GROTHENDIECK HOMOMORPHISMS IN ALGEBRAICALLY CLOSED VALUES FIELDS III: FOURIER TRANSFORM

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ABSTRACT. We continue the study of the Hrushovski-Kazhdan integration theory and consider exponential integrals. The Grothendieck ring is enlarged via a tautological additive character and hence can receive such integrals. We then define the Fourier transform in our integration theory and establish some fundamental properties of it. Thereafter a basic theory of distributions (without differential operators) is also developed. We construct the Weil representation in the end as an application. The results are completely parallel to the classical ones.

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1. INTRODUCTION

The Hrushovski-Kazhdan integration theory [10] is a major development in the theory of motivic integration. The fundamental idea of the theory is to construct homomorphisms between various Grothendieck rings associated with the first-order theory ACVF of algebraically closed valued fields. The simplest of these constructions was presented in [16]. The homomorphism constructed there should not be called an integration but merely a Grothendieck homomorphism since it is a pure isomorphism invariant and is not measure-preserving. On the other hand, this construction serves as the template upon which suitable modifications and extensions may be carried out to achieve additional features, such as volume form and exponential. The case of volume form, that is, the measure-preserving homomorphisms, will be presented in [15]. In this paper, based on the work and ideas in [10], we demonstrate how to extend the construction to include parametrized exponential integrals, typically of the form

\[
\int_{\mathfrak{p} \in VF^n} \int_{\mathfrak{p} \in VF^n} f(\mathfrak{p}, \mathfrak{q}) \exp(g(\mathfrak{p}, \mathfrak{q})),
\]
where the restrictions on the definable functions $f, g$ are very natural.

To describe in a few words how motivic integration is different from classical integration it seems best to begin by pointing out that the ring that provides values for integrals is not the real field but a Grothendieck ring. The latter is traditionally constructed from equivalence classes of algebraic varieties and, more generally in the model-theoretic setting, from equivalence classes of definable subsets. Topological tools that are essential to many classical constructions are no longer available; instead, since it was first introduced by M. Kontsevich in 1995, techniques from first-order model theory of definable sets underlie much of the development of this new kind of integration. In fact, at risk of being overly simple-minded, one may think of motivic integration as classical integration with the topological concepts of “continuity”, “convergence”, etc. replaced everywhere by the model-theoretic concept of “definability”.

To be sure, the class of definable integrals is conceptually narrower than the class of integrals that can be more or less dealt with classically. However, there are many reasons why the motivic approach to integration will play an increasingly important role. We mention two here.

Firstly, the progress in model theory in the last few decades suggests that many natural mathematical properties are subject to first-order treatment. In our context, given the fact that some very complicated integral identities are already motivic (see, for example, [2, 5]), it is reasonable to expect that many other important kinds of integrals are definable in some first-order languages and hence may be studied motivically. We note that, in their recent paper [11], Hrushovski and Kazhdan have developed a partially first-order method to study adelic structures over curves and, in particular, have obtained a global Poisson summation formula.

Secondly, if one is more interested in the structure of a space of functions (for example, functional equations) than actual computation of functions, then constantly worrying about things such as convergence seems to be an unnecessary burden. By this we just mean that there is no need to insist on assigning “numerical values” to integrals, especially when it is not possible, and sometimes working with “geometrical values” is more effective. Definable integrals are of a more geometrical nature and are better behaved, at least before specializing to local fields. Some pathological phenomena afforded by point-set topology are thus avoided. For example, while classically it is possible that two iterative integrals of a function exist but are not equal, this cannot happen to definable integrals. This is our Fubini theorem (Theorem 5.11), which is then stronger than the classical one.

This better behavior of definable integrals mentioned above may be a result of how motivic measure is manufactured: the volume of a geometrical object is somehow provided by the object itself, subject to certain geometrical equivalence relations. The construction of exponential integrals in this paper is very illustrative of this “tautological” nature. The Grothendieck homomorphisms constructed in [16, 15] are unable to accommodate additive characters because the target ring $\tilde{\mathbb{R}}$ is not big enough. To remedy this, we just take a quotient $\Omega$ of the additive structure of the field $\mathbb{V}$ and add it to $\tilde{\mathbb{R}}$, as a set of symbols, to form a group ring (hence additive relation is turned into multiplicative relation). There are certain fundamental properties of additive characters that must hold in this group ring. To achieve these we can take quotient or localize or employ other standard algebraic operations. As one can easily see, this sort of construction is very flexible and
many other desirable features may be incorporated along similar lines. For example, an important thing that is still missing in our integration theory is multiplicative character, and its construction will be carried out in a sequel.

Let us now describe in more detail how the sections of this paper are organized. In Section 2 we resume the study of first-order structural properties of models of ACVF in [16, Section 4]. This section relies on definitions and notations introduced in [16, Section 2]. The results provide technical support for later sections and may be skipped (at least the proofs) without impairing the essential understanding of the rest of the paper. In Section 3 we discuss differentiation in valued fields. The main purpose of this discussion is again to establish some crucial technical lemmas that shall be needed later. Although differentiation in RV is alluded at a few places in the text, it may be ignored. The presence of differentiation in VF allows us to define the Jacobian, which, however, is not needed until Section 8. Some modifications of the Grothendieck ring are carried out in Section 4. It is, in this paper, our analogue of the real field. It is no longer graded and has more invertible elements. It is further enlarged in Section 5 via a tautological additive character and thereby becomes an analogue of the complex field. We then single out a subclass of definable functions that will be the focus of the discussion in the subsequent sections, namely integrable functions. There is flexibility in the concept of integrability and the class can be made larger. But the cutoff line we have adopted seems most natural: desirable properties such as the Fubini theorem and closure under convolution can be proved easily. In Section 6 the Fourier transform is defined, which perhaps should be called the Fourier-Laplace transform. Thereafter various fundamental integral identities involving Fourier transform are established, for example, the convolution formula, the Fourier inversion formula, the Plancherel formula, etc. Although distributions may seem manifestly non-first-order, a basic theory of definable distributions can be developed within our framework, which we shall do in Section 7. In the last section, as an application, we show that the Weil representation exists on the Schwartz spaces associated with algebraically closed valued fields.

We note that on the level of local fields there is also a very general approach to motivic integration, namely the Cluckers-Loeser theory [4]; see [7] for an excellent exposition. Their construction of exponential integrals is contained in [3]. For a general introduction to the development before these general approaches we refer to the article [8].

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2. More on structural properties

We continue the investigation of definable subsets in [16, Section 4]. As in there, we work with an underlying substructure $S$ of the monster model. When we need some restrictions on $S$ below, for example, at a few crucial places $S$ needs to be $(VF, \Gamma)$-generated, we shall always make such restrictions explicit.

Notation 2.1. Let $\bar{\sigma} = (a_1, \ldots, a_n)$, $\bar{\sigma}' = (a_1', \ldots, a_n')$ be tuples in $VF^n$. We write $\text{val}(\bar{\sigma} - \bar{\sigma}')$ for the element $\min \{\text{val}(a_i - a_i') : 1 \leq i \leq n\} \in \Gamma$. For any $\bar{\tau} = \ldots$
\((\gamma_1, \ldots, \gamma_n) \in \Gamma\), the open polyball \(\{ (b_1, \ldots, b_n) : \val(b_i - a_i) > \gamma_i \}\) is denoted as 
\(o(\pi, \tau)\) and the closed polyball \(\{ (b_1, \ldots, b_n) : \val(b_i - a_i) \geq \gamma_i \}\) is denoted as 
\(c(\pi, \tau)\). We set \(o(\pi, \infty) = c(\pi, \infty) = \{ \pi \}\).

For the reader’s convenience, some of the lemmas in [16] that will be used in this paper are stated here without proof.

**Lemma 2.2.** [16] Lemma 4.8] Let \(W\) be a definable subset of \(RV^m\) and \(f : W \rightarrow VF^n\) a definable function. Then \(f(W)\) is finite.

**Lemma 2.3.** [16] Corollary 4.13] Suppose that \(S\) is VF-generated. If \(\Gamma(acl(\emptyset))\) is nontrivial then \(acl(\emptyset)\) is a model of \(\mathcal{ACVF}_S^0\).

**Lemma 2.4.** [16] Lemma 4.14] Suppose that \(S\) is VF-generated. Let \(\pi \in \Gamma\) and \(\mathfrak{B}\) a \(\pi\)-algebraic set of closed balls. Then \(\mathfrak{B}\) has \(\pi\)-definable centers.

**Lemma 2.5.** [16] Lemma 4.16] Suppose that \(S\) is \((VF, \Gamma)\)-generated. Let \(\mathfrak{T} = (t_1, \ldots, t_n) \in RV\) and \(\mathfrak{B}\) a \(\mathfrak{T}\)-algebraic set of closed balls. Then \(\mathfrak{B}\) has \(\mathfrak{T}\)-definable centers.

**Lemma 2.6.** [16] Lemma 6.3] Let \(X \subseteq VF^n \times RV^m\) be a definable subset and \(f : X \rightarrow RV^l\) a definable function. Then \(\dim_{VF} X = \max \{ \dim_{VF} f^{-1}(T) : T \in \text{ran}(f) \}\).

**Lemma 2.7.** [16] Lemma 6.4] Let \(X \subseteq VF^n \times RV^m\) be a definable subset. Suppose that there is an \((\pi, \mathfrak{T}) \in X\) such that the transcendental degree of \(VF(\langle \pi \rangle)\) over \(VF(\emptyset)\) is \(k\). Then \(\dim_{VF} X \geq k\).

**Lemma 2.8.** Suppose that \(S\) is \((VF, \Gamma)\)-generated. Let \(X \subseteq VF\), \(\mathfrak{B}\) an infinite set of pairwise disjoint balls, and \(f : X \rightarrow \mathfrak{B}\) a definable surjective function. Then for some \(b \in \mathfrak{B}\) the fiber \(f^{-1}(b)\) is infinite.

**Proof.** Suppose for contradiction that each \(f^{-1}(b)\) is finite. For each \(a \in X\), by Lemma 2.4 each ball \(f(a)\) contains an \(a\)-definable point. By compactness, there is a definable subset \(B \subseteq \bigcup \mathfrak{B}\) such that, for each \(b \in \mathfrak{B}\), \(B \cap b\) is finite. This contradicts \(C\)-minimality since \(\mathfrak{B}\) is an infinite set of pairwise disjoint balls. \(\square\)

Let \(X\) be a subset and \(Y \subseteq X \times VF^n \times RV^m\). We say that \(Y\) is a subset over \(X\) if the projection of \(Y\) into \(X\) is surjective.

**Notation 2.9.** Let \(X_1, X_2\) be subsets and \(R_1, R_2\) equivalence relations on them, respectively. A subset \(Y \subseteq X_1 \times X_2\) over \(X_1\) may be considered as a function from \(X_1 / R_1\) into the powerset \(P(X_2 / R_2)\) if, for each equivalence class \(C \in X_1 / R_1\) and every \(c_1, c_2 \in C\), there is a \(U \in P(X_2 / R_2)\) such that \(\text{fib}(Y, c_1) = \text{fib}(Y, c_2) = U\). In this case, we sometime do write \(Y\) as a function \(X_1 / R_1 \rightarrow P(X_2 / R_2)\). We are of course only interested in definable ingredients. For example, we will discuss functions of the forms \(VF / M \rightarrow P(RV^m)\), \(VF^n \times \Gamma^l \rightarrow P(RV^m)\), etc.

**Lemma 2.10.** Let \(G\) be a definable additive subgroup of \(VF\) (hence \(G\) is either an open ball around 0 or a closed ball around 0). Let \(f : VF \rightarrow P(RV^m)\) be a definable function. Then

1. There are \(G\)-cosets \(D_1, \ldots, D_n\) such that \(f \upharpoonright (VF \setminus \bigcup_i D_i)\) is a function from \((VF \setminus \bigcup_i D_i) / G\) into \(P(RV^m)\).

2. If either \(G\) is a closed ball or \(S\) is \((VF, \Gamma)\)-generated then there is a definable function \(f_1 : VF / G \rightarrow P(RV^m)\) such that for any \(G\)-coset \(D\) there is a \(d \in D\) such that \(f(d) = f_1(D)\).
Proof. For any $D \in VF / G$ and any $\mathcal{F} \in RV^n$ let $U_\mathcal{F}(D) = \{ d \in D : \mathcal{F} \in f(d) \}$. Let $E_\mathcal{F} = \{ D \in VF / G : U_\mathcal{F}(D) \neq \emptyset \text{ and } U_\mathcal{F}(D) \neq D \}$. Note that $E_\mathcal{F}$ is $\mathcal{F}$-definable. Let $A = \{ \mathcal{F} \in RV^m : E_\mathcal{F} \neq \emptyset \}$, which is definable. If $D \notin E_\mathcal{F}$ for any $\mathcal{F}$ then $f \mid D$ is constant. So, without loss of generality, $A \neq \emptyset$. For any $\mathcal{F} \in A$, by $C$-minimality and compactness, there is a $\mathcal{F}$-definable function $h_\mathcal{F}$ on $E_\mathcal{F}$ such that, for each $D \in E_\mathcal{F}$,

1. $h_\mathcal{F}(D)$ is either the union of the positive boolean components of $U_\mathcal{F}(D)$ or the union of the negative boolean components of $U_\mathcal{F}(D)$,
2. there is a $D$-definable closed ball $b_D \subseteq D$ that properly contains $h_\mathcal{F}(D)$.

Since $h_\mathcal{F}(E_\mathcal{F})$ is $\mathcal{F}$-definable, by $C$-minimality again, $E_\mathcal{F}$ must be finite. By Lemma 2.2 there is a $\mathcal{F}$-definable subset $A_\mathcal{F}$ such that $|A_\mathcal{F} \cap b_D| = 1$. Let $g_D : A \rightarrow VF$ be the $D$-definable function given by $\mathcal{F} \mapsto A_\mathcal{F} \cap b_D$ if $D \in E_\mathcal{F}$ and $\mathcal{F} \mapsto 0$ otherwise. By Lemma 2.2, $g_D(A)$ is finite. Since $g_D(A) \subseteq D \cup \{0\}$, by $C$-minimality, the definable subset $\bigcup_{D \in VF / G} g_D(A)$ must be finite and hence $\bigcup_{\mathcal{F} \in A} E_\mathcal{F}$ is finite. This establishes (1). By Lemma 2.3 or Lemma 2.4 $\bigcup_{\mathcal{F} \in A} E_\mathcal{F}$ has definable centers. This establishes (2). \qed

Remark 2.11. The proof of Lemma 2.10 also works if $G$ is a definable multiplicative subgroup of $VF^\times$, in which case $G$ is either an open ball around 1 or a closed ball around 1.

Lemma 2.12. Let $X \subseteq VF^n$ be a definable subset. Then $\dim_{VF} X = n$ if and only if $X$ contains an open polyball.

Proof. The “if” direction is immediate by Lemma 2.7. For the “only if” direction we do induction on $n$. For the base case $n = 1$, since $X$ is infinite, the lemma simply follows from $C$-minimality. We proceed to the inductive step $n = m + 1$. For each $\mathcal{F} \in pr_{\leq m} X$, let $\Delta_{\mathcal{F}}$ be the subset of those $\gamma \in \Gamma$ such that $\fib(X, \mathcal{F})$ contains an open ball of radius $\gamma$; if $\fib(X, \mathcal{F})$ is finite then $\Delta_{\mathcal{F}} = \{ \infty \}$. Since $\Gamma$ is $\omega$-minimal, some element in $\Delta_{\mathcal{F}}$, say $\gamma_{\mathcal{F}}$, is $(\mathcal{F})$-definable. By quantifier elimination, $\gamma_{\mathcal{F}}$ may be defined by a conjunction of $RV$-sort literals. By compactness, there are definable subsets $Y_1, \ldots, Y_t \subseteq pr_{\leq m} X$ with $\bigcup_{i} Y_i = pr_{\leq m} X$ and conjunctions $\phi_1(\mathcal{F}), \ldots, \phi_t(\mathcal{F})$ of $RV$-sort literals such that, for each $\mathcal{F} \in Y_i$, either $\fib(X, \mathcal{F})$ does not contain an open ball or it contains an open ball whose radius is defined by $\phi_i(\mathcal{F})$. Let $X_i = X \cap (Y_i \times VF)$. Since $\bigcup_{i} X_i = X$, for one of these subsets, say, $X_1$, we have $\dim_{VF} X_1 = m + 1$ and hence $\dim_{VF} Y_1 = m$. Then we may simply assume that $\fib(X, \mathcal{F})$ contains an open ball for every $\mathcal{F} \in Y_1$.

Let $g_1(\mathcal{F}), \ldots, g_k(\mathcal{F})$ be all the polynomials occurring in $\phi_1(\mathcal{F})$ in the form $\rv(g_k(\mathcal{F}))$. Let $f : Y_1 \rightarrow RV^k$ be the definable function given by $\mathcal{F} \mapsto (\rv(g_1(\mathcal{F})), \ldots, \rv(g_k(\mathcal{F})))$.

By Lemma 2.6 for some $\mathcal{F} \in RV^k$, $\dim_{VF} f^{-1}(\mathcal{F}) = m$. By the inductive hypothesis (with respect to the substructure $\langle \mathcal{F} \rangle$), $f^{-1}(\mathcal{F})$ contains an open polyball $p$. Note that, by the construction of $f$, for every $\mathcal{F} \in p$ the formula $\phi_1(\mathcal{F})$ defines the same element $\delta \in \Gamma$. Let $b = (b_1, \ldots, b_m) \in p$. We may assume that $p$ is $b$-definable. Note that, by Lemma 2.7 the VF-dimension of $p$ with respect to the substructure $\langle b \rangle$ is still $m$. Consider the $b$-definable subset $W = \{ (\mathcal{F}, c) \in X : \mathcal{F} \in p \text{ and } \mathcal{F}(c, \delta) \subseteq \fib(X, \mathcal{F}) \}$.
Since there is a $\bar{d} \in W$ such that the transcendental degree of $\text{VF}(\langle \bar{d}, \bar{b} \rangle)$ over $\text{VF}(\langle \bar{b} \rangle)$ is $m + 1$, by Lemma 2.7 again, $\dim_{\text{VF}} W = m + 1$. By compactness, for some $c \in \text{pr}_{m+1} W$, $\dim_{\text{VF}} \text{fib}(W, c) = m$. By the inductive hypothesis (with respect to the substructure $(\bar{b}, c)$), $\text{fib}(W, c)$ contains an open polyball. So $\sigma(c, d) \times q \subseteq X$, as required.

**Lemma 2.13.** Let $X \subseteq \text{VF}^n$ be a definable subset. Suppose that there is a $\gamma \in \Gamma$ such that $\sigma(\bar{x}, \gamma) \cap \sigma(\bar{x}', \gamma) = \emptyset$ for every $\bar{x}', \bar{x}'' \in X$. Then $X$ is finite.

**Proof.** We do induction on $n$. The base case $n = 1$ just follows from $C$-minimality. For the inductive step, consider the subset $\text{pr}_1 X$. If $\text{pr}_1 X$ is finite then by the inductive hypothesis $\text{fib}(X, a)$ is finite for every $a \in \text{pr}_1 X$ and hence $X$ is finite. If $\text{pr}_1 X$ is infinite then by $C$-minimality there is an open ball $b \subseteq \text{pr}_1 X$ with $\text{rad}(b) > \gamma$. For any $a' \in b$, $a'' \in \bar{b}$, $\bar{b} \in \text{fib}(X, a')$, and $\bar{b}'' \in \text{fib}(X, a'')$, if $\sigma(\bar{b}', \gamma) \cap \sigma(\bar{b}', \gamma) \neq \emptyset$ then $\sigma((a', \bar{b}'), \gamma) \cap \sigma((a'', \bar{b}''), \gamma) \neq \emptyset$, contradicting the assumption. Therefore, by the inductive hypothesis again, $\bigcup_{a \in b} \text{fib}(X, a)$ is finite. So there is a $\bar{b} \in \bigcup_{a \in b} \text{fib}(X, a)$ such that $\text{fib}(X, \bar{b}) \cap b$ is infinite, contradiction again.

**Lemma 2.14.** Let $f : \text{VF}^n \rightarrow \text{VF}^m$ be a definable function. Let $X \subseteq \text{VF}^n$ be the definable subset of those $\bar{x} \in \text{VF}^n$ such that there are $\epsilon, \delta \in \Gamma$ with $\sigma(\bar{x}, \delta) \cap f^{-1}(\sigma(f(\bar{x}), \epsilon)) = \{\bar{x}\}$.

Then $\dim_{\text{VF}} X < n$.

**Proof.** For each $\bar{x} \in X$ let $(\epsilon, \delta) \in \Gamma^2$ be an $\bar{x}$-definable pair that satisfies the condition above, which exists by $\alpha$-minimality. Let $h : X \rightarrow \Gamma^2$ be the definable function given by $\bar{x} \mapsto (\epsilon, \delta)$. Suppose for contradiction that $\dim_{\text{VF}} X = n$. Then, by compactness and Lemma 2.12 there is a pair $(\epsilon, \delta) \in \Gamma^2$ such that $h^{-1}(\epsilon, \delta)$ contains an open polyball $p$. Without loss of generality we may assume $\bar{x} \in \text{pr}_1(p)$. Fix an $\bar{x}$-definable $\gamma \geq \delta$. If $\bar{x}', \bar{x}'' \in \sigma(\bar{x}, \gamma)$ are distinct then $\sigma(f(\bar{x}'), \sigma(\bar{x}'')) \cap \sigma(f(\bar{x}''), \sigma(\bar{x}'')) = \emptyset$. By Lemma 2.13 $f(\sigma(\bar{x}, \gamma))$ is finite, which is a contradiction.

Let $X$ be a definable subset with $\dim_{\text{VF}} X = n$. A property holds *almost everywhere on $X$ or almost every element in $X$* if there is a definable subset $Y \subseteq X$ with $\dim_{\text{VF}} Y < n$ such that the property holds with respect to $X \setminus Y$. For example, if $f : \text{VF}^n \rightarrow \text{VF}^m$ is a definable function, then the conclusion of Lemma 2.13 holds almost everywhere on $\text{VF}^n$.

**Lemma 2.15.** Let $f : \text{VF} \rightarrow \text{VF}^m$ be a definable function. For almost all $a \in \text{VF}$ there are $a$-definable $\epsilon, \delta \in \Gamma$ such that either $f \upharpoonright \sigma(a, \delta)$ is constant or, for any $b \in \sigma(a, \delta) \setminus \{a\}$,

$$\text{val}(f(b) - f(a)) = \epsilon + k \text{val}(b - a).$$

**Proof.** Let $X$ be as in Lemma 2.14. Note that $X$ is finite and hence $\text{VF} \setminus X$ is an open subset. Fix an $a \in \text{VF} \setminus X$ such that $f$ is not constant on any open ball around $a$. Let $\phi(x, y)$ be a quantifier-free formula in disjunctive normal form that defines $\text{val}(f(x) - f(a))$ for $x \in \text{VF} \setminus X$, where $y$ is a free RV-sort variable. For each disjunct $\phi_i(x, y)$ of $\phi$, the formula $\exists y \phi_i(x, y)$ defines a subset $Y_i$ of $\text{VF} \setminus X$. Since $X$ is finite, by $C$-minimality, one of these subsets, say, $Y_1$, contains an open ball $b$ punctured at $a$. For any $a + d \in b$ and any term of the form $\text{rv}(g(x))$ in $\phi_1(x, y)$, if $\text{val}(d)$ is sufficiently large then there is an $a$-definable $t \in \text{RV}$ and an integer $k \geq 0$...
such that $r(v(g(a + d)) = t(r(v(d)))^k$. By the choice of $a$, for every $\gamma$, the function $\text{val}(f(x) - f(a))$ is not constant on $\sigma(a, \gamma)$. This means that, for every $a + d \in b$ with $\text{val}(d)$ sufficiently large, $r(v(d)$ is not a redundant parameter in $\phi_1(a + d, y)$. The lemma follows. \qed

**Lemma 2.16.** Let $f : VF^n \longrightarrow VF^m$ be a definable function. For almost all $\bar{\pi} \in VF^n$ there are $\bar{\pi}$-definable $\epsilon, \delta \in \Gamma$ such that, for any $\bar{a} \in o(\bar{\pi}, \delta)$ with $\bar{b} \neq \bar{\pi}$,

$$\text{val}(f(\bar{b}) - f(\bar{\pi})) \geq \epsilon + \text{val}(\bar{b} - \bar{\pi}).$$

**Proof.** We do induction on $n$. The base case $n = 1$ is readily implied by Lemma 2.15.

We proceed to the inductive step. By the inductive hypothesis, for each $b \in VF$, there is a $b$-definable subset $X_b \subseteq VF^{n-1} \times \{b\}$ such that the conclusion of the lemma holds with respect to $f : \{VF^{n-1} \times \{b\}\}$ and $X_b$. Symmetrically, for each $\bar{\pi} \in VF^{n-1}$, there is an $\bar{\pi}$-definable subset $X_{\bar{\pi}} \subseteq VF^n = \{\bar{\pi}\} \times VF$ such that the conclusion of the lemma holds with respect to $f : \{\bar{\pi}\} \times VF$ and $X_{\bar{\pi}}$. Let $X_1 = \bigcup_{b \in VF} X_b$, and $X_2 = \bigcup_{\bar{\pi} \in VF^{n-1}} X_{\bar{\pi}}$. Since dim$_{VF} X_0 < n - 1$ and dim$_{VF} X_{\bar{\pi}} = 0$, by compactness, dim$_{VF}(X_1 \cup X_2) < n$.

