OPTIMAL LAMINATES IN SINGLE-SLIP ELASTOPLASTICITY

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Abstract. Recent progress in the mathematical analysis of variational models for the plastic deformation of crystals in a geometrically nonlinear setting is discussed. The focus lies on the first time-step and on situations where only one slip system is active, in two spatial dimensions. The interplay of invariance under finite rotations and plastic deformation leads to the emergence of microstructures, which can be analyzed in the framework of relaxation theory using the theory of quasiconvexity. A class of elastoplastic energies with one active slip system that converge asymptotically to a model with rigid elasticity is presented and the interplay between relaxation and asymptotics is investigated.

1. Introduction. Problems in elastoplasticity for crystalline materials in general and metals in particular have been a fundamental topic in continuum mechanics, mathematical analysis, and numerical simulation since the 1950s. While the formulation within the context of a linearized setting is well-established, see, e.g., [37], the formulation within the framework of nonlinear elasticity is a very active research topic and many fundamental questions remain open. The cornerstone of the main line of research is a nonlinear decomposition of the deformation gradient $F$ into an elastic part $F_e$ and a plastic part $F_p$, here in a multiplicative fashion, $F = F_e F_p$, which originates in the work of Kröner and Lee [38, 42] and was further developed in [58, 69, 68, 49, 47]. A micromechanical justification was recently presented in [64, 65], see also [32] for a discussion of different types of multiplicative decompositions.

In this contribution we focus on one aspect within this general framework of ideas, namely the strong connections to variational models which may be derived by a time discretization from classical evolution equations related to balance of forces and constitutive equations for internal variables based on the principle of maximal dissipation [63, 11]. In this approach, the free energy density of the variational problem which describes the response of the material in one time step consists of three contributions, namely an elastic stored energy density $W_e$ depending on the
elastic part of the deformation gradient, a plastic potential $W_p$ and a dissipation term $\text{Diss}$ related to the plastic variables. If no additional constraints are imposed, then the decomposition $F = F_e F_p$ is not unique, within a variational framework the optimal one can be obtained by minimization of the effective energy, which contains both the stored energy and the corresponding dissipation. That is, if we define

$$W(F_e, F_p) = W_e(F_e) + W_p(F_p) + \text{Diss}(F_p),$$

(1.1)

then a minimization of the effective energy with fixed deformation gradient $F$ leads to the condensed energy density

$$W_{\text{cond}}(F) = \inf_{F = F_e F_p} W(F_e, F_p) = \inf_{F = F_e F_p} (W_e(F_e) + W_p(F_p) + \text{Diss}(F_p)).$$

(1.2)

The corresponding variational problem for an elasto-plastic material body represented by an open and bounded set $\Omega \subset \mathbb{R}^n$ can be formulated in a specified class of admissible deformations $u \in A$ as follows: find the infimum value of

$$I[u] = \int_{\Omega} W_{\text{cond}}(Du) dx$$

in $A$ and determine whether the infimum is attained. This is a variational problem in the vector-valued setting and it is known that the direct method in the calculus of variations based on weak sequential lower semicontinuity can be applied only if the energy density $W_{\text{cond}}$ satisfies a suitable convexity condition, here quasiconvexity in the sense of Morrey [59], see also [30, 66, 60]. It was observed in [63, 11] that typically $W_{\text{cond}}$ fails to be quasiconvex and that minimizing sequences tend to develop oscillations on small scales. In the weak limit, the information encoded in the oscillations is lost and can only be recovered by more advanced techniques, like the study of the corresponding gradient Young measure. However, this failure of quasiconvexity does not rule out the existence of minimizers, and the adaptation of Gromov’s method of convex integration to Lipschitz functions led to surprising existence results for minimizers, see [61].

An important tool in the analysis of nonconvex problems is the theory of relaxation [30], which replaces the nonconvex problem with its relaxation which is defined as the largest functional which is dominated by $I[\cdot]$ and weakly sequentially lower semicontinuous. If the energy density has $p$-growth from above and below, the relaxed functional is again a variational integral [30, 66, 60]. For extended-valued energy densities the situation is more complex. For example, the constraint of being orientation preserving is often modeled by energies which are infinite if the determinant of the deformation gradient is less than or equal to zero, incompressibility correspondingly modeled by energies which are infinite if the determinant is not one. Recent relaxation results in this setting can be found in [24, 17].

The development of a relaxation theory in a time-continuous setting is a much more demanding problem. A variational framework has been developed by Alexander Mielke and his group and is reviewed in the recent book [54]. This framework permitted in particular to prove global existence results in finite and linearized plasticity [57, 51, 52, 46, 56, 55, 53]. Practical implementations of this approach include the development of specific models for the evolution of laminated microstructures [36]. In this paper we shall not address time-continuous evolution and instead only focus on the deformation theory of plasticity, which corresponds to the first time step in a time-discrete approximation of the time-continuous evolution.

We work with the standard Lagrangian formulation of the continuum mechanics of solids, which is based on the existence of a reference configuration. This is of
course a subtle issue in plasticity, and indeed a body of recent work has been devoted to the development of Eulerian models which permit a reference-configuration-free formulation for the theory of plasticity [43, 44, 40], see also [62] for a related discussion. We do not address this point here.

