RATIONALITY OF TWISTS OF THE SIEGEL MODULAR VARIETY OF GENUS 2 AND LEVEL 3

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Abstract. Let \( \overline{\rho} : G_Q \rightarrow GSp_4(F_3) \) be a continuous Galois representation with cyclotomic similitude character. Equivalently, consider \( \overline{\rho} \) to be the Galois representation associated to the 3-torsion of a principally polarized abelian surface \( A/Q \). We prove that the moduli space \( A_2(\overline{\rho}) \) of principally polarized abelian surfaces \( B/Q \) admitting a symplectic isomorphism \( B[3] \cong \overline{\rho} \) of Galois representations is never rational over \( Q \) when \( \overline{\rho} \) is surjective, even though it is both rational over \( C \) and unirational over \( Q \) via a map of degree 6.

1. Introduction

Let \( p \) be a prime and suppose that \( A/Q \) is an abelian variety of dimension \( g \) with a polarization of degree prime to \( p \). Associated to the action of the absolute Galois group \( G_Q \) on \( A[p] \) there exists a Galois representation \( \overline{\rho} : G_Q \rightarrow GSp_{2g}(F_p) \) such that the corresponding similitude character is the mod-\( p \) cyclotomic character \( \varepsilon \). One can ask, conversely, whether any such representation comes from an abelian variety in infinitely many ways. When \( g = 1 \), this question is well-studied, and has a positive answer exactly for \( p = 2, 3, \) and \( 5 \). Indeed, the corresponding twists \( X(\overline{\rho}) \) of the modular curve \( X(p) \) are rational over \( Q \) for \( p = 2, 3, \) and \( 5 \), and have higher genus for larger \( p \).

In [BCGPT18], this question arose for abelian surfaces \( (g = 2) \) when \( p = 3 \). (The case \( p = 2 \), which is also discussed in that paper, is understood by analyzing the branch points of the hyperelliptic involution.) Let \( A_2(3) \) denote the Siegel modular variety of genus 2 and level 3. It is the moduli space of principally polarized abelian surfaces together with a symplectic isomorphism \( A[3] \cong (Z/3Z)^2 \oplus (\mu_3)^2 \). Given a \( \overline{\rho} \) as above, one can form the corresponding moduli space \( A_2(\overline{\rho}) \) where now one insists that there is a symplectic isomorphism \( A[3] \cong V \), where \( V \) is the representation space of \( \overline{\rho} \) with its symplectic structure. The variety \( A_2(3) \) is well-known to be birational to the Burkhardt quartic, which is rational over \( Q \) [BN18]. It is clear that \( A_2(\overline{\rho}) \) is isomorphic to \( A_2(3) \) over \( C \) (and even over the fixed field of the kernel of \( \overline{\rho} \), and hence \( A_2(\overline{\rho}) \) is geometrically rational. If \( A_2(\overline{\rho}) \) was in fact rational (by which we always mean rational over the base field), then indeed the answer to the question above would be positive, just as for elliptic curves when \( p \leq 5 \).

In [BCGPT18] Prop 10.2.3, a weaker result was established: The variety \( A_2(\overline{\rho}) \) is unirational over \( Q \) via a map of degree at most 6. As a consequence, any such \( \overline{\rho} \) does arise from (infinitely many) abelian surfaces. We refer the reader to [CCR20].

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which produces explicit polynomials describing the universal family over a rational
cover of $A_2(\rho)$ of degree 6. However, the question as to whether $A_2(\rho)$ was actually
rational was left open. We address this question here.

**Theorem 1.** Let $\rho : G_\mathbb{Q} \to \text{GSp}_4(F_3)$ be a representation with cyclotomic similitude
character. Suppose that the order of $\text{im}(\rho)$ is greater than 96. Then $A_2(\rho)$ is
not rational over $\mathbb{Q}$.

