Hearing the shape of inequivalent spin structures and exotic Dirac operators

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Abstract
Exotic spinor fields arise from inequivalent spin structures on non-trivial topological manifolds, \(M\). This induces an additional term in the Dirac operator, defined by the cohomology group \(H^1(M, \mathbb{Z}_2)\) that rules a Čech cohomology class. This formalism is extended for manifolds of any finite dimension, endowed with a metric of arbitrary signature. The exotic corrections to heat kernel coefficients, relating spectral properties of exotic Dirac operators to the geometric invariants of \(M\), are derived and scrutinized.

Keywords: exotic spinors, inequivalent spin structures, Dirac operator, heat kernel, Clifford algebras

1. Introduction

Based on the Lounesto’s spinor field classification [1, 2], new spinor fields, beyond the well-known Dirac, Weyl and Majorana ones, have been proposed, discussed, and scrutinized in the last 15 years. There is a comprehensive list of prominent papers in the field, among which one can highlight the most influential ones. In the context of supergravity compactifications, references [3,4] shed light on new classes of spinor fields, on the \(S^7\) sphere. Besides, new spinor classes in superstring theory were derived and analyzed in reference [5]. The compactification process in AdS/CFT then induced new spinor classes in the bulk, in reference [6]. In addition, new spinor classifications have been proposed [7, 8], that are reciprocal to the Lounesto’s one, also encompassing new gauge theoretical aspects. References [9–12] studied new spinors in a gravity background. Singular spinor fields and their relationship with their appropriate analogue of Lounesto’s classification, were also studied in references [13–24] (and references therein).
On non-trivial topological $M$ manifolds, inequivalent spin structures are categorized by the cohomology group $H^1(M, \mathbb{Z}_2)$ [25], where the standard spin structure corresponds to the trivial cohomology class in $H^1(M, \mathbb{Z}_2)$. If $M$ does satisfy all the Geroch’s theorem hypotheses [26], it has at least one spin structure, whose unicity is ruled by the topology of $M$. Manifolds that are simply-connected hold a unique spin structure since, in this case, the fundamental group is indeed trivial. This does not happen to manifolds that are multiply-connected, which can have many spin structures, labeled by the cohomology group $H^1(M, \mathbb{Z}_2)$. Inequivalent spin structures yielding distinct spin connections were more precisely discussed in references [27, 28]. Reference [27] used inequivalent spin structures to describe Cooper pairs in superconductivity, through exotic spinor fields.

On multiply-connected manifolds, the existence of spin structures, that are not equivalent to the trivial spin structure, yield Dirac operators with an additional exotic term. Up to the results in reference [29], the exotic term in the Dirac operator was thought to be experimentally unrealizable. In fact, for the Dirac spinor field and other regular fermionic fields, the exotic term is driven by elements of the cohomology group $H^1(M, \mathbb{Z}_2)$. Hence, modifications of the Dirac operator, containing exotic terms, were thought to be usually encrypted as a shift of the gauge potential in the corresponding Dirac equation. However, reference [29] showed that when 4D QFT takes in, the dispersion relation obtained in this case is different from that one obtained by taking into account the gauge field interaction. Therefore, the exotic term can be also physically realized for mass dimension three-halves fermionic fields in 4D multiply-connected spacetimes. References [25, 30] already showed that mass dimension one spinor fields, in 4D spacetimes, can probe the exotic terms in the Dirac operator ruling the equations of motion of such spinor fields. Several more applications of inequivalent spin structures in physics were studied, including superconductivity and condensed matter. Reference [31] proposed black hole physics as an origin of spacetime exoticness, naturally generating a non-trivial topology. In this context, the Hawking radiation of exotic fermions was computed, showing that exotic terms in the Dirac operator do alter black hole evaporation rates. Exotic spinor fields were scrutinized in references [25–30, 32–34], where various physical applications were investigated. One of the main results of this paper consists of extending the previous results on exotic inequivalent spin structures to manifolds of any finite dimension and metric signature.

Inequivalent spin structures and exotic spinor fields, arising from manifolds having non-trivial topologies, can engender deep modifications on the heat kernel coefficients associated with the manifold. Due to the important status of this research line currently, we want to investigate the spectral properties of the so-called exotic fermionic fields, relating them to the geometric invariants of the manifold that describes the spacetime, where these fermionic fields live in. For it, the heat kernel coefficients and their exotic counterparts will make it possible to define a quantity, the exotic deviation coefficient, that can probe the spectral properties of spacetimes with non-trivial topology.

Heat kernels will be used in this work, where exotic heat kernel coefficients, encompassing the exotic corrections to the Dirac operator in multiply-connected manifolds in any finite dimension equipped with a metric of arbitrary signature, will be derived and discussed. It extends previous results on exotic inequivalent spin structures in the literature, that is almost entirely valid just for 4D manifolds with Lorentzian metric signature.

Related to zeta functions, heat kernel coefficients are tools to study path integrals in QFT as well as processes of diffusion and partition functions in statistical mechanics [37, 38]. The very core scheme underlying heat kernel coefficients is to devise spectral properties of bosonic and/or fermionic fields, that propagate on a given spacetime, to the inherent geometric invariants of the manifold that represents the spacetime. The heat kernel can be thought of as being
a particular case of spectral functions, having an intimate relationship to the zeta function [35, 37]. Kač argued in reference [35] whether two given domains are congruent, once they do have the same sequence of eigenvalues. His celebrated quotation, whether one can hear the shape of a drum [35], consists of retrieving the geometric and topological properties of a manifold from the spectrum of a given differential operator. Calculating the heat kernel coefficients of a differential operator on a manifold provides the answer to this relevant question. In this context, the heat kernel coefficients associated with exotic Dirac operators on non-trivial topological manifolds can be also derived and studied. On the other hand, the heat kernel is also an adequate tool to study the Atiyah–Singer index theorem. In reference [36] a simple example, using triangles, elucidated Kač’s posed question. This problem can be extended to compact Riemannian manifolds without or with boundaries [37]. With the exotic Dirac operator, computing the deviations around the standard heat kernel coefficients of trivial manifolds can inform us about the features of non-trivial topological manifolds. Hence, it can clarify about the structure of inequivalent generalized spin structures.

This paper is organized as follows: section 2 is devoted to present the Clifford bundle on manifolds of any finite dimension, equipped with a metric of arbitrary signature, approaching inequivalent spin structures and the exotic Dirac operator. Section 3 is dedicated to introduce the heat kernel coefficients and to review the way they can be locally computable concerning the geometric invariants of $M$. The exotic heat kernel coefficients and the respective deviations from the standard heat kernel coefficients have their calculations detailed in section 4, being also discussed and scrutinized. Our final remarks are displayed in section 5.

2. Preliminaries: exotic spin structures

Inequivalent spin structures on Clifford spinor bundles are here reviewed and extended to manifolds of arbitrary finite dimension endowed with a metric of arbitrary signature, to define the complete exotic Dirac operator. The 5-tuple $(M, g, \nabla, \tau, \tau_\nu)$ denotes the spacetime structure [39], for $M$ denoting a $n$-dimensional, compact, pseudo-Riemannian spin manifold; $g$ stands for the spacetime metric, of signature $p - q$, where $p + q = n$; $\nabla$ is the connection associated with the metric $g$; $\tau_\nu$ defines a spacetime orientation, whereas $\tau$ denotes a future-pointing temporal orientation. $T^*M [TM]$ consists of the cotangent [tangent] bundle of $M$. The tangent bundle admits the splitting $TM = (TM)^t \oplus (TM)^s$, into timelike and spacelike subbundles. $\Omega(M)$ is the exterior bundle, constructed on the tangent bundle of $M$. Let $\phi_1, \phi_2, \phi_3 \in \Omega(M)$ be differential form fields on $M$. The left contraction can be defined in terms of the metric and the exterior product, denoted by ‘∧’, as $g(\phi_1 \wedge \phi_2, \phi_3) = g(\phi_2, \phi_1 \wedge \phi_3)$, where the tilde denotes the reversion defined on $\Omega(M)$. Let $F(M)$ be the bundle of frames on $M$ and $\mathcal{P}_{\text{Spin}_{pq}}(M)$ the orthonormal coframe bundle, whereas $\mathcal{P}_{\text{Spin}_{pq}}(M)$ is the spin coframe bundle. The Clifford product a vector field $v \in TM \simeq \Omega^1(TM)$ and a form field $\phi \in \Omega(TM)$ reads $v\phi = v \wedge \phi + v.\phi$. The Grassmann bundle $(\Omega(M), g)$, equipped with the Clifford product, is the Clifford bundle $\mathcal{E}(M, g)$, with Clifford algebras $\mathcal{E}_{pq}$ as sections. The tangent bundle $TM$ has the orthogonal group $O_{pq}$ as its structure group. The $F(M)$ bundle of frames can be endowed with a set of (transition) functions

$$u_{ij} : U_i \cap U_j \rightarrow O_{pq},$$

for $\{U_i\}$ representing open sets in $F(M)$. Besides, functions $f(U_i, U_j) = \det u_{ij} = \pm 1$ in a Čech chain define a so-called cocycle, as

$$u_{ij} \circ u_{jk} \circ u_{ki} = \text{id}_{O_{pq}}.$$
This set of functions is an element of the Čech cohomology class as well, representing the first Stiefel–Whitney class \( w_1 \). The tangent bundle \( TM \) admits a spin bundle structure if and only if \( w_1(M) = \{0\} = w_2(M) \), for \( w_2(M) \) being the second Stiefel–Whitney class.

More precisely, a spin structure on \( M \) consists of a principal bundle

\[
\pi_s : P_{\text{Spin}_{p,q}}(M) \rightarrow M,
\]

with group \( \text{Spin}_{p,q} \), and the 2-fold mapping [40]

\[
s : P_{\text{Spin}_{p,q}}(M) \rightarrow P_{\text{SO}_{p,q}}(M),
\]

satisfying \( s(x \phi) = s(x) \text{ad}_\phi \), for all \( x \in P_{\text{Spin}_{p,q}}(M) \). The adjoint mapping reads [39]

\[
\text{ad} : \text{Spin}_{p,q} \rightarrow \text{Aut}(C_{\ell_{p,q}})
\]

\[
\phi \mapsto \text{ad}_\phi : C_{\ell_{p,q}} \rightarrow C_{\ell_{p,q}}
\]

\[
\Xi \mapsto \text{ad}_\phi \Xi = \phi \Xi \phi^{-1}, \tag{4}
\]

where \( \text{Aut}(C_{\ell_{p,q}}) \) denotes the automorphism group of \( C_{\ell_{p,q}} \). This is equivalent to the commutation of the diagram

![Diagram](https://via.placeholder.com/150)

There is also an onto mapping

\[
\sigma : \text{Spin}_{p,q} \rightarrow \text{SO}_{p,q}, \tag{5}
\]

whose kernel consists of the group \( \mathbb{Z}_2 \). Therefore \( \text{ad}(-1) = \text{id}_{\text{Spin}_{p,q}} \), yielding the adjoint mapping to descend to a representation of \( \text{SO}_{p,q} \). One regards \( \text{ad} \) this representation, consisting of \( \text{ad} : \text{SO}_{p,q} \rightarrow \text{Aut}(C_{\ell_{p,q}}) \). Hence, \( \text{ad}_\phi \Xi = \phi \Xi \phi^{-1} \). In this context, the Clifford bundle \( C\ell(M, g) \), can be made a vector bundle, whose sections are differential form fields and \( C\ell(M, g) = P_{\text{SO}_{p,q}}(M) \times \text{ad} C_{\ell_{p,q}} \).

References [25, 27, 28, 34] introduced inequivalent spin structures on 4D Lorentzian spacetimes with non-trivial topology. Their formal aspects are hereon in this section extended for manifolds of any finite dimension, endowed with a metric of arbitrary signature.

