THE CANONICAL ARITHMETIC HEIGHT OF SUBVARIETIES
OF AN ABELIAN VARIETY
OVER A FINITELY GENERATED FIELD

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INTRODUCTION

This paper is the sequel of [4]. In [4], S. Zhang defined the canonical height of subvarieties of an abelian variety over a number field in terms of adelic metrics. In this paper, we generalize it to an abelian variety defined over a finitely generated field over \( \mathbb{Q} \). Our way is slightly different from his method. Instead of using adelic metrics directly, we introduce an adelic sequence and an adelic structure (cf. §§3.1).

Let \( K \) be a finitely generated field over \( \mathbb{Q} \) with \( d = \text{tr} \deg_{\mathbb{Q}}(K) \), and \( \mathcal{B} = (\mathcal{B}, \mathcal{H}_1, \ldots, \mathcal{H}_d) \) a polarization of \( K \), i.e., \( \mathcal{B} \) is a projective arithmetic variety whose function field is \( K \), and \( \mathcal{H}_1, \ldots, \mathcal{H}_d \) are nef \( C^\infty \)-hermitian line bundles on \( \mathcal{B} \). Let \( A \) be an abelian variety over \( K \), and \( \mathcal{L} \) a symmetric ample line bundle on \( A \). Fix a projective arithmetic variety \( \mathcal{A} \) over \( \mathcal{B} \) and a nef \( C^\infty \)-hermitian \( \mathcal{Q} \)-line bundle \( \mathcal{L} \) on \( \mathcal{A} \) such that \( A \) is the generic fiber of \( \mathcal{A} \rightarrow \mathcal{B} \) and \( \mathcal{L} \) is isomorphic to \( \mathcal{L} \) on \( A \). Then we can assign the naive height \( \hat{h}^{\mathcal{B}}_{(\mathcal{A}, \mathcal{X})}(X) \) to a subvariety \( X \) of \( A_K \). Indeed, if \( X \) is defined over \( K \), \( \hat{h}^{\mathcal{B}}_{(\mathcal{A}, \mathcal{X})}(X) \) is given by

\[
\hat{h}^{\mathcal{B}}_{(\mathcal{A}, \mathcal{X})}(X) = \frac{\deg (c_1(\mathcal{L}|_X))^{\dim X + 1} \cdot c_1(\pi_X^*(\mathcal{H}_1)) \cdot \cdots \cdot c_1(\pi_X^*(\mathcal{H}_d))}{(\dim X + 1) \deg (\mathcal{L}|_X^{\dim X})},
\]

where \( \mathcal{X} \) is the Zarisky closure of \( X \) in \( \mathcal{A} \) and \( \pi_X : \mathcal{X} \rightarrow \mathcal{B} \) is the canonical morphism. The canonical height \( \hat{h}^{\mathcal{B}}_L(X) \) of \( X \) with respect to \( L \) and \( \mathcal{B} \) is characterized by the following properties:

(a) \( \hat{h}^{\mathcal{B}}_L(X) \geq 0 \) for all subvarieties \( X \) of \( A_K \).

(b) There is a constant \( C \) such that

\[
\left| \hat{h}^{\mathcal{B}}_L(X) - \hat{h}^{\mathcal{B}}_{(\mathcal{A}, \mathcal{X})}(X) \right| \leq C
\]

for all subvarieties \( X \) of \( A_K \).

(c) \( \hat{h}^{\mathcal{B}}_L([N](X)) = N^2 \hat{h}^{\mathcal{B}}_L(X) \) for all subvarieties \( X \) of \( A_K \) and all non-zero integers \( N \).

The main result of this paper is the following theorem, which is a generalization of [3].

**Theorem** (cf. Theorem [5, 7]). If the polarization \( \mathcal{B} \) is big (i.e., \( \mathcal{H}_1, \ldots, \mathcal{H}_d \) are nef and big), then, for a subvariety \( X \) of \( A_K \), the following are equivalent.

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(1) $X$ is a translation of an abelian subvariety by a torsion point.
(2) The set $\{ x \in X(\mathbb{K}) \mid h_{\mathbb{K}}^X(x) \leq \epsilon \}$ is Zariski dense in $X$ for every $\epsilon > 0$.
(3) The canonical height of $X$ with respect to $L$ and $\mathcal{B}$ is zero, i.e., $h_{\mathbb{K}}^X(X) = 0$.

Next let us consider a case where a curve and its Jacobian. Let $X$ be a smooth projective curve of genus $g \geq 2$ over $K$, and $J$ the Jacobian of $X$. Let $\Theta$ be a symmetric theta divisor on $J$, and $j : X \to J$ a morphism given by $j(x) = \omega_X - (2g - 2)x$. Then, since $j^*(\mathcal{O}_J(\Theta)) = \omega_X^{2g(g-1)}$, we can assign the canonical adelic structure $\mathcal{X}^1$ to $\omega_X$. As a corollary of the above theorem, we have the following, which is a generalization of [3].

**Corollary** (cf. Corollary 5.4). If the polarization $\mathcal{B}$ is big, then the adelic self intersection number of $\mathcal{X}^1$ with respect to $\mathcal{B}$ is positive, i.e., $\langle \mathcal{X}^1 \cdot \mathcal{X}^1 \rangle_{\mathcal{B}} > 0$.

### 1. Preliminaries

For the basic notation of Arakelov Geometry, we follow the paper [2].

Let $X$ be a projective arithmetic variety with $d = \dim X_{\mathbb{Q}}$, and $\mathcal{L}$ a $C^\infty$-hermitian $\mathbb{Q}$-line bundle on $X$. First we review several kinds of positivity of $\mathcal{L}$.

- **ample**: We say $\mathcal{L}$ is ample if $L$ is ample, $c_1(\mathcal{L})$ is a semipositive form on $X(\mathbb{C})$, and, for a sufficiently large $n$, $H^0(X, L^{\otimes n})$ is generated by $\{ s \in H^0(X, L^{\otimes n}) \mid \|s\|_{\text{sup}} < 1 \}$.

- **nef**: We say $\mathcal{L}$ is nef if $c_1(\mathcal{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes $\Gamma$ of $X$, $\deg (\mathcal{L}_\Gamma) \geq 0$.

- **big**: $\mathcal{L}$ is said to be big if $\rk_{\mathbb{Z}} H^0(X, L^{\otimes n}) = O(m^n)$, and there is a non-zero section $s$ of $H^0(X, L^{\otimes n})$ with $\|s\|_{\text{sup}} < 1$ for some positive integer $n$.

- **$\mathbb{Q}$-effective**: We say $\mathcal{L}$ is $\mathbb{Q}$-effective, denote by $\mathcal{L} \succeq 0$, if there are a positive integer $n$ and a non-zero section $s \in H^0(X, L^{\otimes n})$ with $\|s\|_{\text{sup}} \leq 1$. Moreover, if $U$ is a non-empty Zariski open set of $X$ with $\text{div}(s) \subseteq X \setminus U$, then we use the notation $\mathcal{L} \succeq_U 0$. Let $\mathcal{M}$ be another $C^\infty$-hermitian $\mathbb{Q}$-line bundle on $X$. If $\mathcal{L} \otimes \mathcal{M}^{\otimes -1} \succeq 0$ (resp. $\mathcal{L} \otimes \mathcal{M}^{\otimes -1} \succeq_U 0$), then we denote this by $\mathcal{L} \succeq \mathcal{M}$ (resp. $\mathcal{L} \succeq_U \mathcal{M}$).

**Proposition 1.1.** (1) If $\mathcal{L}$ is a nef $C^\infty$-hermitian $\mathbb{Q}$-line bundle and $\mathcal{A}$ is an ample $C^\infty$-hermitian $\mathbb{Q}$-line bundle, then $\mathcal{L} + \epsilon \mathcal{A}$ is ample for all positive rational numbers $\epsilon$.
(2) If $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ are nef $C^\infty$-hermitian $\mathbb{Q}$-line bundles, then
$$\widehat{\deg} (c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d+1})) \geq 0.$$
(3) If $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are nef $C^\infty$-hermitian $\mathbb{Q}$-line bundles and $\mathcal{M}$ is a $\mathbb{Q}$-effective $C^\infty$-hermitian $\mathbb{Q}$-line bundle, then
$$\widehat{\deg} (c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_d) \cdot c_1(\mathcal{M})) \geq 0.$$
(4) Let $\mathcal{L}_1, \ldots, \mathcal{L}_{d+1}$ and $\mathcal{M}_1, \ldots, \mathcal{M}_{d+1}$ be nef $C^\infty$-hermitian line bundles on $X$. If $\mathcal{M}_i \succeq \mathcal{L}_i$ for every $i$, then
$$\widehat{\deg} (c_1(\mathcal{M}_1) \cdots c_1(\mathcal{M}_{d+1})) \geq \widehat{\deg} (c_1(\mathcal{L}_1) \cdots c_1(\mathcal{L}_{d+1})).$$
Theorem 1.1. Let $X$ be a projective arithmetic variety, and $L$ a big $C^\infty$-hermitian $\mathbb{Q}$-line bundle on $X$. Let $x$ be a (not necessarily closed) point of $X$. Then, there are a positive number $n$ and a non-zero section $s \in H^0(X, L^\otimes n)$ with $s(x) = 0$ and $\|s\|_{\sup} < 1$.

Proof. Since $\text{rk}_\mathbb{Z} H^0(X, L^\otimes m) = O(m^d)$, there are a positive number $n_0$ and a non-zero section $s_0 \in H^0(X, L^\otimes n_0)$ with $s_0(x) = 0$. On the other hand, there is a non-zero section $s_1 \in H^0(X, L^\otimes n_1)$ with $\|s_1\|_{\sup} < 1$ for some positive integer $n_1$. Let $n_2$ be a positive integer with $\|s_0\|_{\sup}^n \|s_1\|^n_{\sup} < 1$.

Thus, if we set $s = s_0 \otimes s_1^\otimes n_2 \in H^0(X, L^\otimes n_0+n_1n_2)$, then we have the desired assertion.

Lemma 1.2. Let $X$ be a projective arithmetic variety, and $L$ a bundle on $X$. Then, we have the following lemma.

Lemma 1.3. Let $B$ be a projective arithmetic variety and $K$ the function field of $B$. Let $X$ be a projective variety over $K$, and $L$ an ample line bundle on $X$. Then, there are a projective arithmetic variety $\mathcal{X}$ over $B$, and an ample $C^\infty$-hermitian $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $X$ is the generic fiber of $\mathcal{X} \to B$ and $\mathcal{L}$ coincides with $L$ in $\text{Pic}(X) \otimes \mathbb{Q}$.

Proof. Choose a sufficiently large integer $n$ such that $\phi|_{L^\otimes n}$ gives rise to an embedding $X \hookrightarrow \mathbb{P}^N_K$. Let $\mathcal{X}$ be the Zariski closure of $X$ in $\mathbb{P}^N_B = \mathbb{P}^N \times B$. Since $\mathcal{O}_{\mathbb{P}^N}(1)$ is relative ample, there is an ample line bundle $Q$ on $B$ such that $\mathcal{A} = \mathcal{O}_{\mathbb{P}^N}(1) \otimes \pi^*(Q)$ is ample, where $\pi$ is the natural projection $\mathbb{P}^N_B \to B$. We choose a $C^\infty$-hermitian metric of $\mathcal{A}$ such that $\mathcal{A} = (\mathcal{A}, \|\cdot\|)$ is ample. Thus, if we set $\mathcal{L} = (\mathcal{A}|_{\mathcal{X}})^{\otimes 1/n}$, then we have our assertion.

Next, let us consider the following relative positivity.

