The short-distance first-order correlation function of the interacting one-dimensional Bose gas

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Abstract. We derive exact closed-form expressions for the first few terms of the short-distance Taylor expansion of the one-body correlation function of the Lieb–Liniger gas.

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1. Introduction

Even though the correlation functions for the Lieb–Liniger gas of δ-interacting one-dimensional bosons [1] have been the object of intensive research in the integrable systems community since the late 1970s [2], the full closed-form expressions are known only in the Tonks–Girardeau limit of infinitely strong interactions [3]. While the scaling properties of the long-range asymptotics of...
the correlation functions can be derived from Haldane’s theory of quantum liquids [4], conformal field theory [5], and the quantum inverse scattering method [2, 6], virtually nothing is known about short-range one-body correlations at finite coupling strength [7]. One of the goals of this paper is to extend the existing knowledge in this direction.

2. System of interest

Consider a one-dimensional gas of $N$ $\delta$-interacting bosons confined in a length $L$ box with periodic boundary conditions. The Hamiltonian of the system reads

$$\hat{H} = \frac{-\hbar^2}{2m} \sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g_{1D} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \delta(z_i - z_j)$$

$$= \int_{-L/2}^{+L/2} \frac{\hbar^2}{2m} \partial_z \hat{\Psi}_1^\dagger \partial_z \hat{\Psi}_1 + \frac{g_{1D}}{2} \hat{\Psi}_1^\dagger \hat{\Psi}_1^\dagger \hat{\Psi}_1 \hat{\Psi}_1,$$

where $m$ is the atomic mass, and $g_{1D}$ is the one-dimensional coupling constant, whose expression for real atomic traps is given in [8]. This Hamiltonian can be diagonalized via the Bethe ansatz [1].

At zero temperature, the energy of the system is given through

$$E / N = \frac{\hbar^2}{2m} n^2 e(\gamma),$$

where the dimensionless parameter $\gamma = 2/n|a_{1D}|$ is inversely proportional to the one-dimensional gas parameter $n|a_{1D}|$, $n$ is the one-dimensional number density of particles, $a_{1D} = -2\hbar^2/m g_{1D}$ is the one-dimensional scattering length introduced in [8], and the function $e(\gamma)$ is given by the solution of the Lieb–Liniger system of equations [1]; it is tabulated in [9]. Note the asymptotic behaviour of $e(\gamma)$ (first computed in [1]):

$$e(\gamma) \xrightarrow{\gamma \to 0} \gamma; \quad e(\gamma) \xrightarrow{\gamma \to \infty} \frac{1}{3} \pi^2 \left( \frac{\gamma}{\gamma + 2} \right)^2,$$

where $\gamma \to 0$ corresponds to the mean-field or Thomas–Fermi regime, whereas $\gamma \to \infty$ corresponds to the Tonks–Girardeau regime.

3. High-$p$ momentum distribution

Our first object of interest is the high-$p$ asymptotics of the one-body momentum distribution in the ground state. To evaluate it, we need two mathematical facts, (a) and (b):

(a) The presence of the delta-function interactions in the Hamiltonian (1) implies that its eigenfunctions undergo, at the point of contact of any two particles $i$ and $j$, a kink in the derivative proportional to the value of the eigenfunction at this point [10]:

$$\Psi(z_1, \ldots, z_i, \ldots, z_j, \ldots, z_N) = \Psi(z_1, \ldots, Z_{ji}, \ldots, Z_{ji}, \ldots, z_N)\{1 - |z_{ji}|/a_{1D} + \varepsilon(|z_{ji}|; \{Z_{ji}\})\},$$

