On the contractivity of the Hilbert-Schmidt distance under open system dynamics

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It is shown that the Hilbert-Schmidt (HS) norm and distance, unlike the trace norm and distance, are generally not contractive for open quantum systems under Lindblad dynamics. Necessary and sufficient conditions for contractivity of the HS norm and distance are given, and explicit criteria in terms of the Lindblad operators are derived. It is also shown that the requirements for contractivity of the HS distance are strictly weaker than those for the HS norm, although simulations suggest that non-contractivity is the typical case, i.e., that systems for which the HS distance between quantum states is monotonically decreasing are exceptional for $N > 2$, in contrast to the case $N = 2$ where it is always monotonically decreasing.

I. INTRODUCTION

The trace norm and induced trace distance play an important role in quantum information theory. One of its most important features, widely used in research papers and popular textbooks [1, 2], is its contractivity for trace-preserving quantum evolution, first proved by Ruskai [3]. It was once conjectured that contractivity would extend to other more intuitive norms such that the Hilbert-Schmidt (HS) norm [4, 5], but flaws in the original argument were soon discovered and an explicit counter-example of a trace-preserving map with non-contractive HS norm was provided by Ozawa [6]. The more general question of when a positive trace-preserving [PTP] map between matrix spaces is contractive with respect to the $p$-norm for $p > 1$ was recently considered in [7], where it was shown that PTP maps are contractive for $p > 1$ in general if only if they are unital. This does not answer the question, however, when a PTP map defined on a subset of a matrix space is contractive with respect to a particular $p$-norm such as the HS norm, and contractivity depends on the subspace. In particular, the HS norm restricted to the trace-zero hyperplane in the space of Hermitian matrices may be contractive while it is not on the whole matrix space.

Contractivity of the distance between quantum states in the sense that the distance between two quantum states is monotonically decreasing, is of particular interest in the area of open system dynamics. We consider under what conditions the evolution of an open quantum system subject to semigroup dynamics governed by a Lindblad master equation is contractive with respect to the HS distance. It can be shown that for $N = 2$ both the HS norm and distance are always contractive, while for $N > 2$ both are contractive only for small subsets of open systems, with the set of open systems for which the HS norm is contractive being strictly smaller than the set of open systems for which the HS distance is contractive.

Aside from the relevance to quantum information, e.g., in the construction of entanglement measures, one important implication of the non-contractivity of the HS norm is in quantum control, where the HS distance between quantum trajectories is a common choice of a Lyapunov function, e.g., for local optimal control [8] [9] [10] [11]. As contractivity is a prerequisite for a Lyapunov function, it means that the HS distance is generally not a suitable candidate for a Lyapunov function for open systems.

II. HILBERT SPACE NORMS AND CONTRACTIVITY

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} = N$ and let $\mathfrak{B}[\mathcal{H}]$ be the bounded operators on $\mathcal{H}$, and $\mathfrak{D}[\mathcal{H}]$ be the trace-1 positive operators on $\mathcal{H}$. It is not difficult to check that

$$\|A\|_\text{TR} = \frac{1}{2} \text{Tr} \sqrt{A^\dagger A}$$

is well-defined for any $A \in \mathfrak{B}[\mathcal{H}]$ as $A^\dagger A$ is positive, and satisfies the axioms of a norm, and induces a metric, the trace distance, in the usual way $d_\text{TR}(A, B) = \|A - B\|_\text{TR}$.

Among the nice features of the trace norm is contractivity under trace-preserving maps [3], i.e., given two density operators $\rho_1, \rho_2 \in \mathfrak{D}[\mathcal{H}]$ and a map $\mathcal{E} : \mathfrak{B}[\mathcal{H}] \to \mathfrak{B}[\mathcal{H}]$ with $\text{Tr}{\mathcal{E}}(A) = \text{Tr}(A)$, then

$$d_\text{TR}(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \leq d_\text{TR}(\rho_1, \rho_2).$$

This implies in particular that any (super)operator that maps density matrices to density matrices cannot increase the trace distance between two states. Furthermore, if $\mathcal{E}_t$ is a quantum dynamical semi-group then $d_\text{TR}(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2))$ is monotonically decreasing, although monotonicity is not always strict. Indeed, in the special case of unitary evolution the trace distance remains constant, $d_\text{TR}(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2)) = d_\text{TR}(\rho_1, \rho_2)$ for all $t$.

