I. INTRODUCTION

The observation of high-energy cosmic neutrinos produced in distant astrophysical sites (or possibly by other, more exotic, mechanisms) has become one of the major challenges of astroparticle physics. In both astrophysical and exotic models, substantial fluxes of electron and muon neutrinos are expected from the disintegration of charged pions (and kaons) produced in the interaction of accelerated particles with ambient matter and radiation, either at the source location or on their way through the universe. Given the large distances traveled by these cosmic neutrinos, approximately equal fluxes in \( \nu_e, \nu_\mu \), and \( \nu_\tau \) are expected on Earth as a result of flavour mixing and oscillations \([1, 2, 3]\).

At ultra-high energies, the Earth becomes opaque to neutrinos as a result of charged- and neutral-current interactions which deplete their energy or convert them into the associated charged lepton in case of charged-current. Tau neutrinos can then be traced through the observation of byproducts of the decay of the associated tau lepton in matter such as ice \([4, 5, 6]\), water \([7, 8]\), or in the air \([9, 10]\).

The tau lepton is subject to energy-dependent radiative processes: Bremsstrahlung, production of \( e^+e^- \) pairs and photonuclear interactions (ionisation is negligible at ultra-high energy). A comprehensive knowledge of the probability that a tau with initial energy \( E_0 \) will not be absorbed during its propagation in matter is thus required to predict the expected rates of tau neutrinos detection by various experiments. In the muon case, because of its large lifetime, the survival probability is entirely determined by the energy losses over the whole energy range. Neglecting the fluctuations in energy losses (or, in other words, adopting the continuous energy loss approach) is known to lead to an overestimate of the survival probability \([11]\). In contrast, due to the short lifetime of the tau, the calculation of the range in that case is determined by the decay up to \( \approx 10^8 \) GeV. Then, the attenuation length becomes smaller than the decay length. Fluctuations in the energy losses may then play an important role - as in the muon case.

In this respect, it has already been put forward that such effects may indeed not be negligible for the tau \([12]\).

The approach adopted in this previous work was to inject tau neutrinos in matter as a source of tau leptons, and then to compute the flux of tau leptons propagated after some fixed distance of matter. This calculation was performed in two ways: the first by using a full simulation of all involved processes through a Monte-Carlo generator to account for the stochasticity of the problem, and the second by using a simplified semi-analytical framework to reproduce the solution within the continuous energy loss framework. Eventual differences in the flux calculations were attributed to the lack of accuracy of the continuous energy loss framework.

Such a strategy however makes it difficult to disentangle the direct study of \( \tau \) propagation from other effects related with the propagation and interactions of the \( \nu_\tau \). The aim of this paper is thus to re-examine the effects of the fluctuations in energy losses due to radiative processes of the tau lepton at ultra-high energy in a framework where we choose to directly inject \( \tau \)'s instead of \( \nu_\tau \)'s. This method allows to simplify the problem and to get a more direct comparison between the continuous and the stochastic treatment of the energy losses.

In section \([11]\) we present the physical processes that we take into account in this study, and the methods we will use to compute the propagated flux within both the continuous energy loss framework and the stochastic one. In section \([11]\) we apply our calculations to the case of mono-energetic beams of tau leptons, whereas in section \([14]\) we do the same in case of a power law injection spectra of tau and compare the results obtained within the two frameworks. Finally, we discuss our conclusions in section \([15]\).
At the energies of interest for this study, the relevant electromagnetic processes that the tau lepton undergoes are the Bremsstrahlung, the pair production and the photonuclear interactions. Whereas the cross sections of the two first ones are well established [13, 14], the description of the photonuclear process relies on the proper modelling of the nucleon structure functions at low x and high $Q^2$, and several calculations of the corresponding cross section can be found in the literature. The differences reside in the parameterisation of the relevant structure functions obtained from different formalisms. A popular treatment uses a combination of the generalised vector dominance model for the non-perturbative regime [15], and of the color dipole model for the perturbative part [16]. Other widely used results are obtained on basis of parameterisations of data using Regge theory [17, 18]. A recent comparative study shows that all these calculations give rates of photonuclear energy loss in good agreement [19], except for [20] which results in a significantly higher rate at energies above $10^8$ GeV. A new calculation based on saturation physics [21] is also presented in [19]. In this case the corresponding energy loss rate is more than a factor of 2 lower than the standard results at the highest ($\sim 10^{12}$ GeV) energies. In the following, we use the parameterisation obtained by Dutta et al. [17] for all the calculations we are presenting.

