The spherically symmetric hedgehog ansatz used in the description of the skyrmion is believed to be inadequate for the rotational states such as the nucleon \((I = J = \frac{1}{2})\) and the \(\Delta \ (I = J = \frac{3}{2})\) due to centrifugal forces. We study here a simple alternative: an oblate spheroidal solution which leads to lower masses for these baryons. As one might expect, the shape of the solution is flatter as one increases \(I = J\) whether the size of the soliton is allowed to change or not.

I. INTRODUCTION

When Skyrme first introduced its model a few decades ago \([1]\) to describe baryons as solitons in a non-linear field theory of mesons, the solution proposed was in the spherically symmetric hedgehog ansatz. There are reasons to believe that this solution is not adequate for the rotational states such as the nucleon \((I = J = \frac{1}{2})\) and the \(\Delta \ (I = J = \frac{3}{2})\) due to centrifugal forces \([2–4]\). Alternative treatments have been proposed in the past with relative success. These approaches generally fall into three classes: (a) The original spherical shape of the solution is modified. This is usually done by making a global deformation along one or more axes \([5]\). (b) The size of the skyrmion is allowed to change. This analysis also led to the identification of breathing modes \([5,6]\) with excited states of the nucleon and \(\Delta\)-isobar. Combining deformations (a) and (b), one gets the following scheme: the nucleon’s and \(\Delta\)-isobar’s ground states have \(K = I + J = 0\) and spherical symmetry. This led to the conclusion that these states are stable against quadrupole deformation \([4,5]\). The \(K \neq 0\) states occur in nearly degenerate doublets for all spin: one oblate and one prolate. (c) Finally, the shape of the solution itself could be improved by including the (iso-) rotational kinetic energy in the minimization of the static Hamiltonian. It turns out that there is no finite static hedgehog solution unless one considers massive pions \([7,8]\). This instability is understood to come from the emission of pions from a rapidly rotating skyrmion. Some progress has been made recently to elucidate the connection between skyrmions and Feynman diagrams in an effective field theory but it is also interesting to note that the solution were characterized by small quadrupole deformations away from the spherical hedgehog ansatz \([9,10]\).

In this work, we take a naive approach and propose a simple alternative. Instead of the spherically symmetric hedgehog solution, we introduce an oblate spheroidal solution. This leads to lower masses and quadrupole deformations for these baryons. Moreover, the shape of the solution is flatter as one increases \(I = J\) whether one allows the size of the soliton to change or not.

II. THE STATIC OBLATE SOLITON

The oblate spheroidal coordinates \((\eta, \theta, \phi)\) are related to Cartesian coordinates through the expressions

\[
\begin{align*}
x &= d \cosh \eta \sin \theta \cos \phi \\
y &= d \cosh \eta \sin \theta \sin \phi \\
z &= d \sinh \eta \cos \theta.
\end{align*}
\]

A surface of constant \(\eta\) corresponds to a sphere of radius \(d\) flattened in the \(z\)-direction by a factor of \(\tanh \eta\). For \(\eta\) small, the shape of the surface is more like that of a pancake of radius \(d\) whereas for large \(\eta\), one recovers a spherical
shell of radius $r = \frac{4d}{\varepsilon}$. Let us note that in the limit $d \to 0$, $\eta \to \infty$ with $r$ remaining finite, the coordinate system becomes

$$(x, y, z) = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

which means that it coincides with the spherical coordinate system. The choice of the parameter $d$ determines at what scale the “oblateness” becomes important. The element of volume is given by

$$dV = -d^3 (\cosh \eta) (\cosh^2 \eta - 1 + \cos^2 \theta) \cdot d\eta d(\cos \theta) d\phi$$  \hspace{1cm} (2)

We would like to replace the hedgehog solution for the Skyrme model by an oblate solution. Writing the Lagrangian for the Skyrme model \[11\]

$$\mathcal{L} = -\frac{F^2}{16} Tr (L_\mu L^\mu) + \frac{1}{32e^2} Tr \left([L_\mu, L_\nu]^2\right)$$  \hspace{1cm} (3)

where $L_\mu = U^* \partial_\mu U$ with $U \in SU(2)$. We get the usual expression for static energy density

$$E = E_2 + E_4 = -\frac{F^2}{16} Tr (L_i L_i) - \frac{1}{32e^2} Tr \left([L_i, L_j]_L^2\right).$$  \hspace{1cm} (4)