One drawback of the trace distance, however, is that it is not the most intuitive distance measure for quantum states. A more natural choice is the Hilbert Schmidt...
(HS) distance \(d_{HS}(A, B) = \|A - B\|_{HS}\) where \(\|A\|_{HS} = \sqrt{\langle A^\dagger A \rangle}\) is the norm induced by the HS inner product

\[
\langle A|B \rangle_{HS} = \text{Tr}(A^\dagger B).
\]

(3)

For Hermitian operators \(A = A^\dagger, B = B^\dagger\) the HS inner product and distance simplify to \(\langle A|A \rangle_{HS} = \text{Tr}(A^2)\) and

\[
d_{HS}(A, B) = \sqrt{\text{Tr}[(A - B)^2]}.
\]

(4)

If we choose an orthonormal basis \(\{\sigma_k\}_{k=1}^{N^2}\) for the Hermitian operators on \(\mathcal{H}\) then the coordinate vector \(a = (a_k)_{k=1}^{N^2}\) with \(a_k = \text{Tr}(\sigma_k A)\) for any Hermitian operator \(A\) on \(\mathcal{H}\) is a vector in \(\mathbb{R}^{N^2}\), and noting that \(\text{Tr}(\sigma_k \sigma_{k'}) = \delta_{kk'}\) by orthonormality, shows that the HS inner product reduces to the standard Euclidean inner product in \(\mathbb{R}^{N^2}\)

\[
\langle A|A \rangle_{HS} = \sum_{k,l=1}^{N^2} a_k a_{l} \text{Tr}(\sigma_k \sigma_{l}) = \sum_{k=1}^{N^2} a_k^2 = \langle \mathbf{a}|\mathbf{a} \rangle.
\]

(5)

Furthermore, if \(A, B\) are two Hermitian operators with \(\text{Tr}(A) = \text{Tr}(B) = 0\) and thus

\[
d_{HS}(A, B) = \sqrt{\langle a - b|a - b \rangle} = \|a - b\|,
\]

(6)

i.e., the HS distance between two Hermitian operators of the same trace class, is simply the Euclidean distance of their associated reduced (real) coordinate vectors, which generalize the Bloch vector for \(N = 2\). This equivalence of HS distance between density operators and the Euclidean distance between their reduced coordinate vectors makes the HS distance a very intuitive and useful distance measure.

III. CRITERIA FOR MONOTONICITY OF HS NORM AND DISTANCE

For \(N = 2\), i.e., a single qubit, it is easy to show that the trace and HS distance agree up to a constant factor, and hence contractivity of one implies contractivity of the other. Let \(\rho_1\) and \(\rho_2\) be two qubit density operators and let \(r_1\) and \(r_2\) be their respective coordinate vectors in \(\mathbb{R}^3\) with respect to the (orthonormal) basis \(\{\sigma_k\}_{k=1}^{4}\) defined above. Then \(\rho_j = \sum_{k=1}^{4} r_{jk} \sigma_k + \frac{1}{2} \mathbb{1}\) for \(j = 1, 2\) and setting \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\)

\[
\rho_1 - \rho_2 = \sum_{k=1}^{3} (r_{1k} - r_{2k}) \sigma_k = (r_1 - r_2) \cdot \sigma.
\]

(7)

Choosing \(\sigma\) to be, e.g., the standard (normalized) Pauli basis, it is easy to verify that the eigenvalues of \(\rho_1 - \rho_2\) are \(\pm \frac{1}{\sqrt{2}} |r_1 - r_2|\), and noting that for Hermitian matrices \(\text{Tr}(A)\) is the sum of the absolute values of the eigenvalues of \(A\), we have

\[
d_{TR}(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}|\rho_1 - \rho_2| = \frac{1}{2} \frac{2}{\sqrt{2}} \|r_1 - r_2\| = \frac{1}{\sqrt{2}} d_{HS}(\rho_1, \rho_2).
\]

(8)

Hence, the HS distance for a single qubit is monotonically decreasing under completely positive maps and quantum semigroup dynamics. This argument does not generalize, however, as the relationship between the eigenvalues of a positive map and the Euclidean norm of the coordinate vector that was used is very specific to \(N = 2\).

