Local coordinates in problem of the orbital stability of pendulum-like oscillations of a heavy rigid body in the Bobylev–Steklov case

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Abstract. In this paper a method is proposed for constructing a nonlinear canonical change of variables, which makes it possible to avoid the singularity when introducing local coordinates in the neighborhood of periodic orbit of the autonomous Hamiltonian system with two degrees of freedom. As an application of this method, orbital stability of pendulum-like oscillations of a heavy rigid body with a fixed point in the Bobylev-Steklov case is investigated. In particular, it is performed a nonlinear study of the orbital stability for the so-called case of degeneracy, when it is necessary to take into account terms of order six in the Hamiltonian expansion in a neighborhood of the unperturbed periodic orbit.

1. Introduction
In classical and celestial mechanics one has often to deal with autonomous Hamiltonian systems, which have natural families of periodic motions. These motions are Lyapunov unstable with respect to perturbations of canonical variables. However, the problem of their orbital stability is of great interest both from theoretical point of view and for applications. In accordance with the general approach for investigating the stability of Hamiltonian systems, it is necessary to introduce the so-called local variables [1] in a neighborhood of periodic motions and to obtain equations of perturbed motion. A rigorous stability analysis of the perturbed system can be performed by using normal form methods [2,3] and KAM theory [4,5]. Introducing local variables in a neighborhood of periodic motions and obtaining an explicit form of equations of perturbed motion may turn out to be not easy. In fact, there are a few approaches for introducing local variables. If an explicit analytic form of the periodic motions is known, then local variables can be introduced by constructing a canonical change of variables allowing a transformation to action-angle variables in a region of periodic motion. This approach has been used, for instance, in some problems of rigid body dynamics [6-12]. However, some technical difficulties can arise in the application of this approach. These difficulties concern both constructing the canonical change mentioned and calculating explicit expressions for coefficients of series expansion of Hamiltonian in a neighborhood of the unperturbed periodic motion.
An approach for introducing local variables based on constructing a linear canonical change of variables was developed in [13]. By using such an approach one can avoid the above-mentioned technical difficulties and solve a wide range of problems of orbital stability. However, in some cases this approach cannot be also applied because of a singularity, which appears in the coefficients of the corresponding linear change of variables. For example, this obstacle prevents application of this approach for introducing local variables in a neighborhood of pendulum-like oscillations.

If equations of motion are integrated by quadratures, then the technique based on using topological methods can be applied for study the orbital stability of periodic motions [14].

In this paper a method, which allows to avoid a singularity when introducing local variables in a neighborhood of a periodic orbit, is proposed. In accordance with this method the local variables are introduced by means of a nonlinear canonical transformation. Such a transformation can be constructed in a form of series in powers of the new variable, which describes the normal perturbation of the periodic orbit in the invariant manifold. A constructive algorithm for calculating the coefficients of these series up to any finite order is given in the section 2. In sections 3 the above-mentioned algorithm is applied to introduce local variables in a neighborhood of pendulum-like oscillations of a heavy rigid body with a fixed point in the Bobylev–Steklov case. In sections 4 the orbital stability is investigated for the so-called case of degeneracy, when it is necessary to take into account the terms of order six in the Hamiltonian expansion.

2. On local coordinates in a neighborhood of periodic motion

Let us consider a canonical system of differential equations describing motions of a mechanical system with two degrees of freedom

\[
\begin{align*}
\frac{dq_i}{dt} & = \frac{\partial H}{\partial p_i}, \\
\frac{dp_i}{dt} & = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2.
\end{align*}
\]

(1)

and assume, it has a one-parameter family of periodic solutions of the form

\[
q_1 = f(t), \quad p_1 = g(t), \quad q_2 = p_2 = 0.
\]

(2)

The existence of the solution (2) means that the expansion of the Hamiltonian as a series in powers of the canonical variables \(q_2, p_2\) does not include the terms of the first order, i.e.

\[
H = H_0(q_1, p_1) + \sum_{i+j=2}^{\infty} h_{ij}(q_1, p_1)q_i^j, p_2^j.
\]

(3)

By change of canonical variables \(q_1, p_1 \to \xi, \eta\) the family of periodic solutions (2) can be represented in the form

\[
\xi = t + \xi(0), \quad \eta = \quad q_2 = \quad p_2 = 0.
\]

