Crystalline color superconductors: A review

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Non-homogeneous superconductors and non-homogeneous superfluids appear in a variety of contexts which include quark matter at extreme densities, fermionic systems of cold atoms, type-II cuprates and organic superconductors. In the present review we shall focus on the properties of quark matter at high baryonic density which can exist in the interior of compact stars. The conditions that are realized in this stellar objects tend to disfavor standard symmetric BCS pairing and may be in favor of a non-homogeneous color superconducting phase. We discuss in details the properties of non-homogenous color superconductors and in particular of crystalline color superconductors. We also review the possible astrophysical signatures associated with the presence of non-homogenous color superconducting phases within the core of compact stars.

We dedicate this paper to the memory of Giuseppe Nardulli for his important collaboration to the general subject of this review.
I. INTRODUCTION

Ideas about color superconducting (CSC) matter date back to more than 30 years ago (Bailin and Love, 1984; Barrois, 1977; Collins and Perry, 1975; Frautschi, 1978), but only recently this phenomenon has received a great deal of consideration (for recent reviews see Alford (2001); Alford et al. (2008); Hong (2001); Hai (2000); Nardulli (2002); Rajagopal and Wilczek (2000); Rischke (2004); and Schafer (2003b)). Color superconductivity is the quark matter analogous of the standard electromagnetic superconductivity and is believed to be the ground state of hadronic matter at sufficiently large baryonic densities. At very high density the naive expectation, due to asymptotic freedom, is that quarks form a Fermi sphere of almost free fermions. However, Bardeen, Cooper and Schrieffer (BCS) (Bardeen et al., 1957a,b; Cooper, 1956) have shown that the Fermi surfaces of free fermions are unstable in presence of an attractive, arbitrary small, interaction between fermions. In Quantum-Chromodynamics (QCD) the attractive interaction between quarks can be due to instanton exchange (Schafer and Shuryak, 1998), at intermediate densities, or to gluon...
exchange in the 3 color channel, at higher densities. Therefore, one expects that at high densities quarks form a coherent state of Cooper pairs.

It should be noticed that the mentioned old papers (Bailin and Love, 1984; Barrois, 1977; Collins and Perry, 1978; Frautschi, 1978) were based on the existence of the attractive 3 color channel and on analogies with ordinary superconductors. The main result of these analyses was that quarks form Cooper pairs with a gap of order a few MeV. In more recent times two papers (Rapp et al., 1998) and (Alford et al., 1998), have brought this result to question. These authors considered diquark condensation arising from instanton-mediated interactions and although their approximations are not under rigorous quantitative control, the result was that gaps can be as large as 100 MeV. However, as pointed out by Hsu (Hsu, 2000): “this is an unfair characterization of Bailin and Love’s results (Bailin and Love, 1984). A value of the strong coupling large enough to justify the instanton liquid picture of Alford et al. (1998) and Rapp et al. (1998) also yields a large gap when substituted in Bailin and Love’s results. After all, instantons are suppressed by an exponential factor exp(−2π/α_s). Bailin and Love merely suffered from the good taste not to extrapolate their results to large values of α_s!”

Color superconductivity offers a clue to the behavior of strong interactions at very high baryonic densities, an issue of paramount relevance for the understanding of the physics of compact stars and of heavy ion collisions. In the asymptotic regime it is possible to understand the structure of the quark condensate from basic considerations. Consider the matrix element

\[ \langle 0 | ψ^α_i ψ^β_j | 0 \rangle , \]

where ψ^α_i, ψ^β_j represent the quark fields, and α, β = 1, 2, 3, s, t = 1, 2, i, j = 1, · · · , N_f are color, spin and flavor indices, respectively. For sufficiently high quark chemical potential, µ, color, spin and flavor structure can be completely fixed by the following arguments:

- Antisymmetry in color indices (α, β) in order to have attraction.
- Antisymmetry in spin indices (s, t) in order to have a spin zero condensate. The isotropic structure of the spin zero condensate is favored with respect to the spin 1 (or higher spin) condensate since a larger portion of the phase space around the Fermi surface is available for pairing.
- Given the structure in color and spin, Fermi statistics requires antisymmetry in flavor indices.

Since the quark spin and momenta in the pair are opposite, it follows that the left(right)-handed quarks can pair only with left(right)-handed quarks. Considering three-flavor quark matter at large baryonic density, the so-called color–flavor locked (CFL) phase turns out to be thermodynamically favored, with condensate

\[ \langle 0 | ψ^α_i ψ^β_j | 0 \rangle = −\langle 0 | ψ^α_i ψ^β_j | 0 \rangle \propto \Delta_{\text{CFL}} \sum_{I=1}^{3} ε^{αβI} ε_{ijI} , \]

where Δ_{\text{CFL}} is the pairing gap, we have suppressed spinorial indices and neglected pairing in the color sextet channel. Pairing in the color sextet channel is automatically induced by the quark color structure, but the condensate in this channel is much smaller than in the color anti-triplet channel and in most cases can be neglected. The CFL condensate in Eq. 2 was introduced in Alford et al. (1999c) and the reason for its name is that only simultaneous transformations in color and in flavor spaces leave the condensate invariant. The symmetry breaking pattern turns out to be

\[ SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \rightarrow SU(3)_{c+L+R} \otimes Z_2 , \]

where SU(3)c+L+R is the diagonal global subgroup of the three SU(3) groups and the Z_2 group means that the quark fields can still be multiplied by -1. According with the symmetry breaking pattern, the 17 generators of chiral symmetry, color symmetry and U(1)_B symmetry are spontaneously broken. The 8 broken generators of the color gauge group correspond to the 8 longitudinal degrees of freedom of the gluons and according with the Higgs-Anderson mechanism these gauge bosons acquire a Meissner mass. The diquark condensation induces a Majorana-like mass term in the fermionic sector which is not diagonal in color and flavor indices. Thus, the fermionic excitations consist of gapped modes with mass proportional to Δ_{\text{CFL}}. The low-energy spectrum consists of 9 Nambu-Goldstone bosons

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1 This is a feature of all homogenous superconducting phases: the fermionic excitations which are charged with respect to the condensate acquire a Majorana-like mass term proportional to the pairing gap.
(NGB) organized in an octet, associated with the breaking of the flavor group, and in a singlet, associated with the breaking of the baryonic number. For non-vanishing quark masses the octet of NGBs becomes massive, but the singlet NGB is protected by the $U(1)_B$ symmetry; it remains massless and determines the superfluid properties of the CFL phase. The effective theory describing the NGBs for the CFL phase has been studied in (Casalbuoni and Gatto, 1999; Son, 2002; Son and Stephanov, 2000a). The CFL condensate also breaks the axial $U(1)_A$ symmetry; given that at very high densities the explicit axial symmetry breaking is weak, one has to include the corresponding pseudo-NGB in the low-energy spectrum.

After the first attempts with instanton-induced interaction many authors tried various approaches in order to calculate the magnitude of gap parameters in the CSC phases (for references see the review by Rajagopal and Wilczek (Rajagopal and Wilczek, 2000)). Dealing with QCD the ideal situation would be if these kind of calculations could fall within the scope of lattice gauge theories. Unfortunately, lattice methods rely on Monte Carlo sampling techniques that are unfeasible at finite density because the fermion determinant becomes complex. Although various approximation schemes have been developed, for instance, Taylor expansion in the chemical potential (Alton et al., 2003), reweighting techniques (Fodor and Katz, 2002), analytical continuation of calculation employing imaginary baryonic chemical potential (Alford et al., 1999) Roberge and Weiss, 1986 or heavy Wilson quarks (Fromm et al., 2012), no definite results have been obtained so far for large values of the baryonic chemical potential and physical quark masses.

In the absence of suitable lattice methods, quantitative analyses of color superconductivity have followed two distinct paths. The first path is semi-phenomenological, and based on simplified models. The main feature of these models is that they should incorporate the most important physical effects but being at the same time tractable within present mathematical techniques. All these models have free parameters that are adjusted in such a way to give rise to a reasonable vacuum physics.

Examples of these kind of techniques include Nambu-Jona Lasinio (NJL) models in which the interaction between quarks is replaced by a four-fermion interaction originating from instanton exchange (Alford et al., 1998; Berges and Rajagopal, 1999, Rapp et al., 1998) or where the four-fermi interaction is modeled by that induced by single-gluon exchange (Alford et al., 1999a, 1998). Random matrix models have been studied in Vanderheiden and Jackson (2000) and instanton liquid models have been investigated in Carter and Diakonov (1999); Rapp et al. (2000, 2001), while renormalization group methods have been used in Evans et al. (1999) and Schafer and Wilczek (1999b). Although none of these methods has a firm theoretical basis, all of them yield results all in fairly qualitative agreement. This is probably due to the fact that what really matters is the existence of an attractive interaction between quarks and that the parameters of the various models are chosen in such a way to reproduce the chirally broken ground state. The gap parameter evaluated within these models varies between tens of MeV up to 100 MeV. The critical temperature is typically the same found in normal superconductivity, that is about one half of the gap.

The second path starts from first principles and relies on the property of asymptotic freedom of QCD. Various results have been obtained starting from the QCD action, employing renormalization group techniques and through the Schwinger-Dyson equation (Brown et al., 2000; Evans et al., 2000; Hong et al., 2000; Pisarski and Rischke, 2000; Schafer and Wilczek, 1999a; Son, 1999). In particular, Son (Son, 1999), using the renormalization group near the Fermi surface has obtained the asymptotic form of the gap. However, it has been argued in Rajagopal and Shuster (2000) that the weak-coupling calculations employed in this case are valid only for extremely large densities corresponding to chemical potentials larger than $10^5$ MeV. Moreover, also in this second path, it is not possible to control the approximation, because we do not know how to evaluate higher order corrections. Interestingly enough, extrapolating the results from asymptotic densities to values of quark chemical potential of order $(400 - 500)$ MeV, one obtains a magnitude of the gap in agreement with the results of more phenomenological approaches.

The result of the analyses of the above-mentioned methods is that the CFL phase is the thermodynamically favored state of matter at asymptotic densities. Qualitatively one can understand this result considering that in the CFL phase quarks of all three-flavors participate coherently in pairing. Since superconductivity is a cooperative phenomenon, the larger the number of fermions that participate in pairing, more energetically favored is the superconducting phase.

In the description of color superconductivity one has to deal with various scales, the chemical potential $\mu$, the gap parameter, which we shall generically indicate with $\Delta$, the strange quark mass $M_s$ and the screening/damping scale $g\mu$, where $g$ is the QCD coupling. One typically has that $\mu \gg g\mu \gg \Delta$, whereas the strange quark mass can be considered as a free parameter, although in some models it can be computed self-consistently.

QCD at high density is conveniently studied through a hierarchy of effective field theories, schematically depicted in Fig. The starting point is the fundamental QCD Lagrangian, then one can obtain the low energy effective Lagrangian through different methods. One way is to integrate out high-energy degrees of freedom as shown in Polchinski (1992). The physics is particularly simple for energies close to the Fermi energy where all the interactions are irrelevant except for a four-fermion interaction coupling pair of fermions with opposite momenta. This is nothing
but the interaction giving rise to BCS condensation, which can be described using the High Density Effective Theory (HDET) \cite{Beane:2000tw, Casalbuoni:2001rs, Hong:2000uc, Nardulli:2002up, Schafer:2003ux}, see Sec. III.B.4 for a brief description of the method. The HDET is based on the fact that at vanishing temperature and large chemical potentials antiparticle fields decouple and the only relevant fermionic degrees of freedom are quasi-particles and quasi-holes close to the Fermi surface. In the HDET Lagrangian the great advantage is that the effective fermionic fields have no spin structure and therefore the theory is particularly simple to handle.

This description is supposed to hold up to a cutoff $P^F + \delta$, with $\delta$ smaller than the Fermi momentum, $P^F$, but bigger than the gap parameter, $\Delta$. Considering momenta much smaller than $\Delta$ all the gapped particles decouple and one is left with the low energy modes as NGBs, ungapped fermions and holes and massless gauged fields according with the symmetry breaking scheme. In the case of CFL and other CSC phases, such effective Lagrangians have been derived in \cite{Casalbuoni:2000ch, Casalbuoni:1999mm, Rischke:2001fc}.

The parameters of the effective Lagrangian can be evaluated at each step of the hierarchy by matching the Green’s functions with the ones evaluated at the upper level. For the CFL phase the effective Lagrangian of the superfluid mode associated with the breaking of $U(1)_B$ may also be determined by symmetry arguments alone as in \cite{Son:2002sd}.

In the high density limit one neglects the quark masses, but in realistic situations the quark chemical potential may be of the same order of magnitude of the strange quark mass. The typical effect of quark masses is to produce a mismatch between Fermi surfaces. Neglecting light quark masses and assuming (for simplicity) that quarks have all the same chemical potentials, the Fermi spheres have now different radii

$$P^F_s = \sqrt{\mu^2 - M_s^2}, \quad P^F_u = P^F_d = \mu, \quad \text{(4)}$$

where $M_s$ is the constituent strange quark mass. Thus, increasing $M_s$ for a fixed value of $\mu$, increases the mismatch between the Fermi surface of strange quarks and the Fermi surfaces of up and down quarks (which in this simplistic case are equal).

The standard BCS mechanism assumes that the Fermi momenta of the fermionic species that form Cooper pairs are equal. When there is a mismatch it is not guaranteed that BCS pairing takes place, because the condensation of fermions with different Fermi momenta has a free-energy cost. For small mismatches there is still condensation \cite{Chandrasekhar:1962zz, Clogston:1962zz}, and in the case at hand it means that the CFL phase is favored. However, for large values of the strange quark mass the assumptions leading to prove that the favored phase is CFL should be reconsidered. According with Eq. (4) if the strange quark mass is about the quark chemical potential, then strange quarks decouple, and the corresponding favored condensate should consist of only up and down quarks. With only two flavors of quarks, and due to the antisymmetry in color, the condensate must necessarily choose a direction in color space and one possible pairing pattern is

$$\langle 0 | \bar{\psi}^\alpha_i \gamma^2 \psi^\beta_j | 0 \rangle \propto \Delta_{2SC} \varepsilon_{\alpha \beta} \varepsilon_{ij3}, \quad \alpha, \beta \in SU_c(2), \quad i, j \in SU(2)_L. \quad \text{(5)}$$

This phase of matter is known as two-flavor color superconductor (2SC) and $\Delta_{2SC}$ is the corresponding gap parameter. This phase is characterized by the presence of 2 ungapped quarks, $q_u$, $q_d$ and 4 gapped quasi-particles given by the combinations $q_r - q_g$ and $q_r - q_y$ of the quark fields, where the color indices of the fundamental representation 1, 2, 3 have been identified with $r, g, b$ (red, green and blue). In case massive strange quarks are present
the corresponding phase is named 2SC+s and eventually strange quarks may by themselves form a spin-1 condensate (Pisarski and Rischke 2000b).

In the 2SC phase the symmetry breaking pattern is completely different from the three-flavor case and it turns out to be

\[ SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B \rightarrow SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B \otimes Z_2. \]  

(6)

The chiral group remains unbroken, meaning that there are no NGBs. The original color symmetry group is broken to \( SU(2)_c \), and since three color generators are unbroken, only five gluons acquire a Meissner mass. Even though \( U(1)_B \) is spontaneously broken there is an unbroken \( U(1)_B \) global symmetry, where \( B \) is given by a combination of \( B \) and of the eighth color generator, \( T_8 \), playing the same role of the original baryonic number symmetry. In particular, this means that unlike CFL matter, 2SC matter is not superfluid. One can construct an effective theory to describe the emergence of the unbroken subgroup \( SU(2)_c \) and the low energy excitations, much in the same way as one builds up chiral effective Lagrangian with effective fields at zero density. This development can be found in Casalbuoni et al. (2000) and Rischke et al. (2004).

The problem of condensation for imbalanced Fermi momenta is quite general and can be described assuming that different populations have different chemical potentials, see e.g. Alford and Rajagopal (2002). We shall present a discussion of a simple two-level system in Sec. II.A, which allows to clarify the main effects of a chemical potential difference, \( \delta \mu \).

Regarding quark matter a similar behavior can be found when one considers realistic condition, \( i.e. \) conditions that can be realized in a compact stellar object (CSO). This is a real possibility since the central densities for these stars could reach \( 10^{15} \, g/cm^3 \), whereas the temperature is of the order of tens of keV, much less than the critical temperature for color superconductivity. Matter inside a CSO should be electrically neutral and in a color singlet state. Also conditions for the \( \beta \)-equilibrium should be fulfilled. As far as color is concerned, it is possible to impose a simpler condition, that is color neutrality, since in Amore et al. (2002) it has been shown (in the two-flavor case) that there is a small free-energy cost in projecting color singlet states out of color neutral ones. If electrons are present (as generally required by electrical neutrality) the \( \beta \)-equilibrium condition forces the chemical potentials of quarks with different electric charges to be different, thus Eqs. 41 are no more valid and more complicated relations hold (Alford and Rajagopal 2002). The effect of the mass of the strange quark, \( \beta \)-equilibrium and color and electric neutrality, is to pull apart the Fermi spheres of up, down and strange quarks and many different phases may be realized, depending on the parameters of the system. Besides the above-mentioned standard 2SC and 2SC+s phases, the two-flavor superconducting phase 2SCus, with pairing between up and strange quarks can be favored; see e.g. Iida et al. (2004) and Ruester et al. (2006a) for different pairing patterns. For very large mismatches among the three flavors of quarks only the inter-species single-flavor spin-1 pairing may take place (Alford et al. 2003, 1998; Bailin and Love, 1973; Buballa et al. 2003; Schafer, 2000; Schmitt, 2005; Schmitt et al. 2002), see e.g. Alford et al. (2008) for an extended discussion on these topics.

In the 2SC phase, the above conditions tend to separate the Fermi spheres of up and down quarks and, as discussed in Sec. II.B for \( |\delta \mu| = \Delta \) gapless modes appear. The corresponding phase has been named g2SC (Huang and Shovkovy, 2003; Shovkovy and Huang, 2003), with “g” standing for gapless. The g2SC phase with pairing between up and down quarks has the same condensate of the 2SC phase reported in Eq. 5, and therefore the ground states of the 2SC and of the g2SC phases share the same symmetry. However, these two phases have a different low energy spectrum, due to the fact that in the g2SC phase only two fermionic modes are gapped. The g2SC phase is energetically favored with respect to the 2SC phase and unpaired quark matter in a certain range of values of the four-fermi interaction strength when one considers \( \beta \)-equilibrium, color and electrical neutrality (Shovkovy and Huang, 2003).

Pinning down the correct ground state of neutral quark matter in \( \beta \)-equilibrium is not simple because another difficulty emerges. This problem, which is already present in the simple two-level system discussed in Sec. II.A has a rather general character (Alford and Wang 2003), and is due to an instability connected to the Meissner mass. In particular, when \( |\delta \mu| = \Delta \) the system becomes magnetically unstable, meaning that the Meissner mass becomes imaginary. In the 2SC phase the color group is broken to \( SU(2)_c \) and 5 out of 8 gluons acquire a mass. Four of these masses turn out to be imaginary in the 2SC phase for \( \Delta/\sqrt{2} < \delta \mu < \Delta \), thus in this range of \( \delta \mu \) the 2SC phase is chromo-magnetically unstable (Huang and Shovkovy, 2004a,b). Increasing the chemical potential difference the instability gets worse, because at the phase transition from the 2SC phase to the g2SC phase all the five gluon masses become pure imaginary.

An analogous phenomenon arises in three-flavor quark matter because the gapless CFL (gCFL) phase Alford et al. (2005a, 2004, 2005c; Fukushima et al. 2005) turns out to be chromo-magnetically unstable (Casalbuoni et al. 2005b; Fukushima, 2005). The gCFL phase has been proposed as the favored ground state for sufficiently large mismatch between up, down and strange quarks and occurs in color and electrically neutral quark matter in \( \beta \)-equilibrium.
for $M^2/2\mu \gtrsim \Delta$. Some properties of the gCFL phase and a brief discussion of the corresponding instability will be presented in Sec. III.A.

There is a variety of solutions that have been proposed for the chromo-magnetic instability and that can be realized depending on the particular conditions considered. As we have already discussed, the chromo-magnetic instability is a serious problem not only for the gapless phases (g2SC and gCFL) but also for the 2SC phase. In the latter case, it has been shown in Gorbar et al. (2006a) that vector condensates of gluons with a value of about 10 MeV can cure the instability. The corresponding phase has been named gluonic phase and is characterized by the non-vanishing values of the chromo-electric condensates $\langle A^y \rangle$ and $\langle A^z \rangle$ which spontaneously break the SO(3) rotational symmetry. It is not clear whether the same method can be extended to the gapless phases. The chromo-magnetic instability of the gapped 2SC phase can also be removed by the formation of an inhomogeneous condensate of charged gluons (Ferrer and de la Incera, 2007).

For the cases in which the chromo-magnetic instability is related to the presence of gapless modes, in Hong (2005) the possibility that a secondary gap opens at the Fermi surface is studied. The solution of the instability is due to a mechanism that stabilizes the system preventing the appearance of gapless modes. However, the secondary gap turns out to be extremely small and at temperatures typical of CSOs it is not able to fix the chromo-magnetic instability (Alford and Wang, 2006). The imaginary value of the Meissner mass can be understood as a tendency of the system toward a non-homogeneous phase (Gubankova et al., 2010; Hong, 2003; Iida and Fukushima, 2006). This can be easily seen in the toy-model system discussed in Sec. III.A for the case of a $U(1)$ symmetry, where one can show that the coefficient of the gradient term of the low energy fluctuations around the ground state of the effective action is proportional to the Meissner mass squared (Gubankova et al., 2010).

For three-flavor quark matter two non-homogenous superconducting phases have been proposed. If kaon condensation takes place in the CFL phase (Bedaque and Schafer, 2002; Kaplan and Reddy, 2002), the chromo-magnetic instability might drive the system toward a non-homogeneous state where a kaon condensate current is generated, balanced by a counter-propagating current in the opposite direction carried by gapless quark quasi-particles. This phase of matter, named curCFL-$K^0$, has been studied in Krivjvsk and turns out to be chromo-magnetically stable.

The second possibility is the crystalline color superconducting (CCSC) phase (Alford et al., 2001; Bowers et al., 2002; Casalbuoni et al., 2001; Mannarelli et al., 2006), which is the QCD analogue of a form of non-BCS pairing first proposed by Larkin, Ovchinnikov, Fulde and Ferrell (LOFF) (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964). This phase is chromo-magnetically stable as we shall discuss in Sec. III.F for two-flavor quark matter, and in Sec. III.B.4 for three-flavor quark matter. The condensate characteristic of this phase is given by

$$\langle 0|\psi^\alpha_L \psi^\beta_L|0\rangle \propto \sum_{l=1}^3 \Delta_i e^{\alpha_1 \beta_1 \varepsilon_{3l}} \sum_{q_I^m \in \{ q_I \}} e^{2i q_I^m \cdot r},$$

which is similar to the condensate reported in Eq. (2) but now there are three gap parameters, each having a periodic modulation in space. The modulation of the $f$th condensate is defined by the vectors $q_I^m$, where $m$ is the index which identifies the elements of the set $\{ q_I \}$. In position space, this corresponds to condensates that vary like $\sum_m \exp(2i q_I^m \cdot r)$, meaning that the $q_I^m$'s are the reciprocal vectors which define the crystal structure of the condensate.

In Sec. III we shall discuss various properties of the the two-flavor CCSC phase. This phase, first proposed in Alford et al. (2001), corresponds to the case where only one gap parameter is non-vanishing. The chromo-magnetic stability of a simple two-flavor periodic structure with a gap parameter modulated by a single plane wave (hereafter we shall refer to this phase as Fulde-Ferrell (FF) structure (Fulde and Ferrell, 1964)) has been considered in Giannakis et al. (2003), Giannakis and Ren (2005a,b) where it has been shown that

- The presence of the chromo-magnetic instability in g2SC is exactly what one needs in order that the FF phase is energetically favored (Giannakis and Ren, 2005a).
- The FF phase in the two-flavor case has no chromo-magnetic instability (though it has gapless modes) at least in the weak coupling limit (Giannakis et al., 2005b; Giannakis and Ren, 2005a,b).
- The stability of the FF phase in the strong coupling case has been studied in Gorbar et al. (2006b), where it is shown that for large values of the gap parameter the FF phase cannot cure the chromo-magnetic instability. In Nickel and Buballa (2009) it has been questioned whether among the possible one-dimensional modulations, the
periodic LOFF solution is the favored one. As we shall discuss in Sec. [II.C] it is found that for two-flavor quark matter, a solitonic-like ground state is favored with respect to FF in the range of values $0.7\Delta \lesssim \delta \mu \lesssim 0.78\Delta$. However, at least in weak coupling, the FF phase is not the crystalline structure one should compare to. The FF phase is slightly energetically favored with respect to unpaired quark matter and 2SC quark matter for $\Delta/\sqrt{2} < \delta \mu < 0.75\Delta$, but more complicated crystalline structures have larger condensation energies in a larger range of values of $\delta \mu$ (Bowers and Rajagopal, 2002).

The stability analysis of the three-flavor CCSC phase is discussed in Sec. [III.B.3] where we report on the results obtained for a simple structure made of two plane waves by a Ginzburg-Landau (GL) expansion (Ciminale et al., 2006). This particular three-flavor CCSC phase turns out to be chromo-magnetically stable, but the stability of more complicated crystalline structures has not been studied, although by general arguments they are expected to be stable, at least in the weak coupling limit.

Whether or not the crystalline color superconducting phase is the correct ground state for quark systems with mismatched Fermi surfaces has not yet been proven. In any case it represents an appealing candidate because in this phase quark pairing has no energy cost proportional to $\delta \mu$. The reason is that pairing occurs between quarks living on their own Fermi surfaces. However, this kind of pairing can take place only if Cooper pairs have nonzero total momentum $2q$ and therefore it has an energy cost corresponding to the kinetic energy needed for the creation of quark currents. Moreover, pairing can take place only in restricted phase space regions, meaning that the condensation energy is smaller than in the homogeneous phase. The vector $q$ has a magnitude proportional to the chemical potential splitting between Fermi surfaces, whereas its direction is spontaneously chosen by the system. In case one considers structures composed by a set of vectors $\{q_i\}$, one has to find the arrangement that minimizes the free-energy of the system (Bowers and Rajagopal, 2002; Rajagopal and Sharma, 2006). This is a rather complicated task which is achieved by analyzing some ansatz structures and comparing the corresponding free-energies. We shall report on this subject in Sec. [II.C] and [II.E] for the two-flavor case and in Sec. [III.C] for the three-flavor case.

Summarizing, we can say that the state of matter at asymptotic densities is well defined and should correspond to the CFL condensate. At intermediate and more realistic densities it is not clear which is the ground state of matter. Our knowledge of the phases of matter can be represented in the so-called QCD phase diagram, which is schematically depicted in Fig. 2. At low-density and low temperature quarks are confined in hadrons but increasing the energy scale quarks and gluons degrees of freedom are liberated. At high temperature this leads to the formation of a plasma of quarks and gluons (QGP), while at large densities matter should be in a color superconducting phase. Apart from the phases we have discussed other possibilities may be realized in the density regime relevant for CSO. Here we only mention that recently it has been proposed one more candidate phase, the so-called quarkyonic phase (McLerran and Pisarski, 2007), which is characterized by a non-vanishing baryon number density and found to be a candidate phase at least for a large number of colors. Another possibility is that the constituent value of the strange quark mass is so small that the CFL phase is the dominant one down to the phase transition to the hadronic phase. In this case, a rather interesting possibility is that there is no phase transition between the CFL phase and the hadronic phase (hyper-nuclear matter), in the so-called quark-hadron continuity scenario (Schafer and Wilczek, 1999a).

In Section IV we shall discuss whether the presence of a non-homogenous color superconducting phase within the core of a compact star may lead to observable effects. The different possible signatures are associated with

1. Gravitational wave emission.
2. Anomalies in the rotation frequency (known as glitches).
3. Cooling processes.
4. Mass-radius relation.

Point 1 is discussed in Sec. [IV.A] and relies on the observation that pulsars can be continuous sources of gravitational waves if their mass distribution is not axis-symmetric. The large shear modulus characteristic of the CCSC phase allows the presence of big deformations of the star, usually called “mountains”.

Regarding point 2 the three-flavor CCSC phase is characterized by an extremely large rigidity (Mannarelli et al., 2007), with a shear modulus which is larger than that of a conventional neutron star crust by a factor of 20 to 1000. This fact makes the crystalline phases of quark matter unique among all forms of matter proposed as candidates for explaining stellar glitches. This topic is detailed in Sec. [IV.B].

Point 3 is discussed in Sec. [IV.C] where we report the results of a first step toward calculating the cooling rate for neutron stars with a CCSC core taken in Anglani et al. (2006), where a simple two plane waves structure was considered. Because the crystalline phases leave some quarks at their respective Fermi surfaces unpaired, their neutrino emissivity
and heat capacity are only quantitatively smaller than those of unpaired quark matter [Iwamoto, 1980, 1981], not parametrically suppressed. This suggests that neutron stars with crystalline quark matter cores will cool down by the direct Urca reactions, i.e. more rapidly than in standard cooling scenarios (Page et al., 2004).

