ON FINDING A BURIED OBSTACLE IN A LAYERED MEDIUM VIA THE TIME DOMAIN ENCLOSURE METHOD

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(Communicated by Andreas Kirsch)

Abstract. An inverse obstacle problem for the wave equation in a two layered medium is considered. It is assumed that the unknown obstacle is penetrable and embedded in the lower half-space. The wave as a solution of the wave equation is generated by an initial data whose support is in the upper half-space and observed at the same place as the support over a finite time interval. From the observed wave an indicator function in the time domain enclosure method is constructed. It is shown that, one can find some information about the geometry of the obstacle together with the qualitative property in the asymptotic behavior of the indicator function.

1. Introduction. The problem of finding an obstacle embedded or hidden in a complicated environment by using the electromagnetic wave appears in, for example, ground penetrating or subsurface radar [4] and the through-wall imaging [2].

In this paper, we consider such type of the problems in a simplest, however, important mathematical model which employs a wave governed by a scalar wave equation in a two homogeneous layered medium over a finite time interval.

Let $\mathbb{R}^3_{\pm} = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \pm x_3 > 0 \}$. Consider $\gamma_0 \in L^\infty(\mathbb{R}^3)$ given by

$$\gamma_0(x) = \begin{cases} \gamma_+, & \text{if } x_3 > 0, \\ \gamma_-, & \text{if } x_3 < 0, \end{cases}$$

where $\gamma_{\pm}$ are positive constants.

Let $D$ be a bounded open subset of $\mathbb{R}^3_-$ with a $C^2$-boundary and satisfy $\overline{D} \subset \mathbb{R}^3_-$. Consider $\gamma \in L^\infty(\mathbb{R}^3)$ given by

$$\gamma(x) = \begin{cases} \gamma_0(x)I_3, & \text{if } x \in \mathbb{R}^3 \setminus D, \\ \gamma_0(x)I_3 + h(x), & \text{if } x \in D, \end{cases}$$

where $h = h(x), x \in D$ is a real symmetric $3 \times 3$-matrix valued function and satisfies that: all the components of $h$ are essentially bounded on $D$; there exists a positive constant $C$ such that $(\gamma_0(x)I_3 + h(x))\xi \cdot \xi \geq C|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in D$.

2010 Mathematics Subject Classification. Primary: 35J25, 35P25, 35B40, 35R30; Secondary: 74J25.

Key words and phrases. Enclosure method, inverse obstacle scattering problem, buried obstacle, wave equation, subsurface radar, ground probing radar.
Let $0 < T < \infty$. Given $f \in L^2(\mathbb{R}^3)$ let $u = u_f(x, t)$ be the weak solution of the initial value problem:

$$
\begin{cases}
(\partial_t^2 - \nabla \cdot \gamma \nabla)u = 0 & \text{in } \mathbb{R}^3 \times ]0, T[,
\\
u(x, 0) = 0 & \text{on } \mathbb{R}^3,
\\\partial_t u(x, 0) = f(x) & \text{on } \mathbb{R}^3.
\end{cases}
$$

(1)

Note that the solution class is taken from $[5]$. As is described in $[10]$, the weak solution $u$ of (1) is in $u \in L^2(0,T;H^1(\mathbb{R}^3))$ with $\partial_t u \in L^2(0,T;H^1(\mathbb{R}^3))$, $\partial_t^2 u \in L^2(0,T;H^1(\mathbb{R}^3))$, and for all $\phi \in H^1(\mathbb{R}^3)$, $u$ satisfies

$$
< \partial_t^2 u(., t), \phi > + \int_{\mathbb{R}^3} \gamma(x) \nabla_x u(x, t) \cdot \nabla_x \phi(x) dx = 0 \quad \text{a.e. } t \in (0,T)
$$

and $u(x, 0) = 0$ and $\partial_t u(x, 0) = f(x)$.

We consider the following problem:

**Problem.** Assume that $\gamma_+ \neq \gamma_-$. Fix a large $T$ (to be determined later). Assume that $\gamma_0$ is known and that both $D$ and $h$ are unknown. Let $B$ be an open ball whose closure is contained in $\mathbb{R}^3$. Fix a $f \in L^2(\mathbb{R}^3)$ with supp $f \subset B$ and satisfying that there exists a positive constant $C_0$ such that $f(x) \geq C_0$ a.e. $x \in B$ or $-f(x) \geq C_0$ a.e. $x \in \mathbb{R}^3$. Generate $u = u_f$ of the solution of (1) by the $f$. Extract information about the location and shape of $D$ from the measured data $u$ on $B$ over the time interval $[0, T]$.

It should be emphasized that the problem asks us to extract information about unknown obstacle $D$ from a single wave observed over a finite time interval at the same place where the wave is generated. There are some studies which consider the time harmonic reduced case in a two layered medium. See $[18]$ for uniqueness issue of impenetrable obstacles using infinitely many incident fields; $[17]$ a reconstruction scheme of an impenetrable obstacle using a far-field pattern corresponding to a single incident plane wave; $[6]$ study of a direct problem with an application to mine detection and propose a numerical reconstruction scheme using near field measurements corresponding to finitely many incident sources. Clearly our problem formulation is different from their one and to our best knowledge there is no result for the problem.

In $[10]$ Ikehata has considered the case when $\gamma_+ = \gamma_- (= 1)$ and the wave is observed on a closed surface $S$ over a finite time interval which encloses the obstacle. He assumed that $\gamma$ satisfies one of the following two conditions:

(A1) there exists a positive constant $C'$ such that $-h(x)\xi \cdot \xi \geq C'|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in D$;

(A2) there exists a positive constant $C'$ such that $h(x)\xi \cdot \xi \geq C'|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in D$.

In Theorem 1.2 of $[10]$ he showed that if $B$ is outside surface $S$, then one can extract the distance dist $(D, B) = \inf_{x \in D, y \in B} |x - y|$ from the observed wave and also can distinguish whether (A1) or (A2) is satisfied by using the signature of an indicator function computed from the observed wave. This is the beginning of the multidimensional version of the time domain enclosure method $[9]$ for inverse obstacle scattering in the time domain. In $[11]$ this idea has been extended to the case when the wave is observed on the same place as the support of an initial data. This is a version of the near field inverse back-scattering problem. One can easily transplant the results in $[10]$ to this case as pointed out in Subsection 1.3 of $[11]$. However,
the case when $\gamma_+ \neq \gamma_-$ is not trivial. Clearly this is a quite interesting case from practical and mathematical point of view. The unknown obstacle is embedded in the lower half-space which has a different refractive index from the upper half-space. Thus the wave generated by an initial data produces reflected and refracted waves at the interface. The produced refracted wave hits the surface of the obstacle and generates reflected and refracted waves. How can one extract information about the geometry of the obstacle from the observed wave? The aim of this paper is to extend the previous result to the case when $\gamma_+ \neq \gamma_-$ using the enclosure method in the time domain.

Now let us describe our main result. Let $\tau > 0$. Let $u$ be the solution of (1). Define
\[ w = w_f(x, \tau) = \int_0^\tau e^{-\tau t} u(x, t) \, dt, \quad x \in \mathbb{R}^3. \]

Let $v \in H^1(\mathbb{R}^3)$ be the weak solution of
\[ (\nabla \cdot \gamma_0 \nabla - \tau^2) v + f = 0 \quad \text{in} \ \mathbb{R}^3. \]

Define
\[ I_f(\tau, T) = \int_{\mathbb{R}^3} (w - v) \, dx. \]

We call the function $\tau \mapsto I_f(\tau, T)$ the indicator function. Note that this symbol follows from that of [12].

Define
\[ l(D, B) = \inf_{x \in D, y \in B} l(x, y), \]

where
\[ l(x, y) = \inf_{z' \in \mathbb{R}^3} l_{x, y}(z'), \]

and
\[ l_{x, y}(z') = \frac{1}{\sqrt{\gamma_+}} |\tilde{z}' - x| + \frac{1}{\sqrt{\gamma_-}} |\tilde{z}' - y| \quad (\tilde{z}' = (z_1, z_2, 0), \ z' = (z_1, z_2)). \]

The quantity $l(D, B)$ corresponds to the optical distance or optical path length between $B$ and $D$ in optics and it is easy to see that we have
\[ l(D, B) = l(D, p) - \frac{\eta}{\sqrt{\gamma_+}}, \]

where $p$ and $\eta$ denote the center and radius of $B$, respectively and
\[ l(D, p) = \inf_{x \in D} l(x, p). \]

Thus the unknown obstacle $D$ is contained in the set
\[ E(D; B, \gamma_+, \gamma_-) = \left\{ x \in \mathbb{R}^3_+ \mid l(x, p) > l(D, B) + \frac{\eta}{\sqrt{\gamma_+}} \right\}. \]

The following theorem is the main result of this paper.

**Theorem 1.1.** Assume that $\gamma_+ < \gamma_-$. Then, we have:
\[ \lim_{\tau \to \infty} e^{\tau T} I_f(\tau, T) = \begin{cases} 0, & \text{if} \ T < 2l(D, B), \\ \infty, & \text{if} \ T > 2l(D, B) \ and \ \gamma \ satisfies \ (A1), \\ -\infty, & \text{if} \ T > 2l(D, B) \ and \ \gamma \ satisfies \ (A2). \end{cases} \]
Moreover, if \( \gamma \) satisfies (A1) or (A2), then for all \( T > 2l(D,B) \)
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I_f(\tau,T)| = -2l(D,B).
\]

From Theorem 1.1 we see that the \( T \) in the problem should be an arbitrary number satisfying \( T > 2l(D,B) \). We think that this is optimal. From the indicator function one gets the value \( l(D,B) \) and thus the set \( E(D;B;\gamma_+;\gamma_-) \) which encloses \( D \). Moreover, one can distinguish whether unknown obstacle \( D \) satisfies (A1) or (A2) which is a qualitative property of \( D \) relative to the surrounding background medium, by checking the asymptotic behavior of the indicator function.

**Remark 1.** Intuitively, any signal emanating from \( B \) reaches \( D \). For these signals to go back to the upper side, we need to catch the refracted waves of the reflected waves by \( D \). Hence we need to take measurement in \( B \) at least till time \( 2l(D,B) \) for which the fastest signals may come back. To check whether signals are exactly coming back, we need to take account of total reflection waves. Assumption \( \gamma_+ < \gamma_- \) means that the propagation speed of waves in the upper side is slower than that of the lower side. Hence, there is no total reflected wave for the incident waves from the lower side. This is the case that we do not need to take care of it. Mathematically, this is appeared as a difficulty for obtaining asymptotics of the refracted wave. As is in (14) and (15) below, it is relatively simple since it does not contain waves for total reflection.

The proof of Theorem 1.1 employs two important facts. The first one is the following lemma.

**Lemma 1.2.** We have, as \( \tau \to \infty \)
\[
I_f(\tau,T) \geq \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma) \nabla v \cdot \nabla v dx + O(\tau^{-1} e^{-\tau T})
\]
and
\[
I_f(\tau,T) \leq \int_{\mathbb{R}^3} \gamma_0 (\gamma_0 I_3 - \gamma) \gamma^{-1/2} \nabla v \cdot \gamma^{-1/2} \nabla v dx + O(\tau^{-1} e^{-\tau T}).
\]

For the proof see Appendix. Combining (6) and (7) under the assumption (A1) or (A2), we can easily see that Theorem 1.1 can be proved if one has the following fact concerning with the asymptotic behavior of \( \nabla v \) on \( D \) as \( \tau \to \infty \).

**Theorem 1.3.** Assume that \( \partial D \) is \( C^1 \) and that \( \gamma_+ < \gamma_- \). Then, there exist positive numbers \( C \) and \( \tau_0 \) such that, for all \( \tau \geq \tau_0 \) we have
\[
C^{-1} \tau^{-4} e^{-2\tau l(D,B)} \leq \int_D |\nabla v(x)|^2 dx \leq C \tau^2 e^{-2\tau l(D,B)}.
\]

This is the second important fact. We found that the proof of Theorem 1.3 is not a simple matter and thus the remaining part of this paper is devoted to the proof. In this sense, the main contribution of this paper to the enclosure method in the time domain is the establishment of the estimate (8). Note that in [16, 3] one can find some formal asymptotic computation of the solution of (3), however, we do not know whether or not their formal theory enables us to derive estimate (8).

In [12], Ikehata considered a mathematical model of the through-wall imaging by using the enclosure method in the time domain. Originally the governing equation should be the Maxwell system, however, as a first step, it is assumed that the governing equation is given by the single wave equation \( \alpha(x) \partial_t^2 u - \Delta u = 0 \) in
$\mathbb{R}^3 \times [0, T]$ with the initial data $u(x, 0) = 0$ and $\partial_t u(x, 0) = f(x)$ in $\mathbb{R}^3$. The assumption on the function $\alpha \in L^\infty(\mathbb{R}^3)$ is that: $\alpha$ has a positive lower bound in $\mathbb{R}^3$ and takes the form

$$
\alpha(x) = \begin{cases} 
\alpha_0(x), & \text{if } x \in \mathbb{R}^3 \setminus D, \\
\alpha_0(x) + h(x), & \text{if } x \in D,
\end{cases}
$$

where the function $\alpha_0$ is essentially bounded in $\mathbb{R}^3$ with a positive lower bound; $D$ is an arbitrary bounded open set of the whole space with a Lipschitz boundary; the function $h(x)$ or $-h(x)$ has a positive essential infimum on $D$.

