High-Frequency Solutions to the Constraint Equations

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Abstract: We construct high-frequency initial data for the Einstein vacuum equations in dimension 3+1 by solving the constraint equations on $\mathbb{R}^3$. Our family of solutions $(\tilde{g}_\lambda, K_\lambda)_{\lambda \in (0,1]}$ is defined through a high-frequency expansion similar to the geometric optics approach and converges in a particular sense to the data of a null dust. In order to solve the constraint equations, we use their conformal formulation and the main challenge of our proof is to adapt this method to the high-frequency context. In particular, the parameters of the conformal formulation are oscillating. The main application of this article is our companion article (Touati in High-frequency solutions to the Einstein vacuum equations: local existence in generalised wave gauge. arXiv:2206.12318, 2022) where we construct high-frequency gravitational waves in generalised wave gauge.

Contents

1. Introduction ........................................ 98
   1.1 Backreaction for the constraint equations ........ 99
   1.2 The conformal method ............................ 100
   1.3 Preliminaries ..................................... 101
       1.3.1 Geometric notations ......................... 101
       1.3.2 Function spaces and asymptotically Euclidean manifold 101
       1.3.3 Elliptic estimates ............................ 103
       1.3.4 High-frequency notations .................... 103
2. Statement of the Results ............................ 104
   2.1 The background ................................. 104
   2.2 Solving the constraint equations ................. 105
   2.3 Application to high-frequency vacuum spacetimes 106
3. Strategy of Proof ................................. 107
   3.1 High-frequency expansion of the parameters ...... 108
   3.2 High-frequency expansion of the solutions ...... 108
1. Introduction

In this article, we construct high-frequency initial data for the Einstein vacuum equations

\[ R_{\mu\nu}(g) = 0, \tag{1.1} \]

where \( R_{\mu\nu}(g) \) is the Ricci tensor of \( g \), a Lorentzian metric on the manifold \([0, 1] \times \mathbb{R}^3\). Initial data for (1.1) on \( \Sigma_0 = \{ t = 0 \} \) are given by a Riemannian metric \( \bar{g} \) and a symmetric 2-tensor \( K \). In the spacetime \(( [0, 1] \times \mathbb{R}^3, g) \) that \(( \Sigma_0, \bar{g}, K \) gives birth to, \( \bar{g} \) will be the restriction of \( g \) to \( \Sigma_0 \) and \( K \) will be the second fundamental form of \( \Sigma_0 \), that is \( K = -\frac{1}{2} L_T g \) where \( T \) is the unit normal to \( \Sigma_0 \) for \( g \). A necessary condition for \(( \bar{g}, K \) to be the set of initial data to a solution of (1.1) is that \(( \bar{g}, K \) solves the following vacuum constraint equations:

\[ R(\bar{g}) + (\text{tr}\, \bar{g} K)^2 - |K|_{\bar{g}}^2 = 0, \tag{1.2} \]
\[ -\text{div} \, \bar{g} K + d\text{tr} \, \bar{g} K = 0, \tag{1.3} \]

where \( R(\bar{g}) \) denotes the scalar curvature of \( \bar{g} \). Equation (1.2) is the Hamiltonian constraint, and equation (1.3) is the momentum constraint. Together, they form a coupled system of non-linear elliptic partial differential equations and we refer to Chapter 7 of [CB09] for a complete presentation of their mathematical study.
1.1. Backreaction for the constraint equations. We construct a family \((\tilde{\bar{g}}_\lambda, K_\lambda)_{\lambda \in (0,1]}\) of solutions to (1.2)–(1.3) which oscillate at frequency \(\lambda^{-1}\). The mean feature of this family is its high-frequency limit, i.e. its behaviour when \(\lambda\) tends to 0. It converges in the following weak sense

\[
\begin{align*}
\tilde{\bar{g}}_\lambda & \rightharpoonup \tilde{\bar{g}}_0, \quad \text{uniformly in } L^\infty, \\
\nabla \tilde{\bar{g}}_\lambda & \rightharpoonup \nabla \tilde{\bar{g}}_0, \quad \text{weakly in } L^2_{loc}, \\
K_\lambda & \rightharpoonup K_0, \quad \text{weakly in } L^2_{loc},
\end{align*}
\]

(1.4)

to a fixed solution \((\tilde{\bar{g}}_0, K_0)\) of the null dust maximal constraint equations:

\[
\begin{align*}
R(\tilde{\bar{g}}_0) - |K_0|_{\tilde{\bar{g}}_0}^2 &= 2|\nabla u_0|_{\tilde{\bar{g}}_0}^2 F_0^2, \\
-\text{div}_{\tilde{\bar{g}}_0} K_0 &= |\nabla u_0|_{\tilde{\bar{g}}_0} F_0^2 du_0, \\
\text{tr}_{\tilde{\bar{g}}_0} K_0 &= 0,
\end{align*}
\]

(1.5)–(1.7)

where \(u_0\) and \(F_0\) are scalar functions defined on \(\mathbb{R}^3\). Therefore, this article provides the first example of backreaction for the constraint equations: a sequence of solutions to the vacuum equations describes, in the high-frequency limit, a matter model.

This phenomenon is at the heart of Burnett’s conjecture in general relativity. In [Bur89], he conjectured that if a sequence \((g_\lambda)_{\lambda}\) of solutions to (1.1) converges weakly in \(H^1_{loc}\) to a metric \(g_0\), then \(g_0\) solves the massless Einstein-Vlasov system

\[
\begin{align*}
R_{\alpha\beta}(g_0) &= \int_{g_0^{-1}(p,p)=0} f(x, p) p_\alpha p_\beta d\mu_{g_0}, \\
p^\alpha \partial_\alpha f - p^\rho p^\beta \Gamma^{\rho}_{\alpha\beta}(g_0) p_\rho \partial_\beta f &= 0,
\end{align*}
\]

(1.8)–(1.9)

for a density \(f\) defined on the tangent bundle and where \(d\mu_{g_0}\) is the measure on the tangent bundle. The second equation is the Vlasov equation for the density of massless particles \(f\). In his article, Burnett also asked the reverse question: given \(g_0\) a solution to (1.8)–(1.9), can one construct a sequence \((g_\lambda)_{\lambda}\) of solutions to (1.1) converging weakly in \(H^1\) to \(g_0\)? The present article provides a positive answer to this question at the level of the spacelike data for (1.1) and for a discrete version of the massless Einstein-Vlasov system, namely the Einstein-null dust system

\[
\begin{align*}
R_{\alpha\beta}(g_0) &= F_0^2 \partial_\alpha u_0 \partial_\beta u_0, \\
g_0^{-1}(du_0, du_0) &= 0, \\
2g_0^{\rho\sigma} \partial_\rho u_0 \partial_\sigma F_0 + (\Box_{g_0} u_0) F_0 &= 0.
\end{align*}
\]

(1.10)–(1.12)

The equations (1.5)–(1.7) are the constraint equations that spacelike data for the system (1.10)–(1.12) need to solve, in the particular case where the initial hypersurface is maximal. This explains why we referred to the system (1.5)–(1.7) as the null dust maximal constraint equations. Note that we slightly abuse notations: in this article the scalar functions \(F_0\) and \(u_0\) will be defined on \(\Sigma_0\) only, while in (1.10)–(1.12) they are defined on a whole spacetime.

The two aspects of Burnett’s conjecture, the direct one and the indirect one, have been studied in several works and different settings. Assuming \(U(1)\) symmetry, Huneau and Luk address the direct conjecture by means of microlocal defect measure in [HL19] (see also [GdC21]). Under the same symmetry, they construct high-frequency vacuum
spacetimes converging to \( N \) null dusts in [HL18]. The first result without symmetry assumptions was obtained by Luk and Rodnianski in [LR20], where they address both sides of Burnett’s conjecture in double null gauge. This choice of gauge restricts the class of kinetic spacetimes they consider to 2 null dusts.

In a companion paper [Tou22], the author constructs high-frequency vacuum spacetimes in generalised wave gauge, paving the way to a proof of Burnett’s conjecture in this gauge. This motivates the need for the high-frequency initial data solving (1.2)–(1.3) provided by this article. The link between this article and [Tou22] and its mathematical implications will be further discussed in Sect. 2.3, after we state the main result in Sect. 2.2.

Finally, note that Burnett’s conjecture and the backreaction it describes for the Einstein vacuum equations falls into the widest category of non-linear effects induced by homogenization, that is through the interaction of multiple scales. This type of effects can be encountered in the study of virtually all non-linear equations coming from physics. Examples are porous media, hydrodynamics, quantum mechanics etc. We refer to [Tar09] for a rich presentation of the field of homogenization.

1.2. The conformal method. The constraint equations (1.2)–(1.3) are underdetermined, and in order to solve them we use the conformal method. Introduced by Lichnerowicz in [Lic44], this method is based on a conformal formulation of the data \((\bar{g}, K)\) and it identifies free parameters. In particular, this method transforms (1.2)–(1.3) into a determined system of equations composed of a vectorial equation and a scalar equation. The idea giving its name to the method is to prescribe the conformal class of \(\bar{g}\), i.e. to fix a Riemannian metric \(\gamma\) on \(\mathbb{R}^3\) and to solve for the scalar function \(\varphi\) such that

\[
\bar{g} = \varphi^4 \gamma.
\] (1.13)

We say that a tensor is a TT-tensor if it is traceless divergence free and symmetric and one can show that the space of TT-tensor depends only on the conformal class of the metric. Therefore, the next step in the conformal method is to decompose the symmetric 2-tensor \(K\) in connection with its trace and divergence features. More precisely we fix a scalar function \(\tau\) and a TT-tensor \(\sigma\) for the metric \(\gamma\) and solve for the vector field \(W\) such that

\[
K = \varphi^{-2}(\sigma + L_\gamma W) + \frac{1}{3} \varphi^4 \gamma \tau
\] (1.14)

where \(L_\gamma W\) is defined in (4.30). The exponents appearing in (1.13) and (1.14) are linked to the dimension of the manifold, here 3, see Chapter 6 of [CB09] for their general expression. Once the parameters \(\gamma, \tau,  \sigma\) are chosen, the constraint equations (1.2)–(1.3) rewrite as the following coupled system of non-linear elliptic equations for \((\varphi, W)\):

\[
8\Delta_\gamma \varphi = R(\gamma) \varphi + \frac{2}{3} \tau^2 \varphi^5 - |\sigma + L_\gamma W|_\gamma^2 \varphi^{-7},
\] (1.15)

\[
\text{div}_\gamma L_\gamma W = \frac{2}{3} \varphi^6 d\tau.
\] (1.16)

A wealth of literature has been produced over the years on the construction of solutions to (1.15)–(1.16) on various Riemannian manifolds. Note that if the mean curvature \(\tau\) is constant (CMC setting), then (1.15) and (1.16) decouple and the construction of
solutions is simplified. The results on (1.15)–(1.16) can thus be categorized into CMC, near CMC (where $\frac{d}{\tau}$ is small) or far from CMC results, where no assumption is made on $\tau$. On compact manifolds, CMC results are obtained in [Ise95]. Near CMC results on asymptotically Euclidean manifolds are obtained in [CBIY00], where the authors also treat the constraint equations with matter. For far from CMC results, we refer the reader to [Max09] and [DIMM14], where the case of compact manifolds and asymptotically Euclidean manifolds are respectively treated.

Our main challenge is to adapt the conformal method to the high-frequency context, i.e choose the parameters so that the resulting solution $(\bar{g}_\lambda, K_\lambda)$ displays the behaviour (1.4). Not only will the solutions $(\varphi, W)$ be defined by high-frequency ansatz, but also the parameters $\gamma, \tau$ and $\sigma$. In particular, we will construct an oscillating TT-tensor, i.e solve $\text{div}_\gamma \sigma = 0$ and $\text{tr}_\gamma \sigma = 0$, with $\gamma$ itself oscillating.

1.3. Preliminaries. We fix here our notations and present the analytic setting of this article.

1.3.1. Geometric notations Throughout this article, the notation $\Sigma_0$ refers to the manifold $\mathbb{R}^3$. On $\Sigma_0$ we consider the usual Euclidean coordinates $x = (x^1, x^2, x^3)$, and we denote by $e$ the standard Euclidean metric, i.e

$$e = \left( dx^1 \right)^2 + \left( dx^2 \right)^2 + \left( dx^3 \right)^2.$$  

Latin indices are used for the Euclidean coordinates and therefore runs from 1 to 3. In this article, repeated indices are always summed over. A second frame adapted to the background structure will be defined in Sect.2.1.

If $f$ is a scalar function, we define its gradient by $\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$. If $h$ is a Riemannian metric on $\Sigma_0$ we define $|\nabla f|^2_h = h^{ij} \partial_i f \partial_j f$.

In the particular case of the Euclidean metric we simply write $|\nabla f|^2$. Moreover if $T$ and $S$ are two symmetric 2-tensors on $\Sigma_0$, we define $|T \cdot S|_h = h^{ij} h^{k\ell} T_{ik} S_{j\ell}$ and $|T|^2_h = |T \cdot T|_h$. The trace of $T$ with respect to $h$ is defined by $\text{tr}_h T = h^{ij} T_{ij}$.

1.3.2. Function spaces and asymptotically Euclidean manifold If $x \in \mathbb{R}^3$ we set $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ with $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. We define the following weighted Sobolev spaces on $\mathbb{R}^3$.

**Definition 1.1** (Weighted Sobolev spaces). For $1 \leq p < +\infty$, $\delta \in \mathbb{R}$ and $k \in \mathbb{N}$ we define the space $W^k_{\delta,p}$ as the completion of $C_c^{\infty}$ for the norm

$$\|u\|_{W^k_{\delta,p}} = \sum_{0 \leq |\alpha| \leq k} \| \langle x \rangle^{\delta + |\alpha|} \nabla^\alpha u \|_{L^p},$$

where the $L^p$ norm is defined with the Euclidean volume element. We extend this definition to tensors of any type by summing over all components in the Euclidean coordinates. Some special cases are $H^k_{\delta} := W^k_{\delta,2}$ and $L^p_{\delta} := W^0_{\delta,p}$.

We also define the following $L^\infty$-based spaces:
Definition 1.2. For $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$ we define $C^k_\delta$ as the completion of $C^\infty_c$ for the norm

$$\|u\|_{C^k_\delta} = \sum_{0 \leq |\alpha| \leq k} \| (x)^{\delta + |\alpha|} \nabla^\alpha u \|_{L^\infty},$$

In the following proposition we recall some useful facts about these spaces (see [CB09] for the proofs).

Proposition 1.1. Let $s, s', s_1, s_2, m \in \mathbb{N}$, $\delta, \delta', \delta_1, \delta_2, \beta \in \mathbb{R}$ and $1 \leq p < +\infty$.

1. If $s \leq \min(s_1, s_2)$, $s < s_1 + s_2 - \frac{3}{\beta}$ and $\delta < \delta_1 + \delta_2 + \frac{3}{\beta}$ we have the continuous embedding

$$W^{s_1, p}_{\delta_1} \times W^{s_2, p}_{\delta_2} \subset W^{s, p}_{\delta}.$$ 

2. If $m < s - \frac{3}{\beta}$ and $\beta \leq \delta + \frac{3}{\beta}$ we have the continuous embedding

$$W^{s, p}_{\delta} \subset C^m_{\beta}.$$ 

The background metric and the solution of the constraint equations we will produce are asymptotically Euclidean, meaning that they converge in some sense to the Euclidean metric at infinity. We give the definition of [CBIY00].

Definition 1.3 (Asymptotically Euclidean initial data). Let $\bar{g}$ be a metric on $\mathbb{R}^3$ and $K$ a symmetric 2-tensor on $\mathbb{R}^3$. If $k > \frac{s}{2}$ and $\delta > -\frac{3}{2}$, we say that the pair $(\bar{g}, K)$ is $H^k_\delta$ asymptotically flat if

$$\bar{g} - \epsilon \in H^k_\delta \quad \text{and} \quad K \in H^{k-1}_{\delta+1}.$$ 

Note that the restrictions on $k$ and $\delta$ in the previous definition ensure that $\bar{g}$ is $C^1$ and $\bar{g} - \epsilon$ tends to 0 at infinity, since Proposition 1.1 implies $H^k_\delta \hookrightarrow C^{1}_{\delta + \frac{1}{2}}$.

Remark 1.1. While the assumptions in Definition 1.3 are adapted to the inversion of elliptic operators, the decay assumption $\delta > -\frac{3}{2}$ is too weak to be able to define the ADM mass of $\bar{g}$. Introduced in [ADM08], the ADM mass of an asymptotically Euclidean manifold is defined by

$$M(\bar{g}) = \frac{1}{16\pi} \lim_{R \to +\infty} \int_{\partial B_R} \sum_{i, j} (\partial_i \bar{g}_{ij} - \partial_j \bar{g}_{ii}) v_j d\mu,$$

where $v$ is the outgoing normal to the spheres $\partial B_R$ and $d\mu$ the measure on these spheres. Since the volume of the spheres $\partial B_R$ grows like $R^2$, we need to ask for $\delta \geq -\frac{1}{2}$ in Definition 1.3 in order to be able to define $M(\bar{g})$ in general. See also Remark 2.1.
1.3.3. Elliptic estimates  Solving the constraint equations with the conformal method requires to invert the elliptic operators $\Delta_\gamma$ and $\text{div}_\gamma L_\gamma$, for $\gamma$ a Riemannian metric. The first one is the Laplace-Beltrami operator acting on scalar functions and the second one is the conformal Laplacian acting on vector field, see Sect. 4.2 for their exact definition. The metric $\gamma$ will be defined in Sect. 4.1 but we can already say that it will be close to a background metric $\bar{g}_0$, itself close to the Euclidean metric $e$ on $\mathbb{R}^3$. We will benefit from this fact and invert $\Delta_\gamma = \Delta_\bar{e}$ and $\text{div}_\gamma L_\gamma$ rather than $\Delta_\gamma$ and $\text{div}_\gamma L_\gamma$, which have oscillating coefficients. The following proposition gives the desired inversion properties. The proof of its first part can be found in [McO79] while the second part is proved in [CBIY00].

**Proposition 1.2.** If $-\frac{3}{2} < \delta < -\frac{1}{2}$ then $\Delta : H^2_\delta \rightarrow L^2_{\delta+2}$ and $\text{div}_e L_e : H^2_\delta \rightarrow L^2_{\delta+2}$ are isomorphisms.

1.3.4. High-frequency notations  In this article we consider high-frequency quantities, i.e tensors of all types including scalar functions, metrics, 1-forms etc. A quantity is said to be high-frequency if it admits an expansion in powers of the small parameter $\lambda$ with coefficients of the form

$$T\left(\frac{u_0}{\lambda}\right) f$$

(1.17)

where $f$ depends only on $x \in \Sigma_0$ and $T$ is an oscillating function, i.e an element of

$$\{\theta \in \mathbb{R} \mapsto \sin(k\theta) \mid k \in \mathbb{N}\} \cup \{\theta \in \mathbb{R} \mapsto \cos(k\theta) \mid k \in \mathbb{N}\}.$$  

(1.18)

When considering a high-frequency quantity such as a tensor $S$, we denote by $S^{(i)}$ the coefficients of $\lambda^i$ in the expansion defining $S$, which thus expands formally as

$$S = \sum_{i \in \mathbb{Z}} \lambda^i S^{(i)}.$$ 

Note that $S^{(i)}$ is a tensor of the same type as $S$. Moreover, if $j \in \mathbb{Z}$ we define $S^{(\geq j)}$ by $S^{(\geq j)} = \sum_{k \geq j} \lambda^{k-j} S^{(k)}$. This allows us to clearly truncate high-frequency expansions at a fixed order as in

$$S = \sum_{k \leq j-1} \lambda^k S^{(k)} + \lambda^j S^{(\geq j)}.$$ 

To emphasize the fact that a high-frequency coefficient $S^{(i)}$ of a tensor $S$ is oscillating we will often write $S^{(i)} \left(\frac{u_0}{\lambda}\right)$ instead of just $S^{(i)}$.