Point 3 is discussed in Sec. IV.D where it is shown that recent observations of very massive compact stars do not exclude the possibility that CSO have a CCSC core.

Finally, studying the damping mechanisms of the radial and non-radial star oscillations (star-seismology) might be useful to infer the properties of the CCSC core, and to have information about the width of the various internal layers of the star. We are not aware of any paper discussing this topic, which however might be relevant for restricting the parameter space of the CCSC phase.

II. THE TWO-FLAVOR NON-HOMOGENEOUS PHASES

The non-homogeneous two-flavor crystalline color superconducting phase is an extension to QCD of the phase proposed in condensed matter systems by Fulde and Ferrell (Fulde and Ferrell, 1964) and by Larkin and Ovchinnikov (Larkin and Ovchinnikov, 1964) (LOFF). Some aspects of this state have been previously discussed in the review Casalbuoni and Nardulli (2004). Therefore, we will focus here on recent results, and in particular we discuss one of the main properties of these phase, namely its chromomagnetic stability. This important property is not shared with other homogeneous gapless color superconducting phases, and therefore it strongly motivates its study.

A. Mismatched Fermi spheres

Before discussing the case of two-flavor quark matter, in order to show how gapless superconductivity may arise, it is useful to consider the simpler case of a non-relativistic fermionic gas avoiding the formal complications due to flavor and color degrees of freedom. We consider a system consisting of two unbalanced populations of different species $\psi_1$ and $\psi_2$, with opposite spin, at vanishing temperature, having the hamiltonian density

$$\mathcal{H} = \sum_{s=1,2} \bar{\psi}_s \left( -\frac{\nabla^2}{2m} - \mu_s \right) \psi_s - G \bar{\psi}_1 \bar{\psi}_2 \psi_2 \psi_1 ,$$

where $G > 0$ is the four fermion coupling constant. The chemical potentials of the two species can be written as $\mu_1 = \mu + \delta \mu$ and $\mu_2 = \mu - \delta \mu$, so that $\mu$ is the average of the two chemical potentials and $2\delta \mu$ their difference. The
Increasing $\delta\mu$ $\delta\mu < \Delta_0$ 

Increasing $\delta\mu$ $\delta\mu > \Delta_0$

FIG. 3 (Color online). Pictorial description of the behavior of the Fermi spheres of two different populations of fermions, with up and down spin, in the presence of an attractive interaction and with increasing $\delta\mu$. For $\delta\mu < \Delta_0/\sqrt{2}$ pairing takes place in the gray (red online) region. For $\delta\mu > \Delta_0/\sqrt{2}$ the BCS homogenous pairing is no more energetically allowed and the system has a first order phase transition to the normal phase.

effect of the attractive interaction between fermions is to induce the di-fermion condensate

$$\langle \psi_s(x)\psi_t(x) \rangle = \frac{\Delta(x)}{G}(\sigma_2)_{st},$$

which spontaneously breaks the global symmetry corresponding to particle number conservation. As a result the fermionic excitation spectrum consists of two Bogolyubov modes with dispersion laws

$$E_a = \delta\mu + \sqrt{\xi^2 + \Delta_0^2}, \quad E_b = -\delta\mu + \sqrt{\xi^2 + \Delta_0^2},$$

with $\xi = -\mu + p^2/2m$ and $\Delta_0$ is the homogenous mean field solution. Without loss of generality we take $\delta\mu > 0$, then from Eq. (10) we infer that tuning the chemical potential difference to values $\delta\mu \geq \Delta_0$, the mode $b$ becomes gapless. This phase corresponds to a superconductor with one gapped and one gapless fermionic mode and is named gapless homogeneous superfluid.

In the above naive discussion we did not take into account that increasing $\delta\mu$ the difference between the free-energy of the superfluid phase, $\Omega_s$, and of the normal phase, $\Omega_n$ decreases; eventually the normal phase becomes energetically favored for sufficiently large $\delta\mu$. In weak coupling it is possible to show that the two free-energies become equal at $\delta\mu = \Delta_0/\sqrt{2}$ (corresponding to the so-called Chandrasekhar-Clogston limit (Chandrasekhar, 1962; Clogston, 1962)), that is before the fermionic excitation spectrum becomes gapless. At this critical value of $\delta\mu$ a first order transition to the normal phase takes place and the superfluid phase becomes metastable. The reason for this behavior can be qualitatively understood as follows. Pairing results in an energy gain of the order of $\Delta_0$, however BCS pairing takes place between fermions with equal and opposite momenta. When a mismatch between the Fermi sphere is present it tends to disfavor the BCS pairing, because in order to have equal momenta, fermions must pay an energy cost of the order of $\delta\mu$. Therefore, when $\delta\mu > c\Delta_0$, where $c$ is some number, pairing cannot take place. In the weak coupling limit, one finds that $c = 1/\sqrt{2}$. This behavior is pictorially depicted in Fig. 3.

Considering homogeneous phases, a metastable superconducting phase exists for $\delta\mu \geq \Delta_0/\sqrt{2}$, but still the system cannot develop fermionic massless modes, because at $\delta\mu = \Delta_0$ various instabilities appear (Gubankova et al., 2010; Mannarelli et al., 2006a; Pao et al., 2006; Sheehy and Radzihovsky, 2006; Wu and Yin, 2003). In order to explain what happens, let us consider the low energy spectrum of the system, which can be described considering the fluctuations of $\Delta(x)$ around the mean field solution $\Delta_0$. The oscillations in the magnitude of the condensate are described by the Higgs mode, $\lambda(x)$, while the phase fluctuations are described by the Nambu-Goldstone (or Andersson-Bogolyubov) mode $\phi(x)$. Integrating out the fermionic degrees of freedom results in the Lagrangian densities (Gubankova et al., 2010)

$$\mathcal{L}_\phi = A(\partial_t \phi(x))^2 - \frac{B}{3}(\nabla \phi(x))^2,$$

$$\mathcal{L}_\lambda = -C\lambda(x)^2 + D(\partial_t \phi(x))^2 - \frac{E}{3}(\nabla \phi(x))^2.$$ 

The stability of the system is guaranteed when all the coefficients $A, B, C, D, E$ are positive. $A$ and $D$ turn out to be always positive, then we define the three stability conditions
1. The Higgs has a positive squared mass: \( C > 0 \)

2. The space derivative of the Higgs must be positive: \( E > 0 \)

3. The space derivative of the NGB must be positive: \( B > 0 \)

For \( \delta \mu > \Delta_0 \) the three conditions above are not simultaneously satisfied. Condition 1 is not satisfied because we are expanding around a maximum of the free-energy, and indeed a free-energy analysis shows that at \( \delta \mu = \Delta_0 \) the homogenous superfluid phase becomes unstable. The fact that the condition 2 is not satisfied signals that the system is unstable toward space fluctuations of the absolute value of the condensate, while condition 3 is not fulfilled when the system is unstable toward space fluctuations of the phase of the condensate. Clearly the conditions 2 and 3 are related and tell us that when a large mismatch between the Fermi sphere is present, the system prefers to move to a non-homogenous phase.

In the present toy-model the three conditions above are simultaneously violated in weak coupling at \( \delta \mu/\Delta_0 = 1 \), but they are violated at different values of this ratio in the strong coupling regime. Moreover, with increasing coupling strength it is possible to force the system into a homogeneous gapless phase, but this happens when \( \mu \sim -\Delta \), deep in the Bose-Einstein condensate (BEC) limit, see e.g. Gubankova et al. (2010).

For \( \delta \mu < \Delta_0 \), the homogenous BCS might also be energetically favored if there is a way of reducing \( \Omega_s \), and this is indeed what happens in the CFL phase where multiple interaction channels are available and the color and electrical neutrality conditions may disfavor the normal phase. This corresponds to the fact that \( C > 0 \) and thus the condition 1 above is satisfied. However, the conditions 2 and 3 must be satisfied as well, but decreasing \( \Omega_s \) does not per se guarantee that those constraints are satisfied. Indeed, it is possible to show that in the weak coupling regime the gapless homogeneous phase is in general not accessible, because when \( \delta \mu > \Delta_0 \), the solution with \( \Delta_0 \neq 0 \) is unstable with respect to the conditions 2 and 3.

Gauging the \( U(1) \) global symmetry, it is possible to show that the condition \( B < 0 \) is equivalent to the condition that the Meissner mass squared of the gauge field becomes negative, which corresponds to a magnetic instability. Therefore the magnetic instability is related to the fact that we are expanding the free-energy around a local maximum. This statement is rather general and indeed in Sec. II.F we shall see that an analogous conclusion can be drawn for the 2SC phase. Notice that increasing the temperature of the system does not help to recover from this instability (Alford and Wang, 2005). Indeed, the effect of the temperature is to produce a smoothing of the dispersion law, which first has the effect of increasing the instability region to values \( \delta \mu < \Delta_0 \).

Summarizing, we have seen that for the simplest case of a weakly interacting two-flavor system, for \( \delta \mu > \delta \mu_1 \) the superfluid homogenous phase is metastable, while for \( \delta \mu > \Delta_0 \), \( \Omega_s \) it does not have a local minimum in \( \Delta_0 \neq 0 \) and it is unstable toward fluctuations of the condensate. In general, the three conditions above should be simultaneously satisfied in order to have a stable (or metastable) vacuum. The gapless phase is only accessible for homogenous superfluids deep in the strong coupling regime, for negative values of the chemical potential.

A different possibility is that gapless modes arise at weak coupling in a non-homogenous superfluid. As we shall see in the following sections, the non-homogenous LOFF phase is energetically favored in a certain range of values of \( \delta \mu \) larger than the Chandrasekhar-Clogston limit, see Sec. II.C it is chromomagnetically stable, see Sec. II.F, and it has gapless fermionic excitations, see Sec. II.D. It is important to remark that the presence of a gapless fermionic spectrum is not in contrast with the existence of superconductivity (de Gennes, 1966, e.g. type II superconductors have gapless fermionic excitations for sufficiently large magnetic fields (de Gennes, 1966; Saint-James et al., 1969).

B. Gapless 2SC phase of QCD

The gapless 2SC phase (g2SC in the following) of QCD was proposed in Shovkovy and Huang (2003) (see also Huang and Shovkovy (2003)) as a color superconducting phase which may sustain large Fermi surface mismatches. However, it was soon realized by the same authors that this phase is chromomagnetically unstable (Huang and Shovkovy, 2004a,b), meaning that the masses of some gauge fields become imaginary. In the following we briefly discuss the properties of the g2SC phase at vanishing temperature, and then we deal with the problem of chromomagnetic instability.

We consider neutral two-flavor quark matter at finite chemical potential. The system is described by the following Lagrangian density:

\[
\mathcal{L} = \bar{\psi} \left( i \gamma_\mu \partial^\mu - m + \mu \gamma_0 \right) \psi + \mathcal{L}_{\text{int}},
\]  

(13)
where $\psi \equiv \psi_0^\alpha$, $i = 1, 2$ and $\alpha = 1, 2, 3$ corresponds to a quark spinor of flavor $i$ and color $\alpha$. The current quark mass is denoted by $m$ (we take the isospin symmetric limit $m_u = m_d = m$), and $\mathcal{L}_{\text{int}}$ is an interaction Lagrangian that will be specified later.

In Eq. (13), $\mu$ is the quark chemical potential matrix with color and flavor indices, given by

$$\mu \equiv \mu_{ij, \alpha \beta} = (\mu \delta_{ij} - \mu_c Q_{ij}) \delta_{\alpha \beta} + \frac{2}{\sqrt{3}} \mu_8 (T_8)_{\alpha \beta} \delta_{ij},$$

(14)

$Q_{ij} = \text{diag}(Q_u, Q_d)$ is the quark electric charge matrix and $T_8$ is the color generator along the direction eight in the adjoint color space; $\mu_c$, $\mu_8$ denote respectively the electron and the color chemical potential. Since $\mu$ is diagonal in color and flavor spaces, we can indicate its element with $\mu_{i \alpha}, \text{ e.g.} \mu_{ub}$ is the chemical potential of up blue quarks. A chemical potential along the third direction of color, $\mu_3$, can be introduced besides $\mu_8$, but, for all the cases that we discuss in this section, we require the ground state to be invariant under the $SU(2)_c$ color subgroup; this makes the introduction of $\mu_3$ unnecessary.

As interaction Lagrangian density we consider the NJL-like model

$$\mathcal{L}_{\text{int}} = G_S \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i\gamma_5 T \psi)^2 \right] + G_D \left[ (\bar{\psi} C \varepsilon \varepsilon i\gamma_5 \psi)_{k\gamma} (i \bar{\psi} C \varepsilon \varepsilon i\gamma_5 \psi^C)_{k\gamma} \right],$$

(15)

where $\psi^C = C \bar{\psi}^T$ denotes the charge-conjugate spinor, with $C = i \gamma_2 \gamma_0$ the charge conjugation matrix. The matrices $\varepsilon$ and $\epsilon$ denote the antisymmetric tensors in flavor and color space, respectively; we used in the second term on the right hand side of Eq. (15) the shorthand notation

$$(\bar{\psi} C \varepsilon \varepsilon i\gamma_5 \psi)_{k\gamma} \equiv (\bar{\psi}^C_{\alpha \beta i} \varepsilon_{j k} \epsilon_{\alpha \beta \gamma} i\gamma_5 \psi_{j \gamma}),$$

(16)

and an analogous expression for the other bilinear. In Eq. (15), two coupling constants are introduced in the scalar-pseudoscalar quark-antiquark channel, denoted by $G_S$, and in the scalar diquark channel, denoted by $G_D$. In Huang and Shovkovy (2003) the parameters of the model are chosen to reproduce the pion decay constant in the vacuum, $f_\pi = 93$ MeV, and the vacuum chiral condensate $\langle \bar{u} u \rangle^{1/3} = (\bar{d} d)^{1/3} = -250$ MeV. Moreover, an ultraviolet cutoff $\Lambda$ is introduced to regularize the divergent momentum integrals. The parameter set of Huang and Shovkovy (2003) is given by

$$\Lambda = 653.3 \text{ MeV}, \quad G_S = 5.0163 \text{ GeV}^{-2},$$

(17)

The relative strength between the couplings in the quark-antiquark and quark-quark channels could be fixed by a Fierz rearranging of the quark-antiquark interaction, see for example Buballa (2003). The Fierz transformation gives $G_D/G_S = 0.75$. However, non perturbative in-medium effects might change this value. Therefore, in Huang and Shovkovy (2003) the ratio of $G_D$ to $G_S$ is considered as a free parameter.

In Huang and Shovkovy (2003) the authors consider only the case $m = 0$ and vanishing chiral condensate. When $m \neq 0$ the chiral condensate in the ground state does not vanish, but its effects are presumably negligible, giving a small shift of the quark Fermi momenta. This shift of Fermi momenta might change the numerical value of the electron chemical potential only of some few percent. Hence, the main results of Huang and Shovkovy (2003) should not change much if a finite value of the current quark mass is considered.

Once the Lagrangian density is specified, the goal is to compute the thermodynamic potential. In the mean field (and one loop) approximation, this can be done easily using standard bosonization techniques. Neglecting the chiral condensate, the mean field Lagrangian density can be written within the Nambu-Gorkov formalism, see Sec. III.B.1 in the compact form

$$\mathcal{L} = \chi^\dagger S^{-1} \chi - \frac{\Delta^2}{4G_D},$$

(18)

where

$$\chi = \begin{pmatrix} \psi \\ \psi^C \end{pmatrix},$$

(19)

is the Nambu-Gorkov spinor and the the gap parameter, $\Delta \equiv \Delta_{2SC} \epsilon^{\alpha \beta \gamma} \varepsilon_{ij} C \gamma_5$, is included in the inverse propagator

$$S^{-1} = \begin{pmatrix} i \gamma_\mu \partial^\mu + \mu \gamma_0 & \Delta \\ \Delta \delta^{ij} & i \gamma_\mu \partial^\mu - \mu \gamma_0 \end{pmatrix},$$

(20)
as an off-diagonal term in the “Nambu-Gorkov space”.

We shall focus on the zero temperature regime (for a discussion of the rather uncommon temperature behavior of the g2SC phase see Huang and Shovkovy (2003)), which is relevant for astrophysical applications. The one loop expression of the thermodynamic potential can be determined from the inverse propagator in Eq. (20); for vanishing temperature it is given by

\[ \Omega = \Omega_0 - \frac{\mu^2}{12\pi^2} + \frac{\Delta^2_{2SC}}{4G_D} - \sum_n \int \frac{dp}{(2\pi)^3} |E_n|, \]  

(21)

for a derivation see e.g. Buballa (2003). In the above equation \( \Omega_0 \) is an irrelevant constant which is chosen in order to have a vanishing pressure in the vacuum. The second addendum corresponds to the electron free-energy (electron masses have been neglected). The last addendum is the contribution due to the quark determinant. The sum runs over the twelve fermion propagator poles, six of them corresponding to quarks and the other six corresponding to antiquarks quasiparticles:

\[ E_{1,2} = |p| \mp \mu_{ab}, \]

(22)

\[ E_{3,4} = |p| \mp \mu_{db}, \]

(23)

\[ E_{5,6} = \delta \mu + \sqrt{(|p| \mp \bar{\mu})^2 + \Delta^2_{2SC},} \]

(24)

\[ E_{7,8} = -\delta \mu + \sqrt{(|p| \mp \bar{\mu})^2 + \Delta^2_{2SC},} \]

(25)

and \( E_9 = E_5, E_{10} = E_6, E_{11} = E_7, E_{12} = E_8 \). Here we have introduced the shorthand notation \( \bar{\mu} = \mu - \frac{\mu_e}{6} + \frac{\mu_S}{3}, \quad \delta \mu = \frac{\mu_e}{2}. \)

(26)

Using the explicit form of the dispersion laws, the previous equation can be written as

\[ \Omega = \Omega_0 - \frac{\mu^2}{12\pi^2} - \frac{\mu^2_{ab}}{12\pi^2} - \frac{\mu^2_{db}}{12\pi^2} - \frac{\Lambda^4}{2\pi^2} - 2\int_0^\Lambda \frac{p^2 dp}{\pi^2} \left( \sqrt{(p - \bar{\mu})^2 + \Delta^2_{2SC}} + \sqrt{(p + \bar{\mu})^2 + \Delta^2_{2SC}} \right) - 2\delta \mu - \Delta_{2SC} \int_{\mu_-}^{\mu_+} \frac{p^2 dp}{\pi^2} \left( \delta \mu - \sqrt{(p - \bar{\mu})^2 + \Delta^2_{2SC}} \right), \]

(27)

where \( \mu_\pm = \bar{\mu} \pm \sqrt{\delta \mu^2 - \Delta^2_{2SC}}. \)

The value of \( \Delta_{2SC} \) is determined by the solution of the equation

\[ \frac{\partial \Omega}{\partial \Delta_{2SC}} = 0, \]

(28)

with the neutrality constraints,

\[ n_s = \frac{\partial \Omega}{\partial \mu_s} = 0, \quad n_Q = \frac{\partial \Omega}{\partial \mu_e} = 0, \]

(29)

which fix the values of \( \mu_e \) and \( \mu_s \).

The numerical analysis of Huang and Shovkovy (2003) shows that \( \mu_s \) is much smaller than \( \mu_e \) and \( \Delta_{2SC} \), both for \( \Delta_{2SC} \geq \delta \mu \) and for \( \Delta_{2SC} < \delta \mu \). As a consequence, it is possible to simplify the equations for the gap parameter and the electron chemical potential, \( \Delta \) and \( \mu_e \) respectively, by putting \( \mu_s = 0 \). Therefore, the properties of the system depend only on the values \( \Delta_{2SC} \) and \( \mu_e \) and on the couplings \( G_D \) and \( G_S \). The result of Huang and Shovkovy (2003) can be summarized as follows:

- For \( G_D/G_S \gtrsim 0.8 \), strong coupling, the 2SC phase is the only stable phase
- For \( 0.7 \lesssim G_D/G_S \lesssim 0.8 \), intermediate coupling, the g2SC phase is allowed for \( \delta \mu > \Delta_{2SC} \)
- For \( G_D/G_S \lesssim 0.7 \), weak coupling, only unpaired quark matter is favored.

In the g2SC phase the quasiparticle fermionic spectrum consists of four gapless modes and two gapped modes, whereas in the 2SC phase there are two gapless fermionic modes and four gapped fermionic modes. In the latter case the only gapless modes correspond to the up and down blue quarks that do not participate in pairing.
1. Meissner masses of gluons in the g2SC phase

The diquark condensate of the 2SC phase induces the symmetry breaking pattern reported in Eq. (6); in particular the group $SU(3)_c \otimes U(1)_{em}$ is broken down to $SU(2)_c \otimes \hat{U}(1)_{em}$, where $\hat{U}(1)_{em}$ is the gauge group corresponding to the rotated massless photon associated to the unbroken generator

$$\hat{Q} = Q \cos \theta - \frac{g}{e} T_8 \sin \theta,$$

where $g$ and $e$ denote the strong and the electromagnetic couplings respectively, $Q$ is the generator of the electromagnetic gauge transformations and $T_8$ is the eighth generator of the color transformations. The mixing coefficients have been determined in Alford et al. (2000b) (see also Gorbar (2000)) and are given by

$$\cos \theta = \frac{\sqrt{3}g}{\sqrt{3g^2 + e^2}}, \quad \sin \theta = \frac{e}{\sqrt{3g^2 + e^2}}.$$

The linear combination

$$T_8 = T_8 \cos \theta + \frac{g}{e} Q \sin \theta,$$

is orthogonal to $\hat{Q}$ and gives the broken generator; which we shall refer to as the $\hat{8}$ mode, acquires a Meissner mass. Actually, the NJL-like Lagrangian in Eqs. (13) and (15), has only global symmetries, but gauging the $SU(3)_c$ group and the $U(1)$ subgroup of $SU(2)_L \times SU(2)_R$, one has that the spontaneous symmetry breaking leads to the generation of Meissner masses for the five gluons associated to the broken generators. In order to compute these masses, we define the gauge boson polarization tensor, see e.g. Le Bellac (2000),

$$\Pi_{ab}^{\mu}(p) = -\frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \Gamma_a^\mu S(q) \Gamma_b^\mu S(q-p) \right],$$

where $S(p)$ is the quark propagator in momentum space, which can be obtained from Eq. (20), and

$$\Gamma_a^\mu = g \, \gamma^\mu \, \text{diag} \left( T_a, -T_8^T \right),$$

$$\Gamma_9^\mu = e \, \gamma^\mu \, \text{diag} \left( Q, -Q \right),$$

are the interaction vertex matrices. The trace in Eq. (33) is taken over Dirac, Nambu-Gorkov, color and flavor indices; $a, b = 1, \ldots, 8$ indicate the adjoint color and we use the convention that the component with $a, b = 9$ corresponds to the photon.

The screening masses of the gauge bosons are defined in terms of the eigenvalues of the polarization tensor and in the basis in which $\Pi_{ab}^{\mu}$ is diagonal the Debye masses and the Meissner masses are respectively defined as

$$M_{D,a}^2 = -\lim_{p^+ \to 0} \Pi_{aa}^{00}(0, p),$$

$$M_{M,a}^2 = -\frac{1}{2} \lim_{p^+ \to 0} \left( g_{ij} + \frac{p_ip_j}{|p|^2} \right) \Pi_{ij}^{00}(0, p).$$

Both masses have been evaluated in the 2SC phase in Rischke (2000b); Rischke and Shovkovy (2002); and Schmitt et al. (2004). The Debye masses of all gluons related to the chromo-electric screening and are always real, therefore do not affect the stability of the 2SC and g2SC phases. The Meissner masses of the gluons with adjoint color $a = 1, 2, 3$ are always zero, because they are associated to the unbroken color subgroup $SU(2)_c$.

For non-vanishing values of $\delta \mu$ the Meissner screening masses have been evaluated in Huang and Shovkovy (2004a,b). Gluons with adjoint color $a = 4, 5, 6, 7$ are degenerate and in the limit $\mu_b = 0$ their Meissner masses are given by

$$M_{M,4}^2 = \frac{4\alpha_s \mu^2}{\pi} \left[ \frac{\Delta_{2SC}^2 - 2\delta \mu^2}{2\Delta_{2SC}^2} + \frac{\delta \mu}{\Delta_{2SC}} \sqrt{\delta \mu^2 - \Delta_{2SC}^2} \theta(\delta \mu - \Delta_{2SC}) \right].$$

The squared Meissner mass turn out to be negative not only in the gapless phase, $\Delta_{2SC}/\delta \mu < 1$, but also in the gapped phase, when $\Delta_{2SC}/\delta \mu < \sqrt{2}$. This result seems in contrast with the result of the previous section, where an imaginary Meissner mass was related to the existence of a local maximum of the free-energy arising at $\delta \mu = 0$. However, from the analysis of the 2SC free-energy of the system, one can see that when $\Delta_{2SC}/\delta \mu = \sqrt{2}$ the state with $\Delta_{2SC} \neq 0$ corresponds to a saddle point in the $\Delta - \delta \mu$ plane. The neutrality condition transforms this saddle point into a local minimum. However, as explained in the previous section, the gauge fields can be related to the fluctuations of the
gap parameter. These fluctuations can probe all the directions in the $\Delta - \delta \mu$ plane around the stationary point and would result in a low energy Lagrangian with dispersion laws akin to the those discussed in Eqs. 11 and 12 with the coefficients $E$ and $B$ negative.

Finally let us consider the 8 mode, associated to the broken generator defined in Eq. 32. The corresponding Meissner mass can be obtained diagonalizing the polarization tensor in Eq. 33 in the subspace $a, b \in \{8, 9\}$, or more directly, by substituting $T_a \rightarrow T_8$ in the vertex factors of the polarization tensor. The squared Meissner mass of the 8 mode turns out to be

$$M_{\delta \mu}^2 = \frac{4(3\alpha_s + \alpha)\bar{\mu}^2}{27\pi} \left(1 - \frac{\delta \mu}{\sqrt{\bar{\mu}^2 - \Delta_{2SC}^2}} \theta(\delta \mu - \Delta_{2SC})\right),$$  

(39)

and becomes negative for $\Delta_{2SC}/\delta \mu < 1$. As shown in Gatto and Ruggieri (2007), the instability in this sector is transmitted to a gradient instability of the pseudo-Goldstone boson related to the $U(1)_A$ symmetry which is broken by the diquark condensate. Although in the vacuum the $U(1)_A$ symmetry is explicitly broken by instantons, at finite chemical potential instantons are Debye screened and $U(1)_A$ can be considered as an approximate symmetry, which is then spontaneously broken by the diquark condensate.

C. The two-flavor crystalline color superconducting phase

Since the homogeneous g2SC phase is chromomagnetically unstable, the question arises of the possible existence of a different superconducting phase for large mismatch between the Fermi spheres. There are several candidate phases, which include the gluonic phase (Gorbar et al. 2006a), the solitonic phase (Nickel and Buballa, 2009) and the crystalline color superconducting phases.

In this section we review some of the main results about the two-flavor crystalline phase. Firstly, we describe the one plane wave ansatz in the framework of the symplectic model discussed in Sec. II.A; then, we turn to the crystalline color superconducting phase and report on the Ginzburg-Landau (GL) analysis of various crystalline structures. We relax the constraint of electrical and color neutrality, and treat the difference of chemical potentials between $u$ and $d$ quarks, $2\delta \mu$, as a free parameter.

1. The one plane wave ansatz

In Sec. II.A we have shown that in weak coupling the Chandrasekhar-Clogston limit (Chandrasekhar, 1962; Clogston, 1962) signals that the standard BCS phase becomes metastable, but does not forbid the existence of different forms of superconductivity. In particular, it does not forbid the existence of Cooper pairs with non-vanishing total momentum. It was shown by Larkin and Ovchinnikov (Larkin and Ovchinnikov, 1964) and Fulde and Ferrell (Fulde and Ferrell, 1964), in the context of electromagnetic superconductivity, that in a certain range of values of $\delta \mu$ it might be energetically favorable to have Cooper pairs with nonzero total momentum. For the simple two-level system discussed in Sec. II.A, Cooper pairs with momentum $2q$ can be described by considering the difermion condensate in Eq. 9 given by

$$\langle \psi_a(x)\psi_b(x) \rangle = \frac{\Delta}{G} i(\sigma_2)_{ab} e^{2i q \cdot x},$$  

(40)

and we shall call this state of matter the FF phase.