Remarkably enough, in [12] a higher regularity more than the essential boundedness of $\alpha_0$ is not assumed. Thus the model covers various background media such as multilayered media with complicated interfaces or unions of various domains with different refractive indexes. He showed that an indicator function computed from the wave observed on the same place as the support of an initial data yields lower and upper estimates of the distance $\text{dist}(\alpha, \partial D)$ or $	ext{dist}(\alpha, \partial B)$ has a positive essential infimum on $D$. The result is based on a system of inequalities similar to (6) and (7) in Lemma 1.2 and explicit upper and lower estimates of the solution of the equation

$$
\Delta v - \alpha_0 \tau^2 v + \alpha_0 f = 0 \quad \text{in } \mathbb{R}^3.
$$

In contrast to this result, Theorem 1.1 tells us that one can extract the exact value of the optical distance $l(D, B)$ from the asymptotic behavior of the indicator function under the assumption that the background medium consists of two isotropic homogeneous layered media. It would be possible to apply the idea of the derivation of the estimate in Theorem 1.3 to the case when $\alpha_0$ takes two different constants, $\alpha_+ \in x_3 > 0$ and $\alpha_- \in x_3 < 0$ provided $\alpha_+ > \alpha_-$. However, a typical case to be considered for the Maxwell system is: the upper layer consists of air and the lower of material, like soil, wall, etc., see [4] and [2]. In our problem setting this corresponds to the case when $\gamma_+ > \gamma_-$. Developing an analysis that covers this case together with application to the Maxwell system belongs to our next project. See also [13] for a survey on recent results for inverse obstacle scattering via the time domain enclosure method and [15] for applications to the inverse boundary value problems for the heat equation in three-dimensional space.

The outline of this paper is as follows. Let $\Phi_\tau(x, y)$ be the fundamental solution of (3), which satisfies

$$
\nabla_x \cdot (\gamma_0(x) \nabla_x \Phi_\tau(x, y)) - \tau^2 \Phi_\tau(x, y) + \delta(x - y) = 0 \quad \text{in } \mathbb{R}^3.
$$

Since the solution $v$ of (3) is written by the convolution of $f$ and $\Phi_\tau(x, y)$ as

$$
v(x) = \int_B \Phi_\tau(x, y)f(y)dy,
$$

we obtain

$$
\int_D |\nabla_x v(x)|^2 dx = \int_B dy \int_B d\xi f(y)f(\xi) \int_D \nabla_x \Phi_\tau(x, y) \cdot \nabla_x \Phi_\tau(x, \xi) dx.
$$

Thus, we need to know an asymptotic behavior of $\nabla_x \Phi_\tau(x, y)$ as $\tau \to \infty$ for $x = (x', x_3)$ with $x_3 < 0, x' \in \mathbb{R}^2$ and $y \in B$. 

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**Inverse Problems and Imaging** Volume 12, No. 5 (2018), 1173–1198
The first step for obtaining the asymptotic behavior of (9) is to show that the fundamental solution \( \Phi(x,y) \) for \( x_3 < 0 \) is given by

\[
\Phi(x,y) = \frac{\tau}{4\pi \gamma_+} \int_{\mathbb{R}^2} E_\tau^+(x,z') \frac{e^{-\tau|x-y|/\sqrt{\gamma_+}}}{|z'|^{3/2}} \, dz',
\]

which is given in Section 2. In (10), \( E_\tau^+(x,z') \) is a function given by

\[
E_\tau^+(x,z') = \frac{\tau}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\tau \xi \cdot (x-z')} \frac{1}{\gamma - \xi^2 + 1} R_-(|\xi'|) \, d\xi \quad (x_3 < 0),
\]

where \( z' = (z',0) \) \((z' \in \mathbb{R}^3)\) is the point on the transmission boundary \( \partial \mathbb{R}^3_\pm \) and \( R_-(|\xi'|) \) is a function of \( |\xi'| \) standing for the transmission coefficient given by

\[
R_-(\rho) = \frac{4\gamma_+ - 1/\gamma_- + \rho^2}{\gamma_+\sqrt{1/\gamma_- + \rho^2} + \gamma_-\sqrt{1/\gamma_- + \rho^2}} \quad (\rho \geq 0).
\]

Note that \( E_\tau^+(x,z') \) can be interpreted as the refracted part of the fundamental solution.

We put

\[
E_\tau^{+0}(x,y) = \frac{1}{4\pi \gamma_+} \frac{e^{-\tau|x-y|/\sqrt{\gamma_+}}}{|x-y|} \quad (x \neq y, \tau > 0),
\]

which is a fundamental solution for the equation corresponding to the case that there is no transmission boundary, i.e. \( \gamma_- = \gamma_+ \), and given by

\[
E_\tau^{+0}(x,y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot (x-y)} \frac{1}{\gamma_+\xi^2 + \tau^2} \, d\xi
\]

\[
= \frac{\tau}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot (x-y)} \frac{1}{\gamma_+\xi^2 + 1} \, d\xi.
\]

Thus, (10) stands for the refraction phenomena by the transmission boundary \( \partial \mathbb{R}^3_\pm \).

As in Proposition 2, for any \( N \in \mathbb{N} \), the refracted wave \( E_\tau^-(x,z') \) is of the form:

\[
E_\tau^-(x,z') = e^{-\tau|x-z'|/\sqrt{\gamma_-}} \left( \sum_{j=0}^{N-1} E_j(x-z') \left( \frac{\sqrt{\gamma_-}}{\tau|x-z'|} \right)^j + \tilde{E}_N(x,z';\tau) \right),
\]

where each \( E_j \) is a \( C^\infty \) function in \( \mathbb{R}^3 \), and \( \tilde{E}_N(x,z';\tau) \) is a continuous for \( x \in \mathbb{R}^3 \) and \( z' \in \mathbb{R}^2 \), and satisfies

\[
|\tilde{E}_N(x,z';\tau)| \leq C_N \left( \frac{\sqrt{\gamma_-}}{\tau|x-z'|} \right)^N \quad (x \in \mathbb{R}^3_-, z' \in \mathbb{R}^2, \tau > 1).
\]

For the gradient \( \nabla_x E_\tau^-(x,z') \), as is in Proposition 2 and Remark 2, we have

\[
\nabla_x E_\tau^-(x,z') = -\frac{\tau e^{-\tau|x-z'|/\sqrt{\gamma_-}}}{4\pi \gamma_-^{3/2}} \left( \sum_{j=0}^{N-1} G_j(x-z') \left( \frac{\sqrt{\gamma_-}}{\tau|x-z'|} \right)^j + \tilde{G}_N(x,z';\tau) \right),
\]

where each \( G_j \) is a \( C^\infty \) function in \( \mathbb{R}^3 \), and \( \tilde{G}_N(x,z';\tau) \) is a continuous for \( x \in \mathbb{R}^3_- \) and \( z' \in \mathbb{R}^2 \), and satisfies

\[
|\tilde{G}_N(x,z';\tau)| \leq C_N \left( \frac{\sqrt{\gamma_-}}{\tau|x-z'|} \right)^N \quad (x \in \mathbb{R}^3_-, z' \in \mathbb{R}^2, \tau > 1).
\]
From (14), (15) and (10), the original problem can be reduced to finding asymptotics of the Laplace integral of the form:

\[
I(\tau; x, y) = \int_{\mathbb{R}^2} e^{-\tau l(x, y)} a(z') dz', \quad (x \in \overline{D}, y \in \overline{B})
\]

where \( a \in \mathcal{B}^\infty(\mathbb{R}^2) \), i.e., the function \( a \) belongs to the space of all \( C^\infty \) functions in \( \mathbb{R}^2 \) of whose all derivatives \( \partial_{z'}^\alpha a \) are bounded functions in \( \mathbb{R}^2 \). For \( N \in \mathbb{N} \cup \{0\} \), we put \( \|a\|_{N, \mathcal{B}^\infty(\mathbb{R}^2)} = \max_{|\beta| \leq N} \sup_{z \in \mathbb{R}^2} |\partial^\beta_z a(z')| \).

By usual Laplace’s method, the main part of the asymptotics for (16) is given by points \( z'(x, y) \in \mathbb{R}^2 \) attaining the minimum \( l(x, y) \) of \( l_{x,y}(z') \). We can check the point \( z'(x, y) \) uniquely exists, which corresponds to Snell’s law in geometrical optics. Further, we can show that \( l_{x,y}(z') \) is a \( C^\infty \) function for \( (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \) and \( \text{Hess}(l_{x,y})(z'(x, y)) \) is positive definite, where \( \text{Hess}(l_{x,y})(z') = (\partial_{z'} \partial_{z'} l_{x,y}(z')) \) (cf. Lemma 4.1). We put \( H(x, y) = \text{Hess}(l_{x,y})(z'(x, y)) \)

\[
\Psi(z') = l_{x,y}(z') - l_{x,y}(z'(x, y)) - (H(x, y))^{-1} (z' - z'(x, y), (z' - z'(x, y)))
\]

Take \( \phi \in C_0^\infty(\mathbb{R}^2) \) with \( 0 \leq \phi \leq 1 \) and \( \phi = 1 \) near the set \( \{z'(x, y) \in \mathbb{R}^2 | x \in \overline{D}, y \in \overline{B}\} \), and divide (16) into two parts,

\[
I(\tau; x, y) = \int_{\mathbb{R}^2} e^{-\tau l_{x,y}(z')} \phi(z') a(z') dz' + \int_{\mathbb{R}^2} e^{-\tau l_{x,y}(z')}(1 - \phi(z')) a(z') dz'.
\]

Note that usual Laplace’s method (cf. Theorem 7.7.5 of Hörmander [7], which is for oscillatory integrals, however, the proof also works for this case) can be applied for the first integral of (17). For the second integral of (17), integration by parts implies that this term is negligible. Hence we obtain, for all \( \tau > 1 \)

\[
I(\tau; x, y) = -\frac{2\pi e^{-\tau l(x, y)}}{\tau \sqrt{|\det H(x, y)|}} \left( \sum_{j=0}^{N} (L_j a)(z'(x, y)) \tau^{-j} + R_{N+1}(x, y, \tau) \right),
\]

where \( L_j \) is a differential operator of order less than or equal to \( 6j \) given by

\[
(L_j a)(z') = \sum_{\ell-k+j, 2\ell \geq 3k} \frac{1}{\ell! k!} \frac{(-1)^k}{2^\ell} (\text{Hess}(z'))^{-1} \partial_{z'} \partial_{z'}^j ((\Psi(z'))^k a)(z'),
\]

and for any \( N \in \mathbb{N} \cup \{0\} \), there exists a constant \( C_N > 0 \) depending also on \( a \) such that

\[
|R_{N+1}(x, y, \tau)| \leq C_N \|a\|_{2N, \mathcal{B}^\infty(\mathbb{R}^2)} \tau^{-(N+1)} \quad (x \in \overline{D}, y \in \overline{B}, \tau \geq 1).
\]

From (18), we can obtain the asymptotic expansion of \( \nabla_x \Phi_\tau(x, y) \) of the form:

**Proposition 1.** Assume \( \gamma_+ < \gamma_- \). Then \( \nabla_x \Phi_\tau(x, y) \) \((k = 0, 1)\) have the following asymptotics:

\[
\nabla_x \Phi_\tau(x, y) = \frac{e^{-\tau l(x, y)}}{8\pi \gamma_+ \gamma_- \sqrt{|\det H(x, y)|}} \left( \frac{-\tau}{\sqrt{\gamma_-}} \right)^k \left( \sum_{j=0}^{N} \tau^{-j} \Phi_j^{(k)}(x, y) + Q_{N, \tau}^{(k)}(x, y) \right),
\]

where \( \Phi_j^{(k)}(x, y) \) \((k = 0, 1)\) are \( C^\infty \) in \( \overline{D} \times \overline{B} \), for any \( N \in \mathbb{N} \cup \{0\} \), \( Q_{N, \tau}^{(k)}(x, y) \) \((k = 0, 1)\) are continuous in \( \overline{D} \times \overline{B} \) with a constant \( C_N > 0 \) satisfying

\[
|Q_{N, \tau}^{(0)}(x, y)| + |Q_{N, \tau}^{(1)}(x, y)| \leq C_N \tau^{-(N+1)} \quad (x \in \overline{D}, y \in \overline{B}, \tau \geq 1).
\]
Moreover, $\Phi_0^{(k)}(x, y)$ $(k = 0, 1)$ are given by
\begin{equation}
\Phi_0^{(0)}(x, y) = \frac{E_0(x - z'(x, y))}{|x - z'(x, y)|}|\hat{\tau}(x, y) - y|,
\end{equation}
and
\begin{equation}
\Phi_0^{(1)}(x, y) = \Phi_0^{(0)}(x, y) \frac{x - z'(x, y)}{|x - z'(x, y)|}.
\end{equation}

Note that in (19), $E_0$ is the function appeared in (14). The form of $E_0$ is given by (47) in Section 3.

Proposition 1 is crucial to obtain Theorem 1.3. A proof of Proposition 1 is given in Section 4. In Section 5, we show Theorem 1.3 by using Proposition 1. This is the outline of this paper.