More refined results can be extracted directly from the table in §3. Since $\rho$ has
cyclotomic similitude character, the restriction of $\rho$ to $G_E$, where $E = \mathbb{Q}(\sqrt{-3})$,
has image contained in $\text{Sp}_4(F_3)$. If we let $H$ denote the projection of $\text{im}(\rho|_{G_E})$ to
the simple group $\text{PSp}_4(F_3)$, then we prove that $A_2(\rho)$ is not rational over $\mathbb{Q}$ for all
but 26 of the 116 conjugacy classes of subgroups of $\text{PSp}_4(F_3)$. With the exception
of three cases (including when $H$ is trivial) where the methods of [BN18] may be
applied (see §2.3), we do not know what happens in the remaining 23 cases, nor do
we even know whether the rationality of $A_2(\rho)$ depends only on $\text{im}(\rho)$ or not. One
easy remark is that, for a quadratic character $\chi$, there is an isomorphism $A_2(\rho) \cong A_2(\rho \otimes \chi)$, and so the rationality of $A_2(\rho)$ depends only on the image of $\rho|_{G_E}$
in $\text{PSp}_4(F_3)$.

The case of a surjective representation $\rho$ is of special interest, since this is what
happens generically for the three-torsion Galois representations of abelian surfaces.

**Theorem 2.** Suppose that $\rho$ is surjective. Then $A_2(\rho)$ is not rational over $\mathbb{Q}$, and
the minimal degree of any rational cover is 6.

In light of the result [BCGP18 Prop 10.2.3] mentioned above, the constant 6 is
best possible in this case.

The key ingredient in our results is the explicit description of the cohomology
of the compactified Siegel modular variety $A^*_2(3)$ given in [HW01]. We use it to
study the Galois module $\text{Pic}_\mathbb{Q}(A^*_2(\rho))$. The Galois action over $E = \mathbb{Q}(\sqrt{-3})$ factors
through the projectivization of $\rho$ turning it into a $H$-module. We then calculate
group cohomology of this module for various subgroups $P \subset H$, and employ a
necessary criterion for rationality (see Theorem 3) to deduce our results.

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to us many of the ideas in section 2.3.

2. **Strategy**

The main idea behind the proof is to follow a strategy employed by Manin
for cubic surfaces. Recall [Man86 §A.1] that a continuous $G_K$-module with the
discrete topology is called a *permutation module* if it admits a finite free $\mathbb{Z}$-basis
on which $G_K$ acts (via a finite quotient) via permutations, and that two $G_K$-
mODULES $M$ and $N$ are *similar* if $M \oplus P \simeq N \oplus Q$ for some permutation modules $P$
and $Q$. In particular, we employ the following theorem.

**Theorem 3.** [Man86 §A.1 Theorem 2] Let $Z$ be a smooth projective algebraic
variety over a number field $K$. Suppose that $Z$ is rational over $K$. Then $\text{Pic}_KZ$
as a $G_K$-module is stably permutation. In other words, it is similar to the zero module.

The Shimura variety $\mathcal{A}_2(3)$ admits a smooth toroidal projective compactification $\mathcal{A}_2^* (3)$, the (canonical) toroidal compactification constructed by Igusa [Igu67]. The automorphism group of $\mathcal{A}_2^* (3)$ over $\overline{Q}$ is the group $G = \text{PSp}_4(F_3)$, the simple group of order 25920, which acts over the field $E = Q(\sqrt{-3})$. It will be convenient from this point onwards to always work over the field $E$. (Certainly rationality over $Q$ implies rationality over $E$, so non-rationality over $E$ implies non-rationality over $Q$.) This action on $\mathcal{A}_2^* (3)$ arises explicitly from the action of $G$ on the 3-torsion $A[3] = (Z/3Z)^2 \oplus (\mu_3)^2 \simeq (Z/3Z)^4$ over $E$. We will apply Theorem 3 to the corresponding twist $\mathcal{A}^*_2(\overline{p})$. We then make crucial use of very explicit description of the cohomology of this compactification given by Hoffman and Weintraub [HW01].

We recall some facts from that paper here now.

