ON THE GEVREY REGULARITY OF SOLUTIONS TO THE 3D IDEAL MHD EQUATIONS

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Abstract. In this paper, we prove the propagation of the Gevrey regularity of solutions to the three-dimensional incompressible ideal magnetohydrodynamics (MHD) equations. We also obtain an uniform estimate of Gevrey radius for the solution of MHD equation.

1. Introduction. The three-dimensional (3D) incompressible ideal MHD equations on the torus $\mathbb{T}^3$ take the form,

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - h \cdot \nabla h + \nabla (p + \frac{1}{2} |h|^2) &= 0, \quad x \in \mathbb{T}^3, \ t > 0, \\
\frac{\partial h}{\partial t} + u \cdot \nabla h - h \cdot \nabla u &= 0, \quad x \in \mathbb{T}^3, \ t > 0, \\
\nabla \cdot u &= 0, \quad \nabla \cdot h = 0, \quad x \in \mathbb{T}^3, \ t \geq 0, \\
u(x, 0) &= u_0(x), \quad h(x, 0) = h_0(x), \quad x \in \mathbb{T}^3,
\end{align*}
\]

where $u(x, t) = (u_1, u_2, u_3)(x, t), h(x, t) = (h_1, h_2, h_3)(x, t)$, represent fluid velocity field, magnetic field at point $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ at time $t$, and $p = p(x, t)$ represents the scalar pressure. Note that the incompressibility $\nabla \cdot h = 0$ needs only be required at $t = 0$, and it then holds for all $t > 0$. As for the classical Euler equation, we transform the equations (1) to the following form after taking curl...
operator on both sides,
\[
\begin{align*}
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega - h \cdot \nabla J - \omega \cdot \nabla u + J \cdot \nabla h &= 0, \\
\frac{\partial J}{\partial t} + u \cdot \nabla J - h \cdot \nabla \omega + \nabla u \cdot A \cdot \nabla h - \nabla h \cdot A \cdot \nabla u &= 0,
\end{align*}
\]
(2)
where \( \mathcal{K} \) is the three dimensional Biot-Savart kernel (see the definition in [13]) and \( A \) is a constant matrix, \( \omega = \text{curl } u \) and \( J = \text{curl } h \) denote the vorticity and current density, see [2, 20]. In the equation (2), we used the notation \( \nabla u \cdot A \cdot \nabla v \triangleq \text{curl}(u \cdot \nabla v) - u \cdot \nabla \text{curl } v \) for vector functions \( u = (u_1, u_2, u_3) \), \( v = (v_1, v_2, v_3) \) and \( u \neq v \), see for instance [2]. Here we remarked that the term \( \nabla u \cdot A \cdot \nabla v \) is a vector function whose component takes the form of the following multiplication of one order derivative of \( u \) and one order derivative of \( v \),
\[
\nabla u \cdot A \cdot \nabla v \triangleq \begin{pmatrix}
\partial_2 u \cdot \nabla v_3 - \partial_3 u \cdot \nabla v_2 \\
\partial_3 u \cdot \nabla v_1 - \partial_1 u \cdot \nabla v_3 \\
\partial_1 u \cdot \nabla v_2 - \partial_2 u \cdot \nabla v_1
\end{pmatrix}.
\]

In magneto-fluid mechanics magnetohydrodynamics equations (MHD) describes the dynamics of electrically conducting fluids arising from plasmas, liquid metals, and salt water or electrolytes, see [10, 16]. There is no global well-posedness for the incompressible MHD equations (1) in general case except for small pertubation near the trivial steady solution (see, for instance [6] and [3]). The local existence and uniqueness of \( H^r \)-solution, for \( r > 5/2 \), of the Cauchy problem (2) was proved in [15] following the method of Temam [17] and Kato and Lai [8]. Caflisch, Klapper and Steele [2] extended the well-known Beal, Kato and Majda criterion [1] for incompressible Euler equations to the cases of incompressible ideal MHD equations. Precisely, they proved that if the maximal time of existence \( T \) is finite, then
\[
\int_0^T \left( \| \omega (\cdot, t) \|_{L^\infty} + \| J (\cdot, t) \|_{L^\infty} \right) dt = \infty.
\]
(3)
For more work about the blow-up criterion, please refer to [4, 22] and references therein. In this paper we study the Gevrey class regularity of the \( H^r \)-solutions to equations (2) on the torus \( T^3 \) using the Fourier space method introduced by Foias and Temam [5]. In that paper, the authors studied the Gevrey class regularity of Navier-Stokes equations and proved that the solutions are analytic in time with values in Gevrey class for inital data only in Sobolev space \( H^1 \) with divergence free. Levermore and Oliver [11] applied this method to study the propagation of analyticity of the solutions to the so-called lake and great lake equations. Later, Kukavica and Vicol [9] improved the results of Levermore and Oliver by showing that the radius of space analyticity decays algebraically on \( \exp \int_0^t \| \nabla u^E (\cdot, s) \|_{L^\infty} ds \), where \( u^E \) is the solution of incompressible Euler equations. The purpose of this paper is to generalize the results of Kukavica and Vicol to 3D incompressible ideal MHD equations. Compared with Euler equation, the additional difficulty for MHD equations arise from the estimates of the coupling nonlinear term which require some new cancellation estimates.

When considering viscous and resistive incompressible MHD equations, Kim [14] had investigated the Gevrey class regularity of the strong solutions and proved a
parallel result as Foias and Temam [5] on Navier-Stokes equations. For regularized MHD equations, Yu and Li [21] studied Gevrey class regularity of the strong solutions to the MHD-Leray-alpha equations and Zhao and Li [23] studied analyticity of the global attractor of the so-called MHD-Voight equations following the method of [7]. In the whole space $\mathbb{R}^3$, Wang and Li [18] studied the global existence of solutions to the viscous and resistive MHD equations in the so-called Lei-Lin-Gevrey space and Weng [19] studied the analyticity of solutions to the Hall-MHD equations. However, these aforementioned works are mainly concerned the viscous and resistive MHD equations (or regularized MHD equations). We see no results of Gevrey class regularity for the ideal MHD equations yet by far, and this is the motivation of our work.

The paper is organized as follows. In Section 2, we will give some notations and state our main results. In Section 3, we first recall some known results and then give some lemmas which are needed to prove the main Theorem. In Section 4, we finish the proof of Theorem 2.1.

Throughout the paper, $C$ denotes a generic constant which may vary from line to line.

2. Notations and main theorem. We give now some notations and function spaces which will be used throughout the following arguments. Let $r \geq 0$ be a constant. Denote by $H^r(\mathbb{T}^3)$ the mean zero vector function of Sobolev space,

$$H^r(\mathbb{T}^3) = \left\{v(x) = (v^1, v^2, v^3) = \sum_{k \in \mathbb{Z}^3} \hat{v}_k e^{ik \cdot x}; \hat{v}_0 = 0, \hat{v}_{-k} = \hat{v}_k, \right\}$$

$$\|v\|_{H^r} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^r |\hat{v}_k|^2 < \infty,$$

where $\hat{v}_k = (\hat{v}_k^1, \hat{v}_k^2, \hat{v}_k^3)$ is the $k$-th vector Fourier coefficient defined by

$$\hat{v}_k = \int_{\mathbb{T}^3} v(x) e^{-ik \cdot x} dx.$$ 

The operator $\Lambda$ is defined as follows

$$\Lambda v(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |k| \hat{v}_k e^{ik \cdot x}, \quad \forall v \in H^1(\mathbb{T}^3),$$

here we used the notation $|k| = |k_1| + |k_2| + |k_3|$. Let $m = 1, 2, 3$, define $\Lambda_m$ and $H_m$ as follows,

$$\Lambda_m v(x) := \sum_{k \in \mathbb{Z}^3 \setminus \{0\}, k_m \neq 0} |k_m| \hat{v}_k e^{ik \cdot x}, \quad H_m v(x) := \sum_{k \in \mathbb{Z}^3 \setminus \{0\}, k_m \neq 0} \text{sgn}(k_m) \hat{v}_k e^{ik \cdot x}.$$ 

Let $s \geq 1$ be a real number. For any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{N}^3$, we denote $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3|$. Usually, we say that a smooth function $f(x) \in C^\infty(\mathbb{R}^3)$ is uniformly of Gevrey class, if there exists $C, \tau > 0$ such that

$$|\partial^\alpha f(x)| \leq C |\alpha|^{s \tau} |x|^{-\tau |\alpha|}, \quad \forall, x \in \mathbb{R}^3, \forall \alpha \in \mathbb{N}^3.$$ 

When $s = 1$, $f$ is real analytic. The constant $\tau$ in (4) is called the Gevrey’s radius. Inspired by Foias and Temam [5], the Gevrey space on the torus can be characterized by the decay of the Fourier coefficients, see for instance [9, 11].
In this paper we inherit the notations of the function space of Gevrey class $s$ used in [9]. For fixed $r, \tau \geq 0$ and $m = 1, 2, 3$, let

$$\mathcal{D}(\Lambda_m^r e^{r\Lambda_m^s}) = \left\{ v \in H^r(\mathbb{T}^3); \text{ div } v = 0, \|\Lambda_m^r e^{r\Lambda_m^s} v\|_{L^2} < \infty \right\},$$

where

$$\|\Lambda_m^r e^{r\Lambda_m^s} v\|_{L^2} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k_m|^{2r} e^{2r|k_m|^{1/s}} |\hat{v}_k|^2.$$ 

For $\tau, r \geq 0$, set

$$X_{r,\tau,s} = \bigcap_{m=1}^3 \mathcal{D}(\Lambda_m^r e^{r\Lambda_m^s}), \quad \|v\|^2_{X_{r,\tau,s}} = \sum_{m=1}^3 \|\Lambda_m^r e^{r\Lambda_m^s} v\|^2_{L^2},$$

and $Y_{r,\tau,s} = X_{r,\tau,s}^{1/2}$. The Gevrey spaces $X_{r,\tau,s}$ are showed to be equivalent with the definition (4), see [9, 11, 14] for detailed description.

We can now state our main results.