Let us now define a static oblate solution by

$$U = e^{i(\tau \cdot \hat{\eta}) f(\eta)}$$  \hspace{1cm} (5)

where $\hat{\eta}$ is the unit vector $\hat{\eta} = \nabla \eta \left| \nabla \eta \right|$. The boundary conditions for the winding number $N = 1$ solution are $f(0) = \pi$ and $f(\infty) = 0$. Note that this is not a priori a solution of the field equations derived from the Skyrme Lagrangian.

Using the oblate ansatz $U = \exp \left[i(\tau \cdot \hat{\eta}) f(\eta)\right]$ we get, after a straightforward but tedious calculation and an integration over the angular variables $\theta$ and $\phi$, the static energy contributions $E_2$ and $E_4$ such that

$$E_2 = \int dV \mathcal{E}_2 = \frac{4\pi \varepsilon \bar{d}}{\lambda} \cdot \frac{\bar{d}}{2} \int_0^\infty d\eta \left( \alpha_{21} f^2 + \alpha_{22} \sin^2 f \right)$$  \hspace{1cm} (6)

with

$$\alpha_{21} (\eta) = 2 \cosh \eta$$

$$\alpha_{22} (\eta) = 2 \left( -2 \cosh \eta + (2 \cosh^2 \eta - 1) L (\eta) \right)$$

where $L (\eta) \equiv \ln \left(\frac{\cosh \eta + 1}{\cosh \eta - 1}\right)$ and

$$E_4 = \int dV \mathcal{E}_4 = \frac{4\pi \varepsilon}{\lambda} \cdot \frac{1}{4d} \cdot \frac{\bar{d}}{4} \int_0^\infty d\eta \left( \alpha_{41} f^2 \sin^2 f + \alpha_{42} f^4 \sin^4 f \right)$$  \hspace{1cm} (7)

with

$$\alpha_{41} (\eta) = 2L (\eta)$$

$$\alpha_{42} (\eta) = \frac{1}{2} \left( \frac{1}{\cosh^2 \eta} (2 \cosh \eta + L (\eta)) + \frac{2 \cosh \eta}{(\cosh^2 \eta - 1)} \right)$$

Here, we have expressed the parameters of the model in terms of the constants:

$$\varepsilon = \frac{1}{\sqrt{2} e} \quad \lambda = \frac{2}{F_\pi} \quad \bar{d} = \frac{eF_\pi}{2\sqrt{2}} d.$$  \hspace{1cm} (8)

It might be required to add a pion mass term to the Skyrme Lagrangian. This term takes the usual form

$$\mathcal{L}_m = \frac{m^2 F_\pi^2}{8} (Tr U - 2)$$  \hspace{1cm} (9)

2
leading to the expression

\[ E_{m\pi} = \frac{4\pi \epsilon}{\lambda} \cdot 32\epsilon^2 \epsilon_\pi \overline{d}^3 \cdot \int_0^\infty d\eta \alpha_m (1 - \cos f) \]  

(10)

where \( \epsilon_\pi = \frac{m_\pi}{\rho_\pi} \) and

\[ \alpha_m (\eta) = \cosh \eta \left( \sinh^2 \eta + \frac{1}{3} \right). \]

Minimizing the static energy with respect to \( f (\eta) \), we need to solve (numerically of course) the non-linear ordinary second-order differential equation which reads:

\[
0 = 32\epsilon^2 \epsilon_\pi \overline{d}^3 \cdot \alpha_m \sin f + \frac{\overline{d}}{2} \left( -2f'' \alpha_{21} - 2f' \alpha'_{21} + 2\alpha_{22} \sin f \cos f \right) \\
+ \frac{1}{4\overline{d}} \cdot \left( (4 \sin^3 f \cos f) \alpha_{42} + (-2f'' \sin^2 f - 2f'^2 \sin f \cos f) \alpha_{41} - (2f' \sin^2 f) \alpha'_{41} \right). 
\]

