Birkhoff’s Theorem for Quasi-Metric Gravity

Dag Østvang

Department of Physics, Norwegian University of Science and Technology (NTNU)
N-7491 Trondheim, Norway

Abstract

Working within the quasi-metric framework (QMF), it is examined if the gravitational field exterior to an isolated, spherically symmetric body is necessarily metrically static; or equivalently, whether or not Birkhoff’s theorem holds for quasi-metric gravity. It is found that it does; however the proof is somewhat different from the general-relativistic case.

1 Introduction

For General Relativity (GR), the validity of Birkhoff’s theorem is well understood to be necessary both physically and mathematically. That is, just as Maxwell’s equations forbid the existence of monopole electromagnetic waves, so do the Einstein field equations forbid the existence of monopole gravitational radiation [1], since according to Birkhoff’s theorem, any time-dependent aspects of the space-time geometry interior to a spherically symmetric source cannot be propagated to the exterior gravitational field.

Since the existence of monopole gravitational radiation is not desirable for observational reasons, any potentially viable alternative theory of gravity should also fulfil Birkhoff’s theorem. In particular this applies to quasi-metric gravity (the QMF is described in detail elsewhere [2, 3]). In this paper, equations relevant for the vacuum exterior to a spherically symmetric (in general metrically nonstatic) source are set up. It is then shown that no acceptable metrically nonstatic solutions exist, proving the validity of Birkhoff’s theorem for quasi-metric gravity.

2 Basic quasi-metric gravity

The QMF has been described in detail elsewhere [2, 3]. Here we include only the bare minimum of motivation and general formulae necessary to do the calculations presented in later sections.

The basic motivation for introducing the QMF is the idea that the cosmic expansion should be described as a general phenomenon not depending on the causal structure
associated with any pseudo-Riemannian manifold. This idea would drastically reduce
the enormous multitude of possibilities regarding cosmic genesis and evolution present
in metric gravity and thereby increase the predictive power of the science of cosmology.
And as we will see in what follows, certain properties intrinsic to quasi-metric space-time
ensure that this alternative way of describing the cosmic expansion is mathematically
consistent and fundamentally different from its counterpart in GR.

Briefly the geometrical basis of the QMF consists of a 5-dimensional differentiable
manifold with topology \( M \times \mathbb{R}_1 \), where \( M = S \times \mathbb{R}_2 \) is a Lorentzian space-time manifold,
\( \mathbb{R}_1 \) and \( \mathbb{R}_2 \) both denote the real line and \( S \) is a compact 3-dimensional manifold (without
boundaries). That is, in addition to the usual time dimension and 3 space dimensions,
there is an extra time dimension represented by the global time function \( t \). (To ensure the
uniqueness of \( t \) (see below), the 3-dimensional manifold \( S \) is compact by definition.) The
reason for introducing this extra time dimension is that by definition, \( t \) parameterizes
any change in the space-time geometry that has to do with the cosmic expansion. By
construction, the extra time dimension is degenerate to ensure that such changes will
have nothing to do with causality. Mathematically, to fulfil this property, the manifold
\( M \times \mathbb{R}_1 \) is equipped with two degenerate 5-dimensional metrics \( \bar{g}_t \) and \( g_t \). The metric
\( \bar{g}_t \) is found from field equations as a solution, whereas the “physical” metric \( g_t \) can be
constructed from \( \bar{g}_t \) in a way described in refs. [2, 3].

The global time function is unique in the sense that it splits quasi-metric space-time
into a unique set of 3-dimensional spatial hypersurfaces called fundamental hypersurfaces
(FHSs) (where each FHS is represented by the 3-manifold \( S \) for some epoch \( t \)). Observers
always moving orthogonally to the FHSs are called fundamental observers (FOs). The
topology of \( M \) indicates that there also exists a unique “preferred” ordinary global time
coordinate \( x^0 \). We use this fact to construct the 4-dimensional quasi-metric space-time
manifold \( N \) by slicing the submanifold determined by the equation \( x^0 = ct \) out of the
5-dimensional differentiable manifold. (It is essential that this slicing is unique since the
two global time coordinates should be physically equivalent; the only reason to separate
between them is that they are designed to parameterize fundamentally different physical
phenomena.) Thus the 5-dimensional degenerate metric fields \( \bar{g}_t \) and \( g_t \) may be regarded
as one-parameter families of Lorentzian 4-metrics on \( N \). Note that there exists a set
of particular coordinate systems especially well adapted to the geometrical structure
of quasi-metric space-time, the global time coordinate systems (GTCSs). A coordinate
system is a GTCS iff the time coordinate \( x^0 \) is related to \( t \) via \( x^0 = ct \) in \( N \).

