Ruled austere submanifolds of dimension four

Marianty Ionela\textsuperscript{a}, Thomas Ivey\textsuperscript{b,∗}

\textsuperscript{a} Institute of Mathematics, Federal University of Rio de Janeiro, Brazil
\textsuperscript{b} Department of Mathematics, College of Charleston, Charleston, SC, USA

1. Introduction

A submanifold \( M \) in Euclidean space \( \mathbb{R}^n \) is \textit{austere} if the eigenvalues of its second fundamental form, in any normal direction, are symmetrically arranged around zero. (More precisely, all odd-degree symmetric polynomials in these eigenvalues vanish.) Thus, austere submanifolds are a subclass of minimal submanifolds. When \( M \) is a surface, austerity is equivalent to minimality, but in higher dimensions the austere condition is stronger. This condition was introduced by Harvey and Lawson [5] in connection with special Lagrangian submanifolds. A special Lagrangian submanifold in \( \mathbb{C}^n \) is a submanifold of real dimension \( n \) that is both Lagrangian and minimal. The importance of special Lagrangian submanifolds lies mainly in the fact that they are area-minimizing. Harvey and Lawson showed that the conormal bundle of an immersed submanifold \( M \subset \mathbb{R}^n \) is special Lagrangian in the cotangent bundle \( T^*\mathbb{R}^n \cong \mathbb{C}^n \), equipped with its canonical symplectic structure and Euclidean metric, if and only if \( M \) is austere. The connection between austerity and special Lagrangian submanifolds has been established in other space forms [8] and in fact the austere condition has been generalized to submanifolds of an arbitrary Riemannian manifold [3].

A systematic study and classification of austere submanifolds of dimension 3 in Euclidean space was first undertaken by Bryant [1], and generalized by Dajczer and Florit [4] to austere submanifolds of arbitrary dimension whose Gauss map has rank two. The case of dimension 4 was open until, in [6], we obtained classification results on austere 4-folds whose second fundamental form is of maximal type. In this paper we are concerned with classifying austere submanifolds of dimension 4 which are ruled by 2-planes. (As we will see below in Corollary 5, austere 4-manifolds ruled by 3-planes are easy to classify, while on the other hand we expect the classification of austere submanifolds ruled by lines to be much more difficult.)

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One family of ruled austere submanifolds defined in [1] are simple to describe. A *generalized helicoid* $M \subset \mathbb{R}^n$ is an $m$-dimensional submanifold swept out by $s$-planes rotating while translating along a fixed axis, with $n = m + s$ and $s < m$. It can be parametrized by

$$
(x_0, \ldots, x_{m-1}) \mapsto \left(\lambda_0 x_0, x_1 \cos(\lambda_1 x_0), x_1 \sin(\lambda_1 x_0), \ldots, x_s \cos(\lambda_s x_0), x_s \sin(\lambda_s x_0), x_{s+1}, \ldots, x_{m-1}\right),
$$

where $\lambda_0, \ldots, \lambda_s$ are constants with $\lambda_1, \ldots, \lambda_s$ nonzero. Of course, this gives the classical helicoid minimal surface when $s = 1$, $m = 2$ and $\lambda_0, \lambda_1 \neq 0$. Notice also that if $m > s + 1$ then this splits as a product of a Euclidean factor with an ‘irreducible’ helicoid swept out by $s$-planes in $\mathbb{R}^{2s+1}$. Furthermore, when $\lambda_0 = 0$, $M$ is a cone over an austere submanifold in $S^{m-1}$ which is ruled by $(s-1)$-dimensional totally geodesic spheres.

Helicoids (classical and generalized) will play a significant role in the latter part of this paper. Among austere submanifolds, they have the following characterization:

**Theorem.** (Bryant [1]) If $M \subset \mathbb{R}^n$ is austere, and $|\mathcal{I}|$ is simple (i.e., at each point $p \in M$ the quadratic forms in $|\mathcal{I}|$ share a common linear factor) then $M$ is congruent to a generalized helicoid.

Here, $|\mathcal{I}| \subset S^2T^*_pM$ is the subspace spanned by the second fundamental form $\mathcal{I}$ of $M$ at $p$ in various normal directions. In more detail, we recall that the second fundamental form of a submanifold $M \subset \mathbb{R}^n$ is a tensor defined by

$$
D_X Y = \nabla_X Y + \mathcal{I}(X, Y),
$$

where $X, Y$ are tangent vector fields on $M$, $D$ is the Euclidean connection on $\mathbb{R}^n$ and the right-hand side is split into the tangential and normal components. Thus, $\mathcal{I}$ is a section of $S^2 \mathcal{T} M \otimes \mathbb{R} M$, where $\mathbb{R} M$ is the normal bundle, and $|\mathcal{I}|$ is the image of $\mathcal{I}$ under contraction with a basis for $\mathbb{R} M$. Then dim $|\mathcal{I}|$ is constant on an open dense subset of each connected component of $M$. We assume that $M$ is connected, and let $\delta$ denote the constant value of dim $|\mathcal{I}|$, referred to as the *normal rank* of $M$.

Using an orthonormal basis to identify $T_p M$ with $\mathbb{R}^4$, we see that $|\mathcal{I}|$ must correspond to a subspace of the space $S^2 \mathcal{T}^4$ of $4 \times 4$ symmetric matrices which is an *austere subspace*, i.e., in which every matrix has eigenvalues symmetrically arranged around zero. The *maximal* austere subspaces $Q \subset S^2 \mathcal{T}^4$ were determined (up to $O(4)$ conjugation) by Bryant [1], and fall into three types which we have labeled as $Q_A$, $Q_B$ and $Q_C$. The precise form of each of these subspaces will be given later, in Sections 3, 4 and 5 respectively.

We say that a 4-dimensional austere submanifold $M$ is of a particular *Type A, B, or C at point $p$* if $|\mathcal{I}|$ lies in the corresponding maximal subspace $Q_A$, $Q_B$ or $Q_C$ for some choice of orthonormal basis for $T_p M$. Note that this notion of type is not unique; for example, all 1-dimensional austere subspaces are equivalent under diagonalization, and so any such subspace is $O(4)$-conjugate to subspaces of each of $Q_A$, $Q_B$ and $Q_C$. Nevertheless, we make the blanket assumption that for each austere submanifold $M$ with $\delta > 1$ there is a choice of Type A, B, or C that is constant on an open dense subset of $M$. In particular, we note that if $M$ is of Type A then it carries a complex structure, denoted by $J$, with respect to which the metric on $M$ is Kähler (cf. Corollary 7 in [6]).

We now summarize our results. In each of the following theorems, we assume that $M$ is an austere submanifold in $\mathbb{R}^n$, ruled by 2-planes. We let $E_p \subset T_p M$ denote the 2-dimensional subspace tangent to the ruling at $p$, and we let $\gamma : M \to \text{Gr}(2, n)$ denote the *ruling map*, which takes $p$ to the 2-dimensional linear subspace in $\mathbb{R}^n$ parallel to $E_p$. (We endow $\text{Gr}(2, n)$ with the usual complex structure.)

**Theorem 1.** Assume that $M$ is of Type A and $\delta \geq 2$. Then necessarily $\delta \leq 4$, $E = J(E)$ and $\gamma$ is holomorphic. If $\delta < 4$ then we may assume without loss of generality that $n = 4 + \delta$.

Furthermore, there is a compact complex homogeneous space $V$, an $\text{SO}(n)$-equivariant holomorphic fibration $\rho : V \to \text{Gr}(2, n)$, and a complex Pfaffian system $\mathcal{K}$ on $V$ with the following properties:

1. If the Gauss map of $M$ is nondegenerate, there is a canonical holomorphic mapping $\Gamma : M \to V$ of real rank 2, such that $\gamma = \rho \circ \Gamma$, and along which the 1-forms in $\mathcal{K}$ pull back to zero.
2. Given a complex integral curve $\mathcal{C}$ of $\mathcal{K}$, and any point $q \in \mathcal{C}$ at which $\rho|_q$ is nonsingular, there is an open neighborhood of $q$ within $\mathcal{C}$ which is the image under $\Gamma$ of an austere submanifold $M$ of this type.

The definitions of $V$, $\mathcal{K}$ and $\Gamma$ depend only on the values of $\delta$ and $n$, but are technical, and are postponed until Section 3.

We refer to the manifolds $V$ as ‘twistor spaces’, with the justification that they are lower-dimensional homogeneous spaces in which we can take solutions of well-known or canonical systems of PDE (e.g., the Cauchy–Riemann equations, in the case of holomorphic curves) and use them to construct austere 4-manifolds with the desired properties.

It is important to note that, when $n$ is even, the submanifolds $M$ in Theorem 1 are not in general holomorphic submanifolds of $\mathbb{C}^{n/2}$; our analysis shows that the space of local solutions is too large for this to be the case.

**Theorem 2.** If $M$ is of Type B, then $\delta \leq 2$. If $\delta = 2$, then $M$ is in $\mathbb{R}^5$, and is either a generalized helicoid, or the product of two classical helicoids in $\mathbb{R}^3$ (in which case the product of $\mathbb{R}^3$’s need not be orthogonal), or is also of Type A.
Theorem 3. If $M$ is of Type C, then $\delta \leq 3$. (This is trivial, since $\dim Q_C = 3$.) If $\delta = 3$ then $M$ is a generalized helicoid in $\mathbb{R}^7$ (corresponding to $s = 3$ in (1)). If $\delta = 2$ then $M$ is also of Type B.

We will establish these results using the techniques of moving frames and exterior differential systems. After setting up our basic tools in Section 2, we discuss Types A, B and C in Sections 3, 4 and 5 respectively. Finally in Section 6 we consider the case of 2-ruled austere submanifolds of normal rank $\delta = 1$, in which we show that they must lie in $\mathbb{R}^5$, and provide an analogous twistor construction for this case.

2. The standard system

Let $\mathcal{F}$ be the semi-orthonormal frame bundle of $\mathbb{R}^n$, whose fiber at a point $p$ consists of all bases $(e_1, e_2, \ldots, e_n)$ of $T_p\mathbb{R}^n$ such that the vectors $(e_i)_{i=1,\ldots,4}$ are orthonormal and orthogonal to the $(e_i)_{a=5,\ldots,n}$. (For the rest of the paper, we will use the index ranges $1 \leq i,j \leq 4$ and $5 \leq a,b,c \leq n$.) Given a submanifold $M^4 \subset \mathbb{R}^n$ we may, on an open set in $M$ near any given point, choose a frame field $(e_1, e_2, \ldots, e_n)$ depending smoothly on $p \in M$ such that $e_1(p), \ldots, e_4(p)$ are an orthonormal basis of the tangent space $T_pM$ for each point $p$ in the open set. Note that, in expressing the second fundamental form of $M$, we will not necessarily choose the normal vectors $e_a$ to be mutually orthogonal. When needed, we will let $\mathcal{F}_{on} \subset \mathcal{F}$ denote the subbundle of completely orthonormal frames on $\mathbb{R}^n$.

