Spins of Primordial Black Holes Formed in the Radiation-dominated Phase of the Universe: First-order Effect

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Abstract

The standard deviation of the initial values of the nondimensional Kerr parameter $a_*$ of primordial black holes (PBHs) that formed in the radiation-dominated phase of the universe is estimated to the first order of perturbation for the narrow power spectrum. Evaluating the angular momentum at turnaround based on linearly extrapolated transfer functions and peak theory, we obtain the expression $\sqrt{\langle a_*^2 \rangle} \simeq 4.0 \times 10^{-3} (M/M_H)^{-1/3} \sqrt{1 - \gamma^2 [1 - 0.072 \log_{10}(\beta_0(M_H)/(1.3 \times 10^{-15})]^{-1} }$, where $M_H$, $\beta_0(M_H)$, and $\gamma$ are the mass within the Hubble horizon at the horizon entry of the overdense region, the fraction of the universe which collapsed to PBHs at the scale of $M_H$, and a quantity that characterizes the width of the power spectrum, respectively. This implies that for $M \approx M_H$, the higher the probability of the PBH formation, the larger the standard deviation of the spins, while PBHs of $M \ll M_H$ that formed through near-critical collapse may have larger spins than those of $M \approx M_H$. In comparison to the previous estimate, the new estimate has an explicit dependence on the ratio $M/M_H$ and no direct dependence on the current dark matter density. On the other hand, it suggests that the first-order effect can be numerically comparable to the second-order one.

Unified Astronomy Thesaurus concepts: Primordial black holes (1292)

1. Introduction

Recently, primordial black holes (PBHs) have been intensively investigated not only as a realistic candidate for dark matter (Carr et al. 2010, 2016, 2017, 2020) but also as a possible origin of black holes of tens of solar masses that are the source of gravitational waves detected by LIGO and Virgo (Nakamura et al. 1997; Bird et al. 2016; Sasaki et al. 2016; Cl esse & García-Bellido 2017; Raidal et al. 2017). Various mechanisms generating PBHs have been proposed. Among them, we will focus on PBHs formed as a result of the collapse of the primordial cosmological perturbation. After inflation generates perturbations at super-horizon scales, the scales successively enter the Hubble horizon in the radiation-dominated phase and the perturbations can collapse to form PBHs if the amplitude of the perturbation exceeds some threshold value. The threshold values have been studied in terms of $\delta_H$, the density perturbation averaged over the overdense region at horizon entry (Carr 1975; Polnarev & Musco 2007; Musco et al. 2009; Harada et al. 2013, 2015; Musco & Miller 2013), although this is currently discussed in a more sophisticated way based on the compaction function (Shibata & Sasaki 1999; Germani & Musco 2019; Musco 2019; Escrivà 2020; Escrivà et al. 2020) and peak theory (Yoo et al. 2018, 2021). Roughly speaking, the mass of the PBH is given by the mass $M_H$ contained within the Hubble horizon at the time of horizon entry $t$, where $M_H \sim c^2 t/G$, although for the near-critical case $\delta_H \simeq \delta_{H,th}$, the scaling law $M/M_H \propto (\delta_{H,th} - \delta_H)^{3/2}$ with $\beta\simeq 0.36$ implies the formation of PBHs of $M \ll M_H$ (Niemeyer & Jedamzik 1999; Musco et al. 2009; Musco & Miller 2013).

Thanks to the uniqueness theorem, isolated stationary black holes in vacuum are perfectly characterized by two parameters, the mass $M$ and the spin angular momentum $S$. Alternatively, we can use the nondimensional spin angular momentum $a_* = S c^2 / GM^2$. The statistical distribution of the spins is a key probe into the origin of black holes. In the gravitational-wave observation of binary black holes by LIGO and Virgo, the effective spin parameter $\chi_{\text{eff}}$ can be measured. Up to now, the observed data for most binary black holes have been consistent with $\chi_{\text{eff}} = 0$ (Abbott et al. 2019), although there are some exceptions (Abbott et al. 2020).

PBHs may have changed their spins from their initial values. PBHs have evaporated away through Hawking radiation if their masses are smaller than $\sim 10^{15}$ g. The spin of the black hole enhances the Hawking radiation and deforms its spectrum. A spinning black hole decreases its nondimensional Kerr parameter $a_* = \sqrt{a_0 \cdot a_*}$ through the Hawking radiation, while a black hole much more massive than $\sim 10^{15}$ g does not significantly change $a_*$ through the Hawking radiation (Page 1976; Arbey et al. 2020; Dasgupta et al. 2020). PBHs change their spins very little in the radiation-dominated phase (Chiba & Yokoyama 2017), while it is proposed that mass accretion could change the spin of black holes in some cosmological scenarios (e.g., De Luca et al. 2020).

In this paper, we investigate the initial values of the spins of PBHs. Recently, this issue has been discussed by many authors from different points of view (Chiba & Yokoyama 2017; Harada et al. 2017; De Luca et al. 2019; He & Suyama 2019; Mirbabayi et al. 2020). Among them, De Luca et al. (2019) apply Heavens & Peacock’s (1988) approach to the first-order effect of perturbation and give a clear expression, $\sqrt{\langle a_*^2 \rangle} \sim \Omega_{\text{dm}} \Omega_{\gamma} \sqrt{1 - \gamma^2} / N$, where $\Omega_{\text{dm}}$, $\Omega_{\gamma}$, and $\gamma = (k^2 / \sqrt{\langle k^2 \rangle})$ are the current ratio of the dark matter component to the critical density, the standard deviation of the
density perturbation at horizon entry of the inverse wavenumber, and a quantity that characterizes the width of the power spectrum, respectively. In this paper, we apply the same approach to this issue but reach a different result. This paper is organized as follows. In Section 2, we define the angular momentum and give its expression to the first order of perturbation in the region that collapses to a PBH. In Section 3, we estimate the angular momentum at the turnaround under the assumption of a narrow spectrum. In Section 4, we estimate the nondimensional Kerr parameter of the PBH. Section 5 is devoted to the summary and a discussion, in particular in comparison to previous works. We use units in which \( c = 1 \) in this paper.

