Developed turbulence: From full simulations to full mode reductions

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Developed Navier-Stokes turbulence is simulated with varying wavevector mode reductions. The flatness and the skewness of the velocity derivative depend on the degree of mode reduction. They show a crossover towards the value of the full numerical simulation when the viscous subrange starts to be resolved. The intermittency corrections of the scaling exponents \( \zeta_p \) of the \( p^{th} \) order velocity structure functions seem to depend mainly on the proper resolution of the inertial subrange. Universal scaling properties (i.e., independent of the degree of mode reduction) are found for the relative scaling exponents \( \rho_{ps} = (\zeta_p - \zeta_p/3)/(\zeta_q - \zeta_q/3) \).

Even today fully developed turbulence is hard to access through full numerical simulations of the Navier-Stokes equations because the number of degrees of freedom increases with the Taylor-Reynolds number roughly as \( \text{Re}^{9/2} \). Consequently, models and approximations of the Navier-Stokes dynamics with a reduced number of degrees of freedom are considered. Models which embody the cascade type structure of turbulence enjoyed increasing popularity in recent years, e.g., the so called GOY model. Closer to the Navier-Stokes dynamics it is its reduced wavevector set approximation (REWA) that allows us a reduced, geometrically scaling subset of wavevectors on which the Navier-Stokes equation is solved. Very high Taylor-Reynolds numbers up to \( \text{Re} \geq 7 \times 10^4 \) can be achieved.

However, a priori it is not clear whether these models and approximations are in the same universality class as the Navier-Stokes dynamics itself, as small scale structures corresponding to the high \( k \) modes are not be fully resolved. If inertial subrange (ISR) scaling properties depend on details of the viscous subrange (VSR) as speculated for the GOY model, a cascade type approach towards fully developed turbulence may not give the correct inertial range scaling properties. Moreover – and as we will see more importantly – in these models the phase space has a different representation than in 3D Navier-Stokes turbulence. Indeed, detailed REWA calculations for the scaling exponents \( \zeta_p \) of the \( p^{th} \) order longitudinal velocity structure functions

\[
D_i^{(p)}(r) = \langle (u_i(x + re_i) - u_i(x))^p \rangle \propto r^{\delta_p},
\]

show much smaller (but non vanishing) deviations \( \delta_p = \zeta_p - p/3 \) from their classical values \( \zeta_p = p/3 \) (“K41”) than those from experimental measurements or full numerical simulations (for Reynolds numbers up to \( \text{Re} \approx 210 \)). Also, the flatness \( F_i = \langle (\partial_i u_i)^2 \rangle / \langle (\partial_i u_i)^4 \rangle \) for all \( \text{Re} \), in contrast to experiments and full simulations where it seems to increase with \( \text{Re} \). Analogous results hold for the skewness \( S_i = \langle (u_i(x + re_i) - u_i(x))^3 \rangle / \langle (\partial_i u_i)^2 \rangle^{3/2} \). On the other hand, REWA may well represent the “correct” large \( \text{Re} \) limit where \( F \) and \( S \) are speculated to become independent of \( \text{Re} \).

In this letter we systematically analyse how the scaling properties change with an increasing degree of wavevector mode reduction, i.e., we examine the transition from full numerical simulations to reduced wavevector set approximations. Since full simulations are possible only for low \( \text{Re} \) values, the present calculations are restricted accordingly, even though REWA was constructed for the large \( \text{Re} \) limit. There is at most a short ISR. However, the extended self-similarity method (ESS) allows us to extract scaling exponents.

The aim of the work is to better understand the origin of intermittency scaling corrections. Two views are discussed: The meanwhile classical multifractal picture in which intermittent fluctuations build up in the ISR and the Leveque-Shen picture in which ISR quantities depend on VSR properties. The importance of the latter mechanism could be shown for the GOY model. However, for 3D Navier-Stokes turbulence with different kind of hyperviscosity, no dependence of the ISR scaling exponents \( \zeta_p \) on the kind of hyperviscosity could be detected. Our analysis seems to support this result. VSR effects on \( \zeta_p \) could not be identified. Our interpretation is that the proper local phase space resolution is of prime importance for the correct representation of the scaling corrections \( \delta_p \).