Let $A = VF^n \setminus X_1$ and consider any $(\bar{\pi}, b) \in A$. Let $(\epsilon_b, \delta_b) \in \Gamma^2$ be an $(\bar{\pi}, b)$-definable pair such that, for any $\bar{\pi}' \in o(\bar{\pi}, \delta_b)$ with $\bar{\pi}' \neq \bar{\pi}$,

$$\text{val}(f(\bar{\pi}', b) - f(\bar{\pi}, b)) \geq \epsilon_b + \text{val}(\bar{\pi}' - \bar{\pi}).$$

Let $h_{\bar{\pi}} : \text{fib}(A, \bar{\pi}) \longrightarrow \Gamma^2$ be the $\bar{\pi}$-definable function given by $(\bar{\pi}, b) \mapsto (\epsilon_b, \delta_b)$. For each $(\epsilon, \delta) \in \Gamma^2$ let $Y_{\epsilon, \delta}$ be the topological interior of $h_{\bar{\pi}}^{-1}(\epsilon, \delta)$. Let

$$Y_{\bar{\pi}} = \bigcup_{(\epsilon, \delta) \in \Gamma^2} Y_{\epsilon, \delta} \quad \text{and} \quad Y = \bigcup_{\bar{\pi} \in PF_{< n} A} \{(\bar{\pi}) \times (\text{fib}(A, \bar{\pi}) \setminus Y_{\bar{\pi}})\}.$$ 

Note that, by $C$-minimality, dim$_{VF}(h_{\bar{\pi}}^{-1}(\epsilon, \delta) \setminus Y_{\epsilon, \delta}) = 0$ for every $(\epsilon, \delta) \in \Gamma^2$ and hence, by compactness, dim$_{VF}(\text{fib}(A, \bar{\pi}) \setminus Y_{\bar{\pi}}) = 0$ and dim$_{VF} Y < n$.

Let $X = X_1 \cup X_2 \cup Y$. Let $(\bar{\pi}_1, b_1) \in VF^n \setminus X$ and $h_{\bar{\pi}_1}(b_1) = (\epsilon_1, \delta_1)$. Since the corresponding interior $Y_{\epsilon_1, \delta_1}$ is nonempty, there are $(\bar{\pi}_1, b_1)$-definable $\delta_2, \epsilon_2 \in \Gamma$ such that $o(b_1, \delta_2) \subseteq Y_{\epsilon_1, \delta_1}$ and, for any $b_2 \in o(b_1, \delta_2)$ with $b_2 \neq b_1$,

$$\text{val}(f(\bar{\pi}_1, b_2) - f(\bar{\pi}_1, b_1)) \geq \epsilon_2 + \text{val}(b_2 - b_1).$$

On the other hand, for any $b_2 \in o(b_1, \delta_2)$ and any $\bar{\pi}_2 \in o(\bar{\pi}_1, \delta_1)$ with $\bar{\pi}_2 \neq \bar{\pi}_1$,

$$\text{val}(f(\bar{\pi}_2, b_2) - f(\bar{\pi}_1, b_2)) \geq \epsilon_1 + \text{val}(\bar{\pi}_2 - \bar{\pi}_1).$$

We then have

$$\text{val}(f(\bar{\pi}_2, b_2) - f(\bar{\pi}_1, b_1)) \geq \min\{\text{val}(f(\bar{\pi}_1, b_2) - f(\bar{\pi}_1, b_1)), \text{val}(f(\bar{\pi}_2, b_2) - f(\bar{\pi}_1, b_2))\}$$

$$\geq \min\{\epsilon_1, \epsilon_2\} + \min\{\text{val}(b_2 - b_1), \text{val}(\bar{\pi}_2 - \bar{\pi}_1)\}$$

$$= \min\{\epsilon_1, \epsilon_2\} + \text{val}(\bar{\pi}_2, b_2) - (\bar{\pi}_1, b_1)).$$

Since dim$_{VF} X < n$, the lemma follows. \qed

Clearly this lemma holds with respect to any definable function $f : X \longrightarrow VF^m$ with $X \subseteq VF^n$ and dim$_{VF} X = n$, since $f$ may be extended to $VF^n$ by sending $VF^n \setminus X$ to any definable tuple in $VF^m$. 

Lemma 2.17. Let \( f : VF^n \to VF^m \) be a definable function. Then there is a definable closed subset \( X \subseteq VF^n \) with \( \dim_{VF} X < n \) such that \( f \upharpoonright (VF^n \setminus X) \) is continuous with respect to the valuation topology.

Proof. Let \( X \subseteq VF^n \) be the definable subset of “discontinuous points” of \( f \); that is, \( \overline{X} \subseteq X \) if and only if there is a \( \gamma \in \Gamma \) such that \( f^{-1}(\phi(\overline{X}, \gamma)) \) fails to contain any open ball around \( \overline{X} \). Let \( \overline{X} \) be the topological closure of \( X \), which is definable, and set \( f_1 = f \upharpoonright (VF^n \setminus \overline{X}) \). For any \( \overline{p} \in VF^n \setminus \overline{X} \) and any \( \gamma \in \Gamma \), since \( f^{-1}(\phi(\overline{p}, \gamma)) \) contains an open ball around \( \overline{p} \), \( f^{-1}(\phi(\overline{p}, \gamma)) \) must also contain an open ball around \( \overline{p} \). So it is enough to show that \( \dim_{VF} \overline{X} < n \), which, by Lemma 2.12, is equivalent to showing that \( \dim_{VF} X < n \).

Suppose for contradiction that \( \dim_{VF} X = n \). Let \( Y \subseteq VF^n \) be the definable subset given by Lemma 2.14 for \( f \). Since \( \dim_{VF}(X \setminus Y) = n \), by Lemma 2.12 again, \( X \setminus Y \) contains an open polyball \( b \). Fix an \( \overline{p} \in b \) and let \( \gamma \in \Gamma \) be such that \( f^{-1}(\phi(\overline{p}, \gamma)) \) fails to contain any open ball around \( \overline{p} \). By Lemma 2.14 there are \( \epsilon, \delta \in \Gamma \) such that

1. \( \phi(\overline{p}, \delta) \subseteq \overline{b} \),
2. \( \epsilon + \delta > \gamma \),
3. for any \( \overline{p} \in \phi(\overline{p}, \delta) \) with \( \overline{b} \not\subseteq \overline{p} \), \( \text{val}(f(\overline{p}) - f(\overline{b})) \geq \epsilon + \delta \).

So \( \phi(\overline{p}, \delta) \subseteq f^{-1}(\phi(\overline{p}, \gamma)) \), contradiction. \( \square \)

Definition 2.18. A function \( f \) on \( VF^n \) is locally constant at \( \overline{p} \) if there is an open subset \( U_{\overline{p}} \subseteq VF^n \) containing \( \overline{p} \) such that \( f \upharpoonright U_{\overline{p}} \) is constant. If \( f \) is locally constant at every point in an open subset \( X \) then \( f \) is locally constant on \( X \).

Lemma 2.19. Let \( f : VF^n \to \mathcal{P}(RV^m) \) be a definable function. Then \( f \) is locally constant almost everywhere.

Proof. We do induction on \( n \). For the base case \( n = 1 \), let \( A \subseteq VF \) be the definable subset of those \( a \in VF \) such that \( f \) is not constant on any \( \phi(a, \gamma) \). Let \( \overline{A} \) be the topological closure of \( A \). It is enough to show that \( \dim_{VF} \overline{A} = 0 \), which, by \( C \)-minimality, is equivalent to showing that \( A \) is finite. Suppose for contradiction that \( A \) is infinite. By \( C \)-minimality again there is a definable \( \gamma \in \Gamma \) such that \( A \) contains infinitely many cosets of \( \phi(0, \gamma) \). By Lemma 2.10 \( f \) fails to be constant on only finitely many cosets of \( \phi(0, \gamma) \), contradiction.

We proceed to the inductive step. For any \( \overline{a} = (a_1, \overline{a}_1) \in VF^n \), let \( (a_{\overline{a}}, \beta_{\overline{a}}) \in \Gamma^2 \) be an \( \overline{a} \)-definable pair such that \( f \) is constant on both \( \phi(a_{\overline{a}}, \alpha_{\overline{a}}, \beta_{\overline{a}}) \) and \( \{a_1\} \times \phi(\overline{a}_1, \beta_{\overline{a}_1}) \). If no such pair exists then set \( \beta_{\overline{a}} = \beta_{\overline{a}_1} = \infty \). Let \( g : VF^n \to \Gamma^2 \) be the function given by \( \overline{a} \mapsto (a_{\overline{a}}, \beta_{\overline{a}}) \). By the inductive hypothesis and compactness, \( \dim_{VF} g^{-1}(\infty, \infty) < n \). For each \( (\alpha, \beta) \in \Gamma^2 \) let \( Y_{\alpha, \beta} \) be the topological interior of \( g^{-1}(\alpha, \beta) \). By Lemma 2.12 \( \dim_{VF}(g^{-1}(\alpha, \beta) \setminus Y_{\alpha, \beta}) < n \). Let \( Y = \bigcup_{(\alpha, \beta) \in \Gamma^2} Y_{\alpha, \beta} \). By compactness, \( \dim_{VF}(VF^n \setminus Y) < n \). For any \( \overline{a} = (a_1, \overline{a}_1) \in Y \), since \( Y_{\alpha_{\overline{a}}, \beta_{\overline{a}}} \) contains an open polyball around \( \overline{a} \), clearly for any sufficiently large \( \gamma \) and any \( (a_1', \overline{a}_1') \in \phi(\overline{a}, \gamma) \) we have \( f(a_1, \overline{a}_1) = f(a_1', \overline{a}_1') = f(a_1', \overline{a}_1') \). So \( f \) is locally constant on \( Y \). \( \square \)

3. Differentiation in VF

This section is an excerpt from the forthcoming paper [15].
**Lemma 3.3.** Let $X \subseteq VF^m \times VF^n$, $\pi \in VF^n$, and $L \subseteq VF^m$. We say that $L$ is a limit set of $X$ at $\pi$, written as $\lim_{(pr_{\leq n})_X - \pi} X \subseteq L$, if for every $\epsilon \in \Gamma$ there is a $\delta \in \Gamma$ such that if $\pi_o \in o(\pi, \delta) \cap ((pr_{\leq n})_X \setminus \pi)$ then $fib(X, \pi_o) \subseteq \bigcup_{\delta \in L} o(\delta, \epsilon)$ for some $L' \subseteq L$.

A limit set $L$ of $X$ at $\pi$ is minimal if no proper subset of $L$ is a limit set of $X$ at $\pi$. Observe that if $\lim_{(pr_{\leq n})_X - \pi} X \subseteq L$ and $b \in L$ is not isolated in $L$ then actually $\lim_{(pr_{\leq n})_X - \pi} X \subseteq L \setminus \{b\}$. So in a minimal limit set every element is isolated. Moreover, if a minimal limit set exists then its topological closure is unique:

**Lemma 3.2.** Let $L_1, L_2 \subseteq VF^m$ be two minimal limit sets of $X$ at $\pi$ and $\overline{L_1}$, $\overline{L_2}$ their topological closures. Then $\overline{L_1} = \overline{L_2}$.

**Proof.** Suppose for contradiction that, say, $\overline{L_1} \setminus \overline{L_2} \neq \emptyset$ and hence there is a $\overline{b} \in L_1 \setminus L_2$. So there is an $\epsilon \in \Gamma$ such that $\overline{\pi_o} \cap L_2 = \emptyset$. Let $\delta \in \Gamma$ be such that, for all $\pi \in o(\pi, \delta) \cap ((pr_{\leq n})_X \setminus \pi)$, $fib(X, \pi_o) \subseteq \bigcup_{\overline{\delta} \in L_2} o(\overline{\delta}, \epsilon)$ for some $L_2' \subseteq L_2$. Since $\overline{\pi_o} \cap o(\overline{\pi}, \epsilon) = \emptyset$ for any $\overline{\delta} \in L_2$, we see that $L_1 \setminus \{\overline{b}\}$ is a limit set of $X$ at $\pi$, contradicting the minimality condition on $L_1$. So $\overline{L_1} \subseteq L_2$ and symmetrically $\overline{L_2} \subseteq \overline{L_1}$.

This lemma justifies the equality $\lim_{(pr_{\leq n})_X - \pi} X = L$ when $L$ is a closed (hence the unique) minimal limit set $L$ of $X$ at $\pi$.

**Lemma 3.3.** If $X_1, X_2 \subseteq VF^m \times VF^n$ with $pr_{\leq n} X_1 = Z$ and $\lim_{Z - \pi} X_i = L_i$, then $\lim_{Z - \pi} (X_1 \cup X_2) = L_1 \cup L_2$.

**Proof.** Let $X = X_1 \cup X_2$ and $L = L_1 \cup L_2$. Clearly $L$ is a closed limit set of $X$ at $\pi$. We need to show that it is minimal. To that end, fix a $\overline{b} \in L_1$. If $\overline{b} \in L_1 \cap L_2$ then, since $\overline{b}$ is isolated in both $L_1$ and $L_2$, there is an $\epsilon \in \Gamma$ such that $\overline{\pi_o} \cap (L \setminus \{\overline{b}\}) = \emptyset$. If $\overline{b} \in L_1 \setminus L_2$ then, since $L_2$ is closed, there is again an $\epsilon \in \Gamma$ such that $\overline{\pi_o} \cap (L \setminus \{\overline{b}\}) = \emptyset$. Now, since $L_1$ is a limit set of $X_1$ at $\pi$, but $L_1 \setminus \{\overline{b}\}$ is not, we may assume that $\epsilon$ is so large that there is a $\tau_1 \in Z$ such that $fib(X_1, \tau_1) \cap o(\tau_1, \epsilon) \neq \emptyset$. So $L \setminus \{\overline{b}\}$ cannot be a limit set of $X$ at $\pi$. This shows that $L$ is minimal.

**Lemma 3.4.** Let $X \subseteq VF \times VF^n$ be a definable subset such that $pr_1 | X$ is a finite-to-one map. Let $k$ be the maximal size of the subsets $fib(X, b)$, where $b$ ranges over $pr_1 X = Z$. Let $a \in VF$ and suppose that there is an open ball $b$ containing a such that $b \setminus \{a\} \subseteq Z$. Suppose that $\lim_{Z - a} X = L$. If $L$ is finite then $|L| \leq k$.

**Proof.** Let $L = \{b_1, \ldots, b_l\}$ and suppose for contradiction that $l > k$. Let $\alpha \in \Gamma$ be such that $o(b_i, \alpha) \cap o(b_j, \alpha) = \emptyset$ whenever $i \neq j$. Without loss of generality we may assume that $Z \setminus \{a\} = b \setminus \{a\}$ and $\bigcup_{c \in Z} fib(X, c) \subseteq \bigcup_{b_i \in B} o(b_i, \alpha)$. For each $B \subseteq L$ with $|B| = k$ let

$$Z_B = \left\{ c \in Z : fib(X, c) \subseteq \bigcup_{b_i \in B} o(b_i, \alpha) \right\}. $$

Each $Z_B$ is $(L, \alpha)$-definable. Let $X_B = X \cap (Z_B \times VF^n)$. By $C$-minimality, some $Z_B \cup \{a\}$ contains an open ball around $a$. Clearly

$$\lim_{Z - a} X = \lim_{Z_B - a} X_B \subseteq B,$$
contradicting the assumption that \( \lim_{x \to a} X = L \).

\[ \square \]

**Lemma 3.5.** Let \( b \subseteq VF \) be a ball containing 0 and \( A \subseteq (b \setminus \{0\}) \times VF^m \) a definable subset over \( b \setminus \{0\} \) such that \( pr_1 \restriction A : A \to b \setminus \{0\} \) is a finite-to-one map. Then one of the following two possibilities occurs:

1. for every \( \epsilon \in \Gamma \) there is a \( \delta \in \Gamma \) such that if \( b \in \sigma(0, \delta) \cap (b \setminus \{0\}) \) then \( \text{val}(\pi) < \epsilon \) for some \( \pi \in \text{fib}(A, b) \);
2. there is a finite subset \( L \subseteq VF^m \) such that \( \lim_{b \setminus \{0\} \to 0} A = L \).

**Proof.** We first consider the basic case \( m = 1 \). Without loss of generality we may assume that \( \text{rad}(b) < \infty \). Let \( \phi \) be a quantifier-free formula in disjunctive normal form that defines \( A \), where we assume that no disjunct of \( \phi \) is redundant. Each disjunct \( \phi_i \) of \( \phi \) defines a subset \( A_i \) of \( A \). Let \( X_i = \text{pr}_1 A_i \). By C-minimality, there is a definable open ball \( c \) containing 0 such that, for each \( X_i \), either \( c \subseteq X_i \cup \{0\} \) or \( c \cap X_i = \emptyset \). Replacing \( b \) with \( c \), we may assume that \( b \) is an open ball and every \( X_i \) is the punctured ball \( b \setminus \{0\} = a \).

We may think of \( A_i \) as a function \( F_i \) from \( a \) to the family of finite subsets of \( VF \) and \( A \) as the function \( F = \bigcup_i F_i \). Suppose that the lemma holds for each \( A_i \). If the first item of the lemma occurs to some \( A_i \), then clearly it occurs to \( A \). If the second item of the lemma occurs to every \( A_i \), then, by Lemma 3.3 it also occurs to \( A \). So we may assume that \( \phi \) is a conjunction of literals and \( F = F_1 \). Since \( pr_1 \restriction A \) is finite-to-one, some conjunct of \( \phi \) a VF-sort equality (possibly after replacing all conjuncts of the form \( \text{rv}(g(x, y)) = \infty \) with \( g(x, y) = 0 \)).

By C-minimality, it is easy to see that if \( \text{ran}(F) \) is finite then \( F \) is constant on some open ball punctured at 0 and hence the second item of the lemma occurs. So let us assume that \( \text{ran}(F) \) is infinite. Let \( g_i(x, y) = 0 \) enumerate all the VF-sort equalities in \( \phi \). Consider \( g_1(x, y) \). Write it as \( y^m h(x)g_1^*(x, y) \), where \( h(x) \in VF(\langle 0 \rangle)[x] \) and \( g_1^*(x, y) \in VF(\langle 0 \rangle)[x, y] \) is of the form

\[ h_n(x)y^n + \cdots + h_0(x), \]

where the polynomials \( h_j(x)y^j \in VF(\langle 0 \rangle)[x, y] \) are relatively prime. Further shrinking \( a \) if necessary, we may assume that \( a \) does not contain any root of \( h(x) \) or \( h_j(x) \). If \( n = 0 \) then clearly \( F \) is a constant function, contradicting our assumption. So \( n > 0 \); that is, for each \( a \in a \), the Newton polygon of the polynomial \( g_1^*(a, y) \) is nontrivial. Now we carry out this procedure for every \( g_i(x, y) \), which produces polynomials \( g_i^*(x, y) \). Then, it is not hard to see that there is a finite definable partition \( Y_j \) of \( a \) such that, for every \( a_1, a_2 \in Y_j \),

1. the complete data of the Newton polygons of the polynomials \( g_1^*(a_1, y) \) are the same as that of the polynomials \( g_1^*(a_2, y) \), which include the lengths of the line segments, the signs of the slopes, the complete (lexicographic) ordering of the slopes, etc.,
2. the distribution of \( F(a_1) \) along the line segments of the Newton polygons of the polynomials \( g_1^*(a_1, y) \) is the same as that of \( F(a_2) \) along the line segments of the Newton polygons of the polynomials \( g_1^*(a_2, y) \).

As above, by C-minimality, one of the pieces of this partition, say, \( Y_1 \), contains a definable open ball \( a' \) punctured at 0. Without loss of generality we may just assume that \( a = a' \). For \( a \in a \), in accordance with the first condition above, let \( E_o(j) \) be a uniform enumeration of the line segments of the Newton polygons of the polynomials \( g_i^*(a, y) \). It is not hard to see that, for any \( j \), if the slopes of the
line segments in \( \{ E_a(j) : a \in A \} \) are not bounded from below, then for every \( \epsilon \in \Gamma \) there is a \( \delta \in \Gamma \) such that, for every \( a \in A \) with \( \text{val}(a) > \delta \), the slope of \( E_a(j) \) is lower than \( \epsilon \). In this situation, if \( F(a) \) contains an element on \( E_a(j) \) for some \( a \in A \) (hence for every \( a \in A \) by the second condition above), then the first item of the lemma occurs.

For each \( a \in A \) let \( \Delta^a_\epsilon = \{ \text{val}(d) : g^*_{\epsilon}(a, d) = 0 \} \). Note that every \( \Delta^a_\epsilon \) is nonempty and finite. We consider two cases. If \( g^*_1(0, y) \) is a constant, that is, if \( g^*_1(x, y) \) is of the form

\[
x y (h_n^*(x) y^{n-1} + \ldots + h_1^*(x)) + h_0(x),
\]

where \( h_1^*(x), \ldots, h_n^*(x) \in VF(\emptyset)[x] \), then there is an \( \alpha \in \Gamma \) such that, for every \( a \in A \) with \( \text{val}(a) \) sufficiently large, \( \text{val}(h_0(a)) = \alpha \). In this case, for every \( \epsilon \in \Gamma \) there is a \( \delta \in \Gamma \) such that, for every \( a \in A \) with \( \text{val}(a) > \delta \), every value in \( \Delta^a_\epsilon \) is lower than \( \epsilon \); that is, the first item of the lemma occurs. On the other hand, if \( g^*_1(0, y) \) is a nontrivial polynomial then, by the two conditions listed above, one of the following two possibilities occurs:

1. the first item of the lemma occurs;
2. every \( g^*_1(0, y) \) is a nontrivial polynomial in \( y \) and, for any \( a \in A \) with \( \text{val}(a) \) sufficiently large, \( g^*_1(a, b) = 0 \) only if there is a root \( d \) of \( g^*_1(0, y) \) such that \( \text{val}(b - d) \) is sufficiently large.

Therefore, if the first item of the lemma does not occur, then there are common roots \( d_1, \ldots, d_m \) of the polynomials \( g^*_1(0, y) \) such that for every \( \epsilon \in \Gamma \) there is a \( \delta \in \Gamma \) such that, for every \( a \in A \) with \( \text{val}(a) > \delta \), \( F(a) \subseteq \bigcup \sigma(d_i, \epsilon) \). By an argument similar to the one in the proof of Lemma 3.4 we see that for some subset \( L \subseteq \{ d_1, \ldots, d_m \} \) we have that \( \lim_{a \to 0} A = L \).