2. Angular Momentum

2.1. Definition

We follow De Luca et al. (2019) for the definition of angular momentum. If the spacetime admits a Killing vector field \( \phi_i \), which is tangent to a spacelike hypersurface and generates a spatial rotation on it, the angular momentum \( S_i(\Sigma) \) contained in the region \( \Sigma \) on the spacelike hypersurface can be defined as a conserved charge in terms of the integral on the boundary \( \partial \Sigma \) as (Wald 1984)

\[
S_i(\Sigma) = -\frac{1}{16\pi G} \int_{\partial \Sigma} \epsilon_{abcd} \nabla^c (\phi_i) d^4x - \frac{1}{8\pi G} \int_{\Sigma} R^{ab} n_a (\phi_i) n_b d\Sigma
\]

where \( n^a \) is the unit vector normal to \( \Sigma \). Using the Einstein equation \( G_{ab} = 8\pi T_{ab} \), Equation (1) transforms to

\[
S_i(\Sigma) = -\int_{\Sigma} T^{ab} n_a (\phi_i) n_b d\Sigma.
\]

Let us use the 3+1 decomposition of the spacetime

\[
ds^2 = -\alpha^2 d\eta^2 + a^2(t) \gamma_{ij} (d\xi^i + \beta^i d\eta)(d\xi^j + \beta^j d\eta).
\]

We assume that the matter field is given by a single perfect fluid described by

\[
T^{ab} = \rho u^a u^b + p(g^{ab} + u^a u^b),
\]

where \( u^a \) is the four-velocity of the fluid element and that the background spacetime is given by a flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime, in which the line element is written in the conformally flat form:

\[
ds^2 = a^2(-d\eta^2 + dx^2 + dy^2 + dz^2).
\]

We can naturally define \( \phi_i^a \), the generator of spatial rotation with respect to the peak of the density perturbation at \( x = x_{pk} \), as

\[
(\phi_i)^a = \epsilon_{ijk} (x - x_{pk})^j \frac{\partial}{\partial x^k}.
\]

To the first order of perturbation from the flat FLRW spacetime, we find

\[
S_i(\Sigma) = (1 + w)a^4 \rho_b \epsilon_{ijk} \int_{\Sigma} (x - x_{pk})^j (\nu - \nu_{pk}) d^3x
\]

in the gauge with \( \beta^k = 0 \), where \( \nu^i = u^i / u^0 \), and we have assumed the equation of state \( p = \omega \rho \). The region \( \Sigma \) should be taken as the region that will collapse into a black hole. Although the determination of \( \Sigma \) is a nontrivial task, following Heavens & Peacock (1988) and De Luca et al. (2019), we assume

\[
\Sigma = \{ x | \delta(x) > 3\delta_{pk} \}.
\]

We truncate the Taylor-series expansion of \( \delta \) around the peak at the second order as

\[
\delta \approx \delta_{pk} + \frac{1}{2} \zeta_{ij} (x - x_{pk})^i (x - x_{pk})^j,
\]

where

\[
\zeta_{ij} \equiv \frac{\partial^2 \delta}{\partial x^i \partial x^j} \bigg|_{x=x_{pk}}.
\]

This truncation is justified provided that physical quantities do not change so steeply within \( \Sigma \). Adjusting the \( x \)-, \( y \)-, and \( z \)-axes to the principal ones, we obtain

\[
\delta \approx \delta_{pk} - \frac{1}{2} \sigma_j \sum_{i=1}^3 \lambda_i ((x - x_{pk})^i)^2,
\]

where \( \sigma_j \) and \( \lambda_i \) are defined in Appendix A. Equations (4) and (5) imply that \( \Sigma \) is given by an ellipsoid with the three axes given by

\[
a_i^2 = \frac{\sigma_0}{\lambda_1} \frac{1 - f}{\sigma_1},
\]

where we have defined \( \nu = \delta_{pk}/\sigma_0 \).

Taking the truncated Taylor-series expansion of \( \nu - \nu_{pk} \) at \( x = x_{pk} \),

\[
\nu^i - \nu_{pk}^i \approx \nu_j^i (x - x_{pk})^j,
\]

we find

\[
S_i(\Sigma) \approx (1 + w)a^4 \rho_b \epsilon_{ijk} \int_{\Sigma} (x - x_{pk})^j (x - x_{pk})^i d^3x
 \]

\[
= (1 + w)a^4 \rho_b \epsilon_{ijk} \nu_j^i J^{jk},
\]

where

\[

\nu_j^i \equiv \frac{\partial \nu^i}{\partial x^j} \bigg|_{x=x_{pk}},
\]

\[
J^{jk} \equiv \int_{\Sigma} (x - x_{pk})^i (x - x_{pk})^j d^3x
\]

\[
= \frac{4\pi}{15} a_1 a_2 a_3 \text{diag} (a_1^2, a_2^2, a_3^2).
\]

Here we concentrate on a growing mode of linear scalar perturbation, which is briefly summarized in Appendix B. According to peak theory (Bardeen et al. 1986; Heavens & Peacock 1988), which is briefly introduced in Appendix A, the distribution of the nondiagonal components of \( \nu_{ij} \) is independent of that of the trace-free part of \( J^{jk} \). Then, we obtain

\[
\sqrt{\langle S_i S_i \rangle} = S_{\text{ref}} \sqrt{\langle \nu_j^i \nu_i^j \rangle},
\]

where

\[
S_{\text{ref}} (\eta) = (1 + w)a^4 \rho_b g(\eta) (1 - f)^{5/2} R_\nu^5,
\]

\[
s_\nu = \frac{16\sqrt{2} \sqrt{3}}{135} \left( \frac{\nu}{\gamma} \right)^{5/2} \frac{1}{\sqrt{\Lambda}} (\alpha_1 \bar{v}_{123} - \alpha_2 \bar{v}_{113} - \alpha_3 \bar{v}_{112}).
\]
\[ \alpha_1 = \frac{1}{\lambda_3} - \frac{1}{\lambda_2}, \quad \alpha_2 = \frac{1}{\lambda_3} - \frac{1}{\lambda_1}, \quad \alpha_3 = \frac{1}{\lambda_2} - \frac{1}{\lambda_0}, \]
\[ \Lambda = \lambda_1 \lambda_2 \lambda_3, \]
and \( R_\ast \) and \( \gamma \) are defined in Appendix A. The quantity \( \gamma \) must satisfy \( 0 \leq \gamma \leq 1 \), and we can usually assume \( 0.8 \leq \gamma \leq 1 \) for PBH formation (De Luca et al. 2019). The function \( g(\eta) \) is defined by
\[ \langle (v^k_\eta(\eta))^2 \rangle = g^2(\eta) \langle (\delta^k_\eta)^2 \rangle \]
for all \((k, l)\), where \( \delta^k_\eta \) is time independent and defined in Equation (A3).\(^6\)

2.2. Long-wavelength Solutions and Near-spherical Approximation

Motivated by inflationary cosmology, we consider cosmological long-wavelength solutions as initial data at \( \eta = \eta_{\text{init}} \), in which the density perturbation in the constant mean curvature (CMC) slicing is written in terms of the curvature perturbation \( \zeta \) in the uniform-density slicing as follows (Harada et al. 2015):
\[ \delta_{\text{CMC}} = -\frac{1}{2\pi^2 \rho_b} e^{5c/2} \Delta e^{-c/2}, \]
where \( \Delta \equiv \delta_{ij} \partial_i \partial_j \) and \( c \) is defined as \( \gamma_{ij} = e^{-c} \delta_{ij} \) in the uniform-density slicing. We assume that the density perturbation is appropriately smoothed at scales smaller than the one under consideration. (See, e.g., Yoo et al. 2018, 2021; Young 2019; Tokeshi et al. 2020 for the possible dependence on the choice of window functions.)

