DEFINITE AND INDEFINITE UNITARY
TIME REPRESENTATIONS
FOR HILBERT AND NON-HILBERT SPACES

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Abstract
Stable states (particles), ghosts and unstable states (particles) are discussed with respect to the time representations involved, their unitary groups and the induced Hilbert spaces. Unstable particles with their decay channels are treated as higher dimensional probability collectives with nonabelian probability groups \( U(n) \) generalizing the individual abelian \( U(1) \)-normalization for stable states (particles).

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\(^1\) Talk presented on the ‘Workshop on Resonances and Time Asymmetric Quantum Theory, Jaca(Spain), 30 May to 4 June (2001)
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1 Dynamics, Unitarity and Time Reflection

A physical dynamics as a complex representation of time defines itself an inner product, sometimes a Hilbert space product, wherein a quantum dynamics is experimentally interpreted with transition and probability amplitudes etc. In this section, the mathematical background structure\(^3\) is given.

As model for time the abelian totally ordered real numbers are used - either multiplicatively, called time group

\[
D(1) = \{e^t \mid t \in \mathbb{R}\}
\]

or additively, called time Lie algebra\(^3\) with the time translations

\[
\mathbb{R} = \log D(1) = \{t\}
\]

Obviously, as Lie groups \((D(1), \cdot)\) and \((\mathbb{R}, +)\) as isomorphic.

A quantum dynamics is a representation of time as group in the automorphisms \(\text{GL}(V)\) and as Lie algebra in the endomorphisms \(\text{AL}(V)\) of a complex vector space \(V\)

- group: \(D : D(1) \longrightarrow \text{GL}(V), e^t \longmapsto D(t), \left\{\begin{array}{l}
D(0) = \text{id}_V \\
D(t + s) = D(t) \circ D(s)
\end{array}\right.\)
- Lie algebra: \(D : \mathbb{R} \longrightarrow \text{AL}(V), t \longmapsto D(t) = tD(1)\)

The represented basis of the time translations is - up to \(i\) - the Hamiltonian

\[
D(1) = iH
\]

States (bound states, scattering states, particles) have to be eigenvectors under time action.

The complex representation space has to come with a conjugation in order to represent the realness of the Lie structure with concepts like hermitian (real)-antihermitian (imaginary) etc. A conjugation of a complex vector space \(V\) is an antilinear isomorphism to its dual space \(V^T\) (vector space with the linear \(V\)-forms)

\[
V \leftrightarrow V^T, \quad v \leftrightarrow v^*, \quad v^{**} = v
\]

Dirac notation: \(v = |v\rangle, \quad v^* = \langle v|\)

For spaces with dimension \(n \geq 2\) there is not only the naive number conjugation (canonical conjugation \(\alpha \leftrightarrow \overline{\alpha}\)). There may exist also more than one conjugation for a vector space, e.g. the conjugation connecting creation and annihilation operators and the conjugation connecting particles and antiparticles.

With a conjugation the \(V\)-endomorphisms \(f \in \text{AL}(V)\) can be conjugated too by using the transposed endomorphisms \(f^T : V^T \longrightarrow V^T\)

\[
\text{AL}(V) \leftrightarrow \text{AL}(V) \text{ with } [f : V \longrightarrow V] \leftrightarrow [f^* : V \longrightarrow V^T \overset{f^T}{\longrightarrow} V^T \overset{*}{\longrightarrow} V]
\]

\(^3\)The Lie algebra for a Lie group \(G\) will be denoted by \(\log G\).
The realness of the time group is implemented by the hermiticity of the Hamiltonian - with respect to the conjugation $*$

$$H = H^*$$

Therewith, the generator $iH$ for the represented real time translations $\mathbb{R}$ is antihermitian and the represented time group unitary - always with respect to the conjugation $^*$

$$D(1)^* = D(-1) = -D(1) \quad -\text{antihermitian for time translations}$$

$$D(t)^* = D(-t) = D(t)^{-1} \quad -\text{unitary for group representation}$$

The induced endomorphism conjugation implements the time reflection

$$t \mapsto -t$$

$$D(t), D(t) \mapsto D(t)^*, D(t)^*$$

With the dual product (bilinear) for the vector space $V$ and its linear forms

$$V^T \times V \longrightarrow \mathbb{C}, \quad (\omega, w) \mapsto \langle \omega, w \rangle$$

a conjugation is equivalent to an inner product (a nonsingular sesquilinear form)

$$V \times V \longrightarrow \mathbb{C}, \quad (v, w) \mapsto \langle v | w \rangle = \langle v^*, w \rangle$$

The unitary group, characterizing the conjugation $*$, is the invariance group of the induced inner product

$$\mathbf{U}(V, *) = \{ u \in \mathbf{GL}(V) \mid \langle u.v | u.w \rangle = \langle v | w \rangle \text{ for all } v, w \in V \}$$

This defines the signature of the unitarity and of the conjugation

$$V \cong \mathbb{C}^n, \quad \langle | \rangle \cong \begin{pmatrix} 1_{n+} & 0 \\ 0 & -1_{n-} \end{pmatrix}, \quad n_+ + n_- = n, \quad \mathbf{U}(V,*) = \mathbf{U}(n_+, n_-)$$

There are as many different types of conjugations as there are signatures - for $n = 1$ only $\mathbf{U}(1)$, for $n = 2$ one has $\mathbf{U}(2)$ and $\mathbf{U}(1, 1)$ etc. With the exception of a Euclidean conjugation $\mathbf{U}(n)$, denoted with the five cornered star $\star$, where one has a scalar product and a Hilbert space structure for the vector space $V$, all conjugations and associated inner products are indefinite. In the following time representations with both definite and indefinite conjugations will be considered.