This concludes the basic case \( A \subseteq (b \setminus \{ 0 \}) \times VF \). In general, suppose that \( A \subseteq (b \setminus \{ 0 \}) \times VF^m \). For each \( i \leq m \), let

\[
A_i = \{ (b, a) : (b, \overline{m}_1, a, \overline{m}_2) \in A \text{ and } a \in \text{pr}_{1+i} \).
\]

If the first item of the lemma occurs to some \( A_i \), then clearly it occurs to \( A \). If for every \( i \) we have that \( \lim_{b \setminus \{ 0 \} \to 0} A_i = L_i \) for some finite \( L_i \), then it is easy to see that \( \lim_{b \setminus \{ 0 \} \to 0} A \subseteq L_1 \times \cdots \times L_m \) and hence, as in the proof of Lemma 3.4, there is an \( L \subseteq L_1 \times \cdots \times L_m \) such that \( \lim_{b \setminus \{ 0 \} \to 0} A = L \). So the general case follows from the basic case.

The proof of the following lemma, which will be useful for integration with additive characters, contains a typical application of Lemma 3.5.

**Definition 3.6.** Let \( X \subseteq VF^m \). A definable function \( p : X \to \Gamma \) is a **volumetric partition** of \( X \) if for any \( \overline{m} \in X \) the function \( p \) is constant on \( \sigma(\overline{m}, p(\overline{m})) \cap X \).

**Lemma 3.7.** Let \( X \subseteq VF^m \) be a definable closed subset such that \( \text{val}(X) \) is bounded from below. Let \( p : X \to \Gamma \) be a volumetric partition of \( X \). Then \( p(X) \) is bounded from above.

**Proof.** Suppose for contradiction that \( p(X) \) is unbounded from above. For each \( \gamma \in \Gamma \) let \( X_\gamma = \{ \overline{m} \in X : p(\overline{m}) > \gamma \} \). Let \( \overline{m} \in VF \) such that \( p \in \langle VF(S), \overline{m} \rangle \)-definable and set \( S^* = \langle VF(S), \overline{m} \rangle \). For each \( c \in M \) let \( S^*_c = \langle c, S^* \rangle \). By Lemma 2.3 \( \text{acl}(S^*_c) \) is a model of \( ACVF^m_{\text{BS}} \), and hence \( \text{val}(c) \cap \text{acl}(S^*_c) \) is nonempty. By compactness, there is an \( S^* \)-definable subset \( Y \subseteq M \times X \) over \( M \) such that \( \text{pr}_1 \restriction Y \) is finite-to-one and, for each \( c \in M \) and each \( \overline{m} \in \text{fib}(Y, c) \), \( p(\overline{m}) > \text{val}(c) \). Since \( \text{val}(X) \) is bounded
from below and $X$ is closed, by Lemma 3.5 there is a finite subset $L \subseteq X$ such that $\lim_{M \rightarrow \{\emptyset\} - 0} Y = L$. So, for any $\overline{\pi} \in L$, there is a $c \in M$ with $\text{val}(c) > p(\overline{\pi})$ such that $\text{fib}(Y, c) \cap a(\overline{\pi}, p(\overline{\pi}))$ is nonempty. Since $p(\text{fib}(Y, c)) > \text{val}(c)$, this contradicts the assumption that $p$ is a volumetric partition of $X$. □

**Definition 3.8.** Let $f : VF^m \rightarrow VF^n$ be a definable function. For any $\overline{\pi} \in VF^n$, we say that $f$ is differentiable at $\overline{\pi}$ if there is a linear map $\lambda : VF^n \rightarrow VF^m$ such that, for any $\epsilon \in \Gamma$, if $\overline{b} \in VF^n$ and $\text{val}(\overline{b})$ is sufficiently large then

$$\text{val}(f(\overline{\pi} + \overline{b}) - f(\overline{\pi}) - \lambda(\overline{b})) - \text{val}(\overline{b}) > \epsilon.$$ 

It is straightforward to check that if such a linear function $\lambda$ exists then it is unique and hence may be called the *derivative of $f$ at $\overline{\pi}$*, which shall be denoted as $\partial_{\overline{\pi}} f$.

For each $1 \leq j \leq m$ let $f_i = p_{ij} \circ f$. For any $\overline{\pi} = (a_i, \overline{\pi}_i) \in VF^n$, if the derivative of the function $f_i \mid VF \times \{\overline{a}_i\}$ at $\overline{a}_i$ exists then we call it the *$i$th partial derivative of $f$ at $\overline{\pi}$* and denote it as $\partial_{\overline{\pi}}^i f$.

The classical differentiation rules, such as the product rule and the chain rule, hold with respect to this definition. Here we only check the chain rule:

**Lemma 3.9.** Let $f : VF^n \rightarrow VF^m$ be differentiable at $\overline{\pi} \in VF^n$ and $g : VF^m \rightarrow VF^i$ differentiable at $f(\overline{\pi})$. Then $g \circ f$ is differentiable at $\overline{\pi}$ and

$$\partial_{\overline{\pi}} (g \circ f) = (d_{f(\overline{\pi})} g) \circ (\partial_{\overline{\pi}} f).$$

**Proof.** Fix an $\epsilon \in \Gamma$. Since $\partial_{\overline{\pi}} f$ is a linear function, there is an $\alpha \in \Gamma$ such that, for every $\overline{b} \in VF^n$, $\text{val}(\partial_{f(\overline{\pi})} f(\overline{b})) - \text{val}(\overline{b}) \geq \alpha$. Similarly there is a $\beta \in \Gamma$ such that, for every $\overline{b} \in VF^m$, $\text{val}(d_{f(\overline{\pi})} g(\overline{b})) - \text{val}(\overline{b}) \geq \beta$. Let $s : VF^n \rightarrow VF^m$ be the function such that, for any $\overline{b} \in VF^n$, $f(\overline{\pi} + \overline{b}) = f(\overline{\pi}) + \partial_{f(\overline{\pi})} f(\overline{b}) + s(\overline{b})$. By assumption, for any $\overline{b} \in VF^n$ with $\text{val}(\overline{b})$ sufficiently large,

$$\text{val}(d_{f(\overline{\pi})} g(s(\overline{b}))) \geq \text{val}(s(\overline{b})) + \beta > \text{val}(\overline{b}) + \epsilon - \beta + \beta = \text{val}(\overline{b}) + \epsilon.$$ 

Therefore, if $\text{val}(\overline{b})$ is sufficiently large then either

$$\text{val}(g(f(\overline{\pi} + \overline{b})) - g(f(\overline{\pi})) - \partial_{f(\overline{\pi})} g(d_{f(\overline{\pi})} f(\overline{b}))) > \text{val}(\overline{b}) + \epsilon$$

or

$$\text{val}(g(f(\overline{\pi} + \overline{b})) - g(f(\overline{\pi})) - \partial_{f(\overline{\pi})} g(d_{f(\overline{\pi})} f(\overline{b})))$$

$$= \text{val}(g(f(\overline{\pi} + \overline{b})) - g(f(\overline{\pi})) - \partial_{f(\overline{\pi})} g(d_{f(\overline{\pi})} f(\overline{b})) - \partial_{f(\overline{\pi})} g(s(\overline{b})))$$

$$= \text{val}(g(f(\overline{\pi})) + \partial_{f(\overline{\pi})} f(\overline{b}) + s(\overline{b})) - g(f(\overline{\pi})) - \partial_{f(\overline{\pi})} g(d_{f(\overline{\pi})} f(\overline{b}) + s(\overline{b})))$$

$$> \text{val}(\partial_{f(\overline{\pi})} f(\overline{b}) + s(\overline{b})) + \min \{\beta, \epsilon - \alpha\}$$

$$\geq \text{val}(\overline{b}) + \epsilon.$$ 

In either case the lemma follows. □

**Lemma 3.10.** Let $f : VF^n \rightarrow VF^m$ be a definable function. Then each $\partial^{ij} f$ is defined almost everywhere.

**Proof.** Let $\overline{\pi} = (a_i, \overline{\pi}_i) \in VF^n$. Let $g_{ij}^{\overline{\pi}} : (VF \setminus \{0\}) \rightarrow VF$ be the $\overline{\pi}$-definable function given by

$$b \mapsto (f_j(a_i + b, \overline{\pi}_i) - f_j(\overline{\pi}_i))/b.$$
By Lemma 2.16 for almost all $$a \in \text{VF}^n$$ there is an $$a$$-definable open ball $$b_a$$ punctured at 0 such that $$\text{val}(g^a_{b_a})$$ is bounded from below. By Lemma 3.5 and Lemma 3.4, $$\lim_{b_a \to 0} g^a = \lambda(a)$$ for some $$\lambda(a) \in \text{VF}$$.

**Corollary 3.11.** Let $$f : \text{VF}^n \to \text{VF}^m$$ be a definable function. Then $$f$$ is continuously partially differentiable almost everywhere.

**Proof.** This is immediate by Lemma 3.10 and Lemma 2.17.

Recall that each definable subset may be treated as a function $$\text{VF}^n \to \mathcal{P}(\text{R V}^m)$$ for some $$n$$, $$m$$. Let $$X$$, $$Y$$ be such definable functions on $$\text{VF}^n$$ and let $$f \subseteq X \times Y$$ a definable subset. We say that $$f$$ is an essential $$\text{VF}$$-bijection if there are definable open subsets $$X'$$, $$Y' \subseteq \text{VF}^n$$ such that

1. $$\text{dim}_{\text{VF}}((\text{VF}^n \setminus X') \cup (\text{VF}^n \setminus Y')) < n$$,
2. for each $$a \in X'$$ there is a $$b \in Y'$$ such that

$$f_{a, b} = f \cap (\text{fib}(X, a) \times Y) = f \cap (X \times \text{fib}(Y, b)) \subseteq \text{fib}(X, a) \times \text{fib}(Y, b)$$

and $$f_{a, b}$$ is a bijection $$\text{fib}(X, a) \to \text{fib}(Y, b)$$.

In this situation, we may assume that $$X'$$, $$Y'$$ is the largest pair of subsets that meet the two conditions. Clearly $$f$$ induces a definable bijection $$f' : \text{fib}(X, X') \to \text{fib}(Y, Y')$$, which is called the core of $$f$$, and the projection of $$f'$$ into the VF-coordinates is a definable bijection $$f'_{\text{VF}} : X' \to Y'$$. The core $$f'$$ of $$f$$ is very often identified with $$f$$. Derivatives and partial derivatives of $$f$$ are defined to be those of $$f'_{\text{VF}}$$. By Lemma 3.10, $$f$$ is partially differentiable almost everywhere.

If all partial derivatives exist at a point then the Jacobian is defined in the usual way and is denoted by $$\text{Jcb}$$.

**Remark 3.12.** There are two types of VF-category considered in the Hrushovski-Kazhdan integration theory: measure-preserving ones (classes up to definable bijection subject to the Jacobian and volume form) and non-measure-preserving ones (classes up to definable bijection). The fundamental constructions for the latter have been presented in [16]. The theory for the former is parallel to that for the latter, with some new aspects in relation to the Jacobian and volume form. This shall be presented in the forthcoming paper [15]. We mention two main differences between the two theories. The first is that, in a VF-category with volume form, isomorphisms are essential VF-bijections and hence isomorphism classes are defined up to definable core (almost everywhere). The second is that, when changing variables in an integral in the measuring-preserving mode, as in the classical theory, one needs to change the volume form, which contains the Jacobian as a factor.

In this paper, we shall only consider constant volume forms, that is, translation invariant volume forms. This means that we only need to compute the Jacobian in changing variables. In fact, for most of the discussion below, it is not necessary to distinguish the two types of VF-category since, except in Section 8, the only type of change of variables we shall use is translation, the Jacobian of which is always 1. Hence, for conceptual and notational simplicity, we shall not mention the Jacobian or volume form explicitly. There are two exceptions. The Jacobian and volume form are present in the change of variables formula (Lemma 4.5 and Lemma 5.12), since it is the main point of the formula. Also, certain coefficients in the Weil representation of $$\text{SL}_2(\text{VF})$$ (see Section 8) where the formalism for integration with
volume form will be explained) depend on the Jacobian in the change of variables formula and hence are computed accordingly.

4. $\mathbb{R}$-valued functions

For the rest of this paper we assume that the substructure $S$ is $(\text{VF}, \Gamma)$-generated.

Throughout this section let $\mathcal{C}$ be an RV-category of definable subsets such that, modulo a canonical congruence relation $I_{\text{sp}}$ as in [16, Theorem 12.2], there is a canonical isomorphism of Grothendieck semirings

$$\int_+ : K_+ \text{VF}_* \rightarrow K_+ \mathcal{C}/I_{\text{sp}}^\times,$$

where $\text{VF}_*$ is a suitable $\text{VF}$-category of definable subsets. As usual, if we allow an additional set $A$ of parameters then the category is accordingly denoted as $\mathcal{C}_A$ and all objects derived from $\mathcal{C}_A$ inherit the subscript, which may be dropped in context if there is no danger of confusion.

Let $K\mathcal{C}$ be the groupification of $K_+ \mathcal{C}$. Recall that an element in the $n$th graded piece $K_+ \mathcal{C}[n]$ of $K_+ \mathcal{C}$ is denoted as $[X]_n$ and a general element of $K_+ \mathcal{C}$ is denoted as $[X]$, where $X \in \mathcal{C}$. For notational ease, canonical images of elements of $K_+ \mathcal{C}$ are denoted as such. Note that, since $I_{\text{sp}}^\times$ is not a homogeneous congruence relation, we do not have a natural gradation for $K_+ \mathcal{C}/I_{\text{sp}}^\times$. However, in the groupification $K\mathcal{C}/\mathcal{I}$ of $K_+ \mathcal{C}/I_{\text{sp}}^\times$, the ideal $\mathcal{I}$ determined by $I_{\text{sp}}^\times$ is generated by the element $[1]_0 \oplus \mathbf{J}$, where

$$\mathbf{J} = [1]_1 - [(\text{RV}_x)^{>1}].$$

So $K\mathcal{C}/\mathcal{I}$ may be embedded into the zeroth graded piece $\mathbb{P}^0$ of the $\mathbb{Z}$-graded ring $K\mathcal{C}[J^{-1}]$ in a canonical way; see the discussion in [16, Section 12]. Note that it is possible to have $[X]_n J^{-n} = [X]_m J^{-m}$ even when $n \neq m$. We have a canonical semiring homomorphism

$$\int_{\tau_0} : K_+ \text{VF}_* \rightarrow \mathbb{P}^0,$$

where the symbol $\tau_0$ indicates that the target ring is $\mathbb{P}^0$. This homomorphism is just the composition of the canonical mappings

$$K_+ \text{VF}_* \xrightarrow{f_*} K_+ \mathcal{C}/I_{\text{sp}}^\times \xrightarrow{\mathcal{I}} K\mathcal{C}/\mathcal{I} \xrightarrow{\tau_0} \mathbb{P}^0.$$

Each element of the commutative ring $\mathbb{P}^0$ may still be represented by a definable subset. To see this, it is enough to consider the images of the graded pieces of $K\mathcal{C}$ separately. An element $[X]_n$ in the $n$th graded piece of $K\mathcal{C}$ may be represented by a function $K_+ \mathcal{C}[n] \rightarrow \mathbb{Z}$ with finite support, where $\mathbb{Z}$ is understood as a subset of the residue field $\mathbf{K}$. We shall write this function as

$$\sum_{i=1}^k n_i [X_i]_n,$$

where $n_i \in \mathbb{Z}$ and $[X_i]_n \in K_+ \mathcal{C}[n]$. So $[X]_n$ may be represented by the definable subset

$$((n_1, X_1), \ldots, (n_k, X_k)).$$

In particular the element $\mathbf{J}$ is represented by the subset

$$J = ((1, 1), (-1, (\text{RV}_x)^{>1})).$$
So \([X]_n J^{-n} \in P^0\) may be represented by the subset
\[\left(\left(n_1, X_{1}\right), \ldots, \left(n_k, X_{k}\right)\right), J^n).\]
In general, since any sum \(\sum |X|_n J^{-n}\) may be written as \(\sum |X|_n J^{n-n} J^{-n}\) for sufficiently large \(n\), we see that any element of \(P^0\) may be represented by a definable subset \(X\) of the form just described. Sometimes the element is denoted as \([X]\) for notational simplicity. If \([X]\) is the image of an element of \(K_+\) then \([X]\) may be expressed as \(\sum |X|_n\), where \([X]|_n \in K_+ C[n_1].\)

Let \(V \subseteq V F^m\) be definable. A definable function \(F : V \rightarrow \mathcal{P}(RV^m)\) may be understood as a representative of a function \([\mu (F)] : V \rightarrow P^0\) if, for every \(\tau \in V, F(\tau)\) is of the form described above. By compactness, there are natural numbers \(m, n\) such that each \(F(\tau)\) may be expressed as
\[\left((-m, X_{-m})\ldots, (m, X_{m})\right), J^n).\]
For each \(-m \leq i \leq m\) we let \(F_i : V \rightarrow \mathcal{P}(RV^m)\) be the definable function such that \(f_i(\tau) = (i, X_i)\). The homomorphism \(\tau_{\tau_0}\) may be extended to all definable functions \([\mu (F)] : V \rightarrow P^0\) such that \(\int \tau_{\tau_0}\) is represented by
\[\left((-m, \int [F_{-m}]), \ldots, (m, \int [F_m]), J^n)\right).\]
It is not hard to see that, by compactness, this definition does not depend on the representative \(F\).

For any \(\tau \in \Gamma\), we write \(\delta\) and \(\alpha\) for the elements \(\int \phi(0, \tau)\) \(\tau_{\tau_0}\) and \(\int \psi(0, \tau)\) \(\tau_{\tau_0}\) in \(P^0\). Note that \(\int \psi(0, \tau) = \alpha_0 = [1]_1\). We also write \(\epsilon, \delta\) for the special elements \(\alpha_0, \alpha_0\) and set \(\epsilon = a\delta\). For the construction of certain integrals below we need to localize \(P^0\) at \(\epsilon\) and obtain the ring \(P^0[\epsilon^{-1}]\).

**Remark 4.1.** The commutative ring \(P^0[\epsilon^{-1}]\) is our motivic analog of the ring \(R\) of real numbers and is henceforth denoted as \(\hat{R}\). Any element in \(\hat{R}\) is of the form \([X] / \epsilon^n\). We may think of the pair \((X, \epsilon^n)\) as a definable subset. The canonical mapping \(P^0 \rightarrow \hat{R}\) is of course not guaranteed to be injective. But a significant part of the structure of \(K C/\mathbb{L}\) does survive in \(\hat{R}\).

Now we describe how to form a group of functions from definable subsets into \(\hat{R}\). Let \(V\) be a definable subset of \(VF^m\). Note that, for every subset \(V' \subseteq V, \langle V'\rangle\) is a \((VF, \Gamma)\)-generated substructure and hence the theory we have developed so far holds with respect to \(\langle V'\rangle\). We form the group of definable functions from \(V\) into \(\hat{R}\), denoted as \(Fn(V, \hat{R})\), as follows. The elements of \(Fn(V, \hat{R})\) are definable sections of the product
\[\prod_{\tau \in V} \hat{R}_{\tau}.\]
Such an element may be represented by a definable subset \(X\) over \(V\), which, as usual, may be conveniently thought of as a function from \(V\) into the class of definable subsets. We may write \([X(\tau)]\) for the image of \(\tau \in V\) in \(\hat{R}_{\tau}\). The element in \(Fn(V, \hat{R})\) represented by \(X\) is denoted as \([X]\). Addition in \(Fn(V, \hat{R})\) is defined coordinatewise. For any \(\tau, \bar{\tau} \in V\), the equality \([X][\tau]\ = \[X][\bar{\tau}\) means that \(X(\tau)\) and \(X(\bar{\tau})\) are representatives of the same element in \(\hat{R}(\tau, \bar{\tau})\).

**Remark 4.2.** As usual, it is better to think of \(Fn(V, \hat{R})\) as an \(\hat{R}\)-module. Each element in \(\hat{R}\) may be treated as a constant function in \(Fn(V, \hat{R})\). However, in
general, we cannot treat every constant function \( f \) in \( \text{Fn}(V, \bar{\mathbb{R}}) \) as an element in \( \bar{\mathbb{R}} \). Of course, we may do so if there is a definable representative of \( f(\bar{\pi}) \) for some \( \bar{\pi} \in V \). In particular, if \( V \) contains a definable point then the subgroup of constant functions may be identified with \( \bar{\mathbb{R}} \). For many interesting definable subsets, for example, definable groups, this is indeed the case.

In the discussion above we have actually constructed a canonical \( \mathbb{P}^0 \)-module homomorphism \( \int_V \tau_{\mathbb{P}^0} : \text{Fn}(V, \mathbb{P}^0) \to \mathbb{P}^0 \). This can now be extended to a canonical \( \bar{\mathbb{R}} \)-module homomorphism \( \text{Fn}(V, \bar{\mathbb{R}}) \to \bar{\mathbb{R}} \) as follows. Let \( f \in \text{Fn}(V, \bar{\mathbb{R}}) \). By compactness, there is a representative \( f' \in \text{Fn}(V, \mathbb{P}^0) \) of \( f(\bar{\pi}) \) for some \( \bar{\pi} \in V \), and a natural number \( k \) such that, for every \( a \in V \),

\[
|f(a)| = \frac{|f'(a)|}{e^k}.
\]

**Definition 4.3.** The \( \bar{\mathbb{R}} \)-module homomorphism \( \int_V \tau_{\bar{\mathbb{R}}} : \text{Fn}(V, \bar{\mathbb{R}}) \to \bar{\mathbb{R}} \) is given by

\[
\int_V f \tau_{\bar{\mathbb{R}}} = \left( \int_V f' \tau_{\mathbb{P}^0} \right) / e^k \in \bar{\mathbb{R}}.
\]

It is not hard to see that this definition does not depend on the representative \( f' \) or the number \( k \).

For any \( i \leq n \) let \( f'_i \in \text{Fn}(\text{pr}_i V, \mathbb{P}^0) \) be such that, for every \( \bar{\pi}_i \in \text{pr}_i V \),

\[
\int_V (f' \mid \text{fib}(V, \bar{\pi}_i)) \tau_{\mathbb{P}^0} = f'_i(\bar{\pi}_i) \in \mathbb{P}^0_{\bar{\pi}_i}.
\]

Now, by the construction of the isomorphism \( \int_+ \) (in particular, [16, Lemma 11.15]) and the definition of the homomorphism \( \int_\tau_{\mathbb{P}^0} \), we have

\[
\int_V f' \tau_{\mathbb{P}^0} = \int_{\text{pr}_i V} f'_i \tau_{\mathbb{P}^0}.
\]

Hence, if we let \( f_i \in \text{Fn}(\text{pr}_i V, \bar{\mathbb{R}}) \) be such that \( f_i(\bar{\pi}_i) = f'_i(\bar{\pi}_i) / e^k \), then

\[
\int_V f \tau_{\bar{\mathbb{R}}} = \left( \int_V f' \tau_{\mathbb{P}^0} \right) / e^k = \left( \int_{\text{pr}_i V} f'_i \tau_{\mathbb{P}^0} \right) / e^k = \int_{\text{pr}_i V} f_i \tau_{\bar{\mathbb{R}}}.
\]

Note that \( f_i \) is the function such that

\[
f_i(\bar{\pi}_i) = \int_{\text{fib}(V, \bar{\pi}_i)} f \tau_{\bar{\mathbb{R}}} \in \bar{\mathbb{R}}_{\bar{\pi}_i}.
\]

**Remark 4.4.** The above discussion shows that, for any \( i, j \leq n \),

\[
\int_{\text{pr}_i V} f_i \tau_{\bar{\mathbb{R}}} = \int_{\text{pr}_j V} f_j \tau_{\bar{\mathbb{R}}}.
\]

This should be understood as the Fubini theorem for \( \bar{\mathbb{R}} \)-valued integrals. A more general version of this will be stated below when the ring \( \bar{\mathbb{R}} \) is further enlarged.

