The solution space of the Einstein’s vacuum field equations for the case of five-dimensional Bianchi Type I (Type 4A$_1$)

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Received 1 March 2018, revised 23 May 2018
Accepted for publication 29 May 2018
Published 19 June 2018

Abstract
We consider the 4+1 Einstein’s field equations (EFE’s) in vacuum, simplified by the assumption that there is a 4D sub-manifold on which an isometry group of dimension four acts simply transitive. In particular, we consider the Abelian group Type 4A$_1$; and thus the emerging homogeneous sub-space is flat. Through the use of coordinate transformations that preserve the sub-manifold’s manifest homogeneity, a coordinate system is chosen in which the shift vector is zero. The resulting equations remain form invariant under the action of the constant Automorphisms group. This group is used in order to simplify the equations and obtain their complete solution space which consists of seven families corresponding to 21 distinct solutions. Apart from the Kasner type all the other solutions found are, to the best of our knowledge, new. Some of them correspond to cosmological solutions, others seem to depend on some spatial coordinate and there are also pp-wave solutions.

Keywords: cosmology, Einstein’s field equations, five dimensional, automorphisms, vacuum solutions, Bianchi Type I, homogeneous submanifolds

1. Introduction

It is commonly known that the concept of symmetry possesses a fundamental role in mathematical physics. Specifically, in the branch of general relativity, symmetry has been used in order to simplify and subsequently solve (EFE’s), as well as classify the solutions, see for example [1, 2]. An interesting example is that of the group of Automorphisms also called rigid symmetries [3]. Ashtekar and Samuel were the first to study the group of Automorphisms
from a geometric viewpoint [4]. In the case of 3+1 Bianchi Types the use of Automorphisms of 3D Lie algebras has proven very useful since eventually leads to the specification of the general solution space for Bianchi Types (I–VII) [5–8]. Furthermore, the Automorphisms provide an algorithm for counting the number of essential constants and therefore invariantly characterize the manifold at hand [9–11].

The existence of extra dimensions seems to appear in mathematical physics since the seminal works of Kaluza and Klein [12, 13]. Their theory was a prototype for more sophisticated theories developed in later years, such as, string theory, brane theory, supergravity, supersymmetry etc. We cite only a few articles dealing with these subjects since there are two many to include them all [14–21].

Much work has been done in the context of higher dimensional cosmology. In a paper of Forgacs and Horvath [22], homogeneous and isotropic universes in the presence of gauge fields and two extra compact dimensions were studied. The idea that the properties of matter in the 4D universe can be purely described geometrically by using an extra dimension, works remarkably well in the case of spatially flat cosmological solutions in the presence of a perfect fluid, as was shown by Wesson [23]. The extension of this work in general spatially FR-LW cosmological solutions was presented in [24]. Also, a class of wave-like solutions were derived in [25]. More recently, non-separable 5D solutions in which the induced 4D metric has the form of FR-LW cosmology, were obtained in [26], Alan Chodos and Steven Detweiler [27] considered a 5D extension of Kasner’s 4D solution [28, 29]. This was the first attempt to study anisotropic cosmological spaces in the presence of extra dimensions. A series of papers concerning higher dimensional anisotropic spacetimes in Einstein’s general relativity, modifications of it like Brans–Dicke theory and supergravity, were presented by Lorenz-Petzold [30–37]. The existence of chaotic solutions in some of the 5D homogeneous spacetimes was studied by Paul Halpern [38]. In the paper [39], Sigbjorn Hervik classifies the 5D cosmological models based on whether the spatial hypersurfaces are connected or simply connected homogeneous Riemannian manifolds. Finally, some homogeneous vacuum plane wave solutions of 5D (EFE’s) are obtained in [40].

In this work, we are interested in the case of 5D manifolds possessing a 4D homogeneous sub-manifold. Specifically, we will concentrate in the Type 4A1. This particular choice makes the 4D homogeneous sub-manifold flat and therefore is the straightforward generalization of the Bianchi Type I. All the other 4D groups result in curved 4D manifolds, a thing that makes the investigation more involved and will be considered in future works. This enumeration can be found in the works [41, 42]. Thus, the group of Automorphisms is represented by the $GL(4, \mathbb{R})$ matrices. The corresponding Automorphisms for all the other real 4D Lie algebras can be found in [10]. Eventually, the Automorphisms will provide us a way to separate the different families of solutions to (EFE’s) and find their solution space.

The paper’s structure is organized as follows. In section 2 an introduction to the canonical formalism and the homogeneous manifolds is provided alongside with the 4+1 form of (EFE’s). The basic idea of transformations which preserve the manifest homogeneity is reviewed in section 3. In section 4, everything mentioned in the previous sections is applied to the case of Type 4A1 and the way that the constant Automorphisms can be used is presented. Two of the solutions are presented in detail in section 5 alongside with tables with all the other solutions. A table concerning the existence of homothetic vector fields, and additional Killing vector fields for specific range of values of the parameters is provided. Also, a table with the invariant relations and the number of essential constants for each family, can be found. At the end of this section some remarks can be found about the solutions. In section 6 one possible physical interpretation in terms of the brane worlds scenario for one of the solutions is presented. Lastly, a discussion of the overall results is given.
2. Canonical formalism

In the context of canonical formalism [43] the line element of a \((d+1)\)-dimensional manifold \(M\), in a coordinate system \(\Sigma\) with coordinates \((t,x^i)\), \(i = 1, \ldots, d\), acquires the form

\[
dr^2_{(d+1)} = \left( -N^2 + N_iN^i \right) dt^2 + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j,
\]

where \(N(t,x^i), N_i(t,x^i)\) are the lapse function and shift vector respectively and \(\gamma_{ij}(t,x^i)\) the metric of the \(d\)-dimensional sub-manifold which is given by \(t = \text{constant}\).

In the light of this, the (EFE’s) in vacuum are decomposed into the following equivalent set,

\[
R^{(d)} + K^2 - K_{ij}K^{ij} = 0, \quad \text{(Quadratic constraint)}
\]

\[
D_iK - D_jK_{ij} = 0, \quad \text{(Linear constraint)}
\]

\[
(\partial_t - \mathcal{L}_\gamma) K_{ij} = N K^{(d)}_{ij} + NK_{ij} - 2NK_{K_iK_j} - D_i D_j N, \quad \text{(Dynamical equations)}
\]

where \(R^{(d)}\), \(K^{(d)}\), \(D_i\) are the Ricci tensor, the Ricci scalar and the covariant derivative constructed out of \(\gamma_{ij}\) correspondingly. Also, \(\mathcal{L}_\gamma\) stands for the Lie derivative along the shift vector while \(K_{ij}\) is the extrinsic curvature tensor given by

\[
K_{ij} = \frac{1}{2N} (D_i N_j + D_j N_i - \partial_t \gamma_{ij}).
\]

Note also that \(K = \gamma^\alpha_{ij} K_{ij}\) is the extrinsic curvature scalar.

