ONE-DIMENSIONAL PROJECTIONS AND MONOTONE PATHS ON CROSS-POLYTOPES AND OTHER REGULAR POLYTOPES

ALEXANDER E. BLACK AND JESÚS A. DE LOERA

Abstract. Projections of regular convex polytopes have been studied for a long time (see work by Coxeter, Petrie, Schläfli, others). One-dimensional linear projections may not appear interesting at first sight, but a linear functional orders the vertices of a polytope and yields an orientation of the graph of the polytope in question. The monotone paths of a polytope are the directed paths on the oriented one-skeleton from the minimum to the maximum. Here we revisit projections for regular polytopes from the lens of monotone paths. Billera and Sturmfels introduced a construction in the early 1990s called the monotone path polytope, which is a polytope whose face lattice encodes the combinatorial structure of certain monotone paths on a polytope. They computed the monotone path polytopes of simplices and cubes, which yield a combinatorial cube and permutohedron respectively. However, the question remains: What are the monotone path polytopes of the remaining regular polytopes? We answer that question in this paper for cross-polytopes, the dodecahedron, the icosahedron, and partially for the 24-cell, 120-cell, and the 600-cell.

1. Introduction

The Platonic solids are some of the most recognizable geometric objects to a general audience. Their study goes back at least two thousand years, yet questions still remain unanswered about them. Similarly, the cross-polytopes or hyperoctahedra are highly recognizable to mathematicians as the unit-balls in $\mathbb{R}^d$ with respect to the $\ell^1$-norm. Both types of polytopes are examples of regular convex polytopes, polytopes with a symmetry group that acts transitively on flags of faces. Characterizing symmetric two dimensional and three dimensional projections of regular polytopes is a classical problem in geometry studied by Schläfli, Schlegel, Petrie, and others (see Chapter XIII of Coxeter’s book [3]). The focus in this paper is studying the regular polytopes in terms of their one-dimensional linear projections.

In the 1990’s Billera and Sturmfels developed a construction that, given a polytope $P$, associates a polytope to the linear projection of $P$, called the fiber polytope (see [2] and Chapter 9 of [12]). Fiber polytopes extract complicated combinatorial structure even from one-dimensional projections. From the perspective of Coxeter, the one-dimensional projects were only intervals and thus not of interest. However, with this tool, we find a natural new characterization problem for one-dimensional projections of these special polytopes. The fiber polytopes of one-dimensional projections are called monotone path polytopes (MPPs) due to their vertices corresponding to certain increasing paths on the one-skeleton of the polytope and edges to polygon flips between those monotone paths. For example, for a generic projection, a monotone path polytope of a simplex is a combinatorial cube, and a monotone path polytope of a cube is a permutahedron. Our Theorem 1.1 provides a classification for fiber polytopes of the cross-polytopes with respect to a generic one-dimensional projection.

Theorem 1.1. For any generic linear functional $\pi$ such that $\pi(e_i) = a_i$ for all $i \in [n],$

1. The total number of monotone paths in $\diamond^n$ is precisely $\frac{2^{2n-1} - 2}{3}$. Not all paths are coherent.
2. The monotone path polytope, the signohedron, $MPP(\diamond^n)$ is combinatorially equivalent to the cubical complex formed by gluing together all unit cubes of dimension $\leq n - 2$ with vertices

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contained in \(\{0, \pm 1\}^{n-1} \setminus \{0\}\). Its face lattice is isomorphic to the poset of intervals in the sign poset \(\{0, +, -\}^{n-1} \setminus \{0\}\) ordered under inclusion.

(3) The f-vector of \(\text{MPP}(\phi^n)\) is given by

\[
f_m(\text{MPP}(\phi^n)) = \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^{k+m} = 2^m \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^k.
\]

Hence, \(\text{MPP}_e(\phi^n)\) has precisely \(3^{n-1} - 1\) vertices. In particular, they correspond to a sign vector in \(\{0, +, -\}^{n-1} \setminus \{0\}\).

(4) Two vertices in \(\text{MPP}(\phi^n)\) are adjacent if and only if their corresponding vectors are distance 1 from one another in the Taxi Cab metric. The diameter of

\[
\text{MPP}_e(\phi^n) = 2(n-1) = (n-1)\text{diam}(\phi^n).
\]

The diameter of the entire flip graph of \(\phi^n\) is \(2(n-1)\). The longest flip distance to the nearest coherent path is \(n-2\).

(5) There is an explicit polyhedral realization of the signohedron \(\text{MPP}(\phi^n)\). In terms of facets, we have for a linear functional such that \(0 < a_1 < a_2 < \cdots < a_n\) that

\[
\text{MPP}_{\pi}(\phi^n) = \{x \in \mathbb{R}^n : \pi(x) = 0 \text{ and } \varphi_{i,\varepsilon}(x) \geq -a_i - a_n \text{ for all } \varepsilon : [n-1] \to \{\pm 1\}, k \in [n-1]\},
\]

where we define \(\varphi_{i,\varepsilon}\) on the basis \(F_1 \cup F_2 \cup \{e_n\}\) by

\[
\varphi_{i,\varepsilon}(e_k) = \begin{cases} 
-a_k - a_n & \text{if } k \in F_1 \\
\frac{a_i + a_n}{a_n - a_2}(a_k - a_n) & \text{if } k \in F_2 \\
0 & \text{if } k = n
\end{cases}
\]

for \(F_1 = \{k : \varepsilon(k)k \leq i\}\) and \(F_2 = \{k : \varepsilon(k)k \geq i\}\).

(6) For a linear functional such that \(0 < a_1 < a_2 < \cdots < a_n\), the set of vertices of the Signohedron \(\text{MPP}(\phi^n)\) is precisely:

\[
\left\{ \left( 1 - \frac{a_{i_k} + a_{i_1}}{2a_n} \right) e_n + \sum_{i=1}^{k} \left( \frac{a_{i_{k-1}} + a_{i_{k+1}}}{2a_n} \right) e_{i_k} : -n = i_0 < \cdots < i_{k+1} = n \text{ and } i_a \neq -i_b \text{ for all } a, b \in [k] \right\}.
\]

The combinatorial type of these monotone path polytopes corresponds exactly to the poset of intervals of the sign poset from oriented matroid theory as one would find in [1]. As a result, we refer to these polytopes as the **signohedra**. This result turns out to be completely analogous to that of the case of the simplex. Namely, the face lattice of the simplex may be thought of as the poset of subsets of \([n]\) under inclusion. The poset of intervals in that poset is precisely the face lattice of a cube, which is combinatorially equivalent to the monotone path polytope of a simplex. Similarly, the face lattice of the cross-polytope may be thought of as the sign poset. The poset of intervals in that poset is another cubical complex that, by Theorem [1.1], is combinatorially equivalent to a monotone path polytopes of the cross-polytope.

Furthermore, via a functorial lemma proven in [2], the monotone path polytope of a projection of a polytope is the projection of that monotone path polytope. The projections of the cross-polytopes are precisely the **centrally symmetric polytopes**, which may be understood as the set of piece-wise linear unit balls in \(\mathbb{R}^d\). Thus, Theorem [1.1] yields the following corollary:
Corollary 1.2. Let $P$ be a centrally symmetric polytope with $2n$ vertices $\pm v_1, \ldots, \pm v_n$ and linear functional $\ell$ such that $0 < \ell(v_1) < \cdots < \ell(v_n)$. Let $a_i = \ell(v_i)$. Then we have the following:

$$
MPP_\ell(P) = \text{conv}\left\{ \left(1 - \frac{a_{i_k} + a_{i_{k+1}}}{2a_n}\right)(v_n) + \sum_{i=1}^{k} \left(\frac{a_{i_{k-1}} + a_{i_{k+1}}}{2a_n}\right)v_{i_k} : -n = i_0 < \cdots < i_{k+1} = n \text{ and } i_a \neq -i_b \text{ for all } a, b \in [k]\right\}.
$$

Understanding the structure of monotone paths on centrally symmetric polytopes could yield insight into the polynomial Hirsch conjecture (see [7]). That conjecture asks for a polynomial bound on the lengths of paths on polytopes in terms of the number of facets and dimension and is of fundamental interest in applications due to its relationship to the run-time of the simplex method for linear programming. The problem remains open for centrally symmetric polytopes.