$$\varepsilon(|z_{ji}|; \{Z_{ji}\}) = \mathcal{O}(|z_{ji}|^2),$$

where $Z_{ji} = (z_i + z_j)/2$ and $z_{ji} = z_j - z_i$ are the centre-of-mass and relative coordinates of the $ij$ pair of particles, respectively, and $\{Z_{ji}\} = \{Z_{ji}, z_1, \ldots, Z_{i-1}, z_{i+1}, \ldots, Z_{j-1}, z_{j+1}, \ldots, z_N\}$ denotes a set consisting of the centre-of-mass coordinate of the $i$th and $j$th particles and the coordinates of all the other particles.
(b) Imagine that a periodic function \( f(z) \), defined on the interval \([-L/2, +L/2]\), has a singularity of the form \( f(z) = |z - z_0|^{\alpha} F(z) \), where \( F(z) \) is a regular function, \( \alpha > -1 \), and \( \alpha \neq 0, 2, 4, \ldots \) Then the leading term in the asymptotics of the Fourier transform of \( f \) reads \[ \int_{-L/2}^{+L/2} dz \, e^{-ikz} f(z) \equiv 2 \cos \left( \frac{\pi}{2}(\alpha + 1) \right) \Gamma(\alpha + 1) e^{-ikz_0} F(z_0) \frac{1}{|k|^\alpha + \cdots} + O \left( \frac{1}{|k|^\alpha + 2} \right) . \] where \( k = (2\pi/L)s \) and \( s \) is an integer. For multiple singular points of the same order, the full asymptotics is the sum of the corresponding partial asymptotics of the form (6).

Let us evaluate, using (5) and (6), the momentum representation of the ground state wavefunction of the Hamiltonian (1) with respect to the first particle:

\[ \Psi(p_1, z_2, \ldots, z_N) = L^{-\frac{1}{2}} \int_{-L/2}^{+L/2} dz_1 \, e^{-ip_1z_1/\hbar} \Psi(z_1, z_2, \ldots, z_N) \]

\[ \forall z_i \geq N \, L^{-\frac{3}{2}} \int_{-L/2}^{+L/2} dz_1 e^{-ip_1z_1/\hbar} \Psi(z_1 = Z_1, \ldots, z_i = Z_i, \ldots, z_N) \times \{ 1 - |z_i|/a_{1D} + \cdots \} \]

\[ \left| p_1 \right| \to \infty \sum_{i=2}^{N} \left( 2L^{-\frac{3}{2}}/a_{1D} \right) e^{-ip_1z_i/\hbar} \Psi(z_1 = z_i, \ldots, z_i, \ldots, z_N) \frac{1}{(p_1/\hbar)^2} . \]

Here \( p_1 = (2\pi \hbar/L)s \), where \( s \) is an integer.

Let us now turn to the one-body momentum distribution per se. After a lengthy but straightforward calculation it takes the form

\[ w(p) \equiv \int_{-L/2}^{+L/2} dz_2 \cdots \int_{-L/2}^{+L/2} dz_N \left| \Psi(p, z_2, \ldots, z_N) \right|^2 \left| p \right| \to \infty 4(N - 1) \rho_2(0, 0, 0, 0) \frac{1}{a_{1D}^2} (p/\hbar)^4 , \]

where \( \rho_2(z_1, z_2; z'_1, z'_2) \) is the two-body density matrix, normalized as

\[ \int_{-L/2}^{+L/2} dz_1 \int_{-L/2}^{+L/2} dz_2 \rho_2(z_1, z_2; z_1, z_2) = 1 , \]

and \( w(p) \) is the momentum distribution, normalized as

\[ \sum_{p = -\infty}^{\infty} w(2\pi \hbar s/L) = 1 . \]

The expression (8) involves the two-body density matrix whose form is unknown for a finite system. However, an elegant thermodynamic limit formula for \( \rho_2(0, 0, 0, 0) \) does exist due to Gangardt and Shlyapnikov [12], who derived it using the Hellmann–Feynman theorem [13]: \( L^2 \rho_2(0, 0, 0, 0) = \delta(\gamma) \). We are now ready to give a closed-form thermodynamic limit expression for the high-\( p \) asymptotics of the one-body momentum distribution for one-dimensional \( \delta \)-interacting bosons in a box with periodic boundary conditions:

\[ W(p) \left| p \right| \to \infty \frac{1}{\hbar n} \frac{\gamma \gamma' \zeta(4)}{2\pi} \left( \frac{\hbar n}{p} \right)^4 . \]

1 The double sum over particles involved in the computation can be split into a diagonal part (proportional to the two-body density matrix) and an off-diagonal part involving Fourier transforms of the three-body density matrix. The latter can be shown to decay faster than \( 1/p^4 \) and thus does not contribute to the final result.

