Hamiltonian systems and Sturm-Liouville equations: Darboux transformation and applications

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Abstract

We introduce GBDT version of Darboux transformation for symplectic and Hamiltonian systems as well as for Shin-Zettl systems and Sturm-Liouville equations. These are the first results on Darboux transformation for general-type Hamiltonian and for Shin-Zettl systems. The obtained results are applied to the corresponding transformations of the Weyl-Titchmarsh functions and to the construction of explicit solutions of dynamical symplectic systems, of two-way diffusion equations and of indefinite Sturm-Liouville equations. The energy of the explicit solutions of dynamical systems is expressed (in a quite simple form) in terms of the parameter matrices of GBDT.

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1 Introduction

This paper is dedicated to the study of the important subclasses of the first order differential systems with a spectral parameter $\lambda$. Namely, we consider
Hamiltonian systems
\[ \frac{d}{dx} y(x, \lambda) = F(x, \lambda) y(x, \lambda), \quad F(x, \lambda) = J(\lambda H_1(x) + H_0(x)), \quad (1.1) \]
where
\[ J^* = -J, \quad H_1(x)^* = H_1(x), \quad H_0(x) = H_0(x)^*, \quad H_1(x) \geq 0; \quad (1.2) \]
and so called Shin-Zettl systems
\[ \frac{d}{dx} y(x, \lambda) = F(x, \lambda) y(x, \lambda), \quad F(x, \lambda) = \begin{bmatrix} r_1(x) & p(x)^{-1} \\ q(x) - \lambda \omega(x) & r_2(x) \end{bmatrix}. \quad (1.3) \]
Here \( J^* \) is the conjugate transpose of the matrix \( J \). We assume that the \( m \times m \) \((m \in \mathbb{N})\) matrix functions \( H_1(x) \) and \( H_0(x) \) in (1.1) and the functions \( p^{-1}, q, r_1, r_2 \) and \( \omega \) in (1.3) are locally summable on \([0, \ell] \) \((\ell \leq \infty)\). The matrix function \( F \) in (1.3) is the \( 2 \times 2 \) Shin-Zettl matrix of general form (see, e.g., § 2 in [13] or in [14]). We note that Shin-Zettl differential expressions were introduced in [47, 50] and were actively studied in regularization and spectral theories (see the books [151], papers [13,14], recent surveys [35,52] and various references therein). The Lagrange-symmetric case
\[ \omega = \overline{\omega}, \quad p = \overline{p}, \quad q = \overline{q}, \quad r_1 = -\overline{r_2} \quad (1.4) \]
and the Lagrange-J-symmetric case
\[ r_1 = -r_2 \quad (1.5) \]
are of special interest [14]. Here \( \overline{\mu} \) stands for the value which is complex conjugate to \( \mu \).

The entries of the \( 2 \times 1 \) vector function \( y \) in (1.3) are denoted by \( y_1 \) and \( y_2 \). When \( r_1 \equiv r_2 \equiv 0 \), we rewrite (1.3) in the form
\[ y_1' = p^{-1} y_2, \quad y_2' = (q - \lambda \omega)y_1 \quad \left( y_k' = \frac{d}{dx} y_k \right), \quad (1.6) \]
which is equivalent to the Sturm-Liouville equation
\[ -\left( p(x)u'(x, \lambda) \right)' + q(x)u(x, \lambda) = \lambda \omega(x)u(x, \lambda), \quad (1.7) \]
where \( u = y_1 \). If \( \omega = \overline{\omega}, \ p = \overline{p} \) and \( \omega \) or \( p \) change signs, one speaks about *indefinite Sturm-Liouville problem*. Quasi-derivatives related to the quasi-derivatives generated by Shin-Zettl systems are used in the study of important modifications of Schrödinger-type operators (see, e.g., \cite{12,46} and references therein) including Schrödinger-type operators with distributional potentials \cite{12}.

On the other hand, Lagrange-symmetric Shin-Zettl systems, where \( \omega \geq 0 \), form also a subclass of Hamiltonian systems. See, for instance, \cite{21} on the representation \((1.1),\ (1.2)\) of Hamiltonian systems and the equivalence of the definite Sturm-Liouville equation to a certain subclass of Hamiltonian systems. We note that the book \cite{2} by Atkinson, the papers by Hinton and Shaw as well the Kac-Krein supplement \cite{23} (to the translation of \cite{2}) presented seminal developments in the theory of Hamiltonian systems and Sturm-Liouville equations. (For recent references on Hamiltonian systems see, e.g., \cite{24,30,34,48}.) In some works, conditions \((3.1)\) are added in the definition of Hamiltonian systems but these conditions are absent in \cite{21} and they are not essential for Darboux transformations, which we will construct here, as well.

In this paper we construct our GBDT version of the Bäcklund-Darboux transformation (see the results and references in \cite{39,41,43}) for the cases of Hamiltonian and Shin-Zettl systems in order to study perturbations of these systems and corresponding transformations of the Weyl-Titchmarsh functions. We construct explicit solutions of the perturbed systems as well. Several versions of Bäcklund-Darboux transformations (see, e.g., \cite{8,20,33,43} and references therein) are a well-known tool for the construction of explicit solutions of linear and integrable nonlinear equations. GBDT as well as Crum-Krein and commutation methods (which are related to Bäcklund-Darboux transformations) are also essential in the study of Weyl-Titchmarsh theory and important spectral problems \cite{10,11,16,17,19,27,30,34,42}.

As far as we know, neither Bäcklund-Darboux transformations nor commutation methods were applied to general-type Hamiltonian systems \((1.1)\) and to Shin-Zettl systems \((1.3)\) before (although commutation and Bäcklund-Darboux transformations for such important particular cases as Schrödinger equations, canonical systems and related Dirac equations are well-known).
We mention an interesting paper [5] on Kummer-Liouville transformation for Shin-Zettl systems but that transformation is different and was applied with different purposes.

Darboux transformation for symplectic and general-type Hamiltonian systems is introduced in Section 2. The corresponding transformations of the Weyl-Titchmarsh functions are considered in Section 3. GBDT for Shin-Zettl systems and Sturm-Liouville equations is introduced in Sections 4-6. Explicit solutions of dynamical symplectic systems and of two-way diffusion equations are constructed in Section 7. Finally, explicit solutions of indefinite Sturm-Liouville equations are considered in Section 8.

As usual, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{C} \) denotes the complex plane, \( \mathbb{C}_+ \) is the open upper half-plane \( \{ \lambda : \Im(\lambda) > 0 \} \) and \( \mathbb{C}_- \) is the open lower half-plane \( \{ \lambda : \Im(\lambda) < 0 \} \). The notation \( I_n \) stands for the \( n \times n \) identity matrix, \( H^* \) is the conjugate transpose of the matrix \( H \), the inequality \( H \geq 0 \) means that \( H = H^* \) and that all the eigenvalues of the matrix \( H \) are nonnegative.

2 GBDT for Hamiltonian systems

1. Our GBDT version of Bäcklund-Darboux transformation for system (1.1) is a particular case of GBDT for systems with rational dependence on spectral parameter (see, e.g., [41] or [43, Sect. 7.2]). We start with introducing GBDT for the first order system of \( m \) differential equations with a linear dependence on the spectral parameter (\( m \in \mathbb{N} \)):

\[
y'(x, \lambda) = F(x, \lambda)y(x, \lambda), \quad F(x, \lambda) = -\left( \lambda Q_1(x) + Q_0(x) \right). \tag{2.1}
\]

For that purpose we fix some initial system (2.1) (i.e., some \( m \times m \) matrix functions \( Q_1(x) \) and \( Q_0(x) \), which are locally summable on \([0, \ell] \)), an integer \( n \in \mathbb{N} \) and five parameter matrices, namely, \( n \times n \) matrices \( A_1, A_2 \) and \( S(0) \), and \( n \times m \) matrices \( \Pi_1(0) \) and \( \Pi_2(0) \) such that the matrix identity

\[
A_1 S(0) - S(0) A_2 = \Pi_1(0) \Pi_2(0)^* \tag{2.2}
\]
holds. Matrix functions $\Pi_1(x)$, $\Pi_2(x)$ and $S(x)$ are introduced by their initial values $\Pi_1(0)$, $\Pi_2(0)$, $S(0)$ and differential equations

$$\Pi_1' = A_1\Pi_1 Q_1 + \Pi_1 Q_0, \quad (\Pi_2^*)' = -Q_1\Pi_2 A_2 - Q_0\Pi_2, \quad S' = \Pi_1 Q_1\Pi_2^*. \quad (2.3)$$

The identity

$$A_1 S(x) - S(x) A_2 = \Pi_1(x)\Pi_2(x)^*, \quad (2.4)$$

for all $x \in [0, \ell)$, is a particular case of [43] f-la (7.18) and easily follows from (2.2) and (2.3).

