Lagrangian duality for nonconvex optimization problems with abstract convex functions

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Abstract We investigate Lagrangian duality for nonconvex optimization problems. To this aim we use the $\Phi$-convexity theory and minimax theorem for $\Phi$-convex functions. We provide conditions for zero duality gap and strong duality. Among the classes of functions, to which our duality results can be applied, are prox-bounded functions, DC functions, weakly convex functions and paraconvex functions.

Keywords Abstract convexity · Minimax theorem · Lagrangian duality · Nonconvex optimization · Zero duality gap · Weak duality · Strong duality · Prox-regular functions · Paraconvex and weakly convex functions

1 Introduction

Lagrangian and conjugate dualities have far reaching consequences for solution methods and theory in convex optimization in finite and infinite dimensional spaces. For recent state-of the-art of the topic of convex conjugate duality we refer the reader to the monograph by Radu Bot [5].

There exist numerous attempts to construct pairs of dual problems in nonconvex optimization e.g., for DC functions [19], [34], for composite functions [8], DC and composite functions [30], [31] and for prox-bounded functions [15].

In the present paper we investigate Lagrange duality for general optimization problems within the framework of abstract convexity, namely, within the theory of $\Phi$-convexity. The class $\Phi$-convex functions encompasses convex l.s.c.

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functions, and, among others, the classes of functions mentioned above. A
comprehensive study of \( \Phi \)-convex functions can be found in the monographs
by Pallaschke and Rolewicz [20] and by Rubinow [29].

The set \( \Phi \) is a set of functions defined on a given space \( X \), called
elementary functions. \( \Phi \)-convex functions are pointwise suprema of elementary
minorizing functions \( \varphi \in \Phi \) (e.g. quadratic, quasi-convex). This corresponds
to the classical fact that proper lower semicontinuous convex functions are
pointwise suprema of affine minorizing functions. \( \Phi \)-convexity provides a uni-
fying framework for dealing with important classes of nonconvex functions, e.g.,
paraconvex (weakly convex), DC, and prox-bounded functions. In the context
of duality theory \( \Phi \)-convexity is investigated in a large number of papers, e.g.
[1], [3], [12], [16], [25].

The underlying concepts of \( \Phi \)-convexity are \( \Phi \)-conjugation and \( \Phi \)-subdifferentiation, which mimic the corresponding constructions of convex analysis, i.e.,
the \( \Phi \)-conjugate function and the \( \Phi \)-subdifferential are defined by replacing, in
the respective classical definitions, the linear (affine) functions with elemen-
tary functions \( \varphi \in \Phi \) which may not be affine, in general. This motivates the
name \textit{Convexity without linearity}, coined for \( \Phi \)-convexity by Rolewicz [25].

The subject of the present investigation is the Lagrangian duality for opti-
mization problems involving \( \Phi \)-convex functions. In particular, we investigate
Lagrangian duality for problems involving \( \Phi_{lsc} \)-convex functions (see (1)), i.e.,
proper lower semicontinuous functions defined on a Hilbert space and mi-
norized by quadratic functions.

The class of \( \Phi_{lsc} \)-convex functions embodies many important classes of
functions appearing in optimization, e.g. prox-bounded functions [23], DC
(difference of convex) functions [35], weakly convex functions [36], paracon-
vex functions [27] and lower semicontinuous convex (in the classical sense)
functions.

Within the framework of \( \Phi \)-convexity, the Lagrange duality has been al-
ready investigated on different levels of generality by [9], [14], [21].

Main contributions of the paper are as follows.

(i) Theorem 3 provides necessary and sufficient condition for zero duality gap
for the pair of Lagrange dual problems \( L_P \) and \( L_D \) with the Lagrangian \( \mathcal{L} \)
satisfying some \( \Phi \)-convexity assumptions, where \( \Phi \) is any class of elemen-
tary functions. This condition, called \textit{the intersection property} is defined in
Definition 4. See also [32].

(ii) Theorem 6 and Theorem 7 provide conditions for zero duality gap for
the pair of Lagrange dual problems \( L_P \) and \( L_D \) with the Lagrangian \( \mathcal{L} \)
satisfying some \( \Phi_{lsc} \)-convexity assumptions, where the class \( \Phi_{lsc} \) is defined
in Example 1.2.

(iii) Theorem 9 provides sufficient conditions for the strong duality for prob-
lems, where the optimal value function \( V \) (see the formula (14)) is para-
convex (weakly convex), and int \( \text{dom} V \neq \emptyset \).
(iv) Theorem [7] shows that for constrained optimization problems with finite optimal values, \( \Psi_{\text{lsc}} \)-convex perturbation functions and \( \Phi_{\text{lsc}} \)-convex Lagrangians, our main condition i.e. intersection property is satisfied.

The organization of the paper is as follows. In Section 2 we recall basic concepts of \( \Phi \)-convexity. We close Section 2 with a minimax theorem from [32] which is the starting point for our investigations. In Section 3 we introduce the Lagrange function and we discuss the Lagrangian duality for optimization problems involving \( \Phi \)-convex functions (Theorem 3), we also deliver conditions for the strong duality (Theorem 4).

In Section 4 and Section 5 we discuss our duality scheme in the class of \( \Phi_{\text{conv}} \)-convex functions nad \( \Phi_{\text{lsc}} \)-convex functions, respectively. In Section 6 we discuss our duality scheme for for a particular form of optimization problem.

2 \( \Phi \)-convexity

Let \( X \) be a set. A function \( f : X \to \hat{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \) is proper if its domain \( \text{dom} f = \{x \in X \mid f(x) < +\infty\} \neq \emptyset \) and \( f(x) > -\infty \) for any \( x \in X \).

Let \( \Phi \) be a set of real-valued functions \( \varphi : X \to \mathbb{R} \) and \( f : X \to \hat{\mathbb{R}} \). The set

\[
\text{supp}(f) := \{ \varphi \in \Phi : \varphi \leq f \}
\]

is called the support of \( f \) with respect to \( \Phi \), where, for any \( g, h : X \to \hat{\mathbb{R}} \),

\[
g \leq h \iff g(x) \leq h(x) \quad \forall \ x \in X.
\]

We will use the notation \( \text{supp}(f) \) whenever the class \( \Phi \) is clear from the context. Elements of class \( \Phi \) are called elementary functions.

Definition 1 ([12], [20], [29]) A function \( f : X \to \hat{\mathbb{R}} \) is called \( \Phi \)-convex on \( X \) if

\[
f(x) = \sup\{\varphi(x) : \varphi \in \text{supp}(f)\} \quad \forall \ x \in X.
\]

If the set \( X \) is clear from the context, we simply say that \( f \) is \( \Phi \)-convex. A function \( f : X \to \hat{\mathbb{R}} \) is called \( \Phi \)-convex at \( x_0 \in X \) if

\[
f(x_0) = \sup\{\varphi(x_0) : \varphi \in \text{supp}(f)\}.
\]

We have \( \text{supp}(f) = \emptyset \) iff \( f \equiv -\infty \). By convention, we consider that the function \( f \equiv -\infty \) is \( \Phi \)-convex (c.f. [29]). A \( \Phi \)-convex function \( f : X \to \hat{\mathbb{R}} \) is proper if \( \text{supp}(f) \neq \emptyset \) and the domain of \( f \) is nonempty, i.e.

\[
\text{dom}(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset.
\]

If \( X \) is a topological space and a given class \( \Phi \) consists of functions \( \varphi : X \to \mathbb{R} \) which are lower semicontinuous on \( X \), then \( \Phi \)-convex functions are lower semicontinuous on \( X \) ([37]). Note that \( \Phi \)-convex functions defined above may admit the value \( +\infty \) which allows us to consider indicator functions within the framework of \( \Phi \)-convexity.
Analogously, we say that \( f : X \to \mathbb{R} \) is \( \Phi \)-concave on \( X \) if
\[
f(x) = \inf \{ \varphi(x) : \varphi \in \Phi, \ f \leq \varphi \}.
\]
Clearly, \( f \) is \( \Phi \)-concave on \( X \) iff \( -f \) is \( -\Phi \)-convex on \( X \).

The following classes of elementary functions are considered.

**Example 1**

1. \( \Phi_{\text{conv}} := \{ \varphi : X \to \mathbb{R}, \ \varphi(x) = \langle \ell, x \rangle + c, \ x \in X, \ \ell \in X^*, \ c \in \mathbb{R} \} \),
   where \( X \) is a topological vector space, \( X^* \) is the dual space to \( X \). It is a well known fact (see for example Proposition 3.1 of [13]) that a proper convex lower semicontinuous function \( f : X \to \mathbb{R} \) is \( \Phi_{\text{conv}} \)-convex. For the analysis of the class \( \Phi \) which generates all convex functions [32].

2. \( \Phi_{\text{lsc}} := \{ \varphi : X \to \mathbb{R}, \ \varphi(x) = -a \|x\|^2 + \langle \ell, x \rangle + c, \ x \in X, \ \ell \in X^*, \ a \geq 0, \ c \in \mathbb{R} \} \), \( (1) \)
   where \( X \) is a normed space. If \( X \) is a Hilbert space, then \( f : X \to \mathbb{R} \) is \( \Phi_{\text{lsc}} \)-convex iff \( f \) is lower semicontinuous and minorized by a quadratic function \( q(x) := -a \|x\|^2 - c \) on \( X \) (e.g. [29], Example 6.2). The class of \( \Phi_{\text{lsc}} \)-convex functions encompasses: prox-bounded functions [22] and weakly convex functions [36], known also under the name paraconvex functions [27] and semiconvex functions [11].