Notice that this ansatz breaks rotational symmetry because there is a privileged direction corresponding to $q$. In the left panel of Fig. 4 the two Fermi spheres of fermions are pictorially shown and the red ribbons correspond to the regions in momentum space where pairing occurs. Pairing between fermions of different spin can only take place in a restricted region of momentum space and this implies that $\Delta < \Delta_0$. The reason why the FF phase is energetically favorable with respect to the normal phase, is that no energy cost proportional to $\delta \mu$ has to be payed for allowing the formation of Cooper pairs. The only energetic price to pay is due to the kinetic energy associated to Cooper pairs. Actually, there is a spontaneous generation of a super-current in the direction of $q$, which is balanced by a current of normal fermions in the opposite direction (Fulde and Ferrell, 1964). The gap parameter in Eq. 40 can be determined solving a gap equation under the constraint that the modulus of the Cooper momentum, $q$, minimizes the free-energy. The result is that at $\delta \mu \approx \delta \mu_1 = \Delta_0/\sqrt{2}$ there is a first order phase transition from the homogeneous BCS phase to the FF phase. Increasing further $\delta \mu$ results in a smooth decreasing of the gap function of the FF phase,
FIG. 4 (Color online). Pictorial description of the LOFF pairing in the weak coupling approximation. When \( \delta \mu > \Delta_0/\sqrt{2} \) the BCS homogenous pairing is not energetically allowed, but pairing between fermions with total nonvanishing momentum can be realized. Left panel: In the FF phase pairing takes place in two ribbons on the top the Fermi spheres of up and down fermions, such that \( \mathbf{P}_u + \mathbf{P}_d = 2\mathbf{q} \), having opening angle \( 2\theta_q \simeq 2 \arccos(\delta \mu/\mathbf{q}) \simeq 67^\circ \), thickness \( \Delta \) and angular width \( \Delta/\mathbf{q} \), see Sec. II.E. Right panel: Structure obtained with two plane waves corresponding to two vectors \( 2\mathbf{q}_1 \) and \( 2\mathbf{q}_2 \) with relative angle \( \phi \). The size and the opening angle of each ribbon is as in the FF phase. The structure with \( \phi = 180^\circ \) is called the “strip”. For illustrative purposes, we have greatly exaggerated the splitting between the Fermi surfaces, relative to the values used in the calculations reported in Sec. II.C.

until at a critical value \( \delta \mu_2 \) a second order phase transition to the normal phase takes place. In the weak coupling limit \( \delta \mu_2 = 0.754\Delta_0 \); the range \( [\delta \mu_1, \delta \mu_2] \) is called the LOFF window. In the LOFF window, the optimal value of \( \mathbf{q} \) turns out to be approximately constant, \( \mathbf{q} \approx 1.2\delta \mu \).

In [Alford et al. (2001)] the two-flavor QCD analog of the FF phase was presented. In this case the condensate has the same color, spin and flavor structure of the 2SC condensate, but with the plane wave space dependence characteristic of the FF phase, that is

\[
\langle \psi_i^\alpha(x) C \gamma_5 \psi_j^\beta(x) \rangle \propto \Delta e^{i2\mathbf{q} \cdot \mathbf{r}} \epsilon^{\alpha \beta \gamma} \epsilon_{ij3} .
\] (41)

As noticed in [Alford et al. (2001)], the FF condensate induces a spin-1 condensate as well; however, its effect is found to be numerically small, and it will be neglected here. As in the simple two-level system, in the two-flavor FF phase it is possible to determine the free-energy employing a NJL-like model with gap parameter in (41), which can be determined solving the corresponding gap equation under the constraint that the value of \( \mathbf{q} \) in Eq. (41) minimizes the free-energy. The results are the same obtained in the two-level system; in particular the LOFF window and \( \mathbf{q} \) have the same expressions reported above (but now \( 2\delta \mu \) is the the difference of chemical potentials between \( u \) and \( d \) quarks).

2. Ginzburg-Landau analysis

From Fig. 4 it seems clear than an immediate generalization of the FF phase, can be obtained adding more ribbons on the top of the Fermi spheres, corresponding to different vectors \( \mathbf{q}_m \), with \( \mathbf{q}_m \in \{\mathbf{q}\} \), where \( \{\mathbf{q}\} \) is some set of vectors to be determined by minimizing the free-energy of the system and \( m \) is a label that identifies the vectors of the set. This in turn, corresponds to consider inhomogeneous CSC phases with a more general ansatz than in Eq. (41), where the single plane wave is replaced by a superposition of plane waves, that is

\[
\Delta e^{i2\mathbf{q} \cdot \mathbf{r}} \rightarrow \sum_{\mathbf{q}^m \in \{\mathbf{q}\}} \Delta_m e^{i2\mathbf{q}^m \cdot \mathbf{r}} .
\] (42)

Assuming that the set of vectors \( \{\mathbf{q}\} \) identifies the vertices of a crystalline structure it follows that at each set \( \{\mathbf{q}\} \) corresponds a particular crystalline phase. We shall assume that the vectors \( \mathbf{q}^m \) have equal length and then we can
write \( q^m = q n^m \), therefore the crystalline structure is determined by the set of unit vectors \( \{n\} \). We shall also consider the simplified case that the coefficients \( \Delta^m \) do not depend on \( q^m \) and we shall indicate their common value with \( \Delta \). In other words, we shall consider condensates with

\[
\Delta(r) = \Delta \sum_{m=1}^{P} e^{2i\mid q m \cdot r},
\]

where \( P \) is the number of vectors \( n^m \). The simplest example is clearly the FF condensate, depicted in the left panel of Fig. 4 characterized by a single plane wave, thus corresponding to \( P = 1 \). The case with \( P = 2 \) is reported in the right panel of Fig. 4 in this case the “crystalline” structure is completely determined by \( \phi \), the relative angle between \( n^1 \) and \( n^2 \); more complicated structures can be pictorially represented in a similar way.

It is important to stress that the crystalline structure is determined by the modulation of the condensate, but the underlying fermions are not arranged in an ordered pattern, indeed fermions are superconducting, that is they form a superfluid of charged carriers.

The computation of the free-energy of a system with a general crystalline condensate cannot be obtained analytically and also the numerical evaluation is quite involved. As a consequence, the use of some approximation is necessary. A viable method is the GL expansion of the free-energy, which is obtained expanding \( \Omega \) in powers of \( \Delta \) (Bowers and Rajagopal, 2002):

\[
\frac{\Omega}{2\mu^2/\pi^2} = P\alpha \Delta^2 + \frac{\beta}{2} \Delta^4 + \frac{\gamma}{3} \Delta^6 + O(\Delta^8),
\]

where the coefficients \( \alpha, \beta \) and \( \gamma \) are computed, in the one-loop approximation using, as microscopic model, a NJL model. The GL expansion is well suited for studying second order phase transitions but might give reasonable results for soft first order phase transitions as well. In the present case the expansion is under control for \( \Delta/q \ll 1 \) and if the coefficient \( \gamma \) is positive, meaning that the free-energy is bounded from below.

For a given crystalline structure, the coefficients in Eq. (44) depend on \( \delta \mu \) and on the magnitude of \( q \); the latter is fixed, in the calculation of (Bowers and Rajagopal, 2002), to the weak coupling value \( q = 1.2\delta \mu \). For any value of \( \delta \mu \) the thermodynamic potential of a given structure is computed by minimization with respect to \( \Delta \) and then the optimal crystalline structure is identified with that with the lowest free-energy. It is possible to compute analytically the GL coefficients only for few structures; in general, they have to be computed numerically. We refer to the appendix of (Bowers and Rajagopal, 2002) for details.

In (Bowers and Rajagopal, 2002), twenty-three crystalline structures have been studied and among them, those with more than nine rings turn out to be energetically disfavored. This has been nicely explained in the weak coupling regime: in this case, as shown in the left panel of Fig. 4, for the FF phase, the pairing regions of the inhomogeneous superconductor can be approximated as rings on the top of the Fermi surfaces; one ring per wave vector \( n \) in the set \( \{n\} \). The computation of the lowest order GL coefficients shows that the intersection of two rings is energetically disfavored (Bowers and Rajagopal, 2002). As a consequence, it is natural to expect that in the most favored structure no intersecting rings appear. Since each ring has an opening angle of approximately 67 degrees, a maximum of nine rings can be accommodated on a spherical surface.

This result can be quantitatively understood as follows. For the case of the two plane waves structure (right panel of Fig. 4), in the weak coupling approximation there is one pairing ring for each of the two wave vectors. The quartic coefficient \( \beta \) depends on the angle \( \phi \) between the two wave vectors and it diverges at

\[
\phi_0 \approx 2\theta_q \approx 2\arccos \frac{\delta \mu}{q} \approx 67^\circ,
\]

which corresponds to the angle at which the two pairing rings are contiguous, meaning that for \( \phi < \phi_0 \) the two rings overlap. The latter case is energetically disfavored because, being \( \beta \) large and positive, the free-energy would be smaller.

The divergence of the coefficient \( \beta(\phi) \) at \( \phi = \phi_0 \) is due to the two limits that have been taken to compute the free-energy, namely the Ginzburg-Landau and weak coupling limits. A detailed explanation of what happens will be given in Sec. 11.2.3 when discussing a simple crystalline structure in the three-flavor case. In any case it is clear that the divergency of a GL coefficient means that the expansion is not under control, or more precisely, that the radius of convergence of the series (44) tends to zero.

Among the structures with no intersecting rings, seven are good candidates to be the most favored structure. Within these seven structures, the octahedron, which corresponds to a crystal with \( P = 6 \) and whose wave vectors point into
the direction of a body-centered-cube (BCC), is the only one with effective potential bounded from below (that is, with \( \gamma > 0 \)). The remaining six structures, with \( P = 7, 8 \) and 9, are characterized by a potential which is unbounded from below, at least at the order \( \Delta^6 \). Even if in this case the free-energy cannot be computed, qualitative arguments given in Bowers and Rajagopal (2002) suggest that the face-centered-cube (FCC), with \( P = 8 \) and with wave vectors pointing towards the vertices of a face-centered-cube, is the favored structure.

Of course, as the authors of Bowers and Rajagopal (2002) admit, their study cannot be trusted quantitatively, because of the several limitations of the GL analysis. First of all, the GL expansion formally corresponds to an expansion in powers of \( \Delta/q \) and therefore it is well suited for the study of second order phase transitions, but the condition that \( \Delta/q \ll 1 \) is not satisfied by all crystalline structures considered in Bowers and Rajagopal (2002). In some cases the GL analysis predicts a strong first-order phase transition to the normal state, with a large value of the gap at the transition point. Moreover, it may happen that the local minimum for small values of \( \Delta/q \) is not a global minimum of the system, as discussed in Sec. II.G. In this case the GL expansion in Eq. (44) underestimates the free-energy of the system and is not able to reproduce the correct order of the phase transition. For a more reliable determination of the ground state one should consider terms of higher power in \( \Delta \), which are difficult to evaluate. Finally, the claimed favored crystalline structure, namely the face-centered-cube, has \( \gamma < 0 \) and a global minimum cannot be found unless the coefficient \( O(\Delta^8) \) (or of higher order) is computed and found to be positive.

Because of these reasons, the quantitative predictions of the GL analysis should be taken with a grain of salt. One should not trust the order of the phase transition obtained by the GL expansion and also the comparison among various crystalline structures may be partially incorrect, because it is not guaranteed that one is comparing the energies of the true ground states.

On the other hand, the qualitative picture that we can draw from it, namely the existence of crystalline structures with lower free-energy than the single plane wave, is quite reasonable: crystalline structures benefit of more phase space available for pairing, thus lowering the free-energy. The symmetry argument is quite solid too, because it is based on the fact that configurations with overlapping pairing regions are disfavored, and as we shall see for one particular configuration in the three-flavor case in Sec. III.B one can prove that this statement is correct without relying on the GL expansion. Also, we shall show an interesting point, that the GL expansion underestimates the free-energy of the crystalline structures. And this happens not only in the presence of a global minimum different from the local minimum around which the GL expansion is performed, see Sec. II.G but also comparing the GL free-energy with the free-energy evaluated without the \( \Delta/q \) expansion.

D. Dispersion laws and specific heats

The thermal coefficients (specific heat, thermal conductivity etc.) of quark matter at very low temperature are of fundamental importance for the transport properties and cooling mechanisms of compact stars. The largest contribution to thermal coefficients comes from the low energy degrees of freedom and it is therefore of the utmost importance to understand whether fermionic modes are gapped or gapless. Indeed, the absence of a gap in the spectrum of fermions implies that quasi-quarks can be excited even at low temperature and therefore the corresponding thermal coefficients are not suppressed by a factor \( \approx e^{-\Delta/T} \) (which is distinctive of homogeneous BCS superconductors).

In this section, we discuss the fermion and phonon dispersion laws in the two-flavor crystalline phases for low values of momenta. Then we employ the obtained dispersion laws for the computation of the specific heats. The results discussed here do not rely on the GL approximation but are obtained by finding the zeros of the full inverse propagator close to the nodes of the dispersion law (Casalbuoni et al., 2003; Larkin and Ovchinnikov, 1964).

1. Fermi quasi-particle dispersion law: general settings

We consider a general di-fermion condensate \( \Delta(r) \), and determine the quasi-particle dispersion laws looking at the zero modes of the inverse propagator of the system. Arranging the fields in the Nambu-Gorkov spinor as follows,

\[
\chi^\alpha_i = \begin{pmatrix} \tilde{G}^\alpha_i \\ -i(\sigma_2)_{\alpha\beta} \tilde{F}^\beta_i \end{pmatrix},
\]

the inverse propagator is given by

\[
(S^{-1})^{ij}_{\alpha\beta} = \begin{pmatrix} \delta^{\alpha\beta} & \delta_{ij} (E + i\mathbf{v} \cdot \mathbf{\nabla}) + \delta \mu (\sigma_3)_{ij} \\ -\varepsilon^{\alpha\beta\gamma} \varepsilon_{ij\gamma} \Delta(r)^* \\ \delta^{\alpha\beta} & \delta_{ij} (E - i\mathbf{v} \cdot \mathbf{\nabla}) + \delta \mu (\sigma_3)_{ij} \end{pmatrix},
\]

where \( \varepsilon^{\alpha\beta\gamma} \varepsilon_{ij\gamma} \Delta(r)^* \) is computed and found to be positive.
where $E$ is the quasi-particle energy and $v$ is the Fermi velocity, that for massless quarks satisfies $v = |v| = 1$. Then, the quasiparticle spectrum can obtained solving the eigenvalue equation

$$
(S^{-1})_{ij}^{\alpha \beta} \chi_j^\beta = 0.
$$

(48)

Performing the unitary transformation

$$
\tilde{G}_i^\alpha = \left(e^{i \delta \mu \sigma_3 v \cdot r/v^2}\right)_{ij} G_j^\alpha, \quad \tilde{F}_i^\alpha = \left(e^{-i \delta \mu \sigma_3 v \cdot r/v^2}\right)_{ij} F_j^\alpha,
$$

it is possible to eliminate the dependence on $\delta \mu$ in the eigenvalue problem and this corresponds to measure the energy of each flavor from its Fermi energy. The resulting equations for $F_i^\alpha$ and $G_i^\alpha$ are independent of color and flavor indices, and therefore these indices will be omitted below. The eigenvalue problem reduces to solve the coupled differential equations:

$$
(E + iv \cdot \nabla)G - i\Delta(r)F = 0,
$$

$$
(E - iv \cdot \nabla)F + i\Delta(r)^*G = 0.
$$

(50)

These equations can be used to find the dispersion laws for any non-homogeneous $\Delta(r)$, and we shall consider here the periodic structures of the form given in Eq. (48) in order to determine whether there are gapless fermionic excitations. We shall prove that for any crystalline structure with real-valued periodic functions $\Delta(r)$ there exists a gapless mode iff the set $\{n\}$ does not contain the vector $n = 0$.

The proof is given below. Here we notice that this theorem does not apply to the case in which $\Delta(r)$ is not real, and indeed we shall show that the single plane wave structure does have a gapless mode. The theorem implies that any antipodal structure has a gapless mode, in particular the “strip” (corresponding to the structure depicted in the right panel of Fig. 4 for $\phi = 180^\circ$) and the cube have gapless modes. On the other hand, the set of vectors which identify a trixonal prism or a hexahedral prisms have a vector with $n = 0$, and the corresponding dispersion laws are gapped.

Proof:

For a periodic $\Delta(r)$, a general solution of the system in (50) is given by the Bloch functions

$$
G(r) = u(r)e^{ik\cdot r} \quad F(r) = w(r)e^{ik\cdot r},
$$

(51)

where $u(r)$ and $w(r)$ are periodic functions. Notice that if $k$ is real, the dispersion law is ungapped, otherwise a gap is present in the excitation spectrum and the eigenfunctions decrease exponentially with $r$.

Taking $E = 0$ in (50), the two solutions of the system of equations are given by $F_+ = G_+$ or by $F_- = -G_-$, where

$$
G_\pm(r) = \exp \left(\pm \int_0^{r_||} \Delta(r') \frac{dr'}{v^2}\right),
$$

(52)

with $r_|| = r \cdot v$.

Comparing this expression with Eq. (51), it is clear that these solutions correspond to Bloch functions with Re$(k) = 0$. If $\Delta(r)$ has a term with $n = 0$, it means that $\Delta(r) = A + f(r)$, where $A$ is a constant and $f(r)$ is a periodic non-constant function. In this case Eq. (52) has an exponential behavior of the type $G(r) \propto \exp(\pm Ar)$ and therefore the spectrum is gapped. On the other hand, if in the expansion of $\Delta(r)$ no term with $n = 0$ is present then the imaginary part of $k$ vanishes and the spectrum is gapless c.v.d.

The fermion dispersion law for the gapless modes can be determined using degenerate perturbation theory for $\xi = p - \mu \ll q$, i.e. close to the Fermi sphere. At the lowest order in $\xi/q$ one finds that

$$
E(v, \xi) = \frac{\xi}{\sqrt{A_+(v)A_-(v)}} = c(v)\xi,
$$

(53)

where $c(v)$ is the velocity of the excitations, and

$$
A_{\pm}(v) = \frac{1}{V_c} \int_{\text{cell}} dr \exp \left[\pm 2 \int \Delta(r') \frac{dr'(r \cdot v)}{v^2}\right],
$$

(54)

where $V_c$ is the volume of a unit cell of the lattice. The energy of the fermionic excitations depends linearly on the residual momentum $\xi$, but the velocity of the excitations is not isotropic.

Let us now specialize (53) to the case of the strip,

$$
\Delta(r) = 2\Delta \cos(2q \cdot r),
$$

(55)
which corresponds to the condensate in Eq. 18 with \( m = 2 \) and \( n^2 = -n^1 = n \) and strictly speaking does not describe a crystal, but a condensate that is modulated in the \( n \)-direction. The coefficients in the dispersion law are given by

\[
A_{\pm}^{\text{strip}} = A_{\text{strip}} = I_0 \left( \frac{2\Delta}{q \cdot v} \right),
\]

where \( I_0(z) \) is the modified Bessel function of the zeroth order. Therefore the velocity of the fermionic quasiparticles has the analytic expression

\[
c_{\text{strip}}(v) = \frac{1}{I_0 \left( \frac{2\Delta}{q \cdot v} \right)},
\]

which has the important property to vanish when \( v \) is orthogonal to \( n \). The reason is that, in the direction orthogonal to \( n \) the gap is constant and its effect is equivalent to a potential barrier. Taking \( n = (0, 0, 1) \), the dispersion law is symmetric with respect to rotations around the \( z \)-axis, for inversions with respect to the plane \( z = 0 \) and depends only on the polar angle \( \vartheta \), between \( v \) and the \( z \)-axis. In Fig. 5 we report a plot of the velocity of fermionic quasiparticles as a function of \( \cos \vartheta \), for three different values of the ratio \( \Delta/(qv) \). In the ultrarelativistic case, \( v = 1 \), the relevant case is \( \Delta/q < 1 \) and we see that the dispersion law of fermionic quasiparticles is not much affected by the condensate for \( \cos \vartheta \gg 0.2\Delta/q \) and it is the same of relativistic fermions. On the other hand for small values of \( \cos \vartheta \), the fermionic velocity is exponentially suppressed and vanishes for \( \Delta = 0 \), meaning that fermionic quasiparticles cannot propagate in the \( x-y \) plane as discussed above.

The fact that the dispersion law is linear in \( \xi \) for small values of the momentum does not assure that it is linear for any value of the momentum. Considering \( v_z \ll 1 \), it is possible to solve the Eq. 50 for \( k \) in 51 along the \( z \)-direction, without restricting to low momenta (Larkin and Ovchinnikov, 1964); the result is that

\[
E^2 = \frac{4\Delta |v_z q|}{\pi} e^{-\frac{4\Delta}{|v_z q|}} \left( 1 - \cos \frac{\pi k}{q} \right),
\]

meaning that the dispersion law is linear in the residual momentum, only for \( k/q \ll 1 \).

For the octahedron, whose six wave vectors point into the direction of a BCC structure, the corresponding gap parameter can be written as

\[
\Delta(r) = 2\Delta \left[ \cos(2qx) + \cos(2qy) + \cos(2qz) \right].
\]

It is easy to show that in this case the integral in Eq. 53 factorizes, and the dispersion law is gapless with velocity

\[
c_{\text{BCC}}(v) = \frac{1}{I_0 \left( \frac{2\Delta}{qv_x} \right) I_0 \left( \frac{2\Delta}{qv_y} \right) I_0 \left( \frac{2\Delta}{qv_z} \right)},
\]
FIG. 6 (Color online). Velocity of the fermionic quasiparticles in the BCC crystalline structure (left panel) and in the FCC crystalline structure (right panel) as a function of the polar angles \( \vartheta \) and \( \varphi \), for \( 2\Delta/(qv) = 0.3 \).

The corresponding plot is reported in Fig. left panel, where the unit vector \( \mathbf{v} \) has been expressed by the polar angles \( \vartheta \) and \( \varphi \). The plot has been obtained for \( 2\Delta/(qv) = 0.3 \) and considering \( \vartheta \in [0, \pi] \) and \( \varphi \in [0, 2\pi] \). Note that according with Eq. (60), the velocity of the fermionic quasiparticles vanishes along the planes \( v_x = 0, v_y = 0 \) and \( v_z = 0 \).

For more complicated crystalline structures it is not possible to have an analytic expression of the fermionic velocity, one notable example is the FCC structure which is defined by the condensate

\[
\Delta(r) = 2\Delta \sum_{m=1}^{4} \cos(2q \mathbf{n}^m \cdot \mathbf{r}),
\]

(61)

where \( \mathbf{n}^1 = \sqrt{\frac{3}{2}}(1, 1, 1), \mathbf{n}^2 = \sqrt{\frac{3}{2}}(1, 1, -1), \mathbf{n}^3 = \sqrt{\frac{3}{2}}(1, -1, 1) \) and \( \mathbf{n}^4 = \sqrt{\frac{3}{2}}(-1, 1, 1) \). Upon plugging this expression in Eq. (64) we obtain that

\[
A^{(\text{FCC})}_\pm = \left( \frac{q}{\pi} \right)^3 \int_{\text{cell}} dV \exp \left\{ \pm \frac{2\Delta}{qv} B \right\},
\]

(62)

where the integration is over the elementary cell of volume \( (\pi/q)^3 \) and

\[
B = \sqrt{3}v \left( \frac{\sin 2q(x+y+z)}{v_x+v_y+v_z} + \frac{\sin 2q(x+y-z)}{v_x+v_y-v_z} + \frac{\sin 2q(x-y+z)}{v_x-v_y+v_z} + \frac{\sin 2q(-x+y+z)}{-v_x+v_y+v_z} \right),
\]

(63)

and it is then easy to show that \( A^{(\text{FCC})}_+ = A^{(\text{FCC})}_- \equiv A^{(\text{FCC})} \).

Expressing the components of the unit vector \( \mathbf{v} \) in \( B \) as functions of the polar angles \( \vartheta \) and \( \varphi \), and upon substituting in \( A^{(\text{FCC})} \), one has that the quasiparticle velocity

\[
c^{\text{FCC}}(\vartheta, \varphi) = \frac{1}{A^{(\text{FCC})}(\vartheta, \varphi)},
\]

(64)

has the behavior reported in Fig. right panel. The plot of \( c^{\text{FCC}}(\vartheta, \varphi) \) has been obtained for \( 2\Delta/(qv) = 0.3 \) and considering \( \vartheta \in [0, \pi] \) and \( \varphi \in [0, 2\pi] \). The velocity of the fermionic quasiparticles is equal to 1 almost everywhere with the exclusion of a restricted region which corresponds to the zeros of \( B(\vartheta, \varphi) \), given by the solutions of \( \mathbf{v} \cdot \mathbf{n}^m = 0 \).

In conclusion, we have shown that for various crystalline structures the fermionic spectrum is gapless. For \( k/q \ll 1 \) the dispersion law is linear in momentum and the fermionic velocity vanishes along the planes orthogonal to the direction of the vertices of the reciprocal lattice. This result remains valid also for massive quarks, since the effect of the quark mass can be accounted for by reducing \( v \) (Casalbuoni et al., 2002a).
Regarding the FF condensate we take the direction of the Cooper pair total momentum $2q$ along the $z-$axis. In this case, the quasi-particle spectrum has the analytical expression

$$E_{\pm} = qv_{z} \pm \sqrt{\xi^{2} + \Delta^{2}}, \quad (65)$$

where the quasi-particle energies are computed from the corresponding Fermi energies $\mu_{u,d}$. Eq. (65) is the dispersion law of quasi-particle ($E_{\pm} \geq 0$) or hole states ($E_{\pm} < 0$). As for the case of the strip and of the FCC, the dispersion law depends on the direction in coordinate space that one is considering. However, contrary to what happens for real crystalline structures, in the FF phase there are directions along which the dispersion laws are gapped and directions along which the dispersion law is gapless.

The zeros of the dispersion laws are located, for a given value of $\cos \vartheta$, at

$$|\xi_{0}| = \sqrt{(qv \cos \vartheta)^{2} - \Delta^{2}} \quad , \quad (66)$$

and from Eq. (65) one can see that $E_{-}$ is gapped for $\cos \vartheta > \Delta/qv$, while $E_{+}$ is gapped for $\cos \vartheta < -\Delta/qv$. Therefore for $(qv \cos \vartheta)^{2} < \Delta^{2}$, both dispersion laws would be gapped but in this case the zeros of the dispersion law becomes imaginary, meaning that both dispersion law cannot be simultaneously gapless. Note that for $\Delta/qv > 1$ the gapless modes disappear.

2. Specific heat of the Fermi quasi-particles

The contribution of the Fermi quasi-particles to the specific heat per unit volume is given by

$$c_{V} = 2 \sum_{j} \int \frac{d^{3}p}{(2\pi)^{3}} \frac{E_{j}}{\partial n(E_{j},T)/\partial T} , \quad (67)$$

$n(E_{j},T)$ is the Fermi distribution function and the sum is over all the fermionic modes. Considering the low temperature range, $T << \Delta$, which is relevant for astrophysical applications, and using Eq. (53) one obtains for a generic two-flavor crystalline structure

$$c_{V} = \frac{4\mu^{2}T}{3} \int \frac{d\Omega}{4\pi} \frac{1}{c(v)} + \frac{2\mu^{2}}{3} T, \quad (68)$$

where the second term on the r.h.s is the contribution of unpaired quarks. This expression can be evaluated in closed form for the strip \cite{Casalbuoni03}

$$c_{V}^{(\text{strip})} = \frac{4\mu^{2}T}{3} \text{I}_{2}(-1/2; 1/2, 1; (\Delta/qv)^2) + \frac{2\mu^{2}}{3} T , \quad (69)$$

where $\text{I}_{2}$ denotes the generalized hypergeometric function \cite{Gradsteyn80}. Differently from the analysis of \cite{Larkin64}, here $v$ is not small and we can take $\Delta/qv \to 0$ near the second order phase transition. Since for small $\Delta/qv$ one has $\text{I}_{2}(-1/2; 1/2, 1; (\Delta/qv)^2) \simeq 1 - (\Delta/qv)^2$, it is easily seen that the normal Fermi liquid result is obtained for $\Delta = 0$. On the other hand, for non-vanishing $\Delta$, the specific heat turns out to be smaller.

In the case of the FF state, the dispersion law of the quasi-particles is given by (65) and using Eq. (67) one obtains that in the small temperature limit ($T \ll \Delta$) and for $\Delta < q$

$$c_{V}^{(\text{FF})} \simeq \frac{4\mu^{2}T}{3} \sqrt{1 - \frac{\Delta^{2}}{(qv)^{2}}} + \frac{2\mu^{2}}{3} T . \quad (70)$$

In the above equation, the first and the second addenda correspond to the contribution of the paired and the unpaired quarks respectively. The paired quarks contribution depends linearly on temperature because the quasi-particle dispersion law (65) gives rise to gapless modes when $\Delta/qv < 1$. There is also a contribution to the specific heat that comes from gapped modes, but this contribution is exponentially suppressed with the temperature.
3. Effective Lagrangian of phonons and contribution to the specific heat

The low energy spectrum of a periodically modulated condensate, besides Fermi quasi-particles consists also of massless NGBs which originate from the spontaneous breaking of rotational symmetries. Note that for two-flavor QCD in any crystalline phase, these are the only NGBs [Casalbuoni et al., 2002b, c], because no global symmetry of the system is spontaneously broken. Since the modulation of the condensate is associated to a crystalline structure, the NGBs describe the vibrations of the crystal and are for this reason called phonons. These phonons are not associated to pressure oscillations, rather it can be shown that they are related to chemical potential oscillations, see e.g. [Anglani et al., 2011]. For any crystalline structure one can deduce the general expression of the phonon Lagrangian from the symmetries of the system, but the various coefficients have to be evaluated by a microscopic theory. These coefficients are related to the elastic properties of the crystalline structure and we shall present an analysis of the shear modulus of various three-flavor crystalline structures in Sec. III.D.