Regarding the frame vectors and basepoint $p$ as $\mathbb{R}^n$-valued functions on the frame bundle $\mathcal{F}$, we let

\[
dp = e_i \omega^i + e_a \omega^a,
\]
\[
d e_i = e_j \omega^j + e_a \omega^a,
\]
\[
d e_a = e_b \omega^b + e_c \omega^c,
\]

(2)

define the canonical 1-forms $\omega^i, \omega^a$ and the connection 1-forms $\omega^i_j, \omega^a_i, \omega^a_b$ and $\omega^a_c$ on $\mathcal{F}$. (Note that the Einstein summation convention will be used throughout this paper.) These forms span the cotangent space of $\mathcal{F}$ at each point, but they are not linearly independent. Differentiating the equations $e_i \cdot e_j = \delta_{ij}$ and $e_i \cdot e_a = 0$ yields the relations

\[
\omega^i_j = -\omega^j_i, \quad \omega^a = -\omega^b_i g_{ba},
\]

(3)

where $g_{ab} = e_a \cdot e_b$. Differentiating the first line of (2), we obtain the structure equations

\[
d \omega^i = -\omega^i_j \wedge \omega^j + \omega^b_i g_{ba} \wedge \omega^a,
\]
\[
d \omega^a = -\omega^a_i \wedge \omega^i - \omega^b_a \wedge \omega^b.
\]

(4)

Differentiating the last two equations of (2) yields

\[
d \omega^i_j = -\omega^i_k \wedge \omega^k_j + \omega^b_i g_{ba} \wedge \omega^b_j,
\]
\[
d \omega^a_i = -\omega^a_k \wedge \omega^k_i - \omega^b_a \wedge \omega^b_i,
\]
\[
d \omega^a_b = \omega^a_i \wedge \omega^b_i g_{cb} - \omega^b_i \wedge \omega^b,
\]

(5)

along with

\[
d g_{ab} = g_{ac} \omega^c_b + g_{bc} \omega^c_a.
\]

(6)

An adapted frame along a submanifold $M \subset \mathbb{R}^n$ gives a section of $\mathcal{F}|_M$. The following fundamental fact characterizes these sections:

A submanifold $\Sigma^4 \subset \mathcal{F}$ is the image of an adapted frame along the submanifold $M^4 \subset \mathbb{R}^{4+\delta}$ if and only if $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 |_{\Sigma} \neq 0$ and $\omega^a |_{\Sigma} = 0$.

The first of these conditions is a nondegeneracy assumption called the independence condition. The second condition implies, by differentiation, that $\omega^1_i |_{\Sigma} = S^1_{ij} \omega^j$ for some functions $S^1_{ij}$. These functions give the components of the second fundamental form in this frame, i.e.,

\[
ll(e_i, e_j) = S^1_{ij} e_a.
\]

(7)

In this paper we classify the austere 4-folds which are ruled by 2-planes. We will now describe a class of exterior differential systems, for later use, whose integral submanifolds are adapted frames along 2-ruled austere submanifolds. (Being an integral submanifold of an EDS $\mathcal{I}$ means that the pullback to the submanifold of any differential forms in $\mathcal{I}$ is zero.)
First, suppose we wish to construct an austere submanifold \( M^d \) in \( \mathbb{R}^n \) such that at each point \( p \), \( |p| \) is conjugate to an austere subspace \( Q_\lambda \) of fixed dimension \( \delta \) depending on parameters \( \lambda^1, \ldots, \lambda^\delta \) which are allowed to vary along \( M \). Let the symmetric matrices \( S^\delta(\lambda), \ldots, S^{d+4}(\lambda) \) be a basis for the subspace \( Q_\lambda \) and suppose the parameters are allowed to range over an open set \( L \subset \mathbb{R}^\delta \). Let
\[
\theta^a := \omega^a - S^a_{ij}(\lambda)\omega^j.
\]
Then on \( \mathcal{F} \times L \) we define the Pfaffian exterior differential system
\[
\mathcal{H} = \{ \omega^a, \theta^a \}
\]
whose integral submanifolds correspond to austere manifolds of type \( Q_\lambda \). Namely, given any austere manifold \( M \) of this kind, we may construct an adapted semiorthonormal frame along \( M \) such that
\[
\Pi(\mathbf{e}_1, \mathbf{e}_j) = S^a_{ij}(\lambda)\mathbf{e}_a
\]
for functions \( \lambda^1, \ldots, \lambda^\delta \) on \( M \). Then the image of the fibered product of the mappings \( p \mapsto (p, \mathbf{e}_1(p), \mathbf{e}_a(p)) \) and \( p \mapsto (\lambda^1(p), \ldots, \lambda^\delta(p)) \) will be an integral submanifold of \( \mathcal{H} \). Conversely, any integral submanifold of \( \mathcal{H} \) satisfying the independence condition gives (by projecting onto the first factor in \( \mathcal{F} \times L \)) a section of \( \mathcal{F}|_M \) which is an adapted frame for an austere manifold \( M \).

Now consider the additional condition that \( M \) is ruled by \( k \)-dimensional planes. We will let \( E \subset TM \) denote a smooth distribution on \( M \) with fiber \( E_p \) at \( p \in M \). The following result characterizes those distributions that are tangent to a ruling of \( M \):

**Proposition 4.** Let \( E \) be a smooth tangent \( k \)-plane field on \( M \). Then \( M \) is ruled by \( k \)-planes in \( \mathbb{R}^n \) tangent to \( E \) if and only if at each \( p \in M \)
\[
\mathcal{D}_p w \in E \quad \forall v \in E_p, \quad w \in \Gamma(E),
\]
where \( \mathcal{D} \) denotes the Euclidean connection in \( \mathbb{R}^n \).

Projecting the condition (10) into the normal bundle, we get
\[
\Pi(v, w) = 0 \quad \forall v, w \in E,
\]
and we see from (9) that this puts extra conditions on \( S^a_{ij} \). In fact, for \( k = 3 \) the condition (11) implies the following:

**Corollary 5.** Any 4-dimensional austere submanifold that is ruled by 3-planes is a generalized helicoid.

**Proof.** If we choose an orthonormal frame in which \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) span the ruling, then (11) implies that \( |\Pi| \) lies in the space of matrices with nonzero entries only in the fourth row and column. Thus, \( |\Pi| \) is simple, and the result follows by Bryant’s theorem quoted in Section 1. \( \square \)

From now on we will consider submanifolds ruled by 2-planes. We will ensure that the Pfaffian system \( \mathcal{H} \) encodes the condition (11) by assuming that \( E \) has a certain basis with respect to the orthonormal frame on \( M \), and choosing the space \( Q_\lambda \) so that this condition holds for those basis vectors. However, encoding the tangential part of (10) requires additional 1-form generators.

Suppose that \( \nu_1, \nu_2 \) are vector fields spanning \( E, \phi_1, \phi_2 \) are 1-forms that annihilate \( E \), and \( w_1, w_2 \) are vector fields that span the orthogonal complement of \( E \) at each point in an open set \( U \subset M \). Then the tangential part of (10) is equivalent to
\[
w_i \cdot \nabla \nu_j \equiv 0 \quad \text{mod} \quad \phi_1, \phi_2
\]
for all \( 1 \leq i, j \leq 2 \). (Here, \( \nabla \) denotes the Riemannian connection of \( M \), so that \( \nabla \nu_i \) is a \((1, 1)\) tensor on \( U \).) We may encode this condition by defining 1-forms
\[
\psi^i_j := w_i \cdot \nabla \nu_j - p^i_{jk} \phi_k,
\]
for arbitrary coefficients \( p^i_{jk} \). (We will give the specific forms for the \( \psi^i_j \) in later sections.)

Thus, we define the **ruled austere system** \( \mathcal{I} \) as the Pfaffian system generated by the forms \( \omega^a \) and \( \theta^a = \omega^a - S^a_{ij}(\lambda)\omega^j \) from \( \mathcal{H} \), plus the extra 1-forms \( \psi^i_j \). The integral submanifolds of this differential ideal
\[
\mathcal{I} = \{ \omega^a, \theta^a, \psi^i_j \}
\]
correspond to 2-ruled austere manifolds of type $Q_3$. The system $J$ is in general defined on the manifold $\mathcal{F} \times L \times \mathbb{R}^8$, due to the introduction of the $p_{jk}^i$ as new variables. However, for specific types we will restrict to submanifolds of $\mathcal{F} \times L \times \mathbb{R}^8$ on which certain necessary integrability conditions hold. We will also sometimes restrict to using orthonormal frames, in which case the ruled system will be denoted by $\mathcal{F}_{on}$.

When analyzing system $I$, we will often need to calculate the space of integral elements, i.e., subspaces of the tangent space of the underlying manifold to which all differential forms in the ideal restrict to be zero. We will consider only 4-dimensional integral elements which satisfy the independence condition, i.e., $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4$ must restrict to be nonzero on the integral element.

3. Type A

Recall from [1] that if a submanifold $M \subset \mathbb{R}^n$ carries an almost complex structure $J$ such that $II(JX, Y) = II(X, JY)$ for all tangent vectors $X, Y$, then $M$ is austere. We refer to these as Type A. On austere 4-folds of this type, we will take orthonormal frames with respect to which the almost complex structure is represented by

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. $$

The maximal austere subspace $Q_A$ consists of symmetric matrices which anticommute with $J$. This subspace is preserved by the action of $U(2) \subset SO(4)$, the group of orthogonal matrices commuting with $J$. In Corollary 7 of [6] we showed that any austere 4-fold of Type A with $\delta \geq 2$ is Kähler with respect to $J$. Consequently, $J$ is integrable and parallel along $M$, and the connection forms must satisfy

$$\omega_1^4 = \omega_3^4, \quad \omega_1 = -\omega_2. $$

Suppose a Type A austere submanifold is 2-ruled. Then $E + J(E)$ is $J$-invariant, so is either 2- or 4-dimensional. We treat these cases separately.

Case A.1: $E = J(E)$

Let $M$ be a 2-ruled austere submanifold of this type. Using the $U(2)$ symmetry, we can choose near each point a semiorthonormal frame with respect to which $II$ lies in $Q_A$ and $E$ is spanned by $e_1, e_2$. Thus, $II$ must lie inside the subspace

$$\mathcal{R} = \left\{ \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & b_1 & -a_1 \\ a_1 & b_1 & a_2 & b_2 \\ b_1 & -a_1 & b_2 & -a_2 \end{bmatrix} : a_1, a_2, b_1, b_2 \in \mathbb{R} \right\}, $$

consisting of matrices in $Q_A$ satisfying the algebraic condition (11). Thus, $\delta \leq 4$.

Proposition 6. The map $\gamma : M \rightarrow Gr(2, n)$, taking $p \in M$ to the subspace parallel to $E_p$, is holomorphic.

