y,f \circ g) \in x^{(\text{set})} \right\}, \)
Definition 2.2. We define $W^{(C,J)}(f)(k^{\text{atom}}) := W^C_\alpha(f)(k^{\text{atom}})$. 

[Case: limit ordinal $\gamma$] For $A \in \text{Ob}(\mathcal{C})$, 

$$W^{(C,J)}_\gamma(A) := \bigcup_{\alpha < \gamma} W^C_\alpha(A).$$

For $f \in \text{Hom}_\mathcal{C}(A,B)$, since the functions $\{W^{(C,J)}_\alpha(f) \mid \alpha < \gamma\}$ are pairwise compatible by the definition, we define 

$$W^{(C,J)}_\gamma(f) := \bigcup_{\alpha < \gamma} W^C_\alpha(f).$$

Definition 2.3. Let $A \in \text{Ob}(\mathcal{C})$ and let $\dot{a} \in W^{(C,J)}(A)$. $\dot{a}$ is atom type if $\dot{a} = x^{\text{(atom)}}$ for some $x$. $\dot{a}$ is set type if $\dot{a} = x^{\text{(set)}}$ for some $x$. When $\dot{a} = x^{\text{(atom)}}$ or $\dot{a} = x^{\text{(set)}}$, we will also write $\dot{a}$ for $x$ if there is no confusion.

Definition 2.4. We define the forcing relation $A \Vdash_{(C,J)} \phi(\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{n-1})$ for a formula $\phi(x_0, x_1, \ldots, x_{n-1})$ of $\mathcal{L}_{\text{atom}}$, $A \in \text{Ob}(\mathcal{C})$, and $\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{n-1} \in W^{(C,J)}(A)$ as follows:

- $A \Vdash_{(C,J)} \dot{a}$: atom” if and only if
  1. $\emptyset \in J(A)$ or
  2. $\dot{a}$ is atom type.

- $A \Vdash_{(C,J)} \dot{a}$: set” if and only if
  1. $\emptyset \in J(A)$ or
  2. $\dot{a}$ is set type.

- $A \Vdash_{(C,J)} \dot{a} \in \dot{b}$” if and only if
  1. $\emptyset \in J(A)$ or
  2. (a) $\dot{b}$ is set type and
     (b) $\exists S \in J(A) \forall f \in S \exists \check{x} \in W^{(C,J)}(\text{dom } f)$
        (i) $(\check{x}, f) \in \dot{b}$ and
        (ii) $\text{dom } f \Vdash_{(C,J)} \dot{a} \cdot f = \check{x}$.

- $A \Vdash_{(C,J)} \dot{a} = \dot{b}$” if and only if
  1. $\emptyset \in J(A)$,
  2. (a) $\dot{a}$ and $\dot{b}$ are atom type and
     (b) $\exists S \in J(A) \forall f \in S (\dot{a} \cdot f = \dot{b} \cdot f)$, or
  3. (a) $\dot{a}$ and $\dot{b}$ are set type and
     (b) $\forall f \in \text{Hom}_\mathcal{C}(any, A) \forall \check{x} \in W^{(C,J)}(\text{dom } f)$
        (i) $(\check{x}, f) \in \dot{a} \Rightarrow \text{dom } f \Vdash_{(C,J)} \dot{a} \cdot f = \check{x} \cdot f$ and
        (ii) $(\check{x}, f) \in \dot{b} \Rightarrow \text{dom } f \Vdash_{(C,J)} \dot{a} \cdot f = \check{x} \cdot f$.

- $A \Vdash_{(C,J)} (\phi \land \psi)(\dot{a}_0, \ldots, \dot{a}_{n-1})$” if and only if
  1. $A \Vdash_{(C,J)} \phi(\dot{a}_0, \ldots, \dot{a}_{n-1})$ and
  2. $A \Vdash_{(C,J)} \psi(\dot{a}_0, \ldots, \dot{a}_{n-1})$.