A spin structure \( (P_{\text{Spin}_{p,q}}(M), s) \) is not uniquely defined, when \( H^1(M, \mathbb{Z}_2) \neq \{0\} \). Then, other inequivalent spin structures, labeled by elements of \( H^1(M, \mathbb{Z}_2) \), arise from \( (P_{\text{Spin}_{p,q}}(M), s) \). Hereon quantities having a ring over them are associated with exotic, inequivalent, spin structures. Two spin structures, \( (P_{\text{Spin}_{p,q}}(M), s) \) and \( (P_{\text{Spin}_{p,q}}(M), \tilde{s}) \), are said to be equivalent when the mapping \( \zeta : (P_{\text{Spin}_{p,q}}(M), s) \rightarrow (P_{\text{Spin}_{p,q}}(M), \tilde{s}) \) exists, making the following diagram

![Diagram](https://via.placeholder.com/150)
to commute.

\[
P_{SO_{p,q}}(M) \quad \circlearrowright \quad (P_{Spin_{p,q}}(M), s) \quad \xrightarrow{\zeta} \quad (\hat{P}_{Spin_{p,q}}(M), \hat{s})
\]

To introduce the exotic corrections to the Dirac operator, one needs to define spinor fields as sections of the spin bundle. A \(\mathbb{C}\)-spinor bundle on \(M\) is the bundle \(S_{\mathbb{C}}(M) = P_{Spin_{p,q}}(M) \times_{\mu} S_{p,q}\), where \(S_{p,q}\) is a complex left module for \(\mathbb{C} \otimes \mathcal{E}_{p,q}\), being \(\mu\) a representation of \(\hat{\text{Spin}}_{p,q}\) in the endomorphism space of \(S_{p,q}\). The particular case where the metric signature has \(p = 1\) and \(q = 3\) yields \(S_{p,q} = \mathbb{C}^4\) and \(\mu\) is the \((1/2, 0) \oplus (0, 1/2)\) representation of \(\text{Spin}_{1,3} \simeq \text{SL}(2, \mathbb{C})\) in \(\text{End}(\mathbb{C}^4)\), what complies with the relativistic quantum mechanical standard approach. In this case, classical spinor fields are sections of the bundle \(P_{\text{Spin}_{1,3}}(M) \times_{\rho} \mathbb{C}^4\), where \(\rho\) stands for the \((1/2, 0) \oplus (0, 1/2)\) representation of the 4D Lorentz group in \(\mathbb{C}^4\). For the most general case for any dimension and any metric \((p, q)\) signature, \(S_{p,q}\) is given by the classical spinors given by table 1, that carry the representations of the respective Clifford algebras in table 2 [2].

Two spin structures \((P_{Spin_{p,q}}(M), s)\) and \((P_{Spin_{p,q}}(M), \hat{s})\) are respectively described by the mappings \(h_{jk} : U_j \cap U_k \rightarrow \text{Spin}_{p,q}\) and \(\hat{h}_{jk} : U_j \cap U_k \rightarrow \text{Spin}_{p,q}\), together with \(u_{jk} : U_j \cap U_k \rightarrow \text{SO}_{p,q}\), satisfying

\[
\sigma \circ h_{ij} = u_{ij}, \quad h_{ij} \circ h_{jk} = h_{ik}, \quad h_{ij} = \text{id}_{U_j \cap U_j}.
\]
Now one defines a mapping $c_{\tilde{r}}$ by the relation $h_{\tilde{r}}(x) = \tilde{h}_{\tilde{r}}(x)c_{\tilde{r}}$ such that $c_{ij} : U_i \cap U_j \to \text{ker } \sigma = \mathbb{Z}_2$, satisfying $c_{ij} \circ c_{\tilde{r}} = c_{ij}$. Given the irreducible representation $\rho : \mathfrak{c}_{\ell pq} \to M(k, \mathbb{C})$ in $P_{\text{Spin}_{pq}(M)} \times \rho S_{pq}$, one has $\rho(c_{ij}(x)) = \pm 1$, since $\rho$ is a faithful representation. Deciding the type of the matrix $M(k, \mathbb{C})$ and the representation space, $S_{pq}$, depends on the dimension of $M$ and on the metric signature as well. In some cases, $M(k, \mathbb{C})$ can also denote the direct sum of two identical matrices, whereas respectively $S_{pq}$ can also consist of the direct sum of two representation spaces. Tables 1 and 2 make it precise which choice one must take into account, respectively, for $S_{pq}$ and $M(k, \mathbb{C})$.

When the cohomology group $H^2(M, \mathbb{Z}_2)$ is devoid of 2-torsion, one can define functions $\xi_i : U_i \to \mathbb{C}$, such that $\xi_i(x) \in U(1)$ [25, 27, 28, 34], and

$$\xi_i(x)(\xi_j(x))^{-1} = \rho(c_{ij}(x)) = \pm 1. \quad (7)$$

In addition, $\xi_i^2(x) = \xi_j^2(x)$, for all $x \in U_i \cap U_j$. Hence, the (local) scalar fields $\xi_i$ induce a unique scalar field $\xi : M \to \mathbb{C}$ such that $\xi(x) = \xi_i^2(x)$, for all $x \in U_i$.

Let one considers an arbitrary spinor field $\psi \in \text{see } P_{\text{Spin}_{pq}(M)} \times \rho S_{pq}$. There is, then, a one-to-one correspondence between elements of the cohomology group $H^1(M, \mathbb{Z}_2)$ and a Dirac operator, $D$. A local spinor field component $\psi_i : U_i \to S_{pq}$ is the mapping that satisfies $\rho(\psi_i, \psi_j(x)) = \psi(x)$, for local spin bundle sections $\psi_i : U_i \to (P_{\text{Spin}_{pq}(M)}(M), s)$. Therefore the transition rule $\psi_i(x) = \rho(h_{ij}(x))\psi_j(x)$, for all $x \in U_i \cap U_j$, does hold.

Given the local spin bundle sections $\psi_i : U_i \to (P_{\text{Spin}_{pq}(M)}(M), s)$, another set constituted of exotic sections $\tilde{\psi}_i : U_i \to P_{\text{Spin}_{pq}(M)}$, corresponding to an inequivalent spin structure, satisfies the following diagram:

Locally, for all $x \in U_i \cap U_j$, it permits the exotic spinor field to have the property

$$\tilde{\psi}_i(x) = \rho(\tilde{h}_i) = \rho[h_{ij}(x)]\rho[c_{ij}(x)]\tilde{\psi}_j(x). \quad (8)$$

Hence $\rho(\xi_i) = \rho(c_{ij}(x))[\rho(\xi_j)]$. Comparing it to equation (8), it is straightforward to realize that the term $\rho(\xi_j)\tilde{\psi}_j$, behaves as $\psi_j \in P_{\text{Spin}_{pq}(M)} \times \rho S_{pq}$, inducing a mapping between the exotic spin bundle to the standard spin bundle, given by

$$\Gamma : P_{\text{Spin}_{pq}(M)} \times \rho S_{pq} \to P_{\text{Spin}_{pq}(M)} \times \rho S_{pq}$$

$$\tilde{\psi}_i \mapsto \Gamma(\tilde{\psi}_i) \equiv \rho(\xi_i)\tilde{\psi}_i = \psi_i \quad (9)$$

such that the exotic Dirac operator can be defined as

$$\hat{D}_X \psi = \Gamma(D_X \tilde{\psi}) + \frac{1}{2}(X, d(\xi^{-1}d\xi))\Gamma(\tilde{\psi}) \quad (10)$$

for all sections $\psi \in P_{\text{Spin}_{pq}(M)} \times \rho S_{pq}$ and all vector fields $X$ on $M$ [25, 27, 28, 34]. In a chart of coordinates on $M$, equation (10) is equivalent to the exotic Dirac operator to be given in
terms of the standard Dirac operator by
\[ \hat{D} X = D X - \frac{1}{2} \left[ X_{\mu} \left( \xi^{-1}(x) d \xi(x) \right) \right]. \quad (11) \]
As the exotic additional term in equation (11) is an integer in a Čech cohomology class, equivalent to \( \frac{1}{2\pi} \xi^{-1}(x) d \xi(x) \) [27, 28, 34], one can rescale \( \xi(x) \mapsto (2\pi)^{-1/2} \xi(x) \), yielding
\[ i \gamma^\mu \hat{D}_\mu = i \gamma^\mu D_\mu + \xi^{-1}(x) d \xi(x), \quad (12) \]
where the \( \gamma \) matrices satisfy the Clifford commutation relations, \( \{ \gamma^\mu, \gamma^\nu \} = 2g^\mu\nu \) id\( _{\mathbb{C}}\) and \( g^\mu\nu \) denotes the metric tensor components. As \( \xi(x) \in U(1) \), one can write \( \xi(x) = e^{i\theta(x)} \), for a \( C^1 \) scalar field \( \theta : U \subset M \to \mathbb{C} \). The exotic spin structure term, in the exotic Dirac operator (12), then reads
\[ \xi^{-1}(x) d \xi(x) = e^{-i\theta(x)}(i \gamma^\mu \partial_\mu \theta(x))e^{i\theta(x)} = i \gamma^\mu \partial_\mu \theta(x). \quad (13) \]

3. Heat kernel coefficients

Let \( R^\rho_{\mu\nu\sigma} \) be the Riemann curvature tensor, \( R^\rho_{\sigma\mu\nu} \) the Ricci tensor, and \( R = R^\rho_{\rho\mu\nu} \) be the scalar curvature. Given a vielbein \( \{ e_i \} \subset TM \), the tetrads are such that \( e^\rho_j e^\mu_k g^\mu\nu = \delta^j_k \). The inverse vielbein is defined by the relation \( e^\rho_j e^\mu_k = \delta^j_k \). It is worth to emphasize that Euclidean spaces make upper and lower indexes to be equivalent. The spin connection \( \sigma^\mu_\nu \) is given by the relation \( \nabla_\mu v^\nu = \partial_\mu v^\nu + \sigma^\nu_\mu v^\rho + \sigma^\rho_\nu v^\mu \), for an arbitrary vector \( v^\nu \). The condition \( \nabla_\mu e^\nu_k = 0 \) yields [37]
\[ \sigma^\rho_k_\mu = g^\mu^a \left( e^\nu_a \Gamma^\rho_\mu^\nu e^\rho_k - e^\rho_a \partial_\mu e^\rho_k \right). \quad (14) \]
Let a semicolon denote covariant differentiation with respect to the Levi–Civita connection of \( M \). The quantity \( \Omega^\rho_\mu = \partial^\rho_\mu \omega_\nu + \omega^\rho_\mu \omega^\nu \) denotes the connection field strength, given by \( \omega^\rho_\mu \):
\[ \Omega^\rho_\mu = \partial^\rho_\mu \omega_\nu + \omega^\rho_\mu \omega^\nu. \quad (15) \]
The action for spinor fields \( \psi \) reads [37]
\[ \mathcal{L} = \int_M d^n x \sqrt{g} \bar{\psi} D \psi, \quad (16) \]
where \( /D \) is the Dirac operator satisfying \( D = /D^2 \), where \( D \) is the Laplace operator. The matrix \( \gamma^5 \) represents the volume element, also used to define the chirality operator in QFT. The usual Dirac operator reads
\[ D = i \gamma^\mu \left( \partial_\mu + \frac{1}{8} [\gamma_\mu, \gamma_\rho] \sigma^{\rho\mu} + A^5_\mu + iA^5_\mu \right) \gamma^5, \quad (17) \]
where \( A_\mu \) [\( A^5_\mu \)] denotes the gauge vector [axial gauge vector] field, which is skew-Hermitian when gauge indexes are taken into account. The gauge vector field \( A_\mu \) corresponds to the twisting of the Dirac operator by an Abelian connection. The operator
\[ D = D^2 \sim g^\mu_\nu \nabla^\mu \nabla^\nu + E \quad (18) \]
is of Laplace type, where

$$\omega_\mu = \frac{1}{8}[\gamma_\nu, \gamma_\rho] A^\mu_\nu + A_\mu + \frac{i}{2}[\gamma_\mu, \gamma_\nu] A^5_\nu A^5_\gamma,$$

$$E = -\frac{1}{4} R + \frac{1}{4}[\gamma_\mu, \gamma_\nu] F_{\mu\nu} + i \gamma_5 D^\mu A^5_\mu - (n - 2) A^5_\mu A^5_\mu
- \frac{1}{4}(n - 3)[\gamma_\mu, \gamma_\nu][A^5_\mu, A^5_\nu],$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

$$D^\mu A^5_\nu = \partial^\mu A^5_\nu - \Gamma^\mu_\rho_\sigma A^5_\rho + [A^5_\mu, A^5_\nu].$$

Besides,

$$\Omega_{\mu\nu} = F_{\mu\nu} - [A^5_\mu, A^5_\nu] - \frac{1}{4} \gamma_\rho \gamma_\sigma R_{\rho\sigma\mu\nu} - i \gamma_5 \gamma_\rho (\gamma_\nu D^\mu A^5_\rho - \gamma_\mu D^\mu A^5_\nu)$$

$$+ i \gamma_5 A^5_\mu A^5_\nu + [A^5_\mu, A^5_\nu] \gamma_\rho \gamma_\sigma - [A^5_\mu, A^5_\nu] \gamma_\rho \gamma_\sigma - \gamma_\rho A^5_\gamma A^5_\nu A^5_\mu + \gamma_\rho A^5_\nu A^5_\gamma A^5_\mu.$$  

where

$$A^5_{\mu\nu} = \partial_\mu A^5_\nu + [A^5_\mu, A^5_\nu].$$

When considering compact Riemannian manifolds $M$ with no boundary, let $V$ be a vector bundle over $M$. Let $f$ denote a smooth function on $M$. The operator $e^{-tD}$, for $t > 0$, represents a trace class on $L^2(V)$, being the operator function $[37],$

$$K(t, f, D) = \text{Tr}_{L^2}(f e^{-tD}),$$

then, well defined, where the operator in equation (25) emulates the heat conduction equation solution,

$$\left(\frac{\partial}{\partial t} + D_x\right) K(t, x, x_1, D) = 0, \quad K(0, x, x_1, D) = \delta(x, x_1).$$

One can express

$$K(t, f, D) = \int_M d^n x \sqrt{g} \lim_{x \rightarrow x_1} K(t, x, x_1, D) f(x).$$

As $K(t, x, y, D)$ can be thought of as being a tensor with gauge group indexes, the operator Tr computes the trace over these indexes.

