Two Estimates for the First Robin Eigenvalue of the Finsler Laplacian with Negative Boundary Parameter

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Abstract
We prove two bounds for the first Robin eigenvalue of the Finsler Laplacian with negative boundary parameter in the planar case. In the constant area problem, we show that the Wulff shape is a maximizer only for values of the boundary parameter, which are close to zero. In the fixed perimeter case, we prove that the Wulff shape is a maximizer of the first eigenvalue for all values of the boundary parameter.

Keywords  Eigenvalue optimization · Finsler Laplacian · Robin boundary condition · Negative parameter · Wulff shape

Mathematics Subject Classification 58J50 · 35P15

1 Introduction

We consider the eigenvalue problem for the Laplace operator with Robin boundary conditions, which is a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions; when the boundary parameter of the Robin problem is zero, then we obtain the Neumann problem; when the boundary parameter is infinity, then we formally obtain the Dirichlet problem. We refer to [1] for a collection of the eigenvalue properties of the Robin Laplacian and for the related proofs.

If we analyze the problem of minimizing the first eigenvalue of the Dirichlet problem under volume constraint, the Faber–Krahn inequality tells us that the unique solutions are given by balls; see [2]. For the case of Neumann boundary conditions, we can find
analogous isoperimetric spectral inequalities in the works of Szegö [3] and Weinberger [4].

If we consider the Robin boundary conditions with positive boundary parameter, we have that the ball minimizes the first Robin eigenvalue among all Lipschitz domains of given volume. This fact was proved by Bossel and Daners [5] and generalized to the $p$-Laplacian by Bucur and Daners in [6] and by Dai and Fu in [7]; this result was also shown to hold on general open sets of finite measure by Bucur and Giacomin; see [8]. Moreover, this inequality is sharp: If the first Robin eigenvalue of our domain is equal to the first Robin eigenvalue of the ball, then our domain is a ball up to a negligible set.

If the boundary parameter is negative and the dimension is strictly greater than two, then it is not true that the $n$-dimensional ball maximizes the first Robin eigenvalue among all bounded and smooth $n$-dimensional sets with given volume. A counterexample is provided in [9]. The above fact is true within the class of Lipschitz sets, which are close to a ball in a Hausdorff metric sense; see for instance [10]. On the other hand, in [9] is proved that, if we have a bounded planar smooth domain with fixed area, then there exists a negative number, depending only on the area, such that the first eigenvalue of this domain is maximized by a ball with the same area and this is true for all boundary parameters between this negative number and zero. This fact is proved by applying the method of parallel coordinates, introduced by Payne and Weinberger in [11]. In the first part of this work, we find an analogous of this spectral inequality in the anisotropic case: we consider the anisotropic version of the Robin eigenvalue problem, that is studied, for instance, in [12–15], and we use the method of parallel coordinates, adapted to the anisotropic case.

In the second part of the work, we generalize to the anisotropic case a result presented in [16], in the case of negative Robin boundary parameter. Here, the authors, using again the methods of parallel coordinates, prove that the disk maximizes the first Robin eigenvalue among all bounded planar and smooth domains with given perimeter.

## 2 Notation and Preliminaries

In the following, we denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean scalar product in $\mathbb{R}^n$ and by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$, for $n \geq 2$. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ [sometimes also denoted by $V(\cdot)$] and by $\mathcal{H}^k$, for $k \in [0, n]$, the $k$-dimensional Hausdorff measure in $\mathbb{R}^n$. If $\Omega \subseteq \mathbb{R}^n$, $\text{Lip}(\partial \Omega)$ [resp. $\text{Lip}(\partial \Omega; \mathbb{R}^n)$] is the class of all Lipschitz functions (resp. vector fields) defined on $\partial \Omega$. If $\Omega$ has Lipschitz boundary, for $\mathcal{H}^{n-1}$—almost every $x \in \partial \Omega$, we denote by $v_{\partial \Omega}(x)$ the outward unit Euclidean normal to $\partial \Omega$ at $x$ and by $T_x(\partial \Omega)$ the tangent hyperplane to $\partial \Omega$ at $x$.

