Total dominator chromatic number of some operations on a graph

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Abstract

Let $G$ be a simple graph. A total dominator coloring of $G$ is a proper coloring of the vertices of $G$ in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic number $\chi^t_d(G)$ of $G$ is the minimum number of colors among all total dominator coloring of $G$. In this paper, we examine the effects on $\chi^t_d(G)$ when $G$ is modified by operations on vertex and edge of $G$.

Keywords: Total dominator chromatic number; Contraction; Graph.

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1 Introduction

In this paper, we consider simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V, E)$ be such a graph and $k \in \mathbb{N}$. A mapping $f : V(G) \longrightarrow \{1, 2, \ldots, k\}$ is called a $k$-proper coloring of $G$ if $f(u) \neq f(v)$ whenever the vertices $u$ and $v$ are adjacent in $G$. A color class of this coloring is a set consisting of all those vertices assigned the same color. If $f$ is a proper coloring of $G$ with the coloring classes $V_1, V_2, \ldots, V_k$ such that every vertex in $V_i$ has color $i$, then sometimes write simply $f = (V_1, V_2, \ldots, V_k)$. The chromatic number $\chi(G)$ of $G$ is the minimum number of colors needed in a proper coloring of a graph. The chromatic number is perhaps the most studied of all graph theoretic parameters. A dominator coloring of $G$ is a proper coloring of $G$ such that every vertex of $G$ dominates all vertices of at least one color class (possibly its own class), i.e., every vertex of $G$ is adjacent to all vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ of $G$ is the minimum number of color classes in a dominator coloring of $G$. Kazemi [3, 7] studied the total dominator coloring, abbreviated TD-coloring. Let $G$ be a graph with no isolated

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vertex, the total dominator coloring is a proper coloring of $G$ in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TD-chromatic number, $\chi_t^d(G)$ of $G$ is the minimum number of color classes in a TD-coloring of $G$. The TD-chromatic number of a graph is related to its total domination number. Recall that a total dominating set of $G$ is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$ and the total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A total dominating set of $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. The literature on the subject on total domination in graphs has been surveyed and detailed in the book [4]. It has proved that the computation of the TD-chromatic number is NP-complete ([6]). The TD-chromatic number of some graphs, such as paths, cycles, wheels and the complement of paths and cycles has computed in [6]. Also Henning in [5] established the lower and upper bounds on the TD-chromatic number of a graph in terms of its total domination number. He has shown that, for every graph $G$ with no isolated vertex satisfies $\gamma_t(G) \leq \chi_t^d(G) \leq \gamma_t(G) + \chi(G)$. The properties of TD-colorings in trees have been characterized in [5]. In [2] considered graphs with specific construction and study their TD-chromatic number. The join $G = G_1 + G_2$ of two graphs $G_1$ and $G_2$ with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V_1$ and $V_2$. For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of $G$ with $|V|$ copies of $H$, by adding edges between the $i$th vertex of $G$ and all vertices of $i$th copy of $H$. In the study of TD-chromatic number of graphs, this naturally raises the question: What happens to the TD-chromatic number, when we consider some operations on the vertices and the edges of a graph? In this paper we would like to answer to this question.

In the next section, examine the effects on $\chi_t^d(G)$ when $G$ is modified by deleting a vertex or deleting an edge. In Section 3, we study the effects on $\chi_t^d(G)$, when $G$ is modified by contracting a vertex and contracting an edge. Also we consider another obtained graph by operation on a vertex $v$ denoted by $G \odot v$ which is a graph obtained from $G$ by the removal of all edges between any pair of neighbors of $v$ in Section 3 and study $\chi_t^d(G \odot v)$.

## 2 Vertex and edge removal

The graph $G - v$ is a graph that is made by deleting the vertex $v$ and all edges connected to $v$ from the graph $G$ and the graph $G - e$ is a graph that obtained from $G$ by simply removing the edge $e$. Our main results in this section are in obtaining a bound for TD-chromatic number of $G - v$ and $G - e$. To do this, we need to consider some preliminaries.