2.1. Picard group. The Picard group of $\mathcal{A}_2^* (3)$ over $\overline{Q}$ is a free $Z$-module of rank 61. It is generated by two natural sets of classes. The first is a 40-dimensional space explained by the 40 connected components of the boundary. The second is a 45-dimensional space explained by divisors coming from Humbert surfaces. These are also in one to one correspondence with the 45 nodes on the Burkhard quartic. Together, these generate the Picard group of $\mathcal{A}_2^* (3)$ over $\overline{Q}$, which is free of rank 61. Indeed, the Betti cohomology of $\mathcal{A}_2^* (3)$ over $Z$ is free of degrees $1, 0, 61, 0, 61, 0, 1$ for $i = 0, \ldots, 6$ by [HW01, Theorem 1.1]. Furthermore, all of these classes are trivial under the action of $G_E$.

Let $\overline{p}: G_Q \to \text{GSp}_4(F_3)$ be a continuous Galois representation with cyclotomic similitude character. The assumption on the similitude character implies that the restriction of $\overline{p}$ to $E$ is valued in $\text{Sp}_4(F_3)$. Let $\rho: G_E \to G = \text{PSp}_4(F_3)$ denote the projectivization of the representation $\overline{p}$ restricted to $E$. The group $G$ acts over $E$ on $\mathcal{A}_2^* (3)$ via automorphisms, and $\mathcal{A}_2^* (\overline{p})$ is the twist of $\mathcal{A}_2^* (3)$ by $\rho$. The group $\text{Pic}_\overline{Q}(\mathcal{A}_2^* (\overline{p}))$ as a $G_E$-module is obtained by considering $\text{Pic}_Q(\mathcal{A}_2^*(3))$ as a $G$-module and then obtaining the Galois action via the map $\rho: G_E \to G$. Thus it remains to closely examine $\text{Pic}_\overline{Q}(\mathcal{A}_2^*(3))$ as a $G$-module over $Z$. In fact, we can quickly prove a weaker version of Theorem 3 by studying this $G$-module over $Q$.

The group $G$ admits a unique conjugacy class $G_{45}$ of subgroups of index 45, but two conjugacy classes of index 40; let $G_{40}$ denote the (conjugacy class of) subgroups which fix a point in the tautological action of $G \subset \text{PGL}_4(F_3)$ on $\mathbb{P}^3(F_3)$. The following is an easy consequence of the calculations of [HW01] (and is also confirmed by our Magma code).

**Lemma 1.** As $Q[G]$-modules, there is an equality of virtual representations

$$H^2(\mathcal{A}_2^*(3), Q) \simeq \text{Pic}_Q(\mathcal{A}_2^*(3)) \otimes Q = Q[G/G_{40}] + Q[G/G_{45}] - [\chi_{24}],$$

where $\chi_{24} \otimes Q C$ is the unique absolutely irreducible 24-dimensional representation of $G$.

Now, assuming that $\rho$ is surjective, we can prove that $\mathcal{A}_2^* (\overline{p})$ is not rational simply by proving that $\chi_{24}$ is not virtually equal to a sum of permutation representations. If $R_Q(G)$ denotes the representation ring of $G$, this is equivalent to
proving that $\chi_{24} \in R_\mathbb{Q}(G)$ does not lie in the Burnside subring generated by permutation representations. But one may compute (using Magma or otherwise) that the Burnside cokernel of $G$ has order 2 and is generated by $\chi_{24}$. This proves a weaker version of Theorem 2, showing that any rational cover of $A_4(\mathbb{F}_3)$ should have degree at least 2, although it is softer in that it only needs the $\mathbb{Q}[G]$-representation rather than the $\mathbb{Z}[G]$-module. This argument also applies if one only assumes that the image of $\varphi$ is $H \subset G$, as long as the restriction of $\chi_{24}$ to $H$ is still non-trivial in the Burnside cokernel, which it is for precisely 8 of the 116 conjugacy classes of subgroups of $G$.

