Single-photon single-ion interaction in free space configuration in front of a parabolic mirror

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We theoretically investigate conditions for an experimental setup consisting of a single two-level ion trapped at the focus of a parabolic metallic mirror, under which the assumption about a free space mode structure of radiation field in vicinity of the atom is justified. We seek for the changes in the spontaneous emission rate of the atom resulting from the presence of the parabolic boundary conditions, within the vectorial model of light by including the polarization degree of freedom. We assume single-photon single-atom interaction.

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I. INTRODUCTION

The interaction of single matter objects with a radiation field has always been inspiring topic based on quantum phenomena. First, it was possible to observe interactions in cavity QED regime for an "isolated" system consisting of an atom and a single light mode. It allowed for exploiting strong atom-light coupling, the key property for building scalable quantum computing and distributed networks [1,2,3]. Recently, a technology development enabled applications such as single photon atom trapping and atom-cavity microscope [4,5] allowing for reconstruction of atomic trajectory. Also different regime of interactions has brought a lot of interest nowadays: atom-light interaction in free space [6,7,8,9], once achieved it would provide us with technologically less involved solutions for quantum communication over large distances.

The dynamics of matter interacting with a radiation field in presence of electromagnetic boundaries is significantly different with respect to free space configuration due to change in the surrounding field mode structure. The most studied example, both theoretically [10] and experimentally [11,12,13], concerns an atom in a small cavity where the radiating atom can excite only one or few radiation modes. Since the cavity modifies the inner field by suppressing a large number of modes and supporting few, we can witness spontaneous emission enhancement or inhibition to the modes which are resonant or non-resonant with the cavity respectively.

An interesting intermediate case, between small cavity limit and free space (continuum of modes), corresponds to a large cavity limit [14] where the atom couples to a large number of modes. A half-cavity, i.e. cavity with one mirror, constitutes another example of such a case. It has been experimentally verified that also in this regime one can witness a change of density of modes near the atom by measuring its spontaneous emission rate [15]. One-dimensional (scalar) model for a laser-driven atom in a half-cavity has been discussed in [16].

In this paper we theoretically investigate conditions for an experimental setup consisting of a single two-level ion trapped at the focus of a parabolic metallic mirror (see Fig. 1) [17], under which the assumption about a free space mode structure of electromagnetic field in vicinity of the atom is justified. In our analysis we focus on single-photon single-atom interaction instead. We included the polarization degree of freedom and thus apply vectorial properties of light. We look for the possible changes in the spontaneous emission rate of the atom resulting from the presence of the parabolic boundary conditions.

FIG. 1: The experimental setup under consideration: a single two-level ion trapped at the focus of a parabolic metallic mirror. The focal length equals $f$. The ion interacts with the light coming from the whole $4\pi$ solid angle.

The atom is located in half-open space, or in a half-cavity. The parabolic shape of the mirror ensures that if a light beam is send parallelly to the mirror axis towards the ion, it interacts with the light coming from the whole $4\pi$ solid angle, and similarly it allows to collect the whole light resulting from its spontaneous decay. Thinking in terms of the classical ray-picture, the incident angle for either incoming or outgoing light is never equal to $\pi/2$ (except from one direction). It means that if the atom emits a photon in the spontaneous emission process, on contrary to a standard cavity case, the radiation is never back reflected to the atom, and thus the atom does not feel the presence of the boundary conditions, just like in free space. Nevertheless, the mirror creates the nodes and anti-nodes in the reflected modes and by that it can change the electromagnetic vacuum structure. Therefore, in the limit where the focal length $f$ of the mirror is comparable to the...
wavelength $\lambda$ of the mode resonant with the atomic transition, similarly as in a small cavity or in front of a planar mirror \[18\], the changes of the spontaneous decay rate should be significant.

We will show that depending on the characteristic parameters of the setup: the focal length $f$, the wavelength $\lambda$ and the orientation of the atomic electric dipole moment, the setup provides us with either free space configuration or a tailored electromagnetic reservoir near the atom.

This paper is organized as follows. In section II we develop the formalism for the normal modes genuine to the parabolic geometry. In section III we analyze the correction to the spontaneous emission rate resulting from presence of the parabolic metallic mirror. We finish the paper with the conclusions.