• $\pi$-nef (nef with respect to a morphism): Let $\pi : X \to B$ be a morphism of projective arithmetic varieties, and $L$ a $C^\infty$-hermitian $\mathbb{Q}$-line bundle on $X$. We say $L$ is nef with respect to $X \to B$ (or $\pi$-nef) if the following properties are satisfied:

(1) For any analytic maps $h : M \to X(\mathbb{C})$ from a complex manifold $M$ to $X(\mathbb{C})$ with $\pi(h(M))$ being a point, $c_1(h^*(L))$ is semipositive.

(2) For every $b \in B$, the restriction $L|_{X_\pi}$ of $L$ to the geometric fiber over $b$ is nef.

Then, we have the following lemma.

Lemma 1.4. Let $\pi : X \to B$ be a morphism of projective arithmetic varieties with $d = \text{dim } B_\mathbb{Q}$ and $e = \text{dim}(X/B)$. Let $H_1, \ldots, H_d$ be nef $C^\infty$-hermitian $\mathbb{Q}$-line bundles on $B$. Then, we have the following.
(1) Let $\mathcal{L}_1, \ldots, \mathcal{L}_e$ be $\pi$-nef $C^\infty$-hermitian $\mathbb{Q}$-line bundles on $X$, and $\mathcal{L}$ a $C^\infty$-hermitian $\mathbb{Q}$-line bundle on $X$. If there is a non-empty Zariski open set $U$ of $B$ with $\mathcal{L} \not\supseteq_{\pi^{-1}(U)} 0$, then

$$\deg(\hat{c}_1(\mathcal{L}_1) \cdots \hat{c}_1(\mathcal{L}_e) \cdot \hat{c}_1(\mathcal{L}) \cdot \hat{c}_1(\pi^*\mathcal{H}_1) \cdots \hat{c}_1(\pi^*\mathcal{H}_d)) \geq 0.$$

(2) Let $\mathcal{T}_1, \ldots, \mathcal{T}_{e+1}$ and $\mathcal{T}_1^{\prime}, \ldots, \mathcal{T}_{e+1}^{\prime}$ be $\pi$-nef $C^\infty$-hermitian $\mathbb{Q}$-line bundles on $X$. If there is a Zariski open set $U$ of $B$ such that $\mathcal{T}_i \not\supseteq_{\pi^{-1}(U)} \mathcal{T}_i^{\prime}$ for all $i$, then

$$\deg(\hat{c}_1(\mathcal{T}_1) \cdots \hat{c}_1(\mathcal{T}_{e+1}) \cdot \hat{c}_1(\pi^*\mathcal{H}_1) \cdots \hat{c}_1(\pi^*\mathcal{H}_d)) \geq \deg(\hat{c}_1(\mathcal{T}_1) \cdots \hat{c}_1(\mathcal{T}_{e+1}^{\prime}) \cdot \hat{c}_1(\pi^*\mathcal{H}_1) \cdots \hat{c}_1(\pi^*\mathcal{H}_d)).$$

Proof. (1) By our assumption, there are a positive integer $n$ and a non-zero section $s \in H^0(X, L^\otimes n)$ such that $\|s\|_{\sup} \leq 1$ and $\text{Supp}(\text{div}(s)) \subseteq X \setminus \pi^{-1}(U)$. Let $\text{div}(s) = a_1 \Delta_1 + \cdots + a_r \Delta_r$ be the decomposition as cycles. Then,

$$\int_{X(\mathbb{C})} \log(\|s\|) c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_e) \wedge c_1(\pi^*\mathcal{H}_1) \wedge \cdots \wedge c_1(\pi^*\mathcal{H}_d).$$

First, by the Fubini’s theorem,

$$\int_{X(\mathbb{C})} \log(\|s\|) c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_e) \wedge c_1(\pi^*\mathcal{H}_1) \wedge \cdots \wedge c_1(\pi^*\mathcal{H}_d)$$

$$= \int_{B(\mathbb{C})} \left( \int_{X(\mathbb{C})/B(\mathbb{C})} \log(\|s\|) c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_e) \right) c_1(\mathcal{H}_1) \wedge \cdots \wedge c_1(\mathcal{H}_d).$$

Here, by the property (1) of “$\pi$-nef”,

$$\int_{X(\mathbb{C})/B(\mathbb{C})} \log(\|s\|) c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_e)$$

is a non-negative locally integrable function on $B(\mathbb{C})$. Thus, the integral part of (1.4.1) is non-negative. Let $b_i$ be the generic point of $\pi(\Delta_i)$. Then, by the projection formula, we can see

$$\deg(\hat{c}_1(\mathcal{L}_1|_{\Delta_i}) \cdots \hat{c}_1(\mathcal{L}_e|_{\Delta_i}) \cdot \hat{c}_1(\pi^*\mathcal{H}_1|_{\Delta_i}) \cdots \hat{c}_1(\pi^*\mathcal{H}_d|_{\Delta_i}))$$

$$= \begin{cases} 
0 & \text{if codim}(\pi(\Delta_i)) \geq 2 \\
\deg(L_1|_{(\Delta_i)_{b_i}} \cdots L_e|_{(\Delta_i)_{b_i}}) \deg(\hat{c}_1(\mathcal{H}_1|_{\pi(\Delta_i)}) \cdots \hat{c}_1(\mathcal{H}_d|_{\pi(\Delta_i)})) & \text{if codim}(\pi(\Delta_i)) = 1
\end{cases}$$

Therefore, we get (1) because

$$\deg(L_1|_{(\Delta_i)_{b_i}} \cdots L_e|_{(\Delta_i)_{b_i}}) \geq 0 \quad \text{and} \quad \deg(\hat{c}_1(\mathcal{H}_1|_{\pi(\Delta_i)}) \cdots \hat{c}_1(\mathcal{H}_d|_{\pi(\Delta_i)})) \geq 0.$$
(2) Since
\[ \widehat{c}_1(\mathcal{L}_1) \cdots \widehat{c}_1(\mathcal{L}_{e+1}) - \widehat{c}_1(\mathcal{L}_1) \cdots \widehat{c}_1(\mathcal{T}_{e+1}) = \sum_{i=1}^{e+1} \widehat{c}_1(\mathcal{L}_1) \cdots \widehat{c}_1(\mathcal{L}_{i-1}) \cdot \left( \widehat{c}_1(\mathcal{L}_i) - \widehat{c}_1(\mathcal{T}_i) \right) \cdot \widehat{c}_1(\mathcal{T}_{i+1}) \cdots \widehat{c}_1(\mathcal{T}_{e+1}), \]

(2) is a consequence of (1). \qed

Finally, let us consider the following lemma.

**Lemma 1.5.** Let \( \pi : X \to B \) be a morphism of projective arithmetic varieties, and \( \mathcal{T} \) a \( C^\infty \)-hermitian line bundle on \( X \). Let \( U \) be a non-empty Zariski open set of \( B \) such that \( B \setminus U = \text{Supp}(D) \) for some effective Cartier divisor \( D \) on \( B \). If there is a non-zero rational section \( s \) of \( L \) with \( \text{Supp}(\text{div}(s)) \subseteq X \setminus \pi^{-1}(U) \), then there are a positive integer \( n \) and a \( C^\infty \)-metric \( \| \cdot \|_{nD} \) of \( \mathcal{O}_B(nD) \) with
\[ \pi^*(\mathcal{O}_B(nD), \| \cdot \|_{nD}) \lesssim_{\pi^{-1}(U)} \mathcal{T} \lesssim_{\pi^{-1}(U)} \pi^*(\mathcal{O}_B(nD), \| \cdot \|_{nD}). \]
Moreover, if \( D \) is ample, then we can choose \( \| \cdot \|_{nD} \) such that \( (\mathcal{O}_B(nD), \| \cdot \|_{nD}) \) is ample.

**Proof.** First, we fix a hermitian metric \( \| \cdot \|_D \) of \( \mathcal{O}_B(D) \). If \( D \) is ample, then we choose \( \| \cdot \|_D \) such that \( (\mathcal{O}_B(D), \| \cdot \|_D) \) is ample. Find a positive integer \( n \) with
\[ -nf^*(D) \leq \text{div}(s) \leq nf^*(D). \]
Let \( l \) be a section of \( \mathcal{O}_Y(nD) \) with \( \text{div}(l) = nD \). We set \( t_1 = l \otimes s^{-1} \) and \( t_2 = l \otimes s \). Then, \( t_1 \) and \( t_2 \) are global sections of \( \mathcal{O}_X(nf^*(D)) \otimes L^{-1} \) and \( \mathcal{O}_X(nf^*(D)) \otimes L \) respectively. Choose a sufficiently small positive number \( c \) such that if we give a norm of \( \mathcal{O}_B(nD) \) by \( c\| \cdot \|_D \), then \( \|t_1\|_{\text{sup}} \leq 1 \) and \( \|t_2\|_{\text{sup}} \leq 1 \). Thus we get our lemma. \qed

## 2. Arithmetic height of subvarieties

Let \( K \) be a finitely generated field over \( \mathbb{Q} \) with \( d = \text{tr. deg}_{\mathbb{Q}}(K) \), and \( \mathcal{L} = (B, \mathcal{L}_1, \ldots, \mathcal{L}_d) \) a polarization of \( K \). Let \( X \) be a projective variety over \( K \), and \( L \) a nef line bundle on \( X \). Let \( \mathcal{X} \) be a projective arithmetic variety over \( B \) such that \( X \) is the generic fiber of \( \mathcal{X} \to B \), and let \( \mathcal{L} \) be a \( C^\infty \)-hermitian \( \mathbb{Q} \)-line bundle on \( \mathcal{X} \) such that \( \mathcal{L} \) coincides with \( L \) in \( \text{Pic}(X) \otimes \mathbb{Q} \). The pair \( (\mathcal{X}, \mathcal{L}) \) is called a \( C^\infty \)-model of \( (X, L) \). We assume that \( \mathcal{L} \) is nef with respect to \( \mathcal{X} \to B \). Note that if \( L \) is ample, then there is a \( C^\infty \)-model \( (\mathcal{X}, \mathcal{E}) \) of \( (X, L) \) such that \( \mathcal{E} \) is ample by Lemma 1.3.

Let \( Y \) be a subvariety of \( X_{\mathcal{L}} \). We assume that \( Y \) is defined over a finite extension field \( K' \) of \( K \). Let \( B^{K'} \) be the normalization of \( B \) in \( K' \), and let \( \rho^{K'} : B^{K'} \to B \) be the induced morphism. Let \( \mathcal{X}^{K'} \) be the main component of \( \mathcal{X} \times_B B^{K'} \). We set the induced morphisms as follows.

\[
\begin{array}{ccc}
\mathcal{X}^{K'} & \xleftarrow{\pi^{K'}} & \mathcal{X}^{K'} \\
\downarrow & & \downarrow
\pi^{K'} \\
B & \xleftarrow{\rho^{K'}} & B^{K'}
\end{array}
\]
Let $Y$ be the Zariski closure of $Y$ in $\mathcal{X}^{K'}$. Then the naive height $h_{(\mathcal{X}, \mathcal{L})}(Y)$ of $Y$ with respect to $(\mathcal{X}, \mathcal{L})$ and $\mathcal{B}$ is defined by

$$h_{(\mathcal{X}, \mathcal{L})}(Y) = \deg \left( \frac{\deg \left( \mathcal{L}^{\mathcal{X}(\mathcal{L})} \right)^{\dim Y + 1}}{[K' : K](\dim Y + 1) \deg(L|Y)} \right).$$

Note that the above definition does not depend on the choice of $K'$ by the projection formula. Here we have the following proposition. By this proposition, we may denote by $h_{L}^N$ the class of $h_{(\mathcal{X}, \mathcal{L})}$ modulo the set of bounded functions. Moreover, we say $h_{L}^N$ is the height function associated with $L$ and $\mathcal{B}$.

