Abstract

This paper continues earlier work on the quantum evaporation of black holes. This work has been concerned with the calculation and understanding of quantum amplitudes for final data perturbed slightly away from spherical symmetry on a space-like hypersurface $\Sigma_F$ at a late Lorentzian time $T$. For initial data, we take, for simplicity, spherically-symmetric asymptotically-flat data for Einstein gravity with a massless scalar field on an initial surface $\Sigma_I$ at time $t = 0$. Together, such boundary data give a quantum analogue of classical Einstein/scalar gravitational collapse to a black hole, perhaps starting from a diffuse, early-time configuration. Quantum amplitudes are calculated following Feynman’s approach, by first rotating: $T \rightarrow |T| \exp(-i\theta)$ into the complex, where $0 < \theta \leq \pi/2$, then solving the corresponding complex classical boundary-value problem, which is expected to be well-posed provided $\theta > 0$, and computing its classical Lorentzian action $S_{\text{class}}$ and corresponding semi-classical quantum amplitude, proportional to $\exp(iS_{\text{class}})$. For a locally-supersymmetric Lagrangian, describing supergravity coupled to supermatter, any loop corrections will be negligible, provided that the frequencies involved in the boundary data are well below the Planck scale. Finally, the Lorentzian amplitude is recovered by taking the limit $\theta \rightarrow 0_+$ of the semi-classical amplitude. In the black-hole case, by studying the linearised spin-0 or spin-2 classical solutions in the above (slightly complexified) case, for the corresponding classical boundary-value problem with the given perturbative data on $\Sigma_F$, one can compute an effective energy-momentum tensor $<T^{\mu\nu}_{\text{EFF}}>$, which has been averaged over several wavelengths of the radiation, and which describes the averaged extra energy-momentum contribution in the Einstein field equations, due to the perturbations. In general, this averaged extra contribution will be spherically symmetric, being of the form of a null fluid, describing the radiation (of quantum origin) streaming radially outwards. The corresponding space-time metric, in this region containing radially outgoing radiation, is of the Vaidya form. This, in turn, justifies the treatment of the adiabatic radial mode equations, for spins $s = 0$ and $s = 2$, which is used elsewhere in this work.

1. Introduction

This paper is concerned with the problem of finding approximate classical Lorentzian (or slightly complexified) solutions of the coupled Einstein gravity/massless-scalar field equations, to describe the region of space-time containing outgoing radiation (both spin-0 and spin-2) in a very large number of modes, generated by quantum-mechanical evaporation, as a result of nearly-spherical gravitational collapse to a black hole. The space-time metric $g_{\mu\nu}$ and scalar field $\phi$ are split into a 'background' spherically-symmetric part ($\gamma_{\mu\nu}, \Phi$), plus perturbations ($h^{(1)}_{\mu\nu}, \phi^{(1)}$), etc., which are typically non-spherical. The
energy-momentum tensor formed from \( \Phi \) provides the ‘matter source’ for an exactly spherical collapse to a black hole.

In recent papers [1-5], the quantum amplitude for a given perturbative configuration (say of the scalar field \( \phi^{(1)} \)) on a final hypersurface \( \Sigma_F \) at a very late time \( T \), was found by rotating \( T \) slightly into the complex: \( T \to |T| \exp(-i\theta) \), for \( 0 < \theta \leq \pi/2 \); then calculating the (complex-valued) Lorentzian classical action \( S_{\text{class}} \) for the corresponding classical boundary-value problem, which is expected to be well-defined; then computing the resulting semi-classical amplitude, proportional to \( \exp(iS_{\text{class}}) \); then finally obtaining the amplitude for real Lorentzian \( T \) by taking the limit of \( \exp(iS_{\text{class}}) \) as \( \theta \to 0^+ \). Typically, the perturbative scalar-field configuration \( \phi^{(1)} \) given on the late-time surface \( \Sigma_F \) will involve an enormous number of modes, both angular and radial, but with a minute coefficient for each such mode. That is, the given \( \phi^{(1)} \) may contain extremely detailed angular structure, and also be spread over a considerable radius from the centre of spherical symmetry of the background \( (\gamma_{\mu\nu}, \Phi) \), again with detailed radial structure. Now consider the corresponding classical Dirichlet boundary-value problem above, in which one takes \( \phi^{(1)} = \phi^{(1)}|_{\Sigma_F} \) as given on \( \Sigma_F \), but chooses \( \phi^{(1)}|_{\Sigma_I} = 0 \) (for simplicity), together with the complex time-interval-at-infinity \( T = |T| \exp(-i\theta) \) for \( 0 < \theta \leq \pi/2 \). The solution for \( \phi^{(1)} \) will gradually decay towards zero, as one moves from the final surface \( \Sigma_F \) to earlier times; the rate of this exponential decay will be extremely slow when \( \theta \) is close to zero. In this case, one will find that, at all times \( t \) with \( 0 < t < T \) (that is, between \( \Sigma_I \) and \( \Sigma_F \)), the classical solution will continue to have complicated angular and radial structure, much as does its boundary value \( \phi^{(1)}|_{\Sigma_F} \).

As a result, one must study Lorentzian (or complexified Lorentzian) classical solutions for the linearised metric and scalar perturbations \( (h^{(1)}_{\mu\nu}, \phi^{(1)}) \), which contain classical spin-0 and spin-2 radiation outgoing from the ‘gravitational collapse’, with detailed structure over (typically) an enormous radial extent. The cumulative effective energy-momentum tensor, formed quadratically from derivatives of these first-order perturbations, and then averaged over several wavelengths of the radiation, so as to produce a smooth averaged \( <T^{\mu\nu}_{\text{EFF}} > \), is expected to be nearly spherically-symmetric, and indeed to have the form appropriate to a radially-outgoing null fluid [6]. This viewpoint simplifies enormously the description of the ‘effective energy-momentum source’ due to the wave-like perturbations, which then feeds back into the spherically-symmetric background solution \( (\gamma_{\mu\nu}, \Phi) \), albeit over a suitably long time-scale. In this description, the effective energy-momentum contribution of the emitted radiation can be reduced to just one spherically-symmetric ‘density of radiation’ function of retarded time, instead of an infinite number of multipole or mode coefficients for the final boundary data \( \phi^{(1)}|_{\Sigma_F} \).

The space-metric metric resulting from such a null-fluid effective \( T_{\mu\nu} \) is precisely of the Vaidya type [7]. This resembles the Schwarzschild geometry, except that the role of the Schwarzschild mass \( M \) is taken by a mass function \( m(t, r) \), which varies extremely slowly with respect both to \( t \) and to \( r \) in the space-time region containing outgoing radiation. In this region, the slowly-varying Vaidya metric provides a valid approximation. Of course, one does not expect such a relatively simple analytic approximation to the metric and scalar field in the strong-field collapse region, where, in the case of a real time-interval \( T \), the classical Lorentzian black-hole solution is highly dynamical.
In Sec.2, we discuss the calculation and form of the averaged energy-momentum tensor \(< T^{\mu \nu}_{\text{EFF}} >\), assuming that both spin-0 perturbations \(\phi^{(1)}\) and spin-2 (graviton) perturbations \(h^{(1)}_{\mu \nu}\) are present. It is consistently assumed that the time-scale associated with typical radiation frequencies is very much less than the time-scale over which the background geometry changes. Thus, the wave-like perturbations in the metric and in the scalar field can be treated in a WKB approximation, leading to an expression for \(< T^{\mu \nu}_{\text{EFF}} >\). One can then verify that this \(< T^{\mu \nu}_{\text{EFF}} >\) generates an extremely slow evolution of the resulting Vaidya metric. The case of spin-1 (Maxwell) perturbations is also discussed. The resulting Vaidya metric in the outgoing-radiation region is described in Sec.3, in different coordinate systems adapted to different aspects of the radiating system; this material has also been covered in part in [20]. A brief Conclusion is included in Sec.4.