The asymptotic expansion, in the limit $t \rightarrow 0$ can be usually employed $[37],$

$$\text{Tr}_{L^2}(f e^{-tD}) \approx \sum_{j \in \mathbb{N} \cup \{0\}} \frac{t^{\frac{j-2}{2}}}{j!} a_j(f, D).$$
Heat kernel coefficients with odd index equal zero. The even index heat kernel coefficients, \( a_j(f, D) \), are locally computable with respect to the geometric invariants of \( M \), as [37]

\[
a_j(f, D) = \text{Tr}_V \int_M d^n x \sqrt{g} \left[ a_j(x, D) f(x) \right] = \sum_{\ell} \text{Tr}_V \int_M d^n x \sqrt{g} \left[ f(x) u^\ell(x) B_j^\ell(x, D) \right],
\]

(29)

where the \( B_j^\ell \) represent all the geometric invariants of the length \( j \), composed by the Riemann tensor, by \( E \) and \( \Omega \) (and their derivatives); the \( u^\ell \) are parameters determined by constraints between heat kernel coefficients. The case \( j = 2 \) yields \( E \) and \( R \) as the only independent invariants.

When \( M = M_1 \times M_2 \) is factorizable, the Laplace type operator \( D \) is, therefore, written as \( D = D_1 \otimes \text{id}_{M_2} + \text{id}_{M_1} \otimes D_2 \), where \( D_a : M_a \to M_a \), for \( a = 1, 2 \), is the Laplace type operator restricted to the manifolds \( M_1 \) and \( M_2 \). Therefore, one can write \( e^{-tD} = e^{-tD_1} \otimes e^{-tD_2} \), yielding [37],

\[
a_j(x, D) = \sum_{r + s = j} a_r(x_1, D_1) a_s(x_2, D_2).
\]

(30)

Equation (30) yields the constants \( u^\ell \) to be dependent on the spacetime dimension \( n = p + q \). As posed in reference [37], when one considers \( M_1 = S^l \), then one can choose \( D_1 = -\partial_{x_1}^2 \). Hence equation (29) yields

\[
a_j(f(x_2), D) = 2\pi \int_{M_2} d^{n-1} x \sqrt{g} \sum_{\ell} \text{Tr}_V \left\{ f(x_2) u_0^\ell B_j^\ell(D_2) \right\},
\]

(31)

where the \( f \) function was split as \( f(x_1, x_2) = f_1(x_1) f_2(x_2) \), for \( x_u \in M_u \). The constants \( u_0^\ell \) depend on the spacetime dimension \( n \) only by the factor \((4\pi)^{-n/2}\).

The heat kernel coefficients read [37]

\[
a_0(f, D) = \frac{1}{(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \text{Tr}_V(f),
\]

(32)

\[
a_2(f, D) = \frac{1}{6(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \text{Tr}_V [f(6E + R)],
\]

(33)

\[
a_4(f, D) = \frac{1}{360(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \text{Tr}_V \left\{ f \left[ (60E_{\alpha\beta\gamma\delta} + 12R_{\alpha\beta} + 5R^2 + 60RE + 180E^2
\right.ight.
\]

\[
- \left. 2R_{\alpha\beta} R_{\rho\sigma} + 2R_{\alpha\beta\rho\sigma} R_{\rho\sigma\alpha\beta} + 30 \Omega_{\rho\sigma} \Omega_{\rho\sigma} \right] \right\},
\]

(34)

\[
a_6(f, D) = \frac{1}{(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \text{Tr}_V \left\{ \frac{f}{7!} \left[ 18 R_{\sigma\beta\gamma\rho\alpha\delta} + 17 R_{\rho\sigma\alpha\beta\gamma\delta} - 2R_{\sigma\beta\rho\alpha\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} - 4R_{\sigma\beta\alpha\gamma\delta} R_{\rho\sigma\alpha\gamma\delta}
\right.ight.
\]

\[
+ 9R_{\sigma\beta\alpha\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} + 28RR_{\epsilon\delta\alpha\gamma} - 8R_{\epsilon\delta\alpha\gamma} R_{\rho\sigma\alpha\gamma\delta} + 24R_{\rho\sigma\alpha\beta\gamma\delta} + 12R_{\rho\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\gamma\delta}
\]

\[
+ \frac{35}{3} R^3 - \frac{14}{3} R R_{\rho\sigma\alpha\beta\gamma\delta} + \frac{14}{3} R R_{\rho\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} - \frac{208}{9} R_{\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} - \frac{64}{9} R_{\rho\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} - \frac{80}{9} R_{\rho\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} R_{\rho\sigma\alpha\beta\gamma\delta} \right\}
\]

9
In the next section we introduce the exotic deviation coefficients, measuring the exotic corrections to the heat kernel coefficients (32)–(35).

4. Exotic heat kernel coefficients

In section 2 the exotic additional terms in the spin connection, defining the Dirac operator, was shown to arise from the non-trivial topology of the manifold \( M \). These exotic terms also must be renormalizable. Any spinor field, on which the Dirac operator acts, realizes the exotic term not only as a shift of the gauge potential that eventually appears in the Dirac equation (or any first-order equation of motion that governs the spinor fields [23, 41]). Inequivalent spin structures, arising from manifolds having non-trivial topologies, can also engender deep modifications on the heat kernel coefficients associated with the manifold. Exotic spinor fields, in 4D spacetimes with non-trivial topology, were scrutinized in Refs. [25, 30].

Due to the form of the exotic Dirac operator in equations (12) and (13), exotic spinor fields, appearing in first order equations of motion, realize it as

\[
\begin{align*}
\hat{A}_\mu &\rightarrow A_\mu + iA_\mu \theta, \\
\hat{A}_5 &\rightarrow A_5^5 + iA_5 \gamma^5,
\end{align*}
\]

where \( \theta : U \subset M \rightarrow \Omega^0(M) \simeq \mathbb{R} \) is a scalar field defined on an open subset \( U \subset M \), whereas \( \theta^5 : U \subset M \rightarrow \Omega^0(M) \) is a pseudoscalar field. Alternatively, equation (37) can be introduced by considering \( \xi(x) \in U(1) \times U(1) = (\mathbb{R} \oplus \mathbb{R}) \times U(1) \) in equation (10), what is equivalent to replicate the formal aspects in section 2 using the field hyperbolic numbers \( \mathbb{R} \oplus \mathbb{R} \), instead of the real numbers. In fact, employing a second copy of \( U(1) \) regards pseudoscalar fields, as shown in references [2, 42]. Therefore, given the usual Dirac operator

\[
D = \gamma^\mu D_\mu = i\gamma^\mu \left( \partial_\mu + \frac{1}{8} [\gamma_\mu, \gamma_\nu] \sigma^{\nu\rho} + A_\mu + iA_5^5 \gamma^5 \right),
\]

we can write the exotic Dirac operator, using equations (36) and (37) as

\[
\gamma^\mu D_\mu = i\gamma^\mu \left( \partial_\mu + \frac{1}{8} [\gamma_\mu, \gamma_\nu] \sigma^{\nu\rho} + A_\mu + iA_5^5 \gamma^5 \right) + i\gamma^\mu \left( i\partial_\mu \theta + \tilde{\partial}_\mu \theta^5 \gamma^5 \right) = \gamma^\mu \left[ D_\mu - (\partial_\mu \theta + i\partial_\mu \theta^5 \gamma^5) \right],
\]

that is, \( \hat{D}_\mu \rightarrow D_\mu - \partial_\mu \tilde{\theta} \), where

\[
\tilde{\theta} \equiv \theta + i\theta^5 \gamma^5.
\]

Thereupon, looking to the exotic field strength, yields

\[
F_{\mu\nu} = F_{\mu\nu} + [A_{[\mu}, i\partial_{\nu]}/\theta].
\]
where \( [A_{i\mu},\partial_\nu \theta] \equiv [A_{i\mu}, i\partial_\nu \theta] - [A_{i\mu}, i\partial_\nu \theta] \).

On the other hand, the field strength (15) must be lifted to the inequivalent spin structure. Thus, the exotic connection reads

\[
\tilde{\omega}_\mu = \omega_\mu + \partial_\mu \theta - \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right] \partial_\nu \theta \gamma^5 \gamma^5 .
\] (42)

The complete proof is accomplished in equation (A2) in appendix A. Collecting the computations in appendix A, implemented by equation (41) and (A4)–(A9), yields the complete form of the exotic full field strength displayed initially in equation (A3),

\[
\Omega_{\mu\nu} = \Omega_{\mu\nu} + i \left[ A_{\mu\nu}, \partial_\sigma \theta \right] + \gamma^5 \left[ A_{\mu\nu}, \partial_\sigma \theta \gamma_5 - \frac{n-3}{4} \left[ \gamma^\mu, \gamma^\nu \right] \left[ A^\sigma_\mu, \tilde{A}^\sigma_\nu \right] \right] + \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \right] - \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \gamma_5 \right] - \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \gamma_5 \right] .
\] (43)

To recover the usual field strength (15), one just sets \( \theta = 0 \) and \( \theta^5 \) to 0.

Next, let us figure out one of the invariants in the heat kernel coefficients. At first, the exotic \( E \) invariant can be written as,

\[
\hat{E} = \frac{1}{4} R + \frac{1}{4} \left[ \gamma^\mu, \gamma^\nu \right] \tilde{F}_{\mu\nu} + \gamma^5 \left[ A^\sigma_\mu, \hat{A}^\sigma_\nu \right] - \frac{n-3}{4} \left[ \gamma^\mu, \gamma^\nu \right] \left[ \hat{A}^\sigma_\mu, \hat{A}^\sigma_\nu \right] = E + \frac{i}{4} \left[ A_{\mu\nu}, \partial_\sigma \theta \right] - \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \right] - \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \gamma_5 \right] - \frac{i}{2} \left[ A_{\mu\nu}, \partial_\sigma \theta \gamma_5 \right] .
\] (44)

Endowed with the expressions (41), (A4) and (44), respectively for the exotic field strengths and the exotic \( E \) invariant, one can now compute the exotic heat kernel expansion.