When the manifold \(M\) admits a \(d\)-dimensional isometry group \(G\) which acts simply transitively on the \(d\)-dimensional sub-manifold \(t = \text{constant}\), there exists an invariant basis of one-forms \(\{\sigma^\alpha\}\) satisfying the curl relations [44]

\[
d\sigma^\alpha = -\frac{1}{2} C_{\beta\gamma}^{\sigma} \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \partial_i \sigma^\alpha_j - \partial_j \sigma^\alpha_i = -C_{\beta\gamma}^{\sigma} \sigma_i^\beta \sigma_j^\gamma,
\]

where the Greek indices run from 1 to \(d\) and \(C_{\beta\gamma}^{\sigma}\) are the structure constants of the Lie algebra of the isometry group. The sub-manifold is then called homogeneous. Under this assumption, a coordinate system \((t,x^i)\) exists such that the line element (1) acquires the manifestly homogeneous form [45]

\[
dr^2_{(d+1)} = \left[ -N(t)^2 + N_i(t) N^i(t) \right] dt^2 + 2N_i(t) \sigma_j^i(x^i) dx^i dt + \gamma_{ij}(t) \sigma_j^i(x^i) \sigma_j^\sigma(x^\sigma) dx^\sigma dx^i,
\]

and (2)-(4) reduce to ordinary differential equations, with \(t\) the independent variable,

\[
R^{(d)} + K^2 - K_{\alpha\beta}K^{\alpha\beta} = 0, \quad \text{(Q. c.)}
\]

\[
K_{\alpha\beta} C^{\alpha}_{\epsilon\beta} - K_{\epsilon\beta} C^{\alpha}_{\alpha\beta} = 0, \quad \text{(L. c.)}
\]

\[
\dot{K}_{\alpha\beta} = N K^{(d)}_{\alpha\beta} - 2NK_{\alpha\epsilon}K^{\epsilon}_{\beta} + NK\dot{K}_{\alpha\beta} - N^\epsilon \left( K_{\alpha\lambda} C^{\lambda}_{\epsilon\beta} + K_{\lambda\beta} C^{\lambda}_{\epsilon\alpha} \right), \quad \text{(D. e.)}
\]

while

\[
K_{\alpha\beta} = -\frac{1}{2N} \left( N^\epsilon C^{\lambda}_{\epsilon\alpha} \gamma_{\lambda\beta} + N^\epsilon C^{\lambda}_{\epsilon\beta} \gamma_{\lambda\alpha} + \dot{\gamma}_{\alpha\beta} \right),
\]
and
\[ R_{\alpha\beta} = -\frac{1}{2} C_{\lambda\alpha} \left( C^\lambda_{\beta\bar{\gamma}} + C^\lambda_{\bar{\gamma}\alpha} \gamma^\nu \gamma_{\alpha\mu} \right) + \frac{1}{4} \gamma^{\mu\nu} \gamma^{\rho\sigma} C_{\mu\rho} C_{\nu\sigma} \gamma_{\alpha\lambda} \gamma_{\beta\gamma} + \frac{1}{2} C^\mu_{\rho\lambda} \gamma^{\nu\rho} \left( C_{\rho\alpha} \gamma_{\beta\epsilon} + C^\rho_{\beta\alpha} \gamma_{\alpha\epsilon} \right). \]

3. Automorphism inducing diffeomorphisms

In the paper [46] a group of coordinate transformations was derived that satisfy the following conditions:

1. Preservation of sub-manifold’s manifest homogeneity
2. They are symmetries of the equations (6)–(8)

A very brief outlined of the basic idea is as follows:

For transformations of the form
\[ t \mapsto \tilde{t} = t, \]
\[ x^i \mapsto \tilde{x}^i = h^i(t, x^l), \]
the restrictions on the functions \( f^i \), in order to satisfy the above conditions, are summarized as follows
\[
\sigma^\alpha (x^l) \frac{\partial x^l}{\partial \tilde{x}^m} = \Lambda^\alpha_{\beta} (\tilde{t}) \sigma^m (\tilde{x}^l), \quad (9)
\]
\[
\sigma^\alpha (x^l) \frac{\partial x^l}{\partial \tilde{t}} = P^\alpha (\tilde{t}). \quad (10)
\]
The relations (9) and (10) must be regarded as the definition of the matrix \( \Lambda^\alpha_{\beta} \) and the vector \( P^\alpha \) respectively.

The line element (5) can then be written as
\[
\text{d}s^2_{(d+1)} = \left[ -\tilde{N} (\tilde{t})^2 + \tilde{N}_\alpha (\tilde{t}) \tilde{N}^\alpha (\tilde{t}) \right] \text{d}\tilde{t}^2 + 2 \tilde{N}_\alpha (\tilde{t}) \sigma^\alpha (\tilde{x}^l) \text{d}x^l \text{d}\tilde{t} + \tilde{\gamma}_{\alpha\beta} (\tilde{t}) \sigma^\alpha (\tilde{x}^l) \sigma^\beta (\tilde{x}^j) \text{d}x^l \text{d}x^j,
\]
with the abbreviations
\[
\tilde{\gamma}_{\alpha\beta} (\tilde{t}) = \Lambda^\mu_{\alpha} (\tilde{t}) \Lambda^\nu_{\beta} (\tilde{t}) \gamma_{\mu\nu} (\tilde{t})
\]
\[
\tilde{N}_\alpha (\tilde{t}) = \Lambda^\mu_{\alpha} (\tilde{t}) \left( N_\mu (\tilde{t}) + P^\rho (\tilde{t}) \gamma_{\rho\mu} (\tilde{t}) \right)
\]
\[
\tilde{N} (\tilde{t}) = N (\tilde{t}). \quad (11)
\]
The existence of local solutions to the equations (9) and (10) is guaranteed by the Frobenious theorem if the following necessary and sufficient conditions hold:
\[
\Lambda^\alpha_{\mu} (\tilde{t}) C^\mu_{\beta\nu} = C^\alpha_{\beta\nu} \Lambda^\mu (\tilde{t}), \quad (12)
\]
\[
\dot{\Lambda}^\alpha_{\beta} (\tilde{t}) = \Lambda^\mu_{\beta} (\tilde{t}) C^\alpha_{\mu\nu} P^\nu (\tilde{t}), \quad (13)
\]
where the dot stands for differentiation with respect to \( \tilde{t} \). The solutions of (12) and (13) form a group.

Due to the transformation of the shift vector under the previous group, we can always choose the vector \( P^\rho \) such that the shift vector in the transformed system is zero. Thus, both the line element and the (EFE’s) acquire a simpler form.
\[ ds^2_{(d+1)} = -N(t)^2 \, dt^2 + \gamma_{\alpha\beta}(t) \, \sigma^\alpha_i(x') \, \sigma^\beta_j(x') \, dx^i \, dx^j. \]

\[ R^{(d)} + K^2 - K_{\alpha\beta} K^{\alpha\beta} = 0, \quad \text{(Q. c.)} \quad (14) \]

\[ K_\alpha^\beta C_\beta^\alpha - K_\beta^\alpha C_\alpha^\beta = 0, \quad \text{(L. c.)} \quad (15) \]

\[ K_{\alpha\beta} = NR^{(d)}_{\alpha\beta} - 2NK_{\alpha\epsilon} K_\beta^\epsilon + NK K_{\alpha\beta}, \quad \text{(D. e.)} \quad (16) \]

while

\[ K_{\alpha\beta} = -\frac{1}{2N} \dot{\gamma}_{\alpha\beta}. \quad (17) \]

The above system of equations still admits the sub-group of constant Automorphisms,

\[ \Lambda^\alpha_\mu C^\mu_\beta = C^\alpha_\mu \Lambda^\beta_\nu, \]

which can also be found as ‘rigid’ symmetries [3]. Given the structure constants of the group, the matrix \( \Lambda^\alpha_\mu \) is determined. The remaining non-zero elements of \( \Lambda \) provide the dimension of the constant Automorphisms group.

4. Type 4A1

In the present work, we are interested in the case of a 5D manifold with a 4D homogeneous sub-manifold of Type 4A1. The structure constants for this Type are

\[ C^\alpha_\beta_\mu = 0, \forall \alpha, \beta, \mu = 1, 2, 3, 4. \]

Under the previous assumption the linear constraints (15) are identically satisfied and \( R^{(4)}_{\alpha\beta} = 0, R^{(4)} = 0 \). Also, the Automorphisms equation is identically satisfied which implies that \( \Lambda^\alpha_\beta \in \text{GL}(4, \mathbb{R}) \). The remaining (EFE’s) (14) and (16) are

\[ K^2 - K_{\alpha\beta} K^{\alpha\beta} = 0, \quad (18) \]

\[ K_{\alpha\beta} + 2NK_{\alpha\epsilon} K_\beta^\epsilon - NK K_{\alpha\beta} = 0. \quad (19) \]

By using (17) and the gauge choice \( N = \sqrt{\text{Det}(\gamma)} \) the dynamical equation (19) are integrated

\[ \partial_t (\gamma^\alpha_\beta \dot{\gamma}_{\alpha\beta}) = 0 \Rightarrow \dot{\gamma}_{\alpha\beta} = \theta^\mu_\alpha \gamma_{\beta\mu}, \text{or in matrix form } \dot{\gamma} = \theta^T \gamma, \quad (20) \]

where \( \theta^\mu_\alpha \) is some constant matrix.