(11)

For calculational purposes, we need to set the value of the parameters of the Skyrme Model. \( F_\pi \) and \( e \) are first chosen to coincide with those of ref. [11]:

\[
F_\pi = 129 \text{ MeV} \quad e = 5.45 \quad m_\pi = 0 \\
F_\pi = 108 \text{ MeV} \quad e = 4.84 \quad m_\pi = 138 \text{ MeV} 
\]

obtained by fitting for the masses of the nucleon and the \( \Delta \) in the hedgehog ansatz.

The solution near \( \eta \to 0 \) has the form

\[ f (\eta) \sim \pi - a_1 \eta \]

whereas in the limit \( \eta \to \infty \), one recovers the spherical symmetry with,

\[ f (\eta) \sim k \left[ \frac{2m_\pi}{d \eta} + \frac{4}{(d \eta)^2} \right] \exp \left( -\frac{m_\pi d \eta}{2} \right). \]

(14)

where \( a_1 \) and \( k \) are constants which depend on \( \overline{d} \) and \( m_\pi \). The solutions of differential equation [11] are presented for several values of \( \overline{d} \) and \( m_\pi = 0 \) in Fig. [5]. For small \( \overline{d} \) (here \( \overline{d} \leq 0.0001 \)), we get exactly the solution of the spherical hedgehog skyrmion. As one increases \( \overline{d} \), we observe a displacement of the function \( f (\eta) \) and the continuous deformation of the soliton from a spherical to an oblate shape. The static energy, \( E_s = E_{m\pi} + E_2 + E_4 \), has a minimum for \( \overline{d} = 0 \) which is expected since it corresponds to the spherical solution (see Fig. [3]).

The masses of the nucleon and of the \( \Delta \)-isobar get contributions both from the static and rotational energy and will generally depend on the choice of \( \overline{d} \). We fix the value of \( \overline{d} \) for each baryon by minimizing its mass with respect to \( \overline{d} \).

### III. COLLECTIVE VARIABLES

Using the oblate solution, we can then compute the masses of the nucleons \( (I = J = \frac{1}{2}) \) and of the \( \Delta \)-isobar \( (I = J = \frac{3}{2}) \). However, several remarks are in order before we go on. When one departs from the spherical symmetry of the hedgehog ansatz, it is customary to introduce extra collective variables for isorotation in addition to those characterizing spatial rotation since these are no longer equivalent, in general. The spin and isospin contributions to the rotational energy are however equal in our case since we use solution [5] and we are only interested in ground states with \( K = J + I = 0 \) (see Appendix). As a result, we need only consider one set of collective variables.

Let us work in the body-fixed system and assume that the time dependence can be introduced using the usual substitution

\[ U \to A(t) U A^\dagger (t) \]

(15)
where $A(t)$ is a time-dependent $SU(2)$ matrix. This transformation leaves the static energy (or mass of the soliton) invariant. We can then go on and treat $A(t)$ approximately as quantum mechanical variables. The calculation procedure is fairly standard (see ref. [11] for example).

Using (13), the Lagrangian gets new terms due to the time dependence of $A$:

$$L_2^t = \int dV L_2^t = -\frac{F_2^2}{16} \int dV \mathrm{Tr} \left( \bar{L}_0 \bar{L}^0 \right)$$

and

$$L_4^t = \int dV L_4^t = -\frac{1}{32\epsilon^2} \int dV \mathrm{Tr} \left( \left[ L_0, L_i \right]^2 \right)$$

where

$$\bar{L}_0 = AU^t A^\dagger \partial_0 \left( AU A^\dagger \right)$$

Following straightforward but lengthy calculations, we get after angular integrations

$$L_2^t = \frac{1}{2} a_2^{ij} \mathrm{Tr} \left[ \tau_i A^t \hat{A} \right] \mathrm{Tr} \left[ \tau_j A^t \hat{A} \right]$$

