THE COMPACTNESS OF COMMUTATORS OF CALDERÓN-ZGYMUND OPERATORS WITH DINI CONDITION

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Abstract. Let $T$ be the $\theta$-type Calderón-Zygmund operator with Dini condition. In this paper, we prove that for $b \in \text{CMO}(\mathbb{R}^n)$, the commutator generated by $T$ with $b$ and the corresponding maximal commutator, are both compact operators on $L^p(\omega)$ spaces, where $\omega$ be the Muchenhoupt $A_p$ weight function and $1 < p < \infty$.

1. Introduction and main results

Let $\theta : [0, 1] \to [0, \infty)$ be a continuous, increasing, subadditive function with $\theta(0) = 0$, we say that $\theta$ satisfies the Dini condition if $\int_0^1 \theta(t)\frac{dt}{t} < \infty$. A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ is said to be a $\theta$-type kernel if it satisfies

1. $|K(x, y)| \leq \frac{C}{|x-y|^n}$ whenever $x \neq y$,
2. $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \theta\left(\frac{|x-x'|}{|x-y|^n}\right)^{\frac{1}{n}}$ whenever $|x-y| \geq 2|x-x'|$.

We say that $T$ is a $\theta$-type Calderón–Zygmund operator if

1. $T$ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;
2. There is a $\theta$-type kernel $K(x, y)$ such that

$$(1.1) \quad Tf(x) := \int_{\mathbb{R}^n} K(x, y)f(y) \, dy$$

for all $f \in C_0^{\infty}(\mathbb{R}^n)$ and for all $x \notin \text{supp} f$, where $C_0^{\infty}(\mathbb{R}^n)$ is the space consisting of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. Historically, The $\theta$-type Calderón–Zygmund operator was introduced by Yabuta in [19] as a nature generalization of the classical Calderón-Zygmund operator. We note that when $\theta(t) = t^\delta$ with $0 < \delta \leq 1$, the $\theta$-type operator is just the classical Calderón–Zygmund operator with standard kernel (see [8, 9]). Given a locally integrable function $b$ defined on $\mathbb{R}^n$, and given a $\theta$-type Calderón–Zygmund operator $T$, the linear commutator $[b, T]$ generated by $b$ and $T$ is defined for smooth, compactly supported function $f$ as

$$(1.2) \quad [b, T]f(x) := b(x)Tf(x) - Tf(bf)(x) = \int_{\mathbb{R}^n} \left[ b(x) - b(y) \right] K(x, y)f(y) \, dy.$$ 

Also, we can define the maximal commutator of $\theta$-type Calderón–Zygmund operator as

$$(b, T^\ast)f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|,$$
where $T_{\varepsilon}f(x) = \int_{|x-y| \geq \varepsilon} [b(x) - b(y)] K(x,y) f(y) dy$ be the truncated part of $T$. Historically, commutators related to singular integral gave a new characterization of BMO function space, see Coifman, Rochberg and Weiss [5] and Janson [12]. Recently, Lerner [15] considered the weighted $L^p(\omega)$ estimates for $T$ with the sharp norm constant with respect to weight function, where $\omega$ be Muchenhoupt weight function and $1 < p < \infty$ (see also Quak-Yang [16] for boundedness without the sharp constant case). As the consequence of [1], $[b,T]$ is bounded also on $L^p(\omega)$.

On the other hand, many researcher were interested in discussing the compact of commutators. Uchiyama [18] proved that the commutator generated by a locally integral function $b$ with the homogenous singular integral (Lipschitz kernel) is compact on $L^p$ if and only if $b \in VMO(\mathbb{R}^n)$. Recently, Chen, Ding, Hu etc. consider the compactness of commutators generated by singular integrals with rougher kernel, see [3]. Torres etc. discuss the multilinear case for compactness. Krantz and Li [14] discuss the compact on $L^p(X)$, where $X$ be the space of homogenous type. application. Recently to study the regularity of solutions to the Beltrami equation, Clop and Cruz [4] proved that when $b$ is a VMO function, the commutator for stand Calderon-Zygmund operator is compact on $L^p(\omega)$. It is nature to ask whether $b$ belongs to VMO is also the sufficient condition for the compactness for the commutator generated by the $\theta$-Calderon-Zygmund operator on $L^p(\omega)$. This note will give a formative answer, moreover, we prove that maximal commutator $[b, T^*]$ share the same result. More precisely, we give here the main result as the following theorem.

**Theorem 1.1.** Let $T$ be a $\theta$-Calderón-Zygmund operator with $\theta$ satisfying the Dini condition and $\omega \in A_p$ with $1 < p < \infty$. If $b \in VMO(\mathbb{R}^n)$, then

1. $[b, T]: L^p(\omega) \to L^p(\omega)$ is compact.
2. $[b, T^*]: L^p(\omega) \to L^p(\omega)$ is compact.

**Remark 1.2.** For the compact of $[b, T]$, our proof is quiet similar as in [1], based on the argument in [14]. However we do some modification by choosing a suitable smooth truncation technique, the idea is coming from Bényi-Dami anMoen-Torres [2] dealing sharp constant for $A_p$ estimate. To some extend, our proof here simplify the one in [1]. On the other hand, to obtain the compact of $[b, T^*]$, we need the boundedness of $[b, T^*]$ on $L^p(\omega)$. We mention that since $[b, T^*]$ is non-linear, it is not the direct consequences of [1]. We prove it in a rather simple way based on some results in [10].

This note is organized as following way. In Section 2, we give some definitions and some lemmas. We deal with $[b, T]$ in Section 3, while $[b, T^*]$ is in Section 4. last but not least, we denote $s' = \frac{s}{s-1}$ and $C$ be a positive constant whose value may change at each appearance.