On the other hand, the quadratic constraint (18) becomes a relation for the \( \theta \) matrix.

\[ \text{Tr}(\theta^T) - (\text{Tr}(\theta))^2 = 0, \]

where \( \text{Tr}() \) the trace. The solution of (20) is

\[ \gamma = e^{\theta^T} \gamma_0, \]

where \( \gamma_0 \) is some real, constant matrix corresponding to the value of \( \gamma \) at \( t = 0 \).

The matrix \( \theta \) has 16 constant elements, thus calculating the exponential is quite difficult. This is the point where we can use the rigid symmetries in order to simplify it.

The action of constant Automorphisms on \( \gamma \) is
\[ \gamma = \Lambda^T \tilde{\gamma} \Lambda. \]

If we use it in (20), the following equation holds
\[ \dot{\tilde{\gamma}} = \tilde{\theta}^T \tilde{\gamma}, \]
if and only if \( \theta \) transforms as follows
\[ \tilde{\theta}^T = (\Lambda^T)^{-1} \theta^T \Lambda^T. \]

Since \( \Lambda \in \text{GL}(4, \mathbb{R}) \) the degree of simplification that can be achieved for the matrix \( \theta \) depends upon it’s eigenvectors. In the light of this, there are families of different solutions which altogether form the complete space of vacuum solutions for the Type 4A1. When there are only real eigenvectors the \( \theta \) matrix transforms into it’s Jordan canonical form, while when complex eigenvectors exist, acquires it’s canonical rational form. A table is presented with all the possible cases and the form of the \( \theta \) matrix in each one of them.

| Eigenvectors | Form of the matrix |
|--------------|--------------------|
| Four linearly independent, real eigenvectors | \( \theta = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \) |
| Three linearly independent, real eigenvectors, the fourth being trivial | \( \theta = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 1 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \) |
| Two linearly independent, real eigenvectors, the other two being trivial | \( \theta = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 1 & 0 \\ 0 & 0 & p_3 & 1 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \) |
| One real eigenvector, the other three being trivial | \( \theta = \begin{pmatrix} p_1 & 1 & 0 & 0 \\ 0 & p_1 & 1 & 0 \\ 0 & 0 & p_1 & 1 \\ 0 & 0 & 0 & p_1 \end{pmatrix} \) |
| Four linearly independent eigenvectors, two being complex conjugate | \( \theta = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & p_4 \\ 0 & 0 & -p_4 & p_3 \end{pmatrix} \) |
| Three linearly independent eigenvectors, two complex conjugate and one trivial | \( \theta = \begin{pmatrix} p_1 & 1 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & p_3 \\ 0 & 0 & -p_3 & p_2 \end{pmatrix} \) |
| Four linearly independent eigenvectors, two pairs of complex conjugate | \( \theta = \begin{pmatrix} p_1 & p_2 & 0 & 0 \\ -p_2 & p_1 & 0 & 0 \\ 0 & 0 & p_3 & p_4 \\ 0 & 0 & -p_4 & p_3 \end{pmatrix} \) |
5. Solutions

We present the detail calculations concerning only two of the seven different families of solutions.

5.1. Three linearly independent, real eigenvectors, the fourth being trivial

In this case the matrix $\theta$ is transformed into its Jordan canonical form

$$
\theta = \begin{pmatrix}
p_1 & 0 & 0 & 0 \\
0 & p_2 & 0 & 0 \\
0 & 0 & p_3 & 1 \\
0 & 0 & 0 & p_4
\end{pmatrix},
$$

with $p_i, (i = 1, 2, 3)$ the eigenvalues. The 4D line element becomes

$$
d_{(4)}^2 = k_1 e^{p_1 t} d\xi^2 + k_2 e^{p_2 t} d\eta^2 + 2k_3 e^{p_3 t} d\zeta dw + (k_4 + k_5 t) e^{p_4 t} dw^2,
$$

where $k_i, (i = 1, 2, 3, 4)$ are integration constants, while the quadratic constraint reduces to

$$
p_1 p_2 + 2p_1 p_3 + 2p_2 p_3 + p_3^2 = 0. \tag{21}
$$

If all the eigenvalues are equal to zero then the resulting line element corresponds to the flat Minkowski space-time. Therefore we will assume that at least one of the eigenvalues, let us choose $p_3$, is different from zero. If we divide the constraint (21) by $p_3^2$ and form the ratios of the eigenvalues ($$\alpha = \frac{p_1}{p_3}, \beta = \frac{p_2}{p_3}$$) the equation becomes

$$
1 + 2\alpha + 2\beta + \alpha\beta = 0. \tag{22}
$$

For a vacuum solution we solve (22) with respect to one of the constants, let us choose $\alpha$.

$$
\alpha = \frac{-1 + 2\beta}{2 + \beta}.
$$

For the branch ($$\beta = -2$$), the constraint equation would be ($$-3 = 0, \forall \alpha$$) which is invalid. The 5D metric acquires the form

$$
d_{(5)}^2 = k_1 k_2 k_3 e^{\phi_3} dt^2 + k_1 e^{\phi_3} d\xi^2 + k_2 e^{\phi_3} d\eta^2 + 2k_3 e^{\phi_3} d\zeta dw + (k_4 + k_5 t) e^{\phi_3} dw^2,
$$

where ($$\phi_3 = \frac{3+2\beta+\beta^2}{2+\beta}, \phi_4 = \frac{1+2\beta}{2+2\beta}$$). The eigenvalues of this metric are

$$
\lambda_\mu = (k_1 k_2 k_3 e^{\phi_3}, k_1 e^{\phi_3}, k_2 e^{\phi_3}, \lambda_4, \lambda_5),
$$

with

$$
\lambda_4 = \frac{1}{2} e^{\phi_3} \left( k_4 + k_5 t - \sqrt{k_4^2 + 2k_3 k_5 t + k_5^2 (4 + t^2)} \right),
$$

$$
\lambda_5 = \frac{1}{2} e^{\phi_3} \left( k_4 + k_5 t + \sqrt{k_4^2 + 2k_3 k_5 t + k_5^2 (4 + t^2)} \right).
$$
The signs of \((k_3, k_4)\) affect only the eigenvalues \((\lambda_4, \lambda_5)\), but always one is positive and the other is negative \((\forall t \in \mathbb{R})\). Overall, the signature of the metric depends only on the signs of \((k_1, k_2)\). Only one case will be presented.

5.1.1. \(k_i > 0, \forall (i = 1, 2)\). By performing a real coordinate transformation and a redefinition of \(p_3\),

\[
t = \frac{i}{p_3}, x = \frac{\sqrt{k_2 k_3}}{p_3} \tilde{x}, y = \frac{\sqrt{k_1 k_3}}{p_3} \tilde{y}, z = \frac{1}{2p_3^{1/2}} (k_4 p_3 \tilde{w} - 2k_3 \tilde{z}), w = -\sqrt{\frac{k_1 k_2 k_3}{p_3}} \tilde{v},
\]

the 5D line element becomes

\[
ds^2 = e^{\phi_3} dt^2 + e^{\phi_4} dx^2 + e^{\beta} dy^2 + 2e^t dz\, dw + te^t dw^2,
\]

where the constant \(m\) appearing in the redefinition of \(p_3\), was absorbed due to the existence of the homothetic vector field

\[
\xi_h = \left[ 1, \frac{2 + \beta}{2} x, \frac{3 y}{4 + 2 \beta}, z \left(1 + \beta + \beta^2\right) - w \left(2 + \beta\right), w \left(1 + \beta + \beta^2\right) \right].
\]

**Signature:** In order to adjudicate for the signature of this non-diagonal metric we have to find it’s eigenvalues

\[
\lambda_\mu = \left[ e^{\phi_4 t}, e^{\phi_4 t}, e^{\beta t}, \frac{1}{2} e^t \left(t - \sqrt{4 + t^2}\right), \frac{1}{2} e^t \left(t + \sqrt{4 + t^2}\right) \right].
\]

It is easy to see that \((\lambda_4 < 0, \forall t \in \mathbb{R})\) while all the others are positive. The signature is Lorentzian \(s = (1, 4)\) and the coordinate with the ‘time’ character is either \(z\) or \(w\).