**Proposition 2.1.** Let $(\mathcal{X}', \mathcal{L}')$ be another model of $(X, L)$ over $B$ such that $\mathcal{L}'$ is nef with respect to $\mathcal{X}' \rightarrow B$. Then, there is a constant $C$ such that

$$\left| h_{(\mathcal{X}, \mathcal{L})}(Y) - h_{(\mathcal{X}', \mathcal{L}')}(Y) \right| \leq C$$

for all subvarieties $Y$ of $X_K$.

**Proof.** Let $U$ be a Zariski open set of $B$ with $\mathcal{X}_U = \mathcal{X}'_U$ and $\mathcal{L}_U = \mathcal{L}'_U$ in Pic($\mathcal{X}_U$) $\otimes \mathbb{Q}$. Let $A$ be an ample line bundle on $B$ and $I$ the defining ideal of $B \setminus U$. Then, there is a non-zero section $t$ of $H^0(B, A^{\otimes m} \otimes I)$ for some positive integer $m$. Thus, $B \setminus U \subseteq \mathrm{Supp}(\mathrm{div}(s))$. Therefore, shrinking $U$, we may assume that there is an effective ample Cartier divisor $D$ on $B$ with $\mathrm{Supp}(D) = B \setminus U$.

Let $\mu : Z \rightarrow \mathcal{X}$ and $\mu' : Z \rightarrow \mathcal{X}'$ be birational morphisms of projective arithmetic varieties such that $\mu$ and $\mu'$ are the identity map over $\mathcal{X}_U$. Then, $h_{(\mathcal{X}, \mathcal{L})}(Y) = h_{(\mathcal{X}, \mathcal{L})}(Y)$ and $h_{(\mathcal{X}', \mathcal{L}')}(Y) = h_{(\mathcal{X}', \mathcal{L}')}(Y)$ for all subvarieties $Y$ of $X_K$. Thus, to prove our proposition, we may assume that $\mathcal{X} = \mathcal{X}'$.

First of all, by Lemma 1.3, there is a nef $C^\infty$-hermitian line bundle $\mathcal{L}$ on $B$ such that

$$(2.1.1) \quad \pi^*(\mathcal{L})^{\otimes -1} \preceq \pi_{-1}(U) \mathcal{L} \otimes \mathcal{L}^{\otimes -1} \preceq \pi_{-1}(U) \pi^*(\mathcal{L}),$$

where $\pi : \mathcal{X} \rightarrow B$ is the canonical morphism. Let $Y$ be a subvariety of $X_K$. We assume that $Y$ is defined over a finite extension field $K'$ of $K$. Let $B^{K'}$ be the normalization of $B$ in $K'$, and $\mathcal{X}^{K'}$ the main components of $\mathcal{X} \times_B B^{K'}$. Let $\mathcal{Y}$ be the closure of $Y$ in $\mathcal{X}^{K'}$. Then,

$$h_{(\mathcal{X}, \mathcal{L})}(Y) = \deg \left( \frac{\deg \left( \mathcal{L}^{\mathcal{X}(\mathcal{L})} \right)^{\dim Y + 1}}{[K' : K](\dim Y + 1) \deg(L|Y)} \right)$$

and

$$h_{(\mathcal{X}', \mathcal{L}')}(Y) = \deg \left( \frac{\deg \left( \mathcal{L}^{\mathcal{X}'(\mathcal{L}')} \right)^{\dim Y + 1}}{[K' : K](\dim Y + 1) \deg(L|Y)} \right),$$

where $\mathcal{L}^{\mathcal{X}(\mathcal{L})}$ and $\mathcal{L}^{\mathcal{X}'(\mathcal{L}')}$ are the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $\mathcal{X}$ and $\mathcal{X}'$, respectively.
where \( \mathcal{L}^{K'}, \mathcal{L}'^{K'} \) and \( \mathcal{H}'_i^{K'} \)'s are pullbacks of \( \mathcal{L}, \mathcal{L}' \) and \( \mathcal{H}'_i \)'s to \( X^{K'} \) respectively. Here, by virtue of (2.1.1),
\[
\mathcal{L}^{K'} \otimes T^{K' \otimes -1} \simeq \mathcal{L}'^{K'} \simeq \mathcal{L}'^{K'} \otimes T^{K'}.
\]
Therefore, by (2) of Lemma [4], we can see that
\[
\left| \text{deg} \left( \hat{c}_1 \left( \mathcal{L}^{K'} \right|_Y \right) - \text{deg} \left( \hat{c}_1 \left( \mathcal{H}'_1^{K'} \right|_Y \right) \cdots \hat{c}_1 \left( \mathcal{H}'_d^{K'} \right|_Y \right) \right| \leq [K': K](\dim Y + 1) \deg(L_{1, Y}^{\text{dim} Y}) \deg(T \cdot H_1 \cdots H_d).
\]
Thus we get our proposition. \( \square \)

3. Adelic sequence and adelic structure

3.1. Adelic sequence, adelic structure and adelic line bundle. Let \( K \) be a finitely generated field over \( \mathbb{Q} \) with \( d = \text{tr} \cdot \deg_{\mathbb{Q}}(K) \), and \( B = (B; H_1, \ldots, H_d) \) a polarization of \( K \).

Let \( X \) be a projective variety over \( K \), and \( L \) a nef line bundle on \( X \).

A sequence of \( C^\infty \)-models \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \) of \( (X, L) \) is called an adelic sequence of \( (X, L) \) (with respect to \( B \)) if \( \mathcal{L}_n \) is nef with respect to \( \mathcal{X}_n \to B \) for every \( n \), and there is a non-empty Zariski open set \( U \) of \( B \) with following properties:

\( (1) \) \( \mathcal{X}_n|_U = \mathcal{X}_m|_U \) (say \( \mathcal{X}_U \)) and \( \mathcal{L}_n|_U = \mathcal{L}_m|_U \) in \( \text{Pic}(\mathcal{X}_U) \otimes \mathbb{Q} \) for all \( n, m \).

\( (2) \) For each \( n, m \), there are a projective arithmetic variety \( \mathcal{X}_{n,m} \) over \( B \), birational morphisms \( \mu_{n,m}^*: \mathcal{X}_{n,m} \to \mathcal{X}_n \) and \( \mu_{m,n}^*: \mathcal{X}_{n,m} \to \mathcal{X}_m \), and a nef \( C^\infty \)-hermitian \( \mathbb{Q} \)-line bundle \( \mathcal{D}_{n,m} \) on \( \mathcal{X}_{n,m} \) such that
\[
\pi_{n,m}^* \left( \mathcal{D}_{n,m}^{\otimes -1} \right) \simeq \pi_{n,m}^* \left( \mathcal{L}_n \right) \otimes \left( \mu_{n,m}^* \right)^* \left( \mathcal{D}_{m}^{\otimes -1} \right) \simeq \pi_{n,m}^* \left( \mathcal{D}_{n,m} \right)
\]
and that
\[
\text{deg} \left( \hat{c}_1 \left( \mathcal{D}_{n,m} \right) \cdot \hat{c}_1 \left( \mathcal{H}_1 \right) \cdots \hat{c}_1 \left( \mathcal{H}_d \right) \right) \to 0
\]
as \( n, m \to \infty \), where \( \pi_{n,m} \) is the natural morphism \( \mathcal{X}_{n,m} \to B \).

The open set \( U \) as above is called a common base of the sequence \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \). Note that if \( U' \) is a non-empty Zariski open set of \( U \), then \( U' \) is also a common base of \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \).

Let \( \{ (\mathcal{Y}_n, \mathcal{M}_n) \} \) be another adelic sequence of \( (X, L) \). We say \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \) is equivalent to \( \{ (\mathcal{Y}_n, \mathcal{M}_n) \} \), denoted by \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \sim \{ (\mathcal{Y}_n, \mathcal{M}_n) \} \), if the concatenated sequence
\[
(\mathcal{X}_1, \mathcal{L}_1), (\mathcal{Y}_1, \mathcal{M}_1), \ldots, (\mathcal{X}_n, \mathcal{L}_n), (\mathcal{Y}_n, \mathcal{M}_n), \ldots
\]
is adelic. In other words, if we choose a suitable common base \( U \), then, for each \( n \), there are a projective arithmetic variety \( \mathcal{Z}_n \) over \( B \), birational morphisms \( \nu_n : \mathcal{Z}_n \to \mathcal{X}_n \) and \( \nu_n : \mathcal{Z}_n \to \mathcal{Y}_n \), and a nef \( C^\infty \)-hermitian \( \mathbb{Q} \)-line bundle \( \mathcal{D}_n \) on \( \mathcal{Z}_n \) such that
\[
\pi_{\mathcal{Z}_n}^* \left( \mathcal{D}_n^{\otimes -1} \right) \simeq \pi_{\mathcal{Z}_n}^* \left( \mathcal{L}_n \right) \otimes \nu_n^* \left( \mathcal{M}_n^{\otimes -1} \right) \simeq \pi_{\mathcal{Z}_n}^* \left( \mathcal{D}_n \right)
\]

Thus, we have
\[
\pi_{\mathcal{Z}_n}^* \left( \mathcal{D}_n^{\otimes -1} \right) \simeq \pi_{\mathcal{Z}_n}^* \left( \mathcal{L}_n \right) \otimes \nu_n^* \left( \mathcal{M}_n^{\otimes -1} \right)
\]
for all \( n \).
and that
\[
\lim_{n \to \infty} \widehat{\deg} \left( \hat{c}_1(D_n) \cdot \hat{c}_1(H_1) \cdots \hat{c}_1(H_d) \right) = 0,
\]
where \( \pi_{Z_n} \) is the natural morphism \( Z_n \to B \).

An equivalent class of adelic sequences of \((X, L)\) is called an *adelic structure of \( L \)* (with respect to \( B \)). Further, a line bundle \( L \) with an adelic structure is called an *adelic line bundle* and is often denoted by \( \mathcal{T} \) for simplicity. If an adelic line bundle \( \mathcal{T} \) is given by an adelic sequence \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \), then we denote this by \( \mathcal{T} = \lim_{n \to \infty} (\mathcal{X}_n, \mathcal{L}_n) \). Moreover, we say \( \mathcal{T} \) is nef if \( \mathcal{T} = \lim_{n \to \infty} (\mathcal{X}_n, \mathcal{L}_n) \) and \( \mathcal{L}_n \) is nef for \( n \gg 0 \).

Let \( g : Y \to X \) be a morphism of projective varieties over \( K \), and \( \mathcal{T} \) an adelic line bundle on \( X \). We assume that \( \mathcal{T} \) is given by an adelic sequence \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \). Let us fix a morphism \( g_n : \mathcal{Y}_n \to \mathcal{X}_n \) of projective arithmetic varieties over \( B \) for each \( n \) with the following properties:

(a) \( g_n : \mathcal{Y}_n \to \mathcal{X}_n \) coincides with \( g : Y \to X \) over \( K \) for every \( n \).

(b) There is a non-empty Zariski open set \( U \) of \( B \) such that \( \mathcal{Y}_n|_U = \mathcal{Y}_m|_U, \mathcal{X}_n|_U = \mathcal{X}_m|_U \), and \( g_n|_U = g_m|_U \) for all \( n, m \).

Then it is not difficult to see that \( \{ (\mathcal{Y}_n, g_n^*(\mathcal{L}_n)) \} \) is an adelic sequence of \( (Y, g^*(L)) \). We denote by \( g^*(\mathcal{T}) \) the adelic structure given by \( \{ (\mathcal{Y}_n, g_n^*(\mathcal{L}_n)) \} \). Note that this adelic structure does not depend on the choice of the adelic sequence \( \{ (\mathcal{X}_n, \mathcal{L}_n) \} \) and the morphisms \( g_n : \mathcal{Y}_n \to \mathcal{X}_n \).