Finally, the tau lepton is also subject to weak interactions. However, using the cross sections derived in [22], the corresponding interaction length is expected to be much larger than for the electromagnetic processes until an energy greater than $10^{12}$ GeV. As we are only interested in studying the effect of the fluctuations of the electromagnetic interactions of the tau, we can thus comfortably neglect its weak interactions. We do as well regarding the regeneration of $\tau$ leptons at lower energy through the chain $\tau \rightarrow \nu_{\tau} \rightarrow \tau$ via a tau decay followed by a charged-current weak interaction of the neutrino.

B. Solving the equation of transport

Within the above assumptions, the transport equation that describes the evolution of the $\tau$ flux $\Phi$ along its path through matter, accounting for all processes of creation or absorption of a tau with an energy $E$ at a position $x$, is given by:

$$\frac{\partial \Phi(E,x)}{\partial x} = -\frac{\Phi(E,x)}{\lambda_{dec}(E)} - \sum_i \frac{\Phi(E,x)}{\lambda_i(E)} - \frac{\rho N}{A} \int dy \frac{d\sigma_i}{dy}(E,y) \langle \sigma_i \rangle,$$

(2)

where $\lambda_{dec}(E)$ is the decay length, $\lambda_i(E)$ and $d\sigma_i/dy(E,y)$ respectively the interaction length and the differential cross section of each process, $y$ the inelasticity of the process considered, $\rho$ the density of matter, $N$ the Avogadro number and $A$ the atomic mass number.

![FIG. 1: Evolution of $\beta$ as a function of the energy for a $\tau$ lepton in standard rock (A=22). From bottom to top: bremsstrahlung, pair production and photonuclear interactions.](image)

1. Continuous energy losses approximation

When the differential cross-sections exhibit a peak near $y=0$, as it is indeed the case for the processes we are dealing with, the integrals are dominated by the behavior of the integrands around 0, in such a way that an expansion of these integrands can be performed [23]. At first order in $y$, this yields the following equation:

$$\frac{\partial \Phi(E,x)}{\partial x} = -\frac{\Phi(E,x)}{\lambda_{dec}(E)} + \rho \frac{d}{dE} \left( E \beta(E) \Phi(E,x) \right),$$

(3)

where we have introduced the standard notation

$$\beta(E) = \frac{N}{A} \sum_i \int_{y_{min}}^{y_{max}} dy \frac{d\sigma_i}{dy}(E,y).$$

(4)

We show on Fig. 1 the evolution of $\beta$ as a function of energy for the three processes described in the previous sub-section. Note that the contribution of the energy loss due to the photonuclear interactions does not reach any asymptotic value because of the expected growth of the photonuclear cross section $\sigma_{\gamma N}$.

Within the approximation of continuous energy losses, the average energy lost per unit distance $dE/dx$ is, at ultra-high energy, assumed to be proportional to the mean inelasticity of each process in the following way:

$$\frac{dE}{dx} = -\frac{\rho N}{A} E \sum_i \langle \sigma_i \rangle.$$  

(5)
The right-hand side of this expression is nothing else but $-\rho E\beta(E)$. This implies that Eqn. 3 can be easily integrated, leading to the following expression:

$$\Phi(E, x) = \Phi_0(\tilde{E}_0) \exp \int_0^x du \left( \frac{\partial}{\partial E} \gamma(\tilde{E}_u) - \frac{1}{\lambda_{\text{dec}}(\tilde{E}_u)} \right)$$

(6)

where $\gamma(E) = \rho E\beta(E)$ and $\tilde{E}_u$ the solution of

$$\int_{E_u}^{E} \frac{dE_{\tau}}{\gamma(E_{\tau})} = v - x.$$  

(7)

In the following, we use Eqn. 6 to compute any propagated flux of tau leptons when referring to as the continuous energy losses approximation. To verify this expression, we performed a Monte-Carlo calculation evaluating at each step the decay probability as well as the continuous energy losses through Eqn. 5. For an incident flux of tau leptons following a $E^{-2}$ spectrum in energy between $10^8$ GeV and $3 \cdot 10^{11}$ GeV, we show the agreement of the two calculations on Fig. 2 after a propagation in 1, 5, and 10 km of rock. Let us point out that, within the continuous energy losses approximation, the sharp cutoffs in energy result from the choice of the maximal energy at $x = 0$, which is univocally related to the propagated energy at depth $x$.