Expressed in an isotropic GTCS, the most general form allowed for the family \( \bar{g}_t \) is
represented by the family of line elements valid on the FHSs (this may be taken as a
The field equations then read (valid on the FHSs using a GTCS, and where a comma turns out that a subset of the Einstein field equations (albeit modified) can be tailored via two different coupling parameters $G^B$ and $G^S$, respectively. This non-universality of the gravitational coupling is required for consistency reasons and yields a modification of the right hand side of the gravitational field equations. (Said modification was missed in the original formulation of quasi-metric gravity.)

Moreover, due to the prior-geometric restriction on $\tilde{h}_t$, a full coupling to space-time curvature of the active stress-energy tensor $T_t$ should not be expected to exist. But it turns out that a subset of the Einstein field equations (albeit modified) can be tailored to $\tilde{g}_t$, so that partial couplings to space-time curvature of $T_t^{(EM)}$ and $T_t^{(MA)}$ exist [2, 3]. The field equations then read (valid on the FHSs using a GTCS, and where a comma means taking a partial derivative)

$$2\ddot{R}_{(t)\perp\perp} = 2(c^{-2}\ddot{a}_{F,i} + c^{-4}\ddot{a}_{F,i}\dddot{a}_F - \dddot{K}_{(t)ik}\dot{K}^{ik} + L_{\dot{n}_i}\dot{K}_t)$$

$$= \kappa^B(T_t^{(EM)} + \dot{T}_t^{(EM)}i) + \kappa^S(T_t^{(MA)} + \dot{T}_t^{(MA)}i),$$

$$c^{-2}\ddot{a}_{F,i} = \ddot{N}_t^i,$$  

(1)
\[ R_{(t)j\perp} + \left( \frac{\bar{h}_{ik}^{(t)}}{N_t} \frac{\partial}{\partial x^0} \bar{h}_{(t)ij} \right) |_k - \left( \frac{\bar{h}_{ik}^{(t)}}{N_t} \frac{\partial}{\partial x^0} \bar{h}_{(t)ik} \right) |_j = \bar{K}_{(t)ji}^{(t)\perp} - \bar{K}_{t,ij}^{(t)\perp} + \frac{\bar{h}_{(t)ij}^{(t)}}{N_t} \frac{\partial}{\partial x^0} \bar{h}_{(t)ij} \]

(4)

Here, \( \bar{R}_t \) is the Ricci tensor family corresponding to the metric family \( \bar{g}_t \), and the symbol \( \perp \) denotes a scalar product with \( -\bar{n}_t \), that is the negative unit normal vector field family of the FHSs. Moreover, \( \mathcal{L}_{\bar{n}_t} \) denotes a projected Lie derivative in the direction normal to the FHSs, \( \bar{K}_t \) denotes the extrinsic curvature tensor family (with trace \( \bar{K}_t \)) of the FHSs, a “hat” denotes an object projected into the FHSs and the symbol \( | \) denotes taking a spatial covariant derivative (compatible with \( \bar{h}_t \)). Finally, \( \kappa^B \equiv 8\pi G^B / c^4 \) and \( \kappa^S \equiv 8\pi G^S / c^4 \), where the values of \( G^B \) and \( G^S \) are by convention chosen as those measured in (hypothetical) local gravitational experiments in an empty universe at epoch \( t_0 \). Note that the left hand side of equation (3) is similar to its counterpart in GR. On the other hand, the left hand side of equation (4) contains extra terms compared to its counterpart in GR (\( \bar{h}_{(t)ij} \) are the components of the spatial metric family \( \bar{h}_t \) intrinsic to the FHSs). Said extra terms must be included for consistency reasons.