Let $F$ be a convex, even, 1-homogeneous and nonnegative function defined on $\mathbb{R}^n$. Then $F$ is a convex function, such that

$$F(t \xi) = |t|F(\xi), \quad t \in \mathbb{R}, \; \xi \in \mathbb{R}^n,$$

and such that

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^n,$$
for some constant $a > 0$. The hypotheses on $F$ imply that there exists $b \geq a$, such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^n.$$  

Moreover, throughout the paper we will assume that $F \in C^2(\mathbb{R}^n \setminus \{0\})$, and

$$[F^p]_{\xi\xi}(\xi)$$
is positive definite in $\mathbb{R}^n \setminus \{0\}$.

with $1 < p < +\infty$. The polar function $F^\circ : \mathbb{R}^n \to [0, +\infty[$ of $F$ is defined as

$$F^\circ(v) := \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F(\xi)}.$$

It is easy to verify that also $F^\circ$ is a convex function, which satisfies properties (1) and (2). $F$ and $F^\circ$ are usually called Finsler norms. Furthermore,

$$F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F^\circ(\xi)}.$$  

The above property implies the following anisotropic version of the Cauchy–Schwarz inequality

$$|\langle \xi, \eta \rangle| \leq F(\xi)F^\circ(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n.$$  

Then, we can introduce the set

$$\mathcal{W} := \{\xi \in \mathbb{R}^n : F^\circ(\xi) < 1\},$$

the so-called Wulff shape centered at the origin. We put $\kappa_n := V(\mathcal{W})$. More generally, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at $x_0$ with measure $\kappa_n r^n$, and we use the notation $\mathcal{W}_r := \mathcal{W}_r(0)$. In particular, when $\mathcal{W}$ is a subset of $\mathbb{R}^2$, we write $\kappa := V(\mathcal{W})$.

We conclude this paragraph reporting the following properties of $F$ and $F^\circ$:

$$\langle \nabla_\xi F(\xi), \xi \rangle = F(\xi), \quad \langle \nabla_\xi F^\circ(\xi), \xi \rangle = F^\circ(\xi), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

$$F(\nabla_\xi F^\circ(\xi)) = F^\circ(\nabla_\xi F(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

$$F^\circ(\xi)\nabla_\xi F(\nabla_\xi F^\circ(\xi)) = F(\xi)\nabla_\xi F^\circ(\nabla_\xi F(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$  

We recall now some basic definitions and theorems concerning the perimeter in the Finsler norm.

**Definition 2.1** Let $\Omega$ be a bounded and open subset of $\mathbb{R}^2$ with Lipschitz boundary, the anisotropic perimeter of $\Omega$ is defined as

$$P_{F}(\Omega) := \int_{\partial \Omega} F(\nu_{\partial \Omega}) \, d\mathcal{H}^1.$$
Clearly, the anisotropic perimeter of $\Omega$ is finite, if and only if the usual Euclidean perimeter of $\Omega$, that we denote by $P(\Omega)$, is finite. Indeed, by the quoted properties of $F$, we obtain that

$$a P(\Omega) \leq P_F(\Omega) \leq b P(\Omega).$$

For example, if $\Omega = W_R$, then

$$P_F(W_R) = 2\kappa R.$$

Moreover, an isoperimetric inequality is proved for the anisotropic perimeter, see for instance [17–19].

**Theorem 2.1** Let $\Omega$ be a subset of $\mathbb{R}^2$ with finite perimeter. Then

$$P_F(\Omega)^2 \geq 4\kappa V(\Omega)$$

and equality holds, if and only if $\Omega$ is homothetic to a Wulff shape.

Moreover, if $K$ is a bounded and convex subset of $\mathbb{R}^2$ and $\delta > 0$, the following Steiner formulas hold (see [20,21]):

$$V(K + \delta W) = V(K) + P_F(K)\delta + \kappa \delta^2;$$

(4)

$$P_F(K + \delta W) = P_F(K) + 2\kappa \delta.$$  (5)

Let $\Omega$ be a bounded and open set of $\mathbb{R}^2$, the anisotropic distance of a point $x \in \Omega$ from the boundary $\partial \Omega$ is defined as

$$d_F(x, \partial \Omega) := \inf_{y \in \partial \Omega} F^0(x - y).$$

By the properties of the Finsler norm $F$, the distance function satisfies

$$F(DD_dF(x)) = 1 \text{ a.e. in } \Omega.$$  (6)

For the properties of the anisotropic distance function, we refer, for instance, to [22].