2. SOME DEFINITIONS AND TECHNICAL LEMMAS

As usually, we denote $\langle f \rangle_E = \frac{1}{|E|} \int_E f(x) dx$. We say $\omega$ is a weight function if $\omega \in L^1_{loc}(\mathbb{R}^n)$ such that $\omega(x) > 0$ almost everywhere. A weight function $\omega$ is said to belong to the Muckenhoupt class $A_p$, $1 < p < \infty$, if

$$[\omega]_{A_p} := \sup_Q \langle \omega \rangle_Q \langle \omega^{-\frac{p'}{p}} \rangle_Q^\frac{p}{p'} < \infty,$$
where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \), and where \( \frac{1}{p} + \frac{1}{p'} = 1 \). By \( L^p(\omega) \) we denote the set of measurable functions \( f \) that satisfy

\[
\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < \infty.
\]

The quantity \( \|f\|_{L^p(\omega)} \) defines a complete norm in \( L^p(\omega) \).

**Definition 2.1.** Suppose that \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( B \in \mathbb{R}^n \) is a ball. For \( a > 0 \), let

\[
\mathcal{M}(f, B) = \frac{1}{|B|} \int_B |f(x) - \langle f \rangle_B| dx \text{ for any ball } B \subset \mathbb{R}^n,
\]

and \( \mathcal{M}_a(f) = \sup_{B = a} \mathcal{M}(f, B) \). We say

1. the function \( f \) is said to belong to \( \text{BMO}(\mathbb{R}^n) \) if there exists a constant \( C > 0 \) such that
   \( \|f\|_{\text{BMO}} := \sup_{a > 0} \mathcal{M}_a \leq C \), and
2. a function \( f \) is said to belong to \( \text{VMO}(\mathbb{R}^n) \) if \( f \in \text{BMO}(\mathbb{R}^n) \),
   \[ \lim_{a \to 0} \mathcal{M}_a(f) = 0 \text{ and } \lim_{a \to +\infty} \mathcal{M}_a(f) = 0. \]

**Remark 2.2.** \( \text{VMO} \) space concides with \( \text{CMO} \) space, where \( \text{CMO} \) space denotes the closure of \( C_c^\infty \) in the \( \text{BMO} \) topology.

We need the following sufficient condition for compactness in \( L^p(\omega) \), \( \omega \in A_p \) and \( 1 < p < \infty \). This lemma was established in [4, Theorem 5].

**Lemma 2.3.** [4] Let \( p \in (1, \infty) \), \( \omega \in A_p \), and let \( \mathcal{F} \subset L^p(\omega) \). Then \( \mathcal{F} \) is totally bounded if it satisfies the next three conditions:

1. \( \mathcal{F} \) is uniformly bounded, i.e. \( \sup_{f \in \mathcal{F}} \|f\|_{L^p(\omega)} < \infty \).
2. \( \mathcal{F} \) is uniformly equicontinuous, i.e. \( \sup_{f \in \mathcal{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p(\omega)} \to 0 \) as \( h \to 0 \).
3. \( \mathcal{F} \) uniformly vanishes at infinity, i.e. \( \sup_{f \in \mathcal{F}} \|f - \chi_{Q(0,R)}f\|_{L^p(\omega)} \to 0 \) as \( \omega \to \infty \),
   where \( Q(0,R) \) is the cube with center at the origin and sidelength \( R \).

Technically, by Remark 2.2 we can approximate \( \text{VMO} \) function by \( C_0^\infty(\mathbb{R}^n) \) function. More precisely, we have the following lemma.

**Lemma 2.4.** For any \( b \in \text{VMO}(\mathbb{R}^n) \), we can approximate the function \( b \) by functions \( b_j \in C_c^\infty(\mathbb{R}^n) \) in the \( \text{BMO} \) norm, such that the following is satisfying

\[
\|b_j, T\|_{L^p(\omega)} \to 0, \quad \text{as } j \to \infty.
\]

Suppose that \( \psi : [0, \infty) \to [0, 1] \) satisfy (a)\( \text{supp} \psi = \{ t : t \geq \frac{1}{2} \} \), (b) \( \psi(t) = 1 \) when \( t > 1 \) and (c) \( |\psi'| \leq C \), where \( \frac{1}{2} < t < 1 \). Then for every \( \eta > 0 \) small enough, let us take a continuous function \( K^\eta \) defined on \( \mathbb{R}^n \times \mathbb{R}^n \) as

\[
K^\eta(x, y) = K(x, y)\psi(|x - y|/\eta).
\]

We can find that \( K^\eta(x, y) \) satisfy

1. \( K^\eta(x, y) = K(x, y) \), if \(|x - y| \geq \eta\),
2. \( |K^\eta(x, y)| \leq \frac{1}{|x-y|^n} \), if \( \frac{1}{2} < |x - y| < \eta \),
3. \( K^\eta(x, y) = 0 \), if \(|x - y| \leq \frac{1}{2}\).

Now, we denote

\[
T^\eta(f(x) = \int_{\mathbb{R}^n} K^\eta(x, y)f(y)dy
\]

and

\[
\left[ b, T^\eta \right]f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))K^\eta(x, y)f(y)dy.
\]
Let is not relies on the kernel condition, the proof is also the same, we omit the detail.

The following lemma shows that if \( b \in C_c^1(\mathbb{R}^n) \), the commutators \([b, T^\alpha]\) approximate \([b, T]\) in the operator norm, this is quiet the same in \([4, \text{Lemma 7}]\). Since the proof is not relies on the kernel condition, the proof is also the same, we omit the detail.

**Lemma 2.5.** Let \( b \in C_c^1(\mathbb{R}^n) \). There exists a constant \( C = C(n, C_0) \) such that
\[
||[b, T]f(x) - [b, T^\alpha]f(x)|| \leq C\eta||b||_{\infty}Mf(x) \quad x \in \mathbb{R}^n \text{ a.e.,}
\]
for every \( \eta > 0 \). As a consequence,
\[
\lim_{\eta \to 0} ||[b, T^\alpha] - [b, T]||_{L^p(\omega) \to L^p(\omega)} = 0,
\]
whenever \( \omega \in A_p \) and \( 1 < p < \infty \).