**Theorem 2.1** ([6])**


Let $P_n$ be a path of order $n \geq 2$. Then

$$\chi^t_d(P_n) = \begin{cases} 2\lfloor \frac{n}{3} \rfloor - 1 & \text{if } n \equiv 1 \pmod{3}, \\ 2\lfloor \frac{n}{3} \rfloor & \text{otherwise.} \end{cases}$$

Let $C_n$ be a cycle of order $n \geq 3$. Then

$$\chi^t_d(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ 4\lfloor \frac{n}{6} \rfloor + r & \text{if } n \neq 4, n \equiv r \pmod{6}, r = 0, 1, 2, 4, \\ 4\lfloor \frac{n}{6} \rfloor + r - 1 & \text{if } n \equiv r \pmod{6}, r = 3, 5. \end{cases}$$

The following theorem gives an upper bound and a lower bound for $\chi^t_d(G - e)$.

**Theorem 2.2** Let $G$ be a connected graph, and $e = vw \in E(G)$ is not a bridge of $G$. Then we have:

$$\chi^t_d(G) - 1 \leq \chi^t_d(G - e) \leq \chi^t_d(G) + 2.$$ 

**Proof.** First we prove the left inequality. We shall present a TD-coloring for $G - e$. If we add the edge $e$ to $G - e$, then we have two cases. If two vertices $v$ and $w$ have the same color in the TD-coloring of $G - e$, then in this case we add a new color, like $i$, to one of them. Since every vertex use the old class for TD-coloring then this is a TD-coloring for $G$. So we have $\chi^t_d(G) \leq \chi^t_d(G - e) + 1$. If two vertices $v$ and $w$ do not have the same color in the TD-coloring of $G - e$, then the TD-coloring of $G - e$ can be a TD-coloring for $G$. So $\chi^t_d(G) \leq \chi^t_d(G - e)$ and therefore we have $\chi^t_d(G) - 1 \leq \chi^t_d(G - e)$.

Now we prove $\chi^t_d(G - e) \leq \chi^t_d(G) + 2$. Suppose that the vertex $v$ has color $i$ and $w$ has color $j$. We have the following cases:

Case 1) The vertex $v$ does not use the color class $j$ and $w$ does not use the color class $i$ in the TD-coloring of $G$. So the TD-coloring of $G$ gives a TD-coloring of $G - e$ and in this case $\chi^t_d(G - e) = \chi^t_d(G)$.

Case 2) The vertex $v$ uses the color class $j$ but $w$ does not use the color class $i$ in the TD-coloring of $G$. Since $v$ used the color class $j$ for the TD-coloring then we have two cases:

(i) If $v$ has some adjacent vertices which have color $j$, then we give the new color $l$ to all of these vertices and this coloring is a TD-coloring for $G - e$.

(ii) If any other vertex does not have color $j$, since $G - e$ is a connected graph, then exists vertex $s$ which is adjacent to $v$. Now we give to $s$ the new color $l$ and this coloring is a TD-coloring for $G - e$.

So for this case, we have $\chi^t_d(G - e) = \chi^t_d(G) + 1$.

Case 3) The vertex $v$ uses the color class $j$ and $w$ uses the color class $i$ in the TD-coloring of $G$. We have three cases:
(i) There are some vertices which are adjacent to \( v \) and have color \( j \). Then we color all of them with color \( l \). And there are some vertices which are adjacent to \( w \) and have color \( i \). We color all of them with color \( k \). So this is a TD-coloring for \( G - e \).

(ii) Any other vertex does not have color \( j \). Then we do the same as Case 2 (ii) and there are some vertices which are adjacent to \( w \) and have color \( i \). Then we do the same as Case 3 (i).

(iii) Any other vertex does not have colors \( i \) and \( j \). Then we do the same as Case 2 (ii) and use two new colors \( l \) and \( k \).

So we have \( \chi^t_d(G - e) \leq \chi^t_d(G) + 2 \). □

**Remark 1.** The lower bound of \( \chi^t_d(G - e) \) in Theorem 2.2 is sharp. It suffices to consider complete graph \( K_3 \) as \( G \). Also the upper bound is sharp, because as we see in Figure 1, \( \chi^t_d(G) = 2 \) and \( \chi^t_d(G - e) = 4 \).

Now we consider the graph \( G - v \), and present a lower bound and an upper bound for the TD-chromatic number of \( G - v \).