**AKVF:** This solution admits additional Killing vector fields for specific values of the ratios.

| \(\alpha\) | \(\beta\) | \(\xi_5\) |
|-----------|-----------|-----------|
| 1         | -1        | (0, w, 0, -x, 0) |
| -1        | 1         | (0, 0, w, -y, 0) |

5.2. Three linearly independent eigenvectors, two complex conjugate and one trivial

The matrix \(\theta\) transforms into it’s canonical rational form

\[
\theta = \begin{pmatrix}
    p_1 & 1 & 0 & 0 \\
    0 & p_1 & 0 & 0 \\
    0 & 0 & p_2 & p_3 \\
    0 & 0 & -p_3 & p_2
\end{pmatrix}.
\]
with \((p_2, p_3)\) the real and imaginary parts of the complex eigenvalues correspondingly. The 4D line element is
\[
dx^2_{(4)} = 2k_1 e^{\rho_t} dx \, dy + (k_2 + k_1 t) e^{\rho_t} dy^2 + e^{\rho_t} [k_3 \cos(p_{3t}) + k_4 \sin(p_{3t})](-dz^2 + dw^2) + 2e^{\rho_t} [k_4 \cos(p_{3t}) - k_3 \sin(p_{3t})] dz \, dw,
\]
and the quadratic constraint reads
\[
p_1^2 + 4p_1 p_2 + p_2^2 + p_3^2 = 0. \tag{23}
\]
The imaginary part of the complex eigenvalue has to be different from zero, \(p_3 \neq 0\). If we divide \(23\) with \(p_3^2\) the ratios are formed, \((\alpha = \frac{p_1}{p_3}, \beta = \frac{p_2}{p_3})\).
\[
1 + \alpha^2 + 4\alpha \beta + \beta^2 = 0. \tag{24}
\]
For vacuum solutions we choose to solve \((24)\) with respect to \(\alpha\).
\[
\alpha = -2\beta + \epsilon \sqrt{-1 + 3\beta^2}, \beta \in \mathbb{R} - \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).
\]
where the symbol \(\epsilon\) stands for \(\pm\) and will be used wherever is needed. This restriction on the ratio \(\beta\) was imposed because \(\alpha\) has to be real.
This relation is already satisfied.

The 5D line element is
\[
dx^2_{(5)} = -k_1^2 (k_3^2 + k_4^2) e^{\rho_t \phi_7} dx^2 + 2k_1 e^{\rho_t \phi_7} dx \, dy + (k_2 + k_1 t) e^{\rho_t \phi_7} dy^2 + e^{\rho_t \phi_7} [k_3 \cos(p_{3t}) + k_4 \sin(p_{3t})](-dz^2 + dw^2) + 2e^{\rho_t \phi_7} [k_4 \cos(p_{3t}) - k_3 \sin(p_{3t})] dz \, dw,
\]
where \((\phi_7 = -2\beta + 2\epsilon \sqrt{-1 + 3\beta^2}, \phi_8 = -2\beta + \epsilon \sqrt{-1 + 3\beta^2})\) and it’s eigenvalues
\[
\lambda_\mu = \begin{bmatrix} -k_1^2 (k_3^2 + k_4^2) e^{\rho_t \phi_7}, \lambda_2, \lambda_3, -\sqrt{k_3^2 + k_4^2} e^{\rho_t \phi_7} \end{bmatrix}.
\]
with
\[
\lambda_2 = \frac{1}{2} e^{\rho_t \phi_7} \left( k_2 + k_1 t - \sqrt{k_3^2 + 2k_1 k_2 t + k_1^2 (4 + t^2)} \right),
\lambda_3 = \frac{1}{2} e^{\rho_t \phi_7} \left( k_2 + k_1 t + \sqrt{k_3^2 + 2k_1 k_2 t + k_1^2 (4 + t^2)} \right).
\]
The eigenvalues \((\lambda_2, \lambda_3)\) depend on the signs of \((k_1, k_2)\), but always one is positive and the other is negative, so they do not affect the signature of the metric. Also, it is easy to observe that neither \((k_3, k_4)\) affect the signature. Therefore, there is only one case. With the coordinate transformation and the redefinition of \(p_3\).
\[ t = \frac{i}{p_3}, x = -\sqrt{\frac{k_1^2 + k_2^2}{k_1} \frac{1}{2p_3^2}} (2k_1 \tilde{x} + k_2 p_3 \tilde{y}), \quad y = \sqrt{\frac{k_1 (k_2^2 + k_3^2)}{p_3} \tilde{y}}. \]

\[ z = k_1 \left[ -k_3 \tilde{w} + \left( -k_3 + \sqrt{k_2^2 + k_3^2} \right) \tilde{z} \right] \sqrt{2 \left( -k_3 + \sqrt{k_2^2 + k_3^2} \right) p_3}, \quad w = k_1 \left[ k_4 \tilde{z} + \left( -k_3 + \sqrt{k_2^2 + k_3^2} \right) \tilde{w} \right] \sqrt{2 \left( -k_3 + \sqrt{k_2^2 + k_3^2} \right) p_3}, \]

\[ p_3 = \sqrt{\frac{k_1^2 + k_2^2}{m^2}}. \]

the line element simplifies to

\[ ds^2 = -e^{\phi t} dt^2 - 2e^{\phi t} dx dy + te^{\phi t} dy^2 + \cos \beta te^{\beta t} (dz^2 - dw^2) + 2 \sin \beta te^{\beta t} dz dw, \]

with homothetic vector field

\[ \xi_h = \left( 1, y + \epsilon x \sqrt{-1 + 3 \beta^2}, \frac{\epsilon y \sqrt{-1 + 3 \beta^2}}{2}, \xi_{h4}, \xi_{h5} \right), \]

\[ \xi_{h4} = -w + 3 \beta z - 2 \epsilon x \sqrt{-1 + 3 \beta^2}, \]

\[ \xi_{h5} = z - 3 \beta w + 2 \epsilon x \sqrt{-1 + 3 \beta^2}. \]

**Signature:** The eigenvalues are

\[ \lambda_{\mu} = \left( -e^{\phi t}, \frac{t - \sqrt{4 + t^2}}{2} e^{\phi t}, \frac{t + \sqrt{4 + t^2}}{2} e^{\phi t}, e^{\beta t}, -e^{\beta t} \right). \]

The signature is \( s = (3, 2) \) with ‘time’ coordinates either \((t, x, w)\) or \((y, z)\).