2.2. Cohomological Obstructions. From now on, we let $H$ denote the image of $\varphi : G_E \to G = \text{PSp}_4(\mathbb{F}_3)$. A second way to prove that a Galois module is not similar to the zero module is to use cohomology. If $M$ is a permutation module of $H$, then the restriction of $M$ to any subgroup $P$ is also a permutation module, and thus a direct sum of $P$-modules of the form $\mathbb{Z}[P/Q]$ for subgroups $Q$ of $P$. (Note that since a permutation module of a group $G$ arises from a finite $G$-set, it always decomposes over $\mathbb{Z}$ into a direct sum of such irreducible permutation modules.) Then, Shapiro’s Lemma implies that $H^1(P, M)$ is a direct sum of groups of the form

$$H^1(P, \mathbb{Z}[P/Q]) = H^1(Q, \mathbb{Z}) = 0,$$

where the second group vanishes because $Q$ is finite. Moreover, the $\mathbb{Z}$-dual $M' = \text{Hom}(M, \mathbb{Z})$ of a permutation module is isomorphic to the same permutation module (a permutation matrix is its own inverse transpose). Thus one immediately has the following elementary criterion.

**Lemma 2** (Cohomological Criterion for non-rationality). Let $M$ denote the $G$-module $	ext{Pic}_G(A_4(3))$. Suppose $A_4(\mathbb{F}_3)$ is rational over $E = \mathbb{Q}(\sqrt{-3})$, and $\varphi|_{G_{E^+}}$ has image $H \subset G$. Then

$$H^1(P, M') = H^1(P, M) = 0$$

for every subgroup $P \subset H$.

We note that this is not an “if and only if” criterion. In the language of [CST77], the lemma is saying that $M$ as a $G_{E^+}$-module is flasque and coflasque respectively. In general, this is weaker than being stably permutation (which itself is not enough to formally imply rationality).

In order to test this criterion in practice, we need an explicit description of $M$ as a $\mathbb{Z}[G]$-module rather than a $\mathbb{Q}[G]$-module. In order to do this, we explain how an explicit description of $M$ can be extracted from Theorem 4.9 of [HW01]. That theorem describes a set of elements which generate both $H_4(A_4^*(3), \mathbb{Z})$ and $H^2(A_4^*(3), \mathbb{Z})$, and explicitly gives the intersection pairing between them. Moreover, the basis comes with a transparent action of the group $G$. Specifically, $H^2(A_4^*(3), \mathbb{Z})$ is given as a quotient of $\mathbb{Z}[G/G_{40}] \oplus \mathbb{Z}[G/G_{45}]$. Hence to compute $H^2(A_4^*(3), \mathbb{Z})$ as a $G$-module, it suffices to compute the quotient of $\mathbb{Z}[G/G_{40}] \oplus \mathbb{Z}[G/G_{45}]$ by the saturated subspace which pairs trivially with all elements of $H_4(A_4^*(3), \mathbb{Z})$. Having carried out this computation, we obtain a free abelian group of rank 61 with an explicit action of $G$. We then do the following for every conjugacy class of subgroups $H \subset G$.

1. Determine whether $\chi_{24}$ is non-trivial in the Burnside cokernel of $H$.
2. Determine whether $H^1(P, M) \neq 0$ for any subgroup $P \subset H$.
3. Determine whether $H^1(P, M') \neq 0$ for any subgroup $P \subset H$. 


If any of these is non-trivial, this proves that $A^*_2(\overline{\mathbb{F}_p})$ is not rational. Moreover, the computation of these cohomology groups allows us to deduce our result about the minimal degree of any rational covering.

**Lemma 3.** Let $M$ denote the $G$-module $\text{Pic}(A^*_2(\overline{\mathbb{F}_p}))$. Suppose $\varphi|_{G_E}$ has image $H \subset G$. Let $n$ denote the least common multiple of the exponents of $H^1(P, M)$ and $H^1(P, M^\vee)$ as $P$ varies over all subgroups of $H$. Suppose $f : X \to A^*_2(\overline{\mathbb{F}_p})$ is a rational cover of degree $d$ defined over $\mathbb{Q}$. Then $n$ divides $d$.