### 3.2. Adelic sequence by an endomorphism.

Let \( K \) be a finitely generated field over \( \mathbb{Q} \) with \( d = \text{tr. deg}_{\mathbb{Q}}(K) \), and \( B = (B; \mathcal{P}_1, \ldots, \mathcal{P}_d) \) a polarization of \( K \). Let \( X \) be a projective variety over \( K \), and \( L \) an ample line bundle on \( X \). We assume that there is a surjective morphism \( f : X \to X \) and an integer \( d \geq 2 \) with \( L^\otimes d \simeq f^*(L) \). Let \( (\mathcal{X}, \mathcal{T}) \) be a \( C^\infty \)-model of \((X, L)\) such that \( \mathcal{T} \) is nef with respect to \( \mathcal{X} \to B \). Note that the existence of a \( C^\infty \)-model \((\mathcal{X}, \mathcal{T})\) of \((X, L)\) with \( \mathcal{L} \) being nef with respect to \( \mathcal{X} \to B \) is guaranteed by Lemma 1.3. Then, there is a Zariski open set \( U \) of \( B \) such that \( f \) extends to \( f_U : \mathcal{X}_U \to \mathcal{X}_U \) and \( \mathcal{L}_U^\otimes d = f_U^*(\mathcal{L}_U) \) in \( \text{Pic}(\mathcal{X}_U) \otimes \mathbb{Q} \). Let \( \mathcal{X}_n \) be the normalization of \( \mathcal{X}_U \xrightarrow{f_U^n} \mathcal{X}_U \to \mathcal{X} \), and \( f_n : \mathcal{X}_n \to \mathcal{X} \) the induced morphism. Then, we have the following proposition.

**Proposition 3.2.1.**

1. \( \{ (\mathcal{X}_n, f_n^*(\mathcal{T})^\otimes d-n) \} \) is an adelic sequence of \((X, L)\). Moreover, if \( \mathcal{T} \) is nef, then the adelic line bundle \( \lim_{n \to \infty} (\mathcal{X}_n, f_n^*(\mathcal{T})^\otimes d-n) \) is nef.
2. Let \( f' : X \to X \) be another surjective morphism with \( L^\otimes d' \simeq f'^*(L) \) for some \( d' \geq 2 \). Let \( (\mathcal{X}', \mathcal{T}') \) be another \( C^\infty \)-model of \((X, L)\) such that \( \mathcal{T}' \) is nef with respect to \( \mathcal{X}' \to B \). Let \( U' \) be a non-empty Zariski open set of \( B \) such that \( f' \) extends to \( f_U' : \mathcal{X}_U' \to \mathcal{X}_U' \) and \( \mathcal{L}_U'^\otimes d' = f_U'^*(\mathcal{L}_U') \) in \( \text{Pic}(\mathcal{X}_U') \otimes \mathbb{Q} \). Let \( \mathcal{X}_n' \) be the normalization of \( \mathcal{X}_U' \xrightarrow{f_U'^n} \mathcal{X}_U' \to \mathcal{X}' \), and \( f_n' : \mathcal{X}_n' \to \mathcal{X}' \) the induced morphism. If \( f \cdot f' = f' \cdot f \), then

\[
\left\{ (\mathcal{X}_n, f_n^*(\mathcal{T})^\otimes d-n) \right\} \sim \left\{ (\mathcal{X}_n', f_n'^*(\mathcal{T})^\otimes d'-n) \right\}.
\]
Definition 3.2.2 (f-adelic structure). The adelic sequence \( \left\{ (\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}}) \right\} \) in the above proposition gives rise to the adelic structure on \( L \), which is called the \( f \)-adic structure of \( L \). The line bundle \( L \) with this adelic structure is denoted by \( \overline{T}^f \), i.e., \( \overline{T}^f = \lim_{n \to \infty} (\mathcal{X}_n, f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}}) \). Considering a case “\( f = f^m \)” in (2), we can see that \( \overline{T}^f \) does not depend on the choice of the \( C^\infty \)-model \((\mathcal{X}, \overline{\mathcal{L}})\). Moreover, (2) says us that if \( f \cdot f' = f' \cdot f \), then \( \overline{T}^f = \overline{T}'^f \). Further, \( \overline{T}^f \) is nef by the second assertion of (1) and Lemma 1.3.

Proof of Proposition 3.2.1 In the same way as in the proof of Proposition 2.1, shrinking \( U \) if necessarily, we may assume that there is an effective ample Cartier divisor \( D \) on \( B \) with \( \Supp(D) = B \setminus U \).

(1) For simplicity, we denote \( f_n^*(\overline{\mathcal{L}})^{\otimes d^{-n}} \) by \( \overline{\mathcal{L}}_n \). From now on, we treat the group structure of the Picard group additively. Note that \( X_0 = \mathcal{X} \) and \( \overline{\mathcal{L}}_0 = \overline{\mathcal{L}} \). Let \( \mathcal{Y} \) be a projective arithmetic variety over \( B \) such that there are birational morphisms \( \rho_0 : \mathcal{Y} \to X_0 \) and \( \rho_1 : \mathcal{Y} \to X_1 \), which are the identity map over \( U \). We fix \( n > m \geq 0 \). Let \( \mathcal{Z} \) be a projective arithmetic variety over \( B \) with the following properties:

(a) \( Z_U = X_U \).
(b) For each \( m \leq i \leq n \), there is a birational morphism \( \mu_i : \mathcal{Z} \to X_i \), which is the identity map over \( U \).
(c) For each \( m \leq j < n \), there is a morphism \( g_j : \mathcal{Z} \to \mathcal{Y} \) which is an extension of \( f_U^* : Z_U \to Y_U \).

Here we claim the following.

Claim 3.2.3. (i) \( \mu_{j+1}(\overline{\mathcal{L}}_{j+1}) = d^{-j}g_j^*(\rho_1^*(\overline{\mathcal{L}}_1)) \) for each \( m \leq j \leq n \).
(ii) \( \mu_j^*(\overline{\mathcal{L}}_j) = d^{-j}g_j^*(\rho_0^*(\overline{\mathcal{L}}_0)) \) for each \( m \leq j < n \).
(iii) \( \mu_m^*(\overline{\mathcal{L}}_m) - \mu_m^*(\overline{\mathcal{L}}_m) = \sum_{j=m}^{n-1} d^{-j}g_j^*(\rho_1^*(\overline{\mathcal{L}}_1) - \rho_0^*(\overline{\mathcal{L}}_0)). \)

(i) Let us consider the following two morphisms between \( \mathcal{Z} \) and \( X_0 \):
\[
\mathcal{Z} \xrightarrow{g_j} \mathcal{Y} \xrightarrow{\rho_1} X_1 \xrightarrow{f_1} X_0 \quad \text{and} \quad \mathcal{Z} \xrightarrow{\mu_{j+1}} X_{j+1} \xrightarrow{f_{j+1}} X_0.
\]
These are same over \( U \). Thus, so are over \( B \). Therefore, \( g_j^*\rho_1^*f_1^*(\overline{\mathcal{L}}) = \mu_{j+1}^*f_{j+1}^*(\overline{\mathcal{L}}) \), which shows us the assertion of (i).

(ii) In the same way as above, we can see \( \mu_j \cdot f_j = \rho_0 \cdot g_j \). Thus we get (ii).

(iii) Since \( \mu_{n}(\overline{\mathcal{L}}_n) - \mu_{m}(\overline{\mathcal{L}}_m) = \sum_{j=m}^{n-1} \mu_{j+1}(\overline{\mathcal{L}}_{j+1}) - \mu_{j}^*(\overline{\mathcal{L}}_{j}) \), this is a consequence of (i) and (ii).

By Lemma 1.3, there is an ample \( C^\infty \)-hermitian line bundle \( \Delta \) on \( B \) such that
\[
-\pi_\mathcal{Y}^*(\Delta) \lesssim_{\pi_\mathcal{Y}^{-1}(U)} \rho_1^*(\overline{\mathcal{L}}_1) - \rho_0^*(\overline{\mathcal{L}}_0) \lesssim_{\pi_\mathcal{Y}^{-1}(U)} \pi_\mathcal{Y}^*(\Delta).
\]
Hence, by (iii) of the above claim, we get
\[- \left( \sum_{j=m}^{n-1} d^{-j} \right) \pi_{\bar{\Delta}}^* (\bar{\Delta}) \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} \mu_n^* (\mathcal{L}_n) - \mu_m^* (\mathcal{L}_m) \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} \left( \sum_{j=m}^{n-1} d^{-j} \right) \pi_{\bar{\Delta}}^* (\bar{\Delta}).\]

Thus, we obtain the first assertion of (1). The second assertion is obvious.

(2) Let us consider the following cases:

Case 1: \( f = f' \).

Case 2: \( \mathcal{X} = \mathcal{X}' \) and \( \bar{\mathcal{L}} = \bar{\mathcal{L}}' \).

Clearly, it is sufficient to check (2) under the assumption Case 1 or Case 2.

**Case 1:** In this case, we assume \( f = f' \). Shrinking \( U \) and \( U' \), we may assume that \( U = U' \), \( \mathcal{X}_U = \mathcal{X}'_U \), and \( \mathcal{L}_U = \mathcal{L}'_U \) in \( \text{Pic}(\mathcal{X}_U) \otimes \mathbb{Q} \). For each \( n \geq 0 \), let \( Z_n \) be a projective arithmetic variety over \( B \) such that there are birational morphisms \( \nu_n : Z_n \to \mathcal{X}_n \) and \( \nu'_n : Z_n \to \mathcal{X}'_n \), which are the identity map over \( U \). We may assume that there is a morphism \( g_n : Z_n \to Z_0 \) such that the following diagrams are commutative:

\[
\begin{array}{c}
Z_0 \leftarrow g_n \downarrow \leftarrow g_n \downarrow \leftarrow g_n \\
\mathcal{X}_0 \leftarrow f_n \downarrow \leftarrow f_n \downarrow \leftarrow f_n
\end{array}
\]

Then,
\[d^{-n} \nu_n^* (f_n^* (\mathcal{L})) - d^{-n} \nu_n' (f_n' (\mathcal{L}')) = d^{-n} g_n^* (\nu_0^* (\mathcal{L}) - \nu_0' (\mathcal{L}')).\]

By Lemma [3], there is an ample \( C^\infty \)-hermitian line bundle \( \bar{\Delta} \) on \( B \) such that
\[- \pi_{\bar{\Delta}}^* (\bar{\Delta}) \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} \nu_0^* (\bar{\mathcal{L}}) - \nu_0' (\bar{\mathcal{L}}') \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} \pi_{\bar{\Delta}}^* (\bar{\Delta}).\]

Therefore, we have
\[-d^{-n} \pi_{\bar{\Delta}}^* (\bar{\Delta}) \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} d^{-n} \nu_n^* (f_n^* (\mathcal{L})) - d^{-n} \nu_n' (f_n' (\mathcal{L}')) \lesssim_{\pi_{\bar{\Delta}}^{-1}(U)} d^{-n} \pi_{\bar{\Delta}}^* (\bar{\Delta}),\]
which shows us our assertion in this case.

**Case 2:** In this case, we assume that \( \mathcal{X} = \mathcal{X}' \) and \( \bar{\mathcal{L}} = \bar{\mathcal{L}}' \). We denote \( f_n^* (\bar{\mathcal{L}})^{\otimes d^{-n}} \) and \( f_n' (\bar{\mathcal{L}})^{\otimes d^{-n}} \) by \( \bar{\mathcal{L}}_n \) and \( \bar{\mathcal{L}}'_n \) respectively. Let \( \mathcal{Y} \) be a projective arithmetic variety over \( B \) such that there are birational morphisms \( \rho : \mathcal{Y} \to \mathcal{X}, \rho_1 : \mathcal{Y} \to \mathcal{X}_1, \) and \( \rho'_1 : \mathcal{Y} \to \mathcal{X}'_1, \) which are the identity map over \( U \). We fix \( n > 0 \). Let \( Z \) be a projective arithmetic variety over \( B \) with the following properties:

(a) \( Z_U = \mathcal{X}_U \).