**AKVF:** For \( \alpha = -\frac{\epsilon}{\sqrt{2}}, \beta = \frac{\epsilon}{\sqrt{2}} \) the AKVF is

\[ \xi_5 = \left( 1, \frac{y + \epsilon x}{2}, \frac{\epsilon y}{2 \sqrt{2}}, -\frac{2w + \epsilon z \sqrt{2}}{4}, \frac{2z - \epsilon w \sqrt{2}}{4} \right). \]

At this point we present the tables 1–7 corresponding to the seven families of solutions. The table 8, in which for every solution the additional Killing vector fields (AKVF) and the homothetic field, can be found. Finally, a table concerning the invariant relations and the number of essential constants for each family is given.
5.3. Tables

**Table 1.** The following abbreviations were used \( \phi_1 = \frac{1+\beta+\gamma+\beta\gamma+\beta^2+\gamma^2}{1+\beta+\gamma} \), \( \phi_2 = -\frac{\beta\gamma}{1+\beta+\gamma} \). The restrictions on the ratios of the eigenvalues are \( \beta \neq \{ -1 - \gamma \} \), \( \forall \gamma \in \mathbb{R} - \{-\frac{1}{2}\} \).

| Name of solution | Line element | Signature/‘Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \( s_1 \)       | \( \phi_2 \)  | (1, 4) / \( t \)              |
| \( s_2 \)       | \( -\phi_2 \) | (1, 4) / \( w \)              |
| \( s_3 \)       | \( \phi_1 \)  | (3, 2) / (\( t, z \), \( w \)) or (\( x, y \)) |
| \( s_4 \)       | \( -\phi_1 \) | (3, 2) / (\( y, z \), \( w \)) or (\( t, x \)) |
| \( s_5 \)       | \( \phi_3 \)  | (5, 0) or (0, 5) / (\( t, x, y, z, w \)) or none |

**Table 2.** The following abbreviations were used \( \phi_3 = \frac{1+2\beta+\gamma}{2-\beta} \), \( \phi_4 = -\frac{1+2\beta}{2+\beta} \). The restrictions on the ratio of the eigenvalues are \( \beta \in \mathbb{R} - \{-2\} \).

| Name of solution | Line element | Signature/‘Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \( s_6 \)       | \( \phi_4 \)  | (1, 4) / \( z \) or \( w \)    |
| \( s_7 \)       | \( -\phi_4 \) | (3, 2) / (\( t, y, z \)) or (\( x, z \)) |
| \( s_8 \)       | \( \phi_3 \)  | (3, 2) / (\( x, y, z \)) or (\( t, z \)) |

**Table 3.** There are only two possible values for the ratio of the eigenvalues for which we acquire a vacuum solution, \( \alpha = 0 \) or \( \alpha = -1 \).

| Name of solution | Line element | Signature/‘Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \( s_9 \)       | \( \phi_2 \)  | (1, 4) / \( y \)              |
| \( s_{10} \)    | \( -\phi_2 \) | (3, 2) / (\( t, z \), \( w \)) or (\( x, y \)) |
| \( s_{11} \)    | \( \phi_1 \)  | (3, 2) / (\( x, z \), \( w \)) or (\( t, y \)) |

**Table 4.** No abbreviations were used.

| Name of solution | Line element | Signature/‘Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \( s_{12} \)    | \( -\phi_2 \) | (3, 2) / (\( t, z \), \( w \)) or (\( x, y \)) |
| \( s_{13} \)    | \( \phi_1 \)  | (3, 2) / (\( x, z \), \( w \)) or (\( t, y \)) |
Table 5. The following abbreviations were used \( \phi_5 = \frac{-1 + \sqrt{1 + 4 \delta}}{2}, \phi_6 = \frac{-1 + \sqrt{4 \beta + 1}}{2} \).

The restrictions on the ratios of the eigenvalues are \( \beta \in \mathbb{R} \setminus \{-2\} \), \( \forall \gamma \in \mathbb{R}, \alpha \in \mathbb{R} \).

| Name of solution | Line element | Signature/’Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \$s_14\$ | \( d_3^2 = e^{\phi_4}dr^2 + e^{\phi_4}ds^2 + e^{\phi_4}dt^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \((1, 4)/w\) |
| \$s_15\$ | \( d_3^2 = -e^{\phi_4}dr^2 + e^{\phi_4}ds^2 - e^{\phi_4}dt^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \((3, 2)/(t, y, w)\) |
| \$s_16\$ | \( d_3^2 = e^{\phi_4}dr^2 - e^{\phi_4}ds^2 - e^{\phi_4}dt^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \(or(x, z)\) |
| \$s_17\$ | \( d_3^2 = e^{\phi_4} (dz^2 - dw^2) + e^{\phi_3} dy^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \((1, 4)/w\) |
| \$s_18\$ | \( d_3^2 = e^{\phi_4} (-dz^2 + dw^2) - e^{\phi_3} dy^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \((3, 2)/(t, y, w)\) |
| \$s_19\$ | \( d_3^2 = e^{\phi_4} (dz^2 - dw^2) - e^{\phi_3} dy^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \(or(x, z)\) |

Table 6. The following abbreviations were used \( \phi_7 = -2 \beta + 2 \epsilon \sqrt{-1 + 3 \beta^2}, \phi_8 = -2 \beta + \epsilon \sqrt{-1 + 3 \beta^2} \).

The restrictions on the ratios of the eigenvalues are \( \beta \in \mathbb{R} \setminus \{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\} \).

| Name of solution | Line element | Signature/’Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \$s_20\$ | \( d_3^2 = -e^{\phi_4}dr^2 - 2e^{\phi_4}dx dy + e^{\phi_4}dy^2 + \cos e^{\phi_3} (dz^2 - dw^2) + 2 \sin e^{\phi_3} dz dw \) | \((3, 2)/(t, x, w)\) |
| \$s_20\$ | \( or(y, z)\) | |

Table 7. The following abbreviations were used \( \phi_9 = -2 \gamma + 2 \epsilon \sqrt{-1 - \beta^2 + 3 \gamma^2}, \phi_{10} = -2 \gamma + \epsilon \sqrt{-1 - \beta^2 + 3 \gamma^2} \).

The restrictions on the ratios of the eigenvalues are \( \beta \in \left( -\sqrt{-1 + 3 \gamma^2}, \sqrt{-1 + 3 \gamma^2} \right), \gamma \in \mathbb{R} \setminus \left\{ -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\} \).

| Name of solution | Line element | Signature/’Time’ coordinate(s) |
|------------------|--------------|-------------------------------|
| \$s_21\$ | \( d_3^2 = -e^{\phi_4}dr^2 + \cos (\beta t) e^{\phi_3} (dz^2 - dw^2) + 2 \sin (\beta t) e^{\phi_3} dx dy + 2 \sin e^{\phi_3} dz dw + \cos e^{\phi_3} (dz^2 - dw^2) \) | \((3, 2)/(t, y, w)\) |
| \$s_21\$ | \( or(x, z)\) | |
Table 8. Additional Killing vector fields