In order to describe concisely the oscillating behaviour of a high-frequency coefficient $S^{(i)}$ of a tensor $S$, we write for $A$ a finite subset of (1.18)

$$S^{(i)} \overset{\text{osc}}{\sim} \sum_{T \in A} T(\theta)$$

if there exists tensors $(S^{(i)}_T)_{T \in A}$ of the same type as $S^{(i)}$ such that

$$S^{(i)} = \sum_{T \in A} T\left(\frac{u_0}{\lambda}\right) S^{(i)}_T.$$
This notation allows us to compute the oscillating behaviour of non-linear quantities without caring too much about the non-oscillating coefficients $S_T^{(i)}$. Note that $S^{(i)} \overset{osc}{\sim} 1$ simply means that $S^{(i)}$ is non-oscillating, i.e does not depend on $\frac{u_0}{\lambda}$.

In terms of derivation, we use the symbol $\theta$ for the derivation with respect to the $\frac{u_0}{\lambda}$ variable. For example, if $f$ is a scalar function and if $g = T\left(\frac{u_0}{\lambda}\right) f$ then

$$\partial_{\theta} g = T'\left(\frac{u_0}{\lambda}\right) f.$$ 

2. Statement of the Results

In this section, we give the assumptions on the background, state our main result and discuss its main application.

2.1. The background. The background metric and 2-tensor $(\tilde{g}_0, K_0)$ solve (1.5)–(1.7) with $F_0$ and $u_0$ two scalar functions defined on $\mathbb{R}^3$. The full background solution is then $(\tilde{g}_0, K_0, F_0, u_0)$ and we make some assumptions on it.

- **Assumptions on $(\tilde{g}_0, K_0)$**. Even though $(\tilde{g}_0, K_0)$ satisfies non-vacuum constraint equations, the sources are compactly supported (see (2.2) below) and we assume that $(\tilde{g}_0, K_0)$ corresponds to asymptotically Euclidean and highly regular initial data. By this we mean that there exists $\delta > -\frac{3}{2}$, a large integer $N \geq 10$ and $\varepsilon > 0$ such that

$$\| \tilde{g}_0 - e \|_{H^{N+1}_{\delta}} + \| K_0 \|_{H^{N+1}_{\delta}} \leq \varepsilon. \quad (2.1)$$

We denote by $\tilde{\mathcal{D}}$ the covariant derivative associated to $\tilde{g}_0$.

- **Assumptions on $F_0$**. The density $F_0$ is supported in a ball of size $R > 0$ in $\mathbb{R}^3$, i.e

$$\text{supp}(F_0) \subset B_R := \{ x \in \mathbb{R}^3 \mid |x| \leq R \}. \quad (2.2)$$

It also enjoys some regularity:

$$\| F_0 \|_{H^N} \leq \varepsilon \quad (2.3)$$

where $\varepsilon$ is defined above.

- **Assumptions on $u_0$**. There exists a constant non-zero vector field $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ such that

$$\| \nabla u_0 - \tilde{z} \|_{H^{N+1}_{\delta}} \leq \varepsilon \quad (2.4)$$

where $\nabla u_0 = (\partial_1 u_0, \partial_2 u_0, \partial_3 u_0)$ is the euclidean gradient of $u_0$. By taking $\varepsilon$ small enough in (2.4) we can assume that $|\nabla u_0|$ is uniformly bounded from below, which implies that $u_0$ has no critical points. Moreover, the level hypersurfaces of $u_0$ defined by

$$P_{0,u} = \{ x \in \mathbb{R}^3 \mid u_0(x) = u \}$$
There exists \( \varepsilon \) High-Frequency Solutions to the Constraint Equations

Moreover:

Moreover:

In this article, we don’t prove that a background solution \((\bar{g}_0, K_0, F_0, u_0)\) solving (1.5)–(1.7) and satisfying the above assumptions exists. We refer to [CBIY00] for the details of how one can solve the constraint equations with sources in the asymptotically Euclidean setting.

2.2. Solving the constraint equations. The following theorem is the main result of this article.

**Theorem 2.1.** Let \((\bar{g}_0, K_0, F_0, u_0)\) be the solution of the maximal constraint equations coupled with a null dust described in Sect. 2.1, and let \( \varepsilon > 0 \) be the smallness threshold. There exists \( \varepsilon_0 = \varepsilon_0(\delta, R) > 0 \) such that if \( 0 < \varepsilon \leq \varepsilon_0 \), there exists for all \( \lambda \in (0, 1] \) a solution \((\bar{g}_\lambda, K_\lambda)\) solution of the vacuum constraint equations (1.2)–(1.3) on \( \mathbb{R}^3 \) of the form

\[
\bar{g}_\lambda = \bar{g}_0 + \lambda \cos \left( \frac{u_0}{\lambda} \right) \bar{F}^{(1)} + \lambda^2 \left( \sin \left( \frac{u_0}{\lambda} \right) \bar{F}^{(2, 1)} + \cos \left( \frac{2u_0}{\lambda} \right) \bar{F}^{(2, 2)} \right) + \lambda^2 \bar{h}_\lambda, 
\]

(2.5)

\[
K_\lambda = K^{(0)}_\lambda + \lambda K^{(1)}_\lambda + \lambda^2 K^{(2)}_\lambda, 
\]

(2.6)

with

\[
K^{(0)}_\lambda = K_0 + \frac{1}{2} \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)},
\]

(2.7)

\[
K^{(1)}_\lambda = \cos \left( \frac{u_0}{\lambda} \right) \bar{K}^{(1, 1)} + \sin \left( \frac{2u_0}{\lambda} \right) \bar{K}^{(1, 2)}. 
\]

(2.8)

Moreover:

(i) the tensors \( \bar{F}^{(1)}, \bar{F}^{(2, 1)} \) and \( \bar{F}^{(2, 2)} \) are supported in \(|x| \leq R\) and there exists \( C_{\text{cons}} = C_{\text{cons}}(\delta, R) > 0 \) such that

\[
\left\| \bar{F}^{(1)} \right\|_{H^1} + \left\| \bar{F}^{(2, 1)} \right\|_{H^{1-1}} + \left\| \bar{F}^{(2, 2)} \right\|_{H^{2-1}} \leq C_{\text{cons}}\varepsilon, 
\]

(ii) the tensor \( \bar{F}^{(1)} \) is \( \bar{g}_0 \)-traceless, tangential to \( P_{0,u} \) and satisfies

\[
\left\| \bar{F}^{(1)} \right\|^2_{\bar{g}_0} = 8 F^2_0, 
\]

(2.9)
(iii) $K^{(1,1)}$ and $K^{(1,2)}$ are given by

$$K^{(1,1)} = \frac{1}{2} \mathcal{N}_0 \bar{F}_{ij}^{(1)} + \frac{1}{2} \left( \tilde{g}_0^{k \ell} (K_0)(i \ell) - \mathcal{N}_0^k \Gamma (\tilde{g}_0)(i) \right) \bar{F}_{ij}^{(1)}$$

$$+ \frac{1}{2|\nabla u_0|_{\tilde{g}_0}} \left( K_0^{k \ell} \partial_k u_0 \partial_\ell u_0 + \tilde{g}_0^{k \ell} \partial_k u_0 \partial_\ell [\nabla u_0]_{\tilde{g}_0} - \frac{1}{2} \bar{g}_0^{k \ell} \partial_k \partial_\ell u_0 \right) \bar{F}_{ij}^{(1)}$$

$$- \frac{1}{2} |\nabla u_0|_{\tilde{g}_0} \bar{F}^{(2,1)},$$

$$K^{(1,2)} = |\nabla u_0|_{\tilde{g}_0} \bar{F}^{(2,2)},$$

(iv) the tensors $\tilde{h}_\lambda$ and $K^{(2)}_\lambda$ belong to the spaces $H_\delta^5$ and $H^{4}_{\delta+1}$ respectively and satisfy

$$\max_{r \in [0, 4]} \lambda^r \left\| \nabla^{r+1} \tilde{h}_\lambda \right\|_{L^2_{\delta+1}} \leq C_{\text{cons}},$$

$$\max_{r \in [0, 4]} \lambda^r \left\| \nabla^{r} K^{(2)}_\lambda \right\|_{L^2_{\delta+1}} \leq C_{\text{cons}}.$$
Remark 2.2. These conditions correspond exactly to the definition of the TT gauge in the linearized gravity setting for a plane wave propagating in the $N_0$ direction. Note that $N_0$ is not a constant vector field, therefore strictly speaking a plane wave can’t propagate in the $N_0$ direction. However, as (2.1) and (2.4) show, $N_0$ is close to the $3$ direction and the analogy with the TT gauge of linearized gravity is thus valid.

In [Tou22], these polarization conditions are propagated by the transport equation that $F^{(1)}$ must satisfy in the spacetime so that $R_{\mu \nu}(g_\lambda) = O(\lambda^1)$, namely

$$-2D_{L_0} F^{(1)} + (\Box_{g_0} u_0) F^{(1)} = 0,$$

where $D$ is the covariant derivative associated to $g_0$ and $L_0 = -g_0^{\mu \nu} \partial_\mu u_0 \partial_\nu$ is the spacetime gradient of $u_0$. Moreover, in [Tou22] we assume without loss of generality that $\partial_t$ is the unit normal to $\Sigma_0$ for $g_0$ and that $u_0$ solves the eikonal equation (see (1.11)), which thus give $\partial_t u_0 = |\nabla u_0|_{\bar{g}_0}$ and $L_0 = |\nabla u_0|_{\bar{g}_0}(\partial_t + N_0)$ on $\Sigma_0$. This also implies that on $\Sigma_0$ we have

$$\partial_t \left( \lambda g^{(1)}_{ij} \right) = -\sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} F^{(1)}_{ij} + \lambda \cos \left( \frac{u_0}{\lambda} \right) \partial_t F^{(1)}_{ij}$$

where $\partial_t F^{(1)}_{ij}$ on $\Sigma_0$ is directly given by the first order transport equation (2.15). Note in particular that thanks to our previous geometric assumption on $\partial_t$ and $u_0$ on $\Sigma_0$ and thanks to the wave gauge condition that $g_0$ is also assumed to satisfy in the spacetime, we can completely rewrite $\partial_t F^{(1)}_{ij} \mid \Sigma_0$ given by (2.15) in terms of $\bar{g}_0, K_\lambda$ and $\bar{F}^{(1)}$. For instance, we can show that

$$-\frac{1}{2} \Box_{g_0} u_0 \mid \Sigma_0 = (K_\lambda)^{k \ell} \partial_k u_0 \partial_\ell u_0 + \bar{g}_0^{k \ell} \partial_k u_0 \partial_\ell |\nabla u_0|_{\bar{g}_0} - \frac{1}{2} \bar{g}_0^{k \ell} \partial_k \partial_\ell u_0.$$

A similar treatment can be applied to all the terms in (2.15). Now, on the one hand, $K_\lambda$ must be the second fundamental form $-\frac{1}{2} \mathcal{L}_{T_\lambda} g_{\lambda}$ of $\Sigma_0$, where $T_\lambda$ is the unit normal to $\Sigma_0$ for $g_{\lambda}$. On the other hand, we choose data in [Tou22] such that $T_\lambda = \partial_t + O(\lambda^2)$. Therefore, the oscillating parts of $K_\lambda^{(0)}$ and $K_\lambda^{(1)}$ must be consistent with (2.16), which explains expressions (2.7) and (2.10).

Remark 2.3. This entanglement between the construction of the data and the evolution equations in the spacetime is characteristic of the present high-frequency setting. Indeed, the spacelike constraint equations give two initial data, roughly $g \mid \Sigma_0$ and $\partial_t g \mid \Sigma_0$, while the oscillating terms in the spacetime ansatz satisfy first order transport equations which only require $g \mid \Sigma_0$ and prescribe, in part, $\partial_t g \mid \Sigma_0$.

Finally, note that the use of Sobolev spaces instead of Hölder spaces in Theorem 2.1 also comes from the application to [Tou22], since Sobolev spaces are more suited to non-linear wave equations. Similarly, the constraint $N \geq 10$ allows us to perform a bootstrap argument for the remainder in [Tou22] using only $L^2 - L^\infty$ estimates.

3. Strategy of Proof

In this section, we present the strategy of the proof of Theorem 2.1, which consists mainly in an adaptation of the conformal method to the high-frequency setting.

Remark 3.1. From now on and until Sect. 7 we drop the index $\lambda$ in the solution $(\bar{g}_\lambda, K_\lambda)$ obtained in Theorem 2.1 and simply write $(\bar{g}, K)$. 

3.1. High-frequency expansion of the parameters. We start by giving and motivating the expansions of the parameters of the conformal method, i.e the metric $\gamma$, the scalar function $\tau$, and the TT-tensor $\sigma$.

Since the metric $\bar{g}$ obtained in Theorem 2.1 will ultimately inherit its properties from the metric $\gamma$ defining the conformal class, $\gamma$ needs to fulfill two requirements: it needs to oscillate and to be close to $\bar{g}_0$. Therefore, we define $\gamma$ by

$$\gamma = \bar{g}_0 + \lambda \gamma^{(1)} \left( \frac{u_0}{\lambda} \right) + \lambda^2 \gamma^{(2)} \left( \frac{u_0}{\lambda} \right),$$

(3.1)

where we recall that the notation $\gamma^{(i)} \left( \frac{u_0}{\lambda} \right)$ is used to emphasize the fact that $\gamma^{(i)}$ is a linear combination of terms of the form (1.17). See Sect. 4.1 for the exact definition of $\gamma^{(1)}$ and $\gamma^{(2)}$.

Similarly, we want $K$ to be close to $K_0$ (in a weak sense since it is at the level of one derivative of the metric, see (1.4)) and $K_0$ is assumed to be $\bar{g}_0$-traceless. We thus define $\tau$ to be of order $\lambda^1$, i.e

$$\tau = \lambda \tau^{(1)} \left( \frac{u_0}{\lambda} \right).$$

(3.2)

The fact that the high-frequency perturbations don’t contribute to the mean curvature at the $\lambda^0$ order will be justified in the proof, but this can already be seen in Theorem 2.1: the contribution at $\lambda^0$ order to $K$ is given by $\frac{1}{2} \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} F^{(1)}$ (see (2.7)), and $F^{(1)}$ is $\bar{g}_0$-traceless.

The definition of the TT-tensor $\sigma$ requires a special construction linked to expressions (2.10)–(2.11). Since it involves also the expansion of the solution $\phi$ given in the next section, we postpone the discussion of this special construction to Sect. 3.3. However, we can already give its high-frequency expansion:

$$\sigma = \sigma^{(0)} \left( \frac{u_0}{\lambda} \right) + \lambda \sigma^{(1)} \left( \frac{u_0}{\lambda} \right) + \lambda^2 \left( \sigma^{(2)} \left( \frac{u_0}{\lambda} \right) + \tilde{\sigma} \right).$$

(3.3)

Here, $\tilde{\sigma}$ is a non-oscillating remainder whose role is to solve the equations defining the class of TT-tensors, that is $\text{div}_\gamma \sigma = 0$ and $\text{tr}_\gamma \sigma = 0$, with $\gamma$ defined by (3.1).

3.2. High-frequency expansion of the solutions. The solutions $(\phi, W)$ need to solve equations (1.15)–(1.16), where the parameters defined above appear as coefficients. To understand the expansions for $(\phi, W)$, let us look at the effect of the expansions (3.1)–(3.2)–(3.3) on the equations (1.15)–(1.16).

A standard feature of high-frequency quantities is that they lose one power of $\lambda$ per derivative. In (1.15), derivatives of the metric $\gamma$ only appear in its scalar curvature $R(\gamma)$ and we can derive from (3.1) that

$$R(\gamma) = \frac{|\nabla u_0|_{\bar{g}_0}^2}{\lambda} \lambda^2 \gamma_{N_0 N_0} \left( \gamma^{(1)} \left( \frac{u_0}{\lambda} \right) \right) + O \left( \lambda^0 \right).$$

(3.4)

Therefore, if $\gamma^{(1)}$ is well-chosen, the RHS of (1.15) is $O \left( \lambda^0 \right)$. This implies that the first oscillating term in $\phi$ must be at order $\lambda^2$, since $\Delta_\gamma$ is a second order elliptic operator. More precisely we choose

$$\phi = 1 + \lambda^2 \left( \phi^{(2)} \left( \frac{u_0}{\lambda} \right) + \tilde{\phi} \right) + \lambda^3 \phi^{(3)} \left( \frac{u_0}{\lambda} \right),$$

(3.5)
where \( \bar{\varphi} \) is a non-oscillating remainder. The constant coefficient in (3.5) ensures that \( \tilde{g}_\lambda = \tilde{g}_0 + O(\lambda) \). Even though \( \varphi^{(3)} \) appears after the remainder \( \bar{\varphi} \) in the ansatz for \( \varphi \), it appears before \( \bar{\varphi} \) in the hierarchy obtained from (1.15), again since \( \Delta_\gamma \) is a second order elliptic operator.

Similarly, (3.2) and (3.5) imply that \( \varphi^6 \) appears after the remainder \( \tilde{\varphi} \) in the ansatz for \( \varphi \), it appears before \( \tilde{\varphi} \) in the hierarchy obtained from (1.15), since \( \Delta_\gamma \) is a second order elliptic operator.

Similarly, (3.2) and (3.5) imply that \( \varphi^6 \) appears after the remainder \( \tilde{\varphi} \) in the ansatz for \( \varphi \), it appears before \( \tilde{\varphi} \) in the hierarchy obtained from (1.15), since \( \Delta_\gamma \) is a second order elliptic operator.

Increasingly, (3.2) and (3.5) imply that \( \varphi^6 \) appears after the remainder \( \tilde{\varphi} \) in the ansatz for \( \varphi \), it appears before \( \tilde{\varphi} \) in the hierarchy obtained from (1.15), since \( \Delta_\gamma \) is a second order elliptic operator.

Even though the operators \( \Delta_\gamma \) and \( \text{div}_\gamma L_\gamma \) are both second order elliptic operators, they depend differently on \( \gamma \). Indeed, the coefficients of \( \Delta_\gamma \) depend on \( \gamma \) and \( \nabla_\gamma \) only, while the coefficients of \( \text{div}_\gamma L_\gamma \) also depend on \( \nabla^2_\gamma \). From (3.1), we thus see that \( \text{div}_\gamma L_\gamma \) loses inherently one power of \( \lambda \). The analysis of the last equations in the hierarchy will therefore differ from the Hamiltonian to the momentum constraint.

