A Construction of Rational Seifert Surface in Lens Space

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Abstract

In this note, we give a method to construct rational Seifert surface for those smooth or piece-wise linear oriented knots in Lens space $L(p, q)$. We assume that the oriented knot has a regular projection on Heegaard torus and then construct rational Seifert surface on twist toroidal diagram.

1 Introduction

The existence of Seifert surface of a null-homologous knot or link is a very interesting problem in topology. In chapter 5.A.4 [1], Rolfsen showed us a direct way to constructing Seifert surface by regular projection of a smooth or piece-wise linear knot. It’s a natural question whether we can generalize Seifert surface of a link. In section 1 of [2], Kenneth Baker and John Etnyre defined rational Seifert surface for a knot which represents a torsion element in homology group $H_1$. Especially, $H_1(L(p, q)) = \mathbb{Z}_p$. Thus, every knot represents a torsion element in homology group. We give a construction of rational Seifert surface for arbitrary smooth knot when it has a regular projection on Heegaard torus of $L(p, q)$. We assume that all knots mentioned in this note are smooth or piece-wise linear.

2 Representation of a smooth knot in $L(p, q)$

Let $V_i (i = 1, 2)$ be two solid torus $D^2 \times S^1$. Its meridian and longitude is denoted by $(\mu_i, \lambda_i)$. Then, in the sense of Heegaard decomposition, a lens space $L(p, q)$ can be described by $V_1 \cup_\phi V_2$ where the gluing map $\phi : \partial V_2 \rightarrow V_1$ is an orientation-reversing diffeomorphism given in standard longitude-meridian coordinates on the torus by the matrix

$$
\begin{pmatrix}
-q & q' \\
p & -p'
\end{pmatrix} \in -SL_2(\mathbb{Z})
$$

In particular, $\phi(\mu_2) = -q \mu_1 + p \lambda_1$. This fact concludes that $H_1(L(p, q)) = \langle \lambda_1 \mid p \lambda_1 = 1 \rangle$.

Let $K$ be a knot in Lens space $L(p, q)$. Of course, after a small perturbation, it can be disjoint from the core $C_i = 0 \times S^1 \subset D^2 \times S^1$ of two solid torus at the same time. Please notice that $V_i \setminus C_i$ deformation retracts to its boundary $\partial V_i$. Thus, the deformation retraction $P : L(p, q) \setminus V_1 \cup V_2 \rightarrow \partial V_1$ projects $K$ onto Heegaard torus $\partial V_1$

**Definition 1. (see chapter 3.E of [1])**

Assume $K$ is a smooth knot. The deformation retraction $P$ is said to be regular for $K$ iff:

$\forall x \in \partial V_1, \ |P^{-1}(x)| = 0, 1, 2$ and if 2, $P(K)$ intersects itself transversely at $x$

**Remark 1.** if $P$ is not regular for $K$, then, after a small perturbation of $K$, $P$ is regular. From now on, We assume $K$ is in the interior of thickened torus $\partial V_1 \times [-1, 1]$ and the natural projection $\partial V_1 \times [-1, 1] \rightarrow \partial V_1$ is regular for $K$. We regard $L(p, q)$ is obtained from $\partial V_1 \times [-1, 1]$ gluing $V_1$ to the lower boundary of this thickened torus and $V_2$ to the upper boundary.

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After above discussions, the reader can realize that such a knot K can be drawn on a fundamental domain of torus \( \partial V_1 \). Notice that \( \partial V_1 = T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \). The usual choice of fundamental domain of this torus is a square \([0, 1] \times [0, 1] \subset \mathbb{R}^2\). In this square, \([0, 1] \times \{0\}\) represents \( \mu_1 \) while \( \{0\} \times [0, 1] \) represents \( \lambda_1 \).

**Definition 2.** (see Def 2.1 of [3])
The twist toroidal diagram of \( \partial V_1 \subset L(p, q) \) is a fundamental domain in \( \mathbb{R}^2 \) bounded by four straight lines:

\[
\begin{aligned}
&x = 0 \\
x = 1 \\
y = -\frac{q}{p} x \\
y = -\frac{q}{p} (x - 1)
\end{aligned}
\]

**Remark 2.** In twist toroidal diagram, it’s also holds that \((0, 1)(0, 0)(1, 0)\) represent a same point in \( \partial V_1 \). The straight line \( y = -\frac{q}{p} x \) has same direction as \( \mu_2 \).