To make the situation clear, we will apply peak theory to this density perturbation field. For \( \nu \gg 1 \), peak theory implies that the density perturbation is nearly spherical near the peak with \( \lambda_i = (\gamma \nu / 3)(1 + \epsilon_i) \) (Bardeen et al. 1986; Heavens & Peacock 1988), where \( \epsilon_i = O(1/\gamma \nu) \), and hence we obtain
\[ a_i \approx r_f = \sqrt{6(1 - f)} \frac{\sigma_0}{\sigma_i}. \]
That is, the region \( \Sigma \) is nearly spherical, and the deviation appears on the order of \( 1/\nu \).

In the following, we assume \( w = 1/3 \) and without loss of generality take \( x_{pk} = 0 \). Linearizing Equation (12), we obtain
\[ \delta_{\text{CMC}} = \frac{2}{3a^2 H^2} \Delta \zeta. \]
Therefore, we find
\[ \sigma_j^2 = \frac{4}{9} \eta_{\text{init}}^4 \int \frac{dk}{k} k^{4+2j} P_c(k), \]
where we have assumed that \( \zeta_\eta(0) \) obeys a homogeneous Gaussian distribution with
\[ \langle \zeta_\eta(0) \zeta_\eta'(0) \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \langle \zeta_\eta(0) \rangle^2 \]
and the power spectrum \( P_c(k) \) is defined as \( P_c(k) \equiv k^3 \langle \zeta_\eta(0) \rangle^2 / (2\pi^2) \).

As for the velocity gradient field, from Equation (B1), we have
\[ v^j(\eta, x) = \left( \frac{\partial v^j}{\partial x^i} \right)(\eta, x) = \int \frac{d^3k}{(2\pi)^3} \xi \zeta_k(\eta) e^{ikx}. \]
Therefore, we obtain the following expression for \( g(\eta) \),
\[ g^2(\eta) = \frac{4}{9} \int \frac{dk}{k} k^2 T_c^2(k, \eta) P_c(k), \]
where \( T_c(k, \eta) \) is a transfer function for \( v_k(\eta) \) defined in Appendix B, and we have used \( \langle \bar{v}^j \bar{v}^j \rangle = 1 \) as seen in Equation (A1).

3. Estimate of the Angular Momentum
3.1. Narrow Power Spectrum

In general, we cannot expect a simple expression for \( g(\eta) \) because it is obtained by a convolution of different modes with different time dependences. In Heavens & Peacock (1988), this is possible because the growth rate function is homogeneous at subhorizon scales in the Einstein–de Sitter universe. In De Luca et al. (2019), they implicitly assume that the perturbation of some single \( k \) effectively determines the angular momentum of the region \( \Sigma \). Here, we assume the same assumption as in De Luca et al. (2019). This is possible if we assume the power spectrum has a narrow peak at \( k = k_0 \) so that
\[ P_c(k) \simeq \sigma^2 c k_0 \delta(k - k_0). \]
Then, Equation (14) implies
\[ \sigma_j \simeq \frac{2}{3} \eta_{\text{init}}^2 k_0^{2+2j} \zeta, \]
and therefore \( \gamma \simeq 1 \). In this case, from Equation (16), we can obtain
\[ g(\eta) \simeq \frac{2}{3} k_0 T_c(k_0, \eta) |\zeta|. \]
In a more general case, \( k_0 \) is identified with \( k \), which dominates the integral on the right-hand side of Equation (16).

According to peak theory, in the case of a narrow power spectrum, the most probable profile is given by a sinc function (Bardeen et al. 1986; Yoo et al. 2018), that is,
\[ \delta_{\text{CMC}}(\eta, r) = \delta_{pk}(\eta) \psi(r), \quad \zeta(\eta, r) = \zeta_{pk}(\eta) \psi(r), \]
\[ \psi(r) = \frac{\sin(k_0 r)}{k_0 r}. \]
Then, we can replace the harmonic function \( Y \) with \( \psi(r) \). We identify \( \delta_{pk}(\eta) \) with \( \delta_{\text{CMC}, k_0}(\eta) \) in Equation (B2). From Equation (14), we obtain
\[ \delta_{\text{CMC}, k_0}(\eta) \simeq \frac{2}{3} \eta^{2+2}(\zeta_{k_0}(0)), \quad D = \frac{4\sqrt{3}}{3} (\zeta_{k_0}(0)), \]
where \( x = k_0 \eta \) and \( D \) is defined in Appendix B.
3.2. PBH Formation Threshold

Under the truncated Taylor-series expansion, because the initial density perturbation profile is given by
\[
\delta_{\text{CMC},k_0}(\eta_{\text{init}}, r) \approx \delta_{\text{CMC},k_0}(\eta_{\text{init}}) \left[ 1 - \frac{1}{6} (k_0 r)^2 \right],
\]
the compaction function \(C_{\text{CMC}}(\eta, r)\) in the CMC slicing is given in the long-wavelength limit by
\[
C_{\text{CMC}}(\eta_{\text{init}}, r) := \left( \frac{\delta M}{\delta r} \right)(\eta_{\text{init}}, r) \\
\approx \frac{1}{3} (k_0 r)^2 \left[ 1 - \frac{1}{10} (k_0 r)^2 \right] (-\zeta_{k_0}(0)),
\]
where \(\delta M\) is the mass excess. This is independent of \(\eta_{\text{init}}\). It takes a maximum value \(C_{\text{max}}\) at \(r = r_m\), where
\[
C_{\text{max}} = \frac{5}{6} (-\zeta_{k_0}(0)), \quad r_m = \sqrt{5} \kappa_0^{-1}.
\]

The threshold value of \(C_{\text{max}}\) for the PBH formation is known to
\[
C_{\text{max}} \approx 0.38 - 0.42 \approx 2/5
\]
from fully nonlinear numerical simulations, and this is fairly stable against different profiles of Gaussian-function or sinc-function shape (Shibata & Sasaki 1999; Harada et al. 2015; Germani & Musco 2019; Musco 2019). Using the threshold value \(C_{\text{max}} \approx 2/5\), we can identify the threshold values for other variables as \(\zeta_{k_0}(0) \approx -12/25\) or \(D \approx 16/\sqrt{3}/25\). For this value of \(\zeta_{k_0}(0)\), we can calculate the density perturbation \(\delta_H\) averaged over the overdense region with the radius \(r_0 = \sqrt{5} \kappa_0^{-1}\) in the long-wavelength limit at horizon entry \(\eta_{\text{init}} = \eta_H\), which we define as \((aH)(\eta_H) r_0 = 1\). The result is the following:
\[
\delta_H = \delta(aHr_0)^2 \approx \frac{2}{5} \frac{2}{3} (k_0 r_0)^2 (-\zeta_{k_0}(0)) \approx \frac{96}{125} = 0.768.
\]

