Gaussian initializations help deep variational quantum circuits escape from the barren plateau

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Abstract

Variational quantum circuits have been widely employed in quantum simulation and quantum machine learning in recent years. However, quantum circuits with random structures have poor trainability due to the exponentially vanishing gradient with respect to the circuit depth and the qubit number. This result leads to a general belief that deep quantum circuits will not be feasible for practical tasks. In this work, we propose an initialization strategy with theoretical guarantees for the vanishing gradient problem in general deep circuits. Specifically, we prove that under proper Gaussian initialized parameters, the norm of the gradient decays at most polynomially when the qubit number and the circuit depth increase. Our theoretical results hold for both the local and the global observable cases, where the latter was believed to have vanishing gradients even for shallow circuits. Experimental results verify our theoretical findings in the quantum simulation and quantum chemistry.

1 Introduction

Quantum computing has attracted great attention in recent years, especially since the realization of quantum supremacy [1, 2] with noisy intermediate-scale quantum (NISQ) devices [3]. Due to mild requirements on the gate noise and the circuit connectivity, variational quantum algorithms (VQAs) [4] become one of the most promising frameworks for achieving practical quantum advantages on NISQ devices. Specifically, different VQAs have been proposed for many topics, e.g., quantum chemistry [5, 6, 7, 8, 9, 10, 11, 12, 13], quantum simulations [14, 15, 16, 17, 18, 19, 20, 21, 22, 23], machine learning [24, 25, 26, 27, 28, 29, 30, 31], numerical analysis [32, 33, 34, 35, 36], and linear algebra problems [37, 38, 39]. Recently, various small-scale VQAs have been implemented on real quantum computers for tasks such as finding the ground state of molecules [8, 11, 12] and exploring promising applications in supervised learning [25], generative learning [30] and reinforcement learning [29].

Typical variational quantum algorithms could be described as a trainable quantum-classical hybrid framework based on parameterized quantum circuits (PQCs) [40]. Similar to classical counterparts such as neural networks [41], first-order methods including the gradient descent [42] and its variants [43] are widely employed in optimizing the loss function of VQAs. However, VQAs may face the trainability barrier when scaling up the size of quantum circuits (i.e., the number of involved qubits or the circuit depth), which is known as the barren plateaus problem [44].

Roughly speaking, the barren plateau describes the phenomenon that the value of the loss function and its gradients concentrate around their expectation values with exponentially small variances.
We remark that gradient-based methods could hardly handle trainings with the barren plateaus phenomenon. Both the machine noise of the quantum channel and the statistical noise induced by measurements could severely degrade the estimation of gradients. Moreover, the optimization of the loss with a flat surface takes much more time using inaccurate gradients than ideal cases. Thus, solving the barren plateaus problem is imperative for achieving practical quantum advantages with VQAs.

1.1 Related work

The barren plateaus phenomenon was first noticed in Ref. [44], which proves that if the circuit distribution forms unitary 2-designs [45], the variance of the gradient of the circuit vanishes to zero with the rate exponential in the qubit number. Subsequently, several positive results are proved for shallow quantum circuits such as the alternating-layered circuit [46, 47] and the quantum convolutional neural network [48] when the observable is constrained in small number of qubits (local observable). For shallow circuits with $N$ qubits and $O(\log N)$ depth, the variance of the gradient has the order poly($N$)$^{-1}$ if gate blocks in the circuit are sampled from local 2-design distributions. Later, several works prove an inherent relationship between the barren plateaus phenomenon and the complexity of states generated from the circuit. Specifically, circuit states that satisfy the volume law could lead to the barren plateaus problem [49]. Expressive quantum circuits, which is measured by the distance between the Haar distribution and the distribution of circuit states, could have vanishing gradients [50]. Since random circuits form approximately 2-designs when they achieve linear depths [45], deep quantum circuits were believed to suffer the barren plateaus problem generally.

The parameterization of quantum circuits is achieved by tuning the time of Hamiltonian simulations, so the gradient of the circuit satisfies the parameter-shift rule [51]. Thus, the variance of the loss in VQAs and that of its gradient have similar behaviors for uniform distributions [44, 52]. One corollary of the parameter-shift rule is that the gradient of depolarized noisy quantum circuits vanishes exponentially with increasing circuit depth [53], since the loss itself vanishes in the same rate. Another corollary is that both gradient-free [54] and higher-order methods [55] could not solve the barren plateaus problem.

Although most existing theoretical and practical results imply the barren plateaus phenomenon in deep circuits, VQAs with deep circuits do have impressive advantages from other aspects. For example, the loss of VQAs is highly non-convex, which is hard to find the global minima [56] for both shallow and deep circuits. Meanwhile, for VQAs with shallow circuits, local minimas and the global minima have considerable gaps [57], which could severely influence the training performance of gradient-based methods. Contrary to shallow cases, deep VQAs have vanishing gaps between local minimas and the global minima [58]. In practice, experiments show that overparameterized VQAs [59] can be optimized towards the global minima. Moreover, VQAs with deep circuits have more expressive power than that of shallow circuits [60, 61, 62], which implies the potential to handle more complex tasks in quantum machine learning and related fields.