**Lemma 4.5.** Let \( V, W \subseteq \text{VF}^n \) and \( \phi : V \to W \) an isomorphism in \( \text{Mor V} \).

For any \( f \in \text{Fn}(W, \bar{\mathbb{R}}) \),

\[
\int_W (f, \mu) \tau_{\bar{\mathbb{R}}} = \int_V (f \circ \phi, \text{rv}(Jcb \phi)(\mu \circ \phi)) \tau_{\bar{\mathbb{R}}}.
\]

**Proof.** This follows immediately from the construction of \( \int_+ \), Definition 4.3 and compactness. \( \square \)
Notation 4.6. For notational simplicity, the group operation and the inverse operation in any definable group are denoted as usual as + and − respectively. This should not cause any confusion in context.

Remark 4.7. With pointwise multiplication, \( \text{Fn}(V, \tilde{\mathbb{R}}) \) is naturally a commutative ring. However, we adopt the convention that if \( V \) is a definable group then the default multiplication of \( \text{Fn}(V, \tilde{\mathbb{R}}) \) is given by convolution: for any \( f, g \in \text{Fn}(V, \tilde{\mathbb{R}}) \), the convolution \( f \ast g \in \text{Fn}(V, \tilde{\mathbb{R}}) \) is given by

\[
(f \ast g)(a) = \int_{x \in V} f(x)g(a-x) \tau_{\tilde{\mathbb{R}}}.
\]

We emphasize here that, unlike the classical cases, convolution in \( \text{Fn}(V, \tilde{\mathbb{R}}) \) does have an identity, which is of course the function

\[
a \mapsto \begin{cases} 
1, & \text{if } a \text{ is the identity element of } V; \\
0, & \text{otherwise}.
\end{cases}
\]

This makes \( \text{Fn}(V, \tilde{\mathbb{R}}) \) into a commutative ring that is different from the one endowed by pointwise multiplication. The convolution identity is in some sense an analogue of the Dirac delta function.

For certain identities between integrals that shall be established below, in particular the Poisson summation formula (Theorem 6.15), we would like to modify \( R \) with respect to a bounded definable subgroup \( H \) of \( \text{VF}^n \) that is possibly not a polyball around 0. As usual, this involves taking quotient and localization. We shall impose a condition on \( H \) here and another one in Section 6. These conditions are designed to make the Fourier inversion formula hold (Theorem 6.13) and are satisfied at least by subgroups that are polyballs.

Let \( \overline{\pi} \cdot \overline{b} \) be the ordinary dot product for any \( \overline{\pi}, \overline{b} \in \text{VF}^n \). Let \( H_\ast \) be the dual group of \( \text{VF}^n / H \) consisting of those \( \overline{\pi} \in \text{VF}^n \) such that the subset \( \{ \overline{\pi} \cdot \overline{b} : \overline{b} \in H \} \) is contained in \( \mathcal{M} \). We also say that \( H_\ast \) is the annihilator group of \( H \). Let \( H_{\ast \ast} \) be the annihilator group of \( H_\ast \). The dual group of \( H_\ast \) is \( \text{VF}^n / H_{\ast \ast} \). Note that if \( H \) is the trivial group then \( H_\ast = \text{VF}^n / H_{\ast \ast} = \text{VF}^n \).

Hypothesis 4.8. For any \( \beta \in \Gamma \), if \( \alpha \in \Gamma \) is sufficiently low then

\[
(H_\ast \cap \sigma(0, \alpha))_{\ast} \subseteq \sigma(0, \beta),
\]

where \((H_\ast \cap \sigma(0, \alpha))_{\ast}\) is the annihilator group of \( H_\ast \cap \sigma(0, \alpha) \). By \( \sigma \)-minimality, there is a largest definable \( \zeta \in \Gamma \) such that \((H_\ast \cap \sigma(0, \alpha))_{\ast} \subseteq \sigma(0, \alpha)\) for every \( \alpha < \zeta \).

Let \( \kappa, \kappa_{\ast \ast} \) be the images of \([H], [H_{\ast \ast}]\) in \( \tilde{\mathbb{R}} \). Note that if \( H \) is the trivial group then \( \kappa = \kappa_{\ast \ast} = 1 \). For every \( \gamma \in \Gamma \), let \( \kappa_{\ast \gamma} \) and \( \kappa_{\ast \ast \gamma} \) be the images of \([H_\ast \cap \sigma(0, \gamma)]\) and \([H_\ast \cap \sigma(0, \gamma)_{\ast}]\) in \( \tilde{\mathbb{R}} \). We shall use these same notations (\( \kappa, \kappa_{\ast \ast}, \kappa_{\ast \ast \gamma} \), and \( \kappa_{\ast \ast \gamma} \) included) for the subsequent images of what they denote in any ring derived from \( \tilde{\mathbb{R}} \).

Definition 4.9. Let \( f : V \rightarrow \tilde{\mathbb{R}} \) be a definable function such that, for every \( \overline{\pi} \in V \), \( f(\overline{\pi}) \) is of the form

\[
\epsilon(\kappa_\alpha \kappa_{\ast \ast} - \kappa_\beta \kappa_{\ast \ast \gamma}),
\]

where \( \alpha, \beta < \zeta \) are \( \pi \)-definable. The element \( \int_V f \tau_{\tilde{\mathbb{R}}} \in \tilde{\mathbb{R}} \) is an \( H \)-rectangular remainder. The \( H \)-rectangular ideal \( \mathcal{H} \subseteq \tilde{\mathbb{R}} \) is generated by \( H \)-rectangular remainders.
The element $\delta^\alpha \delta^{\alpha^*} \in \bar{\mathbb{R}}/\mathcal{H}$ for any definable $\alpha < \zeta$ shall be denoted as $\delta$.

**Remark 4.10.** The $H$-rectangular ideal $\mathcal{H}$ is closed under parametrization in the following sense. If $f : W \longrightarrow \bar{\mathbb{R}}$ is a definable function such that $f(\pi) \in H(\pi)$ for every $\pi \in W$ then, by compactness, $\int_W f \tau_{\bar{\mathbb{R}}} \in H$.

It hardly needs to be pointed out that the construction of $\mathcal{H}$ is modeled on the behavior of Haar measure on local fields. This type of construction is flexible. More classical relations may be accommodated if we enlarge the $H$-rectangular ideal accordingly. For example,

$$\epsilon_{\alpha+\beta} = \epsilon_\alpha \epsilon_\beta \quad \text{and} \quad \epsilon_{\alpha+\beta} = \epsilon_\alpha \epsilon_\beta$$

for any definable $\alpha, \beta \in \Gamma$.

Let $f \in \text{Fn}(V, \bar{\mathbb{R}}/\mathcal{H})$ and $f', f'' \in \text{Fn}(V, \bar{\mathbb{R}})$ two representatives of $f$. By Remark 4.110

$$\int_V (f' - f'') \tau_{\bar{\mathbb{R}}} \in \mathcal{H}$$

and hence we may define an $\bar{\mathbb{R}}/\mathcal{H}$-module homomorphism

$$\int_V \tau_{\bar{\mathbb{R}}/\mathcal{H}} : \text{Fn}(V, \bar{\mathbb{R}}/\mathcal{H}) \longrightarrow \bar{\mathbb{R}}/\mathcal{H}$$

by

$$f \longmapsto \left( \int_V f' \tau_{\bar{\mathbb{R}}} \right) / \mathcal{H}.$$ 

Now let $\bar{\mathbb{R}}^H = \bar{\mathbb{R}}/\mathcal{H}[h^{-1}]$. As in Definition 4.3 we have an $\bar{\mathbb{R}}^H$-module homomorphism

$$\int_V \tau_{\bar{\mathbb{R}}^H} : \text{Fn}(V, \bar{\mathbb{R}}^H) \longrightarrow \bar{\mathbb{R}}^H.$$ 

Much of the development below can be carried out without distinguishing $\bar{\mathbb{R}}$ and $\bar{\mathbb{R}}^H$. For simplicity, except in Section 6 we shall work with $\bar{\mathbb{R}}$.

5. Integrable functions

Let $\Omega = VF/\mathcal{M}$ and $\theta : VF \longrightarrow \Omega$ the quotient map. Note that $\theta$ is treated as a definable subset of $VF^2$. There is a natural map $\Omega \longrightarrow \Gamma$ that is induced by val, which shall also be denoted as val.

For any $p$-adic field $Q_p$, the natural specialization $\Omega(Q_p)$ of $\Omega$ to $Q_p$ may be identified naturally with the $p$th power roots of unity via any additive character $\chi$ such that, for any $p$-adic number $a$ with $\text{val}(a) \leq 0$, $\chi(a)$ is a $p^{-\text{val}(a)+1}$th root of unity. In general, let $K$ be a number field, $\text{val}_p$ the valuation of $K$ with respect to the prime number $p$, $K_p$ the completion of $K$ with respect to $\text{val}_p$, and $\mathcal{O}_p$, $\mathcal{M}_p$ the corresponding valuation ring and its maximal ideal. Let $\chi$ be an additive character of $K_p$. The **conductor** $f_\chi$ of $\chi$ is the largest ideal of the form $\mathcal{M}_p^m$, where $m$ is a nonnegative integer ($\mathcal{M}_p^0 = \mathcal{O}_p$ by definition), such that $\chi(\mathcal{M}_p^m) = 1$. This always exists, see [14, Lemma 4, p. 114]. The integer $m$ is the **multiplicity** of $f_\chi$. Now, if we view $\Omega$ as a motivic analog of the subgroup of roots of unity of the unit circle and $\theta$ a motivic analog of a generic additive character, then motivic integration with respect to $\theta$, when specialized to (non-archimedean) completions of $K$, may be viewed as integration with respect to all additive characters with multiplicity 1 for almost all $p$ at once.
In the classical situation, summation of a character over any compact subgroup vanishes. To accommodate this phenomenon when we integrate with an additive character, a cancellation rule will be introduced in the construction of the ring in which integration with an additive character takes values. The construction of this ring has its geometrical analog on the complex plane, which is the group ring \( \mathbb{R}[S^1] \), where \( S^1 \) is the unit circle on the complex plane. We note here that, in order to avoid undesirable collapses of volumes of definable subsets, we need to work with bounded subsets of \( \Omega \), that is, subsets \( X \subseteq \Omega \) such that \( \text{val}(X) \) is bounded from below.

**Definition 5.1.** A subset \( X \subseteq \mathcal{V}F^n \) is bounded if \( \text{val}(X) \) is bounded from below. A function \( \phi : \mathcal{V}F^n \rightarrow \mathcal{V}F^n \) is bounded if, for every bounded \( X \subseteq \mathcal{V}F^n \), \( \phi(X) \) is also bounded. A function \( f \in \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) has bounded support if \( \text{supp}(f) \) is bounded, where \( \text{supp}(f) \) is the support of \( f \).

The subset \( \mathcal{B}(\mathcal{V}, \mathcal{R}) \) of \( \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) of all functions with bounded support obviously form a subring of \( \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) with respect to the pointwise product. On the other hand, if \( \mathcal{V} \) is a definable group such that its group operation is a bounded function, then \( \mathcal{B}(\mathcal{V}, \mathcal{R}) \) also forms a subring of \( \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) with respect to the convolution product.

**Definition 5.2.** Let \( \mathcal{V} \) be a definable group and \( W \) a definable subgroup of \( \mathcal{V} \). A definable function \( f \in \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) is \( W \)-invariant if, for any \( \bar{v} \in \mathcal{V} \) and any \( \bar{b} \in W \), \( f(\bar{v}) = f(\bar{v} + \bar{b}) \). The subgroup of \( \mathcal{F}n(\mathcal{V}, \mathcal{R}) \) of \( W \)-invariant functions is denoted as \( \mathcal{F}n(\mathcal{V}, \mathcal{R})^W \).

**Lemma 5.3.** The group \( \mathcal{F}n(\mathcal{V}, \mathcal{R})^\mathcal{M} \) is isomorphic to the group \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \).

**Proof.** It suffices to show that, for each element \( [[F]] \in \mathcal{F}n(\mathcal{V}, \mathcal{R})^\mathcal{M} \) with \( F \) a definable function \( \mathcal{V}F \rightarrow \mathcal{P}(\mathcal{R}V^m) \), there is a definable function \( F_1 : \mathcal{O} \rightarrow \mathcal{P}(\mathcal{R}V^m) \) such that for every \( \omega \in \mathcal{O} \) there is an \( a \in \omega \) such that \( F(a) = F_1(\omega) \). This is immediate by Lemma 2.10. \( \square \)

For any \( f, g \in \mathcal{F}n(\mathcal{O}, \mathcal{R}) \), it is clear that \( f * g \in \mathcal{F}n(\mathcal{O}, \mathcal{R}) \). So \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) is a commutative ring with respect to convolution. Let \( e_0 \in \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) be the function

\[
    a \mapsto \begin{cases} 
        1, & \text{if } a \in \mathcal{M}; \\
        0, & \text{otherwise}.
    \end{cases}
\]

Unfortunately, \( e_0 \) is not the convolution identity, since, for any \( f \in \mathcal{F}n(\mathcal{O}, \mathcal{R}) \),

\[
    (f * e_0)(b) = \int_{x \in \mathcal{V}F} f(x)e_0(b-x) \tau_{\mathcal{R}} = \int_{x \in b+\mathcal{M}} f(x) \tau_{\mathcal{R}} = a f(b).
\]

To remedy this, we “normalize” convolution in \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) as follows: for any \( f, g \in \mathcal{F}n(\mathcal{O}, \mathcal{R}) \),

\[
    (f * g)(b) = a^{-1} \int_{x \in \mathcal{V}F} f(x)g(b-x) \tau_{\mathcal{R}}.
\]

With this normalization, \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) becomes a commutative ring with identity.

We shall identify \( \mathcal{F}n(\mathcal{V}, \mathcal{R})^\mathcal{M} \) with \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) below. Similarly \( \mathcal{F}n(\mathcal{O}, \mathcal{R})^\mathcal{O} \) may be identified with \( \mathcal{F}n(\mathcal{O}, \mathcal{R}) \) and \( \mathcal{F}n(\mathcal{V}, \mathcal{R})^\mathcal{O} \) with \( \mathcal{F}n(\mathcal{V}, \mathcal{R})^\mathcal{O} \). We are interested in the bounded subrings \( \mathcal{B}(\mathcal{O}, \mathcal{R}) \), \( \mathcal{B}(\mathcal{K}, \mathcal{R}) \) and the bounded subgroup
\( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \). Of course \( \mathcal{B}(\mathbb{K}, \tilde{\mathbb{R}}) = \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}}) \), which is a subring of \( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \).

Also, \( \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \) is a subgroup of \( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \). Since the convolution identity of \( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \) is not in \( \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \) and, for any \( f \in \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \) and any \( g \in \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \),

\[
 f * g \in \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}),
\]

we see that \( \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \) is actually an ideal of \( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \).

Let \( \mathcal{B}(\Omega, \tilde{\mathbb{R}})_{\text{fin}} \) be the subring of \( \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \) consisting of those functions whose support is covered by a finite number of \( \mathbb{K} \)-cosets.

**Lemma 5.4.** For any \( [[F]] \in \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \) there are \( [[F']] \in \mathcal{B}(\Omega, \tilde{\mathbb{R}})_{\text{fin}} \) and \( [[F'']] \in \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \) such that \( [[F]] = [[F']] + [[F'']] \).

**Proof.** This is immediate by Lemma 2.10 \( \square \)

Next we consider the ring \( \text{Fn}(\mathbb{K} \times \Omega, \tilde{\mathbb{R}}) \), where convolution is normalized in the obvious way induced by the normalized convolution in \( \text{Fn}(\Omega, \tilde{\mathbb{R}}) \) and the Fubini theorem for \( \mathbb{R} \)-valued integrals (see Remark 4.4). We are particularly interested in the subring of \( \text{Fn}(\mathbb{K} \times \Omega, \tilde{\mathbb{R}}) \) of definable functions the projection of whose support into \( \Omega \) is finite. This subring may be identified with the ring of definable functions from \( \Omega \) into \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}}) \) with finite support. Although it is certainly not the entire group ring \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \), we abuse the notation slightly to denote it as such. Also note that if we write a typical element \( f \) in \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) as \( \sum_{\omega \in O} \omega[, F_\omega] \), where \( O \) is a definable finite subset of \( \Omega \), then each subset \( F_\omega \) is \( \omega \)-definable (but may not be definable). We call \( [F_\omega] \) the coefficient of \( \omega \). If each \( [F_\omega] \) is a constant function then \( f \) is a \( \mathbb{K} \)-constant function. Let \( \text{Fn}^1(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) be the subgroup of \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) of \( \mathbb{K} \)-constant functions. Since multiplication is given by convolution in \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}}) \), it is clear that \( \text{Fn}^1(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) is an ideal of \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \).

Since \( \mathbb{K} \) is a subgroup of \( \Omega \), there are two natural actions of \( \mathbb{K} \) on \( \text{Fn}(\mathbb{K} \times \Omega, \tilde{\mathbb{R}}) \): one on the \( \mathbb{K} \)-coordinate and the other on the \( \Omega \)-coordinate. We identify them as follows. For any \( f, g \in \text{Fn}(\mathbb{K} \times \Omega, \tilde{\mathbb{R}}) \), if there are \( t', t'' \in \mathbb{K} \) such that, for every \( (s, \omega) \in \mathbb{K} \times \Omega \),

\[
 g(s, \omega) = f(t' + s, t'' + \omega)
\]

then we write \( A_{(t', t'')} (f) = g \). The anti-diagonal ideal \( \mathcal{A} \) of \( \text{Fn}(\mathbb{K} \times \Omega, \tilde{\mathbb{R}}) \) is generated by elements of the form \( f - A_{(t, -t)} (f) \). The ideal \( \mathcal{A} \cap \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) of \( \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \) is also denoted as \( \mathcal{A} \).

**Theorem 5.5.** There is a canonical ring isomorphism

\[
 \phi : \mathcal{B}(\Omega, \tilde{\mathbb{R}}) / \mathcal{B}(\mathbb{K} / \mathbb{O}, \tilde{\mathbb{R}}) \longrightarrow \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] / (\text{Fn}^1(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] + \mathcal{A}).
\]

**Proof.** For any \( \omega \in \Omega \) and any \( f \in \mathcal{B}(\Omega, \tilde{\mathbb{R}}) \) we denote the translation of \( f \) by \( \omega \) as \( T_\omega (f) \); that is, the \( \omega \)-definable function \( T_\omega (f) \) is given by \( T_\omega (f)(t) = f(t + \omega) \). Let

\[
 \rho : \text{Fn}(\mathbb{K}, \tilde{\mathbb{R}})[\Omega] \longrightarrow \mathcal{B}(\Omega, \tilde{\mathbb{R}})_{\text{fin}}
\]

be the ring homomorphism given by

\[
 \sum_{\omega \in O} \omega f_\omega \longmapsto \sum_{\omega \in O} T_\omega (f_\omega),
\]

where \( O \) is a definable finite subset of \( \Omega \). We claim that \( \rho \) is surjective. To see this, let \( f \in \mathcal{B}(\Omega, \tilde{\mathbb{R}})_{\text{fin}} \) and \( D_1, \ldots, D_n \) the \( \mathbb{K} \)-cosets that cover the support of \( f \). By
Lemma 2.5 there is a definable subset \( \{a_1, \ldots, a_n\} \subseteq \text{VF} \) such that \( a_i \in D_i \). Let \( \omega_i = a_i + M \). So \( O = \{\omega_1, \ldots, \omega_n\} \subseteq \Omega \) is definable. For each \( \omega_i \) let \( f_{\omega_i} = T_{\omega_i}(f) \mid K \). Clearly

\[
\rho(\omega_1 f_{\omega_1} + \cdots + \omega_n f_{\omega_n}) = f.
\]

On the other hand, by Lemma 5.4, there is a canonical surjective homomorphism \( \sigma : \mathcal{B}(\Omega, \tilde{\mathcal{R}})_{\text{fin}} \twoheadrightarrow \mathcal{B}(\Omega, \tilde{\mathcal{R}})/\mathcal{B}(\text{VF}/\mathcal{O}, \tilde{\mathcal{R}}) \).

It is not hard to see that the kernel of the surjective homomorphism \( \sigma \circ \rho \) is precisely the ideal \( \text{Fn}^1(\mathcal{K}, \tilde{\mathcal{R}})[\Omega] + \mathcal{A} \). \( \square \)

Remark 5.6. The quotient ring

\[
\text{Fn}(\mathcal{K}, \tilde{\mathcal{R}})[\Omega]/(\text{Fn}^1(\mathcal{K}, \tilde{\mathcal{R}})[\Omega] + \mathcal{A})
\]

is our motivic analog of the group ring \( \mathbb{R}[S^1] \) and is henceforth denoted as \( \bar{\mathcal{C}} \). Note that members of \( \Omega \) in \( \bar{\mathcal{C}} \) are treated as symbols rather than \( M \)-cosets. The conceptual reason that we do not take the quotient ring \( \mathcal{B}(\Omega, \tilde{\mathcal{R}})/\mathcal{B}(\text{VF}/\mathcal{O}, \tilde{\mathcal{R}}) \) as the analog is that it is not completely free of VF-data, since there members of \( \Omega \) and VF/\( \mathcal{O} \) are treated as subsets of VF. However, technically, it is more effective to work with the latter, which is what we shall do below, since its construction corresponds directly to the cancellation rule mentioned above.

Although multiplication in \( \bar{\mathcal{C}} \) comes from the normalized convolution in \( \mathcal{B}(\Omega, \tilde{\mathcal{R}}) \), for simplicity, we shall not make this explicit notationally when it is not necessary.

For each \([X] \in \tilde{\mathcal{R}}\) let \( e_{[X]} \in \mathcal{B}(\Omega, \tilde{\mathcal{R}}) \) be the function given by

\[
\omega \mapsto \begin{cases} [X], & \text{if } \omega = 0; \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly if \([X] \neq [X']\) then \( e_{[X]} - e_{[X']} \notin \mathcal{B}(\text{VF}/\mathcal{O}, \tilde{\mathcal{R}}) \). So \( \tilde{\mathcal{R}} \) may be treated as a subring of \( \bar{\mathcal{C}} \) via the canonical embedding

\[
[X] \mapsto e_{[X]} + \mathcal{B}(\text{VF}/\mathcal{O}, \tilde{\mathcal{R}}).
\]

Similarly, if for each \( \omega \in \Omega \) we let \( e_{\omega} \in \mathcal{B}(\Omega, \tilde{\mathcal{R}}) \) be the function with support \( \{\omega\} \) and \( e_{\omega}(\omega) = 1 \), then the mapping given by

\[
\omega \mapsto e_{\omega} + \mathcal{B}(\text{VF}/\mathcal{O}, \tilde{\mathcal{R}})
\]

is an injective group homomorphism from \( \Omega \) into the multiplicative group of \( \bar{\mathcal{C}} \). So \( \Omega \) may be considered as a multiplicative subgroup of \( \bar{\mathcal{C}} \).

Notation 5.7. In writing we shall not distinguish \( \tilde{\mathcal{R}} \) and \( \Omega \) from their images in \( \mathcal{B}(\Omega, \tilde{\mathcal{R}}) \) or \( \bar{\mathcal{C}} \). Also, to emphasize the transition from an additively written group to a multiplicatively written group, we shall think of the embedding of \( \Omega \) into \( \mathcal{B}(\Omega, \tilde{\mathcal{R}}) \) or \( \bar{\mathcal{C}} \) as an exponential map and denote it as \( \exp \).