2. High-frequency limit: fields and energy-momentum tensor

In Sec.3 of [3], we expanded out the Einstein field equations in powers of \(\epsilon\), given a perturbative expansion for the classical solution \((g_{\mu \nu}, \phi)\) about a spherically-symmetric reference or 'background' solution \((\gamma_{\mu \nu}, \Phi)\). We write

\[
g_{\mu \nu}(x, \epsilon) = \gamma_{\mu \nu}(x) + \epsilon h^{(1)}_{\mu \nu}(x) + \epsilon^2 h^{(2)}_{\mu \nu}(x) + \ldots ,
\]

\[
\phi(x, \epsilon) = \Phi(\tau, r) + \epsilon \phi^{(1)}(x) + \epsilon^2 \phi^{(2)}(x) + \ldots .
\]

At lowest order \(O(\epsilon^0)\), one has the background Einstein and scalar field equations

\[
R^{(0)}_{\mu \nu} - \frac{1}{2} R^{(0)} \gamma_{\mu \nu} = 8\pi T^{(0)}_{\mu \nu},
\]

\[
\gamma_{\mu \nu} \Phi_{; \mu \nu} = 0 ,
\]

Here, \(R^{(0)}_{\mu \nu}\) denotes the Ricci tensor and \(R^{(0)}\) denotes the Ricci scalar of the background geometry \(\gamma_{\mu \nu}\). Covariant differentiation in the background is denoted by a semi-colon \(\(\_\)_{; \alpha}\) or (below) by \(\nabla_{\alpha}\). The background energy-momentum tensor is denoted by

\[
T^{(0)}_{\mu \nu} = \Phi_{; \mu} \Phi_{; \nu} - \frac{1}{2} \gamma_{\mu \nu} \left( \Phi_{; \alpha} \Phi_{; \beta} \gamma^{\alpha \beta} \right).
\]

The linearised or \(O(\epsilon^1)\) part of the Einstein field equations reads [9]

\[
\bar{h}^{(1)}_{\mu \nu ; \sigma} - 2 \bar{h}^{(1)}_{\sigma (\mu ; \nu)} - 2 R^{(0)}_{\sigma \mu \nu \alpha} \bar{h}^{(1)}_{\sigma \alpha} - 2 R^{(0)}_{\alpha (\mu} \bar{h}^{(1)}_{\nu) \alpha}
\]

\[
+ \gamma_{\mu \nu} \left( \bar{h}^{(1)}_{\alpha \beta} - \bar{h}^{(1)}_{\mu \nu} R^{(0)}_{\alpha \beta} \right) + \bar{h}^{(1)}_{\mu \nu} R^{(0)} = -16\pi T^{(1)}_{\mu \nu} .
\]

where indices on all quantities are raised and lowered using the background metric \(\gamma_{\mu \nu}\). As usual [9], we define

\[
\bar{h}^{(1)}_{\mu \nu} = h^{(1)}_{\mu \nu} - \frac{1}{2} \gamma_{\mu \nu} h^{(1)} ,
\]

where

\[
h^{(1)} = h^{(1)}_{\mu} .
\]
Here, $R^{(0)}_{\sigma\mu\nu\alpha}$ denotes the Riemann tensor of the background geometry $\gamma_{\mu\nu}$, and $T^{(1)}_{\mu\nu}$ denotes the linearisation or $O(\epsilon^1)$ part of the energy-momentum tensor $T_{\mu\nu}(x, \epsilon)$, given explicitly in Eq.(3.23) of [3].

The linearised Einstein equations (2.6) are most easily studied in a 'linearised harmonic gauge' [9] in which, by an infinitesimal coordinate transformation, one has arranged that

$$\bar{h}^{(1)\alpha}_{\alpha\beta} = 0 \quad \text{(2.9)}$$

At very late Lorentzian times, the background Riemann curvature and the background scalar field $\Phi$ will die off rapidly, whence the linearised Einstein equations (2.6) simplify to

$$\bar{h}^{(1)\sigma}_{\mu\nu;\sigma} - 2 \bar{h}^{(1)\sigma}_{\sigma(\mu;\nu)} - 2 R^{(0)}_{\sigma\mu\nu\alpha} \bar{h}^{(1)\sigma\alpha} + \gamma_{\mu\nu} \bar{h}^{(1)\alpha\beta} = 0 \quad \text{(2.10)}$$

that is, the linearised vacuum field equations [9], subject also to Eq.(2.9).

As described in [4], the rate of change with time of the spherically-symmetric background geometry $\gamma_{\mu\nu}$ will be extremely small, during the long quasi-static period when the rate of emission of quantum radiation by the black hole hardly varies with time. Hence, most perturbation modes, for scalar (spin-0) or gravitational (spin-2) oscillations, will be 'adiabatic' or high-frequency. Within the high-frequency approximation, in a space-time without background matter, one may additionally (without loss of generality) impose the traceless gauge condition [9]:

$$\bar{h}^{(1)\sigma}_{\alpha} = 0 \quad \text{(2.11)}$$

In this case, the linearised Einstein field equations (2.6), subject to the transverse-traceless $(TT)$ gauge conditions (2.9,11), reduce further [9] to:

$$\bar{h}^{(1)\sigma}_{\mu\nu;\sigma} - 2 R^{(0)}_{\sigma\mu\nu\alpha} \bar{h}^{(1)\sigma\alpha} = 0 \quad \text{(2.12)}$$

At $O(\epsilon^2)$, the gravitational field equations give the second-order contribution $G^{(2)}_{\mu\nu}$ to the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{(2.13)}$$

Following a lengthy calculation, one finds [3] that the Einstein field equations, up to and including second order in perturbations, can be written as

$$G^{(0)}_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu} + 8\pi T^{(2)}_{\mu\nu} + 8\pi T^{'}_{\mu\nu} - G^{(1)}_{\mu\nu} \quad \text{(2.14)}$$

Here, $T^{(0)}_{\mu\nu}$ is the background energy-momentum tensor (2.5), and $G^{(1)}_{\mu\nu}$ denotes $-\left(\frac{1}{2}\right) \times$ the left-hand side of Eq.(2.6). The quantity $T^{(2)}_{\mu\nu}$ denotes

$$T^{(2)}_{\mu\nu} = \nabla_{\mu} \phi^{(1)} \nabla_{\nu} \phi^{(1)} - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\rho\sigma} \nabla_{\rho} \phi^{(1)} \nabla_{\sigma} \phi^{(1)} + \left(\gamma_{\mu\nu} \bar{h}^{(1)\sigma\rho} - \bar{h}^{(1)\sigma\rho}_{\mu\nu} \gamma^{\sigma\rho}\right) \nabla_{\sigma} \Phi \nabla_{\rho} \phi^{(1)}$$

$$+ \frac{1}{2} \left(\bar{h}^{(1)\rho}_{\mu\nu} \bar{h}^{(1)\sigma\rho} - \gamma_{\mu\nu} \bar{h}^{(1)\sigma\alpha} \bar{h}^{(1)\rho}_{\alpha}\right) \nabla_{\sigma} \Phi \nabla_{\rho} \Phi \quad \text{(2.15)}$$
and $T'_{\mu\nu}$ is defined by

$$8\pi T'_{\mu\nu} = \frac{1}{4} \left( \bar{h}^{(1)\sigma\rho}_{\mu} h^{(1)}_{\sigma\rho\nu} - 2 \bar{h}^{(1):\alpha}_{\mu} \bar{h}^{(1)\sigma}_{(\mu;\nu)} \right) - \frac{1}{2} \bar{h}^{(1)\sigma}_{(\mu} R^{(0)}_{\nu)\rho\sigma\alpha} \bar{h}^{(1)\alpha\rho}_{\mu}$$

$$+ \frac{1}{2} \bar{h}^{(1)}_{\sigma(\mu} R^{(0)\nu)\alpha} \bar{h}^{(1)\alpha\sigma}_{\nu} - \frac{1}{2} \bar{h}^{(1)\sigma}_{(\mu} R^{(0)}_{\nu)\sigma} R^{(0)} - 8\pi T^{(1)} T^{(1)\sigma}_{(\mu} \bar{h}^{(1)\sigma}_{\nu)}$$

$$- 4\pi \gamma_{\mu\nu} \left( 2 \bar{h}^{(1)\rho\sigma} \nabla_\sigma \phi^{(1)} \nabla_\rho \Phi + \phi^{(1)} \nabla_\rho \nabla_\sigma \phi^{(1)} - \bar{h}^{(1)\sigma\rho} h^{(1)\beta}_{\sigma} \nabla_\rho \Phi \nabla_\beta \Phi \right)$$

$$+ C^\sigma_{\mu\nu;\sigma},$$

where the explicit form of $C^\sigma_{\mu\nu}$ will not be needed.