Inspired by various advantages of deep VQAs, some approaches have been proposed recently for solving the related barren plateaus problem in practice. For example, the block-identity strategy [63] initializes gate blocks in pairs and sets parameters inversely, such that the initial circuit is equivalent to the identity circuit with zero depth. Since shallow circuits have no vanishing gradient problem, the corresponding VQA is trainable with guarantees at the first step. However, we remark that the block-identity condition would not hold after the first step, and the structure of the circuit needs to be designed properly. The layerwise training method [64] trains parameters in the circuit layers by layers, such that the depth of trainable part is limited. However, this method implements circuits with larger depth than that of the origin circuit, and parameters in the first few layers are
not optimized. A recent work provides theoretical guarantees on the trainability of deep circuits with certain structures [65]. However, the proposed theory only suits VQAs with local observables, but many practical applications such as finding the ground state of molecules and the quantum compiling [66, 67] apply global observables.

1.2 Contributions

We summarize the contribution of this work in three folds.

- We propose a Gaussian initialization strategy for general deep variational quantum algorithms. By setting the variance $\gamma^2 = O(\frac{1}{L})$ for $L$-depth quantum circuits, we theoretically guarantee the trainability of corresponding VQAs. For $N$-qubit quantum circuits with independent parameters and local observables, the expectation of the gradient norm is lower bounded by $\text{poly}(N, L)^{-1}$ as provided in Theorem 4.1, which outperforms previous $2^{-O(L)}$ results.

- We extend the gradient norm result to the global observable case in Theorem 4.2, which was believed to have the barren plateaus problem even for very shallow circuits. Moreover, our bound holds for correlated parameterized gates, which are widely employed in practical tasks like quantum chemistry and quantum simulations.

- We provide further analysis on the number of necessary measurements for estimating the gradient in Corollary 4.3, which scales as $O(L^2 \epsilon)$. We verify the proposed theorems in numerical experiments, i.e., finding the ground energy and the ground state of the Heisenberg model and the LiH molecule, respectively.

The rest part of this work is organized as follows. We address necessary notations and quantum computing basics in Section 2. We introduce the general framework of variational quantum algorithms in Section 3. Main theoretical results of this work are presented in Section 4. We implement numerical experiments for verifying theorems in Section 5. We make conclusions in Section 6.

2 Notations and quantum computing basics

We denote by $[N]$ the set $\{1, \cdots, N\}$. The form $\| \cdot \|_1$ represents the $\ell_1$ norm of the vector. The form $\| \cdot \|_2$ represents the $\ell_2$ norm for the vector and the spectral norm for the matrix, respectively. We denote by $a_j$ the $j$-th component of the vector $a$. The tensor product operation is denoted as “$\otimes$”. The conjugate transpose of a matrix $A$ is denoted as $A^\dagger$. The trace of a matrix $A$ is denoted as $\text{Tr}[A]$. We denote $\nabla_\theta f$ as the gradient of the function $f$ with respect to the variable $\theta$. We employ notations $O$ to describe complexity notions.

Now we introduce quantum computing knowledge and notations. The pure state of a qubit could be written as $|\phi\rangle = a|0\rangle + b|1\rangle$, where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$, and $\{|0\rangle = (1, 0)^T, |1\rangle = (0, 1)^T\}$. The $n$-qubit space is formed by the tensor product of $n$ single-qubit spaces. For the vector $x \in \mathbb{R}^{2^n}$, the amplitude encoded state $|x\rangle$ is defined as $\frac{1}{\|x\|} \sum_{j=1}^{2^n} x_j |j\rangle$. For pure states, the corresponding density matrix is defined as $\rho = |x\rangle \langle x|$, in which $|x\rangle = (|x\rangle)^\dagger$. We use the density matrix to represent general mixed quantum states, i.e.,

$$\rho = \sum_k c_k |x_k\rangle \langle x_k|,$$

where $c_k \in \mathbb{R}$ and $\sum_k c_k = 1$.

A single-qubit operation to the state behaves like the matrix-vector multiplication and can be referred to as the gate $\text{---}$ in the quantum circuit language. Specifically, single-qubit operations are often
used as $R_X(\theta) = e^{-i\theta X}$, $R_Y(\theta) = e^{-i\theta Y}$, and $R_Z(\theta) = e^{-i\theta Z}$, where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Pauli matrices will be referred to as

$$\{I, X, Y, Z\} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$$

for the convenience. Moreover, two-qubit operations, including the CNOT gate, the CZ gate, and the $\sqrt{i}$SWAP gate, are employed for generating quantum entanglement:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sqrt{i}\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & i/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We could obtain information from the quantum system by performing measurements, for example, measuring the state $|\phi\rangle = a|0\rangle + b|1\rangle$ generates 0 and 1 with probability $p(0) = |a|^2$ and $p(1) = |b|^2$, respectively. Such a measurement operation could be mathematically referred to as calculating the average of the observable $O = \sigma_3$ under the state $|\phi\rangle$:

$$\langle \phi | O | \phi \rangle \equiv \text{Tr}[\sigma_3 |\phi\rangle \langle \phi|] = |a|^2 - |b|^2 = p(0) - p(1).$$

Mathematically, quantum observables are Hermitian matrices. Specifically, the average of a unitary observable under arbitrary states is bounded by $[-1, 1]$. We remark that $O(\frac{1}{\epsilon^2})$ times of measurements could provide an $\epsilon\|O\|_2$-error estimation to the value $\text{Tr}[O\rho]$.

## 3 Framework of general VQAs

In this section, we introduce the framework of general VQAs and corresponding notations. A typical variational quantum algorithm can be viewed as the optimization of the function $f$, which is defined as the expectation of observables. The expectation varies for different initial states and different parameters $\theta$ used in quantum circuits. Throughout this paper, we define

$$f(\theta; \rho_{in}) = \text{Tr} \left[ O V(\theta) \rho_{in} V(\theta)^\dagger \right]$$

as the loss function of VQAs, where $V(\theta)$ denotes the parameterized quantum circuit, the hermitian matrix $O$ denotes the observable, and $\rho_{in}$ denotes the density matrix of the input state.

