The Feynman Propagator from a Single Path

G. N. Ord
M.P.C.S.
Ryerson University
Toronto Ont.

J. A. Gualtieri†
Applied Information Sciences Branch
Global Science and Technology
Code 935 NASA/Goddard Space Flight Center
Greenbelt MD.
(Dated: September 19 2001)

We show that it is possible to construct the Feynman Propagator for a free particle in one dimension, without quantization, from a single continuous space-time path.

The Feynman path-integral formulation of Quantum Mechanics[1, 2] is well known for its utility and intuitive appeal. An interesting history of its development may be found in the article and book by Schweber[3, 4]. Although the mathematics of the path integral encourages us to think of the paths in terms of real space-time trajectories, and there have been very interesting proposals for testing the reality of the paths[5, 6, 7], the formulation itself falls short of providing a full microscopic basis for quantum mechanics. This is in contrast to the Wiener integral which is an abstraction of the microscopic model (Brownian motion) supporting the diffusion equation. In particular, Wiener paths are known to approximate actual physical trajectories of diffusing particles, whereas the relation between Feynman paths and physical particles is not so direct.

There are two main barriers to an association between Feynman paths and any physical trajectory of a real particle. First of all there is a many-to-one correspondence between Feynman paths and the particle being described. Interference effects require this non-uniqueness since individual trajectories carry variable phase but not variable amplitude in the propagator[8]. Thus a physical particle cannot simply traverse a single Feynman path while propagating in space-time.

A second impediment is that, in the path integral formulation, the required reduction of wave functions on measurement is grafted onto the dynamics of propagation; it does not follow in a direct fashion from the paths themselves. As in other formulations of quantum mechanics we need measurement postulates to interpret the theory in terms of the real world.

In this paper we show that in the particular case of the Feynman Chessboard model, one can modify the formulation so that the propagator can be constructed by a single continuous space-time curve. This is done by allowing particles to have trajectories with reversed time segments. Although this might seem conceptually 'expensive', allowing this feature explicitly provides the physical mechanism which creates the phase of a wave function without invoking an analytic continuation. The propagator appears naturally as a pattern created by the (space-time) plane-filling path of a single point-particle. In the new formulation, the many-to-one aspect of Feynman paths is circumvented by sewing together an ensemble of Chessboard paths into a single curve in such a way that formal quantization is unnecessary.

The Chessboard or Checkerboard model[2, 9, 10] extended Feynman’s path integral approach to the relativistic domain in order to incorporate electron spin. In this model, particles hop with speed \( \pm c \) on a discrete space-time lattice with spacing \( \epsilon \). Choosing units in which \( c = 1 \), paths consist of diagonal segments resembling forward bishop’s moves in chess(Fig. 1).

A lattice approximation to the Kernel \( K(b, a) \) for a particle to propagate from position \( a \) at time \( t_a \) to position \( b \) at time \( t_b \) is given by Feynman to be:

\[
K(b, a) = \sum_R N(R)(i\epsilon m)^R
\]  

(1)

![FIG. 1: A Feynman Chessboard trajectory. The x-axis is horizontal and the t-axis vertical. The sign of the contribution changes every two corners in the trajectory. This is indicated in the figure by the different line widths in the different segments.](image-url)
where the sum is over all Chessboard paths and \( N(R) \) is the number of paths with \( R \) corners. Here \( m \) is the mass of the particle in units where \( \hbar = 1 \). If we distinguish between the two directions in space, \( K \) is a 2x2 matrix which converges to the Dirac propagator in the continuum limit. The prescription given in (1) can be modified somewhat for convenience. Gersch, who established the relation between the Chessboard model and the one dimensional Ising model, pointed out that the non-relativistic limit is more direct if \( i \) is replaced by \(-i\) in (1). Kull and Treumann also noted that paths fixed at both ends have \((R-1)\) degrees of freedom, so the \( R \) in (1) may be replaced by \((R-1)\) without interfering with the continuum limit.

Equation (1) is a formal analytic continuation (quantization) of a classical partition function. The \( i \) in the sum, which replaces a real positive weight in the partition function, enforces the quantization. It also partitions the sum into 4 components, each of which is real, i.e.:

\[
K(b, a) = \left( \sum_{R=0, 4, \ldots} N(R)(em)^R - \sum_{R=2, 6, \ldots} N(R)(em)^R \right)
+ i \left( \sum_{R=1, 5, \ldots} N(R)(em)^R - \sum_{R=3, 7, \ldots} N(R)(em)^R \right)
= \Phi_R + i \Phi_I. \tag{2}
\]

Each of the above sums is, by itself, a partition function for a class of random walks in which the term \((em)^R\) is just a Boltzman weight. The interference of alternative paths is a result of the two subtractions in (1). If we replace the minus signs in (2) by plus signs, the resulting propagator is related to the Telegraph equation, which in turn becomes the diffusion equation in the appropriate 'non-relativistic' limit, the remaining \( i \) then being superfluous. The underlying stochastic model for this case has been studied by Kac and its relation to the Dirac equation through analytic continuation has been discussed by Gaveau et.al. and Jacobson and Schulman. With the original minus signs in place, the \( i \) which appears in (2) just expresses \( K \) as a particularly convenient linear combination of the real amplitudes \( \Phi_R/I \), however the actual interference characteristic of quantization is apparent in the oscillatory nature of the \( \Phi \) themselves.