When we deal with $S(x)^{-1}$, our further statements are valid in the points of invertibility of $S(x)$. The questions of invertibility of $S(x)$ are discussed in our sections separately (see, e.g., Remarks 2.3 and 8.1).

According to the subcase $r = 1$, $l = 0$ of [43] Theor. 7.4, the so called Darboux matrix for system (2.1) is given by the formula

$$w_A(x, \lambda) = I_m - \Pi_2(x)^* S(x)^{-1}(A_1 - \lambda I_n)^{-1}\Pi_1(x). \quad (2.5)$$

More precisely, [43] Theor. 7.4 yields that $w_A$ satisfies the following equation

$$\frac{d}{dx}w_A(x, \lambda) = \tilde{F}(x, \lambda)w_A(x, \lambda) - w_A(x, \lambda)F(x, \lambda), \quad (2.6)$$

where

$$\tilde{F}(x, \lambda) := -(\lambda Q_1(x) + \tilde{Q}_0(x)), \quad (2.7)$$

$$\tilde{Q}_0(x) := Q_0(x) - (Q_1(x)X(x) - X(x)Q_1(x)), \quad (2.8)$$

$$X(x) := \Pi_2(x)^*S(x)^{-1}\Pi_1(x). \quad (2.9)$$

We note that (in view of (2.4)) the matrix function $w_A(\lambda)$ of the form (2.5) is (for each $x$) the so called transfer matrix function in Lev Sakhnovich form (see [43–45] and references therein).

System $y' = \tilde{F}y$ is called the transformed (GBDT-transformed) system (recall that (2.1) is the initial system). An important step in the proof of (2.6) is the proof of the equation

$$(\Pi_2^*S^{-1})' = -Q_1\Pi_2^*S^{-1}A_1 - \tilde{Q}_0\Pi_2^*S^{-1}. \quad (2.10)$$
See [43, f-la (7.61)] for the general formula, of which (2.10) is a particular case. We shall use (2.10) in Section 7.1.

Formula (2.6) implies the following theorem.

**Theorem 2.1** Let \( y(x, \lambda) \) satisfy system (2.1) and let \( w_A \) be given by (2.5), where the matrix functions \( \Pi_1, \Pi_2 \) and \( S \) are determined by (2.3) and identity (2.2) holds. Then the function

\[
\tilde{y}(x, \lambda) := w_A(x, \lambda)y(x, \lambda)
\]

satisfies, in the points of invertibility of \( S(x) \), another (transformed) first order system

\[
\frac{d}{dx}\tilde{y}(x, \lambda) = \tilde{F}(x, \lambda)\tilde{y}(x, \lambda),
\]

where \( \tilde{F}(x, \lambda) \) is given by (2.7)–(2.9).

2. The most important subcase of the considered above GBDT-transformations is the subcase of the initial system (2.1) such that

\[
\begin{align*}
Q_1(x) &= -JH_1(x), \quad Q_0(x) = -JH_0(x), \\
J^* &= -J, \quad H_1(x)^* = H_1(x), \quad H_0(x) = H_0(x)^*.
\end{align*}
\]  

(2.13)  

(2.14)

In that subcase we deal with system (1.1), where all the conditions (1.2) on Hamiltonian system, excluding the nonnegativity condition \( H_1(x) \geq 0 \), hold. If \( J^* = J^{-1} \) (e.g., \( J \) has the form (3.1)) conditions (2.14) mean that system (1.1) is symplectic. Further in the paragraphs 2 and 3 we assume that the equalities (2.13) and (2.14) are valid.

We omit indices in \( A_1 \) and \( \Pi_1 \) and set

\[
A = A_1, \quad \Pi = \Pi_1; \quad A_2 = A^*, \quad S(0) = S(0)^*, \quad \Pi_2(0) = -\Pi(0)J.
\]

(2.15)

Using (2.13)–(2.15) we rewrite the first and second equations in (2.3), correspondingly, in the forms

\[
\begin{align*}
(-\Pi J)' &= -A(-\Pi J)H_1J - (-\Pi J)H_0J, \\
(\Pi_2)' &= -A\Pi_2H_1J - \Pi_2H_0J.
\end{align*}
\]
Thus, the equations on \(-\Pi J\) and on \(\Pi_2\) coincide, and, in view of \(\Pi_2(0) = -\Pi(0)J\) we obtain \(\Pi_2(x) \equiv -\Pi(x)J\). In this way, equations (2.3) are reduced to the equations

\[
\Pi' = -A\Pi JH_1(x) - \Pi JH_0(x), \quad S'(x) = \Pi JH_1(x)J^*\Pi(x)^*.
\]  

(2.16)

Since we assume in (2.15) that \(S(0) = S(0)^*\), the second equation in (2.16) yields \(S(x) = S(x)^*\). Thus, we have

\[
\Pi_2(x) \equiv -\Pi(x)J, \quad S(x) = S(x)^*.
\]  

(2.17)

Now, the matrix identity (2.4) and Darboux matrix (2.5) are rewritten in the form

\[
AS(x) - S(x)A^* = \Pi(x)J\Pi(x)^*,
\]  

(2.18)

\[
w_A(x, \lambda) = I_m - J\Pi(x)^*S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x).
\]  

(2.19)

Moreover, using (2.13) and the equalities \(\Pi_2(x) \equiv -\Pi(x)J\) and \(J^* = -J\), we rewrite (2.7)–(2.9) in the form

\[
\tilde{F}(x, \lambda) = J(\lambda H_1(x) + \tilde{H}_0(x)), \quad \tilde{H}_0(x) = H_0(x) + Z(x),
\]  

(2.20)

\[
Z(x) := \Pi(x)^*S(x)^{-1}\Pi(x)JH_1(x) + H_1(x)J^*\Pi(x)^*S(x)^{-1}\Pi(x).
\]  

(2.21)

Formulas (2.14), (2.20) and (2.21) imply that \(\tilde{H}_0 = \tilde{H}_0^*\), that is, \(\tilde{F}\) has the same form as \(F\). Hence, the next proposition follows from Theorem 2.1.

**Proposition 2.2** Let \(y(x, \lambda)\) satisfy system (1.1) (such that (2.14) holds), and let a triple \(\{A, S(0) = S(0)^*, \Pi(0)\}\) of parameter matrices satisfying (2.18) at \(x = 0\) be given. Introduce \(w_A(x, \lambda)\) by (2.19), where the matrix functions \(\Pi(x)\) and \(S(x)\) are determined by (2.16).

Then the function \(\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda)\) satisfies, in the points of invertibility of \(S(x)\), another (transformed) system of the same form as (1.1), namely,

\[
\frac{d}{dx}\tilde{y}(x, \lambda) = \tilde{F}(x, \lambda)\tilde{y}(x, \lambda),
\]  

(2.22)

where \(\tilde{F}(x, \lambda)\) is given by (2.20), (2.21) and the equality \(\tilde{H}_0 = \tilde{H}_0^*\) holds.

If \(H_1(x) \geq 0\) and \(S(0) > 0\), the systems (1.1) and (2.22) are Hamiltonian.
Remark 2.3 If system \((1.1), (2.14)\) is Hamiltonian (i.e., \(H_1(x) \geq 0\)) and, in addition, the inequality \(S(0) > 0\) holds, formula \((2.16)\) shows that \(S(x) > 0\) for all \(x \in [0, \ell)\). Therefore, \(S(x)\) is invertible on \([0, \ell)\). In particular, it follows that the system \((2.22)\) is, indeed, Hamiltonian.