**Theorem 2.1.** Let $r > \frac{5}{2} + \frac{3}{2r}, s \geq 1$ be fixed constants. If $(u_0, h_0)$ are divergence-free and $(\omega_0, J_0) = (\text{curl } u_0, \text{curl } h_0) \in X_{r,\tau_0,s}$ with $\tau_0 > 0$. Then the equation (2) admits an unique solution $(\omega, J) \in L^\infty([0, T]; H^r(\mathbb{T}^3))$ such that,

$$(\omega, J) \in L^\infty([0, T], X_{r,\tau,\cdot,s}),$$

where $T > 0$ is the life-span of $H^r$-solution $(u, h)$ to equations (1). Moreover the Gevrey’s radius $\tau(t)$ is a decreasing function of $t$ with $\tau(0) = \tau_0$ and satisfies, for $0 \leq t < T$,

$$\tau(t) \geq \exp \left[ -C \int_0^t \left( \|\nabla u(\cdot, \sigma)\|_{L^\infty} + \|\nabla h(\cdot, \sigma)\|_{L^\infty} \right) d\sigma \left( \tau_0^{-1} + C_0 t + \frac{C_1}{2} t^2 \right)^{-1},$$

where $C > 0$ is a constant depending only on $r, s$, while $C_0$ and $C_1$ have additional dependence on the initial data.

**Remark 1.** In the case $s = 1$ and $h = \text{constant}$, Theorem 2.1 recovers the result of Kukavica and Vicol [9] for incompressible Euler equation. And we remarked that in the case $s = 1$, we need only $r > \frac{7}{2}$ in Theorem 2.1.

**Remark 2.** The smooth solution criterion (3) in [2] states that the solution remain smooth to $T$ as long as

$$\int_0^T \left( \|\omega(\cdot, t)\|_{L^\infty} + \|J(\cdot, t)\|_{L^\infty} \right) dt < \infty.$$

3. The estimate of the nonlinear terms. We first recall the following results about the local existence and uniqueness of $H^r$-solution of the ideal MHD equations (1).

**Theorem 3.1 (Caflisch-Klapper-Steele, [2]).** Let $r \geq 3$. If $u_0, h_0 \in H^r(\mathbb{T}^3)$ are divergence-free. Then equations (1) admit an unique solution $(u, h)$ such that

$$(u, h) \in C([0, T], H^r(\mathbb{T}^3)) \cap C^1([0, T], H^{r-1}(\mathbb{T}^3))$$

where $0 < T < \infty$ is the maximal existence time of $H^r$-solution, namely $T$ statifies

$$\int_0^T \|\omega(\cdot, t)\|_{L^\infty(\mathbb{T}^3)} + \|J(\cdot, t)\|_{L^\infty(\mathbb{T}^3)} dt = \infty.$$
The proof of Theorem 3.1 can be found in [2]. With this Theorem and the Biot-Savart law, one can easily deduce the existence of solution \((\omega, J) \in C([0, T]; H^r(T^3))\) to equations (2) if the initial data \((\omega_0, J_0) = (\text{curl}\, u_0, \text{curl}\, h_0) \in H^r(T^3)\).

In the following, we state some Lemmas concerning the estimates of the nonlinear terms in equation. First, we recall two useful Lemmas from [9].

**Lemma 3.2** (Lemma 3.1 of [9]). Let \(w \in X_{r, \tau, s}, \) for \(\tau > 0\) and \(r \geq 1.\) Then for \(m = 1, 2, 3\) we have

\[
\|\Lambda_m^r w\|_{L^2} \leq \|\Lambda_m^{r-1} w\|_{L^2} \leq C\|w\|_{H^r}
\]

and

\[
\|\nabla H_m \Lambda_m^{r-1} e^{\tau A_m^{1/2}} w\|_{L^2} \leq \|\Lambda_m^{r-1} e^{\tau A_m^{1/2}} w\|_{L^2} \leq C\|w\|_{X_{r, \tau, s}},
\]

where \(C\) is a positive constant.

For the Biot-Savart law (see [13]), we have.

**Lemma 3.3.** Let \(w \in X_{r, \tau, s},\) for \(\tau > 0\) and \(r \geq 1.\) Then for \(m = 1, 2, 3\) we have

\[
\|\Lambda_m^{r+1} (K * w)\|_{L^2} \leq \|\Lambda_m^r (K * w)\|_{L^2} \leq C\|w\|_{H^r}
\]

and

\[
\|\Lambda_m^{r+1} e^{\tau A_m^{1/2}} (K * w)\|_{L^2} \leq \|\Lambda_m^r e^{\tau A_m^{1/2}} (K * w)\|_{L^2} \leq C\|w\|_{X_{r, \tau, s}},
\]

where \(C\) is a positive constant independent of \(v, w.\)

The proof is standard by Calderón-Zygmund theory and the details can be found in [9], we thus omit the proof. In order to estimate the nonlinear terms in equations (2), we first recall the Lemma 2.5 in [9], in which the authors proved the case of \(s = 1.\) Denote the \(L^2\)-norm and and the inner product by \(\|\cdot\|_{L^2(T^3)}\) and \((\cdot, \cdot)_{L^2(T^3)}\) respectively.

**Lemma 3.4** (Lemma 2.5 of [9]). Let \(s \geq 1, m = 1, 2, 3\) and \(\omega \in Y_{r, \tau, s},\) where \(r > \frac{3}{2} + \frac{1}{2s}\). If \(u = K * \omega,\) where \(K\) is the Biot-Savart kernel, then

\[
\left| \left( u \cdot \nabla \omega, \Lambda_m^{2r} e^{2\tau A_m^{1/2}} \omega \right)_{L^2(T^3)} \right| + \left| \left( \omega \cdot \nabla u, \Lambda_m^{2r} e^{2\tau A_m^{1/2}} \omega \right)_{L^2(T^3)} \right|
\]

\[
\leq C(\tau \|\nabla u\|_{L^\infty} + \tau^2 \|\omega\|_{H^r} + \tau^2 \|\omega\|_{X_{r, \tau, s}}) \|\omega\|_{Y_{r, \tau, s}}^2 + C(\|\nabla u\|_{L^\infty} \|\omega\|_{X_{r, \tau, s}} + (1 + \tau)) \|\omega\|_{H^r}^2 \|\omega\|_{X_{r, \tau, s}},
\]

where the positive constant \(C\) depends on \(r\) and \(s.\)

We remark that, for \(s > 1,\) there are some minor changes in the proof which cause the condition \(r > \frac{3}{2} + \frac{1}{2s},\) and we show the details in the proof of the following Lemma. First we introduce the following notation

\[
\Psi = (\omega, J),
\]

and the corresponding norm

\[
\|\Psi\|_{H^r}^2 = \|\omega\|_{H^r}^2 + \|J\|_{H^r}^2, \quad \|\Psi\|_{X_{r, \tau, s}}^2 = \|\omega\|_{X_{r, \tau, s}}^2 + \|J\|_{X_{r, \tau, s}}^2.
\]

With very similar method as Lemma 2.5 of [9], we can obtain the following Lemma.
Lemma 3.5. Let $s \geq 1, m = 1, 2, 3$ and $\omega, J \in Y_{r, \tau, s}$, where $r > \frac{5}{2} + \frac{3}{2s}$. If $u = K * \omega, h = K * J$, where $K$ is the Biot-Savart kernel, then

$$
\left| \left( u \cdot \nabla J, A^{r} e^{2 \tau \Lambda^{1/s}} \right) \right|_{L^{2}(\mathbb{T}^{3})} + \left| \left( \nabla h \cdot A \cdot \nabla u, A^{r} e^{2 \tau \Lambda^{1/s}} \right) \right|_{L^{2}(\mathbb{T}^{3})} \\
\leq C \left( \| \nabla u \|_{L^{\infty}} + \tau^{2} \| \Psi \|_{H^{1}} + \tau^{2} \| \Psi \|_{X_{r, \tau, s}} \| J \|_{Y_{r, \tau, s}} \right) \\
+ C \left[ \| \nabla u \|_{L^{\infty}} \| J \|_{X_{r, \tau, s}} + \| \nabla h \|_{L^{\infty}} \| \omega \|_{X_{r, \tau, s}} + (1 + \tau) \| \Psi \|_{H^{1}} \right] \| J \|_{X_{r, \tau, s}},
$$

(6)

where $C$ is a positive constant.

Proof. Let $m \in \{1, 2, 3\}$. In order to estimate $\left| \left( u \cdot \nabla J, A^{r} e^{2 \tau \Lambda^{1/s}} \right) \right|_{L^{2}(\mathbb{T}^{3})}$, we appeal to the cancellation property $\left( u \cdot \nabla A^{r} e^{2 \tau \Lambda^{1/s}} J, A^{r} e^{2 \tau \Lambda^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})} = 0$ with notification div $u = 0$. Using Plancherel’s theorem, we obtain

$$
\left( u \cdot \nabla J, A^{r} e^{2 \tau \Lambda^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})} = \left( u \cdot \nabla J, A^{r} e^{2 \tau \Lambda^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})} - \left( u \cdot \nabla A^{r} e^{2 \tau \Lambda^{1/s}} J, A^{r} e^{2 \tau \Lambda^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})}
$$

$$
= i(2\pi)^{3} \sum_{j+k+\ell=0} \left( |\ell|^{r} e^{\tau|k|} - |k|^{r} e^{\tau|k|} \right) \left( \tilde{u}_{j,k} \cdot J_{k} \right) |\ell|^{r} e^{\tau|\ell|}
$$

$$
= i(2\pi)^{3} \sum_{j+k+\ell=0} \left( |\ell|^{r} - |k|^{r} \right) e^{\tau|k|} \left( \tilde{u}_{j,k} \cdot J_{k} \right) |\ell|^{r} e^{\tau|\ell|}
$$

$$
+ i(2\pi)^{3} \sum_{j+k+\ell=0} \left( |\ell|^{r} e^{\tau|\ell|} - e^{\tau|k|} \right) \left( \tilde{u}_{j,k} \cdot J_{k} \right) |\ell|^{r} e^{\tau|\ell|}
$$

$$
:= T^{(1)}_{u,J} + T^{(2)}_{u,J},
$$

(7)

with $j, k, \ell \in \mathbb{Z}^{3}$. Recall that $\tilde{u}_{0} = J_{0} = 0$. In order to estimate $T^{(1)}_{u,J}$, we first expand $|\ell|^{r} - |k|^{r}$ by means of mean value theorem,

$$
|\ell|^{r} - |k|^{r} = r \left( |\ell| - |k| \right) \left( \theta_{m,k,\ell} |\ell| + (1 - \theta_{m,k,\ell}) |k| \right)^{r-1}
$$

$$
= r \left( |\ell| - |k| \right) \left[ \left( \theta_{m,k,\ell} |\ell| + (1 - \theta_{m,k,\ell}) |k| \right)^{r-1} - |k|^{r-1} \right]
$$

$$
+ r \left( |\ell| - |k| \right) |k|^{r-1},
$$

(8)

where $\theta_{m,k,\ell} \in (0, 1)$ is a constant. Since $j + k + \ell = 0$, we have, by the triangle inequality,

$$
\left| r \left( |\ell| - |k| \right) \left[ \left( \theta_{m,k,\ell} |\ell| + (1 - \theta_{m,k,\ell}) |k| \right)^{r-1} - |k|^{r-1} \right] \right|
$$

$$
\leq C |m|^{2} \left( |j|^{r-2} + |k|^{r-2} \right).
$$

Since $j_{m} + k_{m} + \ell_{m} = 0$, we have the following decomposition, introduced by [9],

$$
|\ell_{m} - |k_{m}| = |j_{m} + k_{m}| - |k_{m}|
$$

$$
= j_{m} \text{sgn}(k_{m}) + 2(j_{m} + k_{m}) \text{sgn}(j_{m}) \chi_{\{ \text{sgn}(k_{m} + j_{m}) \text{sgn}(k_{m}) = -1 \}}.
$$