Before we turn to the analytic results, we briefly comment on numerical approaches which have been developed in the context of nonconvex variational problems related to phase transformations in solids and which can be applied also in the framework of plasticity. They mirror the analytical aspects just outlined, construction of minimizing sequences, gradient Young measures, and relaxation.

A first observation is that minimizing sequences can be approximated in finite element spaces if they are sufficiently smooth. More precisely, the approximation of the elements of a minimizing sequence in finite element spaces on finer and finer grids will lead to a new minimizing sequence. Suppose without loss of generality that the infimum of the energy is equal to zero. Then one obtains for an underlying mesh of typical size $h$ for a finite element space $V_h$ an estimate for the minimum of the nonconvex minimization problem $I$ in terms of $h$, see [45] for a review of this strategy and many applications to specific microstructures related to phase transformations.

From a computational point of view, the direct minimization of the nonconvex problem, that is, the minimization in finite element spaces and the search for minimizing sequences, is a subtle issue since the resolution of the oscillations is directly coupled to the quality of the underlying mesh and this leads to sharp lower bounds on the energy, see [16]. A natural approach in this context is the passage from oscillating sequences to minimizing gradient Young measures $\nu$ with center of mass $F$, see, e.g., [6, 14, 15, 39, 67, 66] and the references therein for detailed information on these aspects. Some information about the system which is lost by the passage to gradient Young measures may be recovered in a post-processing step as, e.g., in [3, 10, 7, 13, 9, 19]. In the context of problems in elasticity, an important analytical ingredient in this approach are the results in [2, 12] which allow one to obtain the energy density and the uniquely defined stress field from the underlying gradient Young measure via the representations

$$W^{\text{ef}}(F) = \int_{\mathbb{R}^{m \times n}} W(A) d\nu(A), \quad DW^{\text{ef}}(F) = \int_{\mathbb{R}^{m \times n}} DW(A) d\nu(A).$$

Finally, relaxation methods can also be used in numerical schemes. However, this approach requires an explicit formula for the relaxed variational problem and due to the complexity of the calculations required to derive such an explicit formula, only few examples for quasiconvex envelopes with immediate applications to problems in solid mechanics are known, see, e.g., [22, 34, 41]. Therefore the approach via relaxation needs to be coupled with a mixed analytical-numerical approximation or a full numerical computation of the relaxed energy density. An example for the first approach, a combination of an analytical approximation of the relaxed energy density combined with a finite element simulation can be found in [8, 50, 48]. For the second approach a novel algorithmic approach for the computation of the relaxed energy density with applications to problems in phase transformations and models in plasticity was recently proposed in [26, 27], see also [4, 5] for a different approach. A common theme in all these approaches is that they are based on the computation of an upper bound via optimal laminates, the only exception is [5] where a lower bound is studied.
2. **Notation.** Since the subsequent sections focus on models in \( \mathbb{R}^2 \) we define all notions for this special case. A function \( f : \mathbb{R}^{2 \times 2} \to \mathbb{R} = \mathbb{R} \cup \{ \infty \} \) is said to be rank-one convex, if for all matrices \( F, R \in \mathbb{R}^{2 \times 2} \) with \( \text{rank}(R) = 1 \) the scalar function of one real variable \( t \) given by \( t \mapsto f(F + tR) \) is a convex function in \( t \). The function \( f \) is said to be quasiconvex at \( F \in \mathbb{R}^{2 \times 2} \), if for all Lipschitz functions which vanish on the boundary of the unit cube in \( \mathbb{R}^2 \), i.e., for all \( \phi \in W^{1,\infty}_0((0,1)^2; \mathbb{R}^2) \), the inequality

\[
f(F) \leq \int_{(0,1)^2} f(F + D\phi) \, dx
\]

holds. Finally, \( f \) is said to be polyconvex in \( \mathbb{R}^2 \) if there exists a convex and lower semicontinuous function \( g : \mathbb{R}^5 \to \mathbb{R} \) such that \( f(F) = g(F, \det F) \) holds. If \( f \) fails to be rank-one convex or polyconvex, then the largest (extended-valued) rank-one convex or polyconvex function less than or equal to \( f \) is called the rank-one convex or polyconvex envelope of \( f \) and denoted by \( f^{rc} \) or \( f^{pc} \). Finally \( f^{qc}(F) = \inf \{ \int_{(0,1)^2} f(F + D\phi) \, dx : \phi \in W^{1,\infty}_0((0,1)^2; \mathbb{R}^2) \} \) (2.1) denotes the quasiconvex envelope, see [30, 60] and [23] for a discussion of this definition. It is also referred to as the effective of macroscopic energy density of the system.

3. **Models in crystal plasticity with few active slip system.** In this paper we focus on nonlinear models in crystal plasticity described by effective energy densities \( W \) of the form (1.1) and investigate their macroscopic properties based on an optimization in the underlying microstructures according to (2.1). It turns out that, in general, an explicit characterization of the infimum in (2.1) is not possible. Even in two dimensions and in situations with very restrictive kinematic assumptions only few examples are known for which closed formulas were obtained. In fact, the first results were derived based on the assumption that the elastic energy density is finite if and only if \( F_e \in SO(2) \), that is,

\[
W_e(F_e) = W^{rigid}_e(F_e) = \begin{cases} 
0 & \text{if } F_e \in SO(2), \\
\infty & \text{otherwise},
\end{cases}
\]