II. DECAY RATE IN THE PARABOLIC GEOMETRY

Since the ion is located in a half-open space we work within the framework of the Weisskopf-Wigner model of interaction between a single matter qubit and a quantized radiation field. We begin with the expansion of the electric field operator for the electromagnetic field in the free space using the basis (modes) suitable for the parabolic symmetry of the problem. We follow the results of \[19\] and use the modes given by the formula

$$E_{k,\ell,\mu}(\vec{n}) = \frac{k}{(2\pi)^{3/2}} \int_{S^2} d\vec{n} e^{ik\vec{n} \cdot \vec{r}} h_{\ell,\mu}(\vec{n}) e^\sigma(\vec{n}),$$  \hspace{1cm} (1)

where $\vec{k} = k\vec{n}$ denotes the wavevector, $\sigma = 1, 2$ enumerates polarization states, parameters $\ell = 0, \pm 1, \pm 2, \ldots$ and $\mu \in (-\infty, +\infty)$ are the mode numbers. The unit vector $\vec{n}$ and the polarization vectors $\vec{e}^1(\vec{n})$, $\vec{e}^2(\vec{n})$ constitute the orthonormal basis which ensures the transversality condition

$$\nabla \cdot \vec{E} = 0.$$  \hspace{1cm} (2)

We choose

$$\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta),$$  \hspace{1cm} (3)

$$\vec{e}^1(\vec{n}) = (\sin \varphi, -\cos \varphi, 0),$$  \hspace{1cm} (4)

$$\vec{e}^2(\vec{n}) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta).$$  \hspace{1cm} (5)

The manifest form of the modes $h_{\ell,\mu}(\vec{n})$ reads \[19\]

$$h_{\ell,\mu}(\theta, \varphi) = \chi_\mu(\theta) e^{i\ell \varphi} \sqrt{2\pi}$$  \hspace{1cm} (6)

where

$$\chi_\mu(\theta) = \exp \left( -i\mu \ln|\tan \theta/2| \right) \sqrt{2\pi \sin \theta}.$$  \hspace{1cm} (7)

One can easily check the orthogonality and completeness conditions

$$\sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\mu h_{\ell,\mu}^*(\theta, \varphi) h_{\ell,\mu'}(\theta, \varphi') = \delta(\varphi - \varphi') \frac{\delta(\theta - \theta')}{\sin \theta}.$$  \hspace{1cm} (8)

Combining Eq. (1) with Eq. (5) one obtains the orthogonality of the electric modes

$$\int d^3\vec{r} \vec{E}_{k,\ell,\mu}(\vec{r}) \cdot \vec{E}_{k',\ell',\mu'}(\vec{r}) = \delta(k - k') \delta(\mu - \mu') \delta_{\ell\ell'} \delta_{\sigma\sigma'}.$$  \hspace{1cm} (9)

Let us consider an atomic qubit at a fixed position $\vec{r}$ in free space with transition dipole parallel to the $z$-axis. Its excited and ground state are denoted by $|e\rangle$ and $|g\rangle$ respectively. The atom interacts with the quantized radiation field distributed over a continuum of modes centered around the optical atomic transition frequency $\omega_0$ and given by Eq. (1). The standard dipole-interaction Hamiltonian of such matter-field system reads $H = -\vec{d} \cdot \vec{E}(\vec{r})$ and simplifies for $\vec{d} = d \hat{e}_z$ to the following form

$$H_{int} = -id\sqrt{\frac{\hbar c}{2\omega_0}} \int_0^{+\infty} dk \int_{-\infty}^{+\infty} d\mu \left\{ \frac{E_{k,\ell,\mu}(\vec{r})}{E_{k,\ell,\mu}(\vec{r})} e^{i\sigma \cdot \vec{r}} - h.c. \right\}$$  \hspace{1cm} (10)

where $\sigma^+ = |e\rangle \langle g|$ and $\sigma^- = |g\rangle \langle e|$ are the atomic rising and lowering operators respectively. This Hamiltonian shows that the radiation field couples to the atom only if its polarisation has a component which is parallel to the $z$-axis at the position of the atom so that $E_{\sigma}(\vec{r}) \cdot \hat{e}_z \neq 0$. From now on we fix the frequency of the mode, $k = \omega_0/c$, where $\omega_0$ denotes the atomic transition frequency. The spontaneous emission decay rate for the atom immersed in the electromagnetic field reservoir computed in the standard lowest order (Born) approximation equals

$$\Gamma(k, \vec{r}) = \frac{d^2k^3}{(2\pi)^2} \frac{1}{2\hbar c} \sum_{\ell,\mu} \int_{-\infty}^{+\infty} d\mu \int d\vec{n} \int d\vec{n}' f_{k}(\vec{n}, \vec{r}) f_{k}^*(\vec{n}', \vec{r}') e^{\sigma \cdot \vec{r}} h_{\ell,\mu}(\vec{n}, k) h_{\ell,\mu}(\vec{n}', k).$$  \hspace{1cm} (11)