(9)

In the region $\{ \text{sgn}(k_{m} + j_{m}) \text{sgn}(k_{m}) = -1 \}$, we have $|k_{m}| \leq |j_{m}|$. Then with use of $e^{\xi} \leq e + e^{\frac{\xi}{2}}$ for $\xi = r|k_{m}|^{1/s} \geq 0$, $|\tilde{u}_{j,k}| \leq C |\tilde{u}_{j}| |k|_{1}$ and Plancherel’s theorem
we have, by discrete Cauchy-Schwarz inequality,
\[ T_{u,J,J}^{(1)} \leq C \sum_{j+k+\ell=0} \left\{ (|j_m|^r + |j_m|^2 |k_m|^{r-2}) (e + \tau^2 |k_m|^{2/s} e^{\tau|k_m|^{1/s}}) |\hat{u}_j| |k| |\hat{J}_k| |\hat{J}_\ell| \right\} \times |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ + C \sum_{j+k+\ell=0} j_m \text{sgn}(k_m) |k_m|^{r-1} e^{\tau|k_m|^{1/s}} (\hat{u}_j \cdot k)(\hat{J}_k \cdot \hat{J}_\ell) |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ \leq C \| \nabla u \|_{L^\infty} \| J \|_{X_{r,r,s}} \| \Lambda_m^r e^{\tau \Lambda_m^{1/s}} J \|_{L^2} + C \| \omega \|_{H^r} \| J \|_{H^r} \| \Lambda_m^r e^{\tau \Lambda_m^{1/s}} J \|_{L^2} \]
\[ + C \tau^2 \| \omega \|_{H^r} \| J \|_{Y_{r,r,s}} \| \Lambda_m^{r+\frac{1}{s}} e^{\tau \Lambda_m^{1/s}} J \|_{L^2}, \] (10)
where \( C \) is some constant depending on \( r \). The presence of the supremum of the velocity gradient, the innovative point of [9], is due to the use of Plancherel’s theorem in the following form,
\[ \sum_{j+k+\ell=0} j_m \text{sgn}(k_m) |k_m|^{r-1} e^{\tau|k_m|^{1/s}} (\hat{h}_j \cdot k)(\hat{\omega}_k \cdot \hat{J}_\ell) |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ = \left| (\partial_m h \cdot \nabla H_m \Lambda_m^{r-1} e^{\tau \Lambda_m^{1/s}} \omega, \Lambda_m^r e^{\tau \Lambda_m^{1/s}} J) \right|_{L^2(\mathbb{T}^3)} \]
\[ \leq \| \nabla h \|_{L^\infty} \| \omega \|_{X_{r,r,s}} \| \Lambda_m^r e^{\tau \Lambda_m^{1/s}} J \|_{L^2}. \]

In order to estimate \( T_{u,J,J}^{(2)} \), a little different from Lemma 2.5 of [9], we rewrite it into the sum of the following three terms,
\[ T_{u,J,J}^{(2)} = i(2\pi)^3 \sum_{j+k+\ell=0} \left\{ (\hat{u}_j \cdot k) |\ell_m|^{r-\frac{1}{s}} \left[ e^{\tau(|\ell_m|^{1/s}-|k_m|^{1/s})} - 1 \right] \right\} \]
\[ - \tau (|\ell_m|^{1/s} - |k_m|^{1/s}) e^{\tau|k_m|^{1/s}} (\hat{J}_k \cdot \hat{J}_\ell) |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ + i(2\pi)^3 \sum_{j+k+\ell=0} \left[ \tau (|\ell_m|^{1/s} - |k_m|^{1/s}) e^{\tau|k_m|^{1/s}} (\hat{u}_j \cdot k)(\hat{J}_k \cdot \hat{J}_\ell) \right. \]
\[ \times |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ - i(2\pi)^3 \sum_{j+k+\ell=0} \left[ \tau |k_m|^{1/s} (|\ell_m|^{1/s} - |k_m|^{1/s}) e^{\tau|k_m|^{1/s}} (\hat{u}_j \cdot k) \right. \]
\[ \times (\hat{J}_k \cdot \hat{J}_\ell) |\ell_m|^r e^{\tau|k_m|^{1/s}} \]
\[ := R_{u,J,J}^{(1)} + R_{u,J,J}^{(2)} - R_{u,J,J}^{(3)}. \] (11)

We remark that we may have a different form of the above expression if \( s = 1 \), see [9], however the above identity is valid for all \( s \geq 1 \). For the first term \( R_{u,J,J}^{(1)} \), we appeal to the inequality \( |e^\xi - 1 - \xi| \leq |\xi^2 e^\xi| \), for \( \xi = \tau(|\ell_m|^{1/s} - |k_m|^{1/s}) \in \mathbb{R} \), the triangle inequality \( |\ell_m|^{r-\frac{1}{s}} \leq C(|j_m|^{r-\frac{1}{s}} + |k_m|^{r-\frac{1}{s}}) \) and
\[ \left| |\ell_m|^{1/s} - |k_m|^{1/s}\right| \leq |j_m|^{1/s}, \quad \left| |\ell_m|^{1/s} - |k_m|^{1/s}\right| \leq C \frac{|j_m|}{|\ell_m|^{1-1/s} + |k_m|^{1-1/s}}, \]
\[ (12) \]
where we note that \( |\ell_m|^{1-1/s} + |k_m|^{1-1/s} \neq 0 \). With use of the above inequalities, 
\( R_{u,J}^{(1)} \) can be bounded by

\[
\left| R_{u,J}^{(1)} \right| 
\leq C \tau^2 \sum_{j+k+\ell=0} \left[ \left| \hat{u}_j \right| \left| k_1 \right| \left| j_m \right|^{r-\frac{1}{s}} \right] \left( |j_m| \left| \ell_m \right|^{-1/s} + |k_m| \left| \ell_m \right|^{-1/s} \right) \left( 1 + |j_m| \left| \ell_m \right|^{-1/s} + |k_m| \left| \ell_m \right|^{-1/s} \right) \leq C \tau^2 \sum_{j+k+\ell=0} \left[ \left| \hat{u}_j \right| \left| k_1 \right| \left| j_m \right|^{r+\frac{1}{s}} \right] \left( |j_m| \left| \ell_m \right|^{1-1/s} + |k_m| \left| \ell_m \right|^{1-1/s} \right) \times \left( |\ell_m|^{1-1/s} + |k_m|^{1-1/s} \right).
\]

The first term on the right side of (14) is bounded by \( C |j_m|^2 \left( |j_m|^{r-2+\frac{1}{s}} + |k_m|^{r-2+\frac{1}{s}} \right) \) for some constant \( C \) depending on \( r, s \). For the latter term we use the decomposition (9) again, and note in the region \( \{ \text{sgn}(k_m + j_m) \text{ sgn}(k_m) = -1 \} \) we have \( |k_m| \leq |j_m| \) and \( e^{r|k_m|^{1/s}} \leq 1 + \tau |j_m|^{1/s} e^{r|j_m|^{1/s}} \). Combining these facts, we have

\[
\left| R_{u,J}^{(2)} \right| 
\leq C \tau \sum_{j+k+\ell=0} \left[ |j_m|^2 \left( |j_m|^{r-2+\frac{1}{s}} + |k_m|^{r-2+\frac{1}{s}} \right) \right] \left( 1 + \tau |j_m|^{1/s} e^{r|j_m|^{1/s}} \right) \times \left( |\ell_m|^{1-1/s} + |k_m|^{1-1/s} \right) \leq C \tau |\omega| H^1 \left( |J|H^1 \right) \left( |\Lambda_r^{k,m} e^{r|\Lambda_r^{k,m} J|^{1/s}} \right) \left( |J|H^1 \right) \left( |\Lambda_r^{k,m} e^{r|\Lambda_r^{k,m} J|^{1/s}} \right) \left( |J|H^1 \right) \left( |\omega| H^{-r,s} \right) \left( |J|H^{-r,s} \right),
\]

(15)
where we have used $|\ell_m|^{1/2} \leq |j_m|^{1/2} + |k_m|^{1/2}$ for the estimate of the first term. In order to estimate the third term $R_u^{(3)}$, we once again expand the $|\ell_m|^{1/2} - |k_m|^{1/2}$ by mean value theorem,

$$
|\ell_m|^{1/2} - |k_m|^{1/2} = \left( r - \frac{1}{2s} \right) (|j_m| - |k_m|) \left[ (\theta_{m,k,t}^* |\ell_m| + (1 - \theta_{m,k,t}^*) |k_m|) |r|^{1/2} - |k_m| |r|^{1/2} - 1 \right] \\
+ \left( r - \frac{1}{2s} \right) (|j_m| - |k_m|) |k_m|^{1/2} - |k_m|^{1/2}.
$$

(16)

Using similar method as above, $R_u^{(3)}$ can also be bounded by

$$
\left| R_u^{(3)} \right| \leq C \sum_{j+k+\ell=0} \tau |k_m|^{1/2} |j_m|^2 |k_m|^{1/2} + |k_m| |r|^{1/2} e^\tau |k_m|^{1/2} \\
\times |\hat{u}_j| |j_k| |j_l| |\ell_m|^{1/2} e^{\tau |\ell_m|^{1/2}} \\
+ C \sum_{j+k+\ell=0} \tau |j_m|^{1/2} \left( 1 + \tau |j_m|^{1/2} e^{\tau |j_m|^{1/2}} \right) |\hat{u}_j| |j_k| |j_l| |\ell_m|^{1/2} e^{\tau |\ell_m|^{1/2}} \\
+ C \tau \left( \partial_m u \cdot \nabla H_{m} \Lambda_m^{1/2} J, \left. \Lambda_m^{1/2} e^{\Lambda_{m}^{1/2}} J \right)_{L^2(T^3)} \right| \\
\leq C \tau \left| \nabla h \cdot A \cdot \nabla u \right|_{H^r_T} \left| J \right|_{H^r_T} \left| \Lambda_m^{1/2} e^{\Lambda_{m}^{1/2}} J \right|_{L^2} \\
+ C \tau^2 \left| \nabla h \cdot A \cdot \nabla u \right|_{H^r_T} \left| \Lambda_m^{1/2} e^{\Lambda_{m}^{1/2}} J \right|_{L^2},
$$

(17)

Combining (13), (15), (17) and the estimate (10) on $T_{u,J,J}^{(1)}$ in (7), we have proven that the term $\left| \langle u \cdot \nabla J, \Lambda_m^{2r} e^{2\Lambda_m^{1/2}} J \rangle_{L^2(T^3)} \right|$ is bounded by the (6).