Next, we explain observables, input states, and parameterized quantum circuits in detail in Sections 3.1 and 3.2, which are crucial components of the variational quantum algorithms.

### 3.1 Observables and input states

Both the observable and the density matrix could be decomposed under the Pauli basis with real coefficients. We define the locality of a quantum observable as the maximum number of non-identity Pauli matrices in the tensor product, such that the corresponding coefficient is not zero. Thus, the observable with the constant locality is said to be local, and the observable that acts on all qubits is said to be global.

The observable and the input state in VQAs could have various formulations for specific tasks. For the quantum simulation or the quantum chemistry scenario, observables are constrained to be the system
Hamiltonians, while input states are usually prepared as computational basis states. For example, \((|0\rangle \langle 0|)^\otimes N\) is used frequently in quantum simulations [17, 18]. Hartree–Fock (HF) states [8, 10], which are prepared by the tensor product of \(\{ |0\rangle, |1\rangle \}\), serve as good initial states in quantum chemistry tasks [10, 11, 12, 13]. For quantum machine learning (QML) tasks, initial states encode the information of the training data, which could have a complex form. Many encoding strategies have been introduced in the literature [24, 68, 69]. In contrary with the complex initial states, observables employed in QML are quite simple. For example, \(\pi_0 = |0\rangle \langle 0|\) serves as the observable in most QML tasks related with the classification [24, 25, 26] or the dimensional reduction [70].

3.2 Parameterized quantum circuits

Apart from the input states and the observable choices, parameterized quantum circuits employed in different variational quantum algorithms have various structures, which are also known as ansätze [71, 72, 73]. Specifically, the ansatz in the VQA denotes the initial guess on the circuit structure. For example, alternating-layered ansätze [71, 74] are proposed for approximating the Hamiltonian evolution. Recently, hardware efficient ansätze [7, 75] and tensor-network based ansätze [76, 77], which could utilize parameters efficiently on noisy quantum computers, have been developed for various tasks, including quantum simulations and quantum machine learning. For quantum chemistry tasks, unitary coupled cluster ansätze [78, 79] are preferred since they preserve the number of electrons corresponding to circuit states.

In practice, ansatz is deployed as the sequence of single-qubit rotations \(\{ e^{-i\theta_k \sigma_k}, k \in \{1, 2, 3\} \}\) and two-qubit gates \(\{ \text{CNOT}, \text{CZ} \}\). We remark that the gradient of the VQA satisfies the parameter-shift rule [51, 80, 81]; namely, for independently deployed parameters \(\theta_j\), the corresponding partial derivative is

\[
\frac{\partial f}{\partial \theta_j} = f(\theta_+; \rho_{in}) - f(\theta_-; \rho_{in}),
\]

where \(\theta_+\) and \(\theta_-\) are different from \(\theta\) only at the \(j\)th parameter: \(\theta_j \rightarrow \theta_j \pm \frac{\pi}{4}\), and \(f(\theta_+; \rho_{in})\) and \(f(\theta_-; \rho_{in})\) are defined in Eq. (1). Thus, the gradient of \(f\) could be estimated efficiently, which allows optimizing VQAs through gradient-based methods [82, 83, 84].

4 Theoretical results about Gaussian initialized VQAs

In this section, we provide our theoretical guarantees on the trainability of deep quantum circuits through proper designs for the initial parameter distribution. In short, we prove that the gradient of the \(L\)-layer \(N\)-qubit circuit is upper bounded by \(1/\text{poly}(L, N)\), if initial parameters are sampled from a Gaussian distribution with \(O(1/L)\) variance. Our bounds significantly improve existing results of the gradients of VQAs, which have the order \(2^{-O(L)}\) for shallow circuits and the order \(2^{-O(N)}\) for deep circuits. We prove different results for the local and global observable cases in Section 4.1 and Section 4.2, respectively.

4.1 Independent parameters with local observables

First, we introduce the Gaussian initialization of parameters for the local observable case. We use the quantum circuit illustrated in Figure 1 as the ansatz in this section. The circuit in Figure 1 performs \(L\) layers of single qubit rotations and CZ gates on the input state \(\rho_{in}\), followed by a \(R_X\) layer and a \(R_Y\) layer. We denote the single-qubit gate on the \(n\)-th qubit of the \(\ell\)-th layer as \(e^{-i\theta_{\ell,n} G_{\ell,n}}, \forall \ell \in \{1, \cdots, L + 2\}\) and \(n \in \{1, \cdots, N\}\), where \(\theta_{\ell,n}\) is the corresponding parameter and \(G_{\ell,n}\) is a Hermitian unitary. To eliminate degenerate parameters, we require that single-qubit gates in the first
Then, and the cost function $f$ where $S$ is the number of non-zero elements in $i$, and the index $j = (j_1, j_2, \cdots, j_N)$ such that $j_m = 0, \forall i_m = 0$ and $j_m = 3, \forall i_m \neq 0$. The expectation is taken with the Gaussian distribution $\mathcal{N}(0, \frac{1}{3N(L+2)})$ for the parameters $\theta$.

Compared to existing works [44, 46, 47, 48, 65], Theorem 4.1 provides a larger lower bound of the gradient norm, which improves the complexity exponentially with the depth of trainable circuits. Different from unitary 2-design distributions [44, 46, 47, 48] or the uniform distribution in the parameter space [52, 86, 65] that were employed in existing works, we analyze the expectation of the gradient norm under a depth-induced Gaussian distribution. This change follows a natural idea that the trainability is not required in the whole parameter space or the entire circuit space, but only on the optimization trajectory. Moreover, large norm of gradients could only guarantee the trainability in the beginning stage, instead of the whole optimization, since a large gradient for trained parameters corresponds to non-convergence.