2. The refracted part of the fundamental solution. In what follows, we only treat the case $y = (y', y_3)$, $y' = (y_1, y_2)$, $y_3 > 0$ for large $\tau > 0$. Note that the fundamental solution $\Phi_\tau(x, y)$ is given by
\begin{equation}
\Phi_\tau(x, y) = \left\{ \begin{array}{ll}
v_0(x, y; \tau) + v_+(x, y; \tau) & (x_3 > 0), \\
v_-(x, y; \tau) & (x_3 < 0), \end{array} \right.
\end{equation}
where $v_0 = v_0(x, y; \tau)$ is the solution of
\begin{equation}
\nabla_x \cdot (\gamma + \nabla_x v_0) - \tau^2 v_0 + \delta(x - y) = 0 \quad \text{in } \mathbb{R}^3,
\end{equation}
and $v_\pm = v_\pm(x, y; \tau)$ satisfy
\begin{equation}
\left\{ \begin{array}{ll}
\nabla_x \cdot (\gamma \pm \nabla_x v_\pm) - \tau^2 v_\pm &= 0 \\
v_0 + v_+ = v_-, & \quad \gamma_+ \partial_{x_3} v_0 + \gamma_+ \partial_{x_3} v_+ = \gamma_- \partial_{x_3} v_- \\
(\pm x_3 > 0), & \quad (x_3 = 0).
\end{array} \right.
\end{equation}
In what follows, we also write $x = (x', x_3)$, $x' = (x_1, x_2)$.

For $v_0$, Fourier transform implies
\begin{equation}
v_0(x, y; \tau) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} d\xi' e^{i\xi' \cdot (x - y')} \int_{\mathbb{R}} e^{i\eta \cdot (x_3 - y_3)} \
\end{equation}
\begin{equation}
\cdot e^{-\eta \cdot (x_3 - y_3)} \
\end{equation}
\begin{equation}
\cdot \frac{\tau^2 (\xi')^2 + \tau^2 + \gamma_+ \eta^2}{\gamma_+ (\xi')^2 + \tau^2 + \gamma_+ \eta^2} d\eta.
\end{equation}
We put
\begin{equation}
C_+^{(\xi')} = \left( \frac{\tau^2 (\xi')^2 + \tau^2 + \gamma_+ \eta^2}{\gamma_+ (\xi')^2 + \tau^2 + \gamma_+ \eta^2} \right)^{1/2} > 0.
\end{equation}
Since
\begin{equation}
\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta \cdot (x_3 - y_3)} d\eta = \frac{1}{2\gamma_+ C_+^{(\xi')} e^{-C_+^{(\xi')}|x_3 - y_3|}},
\end{equation}
we obtain
\begin{equation}
v_0(x, y; \tau) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} d\xi' e^{i\xi' \cdot (x' - y')} \int_{\mathbb{R}} e^{i\eta \cdot (x_3 - y_3)} \
\cdot \frac{1}{2\gamma_+ C_+^{(\xi')} e^{-C_+^{(\xi')}|x_3 - y_3|}},
\end{equation}
which is the representation by the partial Fourier transform
\begin{equation}
\hat{v}_0(x_3, y; \xi', \tau) = \int_{\mathbb{R}^2} e^{-i\xi' \cdot x'} v_0(x', x_3, y; \tau) dx' = \frac{e^{-i\xi' \cdot y'}}{2\gamma_+ C_+^{(\xi')} e^{-C_+^{(\xi')}|x_3 - y_3|}}.
\end{equation}
of $v_0(x, y; \tau)$ for the tangential direction $x' \in \mathbb{R}^2$.

To obtain $v_\pm$, we take the partial Fourier transform
\begin{equation}
\hat{v}_\pm(x_3, y; \xi', \tau) = \int_{\mathbb{R}^2} e^{-i\xi' \cdot x} v_\pm(x', x_3, y; \tau) dx',
\end{equation}
which satisfy the partial Fourier transform of (21), that is,

\begin{equation}
\begin{cases}
- (\gamma_+ (\xi')^2 + \tau^2) \hat{v}_+ + \gamma_+ \partial_{\xi'} \hat{v}_+ = 0 \\
\hat{v}_0 + \hat{v}_+ = \hat{v}_-, \quad \gamma_+ (\partial_{\xi'} \hat{v}_0 + \partial_{\xi'} \hat{v}_+) = \gamma_- \partial_{\xi'} \hat{v}_- \\
(\pm x_3 > 0), \\
(\hat{v}_+ (x_3, y; \xi', \tau), \hat{v}_0(0; \xi', y, \tau)), \\
(\hat{v}_- (x_3, y; \xi', \tau), \hat{v}_0(0; \xi', y, \tau)), \\
R_+ (\xi') = 2\gamma_+ C_+ (\xi') \frac{\gamma_+ C_+ (\xi') - \gamma_- C_- (\xi')}{R_+ C_+ (\xi')} + R_- (\xi') = \frac{4\gamma_+ C_+ (\xi') C_+ (\xi')}{} + \gamma_- C_- (\xi').
\end{cases}
\end{equation}

Since \( v_+ \) are bounded, from (23), the solutions of (24) are given by

\begin{equation}
\hat{v}_\pm (x_3, y; \xi', \tau) = e^{i\tau C_+ (\xi') x_3} R_\pm (\xi') \hat{v}_0(0; \xi', y, \tau),
\end{equation}

where

\begin{equation}
R_+ (\xi') = 2\gamma_+ C_+ (\xi') \frac{\gamma_+ C_+ (\xi') - \gamma_- C_- (\xi')}{\gamma_+ C_+ (\xi')} + R_- (\xi') = \frac{4\gamma_+ C_+ (\xi') C_+ (\xi')}{} + \gamma_- C_- (\xi').
\end{equation}

We concentrate on for \( v_- \). From (25), (22) and (23), it follows that

\begin{equation}
\hat{v}_-(x_3, y; \xi', \tau) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi_3 x_3} R_-(\xi') \hat{v}_0(0; \xi', y, \tau) (x_3 < 0),
\end{equation}

which yields

\begin{equation}
v_-(x, y; \tau) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d\xi d\zeta d\tau \int_{\mathbb{R}^3} d\xi' d\zeta' d\tau' \frac{e^{i\tau (\xi' - \xi) + \xi_3 x_3} R_-(\xi') e^{i\tau (\xi' - \xi) - \xi_3 y_3}}{\gamma_+ \xi^2 + \gamma_- \xi'^2 + \tau^2}.
\end{equation}

Since \( \Phi_\tau(x, y) = v_-(x, y; \tau) \) for \( x_3 < 0 \), and \( R_-(\tau \xi') = \tau R_+(\xi') \), we obtain

\begin{equation}
\Phi_\tau(x, y) = \frac{\tau^3}{(2\pi)^3} \int_{\mathbb{R}^3} d\xi d\zeta d\tau R_+(\xi') e^{i\tau (\xi' - \xi) - \xi_3 y_3} \frac{R_-(\xi')}{\gamma_+ \xi^2 + \gamma_- \xi'^2 + \tau^2}.
\end{equation}

for \( x_3 < 0 \). Since (12) implies \( R_+(\xi') = R_-(|\xi'|) \), noting (13), (11) and the above formula of \( \Phi_\tau(x, y) \) for \( x_3 < 0 \), we obtain (10).

We can also obtain the formula of \( \Phi_\tau(x, y) \) for \( x_3 > 0 \), which is for the reflected phenomena. In this case, \( \Phi_\tau(x, y) = v_0(x, y; \tau) + v_-(x, y; \tau) \), which yields

\begin{equation}
\Phi_\tau(x, y) = \frac{\tau}{(2\pi)^3} \int_{\mathbb{R}^3} d\xi e^{i\tau \xi (x-y)} R_+(\xi') e^{i\tau (\xi' - \xi) - \xi_3 y_3} \frac{R_-(\xi')}{\gamma_+ \xi^2 + \gamma_- \xi'^2 + \tau^2},
\end{equation}

for \( x_3 > 0 \). Similarly in this paper, we do not use this formula.

3. Asymptotics of the refracted waves. In this section, we show the asymptotics (14) and (15) for the refracted wave defined by (11). We put \( R(\rho) = R_-(\rho/\sqrt{\gamma_-}) \) and \( \tilde{\tau} = \tau/\sqrt{\gamma_-} \). From (11), it follows that

\begin{equation}
\hat{E}_- (x, z') = \frac{\tau}{(2\pi)^3} \lim_{\varepsilon \to 0} J_\varepsilon (x - z') (x_3 < 0),
\end{equation}

where

\begin{equation}
J_\varepsilon (x - z') = \int_{\mathbb{R}^3} e^{-|\xi'|^2 \varepsilon} e^{i\xi (x - z')} \frac{1}{\varepsilon^2 + 1} R(|\xi'|) d\xi.
\end{equation}
Note that from (22), it follows that
\[
J_{z}(x - z') = \int_{\mathbb{R}^{2}} e^{-i|\xi'|^{2}} e^{i\tilde{x}' \cdot \xi'} e^{i\tilde{z}' \cdot \xi'} \int_{\mathbb{R}} e^{i\tilde{x} \cdot \xi} \frac{1}{\xi^{2} + 1} d\xi_{3} R(|\xi'|) d\xi'
\]
which yields
\[
J_{z}(x - z') = \pi \int_{\mathbb{R}^{2}} e^{-i|\xi'|^{2}} e^{i\tilde{x}' \cdot \xi'} R(|\xi'|) e^{-\tilde{x}' \cdot |x_{3}|/\sqrt{1 + |\xi'|^{2}}} \frac{d\xi'}{\sqrt{1 + |\xi'|^{2}}}
\]
by rotating the coordinate. Thus, we obtain
\[
E_{\tau}^{-}(x, z') = \frac{\tau}{2(2\pi)^{1/2}} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}} e^{-i\tilde{x}' \cdot \xi'} R(|\xi'|) e^{-\tilde{x}' \cdot |x_{3}|/\sqrt{1 + |\xi'|^{2}}} \frac{d\xi_{1}}{\sqrt{1 + |\xi'|^{2}}}
\]
We change the variable \(\zeta_{1} = \sqrt{1 + \zeta_{2}^{2}}\), and have
\[
E_{\tau}^{-}(x, z') = \frac{\tau}{2(2\pi)^{1/2}} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}} e^{i\tilde{x}' \cdot \xi'} R(|\xi'|) e^{-\tilde{x}' \cdot |x_{3}|/\sqrt{1 + |\xi'|^{2}}} \frac{d\xi_{1}}{\sqrt{1 + |\xi'|^{2}}}
\]
which yields
\[
\nabla_{x} E_{\tau}^{-}(x, z') = \frac{\tau^{2}}{2(2\pi)^{1/2}} \int_{\mathbb{R}^{2}} \int_{\mathbb{R}} e^{i\tilde{x}' \cdot \xi'} R(|\xi'|) e^{-\tilde{x}' \cdot |x_{3}|/\sqrt{1 + |\xi'|^{2}}} \frac{d\xi_{1}}{\sqrt{1 + |\xi'|^{2}}}
\]
For \(x \in \mathbb{R}^{3}, z' \in \mathbb{R}^{2}\), we put
\[
I_{\tau, k}(x - z', \zeta_{2}) = \int_{\mathbb{R}} e^{-\tilde{x}' \cdot |x_{3}|/\sqrt{1 + \zeta_{2}^{2}}} e^{-i|x' - z'| \cdot |x_{3}|/\sqrt{1 + \zeta_{2}^{2}}} Q_{k}(\zeta_{1}, \zeta_{2}) \frac{d\zeta_{1}}{\sqrt{1 + \zeta_{1}^{2}}},
\]
where
\[
Q_{0}(\zeta_{1}, \zeta_{2}) = R\left(\sqrt{\zeta_{1}^{2} + \zeta_{2}^{2} + \zeta_{2}^{2}}\right) \quad Q_{0}(\zeta_{1}, \zeta_{2}) = \sqrt{1 + \zeta_{2}^{2}} Q_{0}(\zeta_{1}, \zeta_{2}),
\]
\[
Q_{1}(\zeta_{1}, \zeta_{2}) = i \zeta_{1} R(\zeta_{1}, \zeta_{2}), \quad Q_{2}(\zeta_{1}, \zeta_{2}) = -\sqrt{1 + \zeta_{1}^{2}} \tilde{Q}_{0}(\zeta_{1}, \zeta_{2}).
\]
To obtain the asymptotics of \(E_{\tau}^{-}(x, z')\) and \(\nabla_{x} E_{\tau}^{-}(x, z')\), we need to study the asymptotics of (28). We use the steepest decent method, which is similar to getting the distribution kernel for the usual wave equations in the two dimensional half-space by Hankel functions (cf. [1], p. 286 for example).

We take \(\theta\) satisfying
\[
\sin \theta = \frac{|x' - z'|}{|x - \tilde{z}|}, \quad \cos \theta = \frac{|x_{3}|}{|x - \tilde{z}|} \quad (0 \leq \theta \leq \pi/2),
\]
and put \( r = \tilde{r}|x - \tilde{z}|\sqrt{1 + \zeta_0^2} \) and

\[
\lambda = \lambda(\zeta_1, x, z') = -i \sin \theta \zeta_1 + \cos \theta \sqrt{1 + \zeta_1^2}.
\]

Then, (28) is written by

\[
I_{\tau,k}(x - \tilde{z}', \zeta_2) = \int_{\mathbb{R}} e^{-r\lambda} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}.
\]

From (31), it follows that \( \zeta_1 = i\lambda \sin \theta \pm \sqrt{\lambda^2 - 1} \cos \theta \). Hence, putting \( \lambda = \sqrt{1 + \rho^2} \) for \( \lambda \geq 1 \), we have \( \sqrt{\lambda^2 - 1} = \sqrt{\rho^2} = |\rho| \), which yields

\[
(33) \quad \zeta_1 = \zeta_1(\rho, x, z') = i\sqrt{1 + \rho^2} \sin \theta + \rho \cos \theta \quad (\rho \in \mathbb{R}, x \in \mathbb{R}^3, z' \in \mathbb{R}^2).
\]

We denote by \( \Gamma \) the curve defined by (33). This is the steepest decent curve of integral (32). The contour of (32) should be changed for \( \Gamma \).