For one-dimensional projections, the interpretation in terms of monotone paths also provides an alternative to the study of Petrie polygons and Schlegel diagrams as described in [3]. Instead of characterizing highly symmetric paths on the surfaces of Platonic solids, one could instead characterize the monotone paths on their surfaces. Theorem 1.3 provides a characterization of the combinatorics of monotone paths on the Platonic solids both in terms of the monotone path polytope and the flip graph, a graph of all monotone paths on the surface of a polytope. For the remaining Platonic solids, our results are primarily computational.

**Theorem 1.3.** For a generic linear functional, any Platonic solid has the same orientation. With respect to monotone paths, the statistics for non-coherent paths in Table 1 and pictures from Tables 2 and 4 hold for a generic linear functional. The coherent statistics are computed for the generic linear functional $[7, 7^2, 7^3]$.

| Polytope     | Number of Monotone Paths | Number of Coherent Monotone Paths |
|--------------|--------------------------|-----------------------------------|
| $n$-simplex  | $2^{n-1}$                | $2^n$                             |
| $n$-cube     | $n!$                     | $n!$                              |
| $n$-cross-polytope | $\frac{2^{2n-1} - 2}{3}$ | $3^{n-1} - 1$                    |
| Dodecahedron | 14                       | 12                                |
| Icosahedron  | 62                       | 20                                |

Table 1. Counts of monotone paths and coherent monotone paths (i.e., those monotone paths that correspond to the vertices of the monotone path polytope)

The natural next direction after our results is to turn to higher dimensions. Namely, one could study fiber polytopes of higher dimensional projections of the cross-polytopes and the monotone path polytopes of regular 4–polytopes. In Section 5 we describe some progress and difficulties in pursuing those directions. Section 6 presents some further directions and conclusions.

## 2. Background

Throughout this paper, we rely on a familiarity with convex polytopes, in particular the structure of monotone path polytopes, at the level of [12]. A comprehensive reference for the structure of monotone path polytopes may be found in [9], but we reproduce it here for convenience.
Definition 2.1. A monotone path polytope (MPP) of a polytope $P$ and orientation induced by a linear functional $\ell : P \to \mathbb{R}$ is the fiber polytope induced by the map $\ell : P \to \ell(P)$. As shown in [2], the monotone path polytope is given by

$$MPP_\ell(P) = \text{conv} \left( \int_{\ell(P)} s(x) dx : s \text{ is a section of } \ell \right).$$

Furthermore, Billera and Sturmfels showed in the same paper that the integrals of the sections of monotone paths generate the monotone path polytope. This provides a finite generating set for that infinite space. From this result, we obtain a way to compute monotone path polytopes.

Theorem 2.2 ([2]). For a linear functional $\varphi$ and polytope $P$ with vertices ordered by the linear functional $p_1, p_2, \ldots, p_n$,

$$MPP_\varphi(P) = \text{conv} \left( \sum_{j=1}^{k} \frac{\varphi(p_{i_j} - p_{i_{j-1}})}{2\varphi(p_n - p_1)} (p_{i_{j-1}} + p_{i_j}) : (p_{i_j})_{j \in [k]} p_1 = i_0 < i_1 < \cdots < i_k = p_n \right).$$
The example of an incoherent path from page 295 of [12]. By Lemma 2.5, the path in the top picture is incoherent, because $\operatorname{conv}(v_1,v_5) \cap \operatorname{conv}(v_2,v_3,v_4) \neq \emptyset$. The bottom picture shows visually why that intersection is problematic.

The monotone paths that give rise to vertices in the resulting monotone path polytope are called coherent. Faces of the monotone path polytope correspond to coherent cellular strings, a generalization of coherent paths. We take the following definition directly from [9]:

**Definition 2.3 ([9]).** Fix a $d$-polytope $P$ and orientation $\ell \in (\mathbb{R}^d)^*$. A cellular string is a sequence of faces $F_1,F_2,\ldots,F_n$ that satisfies the following:

(i) $v_{\min} \in F_1$ and $v_{\max} \in F_n$, where $v_{\min}$ and $v_{\max}$ are minimal and maximal vertices respectively with respect to $\ell$.

(ii) $\ell$ is non-constant on any $F_i$.

(iii) For each $i$, the $\ell$-maximizing face of $F_i$ is the $\ell$-minimizing face of $F_{i+1}$.

A cellular string is called coherent if there exists some linear functional $\ell' \in (\mathbb{R}^d)^*$ such that $\bigcup_{i=1}^n F_i$ is the union of all points $x \in \ell(P)$ of the $\ell'$-minimal points in the fibers $\ell^{-1}(x)$. More concretely, a cellular string is coherent if it factors through a shadow map in the projection.

Using the tools we know about fiber polytopes, we may describe the face lattice of monotone path polytopes completely by identifying each face with a coherent cellular string.

**Theorem 2.4 ([2]).** The face lattice of a monotone path polytope is equivalent to the lattice of coherent cellular strings with the partial order induced by the refinement of subdivisions.

Thus, the combinatorics of a MPP of a polytope is completely determined by the structure of that polytope’s coherent cellular strings. We will use this connection to give a complete description of the MPPs of cross-polytopes up to combinatorial equivalence. Applying this result, the coherent monotone paths are the monotone paths whose vertices mapped to the lower vertices of some two dimensional projection. Knowing this property yields a simple geometric obstruction to coherence captured by the Lemma 2.5 and Figure 1.

**Lemma 2.5.** For any polytope $P$, any coherent monotone path $v_1,v_2,\ldots,v_n$ on $P$ must satisfy

$$\operatorname{conv}(v_i,v_j) \cap \operatorname{conv}\left(\bigcup_{k=i+1}^{j-1} v_k\right) = \emptyset.$$ 

**Proof.** The lower vertices in a polygon must satisfy this condition, and it is preserved under projection. □
As a corollary, we obtain a general result for coherent monotone paths on centrally symmetric polytopes. Namely, we have the following:

**Corollary 2.6.** A coherent monotone path in a centrally symmetric polytope cannot contain a pair of antipodes other than its min and max.

**Proof.** Since \( \text{conv}(−m, m) \cap \text{conv}(−v, v) \neq \emptyset \) for any vertex \( v \) and the maximal vertex \( m \), we have by Lemma 2.5 that a coherent path cannot contain both \( −v \) and \( v \).

This intuitive geometric obstruction turns out to be the only obstruction to coherence of monotone paths on the cross-polytopes. We will generalize this result in the next section where we characterize all coherent cellular strings of a cross-polytope. The last general fact we require is the following functorial lemma from [2] that allows for the computation of the monotone path polytope of a projection of a polytope.

**Lemma 2.7 ([2]).** Let \( P \xrightarrow{θ} Q \xrightarrow{π} R \) be a sequence of surjective affine maps of polytopes. Then \( Σ(Q, R) = θ(Σ(P, R)) \). In particular, for \( π \), a linear functional, we find that \( \text{MPP}_ϕ(Q) = π(\text{MPP}_ϕθ(P)) \).

This lemma allows for the computation of monotone path polytopes of any centrally symmetric polytope as the projection of a signohedron and makes Corollary 1.2 immediate from the proof of Theorem 1.1.

### 3. Signohedra: Monotone Paths on Cross-Polytopes

Our goal if this section is to provide as much information as we can about the combinatorics of the signohedra and prove Theorem 1.1. To do this, we require a few basic facts about the structure of cross-polytopes that will be taken for granted for the remainder of this section. First, there is the definition.

**Definition 3.1.** The \( d \)-cross-polytope \( ⋄_d \) is given by \( \text{conv}(\{±e_i : i \in [d]\}) \).

Then the following well known lemma will be used implicitly throughout.

**Lemma 3.2.** Cross-polytopes are simplicial. In particular, a subset of vertices of the cross-polytope is a face if and only if it does not contain pairs of antipodes.