Thus, the barren plateaus problem could be crucial if initial parameters have vanishing gradients, which has been proved for deep VQAs with uniform initializations. In contrary, we could solve the barren plateaus problem if parameters are initialized properly with large gradients, as provided in Theorem 4.1.

### 4.2 Correlated parameters with global observables

Next, we extend the Gaussian initialization framework to general quantum circuits with correlated parameters and global observables. Quantum circuits with correlated parameters have wide applications...
in quantum simulations and quantum chemistry \[10, 11, 12, 13\]. One example is the Givens rotation

\[
R_{\text{Givens}}(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which preserves the number of electrons in parameterized quantum states \[11\]. To analyze VQAs with correlated parameterized gates, we consider the ansatz

\[
V(\theta) = \prod_{j=L}^{1} V_j(\theta_j),
\]

which consists of parameterized gates \(\{V_j(\theta_j)\}_{i=1}^{L}\). Denote by \(h_j\) the number of unitary gates that share the same parameter \(\theta_j\). Thus, the parameterized gate \(V_j(\theta_j)\) consists of a list of fixed and parameterized unitary operations

\[
V_j(\theta_j) = \prod_{k=1}^{h_j} W_{jk} e^{-i \frac{\theta_j}{a_j} G_{jk}}
\]

with the term \(a_j \in \mathbb{R} - \{0\}\), where the Hamiltonian \(G_{jk}\) and the fixed gate \(W_{jk}\) are unitary \(\forall k \in [h_j]\).

Moreover, we consider the objective function

\[
f(\theta) = \text{Tr} \left[ O \prod_{j=L}^{1} V_j(\theta_j) \rho_{\text{in}} \prod_{j=1}^{L} V_j(\theta_j)^\dagger \right],
\]

where \(\rho_{\text{in}}\) and \(O\) denote the input state and the observable, respectively. In practical tasks of quantum chemistry, the molecule Hamiltonian \(H\) serves as the observable \(O\). Minimizing the function \(f(\theta)\) provides the ground energy and the corresponding ground state of the molecule. We provide the bound of the gradient norm of the Gaussian initialized variational quantum circuit in Theorem 4.2 with the proof in Appendix C. Similar to the local observable case, we could bound the norm of the gradient of Eq. (6) if parameters are initialized with \(\mathcal{O}(\frac{1}{L})\) variance.

**Theorem 4.2.** Consider the \(N\)-qubit variational quantum algorithms with the objective function (6). Then the following formula holds for any \(\ell \in \{1, \cdots, L\}\),

\[
\mathbb{E}\left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \geq (1 - \epsilon) \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \bigg|_{\theta=0},
\]

where \(\mathbf{0} \in \mathbb{R}^L\) denotes the zero vector. The expectation is taken with Gaussian distributions \(\mathcal{N}(0, \gamma_j^2)\) for parameters in \(\theta = \{\theta_j\}_{j=1}^{L}\), where the variance \(\gamma_j^2 \leq \frac{a_j^2 \epsilon}{16h_j^2(3h_j(h_j-1)+1)L\|O\|_2^2} \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \bigg|_{\theta=0}, \forall j \in [L].\)

We remark that Theorem 4.2 not only provides an initialization strategy, but also guarantees the update direction during the training. Different from the classical neural network, where the gradient could be calculated accurately, the gradient of VQAs, obtained by the parameter-shift rule (2), is perturbed by the measurement noise. A guide on the size of acceptable measurement noise could be useful for the complexity analysis of VQAs. Specifically, suppose the parameter \(\theta = \theta^{(t)}\) at the \(t\)-th iteration. We could derive the lower bound on the gradient norm around the parameter \(\theta^{(t)}\), as provided in Corollary 4.3. Thus, we need roughly \(\frac{1}{\gamma^2} = \mathcal{O}(\frac{1}{L})\) number of measurements to guarantee the accuracy of the estimated gradient.
Corollary 4.3. Consider the $N$-qubit variational quantum algorithms with the objective function (6). Then the following formula holds for any $\ell \in \{1, \cdots, L\}$,
\[
\mathbb{E} \left( \frac{\partial f}{\partial \theta_{\ell}} \right)^2 \bigg|_{\theta = \theta^{(t)} + \delta} \geq (1 - \epsilon) \left( \frac{\partial f}{\partial \theta_{\ell}} \right)^2 \bigg|_{\theta = \theta^{(t)}}. \tag{8}
\]
The expectation is taken with Gaussian distributions $\mathcal{N}(0, \gamma^2_j)$ for parameters $\delta = \{\delta_j\}_{j=1}^L$, where the variance $\gamma^2_j \leq \frac{a^2e}{160\epsilon^2(3h_j(h_j-1)+1)L||O||^2} \left( \frac{\partial f}{\partial \theta_{\ell}} \right)^2 \bigg|_{\theta = \theta^{(t)}}, \forall j \in [L]$.

Corollary 4.3 is derived by analyzing the gradient of the function $g(\delta) = f(\delta + \theta^{(t)})$ via Theorem 4.2.

5 Experiments

In this section, we analyze the training behavior of two variational quantum algorithms, i.e., finding the ground energy and state of the Heisenberg model and the LiH molecule, respectively. All numerical experiments are provided using the Pennylane package [87].