In the FF phase there is one single phonon field, \( \phi \), and one privileged direction corresponding to \( q \). The leading order (LO) Lagrangian density in the momentum expansion obtained in [Casalbuoni et al., 2002c] is given by

\[
\mathcal{L} = \frac{1}{2} \left( \hat{\phi}^2 - v_\parallel^2 (\nabla_\parallel \phi)^2 - v_\perp^2 |\nabla_\perp \phi|^2 \right),
\]

(71)

where \( \nabla_\parallel = n \cdot \nabla \), \( \nabla_\perp = \nabla - n \nabla_1 \) and \( n \) is the unit vector parallel to \( q \). The breaking of the rotational symmetry in the underlying microscopic theory implies that the velocity of propagation in the direction parallel to \( n \), \( v_\parallel \) is different from the velocity of propagation in the direction orthogonal to \( n \), \( v_\perp \). These velocities have been computed in [Casalbuoni et al., 2002c] employing as microscopic theory a NJL-like model and the result is

\[
v_\parallel = \cos^2 \theta_q, \quad v_\perp = \frac{1}{2} \sin^2 \theta_q,
\]

(72)

with \( \cos \theta_q \approx \delta \mu/q \). The dispersion law, relating the phonon quasi-momentum \( k \) and energy \( \omega \), therefore is

\[
\omega(k) = \sqrt{v_\perp^2 (k_\perp^2 + k_\parallel^2) + v_\parallel^2 k_\parallel^2}.
\]

(73)

The contribution of phonons to the specific heat at small temperatures is given by

\[
c_V^{(FF)} = \frac{8\pi^2}{15v_\parallel^2} T^3 \quad (\text{phonons}).
\]

(74)

Since the strip has the same space symmetries of the FF phase and one phonon field, the Lagrangian density has the same expression reported in Eq. (71), but the values of the longitudinal and transverse velocities are different [Casalbuoni et al., 2002c]. Thus, for the strip the expression of the specific heat is formally the same reported above.

The low energy oscillations the FCC crystal structure are described by three different modes and the LO Lagrangian density for the three phonon fields is given by

\[
\mathcal{L} = \frac{1}{2} \sum_{i=1,2,3} (\hat{\phi}^{(i)})^2 - \frac{a}{2} \sum_{i=1,2,3} |\nabla \phi^{(i)}|^2 - \frac{b}{2} \sum_{i=1,2,3} (\partial_\parallel \phi^{(i)})^2 - c \sum_{i<j=1,2,3} \partial_i \phi^{(i)} \partial_j \phi^{(j)},
\]

(75)

where \( a, b \) and \( c \) are three coefficients to be determined by the microscopic theory. Notice that the Lagrangian is rotationally invariant, therefore it is more symmetric than the underlying theory. The reason is that phonons are long wavelength fluctuations and therefore for the cubic structure they are not sensitive to the modulation of the condensate. This is different from the FF or strip structure, because in that case the microscopic symmetry is the same symmetry of the macroscopic level.

The microscopical calculation of the coefficients in (75) has been done in [Casalbuoni et al., 2002b] employing a NJL model, and the result is

\[
a = 1/12, \quad b = 0, \quad c = (3 \cos^2 \theta_q - 1)/12.
\]

(76)

The corresponding specific heat contribution has been evaluated numerically in [Casalbuoni et al., 2003], and the result is

\[
c_V^{(FCC)} \approx 88 \pi^2 T^3 \quad (\text{phonons}).
\]

(77)
E. Smearing procedure

In Section II.C.2 we have discussed the Ginzburg-Landau expansion of the free-energy for various CCSC phases. Since the GL expansion has several limitations and is reliable in a limited number of cases, it is useful to derive a different approximation scheme. In the previous section we have presented an approximation that allows to deal with the dispersion law of fermionic quasiparticles close to the gapless momentum. Here we shall discuss the smearing procedure, an approximation developed in Casalbuoni et al. (2004) which is valid when \( \Delta/\delta \mu \) is large.

1. Gap equations

In order to define the smearing procedure we first consider the FF phase. Although this case can be solved exactly, it is useful to consider it to fix the notation and introduce some definitions to be used later on. We shall consider Cooper pairing of the massless up and down quarks, chemical potential \( \mu_u, \mu_d \), and we define \( \mu = (\mu_u + \mu_d)/2 \) and \( \delta \mu = |\mu_u - \mu_d|/2 \ll \mu \).

As detailed in Casalbuoni and Nardulli (2004), from the mean field Lagrangian of the FF state

\[
L_{\text{cond}} = -\frac{1}{2} \varepsilon^{\alpha \beta \gamma} \epsilon_{ij} \psi^\alpha_i \psi^\beta_j \Delta(r) + \text{c.c.} + (L \rightarrow R) - \frac{1}{g} \Delta(r) \Delta^*(r),\]

one can deduce the zero temperature gap equation, which in the HDET approximation is given by

\[
1 = \frac{g \rho}{2} \int \frac{d\Omega}{4\pi} \int_0^\delta \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} (1 - \theta(-E_u) - \theta(-E_d)),
\]

where \( \rho = \frac{4\pi^2}{2} \) is the density of states in two-flavor QCD and \( \delta \) is the ultraviolet cutoff taken equal to \( \mu/2 \); moreover it is assumed that \( \delta \gg q \sim \delta \mu \). The quasi-particle dispersion laws have been obtained in Eq. (65), but energies are now measured from the common energy level \( \mu \), and therefore

\[
E_{u,d} = \pm \delta \mu + q \cdot v \pm \sqrt{\xi^2 + \Delta^2}.
\]

In Eq. the contribution of hole excitations is taken into account simply multiplying the contribution of quasi-particles times two. The reason is that the difference between the contribution of quasi-particles and quasi-holes is of order \( \delta \mu/\mu \).

We observe that in general

\[
1 - \theta(-x) - \theta(-y) = \theta(x)\theta(y) - \theta(-x)\theta(-y),
\]

and since \( E_u \) and \( E_d \) cannot be simultaneously negative, then

\[
1 - \theta(-E_u) - \theta(-E_d) = \theta(E_u)\theta(E_d),
\]

so the integration in Eq. is over a restricted region named the pairing region (PR), defined by

\[
\text{PR} = \{ (\xi, v \cdot n) | E_u > 0 \text{ and } E_d > 0; \xi \leq \delta \}.
\]

More explicitly, the pairing region is defined by the condition

\[
\text{Max} \left\{ -1, z_q - \frac{\sqrt{\xi^2 + \Delta^2}}{q} \right\} < v \cdot n < \text{Min} \left\{ 1, z_q + \frac{\sqrt{\xi^2 + \Delta^2}}{q} \right\},
\]

with

\[
z_q = \frac{\delta \mu}{q}.
\]

From the above definition it follows that for small values of \( \Delta \) the pairing region is centered at \( \vartheta = \arccos z_q \), has an angular width of order \( \Delta/q \) and a thickness of order \( \Delta \), corresponding to the upper red ribbon in the left panel of Fig. 4. The lower red ribbon can instead be obtained considering hole excitations and the same reasoning used above. In the limit \( \delta \mu/\mu \rightarrow 0 \) the two ribbons have exactly the same dimensions.
Thus, Eq. (79) can be written in a different way:

\[ \Delta = \frac{g \rho}{2} \int \int_{\text{PR}} \frac{d\Omega}{4\pi} d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} = \frac{g \rho}{2} \int \frac{d\Omega}{4\pi} \int_0^\delta d\xi \frac{\Delta_{\text{eff}}}{\sqrt{\xi^2 + \Delta_{\text{eff}}^2}}, \]

where \( \Delta_{\text{eff}} \equiv \Delta_{\text{eff}}(v \cdot n, \xi) \) is defined as

\[ \Delta_{\text{eff}} = \Delta E(v) \theta(E) = \begin{cases} \Delta & \text{for } (\xi, v \cdot n) \in \text{PR} \\ 0 & \text{elsewhere} \end{cases}, \]

Eq. (86)

The above procedure defines the smearing procedure for the FF phase; it can be extended to the case of \( P \) plane waves, Eq. (43), generalizing the results of the previous equations, assuming that in the mean field Lagrangian one can substitute \( \Delta \) with \( \Delta_{\text{E}}(v, \ell_0) \), where

\[ \Delta_{\text{E}} = \sum_{n=1}^{P} \Delta_{\text{eff}}(v \cdot n^m, \ell_0), \]

Eq. (88)

meaning that \( \Delta_{\text{E}} = n\Delta \), where \( n = (1, \ldots, P) \). We can thus generalize the pairing region to

\[ \mathcal{P}_n = \{(v, \xi) | \Delta_{\text{E}}(v, \epsilon) = n\Delta\}. \]

Eq. (89)

Note that in this equation we have made explicit the dependence on the energy \( \ell_0 \) instead of that on the residual momentum, because the pole position is in general in

\[ \epsilon_n = \sqrt{\xi^2 + n^2 \Delta^2}. \]

Eq. (90)

Correspondingly, the gap equation is

\[ P \frac{\Delta}{\Delta_0} \ln \frac{2\delta}{\Delta_0} = \sum_{n=1}^{P} \int_{\mathcal{P}_n} \frac{d\Omega}{4\pi} d\xi \frac{n\Delta}{\sqrt{\xi^2 + n^2 \Delta^2}}, \]

Eq. (93)

where we have made use of the equation

\[ \frac{2}{g \rho} = \ln \frac{2\delta}{\Delta_0}. \]

Eq. (94)

relating the BCS gap \( \Delta_0 \) to the four fermion coupling \( g \) and the density of states. The first term in the sum, corresponding to the region \( \mathcal{P}_1 \), has \( P \) equal contributions with a dispersion rule equal to the Fulde and Ferrel case. This can be interpreted as a contribution from \( P \) non interacting plane waves. In the other regions the different plane waves have an overlap. Since the definition of the regions \( \mathcal{P}_n \) depends on the value of \( \Delta \), their determination is part of the problem of solving the gap equation.

Said in a different way, in the smearing procedure the dispersion relation of the quasi-particles has several branches corresponding to the values \( n\Delta, \ n = 1, \ldots, P \). Therefore, the following interpretation of the gap equation (93) can be given. Each term in the sum corresponds to one branch of the dispersion law, i.e. to the propagation of a gapped quasi-particle with gap \( n\Delta \), which is defined in the region \( \mathcal{P}_n \). However, the regions \( \mathcal{P}_n \) do not represent a partition of the phase space since it is possible to have at the same point quasi-particles with different gaps.
2. Numerical results: free-energy computation

In Table I we report the results obtained at the Chandrasekhar-Clogston point ($\delta \mu = \delta \mu_1$) for four crystalline structures, respectively the FF ($P = 1$), the strip ($P = 2$), the octahedron (BCC) ($P = 6$), and the FCC ($P = 8$). The table shows that, among the four considered structures, the favored one at $\delta \mu = \delta \mu_1$ is the octahedron, which however does not have the largest gap $\Delta$. The gap parameter determines the extension of the pairing regions, see e.g. Eq. (84), but a free-energy gain may result from having many small non-overlapping pairing regions, as well. Indeed, the strip has the largest gap parameter, but not the largest free-energy, presumably because the pairing occurs only in four large ribbons (two for each vector $q^m$, see the right panel of Fig. 4). Indeed, as we have shown in the previous section, the width and the thickness of the ribbons are proportional to $\Delta$. On the other hand, for the octahedron the pairing occurs in smaller ribbons, but there are twelve pairing regions, which give a large contribution to the free-energy.

For the case $\delta \mu \neq \delta \mu_1$, we report in Fig. 7 the plot of the free-energy of the octahedron (dashed line) and of the FCC (full line) structures as a function of $\delta \mu / \Delta_0$. The octahedron is the favored structure up to $\delta \mu \approx 0.95 \Delta_0$; for $0.95 \Delta_0 < \delta \mu < 1.32 \Delta_0$ the FCC is favored. Here, for each value of $\delta \mu$, the values of $z_q$ and $\Delta$ are those that minimize the free-energy. Adapted from Casalbuoni et al. (2004).

| $P$ | $z_q$ | $\Delta / \Delta_0$ | $\frac{2 \mu}{\Delta_0 \Delta}$ |
|-----|-------|----------------------|-------------------------------|
| 1   | 0.78  | 0.24                 | $-1.8 \times 10^{-1}$          |
| 2   | 1.0   | 0.75                 | -0.08                         |
| 6   | 0.9   | 0.28                 | -0.11                         |
| 8   | 0.9   | 0.21                 | -0.09                         |

TABLE I The gap, $z_q = \delta \mu / \Delta_0$, and the free-energy at $\delta \mu = \delta \mu_1 = \Delta_0 / \sqrt{2}$ for different crystalline structures. From Casalbuoni et al. (2004).
the GL results, the structure with 6 plane waves is more favorable than the structure with 2 plane waves or with 1 plane wave. Therefore, increasing the pairing region leads to an increase of the free-energy (in absolute value) making the configuration more stable. An interesting result of the smearing procedure is that with increasing $\delta \mu$ the configuration with 6 plane waves is superseded by the structure with 8 plane waves, which has a lower $\Delta$ but a larger number of pairing regions. This has to be contrasted with the result of the GL approximation, see Sec.II.C.2, where the octahedron is always disfavored with respect to the FCC (but one should consider that the GL free-energy of the FCC is not bounded from below).

F. Chromomagnetic stability of the two-flavor crystalline phase

As discussed in Section II.B.1, the 2SC phase is chromo-magnetically unstable for $\delta \mu > \delta \mu_1 = \Delta_0 / \sqrt{2}$. Further increasing $\delta \mu$ one eventually enters into the g2SC phase which is chromo-magnetically unstable as well. This could be interpreted as the tendency of the system to generate a net momentum of the quark pair, as shown in Giannakis and Ren (2005a). Thus the chromo-magnetic instability can be interpreted as a tendency to develop quark current, which in turn is equivalent to the FF phase, where diquark carry momentum $2q$.

In this section, we review the results of Giannakis and Ren (2005a), which relate the Meissner mass of the $\bar{8}$ mode of the 2SC to the momentum susceptibility and then the computation of the Meissner tensor in the crystalline phases Giannakis et al. (2005); Giannakis and Ren, 2005b; Reddy and Rupak, 2005).

1. Momentum susceptibility

The response of the thermodynamic potential of the 2SC phase to a small momentum, $2q$, of the quark pair can be computed absorbing the phase of the condensate into the phase of the quark fields; the net effect is a shift of each quark momentum by $q$, see Sec. III.B for more details on this procedure. Therefore, the one loop effective action (in presence of background gauge fields) can be computed using the same steps that lead to Eq. (21). Expanding the thermodynamic potential around $q = 0$ one has, at the lowest order,

$$\Omega = \Omega_{2SC} + \frac{1}{2} \mathcal{K} q^2,$$

(95)

where the momentum susceptibility is given by

$$\mathcal{K} = \frac{i}{6} \sum_{i=1}^{3} \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[ \Gamma_i S(p) \Gamma_i S(p) \right],$$

(96)

where $S(p)$ is the quark propagator in momentum space and $\Gamma = \text{diag}(\gamma, -\gamma)$ is the appropriate vertex factor. Notice that we are considering an expansion of the thermodynamic potential for small $q$, therefore the momentum susceptibility does not depend on $q$, meaning that $S(p)$ is the 2SC quark propagator which can be obtained inverting the expression in Eq.(20). The momentum susceptibility has an expression similar to the space component of the 2SC polarization tensor, see Eq (33), at vanishing momentum. In particular it is possible to show that it is proportional to the squared Meissner mass of the $\bar{8}$ mode. The reason is that the only non-vanishing contribution to both the momentum susceptibility and to the Meissner mass of the $\bar{8}$ mode are determined from the red-green color sector (blue quarks do not carry a condensate and do not mix with red and green quarks). But in this color sector $T_8$ is

| $P$ | $\delta \mu_2 / \Delta_0$ | Order | $z_q$ | $\Delta / \Delta_0$ |
|-----|-----------------|-------|-------|----------------|
| 1   | 0.754           | II    | 0.83  | 0              |
| 2   | 0.83            | I     | 1.0   | 0.81           |
| 6   | 1.22            | I     | 0.95  | 0.43           |
| 8   | 1.32            | I     | 0.90  | 0.35           |

TABLE II The values of $\delta \mu_2$, $z_q = \delta \mu / q$, the discontinuity of $\Delta / \Delta_0$ and the order of the phase transition between the CCSC phase and the normal phases for different crystalline structures. From Casalbuoni et al. (2004).
proportional to the identity and thus the corresponding vertex factor is proportional to $\Gamma$. A more detailed discussion can be found in Giannakis and Ren (2005a), where it is shown that

$$M^2_{M,8} = \frac{1}{12} \left( g^2 + \frac{e^2}{3} \right) K,$$

(97)

where the mass of the $\bar{8}$ mode in the 2SC phase is given in Eq. (39).

At the transition point between 2SC phase and the $g^2$SC phase, $\delta \mu = \Delta$, one finds $M^2_{M,\bar{8}} < 0$ and thus, Eq. (97) implies that the system is unstable towards the formation of pairs with nonvanishing net momentum; indeed a negative $K$ in Eq. (97) implies that there is a gain in free-energy if $q \neq 0$. Therefore, the chromomagnetic instability of the $\bar{8}$ mode leads naturally to the FF state.

Before turning to the computation of the Meissner tensor in the crystalline phase, we comment briefly on the absence of total currents induced by the net momentum of the quark pair. The value of the total momentum is determined minimizing the free-energy, thus $\partial \Omega / \partial q = 0$, and the stationarity condition is equivalent to the vanishing of the tadpole diagram,

$$\langle \bar{\psi} \gamma \psi \rangle = 0,$$

(98)

which implies that no baryon matter current is generated in the ground state. Analogously, one can show that electric and color currents vanish as well. In particular, the residual $SU(2)_c \otimes U(1)_{\tilde{Q}}$ symmetry implies that $\langle \bar{\psi} \Gamma_A \psi \rangle = 0$ for $A \neq 8$, with $\Gamma_A$ defined in Eqs. (34) and (35); moreover, for $A = 8$ one finds

$$J_8 = \frac{\sqrt{3} g^2 + e^2}{6} \langle \bar{\psi} \gamma \psi \rangle,$$

(99)

which vanishes because of the minimum condition, Eq. (98). Therefore, no total current is generated in the FF state.

2. Meissner masses in the FF phase

Since the FF phase has the same gauge symmetry breaking pattern of the 2SC phase, it has five massive gluons. The computation of the Meissner tensor of gluons in the two-flavor FF phase has been done in Giannakis et al. (2005) and Giannakis and Ren (2005b), neglecting neutrality conditions and considering the isospin chemical potential, $\delta \mu = \mu_e / 2$, as a free parameter. The Meissner masses of the gluon fields with adjoint color $a = 4, \ldots, 7$ are all equal and can be written as

$$(M^2_{M,A})_{ij} = A \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) + B \frac{q_i q_j}{q^2},$$

(100)

where we have decomposed the Meissner tensor into longitudinal and transverse components with respect to $q$. The Meissner tensor of the $\bar{8}$ mode can be decomposed in a similar way

$$(M^2_{M,\bar{8}})_{ij} = C \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) + D \frac{q_i q_j}{q^2}.$$

(101)

The coefficients $A$ and $C$ are called the transverse Meissner masses; similarly, $B$ and $D$ are the longitudinal Meissner masses. As discussed in the previous section, the mass of the rotated eighth gluon is related to the variation of the free-energy with respect to $q$. In the FF phase it has been shown by Giannakis et al. (2005) that the precise relation is the following

$$C = \frac{1}{12} \left( g^2 + \frac{e^2}{3} \right) \frac{1}{q} \frac{\partial \Omega}{\partial |q|},$$

(102)

$$D = \frac{1}{12} \left( g^2 + \frac{e^2}{3} \right) \frac{\partial^2 \Omega}{\partial q^2}.$$  

(103)

If the phase with $q \neq 0$ is a minimum of the free-energy, then both the conditions $\partial \Omega / \partial |q| = 0$ and $\partial^2 \Omega / \partial |q|^2 > 0$ must be satisfied. Thus, $C = 0$ and $D > 0$ at the minimum. As a consequence, the Meissner tensor of the $\bar{8}$ mode is purely longitudinal and positively defined.
The coefficients $A$, $B$ and $D$ can be computed analytically. We refer the interested reader to the original article [Giannakis and Ren, 2005b]. Here, it is enough to consider the small gap expansion (Ciminale et al., 2006; Giannakis and Ren, 2005b); in this approximation scheme one has

\[
A = \frac{g^2 \mu^2}{96 \pi^2} \frac{\Delta^4}{\delta \mu^2 (z_q^2 - 1)^2},
\]

(104)

\[
B = \frac{g^2 \mu^2}{8 \pi^2} \frac{\Delta^2}{\delta \mu^2 (z_q^2 - 1)},
\]

(105)

\[
D = \frac{g^2 \mu^2}{6 \pi^2} \left(1 + \frac{\epsilon^2}{3g^2}\right) \frac{\Delta^2}{\delta \mu^2 (z_q^2 - 1)}.
\]

(106)

The message of the above equations is that the Meissner tensor is positively defined for the one plane wave ansatz, within the small gap parameter approximation. The authors Giannakis and Ren (2005b) argue that for a multiple plane wave structure, the situation will be better (within the small gap expansion). As a matter of fact, at the $\Delta^2/\delta \mu^2$ order, the Meissner tensor will be positive definite, being it additive with respect to different terms of the plane wave expansion to order $\Delta^2/\delta \mu^2$. This has been explicitly checked in Ciminale et al. (2006).

Besides, in Giannakis and Ren (2005b) the numerical computation of the Meissner masses, beyond the small gap expansion, has been performed. The results can be summarized as follows. First of all, the FF phase is found to be more stable than the 2SC phase in the range

\[
0.706 \Delta_0 \lesssim \delta \mu \lesssim 0.754 \Delta_0.
\]

(107)

Within this range, the LOFF gap is within the range

\[
0 \leq \Delta \lesssim 0.242 \Delta_0,
\]

(108)

and the longitudinal Meissner masses, $B$ and $D$, turn out to be positive within the range

\[
0 \leq \Delta \lesssim 0.84 \Delta_0;
\]

(109)

moreover, the transverse mass $A$ of the gluons with $a = 4, \ldots, 7$ turns out to be positive within the range

\[
0 \leq \Delta \lesssim 0.38 \Delta_0.
\]

(110)

Since the LOFF window for the FF state (108) is contained in both the intervals (109) and (110), the FF state is free from the chromomagnetic instability, as long as it is energetically favored with respect to the homogeneous phase.

G. Solitonic ground state

The analysis of various crystalline phases has shown that a periodic structure is energetically favored for mismatched Fermi spheres. In Nickel and Buballa (2009) a generalization of the crystalline structure has been proposed in order to explore whether the more complicated periodic condensate (we do not report the color-flavor structure because it is the same of Eq. (41)):

\[
\Delta(z) = \sum_{k \in \mathbb{Z}} \Delta_{q,k} \cos(2kqz),
\]

(111)

may be energetically favored with respect to standard crystalline structures. The wave vector $q$ is taken along the $z$-axis, thus the condensate corresponds to a band structure along the $z$-direction. Assuming that the condensate is real (which is the assumption of Nickel and Buballa (2009)), one has $\Delta_{q,k} = \Delta_{q,-k}^*$ and Eq. (111) can be rewritten as

\[
\Delta(z) = 2 \sum_{n=1}^{\infty} \Delta_n \cos(2nqz).
\]

(112)

Written in this form, it is clear that the ansatz is a generalization of the strip structure and amounts to consider higher harmonics contributions to the gap function, which are not included in the strip.
FIG. 8 (Color online). Left panel: profile for the gap function, at several values of $q$, obtained from the self-consistent solution of the gap equation, at $\delta \mu = 0.7 \Delta_{BCS}$. Right panel: physical wave vector magnitude as a function of $\delta \mu$. From [Nickel and Buballa (2009)].

For any value of $\delta \mu$, the ground state in [Nickel and Buballa (2009)] is determined using a two-steps procedure. Firstly, the magnitude of $q$ in Eq. (112) is fixed, and the profile $\Delta(z)$ is determined by solving the gap equation. Then, among several choices for $q$, the physical value corresponds to the one which minimizes the free-energy. In the left panel of Fig. 8, which is taken from [Nickel and Buballa (2009)], we plot the profile $\Delta(z)$ obtained by the numerical solution of the gap equation at fixed value of $q$. In the plot, $\delta \mu$ is fixed to the numerical value $0.7 \Delta_0$ with $\Delta_0 = 80$ MeV; but changing $\delta \mu$ does not change the picture qualitatively. For $q$ of the same order of $\Delta_0$, the shape of the gap function is very close to that of the strip. In this case, the largest contribution to the gap comes from the lowest order harmonic $n = 1$ in Eq. (112). However, for small values of $q/\Delta_0$, the solution of the gap equation has a solitonic shape. This is evident in the case $q = 0.1 \Delta_0$, in which $\Delta(z) \approx \pm \Delta_0$ for one half-period, then suddenly changes its sign in a narrow interval. In this case, the higher order harmonics play a relevant role in the gap function profile.

In the right panel of Fig. 8 we report the plot of the physical value of $q$ as a function of $\delta \mu$. In the window in which $q = 0$, the ground state is the homogeneous BCS state. At the critical value $\delta \mu \equiv \delta \mu_c \approx 0.695 \Delta_0$, the ground state has $q \neq 0$. This signals the transition to the inhomogeneous phase. Since $q$ can be arbitrary small in proximity of the transition point, from the left panel of Fig. 8 we read that the ground state will consist of a solitonic structure for $\delta \mu \approx \delta \mu_c$. As $\delta \mu$ is increased, the physical value of $q$ increases as well. Therefore, again from the left panel of the figure, we read that the solitonic structure will continuously evolve into the strip structure.

In Fig. 9, taken from [Nickel and Buballa (2009)], we report the plot of the BCS free-energy (solid line) and of the inhomogeneous phase (dashed line) as a function of $\delta \mu$. The dotted line corresponds to the free energy of the FF phase. The baseline corresponds to the free-energy of the normal phase. With increasing mismatch a second order...
phase transition takes place from the BCS to the solitonic structure at a value $\delta \mu / \Delta_{\text{BCS}} \approx 0.7$ which is smaller than the value at which the transition from the BCS phase to the FF phase takes place. On the other hand, the transition from the inhomogeneous phase to the normal one takes place almost simultaneously to the transition from the FF phase to the normal one. In the case of the FF phase, the transition to the BCS state is first order and to the normal phase is second order. On the other hand, the transition from the soliton lattice to the BCS phase is of the second order. This is possible because the gap function $\Delta(z)$ naturally interpolates between the homogeneous state, corresponding to a single soliton with infinite period, and the cosine-shaped solution.

Moreover, the transition to the inhomogeneous state $\Delta(z)$ to the normal phase is found to be of the first order, in agreement with the results of the smearing procedure (Casalbioni et al. 2004). At this transition point, the profile of the gap function is of the cosine type, i.e. the strip profile discussed above.

The results of this analysis are in disagreement with those of the GL expansion (Bowers and Rajagopal, 2002; Larkin and Ovchinnikov, 1964), which predicts that the transition from the strip to the normal phase is of the second order. Since the GL expansion is expected to be exact in proximity of a second order phase transition, it is important to understand the origin of the discrepancy among the GL result and that of Nickel and Buballa (2009). To this end, the authors of Nickel and Buballa (2009) compute the thermodynamic potential, $\Omega$, for a gap function $\Delta(z) = \Delta \cos(2qz)$ with a fixed value of $q$, as it is customary in the GL studies. Their analysis reveals that in this case, beside the local minimum of $\Omega$ located at small values of $\Delta$, and which is captured by the GL expansion, a global minimum appears for larger values of $\Delta$. Then the authors argue that in order to capture this true minimum, which is responsible for the first-order transition to the normal phase, terms of at least eighth order in the GL expansion, which are usually neglected, should be included. In a subsequent paper (Nickel, 2009) Nickel discusses the inhomogeneous phases for lower dimensional modulations in the NJL model and in the quark-meson model. His results confirm the replacement of the first order transition of the phase diagram of the homogeneous NJL phase by two transition lines of second order. These lines are the borders of an inhomogeneous phase and they intersect at the critical point. An interesting point is also the relation to the chiral Gross-Neveu model (Gross and Neveu, 1974). A more complete program would require inclusion in the study of higher dimensional modulations of the inhomogeneity (as done in Abuki et al. (2012) for the chiral condensates), and, as pointed out by Buballa and Nickel (2010), simultaneous study of both superconducting and chiral condensates.

H. An aside: Condensed matter systems

Fermionic systems consisting of two different “flavors” with mismatched Fermi surfaces and non-homogeneous condensates are quite generic and appear in various contexts. Particularly interesting are the population imbalanced superfluid (or superconducting) systems that can be realized and studied in laboratory. Examples of these type are gases of cold atoms, type-II cuprates and organic superconductors. The study of these systems allows to shed light on various aspects of superfluidity of asymmetric systems in a framework under experimental control.