We take any \( \varepsilon_0 \) with \( 0 < \varepsilon_0 < \pi/2 \). Then, for \( \zeta_1 \in \mathbb{C} \) with \( |\arg \zeta_1| < \pi/2 - \varepsilon_0 \) and \( |\zeta_1| \geq (\sin \varepsilon_0)^{-1/2} \),

\[
|\arg(1 + \zeta_1^2)| \leq 2|\arg \zeta_1| + |\arg(1 + \zeta_1^{-2})| \leq \pi - 2\varepsilon_0 + \varepsilon_0 = \pi - \varepsilon_0,
\]

since \( |\arg(1 + \zeta_1^{-2})| \leq \varepsilon_0 \) for \( |\zeta_1| \geq (\sin \varepsilon_0)^{-1/2} \). Hence, we have

\[
(34) \quad \sqrt{1 + \zeta_1^2} = |1 + \zeta_1^2|^{1/2} e^{i \arg(1 + \zeta_1^2) / 2} = \zeta_1(1 + O(\zeta_1^{-2}))
\]

(\( |\zeta_1| \to \infty \) uniformly for \( |\arg \zeta_1| \leq \pi/2 - \varepsilon_0 \))

since \( \sqrt{1 + \zeta_1^2} > 0 \) for \( \zeta_1 \in \mathbb{R} \). Similarly, we also obtain

\[
(35) \quad \sqrt{1 + \zeta_1^2} = |1 + \zeta_1^2|^{1/2} e^{i (2\arg \zeta_1 + 2\pi / 2) / 2} e^{i \arg(1 + \zeta_1^{-2}) / 2} = e^{\pi i} \zeta_1(1 + O(\zeta_1^{-2}))
\]

(\( |\zeta_1| \to \infty \) uniformly for \( |\arg \zeta_1 + \pi| \leq \pi/2 - \varepsilon_0 \))

since in this case, it follows that

\[
|\arg(1 + \zeta_1^2) + 2\pi| \leq 2|\arg \zeta_1 + \pi| + |\arg(1 + \zeta_1^{-2})| \leq \pi - 2\varepsilon_0 + \varepsilon_0 = \pi - \varepsilon_0.
\]

From these asymptotics, it follows that \( \lambda \) defined by (31) satisfies

\[
(36) \quad \Re \lambda = \Im \zeta_1 \sin \theta + \Re \zeta_1 \cos \theta + O(|\zeta_1|^{-1})
\]

(\( |\zeta_1| \to \infty \) uniformly for \( |\arg \zeta_1| \leq \pi/2 - \varepsilon_0 \)),

\[
(37) \quad \Re \lambda = \Im \zeta_1 \sin \theta - \Re \zeta_1 \cos \theta + O(|\zeta_1|^{-1})
\]

(\( |\zeta_1| \to \infty \) uniformly for \( |\arg \zeta_1 + \pi| \leq \pi/2 - \varepsilon_0 \)).

Noting \( 1 + \zeta_1^2 = (\sqrt{1 + \rho^2} \cos \theta + i\rho \sin \theta)^2 \), and (34) and (35), we also have

\[
(38) \quad \sqrt{1 + \zeta_1^2} = \sqrt{1 + \rho^2} \cos \theta + i\rho \sin \theta
\]

for \( \zeta_1 = \zeta_1(\rho, x, z') \).

From (12), it follows that

\[
R(\rho) = R - (\rho / \sqrt{\gamma_-}) = \frac{4\sqrt{\gamma_-} - \sqrt{a_0^2 + \rho^2} \sqrt{1 + \rho^2}}{\sqrt{a_0^2 + \rho^2} + a_0^2 \sqrt{1 + \rho^2}},
\]

where

\[
a_0 = \sqrt{\frac{\gamma_-}{\gamma_+}}.
\]
Since \(1 + (\sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_1^2 \zeta_2^2})^2 = (1 + \zeta_1^2)(1 + \zeta_2^2)\), we have
\[
R \left( \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_1^2 \zeta_2^2} \right) = \frac{4 \sqrt{\gamma - 1 + \zeta_2^2} \sqrt{1 + \zeta_1^2} P(\zeta_1, \zeta_2)}{P(\zeta_1, \zeta_2) + a_0^2 \sqrt{1 + \zeta_1^2}},
\]
where
\[
P(\zeta_1, \zeta_2) = \sqrt{\frac{a_0^2 - 1}{1 + \zeta_2^2} + 1 + \zeta_2^2}.
\]
In what follows, we assume \(\gamma_+ < \gamma_-\), being the case that there is no total reflected wave for incident waves coming from the lower side (cf. Remark 1). In this case, \(a_0 > 1\). Hence, \(P(\zeta_1, \zeta_2)\) and \(\sqrt{1 + \zeta_1^2}\) are holomorphic for \(\zeta_1 \in \mathbb{C} \setminus \{(\pm i \infty, -i] \cup [i, i \infty)\}\). From this, (36) and (37), we can change the contour of (32) for \(\Gamma\), which yields
\[
I_{\tau,k}(x - \bar{z}', \zeta_2) = \int_{\Gamma} e^{-r \lambda} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}
\]
We can express this integral by using the parametrization of \(\Gamma\) given by (33). In this parametrization, \(\lambda = \sqrt{1 + \rho^2}\), and (33) and (38) implies
\[
\frac{d\zeta_1}{d\rho} = \frac{\sqrt{1 + \zeta_1^2}}{\sqrt{1 + \rho^2}}
\]
and we obtain
\[
I_{\tau,k}(x - \bar{z}', \zeta_2) = \int_{\mathbb{R}} e^{-r|\bar{z}' - z'| \sqrt{1 + \zeta_2^2} \sqrt{1 + \rho^2}} Q_k(\zeta_1(\rho, x, z'), \zeta_2) \frac{d\rho}{\sqrt{1 + \rho^2}}.
\]
Note that \(Q_k(\zeta_1(\rho, x, z'), \zeta_2)\) are \(C^\infty\) function of \(\rho \in \mathbb{R}\) since \(a_0 > 1\) implies \(\Gamma \subset \mathbb{C} \setminus \{(\pm i \infty, -i] \cup [i, i \infty)\}\). For simplicity, we write \(\sigma_1 = \rho, \sigma_2 = \zeta_2\) and \(\sigma = (\sigma_1, \sigma_2)\), and put
\[
f(\sigma) = \sqrt{1 + \sigma_1^2} \sqrt{1 + \sigma_2^2},
\]
\[
F_k(\sigma, x, z') = Q_k(\zeta_1(\sigma_1, x, z'), \sigma_2) \frac{1}{\sqrt{1 + \sigma_1^2}},
\]
\[
\bar{F}_+(\sigma, x, z') = \tilde{Q}_0(\zeta_1(\sigma_1, x, z'), \sigma_2) \frac{1}{\sqrt{1 + \sigma_1^2}}
\]
and
\[
F_-(\sigma, x, z') = i \sigma_1 \tilde{Q}_0(\zeta_1(\sigma_1, x, z'), \sigma_2) \frac{1}{\sqrt{1 + \sigma_1^2}}.
\]
Notice that (29), (33) and (38) imply that
\[
\begin{align*}
F_1(\sigma, x, z') &= F_-(\sigma, x, z') \cos \theta - F_+(\sigma, x, z') \sin \theta, \\
F_2(\sigma, x, z') &= -F_+(\sigma, x, z') \cos \theta - F_-(\sigma, x, z') \sin \theta.
\end{align*}
\]
From (26), (27) and (41), it follows that
\[
E^{-}_\tau(x, z') = \frac{\tau}{2(2\pi)^2 \gamma^3/2} \int_{\mathbb{R}^2} e^{-r|\bar{z}' - z'| f(\sigma)} F_0(\sigma, x, z') d\sigma,
\]
\[
\nabla_x E^{-}_\tau(x, z') = \frac{\tau^2}{4(2\pi)^2 \gamma^2} \int_{\mathbb{R}^2} e^{-r|\bar{z}' - z'| f(\sigma)} F_1(\sigma, x, z') d\sigma \frac{x' - z'}{|x' - z'|},
\]
\[
\partial_{x_3} E^{-}_\tau(x, z') = \frac{\tau^2}{4(2\pi)^2 \gamma^2} \int_{\mathbb{R}^2} e^{-r|\bar{z}' - z'| f(\sigma)} F_2(\sigma, x, z') d\sigma \frac{x_3}{|x_3|}.
\]
From (43)-(46), the problem is reduced to finding the asymptotics of
\[ \int_{\mathbb{R}^2} e^{-\frac{1}{2} |x-z'|^2} f(x) K_k(x, z') d\sigma \quad (k = 0, 1, 2 \text{ and } \pm) \]
as \( \tilde{\tau} \to \infty \). For treating these integrals, we need to assume \( \gamma_+ < \gamma_- \), which implies that the amplitude functions \( F_k \) in (44)-(46) are smooth. This allows us to use the Laplace methods to give the asymptotic expansions for \( E_k^\pm(x, z') \) and its gradient.

**Proposition 2.** Assume \( \gamma_+ < \gamma_- \). Then, it follows that
\[ E_k^\pm(x, z') = \frac{e^{-\sqrt{-\tau}}}{4\pi \gamma_{\pm} |x-z'|} \left( \sum_{j=0}^{N-1} E_j(x-z') \left( \frac{\sqrt{-\tau}}{\tau|x-z'|} \right)^j + \tilde{E}_N(x, z'; \tau) \right), \]
and for \( k = 1, 2, 3 \),
\[ \partial_{x_k} E_k^\pm(x, z') = -\frac{\tau e^{-\sqrt{-\tau}|x-z'|}}{4\pi \gamma_{\pm}^{3/2} |x-z'|} \left( \sum_{j=0}^{N-1} G_{k,j}(x-z') \left( \frac{\sqrt{-\tau}}{\tau|x-z'|} \right)^j + \tilde{G}_{k,N}(x, z'; \tau) \right), \]
where \( E_j(x-z'), G_{k,j}(x-z') \) \( (k = 1, 2, 3 \text{ and } j = 0, 1, 2, \ldots) \) are \( C^\infty \) functions for \( x \in \mathbb{R}^3 \) and \( z' \in \mathbb{R}^3 \). Here, the remainder terms \( \tilde{E}_N(x, z'; \tau) \) and \( \tilde{G}_{k,N}(x, z'; \tau) \) \( (k = 1, 2, 3) \) are estimated by
\[ |\tilde{E}_N(x, z'; \tau)| + \sum_{k=1}^{3} |\tilde{G}_{k,N}(x, z'; \tau)| \leq C_N \left( \frac{\sqrt{-\tau}}{|x-z'|} \right)^N (x \in \mathbb{R}^3, z' \in \mathbb{R}^2) \]
for some constant \( C_N > 0 \) depending only on \( N \in \mathbb{N} \). In particular, we have
\[ E_0(x-z') = \frac{4\sqrt{\gamma_-}|x_3|}{|x-z'|^2 (\sqrt{\gamma_-}|x-z'|^2 - |x-z'|^2 + a_0^2|x_3|)}, \]
where \( a_0 > 1 \) is given by (39), and
\[ \begin{align*}
G_{k,0}(x-z') &= E_0(x-z') \frac{x_k - z_k}{|x-z'|} \quad (k = 1, 2) \\
G_{3,0}(x-z') &= E_0(x-z') \frac{x_3}{|x-z'|}.
\end{align*} \]

**Proof.** Note that \( f \) in the integrals of (44)-(46) has only one critical point \( \sigma = 0 \); and \( \text{Hess} f(0) = I \), where \( \text{Hess} f(0) \) is the Hessian of \( f \) at \( \sigma = 0 \) and \( I \) is the \( 2 \times 2 \) unit matrix. Since \( (1 + \sigma_1^2)(1 + \sigma_2^2) \geq 1 + |\sigma|^2 \geq (1 + |\sigma|/3)^2 \) for \( |\sigma| \geq 3/4 \), we have
\[ f(\sigma) \geq 1 + \frac{|\sigma|}{3} \geq \frac{9}{8} + \frac{|\sigma|}{6} \quad (|\sigma| \geq 3/4). \]

Further, there exists a constant \( C > 0 \) such that
\[ |F_k(\sigma, x, z')| \leq C(1 + |\sigma|)^3 \quad (\sigma \in \mathbb{R}^2, x \in \mathbb{R}^3, z' \in \mathbb{R}^2). \]
Take \( \psi \in C_0^\infty(\mathbb{R}^3) \) with \( 0 \leq \psi \leq 1, \psi(\sigma) = 1 \) for \( |\sigma| \leq 1 \), and \( \psi(\sigma) = 0 \) for \( |\sigma| \geq 3/2 \). From (48) and (49), it follows that
\[ \left| \int_{\mathbb{R}^2} e^{-\frac{1}{2} |x-z'|^2} f(x) F_k(\sigma, x, z')(1-\psi(\sigma)) d\sigma \right| \leq C e^{-3|\sigma||x-z'|/8} \int_{\mathbb{R}^2} (1 + |\sigma|)^3 e^{-\left(\frac{1}{2} |x-z'|^2 + |x-z'|^2 / 8\right)} |\sigma| d\sigma \]
\[ \leq C_N e^{-\frac{1}{2} |x-z'|^2} \left( \frac{1}{|x-z'|^2} \right)^N, \]
and usual Laplace’s method as is stated in Introduction implies
\[
\left| e^{i|\tilde{\tau} - \tilde{\tau}'|} \int_{\mathbb{R}^2} e^{-i|\tilde{\tau} - \tilde{\tau}'|f(\sigma)} F_k(\sigma, x, z') \psi(\sigma) d\sigma - \frac{2\pi}{\tilde{\tau} - \tilde{\tau}'} \sum_{j=0}^{N-1} F_{k,j}(x - \tilde{\tau}')(\tilde{\tau} - \tilde{\tau}')^{-j} \right| \leq C_N \| F_k(\cdot, x, z') \|_{2N,B^0(\mathbb{R}^2)} (\tilde{\tau} - \tilde{\tau}')^{-N-1},
\]

(51)

where \(B_2(0) = \{ \sigma \in \mathbb{R}^2 | |\sigma| < 2 \} \) and
\[
\| F_k(\cdot, x, z') \|_{2N,B^0(\mathbb{R}^2)} = \max_{|\beta| \leq 2N} \sup_{\sigma \in B_2(0)} |\partial^\beta F_k(\sigma, x, z')|.
\]

In (51), \( F_{k,j}(x - \tilde{\tau}') \) is given by
\[
(52) \quad F_{k,j}(x - \tilde{\tau}') = \sum_{l-p=j,2l \geq 3p} \frac{1}{l!p!} (-1)^p \frac{(-1)^p}{2^l} \Delta^l((\Psi^p F_k(x, z'))(0) \quad (j = 0, 1, \ldots),
\]

where \( \Psi(\sigma) = f(\sigma) - f(0) - |\sigma|^2/2 \).