For \(N > 2\) we still have \(\|A\|_{HS} \leq \|A\|_{TR}\) from basic functional analysis [12]. Thus the trace norm provides an upper bound on the Hilbert-Schmidt norm. Applied to open quantum systems subject to semi-group dynamics governed by a Lindblad master equation

\[
\dot{\rho}(t) = -i[H, \rho(t)] + \mathcal{L}_D \rho(t),
\]

(9)

with \(\mathcal{L}_D \rho(t) = \sum_d D[V_d] \rho(t), V_d \in \mathfrak{B}[\mathcal{H}]\) and

\[
D[V_d] \rho(t) = V_d \rho(t) V_d^\dagger - \frac{1}{2} (V_d^\dagger V_d \rho(t) + \rho(t) V_d^\dagger V_d),
\]

(10)

this means that if the trace distance between any two quantum states goes to zero for \(t \to \infty\), for instance, then the HS distance must go to zero as well, but the convergence need not be monotonic.

Expanding the density operator \(\rho\) and super-operators \(\mathcal{L}_H\) and \(\mathcal{L}_D\) with respect to an orthonormal basis for the (trace-zero) Hermitian matrices on \(\mathcal{H}\), it is easy to show that the master equation (9) becomes an affine-linear equation for the coordinate vector \(r \in \mathbb{R}^{N^2 - 1}\)

\[
\dot{r}(t) = \mathbf{A} r(t) + \mathbf{c},
\]

(11)

where \(\mathbf{A}\) is a \((N^2 - 1) \times (N^2 - 1)\) real matrix and \(\mathbf{c} \in \mathbb{R}^{N^2 - 1}\) can be computed from the Hamiltonian and Lindblad generators (See Appendix B).

Proposition 1. A necessary and sufficient condition for the HS norm \(\|ho(t)\|_2\) of any quantum state \(\rho(t)\) to be monotonically decreasing under the open system dynamics (11) is that the Bloch equation is linear, i.e., \(\mathbf{c} = 0\).

Proof. If \(T\) is a PTP map acting on the Hermitian operators on a finite-dimensional Hilbert space then \(\|T\rho\|_2 \leq \|ho\|_2\) if and only if \(T\) is unital [7]. For our open system dynamics, the evolution is unital, i.e., preserves the identity \(\mathbb{1}\), and if only if the Bloch equation is homogeneous, i.e., \(\mathbf{c} = 0\), as the identity \(\mathbb{1}\) is mapped to \(r = 0\) in the trace-one hyperplane the density operators live in. □

Proposition 2. A necessary and sufficient condition for the HS distance \(\|\rho_1(t) - \rho_2(t)\|\) between quantum states to
be monotonically decreasing under the dynamics (11) is that the symmetric part $\mathbf{A} + \mathbf{A}^T$ of the evolution operator $\mathbf{A}$ be negative definite, i.e., have no positive eigenvalues.

**Proof.** Denote the set of all physical coordinate vectors as $\mathcal{D}_R[\mathcal{H}]$, corresponding to $\mathcal{D}[\mathcal{H}]$. We can easily see that the eigenvalues of $\mathbf{A}$ must all have non-positive real parts since the dynamical system is invariant on the compact set $\mathcal{D}_R[\mathcal{H}]$. Moreover, if $\rho_1(0) = \rho_1$ and $\rho_2(0) = \rho_2$ are two initial states with coordinate vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, then $\Delta(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t)$ satisfies the linear equation

$$\dot{\Delta}(t) = (\mathbf{A}) \Delta(t) \Rightarrow \Delta(t) = e^{t \mathbf{A}} \Delta(0) \quad (12)$$

and thus we have

$$\frac{d}{dt} \text{d}_{\text{HS}}(\mathbf{r}_1(t), \mathbf{r}_2(t)) = \langle \dot{\Delta}(t) | \Delta(t) \rangle + \langle \Delta(t) | \dot{\Delta}(t) \rangle = (\mathbf{A}) (\mathbf{A} \mathbf{A} + \mathbf{A}) \Delta(t) = \Delta(t)^T (\mathbf{A} + \mathbf{A}) \Delta(t)).$$

As $\mathbf{A} + \mathbf{A}^T$ is real symmetric, its eigenvalues are real, and the HS distance will be monotonically decreasing, if and only if $\mathbf{A} + \mathbf{A}^T$ has only non-negative eigenvalues. □

Note that although we know that the eigenvalues of $\mathbf{A}$ have non-positive real parts, this does not imply that $\mathbf{A} + \mathbf{A}^T$ has non-positive (real) eigenvalues in general.