Of considerable importance for their similarity to quark matter are ultracold systems consisting of fermions of two different species, $\psi^1$ and $\psi^2$, corresponding to two hyperfine states of a fermionic atom, see Giorgini et al. (2008) and Ketterle and Zwierlein (2008) for reviews. These fermions have opposite spin and one can change the number of up and down fermions at will. One of the most intriguing aspects of these systems is that the interaction between fermions can be tuned by employing a Feshbach resonance (Chin et al., 2010), and therefore the crossover between the BCS and the BEC superfluid phases can be studied. This research field has grown in an impressive way in the last two decades and continuous progress in understanding and characterizing the properties of these systems is under way, see e.g. Bedaque et al. (2003); Bulgac et al. (2006); Carlson and Reddy (2005); Castorina et al. (2005); Forbes McNeil et al. (2005); Gubankova et al. (2006, 2003); Lin and Wilczek (2003); Mannarelli et al. (2006a); Muther and Sedrakian (2002); Pao et al. (2000); Partridge et al. (2000); Rizzi et al. (2008); Sheehy and Radzihovsky (2006); Shin et al. (2008); Son and Stephanov (2006); Yang (2005, 2006); Zwierlein and Ketterle (2006); and Zwierlein et al. (2006).

A mismatch between the populations of electrons in a superconductor can also be produced by Zeeman splitting. The magnetic field that couples with the spins of the electrons can be an external one or an exchange field. However, an external magnetic field couples with the orbital motion of the electrons as well, destroying superconductivity or leading to the creation of a vortex lattice structure.

In order to reduce the orbital effect one employs 2D superconductors, i.e. films of superconducting material or systems with a layered structure, and an in-plane magnetic field (Bulaevskii, 1973). Good candidates for LOFF superconductors of this type are heavy-fermion compounds like CeRu$_2$ (Huxley et al., 1993); recently interesting results have been obtained with CeCOIN$_5$ (Matsuda and Shimahara, 2007). Quasi-two-dimensional organic superconductors...
like $\kappa$-(ET) or $\lambda$-(ET) salts, in particular $\lambda$-(BETS)$_2$F$_4$Cl$_4$, are promising candidates for realizing the LOFF phase. High $T_c$ superconductors are good candidates as well. In case one uses 3D materials one has to understand how the vortex lattice structure changes in presence of a LOFF periodic order parameter.

As a final remark, we recall that the gapless CSC phases are the QCD analogue of a condensed matter phase, known as Sarma phase (Forbes McNeil et al., 2003; Gubankova et al., 2003; Liu and Wilets, 2003; Sarma, 1963), found to be unstable (Gubankova et al., 2010; Mannarelli et al., 2006; Pao et al., 2006; Sheehy and Radzihovsky, 2006; Wu and Yip, 2003) in the weak coupling limit.

III. THE THREE-FLAVOR NON-HOMOGENOUS PHASES

In the previous chapter we have discussed the crystalline non-homogenous phases that can be realized in two-flavor quark matter. A plethora of possible structures have been analyzed by a Ginzburg-Landau expansion and by the so-called smearing procedure in order to determine the favored thermodynamic state.

Here we extend the analysis to the three-flavor case which is relevant if the effective strange quark mass, $M_s$, is not too heavy. In compact stars $M_s$ will lie somewhere between its current mass of order 100 MeV and its vacuum constituent mass of order 500 MeV, and therefore is of the order of the quark number chemical potential $\mu$, which is expected to be in the range (400 − 500) MeV. Furthermore, deconfined quark matter, if present in compact stars, must be in weak equilibrium and must be electrically and color neutral. All these factors work to separate the Fermi momenta of the quarks and thus disfavor the cross-species BCS pairing.

In order to understand how the mismatch among Fermi momenta is linked to $\beta$-equilibrium and neutrality, let us consider quark matter composed by $u$, $d$ and $s$ quarks with no strong interactions. We treat the strange quark mass as a parameter and we consider sufficiently long time-scales for which weak equilibrium and electrical neutrality are relevant.

Since we are assuming that color interactions are absent, the chemical potential of quarks is diagonal in the color indices and the free-energy of the system is given by

$$\Omega_{\text{unpaired}}(\mu, \mu_e, M_s) = \frac{3}{\pi^2} \int_0^{P_{u}^F} (p - \mu_u) p^2 dp + \frac{3}{\pi^2} \int_0^{P_{d}^F} (p - \mu_d) p^2 dp + \frac{3}{\pi^2} \int_0^{P_{s}^F} (\sqrt{p^2 + M_s^2} - \mu_s) p^2 dp + \frac{1}{\pi^2} \int_0^{\mu_e} (p - \mu_e) p^2 dp,$$

where the Fermi momenta are given by

$$P_{u}^F = \mu_u, \quad P_{d}^F = \mu_d, \quad P_{s}^F = \sqrt{\mu_s^2 - M_s^2}.$$  \hspace{1cm} (113)

Weak equilibrium relates the chemical potentials of quarks with different flavors as follows

$$\mu_u = \mu - \frac{2}{3} \mu_e, \quad \mu_d = \mu + \frac{1}{3} \mu_e, \quad \mu_s = \mu + \frac{1}{3} \mu_e,$$  \hspace{1cm} (114)

and the electrical chemical potential is obtained from the neutrality constraint

$$\frac{\partial \Omega}{\partial \mu_e} = 0.$$  \hspace{1cm} (115)

Solving Eq. (114) one determines $\mu_e$ and by Eqs. (113) one obtains the values of $P_u^F$, $P_d^F$ and $P_s^F$ for the electrically neutral unpaired phase. The effect of $M_s$ is to reduce the number of strange quarks, which must be compensated by a larger number of down quarks to ensure electrical neutrality. It follows that there is a hierarchy of Fermi momenta

$$P_s^F \leq P_u^F \leq P_d^F,$$  \hspace{1cm} (116)

and in Fig. 10 we report the difference of the Fermi momenta $P_d^F - P_u^F$ and $P_u^F - P_s^F$ as a function of $M_s$, for $\mu = 500$ MeV.

For $M_s \ll \mu$, the effect of a nonzero strange quark mass can be taken into account by treating the strange quark as massless, but with a chemical potential that is lowered by $M_s^2/(2\mu)$ from $\mu + \mu_e/3$. To the order $M_s^2/\mu$ electric

\[2\] Actually, as already discussed in Sec. I, quark matter must be in a color singlet, however it has been shown in [Amore et al., 2002] that projecting out color singlet states into color neutral states does not lead to a large change of the free energy.
neutrality, Eq. (113), requires that $\mu_e \simeq M_s^2/4\mu$. In this case $P_{dF} - P_{uF} \simeq P_{uF} - P_{sF}$, and we need no longer to be careful about the distinction between $P_F$ and $\mu$, as we can simply think of the three flavors of quarks as if they have chemical potentials

$$\mu_d = \mu_u + 2\delta\mu \quad \mu_u = p_F^u \quad \mu_s = \mu_u - 2\delta\mu,$$

(118)

with

$$\delta\mu \equiv \frac{M_s^2}{8\mu},$$

(119)

and we can write the chemical potential matrix as

$$\mu_{ij,\alpha\beta} = \delta_{\alpha\beta} \otimes \text{diag}(\mu_u, \mu_d, \mu_s)_{ij}.$$

(120)

The above results are strictly valid for unpaired matter, but one expects that, if the pairing interaction channel is not exceedingly strong, then it should still be qualitatively correct. In particular, it should happen that the hierarchy of Fermi momenta is the one reported in Eq. (117) and the splittings those reported in Fig. 10.

As discussed in the two-flavor case, a small values of the chemical potential difference $\delta\mu$, i.e., of $M_s$, cannot disrupt the BCS pairing and the CFL phase will be energetically favored. However, when $M_s$ is sufficiently large, the mismatch between the Fermi momenta becomes large disfavoring CFL pairing, that must be superseded by a less symmetric phase.

As shown in Fig. 10 for values of $M_s$ comparable with $\mu$ the strange quarks decouple and only the 2SC phase with the pairing between up and down quarks seems realizable. However, as pointed out in [Alford and Rajagopal (2002)] once the constraints of electrical and color neutrality and $\beta$-equilibrium are imposed, the 2SC phase turns out to be strongly disfavored or even excluded at least at zero-temperature (for finite temperature evaluations see for example [Abuki (2003)]). The authors of [Alford and Rajagopal (2002)], performing an expansion in terms of the strange quark mass (at the leading non-trivial order), found that whenever the 2SC phase is more favored than unpaired matter, then the CFL is even more favored. The point is that the CFL phase is extremely robust, because it allows pairing between all the three-flavor species. This analysis has been redone in [Steiner et al. (2002)] evaluating the density-dependent strange quark mass self-consistently. The results of [Steiner et al. (2002)] almost confirm previous conclusions, finding the 2SC favored in a very narrow region of density, that is likely to disappear once the hadronic phase boundary is properly taken into account. Notice that these results are valid in the weak-coupling approximation. Computation using NJL-like models with a stronger coupling [Abuki and Kumihira (2006)] [Ruester et al. (2006a)], see also Sec. 11.3 find that a large portion of the phase diagram is occupied by the 2SC phase.
A. The gapless CFL phase

Given that the effect of non-zero strange quark mass and electrical neutrality constraints is to pull apart the Fermi spheres of different quark species, there is little motivation to assume the symmetric CFL pairing reported in Eq. (2). Since the gap parameter gives the strength of the interaction channel, one might expect that a mismatch between the Fermi momenta reflects in a reduction of the interaction channel and thus in a reduction of the corresponding gap parameter. Therefore, the gap parameters should be now flavor dependent and one can consider the generalized pairing ansatz

$$
\langle 0 | \psi^\alpha_i \psi^\beta_j | 0 \rangle \propto \sum_{I=1}^{3} \Delta_I \epsilon^{\alpha \beta I} \epsilon_{ijI},
$$

(121)

where $\Delta_1$, $\Delta_2$, and $\Delta_3$, describe down-strange, up-strange and up-down Cooper pairing, respectively. In [Alford et al. 2004, 2005c] the superconducting phase characterized by the ansatz (121) has been studied by a NJL-like model at zero temperature (see Fukushima et al. 2005 for a study at nonvanishing temperature), considering the in-medium strange quark mass as a free parameter, while the light quarks are taken massless. The gap equations, coupled to the neutrality conditions, have been solved and the corresponding free-energy has been determined. Besides the strange quark mass, the final results depend on the quark chemical potential, which is fixed to the numerical value $\mu = 500$ MeV; moreover, the strength of the NJL-like interaction is fixed by the value of the homogenous CFL gap, $\Delta_0$. The numerical value $\Delta_0 = 25$ MeV, corresponding to the weak coupling regime, has been chosen.

Some of the results of [Alford et al. 2004, 2005c] are summarized in Fig. 11. On the left panel of the figure, the solutions of the gap equations as a function of $M_s^2/\mu$ are plotted. As it can be inferred from the figure, at $M_s^2/\mu \approx 47.1$ MeV, there is a continuous phase transition between the CFL phase and a phase characterized by $\Delta_3 > \Delta_2 > \Delta_1 > 0$. Note that the hierarchy of the gap parameters is in agreement with the hierarchy of the Fermi momenta splitting reported in Fig. 10, which substantiates the reasoning presented above about the relation between mismatched Fermi momenta and gap parameters. Actually, the mismatches of chemical potential are not only due to the finite value of the strange quark mass, but as well as to the nonzero values of the color chemical potentials $\mu_3$ and $\mu_8$, which however do not overturn the hierarchy in Fig. 10.

The phase appearing at $M_s^2/\mu \approx 47.1$ MeV is dubbed gapless CFL, or gCFL, phase, and is the three-flavor analogous of the g2SC phase discussed in the previous chapter. The symmetry breaking pattern is the same of the CFL phase, see Eq. (3), but the spectrum of some fermionic excitations is gapless. The possibility of a gapless CFL superconductor was first argued in [Alford et al. 2000a], where a toy-model was used to infer the effect of a heavy $M_s$ on the quasiparticle spectrum and it was found that if the condensates of light and strange quarks are below a certain critical value there is not a minimum excitation energy in the quasiparticle spectrum. As already discussed in the previous...
chapter, the mechanism at the origin of gapless modes in a superconductor is quite general, and is due to a mismatch \( \delta \mu \) between condensing fermions of the order of \( \Delta \).

In case of three-flavor quark matter, the situation is complicated by the presence of several interaction channels. In the gCFL phase the dispersion laws are more complicated than those reported in Eq. (11), because they depend not only on the strange quark mass, but on the nonzero values of the color chemical potentials \( \mu_3 \) and \( \mu_8 \), as well. However, the result is rather similar, as gapless excitations appear in the spectrum.

Comparing the free-energy of the gCFL with other candidate phases, e.g. the 2SC phase, or the 2SC+s, which is a three-flavor model in which it is taken into account the decoupled strange quark, or the so-called 2SCu in which only the \( u, s \) pairing survives (i.e. \( \Delta_2 > 0, \Delta_3 = \Delta_1 = 0 \)) or the g2SC discussed in the previous chapter, the gCFL phase turns out to be energetically favored in a quite large window of the control parameter (Alford et al. 2004, 2005c). The comparison of the gCFL free-energy with the free-energy of some of these phases is reported in Fig. 11. Besides the considered phases, other pairing patterns have been proposed (Iida et al. 2004), corresponding to the uSC phase (with \( \Delta_1 = 0 \) and \( \Delta_2, \Delta_3 \neq 0 \)) and the dSC phase (with \( \Delta_2 = 0 \) and \( \Delta_1, \Delta_3 \neq 0 \)), see (Fukushima et al. 2005; Ruester et al. 2006a) for a comparison of the thermodynamic potentials of the various phases.

However, as discussed in the previous chapter, the fact that a phase is energetically favored with respect to other phases is not a sufficient condition to ensure its stability. Indeed, the analysis of fluctuations around the mean field solutions may reveal that the phase does not correspond to a minimum of the free energy. As in the case of the g2SC phase, the gCFL phase is indeed chromo-magnetically unstable, because four of the eight Meissner masses of gluons become imaginary when gapless modes appear (Casalbuoni et al. 2005b; Fukushima, 2005). Further increasing the mismatch among the Fermi surfaces, the masses of the remaining four gluons become imaginary, as well. The effects of temperature on the Meissner masses in the gCFL phase has been studied in Fukushima (2003), where it is found that for sufficiently high temperatures (of about 10 MeV for the parameter choice of Fukushima (2005)), the gCFL phase becomes stable. However, such a temperature is much larger than the typical temperature of compact stars and therefore cannot be used as an argument in favor of the gCFL phase in these systems. Also in this case the instability can be associated to the tendency of the system to develop supercurrents, that is to realize a FF-like state. Then, one generalizes the FF state to more complicated structures trying to single out the correct ground state among various crystalline phases. We shall investigate various crystalline structures in the following sections.

B. Three-flavor crystalline phase: Two plane waves

In this section we present the evaluation of the simplest crystalline superconducting phase using a modified Nambu-Gorkov formalism. The approach presented is based on the High Density Effective Theory (HDET) of Casalbuoni et al. (2001a) and Nardulli (2002) and the evaluation is performed with and without making a GL expansion. As already discussed in the previous chapter, the GL expansion of the free-energy in powers of the gap parameter \( \Delta \) is under control close to a second order transition, so that we should consider only small value of the gap, or more precisely \( \Delta / \Delta_0 \approx \Delta / \delta \mu \rightarrow 0 \). This assumption implies that pairing does not significantly change any number densities. In this context neutrality is achieved with the same chemical potential \( \mu_u = M_s^2 / (4 \mu) \) and \( \mu_3 = \mu_8 = 0 \) as in unpaired quark matter. We will comment briefly in a next subsection on the latter approximation.

1. Nambu-Gorkov and HDET formalisms

In the framework of the Nambu-Gorkov formalism one defines an multi-component Nambu-Gorkov spinor

\[
\chi(p) = \begin{pmatrix} \psi(p) \\ \bar{\psi}^T(-p) \end{pmatrix},
\]

(122)
such that in this basis the pairing between fermions with momentum \( p \) and \( -p \) is described by an off-diagonal term in the fermion propagator, see e.g. Eq. (20). Therefore in the Nambu-Gorkov space, the quark propagator has a compact form. In the two-flavor case, considering the FF ansatz (11), we are considering a condensate of the form

\[
\langle \psi_u(x) C \gamma_5 \psi_d(x) \rangle \propto \Delta e^{i q \cdot r},
\]

(123)
where we have for simplicity suppressed color indices. In this case we are considering pairing between \( u \) quarks with momentum \( p + q \) and \( d \) quarks with momentum \( -p + q \). This could be described via a propagator with terms in it that are off-diagonal in momentum space, rather than merely off-diagonal in Nambu-Gorkov space. However, it
greatly simplifies the calculation changing to a basis in which the Nambu-Gorkov spinor is written as in Bowers et al. [2001]

\[
\chi(p) = \begin{pmatrix}
\psi_u(p+q) \\
\psi_d(p-q) \\
\bar{\psi}_u^T(-p-q) \\
\bar{\psi}_d^T(-p+q)
\end{pmatrix}.
\] (124)

The pair condensate is then described by terms in the fermion propagator that are off-diagonal in Nambu-Gorkov space, occurring in the \(\psi_u \bar{\psi}_d^T\) and \(\psi_d \bar{\psi}_u^T\) entries. In this basis the fermions that pair have opposite momentum, making the propagator diagonal in \(p\)-space and the calculation tractable.

One must always keep in mind that the fermions that pair have momenta \(p + q\) and \(-p + q\) and not \(p\) and \(-p\). Since \(p\) is an integration variable, we can shift it by \(p \to p - q\) thus rewriting the Nambu-Gorkov spinor in the two-flavor phase as

\[
\chi(p) = \begin{pmatrix}
\psi_u(p) \\
\psi_d(p-2q) \\
\bar{\psi}_u^T(-p) \\
\bar{\psi}_d^T(-p+2q)
\end{pmatrix}.
\] (125)

Once we turn to the three-flavor case, the simplest form of inhomogeneous pairing corresponds to the condensate

\[
\Delta_{ij}^{\alpha \beta} = \sum_{I=1}^{3} \Delta_I e^{2q_I \cdot r} \varepsilon^{\alpha \beta I} \varepsilon_{ijI},
\] (126)

meaning that for each pairing channel we assume a Fulde-Ferrell ansatz, with \(2q_I\) representing the momentum of the pair.

In this case it is not possible to diagonalize the propagator in \(p\)-space, because three different fields are locked together in pairs and thus one can only eliminate in the off-diagonal terms of the propagator two independent momenta. However, as shown in [118] and [119], the separation between the \(s\) and \(d\) Fermi spheres is twice the separation between the \(d\) and \(s\) Fermi spheres and the \(u\) and \(s\) Fermi spheres, thus it is reasonable to expect \(\Delta_1 \ll \Delta_2, \Delta_3\). As a first approximation one can consider \(\Delta_1 = 0\) and in this case it is possible to diagonalize the propagator in momentum space.

The form of the two-flavor Nambu-Gorkov spinor (125) immediately suggests that we analyze the three-flavor crystalline phase with condensate (126) with \(\Delta_1\) set to zero by introducing the Nambu-Gorkov spinor

\[
\chi(p) = \begin{pmatrix}
\psi_u(p) \\
\psi_d(p-2q_3) \\
\bar{\psi}_u^T(-p) \\
\bar{\psi}_d^T(-p+2q_3)
\end{pmatrix}.
\] (127)

It is the clear that it would not be possible to use this method of calculation if \(\Delta_1\) were kept nonzero, except for the special case in which \(q_1 = q_2 - q_3\). (That is, except in this special case which is far from sufficiently generic, it will not be possible to choose a Nambu-Gorkov basis such that one obtains a propagator that is diagonal in some momentum variable \(p\).) Moreover, it seems unlikely that this method can be employed to analyze more complicated crystal structures. Indeed, it seems that the pairing with \(\Delta_1 = 0\) and \(\Delta_2, \Delta_3\) each multiplying a single plane wave, is the most complex example that is currently know how to analyze without using the GL expansion.

We now implement the calculation in the basis (127) using the HDET formalism [Nardulli 2002]. We Fourier decompose the fermionic fields as follows:

\[
\psi_{\alpha}^i(x) = e^{-i k_i \cdot x} \int \frac{d\Omega}{4\pi} e^{-i \mu \cdot x} \left( \psi_{\alpha}^{i + \mu}(x) + \psi_{\alpha}^{i - \mu}(x) \right),
\] (128)

where \(\mu\) is a unit three-vector whose direction is integrated over, \(k_i\) are three vectors, one for each flavor, that we shall specify below and \(\psi_{\alpha}^{i + \mu}(x)\) (resp. \(\psi_{\alpha}^{i - \mu}(x)\)) are positive (resp. negative) energy projections of the fermionic fields.
with flavor $i$ and color $\alpha$ indices, as defined in [Casalbuoni et al. (2001a) and Nardulli (2002)]. Note that these fields are written in a mixed notation, meaning that they depend both on the space coordinates and the unit vector $v$, which points to a particular direction in momentum space. Indeed, in the standard HDET approximation (employed to describe the homogeneous phases), $k_i \equiv 0$ and the field $\psi_{i,0}^\alpha(x)$ is used to describe a quark in a patch in momentum space in the vicinity of momentum $P = \mu v$. The introduction of the $k_i$ vectors means that now $\psi_{i,v}^\alpha(x)$ describes a quarks with momentum in a patch in the vicinity of momentum $\mu v + k_i$ and the chemical potential differences are then given by

$$\delta \mu_i(v) = \mu_i - \mu - k_i \cdot v.$$  \hspace{1cm} (129)

To reproduce (127), then we choose

$$
\begin{align*}
k_0 &= 0 \\
k_d &= 2q_3 \\
k_s &= 2q_2.
\end{align*}
$$

At the leading order in $1/\mu$ (i.e. neglecting the contribution of antiparticles) the QCD Lagrangian can be written as

$$\mathcal{L} = \int \frac{d\Omega}{4\pi} \left[ \psi_{i,v}^\dagger \left( iV \cdot \partial + \delta \mu_i(v) \right) \psi_{i,v}^\alpha \right],$$  \hspace{1cm} (131)

where $\delta \mu_i(v) = P_i^F - \mu - k_i \cdot v$ and the four vectors $V^\nu$ and $\bar{V}^\nu$ (the latter used only below) are defined by $V^\nu = (1, v)$ and $\bar{V}^\nu = (1, -v)$.

To take into account diquark condensation, we add to (131) a NJL-like interaction term, which is then treated in the mean field approximation (as in Alford et al. (2005c)), giving

$$\mathcal{L}_\Delta = -\frac{1}{2} \int \frac{d\Omega}{4\pi} \Delta_\alpha^\beta \psi_{i,-v}^\dagger C \gamma_0 \psi_{j,v}^\beta \psi_{i,-v} e^{-i(k_i + k_j) \cdot r} + h.c.,$$

where the condensate $\Delta_\alpha^\beta$ is given in (129). It is convenient to work in a new basis for the spinor fields defined by

$$\psi_{i,v}^\alpha = \sum_{A=1}^9 (F_A)^\alpha_{\alpha} \psi_A,$$ \hspace{1cm} (133)

where the unitary matrices $F_A$ are given by

$$\begin{align*}
F_1 &= \frac{1}{3} I_0 + T_3 + \frac{1}{\sqrt{3}} T_8, & F_2 &= \frac{1}{3} I_0 - T_3 + \frac{1}{\sqrt{3}} T_8, & F_3 &= \frac{1}{3} I_0 - \frac{2}{\sqrt{3}} T_8, \\
F_{4,5} &= T_1 \pm i T_2, & F_{6,7} &= T_4 \pm i T_5, & F_{8,9} &= T_6 \pm i T_7,
\end{align*}$$

with $T_a = \lambda_a/2$ the $SU(3)$ generators and $I_0$ the identity matrix. Introducing the velocity dependent Nambu-Gorkov fields

$$\chi_v^A = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_v^\alpha \\ C \psi_{-v}^\alpha \end{pmatrix},$$ \hspace{1cm} (134)

and replacing any matrix $M$ in color-flavor space with $M_{AB} = \text{Tr}[F_A^T MF_B]$, the NJL Lagrangian density can be written in the compact form

$$\mathcal{L} = \frac{1}{2} \int \frac{d\Omega}{4\pi} \chi_v^A \chi_v^B S^{-1}_{AB}(v),$$ \hspace{1cm} (135)

where the inverse propagator is given by

$$S^{-1}_{AB} = \begin{pmatrix} (V \cdot \ell + \delta \mu_A(v)) \delta_{AB} & -\Delta_{AB} \\
-\Delta_{AB} & (V \cdot \ell - \delta \mu_A(-v)) \delta_{AB} \end{pmatrix},$$ \hspace{1cm} (136)

where $\ell' = (\ell_0, \ell_\parallel v)$ is a four-vector. Here, $\ell_\parallel$ is the “radial” momentum component of $\psi$, parallel to $v$. In HDET, the three-momentum of a fermion is written as $(\mu + \ell_\parallel) v$, with the integration over momentum space separated into
an angular integration over $\nu$ and a radial integration over $-\delta < \ell_\parallel < \delta$. Here, the cutoff $\delta$ must be smaller than $\mu$ but must be much larger than $\Delta_0$ (and $\delta \mu_i$), see Fig.4.

If $\Delta_1 = 0$, the space dependence in the anomalous terms of the propagator can be eliminated by choosing

$$k_u + k_d = 2q_3$$
$$k_u + k_s = 2q_2,$$  \hspace{1cm} (137)

and the calculation is technically simplified. It has been numerically checked in Mannarelli et al. (2006b) that different choices of $k_u$, $k_d$ and $k_s$ satisfying (137) yield the same results for the gap parameter and free energy. The different choices yield quite different intermediate stages to the calculation; the fact that the final results are the same is a nontrivial check of the numerics.

From the Lagrangian (135), following a derivation analogous to that in Alford et al. (2005a), the thermodynamic potential per unit volume can be evaluated to be

$$\Omega = -\frac{\mu^2}{4\pi^2} \sum_{a=1}^{18} \int_{-\delta}^{+\delta} d\ell_\parallel \int \frac{d\Omega}{4\pi} \left| E_a(\nu, \ell_\parallel) \right| + \frac{2\Delta^2}{G} - \frac{\mu^4}{12\pi^2},$$  \hspace{1cm} (138)

where we have set $\Delta_2 = \Delta_3 = \Delta$ and where $G$ is the dimensionful coupling constant. The dependence on $G$ and of the ultraviolet cutoff $\delta$ can be eliminated by using the CFL gap equation, through the parameter $\Delta_0$ defined by

$$\Delta_0 \equiv 2^{2/3} \delta \exp \left\{ -\frac{\mu^2}{2 G \mu^2} \right\},$$  \hspace{1cm} (139)

where $\Delta_0$ is the CFL gap parameter for $M_0 = 0$ and $\mu_e = 0$. In Eq. (138), the $E_a$ are the energies of the fermionic quasiparticles in this phase, which are given by the 18 roots of $\det S^{-1} = 0$, given in Eq. (136). The doubling of degrees of freedom in the Nambu-Gorkov formalism means that the 18 roots come in pairs whose energies are related by $E_a(\nu, \ell_\parallel) = E_b(-\nu, \ell_\parallel)$. One set of nine roots describes $(\psi_{v,d}, \psi_{v,u})$ and $(\psi_{s,s}, \psi_{s,u})$ pairing, while the other set describes $(\psi_{v,u}, \psi_{v,d})$ and $(\psi_{s,d}, \psi_{s,u})$ pairing. Since $\nu$ is integrated over, the free-energy can be evaluated by doing the sum in (138) over either set of nine roots, instead of over all 18, and multiplying the sum by two.

The lowest free-energy state is determined minimizing the free-energy given in Eq. (138) with respect to the gap parameter $\Delta$ and with respect to $q_2$ and $q_3$. One could also determine self-consistently the values of $\mu_e$, $\mu_3$ and $\mu_8$ which ensure electrical and color neutrality. Moreover, one could allow $\Delta_2 \neq \Delta_3$ and minimize with respect to the two gap parameters separately. However, in the results that we shall present in the next section we shall fix $\Delta_2 = \Delta_3 = \Delta$, $\mu_e = M_2/(4\mu)$ and $\mu_3 = \mu_8 = 0$, as is correct for small $\Delta$.

2. Ginzburg-Landau analysis

As discussed in the two-flavor case, the GL expansion allows to determine the gap parameter and the free-energy for a generic crystalline structure. We shall discuss the GL results for various crystalline phases in Section III.C in the present section we restrict to the case of the condensate reported in Eq. (129). We recall that the GL expansion is controlled by the ratio $\Delta/\Delta_0 \approx \Delta/\delta \mu$ and is reliable in the vicinity of a second order transition; this restriction implies that pairing does not significantly change any number density, and thus one can assume $\mu_3 = \mu_8 = 0$ and $\mu_e \approx M_2/4\mu$ as in the normal phase.