Since it is the occurrence of the minus signs in the propagator which is essential for interference we look for a physical basis for the subtractions. Regarding Fig.1 we can encode the counting and subtractions involved in (2) by colouring the trajectories with two colours, say blue (thick lines in figure) and red (thin in figure). If the trajectories start out blue, they change to red at the second corner, blue at the fourth and so on. The sign of the contribution of a trajectory is then determined by its colour at the end point, + for blue, − for red. Red contributions behave like antiparticles in that they reduce the contribution of the particles, providing interference effects. The ensemble of such coloured paths between \( a \) and \( b \) provides the appropriate contribution to a quantum propagator, but is not explicitly traversed as a single path. What we would like to do is to sew together the Chessboard paths in such a way that they may be traversed by a single path which also provides the alternating colours of the trajectories through the direction in time of the traversal. To this end, we note from Fig. 2 that each Chessboard path has an orthogonal twin.

The orthogonal twin starts from the origin moving in the opposite direction with the opposite colour. It moves the same distance as the second leg of its twin’s path, reverses direction and moves the same distance as its twin’s first leg. Twins meet at every second corner where they change both colour and direction. For paths with an odd number of corners, this is repeated until the twins meet at \( t = t_b \) (for paths with an even number of corners see below). The orthogonal twin is also a Chessboard path with colouring 180° out of phase with the original.

Now consider the following ‘entwined’ traversal of the two paths. Follow the first twin to the first meeting, the second to the second meeting and so on. This path is blue from the origin to the last meeting. From there reverse the direction in \( t \) by proceeding down the remaining red sections. This brings you back to the origin on an entirely red path. This choice of traversal gives a meaning to the original Feynman colouring: the colouring corresponds to the direction in time of an entwined path traversal. Blue corresponds to forward in \( t \), red to backwards. Entwined pairs also conserve charge if we associate opposite charges with reversed time segments.

Each chessboard trajectory in (3) has a unique orthog-
onal twin. Let $P_R$ be an arbitrary $n$-step $R$-cornered Chessboard path. Write $P_R = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_k = \pm 1$ according to the direction of the $k$-th step of the path. If we define a ‘leg’ as a set of contiguous steps all in the same direction and bounded by either corners or ends of a path (i.e. a domain in the Ising analogy), then if $R$ is odd, we may write $P_R = (l_1, l_2, \ldots, l_{R+1})$ with the understanding that $l_1$ stands for the first leg, $l_2$ stands for the second and so on. If $R$ is even then the path ends with the last link in the same direction as the first link. In order to join the path to an orthogonal twin we need to add a final leg in the opposite direction. To do this uniquely we add a final leg the same length as the original last leg but in the opposite direction. Thus the following classical stochastic process gives rise to a properly weighted chessboard ensemble of coloured component Chessboard trajectories. Thus we may cover all paths in orthogonal twins. Thus we may cover all paths in $\Phi_R + \Phi_I$ along the $x$-axis, from the Chessboard model (curve) and the single path simulation (points) at $t = 15$ steps from the origin.

If we allow a walker to cycle through the entwined paths according to the above prescription, we can immediately write down the expected net ‘charge’ accumulated on the lattice. Referring to the kernel in (3), we can define the four components of the $2 \times 2$ matrix as $K_{\sigma,\sigma'}$ where the subscripts refer to the end and beginning directions respectively. If the walker, starting in the $+x$ direction, loops over $N$ entwined pairs and $(x, t)$ is a lattice point within the light cone with $t < t_b$ then the contribution to the $+x$-component of the net charge is proportional to $\rho_+ = N(K_{++}(x, t) - K_{+-}(x, t))$. This is because an entwined loop corresponds to two forward Chessboard paths, one originating from the origin with a positive, blue first leg ($K_{++}$ contribution) and the second from a negative red first leg($K_{+-}$ contribution). Similarly the $-x$-component at $(x, t)$ is proportional to $\rho_- = N(K_{-+}(x, t) - K_{--}(x, t))$. The $\rho$ may be interpreted as particle densities which may be either positive or negative depending on the predominance of entwined trajectories in plus or minus $t$ directions. $\rho_+$ is positive in $(+x, +t)$-rich areas and $\rho_-$ is positive in $(-x, +t)$-rich areas. Note that $\rho_+ - \rho_-$ is proportional to the sum of the real and imaginary part of the Feynman propagator $\Phi_R + \Phi_I$ (Gersch convention for the sign of $i$). Unlike the predecessors of this model, which did not use bound pairs of trajectories, this new model is relatively easy to simulate on a lattice.