(b) For each \( 0 \leq i \leq n \), there are birational morphisms \( \mu_i : Z \to \mathcal{X}_i \) and \( \mu'_i : Z \to \mathcal{X}'_i \), which are the identity map over \( U \).

(c) For each \( 0 \leq j \leq n \), there are morphisms \( g_j : Z \to \mathcal{Y} \) and \( g'_j : Z \to \mathcal{Y} \) which are extensions of \( f_j^U : \mathcal{Z}_U \to \mathcal{Y}_U \) and \( f'_j^U : \mathcal{Z}_U \to \mathcal{Y}_U \) respectively.
Note that $\mu_0 = \mu'_0$. Then, in the same way as in (1) ((iii) of Claim 3.2.3), we can see

\begin{equation}
\mu_n^*(\mathcal{L}_n) - \mu_0^*(\mathcal{E}) = \sum_{j=0}^{n-1} d^{-j} g_j^* (\rho_1^* \mathcal{L}_1 - \rho_0^*(\mathcal{L}))
\end{equation}

and

\begin{equation}
\mu'_n^*(\mathcal{L}'_n) - \mu'_0^*(\mathcal{E}) = \sum_{j=0}^{n-1} d'^{-j} g'_j (\rho'_1^* \mathcal{L}'_1 - \rho'_0^*(\mathcal{L})).
\end{equation}

Let $\mathcal{Z}_n$ (resp. $\mathcal{Z}'_n$) be the normalization of $\mathcal{Z}_U \xrightarrow{f_0} \mathcal{Z}_U \rightarrow \mathcal{Z}$ (resp. $\mathcal{Z}_U \xrightarrow{f'_0} \mathcal{Z}_U \rightarrow \mathcal{Z}$) and let $h_n : \mathcal{Z}_n \rightarrow \mathcal{Z}$ (resp. $h'_n : \mathcal{Z}'_n \rightarrow \mathcal{Z}$) be the induced morphism. Moreover, let $T$ be a projective arithmetic variety over $B$ such that there are birational morphisms $\tau : T \rightarrow \mathcal{Z}_n$, $\tau' : T \rightarrow \mathcal{Z}'_n$, $\sigma : T \rightarrow \mathcal{X}_n$, and $\sigma' : T \rightarrow \mathcal{X}'_n$, which are the identity map over $U$. Now we have a lot of morphisms, so that we summarize them. The following morphisms are birational and the identity map over $U$.

\begin{align*}
\mathcal{Y} \xrightarrow{\rho} \mathcal{X} \quad & \quad \mathcal{Y} \xrightarrow{\rho_1} \mathcal{X}_1 \quad \mathcal{Z} \xrightarrow{\mu} \mathcal{X}_i \quad T \xrightarrow{\tau} \mathcal{Z}_n \quad T \xrightarrow{\sigma} \mathcal{X}_n \\
\mathcal{Y} \xrightarrow{\rho'_1} \mathcal{X}'_1 \quad \mathcal{Z} \xrightarrow{\mu'} \mathcal{X}'_i \quad T \xrightarrow{\tau'} \mathcal{Z}'_n \quad T \xrightarrow{\sigma'} \mathcal{X}'_n
\end{align*}

Moreover, the following morphisms are extensions of the power of $f$ or $f'$.

\begin{align*}
\mathcal{X}_n \xrightarrow{f_n} \mathcal{X} \quad & \quad \mathcal{Z} \xrightarrow{g_j} \mathcal{Y} \quad \mathcal{Z}_n \xrightarrow{h_n} \mathcal{Z} \\
\mathcal{X}'_n \xrightarrow{f'_n} \mathcal{X}' \quad & \quad \mathcal{Z} \xrightarrow{g'_j} \mathcal{Y} \quad \mathcal{Z}'_n \xrightarrow{h'_n} \mathcal{Z}
\end{align*}

Here, $f_n \cdot \mu_n \cdot h_n \cdot \tau' = f'_n \cdot \mu'_n \cdot h_n \cdot \tau$ over $U$ because $f \cdot f' = f' \cdot f$. Hence, so is over $B$ as $T \rightarrow \mathcal{X}$. Thus,

\begin{equation}
d^{-n} \tau^* h_n^*(\mu_n^*(\mathcal{L}_n) - \mu_0^*(\mathcal{E})) - d^{-n} \tau^* h_n^*(\mu'_n^*(\mathcal{L}'_n) - \mu'_0^*(\mathcal{E})) = d^{-n} \tau^* h_n^* \mu_0^*(\mathcal{L}) - d^{-n} \tau'^* h'_n^* \mu'_0^*(\mathcal{L})
\end{equation}

Moreover, since $\mu_0 \cdot h_n \cdot \tau = f_n \cdot \sigma$ and $\mu_0 \cdot h'_n \cdot \tau' = f'_n \cdot \sigma'$, by the above equation, we have

\begin{equation}
d^{-n} \tau^* h_n^*(\mu_n^*(\mathcal{L}_n) - \mu_0^*(\mathcal{L})) - d^{-n} \tau^* h_n^*(\mu'_n^*(\mathcal{L}'_n) - \mu'_0^*(\mathcal{L})) = \sigma^*(\mathcal{L}) - \sigma'^*(\mathcal{L}')
\end{equation}

On the other hand, by Lemma 3.3, we can find ample $C^\infty$-hermitian $\mathbb{Q}$-line bundles $\mathcal{X}$ and $\mathcal{L}'$ such that

\begin{equation}
-\pi_1^*(\Delta) \preceq_{\pi_1^*(\mathcal{U})} \rho_1^*(\mathcal{L}_1) - \rho^*(\mathcal{L}) \preceq_{\pi_1^*(\mathcal{U})} \pi_1^*(\Delta)
\end{equation}

and

\begin{equation}
-\pi_1^*(\Delta) \preceq_{\pi_1^*(\mathcal{U})} \rho'_1^*(\mathcal{L}'_1) - \rho^*(\mathcal{L}) \preceq_{\pi_1^*(\mathcal{U})} \pi_1^*(\Delta)
\end{equation}

Therefore, if we set

\begin{align*}
d_n = \left( d^{-n} \sum_{j=0}^{n-1} d^{-j} \right) \quad & \quad d'_n = \left( d^{-n} \sum_{j=0}^{n-1} d'^{-j} \right)
\end{align*}
then, by (3.2.4) and (3.2.5),
\[-d_n π_τ(Δ) \lesssim π_{τ^{-1}(U)} d''^{-n} \tau^* h^*_n (μ^*_n(\overline{L}) - μ^*_0(\overline{L})) \lesssim π_{τ^{-1}(U)} d_n π_τ(Δ),\]
and
\[-d'_n π_τ(\overline{\Delta}) \lesssim π_{τ^{-1}(U)} d''^{-n} \tau^* h^*_n (μ^*_n(\overline{L}) - μ^*_0(\overline{L})) \lesssim π_{τ^{-1}(U)} d'_n π_τ(\overline{\Delta}).\]

Hence, using (3.2.8),
\[-π_τ \left( d_n Δ + d'_n \overline{\Delta} \right) \lesssim π_{τ^{-1}(U)} \sigma^*(\overline{L}) - σ^*(\overline{L}) \lesssim π_{τ^{-1}(U)} π_τ \left( d_n Δ + d'_n \overline{\Delta} \right).\]

Therefore, we have this case because \(\lim_{n→∞} d_n = \lim_{n→∞} d'_n = 0.\) □

4. Adelic intersection number and adelic height

4.1. Adelic intersection number. Let \(K\) be a finitely generated field over \(\mathbb{Q}\) with \(d = \text{tr. deg}_\mathbb{Q}(K)\), and \(\mathcal{B} = (B; \mathcal{H}_1, \ldots, \mathcal{H}_d)\) a polarization of \(K\).

**Proposition 4.1.1.** Let \(X\) be an \(e\)-dimensional projective variety over \(K\), and let \(L_1, \ldots, L_{e+1}\) be nef line bundles on \(X\). Let \(\{(X^{(i)}_n, \overline{X}^{(i)}_n)\}\) be an adelic sequence of \((X, L_i)\) for each \(1 ≤ i ≤ e + 1\). Let \(\mathcal{Z}_n\) be a projective arithmetic variety over \(B\) such that there are birational morphisms \(μ^{(i)}_n : \mathcal{Z}_n → X^{(i)}_n (i = 1, \ldots, e + 1)\). Then, the limit
\[\overline{\text{deg}} \left( \hat{c}_1(μ^{(1)}_n(\overline{X}^{(1)}_n)) \cdots \hat{c}_1(μ^{(e+1)}_n(\overline{X}^{(e+1)}_n)) \cdot \hat{c}_1(π^*_Z(\mathcal{H}_1)) \cdots \hat{c}_1(π^*_Z(\mathcal{H}_d)) \right)\]
as \(n → ∞\) exists, where \(π_Z : \mathcal{Z}_n → B\) is the natural morphism. Moreover, if \(\{(Y^{(i)}_n, \overline{Y}^{(i)}_n)\}\) is another adelic sequence of \((X, L_i)\) for each \(1 ≤ i ≤ e + 1\), and \(\{(X^{(i)}_n, \overline{X}^{(i)}_n)\}\) is equivalent to \(\{(Y^{(i)}_n, \overline{Y}^{(i)}_n)\}\) for each \(i\), then the limit by \(\{(X^{(i)}_n, \overline{X}^{(i)}_n)\}\) coincides with the limit by \(\{(Y^{(i)}_n, \overline{Y}^{(i)}_n)\}\).

**Proof.** Let \(\mathcal{Z}_{n,m}\) be a projective arithmetic variety over \(B\) such that there are birational morphisms \(\mathcal{Z}_{n,m} → \mathcal{Z}_n\) and \(\mathcal{Z}_{n,m} → \mathcal{Z}_m\). By abuse of notation, we denote birational morphisms \(\mathcal{Z}_{n,m} → X^{(i)}_n\) and \(\mathcal{Z}_{n,m} → X^{(j)}_m\) by \(μ^{(i)}_n\) and \(μ^{(j)}_m\) respectively. First of all, we can see
\[
\hat{c}_1(μ^{(1)}_n(\overline{X}^{(1)}_n)) \cdots \hat{c}_1(μ^{(e+1)}_n(\overline{X}^{(e+1)}_n)) = \sum_{i=1}^{e+1} \hat{c}_1(μ^{(1)}_n(\overline{X}^{(1)}_n)) \cdots \hat{c}_1(μ^{(i)}_n(\overline{X}^{(i)}_n)) \cdots \hat{c}_1(μ^{(e+1)}_m(\overline{X}^{(e+1)}_m)).
\]

Therefore, it is sufficient to show that, for any positive \(ε\), there is a positive integer \(N\) such that if \(n, m ≥ N\), then
\[\overline{\text{deg}} \left( Δ_{n, m, i} \cdot \hat{c}_1(π^*_Z(\mathcal{H}_1)) \cdots \hat{c}_1(π^*_Z(\mathcal{H}_d)) \right) ≤ ε,\]
where \(Δ_{n, m, i} = \hat{c}_1(μ^{(1)}_n(\overline{X}^{(1)}_n)) \cdots \hat{c}_1(μ^{(i)}_n(\overline{X}^{(i)}_n)) \cdots \hat{c}_1(μ^{(e+1)}_m(\overline{X}^{(e+1)}_m)).\) By the definition of adelic sequences, there are a projective arithmetic variety \(X_{n,m}\) over \(B\), a
birational morphism \( \nu_{n,m} : X_{n,m} \to Z_{n,m} \), and a nef \( C^\infty \)-hermitian \( \mathbb{Q} \)-line bundle \( \mathcal{D}_{n,m} \) on \( B \) such that
\[
-\pi^*_{X_{n,m}}(\mathcal{D}_{n,m}) \geq \nu^*_{n,m} \left( \mu^{(i)*}_n(\mathcal{L}^{(i)}_{n}) - \mu^{(i)*}_m(\mathcal{L}^{(i)}_{m}) \right) \geq \pi^*_{X_{n,m}}(\mathcal{D}_{n,m}).
\]
Here, since \( \mathcal{L}^{(i)}_n \)'s are nef with respect to \( X^{(i)}_{n} \to B \) and \( \mathcal{H}_j \)'s are nef, by using Lemma 1.4 together with the projection formula, we can see
\[
\left| \widehat{\deg} \left( \Delta_{n,m,i} \cdot \hat{c}_1(\pi^*_{Z_{n,m}}(\mathcal{H}_1)) \cdots \hat{c}_1(\pi^*_{Z_{n,m}}(\mathcal{H}_d)) \right) \right| 
\leq \deg(L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_{e+1}) \left| \widehat{\deg} \left( \hat{c}_1(\mathcal{D}_{n,m}) \cdot \hat{c}_1(\mathcal{H}_1) \cdots \hat{c}_1(\mathcal{H}_d) \right) \right|.
\]
Thus we get the first assertion. The second one is obvious by the definition of equivalence.