In the high-frequency limit, after Brill-Hartle or Isaacson averaging [9] over many wavelengths (both in space and in time), as summarised in [3], $< T'_{\mu\nu} >$ will give the leading spin-2 (graviton) contribution to $G^{(0)}_{\mu\nu}$, and $T^{(2)}_{\mu\nu}$ will give the contribution quadratic in the scalar fluctuations $\phi^{(1)}$. On averaging over many wave periods and over angles, following Sec.3 of [3], one finds for late Lorentzian times in the high-frequency (Isaacson) approximation:

$$< T'_{\mu\nu} >= \frac{1}{32\pi} < \bar{h}^{(1)\sigma\rho}_{\mu} \bar{h}^{(1)}_{\sigma\rho\nu} - 2 \bar{h}^{(1):\alpha}_{\mu} \bar{h}^{(1)\sigma}_{(\mu;\nu)}> .$$

In the transverse-traceless gauge (2.9,11), appropriate for this region of the space-time, Eq.(2.3) simplifies to give

$$T^{GW}_{\mu\nu} \equiv < T'_{\mu\nu} >_{TT} = \frac{1}{32\pi} < \bar{h}^{(1)\sigma\rho}_{\mu} \bar{h}^{(1)}_{\sigma\rho\nu} >_{TT} ,$$

The Isaacson averaged energy-momentum contribution of the scalar-field fluctuations, taken for simplicity in the late-time region where the background scalar field $\Phi$ is nearly zero, is

$$< T^{(2)}_{\mu\nu} > = < \nabla_\mu \phi^{(1)} \nabla_\nu \phi^{(1)}> .$$

Spin-1 Maxwell field perturbations can also be treated in a similar way [16]. For a perturbative Maxwell vector potential $A^{(1)}_{\mu}$ in the Lorentz gauge

$$\nabla^\mu A^{(1)}_{\mu} = 0 ,$$

the Maxwell field equations read [9]

$$\nabla^\mu \nabla_\mu A^{(1)}_{\nu} - R^{(0)}_{\mu\nu} A^{(1)\mu} = 0 .$$

The averaged Maxwell energy-momentum tensor

$$< T^{\alpha\beta} >_{\text{Maxwell}} = \frac{1}{4\pi} < \nabla^\alpha A^{(1)}_{\sigma} \nabla^\beta A^{(1)}_{\sigma} + R^{(0)}_{\mu\nu} A^{(1)\mu} A^{(1)\beta} >_{\text{Lor}} .$$
in Lorentz gauge can be simplified using the field equation (2.21) and integration by parts
in the Isaacson limit, to give

\[ \langle T^{\alpha\beta} \rangle_{\text{Maxwell}} = \frac{1}{4\pi} \langle \nabla_\mu A^{(1)\alpha} \nabla_\nu A^{(1)}_{\alpha} \rangle_{\text{Lor}} . \]  

(2.23)

For spin-1 Yang-Mills fields, which typically appear when working with locally-supersymmetric
theories of supergravity coupled to supermatter [10], a similar but more complicated treatment
can be given.

Thus Eq.(2.14), averaged over high-frequency fluctuations, and including a spin-1
Maxwell-field contribution, becomes

\[ G^{(0)}_{\mu\nu}(\gamma) = 8\pi \langle \nabla_\mu \phi^{(1)} \nabla_\nu \phi^{(1)} \rangle + 2 \langle \nabla_\mu A^{(1)\alpha} \nabla_\nu A^{(1)}_{\alpha} \rangle_{\text{Lor}} + \frac{1}{4} \langle h^{(1)\sigma\rho;\mu} h^{(1)\sigma;\rho;\nu} \rangle_{TT} . \]  

(2.24)

Further perturbative corrections to Eq.(2.24) are of a relative size \( O(\epsilon) \) smaller, and it must
be understood that one solves Eqs. (2.12,21,24) simultaneously. To ease the notation, we
henceforth drop the labels Lor and TT.

For high-frequency (real) massless perturbations \( \phi^{(1)}, A^{(1)}_\mu, h^{(1)}_{\mu\nu} \), we make an Ansatz
which is natural for late times:

\[ \phi^{(1)}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^\infty d\omega \left[ A_{\omega\ell m}(t, r, \Omega) e^{i\theta_\omega(t, r)/\epsilon} + \text{c.c.} \right] , \]  

(2.25)

\[ A^{(1)}_\mu(x) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sum_P \int_0^\infty d\omega \left[ (A_\mu)_{\omega\ell m P}(t, r, \Omega) e^{i\theta_\omega(t, r)/\epsilon} + \text{c.c.} \right] , \]  

(2.26)

\[ h^{(1)}_{\mu\nu}(x) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sum_P \int_0^\infty d\omega \left[ (A_{\mu\nu})_{\omega\ell m P}(t, r, \Omega) e^{i\theta_\omega(t, r)/\epsilon} + \text{c.c.} \right] , \]  

(2.27)

where \( P = +, \times \) denotes the two orthogonal polarisation states for a radially-travelling
gravitational wave, and also the standard independent polarisation states in the \( \theta \)-and
\( \phi \)-directions for a radially-travelling electromagnetic wave [9]. The quantity \( \theta_\omega(t, r)/\epsilon \) is a
rapidly-varying real phase, in common to all the spins 0, 1 and 2, which precisely allows
for the predominantly radial wave-propagation at late times. Defining \( \psi = \theta_\omega(t, r)/\epsilon \), we
assume that the first derivative of \( \psi \) is large in comparison with first derivatives of the
'amplitude' \( A_{\omega\ell m} \) or of the corresponding tetrad components of \( (A_\mu)_{\omega\ell m P} \) or \( (A_{\mu\nu})_{\omega\ell m P} \)
– see below. Schematically, \( |\partial \psi/\psi| \gg |\partial A/A| \).

One might expect the high-frequency expansions (2.25-27) for the perturbed scalar,
Maxwell and graviton fields to give a good approximation to the radially-outgoing radiation
at late times during the 'long' period when the black hole is radiating in a quasi-static way,
with its mass 'slowly' decreasing from the initial value \( M_I \). The corresponding approximate
behaviour of the overall spherically-symmetric 'background' gravitational field at late times
is expected to be given by the Vaidya metric [7], as treated in Sec.3 below. In particular, the connection between the late-time high-frequency expansions (2.25-27) and the Vaidya metric will be described explicitly in Sec.3. Of course, in our classical boundary-value formulation, with the time-at-infinity \( T \) taken to be of the form \( T = |T|e^{-i\theta} \), for \( 0 < \theta \leq \pi/2 \), the amplitudes \( A_{\omega \ell m}, (A_\mu)_{\omega \ell m P} \) and \( (A_{\mu \nu})_{\omega \ell m P} \) in the late-time high-frequency expansions (2.25-27) are related to the scalar, spin-1 Maxwell and spin-2 graviton data on the final surface \( \Sigma_F \), with time \( T \) at infinity. Conversely, on following the radiation to the past, we reach the strongly-interacting collapse region of the space-time, where both the background spherically-symmetric metric \( \gamma_{\mu \nu} \) and the scalar field \( \Phi \) may vary rapidly with the coordinates \( t \) and \( r \). It is only because the background \( (\gamma_{\mu \nu}, \Phi) \) is the complex solution of the boundary-value problem for the spherically-symmetric Einstein/scalar system, with a complex time-separation-at-infinity \( T = |T|\exp(-i\theta) \), for \( 0 < \theta < \pi/2 \), that large deviations from flatness in the boundary data are expected to be smoothed out in the usual elliptic fashion. This is the distinguishing feature of this complex approach; in contrast, Lorentzian-signature evolution of the Einstein field equations, including matter, generically leads to space-time singularities. If one knew the form of the background solution, then (computationally, at least) one could solve the coupled evolution equations for harmonics of (say) the perturbed scalar and gravitational fields. One would then, by matching of asymptotic expansions [11], have to join the wave-like solutions emerging from the strong-field ‘collapse region’ above onto the high-frequency expansions (2.25-27) for the radiative parts of the fields at late times.