To become familiar with indefinite conjugations a $\mathbf{U}(1, 1)$ inner product is considered

for $\mathbf{U}(1, 1)$: \quad $\langle | \rangle \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Us usual, the inner product matrix depends on the vector space basis: A diagonal matrix with the explicit signature arises for Sylvester bases whereas neutral,
not orthogonal pairs with trivial norm show up in Witt bases. In the example above Sylvester and Witt bases are related to each other with the automorphisms \( w = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \). The inner product defines the \( U(1, 1) \)-conjugation, denoted with \( \times \), which - in the basis with the skew-diagonal matrix for the inner product - interchanges the conjugated diagonal elements in contrast to the familiar \( U(2) \)-conjugation \( \star \) (number conjugation and transposition) which interchanges the conjugated skew-diagonal elements

\[
\begin{align*}
U(2) &= \star : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\star = \begin{pmatrix} \overline{\delta} & \overline{\beta} \\ \overline{\gamma} & \overline{\alpha} \end{pmatrix} \\
U(1, 1) &= \times : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\times = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^\star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\delta} & \overline{\beta} \\ \overline{\gamma} & \overline{\alpha} \end{pmatrix}
\end{align*}
\]

The product of two conjugations gives a vector space automorphism, here \( \times \circ \star = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Obviously, \( U(2) \) and \( U(1, 1) \)-hermiticity or unitarity are different.

## 2 Three Characteristic Time Representations

Complex representations of the time group \( D(1) \) and the time translations \( \mathbb{R} \) are in unitary groups and their Lie algebras. The following three non-decomposable types\(^{[2, 8]}\) are characteristic.

Irreducible representations of time with reflection are definite unitary in \( U(1) \)

\[
\begin{align*}
D(1) \ni e^t &\mapsto e^{iEt} \in U(1) = \exp i\mathbb{R} \\
\mathbb{R} \ni t &\mapsto iEt \in \log U(1) = i\mathbb{R}
\end{align*}
\]

with eigenvalue \( iE \in i\mathbb{R} \)

These time representations are not faithful. They are used for stable states (particles). As well known and repeated in the next section, the definite unitary group leads to a Hilbert space with the quantum characteristic probability amplitudes.

The smallest faithful nondecomposable time representations are indefinite unitary, they are in \( U(1, 1) \)

\[
\begin{align*}
D(1) \ni e^t &\mapsto e^{iEt}\begin{pmatrix} 1 & i\nu t \\ 0 & 1 \end{pmatrix} \in U(1, 1), \ 0 \neq \nu \in \mathbb{R} \\
\mathbb{R} \ni t &\mapsto i\begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix}t \in \log U(1, 1)
\end{align*}
\]

with \( 2 \times 2 \)-Hamilton-matrix: \( H = \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix} \)

with eigenvalue \( iE \in i\mathbb{R} \)

These triangular reducible, but nondecomposable time representations with Jordan matrices are used for ghosts, i.e. interactions without asymptotic states, e.g. for the non-photonic degrees of freedom in gauge fields, i.e. the Coulomb force and the gauge degree of freedom. It will be discussed below in more detail.

Irreducible faithful time representations have no time reflection - they are valid either only for the future or for the past. The representations of the
future cone
\[
\begin{align*}
t \geq 0 : & \quad e^t \mapsto e^{i(E + i\Gamma/2)t} \\
& \in \text{GL}(\mathbb{C}) = \text{U}(1) \times \text{D}(1), \quad \Gamma > 0
\end{align*}
\]
with eigenvalue
\[
\begin{align*}
iE - \frac{\Gamma}{2} \in \mathbb{C}
\end{align*}
\]
with the eigenvalues having a nontrivial real part (width) are used for decaying states, e.g. for unstable particles. The corresponding decomposable indefinite unitary representations are
\[
\begin{align*}
& \text{D}(1) \ni e^t \mapsto e^{iEt} \left( \begin{array}{cc}
\cosh \frac{\Gamma}{2}t & \sinh \frac{\Gamma}{2}t \\
\sinh \frac{\Gamma}{2}t & \cosh \frac{\Gamma}{2}t
\end{array} \right) \\
& \in \text{U}(1, 1)
\end{align*}
\]
\[
\begin{align*}
& \text{I R} \ni t \mapsto \left( \begin{array}{cc}
_i(E + i\frac{\Gamma}{2}) & 0 \\
0 & _i(E - i\frac{\Gamma}{2})
\end{array} \right) t \\
& \sim \left( \begin{array}{cc}
iE & \frac{\Gamma}{2} \\
\frac{\Gamma}{2} & iE
\end{array} \right) t \\
& \in \log \text{U}(1, 1)
\end{align*}
\]

3 Stable States and Particles

Quantum probability as used with the Hilbert space formulation of quantum mechanics is induced by irreducible time representations in \(\text{U}(1)\). The well known construction from \(\text{U}(1)\)-conjugation to Fock-Hilbert space is shortly reviewed.

The harmonic Bose and Fermi oscillator give the quantum representations of the irreducible time representation, starting from representations
\[
\begin{align*}
t \mapsto \pm iEt, \quad e^t \mapsto e^{\pm iEt} \quad \text{with } E \in \text{IR}
\end{align*}
\]
on 1-dimensional dual vector spaces with dual bases \((u, u^*)\), later called creation and annihilation operator, related to each other by the \(\text{U}(1)\)-conjugation
\[
\text{U}(1) : \quad V = \mathbb{C}u \leftrightarrow V^T = \mathbb{C}u^*
\]
\[
\begin{align*}
V \times V \rightarrow \mathbb{C}, \quad \langle u|u \rangle = \langle u^*, u \rangle = 1
\end{align*}
\]
The time translation generator is the basic space identity \(iE \text{id}_V\), which can be written in the dual basic vectors (creation-annihilation operators) as tensor product
\[
iH = iE \text{id}_V = iE u \otimes u^* = iE|u\rangle\langle u|
\]

The noncommutative quantum algebra \(\mathbb{C}^2\) of the direct sum space \(V \oplus V^T \cong \mathbb{C}^2\) - a duality induced quotient structure of the multilinear tensor algebra of \(V \otimes V^T\) - contains as elements all complex polynomials \(\mathbb{C}[u, u^*]\), modulo the duality induced (anti)commutators with the notation \([a, b]_\epsilon = ab + \epsilon ba\). It is finite dimensional for Fermi (Pauli principle) and countably infinite dimensional for Bose
\[
\epsilon = +1 \quad (\text{Fermi}) \quad \epsilon = -1 \quad (\text{Bose})
\]
\[
\begin{align*}
\dim_{\text{U}} Q_\epsilon(\mathbb{C}^2) = \begin{cases} 
4, & (\text{Fermi}) \\
\aleph_0, & (\text{Bose})
\end{cases}
\end{align*}
\]
Therein the Hamiltonian for the $U(1)$-time representation above is implemented with the quantization opposite commutator - its adjoint action gives 
the equations of motion
\[ H = E \frac{[u_0, u^*]}{2} \in Q_\epsilon(\mathbb{C}^2) \Rightarrow \begin{cases} [iH, u] = iEu \\ [iH, u^*] = -iEu^* \end{cases} \]