3. \( \Phi_{\text{lsc}}^+ := \{ \varphi : X \to \mathbb{R}, \ \varphi(x) = a \|x\|^2 + \langle \ell, x \rangle + c, \ x \in X, \ \ell \in X^*, \ a \geq 0, \ c \in \mathbb{R} \} \),
   where \( X \) is a normed space. If \( X \) is a Hilbert space, then \( f : X \to \mathbb{R} \) is \( \Phi_{\text{lsc}}^+ \)-concave iff \( f \) is upper semicontinuous and majorized by a quadratic function \( q(x) := a \|x\|^2 - c \) on \( X \) ([29], Example 6.2).

**2.1 \( \Phi \)-Subgradients**

**Definition 2** An element \( \varphi \in \Phi \) is called a \( \Phi \)-subgradient of a function \( f : X \to \mathbb{R} \) at \( \bar{x} \in \text{dom} f \), if the following inequality holds
\[
f(x) - f(\bar{x}) \geq \varphi(x) - \varphi(\bar{x}) \quad \forall \ x \in X.
\]
(2)
The set of all \( \Phi \)-subgradients of \( f \) at \( \bar{x} \) is denoted as \( \partial \Phi f(\bar{x}) \).

Let \( \varepsilon > 0 \). An element \( \varphi \in \Phi \) is called a \( \Phi \)-\( \varepsilon \)-subgradient of a function \( f : X \to \mathbb{R} \) at \( \bar{x} \in \text{dom} f \), if the following inequality holds
\[
f(x) - f(\bar{x}) \geq \varphi(x) - \varphi(\bar{x}) - \varepsilon \quad \forall \ x \in X.
\]
(3)
The set of all \( \Phi \)-\( \varepsilon \)-subgradients of \( f \) at \( \bar{x} \) is denoted as \( \partial \Phi \varepsilon f(\bar{x}) \).
Definition 3 An element \((a,v) \in \mathbb{R}_+ \times X\) is called a \(\Phi_{lsc}\)-subgradient of a function \(f : X \to \bar{\mathbb{R}}\) at \(\bar{x} \in \text{dom } f\), if the following inequality holds
\[
f(x) - f(\bar{x}) \geq \langle v, x - \bar{x} \rangle - a\|x\|^2 + a\|\bar{x}\|^2, \quad \forall \ x \in X.
\] (4)
The set of all \(\Phi_{lsc}\)-subgradients of \(f\) at \(\bar{x}\) is denoted as \(\partial_{lsc} f(\bar{x})\).

Let \(\varepsilon > 0\). An element \((a,v) \in \mathbb{R}_+ \times X\) is called a \(\Phi_{lsc-\varepsilon}\)-subgradient of a function \(f : X \to \bar{\mathbb{R}}\) at \(\bar{x} \in \text{dom } f\), if the following inequality holds
\[
f(x) - f(\bar{x}) \geq \langle v, x - \bar{x} \rangle - a\|x\|^2 + a\|\bar{x}\|^2 - \varepsilon, \quad \forall \ x \in X.
\] (5)
The set of all \(\Phi_{lsc-\varepsilon}\)-subgradients of \(f\) at \(\bar{x}\) is denoted as \(\partial_{\varepsilon lsc} f(\bar{x})\).

Remark 1 It is easy to show that \(\partial_{lsc} f(\bar{x})\) and \(\partial_{\varepsilon lsc} f(\bar{x})\), \(\varepsilon > 0\), are convex sets for all \(\bar{x} \in \text{dom } f\).

Moreover, \(\partial_{\varepsilon lsc} f(\bar{x}) \neq \emptyset\) for any \(\bar{x} \in \text{dom } f\). Indeed, if \(\bar{x} \in \text{dom } f\), then, by the \(\Phi\)-convexity of \(f\), we have \(f(\bar{x}) = \sup\{\varphi(\bar{x}), \varphi \in \text{supp } f\}\). By the definition of supremum, for any \(\varepsilon > 0\), there exists \(\bar{\varphi} \in \text{supp } f\) such that
\[
\bar{\varphi}(\bar{x}) > f(\bar{x}) - \varepsilon
\]
and consequently, \(f(x) - f(\bar{x}) \geq \bar{\varphi}(x) - \bar{\varphi}(\bar{x}) - \varepsilon\), i.e. \(\bar{\varphi} \in \partial_{\varepsilon lsc} f(\bar{x})\).

2.2 \(\Phi\)-conjugation

Let \(f : X \to \bar{\mathbb{R}}\). The function \(f^*_{\Phi} : \Phi \to \bar{\mathbb{R}}\),
\[
f^*_{\Phi}(\varphi) := \sup_{x \in X} (\varphi(x) - f(x))
\] (6)
is called the \(\Phi\)-conjugate of \(f\). The function \(f^*_{\Phi}\) is \(\Phi\)-convex (c.f. Proposition 1.2.3 of [20]). Accordingly, the second \(\Phi\)-conjugate of \(f\) is defined as
\[
f^{**}_{\Phi}(x) := \sup_{\varphi \in \Phi} (\varphi(x) - f^*_{\Phi}(\varphi)).
\]

Theorem 1 ([20] and Theorem 1.2.6, [29], Theorem 7.1) Function \(f : X \to \bar{\mathbb{R}}\) is \(\Phi\)-convex if and only if
\[
f(x) = f^{**}_{\Phi}(x) \quad \forall \ x \in X.
\]

For any function \(f : X \to \bar{\mathbb{R}}\) the Young inequality holds, i.e.
\[
f(x) + f^*_{\Phi}(x) \geq \varphi(x), \quad \forall \ \varphi \in \Phi, \ x \in X.
\]

Moreover we have the following proposition.

Proposition 1 ([29], Proposition 7.7, [20], Proposition 1.2.4) Let \(\varphi \in \Phi\) and \(x \in X\). The following conditions are equivalent
\[(i) \ f(x) + f^*_{\Phi}(x) = \varphi(x) \quad (ii) \ \varphi \in \partial_{\Phi} f(x)\]
2.3 Minimax Theorem for $\Phi$-convex functions

Minimax theorems for function $a : X \times Y \to \mathbb{R}$ such that for each $y \in Y$ the function $a(\cdot, y) : X \to \mathbb{R}$ is $\Phi$-convex are based on the following property introduced in [33] and investigated in [32], [33].

**Definition 4** Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be any two functions and $\alpha \in \mathbb{R}$. We say that $\varphi_1$ and $\varphi_2$ have the intersection property on $X$ at the level $\alpha \in \mathbb{R}$ if for every $t \in [0, 1]$ defines $[\varphi_1 + (1 - t)\varphi_2 < \alpha] \cap [\varphi_1 < \alpha] = \emptyset$ or $[\varphi_1 + (1 - t)\varphi_2 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$, \quad (7)

where $[\varphi < \alpha] := \{x \in X : \varphi(x) < \alpha\}$.

Let us note that, by Proposition 4 of [4], in the class $\Phi_{lsc}$ the condition (7) takes the form

$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$. \quad (8)

The following minimax theorem was proved in [32].

**Theorem 2** ([32], Theorem 5.1) Let $X$ be a real vector space and $Z$ be a convex subset of a real vector space $U$. Let $a : X \times Z \to \mathbb{R}$ be a function such that for any $z \in Z$ the function $a(\cdot, z) : X \to \mathbb{R}$ is $\Phi$-convex on $X$ and for any $x \in X$ the function $a(x, \cdot) : Z \to \mathbb{R}$ is concave on $Z$.

The following conditions are equivalent:

(i) for every $\alpha \in \mathbb{R}$, $\alpha < \inf_{x \in X} \sup_{z \in Z} a(x, z)$, there exist $z_1, z_2 \in Z$ such that the intersection property holds for $\varphi_1$ and $\varphi_2$ on $X$ at the level $\alpha$,

(ii) $\sup_{z \in Z} \inf_{x \in X} a(x, z) = \inf_{x \in X} \sup_{z \in Z} a(x, z)$.

In the class $\Phi_{lsc}$ the intersection property is related to the following zero subgradient condition (see [33]).

**Definition 5** Let $f, g : X \to \mathbb{R}$ be $\Phi_{lsc}$-convex functions. We say that $f$ and $g$ satisfy the zero subgradient condition at $(x_1, x_2)$, where $x_1 \in \text{dom}(f)$, $x_2 \in \text{dom}(g)$ if

$0 \in \text{co}(\partial_{lsc} f(x_1) \cup \partial_{lsc} g(x_2))$,

where $\text{co}(\cdot)$ is the standard convex hull of a set. If $x_1 = x_2 = \hat{x}$ we say that $f$ and $g$ satisfy the zero subgradient condition at $\hat{x}$.

The following proposition was proved in [33] (Proposition 12), for convenience of the reader we rewrite the proof with $\varepsilon = 0$.

**Proposition 2** Let $X$ be a Hilbert space, $f, g : X \to \mathbb{R}$ be $\Phi_{lsc}$-convex functions, $\alpha \in \mathbb{R}$. Assume that $\hat{x} \in \text{dom}(f) \cap \text{dom}(g)$ and $\hat{x} \in [f \geq \alpha] \cap [g \geq \alpha]$.

If $f$ and $g$ satisfy the zero subgradient condition at $\hat{x}$ then, there exist $\varphi_1 \in \text{supp}(f)$, $\varphi_2 \in \text{supp}(g)$ for which the intersection property holds at the level $\alpha$. 