3.3. **The TT-tensor.** In this section, we give more details on the definition of \( \sigma \), and in particular the first two terms in its high-frequency expansion (3.3), i.e. \( \sigma^{(0)} \) and \( \sigma^{(1)} \). Thanks to (3.5) we have \( \varphi = 1 + O(\lambda^2) \) which implies that

\[
K^{(0)} = \sigma^{(0)} + (L_\gamma W)^{(0)},
\]

\[
K^{(1)} = \sigma^{(1)} + (L_\gamma W)^{(1)} + \frac{1}{3} \tilde{g}_0 \tau^{(1)},
\]

where we used (1.14), (3.1) and (3.2). As explained in Sect. 2.3, the expressions of \( K^{(0)} \) and \( K^{(1)} \) are actually prescribed the application of Theorem 2.1 to the definition of initial data for (1.1). Therefore, (2.7)–(2.11) are used to define \( K^{(0)} \) and \( K^{(1)} \), while \( \sigma^{(0)} \) and \( \sigma^{(1)} \) are defined a posteriori such that (3.7)–(3.8) hold. We are then left with the task of showing that \( \sigma^{(0)} + \lambda \sigma^{(1)} \) defines an approximate TT-tensor and of constructing an exact one, with the help of \( \sigma^{(2)} \) and the remainder \( \tilde{\sigma} \) introduced in (3.3). The full construction of \( \sigma \) is the content of Sects. 5.3 and 5.4.

This procedure goes against the standard conformal method, where one starts by defining the parameters, then solve for the \( (\varphi, W) \), and finally obtain \( (\tilde{g}, K) \) via (1.13)–(1.14). This is another characteristic feature of the high-frequency setting, which originates in the redundancy described in Remark 2.3.

3.4. **Outline of the proof.** We give here the outline of the rest of this article, which proves Theorem 2.1.

- In Sect. 4 we define the metric \( \gamma \) and compute useful high-frequency expansions such as its scalar curvature \( R(\gamma) \) or differential operators depending on \( \gamma \) and appearing in the constraint equations.
- In Sect. 5 we define \( \varphi^{(2)}, \varphi^{(3)} \) and \( W^{(2)} \) such that they solve the first orders of the constraint equations and by doing so we also define the parameter \( \tau \). This section is concluded by the construction of the TT-tensor \( \sigma \).
• In Sect. 6 we fully solve the constraint equations with a fixed point argument for the remainders $\tilde{\varphi}$ and $\tilde{W}$ (we also define $W^{(3)}$ in the process).
• In Sect. 7 we conclude the proof of Theorem 2.1 by proving (2.12) and (2.13).

4. High-Frequency Conformal Class

In this section we define the metric $\gamma$ on $\Sigma_0$, that is the preferred member of the conformal class in which we look for $\bar{g}$ according to (1.13).

4.1. Definitions and first computations. We choose

$$\gamma = \bar{g}_0 + \lambda \gamma^{(1)} + \lambda^2 \gamma^{(2)},$$

(4.1)

where

$$\gamma^{(1)} = \cos \left( \frac{u_0}{\lambda} \right) F_0 \omega^{(1)},$$

(4.2)

$$\gamma^{(2)} = \sin \left( \frac{u_0}{\lambda} \right) \omega^{(2)}.$$  

(4.3)

The two symmetric 2-forms $\omega^{(1)}$ and $\omega^{(2)}$ are directly defined in the frame $(N_0, e_1, e_2)$.

• Definition of $\omega^{(1)}$. The coefficients of $\omega^{(1)}$ in the frame $(N_0, e_1, e_2)$ are constants and satisfy

$$\omega^{(1)}_{N_0i} = 0,$$

(4.4)

$$\omega^{(1)}_{11} + \omega^{(1)}_{22} = 0,$$

(4.5)

$$\left( \omega^{(1)}_{11} \right)^2 + \left( \omega^{(1)}_{12} \right)^2 = 4.$$  

(4.6)

• Definition of $\omega^{(2)}$. The 2-form $\omega^{(2)}$ depends on $\omega^{(1)}$ in the following way: all the coefficients of $\omega^{(2)}$ in the frame $(N_0, e_1, e_2)$ are set to be zero except $\omega^{(2)}_{N_01}$ and $\omega^{(2)}_{N_02}$ which are defined by

$$\omega^{(2)}_{N_0j} = \frac{1}{|\nabla u_0|_{\bar{g}_0}^2} (\text{div} \bar{g}_0 |\nabla u_0|_{\bar{g}_0} F_0 \omega^{(1)})(i-j) - \frac{1}{|\nabla u_0|_{\bar{g}_0}} F_0 \omega^{(1)}_{ij} \left( D_{N_0} N_0^i - \bar{g}^{ik} (K_0)_{N_0k} \right).$$

(4.7)

This expression will be justified in the proof of Lemma 5.3, where we construct the TT-tensor of the conformal method.

Remark 4.1. The properties (4.4) and (4.5) and the symmetry of $\omega^{(1)}$ imply that the tensor $\gamma^{(1)}$ is a linear combination of the tensors $e_1 \otimes e_1 - e_2 \otimes e_2$ and $e_1 \otimes e_2 + e_2 \otimes e_1$.

This is the equivalent of the 2 degrees of freedom (or polarization) of the TT gauge in the linearized gravity setting where $g_0$ is replaced by the Minkowskian metric, see Remark 2.2. The choice of the polarization of $\omega^{(1)}$, i.e the choice of $\omega^{(1)}_{11}$ and $\omega^{(1)}_{12}$ under the constraint (4.6), is the only freedom we have in the definition of the tensors $\bar{F}^{(1)}$, $\bar{F}^{(2,1)}$ and $\bar{F}^{(2,2)}$. 

We define
\[\tilde{F}^{(1)} = F_0 \omega^{(1)}.\] (4.8)

The following lemma summarizes the important properties of \(\tilde{F}^{(1)}\).

**Lemma 4.1.** The symmetric 2-tensor \(\tilde{F}^{(1)}\) is supported in \(B_R, \bar{g}_0\)-traceless and \(P_{0,u}\)-tangential. Moreover, (2.9) holds and
\[\| \tilde{F}^{(1)} \|_{H^N} \lesssim \varepsilon,\] (4.9)
with a constant depending only on \(\delta\) and \(R\).

**Proof.** The support property is implied by the support property of \(F_0\). The property (4.4) implies that \(\tilde{F}^{(1)}\) is \(P_{0,u}\)-tangential, which together with (4.5) implies that \(\tilde{F}^{(1)}\) is \(\bar{g}_0\)-traceless since
\[\text{tr}_{\bar{g}_0} \tilde{F}^{(1)} = \tilde{F}^{(1)}_{N_0N_0} + \tilde{F}^{(1)}_{11} + \tilde{F}^{(1)}_{22}.\]
Moreover, (4.6) and (4.8) imply (2.9). The estimation (4.9) comes from (2.3) and (2.4), the latter allowing us to estimate the coefficients of \(\omega^{(1)}\) in the usual Euclidean coordinates, i.e \(\omega^{(1)}_{ij}\). \(\square\)

The following lemma gives the expansion of the scalar curvature of \(\gamma\). This will be used to solve the Hamiltonian constraint (1.15).

**Lemma 4.2.** We have
\[R(\gamma) = R^{(0)} + \lambda R^{(1)} + \lambda^2 R^{(\geq 2)},\]
with
\[R^{(0)} = R(\bar{g}_0) - |\nabla u_0|^2 \frac{F_0^2}{\bar{g}_0} - 7 \cos\left(\frac{2u_0}{\lambda}\right) |\nabla u_0|^2 \frac{F_0^2}{\bar{g}_0^2} + \sin\left(\frac{u_0}{\lambda}\right) |\nabla u_0| \tilde{F}^{(1)}_{ij} \left(-\frac{\bar{g}_0^{ij}}{\bar{g}_0} \partial_i N_0^j + \frac{1}{2} N_0^i \bar{g}_0^{ij}\right),\] (4.10)
and
\[\left|R^{(1)}\right| + \left|R^{(\geq 2)}\right| \lesssim \left|\gamma^{-2} \partial^2 \gamma\right| + \left|\gamma^{-3} (\partial \gamma)^2\right|.\] (4.11)
Moreover
\[R^{(1)} \overset{osc}{\sim} \cos(\theta) + \sin(2\theta) + \cos(3\theta).\] (4.12)

**Proof.** We recall the definition of the scalar curvature
\[R(\gamma) = \gamma^{ij} \left(\partial_k \Gamma(\gamma)^{ik}_{ij} - \partial_i \Gamma(\gamma)^{kj}_{ij} + \Gamma(\gamma)^{ik}_{jk} \Gamma(\gamma)^{j\ell}_{ij} - \Gamma(\gamma)^{ik}_{ij} \Gamma(\gamma)^{j\ell}_{jk}\right).\]
We start by giving an estimation up to second order in \(\lambda\) of the inverse of \(\gamma\):
\[\gamma^{ij} = \bar{g}_0^{ij} - \lambda \cos\left(\frac{u_0}{\lambda}\right) (\tilde{F}^{(1)})^{ij} + \lambda^2 \left(\cos^2\left(\frac{u_0}{\lambda}\right) \bar{g}_0^{ij} (\tilde{F}^{(1)})^{j\ell} \tilde{F}^{(1)}_{k\ell} - \sin\left(\frac{u_0}{\lambda}\right) (\omega^{(2)})^{ij} + O\left(\lambda^3\right),\] (4.13)
where on the RHS all the inverses are taken with respect to the background metric \( \tilde{g}_0 \). We now expand the Christoffel symbols of \( \gamma \), we obtain

\[
\Gamma(\gamma)^k_{ij} = \Gamma(\tilde{g}_0)^k_{ij} + (\tilde{\Gamma}^{(0)})^k_{ij} + \lambda (\Gamma^{(1)})^k_{ij} + O \left( \lambda^2 \right),
\]

with

\[
(\tilde{\Gamma}^{(0)})^k_{ij} = -\frac{1}{2} \sin \left( \frac{u_0}{\lambda} \right) \tilde{g}_0^{\ell k} \left( \partial_{(i} u_0 \tilde{F}^{(1)}_{\ell j)} - \partial_{\ell} u_0 \tilde{F}^{(1)}_{ij} \right),
\]

\[
(\Gamma^{(1)})^k_{ij} = \frac{1}{2} \cos \left( \frac{u_0}{\lambda} \right) \tilde{g}_0^{\ell k} \left( \partial_{(i} u_0 \omega_{\ell j)} - \partial_{\ell} u_0 \omega_{ij} \right)
+ \frac{1}{4} \sin \left( \frac{2u_0}{\lambda} \right) (\tilde{F}^{(1)})^{k\ell} \partial_{(i} u_0 \tilde{F}^{(1)}_{\ell j)} + \cos \left( \frac{u_0}{\lambda} \right) \tilde{Q}^k_{ij},
\]

where we defined

\[
\tilde{Q}^k_{ij} = \frac{1}{2} \tilde{g}_0^{\ell k} \left( \partial_{(i} \tilde{F}^{(1)}_{\ell j)} - \partial_{\ell} \tilde{F}^{(1)}_{ij} \right) - \frac{1}{2} (\tilde{F}^{(1)})^{k\ell} \left( \partial_{(i} \tilde{g}_0 \tilde{F}^{(1)}_{\ell j)} - \partial_{\ell} \tilde{g}_0 \tilde{F}^{(1)}_{ij} \right).
\]

By using \( \tilde{\Gamma}^{(1)}_{N_0} = 0 \) and \( \text{tr}_{\tilde{g}_0} \tilde{F}^{(1)} = 0 \), we can compute useful contractions of \( \tilde{Q}^k_{ij} \). We first look at \( \tilde{Q}^k_{jk} \) which vanishes thanks to \( \text{tr}_{\tilde{g}_0} \tilde{F}^{(1)} = 0 \):

\[
\tilde{Q}^k_{jk} = \frac{1}{2} \left( \tilde{g}_0^{\ell k} \partial_j \tilde{F}^{(1)}_{\ell k} - (\tilde{F}^{(1)})^{k\ell} \partial_j (\tilde{g}_0)_{\ell k} \right) = \frac{1}{2} \partial_j \text{tr}_{\tilde{g}_0} \tilde{F}^{(1)} = 0.
\]

Using in addition \( \tilde{\Gamma}^{(1)}_{N_0} = 0 \) we have:

\[
\tilde{g}_0^{ij} (N_0)^k_{ij} \tilde{Q}^k_{ij} = \tilde{g}_0^{ij} N_0^\ell \left( \partial_{(i} \tilde{F}^{(1)}_{\ell j)} - \frac{1}{2} \partial_{\ell} \tilde{F}^{(1)}_{ij} \right)
= \tilde{g}_0^{ij} \partial_{(i} \tilde{F}^{(1)}_{\ell j)} - \frac{1}{2} N_0 \text{tr}_{\tilde{g}_0} \tilde{F}^{(1)} - \tilde{g}_0^{ij} \tilde{F}^{(1)}_{\ell j} \partial_{(i} N_0^\ell + \frac{1}{2} \tilde{F}^{(1)}_{\ell j} N_0 \tilde{g}_0^{ij}
= \tilde{F}^{(1)}_{\ell j} \left( -\tilde{g}_0^{ij} \partial_{(i} N_0^\ell + \frac{1}{2} N_0 \tilde{g}_0^{ij} \right).
\]

Since the scalar curvature contains first derivatives of the Christoffel symbols, \( R(\gamma) \) contains \textit{a priori} a \( \lambda^{-1} \) contribution from \( \partial_{\theta} \tilde{\Gamma}^{(0)} \), but thanks to the properties of \( \tilde{\Gamma}^{(1)} \) we can see that it vanishes. Indeed we have

\[
R^{(-1)} = \tilde{g}_0^{ij} \left( \partial_k u_0 \partial_{\theta} (\tilde{\Gamma}^{(0)})^k_{ij} - \partial_{(i} u_0 \partial_{\theta} (\tilde{\Gamma}^{(0)})^k_{\ell j)} \right),
\]

and \( \tilde{\Gamma}^{(1)}_{N_0} = 0 \) and \( \text{tr}_{\tilde{g}_0} \tilde{F}^{(1)} = 0 \) imply that

\[
\tilde{g}_0^{ij} (\tilde{\Gamma}^{(0)})^k_{ij} = 0 \quad \text{and} \quad (\tilde{\Gamma}^{(0)})^k_{jk} = 0.
\]
Let us now look at the $\lambda^0$ terms in $R(\gamma)$. Using again (4.19) we obtain:

\[
R^{(0)} \cdot R(\bar{g}_0) = \bar{g}_0^{ij} \left( \partial_k u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \partial_i u_0 \partial_\theta (\Gamma^{(1)})^k_{jk} \right) + \bar{g}_0^{ij} \partial_\lambda (\bar{\Gamma}^{(0)})^k_{ij} - 2 \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \Gamma(\bar{g}_0)^\ell_{ij} \right.
- \cos \left( \frac{u_0}{\lambda} \right) \left( \bar{F}^{(1)})^{ij}_{ij} \left( \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \partial_i u_0 \partial_\theta (\Gamma^{(1)})^k_{jk} \right) \right.

- \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} - 2 \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \Gamma(\bar{g}_0)^\ell_{ij} \right)

\left. - \cos \left( \frac{u_0}{\lambda} \right) \left( \bar{F}^{(1)})^{ij}_{ij} \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \bar{\Gamma}^{(0)}_{ij} \right), \right.

where we used (4.19) again. We set

\[
I := \bar{g}_0^{ij} \left( \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{jk} \right),

II := \bar{g}_0^{ij} \partial_\lambda (\bar{\Gamma}^{(0)})^k_{ij} - 2 \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \Gamma(\bar{g}_0)^\ell_{ij},

III := - \cos \left( \frac{u_0}{\lambda} \right) \left( \bar{F}^{(1)})^{ij}_{ij} \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \bar{\Gamma}^{(0)}_{ij} \right),

and compute further each of these terms. The term $I$ contains all the contributions from $\Gamma^{(1)}$ and (4.16) implies

\[
I = \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} \left( \omega^{(1)}_{11} + \omega^{(1)}_{22} \right) - \frac{1}{2} \cos \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} \left| \bar{F}^{(1)} \right|^2_{\bar{g}_0}

- \sin \left( \frac{u_0}{\lambda} \right) \bar{g}_0^{ij} \left( \partial_\lambda u_0 \bar{Q}^k_{ij} - \partial_\theta u_0 \bar{Q}_{jk}^k \right)

= -4 \cos \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2 + \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} \left( -\bar{g}_0^{ij} \partial_\lambda N_{ij}^\ell + \frac{1}{2} N_0 \bar{g}_0^{ij} \right),

where we used that the diagonal coefficients of $\omega^{(1)}$ in the frame $(N_0, e_1, e_2)$ vanish in addition to (4.17)–(4.18) and (2.9). The term $II$ contains all the linear terms in $\bar{\Gamma}^{(0)}$. Note that in the notation $\partial_k (\bar{\Gamma}^{(0)})^k_{ij}$ the derivative does not hit the oscillating part of $\bar{\Gamma}^{(0)}$ (i.e. $\sin (\frac{u_0}{\lambda})$ in (4.15)). Using (4.19) we obtain $\bar{g}_0^{ij} \partial_\lambda (\bar{\Gamma}^{(0)})^k_{ij} = - (\bar{\Gamma}^{(0)})^k_{ij} \partial_\lambda \bar{g}_0^{ij}$ and

\[
II = - (\bar{\Gamma}^{(0)})^k_{ij} \partial_\lambda \bar{g}_0^{ij} + 2 \bar{g}_0^{ij} \Gamma(\bar{g}_0)^\ell_{ij} \right) = 0,

where we used $(\bar{\Gamma}^{(0)})^k_{ij} = (\bar{\Gamma}^{(0)})^k_{ij}$. The term $III$ contains quadratic terms in $\bar{\Gamma}^{(0)}$. Using (4.15) and (2.9) we obtain

\[
III = - \cos \left( \frac{u_0}{\lambda} \right) \left( \bar{F}^{(1)})^{ij}_{ij} \partial_\lambda u_0 \partial_\theta (\Gamma^{(1)})^k_{ij} - \bar{g}_0^{ij} (\bar{\Gamma}^{(0)})^k_{ij} \bar{\Gamma}^{(0)}_{ij} \right),

\[
= - \frac{1}{2} \cos^2 \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} \left| \bar{F}^{(1)} \right|^2_{\bar{g}_0} + \frac{1}{4} \sin^2 \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} \left| \bar{F}^{(1)} \right|^2_{\bar{g}_0}

= - |\nabla u_0|^2_{\bar{g}_0} F_0^2 - 3 \cos \left( \frac{u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2.

We conclude the proof of (4.10) by adding \( I \) and \( III \).

