A Short Note on Disjointness Conditions for Triples of Group Subsets Satisfying the Triple Product Property

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1.1 The Triple Product Property (TPP), and Matrix Multiplication via Finite Groups

Three subsets $S, T, U \subseteq G$ of a (nontrivial) finite group $G$ are said to satisfy the triple product property (TPP) iff for any three pairs of elements $s', s \in S, t', t \in T, u', u \in U$ it is the case that:

$$s's^{-1}t't^{-1}u'u^{-1} = 1_G \implies s = s', t = t', u = u'.$$

(See [2] for a basic definition.) The TPP property for the triple $(S, T, U)$ is invariant under permutations of the triple, that is, the permuted triples $(S, U, T), (T, U, S), (U, S, T), (U, T, S)$ all satisfy the TPP iff $(S, T, U)$ satisfies the TPP, ([6], p. 45). We denote by $\mathcal{G}(G)$ the set of all TPP triples of $G$, and by $\mathcal{G}(G)/\text{Sym}_3$ the set of equivalence classes of $\mathcal{G}(G)$ under the equivalence relation that two TPP triples of $G$ are equivalent iff they are permutations of each other. $\mathcal{G}(G)$ is non-empty since it always contains the TPP triple $(G, \{1_G\}, \{1_G\})$, ([6], p. 44).

Let the sizes of $S, T, U$ be $m, p, q \geq 1$ respectively, with $mpq > 1$ (i.e. not all $m, p, q = 1$) and let $n$ be the size $|G|$ of $G$. We label the elements of these subsets by indices $i \in [1..m], j \in [1..p], k \in [1..q]$ respectively. By the assumption of the TPP for subsets $S, T, U$ it can be proved that the following maps:

$$\varepsilon_{m,p} : S \times T \to G, \varepsilon_1(s_i, t_j) = s_i^{-1}t_j,$$

$$\varepsilon_{p,q} : T \times U \to G, \varepsilon_2(t_j, u_k) = t_j^{-1}u_k,$$

$$\varepsilon_{m,q} : S \times U \to G, \varepsilon_3(s_i, u_k) = s_i^{-1}u_k,$$

are all injective ([2], p. 382). We give here a clearer proof of this result than the one given in [2]. Assume that just one of the maps, say $\varepsilon_{m,p}$, is not injective: that there are two distinct pairs $(s, t), (s', t') \in S \times T$, with $s \neq s'$ or $t \neq t'$ or both, which are both mapped by $\varepsilon_{m,p}$ to the same element $s^{-1}t = s'^{-1}t'$. From
the latter we deduce that \( s' s^{-1} t t'^{-1} = 1_G \). We can take any \( u \in U \) and set \( u' = u \), and then we have that \( s' s^{-1} t t'^{-1} u' u^{-1} = 1_G \). The assumption of the TPP for \( S, T, U \) implies that \( s = s', t = t', u = u \), yet earlier we deduced that \( s \neq s' \) or \( t \neq t' \) from assuming that \( \varepsilon_{m,p} \) was not injective - a contradiction. Thus, \( \varepsilon_{m,p} \) must be injective given the TPP for \( S, T, U \), and we can prove this in the same way for the maps \( \varepsilon_{p,q} \) and \( \varepsilon_{m,q} \). By their injectivity, the maps \( \varepsilon_{m,p}, \varepsilon_{p,q}, \varepsilon_{m,q} \) have inverses \( \varepsilon_{-1}^{-1}, \varepsilon_{p,q}^{-1}, \varepsilon_{m,q}^{-1} \) respectively, and it can be proven that if in addition the subsets \( S, T, U \) have the largest possible sizes (a so-called maximal TPP triple) then the following inequalities hold for the product \( mpq \) of their sizes:

\[
n \leq mpq < n^{\frac{3}{2}}
\]

(see [2], also pp. 55-56 in [6]). The lower bound \( n \) is reached by \( mpq \) if \( G \) is Abelian, or, equivalently, if \( mpq > n \) then \( G \) is non-Abelian.