Definition 4.1.2 (Adelic intersection number). Let \( \mathcal{L}_1, \ldots, \mathcal{L}_{e+1} \) be adelic line bundles on \( X \). Then, by the above proposition, the limit of intersection numbers does not depend on the choice of adelic sequences representing each \( \mathcal{L}_i \). Thus, we may define the adelic intersection number \( \langle \mathcal{L}_1 \cdots \mathcal{L}_{e+1} \rangle_B \) to be the limit in Proposition 4.1.1.

Here let us consider the following two propositions. The second proposition is a property concerning the specialization of adelic intersection number.

Proposition 4.1.3. Let \( \mathcal{L}_1, \ldots, \mathcal{L}_{e+1} \) be adelic line bundles on \( X \). Then, we have the following.

1. If \( \mathcal{L}_1, \ldots, \mathcal{L}_{e+1} \) are nef, then \( \langle \mathcal{L}_1 \cdots \mathcal{L}_{e+1} \rangle_B \geq 0 \).
2. Let \( \mathcal{H}_1, \ldots, \mathcal{H}_d \) be nef \( C^\infty \)-hermitian line bundles on \( B \) with \( \mathcal{H}_i \supseteq \mathcal{H}_j \) for all \( i \). If \( \mathcal{L}_1, \ldots, \mathcal{L}_{e+1} \) are nef, then
\[
\langle \mathcal{L}_1 \cdots \mathcal{L}_{e+1} \rangle_{(B; \mathcal{H}_1, \ldots, \mathcal{H}_d)} \geq \langle \mathcal{L}_1 \cdots \mathcal{L}_{e+1} \rangle_{(B; \mathcal{H}_1, \ldots, \mathcal{H}_d)}.
\]
3. Let \( g : Y \to X \) be a generically finite morphism of projective varieties over \( K \). Then,
\[
\langle g^*(\mathcal{L}_1) \cdots g^*(\mathcal{L}_{e+1}) \rangle_B = \deg(g) \langle \mathcal{L}_1 \cdots \mathcal{L}_{e+1} \rangle_B
\]

Proof. (1) is a consequence of (2) of Proposition 1.1. (2) follows from (4) of Proposition 1.1. (3) is a consequence of the projection formula.

Proposition 4.1.4. Let \( \{ \langle X_n, \mathcal{L}_n \rangle \} \) be an adelic sequence of \( (X, L) \) such that \( \mathcal{L}_n \) is nef for every \( n \), and let \( \mathcal{L} \) be a nef adelic line bundle on \( X \) given by the adelic sequence \( \{ \langle X_n, \mathcal{L}_n \rangle \} \). Let \( U \) be a common base of the adelic sequence \( \{ \langle X_n, \mathcal{L}_n \rangle \} \) (cf. the definition of adelic sequences in §3.1). Let \( \gamma \) be a point of codimension one in \( U_Q \) such that \( X_U \) is flat over \( \gamma \) and the fiber \( X_\gamma \) of \( X_U \to U \) over \( \gamma \) is integral. Then, \( X_\gamma \) is a projective variety over the residue field \( \kappa(\gamma) \) at \( \gamma \), and \( L_\gamma = \mathcal{L}|_{X_\gamma} \) is a line bundle on \( X_\gamma \). Let \( \Gamma \) be the Zariski closure of \( \{ \gamma \} \) in \( B \), and \( \mathcal{Z}_n \) the Zariski closure of \( X_\gamma \) in \( X_n \). If \( \mathcal{H}_d \) is big, then we have the following.

1. \( \{ \langle \mathcal{Z}_n, \mathcal{L}_n \rangle \} \) is an adelic sequence of \( (X_\gamma, L_\gamma) \) with respect to \( \langle \Gamma; \mathcal{H}_1|_\Gamma, \ldots, \mathcal{H}_{d-1}|_\Gamma \rangle \).
(2) If we denote by \( \mathcal{T} \) the adelic line bundle arising from the adelic sequence \( \{(\mathcal{Z}_n, \mathcal{L}_n|\mathcal{Z}_n)\} \), then \( \langle \mathcal{T}^{\dim X+1} \rangle_{(B; \mathcal{H}_1, \ldots, \mathcal{H}_d)} = 0 \) implies \( \langle \mathcal{T}_\gamma^{\dim X+1} \rangle_{(\Gamma; \mathcal{H}_1|\Gamma, \ldots, \mathcal{H}_{d-1}|\Gamma)} = 0 \).

Proof. First of all, by using Lemma 1.2, we fix a positive integer \( N \) and a non-zero section \( s \in H^0(B, \mathcal{H}_d^N) \) with \( s(\gamma) = 0 \) and \( \|s\|_{\sup} \leq 1 \). Then, \( \text{div}(s) = \Gamma + \Sigma \) for some effective divisor \( \Sigma \).

(1) To prove (1), it is sufficient to show that
\[
\lim_{n,m \to \infty} \sup \langle \deg (\mathcal{D}_{n,m}|\Gamma) \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle = 0,
\]
where \( \mathcal{D}_{n,m} \) is a nef \( C^\infty \)-hermitian \( \mathbb{Q} \)-line bundle on \( B \) appeared in the definition of adelic sequences (cf. §3.3). First of all,
\[
N \langle \deg (\mathcal{D}_{n,m} \cdot \mathcal{H}_1 \cdots \mathcal{H}_d) \rangle = \langle \deg (\mathcal{D}_{n,m}|\Gamma) \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle + \langle \mathcal{H}_1 \cdots \mathcal{H}_d \rangle + \int_{B(\mathbb{C})} -\log(\|s\|) c_1(\mathcal{D}_{n,m}) \wedge c_1(\mathcal{H}_1) \wedge \cdots \wedge c_1(\mathcal{H}_d).
\]

Here every term is non-negative. Thus, we can see that
\[
\lim_{n,m \to \infty} \sup \langle \deg (\mathcal{D}_{n,m}|\Gamma) \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle = 0.
\]

(2) We can set \( \text{div}(\pi_{\mathcal{X}_n}^*(s)) = \mathcal{Z}_n + \Delta_n \) for some effective divisor \( \Delta_n \). Therefore,
\[
N \langle \deg (\mathcal{Z}_n)^{e+1} \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle = \langle \deg (\mathcal{Z}_n)^{e+1} \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle + \langle \mathcal{H}_1 \cdots \mathcal{H}_d \rangle + \int_{\mathcal{X}_n(\mathbb{C})} -\log(\|\pi_{\mathcal{X}_n}^*(s)\|) c_1(\mathcal{Z}_n)^{e+1} \wedge c_1(\mathcal{H}_1) \wedge \cdots \wedge c_1(\mathcal{H}_d).
\]

Since the last two terms of the above equation are non-negative, we have
\[
N \langle \deg (\mathcal{Z}_n)^{e+1} \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle \geq \langle \deg (\mathcal{Z}_n)^{e+1} \cdot \mathcal{H}_1 \cdots \mathcal{H}_d \rangle.
\]

Thus, taking \( n \to \infty \),
\[
N \langle \mathcal{T}^{\dim X+1} \rangle_{(B; \mathcal{H}_1, \ldots, \mathcal{H}_d)} \geq \langle \mathcal{T}_\gamma^{\dim X+1} \rangle_{(\Gamma; \mathcal{H}_1|\Gamma, \ldots, \mathcal{H}_{d-1}|\Gamma)}.
\]

Therefore, we get (2). \( \square \)
4.2. Adelic height. Let $K$ be a finitely generated field over $\mathbb{Q}$ with $d = \text{tr. deg}_\mathbb{Q}(K)$, and $\mathcal{B} = (B; \mathcal{P}_1, \ldots, \mathcal{P}_d)$ a polarization of $K$. Let $X$ be a projective variety over $K$, and $L$ an ample line bundle on $X$.

Let $\mathcal{T}$ be an adelic line bundle given by an adelic sequence $\{(X_n, \mathcal{L}_n)\}$. Let $K'$ be a finite extension of $K$, $B'$ the normalization of $B$ in $K'$, and let $\rho : B' \to B$ be the induced morphism. Let $X'_n$ be the main component of $X_n \times_B B'$. We set the induced morphisms as follows.

$$X_n \xleftarrow{\tau_n} X'_n$$
$$\pi_n \downarrow \quad \downarrow \pi'_n$$

Then, $\{(X'_n, \rho^*(\mathcal{L}_n))\}$ is an adelic sequence of $(X_{K'}, L_{K'})$. We denote by $\mathcal{T}_{K'}$ the adelic line bundle induced by $\{(X'_n, \rho^*(\mathcal{L}_n))\}$. With this notation, if $\mathcal{T}_1, \ldots, \mathcal{T}_{e+1}$ are adelic line bundles on $X$, then we can see

$$(4.2.1) \quad \langle (\mathcal{T}_1)_{K'} \cdots (\mathcal{T}_{e+1})_{K'} \rangle_{B_{K'}} = [K' : K] \langle (\mathcal{T}_1)_{K'} \cdots (\mathcal{T}_{e+1})_{K'} \rangle_{\mathcal{B}}$$

by virtue of the projection formula, where $\mathcal{B}_{K'} = (B'; \rho^*(\mathcal{P}_1), \ldots, \rho^*(\mathcal{P}_d))$.

Let $Y$ be a subvariety of $X_{K'}$. We assume that $Y$ is defined over $K'$. Let $Y'_n$ be the closure of $Y$ in $X'_n$. Then, $\{Y'_n, \rho^*(\mathcal{L}_n) |_{Y'_n}\}$ is an adelic sequence of $(Y, L_{K'} |_{Y'})$. We denote by $\mathcal{T}_{K'} |_{Y'}$ the adelic line bundle given by $\{Y'_n, \rho^*(\mathcal{L}_n) |_{Y'_n}\}$. We define the height of $Y$ with respect to $\mathcal{T}$ to be

$$h^{\mathcal{B}}_{\mathcal{T}}(Y) = \frac{\langle (\mathcal{T}_{K'} |_{Y'})^{\dim Y + 1} \rangle_{\mathcal{B}}}{[K' : K](\dim Y + 1) \deg (L_{K'} |_{Y'})}.$$

Note that by virtue of $(4.2.1)$, the above does not depend on the choice of $K'$. We call $h^{\mathcal{B}}_{\mathcal{T}}(Y)$ the adelic height of $Y$ with respect to $\mathcal{T}$ and $\mathcal{B}$.