In order to estimate the coupled term $\langle \nabla h \cdot A \cdot \nabla u, \Lambda_m^{2r} e^{2\Lambda_m^{1/2}} J \rangle_{L^2(T^3)}$, we treat it as follows. First of all, by Hölder equality we have

$$
\left| \left( A_m^{r} e^{\Lambda_{m}^{1/2}} \nabla h \cdot A \cdot \nabla u, \Lambda_m^{r} e^{\Lambda_{m}^{1/2}} J \right)_{L^2(T^3)} \right| \\
+ \left| \left( \nabla h \cdot A \cdot \Lambda_m^{r} e^{\Lambda_{m}^{1/2}} \nabla u, \Lambda_m^{r} e^{\Lambda_{m}^{1/2}} J \right)_{L^2(T^3)} \right| \\
\leq C \left| \nabla u \right|_{L^\infty} \left| \Lambda_m^{r} e^{\Lambda_{m}^{1/2}} J \right|_{L^2} + C \left| \nabla h \right|_{L^\infty} \left| \omega \right|_{X_{r,r}, \Lambda_m^{r} e^{\Lambda_{m}^{1/2}} J}_{L^2}.
$$

(18)

Then by Fourier series expansion the vector function $\nabla h \cdot A \cdot \nabla u$ can be expanded by

$$
\nabla h \cdot A \cdot \nabla u = \sum_{\ell \in \mathbb{Z}^3} \mathcal{F}(\nabla h \cdot A \cdot \nabla u)(\ell) e^{ix \cdot \ell},
$$
where \( \mathcal{F}(\nabla h \cdot A \cdot \nabla u)(\ell) \) denotes the \( \ell \)-th vector Fourier coefficient of \( \nabla h \cdot A \cdot \nabla u \) and by use of Fourier inverse Theorem \( \langle \mathcal{F} h_{j} \rangle \) and \( \langle \mathcal{F} u_{k} \rangle \) represent the \( j \)-th and \( k \)-th Fourier coefficient of \( \nabla h \) and \( \nabla u \) respectively. Note that according to the notation of \( \nabla h \cdot A \cdot \nabla u \), the term \( \langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle \) represents a vector but not a matrix here.

Then we substract \( \langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle e^{\lambda_{m}^{1/s}} J_{L^{2}(\mathbb{T}^{3})} \) by

\[
\left( \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} \nabla h \cdot A \cdot \nabla u, \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})} + \left( \nabla h \cdot A \cdot \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} \nabla u, \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})}
\]

and we consider their differences, by Fourier series expansion,

\[
\langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle e^{\lambda_{m}^{1/s}} J_{L^{2}(\mathbb{T}^{3})} - \left( \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} \nabla h \cdot A \cdot \nabla u, \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} J \right)_{L^{2}(\mathbb{T}^{3})}
\]

\[
= i(2\pi)^{3} \sum_{j+k+\ell=0} \left[ (|\ell_{m}| - |j_{m}|) e^{\xi|\ell_{m}|^{1/s}} e^{\tau|k_{m}|^{1/s}} \right] \left( \langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle \cdot \hat{J}_{\ell} \right)
\]

\[
+ i(2\pi)^{3} \sum_{j+k+\ell=0} \left[ (|\ell_{m}| - |k_{m}| - |j_{m}|) e^{\xi|k_{m}|^{1/s}} \right] \left( \langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle \cdot \hat{J}_{\ell} \right)
\]

\[
+ i(2\pi)^{3} \sum_{j+k+\ell=0} \left[ (|j_{m}| - |\ell_{m}|) e^{\xi|\ell_{m}|^{1/s}} \right] \left( \langle \mathcal{F} h_{j} \rangle \cdot A \cdot \langle \mathcal{F} u_{k} \rangle \cdot \hat{J}_{\ell} \right)
\]

\[
:= T_{h,u,J}^{(1)} + T_{h,u,J}^{(2)} + T_{h,u,J}^{(3)}
\]  

(19)

It rest to estimate the right hand side of (19). For the first term \( T_{h,u,J}^{(1)} \), we appeal to the mean value theorem for \( |\ell_{m}| \) and \( |e^{\xi} - 1| \leq |\xi| \), for \( \xi = \tau(|\ell_{m}|^{1/s} - |k_{m}|^{1/s}) \in \mathbb{R} \), and the inequality (12),

\[
\left| (|\ell_{m}| - |j_{m}|) e^{\xi|\ell_{m}|^{1/s}} - e^{\tau|k_{m}|^{1/s}} \right|
\]

\[
\leq C \tau \frac{|j_{m}| - |k_{m}| e^{\tau|j_{m}|^{1/s}} |e^{\xi|\ell_{m}|^{1/s}}}{|\ell_{m}|^{1-1/s} + |k_{m}|^{1-1/s}}
\]

\[
\leq C \tau |j_{m}|^{1/s} e^{\tau|j_{m}|^{1/s}} (1 + \tau|k_{m}|^{1/s} e^{\tau|k_{m}|^{1/s}})
\]

\[
+ C \tau \frac{|\ell_{m}|^{1/s} e^{\tau|\ell_{m}|^{1/s}}}{|\ell_{m}|^{1-1/s} + |k_{m}|^{1-1/s}}.
\]

(20)

Substituting the right of (20) into \( T_{h,u,J}^{(1)} \) and using again the inequality \( e^{\tau|j_{m}|^{1/s}} \leq 1 + \tau|j_{m}|^{1/s} e^{\tau|j_{m}|^{1/s}} \) and \( e^{\tau|k_{m}|^{1/s}} \leq 1 + \tau|k_{m}|^{1/s} e^{\tau|k_{m}|^{1/s}} \) for the order-\( \tau \) term, we have

\[
\left| T_{h,u,J}^{(1)} \right| \leq C \tau \| \omega \|_{H^{r}} \| J \|_{H^{r}} \| \Lambda_{m}^{r} e^{\lambda_{m}^{1/s}} J \|_{L^{2}} + C \tau^{2} \| \omega \|_{H^{r}} \| J \|_{Y_{r,s}}^{2}
\]
+ C\tau^2 \|J\|_{H^r} \|\omega\|_{Y_{r,\tau}, s} + J\|y_{r,\tau}, s + C\tau^2 \|J\|_{X_{r,\tau}, s} \|\omega\|_{Y_{r,\tau}, s} \|\Lambda_m r^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2} \]
+ C\tau^2 \|\omega\|_{X_{r,\tau}, s} \|J\|_{Y_{r,\tau}, s}^2, \tag{21}
where we used the inequalities \(|k_m| \geq |j_m| \frac{1}{r} + |\ell_m| \frac{1}{s}\) and \(|j_m| \frac{1}{r} \leq |k_m| \frac{1}{r} + |\ell_m| \frac{1}{s}\).

For the second term \(T_{h,u,J}^{(2)}\), by the mean value theorem we have
\[
|(|\ell_m|^r - |k_m|^r) - |j_m|^r| \leq C |j_m|(|j_m|^{r-1} + |k_m|^{r-1}) + |j_m|^r.
\]
Using the inequality \(e^x \leq e + x^2 e^x\), for all \(x = \tau/k_m^{1/s}\), then we obtain
\[
\left| T_{h,u,J}^{(2)} \right| \leq C \|J\|_{H^r} \|\omega\|_{H^r} \|\Lambda_m r^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2} \]
+ C\tau^2 \|\omega\|_{Y_{r,\tau}, s} \|\Lambda_m r^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2}, \tag{22}
where we used \(|k_m| \frac{1}{r} \leq |j_m| \frac{1}{r} + |\ell_m| \frac{1}{s}\) in the estimate of the second term on right of (22). For the third term \(T_{h,u,J}^{(3)}\), we use the inequality \(|e^\xi - 1| \leq |\xi| e^{|\xi|}\), for \(\xi = \tau/|j_m|^{1/s} \in \mathbb{R}\), and the inequality \(e^\xi \leq 1 + \xi e^\xi\), for \(\xi = \tau/|j_m|^{1/s}\) and \(\xi = \tau/k_m^{1/s}\), and the triangle inequality \(|j_m| \frac{1}{r} \leq |k_m| \frac{1}{r} + |\ell_m| \frac{1}{s}\). Thus we finally have
\[
\left| T_{h,u,J}^{(3)} \right| \leq C\tau \|J\|_{H^r} \|\omega\|_{H^r} \|\Lambda_m \tau^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2} + C\tau^2 \|\omega\|_{H^r} \|J\|_{Y_{r,\tau}, s}^2 \]
+ C\tau^2 \|\omega\|_{X_{r,\tau}, s} \|\Lambda_m \tau^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2}^2. \tag{23}

Collecting (21), (22), (23) and (18), we have the estimate
\[
\left| \left( \partial_h \partial_u \cdot \nabla u, \Lambda_m e^{2r\Lambda_m^s} J \right) \right|_{L^2(T^3)} \leq C(\|\nabla u\|_{L^\infty} \|J\|_{X_{r,\tau}, s} + \|\nabla h\|_{L^\infty} \|\omega\|_{X_{r,\tau}, s}) \|J\|_{Y_{r,\tau}, s} \]
+ C\tau \|\Psi\|_{H^r} \|\Lambda_m \tau^{\frac{3}{2}} e^{r\Lambda_m^s} J\|_{L^2} + C\tau^2 (\|\Psi\|_{H^r} + \|\Psi\|_{X_{r,\tau}, s}) \|\Psi\|_{Y_{r,\tau}, s} \|J\|_{Y_{r,\tau}, s}.
\]
Obviously the above right hand side is also bounded by the right of (6), thus the proof is complete. \(\square\)

In the following, we give the main Lemma concerning the estimates of the coupled nonlinear terms.