Next, we consider the leading, geometrical-optics limit of the perturbative field equations for spin-0, 1, and 2. For spin-0, one has

\[
\gamma^{\mu \nu} \phi^{(1)}_{,\mu \nu} = 0,
\]

which is the \( O(\epsilon^1) \) part of the scalar field equation Eq.(2.4), in the late-time limit that \( \Phi = 0 \). The spin-1 field equations are given in Eq.(2.21), and the linearised spin-2 field equations in Eq.(2.10). Define

\[
(k_\mu)_\omega = \nabla_\mu \theta_\omega.
\]

Working again in the late-time region and taking \( \Phi = 0 \) there, a straightforward calculation, applying the perturbative field equations (2.10,21,28) to the high-frequency expansions (2.25-27), together with the \( TT \) and Lorentz gauge conditions, leads to the following properties:

\[
(k^\mu)_\omega (k_\mu)_\omega = 0,
\]

\[
(A_\mu)_{\omega \ell m P} (k^\mu)_\omega = 0,
\]

\[
(A_{\mu \nu})_{\omega \ell m P} (k^\mu)_\omega = 0,
\]

\[
\gamma^{\mu \nu} (A_{\mu \nu})_{\omega \ell m P} = 0,
\]

at lowest order in \( \epsilon \). Suppose that the background metric \( \gamma_{\mu \nu} \) in this region, written with respect to coordinates \( (t,r,\theta,\phi) \) in the form

\[
\text{d}s^2 = -e^{b(t,r)} \, \text{d}t^2 + e^{a(t,r)} \, \text{d}r^2 + r^2 \left( \text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2 \right),
\]

\[
\text{d}t = |T| e^{-i\theta},
\]

\[
\text{d}r = e^{a(t,r)} \, \text{d}r,
\]

\[
\text{d}\theta = \sin \theta \, \text{d}\theta,
\]

\[
\text{d}\phi = \sin \theta \, \text{d}\phi,
\]
has the Vaidya form [7], but in the coordinate system described in Eq.(3.18) below:

\[ e^{-a(t,r)} = 1 - \frac{2m(t,r)}{r}, \quad e^{b(t,r)} = \left( \frac{\dot{m}}{f(m)} \right)^2 e^{-a} \] (2.35)

Here, \( m(t,r) \) is a slowly-varying 'mass function', with \( \dot{m} = (\partial m/\partial t) \), and where \( f(m) \) depends on the details of the radiation. Then, for each choice of the integration variable \( \omega \) in Eqs.(2.25-27), Eq.(2.29) has an outgoing-wave solution

\[ \theta_{\omega}(t,r) = \omega (t - r^*) \] (2.36)

where we define

\[ r^* = \int^r d\hat{r} e^{a(t,\hat{r})} \] (2.37)

by analogy with the 'tortoise coordinate' \( r^* = r + 2M \ln\left( (r/2M) - 1 \right) \) in the Schwarzschild solution [9,12]. Because of the slowly-varying nature of the background, one has \( |\partial t r^*| \ll 1 \). Note that a general solution of Eq.(2.30): \( (k^\mu_\omega)(k^\mu_\omega) = 0 \), would involve a general function of \( t \pm r^* \). The outgoing-wave solution (2.36) is picked out because we require the expansions (2.25-27) to reduce to outgoing Fourier expansions at large radius.

In a standard fashion, the application of the linearised field equations and gauge conditions to the high-frequency expansions (2.25-27) can be carried on to the next order, one power of \( \epsilon \) beyond geometrical optics. For the spin-0 perturbations, one finds

\[ A_{\omega \ell m} \nabla^\mu (k^\mu_\omega) + 2 (k^\mu_\omega) \nabla_\mu A_{\omega \ell m} = 0 \] (2.38)

whence

\[ \nabla^\mu \left[ |A_{\omega \ell m}|^2 (k^\mu_\omega) \right] = 0 \] (2.39)

For the spin-1 field, one finds

\[ (A_\nu)_{\omega \ell m P} \nabla^\sigma (k^\sigma_\omega) + 2 (k^\sigma_\omega) \nabla_\sigma (A_\nu)_{\omega \ell m P} = 0 \] (2.40)

Now introduce a polarisation vector \( (e_\mu)_{\omega \ell m P} \) such that

\[ (A_\mu)_{\omega \ell m} = A_{1 \omega \ell m P} (e_\mu)_{\omega \ell m P} \] (2.41)

\[ (e_\mu)_{\omega \ell m P} (e^\mu)_{\omega \ell m P} = \delta_{PP'} \] (2.42)

\[ A_{1 \omega \ell m P} = \left[ (A_\mu)_{\omega \ell m P} (A^\mu)_{\omega \ell m P} \right]^{1/2} \] (2.43)

where a star denotes complex conjugation. The Lorentz condition implies

\[ (e^\mu)_{\omega \ell m P} (k^\mu_\omega) = 0 \] (2.44)
The Maxwell field equations then imply

$$\nabla^\mu \left| A_{1\omega \ell m} \right|^2 (k_\mu, \omega) = 0 . \quad (2.45)$$

Correspondingly, for the spin-2 field (gravitons), one finds

$$(A_{\mu \nu})_{\omega \ell m} (k_\sigma, \omega)^{\sigma} + 2 (k^\sigma, \omega) (A_{\mu \nu})_{\omega \ell m}^{; \sigma} = 0 . \quad (2.46)$$

One then introduces a symmetric polarisation tensor $(e_{\mu \nu})_{\omega \ell m}$ such that

$$(A_{\mu \nu})_{\omega \ell m} = A_{2\omega \ell m} (e_{\mu \nu})_{\omega \ell m} , \quad (2.47)$$

$$(e_{\mu \nu})_{\omega \ell m} (e_{\alpha \beta})_{\omega \ell m}^{*} = \delta_{\alpha \beta} P , \quad (2.48)$$

$$A_{2\omega \ell m} = \left[ (A_{\mu \nu})_{\omega \ell m} (A_{\mu \nu})_{\omega \ell m}^{*} \right]^{\frac{1}{2}} , \quad (2.49)$$

where the last equality is valid up to an unimportant phase. The $TT$ condition implies

$$(e_{\mu \nu})_{\omega \ell m} (k_\mu, \omega) = 0 , \quad (2.50)$$

$$\gamma^{\mu \nu} (e_{\mu \nu})_{\omega \ell m} = 0 . \quad (2.51)$$

Then the linearised spin-2 field equations imply

$$\nabla^\mu \left| A_{2\omega \ell m} \right|^2 (k_\mu, \omega) = 0 . \quad (2.52)$$

For $s = 1$ and $2$, write

$$A_{s \ell m} = |A_{s \omega \ell m}| e^{i \sigma_{s \omega \ell m}} , \quad (2.53)$$

where $\sigma_{s \omega \ell m}$ is a real phase. From the 'evolution equations' (2.45,52), one finds that

$$(k_\mu, \omega) \nabla_\mu \sigma_{s \omega \ell m} = 0 , \quad (2.54)$$

provided that

$$(k_\mu, \omega) \nabla_\mu \left( \ln |A_{s \omega \ell m}| \right) = - \frac{1}{2} \nabla^\mu (k_\mu, \omega) . \quad (2.55)$$