| S  | AKVF | Values for which these appear | Homothetic field |
|----|------|--------------------------------|------------------|
| $s_1$ | $(0, -y, x, 0)/\alpha = 1, \beta = -\frac{1+2\gamma}{2+\gamma}, \gamma = R - \{-2, -1, 1\}$ | | |
|     | $(0, -z, 0, x)/\alpha = -\frac{1+2\gamma}{2+\gamma}, \beta = 1, \gamma = R - \{-2, -1, 1\}$ | | |
|     | $(0, -w, 0, x)/\alpha = -\frac{1+2\gamma}{2+\gamma}, \beta = R - \{-2, -1, 1\}, \gamma = 1$ | | |
| $s_2$ | Same as $s_1$ | Same as $s_1$ |  |
|     | Same as $s_1$ | Same as $s_1$ | |
|     | $(0, w, 0, x)/\alpha = -\frac{1+2\gamma}{2+\gamma}, \beta = R - \{-2, -1, 1\}, \gamma = 1$ | | |
| $s_3$ | Same as $s_1$ | Same as $s_1$ | |
|     | Same as $s_1$ | Same as $s_1$ | |
|     | $(0, z, 0, x)/\alpha = -\frac{1+2\gamma}{2+\gamma}, \beta = 1, \gamma = R - \{-2, -1, 1\}$ | | |
|     | Same as $s_2$ | Same as $s_2$ | |
| $s_4$ | $(0, y, x, 0)/\alpha = 1, \beta = -\frac{1+2\gamma}{2+\gamma}, \gamma = R - \{-2, -1, 1\}$ | | |
|     | Same as $s_3$ | Same as $s_3$ | |
| $s_5$ | Same as $s_1$ | Same as $s_1$ |  |
| $s_6$ | $(0, 0, -x, 0)/\alpha = 1, \beta = -1$ | | |
|     | $(0, 0, -y, 0)/\alpha = -1, \beta = 1$ | | |
| $s_7$ | Same as $s_6$ | | |
|     | $(0, 0, w, y, 0)/\alpha = -1, \beta = 1$ | | |
| $s_8$ | $(0, 0, 0, x)/\alpha = 1, \beta = -1$ | | |
|     | Same as $s_7$ | | |
| $s_9$ | None additional Killing vector field | | |
| $s_{10}$ | None additional Killing vector field | | |
|     | $[1, \frac{\alpha w + y - 2\alpha x}{2}, \frac{-w + 2\alpha x}{2}, \frac{\alpha y + 2\alpha w}{2}, \frac{\alpha w + x}{2}]$ | | |
| $s_{11}$ | None additional Killing vector field | | |
| $s_{12}$ | $(1, -\frac{\beta}{2}, \frac{\beta}{2}, -\frac{\alpha}{2}, 0)$ | | |
| $s_{13}$ | $(1, -\frac{\beta}{2}, -\frac{\beta}{2}, -\frac{\alpha}{2}, 0)$ | | |
| $s_{14}$ | $[1, \frac{1+\gamma + \sqrt{1+2\gamma^2}}{2}, \frac{1+\gamma - \sqrt{1+2\gamma^2}}{2}, \frac{w + 2\gamma}{2}, \frac{\gamma + \sqrt{1+2\gamma^2}}{2}]$ | | |
|     | $\alpha = -\frac{1+\gamma - 2\gamma^2}{2\gamma + \sqrt{1+2\gamma^2}} \gamma \sqrt{1+2\gamma^2}, \gamma = \frac{w + 2\gamma}{2}, \gamma = \frac{1+\gamma - \sqrt{1+2\gamma^2}}{2}$ | | |
|     | $\beta = -\gamma + \epsilon \sqrt{1+2\gamma^2}$ | | |
|     | $-\frac{1}{\sqrt{2}} < \gamma < \frac{1}{\sqrt{2}}$ | | |

(Continued)
\[ \nabla \gamma = z - \beta \]  

None of the constants appears in such a relation is essential since it cannot be altered by any coordinate transformation. Whatever combination of constants appears in such a relation is essential since, it cannot be altered by any coordinate transformation. In this sense the second column of Table 9 is dedicated to prove the essential constants of Table 8. (Continued)

| S   | AKVF/Values for which these appear | Homothetic field |
|-----|-----------------------------------|------------------|
| $s_{17}$ | \( \left( 1, 0, \frac{\alpha}{\sqrt{2}}, -\frac{2w+\alpha\sqrt{2}}{2}, \frac{1}{2} \right) \) | \( \left( 1, 0, y^2 + 2z, \frac{3w-3w-\epsilon y}{6}, \frac{3w+3w-\epsilon y}{6} \right) \) |
| $s_{18}$ | \( \left( 1, 0, \frac{\gamma}{\sqrt{2}}, \frac{2w+\alpha\sqrt{2}}{2}, \frac{1}{2} \right) \) | |
| $s_{19}$ | \( \left( 1, \frac{\gamma}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}, -\frac{2w+\alpha\sqrt{2}}{2}, \frac{1}{2} \right) \) | |
| $s_{20}$ | \( \left( 1, \frac{\gamma}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}, -\frac{2w+\alpha\sqrt{2}}{2}, \frac{1}{2} \right) \) | \( \left( 1, \frac{\gamma + \xi \sqrt{1 - 3\beta^2}}{2}, \frac{\gamma + \xi \sqrt{1 - 3\beta^2}}{2}, \xi \right) \) |
| $s_{21}$ | \( \left( 1, \frac{\gamma}{\sqrt{2}}, \frac{\gamma}{\sqrt{2}}, -\frac{2w+\alpha\sqrt{2}}{2}, \frac{1}{2} \right) \) | \( \left( 1, \frac{\gamma + \xi \sqrt{1 - 3\beta^2}}{2}, \frac{\gamma + \xi \sqrt{1 - 3\beta^2}}{2}, \xi \right) \) |

In the case of cosmology the expected number of essential (non-absorbable by coordinate transformations) constants for a d+1 dimensional space-time is given by the number \( N = (2 \times \# \text{ independent } \gamma_{\alpha\beta}, 2 \times \# \text{ independent constraint equations}) \), which is exactly what was obtained for each family. There are two basic methods in order to adjudicate whether a constant is essential or not. The first one makes use of the concept of invariant relations. Such relations can be obtained by computing at least two curvature and/or higher derivative curvature scalars and eliminate the ‘time’ coordinate. Whatever combination of constants appears in such a relation is essential since, it cannot be altered by any coordinate transformation. In this sense the second column of Table 9 is dedicated to prove the essential

$$ Table 8. $$

| S   | Invariant relations | Number of essential constants |
|-----|---------------------|------------------------------|
| $s_{1}-s_{3}$ | \( Q \left( \mu \right) \left( f_{1} \right) \) | 3 |
| $s_{4}$ | \( Q_{1} \left( \gamma \right) \left( f_{2} \right) \) | 3 |
| $s_{5}$ | \( Q_{2} \left( \beta \right) \) | 3 |
| $s_{6}$ | \( Q_{3} \left( \gamma \right) \) | 3 |
| $s_{7}$ | \( Q_{4} \left( \beta \right) \) | 3 |
| $s_{8}$ | \( Q_{5} \left( \gamma \right) \) | 3 |
| $s_{9}$ | \( Q_{6} \left( \beta \right) \) | 3 |
| $s_{10}$ | \( Q_{7} \left( \gamma \right) \) | 3 |
| $s_{11}$ | \( Q_{8} \left( \beta \right) \) | 3 |
| $s_{12}$ | \( Q_{9} \left( \gamma \right) \) | 3 |
| $s_{13}$ | \( Q_{10} \left( \beta \right) \) | 3 |
| $s_{14}$ | \( Q_{11} \left( \gamma \right) \) | 3 |
| $s_{15}$ | \( Q_{12} \left( \beta \right) \) | 3 |
| $s_{16}$ | \( Q_{13} \left( \gamma \right) \) | 3 |
| $s_{17}$ | \( Q_{14} \left( \beta \right) \) | 3 |
| $s_{18}$ | \( Q_{15} \left( \gamma \right) \) | 3 |
| $s_{19}$ | \( Q_{16} \left( \beta \right) \) | 3 |
| $s_{20}$ | \( Q_{17} \left( \gamma \right) \) | 3 |
| $s_{21}$ | \( Q_{18} \left( \beta \right) \) | 3 |

$$ Table 9. $$

We used the abbreviations \( K = R_{\mu
u\rho\sigma}R_{\mu
u\rho\sigma}, \ Q = g^{\mu\nu}\nabla_\mu K \nabla_\nu K, \ W = \nabla^K R_{\mu
u\rho\sigma}R_{\mu
u\rho\sigma}. \)
character of the constants appearing in the corresponding line elements. One might ask what happens in the case of pp-waves where all curvature and/or higher derivative curvature scalars are zero. The answer is contained in a method presented in [48]; one seeks solutions to the Killing-like equation

\[ \mathcal{L}_\xi g_{\alpha\beta} = \partial_\lambda g_{\alpha\beta} \]  

(25)

where \( \lambda \) the constant to be tested. If such \( \xi \) exists, \( \lambda \) is absorbable and the integral curves of \( \xi \) define the absorbing transformation. If there is no such \( \xi \) then \( \lambda \) is essential. In the cases of the pp-waves presented in this work no such need arose since during the finding of the solutions all the constants that appeared were absorbed by coordinate transformations.

5.4. Remarks

Some remarks concerning the solutions found are noteworthy.

1. To the best of our knowledge, only the first family of solutions is known. This corresponds to the 5D Kasner Type [28, 29].