One may easily verify this lemma by noting that the facet defining relations of the polytope are given by \( ||x|| \leq 1 \) and applying results from Chapter 2 of [12] on the face lattices of polytopes. To start studying the MPPs of cross-polytopes for a generic orientation, we must first clarify what was meant by a generic orientation. Using this notion of generic, we will fix an ordering of the vertices of the cross-polytope that will be used for all remaining computations of the signohedra.

**Theorem 3.3.** Generically, a monotone path polytope of \( ⋄^n \) is affinely equivalent to one obtained from a linear functional with distinct positive values for each \( e_i \).

**Proof.** The vertex generic linear functionals on the cross-polytope are precisely those with distinct nonzero values of coefficients \( a_i \) for each \( e_i \). Furthermore, they each map \( e_i \rightarrow a_i \), where up to a change in indices, \( |a_1| < |a_2| < \cdots < |a_n| \). Then, by applying the reflection map taking \( e_i \rightarrow −e_i \), we may assume that each \( a_i \) is positive. The cross-polytope \( ⋄^n \) has vertices \( ±e_i \), so under this map, the vertices are ordered \( −e_n < −e_{n−1} < \cdots < −e_1 < e_1 < e_2 < \cdots < e_{n−1} < e_n \).

Hence, up to a permutation and reflection, we always obtain the same vertex ordering. Since these symmetries are linear, by Lemma 2.7, the affine isomorphism from the cross-polytope to itself induces an affine isomorphism of the monotone path polytopes.
For cubes and simplices, all monotone paths are coherent. This property makes understanding their monotone path polytopes relatively easy. For cross-polytopes with an orientation given by a generic linear functional, the monotone paths need not all be coherent via the obstruction from Lemma 2.3. The following theorem is the primary technical fact from which all of our remaining work on the characterization follows.

**Theorem 3.4.** A cellular string on $\mathcal{P}^n$ is coherent if and only if the set of vertices contained in some cell in the string only contains one pair of antipodes, namely the maximum and minimum pair.

**Proof.** Without loss of generality, we may choose an orientation such that

$$-e_n < -e_{n-1} < \cdots < -e_1 < e_1 < \cdots < e_n.$$ 

Let $\varphi$ be a linear functional showing that this cellular string is coherent. Since $-e_i$ and $e_i$ are both lower vertices, the slope from $-e_i$ to $e_i$ must be between the slopes from $-e_n$ to $-e_1$ and from $e_1$ to $e_n$ in the projection. Captured in an equation, we must have:

$$\frac{\varphi(e_n) - \varphi(e_i)}{a_n - a_i} = \frac{\varphi(-e_i) - \varphi(-e_n)}{-a_i + a_n} < \frac{\varphi(e_i) - \varphi(-e_i)}{2a_i} < \frac{\varphi(e_n) - \varphi(e_i)}{a_n - a_i},$$

a contradiction.

Suppose instead we have a cellular string satisfying this condition. The vertices in the cellular string are of the form $S_+, S_0 \subseteq \{e_1, e_2, \ldots, e_n\}$ and $S_- \subseteq \{-e_1, -e_2, \ldots, -e_n\}$ such that

$$S_+ \cap -S_- = \{e_n\} \text{ and } S_+ \cap S_0 = -S_- \cap S_0 = \emptyset$$

and $S_+ \cup -S_- \cup S_0 = S_+$. Note that $(S_+ \cup S_-) \setminus \{-e_n\}$ is linearly independent. Let $F_1 < F_2 < \cdots < F_k$ denote the chain of faces in the cellular string. Let $e_{i-1} = -e_i$ and $a_{i-1} = -a_i$ for $i \in [n]$. Then, by linear independence, we may define $\varphi$ however we choose for each vertex $e_i$ of $F_j$. Let $e_{b_j}$ and $e_{c_j}$ denote the minimal and maximal vertices of $F_j$. Define $\varphi(e_{n}) = 0$. Define $\varphi(e_{b_j})$ to be $-\varphi(a_{c_j})$. Then the slope from $(-a_{b_j}, \varphi(-e_n))$ to $(a_{c_j}, \varphi(a_{c_j}))$ will be precisely $-1$. Define $\varphi$ inductively so that the slope from $(a_{b_j}, \varphi(e_{b_j}))$ to $(a_{c_j}, \varphi(e_{c_j}))$ is $\frac{1}{j}$ for all $1 \leq j < k$. For the remaining vertices in each $F_j$, define $\varphi$ as the linear interpolation between $e_{b_j}$ and $e_{c_j}$. Finally, define $\varphi$ to be 0 for all vertices in $S_0$.

This definition of $\varphi$ is precisely what is required in Definition 2.3. To justify this, observe that $\varphi$ is negative for each vertex in $F_j$ and non-negative otherwise. Since $-e_n$ and $e_n$ are both in some $F_j$, it follows that the lower edges of the polygon given by the projection $(\ell, f)$ must be some subset of the $F_i$. By construction, we defined the $F_i$ to each be mapped to an edge and so that the slope of each edge increases as $i$ increases. These edges yield a path from $(-a_{n}, -\varphi(e_n))$ to $(a_{n}, \varphi(e_n))$. Since the slope is increasing, these edges are the lower edges of the convex hull of that path. Furthermore, note that $\varphi$ is larger for any vertex not contained in this path. Thus, any point in the convex hull of an additional vertex to the path not already contained in the convex hull of the path must lie above that path. Hence, the projections of each $F_j$ yield lower edges of the polygon.

Interpreted for cellular strings corresponding to monotone paths, we establish what was suggested in the background section.

**Corollary 3.5.** A monotone path on $\mathcal{P}^n$ is coherent if and only if the only antipodes it contains are the maximum and minimum pair.

Note the assumption that the linear functional is generic is necessary here. For any $S$-hypersimplex, which includes simplices, cubes, and cross-polytopes, the following theorem holds from [3]:

**Theorem 3.6.** Let $S \subseteq [0, d]$ be proper. Then all $\sum_{i=1}^{n} e_i^T$-monotone paths of $\Delta(d, S)$ are coherent.
They also provide an explicit construction of $MPP(\phi^n)$ for that given linear functional. However, since the orientation induced by that linear functional is not generic, our work is still very much relevant. As an immediate corollary of this:

**Corollary 3.7.** A polytope being all-coherent (i.e., such that all monotone paths are coherent as defined in [6]) for one orientation does not imply it is all-coherent for all orientations. Furthermore, a polytope being generically not all-coherent does not imply it is not all-coherent for all orientations.

By Theorem 2.4, the coherent paths correspond exactly to the vertices of the monotone path polytope. From this result, we may immediately compute the number of vertices.

**Corollary 3.8.** For a generic linear functional, $MPP(\phi^n)$ has precisely $3^{n-1} - 1$ vertices. In particular, they correspond to elements of $\{-1, 1, 0\}^{n-1} \setminus 0$.

**Proof.** Without loss of generality, we may choose an orientation such that

$$-e_n < -e_{n-1} < \cdots < -e_1 < e_1 < \cdots < e_n.$$

Recall that any two non-antipodal points of the cross-polytope are connected by an edge. It follows then that the coherent monotone paths consists of choices $e_i, -e_i$ or neither to include in our sequence of points. That gives $3^{n-1}$ possible choices. Since we have to include at least 1 element between $-e_n$ and $e_n$, we obtain that $\phi^n$ has precisely $3^{n-1} - 1$ coherent monotone paths. Since the vertices of $MPP(\phi^n)$ correspond to coherent monotone paths, $MPP(\phi^n)$ has precisely $3^{n-1} - 1$ vertices.

Understanding coherent monotone paths allows us to describe the vertices of the polytope. Again by Theorem 2.4, edges in the monotone path polytope correspond to refinements of pairs of coherent monotone paths. Geometrically, we may interpret this refinement as precisely when two monotone paths agree everywhere except on a single two dimensional face. In the sense of the flip graph, this interpretation means that the two monotone paths differ by a polygonal flip. From this observation, we obtain the following lemma.

**Lemma 3.9.** Two vertices in $MPP(\phi^n)$ are adjacent if and only if their corresponding vectors are distance one from one another in the Taxi Cab metric.