Proof. The usual complex structure on $Gr(2, n)$ is assumed. Define a map $\pi : \mathcal{F} \rightarrow Gr(2, n)$ that takes $(p, e_1, \ldots, e_n)$ to the span of $\{e_1, e_2\}$; then $\gamma = \pi \circ f$, where $f : M \rightarrow \mathcal{F}$ is the section provided by the frame. The pullbacks under $\pi$ of the $(1, 0)$-forms on $Gr(2, n)$ are spanned by $\omega_3^1 - i\omega_2^1, \ldots, \omega_3^n - i\omega_2^n$. Meanwhile, a basis for the $(1, 0)$-forms on $M$ is given by

$$\tau := \omega_1^1 + i\omega_2^1, \quad \zeta := \omega_3^1 + i\omega_4^1. $$

For $a > 4$, the connection forms for the framing satisfy $\omega_1^a = S_1^a \omega_1^1$ for some matrices $S^a$ taking value in $\mathcal{R}$. We then compute that $\omega_1^a - i\omega_2^a = (S_1^a - iS_2^a)\zeta$ along the lift of $M$ into $\mathcal{F}$. Thus, $f^*(\omega_1^a - i\omega_2^a)$ is a multiple of $\zeta$ for each $a > 4$. We must show that this is also true for the pullbacks of $\omega_1^1 - i\omega_2^1$ and $\omega_3^1 - i\omega_4^1$.

The subgroup $U(1) \times U(1) \subset U(2)$ preserves $\mathcal{R}$. Given any nontrivial subspace of $\mathcal{R}$, we may use this symmetry to arrange that the subspace contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & r & 0 \\ 0 & 0 & 0 & -r \\ r & 0 & q & 0 \\ 0 & -r & 0 & -q \end{bmatrix}$$

for $q, r$ not both zero. Thus, along $M$ we can adapt a semiorthonormal frame such that, say, $S^5$ has the form (18).

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1 See [9, Chapter XI, Example 10.6], in which $Gr(2, n)$ is identified with the quadric in $\mathbb{C}^{n+1}$; the complex structure for $Gr(2, n)$ as an Hermitian symmetric space is discussed a few pages later.
The 1-forms $\theta_i^a$, defined by (8), vanish on $\Sigma = f(M)$. Computing their exterior derivatives gives

$$-d\theta_i^a \equiv (dS^a_{ij} - S^a_{ik} \omega^k_j - S^a_{kj} \omega^i_k + S^a_{ij} \omega^k) \wedge \omega^j \mod \omega^a, \theta_i^a.$$  \hspace{1cm} (19)

The ruled condition (12), together with (15), implies that

$$\omega_1^2 = \omega_2^4 = u\omega^3 + v\omega^4, \quad \omega_3^5 = -\omega_4^1 = x\omega^3 + y\omega^4,$$

for some functions $u, v, x, y$ along $\Sigma$. Using these values in (19), we find that

$$0 = (-d\omega_1^2 + i d\omega_2^5) \wedge \zeta = 2r(x - v + i(y + u)) \tau \wedge \omega^3 \wedge \omega^4.$$

The 3-form $\tau \wedge \omega^3 \wedge \omega^4$ is nonvanishing since $\Sigma$ satisfies the independence condition. Thus, wherever $r \neq 0$ we have

$$v = x, \quad u = -y.$$  \hspace{1cm} (21)

Even if $r$ is identically zero on an open set in $\Sigma$, we can compute

$$0 = (-d\omega_1^2 + i d\omega_2^5) \wedge \zeta = q(x - v + i(u + y)) \tau \wedge \omega^3 \wedge \omega^4,$$

again showing that (21) must hold along $\Sigma$. Substituting (21) into (20) shows that

$$\omega_1^2 - i \omega_3^5 = -i(x - iy)\zeta, \quad \omega_1^2 - i \omega_2^5 = -(x - iy)\zeta.$$

Thus, $\gamma$ is holomorphic. $\square$

Let $I$ denote the Pfaffian system (14) for 2-ruled austere submanifolds of Type A satisfying $J(E) = E$. Our choice that $e_1, e_2$ span the ruling implies that $\phi_1 = \omega^3$ and $\phi_2 = \omega^4$ in the 1-form generators $\psi_j^1$. In fact, based on the computations in the proof of Proposition 6, we will take

$$\psi_1^1 = \omega_1^2 + y\omega^3 - x\omega^4, \quad \psi_1^2 = \omega_3^5 - x\omega^3 - y\omega^4,$$

$$\psi_2^1 = \omega_2^4 + x\omega^3 + y\omega^4, \quad \psi_2^2 = \omega_4^5 + y\omega^3 - x\omega^4.$$  \hspace{1cm} (22)

**Definition of twistor space**

Assuming that the Gauss map of $M$ has full rank, then $|\xi|$ is conjugate to a subspace of $R$ spanned by matrices of the form

$$S^5 = \begin{bmatrix} 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & -p & 0 \\ p & 0 & q & 0 & 0 \\ 0 & -p & 0 & q & 0 \end{bmatrix}, \quad S^6 = \begin{bmatrix} 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^\ell = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^j & c_2^j \\ 0 & 0 & c_2^j & -c_1^j & 0 & 0 \end{bmatrix},$$

$$6 < \ell < n.$$  \hspace{1cm} (23)

Along $M$, we choose a semiorthonormal frame so that (9) holds. We then compute

$$d(\psi_1^1 - \psi_2^2) \equiv 2p g_{56}(\omega_1^3 \wedge \omega^4 + \omega_1^5 \wedge \omega^5) + (g_{55} - p^2 g_{56})(\omega_1 \wedge \omega^5 - \omega_2 \wedge \omega^4)$$

modulo $\theta_i^a, \theta_i^a, \psi_j^1$. Because of the independence condition, both coefficients on the right must vanish. Because $p = 0$ would imply that $g_{55} = 0$, we must have $g_{56} = 0$, i.e., vectors $e_5, e_6$ are orthogonal. We now re-scale them so that they are of unit length, and choose the remaining frame vectors $e_t$ for $\ell > 6$ to complete an orthonormal frame. With respect to this frame, the coefficient matrices of the second fundamental form are now

$$S^5 = \begin{bmatrix} 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & -r & 0 \\ r & 0 & 0 & q & 0 \\ 0 & -r & 0 & 0 & q \end{bmatrix}, \quad S^6 = \begin{bmatrix} 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^\ell = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^j & c_2^j \\ 0 & 0 & c_2^j & -c_1^j & 0 & 0 \end{bmatrix},$$

$$6 < \ell < n.$$  \hspace{1cm} (24)

for some functions $q, r, s, t$ with $r \neq 0$. (When $\delta = 2$ and $n = 6$, the matrices $S^\ell$ are omitted.) Note that the orthonormal vectors $e_1, \ldots, e_6$ are uniquely defined, up to simultaneously rotating in the $e_1$-$e_2$ plane, the $e_3$-$e_4$ plane, and the $e_5$-$e_6$ plane.

We may now define the map $\Gamma$ to the twistor space, referred to in Theorem 1. At each point $p \in M$, let $F_p$ be the 6-dimensional oriented subspace in $\mathbb{R}^a$ spanned by the oriented basis $e_1, \ldots, e_5$, and let $J_p$ be the complex structure on $F_p$, taking $e_1$ to $e_2$, $e_3$ to $e_4$, and $e_5$ to $e_6$. (This extends the complex structure on $T_pM$.) Define $V$ as the space of triples $(E, F, J)$ such that $E, F$ are oriented subspaces of $\mathbb{R}^a$ of dimension 2 and 6 respectively, $E \subset F$, and $J$ is an orthogonal complex structure on $F$ preserving $E$. Then we define a smooth mapping $\Gamma : M \rightarrow V$ sending $p$ to $(E_p, F_p, J_p)$.

By adapting orthonormal frames to each triple $(E, F, J)$, we identify $V$ with the homogeneous space $SO(5)/SO(2) \times U(2)$, where the subgroups $SO(2)$ and $U(2) \subset SO(4)$ sit in blocks along the diagonal in $SO(n)$. The submersion $\rho : (E, F, J) \rightarrow E$ to the Grassmannian corresponds to replacing $U(2)$ by the diagonal $SO(n - 2)$ in the quotient.
The mapping $\Gamma$

We will give $V$ an integrable complex structure so that $\Gamma$ is holomorphic. To this end, define the following complex-valued 1-forms on $SO(n)$:

$$\eta^\ell := \omega^\ell_j - i\omega^\ell_k, \quad 2 < \ell \leq n,$$

$$v^\ell := \omega^\ell_j - i\omega^\ell_k, \quad 4 < \ell \leq n,$$

$$\sigma^\ell := \omega^\ell_j - i\omega^\ell_k, \quad 6 < \ell \leq n.$$ 

Within the span of these forms, the following annihilate the Lie subalgebra $a\mathfrak{o}(2) \oplus u(2) \subset a\mathfrak{o}(n)$, and thus are semibasic for the quotient map $\pi : SO(n) \to V$:

$$\eta^3, \ldots, \eta^n, v^5 - iv^6, v^\ell, \sigma^\ell, \quad 6 < \ell \leq n. \tag{25}$$

In fact, these span the pullback to $SO(n)$ of the bundle of $(1,0)$-forms for the complex structure on $V$.

Let $I_{\Omega_0}$ denote the ruled system defined using orthonormal frames, whose generators are $\omega^a$ for $a > 4, \theta^a_1$ as defined by (8) with matrices given by (24), and $\psi_j$ as defined by (22). The generators of $I_{\Omega_0}$ are related to the $(1,0)$-forms of $V$ as follows:

$$\psi_1^3 - i\psi_2^3 = \eta^3 + (y + ix)\xi, \quad \psi_1^4 - i\psi_2^4 = \eta^4 + (x - iy)\xi,$$

$$\theta_1^3 - i\theta_2^3 = \eta^5 - r\xi, \quad \theta_3^3 - i\theta_4^3 = v^5 - rt - q\xi,$$

$$\theta_1^5 - i\theta_2^5 = \eta^6 + ir\xi, \quad \theta_3^5 - i\theta_4^5 = v^6 + irt - (s - it)\xi,$$

$$\theta_1^7 - i\theta_2^7 = \eta^n, \quad \theta_3^7 - i\theta_4^7 = v^\ell - (c^\ell_1 - ic^\ell_2)\xi, \quad 6 < \ell \leq n. \tag{26}$$

where $\tau$ and $\xi$ are as in (17). We compute that

$$d(\theta_1^\ell - i\theta_2^\ell) \equiv r\sigma^\ell \wedge \xi, \quad 6 < \ell \leq n,$$

modulo the 1-forms of $I_{\Omega_0}$. Hence, on any integral submanifold of $I_{\Omega_0}$, all the 1-forms in (25) pull back to be multiples of $\xi$. (Notice that the $\tau$ terms cancel out in the linear combination $v^5 - iv^6$.) Therefore, the mapping $\Gamma$ is holomorphic. Since $r$ is nonzero, it has real rank 2.