We can define also the anisotropic inradius of $\Omega$ as

$$r_F(\Omega) := \sup\{d_F(x, \partial \Omega) : x \in \Omega\}.$$  (7)

We use the following notation

$$\tilde{\Omega}_t := \{x \in \Omega : d_F(x, \partial \Omega) > t\},$$

with $t \in [0, r_F(\Omega)]$. The general Brunn–Minkowski theorem (see [21]) and the concavity of the anisotropic distance function give that the function $P_F(\tilde{\Omega}_t)$ is concave in $[0, r_F(\Omega)]$, hence it is decreasing and absolutely continuous. In [14] the following result is stated.
Lemma 2.1 For almost every $t \in [0, r_F(\Omega)]$, 

$$-\frac{d}{dt} V(\tilde{\Omega}_t) = P_F(\tilde{\Omega}_t).$$

3 The Robin Problem in the Anisotropic Case

Let $\Omega$ be a bounded and open subset of $\mathbb{R}^2$ of class $C^2$. We consider the anisotropic eigenvalue problem with Robin boundary conditions. We fix a negative number $\alpha$, and we study the following problem:

$$\lambda_{1,F}(\alpha, \Omega) := \min_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} J(u), \quad (7)$$

where

$$J(u) := \frac{\int_{\Omega} [F(Du)]^2 \, dx + \alpha \int_{\partial \Omega} |u|^{2} F(v_{\partial \Omega}) \, d\mathcal{H}^1}{\int_{\Omega} |u|^2 \, dx} \quad (8)$$

and $v_{\partial \Omega}$ is the outer normal to $\partial \Omega$. Using a constant as test function, we obtain the following inequality

$$\lambda_{1,F}(\alpha, \Omega) \leq \alpha \frac{P_F(\Omega)}{|\Omega|} \leq 0. \quad (9)$$

The minimizers $u$ of problem (7) satisfy the following eigenvalue problem

$$-\text{div} \left( F(Du) F_\xi(Du) \right) = \lambda_{1,F}(\alpha, \Omega) u \quad \text{in } \Omega$$

$$\langle F(Du) F_\xi(Du), v_{\partial \Omega} \rangle + \alpha F(v_{\partial \Omega}) u = 0 \quad \text{on } \partial \Omega, \quad (10)$$

that is, in the weak sense,

$$\int_{\Omega} F(Du) [D_\xi F(Du), D\varphi] \, dx + \alpha \int_{\partial \Omega} u \varphi F(v_{\partial \Omega}) \, d\mathcal{H}^1 = \lambda_{1,F}(\alpha, \Omega) \int_{\Omega} u \varphi \, dx, \quad (11)$$

for all $\varphi \in W^{1,2}(\Omega)$. The following proposition is proved in [13].

**Proposition 3.1** There exists a function $u \in C^{1,\alpha}(\Omega) \cap C(\tilde{\Omega})$, which realizes the minimum in (7) and satisfies the anisotropic Robin eigenvalue problem (10). Moreover, $\lambda_{1,F}(\alpha, \Omega)$ is the first eigenvalue of the Robin problem and the first eigenfunctions are positive (or negative) in $\Omega$.

4 Isoperimetric Estimates with a Volume Constraint

In the following, we are fixing a Finsler norm $F$. 

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Theorem 4.1 For bounded and planar domains of class $C^2$ and fixed area, there exists a negative number $\alpha_*$, depending only on the area, such that the following inequality holds $\forall \alpha \in [\alpha_*, 0]$:

$$\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, \mathcal{W}_\Omega^*),$$

where $\mathcal{W}_\Omega^*$ is the Wulff shape of the same area as $\Omega$.

In order to prove Theorem 4.1, we adapt in the anisotropic case the proof of Freitas and Krejcirík contained in [9]. This proof makes use of the classical method of parallel coordinates, developed for the Euclidean case in [11] and for the Riemannian case in [23].

We assume that $\partial \Omega$ is composed by a finite union of $C^2$ Jordan curves $\Gamma_0, \ldots, \Gamma_N$, where $\Gamma_0$ is the outer boundary of $\Omega$, i.e., $\Omega$ lies in the interior $\Omega_0$ of $\Gamma_0$. We observe that, if $N = 0$, then $\Omega$ is simply connected and $\Omega = \Omega_0$. We denote by

$$L^F_0 := P_F(\Omega_0)$$

the outer anisotropic perimeter. Therefore, by the anisotropic isoperimetric inequality, we have

$$(L^F_0)^2 \geq 4\kappa A_0,$$  \hspace{1cm} (12)

where $A_0 = V(\Omega)$ denotes the area of $\Omega$ (not of $\Omega_0$).