For any definable subset \( V \subseteq \text{VF}^n \), as the formulation of \( \text{Fn}(V, \tilde{\mathcal{R}}) \), we can form the \( \bar{\mathcal{C}} \)-module of definable functions \( V \rightarrow \bar{\mathcal{C}} \), which is denoted as \( \text{Fn}(V, \bar{\mathcal{C}}) \). Clearly \( \text{Fn}(V, \bar{\mathcal{C}}) \) contains \( \text{Fn}(V, \tilde{\mathcal{R}}) \) as a subgroup (or a sub-\( \tilde{\mathcal{R}} \)-module). Similarly \( \mathcal{B}(V, \bar{\mathcal{C}}) \) contains \( \mathcal{B}(V, \tilde{\mathcal{R}}) \) as a subgroup.
Notation 5.8. Let \( g : V \rightarrow \text{Fn}(\Omega, \bar{\mathbb{R}}) \) be a definable function. For each \( \omega \in \Omega \), the function \( \text{Leb}_\omega g : V \rightarrow \bar{\mathbb{R}} \) is given by \( \overline{\tau} \mapsto g(\overline{\tau})(\omega) \). Let \( \text{Leb} g \in \text{Fn}(\Omega, \bar{\mathbb{R}}) \) be the function given by \( \omega \mapsto \int_V (\text{Leb}_\omega g) \tau_{\bar{R}} \).

Definition 5.9. A function \( f \in \text{Fn}(V, \bar{\mathbb{C}}) \) is integrable if there is a representative \( f' : V \rightarrow \mathcal{B}(\Omega, \bar{\mathbb{R}}) \) of \( f \) that is uniformly bounded, that is, there is a \( \gamma \in \Gamma \) such that \( \text{supp}(f'(\overline{\tau})) \subseteq \sigma(0, \gamma) \) for every \( \overline{\tau} \in V \). In this case, we say that \( f' \) is an integrable representative of \( f \).

It is not hard to see that a function \( f \) is integrable if and only if there is a definable \( \gamma \in \Gamma \) such that, for every representative \( f' : V \rightarrow \mathcal{B}(\Omega, \bar{\mathbb{R}}) \) of \( f \) and every \( \overline{\tau} \in V \), \( f'(\overline{\tau}) \) is \( \mathcal{O} \)-invariant outside of \( \sigma(0, \gamma) \). The subset of \( \text{Fn}(V, \bar{\mathbb{C}}) \) of integrable functions is denoted as \( \mathcal{I}(V, \bar{\mathbb{C}}) \). Obviously \( \mathcal{I}(V, \bar{\mathbb{C}}) \) is a subgroup of \( \text{Fn}(V, \bar{\mathbb{C}}) \) and is closed under pointwise multiplication.

Proposition 5.10. There is a canonical \( \bar{\mathbb{C}} \)-module homomorphism

\[
\int_V : \mathcal{I}(V, \bar{\mathbb{C}}) \rightarrow \bar{\mathbb{C}}.
\]

Proof. Let \( f', f'' : V \rightarrow \mathcal{B}(\Omega, \bar{\mathbb{R}}) \) be two integrable representatives of an integrable function \( f \in \mathcal{I}(V, \bar{\mathbb{C}}) \). For any \( \mathbb{K} \)-coset \( D \) and any \( \omega_1, \omega_2 \in D \), since

\[
\text{Leb}_{\omega_1} f'(\overline{\tau}) - \text{Leb}_{\omega_1} f''(\overline{\tau}) \quad \text{and} \quad \text{Leb}_{\omega_2} f'(\overline{\tau}) - \text{Leb}_{\omega_2} f''(\overline{\tau})
\]

can be represented by the same \( (\overline{\tau}, D) \)-definable subset, we have

\[
\int_V \text{Leb}_{\omega_1} f' \tau_{\bar{R}} - \int_V \text{Leb}_{\omega_1} f'' \tau_{\bar{R}} = \int_V (\text{Leb}_{\omega_1} f' - \text{Leb}_{\omega_1} f'') \tau_{\bar{R}}
\]
\[
= \int_V (\text{Leb}_{\omega_2} f' - \text{Leb}_{\omega_2} f'') \tau_{\bar{R}}
\]
\[
= \int_V \text{Leb}_{\omega_2} f' \tau_{\bar{R}} - \int_V \text{Leb}_{\omega_2} f'' \tau_{\bar{R}},
\]

which is an element in \( \bar{\mathbb{R}}_D \). So \( \text{Leb} f' - \text{Leb} f'' \in \mathcal{B}(VF / \mathcal{O}, \bar{\mathbb{R}}) \). So if we set

\[
\int_V f = \text{Leb} f' + \mathcal{B}(VF / \mathcal{O}, \bar{\mathbb{R}})
\]

then \( \int_V f \) is a well-defined \( \bar{\mathbb{C}} \)-module homomorphism. \( \square \)

Note that, for convenience, we have omitted the symbol \( \tau_{\bar{C}} \) in the notation for the \( \bar{\mathbb{C}} \)-module homomorphism in Proposition 5.10.

Let \( E \) be a nonempty subset of \( \{1, \ldots, n\} \). A function \( f \in \text{Fn}(V, \bar{\mathbb{C}}) \) is iteratively integrable on \( E \) if for every \( \overline{\tau} \in \text{pr}_E V \)

\[
\int_{\overline{\tau} \in \text{pr}_E V} f \mid \text{fib}(V, \overline{\tau}) \in \mathcal{I}(\text{fib}(V, \overline{\tau}), \bar{\mathbb{C}}_{\overline{\tau}}) \quad \text{and} \quad \text{pr}_E f \in \mathcal{I}(\text{pr}_E V, \bar{\mathbb{C}}),
\]

where \( \text{pr}_E f \) is the function given by

\[
\overline{\tau} \mapsto \int_{\text{fib}(V, \overline{\tau})} f.
\]

In other words, \( f \) is iteratively integrable on \( E \) if and only if the iterated integral

\[
\int_{\overline{\tau} \in \text{pr}_E V} \int_{\overline{\eta} \in \text{fib}(V, \overline{\tau})} f(\overline{\tau}, \overline{\eta})
\]
is defined. It clearly follows from Definition 5.9 and compactness that the iterated integral is defined if and only if there is a representative \( f' : V \to \mathcal{B}(\Omega, \check{R}) \) of \( f \) and a \( \gamma \in \Gamma \) such that for every \( \overline{\pi} \in \text{pr}_E V \)

1. \( f' \mid \text{fib}(V, \overline{\pi}) \) is uniformly bounded,
2. \( \text{Leb}(f' \mid \text{fib}(V, \overline{\pi})) \) is \( \mathcal{O} \)-invariant outside of \( \sigma(0, \gamma) \).

Let \( \mathcal{I}_E(V, \check{C}) \) be the group of functions that are iteratively integrable on \( E \). We have that \( \mathcal{I}(V, \check{C}) \subseteq \mathcal{I}_E(V, \check{C}) \) for any \( E \) and if \( E = \{1, \ldots, n\} \) then \( \mathcal{I}_E(V, \check{C}) = \mathcal{I}(V, \check{C}) \).

It is clear that if \( f \) is iteratively integrable on some \( E \) then \( f \) may be iteratively integrated with respect to any suitable permutation of the variables, that is, one variable at a time. For such a permutation \( \sigma \) we write \( \int_{V}^{\sigma} f \) for the corresponding iterative integral. Different iterative integrals always give the same answer:

**Theorem 5.11** (Fubini theorem). For any nonempty \( E_1, E_2 \subseteq \{1, \ldots, n\} \) and any \( f \in \mathcal{I}_{E_1}(V, \check{C}) \cap \mathcal{I}_{E_2}(V, \check{C}) \),

\[
\int_{\text{pr}_{E_1} V} \text{pr}_{E_1} f = \int_{\text{pr}_{E_2} V} \text{pr}_{E_2} f.
\]

**Proof.** This follows immediately from the definition of \( \int_V \) in Proposition 5.10 and the Fubini theorem for \( \check{R} \)-valued integrals (see Remark 4.4).

Let \( V, W \subseteq \text{VF}^n \) and \( \phi : V \to W \) an isomorphism in \( \text{MorVF}_* \). Clearly the mapping \( \text{Fn}(V, \check{C}) \to \text{Fn}(W, \check{C}) \) given by \( f \to f \circ \phi^{-1} \) is a \( \check{C} \)-module isomorphism. Moreover, if \( f \in \mathcal{I}(V, \check{C}) \) then \( f \circ \phi^{-1} \in \mathcal{I}(W, \check{C}) \). As in Lemma 4.9 we have

**Lemma 5.12** (Change of variables). For any \( f \in \mathcal{I}(V, \check{C}) \),

\[
\int_V (f, \mu) = \int_W (f \circ \phi^{-1}, \text{rv}(\text{Jcb} \phi^{-1})(\mu \circ \phi^{-1})).
\]

**Lemma 5.13.** Let \( \phi : V \to \text{VF}^n \times \Gamma^l \) be a definable function. For any integrable functions \( f, g \in \mathcal{I}(V, \check{C}) \), if \( \int_{\phi^{-1}(\overline{\pi}, \overline{\gamma})} f = \int_{\phi^{-1}(\overline{\pi}, \overline{\gamma})} g \) for every \( (\overline{\pi}, \overline{\gamma}) \in \text{VF}^n \times \Gamma^l \) then \( \int_V f = \int_V g \).

**Proof.** This is immediate by Proposition 5.10 and compactness.

For any \( V \subseteq \text{VF}^n \), its volume, denoted as \( \text{vol}(V) \), is by definition \( \int_V 1 \). Conceptually, \( \text{vol}(V) \) is “finite” if \( V \) is bounded and “infinite” if \( V \) is unbounded. However, this distinction only makes sense if we work with a measure-preserving VF-category. We also note that, in any VF-category \( \text{VF}_* \) we have considered, translation by VF-sort elements is always an isomorphism in \( \text{MorVF}_* \). Therefore, for any \( \overline{\pi} \in \text{VF}^n \) and any \( \overline{\pi} \)-definable \( \overline{\gamma} \in \Gamma \), \( \text{vol}(\sigma(0, \overline{\gamma})) = \text{vol}(\sigma(\overline{\pi}, \overline{\gamma})) = \overline{\gamma} \). Similarly \( \text{vol}(\sigma(0, \overline{\gamma})) = \text{vol}(\sigma(\overline{\pi}, \overline{\gamma})) = \overline{\gamma} \). By Corollary 5.5 below, \( \sigma_\gamma \) and \( \epsilon_\gamma \) are invertible for any definable \( \gamma \).

**Lemma 5.14** (Averaging formula). Let \( f \in \mathcal{I}(V, \check{C}) \) and \( p : V \to \Gamma \) a volumetric partition such that \( \sigma(\overline{\pi}, p(\overline{\pi})) \subseteq V \) for every \( \overline{\pi} \in V \). Then

\[
\int_{\overline{\pi} \in V} f(\overline{\pi}) = \int_{\overline{\pi} \in V} \left( \sigma^{-n}_{p(\overline{\pi})} \int_{\overline{\tau} \in \sigma(\overline{\pi}, p(\overline{\pi}))} f(\overline{\tau}) \right).
\]
Proof. Let \( W = \bigcup_{\pi \in V} \{(\pi) \times \mathfrak{a}(\pi, p(\pi))\} \) and \( f^* \) the function on \( W \) given by \((\pi, h) \mapsto \sigma_{\pi}^{|\pi|} f(h)\). It is easy to see that any integrable representative of \( f \) gives rise to an integrable representative of \( f^* \) and hence \( f^* \) is integrable. So the righthand side of the equation is defined.

Now, by Lemma 5.13 it is enough to show that, for any \( \gamma \in \Gamma \),

\[
\int_{p^{-1}(\gamma)} f = c_{\gamma}^{-1} \int_{\pi \in p^{-1}(\gamma)} \int_{\mathfrak{a}(\pi, \gamma)} f(\overline{\gamma}).
\]

Let \( g \) be the function on \( p^{-1}(\gamma) \times \mathfrak{a}(0, \gamma) \) such that \( g(\pi, \overline{\gamma}) = f^*(\pi, \overline{\pi} + \overline{\gamma}) \). Clearly, \( g \) is integrable and, by change of variables,

\[
c_{\gamma}^{-1} \int_{\mathfrak{a}(\pi, \gamma)} f(\overline{\gamma}) = \int_{\mathfrak{a}(\pi, \gamma)} g(\pi, \overline{\gamma}).
\]

Notice that, for every \( \overline{\gamma} \in \mathfrak{a}(0, \gamma) \),

\[
\int_{\mathfrak{a}(\pi, \gamma)} g(\pi, \overline{\gamma}) = \int_{\pi \in p^{-1}(\gamma)} f^*(\pi, \overline{\pi} + \overline{\gamma}) = c_{\gamma}^{-1} \int_{\pi \in p^{-1}(\gamma)} f^*(\pi + \overline{\gamma}) = c_{\gamma}^{-1} \int_{p^{-1}(\gamma)} f,
\]

where the last equality is by change of variables again. Finally, by the Fubini theorem, we have

\[
\int_{p^{-1}(\gamma)} g = \int_{\mathfrak{a}(0, \gamma)} \int_{p^{-1}(\gamma)} g = c_{\gamma}^{-1} c_{\gamma}^{-1} \int_{p^{-1}(\gamma)} f = \int_{p^{-1}(\gamma)} f. \quad \square
\]

A definable group \( V \) is admissible if translation by any \( a \in V \) induces an isomorphism \( V \to V \) in the corresponding category. Integrating averages can also be done with respect to a definable subgroup of an admissible group.

**Lemma 5.15.** Suppose that \( V \) is an admissible group and \( H \) a definable subgroup of \( V \). For every \( f \in \mathcal{J}(V, \check{\mathcal{C}}) \),

\[
\text{vol}(H) \int_{\pi \in V} f(\pi) = \int_{\pi \in V} \int_{\pi \in H} f(\pi).
\]

**Proof.** First note that, since \( V \) is admissible, \( \text{vol}(a + H) = \text{vol}(H) \in \check{\mathcal{C}}_a \) for every \( a \in V \). Then it is clear that the argument in the proof of Lemma 5.13 works. \( \square \)

If \( V \) is a definable group then convolution in \( \text{Fn}(V, \check{\mathcal{C}}) \) is defined in the usual way:

\[
(f \ast g)(\pi) = \int_{\pi \in V} f(\pi) g(\pi - \pi),
\]

provided that the righthand side exists.

**Proposition 5.16.** The convolution product of two integrable functions always exists and is also an integrable function. Moreover, the convolution map

\[
* : \mathcal{J}(V, \check{\mathcal{C}})^2 \to \mathcal{J}(V, \check{\mathcal{C}})
\]

is \( \check{\mathcal{C}} \)-bilinear, associative, and commutative.

**Proof.** Let \( f, g \in \mathcal{J}(V, \check{\mathcal{C}}) \). Let \( f', g' \) be two integrable representatives of \( f, g \) that are uniformly bounded by \( \alpha, \beta \in \Gamma \), respectively. Let \( h_{\pi}^{\alpha} \) be the function on \( V \) given by \( \overline{h} \mapsto f'(\overline{h}) g'(\pi - \overline{h}) \). Clearly every \( \text{supp}(h_{\pi}^{\alpha}(\overline{h})) \) is bounded by \( \min \{\alpha, \beta\} \) and hence \( \text{supp}(\text{Leb} h_{\pi}^{\alpha}) \) is bounded by \( \min \{\alpha, \beta\} \). So the function given by \( \pi \mapsto \text{Leb} h_{\pi}^{\alpha} \) is an integrable representative of \( f \ast g \).
From the definition of convolution, $\mathcal{C}$-bilinearity is clear. For associativity, we have
\[
((f * g) * h)(\overline{\gamma}) = \int_{\gamma \in V} \left( \int_{\gamma \in V} f(\overline{\gamma}) g(\overline{\gamma} - \overline{\nu}) h(\overline{\nu} - \overline{\gamma}) \right) d\overline{\gamma} = \int_{\gamma \in V} f(\overline{\gamma}) \left( \int_{\gamma \in V} g(\overline{\gamma} - \overline{\nu}) h(\overline{\nu} - \overline{\gamma}) \right) d\overline{\gamma} = \int_{\gamma \in V} f(\overline{\gamma}) \left( \int_{\gamma \in V} g(\overline{\gamma}) h(\overline{\gamma} - \overline{\nu}) \right) d\overline{\gamma} = (f * (g * h))(\overline{\gamma}).
\]

Commutativity may be proved in a similar way, since the standard proof also only makes use of the Fubini theorem and change of variables. \(\square\)

As in Remark 4.7, we note that convolution in $\mathcal{S}(V, \hat{G})$ has an identity. Set $\mathcal{S}(V, \hat{G}) = \mathcal{S}(V, \hat{C}) \cap \mathcal{B}(V, \hat{C})$. If $V$ is an unbounded group but its group operation is a bounded function then, for any two functions $f, g \in \mathcal{S}(V, \hat{G})$ and any $\overline{\gamma}, \overline{\chi} \in V$ with $\text{val}(\overline{\gamma})$ sufficiently low, either $f(\overline{\gamma}) = 0$ or $g(\overline{\gamma} - \overline{\chi}) = 0$, and hence $\int_{\gamma \in V} f(\overline{\gamma}) g(\overline{\gamma} - \overline{\chi}) = 0$. Therefore,

**Corollary 5.17.** The group $\mathcal{S}(V, \hat{G})$ is closed under convolution.

### 6. Integration with an additive character

In this section we shall define the Fourier transform for definable functions with respect to a fixed subgroup of $\text{VF}^n$. We point out two things here that make this definition work. The first is self-duality of local fields, that is, the group of additive characters of any local field $K$ may be identified with the additive group of $K$. This fact makes integrating over the group of definable characters possible in our first-order setting. The second is that we have included in $\hat{C}$ a tautological image $\Omega$ of $\text{VF}$ under the generic additive character of $\text{VF}$. Because of this, our construction below is not supposed to work for definable functions on nonabelian definable groups, for example, $\text{SL}_2(\text{VF})$. On the other hand, the construction below may be adapted for any definable abelian group $G$ as long as the group of additive characters of $G$ can be handled in a first-order fashion and the ring $\hat{G}$ contains the image of $G$ under any definable character.

**Definition 6.1.** A function $f \in \text{Fn}(V, \hat{C})$ is *almost integrable* if $f \upharpoonright (V \cap o(0, \gamma))$ is integrable for every $\overline{\gamma}$-definable $\gamma \in \Gamma$, where $\overline{\gamma} \in V$. If for every $\overline{\gamma} \in V$ there is an $\overline{\theta}$-definable $\gamma \in \Gamma$ such that $f \upharpoonright (V \cap o(\overline{\theta}, \gamma))$ is integrable then $f$ is *locally integrable*.

The subsets of $\text{Fn}(V, \hat{C})$ of almost integrable functions and locally integrable functions are respectively denoted as $\mathcal{A}(V, \hat{C})$ and $\mathcal{L}(V, \hat{C})$. These are clearly subgroups of $\text{Fn}(V, \hat{C})$. We have $\mathcal{S}(V, \hat{C}) \subseteq \mathcal{A}(V, \hat{C}) \subseteq \mathcal{L}(V, \hat{C})$.

**Definition 6.2.** Let $V$ be a definable subgroup of $\text{VF}^n$. A function $f \in \text{Fn}(V, \hat{C})$ is a *definable additive character* of $V$ if $f$ is a definable group homomorphism $V \to \Omega$.

For any $\overline{b} \in \text{VF}^n$ let $\lambda_{\overline{b}}$ be the $\overline{b}$-definable map given by $\overline{a} \mapsto \theta(\overline{a} \cdot \overline{b})$, where $\overline{a} \cdot \overline{b}$ is the ordinary dot product. Clearly $\text{exp}_{\overline{\gamma}} = \text{exp} \circ \lambda_{\overline{\gamma}}$ is an additive character of $\text{VF}^n$. Note that $\text{exp}_{\overline{\gamma}}$ is almost integrable but not integrable.
Notation 6.3. The dual ball of $\sigma(a, \gamma)$ is $\sigma(a, -\gamma)$ and the dual ball of $\gamma(a, \gamma)$ is $\sigma_0(a, -\gamma)$. The dual polyball of a polyball $b \subseteq VF^n$ around $(a_1, \ldots, a_n)$ is the polyball $b^\circ$ around $(a_1, \ldots, a_n)$ such that each $\text{pr}_i b^\circ$ is the dual of $\text{pr}_i b$ with respect to $a_i$.

**Lemma 6.4.** Suppose that $t \in \Omega$ is definable. For any $\beta, \beta' \in VF^n$ and any polyball $b$ around $\beta$,

$$
\int_{\beta \in b} \exp(\chi_{\beta'}(\beta) + t) = \text{vol}(b) \exp(\chi_{\beta'}(\beta) + t), \quad \text{if } \beta' \in b^\circ - \beta;
$$

$$
\int_{\beta \in b} \exp(\chi_{\beta'}(\beta) + t) = 0, \quad \text{if } \beta' \notin b^\circ - \beta.
$$

**Proof.** To simplify the argument, we shall just show the case that $b$ is an open polyball $\sigma(\beta, \beta')$. The proof for the other cases are almost identical.

For the first item, since $\beta \in c(0, -\beta)$, clearly $\chi_{\beta}(\beta') = \chi_{\beta'}(\beta)$ for every $\beta' \in \sigma(\beta, \beta')$. So the equality is clear by the definition of $\int_{\sigma(\beta, \beta')}$. For the second item, we first consider the case $n = 1$, where $\beta, \beta', \beta^\circ$ are simply written as $\gamma, a, b$. Observe that, since $\text{val}(b) < -\gamma$ and $\sigma_0(a, \gamma) = \sigma(ba, \text{val}(b) + \gamma)$, the set $\chi_0(\sigma(a, \gamma))$ is a union of $K$-cosets. We have $(\omega - t)/b \subseteq \sigma_0(a, \gamma)$ for any $\omega - t \in \chi_0(\sigma(a, \gamma))$. Note that, for any $a' \in \sigma(a, \gamma)$, $\chi_0(a') + t = \omega$ if and only if $a' \in (\omega - t)/b$. Let $h : \sigma(a, \gamma) \rightarrow \mathcal{B}(\Omega, \bar{R})$ be an integrable representative of $(\chi_0 + t) \downarrow \sigma(a, \gamma)$. Then it is enough to show that $\text{Leb} h \in \mathcal{B}(VF/O, \bar{R})$. In fact, $\text{Leb} h$ is a constant function on its support $\chi_0(\sigma(a, \gamma)) + t$. To see this, observe that, for any $d \in VF$ with $\theta(d) \in \chi_0(\sigma(a, \gamma)) + t$,

$$
\text{Leb} h(\theta(d)) = \int_{\sigma(a, \gamma)} \text{Leb}_{\theta(d)} h \tau_{\bar{R}} = \int [(\theta(d) - t)/b] _{\tau_{\bar{R}}} \in \bar{R}_d.
$$

By Lemma 2.4, $t$ has a definable center. So, for any $d', d'' \in VF$, the obvious $\langle d', d'' \rangle$-definable bijection between $(\theta(d') - t)/b$ and $(\theta(d'') - t)/b$ is an isomorphism, which implies that

$$
\int [(\theta(d') - t)/b] _{\tau_{\bar{R}}} = \int [(\theta(d'') - t)/b] _{\tau_{\bar{R}}} \in \bar{R}_{d', d''}.
$$

We now consider the case $n > 1$. Let $\beta = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$. Without loss of generality we may assume $\text{val}(b_1) < -\gamma_1$. Let $\beta_1 = (b_2, \ldots, b_n)$. For each $\beta_1 \in \text{pr}_1 \sigma(\beta, \beta_1)$, by the case $n = 1$ above, we have

$$
\text{pr}_{\sigma_1}(\exp(\chi_{\beta_1} + t))(\beta_1) = \int_{\sigma(\beta_1, \gamma_1)} \exp(\chi_{\beta_1} + \theta(\beta_1) + t) = 0.
$$

By the Fubini theorem,

$$
\int_{\sigma(\beta, \beta_1)} \exp(\chi_{\beta_1} + t) = \int_{\sigma(\beta_1, \gamma_1)} \text{pr}_{\sigma_1}(\exp(\chi_{\beta_1} + t)) = 0.
$$

**Corollary 6.5.** For any definable polyball $b \subseteq VF^n$ around $0$, $\text{vol}(b) \text{vol}(b^\circ) = \epsilon^n$ and hence both $\text{vol}(b)$ and $\text{vol}(b^\circ)$ are invertible.