The image $\mathcal{K}$ of $\Gamma : M \to V$ is not a generic holomorphic curve; in fact, it is an integral of a well-defined complex Pfaffian system on $V$. For, within the span of the semibasic forms listed in (25), the 1-forms in the following subsystem vanish on any integral of $I_{\Omega_0}$:

$$\mathcal{K} := \{ \eta^3 - i\eta^4, \eta^5 - i\eta^6, \eta^\ell \}.$$ 

Then one calculates that

$$d(\eta^3 - i\eta^4) \equiv (v^5 - iv^6) \wedge \eta^5,$$

$$d(\eta^5 - i\eta^6) \equiv -(v^5 - iv^6) \wedge \eta^3,$$

$$d\eta^\ell \equiv \eta^3 \wedge v^\ell + \eta^5 \wedge \sigma^\ell$$

modulo the 1-forms in $\mathcal{K}$. The fact that the right-hand sides are pure wedge products of semibasic forms for $\pi$ indicates that $\mathcal{K}$ is the pullback of a well-defined Pfaffian system $\mathcal{K}$ on $V$ (see [7, Proposition 6.1.19]).

Remark 7. The Pfaffian system $\mathcal{K}$ may be characterized in two ways.

First, note that $V$ is the total space of a double fibration over more familiar complex homogeneous spaces, with mappings $\rho : V \to Gr(2, \mathbb{R}^n)$ and $\rho' : V \to SO(n)/U(3)$. (The latter is the space of 6-planes in $\mathbb{R}^n$ carrying an orthogonal complex structure, and the mapping $\rho'$ is defined by replacing $SO(2) \times U(2)$ with $U(3)$ in the quotient.) Then $\mathcal{K}$ is spanned by the intersection of the pullbacks of $(1,0)$-forms on the Grassmannian via $\rho$ and the pullbacks of $(1,0)$-forms on the second space via $\rho'$.

Next, recall that the tangent space to $Gr(2, \mathbb{R}^n)$ at $E$ is naturally identified with $E^* \otimes \mathbb{R}^6/E$. Thus, for any tangent vector $v \in T_pM$, $\gamma_*v$ is an element of $(E_p)^* \otimes \mathbb{R}^6/E_p$. However, the form of the matrices $S^a$ in this case, and the vanishing of the 1-forms $\psi_j^1$ on $M$, imply that $\Gamma : M \to V$ satisfies the following

Contact Condition: For any $v \in T_pM$, $\Gamma_*v$ takes value in $(E_p)^* \otimes F_\rho/E_p$ and its value, as a mapping, is complex-linear with respect to $F_\rho$.

In fact, any holomorphic mapping into $V$ is an integral of $\mathcal{K}$ if and only if it satisfies this contact condition.
In the remainder of this subsection, we will discuss results for specific values of \( \delta \).

(a) \( \delta = 2 \). The analysis in Section 5 of our previous paper [6] shows that when \( M \) is an austere 4-fold of Type A, with \( \delta = 2 \), then \( \| \| \) has a two-dimensional nullspace \( E \) only if either the Gauss map is degenerate, or \( \| \| \) is conjugate to the span of the matrices \( S^5, S^6 \) in (23). In the first case, \( M \) belongs to the class of elliptic austere manifolds investigated by Dajczer and Florit [4], which are ruled by totally geodesic submanifolds of codimension two in \( M \).

In the second case, \( M \) lies in \( \mathbb{R}^5 \), and the twistor space \( V \) is \( SO(6)/SO(2) \times U(2) \). There is an obvious submersion to a lower-dimensional space \( SO(6)/U(3) \), which is the Hermitian symmetric space of orthogonal complex structures on \( \mathbb{R}^6 \) and is biholomorphic to \( \mathbb{C}P^3 \). Let \( \hat{F} : M \to \mathbb{C}P^3 \) denote the composition of \( F \) with this submersion. Remarkably, the image of \( M \) under \( \hat{F} \) is an arbitrary holomorphic curve. For, in Theorem 16 of [6] it is shown that, given any holomorphic curve \( \mathcal{C} \) in \( \mathbb{C}P^3 \), and a non-planar point \( p \in \mathcal{C} \), we can construct (by solving a first-order system of PDE) an austere \( M \subset \mathbb{R}^6 \) of this type, such that \( \hat{F}(M) \) is an open set in \( \mathcal{C} \) containing \( p \).

(b) \( \delta = 3 \). It is easy to check that the first prolongation (as a tableau\(^2\)) of any 3-dimensional subspace of \( \mathcal{R} \) is zero. Hence, by Proposition 3 in [6], \( M \) lies in a totally geodesic copy of \( \mathbb{R}^7 \) when \( \delta = 3 \).

In this case, the system \( \mathcal{I}_{\text{on}} \) is defined using basis matrices \( S^5, S^6, S^7 \) given by (24) with \( n = 7 \). Thus, the manifold on which \( \mathcal{I}_{\text{on}} \) is defined is \( \mathcal{I}_{\text{on}} \times L \), where \( L \subset \mathbb{R}^8 \) is the open set with coordinates \( x, y, q, r, s, t, c_1 = c_1^7, c_2 = c_2^5 \) such that \( r \not= 0 \). The twistor space \( V \) is \( SO(7)/(SO(2) \times U(2)) \) and the complex system \( \mathcal{X} \) has rank 3.

We will show that \( M \) can be locally reconstructed from the image of \( \Gamma' \).

Theorem 8. Let \( \mathcal{C} \) be a holomorphic curve in \( V \) which is an integral of \( \mathcal{X} \). Given any point \( p \in \mathcal{C} \) at which \( \rho|_{\mathcal{C}} \) is nonsingular, there is an open neighborhood \( U \subset \mathcal{C} \) containing \( p \) and an austere \( M^4 \subset \mathbb{R}^7 \) which is of Type A, 2-ruled with \( J(E) = E \), such that \( \Gamma'(M) = U \).

Proof. Let \( N \subset SO(7) \) be the inverse image of \( \mathcal{C} \). Let \( z \) be a local holomorphic coordinate defined on open set \( U \subset \mathcal{C} \). Then there are functions \( f_1, \ldots, f_5 \) on \( \rho^{-1}(U) \subset N \) such that

\[
\begin{align*}
\eta^3 &= f_1 dz, \\
\eta^5 &= f_2 dz, \\
\nu^5 - i\nu^6 &= f_3 dz, \\
\nu^7 &= f_4 dz, \\
\sigma^7 &= f_5 dz.
\end{align*}
\]

By hypothesis, \( f_1 \) and \( f_2 \) are never both zero. By substituting (27) into (26), we see that to construct an integral manifold of \( \mathcal{I}_{\text{on}} \) on which \( \zeta \neq 0 \) we will need \( f_2 \) to be nonzero. However, the action of \( SO(2) \times U(2) \) on the fiber of \( \rho \) ensures that there is an open subset \( N_0 \subset \rho^{-1}(U) \) where this condition holds.

Let \( W \subset \mathcal{I}_{\text{on}} \times L \) be the inverse image of \( N_0 \) under the projection to \( SO(7) \). The restriction of \( \mathcal{I}_{\text{on}} \) to \( W \) is spanned by the \( \omega^\theta \) and the real and imaginary parts of \( \sigma^\gamma \) and the 1-forms on the right-hand sides in (26). However, these forms are now linearly dependent, because \( \eta^4 = i\eta^3 \), \( \eta^6 = i\eta^5 \) and \( \eta^7 = 0 \) on \( W \), and the rest satisfy the following relationships:

\[
\begin{align*}
\eta^3 + (y + ix)\zeta &= f_1 dz + (y + ix)\zeta, \\
\eta^5 - r\zeta &= f_2 dz - r\zeta, \\
(v^5 - iv^6 + i(r\zeta + s + i(t + q + is)) &= f_3 dz + (t + q + is)\zeta, \\
\nu^7 - (c_1 - ic_2)\zeta &= f_4 dz - (c_1 + ic_2)\zeta.
\end{align*}
\]

Because the left-hand sides are in the ideal, we need the right-hand sides to vanish; in particular, the second line implies that \( dz = (t/f_2)\zeta \) on solutions. Thus, we restrict to the submanifold \( W' \subset W \) on which

\[
y + ix = -rf_1/f_2, \quad t - q + is = -rf_3/f_2, \quad c_1 - ic_2 = rf_4/f_2.
\]

On \( W' \), the remaining linearly independent 1-form generators of \( \mathcal{I}_{\text{on}} \) are the \( \omega^\theta \) and the real and imaginary parts of \( \beta_1 := r\zeta - f_2 dz, \quad \beta_2 := r\tau + q\zeta - v^5 \).

Because (28) determines \( x, y, s, t, c \) in terms of \( q, r \) and the functions \( f_j \), a coframe for \( W' \) is given by these 1-forms of \( \mathcal{I}_{\text{on}} \), together with \( \omega^1, \ldots, \omega^4, dq, dr, \omega^6, \alpha^7 \) and \( \omega^6 \).

To test the Pfaffian system for involutivity, we compute on \( W' \) that

\[
\begin{align*}
d\beta_1 &\equiv \pi_1 \wedge \zeta \\
d\beta_2 &\equiv \pi_1 \wedge \tau + \pi_2 \wedge \zeta \mod \omega^\theta, \beta_1, \beta_2.
\end{align*}
\]

where

\[
\begin{align*}
\pi_1 &\equiv d\tau + i(r(\omega^6 - \alpha^7) \mod \tau, \zeta, \\
\pi_2 &\equiv dq - 2i(\omega^6 - \alpha^7/(q - rf_3/f_2)) \mod \zeta, \bar{\zeta}.
\end{align*}
\]

\(^2\) Given a linear subspace \( \mathcal{R} \subset V^* \otimes W \), the prolongation \( \mathcal{R}^{(1)} \) is the intersection of \( V^* \otimes \mathcal{R} \) with the subspace \( S^V V^* \otimes W \). In our case, \( V = W = R^4 \).
(Note that these are the only 2-form generators for $\mathcal{I}_{on}$ on $W'$, since the $\omega^q$ lie in the first derived system of $\mathcal{I}_{on}$.) The real and imaginary parts of $\pi_1, \pi_2$ will be linearly independent provided that either $q$ or $f_3$ is nonzero. Accordingly, we restrict our attention to the open set where this is the case. Then the system is involutive with terminal Cartan character $s_1 = 4$. It follows by applying the Cartan–Kähler Theorem (see, e.g., [7]) that through any point of $W'$ there exists an integral 4-fold of $\mathcal{I}_{on}$, satisfying the independence condition, which projects to an austere submanifold $M \subset \mathbb{R}^7$. □

**Remark 9.** The starting ingredient in Theorem 8 is an integral curve of system $\mathcal{I}$ on the homogeneous space $V$ of complex dimension 8. Since $\mathcal{I}$ is generated by holomorphic 1-forms on $V$, this system is equivalent to an underdetermined system of three ordinary differential equations in local complex coordinates. Hence, it is possible that explicit solutions could be written down in terms of 4 arbitrary holomorphic functions.