and

$$L_4^t = -\frac{1}{2} a_4^{ij} \mathrm{Tr} \left[ \tau_i A^t \hat{A} \right] \mathrm{Tr} \left[ \tau_j A^t \hat{A} \right]$$

where

$$a_2^{ij} = \frac{\lambda}{4\pi\epsilon} \cdot 128\pi^2 \epsilon^4 \bar{d}^2 \cdot \frac{d}{2} \int_0^\infty \mathrm{d} \eta \cosh \eta \sin^2 f \ A_{ij}$$

$$a_4^{ij} = \frac{\lambda}{4\pi\epsilon} \cdot 128\pi^2 \epsilon^4 \bar{d}^2 \cdot \frac{1}{4d} \int_0^\infty \mathrm{d} \eta \cosh \eta \sin^2 f \ [C_{ij} \sin^2 f + B_{ij} f^2]$$

with

$$A_{11} = A_{22} = \left( \cosh^2 \eta - \frac{1}{2} \right) \left( -\frac{2}{3} (3 \cosh^2 \eta - 4) + \cosh \eta \sinh^2 \eta \ L(\eta) \right)$$

$$A_{33} = 4 \cosh^4 \eta - \frac{10}{3} \cosh^2 \eta - (2 \cosh^5 \eta - 3 \cosh^3 \eta + \cosh \eta) \ L(\eta)$$

$$B_{11} = B_{22} = 2 - \cosh^2 \eta - \frac{1}{2} \cosh \eta \ (1 - \cosh^2 \eta) \ L(\eta)$$

$$B_{33} = 2 \cosh^2 \eta + \cosh \eta \ (1 - \cosh^2 \eta) \ L(\eta)$$

$$C_{11} = C_{22} = \left( -\frac{1}{4 \cosh \eta} \right) (10 \cosh \eta - 8 \cosh^3 \eta + (4 \cosh^4 \eta - 9 \cosh^2 \eta + 3) \ L(\eta))$$

$$C_{33} = \frac{1}{2 \cosh \eta} \left( (4 \cosh^4 \eta - \cosh^2 \eta - 1) \ L(\eta) - 8 \cosh^3 \eta + 2 \cosh \eta \right).$$

Non-diagonal terms for $A_{ij}$, $B_{ij}$ and $C_{ij}$ give zero contribution upon $\phi$ integration due to the axial symmetry of the solution.

Let us now consider the quantity

$$\mathrm{Tr} \left[ \tau_i A^t \hat{A} \right] \mathrm{Tr} \left[ \tau_j A^t \hat{A} \right] a_n^{ij}$$

for an axially symmetric system where $a_n^{11} = a_n^{22} \neq a_n^{33}$ and $a_n^{ij} = 0$ for $i \neq j$. We can rewrite

$$\mathrm{Tr} \left[ \tau_i A^t \hat{A} \right] \mathrm{Tr} \left[ \tau_j A^t \hat{A} \right] a_n^{ij} = \mathrm{Tr} \left( \hat{A} \hat{A}^t \right) a_n^{11} + \left( \mathrm{Tr} \left[ \tau_3 A^t \hat{A} \right] \right)^2 (a_n^{33} - a_n^{11}).$$

In terms of the Euler angles $\Theta$, $\Phi$ and $\Psi$, the traces correspond to the expressions
Here and where in space and isospace number denoted by $m$.

which finally leads to

$$L_2 + L_4 = \frac{b}{2} \left( \hat{\Theta}^2 + \sin^2 \Theta \hat{\Phi}^2 \right) + \frac{c}{2} \left( \hat{\Psi} + \cos \Theta \hat{\Phi} \right)^2$$

$$= \frac{b}{2} \Omega_1^2 + \frac{c}{2} \Omega_2^2$$

where

$$b = (a_2^{11} + a_4^{11})$$

$$c = (a_2^{31} + a_4^{33})$$

Here $b$ and $c$ play the role of principal moment of inertia.