For the condensate in (129), the GL expansion of the free-energy is given by Casalbuoni et al. (2005a), 2006b)

$$\Omega = \Omega_n + \sum_{I=1}^{3} \left( \frac{\alpha_I}{2} \Delta_I^2 + \frac{\beta I}{4} \Delta_I^4 + \sum_{J \neq I} \frac{\beta_{IJ}}{4} \Delta_I^2 \Delta_J^2 \right) + O(\Delta^6),$$  \hspace{1cm} (140)

where $\Omega_n$ is the free-energy of the normal phase and $\alpha_I$, $\beta_I$ and $\beta_{IJ}$ can be determined by the microscopic theory. Employing the HDET formalism, it has been shown in Casalbuoni et al (2005a) that $\alpha_I$ and $\beta_I$ have the same formal expression derived in the GL analysis of the two-flavor FF structure (Bowers and Rajagopalan, 2002), meaning that

$$\alpha_I \equiv \alpha_{FF}(q_I, \delta \mu_I) = \frac{4 \mu^2}{\pi^2} \left( -1 + \frac{\delta \mu_I}{2 q_I} \log \frac{|q_I + \delta \mu_I|}{|q_I - \delta \mu_I|} + \frac{1}{2} \log \frac{4(q_I^2 - \delta \mu_I^2)}{(2\pi \Delta_0)^2} \right),$$  \hspace{1cm} (141)

$$\beta_I \equiv \beta_{FF}(q_I, \delta \mu_I) = \frac{\mu^2}{\pi^2} \frac{1}{q_I^2 - \delta \mu_I^2},$$  \hspace{1cm} (142)
and in the weak coupling limit
\[ \delta \mu_1 = -\mu_e, \quad \delta \mu_2 = -\delta \mu_3 = -\frac{\mu_s}{2}. \]  

(143)

The terms \( \beta_{IJ} \), with \( I \neq J \), are instead characteristic of the three-flavor case as they represent the interaction term between different condensates. We have that
\[ \beta_{12} = -\frac{\mu^2}{2\pi^2} \int \frac{d\nu}{4\pi} \frac{1}{(i\varepsilon - 2q_1 \cdot \nu - 2\delta \mu_1)(i\varepsilon - 2q_2 \cdot \nu - 2\delta \mu_2)}, \]

(144)

and \( \beta_{13} \) is obtained from \( \beta_{12} \) by the exchange \( q_2 \rightarrow q_3 \) and \( \mu_s \leftrightarrow \mu_d \) and \( \beta_{23} \) from \( \beta_{12} \) by \( q_1 \rightarrow q_3 \) and \( \mu_s \leftrightarrow \mu_u \).

These are the only terms in which a dependence of the free-energy from the relative orientation of the \( q \) flavors and therefore gives the result
\[ \Delta \approx \Omega \]

This is in general a complex task as it would require the simultaneous minimization of the free-energy as a function of the angles between the three different \( q \) orientations of the three vectors \( q \).

One should fix the norms \( q \) by the minimization procedure and finally look for the most energetically favored orientations of the three vectors \( q \). This is in general a complex task as it would require the simultaneous minimization of the free-energy as a function of the angles between the three different \( q \). What is usually done to circumvent this complication is to propose different definite structures for the \( q \), and choose among them the one with the lowest free energy.

It has been found in Casalbuoni et al. (2005a) that the energetically favored solution corresponds to \( \Delta_1 = 0 \) and \( \Delta_2 \approx \Delta_3 \) with \( q_2 \) parallel to \( q_3 \). As to the norms \( q \), since we work in the GL approximation, it is possible to neglect the \( O(\Delta^2) \) terms in the minimization of \( \Omega \). Thus, one has that \( \partial \Delta_1/\partial q_I = 0 \), which is identical to the condition for two flavors and therefore gives the result \( q_I = \eta |\delta \mu_I| \) where \( \eta = 1/zq \approx 1.2 \) Larkin and Ovchinnikov (1964), where \( zq \) was defined in Eq. 15a; considering Eq. 14b) this implies that \( uS \) and \( duS \) quark pair with equal momenta
\[ q \equiv |q_2| = |q_3|. \]

The free-energy can now be written as
\[ \Omega = \Omega_n + \alpha \Delta^2 + \frac{1}{2}(\beta + \beta_{23})\Delta^4, \]

where
\[ \Delta \equiv \Delta_2 = \Delta_3, \]

is the only independent gap parameter. Minimizing the free-energy expression with respect to \( \Delta \) we obtain for values of \( \delta \mu \) where \( \alpha \) is negative, the solution
\[ \Delta^2 = \frac{|\alpha|}{\beta + \beta_{23}}, \]

(148)

and the free-energy at the minimum is given by
\[ \Omega = \Omega_n - \frac{\alpha^2}{2(\beta + \beta_{23})}, \]

(149)

which still depend on the angle, \( \phi \), see Fig. 13 between \( q_2 \) and \( q_3 \), by \( \beta_{23} \).

3. Testing the Ginzburg-Landau approximation

We are now able to evaluate the gap parameter and the free-energy for the “crystalline” color superconducting phase
\[ \Delta^{ij}_{\eta} = \Delta(e^{2in_2 \cdot r \varepsilon^{\alpha \beta_2} \xi_{ij2}} + e^{2in_3 \cdot r \varepsilon^{\alpha \beta_3} \xi_{ij3}}), \]

(150)

with and without the GL approximation (Mannarelli et al. 2006a); here we have written \( q_2 = q n_2 \) and \( q_3 = q n_3 \). We report the results obtained for \( \mu = 500 \) MeV and \( \Delta_0 = 25 \) MeV. The calculations are made varying \( M_s \), but we plot quantities versus \( M_s^2/\mu \) because the most important effect of nonzero \( M_s \) is the splitting between the \( d, u \) and \( s \) Fermi momenta given in Eq. 14b), and we report the results for four values of the angle between \( n_2 \) and \( n_3 \): \( \phi = 0, \ 2\pi/3, \ 7\pi/8 \) and \( 31\pi/32 \). The lines correspond to the GL analysis, where we have plotted \( \Delta \) and \( \Omega - \Omega_n \) of Eqs. 14b).
and (149) and using Eq. (141) to relate $\alpha$ to $\delta\mu$ and hence to $M_s^2/\mu$. The points correspond to the NJL calculation without GL, where $\Omega$ of Eq. (138) has been minimized with respect to $\Delta$.

The NJL calculation has a second order transition at $M_s^2/\mu \sim 151$ MeV, see Fig. 12 corresponding to $\delta\mu \sim 0.754\Delta_0$, for all values of the angle $\phi$. This result is in agreement with the GL calculation, in which the location of the phase transition depends only on $\alpha$, which is independent of $\phi$. Near the phase transition, where $\Delta/\Delta_0$ and hence $\Delta/\delta\mu$ are small, there is good agreement between the NJL calculation and the GL approximation, as expected. When the GL approximation breaks down, it does so conservatively, under-predicting both $\Delta$ and $\delta\mu$. For example, with $\phi = 31\pi/32$ the GL calculation gives results in quantitative agreement with the full NJL calculation only for $\Delta/\Delta_0 \lesssim 0.04$, corresponding to $\Delta/\delta\mu \lesssim 0.05$ (Mannarelli et al., 2006b).

The free-energy of the favored crystalline phase, corresponding to $n_2 = n_3$, has a lower free-energy of the CFL and gCFL phases in a window 128 MeV $< M_s^2/\mu < 151$ MeV. Since the GL approximation is reliable over the widest domain precisely for the case with $n_2 \parallel n_3$, we note from Fig. 12 that the GL analysis provides good approximation to the results of the full NJL calculations. These results strongly motivates the study of more complicated crystalline structures employing the GL approximation.

We can now get a deeper insight about the favored orientation of the $q_I$ from the analysis of $\beta_{23}$. As explained above, this is the only term in the GL expansion that depends on the relative orientation between $n_2$ and $n_3$ and can be rewritten as

$$\beta_{23} = \frac{2\mu^2}{\pi^2\delta\mu^2} I(\phi),$$

(151)

where

$$I(\phi) = \Re \int \frac{d\Omega}{4\pi} \frac{-1}{(ie - \eta v \cdot n_2 - 1)(ie - \eta v \cdot n_3 + 1)}$$

(152)

can be evaluated numerically and the result is plotted in Fig. 13 showing that the minimum value occurs at $\phi = 0$, that is for $n_1 = n_2$, in agreement with the fact that the configuration that minimizes $I(\phi)$ does as well minimize the
FIG. 13 $I(\phi)$, defined in Eq. (152), versus the angle $\phi$ between the wave vectors $\mathbf{q}_2$ and $\mathbf{q}_3$.

free-energy, see Eq. (149). We note that although $I(\phi)$ is an increasing function of $\phi$ it depends very slightly on the angle except close to the value $\phi = \pi$, i.e. when the two vectors are antiparallel and $I(\phi)$ diverges.

Although the divergence of $\beta_{23}$ at $\phi = \pi$ is not physically relevant, since it is not the value that minimizes the free-energy, it is worth considering this case to gain a qualitative understanding of the behavior of the GL approximation. We see in Fig. 14 that there are two pairing rings on the up quark Fermi surface, because some up quarks pair with down quarks forming Cooper pairs with wave vector $2\mathbf{q}_3$ and other up quarks pair with strange quarks forming Cooper pairs with wave vector $2\mathbf{q}_2$. However, as shown in the right panel of Fig. 14 if $\phi = \pi$ the two pairing rings on the up quark Fermi surface are close to be coincident. In the weak-coupling limit in which $\delta \mu/\mu \to 0$ (and $\Delta_0 \to 0$ with $\delta \mu/\Delta_0$ fixed) these two rings become precisely coincident. We attribute the divergence in $\beta_{23}$ to the fact that antiparallel wave vectors pay an infinite free-energy price and hence are forbidden, because of the coincidence of these two pairing rings. In contrast, if $\phi = 0$, the two pairing rings on the up Fermi surface are as far apart as they can be, and $\beta_{23}$ and the free-energy of the state are minimized. This qualitative understanding also highlights that it is only in the strict GL and weak coupling limits that the cost of choosing antiparallel wave vectors diverges. If $\Delta/\delta \mu$ is small but nonzero, the pairing regions are ribbons on the Fermi surfaces instead of lines. And, if $\delta \mu/\mu$ is small but

FIG. 14 (Color online). Sketch showing where on the Fermi surfaces pairing occurs for condensates in which $\mathbf{q}_2$ and $\mathbf{q}_3$ are at an angle $\phi$. The light gray (red online) ribbons on the $d$ and $u$ Fermi surfaces indicate those quarks that contribute the most to the $\langle ud \rangle$ condensate with gap parameter $\Delta_3$ and wave vector $\mathbf{q}_3$, which points upward in both panels. The dark gray (blue online) ribbons on the $u$ and $s$ Fermi surfaces indicate those quarks that contribute most to the $\langle us \rangle$ condensate with gap parameter $\Delta_2$ and wave vector $\mathbf{q}_2$. For descriptive reasons, in the left panel it is shown the case $\phi = 70^\circ$, and in the right panel the antipodal case, with the two vectors pointing in opposite directions. We have greatly exaggerated the splitting between the Fermi surfaces, relative to the values used in the calculations reported in Sec. III.B.
not taken to zero (as of course is the case in Fig. [14]) then the two ribbons on the up Fermi surface will have slightly different diameter, as the figure indicates. This means that we expect that if we do a calculation at small but nonzero $\Delta_0 \sim \delta \mu$, and do not make a GL expansion, we should find some free-energy penalty for choosing $\phi = \pi$, but not a divergent one. This is indeed the result that one obtains without using the GL expansion, see Fig. [12]. These results also explain why the breakdown of the GL expansion in Fig. [12] happens for very small values of $\Delta$ for $\phi \approx \pi$, while only for larger values of $\Delta$ for $\phi = 0$. Indeed, increasing $\phi$ from 0 to $\pi$, the $\beta_{23}$ term in the GL expansion increases, and therefore the radius of convergence of the GL expansion decreases.

A more quantitative study of the radius of convergence of the GL approximation would require evaluating (at least) the $\Delta_6$ terms, whose coefficients we shall generically call $\gamma$. Because we are working in the vicinity of a point where $\alpha = 0$, the first estimator of the radius of convergence that we can construct comes by requiring $\gamma \Delta_6 \lesssim (\beta + \beta_{23}) \Delta_4$. Thus, the results of the comparison in Fig. [12] are not conclusive on this point, but they certainly indicate that the radius of convergence in $\Delta$ decreases with increasing $\phi$, and tends towards zero for $\phi \to \pi$.

4. Chromo-magnetic stability of the three-flavor crystalline phase

Given that the three-flavor crystalline phase in Eq. (150) is thermodynamically favored with respect to the CFL and normal phases, it remains to be proven that it is chromo-magnetically stable. This issue has been discussed within the GL expansion in Ciminale et al. (2006), where the Meissner masses of gluons have been evaluated and it has been shown that this phase is indeed chromo-magnetically stable. To take into account gluons, the HDET Lagrangian has to be extended to the next-leading order (NLO) in $1/\mu$, obtaining in momentum space:

$$\mathcal{L} = \bar{\psi}_{i, \alpha}^\dagger(\ell) \left( V \cdot \ell_{ij}^{\alpha\beta} + \mu_{ij}^{\alpha\beta} + P_{\mu\nu} \left[ \ell_{\mu}^{\alpha} \ell_{\nu}^{\beta} \right] \right) \psi_{j, \beta}(\ell), \quad (153)$$

where

$$\langle \ell^{\mu} \rangle_{ij}^{\alpha\beta} = \ell^{\mu} \delta_{ij} \delta^{\alpha\beta} - g A_a T^{\alpha\beta}_{ij} \delta_{ij}, \quad (154)$$

and

$$P^\mu_{\nu} = g^{\mu\nu} - \frac{1}{2} \left( V_{\mu} V^{\nu} + \tilde{V}_{\mu} \tilde{V}^{\nu} \right). \quad (155)$$

In the HDET there are two effective vertices describing the coupling between gluons and quarks. Either one gluon couples to a quark and an antiquark (three-body vertex, coupling $\sim g$) or two quarks couple to two gluons (four-body vertex, coupling $\sim g^2$). These two vertices originate from the terms in Eq. (151) with one and two momenta $\ell$.

FIG. 15 (Color online). Squared Meissner masses of the gluons $\tilde{A}_3$ (solid blue line), $A_6$ (dotted green line), $\tilde{A}_8$ (dashed black line), and $A_1$ (dot-dashed red line), in units of the CFL squared Meissner mass, vs $M^2_s/\mu$. Left panel: Longitudinal masses. Right panel: Transverse masses. Adapted from Ciminale et al. (2006).
Theses two vertices lead, at the order of $g^2$, to two distinct contribution to the Meissner mass of gluons. The four-body coupling gives rise to the contribution $g^2 \mu^2 / (2\pi^2)$, which is the same for all the eight gluons and is the same one has in the CFL phase.

The three-body coupling gives rise to the polarization tensor:

$$i\Pi_{\mu\nu}^{ab}(x, y) = -\text{Tr}[i S(x, y) i H^a_\mu i S(y, x) i H^b_\nu],$$

where the trace is over all the internal indexes; $S(x, y)$ is the quark propagator, and

$$H^a_\mu = ig \left( i \cdot V^a_\mu \right) - i \cdot \tilde{V}^a_\mu h_a^*,$$

is the vertex matrix in the HDET formalism.

The Meissner masses are defined in terms of the gluon self-energy in momentum space, see Eq. (37), and in this case the mass matrix of gluons in the adjoint sector $a = 3, 8$ is not diagonal. Therefore we introduce the two linear combinations

$$\tilde{A}_{i3} = \cos \theta_i A_{i3} + \sin \theta_i A_{i8}, \quad \tilde{A}_{i8} = -\sin \theta_i A_{i3} + \cos \theta_i A_{i8},$$

which are eigenstates of the mass matrix. In (158) the subscript $i$ denotes the spatial component of the gluon field; it is easily shown that the mixing angle satisfies the equation

$$\tan 2\theta_i = \frac{2M_{i,38}^2}{M_{i,33}^2 - M_{i,88}^2};$$

the corresponding Meissner masses are the eigenvalues of the matrix

$$\begin{pmatrix}
M_{i,33}^2 & M_{i,38}^2 \\
M_{i,38}^2 & M_{i,88}^2
\end{pmatrix},$$

which turn out to be positive and are reported in Fig. 15.

On the left panel of Fig. 15 we report the longitudinal (i.e. $zz$) components of the squared Meissner masses against $M_s^2 / \mu$, in units of the CFL squared mass (Rischke, 2000a; Son and Stephanov, 2000b), at the $O(\Delta^4)$; on the right panel the results for the transverse (i.e. $xx$) squared Meissner masses are given.

These results are the analogous of those obtained in the two-flavor case in Giannakis and Ren (2005b). Moreover, the transverse mass of $\tilde{A}_8$, although positive, is almost zero, being three order of magnitude smaller than the other ones. The conclusion is that the three-flavor condensate in Eq. (150) has no chromo-magnetic instability, at least within the GL approximation. The results of Ciminale et al. (2006) have not been extended to more complicated structures, however, for more complicated crystalline structures (Rajagopal and Sharma, 2006) the Meissner tensor should be positive definite for small values of $\Delta$, since it is additive with respect to different terms of order $\Delta^2$ in the GL expansion (Giannakis and Ren, 2005b). These considerations suggest that crystalline structures can be good candidates in removing the chromo-magnetic instability of the homogeneous gapless superconducting phases of QCD.

5. Influence of $O(1/\mu)$ corrections

Since the Meissner masses have been determined including the $O(1/\mu)$ corrections in the HDET Lagrangian, for a consistent calculation it is necessary considering the effect of $O(1/\mu)$ corrections on the gap parameters and the free-energy (Casalbuoni et al. 2006). These corrections amount to a shift of the strange quark Fermi momentum to lower values with respect to the corresponding chemical potential

$$p_s^F \simeq \mu_s - \frac{M_s^2}{2\mu_s} - \frac{1}{2\mu} \left( \frac{M_s^2}{2\mu} \right)^2,$$

and thus increasing the difference between the $u$ and $s$ strange chemical potential, without affecting the $d$ chemical potential. Thus (neglecting corrections proportional to $\mu_3$ and $\mu_8$ which are of order $O(\Delta^6)$) one has that

$$\delta \mu_2 = \frac{1}{2} \left[ \frac{M_s^2}{2\mu} + \frac{1}{3\mu} \left( \frac{M_s^2}{2\mu} \right)^2 - \mu_c \right].$$
FIG. 16 (Color online). Left panel: Quark chemical potential difference as a function of $M_s^2/\mu$. The dashed blue line corresponds to $\delta \mu_2$ and the dash-dotted red line corresponds to $\delta \mu_3$; their common value $M_s^2/(8\mu)$, obtained neglecting $O(1/\mu)$ corrections, corresponds to the solid black line. Right panel: gap parameters as a function of $M_s^2/\mu$. The solid black line represents the solution $\Delta = \Delta_2 = \Delta_3$ obtained neglecting $O(1/\mu)$ corrections; the dashed blue line and the dash-dotted red line represent respectively $\Delta_2$ and $\Delta_3$ obtained including $O(1/\mu)$ corrections. All the gap parameters are normalized to the CFL homogeneous gap $\Delta_0 = 25$ MeV. Adapted from Casalbuoni et al. (2006).

$\delta \mu_3 = \frac{\mu_e}{2}$, (162)

and $\delta \mu_2 > \delta \mu_3$ (neglecting higher order effects, as in Casalbuoni et al. (2005a), one would get the result $\delta \mu_2 = \delta \mu_3$ since in this limit $\mu_e = M_s^2/4\mu$). The results for the splitting of chemical potentials are reported in the left panel of Fig. 16 together with the result obtained in the large $\mu$ limit.

As a consequence of the $O(1/\mu)$ corrections, the two gap parameters are not equal and $\Delta_2 < \Delta_3$, as shown in the right panel of Fig. 16. This effect is akin to the one considered in the gCFL phase, see Fig. 11, but in that case the chromo-magnetic instability prevented the splitting of the gap parameters. The effect of the corrections considered here is to enlarge the LOFF window, as it can be seen from the right panel of Fig. 16. Neglecting $O(1/\mu)$ corrections the crystalline phase was characterized by one gap, and by condensation in two channels: $u - s$ and $u - d$; including $O(1/\mu)$ corrections there is pairing in both channels only for small values of $M_s^2/\mu$, but the crystalline phase extends to larger values of $M_s^2/\mu$ where only $\Delta_3 \neq 0$ and therefore there is no $u - s$ pairing. This phase is the analogous of the two-flavor FF phase, with a chemical potential difference determined by the ratio of the strange quark mass to the average quark chemical potential.

C. Ginzburg-Landau analysis of crystalline structures

As in the two-flavor case, more complicated crystalline structure can be considered and analyzed by the GL expansion (Rajagopal and Sharma, 2006). The most general crystalline diquark condensate was given in Eq. 7, which we report here for convenience

$$\langle 0 | \bar{\psi}_i^\alpha \psi_j^\beta | 0 \rangle \propto \sum_{l=1}^3 \Delta_l \varepsilon^\alpha^\beta^l \varepsilon_{ij} \sum_{q^m_i \in \{q_i\}} e^{2\pi q^m_i \cdot r}.$$ (163)

This condensate is antisymmetric in color ($\alpha, \beta$), spin, flavor ($i, j$) indices, and corresponds to a inhomogeneous generalization of the CFL ansatz, but now each gap parameter has a periodic modulation in space corresponding to a crystalline structure, e.g. $\Delta_2$ is associated to a crystal due to $u-s$ pairing which is described by the vectors $q^2_m$, where $m$ is the index which identifies the elements of the set $\{q_2\}$, meaning that the $q^m_2$’s are the reciprocal vectors which define the crystal structure of the condensate.

Self-consistent computations based on Eq. (163) are very complicated even within the GL calculation scheme; as a consequence, several assumptions and simplifications have been used in Rajagopal and Sharma (2006). As discussed in the previous section, the pairing between $d$ and $s$ quarks can been neglected; this is a very reasonable approximation,
since the difference of chemical potentials between $d$ and $s$ quarks is approximately twice the imbalance of chemical potentials between $u - s$ and $u - d$. Therefore, $\Delta_1 = 0$, and one can take $\Delta_2 = \Delta_3$, neglecting $O(1/\mu)$ corrections. Another simplification is to consider only crystalline structures with wave vectors $\{q^m_2\}$ and $\{q^m_3\}$ with equal modulus. This means that any phase is determined by one gap parameter, by the modulus of the total pair momentum, $q$, and by the unit vectors which determine the two crystalline structures. \cite{Rajagopal and Sharma 2006}. The corresponding pairing ansatz simplifies to

$$\langle 0| \psi^\alpha_i \psi^\beta_j | 0 \rangle \propto \Delta \sum_{I=2}^3 \varepsilon^{\alpha \beta I} \varepsilon_{ijI} \sum_{n_I^m \in (n_I)} \epsilon^{2i\alpha \beta I \cdot r}.$$  \hspace{1cm} (164)

A further simplification of \cite{Rajagopal and Sharma 2006} is to consider only structures which are exchange symmetric, which means that $\{n_2\}$ and $\{n_3\}$ can be exchanged by some combination of rotations and reflections applied simultaneously to all the wave vectors.

The thermodynamic potential within the GL expansion up to the sextic order in the gap parameters is given by

$$\Omega(\Delta_2, \Delta_3) = \frac{2\mu^2}{\pi^2} \left[ P_0\left(\left|\Delta_2\right|^2 + \left|\Delta_3\right|^2\right) + \frac{\beta}{2}\left(\left|\Delta_2\right|^4 + \left|\Delta_3\right|^4\right) + \frac{\gamma}{3}\left(\left|\Delta_2\right|^6 + \left|\Delta_3\right|^6\right)\right]$$

$$\hspace{1cm} + \frac{2\mu^2}{\pi^2} \left[ \frac{\beta_23}{2}\left|\Delta_2\right|^2\left|\Delta_3\right|^2 + \frac{\gamma_233}{3}\left|\Delta_2\right|^4\left|\Delta_3\right|^4 + \frac{\gamma_2333}{3}\left|\Delta_2\right|^6\left|\Delta_3\right|^6\right],$$  \hspace{1cm} (165)

where we have explicitly diversified $\Delta_2$ and $\Delta_3$ in order to show the interaction between the condensates in the $(ud)$ and the $(us)$ channels. The coefficients $\alpha, \beta$ and $\gamma$ are the same coefficients computed for the two-flavor case in Bowers and Rajagopal \cite{Bowers and Rajagopal 2002}. The expression in Eq. 165, the remaining terms are peculiar of the three-flavor case. In the numerical computations $\Omega(\Delta, \Delta)$ is minimized to compute the physical value of $\Delta$ and for any crystalline structure considered it is found that the thermodynamic potential is always bounded from below. Therefore, it is possible to compute the numerical value of $\Delta$ and the value of the free-energy density at the minimum.

Among the many crystalline structures considered in \cite{Rajagopal and Sharma 2006}, two of them have been found to have the lowest free-energy: they are called the CubeX and the 2Cube45z structures. In the CubeX crystal, $\{n_2\}$ and $\{n_3\}$ each contains four wave vectors, pointing to the vertices of a square; the eight vectors together point toward the vertices of a cube. In the 2Cube45z crystal, $\{n_2\}$ and $\{n_3\}$ each contains eight wave vectors, pointing to the corners of a cube; the two cubes are rotated by 45 degrees about the $z$-axis. The computation of the free-energy density at the global minimum shows that the CubeX crystal is favorable over all the structures considered in \cite{Rajagopal and Sharma 2006} in the range

$$2.9\Delta_0 < \frac{M_a^2}{\mu} < 6.4\Delta_0;$$  \hspace{1cm} (166)

similarly for the 2Cube45z crystal it is found that

$$6.4\Delta_0 < \frac{M_a^2}{\mu} < 10.4\Delta_0.$$  \hspace{1cm} (167)

Putting together the above results, the crystalline superconducting phase is favored with respect to the homogeneous CFL, gCFL and unpaired phases in the range

$$2.9\Delta_0 < \frac{M_a^2}{\mu} < 10.4\Delta_0.$$  \hspace{1cm} (168)

The above condition can be translated to a condition on $\mu$ only if the constituent strange quark mass as a function of $\mu$ is known. This topic will be discussed in the next section.

We remark that from the quantitative point of view, the result in Eq. 168 should be taken with care. The GL expansion is reliable only if the gap parameter $\Delta$ is small, compared to $\delta \mu$. On the other hand, the numerical results of \cite{Rajagopal and Sharma 2006} show that within this expansion, the ratio $\Delta/\delta \mu$ turns out to be of order one. This clearly signals that the results lie beyond the validity of the expansion itself. Moreover, the coefficients in the expansion in Eq. 165 depend on the microscopic model. Therefore, Eq. 168 is certainly model dependent. Nevertheless, the qualitative picture which arises in \cite{Rajagopal and Sharma 2006} seems to be quite robust: the crystalline superconductor has lower free-energy with respect to the single plane wave state. This conclusion seems very reasonable and model independent: in fact, the phase space for pairing in the case of CubeX and 2Cube45z is larger than the one corresponding to the plane wave structure; this suggests that the free-energy gain for multiple-plane-waves pairings is larger than the ones obtained for the single plane wave state.
1. LOFF window in the QCD phase diagram

It is important to identify, within a self-consistent computation, the range of temperature/chemical potential in which the crystalline phase is expected to be thermodynamically favored. In the previous sections we have been able to identify the LOFF window in terms of the dimensionless ratio \(M^2/\mu^2\); in order to translate this window to a window in \(\mu\), the computation of the in-medium strange quark mass is necessary. This investigation has been started in \cite{ippolito2007} considering vanishing temperature. In that article, the in-medium quark masses are computed self-consistently within a NJL model. This model offers a quite simple theoretical framework to discuss simultaneously dynamical breaking of chiral symmetry, as well as color superconducting pairing.

In the simple case of two plane waves, \cite{150}, it was found in \cite{casalbuoni2005a, maniarelli2006b, rajagopal2006} that this state is energetically favored for\(^{4.80\Delta_0 \leq \frac{M^2}{\mu} \leq 7.56\Delta_0}, \quad (169)\)

where

\[
\Delta_0 = 2^{2/3}(\Lambda_0 - \mu) \exp\left(\frac{-\pi^2}{6G(\Lambda_0)\mu^2}\right) \quad (170)
\]

is the CFL gap in the chiral limit evaluated by the NJL model. At the reference point considered in \cite{ippolito2007}, \(\Lambda_0 \simeq 643\) MeV, \(G(\Lambda_0) \simeq 1.81/\Lambda_0^2\) have been determined and taking \(\mu = 500\) MeV one has that \(\Delta_0 \approx 50\) MeV. Using the results for the constituent quark masses computed in \cite{ippolito2007}, equation \(169\) can be translated to a window for the existence of the LOFF phase in the quark number chemical potential, namely

\[
467\text{ MeV} < \mu < 488\text{ MeV} \quad (171)
\]

Therefore there exists a small but finite window in \(\mu\) where the structure \(150\), is favored in comparison with the normal state.

As discussed in the previous section, the analysis of \cite{rajagopal2006} shows that more complicated crystalline structures can lead to a smaller free energy. In \cite{rajagopal2006} it was found that a CCSC phase exists in the interval reported in \(168\) which using the self-consistent treatment by \cite{ippolito2007} translates in

\[
442\text{ MeV} < \mu < 515\text{ MeV} \quad (172)
\]

This result is certainly model dependent, however it shows that the actual extension of the LOFF window might not be very large and therefore it may occupy only a fraction of the volume of hybrid compact stars with a quark core. We shall discuss more on this topic in the next chapter.