We now introduce the anisotropic parallel coordinate method based at the outer boundary $\Gamma_0$. Let $\rho_F : \Omega_0 \to ]0, \infty[ $ be the anisotropic distance function from the outer boundary $\Gamma_0$:

$$\rho_F(x) := d_F(x, \Gamma_0).$$

Let

$$A_F(t) := V([x \in \Omega : 0 < \rho_F(x) < t])$$

denote the area of $\Omega_t = \Omega \setminus \tilde{\Omega}_t$, and let

$$L_F(t) := \int_{\rho_F^{-1}(t) \cap \Omega} F(v_{\partial \Omega}(x)) \, d\mathcal{H}^1(x).$$

Remark 4.1 By Lemma 2.1, we obtain that, for almost every $t \in [0, r_F(\Omega_0)]$,

$$A'_F(t) = L_F(t).$$  \hspace{1cm} (13)
4.1 Step 1: Use of the Anisotropic Parallel Coordinates

Let $\phi : [0, r_F(\Omega)] \to \mathbb{R}$ be a smooth function and consider the test function

$$u := \phi \circ A_F \circ \rho_F,$$

which is Lipschitz in $\Omega$. Using the anisotropic parallel coordinates, the coarea formula and the fact that $F(D\rho_F) = 1$, we obtain the following relations:

$$||u||^2_{L^2(\Omega)} = \int_{\Omega} u^2(x) \, dx = \int_{\Omega} (\phi \circ A_F \circ \rho_F(x))^2 \, dx$$

$$= \int_0^{r_F(\Omega)} \left( \int_{\{\rho_F(x) = t\}} (\phi \circ A_F \circ \rho_F(x))^2 \frac{1}{|D\rho_F(x)|} \, d\mathcal{H}^1(x) \right) \, dt$$

$$= \int_0^{r_F(\Omega)} \phi(A_F(t))^2 P_F(\{|\rho_F(x) < t\}) \, dt$$

$$= \int_0^{r_F(\Omega)} \phi(A_F(t))^2 A_F(t) \, dt,$$

$$\int_{\Omega} \left( F^2(Du(x)) \right) \, dx = \int_{\Omega} F^2 \left( \phi' (A_F \circ \rho_F (x)) A'_F (\rho_F (x)) D\rho_F (x) \right) \, dx$$

$$= \int_{\Omega} \left( \phi' (A_F \circ \rho_F (x)) \right)^2 \left( A'_F (\rho_F (x)) \right)^2 \, dx$$

$$= \int_0^{r_F(\Omega)} \left( \phi' (A_F (t)) \right)^2 \left( A'_F (t) \right)^3 \, dt,$$

$$\int_{\partial \Omega} |u(x)|^2 F(v_{\partial \Omega} (x)) \, d\mathcal{H}^1(x) = \int_{\partial \Omega} (\phi \circ A_F \circ \rho_F (x))^2 F(v_{\partial \Omega} (x)) \, d\mathcal{H}^1(x)$$

$$= (\phi \circ A_F (0))^2 P_F(\Omega) \geq \phi^2(0) L_0.$$

Therefore, we have that

$$\lambda(\Omega) \leq \frac{\int_0^{r_F(\Omega)} \left( \phi' (A_F (t)) \right)^2 \left( A'_F (t) \right)^3 \, dt + \alpha \phi^2(0) L_0^F}{\int_0^{r_F(\Omega)} \phi(A_F(t))^2 A'_F(t) \, dt}.$$  (14)
4.2 Step 2: From Domains to Annuli

We adapt in the anisotropic case the idea contained in [11]. We consider the following change of variables:

\[ R(t) := \frac{\sqrt{(L_0^F)^2 - 4\kappa A_F(t)}}{2\kappa} \quad (15) \]

on the interval \([r_1, r_2]\), where

\[ r_1 := R(r_F(\Omega)) = \frac{\sqrt{(L_0^F)^2 - 4\kappa A_0}}{2\kappa}, \quad r_2 := R(0) = \frac{L_0^F}{2\kappa}. \quad (16) \]