On the other hand, the exotic counterparts of the heat kernel coefficients take into account all the exotic contributions, that measure the topological non-triviality of \( M \). Clearly, exclusively geometric terms, involving the Riemann tensor and its contractions, remain invariant, as they are not topological terms. The exotic heat kernel coefficients are then given by

\[
\hat{a}_0(f, D) = \frac{1}{(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \Tr \left( f \right).
\] (45)

\[
\hat{a}_2(f, D) = \frac{1}{6(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \Tr \left( f \left( 6\hat{E} + R \right) \right).
\] (46)

\[
\hat{a}_4(f, D) = \frac{1}{360(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \Tr \left( f \left( 60\hat{E}_{\alpha\beta\gamma} + 12R_{\alpha\beta} + 5R^2 + 60\hat{E} + 180\hat{E}^2 + 12R_{\alpha\beta} + 5R^2 - 2R_{\rho\sigma} R_{\rho\sigma} + 2R_{\rho\sigma\beta\gamma} R_{\rho\sigma\beta\gamma} + 30\hat{A}_{\rho\sigma} \hat{A}_{\rho\sigma} \right) \right).
\] (47)

\[
\hat{a}_6(f, D) = \frac{1}{(4\pi)^{n/2}} \int_M d^n x \sqrt{g} \Tr \left( f \left( 18R_{\rho\sigma\tau\sigma} + 17R_{\rho\tau} R_{\rho\tau} - 2R_{\rho\sigma\tau} R_{\rho\sigma\tau} - 4R_{\rho\sigma\tau} R_{\sigma\tau\zeta} + 9R_{\rho\sigma\tau\beta\gamma} R_{\rho\sigma\tau\beta\gamma} + 28RR_{\zeta\zeta} - 8R_{\rho\tau} R_{\rho\tau\zeta} \right) \right).
\]
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\[ + 24R_{\alpha\sigma}R_{\gamma\zeta\alpha} + 12R_{\rho\sigma\alpha\beta}R_{\rho\sigma\alpha\beta} + \frac{35}{9} R^3 - \frac{14}{3} RR_{\rho\sigma}R_{\rho\sigma} \\
+ \frac{14}{3} RR_{\rho\sigma\alpha\beta}R_{\rho\sigma\alpha\beta} - \frac{208}{9} R_{\sigma\alpha}R_{\zeta\alpha} - \frac{64}{3} R_{\rho\sigma}R_{\alpha\beta}R_{\rho\sigma\alpha\beta} \\
- \frac{16}{3} R_{\sigma\alpha}R_{\tau\gamma\beta\rho}R_{\alpha\gamma\beta\rho} - \frac{44}{9} R_{\rho\sigma\alpha\beta}R_{\rho\sigma\beta\tau}R_{\alpha\gamma\beta\tau} - \frac{80}{9} R_{\rho\sigma\alpha\beta}R_{\rho\sigma\beta\tau}R_{\alpha\gamma\beta\tau} \]

+ \frac{f}{360} \left( 8\Omega_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} + 2\Omega_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} + 12\Omega_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} - 12\Omega_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} \right)

- 6R_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} - 4R_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} + 5R_{\rho\sigma\alpha\beta}\Omega_{\rho\sigma\alpha\beta} + 6\tilde{E}_{\rho\sigma\alpha\beta} + 60\tilde{E}_{\rho\sigma\alpha\beta} \\
+ 30\tilde{E}_{\rho\sigma\alpha\beta} + 60\tilde{E}_{\rho\sigma\alpha\beta} + 30\tilde{E}_{\rho\sigma\alpha\beta} + 10R_{\rho\sigma\alpha\beta} + 4R_{\rho\sigma\alpha\beta}E_{\rho\sigma\alpha\beta} + 12R_{\rho\sigma\alpha\beta}E_{\rho\sigma\alpha\beta} \\
+ (30E^2 + 12\tilde{E}_{\rho\sigma\alpha\beta} + 5E^2 - 2E_{\rho\sigma\alpha\beta} + 2\tilde{E}_{\rho\sigma\alpha\beta}R_{\rho\sigma\alpha\beta}) \right). \tag{48} \]

To measure the deviations of the exotic heat kernels coefficients (45)–(48), with respect to the standard ones (32)–(35) associated with topologically trivial manifolds, we define the exotic deviation coefficient, \( \delta a_j(f, D) \), by

\[ \delta a_j(f, D) = \hat{a}_j(f, D) - a_j(f, D). \tag{49} \]

The results for the leading heat kernel coefficients are explicit in what follows. First, the exotic deviation to the zeroth heat kernel coefficient reads

\[ \hat{a}_0(f, D) = \frac{1}{(4\pi)^{n/2}} \int d^nx \sqrt{g} \text{Tr}_V \{ f \} = a_0(f, D). \tag{50} \]

Hence \( \delta a_0(f, D) = 0 \) and there is no effects due to the exotic topology in the heat kernel coefficient (50).

Using equations (33) and (44), the second exotic deviation coefficient reads

\[ \hat{a}_2(f, D) = a_2(f, D) + \delta a_2(f, D), \tag{51} \]

so that

\[ \delta a_2(f, D) = \frac{1}{(4\pi)^{n/2}} \int d^nx \sqrt{g} \text{Tr}_V [ f (\delta E) ], \tag{52} \]

where

\[ \delta E = E - E \]

\[ = \frac{i}{4} \{ A_\mu, \partial_j \theta \} [ \gamma^\mu, \gamma^\nu ] + \gamma^5 \left( - D_\mu \partial^\nu \theta^5 - i \partial^\mu \partial^5 \partial_\mu \theta^5 \right) - i(n - 2) \left( 2A_\mu^5 \partial^\nu \theta^5 - \partial_\nu \theta^5 \partial^\mu \theta^5 \right) - \frac{i(n - 3)}{4} [ \gamma^\mu, \gamma^\nu ] [ A_\mu^5, \partial_\nu \theta^5 ]. \tag{53} \]

Besides, the exotic correction to the fourth heat kernel coefficient reads

\[ \hat{a}_4(f, D) = a_4(f, D) + \delta a_4(f, D), \tag{54} \]
where the fourth exotic deviation coefficient is given by

\[
\delta a_4 (f, D) = \frac{1}{6(4\pi)^{n/2}} \int_{M} d^{n}x \sqrt{g} \text{Tr}_V \left\{ f \left[ (\delta E)_{\alpha\alpha} + R (\delta E) + 3(\delta E)^2 \right] \right. \\
\left. + 6E (\delta E) + \frac{1}{2} \left( \delta \Omega_{\mu\nu} \right) \left( \delta \Omega_{\mu\nu} \right) + \Omega_{\mu\nu} \left( \delta \Omega_{\mu\nu} \right) \right\},
\]

(55)

The terms in (55) are expressed as

\[
\delta \Omega_{\mu\nu} = \tilde{\Omega}_{\mu\nu} - \Omega_{\mu\nu}
\]

\[
= i \left[ A_{[\mu}, \partial_\nu \theta \right] + i \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \gamma^5 \gamma^\mu \left[ \gamma_\nu D_\rho + \partial_\rho \theta \right] \theta^5 - i \gamma_\nu \partial_\rho \tilde{\theta} A^5_{\rho} \right)
\]

\[
+ i \gamma^5 \left[ \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \left[ \partial_\nu \theta, A^5_{\rho} \right] \right] + i \left( \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] - \left[ A^5_{\rho}, \partial_\nu \theta \right] \right) \right) \gamma^\mu \gamma_\nu \]

\[- i \gamma^5 \left( A^5_{[\mu} \gamma^\nu \partial_\rho \theta \left( \partial_\nu \gamma_\rho \right) + \partial_\rho \theta \left( \partial_\nu \gamma_\rho \right) A^5_{[\rho} \gamma_\nu \right) \right),
\]

(56)

\[
(\delta E)_{\alpha\alpha} = \frac{i}{4} \left[ \left[ A_{[\mu}, \partial_\nu \theta \right] \right]_{\alpha\alpha} \left[ \gamma^\mu, \gamma^\nu \right] + \gamma^5 \left( -D_\rho \theta \gamma_\rho \theta^5 - i \gamma^\nu \tilde{\theta} A^5_{\nu} + \theta \gamma^\nu \partial_\nu \theta \right) \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right]_{\alpha\alpha}
\]

\[- i(n-2) \left( 2A^5_{\rho} \partial_\nu \theta \theta^5 - \partial_\nu \theta \theta^5 \partial_\nu \theta \right) \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right];
\]

(57)

\[
(\delta E)^2 = \left\{ \frac{i}{4} \left[ \left[ A_{[\mu}, \partial_\nu \theta \right] \right] \left[ \gamma^\mu, \gamma^\nu \right] + \gamma^5 \left( -D_\rho \theta \gamma_\rho \theta^5 - i \gamma^\nu \tilde{\theta} A^5_{\nu} + \theta \gamma^\nu \partial_\nu \theta \right) \right] \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right]
\]

\[- i(n-2) \left( 2A^5_{\rho} \partial_\nu \theta \theta^5 - \partial_\nu \theta \theta^5 \partial_\nu \theta \right) \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right] \right\},
\]

(58)

\[
E(\delta E) = \left\{ \frac{1}{4} R + \frac{1}{4} \left[ \gamma^\mu, \gamma^\nu \right] F_{\mu\nu} + i \gamma^5 D^\rho A^5_{\rho} - (n-2) A^5_{\rho} A^5_{\rho} - \frac{(n-3)}{4} \left[ \gamma^\mu, \gamma^\nu \right] \left[ A^5_{\rho}, A^5_{\rho} \right] \right\}
\]

\times \left\{ \frac{i}{4} \left[ \left[ A_{[\mu}, \partial_\nu \theta \right] \right] \left[ \gamma^\mu, \gamma^\nu \right] + \gamma^5 \left( -D_\rho \theta \gamma_\rho \theta^5 - i \gamma^\nu \tilde{\theta} A^5_{\nu} + \theta \gamma^\nu \partial_\nu \theta \right) \right] \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right]
\]

\[- i(n-2) \left( 2A^5_{\rho} \partial_\nu \theta \theta^5 - \partial_\nu \theta \theta^5 \partial_\nu \theta \right) \left[ \left[ A^5_{\rho}, \partial_\nu \theta \theta^5 \right] \right] \right\},
\]

(59)

\[
(\delta \Omega_{\mu\nu}) (\delta \Omega_{\mu\nu}) = \left\{ i \left[ A_{[\mu}, \partial_\nu \theta \right] + i \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \gamma^5 \gamma^\mu \left( \gamma_\nu D_\rho + \partial_\rho \theta \right) \theta^5 - i \gamma_\nu \partial_\rho \tilde{\theta} A^5_{\rho} \right)
\]

\[+ i \gamma^5 \left[ \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \left[ \partial_\nu \theta, A^5_{\rho} \right] \right] + i \left( \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] - \left[ A^5_{\rho}, \partial_\nu \theta \right] \right) \right) \gamma^\mu \gamma_\nu \]

\[+ i \gamma^5 \left[ \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \left[ \partial_\nu \theta, A^5_{\rho} \right] \right] + i \left( \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] - \left[ A^5_{\rho}, \partial_\nu \theta \right] \right) \right) \gamma^\mu \gamma_\nu
\]

\[+ i \gamma^5 \left[ \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] + \left[ \partial_\nu \theta, A^5_{\rho} \right] \right] + i \left( \left[ A_{[\mu}, \partial_\nu \theta \theta^5 \right] - \left[ A^5_{\rho}, \partial_\nu \theta \right] \right) \right) \gamma^\mu \gamma_\nu
\]
\[ -i\gamma^\alpha \left( A^5_{\mu\nu} \gamma^\rho \partial_{\mu\nu} \Theta^5_{\gamma\rho} + \partial_{\mu\nu} \Theta^5_{\gamma\rho} A^5_{\mu\nu} \right) \]  

(60)

and

\[
\Omega_{\rho\sigma} (\delta\Omega_{\rho\sigma}) = \left\{ F_{\rho\sigma} - [A_{\rho}, A_{\sigma}] - \frac{1}{4} \gamma^\mu \gamma^\nu R_{\mu\nu\rho\sigma} - i\gamma^\mu \gamma^\nu (\gamma_{\mu\sigma} D_{\nu\rho} A^5_{\mu\rho}) \\
+ i\gamma^5 A^5_{\rho\sigma} + [A^5_{\rho\sigma}, A^5_{\mu\nu}] \gamma_{\mu\nu} - \gamma^\mu A_5^5 \gamma_{\mu\nu} A^5_{\nu\rho} \gamma_{\mu\sigma} \right\} \times \left\{ i \left[ A_{\mu\rho}, \partial_\mu \Theta^5 \right] + i \left[ A^5_{\mu\rho}, \partial_\mu \Theta^5 \right] + \gamma^5 \gamma^\alpha \left( \gamma_\mu D_\rho \partial_\sigma \Theta^5 - i\gamma_\mu \partial_\rho \partial_\alpha \right) \right\} \times \left\{ \right. \\
+ i\gamma^5 \left( [A_{\mu\rho}, \partial_\mu \Theta^5] + [\partial_\mu \Theta, A^5_{\mu\rho}] \right) + i \left( [A^5_{\mu\rho}, \partial_\mu \Theta^5] - [A^5_{\mu\rho}, \partial_\mu \Theta^5] \right) \gamma^\alpha \gamma_\alpha \right\} \\
- i\gamma^\alpha \left( A^5_{\mu\nu} \gamma^\rho \partial_{\mu\nu} \Theta^5_{\gamma\rho} + \partial_{\mu\nu} \Theta^5_{\gamma\rho} A^5_{\mu\nu} \right) \right\} 
\]