**Proof.** Recall that $\phi^n$ is simplicial. It follows that a polygonal flip either deletes a vertex from a path or adds a single, new vertex. Deleting or adding a vertex corresponds to changing a 1 or $-1$ to a 0 or a 0 to a 1 or $-1$ in the sequence bijection from Corollary 3.8. The connectivity of $\phi^n$ allows us to perform this operation for any element of the sequence. Two sequences of 1’s, 0’s, and $-1$’s are at distance 1 in the Taxi-cab metric if and only if they agree on all but one entry in which one is a 0 and the other is a $-1$ or 1, which yields the result.

Via explicit computation, we may then compute the diameter of $MPP(\phi^n)$:

**Corollary 3.10.** The diameter of $MPP(\phi^n) = 2(n-1) = (n-1)diam(\phi^n)$.

**Proof.** By the triangle inequality,

$$\sup_{x, y \in \text{Verts}(MPP(\phi^n))} |x - y|_1 \geq \left| \sum_{i=1}^{n-1} e_i - \sum_{i=1}^{n} -e_i \right|_1 \geq 2(n-1).$$

Since each step along an edge can change the distance by at most one, we have the diameter is at least $2(n-1)$. To see that it is at most $2(n-1)$ may be seen by taking two vectors and changing the coordinates by $\pm 1$ until one vector equals the other. Each coordinate change represents a single step, and the path requires at most $2(n-1)$ coordinate changes. The only detail is avoiding the origin, and that is also easy.
Using our characterization of coherent cellular strings, we obtain a complete characterization of the face lattice of $MPP(o^n)$ in terms of the sign poset on $\{+, -, 0\}^{n-1} \setminus 0$.

**Theorem 3.11.** We have $MPP(o^n)$ is combinatorially equivalent to the cubical complex formed by gluing together all unit cubes of dimension $\leq n - 2$ with vertices contained in $\{\pm 1, 0\}^{n-1} \setminus \{0\}$. Equivalently, its face lattice is isomorphic to the poset of intervals in the poset $\{+, -, 0\}^{n-1} \setminus \{0\}$ ordered under inclusion.

**Proof.** By Theorem 2.4 the face lattice of the monotone paths corresponds to the poset of coherent cellular strings under refinement. Note that a coherent cellular string is uniquely determined by two convex sets of vertices contained in $\{\pm 1, 0\}^{n-1} \setminus \{0\}$. From Theorem 3.11, faces correspond exactly to elements of the poset of coherent cellular strings under refinement. Note that each interval $I = [a, b]$ in $\{0, -, +\}^{n-1} \setminus \{0\}$ is Boolean. Furthermore, conv($[a, b]$) $\cong$ conv($[a, b] - a$) $\cong$ conv($[0, b - a]$), where $0$ is adjoined in the natural way to the partial order. Let $k$ be the length of the interval. Then $b - a$ will have precisely $k$ nonzero entries of either 0’s or 1’s. Let $A$ be the linear map defined by $A(e_i) = \sigma(i)e_i$, where $\sigma(i)$ denotes the sign of $e_i$ in $b - a$. Let $S = \{i \in [n - 1] : \sigma(i) \neq 0\}$. Then $A([0, b - a]) = [0, \sum_{i \in S} e_i]$. By construction $A$ is then an isometry taking conv($I$) to the unit cube. Hence, each interval corresponds to a unit cube of dimension $\leq n - 2$ with vertices contained in $\{\pm 1, 0\}^{n-1} \setminus \{0\}$. The converse is similar, since each unit cube has vertices corresponding to an interval in the poset $\{+, -, 0\}^{n-1} \setminus \{0\}$. 

From this characterization of the face lattice, we have a combinatorial framework for calculating the $f$-vector explicitly.

**Corollary 3.12.** We may compute the $f$-vector of the $MPP(o^n)$ explicitly as:

$$f_m(MPP(o^n)) = \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^{k+m} = 2^m \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^k.$$  

**Proof.** From Theorem 3.11 faces correspond exactly to elements of the poset of intervals in the sign poset. Namely, they are identified precisely by pairs of elements of comparable elements of the poset. An $m$ face corresponds to pairs of elements of distance $m$ from each other. That is one has $k$ nonzero entries, and the other has $k + m$ nonzero entries including the $k$ nonzero entries of the starting point. Thus, the faces correspond to flags of length two of subsets of $n - 1$ of subsets of size $k$ and $k + m$ counted by $\binom{n-1}{k, m, n-k-m-1}$ together with $2^{k+m}$ choices of signs for each vertex contained in the flag. Then we have that

$$f_m(MPP(o^n)) = \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^{k+m} = 2^m \sum_{k=1}^{n-m-1} \binom{n-1}{k, m, n-k-m-1} 2^k.$$  

Furthermore, in [12], Ziegler provides a method for computing the facets of a MPP given an understanding of the coherent cellular strings that apply to them. We apply that methodology in this case and give a complete set of facet defining relations for $MPP(o^n)$.

**Theorem 3.13.** In terms of facets, we have that

$$MPP(o^n) = \{x \in \mathbb{R}^n : \varphi_{i, \epsilon}(x) \geq -a_i - a_n \text{ for all } \epsilon : [n - 1] \to \{\pm 1\}, k \in [n - 1] \} \cap \{x \in \mathbb{R}^d : \pi(x) = 0\},$$
where we define $\varphi_{i, \varepsilon}$ on the basis $F_1 \cup F_2 \cup \{e_n\}$ by

$$
\varphi_{i, \varepsilon}(e_k) = \begin{cases} 
-a_k - a_n & \text{if } k \in F_1 \\
\frac{a_{k+1} + a_n}{a_n - a_i} (a_k - a_n) & \text{if } k \in F_2 \\
0 & \text{if } k = n
\end{cases}
$$

for $F_1 = \{k : \varepsilon(k)k \leq i\}$ and $F_2 = \{k : \varepsilon(k)k \geq i\}$.

**Proof.** To find the facet defining relations for the MPP, we follow the method outlined in Section 9.1 of [12]. Namely, the facet defining inequalities are obtained from the linear functional that yields the polygon whose lower vertices correspond to maximal coherent subdivisions. The inequality is precisely that for a linear functional of that form $\varphi$, the evaluation must be at least that of the lowest vertex. From the combinatorial characterization in Theorem 3.11, the facets correspond precisely the maximal intervals in the sign poset. Maximal intervals are obtained from starting with a choice of a separating vertex $e_i$ and choosing a maximal length monotone path through $e_i$. In the notation from the proof of Theorem 3.4, these are precisely the subdivisions corresponding $F_1 < F_2$ such that $\pm F_1 \cup \pm F_2 = \{k : k \in \pm [n-1]\}$, where we remove $e_n$ from $F_2$ and $-e_n$ from $F_1$ and identify each $e_i$ with $i$. To obtain a lifting functional, we define

$$
\varphi(e_n) = 0 \text{ and } \varphi(e_i) = -a_i - a_n.
$$

Then for the remaining vertices in $F_1$ we linearly interpolate between $0$ and $-a_i - a_n$. For the vertices in $F_2$, we provide a similar interpolation. In particular, $\varphi(e_k) = -a_k - a_n$ if $k \in F_1$ and $\varphi(e_k) = \frac{a_{k+1} + a_n}{a_n - a_i} (a_k - a_n)$ if $k \in F_2$

$$
\varphi(e_k) = \begin{cases} 
-a_k - a_n & \text{if } k \in F_1 \\
\frac{a_{k+1} + a_n}{a_n - a_i} (a_k - a_n) & \text{if } k \in F_2 \\
0 & \text{if } k = n
\end{cases}
$$

We denote any functional of this form as $\varphi_{i, \varepsilon}$, where $i \in [n-1]$ is the choice of the splitting vertex, and $\varepsilon : [n-1] \to \pm 1$ denotes the sign sequence of the vertices. Then $F_1 = \{k : \varepsilon(k)k \leq i\}$ and $F_2 = \{k : \varepsilon(k)k \geq i\}$, which yields the result. 

Then, to find the vertices of the signohedra, all we need to do is compute.