Fig. (3) shows an example of such a simulation, where the sum of the real and imaginary parts of the propagator, $\Phi_R + \Phi_I$, at fixed $t$, are plotted versus $x$. The expected results from the Chessboard model are plotted (continuous curve) at the same lattice resolution as the results of a simulation with a single path which loops over the lattice $10^8$ times. In the figure, $t$ is 15 steps from the origin, with an average of two steps between corners or a probability of $1/2$ for a direction change at each step. At smaller values of $t$, the simulation and the Chessboard
model are indistinguishable on the scale of the figure, at larger values of \( t \) the single path gives sparser coverage of the chessboard ensemble and the scatter increases. The individual real and imaginary parts of the propagator may be calculated using the symmetry of the solutions, or by recording the \( \rho_\pm \) in two components to separate contributions from the original chessboard path and its orthogonal twin.

Although we do not know how much of the above can survive inclusion of an external field and/or extension to three space dimensions, we do think the result reveals several qualitatively appealing features of the simplest case of a free particle in one dimension. First, the Feynman propagator has an independent existence as an expected net charge over an ensemble of entwined paths which can be joined into a single trajectory. In this context, the propagator has an underlying classical stochastic model which is in effect self-quantizing and produces real densities in place of amplitudes.

A second feature is that the above model provides a bridge between two distinct views of quantum mechanics in this case. Regarding Fig. 2, we may view the two trajectories in three ways. We can consider them as two separate chessboard trajectories, coloured according to Feynman’s corner rule. An ensemble of such trajectories builds a quantum propagator as a sum-over-histories. This is the conventional view. A second picture is to note that an entwined pair forms a chain of creation/annihilation events. An ensemble of these would provide a vacuum of virtual particles upon which an excitation could presumably propagate. This is close to a field theory perspective.

The third picture, which is suggested by the new formulation, is the continuous loop in space-time, coloured according to direction of motion in time. In this picture, the phase of the wave function, ‘zitterbewegung’, and the presence of virtual particles are all manifestations of a single path which forms entwined space-time loops. In many respects, this picture is an implementation of the original Wheeler–Feynman one-electron-universe\(^4\), scaled down to provide a single-path electron. Here the multiple tracks in space-time create a ‘Dirac sea’ rather than the multitude of electrons in the universe.

Finally, the entwined path formulation allows an analog of wavefunction collapse to be associated with the system. Suppose that we impose a minimal requirement that a ‘measurement’ at time \( t = t_m \) must fix the wavefunction at all times \( t < t_m \). This requirement eventually (in a local ‘time’ parameter of the particle which measures distance along the full space-time trajectory) forces the point particle which draws the propagator to stay in the region \( t > t_m \), thus making it redraw the ‘future’ propagator in a manner that is consistent with an ‘initial’ condition at \( t = t_m \). This change in the wavefunction at \( t_m \) need not be unitary and may provide the analog of collapse. An interesting next step would be to see if a traversal and measurement scheme could be found that would initiate collapse in a manner consistent with the Born postulate.

This work was partly funded by NSERC (GNO). The authors are grateful for helpful discussions and computational expertise from John Dorband and Scott Antonille at NASA-GSFC.

\[ \ast \] corresponding author gord@acs.ryerson.ca

\[ \ast \ast \] gualt@peep.gsfc.nasa.gov

1. R. P. Feynman, Rev. Mod. Physics. 20, 367 (1948).
2. R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals (New York: McGraw-Hill, 1965).
3. S. Schweber, Review of Modern Physics 58, 449 (1986).
4. S. Schweber, QED and The Men Who Made It (Princeton University Press, 1994).
5. H. Kroger, Phys. Rev. A. 55, 951 (1997).
6. H. Kroger, Phys. Lett. A 226, 127 (1997).
7. H. Kroger, Phys. Rep 323, 81 (2000).
8. However, a recent result by Kroger has motivated a conjecture that for a particular class of non-relativistic path integrals, the sum-over-paths for a particular transition amplitude may be replaced by a sum over a single path with a renormalized action. Jirari et al., Phys. Rev. Lett. 86 (2001) 187, Jirari et al. Phys. Lett. A 281 (2001) 1
9. H. Gersch, Int. J. Theor. Physics 20, 491 (1981).
10. T. Jacobson and L. Schulman, J. Physics A 17, 375 (1984).
11. A. Kull and R. Treumann, Int. J. Theo. Phys. 38, 1423 (1999).
12. G. N. Ord, J. Stat. Phys. 66, 647 (1992).
13. M. Kac, Rocky Mountain Journal of Mathematics p. 4 (1974).
14. B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, Phys. Rev. Lett. 53, 419 (1984).
15. G. N. Ord, Int. J. Theor. Physics 31, 1177 (1992).
16. D. G. C. McKeon and G. N. Ord, Phys. Rev. Lett. 69, 3 (1992).
17. G. N. Ord, Phys. Lett.A 173, 343 (1993).