**Proof.** The induced pullback map $f^* : \text{Pic}(A^*_2(\overline{\mathbb{F}_p})) \to \text{Pic}(X)$ and pushforward map $f_* : \text{Pic}(X) \to \text{Pic}(A^*_2(\overline{\mathbb{F}_p}))$ are Galois equivariant since $f$ is defined over $\mathbb{Q}$. The composite map $g = f_* \circ f^*$ on $\text{Pic}(A^*_2(\overline{\mathbb{F}_p}))$ is multiplication by $d$. The discussion in [21] shows that the $G_E$-module $\text{Pic}(A^*_2(\overline{\mathbb{F}_p}))$ can be thought of as the $H$-module $M$.

By Theorem 3, we know that $\text{Pic}(X)$ is stably permutation as a Galois module and hence the Galois cohomology group $H^1(G, \text{Pic}(X)) = 0$. Therefore, the maps induced by $g$ on the cohomology groups $H^1(P, M)$ and $H^1(P, M^\vee)$ are the zero maps for every subgroup $P \subset H$. Since the map $g$ is multiplication by $d$, the induced map on cohomology is also multiplication by $d$, and hence we deduce that the exponent of each of these cohomology groups divides $d$. □

We give one final statement which can be extracted from Magma using the code given in [CC21], but not directly from the table. In order to represent elements of $G = \text{PSp}_4(\mathbb{F}_3)$ by matrices, we follow the conventions of Magma by fixing $\text{Sp}_4(\mathbb{F}_3) \subset \text{GL}_4(\mathbb{F}_3)$ to be the matrices preserving the symplectic form

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$ 

**Lemma 4.** Suppose that the image of $\overline{\mathbb{F}_p}$ contains an element conjugate in $\text{PSp}_4(\mathbb{F}_3)$ to

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Then $A^*_2(\overline{\mathbb{F}_p})$ is not rational, and the minimal degree of any rational cover is divisible by 3.

**Proof.** It suffices to note that this element generates the subgroup labelled as subgroup 6 in the table below, and then to apply Lemma 3 □

2.3. Other cases where rationality can be established. The analysis of Baker’s parametrization [Bak46] undertaken in [BN18, §4] allows one to deduce the rationality of certain twists of the Burkhardt quartic $B$ (and hence of $A^*_2(\overline{\mathbb{F}_p})$) in a few more cases. (We thank Nils Bruin for pointing this out to us, as well as explaining the geometric construction below.) The rational parametrization $\mathbb{P}^3 \dashrightarrow B$ over $\mathbb{Q}$ constructed in [BN18] is not equivariant with respect the action of $\text{PSp}_4(\mathbb{F}_3)$. If it were, then the twists $A^*_2(\overline{\mathbb{F}_p})$ we are considering would all be birational to Brauer–Severi varieties. However, because they are also unirational over $\mathbb{Q}$ by [BCGPT18 Prop 10.2.3], they would be rational over $\mathbb{Q}$, which we prove in this paper to be...
false in general. On the other hand, the parametrization $P^3 \to B$ is equivariant with respect to the (unique up to conjugacy) cyclic group of order 9 \cite[§4.3]{BN18}, and also with respect to the corresponding group scheme over $Q$ whose $E$ points are this group of order 9 (c.f. \cite[§2.3]{CCR20}), which controls the descent from $E$ to $Q$. In particular, the same argument implies that $A_2(\mathfrak{p})$ is rational in two further cases, namely, the subgroups labelled $n = 4$ (of order 3) and $n = 24$ (of order 9) in the table below. One can also arrive at this rational parametrization more geometrically, following \cite[§4]{BN18}, whose notation we now freely follow.