(4)

The problem of the orbital stability of the periodic solution (4) is equivalent to the problem of Lyapunov stability with respect to the variables \(\eta, q_2, p_2\), which we shall call local coordinates. The variables \(\xi, \eta\) can be introduced by different ways. The proper choice of these variables can essentially simplify further investigation of the system behavior in a neighborhood of its periodic motion. A method for introducing the local coordinates has been proposed in [15]. The idea of this method is to construct the canonical change of variables \(q_1, p_1 \to \xi, \eta\) in the form of series in powers of \(\eta\):

\[
\begin{align*}
q_1 & = f(\xi) + a_1(\xi)\eta + a_2(\xi)\eta^2 + a_3(\xi)\eta^3 + \ldots, \\
p_1 & = g(\xi) + b_1(\xi)\eta + b_2(\xi)\eta^2 + b_3(\xi)\eta^3 + \ldots.
\end{align*}
\]

(5)

The coefficients \(a_i(\xi), b_i(\xi)\) \((i = 1, 2, \ldots)\) can be always chosen such that the change of variables is canonical and periodic with respect to \(\xi\). Moreover, such a choice is not unique. In particular, to avoid
a singularity in (5) the coefficients $a_i(\xi), b_i(\xi) \ (i = 1,2,3)$ can be chosen in the following explicit form [15]

$$
a_1(\xi) = \frac{1}{V^2} \frac{dg}{d\xi}, \quad b_1(\xi) = \frac{1}{V^2} \frac{df}{d\xi}, \quad a_2(\xi) = \frac{1}{2V^4} \frac{d^2f}{d\xi^2}, \quad b_2(\xi) = \frac{1}{2V^4} \frac{d^2g}{d\xi^2},
$$

$$
a_3(\xi) = \frac{1}{6V^4} \frac{d^3g}{d\xi^3} - \frac{W^2}{3V^8} \frac{dg}{d\xi}, \quad b_3(\xi) = -\frac{1}{6V^4} \frac{d^3f}{d\xi^3} + \frac{W^2}{3V^8} \frac{df}{d\xi},
$$

(6)

where

$$V^2 = \left( \frac{df}{d\xi} \right)^2 + \left( \frac{dg}{d\xi} \right)^2, \quad W^2 = \left( \frac{d^2f}{d\xi^2} \right)^2 + \left( \frac{d^2g}{d\xi^2} \right)^2.
$$

(7)

Let us note that the function $V^2$ cannot vanish. Indeed, it follows from the equation $V^2 = 0$ that $\frac{df}{d\xi} = \frac{dg}{d\xi} = 0$. However, this is impossible since the functions $f(\xi)$ and $g(\xi)$ define the periodic solution to the system (1) and their derivatives cannot vanish simultaneously. Making a change of variables by formulae (5), we obtain the following expansion of the Hamiltonian function in the neighborhood of $\eta = q_2 = p_2 = 0$:

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4.
$$

(8)

where

$$\Gamma_2 = \eta + \varphi_2(q_2,p_2,\xi), \quad \Gamma_3 = \varphi_3(q_2,p_2,\xi),
$$

$$\Gamma_4 = \chi(\xi)\eta^2 + \psi_2(q_2,p_2,\xi)\eta + \varphi_4(q_2,p_2,\xi).
$$

(9)

The functions $\chi, \psi_2, \varphi_i \ (i = 2,3,4)$ are given by the following explicit expressions:

$$\chi(\xi) = \frac{1}{2V^2} \left\{ \left( \frac{\partial^2H_0}{\partial q_1^2} - \frac{\partial^2H_0}{\partial p_1^2} \right) \left( \frac{\partial^2H_0}{\partial q_1^2} - \frac{\partial^2H_0}{\partial p_1^2} \right) + 4 \frac{\partial^2H_0}{\partial q_1 \partial p_1} \frac{\partial^2H_0}{\partial q_1 \partial p_1} \right\},
$$

(10)

$$\psi_2(q_2, p_2, \xi) = \frac{1}{V^2} \sum_{i+j=2} \frac{\partial h_{ij}}{\partial p_1} \frac{\partial h_{0}}{\partial p_1} - \frac{\partial h_{ij}}{\partial q_1} \frac{\partial h_{0}}{\partial q_1} q_2 p_2^j
$$

(11)

$$\psi_k(q_2, p_2, \xi) = \sum_{i+j=k} h_{ij}(f(\xi),g(\xi)) q_2^i p_2^j, \quad k = 2,3,4.
$$

(12)

The functions $h_{ij}$ in (12) are the coefficients in the series expansion (3) of the initial Hamiltonian. The partial derivatives of the functions $h_{ij}$ and $H_0$ in the expressions (10) and (11) are calculated for $q_1 = f(\xi), p_1 = g(\xi)$.