**Example 1.** Consider the system $\dot{\rho}(t) = \mathcal{D}[V] \rho$ with the simple Lindblad generator

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Choosing $\{\sigma_k : k = 1, \ldots, 8\}$ to be the standard orthonormal basis for the trace-zero Hermitian matrices $[A]_N$ for $N = 3$, and $\sigma_N = \frac{1}{\sqrt{3}}$, we obtain

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & \sqrt{3} & 0 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & -1 & -\frac{1}{2} & \sqrt{3} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \sqrt{3} \\ 0 & \frac{1}{2} & -1 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & -1 & -\frac{1}{2} & \sqrt{3} & 0 & -1 & -\frac{1}{2} \\ 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \sqrt{3} & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

as well as $c = \frac{\sqrt{2}}{3}(1, 0, 0, 1, \sqrt{3}, 0, 0, -1)^T$. It is easy to check that $\mathbf{A}$ is invertible, its eigenvalues have negative real parts, and the system has a unique steady state $\mathbf{r}_{ss} = -\mathbf{A}^{-1} c$ corresponding to

$$\rho_{ss} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

However, $[A, A^T] \neq 0$, i.e., $\mathbf{A}$ is not normal and $\mathbf{A} + \mathbf{A}^T$ has a positive eigenvalue $\gamma = 0.1914$ with eigenvector $\mathbf{v}$. Hence, setting $\Delta(0) = \alpha \mathbf{v}$ with $\alpha > 0$ chosen such that $\mathbf{r}(0) = \mathbf{r}_{ss} + \alpha \mathbf{v} \in \mathcal{D}_R[\mathcal{H}]$ gives a trajectory for which

$$\frac{d}{dt} \text{d}^2 = \alpha^2 v^T (A^T + A) v = \gamma \alpha^2 \Vert v \Vert^2 > 0, \quad (13)$$

and thus the distance from the steady state increases at least initially. Indeed Fig. 2 shows that for $\Delta(0) = \frac{1}{2} \mathbf{v}$ and $\mathbf{s}(0) \in \mathcal{D}_R[\mathcal{H}]$ the HS distance $d_{\text{HS}}(t) = \|\Delta(t)\|$ eventually converges to 0—as it must as $\mathbf{A}$ has no eigenvalues with real part 0—but it increases initially.

Whenever $\mathbf{A} + \mathbf{A}^T$ has a positive eigenvalue $\gamma$ with eigenvalue $\mathbf{v}$ then the distance between any two initial states $\rho_1(0)$ and $\rho_2(0)$, whose corresponding real coordinate vectors satisfy $\mathbf{r}_1 - \mathbf{r}_2 = \alpha \mathbf{v}$, will increase at least initially. Furthermore, if the system has a steady state $\rho_{ss}$ in the interior of $\mathcal{D}[\mathcal{H}]$ then there are initial states that do not converge to the steady state monotonically with respect to the HS distance. This is easy to see since for a point in the interior, it is always possible to choose $\alpha > 0$ such that $\mathbf{s}(0) = \mathbf{s}_{ss} + \alpha \mathbf{v} \in \mathcal{D}_R[\mathcal{H}]$, where $\mathcal{D}_R[\mathcal{H}]$ is the subset of $\mathbb{R}^{N^2-1}$ corresponding to physical states, for any $\mathbf{v}$.

**IV. SUFFICIENT CONDITIONS ON LINDBLAD OPERATORS FOR MONOTONICITY**

A necessary and sufficient condition for the HS norm of a quantum state to be monotonically decreasing is that $\mathcal{L}_D(\Delta) = 0$. Inserting this into the LME (1) leads to $\sum_d [V_d, V_d^\dagger] = 0$, which gives the following explicit result:

**Proposition 3.** The HS norm of any quantum state is monotonically decreasing under the open system dynamics (11) if and only if the Lindblad operators $V_d$ satisfy $\sum_d [V_d, V_d^\dagger] = 0$.  