(c) $\delta = 4$. In this case, $\|\|$ is all of the space $\mathcal{R}$ defined in (16). We will state the analogue of Theorem 8 for this case. Note that because the prolongation $\mathcal{R}^{(1)}$ is nonzero, we cannot assume that $M$ lies in $\mathbb{R}^{4+\delta}$; in fact, austere 4-folds of this type exist which are substantial\(^3\) in $\mathbb{R}^n$ for any $n \geq 8$.

**Theorem 10.** Let $\mathcal{E}$ be a holomorphic curve in $V$ which is an integral of $\mathcal{I}$, and has a nonsingular projection onto $\text{Gr}(2, \mathbb{R}^n)$. Given any point $p \in \mathcal{E}$, there is an open neighborhood $U \subset \mathcal{E}$ containing $p$ and an austere $M^4 \subset \mathbb{R}^8$ which is of Type A, 2-ruled satisfying $J(E) = E$, such that $\Gamma'(M) = U$.

The proof is completely analogous to that of Theorem 8; at the last stage, the Cartan–Kähler Theorem is required to construct an integral for an involutive system with terminal Cartan character $s_1 = 4$.

**Case A.2:** $E \cap J(E) = 0$

In this case we can use the $U(2)$ symmetry to choose orthonormal frames so that $E$ is spanned by $e_1$ and $e_2 + ke_3 + me_4$ for some functions $k, m$. Then the subspace $\mathcal{R} \subset \mathcal{Q}_A$ of matrices satisfying (11) is 3-dimensional. One can check that $\mathcal{R}^{(1)} = 0$, so that $\delta = \dim \|\|$ is the effective codimension\(^4\) of $M$.

**Theorem 15 in [6]** implies that any 2-ruled submanifolds of Type A with $\delta = 2$ are those with $J(E) = E$, discussed in Case A.1(a) above. In the case where $\delta = 3$, an extensive analysis of the exterior differential system $\mathcal{I}$ yields integrability conditions which imply that no such submanifolds exist. To give the reader a sense of this analysis, we outline the proof in the special case where $E$ and $J(E)$ are assumed to be orthogonal.

Along $M$ we choose near each point a semiorthonormal frame with respect to which $\|\|$ lies in $\mathcal{Q}_A$, and $E$ is spanned by $e_1$ and $e_3$. Then the algebraic condition (11), together with the assumption $\delta = 3$, implies that $\|\|$ is spanned by

\[
S^5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
S^6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\
S^7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]

(29)

Let $\mathcal{I}$ be the ruled system with this choice of matrices $S^q$. The generators $\psi_j^\mathcal{I}$ are given by

\[
\psi_1^\mathcal{I} = \omega^2 - x_1\omega^2 - x_3\omega^4, \\
\psi_2^\mathcal{I} = \omega^2 - y_1\omega^2 - y_2\omega^4, \\
\psi_3^\mathcal{I} = \omega^4 - y_1\omega^2 - y_2\omega^4, \\
\psi_4^\mathcal{I} = \omega^4 - z_1\omega^2 - z_2\omega^4.
\]

Note that the coefficients in $\psi_1^\mathcal{I}$ and $\psi_2^\mathcal{I}$ are the same because of the second part of the Kähler condition (15); to encode the rest of (15), we add to $\mathcal{I}$ the additional generator 1-form

\[
\psi_0^\mathcal{I} = \omega^2 - \omega^3,
\]

and let $\mathcal{I}^+$ denote the resulting Pfaffian system, which is defined on $\mathcal{I} \times \mathbb{R}^6$ with coordinates $x_1, x_2, y_1, y_2, z_1, z_2$ on the second factor. We calculate that

\[
\begin{align*}
d\theta_5^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv (y_1 - 2x_2^2)\Omega, \\
d\theta_4^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv -y_2\Omega, \\
d\theta_6^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv (x_1 - 2y_2 + z_1)\Omega, \\
d\theta_5^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv (-x_2 + 2y_1 - z_2)\Omega, \\
d\theta_7^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv y_1\Omega, \\
d\theta_6^\mathcal{I} \wedge \omega^1 \wedge \omega^3 &\equiv (2z_1 - y_2)\Omega.
\end{align*}
\]

\(^3\) A submanifold is substantial in $\mathbb{R}^n$ if it does not lie in a smaller dimension totally geodesic submanifold.

\(^4\) The effective codimension of a submanifold $M$ is its codimension within the smallest totally geodesic submanifold that contains it.
modulo the 1-forms of \( I^+ \), where \( Q := \omega_1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \). Thus, we obtain integrability conditions that imply that \( x_1, x_2, y_1, y_2, z_1, z_2 \) must all be zero. Restricting to the submanifold where these conditions hold, we then compute (for example) that

\[
\psi_1 = g_{55} \omega_1 \wedge \omega^2 + g_{56} \left( \omega_1 \wedge \omega^4 - \omega^2 \wedge \omega^3 \right) + g_{66} \omega_3 \wedge \omega^4.
\]

Thus, integrability conditions imply that \( g_{55} = 0 \) and \( g_{66} = 0 \), which is impossible since the metric \( g_{ab} \) on the normal bundle must be positive definite.

4. Type B ruled austere submanifolds

In this section we discuss the 2-ruled austere 4-folds of Type B. Recall from [6] that the maximal austere subspace of Type B is given by

\[
Q_B = \left\{ \begin{bmatrix} m & 0 & b_1 & b_2 \\ 0 & m & b_3 & b_4 \\ b_1 & b_3 & -m & 0 \\ b_2 & b_4 & 0 & -m \end{bmatrix} \mid m, b_1, \ldots, b_4 \in \mathbb{R} \right\},
\]

and may be characterized as the span of a matrix representing a reflection \( R \) that fixes a 2-plane in \( \mathbb{R}^4 \), together with the symmetric matrices that commute with that reflection. Thus, austere 4-folds of Type B carry a reflection automorphism of the tangent space (well-defined up to multiple) which we also denote by \( R \). Depending on the position of the ruling plane \( E \) relative to the eigenspaces of \( R \), the intersection \( R(E) \cap E \) is a vector space of dimension equal to 0, 1 or 2. We examine these cases separately.

Case B.1: \( \dim R(E) \cap E = 2 \)

In this case, \( E \) can be an eigenspace of \( R \) or can be a sum of the +1 and -1 eigenspaces of \( R \).

(a) \( E \) is an eigenspace of \( R \). The symmetry group of \( Q_B \) is generated by conjugation by \( O(2) \times O(2) \subset O(4) \) and the permutation \( e_1 \leftrightarrow e_3, e_2 \leftrightarrow e_4 \). Using the permutation we can assume \( E = \langle e_1, e_2 \rangle \). Since the second fundamental form vanishes on the ruling \( E \), this implies that \( m = 0 \). Therefore the subspace of \( Q_B \) satisfying the algebraic condition (11) is

\[
R = \left\{ \begin{bmatrix} 0 & B \\ \cdot & 0 \end{bmatrix} \mid B \in M_{2 \times 2}(\mathbb{R}) \right\}.
\]

Since \( Q(1) \) is zero, the effective codimension of \( M \) is equal to \( \delta \). We break into subcases according to the value of \( \delta \). If \( \delta = 4 \) or \( \delta = 3 \), an analysis of the EDS \( I \) shows that there are no integral manifolds satisfying the independence condition. This somewhat lengthy analysis is omitted, but we explain the case \( \delta = 2 \) in more detail.

In this case, \( \|I\| \) is a 2-dimensional subspace of \( R \), determined by the 2-dimensional subspace \( B \subset M_{2 \times 2}(\mathbb{R}) \) that is spanned by the matrices \( B \) in the upper right corner. The conjugation action of \( O(2) \times O(2) \) induces an action on \( M_{2 \times 2}(\mathbb{R}) \) given by \( (S, T) \cdot B = S B T^{-1} \). Note that the determinant is invariant up to sign under this action; we distinguish several subcases depending on the type of the quadratic form \( \det \) given by the determinant restricted to \( E \):

(a.i) \( \det |_E \) has rank 2. In this case we normalize the space of second fundamental forms so that \( B \) is spanned by \( \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \) and \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) where \( pq < 0 \) if \( \det \) is definite and \( pq > 0 \) if \( \det \) is indefinite.

With respect to the orthonormal basis \( (e_1, e_2, e_3, e_4) \) of the tangent space to \( M \), \( \|I\| \) is spanned by the matrices

\[
S^5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{bmatrix}, \quad S^6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Let \( I \) be the ruled Pfaffian system, described in Section 2, for this choice of basis matrices. Because the forms \( \phi_1 = \omega_3^2 \), \( \phi_2 = \omega_4^2 \) span the annihilator of \( E \), the generator 1-forms \( \psi_1^i \) of \( I \) which encode the tangential part (12) of the ruling condition are

\[
\psi_1^1 = \omega_3^2 - u_1 \omega_3^2 - u_2 \omega_4^2, \quad \psi_1^2 = \omega_3^2 - v_1 \omega_3^2 - v_2 \omega_4^2, \\
\psi_2^1 = \omega_4^2 - x_1 \omega_3^2 - x_2 \omega_4^2, \quad \psi_2^2 = \omega_4^2 - y_1 \omega_3^2 - y_2 \omega_4^2.
\]

Thus, the system \( I \) is defined on \( \mathcal{F} \times \mathbb{R}^{16} \) with \( p, q, u_1, u_2, v_1, v_2, x_1, x_2, y_1, y_2 \) as coordinates on the second factor, and the restriction \( pq \neq 0 \).

