Modelling gap-size distribution of parked cars using random-matrix theory

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We apply the random-matrix theory to the car-parking problem. For this purpose, we adopt a Coulomb gas model that associates the coordinates of the gas particles with the eigenvalues of a random matrix. The nature of interaction between the particles is consistent with the tendency of the drivers to park their cars near to each other and in the same time keep a distance sufficient for manoeuvring. We show that the recently measured gap-size distribution of parked cars in a number of roads in central London is well represented by the spacing distribution of a Gaussian unitary ensemble.

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1. Introduction

Random-matrix theory (RMT) models quantal chaotic systems in terms of three canonical ensembles of random matrices. The Gaussian orthogonal and symplectic ensembles (GOE and GSE) are used to describe time-reversal invariant systems with integer and half-integer spins, respectively, while the Gaussian unitary ensemble (GUE) is connected with systems without time reversibility. The theory was originally proposed to describe the fluctuation properties of energy spectra of atomic nuclei. Bohigas et al. have established that the fluctuation properties of the canonical random matrices are generic and therefore applicable for the spectral statistics of a wide variety of quantal systems. The broad range of applicability of RMT has recently been the subject of many excellent reviews, e.g.

RMT is also applicable for classical one-dimensional interacting many particle systems. Long time ago, Dyson realized that eigenvalues of Gaussian random matrices do indeed behave like charges, repelling each other with a force varying inversely with the first power of distance. The matrix eigenvalues describe the positions of the charged particles. He showed that the joint eigenvalue-distribution of Gaussian matrices have exactly the same mathematical structure as the Boltzmann factor for a one-dimensional classical Coulomb gas. The same holds true also for other potentials. An example is the Pechukas gas, where one-dimensional particles interact by a potential inversely proportional to their mutual distance. Recently, there have been numerous fascinating applications of RMT outside quantum chaology. Representative examples are the price fluctuations in a stock markets, the statistical properties of bus arrivals, the atmospheric correlations and the statistics of geometrical resonances in disordered binary networks.
The present paper proposes an application of RMT to the analysis of recent empirical data on the gap distribution of parked cars in the streets of London [13]. These authors could not reproduce their data using different versions of the random parking model, in which a car can park at any space greater than or equal to its length [14][15][16]. They had to introduce several additional assumptions in order to achieve qualitative agreement with the empirical results. Alternatively, we apply Dyson’s Coulomb gas model [6], which allows for using the results of RMT. Section 2 is a brief review of the model with a special emphasis on its relevance to real car parking. Section 3 shows a comparison between the empirical gap-size distribution measured in Ref. [13] and the level-spacing distribution for GUE. The conclusion of this work is formulated in Section 4.

2. Coulomb gas model

The purpose of this section is to show that Dyson’s one-dimensional gas model is relevant to the car parking problem. The model considers a gas of \( N \) point charges with positions \( x_1, x_2, ..., x_N \) free to move on the infinite straight line \(-\infty < x < \infty\). The potential energy of the gas is given by

\[
V = \frac{1}{2} \sum_i x_i^2 - \sum_{i<j} \ln |x_i - x_j|.
\]  

(1)

The first term represents a harmonic potential that attracts each charge independently towards the coordinate origin. The second term represents an electrostatic repulsion between each pair of charges. The logarithm function comes in if we assume the universe to be two-dimensional. Nevertheless, the methods of the statistical physics remain valid also for different shapes of the particle-particle interaction. For example, Krbaček and Šeba [10] have numerically evaluated the equilibrium distributions of one dimensional gas interacting via two body potentials \( v_{ij} \approx 1/|x_i - x_j|^a \) with \( a < 2 \) and found that the resulting distributions belong to the same class as the Dyson case. Now, let the charged gas be in thermodynamic equilibrium. The probability density function for the position of the charges is given by

\[
P(x_1, x_2, ..., x_N) = Ce^{-\beta V},
\]

(2)

where \( C \) is a normalization constant, \( \beta = 1/kT \) and \( k \) is the Boltzmann constant. Substituting (2) into (1), we obtain

\[
P(x_1, x_2, ..., x_N) = C \prod_{i<j} |x_i - x_j|^{-\beta} \exp \left( -\frac{1}{2} \beta \sum_k x_k^2 \right).
\]

(3)

This is exactly the joint probability density function for the eigenvalues \( x_1, x_2, ..., x_N \) of matrices from a GOE, GUE or GSE if \( \beta = 1, 2 \) or 4, respectively. Thus thus the role of the inverse temperature in the Coulomb gas model is played by the level-repulsion power of eigenvalues of the random matrices, which is specified by the symmetry of the ensemble with respect to time reversibility.