3. If in the system \((1.1)\) we have \(J = -J^* = -J^{-1}\) and \(H_0 \equiv 0\), we come to the important class of canonical systems. See GBDT for canonical system and its applications to Weyl-Titchmarsh theory in \([40]\). For the case of Hamiltonian systems with invertible \(J\) we can (similar to the case of canonical systems) consider transformation slightly different from \((2.20), (2.21)\). More precisely, we introduce matrix functions \(\hat{w}(x)\) and \(v(x, \lambda)\) by the formulas

\[
\hat{w}'(x) = -\hat{w}(x)JZ(x), \quad \hat{w}(0) = I_m; \quad v(x, \lambda) = \hat{w}(x)w_A(x, \lambda).
\]  

(2.23)

It is easy to see that \(\hat{w}(x)J\hat{w}(x)^* = J\), and so

\[
\hat{w}(x)^{-1} = J\hat{w}(x)^*J^{-1} = J^*\hat{w}(x)^*(J^*)^{-1}.
\]  

(2.24)

In view of Proposition 2.2 and relations \((2.23)\) and \((2.24)\), if \(y(x, \lambda)\) satisfies \((1.1)\), then the matrix function \(\hat{y}(x, \lambda) = v(x, \lambda)y(x, \lambda)\) satisfies the system

\[
\frac{d}{dx}\hat{y}(x, \lambda) = \hat{F}(x, \lambda)\hat{y}(x, \lambda), \quad \hat{F}(x, \lambda) = J(\lambda\hat{H}_1(x) + \hat{H}_0(x)),
\]  

(2.25)

where

\[
\hat{H}_1 = J^{-1}\hat{w}JH_1\hat{w}^{-1} = J^{-1}\hat{w}JH_1J^*\hat{w}^*(J^*)^{-1} = \hat{H}_1^*,
\]  

(2.26)

\[
\hat{H}_0 = J^{-1}\hat{w}J(\hat{H}_0 - Z)\hat{w}^{-1} = J^{-1}\hat{w}JH_0J^*\hat{w}^*(J^*)^{-1} = \hat{H}_0^*.
\]  

(2.27)

In the special case \(H_0 = icJ^{-1} (c = \overline{c})\), the formula \((2.27)\) is simplified and we obtain \(\hat{H}_0 \equiv icJ^{-1}\).

3 Darboux transformations of Weyl-Titchmarsh functions

In his important paper \([28]\), Krall introduced Weyl-Titchmarsh (or simply Weyl) \(M(\lambda)\)-functions of Hamiltonian systems in the classical terms of “Weyl
circle” inequalities. Here, Weyl circles of system (1.1) on the intervals \([0, \ell']\) \((\ell' < \ell)\) and the values \(\lambda\) in the upper half-plane \(\lambda \in \mathbb{C}_+\) (i.e., \(\Im(\lambda) > 0\)) are considered. The Weyl circles in the lower half-plane \(\mathbb{C}_-\) are treated in a quite similar way and we omit that case.

Krall required that \(m\) is even and that \(J\) in (1.1) has a special form:

\[
m = 2r \quad (r \in \mathbb{N}), \quad J = \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}. \tag{3.1}
\]

In fact, Hamiltonian system in [28] is written in a slightly different from (1.1) way and our \(J^*\) stands for \(J\) in an equivalent to (1.1) system in [28]. Rewriting correspondingly the inequality for the Weyl circle (of matrices \(M(\lambda)\) with \(\lambda \in \mathbb{C}_+\)) from [28, p. 670], we obtain

\[
i \begin{bmatrix} I_r & M(\lambda)^* \end{bmatrix} Y(\ell', \lambda)^* J Y(\ell', \lambda) \begin{bmatrix} I_r \\ M(\lambda) \end{bmatrix} \leq 0. \tag{3.2}
\]

Here \(Y(x, \lambda)\) is the fundamental \(m \times m\) solution of the Hamiltonian system (1.1) (such that (1.2) and (3.1) are valid), normalized by the initial condition

\[
Y(0, \lambda) = E \quad (EJ = JE, \quad E^*E = I_m). \tag{3.3}
\]

According to Proposition 2.2, the fundamental solution \(\tilde{Y}(x, \lambda)\) (normalized by \(\tilde{Y}(0, \lambda) = E\)) of the transformed Hamiltonian system (2.22) is given by the formula

\[
\tilde{Y}(x, \lambda) = w_A(x, \lambda) Y(x, \lambda) E^*w_A(0, \lambda)^{-1}E. \tag{3.4}
\]

Let us set

\[
\mathcal{U}(\lambda) = \{U_{ij}(\lambda)\}_{i,j=1}^2 := E^*w_A(0, \lambda)E, \tag{3.5}
\]

where \(U_{ij}(\lambda)\) are \(r \times r\) blocks of \(\mathcal{U}\). In view of (3.4) and (3.5), the Weyl circle (of matrices \(\tilde{M}(\lambda)\)) for the transformed system on \([0, \ell']\) and for \(\lambda \in \mathbb{C}_+\) is determined by the inequality

\[
i \begin{bmatrix} I_r & \tilde{M}(\lambda)^* \end{bmatrix} \left(\mathcal{U}(\lambda)^{-1}\right)^* Y(\ell', \lambda)^* w_A(x, \lambda)^* J w_A(x, \lambda) Y(\ell', \lambda) \mathcal{U}(\lambda)^{-1} \begin{bmatrix} I_r \\ \tilde{M}(\lambda) \end{bmatrix} \leq 0. \tag{3.6}
\]
Relations (2.18) and (2.19) yield the following identity \[ f-la (1.88) \]:
\[
iw_A(x, \lambda) \ast Jw_A(x, \lambda) = iJ + i(\lambda - \overline{\lambda})\Pi(x) \ast (A^* - \overline{A}I_n)^{-1}S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x).
\tag{3.7}
\]

In this section we consider Hamiltonian systems and assume that \( S(0) > 0 \). Hence, according to Remark 2.3 we have \( S(x) > 0 \). Now, it is immediate from (3.7) that
\[
iw_A(x, \lambda) \ast Jw_A(x, \lambda) \leq iJ \quad (\lambda \in \mathbb{C}_+).
\tag{3.8}
\]

Formula (3.8) implies that
\[
i\left[ I_r \ M(\lambda)^* \right] (U(\lambda)^{-1})^*Y(\ell', \lambda)^*w_A(x, \lambda)^*Jw_A(x, \lambda)Y(\ell', \lambda)U(\lambda)^{-1} \left[ \frac{I_r}{M(\lambda)} \right]
\leq i\left[ I_r \ M(\lambda)^* \right] (U(\lambda)^{-1})^*Y(\ell', \lambda)^*JY(\ell', \lambda)U(\lambda)^{-1} \left[ \frac{I_r}{M(\lambda)} \right] \tag{3.9}
\]

Using (3.9), we derive the next theorem.

**Theorem 3.1** Let Hamiltonian system (1.1) (such that (1.2) and (3.1) are valid) be given. Let its GBDT transformation be determined by the triple of matrices \( \{A, S(0), \Pi(0)\} \) such that \( S(0) > 0 \) and that the matrix identity
\[
AS(0) - S(0)A^* = \Pi(0)J\Pi(0)^*
\tag{3.10}
\]
holds. Assume that \( M(\lambda) \ (\lambda \in \mathbb{C}_+) \) belongs to the Weyl circle (3.2) of the system (1.1) and that
\[
det (U_{11}(\lambda) + U_{12}(\lambda)M(\lambda)) \neq 0,
\tag{3.11}
\]
where \( U \) is defined in (3.5). Then
\[
\tilde{M}(\lambda) = (U_{21}(\lambda) + U_{22}(\lambda)M(\lambda))(U_{11}(\lambda) + U_{12}(\lambda)M(\lambda))^{-1}
\tag{3.12}
\]
belongs to the Weyl circle of the transformed system.
Proof. Taking into account (3.11) and (3.12), we obtain

\[
I_r \widetilde{M}(\lambda) = U(\lambda) \left[ \begin{array}{c} I_r \\ M(\lambda) \end{array} \right] \left( U_{11}(\lambda) + U_{12}(\lambda)M(\lambda) \right)^{-1}.
\] \hspace{1cm} (3.13)

Now, substitute (3.13) into the right-hand side of (3.9) and use (3.2) in order to see that (3.6) is valid. ■

According to [28, p. 671], we have \(i(M(\lambda) - M(\lambda)^*) \leq 0\). Moreover, we have

\[i(M(\lambda) - M(\lambda)^*) < 0, \] \hspace{1cm} (3.14)

if only \(\int_0^{\lambda'} y(x, \lambda)^*H_1(x)y(x, \lambda)\,dx > 0\) for each nontrivial solution \(y\) of (1.1).