- $A \Vdash_{(C,J)} (\phi \lor \psi)(\dot{a}_0, \ldots, \dot{a}_{n-1})$” if and only if
\[ \exists S \in J(A) \forall f \in S 
\begin{align*}
\quad (a) \text{ dom } f \Vdash (C,J) \text{ " } \phi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ " or} \\
\quad (b) \text{ dom } f \Vdash (C,J) \text{ " } \psi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "}. \\
\end{align*}
\]

- \( A \Vdash (C,J) \text{ " } \phi(\phi \rightarrow \psi)(a_0, \ldots, a_{n-1}) \text{ " if and only if} \)
\[ \forall f \in \text{Hom}(\text{any, } A) 
\begin{align*}
\quad (a) \text{ dom } f \Vdash (C,J) \text{ " } \phi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ " implies} \\
\quad (b) \text{ dom } f \Vdash (C,J) \text{ " } \psi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "}. \\
\end{align*}
\]

- \( A \Vdash (C,J) \text{ " } \neg \phi(a_0, \ldots, a_{n-1}) \text{ " if and only if} \)
\[ \forall f \in \text{Hom}(\text{any, } A) 
\begin{align*}
\quad (a) \text{ dom } f \Vdash (C,J) \text{ " } \phi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ " implies} \\
\quad (b) \emptyset \in J(\text{dom } f). \\
\end{align*}
\]

- \( A \Vdash (C,J) \text{ " } \forall x \phi(x, a_0, \ldots, a_{n-1}) \text{ " if and only if} \)
\[ \forall f \in \text{Hom}(\text{any, } A) \forall x \in \text{W}(C,J)(\text{dom } f) 
\begin{align*}
\quad \text{ dom } f \Vdash (C,J) \text{ " } \phi(x, a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "}. \\
\end{align*}
\]

- \( A \Vdash (C,J) \text{ " } \exists x \phi(x, a_0, \ldots, a_{n-1}) \text{ " if and only if} \)
\[ \exists S \in J(A) \forall f \in S \exists x \in \text{W}(C,J)(\text{dom } f) 
\begin{align*}
\quad \text{ dom } f \Vdash (C,J) \text{ " } \phi(x, a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "}. \\
\end{align*}
\]

2.2. Soundness.

**Proposition 2.5.** Let \( \phi(x_0, \ldots, x_{n-1}) \) be a formula of \( \mathcal{L}_{\text{atom}} \) and let \( A \in \text{Ob}(\mathcal{C}) \). Let \( a_0, a_1, \ldots, a_{n-1} \in \text{W}(C,J)(A) \).

1. If \( A \Vdash (C,J) \text{ " } \phi(a_0, \ldots, a_{n-1}) \text{ " holds, then} \)
\[ \text{ dom } f \Vdash (C,J) \text{ " } \phi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "} \]
holds for every \( f \in \text{Hom}(\text{any, } A) \).

2. If there exists \( S \in J(A) \) for which
\[ \text{ dom } f \Vdash (C,J) \text{ " } \phi(a_0 \cdot f, \ldots, a_{n-1} \cdot f) \text{ "} \]
holds for every \( f \in S \), then \( A \Vdash (C,J) \text{ " } \phi(a_0, \ldots, a_{n-1}) \text{ " holds.} \)

**Proof.** By induction on \( \phi(x_0, \ldots, x_{n-1}) \). \( \square \)

The following complete Heyting algebra \( \Omega^{(C,J)}(A) \) is convenient for describing propositions on the forcing relation.

**Definition 2.6.** Let \( A \in \text{Ob}(\mathcal{C}) \). A sieve \( S \) on \( A \) is \( J \)-closed if for every \( f \in \text{Hom}(\text{any, } A) \), \( f^*(S) \in J(\text{dom } f) \) implies \( f \in S \). We define \( \Omega^{(C,J)}(A) \) to be the set of all \( J \)-closed sieves on \( A \).

**Proposition 2.7.** For every \( A \in \text{Ob}(\mathcal{C}) \), the poset \( (\Omega^{(C,J)}(A), \subseteq) \) is a complete Heyting algebra in which the following properties hold:

1. \( \bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i \),
2. \( \bigvee_{i \in I} S_i = \{ f \in \text{Hom}(\text{any, } A) \mid f^*(\bigcup_{i \in I} S_i) \in J(\text{dom } f) \} \),
3. \( S_0 \to S_1 = \{ f \in \text{Hom}(\text{any, } A) \mid f^*(S_0) \subseteq f^*(S_1) \} \),
4. \( 1 = \text{Hom}(\text{any, } A) \),
5. \( 0 = \{ f \in \text{Hom}(\text{any, } A) \mid \emptyset \in J(\text{dom } f) \} \).