A similar equation holds for the evolution of the spin-0 coefficients $A_{\omega \ell m}$ . Now define a preferred affine parameter $\lambda$ along the null rays such that

$$(k_\mu, \omega) = \frac{dx^\mu}{d\lambda} , \quad (2.56)$$

and the $x^\mu(\lambda)$ are affinely parametrised null geodesics. This can alternatively be written in the form

$$(k_\mu, \omega) \nabla^\mu (k_\nu, \omega) = 0 ; \quad (2.57)$$
that is, that the high-frequency waves move along null geodesics. From Eq. (2.55), one sees that the amplitude $|A_{s\omega\ell m P}|$ decreases if $\nabla^\mu (k_\mu)_{\omega} > 0$, that is, if the null rays diverge. In the arguments leading to Eqs. (2.45,53), one finds also that the corresponding polarisation tensors are parallely transported along the null geodesic $x^\mu (\lambda)$. That is,

$$
(k^\sigma)_\omega \nabla_\sigma (e_\mu)_{\omega \ell m P} = 0 \ , \quad (k^\sigma)_\omega \nabla_\sigma (e_{\mu \nu})_{\omega \ell m P} = 0 \ .
$$

(2.58, 2.59)

Of course, such a description in terms of a family of null geodesics will only be valid in a comparatively late-time, large-distance region of the space-time. Where space-time becomes highly curved, caustics would be expected to develop and the geometrical-optics approach would break down.

Turning again to the Einstein field equations, we calculate the quantities on the right-hand side of Eq. (2.24), being the contributions to $G^{(0)}_{\mu \nu} (\gamma)$ which are quadratic in the spin-0, spin-1, and spin-2 fluctuations. As earlier, $<>$ denotes an Isaacson average over times and angles. For an incoherent source of waves, comprising a large number of roughly stationary sources (essential for near-spherical symmetry), only terms in Eqs. (2.26,27) with $\ell = \ell'$, $m = m'$ contribute to the average. In the context of Eqs. (2.26,27), $<>$ is also an average over the random phase $\theta_\omega$, since a time average. Therefore, at leading order $O(\epsilon^{-2})$ with respect to the high-frequency approximations (2.25-27), one has

$$
< \nabla_\mu \phi^{(1)} \nabla_\nu \phi^{(1)} > = \frac{2}{\epsilon^2} \sum_{\ell m} \int_0^\infty d\omega \ (k_\mu)_\omega (k_\nu)_\omega |A_{\omega \ell m (t,r)}|^2 \ ,
$$

(2.60)

and

$$
< \nabla_\mu A^\sigma \nabla_\nu A_\sigma > = \frac{2}{\epsilon^2} \sum_{\ell m P} \int_0^\infty d\omega \ (k_\mu)_\omega (k_\nu)_\omega |A_{1\omega \ell m P (t,r)}|^2 \ .
$$

(2.61)

Further,

$$
< \nabla_\mu h^{(1)}_{\sigma \rho} \nabla_\nu h^{(1)\sigma \rho} > = \frac{2}{\epsilon^2} \sum_{\ell m P} \int_0^\infty d\omega \ (k_\mu)_\omega (k_\nu)_\omega |A_{2\omega \ell m P (t,r)}|^2 \ .
$$

(2.62)

Here, we define the quantity $|A_{s\omega \ell m P (t,r)}|^2$ to be

$$
|A_{s\omega \ell m P (t,r)}|^2 = \frac{1}{4\pi} \int d\Omega \ < |A_{s\omega \ell m P (x)}|^2 >_\theta \ ,
$$

(2.63)

where $<>_\theta$ denotes a time or phase average. Define further

$$
<T_{\mu \nu} > = < T^{(2)}_{\mu \nu} > + < T_{\mu \nu} >_{\text{Maxwell}} + T_{\mu \nu}^{GW} \ .
$$

(2.64)

Combining the high-frequency approximation with Isaacson averaging leads to the tensor (at leading order)

$$
<T_{\mu \nu} > = \frac{2}{\epsilon^2} \sum_{s \ell m P} c_s \int_0^\infty d\omega \ (k_\mu)_\omega (k_\nu)_\omega |A_{s\omega \ell m P (t,r)}|^2 \ ,
$$

(2.65)
where

\[ c_0 = 1, \quad c_1 = \frac{1}{4\pi}, \quad c_2 = \frac{1}{32\pi}. \] (2.66)

As one would expect for a null fluid, one has

\[ < T^\sigma_\sigma > = 0 \] (2.67)

to leading order. The quantity \( \epsilon \) was regarded as a free parameter above, which helps in keeping track of the magnitudes of different quantities in the high-frequency approximation. But, given that the terms denoted by \( \exp\left(i\theta_\omega(t,r)/\epsilon\right) \) in Eqs.(2.25-27) are indeed of high frequency, one may then set \( \epsilon = 1 \) in future calculations, without loss of generality.

One can readily show that \( < T_{\mu\nu} > \) transforms as a tensor under \((t,r)\)-dependent 'background' coordinate transformations. Further, the equations of continuity (2.39,45,52) imply the conservation equation \( \nabla^\nu < T_{\mu\nu} > = 0 \). It is then natural to regard \( c_s |A_{sw\ell mP}|^2 \) the total intensity in the high-frequency perturbations, as a measure of the total energy density. But while the total energy is independent of the choice of space-like hypersurface, the notion of energy density only has significance with respect to a particular space-like hypersurface. Denoting the unit future-directed time-like normal vector to the hypersurface by \( n^{(0)}\), the energy density measured locally by an observer with 4-velocity \( n^{(0)}\) is

\[ \rho = n^{(0)\mu} n^{(0)\nu} < T_{\mu\nu} > = 2 \sum_{s\ell mP} \int_0^\infty d\omega (n.k_\omega)^2 c_s |A_{sw\ell mP}(t,r)|^2. \] (2.68)

As expected in perturbation theory about the spherically-symmetric background \((\gamma_{\mu\nu}, \Phi)\), the mass-energy of the massless waves is quadratic in their amplitude, for small \( |A_{sw\ell mP}| \). A further consequence of Eqs.(2.39,45,52) is that the quantity

\[ N_\omega = \sum_{s\ell mP} \int_\Sigma d^3x \sqrt{\left(3\gamma\right)} (n.k_\omega) |A_{sw\ell mP}|^2 \] (2.69)

is the conserved total number density (independent of space-like hypersurface) of massless waves (massless-scalar particles, photons and gravitons) passing through the space-like hypersurface \( \Sigma \).

3. Solution of background field equations

The Einstein field equations for a spherically-symmetric geometry of Lorentzian signature, of the form (2.34), may be derived from the Riemannian field equations, as given in Eq.(3.5-11) of [3], on replacing \( e^b \) by \((-e^b)\):

\[ a' = 8\pi r T_{rr} + \frac{(1-e^a)}{r}, \] (3.1)

\[ b' = 8\pi r e^{a-b} T_{tt} - \frac{(1-e^a)}{r}, \] (3.2)

\[ \dot{a} = 8\pi r T_{tr} , \] (3.3)
\[
1 - e^{-a} + \frac{1}{2} r e^{-a} (a' - b') - \frac{1}{2} r^2 R^{(0)} = 8\pi T_{\theta \theta} = \frac{8\pi T_{\phi \phi}}{\sin^2 \theta}, \quad (3.4)
\]

where

\[
R^{(0)} = -\frac{2}{r^2} (1 - e^{-a}) + e^{-\frac{1}{2}(a+b)} \left[ \partial_t \left( \dot{a} e^{\frac{1}{2}(a-b)} \right) - \partial_r \left( b' e^{\frac{1}{2}(b-a)} \right) \right] = -8\pi T_{\mu \mu}. \quad (3.5)
\]

As usual, for a massless scalar field \( \Phi(t,r) \), one has

\[
T_{\mu \nu} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} \gamma_{\mu \nu} (\Phi_{,\alpha} \Phi_{,\beta} \gamma^{\alpha \beta}).
\]