and that only one slip system is active, either with no hardening or linear hardening. This means that there exist orthonormal vectors \( s \) and \( m \) such that the plastic part of the deformation is of the form \( F_p = I + \gamma s \otimes m \) with \( \gamma \in \mathbb{R} \) and that the plastic potential and the dissipative term are given by

\[
W_p(F_p) = \frac{h}{2} \gamma^2 \quad \text{and} \quad \text{Diss}(F_p) = \tau |\gamma|
\]

with constants \( h, \tau \geq 0 \). In this case the condensed energy density, which is obtained as in (1.2), becomes

\[
W_{\text{rigid}}(F) = \begin{cases} 
\frac{h}{2} |\gamma|^2 + \tau |\gamma| & \text{if } F = Q(I + \gamma s \otimes m), \ Q \in SO(2), \ \gamma \in \mathbb{R}, \\
\infty & \text{otherwise}.
\end{cases}
\]

Here and below we drop the superscript \text{cond} from the energy densities for notational simplicity. One can easily show that this condensed energy density is finite only on the set

\[
\mathcal{M} = \{ F : \det F = 1, |Fs| = 1 \}.
\]
slip systems

\[ \begin{array}{ccc}
N = 1 & h = 0 & \tau = 0 \\
N = 2 \text{ at } 90^\circ & W^{rc} \neq W^{pc} [18] & W^{rc} = W^{qc} = W^{pc} [28] \\
N = 3 \text{ at } 120^\circ & \text{no results partial results [25]} & \text{no results partial results [25]}
\end{array} \]

Table 1. Relaxation results in the literature with rigid elasticity and plastic energy density proportional to \(|\gamma|^\alpha\), \(\alpha = 1, 2\) and \(h\tau = 0\). In the three cases with \(W^{rc} = W^{qc} = W^{pc}\) an explicit formula is given in the mentioned papers, in the others there are only partial results.

The relaxed energy density turns out to be finite on the larger set [29]

\[ \mathcal{N} = \{ F : \det F = 1, |Fs| \leq 1 \} \].

Indeed, \(\mathcal{N}\) is the quasiconvex hull of \(\mathcal{M}\) (see [30, 60] for the precise definition).

Since the relaxed energy density is finite only on \(\mathcal{N}\) it suffices to find a formula for \(W^{qc}\) on \(\mathcal{N}\), which was accomplished for a model with dissipation only \((h = 0)\) in [29] and for a model with linear hardening only \((\tau = 0)\) in [18]. For future reference, we note that the explicit formula for the quasiconvex envelope of \(W_{\text{rigid}}^{qc}\) with \(\tau = 0\), which we denote by \(W_{\text{rigid},0}^{qc}\) is given by

\[ W_{\text{rigid},0}^{qc}(F) = \begin{cases} 
\frac{h}{2} |Fm|^2 - 1 & \text{if } F \in \mathcal{N}, \\
\infty & \text{otherwise}.
\end{cases} \] (3.3)

This model can be extended to several slip systems assuming infinite cross-hardening so that at each material point only one slip system is active. More precisely, suppose that these slip systems are given by pairs of orthonormal vectors \((s_i, m_i)\) with corresponding plastic deformations \(F_p = I + \gamma s_i \otimes m_i, i = 1, \ldots, N\), and that the condensed energy density is

\[ W^{(N)}(F) = \begin{cases} 
\frac{h}{2} |\gamma|^2 + \tau |\gamma| & \text{if } F = Q(I + \gamma s_i \otimes m_i) \text{ for some } i \in \{1, \ldots, N\}, \\
\infty & \text{otherwise}.
\end{cases} \]

more precisely, we focus on the two situations \(N = 2\) with slip-systems at \(90^\circ\) and \(N = 3\) with slip systems at \(120^\circ\). The available relaxation results in this setting are summarized in Table 1; they all assume that \(h\tau = 0\) [29, 18, 1, 28, 25].

Because of its importance in engineering applications, we briefly review the elastic energy density in [11] with constants \(\mu, \tau, h > 0\),

\[ W_e(F_e) = W_{\text{CHM}}^{qc}(F_e) = \frac{\mu}{2}(|F_e|^2 - 2) + U(\det F_e), \] (3.4)

where \(U\) is a convex and lower semicontinuous function \(U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}\). Hence the total elastoplastic energy density is given, for \(F_p = I + \gamma s \otimes m\), by

\[ \frac{\mu}{2}(|F_e|^2 - 2) + U(\det F_e) + \frac{h}{2} \gamma^2 + \tau |\gamma| \]

which can be rewritten in view of \(F_e = FF_p^{-1}\) and the specific form of \(F_p\) as

\[ \frac{\mu}{2}(|F|^2 - 2 - 2\gamma Fm \cdot Fs + \gamma^2 |Fs|^2) + U(\det F) + \frac{h}{2} \gamma^2 + \tau |\gamma| . \]

This expression is a piecewise smooth function in \(\gamma\). An explicit computation gives for the condensed energy density (defined as in (1.2)) the expression [11],
Figure 1. $W_{\text{CHM}}$ with $\mu = 2$ along the rank-one line (3.6) with $h = 0.1$ and $\tau = 1$ (left panel) and $h = 1$ and $\tau = 0$ (right panel), see also [11, Figure 1] for the plot with $h = 0$.

