THE OPENNESS CONJECTURE FOR PROJECTIVE MANIFOLDS

BO BERNDTSSON

ABSTRACT. We give a proof of the openness conjecture of Demailly and Kollár for positively curved singular metrics on ample line bundles over projective varieties. As a corollary it follows that the openness conjecture for plurisubharmonic functions with isolated singularities holds.

1. INTRODUCTION

Let \( u \) be a plurisubharmonic function defined in a neighbourhood of the origin of \( \mathbb{C}^n \) such that \( e^{-u} \) lies in \( L^1 \). The openness conjecture, first proposed by Demailly and Kollár in [5], says that then there is a number \( p > 1 \) such that \( e^{-u} \) lies in \( L^p \), possibly after shrinking the neighbourhood. This conjecture has attracted a good deal of attention; in particular it has been completely proved in dimension 2 by Favre and Jonsson, [6]. In arbitrary dimension it is still open (no pun intended), but has been reduced to a purely algebraic statement in [7].

In this paper we will prove a global version of the openness conjecture for metrics on line bundles over projective manifolds.

**Theorem 1.1.** Let \( X \) be a projective manifold and let \( L \) be a positive line bundle over \( X \). Let \( \phi \) be a possibly singular metric of nonnegative curvature on \( L \), and let \( \phi_0 \) be a smooth positively curved metric on \( L \). Assume that

\[
\int_X e^{-(\phi - \phi_0)} d\mu < \infty
\]

where \( \mu \) is some smooth volume form on \( X \). Then there is a number \( p > 1 \) such that

\[
\int_X e^{-p(\phi - \phi_0)} d\mu < \infty.
\]

In particular the theorem applies to projective space, \( \mathbb{P}^n \). By a simple max-construction one can show that any local plurisubharmonic function with an isolated singularity (by this we mean that \( \phi(0) = -\infty \) and \( \phi \) is bounded for \( |z| > 1/2 \)) can be extended to a metric with positive curvature on some \( \mathcal{O}(k) \) over \( \mathbb{P}^n \), with no additional singularities. As a consequence we see that the openness conjecture holds in any dimension for functions with isolated singularities.

**Corollary 1.2.** Let \( u \) be a plurisubharmonic function in the unit ball of \( \mathbb{C}^n \), \( B \), with an isolated singularity at 0. Assume that

\[
\int_B e^{-u} < \infty.
\]

Then there is a number \( p > 1 \) such that

\[
\int_B e^{-pu} < \infty.
\]
The proof of the theorem is inspired by a result from [1]. There we proved that the Schwarz symmetrization, \( u^* \), of an \( S^1 \)-invariant plurisubharmonic function in the ball, \( u \), is again plurisubharmonic. Here \( S^1 \)-invariance means that \( u(e^{i\theta}z) = u(z) \) for any \( e^{i\theta} \) on \( S^1 \). Since the Schwarz symmetrization of \( u \) is equidistributed with \( u \), it holds that

\[
\int_B F(u^*) = \int_B F(u)
\]

for any (measurable) function \( F \). Choosing \( F(t) = e^{-t} \) and \( F(t) = e^{-\epsilon t} \), this reduces the openness problem for \( S^1 \)-invariant functions to the case of radial functions. As a consequence we get

**Proposition 1.3.** The openness conjecture holds for any \( S^1 \)-invariant plurisubharmonic function in the unit ball.

The proof of Proposition 1.3, via the result from [1], depends ultimately on a complex variant of the Brunn-Minkowski inequality from [3]. The argument can be rephrased in the following form. A consequence of the 'Brunn-Minkowski'-inequality is that if \( u \) is plurisubharmonic and \( S^1 \)-invariant, the volume of the sublevel sets

\[
\Omega(s) := \{ z; u(z) < -s \}
\]

is a logconcave function of \( s \). The integral of \( e^{-u} \) can be written as

\[
\int_0^\infty e^s|\Omega(s)|ds + \omega_n,
\]

with \( \omega_n \) the volume of the unit ball, and the logconcavity is the key to studying the convergence of this integral. In fact, if \( |\Omega(s)| \) is logconcave, the integral (1.1) converges (if and) only if \( |\Omega(s)| \) decreases like \( e^{-(1+\epsilon)s} \) at infinity (cf Theorem 3.1).

In the situation of Theorem 1.1, instead of looking at just volumes of sets or integrals of functions, we look at the \( L^2 \)-norms on the space \( H^0(X, K_X + kL) \) induced by our metric \( \phi \) in section 2. We then find a representation of such an \( L^2 \)-norm as an integral over \((0, \infty)\) of weaker norms depending on the variable \( s \). These weaker norms have a property analogous to logconcavity - they define an hermitean metric on a certain vector bundle of positive curvature. This positivity property is finally shown in the last section to imply Theorem 1.1.