**Lemma 3.6.** Let \(s \geq 1, m = 1, 2, 3\). Let \(\tau > 0, r > \frac{5}{3} + \frac{2}{3r}\), and \(u = K\omega, h = K^r J\) with \(\omega, J \in Y_{r,\tau, s}\). Then we have the following upper bounded estimates :
\[
\left| \left( \partial_h \partial_u \cdot \nabla J, \Lambda_m e^{2r\Lambda_m^s} \omega \right) \right|_{L^2(T^3)} + \left| \left( \partial_h \partial_u \cdot \nabla \omega, \Lambda_m e^{2r\Lambda_m^s} J \right) \right|_{L^2(T^3)} \leq C(\tau \|\nabla h\|_{L^\infty} + \tau^2 \|\Psi\|_{H^r} + \tau^2 \|\Psi\|_{X_{r,\tau}, s}) \|\Psi\|_{Y_{r,\tau}, s} \]
+ C(\|\nabla h\|_{L^\infty} \|\Psi\|_{X_{r,\tau}, s} + (1 + \tau) \|\Psi\|_{H^r} \|\Psi\|_{X_{r,\tau}, s}, \tag{24}
\]
\[
\left| \left( \partial_h \partial_u \cdot \nabla h, \Lambda_m e^{2r\Lambda_m^s} \omega \right) \right|_{L^2(T^3)} + \left| \left( \partial_u \cdot \nabla u, \Lambda_m e^{2r\Lambda_m^s} J \right) \right|_{L^2(T^3)} \leq C(\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}) \|\Psi\|_{X_{r,\tau}, s} + C\tau \|\Psi\|_{H^r} \|\Psi\|_{X_{r,\tau}, s} \]
+ C\tau^2 (\|\Psi\|_{H^r} + \|\Psi\|_{X_{r,\tau}, s}) \|\Psi\|_{Y_{r,\tau}, s}^2, \tag{25}
where \(C\) is a constant depending only on \(r, s\).
We note that the key point in the proof of Lemma 3.6 is that the coefficients of \(\tau\) and \(\tau^s\) are carefully arranged such that on one hand we can obtain an upper bound of \(\|\omega\|_{X_{\tau^s}}\), on the other hand we can obtain a lower bound of \(\|\nabla u\|_{L^\infty}\) and \(\|\nabla h\|_{L^\infty}\).

**Proof of (24).** Since \(h = K * J\) is divergence-free, we have the following cancellation property, by integration by parts and the symmetry structure,

\[
\left( h \cdot \nabla \Lambda_m e^{\tau \Lambda_m^s J}, \Lambda_m e^{\tau \Lambda_m^s \omega} \right)_{L^2(\Omega)} + \left( h \cdot \nabla \Lambda_m e^{\tau \Lambda_m^s \omega}, \Lambda_m e^{\tau \Lambda_m^s J} \right)_{L^2(\Omega)} = 0.
\]

Thus we have

\[
\left( h \cdot \nabla J, \Lambda_m e^{2\tau \Lambda_m^s \omega} \right)_{L^2(\Omega)} + \left( h \cdot \nabla \omega, \Lambda_m e^{2\tau \Lambda_m^s J} \right)_{L^2(\Omega)} = 0.
\]

Due to the symmetry of \(h, \omega, J\) on the right hand side, it suffices to estimate one of them and the other one can be estimated in a similar way.

Let us consider for example \(T_{h, \omega, J}\), which takes the following form

\[
T_{h, \omega, J} = i(2\pi)^3 \sum_{j+k+\ell = 0} \left( \hat{h}_j \cdot k \right) \left( |\ell_m|^r e^{\tau |\ell_m|^s} - |k_m|^r e^{\tau |k_m|^s} \right) \left( \hat{\omega}_k \cdot J_\ell \right) \times |\ell_m|^r e^{\tau |\ell_m|^s}.
\]

It also can be split into the summation of two terms \(T_{h, \omega, J} = T_{h, \omega, J}^{(1)} + T_{h, \omega, J}^{(2)}\), where

\[
T_{h, \omega, J}^{(1)} = i(2\pi)^3 \sum_{j+k+\ell = 0} \left( \hat{h}_j \cdot k \right) \left( |\ell_m|^r - |k_m|^r \right) e^{\tau |k_m|^s} |\omega_k \cdot J_\ell| \times |\ell_m|^r e^{\tau |\ell_m|^s},
\]

\[
T_{h, \omega, J}^{(2)} = i(2\pi)^3 \sum_{j+k+\ell = 0} \left( \hat{h}_j \cdot k \right) |\ell_m|^r e^{\tau |\ell_m|^s} - e^{\tau |k_m|^s} |\omega_k \cdot J_\ell| \times |\ell_m|^r e^{\tau |\ell_m|^s}.
\]

In order to estimate \(T_{h, \omega, J}^{(1)}\), we appeal to the expansion of (8), (9) and the arguments of (10). Then we immediately have

\[
|T_{h, \omega, J}^{(1)}| \
\leq C \sum_{j+k+\ell = 0} \left\{ (|j_m|^r + |j_m|^2 |k_m|^{-2})(\epsilon + \tau^2 |k_m|^{2s} e^{\tau |k_m|^s}) |\hat{h}_j| |\ell_m|^r |\hat{\omega}_k| |J_\ell| \times |\ell_m|^r e^{\tau |\ell_m|^s} \right\} \\
+ C \sum_{j+k+\ell = 0} j_m sgn(k_m) |k_m|^{-1} e^{\tau |k_m|^s} (\hat{h}_j \cdot k) |\omega_k \cdot J_\ell| \times |\ell_m|^r e^{\tau |\ell_m|^s}.
\]
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \sum_{m} & \frac{1}{|k_m|} \left| \sum_{j+k+\ell=0} \hat{h}_j \cdot k, \tau \right| e^{\tau |k_m|^{1/\delta}} \left| \hat{\omega}_k \cdot \hat{J}_\tau \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}}} \\
& \leq C \left\| \nabla h \right\|_{L^\infty} \left\| \omega \right\|_{X_{r,\tau,s}} \left\| J \right\|_{X_{r,\tau,s}} + C \left\| \omega \right\|_{H^r} \left\| J \right\|_{H^r} \left\| J \right\|_{X_{r,\tau,s}} \\
& + C \tau^2 \left\| \nabla h \right\|_{H^r} \left\| \omega \right\|_{Y_{r,\tau,s}} \left\| J \right\|_{Y_{r,\tau,s}}.
\end{align*}
\]

Still the supremum of gradient of \( h \) on the right hand side come from the use of Plancherel’s theorem as follows,

\[
\left| \sum_{j+k+\ell=0} \hat{h}_j \cdot k, \tau \right| e^{\tau |k_m|^{1/\delta}} \left( \hat{\omega}_k \cdot \hat{J}_\tau \right) \left| \ell_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}}}
\leq \left| \left( \partial_m h \cdot \nabla H_m \Lambda_m^{-1} e^{\tau \Lambda_m^{1/\epsilon}} \omega, \Lambda_m e^{\tau \Lambda_m^{1/\epsilon}} J \right) \right|_{L^2(T^3)}
\leq \| \nabla h \|_{L^\infty} \| \omega \|_{X_{r,\tau,s}} \| \Lambda_m e^{\tau \Lambda_m^{1/\epsilon}} J \|_{L^2}.
\]

To estimate \( T_h^{(2)} \), like (11), we rewrite it into the sum of the following three terms,

\[
T_h^{(2)} = i(2\pi)^3 \sum_{j+k+\ell=0} \left[ (\hat{h}_j \cdot k) \left| \ell_m \right| e^{\tau |k_m|^{1/\delta}} \left( e^{\tau \left| (\ell_m |^{1/\delta} - |k_m|^{1/\delta} \right)} - 1 \\
- \tau (|k_m|^{1/\delta} - |k_m|^{1/\delta}) e^{\tau |k_m|^{1/\delta}} (\hat{\omega}_k \cdot \hat{J}_\tau) \left| \ell_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}}} \\
+ i(2\pi)^3 \sum_{j+k+\ell=0} \left[ \tau \left| \ell_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}} - |k_m|^{1/\delta}} \right) e^{\tau |k_m|^{1/\delta}} (\hat{h}_j \cdot k) \right] \left| \ell_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}}}
\right]
\]

The three terms \( R_h^{(1)} \), \( R_h^{(2)} \) and \( R_h^{(3)} \) are estimated with the same arguments with (13), (15) and (17), thus we immediately have from the arguments of (12) and (13),

\[
\left| R_h^{(1)} \right| \leq C \tau^2 \sum_{j+k+\ell=0} \left[ \left| \hat{h}_j \right| \left| k_1 \right| \left( |j_m|^{1/\delta} + |k_m|^{1/\delta} \right) \left| j_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}} - |k_m|^{1/\delta}} |j_m| e^{\tau |k_m|^{1/\delta}} \left| \hat{\omega}_k \right| \right| \ell_m \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}}} \left| \hat{J}_\ell \right|
\leq C \tau^2 \sum_{j+k+\ell=0} \left[ \left( |j_m|^{1/\delta} + 1 \right) \left| k_1 \right| e^{\frac{1}{\delta} \frac{1}{|k_m|^{1/\delta}} - |k_m|^{1/\delta}} \left| \hat{h}_j \right| \right| \left| k_1 \right| e^{\tau |k_m|^{1/\delta}} \left| \hat{\omega}_k \right| \right]
\leq C \tau^2 \left\| J \right\|^2_{Y_{r,\tau,s}} \left\| \omega \right\|_{X_{r,\tau,s}} + C \tau^2 \left\| J \right\|^2_{X_{r,\tau,s}} \left\| \omega \right\|_{Y_{r,\tau,s}} \left\| J \right\|_{Y_{r,\tau,s}}.
\]