The variety of lines $L_{J_1, J_2, J_3}$ incident with 3-distinct planes $J_i \subset P^4$ is geometrically rational. If these planes are mutually skew and lie on $B$, there is a dominant map $L_{J_1, J_2, J_3} \to B$ defined by noting that a line will generically intersect $B$ in four points and each $J_i$ in one point, and hence one can send the line to the fourth point of intersection with $B$. There are 40 Jacobi planes $J_i$ on $B$, and 2880 triples of mutually skew such planes. The stabilizer under $\text{PSp}_4(F_3)$ on these 2880 triples is the cyclic group of order 9. The assumption that $H$ is contained inside this group then implies that there exists a triple of $\text{Gal}(\overline{Q}/Q)$-invariant mutually skew planes on the twist of $B$ corresponding to $\mathfrak{p}$. The result then follows after noting that $L_{J_1, J_2, J_3}$ is rational over $Q$ whenever this triple is defined over $Q$. (We omit a direct proof of this last claim in light of the alternate argument given above.)

3. Computation

Let $M$ denote the $G$-module $\text{Pic}(A_2^*(3)) \simeq H^2(A_2^*(3), \mathbb{Z})$. We have, by Poincaré duality, an isomorphism $M^\vee = H^4(A_2^*(3), \mathbb{Z})$. Below we present in a table the result of our computation for all 116 conjugacy classes of subgroups $H \subset G$, indicating the following data:

1. An ordering $n = 1 \ldots 116$ of the conjugacy class of the subgroup $H$ as determined by Magma.
2. The group $H$ in the small groups database \cite{BEO01}. The first element of the pair gives the order of $H$.
3. The order of $M$ in the Burnside cokernel of $H$ over $Q$ (if it is non-trivial). If this is greater than 1, then the corresponding twist is not rational over $E$ (or $Q$).
4. The least common multiple of the exponents of $H^1(P, M)$ and $H^1(P, M^\vee)$ as $P$ ranges over subgroups $P \subset H$. If this is greater than 1, then the corresponding twist is not rational over $E$ (or $Q$). In particular, the fact that this number is 6 for $G$ itself proves Theorem 2.
5. The pre-image of $H$ in $\text{Sp}_4(F_3)$ acts on $F_3^4$. Is this action absolutely irreducible? (That is, is the action on $F_3^4$ irreducible.)
6. A list of the conjugacy class of maximal subgroups of $H$ (as indexed in the table). This allows one to compute the LCM column directly. The table is separated into blocks to reflect the geometry of the corresponding poset of subgroups. In particular, all maximal subgroups of $H$ occur in blocks before that of $H$.
7. The last two columns give $H^1(H, M)$ and $H^1(H, M^\vee)$.

One must be careful while reading the table because the ordering of the conjugacy classes of subgroups is not canonical. The Small Group tag and the indices of the maximal subgroups given in the second and sixth columns of the table do, however,
determine the ordering uniquely once we distinguish between the conjugacy classes indexed by \( n = 2, 3, n = 4, 5, 6, n = 9, 11, \) and \( n = 10, 12. \) This can be done by considering the length of each of these conjugacy classes (i.e., the number of subgroups in each conjugacy class) as shown in the following table.

| \( n \) | Length |
|--------|--------|
| 2      | 45     |
| 3      | 270    |
| 4      | 40     |
| 5      | 120    |
| 6      | 240    |

The Magma code available at [CC21] computes \( G \) and \( M \) directly from the description given by Hoffman and Weintraub [HW01]. This leads to a representation of \( G \) as generated by two sparse \( 61 \times 61 \) matrices \( x \) and \( y \) in \( \text{GL}_{61}(\mathbb{Z}) \) such that the underlying module on which \( G \) acts (on the right, by Magma conventions) is \( M \). The matrices \( x \) and \( y \) are also printed in the output file of our Magma script.