3. **Proof of part (1) in Theorem 1.1.**

We mention that \([b, T]\) is bounded on \( L^p(\omega) \) for \( \omega \in L^p(\omega)(1 < p < \infty) \) in Section 1. Now we denote
\[
\mathfrak{F} = \{ [b, T^\alpha] f; f \in L^p(\omega), ||f||_{L^p(\omega)} \leq 1, b \in C_c^1(\mathbb{R}^n) \}.
\]
Thanks to Lemma 2.2 and 2.3, to prove part (1) of Theorem 1.1, it is suffice to check that \( \mathfrak{F} \) satisfying the condition (2) and (3) in Lemma 2.1. This is to say, we need to prove the following two equations,
\[
\lim_{h \to 0} \sup_{f \in \mathfrak{F}} ||[b, T^\alpha] f(\cdot) - [b, T^\alpha] f(\cdot + h) ||_{L^p(\omega)} = 0,
\]
and
\[
\lim_{R \to \infty} \sup_{f \in \mathfrak{F}} \left( \int_{|x| > R} ||[b, T^\alpha] f(x)||^p \omega(x) dx \right)^{\frac{1}{p}} = 0.
\]

To prove (3.1). Indeed, for \( b \in C_c^1(\mathbb{R}^n) \)
\[
[b, T^\alpha] f(x) - [b, T^\alpha] f(x + h) = b(x) T^\alpha f(x) - T^\alpha (bf)(x) - b(x + h) T^\alpha f(x + h) + T^\alpha (bf)(x + h)
\]
\[
= b(x) T^\alpha f(x) - T^\alpha (bf)(x) - b(x + h) T^\alpha f(x + h) + T^\alpha (bf)(x + h)
\]
\[
+ T^\alpha (bf)(x + h) - b(x + h) T^\alpha f(x) + b(x + h) T^\alpha f(x)
\]
\[
=: A(x, h) + B(x, h),
\]
where
\[
A(x, h) = b(x) T^\alpha f(x) - b(x + h) T^\alpha f(x)
\]
\[
= (b(x) - b(x + h)) \int_{\mathbb{R}^n} K^\alpha(x, y) f(y) dy
\]
and
\[
B(x, h) = b(x + h) T^\alpha f(x) - T^\alpha (bf)(x) - b(x + h) T^\alpha f(x + h) + T^\alpha (bf)(x + h)
\]
\[
= \int_{\mathbb{R}^n} (b(x + h) - b(y))(K^\alpha(x, y) - K^\alpha(x + h, y)) f(y) dy
\]
\[
= \int_{\mathbb{R}^n} (b(x + h) - b(y)) \left[ K(x, y) \psi \left( \frac{|x - y|}{\eta} \right) - K(x + h, y) \psi \left( \frac{|x + h - y|}{\eta} \right) \right] f(y) dy.
\]
For \( A(x, h) \), using the regularity of the function \( b \) and the definition of the operator \( T^\alpha \)
\[
|A(x, h)| \leq ||\nabla b||_{\infty} |h| \left( \int_{|x - y| > \frac{\eta}{2}} (K^\alpha(x, y) - K(x, y)) f(y) dy + \int_{|x - y| > \frac{\eta}{2}} K(x, y) f(y) dy \right)
\]
\[ \| \nabla b \|_\infty |h| \left( \int_{|x-y| > \frac{\eta}{2}} |K^0(x, y) - K(x, y)| |f(y)| dy + T^* f(x) \right) \]
\[ \leq \| \nabla b \|_\infty |h| \left( \int_{|x-y| > \frac{\eta}{2}} \frac{|f(y)|}{|x-y|^n} dy + T^* f(x) \right) \]
\[ \leq \| \nabla b \|_\infty |h|(CMf(x) + T^* f(x)) \]
for some constant \( C > 0 \) independent of \( h \). Therefore
\[ (3.3) \quad (\int |A(x, h)|^p \omega(x) dx) \leq C\| h \|_{L^p(\omega)}, \]
for \( C \) independent of \( f \) and \( h \). Here we used the boundedness of \( M \) and \( T^* \) on \( L^p(\omega) \) (see [11]).

Suppose \( |h| < \frac{\eta}{4} \), then
\[ |B(x, h)| \leq |B_1(x, h)| + |B_2(x, h)|, \]
where
\[ |B_1(x, h)| = \left| \int_{\mathbb{R}^n} (b(x + h) - b(y))(K(x, y) - K(x + h, y)) \psi \left( \frac{|x-y|}{\eta} \right) f(y) dy \right| \]
and
\[ |B_2(x, h)| = \left| \int_{\mathbb{R}^n} (b(x + h) - b(y))K(x + h, y)\left( \psi \left( \frac{|x-y|}{\eta} \right) - \psi \left( \frac{|x+h-y|}{\eta} \right) \right) f(y) dy \right|. \]

For \( |B_1(x, h)| \), we have
\[ |B_1(x, h)| \leq \int_{|x-y| > \frac{\eta}{2}} |b(x + h) - b(y)||K(x, y) - K(x + h, y)||f(y)| dy \]
\[ \leq C\| b \|_{\infty} \int_{|x-y| > \frac{\eta}{2}} \theta \left( \frac{|h|}{|x-y|} \right) \frac{1}{|x-y|^n} |f(y)| dy \]
\[ = C\| b \|_{\infty} \int_{|y| > \frac{\eta}{2}} \theta \left( \frac{|h|}{|y|} \right) \frac{1}{|y|^n} |f(x-y)| dy \]
\[ = C\| b \|_{\infty} \sum_{k=1}^{\infty} \int_{\frac{2^{k-1}|y|}{2} < |y| < \frac{2^k|y|}{2}} \theta \left( \frac{|h|}{|y|} \right) \frac{1}{|y|^n} |f(x-y)| dy \]
\[ \leq C\| b \|_{\infty} \sum_{k=1}^{\infty} \theta \left( \frac{|h|}{\frac{2^{k-1}}{2}} \right) \frac{1}{\left( \frac{2^k}{2^{k-1}} \right)^n} \int_{|y| < \frac{2^k|y|}{2}} |f(x-y)| dy \]
\[ = C\| b \|_{\infty} \sum_{k=1}^{\infty} \theta \left( \frac{|h|}{\frac{2^{k-1}}{2}} \right) \frac{2^n}{\left( \frac{2^k}{2^{k-1}} \right)^n} \int_{|y| < \frac{2^k|y|}{2}} |f(x-y)| dy \]
\[ = C2^n\| b \|_{\infty} Mf(x) \sum_{k=1}^{\infty} \theta \left( \frac{4|h|}{|y| \cdot 2^k} \right) \]
\[ \leq C\| b \|_{\infty} Mf(x) \int_0^{\frac{4|h|}{2^k}} \theta(t) \frac{1}{t} dt \]