Quantization of (26) is straightforward. It indeed represents a symmetrical top with the rotational kinetic energy in space and isospace

$$E_{rot}^{J,J_3} = \frac{1}{2b} \left( |J|^2 + |I|^2 \right) + \frac{1}{2} \left( \frac{1}{c} - \frac{1}{b} \right) J_3^2.$$  

where $|J|^2$ and $|I|^2$ are the spin and the isospin respectively and, $J_3$, the z-component of the spin. We have already used the relation $J_3 = -I_3$ here which follows from axial symmetry. Added to the static energy $E_s$, it leads to the total energy

$$M^{J,J_3} = E_s + E_{rot}^{J,J_3}. \quad (30)$$

Up to now, we have analyzed the rotational and isorotational kinetic energy from the point of view of the body-fixed frame. Observables states, however, must be eigenstates of $|J|^2$, $J_3$, $|I|^2$, $I_3$ with eigenvalues $J(J+1)$, $m_J$, $I(I+1)$, $m_I$ where the operators now refer to the laboratory system (as opposed to body-fixed operators in (29) and above). These eigenstates can be represented by direct products of rotation matrices

$$\langle \alpha, \beta, \gamma | J, m_J, m \rangle \langle \rho, \sigma, \tau | I, m_I, -m \rangle = D_{mJm}^J (\alpha, \beta, \gamma) D_{mIm}^I (\rho, \sigma, \tau) \quad (31)$$

where $(\alpha, \beta, \gamma)$ and $(\rho, \sigma, \tau)$ are, respectively, the Euler angles for the rotation and isorotation from the body-fixed frame to the laboratory system. Since we have axial symmetry in the body-fixed system where $J_3 = -I_3$, the quantum number denoted by $m$ must be opposite in sign in space and isospace rotation matrices. It is convenient to label the basis by the sum of the body-fixed spin and isospin, $K$. The eigenstates of $K^2$ are linear combinations of the basis states 

with Clebsch-Gordan coefficients $\langle J, m; I, -m | K, 0 \rangle$. The explicit calculation of the energy of rotation requires in general the diagonalization $E_{rot}^{J,J_3}$. (see ref. [3] for more details). Since we are only interested in the ground states here, i.e. the nucleon and $\Delta$-isobar, we set $K = 0$ which simplifies much of the above procedure.

We proceeded with the case $m_x = 0$. Numerically, the minimization of the static energy for the spherical symmetric ansatz gives $E_s = \frac{4\pi}{3} \cdot [8.20675]$, $M_N = \frac{4\pi}{3} \cdot [8.906]$ and $M_\Delta = \frac{4\pi}{3} \cdot [11.703]$. For the oblate spheroidal ansatz, the solution for $f(\eta)$ is found from (11) and the parameter $\tilde{d}$ is chosen in order to minimize the mass of the corresponding baryon. In general, as $\tilde{d}$ increases, the static energy $E_s$ deviates from its lowest energy value given by the spherical hedgehog configuration. On the other hand, oblate configurations have larger moment of inertia which tends to decrease the rotational kinetic energy (see Fig. 2). The existence of a non-trivial oblate spheroidal ground state for the nucleon and the $\Delta$-isobar, as it turns out, depends mostly on the relative importance of static and rotational energy.

Our results are summarized in Table 1. We find that the ground state for the nucleon is almost spherical but nonetheless oblate with $\tilde{d} = 0.0013$ thus exhibiting a small quadrupole deformation and a slightly lower mass with respect to a spherical configuration. For the $\Delta$-isobar, the oblateness or quadrupole deformation is even more important and accounts for a 4% decrease in mass. We obtain a minimum for the $\Delta$ mass for a value of $\tilde{d} = 0.32$ with $M_\Delta = \frac{4\pi}{3} \cdot [11.293]$. 

\[ TR \left( \hat{A} \hat{A}^\dagger \right) = \frac{1}{2} \left( \hat{\Theta}^2 + \hat{\Phi}^2 + \hat{\Psi}^2 + 2 \cos \Theta \hat{\Phi} \hat{\Psi} \right) \quad (24) \]

\[ TR \left[ \tau_3 \hat{A} \hat{A}^\dagger \right] = i \left( \hat{\Psi} + \cos \Theta \hat{\Phi} \right) \quad (25) \]

where

\[ L_2 + L_4 = \frac{b}{2} \left( \hat{\Theta}^2 + \sin^2 \Theta \hat{\Phi}^2 \right) + \frac{c}{2} \left( \hat{\Psi} + \cos \Theta \hat{\Phi} \right)^2 \]

\[ = \frac{b}{2} \Omega_1^2 + \frac{c}{2} \Omega_2^2 \quad (26) \]