The proof of (4.12) comes from the schematic formula

\[
R(\gamma) = \gamma^{-2}\partial^2\gamma + \gamma^{-3}(\partial\gamma)^2,
\]

which gives schematically

\[
R^{(1)} = \left(\gamma^{-1}(2)\gamma^{-1}(0) + \left(\gamma^{-1}(1)\right)^2\right)(\partial^2\gamma)^{(-1)} + \left(\gamma^{-1}(1)\gamma^{-1}(0)\right)\left(\partial^2\gamma\right)\left(\partial\gamma\right)^{(0)}
+ \left(\gamma^{-1}(0)\right)^2(\partial^2\gamma)^{(1)} + \left(\gamma^{-1}(1)\right)\left(\gamma^{-1}(0)\right)\left(\partial\gamma\right)^{(0)}
+ \left(\gamma^{-1}(0)\right)^3\left(\partial\gamma\right)^{(1)}\left(\gamma^{-1}(0)\right),
\]

(4.20)

where we recall that our high-frequency notations introduced in Sect. 1.3.4 give for instance

\[
(\partial^2\gamma)^{(1)} = \cos\left(\frac{u_0}{\lambda}\right)\partial^2\tilde{F}^{(1)} + 2\cos\left(\frac{u_0}{\lambda}\right)\partial\omega^{(2)},
\]

and

\[
(\gamma^{-1})^{(1)} = -\cos\left(\frac{u_0}{\lambda}\right)\tilde{F}^{(1)}.
\]

Using (4.1) and (4.13) we obtain

\[
(\gamma^{-1})^{(0)} \overset{osc}{\sim} 1,
(\gamma^{-1})^{(1)} \overset{osc}{\sim} \cos(\theta),
(\gamma^{-1})^{(2)} \overset{osc}{\sim} 1 + \sin(\theta) + \cos(2\theta),
\]

and

\[
(\partial\gamma)^{(0)}, \ (\partial^2\gamma)^{(0)} \overset{osc}{\sim} 1 + \sin(\theta),
(\partial\gamma)^{(1)}, \ (\partial^2\gamma)^{(-1)}, \ (\partial^2\gamma)^{(1)} \overset{osc}{\sim} \cos(\theta).
\]

Using these oscillating behaviours we can prove by a direct computation that \( R^{(1)} \) defined by (4.20) satisfies (4.12). This concludes the proof of the lemma.

\[\square\]

Remark 4.2. Note that we denote by \( \gamma^{-k} \) any product of \( k \) coefficients of the inverse metric \( \gamma^{-1} \). This also applies to the background inverse metric \( \tilde{g}_0^{-1} \).

4.2. Expansion of differential operators. The equations (1.15) and (1.16) involve differential operators depending on the metric \( \gamma \). When they are applied to high-frequency quantities, we need to take into account the expansion (4.1) affecting the coefficients of the operators as well as the expansion of the quantities themselves. In this section we compute such expansions for all the operators involved, that is the Laplace-Beltrami operator, the divergence operator, the conformal Killing operator and the conformal Laplacian.
In terms of notation, if $P_\gamma$ is a differential operator acting on tensors of any type and whose coefficients depend on $\gamma$, we can formally obtain an expansion of $P_\gamma(T)$ of the form

$$P_\gamma(T) = \sum_k \lambda^k P_\gamma^{[k]}(T).$$

where the previous sum has finite support. The bracket notation $[k]$ thus plays the same role for differential operators as the parenthesis notation $(i)$ introduced in Sect. 1.3.4 for tensors.

Moreover we can mix the two cases, i.e apply differential operators $P_\gamma$ depending on $\gamma$ to oscillating tensors $T (u_0^0 \lambda)$. The expansion of $P_\gamma(T)$ then depends on the order of $T \mapsto P_\gamma(T)$ and also $\gamma \mapsto P_\gamma(T)$. By the order of $\gamma \mapsto P_\gamma(T)$, we simply mean the top derivative of $\gamma$ appearing in the coefficients of $P_\gamma$. Since both oscillating and non-oscillating terms appear in the expansions for the parameters and the unknowns of the conformal method (see Sect.3.1), we make a difference between the expansions for $P_\gamma(T)$ when $T$ is non-oscillating and when $T$ is oscillating. Usual capital letters are used in the first case and bold capital letters in the second case, i.e

$$P_\gamma(T) = \sum_k \lambda^k P_\gamma^{[k]}(T) \quad \text{and} \quad P_\gamma \left( T \left( \frac{u_0^0}{\lambda} \right) \right) = \sum_k \lambda^k P_\gamma^{[k]}(T),$$

where the support of the two previous finite sums are a priori different, depending on $T$ and $P_\gamma$. This explains the difference between Lemmas 4.6 and 4.7 below.

### 4.2.1. The Laplace-Beltrami operator

The only differential operator in the hamiltonian constraint (1.15) is the Laplace-Beltrami operator associated to $\gamma$. If $h$ is a generic Riemannian metric on $\Sigma_0$, we define its Laplace-Beltrami operator by

$$\Delta_h f = h^{ij} \left( \partial_i \partial_j f - \Gamma(h)^{ij}_k \partial_k f \right),$$

for $f$ a scalar function. Note that $\Delta_e$ is the usual Laplacian operator on $\mathbb{R}^3$ and is denoted by $\Delta$ in this article. Since $\gamma = O (\lambda^0)$ and $\partial \gamma = O (\lambda^0)$, if $f$ does not admit a high-frequency expansion then

$$\Delta_\gamma f = O (\lambda^0).$$

If $f$ admit a high-frequency expansion we have the following lemma.

**Lemma 4.3.** For $f \left( \frac{u_0^0}{\lambda} \right)$ an oscillating scalar function we have

$$\Delta_\gamma \left( f \left( \frac{u_0^0}{\lambda} \right) \right) = \frac{1}{\lambda^2} H^{[-2]}(f) + \frac{1}{\lambda} H^{[-1]}(f) + H^{[\geq 0]}(f),$$

with

$$H^{[-2]}(f) = |\nabla u_0|^2 \bar{g}_0^{ij} \partial_\theta^2 f, \quad (4.21)$$

$$H^{[-1]}(f) = 2 \bar{g}_0^{ij} \partial_i u_0 \partial_j \theta f + (\bar{\Delta}_{\bar{g}_0} u_0) \partial_\theta f - \bar{g}_0^{ij} \Gamma(\bar{g}_0)^k_{ij} \partial_k u_0 \partial_\theta f, \quad (4.22)$$

and

$$\left| H^{[\geq 0]}(f) \right| \lesssim \left| \gamma^{-1} \partial^2 f \right| + \left| \gamma^{-2} \partial \gamma \partial f \right|. \quad (4.23)$$
Proof. We start with the definition of the Laplace-Beltrami operator:

\[ \Delta_\gamma \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) = \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right). \]

Using the expansion of the inverse of \( \gamma \) and \( \bar{F}^{(1)}_{N_0} = 0 \) we have

\[ \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) = \frac{1}{\lambda^2} \left| \nabla u_0 \right| \gamma \partial_\theta \left( \rho A_0 \right) \partial_\theta \varphi + \frac{1}{\lambda} \left( 2 \gamma^{ij} \partial_i u_0 \partial_j \varphi \right) + O \left( \lambda^0 \right) \]

where \( \tilde{\Delta}_h = h^{ij} \partial_i \partial_j \). Moreover from the decomposition of the Christoffel symbols (4.14) and (4.19) we obtain

\[ \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) = \frac{1}{\lambda^2} \nabla \left( \rho A_0 \right) \partial_\theta \varphi + \frac{1}{\lambda} \left( 2 \bar{g}_0^{ij} \partial_i u_0 \partial_j \varphi \right) + O \left( \lambda^0 \right). \]

The estimate (4.23) simply comes from the definition of \( \Delta_\gamma \). \( \square \)

4.2.2. The divergence operator The divergence operator appears in the momentum constraint (1.16) but also in the definition of the parameter \( \sigma \), which in particular needs to be a divergence free tensor for the metric \( \gamma \). Recall that if \( h \) is a Riemannian metric on \( \Sigma_0 \) and if \( D^{(h)} \) denotes the covariant derivative associated, then \( \text{div}_h A = h^{k\ell} D_k^{(h)} A_\ell \) and \( (\text{div}_h B)_i = h^{k\ell} D_k^{(h)} B_{i\ell} \) for \( A \) a 1-tensor and \( B \) a 2-tensor. The divergence operator only depends on first derivatives of \( \gamma \) which are \( O \left( \lambda^0 \right) \) so we only need an expansion when the tensor on which it acts is itself oscillating.

Lemma 4.4. For \( A_{ij} \left( \frac{u_0}{\lambda} \right) \) an oscillating symmetric 2-tensor we have

\[ \text{div}_\gamma A \left( \frac{u_0}{\lambda} \right)_\ell = \frac{1}{\lambda} \left( \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right) + \lambda \left( \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right), \]

with

\[ \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) = \left( \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right), \]

and

\[ \left| \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right| = \left| \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right|. \]

Moreover

\[ \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) = \left( \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right), \]

\[ \left| \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right| = \left| \gamma^{ij} \partial_i \partial_j \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) - \gamma^{ij} \Gamma(\gamma)^k_{ij} \partial_k \left( \varphi \left( \frac{u_0}{\lambda} \right) \right) \right|. \]
Proof. We start by using the expansion of the Christoffel’s symbols from (4.14):

\[
\mathbf{D}^{(\nu)}_i A \left( \frac{u_0}{\lambda} \right)_{j\ell} = \frac{1}{\lambda} \partial_i u_0 \partial_\theta A_{j\ell} + \tilde{D}_i A_{j\ell} - (\hat{\Gamma}^{(0)})^a_{i(j} A_{a\ell)} - \lambda (\Gamma^{(1)})^a_{i(j} A_{a\ell)} + O \left( \lambda^2 \right).
\]

We now use the expansion of the inverse of \( A \):

\[
\bar{L} \expansion of \gamma \text{ defined by (1.14), we need to expand the conformal Killing operator defined by}

\[
\bar{L} \expansion of \frac{1}{\lambda} (\tilde{\gamma}^{-1}) = \frac{1}{\lambda} \partial_i u_0 \partial_\theta A_{N\ell} + \text{div} \tilde{g}_0 A_{N\ell} - \tilde{g}_0 (\tilde{\gamma}^{(0)})_{i(j} A_{a\ell)}^a - \lambda (\Gamma^{(1)})_{i(j} A_{a\ell)}^a + O \left( \lambda^2 \right).
\]

This concludes the proof, using \( \hat{F}^{(1)}_{N\ell} = 0 \) and \( \tilde{g}_0 (\tilde{\gamma}^{(0)})_{i(j} A_{a\ell)}^a = 0 \). The estimate (4.26) just comes from the definition of the divergence. The oscillating behaviours (4.27) and (4.28) can be directly read on (4.29).

4.2.3. The conformal Killing operator In order to compute \( L_\gamma W \) in the ansatz for \( K \) (1.14), we need to expand the conformal Killing operator defined by

\[
(L_h A)_{ij} = \mathbf{D}^{(h)}_{ij} A_{j} - \frac{2}{3} (\text{div}_h A) h_{ij},
\]

where \( A \) is a 1-tensor. Note that \( L_h A \) is a symmetric 2-tensor traceless with respect to \( h \). As for the divergence operator, \( L_\gamma \) only depends on first derivatives of \( \gamma \) so only an expansion of \( L_\gamma W \) with \( W \) oscillating is necessary.

Lemma 4.5. Let \( W \left( \frac{u_0}{\lambda} \right) \) be an oscillating 1-form. We have

\[
L_\gamma \left( W \left( \frac{u_0}{\lambda} \right) \right)_{ij} = \frac{1}{\lambda} K^{[-1]}_{ij} (W) + K^{[0]}_{ij} (W) + \lambda K^{[\geq 1]}_{ij} (W),
\]

where

\[
K^{[-1]}_{ij} (W) = \partial_i u_0 \partial_\theta W_j + \frac{2}{3} |\nabla u_0| \tilde{g}_0 (\tilde{g}_0)_{ij} \partial_\theta W_{N0},
\]

\[
K^{[0]}_{ij} (W) = \tilde{D}_i W_j - 2 W_k (\tilde{\gamma}^{(0)})^{k} \partial_i u_0 \partial_\theta W_{N0} + \frac{2}{3} \cos \left( \frac{u_0}{\lambda} \right) |\nabla u_0| \tilde{F}^{(1)}_{ij} \partial_\theta W_{N0} - \frac{2}{3} \text{div} \tilde{g}_0 W (\tilde{g}_0)_{ij},
\]

and

\[
|K^{[\geq 1]}_{ij} (W)| \lesssim |\partial W| + \|\gamma^{-1} \partial_\gamma W\|.
\]
The following hold:

\[
\text{tr}_{\tilde{g}_0} K^{-1}(W) = 0, \tag{4.34}
\]

\[
\text{tr}_{\tilde{g}_0} K^{[0]}(W) = \cos \left( \frac{u_0}{\lambda} \right) \text{tr}_{\tilde{F}^{(1)}} K^{-1}(W). \tag{4.35}
\]

The following oscillating behaviour holds

\[
K^{[0]}(W) \overset{\text{osc}}{\sim} (1 + \sin(\theta)) W + \cos(\theta) \partial_{\theta} W. \tag{4.36}
\]

**Proof.** We start with the covariant derivative of the oscillating 1-form \( W(\frac{u_0}{\lambda}) \):

\[
D_i^{(\gamma)} \left( W \left( \frac{u_0}{\lambda} \right) \right)_j = \frac{1}{\lambda} \partial_i u_0 \partial_{\theta} W_j + \tilde{D}_i W_j \left( \frac{u_0}{\lambda} \right) \\
- W_k \left( (\tilde{\Gamma}^{(0)})_{ij}^k + \lambda (\Gamma^{(1)})_{ij}^k + \lambda^2 (\Gamma^{(2)})_{ij}^k + \cdots \right),
\]

which also gives us the divergence

\[
\text{div}_\gamma \left( W \left( \frac{u_0}{\lambda} \right) \right) = \gamma^{ij} D_i^{(\gamma)} \left( W \left( \frac{u_0}{\lambda} \right) \right)_j \\
= \frac{1}{\lambda} (\tilde{g}_0)^{k\ell} \partial_k u_0 \partial_{\theta} W_{\ell} - \cos \left( \frac{u_0}{\lambda} \right) (\tilde{F}^{(1)})_{ij}^{k\ell} \partial_k u_0 \partial_{\theta} W_{\ell} \\
+ \text{div}_{\tilde{g}_0} W - (\tilde{g}_0)^{k\ell} (\tilde{\Gamma}^{(0)})_{ji}^{k\ell} W_a + O(\lambda).
\]

We conclude the proof of (4.31)–(4.32) with

\[
L_{\gamma} \left( W \left( \frac{u_0}{\lambda} \right) \right)_{ij} = D_i^{(\gamma)} \left( W \left( \frac{u_0}{\lambda} \right) \right)_j - \frac{2}{3} \text{div}_\gamma \left( W \left( \frac{u_0}{\lambda} \right) \right) \gamma_{ij},
\]

and \( \tilde{\mathcal{F}}_{Nai} = 0 \) and \((\tilde{g}_0)^{ij} (\tilde{\Gamma}^{(0)})_{ji}^a = 0 \). The trace identities (4.34)–(4.35) and the oscillating behaviour (4.36) follows directly from (4.31)–(4.32).

**Remark 4.3.** In the previous lemma we used the notation \( \text{tr}_{\tilde{F}^{(1)}} K^{-1}(W) \) to denote

\[
(\tilde{F}^{(1)})^{ij} K_{ij}^{-1}(W) = \tilde{g}_0^{ik} \tilde{g}_0^{j\ell} \tilde{F}_{k\ell}^{(1)} K_{ij}^{-1}(W),
\]

even though \( \tilde{F}^{(1)} \) is not a Lorentzian metric.

### 4.2.4. The conformal Laplacian

The conformal Laplacian associated to a Riemannian metric \( h \) is the operator \( \text{div}_h L_h \) acting on 1-tensor. It is the only operator considered in this article depending on \( \partial^2 \gamma \), i.e second derivatives of the metric \( \gamma \). Indeed \( L_{\gamma} \) contains the Christoffel symbols of \( \gamma \) through the covariant derivative, and they are differenciated once by the divergence. Since \( \partial^2 \gamma = O(\lambda^{-1}) \), this a major difference with the Laplace-Beltrami operator or the divergence operator from the point of view of expanding quantities in powers of \( \lambda \). Indeed, this implies that even if the 1-form \( \beta \) is not oscillating, the quantity \( \text{div}_\gamma L_{\gamma} \beta \) still loses one power of \( \lambda \). If \( \beta \) is oscillating, then \( \text{div}_\gamma L_{\gamma} \beta \) loses two powers of \( \lambda \) since the conformal Laplacian is a second order operator. This explains the two following lemmas.
Lemma 4.6. Let \( \beta \) be a 1-form. We have

\[
\text{div}_\gamma L_{\gamma} \beta = \frac{1}{\lambda} M[\lambda^{-1}(\beta)] + M[\lambda^{\geq 0}(\beta)],
\]

with

\[
M[\lambda^{-1}](\beta) = |\nabla u_0|^2_{\tilde{g}_0} \cos \left( \frac{u_0}{\lambda} \right) \tilde{g}^{ij}_0 \tilde{F}^{(1)}_{i\ell} \beta_j,
\]

and

\[
\left| M[\lambda^{\geq 0}](\beta) - \text{div}_e L_e \beta \right| \lesssim \left| \gamma^{-1} - e^{-1} \right| |\partial^2 \beta| + \left| \gamma^{-2} \partial \gamma \right| \left| 1 + \gamma^{-1} \right| |\partial \beta| + \left| \gamma^{-3} (\partial \gamma)^2 + \gamma^{-2} (\partial^2 \gamma)^{(\geq 0)} \right| \left| 1 + \gamma^{-1} \right| |\beta|.
\]

Proof. Let \( \tilde{g} \) be any Riemannian metric on \( \mathbb{R}^3 \), we have

\[
\text{div}_\gamma L_{\gamma} \beta = \tilde{g}^{ij} \left( \partial_i \partial_j \beta_\ell - \frac{2}{3} \partial_i \partial_\ell \beta_j \right) + \beta_k \left( -2 \tilde{g}^{ij} \partial_i \Gamma(\tilde{g})^{k}_j \ell \beta_j + \frac{2}{3} \partial_i \left( \tilde{g}^{ij} \Gamma(\tilde{g})^{k}_{ij} \right) \right)
\]

\[
- \frac{2}{3} \partial_i \partial_j \beta_\ell \left( \tilde{g}^{ij} \beta_k + \frac{2}{3} \text{div}_e \beta \right) + \frac{2}{3} \left( \partial_i \beta_j \partial_\ell \tilde{g}^{ij} + \tilde{g}^{ij} \Gamma(\tilde{g})^{k}_{ij} \partial_\ell \beta_k \right)
\]

\[
- \tilde{g}^{ij} \Gamma(\tilde{g})^{k}_{ij} \tilde{L}_\ell \tilde{\beta}_k \ell.
\]

In the case of the high-frequency metric \( \gamma \), the only terms loosing \( \frac{1}{\lambda} \), i.e contributing to \( M[\lambda^{-1}](\beta) \), are the \( \gamma^{-2} \partial^2 \gamma \beta \). More precisely, it concerns terms involving \( \partial \tilde{\Gamma}^{(0)} \), and since \( \tilde{g}^{ij}_0 (\tilde{\Gamma}^{(0)})^{k}_{ij} = 0 \), the only contribution is

\[
M[\lambda^{-1}](\beta) = -2 \beta_k \tilde{g}^{ij} \partial_i \partial_\ell \beta_j + \tilde{g}^{ij} \tilde{F}^{(1)}_{i\ell} \beta_j,
\]

where we used (4.15) and \( \tilde{F}^{(1)}_{N_0} = 0 \). From (4.39) we also get

\[
\text{div}_e L_e \beta = e^{ij} \left( \partial_i \partial_j \beta_\ell - \frac{2}{3} \partial_\ell \partial_j \beta_j \right),
\]

which concludes the proof of (4.38). \( \square \)

The estimation (4.38) will allow us to invert the operator \( M[\lambda^{\geq 0}] \), since we know how to invert \( \text{div}_e L_e \) (see Proposition 1.2) and since we gain a smallness constant \( \varepsilon \) in front of \( \partial^2 \beta \) thanks to (2.1) and (4.13). We now state the final lemma of this section, which will allow us to construct high-frequency solutions of the momentum constraint. The proof is left to the reader since it follows from Lemmas 4.4 and 4.5.