Formula (6.6)

$$W_{\text{CHM}}(F) = \frac{\mu}{2} |F|^2 - 2 + U(\det F) - \frac{(\mu |F_s \cdot Fm| - \tau)^2}{2(\mu |F_s|^2 + h)}.$$  \hspace{1cm} (3.5)

The crucial observation in [11] is that $W_{\text{CHM}}$ fails to be convex along the rank-one line

$$t \mapsto F(t) = I + \frac{t}{2}(m + s) \otimes (m - s),$$  \hspace{1cm} (3.6)

along which the energy density can be written as

$$t \mapsto \frac{\mu}{2} t^2 - \frac{(\frac{\mu}{2} t^2 - \tau)^2}{2h + \mu(2 - 2t + t^2)}$$

see Fig. 1. Since for finite-valued functions quasiconvexity implies rank-one convexity, $W_{\text{CHM}}$ fails to be quasiconvex. It is an open problem to find the relaxation of $W_{\text{CHM}}$. Numerical results in [3, 8] indicate that at least second-order laminates are necessary for the rank-one convex relaxation while the results in [27] suggest that even third order laminates are required. These complex structures are caused by the interplay between the linear and quadratic term in $|\gamma|$ which favour different microstructures as already observed in [29, 18]. Apart from these numerical results, information about the full relaxation or other qualitative properties of $W_{\text{CHM}}$ is lacking. One exception are the results in [20] which show that $W_{\text{CHM}}$ with hardening $h > 0$ is positive outside of $SO(2)$. Thus the relaxed energy density vanishes exactly on $SO(2)$ and there are no spurious zero-energy states except rigid body motions.

Despite of these intrinsic difficulties, it was possible to find an explicit formula for the relaxation of $W_{\text{CHM}}$ with positive hardening modulus, $h > 0$, and no dissipative terms, $\tau = 0$, see [23] for details.

**Theorem 3.1.** Suppose that $W_{\text{CHM}}$ is given by (3.5) and $W_{\text{CHM}}^\text{qc}$ by (3.4), both with $\tau = 0$. Assume that $U : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is convex and lower semicontinuous, with $\{U < \infty\}$ open or equal to $\{1\}$. Let $f : \mathbb{R}^{2 \times 2} \to \mathbb{R} \cup \{\infty\}$ be defined by

$$f(F) = \min_{F_p \in \mathcal{N}} \left\{W_{\text{CHM}}^\text{qc}(FF_p^{-1}) + \frac{h}{2}(|F_p m|^2 - 1)\right\}.$$ \hspace{1cm} (3.7)

Then $f$ is the quasiconvex envelope of $W_{\text{CHM}},$

$$W_{\text{CHM}}^\text{qc}(F) = f(F).$$

The proof is based on the following lemma, which provides an explicit formula for the minimum in the definition of $f$. We refer to [23] for details. For simplicity we write $a \times b = a^\perp \cdot b = a_1 b_2 - a_2 b_1$ for two vectors $a, b \in \mathbb{R}^2$. 
Lemma 3.2. Let \( W_{\text{CHM}}^\varepsilon \) be as in (3.4), \( f \) as in (3.7). Then
\[
f(F) = U(\det F) + \psi(\sigma(F), \tau(F)) - \frac{h}{2}
\] (3.8)
where \( \psi : [0, \infty)^2 \to (0, \infty) \) is defined by
\[
\Psi(\sigma, \tau) = \begin{cases} 
\tau^2/(2\sigma) + \sigma/2 & \text{if } \tau < \sigma , \\
\tau & \text{if } \sigma \leq \tau ,
\end{cases}
\]
with
\[
\sigma(F) = \mu |Fs|^2 + h \quad \text{and} \quad \tau(F) = \sqrt{\mu^2(Fs \times Fm)^2 + h\mu |Fm|^2}.
\] (3.9)

4. \( \Gamma \)-convergence in elasto-plasticity. In this section we continue the analysis of \( W_{\text{CHM}} \) with \( \tau = 0 \) in the two-dimensional setting. From a physical point of view, the assumption (3.1) that only rigid body motions carry finite elastic energy is a strong simplification. However, this assumption can be justified by means of \( \Gamma \)-convergence [33, 35] if one considers the limit of large elastic constants. Define for \( \varepsilon > 0 \) the elastic energy density by
\[
I_{\varepsilon}^{\text{CDK}}[u] = \begin{cases} 
\int_{\Omega} W_{\varepsilon}^{\text{CDK}}(Du)dx & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^2) , \\
\infty & \text{otherwise}
\end{cases},
\]
and
\[
W_{\varepsilon}^{\text{CDK}}(F) = \inf_{\gamma \in \mathbb{R}} \left\{ \frac{1}{\varepsilon} \text{dist}^2(F(I - \gamma s \otimes m), SO(2)) + \frac{h}{2} \gamma^2 \right\} .
\] (4.1)
In [21] it was shown that the functionals
\[
I_{\varepsilon}^{\text{CDK}}[u] = \begin{cases} 
\int_{\Omega} W_{\varepsilon}^{\text{CDK}}(Du)dx & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^2) , \\
\infty & \text{otherwise}
\end{cases},
\]
converge in the sense of \( \Gamma \)-convergence with respect to the strong topology in \( L^1(\Omega) \) to the functional
\[
I_0[u] = \begin{cases} 
\int_{\Omega} W_{\varepsilon_{\text{rigid},0}}^{\text{rigid}}(Du)dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2) , Du \in \mathcal{N} \text{ a.e. in } \Omega , \\
\infty & \text{otherwise ,}
\end{cases}
\]
where \( \mathcal{N} \) has been defined in (3.2) and \( W_{\varepsilon_{\text{rigid},0}}^{\text{rigid}} \) is the relaxation of the energy density with hardening but no dissipation in (3.3).