In addition to the directly coupled field equations (3) and (4), we also have a third field equation not involving any extra direct coupling to \( T_l \), i.e., [2, 3]

\[ \bar{C}_{(t)\perp} = \bar{H}_{(t)ij} + \frac{1}{\Xi_0} \bar{h}_{(t)ij} \]

(5)

where \( \bar{C}_t \) is the Weyl tensor family in \( (N, \bar{g}_t) \), and \( \bar{H}_t \) is the spatial Einstein tensor family calculated from \( \bar{h}_t \). (Note the last term on the right hand side of equation (5) yields a prior-geometric restriction on \( \bar{h}_t \) since it implies that the corresponding spatial Ricci scalar family \( \bar{P}_t = \frac{\bar{g}}{\Xi_0} \) is a constant.) Equation (5) may be written in the form [2, 3]

\[ \frac{1}{N_t} \mathcal{L}_{\bar{N}_t \bar{n}_t} \bar{K}_{(t)ij} + \bar{K}_t \bar{K}_{(t)ij} - \bar{H}_{(t)ij} = \frac{1}{3} \left[ \mathcal{L}_{\bar{n}_t} \bar{K}_t + \bar{K}_t^2 - 2 \bar{K}_{(t)ks} \bar{K}_{(t)}^{ks} + \frac{3}{(c t \bar{N}_t)^2} \right] \bar{h}_{(t)ij} \]

(6)

An explicit coordinate expression for \( \bar{K}_t \) may be calculated from equation (1). This expression reads (in a GTCS) [2, 3]

\[ \bar{K}_{(t)ij} = \frac{1}{2 N_t} \left[ \frac{t}{t_0} (\bar{N}_{(t)ij} + \bar{N}_{(t)ji}) - \frac{\partial}{\partial x^0} \bar{h}_{(t)ij} \right] \]

(7)

\[ \bar{K}_t = \frac{t_0}{t} \bar{N}_{(t)ij} + \frac{1}{2 N_t} \bar{h}_{(t)ij} \frac{\partial}{\partial x^0} \bar{h}_{(t)ij} \]

(8)

and equations (7) and (8) have well-known counterparts in GR.
3 Spherically symmetric exteriors in general

We now set up the most general form for \( \tilde{g}_t \) compatible with the spherically symmetric condition. Introducing a spherically symmetric GTCS \( \{ x^0, \rho, \theta, \phi \} \) where \( \rho \) is an isotropic radial coordinate, the spherically symmetric condition means that any shift vector field must point in the \( \pm \rho \)-direction and that all unknown quantities at most depend on \( t, x^0 \) and \( \rho \). Then equation (1) yields the family of line elements

\[
\overline{ds}^2 = [\bar{N}_o(t)](dx^0)^2 + 2\frac{t}{t_0} \bar{N}_o(t)\rho d\rho dx^0 + \frac{t^2}{t_0^2} \bar{N}_1^2 \left( \frac{\dot{A}d\rho^2}{1 - \frac{\rho^2}{\Xi}} + \rho^2 d\Omega^2 \right)
\]

\[
= \tilde{B} \left[ - \left( 1 - \frac{\bar{N}_o(t)}{1 - \frac{\rho^2}{\Xi}} \right) (dx^0)^2 + 2\frac{t}{t_0} \frac{\bar{N}_o(t)}{1 - \frac{\rho^2}{\Xi}} d\rho dx^0 + \frac{t^2}{t_0^2} \left( \frac{\dot{A}d\rho^2}{1 - \frac{\rho^2}{\Xi}} + \rho^2 d\Omega^2 \right) \right],
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( \bar{N}_o(t) = \bar{N}_0(t)(x^0, \rho, t) \), \( \tilde{A} = \tilde{A}(\rho, t) \) and \( \tilde{B} = \tilde{B}(x^0, \rho, t) \equiv \bar{N}_1^2 \). Note that the line element family (9) is by definition metrically static iff \( \tilde{A} = \tilde{A}(\rho), \tilde{B} = \tilde{B}(\rho) \) and \( \bar{N}_o(t) \equiv 0 \).