**Corollary 3.14.** The vertices of $MPP(\phi^n)$ are precisely:

$$
\left\{ \left(1 - \frac{a_{i_k} + a_{i_1}}{2a_n}\right)(e_n) + \sum_{i=1}^{k} \left(\frac{a_{i_{k-1}} + a_{i_{k+1}}}{2a_n}\right)e_{i_k} : 
\begin{align*}
-n &= i_0 < \cdots < i_{k+1} = n \text{ and } i_a \neq -i_b \text{ for all } a, b \in [k],
\end{align*}
\right. \right\},
$$

where $a_i$ is defined as Theorem 3.13.

**Proof.** Use the characterization of coherent monotone paths in Theorem 3.4 in combination with Theorem 2.2 and the result is immediate. 

At this point, we may prove Corollary 1.2

**Proof.** (of Corollary 1.2) Apply Lemma 2.7 to the result of Corollary 3.14

Now that we understand $MPP(\phi^n)$, to obtain a more complete description of the space of monotone paths, we will describe the flip graph. The flip graph has as its vertices the set of all monotone paths on a polytope with edges given by polygon flips. We start describing it by first enumerating its vertices.
Theorem 3.15. The number of monotone paths in $\phi^n$ is precisely $\frac{2^{2n-1} - 2}{3}$.

Proof. A monotone path corresponds to a subsequence $s_n$ of

$$(-e_{n-1}, -e_{n-2}, \ldots, -e_1, e_2, \ldots, e_{n-1})$$

such that $s_k + s_{k+1} \neq 0$, since all vertices are connected to all vertices other than their antipodes. There are $2^{2(n-1)} - 1$ non-empty subsets of $\{e_i : i \in \{\pm 1, \pm 2, \ldots, \pm n-1\}\}$. Then $2^{2(n-2)}$ of those subsets include $-e_1, e_1$, $2^{2(n-3)}$ include $-e_2, e_2$ but neither of $-e_1, e_1$, and in general $2^{2(n-1-k)}$ include $-e_k, e_k$ but none of $e_j$, where $|j| < |k|$. Hence, via standard results for geometric series, the resulting number of possible sequences is

$$2^{2(n-1)} - 1 - \sum_{k=1}^{n-1} 2^{2(n-k-1)} = 4^{n-1} - \sum_{k=1}^{n-1} 4^{n-k-1} = 4^{n-1} - 1 - 4^{n-1} \left(\sum_{k=1}^{n-1} 4^{-k}\right)$$

$$= 4^{n-1} - 4^{n-1} \sum_{k=1}^{n-1} 4^{-k} - 1 = 4^{n-1} \left(1 - \frac{1 - (1/4)^{n-1}}{4(1 - 1/4)}\right) - 1$$

$$= 4^{n-1} \left(1 - \frac{1 - (1/4)^{n-1}}{3}\right) - 1 = 4^{n-1} \left(\frac{1}{3(4^{n-1})} + \frac{2}{3}\right) - 1$$

$$= \frac{2(4^{n-1}) + 1}{3} - 1 = \frac{2^{2n-1} - 2}{3} \square$$

Thus, the total number of monotone paths grows exponentially faster than the number of coherent monotone paths at a rate of $\Theta(4^n)$ in place of $\Theta(3^n)$. Alongside enumerating vertices of a flip graph, we may compute the diameter of the flip graph to contrast the overall flip graph with the signohedra graphs.

Theorem 3.16. The diameter of the flip graph of $\phi^n$ is $2(n-1)$. The longest distance to the nearest coherent path is $n - 2$.

Proof. Since $\phi^n$ is simplicial, two monotone paths are adjacent if and only if they differ by the addition or removal of a single vertex. A vertex may only be added or removed if it does not introduce consecutive antipodal points. Note that given these restrictions, the distance between the path given by $e_1$ and the path $(-e_{n-1}, -e_{n-2}, \ldots, -e_1, e_2, e_3, \ldots, e_n)$ is at least $2(n-1)$. By starting with removing all negative vertices starting with $-e_{n-1}$ and ending with $-e_1$ and then adding $e_1$ and removing $e_2$ through $e_n$ we achieve a sequence of flips going between these paths of length precisely $2(n-1)$. Hence, the distance between those points is precisely $2(n-1)$.

Let $e_{s_-}$ and $e_{t_+}$ be two different paths. Let $s_-$ and $s_+$ denote the maximal negative element and minimal positive element of $s$ respectively. define $t_-$ and $t_+$ similarly. If $s_- < t_-$ and $s_+ = t_+$, we may go from $s$ to $t$ by adding elements from $t$ that are not in $s$ and taking away elements from $s$ that are not in $t$. Such a path will have length at most $2(n-2)$.

If $s_- < t_-$ and $s_+ = t_+$, then we may add $t_-$ to $s$ and follow the same strategy. This will result in a sequence of moves of length at most $2(n-2) + 1$. A similar idea works for any of the possible cases in which $s_- = t_-$ or $s_+ = t_+$.

Suppose that $s_- < t_-$ and $s_+ < t_+$. If $t_- \neq -s_+$, we may add $t_-$ to the list $s_-$. Then we may follow the same strategy for the remaining list keeping $s_+$ and $t_-$. Then, at the end, we remove $s_+$. The result must take fewer that $2(n-2) + 2 \leq 2(n-1)$ moves. Suppose instead that $s_- < t_- = -s_+ < s_+ < t_+$. First modify all elements of $s$ greater that $s_+$ to agree with $t$. If $t_+ = -s_-$, remove $t_+$. Otherwise, leave it. In both cases, add $t_-$, add $t_+$ back and modify all elements $< t_-$ to agree with $t$. The result takes $\leq 2(n-2) - 1 + 3 = 2(n-1)$ moves, since $e_1 < t_+$.  

The only remaining case is that \( s_- < t_- \) and \( s_+ > t_+ \). In that case, add \( t_- \) and \( t_+ \) to \( s \) and make the required changes. The result takes fewer than \( 2(n-2) + 2 \) moves. Hence, the diameter of the flip graph is \( 2(n-1) \).

The longest distance to the nearest coherent path for a monotone path may be computed similarly. Start with a path \( s \) with \( s_- \) and \( s_+ \) defined as before. If \( s_+ < -s_- \), remove all parts of antipodal pairs after \( s_- \). Otherwise, remove all parts of antipodal pairs before \( s_- \) and \( s_+ \). Since there are at most \( n-2 \) elements before \( s_- \) and \( s_+ \), the distance to the nearest coherent path is at most \( n-2 \). This bound is attained for the path \((-e_{n-1}, -e_{n-2}, \ldots, -e_1, e_2, e_3, \ldots, e_{n-1})\).

That last proof concludes our description of the structure of monotone paths on the cross-polytopes and our proof of Theorem 1.1.

4. The Remaining Platonic solids

With our discussion of the cross-polytope, we now have a complete description of the monotone paths on all three infinite classes of regular convex polytopes and have proven Theorem 1.1. However, to finish off the Platonic solids, we still have two more cases to understand in detail: the dodecahedron and icosahedron. To study these cases, we will enumerate monotone paths and coherent monotone paths numerically but also provide a more direct description of how and why they arise. Before that, we must justify that the computation will yield the same result for any generic linear functional.

Lemma 4.1. Any two linear orientations on the graph of a Platonic solid are equivalent up to a linear symmetry. That is, the induced directed graphs are isomorphic.

Proof. This result is well known for the cube and the simplex, and we proved it earlier for the cross-polytope. Thus, it suffices to prove it for the remaining two cases. Throughout, we will apply well known results from [3] about the explicit coordinates of the Platonic solids.

We start with the icosahedron. As stated in Chapter 3 Section 8 of [3], the coordinates are given by \((0, \pm 1, \pm \varphi), (\pm \varphi, 0, \pm 1), \) and \((\pm 1, \pm \varphi, 0)\). By vertex transitivity we may choose the minimal vertex. We may also choose the next greatest vertex by edge transitivity. Suppose that the maximal vertex is \((0, 1, \varphi)\). Note that an icosahedron is formed by taking the pyramid over a regular pentagon, flipping it up side down and taking the convex hull of the two pieces. The top part of the pyramid over the regular pentagon has vertices

\[(0, 1, \varphi), (0, -1, \varphi), (1, \varphi, 0), (-1, \varphi, 0), (\varphi, 0, 1), (-\varphi, 0, 1),\]

where \((0, 1, \varphi)\) is the pyramid point. The remaining vertices are the negatives of all these vertices, which form another pyramid over a regular pentagon. Note that a pentagon has a unique orientation, and a pyramid over a pentagon also has a unique orientation up to symmetry with the pyramid point maximal. The orientation on the top pyramid induces the reversed orientation on the negative points for the bottom pyramid.