The Hilbert space is constructed from the quantum algebra by canonical extension of the scalar product $(u^*, u) = 1$ for the conjugation group $U(1)$ on the vector space $V$ to the quantum algebra as the inner product
\[ \langle u^* | u \rangle = \langle u^* | u \rangle = 1 \quad \text{for } k = 0, 1, \ldots \]
\[ \langle (u^*)^k(u) \rangle = 0 \text{ for } k \neq l \]
The inner product is positive semidefinite $\langle a | a \rangle \geq 0$. It has the annihilation left ideal in the quantum algebra
\[ \{ n \in Q_\epsilon(\mathbb{C}^2) | \langle a | n \rangle = 0 \text{ for all } a \in Q_\epsilon(\mathbb{C}^2) \} = Q_\epsilon(\mathbb{C}^2)u^* \]
The classes of the quantum algebra with respect to the annihilation left ideal
\[ Q_\epsilon(\mathbb{C}^2) \longrightarrow FOCK_\epsilon(\mathbb{C}^2), \quad a \mapsto |a\rangle = a + Q_\epsilon(\mathbb{C}^2)u^* \]
constitute a complex vector space with definite scalar product, the Fock space, 2-dimensional for Fermi and $\aleph_0$-dimensional for Bose

\[ FOCK_\epsilon(\mathbb{C}^2) = \frac{Q_\epsilon(\mathbb{C}^2)}{Q_\epsilon(\mathbb{C}^2)u^*} \approx \begin{cases} \mathbb{C}^2, & (\text{Fermi}) \\ \mathbb{C}^{\aleph_0}, & (\text{Bose}) \end{cases} \]
\[ FOCK_\epsilon(\mathbb{C}^2) \times FOCK_\epsilon(\mathbb{C}^2), \quad \langle a | b \rangle = \langle a^* b \rangle, \quad \langle a | a \rangle = 0 \iff |a\rangle = 0 \]
The Fock space can be spanned by the energy eigenvectors of the $U(1)$-time representation as implemented in the quantum algebra

\[ \text{basis of } FOCK_\epsilon(\mathbb{C}^2) : \quad \{|k\rangle = \frac{(u_k)}{\sqrt{k!}} + Q_\epsilon(\mathbb{C}^2)u^* \mid \begin{cases} k = 0, 1 \} & (\text{Fermi}) \\ k = 0, 1, 2 \ldots \} & (\text{Bose}) \end{cases} \]
\[ FOCK_\epsilon(\mathbb{C}^2) \times FOCK_\epsilon(\mathbb{C}^2) \longrightarrow \mathbb{C}, \quad \langle k | l \rangle = \delta_{kl} \]

Its Cauchy completion defines a Hilbert space.

In quantum mechanics, the position representation for the Bose case
\[ x = u^* \frac{u}{\sqrt{2}}, \quad ip = \frac{d}{dx} = u^* \frac{u}{\sqrt{2}}, \quad H = E \frac{[u_0, u^*]}{2} = E \frac{\psi^2 + x^2}{2} \]
gives the square integrable functions $FOCK_\epsilon(\mathbb{C}^2) \cong L^2_{dx}(\mathbb{R}, \mathbb{C})$ with the Hermite polynomials $H_k$
\[ \frac{u^k}{\sqrt{k!}} |0\rangle = |k\rangle \cong \psi_k : \quad \begin{cases} u^* |0\rangle = 0 \Rightarrow (x + \frac{d}{dx}) \psi_0(x) = 0 \\ \psi_k(x) = \frac{1}{\sqrt{k!}} \binom{x}{\frac{k}{2}} \psi_0(x) = \frac{1}{\sqrt{2^{2k}k!\pi}} e^{x^2} \left( -\frac{d}{dx} \right)^k e^{-x^2} \\ = \frac{1}{\sqrt{2^{2k}k!\pi}} e^{-\frac{x^2}{2}} H_k(x) \end{cases} \]
All particle quantum fields are built with harmonic oscillators where the
creation and annihilation operators are indexed with momenta \( \vec{q} \in \mathbb{R}^3 \)

\[
V_{\vec{q}} = \Phi u(\vec{q}) \leftrightarrow V_{\vec{q}}^T = \Phi u^*(\vec{q}) \quad \text{with} \quad \begin{cases} [u^*(\vec{p}), u(\vec{q})]_\epsilon = (2\pi)^3 \delta(\vec{q} - \vec{p}) \\ [u(\vec{p}), u(\vec{q})]_\epsilon = 0 \\ [u^*(\vec{p}), u^*(\vec{q})]_\epsilon = 0 \\ \langle u^*(\vec{p})u(\vec{q}) \rangle = (2\pi)^3 \delta(\vec{q} - \vec{p}) \end{cases}
\]
e.g. for a stable spinless \( \pi^0 \) with a Lorentz scalar field \( \Phi \), for a stable spin 1 \( Z^0 \) with a Lorentz vector field \( Z \) or for the spin \( \frac{1}{2} \) electron-positron with particles \((u, u^*)\) and antiparticles \((a, a^*)\) in the left and right handed contributions with a Dirac field \( \Psi = (\gamma^A, \gamma^4) \) - all given by direct integrals \( \int^\oplus \) over the \( \vec{q} \)-indexed subspaces

\[
\Phi(x) = \int^\oplus \frac{d^3q}{(2\pi)^3} \frac{u(\vec{q})e^{iqx} + u^*(\vec{q})e^{-iqx}}{\sqrt{2}}
\]
\[
Z^j(x) = \int^\oplus \frac{d^3q}{(2\pi)^3} \Lambda(\frac{q}{m})_j^i a \frac{u^*(\vec{q})e^{iqx} + u(\vec{q})e^{-iqx}}{\sqrt{2}}, \quad \begin{cases} j = 0, 1, 2, 3 \\ a = 1, 2, 3 \end{cases}
\]
\[
r^A(x) = \int^\oplus \frac{d^3q}{(2\pi)^3} s(\frac{q}{m})^A_\alpha a \frac{u^*(\vec{q})e^{iqx} + u(\vec{q})e^{-iqx}}{\sqrt{2}}, \quad \begin{cases} A, \hat{A} = 1, 2 \\ \alpha = 1, 2 \end{cases}
\]
The nonscalar fields involve the corresponding representations of the boosts \( SO_0(1, 3)/SO(3) \cong SL(\mathbb{C}^2)/SU(2) \) - here the Weyl and vector representations \( \{s(\frac{q}{m}), \Lambda(\frac{q}{m})\} \).