The nonvanishing components of the extrinsic curvature tensor \( \mathbf{K}_t \) become (from equations (7), (8) and (9))

\[
\bar{K}^\rho_{(t)\rho} = \frac{t_0}{t\sqrt{B}} \left[ \bar{N}_o(t, \rho) + \bar{N}_o(t) \left( \frac{\dot{A}_\rho}{2A} + \frac{\dot{B}_\rho}{2B} + \frac{\rho}{\Xi_0(1 - \frac{\rho^2}{\Xi})} \right) \right] - \frac{\dot{A}_0}{2\sqrt{BA}} - \frac{\dot{B}_0}{2B^{3/2}},
\]

\[
\bar{K}^\theta_{(t)\rho} = \bar{K}^\phi_{(t)\rho} = \frac{t_0}{t\sqrt{B}} \bar{N}_o(t) \left( \frac{\dot{B}_\rho}{2B} + \frac{1}{\rho} \right) - \frac{\dot{B}_0}{2B^{3/2}},
\]

\[
\bar{K}_t = \frac{t_0}{t\sqrt{B}} \left[ \bar{N}_o(t, \rho) + \bar{N}_o(t) \left( \frac{\dot{A}_\rho}{2A} + \frac{3\dot{B}_\rho}{2B} + \frac{2}{\rho} + \frac{\rho}{\Xi_0^2(1 - \frac{\rho^2}{\Xi})} \right) \right] - \frac{\dot{A}_0}{2\sqrt{BA}} - \frac{3\dot{B}_0}{2B^{3/2}}.
\]

Now the constraint equation (4) for spherically symmetric vacuum yields, after some straightforward calculations, that

\[
\bar{N}_o(t) \left[ \frac{\dot{B}_\rho}{B} - \frac{3}{2} \left( \frac{\dot{B}_\rho}{B} \right)^2 - \left( \frac{\dot{B}_\rho}{2B} + \frac{1}{\rho} \right) \left( \frac{\dot{A}_\rho}{A} + \frac{2\rho}{\Xi_0^2(1 - \frac{\rho^2}{\Xi})} \right) \right]
\]

\[
+ \frac{t}{t_0} \left[ \frac{\dot{B}_\rho}{B} - \frac{3\dot{B}_0 \dot{B}_\rho}{2B^2} - \left( \frac{\dot{B}_\rho}{2B} + \frac{1}{\rho} \right) \frac{\dot{A}_\rho}{A} \right] = 0.
\]

Equation (13) is a (nonlinear) partial differential equation involving three unknown functions \( \tilde{A}, \tilde{B} \) and \( \bar{N}_o(t) \). To fulfil Birkhoff’s theorem, no vacuum solution exterior to an isolated, spherically symmetric source should exist for this equation, besides the trivial metrically static solution \( \tilde{A} = \tilde{A}^{ms} = 1, \tilde{B} = \tilde{B}^{ms}(\rho), \bar{N}_o(t) = 0 \) (see below).
4 Birkhoff’s theorem

The first step in proving Birkhoff’s theorem in GR involves elimination of the nonzero offdiagonal components of the metric. This can be done by performing a simple coordinate transformation to a new time coordinate (see, e.g., [4]). Similarly, a new time coordinate \( x_0' \) defined from the differentials

\[
\begin{align*}
    dx_0' &= \eta(\rho, x_0, t) \left[ 1 - \frac{\tilde{N}_0^\rho \tilde{N}_0^\rho \tilde{A}}{1 - \frac{\rho^2}{\Xi_0^2}} dx_0 - \frac{t}{t_0} \frac{\tilde{N}_0^\rho \tilde{A}}{1 - \frac{\rho^2}{\Xi_0^2}} d\rho \right],
\end{align*}
\]

where \( \eta(\rho, x_0, t) \) is an integrating factor satisfying the condition

\[
\frac{\partial}{\partial \rho} \left[ \eta \left(1 - \frac{\tilde{N}_0^\rho \tilde{N}_0^\rho \tilde{A}}{1 - \frac{\rho^2}{\Xi_0^2}} \right) \right] = -\frac{\partial}{\partial x_0'} \left[ \eta \frac{t}{t_0} \frac{\tilde{N}_0^\rho \tilde{A}}{1 - \frac{\rho^2}{\Xi_0^2}} \right].
\]