5.1 Heisenberg model

In the first task, we aim to find the ground state and the ground energy of the Heisenberg XXX model [88]. The corresponding Hamiltonian matrix is
\[
H = \sum_{i=1}^{N-1} X_iX_{i+1} + Y_iY_{i+1} + Z_iZ_{i+1}, \tag{9}
\]
where $N$ is the number of qubit, $X_i = I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(N-1)}$, $Y_i = I^{\otimes(i-1)} \otimes Y \otimes I^{\otimes(N-1)}$, and $Z_i = I^{\otimes(i-1)} \otimes Z \otimes I^{\otimes(N-1)}$. We employ the loss function defined by Eq. (1) with the input state $|0\rangle \langle 0| \otimes^N$ and the observable (9). Thus, by minimizing the function (1), we can obtain the least eigenvalue of the observable (9), which is the ground energy.

We adopt the ansatz with $N = 15$ qubits, which consists of $L_1 = 10$ layers of $R_Y R_X CZ$ blocks. In each block, we first employ the CZ gate to neighboring qubits pairs $\{(1, 2) \cdots, (N, 1)\}$, followed by $R_X$ and $R_Y$ rotations for all qubits. Overall, the quantum circuit has 300 parameters. We consider three initialization methods for comparison, i.e., initializations with the Gaussian distribution $\mathcal{N}(0, \gamma^2)$ and the uniform distribution in $[0, 2\pi]$, respectively, and the zero initialization (all parameters equal to 0 at the initial point). We remark that each term in the observable (9) contains at most $S = 2$ non-identity Pauli matrices, which is consistent with the $(S, L) = (2, 18)$ case of Theorem 4.1. Thus, we expect that the Gaussian initialization with the variance $\gamma^2 = \frac{1}{4S(L+2)} = \frac{1}{110}$ could provide trainable initial parameters.

In the experiment, we train VQAs with gradient descent (GD) [89] and Adam optimizers [90], respectively. The learning rate is 0.01 and 0.01 for both GD and Adam cases. Since the estimation of gradients on real quantum computers could be perturbed by statistical measurement noise, we compare optimizations using accurate and noisy gradients. For the latter case, we set the variance of measurement noises to be $\frac{1}{110}$.

The numerical results of the Heisenberg XXX model are shown in Figures 2 and 3. The loss during the training with gradient descents is shown in Figures 2(a) and 2(b) for the accurate and the noisy gradient cases, respectively. The Gaussian initialization outperforms the other two initializations with faster convergence rates. Figures 2(c) and 2(d) verify that Gaussian initialized VQAs have larger gradients in the early stage, compared to that of uniformly initialized VQAs. We notice that zero
Figure 2: Numerical results of finding the ground state energy of the Heisenberg model using the gradient descent optimizer. Figures 2(a) and 2(b) illustrate the loss function corresponding to the Hamiltonian of the heisenberg model (9) during the optimization with accurate and noisy gradients, respectively. Figures 2(c) and 2(d) show the $\ell_2$ norm of corresponding gradients during the optimization. Red, blue, and black lines denote the average of 5 rounds of optimizations with the Gaussian initialization, the uniform initialization, and the zero initialization, respectively.

initialized VQAs cannot be trained with accurate gradient descent, since the initial gradient equals to zero. This problem is alleviated in the noisy case, as shown in Figures 2(b) and 2(d). Since the gradient is close to zero at the initial stage, the update direction mainly depends on the measurement noise, which forms the Gaussian distribution. Thus, the parameter in the noisy zero initialized VQAs is expected to accumulate enough variances, which takes around 10 iterations based on Figure 3(d). As illustrated in Figure 2(b), the loss function corresponding to the zero initialization decreases quickly after the variance accumulation stage. Results in Figures 3(a) and 3(d) show similar training behaviors using the Adam optimizer.

5.2 Quantum chemistry

In the second task, we aim to find the ground energy and the corresponding quantum state of the LiH molecule. We follow settings on the ansatz in Refs. [12, 13]. For the molecule with $n_e$ active electrons and $n_o$ free spin orbitals, the corresponding VQA contains $N = n_o$ qubits, which employs the HF
Figure 3: Numerical results of finding the ground state energy of the Heisenberg model using the Adam optimizer. Figures 3(a) and 3(b) illustrate the loss function corresponding to the Hamiltonian of the Heisenberg model (9) during the optimization with accurate and noisy gradients, respectively. Figures 3(c) and 3(d) show the $\ell_2$ norm of corresponding gradients during the optimization. Red, blue, and black lines denote the average of 5 rounds of optimizations with the Gaussian initialization, the uniform initialization, and the zero initialization, respectively.

The ground state $|\phi_{HF}\rangle$ as the input state. We construct the parameterized quantum circuit with Givens rotation gates [12], where each gate is implemented on 2 or 4 qubits with one parameter. Specifically, for the LiH molecule, the number of electrons $n_e = 2$, the number of free spin orbitals $n_o = 10$, and the number of different Givens rotations is $L = 24$ [13]. We follow the molecule Hamiltonian $H_{\text{LiH}}$ defined in Ref. [13]. Thus, the loss function for finding the ground energy of LiH is defined as

$$f(\theta) = \text{Tr} \left[ H_{\text{LiH}} V_{\text{Givens}}(\theta) |\phi_{HF}\rangle \langle \phi_{HF}| V_{\text{Givens}}(\theta)^\dagger \right],$$

where $V_{\text{Givens}}(\theta) = \prod_{i=1}^{L} H_{\text{Givens}}^{i}(\theta_i)$ denotes the product of all parameterized Givens rotations of the LiH molecule. By minimizing the function (10), we can obtain the least eigenvalue of the Hamiltonian $H_{\text{LiH}}$, which is the ground energy of the LiH molecule.
Figure 4: Numerical results of finding the ground state energy of the molecule LiH. The first and second lines show training results with the gradient descent and the Adam optimizer, respectively. The left, the middle, and the right lines show the loss during the training using accurate gradients, noisy gradients with adaptive-distributed noises, and noisy gradients with constant-distributed noises. The variance of noises in the middle line (Figures 4(b) and 4(e)) follows Eq. (11), while the variance of noises in the right line (Figures 4(c) and 4(f)) is 0.001. Red, blue, and black lines denote the average of 5 rounds of optimizations with the Gaussian initialization, the uniform initialization, and the zero initialization, respectively.