**FIG. 1: Non-monotonic convergence to steady state.**
We know that contractivity of the HS norm is a sufficient condition for contractivity of the HS distance, but we can derive other sufficient conditions.

**Proposition 4.** The distance between any two quantum states is monotonically decreasing under the open system dynamics \([11]\) if \(A\) is normal.

*Proof.* If \(A\) is normal, i.e., \([A, A^T] = 0\), then there exists a unitary transformation \(U \in SU(N^2 - 1)\) such that \(A = UDU^\dagger\), where \(D\) is a diagonal matrix. Hence, noting that \(A\) is real, \(A^T = A^\dagger = UDU^\dagger\), and thus \(A + A^T = U(D + D^\dagger)U^\dagger\) shows that the eigenvalues of \(A + A^T\) are twice the real parts of those of \(A\), i.e., non-positive. \(\square\)

For simple Lindblad equations we can further show that normality of the Lindblad operators is a sufficient condition for normality of \(A\).

**Proposition 5.** For a system governed by the purely dissipative LME \(\dot{\rho}(t) = D[V]\rho(t)\) the superoperator \(A + A^T\) is normal if \(V\) is normal.

*Proof.* Let \(a_{\ell k}\) be the \((\ell, k)\)th component of \(A_V\). From the definition of the Bloch equation it follows

\[
a_{\ell k} = \text{Tr}(V^\dagger\sigma_\ell V\sigma_k) - \frac{1}{2}\text{Tr}(V^\dagger V\{\sigma_\ell, \sigma_k\}),
\]

where \(A, B = AB + BA\) is the anticommutator, and thus the \((\ell, k)\)th component of \(A_V A_V^T\) is equal to \(\sum_j a_{\ell j}a_{kj}\) and the \((\ell, k)\)th component of \(A_V^T A_V\) is \(\sum_j a_{\ell j}a_{jk}\). If \(H_1\) and \(H_2\) are two Hermitian matrices and at least one has zero trace, we have the identity

\[
\sum_j \text{Tr}(H_1 \sigma_j) \text{Tr}(H_2 \sigma_j) = \text{Tr}(H_1 H_2),
\]

and thus normality of \(A_V\) is equivalent to

\[
0 = \text{Tr}(\sigma_\ell V\sigma_k V^\dagger [V, V^\dagger]) + \text{Tr}(\sigma_k V\sigma_\ell V^\dagger [V, V^\dagger]) + \text{Tr}(V\sigma_\ell V^\dagger \sigma_k [V, V^\dagger]) + \text{Tr}(V\sigma_k V^\dagger \sigma_\ell [V, V^\dagger]) + \text{Tr}(V^\dagger V\sigma_\ell V^\dagger \sigma_k) - \text{Tr}(V^\dagger V\sigma_k V^\dagger \sigma_\ell),
\]

for all \(k, \ell\), which is satisfied if \([V, V^\dagger] = 0\). \(\square\)

If there are multiple Hamiltonian and Lindblad terms then by linearity of the master equation, the superoperator splits, i.e., \(A = A_H + \sum_d A_{V_d}\), where \(A_H\) is associated with the Hamiltonian dynamics and \(A_{V_d}\) corresponds to the decoherence operator \(V_d\). It is easy to see that \(A_H\) is real-antisymmetric, and thus \(A_H + A_H^T = 0\). Since the sum of negative semi-definite matrices is negative semi-definite, \(A + A^T\) is negative semi-definite if \(A_{V_1} + A_{V_1}^T\) is negative semi-definite for all \(d\), and the latter is the case if \(A_{V_d}\) is normal. Thus we can conclude that for a system governed by the LME \([9]\) the superoperator \(A + A^T\) is negative definite if all Lindblad operators \(V_d\) are normal. This is of course consistent with the previous observation that \(\sum_d [V_d, V_d^\dagger] = 0\) implies contractivity of the HS norm, and hence contractivity of the HS distance between quantum states to either of the two ground states.