To test the accuracy of results obtained by using the continuous energy losses approximation, we need to account for the stochastic nature of the transport equation. A convenient way to solve Eqn. 2 is to use a Monte-Carlo generator sampling all the interactions. However, to limit CPU time spent where the cross sections are large but the energy losses are small, a standard way to proceed is to separate the losses into two components \[12, 24\]: a continuous one where the rate of the losses is large ($y \in [y_{\text{min}}, y_{\text{cut}}]$), and a stochastic one where the differential cross sections lead to more catastrophic losses but with a weaker rate ($y \in [y_{\text{cut}}, y_{\text{max}}]$). In practice, this means that within an elementary step, the Monte-Carlo samples all the interactions according to cross sections computed for $y \geq y_{\text{cut}}$ together with the probability to decay. At the same time, a continuous energy loss is applied according to Eqn. 4, except that the upper bound of the $\beta$ coefficient is replaced by the cut $y_{\text{cut}}$. In that way, the $y$ range where the stochastic nature of the interactions can be relevant is taken into account, and at the same time, the frequent small losses due to the peaking of the cross sections near $y = 0$ are applied. A good compromise to reproduce the stochastic features using a reasonably fast code is to take $y_{\text{cut}} = 10^{-3}$.

To get a feeling of the effects which may be induced by fluctuations in energy losses, we plot on Fig. 3 the interaction length of each process, but restricting the $y$ range to $y \in [y_{\text{cut}}, y_{\text{max}}]$. The interaction length of the Bremsstrahlung is clearly the higher, so that the effects
induced by this process are marginal. On the other hand, the interaction lengths induced by pair production and photonuclear interactions are lower and rather close to each other, leading to higher rates. Stochastic effects are not expected to be strong for pair production processes, as the corresponding differential cross-section behaves as \( \approx y^{-2} \). On the contrary, the rate of photonuclear interactions is never negligible, and in particular, above \( \approx 6 \) EeV, this process becomes the dominant one. Its differential cross section behaves as \( y^{-1+\alpha} \) with \( \alpha \) slightly decreasing with energy from 0.1 at \( 10^8 \) GeV to 0.05 at \( 10^{12} \) GeV. Hence, fluctuations in energy losses may become important due to this interaction.

### III. MONO-ENERGETIC TAU

#### A. Propagated energy spectra

Let us first consider the case of a mono-energetic beam of tau leptons propagating in rock. The simulated log\(_{10}\)-distributions of the energy of an incident \( 3 \cdot 10^9 \) GeV tau beam after crossing 1, 5, and 10 km of standard rock are displayed in Fig. 4 (top). We show on the same plots the means of the energy distributions, which are the values used within the continuous energy losses approximation. The most probable values are always greater than the mean values of the distributions. After 1 km, a good part of the simulated events still carry a large fraction of the initial energy \( E_0 \), meaning that this fraction of particles did not undergo many interactions. However, the distribution is asymmetric and there is already a long tail of events undergoing hard losses. For larger paths in rock, fluctuations in the energy losses increase, resulting in a broadening of the distribution and a smoothening of the high-energy cutoff.

The bottom panel of Fig. 4 shows the same quantities for a higher initial energy of the incident \( \tau \) beam \( (E_0 = 3 \cdot 10^{10} \) GeV). As expected from the discussion about the effective interaction lengths for \( y \geq y_{\text{cut}} \), those distributions reflect the fact that the fluctuations are larger at higher energies due to the higher rate of photonuclear interactions, which are precisely those leading to harder losses.