Remark 3.2 If (3.14) is valid, then the inequality (3.11) holds automatically. Indeed, if \(\det (U_{11}(\lambda) + U_{12}(\lambda)M(\lambda)) = 0\), then there is a vector \(f \neq 0\) such that \((U_{11}(\lambda) + U_{12}(\lambda)M(\lambda))f = 0\). Therefore, recalling that \(J\) has the form (3.1), we obtain

\[ if^*[I_r \ M(\lambda)^*] U(\lambda)^*J U(\lambda) \left[ \begin{array}{c} I_r \\ M(\lambda) \end{array} \right] f = 0 \quad (f \neq 0). \] \hspace{1cm} (3.15)

On the other hand, relations (3.5) and (3.8) (together with the properties of \(E\) from (3.3)) imply that \(U(\lambda)^*J U(\lambda) \leq iJ\). Hence, using (3.14), we derive

\[ i[I_r \ M(\lambda)^*] U(\lambda)^*J U(\lambda) \left[ \begin{array}{c} I_r \\ M(\lambda) \end{array} \right] \leq i[I_r \ M(\lambda)^*] J \left[ \begin{array}{c} I_r \\ M(\lambda) \end{array} \right] < 0, \] \hspace{1cm} (3.16)

which contradicts (3.15).

In the limit point case (see, e.g., the discussions in [22, 29]) there is a unique holomorphic in \(C_+\) Weyl function \(M(\lambda)\) the values of which belong to all the Weyl circles (3.2) such that \(\ell' < \ell\) (\(\lambda \in C_+\)). We note that \(M(\lambda)\) is the limit of the values of \(M(\lambda)\) when \(\ell'\) tends to \(\ell\). Thus, formula (3.12) shows that

\[
\widetilde{M}(\lambda) := (U_{21}(\lambda) + U_{22}(\lambda)M(\lambda))(U_{11}(\lambda) + U_{12}(\lambda)M(\lambda))^{-1}
\] \hspace{1cm} (3.17)

is a Weyl function of the transformed system considered on \([0, \ell]\).
4 GBDT for Shin-Zettl systems

Shin-Zettl systems (1.3) present (as well as Hamiltonian systems) an important subclass of systems (2.1). Matrices $Q_1$ and $Q_2$, in the case of Shin-Zettl systems, have the form

$$Q_1(x) = \begin{bmatrix} 0 & 0 \\ \omega(x) & 0 \end{bmatrix}, \quad Q_0(x) = -\begin{bmatrix} r_1(x) & p(x)^{-1} \\ q(x) & r_2(x) \end{bmatrix}. \quad (4.1)$$

Recall that GBDT is determined by the parameter matrices $A_1$, $A_2$, $S(0)$, $\Pi_1(0)$ and $\Pi_2(0)$ such that (2.2) holds. For the Shin-Zettl systems, we have $m = 2$, and so matrices $\Pi_1(0)$ and $\Pi_2(0)$ are $n \times 2$ matrices. Using the second equality in (1.3) and the first equality in (4.1), we rewrite $\tilde{F}$ given by (2.7)–(2.9) in the Shin-Zettl form

$$\tilde{F}(x, \lambda) = \begin{bmatrix} \tilde{r}_1(x) \\ \tilde{q}(x) - \lambda \tilde{\omega}(x) \\ \tilde{r}_2(x) \end{bmatrix}, \quad \tilde{\omega} = \omega, \quad \tilde{p} = p; \quad (4.2)$$

$$\tilde{r}_1 = r_1 - \omega X_{12}, \quad \tilde{r}_2 = r_2 + \omega X_{12}, \quad \tilde{q} = q + \omega (X_{11} - X_{22}). \quad (4.3)$$

where $X_{ik}(x)$ are the entries of $X(x)$. Now, the following proposition is immediate from Theorem 2.1.

**Proposition 4.1** Let $y(x, \lambda)$ satisfy Shin-Zettl system (1.3) and let $w_A$ be given by (2.5), where the matrix functions $\Pi_1$, $\Pi_2$ and $S$ are determined by (2.3) and identity (2.2) holds. Then the function $\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda)$ satisfies, in the points of invertibility of $S(x)$, the transformed Shin-Zettl system (2.12), where $\tilde{F}(x, \lambda)$ is given by (4.2) and (4.3).

The next corollary easily follows from Proposition 4.1.

**Corollary 4.2** Let the conditions of Proposition 4.1 hold and let the initial system (1.3) be Lagrange-J-symmetric (i.e., let (1.5) be valid). Then the transformed system is Lagrange-J-symmetric as well, that is, the equality $\tilde{r}_1 = -\tilde{r}_2$ holds.

In the next section, we consider the Lagrange-symmetric case (i.e., the case (1.4)).
5 Lagrange-symmetric case

Further we assume that (1.4) is fulfilled and rewrite (4.1) for that case:

\[ Q_1(x) = \begin{bmatrix} 0 & 0 \\ \omega(x) & 0 \end{bmatrix}, \quad Q_0(x) = - \begin{bmatrix} r(x) & p(x)^{-1} \\ q(x) & -r(x) \end{bmatrix}, \quad (5.1) \]
\[ r(x) := r_1(x) = -r_2(x). \quad (5.2) \]

Now, system (1.3) may be rewritten as the quasi-differential equation:

\[ - (u^{[1]})' - \tau u^{[1]} + qu = \lambda \omega u, \quad u^{[1]} := p(u' - ru), \quad (5.3) \]

where \( y_1(x) = u(x), y_2(x) = u^{[1]}(x) \) and the quasi-differential expression

\[ Mu = - (u^{[1]})' - \tau u^{[1]} + qu - \lambda \omega u \]

is symmetric (see, e.g., [14]). See also [6, 37, 51] and references therein on symmetric expressions \((- (u^{[1]})' - \tau u^{[1]} + qu)/\omega\) in the weighted spaces \(L^2_{\omega}|0, \ell\) and \(L^2_{\omega}[0, \ell]\). Using the quasi-derivative \(u^{[1]}\) one may consider Sturm-Liouville equations (including self-adjoint Sturm-Liouville equations) with non-smooth coefficients (see, e.g., the discussions in [50, p. 455] and in [51, p. 25]).

We note that \(Q_1\) and \(Q_0\) given by (5.1) admit representation (2.13), where

\[ J = i\sigma_2, \quad H_1(x) = \begin{bmatrix} \omega(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad H_0(x) = \begin{bmatrix} -q(x) & r(x) \\ r(x) & p(x)^{-1} \end{bmatrix}, \quad (5.4) \]

\[ \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \]

is a Pauli matrix, and (2.13) holds. In fact, conditions (2.13) and (2.14) (in the Shin-Zettl case and with \( J = i\sigma_2 \)) are equivalent to the conditions (1.4) of Lagrange symmetry. (Clearly, when \( \omega \geq 0 \) we deal with a subclass of Hamiltonian systems.) Thus, omitting the indices in \(A_1\) and \(\Pi_1\) and rewriting (2.15) in the form

\[ A = A_1, \quad \Pi = \Pi_1; \quad A_2 = A^*, \quad S(0) = S(0)^*, \quad \Pi_2(0) = -i\Pi(0)\sigma_2, \quad (5.5) \]

we see that the formulas of §2 in Section 2 are valid for Lagrange-symmetric case.
Since $\Pi_2(x) = -i\Pi(x)\sigma_2$, formula (2.9) for $X$ may be rewritten as
\[ X(x) = J\Pi(x)^*S(x)^{-1}\Pi(x), \quad J = i\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (5.6) \]
and we obtain
\[ X_{12}(x) = \overline{X_{12}(x)}, \quad X_{22}(x) = -\overline{X_{11}(x)}. \quad (5.7) \]
Recall that $\Pi(x)$ and $S(x)$ are given by the equations
\[ \Pi' = -A\Pi JH_1 - \Pi JH_0, \quad S' = \Pi JH_1 J^*\Pi, \quad J = i\sigma_2. \quad (5.8) \]
Formula (2.19) for the Darboux matrix takes the form
\[ w_A(x, \lambda) = I_2 - i\sigma_2\Pi(x)^*S(x)^{-1}(A - \lambda I_n)^{-1}\Pi(x). \quad (5.9) \]
Using (5.4) and (5.7), we derive from Propositions 2.2 and 4.1 the next corollary.