The orientation is induced by a linear functional \( \ell \) defined by \( \ell(x) = ax_1 + bx_2 + cx_3 \). If \( \ell \) is positive on all points on the top pyramid, it must be negative on all points of the bottom pyramid. Thus, any edge from the bottom pyramid to the top pyramid must be oriented from the bottom pyramid to the top. Suppose that \( \ell \) is negative at some point in the top pyramid. Due to symmetry, we may assume that \((0, 1, \varphi)\) is the maximal point, and the negative point is \((0, -1, \varphi)\).

Recall that for the golden ratio: \( \varphi^2 - \varphi = 1 \), so \( \varphi - 1 = \frac{1}{\varphi} \). By the maximality, of \((0, 1, \varphi)\), we have the following inequalities:

\[
a + \varphi b < b + \varphi c, \quad a < \varphi c - (\varphi - 1)b, \quad a < \varphi c - \left(\frac{1}{\varphi}\right)b, \quad \varphi a < \varphi^2 c - b, \\
b < \varphi^2 c - \varphi a, \quad \varphi a < (\varphi + 1)c - b, \quad c + \varphi a < b + \varphi c, \quad \varphi a < b + (\varphi - 1)c.
\]
Furthermore, since \((0, -1, \varphi)\) is negative, we have \(\varphi c < b\). Then
\[|a| < \varphi c - (\varphi - 1)b < b - (\varphi - 1)b = (2 - \varphi)b.\]
Thus, \(\pm a + \varphi b > 0\).
Furthermore, note that
\[-a + (\varphi - 1)b > -(2 - \varphi b) + (\varphi - 1)b = (2\varphi - 3)b > 0,\]
so
\[\pm a + (\varphi - 1)b > -\varphi c\]
meaning that \(\pm a + \varphi b > |b - \varphi c|\).
Observe also that
\[\pm a + \varphi b < b + \varphi c, \pm a < \varphi c - (\varphi - 1)b,\]
\[\pm a < \varphi c - \left(\frac{1}{\varphi}\right)b, \pm a < \varphi^2 c - b,\]
\[b < (1 + \varphi)c \pm \varphi a, \text{ and } b - \varphi c < c \pm \varphi a.\]
Hence, \(c \pm \varphi a > |b - \varphi c| > 0\). Thus, at most one point in the top pyramid may have a negative result with respect to the objective function. Furthermore, the absolute value of the other points is greater than that of the one negative point. The edges from a bottom vertex that is negative to a top vertex that is positive are oriented from the bottom pyramid toward the top pyramid. The only remaining edges to consider are from a negative bottom vertex to the negative top vertex and from the positive bottom vertex to positive top vertices. Since the absolute value of \(\ell(x)\) is minimized at the one negative vertex in the top pyramid, and the two adjacent vertices in the bottom pyramid must also be negative, the orientation of those edges also goes from the bottom pyramid to the top. The same occurs for the orientation from the positive bottom vertex to the positive top vertices. Hence, the orientation on the icosahedron is always given by the orientation of the top pyramid with its reverse on the bottom pyramid, and every remaining edge oriented from the bottom pyramid to the top.

Next is the dodecahedron, which has coordinates given by \((\pm \varphi, \pm \varphi, \pm \varphi), (\pm 1, 0, \pm \varphi^2), (\pm \varphi^2, \pm 1, 0),\) and \((0, \pm \varphi^2, \pm 1)\) as one may also find Chapter 3 Section 8 of [3]. Consider a generic linear functional given by \((a, b, c)\). Note that there will be a minimal vertex with respect to this orientation. Furthermore, that vertex will be adjacent to three other vertices. Each pair of those 3 vertices is contained in a facet. By flag transitivity, we may assume that the minimal vertex is \((-1, -1, -1)\) and the ordering on the three adjacent vertices. Note that, for a regular pentagon, the ordering on the first three vertices induces the ordering on the remaining vertices with respect to a linear functional. Thus, we have an ordering on the three regular pentagons containing the minimal vertex. By central symmetry, this induces an ordering on the three pentagons containing the top vertex. All remaining edges are from the bottom pentagons to the top pentagons. All of those edges will be oriented upward by similar reasoning to the case of the icosahedron.

Having a generic directed graph already completes the proof for enumerating monotone paths without requiring coherence. Similarly, there is no change to the flip graph.

**Corollary 4.2.** With respect to a generic linear functional, the number of monotone paths on a Platonic solid will always be the same.

**Proof.** The monotone paths are induced by the orientation of a polytope. By Lemma 4.1 all directed graphs induced by linear orientations on Platonic solids are isomorphic. The number of monotone paths is completely determined by the isomorphism class of that directed graph.
To compute the number of monotone paths, we first found the graph and a topological ordering induced by the linear functional. Then we could count the number of monotone paths from each vertex to the final vertex starting from the end and moving backwards in the topological ordering by noting that

$$P(u) = \sum_{v > u, (u,v) \in E} P(v),$$

where $P(u)$ denotes the number of monotone paths from vertex $u$ to the final vertex and $E$ is the set of edges in the graph. This may easily be converted into a dynamic program that counts the number of paths efficiently in $O(|V|^2)$-time. This algorithm was used to fill the column of Theorem 1.3 for numbers of monotone paths.

To find the number of coherent monotone paths and the monotone path polytopes, there are two possible approaches. The first is to take the Minkowski sums of pre-images with respect to the linear functional via a result in [2], and the second was to compute the integral of the section for each monotone path and take the convex hull using SageMath (see [11]). The latter method tended to be more efficient and was used to fill out the final column of the table. Interestingly, the number of coherent paths for both the icosahedron and dodecahedron correspond to the number of facets of each. For coherent monotone paths, there is more to do. Namely, we need to insure that for two linear functionals that induce the same directed graph, the same paths will end up being coherent.

**Open Problem 4.3.** Let $P$ be a polytope, and let $\varphi$ and $\psi$ be two linear functionals. Under what conditions are $\text{MPP}_\varphi(P)$ and $\text{MPP}_\psi(P)$ are combinatorially equivalent?

Computational evidence suggests that if $\varphi$ and $\psi$ induce the same orientation on the polytope, then the monotone path polytopes are combinatorially equivalent for the regular polytopes. This is, for example, true of simplices, cross-polytopes, and cubes. However, there exist two equivalent orientations of the three dimensional permutahedron that yield different monotone path polytopes. Namely, the monotone path polytope induced by the linear functional $[0, 1, 2, 3]$ has 12 vertices, while the $\text{MPP}$ for the linear functional $[1, 7, 7^2, 7^3]$ has 14 vertices. The first is edge-generic but not vertex generic, while the latter is vertex generic. A weaker form of this open problem is at least clear:

**Lemma 4.4.** If two linear functionals are equal on a polytope up to some affine symmetry then the resulting monotone path polytopes are isomorphic.

**Proof.** This is immediate by applying the projection lemma to $P$ and $A(P)$, where $P$ is a polytope and $A$ is the map that gives the affine symmetry. $\square$

For regular polytopes, applying this lemma allows us to choose a great deal about the ordering of vertices in a given orientation. However, we still do not understand whether it is resilient to perturbation when we fix a given ordering.

5. Higher Dimensions and Open Questions

After the Platonic solids, hypercubes, simplices, and cross-polytopes, there are only three remaining convex regular polytopes as described in [3]: The 24-cell, the 120-cell, and 600-cell. For the dodecahedron and icosahedron, we made substantial use of how small they were to study them. For the 120-cell and 600-cell such methods become a problem. We computed the structure of monotone paths for the vertex and edge-generic linear functional $[7, 7^2, 7^3, 7^4]$ and stored them in Table 3. For our final result, we will show that the 24-cell has generic orientation. This problem for the 120-cell and 600-cell remains open, but the 24-cell is an example of a different class of polytopes from regular polytopes that are more straightforward to show have generic orientation. Consider the following objects:
Theorem 5.2. Let \( S_{\pm d} \) be a polytope given by the relations \( ||x||_{\pm} \leq 1 \) and \( ||x||_{\pm} \leq k \) for \( 1 \leq k \leq d \) and \( k, d \in \mathbb{Z} \). Under a different name, these polytopes were recently studied as polytopal unit balls in \([4]\). Geometrically, this is the intersection of a cube with a scaled cross-polytope. In particular, \( \diamond(d, k) \) is the \( d \)-cube, \( \diamond(d, 1) \) is the \( d \)-cross-polytope, and the remaining \( \diamond(d, k) \) are interpolations of these two objects.