**Remark 4.2** Thanks to (3), the transformation (15) is well defined on the set \([0, r_F(\Omega)]\).

We introduce now the function

\[ \psi(r) := \phi \left( \frac{(L_0^F)^2}{4\kappa} - \kappa r^2 \right) \]

and we obtain the following expressions:

\[ \int_{\Omega} u^2(x) \, dx = 2\kappa \int_{r_1}^{r_2} (\psi(r))^2 r \, dr, \]

\[ \int_{\Omega} \left( F^2(Du(x)) \right) \, dx = 2\kappa \int_{r_1}^{r_2} (\psi'(R))^2 (R'(t))^2 R \, dR, \]

\[ \int_{\partial\Omega} |u(x)|^2 F(v_{\partial\Omega}(x)) \, d\mathcal{H}^1(x) \geq L_0^F \psi(r_2)^2. \]

**Remark 4.3** The radii in (16) are such that the \(F\)-annulus \(A_{r_1, r_2}^F := \mathcal{W}_{r_2} \setminus \mathcal{W}_{r_1}\) has the same area \(A_0\) as the original domain \(\Omega\). We observe that the transformation (15) maps \(\partial\Omega\) into the boundary of the Wulff shape of radius \(R(t)\); so \(\Gamma_0\) is mapped into the Wulff shape of equal anisotropic perimeter. Moreover, \(\Omega_t\) is mapped in the anisotropic annulus of area \(A_F(t)\).

**Proposition 4.1** Let \(\Omega\) be a bounded and planar domain of class \(C^2\), then

\[ |R'(t)| \leq 1, \]

where \(R\) is defined in (15).
Proof From (13) follows that, for almost every $t \in [0, r_F(\Omega)]$ we have

$$R'(t) = -\frac{L_F(t)}{\sqrt{(L_F^0)^2 - 4\kappa A_F(t)}}. \quad (17)$$

Using the Steiner formula, we obtain, for almost every $t \in [0, r_F(\Omega)]$,

$$L_F(t) \leq L_F^0 - 2\kappa t,$$

$$A_F(t) = \int_0^t L_F(v) \, dv \leq L_F^0 t - \kappa t^2.$$

Therefore,

$$L_F(t)^2 \leq \left( L_F^0 \right)^2 - 4\kappa A_F(t),$$

and, putting this in (17), the thesis follows. \qed

We obtain this upper bound

$$\lambda_{1,F}(\alpha, \Omega) \leq \inf_{\psi \neq 0} \frac{\int_{r_1}^{r_2} \psi'(r)^2 r \, dr + \alpha r_2 \psi(r_2)^2}{\int_{r_1}^{r_2} \psi(r)^2 r \, dr} =: \mu(\alpha, A_{r_1,r_2}^F), \quad (18)$$

so the infimum is attained for the first eigenfunction of the Laplacian in $A_{r_1,r_2}^F$, with anisotropic Robin boundary conditions on $\partial\mathcal{W}_2$ and anisotropic Neumann boundary conditions on $\partial\mathcal{W}_1$. Therefore, we have proved the following proposition.

**Proposition 4.2** Let $\alpha \leq 0$. For any bounded and planar domain $\Omega$ of class $C^2$, we have

$$\lambda_{1,F}(\alpha; \Omega) \leq \mu(\alpha, A_{r_1,r_2}^F),$$

where $A_{r_1,r_2}^F$ is the anisotropic annulus of the same area as $\Omega$ with radii given by (16).

### 4.3 Step 3: From Annuli to Disks

Let $\mathcal{W}_{r_1,r_2}$ be the Wulff shape of the same area as the anisotropic annulus $A_{r_1,r_2}^F$, which has the same area $A_0$ as $\Omega$. So, we have that

$$r_3 = \sqrt{\frac{A_0}{\kappa}}, \quad (19)$$

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where $r_3$ is the radius of $W_{r_1, r_2}$. In [9], we find the following asymptotics as $\alpha \to 0$:

$$
\lambda_{1,F}(\alpha, W_{r_1, r_2}) = 2\alpha \frac{r_3}{r_2^2} + O(\alpha^2) \quad \text{(Robin Wulff)}; \quad (20)
$$

$$
\mu(\alpha, A_{r_1, r_2}^F) = 2\alpha \frac{r_2}{r_3^2} + O(\alpha^2) \quad \text{(Neumann–Robin annulus).} \quad (21)
$$