### 2. Hermitean norms on \( H^0(K_X + kL) \)

Let \( X \) be a projective manifold and let \( L \) be a positive line bundle over \( X \). We will consider a possibly singular metric, \( \phi \) with \( i\partial\bar{\partial}\phi \geq 0 \), and we also let \( \phi_0 \) be a smooth positively curved reference metric on \( L \). For \( \sigma \), an element in \( H^0(K_X + kL) \) we define the \( L^2 \)-norm

\[
\|\sigma\|^2 := c_n \int_X \sigma \wedge \bar{\sigma} e^{-(\phi-(k-1)\phi_0)} = c_n \int_X \sigma \wedge \bar{\sigma} e^{-(\phi-\phi_0)-k\phi_0}.
\]

Clearly, this norm is finite for any \( \sigma \) in \( H^0(K_X + kL) \) if and only if \( e^{-\phi} \) is locally integrable, provided that \( k \) has been chosen so large that \( K_X + kL \) is base point free. For \( s > 0 \) we define a regularization of \( \phi \) by

\[
\phi_s := \max(\phi + s, \phi_0).
\]
If we normalize so that $\phi \leq \phi_0$, we get that $\phi_s = \phi_0$ for $s = 0$ so there is no conflict in notation. With $\phi_s$ we associate the norms

$$\|\sigma\|^2_s := c_n \int_X \sigma \wedge \bar{\sigma} e^{-2\phi_s - (k-2)\phi_0} = c_n \int_X \sigma \wedge \bar{\sigma} e^{-2(\phi_s - \phi_0) - k\phi_0}. \tag{2.3}$$

(Notice the factor 2 in front of $(\phi_s - \phi_0)$ as opposed to 1 in the formula in (2.1).)

**Proposition 2.1.** If $\sigma$ is an element of $H^0(K_X + kL)$,

$$2\|\sigma\|^2 = \int_0^\infty e^s \|\sigma\|^2_s ds + \|\sigma\|^2_0,$$

For the proof we use the following lemma.

**Lemma 2.2.** If $x < 0$

$$\int_0^\infty e^s e^{-2\max(x+s,0)} ds + 1 = 2e^{-x}.$$

More generally, if $0 < p < 2$,

$$\int_0^\infty e^{ps} e^{-2\max(x+s,0)} ds + 1/p = C_p e^{-px}.$$

**Proof.**

$$\int_0^\infty e^s e^{-2\max(x+s,0)} ds = \int_0^{-x} e^s ds + e^{-2x} \int_{-x}^\infty e^{-s} ds = 2e^{-x} - 1.$$

This proves the first part; the second part is of course also proved by direct computation. \hfill \square

To prove the proposition we use

$$\|\sigma\|^2_s = c_n \int_X \sigma \wedge \bar{\sigma} e^{-2(\phi_s - \phi_0) - k\phi_0}.$$

Note that $\phi_s - \phi_0 = \max(\phi - \phi_0 + s, 0)$. By the lemma

$$\int_0^\infty e^s \|\sigma\|^2_s ds = 2c_n \int_X \sigma \wedge \bar{\sigma} e^{-\phi_s - \phi_0} - c_n \int_X \sigma \wedge \bar{\sigma} e^{-k\phi_0}.$$

This completes the proof. \hfill \square

For later reference we note that by the last part of the lemma, if $0 < p < 2$,

$$\int_0^\infty e^{ps} \|\sigma\|^2_s ds = C_p c_n \int_X \sigma \wedge \bar{\sigma} e^{-p(\phi_s - \phi_0) - k\phi_0} - (1/p) \|\sigma\|^2_0. \tag{2.4}$$

We finally quote a particular case of a result from [4] that is the most important ingredient in the proof of Theorem 1.1. We let $D$ be a domain in $\mathbb{C}$ and let $E := H^0(X, K_X + F)$, where $F$ is a positive line bundle over $X$. Denote by $\mathcal{E} := E \times_{D} E$, the trivial vector bundle with fiber $E$ over $D$. Let for $\zeta$ in $D$, $\psi_\zeta$ be a metric on $F$. 

Theorem 2.3. With notation as above, define an hermitean metric on $E$ by

$$(2.5) \quad \|\sigma\|_c^2 := c_n \int_X \sigma \wedge \bar{\sigma} e^{-\psi_c}.$$ 

Assume that $i\partial\bar{\partial}X \psi_c \geq 0$, i.e., $\psi_c$ is plurisubharmonic on $D \times X$. Then the curvature of the metric (2.5) on $E$ is nonnegative.

This result applies in particular to our present setting, with $D$ equal to the right half plane and $F = kL$. Then

$$\psi_c = 2\phi_s + (k - 2)\phi_0$$

for $s = \text{Re} \zeta$. By the definition (2.2), $\phi_{\text{Re} \zeta}$ and hence $\psi_c$ are plurisubharmonic on all of $D \times X$. Hence, by Theorem 2.3, the norms $\|\sigma\|_s$ on $H^0(K_X + kL)$ define a metric on $E$ of positive curvature. Moreover, they depend only on $s = \text{Re} \zeta$. It is easily checked that if all $\|\sigma\|_s$ can be simultaneously diagonalised in some fixed basis, with diagonal entries $\omega_j(s)$, then the positivity of the curvature means that all $\omega_j(s)$ are logconcave. This is our substitute for the logconcavity of $|\Omega(s)|$, mentioned in the introduction.