**Proof.** Straightforward. \( \square \)
Definition 2.8. Let $\phi(x_0, \ldots, x_{n-1})$ be a formula of $\mathcal{L}_{atom}$ and let $A \in \text{Ob}(C)$. Let $\dot{a}_0, \ldots, \dot{a}_{n-1} \in W^{(C,J)}(A)$.

\[ \Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} := \{ f \in \text{Hom}_C(\text{any}, A) \mid \text{dom } f \models_{(C,J)} \phi(\dot{a}_0 \cdot f, \ldots, \dot{a}_{n-1} \cdot f) \}. \]

Corollary 2.9. Let $\phi(x_0, \ldots, x_{n-1})$ be a formula of $\mathcal{L}_{atom}$ and let $A \in \text{Ob}(C)$. Let $\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{n-1} \in W^{(C,J)}(A)$. Then $\Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}$ is a $J$-closed sieve on $A$ i.e. $\Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \in \Omega^{(C,J)}(A)$.

Proof. Immediate from Proposition

Proposition 2.10. Let $\phi(x_0, \ldots, x_{n-1})$ and $\psi(x_0, \ldots, x_{n-1})$ be formulas of $\mathcal{L}_{atom}$. Let $A \in \text{Ob}(C)$ and let $\dot{a}_0, \dot{a}_1, \ldots, \dot{a}_{n-1} \in W^{(C,J)}(A)$. Then in the complete Heyting algebra $\Omega^{(C,J)}(A)$,

1. \[ \Vert (\phi \lor \psi)(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} = \Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \lor \Vert \psi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
2. \[ \Vert (\phi \land \psi)(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} = \Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \land \Vert \psi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
3. \[ \Vert (\phi \rightarrow \psi)(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} = \Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \rightarrow \Vert \psi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
4. \[ \Vert \neg \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} = \neg \Vert \phi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}. \]

Proof. Straightforward by the definition of the forcing relation.

Proposition 2.11. Let $\phi(x, y_0, \ldots, y_{n-1})$ and $\psi(y_0, \ldots, y_{n-1})$ be formulas of $\mathcal{L}_{atom}$. Let $A \in \text{Ob}(C)$ and let $\dot{a}_0, \ldots, \dot{a}_{n-1}, \dot{b} \in W^{(C,J)}(A)$.

1. \[ \Vert \forall x \phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \leq \Vert \phi(\dot{b}, \dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
2. \[ \Vert \forall x \psi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)} \leq \exists x \phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
3. \[ \Vert \forall x (\phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \rightarrow \psi(x, \dot{a}_0, \ldots, \dot{a}_{n-1})) \Vert_A^{(C,J)} \]
   \[ \leq \Vert \psi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \rightarrow \forall x \phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}; \]
4. \[ \Vert \forall x (\phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1}) \rightarrow \psi(\dot{a}_0, \ldots, \dot{a}_{n-1})) \Vert_A^{(C,J)} \]
   \[ \leq \Vert (\exists x \phi(x, \dot{a}_0, \ldots, \dot{a}_{n-1})) \rightarrow \psi(\dot{a}_0, \ldots, \dot{a}_{n-1}) \Vert_A^{(C,J)}. \]

Proof. Straightforward by the definition of the forcing relation.

Proposition 2.12. Let $\phi(x, y_0, \ldots, y_{n-1})$ be a formula of $\mathcal{L}_{atom}$. If

\[ \Vert \phi(\dot{a}, \dot{b}_0, \ldots, \dot{b}_{n-1}) \Vert_A^{(C,J)} = 1 \]

holds for every $A \in \text{Ob}(C)$ and every $\dot{a}, \dot{b}_0, \ldots, \dot{b}_{n-1} \in W^{(C,J)}(A)$, then

\[ \Vert \forall x \phi(x, \dot{b}_0, \ldots, \dot{b}_{n-1}) \Vert_A^{(C,J)} = 1 \]

holds for every $A \in \text{Ob}(C)$ and every $\dot{b}_0, \ldots, \dot{b}_{n-1} \in W^{(C,J)}(A)$.