In practice, we initialize parameters in the VQA (10) with three distributions for comparison, i.e., the Gaussian distribution $\mathcal{N}(0, \gamma^2)$, the zero distribution (all parameters equal to 0), and the uniform distribution in $[0, 2\pi]$. For 2-qubit Givens rotations, the term $(h, a) = (2, 2)$ as shown in Eq. (5). For 4-qubit Givens rotations, the term $(h, a) = (8, 8)$ [91]. Thus, we set the variance in the Gaussian distribution $\gamma^2 = \frac{8^2 \gamma^2}{24^2 + 8^2 + 24}$, which matches the $(L, h, a, \epsilon) = (24, 8, 8, \frac{1}{2})$ case of Theorem 4.2. Similar to the task of the Heisenberg XXX model, we consider both the accurate and the noisy gradient cases, where the variance of noises in the latter case is the constant 0.001. Moreover, we consider the noisy case with adaptive noises, where the variance of the noise on each partial derivative $\frac{\partial f}{\partial \theta^s} \bigg|_{\theta = \theta^{(t)}}$ in the $t$-th iteration is

$$\gamma^2 = \frac{1}{96 \times 24 \times 8^2 \|H_{\text{LiH}}\|_2^2} \left( \frac{\partial f}{\partial \theta^s} \right)^2 \bigg|_{\theta = \theta^{(t-1)}}. \quad (11)$$

The variance in Eq. (11) matches the $(L, h, a, \epsilon) = (24, 8, 8, \frac{1}{2})$ case of Corollary 4.3 when the VQA is nearly converged:

$$\frac{\partial f}{\partial \theta^s} \bigg|_{\theta = \theta^{(t)}} \approx \frac{\partial f}{\partial \theta^s} \bigg|_{\theta = \theta^{(t-1)}}.$$

In the experiment, we train VQAs with gradient descent and Adam optimizers. Learning rates are set to be 0.1 and 0.01 for GD and Adam cases, respectively. The loss (10) during training iterations
is shown in Figure 4. Optimization results with gradient descents are shown in Figures 4(a)-4(c) for the accurate gradient case, the adaptive noisy gradient case, and the noisy gradient case with the constant noise variance 0.001, respectively. The variance of the noise in the adaptive noisy gradient case follows Eq. (11). Figures 4(a) and 4(b) show similar performance, where the loss $f$ with the Gaussian initialization and the zero initialization converge to $10^{-4}$ over the global minimum $f_\star$. The loss with the uniform initialization is higher than $10^{-1}$ over the global minimum. Figure 4(c) shows the training with constantly perturbed gradients. The Gaussian initialization and the zero initialization induce the $10^{-3}$ convergence, while the loss function with the uniform initialization is still higher than $10^{-1}$ over the global minimum. Figures 4(d)-4(f) show similar training behaviors using the Adam optimizer. Based on Figures 4(a)-4(f), the Gaussian initialization and the zero initialization outperform the uniform initialization in all cases. We notice that optimization with accurate gradients and optimization with adaptive noisy gradients have the same convergence rate and the final value of the loss function, which is better than that using constantly perturbed gradients. We remark that the number of measurements $T = O\left( \frac{1}{\text{Var}(\text{noise})} \right)$. Thus, finite number of measurements with the noise (11) for gradient estimation is enough to achieve the performance of accurate gradients, which verifies Theorem 4.2 and Corollary 4.3.

6 Conclusions

In this work, we provide a Gaussian initialization strategy for solving the vanishing gradient problem of deep variational quantum algorithms. We prove that the gradient norm of $N$-qubit quantum circuits with $L$ layers could be lower bounded by $\text{poly}(N,L)^{-1}$, if the parameter is sampled independently from the Gaussian distribution with the variance $O\left( \frac{1}{L} \right)$. Our results hold for both the local and the global observable cases, and could be generalized to VQAs employing correlated parameterized gates. Moreover, we show that the necessary number of measurements, which scales $O\left( \frac{1}{\epsilon} \right)$, suffices for estimating the gradient during the training.

We provide numerical experiments on finding the ground energy and state of the Heisenberg model and the LiH molecule, respectively. Experiments show that the proposed Gaussian initialization method outperforms the uniform initialization method with faster convergence rate, and the training using gradients with adaptive noises shows the same convergence compared to the training using noiseless gradients.

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A Technical Lemmas

In this section we provide some technical lemmas.