Proof By Remark 1, we only need to consider the case where \((a_1, v_1) \in \partial_{lsc} f(\bar{x})\) and \((a_2, v_2) \in \partial_{lsc} g(\bar{x})\) are such that \(\lambda a_1 + \mu a_2 = 0, \lambda v_1 + \mu v_2 = 0,\) for some \(\lambda, \mu \geq 0, \lambda + \mu = 1.\)

If \(\lambda = 0,\) then \(0 \in \partial_{lsc} g(\bar{x})\) and consequently, \(g(x) \geq g(\bar{x})\) for all \(x \in X.\) By assumption, \(g(\bar{x}) \geq \alpha,\) so \(\phi_1 \equiv \alpha\) is in the support of \(g,\) \(\phi_1 \in \text{supp}(g),\) hence, \(\phi_1\) and any function \(\phi_2 \in \text{supp}(f)\) have the intersection property at the level \(\alpha,\) since \([\phi_1 < \alpha] = \emptyset.\) By analogous reasoning, we get the desired conclusion if \(\mu = 0.\)

Now assume that \(\lambda > 0\) and \(\mu > 0.\) This implies that \(a_1 = a_2 = 0,\) since \(a_1, a_2 \geq 0.\) Let us take \(\phi_1 \in \text{supp}(f)\) and \(\phi_2 \in \text{supp}(g)\).

\[
\phi_1(x) := \langle v_1, x - \bar{x} \rangle + f(\bar{x}) \quad \text{and} \quad \phi_2(x) := \langle v_2, x - \bar{x} \rangle + g(\bar{x})
\]

for all \(x \in X.\) We show that \(\phi_1\) and \(\phi_2\) have the intersection property at the level \(\alpha.\) Let \(x_1 \in [\phi_1 < \alpha],\) we have

\[
\phi_1(x_1) < \alpha \Leftrightarrow \langle v_1, x_1 - \bar{x} \rangle + f(\bar{x}) < \alpha \Leftrightarrow \langle v_1, x_1 - \bar{x} \rangle < \alpha - f(\bar{x})
\]

By assumption that \(\bar{x} \in [f \geq \alpha] \cap [g \geq \alpha],\) we have \(\alpha - f(\bar{x}) \leq 0,\) so

\[
\langle v_1, x_1 - \bar{x} \rangle < 0.
\]

Since, \(x_1\) is chosen arbitrarily, we get \([\phi_1 < \alpha] \subset [\langle v_1, \cdot - \bar{x} \rangle < 0].\) By similar calculations we get \([\phi_2 < \alpha] \subset [\langle v_2, \cdot - \bar{x} \rangle < 0].\)

Since \(\lambda v_1 + \mu v_2 = 0\) we have

\[
[\langle v_2, \cdot - \bar{x} \rangle < 0] = [-\langle v_1, \cdot - \bar{x} \rangle < 0] = [\langle v_1, \cdot - \bar{x} \rangle > 0]
\]

and consequently

\[
[\phi_1 < \alpha] \cap [\phi_2 < \alpha] = \emptyset
\]

which completes the proof.

3 Duality for \(\Phi\)-convex Lagrangian.

Let \(X\) be a vector space and let \(\Phi\) be a class of elementary functions \(\varphi : X \to \mathbb{R}.\) In this section we investigate Lagrange duality for minimization problem

\[
\text{Min} \quad f(x) \quad x \in X,
\]

where \(f : X \to \mathbb{R}\) is a proper function.

We introduce the Lagrange function for problem \((P)\) by applying perturbation/parametrization approach, see e.g. [2], [7], [34].

Let \(Y\) be a vector space. A function \(p : X \times Y \to \mathbb{R}\) satisfying

\[
p(x, y_0) = f(x), \quad \text{for some } \ y_0 \in Y
\]
is called a perturbation function to problem \((P)\). Often we take \(y_0 = 0\). Clearly, \(\text{dom } p \neq \emptyset\) since \(\text{dom } f \neq \emptyset\) but, in general, \(p\) need not to be proper if \(f\) is a proper.

By using function \(p(\cdot, \cdot)\), the family of parametric problems \((P_y)\) is defined as

\[
\begin{align*}
\text{Min} & \quad p(x,y), \quad x \in X, \\
\end{align*}
\]

where \((P)\) coincides with \((P_{y_0})\).

Let \(\Psi\) be a class of elementary functions \(\psi : Y \rightarrow \mathbb{R}\). We consider the Lagrangian \(L : X \times \Psi \rightarrow \mathbb{R}\) defined as

\[
L(x, \psi) := \psi(y_0) - p^*_\Psi(x),
\]

(9)

where \(p^*_\Psi : \Psi \rightarrow \mathbb{R}\), and

\[
p^*_\Psi(\psi) = \sup_{y \in Y} \{\psi(y) - p(x,y)\}
\]

is the \(\Psi\)-conjugate of the function \(p_\Psi(\cdot) := p(x,\cdot), \ x \in X\). When \(Z_1 := Y, Z_2 := \Psi, c(y, \psi) := \psi(y)\), and \(y_0 = 0\), Lagrangian defined by (9) coincides with Lagrangian \(L(x,y)\) as defined in Proposition 1 of [21]. Also the Lagrangian given by formula 2.1 of [10] coincides with (9).

Analogous definitions of Lagrangian have been introduced in convex case e.g. [7], in DC case [34] and in general \(\Phi\)-convex case [10], [12], [18] and [20], Section 1.7.

Consider the Lagrangian primal problem

\[
\text{val}(LP) := \inf_{x \in X} \sup_{\psi \in \Psi} L(x, \psi).
\]

\((LP)\)

**Proposition 3** Problems \(P\) and \((LP)\) are equivalent in sense that

\[
\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} L(x, \psi),
\]

if and only if \(p(\cdot, \cdot)\) is \(\Psi\)-convex function at \(y_0 \in Y\) for all \(x \in X\).

**Proof** It is enough to observe that the following equality holds

\[
\sup_{\psi \in \Psi} L(x, \psi) = \sup_{\psi \in \Psi} \{\psi(y_0) - p^*_\Psi(\psi)\} = p^*_\Psi(y_0) = p(x, y_0).
\]

where the latter equality follows from Theorem [11].

The dual problem to \((LP)\) is defined as

\[
\text{val}(LD) := \sup_{\psi \in \Psi} \inf_{x \in X} L(x, \psi).
\]

\((LD)\)

Problem \((LD)\) is called the Lagrangian dual. The inequality

\[
\text{val}(LD) \leq \text{val}(LP)
\]

(10)
always holds. We say that the zero duality gap holds for problems \( (L_P) \) and \( (L_D) \) if the equality \( \text{val}(L_P) = \text{val}(L_D) \) holds.

The following theorem provides sufficient and necessary conditions for the zero duality gap, within the framework of \( \Phi \)-convexity, where \( \Phi \) is any class of elementary functions. Up to our knowledge, this is the first result on this level of generality.

We start with some preliminary observations. We have

\[
\text{dom } L := \{(x, \psi) \in X \times \Psi \mid L(x, \psi) < +\infty\} = \{(x, \psi) \in X \times \Psi \mid \inf_{\psi \in \Psi} \{p_\psi(y) - \psi(y)\} < +\infty\}.
\]

Observe that \( \text{dom } p_\psi \neq \emptyset \) for \( x \in \text{dom } f \) and, for any \( \psi \in \Psi \),

\[ p_\psi^*(\psi) \geq \psi(y_0) - p(x, y_0) = \psi(y_0) - f(x), \]

i.e. \( p_\psi^*(\cdot) > -\infty \) for any \( x \in \text{dom } f \). Since \( f \) is proper, \( \text{dom } f \neq \emptyset \) and \( \text{dom } f \subset \text{dom } L(\cdot, \psi) \) for any \( \psi \in \Psi \).

On the other hand, under the assumptions of Proposition \( \text{(iii)} \) since \( f \) is proper we have \( f(x) = \sup_{\psi \in \Psi} L(x, \psi) > -\infty \) for any \( x \in X \), i.e. \( L(x, \psi) > -\infty \) for some \( \psi \in \Psi \) which means that among functions \( L(\cdot, \psi), \psi \in \Psi \) may exist proper functions. In conclusion,

\[ \text{dom } L(\cdot, \psi) \neq \emptyset \text{ for every } \psi \in \Psi \text{ and } \text{supp } L(\cdot, \psi) \neq \emptyset \text{ for some } \psi \in \Psi. \quad (11) \]

Moreover, the following fact holds.

\[ \exists x \in X \exists \bar{x} \in \Psi \exists \bar{\psi} \in \Psi L(\bar{x}, \bar{\psi}) = +\infty \Leftrightarrow \forall \psi \in \Psi L(x, \psi) = +\infty. \quad (12) \]

To see this it is enough to note that the condition \( L(x, \psi) = \bar{\psi}(y_0) - p_\psi^*(\bar{\psi}) = +\infty \) for some \( x \in X \) and \( \psi \in \Psi \) can be rewritten as \( \inf_{\psi \in \Psi} \{p_\psi(y) - \psi(y)\} = +\infty \) which means that \( p_\psi(y) = p(x, y) = +\infty \) for any \( y \in Y \) i.e. \( \text{dom } p_\psi = \emptyset \) and consequently \( \forall \psi \in \Psi \) \( L(x, \psi) = +\infty \).