Proof. Straightforward by the definition of the forcing relation.

Proposition 2.13. Let $A \in \text{Ob}(C)$ and let $\dot{a}, \dot{b}, \dot{c} \in W^{(C,J)}(A)$.

1. \[ \Vert \dot{a} = \dot{a} \Vert_A^{(C,J)} = 1, \]
2. \[ \Vert \dot{a} = \dot{b} \Vert_A^{(C,J)} \leq \Vert \dot{b} = \dot{a} \Vert_A^{(C,J)}. \]
(3) \(\|\hat{a} = \hat{b}\|_{A}^{(c,j)} \land \|\hat{b} = \hat{c}\|_{A}^{(c,j)} \leq \|\hat{a} = \hat{c}\|_{A}^{(c,j)}\),
(4) \(\|\hat{a} \in \hat{b}\|_{A}^{(c,j)} \land \|\hat{a} = \hat{c}\|_{A}^{(c,j)} \leq \|\hat{e} \in \hat{b}\|_{A}^{(c,j)}\),
(5) \(\|\hat{a} \in \hat{b}\|_{A}^{(c,j)} \land \|\hat{b} = \hat{c}\|_{A}^{(c,j)} \leq \|\hat{a} \in \hat{c}\|_{A}^{(c,j)}\),
(6) \(\|\hat{a}: \text{atom}\|_{A}^{(c,j)} \land \|\hat{a} = \hat{b}\|_{A}^{(c,j)} \leq \|\hat{b}: \text{atom}\|_{A}^{(c,j)}\),
(7) \(\|\hat{a}: \text{set}\|_{A}^{(c,j)} \land \|\hat{a} = \hat{b}\|_{A}^{(c,j)} \leq \|\hat{b}: \text{set}\|_{A}^{(c,j)}\).

**Proof.** (1): By induction on \(\hat{a}\).

(2), (6), (7): Straightforward by the definition of the forcing relation.

(3), (4), (5): By simultaneous induction on \(\hat{a}, \hat{b}, \hat{c}\). 

**Theorem 2.14** (Soundness). Let \(\phi(x_{0}, \ldots, x_{n-1})\) be a formula of \(L_{\text{atom}}\). If \(\phi\) is provable in intuitionistic first-order logic with equality, then

\[ A \models_{(c,j)} \text{"} \phi(\hat{a}_{0}, \ldots, \hat{a}_{n-1}) \text{"} \]

holds for every \(A \in \text{Ob}(C)\) and every \(\hat{a}_{0}, \ldots, \hat{a}_{n-1} \in W^{(c,j)}(A)\).

**Proof.** It is sufficient to show that \(\|\phi(\hat{a}_{0}, \ldots, \hat{a}_{n-1})\|_{A}^{(c,j)} = 1\) for all \(A \in \text{Ob}(C)\) and all \(\hat{a}_{0}, \ldots, \hat{a}_{n-1} \in W^{(c,j)}(A)\), but it is straightforward by Propositions 2.10, 2.11, 2.12 and 2.13. 

2.3. Check operator.

**Definition 2.15** (Check operator). For a set \(x\) and an object \(A \in \text{Ob}(C)\), we recursively define \(\hat{x}^{A}\) (or \((x)^{A}\) \(\in W^{(c,j)}(A)\)) by

\[ \hat{x}^{A} = \{ (y^{\text{dom}} f, f) \mid y \in x, f \in \text{Hom}_{C}(\text{any}, A) \}. \]

**Proposition 2.16.** Let \(x\) be a set and let \(A \in \text{Ob}(C)\). Then \(\hat{x}^{A} \cdot f = \hat{y}^{\text{dom}} f\) for all \(f \in \text{Hom}_{C}(\text{any}, A)\).