Using them, we can prove that, for $\alpha < 0$ small enough,

$$
\mu(\alpha, A_{r_1, r_2}^F) \leq \lambda_{1,F}(\alpha, W_{r_1, r_2}), \quad (22)
$$

where $W_{r_1, r_2}$ is the Wulff shape of the same area as the anisotropic annulus $A_{r_1, r_2}^F$. Thus, we have proved the following theorem.

**Proposition 4.3** For any bounded and planar domain $\Omega$ of class $C^2$, there exists a negative number $\alpha_0 = \alpha_0(A_0, L_0^F)$, such that

$$
\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, W_{\Omega}^*)
$$

holds $\forall \alpha \in [\alpha_0, 0]$, where $W_{\Omega}^*$ is the Wulff shape of the same area as $\Omega$.

**Remark 4.4** Using (7), we can show that

$$
\frac{d}{d\alpha} \lambda_{1,F}(\alpha, \Omega)|_{\alpha=0} = \frac{\mathcal{H}^1(\partial \Omega)}{|\Omega|}.
$$

### 4.4 Step 4: Uniform Behavior and Conclusion

In order to complete the proof of Theorem 4.1, it remains only to show the following fact.

**Proposition 4.4** The constant $\alpha_0$ of Proposition 4.3 is independent of $L_0^F$.

Following [9], we need to show that the neighborhood of zero, in which (22) holds, does not degenerate either when $r_1 \to 0$ or when $r_2 \to +\infty$. So, we are going to prove that $\alpha_0$ remains bounded away from 0 uniformly in these two instances.

We fix $\epsilon > 0$ and we consider

$$
r_1 = \sqrt{(2\epsilon r_3 + \epsilon^2)}, \quad r_2 = r_3 + \epsilon,
$$

where $r_3$ is fixed and equal to $\sqrt{A_0/\kappa}$. In an analogous way to the one reported in [9], it can be proved that there exists $\alpha^* < 0$, such that the curve $\Gamma_A: \alpha \mapsto \mu(\alpha, A_{r_1, r_2}^F)$ stays below the curve $\Gamma_B: \alpha \mapsto \lambda_{1,F}(\alpha, W_{r_3})$, for all $\epsilon > 0$ and for all $\alpha \in [\alpha^*, 0[$.

Because of the simplicity of the eigenvalues, both the curves are analytic. Moreover, taking into account the asymptotics (20) and (21), we have that

$$
\frac{d}{d\alpha} \mu(\alpha, W_{r_1, r_2}) \leq \frac{d}{d\alpha} \lambda_{1,F}(\alpha, A_{r_1, r_2}^F).
$$
Remark 4.5 We prove that the curves \( \Gamma_A \) are concave in \( \alpha \). Let \( \epsilon > 0 \) and let \( \psi \) be the first eigenfunction related to \( \mu(\alpha + \epsilon, A_{r_1, r_2}^F) \) of the Laplacian in the anisotropic annulus. We can choose \( \psi \) normalized to 1, so we have

\[
\mu(\alpha + \epsilon, A_{r_1, r_2}^F) = \int_{r_1}^{r_2} \psi'(r)^2 r \, dr + (\alpha + \epsilon) r_2 \psi(r_2)^2. \tag{23}
\]

Let \( \phi \) be the first eigenfunction related to \( \mu(\alpha, A_{r_1, r_2}^F) \) normalized to 1:

\[
\mu(\alpha, A_{r_1, r_2}^F) = \int_{r_1}^{r_2} \phi'(r)^2 r \, dr + \alpha r_2 \phi(r_2)^2. \tag{24}
\]

Now, putting \( \phi \) as a test function in the variational formula of \( \mu(\alpha + \epsilon, A_{r_1, r_2}^F) \), we obtain

\[
\mu(\alpha + \epsilon, A_{r_1, r_2}^F) \leq \int_{r_1}^{r_2} \phi'(r)^2 r \, dr + (\alpha + \epsilon) r_2 \phi(r_2)^2 = \mu(\alpha, A_{r_1, r_2}^F) + \epsilon r_2 \phi(r_2)^2.
\]

In order to prove our claim, we need only to show that

\[
\frac{d}{d\alpha} \mu(\alpha, A_{r_1, r_2}^F) = r_2 \phi(r_2)^2.
\]

We prove the following more general result.