3. Introducing local coordinates in a neighborhood of pendulum-like oscillations and rotations of a heavy rigib body in the Bobylev–Steklov case

In this section the above-described method is used for introducing local coordinates in a neighborhood of pendulum-like oscillations and rotations of a heavy rigid body with one fixed point in the Bobylev–Steklov case.

Let us consider a heavy rigid body, which has a fixed point $O$. Denote by $mg$ the weight of the body and by $l$ the distance between the fixed point $O$ and the mass center of the body. To describe the motion of the body, two following coordinate systems with origins in the point $O$ are introduced. The fixed coordinate system $OXYZ$ with the axis $Z$ directed along with the vertical, and the body coordinate system $Oxyz$, with the axes directed along the principal axes of inertia of the body for the point $O$. 


The following notations are used. \( A, B \) and \( C \) are the moments of inertia corresponding to axes \( x, y \) and \( z \). The coordinates of the mass center of the body in coordinate system \( Oxyz \) is denoted by \( x, y, z \). We choose three Euler angles \( \psi, \theta, \phi \) as the generalized coordinates defining the orientation of the body system \( Oxyz \) with respect to the fixed coordinate system \( OXYZ \).

In what follows it is assumed that the so-called Bobylev–Steklov case takes place. That is, the moments of inertia and coordinates of the mass center of the body satisfy the relations \( A = 2C \) and \( x_s = l, \ y_s = z_s = 0 \). The Bobylev–Steklov case is remarkable due to the existence of a family of periodic motions [16,17], which can be expressed in terms of elliptic Jacobi functions (see, e.g., [18]). In the Bobylev–Steklov case, the equations of motion also admit two one-parametric families of partial solutions describing pendulum-like motions of the body about its principal axis of inertia. Such motions are possible either about the axis \( z \) or about the axis \( y \).

We shall consider pendulum-like motions about the axis of inertia \( z \), which keeps an invariable horizontal position in the space. On this motion, \( \theta = \pi/2, \psi = \text{const} \), and the angle \( \varphi \) satisfies the physical pendulum equation:

\[
\frac{d^2}{dt^2} \varphi + \mu^2 \cos \varphi = 0, \quad \mu^2 = \frac{mg l}{C}.
\] (13)

Thus, in this motion the body either performs pendulum-like periodic motions about the axis \( z \) or asymptotically approaches the unstable equilibrium position. Let us note that in the Bobylev–Steklov case \( C \) is the smallest moment of inertia, thus in what follows oscillations and rotations about the largest principal axis of inertia of the body for fixed point \( O \) are considered.

By introducing generalized momenta \( p_\psi, p_\theta, p_\phi \), corresponding to Euler angles \( \psi, \theta, \phi \) one can write the equations of motion in Hamiltonian form. The angle \( \psi \) is a cyclic coordinate, thus is the corresponding momentum \( p_\psi \) is a first integral, which is equal to zero on the unperturbed pendulum-like periodic motions. It is supposed that \( p_\psi = 0 \) for the perturbed motion as well. Let us pass to the new independent variable \( \tau = \mu t \) (dimensionless time) and introduce dimensionless coordinates \( q_1, q_2 \), and momenta \( p_1, p_2 \)

\[
q_1 = \varphi - \frac{3 \pi}{2}, \quad q_2 = \theta - \frac{\pi}{2}, \quad p_1 = \frac{p_\varphi}{C \mu}, \quad p_2 = \frac{p_\theta}{C \mu}.
\] (14)

In the new canonical variables the Hamiltonian reads [10]

\[
H = \frac{1}{4} \left( 2\alpha - 1 \right) \sin^2 q_1 \tan^2 q_2 + \tan^2 q_2 + 2p_1^2 + \frac{1}{4} \left( 2\alpha - 1 \right) \sin 2q_1 \tan q_2 - p_1 p_2 + \\
\frac{1}{4} \left[ 2\alpha - \left( 2\alpha - 1 \right) \sin^2 q_1 \right] p_2^2 - \cos q_1 \cos q_2,
\] (15)

where \( \alpha = C / B, \left( \frac{1}{3} \leq \alpha < 1 \right) \).