As noted in Remark 1 following Theorem 1.1 in [21], the \( \Gamma \)-convergence result holds for elastic energies \( W_{\varepsilon} \) which satisfy an estimate of the type
\[
c \text{dist}^2(F, SO(2)) \leq W_{\varepsilon}(F) \leq C \text{ dist}^2(F, SO(2))
\] (4.2)
for all \( F \in \mathbb{R}^{2\times 2} \), with positive constants \( c, C > 0 \). We shall present below an explicit example of a volumetric energy density \( U \) such that the present model has this property.

Theorem 4.1. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz set, and assume that the elastic stored energy density \( W_{\varepsilon} : \mathbb{R}^{2\times 2} \to \mathbb{R} \) is a continuous function with the growth stated in (4.2). Let \( h > 0 \) and define the condensed energy density
\[
W_{\varepsilon}(F) = \inf_{\gamma \in \mathbb{R}} \left\{ \frac{1}{\varepsilon} W_{\varepsilon}(F(I - \gamma s \otimes m)) + \frac{h}{2} \gamma^2 \right\} .
\] (4.3)
Then the functionals
\[ I_\epsilon[u] = \begin{cases} \int_\Omega W_\epsilon(Du)dx & \text{if } u \in W^{1,1}(\Omega; \mathbb{R}^2), \\ \infty & \text{otherwise} \end{cases} \]
converge, in the sense of \( \Gamma \)-convergence with respect to the strong topology in \( L^1 \), to \( I_0 \) as \( \epsilon \searrow 0 \).

**Proof.** Follows from [21, Th. 1.1 and Rem. 1]. \qed

We remark that the energies in (4.1) have for a suitable choice of \( U \) the same growth as the energy density (3.5) with \( \tau = 0 \), and they fit into the framework of the asymptotic relaxation results in [21]. As a specific example we choose
\[ W_e(F_e) = W_e^{\kappa,\mu}(F_e) = \frac{1}{2} \mu |F_e|^2 + U_{\kappa,\mu}^e(\det F_e) \quad (4.4) \]
where
\[ U_{\kappa,\mu}(t) = \frac{1}{2} (\kappa + \mu) u(t) - \mu t, \quad u(t) = \begin{cases} |t-1|^2 & \text{if } |t-1| \leq \frac{1}{2}, \\ |t-1| - \frac{1}{4} & \text{if } |t-1| > \frac{1}{2}, \end{cases} \]
and \( \kappa > 0, \mu > 0 \) are two parameters. This functional form has been chosen so that it has quadratic growth at infinity, and the same behavior (up to second order) as the one given in [11] close to the minimum. We proceed to show that \( W_e(F_e) = 0 \) if and only if \( F_e \in SO(2) \) and that \( W_e \) has quadratic growth from above and below, in the sense of (4.2).

**Lemma 4.2.** Assume \( \kappa, \mu > 0 \), and let \( W_e \) be as defined in (4.4). Then there are \( c, C > 0 \) such that (4.2) holds.

**Proof.** We divide the proof into three steps.

**Step 1. Preliminary estimates.** Any matrix \( F \in \mathbb{R}^{2 \times 2} \) can be decomposed into a conformal and an anticonformal part,
\[ F^+ = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad F^- = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}, \]
with
\[ a = \frac{F_{11} + F_{22}}{2}, \quad c = \frac{F_{11} - F_{22}}{2}, \quad b = \frac{-F_{12} + F_{21}}{2}, \quad d = \frac{F_{12} + F_{21}}{2}. \]
Since this decomposition is orthogonal in the Euclidean inner product in \( \mathbb{R}^{2 \times 2} \), one can compute
\[ \text{dist}^2(F, SO(2)) = \min_{Q \in SO(2)} |F - Q|^2 = \frac{1}{2} \left( |F^+|^2 + |F^-|^2 \right)^2. \quad (4.5) \]
We observe that \( |F|^2 = |F^+|^2 + |F^-|^2 \) and
\[ 2 \det F = 2(a + c)(a - c) - 2(d + b)(d - b) = |F^+|^2 - |F^-|^2. \quad (4.6) \]
This immediately implies that \( 2 \det F \leq |F|^2 \). We compute
\[ W_e^{\kappa,\mu}(F) = \frac{1}{2} \mu |F|^2 - 2 \det F + \frac{1}{2} \kappa \mu u(\det F) \]
\[ = \mu |F^-|^2 + \frac{1}{2} (\kappa + \mu) u(\det F). \]
Both terms are nonnegative, and therefore
\[
\min\{\kappa, \mu\} W^{1, 1}_e \leq W^{e, e}_e \leq \max\{\kappa, \mu\} W^{1, 1}_e.
\]
For the rest of the proof we shall assume \( \kappa = \mu = 1 \) and write \( W_e \) for \( W^{1, 1}_e \).

**Step 2. Lower bound in (4.2).** We first show that there is \( x \) such that
\[
\min\{\kappa, \mu\} W^{1, 1}_e \leq W^{e, e}_e \leq \max\{\kappa, \mu\} W^{1, 1}_e.
\]
In this proof we write for brevity \( x \) and \( \kappa \).