**Theorem 3** Let \( X, U \) be vector spaces. Let \( \Psi \subset U \) be a convex set of elementary functions \( \psi : Y \to \mathbb{R} \) and let the function \( L(\cdot, \psi) : X \to \mathbb{R} \), given by \( (\ref{L}) \), be \( \Phi \)-convex on \( X \) for any \( \psi \in \Psi \). Assume that \( p(x, \cdot) \) is \( \Psi \)-convex function at \( y_0 \in Y \) for all \( x \in X \). The following are equivalent:

(i) for every \( \alpha < \inf_{x \in X, \psi \in \Psi} L(x, \psi) \) there exist \( \psi_1, \psi_2 \in \Psi \) and \( \varphi_1 \in \text{supp } L(\cdot, \psi_1) \) and \( \varphi_2 \in \text{supp } L(\cdot, \psi_2) \) such that functions \( \varphi_1 \) and \( \varphi_2 \) have the intersection property at the level \( \alpha \);

(ii)

\[ \inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} L(x, \psi) = \sup_{\psi \in \Psi} \inf_{x \in X} L(x, \psi). \]

**Proof** The first equality follows from Proposition \( \text{(iii)} \). To prove the second equality we apply Theorem \( \text{(ii)} \) i.e. we need to show that \( L(x, \cdot) \) is a concave function of \( \psi \) for all \( x \in X \), i.e.

\[ L(x, t\psi_1 + (1 - t)\psi_2) \geq tL(x, \psi_1) + (1 - t)L(x, \psi_2) \quad (13) \]
for any \( t \in [0,1], \) \( x \in X \) and any \( \psi_1, \psi_2 \in \Psi \).

Take any \( \psi_1, \psi_2 \in \Psi \). \( \mathcal{L}(x, \psi_1) \) and \( \mathcal{L}(x, \psi_2) \) are finite. Let \( t \in [0,1] \). We have

\[
\mathcal{L}(x, t\psi_1 + (1-t)\psi_2) = \\
t\psi_1(y_0) + (1-t)\psi_2(y_0) - \sup_{y \in Y} \{ t\psi_1(y) + (1-t)\psi_2(y) - p(x, y) \} \\
= t\psi_1(y_0) + (1-t)\psi_2(y_0) - \sup_{y \in Y} \{ t\psi_1(y) + (1-t)\psi_2(y) - tp(x, y) - (1-t)p(x, y) \} \\
\geq t\psi_1(y_0) + (1-t)\psi_2(y_0) - t \sup_{y \in Y} \{ \psi_1(y) - p(x, y) \} - (1-t) \sup_{y \in Y} \{ \psi_2(y) - p(x, y) \} \\
= t\mathcal{L}(x, \psi_1) + (1-t)\mathcal{L}(x, \psi_2) .
\]

Hence, \( \mathcal{L}(x, \cdot) \) is concave on \( \Psi \). Observe that \( \mathcal{L}(x, \psi_1) \) and \( \mathcal{L}(x, \psi_2) \) can take infinite values.

**Remark 2** Assume that \( p(x, \cdot) \) is \( \Psi \)-convex function at \( y_0 \in Y \) for all \( X \). By Proposition 3, \( f(x) = \sup_{\psi \in \Psi} \mathcal{L}(x, \psi) \). Since \( f \) is proper, it may only happen that

\[
\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} \mathcal{L}(x, \psi) = -\infty \quad \text{or} \quad \inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} \mathcal{L}(x, \psi) \quad \text{is finite}.
\]

In the first case, when \( \text{val}(L_P) = -\infty \), in view of (13), there is nothing to prove and the condition (i) of Theorem 4 is automatically satisfied.

Theorem 4 can also be formulated in terms of the optimal value function of the problem \( (P_y) \). The optimal value function of \( (P_y) \), \( V : Y \to \mathbb{R} \) is defined as

\[
V(y) := \inf_{x \in X} p(x, y) .
\]

In Proposition 4 below the convexity of elementary function set \( \Psi \) is not required.

**Proposition 4** Assume that \( p_x = p(x, \cdot) \) is \( \Psi \)-convex function on \( Y \) for all \( x \in X \). The following are equivalent:

(i) \( \inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} \mathcal{L}(x, \psi) = \sup_{\psi \in \Psi} \inf_{x \in X} \mathcal{L}(x, \psi) . \)

(ii) \( V(y_0) = V^{**}(y_0) \),

where \( y_0 \in Y \) and \( p(x, y_0) = f(x) \), for any \( x \in X \).

**Proof** For any \( \psi \in \Psi \), we have

\[
V^*(\psi) = \sup_{y \in Y} \{ \psi(y) - \inf_{x \in X} p(x, y) \} = \sup_{y \in Y} \sup_{x \in X} \{ \psi(y) - p(x, y) \} = \sup_{x \in X} p^*_x(\psi).
\]

On the other hand, for the dual function we have

\[
\inf_{x \in X} L(x, \psi) = \inf_{x \in X} \{ \psi(y_0) - p^*_x(\psi) \} = \psi(y_0) - \sup_{x \in X} \{ p^*_x(\psi) \} .
\]
By (15),
\[ \inf_{x \in X} L(x, \psi) = \psi(y_0) - V^*(\psi), \]
and for the dual \( (L^D) \) we have
\[ \sup_{\psi \in \Psi} \inf_{x \in X} L(x, \psi) = \sup_{\psi \in \Psi} \{ \psi(y_0) - V^*(\psi) \} = V^{**}(y_0). \] (16)

By Proposition 3 when the perturbation function \( p(x, \cdot) \) is \( \Psi \)-convex for each \( x \in X \) then
\[ \inf_{x \in X} \sup_{\psi \in \Psi} L(x, \psi) = V(y_0) \]
which completes the proof.

In reflexive Banach spaces for some particular coupling functions, conditions ensuring \((ii)\) were proved in Theorem 4.1 and Proposition 4.1 of [10].

**Corollary 1** Assume that \( p_x = p(x, \cdot) \) is \( \Psi \)-convex function on \( Y \) for all \( x \in X \). The following are equivalent:

(i) \[ \inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi} L(x, \psi) = \sup_{\psi \in \Psi} \inf_{x \in X} L(x, \psi). \]

(ii) \( V \) is \( \Psi \)-convex at \( y_0 \) i.e. \( V(y_0) = \sup \{ \psi(y_0), \ \psi \in \text{supp} V(y) \} \) (cf. Theorem 7).

**Proof** Follows directly from Theorem 3 and Proposition 4. To see the implication \((i) \Rightarrow (ii)\) we need to show the following.

(a) If \( V(y_0) = \inf_{x \in X} f(x) = -\infty \), then \( V(y) = -\infty \) for all \( y \in Y \).

(b) If \( V(y_0) \) is finite, then \( \text{supp} \neq \emptyset \) and
\[ V(y) = \sup \{ \psi(y) \mid \psi \in \text{supp} V \}. \] (17)
Ad (a). In this case, supp $V = \emptyset$, i.e. $V \equiv -\infty$.

Ad (b). Assume that $V(y_0) = \inf_{x \in X} \sup_{\psi \in \Psi} \mathcal{L}(x, \psi)$. By (i), in view of Theorem 3 and Proposition 4, we have $V^{**}(y_0) = \sup_{\psi \in \Psi} \sup_{x \in X} \mathcal{L}(x, \psi) = V(y_0)$. By Theorem 1, $V$ is $\Psi$-convex at $y_0$.

The following theorem corresponds to the classical fact for convex functions (see i.e. [7], Theorem 2.142 [5], Theorem 1.6). For abstract convex functions this theorem was proved in Theorem 5.2 of [28] (see also Proposition 2.1 of [10]).

We say that an element $\psi_0 \in \Psi$ is a solution to the Lagrangian dual problem $(L_D)$ if

$$\sup_{\psi \in \Psi} \inf_{x \in X} \mathcal{L}(x, \psi) = \inf_{x \in X} \mathcal{L}(x, \psi_0).$$

(18)

The function $q : \Psi \to \hat{\mathbb{R}}$ defined as

$$q(\psi) := \inf_{x \in X} \mathcal{L}(x, \psi)$$

is called the dual function. With this notation, $\psi_0 \in \Psi$ solves the Lagrangian dual $(L_D)$ iff

$$q(\psi_0) = \sup_{\psi \in \Psi} q(\psi).$$

**Theorem 4** The following statements hold:

(i) If $\partial_\Psi V(y_0) \neq \emptyset$, then $val(L_P) = val(L_D)$, and the solution set to the Lagrangian dual problem $(L_D)$ coincides with $\partial_\Psi V(y_0)$.

(ii) If $val(L_P) = val(L_D)$, and both values are finite, then the (possibly empty) optimal solution set of $(L_D)$ coincides with $\partial_\Psi V(y_0)$.

**Proof** By Proposition 4, the following equivalence holds

$$\psi \in \partial_\Psi V(y_0) \iff \psi(y_0) - V^*(\psi) = V(y_0).$$

Consequently,

$$val(L_P) = V(y_0) = \psi(y_0) - V^*(\psi) \leq \sup_{\psi \in \Psi} \{\psi(y_0) - V^*(\psi)\} = V^{**}(y_0) = val(L_D),$$

which means that $val(L_P) = val(L_D)$ if and only if $\psi \in \partial_\Psi V(y_0)$. We have proved (i) and (ii).
4 Duality for $\Phi_{conv}$-Lagrangian

Let $X$ be a Banach space with the dual $X^*$. In the present section we analyse the Lagrangian duality for the $\Phi_{conv}$ Lagrangian, where the class $\Phi_{conv}$ defined in Example 1,

$$\Phi_{conv} := \{\varphi : X \to \mathbb{R}, \varphi(x) = \langle \ell, x \rangle + c, \ x \in X, \ \ell \in X^*, \ c \in \mathbb{R}\},$$

and in the original problem \(^{13}\), the minimized function $f : X \to \overline{\mathbb{R}}$ is a proper convex and lsc function of the form

$$f(x) := g(x) + h(x)$$

where $g, h : X \to \overline{\mathbb{R}}$ are convex proper lsc functions with $\text{dom } g \cap \text{dom } h \neq \emptyset$.