**Proof.** Clearly it is enough to show the case $n = 1$. Without loss of generality let $b$ be a ball around $0$ such that $\text{rad}(b) < 0$. By Lemma 6.4 and the Fubini theorem,

$$
\int_{(x, y) \in b \times M} \exp(xy) = \int_{x \in b} \int_{y \in M} \exp_x(y) = \int_{x \in O} a = \epsilon
$$
and
\[
\int_{(x,y) \in b \times M} \exp(xy) = \int_{y \in M} \int_{x \in b} \exp_y(x) = \int_{y \in b} \text{vol}(b) = \text{vol}(b) \text{vol}(b'). \qed
\]

We would like to consider a more general setting, namely integrable functions on \(VF^n / H\), where \(H\) is a definable subgroup of \(VF^n\) that is possibly not a polyball. For that purpose we need to modify the ring \(\tilde{C}\) so that Lemma 6.4, which corresponds to the cancellation rule mentioned above, holds with respect to the dual group of \(VF^n / H\) and the dual group of the dual group of \(VF^n / H\). This modification is very similar to the construction of the \(H\)-rectangular ideal in Section 4.

Remark 6.6. Because of Corollary 6.5, the construction of the \(H\)-rectangular ideal in Section 4 is redundant if \(H\) is a polyball.

For the rest of this section, fix a bounded definable subgroup \(H\) of \(VF^n\) that satisfies Hypothesis 4.8. Let \(H^*, k^*, \zeta, \iota, \tilde{R}^H\), etc. be as in Section 4. Note that, by Corollary 6.5, if \(H = \{0\}\), then \(\iota = e^n\). The ring \(\tilde{C}\) is reset as
\[\mathcal{B}(\Omega, \tilde{R}^H) / \mathcal{B}(VF / \mathcal{O}, \tilde{R}^H).\]

The group \(\mathcal{F}(VF^n, \tilde{C})\) is still defined in the same way. It is easy to see that Proposition 5.10 still holds.

Definition 6.7. A definable function \(f : VF^m \rightarrow \tilde{C}\) is \(H^*\)-segmental if
\(f\) is iteratively integrable.

In this case the element \(\int_{VF^n} f \in \tilde{C}\) is an \(H^*\)-segment, where \(\sigma\) is any suitable permutation of the variables.

It is easily seen that the set \(\mathcal{H}_*\) of all \(H^*\)-segments forms an ideal of \(\tilde{C}\). Of course, we do need to avoid trivialities:

Hypothesis 6.8. For every \(\pi \in VF^n\), \(\mathcal{H}_*\pi\) is a proper ideal of \(\tilde{C}_\pi\).

For any \(f, g \in \mathcal{F}(VF^n, \tilde{C})\) such that \(f(\pi) - g(\pi) \in \mathcal{H}_*\pi\) for every \(\pi \in VF^n\), since \(f - g \in \mathcal{F}(VF^n, \tilde{C})\), by the construction of \(\mathcal{H}_*\) and compactness, we have \(\int_{VF^n} (f - g) \in \mathcal{H}_*\). Now set
\[\tilde{C}^H = \tilde{C} / \mathcal{H}_*.\]

A definable function \(f : VF^n \rightarrow \tilde{C}^H\) is integrable if there is an integrable representative \(f' \in \mathcal{F}(VF^n, \tilde{C})\) of \(f\). We then have a canonical \(\tilde{C}^H\)-module homomorphism
\[\int_{VF^n} : \mathcal{F}(VF^n, \tilde{C}^H) \rightarrow \tilde{C}^H\]
given by
\[f \mapsto \left(\int_{VF^n} f'\right) / \mathcal{H}_*\]
for any integrable representative \(f'\) of \(f\).
After examining the proofs in Section 5, we see that the definitions and the results there work with respect to the homomorphism \( \int_{VF^n}^H \). In this section, when we refer to the definitions and the results in Section 5, we actually mean their accordingly modified versions for the homomorphism \( \int_{VF^n}^H \).

For the Poisson summation formula (Theorem 6.15) we need to discuss Fourier transform with respect to both \( H \) and the trivial group. Since the results for the two groups are almost identical, we can prove them together. To that end, let \( G \) be either \( H \) or the trivial group for the rest of this section.

**Definition 6.9.** The Fourier transform of a function \( f \in \text{Fn}(VF^n/G, \tilde{C}^H) \) is the function \( \hat{f} \in \text{Fn}(G^\ast, \tilde{C}^H) \) such that

\[
\hat{f}(\mathfrak{b}) = \text{vol}(G)^{-1} \int_{\mathfrak{t} \in VF^n} f(\mathfrak{t}) \exp(\mathfrak{b}(\mathfrak{t})),
\]

provided that the integral is defined for every \( \mathfrak{b} \in G^\ast \). Similarly, the Fourier transform of a function \( g \in \text{Fn}(G^\ast, \tilde{C}^H) \) is the function \( \hat{g} \in \text{Fn}(VF^n/G^\ast, \tilde{C}^H) \) such that

\[
\hat{g}(\mathfrak{b}) = \int_{\mathfrak{t} \in G^\ast} g(\mathfrak{t}) \exp(\mathfrak{b}(\mathfrak{t})),
\]

provided that the integral is defined for every \( \mathfrak{b} \in VF^n \).

Sometimes the Fourier transform of a function \( f \) is also denoted as \( \mathcal{F}_G(f) \). This is more suggestive if we understand the Fourier transform as a (partial) linear operator between the corresponding spaces of functions. We write \( \mathcal{F}_0 \) and \( \mathcal{F}_H \) for the two cases.

We can actually define iterative Fourier transform of any finite length in the obvious way. The ring \( \tilde{C}^H \) may be modified with respect to the annihilator groups so that the results below will also hold for iterative Fourier transform. For conceptual simplicity, we shall not explore this here.

**Lemma 6.10.** If \( f \in \mathcal{I} \mathcal{B}(VF^n, \tilde{C}^H) \) then the integral

\[
\int_{\mathfrak{t} \in VF^n} f(\mathfrak{t}) \exp(\mathfrak{t})
\]

is defined for every \( \mathfrak{b} \in VF^n \).

**Proof.** Let \( f \in \mathcal{I} \mathcal{B}(VF^n, \tilde{C}^H) \) and \( f'' : VF^n \rightarrow \mathcal{B}(\Omega, \tilde{R}^H) \) an integrable representative of \( f \). Let \( \mathfrak{o}(0, \alpha) \) be a definable open polyball that contains both \( \text{supp}(\text{Leb} f'') \) and \( \text{supp}(f) \). For any \( \mathfrak{b} \in VF^n \), clearly there is a \( \beta \in \Gamma \), which depends on \( \alpha \) and \( \text{val}(\mathfrak{b}) \), such that \( \text{supp}(f''(\mathfrak{t}) \exp(\mathfrak{b}(\mathfrak{t}))) \subseteq \mathfrak{o}(0, \beta) \) for every \( \mathfrak{t} \in \mathfrak{o}(0, \alpha) \). So the integral in question is defined. \( \square \)

**Remark 6.11.** Since \( H \) is bounded, \( \mathcal{I} \mathcal{B}(VF^n/G, \tilde{C}^H) \) is not the trivial group. Lemma 6.10 shows that the Fourier transform of any \( f \in \mathcal{I} \mathcal{B}(VF^n/G, \tilde{C}^H) \) exists. On the other hand, because we need to consider \( \text{val}(\mathfrak{b}) \), there is no guarantee that \( f \) is integrable. Of course, if \( f \) has bounded support then it is integrable. In other words, \( f \) is always an almost integrable function. The map

\[
\mathcal{F}_G : \mathcal{I} \mathcal{B}(VF^n/G, \tilde{C}^H) \rightarrow \mathcal{A}(G^\ast, \tilde{C}^H)
\]

is clearly a \( \tilde{C}^H \)-module homomorphism.
In order not to lose the identity, convolution in $\mathcal{F}(V^n, G, \hat{C}^H)$ is adjusted in the obvious way:

$$(f * g)(\overline{v}) = \text{vol}(G)^{-1} \int_{V^n} f(\overline{u})g(\overline{u} - \overline{v}).$$

**Proposition 6.12** (Convolution formula). Let $f, g \in \mathcal{F}(V^n, G, \hat{C}^H)$. Then

$$\mathcal{F}_G(f * g) = \mathcal{F}_G(f) \mathcal{F}_G(g).$$

**Proof.** Note that, by Corollary 5.17, $f * g \in \mathcal{F}(V^n, G, \hat{C}^H)$ and hence the transform $\mathcal{F}_G(f * g)$ exists. For any definable $\gamma \in \Gamma$ that is sufficiently low, we have

$$\mathcal{F}_G(f * g)(\overline{b}) = \text{vol}(G)^{-2} \int_{V^n} f(\overline{u})g(\overline{u} - \overline{b}) \exp(\overline{u} - \overline{b})$$

By the Fubini theorem and change of variables, $\mathcal{F}_G(f * g)(\overline{b})$ is equal to

$$\text{vol}(G)^{-2} \int_{V^n} f(\overline{u})g(\overline{u} - \overline{b}) \exp(\overline{u} - \overline{b})$$

A function $f \in \text{Fn}(V, \hat{C}^H)$ is **locally constant at** $\overline{v} \in V$ if there is a $\gamma \in \Gamma$ such that $f \upharpoonright (\sigma(\overline{v}, \gamma) \cap V)$ is constant. It is **locally constant** if it is locally constant at every $\overline{v} \in V$. Obviously if $V$ is discrete then every function in $\text{Fn}(V, \hat{C}^H)$ is locally constant. If $f$ is locally constant at $\overline{v} \in V$ then, by compactness and $\omega$-minimality, there is an $\overline{v}$-definable $\iota_f(\overline{v}) \in \Gamma$ such that $f \upharpoonright (\sigma(\overline{v}, \iota_f(\overline{v})) \cap V)$ is constant. If $f$ is locally constant then we can associate with $f$ a definable function $\iota_f : V \rightarrow \Gamma$ as follows. By compactness, there is a definable function $\gamma : V \rightarrow \Gamma$ such that $f \upharpoonright (\sigma(\overline{v}, \gamma(\overline{v})) \cap V)$ is constant. For each $\overline{v} \in V$ let

$$B_{\overline{v}} = \{ \overline{b} \in V : \overline{v} \in \sigma(\overline{b}, \gamma(\overline{b})) \text{ and } \text{val}(\overline{v}) \leq \gamma(\overline{b}) \leq \gamma(\overline{v}) \} .$$

By $\omega$-minimality, $\gamma(B_{\overline{v}})$ have finitely many $\overline{v}$-definable endpoints. Let $\iota_f(\overline{v})$ be the lowest endpoint of $\gamma(B_{\overline{v}})$. It is clear that $\iota_f(\overline{v}) = \iota_f(\overline{v})$ for any $\overline{v} \in \sigma(\overline{v}, \iota_f(\overline{v})) \cap V$. That is, $\iota_f$ is a volumetric partition of $V$. If $f$ has bounded support then we assume that, for some definable open polyball $\sigma(0, \alpha)$ that contains $\text{supp}(f)$ and every $\overline{v} \in V \setminus \sigma(0, \alpha)$, $\iota_f(\overline{v}) = \text{val}(\overline{v})$. If $f$ is the characteristic function of a definable subset $W \subseteq V$ then we write $\iota_W(\overline{v})$ if $f$ is locally constant at $\overline{v} \in V$ and $\iota_W : V \rightarrow \Gamma$ if $f$ is locally constant. We note here that the results below do not depend on the choice of each $\iota_f(\overline{v})$.

**Theorem 6.13** (Fourier inversion formula). For any $f \in \mathcal{F}(V^n, G, \hat{C}^H)$, if $\hat{f} \in \mathcal{F}(G, \hat{C}^H)$ then, for every $\overline{v} \in V^n$ where $f$ is locally constant,

$$\hat{f}(\overline{v}) = 2 \text{vol}(G)^{-1} f(\overline{v}).$$
where \( t = e^n \) if \( G \) is the trivial group and \( t = \delta \) if \( G = H \).

**Proof.** Since both \( f \) and \( \hat{f} \) have bounded support, for every \( \pi \in \text{VF}^n \) and every definable \( \gamma \in \Gamma \) that is sufficiently low (in particular \( \gamma < \min \{ \zeta, \text{val}(\pi) \} \)),

\[
\hat{f}(-\pi) = \int_{\pi \in G} \left( \text{vol}(G)^{-1} \int_{\pi \in \text{VF}^n} f(\bar{y}) \exp_{\gamma}(\bar{y}) \right) \exp_{-\gamma}(\pi)
\]

\[
= \text{vol}(G)^{-1} \int_{\pi \in \text{VF}^n} \left( \int_{\pi \in (0, \gamma)} f(\bar{y}) \exp_{\gamma}(\bar{y}) \right) \exp_{-\gamma}(\pi)
\]

\[
= \text{vol}(G)^{-1} \int_{\pi \in (0, \gamma)} \left( \int_{\pi \in G} f(\bar{y}) \exp_{-\gamma}(\pi) \right),
\]

where the last equality is by the Fubini theorem. Suppose that \( f \) is locally constant at \( \pi \). Then we may assume \( (G \cap \mathfrak{o}(0, \gamma))_+ \subseteq \mathfrak{o}(0, t_f(\pi)) \). By Lemma 6.14, the construction of the ideal \( \mathcal{H}_* \), and change of variables,

\[
\int_{\pi \in \mathfrak{o}(0, \gamma)} \int_{\pi \in G \cap \mathfrak{o}(0, \gamma)} f(\bar{y}) \exp_{-\gamma}(\pi) = 0.
\]

Therefore,

\[
\text{vol}(G) \hat{f}(-\pi) = \int_{\pi \in \mathfrak{o}(0, \gamma)} \int_{\pi \in G \cap \mathfrak{o}(0, \gamma)} f(\bar{y}) \exp_{-\gamma}(\pi) = t f(\pi),
\]

where \( t = e^n \) if \( G \) is the trivial group and, by the construction of the ring \( \hat{R}^H \),

\[ t = k_\ast^\gamma k_\ast^\gamma = \delta \] if \( G = H \). \( \square \)

**Remark 6.14.** By Lemma 2.19, Theorem 6.13 holds almost everywhere.

**Theorem 6.15** (Poisson summation formula). Let \( f \in \mathcal{S}(\text{VF}^n, \hat{\mathcal{C}}^H) \) be such that

1. the function \( g(\underline{y}) = \int_{\pi \in H} f(\pi + \underline{y}) \) is in \( \mathcal{S}(\text{VF}^n / H, \hat{\mathcal{C}}^H) \),
2. \( g \) is locally constant at 0,
3. \( f_H(g) \) is in \( \mathcal{S}(H_*, \hat{\mathcal{C}}^H) \).

Then

\[
\frac{d}{d \delta} \int_{H_\ast} f = \int_{H_\ast} \mathcal{F}_0(f).
\]

**Proof.** For any \( \bar{b} \in H_\ast \) and any \( \gamma \in \Gamma \) that is sufficiently low, we have

\[
\mathcal{F}_H(g)(\bar{b}) = k^{-1} \int_{\pi \in (0, \gamma)} \left( \int_{\pi \in H} f(\pi + \underline{y}) \right) \exp_{\bar{y}}(\pi)
\]

\[
= k^{-1} \int_{\pi \in (0, \gamma)} \left( \int_{\pi \in H} f(\pi + \underline{y}) \exp_{\bar{y}}(\pi + \underline{y}) \right)
\]

since \( \exp_{\bar{y}} \mid H \) is constant

\[
= k^{-1} \int_{\pi \in (0, \gamma)} \left( \int_{\pi \in H} f(\pi) \exp_{\bar{y}}(\pi) \right)
\]

by change of variables

\[
= \int_{\pi \in (0, \gamma)} f(\underline{y}) \exp_{\bar{y}}(\pi)
\]

by Lemma 6.15

\[
= \mathcal{F}_0(f)(\bar{b}).
\]
So, by the Fourier inversion formula,
\[ \frac{d}{dx} \int_{-\pi}^{\pi} f(x) e^{-2\pi i x y} \, dx = \int_{-\pi}^{\pi} \hat{f}(y) e^{2\pi i x y} \, dy = \int_{-\pi}^{\pi} \hat{f}(y) e^{2\pi i x y} \, dy = \int_{-\pi}^{\pi} f(x) e^{-2\pi i x y} \, dx. \]
\[ \square \]

As in the classical integration theory, there is a submodule of \( \mathcal{S}(\mathbb{R}^n / G, \mathcal{C}^\infty) \) whose image under \( \mathcal{F}_G \) is contained in \( \mathcal{S}(G^\ast, \mathcal{C}^\infty) \) and hence the double Fourier transform of any function in the module exists.

**Definition 6.16.** The submodule of \( \mathcal{S}(\mathbb{R}^n / G, \mathcal{C}^\infty) \) consisting of Schwartz-Bruhat functions, that is, locally constant integrable functions with bounded support, is called the *Schwartz space* on \( V \) and is denoted as \( \mathcal{S}(V, \mathcal{C}^\infty) \).

Clearly \( \mathcal{S}(V/E, \mathcal{C}^\infty) \) is closed under pointwise multiplication, where \( E \) is any definable equivalence relation on \( V \). If \( V \) is a definable group and \( \mathcal{S}(V, \mathcal{C}^\infty) \) forms a commutative ring with respect to convolution (see Corollary 5.17) then, for any definable subgroup \( D \) of \( V \), \( \mathcal{S}(V/D, \mathcal{C}^\infty) \) is an ideal of \( \mathcal{S}(V, \mathcal{C}^\infty) \), unless they are identical.

**Lemma 6.17.** Suppose that \( V \) is a closed subset. For any \( f \in \mathcal{S}(V, \mathcal{C}^\infty) \), \( \iota_f(V) \) is bounded from above and hence, by \( \alpha \)-minimality, there is a definable upper bound \( \beta \in \Gamma \) of \( \iota_f(V) \).

**Proof.** Let \( (0, \gamma) \) be a definable open polyball containing \( \text{supp}(f) \) such that, if \( \bar{a} \in V \setminus (0, \gamma) \), then \( \iota_f(\bar{a}) = \text{vol}(\bar{a}) \leq \gamma \). Since \( \iota_f \upharpoonright (V \setminus (0, \gamma)) \) is a volumetric partition of the bounded closed subset \( V \setminus (0, \gamma) \), by Lemma 6.17 \( \iota_f(V \setminus (0, \gamma)) \) is bounded from above.

**Proposition 6.18.** The Fourier transform of a function in \( \mathcal{S}(\mathbb{R}^n / G, \mathcal{C}^\infty) \) is a function in \( \mathcal{S}(G^\ast, \mathcal{C}^\infty) \).

**Proof.** Fix a nonzero \( f \in \mathcal{S}(\mathbb{R}^n / G, \mathcal{C}^\infty) \). Let \( \beta \in \Gamma \) be a definable upper bound of \( \iota_f(\mathbb{R}^n) \). By Lemma 6.14 for any \( \bar{b} \in G^\ast \) we have
\[
\hat{f}(\bar{b}) = \text{vol}(G)^{-1} \int_{\bar{y} \in \mathbb{R}^n} \int_{\bar{y} \in G^\ast \iota_f(\mathbb{R}^n)} f(\bar{y}) e^{2\pi i \bar{b} \cdot \bar{y}} d\bar{y}
= \text{vol}(G)^{-1} \int_{\bar{y} \in \mathbb{R}^n} \int_{\bar{y} \in G^\ast \iota_f(\mathbb{R}^n)} f(\bar{y}) e^{2\pi i \bar{b} \cdot \bar{y}} d\bar{y}.
\]
If \( \bar{b} \notin \iota(0, -\beta) \) then, by Lemma 6.14 for any \( \bar{a} \in G^\ast \),
\[
\int_{\bar{y} \in G^\ast \iota_f(\mathbb{R}^n)} e^{2\pi i \bar{b} \cdot \bar{y}} d\bar{y} = 0
\]
and hence \( \hat{f}(\bar{b}) = 0 \). So \( \hat{f} \) has bounded support. So, by Remark 6.11 \( \hat{f} \) is integrable.

Choose a definable \( \alpha \in \Gamma \) such that \( \iota(0, \alpha) \) contains \( \text{supp}(f) \). Let \( \bar{b} \in G^\ast \) and \( \bar{b}' \in G^\ast \cap \iota(\bar{b}, -\alpha) \). If \( \bar{a} \notin \text{supp}(f) \) then obviously
\[
f(\bar{a}) e^{2\pi i \bar{b} \cdot \bar{a}} = f(\bar{a}) e^{2\pi i \bar{b} \cdot \bar{a}} = 0.
\]
If \( \bar{a} \in \text{supp}(f) \) then, for any \( \bar{a}' \in \iota(\bar{a}, \iota_f(\mathbb{R}^n)) \), since \( \iota(\bar{a}, \iota_f(\mathbb{R}^n)) \subseteq \iota(0, \alpha) \), we have
\[
\bar{b} \cdot \bar{a}' - \bar{b}' \cdot \bar{a}' = (\bar{b} - \bar{b}') \cdot \bar{a}' = 0.
\]
and hence
\[ \int_0^H \exp_{\overline{\pi}} = \int_0^H \exp_{\overline{\eta}}. \]
So, for every \( \overline{\pi} \in \mathcal{V}F^n \),
\[ f(\overline{\pi}) \int_0^H \exp_{\overline{\pi}} = f(\overline{\eta}) \int_0^H \exp_{\overline{\eta}}. \]
So \( \hat{f} \) is also locally constant. \( \square \)

**Corollary 6.19.** The restriction of \( \mathcal{F}_G \) is a \( \mathcal{C}^\infty \)-module homomorphism
\[ \mathcal{J}(\mathcal{V}F^n / G, \mathcal{C}H) \longrightarrow \mathcal{J}(G_*, \mathcal{C}H). \]

**Remark 6.20.** If \( G = H \) then \( \mathcal{F}_H \mid \mathcal{J}(\mathcal{V}F^n / G, \mathcal{C}H) \) may not be injective since \( \mathcal{A} \) is possibly not invertible. If \( G \) is the trivial group then, since \( \epsilon \) is invertible, \( \mathcal{F}_0 \mid \mathcal{J}(\mathcal{V}F^n, \mathcal{C}H) \) is a \( \mathcal{C}^\infty \)-module automorphism, as in the classical theory.

For any function \( f \in \mathcal{F}_n(\mathcal{V}F^n, \mathcal{C}H), \) \( \hat{f} \) is the function such that \( f(-\overline{\pi}) = \hat{f}(\overline{\pi}). \)

**Corollary 6.21.** For any \( f, g \in \mathcal{J}(\mathcal{V}F^n, \mathcal{C}H), \)
\[ \mathcal{F}_0(fg) = \epsilon^n(\mathcal{F}_0(f) * \mathcal{F}_0(g)). \]

**Proof.** By the convolution formula, the Fourier inversion formula, and the fact that \( \mathcal{J}(\mathcal{V}F^n, \mathcal{C}H) \) is closed under convolution, we have
\[
\epsilon^n(\mathcal{F}_0(f) * \mathcal{F}_0(g))(\overline{\pi}) = \mathcal{F}_0(\mathcal{F}_0(f) * \mathcal{F}_0(g))(\overline{\pi}) \\
= \mathcal{F}_0(\mathcal{F}_0(f)) \mathcal{F}_0(\mathcal{F}_0(g))(\overline{\pi}) \\
= \epsilon^n(\hat{f}(\overline{\pi}) \hat{g}(\overline{\pi})) \\
= \epsilon^n(\mathcal{F}_0(fg))(\overline{\pi}). \]
\( \square \)

**Theorem 6.22** (Plancherel formula). Suppose that \( f \in \mathcal{J}(\mathcal{V}F^n / G, \mathcal{C}H) \) and \( g \in \mathcal{S}(\mathcal{V}F^n / G, \mathcal{C}H) \). Then
\[ t \ vol(G)^{-2} \int_{\overline{\pi} \in \mathcal{V}F^n} f(\overline{\pi})g(\overline{\pi}) = \int_{\overline{\tau} \in G_*} \hat{f}(\overline{\tau}) \hat{g}(\overline{\tau}) \]
where \( t \) is as in the Fourier inversion formula.