The importance of the TPP property for the subsets \( S, T, U \), and the related embedding maps \( \varepsilon_{m,p}, \varepsilon_{p,q}, \varepsilon_{m,q} \) is that it allows \( G \) to “realize” or “support” matrix multiplication of dimensions \( m \times p \) by \( p \times q \) via its regular group algebra \( C_G \), in which case \( G \) is said to realize the tensor \( \langle m, p, q \rangle \) describing the (bi-linear) matrix multiplication map \( C^{m \times p} \times C^{p \times q} \to C^{m \times q} \), and the product \( mpq \) is called the (multiplicative) size \( z(\langle m, p, q \rangle) \) of \( \langle m, p, q \rangle \) and also of the corresponding TPP triple \( (S, T, U) \), (see [6], pp. 48-49 and pp. 51-55). This is represented by the following commutative diagram:

\[
\begin{array}{cccc}
C^{m \times p} & \times & C^{p \times q} & \rightarrow C^{m \times q} \\
\varepsilon_{m,p} & & \varepsilon_{p,q} & \varepsilon_{m,q} \\
C_G & \times & C_G & \rightarrow C_G \\
\oplus_{d_\varphi \times d_\varphi} & \times & \oplus_{d_\varphi \times d_\varphi} & \rightarrow \oplus_{d_\varphi \times d_\varphi} \\
\end{array}
\]

(Here \( C_G \) is the regular group algebra of \( G \), and \( \oplus_{\varphi \in \text{Irrep}(G)} C^{d_\varphi \times d_\varphi} \) is the isomorphic image of \( C_G \) under the discrete group Fourier transform \( F \), and the \( \varphi \) are the distinct irreducible representations of \( G \) of dimensions \( d_\varphi \) such that \( \text{Dim } C_G = |G| = \sum_{\varphi \in \text{Irrep}(G)} d_\varphi^2 \) ([4], pp. 46-47)). This means that if \( A \) is an \( m \times p \) matrix and \( B \) is a \( p \times q \) matrix their \( m \times q \) product \( AB \) can be computed by
the composite map $\varepsilon_m^{-1} \circ F^{-1} \circ \left( \oplus_{\varrho \in \text{Irrep}(G)} \langle d_e, d_e, d_e \rangle \right) \circ (F \times F) \circ (\varepsilon_m \times \varepsilon_p \times \varepsilon_{p,q})$. Algebraically, this means that the map $\langle m, p, q \rangle$ is smaller than the multiplication map $mCG$ of $CG$, and that we can compute this small map $\langle m, p, q \rangle$ by restricting the larger map $mCG \cong \oplus_{\varrho \in \text{Irrep}(G)} \langle d_e, d_e, d_e \rangle$, and therefore that the complexity of $m \times p$ by $p \times q$ matrix multiplication is at most the complexity of group algebra multiplication in $CG$. So, formally, if $G$ realizes the tensor $\langle m, p, q \rangle$ then:

$$\langle m, p, q \rangle (A, B) = \varepsilon_m^{-1} \circ F^{-1} \circ \left( \oplus_{\varrho \in \text{Irrep}(G)} \langle d_e, d_e, d_e \rangle \right) \circ (F \times F) \circ (\varepsilon_m \times \varepsilon_p \times \varepsilon_{p,q})(A, B) \in \mathbb{C}^{m \times q},$$

for any pair of matrices $A \in \mathbb{C}^{m \times p}, B \in \mathbb{C}^{p \times q}$, and

$$\mathcal{R}(\langle m, p, q \rangle) \leq \mathcal{R}(mCG)$$

where the $\mathcal{R}$s are the rank functions of these multiplication maps, (see [6], pp. 51-53 for a detailed proof.)