D. Shear modulus and Nambu-Goldstone modes

The three-flavor CCSC phase has several low energy excitations; besides gapless quasifermions, there are scalar modes which are related to the spontaneous breaking of translational symmetry, the phonons, of the chiral symmetry, the pseudoscalar NB bosons \cite{anglani2007} and to the breaking of \(U(1)_B\) symmetry, the \(H-\)phonon.

In view of the astrophysical applications of the crystalline phases that we shall discuss in the next chapter, the low energy effective actions of the low energy scalar modes of the LOFF phase have to be computed. The phonon Lagrangian for several two-flavor crystalline structures has been determined in \cite{casalbuoni2002, casalbuoni2001a}, and discussed in \(\text{II.D.3}\) while for the three-flavor case it has been studied in \cite{maniarelli2007}. Regarding the NG bosons, we shall discuss the results of \cite{anglani2007} obtained for three-flavor crystalline phases.

1. Phonons Effective Action and Shear Modulus

According to the basic theory of elastic media \cite{landau1959}, the elastic moduli are related to the potential energy cost of small deformations of the crystal. Therefore, the evaluation of the shear modulus requires the knowledge of the low energy Lagrangian for the displacement fields \cite{maniarelli2007}. These displacements can be described by phonons that originate from the spontaneous breaking of rotational invariance by means of spatial
modulation of the gap parameter $\Delta$. More in detail, phonons are small position and time dependent displacements of the condensate and, since the three condensates that characterize the crystalline phase can oscillate independently, we expect there to be three sets of displacement fields $\mathbf{u}_I(x)$. In the presence of the phonons, then,

$$\Delta_I(r) \to \Delta_I^w(x) = \Delta_I(r - \mathbf{u}_I(x)),$$

(173)

and the Lagrangian that includes fluctuation on the top of the mean field solution can be written as

$$\mathcal{L} = \frac{1}{2} \chi \left[ i \bar{\phi} + \phi \begin{pmatrix} \Delta^u(x) \\ \bar{\Delta}^u(x) \end{pmatrix} \right] \chi + \frac{1}{16G} Tr \left( (\Delta^u)^T \Delta^u \right),$$

(174)

where $\chi$ stands for the trace over color, flavor and Dirac indices.

To find the low energy effective action which describes the phonons and the gapless fermionic excitations, one should integrate out the high energy fermion fields. The procedure is detailed in Mannarelli et al. (2007), where the effective action is derived starting from a NJL-like microscopic model. The final form of the effective action, of the phonon fields, is given by

$$i \mathcal{S}[\mathbf{u}] = \log(\mathcal{Z}[\mathbf{u}]) = i \int d^4x \left[ \frac{1}{16G} Tr \left( (\Delta^u)^T \Delta^u \right) + \frac{1}{2} Tr_{ng} \log (S^{-1}) \right] +$$

(175)

where $Tr_{ng}$ stands for the trace over the Nambu-Gorkov, color, flavor and Dirac indices and a further trace over a set of functions on space-time containing all energy modes. The full inverse propagator, is given by

$$S^{-1} = \left( \begin{pmatrix} i \bar{\phi} + \phi \Delta^u(x) \\ \bar{\Delta}^u(x) \end{pmatrix} (i \bar{\phi} - \phi)^T \right),$$

(176)

and it includes interactions of the phonon fields.

For the crystal structures CubeX and 2Cube45z, the full inverse propagator cannot be inverted, so a GL expansion has been performed in order to obtain the effective action for the phonon field, first separating the full inverse propagator into the free part, $S_0^{-1}$, and a part containing the condensate, $\Sigma$, as follows: $S^{-1} = S_0^{-1} + \Sigma$, where

$$S_0^{-1} = \left( \begin{pmatrix} i \bar{\phi} + \phi & 0 \\ 0 & (i \bar{\phi} - \phi)^T \end{pmatrix} \right)$$

(177)

and

$$\Sigma = \left( \begin{pmatrix} 0 & \Delta^u(r) \\ \bar{\Delta}^u(r) & 0 \end{pmatrix} \right).$$

(178)

Then, one can expand the term $\log(S^{-1})$ that appears on the right-hand side of Eq. (176) as

$$Tr_{ng} \left( \log(\mathcal{Z}[\mathbf{u}] + \Sigma) \right) = Tr_{ng} \left( \log(\mathcal{Z}[\mathbf{u}]_0) + Tr_{ng} (\mathcal{Z}[\mathcal{Z}[\mathbf{u}]_0)] - \frac{1}{2} Tr_{ng} (\mathcal{Z}[\mathcal{Z}[\mathbf{u}]_0]^2 + \ldots. \right)$$

(179)

The $Tr_{ng}(\log S_0^{-1})$ term is related to the free-energy of unpaired (normal) quark matter. Furthermore, it is easily found that only even powers of $(\mathcal{Z}[\mathcal{Z}[\mathbf{u}]_0])$ contribute to the trace over Nambu-Gorkov indices. Expanding the effective action in powers of $\phi$, and including the first non-trivial quadratic term, which is calculated to order $\Delta^2$, one obtains the effective action

$$\mathcal{S}^2 \Delta^2 = \sum_I \sum_{m} \int \frac{d^4k}{(2\pi)^4} \phi^m_I(k) \phi^m_I(-k) \Delta^I \Delta^I P^m_I(k),$$

(180)

where $k = k_2 - k_1$ is the four momentum of the phonon and

$$P^m_I(k) = i \sum_{j \neq i} \int \frac{d^4p}{(2\pi)^4} \left[ \frac{1}{(\bar{p} + \bar{\phi}^m_I + \bar{\phi} + \phi^m_I) \bar{\phi}^m_I + \phi^m_I - \bar{\phi} - \phi} \right] \left[ \frac{1}{(\bar{p} + \bar{\phi}^m_I + \bar{\phi} + \phi^m_I) \bar{\phi}^m_I - \bar{\phi} - \phi} \right].$$

(181)
where the trace is over Dirac indices. Integrating the expression above one obtains

\[ S_{\Delta^2} = \frac{1}{2} \int d^4x \sum_I \kappa_I \sum_m [\partial_0(n_I^m \cdot u_I)\partial_0(n_I^m \cdot u_I) - (n_I^m \cdot \partial)(n_I^m \cdot u_I)(n_I^m \cdot \partial)(n_I^m \cdot u_I)] , \]  

(182)

where

\[ \kappa_I = \frac{2\mu^2|\Delta_I|^2}{\pi^2(1 - z_q^2)} , \]  

(183)

where \( z_q \) was defined in Eq. (85).

This is the low energy effective action for phonons in any crystalline color superconducting phase, valid to second order in derivatives, to second order in the gap parameters \( \Delta_I \)'s and to second order in the phonon fields \( u_I \). Note that there are no terms in (182) that “mix” the different \( u_I(k) \), meaning that at this order the displacement of the various crystals can be considered separately. This follows from the fact that the Lagrangian conserves particle number for every flavor of quarks, which corresponds to symmetry under independent global phase rotations of quark fields of the three-flavors, meaning independent phase rotations of the three \( \Delta_I \). The effective action should be invariant under these rotations and hence \( \Delta_I \) can only occur in the combination \( \Delta_I^* \Delta_I \). The mixing between different fluctuation can only appear in higher order terms in \( \Delta_I \), e.g. by by terms like \( \mu^2|\Delta_I\Delta_J|^2\partial u_I\partial u_J/\delta\mu^2 \).

The coefficients \( \kappa_I \) determine the potential energy cost of a fluctuation and therefore they are related to the shear modulus. In particular in Mannarelli et al. (2007) it is found that for the the two favored structures, 2Cube45z and CubeX, the shear modulus is a \( 3 \times 3 \) non diagonal matrix in coordinate space with entries proportional to

\[ \nu_{\text{CQM}} = \frac{16}{9} \kappa . \]  

(184)

Evaluating \( \kappa \) yields

\[ \nu_{\text{CQM}} = 2.47 \text{ MeV/fm}^3 \left( \frac{\Delta}{10 \text{ MeV}} \right)^2 \left( \frac{\mu}{400 \text{ MeV}} \right)^2 \]  

(185)

and considering

\[ 350 \text{ MeV} < \mu < 500 \text{ MeV} . \]  

(186)

one has that

\[ 0.47 \text{ MeV/fm}^3 < \nu_{\text{CQM}} < 24 \text{ MeV/fm}^3 . \]  

(188)

The standard neutron star crust, which is a conventional crystal of positively charged ions immersed in a fluid of electrons (and, at sufficient depth, a fluid of neutrons) has a shear modulus approximately given by \( \nu_{\text{NM}} \) in Strohmayer et al. (1991)

\[ \nu_{\text{NM}} = 0.092 \text{ keV/fm}^3 \]  

(189)

thus, the crystalline quark matter is more rigid than the conventional neutron star crust by at least a factor of 20-1000. Note that the three-flavor CCSC phase is also a superfluid, by picking a phase its order parameter does indeed break the quark-number \( U(1)_B \) symmetry spontaneously. These results demonstrate that this superfluid phase of matter is at the same time a rigid solid and a superfluid.

2. Goldstone modes

The superfluid property of the CCSC phase derives from the existence of a massless NGB associated to the breaking of \( U(1)_B \). Actually, the crystalline condensate breaks the same global (and local) symmetries of the CFL phase (Alford et al., 1999e), leaving unbroken a global symmetry group: \( SU(3)_c \times U(1)_B \), see (3). Therefore, there
are nine NGBs due to the spontaneous breaking of $U(1)_B$ and of the chiral symmetry, but only the $U(1)_B$ boson ($H$-phonon) is massless. Indeed, the pseudo-scalars associated to chiral symmetry have mass, because this group is explicitly broken by quark mass terms and by chemical potential differences.

Alike the phonons described in the previous section, the NGBs discussed here describe the fluctuations of the condensate, see e.g. Eguchi (1976). The pseudo-NGBs are related to fluctuations in flavor space, while the $H$-phonon, $\phi$, describes fluctuations in baryonic number. Here we shall focus on the $H$-phonon and introduce it by means of the transformation $\psi \rightarrow U^\dagger \psi$ with $U = \exp \{i\phi/f\}$, where $f$ is its decay constant. Taking into account the unitary rotations, the HDET Lagrangian takes the form (Anglani et al. 2007)

$$\mathcal{L} = \frac{1}{2} \int \frac{d\Omega}{4\pi} \chi_A^+ \left( (V \cdot \ell + \delta \mu_A (v)) \delta_{AB} - \Xi_{AB} (V \cdot \ell - \delta \mu_A (-v)) \right) \chi_B + (L \rightarrow R),$$

where all the quantities have been defined in Sec. III.B.1, see Eq. (136), except $\Xi_{AB}$, which is given by

$$\Xi_{AB} = \Delta_I^r (r) \text{Tr} [\epsilon_I (F_A U^\dagger)^T \epsilon_I F_B U^\dagger].$$

Expanding the Lagrangian in the scalar field one derives three body and four body interaction vertices. Then, at the leading order in $\Delta$, there are two contributions to the $H$-phonon self-energy, due to a tadpole-like and a charm-like Feynman diagrams. Technically the calculation is very similar to the one sketched in the previous section and the result is that the charm-like diagram contributes to the self-energy by

$$S_{s.e.} = -i \frac{2}{f^2} \sum_{l=2}^3 \Delta_I^2 \sum_{q_I^m} \int \frac{d^4k}{(2\pi)^4} \phi(-k)\phi(k) \mathcal{P}_I^m (k_0, k),$$

with $k = (k_0, k)$ and $\mathcal{P}_I^m (k_0, k)$ corresponds to the HDET version of Eq. (191),

$$\mathcal{P}_I^m (k_0, k) = -2 \int \frac{d\Omega}{4\pi} \int \frac{d^4\ell}{(2\pi)^4} \left[ \frac{1}{(V \cdot \ell + \delta \mu - q_I^m \cdot v) [V \cdot (\ell + k) + \delta \mu - q_I^m \cdot v]} + \frac{1}{(V \cdot \ell - \delta \mu - q_I^m \cdot v) [V \cdot (\ell + k) - \delta \mu - q_I^m \cdot v]} \right] + \delta \mu \rightarrow -\delta \mu;$$

the tadpole-like contribution is given by

$$S_{tad} = i \frac{2}{f^2} \sum_{l=2}^3 \Delta_I^2 \sum_{q_I^m} \int \frac{d^4k}{(2\pi)^4} \phi(-k)\phi(k) \mathcal{P}_I^m (k_0 = 0, k = 0).$$

We do not provide further details of the calculation which can be found in Anglani et al. (2007), and we only quote the final result valid at small momenta:

$$\mathcal{L}(k) = \frac{1}{2} \phi(-k) \left[ k_0 I_0 - k_i k_j V_{ij} \right] \phi(k),$$

where

$$I_0 = -\frac{\mu^2}{f^2} \sum_{l=2}^3 \Delta_I^2 \sum_{q_I^m} \text{Re} \int \frac{d\Omega}{4\pi} \frac{1}{(\delta \mu - q_I^m \cdot v + i0^+)^2} + (\delta \mu \rightarrow -\delta \mu),$$

$$V_{ij} = -\frac{\mu^2}{f^2} \sum_{l=2}^3 \Delta_I^2 \sum_{q_I^m} \text{Re} \int \frac{d\Omega}{4\pi} \frac{v_i v_j}{(\delta \mu - q_I^m \cdot v + i0^+)^2} + (\delta \mu \rightarrow -\delta \mu).$$

Specializing these results to the case $\Delta_3 = \Delta_3 |q_I^2| = |q_3^2| = q$ (Rajagopal and Sharma, 2006), and requiring canonical normalization of the Lagrangian in Eq. (195), leads to

$$f^2 = \frac{4P \mu^2}{\pi} \frac{\Delta^2}{\delta \mu^2 (1 - \Delta^2)},$$

(198)
where $P$ is the number of plane waves comprising each crystal. Upon substituting this result in Eq. (197), and specializing it to $P = 1$, we find $V_{ij} = \text{diag}(0, 0, 1)$, where we have chosen $q_2$ and $q_3$ along the positive $z$-axis. For the two cubic structures corresponding to the values $P = 4$ (CubeX) and $P = 8$ (2cube45z) we find $V_{ij} = \delta_{ij}/3$, i.e. the velocity is isotropic and has the value $1/\sqrt{3}$.

The extension of the above procedure to the octet of pseudo-NGBs is straightforward, and is presented in Anglani et al. (2007). Here we just recall that one of the important results is that the squared masses of the pseudo-NGBs are always positive, thus kaon condensation for the considered crystalline phases is excluded, at least at the order $\Delta^2$.

**IV. ASTROPHYSICS**

The artificial creation of the low temperature-high density conditions appropriate for testing the properties of color superconductors is one of the challenging aims of high-energy experiments (Klahn et al., 2012). To date, however, the unique laboratories which may produce these extreme conditions are CSOs. Note that even if the relevant densities and low temperatures conditions were reached in a terrestrial laboratory, the conditions realized in a CSO are different from those produced in an accelerator, because in the former case quark matter is long-lived, charge neutral and in $\beta$-equilibrium.

In the inner core of a CSO, the extremely high density and low temperatures may favor the transition from nuclear to quark matter (Baym and Chin, 1974; Collins and Perry, 1975; Ivanenko and Kurdgelaidze, 1969, 1965) and in turn, to the color superconducting phase. Indeed, if matter is compressed at densities about a factor 5 larger than the density of an ordinary nucleus, a simple geometrical reasoning suggests that baryons are likely to lose their identity and dissolve into deconfined quarks (Weber, 1999). In this case compact (hybrid) stars featuring quark cores surrounded by a nuclear mantle would exist and they could provide a unique window on the properties of QCD at high baryon densities.

Whether matter in the interior of compact stars is compressed by gravity to such extreme densities is still an open question. Recent astronomical observation of very massive CSO (Demorest et al., 2010) seem to disfavor this possibility (Logoteta et al., 2012), but the results depend on the poorly known equation of state (EoS) of matter at high density, and the possibility that hybrid stars of about $2M_\odot$ have a quark matter core (Alford et al., 2005a) or a CCSC core (Ippolito et al., 2008) cannot be excluded.

Assuming that deconfined matter is present, it should be in a color superconducting phase, because its critical temperature (in weak coupling) is given by $T_c \approx 0.57\Delta$, and, as we have seen in the previous chapters we expect that $\Delta \approx 5 \div 100$ MeV. Lower values of the gap parameter, of the order of the keV up to few MeV, are realized in the spin-1 single flavor pairing phase (Alford et al., 2003, 1998; Buballa et al., 2003; Schafer, 2000; Schmitt, 2005; Schmitt et al., 2002). In any case, for the greatest part of the compact star lifetime, the temperature is much lower than this critical temperature and the color superconducting state is thermodynamically favored.

The presence of color superconducting phases in the interior of the CSOs can be probed by a number of astrophysical observables linked to the microscopic internal properties. Unfortunately, the available observational data do not allow us to infer in a unique way the internal structures of CSOs, but the investigation of the astrophysical signatures can help to pave the path connecting theoretical models to real observations. Searching for “the smoking gun” of deconfined matter is a hard task that should take into account several results about mass-radius relations, equations of state, thermodynamic observables, $r$-modes, glitches and magnetic fields. In the following we will present a brief state-of-the-art of some astrophysical implications of color superconductivity and in particular, of a crystalline color superconducting core.

**A. Gravitational Waves**

As reported in Section III.D the crystalline color superconducting phase is extremely rigid, with a shear modulus larger than the one of standard neutron crust by at least a factor 20. The existence of such a rigid core within neutron stars may have a variety of observable consequences. If some agency (for example magnetic fields not aligned with the rotation axis) could maintain the rigid core in a shape that has a nonzero quadrupole moment, gravity waves would be emitted by a spinning CSO. The size of the distortion can be measured by the equatorial ellipticity

$$\epsilon = \frac{I_{xx} - I_{yy}}{I_{zz}},$$

(199)
FIG. 17 (Color online). Exclusion plots in the $\Delta$-$\mu$ plane for the Crab (left panel) and the pulsar J2124-3358 (right panel) obtained assuming that these stars have mass $M = 1.4 \ M_\odot$, radius $R = 10$ km, uniform mass distribution and are maximally strained. The area above each line is excluded by the corresponding model. The rectangular box is the theoretically allowed $\mu$ region of Haskell et al. (2007), and the results of the S5 LIGO scientific run (Abbott et al., 2010) and considering the case of $\sigma_{\text{max}} = 10^{-3}$ (upper heavy red solid line) and $\sigma_{\text{max}} = 10^{-2}$ (lower light blue solid line). Right panel: Dependence of the exclusion region on the size of the crystalline core for the pulsar J2124-3358 obtained using the S5 LIGO data (Abbott et al. 2010) and considering the case of $\sigma_{\text{max}} = 10^{-3}$. The three lines correspond to different values of $x = R_c/R$, where $R_c$ is the radius of the crystalline color superconducting core.

which is defined in terms of the star principal moments of inertia. Standard neutron stars are expected to have maximum ellipticity of the order of $\epsilon_{\text{max}} \approx 10^{-6}$, meaning that this is the largest deformation sustainable by the crust before breaking. For the three-flavor crystalline phases considered in Section III.D deformed to the maximum extent allowed by the shear modulus, the maximum equatorial ellipticity sustainable could be as large as $\epsilon_{\text{max}} \approx 10^{-2}$ (Haskell et al. 2007; Lin, 2007). Different models of exotic stars composed of quark-clusters have a magnitude of the maximum ellipticity which is smaller than the CCSC phase by two orders of magnitude (Owen, 2005).

The gravitational waves emitted by an optimally oriented compact star spinning at a frequency $\nu$, would produce a strain on Earth-based interferometric detectors (such as LIGO, VIRGO, GEO600 and TAMA300) given by

$$h_0 = \frac{16\pi^2 G I_{zz} \nu^2}{c^2 r},$$  \hspace{1cm} (200)

where $G$ is the gravitational constant and $r$ is the distance. If the standard value $I_{zz} = 10^{38} \text{ kg m}^2$ is assumed, the above equation allows to relate the ellipticity to the spinning frequency and distance. The LIGO non-detection of gravity waves from nearby neutron stars already limits the parameter space of the model. A first analysis of CSOs which include the crystalline phase was made in Haskell et al. (2007), Lin (2007) using the results of the S3/S4 LIGO scientific run (Abbott et al. 2007) and in Knippel and Sedrakian (2009) using the EoS discussed in Sec. IV.E and the results of the S5 LIGO scientific run (Abbott et al. 2008) on gravitational wave emission from the Crab pulsar. In Fig. 17 we present the exclusion plots for the Crab (left panel) and for the pulsar J2124-3358 (right panel), assuming maximum strain and that these stars are constituted by incompressible and uniformly distributed matter. The dashed lines on the left panel are those obtained in Lin (2007) for the Crab pulsar using the S3/S4 LIGO data assuming that the whole star is in the crystalline color superconducting phase. We have determined the solid lines using the S5 LIGO data of Abbott et al. (2010). The parameter space results to be extremely reduced and even more severe constraints can be obtained from the pulsar J2124-3358, which leads to the exclusion plots reported on the right panel of Fig. 17 considering three different values of $x = R_c/R$, where $R_c$ is the CCSC core radius.

Unfortunately, the maximum deformation of a star depends as well on the breaking strain, $\sigma_{\text{max}}$, which measures the largest shear stress deformation sustainable by a rigid body before breaking. For standard neutron stars it is assumed that $10^{-5} \leq \sigma_{\text{max}} \leq 10^{-2}$. The results shown on the left panel of Fig. 17 have been obtained with $\sigma_{\text{max}} = 10^{-3}$ (upper red dashed line) and $\sigma_{\text{max}} = 10^{-2}$ (lower blue dashed line). The plots on the right panel are instead obtained
assuming $\sigma_{\text{max}} = 10^{-3}$. Considering lower values of $\sigma_{\text{max}}$ would move the curves up. It is important to remark that there seems to be little motivation for assuming that the breaking strain of the CCSC phase is of the same order of standard neutron star crust and only a study of the breaking mechanism of the crystalline pattern may allow to estimate this parameter.

From Fig. 17 we infer that the S5 LIGO data give some hints about the properties of quark matter (under the assumption discussed above), because if it is present it should have a breaking strain $\sigma_{\text{max}} \leq 10^{-3}$ and that if $\sigma_{\text{max}} \sim 10^{-3}$, it is unlikely that the pulsar J1214-3358 is a crystalline quark star; although, it could still contain a small crystalline quark core. With Advanced LIGO and Virgo detectors the spin-down limit on GW emission from known CSOs will reach $\epsilon \approx 10^{-5}$, and objects with ellipticity of the order of $10^{-6}$ would be detectable up to the Galactic center \cite{Palomba2012}. These observation will put even more severe limitations on the parameters of the crystalline phases and on the EoS of highly compressed matter.

### B. Glitches

The rigidity of the crystalline phase may also be put in relation with the anomalies in the frequency of rotation of CSOs \cite{Alford2001,Mannarelli2007}. A spinning neutron star observed as a pulsar gradually spins down as it loses rotational energy to electromagnetic radiation. But, every once in a while the angular velocity at the crust of the star is observed to increase suddenly in a dramatic event called a glitch.

Pulsar glitches are rare events first observed in the Vela pulsar \cite{Radhakrishnan1969,Reichley1969} and in the Crab pulsar \cite{Bovnton1969,Richards1969} which manifest in a variety of sizes \cite{Espinoza2011}, with an activity changing with time and peaking for pulsars with a characteristic age of about 10 kyr. Although a number of models have been proposed including two-fluid models \cite{Alpar1977,Anderson1975} and “crust-quake” models \cite{Baym1973,Ruderman1969}, these phenomena still remain to be well understood, and their underlying mechanism is not yet completely clear. The standard explanation of pulsar glitches \cite{Alpar1977,Anderson1975} requires the presence of two basic ingredients: a superfluid in some region of the star and a rigid structure that can pin the vortices without deforming when the vortices are under tension. As a spinning pulsar slowly loses angular momentum by radiation emission, the superfluid vortices tend to move toward the surface of the star — this happens because the angular momentum of a superfluid is proportional to the density of vortices— but, if the vortices are pinned to a rigid structure, they cannot move and after, as time passes, the lag between the superfluid component of the star and the rest of the star increases. This state persists until the stress exerted by the pinned vortices reaches a critical value, then there is a sudden avalanche in which vortices unpin, move outwards reducing the angular momentum of the superfluid, and then re-pin. As this superfluid suddenly loses angular momentum, the rest of the star, including in particular the surface whose angular velocity is observed, speeds up.

The crystalline phases are rigid as well as superfluid, and it is reasonable to expect that the superfluid vortices within the rotating CCSC matter will have lower free energy if they are centered along the intersections of the nodal planes of the underlying crystal structure, \textit{i.e.} along lines along which the condensate vanishes even in the absence of a rotational vortex. A crude estimate of the pinning force on vortices within CCSC quark matter \cite{Mannarelli2007} indicates that it is comparable to that on neutron superfluid vortices within a conventional neutron star crust \cite{Alpar1984}. So, the basic requirements for superfluid vortices pinning to a rigid structure are all present. The central questions that remain to be addressed are the explicit construction of vortices in the crystalline phase and the calculation of their pinning force, as well as the calculation of the timescale over which sudden changes in the angular momentum of the core are communicated to the surface, presumably either via the common electron fluid or via magnetic stresses. Much theoretical work remains before the hypothesis that pulsar glitches originate within a CCSC core is developed fully enough to allow it to confront data on the magnitudes, relaxation timescales, and repeat rates of the observational data.

The same mechanism outlined above might work for a star with a CFL inner core and an outer core in the CCSC phase. In this case superfluid vortices lying in the CFL phase are pinned to the periodic structure of the crystalline phase.

### C. Cooling and Urca processes

Neutron stars cool down by neutrino emission and by photon emission, the latter dominating at late ages ($t \gtrsim 10^6$ yr). The cooling rate of a CSO may give information on its interior constitution, because different phases of hadronic
matter cool down in a rather different way, see e.g. Pethick (1992) for a brief review.

The neutron star cooling is governed by the following differential equation:

$$\frac{dT}{dt} = \frac{L_\nu + L_\gamma}{V_n c_s^4 + V_q c_s^4} = \frac{V_n \varepsilon_\nu^0 + V_q \varepsilon_q^0 + L_\gamma}{V_n c_s^4 + V_q c_s^4},$$

(201)

where \( T \) is the inner temperature at time \( t \) and \( L_\nu \) and \( L_\gamma \) are respectively neutrino and photon luminosities, i.e. heat losses per unit time. After a very short epoch, when the temperature of the compact star is of the order of \( \sim 10^{11} \text{K} \) and neutrinos are trapped in the stellar core (Prakash et al., 2001; Ruster et al., 2006b; Shapiro and Teukolsky, 1983; Steiner et al., 2002), the temperature drops and the mean free path of neutrinos becomes larger than the star radius. For this reason the neutrino luminosity is obtained multiplying the emissivity by the corresponding volume, however one must distinguish between the emissivity \( \varepsilon_n^0 \) of nuclear matter, which is present in the volume \( V_n \), from the emissivity due to quarks, \( \varepsilon_q^0 \) if they are present in some volume \( V_q \). \( \varepsilon_n \) and \( \varepsilon_q \) denote specific heats of the two forms of hadronic matter. We assume a common inner temperature \( T \), which is appropriate for sufficiently old compact stars (Lattimer et al., 1994), see e.g. Ho et al. (2012) for a recent discussion.

The luminosity by photon emission is instead a surface effect and can be estimated by the black-body expression

$$L_\gamma \sim 4\pi R^2 \sigma T^4,$$

(202)

where \( R \) is the radius of the star, \( \sigma \) is the Stefan-Boltzmann constant and the surface temperature is given by

$$T_s \simeq 0.87 \times 10^6 \left( \frac{g_s}{10^4 \text{cm}^2/\text{s}^2} \right)^{1/4} \left( \frac{T_b}{10^8 \text{K}} \right)^{0.55} \text{K},$$

(203)

see Gudmundsson et al. (1982) and Page et al. (2004), where \( T_b \simeq T \) is the temperature at the basis of the stellar envelope and \( g_s = G_N M/R_s^2 \) is the surface gravity.

For what concerns the neutrino luminosity one has to consider the relevant weak processes. When kinematically allowed, direct Urca processes are the most efficient cooling mechanism for a CSO in the early stage of its lifetime (Shapiro and Teukolsky, 1983). However, the neutrino emission via the (nuclear) direct Urca processes, \( n \rightarrow p + e^- + \nu_e \) and \( e^- + p \rightarrow n + \nu_e \), is only allowed for certain EoS (Lattimer et al., 1991) having a sufficiently large proton abundance to guarantee energy-momentum conservation (Bahcall and Wolf, 1963; Chiu and Salpeter, 1964). Therefore, considering nuclear matter, only modified Urca processes are in general considered, where a bystander particle allows energy-momentum conservation. The resulting cooling is less rapid and the emissivity turns out to be

$$\varepsilon_\nu^0 = (1.2 \times 10^4 \text{erg cm}^{-3} \text{s}^{-1}) \left( \frac{n}{n_0} \right)^{2/3} \left( \frac{T}{10^7 \text{K}} \right)^8,$$

(204)

see e.g. Shapiro and Teukolsky, 1983, much smaller than the emission rate \( \varepsilon_\nu^0 \sim T^6 \) due to direct Urca processes. Here \( n \) is the number density and \( n_0 = 0.16 \text{ fm}^{-3} \) is the nuclear equilibrium density.