This is fairly consistent with the numerical value \(\approx 0.63 - 0.84\) in the CMC slicing, which is obtained by converting the threshold value \(\approx 0.42 - 0.56\) in the conformal Newtonian gauge at the decoupling from the cosmological expansion. Here, we will estimate the angular momentum of the black hole by that of the region \(\Sigma\) at turnaround, after which the evolution of the region decouples from the cosmological expansion, and the mass and the angular momentum of the collapsing region should be approximately conserved. However, it is not a trivial task to determine this moment. Strictly speaking, turnaround is beyond the regime of linear perturbation. However, because it can be regarded as still being in a quasi-linear regime, we should be able to apply an extrapolation of linear perturbation theory. We here identify the condition \(\delta_{\text{CMC}} \approx 1\) in the CMC slicing as the decoupling condition because this implies that the local density perturbation becomes so large that the expansion should be about to turn around.

To go beyond the turnaround, CMC slicing will not be appropriate because the maximum expansion means a vanishing mean curvature, while there exists a far region where the mean curvature is nonvanishing due to Hubble expansion. To avoid this difficulty, we will shift to the conformal Newtonian gauge. It is expected that the dynamics should fit a usual Newtonian picture later. For this reason, we evaluate \(T_r(\eta_{\text{init}})\) in Equation (16) for \(\gamma_{\text{CMN}}\), the velocity perturbation in the conformal Newtonian gauge at the decoupling from the cosmological expansion.

In Figure 1, we can see that the turnaround occurs at \(x = x_\text{ta} \approx 2.14\) for \(D = 16\sqrt{3}/25\). The value of the transfer function at the turnaround \(x = x_\text{ta}\) is calculated to give
\[
T_{\gamma_{\text{CMN}}}(k_0, \eta_{\text{ta}}) = \frac{\gamma_{\text{CMN}}(x_{\text{ta}})}{\Phi_k(0)} \approx 0.622,
\]
where we have used \(\gamma_{\text{CMN}}(x_{\text{ta}}) \approx -0.199\). Thus, from Equation (20), the value of \(g_{\gamma_{\text{CMN}}}(\eta_{\text{ta}})\) is given by
\[
g_{\gamma_{\text{CMN}}}(\eta_{\text{ta}}) \approx 0.104 k_0 \sigma_H.
\]

Although there is some ambiguity in the choice of the decoupling condition and the gauge condition, it will not change the estimate by orders of magnitude as seen from Figure 1 if we choose \(x_{\text{ta}}\) between \(\approx 1.5\) and \(\approx 3\).

4. Estimate of the Nondimensional Kerr Parameter

4.1. Estimate of \(A_{\text{ref}}\)

Let us estimate the reference spin value at turnaround:
\[
A_{\text{ref}}(\eta_{\text{ta}}) = \frac{S_{\text{ref}}(\eta_{\text{ta}})}{G M_{\text{ta}}} = \frac{4}{5} \left[ \rho_s g_{\gamma_{\text{CMN}}}(\eta_{\text{ta}}) (1 - f)^{5/2} R_s^5 \right] \frac{G M_{\text{ta}}}{S_{\text{ref}}(\eta_{\text{ta}})}.
\]
where the black hole mass \( M \) is identified with the mass within the region \( \Sigma \) at turnaround,
\[
M_{\text{ia}} = (\rho_0 a^3)(\eta_{\text{ia}}) \cdot \frac{4}{3} \pi r_0^3.
\]
This is different from \( M_H \), which we define to be the mass within the horizon at the horizon entry of the overdense region. The condition for the horizon entry \( H^{-1}(\eta_H) = ar_0 \) implies \( \eta_H = r_0 \) or \( x = \sqrt{6} \). Because \( a(\eta) \propto \eta \), we have
\[
\frac{a(\eta_H)}{a(\eta_{\text{ia}})} = \frac{\eta_{\text{ia}}}{r_0} = \frac{x_{\text{ia}}}{\sqrt{6}}.
\]
Using \( \rho_0 a^3 \propto a^{-1} \), we find
\[
M_{\text{ia}} \simeq \frac{\sqrt{6}}{x_{\text{ia}}} (1 - f)^{3/2} M_H.
\]
Using Equation (20) and \( 2GM_H = a(\eta_H) r_0 \), we obtain a simple expression:
\[
A_{\text{ref}}(\eta_{\text{ia}}) \approx \frac{1}{24 \sqrt{3} \pi} x_{\text{ia}}^2 (1 - f)^{1/2} |T_{\text{ov}}(k_0, \eta_{\text{ia}})| \sigma_H. \tag{22}
\]

### 4.2. Estimate of \( a_* \)

As for the distribution of \( s_e \), we just quote the result of Heavens & Peacock (1988) with the correction by De Luca et al. (2019). For the large \( \nu \) limit, if we define \( h \) by
\[
s_e := s_e - \hat{s}_e = 2^{9/2} \pi \frac{2}{5 \gamma \nu} \sqrt{1 - \gamma^2} h,
\]
the probability distribution of \( h \) is approximately given by
\[
P(h) dh \approx \exp \left[ -2.37 - 4.12 \ln h - 1.53 (\ln h)^2 - 0.13 (\ln h)^3 \right] dh.
\]
P\( (h) \) takes a maximum at \( h \approx 0.178 \), while \( \langle h^2 \rangle \approx 0.419 \). Using
\[
P(s_e | \nu) ds_e = P(h) \frac{dh}{ds_e},
\]
we have
\[
\sqrt{\langle s_e^2 \rangle} \approx 5.96 \frac{1 - \gamma^2}{\gamma \nu}.
\]
Putting \( a = A_{\text{ref}} s_e = C \), we have
\[
P_s(a) da = P(C^{-1}a) C^{-1} da.
\]
From the above argument and the equation
\[
\sqrt{\langle a^2 \rangle} = A_{\text{ref}}(\eta_{\text{ia}}) \sqrt{\langle s_e^2 \rangle},
\]
we find the expression for the initial spin of PBHs for \( \gamma \approx 1 \):
\[
\sqrt{\langle a^2 \rangle} \approx \frac{5.96}{24 \sqrt{3} \pi} x_{\text{ia}}^2 (1 - f)^{-1/2} T_{\text{ov}}(k_0, \eta_{\text{ia}}) \sigma_H \sqrt{1 - \gamma^2} \nu^{-1}.
\tag{23}
\]
Putting \( x_{\text{ia}} = 2.14 \), \( T_{\text{ov}}(k_0, \eta_{\text{ia}}) = 0.622 \), \( \delta_H = \tilde{\nu} \sigma_H \), \( \nu = (5/2) \tilde{\nu} \), and \( \delta_H \approx 0.768 \), we find
\[
\sqrt{\langle a^2 \rangle} \approx 3.90 \times 10^{-3} (1 - f)^{-1/2} \sqrt{1 - \gamma^2} \left( \frac{\nu}{8} \right)^{-2} \tag{24}
\]
for the PBH mass
\[
M \approx 1.14(1 - f)^{3/2} M_H. \tag{25}
\]
Eliminating \( f \) from Equations (24) and (25), we obtain the following simple expression:
\[
\sqrt{\langle a^2 \rangle} \approx 4.01 \times 10^{-3} \left( \frac{M}{M_H} \right)^{-1/3} \sqrt{1 - \gamma^2} \left( \frac{\nu}{8} \right)^{-2}. \tag{26}
\]
Although \( f \) or \( M \) is a free parameter in the present scheme, numerical simulations strongly suggest \( M \approx M_H \) except for the near-critical case in which \( M \ll M_H \) (Musco & Miller 2013; Escrivà 2020). If we put \( M = M_H \), \( \gamma = 0.85 \), and \( \nu = 8 \), the above expression yields \( \sqrt{\langle a^2 \rangle} \approx 2.14 \times 10^{-3} \). Therefore, we conclude that \( \sqrt{\langle a^2 \rangle} = O(10^{-3}) \) or even smaller for \( M \approx M_H \).