On the considered pendulum-like motions we have \( q_2 = p_2 = 0 \) and evaluation of \( q_1, p_1 \) is described by a canonical system with the Hamiltonian \( H_0 = 1 / 2p_1^2 - \cos q_1 \). Depending on the value of the energy integral \( H_0 = h \), the pendulum-like motions are either asymptotic \((h = 1)\) to the unstable equilibrium point \( \varphi = \pi/2 \) of the rigid body or periodic motions: pendulum-like oscillations \((|h| < 1)\) in a neighborhood of the stable equilibrium point \( \varphi = 3\pi/2 \) or rotations \((|h| > 1)\) about the axis \( Oz \).

Let \( q_1 = f(\tau - \tau_0, h), p_1 = g(\tau - \tau_0, h) \) be a solution of the system with Hamiltonian \( H_0 \). Then, the following equalities hold:

\[
\frac{df}{d\tau} = g, \quad \frac{dg}{d\tau} = -\sin f.
\] (16)

Differentiation both sides of equations (16) yields
The coefficients of the forms
\[ \frac{d^2 f}{dt^2} = -\sin f , \quad \frac{d^2 g}{dt^2} = -g \cos f . \]  

By the similar way from (17) one can obtain
\[ \frac{d^3 f}{dt^3} = -g \cos f , \quad \frac{d^3 g}{dt^3} = \sin f (\cos f + g^2) . \]  

Finally by substituting (16)–(18) in (5), we have the following of the canonical transformation which introduces the local variables:
\[ q_1 = f + \frac{\sin f}{V^2} \eta - \frac{\sin f}{2V^4} \eta^2 + \left[ \frac{\sin f (\cos^2 f + g^2)}{6V^6} + \frac{W^2 \sin f}{3V^8} \right] \eta^3 + O(\eta^4) \]
\[ p_1 = g + \frac{g \cos f}{V^2} \eta - \frac{g \cos f}{2V^4} \eta^2 + \left[ \frac{g \cos f}{6V^6} + \frac{W^2 g}{3V^8} \right] \eta^3 + O(\eta^4), \]
where \( V^2 = g^2 + \sin^2 f, W^2 = g^2 \cos^2 f + \sin^2 f \).

The functions \( f \) and \( g \) in (19) depends on new canonical coordinate \( \xi \). If \( |h| < 1 \) (the case of pendulum-like oscillations), then the explicit expressions for \( f \) and \( g \) read [8]:
\[ f(\xi) = 2 \arcsin[k_1 \sin(\xi, k_1)], \quad g(\xi) = 2k_1 \cos(\xi, k_1), \quad k_1^2 = (h + 1)/2 , \]
In the case \( |h| > 1 \) (pendulum-like rotations) we have:
\[ f(\xi) = 2 \arccos[k_2 \cos(\xi, k_2)], \quad g(\xi) = 2k_2^{-1} \cos(\xi, k_2), \quad k_2^2 = 2/(h + 1) . \]

In (20)–(21), usual notations are used for the elliptic functions [19]. The period of functions \( f(\xi), g(\xi) \) is equal to \( 2\pi/\omega \), where \( \omega = \pi/(2K(k_1)) \) in the case of oscillations and \( \omega = \pi/(k_2 K(k_2)) \) in the case of rotations. By \( K \) the complete elliptic integral of the first kind is denoted.

Now we perform the following canonical transformation \( \xi, \eta \to w, r \)
\[ \xi = \frac{1}{\omega} w, \quad \eta = \omega r . \]

It is worth noting that the Hamiltonian of the problem is a \( \pi \)-periodic function of \( w \) in the case of oscillations, and it is a \( 2\pi \)-periodic function of \( w \) in the case of rotations.