(26)
By use of the expansion (14) and similar arguments as (15), we have

\[ |R_{h,\omega,J}^{(2)}| \]
\[ \leq C \tau \left( \left| \partial_m h \cdot \nabla H_m \Lambda_m^{-1} \frac{1}{2} e^{\tau \Lambda_m^{1/2}} \omega, \Lambda_m^{1/2} e^{\tau \Lambda_m^{1/2}} h J \right|_{L^2(\Omega)} \right) \]
\[ + C \tau \sum_{j+k+\ell=0} \left| \hat{h}_j \right| |k_1| \left| \hat{\omega}_k \right| |\ell_m|^{r+\frac{1}{2}} e^{\tau \ell_m^{1/2}} \hat{J}_\ell \]
\[ + C \tau \sum_{j+k+\ell=0} \left| \hat{h}_j \right| \left| \hat{\omega}_k \right| \left| \hat{J}_\ell \right| |\ell_m|^{r+\frac{1}{2}} e^{\tau \ell_m^{1/2}} \hat{J}_\ell \]
\[ \leq C \tau \|J\|_{H} \|\omega\|_{H} \|J\|_{X_{r,\tau,s}} + C \tau \|\nabla h\|_{L=\infty} \|\omega\|_{Y_{r,\tau,s}} \|J\|_{Y_{r,\tau,s}} \]
\[ + C \tau^2 \|J\|_{H} \|\omega\|_{Y_{r,\tau,s}} \|J\|_{Y_{r,\tau,s}} + C \tau^2 \|\omega\|_{H} \|J\|_{Y_{r,\tau,s}}^2 , \quad (27) \]

where \( C \) is a positive constant. By use of the expansion (16) and similar arguments as (17), we have

\[ |R_{h,\omega,J}^{(3)}| \]
\[ \leq C \sum_{j+k+\ell=0} \tau \left| \hat{h}_j \right| |k_1| \left| \hat{\omega}_k \right| \left| \hat{J}_\ell \right| |\ell_m|^{r+\frac{1}{2}} e^{\tau \ell_m^{1/2}} \hat{J}_\ell \]
\[ + C \tau \sum_{j+k+\ell=0} \left| \hat{h}_j \right| \left| \hat{\omega}_k \right| \left| \hat{J}_\ell \right| |\ell_m|^{r+\frac{1}{2}} e^{\tau \ell_m^{1/2}} \hat{J}_\ell \]
\[ + C \tau \left( \left| \partial_m h \cdot \nabla H_m \Lambda_m^{1/2} e^{\tau \Lambda_m^{1/2}} \omega, \Lambda_m^{1/2} e^{\tau \Lambda_m^{1/2}} h J \right|_{L^2(\Omega)} \right) \]
\[ \leq C \tau \|J\|_{H} \|\omega\|_{H} \|J\|_{X_{r,\tau,s}} + C \tau \|\nabla h\|_{L=\infty} \|\omega\|_{Y_{r,\tau,s}} \|J\|_{Y_{r,\tau,s}} \]
\[ + C \tau^2 \|J\|_{H} \|\omega\|_{Y_{r,\tau,s}} \|J\|_{Y_{r,\tau,s}} + C \tau^2 \|\omega\|_{H} \|J\|_{Y_{r,\tau,s}}^2 , \quad (28) \]

where \( C \) is a constant depending only on \( r, s \) for \( r > \frac{3}{2} + \frac{5}{2} \). Combining (26), (27) and (28), we have the estimate of \( T_{h,\omega,J}^{(2)} \). Then with the estimate of \( T_{h,\omega,J}^{(1)} \). We then proved (24).
Proof of (25). It suffices to estimate \( (J \cdot \nabla h, \Lambda_m^r e^{2\pi \Lambda_m^{1/s}} \omega)_{L^2(T^3)} \), since the other term \( (\nabla u \cdot A \cdot \nabla h, \Lambda_m^r e^{2\pi \Lambda_m^{1/s}} J)_{L^2(T^3)} \) can be estimated in similar way (replacing the position of \( \nabla u \) and \( \nabla h \)). First of all, we note that \( \| J \|_{L^\infty} \leq \| \nabla h \|_{L^\infty} \) and Lemma 3.3,

\[
\left| \left( \Lambda_m^r e^{\pi \Lambda_m^{1/s}} J \cdot \nabla h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)} \right| + \left| \left( J \cdot \nabla \Lambda_m^r e^{\pi \Lambda_m^{1/s}} h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)} \right| \\
\leq C \| \nabla h \|_{L^\infty} \| J \|_{X_{r,r,s}} \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2}.
\]

(29)

Then like (19), we subtract \( (J \cdot \nabla h, \Lambda_m^r e^{2\pi \Lambda_m^{1/s}} \omega)_{L^2(T^3)} \) by

\[
\left( \Lambda_m^r e^{\pi \Lambda_m^{1/s}} J \cdot \nabla h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)} + \left( J \cdot \nabla \Lambda_m^r e^{\pi \Lambda_m^{1/s}} h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)}
\]

and we consider their differences

\[
\left( J \cdot \nabla h, \Lambda_m^r e^{2\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)} - \left( \Lambda_m^r e^{\pi \Lambda_m^{1/s}} J \cdot \nabla h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)}
\]

\[
- \left( J \cdot \nabla \Lambda_m^r e^{\pi \Lambda_m^{1/s}} h, \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \right)_{L^2(T^3)}
\]

\[
i(2\pi)^3 \sum_{j+k+\ell=0} (|\ell_m|^r - |j_m|^r) (e^{\tau |k_m|^{\frac{1}{s}}} - e^{\tau |k_m|^{\frac{1}{s}}}) (\hat{J}_\ell \cdot k) (\hat{h}_k \cdot \hat{w}_\ell) |\ell_m|^r e^{\tau |\ell_m|^{\frac{1}{s}}}
\]

\[
i(2\pi)^3 \sum_{j+k+\ell=0} (|\ell_m|^r - |j_m|^r) e^{\tau |k_m|^{\frac{1}{s}}} (\hat{J}_\ell \cdot k) (\hat{h}_k \cdot \hat{w}_\ell) |\ell_m|^r e^{\tau |\ell_m|^{\frac{1}{s}}}
\]

\[
i(2\pi)^3 \sum_{j+k+\ell=0} |j_m|^r (e^{\tau |\ell_m|^{\frac{1}{s}}} - e^{\tau |j_m|^{\frac{1}{s}}}) (\hat{J}_\ell \cdot k) (\hat{h}_k \cdot \hat{w}_\ell) |\ell_m|^r e^{\tau |\ell_m|^{\frac{1}{s}}}
\]

\[
:= T_{j,h,\omega}^{(1)} + T_{j,h,\omega}^{(2)} + T_{j,h,\omega}^{(3)}.
\]

Analogue to (19), the three terms \( T_{j,h,\omega}^{(1)}, T_{j,h,\omega}^{(2)} \) and \( T_{j,h,\omega}^{(3)} \) are estimated in the same way. Then we directly have

\[
\left| T_{j,h,\omega}^{(1)} \right| \leq C \tau \| J \|_{H^r}^2 \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2}
\]

\[
+ C \tau^2 \| J \|_{X_{r,r,s}} \| J \|_{Y_{r,r,s}} \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2},
\]

and

\[
\left| T_{j,h,\omega}^{(2)} \right| \leq C \| J \|_{H^r}^2 \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2} + C \tau^2 \| J \|_{H^r} \| J \|_{Y_{r,r,s}} \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2},
\]

and

\[
\left| T_{j,h,\omega}^{(3)} \right| \leq C \tau \| J \|_{H^r}^2 \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2}
\]

\[
+ C \tau^2 \| J \|_{X_{r,r,s}} \| J \|_{Y_{r,r,s}} \| \Lambda_m^r e^{\pi \Lambda_m^{1/s}} \omega \|_{L^2}.
\]
We get then the estimate
\[
\left| \left( J \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right) \right|_{L^2(T^3)} \leq C \| \nabla h \|_{L^\infty} \| J \|_{X_{r,\tau,s}} \| \Lambda^r e^{\tau \Lambda^1 / s} \omega \|_{L^2} + C \| \Psi \|_{H^1}^2 \| \Lambda^r e^{\tau \Lambda^1 / s} J \|_{L^2} + C \tau^2 (\| J \|_{H^s} + \| J \|_{X_{r,\tau,s}}) \| \Lambda^r e^{\tau \Lambda^1 / s} \omega \|_{L^2}.
\]

Symmetrically, we have
\[
\left| \left( \nabla u \cdot A \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right) \right|_{L^2(T^3)} \leq C \| \nabla h \|_{L^\infty} \| \omega \|_{X_{r,\tau,s}} \| \Lambda^r e^{\tau \Lambda^1 / s} J \|_{L^2} + C \| \nabla u \|_{L^\infty} \| J \|_{X_{r,\tau,s}} \| \Lambda^r e^{\tau \Lambda^1 / s} \omega \|_{L^2} + C \tau^2 (\| \Psi \|_{H^s} + \| \Psi \|_{X_{r,\tau,s}}) \| \Lambda^r e^{\tau \Lambda^1 / s} J \|_{L^2}.
\]

Finally we get
\[
\left| \left( J \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right) \right|_{L^2(T^3)} + \left| \left( \nabla u \cdot A \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right) \right|_{L^2(T^3)} \leq C (\| \nabla u \|_{L^\infty} + \| \nabla h \|_{L^\infty}) \| \Psi \|_{X_{r,\tau,s}}^2 + C \tau^2 (\| \Psi \|_{H^s} + \| \Psi \|_{X_{r,\tau,s}}) \| \Psi \|_{X_{r,\tau,s}}^2.
\]

Then (25) is proved. \(\square\)

4. **Proof of Theorem 2.1.** In this Section, we will give the proof of the main theorem. Here we only present a priori estimate, since the rigorous construction of the solution follows from the standard Galerkin approximation which we present in the Appendix.

**Remark 3.** In a recent paper [12], the authors obtained an uniform Gevrey norm estimate with respect to the viscous parameters of the regularized (or viscous) MHD equations which can also be used to prove the existence of the smooth solutions for the ideal MHD equations.