**Corollary 5.1** Assume that the initial Shin-Zettl system is Lagrange-symmetric (i.e., that (1.4) holds). Let the matrices $A$, $\Pi(0)$ and $S(0)$ be chosen so that $S(0) = S(0)^*$ and $AS(0) - S(0)A^* = i\Pi(0)\sigma_2\Pi(0)^*$, and let $\Pi(x)$, $S(x)$ and $X(x)$ be determined by (5.8) and (5.6), respectively.

Then the corresponding transformed Shin-Zettl system
\[ \tilde{y}'(x, \lambda) = \tilde{F}(x, \lambda)\tilde{y}(x, \lambda) \]
with
\[ \tilde{r}_1 = -\overline{r_2} = r - \omega X_{12}, \quad \tilde{q} = q + \omega(X_{11} + \overline{X_{11}}). \quad (5.10) \]
This transformed system is Lagrange-symmetric as well. Moreover, the function $\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda)$, where $w_A$ has the form (5.9), satisfies the transformed system.

### 6 Sturm-Liouville equations

In this section we consider Sturm-Liouville equation (1.7). GBDT for its particular case (namely, for Schrödinger equation where $p \equiv \omega \equiv 1$) was dealt with in [18] but the general equation (1.7) contains other interesting subcases, where GBDT could be useful as well.
Proposition 6.1 Let the function \( p \omega \) be differentiable and its derivative \((p \omega)\)' as well as the functions \( p^{-1}, q \) and \( \omega \) be locally summable on \([0, \ell)\).
Assume that
\[
\omega = \overline{\omega}, \quad p = \overline{p}, \quad q = \overline{q}, \quad r \equiv 0, \tag{6.1}
\]
and set
\[
\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda), \tag{6.2}
\]
where \( w_A \) is given by the relations (5.9) and (5.8), \( H_0 \) and \( H_1 \) (in (5.8)) are given by (5.4) and \( y \) satisfies the initial Lagrange-symmetric Shin-Zettl equation
\[
y'(x, \lambda) = J(\lambda H_1(x) + H_0(x))y(x, \lambda). \tag{6.3}
\]
Then the entry \( \tilde{y}_1 \) of \( \tilde{y} \) satisfies the transformed Sturm-Liouville equation
\[
-(p(x)\tilde{y}'_1(x, \lambda))' + \tilde{q}(x)\tilde{y}_1(x, \lambda) = \lambda \omega(x)\tilde{y}_1(x, \lambda), \tag{6.4}
\]
where
\[
\tilde{q} = q + 2\omega(X_{11} - X_{22}) + 2p(\omega X_{12})^2 - (p \omega)'X_{12}, \tag{6.5}
\]
and \( X_{ik} \) are the entries of \( X \) given by (5.6).

Proof. Recall that in Section 5 we rewrote Shin-Zettl system in the form (5.3) where \( u = y_1 \). In the notations of the transformed system it means
\[
-(p(\overline{y}'_1 - \overline{r}\overline{y}_1))' - \overline{r}p(\overline{y}'_1 - \overline{r}\overline{y}_1) + \overline{q}\overline{y}_1 = \lambda \omega \overline{y}_1, \tag{6.6}
\]
where \( \overline{r} := \overline{r}_1(x) = -\overline{r}_2(x) \). Using the identity \( r_1 \equiv r_2 \equiv 0 \) and equalities (4.3) and (5.7) we present (6.6) in the form
\[
-(p\tilde{y}'_1)' - (p\omega X_{12}\tilde{y}_1)' + p\omega X_{12}\tilde{y}_1' + p(\omega X_{12})^2\tilde{y}_1 \\
+ (q + \omega(X_{11} - X_{22}))\tilde{y}_1 = \lambda \omega \tilde{y}_1, \tag{6.7}
\]
which is equivalent to (6.4) with
\[
\tilde{q} = q + \omega(X_{11} - X_{22}) + p(\omega X_{12})^2 - (p \omega)'X_{12} - p\omega X_{12}'. \tag{6.8}
\]
Finally, in order to show that the functions $\dot{q}$ given by (6.5) and (6.8) coincide, let us differentiate $X_{12}$. Taking into account (5.6) and (5.8), we obtain:

$$X' = J(H_1 J\Pi^* A^* S^{-1} \Pi + H_0 J\Pi^* S^{-1} \Pi) - J\Pi^* S^{-1} \Pi JH_1 J^* \Pi^* S^{-1} \Pi$$

$$- J\Pi^* S^{-1} (\Pi JH_1 + \Pi JH_0).$$

In particular, for $X_{12}$ we obtain

$$X_{12}' = p^{-1}(X_{22} - X_{11}) - \omega X_{12}^2. \quad (6.9)$$

Here we again took into account that $r_1 \equiv r_2 \equiv 0$. Equalities (6.8) and (6.9) imply (6.5). ■

**Remark 6.2** In view of (5.7), (6.1) and (6.5), the equality $\Im(\dot{q}) \equiv 0$ is valid. Thus, the coefficients of the transformed Sturm-Liouville equation (6.4) are real-valued. It is easy to see that the function $\dot{q}$ is locally summable on $[0, \ell)$ if the conditions of Proposition 6.1 hold and $S(x)$ is invertible on $[0, \ell)$.

### 7 Dynamical systems

#### 7.1 Dynamical symplectic system

Formally applying Laplace transform to the system (1.1) (satisfying (2.14)), we come to the interesting dynamical system

$$\frac{\partial}{\partial x} z(x, t) = J \left( -H_1(x) \frac{\partial}{\partial t} z(x, t) + H_0(x) z(x, t) \right). \quad (7.1)$$

When $J^* = J^{-1}$ system (7.1) is a dynamical symplectic system.

In order to construct Darboux transformation of system (7.1) and solutions of the transformed system, we use (2.17) and rewrite (2.10) (for our case where the relations (2.13)–(2.15) are valid) in the form

$$(J\Pi^* S^{-1})' = J(H_1 J\Pi^* S^{-1} A + \tilde{H}_0 J\Pi^* S^{-1}), \quad (7.2)$$

$$\tilde{H}_0 = H_0 - X^* H_1 - H_1 X, \quad X = J\Pi^* S^{-1} \Pi. \quad (7.3)$$

We note that (7.3) is equivalent to the second equality in (2.20).
Proposition 7.1 Let \( J, H_1(x) \) and \( H_0(x) \) satisfying (2.14), as well as the triple \( \{A, S(0) = S(0)^*, \Pi(0)\} \) satisfying (3.10), be given. Let the matrix functions \( \Pi(x) \) and \( S(x) \) be determined by (2.16). Then the vector functions

\[
\tilde{z}(x, t) = J\Pi(x)^*S(x)^{-1}e^{-tA}h \quad (h \in \mathbb{C}^m)
\]  

satisfy, in the points of invertibility of \( S(x) \), the transformed dynamical system (of the same form as (7.1)). More precisely, we have

\[
\frac{\partial}{\partial x}\tilde{z}(x, t) = J \left( -H_1(x) \frac{\partial}{\partial t}\tilde{z}(x, t) + \tilde{H}_0(x)\tilde{z}(x, t) \right),
\]  

where \( \tilde{H}_0 \) is given by (7.3).

Proof. In view of (7.2) and (7.4), both sides of (7.5) equal \( J(H_1J\Pi^*S^{-1}A + \tilde{H}_0J\Pi^*S^{-1})e^{-tA}h \).

When \( H_1 \geq 0 \), the energy \( E_z(t) \) of the solutions \( z \) of system (7.1) on \([0, a][0 < a < \ell]\) is given by the formula

\[
E_z(t)^2 = \int_0^a z(x, t)^*H_1(x)z(x, t)dx.
\]  

The energy of the transformed solutions \( \tilde{z} \) of the form (7.4) is expressed via \( A \) and \( S(x) \).