For problem \(^{13}\) with $f$ given by \(^{20}\), we consider the perturbation function $p : X \times X \to \overline{\mathbb{R}}$,

$$p(x, y) := g(x) + h(x - y), \quad p_x(\cdot) := p(x, \cdot) : X \to \overline{\mathbb{R}}.$$ (21)

Clearly, $y_0 = 0$, $0 \in \text{dom } p_x(\cdot) \subset X$ for any $x \in \text{dom } f$ and $p$ is a proper function.

Hence, we take $\Psi_{conv} := \Phi_{conv}$. In the sequel, we identify $X$ and $Y$ (i.e. $X = Y$) and the class $\Phi_{conv}$ with the Cartesian product $X^* \times \mathbb{R}$.

For any $x \in X$ the conjugate $p^*_x : X^* \times \mathbb{R} \to \widehat{\mathbb{R}}$, is given by the formula

$$p_x^*(\psi) := p_x^*(y^*, c) = \sup_{y \in X} \{y^*(y) + c - g(x) - h(x - y)\}. $$ (22)

This definition of conjugacy coincides with the classical one when $c = 0$. Clearly, the conjugate function \(^{22}\), is convex and lsc on $X^* \times \mathbb{R}$. The reason for defining conjugacy in the above (a bit unusual way) way is motivated by the importance of constant $c \in \mathbb{R}$ in the intersection property, Definition \(^3\).

Let us observe, that the appearance of the constant $c$ above (which amounts to considering affine functionals rather than linear) does not influence the values of the Lagrangian.

The Lagrangian $L : X \times X^* \times \mathbb{R} \to \overline{\mathbb{R}}$ given by \(^{9}\) takes the form

$$L(x, \psi) = L(x, y^*, c) = L(x, y^*) = - \sup_{y \in X} \{\langle y^*, y \rangle - g(x) - h(x - y)\}$$

$$= g(x) + \langle y^*, x \rangle - \langle y^*, x \rangle - \sup_{y \in X} \{\langle y^*, y \rangle - h(x - y)\}$$

$$= g(x) + \langle -y^*, x \rangle - \sup_{y \in X} \{\langle -y^*, y \rangle + \langle -y^*, x \rangle - h(x - y)\}$$

$$= g(x) - \langle y^*, x \rangle - h^*(-y^*).$$

(23)

The Lagrangian $L(\cdot, y^*)$ is lsc and convex for any $y^* \in Y$ and $L(x, \cdot)$ is usc and concave for any $x \in X$. 
In particular, when \( h \) is the indicator function of a convex set \( A = \{ x \in X \mid G(x) \in K \} \), where \( G : X \rightarrow Z \), \( Z \) is a Banach space, \( K \subset Z \) is closed convex cone in \( Z \), then

\[
L(x, y^*, c) = g(x) - \sup_{y \in Y} \{ \langle y^*, y \rangle - \text{ind}_A(x - y) \} \quad u := x - y
\]

\[
= g(x) - \langle y^*, x \rangle - \sup_{u \in Y} \{ \langle -y^*, u \rangle - \text{ind}_A(u) \}
\]

\[
= g(x) - \langle y^*, x \rangle - \sup_{u \in A} \{ \langle -y^*, u \rangle \}
\]

For any \((y^*, c) \in X^* \times \mathbb{R}^*\),

\[
\text{supp} \mathcal{L}(\cdot, y^*, c) := \{ (z^*, d) \in X^* \times \mathbb{R} \mid (z^*, d) \leq \mathcal{L}(\cdot, y^*, c) \}.
\]

Since \( f \) is proper, by \( (\mathbf{1}) \), there exist \((y^*, c) \in X^* \times \mathbb{R} \) such that \( \text{supp} \mathcal{L}(\cdot, y^*, c) \neq \emptyset \).

Suppose now that our original problem \( \mathbf{P} \) is the classical constrained optimization problem

\[
\min_{x \in A} \ g(x),
\]

where \( A := \{ x \in X \mid g_x \leq 0, \ i = 1, \ldots, m \} \), \( g_i : X \rightarrow \mathbb{R}, \ i = 1, \ldots, m \) and \( p : X \times \mathbb{R}^m \rightarrow \mathbb{R} \),

\[
p(x, y) := \begin{cases} f(x) \text{ when } g_i(x) + y_i \leq 0, \ i = 1, \ldots, m \\ +\infty \text{ otherwise} \end{cases}
\]

Hence,

\[
f(x) := g(x) + \text{ind}_K(G(x)) ,
\]

where \( K := \{ y = (y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_i \leq 0, \ i = 1, \ldots, m \} \), \( G(x) := (g_1(x), g_2(x), \ldots, g_m(x)) \) and the perturbation function is of the form

\[
p(x, y) = g(x) + \text{ind}_K(G(x) + y), \quad p(y_0) = p(0) = f(x), \quad x \in X, \ y \in \mathbb{R}^m
\]

with \( p^*_y(y^*, c) := \sup_{y \in Y} \{ \langle y^*, y \rangle + c - g(x) - \text{ind}_K(G(x) + y) \} \), \( y^* \in \mathbb{R}^m, \ c \in \mathbb{R} \).

Note that here \( \Phi_{\text{cone}} \) is identified with \( X^* \times \mathbb{R} \) and \( \Psi \) is identified with \( \mathbb{R}^m \times \mathbb{R} \).

The Lagrangian \( \mathbf{11} \), \( \mathcal{L} : X \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R} \), takes the form

\[
\mathcal{L}(x, y, c) = \mathcal{L}(x, y^*, c) = -\sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle - g(x) - \text{ind}_K(G(x) + y) \}.
\]

We have

\[
\mathcal{L}(x, y^*, c) = \mathcal{L}(x, y^*) = -\sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle - g(x) - \text{ind}_K(G(x) + y) \}
\]

\[
= g(x) - \sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle - \text{ind}_K(G(x) + y) \}
\]

\[
= g(x) - \sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle + \langle y^*, G(x) \rangle - \langle y^*, G(x) \rangle - \text{ind}_K(G(x) + y) \}
\]

\[
= g(x) + \langle y^*, G(x) \rangle - \sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle + \langle y^*, G(x) \rangle - \text{ind}_K(G(x) + y) \}
\]

\[
= g(x) + \langle y^*, G(x) \rangle - \sup_{y \in \mathbb{R}^m} \{ \langle y^*, y \rangle + \langle y^*, G(x) \rangle - \text{ind}_K(G(x) + y) \}
\]
Consequently,

\[
\mathcal{L}(x, y^*) = \begin{cases} 
  g(x) + \langle y^*, G(x) \rangle & \text{when } y^* \geq 0 \\
  -\infty & \text{otherwise}
\end{cases}
\]

For problem (20) with perturbation function (21), Theorem 3 takes the following form.

**Theorem 5** Let \( X \) be a Banach space with the dual \( X^* \). Let the function \( \mathcal{L}(x, x^*, c) : X \times X^* \times \mathbb{R} \to \mathbb{R} \) be given by (23). The following are equivalent:

(i) for every \( \alpha < \inf_{x \in X} \sup_{(x^*, c) \in X^* \times \mathbb{R}} \mathcal{L}(x, x^*, c) \) there exist \( (x_1^*, c_1), (x_2^*, c_2) \in X^* \times \mathbb{R} \) and \( \varphi_1 \in \text{supp} \mathcal{L}(\cdot, x_1^*, c_1) \) and \( \varphi_2 \in \text{supp} \mathcal{L}(\cdot, x_2^*, c_2) \) such that

\[
|\varphi_1 - \alpha| \cap |\varphi_2 - \alpha| = \emptyset;
\]

(ii) 

\[
\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\varphi \in \varphi} \mathcal{L}(x, \varphi) = \sup_{\varphi \in \varphi} \inf_{x \in X} \mathcal{L}(x, \varphi).
\]

**Proof** The proof follows from Proposition 4.1 and Theorem 3. Let the function \( p_x : X^* \times \mathbb{R} \to \mathbb{R}, p_x(y) = p(x, y) \), with \( p \) is given by (21) be \( \Phi_{\text{conv-conv}} \)-convex on \( Y \). Let the function \( \mathcal{L}(\cdot, x^*, c) : X \to \mathbb{R} \), given by (23), be \( \Phi_{\text{conv-conv}} \)-convex on \( X \) for any \( (x^*, c) \in X^* \times \mathbb{R} \).

**Proposition 5** Assume that \( \inf_{x \in X} f(x) \) finite. The following are equivalent.

(I) Condition (i) of Theorem 5 (the intersection property).

(II) For every \( \varepsilon > 0 \) there exist \( (y_1^*, c_1), (y_2^*, c_2) \in Y^* \times \mathbb{R} \) and \( \varphi_1(x) := (z_1^*, d_1) \in \text{supp} \mathcal{L}(\cdot, y_1^*, c_1), \varphi_2(x) := (z_2^*, d_2) \in \text{supp} \mathcal{L}(\cdot, y_2^*, c_2) \) such that

\[
\begin{align*}
& (a) \ t_0d_1 + (1 - t_0)d_2 \geq \inf_{x \in X} f(x) - \varepsilon, \\
& (b) \ t_0z_1^* + (1 - t_0)z_2^* = 0
\end{align*}
\] (26)

are satisfied for some \( t \in [0, 1] \).