**Lemma 4.1** Let \( \Omega \) be a bounded subset of \( \mathbb{R}^2 \) and let \( u_\alpha \) be an eigenfunction related to the eigenvalue \( \lambda_{1,F}(\alpha, \Omega) \), defined in (7), such that \( \| u_\alpha \|_{L^2(\Omega)} = 1 \). Then,

\[
\lambda'_{1,F}(\alpha, \Omega) := \frac{d\lambda_{1,F}(\alpha, \Omega)}{d\alpha} = \int_{\partial\Omega} u_\alpha^2 F(v_\beta \Omega) d\mathcal{H}^1. \tag{25}
\]

**Proof** From the variational characterization (7) and using the fact that \( \| u_\alpha \|_{L^2(\Omega)} = 1 \), we have

\[
\lambda_{1,F}(\alpha, \Omega) = \int_{\Omega} F^2(Du_\alpha) \, dx + \alpha \int_{\partial\Omega} u_\alpha^2 F(v_\beta \Omega) \, d\mathcal{H}^1. \tag{26}
\]

Deriving both sides of (26) with respect to \( \alpha \), we obtain

\[
\lambda'_{1,F}(\alpha, \Omega) = 2 \int_{\Omega} F(Du_\alpha) D_\xi F(Du_\alpha) Du_\alpha' \, dx + \int_{\partial\Omega} u_\alpha^2 F(v_\beta \Omega) \, d\mathcal{H}^1
\]
\[ +2\alpha \int_{\partial \Omega} u_\alpha' F(v_{\beta \Omega}) \, d\mathcal{H}^1. \]  

(27)

Using the weak formulation (11) of the problem in the equation (27), remembering that \( u_\alpha' \) is the derivative with respect to \( \alpha \) and it is in the set of the test functions by standard elliptic regularity theory, we obtain

\[ \lambda_{1,F}'(\alpha, \Omega) = 2\lambda_{1,F}(\alpha, \Omega) \int_{\Omega} u_\alpha u_\alpha' \, dx + \int_{\partial \Omega} u_\alpha^2 F(v_{\beta \Omega}) \, d\mathcal{H}^1 \]  

(28)

and, having in mind that, from the condition \( \|u_\alpha\|_{L^2(\Omega)} = 1 \),

\[ \int_{\Omega} u_\alpha u_\alpha' \, dx = 0, \]

we get, from (28), Eq. (25).

Therefore, since the curves \( \Gamma_A \) are concave in \( \alpha \) and their derivatives with respect to \( \alpha \) are increasing with \( \epsilon \), we have that the tangent to the curve corresponding to a specific anisotropic annulus intersects \( \Gamma_B \) at one and only one point, that we call \( \alpha_1 \), to the left of zero. Thanks to the concavity, we can say that, for larger values of \( \epsilon \), any \( \Gamma_A \) that intersects \( \Gamma_B \) must do so to the left of \( \alpha_1 \).

As far as the case when \( \epsilon \) is small, we follow closely the proof presented in [9]. We study the intersection points of the two curves \( \Gamma_A \) and \( \Gamma_B \), comparing the following two equations; the first equation is the equation of the Wulff shape

\[ kI_1(kr_3) + \alpha I_0(kr_3) = 0; \]  

(29)

the second equation is the one of the Neumann–Robin anisotropic annulus

\[ K_1 \left( k\sqrt{2\epsilon r_3 + \epsilon^2} \right) [kI_1(k(r_3 + \epsilon)) + \alpha I_0(k(r_3 + \epsilon))] 
- I_1 \left( k\sqrt{2\epsilon r_3 + \epsilon^2} \right) [kK_1(k(r_3 + \epsilon)) - \alpha K_0(k(r_3 + \epsilon))] = 0, \]

where \( k := \sqrt{-\mu(\alpha, A^F_{r_1,r_2})} \). We denote here by \( I_\nu \) and \( K_\nu \) the modified Bessel functions (for their properties we refer to [24]). The solution in \( \alpha \) of the intersection is given by

\[ \alpha = -k \frac{I_1(kr_3)}{I_0(kr_3)}. \]

The proof, that there are no intersections between \( \Gamma_A \) and \( \Gamma_B \), for \( \alpha \) close to zero, is the same as the one presented in [9]. In this way, we have proved Proposition 4.4.
5 Isoperimetric Estimates with a Perimeter Constraint

Using the method of parallel coordinates, we are able to prove also the following theorem.