**Proof.** Straightforward. 

**Theorem 2.17.** Let \(\phi(x_{0}, \ldots, x_{n-1})\) be a \(\Delta_{0}\)-formula of \(L_{\bar{e}}\) and let \(A \in \text{Ob}(C)\). Let \(a_{0}, \ldots, a_{n-1}\) be sets. Then

\[ \|\phi(\hat{a}_{0}^{A}, \ldots, \hat{a}_{n-1}^{A})\|_{A}^{(c,j)} = \begin{cases} 1 & \text{if } \phi(a_{0}, \ldots, a_{n-1}) \text{ holds in } V, \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** By induction on \(\phi(x_{0}, \ldots, x_{n-1})\). 

2.4. Maximum principle.

**Definition 2.18.** Let \(A \in \text{Ob}(C)\) and let \(S\) be a sieve on \(A\). A function \(F\) on \(S\) is called a matching function for \(S\) if the following conditions hold:

1. \(F(f)\) is a nonempty subset of \(W^{(c,j)}(\text{dom } f)\) for every \(f \in S\),
2. For every \(f \in S\) and every \(g \in \text{Hom}_{C}(\text{any}, \text{dom } f)\), if \(\hat{a} \in F(f)\) and \(\hat{b} \in F(f \circ g)\), then \(\text{dom } g \models_{(c,j)} \text{"} \hat{a} \cdot g = \hat{b} \text{"}\).

**Definition 2.19.** Let \(A \in \text{Ob}(C)\) and let \(S\) be a sieve on \(A\). Let \(F\) be a matching function for \(S\). We assume that all elements of \(F(f)\) are set type for every \(f \in S\). Then we define the amalgamation of \(F\) by

\[ \text{ama } F := \{ (\hat{x}, f \circ g) \mid f \in S, \exists \hat{a} \in F(f) \left( ((\hat{x}, g) \in \hat{a}) \right) \} \in W^{(c,j)}(A). \]
**Definition 2.22.** Intuitionistic Zermelo-Fraenkel set theory with atoms (or IZFA) is the theory in \( \mathcal{L}_{\text{atom}} \) based on the following axioms:

1. Set existence
   \[ \exists x \ (x: \text{set}). \]
2. Extensionality
   \[ \forall x: \text{set} \ \forall y: \text{set} \ (\forall z \ (z \in x \leftrightarrow z \in y) \rightarrow x = y). \]
3. Separation
   \[ \forall u: \text{set} \ \exists v: \text{set} \ (\forall x \ (x \in v \leftrightarrow x \in u \land \phi(x)), \]
   where \( v \) is not free in the formula \( \phi(x) \) of \( \mathcal{L}_{\text{atom}} \).
4. Collection
   \[ \forall u: \text{set} \ (\forall x \in u \ \exists y \ \phi(x, y) \rightarrow \exists v: \text{set} \ \forall x \in u \ \exists y \in v \ \phi(x, y)), \]
   where \( v \) is not free in the formula \( \phi(x, y) \) of \( \mathcal{L}_{\text{atom}} \).
5. Pairing
   \[ \forall x \ \forall y \ \exists z \ \forall w \ (w \in z \leftrightarrow w = x \lor w = y). \]
6. Union
   \[ \forall u: \text{set} \ \exists v: \text{set} \ (\forall x \ (x \in v \leftrightarrow \exists y \in u \ (x \in y)). \]
7. Power set
   \[ \forall u: \text{set} \ \exists v: \text{set} \ (\forall x \ (x \in v \leftrightarrow \forall y \in x \ (y \in u)). \]
8. Infinity
   \[ \exists u \ (\emptyset \in u \land \forall x \in u \ (x \cup \{x\} \in u)). \]
9. \( \in \)-induction
   \[ \forall x \ (\forall y \in x \ \phi(y) \rightarrow \phi(x)) \rightarrow \forall x \ \phi(x), \]
   where \( y \) is not free in the formula \( \phi(x) \) of \( \mathcal{L}_{\text{atom}} \).
10. Atom
    \[ \forall x: \text{atom} \ \forall y \ (y \notin x), \]
    \[ \forall x \ (x: \text{atom} \lor x: \text{set}), \]
    \[ \forall x \lnot (x: \text{atom} \land x: \text{set}). \]