(61)

Finally, the sixth exotic deviation coefficient can be derived from the analysis of the sixth exotic heat kernel coefficient, (48), as

\[ \hat{a}_6(f, D) = a_6(f, D) + \delta a_6(f, D), \]

(62)

where

\[
\delta a_6(f, D) = \frac{1}{360(4\pi)^{1/2}} \int_M \mathrm{d}^n x \sqrt{g} \operatorname{Tr}_V \left\{ f \left\{ 8 \left[ (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu} (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu} + 2(\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu} \Omega_{\rho\sigma,\mu\nu} \right] + 2 \left[ (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu,\gamma\rho\mu\nu} + 2(\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu,\gamma\rho\mu\nu} + 12 \left[ (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu,\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu,\gamma\rho\mu\nu} \right. \right. \right. \right. \\
+ (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu,\gamma\rho\mu\nu} - 12 \left[ 3\Omega_{\rho\sigma,\gamma\rho\mu\nu}(\delta\Omega_{\rho\sigma}) + 3\Omega_{\rho\sigma,\gamma\rho\mu\nu}(\delta\Omega_{\rho\sigma}) + (\delta\Omega_{\rho\sigma})_5 \right] + 4\Omega_{\rho\sigma,\gamma\rho\mu\nu} \right] + 4R_{\rho\sigma,\gamma\rho\mu\nu} \right] + 6 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + 6 \left[ (\delta\Omega_{\rho\sigma})_{\gamma\rho\mu\nu} + 2\Omega_{\rho\sigma,\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 30 \left[ 2\Omega_{\rho\sigma,\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 6 \left[ (\delta\Omega_{\rho\sigma})_5 + (\delta\Omega_{\rho\sigma})_5 \right] \\
+ 30 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + 2\Omega_{\rho\sigma,\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 30 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 30 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 30 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] + 30 \left( \delta\Omega_{\rho\sigma} \right)_{\gamma\rho\mu\nu} + (\delta\Omega_{\rho\sigma})_5 \right] \right\} , 
\]

(63)

where the exotic terms in equation (63) are displayed in appendix A, throughout equations (A10)–(A43).

All the even exotic \( \hat{a}_2(f, D) \) heat kernel coefficients can be similarly computed for any manifold with non-trivial topology. The further steps consist of computing the coefficient \( \hat{a}_6(f, D) \), based on reference [45], whereas \( \hat{a}_{10}(f, D) \) can be analogously derived, using the general formula of the coefficient \( a_{10}(f, D) \) in reference [44]. However, it is not our aim here, since the computations of the \( a_{10}(f, D) \) coefficient involved already ten pages of appendix A.

5. Conclusions

This work was motivated by the heat kernel spectral approach to understanding specific geometrical and topological properties of non-trivial manifolds without boundaries, employing
exotic spin structures. It was implemented through the exotic additional term described by an integer in a Čech cohomology. The form of the exotic Dirac operator is made explicit in equation (39). Therefore one can use this form to describe the equations of motion, whatever the spinor field is [25,30]. The formalism regarding exotic spinor fields that arise from inequivalent spin structures, on non-trivial topological manifolds endowed with a metric of arbitrary signature, was extended for any finite dimension. The exotic corrections to the heat kernel coefficients, that relate spectral properties of exotic Dirac operators to the geometric invariants of $M$, were implemented by the exotic heat kernel deviation coefficients. The four first exotic heat kernel even coefficients were derived and their respective deviations from the standard ones were calculated, involving equations (52), (54) and (62).

The terms carrying the exotic fields are signatures of the influence of the exotic spin structures, which is meticulously demonstrated in equations (53), (57)–(61), (A10)–(A43). These derivations permit, as expected, to recover the standard heat kernel coefficients when the exotic corrections vanish when the manifolds are simply-connected. One of the most important perspectives of this work, consequently, is the future possibility to investigate the respective dynamics of such exotic contributions in the equations of motion governing fermionic fields, which were explicitly computed.

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Appendix A. Auxiliary computations to the exotic heat kernel coefficients

The exotic field strength can be obtained as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [\bar{A}_\mu, A_\nu]$$

$$= \partial_\mu (A_\nu + i\partial_\nu \theta) - \partial_\nu (A_\mu + i\partial_\mu \theta) + [A_\mu + i\partial_\mu \theta, A_\nu + i\partial_\nu \theta]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] + [\bar{A}_\mu, i\partial_\nu \theta] + [i\partial_\mu \theta, A_\nu] - [\bar{A}_\mu, A_\nu]$$

$$= F_{\mu\nu} + [A_\mu, i\partial_\nu \theta]$$  \hspace{1cm} (A1)

Thus, the exotic connection must be defined as

$$\bar{\omega}_\mu = \frac{1}{8} \left[ \gamma_\nu, \gamma_\rho \right] \sigma_{\mu}^{\nu \rho} + \bar{A}_\mu + i \left[ \gamma_\mu, \gamma_\nu \right] \bar{A}^{\nu} \gamma^5$$

$$= \frac{1}{8} \left[ \gamma_\nu, \gamma_\rho \right] \sigma_{\mu}^{\nu \rho} + \bar{A}_\mu + i\partial_\mu \theta + \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right] \left( \bar{A}^{\nu} + i\partial_\nu \theta^5 \right) \gamma^5$$

$$= \omega_\mu + \partial_\mu \theta - \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right] \partial_\mu \theta^5 \gamma^5.$$  \hspace{1cm} (A2)
Besides, the exotic version of the field strength (15) reads
\[
\hat{\Omega}_{\mu \nu} = \hat{\tilde{F}}_{\mu \nu} - \left[ \hat{A}_\mu^5, \hat{A}_\nu^5 \right] - \frac{1}{4} \gamma^\rho \gamma^\sigma \hat{R}_{\rho \sigma \mu \nu} - i \gamma^5 \gamma^\rho \gamma_{(\nu} \hat{D}_{\mu)} \hat{A}_5^\rho + i \gamma^5 \hat{A}_{\mu \nu}^5 \\
+ \left[ \hat{A}_\mu^5, \hat{A}_\nu^5 \right] \gamma^\rho \gamma_{\nu|} - \gamma^\rho \hat{A}_5^\rho \gamma_{(\mu} \hat{A}_5^5 \gamma_{\nu|} .
\]
(A3)

As well as the field strength calculated previously, we go now to determine each term of the full field strength separately, as follows:

\[
\left[ \hat{A}_\mu^5, \hat{A}_\nu^5 \right] = \left[ A_\mu^5 + i \partial_\mu \theta^5, A_\nu^5 + i \partial_\nu \theta^5 \right] \\
= \left[ A_\mu^5, A_\nu^5 \right] + i \left( \left[ A_\mu^5, \partial_\nu \theta^5 \right] + \left[ \partial_\mu \theta^5, A_\nu^5 \right] \right) = \left[ A_\mu^5, A_\nu^5 \right] + i \left[ A_\mu^5, \partial_\nu \theta^5 \right] .
\]
(A4)

First, the commutator,

\[
\hat{A}_{\mu \nu} = \partial_{[\mu} \hat{A}_{\nu]}^5 \\
= \partial_{[\mu} \hat{A}_5^\rho + i \partial_{[\nu} \theta^5 \right] + \left[ A_{[\mu}, \theta^5 \right] \\
= \partial_{[\mu} \hat{A}_5^\rho + i \partial_{[\nu} \theta^5 \right] + \left[ A_{[\mu}, \theta^5 \right] \\
+ i \partial_{[\mu} \theta^5 \right] + \left[ \partial_{[\nu} \theta^5 \right] = A_{\mu \nu}^5 + i \left( \left[ A_{\mu}, \theta^5 \right] + \left[ \partial_{[\nu} \theta^5 \right] \right) \right) .
\]
(A5)

Second, considering

\[
\hat{D}_\mu \hat{A}_5^\rho = \left( D_\mu - i \partial_\mu \theta \right) \left( A_5^\rho + i \partial_\mu \theta^5 \right) \\
= D_\mu A_5^\rho + i D_\mu \partial_\mu \theta^5 - \partial_\mu \hat{\theta} A_5^\rho - i \partial_\mu \hat{\theta} \partial_\mu \theta^5 ,
\]
(A6)

yields

\[
-i \gamma^5 \gamma^\rho \gamma_{(\nu} \hat{D}_{\mu)} \hat{A}_5^\rho = - i \gamma^5 \gamma^\rho \gamma_{(\nu} \left( D_\mu A_5^\rho + i D_\mu \partial_\mu \theta^5 - \partial_\mu \hat{\theta} A_5^\rho \right) \\
= - i \gamma^5 \gamma^\rho \gamma_{(\nu} \left( D_\mu A_5^\rho + \gamma^\rho \gamma_{(\nu} \partial_\mu \theta^5 - i \gamma^5 \gamma^\rho \gamma_{(\nu} \partial_\mu \hat{\theta} A_5^\rho \right) .
\]
(A7)

Using the commutator obtained in (A4), we get

\[
\left[ \hat{A}_\mu^5, \hat{A}_\nu^5 \right] \gamma^\rho \gamma_{\nu|} = \left[ A_\mu^5, A_\nu^5 \right] \gamma^\rho \gamma_{\nu|} + i \left( \left[ A_\mu^5, \theta^5 \right] + \left[ A_\nu^5, \theta^5 \right] \right) \gamma^\rho \gamma_{\nu|} .
\]
(A8)

The last computation becomes

\[
- \gamma^\rho \hat{A}_\mu^5 \gamma^\rho \gamma_{\nu|} \hat{A}_5^\rho \gamma_{\nu|} = - \gamma^\rho \left( A_\mu^5 + i \partial_\mu \theta^5 \right) \gamma_{\nu|} \hat{A}_5^\rho \gamma_{\nu|} \\
= - \gamma^\rho A_\mu^5 \gamma^\rho \gamma_{\nu|} \gamma_{\nu|} - i \gamma^\rho A_\mu^5 \gamma^\rho \gamma^\rho \gamma_{\nu|} \\
- i \gamma^\rho \partial_\mu \theta^5 \gamma_{\nu|} \gamma^\rho \gamma_{\nu|} .
\]
(A9)
Now, the terms involving sixth exotic deviation coefficient \((48)\), read

\[
(\delta\Omega_{\rho\sigma})_{\alpha\beta} = \left\{ i \left[ \left[ A_{\mu\nu}, \partial_\alpha \theta \right] \right]_{\alpha\beta} + i \left[ A_{\mu\nu}^5, \partial_\alpha \theta^5 \right]_{\alpha\beta} + \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta^5 \right)_{\alpha\beta} + i \left[ A_{\mu\nu}, \partial_\beta \theta \right]_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} \right\} 
\]

\[
(\delta\Omega_{\rho\sigma})_{\alpha\beta} = \left\{ i \left[ \left[ A_{\mu\nu}, \partial_\alpha \theta \right] \right]_{\alpha\beta} + i \left[ A_{\mu\nu}^5, \partial_\alpha \theta^5 \right]_{\alpha\beta} + \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta^5 \right)_{\alpha\beta} + i \left[ A_{\mu\nu}, \partial_\beta \theta \right]_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} \right\} 
\]