Proposition 11. The only austere 4-manifolds of this kind are those where \( pq \) is identically equal to 1.
**Proof.** We analyze the structure equations of $\mathcal{I}$ with the 1-forms $\psi^j_i$ given above. We compute:

\[
\begin{align*}
\delta\theta^5 &= \pi_1 \wedge \theta^3 + \pi_2 \wedge \theta^4, \\
\delta\theta^5 &= \pi_1 \wedge \theta^1 + \pi_3 \wedge \theta^2 + (v_1 p + x_1 - 2u_2)\theta^3 \wedge \theta^4, \\
\delta\theta^5 &= \pi_3 \wedge \theta^3 + \pi_4 \wedge \theta^4, \\
\delta\theta^5 &= \pi_2 \wedge \theta^1 + \pi_4 \wedge \theta^2 + (2y_1 p - x_2 - v_2 p)\theta^3 \wedge \theta^4, \\
\delta\theta^5 &= \pi_5 \wedge \theta^3 + \pi_6 \wedge \theta^3, \\
\delta\theta^5 &= \pi_5 \wedge \theta^1 + \pi_7 \wedge \theta^2 + (u_1 + qy_1 - 2qv_2)\theta^3 \wedge \theta^4, \\
\delta\theta^5 &= \pi_7 \wedge \theta^3 + \pi_8 \wedge \theta^4, \\
\delta\theta^5 &= \pi_8 \wedge \theta^1 + \pi_8 \wedge \theta^2 + (2x_1 - u_2 - qy_2)\theta^3 \wedge \theta^4,
\end{align*}
\]

modulo $\omega^0, \omega^1, \omega^2, \omega^3$, where $\pi_1, \ldots, \pi_8$ are certain 1-forms that are linearly independent combinations of the connection forms $\omega^j_i, \omega^0_i, dp, dq$ and $\theta^i$. In each equation, the 2-form on the right-hand side must vanish on any integral element. Wedging the 2-forms in the right-hand column with $\omega^1 \wedge \omega^2$ gives the following integrability conditions:

\[
v_1 p + x_1 - 2u_2 = 0, \quad 2y_1 p - x_2 - v_2 p = 0, \quad u_1 + qy_1 - 2qv_2 = 0, \quad 2x_1 - u_2 - qy_2 = 0.
\]

Moreover, the vanishing of all the 2-forms on the right-hand sides of (32) implies that all the forms $\pi_1, \ldots, \pi_8$ must vanish on any integral 4-plane. (For example, the vanishing of the 2-form $\delta\theta^5$ implies that, on any integral element, $\pi_1$ must be a linear combination of $\omega^3$ and $\omega^4$, while the vanishing of $\delta\theta^5$ implies—with the integrability conditions (33) taken into account—that $\pi_1$ must be a linear combination of $\omega^1$ and $\omega^2$. Therefore, $\pi_1 = 0$ on any integral element.)

Let $\mathfrak{I}$ be the differential ideal obtained from adding the forms $\pi_1, \ldots, \pi_8$ to the original ideal $\mathcal{I}$, and restricting to the submanifold where the integrability conditions (33) hold. (We solve these equations for $u_1, u_2, x_1, x_2$ in terms of the rest.) While $\delta\theta^5 \equiv 0$ modulo the one-forms in $\mathfrak{I}$, we also compute that

\[
\begin{align*}
\delta\psi^1 &= (2q_1 \pi_1 - q_2 \pi_1 + F_1 \omega^1 + F_2 \omega^2) \wedge \omega^3 + (\frac{1}{2}q_2 \pi_1 + \frac{1}{2}q_2 \pi_2 + F_3 \omega^1 + F_4 \omega^2) \wedge \omega^4, \\
\delta\psi^2 &= \pi_9 \wedge \omega^3 + \pi_{10} \wedge \omega^4, \\
\delta\psi^3 &= (\frac{1}{2}q_2 \pi_9 + \frac{1}{2}q_2 \pi_12 + F_5 \omega^1 + F_6 \omega^2) \wedge \omega^3 + (2p \pi_9 - p \pi_10 + F_7 \omega^1 + F_8 \omega^2) \wedge \omega^4, \\
\delta\psi^4 &= \pi_11 \wedge \omega^3 + \pi_{12} \wedge \omega^4,
\end{align*}
\]

where $\pi_9, \ldots, \pi_{12}$ are linearly independent combinations of $dv_1, dv_2, dy_1, dy_2$ and the $\omega^i$, and $F_1, \ldots, F_8$ are certain polynomial functions in $p, q, v_1, v_2, y_1, y_2, g_{55}, g_{56}, g_{66}$. In Eqs. (34), the forms $g_{55}, g_{56}, g_{66}$ are not unique, but can be adjusted only by multiples of the forms $\omega^3$ and $\omega^4$; therefore all the polynomials $F_1, \ldots, F_8$ must vanish at points where integral elements of the ideal $\mathfrak{I}$ exist. Setting these equal to zero gives a system of 8 equations which are linear in $g_{55}, g_{56}, g_{66}$ with coefficients depending on $p, q, v_1, v_2, y_1, y_2$. Eliminating the $g_{ab}$ gives 5 homogeneous quadratic equations in $v_1, v_2, y_1, y_2$ with coefficients depending on $p$ and $q$. Of these, three equations constitute a homogeneous linear system in $v_1 v_2, v_1 y_1, y_1 y_2, v_2 y_2$. The coefficient matrix of this system has rank 3, unless $pq = 1$ or $pq = -1$. The latter cases will be considered separately later on; for now, suppose that $p + q \neq 0$ and $pq \neq \pm 1$. Then the vector $(v_1 v_2, v_1 y_1, y_1 y_2, v_2 y_2)$ must be a multiple of $(q, q, p, p)$, which spans the kernel of the matrix.

We distinguish two possible subcases. First, suppose that $v_1$ and $v_2$ are identically zero on an integral submanifold. Then $F_2 = 0$ implies that $g_{56} = 0$, and the 2-forms (34) determine the values of $dv_2$ and $dy_1$ uniquely on an integral element. Setting $d^2 v_2 = 0$ and $d^2 y_1 = 0$ implies that $g_{55} = g_{66} = 0$, which is impossible. Second, if $v_1$ or $y_2$ is nonzero, then $v_1 = (q/p)y_2$ and $v_2 = y_1$. Then $F_1 = \ldots = F_8 = 0$ implies that $g_{55} = g_{66} = 0$, which again is impossible.

In the case that $p + q = 0$, a computation of integrability conditions similar to (32) gives that $v_1 = y_2$ and $v_2 = -y_1$; then, restricting to the submanifold where these relations hold and computing $d\psi^j_i$ yields contradictory integrability conditions. We eliminate the case $pq = 1$ using similar computations. □

In what follows, we deal with the remaining case, when $pq = 1$. By replacing $e_3, e_4$ by linear combinations of themselves, we can assume that $||l||$ is spanned by matrices

\[
S^5 = \begin{pmatrix} 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ r & 0 & 0 & 0 \end{pmatrix}, \quad S^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -r \\ 0 & 1 & 0 & 0 \\ 0 & -r & 0 & 0 \end{pmatrix}.
\]

where $r$ is some nonzero function on $M$. The corresponding basis for the normal space is characterized by the fact that the components of $l$ in the direction of each of $e_5, e_6$ has rank 2, with a 3-dimensional nullspace. Within $E$, the lines spanned by $e_1$ and $e_2$ are characterized as intersections of $E$ with these nullspaces. In fact, with respect to the orthonormal coframe ($\omega^1, \omega^2, \omega^3, \omega^4$) on the tangent space, we may write

\[
ll = e_5 \otimes \omega^1 \circ \kappa_1 + e_6 \otimes \omega^2 \circ \kappa_2.
\]
where, for later convenience, we define
\[ \kappa_1 := \omega^3 + r\omega^4, \quad \kappa_2 := \omega^3 - r\omega^4. \]

We now take \( I \) to be the ruled austere system with the choice (35) of basis matrices, defined on \( \mathcal{F} \times \mathbb{R}^3 \) with coordinates \( r, u_1, u_2, v_1, v_2, x_1, x_2, y_1, y_2 \) on the second factor and \( r \neq 0 \). By computing \( d\theta^2 \) we obtain integrability conditions analogous to (33):
\[ \begin{align*}
    v_1 &= -\frac{y_1}{r}, & y_2 &= -ry_1, & u_1 &= \frac{x_1}{r}, & x_2 &= rx_1, & u_2 &= x_1, & v_2 &= y_1.
\end{align*} \]

We restrict \( I \) to the subset \( \mathcal{F} \times \mathbb{R}^3 \) on which these integrability conditions hold, with \( r, x_1, y_1 \) as coordinates on the second factor. (We will abbreviate \( x_1 \) by \( x \) and \( y_1 \) by \( y \) in what follows.) The generators \( \psi_j \) now take the form
\[ \begin{align*}
    \psi_1^1 &= \omega_1^3 - \frac{x}{r} \kappa_1, & \psi_2^1 &= \omega_2^3 + \frac{y}{r} \kappa_2, \\
    \psi_1^2 &= \omega_2^4 - x \kappa_1, & \psi_2^2 &= \omega_2^4 - y \kappa_2.
\end{align*} \]

By computing \( (d\psi_1^3 \wedge \omega_3 + d\psi_1^4 \wedge \omega_4) \wedge \omega^1 \) modulo the 1-forms of \( I \), we conclude that integral elements occur only at points where the following extra integrability condition holds:
\[ g_{56} = xy(r^{-2} - 1). \] (36)

Restricting to the codimension-one submanifold \( N \subset \mathcal{F} \times \mathbb{R}^3 \) where this condition holds yields an EDS \( I' \) with a unique integral element at each point. Adjoining to \( I' \) the 1-forms that vanish on these yields a Frobenius system \( \beta \) on \( N \). (Thus, \( \beta \) is locally equivalent to a compatible system of total differential equations on \( N \), and the integral 4-folds of \( \beta \) foliate \( N \).)

We now wish to interpret the solutions:

**Proposition 12.** Let \( M \) be a connected, 2-ruled austere submanifold of Type B, substantial in \( \mathbb{R}^6 \), such that \( E \) is preserved by the reflection \( B \) and the rank of \( d\theta|_{\mathcal{B}} \) is 2. Then \( M \) is congruent to an open subset of a Cartesian product \( H_1 \times H_2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3 \) where \( H_1 \subset \mathbb{R}^3 \) is a helicoid. The rulings of the two helicoids are mutually orthogonal, but the axes of the helicoids are not necessarily orthogonal.

**Proof.** Let \( (e_1, \ldots, e_4, e_5, e_6) \) be a semiorthonormal framing on \( M \), which induces an integral \( \Sigma \) of system \( \beta \). If we define vectors
\[ f_5 := e_5 + x(e_4 + r^{-1}e_3), \quad f_6 := e_6 + y(e_4 - r^{-1}e_3) \]
then
\[ \begin{align*}
    de_1 &= f_5 \kappa_1, & de_2 &= f_6 \kappa_2, \\
    d\left(\frac{f_5}{L_1}\right) &= -L_1 e_1 \kappa_1, & d\left(\frac{f_6}{L_2}\right) &= -L_2 e_2 \kappa_2 \\
    \end{align*} \]
mod \( \beta \).

where \( L_1 = |f_5| \) and \( L_2 = |f_6| \). (Note that (36) implies that \( f_5 \) and \( f_6 \) are orthogonal.) This shows that the orthogonal planes \( P_1, P_2 \) through the origin in \( \mathbb{R}^6 \), spanned by \( \{e_1, f_5\} \) and \( \{e_2, f_6\} \) respectively, are fixed, independent of the choice of point \( p \in M \). Within each ruling plane, the lines spanned by \( e_1 \) and \( e_2 \) are parallel to \( P_1 \) and \( P_2 \), respectively.