2. For the solutions \( s_1 - s_5 \) there is the possibility to choose either of the ratios to be equal to zero, suppose \( \gamma = 0 \). These represent the 4D Kasner solutions [28, 29] embedded in a 5D manifold.

3. In the case of \( (s_9, s_{10}, s_{11}) \), for one of the two possible values of the ratio \( \alpha = 0 \), the Killing vector field \( \xi = \partial_y \) satisfies the following properties,

\[ \xi_\mu \xi^\mu = 0, \]
\[ \nabla_\mu \xi_\nu = 0. \]

This is the definition of pp-waves [47] which are solutions to (EFE’s), representing strong gravitational waves propagating along null trajectories, generated by some null vector field. In our case this field is \( \xi = \partial_y \).

4. The solutions \( (s_{12}, s_{13}) \) are also pp-waves, with \( \xi = \partial_x \) being the null vector field.

6. Physical interpretation

In this section we would like to comment on a possible physical interpretation of our results. The acquisition of the entire solution space can be used in the framework of braneworld scenarios. The physical character of the solutions at hand can be revealed by examining the effective gravitational equations on the brane, induced by the Gauss Codazzi equations [52], and the Israel junction conditions along with the thin shell formalism [50, 51].

In the brane world scenario the 4D world lies in a hypersurface \( (\Sigma, h_{ab}) \) (the brane or a domain wall or a thin shell) embedded in a 5D spacetime \( (V, g_{\mu\nu}) \) (the bulk). If we denote the normal to the brane as \( n^\alpha \) and its tangent vectors as \( e^a_\alpha \), then we can define the induced metric \( h_{ab} \) (first fundamental form) and the extrinsic curvature \( K_{ab} \) (second fundamental form) as

\[ h_{ab} = g_{\alpha\beta} e^a_\alpha e^b_\beta, \]
\[ K_{ab} = n^\alpha e^a_\alpha e^b_\beta = \frac{1}{2} (\mathcal{L}_n g_{\alpha\beta}) e^a_\alpha e^b_\beta, \]

where the semicolon denotes the covariant derivative in \( V \). The stress–energy tensor in the bulk \( T^{\alpha\beta} \) decomposes into two parts \( T_V^{\alpha\beta} \) and \( T_\Sigma^{\alpha\beta} \) describing, respectively the stress–energy tensors outside the brane and in it.
\[ T^{\alpha\beta} = T_V^{\alpha\beta} + T_\Sigma^{\alpha\beta} \]

where \( \ell \) a parameter along a geodesic which intersects orthogonally \( \Sigma \), with \( \ell = 0 \) on the brane. The stress–energy tensor \( S^{\alpha\beta} \) is related to the actual stress–energy tensor \( S_{ab} \) in the brane by

\[ S_{ab} = S_{\alpha\beta} e_\alpha^a e_\beta^b. \]

The presence of the delta function \( \delta(\ell) \) in \( T_\Sigma^{\alpha\beta} \), gives rise to junction conditions

\[ S_{ab} = -\varepsilon \frac{\kappa}{\epsilon} ([K_{ab}] - h_{ab}[K]) \Rightarrow [K_{ab}] = -\varepsilon \kappa \left( S_{ab} - \frac{1}{3} h_{ab}S \right), \tag{26} \]

where \([A] = A (\ell \rightarrow 0^+) - A (\ell \rightarrow 0^-)\), is the jump of a tensor \( A \) between the two sides of the brane, \( \kappa \) the appropriate Einstein’s constant and \( \epsilon = \pm 1 \), characterizing the nature of \( n^\alpha n_\alpha = \epsilon \). The most common choice for the jump of the extrinsic curvature is a \( \mathbb{Z}_2 \) symmetry, meaning that

\[ K_{ab}^+ = -K_{ab}^- = \frac{1}{2} [K_{ab}] . \]

In our case, since we solve the field equations in vacuum, \( T_V^{\alpha\beta} = 0 \); furthermore we have at our disposal the choice of \( S_{ab} \) along with the choice of one of the 21 distinct solutions \( s_1, \ldots, s_{21} \). Here we choose to analyze the solution \( s_1 \) with \( S_{ab} \) describing a perfect fluid; the remaining possibilities will be presented elsewhere.

To start our analysis we assume that the brane is moving with velocity \( w = F(t) \), a condition that also defines the brane as a hyper-surface \( \Sigma \) of \( V \); thus the line element on the brane is

\[ ds^2_{(4)} = -\left( e^{\phi_1} - e^{\phi_2} F'(t)^2 \right) dt^2 + e^{\phi_1} dt^2 + e^{\phi_2} dy^2 + e^{\phi_3} dz^2. \]

The tangent vectors \( e_{(i)}^\alpha \), along with the space-like normal \( n^\alpha \) are

\[ e_{(1)}^\alpha = \partial_t + F'(t) \partial_x, \quad e_{(2)}^\alpha = \partial_x, \quad e_{(3)}^\alpha = \partial_y, \quad e_{(4)}^\alpha = \partial_z, \]

\[ n^\alpha = -\frac{1}{\sqrt{e^{-\phi_1} - e^{-\phi_2} F'(t)^2}} \left( e^{-\phi_1} F'(t) \partial_t + e^{-\phi_2} F'(t) \partial_x \right), \]

with \( e^{-\phi_1} - e^{-\phi_2} F'(t)^2 \neq 0 \) (since otherwise the hyper-surface would be null), and the stress–energy tensor \( S_{ab} \) reads

\[ S_{ab} = (\rho + p) u_a u_b + ph_{ab}. \]

With the above prescriptions the junction conditions (26) fix the constants \( \beta, \gamma \) to the values

\[ \beta = 1, \quad \gamma = -1, \]

while the energy density \( \rho \) and the pressure \( p \) are given by the relations

\[ \rho = -\frac{3 \sqrt{\varepsilon^2 - e^{-2} F'(t)^2} \left( F'(t)^3 - 2 e^{\phi_1} \left( 2 F'(t) - F''(t) \right) \right)}{2 \kappa \left( e^{\phi_1} - F'(t)^2 \right)^2}, \tag{27a} \]

\[ p = -\frac{\rho}{2 \kappa \sqrt{\varepsilon^2 - e^{-2} F'(t)^2}}. \tag{27b} \]

The interesting feature of the above construction is that in order to have a perfect fluid solution on the brane the metric of it must be isotropic, since its line element is...
\[ ds^2_{(4)} = -\left( e^{2t} - e^{-t} F'(t)^2 \right) dt^2 + \epsilon' \left( dx^2 + dy^2 + dz^2 \right), \]
a feature that was also noted in \[49\]; notice that the velocity of the brane \( F(t) \) is left arbitrary. If we assume a barotropic equation of state \( p = \rho \) then we can determine the function \( F(t) \). From equation (27a) we have
\[ F'(t) = \frac{e^{-3t} F'(t)}{2Q} \left( e^{3t}(4Q - 1) - (Q - 1) F'(t)^2 \right), \]
where the branch with \((Q = 0)\) leads to a standing brane i.e. one with velocity \( F(t) = \text{const.} \), and as we observe it cannot withstand a perfect fluid since it corresponds to zero pressure and energy density. The equation (28) can be considerably simplified with the aid of the transformation. Finally the metric on the brane reads
\[ F''(\xi) = -\mu \xi^2 F'(\xi)^3, \quad \mu = \frac{Q - 1}{4Q} \ln \xi, \]
leading to
\[ \mu = \frac{2Q + 1}{-4Q + 1}. \]
The above ODE can be easily integrated and its solution is
\[ F(t) = k_2 + 2\sqrt{\kappa_1} e^{(\alpha + 3)t/2} 2F_1 \left[ \frac{1}{2}, \frac{3(\alpha + 1)}{2\alpha}, -\kappa_1 e^{\alpha t} \right], \]
where \( \alpha = (Q - 1)/Q, k_1, k_2 \) are constants of integration and \( 2F_1 \) is the hypergeometric function. Finally the metric on the brane reads
\[ ds^2_{(4)} = -\frac{e^{2t}}{1 + k_1 e^{\alpha t}} dt^2 + \epsilon' \left( dx^2 + dy^2 + dz^2 \right), \]
while the pressure and energy density are given as
\[ p = \pm \frac{3}{4\kappa} e^{(\alpha - 3)t/2} \sqrt{\kappa_1}, \]
\[ \rho = \pm \frac{3}{4\kappa} e^{(\alpha - 2)t/2} \sqrt{\kappa_1(1 - \alpha)}, \]
(where \( \pm \) corresponds to \( Q < \frac{1}{4} \) and \( Q > \frac{1}{4} \)). Also, we notice that in order to have real pressure and energy density the constant \( k_1 \) has to be positive (\( k_1 > 0 \)).