**Proof.** First note that, since \( \hat{f} \in \mathcal{J}(G_*, \mathcal{C}H) \) and \( \hat{g} \in \mathcal{S}(G_*, \mathcal{C}H) \) and hence the righthand side of the equality is defined. For any definable \( \gamma \in \Gamma \) that is sufficiently low,
\[
\int_{\overline{\tau} \in G_*} \hat{f}(\overline{\tau}) \hat{g}(\overline{\tau}) = \ vol(G)^{-1} \int_{\overline{\tau} \in G_* \cap \overline{o}(0, \gamma)} \hat{f}(\overline{\tau}) \int_{\overline{\tau} \in \overline{o}(0, \gamma)} g(\overline{\tau}) \exp_{\overline{\tau}}(-\overline{\tau}) \\
= \ vol(G)^{-1} \int_{\overline{\tau} \in \overline{o}(0, \gamma)} g(\overline{\tau}) \int_{\overline{\tau} \in G_* \cap \overline{o}(0, \gamma)} \hat{f}(\overline{\tau}) \exp_{\overline{\tau}}(\overline{\tau}) \\
= \ t \ vol(G)^{-2} \int_{\overline{\tau} \in \overline{o}(0, \gamma)} g(\overline{\tau}) f(\overline{\tau}).
\]
where the second equality is by the Fubini theorem and the third equality is by the Fourier inversion formula. \( \square \)
We note that, for any \( f, g \in S(VF^n, \mathcal{C}) \), a straightforward computation using only the Fubini theorem shows that the following classical version of the Plancherel formula holds:

\[
\int_{VF^n} F_0(f) g = \int_{VF^n} f F_0(g).
\]

7. Definable distribution

Our main references for the classical theory of distributions are [6] and [9].

The development in this section is valid with respect to any group \( G \) that satisfies Hypothesis \([4,5,6,8]\) and Hypothesis \([4,6,8]\). For notational simplicity, we shall just deal with the trivial group. We shall then switch back to the simpler notations in Section 5. The Schwartz space \( S(VF^n, \mathcal{C}) \) is now simply written as \( S_n \) and the Fourier transform operator as \( F \).

By a definable function \( VF^n \times G \rightarrow \tilde{\mathcal{C}} \) we mean a function in \( F \in Fn(VF^{n+1}, \tilde{\mathcal{C}}) \) such that, for every \((\pi, b_1), (\pi, b_2) \in VF^{n+1}\) with \( val(b_1) = val(b_2) \), \( f(\pi, b_1) = f(\pi, b_2) \). We write \( Fn(VF^n \times G, \mathcal{C}), \mathcal{B}(VF^n \times G, \mathcal{C}), S(VF^n \times G, \mathcal{C}) \), etc. for the corresponding subgroups of functions. If \( f \in Fn(VF^n \times G, \mathcal{C}) \) then the function \( f_\gamma : VF^n \rightarrow \mathcal{C} \) is given by \( \overline{a} \mapsto f(\overline{a}, \gamma) \).

**Definition 7.1.** A predistribution on \( VF^n \) is a function \( \mathcal{D} \in Fn(VF^n \times G, \mathcal{C}) \) such that

1. \( \mathcal{D}_\gamma \) is an almost integrable function for every \( \gamma \in G \),
2. for every \((\pi, \gamma), (\pi', \gamma) \in VF^n \times G\), if \( val(\pi - \pi') > \gamma \), that is, if \( o(\pi, \gamma) = o(\pi', \gamma) \), then \( \mathcal{D}(\pi, \gamma) = \mathcal{D}(\pi', \gamma) \),
3. \( \mathcal{D} \) is coherent, that is, for every \((\pi, \gamma), (\pi, \gamma') \in VF^n \times G\), if \( \gamma' \geq \gamma \) then

\[
\mathcal{D}(\pi, \gamma) = \int_{\pi' \in o(\pi, \gamma)} o^{-1}(\pi', \gamma') \mathcal{D}(\pi', \gamma').
\]

Let \( f \in \mathcal{A}(VF^n, \mathcal{C}) \). Then, as in the classical distribution theory, the almost integrable function \( \mathcal{D}_f \) on \( VF^n \times G \) given by

\[
\mathcal{D}_f(\pi, \gamma) = \int_{o(\pi, \gamma)} f
\]

is a predistribution on \( VF^n \), which shall be called a regular predistribution. That \( \mathcal{D}_f \) is coherent is clear by the averaging formula (Lemma 5.14).

**Lemma 7.2.** Let \( \mathcal{D}_1, \mathcal{D}_2 \) be two predistributions. Suppose that for every \( \overline{a} \in VF^n \) there is a \( \gamma_{\overline{a}} \in G \) such that \( \mathcal{D}_1(\overline{a}, \gamma) = \mathcal{D}_2(\overline{a}, \gamma) \) for every \( \gamma \geq \gamma_{\overline{a}} \) and every \( \overline{b} \in o(\overline{a}, \gamma_{\overline{a}}) \). Then \( \mathcal{D}_1 = \mathcal{D}_2 \).

**Proof.** For each \( \overline{a} \in VF^n \) let \( G_{\overline{a}} \subseteq G \) be the subset of all the values that satisfy the said property. Note that \( G_{\overline{a}} \) is \( \overline{a} \)-definable. Suppose for contradiction that there is a \( (\overline{d}, \alpha) \) such that \( \mathcal{D}_1(\overline{d}, \alpha) \neq \mathcal{D}_2(\overline{d}, \alpha) \). Then every \( G_{\overline{a}} \) is bounded from below. By coherence of \( \mathcal{D}_1, \mathcal{D}_2 \) and \( o \)-minimality, there is an \( \overline{a} \)-definable least element \( \beta_{\overline{a}} \) in every \( G_{\overline{a}} \). Let \( p : VF^n \rightarrow \Gamma \) be the definable function given by \( \bar{a} \mapsto \beta_{\overline{a}} \). Note that if \( \overline{b} \in o(\overline{a}, p(\overline{a})) \) then \( p(\overline{b}) = p(\overline{a}) \) and hence \( p \) is a volumetric partition of \( VF^n \). So \( p \upharpoonright o(\overline{d}, \alpha) \) is a volumetric partition of \( o(\overline{d}, \alpha) \). By Lemma 5.7 there is a
\((\overline{d}, \alpha)\)-definable \(\alpha' \in \Gamma\) such that \(p(\overline{d}, \alpha)) < \alpha'\). Since \(\alpha < p(\overline{d}) < \alpha'\), by coherence of \(\mathcal{D}_1, \mathcal{D}_2\),

\[
\mathcal{D}_1(\overline{d}, \alpha) = \int_{\Gamma \in \mathcal{O}(\overline{d}, \alpha)} \alpha'^{-n} \mathcal{D}_1(\overline{\Gamma}, \alpha') = \int_{\Gamma \in \mathcal{O}(\overline{d}, \alpha)} \alpha_1^{-n} \mathcal{D}_2(\overline{\Gamma}, \alpha') = \mathcal{D}_2(\overline{d}, \alpha),
\]

contradiction.

\(\square\)

**Lemma 7.3.** Let \(\mathcal{D}\) be a predistribution and \(f\) an almost integrable locally constant function on \(\text{VF}^n\). Suppose that \(p : o(\overline{\Gamma}, \gamma) \rightarrow \Gamma\) is a volumetric partition such that \(p(\overline{b}) \geq \iota_f(\overline{\beta}) \geq \gamma\) for every \(\overline{b} \in o(\overline{\Gamma}, \gamma)\). Then

\[
\int_{\Gamma \in o(\overline{\Gamma}, \gamma)} f(\overline{x}) \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, p(\overline{x})) \quad \text{and} \quad \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} f(\overline{x}) \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \iota_f(\overline{x}))
\]

are defined and they are equal.

**Proof.** Let \(\beta \in \Gamma\) be an \((\overline{\Gamma}, \gamma)\)-definable bound of \(p(o(\overline{\Gamma}, \gamma)) \cup \iota_f(o(\overline{\Gamma}, \gamma))\), which exists by Lemma 3.7. For any \(\overline{b} \in o(\overline{\Gamma}, \gamma)\), by coherence,

\[
\mathcal{D}(\overline{b}, p(\overline{b})) = \int_{\overline{x} \in o(\overline{\Gamma}, \gamma)} \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \beta) \quad \text{and} \quad \mathcal{D}(\overline{b}, \iota_f(\overline{b})) = \int_{\overline{x} \in o(\overline{\Gamma}, \iota_f(\overline{b}))} \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \beta).
\]

Therefore,

\[
\alpha_1^{-n} \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} f(\overline{x}) \mathcal{D}(\overline{\Gamma}, \beta) = \alpha_1^{-n} \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} \alpha_1^{-n} \int_{\overline{y} \in o(\overline{\Gamma}, \gamma)} \int_{\overline{y} \in o(\overline{\Gamma}, p(\overline{x}))} f(\overline{y}) \mathcal{D}(\overline{\Gamma}, \beta)
\]

\[
= \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} \int_{\overline{y} \in o(\overline{\Gamma}, p(\overline{x}))} f(\overline{y}) \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \beta)
\]

\[
= \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} \mathcal{D}(\overline{\Gamma}, p(\overline{x})),
\]

where the first equality is by the averaging formula. The same computation shows

\[
\alpha_1^{-n} \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} f(\overline{x}) \mathcal{D}(\overline{\Gamma}, \beta) = \int_{\Gamma \in o(\overline{\Gamma}, \gamma)} f(\overline{x}) \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \iota_f(\overline{x})). \quad \square
\]

**Corollary 7.4.** Let \(\mathcal{D}\) be a predistribution and \(p : o(\overline{\Gamma}, \gamma) \rightarrow \Gamma\) a volumetric partition. Then

\[
\mathcal{D}(\overline{\Gamma}, \gamma) = \int_{\overline{x} \in o(\overline{\Gamma}, \gamma)} \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, p(\overline{x})).
\]

Therefore, if \(p\) is a volumetric partition of \(\text{VF}^n\) then the function given by \(\overline{b} \mapsto \alpha_1^{-n} \mathcal{D}(\overline{b}, p(\overline{b}))\) is an almost integrable function.

By Lemma 7.3 the following is well-defined:

**Definition 7.5.** Let \(\mathcal{D}\) be a predistribution. A distribution induced by \(\mathcal{D}\) is a linear functional \(\mathcal{D}' : \mathcal{F}^n \rightarrow \mathcal{C}\) given by

\[
\mathcal{D}'(f) = \int_{\overline{x} \in \text{VF}^n} f(\overline{x}) \alpha_1^{-n} \mathcal{D}(\overline{\Gamma}, \beta),
\]

where \(\beta\) is any definable upper bound of \(\iota_f(\text{VF}^n)\), which exists by Lemma 6.17.
Note that $\mathcal{D}'(o(\overline{\tau}, \gamma)) = \mathcal{D}(\overline{\tau}, \gamma)$ for every $(\overline{\tau}, \gamma) \in VF^n \times \Gamma$. Therefore, we shall not distinguish notationally a predistribution from the distribution induced by it.

If $\mathcal{D}$ is regular then $\mathcal{D}'$ is a regular distribution; otherwise $\mathcal{D}'$ is a singular distribution. If $h \in \mathcal{A}(VF^n, \mathcal{C})$ then, by the averaging formula, we have

$$\mathcal{D}_h(f) = \int_{\overline{\tau} \in VF^n} f(\overline{\tau})h(\overline{\tau}).$$

We see that the regular distribution $\mathcal{D}_h$ is naturally defined on a larger domain, namely $\mathcal{A}(VF^n, \mathcal{C})$; that is, $\mathcal{D}_h : \mathcal{A}(VF^n, \mathcal{C}) \rightarrow \mathcal{C}$ is a linear functional given by the above integral.

**Lemma 7.6.** Let $\mathcal{D}$ be a distribution. Then there is a definable open subset $V \subseteq VF^n$ and an almost integrable function $f \in \mathcal{A}(V, \mathcal{C})$ such that $\dim_{VF}(VF^n \setminus V) < n$ and, for every bounded open definable subset $U \subseteq V$,

$$\mathcal{D}(U) = \int_U f.$$

**Proof.** For each $\overline{\tau} \in VF^n$, $\mathcal{D}$ induces an $\overline{\tau}$-definable function $\mathcal{D}_{\overline{\tau}} : \Gamma \rightarrow \mathcal{C}$ given by $\gamma \rightarrow \mathcal{D}(o(\overline{\tau}, \gamma))$. By quantifier elimination, $\mathcal{D}_{\overline{\tau}}$ may be defined by an RV-sort formula in disjunctive normal form. By compactness, there are pairwise disjoint definable subsets $Y_1, \ldots, Y_l \subseteq VF^n$ that cover $VF^n$ and RV-sort formulas $\phi_1(\overline{\tau}), \ldots, \phi_l(\overline{\tau})$ that are in disjunctive normal form such that, for each $\overline{\tau} \in Y_i$, $\mathcal{D}_{\overline{\tau}}$ is defined by $\phi_i(\overline{\tau})$. For each $i \leq l$ let $g_1(\overline{\tau}), \ldots, g_{k_i}(\overline{\tau})$ be all the polynomials occurring in $\phi_i(\overline{\tau})$ in the form $rv(\overline{g}_i(\overline{\tau}))$. Let $f'_i : Y_i \rightarrow RV^k$ be the definable function given by

$$\overline{\tau} \mapsto (rv(g_1(\overline{\tau})), \ldots, rv(g_{k_i}(\overline{\tau}))).$$

Let $f'$ be the union of $f'_1, \ldots, f'_l$.

By Lemma 2.12 and compactness, there is a definable open subset $V \subseteq VF^n$ such that $\dim_{VF}(VF^n \setminus V) < n$ and, for every $\overline{\tau} \in V(f'V)$, $f'^{-1}(f'V) \cap V$ is an open subset. By the construction of $f'$, for any $\overline{\tau} \in V$, any $\overline{\beta} \in f'^{-1}(f'V)$, and any $\gamma \in \Gamma$, $\mathcal{D}(o(\overline{\tau}, \gamma)) = \mathcal{D}(o(\overline{\beta}, \gamma))$. Let $\alpha \in \Gamma$ be such that $o(\overline{\tau}, \alpha) \subseteq f'^{-1}(f'V) \cap V$.

By coherence, for any $\beta \geq \alpha$,

$$\mathcal{D}(\overline{\tau}, \alpha) = \int_{\overline{\tau} \in o(\overline{\tau}, \alpha)} \epsilon_{\beta}^{-n} \mathcal{D}(\overline{\tau}, \beta) = \int_{\overline{\tau} \in o(\overline{\tau}, \alpha)} \epsilon_{\beta}^{-n} \mathcal{D}(\overline{\tau}, \beta) = \epsilon_{\alpha}^{-n} \mathcal{D}(\overline{\tau}, \beta)$$

and hence

$$\epsilon_{\alpha}^{-n} \mathcal{D}(\overline{\tau}, \alpha) = \epsilon_{\beta}^{-n} \mathcal{D}(\overline{\tau}, \beta).$$

Let $p : V \rightarrow \Gamma$ be the definable function such that $o(\overline{\tau}, p(\overline{\tau}))$ is the largest open polyball around $\overline{\tau}$ of this form contained in $f'^{-1}(f'V) \cap V$. Obviously $p$ is a volumetric partition of $V$. Let $f$ be the function on $V$ given by $\overline{\tau} \mapsto \epsilon_{p(\overline{\tau})}^{-n} \mathcal{D}(\overline{\tau}, p(\overline{\tau}))$, which, by Corollary 2.14, is an almost integrable function.

Let $U \subseteq V$ be a bounded open definable subset. Let $\beta \in \Gamma$ be a definable upper bound of $u(\overline{\tau}/VF^n) \cup p(U)$. Then

$$\mathcal{D}(U) = \int_{\overline{\tau} \in U} \epsilon_{\beta}^{-n} \mathcal{D}(\overline{\tau}, \beta) = \int_{\overline{\tau} \in U} f(x). \quad \square$$

Let $h_{o(\overline{\tau}, \gamma)}$ be the characteristic function of the open polyball $o(\overline{\tau}, \gamma)$. Its Fourier transform $\hat{h}_{o(\overline{\tau}, \gamma)}$ is the function $\epsilon_{\gamma}^{\tau} \exp_{\overline{\tau}} \hat{h}_{i(0, -\gamma)}$, where $h_{i(0, -\gamma)}$ is the characteristic function of the closed polyball $c(0, -\gamma)$. By Proposition 6.18, $\hat{h}_{o(\overline{\tau}, \gamma)} \in \mathcal{S}$. If
val(\(\mathbf{\pi}\)) > \gamma \) then we may assume \(t_{\hat{h}_{\alpha(\mathbf{\pi},\gamma)}}(\mathbf{b}) = -\gamma\) for every \(\mathbf{b} \in \epsilon(0, -\gamma)\); if \(\text{val}(\mathbf{\pi}) \leq \gamma\) then we may assume \(t_{\hat{h}_{\alpha(\mathbf{\pi},\gamma)}}(\mathbf{b}) = -\text{val}(\mathbf{\pi})\) for every \(\mathbf{b} \in \epsilon(0, -\gamma)\).

**Lemma 7.7.** For any distribution \(\mathcal{D}\), the function \(\hat{\mathcal{D}}\) on \(\text{VF}^{n} \times \Gamma\) given by

\[
\hat{\mathcal{D}}(\mathbf{\pi}, \gamma) = \mathcal{D}(\hat{h}_{\alpha(\mathbf{\pi},\gamma)})
\]

is a predistribution.

**Proof.** Let \(\beta \geq -\gamma\) be \(\gamma\)-definable. Then, for every \(\mathbf{\pi} \in \text{VF}^{n}\),

\[
\mathcal{D}(\hat{h}_{\alpha(\mathbf{\pi},\gamma)}) = \alpha_{\gamma} \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \exp(\mathbf{\pi}) \epsilon_{\beta}^{-n} \mathcal{D}(\mathbf{\pi}, \beta).
\]

Since \(\mathcal{D}_{\beta}\) is an almost integrable function and the function given by \((\mathbf{y}, \mathbf{\pi}) \mapsto \exp(\mathbf{\pi})\) is also almost integrable, clearly \(\mathcal{D}_{\gamma}\) is an almost integrable function as well. Next, if \(\mathbf{\pi} = \mathbf{\pi}(\mathbf{y}, \gamma)\) then \(\exp(\mathbf{\pi}) \upharpoonright \epsilon(0, -\gamma) = \exp(\mathbf{\pi}) \upharpoonright \epsilon(0, -\gamma)\) and hence

\[
\mathcal{D}(\hat{h}_{\alpha(\mathbf{\pi},\gamma)}) = \mathcal{D}(\hat{h}_{\alpha(\mathbf{\pi},\gamma)}).\]

Lastly, let \(\gamma' \geq \gamma\). Then

\[
\mathcal{D}(\mathbf{\pi}, \gamma) = \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \alpha_{\gamma} \mathcal{D}(\mathbf{y}, \beta) \exp(\mathbf{\pi})
\]

\[
= \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \alpha_{\gamma} \mathcal{D}(\mathbf{y}, \beta) \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \exp(\mathbf{\pi})
\]

\[
= \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \alpha_{\gamma} \mathcal{D}(\mathbf{y}, \beta) \int_{\mathbf{\pi} \in \epsilon(0, -\gamma)} \exp(\mathbf{\pi}) \epsilon_{\beta}^{-n} \mathcal{D}(\mathbf{\pi}, \beta)
\]

\[
= \mathcal{D}(\hat{f}),
\]

where the last equality holds since \(\beta \geq -\gamma'\). \(\square\)

**Definition 7.8.** The Fourier transform of the distribution induced by the predistribution \(\mathcal{D}\) is the distribution induced by the predistribution \(\hat{\mathcal{D}}\). Sometimes \(\hat{\mathcal{D}}\) is also written as \(\mathcal{F}(\mathcal{D})\).

**Theorem 7.9.** For every \(f \in \mathcal{S}_{n}\), \(\hat{\mathcal{D}}(\hat{f}) = \hat{\mathcal{D}}(f)\).

**Proof.** By Proposition 6.18, \(\hat{f} \in \mathcal{S}_{n}\). Let \(\beta\) be a definable upper bound of \(\epsilon_{f}(\text{VF}^{n})\) such that \(\text{supp}(\hat{f}) \subseteq \epsilon(0, -\beta)\). Let \(\delta \geq -\beta\) be a definable upper bound of \(\epsilon_{f}(\text{VF}^{n})\). Then, for every definable \(\gamma \in \Gamma\) that is sufficiently low,

\[
\hat{\mathcal{D}}(f) = \int_{\mathbf{\pi} \in \epsilon(0, \gamma)} f(\mathbf{\pi}) \alpha_{\gamma} \mathcal{D}(\mathbf{y}, \beta) \int_{\mathbf{\pi} \in \epsilon(0, -\beta)} \exp(\mathbf{\pi}) \epsilon_{\beta}^{-n} \mathcal{D}(\mathbf{\pi}, \delta)
\]

\[
= \int_{\mathbf{\pi} \in \epsilon(0, -\beta)} \left( \int_{\mathbf{\pi} \in \epsilon(0, \gamma)} f(\mathbf{\pi}) \exp(\mathbf{\pi}) \right) \epsilon_{\beta}^{-n} \mathcal{D}(\mathbf{\pi}, \delta)
\]

\[
= \mathcal{D}(\hat{f}),
\]

where the second equality is by the Fubini theorem. \(\square\)

An analog of Bernstein’s theorem in [1] for local fields is proved in [10]. It says the following: For any integers \(n, d\), any local field \(L\) of sufficiently large residue characteristic, and any polynomial \(G \in L[X_{1}, \ldots, X_{n}]\) of degree \(\leq d\), there is a proper variety \(V_{G} \subseteq L^{n}\) such that the Fourier transform \(\mathcal{F}(G)\) agrees with a locally constant function outside of \(V_{G}\). Although this is a direct consequence of
Lemma 7.6 and Lemma 2.19, some discussion on specialization to local fields is also needed. A presentation of this will appear in a sequel.

Let $\mathcal{D}$ be a distribution on $\mathcal{V}^n$. We say that $\mathcal{D}$ **vanishes at** $\bar{\pi} \in \mathcal{V}^n$ if there is a $\gamma \in \Gamma$ such that $\mathcal{D}(\bar{\pi}, \gamma') = 0$ for every $\bar{\pi}' \in \phi(\bar{\pi}, \gamma)$ and every $\gamma' \geq \gamma$. By Definition 7.5, if $\mathcal{D}$ vanishes at $\bar{\pi}$ then $\mathcal{D}(f) = 0$ for every $f \in \mathcal{X}_n$ with $\text{supp}(f) \subseteq \phi(\bar{\pi}, \gamma)$. If $\mathcal{D}$ does not vanish at $\bar{\pi}$ then $\bar{\pi}$ is an **essential point** of $\mathcal{D}$.

**Definition 7.10.** The **support** of $\mathcal{D}$ is the definable subset $\text{supp}(\mathcal{D}) \subseteq \mathcal{V}^n$ of essential points of $\mathcal{D}$.