Here $b$ and $c$ play the role of principal moment of inertia.
Since the minimum of the ground state is affected by the oblate shape of the solution, the parameters $F_\pi$ and $e$ as given in ref. [11] no longer reproduce the quantities they were designed to fit. However, the existence and the form of an oblate ground state for baryons depends on the precise value of $F_\pi$, $e$ and $d$ through eq. (11). Therefore in order to fit for $M_N = 939$ MeV and $M_\Delta = 1232$ MeV, we must readjust $F_\pi$ and $e$ which determine the value of $d$ for the nucleon and $\Delta$-isobar respectively. After several iterations, we find $F_\pi = 118.4$ MeV and $e = 5.10$ with $d = 0.0014$ ($\bar{d} = 0.40$) for the nucleon ($\Delta$-isobar).

The numerical calculations for $m_\pi \neq 0$ lead to similar conclusions. Starting from input values for $F_\pi$ and $e$ in (13), we get a deformation parameter of $\bar{d} = 0.0009$ for the nucleon and $\bar{d} = 0.18$ for the $\Delta$-isobar leading to small decreases in their respective masses. The deformation parameters here are significantly smaller than what is observed in the $m_\pi = 0$ case, which is partly explained by the sensitivity of $\bar{d}$ with respect to $F_\pi$ and $e$. But since the chiral symmetry breaking term contributes here (i.e. $E_{m_\pi}$), this is also connected to the relative importance of the rotational energy contribution to the baryon mass and perhaps more importantly to how each contribution depends on $\bar{d}$.

IV. DISCUSSION

Quadrupole deformations were found previously [11] in the context of rotationally improved skyrmions. Contrary to our variational approach, these solutions involve the minimization of a Hamiltonian which includes the (iso-) rotational kinetic energy, i.e. eq. (11) with contributions from the (iso) rotational kinetic energy. Nonetheless, we found that the oblate spheroidal ansatz gives lower energy than the spherical one for baryon ground states.

Of course, ansatz (5) is not necessarily the lowest energy solution, the latter being obtained in principle by solving the integro-differential equation of ref. [10]. Unfortunately, only large-distance asymptotics of this solution can be written in a closed form. Moreover, the most relevant physical quantity here, the mass of the baryons, gets negligible contributions from that region and so it is not very sensitive to the exact form of the solution at large distances. Yet, it may be interesting to consider deformations of the oblate skyrmions under scaling of the unitary transformation $U(r)$ in which the pion field reads

$$\pi(r, J) \sim \frac{B}{J^2} \left\{ \frac{m_\pi}{r} + \frac{1}{r^2} \right\} \exp \left( -m_\pi r \right) \left( \tilde{J} \cdot r \right) \tilde{J}$$

$$+ \left[ \frac{\sqrt{m_\pi^2 - \tilde{J}^2}}{r} + \frac{1}{r^2} \right] \exp \left( -\frac{m_\pi^2 - \tilde{J}^2 r}{r} \right) \left( \tilde{J} \times r \times \tilde{J} \right).$$

(32)