\[
(\delta\Omega_{\rho\sigma})_{\alpha\beta} = \left\{ i \left[ \left[ A_{\mu\nu}, \partial_\alpha \theta \right] \right]_{\alpha\beta} + i \left[ A_{\mu\nu}^5, \partial_\alpha \theta^5 \right]_{\alpha\beta} + \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta^5 \right)_{\alpha\beta} + i \left[ A_{\mu\nu}, \partial_\beta \theta \right]_{\alpha\beta} - i \gamma^5 \gamma^5 \left( \gamma_{\alpha} \partial_\beta \partial_\mu \partial_\rho \theta \right)_{\alpha\beta} \right\} 
\]
\[ \begin{aligned}
(\delta \Omega_{\mu \nu})_{\alpha \beta \gamma \delta} &= \left\{ i \left[ A_{\mu\nu}, \partial_{\alpha} \theta \right]_{\gamma \delta} + i \left[ A_{\mu\nu}, \theta, \partial_{\alpha} \theta \right]_{\gamma \delta} + \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \\
&+ i \gamma^2 \left( \left[ A_{\mu\nu}, \theta, \partial_{\alpha} \theta \right]_{\gamma \delta} + \left[ \partial_{\alpha} \theta, A_{\mu\nu} \right]_{\gamma \delta} \right) + i \left( \left[ A_{\mu\nu}, \partial_{\alpha} \theta \right]_{\gamma \delta} - \left[ A_{\mu\nu}, \partial_{\alpha} \theta \right]_{\gamma \delta} \right) \right\} \\
&+ \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \right\} \\
&\times \left\{ F_{\rho\sigma\alpha\beta} - [A_{\mu\nu}, A_{\mu\nu}]_{\gamma \delta} - \frac{1}{4} \gamma^2 \gamma^2 \gamma^2 R_{\rho\sigma\alpha\beta\gamma \delta} - i \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \right\} \\
&+ \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \right\} \\
&\times \left\{ F_{\rho\sigma\alpha\beta} - [A_{\mu\nu}, A_{\mu\nu}]_{\gamma \delta} - \frac{1}{4} \gamma^2 \gamma^2 \gamma^2 R_{\rho\sigma\alpha\beta\gamma \delta} - i \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \right\} \\
&+ \gamma^2 \gamma^2 \left( \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} - i \gamma_{\nu\sigma D\rho \beta \gamma \delta \gamma \alpha} \right) \right\} .
\end{aligned} \]
\[ \Omega_{\rho\sigma}(\Omega_{\sigma\alpha}) = \left\{ F_{\rho\sigma} - \left[ A^5_{\mu}, A^5_{\nu} \right] - \frac{1}{4} \epsilon^{\mu\nu\kappa\lambda} R_{\rho\mu\nu\sigma} - i\gamma^5 \gamma^\mu \left( \gamma_{[\mu} D_{\nu]} A^5_{\sigma]} \right) \right\} \\
+ i\gamma^5 A^5_{\rho\sigma} + \left[ A^5_{\nu}, A^5_{\mu} \right] \gamma^\nu \gamma_{\sigma} - \gamma^\mu A^5_{\rho\sigma} \gamma_{[\mu} \gamma^\nu A^5_{\nu]} [\gamma_{\sigma}] \right\} \\
\times \left\{ F_{\sigma\alpha} = \left[ A^5_{\alpha}, A^5_{\beta} \right] - \frac{1}{4} \epsilon^{\alpha\beta\kappa\lambda} R_{\sigma\alpha\beta\gamma} - i\gamma^5 \gamma^\beta \left( \gamma_{[\alpha} D_{\beta]} A^5_{\beta]} \right) \right\} \\
+ i\gamma^5 A^5_{\sigma\alpha} + \left[ A^5_{\beta}, A^5_{\gamma} \right] \gamma^\beta \gamma_{\alpha} - \gamma^\beta A^5_{\sigma\alpha} \gamma_{[\beta} \gamma^\gamma A^5_{\gamma]} [\gamma_{\alpha}] \right\} \\
\times \left\{ i \left[ A_{\alpha\gamma}, \partial_{\alpha} \gamma \right] + i \left[ A_{\alpha\gamma}, \partial_{\alpha} \gamma \right] + \gamma^5 \gamma^\beta \left( \gamma_{[\alpha} D_{\beta]} \partial_{\gamma} \gamma^\beta - i\partial_{[\alpha} \gamma \partial_{\beta]} A^5_{\gamma} \right) \right\} \\
+ i\gamma^5 \left( \left[ A_{\alpha\gamma}, \partial_{\alpha} \gamma \right] + \left[ \partial_{\alpha} \gamma A^5_{\gamma} \right] \right) + i \left( \left[ A_{\alpha\gamma}, \partial_{\alpha} \gamma \right] - \left[ A_{\gamma \alpha}, \partial_{\alpha} \gamma \right] \right) \gamma^\beta \gamma_{\beta} \\
- i\gamma^5 \left( A^5_{\gamma \alpha} \gamma_{[\gamma} \gamma^\beta \partial_{[\alpha} \gamma^\beta + \partial_{[\alpha} \gamma^\beta \gamma_{\alpha]} \right) \right\} . \tag{A17} \]

\[ \Omega_{\rho\sigma}(\partial\Omega_{\sigma\alpha})(\partial\Omega_{\tau\nu}) = \left\{ F_{\rho\sigma} - \left[ A^5_{\mu}, A^5_{\nu} \right] - \frac{1}{4} \epsilon^{\mu\nu\kappa\lambda} R_{\rho\mu\nu\sigma} - i\gamma^5 \gamma^\mu \left( \gamma_{[\mu} D_{\nu]} A^5_{\sigma]} \right) \right\} \\
+ i\gamma^5 A^5_{\rho\sigma} + \left[ A^5_{\nu}, A^5_{\mu} \right] \gamma^\nu \gamma_{\sigma} - \gamma^\mu A^5_{\rho\sigma} \gamma_{[\mu} \gamma^\nu A^5_{\nu]} [\gamma_{\sigma}] \right\} \\
\times \left\{ i \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + i \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + \gamma^5 \gamma^\beta \left( \gamma_{[\alpha} D_{\beta]} \partial_{\beta} \beta - i\partial_{[\alpha} \beta \partial_{\beta]} A^5_{\beta} \right) \right\} \\
+ i\gamma^5 \left( \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + \left[ \partial_{\alpha} \beta A^5_{\beta} \right] \right) + i \left( \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] - \left[ A_{\beta \alpha}, \partial_{\alpha} \beta \right] \right) \gamma^\beta \gamma_{\beta} \\
- i\gamma^5 \left( A^5_{\beta \alpha} \gamma_{[\beta} \gamma^\beta \partial_{[\alpha} \beta^\beta + \partial_{[\alpha} \beta^\beta \gamma_{\beta]} \right) \right\} . \tag{A18} \]

\[ (\partial\Omega_{\rho\sigma})(\partial\Omega_{\sigma\alpha})(\partial\Omega_{\tau\nu}) = \left\{ i \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + i \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + \gamma^5 \gamma^\beta \left( \gamma_{[\alpha} D_{\beta]} \partial_{\beta} \beta - i\partial_{[\alpha} \beta \partial_{\beta]} A^5_{\beta} \right) \right\} \\
+ i\gamma^5 \left( \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] + \left[ \partial_{\alpha} \beta A^5_{\beta} \right] \right) + i \left( \left[ A_{\alpha\beta}, \partial_{\alpha} \beta \right] - \left[ A_{\beta \alpha}, \partial_{\alpha} \beta \right] \right) \gamma^\beta \gamma_{\beta} \\
- i\gamma^5 \left( A^5_{\beta \alpha} \gamma_{[\beta} \gamma^\beta \partial_{[\alpha} \beta^\beta + \partial_{[\alpha} \beta^\beta \gamma_{\beta]} \right) \right\} . \tag{A19} \]
\[ + i \gamma^5 \left( \left[ A_{\mu\rho}, \partial_\rho \theta^5 \right] + \left[ \partial_\mu \theta, A^5_{\rho} \right] \right) + i \left( A^5_{\mu}, \partial_\mu \theta^5 \right) - \left[ A^5_{\mu}, \partial_\mu \theta^5 \right] \right) \gamma^5 \gamma_\rho \]) \\
+i \gamma^5 (A^5_{\mu}, \partial_\mu \theta^5) - i \gamma^5 \gamma_\rho \sigma (\gamma_\rho \sigma D_\rho A^5_{\rho}) \right), \quad (A19) \]

\[ R_{\rho\mu\nu\sigma} \left[ 2 \Omega_{\rho\mu} (\partial \Omega_{\nu\sigma}) \right] = 2 R_{\rho\mu\nu\sigma} \left( F_{\rho\mu} - [A^5_{\rho}, A^5_{\mu}] - \frac{1}{4} \gamma^\rho \gamma^\sigma R_{\rho\mu\nu\sigma} + i \gamma^5 \gamma^\rho (\gamma_\rho \sigma D_\rho A^5_{\rho}) \right) \\
+ i \gamma^5 A^5_{\rho\mu} + \left[ A^5_{\rho\mu}, A^5_{\rho\mu} \right] \gamma^\rho \gamma_\rho \sigma A^5_{\rho\mu} \right) \gamma_\rho \sigma \right) \\
\times \left\{ i \left[ A_{\mu\rho}, \partial_\rho \theta^5 \right] + i \left[ A^5_{\mu\rho}, \partial_\rho \theta^5 \right] + i \gamma^5 \gamma^\rho \sigma \left( \gamma_\rho \sigma D_\rho \partial_\rho \theta^5 - i \gamma_\rho \sigma \partial_\rho \theta^5 \right) \right\} \\
+ i \gamma^5 \left( \left[ A_{\mu\rho}, \partial_\rho \theta^5 \right] + \left[ \partial_\mu \theta, A^5_{\rho} \right] \right) + i \left( \left[ A^5_{\mu\rho}, \partial_\rho \theta^5 \right] - \left[ A^5_{\mu\rho}, \partial_\rho \theta^5 \right] \right) \gamma^5 \gamma_\rho \sigma \right) \\
- i \gamma^5 \left( A^5_{\rho\mu} \gamma_\rho \sigma \partial_\rho \theta^5 \gamma_\rho \sigma + \partial_\rho \theta^5 \gamma_\rho \sigma A^5_{\rho} \gamma_\rho \sigma \right) \right\} \right), \quad (A20) \]

\[ R_{\rho\nu\alpha\sigma} \left( \partial \Omega_{\rho\mu} (\partial \Omega_{\nu\alpha}) \right) = R_{\rho\nu\alpha\sigma} \left\{ i \left[ A_{\rho\nu}, \partial_\nu \theta^5 \right] + i \left[ A^5_{\rho\nu}, \partial_\nu \theta^5 \right] + i \gamma^5 \gamma_\rho \sigma \left( \gamma_\rho \sigma D_\rho \partial_\rho \theta^5 - i \gamma_\rho \sigma \partial_\rho \theta^5 \right) \right\} \\
+ i \gamma^5 \left( \left[ A_{\rho\nu}, \partial_\nu \theta^5 \right] + \left[ \partial_\rho \theta, A^5_{\nu} \right] \right) + i \left( \left[ A^5_{\rho\nu}, \partial_\nu \theta^5 \right] - \left[ A^5_{\rho\nu}, \partial_\nu \theta^5 \right] \right) \gamma^5 \gamma_\rho \sigma \right) \\
- i \gamma^5 \left( A^5_{\rho\nu} \gamma_\rho \sigma \partial_\rho \theta^5 \gamma_\rho \sigma + \partial_\nu \theta^5 \gamma_\rho \sigma A^5_{\nu} \gamma_\rho \sigma \right) \right\} \right) \right), \quad (A21) \]