We denote here by $f_{k}(\vec{n}, \vec{r})$ the plane wave $e^{ik\vec{n} \cdot \vec{r}}$ and $e^{\sigma \cdot \vec{r}}$ and use the fact the the summation over $\ell$ produces $\delta(\varphi - \varphi')$ (see Eq. (9)). Therefore relevant $\vec{n}$, $\vec{n}'$ and the $z$-axis belong to the same plane what leads to the formula

$$\sum_{\sigma} e^{\sigma \cdot \vec{n}} e^{\sigma \cdot \vec{n}'} = e^{\sigma \cdot \vec{n}} e^{\sigma \cdot \vec{n}'} = \sin \theta \sin \theta'.$$  \hspace{1cm} (12)

Taking into account that the integration over $\mu$ yields another Dirac delta $\delta(\theta - \theta')$ we obtain

$$\Gamma(k; x, y, z) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^3 \theta |f_k(x, y, z; \varphi, \theta)|^2$$  \hspace{1cm} (13)

where $\vec{r} \equiv (x, y, z)$ and $f_k(\vec{n}, \vec{r}) \equiv f_k(x, y, z; \varphi, \theta)$. Knowing that $|f_k(x, y, z; \varphi, \theta)| = \Gamma_0(k)$ one recovers the standard result

$$\Gamma_0(k) = \frac{1}{3\pi} \frac{d^2k}{\hbar c}.$$  \hspace{1cm} (14)
III. DECAY RATE AT THE PRESENCE OF MIRROR

The presence of the conducting parabolic mirror leads to the boundary conditions which should be imposed on the modes of the electric field given by Eq. (11). Since it is very challenging to solve the Helmholtz equation while keeping the zero-divergence condition (Eq. (2) and the boundary conditions satisfied at the same time [20], on contrary to the reference [19] we decided to keep the transversality condition at the mirror surface. To discuss this problem we introduce the parabolic coordinates $(\xi, \eta, \varphi)$ related to Cartesian ones in the following way

\begin{align*}
x &= 2\sqrt{\xi \eta} \cos \varphi, \\
y &= 2\sqrt{\xi \eta} \sin \varphi, \\
z &= \xi - \eta.
\end{align*}

The shape of the parabolic mirror is given by the equation

\[ \eta = f. \tag{19} \]

The normal modes (in fact their scalar counterparts) obtained by separation method and expressed in parabolic coordinates are given by products of the functions of $\xi, \eta, \varphi$ respectively [19]. We are interested in the $\eta$-dependent part $F_{\ell,\mu}(k; \eta)$ which possesses the following asymptotic behaviour

\[ F_{\ell,\mu}(k; \eta) \sim \frac{\cos \left\{ \mu \ln 2k\eta + k\eta - \alpha_{\ell,\mu} \right\}}{\sqrt{\eta}}, \tag{20} \]

where $\alpha_{\ell,\mu}$ is a certain phase. The boundary condition imposed on Eq. (20) at the value $\eta = f$ can be satisfied for a discrete set of $\mu_m$ only, which is related to the periodicity of the cosine function. Take for the illustration the simplest choice

\[ kf - \alpha_{\ell,\mu} = 0, \quad \mu_m \ln 2k f = m\pi, \quad m = 1, 2, 3, \ldots \tag{21} \]

This periodic condition is consistent with the replacement of the continuous set of modes given by Eq. (7) by a discrete one

\[ \hat{\chi}_m(\theta) = \frac{\sin \left( \frac{m\pi \ln(\tan \theta/2)}{2kf} \right)}{2\pi \ln 2kf \sin \theta} \quad \text{for} \quad \theta \in [\theta_0, \pi - \theta_0] \tag{22} \]

\[ = 0 \quad \text{otherwise} \]

such that

\[ \tan \frac{\theta_0}{2} = \frac{1}{2kf} = \frac{1}{4\pi f}. \tag{23} \]