In addition, we compute that \( \omega^1, \omega^2, L_1 \kappa_1, L_2 \kappa_2 \) are all closed modulo \( \beta \), so we may introduce coordinates \( s, t, u, v \) along \( M \) such that
\[ \begin{align*}
    ds &= \omega^1, & dt &= \omega^2, & du &= L_1 \kappa_1, & dv &= L_2 \kappa_2.
\end{align*} \]

Modulo \( \beta \), we compute that \( dr, dL_1, dL_2 \) are in the span of \( \omega^1 \) and \( \omega^2 \), so that \( r, L_1, L_2 \) are functions of \( s \) and \( t \) on \( M \). In fact,
\[ \begin{align*}
    d\left(\frac{x}{rL_1^2}\right) &= \omega^1, & d\left(\frac{y}{rL_2^2}\right) &= -\omega^2 \mod \beta.
\end{align*} \]

Hence we may choose the arclength coordinates \( s, t \) along the \( e_1 \)- and \( e_2 \)-lines to satisfy \( x = rL_1^2 s \) and \( y = -rL_2^2 t \).

In terms of these coordinates, the derivative of the basepoint map \( p : \mathcal{F} \to \mathbb{R}^6 \) is
\[ dp = e_1 \, ds + e_2 \, dt + f_5 \, du + f_4 \, dv \mod \beta, \]
(37)
where
\[ \begin{align*}
    f_3 &= \frac{1}{2L_1} (e_3 + r^{-1}e_4), & f_4 &= \frac{1}{2L_2} (e_3 - r^{-1}e_4).
\end{align*} \]
The vectors $f_3 - (s/L_1)f_5$ and $f_4 - (t/L_2)f_6$ are the projections of $f_3, f_4$ onto the orthogonal complements of $f_5$ and $f_6$ respectively, and we compute that these vectors are constant along $M$.

Now consider the splitting
\[
\mathbb{R}^6 = \{e_1, f_3, f_5\} \oplus \{e_2, f_4, f_6\},
\]
and let $\pi_1, \pi_2$ be the projections onto the summands on the right, which are fixed subspaces of $\mathbb{R}^6$. Then (37) shows that $\pi_1|\mathcal{I}$ and $\pi_2|\mathcal{I}$ have rank two, with coordinates $s, u$ and $t, v$ respectively on the images. In fact, the images are open subsets of classical helicoids. For example, on the surface $\pi_1(M)$ the vectors $e_1$ and $f_3$ span the tangent space of the surface, and are tangent to the $s$- and $u$-coordinate curves respectively. The $s$-coordinate curves are straight lines. Since $r, L_1, L_2$ are functions of $s, t$ only along $M$, $\{f_3\}$ is constant along the $u$-coordinate curves. We compute
\[
\frac{\partial}{\partial u} f_3 = -se_1, \quad \frac{\partial}{\partial u} e_1 = \frac{1}{L_1} f_5,
\]
which shows that the $u$-coordinate curves are helices with curvature $s/|f_3|$ and torsion $1/|f_3|$.

We note that the splitting (38) is not orthogonal, unless $r$ is identically equal to $\pm 1$. In fact, the inner product of the fixed vectors pointing along the axes of the helicoids is
\[
\left( f_3 - \frac{s}{L_1} f_5 \right) \cdot \left( f_4 - \frac{t}{L_2} f_6 \right) = 1 - r^{-2} \frac{L_1}{4L_1 L_2}.
\]

(a.ii) $\det|\mathcal{I}$ has rank 1. In this case we can normalize so that $B$ is spanned by \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \) for $r \neq 0$. Thus, with respect to an orthonormal basis for the tangent space, $|\mathcal{I}|$ is spanned by the matrices
\[
S_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \\ 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \end{pmatrix}.
\]

Let $\mathcal{I}$ be the ruled austere system for this choice of basis matrices; the 1-forms encoding the tangential ruled condition are as in (31). Computing the 2-forms of this system yields immediate integrability conditions
\[
u_2 = 0, \quad x_1 = -rv_1, \quad x_2 = 3rv_2, \quad y_1 = 2v_2, \quad y_2 = 0.
\]
We restrict to the submanifold where these conditions hold and derive further integrability conditions on the remaining variables, which imply that $v_2 = 0$ and either $u_1 = 0$ or $v_1 = 0$. In both cases, new integrability conditions arise that imply that $g_{66} = 0$, which is impossible. Therefore, there are no integral submanifolds in this case.

(a.iii) $\det|\mathcal{I} = 0$. In this case we can normalize so that $B = \begin{pmatrix} (a & 0) \\ (b & 0) \end{pmatrix}$ or $\begin{pmatrix} (a \ b) \\ (0 \ 0) \end{pmatrix}$. In either case, the quadratic forms in $|\mathcal{I}|$ have a common linear factor, hence $|\mathcal{I}|$ is simple, and $M$ is a generalized helicoid. We will determine what are the particular values for the constants in (1).

We consider the first case, namely when $B$ is of a form $\begin{pmatrix} (a & 0) \\ (0 & 0) \end{pmatrix}$. Integrability conditions imply that the extra 1-forms (31) of the ideal $\mathcal{I}$ must have $u_2, v_2, x_1, x_2, y_1, y_2$ all zero. The first prolongation $\mathcal{I}^{(1)}$ of the ideal is involutive—in fact, Frobenius. Moreover, we compute that
\[
d e_4 \equiv 0 \mod \mathcal{I}^{(1)},
\]
indicating that $e_4$ is parallel to a fixed line. Thus, $M$ is congruent to the product of a line with a generalized helicoid in $\mathbb{R}^5$ (i.e., $s = 2$ in (1), with $\lambda_0 \neq 0$).

In the second case, $B$ is of the form $\begin{pmatrix} (a \ b) \\ (0 \ 0) \end{pmatrix}$. In this case, integrability conditions imply that all coefficients in (31) vanish except for $y_2$. Again, the first prolongation is Frobenius. However, in this case $M$ is a cone, with the vector $e_2$ tangent to the generators, and is ruled by 3-planes spanned by $\{e_2, v_3, e_4\}$. This corresponds to $s = 3, \lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3$ in (1). Thus, $M$ is a cone over a generalized Clifford torus, the embedding of the product of the unit spheres $S^1 \subset \mathbb{R}^2$ and $S^2 \subset \mathbb{R}^3$ into $S^5 \subset \mathbb{R}^6$.

We summarize this case as follows:

**Proposition 13.** Let $M$ be a connected, 2-ruled austere submanifold of Type B, substantial in $\mathbb{R}^6$, such that $E = \pi(E)$ and $\delta = 2$. Then $M$ is congruent to an open subset of one of the following: a product of helicoids in $\mathbb{R}^3$, a generalized helicoid in $\mathbb{R}^6$ with $s = 2$ and $\lambda_0 \neq 0$, or a cone over $S^1 \times S^2 \subset S^5$.

(b) $E$ is a sum of $+1$ and $-1$ eigenspaces of $R$. In this case we can assume that $E = \{e_1, e_3\}$. This implies that $m = b_1 = 0$ in (30) and therefore $\delta \leq 3$. Analyzing the standard system $\mathcal{I}$ in either case $\delta = 3$ and $\delta = 2$ leads to impossible integrability conditions. Thus, no submanifolds of this type exist.
5. Type C ruled austere submanifolds

In this section we discuss the ruled austere 4-folds of Type C. We recall from [1] that the family of maximal austere submanifolds of Type C is described by

\[ Q_C = \begin{cases} 
0 & x_1 x_2 x_3 \\
0 & 0 \lambda_3 x_3 \lambda_2 x_2 \\
x_2 & \lambda_3 x_3 0 \lambda_1 x_1 \\
x_3 & \lambda_2 x_2 \lambda_1 x_1 0 
\end{cases}, \quad x_1, x_2, x_3 \in \mathbb{R}, \tag{39} \]

where \( \lambda_1, \lambda_2, \lambda_3 \) are real parameters satisfying

\[ \lambda_1 \lambda_2 \lambda_3 + \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{40} \]

An austere \( 4 \)-manifold \( M \) of Type C if near any point there is an orthonormal frame field \( (e_1, e_2, e_3, e_4) \) with respect to which \( |I| \subset Q_C \), for parameters \( \lambda_1 \) which may vary smoothly along \( M \). Notice that this condition is invariant under simultaneously permuting the frame vectors \( (e_2, e_3, e_4) \) and the parameters \( (\lambda_1, \lambda_2, \lambda_3) \).

We distinguish two cases, depending on whether or not \( e_1 \) is orthogonal to the ruling plane \( E \).

Case CI: \( e_1 \cdot E = 0 \)

In this case, the algebraic condition (11) says that \( E \), lying in the span of \( e_2, e_3, e_4 \), is a 2-dimensional nullspace for every matrix in \( |I| \). Thus, matrices in \( |I| \) of the form in (39) must satisfy

\[ 0 = \det \begin{bmatrix} 0 & \lambda_3 x_3 & \lambda_2 x_2 \\
\lambda_3 x_3 & 0 & \lambda_1 x_1 \\
\lambda_2 x_2 & \lambda_1 x_1 & 0 
\end{bmatrix} = 2\lambda_1 \lambda_2 \lambda_3 x_1 x_2 x_3, \]

implying that either one of \( x \)'s is zero, or one of the \( \lambda \)'s; moreover, since \( M \) is connected, one of these conditions must hold on all of \( M \).

In the first case, we can assume that \( |I| \subset Q_C \) is a 2-dimensional subspace defined by \( x_1 = 0 \) at each point. Then the ruling plane is spanned by \( (e_2, e_4) \), and \( |I| \) is also a subspace of \( Q_B \). Swapping \( e_2, e_4 \) with \( e_1, e_2 \) respectively, we see that \( M \) falls into Case B.1 above.

In the second case, we can assume that \( \lambda_1 \) is identically zero along \( M \). Assuming first that \( \delta = 3 \), an analysis of the standard system \( I \) shows that the remaining parameters must be constant along \( M \). Then Proposition 14 in [6] implies that \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) and \( M \) is a generalized helicoid, swept out by 3-planes in \( \mathbb{R}^7 \). On the other hand, if \( \delta = 2 \) then \( E = (e_3, e_4) \) and the coordinates \( x_1, x_2, x_3 \) are linearly related at each point. An analysis of the system \( I \) in this case, similar to that in the proof of Proposition 11, shows that no such manifolds exist.

Case CII: \( e_1 \cdot E \neq 0 \)

In this case, by permuting the frame vectors, we can assume that \( E \) is spanned by \( e_1 + a_3 e_3 + a_4 e_4 \) and \( e_2 + b_3 e_3 + b_4 e_4 \) for some functions \( a_3, a_4, b_3, b_4 \). The algebraic condition (11) implies that \( \delta \leq 2 \). We assume that \( \delta = 2 \). Requiring that a 2-dimensional subspace of \( Q_C \) satisfy the algebraic condition leads to the following subcases:

(i) \( a_3 = a_4 = 0, b_3 \neq 0, b_4 \neq 0 \). In this case, (40) implies that \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \). Then a change of basis shows that \( |I| \) is of Type B and simple, falling into Case B.1(iii) above; in particular \( M \) is a generalized helicoid cone in \( \mathbb{R}^6 \).