RMT has elaborate methods for calculating the spectral characteristics of the three canonical ensembles. For example, Mehta [1] expresses the nearest-neighbor spacing distribution \( P(s) \) as second derivatives of an infinite product of the factors \( [1 - \lambda_i (s)] \), where
$\lambda_n(s)$ are the eigenvalues of certain integral equations. Here $s_i = (x_{i+1} - x_i)/D$ and $D$ is the mean level separation and $s$ is a randomly chosen $s_i$. These approaches result in tabulated numerical values, series expansions and asymptotic expressions for the spacing distribution. They unfortunately do not lead to closed-form expressions that can be in the analysis of experimentally observed or numerically calculated discrete data.

In many cases, the empirical spacing distributions are compared to the so-called Wigner surmise. The latter follows from the exact solution for $2 \times 2$ matrices but still presents an accurate approximation to the exact results obtained for ensembles of large matrices. The Wigner surmise for GUE is given by

$$P(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi} s^2}. \quad (4)$$

To demonstrate the accuracy of the Wigner surmise, we expand this distribution in powers of $s$ to obtain

$$P(s) = \frac{32}{\pi^2} s^2 \left( 1 - \frac{4}{\pi} s^2 + \cdots \right) \approx 3.242 s^2 - 4.128 s^4 + \cdots, \quad (5)$$

while the power-series expansion of the corresponding exact distribution \[1\] yields

$$P_{\text{exact}}(s) = \frac{\pi^2}{3} s^2 - \frac{2\pi^4}{45} s^4 + \cdots \approx 3.290 s^2 - 4.329 s^4 + \cdots. \quad (6)$$

The question which remains to be answered is whether the Coulomb model is suitable for the car-parking problem. We see no problem in representing cars of different size by point particles since we are eventually interested in the gap-size distribution. We even dare to consider this as an advantage because the results do not involve the (average) car length as in other formalisms. The gap distribution, when represented by Eq. (4), has no adjustable parameters once the gaps $s$ are measured in units of their mean size $D$. Furthermore, we represent the cars by classical electrically-charged particles in the state of thermodynamic equilibrium using the Boltzmann factor (2). This factor is obtained from the Gibbs-Boltzmann canonical distribution by integration over the momenta of the particles. Namely this kind of averaging over the momenta is what allows to link the stationary parked cars with the moving gas particles. The particles are under the influence of a confining potential, which reflects the preference of the drivers to park their cars close to the target of the journey. A potential consisting of a sum of a single- and two-body terms, as in Eq. (1), seems to provide a reasonable representation for the car parking process. The single-particle term reflects the general tendency of the drivers to park their cars not too far from each other. The repulsive two-body term is meant as an expression of the tendency of the driver to keep a distance from the other parked cars in order to allow for parking manoeuvring. Thus, the superposition of these two potential, which is repulsive for small gaps and attractive for the large ones, expresses the fact that it is unlikely to see too small or too large gaps between parked car. The proposed model may thus be viewed as a generalization of the random car parking model, in which the simple geometric exclusion is replaced by dynamical coupling through a repulsive potential. The correspondence between RMT and the Coulomb-gas model is technical and based on identification of the inverse temperature $\beta$ of the gas with degree of level
repulsion as mentioned above. The car parking problem has no time reversal symmetry. For this reason, we assume GUE which violates time reversibility is more suitable for the car parking problem than the other ensembles, and set $\beta = 2$.