We now describe our analysis in detail. The 3D incompressible Navier-Stokes equations are numerically solved on a \( N^3 \) grid with periodic boundary conditions. Spherical truncation is used to reduce aliasing. We force the system on the largest scale (wavevectors \( k = (0,0,1)/L \) and permutations thereof) as e.g. described in ref. \( 6 \). Units are fixed by picking the length scale \( L = 1 \) and the average energy input rate (= the energy dissipation rate) \( \epsilon = 1 \). The Taylor-Reynolds number is defined as \( \text{Re} = u_{1,\text{rms}} \lambda / \nu \), where \( \lambda = u_{1,\text{rms}} / (\partial_1 u_1)_{\text{rms}} \) is the Taylor length and \( \nu \) the viscosity. Our results refer to \( N = 60 \) and \( \nu = 0.009 \), corresponding to a resolution of scales \( r > 2\pi L / N \approx 3.6n \) and \( \text{Re} \approx 100 \). Time integrations of about 60 large eddy turnover times are performed. Averages are taken over space and time. We also did shorter runs for \( N = 80 \) and longer runs for \( N = 48 \) which gave the same results.

As our key parameter we now introduce the wavenum-
ber \( k_B \) with \( 1 < k_B \leq k_{\text{max}} = N/2 \), characterizing the degree of mode reduction: For a simulation with given \( k_B \) all wavevectors with \( |k| \leq k_B \) and scaled replica \( 2^l k \), \( l = 1, 2, 3, \ldots \), thereof are considered: the mode amplitudes of the remaining wavevectors are put to zero. The choice \( k_B = k_{\text{max}} = N/2 \) corresponds to a full simulation, \( k_B \approx 2 \) is our former REWA calculation [6]. For those calculations a pure spectral code could be used; here, because of the huge increase of couplings, a pseudospectral code as described in [18,14] was employed.

The error bars express the statistical differences of the values \( S_i \) and \( F_i \) for the three space directions \( i = 1, 2, 3 \). With the chosen lattice resolution \( S \) and \( F \) roughly reach their saturation value. The crosses on the very left refer to the REWA calculation with \( k_B = 3 \), but with a full VSR resolution for \( k > 9 \).

Figure 1 shows the skewness \( S \) and the flatness \( F \) as a function of \( k_B \). A crossover at \( k_D \approx 1/(4\eta) \approx 9 \) can be identified. Here, \( k_D \) denotes the wave number with maximal dissipation rate, where massive viscous damping starts in the spectrum. For \( k_B < k_D \) the flatness and the skewness essentially remain on their REWA values. But at \( k_B > k_D \) they start to drastically increase towards their saturated values corresponding to the full simulation.

Figure 2 shows the compensated structure function \( D^{(6)} (r)/[D^{(3)}(r)]^2 \) vs \( D^{(3)}(r) \). This kind of plot allows for a better detection of local deviations from scaling than the standard ESS [3] plot \( D^{(6)} \) vs \( D^{(3)} \). We find that for \( k_B \geq 5 \) the value \( \delta_{\zeta_0} \approx -0.22 \) is always a good fit in the large \( r \) regime between \( 2\pi/k_B \) and \( L \). This scaling regime shrinks for decreasing \( k_B \) and vanishes below \( k_B \approx 5 \) as then \( 2\pi/k_B \) essentially collapses with the external length scale \( L \).