4 Ghosts without Particles

Nondiagonalizable time representation in the indefinite unitary group \( U(1, 1) \) are used for the nonparticle ghost degrees of freedom in gauge and Fadeev-Popov fields. The indefinite unitary time representation will be considered, first in an algebraic matrix model, then in the associated quantum algebras and, finally, in the relativistic gauge fields.

4.1 Indefinite Unitarity for Ghosts

The dynamics of a Newtonian free mass point with Hamiltonian \( H = \frac{p^2}{2M} \)

\[
\begin{align*}
(x_{ip})(t) &= \begin{pmatrix} 1 & -i/M \\ 0 & 1 \end{pmatrix} (x_{ip})(x) & \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} &= \begin{pmatrix} x(0) + \frac{i}{M}p(0) \\ p(0) \end{pmatrix} \\
\frac{d}{dt}(x_{ip}) &= i\begin{pmatrix} 0 & -M \\ 0 & 0 \end{pmatrix} (x_{ip})
\end{align*}
\]
is a faithful nondiagonalizable time representation, in general

\[
e^t \mapsto e^{iEt} \begin{pmatrix} 1 & ivt \\ 0 & 1 \end{pmatrix} \quad \text{with} \ E, v \in \mathbb{R}, \ v \neq 0
\]
\[
t \mapsto i\begin{pmatrix} E & v \\ 0 & E \end{pmatrix}
\]
For the Newtonian mass point momentum is an eigenvector with trivial time translation eigenvalue, position is a nilvector (principal vector, no eigenvector).

The Hamilton-matrix is $\mathbf{U}(1,1)$-hermitian and the group representation $\mathbf{U}(1,1)$-unitary (1st section)

$$H = \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix} = H^\times$$
$$D(t) = e^{iEt} \begin{pmatrix} 1 & ivt \\ 0 & 1 \end{pmatrix} = D(-t)^\times$$

The Hamilton-matrix is the sum of two commuting transformations, the semisimple and the nilpotent part, called nil-Hamiltonian

$$H = E\mathbf{1}_2 + \nu N, \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad [H, N] = 0, \quad N^2 = 0$$

The representation space cannot be spanned by energy eigenvectors alone which are characterized by the trivial action of the nil-Hamiltonian

$$|G\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle G| = (1, 0), \quad \left\{ \begin{array}{l} H|G\rangle = E|G\rangle, \quad N|G\rangle = 0 \\ (H - E)|G\rangle = 0 \\ |G\rangle(t) = e^{iEt}|G\rangle \end{array} \right.$$  

In addition to the eigenvector one needs another principal vector

$$|B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle B| = (0, 1), \quad \left\{ \begin{array}{l} H|B\rangle = E|B\rangle + \nu|G\rangle, \quad N|B\rangle = \nu|G\rangle \\ (H - E)^2|B\rangle = 0 \\ |B\rangle(t) = e^{iEt}|B\rangle + ivte^{iEt}|G\rangle \end{array} \right.$$  

For the $\mathbf{U}(1,1)$-inner product both eigenvectors and nilvectors are ghosts, i.e. their $\mathbf{U}(1,1)$-norm vanishes in Witt bases

$$\mathbf{U}(1,1) : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} \langle G|G\rangle = 0, \quad \langle G|B\rangle = 1 \\ \langle B|G\rangle = 1, \quad \langle B|B\rangle = 0 \end{array} \right.$$  

$$H = E(|B\rangle\langle G| + |G\rangle\langle B|) + \nu|G\rangle\langle G|$$

Eigen- and nilvectors are not $\mathbf{U}(1,1)$-orthogonal. Obviously the inner product gives no Hilbert space structure as seen in Sylvester bases

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} \langle G + B|G + B\rangle = 2, \quad \langle G + B|G - B\rangle = 0 \\ \langle G - B|G + B\rangle = 0, \quad \langle G - B|G - B\rangle = -2 \end{array} \right.$$  

### 4.2 Ghosts in Quantum Structures

The quantum structure of the $\mathbf{U}(1,1)$-time representation in the last subsection becomes somewhat more complicated since the nilpotency $N^2 = 0$ for the matrix product (endomorphism product) cannot be implemented with the quantum algebra product using Bose degrees of freedom alone. This requires the introduction of twin-like Fermi degrees of freedom as done with the Fadeev-Popov fields as Fermi twins for the nonparticle degrees of freedom in the Bose gauge fields.
By using a doubling of the basic representation space and constructing a \( \mathbb{Z}_2 \)-graded quantum algebra (last section) with both a Bose (capital letters \( G, B \)) and a Fermi (small letters \( g, b \)) factor

\[
Q^- (\mathcal{F}^4) \otimes Q^+ (\mathcal{F}^4) \cong \mathcal{C}[B, G, B^x, G^x] \otimes \mathcal{C}[b, g, b^x, g^x]
\]

with \[
\begin{align*}
\{ G^x, B \} &= 1, & \{ B^x, G \} &= 1 \\
\{ g^x, b \} &= 1, & \{ b^x, g \} &= 1
\end{align*}
\]

the time development is implemented by the doubled Hamiltonian

\[
H_{BF} = H_B + H_F, \quad \begin{cases}
H_B &= E G^x B^x + \nu G G^x \\
H_F &= E g^x b^x + \nu g g^x
\end{cases}
\]

By products of Bose with Fermi operators a nilquadratic BRS-operator\[1, 9\] of Fermi type can be constructed

\[
N_{BF} = g G^x + G g^x \Rightarrow [H_{BF}, N_{BF}] = 0, \quad N_{BF}^2 = 0
\]

The quantum product \( N_B = G G^x \) of Bose operators is not nilpotent.