We compute, using (4.6),
\[
W_e(F) = y^2 + (\det F - 1)^2.
\]
For the rest of the proof we shall assume \( \kappa \).

Both terms are nonnegative, and therefore \( \min\{\kappa, \mu\} W^{1, 1}_e \leq W^{e, e}_e \leq \max\{\kappa, \mu\} W^{1, 1}_e \).

**Step 3. Upper bound in (4.2).** We now turn to the second inequality in (4.2). Let \( d = \text{dist}(F, SO(2)) \). We fix \( Q \in SO(2) \) such that \( d = |F - Q| \) and...
define $\hat{F} = Q^T F$, so that $W_{c}(\hat{F}) = W_{c}(F)$ and $|\hat{F} - I| = d$. We estimate with

$$\hat{F}^+ - I = (\hat{F} - I)^+$$

$$|\hat{F}^+| \leq |I| + |\hat{F} - I| \leq \sqrt{2} + d$$

and with $\hat{F}^- = (\hat{F} - I)^-$

$$|\hat{F}^-| \leq |\hat{F} - I| \leq d.$$ 

Since $\text{cof} = D \det$ is a linear function,

$$|\det \hat{F} - 1| \leq \max_{G \in [I, \hat{F}]} |\text{cof} G||\hat{F} - I| \leq d(\sqrt{2} + d) \quad (4.9)$$

where we used that $|\text{cof} G|=|G| \leq |I| + |\hat{F} - I|$ for all $G \in [I, \hat{F}]$. Here $[I, \hat{F}]$ denotes the segment with endpoints $I$ and $\hat{F}$.

If $d \leq 1/4$ then $\sqrt{2} + d \leq 2$, which in particular implies $|\det \hat{F} - 1| \leq 1/2$ and therefore $\det \hat{F} \in [1/2, 3/2]$. Then by (4.7)

$$W_{c}(F) = W_{c}(\hat{F}) = |\hat{F}^-|^2 + (\det \hat{F} - 1)^2 \leq d^2 + 4d^2 = 5d^2.$$ 

If instead $d \geq 1/4$ then we use $u(t) \leq |t - 1|$ and (4.6) to obtain

$$W_{c}(F) = W_{c}(\hat{F}) \leq \frac{1}{2} |\hat{F}^-|^2 - \det \hat{F} + |\det \hat{F} - 1|
\leq |\hat{F}^-|^2 + |\det \hat{F} - 1| \leq d^2 + d(\sqrt{2} + d) \leq 8d^2.$$ 

We conclude that the upper bound in (4.2) holds with $C = 8 \max\{\kappa, \mu\}$. \hfill \Box

5. **Pointwise and variational convergence to the elastically rigid case.** In this section we address the approximation of the rigid-elastic energy density of (3.1) by a sequence of finite-valued elastic energies, of the type introduced in (3.4) and discussed in the previous section. We formulate these results for a sequence of energies $(W_j)_{j \in \mathbb{N}}$ in a slightly more general framework that allows for different elastic moduli and not only for a rescaling with $\epsilon$ in the elastic part of the energy density. As in the previous section we focus on the case of one active slip system.

Specifically, we consider sequences $\mu_j \to \infty, \kappa_j \to \infty$ and define

$$W_j(F) = \min_{\gamma \in \mathbb{R}} \left[ \frac{1}{2} \mu_j |F(I - \gamma s \otimes m)|^2 + \frac{h}{2} \gamma^2 \right] + U_j(\det F) \quad (5.1)$$

where

$$U_j(t) = \frac{1}{2} (\kappa_j + \mu_j) u(t) - \mu_j t, \quad u(t) = \begin{cases} |t - 1|^2 & \text{if } |t - 1| \leq \frac{1}{2} \\ |t - 1| - \frac{1}{4} & \text{if } |t - 1| > \frac{1}{2} \end{cases}$$

We shall show that the energies $W_j$ converge, both before and after relaxation, to the rigid case. The results are summarized in Figure 2. We recall that explicit formulas for $W_{rc}$ and $W_{rigid,0}$ were given in Theorem 3.1 and in (3.3), respectively.

**Theorem 5.1.** If $\mu_j \to \infty$ and $\kappa_j \to \infty$ then for any $F \in \mathbb{R}^{2 \times 2}$ one has

$$\lim_{j \to \infty} W_j(F) = W_{\text{rigid,0}}(F) \quad (5.2)$$

and

$$\lim_{j \to \infty} W_{rc,j}(F) = W_{rc,0}(F). \quad (5.3)$$
Proof. Fix $F \in \mathbb{R}^{2 \times 2}$. By growth and continuity, for any $j$ there is $\gamma_j$ such that

$$W_j(F) = \frac{1}{2} \mu_j |F|^2 - 2 \mu_j \det F + \frac{1}{2} (\kappa_j + \mu_j) u(\det F) + \frac{h}{2} \gamma_j^2,$$

where $F_j = F(I - \gamma_j s \otimes m)$. We observe that $\det F = \det F_j$. Since $|F_j|^2 \geq 2 \det F_j$ for any matrix $F_j$, each of the three terms is nonnegative.