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where \( W(p) = (L/2\pi\hbar)w(p) \) is normalized as \( \int_{-\infty}^{+\infty} dp \ W(p) = 1 \). Notice that this asymptotics is universally described by a \( 1/p^4 \) law for all values of the coupling strength \( \gamma \). (Note that for \( \gamma \to \infty \), this law was predicted in [14].)

4. Short-range expansion for the correlation function

Let us now return to the ground state one-body correlation function

\[
g_1(z) = \langle \hat{\Psi}^\dagger(z)\hat{\Psi}(0) \rangle, \tag{10}
\]

and in particular to its Taylor expansion around zero:

\[
g_1(z)/n = 1 + \sum_{i=1}^{\infty} c_i |nz|^{i}. \tag{11}
\]

In the limit of infinitely strong interactions \( \gamma \to \infty \), this expansion is known to all orders [3]:

\[
c_1^{\text{TG}} = 0; \quad c_2^{\text{TG}} = -\frac{\pi^2}{6}; \quad c_3^{\text{TG}} = \frac{\pi^2}{9}; \quad c_4^{\text{TG}} = \frac{\pi^4}{120}; \quad \cdots. \tag{12}
\]

Our goal now is to obtain the first few (through the order \( |z|^3 \)) coefficients of the expansion (11) for an arbitrary interaction strength \( \gamma \).

The knowledge of the momentum distribution (9) is crucial for determining the \( c_1 \)- and \( c_3 \)-coefficients. Let us look at the relation between the momentum distribution and the correlation function, where the former is simply the Fourier transform of the latter:

\[
W(p) = (2\pi\hbar)^{-1} \int_{-\infty}^{+\infty} dz \ e^{-ipz/\hbar} g_1(z). \]

Since the leading term in the asymptotics of \( W(p) \) is \( 1/p^4 \) we may conclude, using the Fourier analysis theorem (6), that the lowest \( \text{odd} \) power in the short-range expansion of the correlation function \( g_1(z) \) is \( |z|^3 \), and therefore the \( |z| \) term is absent from the expansion:

\[
c_1 = 0. \tag{13}
\]

Furthermore, the theorem (6) allows one to deduce the coefficient \( c_3 \) from the momentum distribution (9):

\[
c_3 = \frac{1}{\pi^3} \gamma^2 e'(\gamma). \tag{14}
\]

To obtain the coefficient \( c_2 \), we employ the Hellmann–Feynman theorem [13] again. Let a Hamiltonian \( \hat{H}(w) \) depend on a parameter \( w \). Let \( E(w) \) be an eigenvalue of this Hamiltonian. Then the mean value of the derivative of the Hamiltonian with respect to the parameter can be expressed through the derivative of the eigenvalue:

\[
\langle \Psi_E(w) | (d/dw) \hat{H}(w) | \Psi_E(w) \rangle = (d/dw)E(w). \]

Let us now denote the fraction \( \hbar^2/\mu \) as \( \kappa \) and differentiate the Hamiltonian (2) with respect to \( \kappa \). According to the Hellmann–Feynman theorem, we get

\[
\frac{1}{2} \int_{-L/2}^{+L/2} dz \ \frac{\partial}{\partial z} \langle \hat{\Psi}^\dagger(z)\hat{\Psi}(z') \rangle \big|_{z=z'} = dE/d\kappa. \]

Now, using

\[
\langle \hat{\Psi}^\dagger(z)\hat{\Psi}(z') \rangle = \langle \hat{\Psi}^\dagger(z-z')\hat{\Psi}(0) \rangle, \]

we obtain

\[
c_2 = -\frac{1}{2} \{ e(\gamma) - \gamma e'(\gamma) \}, \tag{15}
\]

where we have used the known expression for the energy (3).

Note that, as expected, our expressions for the coefficients \( c_{1-3} \) converge, in the limit \( \gamma \to \infty \), to the known results for the impenetrable bosons (12). This can be easily verified using the \( \gamma \to \infty \) expansion for the function \( e(\gamma) \) (4).

Expressions (13)–(15) constitute the main result obtained in our paper.
5. Summary

In this paper, we present a short-range Taylor expansion (up to the order $|z|^3$) for the zero-temperature correlation function $g_1(z)$ of a one-dimensional $\delta$-interacting Bose gas (see equations (11), (13)–(15)). As an intermediate result we compute the leading term in the high-$p$ asymptotics of the atomic momentum distribution (equation (9)).

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