The quantity $(mpq)^{\frac{1}{3}} = z(\langle m, p, q \rangle)^{\frac{1}{3}}$ is called the (geometric) mean size of the tensor $\langle m, p, q \rangle$ realized by $G$. It is important to note that $G$ realizes a tensor $\langle m, p, q \rangle$ iff it realizes any permuted tensor $\langle \pi(m), \pi(p), \pi(q) \rangle$ where $\pi \in \text{Sym}_3$, ([6], p. 49). This is because two tensors which are permutations of each other describe the same (bilinear) matrix multiplication map, and have the same complexity. How small or large can the components $m, p, q$ of this tensor be? This is determined by how small or large are the three underlying subsets $S, T, U \subseteq G$ corresponding to the tensor. We are interested in studying the complexity of multiplication of nontrivial matrices, that is, of dimensions $m \times p$ by $p \times q$ where all $m, p, q \geq 2$, since otherwise, if, say, only $m, p \geq 2$ and $q = 1$ then the corresponding tensor $\langle m, p, 1 \rangle$ would describe multiplication of a column vector with $p$ rows by an $m \times p$ matrix; or if, say, only $m \geq 2$ and $p = q = 1$ the the corresponding tensor $\langle m, 1, 1 \rangle$ would describe the multiplication of a column vector with $m$ rows by a constant. This means that we must impose the lower bound $m, p, q \geq 2$ on the individual sizes of any three subsets of $G$ forming a TPP triple. We must for the moment merely state that we require that $m, p, q \leq n - 1$, although an explanation will be given later on. In answer to the question opening this paragraph, then, a TPP triple of $G$ is defined to be such that it must be composed of proper subsets of $G$, although for proving multiplicative bounds for sizes of TPP triples in the next section we may relax these individual size bounds without affecting their validity.

1.2 Minimal Disjointness Conditions on TPP Triples and Disjointness Types

Logically, if at least one of the maps $\varepsilon_{m,p}, \varepsilon_{p,q}, \varepsilon_{m,q}$ is not injective for any three subsets $S, T, U \subseteq G$ then they cannot satisfy the TPP. Now, we investigate
how this may arise from assuming that some pair(s) from this triple \(S,T,U\) has a nonempty intersection, our objective being to identify the minimal disjointness condition(s) on triples of subsets \(S,T,U\) which satisfy the TPP, so that in searching for a maximal TPP triple of \(G\) we rule out those triples which do not meet the minimal disjointness condition(s).

Assume that among the subsets \(S,T,U\) only two of them, say \(S\) and \(T\), have a non-empty intersection, i.e. intersect in some \(a\) number of elements where \(1 \leq a \leq \min(m,p)\), and that all other pairs \(T,U\) and \(S,U\) are disjoint. We can always assume that these \(a\) common elements of \(S\) and \(T\) occur as their first \(a\) elements:

\[
s_1 = t_1, s_2 = t_2, \ldots, s_a = t_a.
\]

Then \(\text{Im} \varepsilon_{m,p}\) is the following union of disjoint sets:

\[
\text{Im} \varepsilon_{m,p} = \{s_1^{-1}t_1, \ldots, s_a^{-1}t_a\} \cup \{s_i^{-1}t_j\}_{1 \leq i \leq a; 1 \leq j \leq p, j \neq i} \cup \{s_i^{-1}t_j\}_{a+1 \leq i \leq m; 1 \leq j \leq p}.
\]

The number of elements in \(\text{Im} \varepsilon_{m,p}\) is given by:

\[
|\text{Im} \varepsilon_{m,p}| = |\{1_G\} \cup \{s_i^{-1}t_j\}_{1 \leq i \leq a; 1 \leq j \leq p, j \neq i} \cup \{s_i^{-1}t_j\}_{a+1 \leq i \leq m; 1 \leq j \leq p}|
\]

\[
= 1 + \left| \{s_i^{-1}t_j\}_{1 \leq i \leq a; 1 \leq j \leq p, j \neq i} \right| + \left| \{s_i^{-1}t_j\}_{a+1 \leq i \leq m; 1 \leq j \leq p} \right|
\]

\[
= 1 + (m-a)p + ap - 1
\]

\[
= mp - (a - 1)
\]

\[
\leq mp.
\]

We can see that \(|\text{Im} \varepsilon_{m,p}| = mp\) iff \(a = 1\), i.e. iff \(S\) and \(T\) have just one element in common, including the possibility that either \(T\) is a singleton subset of \(S\), or \(S\) is a singleton subset of \(T\), or both in which case \(S = T\) are both identical singleton subsets of \(G\). Otherwise, \(|\text{Im} \varepsilon_{m,p}| < mp\) iff \(a \geq 2\), in which we would have a contradiction to the assumption that \(\varepsilon_{m,p}\) is injective, which would in turn contradict the assumption that the triple \(S,T,U\) satisfies the TPP. If two or even all three distinct pairs in the triple \(S,T,U\) had nonempty intersections of size 1, including the possibility that among some pairs \(X,Y\) one was a singleton subset of the other, or even the extreme possibility (which we shall exclude later) that \(S,T,U\) are all identical singleton subsets of \(G\), this would not preclude the possibility that the triple satisfies the TPP. On the other hand, there is certainly no contradiction in assuming that \(S,T,U\) are pairwise disjoint while satisfying the TPP, including the extreme possibility that \(S,T,U\) are all distinct singleton subsets of \(G\).