Successively substituting (19) and (22) into (15), we obtain the following Hamiltonian of the system, which describes perturbed motion in a neighborhood of an unperturbed periodic orbit
\[ \Gamma = \Gamma_2 + \Gamma_4 + \cdots + \Gamma_{2m} + \cdots , \]
where \( \Gamma_{2m} \) is a form of degree \( 2m \) in \( q_2, p_2, \eta^{1/2} \). The forms \( \Gamma_2 \) and \( \Gamma_4 \) read
\[ \Gamma_2 = \omega r + \varphi_2(q_2, p_2, w), \]
\[ \Gamma_4 = \omega^2 \chi(w) r^2 + \omega \psi_2(q_2, p_2, w) r + \varphi_4(q_2, p_2, w) , \]
where
\[ \psi_2(q_2, p_2, w) = \sum_{i+j=k} \psi_{ij} q_2^i p_2^j, \quad \varphi_k(q_2, p_2, w) = \sum_{i+j=k} \varphi_{ij} q_2^i p_2^j, \quad k = 2, 4 . \]

The coefficients of the forms \( \psi_2, \varphi_2, \varphi_4 \) read
\[ \chi(w) = \frac{1}{2V^4} (\cos f, -1)(\sin^2 f, -g^2) \]
\[ \psi_{20}(w) = \frac{1}{2V^2} [(\sin^2 f, (\cos f, +1)(2\alpha - 1) + 1)g^2 - \sin^2 f,] \]
\[ \psi_{20}(\omega) = \frac{1}{2\sqrt{2}} [(\sin^2 f_1 \cos f_1 + 1)(2\alpha - 1) + 1) g^2 - \sin^2 f_1], \]
\[ \psi_{11}(\omega) = -\frac{1}{2\sqrt{2}} (2\alpha - 1) g_1 \sin f_1 [\sin^2 f_1 - \cos f_1 (\cos f_1 + 1)], \]
\[ \psi_{02}(\omega) = -\frac{1}{2\sqrt{2}} (2\alpha - 1) \sin^2 f_1 \cos f_1, \quad \varphi_{20}(\omega) = \frac{1}{4} [(2\alpha - (2\alpha - 1) \cos^2 f_1) g_2^2 + 2 \cos f_1], \]
\[ \varphi_{11}(\omega) = \frac{1}{2} (2\alpha - 1) g_1 \sin f_1 \cos f_1, \quad \varphi_{02}(\omega) = \frac{1}{4} [(2\alpha - 1) \cos^2 f_1 + 1], \]
\[ \varphi_{40}(\omega) = \frac{1}{6} [(2\alpha \sin^2 f_1 + \cos^2 f_1) g_4^2 - \frac{1}{4} \cos f_1], \quad \varphi_{31}(\omega) = \frac{1}{6} (2\alpha - 1) g_1 \sin f_1 \cos f_1, \]

where \( f_1(\omega) = f(\omega^{-1}w), \quad g_1(\omega) = g(\omega^{-1}w) \). Let us note that the period of \( f_1, g_1 \) with respect to \( w \) is equal to \( \pi \) in the case of oscillations and \( 2\pi \) in case of rotations.

Consider the motion on the zero isoenergetic level. Hence the coordinate \( w \) is an increasing function of the variable \( \tau \), it can play the role of a new time. Then, from the equation \( \Gamma = 0 \) for small \( q_2, p_2, r \) we have \( r = -K(q_2, p_2, w) \), where
\[ K = K_2 + K_4 + \cdots + K_k + \cdots, \]

\( K_n \) is a form of degree \( n \) in \( q_2, p_2 \) with coefficients \( T \)-periodic in \( w \). The forms \( K_2 \) and \( K_4 \) read:
\[ K_2 = \frac{1}{\omega} \varphi_2(q_2, p_2, w), \]
\[ K_4 = \frac{1}{\omega} [\chi(w) \varphi_4^2(q_2, p_2, w) - \varphi_2(q_2, p_2, w) \varphi_2(q_2, p_2, w) + \varphi_4(q_2, p_2, w)]. \]

On the isoenergetic level \( \Gamma = 0 \) the equations of motion have canonical form with Hamiltonian \( K \)
\[ \frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}. \]