eliminates the nonzero offdiagonal elements in equation (9). However, the coordinate system \( \{x_0', \rho, \theta, \phi\} \) is not a GTCS since we in general will have \( x_0' \neq ct \) on the FHSs, i.e., the hypersurfaces \( x_0' = \text{constant} \) cannot be identified with the FHSs. This would be inconvenient in the further analysis since quasi-metric spacetime directly involves its foliation into the FHSs and not any other hypersurfaces. That is, since the basic formulae listed in section 2 will not in general be valid for a metric family foliated by hypersurfaces other than the FHSs, said elimination would not be useful (but see [2, 3] for the possibility of having alternative foliations of quasi-metric space-time as a weak-field approximation for isolated systems in the limiting case \( \Xi_0 \to \infty \)). For this reason we will rather use equation (13) to prove that \( \tilde{N}_0^\rho \) must necessarily vanish for the gravitational field in vacuum outside an isolated spherically symmetric source.

To do that, we notice that equation (2) yields the condition

\[
\frac{\partial}{\partial t} \left( \tilde{N}_0^\rho \tilde{N}_0^\rho \tilde{A} \right) = 0.
\]

But the \( t \)-dependence obtained from equation (13) is consistent with equation (16) only if \( \tilde{A} \) is of the form \( \tilde{A} = \frac{\tilde{t}}{\rho^2} \tilde{a} \), where \( \tilde{a} \) does not depend on \( t \). But this form of \( \tilde{A} \) is inconsistent with the general form (1) of the metric family \( \tilde{g}_t \). Thus we cannot have \( \tilde{N}_0^\rho \neq 0 \), provided that the expressions in the square brackets of equation (13) do not vanish. On the other hand, if these expressions do vanish, i.e., if

\[
\begin{align*}
    \frac{\tilde{B}_{\rho \rho}}{B} - 3 \left( \frac{\tilde{B}_\rho}{B} \right)^2 - \left( \frac{\tilde{B}_\rho}{2B} + \frac{1}{\rho} \right) \left( \frac{\tilde{A}_\rho}{\tilde{A}} + \frac{2\rho}{\Xi_0^2(1 - \frac{\rho^2}{\Xi_0^2})} \right) &= 0, \\
    \frac{\tilde{B}_{00}}{B} - \frac{3}{2} \frac{\tilde{B}_{00} \tilde{B}_\rho}{B^2} - \left( \frac{\tilde{B}_\rho}{2B} + \frac{1}{\rho} \right) \frac{\tilde{A}_\rho}{\tilde{A}} &= 0,
\end{align*}
\]

eliminates the nonzero offdiagonal elements in equation (9). However, the coordinate system \( \{x_0', \rho, \theta, \phi\} \) is not a GTCS since we in general will have \( x_0' \neq ct \) on the FHSs, i.e., the hypersurfaces \( x_0' = \text{constant} \) cannot be identified with the FHSs. This would be inconvenient in the further analysis since quasi-metric spacetime directly involves its foliation into the FHSs and not any other hypersurfaces. That is, since the basic formulae listed in section 2 will not in general be valid for a metric family foliated by hypersurfaces other than the FHSs, said elimination would not be useful (but see [2, 3] for the possibility of having alternative foliations of quasi-metric space-time as a weak-field approximation for isolated systems in the limiting case \( \Xi_0 \to \infty \)). For this reason we will rather use equation (13) to prove that \( \tilde{N}_0^\rho \) must necessarily vanish for the gravitational field in vacuum outside an isolated spherically symmetric source.

To do that, we notice that equation (2) yields the condition

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But the \( t \)-dependence obtained from equation (13) is consistent with equation (16) only if \( \tilde{A} \) is of the form \( \tilde{A} = \frac{\tilde{t}}{\rho^2} \tilde{a} \), where \( \tilde{a} \) does not depend on \( t \). But this form of \( \tilde{A} \) is inconsistent with the general form (1) of the metric family \( \tilde{g}_t \). Thus we cannot have \( \tilde{N}_0^\rho \neq 0 \), provided that the expressions in the square brackets of equation (13) do not vanish. On the other hand, if these expressions do vanish, i.e., if