In the basic vector space formulation there arises as Hamilton-matrix and BRS matrix

\[
H_{BF} = H_B \oplus H_F = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \begin{pmatrix} E & \nu \\ 0 & E \end{pmatrix}
\]

\[
N_{BF} = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

The BRS-operator effects - in analogy to the nil-Hamiltonian action in the \( 2 \times 2 \)-matrix formulation - the projection to a subspace, spanned by time translation eigenvectors. The graded adjoint action with the BRS-operator - replacing the classical gauge transformation

\[
\text{ad} N_{BF}(a) = \begin{cases} [N_{BF}, a] & \text{for a Bose, e.g. } [N_{BF}, G] = 0 \\ \{ N_{BF}, a \} & \text{for a Fermi, e.g. } \{ N_{BF}, g \} = 0 \end{cases}
\]

defines the unital subalgebra with the linear combinations of the time translation eigenvectors - replacing the gauge invariant operators

\[
\text{INV}_{N_{BF}} Q^\pm(\mathcal{F}^8) = \{ p \in Q^- (\mathcal{F}^4) \otimes Q^+ (\mathcal{F}^4) \mid \text{ad} N_{BF}(p) = 0 \}
\]

The product Fock space for \( Q^- (\mathcal{F}^4) \otimes Q^+ (\mathcal{F}^4) \), as constructed for the \( \mathbb{U}(1) \)-time representations in the last section, has an indefinite metric

\[
\text{FOCK}^- (\mathcal{F}^2) \otimes \text{FOCK}^+ (\mathcal{F}^2) \quad \begin{cases} \langle G \pm B | G \pm B \rangle = \pm 2 \\ \langle g \pm b | g \pm b \rangle = \pm 2 \end{cases}
\]

The subspace with the time translation eigenvectors - i.e. the classes for the BRS-invariance algebra \( \text{INV}_{N_{BF}} Q^\pm(\mathcal{F}^8) \) above, replacing the gauge invariant states

\[
\{ |p\rangle \in \text{FOCK}^- (\mathcal{F}^2) \otimes \text{FOCK}^+ (\mathcal{F}^2) \mid N_{BF}|p\rangle = 0 \}\]
contains - up to $|0\rangle$ (the class of the quantum algebra unit 1) with $\langle 0|0 \rangle = 1$ - only normless vectors (ghosts), e.g. $\langle g|g \rangle = 0 = \langle G|G \rangle$. Its metric is semidefinite. The associated Hilbert space with the definite classes is 1-dimensional $\mathbb{C}|0 \rangle$, i.e. it contains only the classes of the scalars. From the whole operator quantum algebra for the $U(1, 1)$-time representation there is - apart form the scalars $\mathbb{C}|0 \rangle$ - no vector left for the asymptotic Hilbert space - ghosts have no asymptotic states, i.e. they have no particle projections.

### 4.3 Ghosts in Gauge Theories

$U(1, 1)$-time representations with the characteristic ghost pairs $(B, G)$ arise in gauge fields which embed via the spacetime translation representations the indefinite Lorentz group into an indefinite unitary group $\text{SO}_0(1, 3) \subset U(1, 3)$. Orthogonal time-space bases (Sylvester bases) and lightlike bases (Witt bases) reflect the two bases for eigen- and nilvector used above

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_0^2 - x_3^2 = (x_0 + x_3)(x_0 - x_3)$$

The algebraic concepts used in the $(2 \times 2)$-matrix language above are the left hand side of the following dictionary for the gauge field language

- nilconstant $\nu \neq 0 \sim$ gauge fixing constant
- nil-Hamiltonian $N$ with $N^2 = 0 \sim$ Becchi-Rouet-Stora charge
- nil-Hamiltonian action, e.g. $N|B \rangle = \nu |G \rangle \sim$ gauge transformation
- time translation eigenvectors, e.g. $N|G \rangle = 0 \sim$ gauge invariant vectors
- eigenvectors with nontrivial norm $\sim$ asymptotic particles
- ghost pairs with trivial norm $\sim$ interaction without particles

For a free massless electromagnetic gauge field the quantization

$$[A^k, A^j](x) = \int \frac{d^4q}{(2\pi)^4} \epsilon(q_0) \left[-\eta^{kj} + 2\nu q^k q^j \frac{\partial}{\partial q^2}\right] \delta(q^2) e^{iqx}$$

is contrasted with the quantization of a free massive vector field

$$[Z^k, Z^j](x) = \int \frac{d^4q}{(2\pi)^4} \epsilon(q_0) \left[-\eta^{kj} + \frac{q^k q^j}{q^2}\right] \delta(q^2 - m^2) e^{iqx}, \quad m^2 > 0$$

The gauge field employs the characteristic Dirac function derivative $\delta'(q^2)$, multiplied with the gauge fixing constant $\nu$.

The harmonic analysis of the massive vector field with respect to the time representations with spin 1 involves the Lorentz transformation $\Lambda(\frac{q}{m})$ to a rest system with $\text{SO}(3)$ the ‘little group’ for energy-momenta with $q^2 = m^2 > 0$

$$Z^j(x) = \int \frac{d^4q}{(2\pi)^4} \Lambda(\frac{q}{m}) \frac{w^a(q)e^{iqx} + w^a(q)e^{-iqx}}{\sqrt{2}} \text{ with } q_0 = \sqrt{m^2 + \vec{q}^2}$$

and $\Lambda(\frac{q}{m}) = \frac{1}{m} \begin{pmatrix} q_0 & \vec{q} \\ -\vec{q} & m + q_0 \end{pmatrix} \in \text{SO}_0(1, 3)$, $\Lambda(\frac{q}{m}) \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix} = q$. 

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The time representations in the gauge field

\[
A^j(x) = \int \frac{d^4q}{(2\pi)^4} h(\frac{q}{|q|})^j \frac{1}{\sqrt{q_0}} \left[ \begin{array}{c} \frac{\sqrt{2}}{\sqrt{q_0}} \left( u^2(q)e^{iqx} + u^2(q)e^{-iqx} \right) \\
\frac{\sqrt{2}}{\sqrt{q_0}} \left( v^2(q)e^{iqx} + v^2(q)e^{-iqx} \right) \\
\frac{\sqrt{2}}{\sqrt{q_0}} \left( (1+\nu)G(q)e^{iqx} + (1+\nu)G(q)e^{-iqx} \right) \end{array} \right]
\]

with \( q_0 = |q| \)

involve - in addition to the two photonic particle degrees of freedom in the 1st and 2nd component with two time representations in \( U(1) \) - the Coulomb interaction and gauge degree of freedom in the 0th and 3rd component. \( h(\frac{q}{|q|}) \) is a representative of \( SO_0(1,3)/SO(2) \) to transform to the polarization group \( SO(2) \) (‘little group’ for energy-momenta with \( q^2 = 0, q \neq 0 \))

\[
h(\frac{q}{|q|}) = \left( \begin{array}{cc} 1 & 0 \\
0 & O(\frac{q}{|q|}) \end{array} \right) \circ w, \quad w = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{array} \right)
\]

\[O(\frac{q}{|q|}) \in SO(3) \text{ with } O(\frac{q}{|q|}) \left( \begin{array}{c} 0 \\
0 \\
0 \end{array} \right) = \frac{q}{|q|}\]

The \( U(1,1) \)-time representation structure of the Faddev-Popov Fermi scalar fields is similar and will not be given here explicitly.