Since \((\Psi(\sigma))^p = O(|\sigma|^{4p})\), for
\[
\Psi(\sigma) = \frac{(\sigma_1 \sigma_2)^2}{4} - \frac{\sigma_1^4 + \sigma_2^4}{8} + O(|\sigma|^6) \quad (|\sigma| \to 0),
\]

we have \( \Delta^l((\Psi(\sigma))^p)(0) = 0 \) for \( 2l < 4p \), which yields that \( p \leq j \) holds for \( l - p = j \) with \( \Delta^l((\Psi(\sigma))^p)(0) \neq 0 \). Thus, the summation of (52) is not taken from all pairs \((p, l)\) in (52). It consists only from pairs \((p, l)\) with \( 0 \leq p \leq j \) and \( l = p + j \), which implies
\[
(53) \quad F_{k,j}(x - \tilde{\tau}') = \sum_{p=0}^{j} \frac{1}{(p+j)!p!} (-1)^p \frac{(-1)^p}{2^{p+j}} \Delta^{p+j}((\Psi^p F_k(x, z'))(0) \quad (j = 0, 1, \ldots).
\]

From (30) and (43), it follows that
\[
\begin{align*}
F_{1,j}(x - \tilde{\tau}') &= F_{-j}(x - \tilde{\tau}') \frac{|x_3|}{|x - \tilde{\tau}'|} - F_{+j}(x - \tilde{\tau}') \frac{|x' - \tilde{\tau}'|}{|x - \tilde{\tau}'|}, \quad \\
F_{2,j}(x - \tilde{\tau}') &= -F_{+j}(x - \tilde{\tau}') \frac{|x_3|}{|x - \tilde{\tau}'|} - F_{-j}(x - \tilde{\tau}') \frac{|x' - \tilde{\tau}'|}{|x - \tilde{\tau}'|}.
\end{align*}
\]

Since (53) and (33) imply that for \( k = 0, 1, 2 \),
\[
F_{k,0}(x - \tilde{\tau}') = Q_k(0, x, z'), 0) = Q_k(i \sin \theta, 0).
\]

From (29) and (40), it follows that which yields
\[
(55) \quad F_{0,0}(x - \tilde{\tau}') = \frac{4 \sqrt{\gamma_0} \cos \theta \sqrt{a_0^2 - \sin^2 \theta}}{\sqrt{a_0^2 - \sin^2 \theta + a_0^2 \cos \theta}},
\]

(56) \quad F_{1,0}(x - \tilde{\tau}') = -F_{0,0}(x - \tilde{\tau}') \sin \theta, \quad F_{2,0}(x - \tilde{\tau}') = -F_{0,0}(x - \tilde{\tau}') \cos \theta,
\]

where the relations between \( \theta \) and \( x - \tilde{\tau}' \) is given by (30).

Combining (44)-(46) with (50) and (51), we obtain the asymptotics stated in Proposition 2 except the properties of the coefficients functions \( E_j \) and \( G_{k,j} \). Notice that \( E_j \) and \( G_{k,j} \) are given by
\[
(57) \quad E_j(x - \tilde{\tau}') = F_{0,j}(x - \tilde{\tau}')
\]

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From (55), (56) and (30), we also have the forms of $E_0$, $G_{1,0}$, $G_{2,0}$ and $G_{3,0}$. The remaining parts are to prove smoothness of the coefficients.

For $N \in \mathbb{N} \cup \{0\}$, we denote by $\mathcal{P}_N$ the set consisting of functions $p$ of $\theta$ of the form:

$$
p(\theta) = \sum_{j+2k \leq 2N} a_{jk}(\sin^2 \theta)(\sin^2 \theta - \cos^2 \theta)^j(\cos \theta \sin \theta)^{2k},$$

where $a_{jk}(t)$ are $C^\infty$ for $|t| < 1 + \delta$ with some positive $\delta > 0$. Note that any $p \in \mathcal{P}_N$ is regarded as a $C^\infty$ function in $x - x' \in \mathbb{R}^3$ by relations (30).

First, we show smoothness of $E_j$. Since $\Psi$ is an even function with respect to each of $\sigma_1$ and $\sigma_2$, (i.e. $\Psi(-\sigma_1, \sigma_2) = \Psi(\sigma_1)$ and $\Psi(\sigma_1, -\sigma_2) = \Psi(\sigma_1)$), $Q_0(\zeta_1, \zeta_2)$ is a function for $\zeta_1$ and $\zeta_2$, for $0 \leq p \leq j$, $(\Psi(\sigma))^p F_0(\sigma, x, z')$ is of the form:

$$(59) \quad (\Psi(\sigma))^p F_0(\sigma, x, z') = A(\sigma, \sigma_2^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\sigma_1) \cos \theta \sin \theta),$$

where $A = A(\sigma, \eta)$ for $\sigma, \eta \in \mathbb{R}^2$ is a $C^\infty$ function for $|\sigma| < \delta, |\eta| < 1 + \delta$ with a sufficiently small $\delta > 0$, and $A$ is an even function with respect to each of $\sigma_1$ and $\sigma_2$, (i.e. $A(-\sigma_1, -\sigma_2) = A(\sigma_1, -\sigma_2)$). $\varphi$ is a $C^\infty$ and an odd function in a neighborhood of $\sigma_1 = 0$. From (53) and (57), it suffices to show for any $|\alpha| \leq 2j$,

$$(60) \quad \partial_{\alpha}^2 ((\Psi(\sigma))^p F_0(\sigma, x, z')) |_{\sigma = 0} \in \mathcal{P}_{2j}.$$

Since $A$ is an even function with respect to $\sigma_1$ and $\sigma_2$, it follows that

$$(61) \quad \partial_{\alpha}^2 ((\Psi(\sigma))^p F_0(\sigma, x, z')) |_{\sigma = 0} = \sum_{l=0}^{\alpha_1} \frac{(2\alpha_1)!}{(2l)![(2(\alpha_1 - l))]} A_{l,\alpha_1,\alpha_2}(\theta),$$

where $A_{l,\alpha_1,\alpha_2}$ is defined by

$$
A_{l,\alpha_1,\alpha_2}(\theta) = \partial_{\rho}^2 \left( (\partial_{\rho}^{2\gamma_1} \partial_{\rho}^{2\gamma_2} \partial_{\rho}^{2\alpha_1 - 2l} \partial_{\rho}^{2\alpha_2} A)(0, \rho^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\rho) \cos \theta \sin \theta) \right) |_{\rho = 0}
$$

for $0 \leq l \leq \alpha_1$ and $0 \leq \alpha_2$. Since $\varphi(\rho)$ is odd, $\partial_{\rho}^2 (\rho^{2\gamma_1} (\varphi(\rho))^{2\gamma_2}) |_{\rho = 0} = 0$ for any odd $\gamma_2$, Taylor’s theorem implies $A_{l,\alpha_1,\alpha_2} \in \mathcal{P}_{2l}$ since $A_{l,\alpha_1,\alpha_2}$ can be written as

$$A_{l,\alpha_1,\alpha_2}(\theta) = \sum_{\gamma_1 + 2\gamma_2 \leq 2l} \frac{(\partial_{\rho}^{2\gamma_1} \partial_{\rho}^{2\gamma_2} \partial_{\rho}^{2\alpha_1 - 2l} \partial_{\rho}^{2\alpha_2} A)(0, \sin^2 \theta, 0)}{\gamma_1! (2\gamma_2)!} \partial_{\rho}^{2l} \left( (\rho^{2\gamma_1} (\varphi(\rho))^{2\gamma_2}) \right) |_{\rho = 0}$$

$\times (\sin^2 \theta - \cos^2 \theta)^{\gamma_1} (\cos \theta \sin \theta)^{2\gamma_2}.$

Combining $A_{l,\alpha_1,\alpha_2} \in \mathcal{P}_{2l}$ shown in the above, (59) with (61), we obtain (60), which yields smoothness of $E_j(x - x')$ in $x \in \mathbb{R}^3$ and $z \in \mathbb{R}^2$ for (53).

Next, we show smoothness of $G_{k,j}$. From (54) and (58), $G_{k,j}$ are given by

$$G_{k,j}(x - z') = - \left( F_{-j}(x - z') \frac{|x_3|}{|x' - z'|} - F_{+j}(x - z') \right) \frac{x_k - z_k}{|x' - z'|} \quad (k = 1, 2),$$

$$G_{3,j}(x - z') = - \left( -F_{+j}(x - z') - F_{-j}(x - z') \right) \frac{|x' - z'|}{|x_3|} \frac{x_3}{|x' - z'|}.$$
Since $\tilde{Q}_0(\zeta_1, \zeta_2)$ is also a function for $\zeta_1^2$ and $\zeta_2^2$, $(\Psi(\sigma))^p F_+(\sigma, z')$ is the same form as (59), which yields that $F_+(x - z')$ is $C^\infty$ for $x \in \mathbb{R}_-$ and $z \in \mathbb{R}^2$. Thus, it suffices to show that $F_-(x - z') \frac{|x|}{x - z'}$ is $C^\infty$ for $x \in \mathbb{R}_-$ and $z \in \mathbb{R}^2$.

From (42), by using a function $A_-(\sigma, \eta)$ with the same property as for $A$ in (59), we can write $(\Psi(\sigma))^p F_-(\sigma, z', x)$ as

\[
(\Psi(\sigma))^p F_-(\sigma, z', x) = \sigma_1 A_-(\sigma, \sigma_1^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\sigma_1) \cos \theta \sin \theta).
\]

Since there is $\sigma_1$ in (62) for $|\alpha| \leq j$, we have

\[
\partial_\sigma^{2\alpha} (\sigma_1 A_-(\sigma, \sigma_1^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\sigma_1) \cos \theta \sin \theta))|\sigma=0 = \sum_{i=1}^{\alpha_1} \frac{(2\alpha_1)!}{(2l-1)!(2\alpha_1-2l)!} \times \partial_\rho^{2l-1} \left\{ (\partial^{2\alpha_1-2l}_\sigma \partial^{2\alpha_2}_\rho A_-)(0, 0, \rho^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\rho) \cos \theta \sin \theta) \right\}|\rho=0.
\]

Moreover, Taylor’s theorem implies

\[
\partial_\rho^{2l-1} \left\{ (\partial^{2\alpha_1-2l}_\sigma \partial^{2\alpha_2}_\rho A_-)(0, \rho^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\rho) \cos \theta \sin \theta) \right\}|\rho=0 = \sum_{\gamma_1+2\gamma_2+1 \leq 2l-1} \frac{\gamma_1!(2\gamma_2+1)!}{\gamma_1!(2\gamma_2+1)!} \times \partial_\rho^{2l-1} \left[ (\rho^{2\gamma_1}(\varphi(\rho))^{2\gamma_2+1}) \right]|_{\rho=0} \left( \sin^2 \theta - \cos^2 \theta \right) \gamma_1 (\cos \theta \sin \theta)^{2\gamma_2+1}
\]

since $\partial_\rho^{2l-1} \left[ (\rho^{2\gamma_1}(\varphi(\rho))^{2\gamma_2}) \right]|_{\rho=0} = 0$ for even $\gamma_2$. From these equalities, we obtain

\[
\partial_\sigma^{2\alpha} (\sigma_1 A_-(\sigma, \sigma_1^2(\sin^2 \theta - \cos^2 \theta) + \sin^2 \theta, \varphi(\sigma_1) \cos \theta \sin \theta))|_{\sigma=0} = Y(\theta) \cos \theta \sin \theta
\]

with some $Y \in \mathbb{P}_2$. From this property and (53), $F_-(x - z')$ is of the form:

\[
F_-(x - z') = \tilde{F}_-(x - z') \frac{|x|}{|x - z'|^2}
\]

with some $C^\infty$ function $\tilde{F}_-(x - z')$ for $x \in \mathbb{R}_-$ and $z \in \mathbb{R}^2$. Hence, we have

\[
\begin{cases}
F_-(x - z') \frac{|x|}{|x - z'|} = \tilde{F}_-(x - z') \frac{|x|}{|x - z'|^2}, \\
F_-(x - z') \frac{|x' - z'|}{|x'|} = \tilde{F}_-(x - z') \frac{|x' - z'|}{|x - z'|^2},
\end{cases}
\]

which complete the proof of Proposition 2.