**Example 2.** Consider the a three-level system with \(H = 0\) and two Lindblad operators

\[
V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

It is easy to check that \([V_d, V_d^\dagger] \neq 0\) for \(d = 1, 2\) but \(\sum_{d=1,2} [V_d, V_d^\dagger] = 0\) and \(c = 0\). Thus the evolution is unital and both the HS norm and distance are contractive, but the superoperator \(A\) is not normal.

Hence, the evolution of the system may be unital even if \(A\) is not normal. Similarly, the superoperator \(A\) of an open system may be normal even if the evolution is not unital, showing that the condition \(\sum_d [V_d, V_d^\dagger] = 0\) is not a necessary condition for the HS distance between quantum states to be monotonically decreasing.

**Example 3.** Consider a three-level \(\Lambda\) system with decay from the excited state to the ground state as pictured in Fig. 2. Neglecting the Hamiltonian part, the evolution equation given by two spontaneous emission processes, characterized by Lindblad operators

\[
V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

It is easy to check that \([V_1, V_1^\dagger] = \text{diag}(1, 0, -1)\) and \([V_2, V_2^\dagger] = \text{diag}(0, 1, -1)\) and thus \(\sum_d [V_d, V_d^\dagger] \neq 0\). Thus the evolution is not unital, and neither of the corresponding superoperators \(A_{V_1}\) and \(A_{V_2}\) are normal, but if both decay processes are equally likely, the off-diagonal terms in the sum \(A_{V_1} + A_{V_2}\) cancel, and the resulting superoperator \(A\) is diagonal, hence normal. Thus the HS distance between quantum states under this semi-group dynamics is monotonically decreasing although the HS norm is not.

V. MONOTONICITY OF HS DISTANCE FOR NON-NORMAL \(A\), NON-UNITAL EVOLUTION

We have shown that contractivity of the HS norm is a sufficient but not a necessary requirement for contrac-
Furthermore, we have \( c \neq 0 \), i.e., the evolution is not unital. Nonetheless, we can check that the eigenvalues of \( A + A^T \) are \(-2 \pm \frac{2}{3} \sqrt{3}, -\frac{3}{2} \pm \frac{1}{2} \sqrt{5}\) and \(-1\), all of which are negative. (There are only five distinct eigenvalues as the last three occur with multiplicity 2.) Thus the HS distance between any two states is monotonically decreasing, and we can easily verify that the system has a unique steady state \( \rho_{ss} = |1\rangle\langle 1| \) at the boundary of \( \mathcal{D}[\mathcal{H}] \). The HS norm, of course, is not contractive. E.g., if we start in the state \( \rho_0 = \text{diag}(0, 0, 1) \) then the HS norm \( \|\rho(t)\|_2 = \sqrt{\text{Tr}(\rho(t)^2)} \) first decreases and then increases as shown in Fig. 3.

However, examples when \( A \) is not normal, the evolution not unital and the HS norm non-contractive, but the HS distance is still monotonically decreasing appear to be increasingly hard to find in higher dimensions. Numerical tests with randomly generated simple Lindblad generators \( V \) suggest that when the dissipative part of \( A \) is not normal then there is a high probability that the symmetric part \( A + A^T \) will have at least one positive eigenvalue. Moreover, both the probability of a positive eigenvalue and the number of positive eigenvalues of \( A + A^T \) appear to increase with the system dimension (See Table I).

### VI. CONCLUSION

Unlike the trace distance, and perhaps contrary to intuition, the HS norm is generally not contractive under positive trace-preserving maps, except for \( N = 2 \). For open systems governed by a Lindblad master equation the necessary and sufficient conditions for contractivity of the HS norm translate into a necessary and sufficient condition for the Lindblad generators. This condition is also sufficient to ensure that the HS distance between quantum states is monotonically decreasing, but it is not necessary. We derive alternative necessary and sufficient conditions for monotonicity of the HS distance in terms of the spectrum of the symmetric part of the super-operator, and show that they lead to alternative sufficient conditions for monotonicity of the HS distance, which are strictly weaker than those for monotonicity of the HS norm. This means that the HS distance between any two quantum states under Lindblad dynamics can be monotonically decreasing even if the HS norm \( \|\rho(t)\|_2 \) of quantum states is not monotonic under this evolution. Although the criteria for monotonicity of the HS distance are weaker, in general it is not monotonically decreasing if the Hilbert space dimension is greater than 2, even
for systems that have a unique steady state. The non-monotonicity of the HS distance has important implications for, e.g., quantum control, showing that unlike for Hamiltonian systems, it is not a suitable Lyapunov function for generic open systems. It is a suitable candidate for a Lyapunov function only in very special cases, e.g., when all the Lindblad decoherence operators are normal. Although such systems are only a small subset of possible systems in higher dimensions, they do include the important special case where the decoherence is induced by measurement of a Hermitian observable.