Lemma 4.7. Let \( W \left( \frac{u_0}{\lambda} \right) \) be an oscillating 1-form. We have

\[
\text{div}_\gamma L_{\gamma} \left( W \left( \frac{u_0}{\lambda} \right) \right) = \frac{1}{\lambda^2} M[\lambda^{-2}](W) + \frac{1}{\lambda} M[\lambda^{-1}](W) + M[\lambda^{\geq 0}](W),
\]

with

\[
M[\lambda^{-2}](W) = d[\lambda^{-1}](\mathcal{K}[\lambda^{-1}](W)),
\]

\[
M[\lambda^{-1}](W) = d[\lambda^{-1}](\mathcal{K}[\lambda^{0}](W)) + d[\lambda^{0}](\mathcal{K}[\lambda^{-1}](W)).
\]
The following oscillating behaviour holds

\[ M^{[-1]}(W) \sim \cos(\theta) W + \cos(\theta) \partial_\theta^2 W + (1 + \sin(\theta)) \partial_\theta W. \] (4.42)

From (4.24), (4.31) and (4.40) we obtain

\[ M^{[-2]}(W) = |\nabla u_0|_{\bar{g}_0}^2 \left( \partial_\theta^2 W_\ell + \frac{1}{3} (N_0)_\ell \partial_\theta^2 W_{N_0} \right). \] (4.43)

In order to solve the momentum constraint, we need to invert this operator. This is done in the following simple lemma.

**Lemma 4.8.** If \( \alpha \) and \( \beta \) are 1-forms such that

\[ \alpha_\ell + \frac{1}{3} (N_0)_\ell \alpha_{N_0} = \beta_\ell, \] (4.44)

then

\[ \alpha_\ell = \beta_\ell - \frac{1}{4} (N_0)_\ell \beta_{N_0}. \]

**Proof.** We contract (4.44) with the vector field \( N_0 \) to obtain

\[ \frac{4}{3} \alpha_{N_0} = \beta_{N_0}. \]

Inserting this into (4.44) concludes the proof. \( \square \)

5. **Approximate Solution to the Constraint Equations**

In this section, we construct an approximate solution to the constraint equations (1.15) and (1.16) by solving the \( \lambda^0 \) and \( \lambda^1 \) Hamiltonian levels and the \( \lambda^0 \) momentum level. In the process we will define the parameter \( \tau \) and in Sects. 5.3 and 5.4 we construct the parameter \( \sigma \).

5.1. **The approximate Hamiltonian constraint.** We solve the first two levels of the Hamiltonian constraint by choosing \( \varphi^{(2)} \) and \( \varphi^{(3)} \). At those levels, (1.15) decouples from (1.16) since we replace \( \sigma + L_\gamma W \) (\( \leq 1 \)) by \( \left( K - \frac{1}{3} \tau \gamma \right) (\leq 1) \), where \( K (\leq 1) \) and \( \tau (\leq 1) \) will be defined along the way.

5.1.1. **The \( \lambda^0 \) Hamiltonian level** We want \( \varphi^{(2)} \) to solve the \( \lambda^0 \) Hamiltonian level. Using the expansion of the Laplace-Beltrami operator \( \Delta_\gamma \) (see Lemma 4.3), this is equivalent to the following equation:

\[ 8 |\nabla u_0|_{\bar{g}_0}^2 \partial_\theta^2 \varphi^{(2)} = R^{(0)} + \frac{2}{3} \left( \tau^{(0)} \right)^2 - \left| K^{(0)} - \frac{1}{3} \tau^{(0)} \bar{g}_0 \right|^2. \] (5.1)

where \( K^{(0)} \) is defined by

\[ K^{(0)} = K_0 + \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \tilde{F}^{(1)}, \] (5.2)
following the discussion of Sect. 3.3. As the parameter $\tau$ is the trace with respect to $\gamma$ of $K$, we have
\[
\tau^{(0)} = \text{tr}_{\bar{g}_0} K^{(0)}
= \text{tr}_{\bar{g}_0} K_0 + \frac{1}{2} \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \text{tr}_{\bar{g}_0} \bar{F}^{(1)}
= 0,
\] (5.3)
where we used (1.7) and Lemma 4.1. Therefore, (5.1) rewrites
\[
8|\nabla u_0|^2_{\bar{g}_0} \partial_\theta \varphi^{(2)} = R^{(0)} - \left| K^{(0)} \right|^2_{\bar{g}_0}.
\] (5.4)
The LHS of (5.4) being a derivative with respect to $\theta$, we need the RHS to be purely oscillating. Since this shows how the creation of non-oscillating terms is absorbed by the background constraint equations, we state this in a separate lemma.

**Lemma 5.1.** We have
\[
R^{(0)} - \left| K^{(0)} \right|^2_{\bar{g}_0} = \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} \left( -\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} N_0 \bar{g}^{\ell j}_0 - K_0^{\ell j} \right)
- 6 \cos \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2.
\]

**Proof.** Let us first expand $\left| K^{(0)} \right|^2_{\bar{g}_0}$ using (5.2) and (2.9):
\[
\left| K^{(0)} \right|^2_{\bar{g}_0} = |K_0|^2_{\bar{g}_0} + |\nabla u_0|^2_{\bar{g}_0} F_0^2 - \cos \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2
+ \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} \cdot K_0_{\bar{g}_0}.
\]
We use now (4.10) to compute the full RHS of (5.4):
\[
R^{(0)} - \left| K^{(0)} \right|^2_{\bar{g}_0} = R(\bar{g}_0) - |K_0|^2_{\bar{g}_0} - 2|\nabla u_0|^2_{\bar{g}_0} F_0^2
+ \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \left( \bar{F}^{(1)}_{\ell j} \left( -\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} N_0 \bar{g}^{\ell j}_0 \right) \right)
- \left| \bar{F}^{(1)} \cdot K_0 \right|_{\bar{g}_0}
- 6 \cos \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2.
\]
Using the background Hamiltonian constraint (1.5) to cancel the non-oscillating term, we are left with
\[
R^{(0)} - \left| K^{(0)} \right|^2_{\bar{g}_0} = \sin \left( \frac{u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \left( \bar{F}^{(1)}_{\ell j} \left( -\bar{g}_0^{ij} \partial_i N_0^\ell + \frac{1}{2} N_0 \bar{g}^{\ell j}_0 \right) \right)
- \left| \bar{F}^{(1)} \cdot K_0 \right|_{\bar{g}_0}
- 6 \cos \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|^2_{\bar{g}_0} F_0^2,
\]
which concludes the proof. □
This lemma allows us to solve (5.4) by simply setting
\[\varphi^{(2)} = \frac{1}{8|\nabla u_0|_{g_0}} \sin \left( \frac{u_0}{\lambda} \right) \bar{F}^{(1)}_{ij} \left( g_0^{ij} \partial_i N_0 - \frac{1}{2} N_0 g_0^{\ell j} + K_0^{\ell j} \right) + \frac{3}{16} \cos \left( \frac{2u_0}{\lambda} \right) F_0^2.\] (5.5)

Now that \(\varphi^{(2)}\) is defined, we can set
\[\bar{F}^{(2,1)} = \frac{\bar{F}^{(1)}_{ij}}{2|\nabla u_0|_{g_0}} \left( g_0^{ij} \partial_i N_0 - \frac{1}{2} N_0 g_0^{\ell j} + K_0^{\ell j} \right) g_0 + \omega^{(2)},\] (5.6)
\[\bar{F}^{(2,2)} = \frac{3}{4} F_0^2 g_0.\] (5.7)

These definitions are consistent with (1.13) and the following lemma summarizes the properties of \(\bar{F}^{(2,1)}\) and \(\bar{F}^{(2,2)}\):

Lemma 5.2. The symmetric 2-tensors \(\bar{F}^{(2,1)}\) and \(\bar{F}^{(2,2)}\) are supported in \(B_R\) and the following holds
\[\left\| \bar{F}^{(2,1)} \right\|_{H^{N-1}} + \left\| \bar{F}^{(2,2)} \right\|_{H^{N-1}} \lesssim \varepsilon,\] (5.8)
with a constant depending only on \(\delta\) and \(R\).

Proof. The support property follows from the support properties of \(F_0, \bar{F}^{(1)}\) and \(\omega^{(2)}\).

The estimate (5.8) follows from the definitions (5.6) and (5.7) and the estimates (2.1), (2.3) and (4.9) (recall also (4.7) for the definition of \(\omega^{(2)}\)). \(\square\)

5.1.2. The \(\lambda^1\) Hamiltonian level We now turn to the \(\lambda^1\) Hamiltonian level, which is solved thanks to \(\varphi^{(3)}\). More precisely, since \(\tau = O(\lambda)\) we need \(\varphi^{(3)}\) to satisfy
\[8|\nabla u_0|_{g_0}^2 \partial_0^2 \varphi^{(3)} = -8H^{[-1]} \left( \varphi^{(2)} \right) + R^{(1)} - \left| K^{(0)} \cdot \left( K^{(1)} - \frac{1}{3} \tau^{(1)} g_0 \right) \right|_{g_0} + \cos \left( \frac{u_0}{\lambda} \right) |K^{(0)}|^2 F^{(1)}_{1} (5.9)\]

where \(R^{(1)}\) is defined in Lemma 4.2 and \(H^{[-1]} (\varphi^{(2)})\) in Lemma 4.3. In the RHS of (5.9), it remains to define \(K^{(1)}\) and \(\tau^{(1)}\). Following the discussion of Sect. 3.3, we define
\[K^{(1)}_{ij} = -\frac{1}{2} \cos \left( \frac{u_0}{\lambda} \right) \left( -N_0 \bar{F}^{(1)}_{ij} + \left( -\bar{g}^{k\ell} (K_0)_{(i\ell} + N_0^\ell \Gamma (g_0)_{(i\ell}^{k)} \right) \bar{F}^{(1)}_{j)k} + \frac{1}{|\nabla u_0|_{g_0}} \left( -(K_0)^{k\ell} \partial_k u_0 \partial_\ell u_0 - \bar{g}^{k\ell} \partial_k u_0 \partial_\ell |\nabla u_0|_{g_0} + \frac{1}{2} \bar{g}^{k\ell} \partial_k \partial_\ell u_0 \right) \bar{F}^{(1)}_{ij} \right) - \frac{1}{2} \frac{1}{|\nabla u_0|_{g_0}} \left( \cos \left( \frac{u_0}{\lambda} \right) \bar{F}^{(2,1)}_{ij} - 2 \sin \left( \frac{2u_0}{\lambda} \right) \bar{F}^{(2,2)}_{ij} \right).\] (5.10)

Since \(\tau\) is the trace with respect to \(\gamma\) of \(K\) we set
\[\tau^{(1)} = -\cos \left( \frac{u_0}{\lambda} \right) \left| \bar{F}^{(1)}_{ij} \cdot K^{(1)}_{ij} \right|_{g_0} + \text{tr}_{\bar{g}_0} K^{(1)} \] (5.11)
where we used the expansion of the inverse of $\gamma$ given by (4.13). This actually fully defines the parameter $\tau$, i.e

$$\tau = \lambda \tau^{(1)}. \quad (5.12)$$

As above, we need to show that the RHS of (5.9) is purely oscillating in order to find $\varphi^{(3)}$ solution of this equation. From (5.2), (5.10) and (5.11) we obtain

$$K^{(0)} \overset{osc}{\sim} 1 + \sin(\theta) \quad \text{and} \quad K^{(1)}, \tau^{(1)} \overset{osc}{\sim} \cos(\theta) + \sin(2\theta),$$

which together with (4.12), (4.22) and (5.5) gives

$$\text{RHS of (5.9)} \overset{osc}{\sim} \cos(\theta) + \sin(2\theta) + \cos(3\theta).$$

This shows that we can solve (5.9) by setting

$$\varphi^{(3)} = \cos\left(\frac{u_0}{\lambda}\right) \varphi^{(3,1)} + \sin\left(\frac{2u_0}{\lambda}\right) \varphi^{(3,2)} + \cos\left(\frac{3u_0}{\lambda}\right) \varphi^{(3,3)}, \quad (5.13)$$

for some $\varphi^{(3,i)}$ supported in $B_R$ and satisfying

$$\sum_{i=1,2,3} \left\| \varphi^{(3,i)} \right\|_{H^{N-3}} \lesssim \varepsilon. \quad (5.14)$$

The estimate (5.14) follows from (4.11) and (4.7). Since the ansatz constructed in this article is an order 2 ansatz, the term $\varphi^{(3)}$ will be ultimately hidden in the remainder $\bar{h}_\lambda$ (see (2.5) in Theorem 2.1) and thus we don’t need a precise expression of $\varphi^{(3,i)}$.

### 5.2. The approximate momentum constraint.

In this section we solve the first level of the momentum constraint. Since $\tau = \lambda \tau^{(1)}$ we have $d\tau = \frac{1}{\lambda} du_0 \partial_\theta \tau^{(1)} + \lambda d\tau^{(1)}$ where the derivatives in $d\tau^{(1)}$ don’t hit the oscillating parts of $\tau^{(1)}$. Therefore the first momentum level corresponds to the $\lambda^0$ level of (1.16), solved by $W^{(2)}$.

**Remark 5.1.** This is where the assumption $\text{tr} \bar{g}_0 K_0 = 0$ simplifies our construction. If $\text{tr} \bar{g}_0 K_0 \neq 0$ then $\tau$ needs to include a non-oscillating $\lambda^0$ term in $\tau^{(0)}$. This would require a non-oscillating term in $W$ at the $\lambda^0$ order to absorb it

$$W = w^{(0)} + O(\lambda),$$

where we use a lowercase letter to emphasize that $w^{(0)}$ is non-oscillating. However, as explained in Sect. 4.2.4, this term would produce a $\lambda^{-1}$ term in $\text{div}_\gamma L_\gamma W$ (precisely $M^{[-1]}(w^{(0)})$, see Lemma 4.6) and thus would require an oscillating term $W^{(1)}$ in $W$ to absorb it. We make the assumption $\text{tr} \bar{g}_0 K_0 = 0$ precisely to avoid these technicalities.

More precisely, $W^{(2)}$ needs to solve

$$M^{[-2]}(W^{(2)}) = \frac{2}{3} \partial_\ell u_0 \partial_\theta \tau^{(1)}. \quad (5.15)$$
where we used the expansion obtained in Lemma 4.7. Thanks to Lemma 4.8 this equation rewrites as

$$\partial_\theta^2 W^{(2)}_\ell = -\frac{1}{2|\nabla u_0|\bar{g}_0} (N_0)_\ell \partial_\theta \tau^{(1)}. \tag{5.16}$$

Since the RHS of this equation is a $\partial_\theta$ derivative it is purely oscillating and we can integrate twice to obtain $W^{(2)}$. Using (5.11) this also gives

$$W^{(2)} \overset{osc}{\sim} \sin(\theta) + \cos(2\theta). \tag{5.17}$$

Note that as opposed to the Hamiltonian constraint, we don’t solve the $\lambda^1$ momentum level here. Indeed, as Lemma 4.6 shows, the operator $\text{div}_\gamma L_\gamma$ loses one $\lambda$ power even when applied to a non-oscillating field like $\tilde{W}$ (the remainder in (3.6)). Therefore the $\lambda^1$ momentum level also involves $\tilde{W}$ and the equation for $W^{(3)}$ is coupled with the remainder in the ansatz (3.6), as opposed to $\varphi^{(3)}$ which is not coupled to $\tilde{\varphi}$ (the remainder in (3.5)), since $\Delta_\gamma \tilde{\varphi} = O(\lambda^0)$.

5.3. An almost TT-tensor. In this section we define the first terms in the expansion of the parameter $\sigma$ of the conformal method, i.e $\sigma^{(0)}$ and $\sigma^{(1)}$. As explained in Sect. 3.3, the definition of the first orders of $\sigma$ follows simply from our constraint

$$\left(\sigma + L_\gamma W\right)^{(\leq 1)} = \left(K - \frac{1}{3} \tau\gamma\right)^{(\leq 1)}. \tag{5.18}$$

Since $W$ is given by $W = \lambda^2 (W^{(2)} + \tilde{W}) + O(\lambda^3$) (with $W^{(2)}$ defined by (5.16) and $\tilde{W}$ non-oscillating), we use Lemma 4.5 to compute $(L_\gamma W)^{(\leq 1)}$ and (5.18) forces us to define

$$\sigma^{(0)} = K^{(0)}, \tag{5.19}$$

$$\sigma^{(1)} = K^{(1)} - \frac{1}{3} \tau^{(1)} \bar{g}_0 - K^{-1}(W^{(2)}), \tag{5.20}$$

where $K^{(0)}$, $K^{(1)}$ and $\tau^{(1)}$ are given by (5.2), (5.10) and (5.11) respectively and $K^{-1}(W^{(2)})$ is defined in Lemma 4.5.

In this section, we prove that $\sigma^{(0)} + \lambda \sigma^{(1)}$ is almost a TT-tensor, that is

$$\text{tr}_\gamma \left(\sigma^{(0)} + \lambda \sigma^{(1)}\right) = O(\lambda^2) \quad \text{and} \quad \text{div}_\gamma \left(\sigma^{(0)} + \lambda \sigma^{(1)}\right) = O(\lambda). \tag{5.21}$$

Using the expansion of $\gamma^{ij}$ given by (4.13) we have

$$\text{tr}_\gamma \left(\sigma^{(0)} + \lambda \sigma^{(1)}\right) = \text{tr}_\bar{g}_0 \sigma^{(0)} + \lambda \left( -\cos\left(\frac{\mu_0}{\lambda}\right) \text{tr}_\bar{\gamma}^{(1)} \sigma^{(0)} + \text{tr}_\bar{g}_0 \sigma^{(1)}\right) + O(\lambda^2).$$

Using now the expansion of the divergence operator $\text{div}_\gamma$ obtained in Lemma 4.4 we have

$$\text{div}_\gamma \left(\sigma^{(0)} + \lambda \sigma^{(1)}\right) = \frac{1}{\lambda} D[-1](\sigma^{(0)}) + D[0](\sigma^{(0)}) + D[-1](\sigma^{(1)}) + O(\lambda).$$

The following lemma, which shows that $\sigma^{(0)} + \lambda \sigma^{(1)}$ is almost a TT-tensor, is crucial since it validates a posteriori our whole approximate construction.
Lemma 5.3. We have

\[
\text{tr}_{\bar{g}_0} \sigma^{(0)} = 0, \quad (5.22)
\]

\[
- \cos \left( \frac{\mu_0}{\lambda} \right) \text{tr}_{\bar{F}^{(1)}} \sigma^{(0)} + \text{tr}_{\bar{g}_0} \sigma^{(1)} = 0, \quad (5.23)
\]

and

\[
d_{[-1]}^{\text{(0)}}(\sigma^{(0)}) = 0, \quad (5.24)
\]

\[
d_{[0]}^{\text{(0)}}(\sigma^{(0)}) + d_{[-1]}^{\text{(0)}}(\sigma^{(1)}) = 0. \quad (5.25)
\]

**Proof.** We start with the trace identities. Since \( \sigma^{(0)} = K^{(0)} \), (5.22) follows from (1.7) and \( \text{tr}_{\bar{g}_0} \bar{F}^{(1)} = 0 \). Moreover, from (5.19) and (5.20) we have

\[
- \cos \left( \frac{\mu_0}{\lambda} \right) \text{tr}_{\bar{F}^{(1)}} \sigma^{(0)} + \text{tr}_{\bar{g}_0} \sigma^{(1)} = - \cos \left( \frac{\mu_0}{\lambda} \right) \text{tr}_{\bar{F}^{(1)}} K^{(0)} + \text{tr}_{\bar{g}_0} K^{(1)} - \tau^{(1)}
\]

\[
- \text{tr}_{\bar{g}_0} K_{[-1]}^{(1)}(W^{(2)})
\]

\[
= - \cos \left( \frac{\mu_0}{\lambda} \right) \text{tr}_{\bar{F}^{(1)}} K^{(0)} + \text{tr}_{\bar{g}_0} K^{(1)} - \tau^{(1)}
\]

\[
= 0,
\]

where we use (4.34) and (5.11). This proves (5.23).