We may obtain a vertex description via the following result.

Theorem 5.2. \( S_{\pm d} = \{ A \in GL(d, \mathbb{Z}) : Ae_i = \pm e_j \text{ for some } j \text{ and } Ae_i \neq \pm Ae_j \text{ for any } k \} \).

That is \( S_{\pm d} \) is the set of permutation matrices if we allow a coordinate to be \( \pm 1 \). Then we have the vertices of \( \diamond(d, k) \) are precisely the orbit of \( \sum_{i=1}^{k} e_i \) under the action of \( S_{\pm d} \).

Proof. Recall that a polytope is the convex hull of its extreme points. Let \( c^T \) be a linear functional, where \( c = \sum_{i=1}^{n} c_i e_i \). Then generically, we may assume that all \( |c_i| \) are distinct. Let \( \sigma : [d] \rightarrow [d] \) be the permutation such that \( |c_{\sigma(i)}| > |c_{\sigma(j)}| \) whenever \( i > j \) for \( i, j \in [d] \).

Consider \( x^* = \sum_{i=1}^{k} \text{sign}(c_{\sigma(i)}) e_{\sigma(i)} \). Let \( x \) be some other point in the polytope. Without loss of generality, we assume that \( \text{sign}(x_{\sigma(i)}) = \text{sign}(c_{\sigma(i)}) \). Then \( ||x||_{\pm} \leq 1 \) and \( ||x||_{\pm} \leq k \), so

\[
\begin{align*}
    c^T x^* - c^T x &= \sum_{i=1}^{n} c_{\sigma(i)} (x_{\sigma(i)}^* - x_{\sigma(i)}) \\
                   &= \sum_{i=1}^{k} |c_{\sigma(i)}| - x_{\sigma(i)} c_{\sigma(i)} - \sum_{i=k+1}^{n} x_{\sigma(i)} c_{\sigma(i)} \\
                   &\geq \sum_{i=1}^{k} (1 - |x_{\sigma(i)}|) c_{\sigma(i)} - \sum_{i=k+1}^{n} |x_{\sigma(i)}| c_{\sigma(i)} \\
                   &\geq \sum_{i=1}^{k} (1 - |x_{\sigma(i)}|) c_{\sigma(k)} - \sum_{i=k+1}^{n} |x_{\sigma(i)}| c_{\sigma(k+1)} \\
                   &= \left( k - \sum_{i=1}^{k} |x_{\sigma(i)}| \right) c_{\sigma(k)} - \sum_{i=k+1}^{n} |x_{\sigma(i)}| c_{\sigma(k+1)} \\
                   &\geq \left( \sum_{i=k+1}^{n} |x_{\sigma(i)}| \right) (|c_{\sigma(k)}| - |c_{\sigma(k+1)}|) \geq 0.
\end{align*}
\]

Hence, \( c^T x^* \geq c^T x \) for all \( x \in P \), so \( x^* \) is maximal with respect to \( c \). Note that the set \( \sigma_{\text{sign}}(c_{\sigma(i)}) \) and permutations \( \sigma \) is precisely \( S_{\pm d} \).

The vertices then have a natural interpretation as signed combinations. That is, they \( k \)-subsets together with a sign pattern. In that sense, \( \diamond(d, k) \) may be thought of a signed version of the hypersimplices \( \Delta(d, k) \) describe in Chapter 3 of \([12]\). Pushing this analogy further, \( \Delta(d, k) \) may be thought of as the convex hull of the set of coherent monotone paths on the \( (d+1) \)-simplex of fixed length \( k+1 \), and \( \diamond(d, k) \) may be thought of as the convex hull of the set of coherent monotone paths on the cross-polytope \( \diamond^{d+1} \) of fixed length \( k+2 \). In relation to the 24-cell, from Chapter 3 Section 8 of \([3]\), we have the following corollary connecting hypercross-polytopes to the 24-cell.
Corollary 5.3. The 24–cell is given by $\odot(4, 2)$.

Hence, if we show $\odot(d, k)$ has generic orientation for any $d$ and $k$, we find that the 24–cell has generic orientation. To understand the face lattice, we combine our vertex description with a list of facet defining inequalities.

Theorem 5.4. In terms of inequalities, $\odot(d, k)$ is given by the set of $x \in \mathbb{R}^d$ that satisfy

$$\sum_{i=1}^{d} \varepsilon(i)x_i \leq k$$

$$-1 \leq x_i \leq 1.$$  

for all $\varepsilon : [d] \rightarrow \{\pm 1\}$. Furthermore, this set is irredundant when $1 < k < d$.

Proof. Suppose that we remove the inequality $x_i \leq 1$ and the rest remain. Then $ke_i$ satisfies all the remaining inequalities but is not in $\odot(d, k)$ due to not being in the cube since $k > 1$. The same argument holds if we instead removed $x_i \geq 1$.

Suppose that we remove the inequality $\sum_{i=1}^{d} \varepsilon(i)x_i \leq k$. Consider the vector $v = \sum_{i=1}^{d} k \varepsilon(i)e_i$. Note that all $\varepsilon \neq \varepsilon' \in \{f : [d] \rightarrow \{\pm 1\}\}$ must differ from $\varepsilon$ on at least one coordinate. It follows that

$$\langle v, \sum_{i=1}^{d} \varepsilon'(i)x_i \rangle = \frac{k}{d-1} \sum_{i=1}^{d} \varepsilon(i)\varepsilon'(i) \leq \frac{k}{d-1}(d-1) = k.$$  

Since $k < d$ and $k \in \mathbb{Z}$, $-k \leq d-1$ meaning that $v$ satisfies $-1 \leq x_i \leq 1$ for all $i \in [d]$. Thus, $v$ satisfies all the remaining relations, but

$$||v||_1 = \sum_{i=1}^{d} \frac{k}{d-1} = k \frac{d}{d-1} > k,$$

so $v \notin \odot(d, k)$. Therefore, the set of inequalities is irredundant. \hfill $\square$

From this theorem, we obtain a description of the facets. They come in two types, which we will use to compute the graph.

Lemma 5.5. The facets of $\odot(d, k)$ for $1 < k < d$ come in two types. The first is given by $x_i = 1$ or $x_i = -1$. These facets are all isometric to $\odot(d - 1, k - 1)$. The remaining facets are given by $\sum_{i=1}^{d} \varepsilon(i)e_i = k$. These are all isometric to

$$\text{conv} \left( \sum_{i \in S} e_i : S \subseteq [d], |S| = k \right).$$

Note that when $k = 1$ and $k = d$, they come in precisely one type.

Proof. For the first part, note that if we fix a coordinate $x_i = 1$, the remaining vertices are given by $k$–subsets of $d-1$ with all possible sign patterns. The orthogonal projection that kills off $e_i$ is an isometry that takes those vertices to the vertices of $\odot(d - 1, k - 1)$, which yields the result.

For the second, note that the vertices that satisfy $\sum_{i=1}^{d} \varepsilon(i)e_i = k$ are precisely

$$\left\{ \sum_{i \in S} \varepsilon(i)e_i : S \subseteq [d], |S| = k \right\}.$$  

Via the isometry taking $e_i \mapsto \varepsilon(i)e_i$, this vertex set is taken to $\left\{ \sum_{i \in S} e_i : S \subseteq [d], |S| = k \right\}$, which yields the result. All facets are given by those inequalities, so up to isometry, $\odot(d, k)$ has precisely two types of facets. \hfill $\square$
Thus, a facet of $\sigma(d, k)$ is either isometric to $\sigma(d - 1, k - 1)$ or the $(d, k)$–hypersimplex
\[
\Delta(d, k) = \text{conv}\left(\left\{\sum_{s \subseteq [d]} e_s : S \subseteq [d], |S| = k\right\}\right).
\]
The following lemma is well known from the study of the face lattice of hypersimplices and also straightforward to verify.