4. The study of orbital stability for pendulum-like oscillations in the case of degeneracy

The orbital stability of pendulum-like motions of a heavy rigid body in the Bobylev–Steklov case was studied in [9,10,15]. The problem of orbital stability of pendulum-like periodic motions about the axis \( y \) was considered in [9]. The considered here case of pendulum-like motions about the axis \( z \) was studied in [10], where action-angle variables has been introduced in the region corresponding to unperturbed periodic orbits. A perturbation of the action variable and perturbations normal to the invariant manifold of periodic orbits were chosen as local coordinates. In such an approach, in order to obtain a series expansion of the perturbed Hamiltonian, it is necessary to obtain an explicit form of partial derivatives of the unperturbed Hamiltonian with respect to the action variable. This circumstance leads to rather cumbersome and time-consuming calculations. The above difficulties essentially increase in degeneracy cases, when solving the problem of orbital stability requires an analysis including terms of order higher than four in the expansion of the Hamiltonian in a neighborhood of the unperturbed periodic orbit. That is why the degeneracy case was not studied in [10]. The method of introduction of local coordinates described in section 2 allows to overcome the above obstacle. In [15] the above-mentioned method has been applied to investigate orbital stability in the degeneracy case for small amplitudes of oscillations. In this section this method is applied to study the orbital stability in the case of degeneracy at any values of amplitudes.

The orbital stability diagram obtained in [10] is shown in figure 1. In the hatched domains, the pendulum-like periodic motions are orbital unstable. In the unhatched domains, orbital stability takes place. For parameter values on curve \( \Gamma \), where the degeneracy case takes place, the problem on orbital stability of periodic oscillations was not considered in [10].
It is worth noting that a simple geometrical interpretation can be done for condition of orbital instability of pendulum-like oscillations. Indeed, let us denote by $q_1^*$ the maximum value of $q_1(w)$ on an unperturbed periodic orbit. In other words, $q_1^*$ can be regarded as an amplitude of the oscillations. It easy to see that $\cos q_1^* = h$. The pendulum-like oscillations are orbital unstable for $0 < h < 1$ (see stability diagram in figure 1). It means, that the pendulum-like oscillations are orbital unstable if their amplitude exceed $\pi/2$.

\[ \text{Figure 1. Stability diagram for pendulum-like motions of a rigid body in the Bobylev–Steklov case.} \]

In this section we show that the pendulum-like oscillations are orbital stable if their amplitude is lower than $\pi/2$. To prove this fact, it is necessary to complete stability study for parameter values corresponding to the degeneracy curve $\Gamma$.

We perform the stability analysis by using the explicit expression for the perturbed Hamiltonian obtained in section 3 and apply the methods developed in [20–22]. In particular, we construct a symplectic map generated by the system of nonlinear equations (31) and explore the stability of its fixed point. The problem of stability of the fixed point of this map is equivalent to the problem of stability of the equilibrium position of the system (31).

At first, let us pass to new canonical coordinates $Q, P$ by means of the linear change of variables

\[ q_2 = n_{11} Q + n_{12} P, \quad p_2 = n_{21} Q + n_{22} P, \]  

where

\[ n_{11} = x_{12}(\pi), \quad n_{12} = 0, \quad n_{21} = A - x_{11}(\pi), \quad n_{22} = \sqrt{1 - A^2}, \quad A = \frac{[x_{11}(\pi) + x_{22}(\pi)]}{2}. \]  

Functions $x_{11}(w), x_{12}(w), x_{21}(w), x_{22}(w)$ are elements of the matrizer $X(w)$ of linear system with the Hamiltonian (29).

The new Hamiltonian $K^*(Q, P, w)$ is obtained by substituting (33) in (28). It is also noting that in the variables $Q, P$ the linear part of the symplectic map generated by the canonical system with Hamiltonian $K^*(Q, P, w)$ has the simplest form.

The symplectic map can be constructed by the following way. Let $Q^{(0)}, P^{(0)}$ be the initial values of the variables $Q, P$, and $Q^{(1)}, P^{(1)}$ be their values for $w = \pi$. Then, the symplectic map generated by the canonical system with Hamiltonian $K^*(Q, P, w)$ reads
singularity when introducing local coordinates

The 5. low pendulum values of parameters on the curve $f$ of the map (34) is stable draw conclusions

where $G_0$ terms of degree six or higher are denoted; $F_m = \Phi_m(Q^{(0)}, P^{(0)}, \pi)$, and $\Phi_k(Q^{(0)}, P^{(0)}, w)$ are forms of degree $m \ (m = 4, 6)$ satisfying the equalities:

$$
\frac{\partial \Phi_4}{\partial w} = -G_4, \quad \frac{\partial \Phi_6}{\partial w} = -G_6 - \frac{\partial G_4}{\partial P_0} \frac{\partial \Phi_4}{\partial Q_0}. \quad (35)
$$