We first assume that $\lim_{j \to \infty} \gamma_j = 1$, this implies immediately $\gamma_j \to \gamma_s$, after extracting a subsequence we can assume $\gamma_j \to \gamma_s$. Correspondingly the $F_j$ converge to $F^*_e = F(I - \gamma_s s \otimes m)$. By a similar argument, $\lim_{j \to \infty} W_j(F) < \infty$ implies

$$|F^*_e|^2 - 2 \det F^*_e = \lim_{j \to \infty} (|F^*_e|^2 - 2 \det F^*_e) = 0.$$  \quad (5.4)

Since $\det F^*_e = 1$, this implies $F^*_e \in SO(2)$. In particular, we have shown that $\lim_{j \to \infty} W_j(F) = \infty$ unless $F = F^*_e(I + \gamma_s s \otimes m)$ for some $F^*_e \in SO(2)$, $\gamma_s \in \mathbb{R}$.

Assume now $F = F^*_e(I + \gamma_s s \otimes m)$, $F^*_e \in SO(2)$, so that $W_{\text{rigid,0}}(F) = \frac{1}{2} h \gamma_s^2$. Taking $\gamma = \gamma_s$ in the definition (5.1) of $W_j$ yields $W_j(F) \leq \frac{1}{2} \gamma_s^2$ for any $j$. At the same time,

$$\lim_{j \to \infty} W_j(F) \geq \lim_{j \to \infty} \frac{h}{2} \gamma_j^2 = \frac{h}{2} \gamma_s^2.$$

This concludes the proof of the pointwise convergence of $W_j$ to $W_{\text{rigid,0}}$.

For the second part, we again fix $F$ and observe that by Theorem 3.1 we have

$$W_{\text{qc}}(F) = \min_{F_p \in \mathcal{N}} \left( \frac{1}{2} \mu_j |F_F^{-1}|^2 - \mu_j \det F + \frac{1}{2} (\kappa_j + \mu_j) u(\det F) + \frac{h}{2} (|F_p m|^2 - 1) \right).$$  \quad (5.5)

As above, we denote by $F^*_e \in \mathcal{N}$ a minimizer and define $F^*_e = F(F_p^*)^{-1}$, so that

$$W_{\text{qc}}(F) = \frac{1}{2} \mu_j |F^*_e|^2 - 2 \mu_j \det F^*_e + \frac{1}{2} (\kappa_j + \mu_j) u(\det F^*_e) + \frac{h}{2} (|F_p^* m|^2 - 1).$$

Each of the three terms is nonnegative, and as above we start from the case $\lim_{j \to \infty} W_{\text{qc}}(F) < \infty$. This implies that $\det F = 1$ and that $F^*_e m$ is bounded.

From the definition of $\mathcal{N}$ we obtain $|F^*_p|^2 \leq 1$ and therefore, after passing to a subsequence, we have $F^*_p \to F^*_e$ and $F^*_p \to F^*_e$, with $F = F^*_e F^*_p$, $F^*_p \in \mathcal{N}$. By the same argument as in (5.4) we deduce $F^*_e \in SO(2)$. As above, we conclude that the limit is infinite unless $F = F^*_e F^*_p$ for some $F^*_e \in SO(2)$, $F^*_p \in \mathcal{N}$.
Assume now that \( F \) has this form. As above, inserting \( F_p = F_p^* \) in (5.5) one obtains \( W_{qc}^{rigid,0}(F) \leq W_{qc}^{rigid,0}(F) \). At the same time,
\[
\liminf_{j \to \infty} W_{qc}^{rigid,0}(F) \geq \liminf_{j \to \infty} \frac{h}{2}(|F_j|^2 - 1) = \frac{h}{2}(|F_p^*|^2 - 1) = W_{qc}^{rigid,0}(F).
\]
This concludes the proof of the lemma.

We now turn to the corresponding functionals. We consider the sequence of functionals
\[
E_j[u] = \int_{\Omega} W_j(Du) \, dx
\]
and the ones constructed with the quasiconvex envelope of \( W_j \),
\[
E_j^*[u] = \int_{\Omega} W_{qc}^j(Du) \, dx.
\]
We recall that, if \( W_j \) had \( p \)-growth, \( p \in (1, \infty) \), then standard results on the relaxation of integral functionals \([30, 60]\) would show that \( E_j^* \) is the relaxation of \( E_j \). For the present case with mixed growth a relaxation result is, to the best of our knowledge, still unknown.

We then define the limiting functionals,
\[
E_{rigid,0}[u] = \int_{\Omega} W_{rigid,0}(Du) \, dx
\]
and
\[
E_{rigid,0}^*[u] = \int_{\Omega} W_{qc}^{rigid,0}(Du) \, dx.
\]
Since these integrands are extended valued, standard relaxation results do not apply and in particular it is not known if \( E_{rigid,0}^* \) is the relaxation of \( E_{rigid,0} \). All these functionals are interpreted as \( \infty \) if \( u \) is not weakly differentiable or the integrand is not integrable.

We stress that there is no reason to expect that the relaxation of the pointwise limit coincides with the pointwise limit of the relaxation. This is not even true for functions of one real variable, where quasiconvexity is replaced by convexity. Indeed, it suffices to consider a sequence
\[
f_j(t) = \min\{t^2, j^3(t - 1 - \frac{1}{j})^2\}.
\]
Since \( f_j(0) = f_j(1 + 1/j) = 0 \) one obtains that the convex envelope of \( f_j \) vanishes on \([0, 1 + 1/j]\), which implies that the pointwise limit of the convex envelope vanishes on \([0, 1]\). At the same time, for any \( t \in \mathbb{R} \) we have \( f_j(t) \to t^2 \), therefore the convex envelope of the limit vanishes only in 0.