Lemma A.1. Let $\theta$ be a variable with Gaussian distribution $\mathcal{N}(0, \gamma^2)$. Let $\rho = \sum_k c_k \rho_k$ be the linear combination of density matrices $\{\rho_k\}$ with real coefficients $\{c_k\}$. Let $G$ be a hermitian unitary and $V = e^{-iG}$. Let $O$ be an arbitrary hermitian quantum observable that anti-commutes with $G$. Then

$$
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] \right)^2 \geq (1 - 4\gamma^2)\text{Tr} [O \rho]^2 + 4\gamma^2 (1 - 4\gamma^2)\text{Tr} [iG O \rho]^2.
$$

(12)

Proof. By replacing the term $V = e^{-iG} = I \cos \theta - iG \sin \theta$, we have

$$
\text{Tr} \left[ OV \rho V^\dagger \right] = \text{Tr} \left[ O(I \cos \theta - iG \sin \theta) \rho(I \cos \theta + iG \sin \theta) \right] = \cos 2\theta \text{Tr} [O \rho] + \sin 2\theta \text{Tr} [iG O \rho],
$$

(13)

where Eq. (13) follows from the condition $O G + G O = 0$. Since $O$ anti-commutes with $G$, $iG O$ could be served as a hermitian observable. Based on Eq. (13), we have

$$
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] \right)^2 = \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \cos 2\theta \text{Tr} [O \rho] + \sin 2\theta \text{Tr} [iG O \rho] \right)^2
$$

$$
= \frac{1 + e^{-8\gamma^2}}{2} \text{Tr} [O \rho]^2 + \frac{1 - e^{-8\gamma^2}}{2} \text{Tr} [iG O \rho]^2
$$

$$
\geq (1 - 4\gamma^2)\text{Tr} [O \rho]^2 + 4\gamma^2 (1 - 4\gamma^2)\text{Tr} [iG O \rho]^2,
$$

(14)

(15)

where Eq. (14) is obtained by calculating expectation terms. Equation (15) holds since $1 - 8\gamma^2 \leq e^{-8\gamma^2} \leq 1 - 8\gamma^2 + 32\gamma^4$. Thus, we have proved Eq. (12).

\[\square\]

Lemma A.2. Let $\theta$ be a variable with Gaussian distribution $\mathcal{N}(0, \gamma^2)$. Let $\rho$ be the density matrix of a quantum state. Let $G$ be a hermitian unitary and $V = e^{-iG}$. Let $O$ be an arbitrary hermitian quantum observable that anti-commutes with $G$. Then

$$
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] \right)^2 \geq (1 - 4\gamma^2) \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] \right)^2 \bigg|_{\theta = 0} + 16\gamma^2 (1 - 4\gamma^2)\text{Tr} [O \rho]^2.
$$

(16)

Proof. By calculating the gradient for both sides of Eq. (13), we obtain

$$
\frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] = -2 \sin 2\theta \text{Tr} [O \rho] + 2 \cos 2\theta \text{Tr} [iG O \rho].
$$

(17)

Let $\theta = 0$ in Eq. (17), we obtain

$$
\frac{\partial}{\partial \theta} \text{Tr} \left[ OV \rho V^\dagger \right] \bigg|_{\theta = 0} = 2 \text{Tr} [iG O \rho].
$$

(18)

Now we proceed to prove Lemma A.2.

The left part of Eq. (16) = $\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} (-2 \sin 2\theta \text{Tr} [O \rho] + 2 \cos 2\theta \text{Tr} [iG O \rho])^2$
Before the proof, we define several notations for convenience. We define $O$ an arbitrary hermitian quantum observable. Then

$$
\mathbb{E}_{\theta \sim N(0,\gamma^2)} \text{Tr} \left[ (OV_h(\theta)\rho V_h(\theta)) \right]^2 \geq \text{Tr} \left[ (OV_h(0)\rho V_h(0)) \right]^2 - [12h(h-1) + 4] \gamma^2 \|e\|_2^2 \|O\|_2^2 ,
$$

where $\|e\|_1 = \sum_k |c_k|$ denotes the $\ell_1$ norm of $e$, $\|O\|_2$ denotes the spectral norm of $O$, and the variance $\gamma^2 \leq \frac{1}{12h^2}$.

**Proof.** Before the proof, we define several notations for convenience. We define $V_0 = I$ and

$$
V_j(\theta) = W_{h+1-j}e^{-i\theta G_{h+1-j}} \cdots W_h e^{-i\theta G_h}, \forall j \in \{1, \cdots, h\}.
$$

We denote $0_k$, $1_k$, and $2_k$ as $k$-dimensional vectors with components 0, 1, and 2, respectively. We define $O_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} = O$ for the $k = 0$ case and

$$
O_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} = \begin{cases} 
W_k^* O_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} W_k, & \text{if } i_k = 0, j_k = 0, \\
\frac{1}{2} G_k \left\{ G_k, W_k^* O_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} W_k \right\}, & \text{if } i_k = 1, j_k = 0 \\
\frac{1}{2} G_k \left[ G_k, W_k^* O_{i_1, \cdots, i_k}^{j_1, \cdots, j_k} W_k \right], & \text{if } i_k = 2, j_k = 0 \\
i_k G_k O_{i_1, \cdots, i_k, i_{k+1}}^{j_1, \cdots, j_{k+1}, 0} & \text{if } i_k = 1,
\end{cases}
$$

for increasing $k \in \{1, \cdots, h\}$, where $i_k \in \{0, 1, 2\}$ and $j_k \in \{0, 1\}$.