So, we can get
\[ (\int_{\mathbb{R}^n} |B_1(x, h)|^p \omega(x) dx)^{\frac{1}{p}} \leq C\| b \|_{\infty} \| f \|_{L^p(\omega)} \int_0^{\frac{4|h|}{\eta}} \theta(t) \frac{1}{t} dt \]
For $|B_2(x, h)|$, we know
\[
|B_2(x, h)| \leq \left| \int_{|x-y|>\frac{R}{2}} (b(x+h) - b(y)) K(x+h, y) f(y) dy \right|
\]
\[
\leq C\|\nabla b\|_\infty \int_{|x-y|>\frac{R}{2}} \frac{1}{|x-h-y|^{n-1}} |f(y)| dy
\]
\[
\leq C\|\nabla b\|_\infty \sum_{j=1}^{+\infty} \int_{|x-y|<2^{-j-1}\eta} \frac{1}{|x-h-y|^{n-1}} |f(y)| dy
\]
\[
\leq C\|\nabla b\|_\infty m M(f)(x) \sum_{j=1}^{+\infty} 2^{-j} \eta
\]
\[
\leq C\eta \|\nabla b\|_\infty M(f)(x).
\]
Thus
\[
\left( \int_{R^n} |B_2(x, h)|^p \omega(x) dx \right)^\frac{1}{p} \leq C\eta \|\nabla b\|_\infty \left( \int_{R^n} M(f)(x)^p \omega(x) dx \right)^\frac{1}{p}
\]
\[
\leq C\eta \|\nabla b\|_\infty \|f\|_{L^p(\omega)}
\]
\[
\lim_{h \to 0} \sup_{f \in \mathcal{F}} ||[b, T^\eta] f(\cdot + h) - [b, T^\eta] f(\cdot)||_{L^p(\omega)} = 0.
\]

Finally, we show the decay at infinity of the elements of $\mathcal{F}$. Let $x$ be such that $|x| > R > 2R_0$, suppose that $\text{supp } b \subset B(0, R_0)$. Then, $b \in C_c^\infty$, $x \notin \text{supp } b$, and
\[
|[b, T^\eta] f(x)| = \left| \int_{\mathbb{R}^n} (b(x) - b(y)) K^\eta(x, y) f(y) dy \right|
\]
\[
\leq C_0 \|b\|_\infty \int_{\text{supp } b} \frac{|f(y)|}{|x-y|^n} dy
\]
\[
\leq \frac{C \|b\|_\infty}{|x|^n} \int_{\text{supp } b} |f(y)| dy
\]
\[
\leq \frac{C \|b\|_\infty}{|x|^n} \|f\|_{L^p(\omega)} \left( \int_{\text{supp } b} \omega(y)^{-\frac{p'}{p}} dy \right)^\frac{1}{p'},
\]
whence
\[
(3.4) \quad \left( \int_{|x|>R} |[b, T^\eta] f(x)|^p \omega(x) dx \right)^\frac{1}{p} \leq C \|b\|_\infty \|f\|_{L^p(\omega)} \left( \int_{|x|>R} \omega(x) |x|^n dx \right)^\frac{1}{p}.
\]
By [?, Lemma 2.2], we have
\[
(3.5) \quad \int_{|x|>R} \frac{\omega(x)}{|x|^n} dx \leq \sum_{j=1}^{\infty} (2^{-1} R)^{-n} (2^j R)^n \omega(B(0, 1)) = \frac{C}{R^{n(p-q)}} < \infty.
\]
The right hand side above converges to 0 as $R \to \infty$, due to (3.5).

Thus the proof of part (1) in Theorem 1.1 follows.
4. Proof of part (2) in Theorem 1.1

Lemma 4.1. Suppose that \( b \in \text{BMO}, \ w \in A_p \) and \( 1 < p < \infty \), then
\[
\|[b, T^*]f\|_{L^p(\omega)} \leq C\|[b]_{\text{BMO}}\|_{L^p(\omega)}.
\]

Proof. In \( \mathbb{R}^n \) we define the unit cube, open on the right, to be the set \([0,1)^n\), and we let \( \Omega_0 \) be the collection of cubes in \( \mathbb{R}^n \) which are congruent to \([0,1)^n\) and whose vertices lie on the lattice \( Z^n \). If we dilate this family of cubes by a factor of \( 2^{-k} \) we get the collection \( \Omega_k, k \in Z \); that is, \( \Omega_k \) is the family of cubes, open on the right, whose vertices are adjacent points of the lattice \((2^{-k}Z)^n\). The cubes in \( \bigcup_k \Omega_k \) are called dyadic cubes. Give a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), define
\[
E_k f(x) = \sum_{Q \in \Omega_k} \frac{1}{|Q|} \int_Q f \chi_Q(x);
\]
\( E_k f \) is the conditional expectation of \( f \) with respect to the \( \sigma \)-algebra generated by \( \Omega_k \). It satisfies the following fundamental identity: if \( S \) is the union of cubes in \( \Omega_k \), then
\[
\int_S E_k f = \int_S f.
\]
Define the dyadic maximal function by
\[
M_d f(x) = \sup_k |E_k f(x)|.
\]
We define the sharp maximal function by
\[
M^{\ast} f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f - \langle f \rangle_Q|,
\]
where \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) the supremum is taken over all cubes \( Q \) containing \( x \).