The inner product structure can be seen in the decomposition of the indefinite time representation containing unitary group \( U(1,3) \) extending the Lorentz group - for massive vector fields with Sylvester bases

\[
\text{massive particles, e.g. stable } Z: \quad \text{with} \quad SO_0(1,3) \hookrightarrow U(1,3) \supset U(1) \times U(3)
\]

\[
\text{Lorentz metric: } \left( \begin{array}{ccc} -1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \end{array} \right) = -\eta
\]

and for massless gauge fields with Witt bases

\[
\text{massless ghosts and particles, e.g. photons } \gamma: \quad \text{with} \quad SO_0(1,3) \hookrightarrow U(1,3) \supset U(1,1) \times U(2)
\]

\[
\text{Lorentz metric: } \left( \begin{array}{ccc} 0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \end{array} \right) = -w^T \circ \eta \circ w
\]

5 Unstable States and Particles

Decaying states (particles) can be considered in a Hilbert space where they form, together with other states - stable or unstable - multidimensional probability collectives. The 2-dimensional neutral kaon system with the short and long lived unstable neutral kaon as an illustration leads to the general algebraic formulation.
5.1 The Neutral Kaons as a Probability Collective

The system of the two neutral $K$-meson states $|K_{S,L}\rangle$ shows - on the one hand - the phenomenon of CP-violation (treated in this subsection) and - on the other hand - is unstable and decays into many channels (treated with the general formalism in the next subsection).

The kaon particles are no CP-eigenstates $|K_{\pm}\rangle$ to which they can be transformed by an invertible $(2 \times 2)$-matrix

$$
\left( \begin{array}{c} |K_{S}\rangle \\ |K_{L}\rangle \end{array} \right) = T \left( \begin{array}{c} |K_{+}\rangle \\ |K_{-}\rangle \end{array} \right), \quad T \in \text{GL}(\mathbb{C}^2)
$$

The CP-eigenstates are fictive in the sense that there are no observable particles connected with them. Under the assumption of CPT-invariance the matrix is symmetric and parametrizable by two complex numbers wherefrom - with irrelevant $U(1)$-phases - the normalization $N_K$ can be chosen to be real

$$
T = T^T = \frac{1}{N_K \sqrt{1+|\epsilon|^2}} \left( \begin{array}{cc} 1 & \epsilon \\ \epsilon & 1 \end{array} \right), \quad \epsilon \in \mathbb{C}, \; N_K \in \mathbb{R}
$$

The time development is implemented by a Hamiltonian, non-hermitian for the unstable states $H_K \neq H_K^*$

$$
\text{for } t \geq 0 : \quad \frac{d}{dt} \left( \begin{array}{c} |K_{+}\rangle \\ |K_{-}\rangle \end{array} \right) = iH_K \left( \begin{array}{c} |K_{+}\rangle \\ |K_{-}\rangle \end{array} \right), \quad \frac{d}{dt} \left( \begin{array}{c} |K_{S}\rangle \\ |K_{L}\rangle \end{array} \right) = i \text{diag} \; H_K \left( \begin{array}{c} |K_{S}\rangle \\ |K_{L}\rangle \end{array} \right)
$$

with the diagonal form for the energy eigenstates

$$
\text{diag} \; H_K = \left( \begin{array}{cc} M_S & 0 \\ 0 & M_L \end{array} \right), \quad M = m + i\frac{\Gamma}{2}, \quad m, \; \Gamma > 0
$$

$$
H_K = T^{-1} \text{diag} \; H_K \; T = \left( \begin{array}{cc} M_S - \epsilon^2 M_L & \epsilon(M_S - M_L) \\ \epsilon(M_L - M_S) & M_L - \epsilon^2 M_S \end{array} \right)
$$

The scalar product, constructed for $t = 0$, is time independent. The CP-eigenstates constitute an orthogonal basis in the complex 2-dimensional Hilbert space

$$
\text{CP-eigenstates: } \left( \begin{array}{cc} \langle K_+ | K_+ \rangle & \langle K_+ | K_- \rangle \\ \langle K_- | K_+ \rangle & \langle K_- | K_- \rangle \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
$$

wherefrom there arises the positive non-diagonal $T^*T$ for the time translation eigenstates

$$
\text{energy eigenstates: } \left( \begin{array}{cc} \langle K_S | K_S \rangle & \langle K_S | K_L \rangle \\ \langle K_L | K_S \rangle & \langle K_L | K_L \rangle \end{array} \right) = T^*T = (T^*T)^* = \frac{1}{N_K} \left( \begin{array}{cc} 1 & \delta \\ \delta & 1 \end{array} \right)
$$

The experiments give a nontrivial transition between the short and long lived kaon - the real part of $\epsilon$. Therefore $T$ is not definite unitary

$$
\delta = \frac{\epsilon + \overline{\epsilon}}{1+|\epsilon|^2} \sim 0.327 \times 10^{-2} \Rightarrow T \notin U(2)
$$