| \( n \) | SmallGroup | B | LCM | irreducible | maximal subgroups | \( H^1(M) \) | \( H^1(M^\vee) \) |
|--------|------------|---|-----|-------------|------------------|----------|-----------------|
| 1      | \( <1,1> \) | 1 | no  | 1           |                  |          |                 |
| 2      | \( <2,1> \) | 1 | no  | 1           |                  |          |                 |
| 3      | \( <2,1> \) | 1 | no  | 1           |                  |          |                 |
| 4      | \( <3,1> \) | 1 | no  | 1           |                  |          |                 |
| 5      | \( <3,1> \) | 1 | no  | 1           |                  |          |                 |
| 6      | \( <3,1> \) | 3 | no  | 1           | \( \mathbb{Z}/3\mathbb{Z} \) | \( \mathbb{Z}/3\mathbb{Z} \) |
| 7      | \( <5,1> \) | 1 | no  | 1           |                  |          |                 |
| 8      | \( <4,1> \) | 1 | no  | 2           |                  |          |                 |
| 9      | \( <4,2> \) | 1 | no  | 2 3         |                  |          |                 |
| 10     | \( <4,2> \) | 2 | no  | 3           | \( \mathbb{Z}/2\mathbb{Z} \)² |          |                 |
| 11     | \( <4,2> \) | 2 | no  | 2 3         | \( \mathbb{Z}/2\mathbb{Z} \) |          |                 |
| 12     | \( <4,2> \) | 1 | no  | 3           |                  |          |                 |
| 13     | \( <4,1> \) | 1 | no  | 3           |                  |          |                 |
| 14     | \( <6,1> \) | 3 | no  | 2 6         | \( \mathbb{Z}/3\mathbb{Z} \) |          |                 |
| 15     | \( <6,2> \) | 1 | no  | 2 4         |                  |          |                 |
| 16     | \( <6,2> \) | 3 | no  | 2 6         |                  |          |                 |
| 17     | \( <6,1> \) | 3 | no  | 3 6         | \( \mathbb{Z}/3\mathbb{Z} \) |          |                 |
| 18     | \( <6,1> \) | 1 | no  | 3 5         |                  |          |                 |
| 19     | \( <6,2> \) | 1 | no  | 2 5         |                  |          |                 |
| 20     | \( <6,2> \) | 1 | no  | 3 5         |                  |          |                 |
| 21     | \( <9,2> \) | 3 | no  | 5 6         | \( \mathbb{Z}/3\mathbb{Z} \)² |          |                 |
| 22     | \( <9,2> \) | 3 | no  | 4 6         | \( \mathbb{Z}/3\mathbb{Z} \)² |          |                 |
| 23     | \( <9,2> \) | 3 | no  | 4 5 6       |                  |          |                 |
| 24     | \( <9,1> \) | 1 | no  | 4           |                  |          |                 |
| 25     | \( <10,1> \) | 1 | no  | 3 7         |                  |          |                 |
| 26     | \( <8,4> \) | 1 | no  | 8           |                  |          |                 |
| 27     | \( <8,5> \) | 2 | no  | 11 12       | \( \mathbb{Z}/2\mathbb{Z} \)² |          |                 |