3. Integrals of quadratic forms and the proof of Theorem 1.1

We continue the discussion from the previous section and specialize to $D = U$, the right half plane.

Theorem 3.1. Let $\| \cdot \|_s$ be a family of Hilbert norms on some finite dimensional vector space $E$ such that the induced hermitean metric on the trivial vector bundle $E := U \times E$, $\| \cdot \|_{\text{Re} \zeta}$, has positive curvature over $U$. Then the integrals

$$\int_0^\infty e^s \|\sigma\|_s^2 ds$$

converge for all $\sigma$ in $E$ if and only if there are $\epsilon > 0$ and $s_0$ such that

$$(3.1) \quad \|\sigma\|_s^2 \leq e^{-(1+\epsilon)s} \|\sigma\|_0^2$$

for $s > s_0$ and any $\sigma$ in $E$.

Proof. One direction is of course clear; if (3.1) holds the integral converges. So, assume that (3.1) does not hold. Then, for any $\epsilon > 0$ we can find $s > 1/\epsilon$ and some $\sigma$ in $E$ such that

$$(3.2) \quad \|\sigma\|_s^2 > e^{-(1+\epsilon)s} \|\sigma\|_0^2.$$ 

By the spectral theorem we can choose an orthonormal basis, $e_j$, for $\| \cdot \|_0$ that diagonalises $\| \cdot \|_s$. If

$$\sigma = \sum c_j e_j$$

we can write

$$\|\sigma\|_0^2 = \sum |c_j|^2 \quad \text{and} \quad \|\sigma\|_s^2 = \sum |c_j|^2 e^{s\lambda_j}$$

since the eigenvalues are positive. By (3.2), at least one $\lambda_j$ - say $\lambda_0$ - is larger than $-(1+\epsilon)$.
We now define another family of norms $| \cdot |_t$ for $0 \leq t \leq s$ by

$$|\sigma|_t^2 = \sum |c_j|^2 e^{t\lambda_j}.$$  

Since $e_j(\zeta) = e_j e^{-\lambda_j \zeta/2}$ defines a global holomorphic orthonormal frame, $|\sigma|_{Re \zeta}^2$ defines a hermitian metric on $E$ of zero curvature. Moreover the new norms agree with the previous ones for $t = 0$ and $t = s$. By the maximum principle for positive metrics (see e.g. [2], Lemma 8.11) we have

$$\|\sigma\|_t^2 \geq |\sigma|_t^2$$

for $0 \leq t \leq s$. Choose $\sigma = e_0$. Then we conclude that

$$\|e_0\|_t^2 \geq \|e_0\|_{t_0}^2 e^{-(1+\epsilon)t}.$$  

Therefore

$$\int_0^s e^{t} \|e_0\|_t^2 dt \geq \|e_0\|_{t_0}^2 \int_0^s e^{-t} dt \geq \|e_0\|_{t_0}^2 (1 - e^{-1})/\epsilon,$$

since $s > 1/\epsilon$. Since $\epsilon$ can be taken arbitrarily small we see that there is no constant such that

$$\int_0^\epsilon e^s \|\sigma\|_s^2 ds \leq C\|\sigma\|_{t_0}^2$$

for all $\sigma$ in $E$. Since all norms on a finite dimensional vector space are equivalent, the integral in the right hand side cannot converge, which completes the proof.

□

We can now combine Proposition 2.1 and Theorem 3.1 to prove Theorem 1.1. The hypothesis implies that

$$\|\sigma\|^2 = c_n \int_X \sigma \wedge \bar{\sigma} e^{-(\phi - \phi_0)} e^{-k\phi_0}$$

is finite for any $\sigma$ in $H^0(K_X + kL)$. By Proposition 2.1 this implies that the integrals

$$\int_0^\infty e^s \|\sigma\|_s^2 ds$$

converge for any $\sigma$ in $H^0(K_X + kL)$. By Theorem 3.1 there are $\epsilon > 0$ and $s_0$ such that

$$\|\sigma\|_s^2 \leq e^{-(1+\epsilon)s} \|\sigma\|_{t_0}^2$$

if $s > s_0$. Hence there is some $p > 1$ such that

$$\int_0^\infty e^{ps} \|\sigma\|_s^2 ds$$

converges as well. By (2.4) we then have that

$$c_n \int_X \sigma \wedge \bar{\sigma} e^{-p(\phi - \phi_0)} e^{-k\phi_0}$$

is finite. If $k$ is so large that $K_X + kL$ is base point free this gives that

$$\int_X e^{-p(\phi - \phi_0)} d\mu < \infty,$$
so we are done.

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DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY,
E-mail address: bob@chalmers.se