\footnote{Any matrix product $f^*f$ is unitarily equivalent to a positive diagonal matrix.}
The normalization $N_K$, usually chosen, normalizes individually the particles states, i.e. vectors with $\langle K_S|K_S \rangle = \langle K_L|K_L \rangle = 1$. The kaon system is a 2-dimensional probability collective, i.e. a complex 2-dimensional Hilbert space. Therefore it will be collectively normalized via the discriminant (determinant)

$$\det T^*T = |\det T|^2 = \langle \det T | \det T \rangle = \langle K_S|K_S \rangle \langle K_L|K_L \rangle - |\langle K_S|K_L \rangle|^2 = 1$$

$$\Rightarrow N^2_K = \sqrt{(1-e^2)(1-e^2)} = 1 - \delta^2$$

Since the energy eigenstates are not orthogonal, i.e. the transformation $T$ is not unitary $T^* \neq T^{-1}$, there exists a 3rd basis of the 2-dimensional Hilbert space

$$\begin{pmatrix} |K_+\rangle \\ |K_-\rangle \end{pmatrix} = T^{-1} \begin{pmatrix} |K_+\rangle \\ |K_-\rangle \end{pmatrix} = (TT^*)^{-1} \begin{pmatrix} |K_S\rangle \\ |K_L\rangle \end{pmatrix}$$

also not orthogonal, but orthogonal with the energy eigenstates

$$\begin{pmatrix} \langle K_S|K_S \rangle & \langle K_S|K_L \rangle \\ \langle K_L|K_S \rangle & \langle K_L|K_L \rangle \end{pmatrix} = (TT^*)^{-1}, \quad \begin{pmatrix} \langle K_S|K_S \rangle & \langle K_S|K_L \rangle \\ \langle K_L|K_S \rangle & \langle K_L|K_L \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A decomposition of the unit can be written with the orthogonal and non-orthogonal bases

$$1_2 = |K_+\rangle \langle K_+ | + |K_-\rangle \langle K_- |$$

$$= |K_S\rangle \langle K_S^+ | + |K_L\rangle \langle K_L^+ | = |K_+\rangle \langle K_S^+ | + |K_-\rangle \langle K_L^+ |$$

$$= |K_S\rangle \langle K_S | - \delta |K_S\rangle \langle K_L | - \delta |K_L\rangle \langle K_S | + |K_L\rangle \langle K_L |$$

$$\quad \text{for } N^2_K \frac{1}{1-\delta^2} = 1$$

### 5.2 Non-Orthogonal Decaying States

The possibility to have nontrivial transition elements between particles, as $\langle K_S|K_L \rangle$ above, can be connected to the deviation from the definite unitary structures for unstable states. The following well known theorems with respect to unitary equivalence are relevant for the situation.

An $(n \times n)$-matrix $H$ acting on a vector space $V \cong \mathbb{C}^n$, e.g. a Hamilton-matrix for the time translations, is unitarily equivalent to a diagonal matrix if, and only if it is normal - all concepts with respect to a definite $U(n)$-conjugation

$$H = U \circ \text{diag } H \circ U^* \quad \text{with } U \in U(n) \iff H \circ H^* = H^* \circ H$$

In this case, the vector space can be decomposed as orthogonal sum of the eigenspaces for $N$ different eigenvalues $\text{spec } H = \{M_k\}$

$$V = \bigoplus_{k=1}^N V_k, \quad \text{diag } H = \bigoplus_{k=1}^N M_k \text{id}_{V_k}, \quad \langle V_k|V_l \rangle = \{0\} \quad \text{for } M_k \neq M_l$$

For $U(n)$-hermitian operators $H = H^*$ the eigenvalues are real $M = E \in \mathbb{R}$.

An analogue (real) diagonal structure holds for (selfadjoint) normal operators on infinite dimensional Hilbert spaces.
Hamiltonians acting on a Hilbert space with complex eigenvalues $E + i\frac{\Gamma}{2}$, $\Gamma > 0$, have to be $U(n)$-nonhermitian, $H \neq H^*$. Only with at least one non-real energy involved, i.e. one unstable particle, two time translation eigenvectors (particles) with different energies can have a nontrivial transition element - unstable particles have not to be orthogonal to other particles.

This structure raises a basic question, discussed already in the 1st section: A Hamiltonian implements the real time translations as acting upon a complex vector space. To recognize the realness also in the complex structure by $H = H^*$, the complex vector space has to come with a conjugation or - equivalently - with an inner product, characterized by a unitary invariance group. The conjugation above for the probability structure has to be definite with invariance group $U(n)$. Possibly, the real spacetime translations for unstable states with complex eigenvalues are implemented by operators which are hermitian under an indefinite unitary group where the definite probability group comes as a subgroup. The Lorentz embedding group $U(1, 3)$ with probability subgroups $U(3)$ for massive and $U(2)$ for massless particles has been mentioned above. Another example is the Dirac spinor conjugation group $U(2, 2)$ as familiar from the left-right handed conjugation. The embedded definite $U(2)$-conjugation is used for the probability structure, the indefinite one $U(1, 1)$ for the particle-antiparticle reflection. The indefinite unitary structures for unstable states in general are not discussed in this paper.

5.3 Probability Collectives for Decaying Particles

The two translation eigenstates (particles) for unstable kaons are generalized to $q$ eigenstates $|M\rangle$ (particles) with the eigenvalues $M = m + i\frac{\Gamma}{2}$ involving at least one unstable state $\Gamma > 0$. An orthogonal basis, related to $|M\rangle$ by a non-unitary transformation $T \notin U(q)$ is denoted by $|U\rangle$ - generalizing the CP-eigenstates of the kaon collective. In addition all stable decay modes, assumed to be $p$ translation eigenstates $|E\rangle$ (particles) with real eigenvalues $E$ are included, e.g. $|\pi, \pi\rangle, |\pi, \pi, \pi\rangle, |\pi, l, \nu_l\rangle$ for the kaon collective

$$|M\rangle = \left(\left|m_j + i\frac{\Gamma_j}{2}\right]\right)_{j=1}^q \quad \text{eigenstates with at least one decaying channel}$$

$$|U\rangle = \left(|U_j\rangle\right)_{j=1}^q \quad \text{orthogonal states}$$

$$|E\rangle = \left(|E_i\rangle\right)_{i=1}^p \quad \text{stable eigenstates (decay channels)}$$

In a more general formulation also an infinite dimensional momentum dependence can be included.