Figure 2 suggests that at least for small \( Re_\lambda \) for \( 10\eta < 2\pi/k_B < L \) (a condition which never is really reached in our small \( Re_\lambda \) simulations; the simulation for \( k_B = 6 \) is closest to it, see in particular figure 2b) there are three ranges: The (underresolved) VSR \( r < 10\eta \) where of course \( D^{(6)} \propto (D^{(3)})^2 \), a REWA ISR in the underresolved regime \( [10\eta, 2\pi/k_B] \) with very small but nonvanishing (note the nonzero slope in figure 2b) in that regime intermittency corrections \( \delta_{\zeta_0} \), and the fully mode resolved Navier-Stokes ISR \( [2\pi/k_B, L] \) with the intermittency corrections \( \delta_{\zeta_0} = -0.22 \) as in full numerical simulations. This prompts the conclusion that it is the local phase space resolution and not a proper VSR resolution which is essential for the correct representation of scaling corrections.

To further support this statement, we performed a simulation with \( k_B = 3 \), but in addition a full resolution of all modes \( k > 9 \), i.e., of the VSR. This curve is also shown in figure 2a. Indeed, there are hardly any scaling corrections in the ISR, \( \delta_{\zeta_0} \approx 0 \). On the other hand, as expected both the flatness and the skewness are much bigger than in the REWA calculations [3] as now the VSR is better resolved. We added the corresponding data points in figure 2b.

As is well known, the Navier-Stokes intermittency corrections are well fitted by the She-Leveque (SL) model [19,20,21].
\[
\zeta_p = \frac{p}{3} - C_0 \left( \frac{p}{3}(1 - \beta^3) - (1 - \beta^p) \right). 
\]

with the parameters \(\beta\) and \(C_0\), which in ref. \[10\] were suggested to be \(\beta = (2/3)^{1/3}\) and \(C_0 = 2\). In eq. \[6\] we already used the restriction \(\zeta_3 = 1\) to eliminate a third parameter which was introduced in the original work \[11,22\]. The parameter \(C_0\) was related to the rate at which the probability to find the most intermittent events decays in the large \(k\) limit and also interpreted as the codimension of the dissipative structures \[11,22\]. If in 3D Navier-Stokes turbulence these are 1D filaments, we have \(C_0 = 2\). This interpretation also works for REWA turbulence \((k_B = 3)\) \[13\]: The dissipative structures are nearly 3D because of the lack of large wavevector resolution. Therefore, \(C_0 \approx 0\) and according to \[2\], \(\zeta_3 = p/3\), in agreement with the numerical results \[7,9\]. However, the interpretation of \(C_0\) as codimension of the dissipative structures seems to be at variance with the simulation of Chen and Cao \[20\]. The scaling corrections \(\delta\zeta_p\) are hardly presented here, the ISR scaling properties of minor importance. How come that the energy flux reaches so far in the ISR for the GOY model but apparently not for 3D Navier-Stokes turbulence? We speculate that in 3D the phases of the modes are subjected to far more fluctuations than in the 1D GOY model. Therefore, coherences get destroyed easier. Some coherence however must remain, otherwise no energy could be transported downscape.

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FIG. 4. Relative scaling exponent $\rho_{6,4}$ as a function of our mode reduction parameter $k_B$. Here, $\eta$ is the Kolmogorov inner scale. The error bars take care of both the quality of the linear regression of the scaling laws and the deviations from isotropy. The triangle on the very left refers to the REWA calculation from ref. [3] for much larger $Re_A$. The dotted line shows the SL prediction $[19]$ from equation (5) with $\beta = (2/3)^{1/3} \approx 0.8736$, i.e., $\rho_{6,4} = 4.14$, the dashed one the prediction from the LN model $\rho_{6,4} = 4.5$, and the solid line is the best fit $\rho_{6,4} = 4.33 \pm 0.05$, implying $\beta = 0.94 \pm 0.02$ in the SL model, very close to the value $\beta = (7/9)^{1/3} \approx 0.92$ suggested in ref. [1]. For different pairs $(p,q)$ the universality and the agreement with SL and LN is correspondingly good.

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