**Theorem 2.3** Let \( G \) be a connected graph, and \( v \in V(G) \) is not a cut vertex of \( G \). Then we have:

\[
\chi^t_d(G) - 2 \leq \chi^t_d(G - v) \leq \chi^t_d(G) + \deg(v) - 1.
\]

**Proof.** First we prove \( \chi^t_d(G) - 2 \leq \chi^t_d(G - v) \). We shall present a TD-coloring for \( G - v \). If we add vertex \( v \) and all the corresponding edges to \( G - v \), then it suffices to give the new color \( i \) to vertex \( v \) and the new color \( j \) only to one of the adjacent vertices of \( v \) like \( w \) and do not change all the other colors. Since every vertices except \( v \) and \( w \) use the old classes for TD-coloring and \( v \) uses the color class \( j \) and \( w \) uses the color class \( i \) so we have a TD-coloring of \( G \). Therefore we have \( \chi^t_d(G) \leq \chi^t_d(G - v) + 2 \) and we have the result.

Now we prove \( \chi^t_d(G - v) \leq \chi^t_d(G) + \deg(v) - 1 \). First we give a TD-coloring to \( G \). Suppose that the vertex \( v \) has the color \( i \). So we have the following cases:
Case 1) There is another vertex with color $i$. In this case every vertex uses the old class for TD-coloring and then this is a TD-coloring for $G - v$. So $\chi_d^t(G - v) \leq \chi_d^t(G)$.

Case 2) There is no other vertex with color $i$. In this case we give the new colors $i, a_1, a_2, \ldots, a_{\text{deg}(v)} - 1$ to all the adjacent vertices of $v$. Obviously, this is a TD-coloring for $G - v$. Therefore $\chi_d^t(G - v) \leq \chi_d^t(G) + \text{deg}(v) - 1$. □

**Remark 2.** The lower bound in Theorem 2.3 is sharp. Consider the cycle $C_{10}$, as $G$. For every $v \in V(C_{10})$ we have $C_{10} - v = P_9$ which is a path graph of order 9. Then by the Theorem 2.1 we have $\chi_d^t(C_{10}) = 8$ and $\chi_d^t(P_9) = 6$.

To obtain more results, we consider the corona of $P_n$ and $C_n$ with $K_1$. The following theorem gives the TD-chromatic number of these kind of graphs:

**Theorem 2.4**

(i) For every $n \geq 2$, $\chi_d^t(P_n \circ K_1) = n + 1$.

(ii) For every $n \geq 3$, $\chi_d^t(C_n \circ K_1) = n + 1$.

**Proof.**

(i) We color the $P_n \circ K_1$ with numbers $1, 2, \ldots, n + 1$, as shown in the Figure 2. Observe that, we need $n + 1$ color for TD-coloring. We shall show that we are not able to have TD-coloring with less colors.

![Figure 2: Total dominator coloring of $P_n \circ K_1$ and $C_n \circ K_1$, respectively.](image)

Obviously we have $\chi_d^t(P_2 \circ K_1) = 3$. Now we consider $P_3 \circ K_1$. As we see in Figure 3 we can not give number 1 to vertex $v$, because there is no number to color vertex $w$. Also we can’t consider number 2 for vertex $v$ since the vertex which has color 1 and is adjacent to vertex with number 2, is not adjacent with $v$. Since the coloring is proper, we cannot use color 3 too for this vertex. So we give number 4 to vertex $v$. Between used colors, we can use only number 1 for vertex $w$. Therefore $\chi_d^t(P_3 \circ K_1) = 4$. Similarly, we color $P_i \circ K_1$ from $P_{i-1} \circ K_1$ when $i \geq 3$. Any other kinds of coloring of this graph needs more colors. So we have the result.
We end this section with the following theorem:

**Theorem 2.5** There is a connected graph $G$, and a vertex $v \in V(G)$ which is not a cut vertex of $G$ such that $|\chi^t_d(G) - \chi^t_d(G - v)|$ can be arbitrarily large.

