Brief Comments on “The Shapiro Conjecture, Prompt or Delayed Collapse ?” by Miller, Suen and Tobias

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Recent numerical simulations address a conjecture by Shapiro that when two neutron stars collide head-on from rest at infinity, the two recoil shocks which propagate back into each star following contact might generate sufficient thermal pressure to hold up the remnant against gravitational collapse, at least until it cools via neutrino emission. This quasi-equilibrium state will last many dynamical timescales (∼ msec) because the neutrinos, which eventually carry off the thermal energy, leak out slowly, on the neutrino diffusion timescale (∼ 10 sec). The argument was independent of the total mass of the progenitors and applies even if the mass of the remnant greatly exceeds the maximum mass of a cold neutron star. A simple, analytic analysis using relativistic polytropes served to motivate the conjecture. The analysis assumed strict conservation of rest-mass and total mass-energy in the collision, as well as relaxation, following merger, to the same polytropic density profile in the remnant as in the progenitors. Given these assumptions it was possible to show that a stable equilibrium solution always exists for the remnant, independent of mass. In evaluating these assumptions, Shapiro argued that while they greatly oversimplify the collision in order to permit an analytic proof, they do provide a reasonable approximate description of the idealized head-on scenario under consideration. He did show that the loss of energy due to gravitational radiation, though small, would rule out an equilibrium solution for configurations arbitrarily close to the maximum mass, if the collision were not accompanied by loss of rest-mass. More significantly, he cautioned that a dynamical collision need not relax to a stable, equilibrium solution after all, even though one exists, but could instead overshoot the equilibrium state and collapse to black hole. But he put forward the possibility of delayed collapse as a plausible outcome of head-on collisions from rest at infinity and one which might be realized in numerical simulations.

Numerical simulations to test this conjecture have been reported recently in Ref [2]. The simulations treat colliding Γ = 2 polytropes in a 3+1 numerical scheme that solves Einstein’s field equations of general relativity coupled to the equations of relativistic hydrodynamics. The simulations employed Γ = 2 polytropic models that satisfy the TOV equilibrium equations for isolated, spherical stars. The stars are placed at finite separation, no more than 3R apart, where R is the stellar radius. These configurations are then boosted towards each other with the Newtonian freefall velocity [3]. No attempt is made to correct for the distortions in the initial matter density or pressure profiles that arise at such close separation due to tidal interactions. The simulations described in Ref [2] deal with collisions of 1.4M⊙ stars. We shall assume below that this quoted mass represents the total mass-energy of each of the progenitors, although no distinction is made in Ref [2] between rest-mass M⊙ and total mass-energy M, nor is there any discussion about how the mass in the colliding system is actually measured and monitored numerically. The subsequent evolution of this model produces a black hole a few dynamical timescales following contact and merger. This example, it is claimed in Ref [2], is in contradiction to the conjecture of Shapiro.

Two aspects of the numerical simulations are crucial to note. One is that the maximum stable mass of the polytropic equation of state is 1.46M⊙ in the units of Ref [2]. The other, alluded to but not reported in Ref [2], is that a second set of simulations using the same equation of state for the head-on collision of 0.8M⊙ configurations did not result in collapse following merger [4]. In this case the simulations show that the merged remnant is supported in stable equilibrium by the thermal pressure generated by collision-induced shock heating, in apparent accord with the Shapiro conjecture. Interestingly, in both cases the total mass of the merged remnant exceeds the maximum mass of a cold star.

Does the simulation with 1.4M⊙ stars show unambiguously that a delayed collapse does not occur for stars of this mass? Insight into what could be happening may be gleaned from the original discussion in Ref [1]. Section VI of that paper, “Discussion and Caveats” already anticipated numerical subtleties that would be encountered in simulating a collision of stars that are initially at rest at infinity. To obtain reliable results, the requirements for computational accuracy become very stringent.
for high-mass stars near the maximum mass. The reason is clear from the analytic argument for the existence of a stable, equilibrium solution for the remnant. For such a solution to exist, it is necessary that the binding energy of the final remnant reside along the stable branch of the TOV equilibrium curve. The ratio $M/M_0$ measures the specific binding energy of a star according to $E_{\text{bind}}/M_0 = 1 - M/M_0$. This ratio monotonically decreases with increasing central density $\rho_c$ along the stable branch of the TOV curve, whereas the mass and rest-mass monotonically increase along this branch. The turning point on the curve marks the onset of radial instability; there $M$ and $M_0$ assume their maximum values and the binding energy ratio assumes its minimum value, $(M/M_0)_{\text{min}}$. In the idealized scenario in which the stars begin from rest at infinity, strict conservation of mass and rest-mass imply that for the hot remnant

$$M_{0,\text{hot}} = 2M_0 \quad \text{and} \quad M_{\text{hot}} = 2M.$$  \hfill (1)