\[
\begin{align*}
    \frac{\tilde{B}_{\rho \rho}}{B} - 3 \left( \frac{\tilde{B}_\rho}{B} \right)^2 - \left( \frac{\tilde{B}_\rho}{2B} + \frac{1}{\rho} \right) \left( \frac{\tilde{A}_\rho}{\tilde{A}} + \frac{2\rho}{\Xi_0^2(1 - \frac{\rho^2}{\Xi_0^2})} \right) &= 0, \\
    \frac{\tilde{B}_{00}}{B} - \frac{3}{2} \frac{\tilde{B}_{00} \tilde{B}_\rho}{B^2} - \left( \frac{\tilde{B}_\rho}{2B} + \frac{1}{\rho} \right) \frac{\tilde{A}_\rho}{\tilde{A}} &= 0,
\end{align*}
\]
one might still have that \( \hat{N}_0^0 \neq 0 \), however. But this possibility only works if a limiting solution of equation (17) in the case of no time dependence is the metrically static vacuum solution \( \tilde{A}^{\text{ms}} = 1 \), \( \tilde{B}^{\text{ms}}(\rho) \) found in [5] (see equation (27) below). Since said metrically static vacuum solution is not a solution of equation (17) in the metrically static limit, the necessary correspondence does not exist and we must necessarily have \( \hat{N}_0^0 = 0 \).

To complete the proof of Birkhoff’s theorem for the QMF, it remains to show that the metrically static solution found in [5] (see equation (27) below), is the only possible vacuum solution exterior to a spherically symmetric body. To achieve this, we will use equations (3) and (5) with \( \hat{N}_0^0 = 0 \).

Now we notice that since the Weyl tensor is conformally invariant, we have that \( \tilde{C}_\alpha^\beta \gamma^\delta = \tilde{C}_\alpha^\gamma^\beta^\delta \) and thus \( \tilde{C}_{(t)i}\tilde{i}\tilde{j} = \tilde{C}_{(t)\hat{i}\hat{j}} \), where \( \tilde{\mathbf{C}}_t \) is the Weyl tensor family calculated from the metric family \( \tilde{\mathbf{g}}_t \equiv \hat{N}_t^{-2} \tilde{\mathbf{g}}_t \). The counterpart to equation (6) obtained from equation (5) with \( \tilde{C}_{(t)\hat{i}\hat{j}} \) substituted for \( \tilde{C}_{(t)i}\tilde{i}\tilde{j} \) then reads

\[
\mathcal{L}_{\tilde{n}_i} \tilde{K}_{(t)ij} + \tilde{K}_t \tilde{K}_{(t)ij} - \tilde{H}_{(t)ij} = \frac{1}{3} \left[ \mathcal{L}_{\tilde{n}_i} \tilde{K}_t + \tilde{K}_t^2 - 2 \tilde{K}_{(t)ks} \tilde{K}_{(t)ks}^s + \frac{3}{c^4t^2} \frac{t^2}{\rho^2} \right] \tilde{h}_{(t)ij},
\]

where \( \tilde{\mathbf{K}}_t \) is the extrinsic curvature tensor family of the FHSs in the metric family \( \tilde{\mathbf{g}}_t \). Now the \( \rho \rho \)-component of equation (19) yields (with \( \hat{N}_0^0 = 0 \))

\[
\frac{t^2}{\rho^2} \left[ \frac{1}{3} \frac{\tilde{A}_{00}}{A} - \frac{1}{6} \left( \frac{\tilde{A}_0}{A} \right)^2 \right] \rho^2 = \tilde{H}_{(t)\rho\rho} + \frac{1}{\xi_0} \tilde{h}_{(t)\rho\rho} = \frac{1}{\xi_0} \left( 1 - \tilde{A} \right),
\]

whereas twice the \( \theta \theta \)-component (or equivalently, twice the \( \phi \phi \)-component) yields

\[
\frac{t^2}{\rho^2} \left[ \frac{1}{3} \frac{\tilde{A}_{00}}{A} - \frac{1}{6} \left( \frac{\tilde{A}_0}{A} \right)^2 \right] \rho^2 = \tilde{H}_{(t)\theta\theta} + \frac{2}{\xi_0} \tilde{h}_{(t)\theta\theta} = -\rho \frac{\tilde{A}_0}{(A)^2} (1 - \frac{\rho^2}{\xi_0}) + 2 \left( 1 - \frac{1}{A} \right) \rho^2.
\]