We now look at the divergence identities. From (5.19), (5.2) and (4.24) we obtain

\[
d_{\ell}^{[-1]}(\sigma^{(0)}) = \frac{1}{2} |\nabla u_0|_{\bar{g}_0}^2 \cos \left( \frac{\mu_0}{\lambda} \right) \bar{F}_{N_0}^{(1)} = 0,
\]

which proves (5.24). We now compute the two parts of (5.25). Using (4.25) and (5.19) we obtain

\[
d_{\ell}^{[0]}(\sigma^{(0)}) = \text{div}_{\bar{g}_0} K_{\ell}^{(0)} - \bar{g}^{ij} (\tilde{\Gamma}^{(0)})_{i_{\ell}}^{a} K_{a_{j}}^{(0)}
\]

\[
= (\text{div}_{\bar{g}_0} K_{0})_{\ell} + \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) \left( \text{div}_{\bar{g}_0} |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} \right)_{\ell}
\]

\[
- \bar{g}^{ij} (\tilde{\Gamma}^{(0)})_{i_{\ell}}^{a} (K_{0})_{a_{j}} - \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \bar{g}^{ij} (\tilde{\Gamma}^{(0)})_{i_{\ell}}^{a} \bar{F}_{a_{j}}^{(1)}.
\]

We use (4.15) and (2.9) to rewrite the second line in the previous expression and obtain

\[
d_{\ell}^{[0]}(\sigma^{(0)}) = (\text{div}_{\bar{g}_0} K_{0})_{\ell} + \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) \left( \text{div}_{\bar{g}_0} |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} \right)_{\ell}
\]

\[
+ \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) \partial_{\ell} u_0 |\bar{F}^{(1)} \cdot K_{0}|_{\bar{g}_0} + 2 \sin^2 \left( \frac{\mu_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \partial_{\ell} u_0 F_0^{(2)}.
\]

Using the background momentum constraint (1.6), we see that the non-oscillating terms cancel each other and we are left with

\[
d_{\ell}^{[0]}(\sigma^{(0)}) = \frac{1}{2} \sin \left( \frac{\mu_0}{\lambda} \right) \left( \left( \text{div}_{\bar{g}_0} |\nabla u_0|_{\bar{g}_0} \bar{F}^{(1)} \right)_{\ell} + \partial_{\ell} u_0 |\bar{F}^{(1)} \cdot K_{0}|_{\bar{g}_0} \right)
\]

\[
- \cos \left( \frac{2\mu_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} \partial_{\ell} u_0 F_0^{(2)}. \quad (5.26)
\]
We now look at \( d^{[-1]}_\ell (\sigma^{(1)}) \). Using (4.24) and (5.20) we obtain
\[
 d^{[-1]}_\ell (\sigma^{(1)}) = -\nabla u_0|_{\bar{g}_0} \partial_\theta K^{(1)}_{N_0\ell} - \frac{1}{3} \partial_\ell u_0 \partial_\theta \tau^{(1)} - M^{[-2]}_\ell (W^{(2)})
\]
\[
= \partial_\theta \left( -\nabla u_0|_{\bar{g}_0} K^{(1)}_{N_0\ell} - \partial_\ell u_0 \tau^{(1)} \right),
\]
where we use the equation satisfied by \( W^{(2)} \) (see (5.15)). We use (5.10), (5.6) and (5.7) to obtain
\[
 K^{(1)}_{N_0\ell} = -\cos \left( \frac{u_0}{\lambda} \right) \left( \tilde{F}^{(1)}_{kj} \left( \frac{\tilde{g}^{ij}}{\bar{g}_0} \partial_i N^k_0 - \frac{1}{2} N_0 \tilde{g}^{jk} + K^{(1)}_k \right) \right) \frac{3}{4} \sin \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} F^2_0 (N_0) \ell.
\]

Using (5.11) and \( \text{tr} \tilde{g}_0 \omega^{(2)} = 0 \) we also obtain
\[
\tau^{(1)} = -\frac{3}{4} \cos \left( \frac{u_0}{\lambda} \right) \tilde{F}^{(1)}_{kj} \left( \frac{\tilde{g}^{ij}}{\bar{g}_0} \partial_i N^k_0 - \frac{1}{2} N_0 \tilde{g}^{jk} + K^{(1)}_k \right) + \frac{1}{4} \sin \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} F^2_0 \partial_\ell u_0.
\]
This gives
\[
- |\nabla u_0|_{\bar{g}_0} K^{(1)}_{N_0\ell} - \partial_\ell u_0 \tau^{(1)}
\]
\[
= |\nabla u_0|_{\bar{g}_0} \cos \left( \frac{u_0}{\lambda} \right) \left( \tilde{F}^{(1)}_{kj} \left( \frac{\tilde{g}^{ij}}{\bar{g}_0} \partial_i N^k_0 - \frac{1}{2} N_0 \tilde{g}^{jk} + K^{(1)}_k \right) \right) \frac{3}{4} \sin \left( \frac{2u_0}{\lambda} \right) |\nabla u_0|_{\bar{g}_0} F^2_0 \partial_\ell u_0.
\]

Adding this to the expression of \( d^{[0]}_\ell (\sigma^{(0)}) \) given by (5.26) we notice that the terms oscillating like \( 2\theta \) cancel (see Remark 5.2 below) and we obtain
\[
 d^{[0]}_\ell (\sigma^{(0)}) + d^{[-1]}_\ell (\sigma^{(1)})
\]
\[
= \sin \left( \frac{u_0}{\lambda} \right) \left[ \frac{1}{2} \left( \text{div}_{\bar{g}_0} |\nabla u_0|_{\bar{g}_0} \tilde{F}^{(1)} \right) \ell - |\nabla u_0|_{\bar{g}_0} \tilde{F}^{(1)}_{kj} \left( \frac{\tilde{g}^{ij}}{\bar{g}_0} \partial_i N^k_0 - \frac{1}{2} N_0 \tilde{g}^{jk} + K^{(1)}_k \right) \right] - \frac{1}{2} |\nabla u_0|_{\bar{g}_0} \omega^{(2)} - \frac{1}{2} \partial_\ell u_0 \tilde{F}^{(1)}_{kj} \bar{g}_0 \partial_i N^k_0 + \frac{1}{4} \partial_\ell u_0 \tilde{F}^{(1)}_{kj} N_0 \tilde{g}^{jk}_0.
\]

Using \( \tilde{F}^{(1)}_{N_0\ell} = 0 \) we can compute the divergence of \( |\nabla u_0|_{\bar{g}_0} \tilde{F}^{(1)} \) and using in addition that \( \omega^{(2)}_{N_0 N_0} = 0 \) we obtain
\[
 d^{[0]}_{N_0} (\sigma^{(0)}) + d^{[-1]}_{N_0} (\sigma^{(1)}) = 0.
\]
The tangential components of \( d^{[0]}(\sigma^{(0)}) + d^{[-1]}(\sigma^{(1)}) \) are given by
\[
d^{[0]}(\sigma^{(0)}) + d^{[-1]}(\sigma^{(1)}) = \sin \left( \frac{u_0}{\lambda} \right) \left[ \frac{1}{2} \left( \text{div}_{\bar{g}_0} |\nabla u_0|_{\bar{g}_0} \tilde{F}^{(1)} \right)_j - \frac{|\nabla u_0|_{\bar{g}_0}}{2} \tilde{F}^{(1)} \right]_{kj} \left( D_{N_0} N_{0k} - \bar{g}_0^{kj} (K_0)_{N_0j} \right) - \frac{1}{2} |\nabla u_0|^2_{\bar{g}_0} \omega^{(2)}_{N_0j} \right].
\]

This previous expression vanishes thanks to the choice of \( \omega^{(2)} \) made in (4.7). This concludes the proof of (5.25).

\( \square \)

**Remark 5.2.** The cancellation of the \( \cos \left( \frac{2u_0}{\lambda} \right) \) terms coming from \( d^{[0]}(\sigma^{(0)}) \) and \( d^{[-1]}(\sigma^{(1)}) \) in their sum seems to be linked to the weak polarized null condition satisfied by the semi-linear terms in the Einstein equations, which involved products of derivatives of the metric (see Section 3.1.3 in [Tou22] for the definition of this condition). Indeed, these terms correspond to terms of the form \( \Gamma K, \Gamma \Gamma \) or \( KK \) with \( \Gamma \) the Christoffel symbols and \( K \) the second fundamental form, i.e terms of the form \( \partial g \partial g \).

### 5.4. An exact TT-tensor

In the next section, we are going to solve completely the constraint equations, i.e solve for the remainders in the high-frequency ansatz (3.5)–(3.6). We will thus need the full expression of the parameters of the conformal method. While \( \gamma \) and \( \tau \) are already fully defined, \( \sigma \) is only partially known yet and is only an almost TT-tensor, as it was shown in the previous section. In this section we finish the construction of \( \sigma \). We choose the following ansatz
\[
\sigma = \sigma^{(0)} + \lambda \sigma^{(1)} + \lambda^2 \left( \sigma^{(2)} + L_\gamma Y \right) + \frac{\lambda^3}{3} \bar{f} \gamma.
\]

(5.28)

In this expression, \( \sigma^{(0)} \) and \( \sigma^{(1)} \) are given by (5.19) and (5.20) and \( \sigma^{(2)} \), \( Y \) and \( \bar{f} \) are yet to be defined such that
\[
\text{tr}_\gamma \sigma = 0 \quad \text{and} \quad \text{div}_\gamma \sigma = 0.
\]

(5.29)

Let us explain the ansatz (5.28). Since we need to satisfy the compatibility with the spacetime ansatz (5.18), we can’t modify the order \( \lambda^0 \) and \( \lambda^1 \) of \( \sigma \). Thus, a non-oscillating remainder can only appear at the order \( \lambda^2 \). However, such a remainder would not be able to solve the \( \lambda^1 \) level of \( \text{div}_\gamma \sigma = 0 \) (recall (5.21)). Therefore we need to add an oscillating field at the order \( \lambda^2 \), i.e \( \sigma^{(2)} \). This field will also be able to solve the \( \lambda^2 \) level of \( \text{tr}_\gamma \sigma = 0 \). Finally the remainder is chosen of the form \( L_\gamma Y + \frac{2}{3} \bar{f} \gamma \), where the vector field \( Y \) ensures \( \text{div}_\gamma \sigma = 0 \) and the scalar function \( \bar{f} \) ensures \( \text{tr}_\gamma \sigma = 0 \).

We now derive the equations for \( \sigma^{(2)}, Y \) and \( \bar{f} \), which illustrates the above discussion. Thanks to Lemma 5.3 the equations (5.29) rewrite as
\[
\lambda^2 \left( \text{tr}_{\gamma (\geq 2)} \sigma^{(0)} + \text{tr}_{\gamma (\geq 1)} \sigma^{(1)} + \text{tr}_\gamma \sigma^{(2)} \right) + \frac{\lambda^3}{3} \bar{f} = 0,
\]
and
\[
\lambda \left( d^{[1]}(\sigma^{(0)}) + d^{[0]}(\sigma^{(1)}) + d^{[-1]}(\sigma^{(2)}) + M^{[-1]}(Y) \right)
+ \lambda^2 \left( d^{[\geq 2]}(\sigma^{(0)}) + d^{[\geq 1]}(\sigma^{(1)}) + d^{[\geq 0]}(\sigma^{(2)}) + M^{[\geq 0]}(Y) + \frac{\lambda}{3} d \bar{f} \right) = 0.
\]
where $d\!f$ also includes derivatives of the oscillating parts of $f$, which implies in particular that $d\!f = O(\lambda^{-1})$. In order to solve these two equations, we want $\sigma^{(2)}, Y$ and $f$ to satisfy the following coupled system (recall the expression of $d[-1] \sigma$ given by (4.24)):

\begin{align}
\text{tr}_{\bar{g}_0} \sigma^{(2)} &= -\text{tr}_{\gamma^{(2)}} \sigma^{(0)} - \text{tr}_{\gamma^{(1)}} \sigma^{(1)}, \\
\bar{f} &= -\text{tr}_{\gamma^{(\geq 1)}} \sigma^{(2)} - \text{tr}_{\gamma^{(\geq 2)}} \sigma^{(1)} - \text{tr}_{\gamma^{(\geq 3)}} \sigma^{(0)}, \\
-|\nabla u_0|_{\bar{g}_0} &\partial_\theta \sigma^{(2)} = -M^{[-1]}_\ell(Y) - d^{[1]}_\ell(\sigma^{(0)}) - d^{[0]}_\ell(\sigma^{(1)}), \\
M^{[\geq 0]}(Y) &= -d^{[\geq 2]}(\sigma^{(0)}) - d^{[\geq 1]}(\sigma^{(1)}) - d^{[\geq 0]}(\sigma^{(2)}) - \frac{\lambda}{3} d\!f. \tag{5.33}
\end{align}

Equations (5.30) and (5.31) ensure $\text{tr}_\gamma \sigma = 0$ while (5.32) and (5.33) ensure $\text{div}_\gamma \sigma = 0$. The rest of this section is devoted to the resolution of the system (5.30)–(5.31)–(5.32)–(5.33). It presents a triangular structure, despite the term $M^{[-1]}(Y)$ in (5.32).

5.4.1. Definition of $\sigma^{(2)}$ and $f$ We start by solving the non-differential equations of the previous system, that is (5.30)–(5.31)–(5.32). The first step is to show that the RHS of (5.32) is purely oscillating, which is a necessary condition since the LHS is a $\partial_\theta$ derivative. Thanks to (4.37), the first term in the RHS is oscillating, and the next lemma deals with the last two.

**Lemma 5.4.** The following oscillating behaviour holds

$$
|\nabla u_0|_{\bar{g}_0} \partial_\theta \sigma^{(2)} \sim \cos(\theta) + \sin(2\theta) + \cos(3\theta).
$$

**Proof.** From (4.27) we have

$$
d^{[0]}(\sigma^{(1)}) \sim (1 + \sin(\theta))\sigma^{(1)},
$$

and from (5.20) we have

$$
\sigma^{(1)} \sim K^{(1)} + \cos(\theta) + \tau^{(1)} + K^{[-1]}(W^{(2)}) \sim \cos(\theta) + \sin(2\theta) + \partial_\theta W^{(2)},
$$

where we used (5.10) and (5.11). Now using (5.17) we conclude that

$$
d^{[0]}(\sigma^{(1)}) \sim \cos(\theta) + \sin(2\theta).
$$

Now, from (4.28) we have

$$
d^{[1]}(\sigma^{(0)}) \sim \sin(\theta)\partial_\theta \sigma^{(0)} + (\cos(\theta) + \sin(2\theta))\sigma^{(0)},
$$

and from (5.19) and (5.2) we have

$$
\sigma^{(0)} \sim 1 + \sin(\theta).
$$

This concludes the proof of the lemma. \qed
We have shown that the RHS of (5.32) is purely oscillating, which allows us to formally integrate this equation in θ and obtain σ^(2). More precisely, thanks to \( F_{iN_0}^{(1)} = 0 \), (4.37) implies that \( M^{-1}_{N_0}(Y) = 0 \). Therefore, (5.32) gives us \( \sigma^{(2)}_{N_0N_0} \) as a function of lower order terms in the construction, i.e. \( \sigma^{(0)} \) and \( \sigma^{(1)} \). Then, (5.30) gives us \( \sigma^{(2)}_{11} \) and \( \sigma^{(2)}_{22} \) as a function of \( \sigma^{(2)}_{N_0N_0}, \sigma^{(0)} \) and \( \sigma^{(1)} \). All together, the diagonal components of \( \sigma^{(2)} \) in the frame \((N_0, e_1, e_2)\) are functions of \( \sigma^{(0)} \) and \( \sigma^{(1)} \) satisfying

\[
\left| \sigma^{(2)}_{N_0N_0} \right| + \left| \sigma^{(2)}_{11} \right| + \left| \sigma^{(2)}_{22} \right| \lesssim \left| d^{[1]}(\sigma^{(0)}) \right| + \left| d^{[0]}(\sigma^{(1)}) \right| + \left| (\gamma^{-1})^2(\sigma^{(0)}) \right| + \left| (\gamma^{-1})^3(\sigma^{(1)}) \right|,
\]

with a high-frequency behaviour, meaning that we lose one power of \( \lambda \) for each derivatives.

Remark 5.3. Note that we don’t impose conditions on \( \sigma^{(2)}_{11} \) and \( \sigma^{(2)}_{22} \) separately but only on their sum.

The other components of \( \sigma^{(2)} \) in the frame \((N_0, e_1, e_2)\) as well as the scalar function \( f \) depends on the vector field \( Y \), which is yet to be defined. The Eq. (5.32) allows us to define \( \sigma^{(2)}_{N_0} \) as a linear function of \( Y \) and \( \sigma^{(0)} \) and \( \sigma^{(1)} \). More precisely we obtain

\[
\left| \sigma^{(2)}_{N_01}(Y) \right| + \left| \sigma^{(2)}_{N_02}(Y) \right| \lesssim \left| F^{(1)}Y \right| + \left| d^{[1]}(\sigma^{(0)}) \right| + \left| d^{[0]}(\sigma^{(1)}) \right|,
\]

with a high-frequency behaviour. Since this component doesn’t appear in the equations \( \sigma^{(2)} \) needs to solve, we set \( \sigma^{(2)}_{12} = 0 \).