We let
\[
T_j f(x) = \int_{|x-y|>2^j} K(x,y)f(y)dy, \ j \in \mathbb{Z},
\]
and \([b, T_j]f\) is defined as \([b, T]f\), we also define
\[
[b, T^{**}]f(x) = \sup_{j \in \mathbb{Z}} |T_j b f(x)|
\]
and
\[
M_b f(x) = \sup_{\varepsilon>0} r^{-n} \int_{|x-y|<\varepsilon} |b(x) - b(y)||f(y)|dy
\]
It is easy to check
\[
[b, T^*]f(x) \leq [b, T^{**}]f(x) + M_b f(x).
\]
We will show that, when \( \omega \in A_p(\mathbb{R}^n), \ p \in (1, +\infty), \)
\[
\|[b, T^{**}]f\|_{L^p(\omega)} + \|M_b f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.
\]
First, we prove that
\[
\|[b, T^{**}]f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.
\]
Similar to [8] Lemma 5.15, p102-104, we can obtain the following Cotlar’s inequality: for any \( \gamma \in (0,1] \) and any \( f \in C_c^\infty \),
\[
\sup |T_j f(x)| \leq C(M(|T f|^\gamma)(x)^{1/\gamma} + M f(x)).
\]
As a consequence, for \( \omega \in A_p(\mathbb{R}^n), \ p \in (1, +\infty) \)
\[
\|\sup |T_j f|\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.
\]
So we proved that $\mathcal{S} = \{T_j\}$ is bounded from $L^p(\omega)$ to $L^p(l^\infty, \omega)$, where
\[ L^p(l^\infty, \omega) = \{ \{f_j\}_{j \in \mathbb{Z}} : \| \sup_{j \in \mathbb{Z}} |T_j f| \|_{L^p(\omega)} < \infty \} \].

This result combine the argument in [7], we have
\[ \|[b, T^*]f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}. \]

Next, we will prove the boundedness of $M_b f$, for $0 < q < 1$,
\[ \|M_b f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} (|M_b f(x)|^q)^{\frac{1}{q}} \omega(x) dx \right)^\frac{1}{p} \]
\[ \leq C \left( \int_{\mathbb{R}^n} [M_d((M_b f)^q)]^{\frac{1}{q}} \omega(x) dx \right)^\frac{1}{p} \]
\[ \leq C \left( \int_{\mathbb{R}^n} [M^2((M_b f)^q)]^{\frac{1}{q}} \omega(x) dx \right)^\frac{1}{p} \]
\[ = C \left( \int_{\mathbb{R}^n} [M^2_b f(x)]^p \omega(x) dx \right)^\frac{1}{p} \]
\[ = C \|M^2_b f\|_{L^p(\omega)}. \]

By [10] Lemma2.3 for $0 < q < s < 1$, we can know that
\[ M^2_q(\tilde{M}_b f)(x) \leq C \|b\|_s [M_s(\tilde{M} f)(x) + M_{L(\log L)} f(x)], \]
due to there exists some constant $C \geq 1$ such that for all $x \in \mathbb{R}^n$
\[ C^{-1} \tilde{M}_b f(x) \leq M_b f(x) \leq C \tilde{M}_b f(x) \]
and
\[ C^{-1} \tilde{M} f(x) \leq M f(x) \leq C \tilde{M} f(x), \]
where the definitions of $\tilde{M}_b$ and $\tilde{M}$ have given in [10], because of the limitation of length, no more tautology here. So we can have
\[ \|M^2_q(M_b f)\|_{L^p(\omega)} \leq C \|b\|_{BMO} \|M_s(\tilde{M} f)\|_{L^p(\omega)} + \|M_{L(\log L)} f\|_{L^p(\omega)}. \]

In [7], we know $M_{L(\log L)} \sim M^2$, so we have that $M_{L(\log L)} : L^p(\omega) \to L^p(\omega)$ is bounded, so we only need prove the bounded of $M_s(\tilde{M} f)$, now we give the proof,
\[ \|M_s(\tilde{M} f)\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} (M_s(\tilde{M} f)(x))^p \omega(x) dx \right)^\frac{1}{p} \]
\[ \leq C \int_{\mathbb{R}^n} (\tilde{M} f(x))^p \omega(x) dx \]
\[ \leq C \int_{\mathbb{R}^n} (M f(x))^p \omega(x) dx \]
\[ \leq C \|f\|_{L^p(\omega)}. \]

Thus, we can get
\[ \|M_b f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}. \]

Whence
\[ \|[b, T^*]f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}. \]
Now, we will give the proof of \([b, T^*]\) uniformly vanishes at infinity.

**Lemma 4.2.** For any \(b \in \text{VMO}(\mathbb{R}^n)\), there exists \(\{b_j\} \subset C_c^\infty(\mathbb{R}^n)\) and satisfy \(b = \lim_{j \to +\infty} b_j\) in BMO, such that the following is satisfying
\[
\|[b, T^*]f - [b_j, T^*]f\|_{L^p(\omega)} \to 0 \text{ as } j \to +\infty.
\]

**Proof.** As we know, For any \(b \in \text{VMO}\) and for any \(\varepsilon > 0\), there exists \(b_j \in C_c^\infty\) such that
\[
\|b - b_j\|_{\text{BMO}} < \varepsilon.
\]

It is easy to see that
\[
\|[b, T^*]f(x) - [b_j, T^*]f(x)\| = \sup_{\delta > 0} \left| \int_{|x - y| > \delta} (b(x) - b(y))K(x, y)f(y)dy \right|
\]
\[- \sup_{\delta > 0} \left| \int_{|x - y| > \delta} (b_j(x) - b_j(y))K(x, y)f(y)dy \right|
\]
\[
\leq \sup_{\delta > 0} \left| \int_{|x - y| > \delta} (b(x) - b(y))K(x, y)f(y)dy \right|
\]
\[- \int_{|x - y| > \delta} (b_j(x) - b_j(y))K(x, y)f(y)dy \right|
\]
\[
\leq \sup_{\delta > 0} \left| \int_{|x - y| > \delta} (b(x) - b(y))K(x, y)f(y)dy \right|
\]
\[- \int_{|x - y| > \delta} (b_j(x) - b_j(y))K(x, y)f(y)dy \right|
\]
\[
= [b - b_j, T^*]f(x).
\]

So we can get
\[
\|[b, T^*]f - [b_j, T^*]f\|_{L^p(\omega)} \leq \|[b - b_j, T^*]f\|_{L^p(\omega)} \leq C\varepsilon\|f\|_{L^p(\omega)}
\]
and
\[
\|[b, T^*]f\|_{L^p(\omega)} \leq \|[b_j, T^*]f\|_{L^p(\omega)} + C\varepsilon\|f\|_{L^p(\omega)}.
\]