Equations (3.4,5) imply that

\[
T_{rr} = e^{(a-b)} T_{tt} \quad (3.6)
\]

whence, by Eqs.(3.1,2),

\[
\frac{1}{2} \left( a' - b' \right) = \left( 1 - \frac{e^a}{r} \right) \quad (3.7)
\]

We now derive the Vaidya metric, corresponding to a spherically-symmetric null-fluid source, in the form (2.34,35) quoted above. Taking the metric form (2.34), we define the function \( m(t,r) \) by

\[
e^{-a(t,r)} = 1 - \frac{2m(t,r)}{r} \quad (3.8)
\]

Using Eq.(3.7), we deduce an expression for \( e^b(t,r) \):

\[
e^b(t,r) = \left( 1 - \frac{2m(t,r)}{r} \right) \exp \left[ 4 \int_{\hat{r}}^{r} \frac{m'(t,\bar{r})}{(\bar{r} - 2m(t,\bar{r}))} \right] \quad (3.9)
\]

for some \( \hat{r} \). By elementary flatness at the origin, one must have \( a \to 0 \) as \( r \to 0 \) on each space-like hypersurface. Asymptotic flatness requires setting \( \hat{r} = R_\infty \) at the outer boundary, and then taking the limit \( R_\infty \to \infty \). Eq.(3.1) is the Hamiltonian constraint equation \([9,13]\). Use of Eq.(3.8) shows that Eq.(3.1) can be written as a first-order differential equation for the mass \( m(r) \) inside a radius \( r \) at time \( t = t_0 \), say:

\[
\frac{\partial m}{\partial r} = 4\pi r^2 \rho \quad (3.10)
\]

where

\[
\rho = e^{-b} T_{tt} \quad (3.11)
\]

is the energy density. Eq.(3.3) is the momentum constraint equation \([9,13]\).

We can now determine the background metric at late times, when the energy-momentum tensor is that of the black-hole radiation, following Vaidya \([7]\). As in Sec.2, we study the gravitational field produced by perturbations whose averaged energy-momentum tensor is \( <T_{\mu \nu}> \). (Later, we shall move to a coordinate system more suited to retarded radiation.) Since the direction \( (k^\mu)_\omega \) of propagation of the radiation in Sec.2 is null, we choose

\[
(k_r)_\omega e^{\frac{1}{2}(b-a)} + (k_t)_\omega = 0 \quad (3.12)
\]
which corresponds to an outgoing-wave boundary condition at large $r$. Eq. (3.3) also implies

$$< T^t_{\ t} > e^{\frac{b-a}{2}} + < T^t_{\ t} > = 0 \ .$$  \hspace{1cm} (3.13)

The field equations in terms of the metric functions $a$ and $b$ are as in Eqs. (3.1-5), but with $T_{\mu\nu}$ replaced by $< T_{\mu\nu} >$. Using the momentum constraint (3.3), the Hamiltonian constraint (3.1), with Eq. (3.13) and $\rho = - < T^t_{\ t} >$, we find

$$a' + \left( \frac{e^a - 1}{r} \right) + \dot{a} e^{(a-b)/2} = 0 \ .$$  \hspace{1cm} (3.14)

Using Eq. (3.8), one has

$$e^{(b-a)/2} = - \frac{\dot{m}}{m'} = - (k_r)_\omega (k_t)_\omega e^{-a} \ ,$$  \hspace{1cm} (3.15)

that is,

$$e^{b(t,r)} = \left( \frac{\dot{m}}{m'} \right)^2 \left( 1 - \frac{2m(t,r)}{r} \right)^{-1} \ .$$  \hspace{1cm} (3.16)

Then Eqs. (3.12,15) imply that

$$\theta_\omega = \theta_\omega(m) \ ,$$  \hspace{1cm} (3.17)

denoting an arbitrary function of $m$. Finally, one arrives at the Vaidya solution [7,14], in the form quoted in Eq. (2.34,35),

$$ds^2 = - \left( \frac{\dot{m}}{m'} \right)^2 \left( 1 - \frac{2m(t,r)}{r} \right)^{-1} dt^2 + \left( 1 - \frac{2m(t,r)}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \ ,$$  \hspace{1cm} (3.18)

describing the background space-time $\gamma_{\mu\nu}$ which results from the energy-momentum tensor of the high-frequency black-hole radiation.

A change of variables: $(t,r) \rightarrow (u,r)$ can also be found (see below), such that the line element [7] is of the Eddington-Finkelstein type [9]:

$$ds^2 = - \left( 1 - \frac{2m(u)}{r} \right) du^2 - 2 du \, dr + r^2 d\Omega^2 \ .$$  \hspace{1cm} (3.19)

Then radially-outgoing null geodesics are precisely paths of constant $u$. The function $m$ is now independent of $r$ and constant along outgoing null rays. In the generic case that $(dm/du)$ is not known, it has proved impossible to diagonalise the Vaidya metric and to write $u$ as an explicit function of $t$ and $r$. Since $\dot{m} < 0$ and $m' = (\partial m/\partial r) > 0$, one finds that, along lines $\{ u = \text{constant} \}$, $r$ increases with increasing $t$. As in the fixed-mass Schwarzschild solution, the Vaidya metric, in the form (3.18), has a coordinate singularity where $r = 2m(t,r)$. But, from the $(u,r)$ form (3.19), one can see that the apparent singularity in the metric (3.18) at $r = 2m(u)$ is only a coordinate singularity [14]. Further [14], the surface $\{ r = 2m(u) \}$ is space-like, lying to the past of the region $\{ r > 2m(u) \}$, since $(dm/du) > 0$. In fact, the geometry in the region $\{ r < 2m(u) \}$ (if
such a region exists in the `space-times’ considered here, as generated through solution of a boundary-value problem) would gradually deviate from the Vaidya form, as one moves to the past by (say) reducing $u$ while holding $r$ fixed, since one would reach the region of strong-field gravitational collapse. This region can still be described by the diagonal metric (2.34), but the full field equations enforce a more complicated coupled solution. Provided that the complexified boundary-value problem, outlined in Sec.1, is well-posed for a time-separation-at-infinity $T = |T| \exp(-i\theta)$, for $0 < \theta \leq \pi/2$, then the full (complex) Einstein/scalar classical solution studied here will be regular at the spatial origin $r = 0$. Indeed, a solution would then be regular everywhere (with respect to suitable coordinate charts) in the region between the initial hypersurface $\Sigma_I$ and the final hypersurface $\Sigma_F$. Since we are considering the case in which both $\Sigma_I$ and $\Sigma_F$ are diffeomorphic to $\mathbb{R}^3$, the solution should be regular on a region of the form $I \times \mathbb{R}^3$, where $I$ denotes the closed interval $[0, |T|]$. Note further that the regularity of the boundary data $(h_{ij}, \phi)$, as posed on $\Sigma_I$ and $\Sigma_F$, in the spherically-symmetric case, implies that the boundary value $m_{I,F}(r)$ obeys $2m_{I,F}(r) < r$ for all $r > 0$. Equality only holds at the centre of symmetry $r = 0$.

We now relate the Vaidya metric, as given in Eq.(3.18), to other coordinate forms of the Vaidya geometry. From Eq.(3.7) and from differentiating Eq.(3.15) with respect to $r$, we find

$$\left(\frac{m''}{m'} - \frac{\dot{m}}{\dot{m}} \right) \left(1 - \frac{2m}{r}\right) = - \frac{2m}{r^2} .$$

(3.20)

This can be rearranged in the form

$$\frac{\partial_t \left( m' \left(1 - \frac{2m}{r}\right) \right)}{\partial_r \left( m' \left(1 - \frac{2m}{r}\right) \right)} = \frac{\dot{m}}{m'} ,$$

(3.21)

which has the solution

$$m' \left(1 - \frac{2m}{r}\right) = f(m) ,$$

(3.22)

where $f(m) \geq 0$ is arbitrary [7]. Eq.(3.16) can now be rewritten using Eq.(3.22), to give

$$e^{b(t,r)} = e^{2\psi(t,r)} \left(1 - \frac{2m(t,r)}{r}\right) ,$$

(3.23)

where $e^{2\psi(t,r)}$ is defined as

$$e^{2\psi(t,r)} = \left(\frac{\dot{m}}{f(m)}\right)^2 .$$

(3.24)

Hence, the 4-metric can be written in the form

$$ds^2 = - e^{2\psi(t,r)} \left(1 - \frac{2m(t,r)}{r}\right) dt^2 + \left(1 - \frac{2m(t,r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 .$$

(3.25)
The Vaidya model in the context of black-hole radiation has, for example, been studied by Hiscock [15]. If one chooses \( f(m) = -\dot{m} \), then in Eq.(3.24) one has \( \psi(t,r) = 0 \), whence the Vaidya metric of Eq.(3.25) takes a particularly simple form.