The scalar function \( f \) is actually already defined by (5.31), but as \( \sigma^{(2)} \) is a function of \( Y \), so is \( f \). Therefore, thanks to (5.34) and (5.35) \( f \) satisfies

\[
|f(Y)| \lesssim |(\gamma^{-1})_{(\geq 1)}(\sigma^{(2)})| + |(\gamma^{-1})_{(\geq 2)}(\sigma^{(1)})| + |(\gamma^{-1})_{(\geq 3)}(\sigma^{(0)})| \\
\lesssim |(\gamma^{-1})_{(\geq 1)}F^{(1)}Y| + |(\gamma^{-1})_{(\geq 1)}d^{[1]}(\sigma^{(0)})| + |(\gamma^{-1})_{(\geq 1)}d^{[0]}(\sigma^{(1)})| \\
+ |(\gamma^{-1})_{(\geq 1)}(\gamma^{-1})_{(\geq 2)}(\sigma^{(0)})| + |(\gamma^{-1})_{(\geq 1)}(\gamma^{-1})_{(\geq 1)}(\sigma^{(1)})| \\
+ |(\gamma^{-1})_{(\geq 2)}(\sigma^{(1)})| + |(\gamma^{-1})_{(\geq 3)}(\sigma^{(0)})|,
\]

with a high-frequency behaviour.

5.4.2. Solving for \( Y \) To conclude the construction of \( \sigma \), it remains to solve (5.33). As explained after Lemma 4.6, this is done by actually "replacing" the operator \( M_{[\geq 0]} \) by \( \text{div}_e L_e \) together with a fixed point argument. More precisely, we define a map \( \Psi \)

\[
\Psi : B \longrightarrow B
\]

such that \( \Psi(Y) \) is the solution of

\[
\text{div}_e L_e \Psi(Y) = \text{div}_e L_e Y - M_{[\geq 0]}(Y) - d_{[\geq 2]}(\sigma^{(0)}) \\
- d_{[\geq 1]}(\sigma^{(1)}) - d_{[\geq 0]}(\sigma^{(2)}(Y)) - \frac{\lambda}{3} f(Y),
\]

(5.37)
and where
\[ B = \left\{ Z \in H^2_\delta \mid \|Z\|_{H^2_\delta} \leq C_1 \varepsilon \right\}. \]  
(5.38)

with \( C_1 > 0 \) to be chosen later. Note that any fixed point of \( \Psi \) is a solution of (5.33). In order to prove the existence of a fixed point, we need to show that \( \Psi \) is well-defined and is a contraction.

**Proposition 5.1.** If \( C_1 \) is large enough and \( \varepsilon \) is small enough, then the map \( \Psi \) is well-defined and is a contraction.

**Proof.** Let \( Y \in B \). We bound the RHS of (5.37) in \( L^2_\delta \):
\[
\|\text{RHS of (5.37)}\|_{L^2_\delta} \lesssim \left\| \text{div}_e L_e Y - M^{(\geq 0)}(Y) \right\|_{L^2_\delta} + \left\| d^{(\geq 0)}(\sigma^{(2)}(Y)) \right\|_{L^2_\delta} + \lambda \left\| d[\gamma(Y)] \right\|_{L^2_\delta} + \left\| d^{(\geq 2)}(\sigma^{(0)}) + d^{(\geq 1)}(\sigma^{(1)}) \right\|_{L^2},
\]
where we omitted the weights for the last term since it is compactly supported. For (4.38) gives
\[ A \lesssim \left\| (\gamma^{-1} - e^{-1}) \partial^2 Y \right\|_{L^2_\delta} + \left\| \partial \gamma \partial Y \right\|_{L^2_\delta} + \left\| (\partial \gamma)^2 Y \right\|_{L^2_\delta} + \left\| (\partial^2 \gamma)^{(\geq 0)} Y \right\|_{L^2_\delta}, \]
where we also used the fact that the coefficients of \( \gamma \) are bounded. We bound all the metric terms in \( L^\infty \) using the background regularity (2.1) and (4.9). More precisely, we have
\[
\left\| \gamma^{-1} - e^{-1} \right\|_{L^\infty} \lesssim \left\| \widetilde{g}_0 - e \right\|_{L^\infty} + \lambda \left\| \widetilde{F}^{(1)} \right\|_{L^\infty} + \lambda^2 \left\| \omega^{(2)} \right\|_{L^\infty}, \]  
(5.39)
\[
\left\| \partial \gamma \right\|_{L^\infty} \lesssim \left\| \partial \widetilde{g}_0 \right\|_{L^\infty} + \left\| \widetilde{F}^{(1)} \right\|_{L^\infty} + \lambda \left( \left\| \partial \widetilde{F}^{(1)} \right\|_{L^\infty} + \left\| \omega^{(2)} \right\|_{L^\infty} + \lambda^2 \left\| \partial \omega^{(2)} \right\|_{L^\infty} \right), \]  
(5.40)
\[
\left\| (\partial^2 \gamma)^{(\geq 0)} \right\|_{L^\infty} \lesssim \left\| \partial^2 \widetilde{g}_0 \right\|_{L^\infty} + \left\| \partial \widetilde{F}^{(1)} \right\|_{L^\infty} + \lambda \left( \left\| \partial \widetilde{F}^{(1)} \right\|_{L^\infty} + \left\| \partial \omega^{(2)} \right\|_{L^\infty} + \lambda \left\| \partial^2 \omega^{(2)} \right\|_{L^\infty} \right). \]  
(5.41)

Using (2.1), (4.9), (4.7) and \( \|Y\|_{H^2_\delta} \leq C_1 \varepsilon \), the estimates (5.39)–(5.40)–(5.41) then imply that \( A \lesssim C_1 \varepsilon^2 \).

The term \( D \) depends only on previous terms of the construction so (4.26) simply gives \( D \lesssim \varepsilon \). The maps \( Y \mapsto \sigma^{(2)}(Y) \) and \( Y \mapsto f(Y) \) are affine with coefficients as \( D \) (see (5.34), (5.35) and (5.36)) so a combination of the two previous arguments gives \( B + C \lesssim C_1 \varepsilon^2 + \varepsilon \). Note that for \( C \) we need to compensate the loss in power of \( \lambda \) when differentiating the oscillating parts of \( f \) with the \( \lambda \) in front.

We have proved that \( A + B + C + D \lesssim C_1 \varepsilon^2 + \varepsilon \). In particular, this allows us to use the second part of Proposition 1.2 to prove that there exists a unique \( \Psi(Y) \in H^2_\delta \) solving (5.37). Moreover we have
\[ \|\Psi(Y)\|_{H^2_\delta} \lesssim C_1 \varepsilon^2 + \varepsilon. \]
Therefore, taking $C_1$ large compared to the numerical constant appearing in these estimates and $\varepsilon$ small compared to 1 proves that $\Psi$ is well-defined and maps $B$ to itself.

To prove that $\Psi$ is a contraction, we consider $Y_a$ and $Y_b$ two elements of $B$. By substracting the equations satisfied by $\Psi(Y_a)$ and $\Psi(Y_b)$ we obtain the equation for their difference

$$\text{div} \, L_e \left( \Psi(Y_a) - \Psi(Y_b) \right) = \left( \text{div} \, L_e - M^{[\geq 0]} \right)(Y_a - Y_b)$$

$$- d^{[\geq 0]} \left( \sigma^{(2)}(Y_a) - \sigma^{(2)}(Y_b) \right) - \frac{\lambda}{3} d (f(Y_a) - f(Y_b)).$$

(5.42)

Using again the fact that $Y \mapsto -\sigma^{(2)}(Y)$ and $Y \mapsto -f(Y)$ are affine and using (4.38) for $\text{div} \, L_e - M^{[\geq 0]}$, we can prove that

$$\|\text{RHS of (5.42)}\|_{L_{\delta+2}^2} \lesssim \varepsilon \|Y_a - Y_b\|_{H_{\delta}^2}.$$ 

Therefore, taking $\varepsilon$ small enough ensures that $\Psi$ is a contraction.

Thanks to this proposition, the Banach fixed point theorem implies the existence of $Y \in B$ solving (5.33), and therefore $(\sigma^{(2)}(Y), f(Y), Y)$ solves (5.30)–(5.31)–(5.32)–(5.33).

We can also prove that $Y$ enjoys higher regularity. Indeed we can bound the RHS of (5.37) in higher order Sobolev spaces and use elliptic estimates for $\text{div} \, L_e$ as in the previous proposition. The worse term in (5.37) is given by $\nabla \sigma^{(1)}$ which is bounded in $H^{N-2}$ (see (5.20), (5.10) and (5.8)) and in terms of decay the worse term is $\nabla \sigma^{(0)}$ (see (5.19), (5.2) and (2.1)). Therefore we obtain a solution $Y$ of (5.33) such that $Y \in H_{\delta}^N$ and

$$\|Y\|_{H_{\delta+1}^{k+2}} \lesssim \frac{\varepsilon}{\lambda^k},$$

(5.43)

for $k \in [0, N-2]$, where the loss of $\lambda$ powers is due to the high-frequency character of each term in (5.37). We summarize what we know on the parameter $\sigma$ in the following corollary.

**Corollary 5.1.** The tensor $\sigma$ defined by (5.28) is a TT-tensor for the metric $\gamma$, belongs to $H_{\delta+1}^{N-1}$ and satisfies

$$\max_{k \in [0, N-3]} \lambda^{k+2} \|\sigma\|_{H_{\delta+1}^{k+2}} + \max_{k \in [0, N-3]} \lambda^k \|\nabla^k \sigma\|_{L^\infty} \lesssim \varepsilon.$$ 

(5.44)

**Proof.** The oscillating terms $\sigma^{(0)}$ and $\sigma^{(1)}$ lose one $\lambda$ power for each derivatives and we can estimate the actual tensors by estimating (5.19)–(5.20) directly in weighted Sobolev spaces or in $L^\infty$ using Sobolev embeddings of Proposition 1.1. Moreover, we can neglect $\sigma^{(2)}(Y)$ and $f(Y)$ and focus on $L_e Y$ in (5.28) which rewrites broadly as a $\nabla Y$ term since we can put the $\gamma$ and $\partial \gamma$ term in $L^\infty$. Therefore, the estimate (5.44) follows directly from (5.43) and Sobolev embeddings. \qed
6. Exact Solution to the Constraint Equations

We are now ready to solve the constraint equations (1.15) and (1.16). The parameters of these equations are $\gamma$, $\tau$ and $\sigma$. The metric $\gamma$ and the scalar function $\tau$ are fully known thanks to Sect. 4.1 and (5.11). The TT-tensor $\sigma$ has been defined in Sects. 5.4 and 5.3. Recall that the solutions of (1.15)–(1.16) are of the form

$$W = \lambda^2 \left( W^{(2)} \left( \frac{u_0}{\lambda} \right) + \tilde{W} \right) + \lambda^3 W^{(3)} \left( \frac{u_0}{\lambda} \right),$$

$$\varphi = 1 + \lambda^2 \left( \varphi^{(2)} \left( \frac{u_0}{\lambda} \right) + \tilde{\varphi} \right) + \lambda^3 \varphi^{(3)} \left( \frac{u_0}{\lambda} \right),$$

where $\varphi^{(2)}$, $\varphi^{(3)}$ and $W^{(2)}$ are defined in Sects. 5.1.1, 5.1.2 and 5.2 respectively. Therefore, it remains to construct $\tilde{\varphi}$, $\tilde{W}$ and $W^{(3)}$.

6.1. System for the remainders. The construction of Sects. 5.1.1, 5.1.2 and 5.2 ensures that the constraint equations (1.15) and (1.16) are partly solved. More precisely, it remains to solve the $\lambda \geq 2$ levels of (1.15) and the $\lambda \geq 1$ levels of (1.16). In this section, we compute the exact equations this gives for $\tilde{\varphi}$, $\tilde{W}$ and $W^{(3)}$.

6.1.1. Definition of $W^{(3)}(\tilde{W})$ The purpose of the oscillating vector field $W^{(3)}$ is to solve the $\lambda^1$ momentum level. However, since the conformal Laplacian $\text{div}_\gamma L_\gamma$ loses one power of $\lambda$ even when applied to a non-oscillating field such as $\tilde{W}$ (see Lemma 4.6), the latter is a source term in the equation for $W^{(3)}$. This explains why $W^{(3)}$ is considered as part of the remainders, when $\varphi^{(3)}$ is not.

We define $W^{(3)}$ as a function of $\tilde{W}$. The $\lambda^1$ momentum level writes

$$M^{[-2]}_\ell(W^{(3)}) + M^{[-1]}_\ell(\tilde{W}) + M^{[-1]}_\ell(W^{(2)}) = \frac{2}{3} \partial_\ell \tau^{(1)}. \tag{6.1}$$

Thanks to (4.43) and (4.37) this is equivalent to

$$\partial_\theta^2 W^{(3)} + \frac{1}{3} (N_0)_\ell \partial_\theta^2 W^{(3)}_{N_0} = - \cos \left( \frac{u_0}{\lambda} \right) g^{ij} \tilde{F}_{i\ell}^{(1)} \tilde{W}_j + \frac{1}{|\nabla u_0|^2} \left( \frac{2}{3} \partial_\ell \tau^{(1)} - M^{[-1]}_\ell(W^{(2)}) \right).$$

Lemma 4.8 then gives

$$\partial_\theta^2 W^{(3)} = - \cos \left( \frac{u_0}{\lambda} \right) g^{ij} \tilde{F}_{i\ell}^{(1)} \tilde{W}_j + \frac{1}{|\nabla u_0|^2} \left[ \frac{2}{3} \partial_\ell \tau^{(1)} - M^{[-1]}_\ell(W^{(2)}) - \frac{1}{4} (N_0)_\ell \left( \frac{2}{3} N_0 \tau^{(1)} - M^{[-1]}_{N_0}(W^{(2)}) \right) \right]. \tag{6.2}$$

Let us check that the RHS of this equation is purely oscillating. Since $\tau^{(1)}$ is purely oscillating (see (5.27)), we only need to check $M^{[-1]}_\ell(W^{(2)})$. For this we use first (4.42) and then (5.17), this gives

$$M^{[-1]}_\ell(W^{(2)}) \sim \cos(\theta) W^{(2)} + \cos(\theta) \partial_\theta^2 W^{(2)} + (1 + \sin(\theta)) \partial_\theta W^{(2)} \sim \cos(\theta) + \sin(2\theta) + \cos(3\theta).$$
The RHS of (6.2) is thus purely oscillating and we can integrate it twice with respect to $\theta$. We obtain

$$W^{(3)}(\tilde{W}) = \cos \left( \frac{u_0}{\lambda} \right) g^{ij} f_{ij(1)} \tilde{W} + W^{(3, \text{rest})}, \quad (6.3)$$

where $W^{(3, \text{rest})}$ satisfies

$$\left| W^{(3, \text{rest})} \right| \lesssim \left| \partial_\tau \right| (1) + \left| M[-1](W^{(2)}) \right|, \quad (6.4)$$

with a high-frequency behaviour.

6.1.2. The system for $\tilde{W}$ and $\tilde{\varphi}$ In this section, we will expand in the most concise way the non-linearities involved in the equations for $\tilde{\varphi}$ and $\tilde{W}$. We start with the equation for $\tilde{W}$, which, if we drop the vectorial notation, writes

$$M^{[0]}(\tilde{W}) = -M^{[-1]}(W^{(3)}(\tilde{W})) - M^{[0]}(W^{(2)}) + \frac{2}{3}(\varphi^6 \partial_\tau)^{(\geq 2)}, \quad (6.5)$$

The following lemma expands the non-linearity in (6.5).

**Lemma 6.1.** We have

$$\frac{2}{3}(\varphi^6 \partial_\tau)^{(\geq 2)} = a_0 + \sum_{k=1}^{6} \lambda^{2(k-1)} a_k \tilde{\varphi}^k, \quad (6.6)$$

where for $k \in [0, 6]$, $a_k$ is supported in $B_R$ and

$$\max_{i \in [0, N-5]} \lambda^i \left\| \nabla^i a_k \right\|_{L^\infty} \lesssim \varepsilon. \quad (6.7)$$

**Proof.** Recall that $\tau = \lambda \tau^{(1)}$ implies $\partial \tau = O(\lambda^0)$, thus we only need to expand

$$\left( 1 + \lambda^2 \left( \varphi^{(2)} + \tilde{\varphi} \right) + \lambda^3 \varphi^{(3)} \right)^6,$$

and only keep the terms of order $\lambda^2$ or more, which only excludes the term 1. The coefficient $a_0$ in (6.6) contains all the terms where $\tilde{\varphi}$ doesn’t appear, it is thus a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with no constant term and multiplied by $\partial \tau$. Therefore, $a_0$ shares the same support property as $\varphi^{(2)}$ and $\varphi^{(3)}$ and the estimate (6.7) follows from (5.5), (5.14) and (5.11). If $k \in [0, 6]$, the same reasoning applies but $a_k$ is now a polynomial in $\varphi^{(2)}$ and $\varphi^{(3)}$ with a constant term. But as this polynomial is still multiplied by $\partial \tau$, the support property and the estimate still hold. \qed

The equation for $\tilde{\varphi}$ writes

$$8\Delta_\gamma \tilde{\varphi} = -8 \sum_{i=2}^{3} H^{[2-i]}(\varphi^{(i)}) + R^{(\geq 2)} + R(\gamma) \left( \varphi^{(2)} + \tilde{\varphi} + \lambda \varphi^{(3)} \right)$$

$$+ \frac{2}{3}(\varphi^6 \partial_\tau)^{(\geq 2)} - \left( |\sigma + L_\gamma W|_{\gamma}^2 \varphi^{-\gamma} \right)^{(\geq 2)}. \quad (6.8)$$

The next two lemmas expand the non-linearities in (6.8).
Lemma 6.2. We have
\[ \frac{2}{3} (\tau^2 \varphi^5)(\geq 2) = b_0 + \sum_{k=1}^{s} \lambda^{2k} b_k \tilde{\varphi}^k, \]
where for \( k \in [0, 5] \), \( b_k \) is supported in \( B_R \) and
\[ \max_{i \in [0, N-5]} \lambda^i \left\| \nabla^i b_k \right\|_{L^\infty} \lesssim \varepsilon. \tag{6.9} \]

The proof of Lemma 6.2 is left to the reader since it is very similar to the one of Lemma 6.1. We now expand the non-linearities with a negative power of \( \varphi \).