### 3 Construction of rational Seifert surface

#### 3.1 Basic Idea

By remark 1, we can draw \( K \) on the twist toroidal diagram of \( \partial V_1 \). We want to find a "cobordism" surface (inside of \( \partial V_1 \times [-1, 1] \)) from \( rK \) to a link \( L' \) which is the union of several \((\pm \mu_2) - \text{knot in } \partial V_1 \times \{1\}\) and \((\pm \mu_1) - \text{knot in } \partial V_1 \times \{-1\}\). Then we attach several meridian discs of \( V_1 \) to this "cobordism", this so called "cobordism" should be a real rational Seifert surface of \( K \). We will see later that \( L' \) may contain several null-homologous component on the upper boundary of \( \partial V_1 \times [-1, 1] \).

#### 3.2 Details of the construction

The construction is divided into following steps:

1. Replace crossings of \( P(K) \) by short-cut arcs on the twist toroidal diagram. Or equivalently, cut the crossing point \( A \) into two points \( A_{0,1} \). Then, we get a torus link \( L \subset \partial V_1 \times \{0\} \)

   ![Make a crossing apart](image)

   **Figure 1: Make a crossing apart**

2. Computations:
   
   Compute \([K] = [L] \in H_1(\partial V_1)\) in coordinate \((\mu_1, \lambda_1)\). Assume that \([L] = n(a \mu_1 + b \lambda_1)\) where \( n, a, b \in \mathbb{Z}, g.c.d.(a, b) = 1 \). The coefficient \( na(nb) \) and can be obtained by counting the algebraic intersection numbers of \( L \) and \( \lambda_1(\mu_1)\)-curve.

   Also, Compute order \( r \) of \([K] = [L] \in H_1(L(p, q)) = (\lambda_1|p\lambda_1)\).

   \[
r = \frac{p}{g.c.d.(p, nb)}
   \]
Then,
\[ r[L] = r n a \mu_1 + r n b \lambda_1 = r n a \mu_1 + \frac{r n b}{p} (p \lambda_1) = r n a \mu_1 + \frac{r n b}{p} (q \mu_1 + \mu_2) = (r n a + \frac{r n b q}{p}) \mu_1 + \frac{r n b}{p} \mu_2 \]

3. Construct "cobordism" from link \( L \) to \( L' \) noticed above.

(a) draw torus link \((r n a + \frac{r n b q}{p}) \mu_1\) on \( \partial V_1 \times \{-1\}\) (denoted by \( L^- \)) and \(-(r n a + \frac{r n b q}{p}) \mu_1\) on \( \partial V_1 \times \{1\}\) s.t both torus link avoid a connected neighborhood of each crossing of \( P(K) \) in the diagram where the crossing is now replaced by short-cut arcs.

![Diagram](image1)

Figure 2: Here is a knot \( K \) in \( L(3,1) \), \([L] = 2 \lambda_1, r = 3, r[L] = 2 \mu_1 + 2 \mu_2\). The blue line \( L^- a \)

For convenient, \(-(r n a + \frac{r n b q}{p}) \mu_1\) on \( \partial V_1 \times \{1\}\) should be drawn a little bit above the \((r n a + \frac{r n b q}{p}) \mu_1\) on the diagram.

![Diagram](image2)

Figure 3: the red line of homotopy type \(-2 \mu_1\) is not far away from the blue.

(b) draw torus link \( rL \) on \( \partial V_1 \times \{1\}\). Here, \( rL \) is \( r \) parallel copies of \( L \). For convenience, one shouldn’t draw \( rL \) too far away from \( L \).

![Diagram](image3)

Figure 4: the red line \( rL \) is far from \( L \) in the diagram we draw on.

(c) At each intersection of \(-(r n a + \frac{r n b q}{p}) \mu_1\) and \( rL \) on \( \partial V_1 \times \{1\}\), replace intersection by smooth arc shown by the graph below.
Then, we get a link $L^+$ on $\partial V_1 \times \{1\}$ with homology class $[L^+] = r[L] - (rna + \frac{rn_b q}{p})\mu_1 = \frac{rn_b}{p}\mu_2$. Therefore, its components is torus knot of $\pm \mu_2$ type or null-homologous (simple closed curve on torus). $L'$ is the union of $L^+$ and $L^-$.