As a result, the binding energy of the hot remnant is identical to the binding energy of each of the progenitor stars, and this guarantees the existence of an equilibrium solution (cf. Eqs. 8 - 10 of Ref [8]).

Now consider an alternative scenario in which Eq. (1) is replaced by

$$M_{0,\text{hot}} = 2M_0 \quad \text{and} \quad M_{\text{hot}} = 2M(1-f).$$  \hfill (2)

As discussed in Ref [1], the fractional change in the total mass-energy $f$ might be the result of a true physical departure from the conditions or assumptions of the idealized collision scenario. The quantity $f$ may account, for example, for the energy loss due to gravitational radiation or for the release of the progenitor stars from rest at finite separation rather than from infinity. Alternatively, $f$ might represent the degree of numerical error arising in a numerical simulation. Such error might be present due to imprecise initial data, including, for example, initial data which do not exactly correspond to the true solution at finite separation for two stars which actually begin from rest at infinity [8]. Numerical error might also result from finite-difference integration error in the evolution due to finite spatial grid and time-step resolution, imprecise outer boundary conditions or outer boundary points that don’t extend sufficiently far into the radiation zone, global instabilities, etc [8]. To illustrate, suppose we assume that for whatever reason, numerical error results in a spurious fractional decrease in $M$ in the course of the simulation. We then estimate the the maximum tolerable error $f_{\text{tol}} \geq 0$ which still admits a stable, equilibrium solution for the remnant [8]. According to Eq. (2), relaxation to a stable equilibrium state requires

$$\frac{M_{\text{hot}}}{M_{0,\text{hot}}} = (1-f) \frac{M}{M_0} \geq \left( \frac{M}{M_0} \right)_{\text{min}} \Rightarrow f_{\text{tol}} \leq 1 - \left( \frac{M}{M_0} \right)_{\text{min}}.$$  \hfill (3)

Eqn. (3) demonstrates the difficulty of simulating accurately a head-on collision from rest at infinity as the progenitor masses approach the maximum mass. As $M \to M_{\text{max}}$, we have $(M/M_0) \to (M/M_0)_{\text{min}}$ and, hence $f_{\text{tol}} \to 0$.

To appreciate the severity of the problem, consider the actual simulations performed for $\Gamma = 2$, for which $(M/M_0)_{\text{min}} = 0.910$ (see Fig 3 in Ref [8]). This minimum occurs at $1.46M_\odot$, the maximum mass. We then find $(M/M_0) = 0.915$ for the $M = 1.4M_\odot$ star and $(M/M_0) = 0.959$ for the $M = 0.8M_\odot$ star. According to Eq. (3) these values imply

$$f_{\text{tol}}(0.8M_\odot) = 0.05 \quad \text{and} \quad f_{\text{tol}}(1.4M_\odot) = 0.005.$$  \hfill (4)

Eqn. (4) suggests that determining whether stable equilibrium state forms following the merger of two $1.4M_\odot$ stars requires that the simulations be accurate to better than 0.5%. Considerably less accuracy is required as one moves away from the maximum mass; it is only 5% for the $0.8M_\odot$ stars. Given the potential multiple sources of numerical error, from the initial data to the numerical integrations, it is by no means evident that existing 3 + 1 numerical relativity codes can set up and track such a strong-field collision to 0.5% accuracy [8].

The conjecture put forth in Ref [1] that when two neutron stars collide head-on from rest at infinity, sufficient thermal pressure may be generated to support the hot remnant in quasi-static equilibrium against collapse may be generally correct, despite the report in Ref [1] of a numerical example to the contrary. The simulation cited involves the collision of high-mass stars near the maximum mass [1], which requires very high numerical accuracy to simulate reliably [13]. In the unreported case involving lower-mass stars whose combined mass still exceeds the maximum mass, the conjecture appears to be corroborated. The accuracy requirements for the lower-mass case are far less stringent, and presumably more easily attainable. Evaluating the conjecture at high mass may have to await more sophisticated code development and/or larger machines [14]. Even with such advances, evaluating the conjecture in this domain may still require a careful limiting procedure based on a sequence of runs with increasing initial separation between the initial stars to be confident that the initial data correctly corresponds to configurations which infall from rest at infinity.