| 28     | \( <8,5> \) | 2 | no  | 10 11       | \( \mathbb{Z}/2\mathbb{Z} \)² |          |                 |
| 29     | \( <8,5> \) | 2 | no  | 9 10 11     |                  |          |                 |
| 30     | \( <8,2> \) | 2 | no  | 8 11        |                  |          |                 |
| 31     | \( <8,2> \) | 2 | no  | 11 13       | \( \mathbb{Z}/2\mathbb{Z} \) | \( \mathbb{Z}/2\mathbb{Z} \) |
|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 32 | <8,3> | 2 | no | 8 11 | $Z/2Z$ |
| 33 | <8,3> | 2 | no | 10 12 13 | $Z/2Z$ |
| 34 | <8,3> | 1 | no | 9 12 13 | $Z/2Z$ |
| 35 | <12,3> | 2 | no | 5 10 | $Z/3Z$ |
| 36 | <12,3> | 3 | no | 6 12 | $Z/3Z$ |
| 37 | <12,4> | 3 | no | 9 14 16 17 | $(Z/3Z)^2$ |
| 38 | <12,5> | 1 | no | 9 19 20 | $(Z/3Z)^2$ |
| 39 | <12,1> | 1 | no | 13 20 | $(Z/3Z)^2$ |
| 40 | <12,2> | 1 | no | 8 15 | $(Z/3Z)^2$ |
| 41 | <12,4> | 1 | no | 12 18 20 | $(Z/3Z)^2$ |
| 42 | <18,4> | 3 | no | 17 18 21 | $(Z/3Z)^2$ |
| 43 | <18,3> | 3 | no | 14 16 21 | $(Z/3Z)^2$ |
| 44 | <18,3> | 3 | no | 14 19 23 | $(Z/3Z)^2$ |
| 45 | <18,3> | 3 | no | 14 15 22 | $(Z/3Z)^2$ |
| 46 | <18,3> | 3 | no | 18 20 21 | $(Z/3Z)^2$ |
| 47 | <18,3> | 3 | no | 17 20 23 | $(Z/3Z)^2$ |
| 48 | <18,5> | 3 | no | 15 16 19 23 | $(Z/3Z)^2$ |
| 49 | <20,3> | 1 | yes | 13 25 | $(Z/3Z)^2$ |
| 50 | <27,5> | 3 | no | 21 22 23 | $(Z/3Z)^2$ |
| 51 | <27,3> | 3 | no | 22 | $(Z/3Z)^2$ |
| 52 | <27,4> | 3 | no | 22 24 | $(Z/3Z)^2$ |
| 53 | <16,14> | 2 | yes | 28 29 | $(Z/3Z)^2$ |
| 54 | <16,13> | 2 | no | 26 30 32 | $(Z/3Z)^2$ |
| 55 | <16,11> | 2 | yes | 27 28 30 32 | $(Z/3Z)^2$ |
| 56 | <16,3> | 2 | no | 28 31 | $(Z/3Z)^2$ |
| 57 | <16,11> | 2 | yes | 27 29 31 33 34 | $(Z/3Z)^2$ |
| 58 | <16,3> | 2 | no | 29 30 31 | $(Z/3Z)^2$ |
| 59 | <24,3> | 1 | no | 15 26 | $(Z/3Z)^2$ |
| 60 | <24,13> | 2 | no | 20 29 35 | $(Z/3Z)^2$ |
| 61 | <24,3> | 3 | no | 16 26 | $(Z/3Z)^2$ |
| 62 | <24,3> | 1 | no | 19 26 | $(Z/3Z)^2$ |
| 63 | <24,11> | 2 | 1 | no | 26 40 | $(Z/3Z)^2$ |
| 64 | <24,13> | 2 | no | 19 28 35 | $(Z/3Z)^2$ |
| 65 | <24,13> | 6 | no | 16 27 36 | $(Z/3Z)^2$ |
| 66 | <24,12> | 2 | no | 18 33 35 | $(Z/3Z)^2$ |
| 67 | <24,12> | 6 | no | 17 33 36 | $(Z/3Z)^2$ |
| 68 | <24,12> | 3 | no | 14 34 36 | $(Z/3Z)^2$ |
| 69 | <24,8> | 1 | no | 34 38 39 41 | $(Z/3Z)^2$ |
| 70 | <36,10> | 3 | no | 37 42 43 | $(Z/3Z)^2$ |
| 71 | <36,10> | 3 | no | 41 42 46 | $(Z/3Z)^2$ |
| 72 | <36,9> | 3 | no | 13 42 | $(Z/3Z)^2$ |
| 73 | <36,12> | 3 | no | 37 38 44 47 48 | $(Z/3Z)^2$ |
| 74 | <54,8> | 3 | no | 45 51 | $(Z/3Z)^2$ |
| 75 | <54,13> | 3 | no | 42 46 47 50 | $(Z/3Z)^2$ |
| 76 | <54,12> | 3 | no | 43 44 45 48 50 | $(Z/3Z)^2$ |
| 77 | <60,5> | 2 | no | 18 25 35 | $(Z/3Z)^2$ |
| 78 | <60,5> | 3 | no | 17 25 36 | $(Z/3Z)^2$ |
| 79 | <81,7> | 3 | no | 50 51 52 | $(Z/3Z)^2$ |
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