**Theorem 5.1** Let $\alpha \leq 0$ and let $\Omega \subseteq \mathbb{R}^2$ a bounded domain of class $C^2$. Then

$$\lambda_{1,F}(\alpha, \Omega) \leq \lambda_{1,F}(\alpha, \tilde{W}_\Omega),$$

where $\tilde{W}_\Omega$ is the Wulff shape with the same perimeter as $\Omega$.

The crucial step, in order to prove this theorem, is given by the following proposition.

**Proposition 5.1** Let $\alpha < 0$. For any $0 < r_1 < r_2$, we have

$$\mu(\alpha, A_{r_1,r_2}^F) \leq \lambda_{1,F}(\alpha, W_{r_2}).$$

**Proof** By symmetry, $\lambda_{1,F}(\alpha, W_{r_2})$ is the smallest eigenvalue of the following one-dimensional problem

$$-r^{-1}[r \phi'(r)]' = \lambda_{1,F}(\alpha, W_{r_2}) \phi(r), \quad r \in [0, r_2]$$

$$\phi'(0) = 0$$

$$\phi'(r_2) + \alpha \phi(r_2) = 0. \quad (30)$$

We can choose the associated function $\phi_1$ to be positive and normalized to 1 and this eigenfunction can be used as a test function in (18). Integrating by parts, we obtain

$$\mu(\alpha, A_{r_1,r_2}^F) \leq \lambda_{1,F}(\alpha, W_{r_2}) - r_1 \phi_1(r_1) \phi'_1(r_1). \quad (31)$$

Since $\phi_1$ satisfies (30), we have for all $r \in [0, r_2]$

$$[r \phi_1(r) \phi'_1(r)]' = -\lambda_{1,F}(\alpha, W_{r_2}) r \phi_1(r)^2 + r \phi'_1(r)^2 \geq 0$$

and the inequality is due to (9). From the above inequality the function $g(r) := r \phi_1(r) \phi'_1(r)$ is non-decreasing and, using (31), we obtain the desired result. \qed

**Remark 5.1** The following monotonicity result holds true. Let $W_R$ be a Wulff shape of radius $R$. If $\alpha < 0$, then

$$R \mapsto \lambda_{1,F}(\alpha, W_R)$$

is strictly increasing. The above result is proved for the disks in [16] and for the annuli in [25].

**Proof of Theorem 5.1** Firstly, we observe that the measure of $W_{r_2}$ is greater than the measure of $A_{r_1,r_2}^F$ and the perimeter of $W_{r_2}$, which is equal to $L_0$, is less than the perimeter of $A_{r_1,r_2}$. Using Theorem 4.2 and Proposition 5.1, we obtain the thesis for simply connected domains, i.e., when $L_0 = P_F(\Omega)$. 

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Concerning the general case, when there are multiple connected domains, thanks to remark 5.1, we have that
\[ \lambda_{1,F}(\alpha, R_{T_2}) \leq \lambda_{1,F}(\alpha, R_{T_3}), \]
where \( r_3 = P_{F}(\Omega)/2\kappa \) for all \( \alpha \leq 0 \). \hfill \Box

6 Conclusions

In conclusion, we have considered the first eigenvalue of the Robin problem and we have proved an isoperimetric estimates with a volume constraint and an isoperimetric estimates with a perimeter constraint, both in dimension two and in the anisotropic case.

We recall that in [26] the authors prove that, if the Robin boundary parameter is negative and if the set is convex, then the first Robin eigenvalue of the set is maximized by the ball with the same perimeter. More precisely, the last inequality holds true in dimension equal or greater than two, if we restrict to the class of Lipschitz sets that can be written as the set difference between an open and convex set and a closed set. Moreover, in [27] the authors prove that the second eigenvalue of the Robin problem related to the Laplacian is maximal for the ball among domains of fixed volume. Further developments of the present work could be the generalizations of these facts in the anisotropic case.

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