Moreover, combining equations (20) and (21) yields an equation which can be integrated to obtain an expression for \( \tilde{A} \), i.e.,

\[
\frac{\tilde{A}_\rho}{A(1 - \tilde{A})} = \frac{1}{\rho} \left( 1 - \frac{3\tilde{\xi}^2}{\xi_0} \right), \quad \Rightarrow \quad \tilde{A} = \left( 1 + \tilde{\xi} \frac{(1 - \rho^2)}{\xi_0} \right)^{-1},
\]

where the function \( \tilde{\xi} = \tilde{\xi}(x^0, t) \) does not depend on \( \rho \). Substituting the expression (22) for \( \tilde{A} \) back into equation (21) then yields an ordinary differential equation for \( \tilde{\xi} \), i.e.,

\[
\tilde{\xi}_{00} - \frac{3}{2} \frac{\tilde{\xi}_{00}}{\rho(1 + \frac{\tilde{\xi}}{\rho} - \frac{\rho^2}{\xi_0})} = 3 \frac{t^2}{\rho^2} \tilde{\xi} (1 + \tilde{\xi} - \frac{\rho^2}{\xi_0}).
\]

However, it is straightforward to see that this equation does not have any other solutions than the trivial solution \( \tilde{\xi} = 0 \). This implies that that we must have that \( \tilde{A} = \tilde{A}^{\text{ms}} = 1 \).
Next, with $\tilde{A} = 1$ equation (18) yields
\begin{equation}
\frac{\dot{B}_\rho}{B} - \frac{3}{2} \frac{\dot{B}_\rho B^\rho}{B^2} = 0, \quad \Rightarrow \quad \dot{B}(\rho, x^0, t) = \frac{C(t)}{[f(\rho, t) + g(x^0, t)]^2},
\end{equation}
where the solution is found from MAPLE. Besides, equation (3) yields (with $\tilde{A} = 1$ and $\bar{N}_0 = 0$)
\begin{equation}
(1 - \frac{\rho^2}{\Xi_0^2}) \frac{\ddot{B}_{\rho\rho}}{B} + (2 - 3 \frac{\rho^2}{\Xi_0^2}) \frac{\ddot{B}_\rho}{B} + 3 \frac{\rho^2}{t_0^2} \left[ \left( \frac{\dot{B}_0}{B} \right)^2 - \frac{\ddot{B}_{00}}{B} \right] = 0.
\end{equation}
The general solution of equation (25) is on the form (from MAPLE)
\begin{equation}
\dot{B}(\rho, x^0, t) = F(\rho, t)G(x^0, t).
\end{equation}
However, this form is inconsistent with the form of $\dot{B}$ found in equation (24) and the metrically static solution (27) below unless $\dot{B} = \dot{B}(\rho)$. That is, no changes in the interior gravitational field of a spherically symmetric source can propagate to the exterior vacuum. Nor can $\dot{B}$ depend on $t$ since there is no such dependence for a metrically static source. Thus the unique solution for the spherically symmetric vacuum exterior to a spherically symmetric body with coordinate radius $\rho_s$ can be found from equation (25) with $\dot{B} = \dot{B}(\rho)$, yielding the metrically static solution [5]
\begin{equation}
\dot{B}_{\text{ms}}(\rho) = 1 - \frac{r_{s0}}{\rho} \sqrt{1 - \frac{\rho^2}{\Xi_0^2}}, \quad \frac{r_{s0}}{\sqrt{1 + \frac{r_{s0}^2}{\Xi_0^2}}} < \rho < \Xi_0.
\end{equation}
(Note that it is not meaningful to extend the solution (27) to beyond $\rho = \Xi_0$ since the transformation $\bar{g}_i \to g_i$ becomes singular for $\rho = \Xi_0$ [5].) Here, $r_{s0} = \frac{2M_{t_0}^{(\text{MA})}G}{c^2} + \frac{2M_{t_0}^{(\text{EM})}G}{c^2}$ is the quasi-metric counterpart to the Schwarzschild radius at epoch $t_0$ and
\begin{align}
M_t^{(\text{MA})} &= c^{-2} \int \int \int \sqrt{\bar{B}} \left[ T_{(t)\bot\bot}^{(\text{MA})} + T_{(t)i}^{(\text{MA})i} \right] d\bar{V}_t, \\
M_t^{(\text{EM})} &= c^{-2} \int \int \int \sqrt{\bar{B}} \left[ T_{(t)\bot\bot}^{(\text{EM})} + T_{(t)i}^{(\text{EM})i} \right] d\bar{V}_t,
\end{align}
are Komar masses corresponding to a metrically static source’s content of material particles and electromagnetic fields, respectively [5]. If the source is not metrically static the solution (27) is still valid, but not equation (28) for the active masses. Nevertheless, $r_{s0}$ represents the active mass of the source at epoch $t_0$ as measured by distant orbiters. For some later epoch $t_1 > t_0$ the active mass measured will be represented by $r_{s1} = \frac{t_1}{t_0}r_{s0}$, i.e., active mass increases linearly with epoch independent of whether the source is metrically
static or not. Besides, performing a scaling of the radial coordinate $\rho\rightarrow\rho' = \frac{t_1}{t_0}\rho$, the form of equation (9) will be preserved with $\tilde{N}_{(t)}\equiv 0$ and $\Xi_0\rightarrow\Xi_1 = \frac{t_1}{t_0}\Xi_0$. This means that the secular increase of active mass does not depend on any form of communication between source and external gravitational field. Rather, the secular increase of active mass is just another facet of the global cosmic expansion as described within the QMF, i.e., a systematically changing relationship between dimensionful units as defined operationally from gravitational and atomic systems, respectively. Thus there is no conflict between the secular increase of active mass and the validity of Birkhoff’s theorem for quasi-metric gravity.