Proposition 7.2 Let the conditions of Proposition 7.1 hold and assume additionally that \( H_1 \geq 0 \) and \( S(0) > 0 \). Then the energy \( E_{\tilde{z}} \), where \( \tilde{z} \) has the form (7.4), is given by the formula

\[
E_{\tilde{z}}(t) = \sqrt{h^*e^{-tA^*}(S(0)^{-1} - S(a)^{-1})e^{-tA}h}.
\]  

Proof. Taking into account (5.8) and (7.4) we see that

\[
\tilde{z}(x, t)^*H_1(x)\tilde{z}(x, t) = -h^*e^{-tA^*}(S(x)^{-1})'e^{-tA}h.
\]  

Formula (7.7) follows from (7.6) and (7.8). ■
7.2 Two-way diffusion equation

In this subsection, we consider the important case when \( J, H_1 \) and \( H_0 \) have the form (5.4) (i.e., the same form as in Lagrange-symmetric Shin-Zettl system) and \( \omega = \overline{\omega}, p = \overline{p}, q = \overline{q} \). In that case we set

\[
    z(x, t) = \begin{bmatrix} z_1(x, t) \\ z_2(x, t) \end{bmatrix}, \quad \tilde{z}(x, t) = \begin{bmatrix} \tilde{z}_1(x, t) \\ \tilde{z}_2(x, t) \end{bmatrix},
\]

(7.9) and rewrite (7.1) in the form

\[
    z_1'(x, t) = rz_1 + p - 1 z_2, \quad z_2'(x, t) = \omega \frac{\partial}{\partial t} z_1 + q z_1 - r z_2 \quad \left( z_i' = \frac{\partial}{\partial x} z_i \right). \tag{7.10}
\]

Next, we rewrite the first equality in (7.10) as \( z_2 = p(z_1' - rz_1) \), substitute the expression for \( z_2 \) into the second equality in (7.10) and obtain

\[
    \omega \frac{\partial}{\partial t} z_1 = (p(z_1' - rz_1))' - q z_1 + \overline{p}(z_1' - rz_1). \tag{7.11}
\]

In particular, when \( r = 0 \), equation (7.11) takes the form

\[
    \omega \frac{\partial}{\partial t} z_1 = (p(z_1'))' - q z_1. \tag{7.12}
\]

We note that equation (7.12) coincides (in the case of sign-indefinite \( \omega \)) with the two-way diffusion equation (6.1) in [26]. See also various references in [4, 15, 26] on the literature related to the two-way diffusion equation.

According to Corollary 5.1, \( \tilde{H}_0 \) has the same form as \( H_0 \). More precisely, we have (see (5.10) or (7.3)):

\[
    \tilde{H}_0(x) = \begin{bmatrix} -\tilde{q}(x) & \tilde{r}(x) \\ \tilde{r}(x) & p(x)^{-1} \end{bmatrix}, \quad \tilde{r} = r - \omega X_{12}, \quad \tilde{q} = q + \omega (X_{11} + X_{11}). \tag{7.13}
\]

In the same way as (7.1) yields (7.11), equation (7.5) implies that the entry \( \tilde{z}_1 \) of the solution \( \tilde{z} \) given by (7.4) satisfies the equation

\[
    \omega \frac{\partial}{\partial t} \tilde{z}_1 = (p(\tilde{z}_1' - \tilde{r} \tilde{z}_1))' - \tilde{q} \tilde{z}_1 + \tilde{r} p(\tilde{z}_1' - \tilde{r} \tilde{z}_1). \tag{7.14}
\]
Assuming \( r \equiv 0 \), we see that

\[
\tilde{r} = -\omega X_{12}, \quad \tilde{q} = q + \omega (X_{11} + \overline{X_{11}}).
\]  

(7.15)

Multiplying the left-hand side of (6.6) by “\(-1\)” and substituting there \( \tilde{y}_1 = \tilde{z}_1 \) we obtain the right-hand side of (7.14). Hence, the proof of Proposition of 6.1 shows that

\[
(p(\tilde{z}_1' - \tilde{r}_1))' - \tilde{q}\tilde{z}_1 + \tilde{r}\tilde{p}(\tilde{z}_1' - \tilde{r}_1) = (p\tilde{z}_1')' - \tilde{q}\tilde{z}_1, 
\]  

(7.16)

\[
\tilde{q} = q + 2\omega (X_{11} - X_{22}) + 2p(\omega X_{12})^2 - (p\omega)'X_{12}.
\]  

(7.17)

From (7.9), (7.14) and (7.16), the next proposition is immediate.

**Proposition 7.3** Let \( J \), \( H_1 \) and \( H_0 \) have the form (5.4), and let the function \( p\omega \) be differentiable and its derivative \((p\omega)'\) as well as the functions \( p^{-1} \), \( q \) and \( \omega \) be locally summable on \([0, \ell)\). Assume that (6.1) holds and that the triple \( \{A, S(0) = S(0)^*, \Pi(0)\} \) satisfies (3.10). Introduce \( \Pi(x) \) and \( S(x) \) via (2.16).

Then the function \( \tilde{z}_1 \) (given by (7.4) and (2.16)) satisfies, in the points of invertibility of \( S(x) \), the dynamical equation

\[
\omega \frac{\partial}{\partial t} \tilde{z}_1 = (p\tilde{z}_1')' - \tilde{q}\tilde{z}_1,
\]  

(7.18)

where \( \tilde{q} \) is given by (7.17).

Recall that (7.18) is an equation of the form (7.12).

**Remark 7.4** It is important that [43, Theorem 7.4] and our Theorem 2.1, in particular, is valid on any interval \( I \) such that \( 0 \in I \). Thus, the previous statements of the paper, excluding the last sentence in Proposition 2.2, Remark 2.3, Proposition 7.2 and the statements from Section 3 (where the condition \( S(x) > 0 \) is essential), are also valid on the intervals \( I \) such that \( 0 \in I \). The interval \([0, \ell)\) was chosen for simplicity but the interval \((-\ell, \ell)\) is sometimes more convenient in the two-way diffusion equation and in the indefinite Sturm-Liouville case.
Symplectic systems and indefinite Sturm-Liouville equations are of growing interest in the literature (see, e.g., [3, 6, 7, 31, 32, 38] and references therein). Therefore, in this section of the paper we shall consider some examples of Darboux transformation for the Lagrange-symmetric Shin-Zettl system and indefinite Sturm-Liouville equation considered on the interval \((-\ell, \ell)\), see Remark 7.4.

More precisely, we shall construct explicit solutions for the interesting model case

\[ \omega(x) = \text{sgn}(x), \quad p(x) \equiv 1, \quad (8.1) \]

which was studied in [25]. First, we consider Shin-Zettl system (1.1), (5.4) and assume that the equalities (8.1) and

\[ q(x) = r(x) \equiv 0 \quad (8.2) \]

hold for the initial system. We consider Darboux transformations determined by the triples of matrices \(\{A, S(0), \Pi(0)\}\) of the form

\[ A = \alpha^2, \quad S(0) = 0, \quad \Pi(0) = \begin{bmatrix} -2i\alpha g & 2\mu \alpha g \end{bmatrix}, \quad (8.3) \]

where \(\alpha\) are \(n \times n\) matrices, \(g \in \mathbb{C}^n\) are vector columns, \(\mu\) are purely imaginary values (i.e. \(\bar{\mu} = -\mu\)), and

\[ \det(\mu \alpha \pm I_n) \neq 0, \quad \det(\mu \alpha \pm iI_n) \neq 0. \quad (8.4) \]

It is easily checked that the third equality in (8.3) yields \(\Pi(0)J\Pi(0)^* = 0\) \((J = i\sigma_2)\), and so the matrix identity (3.10), which is required in GBDT, holds for the triple of the form (8.3).

Next, we partition \(\Pi(x)\) into two columns \(\Pi(x) = \begin{bmatrix} \Lambda_1(x) & \Lambda_2(x) \end{bmatrix}\), and (taking into account (8.1)–(8.3)) rewrite the first system in (5.8) in the form

\[ \Lambda'_1 = \begin{cases} \alpha^2 \Lambda_2 & \text{for } x > 0 \\ -\alpha^2 \Lambda_2 & \text{for } x < 0 \end{cases}; \quad \Lambda'_2 = -\Lambda_1. \quad (8.5) \]
It is immediate that the vector functions
\[
\Lambda_1(x) = -i\alpha\left(e^{ix\alpha}(\mu\alpha + I_n)g - e^{-ix\alpha}(\mu\alpha - I_n)g\right) \\
\Lambda_2(x) = e^{ix\alpha}(\mu\alpha + I_n)g + e^{-ix\alpha}(\mu\alpha - I_n)g
\]
for \( x \geq 0 \); (8.6)
\[
\Lambda_1(x) = -\alpha\left(e^{x\alpha}(\mu\alpha + iI_n)g - e^{-x\alpha}(\mu\alpha - iI_n)g\right) \\
\Lambda_2(x) = e^{x\alpha}(\mu\alpha + iI_n)g + e^{-x\alpha}(\mu\alpha - iI_n)g
\]
for \( x \leq 0 \) (8.7)
satisfy (8.5) and the third equality in (8.3).