Let us now discuss small PBHs formed in the near-critical case. In this case, only a small fraction of PBHs are produced through critical collapse, while the rest have \( M \sim M_H \). Therefore, we should fix \( \nu \) at the scale of \( M_H \). Using Equation (26), for example, we find \( \sqrt{\langle a^2 \rangle} \approx 2.14 \times 10^{-2} \) for \( M = 10^{-3} M_H \).
The standard deviation of the initial spins of the PBH, $\sqrt{\langle a^2 \rangle}$, as a function of the PBH mass $M$ with fixed $f_{\text{PBH}}$, where we have assumed that the PBH mass is equal to the horizon mass, i.e., $M = M_H$ and $\gamma = 0.85$.

$\gamma = 0.85$, and $\bar{\nu} = 8$. It also strongly suggests that the angular momentum will play an important role and may significantly suppress the formation of PBHs of $M \lesssim 10^{-8} M_H$, for which $\sqrt{\langle a^2 \rangle} \gtrsim 1$.

4.3. Implications

Because our expression is given in terms of $\nu$, the initial spin directly depends on the fraction $\beta_0(M_H)$ of the universe that collapsed into black holes. If we use the Press–Schechter approximation as a rough estimate of $\beta_0(M_H)$ (Carr 1975),

$$\beta_0(M_H) \approx \frac{2}{\pi} \frac{\nu_{\text{th}}}{\theta_{\text{th}}}^{1/2} \exp \left[ - \frac{5}{2} \frac{\theta_{\text{th}}^2}{2 \sigma_{\text{H}}} \right].$$

we find a simple expression

$$\sqrt{\langle a^2 \rangle} \approx 4.01 \times 10^{-3} \left( \frac{M}{M_H} \right)^{-1/3} \sqrt{1 - \gamma^2} \times \left[ 1 - 0.072 \log_{10} \left( \frac{\beta_0(M_H)}{1.3 \times 10^{-15}} \right) \right]^{-1},$$

where $\nu$ is identified with $\nu_{\text{th}}$ and a weak dependence on $\nu_{\text{th}}$ in the logarithm is ignored. For simplicity, let us concentrate on PBHs of $M \approx M_H$. Using the relation between $\beta_0(M)$ and $f_{\text{PBH}}(M)$ (Carr et al. 2010),

$$\Omega_{\text{dm}} f_{\text{PBH}}(M) \approx 10^{18} \beta_0(M) \left( \frac{M}{10^{15} \text{ g}} \right)^{-1/2},$$

we further obtain

$$\sqrt{\langle a^2 \rangle} \approx 4.01 \times 10^{-3} \times \sqrt{1 - \gamma^2} \times \left[ 1 + 0.036 \left( 21 - 2 \log_{10} \left( \frac{f_{\text{PBH}}(M)}{10^{-7}} \right) - \log_{10} \left( \frac{M}{10^{15} \text{ g}} \right) \right) \right].$$

(27)

We plot Equation (27) in Figure 2. In this figure, we can see that the larger $f_{\text{PBH}}(M)$ and $M$ are, the larger $\sqrt{\langle a^2 \rangle}$. For example, $\sqrt{\langle a^2 \rangle}$ of PBHs for $M = 50 M_\odot$ and $f_{\text{PBH}} = 1$ is about 3.3 times larger than that for $M = 10^{15} \text{ g}$ and $f_{\text{PBH}} = 10^{-7}$.

5. Summary and Discussion

We have applied Heavens & Peacock’s (1988) approach to the first-order effect on the spins of PBHs. Although we have presented numerical values with two or three significant digits, at present we admit that there is large uncertainty in modeling PBH formation. Nevertheless, we would like to claim that the standard deviation of the initial spins is given by $\sqrt{\langle a^2 \rangle} = O(10^{-5})$ or even smaller for $M \approx M_H$ based on peak theory. We have obtained the expression

$$\sqrt{\langle a^2 \rangle} \approx 4.0 \times 10^{-3} \left( \frac{M}{M_H} \right)^{-1/3} \sqrt{1 - \gamma^2} \left( \nu_{\text{th}} \right)^2 \times \left[ 1 - 0.072 \log_{10} \left( \frac{\beta_0(M_H)}{1.3 \times 10^{-15}} \right) \right]^{-1}.$$
spins can be larger than those formed in the radiation-dominated phase.

It would be interesting to remove the assumption of a narrow power spectrum as the broad mass function of PBHs is intensively discussed from an observational point of view (Carr et al. 2016; Carr & Kühnel 2019), although the deviation of $\gamma$ from unity might not change the result by orders of magnitude. Note also that although we have investigated the first-order effect on the angular momentum, the obtained result is apparently second order in terms of $\sigma_H$ as we can see in Equations (23) and (24), where $v^{-1}$ and $b^{-1}$ are of the order of $\sigma_H$ because the threshold value of the PBH formation for the perturbation amplitude is of the order of unity. This means that the first-order effect investigated here might be comparable to the second-order effect. In fact, in Mirbabayi et al. (2020), the second-order effect is estimated to be $\sqrt{\sigma_H^2} \approx \langle \zeta^2 \rangle$. Finally, it should be noted that our analysis is based on linear perturbation theory, which is not completely justified for perturbations that can generate PBHs. In particular, the behaviors of the solutions at the final stage of black hole formation are highly nonlinear and cannot be predicted by linear perturbation theory. Along this line, the assumption of the conservation of the nondimensional Kerr parameter after the decoupling from the cosmological expansion should be confirmed by numerical simulations. It is clear that further investigations are necessary to answer the problem of how large PBH spins are.