**Remark 2.** We put $G_j(x - z') = t(G_{1,j}(x - z'), G_{2,j}(x - z'), G_{3,j}(x - z'))$. From the proof of Proposition 2, the vector valued functions $G_j(x - z')$ are of the form

\[
G_j(x - z') = F_+(x - z') \frac{x - z'}{|x - z'|} - \tilde{F}_-(x - z') \frac{|x|}{|x - z'|^2}
\]

Using these $G_j$ and putting

\[
\tilde{G}_N(x, z'; \tau) = t(\tilde{G}_{1,N}(x, z'; \tau), \tilde{G}_{2,N}(x, z'; \tau), \tilde{G}_{3,N}(x, z'; \tau)),
\]

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we obtain (15). Further,
\[ G_0(x - z') = E_0(x - z') \frac{x - z'}{|x - z'|}. \]

4. Snell’s law and the asymptotics of \( \Phi_x \) and \( \nabla_x \Phi_x \). In this section, the Laplace integral (16) is treated. First, we check the properties of function (5), which describes Snell’s law.

Lemma 4.1. The function \( l_{x,y} \) in \( \mathbb{R}^2 \) defined by (5) satisfies the following properties:

1. Fix \( x, y \) with \( x_3 < 0 \) and \( y_3 > 0 \). For the function \( l(x, y) \) defined by (4), there exists the unique point \( z' \in \mathbb{R}^2 \) satisfying \( l(x, y) = l_{x,y}(z') \). This point \( z' \) is denoted by \( z'(x, y) \). This point \( z'(x, y) \) is on the line segment \( x'y' \).

2. There exists a constant \( C > 0 \) such that
\[ \sum_{i,j=1}^{2} \frac{\partial^2 l_{x,y}}{\partial x_i \partial y_j}(z'(x, y)) \xi_i \xi_j \geq C |\xi'|^2 \quad (\xi' \in \mathbb{R}^2, x \in \overline{D}, y \in \overline{B}). \]

3. The point \( z'(x, y) \) is \( C^\infty \) for \( x, y \in \mathbb{R}^3 \) with \( x_3 < 0 \) and \( y_3 > 0 \).

Proof. In the beginning, we show
\[ l(x, y) = \inf \{|l_{x,y}(z')| : z' \in \mathbb{R}^2, |z' - x'| \leq |z' - y'|, |z' - y'| \leq |z' - y'| \} \]
To obtain (63), it suffices to show \( l_{x,y}(z') > l_{x,y}(y') \) for \( z' \in \mathbb{R}^2 \) satisfying \( |z' - x'| > |z' - y'| \), and \( l_{x,y}(z') > l_{x,y}(x') \) for \( |z' - y'| > |z' - y'| \). Suppose \( |z' - x'| > |z' - y'| \), then it follows that
\[ |x - z'| = \sqrt{|x' - z'|^2 + x_3^2} > \sqrt{|x' - y'|^2 + x_3^2} = |x - y'|. \]
Since \( |y - z'| = \sqrt{|y' - z'|^2 + y_3^2} \geq |y_3| = |y - y'| \), we have
\[ l_{x,y}(z') = \frac{1}{\gamma_+} |z' - x| + \frac{1}{\gamma_-} |z' - y| > \frac{1}{\gamma_+} |y' - x| + \frac{1}{\gamma_-} |y' - y| = l_{x,y}(y'). \]

For \( |z' - y'| > |z' - y'| \), \( l_{x,y}(z') > l_{x,y}(x') \) is shown similarly, which yields (63).

From (63) and \( l_{x,y} \in C^\infty(\mathbb{R}^2) \), there exists a point \( z' \in \mathbb{R}^2 \) attaining the minimum \( l(x, y) \) of \( l_{x,y} \). Since this point \( z' \) satisfies \( \partial_x l_{x,y}(z') = 0 \),
\[ \sin \theta_- \frac{z' - x}{\sqrt{\gamma_-} |z' - x|} + \sin \theta_+ \frac{z' - y'}{\sqrt{\gamma_+} |z' - y'|} = 0, \]
where \( 0 \leq \theta \leq \pi/2 \) is taken by
\[ \sin \theta_- = \frac{|z' - x|}{|z' - x|}, \quad \sin \theta_+ = \frac{|z' - y'|}{|z' - y'|}. \]
From (64), it follows that \( z' \) is on the segment \( x'y' \), and satisfies Snell’s law,
\[ \sin \theta_- = \sin \theta_+. \]

We show that this point \( z' \) is unique. If \( x' = y' \), then \( z' = x' = y' \), which yields \( \theta = 0 \). Thus, \( z' \) is uniquely determined. If \( x' \neq y' \), this point is expressed by \( z' = x' + t_0(y' - x') \) for some \( 0 < t_0 < 1 \). We define \( \varphi(t) \) by \( \varphi(t) = l_{x,y}(x' + t(y' - x')). \)
Note that \( t_0 \) satisfies \( \varphi'(t_0) = 0 \). Since \( \varphi'(0) < 0 \), \( \varphi'(1) > 0 \) and \( \varphi''(t) > 0 \) for \( 0 \leq t \leq 1 \), there exists only one \( 0 < t_0 < 1 \) with \( \varphi'(t_0) = 0 \), which yields uniqueness of \( z' \). Thus, we obtain (1) of Lemma 4.1.
Combining the above estimate with (67), we obtain
\[
\begin{align*}
\sum_{i,j=1}^2 \frac{\partial^2 l_{x,y}}{\partial z_i \partial z_j} (z') &= \frac{1}{\sqrt{\gamma - |z' - x|}} \left\{ \delta_{ij} \left( \left| \frac{z' - x'}{|z' - x|} \right| - \frac{(z_j - x_j)(z_i - x_i)}{|z' - x'|^2} \right) \right. \\
&\quad + \frac{1}{\sqrt{\gamma + |z' - y|}} \left\{ \delta_{ij} \left( \left| \frac{z' - y'}{|z' - y|} \right| - \frac{(z_j - y_j)(z_i - y_i)}{|z' - y'|^2} \right) \right. \\
&\quad \left. \right\}.
\end{align*}
\]

We put \(e = (z' - x')/|x' - z'|\). Then, (64) and (65) implies \(|y' - z'| = -e\) since \(z' = z(x, y)\) is in the line segment \(x'y'\), which yields
\[
\begin{align*}
\sum_{i,j=1}^2 \frac{\partial^2 l_{x,y}}{\partial z_i \partial z_j} (z'(x, y)) \xi_i \xi_j &= \frac{1}{\sqrt{\gamma - |z'|}} \left( |\xi'|^2 - (\sin \theta_-)^2 (e \cdot \xi')^2 \right) \\
&\quad + \frac{1}{\sqrt{\gamma + |z'|}} \left( |\xi'|^2 - (\sin \theta_+)^2 (e \cdot \xi')^2 \right).
\end{align*}
\]

Hence, we have
\[
\begin{align*}
\sum_{i,j=1}^2 \frac{\partial^2 l_{x,y}}{\partial z_i \partial z_j} (z'(x, y)) \xi_i \xi_j &= \frac{1 - (\sin \theta_-)^2}{\sqrt{\gamma - |x_3|}} |\xi'|^2 + \frac{1 - (\sin \theta_+)^2}{\sqrt{\gamma + |y_3|}} |\xi'|^2.
\end{align*}
\]

Since \(\overline{D} \times \overline{B} \subset \mathbb{R}^3_+ \times \mathbb{R}^3_+\) is bounded, there exist constants \(L > 0\) and \(A > 0\) such that
\[
|x' - y'| \leq L, A \leq |x_3| \leq A^{-1}, A \leq |y_3| \leq A^{-1} \quad (x \in \overline{D}, y \in \overline{B}).
\]

Note that \(t \mapsto \frac{L}{\sqrt{\gamma + At}}\) is monotone increasing for \(t \geq 0\), for \(z' = z'(x, y)\), it follows that
\[
0 \leq \sin \theta_\pm \leq \frac{L}{\sqrt{L^2 + A^2}}, \quad (x \in \overline{D}, y \in \overline{B}).
\]

Combining the above estimate with (67), we obtain
\[
\begin{align*}
\sum_{i,j=1}^2 \frac{\partial^2 l_{x,y}}{\partial z_i \partial z_j} (z'(x, y)) \xi_i \xi_j &\geq \left( \frac{1}{\sqrt{\gamma_-}} + \frac{1}{\sqrt{\gamma_+}} \right) \frac{A}{L^2 + A^2} |\xi'|^2 \quad (x \in \overline{D}, y \in \overline{B}, \xi' \in \mathbb{R}^2),
\end{align*}
\]

which yields (2).

Last, we show (3). We put \(F(x, y, z') = l_{x,y}(z')\), which is \(C^\infty\) for \((x, y, z') \in \mathbb{R}^3_+ \times \mathbb{R}^3_+\), and \(F(x, y, z'(x, y)) = 0\). Further, from (2), we obtain \(\det \left( \frac{\partial^2 F}{\partial z_i \partial z_j}(z'(x, y)) \right) \neq 0\). Hence implicit function theorem yields smoothness of \(z'(x, y)\). \(\square\)

Note that (66) implies
\[
\begin{align*}
\sum_{i,j=1}^2 \frac{\partial^2 l_{x,y}}{\partial z_i \partial z_j} (z'(x, y)) \eta_i \xi_j &= \frac{1}{\sqrt{\gamma - |z'|}} \left( \eta' \cdot \xi' - (\sin \theta_-)^2 (e \cdot \eta')(e \cdot \xi') \right) \\
&\quad + \frac{1}{\sqrt{\gamma + |z'|}} \left( \eta' \cdot \xi' - (\sin \theta_+)^2 (e \cdot \eta')(e \cdot \xi') \right)
\end{align*}
\]
for $\xi' = (\xi_1, \xi_2)$ and $\eta' = (\eta_1, \eta_2)$. Hence, the eigenvalues and eigenvectors of the Hessian $H(x, y) = \text{Hess}(l_{x,y})(z'(x, y))$ are given by

$$
\begin{align*}
H(x, y) z' - x' &= \left( \frac{x^3_3}{\sqrt{\gamma_+ - z' - x^3}} + \frac{y^3_3}{\sqrt{\gamma_+ - z' - y^3}} \right) z' - x', \\
H(x, y)e' &= \left( \frac{1}{\sqrt{\gamma_+ - z' - x^3}} + \frac{1}{\sqrt{\gamma_+ - z' - y^3}} \right)e',
\end{align*}
$$

where $e'$ is a unit vector with $e' \cdot (x' - y')/|x' - y'| = 0$. Thus, we also obtain

$$
\det H(x, y) = \left( \frac{1}{\sqrt{\gamma_+ - z' - x^3}} + \frac{1}{\sqrt{\gamma_+ - z' - y^3}} \right)^2.
$$

Now we are in the position to show Proposition 1. Here, we need to assume $\gamma_+ < \gamma_-$. 

**Proof of Proposition 1.** From (10) and (14), we have

$$
\Phi_+(x, y) = \sum_{j=0}^{N-1} \frac{\gamma_j/2}{(4\pi)^2 \gamma_+} \int_{\mathbb{R}^2} \frac{e^{-\tau l_{x,y}(z')}}{|x - z'||z' - y|} f_j(z', x, y) dz' + \frac{\tau}{(4\pi)^2 \gamma_+} \int_{\mathbb{R}^2} \frac{e^{-\tau l_{x,y}(z')}}{|x - z'||z' - y|} E_N(x, z'; \tau) dz',
$$

where

$$
f_j(z', x, y) = \frac{\gamma_j/2}{|x - z'||z' - y|} E_j(x - z') \ (j = 0, 1, \ldots).
$$

From Proposition 2, $E_j(x - z')$ are $C^\infty$ for $x \in \mathbb{R}^3_+$ and $z' \in \mathbb{R}^2$, which yields $f_j \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^3_+ \times \mathbb{R}^3_+)$. Hence, we can apply (18) for each Laplace integral with $f_j$, and obtain the asymptotics. For the remainder term of (68),

$$
\left| \int_{\mathbb{R}^2} \frac{e^{-\tau l_{x,y}(z')}}{|x - z'||z' - y|} E_N(x, z'; \tau) dz' \right| \leq C_N \frac{\gamma_+^{N/2} e^{-\tau l(x, y)}}{\tau^N} \int_{\mathbb{R}^2} \frac{dz'}{|x - z'||z' - y|}
$$

$$
\leq C_N \frac{\gamma_+^{N/2} e^{-\tau l(x, y)}}{\tau^N} \ (x \in \mathcal{D}, y \in \mathcal{B}, \tau \geq 1).
$$

Thus, we obtain the asymptotics for $\Phi_+(x, y)$ in Proposition 1 uniformly in $x \in \mathcal{D}, y \in \mathcal{B}$, where $\Phi_j^{(0)}(x, y)$ is given by $\Phi_j^{(0)}(x, y) = \sum_{p=0}^{j} (L_p f_j - p(x, y))(z'(x, y))$. Hence, we also have $\Phi_0^{(0)}(x, y) = f_0(z'(x', y); x, y)$, which yields (19).