By $\alpha$-minimality, if $\mathcal{D}$ vanishes at $\bar{\pi}$ then there is a least $\bar{\pi}$-definable $\iota_{\mathcal{D}}(\bar{\pi}) \in \Gamma$ such that $\mathcal{D}(\bar{\pi'}, \gamma') = 0$ for every $\bar{\pi}' \in \phi(\bar{\pi}, \iota_{\mathcal{D}}(\bar{\pi}))$ and every $\gamma' \geq \iota_{\mathcal{D}}(\bar{\pi})$. Clearly if $\bar{\pi}' \in \phi(\bar{\pi}, \iota_{\mathcal{D}}(\bar{\pi}))$ then $\mathcal{D}$ vanishes at $\bar{\pi}'$ and $\iota_{\mathcal{D}}(\bar{\pi}') = \iota_{\mathcal{D}}(\bar{\pi})$. So $\mathcal{V}^n \setminus \text{supp}(\mathcal{D})$ is an open subset and

$$\iota_{\mathcal{D}} : \mathcal{V}^n \setminus \text{supp}(\mathcal{D}) \rightarrow \Gamma$$

is a volumetric partition. However, in general we cannot apply Lemma 3.7 to any $\alpha \in \Gamma$ to get an $\alpha$-definable upper bound of $\iota_{\mathcal{D}}((\mathcal{V}^n \setminus \text{supp}(\mathcal{D})) \cap \phi(0, \alpha))$ because $\mathcal{V}^n \setminus \text{supp}(\mathcal{D})$ may not be closed. We do have the following:

**Lemma 7.11.** For any $\phi(\bar{\pi}, \gamma) \subseteq \mathcal{V}^n \setminus \text{supp}(\mathcal{D})$, $\mathcal{D}(\bar{\pi}, \gamma) = 0$. In particular, $\mathcal{D}$ has no essential point if and only if $\mathcal{D} = 0$.

**Proof.** The first assertion follows from Lemma 3.7 and coherence. The second assertion is the special case $\text{supp}(\mathcal{D}) = \emptyset$.

It follows that, for any $\bar{\pi} \in \mathcal{V}^n \setminus \text{supp}(\mathcal{D})$, $\phi(\bar{\pi}, \iota_{\mathcal{D}}(\bar{\pi}))$ is the largest open polyball around $\bar{\pi}$ that is disjoint from $\text{supp}(\mathcal{D})$. In particular, if $\mathcal{D} \neq 0$ and $\text{supp}(\mathcal{D}) \subseteq \phi(0, \gamma)$ then $\iota_{\mathcal{D}}(\bar{\pi}) = \text{val}(\bar{\pi})$ for every $\bar{\pi} \notin \phi(0, \gamma)$.

**Lemma 7.12.** If $\mathcal{D}$ has bounded support then there is an almost integrable locally constant function $h$ such that $\mathcal{D} = \mathcal{D}_h$.

**Proof.** Let $\phi(0, \gamma)$ be the smallest definable closed polyball around 0 containing $\text{supp}(\mathcal{D})$. Let $h$ be the definable function on $\mathcal{V}^n$ given by

$$\bar{\pi} \mapsto \mathcal{D}(\exp_{\bar{\pi}} \phi(0, \gamma)) = \int_{\bar{\pi} \in \phi(0, \gamma)} \exp_{\bar{\pi}}(\bar{\pi}) \phi_{\beta}^{-n} \mathcal{D}(\bar{\pi}, \beta),$$

where $\beta = \max \{ \gamma, -\text{val}(\bar{\pi}) \}$. It is easy to see that $h$ is an almost integrable locally constant function. Then it is enough to show that

$$\mathcal{D}(f) = \mathcal{D}_h(f) = \int_{\bar{\pi} \in \mathcal{V}^n} f(\bar{\pi}) h(\bar{\pi})$$

for every $f \in \mathcal{X}_n$. For every definable $\alpha \in \Gamma$ that is sufficiently low and every definable $\beta$ that is sufficiently large, by the Fubini theorem, we have

$$\mathcal{D}_h(f) = \int_{\bar{\pi} \in \phi(0, \alpha)} f(\bar{\pi}) \int_{\bar{\pi} \in \phi(0, \gamma)} \exp_{\bar{\pi}}(\bar{\pi}) \phi_{\beta}^{-n} \mathcal{D}(\bar{\pi}, \beta)$$

$$= \int_{\bar{\pi} \in \phi(0, \gamma)} \hat{f}(\bar{\pi}) \phi_{\beta}^{-n} \mathcal{D}(\bar{\pi}, \beta).$$

By Lemma 7.11, for every $\bar{\pi} \notin \phi(0, \gamma)$, $\iota_{\mathcal{D}}(\bar{\pi}) = \text{val}(\bar{\pi})$ and hence $\mathcal{D}(\bar{\pi}, \beta) = 0$. Therefore $\mathcal{D}_h(f) = \mathcal{D}(f)$.

□
For any \( f \in \mathcal{A}(VF^n, \check{C})\), let \( fD = Df \) be the function in \( \text{Fn}(VF^n \times \Gamma, \check{C}) \) given by \((\bar{\pi}, \gamma) \mapsto f(\bar{\pi}D(\bar{\pi}, \gamma))\), which is a predistribution. The distribution induced by \( Df \) is also denoted as \( Df \).

**Lemma 7.13.** Let \( f, g \in \mathcal{A}(VF^n, \check{C}) \) such that \( Df = Dg \). Then \( Df = Dg \).

**Proof.** For any \( h \in \mathcal{S}_n \) and any \( \beta \in \Gamma \), the function given by \( \bar{\pi} \mapsto h(\bar{\pi})\epsilon_{\check{\beta}}^{-n}D(\bar{\pi}, \beta) \) is in \( \mathcal{S}_n \). So the assertion is clear. \( \square \)

Therefore, for any regular distribution \( D' \), it makes sense to write \( D \mathcal{D}' \) or \( \mathcal{D}' \mathcal{D} \), where the distribution is given by \( Df \) for any \( f \in \mathcal{A}(VF^n, \check{C}) \) such that \( Df = Df' \).

Let \( D_1 \) be a predistribution on \( VF^{n_1} \) and \( D_2 \) a predistribution on \( VF^{n_2} \). Let \( D_1 \otimes D_2 \) be the definable function on \( VF^{n_1+n_2} \times \Gamma \) given by

\[
(\bar{\pi}_1, \bar{\pi}_2, \gamma) \mapsto D_1(\bar{\pi}_1, \gamma)D_2(\bar{\pi}_2, \gamma).
\]

It is routine to check that \( D_1 \otimes D_2 \) is a predistribution on \( VF^{n_1+n_2} \) and

\[
\mathcal{F}(D_1 \otimes D_2) = \hat{D}_1 \otimes \hat{D}_2.
\]

**Definition 7.14.** The *tensor product* of the distributions induced by \( D_1 \) and \( D_2 \) is the distribution induced by \( D_1 \otimes D_2 \), which is also written as \( D_1 \otimes D_2 \).

Note that \( \text{supp}(D_1 \otimes D_2) = \text{supp}(D_1) \times \text{supp}(D_2) \).

If \( n_1 = n_2 = n \), we wish to define the convolution \( D_1 \ast D_2 \) of \( D_1 \) and \( D_2 \) as the linear functional on \( \mathcal{S}_n \) given by

\[
(D_1 \ast D_2)(f) = (D_1 \otimes D_2)(f^\dagger),
\]

where \( f^\dagger \) is the locally constant integrable function on \( VF^{2n} \) given by

\[
(\bar{\pi}_1, \bar{\pi}_2) \mapsto f(\bar{\pi}_1 + \bar{\pi}_2).
\]

However, as in the classical theory, \( f^\dagger \) is not guaranteed to be a Schwartz-Bruhat function since its support may not be bounded. On the other hand, if either \( D_1 \) or \( D_2 \) has bounded support then \( \text{supp}(f^\dagger) \cap \text{supp}(D_1 \otimes D_2) \) is bounded. So the classical remedy works:

**Definition 7.15.** Suppose that \( D_1 \) (or \( D_2 \)) has bounded support. Then the *convolution* \( D_1 \ast D_2 \) of \( D_1 \) and \( D_2 \) is the linear functional on \( \mathcal{S}_n \) given by

\[
(D_1 \ast D_2)(f) = \int_{(\bar{\pi}_1, \bar{\pi}_2) \in \imath(0, \gamma)} f(\bar{\pi}_1 + \bar{\pi}_2)\epsilon_{\check{\beta}}^{-2n}(D_1 \otimes D_2)(\bar{\pi}_1, \bar{\pi}_2, \beta),
\]

where \( \imath(0, \gamma) \) is a definable open polyball containing \( \text{supp}(f^\dagger) \cap \text{supp}(D_1 \otimes D_2) \) and \( \beta \) is any definable upper bound of \( \imath_{f^\dagger}(0, \gamma)(VF^{2n}) \).

Note that, by Lemma 7.3 and Lemma 7.11, this definition does not depend on the choices of \( \gamma \) and \( \beta \).

**Lemma 7.16.** The definable function \( VF^n \times \Gamma \to \check{C} \) induced by \( D_1 \ast D_2 \) is a predistribution and hence \( D_1 \ast D_2 \) is a definable distribution.

**Proof.** The proofs of Lemma 7.7 and Theorem 7.9 may be easily adapted here. \( \square \)

The following theorem is a generalization of the convolution formula for bounded integrable functions.

**Theorem 7.17.** \( \mathcal{F}(D_1 \ast D_2) = \hat{D}_1 \hat{D}_2 \).
Proof. Note that $\hat{D}_1 \hat{D}_2$ is defined since, by Lemma 7.12, $\hat{D}_1$ is regular. For any $f \in \mathcal{S}_n$, any definable $\gamma \in \Gamma$ that is sufficiently low, and any definable $\beta \in \Gamma$ that is sufficiently large, $(\mathcal{D}_1 \ast \mathcal{D}_2)(\hat{f})$ is equal to

$$\int_{(\tau_1, \tau_2) \in o(0, \gamma)} \left( \int_{\gamma \in o(0, \gamma)} f(\gamma) \exp_{\tau_1, \tau_2}(\gamma) \right) \sigma_{\beta}^{-2n} \mathcal{D}_1(\tau_1, \beta) \mathcal{D}_2(\tau_2, \beta)$$

$$= \hat{D}_1(f \exp_{\tau_2}) \sigma_{\beta}^{-n} \mathcal{D}_2(\tau_2, \beta).$$

Choose any $h \in \mathcal{A}(\text{VF}^n, \hat{C})$ such that $\hat{D}_1 = \mathcal{D}_h$. Then we have

$$(\mathcal{D}_1 \ast \mathcal{D}_2)(\hat{f}) = \int_{\tau_2 \in o(0, \gamma)} \left( \int_{\gamma \in o(0, \gamma)} f(\gamma) \exp_{\tau_2}(\gamma) h(\gamma) \right) \sigma_{\beta}^{-n} \mathcal{D}_2(\tau_2, \beta)$$

$$= \hat{D}_2(fh) = \hat{D}_1 \hat{D}_2(f). \quad \square$$

8. The Weil Representation

The main references for this section are [12, Section 1.1] and [13, Chapter XI]. We shall only work with definable functions on VF$^2$.

Up till now, our discussion does not really distinguish VF-categories without the Jacobian from those with the Jacobian. As mentioned in Remark 3.12, the reason is that in changing variables only translation is used, for which the Jacobian is redundant.

In this section we shall employ other kinds of transformation in changing variables, namely scalars and involution, for which the Jacobian is easily computed but is not trivial. We shall first briefly describe the formalism of integration with volume form, which is already used in the change of variables formula above without explanation. Each object in a VF-category with volume form is a triple $(f, \mu \colon \text{VF} \rightarrow \text{RV}^2, \hat{C})$. The function $\mu$ should be thought of as a definable volume form $\mu dxdy$. In the corresponding RV-category, each object is a triple $(U, f, \mu)$, where $f : U \rightarrow \text{RV}^2$ is a definable finite-to-one function and $\mu : \text{RV}^2 \rightarrow \text{RV}$ a definable function. Let $\mu \text{Fn}(\text{VF}^2, \hat{C})$ be the set of definable functions with volume form. A function in $\mu \text{Fn}(\text{VF}^2, \hat{C})$ may also be written as a pair $(f, \mu)$, where, for simplicity, we just think of $f$ as a function in $\text{Fn}(\text{VF}^2, \hat{C})$. Then integrating $(f, \mu)$ may simply be taken to mean

$$\int_{\text{VF}^2} (f, \mu) = \left( \int_{\text{VF}^2} f, \text{rv}(\mu) \right),$$

where, on the righthand side, the integral does not carry volume form and $\text{rv}(\mu)$ is a definable function $\text{RV}^2 \rightarrow \text{RV}$ whose lift is $\mu$. Note that the group structure of $\mu \text{Fn}(\text{VF}^2, \hat{C})$ is given by disjoint sum. This means that, technically speaking, the domain of a function $(f, \mu) \in \mu \text{Fn}(\text{VF}^2, \hat{C})$ is a subset of the form $\text{VF}^2 \times I$, where $I \subseteq \{1, \infty\} \subseteq \text{RV}^n$. However, for notational ease, we shall never mention this index set $I$.

A constant or translation invariant volume form is a disjoint sum of constant functions $\mu : \text{VF}^2 \rightarrow \text{RV}$. For the rest of this section, we shall only work with constant volume forms. Let $\hat{\mu}_\mathcal{Z}_2$ be the group of Schwartz-Bruhat functions with constant volume form.
Example 8.1. We give an example here to illustrate the difference between integrating without volume form and integrating with volume form. Actually this example will be used below in the construction of the Weil representation. Let \((f, \mu) \in \hat{\mathcal{M}}_2, a \in \mathcal{V}_F, a(0, \gamma)\) a definable open polyball containing \(a^{-1} \text{supp}(f)\). Consider the bijection
\[
\tau_a : a(0, \gamma) \longrightarrow a(0, \gamma + \text{val}(a))
\]
given by \(b \longmapsto ab\). Without volume form, we simply have
\[
\text{vol}(a(0, \gamma)) = \text{vol}(a(0, \gamma + \text{val}(a)))
\]
and hence
\[
\int_{x \in a(0, \gamma)} f(a \bar{x}) \exp_{\bar{x}}(x) = \int_{x \in a(0, \gamma + \text{val}(a))} f(x) \exp_{a^{-1}}(x) = \hat{f}(a^{-1} b).
\]
With a constant volume form \(\mu\) on \(\mathcal{V}_F^2\), since \(\text{Jcb} \tau_a\) is the constant function \(a^2\), by change of variables,
\[
\text{vol}(a(0, \gamma), \text{rv}(a^2) \mu) = \text{vol}(a(0, \gamma + \text{val}(a)), \mu)
\]
and hence
\[
\int_{x \in a(0, \gamma)} (f(a \bar{x}) \exp_{\bar{x}}(x), \text{rv}(a) \mu) = \int_{x \in a(0, \gamma + \text{val}(a))} (f(x) \exp_{a^{-1}}(x), \text{rv}(a^{-1}) \mu).
\]

If all the integrals in a context use the same volume form then we shall not indicate which form is being used. Also, as in the previous sections, if a result holds with or without volume form in the same way then we shall only state it in the simpler formalism.

Each element \(b \in \mathcal{V}_F^\times\) gives rise to a character of \(\mathcal{V}_F^2\) of second order, which is the almost integrable function \(\nu_b : \mathcal{V}_F^2 \longrightarrow \hat{C}\) given by
\[
(a_1, a_2) \longmapsto \exp_b(a_1 a_2).
\]

For any \(\bar{a} = (a_1, a_2) \in \mathcal{V}_F^2\) and any \(\gamma \in \Gamma\), since \(\nu_b \upharpoonright a(\bar{a}, \gamma)\) is an integrable function, by the Fubini theorem, we have
\[
\int_{\bar{x} \in a(\bar{a}, \gamma)} \nu_b(\bar{x}) = \int_{x_1 \in a(\bar{a}_1, \gamma)} \int_{x_2 \in a(\bar{a}_2, \gamma)} \exp_{a^{-1}}(x_2).
\]
If \(\max \{\text{val}(a_1), \text{val}(ba_1)\} < \min \{\gamma, -\gamma\}\) then, since \(\text{val}(ba_1') = \text{val}(ba_1)\) for every \(a' \in a(1, \gamma)\), by Lemma 6.4
\[
\int_{\bar{x} \in a(\bar{a}, \gamma)} \nu_b(\bar{x}) = \int_{a(1, \gamma)} 0 = 0.
\]

Lemma 8.2. For any \(f \in \mathcal{F}_2\), \(f \ast \nu_b \in \mathcal{H}(\mathcal{V}_F^2, \hat{C})\) and hence \(\mathcal{F}(f \ast \nu_b)\) exists.

Proof. Since \(f\) has bounded support and \(\nu_b\) is an almost integrable function, an easy argument similar to the one in the proof of Proposition 6.10 shows that \(f \ast \nu_b\) is also an almost integrable function. So it is enough to show that \(f \ast \nu_b\) has bounded support. Let \(a(0, \gamma)\) be a definable open polyball containing \(\text{supp}(f)\) and \(\beta > 0\) a definable upper bound of \(\text{Jcb}(\mathcal{V}_F^2)\). For any \(\bar{a} = (a_1, a_2) \in \mathcal{V}_F^2\), by the averaging formula,
\[
\int_{x \in \mathcal{V}_F^2} f(x) \nu_b(\bar{a} - x) = \int_{x \in a(0, \gamma)} \left( a_{\beta}^{-2}(x) \int_{y \in a(x, \beta/(\mathcal{Y}))} \nu_b(\bar{a} - y) \right).
\]
If \( \max \{ \text{val}(\pi), \text{val}(b\pi) \} < \min \{ \gamma, -\beta \} \) then, by change of variables and the discussion above,
\[
\int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \iota(\pi))} \nu_b(\pi - \mathbf{\Sigma}) = \int_{\pi \in \mathfrak{a}(\pi, \iota(\pi))} \nu_b(\mathbf{\Sigma}) = 0. \tag*{\Box}
\]

The following lemma is an extension of the convolution formula for integrable functions on \( \text{VF}^2 \).

**Lemma 8.3.** For any \( (f, \mu) \in \mu \mathcal{F}_2 \),
\[
(\mathcal{F}(f \ast \nu_b)(b\pi), \mu) = (\varepsilon \hat{f}(b\pi) \nu_{-b}(\pi), \text{rv}(b^{-1}) \mu).
\]

**Proof.** For any \( \pi = (a_1, a_2) \in \text{VF}^2 \) and any \( \pi \)-definable \( \gamma < \text{val}(\pi) \) that is sufficiently low, by the Fubini theorem and change of variables,
\[
\mathcal{F}(f \ast \nu_b)(b\pi) = \int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \nu_b(\mathbf{\Sigma}) f(\pi - \mathbf{\Sigma}) \exp_{b\mathfrak{s}}(\mathbf{\Sigma})
\]
\[
= \int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \nu_b(\mathbf{\Sigma}) \int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} f(\pi) \exp_{b\mathfrak{s}}(\pi + \mathbf{\Sigma})
\]
\[
= \hat{f}(b\pi) \int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \nu_b(\mathbf{\Sigma}) \exp_{b\mathfrak{s}}(\mathbf{\Sigma}).
\]

If we write \( \mathbf{\Sigma} \) as \( (y_1, y_2) \) then, by the Fubini theorem again,
\[
\int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \nu_b(\mathbf{\Sigma}) \exp_{b\mathfrak{s}}(\mathbf{\Sigma}) = \int_{(y_1, y_2) \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \exp_b(y_1 y_2 + a_1 y_1 + a_2 y_2)
\]
\[
= \int_{y_1 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \exp_{\text{a}_1}(b y_1) \int_{y_2 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \exp_{b(y_1 + a_2)}(y_2).
\]
Changing variables via the map \( (y_1, y_2) \mapsto (b y_1, y_2) \), we have
\[
\int_{y_1 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \exp_{\text{a}_1}(b y_1) \int_{y_2 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} \exp_{b(y_1 + a_2)}(y_2), \mu
\]
\[
= \int_{y_1 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi), \text{val}(b))} \exp_{\text{a}_1}(y_1) \int_{y_2 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} ((\exp_{y_1 + b a_2}(y_2), \text{rv}(b^{-1}) \mu)
\]
\[
= \int_{y_1 \in \mathfrak{a}(\pi, \mathfrak{s}(\pi), \text{val}(b))} (\exp_{y_1 + b a_2}(y_2), \text{rv}(b^{-1}) \mu)
\]
\[
= \exp_{-\text{b}}(a_1 a_2), \text{rv}(b^{-1}) \mu
\]
where the second equality is by Lemma 6.4. Therefore,
\[
\int_{\mathbf{\Sigma} \in \mathfrak{a}(\pi, \mathfrak{s}(\pi))} (\hat{f}(b\pi) \nu_b(\mathbf{\Sigma}) \exp_{b\mathfrak{s}}(\mathbf{\Sigma}), \mu) = (\varepsilon \hat{f}(b\pi) \nu_{-b}(\pi), \text{rv}(b^{-1}) \mu). \tag*{\Box}
\]

The group \( \text{SL}_2(\text{VF}) \) is generated by elements of the form
\[
u_b(a) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad s(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
where \( a, b \in \text{VF} \) and \( a \neq 0 \), subject to the relations
(1) \( \nu_b \) is an additive homomorphism,
(2) \( s \) is a multiplicative homomorphism,
(3) \( ws(a) = s(a^{-1}) w \),
(4) \( w^2 = s(-1) \),
(5) \( wu(a)w = s(-a^{-1})u(-a)wu(-a^{-1}) \).

For any \( \pi = (a_1, a_2) \in \mathcal{V}^2 \), let \( \overline{\pi} = (a_2, a_1) \), which is an involution on \( \mathcal{V}^2 \).

**Theorem 8.4.** Suppose that the underlying substructure \( S \) has a square root \( \pi \) of \(-1 \in \mathcal{V} \). Then there is a unique representation \( r \) of \( SL_2(\mathcal{V}) \) on \( \mu \mathcal{S} \) satisfying the following conditions:

1. \( r(s(a))(f, \mu)(\pi) = (f(a\pi), rv(a)\mu) \),
2. \( r(u(a))(f, \mu)(\pi) = (\nu_{a\pi}(\pi f (\pi), \mu) \),
3. \( r(w)(f, \mu)(\pi) = (\epsilon^{-1}f(\pi)\mu, rv(-\pi^{-1})\mu) \).

**Proof.** The uniqueness of \( r \) is clear. To show the existence, we just need to verify that the mapping \( r \) specified by the three conditions preserves the five relations between the generators of \( SL_2(\mathcal{V}) \). The first two are obvious and the third one is just the example given above. Since the Jacobian of the scalar \( \pi \) is the constant function \(-1 \), the fourth relation is easily seen to follow from the proof of the Fourier inversion formula and Remark 6.20.

For any \( \overline{\pi} = (c_1, c_2) \in \mathcal{V}^2 \), the righthand side in the fifth relation becomes the pair \((g(\overline{\pi}), rv(a^{-1}\pi^{-1})\mu)\), where

\[
g(\overline{\pi}) = \epsilon^{-1}\nu_{a^{-1}\pi}(\pi) \int_{\mathcal{V}} \nu_{a^{-1}\pi}(x) f(x) \exp_{-a^{-1}\pi}(x).
\]

Since the Jacobian of the involution is the constant function \(-1 \), by Lemma 8.3,

\[
(g(\overline{\pi}), rv(a^{-1}\pi^{-1})\mu) = (g^*(\overline{\pi}), rv(-a^{-2})\mu),
\]

where

\[
g^*(\overline{\pi}) = \epsilon^{-2} \int_{\mathcal{V}} f(x) \exp_{a^{-1}\pi}(y_1y_2 - x_1y_1 - x_2y_2) \exp_{-a^{-1}\pi}(y).
\]

Then, applying the change of variables formula to \(-a^{-1}y\), we deduce

\[
(g^*(\overline{\pi}), rv(-a^{-2})\mu) = \int_{\mathcal{V}} \epsilon^{-2} \nu_{a\pi}(y) f(\pi y) \exp_{\pi^2}(y), rv(-1)\mu = r(wu(a)w)(f, \mu)(\pi).
\]

Note that the second order character used in Theorem 8.4 is \( \nu_1 \). It is possible to extend the result to an arbitrary \( \nu_0 \), which, however, requires changing accordingly the theory developed in Section 3 and Section 6.

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