Different choices of the function \( f(m) \) correspond to different physical models; once \( f(m) \) is specified, one determines \( m \) as a function of \( t \) and \( r \). One can consider the complexified boundary-value problem in the case (say) that exactly spherically-symmetric initial data \((\gamma_{ij}, \Phi)\) are given on the initial hypersurface \( \Sigma_I \), whereas on the final hypersurface \( \Sigma_F \), the data consist of a background spherically-symmetric part \((\gamma_{ij}, \Phi)_F\), together with weak 'linear-order' fluctuations \((h_{ij}^{(1)}, \phi^{(1)})_F\), which correspond in the classical theory to emitted gravitons and massless scalar particles. From experience with real elliptic partial differential equation theory, one might not unreasonably expect a unique classical solution to this Dirichlet boundary-value problem [16,17]. Hence, in particular, in the context of the high-frequency approximation of Sec.2, and of the Vaidya description of the corresponding outgoing-wave-source gravitational field of the present Section 3, one would expect that (for example) the detailed high-frequency coefficients \( A_{\omega \ell m}(t,r,\Omega) \), \( (A_{\mu})_{\omega \ell m \phi}(t,r,\Omega) \) and \( (A_{\mu \nu})_{\omega \ell m \phi}(t,r,\Omega) \) of Eqs.(2.25-27) should be determined by the above Dirichlet boundary data. Similarly, the 'free' function \( f(m) \) of Eq.(3.22) in the Vaidya description should also be determined, and indeed \( f(m) \) should be related to the detailed quantities \( A_{\omega \ell m} \), etc., above.

At late times and at correspondingly large radii \( r \), the semi-classical mass-loss formula should hold to great accuracy:

\[
-\dot{m} = \frac{\alpha(m)}{m^2},
\]

(3.26)

where \( \alpha(m) \) effectively accounts for the number of particles light enough to be emitted by a hole of mass \( m \), and \( \alpha(m) \) increases with decreasing \( m \) (here allowing for a more general model than our Einstein/massless-scalar case). This follows since, as described in Sec.3 of [5], the Bogoliubov coefficients for the quantum evaporation of the black hole [17a] are given, for practical purposes, by the standard expression

\[
|\beta_{s\omega \ell m}|^2 = \Gamma_{s\omega \ell m}(\tilde{m}) \left(e^{4\pi \tilde{m}} - (-1)^{2s}\right)^{-1}.
\]

(3.27)

Here, \( \Gamma_{s\omega \ell m}(\tilde{m}) \) is the transmission probability over the centrifugal barrier of the black hole for a mode with spin \( s \), frequency \( \omega \) and angular quantum numbers \((\ell, m)\), and \( \tilde{m} = 2M \omega \) is dimensionless, \( M \) being the space-like or total ADM (Arnowitt-Deser-Misner) mass of the space-time [9]. The original derivation of Eq.(3.27) was in the context where the black-hole singularity was taken to persist at late times. But, because of the very-high-frequency (adiabatic) method through which the above expression for \( |\beta_{s\omega \ell m}|^2 \) was calculated, it should still be valid (up to minute corrections) in the case presently being studied, in which there is assumed to exist a smooth final boundary \( \Sigma_F \) with topology \( \mathbb{R}^3 \). The derivation of Eq.(3.26) then follows as usual.

In particular, consider the late-time behaviour appropriate to our massless-field model, in which \( \alpha(m) = \alpha_0 = \text{constant} \). The large-\( r \) solution to Eq.(3.26), in the region where
where \( M_I \) and \( u_2 \) are constants and \( u \sim (t - r) \) at large \( r \). We set \( m(u_2) = M_I \) for some fixed \( u_2 \), so that, as \( u_2 \to -\infty \), the space-like (ADM) and null (Bondi) masses [9] agree.

Now introduce a null coordinate \( u = u(t, r) \), which agrees asymptotically with the above requirement \( u \sim (t - r) \), but which is defined everywhere, via the transformation

\[
 du = \left( 1 - \frac{2m(t,r)}{r} \right)^{-1} \left( -\frac{\dot{m}}{m'} dt - dr \right) = -\frac{\dot{m}}{f(m)} dt - \left( 1 - \frac{2m(t,r)}{r} \right)^{-1} dr .
\]  

It may be verified that this definition is integrable, as follows: Using Eq.(3.22), one can re-write Eq.(3.29) in the form

\[
du = -\left( \frac{\dot{m}}{f(m)} \right) dt - \left( \frac{m'}{f(m)} \right) dr = -\left( \frac{dm}{f(m)} \right) = d(\mu(m)) ,
\]

where we define

\[
\mu(m) = -\int \frac{dm}{f(m)} .
\]

It is then straightforward to apply the coordinate transformation implicit in Eq.(3.30), to derive the 'null form' (3.19) of the Vaidya metric from the alternative diagonal form (3.18).

We are now in a position to make further contact with the more detailed treatment in Sec.2 of the high-frequency expansions (2.25-27) for massless spin-0, spin-1, and spin-2 fields. In the coordinate system \((u, r, \theta, \phi)\) of Eq.(3.19), we write out Eqs.(2.25-27) in the form

\[
 \phi^{(1)}(u, r, \Omega) = \sum_{\ell m} \int_0^\infty d\omega \left[ A_{\omega \ell m}(u, r, \Omega) e^{i\omega(u,r)} + c.c. \right] ,
\]

etc. The only non-zero component of the null vector \((k^\mu)_{\omega}\) is \((k^r)_{\omega}\) [see Eq.(3.34) below], which, by Eq.(2.57), is in principle an arbitrary function of \( u \). Thus, the radiation, corresponding to outgoing waves at null infinity, travels freely along \( \{ u = \text{constant} \} \) light cones. Further, at any point in the Vaidya space-time, a local observer finds only one direction in which the radiant energy is flowing. Eq.(2.38) can now be solved to give

\[
 A_{s\omega \ell m P}(r, u, \Omega) = \frac{h_{s\omega \ell m P}(u, \Omega)}{r} ,
\]

where \( h_{s\omega \ell m P}(u, \Omega) \) is an arbitrary, dimensionless complex function. By this means, the coefficients \( A_{s\omega \ell m P} \) can be related to the distribution of weak-field massless-scalar, Maxwell and spin-2 graviton data on the final surface \( \Sigma_F \).

In the \((u, r, \theta, \phi)\) coordinate system, the only non-zero component of the Ricci tensor is

\[
 R_{uu} = -\frac{2}{r^2} \frac{dm}{du} = 8\pi <T_{uu}> .
\]
Then Eq.(2.65) gives
\[
-\frac{1}{4\pi r^2} \frac{dm}{du} = 2 \sum_{s\ell mP} c_s \int_0^\infty d\omega \left[ (k_u)_{s\ell mP} \right]^2 |A_{s\omega \ell mP}|^2 = \frac{2}{r^2} \sum_{s\ell mP} c_s \int_0^\infty d\omega \left[ (k_u)_{s\ell mP} \right]^2 |h_{s\omega \ell mP}(u, \Omega)|^2.
\]

(3.35)

Hence, \( m(u) \) can only decrease as \( u \) increases; the perturbation amplitudes are non-zero if and only if the mass \( m \) is changing. Just as the description in Eq.(3.33) of the coefficients \( A_{s\omega \ell mP} \) in the high-frequency approximations (2.25-27) leads to a relation between \( h_{s\omega \ell mP} \) and the perturbative spin-0, 1 and 2 data on the final surface \( \Sigma_F \), so Eq.(3.35) gives \( (dm/du) \) and hence \( m(u) \) for the 'background part' of the classical solution, in terms of the coefficients \( A_{s\omega \ell mP} \). That is, in setting up, as final data for gravitational collapse, the 'background part' \( (\gamma_{ij}, \Phi)_{\Sigma_F} \), together with the perturbative part, one should choose the radial dependence of the late-time background part \( \gamma_{ij} \) to allow for the mass function \( m(u) \) corresponding to Eq.(3.33) for the given particle species and spins.