For $\nabla_x \Phi_+(x, y)$, differentiating (10), and using (15), we obtain

$$
\nabla_x \Phi_+(x, y) = \frac{-\tau^2}{(4\pi)^2 \gamma_+} \sum_{j=0}^{N-1} \frac{\gamma_j/2}{|x - z'||z' - y|} \int_{\mathbb{R}^2} \frac{e^{-\tau l_{x,y}(z')}}{|x - z'||z' - y|} g_j(z', x, y) dz' + \frac{\tau^2}{(4\pi)^2 \gamma_+} \int_{\mathbb{R}^2} \frac{e^{-\tau l_{x,y}(z')}}{|x - z'||z' - y|} \tilde{G}_N(x, z'; \tau) dz',
$$

where

$$
g_j(z', x, y) = \frac{\gamma_j/2}{|x - z'||z' - y|} G_j(x - z') \ (j = 0, 1, \ldots).
$$

From Proposition 2, $G_j(x - z')$ are $C^\infty$ for $x \in \mathbb{R}^3_+$ and $z' \in \mathbb{R}^2$, which yields $g_j \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^3_+ \times \mathbb{R}^3_+)$. Hence, as for $\Phi_+(x, y)$, we obtain the asymptotics.
for $\nabla x \Phi_r(x, y)$ described in Proposition 1. In this case, $\Phi_j^{(1)}(x, y)$ is given by $\Phi_j^{(1)}(x, y) = \sum_{p=0}^j (L_p g_j - p; x, y))(z'(x, y))$. From this and Remark 2, we obtain

$$\Phi_0^{(1)}(x, y) = g_0(z'(x, y); x, y) = E_0(x - \tilde{z}'), \quad \frac{x - \tilde{z}'}{|x - \tilde{z}'|^2 |y - \tilde{z}'|},$$

which yields (20). This completes the proof of Proposition 1. \hfill \Box

5. Proof of Theorem 1.3. We put $L(x, y, \xi) = l(x, y) + l(x, \xi)$,

$$\tilde{K}(x, y, \xi, \tau) = \tau (\Phi_0^{(1)}(x, y) - Q_0^{(1)}(x, \xi)),$$

$$J(x, y, \xi) = -\frac{\sqrt{\det H(x, y) \det H(x, \xi)}}{\sqrt{\gamma + 2 \xi - \gamma - 3}}.$$

and $h_1(x, y, \xi) = \Phi_0^{(0)}(x, y) \Phi_0^{(0)}(x, \xi)$. From Proposition 1 for $N = 0$, it follows that, for all $x \in \mathbb{R}^3, y, \xi \in \mathbb{R}^3$,

$$\nabla x \Phi_0^{(0)}(x, y) \cdot \nabla x \Phi_0^{(0)}(x, \xi) = \tau^2 e^{-\tau L(x, y, \xi)} J(x, y, \xi) K(x, y, \xi, \tau),$$

where $K(x, y, \xi, \tau) = h_0(x, y, \xi) h_1(x, y, \xi) + \tau^{-1} \tilde{K}(x, y, \xi, \tau)$. From (47), there exist constants $0 < C_1 < C_2$ such that

$$C_1 \leq J(x, y, \xi) \leq C_2 \quad (x \in \overline{D}, y, \xi \in \overline{B}),$$

$$C_1 \leq h_1(x, y, \xi) \leq C_2 \quad (x \in \overline{D}, y, \xi \in \overline{B}),$$

$$|\tilde{K}(x, y, \xi, \tau)| \leq C_2 \quad (x \in \overline{D}, y, \xi \in \overline{B}, \tau > 1).$$

Lemma 5.1. Put $l_0 = \min_{x \in \overline{D}, y, \xi \in \overline{B}} l(x, y) > 0$ and $l_1 = \min_{x, \xi \in \overline{B}, x \in \overline{D}} L(x, y, \xi)$.

(1) $l_1 = 2 l_0$. Further, if $x_0 \in \overline{D}, y_0 \in \overline{B}$ satisfy $l_0 = l(x_0, y_0)$, then $L(x_0, y_0, y_0) = l_1$.

(2) If $x_0 \in \overline{D}, y_0 \in \overline{B}$ satisfy $l_0 = l(x_0, y_0)$, then $x_0 \in \partial D$ and $y_0 \in \partial B$. Further,

$$\nu_{x_0} = \frac{\tilde{z}'(x_0, y_0) - x_0}{|\tilde{z}'(x_0, y_0)|}, \quad \nu_{y_0} = \frac{\tilde{z}'(x_0, y_0) - y_0}{|\tilde{z}'(x_0, y_0)|},$$

where $\nu_{x_0}$ and $\nu_{y_0}$ are the unit outer normal of $\partial D$ and $\partial B$ at $x_0$ and $y_0$, respectively.

(3) If $x_1 \in \overline{D}, y_1, \xi_1 \in \overline{B}$ satisfy $L(x_1, y_1, \xi_1) = l_1$, then $y_1 = \xi_1$ and $l(x_1, y_1) = l_0$.

Proof. For any $y, \tilde{y} \in \overline{B}$ and $x \in \overline{D}, l(x, y, \tilde{y}) = l(x, y) + l(x, \tilde{y}) \geq 2 l_0$. Since $l(x, y)$ is continuous on the compact set $\overline{D} \times \overline{B}$, we can take points $x_0 \in \overline{D}$ and $y_0 \in \overline{B}$ satisfying $l_0 = l(x_0, y_0)$, which implies $2 l_0 \leq L(x_0, y_0, y_0) = 2 l_0$. Thus, we obtain (1).

To show (2), assume that $x_0 \in \overline{D}, y_0 \in \overline{B}$ satisfy $l_0 = l(x_0, y_0)$. If $x_0 \notin \partial D$, there exists $\delta > 0$ such that $B_2(x_0, \delta) \subset D$, where for $a \in \mathbb{R}^3$ and $r > 0$, we put $B_r(a) = \{x \in \mathbb{R}^3 | |x - a| < r\}$. We put $z_0' = z'(x_0, y_0), z_0 = (z_0, 0) \in \mathbb{R}^3, e = (z_0' - x_0)/|z_0' - x_0|$ and $x_1 = x_0 + \delta e$. Then, $x_1 \in D$ for $|x_1 - x_0| < \delta$, and $z_0' - x_1 = z_0' - (x_0 + \delta e) = |z_0' - x_0| e - \delta e = (|z_0' - x_0| - \delta) e$. These imply

$$l_{x_1, y_0}(z_0') = \frac{1}{\sqrt{\gamma}} |z_0' - x_1| + \frac{1}{\sqrt{\gamma + 2}} |z_0' - y_0|$$

$$= \frac{1}{\sqrt{\gamma}} (|z_0' - x_0| - \delta) + \frac{1}{\sqrt{\gamma + 2}} |z_0' - y_0|$$

$$< l_{x_0, y_0}(z_0') = l_0,$$

where $\mathbb{R}$.
which is contradiction. Thus, we obtain \(x_0 \in \partial D\). Similarly, we have \(y_0 \in \partial B\).

Next, we show \(e\) is a unit outer normal of \(\partial D\) at \(x_0\). Take any \(C^1\) class curve \(c : (-\varepsilon, \varepsilon) \to \partial D\) with \(c(0) = x_0\). Since

\[
l_0 \leq l(c(t), y_0) \leq l(c(t), y_0)(z'_0) = \frac{1}{\sqrt{7^-}} |z'_0 - c(t)| + \frac{1}{\sqrt{7^+}} |z'_0 - y_0|
\]

and \(l(c(0), y_0)(z'_0) = l(x_0, y_0)(z'_0) = l_0\), the function \((-\varepsilon, \varepsilon) \ni t \mapsto l(c(t), y_0)(z'_0)\) take a minimum at \(t = 0\). This implies

\[
0 = \frac{d}{dt}(l(c(t), y_0)(z'_0))|_{t=0} = \frac{1}{\sqrt{7^-}} \frac{c(0) - z'_0}{|z'_0 - c(0)|} \cdot c'(0) = \frac{1}{\sqrt{7^-}} e \cdot c'(0),
\]

which yields \(e\) is a unit normal of \(\partial D\).

To obtain \(e\) is outward, it suffices to show \(x_0 + \delta e \notin D\) for \(\delta > 0\) small enough. For any \(0 < \delta < |x_0 - y'_0|\), it follows that

\[
l_{x_0 + \delta e, y_0}(z'_0) = \frac{1}{\sqrt{7^-}} |x_0 + \delta e - z'_0| + \frac{1}{\sqrt{7^+}} |z'_0 - y_0|
\]

\[
= \frac{1}{\sqrt{7^-}} (|x_0 - z'_0| - \delta) + \frac{1}{\sqrt{7^+}} |z'_0 - y_0| < l_0,
\]

which yields \(x_0 + \delta e \notin D\). For \(y_0 \in \partial B\), we can show similarly, which obtain (2).

Last, we show (3). Take \(x_1 \in \overline{D}\) and \(y_1, \xi_1 \in \overline{B}\) with \(L(x_1, y_1, \xi_1) = l_1\). From \(l_0 \leq l(x_1, y_1), l_0 \leq l(x_1, \xi_1)\), it follows that

\[
l_0 \leq l(x_1, y_1) = L(x_1, y_1, \xi_1) - l(x_1, \xi_1) = 2l_0 - l(x_1, \xi_1) \leq 2l_0 - l_0 = l_0,
\]

which yields \(l(x_1, y_1) = l_0\). We can obtain \(l(x_1, \xi_1) = l_0\) similarly. To finish the proof, it suffices to show \(y_1 = \xi_1\).

We put \(z'_1 = z'(x_1, y_1) \in \mathbb{R}^2\), \(z''_1 = (z'_1, 0), \eta'_1 = \eta'(x_1, \xi_1) \in \mathbb{R}^2\) and \(\eta''_1 = (\eta'_1, 0)\), where \(z'(x_1, y_1)\) and \(z'(x_1, \xi_1)\) are determined by (1) of Lemma 4.1, respectively. From (2) of Lemma 5.1, \(\nu_{x_1} = (z'_1 - x_1)/|z'_1 - x_1| = (\eta'_1 - x_1)/|\eta'_1 - x_1|\).

Taking the inner product of this vector and (0, 0, 1) \(\in \mathbb{R}^3\), we obtain \(x_{1,3}/|z''_1 - x_1| = x_{1,3}/|\eta''_1 - x_1|\), where \(x_{1,3} = (x_{1,1}, x_{1,2}, x_{1,3})\). Since \(x_{1,3} \neq 0\), it follows that \(|z'_1 - x_1| = |\eta'_1 - x_1|\), which yields \(z'_1 = \eta'_1\). From this and \(l(x_1, y_1) = l(x_1, \xi_1) = l_0, |z'_1 - y_1| = |\eta'_1 - \xi_1|\) also follows.

Now, we remember Snell’s law,

\[
\frac{1}{\sqrt{7^-}} |z'_1 - x'_1| + \frac{1}{\sqrt{7^+}} |z'_1 - y'_1| = 0, \quad \frac{1}{\sqrt{7^-}} |\eta'_1 - x'_1| + \frac{1}{\sqrt{7^+}} |\eta'_1 - \xi'_1| = 0,
\]

which are derived from (64) and (65). These relations imply \(y'_1 = \xi'_1\). Since \(|z'_1 - y_1| = |\eta'_1 - \xi_1|\), it follows that \(y''_{1,3} = \xi''_{1,3}\), which yields \(y_{1,3} = \xi_{1,3}\) since both are positive. Thus, we obtain \(y_1 = \xi_1\), which completes the proof of Lemma 5.1. \(\square\)

Now, we are in a position to show Theorem 1.3. Combining (69)-(72) and (9) with (1) of Lemma 5.1, we obtain the estimate of the right side in (8). The problem is to show the left side of (8).

Since \(h_0(x, y, y) = 1\) for \(x \in \mathbb{R}^3_+, y \in \mathbb{R}_+^3\), and \(h_0(x, y, \xi)\) is continuous for \(x \in \mathbb{R}^3, y, \xi \in \mathbb{R}_+^3\), which is from Lemma 4.1, there exists a constant \(\delta > 0\) such that

\[
h_0(x, y, \xi) \geq 1/2 \text{ for } |y - \xi| < 3\delta, y, \xi \in \ol{B}, x \in \ol{D}.
\]

We can take \(\delta > 0\) in (73) sufficiently small to be \(B_{4\delta}(y) \subset \mathbb{R}^3_+ (y \in \ol{B})\).
Put $E = \{(x_0, y_0, \xi_0) \in \overline{D} \times B \times \overline{B} | l(x_0, y_0, \xi_0) = 2l_0\}$ and $E_0 = \{(x_0, y_0) \in \overline{D} \times B | l(x_0, y_0) = l_0\}$. (1) and (3) of Lemma 5.1 imply $E \neq \emptyset$ and $E_0 \neq \emptyset$, and $E = \{(x_0, y_0, \xi_0) \in \overline{D} \times B \times \overline{B} | l(x_0, y_0) = l_0, y_0 = \xi_0\}$. Since $E_0$ is a compact set, there exist finite points $(x^{(k)}_0, y^{(k)}_0) \in E_0$ ($k = 1, 2, \ldots, N$) such that $E_0 \subset \bigcup_{k=1}^{N} B_\delta(x^{(k)}_0) \times B_\delta(y^{(k)}_0)$, where $\delta > 0$ is given in (73). Then, $E \subset \bigcup_{k=1}^{N} B_\delta(x^{(k)}_0) \times B_\delta(y^{(k)}_0) \times B_\delta(y^{(k)}_0)$ holds.