\[ R_{\rho\alpha\nu\sigma} \left[ 2 \Omega_{\rho\mu} (\partial \Omega_{\nu\sigma}) \right] = 2 R_{\rho\alpha\nu\sigma} \left( F_{\rho\alpha} - [A^5_{\rho}, A^5_{\alpha}] - \frac{1}{4} \gamma^\rho \gamma^\alpha R_{\rho\mu\nu\sigma} - i \gamma^5 \gamma^\rho \sigma \left( \gamma_\rho \sigma D_\rho A^5_{\rho} \right) \right) \\
+ i \gamma^5 A^5_{\rho\alpha} + \left[ A^5_{\rho\alpha}, A^5_{\rho\alpha} \right] \gamma^\rho \gamma_\rho \sigma A^5_{\rho\alpha} \right) \gamma_\rho \sigma \right) \right) \\
\times \left\{ i \left[ A_{\rho\alpha}, \partial_\alpha \theta^5 \right] + i \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] + i \gamma^5 \gamma_\rho \sigma \left( \gamma_\rho \sigma D_\rho \partial_\rho \theta^5 - i \gamma_\rho \sigma \partial_\rho \theta^5 \right) \right\} \\
+ i \gamma^5 \left( \left[ A_{\rho\alpha}, \partial_\alpha \theta^5 \right] + \left[ \partial_\rho \theta, A^5_{\alpha} \right] \right) + i \left( \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] - \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] \right) \gamma^5 \gamma_\rho \sigma \right) \\
- i \gamma^5 \left( A^5_{\rho\alpha} \gamma_\rho \sigma \partial_\rho \theta^5 \gamma_\rho \sigma + \partial_\alpha \theta^5 \gamma_\rho \sigma A^5_{\alpha} \gamma_\rho \sigma \right) \right\} \right) \right), \quad (A22) \]

\[ R_{\rho\alpha\nu\sigma} \left( \partial \Omega_{\rho\mu} (\partial \Omega_{\nu\alpha}) \right) = R_{\rho\alpha\nu\sigma} \left\{ i \left[ A_{\rho\alpha}, \partial_\alpha \theta^5 \right] + i \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] + i \gamma^5 \gamma_\rho \sigma \left( \gamma_\rho \sigma D_\rho \partial_\rho \theta^5 - i \gamma_\rho \sigma \partial_\rho \theta^5 \right) \right\} \\
+ i \gamma^5 \left( \left[ A_{\rho\alpha}, \partial_\alpha \theta^5 \right] + \left[ \partial_\rho \theta, A^5_{\alpha} \right] \right) + i \left( \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] - \left[ A^5_{\rho\alpha}, \partial_\alpha \theta^5 \right] \right) \gamma^5 \gamma_\rho \sigma \right) \\
- i \gamma^5 \left( A^5_{\rho\alpha} \gamma_\rho \sigma \partial_\rho \theta^5 \gamma_\rho \sigma + \partial_\alpha \theta^5 \gamma_\rho \sigma A^5_{\alpha} \gamma_\rho \sigma \right) \right\} \right) \right) \right), \quad (A23) \]
\[ R \left[ 2 \Omega_{\alpha\beta}(\partial \Omega_{\alpha\beta}) \right] = 2R \left\{ F_{\alpha\nu} - [A^5_{\alpha\nu}, A^5_{\mu\delta} \right\} + i \gamma^5 \left( \left[A^5_{\alpha\nu}, \partial_\mu \partial_\nu \right] - \left[A^5_{\nu\mu}, \partial_\nu \partial_\nu \right] - \left[A^5_{\mu\nu}, \partial_\nu \partial_\nu \right] \right) \]
\[
\times \left\{ i \left[ A^5_{\alpha\nu}, \partial_\mu \partial_\mu \right] + i \left[ A^5_{\mu\nu}, \partial_\mu \partial_\mu \right] + \gamma \partial_\nu \theta^5 \right\} \left( \partial \Omega_{\alpha\beta}(\partial \Omega_{\alpha\beta}) \right)
\]
\[
+ i \gamma^5 \left( \left[A^5_{\alpha\nu}, \partial_\mu \partial_\mu \right] + i \left[ A^5_{\mu\nu}, \partial_\mu \partial_\mu \right] + \gamma \partial_\nu \theta^5 \right) \left( \partial \Omega_{\alpha\beta}(\partial \Omega_{\alpha\beta}) \right)
\]
\[
- i \gamma^5 \left( \left[A^5_{\alpha\nu}, \partial_\mu \partial_\mu \right] + i \left[ A^5_{\mu\nu}, \partial_\mu \partial_\mu \right] + \gamma \partial_\nu \theta^5 \right) \left( \partial \Omega_{\alpha\beta}(\partial \Omega_{\alpha\beta}) \right)
\]
\[
+ \frac{i}{4} \left\{ \gamma^\mu \gamma^\nu \right\} \left[ \gamma^\alpha \gamma^\beta \right] - \frac{i}{4} \left\{ \gamma^\mu \gamma^\nu \right\} \left[ \gamma^\alpha \gamma^\beta \right]
\]
\[
- \frac{1}{4} \left\{ R_{\mu\nu} \right\} + \frac{i}{4} \left\{ \gamma^\mu \gamma^\nu \right\} \left[ A^5_{\mu\nu}, \partial_\nu \partial_\nu \right] - \frac{i}{4} \left\{ \gamma^\mu \gamma^\nu \right\} \left[ A^5_{\mu\nu}, \partial_\nu \partial_\nu \right]
\]
\[
- \frac{n-3}{4} \left\{ \gamma^\mu \gamma^\nu \right\} \left[ A^5_{\mu\nu}, A^5_{\mu\nu} \right]
\]
\[ E(\delta E)_{\rho\rho} = \left\{ -\frac{1}{4} R + \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu} + i \gamma^5 D^\mu A^\mu_{\rho\rho} - (n - 2) A^5_{\rho\rho} A^5_{\rho\rho} - \frac{(n - 3)}{4} [\gamma^\mu, \gamma^\nu] [A^5_{\rho\rho}, A^5_{\rho\rho}] \right\} \times \left\{ \frac{i}{4} ([A_{\alpha\beta}, \partial_{\beta}]_{\rho\rho})_{\rho\rho} [\gamma^\alpha, \gamma^\beta] + \gamma^5 \left( -D_\alpha \partial^\mu \theta^5 - i \theta^\mu \bar{\theta} A^5_{\alpha} + \partial^\mu \bar{\theta} \partial_\alpha \theta^5 \right)_{\rho\rho} - i(n - 2) (2 A^5_{\rho\rho} \partial^\mu \theta^5 - \partial_\alpha \theta^5 \partial^\mu \theta^5)_{\rho\rho} - \frac{i(n - 3)}{4} [\gamma^\alpha, \gamma^\beta] [A^5_{\alpha}, \partial_{\beta}]_{\rho\rho} \right\}, \]

(A28)

\[ (\delta E)(\delta E)_{\rho\rho} = \left\{ \frac{i}{4} [A_{\mu}, \partial_{\mu}] [\gamma^\mu, \gamma^\rho] + \gamma^5 \left( -D_\mu \partial^\rho \theta^5 - i \theta^\rho \bar{\theta} A^5_{\mu} + \partial^\rho \bar{\theta} \partial_\mu \theta^5 \right) \right\} \times \left\{ \frac{i}{4} ([A_{\alpha\beta}, \partial_{\beta}]_{\rho\rho})_{\rho\rho} [\gamma^\alpha, \gamma^\beta] + \gamma^5 \left( -D_\alpha \partial^\rho \theta^5 - i \theta^\rho \bar{\theta} A^5_{\alpha} + \partial^\rho \bar{\theta} \partial_\alpha \theta^5 \right)_{\rho\rho} - i(n - 2) (2 A^5_{\rho\rho} \partial^\rho \theta^5 - \partial_\alpha \theta^5 \partial^\rho \theta^5)_{\rho\rho} - \frac{i(n - 3)}{4} [\gamma^\alpha, \gamma^\beta] [A^5_{\alpha}, \partial_{\beta}]_{\rho\rho} \right\}, \]

(A29)

\[ 2E_{\rho\rho}(\delta E)_{\rho\rho} = 2 \left\{ -\frac{1}{4} R_{\rho\rho} + \frac{1}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu,\rho\rho} + i \gamma^5 (D^\mu A^5_{\rho\rho})_{\rho\rho} - (n - 2) (A^5_{\rho\rho} A^5_{\rho\rho})_{\rho\rho} \right\} \times \left\{ \frac{(n - 3)}{4} [\gamma^\mu, \gamma^\nu] [A^5_{\rho\rho}, A^5_{\rho\rho}]_{\rho\rho} \right\} \times \left\{ \frac{i}{4} ([A_{\alpha\beta}, \partial_{\beta}]_{\rho\rho})_{\rho\rho} [\gamma^\alpha, \gamma^\beta] + \gamma^5 \left( -D_\alpha \partial^\mu \theta^5 - i \theta^\mu \bar{\theta} A^5_{\alpha} + \partial^\mu \bar{\theta} \partial_\alpha \theta^5 \right)_{\rho\rho} - i(n - 2) (2 A^5_{\rho\rho} \partial^\mu \theta^5 - \partial_\alpha \theta^5 \partial^\mu \theta^5)_{\rho\rho} - \frac{i(n - 3)}{4} [\gamma^\alpha, \gamma^\beta] [A^5_{\alpha}, \partial_{\beta}]_{\rho\rho} \right\}, \]

(A30)

\[ (\delta E)_{\rho\rho}(\delta E)_{\rho\rho} = \left\{ \frac{i}{4} ([A_{\mu\nu}, \partial_{\lambda}]_{\rho\rho})_{\rho\rho} [\gamma^\mu, \gamma^\nu] + \gamma^5 \left( -D_\mu \partial^\nu \theta^5 - i \theta^\nu \bar{\theta} A^5_{\mu} + \partial^\nu \bar{\theta} \partial_\nu \theta^5 \right)_{\rho\rho} \right\} \times \left\{ \frac{i}{4} ([A_{\alpha\beta}, \partial_{\beta}]_{\rho\rho})_{\rho\rho} [\gamma^\alpha, \gamma^\beta] + \gamma^5 \left( -D_\alpha \partial^\nu \theta^5 - i \theta^\nu \bar{\theta} A^5_{\alpha} + \partial^\nu \bar{\theta} \partial_\alpha \theta^5 \right)_{\rho\rho} - i(n - 2) (2 A^5_{\rho\rho} \partial^\nu \theta^5 - \partial_\alpha \theta^5 \partial^\nu \theta^5)_{\rho\rho} - \frac{i(n - 3)}{4} [\gamma^\alpha, \gamma^\beta] [A^5_{\alpha}, \partial_{\beta}]_{\rho\rho} \right\}, \]

(A31)
\[3E_2(\delta E) = 3 \left\{ -\frac{1}{4} R + \frac{1}{4} [\gamma^{\mu}, \gamma^{\nu}] F_{\mu\nu} + i \gamma^{5} D^{\mu} A^{5}_{\mu} - (n - 2) A^{5}_{\mu} A^{5} - \frac{(n - 3)}{4} \left[ \gamma^{\mu}, \gamma^{\nu} \right] [A^{5}, A^{5}] \right\} \]
\times \left\{ -\frac{1}{4} R + \frac{1}{4} [\gamma^{\xi}, \gamma^{\chi}] F_{\xi\chi} + i \gamma^{5} D^{\xi} A^{5}_{\xi} - (n - 2) A^{5}_{\xi} A^{5} - \frac{(n - 3)}{4} \left[ \gamma^{\xi}, \gamma^{\chi} \right] [A^{5}, A^{5}] \right\} \]
\times \left\{ i \left[ A_{\mu}, \partial_{\nu} \partial \right] \left[ \gamma^{\mu}, \gamma^{\nu} \right] + \gamma^{5} \left( -D_{\alpha} \partial_{\beta} \theta^{5} - i \partial_{\alpha} \partial_{\beta} A^{5}_{\mu} + \partial_{\alpha} \partial_{\beta} \partial_{\mu} \theta^{5} \right) \right. \\
\left. - i(n - 2) (2 A^{5}_{\alpha} \partial_{\mu} \theta^{5} - \partial_{\mu} \theta^{5} \partial_{\nu} \theta^{5}) - \frac{i(n - 3)}{4} \left[ \gamma^{\alpha}, \gamma^{\beta} \right] [A^{5}_{\alpha}, \partial_{\beta} \theta^{5}] \right\} \right), \quad (A32) \]