**Lemma 5.6.** Two vertices of $\Delta(d, k)$ are adjacent if and only if they differ on precisely two coordinates.

This result extends to $\sigma(d, k)$ via an induction

**Lemma 5.7.** For $1 \leq k < d$, two vertices $u$ and $v$ of $\sigma(d, k)$ are adjacent if and only if they agree on all except for two coordinates $i$ and $j$ with $u_i = 0$, $v_j = 0$, and $|u_j| = |v_i| = 1$.

**Proof.** Note that $\sigma(d, 1) = \sigma^d$ and 2 vertices $\varepsilon(i)e_i$ and $\varepsilon(j)e_j$ are adjacent if and only if $i \neq j$. This happens precisely when they meet the condition we assert for adjacency $\sigma(d, k)$.

Note that $\sigma(d, 1)$ is the $d$–cross-polytope. Suppose that the statement holds for all $(m, j)$ such that $1 \leq j < m$ and $2 \leq j < d$. If $k = 1$, then the statement follows by what we just showed. Otherwise, $k \geq 2$, so $k - 1 \geq 1$. Then the facets are given by $\sigma(d - 1, k - 1)$ and $\Delta(d, k)$ by Lemma 5.5. This property holds for $\sigma(d - 1, k - 1)$ by induction and $\Delta(d, k)$ by Lemma 5.6 so it holds for all $\sigma(d, k)$ by induction.

This description of the graphs of both $\sigma(d, k)$ and $\Delta(d, k)$ allows us to prove that they have generic orientation.

**Theorem 5.8.** Both $\Delta(d, k)$ and $\sigma(d, k)$ have generic orientation. Furthermore, $\Delta(d, k)$ is isomorphic to a directed graph on $\binom{[d]}{k}$ with edges given by two subsets that share all but 1 element. An edge denotes an exchange of 1 element for another, and the orientation is from the set with the smaller element to the set with the larger 1. For example, $\{1, 2, 4\}$ is adjacent to $\{1, 3, 4\}$ in the graph associated to $\Delta(5, 3)$ with the edge oriented from $\{1, 2, 4\}$ to $\{1, 3, 4\}$.

Similarly, for $\sigma(d, k)$, the graph is isomorphic to a directed graph on sign patterned $k$–subsets of $[d]$. That is a $k$–subset of signed elements of $[d]$. For example $\{1, -2, -4\}$ and $\{1, 3, -4\}$ are in the graph for $\sigma(5, 3)$. Furthermore, two vertices are adjacent if the agree on all but two elements. In which case, there is an exchange of those elements, and the orientation is from the subset with the smaller element to the subset with the larger. For example $\{1, -2, -4\}$ is adjacent to $\{1, 3, -4\}$ with the orientation going from $\{1, -2, -4\}$ to $\{1, 3, -4\}$.

**Proof.** Let $\varphi$ be a linear functional given by $c^T = (c_1, c_2, \ldots, c_d)$. Then given two vertices $v_S = \sum_{i \in S} e_i$ and $v_T = \sum_{j \in T} e_j$ in $\Delta(d, k)$ we have $v_1 \leq v_2$ if and only if $\sum_{i \in S} c_i \leq \sum_{j \in S_2} c_j$. By Lemma 5.6, 2 vertices are adjacent if and only if $|S \cap T| = k - 1$. Let $\{s\} = S \setminus (S \cap T)$ and $t$ be defined similarly. The edge is then oriented from $v_S$ to $v_T$ if and only $c_s < c_t$. Thus, the orientation is imposed by the ordering on $c_i$. Since $\Delta(d, k)$ is invariant under action by $S_d$, the ordering is generically given by some choice of $c$ such that $c_1 < c_2 < \cdots < c_d$. In that case, we can move by exchange $e_i$ for $e_j$ where $j > i$. This graph is then isomorphic to the graph on $\binom{[d]}{k}$, where two sets are adjacent if you exchange one element for another, and it is oriented such that an edge points from the vertex that traded the smaller to the vertex that traded the larger one. This isomorphism is independent of the choice of $c$, so $\Delta(d, k)$ has generic orientation.

For $\sigma(d, k)$, the argument is similar. Up to an action by $S_{d,d}$, we may assume that $0 < c_1 < c_2 < \cdots < c_d$. Then, by Lemma 5.7, the directed graph is isomorphic to the graph on signed $k$–subsets of $[d]$, where 2 subsets are adjacent by an exchange of one element to another up to a change in sign with the arrow from the smaller element point to the arrow for the larger one.

The final case is $\sigma(d, d)$, which has generic orientation, because it is affinely isomorphic to $C_d$. □
Our final result is then obtained as a corollary.

**Corollary 5.9.** *The 24-cell has generic orientation. Hence, the computation of its number of monotone paths in Table 3 holds for a generic linear functional.*

As one can see from Table 4, the monotone path polytope of the 24-cell is remarkably symmetric and has a great deal of structure worth understanding. The structure of the MPP of the 24-cell is likely also best understood in terms of the more general structure of $\diamondsuit(d, k)$.

| Polytope         | Monotone Path Polytope |
|------------------|------------------------|
| 4-cross-polytope | ![4-cross-polytope](image) |
| 24-cell          | ![24-cell](image)      |

**Table 4.** The new MPPs that we have computed for the regular polytopes in Sage. We know the vertices for the 600-cell, but it is too computationally intense to plot in Sage. Note the remarkable symmetry for the MPP of the 24-cell and that these polytopes are centrally symmetric, so the back of the polytope in the image looks the same as the front.

6. Conclusion

For regular polytopes, we suspect that there may be a simple but more abstract proof opposed to Theorem 5.8 that they have generic orientation more directly from their symmetry.

**Conjecture 6.1.** *There is a simple proof that if the symmetry group of a polytope acts transitively on its flags, then it has generic orientation.*
Such a result would tell us that the 120-cell and 600-cell have generic orientation, which remains open. The problem of studying which polytopes have generic orientation in this sense seems unexplored and worthy of study. We proved it for hypersimplices and hypercross-polytopes, but it likely holds in more general settings.

**Open Problem 6.2.** Define an orbitope to be a polytope in which a linear group acts transitively on its vertices, a special case of the definition in [10]. Do all orbitopes have generic orientation?

We already stated that a few regular polytopes are still missing to understand their monotone path polytopes. The primary issue is that we do not have conditions for when the monotone path polytope is the same for different linear functionals. We hope to settle this later.

Another natural next step is to study the fiber polytope of higher dimensional projections of the regular polytopes. For example, associahedra are combinatorially equivalent to the fiber polytopes of projections of a simplex to an $n$-gon. What fiber polytope do we obtain from symmetric two-dimensional projections of regular polytopes. We know the associated subdivisions must use the vertices of Petrie nets. What is the projection with the most boundary vertices? This is a problem left open in [5].

**Figure 2.** The two centrally regular triangulations of a hexagon.

Even two-dimensional projections of the cross-polytope to a regular $2n$-gon are interesting. Billera and Sturmfels showed in [2] that the fiber polytopes of projections of the cross-polytope correspond to “centrally regular triangulations,” triangulations that occur as the shadow map of some centrally symmetric polytope. Understanding the structure of centrally regular triangulations on $2n$-gons would be the place to start. For example, there are precisely two centrally regular triangulations of a hexagon as depicted in Figure 2. Unfortunately, it is not obvious which triangulations of a $2n$-gon are centrally regular. Furthermore, the set of centrally regular triangulations depends on the choice of the polygon to project to. These two conditions make understanding the fiber polytopes of these projections much more difficult than for projections of simplices.

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(AB) DEPT. MATH., UC DAVIS, DAVIS, CA 95616, USA
Email address: aeblack@ucdavis.edu

(JDL) DEPT. MATH., UC DAVIS, DAVIS, CA 95616, USA
Email address: deloera@ucdavis.edu