(ii) \( a_3 = a_4 = b_4 = 0, b_3 \neq 0, \lambda_3 = 0 \). Then, an analysis of \( I \) shows that the only possibility is that \( \lambda_1, \lambda_2 \) vanish identically, in which case \( M \) is again a generalized helicoid cone.
(iii) \( a_2 = a_4 = b_2 = b_4 = 0 \). In this case, \(|\mathcal{II}|\) is of Type B, falling into Case B.1 above.
(iv) \( a_4 = b_2 = 0, a_3 \neq 0, b_4 \neq 0, \lambda_1 = -1/(a_3 b_4), \lambda_2 = -b_4/a_3 \). A lengthy analysis of the integrability conditions for this case shows that the only possibility is that \( a_3 = \pm 1 \) and \( b_4 = \pm 1 \). Then an orthogonal change of basis shows that \(|\mathcal{II}|\) is precisely of the type described in Proposition 11, and thus \( M \) is a product of helicoids as described by Proposition 12.

The classification of 2-ruled austere 4-folds of Type C is summarized in Theorem 3 in the Introduction.

6. Normal rank one

In this section we investigate austere submanifolds \( M^4 \) where \(|\mathcal{II}|\) is one-dimensional at each point, and the Gauss map has rank two. As we will see, this implies that \( M \) is a hypersurface in \( \mathbb{R}^5 \) ruled by 2-planes; however, it is not known if a 2-ruled austere hypersurface of this dimension must have a degenerate Gauss map. For a submanifold in Euclidean space, the kernel of the differential of the Gauss map is known as the relative nullity distribution (see, e.g., [2]), and consists at \( p \in M \) of those vectors \( X \) such that \( \mathcal{II}(X, Y) = 0 \) for all \( Y \in T_p M \). Thus, if \( M^4 \) is austere with a rank two Gauss map, then we may choose an orthonormal frame so that \( e_1, e_2 \) span the relative nullity distribution, and \(|\mathcal{II}|\) is spanned by matrices of the form

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
p & 0 & q & 0 \\
0 & q & -p & 0
\end{bmatrix}.
\]

(41)

Accordingly, let \( \mathcal{H} \) be the standard system for austere submanifolds (described in Section 2) defined on \( \mathcal{I} \times \mathbb{R}^2 \) with \( p, q \) not both zero as coordinates on the second factor, and where matrix \( S^5 \) is given by (41) and \( S^5, \ldots, S^n \) are zero. Any austere \( M^4 \subset \mathbb{R}^n \) equipped with an orthonormal frame such that \( e_1, e_2 \) span the relative nullity distribution, gives an integral of \( \mathcal{H} \), and conversely.

**Theorem 14.** \( M \) is ruled by planes tangent to the relative nullity distribution, and is contained in a totally geodesic \( \mathbb{R}^5 \).

**Proof.** The ‘normal part’ of the ruled condition (11) holds automatically for the relative nullity distribution. The remaining requirement (12) for \( e_1, e_2 \) to span a ruling is that \( \omega_3^3, \omega_4^3, \omega_2^4 \) be multiples of \( \omega^3, \omega^4 \) along any integral manifold.

We will use complex-valued 1-forms to compute the structure equations of \( \mathcal{H} \) and other related systems. For example, when \( n = 5 \) the 1-forms generating \( \mathcal{H} \) are \( \omega_3, \theta_1^3 = \omega_1^3, \theta_2^3 = \omega_2^3 \) and

\[
\theta_3^5 - i \omega_4^5 = (\omega_3^5 - i \omega_4^5) - (p - iq)(\omega^3 + i \omega^4).
\]

Defining complex-valued 1-forms

\[
\zeta := \omega^3 + i \omega^4, \quad \eta_1 := \omega_1^3 + i \omega_4^3, \quad \eta_2 := \omega_2^3 + i \omega_2^4,
\]

we compute the nontrivial 2-forms of \( \mathcal{H} \), modulo the 1-forms, as

\[
\begin{align*}
d \theta_3^5 &\equiv \text{Re}[ (p - iq) \eta_1 \wedge \zeta ], \\
d \theta_2^3 &\equiv \text{Re}[ (p - iq) \eta_2 \wedge \zeta ], \\
d (\theta_3^5 - i \omega_4^5) &\equiv -(d(p - iq) + 2i(p - iq) \omega_3^5) \wedge \zeta + (p - iq)(\eta_1 \wedge \omega^3 + \eta_2 \wedge \omega^4).
\end{align*}
\]

(43)

By expanding out the wedge products in the first two 2-forms, it follows that on any integral element the real and imaginary parts of \( \eta_1 \) and \( \eta_2 \) must be linear combinations of \( \omega^3 \) and \( \omega^4 \); in fact, by wedging the last 2-form with \( \zeta \), it follows that

\[
\eta_1 = T_1 \zeta, \quad \eta_2 = T_2 \zeta
\]

(44)

for some complex-valued functions \( T_1, T_2 \). In particular, \( M \) is ruled.

When \( n > 5 \), the additional 1-form generators are \( \omega^b \) and \( \omega_1^b, \omega_2^b \) for \( b > 5 \). It is easy to check that \( d \omega^b, d \omega_1^b, d \omega_2^b \) are congruent to zero modulo the 1-forms of \( \mathcal{H} \), while

\[
d(\omega_1^b - i \omega_2^b) \equiv -(p - iq) \omega_3^b \wedge \zeta.
\]

We deduce that the real-valued 1-forms \( \omega_3^b \) must vanish on all integral elements. From the vanishing of \( \omega_1^b, \ldots, \omega_2^b \) it follows, using (2), that the subspace spanned by \( e_1, \ldots, e_5 \) is fixed, and thus \( M \) is contained in a 5-dimensional plane in \( \mathbb{R}^5 \). □
Without loss of generality, we will restrict our attention to austere hypersurfaces in $\mathbb{R}^5$ for the rest of this section. Let $I_{a0}$ be the Pfaffian system obtained by adding to $\mathcal{H}$ the 1-forms encoding the tangential part of the ruling condition; these are obtained from the real and imaginary parts of the equations in (44). Altogether, the 1-form generators of $I_{a0}$ are

\begin{align*}
\omega^5, \omega^5_1, \omega^5_2, \omega^5_3 - pq\omega^3 - qo\omega^4, \omega^4 - qo\omega^3 + pq\omega^4, \\
\omega^3_1 - s_1\omega^3 + t_1\omega^4, \omega^3_4 - t_1\omega^3 - s_1\omega^4, \omega^2_2 - s_2\omega^3 + t_2\omega^4, \omega^2_5 - t_2\omega^3 - s_2\omega^4,
\end{align*}

where $s_i, t_i$ are the real and imaginary parts of $T_i, i = 1, 2$. Then $I_{a0}$ is defined on $\mathcal{I}_{a0} \times \mathbb{R}_+^6 \times \mathbb{R}^4$, with the $s_i, t_i$ added as new variables. It is easy to check that $I_{a0}$ is involutive with Cartan character $s_1 = 6$. Below, we will use the involutivity to show that such hypersurfaces exist, passing through any generic curve in $\mathbb{R}^5$.

**Remark 15.** An analogue of the twistor spaces $V$, used to construct various Type A1 ruled submanifolds in Section 3, exists for this hypersurface case. Namely, let $V$ be the space of flags $E \subset F$ in $\mathbb{R}^3$, where the subspaces have dimensions 2 and 4 respectively. (This manifold is a homogeneous 8-dimensional quotient of $SO(5).$) Given an austere hypersurface $M$ with rank 2 Gauss map, we define a rank 2 mapping $\Gamma : M \to V$ taking $p \in M$ to $(E_p, T_p M)$, where $E_p$ is parallel to the ruling through $p$. Then $\Gamma(M)$ is an integral surface of a certain exterior differential system $J$ on $V$, and any such surface is the locally the image $\Gamma(M)$ of some austere hypersurface $M$ of this type. (Details are left to the interested reader.) The main difference between this case and the Type A twistor spaces is that there is no homogeneous complex structure on $V$ with respect to which $\Gamma(M)$ is a holomorphic curve.

In closing, we consider a Cauchy problem for austere hypersurfaces. Let $f(s)$ be a regular real-analytic curve, parametrized by arclength, which is substantial in $\mathbb{R}^5$. It follows that there exist real-analytic orthonormal vectors $T, N_1, N_2$ and real-analytic functions $k_1, k_2$ such that

\[
\frac{df}{ds} = T, \quad \frac{dT}{ds} = k_1 N_1, \quad \frac{dN_1}{ds} = -k_1 T + k_2 N_2.
\]

We lift $f$ to obtain a real-analytic integral curve of $I_{a0}$ in $\mathcal{I}_{a0} \times \mathbb{R}^6$ as follows. First, letting $e_3 = T, e_5 = N_1, e_4 = N_2$, and letting $e_1, e_2$ be any analytic orthonormal vectors along $f$ that are perpendicular at each point to the plane spanned by $(T, N_1, N_2)$, gives a lift into $\mathcal{I}_{a0}$ which is an integral of $\omega^5, \omega^5_1, \omega^5_2$. (As well, $\omega^3, \omega^3_1, \omega^3_2$ pull back to be zero along the lifted curve, while $\omega^3_3$ pulls back to equal $ds.$) Next, we choose values for the functions $p, q, s_1, t_1, s_2, t_2$ along $f$ so as to give a lift into $\mathcal{I}_{a0} \times \mathbb{R}^6$ which is an integral curve of the remaining generators of $I_{a0}$. (For example, since $\omega^5_3 = e_5 \cdot (de_3/ds)$ and $\omega^5_2 = e_5 \cdot (de_4/ds)$, we set $p = k_1$ and $q = -k_2.$)

It follows by the Cartan–Kähler Theorem that there exists an integral 4-manifold of $I_{a0}$, satisfying the independence condition, containing the lift of $f$. Our hypersurface $M$ is the projection of this 4-manifold into $\mathbb{R}^5$. Note that the fact that $N_1$ is orthogonal to the hypersurface means that $f$ is a geodesic in $M$.

**Proposition 16.** Let $f$ be a real-analytic curve, substantial in $\mathbb{R}^5$. Then there exists an austere 4-fold $M^4 \subset \mathbb{R}^5$ with rank 2 Gauss map, such that $f$ is a geodesic in $M$ and along $f$ the ruling plane is orthogonal to the span of $f', f'', f'''.$ $M$ is unique in the sense that any two such hypersurfaces must coincide in a neighborhood of $f$.

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