Therefore, we can conclude that if a triple of subsets \(S,T,U \subseteq G\) satisfy the TPP in a finite group \(G\) then they are either pairwise disjoint or one or more pairs among them intersect in just one element. Or equivalently, if subsets \(S,T,U \subseteq G\) have the TPP then any two of them intersect in at most one element. We call this the minimal disjointness property for TPP triples. We have also proved a useful precondition for a search algorithm on the space of...
subset triples of $G$: if a subset triple of $G$ does not have the minimal disjointness property then the maps $\varepsilon_{m,p}, \varepsilon_{p,q}, \varepsilon_{m,q}$ for it cannot be injective which means it cannot satisfy the TPP. There are nine mutually exclusive ways in which a TPP triple can satisfy the minimal disjointness property, and they are represented below in the form of Venn diagrams, for hypothetical sizes $m = p = q = 3$:

The black dots represent distinct elements of the sets they occur in, while the red dots represent elements shared between the pair or pairs of subsets in which they occur. The nine different disjointness cases, which are labelled by Roman numbers, have been placed into four different groups, which are labelled by bold decimal numbers, based on the total number of elements of $G$ which are shared between any pair of subsets occurring in a case. We let $V \subset G$ be a remainder set $V = G - (S \cup T \cup U)$ of size $|V| = r \geq 0$ consisting of all those elements of $G$ which are not chosen to be in $S, T$ or $U$. How big can $r$ be? The smallest number of elements of $G$ which are necessary to form three subsets of $G$ satisfying any of the disjointness conditions is 3 - this is the case when, say $G = C_6 = \langle g^k | g^6 = e \rangle$ and $S = \{e, g\}, T = \{g, g^2\}, U = \{g^2, e\}$, this triple corresponding to case (ix), and $|S \cup U \cup T| = 3$ and $r = 6 - 3 = 3$. So $r \leq n - 3$ in general. We also note that $|S \cup U \cup T| = |S| + |T| + |U| = m + p + q$ iff $S, T, U$ are pairwise disjoint, which is case (i), and in other cases we have $|S \cup U \cup T| < |S| + |T| + |U| = m + p + q$. If we let $w = |S| + |T| + |U| - |S \cup U \cup T|$ then we see that $w = 0$ for case (i), $w = 1$ for cases (ii)-(iv), $w = 2$ for case (v)-(vii), and finally $w = 3$ for case (ix). We call the number $w := w(S, T, U)$ so defined for any TPP triple $S, T, U$ its disjointness type number. In all of the cases (i)-(ix) we have a partition $G = (S \cup T \cup U) \cup V$ of $G$ by the two disjoint sets $(S \cup T \cup U)$ and $V$, where $n = (m + p + q - w) + r$ is the corresponding partition of $|G| = n$, and we can now regroup these cases in terms of shared
values of the disjointness type number $w$:

| $w$ | Cases | Partition of $n$ |
|-----|-------|------------------|
| 0   | (i)   | $n = (m + p + q) + r$ |
| 1   | (ii)-(iv) | $n = (m + p + q - 1) + r$ |
| 2   | (v)-(viii) | $n = (m + p + q - 2) + r$ |
| 3   | (ix)  | $n = (m + p + q - 3) + r$ |

We see that case (v) leads to the same partition of $n$ as cases (vi)-(viii), and so we group these together. The new general classification diagram of disjointness cases (i)-(ix) for a TPP triple $(S, T, U)$ in terms of the disjointness type number $w$ is given below:

1.3 Additive Size Bounds

Before we derive lower and upper bounds for the sum $m + p + q$ of sizes $m, p, q$ respectively of a TPP triple of $G$, let us first derive individual upper bounds on the subsets forming any TPP triple. Assume that $S$ is the largest subset in a TPP triple $S, T, U$ and that $S$ consists of all $n$ elements of $G$, that is $m = |G| = n$. By the disjointness property of a TPP triple of $G$ it must be that any distinct pair among $S, T, U$ can intersect in only one element of $G$. So if either $T$ or $U$ (or both) had 2 or more elements, that is, $p \geq 2$ or $q \geq 2$, we would have a contradiction to the disjointness requirement since then $T$ or $U$ (or both) would intersect with $S$ in at least 2 elements of $G$. So if $S$ has $n$ elements then $T$ and $U$ must both be singleton subsets of $G$ having just 1 element. But we have ruled out this case by the requirement that all $m, p, q \geq 2$. So the largest subset $S$ of a TPP triple $S, T, U$ of $G$ cannot have all $n$ elements of $G$, that is, $m \leq n - 1$. Since by the maximality of $S$ we have that $p, q \leq m$
it follows that we require $p, q \leq n - 1$ as well. So the lower and upper bounds for the individual sizes $m, p, q$ of a TPP triple $S, T, U$ of $G$ are expressed by the inequalities $2 \leq m, p, q \leq n - 1$, that is, they must be proper subsets of $G$. The number of proper subset triples of $G$ is $[2^n - (n + 1)]^3$. Though, we shall see that $m, p, q$ cannot all necessarily attain their minimum or maximum values simultaneously if their underlying subset triple $S, T, U$ is assumed to be a TPP triple or even a maximal TPP triple. An absolute lower bound for $m + p + q$ is 6, since all $m, p, q \geq 2$. The lower bound can be made a function of $|G| = n$ if the TPP triple is additionally a maximal one as follows. The arithmetic mean $\frac{1}{3}(x + y + z)$ of three positive integers $x, y, z$ is greater than or equal to their geometric mean $\sqrt[3]{xyz}$, with equality iff $x = y = z$. Thus, $\frac{1}{3}(m + p + q) \geq (mpq)^{\frac{1}{3}}$, with equality iff $m = p = q$. For a maximal TPP triple we know that $mpq \geq n$, which is equivalent to $(mpq)^{\frac{1}{3}} \geq n^{\frac{1}{3}}$. Thus the lower bound for the sum $m + p + q$ of sizes of subsets forming a maximal TPP triple of $G$ is given by $m + p + q \geq 3n^{\frac{1}{3}}$. Since $m + p + q$ is always an integer we can replace $3n^{\frac{1}{3}}$ by $\left\lfloor n^{\frac{1}{3}} \right\rfloor$. A general upper bound for $m + p + q$ (where the TPP triple need not be maximal) is $n + 3$: to see this, we note that for any TPP triple $S, T, U \subset G$ it is the case that $m + p + q = n + w - r$, where $w = |S| + |T| + |U| - |S \cup T \cup U|$ is its disjointness type number and $r$ is the size of the remainder set $V = G - (S \cup T \cup U)$. Since $w$ and $r$ are independent, we set $w$ to its maximum value 3 and $r$ to its minimum value 0, leading to the upper bound $n + 3$ for $m + p + q$. So, in summary, for a maximal tensor $\langle m', p', q' \rangle$ of $G$ it is true that:

$$3n^{\frac{1}{3}} \leq m' + p' + q' \leq n + 3.$$  

References

[1]. Burgisser, P. et. al., *Algebraic Complexity Theory*, Springer, Berlin, 1997.

[2]. Cohn, H., Umans, C., ‘Group-theoretic Algorithms for Fast Matrix Multiplication’, *Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science 2003*, IEEE Computer Society, 2003, pp. 379-388.

[3]. Cohn, H. & Umans, C. & Kleinberg, R. & Szegedy, B., ‘A Group-theoretic Approach to Fast Matrix Multiplication’, *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science 2005*, IEEE Computer Society, 2005, pp. 438-449.

[4]. Huppert, Bertram, *Character Theory of Finite Groups*, Walter de Gruyter, Berlin, 1998 (1927 original print).
[5]. James, G. & Liebeck, M., *Representations and Characters of Groups*, Cambridge University Press, Cambridge, 2001.

[6]. Murthy, S., 'Group-theoretic Methods for the Complexity of Fast Matrix Multiplication', Institute for Logic, Language and Computation (ILLC), Universiteit van Amsterdam, 2007.