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$\textbf{Appendix A}$

$\textbf{Peak Theory}$

We briefly review peak theory based on Heavens & Peacock (1988). We treat the following fields as probability variables:

$$\delta, \quad \zeta_i = \partial \delta / \partial x^i, \quad \zeta_j = \partial^2 \delta / \partial x_i \partial x_j, \quad v_j = \partial \nu / \partial x_j.$$

The correlations of the above variables are given by

$$\langle \delta \rangle^2 = \sigma_0^2, \quad \langle \delta \zeta_1 \rangle = - \langle \zeta_1 \zeta_2 \rangle = \cdots = - \frac{\sigma_0^2}{3},$$

$$\langle \delta \zeta_1 \rangle = \cdots = - \frac{\sigma_0^2}{3},$$

$$\langle \zeta_1 \rangle = \cdots = \frac{\sigma_0^2}{3},$$

$$\langle \zeta_1 \zeta_2 \rangle = 3 \langle \zeta_1 \rangle \langle \zeta_2 \rangle = \cdots = \frac{\sigma_0^2}{5},$$

$$\langle \zeta_1 \zeta_2 \rangle = 3 \langle \zeta_1 \zeta_2 \rangle = \cdots = \frac{\sigma_0^2}{5},$$

$$\langle \zeta_1 \zeta_2 \rangle = 3 \langle \zeta_1 \zeta_2 \rangle = \cdots = \frac{1}{5}. \quad \langle \zeta_1 \zeta_2 \rangle = 3 \langle \zeta_1 \zeta_2 \rangle = \cdots = \frac{1}{5}. \quad (A1)$$

and all other correlations vanish, where

$$\sigma_j^2 = \int \frac{d^3k}{(2\pi)^3} k^2 | \delta_k |^2, \quad (A2)$$

$$v_j' = - \frac{1}{\sigma_0} \int \frac{d^3k}{(2\pi)^3} \frac{k_j k}{k^2} \delta_k e^{ikx}. \quad (A3)$$

Putting the eigenvalues of $-\zeta_1 / \sigma_2$ as $\lambda_1$, $\lambda_2$, and $\lambda_3$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3$), and

$$\nu = \delta / \sigma_0, \quad \zeta_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad \zeta_2 = \frac{1}{2}(\lambda_1 - \lambda_3),$$

$$\zeta_3 = \frac{1}{2}(\lambda_1 - 2 \lambda_2 + \lambda_3), \quad w_1 = \bar{v}_1, \quad w_2 = \bar{v}_2, \quad w_3 = \bar{v}_3,$$

the probability distribution of $\nu$, $\lambda$, and $\omega$ at a peak is given by

$$N_{pk}(\nu, \lambda, \omega) dv d\lambda d\omega = \frac{B}{R_*} \exp(-Q_4) F(\lambda) d\lambda d\omega,$$

where

$$B = \frac{3^{9/2} 2^{5/2}}{2^{11/2} \pi^{9/2} (1 - \gamma^2)^2},$$

$$2Q_4 = \nu^2 + \left( \frac{\zeta_1 - \gamma \nu}{1 - \gamma^2} \right)^2 + 15 \xi^2 + 5 \lambda_3^2 + 15 \omega_1^2 + \omega_2^2 + \omega_3^2,$$

$$F(\lambda) = \frac{27}{2} \lambda_1 \lambda_2 \lambda_3 (\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2),$$

$$R_* := \frac{\sqrt{3} \bar{v}_1}{\sigma_2}, \quad \gamma = \sigma_j^2 / (\sigma_0 \sigma_2).$$

We can see that the distribution of $\omega$ is independent from other variables.

$\textbf{Appendix B}$

$\textbf{Cosmological Linear Perturbations}$

Here we briefly review the result of cosmological linear perturbation theory that is necessary for the present paper. We basically follow the notation of Kodama & Sasaki (1984). We would like readers to refer to Kodama & Sasaki (1984) or other reference for the derivation. The scalar, vector, and tensor harmonic functions $Y, Y_r, Y_g$ in flat space for scalar perturbations are defined as follows:

$$Y = C e^{ikx}, \quad Y_r = -k^{-1} Y_i, \quad Y_g = k^{-2} \left( Y_{ij} - \frac{1}{3} \delta_{ij} \Delta Y \right)$$

$$=- \left( \frac{k_j k_j}{k^2} - \frac{1}{3} \delta_{ij} \right) Y,$$

where the roman indices are raised and lowered by $\delta_{ij}$ and $\delta_{ij}$, respectively, and $Y$ satisfies

$$(\Delta + k^2) Y = 0.$$

The Fourier decomposition of the perturbations is given by

$$\delta(\eta, x) = \int \frac{d^3k}{(2\pi)^3} \hat{\delta}_k(\eta) e^{ikx}, \quad \delta_k(\eta) = \int \frac{d^3x}{(2\pi)^3} \hat{\delta}_k(\eta, x) e^{-ikx},$$