3. Comparison with empirical results

Rawal and Rodger [13] measured the gaps between parked cars on four connected streets without any driveway or side streets in central London. The data were collected in the late evening so that there were few places capable of taking additional cars. In total, 500 gaps were measured. The average gap size was 154.2 cm. Figure 1 shows by a histogram the probability distribution function $P(s)$, normalized to unity, for the gaps measured by Rawal and Rogers [13]. The spacings $s$ are defined as the ratio of the gaps to their mean value. The normalization is done in such a way that the overall surface under the data is 1. These modifications of the original data are introduced in order to make the comparison with the nearest-neighbor spacing distribution for GOE, which is represented in Fig. 1 by a solid curve. We can go further by regarding the renormalized data as a probability distribution for gaps in an ensemble of equivalent streets if the streets involved in the experiment by Rawal and Rodger do not represent a special case. In the opposite case, the results of this experiment would be of low value. We can then assign a statistical error for each gap interval equal to $\sqrt{n_i}$, where $n_i$ is the number of car pairs separated by the corresponding gap. The resulting statistical uncertainties are shown by error bars.

The empirical gap-size distribution [13] does not agree with the random car parking model. In this model, a particle of length $l$ is randomly deposited along a linear chain if it finds an empty place of length greater than or equal to $l$. The frequency of small gaps obtained in this model is extremely high and monotonously decreases as the gap size increases [13]. The empirical distribution behaves in a different way. The frequency of small gaps are small and increases as the gap size increases until it reaches a maximum at a gap size slightly less than the average size. In order to fit the empirical distribution Rawal and Rodger proposed two generalizations of the car-parking model, which were called model A and model B. In model A, cars are allowed to park only if the space is larger than $l + \varepsilon$, where the extra space $\varepsilon$ give a room to manoeuvre. They obtained a distribution that takes a nearly constant value at small gaps and then decreases after the gap size reaches a value of approximately $0.2l$. Their model B suggests that car parking is governed by two competing mechanisms. One is from people who just park anywhere and the other from those who maneuver to make better use of the available space. Thus, in each time step, a car is parked in an empty space and remains with probability $p$, or further derives, with probability $1 - p$, to a distance $y$ to the nearest car with a probability given by an arbitrary function $f(y)$. Model B provides a reasonable description of the data for the choice $p = 0.3$ and $f(y) = 6y(l - y)$, which leads to the dashed curve in Fig. 1. However, the disagreement at small gap sizes is still statistically significant.

We here show that the empirical gap size distribution of parked cars agrees with the prediction of the Coulomb gas model as formulated in the previous Section, without any additional assumptions. As mentioned above, GUE is meant to describe systems which are chaotic and violate time reversibility. The car parking process has both of these two features. The chaotic behavior is caused by the human behavior of the drivers. The
absence of reversibility. is an essential feature of all random adsorption processes. The solid curve in Fig. 1 is calculated using Eq. (4) for the spacing distribution for a GUE. The calculation is done without any adjustable parameter. The agreement between the predictions of RMT and the empirical data is excellent. In comparison with traditional approaches to the car parking problem, the Coulomb gas model includes fewer gaps with sizes tending to zero. This suggests that ”level repulsion” is an important mechanism overlooked in the tradition car-parking process, and RMT may be useful in understanding this process.

4. Discussion

We applied Dyson’s Coulomb gas model to the car parking problem. This model is relevant to real car parking as it represents the tendency of the driver to keep a distance from the nearest parked car by repulsion between gas particles and the tendency of the divers to park their cars close to each other by the single-particle harmonic-oscillator potential. We analyzed the empirical data on the distribution of the relative size of gaps between parked cars, measured by Rawal and Rodger. We found an excellent agreement between the empirical data and the spacing distribution for Gaussian unitary ensemble of random matrices.

The significance of the present results is that they suggest to add the random car parking problem to the long list of systems with RMT-like fluctuations. The latter is the one-dimensional continuum version of the random sequential adsorption process, in which particles land successively and randomly on a surface. In this model, if an incoming particle overlaps a previously deposited one, it is rejected due to the geometrical exclusion effect. Without additional assumptions, the random sequential process cannot explain gap size distribution as shown by Rawal and Rodger. In this respect, the Coulomb gas model has the advantage of extending the repulsion between parked cars further than their geometrical size. The random sequential adsorption model enjoys a wide interest in physics, chemistry, biology and in many other branches of science and technology [17,18]. Obviously, the proposed model is no substitute for the elaborate investigations of the random sequential adsorption process. Nevertheless, the powerful methods of RMT may be useful in understanding some aspects of this important process.

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Fig. 1. The gap size distribution for parked cars, measured by Rawal and Rodgers [13] compared with the spacing distribution for GUE (solid curve). The dashed curve is calculated using model B of Ref. [13].