Lemma 6.3. There exists a universal constant \( C_{emb} > 0 \) such that if \( \| \tilde{\varphi} \|_{H_\delta^2} < C_{emb}^{-1} \) and if \( \varepsilon \) is small enough, then we have
\[ \left( |\sigma + L_\gamma W|_\gamma^2 \varphi^{-7} \right)(\geq 2) = \left( |\sigma + L_\gamma W|_\gamma^2 \right)(\geq 2) + |\sigma + L_\gamma W|_\gamma^2 \left( c_0 + \sum_{k \geq 1} c_k \lambda^{2(k-1)} \tilde{\varphi}^k \right), \]
where the \( c_k \) satisfy:

- \( c_0 \) is supported in \( B_R \) and we have
  \[ \max_{i \in [0, N-5]} \lambda^i \left\| \nabla^i c_0 \right\|_{L^\infty} \lesssim \varepsilon, \tag{6.10} \]
- if \( k \geq 1 \), we have
  \[ \max_{i \in [0, N-5]} \lambda^i \left\| \nabla^i c_k \right\|_{L^\infty} \lesssim 1. \tag{6.11} \]

Proof. The constant \( C_{emb} \) is the one appearing in the embedding \( H_\delta^2 \hookrightarrow L^\infty \) (see Proposition 1.1), i.e.
\[ \| u \|_{L^\infty} \leq C_{emb} \| u \|_{H_\delta^2}, \]
for all \( u \in H_\delta^2 \). Now if \( \varepsilon \) is small enough and if \( \| \tilde{\varphi} \|_{H_\delta^2} < C_{emb}^{-1} \), we have
\[ \| \varphi^{(2)} + \tilde{\varphi} + \lambda \varphi^{(3)} \|_{L^\infty} \leq 1. \]
This allows us to expand \( \varphi^{-7} = (1 + \lambda^2 (\varphi^{(2)} + \tilde{\varphi} + \lambda \varphi^{(3)}))^{-7} \). Indeed there exists a sequence \( (c_k)_{k \in \mathbb{N}} \) such that
\[ \varphi^{-7} = 1 + \lambda^2 \left( c_0 + \sum_{k \geq 1} c_k \lambda^{2(k-1)} \tilde{\varphi}^k \right), \]
where \( c_0 \) is a polynomial in \( \varphi^{(2)} \) and \( \varphi^{(3)} \) with no constant term and \( c_k \) for \( k \geq 1 \) is a polynomial in \( \varphi^{(2)} \) and \( \varphi^{(3)} \) with a constant term bounded but not compactly supported. This justifies the estimates (6.10) and (6.11). Therefore, we have
\[ \left( |\sigma + L_\gamma W|_\gamma^2 \varphi^{-7} \right)(\geq 2) = \left( |\sigma + L_\gamma W|_\gamma^2 \right)(\geq 2) + |\sigma + L_\gamma W|_\gamma^2 \left( c_0 + \sum_{k \geq 1} c_k \lambda^{2(k-1)} \tilde{\varphi}^k \right), \]
which concludes the proof. \( \square \)
Putting Lemmas 6.1, 6.2 and 6.3 together, we obtain the final form of the system solved by $\tilde{\varphi}$ and $\tilde{W}$:

$$
M^{[\geq 0]}(\tilde{W}) = -M^{[\geq -1]}(W^{(3)}(\tilde{W})) + \sum_{k=1}^{6} \lambda a_{k} \tilde{\varphi}^{k} + \mathcal{R}_{\text{mom}},
$$

(6.12)

$$
8 \Delta_{\gamma} \tilde{\varphi} = R(\gamma) \tilde{\varphi} + \sum_{k=1}^{5} \lambda^{2k} b_{k} \tilde{\varphi}^{k} - \left( |\sigma + L_{\gamma} W^{2}_{\gamma}|^{(2)} \right) - |\sigma + L_{\gamma} W^{2}_{\gamma}|^{2} \left( c_{0} + \sum_{k \geq 1} c_{k} \lambda^{2(k-1)} \tilde{\varphi}^{k} \right) + \mathcal{R}_{\text{Ham}},
$$

(6.13)

where we define the following remainders

$$
\mathcal{R}_{\text{mom}} = -M^{[\geq 0]}(W^{(2)}) + a_{0},
$$

(6.14)

$$
\mathcal{R}_{\text{Ham}} = -8 \sum_{i=2}^{3} H^{[\geq -i]}(\varphi^{(i)}) + R^{(\geq 2)} + R(\gamma) \left( \varphi^{(2)} + \lambda \varphi^{(3)} \right) + b_{0}.
$$

(6.15)

6.2. Fixed point argument. In this section, we solve (6.12) and (6.13) by a fixed point argument. As in Sect. 5.4.2, the idea is to replace the operators depending on $\gamma$ by their Euclidean equivalent and use the smallness of $\gamma - \varepsilon$. We introduce the map $\Phi$

$$
\Phi : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B} \ \ \ \ (\tilde{\varphi}, \tilde{W}) \mapsto (\Phi_{1}(\tilde{\varphi}), \Phi_{2}(\tilde{W})),
$$

such that $\Phi_{1}(\tilde{\varphi})$ and $\Phi_{2}(\tilde{W})$ are solutions of the coupled system

$$
\text{div}_{e} L_{e} \Phi_{2}(\tilde{W}) = \text{div}_{e} L_{e}(\tilde{W}) - M^{[\geq 0]}(\tilde{W})
$$

$$
- M^{[\geq -1]}(W^{(3)}(\tilde{W})) + \sum_{k=1}^{6} \lambda^{2(k-1)} a_{k} \tilde{\varphi}^{k} + \mathcal{R}_{\text{mom}},
$$

(6.16)

$$
8 \Delta \Phi_{1}(\tilde{\varphi}) = 8 \Delta \tilde{\varphi} - 8 \Delta_{\gamma} \tilde{\varphi} + R(\gamma) \tilde{\varphi} + \sum_{k=1}^{5} \lambda^{2k} b_{k} \tilde{\varphi}^{k} - \left( |\sigma + L_{\gamma} W^{2}_{\gamma}|^{(2)} \right) - |\sigma + L_{\gamma} W^{2}_{\gamma}|^{2} \left( c_{0} + \sum_{k \geq 1} c_{k} \lambda^{2(k-1)} \tilde{\varphi}^{k} \right) + \mathcal{R}_{\text{Ham}},
$$

(6.17)

and where $\mathcal{B}$ is defined in (5.38). Note that a fixed point of $\Phi$ solves (6.12) and (6.13). In order to apply the Banach fixed point theorem and prove the existence of a fixed point, we prove in the next proposition that $\Phi$ is well-defined and is a contraction.

**Proposition 6.1.** If $C_{1}$ is large enough and $\varepsilon$ is small enough, then $\Phi$ is well-defined and is a contraction.
Proof. Let $({\tilde{\varphi}}, {\tilde{W}}) \in B \times B$. We start by estimating the $L^2_{s+2}$ norm of the RHS of (6.16):

$$\|\text{RHS of (6.16)}\|_{L^2_{s+2}} \lesssim \left\| \text{div}_e L_e({\tilde{W}}) - M^{[\geq 0]}({\tilde{W}}) \right\|_{L^2_{s+2}} + \left\| M^{[\leq -1]}(W^{(3)}({\tilde{W}})) \right\|_{L^2_{s+2}}$$

$$+ \left\| \sum_{k=1}^6 \lambda^{2(k-1)} a_k {\tilde{\varphi}}^k \right\|_{L^2} + \| \mathcal{R}_{\text{mom}} \|_{L^2}$$

$$=: A + B + C + D,$$

where we omitted the weights for the last two terms since they are compactly supported. As in the proof of Proposition 5.1, we obtain $A \lesssim C_1 \varepsilon^2$. For $B$, we note that the operator $M^{[\leq -1]}$ is linear and has bounded coefficients and involves up to two derivatives of $W^{(3)}({\tilde{W}})$, recall Lemma 4.7. Moreover, thanks to (6.3) $W^{(3)}({\tilde{W}})$ is compactly supported so we obtain $B \lesssim \| W^{(3)}(W) \|_{H^2}$. Using (4.9) and (6.4) this implies $B \lesssim C_1 \varepsilon^2 + \varepsilon$.

For $C$, we simply estimate $\tilde{\varphi}$ in $L^\infty$ using the embedding $H^2 \hookrightarrow L^\infty$ (see Proposition 1.1) and together with (6.7) this gives $C \lesssim C(C_1) \varepsilon^2$, where $C(C_1)$ denotes a numerical constant depending on $C_1$. Using (6.14), (5.16) and (6.7) again we also obtain $D \lesssim \varepsilon$. This discussion proves that

$$\|\text{RHS of (6.16)}\|_{L^2_{s+2}} \lesssim C(C_1) \varepsilon^2 + \varepsilon. \tag{6.18}$$

We now estimate the RHS of (6.17):

$$\|\text{RHS of (6.17)}\|_{L^2_{s+2}}$$

$$\lesssim \left\| \Delta \tilde{\varphi} - \Delta_\gamma \tilde{\varphi} \right\|_{L^2_{s+2}} + \| R(\gamma) \tilde{\varphi} \|_{L^2_{s+2}} + \left\| \sum_{k=1}^5 \lambda^{2k} b_k {\tilde{\varphi}}^k \right\|_{L^2} + \| \mathcal{R}_{\text{Ham}} \|_{L^2_{s+2}}$$

$$+ \left\| \left( |\sigma + L_\gamma W|_{\gamma}^2 \right)^{(\geq 2)} \right\|_{L^2_{s+2}} + \| |\sigma + L_\gamma W|_{\gamma}^2 \left( c_0 + \sum_{k \geq 1} c_k \lambda^{2(k-1)} \tilde{\varphi}^k \right) \right\|_{L^2_{s+2}}$$

$$=: A + B + C + D + E + F,$$

where we omitted the weights for the third and sixth terms since they are compactly supported. For $A$ we use the expansion defining $\gamma$, similarly as in (4.38):

$$A \lesssim \left\| (\gamma^{-1} - e^{-1}) \partial^2 \tilde{\varphi} \right\|_{L^2_{s+2}} + \| \partial_\gamma \partial \tilde{\varphi} \|_{L^2_{s+2}} \lesssim C_1 \varepsilon^2,$$

where we bound the metric coefficients and their derivatives in $L^\infty$ using (5.39) and (5.40). The terms $B$ and $C$ only contains $\tilde{\varphi}$ with zero derivatives, which we simply bound in $L^\infty$ using $H^2 \hookrightarrow L^\infty$. We then use Lemma 4.2 and (6.9) to obtain $B + C \lesssim C(C_1) \varepsilon^2$. Similar arguments lead to $D \lesssim \varepsilon$.

We now estimate $E$ and $F$. It involves the TT-tensor $\sigma$ but thanks to the estimate (5.44) we can put it in $L^\infty$ and thus focus on $L_\gamma W$. For the same reason, we neglect $W^{(2)}$ and $W^{(3)}(\tilde{W})$. Since the term

$$c_0 + \sum_{k \geq 1} c_k \lambda^{2(k-1)} \tilde{\varphi}^k,$$
can be bounded in $L^\infty$ by $C(C_1)\varepsilon$ (using (6.10)–(6.11) and $H^2_\delta \hookrightarrow L^\infty$ for the powers of $\tilde{\varphi}$), in order to estimate $E$ and $F$ it is enough to estimate $\| (L_Y \tilde{W})^2 \|_{L^2_{\delta+2}}$. Since $L_Y \tilde{W}$ contains derivatives of $\gamma$ we can’t directly use the product law $L^1_{\delta+1} \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$ of Proposition 1.1 without losing one $\lambda$ power. Instead we expand

$$\| (L_Y \tilde{W})^2 \|_{L^2_{\delta+2}} \lesssim \| (\partial \tilde{W})^2 \|_{L^2_{\delta+2}} + \| (\partial \gamma)^2 (\tilde{W})^2 \|_{L^2_{\delta+2}} + \| \partial \gamma \tilde{W}^2 \|_{L^2_{\delta+2}}.$$ 

For the first term we use the product law $H^1_{\delta+1} \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$ of Proposition 1.1. For the second and third terms, we bound $\partial \gamma$ in $L^\infty$ (recall (5.40)) and use the product laws $H^2_{\delta} \times H^2_{\delta} \hookrightarrow L^2_{\delta+2}$ and $H^2_{\delta} \times H^1_{\delta+1} \hookrightarrow L^2_{\delta+2}$. We obtain $\| (L_Y \tilde{W})^2 \|_{L^2_{\delta+2}} \lesssim C(C_1)\varepsilon^2$ and

$$E + F \lesssim C(C_1)\varepsilon^2 + \varepsilon.$$ 

This discussion proves that

$$\| \text{RHS of (6.17)} \|_{L^2_{\delta+2}} \lesssim C(C_1)\varepsilon^2 + \varepsilon. \quad (6.19)$$

Using the first part of Proposition 1.2, (6.18) and (6.19) prove that there exists a unique $(\Phi_1(\tilde{\varphi}), \Phi_2(\tilde{W})) \in H^2_{\delta} \times H^2_{\delta}$ solving (6.16)–(6.17) and satisfying

$$\| \Phi_1(\tilde{\varphi}) \|_{H^2_{\delta}} + \| \Phi_2(\tilde{W}) \|_{H^2_{\delta}} \lesssim C(C_1)\varepsilon^2 + \varepsilon.$$ 

Therefore, if we take $C_1$ larger than the numerical constant appearing in these estimates and $\varepsilon$ small compared to $C_1$, then $(\Phi_1(\tilde{\varphi}), \Phi_2(\tilde{W})) \in B \times B$. This shows that $\Phi$ is well-defined.

In order to show that $\Phi$ is a contraction we consider the equations satisfied by the differences $\Phi_1(\tilde{\varphi}_a) - \Phi_1(\tilde{\varphi}_b)$ and $\Phi_2(\tilde{W}_a) - \Phi_2(\tilde{W}_b)$, where $(\tilde{\varphi}_a, \tilde{W}_a)$ and $(\tilde{\varphi}_b, \tilde{W}_b)$ are two elements of $B \times B$. Together with non-linear inequalities of the form

$$|x^k - y^k| \lesssim \sup_{0 \leq p, q \leq k-1} \{|x|^p, |y|^q\} \times |x - y|,$$

we can mimic the previous arguments leading to (6.18) and (6.19) and prove that by taking $C_1$ larger and $\varepsilon$ smaller if necessary the map $\Phi$ is a contraction. We omit the details.

The Banach fixed point theorem then implies that there exists $(\tilde{\varphi}, \tilde{W}) \in B \times B$ solving (6.12) and (6.13). We can also prove that $\tilde{\varphi}$ and $\tilde{W}$ enjoy higher regularity, as we did for $Y$ in Sect. 5.4.2. We obtain $\tilde{\varphi}, \tilde{W} \in H^{N-3}_\delta$ with

$$\| \tilde{\varphi} \|_{H^{k+2}_\delta} + \| \tilde{W} \|_{H^{k+2}_\delta} \lesssim \frac{\varepsilon}{\lambda^k}, \quad (6.20)$$

for $k \in \llbracket 0, N-5 \rrbracket$. This concludes the construction of high-frequency solutions to (1.15)–(1.16).
7. Proof of the Main Theorem

In this section we conclude the proof of Theorem 2.1. The solution of the constraint equations \((\tilde{g}_\lambda, K_\lambda)\) we constructed through the conformal method is given by

\[
\tilde{g}_\lambda = \varphi^4 \gamma
\]

\[
K_\lambda = \varphi^{-2} (\sigma + L_\gamma W) + \frac{1}{3} \varphi^4 \gamma \tau
\]

where \(\gamma, \tau, \sigma, W\) and \(\varphi\) are the parameters and unknowns of the conformal method and are defined in the previous sections. Let us check that the two previous expressions match the expressions of Theorem 2.1 and the estimates therein.

7.1. The metric \(\tilde{g}_\lambda\) and proof of (2.12). We start with the induced metric. Thanks to (3.5) and (7.1) we have

\[
\tilde{g}_\lambda = \tilde{g}_0 + \lambda \gamma(1) + O(\lambda^2)
\]

which matches (2.5) using (4.2) and (4.8). If we now look at the order \(\lambda^2\) or higher in \(\varphi^4 \gamma\), we see that it is composed of oscillating terms and terms satisfying better estimates:

\[
\left(\varphi^4 \gamma\right)^{(\geq 2)} = 4\varphi(2) \tilde{g}_0 + 4\tilde{\varphi} \tilde{g}_0 + \gamma(2) + \lambda \left(4\varphi(3) \tilde{g}_0 + 4 \left(\tilde{\varphi} + \varphi(2)\right) \gamma(1)\right) + O(\lambda^2)
\]

(7.3)

where the \(O(\lambda^2)\) is a polynomial in terms of \(\varphi(2), \tilde{\varphi}, \varphi(3), \tilde{g}_0, \gamma(1)\) and \(\gamma(2)\). Using (5.5), (4.3) and (5.6)–(5.7) we see that

\[
\varphi(2) \tilde{g}_0 + \gamma(2) = \sin\left(\frac{\mu_0}{\lambda}\right) \tilde{F}(2.1) + \cos\left(\frac{2\mu_0}{\lambda}\right) \tilde{F}(2.2)
\]

Therefore, by setting

\[
\tilde{h}_\lambda = \left(\varphi^4 \gamma\right)^{(\geq 2)} - 4\varphi(2) \tilde{g}_0 - 4\gamma(2)
\]

we prove that \(\tilde{g}_\lambda\) is indeed given by the expression (2.5). We now prove estimate (2.12). Thanks to (7.3) we have

\[
\tilde{h}_\lambda = 4\tilde{\varphi} \tilde{g}_0 + \lambda \left(4\varphi(3) \tilde{g}_0 + 4 \left(\tilde{\varphi} + \varphi(2)\right) \gamma(1)\right) + O(\lambda^2)
\]

(7.4)

The regularity of each term in \(\tilde{h}_\lambda\) (recall (6.20)) and the decay of \(\tilde{\varphi}\) and \(\tilde{g}_0\) at infinity imply easily that the amount of derivatives together with the weights in (2.12) are allowed. The only part of (2.12) that remains to be checked is the \(\lambda\) behaviour. From this perspective, \(\varphi(3), \varphi(2)\) and \(\gamma(1)\) are the worse terms since they lose one \(\lambda\) power for each derivative. As they are already multiplied by \(\lambda\) in (7.4), this concludes the justification of (2.12).
7.2. The tensor $K_\lambda$ and proof of (2.13). For the tensor $K_\lambda$, we first prove that (7.2) matches the expression (2.6). Since $\varphi = 1 + O(\lambda^2)$, we have $\varphi^{-2} = 1 + O(\lambda^2)$ and $\varphi^4 = 1 + O(\lambda^2)$. Therefore from (7.2) we obtain

$$K_\lambda = \sigma^{(0)} + (L_\gamma W)^{(0)} + \lambda \left( \sigma^{(1)} + (L_\gamma W)^{(1)} + \frac{1}{3} g_0 \tau^{(1)} \right) + O(\lambda^2).$$

We now use the ansatz for $W$ (see (3.6)) and the expansion of Lemma 4.5 to obtain $(L_\gamma W)^{(0)} = 0$ and $(L_\gamma W)^{(1)} = K^{[-1]}(W^{(2)})$. This gives

$$K_\lambda = \sigma^{(0)} + \lambda \left( \sigma^{(1)} + K^{[-1]}(W^{(2)}) + \frac{1}{3} g_0 \tau^{(1)} \right) + O(\lambda^2)$$

$$= K^{(1)}_\lambda + \lambda K^{(1)} + O(\lambda^2),$$

where we used the definition of $\sigma^{(0)}$ and $\sigma^{(1)}$, see (5.19) and (5.20). Therefore, the solution $K_\lambda$ matches the expression (2.6). The remainder $K^{(\geq 2)}_\lambda$ satisfies

$$K^{(\geq 2)}_\lambda = \sigma^{(2)}(\gamma) + L_\gamma Y + K^{[\geq 0]}(W^{(2)}) + K^{[\geq -1]}(W^{(3)})(\tilde{W})) + L_\gamma \tilde{W}$$

$$- 2 \left( \tilde{\varphi} + \varphi^{(2)} \right) \sigma^{(0)} + \frac{1}{3} \gamma^{(1)} \tau^{(1)} + O(\lambda^3).$$

(7.5)

The estimate (2.13) then follows from estimating directly all the oscillating terms in (7.5) and using (5.43) and (6.20) for $L_\gamma Y$, $L_\gamma \tilde{W}$ or $\tilde{\varphi}$. This concludes the proof of Theorem 2.1.

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