\(\Box\)

**Lemma 4.3.** Suppose that \(\omega \in A_p\) for \(1 < p < \infty\), then
\[
(4.1) \quad \lim_{|h| \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(\omega)} = 0.
\]

**Proof.** Since \(C_c^\infty(\mathbb{R}^n)\) is dense in \(L^p(\omega)\) when \(w \in A_p\) for \(1 < p < \infty\), we need only to prove that for any \(f \in C_c^\infty(\mathbb{R})\), \(f\) satisfying \((1.1)\). In fact we can let \(\text{supp} f \subset B(0, R)\), so \(\text{supp}(\cdot + h) \subset B(0, 2R)\) when \(|h|\) small enough, thus
\[
\int_{\mathbb{R}^n} |f(x + h) - f(x)|^p \omega(x)dx
\]
\[
= \int_{B(0,2R)} |f(x + h) - f(x)|^p \omega(x)dx
\]
\[
\leq \|\nabla f\|_\infty |h| \omega(B(0, 2R)).
\]

Since \(\omega \in A_p\), it is obviously locally integrable, we have \(\omega(B(0, 2R)) < \infty\), then we let \(h \to 0\), the Lemma is proved. \(\Box\)
Thus, we only need prove \([b, T^*]\) uniformly vanishes at infinity, where \(b \in C_c^\infty\). We can suppose that \(\text{supp} b \subset \{ x \in \mathbb{R}^n : |x| < R_0 \}\) and \(R > 3R_0\), where \(R_0 > 1\), note that, when \(|x| > R\), we have

\[
\| [b, T^*] f(x) \| = \sup_{\delta > 0} \int_{|x-y| > \delta} (b(x) - b(y)) K(x, y) f(y) dy
\]

\[
\leq C \| b \|_\infty \int_{|y| \leq R_0} \frac{|f(y)|}{|x-y|^n} dy
\]

\[
= C \| b \|_\infty \int_{|y| \leq R_0} \frac{|f(y)|}{|x-y|^n} dy
\]

\[
\leq C \| b \|_\infty \frac{1}{|x|^n} \int_{|y| \leq R_0} |f(y)| dy
\]

\[
\leq C \| b \|_\infty \frac{1}{|x|^n} \| f \|_{L^p(\omega)} \left( \int_{|y| \leq R_0} \omega(y)^{-\frac{np}{p}} dy \right)^\frac{1}{p}.
\]

Whence

\[
\left( \int_{|x| > \alpha} | [b, T^*] f(x) |^p \omega(x) dx \right)^\frac{1}{p}
\]

\[
\leq C \| b \|_\infty \| f \|_{L^p(\omega)} \left( \int_{|y| \leq R_0} \omega(y)^{-\frac{np}{p}} dy \right)^\frac{1}{p} \left( \int_{|x| > R} \frac{\omega(x)}{|x|^np} dx \right)^\frac{1}{p}
\]

\[
\leq C \| b \|_\infty \| f \|_{L^p(\omega)} \frac{1}{R^{n(p-q)}}
\]

where \(q < p\), so we have

\[
\lim_{R \to +\infty} \left( \int_{|x| > R} | [b, T^*] f(x) |^p \omega(x) dx \right)^\frac{1}{p} = 0.
\]

To prove the uniform equicontinuity of \([b, T^*]\), we must see that

\[
\lim_{h \to 0} \sup_{f \in L^p(\omega)} \| [b, T^*] f(\cdot) - [b, T^*] f(\cdot + h) \|_{L^p(\omega)} = 0.
\]

In fact, for any \(h \in \mathbb{R}^n\), we define \(K_\delta(x, y) = K(x, y) \chi_{\{y : |x-y| > \delta\}}(y)\), so

\[
| [b, T^*] f(x + h) - [b, T^*] f(x) |
\]

\[
= \sup_{\delta > 0} \int_{|x-y| > \delta} (b(x + h) - b(y)) K(x + h, y) f(y) dy
\]

\[
- \sup_{\delta > 0} \int_{|x-y| > \delta} (b(x) - b(y)) K(x, y) f(y) dy
\]

\[
\leq \sup_{\delta > 0} \int_{|x-y| > \delta} (b(x + h) - b(y)) K(x + h, y) f(y) dy
\]

\[
- \int_{|x-y| > \delta} (b(x) - b') K(x, y) f(y) dy
\]

\[
= \sup_{\delta > 0} \int_{\mathbb{R}^n} (b(x + h) - b(y)) K_\delta(x + h, y) f(y) dy
\]
now, we can divided $\mathbb{R}^n$ into $|x - y| > \varepsilon^{-1}|h|$ and $|x - y| \leq \varepsilon^{-1}|h|$, so we can have

$$
\|[b, T^*] f(x + h) - [b, T^*] f(x)\|
\leq \sup_{\delta > 0} \left\{ \int_{|x - y| > \varepsilon^{-1}|h|} K_\delta(x, y) f(y)(b(x + h) - b(x)) dy \right. \\
+ \left. | \int_{|x - y| > \varepsilon^{-1}|h|} (K_\delta(x + h, y) - K_\delta(x, y))(b(x + h) - b(y)) f(y) dy \right| \\
+ | \int_{|x - y| \leq \varepsilon^{-1}|h|} K_\delta(x, y)(b(x) - b(y)) f(y) dy | \\
+ | \int_{|x - y| \leq \varepsilon^{-1}|h|} K_\delta(x + h, y)(b(x + h) - b(y)) f(y) dy | \\
\leq \sup_{\delta > 0} \left\{ \int_{|x - y| > \varepsilon^{-1}|h|} K_\delta(x, y) f(y)(b(x + h) - b(x)) dy \right. \\
+ \sup_{\delta > 0} \left| \int_{|x - y| > \varepsilon^{-1}|h|} (K_\delta(x + h, y) - K_\delta(x, y))(b(x + h) - b(y)) f(y) dy \right| \\
+ \sup_{\delta > 0} \left| \int_{|x - y| \leq \varepsilon^{-1}|h|} K_\delta(x, y)(b(x) - b(y)) f(y) dy \right| \\
+ \sup_{\delta > 0} \left| \int_{|x - y| \leq \varepsilon^{-1}|h|} K_\delta(x + h, y)(b(x + h) - b(y)) f(y) dy \right| \\
= E_1 + E_2 + E_3 + E_4
$$