We put $W = \bigcup_{k=1}^{N} B_\delta(x^{(k)}_0) \times B_\delta(y^{(k)}_0) \times B_\delta(y^{(k)}_0) \subset \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^3$. Since $|y - \xi| \leq |y^{(k)} - \xi| < 2\delta$ for $(y, \xi) \in B_\delta(y^{(k)}_0) \times B_\delta(y^{(k)}_0)$, from (73), it follows that
\[
h_0(x, y, \xi) \geq 1/2 \quad ((x, y, \xi) \in W)
\]
Since $E \subset W$, $L(x, y, \xi) = l(x, y) + l(x, \xi) > l_1 = 2l_0$ on the compact set $\overline{D} \times B \times \overline{B} \setminus W$. Thus, there exists $c_0 > 0$ such that
\[
(75) \quad L(x, y, \xi) \geq 2l_0 + c_0 \quad ((x, y, \xi) \in \overline{D} \times B \times \overline{B} \setminus W).
\]
From (74), (71) and (72), for any $(x, y, \xi) \in W$, it follow that
\[
K(x, y, \xi, \tau) = h_0(x, y, \xi)h_1(x, y, \xi) + \tau^{-1}K(x, y, \xi, \tau) \geq C_1 \frac{1}{2} - \tau^{-1}C_2,
\]
which yields that there exists a constant $\tau_0 \geq 1$ such that $K(x, y, \xi, \tau) \geq C_1/4$ for $(x, y, \xi) \in W, \tau \geq \tau_0$. From (71) and (72), for any $(x, y, \xi) \in \overline{D} \times B \times \overline{B}$, it follows that
\[
|K(x, y, \xi, \tau)| \leq |h_0(x, y, \xi)||h_1(x, y, \xi)| + \tau^{-1}|K(x, y, \xi, \tau)| \leq C_2(1 + \tau^{-1}),
\]
which implies $|K(x, y, \xi, \tau)| \leq 2C_2 ((x, y, \xi) \in \overline{D} \times B \times \overline{B}, \tau \geq 1)$.

From the above estimates of $K$, and (69), (75), (70), (72), (9) and the assumption for $f$, it follows that
\[
\int_{\overline{D}} |\nabla v(x)|^2 dx \geq \tau^2 C_0^2 C_1 C_4 \int_{\overline{W}} e^{-\tau L(x, y, \xi)} dy d\xi dx
- \tau^2 e^{-\tau(2l_0 + c_0)} 2C_2^2 \int_{B \times B} d\xi |f(y)||f(\xi)| \int_{\overline{D}} dx
\geq \tau^2 \left( C_3 C_1 C_4 \int_{\overline{W}} e^{-\tau L(x, y, \xi)} dy d\xi dx - C_3 e^{-\tau(2l_0 + c_0)} \right) \quad (\tau \geq \tau_0),
\]
where $C_3 = 2C_2^2 \text{Vol}(D)\text{Vol}(B)|f|_{L^2(B)}$. Thus, to obtain the left hand of (8), it suffices to show the following estimate:

**Lemma 5.2.** There exists $\tau_1 \geq 1$ and $C > 0$ such that
\[
\int_{\overline{W}} e^{-\tau L(x, y, \xi)} dy d\xi dx \geq C \tau^{-6} e^{-2l_0 \tau} \quad (\tau \geq \tau_1).
\]

**Proof.** In what follows, we write $x^{(1)}_0$ and $y^{(1)}_0$ as $x_0$ and $y_0$, respectively. Since (2) of Lemma 5.1 implies $x_0 \in \partial D$ and $y_0 \in \partial B$, and $\partial D$ and $\partial B$ are $C^1$ surfaces, there exist $p_0, q_0 \in \mathbb{R}^3$ and $r > 0$ such that $B_r(p_0) \subset B_\delta(x_0), B_r(q_0) \subset B_\delta(y_0), x_0 \in \partial B_r(p_0)$ and $y_0 \in \partial B_r(q_0)$. Then, $W \supset B_\delta(x_0) \times B_\delta(y_0) \times B_\delta(y_0) \supset B_r(p_0) \times B_r(q_0) \times B_r(q_0)$, and
\[
\int_{\overline{W}} e^{-\tau L(x, y, \xi)} dy d\xi dx \geq \int_{B_r(p_0) \times B_r(q_0) \times B_r(q_0)} e^{-\tau l(x, y) + l(x, \xi)} dy d\xi dx
= \int_{B_r(p_0)} dx \left( \int_{B_r(q_0)} e^{-\tau l(x, y)} dy \right)^2.
\]
We put $z'_0 = z'(x_0, y_0) \in \mathbb{R}^2$ and $\tilde{z}_0 = (z'_0, 0) \in \partial \mathbb{R}_+^3$. Since

$$l(x, y) \leq l_{x,y}(z'_0) = \frac{1}{\sqrt{\gamma_-}} |\tilde{z}_0 - x| + \frac{1}{\sqrt{\gamma_+}} |\tilde{z}_0 - y|,$$

it follows that

$$\int_{\mathcal{W}} e^{-\tau L(x,y, \xi)} dyd\xi dx \geq \int_{B_r(p_0)} e^{-\frac{2\tau}{\sqrt{\gamma_-}} |z_0 - x|} dx \left( \int_{B_r(q_0)} e^{-\frac{2\tau}{\sqrt{\gamma_+}} |z_0 - y|} dy \right)^2.$$ 

As in (2) of Lemma 5.1, $\nu_{y_0} = (\tilde{z}_0 - y_0)/|\tilde{z}_0 - y_0|$, which yields $\inf_{y \in B_r(q_0)} |\tilde{z}_0 - y| = |\tilde{z}_0 - y_0|$. Then, from Proposition 3.2 of [14], it follows that there exist constants $C > 0$ and $\tau_2 > 0$ such that

$$\int_{B_r(q_0)} e^{-\frac{2\tau}{\sqrt{\gamma_-}} |z_0 - x|} dx \geq C\tau^{-2} e^{-\frac{2\tau}{\sqrt{\gamma_+}} |z_0 - y_0|} \quad (\tau \geq \tau_2).$$

Similarly, taking the constants $C$ and $\tau_2$ larger if necessary, we also obtain

$$\int_{B_r(p_0)} e^{-\frac{2\tau}{\sqrt{\gamma_+}} |z_0 - x|} dx \geq C\tau^{-2} e^{-\frac{2\tau}{\sqrt{\gamma_-}} |z_0 - x_0|} \quad (\tau \geq \tau_2).$$

Hence, we have

$$\int_{\mathcal{W}} e^{-\tau L(x,y, \xi)} dyd\xi dx \geq C^3 \tau^{-6} \left( e^{-\frac{2\tau}{\sqrt{\gamma_-}} |z_0 - x_0|} \right)^2$$

since

$$\frac{|\tilde{z}_0 - x_0|}{\sqrt{\gamma_-}} + \frac{|\tilde{z}_0 - y_0|}{\sqrt{\gamma_+}} = l_{x_0, y_0}(z'_0) = l(x_0, y_0) = l_0,$$

which completes the proof of Lemma 5.2. \hfill \square

Appendix. Proof of Lemma 1.2. The function $w$ defined by (2) satisfies

$$\nabla \cdot (\gamma \nabla - \tau^2)w + f = e^{-\tau T} F \quad \text{in} \ \mathbb{R}^3,$$

where

$$F = F(x, \tau) = \partial_t u(x, T) + \tau u(x, T).$$

Define

$$R = w - v.$$

It is easy to derive the following decomposition formula of the indicator function which formally corresponds to the case when $\Omega = \mathbb{R}^3$ on (3.2) of Proposition 3.1 in [10].

**Proposition 3.** We have

$$\int_{\mathbb{R}^3} f dx = \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma) \nabla v \cdot \nabla v dx + \int_{\mathbb{R}^3} \gamma \nabla R \cdot \nabla R dx + \tau^2 \int_{\mathbb{R}^3} |R|^2 dx$$

$$+ e^{-\tau T} \left( \int_{\mathbb{R}^3} FR dx - \int_{\mathbb{R}^3} Fv dx \right).$$

It follows from (76) that $w$ satisfies

$$\nabla \cdot (\gamma \nabla - \tau^2)w + \tilde{f} = 0 \quad \text{in} \ \mathbb{R}^3,$$

where

$$\tilde{f} = f - e^{-\tau T} F.$$

And also $v$ satisfies

$$\nabla \cdot (\gamma_0 \nabla - \tau^2)v + \tilde{f} = e^{-\tau T} \tilde{F} \quad \text{in} \ \mathbb{R}^3,$$
where
\[ \tilde{F} = -F. \]

Thus, changing the role of \( v \) and \( w \) in (78), we obtain
\[
\int_{\mathbb{R}^3} \tilde{f}(R) \, dx = - \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma_0) \nabla w \cdot \nabla w \, dx + \int_{\mathbb{R}^3} \gamma_0 \nabla R \cdot \nabla R \, dx + \tau^2 \int_{\mathbb{R}^3} |R|^2 \, dx \\
+ e^{-\tau T} \left( \int_{\mathbb{R}^3} \tilde{F}(R) \, dx - \int_{\mathbb{R}^3} \tilde{F} \, wdx \right).
\]

This is noting but the following formula.

**Proposition 4.** We have
\[
(79) \\
- \int_{\mathbb{R}^3} f R \, dx = \int_{\mathbb{R}^3} (\gamma - \gamma_0 I_3) \nabla w \cdot \nabla w \, dx + \int_{\mathbb{R}^3} \gamma_0 \nabla R \cdot \nabla R \, dx + \tau^2 \int_{\mathbb{R}^3} |R|^2 \, dx \\
+ e^{-\tau T} \left( \int_{\mathbb{R}^3} F R \, dx + \int_{\mathbb{R}^3} F v \, dx \right).
\]

Now we are ready to prove (6) and (7). It follows from (3) that
\[
\int_{\mathbb{R}^3} (\gamma_0 \nabla v \cdot \nabla v + \tau^2 v^2 - f v) \, dx = 0,
\]
that is
\[
\int_{\mathbb{R}^3} \left\{ \gamma_0 \nabla v \cdot \nabla v + \left( \tau v - \frac{f}{2\tau} \right)^2 \right\} \, dx = \frac{1}{4\tau^2} \int_{\mathbb{R}^3} f^2 \, dx.
\]

This yields, as \( \tau \to \infty \)
\[
\int_{\mathbb{R}^3} (\gamma_0 \nabla v \cdot \nabla v + \tau^2 v^2) \, dx = O(\tau^{-2}).
\]

Hence we obtain, as \( \tau \to \infty \)
\[
\|v\|_{L^2(\mathbb{R}^3)} = O(\tau^{-2})
\]
and
\[
\|\nabla v\|_{L^2(\mathbb{R}^3)} = O(\tau^{-1}).
\]

Rewrite (78) as
\[
(82) \\
\int_{\mathbb{R}^3} \gamma \nabla R \cdot \nabla R \, dx + \int_{\mathbb{R}^3} \left( \tau R - \frac{f - e^{-\tau T} F}{2\tau} \right)^2 \, dx \\
= - \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma) \nabla v \cdot \nabla v \, dx \\
+ e^{-\tau T} \int_{\mathbb{R}^3} F v \, dx + \frac{1}{4\tau^2} \int_{\mathbb{R}^3} (f - e^{-\tau T} F)^2 \, dx.
\]

Here from (77) we have
\[
(83) \quad \|F\|_{L^2(\mathbb{R}^3)} = O(\tau).
\]

This together with (80) and (81) yields that the right-hand side on (82) has a bound \( O(\tau^{-2}). \) Thus we obtain
\[
\int_{\mathbb{R}^3} \left( \gamma \nabla R \cdot \nabla R + \tau^2 R^2 \right) \, dx = O(\tau^{-2})
\]
and, in particular,
\[
\| R \|_{L^2(\mathbb{R}^3)} = O(\tau^{-2}).
\]
Now applying (80), (83) and (84) to the fourth-term in the right-hand side on (78) we obtain
\[
\int_{\mathbb{R}^3} f R dx = \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma) \nabla v \cdot \nabla v dx
\]
\[
+ \int_{\mathbb{R}^3} \gamma \nabla R \cdot \nabla R dx + \tau^2 \int_{\mathbb{R}^3} |R|^2 dx + O(\tau^{-1} e^{-\tau T}).
\]
This immediately yields (6).

For the proof of (7) we recall the following inequality (see [8])
\[
(\gamma - \gamma_0 I_3) A \cdot A + \gamma_0 (A - B) \cdot (A - B)
\]
\[
= \gamma A \cdot A - 2\gamma_0 A \cdot B + \gamma_0 B \cdot B
\]
\[
\geq \gamma_0 \gamma - \gamma_0 I_3 \gamma - \gamma_1/2 B \cdot \gamma - 1/2 B
\]
\[
= \gamma_0 (\gamma - \gamma_0 I_3) \gamma - 1/2 B \cdot \gamma - 1/2 B,
\]
where \( A \) and \( B \) are real vectors. Applying this to the first and second term in the right-hand side on (79), we obtain
\[
- \int_{\mathbb{R}^3} f R dx \geq \int_{\mathbb{R}^3} \gamma_0 (\gamma - \gamma_0 I_3) \gamma - 1/2 \nabla v \cdot \gamma - 1/2 \nabla v dx
\]
\[
+ e^{-\tau T} \left( \int_{\mathbb{R}^3} F R dx + \int_{\mathbb{R}^3} F v dx \right).
\]
Now applying (80), (83) and (84) to the second term in this right-hand side we obtain (7).

Acknowledgments. The first author is partially supported by JSPS KAKENHI Grant Number JP17K05331. The second author is partially supported by JSPS KAKENHI Grant Number JP16K05232. This work was also partly supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

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Received June 2017; revised February 2018.

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