**Proof of Theorem 2.1.** For simplicity of presentation we suppress the time dependence of \(\tau, u, h, \omega\) and \(J\) on \(t\). As usual, let \(m \in \{1, 2, 3\}\), let us take the \(L^2\) inner product of the first equation of (2) with \(\Lambda^2 e^{2\tau \Lambda^1 / s} \omega\), and the second equation of (2) with \(\Lambda^2 e^{2\tau \Lambda^1 / s} J\) respectively,
\[
\left( \partial_t \omega, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right)_{L^2(T^3)} + \left( u \cdot \nabla \omega - \omega \cdot \nabla u, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right)_{L^2(T^3)} - \left( h \cdot \nabla J, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right)_{L^2(T^3)} + \left( J \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} \omega \right)_{L^2(T^3)} = 0, \quad (30)
\]

and
\[
\left( \partial_t J, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right)_{L^2(T^3)} + \left( u \cdot \nabla J - \nabla h \cdot A \cdot \nabla u, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right)_{L^2(T^3)} - \left( h \cdot \nabla \omega, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right)_{L^2(T^3)} + \left( \nabla u \cdot A \cdot \nabla h, \Lambda^2 e^{2\tau \Lambda^1 / s} J \right)_{L^2(T^3)} = 0. \quad (31)
\]
Adding (30) and (31) together, we have
\[
\frac{1}{2} \frac{d}{dt} \left\| \Lambda_m^r e^{\tau Lambda_{m}^{1/\epsilon}} \Psi \right\|_{L^2}^2 = \dot{\tau}(t) \left\| \Lambda_m^r \frac{d}{dt} e^{\tau Lambda_{m}^{1/\epsilon}} \Psi \right\|_{L^2}^2 + K_1 + K_2 + K_3,
\]
where \(K_1, K_2, K_3\) are as follows,
\[
K_1 = -\left( u \cdot \nabla \omega - \omega \cdot \nabla u, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} \Psi \right)_{L^2(T^3)}
- \left( u \cdot \nabla J - \nabla h \cdot A \cdot \nabla u, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} J \right)_{L^2(T^3)}
K_2 = \left( h \cdot \nabla J, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} \Psi \right)_{L^2(T^3)} + \left( h \cdot \nabla \omega, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} J \right)_{L^2(T^3)}
K_3 = -\left( J \cdot \nabla h, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} \Psi \right)_{L^2(T^3)} - \left( \nabla u \cdot A \cdot \nabla h, \Lambda_m^r e^{2\tau Lambda_{m}^{1/\epsilon}} J \right)_{L^2(T^3)}.
\]
By the (5) in Lemma 3.4 and (6) in Lemma 3.5, we have
\[
|K_1| \leq C(\tau \left\| \nabla u \right\|_{L^\infty} + \tau^2 \left\| \Psi \right\|_{H^r} + \tau^2 \left\| \Psi \right\|_{X_{r,r,s}}) \left\| \Psi \right\|^2_{Y_{r,r,s}}
+ C\left( \left\| \nabla u \right\|_{L^\infty} \left\| \Psi \right\|_{X_{r,r,s}} + (1 + \tau) \left\| \Psi \right\|^2_{H^r} \right) \left\| \Psi \right\|_{X_{r,r,s}}.
\]
By (24) in the Lemma 3.6, we have
\[
|K_2| \leq C(\tau \left\| \nabla h \right\|_{L^\infty} + \tau^2 \left\| \Psi \right\|_{H^r} + \tau^2 \left\| \Psi \right\|_{X_{r,r,s}}) \left\| \Psi \right\|^2_{Y_{r,r,s}}
+ C\left( \left\| \nabla h \right\|_{L^\infty} \left\| \Psi \right\|_{X_{r,r,s}} + (1 + \tau) \left\| \Psi \right\|^2_{H^r} \right) \left\| \Psi \right\|_{X_{r,r,s}}.
\]
By (25) in the Lemma 3.6, we have
\[
|K_3| \leq C \left\| \nabla h \right\|_{L^\infty} \left\| \Psi \right\|^2_{X_{r,r,s}} + C\tau \left\| \Psi \right\|^2_{H^r} \left\| \Psi \right\|_{X_{r,r,s}}
+ C\tau^2 \left( \left\| \Psi \right\|_{H^r} + \left\| \Psi \right\|_{X_{r,r,s}} \right) \left\| \Psi \right\|^2_{Y_{r,r,s}}.
\]
Substituting \(K_1, K_2, K_3\) into (32), we have
\[
\frac{1}{2} \frac{d}{dt} \left\| \Lambda_m^r e^{\tau Lambda_{m}^{1/\epsilon}} \Psi \right\|_{L^2}^2 \leq \dot{\tau}(t) \left\| \Lambda_m^r \frac{d}{dt} e^{\tau Lambda_{m}^{1/\epsilon}} \Psi \right\|_{L^2}^2
+ C\left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) \left\| \Psi \right\|_{X_{r,r,s}} + (1 + \tau) \left\| \Psi \right\|^2_{H^r} \left\| \Psi \right\|_{X_{r,r,s}}
+ C\left[ \tau \left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) \left\| \Psi \right\|_{X_{r,r,s}} + \tau^2 \left( \left\| \Psi \right\|_{H^r} + \left\| \Psi \right\|_{X_{r,r,s}} \right) \right] \left\| \Psi \right\|^2_{Y_{r,r,s}}.
\]
Taking summation from \(m = 1\) to \(m = 3\), we have
\[
\frac{1}{2} \frac{d}{dt} \left\| \Psi \right\|^2_{X_{r,r,s}} \leq C\left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) \left\| \Psi \right\|_{X_{r,r,s}} + (1 + \tau) \left\| \Psi \right\|^2_{H^r} \left\| \Psi \right\|_{X_{r,r,s}}
+ \dot{\tau} + C\tau \left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) + C\tau^2 \left( \left\| \Psi \right\|_{H^r} + \left\| \Psi \right\|_{X_{r,r,s}} \right) \left\| \Psi \right\|^2_{Y_{r,r,s}}.
\]
If \(\tau(t)\) is a decreasing function of \(t\) such that
\[
\dot{\tau} + C\tau \left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) + C\tau^2 \left( \left\| \Psi \right\|_{H^r} + \left\| \Psi \right\|_{X_{r,r,s}} \right) \leq 0
\]
Then we have
\[
\frac{d}{dt} \left\| \Psi \right\|_{X_{r,r,s}} \leq C\left( \left\| \nabla u \right\|_{L^\infty} + \left\| \nabla h \right\|_{L^\infty} \right) \left\| \Psi \right\|_{X_{r,r,s}} + C(1 + \tau(0)) \left\| \Psi \right\|^2_{H^r}.
\]
By standard \(H^r\)-energy estimate and noting that \(r > 5/2\) one can obtain that there exists a constant \(C > 0\) depending on \(r\) such that
\[
\left\| \Psi(\cdot,t) \right\|_{H^r} \leq \left\| \Psi_0 \right\|_{H^r} \exp \left( \int_0^t C(\left\| \nabla u(\cdot,\sigma) \right\|_{L^\infty} + \left\| \nabla h(\cdot,\sigma) \right\|_{L^\infty}) d\sigma \right),
\]
for $0 < t < T$, for details one can appeal to [2] and the reference therein. We now let the constant $C$ large enough such that (35) holds. By Grownwall’s inequality in (34), we have
\[
\|\Psi(\cdot, t)\|_{X_{r, r(t), s}} \leq G(t) \left[ \|\Psi_0\|_{X_{r, r(0), s}} + C(1 + \tau(0)) \int_0^t \|\Psi(\cdot, \sigma)\|^2_{H^r} G^{-1}(\sigma)d\sigma \right] \\
\leq G(t) \left( \|\Psi_0\|_{X_{r, r_0, s}} + C_{r_0} \|\Psi_0\|^2_{H^r, t} \right) \triangleq M(t),
\]
where we denote
\[
G(t) = \exp \left( \int_0^t C(\|\nabla u(\cdot, \sigma)\|_{L^\infty} + \|\nabla h(\cdot, \sigma)\|_{L^\infty})d\sigma \right),
\]
and $C_{r_0} = C(1 + \tau(0))$. A sufficient condition for (33) to hold is that $\tau$ satisfies
\[
\dot{\tau}(t) + C_1 \tau(t) \geq 0 \quad \text{for } 0 < t < T,
\]
which indeed solves the ordinary differential equation (36) for $\tau$. In particular, since $\|\Psi(\cdot, t)\|^2_{H^r} \leq \|\Psi_0\|^2_{H^r, t} G(t)$, we obtain
\[
\tau(t) \geq G(t)^{-1} \left( \tau_0^{-1} + C \int_0^t \left( \|\Psi_0\|_{H^r} + \|\Psi_0\|_{X_{r_0, r, s}} + C_{r_0} \|\Psi_0\|^2_{H^r, \sigma} \right) d\sigma \right)^{-1} \\
\geq G(t)^{-1} \left( \tau_0^{-1} + C_0 t + C_1 t^2 \right)^{-1}
\]
where $C_0 = C(\|\Psi_0\|_{H^r} + \|\Psi_0\|_{X_{r_0, r, s}})$ and the constant $C_1 = \frac{C C_{r_0} \|\Psi_0\|^2_{H^r}}{2}$. □

Appendix.

The Biot-Savart law. The well-known Biot-Savart law in [13] indicates that the velocity $u$ and magnetic field $h$ can be recovered from the vorticity $\omega$ and the current $J$ respectively by a nonlocal operator.