Here $\tilde{J}^k = J^k (I_{mk})^{-1}$ where $J^k$ and $I_{mk}$ are the Skyrmion classical angular momentum and moment of inertia tensor respectively. As one might expect, we recover the hedgehog solution form in the limit $\tilde{J}^2 \rightarrow 0$ and $r \rightarrow \infty$. For $\tilde{J}^2 \neq 0$, the second term in (32) dominates which can be interpreted as a swelling of the Skyrmion with the pion field pointing in a direction perpendicular to $\tilde{J}$ due to centrifugal forces. Unfortunately, the exact magnitude and direction of $\tilde{J}$ can only be obtained by solving the full integro-differential equation. On the other hand, for the oblate ansatz the pion field takes the form $\pi(\eta) = \hat{n} f(\eta)$ and coincide with the hedgehog solution in the limit of large distances. The magnitude of $k$ in (13) is found by solving (11) and optimizing for the deformation parameter $\bar{d}$. Our numerical calculations show that $k$ increases slowly with $\bar{d}$, which suggests that the configuration of the Skyrmion at large distances is hedgehog-like and swelling for increasing isospin. This is in partial agreement with the qualitative features of (12). Be that as it may, we recall that the purpose was mainly to look at possible deformations at medium-range distances since this is where energy and baryonic densities are the largest. Ansatz (5) turned out to be a rather simple, intuitive and efficient trial solution.

It may also be interesting to consider deformations of the oblate skyrmions under scaling of the unitary transformations $U(r)$ such that

$$U(r) = U_0 (\rho r)$$

(33)

to minimize the total energy of the nucleon and $\Delta$-isobar. This corresponds to skyrmions which are allowed to change in size. Recall that the previous calculations proposed a change in shape (oblate spheroidal vs spherical). The treatment is straightforward and indeed very similar to that of ref. [5]. In our calculations the scale transformation is equivalent to the substitution

$$\bar{d} \rightarrow \frac{\bar{d}}{\rho}.$$  

(34)
Rewriting the expression for the masses in the body-fixed frame as:

\[ M^{J,J_3} = E_m + E_2 + E_4 + \frac{(J (J + 1) - J_3^2)}{2 (a_{11}^2 + a_{22}^2)} + \frac{J_3^2}{2 (a_{33}^2 + \rho^2 a_{44}^2)} \]  

(35)

we see that under the scaling transformation we have:

\[ M^{J,J_3}(\rho) = \frac{E_m}{\rho^3} + \frac{E_2}{\rho} + \rho E_4 + \frac{\rho^3 (J (J + 1) - J_3^2)}{2 (a_{11}^2 + \rho^2 a_{22}^2)} + \frac{\rho^3 J_3^2}{2 (a_{33}^2 + \rho^2 a_{44}^2)}. \]  

(36)

The total energies \( M_N(\rho) \) and \( M_\Delta(\rho) \), computed in the laboratory system, can be minimized with respect to the \( \rho \) parameter, i.e. to the energically favored size of the oblate skyrmion. The results are shown in Table I for both the oblate and spherical cases. The baryon ground states are now swollen oblate solutions. Again, one should in principle readjust the \( F_\pi \) and \( \epsilon \) parameters to fit the masses of the nucleon and \( \Delta \)-isobar. It would also be interesting to readdress the problem of breathing modes with these oblate skyrmions. This is a problem for further research.

ACKNOWLEDGMENTS

We are indebted to Dr. N.N. Scoccola and M. Paranjape for useful comments and discussions. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by the Fonds pour la Formation de Chercheurs et l’Aide à la Recherche du Québec.

APPENDIX:

The rotational energy for an axially symmetric system is given by

\[ L' = \frac{A}{2} (\omega_1^2 + \omega_2^2) + \frac{B}{2} (\Omega_1^2 + \Omega_2^2) + \frac{C}{2} (\omega_1 - \Omega_3)^2 - D (\omega_1 \Omega_1 + \omega_2 \Omega_2) \]

where \( \omega \) and \( \Omega \) are the angular velocities in coordinate and isospin space. \( A, B, C \) and \( D \) are positive quantities corresponding to moments of inertia. The previous expression can be written in terms of body-fixed angular momentum \( J \) and isospin \( I \):

\[ L' = \frac{1}{AB - D^2} \left[ A (I_1^2 + I_2^2) + B (J_1^2 + J_2^2) + 2D (I_1 J_1 + I_2 J_2) \right] + \frac{J_3^2}{C} \]