**Definition 2.23.** For \( A \in \text{Ob}(\mathcal{C}) \) and \( \dot{a}, \dot{b} \in W^{(C,J)}(A) \), we define the unordered pair and the ordered pair of \( \dot{a}, \dot{b} \) in \( W^{(C,J)}(A) \) as follows:

- \( \{ \dot{a}, \dot{b} \}_A := \{ (\dot{a} \cdot f, f) \mid f \in \text{Hom}_C(\text{any}, A) \} \cup \{ (\dot{b} \cdot f, f) \mid f \in \text{Hom}_C(\text{any}, A) \} \),
\( (\hat{a}, \hat{b})_A := \{\{\hat{a}, \hat{a}\}_A, \{\hat{a}, \hat{b}\}_A\}_A \).

**Theorem 2.24.** For every axioms \( \phi \) of IZFA and every \( A \in \text{Ob}(C) \),
\[
A \models_{(C,J)} "\phi"
\]
holds.

**Proof.** Easy. For example, \( \hat{z} := \{\hat{x}, \hat{y}\}_A \) is a witness for \([5]\) Pairing. \( \square \)

3. Toposes from forcing

Let \((C, J)\) be a site. In this section, we define the category \( \text{Set}^{(C, J)} \) of ‘sets in the forcing model \((W^{(C, J)}, \models_{(C, J)})\)' and prove the main theorem that the categories \( \text{Sh}(C, J) \) and \( \text{Set}^{(C, J)} \) are equivalent by constructing a fully faithful and essentially surjective functor \( L: \text{Set}^{(C, J)} \to \text{Sh}(C, J) \). Henceforth, for each \( A \in \text{Ob}(C) \), we identify elements \( \hat{a}, \hat{b} \) of \( W^{(C, J)}(A) \) when \( A \models_{(C,J)} "\hat{a} = \hat{b}" \).

3.1. **Category** \( \text{Set}^{(C, J)} \) of ‘sets in \((W^{(C, J)}, \models_{(C, J)})\)’.

**Definition 3.1.**
- A \((C, J)\)-sequence is a sequence \((\hat{a}_A)_{A \in \text{Ob}(C)}\) of which each \( \hat{a}_A \) is an element of \( W^{(C, J)}(A) \).
- A \((C, J)\)-sequence \((\hat{a}_A)_{A \in \text{Ob}(C)}\) is called stable if
  \[
  \text{dom } f \models_{(C, J)} "\hat{a}_{\text{cod } f} \cdot f = \hat{a}_{\text{dom } f}"
  \]
  holds for every \( f \in \text{Arr}(C) \).
- A \((C, J)\)-set is a stable \((C, J)\)-sequence \((\hat{a}_A)_{A \in \text{Ob}(C)}\) for which
  \[
  A \models_{(C,J)} "\hat{a}_A: \text{set}" \]
  holds for every \( A \in \text{Ob}(C) \).

**Definition 3.2.** We define a category \( \text{Set}^{(C, J)} \) as follows:
- the objects of \( \text{Set}^{(C, J)} \) are the \((C, J)\)-sets,
- the arrows of \( \text{Set}^{(C, J)} \) from \((\hat{a}_A)_{A \in \text{Ob}(C)}\) to \((\hat{b}_A)_{A \in \text{Ob}(C)}\) are those \((C, J)\)-sets \((\hat{f}_A)_{A \in \text{Ob}(C)}\) for which \( A \models_{(C,J)} "\hat{f}_A \text{ is a function from } \hat{a}_A \text{ to } \hat{b}_A" \) for every \( A \in \text{Ob}(C) \),
- the composition of two arrows \((\hat{f}_A)_{A \in \text{Ob}(C)}\) and \((\hat{g}_A)_{A \in \text{Ob}(C)}\) of \( \text{Set}^{(C, J)} \) is the unique arrow \((\hat{h}_A)_{A \in \text{Ob}(C)}\) for which \( A \models_{(C,J)} "\hat{f}_A \circ \hat{g}_A = \hat{h}_A" \) for every \( A \in \text{Ob}(C) \). (Such \( \hat{h}_A \) exists by Theorem 2.21)

3.2. **Functor** \( L: \text{Set}^{(C, J)} \to \text{Sh}(C, J) \).