For $E_1$, we have

$$
E_1 = \sup_{\delta > 0} \left| \int_{|x - y| > \varepsilon^{-1}|h|} K_\delta(x, y) f(y)(b(x + h) - b(x)) dy \right| \\
\leq |h|\|\nabla b\|_\infty \sup_{\delta > 0} \left| \int_{|x - y| \leq \varepsilon^{-1}|h|,|x - y| > \delta} K(x, y) f(y) dy \right| \\
\leq |h|\|\nabla b\|_\infty T^* f(x).
$$

Thus

$$
\|E_1\|_{L^p(\omega)} \leq |h|\|\nabla b\|_\infty \|T^* f\|_{L^p(\omega)} \\
\leq C|h|\|\nabla b\|_\infty \|f\|_{L^p(\omega)}.
$$

For $E_2$, we can know that

$$
E_2 \leq E_{21} + E_{22},
$$

where

$$
E_{21} = \sup_{\delta > 0} \left| \int_{|x - y| > \varepsilon^{-1}|h|} (K(x + h, y) - K(x, y))\chi_{|x + h - y| > \delta}(y)(b(x + h) - b(y)) f(y) dy \right| \\
\text{and}
$$

$$
E_{22} = \sup_{\delta > 0} \left| \int_{|x - y| > \varepsilon^{-1}|h|} K(x, y)(\chi_{|x + h - y| > \delta}(y) - \chi_{|x - y| > \delta}(y))(b(x + h) - b(y)) f(y) dy \right|.
$$

On the one hand, we will give the estimation of $E_{21}$,

$$
E_{21} \leq \int_{|x - y| > \varepsilon^{-1}|h|} |K(x + h, y) - K(x, y)||b(x + h) - b(y)||f(y)| dy
$$
On the other hand, for
\[ \frac{1}{|x-y|} \leq \frac{1}{|x-y|^n} f(y)dy \]
and
\[ C\|b\|\infty \frac{1}{|y|} \frac{1}{|y|^{2k-1}} \epsilon \frac{1}{|y|^{2k-1}} f(y)dy \]

Further, we are going to estimate
\[ C\|b\|\infty \sum_{k=1}^{\infty} \frac{1}{|y|} \frac{1}{|y|^{2k-1}} \epsilon \frac{1}{|y|^{2k-1}} f(y)dy \]

and
\[ C\|b\|\infty \sum_{k=1}^{\infty} \frac{1}{|y|} \frac{1}{|y|^{2k-1}} \epsilon \frac{1}{|y|^{2k-1}} f(y)dy \]

and
\[ C\|b\|\infty Mf(x) \leq C\|b\|\infty Mf(x) \int_{0}^{t} \frac{dt}{t} \]

therefore
\[ \|E_{21}\|_{L_p(\omega)} \leq C\|b\|\infty \|f\|_{L_p(\omega)} \int_{0}^{t} \frac{dt}{t} \]

On the other hand, for \( E_{22} \), we have
\[ E_{22} = \sup_{\delta > 0} \left| \int_{|x-y| > \epsilon^{-1}|h|} K(x, y)(\chi_{|x+h-y| > \delta(y)} - \chi_{|x-y| > \delta(y)})(b(x + h) - b(y))f(y)dy \right| \]

\[ \leq \sup_{\delta > 0} \left| \int_{|x-y| > \epsilon^{-1}|h|, |x+h-y| > \delta, |x-y| < \delta} K(x, y)(b(x + h) - b(y))f(y)dy \right| \]

\[ + \sup_{\delta > 0} \left| \int_{|x-y| > \epsilon^{-1}|h|, |x+h-y| < \delta, |x-y| < \delta} K(x, y)(b(x + h) - b(y))f(y)dy \right| \]

\[ \leq E_{221} + E_{222} \]

Further, we are going to estimate \( E_{221} \) and \( E_{222} \), for \( E_{221} \), when \( |x-y| > \epsilon^{-1}|h| \), \( |x+h-y| > \delta \) and \( 0 < \epsilon < \frac{1}{4} \), then \( |x-y| > \frac{|h|}{\epsilon} > \frac{\delta}{\epsilon} \), so we have \( |x-y| > \frac{\delta}{\epsilon + 1} \), and
\[ E_{221} = \sup_{\delta > 0} \left| \int_{|x-y| > \epsilon^{-1}|h|, |x+h-y| > \delta, |x-y| < \delta} K(x, y)(b(x + h) - b(y))f(y)dy \right| \]

\[ \leq C \|b\|\infty \sup_{\delta > 0} \left| \int_{\frac{\delta}{\epsilon + 1} < |y| < \delta} f(y)dy \right| \]

\[ \leq C \|b\|\infty \sup_{\delta > 0} \left| \int_{\frac{\delta}{\epsilon + 1} < |y| < \delta} f(y)dy \right| \]

\[ \leq C \|b\|\infty \sup_{\delta > 0} \left( \int_{\frac{\delta}{\epsilon + 1} < |y| < \delta} \frac{f(y)dy}{|y|^n} \right)^{\frac{1}{r}} \times \sup_{\delta > 0} \left( \int_{\frac{\delta}{\epsilon + 1} < |y| < \delta} \frac{1}{|y|^n} dy \right)^{\frac{1}{r}} \]

where \( 1 < r < p \), due to
\[ \int_{\frac{\delta}{\epsilon + 1} < |y| < \delta} \frac{1}{|y|^n} dy = \int_{S^{n-1}} \int_{\frac{\delta}{\epsilon + 1}}^{\delta} \frac{1}{r} dr d\sigma \]

\[ \leq C \ln(1 + \epsilon) \]
and

\[
\sup_{\delta > 0} \left( \int_{\frac{\delta}{1 + \epsilon} < |y| < \delta} \frac{|f(x - y)|^r}{|y|^n} dy \right)^{\frac{1}{r}} \leq \sup_{\delta > 0} \left( (1 + \epsilon)^n \delta^{-n} \int_{|y| < \delta} |f(x - y)|^r dy \right)^{\frac{1}{r}} \leq (1 + \epsilon)^n M(|f|^r)(x)^{\frac{1}{r}},
\]

hence

\[
\|E_{221}\|_{L^p(\omega)} \leq C \varepsilon \left( 1 + \varepsilon \right)^n \|b\|_{\infty} \|M(|f|^r)\|_{L^p(\omega)}^{\frac{1}{r}} \leq C \varepsilon \left( 1 + \varepsilon \right)^n \|b\|_{\infty} \left( \int_{\mathbb{R}^n} |f|^p \omega(x) dx \right)^{\frac{1}{p}} \leq C \varepsilon \left( 1 + \varepsilon \right)^n \|b\|_{\infty} \|f\|_{L^p(\omega)}.
\]