**Proof.** Consider the graph $G$ in Figure 4. We color the vertices $a_1, a_2, \ldots, a_n$ with $\chi^t_d(P_n)$ colors. Then we give the new color $\chi^t_d(P_n) + 1$ to all the adjacent vertices of $v$ and $\chi^t_d(P_n) + 2$ to $v$. Obviously this is a TD-coloring for $G$. So we have:

$$\chi^t_d(G) = 2 + \chi^t_d(P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 1 \pmod{3}, \\ 2\left\lceil \frac{n}{3} \right\rceil + 2 & \text{otherwise.} \end{cases}$$

Now by removing the vertex $v$, we have $G - v = P_n \circ K_1$ and by Theorem 2.4 we have $\chi^t_d(G - v) = n + 1$. So we conclude that $|\chi^t_d(G) - \chi^t_d(G - v)|$ can be arbitrarily large. □

### 3 Vertex and edge contraction

Let $v$ be a vertex in graph $G$. The contraction of $v$ in $G$ denoted by $G/v$ is the graph obtained by deleting $v$ and putting a clique on the (open) neighbourhood of $v$. Note that this operation does not create parallel edges; if two neighbours of $v$ are already adjacent, then they remain simply adjacent (see [8]). In a graph $G$, contraction of an
edge e with endpoints u, v is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph G/e has one less edge than G (R). We denote this graph by G/e. In this section we examine the effects on χ_d(G) when G is modified by an edge contraction and vertex contraction. First we consider edge contraction:

**Theorem 3.1** Let G be a connected graph and e ∈ E(G). Then we have:

$$\chi_d^t(G) - 2 \leq \chi_d^t(G/e) \leq \chi_d^t(G) + 1.$$  

**Proof.** First, we find a TD-coloring for G. Suppose that the end points of e are u and v. The vertex u has the color i and the vertex v has the color j. We give all the used colors in the previous coloring to the vertices E(G) − {u, v}. Now we give the new color k to u = v. Every vertices on the edges of E(G) − {u, v} can uses the previous color class (or even k) in this coloring. The vertex u = v uses the color class which used for u or v unless u used the color class j and v used the color class i. In this case, if there is another vertex with color i, then u = v uses color class i and if there is another vertex with color j, then u = v uses color class j. If any other vertex does not have the color i and j, then it suffices to give color i to one of the adjacent vertices of u (or v) in G. Then this is a TD-coloring for G/e. So we have \(\chi_d^t(G/e) \leq \chi_d^t(G) + 1\).

To find the lower bound, we shall give a TD-coloring to G/e. We add the removed vertex and all the corresponding edges to G/e and keep the old coloring for the new graph. Now we consider the endpoints of e and remove the used color. Now add new colors i and j to these vertices. All the vertices of edges in E(G) − {u, v} can use the previous color class and u can use color class j and v can use color class i. So this is a TD-coloring and we have \(\chi_d^t(G/e) \leq \chi_d^t(G/e) + 2\). Therefore \(\chi_d^t(G) - 2 \leq \chi_d^t(G/e)\). □

**Remark 3.** The bounds in Theorem 3.1 are sharp. For the upper bound consider the cycle C_4 as G and for the lower bound consider cycle C_5.

**Corollary 3.2** Suppose that G is a connected graph and e ∈ E(G) is not a bridge of G. We have:

$$\frac{\chi_d^t(G - e) + \chi_d^t(G/e) - 3}{2} \leq \chi_d^t(G) \leq \frac{\chi_d^t(G - e) + \chi_d^t(G/e) + 3}{2}$$

**Proof.** It follows from Theorems 2.2 and 3.1. □

Now we consider the vertex contraction of graph G and examine the effect on \(\chi_d^t(G)\) when G is modified by this operation:

**Theorem 3.3** Let G be a connected graph and v ∈ V(G). Then we have:

$$\chi_d^t(G) - 2 \leq \chi_d^t(G/v) \leq \chi_d^t(G) + \deg(v) - 1.$$  

**Proof.** First we present a TD-coloring for G. We remove the vertex v and create G/v. We consider one of the adjacent vertices of v like u and do not change its color and
Figure 5: TD-coloring of the graph $G$ and $G \odot v$.

give the new colors $i, i + 1, \ldots, i + \deg(v) - 1$ to other adjacent vertices of $v$. Now each vertex which was not adjacent to $v$ can use the previous color class (or if the color class changed, the new color class we give to adjacent vertices of $v$). Therefore we have

$$\chi^t_d(G/v) \leq \chi^t_d(G) + \deg(v) - 1.$$ 

To find the lower bound, at first we shall give a TD-coloring to $G/v$. We add the vertex $v$, add all the removed edges and remove all the added edges. It suffices to give the vertex $v$ the new color $i$ and only to one of its adjacent vertices like $w$ the new color class $j$. All the vertices which are not adjacent to $v$ can use the previous color classes. All the adjacent vertices of $v$ can use the color class $i$ and $v$ can use the color class $j$. So we have