Since we work in periodic domain and the velocity $u$ and the magnetic field $h$ are divergence free, we have that the vorticity $\omega$ and the current $J$ are divergence free too. With this hypothesis, we have
\[
u = K \ast \omega, \quad h = K \ast J,
\]
where $K$ is a $3 \times 3$ matrix kernel which is defined
\[
K(x) z = \frac{1}{4\pi} \frac{x \times z}{|x|^3}, \quad z \in \mathbb{T}^3.
\]
The velocity $u$ is recovered from the vorticity $\omega$ by
\[
u(x) = \int_{\mathbb{T}^3} K(x - y) \omega(y) dy
\]
where $K$ is homogenous of degree $-2$, i.e., $K(\lambda x) = \lambda^{-2} K(x)$, $x \neq 0$. By the theory of Singular integrals operator, we have for $1 < p < \infty$,
\[
\|\nabla u\|_{L^p} \leq C \|\omega\|_{L^p}.
\]
Galerkin approximation. In this section we give the details of the Galerkin approximation. Denote $L^2_a(T^3)$ the space of divergence-free vector fields in $L^2(T^3)$. Let $\mathcal{P}$ be the well-known Leray projector. Denote $\mathcal{A} = -\mathcal{P}\Delta$ the Stokes operator. It is well known that the eigenvectors $\{E_j\}_{j=1}^\infty$ of $A$ constitute the orthonormal basis of $L^2_a(T^3)$. Let $\{\lambda_j\}_{j=1}^\infty$ be the corresponding eigenvalue

$$AE_j = \lambda_j E_j, \quad j = 1, 2, \ldots.$$  

And they satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots.$$  

We first approximate the equation (1) by the Galerkin method. For fixed integer $m \in \mathbb{N}$, let $\mathbb{P}_m$ be the projection from $L^2_a(T^3)$ onto the subspace $W_m$ spanned by $\{E_j, 1 \leq j \leq m, j \in \mathbb{Z}\}$. We look for solutions $(u_m(t, x), h_m(t, x))$ to the following approximate equations

$$\begin{cases}
\partial_t u_m + \mathbb{P}_m \mathcal{P}(u_m \cdot \nabla u_m) - \mathbb{P}_m \mathcal{P}(h_m \cdot \nabla h_m) = 0, \\
\partial_t h_m + \mathbb{P}_m (u_m \cdot \nabla h_m) - \mathbb{P}_m (h_m \cdot \nabla u_m) = 0, \\
u_m|_{t=0} = \mathbb{P}_m u_0, \quad h_m|_{t=0} = \mathbb{P}_m h_0,
\end{cases} \quad (37)$$

where $u_m(t, x) = \sum_{1 \leq j \leq m} \xi_{j,m}(t)E_j$, $h_m(t, x) = \sum_{1 \leq k \leq m} \eta_{k,m}(t)E_k$.

After taking $L^2$-inner product of both sides of the first two equations of (37) with $E_j$ for $1 \leq j \leq m$. The Cauchy problem (37) is equivalent to the following Cauchy problem of ordinary differential system

$$\begin{cases}
d\xi_{j,m}(t) + \sum_{1 \leq a, b \leq m} A_{a,b,j} \xi_{a,m}(t)\xi_{b,m}(t) - \sum_{1 \leq c, d \leq m} A_{c,d,j} \eta_{c,m}(t)\eta_{d,m}(t) = 0, \\
d\eta_{k,m}(t) + \sum_{1 \leq p, q \leq m} A_{p,q,k} \xi_{p,m}(t)\eta_{q,m}(t) - \sum_{1 \leq l, n \leq m} A_{l,n,k} \eta_{l,m}(t)\xi_{n,m}(t) = 0, \\
\xi_{j,m}(0) = (u_0, E_j), \quad \eta_{k,m}(0) = (h_0, E_k),
\end{cases} \quad (38)$$

where

$$A_{j,k,\ell} = (E_j \cdot \nabla E_k, E_\ell)_{L^2(T^3)}, \quad 1 \leq j, k, \ell \leq m.$$  

The standard ordinary differential equations theory indicates that the ODE system (38) admit a unique local solution $(\xi_{j,m}(t), \eta_{k,m}(t))_{1 \leq j, k \leq m}$ on some interval $[0, T_m]$.

Now we want to prove that the local solution $(\xi_{j,m}(t), \eta_{k,m}(t))_{1 \leq j, k \leq m}$ of the ODE system (38) can be extended to a global in time solution for any fixed $m$. To show this, we note that each $E_j$ are analytical. Then we can perform integration by parts to obtain

$$A_{a,b,j} = (E_a \cdot \nabla E_b, E_j)$$

$$= -(E_a \cdot \nabla E_j, E_b)$$

$$= -A_{a,b,j}, \quad \forall 1 \leq a, b, j \leq m \quad (39)$$

where we used the fact $\nabla \cdot E_a = 0$. Now we perform the energy estimate for the ODE system (38). We multiply the first equation of (38) by $\xi_{j,m}(t)$ and take sum
over \(1 \leq j \leq m, j \in \mathbb{Z}\), then we obtain
\[
\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{m} \xi_{j,m}(t)^2 + \sum_{1 \leq a,b,j \leq m} A_{a,b,j} \xi_{a,m}(t) \xi_{b,m}(t) \xi_{j,m}(t) - \sum_{1 \leq c,d,j \leq m} A_{c,d,j} \eta_{c,m}(t) \eta_{d,m}(t) \xi_{j,m}(t) = 0
\] (40)

We infer from (39) the following fact
\[
\sum_{1 \leq a,b,j \leq m} A_{a,b,j} \xi_{a,m}(t) \xi_{b,m}(t) \xi_{j,m}(t) = - \sum_{1 \leq a,j,b \leq m} A_{a,j,b} \xi_{a,m}(t) \xi_{j,m}(t) \xi_{b,m}(t) = 0
\] (41)

If we multiply the second equation of (38) by \(\eta_{k,m}(t)\) and take sum over \(1 \leq k \leq m, k \in \mathbb{Z}\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \sum_{k=1}^{m} \eta_{k,m}(t)^2 + \sum_{1 \leq p,q,k \leq m} A_{p,q,k} \xi_{p,m}(t) \eta_{q,m}(t) \eta_{k,m}(t) - \sum_{1 \leq \ell,n,k \leq m} A_{\ell,n,k} \eta_{\ell,m}(t) \eta_{n,m}(t) \eta_{k,m}(t) = 0
\] (42)

From (39) we also have
\[
\sum_{1 \leq p,q,k \leq m} A_{p,q,k} \xi_{p,m}(t) \eta_{q,m}(t) \eta_{k,m}(t) = - \sum_{1 \leq p,k,q \leq m} A_{p,k,q} \xi_{p,m}(t) \eta_{k,m}(t) \eta_{q,m}(t) = 0
\]
and
\[
\sum_{1 \leq c,d,j \leq m} A_{c,d,j} \eta_{c,m}(t) \eta_{d,m}(t) \xi_{j,m}(t) + \sum_{1 \leq \ell,n,k \leq m} A_{\ell,n,k} \eta_{\ell,m}(t) \eta_{n,m}(t) \eta_{k,m}(t) = 0
\]

With the above observations, we take summation of (40) and (42) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{j=1}^{m} \xi_{j,m}(t)^2 + \eta_{j,m}(t)^2 \right) = 0.
\] (43)

The equation (43) implies that, for any \(m \in \mathbb{N}\),
\[
\|u_m(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 + \|h_m(t, \cdot)\|_{L^2(\mathbb{T}^3)}^2 \leq \|u_0\|_{L^2(\mathbb{T}^3)}^2 + \|h_0\|_{L^2(\mathbb{T}^3)}^2.
\]

The above priori estimates show that the solution \((\xi_{j,m}(t), \eta_{k,m}(t))\) is bounded only by the initial data, then we can repeat the arguments above to extend the local solution to arbitrary time interval \([0,T]\). Thus for fixed integer \(m\), the Cauchy problem of the ODE system (38) possess a unique solution \((\xi_{j,m}(t), \eta_{k,m}(t)) \in C^1([0,T])\) for all \(T > 0\). Equivalently, we obtain that the solution \((u_m(t,x), h_m(t,x))\) to the approximate equations satisfies
\[
u_m(t, \cdot), h_m(t, \cdot) \in C^1([0,T], C^\infty(\mathbb{T}^3)), \quad \forall T > 0,
\]

for fixed \(m\).

We have obtained the existence of the solution \((u_m(t,x), h_m(t,x))\) to the approximate equation (37). If we let \(\omega_m(t,x) = \text{curl} \, u_m(t,x), J_m(t,x) = \text{curl} \, h_m(t,x),\) then \((\omega_m(t,x), J_m(t,x))\) satisfies the equation
\[
\begin{aligned}
\partial_t \omega_m + \text{curl} \, \mathbb{P}_m \mathbb{P}(u_m \cdot \nabla u_m) - \text{curl} \, \mathbb{P}_m \mathbb{P}(h_m \cdot \nabla h_m) &= 0, \\
\partial_t J_m + \text{curl} \, \mathbb{P}_m (u_m \cdot \nabla h_m) - \text{curl} \, \mathbb{P}_m (h_m \cdot \nabla u_m) &= 0, \\
\omega_m|_{t=0} = \text{curl} \, \mathbb{P}_m u_0, & \quad J_m|_{t=0} = \text{curl} \, \mathbb{P}_m h_0,
\end{aligned}
\] (44)
where
\[ \omega_m = \sum_{1 \leq j, m \leq m} \xi_{j,m}(t) \text{curl} E_j, \quad J_m = \sum_{1 \leq k, m \leq m} \eta_{k,m}(t) \text{curl} E_k. \]

We note that \( \{E_j, j = 1, 2, \ldots\} \) are orthonormal in \( L^2_\sigma \), which means
\[ (E_j, E_k)_{L^2(T^3)} = \delta_{jk}, \quad j, k = 1, 2, 3, \ldots. \]

Then we obtain that
\[ (\text{curl} E_j, \text{curl} E_k)_{L^2(T^3)} = (E_j, -\Delta E_k)_{L^2(T^3)} = \lambda_k (E_j, E_k) = \lambda_k \delta_{jk}, \]
where \( \lambda_k \) is the \( k \)-th eigenvalue of the Stokes operator.

We note that it has already been proved that \( (u_m, h_m) \) converges to a solution \( (u, h) \) in \( L^\infty(0, T^\infty) \) as \( m \) goes to infinity, see [16]. If we choose \( \tau(t) \) to satisfy the condition \( (36) \), and then perform the Gevrey norm energy estimate for the approximate equation \( (44) \) to obtain
\[ (\omega_m(t, x), J_m(t, x)) \in L^\infty([0, T], X_{r, \tau, s}), \]
and
\[ (\partial_t \omega_m(t, x), \partial_t J_m(t, x)) \in L^\infty([0, T], X_{0, \tau, s}). \]

Then by the compactness method we obtain the limit
\[ (\omega(t, \cdot), J(t, \cdot)) \in L^\infty([0, T], X_{r, \tau(t), s}). \]

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