where \( I_i \) and \( J_i \) are the body-fixed components of \( J \) and \( I \). Here we have already used the fact that \( J_3 = -I_3 \) because of axial symmetry. Clearly, the nucleon and \( \Delta \)-isobar ground states are obtained when \( K = J + I = 0 \) since the term \( 2D (I_1 J_1 + I_2 J_2) \) is then negative. The general expressions for \( A, B, C \) and \( D \) are rather lengthy and therefore not given here (see ref. [13]). However, they become much simpler for a solution of the form (5) in which case \( A = B = D \) such that the rotational energy of the nucleon and \( \Delta \)-isobar ground states are given by

\[ L' = \frac{1}{2A} \left( |J|^2 - J_3^2 \right) + \frac{1}{2C} J_3^2 + \frac{1}{2A} \left( |I|^2 - I_3^2 \right) + \frac{1}{2C} I_3^2. \]

The spin and isospin contributions to the rotational energy are equal in this case.

[1] T.H.R. Skyrme, Proc. R. Soc. London A260, 127 (1961).
[2] J.P. Blaizot and G. Ripka, Phys. Rev. D38, 1556 (1988).
[3] B.A. Li, K.F. Liu and M.M. Zhang, Phys. Rev. D35, 1693 (1987).
[4] J. Wambach, H.W. Wyld, H.M. Sommermann, Phys. Lett. B186, 272 (1987).
[5] Ch. Hajduk and B. Schwesinger, Phys. Lett. B140, 172 (1984); B145, 171 (1984); Nucl. Phys. A453, 620 (1986).
[6] L.C. Biedenharn, Y. Dothan and M. Tarlini, Phys. Rev. D31, 649 (1985).
[7] E. Braaten and J.P. Ralston, Phys. Rev. D31, 598 (1985).
[8] K.F. Liu, J.S. Zhang and G.R.E. Black, Phys. Rev. D30, 2015 (1984).
[9] B.J. Schroers, Z. Phys. C61, 479 (1994)
[10] N. Dorey, J. Hughes and M.P. Mattis, Phys. Rev. D50, 5816 (1994).
[11] G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228, 552 (1983); G.S. Adkins, C.R. Nappi, Nucl. Phys. B233, 109 (1984).
[12] Particle Data Goup, Phys. Rev. D54, 1 (1996).
[13] V.G. Mankhankov, Y.P. Rybakov and V.I. Sanyuk, The Skyrme model: fundamentals, methods, applications. Springer-Verlag, pages 120-123 (1993).
FIG. 1. Solutions of differential equation (11) for several values of \( \tilde{d} \) \( (F_\pi = 129 \text{ MeV}, \, e = 5.45 \text{ and } m_\pi = 0) \). For \( \tilde{d} = 0.0001 \), we get exactly the solution of the spherical hedgehog skyrmion.

FIG. 2. Static and rotational energies for the nucleon as a function of \( \tilde{d} \) in units of \( \frac{\hbar}{\sqrt{F_\pi}} \).

TABLE I. Ground states for the nucleon and \( \Delta \)-isobar. The results are shown for both the minimum oblate spheroidal configuration and the spherically symmetric ansatz for comparison. The values of \( M^{J, J_3} \) are defined according to eq. (30) whereas \( M^{J, J_3}(\rho_{\min}) \) is minimized with respect to the scaling parameter \( \rho \) (see eq. (36)). All masses are expressed in units of \( \frac{\hbar}{\sqrt{F_\pi}} \) with parameters \( F_\pi = 129 \text{ MeV}, \, e = 5.45 \text{ and } m_\pi = 0 \).

|                 | Oblate \((\tilde{d} > 0)\) |    | Spherical \((\tilde{d} = 0)\) |    |
|-----------------|-------------------------|---|-------------------------|---|
|                 | \( d \) | \( M^{J, J_3} \) | \( \rho_{\min} \) | \( M^{J, J_3}(\rho_{\min}) \) | \( M^{J, J_3} \) | \( \rho_{\min} \) | \( M^{J, J_3}(\rho_{\min}) \) |
| Nucleon         | 0.0013 | 8.904 | 0.808 | 8.797 | 8.906 | 0.867 | 8.799 |
| \( \Delta \)    | 0.32   | 11.312 | 0.670 | 10.064 | 11.703 | 0.668 | 10.238 |