Provided that the Lorentzian time-interval \( T \) at infinity is sufficiently large, one expects to study background 3-geometries \( \gamma_{ij} \) on the final hypersurface \( \Sigma_F \), which are nearly flat out to a certain large radius \( r = R_1 \), corresponding to the edge of the region in which the radiation reaches \( \Sigma_F \). For \( r > R_1 \), one expects \( \gamma_{ij} \) to correspond to a slowly-varying Vaidya metric, with \( m(u) = m(T - r) \) gradually increasing out to a radius \( R_2 \) which corresponds roughly to the beginning of the radiation. At radii \( r > R_2 \) on the final surface \( \Sigma_F \), the function \( m(u) \) should be approximately equal to \( M_I \), the conserved ADM mass of the system.

At the high-energy end of the emission spectrum, when the black hole approaches the Planck scale, the (thermal) mass-loss rate as given by Eq.(3.26) breaks down. The amount of energy emitted by the black hole in the final stages of the evaporation will be comparable to its mass, \( \omega \sim m \). To account for the small-mass behaviour of the black hole, therefore, the micro-canonical decay rate must be considered. The micro-canonical approach is generally more desirable, as the thermal equilibrium between a black hole and the exterior radiation is unstable, due to a negative specific heat in the canonical ensemble [18]. In addition, there is no information loss in the black-hole evaporation in the micro-canonical picture, as energy is conserved.

For the low-frequency quanta \( (\omega \ll M) \) characteristic of the majority of the evaporation process, however, the canonical and micro-canonical ensembles are almost equivalent, and one obtains a Planck-like number spectrum and the decay rate Eq.(3.26). The micro-canonical decay rate for small \( m \) has the form [19]
\[
-\dot{m} = 3 f(m) \sim \left( \frac{\lambda m_{pl}}{t_{pl}} \right) = \left( \frac{m}{m_{pl}} \right)^6 \exp\left( -4\pi \frac{m^2}{m_{pl}^2} \right),
\]

(3.36)
where \( \lambda \) is a numerical constant and \( m_{pl}, t_{pl} \) denote the Planck mass and Planck time, respectively. This further equation does not have the bad behaviour as \( m \to 0 \), associated with Eq.(3.26). Indeed, the free function \( f(m) \) will naturally have the corresponding form for small \( m \), as follows from a dimensional analysis of the field amplitudes.

4. Conclusion

We have seen in Sec.2 how the averaged effective energy-momentum tensor \(<T^{\mu\nu}>_{EFF}\) is calculated, describing, over scales of several radiation wavelengths, the way in which wave-like fluctuations in the spin-0 (scalar) field \( \phi^{(1)} \) and the spin-2 (graviton) part of the linearised gravitational field \( h_{\mu\nu}^{(1)} \) contribute quadratically as sources for the ‘background’ spherically-symmetric 4-metric and scalar field \( (\gamma_{\mu\nu}, \Phi) \). A similar description holds for the spin-1 Maxwell field. While this contribution is small at any one time, it persists with a comparable magnitude during the whole time \( t_0 \) during which the black hole radiates. Thus, its effects, particularly on the spherically-symmetric background metric \( \gamma_{\mu\nu} \), accrue secularly; indeed, the averaged contribution \(<T^{\mu\nu}>_{EFF}\) is precisely such as to determine the rate of loss of mass \((-\dot{m})\) in the familiar fashion, leading to the eventual disappearance of the central concentration of mass, when one works with a complexified time-interval \( T = |T| \exp(-i\theta) \), with \( 0 < \theta \leq \pi/2 \), for which one expects a classical solution which is regular between the initial hypersurface \( \Sigma_I \) and final hypersurface \( \Sigma_F \).

Such an averaged effective energy-momentum source leads to an approximate space-time geometry \( g_{\mu\nu} \) of the Vaidya type, as described in Sec.3, valid in the space-time region containing the outgoing radiation. This Vaidya description is in turn essential in the treatment of adiabatic radial mode equations, as in [3-5]. Subsequently, in [20], we have generalised the boundary-value treatment in [1-4], which refers to quantum amplitudes with only spin-0 perturbative data on the final hypersurface \( \Sigma_F \), at a late time \( T \). In [20], we treat the other bosonic cases of spin-1 and spin-2 final data, by means of similar but more complicated methods. The fermionic massless spin-1/2 case is treated in [21], and a treatment of the remaining fermionic spin-3/2 case is in preparation [22]; this is needed as part of the treatment of locally-supersymmetric models. In all these examples, understanding of the Vaidya description is essential.

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References

[1] A.N.St.J. Farley and P.D. D’Eath, Phys Lett. B 601, 184 (2004).
[2] A.N.St.J. Farley and P.D. D’Eath, Phys Lett. B 613, 181 (2005).
[3] A.N.St.J. Farley and P.D. D’Eath, ‘Quantum Amplitudes in Black-Hole Evaporation: I. Complex Approach’, submitted for publication (2005).
[4] A.N.St.J. Farley and P.D. D’Eath, ‘Quantum Amplitudes in Black-Hole Evaporation: II. Spin-0 Amplitude’, submitted for publication (2005).
[5] A.N.St.J. Farley and P.D. D’Eath, ‘Bogoliubov Transformations in Black-Hole Evaporation’, submitted for publication (2005).
[6] H. Stephani et al., Exact Solutions to Einstein’s Field Equations, 2nd. ed, (Cambridge University Press, Cambridge) (2003).
[7] P.C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951).
[8] R. Isaacson, Phys. Rev. **166**, 1263, 1272 (1968).
[9] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, (Freeman, San Francisco) (1973).
[10] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd. edition, (Princeton University Press, Princeton) (1992).
[11] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York) (1999).
[12] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[13] P.D. D’Eath, *Supersymmetric Quantum Cosmology*, (Cambridge University Press, Cambridge) (1996).
[14] R.W. Lindquist, R.A. Schwartz and C.W. Misner, Phys. Rev. **137**, 1364 (1965).
[15] W.A. Hiscock, Phys. Rev D **23**, 2813, 2823 (1981).
[16] P.R. Garabedian, *Partial Differential Equations*, (Wiley, New York) (1964).
[17] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, (Cambridge University Press, Cambridge) (2000); O. Reula, ’A configuration space for quantum gravity and solutions to the Euclidean Einstein equations in a slab region’, Max-Planck-Institut für Astrophysik, **MPA 275** (1987).
[17a] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975); N.D. Birrell, P.C.W. Davies, *Quantum Fields in Curved Space*, (Cambridge University Press, Cambridge) (1982); V.P. Frolov, I.D. Novikov, *Black Hole Physics*, (Kluwer Academic, Dordrecht) (1998).
[18] S.W. Hawking, Phys. Rev. D **13**, 191 (1976).
[19] R. Casadio, B. Harms, and Y. Leblanc, Phys. Rev. D **58**, 044014 (1998).
[20] A.N. St. J. Farley and P.D. D’Eath, Class. Quantum Grav. **22**, 2765 (2005).
[21] A.N. St. J. Farley and P.D. D’Eath, Class. Quantum Grav. **22**, 3001 (2005).
[22] A.N. St. J. Farley and P.D. D’Eath, ’Spin-3/2 Amplitudes in Black-Hole Evaporation’, in progress.