The second system in (5.8) takes the form
\[
S' = \omega \Lambda_2 \Lambda_2^*.
\]
Hence, using \( S(0) = 0 \), we see that
\[
S(x) = \int_0^x \Lambda_2(t)\Lambda_2(t)^* dt \geq 0 \quad (x > 0),
\]
\[
S(x) = \int_x^0 \Lambda_2(t)\Lambda_2(t)^* dt \geq 0 \quad (x < 0).
\]

**Remark 8.1** It follows from (8.8) and (8.9) that usually we have
\[
S(x) > 0 \quad \text{for} \quad x \neq 0.
\]

In particular, (8.10) holds when the pair \( \{\hat{\alpha}, \hat{g}\} \), where
\[
\hat{\alpha} := \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}, \quad \hat{g} := \begin{bmatrix} g \\ g \end{bmatrix},
\]
is controllable. Indeed, if (8.10) is not valid, then there is \( f \in \mathbb{C}^n \) (\( f \neq 0 \)) such that \( f^*\Lambda_2(x) = 0 \) either for all \( x > 0 \) or for all \( x < 0 \). In view of (8.4), (8.6) and (8.7) it means that \( \hat{f}^*e^{cx\hat{\alpha}}\hat{g} \equiv 0 \) for some \( \hat{f} \in \mathbb{C}^{2n} \) and \( c \in \mathbb{C} \) (\( \hat{f} \neq 0, \ c \neq 0 \)). However, this contradicts the controllability of \( \{\hat{\alpha}, \hat{g}\} \) (see, e.g., [9]).

Formulas (8.6)–(8.9) present explicit expressions for \( \Pi(x) \) and \( S(x) \), and so the Darboux matrix \( w_A(x, \lambda) \) of the form (5.9) is constructed explicitly.

In order to use Corollary 5.1 we also solve explicitly the initial Shin-Zettl system (1.1), (5.4), where (8.1) and (8.2) hold. Namely, we introduce
matrices

\[
T_+(\lambda) = \begin{bmatrix}
1 & 1 \\
i\sqrt{\lambda} & -i\sqrt{\lambda}
\end{bmatrix},
\quad D_+(\lambda) = \begin{bmatrix}
i\sqrt{\lambda} & 0 \\
0 & -i\sqrt{\lambda}
\end{bmatrix},
\]

\[
T_-(\lambda) = \begin{bmatrix}
1 & 1 \\
\sqrt{\lambda} & -\sqrt{\lambda}
\end{bmatrix},
\quad D_-(\lambda) = \begin{bmatrix}
\sqrt{\lambda} & 0 \\
0 & -\sqrt{\lambda}
\end{bmatrix},
\]

(8.11)

(8.12)

where \(\sqrt{\lambda}\) is any fixed branch of the square root of \(\lambda\). It is easy to see that (in our case) \(F\) given in (1.1) satisfies the equalities \(FT_+ = T_+D_+\) for \(x > 0\) and \(FT_- = T_-D_-\) for \(x < 0\). Therefore, solutions \(y\) of the initial Shin-Zettl system (1.1) are given by the formulas

\[
y(x, \lambda) = T_+(\lambda)e^{xD_+(\lambda)}T_+(\lambda)^{-1}h \quad (x > 0),
\]

(8.13)

\[
y(x, \lambda) = T_-(\lambda)e^{xD_-(\lambda)}T_-(\lambda)^{-1}h \quad (x < 0)
\]

(8.14)

with any vectors \(h \in \mathbb{C}^2\). Now, Corollary 5.1 and Remarks 7.4 and 8.1 yield our next corollary.

**Corollary 8.2** Assume that the initial Shin-Zettl system on \((-\ell, \ell)\) has the form (1.1), (5.4) and that equalities (8.1) and (8.2) hold. Let the matrices \(A, \Pi(0)\) and \(S(0)\) have the form (8.3). Then the corresponding GBDT-transformed Shin-Zettl system \(\tilde{y}'(x, \lambda) = J(\lambda H_1(x) + \tilde{H}_0(x))\tilde{y}(x, \lambda)\) is Lagrange symmetric and we have \(\tilde{H}_0(x) = \begin{bmatrix}
-q(x) & -r(x) \\
r(x) & 1
\end{bmatrix}\), where

\[
\tilde{r}(x) = -\text{sgn}(x)X_{12}(x), \quad \tilde{q}(x) = \text{sgn}(x)(X_{11}(x) + \overline{X}_{11}(x)),
\]

(8.15)

\(X_{ij}\) are the blocks of \(X = J\Pi^*S^{-1}\Pi\), and explicit expressions for \(S\) and \(\Pi\) are given in (8.6)–(8.9). The controllability of the pair \(\{\tilde{\alpha}, \tilde{\gamma}\}\) is a sufficient condition of the invertibility of \(S(x)\) at \(x \neq 0\). If, indeed, \(\det S(x) \neq 0\) for \(x \neq 0\), then \(X(x)\) and the Darboux matrix \(w_A(x, \lambda)\) of the form (5.9) are well defined and explicitly expressed via \(\Pi(x)\) and \(S(x)\) at \(x \neq 0\). Moreover, the solutions \(\tilde{y}\) of the GBDT-transformed system are explicitly expressed via the formula \(\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda)\), where \(y\) is given by (8.13) and (8.14).

By virtue of Proposition 6.1, Remark 7.4 and Corollary 8.2 we obtain explicit solutions of indefinite Sturm-Liouville systems

\[
-\tilde{y}_1''(x, \lambda) + \tilde{q}(x)\tilde{y}_1(x, \lambda) = \lambda\text{sgn}(x)\tilde{y}_1(x, \lambda) \quad (-\ell < x < \ell).
\]

(8.16)
Corollary 8.3 Let $\Pi(x)$ and $S(x)$ be given by (8.6)–(8.9) and assume that $\det S(x) \neq 0$ for $x \neq 0$. Set $\tilde{y}(x, \lambda) = w_A(x, \lambda)y(x, \lambda)$ where explicit expressions for $w_A(x, \lambda)$ (with $A = \alpha^2$) and $y(x, \lambda)$ are given by (5.9) and (8.13), (8.14), respectively. Then the first entry $\tilde{y}_1$ of $\tilde{y}$ satisfies the indefinite Sturm-Liouville system (8.16) where

$$\tilde{q}(x) = 2\text{sgn}(x)(X_{11}(x) - X_{22}(x)) + 2X_{12}(x)^2$$

(8.17)

and $X_{ij}$ are the blocks of $X = J\Pi^*S^{-1}\Pi$.

The singularity of $\tilde{q}(x)$ at $x = 0$ is of interest. Some particular cases (but in greater detail) were considered in [27, Section 5], and it was proved for those cases that $\tilde{q}(x) = O(x^{-2})$ when $x$ tends to 0.

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References

[1] W.O. Amrein, A.M. Hinz and D.B. Pearson (eds), Sturm-Liouville theory. Past and present (Birkhäuser, Basel, 2005).

[2] F.V. Atkinson, Discrete and continuous boundary problems. Mathematics in Science and Engineering, Vol. 8 (Academic Press, New York-London, 1964).

[3] F.V. Atkinson, W.N. Everitt and K.S. Ong, ‘On the $m$-coefficient of Weyl for a differential equation with an indefinite weight function’, Proc. London Math. Soc. (3) 29 (1974) 368–384.

[4] R. Beals, ‘Partial-range completeness and existence of solutions to two-way diffusion equations’, J. Math. Phys. 22 (1981) 954–960.

[5] H. Behncke and D.B. Hinton, ‘Transformation Theory of Symmetric Differential Expressions’, Adv. Differential Equations 11 (2006) 601–626.