**Proposition 4.2.2.** Let $X$ be a projective variety over $K$, and $L$ an ample line bundle on $X$. We assume that there is a surjective morphism $f : X \to X$ and an integer $d \geq 2$ with $L^{\otimes d} \simeq f^*(L)$. Let $\mathcal{T}^f$ be the adelic line bundle with the $f$-adelic structure. Then, we have the following.

1. $h^{\mathcal{B}}_{\mathcal{T}^f}(Y) \geq 0$ for all subvarieties $Y$ of $X_{K'}$.
2. For a $C^\infty$-model $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ with $\mathcal{L}$ being nef with respect to $\mathcal{X} \to B$, there is a constant $C$ such that

$$|h^{\mathcal{B}}_{\mathcal{T}^f}(Y) - h^{\mathcal{B}}_{(\mathcal{X}, \mathcal{L})}(Y)| \leq C$$

for any subvarieties $Y$ of $X_{K'}$.
3. $h^{\mathcal{B}}_{\mathcal{T}^f}(f(Y)) = dh^{\mathcal{B}}_{\mathcal{T}^f}(Y)$ for any subvarieties $Y$ of $X_{K'}$.

Moreover, $h^{\mathcal{B}}_{\mathcal{T}^f}$ is characterized by the above properties (1), (2) and (3).
Proof. (1) Since $\mathcal{L}^f$ is nef by Proposition 3.2.1, (1) is a consequence of (1) of Proposition 4.1.3.

(2) We choose a Zariski open set $U$ of $B$ such that $f$ extends to $f_U : \mathcal{X}_U \to \mathcal{X}$ and $\mathcal{L}^d_f = f^*(\mathcal{L}_U)$ in $\text{Pic}(\mathcal{X}_U) \otimes \mathbb{Q}$. Let $\mathcal{X}_n$ be the normalization of $\mathcal{X}_U$, $f^n_\mathcal{X} : \mathcal{X}_n \to \mathcal{X}$ the induced morphism. We denote $f_n^*(\mathcal{L})(\otimes d-n)$ by $\mathcal{L}_n$. Then, as in proof of (1) of Proposition 3.2.1, there are a projective arithmetic variety $\mathcal{Z}_n$ over $B$, birational morphisms $\mu_n : \mathcal{Z}_n \to \mathcal{X}_n$ and $\nu_n : \mathcal{Z}_n \to \mathcal{X}$ (which are the identity map over $U$), and an ample $C^\infty$-hermitian line bundle $\mathcal{D}$ on $B$ such that

$$-d_n \pi_n^*(\mathcal{D}) \geq \pi_n^{-1}(U) \mu_n^*(\mathcal{L}_n) - \nu_n^*(\mathcal{D}) \geq \pi_n^{-1}(U) d_n \pi_n^*(\mathcal{D}),$$

where $d_n = \sum_{j=0}^{n-1} d^{-j}$.

Let $Y$ be a subvariety of $\mathcal{X}_K$. We assume that $Y$ is defined over a finite extension field $K'$ of $K$. Let $B'$ be the normalization of $B$ in $K'$, and let $\rho : B' \to B$ be the induced morphism. We denote by $\mathcal{X}'$, $\mathcal{X}'_n$ and $\mathcal{Z}'_n$ the main components of $\mathcal{X} \times_B B'$, $\mathcal{X} \times_B B'$ and $\mathcal{Z} \times_B B'$ respectively. We set the induced morphisms as follows.

$$\begin{align*}
\mathcal{X} & \leftarrow \pi \mathcal{X}' \leftarrow \mathcal{X}'_n \leftarrow \mathcal{Z}'_n \leftarrow \mathcal{Z}'_n \\
\mathcal{X}_n & \leftarrow \pi_n \mathcal{X}'_n \leftarrow \mathcal{X}'_n \leftarrow \mathcal{Z}'_n \leftarrow \mathcal{Z}'_n \\
B & \leftarrow \rho \mathcal{B}' \leftarrow \mathcal{B}' \leftarrow \mathcal{B}' \leftarrow \mathcal{B}'
\end{align*}$$

We also have the induced morphisms $\mu'_n : \mathcal{Z}'_n \to \mathcal{X}'_n$ and $\nu'_n : \mathcal{Z}'_n \to \mathcal{X}'. Then,

$$-d_n \pi_n^*(\rho^*\mathcal{D}) \geq \pi_n^{-1}(U) \mu'_n^*(\tau_n^*\mathcal{L}_n) - \nu'_n^*(\tau_n^*\mathcal{D}) \geq \pi_n^{-1}(U) d_n \pi_n^*(\rho^*\mathcal{D}),$$

On the other hand, since

$$\hat{c}_1(\mu'_n(\tau_n^*\mathcal{L}_n))^{\dim Y+1} - \hat{c}_1(\nu'_n(\tau_n^*\mathcal{D}))^{\dim Y+1} = \sum_{i=1}^{\dim Y+1} \hat{c}_1(\mu'_n(\tau_n^*\mathcal{L}_n))^{-i-1} \cdot (\hat{c}_1(\mu'_n(\tau_n^*\mathcal{L}_n)) - \hat{c}_1(\nu'_n(\tau_n^*\mathcal{D}))) \hat{c}_1(\nu'_n(\tau_n^*\mathcal{D}))^{\dim Y-i+1},$$

by using Lemma 4.4, we have

$$\left| \hat{\deg} \left( \hat{c}_1(\tau_n^*\mathcal{L}_n) \right)^{\dim Y+1} \cdot \hat{c}_1(\pi_j^*\rho^*\mathcal{H}_1) \cdots \hat{c}_1(\pi_j^*\rho^*\mathcal{H}_d) \right|$$

$$-\hat{\deg} \left( \hat{c}_1(\tau^*\mathcal{L}) \right)^{\dim Y+1} \cdot \hat{c}_1(\pi_j^*\rho^*\mathcal{H}_1) \cdots \hat{c}_1(\pi_j^*\rho^*\mathcal{H}_d)$$

$$\leq d_n[K' : K] \left( \dim Y + 1 \right) \deg(L_Y^{\dim Y}) \hat{\deg} \left( \hat{c}_1(\mathcal{D}) \cdot \hat{c}_1(\mathcal{H}_1) \cdots \hat{c}_1(\mathcal{H}_d) \right),$$

where $\mathcal{Y}$ and $\mathcal{Y}_n$ are the Zariski closures of $Y$ in $\mathcal{X}'$ and $\mathcal{X}'_n$ respectively. Thus we get (2).

(3) Clearly, we may assume $Y$ is defined over $K$. Let $(\mathcal{X}, \mathcal{L})$ be a $C^\infty$ model of $(\mathcal{X}, L)$. Let us consider a sequence of morphisms of projective arithmetic varieties over $B$:

$$\mathcal{X} = \mathcal{X}_0 \leftarrow f_1 \mathcal{X}_1 \leftarrow f_2 \cdots \leftarrow f_{n-1} \mathcal{X}_{n-1} \leftarrow f_n \mathcal{X}_n \leftarrow f_{n+1} \mathcal{X}_{n+1} \leftarrow f_{n+2} \cdots.$$
such that $X$ is the generic fiber of $X_n \to B$ for every $n$, and that $f_n : X_n \to \overline{X}_{n-1}$ is an extension of $f$ for each $n$. Let $\overline{Y}_n$ be the Zariski closure of $Y$ in $X_n$. Then, $f_{n+1}(Y_{n+1})$ is the Zariski closure of $f(Y)$ in $X_n$. By the definition of the height,

\begin{equation}
(4.2.2.1) \quad h\overline{f}\overline{L}/(Y) = \lim_{n \to \infty} \frac{\text{deg} \left( \widehat{c}_1(f_n^* f_n^* \cdots f_1^*(\overline{L})) \cdot \widehat{c}_1(f_{n+1}^* \overline{X}_n(\overline{H}_1)) \cdots \widehat{c}_1(f_{n+1}^* \overline{X}_n(\overline{H}_d)) \right) \cdot (Y_{n+1}, 0)}{(\dim Y + 1) \text{deg}(L_{\dim Y}^{\overline{f}(Y)} d^{(n+1)(\dim Y + 1)})}.
\end{equation}

On the other hand, by the projection formula,

\begin{equation}
(4.2.2.2) \quad \text{deg} \left( \widehat{c}_1(f_n^* f_n^* \cdots f_1^*(\overline{L})) \cdot \widehat{c}_1(f_{n+1}^* \overline{X}_n(\overline{H}_1)) \cdots \widehat{c}_1(f_{n+1}^* \overline{X}_n(\overline{H}_d)) \cdot (Y_{n+1}, 0) \right) = \text{deg}(f|_Y) \text{deg} \left( \widehat{c}_1(f_n^* \cdots f_1^*(\overline{L})) \cdot \widehat{c}_1(\overline{X}_n(\overline{H}_1)) \cdots \widehat{c}_1(\overline{X}_n(\overline{H}_d)) \cdot (f_{n+1}(Y_{n+1}), 0) \right).
\end{equation}

Here, since $L^\otimes d \simeq f^*(L)$, we have $L^\otimes d|_Y \simeq (f|_Y)^* \left( L|_{f(Y)} \right)$, which implies

\begin{equation}
(4.2.2.3) \quad d^{\dim Y} \text{deg}(L|_{\dim Y}) = \text{deg}(f|_Y) \text{deg}(L|_{f(Y)}).
\end{equation}

Moreover,

\begin{equation}
(4.2.2.4) \quad h\overline{f}\overline{L}/(f(Y)) = \lim_{n \to \infty} \frac{\text{deg} \left( \widehat{c}_1(f_n^* \cdots f_1^*(\overline{L})) \cdot \widehat{c}_1(\overline{X}_n(\overline{H}_1)) \cdots \widehat{c}_1(\overline{X}_n(\overline{H}_d)) \cdot (f_{n+1}(Y_{n+1}), 0) \right)}{(\dim Y + 1) \text{deg}(L|_{f(Y)}^{\dim Y}) d^{(n+1)(\dim Y + 1)}}.
\end{equation}

Therefore, by (4.2.2.1), (4.2.2.2), (4.2.2.3), and (4.2.2.4), we obtain

\[ h\overline{f}\overline{L}/(f(Y)) = dh\overline{f}\overline{L}/(Y). \]

Finally, the last assertion is obvious. For, by (2) and (3), we can see

\[ h\overline{f}\overline{L}/(Y) = \lim_{n \to \infty} \frac{h\overline{f}\overline{L}/(\overline{X}/\overline{L}) (f^n(Y))}{d^n}. \]

\[ \square \]

5. The canonical height of subvarieties of an abelian variety over finitely generated fields

Let $K$ be a finitely generated field over $\mathbb{Q}$ with $d = \text{tr. deg}_{\mathbb{Q}}(K)$, and $\overline{E} = (B; \overline{H}_1, \ldots, \overline{H}_d)$ a polarization of $K$. Let $A$ be an abelian variety over $K$, and $L$ a symmetric ample line bundle on $A$. Since $[2]^*(L) \simeq L^\otimes 4$, we have an adelic line bundle $\overline{L}^{[2]}$ with the $[2]$-adic structure. Let $f : A \to A$ be an endomorphism with $f^*(L) \simeq L^\otimes d$ for some $d \geq 2$. Then, since $f \cdot [2] = [2] \cdot f$, by (2) of Proposition 3.2.1, $\overline{L}^f = \overline{L}^{[2]}$. Thus, the adelic structure does not depend on the choice of the endomorphism. In this sense, we have the line bundle $\overline{L}^{an}$ with the canonical adelic structure.
Let $X$ be a subvariety of $A_{\overline{K}}$. We denote by $\hat{h}_L^*(X)$ the adelic height $h_{L,can}^*(X)$ of $X$ with respect to the line bundle $L_{can}$ with the canonical adelic structure. Then, by Proposition 4.2.2, we can see the following:

(a) $\hat{h}_L^*(X) \geq 0$ for all subvarieties $X$ of $A_{\overline{K}}$.