**Proof** Assume that (I) holds. Condition (ii) of Theorem 5 reads as follows: for every \( \varepsilon > 0 \) one can find \( (y_1^*, c_1), (y_2^*, c_2) \in Y^* \times \mathbb{R} \) and

\[
(z_1^*, d_1) \in \text{supp} \mathcal{L}(\cdot, y_1^*, c_1) \quad \text{and} \quad (z_2^*, d_2) \in \text{supp} \mathcal{L}(\cdot, y_2^*, c_2)
\] (27)

satisfying

\[
[x | \langle z_1^*, x \rangle + d_1 \leq \inf_{x \in X} f(x) - \varepsilon] \cap [x | \langle z_2^*, x \rangle + d_2 \leq \inf_{x \in X} f(x) - \varepsilon] = \emptyset.
\] (28)

Equivalently, (see Lemma 4.1 of [32]) there exists \( t_0 \in [0, 1] \) satisfying

\[
\langle t_0z_1^* + (1 - t_0)z_2^*, x \rangle + t_0d_1 + (1 - t_0)d_2 \geq \inf_{x \in X} f(x) - \varepsilon \quad \text{for all } x \in X.
\] (29)
i.e. it must be 
\[ t_0z_1^* + (1-t_0)z_2^* = 0, \]
and consequently
\[ t_0d_1 + (1-t_0)d_2 \geq \inf_{x \in X} f(x) - \varepsilon \quad \text{for all } x \in X. \tag{30} \]

Assume that (II) holds. Then the following inequality holds
\[ \langle t_0z_1^* + (1-t_0)z_2^*, x \rangle + t_0d_1 + (1-t_0)d_2 \geq \inf_{x \in X} f(x) - \varepsilon \quad \text{for all } x \in X. \tag{31} \]

Which is equivalent to
\[ t_0\varphi_1(x) + (1-t_0)\varphi_2(x) \geq \inf_{x \in X} f(x) - \varepsilon \quad \text{for all } x \in X. \]

where \((z_1^*, d_1) \in \text{supp } \mathcal{L}(\cdot, y_1^*, c_1) \) and \((z_2^*, d_2) \in \text{supp } \mathcal{L}(\cdot, y_2^*, c_2)\). By Lemma 4.1 of [32] functions \(\varphi_1\) and \(\varphi_2\) have the intersection property at the level \(\inf_{x \in X} f(x) - \varepsilon\). Since \(\varepsilon > 0\) was chosen arbitrarily, we get that functions \(\varphi_1\) and \(\varphi_2\) have the intersection property at every level \(\alpha < \inf_{x \in X} f(x)\).

We close this section with a comparison between the intersection property and two types of optimality conditions which are studied in the literature, namely, the so-called generalized interior point-condition and closedness-type condition. These conditions are thoroughly researched in [6].

The first example shows that the generalized interior point-condition fails, but the closedness-type one holds, as was proved in [6]. We show that the intersection property is fulfilled.

**Example 2** ([6], Example 21 and Example 23) Let \((X, \| \cdot \|)\) be a real reflexive Banach space, \(x_0^* \in X^* \setminus \{0\}\) and the functions \(f, g : X \to \mathbb{R}\) be defined by \(f(\cdot) = \text{ind}_{\ker x_0^*}(\cdot)\) and \(g(\cdot) = \| \cdot \| + \text{ind}_{\ker x_0^*}(\cdot)\). We have
\[ \beta = \inf_{x \in X} \sup_{y^* \in X^*} \mathcal{L}(x, y^*) = \inf_{x \in X} \{f(x) + g(x)\} = \inf_{x \in \ker x_0^*} \|x\| = 0 \]
and \(g^*(y^*) = \text{ind}_{B_{\|y^*\|} + \mathbb{R}x_0^*}(y^*)\). Let \(y_1^* = ax_0^*, a \in \mathbb{R}\). We have \(-y_1^* \in B_{\|y_1^*\|} + \mathbb{R}x_0^*, \) and \(g^*(-y_1^*) = 0,\) hence
\[ \mathcal{L}(x, y_1^*) = f(x) - \langle ax_0^*, x \rangle = \begin{cases} 0 - \langle ax_0^*, x \rangle, & \text{if } x \in \ker x_0^*, \\ +\infty - \langle ax_0^*, x \rangle, & \text{if } x \notin \ker x_0^* \end{cases} \]

Let \(y_2^* = bx_0^*, b \in \mathbb{R}, b \neq a\). Let \(\varphi_1, \varphi_2 \in \Phi_{\text{conv}}\) and \(\varphi_1 = \varphi_2 \equiv 0\), then \(\varphi_1 \in \text{supp } \mathcal{L}(\cdot, y_1^*)\) and \(\varphi_2 \in \text{supp } \mathcal{L}(\cdot, y_2^*).\) Functions \(\varphi_1\) and \(\varphi_2\) have the intersection property at the level 0, hence at every level \(\alpha < 0\).

The next example shows that the generalized interior point-condition hold, but closedness-type one fails, as was shown in [6]. We show that the intersection property is fulfilled.
**Example 3** ([2], Example 25) Let $X = \ell^2(N)$ and let the set $C, S$ be such that

$$C = \{(x_n)_{n \in N} \in \ell^2 : x_{2n-1} + x_{2n} = 0 \forall n \in N\},$$

and

$$S = \{(x_n)_{n \in N} \in \ell^2 : x_{2n} + x_{2n+1} = 0 \forall n \in N\},$$

hence $S \cap C = \{0\}$. Define the functions $f, g : \ell^2 \to \mathbb{R}$ by $f(\cdot) = \text{ind}_C(\cdot)$, $g = \text{ind}_S(\cdot)$, so

$$\beta = \inf_{x \in \ell^2} \sup_{y^* \in \ell^2} \mathcal{L}(x, y^*) = \inf_{x \in \ell^2} \{f(x) + g(x)\} = \{0\}. $$

We have $g^* = \text{ind}_{S^\perp}$, where

$$S^\perp = \{(x_n)_{n \in N} \in \ell^2 : x_1 = 0, x_{2n} = x_{2n+1} \forall n \in N\}. $$

Let $y_1^* = (0, 0, \ldots, 0, \ldots)$, then we have

$$\mathcal{L}(x, y_1^*) = f(x).$$

Let $y_2^*$ be such that $-y_2^* \in S^\perp$, then

$$\mathcal{L}(0, y_2^*) = f(0) - g^*(-y_2^*) - \langle y_2^*, 0 \rangle = -g^*(-y_2^*) = 0.$$ 

Let $\varphi_1 \in \text{supp}\mathcal{L}(\cdot, y_1^*)$ and $\varphi_2 \equiv 0$, hence $\varphi_2 \in \text{supp}\mathcal{L}(\cdot, y_2^*)$. Functions $\varphi_1$ and $\varphi_2$ have the intersection property at the level 0, hence at every level $\alpha < 0$.

### 5 Duality for $\Phi_{\text{lsc}}$-convex Lagrangian.

Let $X, Y$ be Hilbert spaces. Let $\Phi_{\text{lsc}}$ and $\Psi_{\text{lsc}}$ be defined by ([1]) on $X$ and $Y$, respectively.

In the present section we consider problem ([P]) of Section 3 with the perturbation function $p$ such that $p_x = p(x, \cdot) : \Psi_{\text{lsc}} \to \mathbb{R}$ is $\Psi_{\text{lsc}}$-convex for any $x \in X$ and with the Lagrangian $\mathcal{L}$, $\mathcal{L}(\cdot, \psi) : X \to \mathbb{R}$, which is $\Phi_{\text{lsc}}$-convex for any $\psi \in \Psi$.

In the considered case the Lagrangian $\mathcal{L} : X \times \Psi_{\text{lsc}} \to \mathbb{R}$ defined by ([32]) takes the form

$$\mathcal{L}(x, \psi) = -a\|y_0\|^2 - \langle v, y_0 \rangle + c - p_x^*(\psi), \tag{32}$$

where $\psi(y) := -a\|y\|^2 + \langle v, y \rangle + c$ and

$$p_x^*(\psi) = \sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle + c - p(x, y)\} = c + \sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle - p(x, y)\}.$$
Remark 3 Observe that the right-hand side of (32) is independent of $c$, i.e. for all functions $\psi$ which differs only be $c$ (i.e. have the same $a$ and $v$) the right-hand side of (32) is the same. In view of this

$$L(x, \psi) = L(x, a, v) = -a\|y_0\|^2 + \langle v, y_0 \rangle - p^*_x(a, v),$$

where

$$p^*_x(a, v) = \sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle - p(x, y)\},$$

i.e. the Lagrangian $L$ can be equivalently regarded as a function defined on $X \times \mathbb{R}_+ \times Y^*$.

According to Proposition 3, problem (P) is equivalent to the Lagrangian primal problem

$$\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} L(x, \psi)$$

provided $p(x, \cdot)$ is $\Psi_{lsc}$-convex for any $x \in X$.

The following theorem is based on Theorem 3.