There are two branches that we have to consider based on the form of the function \( F(t) \). The case where \( \alpha = -3 \Rightarrow Q = \frac{1}{4} \) leads to a line element of the same form as (29) but the density and pressure are imaginary and therefore this case is rejected. The other one is \( \alpha = 0 \Rightarrow Q = 1 \) which leads to the following expressions for the velocity of the brane, the pressure and energy density and the line element.
\[ F(t) = \frac{2}{3} e^{3t/2} \sqrt{\frac{k_1}{1 + k_1}} + k_2, \]
\[ \rho = \frac{3}{4\kappa} e^{-t} \sqrt{k_1}, \]
\[ p = \frac{3}{4\kappa} e^{-t} \sqrt{k_1}, \]
\[ ds^2_{(4)} = -\frac{e^{2t}}{1 + k_1 dt^2 + \epsilon' \left( dx^2 + dy^2 + dz^2 \right)}. \]

At last we would like to note that the original solution had two essential constants, \( \beta, \gamma \) which were determined in order for the brane to support an ideal fluid. The line element (29)
has again two essential constants. Lastly, we remark that other 4D matter content might also be considered (e.g. non-perfect fluids, etc).

7. Discussion

As we have seen, the group of transformations that preserve the sub-manifold’s manifest homogeneity provides us with a way to put the shift vector equal to zero without doing it a priori. Also, we use the gauge freedom of choosing the lapse function in order to simplify further the (EFE’s). The task of finding the explicit form of the general solution of the simplified (EFE’s) remained difficult even in the simple case of Type 4A1. The use of the remaining symmetry, consisting of the group of constant Automorphisms, allowed us to overcome this difficulty. This was due to the fact that the solution space was broken down into different families of solutions with the aid of the constant Automorphisms. By this procedure, we were able to find out all the known solutions as well as all the possible new, to the best of our knowledge. Of course, the solutions found span the entire space-time metrics with non-zero shift and arbitrary lapse; one only has to invert the transformations (11) which enable us to reduce the initial metric to the form which has been used for the subsequent finding of the solutions.

For the future, we aim to use this method in the rest of the 5D manifolds which admit a homogeneous sub-manifold of dimension four, where the group of constant Automorphisms has a lower dimension. Also a matter content may be included.

A case of particular interest is that when a non-vanishing cosmological constant is taken into account. A first conclusion that can be immediately drawn is that all pp-wave solutions are extinguished: The trace of (EFE’s) becomes

$$2 - \frac{D}{2}R = -\Lambda D,$$

where \(D\) is the space-time dimension. Now, one of the key properties for a space-time to be pp-wave is to have all curvature scalars zero, a property that, in view of the above equation, cannot be fulfilled unless \(\Lambda = 0\).

Apart of this easy observation, further description of the solution space is also possible: Let us consider the modified quadratic constraint (18) and the dynamical equation (19) (in the same gauge),

$$\left(\frac{\dot{\gamma}}{\gamma}\right)^2 - \dot{\gamma}_{\sigma \rho} \gamma^{\sigma \alpha} \gamma^{\rho \beta} \dot{\gamma}_{\alpha \beta} = 8 \gamma \Lambda, \quad (31)$$

$$\partial_t \left( \gamma^{\sigma \rho} \dot{\gamma}_{\rho \kappa} \right) = \frac{4}{3} \Lambda \gamma \delta^\sigma_\kappa, \quad (32)$$

where \(\gamma\) is the determinant of the 4D metric. If we take the trace of (32) and use it back in the same equation we arrive at

$$\partial_t \left( \gamma^{\sigma \rho} \dot{\gamma}_{\rho \kappa} - \frac{1}{4} \frac{\dot{\gamma}}{\gamma} \delta^\sigma_\kappa \right) = 0,$$

which can be integrated to give

$$\dot{\gamma}_{\sigma \rho} = \frac{1}{4} \frac{\dot{\gamma}}{\gamma} \gamma_{\sigma \rho} + \gamma_{\sigma \kappa} \delta^\kappa_\rho, \quad (33)$$
where $\tilde{\theta}_{\kappa}^{\rho}$ is now some trace-less constant matrix (as it can be easily seen by the trace of (33)). The equation (33) can be used in the quadratic constraint, which leads to a first order ordinary differential equation for the determinant of the metric,

$$\frac{3}{4} \left( \frac{\dot{\gamma}}{\gamma} \right)^2 - \text{Tr}(\dot{\theta}^2) = 8\gamma\Lambda,$$

(34)

with solution

$$\gamma = -\frac{\text{Tr}(\dot{\theta}^2)}{8\Lambda} \text{sech} \left[ \frac{1}{2} \sqrt{\text{Tr}(\dot{\theta}^2)} \left( m - \frac{2}{\sqrt{3}}t \right) \right]^2,$$

(35)

and $m$ an integration constant. The use of (35) in (33) results, in matrix form

$$\dot{\gamma} = f(t)\gamma + \tilde{\theta}^T \gamma,$$

(36)

which is the modification of (20). The solution is easily given by

$$\gamma = H(t)e^{\tilde{\theta}^T t c},$$

(37)

The function $f(t)$ and the matrix $H(t)$ where used only as abbreviations and their explicit form can be found below

$$f(t) = \frac{1}{\sqrt{3}} \sqrt{\text{Tr}(\dot{\theta}^2)} \tanh \left[ \frac{1}{2} \sqrt{\text{Tr}(\dot{\theta}^2)} \left( m - \frac{2}{\sqrt{3}}t \right) \right],$$

$$H(t) \equiv e^\int f(t)dt = \frac{1}{\sqrt{\cosh \left[ \frac{1}{2} \left( m - \frac{2}{\sqrt{3}}t \right) \text{Tr}(\dot{\theta}^2) \right]}} I,$$

where $I$ stands for the 4D identity matrix. The determinant of (37) should be equated to (35), providing a relation between the elements of the constant matrix $c$ and the cosmological constant. Furthermore, the constant Automorphisms can still be used in order to simplify the matrix $\tilde{\theta}$.

Now that we have the general form of the solution we can see that one modification brought by the presence of $\Lambda$ is the multiplication of the metric components by the factor $H(t)$. Thus, for example, in the large ‘time’ limit this factor scales as $\left( \sim e^{-\left(\text{Tr}(\dot{\theta}^2)/2\sqrt{3}t\right)} \right)$ which can damp some of the $\left( \sim e^{\theta t} \right)$ factors. The anisotropy of the model, does not permit this damping to be complete, as in the case of an FLRW. For the future we aim to use this results in order to study models of brane cosmologies with non-compact extra dimension.

Finally, we gave a physical interpretation for the only known solution $s_1$ (the embedded Kasner), in the context of brane cosmology. A brane with a perfect fluid were embedded inside the solution. It would be interesting to see if a non-perfect fluid can also be embedded in that particular solution. At the end we would like also to find out if any of the other solutions are able to withstand a brane with a perfect or non-perfect fluid.

**Acknowledgments**

The co-author T Pailas thanks the General Secretariat for Research and Technology (GSRT) and the Hellenic Foundation for Research and Innovation (HFRI) of the Greek Ministry of Education for supporting his PhD fellowship.
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