The translation eigenstates have the time development with a diagonal Hamiltonian

$$\text{for } t \geq 0: \quad \frac{d}{dt} \left(|U\rangle \left|E\rangle\right) \right) = iH \left(|U\rangle \left|E\rangle\right) \right), \quad WHW^{-1} = \text{diag } H = \begin{pmatrix} M & 0 \\ 0 & E \end{pmatrix}$$
The equivalence transformation $W \notin U(q+p)$ is the product of a triangular matrix with a $(p \times q)$-matrix $w$ from the decay $|M\rangle \rightarrow |E\rangle$, called Wigner-Weisskopf\[7\] matrix, and a matrix with a $(q \times q)$-matrix $T$ for the transformation $|M\rangle = T|U\rangle$

$$W = \frac{1}{N} \begin{pmatrix} 1_q & w \\ 0 & 1_p \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & 1_p \end{pmatrix} = \frac{1}{N} \begin{pmatrix} T & w \\ 0 & 1_p \end{pmatrix}$$

$$H = \begin{pmatrix} T^{-1} & 0 \\ 0 & 1_p \end{pmatrix} \begin{pmatrix} M & w(M-E) \\ 0 & 1_p \end{pmatrix} = \begin{pmatrix} H_U & T^{-1}w(M-E) \\ 0 & 1_p \end{pmatrix}$$

The eigenstates $|M\rangle$ have projections both on the orthogonal states and on the decay channels

$$\text{eigenstates: } \left( \begin{pmatrix} |M\rangle \\ |E\rangle \end{pmatrix} \right) = \frac{1}{N} \begin{pmatrix} T & w \\ 0 & 1_p \end{pmatrix} \begin{pmatrix} |U\rangle \\ |E\rangle \end{pmatrix} = \frac{1}{N} \begin{pmatrix} T|U\rangle + w|E\rangle \\ |E\rangle \end{pmatrix}$$

The scalar product matrix for the probability collective with the $(q+p)$ translation eigenstates arises from the diagonal matrix with the orthogonal states and the decay channels

$$\left( \begin{pmatrix} \langle U|U \rangle & \langle U|E \rangle \\ \langle E|U \rangle & \langle E|E \rangle \end{pmatrix} \right) = \begin{pmatrix} 1_p & 0 \\ 0 & 1_q \end{pmatrix}, \quad \left( \begin{pmatrix} \langle M|M \rangle & \langle M|E \rangle \\ \langle E|M \rangle & \langle E|E \rangle \end{pmatrix} \right) = W^*W = \frac{1}{N^2} \begin{pmatrix} T^*T & T^*w \\ w^*T & 1_p + w^*w \end{pmatrix}$$

with the collective normalization

$$\langle \det W | \det W \rangle = \langle M|M \rangle \langle E|E \rangle - \langle M|E \rangle \langle E|M \rangle = \det T^*T \left[ \det (1_p + w^*w) - \det w^*w \right] = N^2 = 1$$

### 5.4 Probabilities with Nonabelian Groups $U(n)$

The scalar product $\langle \ | \rangle$ of the complex space $V \cong \mathbb{C}^n$, $n = q + p$, with the translation eigenstates (particles) involving at least one unstable state is a positive matrix, not the unit matrix. It can be factorized with a nonunitary matrix $W \notin U(n)$ chosen\[5\] from the orientation manifold $GL(\mathbb{C}^n)/U(n)$

$$1_n \neq \langle \ | \rangle = W^*W, \ W \in GL(\mathbb{C}^n)/U(n)$$

The individual probability normalization for one state by the scalar product

$$\text{for } U(1): \quad \langle u|u \rangle = 1$$

is generalized to a collective normalization by the discriminant (determinant)

$$\text{for } U(n): \quad \langle \det W | \det W \rangle = \det \langle \ | \rangle = 1$$

The invariance group of a scalar product in diagonal bases

$$U(n) = \{ U \in GL(\mathbb{C}^n) | U^*1_nU = 1_n \}$$

\[5\]The literally wrong notation $g \in G/H$ denotes a coset representative, correct: $g \in gH \in G/H$. 

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is equivalent to the invariance group of the scalar product, reoriented by $W$ in a particle basis to $W^* \mathbf{1}_n W = \langle | \rangle$

$$\{G \in \text{GL}(\mathbb{C}^n) \mid G^* \langle | \rangle G = \langle | \rangle\} = W^{-1} \mathbf{U}(n) W$$

i.e. $W$ is determined up to $\mathbf{U}(n)$.

A remark with respect to the general structure which is used at different points in established physical theories: An orientation manifold $G/H$ arises with the distinction of a subgroup $H$ in a group $G$. Prominent examples are - for spacetime operations with the distinction of a Lorentz group $\text{SO}_0(1, 3)$ - the manifold of Lorentz metrics $g$ as factorized by the tetrad

$$g_{\mu\nu}(x) = h^i_\mu(x) \eta_{ij} h^j_\nu(x), \quad h^i_\mu(x) \in \text{GL}(\mathbb{R}^4)/\text{SO}_0(1, 3)$$

and - for internal operations in the standard model of elementary particles - the Goldstone manifold $\mathbf{U}(2)/\mathbf{U}(1)_+$ arising with the distinction of an electromagnetic group $\mathbf{U}(1)_+$ in the hyperisospin group $\mathbf{U}(2)$ as illustrated by the Higgs isodoublet $\Phi$ as related to the Goldstone transmutator $U$ (discussed in more detail in [1])

$$\Phi^* \Phi = (U^*)^a_b (\begin{pmatrix} 1 & r \\ 0 & -1 \end{pmatrix})^b_a U^b_b, \quad U^b_b \in \mathbf{U}(2)/\mathbf{U}(1)_+$$

As well as, e.g., a spacetime translation in Minkowski space $x \in \mathbb{R}^4$ has to be seen as a ‘whole’ where vector components make sense only if an additional physical structure distinguishes a coordinate system, e.g. rest systems for massive particles, a probability collective including decaying particles (translation eigenstates with complex energies), e.g. the two neutral kaons, together with their decay products, has a holistic identity. In addition to the particle bases which have a physical relevance there exist many orthogonal bases, e.g. CP-eigenstates for the kaon system, which are not related to particles. What can be really measured are transition amplitudes between particle states, not between fictive other states. This is in analogy to the many special relativistic spacetime decompositions into time and space or into lightlike and spacelike subspaces, as illustrated above with Sylvester and Witt bases, which have sense for measurements only if, e.g., base determining particles is considered. Via the nonvanishing transition elements the identity of the energy eigenstates is spread over the whole collective. Obviously, for a small width $\frac{E}{m} \ll 1$, the uncomplete identity of a decaying particle - its collective property - will be difficult to discover. However, it is to be expected that there are experiments which can test the difference between an individualistic probability interpretation of decaying particles and their collective probability interpretation where the collective discriminant normalization may be relevant.

Acknowledgement

I am indebted to Walter Blum for discussions leading to the concept of probability collectives for unstable particles.


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