These considerations apply to stars containing standard nuclear matter; faster cooling can be determined by the presence of a pion (Bahcall and Wolf, 1967; Maxwell et al., 1977; Muto and Tatsumi, 1988) or kaon (Brown et al., 1988) condensate. Moreover, if the central region of the star consists of deconfined quark matter direct Urca processes involving quarks, i.e. the processes \( d \rightarrow u + e^- + \bar{\nu}_e \) and \( u + e^- \rightarrow d + \nu_e \), may take place and largely contribute to the cooling rate. It has been shown by Iwamoto (Iwamoto, 1980, 1981, 1982) that quark direct Urca processes are kinematically allowed and the corresponding emission rate for massless quarks is of the order \( \alpha_s T^6 \), where \( \alpha_s \) is the strong coupling constant. This result is valid if quark matter is a normal Fermi liquid, but in the color superconducting phase the expression above is not correct because quarks form Cooper pairs and fermionic excitations are gapped. If the color superconductor is uniform, the corresponding neutrino emissivity and specific heat are suppressed by a factor \( e^{-\Delta/T} \), where \( \Delta \) is the quasiparticle gap. This suppression is particularly strong in the CFL phase, because \( \Delta \) is large and all quarks are paired.

However, the ground state of quark matter in realistic conditions might not be the CFL phase, but the 2SC or the CCSC phase. In the following we shall evaluate the contribution of the latter phase to the neutrino emissivity and specific heat.

1. Neutrino emissivity

The transition rate for the \( \beta \)-decay of a down quark \( d_\alpha \), of color \( \alpha = r, g, b \), into an up quark \( u_\alpha \)

$$d_\alpha(p_1) \rightarrow \bar{\nu}_e(p_2) + u_\alpha(p_3) + e^-(p_4)$$

(205)
\[ W_n = V(2\pi)^4 \delta^4(p_1 - p_2 - p_3 - p_4)|\mathcal{M}|^2 \prod_{i=1}^{4} \frac{1}{2E_i V}, \]  

(206)

where \( V \) is the available volume and \( \mathcal{M} \) is the invariant amplitude. Neglecting quark masses the squared invariant amplitude averaged over the initial spins and summed over spins in the final state is

\[ |\mathcal{M}|^2 = 64G_F^2 \cos^2 \theta_c (p_1 \cdot p_2)(p_3 \cdot p_4), \]  

(207)

where \( G_F \) is the Fermi constant and \( \theta_c \) the Cabibbo angle; we will neglect the strange-quark \( \beta \)-decay whose contribution is smaller by a factor of \( \tan^2 \theta_c \) in comparison with (204). Since for relatively aging stars there is no neutrino trapping, the neutrino momentum and energy are both of the order \( k_B T \). The magnitude of the other momenta is of the order of the corresponding Fermi momenta \( p_1 \sim p_F^1 \sim \mu, p_3 \sim p_F^3 \sim \mu \) and \( p_4 \sim p_F^4 \sim \mu_v \), which is smaller, but still sizable (see the discussion in Sec. III). It follows that the momentum conservation can be implemented neglecting \( p_2 \) and one can depict the 3-momentum conservation for the decay (205) as a triangle (Iwamoto, 1980, 1981, 1982) having for sides \( p_1, p_3 \) and \( p_4 \). It follows that we can approximate

\[ (p_1 \cdot p_2)(p_3 \cdot p_4) \simeq E_1E_2E_3E_4(1 - \cos \theta_{12})(1 - \cos \theta_{34}), \]  

(208)

where \( E_j \) are the energies and \( \theta_{12} \) (resp. \( \theta_{34} \)) is the angle between momenta of the down quark and the neutrino (resp. between the up quark and the electron). In the color superconducting phase one has to take into account that the neutrino emissivity

\[ \varepsilon_\nu^V = \sum_{\alpha=r,g,b} \varepsilon_\nu^\alpha = \frac{2}{V} \prod_{j=1}^{4} \int \frac{d^3 p_j}{(2\pi)^3} E_2 W_n(n(p_1) [1 - n(p_3)] [1 - n(p_4)] B_{u,n}^2(p_1)B_{d,n}^2(p_3)), \]  

(209)

depends on the Bogolyubov coefficients \( B_{u,n} \) and \( B_{d,n} \) which are functions of the quasiparticle dispersion laws (Alford et al., 2005b). In Eq. (209) the quark thermal equilibrium Fermi distributions

\[ n(p_j) = \left( 1 + \exp \frac{E_j(p_j) - \mu_j}{T} \right)^{-1}, \]  

(210)

appears because strong and electromagnetic processes establish thermal equilibrium much faster than weak interactions. The overall factor of 2 in Eq. (209) keeps into account the electron capture process.

The cooling of the CCSC matter with condensate (150) was studied in Anglani et al. (2006), and in this case the largest contribution to the emissivity stems from the phase space region around the quark gapless modes, while the relevant momentum for the electron is its Fermi momentum, thus we have that

\[ \int d^3 p_1 \int d^3 p_3 \int d^3 p_4 \simeq \int \mu_\nu^2 dp_4 d\Omega_4 P_1^2 dp_1 d\Omega_1 P_3^2 dp_3 d\Omega_3, \]  

(211)

with \( d\Omega_j = \sin \theta_j \, d\theta_j \, d\phi_j \) and \( P_1 \) (resp. \( P_3 \)) is the quark down (resp. quark up) momentum where the corresponding quasi-particle energy vanishes. The gapless momenta \( P_1 \) and \( P_3 \) depend on the angle \( \vartheta_j \) that quark momenta form with the pair momentum \( 2q \). In order to simplify the expression of the integral in Eq. (209) we expand around the gapless modes, \( E_j(p) \simeq \mu_j + v_j (p - P_j) \) (for \( j = 1 \) and 3), with the quasiparticle velocity given by

\[ v_j = \left. \frac{\partial E_j}{\partial p} \right|_{p=P_j}. \]  

(212)

In the three-flavor case the dispersion law of each quasiparticles has from one to three gapless modes, thus one has to expand the corresponding dispersion laws around each gapless momentum.

Employing the above approximations the neutrino emissivity for each pair of gapless momenta \( P_1, P_3 \), can be written as

\[ \varepsilon_\nu^V \simeq \frac{G_F^2 \cos^2 \theta_c \mu_\nu^2 T^6}{32\pi^8} I \prod_{j=1}^{4} d\Omega_j \frac{P_1^2 P_2^2 P_3^2 B_{u,n}^2(P_1)B_{d,n}^2(P_3)}{|v_1| |v_3|} \delta(3)(p_1 - p_3 - p_4 - q)(1 - \cos \theta_{12})(1 - \cos \theta_{34}), \]  

(213)

where \( I = \frac{457\pi^6}{3048}. \) Some of the angular integrations appearing in (213) can be performed analytically, see Anglani et al. (2006) for more details, and the numerical computation can be reduced observing that even if each quasiparticle
shown. The solid black line refers to Model I; the dashed red line refers to Model II; the dotted blue line refers to $R_{\text{MeV}}$. The outer part of the star is made of unpaired nuclear matter. Since the value of the gap parameter in the "nuclear" matter (neutrons, protons and electrons) with mass $M_{\beta}$ models comprising neutral and in the crystalline phase with gap parameter given in (150), with $\Delta_{2}=10$ km and uniform number density $R_{\text{MeV}}$ of unpaired quark matter with $\beta$ (Shapiro and Teukolsky, 1983): where $p_{\text{disp}}$ dispersion law is characterized by various gapless momenta, not all of them satisfy the conservation of momentum $p_{1}-p_{2}-p_{3}-q=0$.

From the expression (213) one can deduce that the largest contribution to the emissivity is due to blue quarks, that is to the process in Eq. (206) with $\alpha=b$. The reason is that according to the Ginzburg-Landau analysis of Casalbuoni et al. (2005a) the gap parameter $\Delta_{1}$ vanishes and therefore the down blue quark is ungapped (see e.g. the discussion after Eq. (121)). Then, the corresponding dispersion law is $E_{1}(p)=p$, with gapless momentum $P_{1} \simeq \mu$ independent of $\theta_{1}$, mixing coefficient $B_{d}(P_{1}) = 1$ and $\epsilon_{1}=1$. The up-blue quark is instead paired with the strange-red quark with gap parameter $\Delta_{2} = \Delta$ and has two gapless momenta depending on $\theta_{3}$, see Anglani et al. (2000) for an explicit expression. Of these two gapless momenta one satisfies the momentum conservation and thus the decay of a down-blue quark in an up-blue quark is not suppressed with respect to the analogous decay in ungapped quark matter, giving the leading contribution to the emissivity. Indeed, the case of quarks with colors red and green can be treated in a similar way, but for these colors neither the down nor the up quarks are unpaired and the corresponding processes are thus suppressed. These results are confirmed by the numerical analysis of Anglani et al. (2006).

2. Specific heats

As already discussed in Sec. II.D for the two-flavor case, at low temperature the largest contribution to specific heat is determined by the fermionic quasi-particles. The specific heat of the three-flavor crystalline phase is given by the same formal expression given in Eq. (67), but with the quasiparticle dispersion laws of the three-flavor CCSC phase. The computation can be simplified following the same reasoning used above: the contributions of gapped modes are exponentially suppressed and each gapless mode contributes by a factor $\propto T$. This result follows from the evaluation of the integral in Eq. (67) employing the saddle point method and assuming that the quasi-particle dispersion laws are linear close the gapless momenta. Then, also the angular integral can be simplified because the dispersion laws are gapless only in a restricted angular region. The numerical analysis confirms the above results, see Anglani et al. (2006) for a discussion and for an expression of the specific heat.

For unpaired nuclear matter and unpaired quark matter the contribution of each fermionic species can be approximated by the fermionic ideal gas result

$$c_{V} = \frac{k_{B}^{2} T}{m_{F}^{2} c} \sqrt{m_{F}^{2} c^{2} + (p_{F})^{2}},$$

(214)

where $m$ and $p_{F}$ are the appropriate fermionic mass and Fermi momentum, respectively. For unpaired nuclear matter, the three species are neutrons, protons and electrons with Fermi momenta evaluated as in neutral matter in weak equilibrium (Shapiro and Teukolsky, 1983):

$$p_{F}^{n} \simeq (340 \text{ MeV}) \left(\frac{n}{n_{0}}\right)^{1/3}, \quad p_{F}^{p} = p_{F}^{e} \simeq (60 \text{ MeV}) \left(\frac{n}{n_{0}}\right)^{2/3}.$$  

(215)

For neutral unpaired quark matter in weak equilibrium, the nine quark species have Fermi momenta independent of color and given by $p_{F}^{d} = \mu + \frac{m_{d}^{2}}{2\mu}$, $p_{F}^{u} = \mu - \frac{m_{u}^{2}}{2\mu}$ and $p_{F}^{s} = \mu - \frac{5m_{s}^{2}}{12\mu}$.

3. Cooling by neutrino emission

In order to evaluate the effect on the compact star cooling of the CCSC phase we consider three different star toy models comprising neutral and $\beta$ equilibrated matter (Anglani et al. 2006). Model I is a star consisting of unpaired “nuclear” matter (neutrons, protons and electrons) with mass $M = 1.4M_{\odot}$ (where $M_{\odot}$ is the solar mass), radius $R = 12$ km and uniform number density $n = 1.5n_{0}$. Model II is a star containing a core of radius $R_{c} = 5$ km of unpaired quark matter with $\mu = 500$ MeV, with a mantle of unpaired nuclear matter with uniform density $n$. Assuming a star mass $M = 1.4 M_{\odot}$ from the solution of the Tolman-Oppenheimer-Volkov equations one gets a star radius $R = 10$ km. Model III is a compact star containing a core of electric and color neutral three-flavor quark matter in the crystalline phase with gap parameter given in (150), with $\Delta_{2} = \Delta_{3} \simeq 6$ MeV and $\mu = 500$ MeV, $m_{d}^{2}/\mu = 140$ MeV. The outer part of the star is made of unpaired nuclear matter. Since the value of the gap parameter in the Model III is small, the radius of the CSO and of the quark core do not differ appreciably from those of a star with a core of unpaired quark matter, i.e. $R_{c} = 5$ km and $R = 10$ km (also in these cases $M = 1.4M_{\odot}$).

In Fig. 18 the cooling curves of the surface temperature as a function of time for the various models of star are shown. The solid black line refers to Model I; the dashed red line refers to Model II; the dotted blue line refers to
FIG. 18 (Color online). Surface temperature, in Kelvin, as a function of time, in years, for three toy models of pulsars. Solid black curve refers to model I; dashed red curve refers to model II; dotted curve blue curve refers to model III. Model I is a neutron star formed by nuclear matter with uniform density $n = 0.24 \text{ fm}^{-3}$ and radius $R = 12 \text{ Km}$; model II corresponds to a star with $R = 10 \text{ km}$, having a mantle of nuclear matter and a core of radius $R_c = 5 \text{ Km}$ of unpaired quark matter, interacting via gluon exchange; model III is like model II, but in the core there is quark matter in the LOFF state, see text for more details. All stars have $M = 1.4 M_\odot$. Parameters for the core are $\mu = 500 \text{ MeV}$ and $m_s^2/\mu = 140 \text{ MeV}$. From Anglani et al. (2006).

Model III. Regarding the Model III, note that with increasing values of $M_s$ the neutrino emissivity decreases (this is due to the fact that $\Delta$ decreases as one approaches the second order phase transition to the normal state) and in this case the quark matter in the star tends to become an unpaired Fermi liquid, for which the description of Iwamoto which includes Fermi liquid effects, should be used. For unpaired quark matter we use $\alpha_s \simeq 1$, the value corresponding to $\mu = 500 \text{ MeV}$ and $\Lambda_{\text{QCD}} = 250 \text{ MeV}$. The use of perturbative QCD at such small momentum scales is however questionable. Therefore the results for the Model II should be considered with some caution and the curve is plotted only to allow a comparison with the other models. In particular the similarity between the LOFF curve and the unpaired quark curve follows from the fact that in the CCSC phase the quasi-particle dispersion laws are linear and gapless, so that the scaling laws $c_V \sim T$ and $\varepsilon_\nu \sim T^6$ are analogous to those of the unpaired quark matter.

From Fig. 18 we can see stars with a CCSC core cool down faster than ordinary neutron stars. This might have interesting phenomenological consequences because observational results on the cooling of pulsars are being accumulated at an increasing rate. Some data indicate that stars with an age in the range $10^3 - 10^4$ years have a temperature significantly smaller than expected on the basis of the modified Urca processes. It is difficult however to infer, from these data, predictions on the star composition, as these stars may have different masses. But, quite recently, the rapid cooling of the neutron star in Cassiopeia A (Heinke and Ho, 2010; Shternin et al., 2011) has been observed and explained as an effect of neutron superfluidity (Page et al., 2004; 2011; Shternin et al., 2011); it would be interesting to check whether a similar behavior might be due to the presence of the CCSC phase. Note that the rapid cooling of neutron superfluidity is due to difermion pair-breaking effect, which enhances the emissivity of nuclear matter; the presence of blocking regions has a very similar effect for quark matter paired in a crystalline pattern.

The analysis presented above is not conclusive because the thermodynamically favored phase should have a crystalline pattern and not the simple modulation of (150), for which, however, the identification of the quasiparticle dispersion laws is still lacking. Nevertheless some qualitative assessments can be made from the obtained results. Slow cooling is typical of stars containing only nuclear matter or of stars with a uniform color superconducting phase (like CFL), thus the observed cooling rate of the neutron star in Cassiopeia A (Heinke and Ho, 2010; Shternin et al., 2011) seems to disfavor both possibilities. This is compatible with the result that for intermediate densities the quark normal state and the CFL are less favored than the CCSC phase for a wide range of values of the baryonic chemical potential $\mu$ and the strange quark mass $M_s$. Thus the presence of crystalline quark matter should be accompanied by a rapid cooling of the CSO. Note that the fast cooling of relatively young stars with a CCSC core is a consequence of the scaling laws for neutrino emissivity and specific heat. They depend on the existence of gapless points and follow from the existence of blocking regions in momentum space. Since this property is typical of any CCSC phase, independently of detailed form of the condensate, a rapid cooling should be appropriate not only for the simple ansatz assumed in Eq. (150), but also for condensates consisting of more plane waves. Note however, that the results reported
above do not take into account the transport properties inside the star, because it is assumed that at any time the external temperature is a given function of the internal one. It would be interesting to include in the analysis the transport properties to have a detailed simulations of the CSO cooling, as e.g. in [Ho et al. (2012)].

D. Mass-radius relation

Since any phase transition leads to a softening of the equation of state [Haensel (2003); Lattimer and Prakash (2001)], it is in general expected that hybrid stars — having a quark matter core and an envelope of baryonic matter — should have mass $M \lesssim 1.7 M_\odot$, see e.g. [Alford and Reddy (2003); Buballa et al. (2004); Maieron et al. (2004)]. For CSOs with larger masses, the deconfinement phase transition from baryonic to quark matter would reduce the central pressure to the point of instability towards black hole collapse.

Some evidence for massive neutron stars with $M \sim 2 M_\odot$ has been inferred from various astronomical observations: A compact star may exist in the LMXB (Low Mass X-ray Binary) 4U 1636-536 with $2.0 \pm 0.1 M_\odot$ [Barret et al. (2005)]. A measurement on the pulsar PSR B1516+02B in the Globular Cluster M5 gave $M = 2.08 \pm 0.19 M_\odot$ [Freire et al. (2008)]. The millisecond pulsar J1614-2230 has a mass $(1.97 \pm 0.04) M_\odot$ accurately measured by Shapiro delay [Demorest et al. (2010)]. Although these observations seem to disfavor the presence of quark matter [Logoteta et al. (2012)], the details of a stability analysis depend on the theoretical model employed for the description of the hadronic phase, of deconfinement, and of the CSC matter. We shall show below that in the presence of the crystalline phase large masses can indeed be reached [Ippolito et al. (2008)]. The drawback, as we shall see, is that if $M \simeq 2 M_\odot$ CSOs have a CCSC core, then ordinary $M \simeq 1.4 M_\odot$ CSOs unlikely have a CCSC core.

The description of quark matter relies on three different kind of models: the MIT bag model, the NJL models and the chromodielectric models. The results obtained within these three models may differ in a sizable way, and do as well depend on the detailed form of the model considered.

Recent phenomenological studies of hybrid stars based on the MIT bag model where carried out using a generic parameterization of the quark matter EoS in [Alford et al. (2005a)] and it was shown that hybrid stars may actually masquerade as neutron stars. In [Alford et al. (2005a)] non-perturbative QCD corrections to the equation of state of the Fermi gas were parameterized in a rather general way, with reasonable estimates [Fraga et al. (2001)]. Taking into account these corrections, stable hybrid stars containing CFL quark matter may exist with maximum mass of about $2 M_\odot$.

Various studies of very massive hybrid stars within the three-flavor NJL model displayed a general instability towards collapse into a black hole [Buballa et al. (2004)]. Stable stars featuring the 2SC phase were obtained with typical maximum masses $M \sim 1.7 M_\odot$, assuming reasonable values of the constituent quark masses [Shovkovy et al. (2003)] or by replacing the hard NJL cut-offs by soft form factors with parameters fitted to a certain set of data [Blaschke et al. (2007); Grigorian et al. (2004)]. Heavier objects can be obtained if a repulsive vector interaction is introduced in the NJL Lagrangian [Klahn et al. (2007)], which makes the equation of state stiffer, but at the same time, reducing the amount of deconfined quark matter in the core of the star.

Another source of uncertainty comes from the nuclear equation of state at high densities, which can be constructed starting from a number of different principles [Sedrakian (2007); Weber (1999)] and a large number of EoS have been proposed for hybrid star configurations. In the analysis presented below, of the considered EoS, only the stiffest were found to be admissible for phase equilibrium between nuclear and CCSC matter.

The inclusion of hyperonic matter in the EoS of compact stars although reasonable is certainly troublesome, because the nucleon-hyperon and hyperon-hyperon interactions are not well known, even though some progress has been done mainly by lattice simulations [Aoki et al. (2012)], and introduce new parameters. Hyperons would certainly soften the equation of state [Glendenning and Kettner (2000)], making the comparison with the observed $2 M_\odot$ CSOs problematic [Lattimer and Prakash (2007)].

1. Matching the equation of states

The self-consistent computation of the strange quark mass given in [Ippolito et al. (2007)] consents to obtain the pressure as a function of the quark chemical potential $\mu$. Thus, varying the quark chemical potential, the phase equilibrium between the confined and the color-superconducting phase can be constructed.

A normalization of the quark pressure in the NJL model is obtained by requiring that the pressure vanishes at zero density and temperature [Buballa and Oertel (1999); Sandin and Blaschke (2007)]. In the terminology of the MIT bag model, this is equivalent to a subtraction of the bag constant from the thermodynamic potential [Alford et al. (2003)].
Since the value of the bag constant is related to confinement, which is absent in the NJL model, it appears reasonable changing its value, and hence the normalization of the pressure. The simplest option is to consider the case of a constant shift in the asymptotic value of the pressure; alternatives include the use of form factors for the bag constant (Grigorian et al., 2004) or the use of the Polyakov loop at nonzero density (Fukushima, 2004; Ratti et al., 2006).

Regarding the matching between the nuclear and quark matter EoS, it can be performed without changing the bag constant in the strong coupling limit, corresponding to $G_D/G_S = 1$ (Ruester et al., 2005; Sandin and Blaschke, 2007) (see Eq. (15) and the discussion after Eq. (29)). The transition from the confined phase to quark matter happens when the baryo-chemical potential at which the pressures of the two phases are equal, meaning that the chemical potential curves $P(\mu)$ for these phases cross. This happens only in the strong coupling case, while at intermediate coupling and weak coupling, corresponding respectively to $G_D/G_S \sim 0.75$ and $G_D/G_S \lesssim 0.7$, the $P(\mu)$ curves for the equations of state of nuclear and quark matter do not cross; the models are thus incompatible, meaning that they cannot describe the desired transition between nuclear and quark matter.

In the following we shall report on the analysis of Ippolito et al. (2008), where the low-density equation of state of nuclear matter and the high-density equation of state of the CCSC matter are matched at an interface via the Maxwell construction. In the analysis the high-density regime is described by two EoS for crystalline color superconductivity which differ by the normalization of pressure at zero density (or, equivalently, the value of the bag constant). Of the various nuclear EoS considered only two are suitable to match with the CCSC equations of state. These are shown in Fig. 19 as dashed blue and dotted blue lines. The selected equations of state are the hardest two in the collection (Weber, 1999; Weber et al., 2007).

Three different hybrid models are thus considered. For the models A and A1 the nuclear (low density) EoS is the same and corresponds to the curve labeled as “nuclear 1” in Fig. 19. For the model B, the nuclear (low density) EoS corresponds to the one labeled as “nuclear 2” in Fig. 19. The model A1 has a high density EoS determined by the CCSC matter and normalized such that the pressure vanishes at zero-density. For the models A and B the high density behavior is determined by the same CCSC EoS used for model A, but the zero density pressure is shifted by an amount $\delta p = 10$ MeV/fm$^3$. This is equivalent to a variation of the bag constant whose value is related to confinement, which is unspecified within the NJL model, and is uncertain in general. (Another possibility is to set $\delta p = 0$, but varying the value of the constituent masses of the light quarks in the fit of the parameters of the NJL model; for small values of the light quark masses the matching between quark and nuclear equations of state is facilitated (Buballa et al., 2004)).
Note that because of the Maxwell construction of the deconfinement phase transition, there is a jump in the density at constant pressure as illustrated in the left panel of Fig. 19.

2. Results

Given the equation of state, the spherically symmetric solutions of Einstein’s equations for self-gravitating fluids are given by the well-known Tolman-Opennheimer-Volkoff equations (Oppenheimer and Volkoff, 1939; Tolman, 1939).

A generic feature of these solutions is the existence of a maximum mass for any equation of state; as the central density is increased beyond the value corresponding to the maximum mass, the star becomes unstable towards collapse to a black hole. One criterion for the stability of a sequence of configurations is the requirement that $dM/d\rho_c > 0$, meaning that the mass of the star should be an increasing function of the central density. At the point of instability the fundamental (pulsation) modes become unstable. If stability is regained at higher central densities, the modes by which the stars become unstable towards the eventual collapse belong to higher-order harmonics.

For configurations constructed from a purely nuclear equation of state the stable sequence extends up to a maximum mass of the order $2 M_\odot$; the value of the maximum mass is large, since the chosen equations of state are rather stiff. The hybrid configurations branch off from the nuclear configurations when the central density reaches that of the deconfinement phase transition. The jump in the density at constant pressure causes a plateau of marginal stability beyond the point where the hybrid stars bifurcate. This is followed by an unstable branch ($dM/d\rho_c < 0$). Most importantly, the stability is regained at larger central densities: a stable branch of hybrid stars emerges in the range of central densities $1.3 \leq \rho_c/(10^{15}\text{g cm}^{-3}) \leq 2.5$. Since the models A and B feature the same high-density quark matter, whereas the models A and A1 the same nuclear equation of state, it is seen that the effect of having different nuclear EoS (the models A and B) at intermediate densities is substantial. At the same time, the small shift $\delta p$ by which the models A and A1 differ does not influence the masses of stable hybrid stars, although it is necessary for matching of nuclear and quark EoS in the models A and B. It is evident that there will exist purely nuclear and hybrid configurations with different central densities but the same masses. This is reminiscent of the situation encountered in non-superconducting hybrid stars (Glendenning and Kettner, 2000); the second branch of hybrid stars was called twin, since for each hybrid star there always exists a counterpart with the same mass composed entirely of nuclear matter.

The right panel of Fig. 19 displays the astronomical bounds on the masses of CSOs (horizontal dashed lines) along with the results obtained with the previously discussed models. The upper bound corresponds to a mass of about $M \sim 2 M_\odot$, while the lower bound on the neutron star mass (1.249 ± 0.001)M$_\odot$ is inferred from the millisecond binary J0737-3039 (Lyne et al., 2004). Note that both the hybrid stars and their nuclear counterparts have masses and radii within these bounds. The hybrid configurations are more compact than their nuclear counterparts, i.e. they have smaller radii (an exception are those configurations which belong to the metastable branch). Contrary to the case of self-bound quark stars, whose radii could be much smaller than the radii of purely nuclear stars, the differences between the radii of hybrid and nuclear stars are tiny ($\leq 1$ km) and cannot be used to distinguish these two classes by means of current astronomical observations.

The reported results differ from those described in Alford et al. (2007) and Grigorian et al. (2004) which have a narrow region of stable hybrid stars, branching off at the bifurcation point from the purely nuclear sequence. Indeed, the reported results cover a broad range of central densities, comparable to those covered by purely nuclear equations of state.

For sequences constructed from models A and A1 there is a range of masses and radii that corresponds to a family of stars that is separated from the stable nuclear sequence by an instability region (Ippolito et al., 2008). The sequences corresponding to model B do not show such an instability region.

The models A and A1 are consistent with the bounds of $M \sim 2 M_\odot$ and for the model B these bounds correspond to the stable configuration with the largest mass. It should be noted that canonical 1.4M$_\odot$ CSOs will be purely nuclear if they are described by the three considered models. According with the presented analysis the CCSC matter appears only in sufficiently massive CSOs, with mass $M \gtrsim 1.7 M_\odot$ for model B, corresponding to the solid horizontal line in the right panel of Fig. 19.
ACRONYMS AND SYMBOLS

For clarity we report below a list of the most used acronyms and symbols.

**BCC:** Body centered cube

**BCS:** Bardeen-Cooper-Schrieffer

**BEC:** Bose-Einstein condensate

**CCSC:** Crystalline color superconducting

**CFL:** Color-flavor locked

**CSC:** Color superconducting

**CSO:** Compact stellar object

**EoS:** Equation of state

**FCC:** Face centered cube

**FF:** Fulde-Ferrell

**GL:** Ginzburg-Landau

**GW:** Gravitational waves

**HDET:** High Density Effective Theory

**LO:** Larkin-Ochinnikov

**LOFF:** Larkin-Ochinnikov-Fulde-Ferrell

**NGB:** Nambu-Goldstone boson

**NJL:** Nambu-Jona Lasinio

**QCD:** Quantum chromodynamics

**QGP:** Quark-gluon plasma

**2SC:** Two flavor color superconducting

**gCFL:** Gapless color-flavor locked

**g2SC:** Gapless two flavor color superconducting

**\( a, b \):** Adjoint color indices

**\( i, j, k \):** Flavor indices

**\( s, t \):** Spin indices

**\( G_D \):** Quark-quark coupling constant

**\( G_S \):** Quark-antiquark coupling constant

**\( P^F \):** Fermi momentum

**\( M_s \):** Constituent strange quark mass

**\( M_\odot \):** Solar mass

**\( \alpha, \beta, \gamma \):** Fundamental color indices

**\( \delta \mu \):** Chemical potential difference

**\( \Delta \):** Pairing gap

**\( \Omega \):** Free-energy

**\( \mu \):** Quark chemical potential

**\( \mu_e \):** Electron chemical potential

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