and so on. In what follows in this section, we abbreviate $\hat{\delta}_k(\eta)$ as $\delta$ and so on. In Equation (2), we write the scalar perturbation
of the metric tensor as follows:
\[
\alpha = a(1 + AY), \quad \beta_i = -a^2BY, \quad \gamma_0 = \delta_0 + 2H_L \delta_0 + 2H_T Y_0.
\]
The trace of the extrinsic curvature of the constant \(\eta\) hypersurface is written as
\[
K = K_b(1 + K_x Y).
\]
The perturbed quantities of the perfect fluid are written as
\[
\rho = \rho_b(1 + \delta Y), \quad \rho = \rho_b(1 + \pi_L Y), \quad \nu^i = \frac{\mu^i}{\nu^0} = \nu Y^i.
\]
\[\text{(B1)}\]
In the adiabatic process with the \(p = w \rho\) equation of state, we have \(\pi_L = \delta\). For the scalar perturbation, the infinitesimal coordinate transformation is given by
\[
\tilde{\eta} = \eta + T(\eta)Y, \quad x^i = x^i + L(\eta)Y^i,
\]
where \(T\) and \(L\) are arbitrary functions of \(\eta\). Under this coordinate transformation, the metric perturbation quantities transform as follows:
\[
\tilde{A} = A - T' - \mathcal{H}T, \quad \tilde{B} = B + L' + kT,
\]
\[
\tilde{H}_L = H_L - \frac{k}{n}L - \mathcal{H}T, \quad \tilde{H}_T = H_T + kL,
\]
where \(n\) is the dimension of the space, \(\mathcal{H} := a'/a\), and the prime denotes the derivative with respect to \(\eta\). On the other hand, matter perturbation quantities transform as follows:
\[
\tilde{\nu} = \nu + L', \quad \tilde{\delta} = \delta + n(1 + w)\mathcal{H}T,
\]
\[
\pi_L = \pi_L + 3\frac{c_s^2}{w}(1 + w)\mathcal{H}T,
\]
where \(c_s^2\) is the sound speed. From the above, we can construct gauge-invariant perturbation quantities corresponding to \(\delta\) and \(\nu\) as follows:
\[
\Delta = \delta + 3(1 + w)\mathcal{H}k^{-1}(\nu - B), \quad V = \nu - k^{-1}H_T'.
\]
From the Einstein equation, we can derive the equations for the gauge-invariant variables \(\Delta\) and \(V\). We present the solutions for the radiation-dominated phase of the universe below:
\[
\Delta(x) = D\sqrt{3}\left(\frac{\sin z}{z} - \cos z\right),
\]
\[
V(x) = D\left[\frac{3}{4}\left(\frac{2}{z^2} - 1\right)\sin z - \frac{3\cos z}{2z}\right],
\]
where \(D\) is an arbitrary constant, \(z := x/\sqrt{3}\), \(x := k\eta\), and a decaying mode is omitted.

In the CMC (\(K_x = 0\)) slicing with \(B = 0\), using the above solutions for \(\Delta(x)\) and \(V(x)\), we can obtain
\[
\delta = D\frac{\sqrt{3}}{z^2 + 2}\left(\frac{2}{z^2} - 1\right)2\sin z - \cos z, \quad \text{(B2)}
\]
\[
\nu = \frac{3}{4}D\frac{(z^2 - 2)\sin z + 2z\cos z}{z^2 + 2}, \quad \text{(B3)}
\]
\[
\mathcal{R} = D\frac{\sqrt{3}}{2}\frac{1}{z^2 + 2}\left(\frac{2}{z^2} - 1\right)2\sin z - \cos z, \quad \text{(B4)}
\]
where \(\mathcal{R} = H_L + \frac{1}{2}H_T\). In this gauge, \(A\) and \(\mathcal{R}\) are completely fixed, while \(H_T\) and \(H_L\) are fixed only up to a constant.

In the conformal Newtonian gauge, in which \(H_T = B = 0\), we can obtain
\[
\delta = \sqrt{3}D\frac{2(\sin^2 z - 1)\sin z + (2 - z^2)\cos z}{z^3}, \quad \text{(B5)}
\]
\[
\nu = \frac{3}{4}D\frac{(2 - z^2)\sin z - 2z\cos z}{z^2}, \quad \text{(B6)}
\]
\[
\mathcal{F} = -\frac{\sqrt{3}}{2}D\frac{\sin z - z\cos z}{z^3}, \quad \text{(B7)}
\]
where \(H_L = -\Phi\) and \(A = \Phi\). Thus, all perturbation quantities are completely fixed in this gauge.

We can define the transfer functions \(T_{\text{CMC}}, T_{\text{CN}}, T_{\text{CN}}\), and \(T_{\text{CN}}\) as follows:
\[
\delta_{\text{CMC}}(\eta) = T_{\text{CMC}}(k, \eta)\Phi(0), \quad \nu_{\text{CMC}}(\eta) = T_{\text{CMC}}(k, \eta)\Phi(0),
\]
\[
\delta_{\text{CN}}(\eta) = T_{\text{CN}}(k, \eta)\Phi(0), \quad \nu_{\text{CN}}(\eta) = T_{\text{CN}}(k, \eta)\Phi(0),
\]
where we can see \(\Phi(0) = -D/(2\sqrt{3}) = -(2/3)\mathcal{R}(0)\) from Equations (B4) and (B7). We should note that \(\mathcal{R} = -\zeta(0)\), where \(\zeta\) is the curvature perturbation in the uniform-density slicing.

**Appendix C**

**Nonexistence of the Overall Factor \(\Omega_{\text{dm}}\)**

Here, we show that the overall factor \(\Omega_{\text{dm}}\) in De Luca et al.’s (2019) expression should be removed. Although their notation is slightly different from ours, we consistently continue to use our notation.

In the following, we follow the process of calculation in De Luca et al. (2019). They estimate the angular momentum at the horizon entry of the inverse wavenumber, saying that the turnaround is just after the horizon entry. Their analysis is confined to the CMC slicing, where \(g(\tilde{\eta}_H) = g_{\text{CMC}}(\tilde{\eta}_H)\) was estimated to be
\[
g_{\text{CMC}}(\tilde{\eta}_H) \sim \left| \frac{T_{\text{CMC}}(k_0, \tilde{\eta}_H)}{T_{\text{CMC}}(k_0, \tilde{\eta}_H)} \right| k_0\tilde{\eta}_H, \quad \text{(C1)}
\]
where \(k_0\) is identified with \(k_H\) in De Luca et al. (2019), \(\tilde{\eta}_H = k_0^{-1}\), and \(\tilde{\eta}_H = \eta_0\) with the long-wavelength limit. Although our calculation does not reproduce their numerical value \(|T_{\text{CMC}}(k_0, \tilde{\eta}_H)/T_{\text{CMC}}(k_0, \tilde{\eta}_H)| \sim 0.5\) but gives a much smaller value \(\sim 0.0714\) at \(x = 1\), this is not the origin of the factor \(\Omega_{\text{dm}}\). Because \(\mathcal{H} \propto a^{-1}\) in the radiation-dominated era and \(\mathcal{H} \propto a^{-1/2}\) in the matter-dominated era, they probably inferred that
\[
\mathcal{H}(\eta_{\text{eq}}) = \frac{\mathcal{H}_0}{(a(\eta_{\text{eq}})/a_0)^{1/2}}, \quad \text{(C2)}
\]
where we have put \(a(\eta_0) = a_0\) and \(\mathcal{H}(\eta_0) = \mathcal{H}_0\) and \(\eta_0\) is the present conformal time. This corresponds to Equation (5.4) in De Luca et al. (2019). Then, using
\[
\frac{a(\tilde{\eta}_H)}{a_0} = \frac{a(\eta_{\text{eq}})}{a_0}\left(\frac{\mathcal{H}(\eta_{\text{eq}})}{\mathcal{H}(\tilde{\eta}_H)}\right) = \left(\frac{a(\eta_{\text{eq}})}{a_0}\right)^{1/2} \left(\frac{\mathcal{H}_0}{\mathcal{H}(\tilde{\eta}_H)}\right), \quad \text{(C3)}
\]
and defining $\bar{M}_H$ as the mass within the Hubble horizon at $\eta = \eta_H$, we find
\begin{equation}
\mathcal{H}(\eta_H) = \frac{a(\eta_H) \Omega_{\text{rad}}}{2GM_H} = \left(\frac{a(\eta_{eq})}{a_0}\right)^{1/2} \frac{\mathcal{H}_0}{\mathcal{H}(\eta_H)} a_0 \frac{a(\eta_{eq})}{2GM_H} (C4)
\end{equation}
and
\begin{equation}
k_0 = \left(\frac{a(\eta_{eq})}{a_0}\right)^{1/4} \sqrt{\frac{\Omega_{\text{rad}}}{2GM}}. \quad (C5)
\end{equation}
Moreover, using
\begin{align}
\rho_\text{rad}(\eta_H) &\approx \rho_\text{rad}(\eta_{eq}) \left(\frac{a(\eta_H)}{a_0}\right)^4 \\
&\approx \rho_\text{rad}(\eta_{eq}) \frac{3H_0^2}{8\pi G a_0^2}, \quad \text{and the cosmological constant, the Friedmann equation implies}
\end{align}