**Definition 3.3.** Let \( a = (\hat{a}_A)_{A \in \text{Ob}(C)} \) be a \((C, J)\)-set. We define a presheaf \( L^\text{pre}_a \) on \( C \) as follows:
- \( L^\text{pre}_a(A) := \{ \hat{c} \in W^{(C, J)}(A) \mid A \models_{(C,J)} "\hat{c} \in \hat{a}_A" \} \) for \( A \in \text{Ob}(C) \),
- for \( f \in \text{Hom}_C(A, B) \), a function \( L^\text{pre}_a(f): L^\text{pre}_a(B) \to L^\text{pre}_a(A) \) is defined by \( L^\text{pre}_a(f)(\hat{c}) = \hat{c} \cdot f \).

Let \( L_a \) be the sheafification of \( L^\text{pre}_a \) and let \( i_a = (i_{a,A})_{A \in \text{Ob}(C)}: L^\text{pre}_a \to L_a \) be its canonical map. Since \( L^\text{pre}_a \) is a separated presheaf, \( i_a \) is a monomorphism.
Definition 3.4. Let \( f = (\hat{f}_A)_{A \in \text{Ob}(\mathcal{C})} \in \text{Hom}_{\text{Set}^{(C,J)}}(a, b) \). We define a natural transformation \( L^\pre_{f} = \left( L^\pre_{f,A} \right)_{A \in \text{Ob}(\mathcal{C})} : L^\pre_a \rightarrow L^\pre_b \) by:

\[
L^\pre_{f,A}(\hat{c}) = \hat{d} \quad \text{if and only if} \quad A \models_{(C,J)} " f_A(\hat{c}) = \hat{d} ".
\]

Let \( L_f \) be the unique natural transformation \( \sigma : L^\pre_a \rightarrow L^\pre_b \) for which the following diagram commutes:

\[
\begin{array}{ccc}
L^\pre_a & \xrightarrow{L^\pre_f} & L^\pre_b \\
\downarrow \sigma & & \downarrow \\
L^\pre_b & \xrightarrow{\sigma} & L^\pre_b
\end{array}
\]

Definition 3.5. We define a functor \( L : \text{Set}^{(C,J)} \rightarrow \text{Sh}(C, J) \) as follows:

- \( L(a) := L_a \) for \( a \in \text{Ob}(\text{Set}^{(C,J)}) \),
- \( L(f) := L_f \) for \( f \in \text{Hom}_{\text{Set}^{(C,J)}}(a, b) \).

3.3. Representation of sheaves on \((C, J)\) by \((C, J)\)-sets. Before proving the main theorem, we construct the following \((C, J)\)-set \( (\bar{F}^A)_{A \in \text{Ob}(\mathcal{C})} \) for each sheaf \( F \) on \((C, J)\), which is used for showing that the functor \( L \) is essentially surjective.

Definition 3.6. Let \( F \) be a sheaf on \((C, J)\) and \( A \in \text{Ob}(\mathcal{C}) \). For each \( a \in F(A) \), we define

\[
\pi^{F,A} := \left\{ \left( x^{\text{dom}}_g, f^{(\text{atom})} \right)_{\text{dom} \ g} , g \right\} \quad f \in \text{Arr}(\mathcal{C}), \ x \in F(\text{cod} f),
\]

\[
g \in \text{Hom}_\mathcal{C}(\text{dom} f, A), \ F(f)(x) = F(g)(a) \}
\]

and let \( \bar{F}^A := \{ (\pi^{F,a}_f, f) \mid f \in \text{Hom}_\mathcal{C}(\text{any}, A), \ a \in F(\text{dom} f) \} \in W^{(C,J)}(A) \).

Then these elements \( \pi^{F,A}, \bar{F}^A \) of \( W^{(C,J)}(A) \) represent the behavior of a sheaf \( F \) well as follows:

Proposition 3.7. Let \( F \) be a sheaf on \((C, J)\) and let \( A \in \text{Ob}(\mathcal{C}) \).