(b) For a $C^\infty$-model $(\mathcal{A}, \overline{\mathcal{L}})$ of $(A, L)$ with $\overline{\mathcal{L}}$ being nef with respect to $A \to B$, there is a constant $\hat{C}$ such that

$$\left| \hat{h}_L^*(X) - \hat{h}_{(\mathcal{A}, \overline{\mathcal{L}})}^*(X) \right| \leq \hat{C}$$

for all subvarieties $X$ of $A_{\overline{K}}$.

(c) $\hat{h}_L^*([N](X)) = N^2 \hat{h}_L^*(X)$ for all subvarieties $X$ of $A_{\overline{K}}$ and all non-zero integers $N$.

The purpose of this section is to prove the following theorem.

**Theorem 5.1.** Let $A$ be an abelian variety over $K$, and $L$ a symmetric ample line bundle on $A$. Let $X$ be a subvariety of $A_{\overline{K}}$. If the polarization $\overline{B}$ is big, then the following are equivalent.

1. $X$ is a translation of an abelian subvariety by a torsion point.
2. The set $\{ x \in X(\overline{K}) \mid \hat{h}_L^*(x) \leq \epsilon \}$ is Zariski dense in $X$ for every $\epsilon > 0$.
3. $\hat{h}_L^*(X) = 0$.

**Proof.** Let us begin with the following two lemmas.

**Lemma 5.2.** Let $A$ be an abelian subvariety over $K$, $C$ an abelian subvariety of $A$, and $\rho : A \to A' = A/C$ the natural homomorphism. Let $X$ be a subvariety of $A$ such that $X = \rho^{-1}(\rho(X))$. Let $L$ and $L'$ be symmetric ample line bundles on $A$ and $A'$ respectively. If $\hat{h}_L^*(X) = 0$, then $\hat{h}_{L'}^*(Y) = 0$, where $Y = \rho(X)$.

**Proof.** Replacing $L$ by $L^{\otimes n}$ ($n > 0$), we may assume that $L \otimes \rho^*(L')^{\otimes -1}$ is generated by global sections. Let $(\mathcal{A}, \overline{\mathcal{L}})$ and $(\mathcal{A}', \overline{\mathcal{L}}')$ be $C^\infty$-models of $(A, L)$ and $(A', L')$ over $B$ with the following properties:

1. $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}'$ are nef and big.
2. There is a morphism $\mathcal{A} \to A'$ over $B$ as an extension of $\rho : A \to A'$. (By abuse of notation, the extension is also denoted by $\rho$.)

Let $\pi : \mathcal{A} \to B$ be the canonical morphism. Replacing $\mathcal{L}$ by $\mathcal{L} \otimes \pi^*(Q)$ for some ample line bundle $Q$ on $B$, we may assume that $\pi_*(\mathcal{L} \otimes \rho^*(\mathcal{L}')^{\otimes -1})$ is generated by global sections. Thus, there are sections $s_1, \ldots, s_r$ of $H^0(\mathcal{L} \otimes \rho^*(\mathcal{L}')^{\otimes -1})$ such that $\{ s_1, \ldots, s_r \}$ generates $L \otimes \rho^*(L')^{\otimes -1}$ on $A$. Moreover, replacing the metric of $\overline{\mathcal{L}}$, we may assume that $s_1, \ldots, s_r$ are small sections, i.e., $\| s_i \|_{\text{sup}} < 1$ for all $i$.

Let $\mathcal{A}_n$ (resp. $\mathcal{A}'_n$) be the normalization of $A \xrightarrow{[2^n]} A \hookrightarrow \mathcal{A}$ (resp. $A' \xrightarrow{[2^n]} A' \hookrightarrow \mathcal{A}'$). Then, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{A} & \xleftarrow{f_n} & \mathcal{A}_n \\
\rho \downarrow & & \downarrow \rho_n \\
\mathcal{A}' & \xleftarrow{f'_n} & \mathcal{A}'_n
\end{array}
$$
where $f_n$ and $f'_n$ are extension of $[2^n]$. Here the adelic structure of $\mathcal{L}$ (resp. $\mathcal{T}$) is induced by $\{4^{-n}f_n^*(\mathcal{L})\}$ (resp. $\{4^{-n}f'_n^*(\mathcal{L})\}$). Let $X_n$ (resp. $Y_n$) be the Zariski closure of $X$ in $\mathcal{A}_n$ (resp. $Y$ in $\mathcal{A}'_n$). Then, since $f^*_n(s_1), \ldots, f^*_n(s_r)$ generate $f^*_n(L \otimes \rho^*(L')^{\otimes -1})$ on $A$, we can find $f^*_n(s_i)$ such that $f^*_n(s_i) \neq 0$ on $X_n$. This means that $f^*_n(\mathcal{L})|_{X_n} \otimes \rho^*_n(f^*_n(\mathcal{L}))^{\otimes -1}|_{X_n}$ is effective. Therefore, if we denote $\dim X$ and $\dim Y$ by $e$ and $e'$ respectively, then, by virtue of (4) of Proposition 1.1 together with the projection formula,

$$\deg \left( \widehat{\deg} \left( \hat{c}_1(f^*_n(\mathcal{L})|_{X_n})^{e+1} \cdot \hat{c}_1(\pi^*_n, \mathcal{H}) \right) \right)$$

$$\geq \deg \left( \hat{c}_1(\rho^*_n f'_n^*(\mathcal{L})|_{X_n})^{e'+1} \cdot \hat{c}_1(f^*_n(\mathcal{L})|_{X_n})^{e-e'} \cdot \hat{c}_1(\rho^*_n \pi^*_n, \mathcal{H}) \right)$$

$$= 4^{n(e-e')} \deg(L|_{\mathcal{C}}^{e-e'}) \hat{c}_1(f^*_n(\mathcal{L})|_{X_n})^{e+1} \cdot \hat{c}_1(\pi^*_n, \mathcal{H})$$

Hence,

$$\hat{h}_{\mathcal{L}}^F(X) \geq \frac{(e'+1) \deg(L|_{\mathcal{C}}^{e'}) \deg(L|_{\mathcal{C}}^{e-e'}) \hat{h}_{\mathcal{L}}^F(Y)}{(e+1) \deg(L|_{X}^e) \hat{h}_{\mathcal{L}}^F(Y)}$$

Thus we get our assertion. \qed

**Lemma 5.3.** Let $A$ and $S$ be algebraic varieties over a field of characteristic zero, and let $f : A \to S$ be an abelian scheme. Let $X$ be a subvariety of $A$ such that $f|_X : X \to B$ is proper and flat. Let $s$ be a point of $S$. If $X_s$ is a translation of an abelian subvariety of $A_s$, then there is a Zariski open set $U$ of $S$ such that (1) $s \in U$ and (2) $X_t$ is a translation of an abelian subvariety of $A_t$ for all $t \in U$. In particular, the geometric generic fiber $X_{\eta}$ is a translation of an abelian subvariety.

**Proof.** Since $X_s$ is smooth and $q(X_s) = \dim(X/S)$, there is a Zariski open set $U$ of $S$ such that $s \in U$, $X_U$ is smooth over $U$, and that $q(X_t) \leq \dim(X/S)$ for all $t \in U$. By Ueno’s theorem (cf. [4, Theorem 10.12]), $q(X_t) \geq \dim(X/S)$ and the equality holds if and only if $X_t$ is a a translation of an abelian subvariety. Thus we get our lemma. \qed

Let us start the proof of Theorem 5.1. First of all, we may assume that $X$ is defined over $K$.

“(1) $\implies$ (2)” is obvious. “(2) $\implies$ (1)” is nothing more than Bogomolov’s conjecture solved in [4].

“(1) $\implies$ (3)”: We set $X = A' + x$, where $A'$ is an abelian subvariety of $A_{\mathcal{L}}$ and $x$ is a torsion point. Let $N$ be a positive integer with $Nx = 0$ and $N \geq 2$. Then, $[N](X) = A' = [N](A')$. Thus, by Proposition 4.2.2,

$$\hat{h}_{\mathcal{L}}^F(X) = (1/N^2) \hat{h}_{\mathcal{L}}^F([N](X)) = (1/N^2) \hat{h}_{\mathcal{L}}^F([N](A')) = \hat{h}_{\mathcal{L}}^F(A')$$

On the other hand,

$$\hat{h}_{\mathcal{L}}^F(A') = \hat{h}_{\mathcal{L}}^F([N](A')) = N^2 \hat{h}_{\mathcal{L}}^F(A')$$

Therefore, $\hat{h}_{\mathcal{L}}^F(X) = \hat{h}_{\mathcal{L}}^F(A') = 0$.
“(3) \implies (1)”: Let $\overline{H}$ be an ample $C^\infty$-hermitian line bundle on $B$. Then, there is a positive integer $n$ such that $\overline{H}_i^{\otimes n} \otimes \overline{H}_{i-1} \geq 0$. for all $i$. Then, by using (4) of Proposition 1.1, we can see that an adelic sequence with respect to $(B; \overline{H}_1, \ldots, \overline{H}_d)$ is an adelic sequence with respect to $(B; \overline{H}, \ldots, \overline{H})$, and that

$$0 \leq \hat{h}_L^{(\overline{H}_1, \ldots, \overline{H})}(X) \leq n^d \hat{h}_L^{(\overline{H}_1, \ldots, \overline{H}_d)}(X).$$

Thus, we may assume that $\overline{H}_1, \ldots, \overline{H}_d$ are ample. We prove the assertion “(3) \implies (1)” by induction on $d = \text{tr. deg}_Q(K)$. If $d = 0$, then this was proved by Zhang [5]. We assume $d > 0$. Then, by the above lemma together with hypothesis of induction and Proposition 4.1.4, $X$ is a translation of an abelian subvariety $C$. Let us consider $\pi : A \to A' = A/C$. Then, $\pi(X)$ is a point, say $P$. Then, by Lemma 5.2, $\hat{h}_L(P) = 0$ for a symmetric ample line bundle $L'$ on $A'$. Thus, $P$ is a torsion point by [2, Proposition 3.4.1]. Therefore, we can see that $X$ is a translation of $C$ by a torsion point.

Let $X$ be a smooth projective curve of genus $g \geq 2$ over $K$. Let $J$ be the Jacobian of $X$ and $L_\Theta$ a line bundle given by a symmetric theta divisor $\Theta$ on $J$, i.e., $L_\Theta = \mathcal{O}_J(\Theta)$. Let $j : X \to J$ be a morphism given by $j(x) = \omega_X - (2g - 2)x$. Then, it is well known that $j^*(L_\Theta) = \omega_X^{\otimes 2g - 2}$. Let $\mathcal{L}_\Theta^{\text{can}}$ be the canonical adelic structure of $L_\Theta$. Thus, we have the adelic line bundle $j^*(\mathcal{L}_\Theta^{\text{can}})$ on $X$. In terms of this, we can give the canonical adelic structure on $\omega_X$. We denote this by $\omega_X^a$. Then, as a corollary of Theorem 5.1 and (3) of Proposition 4.1.3, we have the following.

**Corollary 5.4.** If the polarization $\mathcal{B}$ is big, then $\langle \omega_X^a, \omega_X^a \rangle_{\mathcal{B}} > 0$.

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