\begin{equation}
H^2 = H_0^2 \left[ \Omega_{\text{rad}} \left(\frac{a_0}{a}\right)^{4} + \Omega_{\text{dm}} \left(\frac{a_0}{a}\right)^{3} + \Omega_{\Lambda} \right]. \quad (C8)
\end{equation}

Moreover, we assume that $\Omega_{\text{rad}} \ll \Omega_{\text{dm}}$, $\Omega_{\text{dm}} \approx 0.3$, and $\Omega_{\Lambda} \approx 0.7$. Then, we can safely ignore $\Omega_{\Lambda}$ at the matter–radiation equality $\eta = \eta_{eq}$, when $\rho_{\text{rad}} \approx \rho_{\text{dm}}$. This immediately implies
\begin{equation}
\frac{\Omega_{\text{rad}}}{\Omega_{\text{dm}}} = \left(\frac{a(\eta_{eq})}{a_0}\right)^{4}. \quad (C8)
\end{equation}

Therefore, Equation (C8) implies
\begin{equation}
\mathcal{H}(\eta_{eq}) = \frac{\mathcal{H}_0}{\left(\frac{a(\eta_{eq})}{a_0}\right)^4 \sqrt{2\Omega_{\text{dm}}}}. \quad (C9)
\end{equation}

This corrects Equation (C2) or Equation (5.4) in De Luca et al. (2019). This gives a factor $\sqrt{2\Omega_{\text{dm}}}$ on the rightmost side of Equations (C3) and (C4) and a factor $(2\Omega_{\text{dm}})^{1/4}$ on the rightmost side of Equation (C5). Thus, there appears a factor $(2\Omega_{\text{dm}})^{-1}$ on the rightmost side of Equation (C6), and this $\Omega_{\text{dm}}^{-1}$ cancels out the factor $\Omega_{\text{dm}}$ from $\rho_{\text{H}}$. Then, we reach the same expression as in Equation (C7).

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References
Abbott, B. P., LIGO Scientific & Virgo, et al. 2019, ApJL, 882, L24
Abbott, B. P., LIGO Scientific & Virgo, et al. 2020, PhRvD, 102, 043015
Arbey, A., Auffinger, J., & Silk, J. 2020, MNRAS, 494, 1257
Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15
Bird, S., Cholis, I., Muñoz, J. B., et al. 2016, PhRvL, 116, 201301
Carr, B. J. 1975, ApJ, 201, 1
Carr, B. J., Kohri, K., Sendouda, Y., & Yokoyama, J. 2010, PhRvD, 81, 104019
Carr, B. J., Kohri, K., Sendouda, Y., & Yokoyama, J. 2020, arXiv:2002.12778
Carr, B. J., & Kühnel, F. 2019, PhRvD, 99, 105353
Carr, B. J., Kühnel, F., & Sandstad, M. 2016, PhRvD, 94, 083504
De Luca, V., Desjacques, V., Franciolini, G., Malhotra, A., & Riotto, A. 2019, JCAP, 05, 018
De Luca, V., Franciolini, G., Pan, P., & Riotto, A. 2020, JCAP, 04, 052
De Luca, V., Franciolini, G., Pan, P., & Riotto, A. 2020, JCAP, 04, 052
De Luca, V., Franciolini, G., Pan, P., & Riotto, A. 2020, JCAP, 04, 052
Escrivá, A. 2020, PDU, 27, 100466
Escrivá, A., Germani, C., & Sheth, R. K. 2020, PhRvD, 101, 044022
Germani, C., & Musco, I. 2019, PhRvD, 122, 141302
Harada, T., Yoo, C. M., & Kohri, K. 2013, PhRvD, 88, 084051
Harada, T., Yoo, C. M., Kohri, K., & Nakao, K. I. 2017, PhRvD, 96, 083517
Harada, T., Yoo, C. M., Nakama, T., & Koga, Y. 2015, PhRvD, 91, 084057
He, M., & Suyama, T. 2019, PhRvD, 100, 063520
Heavens, A., & Peacock, J. 1988, MNRAS, 232, 339
Kodama, H., & Sasaki, M. 1984, PThPS, 78, 1
Mirbabayi, M., Gruzinov, A., & Norela, J. 2020, JCAP, 03, 017
Musco, I. 2019, PhRvD, 100, 123524
Musco, I., & Miller, J. C. 2019, CQGra, 30, 145009
Musco, I., Miller, J. C., & Polnarev, A. G. 2009, CQGra, 26, 235001
Nakamura, T., Sasaki, M., Tanaka, T., & Thorne, K. S. 1997, ApJL, 487, L139
Niemeyer, J. C., & Jedamzik, K. 1999, PhRvD, 59, 124013
Page, D. N. 1976, PhRvD, 14, 3260
Polnarev, A. G., & Musco, I. 2007, CQGra, 24, 1405
Raidal, M., Vaskonen, V., & Veermäe, H. 2017, JCAP, 09, 037
Sasaki, M., Suyama, T., Tanaka, T., & Yokoyama, S. 2016, PhRvL, 117, 061101
Shibata, M., & Sasaki, M. 1999, PhRvD, 60, 084002
Takahashi, K., Inomata, K., & Yokoyama, J. 2020, JCAP, 12, 038
Wald, R. M. 1984, General Relativity (Chicago: Chicago Univ. Press) doi:10.7208/chicago/9780226870373.001.0001
Yoo, C. M., Harada, T., Garriga, J., & Kohri, K. 2018, PhRvD, 108, 083002
Yoo, C. M., Harada, T., Hirano, S., & Kohri, K. 2021, PhRvD, 201, 134002
Young, S. 2019, JMPD, 29, 030002