1. \( \left( \bar{F}^A \right)_{A \in \text{Ob}(\mathcal{C})} \) is a \((C, J)\)-set.
2. \( A \models_{(C,J)} " a^{F,A} = b^{F,A} \quad \text{for all} \quad a \in F(A). \)
3. \( \pi^{F,A} : h = F(h)(a) \quad \text{for all} \quad a \in F(A) \quad \text{and all} \quad h \in \text{Hom}_\mathcal{C}(\text{any}, A). \)
4. For \( a, b \in F(A) \), if \( A \models_{(C,J)} " a^{F,A} = b^{F,A} \), then \( a = b. \)
5. For \( \hat{x} \in W^{(C,J)}(A) \), if \( A \models_{(C,J)} " \hat{x}^{F,A} = \hat{x}^{F,A} \), then there exists \( a \in F(A) \) for which \( A \models_{(C,J)} " \hat{x} = \pi^{F,A} \).

Proof. (1), (2), (3): Immediate.

(4): Straightforward since \( A \models_{(C,J)} " \left( a^{F,A}, 1^{(\text{atom})}_a \right) \in \pi^{F,A}. \)

(5): Straightforward by (3) and (4) since \( F \) is a sheaf on \((C, J). \)

□
3.4. Main theorem. Now we will prove the main theorem:

Theorem 3.8. The functor $L: \text{Set}^{(C, J)} \to \text{Sh}(C, J)$ is fully faithful and essentially surjective. Thus, $\text{Set}^{(C, J)}$ and $\text{Sh}(C, J)$ are equivalent.

Proof. [Fullness of $L$]: Let $a$ and $b$ be $(C, J)$-sets and let $\sigma = (\sigma_A)_{A \in \text{Ob}(C)}$ be a natural transformation from $L_a$ to $L_b$. For each $A \in \text{Ob}(C)$, we define

$$j_A := \left\{ (\hat{c}, \hat{d})_A : g, \hat{d} \in L^\text{pre}_b(A), \hat{d} \in L^\text{pre}_b(A), \right\} \in W^{(C, J)}(A).$$

and let $f := (j_A)_{A \in \text{Ob}(C)}$. Then we can prove easily that $f$ is an arrow of $\text{Set}^{(C, J)}$ from $a$ to $b$. By the definitions of $L_f$, $L^\text{pre}_f$ and $f$, it holds that

$$L_f \circ i_a = i_b \circ L^\text{pre}_f = \sigma \circ i_a.$$

Hence, $L_f = \sigma$ by the universal property for the canonical map $i_a: L^\text{pre}_a \to L_a$.

[Faithfulness of $L$]: Let $f$ and $g$ be arrows of $\text{Set}^{(C, J)}$ from $a$ to $b$ for which $L_f = L_g$ holds. Then

$$i_b \circ L^\text{pre}_f = L_f \circ i_a = L_g \circ i_a = i_b \circ L^\text{pre}_g.$$

Since $i_b$ is a monomorphism, $L^\text{pre}_f = L^\text{pre}_g$. Hence, by the definition of $L^\text{pre}_f$ and $L^\text{pre}_g$, $f = g$ holds.

[Essential surjectivity of $L$]: Fix a sheaf $F$ on $(C, J)$. Let $\overline{F} := \left(\mathcal{F}^A\right)_{A \in \text{Ob}(C)}$. By Proposition 3.7 [11] and [2], we can define a function $\sigma_A: F(A) \to L^\text{pre}_F(A)$ for each $A \in \text{Ob}(C)$ by

$$\sigma_A(a) = \mathcal{F}^A(a).$$

Then $\sigma := (\sigma_A)_{A \in \text{Ob}(C)}$ has the following properties:

- $\sigma$ is a natural transformation from $F$ to $L^\text{pre}_F$ by Proposition 3.7 [3],
- each $\sigma_A: F(A) \to L^\text{pre}_F(A)$ is injective by Proposition 3.7 [4],
- it is also surjective by Proposition 3.7 [5].

Hence, $\sigma$ is a natural isomorphism from $F$ to $L^\text{pre}_F$. Since $F$ is a sheaf, $L^\text{pre}_F$ is also a sheaf, which is isomorphic to its sheafification $L_F$. Therefore, $F$ is isomorphic to $L_F$. \qed

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