The $0.8M_\odot$ simulation involves stars that have masses far below the maximum mass. As a result, this simulation is likely to be more reliable. If the delayed collapse outcome found for this collision is indeed correct, then we can already speculate that for neutron stars governed by realistic equations of state (14) which currently give maximum masses in the range $1.8M_\odot - 2.3M_\odot$, the collision of two realistic (low-mass) $1.4M_\odot$ stars from rest at infinity likely will lead to delayed collapse in accord with the conjecture.
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[1] S.L. Shapiro, Phys. Rev. D 58, 103002 (1998).
[2] M. Miller, W.M. Suen and M. Tobias, (1999), gr-qc/9904041.
[3] Simulations with 10% higher initial speeds were also performed, with little change in the reported final outcome.
[4] For polytropes the assignment of a physical mass to a configuration is arbitrary and has no significance, because of the scale freedom to choose the gas constant $K = P/\rho \Gamma_0$ in the equation of state arbitrarily. Physically significant features of a polytrope include its specific binding energy, which is scale invariant, and its proximity to the the maximum mass configuration, as measured by the ratio of its mass to the mass of the maximum mass configuration or by the ratio of their binding energies; see Ref [1] and text below. Had the authors of Ref 4 scaled $K$ to give a more realistic maximum mass like $2.2M_\odot$ instead of $1.46M_\odot$, the simulation they reported would have been for stars of mass $2.1M_\odot$, not $1.4M_\odot$.
[5] W.M. Suen private communication.
[6] Accounting for these effects places restrictions on the possibility of delayed collapse, as discussed in Ref 4.
[7] To test the conjecture, the initial data must not only satisfy the constraint equations to high precision, as they are reported to do in Ref 4, but must also reliably represent the matter and fields at finite separation that arise from infall from rest at infinity. The final outcome is increasingly sensitive to the approximations, like those adopted in Ref 4, as the configurations approach the maximum mass; see text below.
[8] Particularly worrisome about the simulation discussed in Ref 4 are the possible errors introduced by the initial data approximation, the growing violation of the constraint equations with time shown in the plots and the difficulties encountered holding a relativistic polytrope near the maximum mass in stable equilibrium for many dynamical (oscillation) timescales (cf J.A. Font, M. Miller, W. Suen and M. Tobias 1998, gr-qc/9811013). In the case of collapse to a black hole, as reported for the $1.43M_\odot$ collision, the code breakdown may make it difficult to measure the total gravitational radiation loss, which is important in evaluating the conjecture near the maximum mass; see 4 and 11 for discussion.
[9] The argument for spurious mass increase, or for spurious changes in the the rest mass, is similar.

[10] The assessment of Miracle Max, the Miracle Man, upon dispensing his latest miracle pill in the movie *The Princess Bride*, comes immediately to mind: Valerie [Max’s wife]: “Think it will work [sic: work]?” Max: “It’ll take a miracle!”
[11] A rough estimate of the gravitational radiation mass-energy loss is $f = \Delta E/M \sim 10^{-3}$ for the $1.4M_\odot$ collision; see Ref 4. This value of $f$ is not much smaller than the maximum tolerable value for which an equilibrium solution and delayed collapse is possible; cf. Eqn. 4.
[12] It is not clear how useful it is to attempt to distinguish, as in Ref 4, the various timescales associated with the infall velocity, sound velocity, shock velocity, and thermalization speed in attempting to account for the observed simulation outcome. To the accuracy that these timescales can be defined, they must all be comparable on dimensional grounds, given a head-on collision of equilibrium stars from rest at infinity: $t \sim R/(M/R)^{1/2}$.
[13] The use of adaptive gridding and/or a high-level axisymmetric code that treats the collision in 2 + 1 dimensions rather than 3 + 1 dimensions, like the one under development by the authors of Ref 4, may prove necessary.
[14] A. Akmal, V.R. Pandharipande and D.G. Ravenhall, Phys. Rev. C 58, 1804 (1998).