#### B. Tau-lepton range

For a fixed initial energy of the tau, it is interesting to calculate the survival probability of the tau as a function of the traversed depth. In case of continuous energy loss, the survival probability is simply given by

\[
P_{\text{surv}}(E_0, x) = \exp \left( -\int_0^x \frac{du}{\lambda_{\text{dec}}(E_u)} \right)
\]  

(8)

(9)

with \( \overline{E}_u \) the solution of

\[
\int_{E_0}^{\overline{E}_u} \frac{dE_\tau}{\gamma(E_\tau)} = -u.
\]

We also computed \( P_{\text{surv}} \) for the case of stochastic losses. We plot in Fig. 5 the resulting curves, showing that in addition to broadening the distributions, the effect of the
fluctuations is globally to reduce the survival probabilities with increasing energy. Note also the presence of a small tail corresponding at higher ranges, which reflects the cases for which the number of interactions is lower. This decrease can be quantified through the range $R(E_0)$ of the tau lepton which is simply:

$$R(E_0) = \int_0^\infty dx \ P_{\text{surv}}(E_0, x)$$  \hspace{1cm} (10)

We show in Fig. 6 the corresponding range of the tau lepton in standard rock. Let us remind here that the weak interactions of the tau, which we have neglected thorough the whole paper, might change a little bit the picture at the highest energies. However, rather than giving numbers to be taken at face values, we are interested here in analyzing the effect of the fluctuations, which appear to slightly reduce the range of the tau as its energy increases.

**IV. POWER LAW INJECTION SPECTRA**

In this section, we consider the case of $E^{-1}$ and $E^{-2}$ tau injection spectra. We restrict ourselves to these generic fluxes to study possible distortions on the propagated fluxes due to the stochastic effects. All the calculations we present here assume a continuous injection spectrum following a power law between $10^8$ GeV and $3 \cdot 10^{11}$ GeV. The cutoff for the maximum energy is chosen to be sharp to exhibit most clearly the different behaviors of the propagated fluxes.

In Figs. 7 and 8 we show the results obtained respectively from $E^{-1}$ and $E^{-2}$ injection spectra. For a better lecture of the different structures, we choose to plot the fluxes corrected for the injection spectrum. This means that in a trivial situation with no losses and no decay of the tau, all results would be 1. Continuous lines are from the semi-analytical solution in the approximation of continuous energy losses, whereas squared dots are the results of the Monte-Carlo simulation including stochastic effects. Again, the sharp cutoff present in the semi-analytical solutions results from the choice of the maximal energy together with the univocal relation between this maximal energy at $x = 0$ and the propagated energy at a depth $x$. On the other hand, the stochastic treatment of the tau propagation has a smoothening effect on this cutoff: there are indeed fluctuations affecting a small fraction of particles which undergo less interactions and less hard losses. The energy range on which this broadening occurs clearly increases with the depth traversed by the tau, and it is also more pronounced in the case of an $E^{-1}$ flux. At the same time, we can see from both figures that this effect is the only important distortion induced by the stochastic processes. This means that, as far we are dealing with continuous incident spectra behaving as power laws, there is a compensation between positive and negative fluctuations of the energy losses everywhere in the considered energy range, except near the high-energy
FIG. 7: Comparison of the semi-analytical solution (solid line) of the transport equation within the continuous energy losses approximation (Eqn.6) with respect to a Monte-Carlo calculation (squared points) taking into account the stochastic effects of the radiative processes. An $E^{-1}$ flux between $10^8$ GeV and $3 \cdot 10^{11}$ GeV is used at injection. Three distances of propagation in rock are shown: 1 km, 5 km and 10 km (from right to left).

border. This compensation allows to use the continuous energy loss approximation as the correct mean value of the propagated spectrum.

V. CONCLUSIONS

In this paper, we have shown that the stochastic nature of the radiative processes undergone by tau leptons at ultra-high energy is indeed responsible for large fluctuations in the tau energy losses. At the same time, however, these fluctuations are not so large as to blur the picture with respect to the continuous energy loss approximation, as far as the calculation concerns power-law injection spectra in a given, continuous energy range.

As already pointed out in Sec. II A, significant theoretical uncertainties exist in the calculations of the cross section for photonuclear interactions, which are the most relevant processes for tau energy losses at the ultra-high energies we are interested in. The present study was done on basis of one particular model [17] which is rather pessimistic in the $y$ range where the stochastic effects can lead to hard losses, in the sense that it leads to a fairly high rate of interactions. Any other model leading to a lower rate would not change the picture. Moreover, any other model leading to comparable or slightly greater rate would have to present a significantly harder differential cross section in term of $y$ to challenge our conclusions.

Acknowledgments

O.B.B. is supported by the Ministerio de Educacion y Cienca of Spain through the postdoctoral grant program. V.V.E. acknowledges support from the European Community 6th Framework program through the Marie Curie Fellowship MEIF-CT-2005 025057.
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