$G_m(Q^{(0)}, P^{(0)}, w)$ are forms, which are obtained from $K_m^*(Q, P)$ by the change of variables

$$
\left\| \frac{Q}{P} \right\| = X^*(w) \left\| \frac{Q^{(0)}}{P^{(0)}} \right\|. \quad (36)
$$

where $X^*(w)$ is the matrizer of the linear system with the Hamiltonian $K_0^*(P, Q)$. Forms $K_m^*(Q, P)$ are obtained by substituting (33) in forms $K_m^*(Q, P)$ of the expansion (28). The matrix $G$ reads

$$
G = \begin{bmatrix}
\cos \pi \beta & \sin \pi \beta \\
-\sin \pi \beta & \cos \pi \beta
\end{bmatrix}, \quad \beta = \frac{1}{\pi} \arccos A. \quad (37)
$$

Equating coefficients of the same powers in both sides of equality (35), we obtain five ordinary differential equations for coefficients of the forms $\Phi_m \ (m = 4, 6)$. The right-hand sides of these equations depend on $x_{ij}(\nu)$, which are entries of the matrix $X^*(w)$. Thus, integrating the system of sixteen equations (twelve equations for coefficients of the form $\Phi_m$ and four equations for $x_{ij}(\nu)$) in the interval $[0; T]$, we obtain the coefficients of the form $F_m$. In the general case, the above system should be solved numerically. Let us introduce the following notation:

$$
\sigma = 3f_{40} + f_{22} + 3f_{04},
$$

$$
\gamma = -\frac{1}{2} f_{06} - \frac{1}{2} f_{24} - \frac{1}{2} f_{60} - \frac{1}{2} f_{42} + f_{31}(4f_{04} + 2f_{22} + 5f_{40}) + f_{13}(4f_{40} + 2f_{22} + 5f_{04}) - \frac{1}{2} [f_{04} - 2f_{22} + 4f_{40}]^2 + [f_{13} - 3f_{31}]^2 \cot 2\pi \beta - [f_{13} + 3f_{31}]^2 + 4(f_{04} - f_{40})^2 \cot \pi \beta, \quad (38)
$$

where $f_{ij}$ are coefficients of the form $F_m \ (m = 4, 6)$.

If the inequality $\sigma \neq 0$ is satisfied, the fixed point of the map (34) is stable [20]. Otherwise, if $\sigma = 0$, then the case of degeneracy takes place and an additional nonlinear analysis must be performed to draw conclusions on stability. In particular, it is necessary to calculate $\gamma$. If $\gamma \neq 0$, then the fixed point of the map (34) is stable [21, 22]. Otherwise, if $\gamma = 0$, then, terms of degree eight or higher must be taken into account to answer the question on stability.

As it was already mentioned, the case of degeneracy appears in our problem on curve $\Gamma$ (see figure 1). Numerical calculations of the coefficients of the map (34) have shown that $\gamma \neq 0$ for any values of parameters on the curve $\Gamma$. It yields the stability of the fixed point of the map (34). Thus, the pendulum-like oscillations are orbitally stable for values of parameters on the curve $\Gamma$.

Connecting the above result with the results of previous study performed in [10] one can formulate the following condition of orbital stability in Bobylev–Steklov case. The pendulum-like oscillations of a heavy rigid body about the largest principal axis of inertia are orbital stable if their amplitude is lower than $\pi/2$ and orbital unstable if the amplitude exceeds $\pi/2$.

5. Conclusions

The method of introducing local coordinates described in this paper makes it possible to avoid a singularity when introducing local coordinates. The above-mentioned method is especially useful in
cases of degeneracy and can be applied for study the orbital stability of periodic motions both in the presence of their analytic representation and in the case when periodic motions can be found only numerically. By application of this method the orbital stability of pendulum-like oscillations of a heavy rigid body has been studied for previously unconsidered parameters values, corresponding to the degeneracy case. It allows to complete the study of orbital stability of pendulum-like periodic motions in the Bobylev–Steklov case.

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