The limitation for the $\theta$ angle results from the quantization condition and the fact that at the boundary the normal modes vanish $\hat{\chi}_m(\theta_0) = 0$. This anzatz modifies the formula for decay rate because the completeness condition now reads

\[ \sum_{m} \hat{\chi}_m(\theta)\hat{\chi}_m(\theta') = I_{[\theta_0, \pi - \theta_0]}(\theta) \frac{\delta(\theta - \theta')}{\sin \theta}, \tag{24} \]

where $I_A$ denotes the indicator function of the set $A$. Therefore, the integration over $\theta$ in Eq. (14) should be performed over the interval $[\theta_0, \pi - \theta_0]$. This, however leads to the correction of the order of $(kf)^{-4}$ which is completely irrelevant from the experimental point of view. It is easy to notice that while calculating the integral in Eq. (14) in the intervals $[0, \theta_0]$, and $[\pi - \theta_0, \pi]$. In our experiment the focal length is of order of $f = 2$mm and the wavelength of $\lambda = 250$nm, which amounts to $kf \simeq 10^4$ and thus $\theta_0 = 0$. Therefore we can replace $\sin \theta$ by $\theta$. It is rather obvious that the same is true for any reasonable choice of the boundary conditions because the smallness of this correction is entirely due to the large value of $kf$.

Hence the only relevant modification of the spontaneous emission rate due to the presence of the mirror is the replacement of the plane traveling waves $f_k(\vec{n}, \vec{r}) = e^{ik\vec{n} \cdot \vec{r}}$ by the standing waves

\[ f_k(\vec{n}, \vec{r}) = \sqrt{2} \sin \left( k\vec{n} \cdot (\vec{r} - \vec{f}) \right), \tag{25} \]

where the factor $\sqrt{2}$ ensures the completeness condition. The choice in Eq. (25) implies that the electric field for any mode $\vec{n}$, $\vec{r}$ satisfying the Helmholtz equation while keeping the transversality condition at the boundary the normal modes vanish at the point $P$. It leads to the final expression for the spontaneous emission rate in the presence of conducting parabolic mirror

\[ \hat{\Gamma}(k; x, y, z) = \frac{1}{2\pi^2 2\hbar\epsilon_0} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3 \theta \]

\[ \sin^2 \{k[(x \cos \varphi + y \sin \varphi) \sin \theta + (z + f) \cos \theta]\} \tag{26} \]

If the atom is placed at the distance from the point $P$ much larger than $\lambda$ the interference factor $\sin^2(...) \sin^2(\ldots)$ is averaged to $1/2$ and the standard free space result (Eq. (13)) is recovered. For the atom located at the mirror axis $x = y = 0$ the integral in Eq. (26) simplifies to

\[ \hat{\Gamma}(k; z) = \eta \Gamma_0(k), \tag{27} \]

where the correction to the free space decay rate is equal to

\[ \eta = \left( 1 + \frac{3 \cos(2k(z + f))}{4k^2(z + f)^2} - \frac{3}{8k^2(z + f)^2} \right). \tag{28} \]

The correction $\eta$ is evaluated for $f = 2$mm and depicted on Fig. 2. Its value at the focal point becomes significant for small values of the wavevector $k$ (thus large values of the wavelength $\lambda$), corresponding to the condition $|z + f| < \lambda$, i.e. for the atom which is close the the mirror surface (within the distance of $\lambda$). However, if $k$ gets large, then the $\eta$ fluctuations are shifted towards the mirror surface and take place only at a length of order of the wavelength. Far away from the mirror all the fluctuations vanish, $\eta = 1$, and thus we observe free space decay rate, also at the focus. According to the figures, in our experiment the changes in the decay rate could be observable on a scale of 100nm but only within the distance of the wavelength from the mirror surface.

However, the energy distribution among different modes is sensitive to the precise position of the atom. In particular one
should observe interference effects on the screen perpendicular to the mirror axis away from the focal point, please see Fig. 3. Since the dipole radiation has to obey the boundary condition: the field has to vanish on the mirror surface, only those modes will contribute to the pattern on the screen which fulfill this condition. The other modes will be suppressed and will give rise to the dark fringes. The detailed structure of the fringes depends on the value $k f$ and the precise position of the atom.

IV. CONCLUSIONS

We rigorously analyzed the modification of the electromagnetic vacuum structure around an atom trapped at the focus of a parabolic metallic mirror. We assumed that the atomic dipole moment is parallel to the mirror axis. For the focal length large compared to the wavelength of the photon emitted during the atomic transition the total spontaneous emission rate essentially does not differ from its free space value. However, the interference effects in perpendicular plain to the mirror axis resulting from the presence of the boundaries are expected. They are analogical to those observed in the experiment with an atom trapped in front of a flat mirror and described in ref. [15].

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