Furthermore, just as for GR [1], in quasi-metric gravity Birkhoff’s theorem holds for spherically symmetric electrovacuum outside an isolated charged source. This follows from the fact that $T^{(EM)}_{(t),\perp} = 0$ for this case (no radiation), so that all equations used showing the validity of the results $\tilde{N}_0\equiv 0$ and $\tilde{A} = 1$ still hold. The only difference from the vacuum case is that equation (3) now has a source term, so that equation (25) gets a term $\frac{r_0^2}{\rho}\frac{Q}{c^2}$ on the right hand side. That is, equation (25) changes to

$$
\left(1 - \frac{\rho^2}{\Xi_0^2}\right)\frac{\rho}{B} + \left(2 - 3\frac{\rho^2}{\Xi_0^2}\right)\frac{\rho}{\rho B} + 3\frac{t_0^2}{t_0^2}\left[\left(\frac{\rho}{B}\right)^2 - \frac{\rho}{\rho B}\frac{\rho}{B}\right] = \frac{r_0^2}{\rho}\frac{Q}{c^2}, \quad r_0\equiv\sqrt{2G\rho B}|Q|, \quad (29)
$$

where $Q$ is the (passive) charge of the source [6]. But the solution of equation (29) is still of the general form shown in equation (26) and inconsistent with the solution form found in equation (24) if there is any dependence on $x^0$. This again means that $\tilde{B} = \tilde{B}(\rho)$ and it must be equal to the metrically static solution found in [6]. Note that in addition to the secular increase of active mass, the solution of equation (29) also implies a secular (linear) increase of active charge [6] contributing to $T^{(EM)}_t$.

### 5 Conclusion

In this paper it has been shown that Birkhoff’s theorem is valid for quasi-metric gravity. It also holds for electrovacuum exterior to a charged, spherically symmetric, isolated source. There is no conflict between this result and the prediction that active mass as measured by distant test orbiters increases secularly with epoch [5] (see also active charge [6]); similar to the quasi-metric cosmic expansion, said prediction is a global phenomenon not depending on any form of communication between the source and the external field.

Moreover, in quasi-metric gravity spherically symmetric exterior fields are not only isometric to the metrically static cases; in addition Birkhoff’s theorem says that for said exterior fields, the FOs move exactly as for the metrically static cases.
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