In a similar way, for \( E_{222} \), when \(|x - y| > \varepsilon^{-1}|h|, |x + h - y| < \delta\) and \(0 < \varepsilon < \frac{1}{2}\), then

\[
|x - y| < |x + h - y| + |h| < \delta + \varepsilon|x - y|,
\]

so we have \( |x - y| < \frac{\delta}{1 - \varepsilon} \), and

\[
E_{222} = \sup_{\delta > 0} \left| \int_{|x - y| > \varepsilon^{-1}|h|, |x + h - y| < \delta, |x - y| > \delta} K(x, y)(b(x + h) - b(y))f(y) dy \right|
\]

\[
\leq C \|b\|_{\infty} \sup_{\delta > 0} \int_{|y| < \frac{\delta}{1 + \epsilon}} \frac{|f(y)|}{|y|^n} dy
\]

\[
\leq C \|b\|_{\infty} \sup_{\delta > 0} \int_{|y| < \frac{\delta}{1 + \epsilon}} \frac{|f(x - y)|}{|y|^n} dy
\]

\[
\leq C \|b\|_{\infty} \sup_{\delta > 0} \left( \int_{|y| < \frac{\delta}{1 + \epsilon}} \frac{|f(x - y)|^r}{|y|^n} dy \right)^{\frac{1}{r}} \times \sup_{\delta > 0} \left( \int_{|y| < \frac{\delta}{1 + \epsilon}} \frac{1}{|y|^n} dy \right)^{\frac{n}{r}},
\]

due to

\[
\int_{\delta < |y| < \frac{\delta}{1 + \epsilon}} \frac{1}{|y|^n} dy = \int_{S^{n-1}} \frac{1}{r} dr d\sigma
\]

\[
\leq C \ln \left( \frac{1}{1 - \varepsilon} \right)
\]

and

\[
\sup_{\delta > 0} \left( \int_{\delta < |y| < \frac{\delta}{1 + \epsilon}} \frac{|f(x - y)|^r}{|y|^n} dy \right)^{\frac{1}{r}} \leq C \left( \ln \left( \frac{1}{1 - \varepsilon} \right) \right)^{\frac{1}{r}} (1 - \varepsilon)^{-\frac{n}{r}} \|\nabla b\|_{\infty} \|f\|_{L^p(\omega)}.
\]

Next, we consider the \( E_3 \), we know

\[
E_3 \leq C \|\nabla b\|_{\infty} \int_{|x - y| \leq \varepsilon^{-1}|h|} |K(x, y)| |f(y)| dy
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|x - y| \leq \varepsilon^{-1}|h|} \frac{|f(y)|}{|x - y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|y| \leq \varepsilon^{-1}|h|} \frac{|f(x - y)|}{|y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} \sum_{k=1}^{\infty} \int_{e^{-1}|h|2^{k-1} \leq |y| \leq e^{-1}|h|2^{k+1}} \frac{|f(x - y)|}{|y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} Mf(x) \sum_{k=1}^{\infty} e^{-1}|h|2^{-k}
\]
\[
\leq C \| \nabla b \|_{\infty} \varepsilon^{-1}|h|Mf(x),
\]
so, we have
\[
\| E_3 \|_{L^p(\omega)} \leq C \varepsilon^{-1}|h| \| \nabla b \|_{\infty} \| f \|_{L^p(\omega)}.
\]
Finally, let’s give an estimate of \( E_4 \),
\[
E_4 = \sup_{\delta > 0} \left| \int_{|x - y| \leq \varepsilon^{-1}|h|} K_\delta(x + h, y)(b^\varepsilon(x + h) - b^\varepsilon(y))f(y) dy \right|
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|x - y| \leq \varepsilon^{-1}|h|} |K(x + h, y)||f(y)||dy|
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|x - y| \leq \varepsilon^{-1}|h|} \frac{|f(y)|}{|x + h - y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|y| \leq \varepsilon^{-1}|h|} \frac{|f(x - y)|}{|h + y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} \int_{|y| \leq (\varepsilon^{-1} + 1)|h|} \frac{|f(x - h - y)|}{|y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} \sum_{k=1}^{\infty} \int_{(\varepsilon^{-1} + 1)|h|2^{-k} \leq |y| \leq (\varepsilon^{-1} + 1)|h|2^{k+1}} \frac{|f(x + h - y)|}{|y|^{n-1}} dy
\]
\[
\leq C \| \nabla b \|_{\infty} Mf(x + h) \sum_{k=1}^{\infty} (\varepsilon^{-1} + 1)|h|2^{-k}
\]
\[
\leq C \| \nabla b \|_{\infty} (\varepsilon^{-1} + 1)|h|Mf(x + h).
\]
By Lemma 4.3, we can have
\[
\| E_4 \|_{L^p(\omega)} \leq C \| \nabla b \|_{\infty} (\varepsilon^{-1} + 1)|h|Mf(\cdot + h)\|_{L^p(\omega)}
\]
\[
\leq C \| \nabla b \|_{\infty} (\varepsilon^{-1} + 1)|h|\| f \|_{L^p(\omega)}.
\]
If let \( |h| = \frac{\varepsilon^2}{\varepsilon^{-1} + 1} \), then
\[
\lim_{\varepsilon \to 0} \| E_3 \|_{L^p(\omega)} = \lim_{\varepsilon \to 0} \| E_4 \|_{L^p(\omega)} = 0.
\]
COMPACTNESS

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