\[3E(\delta E)^{2} = 3 \left\{ -\frac{1}{4} R + \frac{1}{4} [\gamma^{\xi}, \gamma^{\chi}] F_{\xi\chi} + i \gamma^{5} D^{\xi} A^{5}_{\xi} - (n - 2) A^{5}_{\xi} A^{5} - \frac{(n - 3)}{4} \left[ \gamma^{\xi}, \gamma^{\chi} \right] [A^{5}, A^{5}] \right\} \]
\times \left\{ i \left[ A_{\mu}, \partial_{\nu} \partial \right] \left[ \gamma^{\mu}, \gamma^{\nu} \right] + \gamma^{5} \left( -D_{\alpha} \partial_{\beta} \theta^{5} - i \partial_{\alpha} \partial_{\beta} A^{5}_{\mu} + \partial_{\alpha} \partial_{\beta} \partial_{\mu} \theta^{5} \right) \right. \\
\left. - i(n - 2) (2 A^{5}_{\alpha} \partial_{\mu} \theta^{5} - \partial_{\mu} \theta^{5} \partial_{\nu} \theta^{5}) - \frac{i(n - 3)}{4} \left[ \gamma^{\alpha}, \gamma^{\beta} \right] [A^{5}_{\alpha}, \partial_{\beta} \theta^{5}] \right\} \right), \quad (A33) \]

\[(\delta E)^{3} = \left\{ i \left[ A_{\mu}, \partial_{\nu} \partial \right] \left[ \gamma^{\mu}, \gamma^{\nu} \right] + \gamma^{5} \left( -D_{\alpha} \partial_{\beta} \theta^{5} - i \partial_{\alpha} \partial_{\beta} A^{5}_{\mu} + \partial_{\alpha} \partial_{\beta} \partial_{\mu} \theta^{5} \right) \right. \\
\left. - i(n - 2) (2 A^{5}_{\alpha} \partial_{\mu} \theta^{5} - \partial_{\mu} \theta^{5} \partial_{\nu} \theta^{5}) - \frac{i(n - 3)}{4} \left[ \gamma^{\alpha}, \gamma^{\beta} \right] [A^{5}_{\alpha}, \partial_{\beta} \theta^{5}] \right\} \right), \quad (A34) \]

\[(\delta E)\Omega_{\rho\sigma}, \Omega_{\rho\sigma} = \left\{ i \left[ A_{\mu}, \partial_{\nu} \partial \right] \left[ \gamma^{\mu}, \gamma^{\nu} \right] + \gamma^{5} \left( -D_{\alpha} \partial_{\beta} \theta^{5} - i \partial_{\alpha} \partial_{\beta} A^{5}_{\mu} + \partial_{\alpha} \partial_{\beta} \partial_{\mu} \theta^{5} \right) \right. \\
\left. - i(n - 2) (2 A^{5}_{\alpha} \partial_{\mu} \theta^{5} - \partial_{\mu} \theta^{5} \partial_{\nu} \theta^{5}) - \frac{i(n - 3)}{4} \left[ \gamma^{\alpha}, \gamma^{\beta} \right] [A^{5}_{\alpha}, \partial_{\beta} \theta^{5}] \right\} \right), \quad (A35) \]
\[
\times \left\{ F_{\rho\sigma} - \left[ A_\mu^5, A_\sigma^5 \right] - \frac{1}{4} \gamma^\rho \gamma^\sigma R_{\lambda\rho\sigma\tau} - i \gamma^\rho \gamma^\sigma \left( \gamma_{[\rho} D_{\sigma]} A_\lambda^5 \right) \\
+ i \gamma^\rho A_{\rho\sigma}^5 \left[ A_\mu^5, A_\lambda^5 \right] \gamma_{[\sigma} - \gamma^\rho A_5 \gamma_{[\sigma} \gamma A_5 \gamma_{\rho]} \right\},
\] (A35)

\[
(\delta E) \left[ 2 \Omega_{\rho\sigma}(\delta \Omega_{\rho\sigma}) \right] = 2 \left\{ \frac{i}{4} \left[ A_\rho^5, D_\rho^5 \right] \left[ \gamma^\alpha, \gamma^\beta \right] + \gamma^5 \left( -D_\rho^5 \partial^\rho \theta^5 - iD_\rho^5 \partial^\rho \partial_\rho \theta^5 \right) \\
- i(n - 2) \left( 2 A_5 \partial^\rho \theta^5 \partial_\rho \theta^5 \right) - \frac{i(n - 3)}{4} \left[ [\gamma^\alpha, \gamma^\beta] \left[ A_\rho^5, D_\rho^5 \right] \right] \right\}
\times \left\{ F_{\rho\sigma} - \left[ A_\mu^5, A_\sigma^5 \right] - \frac{1}{4} \gamma^\rho \gamma^\sigma R_{\lambda\rho\sigma\tau} - i \gamma^\rho \gamma^\sigma \left( \gamma_{[\rho} D_{\sigma]} A_\lambda^5 \right) \\
+ i \gamma^\rho A_{\rho\sigma}^5 \left[ A_\mu^5, A_\lambda^5 \right] \gamma_{[\sigma} - \gamma^\rho A_5 \gamma_{[\sigma} \gamma A_5 \gamma_{\rho]} \right\}
\times \left\{ i \left[ A_\mu^5, D_\mu^5 \right] + i \left[ A_\mu^5, \partial_\mu \theta^5 \right] + \gamma^\mu \gamma^5 \left( \gamma_{\mu} D_{\mu} \partial_\mu \theta^5 - i\gamma_{\mu} \partial_\mu \partial_\mu \theta^5 \right) \\
i(5) \left[ A_\mu^5, \partial_\mu \theta^5 \right] + \left[ \partial_\mu \theta^5, A_\sigma^5 \right] \right\} + i \left( \left[ A_\mu^5, \partial_\mu \theta^5 \right] - \left[ A_\mu^5, \partial_\mu \theta^5 \right] \right) \gamma^\mu \eta_{\sigma} \\
i \gamma^\mu \left( A_5 \gamma_{\mu} \gamma^5 \partial_\mu \partial_\mu \theta^5 + \partial_\mu \partial_\mu \theta^5 \gamma A_5 \gamma_{\mu} \eta_{\sigma} \right) \\
\times \left\{ i \left[ A_\mu^5, D_\mu^5 \right] + i \left[ A_\mu^5, \partial_\mu \theta^5 \right] + \gamma^\mu \gamma^5 \left( \gamma_{\mu} D_{\mu} \partial_\mu \theta^5 - i\gamma_{\mu} \partial_\mu \partial_\mu \theta^5 \right) \\
i(5) \left[ A_\mu^5, \partial_\mu \theta^5 \right] + \left[ \partial_\mu \theta^5, A_\sigma^5 \right] \right\} + i \left( \left[ A_\mu^5, \partial_\mu \theta^5 \right] - \left[ A_\mu^5, \partial_\mu \theta^5 \right] \right) \gamma^\mu \eta_{\sigma} \\
i \gamma^\mu \left( A_5 \gamma_{\mu} \gamma^5 \partial_\mu \partial_\mu \theta^5 + \partial_\mu \partial_\mu \theta^5 \gamma A_5 \gamma_{\mu} \eta_{\sigma} \right) \right\}
\] (A36)

\[
R(\delta E)_{\alpha\beta} = R \left\{ \frac{i}{4} \left( \left[ A_\mu^5, \partial_\mu \theta^5 \right] \right)_{\alpha\beta} \left[ \gamma^\mu, \gamma^\nu \right] + \gamma^5 \left( -D_\rho^5 \partial_\rho \theta^5 - iD_\rho^5 \partial_\rho \partial_\rho \theta^5 \right)_{\alpha\beta} \\
- i(n - 2) \left( 2 A_5 \partial^\rho \theta^5 \partial_\rho \theta^5 \right)_{\alpha\beta} - \frac{i(n - 3)}{4} \left[ [\gamma^\mu, \gamma^\nu] \left( \left[ A_\rho^5, \partial_\rho \theta^5 \right] \right) \right]_{\alpha\beta} \right\}
\] (A37)

\[
\] (A38)
\[ R_{\alpha \sigma} (\delta E)_{\sigma \alpha} = R_{\alpha \sigma} \left\{ \frac{i}{4} \left( [A_{\mu}^\nu, \partial_\sigma \theta] \right)_{\sigma \alpha} [\gamma^\mu, \gamma^\nu] + \gamma^5 \left( -D_\mu \partial^\mu \theta^5 - i \partial^\mu \bar{\theta} A^\mu_5 + \partial^\mu \theta \partial_\mu \theta^5 \right)_{\sigma \alpha} \right. \\
\left. - i(n - 2)(2A^5_\sigma \partial^\mu \theta^5 - \partial_\sigma \theta^5 \partial^\mu \theta^5)_{\sigma \alpha} - \frac{i(n - 3)}{4} [\gamma^\mu, \gamma^\nu] \left( [A^5_\mu, \partial_\sigma \theta^5] \right)_{\sigma \alpha} \right\} , \quad (A39) \]

\[ R_{\sigma \alpha} (\delta E)_{\alpha \sigma} = R_{\sigma \alpha} \left\{ \frac{i}{4} \left( [A_{\mu}^\nu, \partial_\sigma \theta] \right)_{\alpha \sigma} [\gamma^\mu, \gamma^\nu] + \gamma^5 \left( -D_\mu \partial^\mu \theta^5 - i \partial^\mu \bar{\theta} A^\mu_5 + \partial^\mu \theta \partial_\mu \theta^5 \right)_{\alpha \sigma} \right. \\
\left. - i(n - 2)(2A^5_\sigma \partial^\mu \theta^5 - \partial_\sigma \theta^5 \partial^\mu \theta^5)_{\alpha \sigma} - \frac{i(n - 3)}{4} [\gamma^\mu, \gamma^\nu] \left( [A^5_\mu, \partial_\sigma \theta^5] \right)_{\alpha \sigma} \right\} , \quad (A40) \]

\[ R[2E(\delta E)] = 2R \left\{ - \frac{1}{4} R + \frac{1}{4} \left( [\gamma^\xi, [\gamma^\xi, \gamma^\lambda] \right) \right. \\
\left. \times \left( \frac{i}{4} \left( [A_{\mu}^\nu, \partial_\xi \theta] \right)_{\xi \sigma} [\gamma^\mu, \gamma^\nu] + \gamma^5 \left( -D_\mu \partial^\mu \theta^5 - i \partial^\mu \bar{\theta} A^\mu_5 + \partial^\mu \theta \partial_\mu \theta^5 \right)_{\xi \sigma} \right) \right. \\
\left. - i(n - 2)(2A^5_\sigma \partial^\mu \theta^5 - \partial_\sigma \theta^5 \partial^\mu \theta^5)_{\xi \sigma} - \frac{i(n - 3)}{4} [\gamma^\mu, \gamma^\nu] \left( [A^5_\mu, \partial_\sigma \theta^5] \right)_{\xi \sigma} \right\} , \quad (A41) \]

\[ R(\delta E)^2 = R \left\{ \frac{i}{4} \left( [A_{\mu}^\nu, \partial_\lambda \theta] \right)_{\lambda \sigma} [\gamma^\xi, \gamma^\lambda] + \gamma^5 \left( -D_\xi \partial^\xi \theta^5 - i \partial^\xi \bar{\theta} A^\xi_5 + \partial^\xi \theta \partial_\xi \theta^5 \right)_{\lambda \sigma} \right. \\
\left. - i(n - 2)(2A^5_\sigma \partial^\mu \theta^5 - \partial_\sigma \theta^5 \partial^\mu \theta^5)_{\lambda \sigma} - \frac{i(n - 3)}{4} [\gamma^\mu, \gamma^\nu] \left( [A^5_\mu, \partial_\sigma \theta^5] \right)_{\lambda \sigma} \right\} , \quad (A42) \]

\[ (\delta E)\Theta (R) = \left\{ \frac{i}{4} \left( [A_{\mu}^\nu, \partial_\lambda \theta] \right)_{\lambda \sigma} [\gamma^\xi, \gamma^\lambda] + \gamma^5 \left( -D_\xi \partial^\xi \theta^5 - i \partial^\xi \bar{\theta} A^\xi_5 + \partial^\xi \theta \partial_\xi \theta^5 \right)_{\lambda \sigma} \right. \\
\left. - i(n - 2)(2A^5_\sigma \partial^\mu \theta^5 - \partial_\sigma \theta^5 \partial^\mu \theta^5)_{\lambda \sigma} - \frac{i(n - 3)}{4} [\gamma^\mu, \gamma^\nu] \left( [A^5_\mu, \partial_\sigma \theta^5] \right)_{\lambda \sigma} \right\} \times \Theta (R) , \quad (A43) \]

since \( \Theta (R) = 12R_{\sigma \alpha} + 5R^2 - 2R_{\rho \sigma} R_{\rho \sigma} + 2R_{\rho \sigma \alpha} R_{\rho \sigma \alpha} \).

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