Theorem 6 Let $X, Y$ be Hilbert spaces. Let $p(x, \cdot)$ be $\Psi_{lsc}$-convex function at $y_0 \in Y$ for all $x \in X$. Let $L : X \times \Psi_{lsc} \to \mathbb{R}$ be the Lagrangian defined by (32), i.e.

$$L(x, \psi) = -a\|y_0\|^2 + \langle v, y_0 \rangle + c - p^*_x(\psi).$$

Assume that for any $\psi \in \Psi_{lsc}$ the function $L(\cdot, \psi) : X \to \mathbb{R}$ is $\Phi_{lsc}$-convex on $X$. The following are equivalent:

(i) for every $\alpha < \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} L(x, \psi)$ there exist $\psi_1, \psi_2 \in \Psi$ and $\varphi_1 \in \text{supp}L(\cdot, \psi_1)$ and $\varphi_2 \in \text{supp}L(\cdot, \psi_2)$ such that

$$[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset$$

(ii) 

$$\inf_{x \in X} f(x) = \sup_{\psi \in \Psi_{lsc}} \inf_{x \in X} L(x, \psi) = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} L(x, \psi).$$

Proof Follows from formula (33), Proposition 3 and Theorem 3.

Let $\beta := \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} L(x, \psi)$. Since the intersection property follows from the zero subgradient condition (Proposition 2), we can prove the following sufficient conditions for zero duality gap.

Theorem 7 Let $X, Y$ be Hilbert spaces. Let $p(x, \cdot)$ be $\Psi_{lsc}$-convex function at $y_0 \in Y$ for all $x \in X$. Let $L : X \times \Psi_{lsc} \to \mathbb{R}$ be the Lagrangian defined by (32), i.e.

$$L(x, \psi) = -a\|y_0\|^2 + \langle v, y_0 \rangle + c - p^*_x(\psi).$$

Assume that for any $\psi \in \Psi_{lsc}$ the function $L(\cdot, \psi) : X \to \mathbb{R}$ is $\Phi_{lsc}$-convex on $X$. 

If there exist $\psi_1, \psi_2 \in \Psi_{lsc}$ and $\bar{x} \in \text{dom } f$, $\bar{x} \in [\mathcal{L}(\cdot, \psi_1) \geq \beta] \cap [\mathcal{L}(\cdot, \psi_2) \geq \beta]$ such that 

$$0 \in \partial_{lsc} \mathcal{L}(\bar{x}, \psi_1) \cup \partial_{lsc} \mathcal{L}(\bar{x}, \psi_2),$$

then

$$\inf_{x \in X} f(x) = \sup_{\psi \in \Psi_{lsc}} \inf_{x \in X} \mathcal{L}(x, \psi) = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi).$$

Proof Since $\text{dom } \mathcal{L}(\cdot, \psi) \supset \text{dom } f$, for all $\psi \in \Psi_{lsc}$, proof follows immediately from Proposition 2, Proposition 3 and Theorem 2.

We say that a function $f : Y \to \bar{\mathbb{R}}$ is paraconvex (in the literature paraconvex functions are known also under the name weakly convex [36] and semiconvex [11]) on $Y$ if there exists $c > 0$ such that $f + c\| \cdot \|_2$ is convex. Equivalently, a function $f : Y \to \bar{\mathbb{R}}$ is called paraconvex on $X$ if there exists $C > 0$ such that for all $x, y \in Y$ and $t \in [0, 1]$ the following inequality holds

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + C\| x - y \|^2,$$

see e.g. [27]. It was shown in [17] and [26] that (36) is equivalent to

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + C(t - t)\| x - y \|^2.$$

It was shown in [33], Proposition 3 that every lower semicontinuous paraconvex function is $\Phi_{lsc}$-convex. Moreover, by Proposition 5 of [33], if $f$ is proper lsc and paraconvex on $Y$, then for every $y \in \text{int } \text{dom}(f)$ the subdifferential $\partial_{lsc} f(y)$ is nonempty.

Proposition 5 of [33] allows to obtain the strong duality theorem for the pair of dual problems problem $P$ with paraconvex optimal value function $V$ defined by (14), i.e.

$$V(y) := \inf_{x \in X} p(x, y).$$

As usual, we say that the strong duality holds for dual problems $L_P$ and $L_D$ when the zero duality gap holds and the dual problem $L_D$ is solvable.

The following strong duality result holds for problems with the paraconvex (weakly convex) optimal value function $V$ and $\text{int } \text{dom } V \neq \emptyset$.

**Theorem 8** Let $X$ and $Y$ be Hilbert spaces. Assume that the perturbation function $p(x, \cdot)$ (as a function of $y$) is $\Psi_{lsc}$-convex at $y_0$ for all $x \in X$.

If the optimal value function $V(y)$ of the problem (P) is proper, lsc, and paraconvex on $Y$ and $y_0 \in \text{int } \text{dom}(V)$, then

$$\inf_{x \in X} f(x) = \sup_{\psi \in \Psi_{lsc}} \inf_{x \in X} \mathcal{L}(x, \psi) = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi),$$

i.e. $\inf_{x \in X} f(x) = \text{val}(L_P) = \text{val}(L_D)$ and the solution set of the dual problem $L_D$ is nonempty and coincides with $\partial_{lsc} V(y_0)$. 
Proof By Proposition 3
\[
\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi).
\]
By paraconvexity of the function \(V\) and by Proposition 5 of [33], we get \(\partial_{lsc} V(y_0) \neq \emptyset\). Hence, by Theorem 4(i) we get
\[
\inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi) = \sup_{\psi \in \Psi_{lsc}} \inf_{x \in X} \mathcal{L}(x, \psi)
\]
and the solution set to the dual problem (L) is nonempty and coincides with \(\partial_{lsc} V(y_0)\).

Let us note that in the above theorem it is possible to replace the \(\text{int dom}(V)\) by the so called quasi-relative interior \(\text{qri dom}(V)\) see i.e Corollary 9 in [38]. Theorem 6 and Theorem 7 refer to generic perturbation function \(p\). For some particular choices of \(p\) the intersection property is automatically satisfied. One such choice is considered below.

6 Special case

In the present section we investigate Lagrangian duality for constrained optimization problems within the framework of \(\Psi_{lsc}\)-convexity.

Let \(X\) be a Hilbert space. Consider the constrained optimization problem of the form
\[
\min f(x) \quad x \in A(y_0),
\]
where, as previously, \(f : X \to \mathbb{R}\) is a proper function in the sense that \(\text{dom } f \neq \emptyset\) and \(f > -\infty\), and \(A : Y \rightrightarrows X\) is a set-valued mapping. The corresponding family of parametrized/perturbed problems \((P_y)\)
\[
\min p(x, y) \quad x \in X
\]
is based on the perturbation function \(p : X \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\) defined as (see [24])
\[
p(x, y) = \begin{cases} f(x), & x \in A(y) \\
+\infty, & x \notin A(y) \end{cases}
\]
Problem (38) casts into general form (P) with the minimized function \(\tilde{f}(x) := f(x) + \text{ind}_{A(y_0)}(\cdot)\), where \(\text{ind}_{A}(\cdot)\) is the indicator function of set \(A\). We adopt the convention that \(\text{ind}_A \equiv +\infty\) whenever \(A = \emptyset\). Assume that \(\beta = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi)\) is finite.

Theorem 9 Let \(X,Y\) be Hilbert spaces and let \(\inf_{x \in X} f(x)\) be finite. Let \(\mathcal{L} : X \times \Psi_{lsc} \to \mathbb{R}\) be the Lagrangian defined by (32) with \(y_0 = 0\),
\[
\mathcal{L}(x, \psi) = c - p^*_\psi(\psi),
\]
where
\[
\tilde{f}(x) := f(x) + \text{ind}_{A(y_0)}(\cdot),
\]
and the solution set to the dual problem (L) is nonempty and coincides with \(\partial_{lsc} V(y_0)\).

Problem (38) casts into general form (P) with the minimized function \(\tilde{f}(x) := f(x) + \text{ind}_{A(y_0)}(\cdot)\), where \(\text{ind}_{A}(\cdot)\) is the indicator function of set \(A\). We adopt the convention that \(\text{ind}_A \equiv +\infty\) whenever \(A = \emptyset\). Assume that \(\beta = \inf_{x \in X} \sup_{\psi \in \Psi_{lsc}} \mathcal{L}(x, \psi)\) is finite.

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\[
\mathcal{L}(x, \psi) = c - p^*_\psi(\psi),
\]
where
where, as previously, \( \psi(y) := -a\|y\|^2 + \langle v, y \rangle + c \) and
\[
p^*_2(\psi) = \sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle + c - p(x, y)\} = c + \sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle - p(x, y)\}.
\]

If, for any \( \psi \in \Psi_{\text{lsc}} \), the function \( \mathcal{L}(\cdot, \psi) : X \to \mathbb{R} \) is \( \Phi_{\text{lsc}} \)-convex on \( X \) and the perturbation function \( p(x, \cdot) \) is \( \Phi_{\text{lsc}} \)-convex at \( y_0 \) for all \( x \in X \), then
\[
\inf_{x \in X} f(x) = \sup_{\psi \in \Psi_{\text{lsc}}} \inf_{x \in X} \mathcal{L}(x, \psi) = \inf_{x \in X} \sup_{\psi \in \Psi_{\text{lsc}}} \mathcal{L}(x, \psi).
\]

**Proof** The first equality follows from Proposition 3 applied to function \( p_x \), \( p_x(y) = p(x, y), \) \( x \in X \), \( y \in Y \) given by (10).