We recall the following abstract result from \([31, Rem. 5.5]\). If \( (F_j) \) is an increasing sequence of lower semicontinuous functionals which converge pointwise to a functional \( F \), then \( F \) is lower semicontinuous and \( (F_j) \) \( \Gamma \)-converges to \( F \). This does not apply to the sequence \( E_j \), since these functionals are not lower semicontinuous. The functionals \( E_j^* \) could, as far as we know, be lower semicontinuous on \( W^{1,1}(\Omega; \mathbb{R}^2) \), but this is not proven yet. Since they have, in some directions, linear growth, coercivity only holds with respect to a subset of \( BV(\Omega; \mathbb{R}^2) \). We do not know if there is an extension of \( E_j^* \) to \( BV(\Omega; \mathbb{R}^2) \) which is lower semicontinuous with respect to the weak topology of this space.

We obtain the following results, which are summarized in Figure 2.
**Theorem 5.2.** If \( \kappa_j \to \infty \) and \( \mu_j \to \infty \), then \( E_j \to E_{\text{rigid},0} \) and \( E_{j}^{\ast} \to E_{\ast \text{rigid},0}^{\ast} \) pointwise in \( L^1 \), whereas \( E_j \to E_{\ast \text{rigid},0}^{\ast} \) in the sense of \( \Gamma \)-convergence with respect to the strong topology in \( L^1 \).

**Proof.** The pointwise convergences follow from the pointwise convergence of the integrands in Theorem 5.1, together with the fact that for any fixed \( F \in \mathbb{R}^{2 \times 2} \) the sequences \( j \mapsto W_{j}(F) \) and \( j \mapsto W_{j}^{qc}(F) \) are nondecreasing.

If \( \kappa_j = \alpha \mu_j \) for some \( \alpha > 0 \) then the last assertion follows from [21], as stated in Theorem 4.1, using the quadratic growth of the elastic energy density proven in Lemma 4.2. The general case can be obtained using the bound

\[
\min\{\kappa_j, \mu_j\} W_{\epsilon_{j}}^{1,1}(F) \leq W_{\epsilon_{j}}^{\kappa_j,\mu_j}(F) \leq \max\{\kappa_j, \mu_j\} W_{\epsilon_{j}}^{1,1}(F),
\]

see the proof of Lemma 4.2, and monotonicity in the definition of the condensed energy density and in the \( \Gamma \)-limit, see [31, Prop. 6.7]. In fact, one applies Theorem 4.1 to

\[
E_{j}^{\prime}[u] = \int_{\Omega} \inf_{\gamma \in \mathbb{R}^{2}} \left[ \frac{1}{\epsilon_{j}} W_{\epsilon_{j}}^{1,1}(Du(I - \gamma s \otimes m)) + \frac{h}{2} \gamma^{2} \right] dx, \quad (\epsilon_{j}^{\prime})^{-1} = \min\{\kappa_j, \mu_j\},
\]

\[
E_{j}^{\prime}[u] = \int_{\Omega} \inf_{\gamma \in \mathbb{R}^{2}} \left[ \frac{1}{\epsilon_{j}} W_{\epsilon_{j}}^{1,1}(Du(I - \gamma s \otimes m)) + \frac{h}{2} \gamma^{2} \right] dx, \quad (\epsilon_{j}^{\prime})^{-1} = \min\{\kappa_j, \mu_j\}
\]

and obtains

\[
E_{\ast \text{rigid},0}^{\ast} = \Gamma \text{-lim inf} \sup_{j \to \infty} E_{j}^{\prime} \leq \Gamma \text{-lim inf} \inf_{j \to \infty} E_{j} \leq \Gamma \text{-lim sup} \sup_{j \to \infty} E_{j}^{\prime} = E_{\ast \text{rigid},0}^{\ast}.
\]

Thus \( E_j \) does converge in the sense of \( \Gamma \)-convergence with respect to strong convergence in \( L^1 \) and the \( \Gamma \)-limit coincides with \( E_{\ast \text{rigid},0}^{\ast} \).

**6. Conclusions.** We have reviewed recent progress on the variational modeling of single-slip elastoplastic deformation of crystals with finite kinematics, and analyzed the transition between models with asymptotically large elastic constants and models with ideally rigid elasticity.

The computation of quasiconvex envelopes becomes substantially simpler in the limit of rigid elasticity, and indeed the study of models with rigid elasticity has been an important method to gain understanding into microstructure formation during elastoplastic deformation. A general theory of relaxation, relating the relaxation of the original problem to a new integral functional written in terms of a convex envelope, is only available for energies which are significantly more regular. Numerical work also typically focuses on continuous energy densities.

We discussed here if the relaxation process, or the process of passing from an energy density to its quasiconvex envelope, commutes with the limiting process of passing to the limit of rigid elasticity. We could show that, at least for one specific model of mechanical interest, the two steps commute for the energy density. For the integral functionals only partial results are available, due in particular to the lack of a general integral representation result in the extended-valued case. Extension of the current results to situations with multiple slip systems remains a direction for possible future work.
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