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APPENDIX A: STANDARD BASIS FOR HERMITIAN MATRICES

A standard orthonormal basis for the trace-zero Hermitian matrices for any $N$ is given by $\{\sigma_k\}_{k=1}^{N^2-1}$ where $\sigma_k = \sigma_{k(r,s)}$ with $k = r + (s-1)N$ and

\[
\begin{align*}
\sigma_{rs} &= \frac{1}{\sqrt{2}}(|r\rangle\langle s| + |s\rangle\langle r|) \\
\sigma_{sr} &= i\frac{1}{\sqrt{2}}(-|r\rangle\langle s| + |s\rangle\langle r|) \\
\sigma_{rr} &= \frac{1}{\sqrt{r+s}}(\sum_{k=1}^{N^2} |k\rangle\langle k| - r|r+1\rangle\langle r+1|)
\end{align*}
\]

for $1 \leq r \leq N-1$ and $r < s \leq N$.

APPENDIX B: BLOCH REPRESENTATION

Let $\{\sigma_k\}_{k=1}^{N^2}$ be a basis for the Hermitian matrices. With respect to this basis the master equation \cite{9} with dissipation term \cite{10} can be written in coordinate form as a linear matrix differential equation (DE)

\[
\dot{r} = (L + \sum_d D^{(d)}_r) r, \quad \text{where} \quad r = (r_n) \in \mathbb{R}^{N^2} \quad \text{with} \quad r_n = \text{Tr}(\rho \sigma_n) \quad \text{and} \quad L \text{ and } D^{(d)} \text{ are } N^2 \times N^2 \text{ (real) matrices with entries}
\]

\[
L_{mn} = \text{Tr}(i[H_{\sigma_m}, \sigma_n]) \quad \text{(B1a)}
\]

\[
D^{(d)}_{mn} = \text{Tr}(V_d^\dagger \sigma_m V_d \sigma_n) - \frac{1}{2} \text{Tr}(V_d^\dagger V_d \sigma_m \sigma_n) \quad \text{(B1b)}
\]

where $\{A, B\} = AB + BA$ is the usual anticommutator.

If we choose the basis such that $\sigma_{N^2} = \frac{1}{\sqrt{N}} I$ and the remaining basis elements form a basis for the trace-zero Hermitian matrices, then, noting that $r_N = \frac{1}{\sqrt{N}} \text{Tr}(\rho) = \frac{1}{\sqrt{N}}$ is constant and thus $\dot{r}_N = 0$, we can define the reduced so-called Bloch vector $s = (r_1, \ldots, r_{N^2-1})^T$, and rewrite the linear matrix DE for $r$ as affine-linear matrix DE $\dot{s}(t) = A s(t) + c$ for $s$, where $A$ is an $(N^2 - 1) \times (N^2 - 1)$ real matrix with $A_{mn} = L_{mn} + \sum_d D^{(d)}_{mn}$ and $c_m = \frac{1}{\sqrt{N}} \sum_d D^{(d)}_{mN} = \frac{1}{N} \sum_d \text{Tr}(V_d^\dagger V_d) \sigma_m$.

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[13] If we used the unnormalized Pauli matrices as a basis we would obtain a factor of $\frac{1}{2}$ instead, but using an orthonormal basis the Bloch sphere for a single qubit has radius $\sqrt{\frac{1}{2}}$ instead of 1, whence we only a factor of $\sqrt{\frac{1}{2}}$.
[14] The Hamiltonian part of the dynamics does not contribute to the symmetric part $A + A^T$ of $A$ and thus is of no concern here.