In view of Theorem 6, to prove the second equality, we need to show that for every \( \alpha < \inf_{x \in X} \sup_{\psi \in \Psi_{\text{lsc}}} \mathcal{L}(x, \psi) \) there exist \( \psi_1, \psi_2 \in \Psi_{\text{lsc}} \) and \( \varphi_1 \in \text{supp} \mathcal{L}(\cdot, \varphi_1) \) and \( \varphi_2 \in \text{supp} \mathcal{L}(\cdot, \varphi_2) \) such that
\[
[\varphi_1 < \alpha] \cap [\varphi_2 < \alpha] = \emptyset.
\]

Let \( \tilde{\psi} \in \Psi_{\text{lsc}}, \tilde{\psi}(y) = -a\|y\|^2 + c \) where \( a > 0 \) and \( c \in \mathbb{R} \). Let \( x \in A_{y_0} \). By (32),
\[
\mathcal{L}(x, \tilde{\psi}) = c - \sup_{y \in Y} \{-a\|y\|^2 + c - p(x, y)\} = -\sup_{y \in Y} \{-a\|y\|^2 - f(x)\} = f(x) - \sup_{y \in Y} \{-a\|y\|^2\} = f(x) \geq \inf(P) = \beta.
\]

If \( x \notin A(y) \), then \( \mathcal{L}(x, \tilde{\psi}) = +\infty \). Hence, for all \( x \in X \)
\[
\mathcal{L}(x, \tilde{\psi}) \geq f(x) \geq \inf(P) = \beta.
\]

The function \( \varphi_1 \equiv \beta \) belongs to the support set of \( \mathcal{L}(\cdot, \tilde{\psi}) \). Let \( \varphi_2 \) be any function from the set \( \text{supp} \mathcal{L}(\cdot, \psi_2) \), for \( \psi_2 \in \Psi_{\text{lsc}} \), of the form \( \psi_2(y) = -a_2\|y\|^2 + c_2 \), where \( a_2 > 0 \) and \( c_2 \in \mathbb{R} \), then \( \varphi_1 \) and \( \varphi_2 \) have the intersection property at the level \( \beta \), hence \( \varphi_1 \) and \( \varphi_2 \) have the intersection property at every level \( \alpha < \beta \).

Let \( g_i : X \to \mathbb{R}, i = 1, \ldots, m \) be given functions. Let \( A : \mathbb{R}^m \rightrightarrows X \) be given as
\[
A(y) := \{ x \in X : g_i(x) \leq y, i = 1, \ldots, m \}, \quad (42)
\]
\( y_0 = 0, A := A(0) \). Consider the problem
\[
\text{Min } f(x), \quad x \in A. \quad (\text{AP})
\]

Problem (AP) can be equivalently rewritten as
\[
\text{Min}_{x \in X} \tilde{f}(x) \quad (\text{AP1})
\]
where \( \tilde{f}(x) := f(x) + \text{ind } A(x) \) and \( \text{ind } A(\cdot) \) is the indicator function of the set \( A \).
According to (40), the perturbation function \( p : X \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) takes the form (see e.g. [24]) as

\[
p(x, y) = \begin{cases} 
    f(x), & x \in A(y) \\
    +\infty, & x \notin A(y)
\end{cases}
\]  

(43)

where \( A(y) \) are given by (42). Let \( \psi(y) = -a\|y\|^2 + \langle v, y \rangle + c \). According to (32), the Lagrangian for problem \((AP)\), \( L : X \times \Psi_{lsc} \to \mathbb{R} \), with \( y_0 = 0 \), is of the form

\[
L(x, \psi) = -p^*_\psi(x) = -\sup_{y \in Y} \{-a\|y\|^2 + \langle v, y \rangle - p(x, y)\} = -\sup_{y \in Y, \ y \geq g(x)} \{-a\|y\|^2 + \langle v, y \rangle - f(x)\} = f(x) + \inf_{y \in Y, \ y \geq g(x)} \{a\|y\|^2 - \langle v, y \rangle\}.
\]  

It is easy to check that

\[
\arg \inf_{y \in Y, \ y \geq g(x)} \{a\|y\|^2 - \langle v, y \rangle\} = (y_i)_{i=1}^m = (\max\{g_i(x), \frac{v_i}{2a}\})_{i=1}^m
\]

where \( v = (v_1, ..., v_m) \) and hence

\[
L(x, \psi) = f(x) + \sum_{i=1}^m [-v_i \max\{g_i(x), \frac{v_i}{2a}\} + a(\max\{g_i(x), \frac{v_i}{2a}\})^2].
\]  

(44)

Similar Lagrangian is defined in [2] and [24]. As observed above, we can write \( L(x, \psi) = L(x, a, v) \), for \((a, v) \in \mathbb{R}_+ \times \mathbb{R}^m\).

For any \( x \in X \), if \( \max\{g_i(x), \frac{v_i}{2a}\} = \frac{v_i}{2a} \) for \( i = 1, ..., m \), then

\[
\sup_{(a, v) \in \mathbb{R}_+ \times \mathbb{R}^m} L(x, a, v) = f(x).
\]

Moreover,

\[
\sup_{(a, v) \in \mathbb{R}_+ \times \mathbb{R}^m} L(x, a, v) = \begin{cases} 
    f(x) & \text{whenever } g_i(x) \leq 0 \text{ & } v_i \geq 0 \ \forall \ i = 1, ..., m \\
    +\infty & \text{otherwise}
\end{cases}
\]

(45)

Let \( \beta = \inf_{x \in X, \psi \in \Psi_{lsc}} L(x, \psi) \) be finite number. By [24] we have the following equality

\[
\inf(P) = \inf_{x \in X, \psi \in \Psi_{lsc}} L(x, \psi).
\]

**Corollary 3** Consider the problem \((AP)\) with the perturbation function given by (43). Assume that the functions \( f, g_i : X \to \mathbb{R} \) are \( \Phi_{lsc} \)-convex on \( X \). Then

\[
\inf_{x \in X} \tilde{f}(x) = \inf_{x \in X} \sup_{(a, v) \in \mathbb{R}_+ \times \mathbb{R}^m} L(x, a, v) = \sup_{(a, v) \in \mathbb{R}_+ \times \mathbb{R}^m} \inf_{x \in X} L(x, a, v),
\]

with the Lagrangian \( L \) given by (44).
Proof By (45),
\[
\sup_{(a,v)\in \mathbb{R}_+ \times \mathbb{R}^m} \mathcal{L}(x, a, v) = \sup_{(a,v)\in \mathbb{R}_+ \times \mathbb{R}^m} \mathcal{L}(x, a, v).
\]
In view of Theorem 9 we need to show that the function \(p_x\) is \(\Psi_{lsc}\) on \(Y\) and the Lagrangian \(\mathcal{L}(\cdot, a, v)\) given by (44) is \(\Phi_{lsc}\)-convex for all \((a, v) \in \mathbb{R}_+ \times \mathbb{R}^m\).

To show that the function \(p_x: \mathbb{R}^m \to \bar{\mathbb{R}}\) is \(\Psi_{lsc}\) on \(\mathbb{R}^m\) observe that for any \(y = (y_i)_{i=1}^m \in \mathbb{R}^m\) and \(x \in X\)
\[
P_x(y) = f(x) + \text{ind}_B(y),
\]
where for any fixed \(x \in X\) we put \(B = \{ y \in \mathbb{R}^m \mid g(x) \leq y \}\), where \(g(x) = (g_i(x))_{i=1}^m\). Since \(B\) is a convex set the function \(p_x\) is convex for any \(x \in X\).

It is enough to observe that the function \(\max\{h(x), \text{const}\}\) is \(\Phi_{lsc}\)-convex whenever \(h\) is \(\Phi_{lsc}\). By taking \(u := -v\), the formula (44) can be equivalently rewritten as
\[
\mathcal{L}(x, a, u) = f(x) + \sum_{i=1}^m [u_i \max\{g_i(x), -\frac{u_i}{2a}\} + a(\max\{g_i(x), -\frac{u_i}{2a}\})^2]
\]
which coincides with the formula 1.3 of [24] (see also Example 2b of [2]).

In view of Corollary 3 zero duality gap holds for the Lagrangian of (24) for \(\Phi_{lsc}\)-convex problem (P), where the function \(f, g_i, i = 1, \ldots, m\) are \(\Phi_{lsc}\)-convex. By the proof of Corollary 3 the perturbation function \(p(x, \cdot) = p_x : \mathbb{R}^m \to \mathbb{R}\) defined by (43) is convex. Hence, by Theorem 8 we obtain the following fact.

Corollary 4 Let \(X\) be a Hilbert space and \(Y = \mathbb{R}^m\).

If the optimal value function \(V(y)\) of the problem \((P)\) is proper, \(lsc\), and paracconvex on \(Y\) and \(y_0 \in \text{int dom}(V)\), then
\[
\inf_{x \in X} f(x) = \sup_{(a,v)\in \mathbb{R}_+ \times \mathbb{R}^*} \inf_{x \in X} \mathcal{L}(x, a, v) = \inf_{x \in X} \mathcal{L}(x, a, v),
\]
i.e. \(\text{val}(L_P) = \text{val}(L_D)\) and the solution set of the dual problem \((L_D)\) is nonempty and coincides with \(\partial_{lsc} V(y_0)\).

Proof Follows directly from Corollary 3 and Theorem 8.

7 Conclusions

In Theorem 4 and Theorem 6 we provide sufficient and necessary conditions for zero duality gap for pairs of dual optimization problems involving \(\Phi\)-convex and \(\Phi_{lsc}\)-convex functions. In particular, our results apply to optimization problems where the considered Lagrangian, and the function \(p(\cdot, \cdot)\) are paracconvex, or prox-bounded, or DC functions.

Let us observe that Theorem 2, provides considerable flexibility in choosing Lagrange function \(\mathcal{L}\). Sufficient and necessary conditions of Theorem 4 and Theorem 6 are based on the intersection property (Definition 4), which, in contrast to many existing in the literature conditions, is of purely algebraic character.
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