(d) The "cobordism" of $L$ is actually bounded by $L$ and $L'$. Near the intersection of $L$ and $(rna + \frac{rn_b q}{p})\mu_1$ link on the diagram, the "cobordism" is glued by the bands below. Outside the neighborhood, the "cobordism" is obtained by gluing $r$ bands along $L$.

(e) For a very special case when $[L] = 0 \in H_1(\partial V_1)$, $L' = \emptyset$ and $L$ consists of $m(m \geq 0)$ non-trivial torus knots of type $a\mu_1 + b\lambda_1$, $m$ torus knots of type $-(a\mu_1 + b\lambda_1)$ and several null-homologous knots on torus. We construct disjoint $m$ bands (i.e $S^1 \times I$) and several discs bounded by null-homologous components of $L$.

4. Construct $r$-cover half-twist band as follow. Let $I \times I \times \{1, 2, \ldots, r\}$ be $k$-copies of a square. Define equivalent relationship $\sim$ by: $(x, 0, 1) \sim (x, 0, k)$ and $(x, 1, 1) \sim (x, 1, k)$.
Then do a half-twist along straight line $I \times \{ \frac{1}{2} \} \times \{0\}$ on the quotient space $I \times I \times \{1, 2, \ldots, r\}/ \sim$, the construction of $r$-cover half-twist band is done. Name arc $\{i\} \times I \times \{k\}$ by $c^k_i$ where $i = 0, 1; k = 1, 2, \ldots, r$.

5. In the first step, we cut apart the crossings (denoted by $A$) of $P(K)$ into two points $A_{0,1}$.

Now we cut off a 3-ball $B_i$ of a very small radius centered at each $A_{i=0,1}$ from the "cobordism"
constructed above. The boundary of 3-ball $\partial B_i$ intersects the cobordism at $r$ arcs with same
endpoints. These arcs is denoted by $\gamma^k_i$ where $i = 0, 1; k = 1, 2, \ldots, r$.

![Figure 11: $\gamma^k_i$ is marked in the figure](image)

Now we attach $r$-cover half-twist band to the punctured cobordism described above by regarding
$\gamma^0_i$ as $c^0_i$ and $\gamma^k_i$ as $-c^k_i$, $k = 1, 2, \ldots, r$. One should take care that the type of $r$-cover half-twist
band to be glued is depended on the writhe of this crossing. Then we get the cobordism from
$rK$ to $L'$.

6. Now we get the cobordism from $rK$ to $L'$. We gluing meridian discs of $V_1$ along $L^-$, and meridian
discs of $V_2$ along the $\pm\mu_2$-type component of $L^+$. For those null-homologous component of $L^+$,
we glue the discs bounded by them, probably with a little push off the diagram s.t. the discs are
disjoint.

Now we get a rational Seifert surface of $K$. It’s not hard to compute its Euler characteristic. Also, we
can find out how it wraps on $K$. See corollary below

**Corollary 1.** Let $K$ be a knot in the interior of $\partial V_1 \times I$ with homotopy type $[K] = n(\mu_1 + b\lambda_1)$ where
$n, a, b \in \mathbb{Z}, \text{g.c.d.}(a, b) = 1$. Let $NK$ be a tubular neighborhood of $K$ with framing $(\mu_{NK}, \lambda_{NK})$. Choose
the longitude $\lambda_{NK}$ of $NK$ to be the one induced from the push-off of $K$ along the positive direction of
$I$. Then, the rational Seifert surface of $K$ intersects $\partial NK$ at a torus link with homology type:

$$r\lambda_{NK} - (rn^2(a + \frac{bq}{p})b + r\text{writhe}(K))\mu_{NK}$$

where the writhe of $K$ is the sum of index defined in the graph of the first step.

**Proof.** The proof is not difficult noticing that the construction of cobordism of $L$ devotes

$$-rn^2(a + \frac{bq}{p})b\mu_{NK}$$

and the attachment of $r$-cover half-twist bands devotes

$$-r\text{writhe}(K)\mu_{NK}.$$


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References

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