$$\chi^t_d(G) \leq \chi^t_d(G/v) + 2.$$ 

Therefore we have the result. \hfill \Box

**Remark 4.** The bounds in Theorem 3.3 are sharp. For the upper bound consider the complete bipartite graph $K_{2,4}$ as $G$. We have $\chi^t_d(K_{2,4}) = 2$. By choosing a vertex which is adjacent to four vertices as $v$, we have $K_{2,4}/v = K_5$ which is the complete graph of order 5 and $\chi^t_d(K_5) = 5$. For the lower bound, we consider cycle graph $C_5$. For every $v \in V(C_5)$ we have $C_5/v = C_4$. Now by Theorem 2.1 we have the result.

**Corollary 3.4** Let $G$ be a connected graph. For every $v \in V(G)$ which is not cut vertex of $G$, we have:

$$\frac{\chi_d^t(G-v) + \chi^t_d(G/v)}{2} - \deg(v) + 1 \leq \chi^t_d(G) \leq \frac{\chi^t_d(G-v) + \chi^t_d(G/v)}{2} + 2.$$ 

**Proof.** It follows from Theorems 2.3 and 3.3. \hfill \Box

Here we consider another operation on vertex of a graph $G$ and examine the effects on $\chi^t_d(G)$ when we do this operation. We denote by $G \odot v$ the graph obtained from $G$ by the removal of all edges between any pair of neighbors of $v$, note $v$ is not removed from the graph \cite{1}. The following theorem gives upper bound and lower bound for $\chi^t_d(G \odot v)$.

**Theorem 3.5** Let $G$ be a connected graph and $v \in V(G)$. Then we have:

$$\chi^t_d(G) - \deg(v) + 1 \leq \chi^t_d(G \odot v) \leq \chi^t_d(G) + 1.$$
Proof. First we prove $\chi_d^t(G \odot v) \leq \chi_d^t(G) + 1$. We give a TD-coloring for the graph $G$. Suppose that the vertex $v$ has the color $i$. We have the following cases:

Case 1) The color $i$ uses only for the vertex $v$. In this case, adjacent vertices of the vertex $v$, can use the color class $i$ and all the other vertices can use the old color class. So we have $\chi_d^t(G \odot v) \leq \chi_d^t(G)$.

Case 2) The color $i$ uses for another vertex except $v$. In this case, we give the new color $j$ to all of these vertices (except $v$). This is a TD-coloring for $G \odot v$, because if a vertex is adjacent to $v$, it can use the color class $i$ and all the other vertices can use old color class and if the old color class changes to $j$ can use $j$ as new color class. So we have $\chi_d^t(G \odot v) \leq \chi_d^t(G) + 1$.

Now we prove $\chi_d^t(G) - \deg(v) + 1 \leq \chi_d^t(G \odot v)$. Consider the graph $G \odot v$ and shall find a TD-coloring for it. We make $G$ from $G \odot v$ and just change the color of all the adjacent vertices of $v$ except one of them like $w$ to the new colors $a_1, a_2, \ldots, a_{\deg(v)} - 1$ and do not change the color of $v, w$ and other vertices. This is a TD-coloring for $G$, because $v$ can use the the color class $a_1$. Adjacent vertices of $v$, can use the old color class of the TD-coloring of $G \odot v$, and other vertices can use old color class and if the old color classes changes to $a_1$ or $a_2$ or $\ldots$ or $a_{\deg(v)} - 1$ can use $a_1$ or $a_2$ or $\ldots$ or $a_{\deg(v)} - 1$ as new color classes. So we have $\chi_d^t(G) \leq \chi_d^t(G \odot v) + deg(v) - 1$. Therefore we have the result. □

Remark 5. The bounds in Theorem 3.5 are sharp. For the upper bound consider the graph $G$ in Figure 5. It is easy to see that these colorings are TD-coloring. For the lower bound consider to the complete graph $K_n$ as $G$ $(n \geq 3)$. $\chi_d^t(K_n) = n$. Now for every $v \in V(K_n)$, $K_n \odot v$ is the star graph $S_n$ and we have $\chi_d^t(S_n) = 2$. By this example we have the following result:

Corollary 3.6 There is a connected graph $G$ and $v \in V(G)$ such that $\frac{\chi_d^t(G)}{\chi_d^t(G \odot v)}$ can be arbitrarily large.

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