[6] J. Behrndt, F. Philipp and C. Trunk, ‘Bounds on the non-real spectrum of differential operators with indefinite weights’, Math. Ann. 357 (2013) 185–213.

[7] P. Binding, H. Langer and M. Möller, ‘Oscillation results for Sturm-Liouville problems with an indefinite weight function’, J. Comput. Appl. Math. 171 (2004) 93–101.

[8] J.L. Cieslinski, ‘Algebraic construction of the Darboux matrix revisited’, J. Phys. A 42:404003 (2009) 40 pp.

[9] W.A. Coppel, ‘Matrix quadratic equations’, Bull. Austral. Math. Soc. 10 (1974) 377–401.

[10] M.M. Crum. ‘Associated Sturm–Liouville systems’, Quart. J. Math., Oxford II Ser. 6 (1955) 121–127.

[11] P.A. Deift, ‘Applications of a commutation formula’, Duke Math. J. 45 (1978) 267–310.
[12] J. Eckhardt, F. Gesztesy, R. Nichols, A.L. Sakhnovich and G. Teschl, ‘Inverse spectral problems for Schrödinger-type operators with distributional matrix-valued potentials’, Differential Integral Equations 28 (2015) 505–522.

[13] W.N. Everitt and D. Race, ‘Some remarks on linear ordinary quasi-differential expressions’, Proc. London Math. Soc. (3) 54 (1987) 300–320.

[14] W.N. Everitt and D. Race, ‘The regular representation of singular second-order differential expressions using quasi-derivatives’, Proc. London Math. Soc. (3) 65 (1992) 383–404.

[15] N.J. Fisch and M.D. Kruskal, ‘Separating variables in two-way diffusion equations’, J. Math. Phys. 21 (1980) 740–750.

[16] F. Gesztesy, ‘A complete spectral characterization of the double commutation method’, J. Funct. Anal. 117 (1993) 401–446.

[17] F. Gesztesy and G. Teschl, ‘On the double commutation method’, Proc. Amer. Math. Soc. 124 (1996) 1831–1840.

[18] I. Gohberg, M.A. Kaashoek and A.L. Sakhnovich, ‘Sturm–Liouville systems with rational Weyl functions: explicit formulas and applications’, Integral Equations Operator Theory 30 (1998) 338–377.

[19] I. Gohberg, M.A. Kaashoek and A.L. Sakhnovich, ‘Scattering problems for a canonical system with a pseudo-exponential potential’, Asymptotic Analysis 29 (2002) 1–38.

[20] C.H. Gu, H. Hu and Z. Zhou, Darboux transformations in integrable systems. Theory and their applications to geometry, Mathematical Physics Studies 26 (Springer, Dordrecht, 2005).

[21] D.B. Hinton and J.K. Shaw, ‘On Titchmarsh-Weyl $M(\lambda)$-functions for linear Hamiltonian systems’, J. Differential Equations 40 (1981) 316–342.
[22] D.B. Hinton and A. Schneider, ‘On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems II’, Math. Nachr. 185 (1997) 67–84.

[23] I.S. Kac and M.G. Krein, ‘On the spectral functions of the string’, Amer. Math. Soc. Transl. (2) 103 (1974) 19–102.

[24] B. Jacob and H.J. Zwart ‘Linear port-Hamiltonian systems on infinite-dimensional spaces’, in: Operator Theory: Advances and Applications 223 (Birkhäuser, Basel, 2012).

[25] I. Karabash and C. Trunk, ‘Spectral properties of singular Sturm-Liouville operators with indefinite weight $\text{sgn} \, x$’, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009) 483–503.

[26] A. Kostenko, ‘The similarity problem for indefinite Sturm-Liouville operators and the HELP inequality’, Adv. Math. 246 (2013) 368–413.

[27] A. Kostenko, A. Sakhnovich and G. Teschl, ‘Commutation Methods for Schrödinger Operators with Strongly Singular Potentials’, Math. Nachr. 285 (2012) 392–410.

[28] A.M. Krall, ‘$M(\lambda)$ theory for singular Hamiltonian systems with one singular point’, SIAM J. Math. Anal. 20 (1989) 664–700.

[29] A.M. Krall, ‘A limit-point criterion for linear Hamiltonian systems’, Appl. Anal. 61 (1996) 115–119.

[30] M.G. Krein, ‘On a continual analogue of a Christoffel formula from the theory of orthogonal polynomials’ (Russian) Dokl. Akad. Nauk SSSR (N.S.) 113 (1957) 970–973.

[31] M. Langer and H. Woracek, ‘Indefinite Hamiltonian systems whose Titchmarsh-Weyl coefficients have no finite generalized poles of non-positive type’, Oper. Matrices 7 (2013) 477–555.
[32] M. Levitin and M. Seri, ‘Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential’, Oper. Matrices 10 (2016) 223–245.

[33] V.B. Matveev and M.A. Salle, Darboux transformations and solitons (Springer, Berlin, 1991).

[34] R. Mennicken, A.L. Sakhnovich and C. Tretter, ‘Direct and inverse spectral problem for a system of differential equations depending rationally on the spectral parameter’, Duke Math. J. 109 (2001) 413–449.

[35] K.A. Mirzoev, ‘Sturm-Liouville operators’, Trans. Moscow Math. Soc. (2014) 281–299.

[36] V. Mogilevskii, ‘Spectral and pseudospectral functions of Hamiltonian systems: development of the results by Arov-Dym and Sakhnovich’, Methods Funct. Anal. Topology 21 (2015) 370–402.

[37] M.A. Naimark, Linear Differential Operators (Frederick Ungar Publishing Co., New York, 1968).

[38] J. Qi, B. Xie and S. Chen, ‘The upper and lower bounds on non-real eigenvalues of indefinite Sturm-Liouville problems’, Proc. Amer. Math. Soc. 144 (2016) 547–559.

[39] A.L. Sakhnovich, ‘Dressing procedure for solutions of nonlinear equations and the method of operator identities’, Inverse Problems 10 (1994) 699–710.

[40] A.L. Sakhnovich, ‘Iterated Bäcklund–Darboux transform for canonical systems’, J. Funct. Anal. 144 (1997) 359–370.

[41] A.L. Sakhnovich, ‘Generalized Bäcklund–Darboux transformation: spectral properties and nonlinear equations’, J. Math. Anal. Appl. 262 (2001) 274–306.
[42] A.L. Sakhnovich. ‘Dirac type system on the axis: explicit formulas for matrix potentials with singularities and soliton-positon interactions’, Inverse Problems 19 (2003) 845–854.

[43] A.L. Sakhnovich, L.A. Sakhnovich and I.Ya. Roitberg, Inverse Problems and Nonlinear Evolution Equations. Solutions, Darboux Matrices and Weyl–Titchmarsh Functions, De Gruyter Studies in Mathematics 47 (De Gruyter, Berlin, 2013).

[44] L.A. Sakhnovich, ‘On the factorization of the transfer matrix function’, Sov. Math. Dokl. 17 (1976) 203–207.

[45] L.A. Sakhnovich, Spectral theory of canonical differential systems, method of operator identities, Oper. Theory Adv. Appl. 107 (Birkhäuser, Basel, 1999).

[46] H. Schmid and C. Tretter, ‘Singular Dirac systems and Sturm-Liouville problems nonlinear in the spectral parameter’, J. Differential Equations 181 (2002) 511–542.

[47] D. Shin, ‘On quasi-differential operators in Hilbert space’, Dokl. Akad. Nauk SSSR 18 (1938) 523–526.

[48] P. Šepitka, Peter and R. Simon Hilscher, ‘Genera of conjoined bases of linear Hamiltonian systems and limit characterization of principal solutions at infinity’, J. Differential Equations 260 (2016) 6581–6603.

[49] R. Šimon Hilscher and P. Zemánek, ‘Time scale symplectic systems with analytic dependence on spectral parameter’, J. Difference Equ. Appl. 21 (2015) 209–239.

[50] A. Zettl, ‘Formally self-adjoint quasi-differential operators’, Rocky Mountain J. Math. 5 (1975) 453–474.

[51] A. Zettl, Sturm–Liouville theory (Amer. Math. Soc., Providence, RI, 2005).
[52] A. Zettl and J. Sun, ‘Survey article: Self-adjoint ordinary differential operators and their spectrum’, Rocky Mountain J. Math. 45 (2015) 763–886.

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