For all $1 \leq k \leq \ell \leq h$, the definition (24) provides the commuting and anti-commuting parts of $O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}}$ with respect to $G_k$, respectively, i.e.

$$
O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} = O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} + O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} G_k,
$$

$$
G_k O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} = O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} G_k,
$$

$$
G_k O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} = -O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} G_k.
$$

Since for all $k \in \{L\}$, $G_k$ is a unitary matrix, $O^{j}_i$ is a hermitian observable for all $i \in \{0, 1, 2\}^\ell$, $j \in \{0, 1\}^\ell$, and $\ell \in \{L\}$. Meanwhile, the spectral norm of $O_j^i$ is bounded,

$$
\left\| O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} \right\|_2 \leq \left\| O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} \right\|_2 \leq \frac{1}{2} \left\| O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} \right\|_2 + \frac{1}{2} \left\| O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} \right\|_2
\geq \left\| O_{i_1, \cdots, i_k, 0_jk_{k+1}, \cdots}^{j_1, \cdots, j_{k+1}, \cdots, j_{\ell}} \right\|_2 \leq \left\| O \right\|_2 \quad \text{for all } j \in \{0, 1\}^\ell, \quad (25)
$$
where $\|A\|_2$ denotes the spectral norm of the matrix $A$. Moreover, for all $k, \ell \geq 0$ such that $k + \ell \leq h$, the observable $O_{h-k, \ell}^{j_1, \cdots, j_h} O_{h-k, \ell+1}^{j_1, \cdots, j_h}$ could be recovered by

$$
\sum_{n=k+1}^{h-\ell} \sum_{i_n=1}^{2} O_{i_1, \cdots, i_{n-1}, j_1, \cdots, j_h}^{h-k, \ell} = O_{i_1, \cdots, i_{n-1}, j_1, \cdots, j_h}^{h-k, \ell+1}.
$$

(26)

Now we begin the proof. To analyze the expectation with respect to the parameter $\theta$, we need the detailed formulation of $\text{Tr} [OV_h \rho V_h^\dagger]$ as the function of $\theta$. In fact, for all $h' \in \{0, 1, \cdots, h\}$ and all $i \in \{0, 1, 2\}^{h-h'}, j \in \{0, 1\}^{h-h'}$, we have

$$
\text{Tr} \left[ O_i^{j'} V_{h'} \rho V_{h'}^\dagger \right] = \sum_{j'':0}^{1} \sum_{j''':0}^{1} \theta \| i' \|_1 - \| j' \|_1 + \| j'' \|_1 (\sin 2\theta) \| j'' \|_1 \text{Tr} \left[ O_i^{j'} O_j^{j''} \rho \right],
$$

(27)

where $\| i \|_1 \equiv \sum_{k=1}^{\dim(i)} |i_k|$ denotes the $\ell_1$ norm of the vector $i$. Equation (27) could be proved inductively. First, for the case $h'=0$, Eq. (27) holds trivially. Next, we assume that Eq. (27) holds for the $h'=k$ case. Then for all $i \in \{0, 1, 2\}^{h-k-1}, j \in \{0, 1\}^{h-k-1}$, we have

$$
\text{Tr} \left[ O_i^{j} V_{k+1} \rho V_{k+1}^\dagger \right] = \text{Tr} \left[ O_i^{j} W_{h-k} (I \cos \theta - iG_{h-k} \sin \theta) V_k \rho V_k^\dagger (I \cos \theta + iG_{h-k} \cos \theta) W_{h-k}^\dagger \right]
$$

$$
\cos^2 \theta \text{Tr} \left[ O_i^{j} O_i^{j} V_k \rho V_k^\dagger \right] + \sin^2 \theta \text{Tr} \left[ G_{h-k} O_i^{j} G_{h-k} V_k \rho V_k^\dagger \right]
$$

$$
\sin \theta \cos \theta \text{Tr} \left[ G_{h-k} O_i^{j} V_k \rho V_k^\dagger \right] - \sin \theta \cos \theta \text{Tr} \left[ O_i^{j} G_{h-k} V_k \rho V_k^\dagger \right]
$$

$$
= \text{Tr} \left[ O_i^{j} V_k \rho V_k^\dagger \right] + \cos 2\theta \text{Tr} \left[ O_i^{j} V_k \rho V_k^\dagger \right] + \sin 2\theta \text{Tr} \left[ O_i^{j} V_k \rho V_k^\dagger \right],
$$

(28)

(29)

where Eqs. (28) and (29) are derived by using the definition (24). We proceed by employing the $h'=k+1$ case of Eq. (27), such that

$$
\text{Eq. (29)} = \sum_{j'':0}^{1} \sum_{j''':0}^{1} (\cos 2\theta) \| i' \|_1 - \| j' \|_1 + \| j'' \|_1 (\sin 2\theta) \| j'' \|_1 \text{Tr} \left[ O_i^{j'} O_j^{j''} \rho \right]
$$

$$
+ \cos 2\theta \sum_{j'':0}^{1} \sum_{j''':0}^{1} (\cos 2\theta) \| i' \|_1 - \| j' \|_1 + \| j'' \|_1 (\sin 2\theta) \| j'' \|_1 \text{Tr} \left[ O_i^{j'} O_j^{j''} \rho \right]
$$

$$
+ \sin 2\theta \sum_{j'':0}^{1} \sum_{j''':0}^{1} (\cos 2\theta) \| i' \|_1 - \| j' \|_1 + \| j'' \|_1 (\sin 2\theta) \| j'' \|_1 \text{Tr} \left[ O_i^{j'} O_j^{j''} \rho \right]
$$

$$
= \sum_{j''':0}^{1} \sum_{j''':0}^{1} (\cos 2\theta) \| i' \|_1 - \| j' \|_1 + \| j'' \|_1 (\sin 2\theta) \| j'' \|_1 \text{Tr} \left[ O_i^{j'} O_j^{j''} \rho \right],
$$

(30)

which matches the formulation of the $h'=k+1$ case of Eq. (27). Thus, Eq. (27) has been proved.
Now we begin to prove Eq. (22). Employing the \( h' = h \) case of Eq. (27) could yield

\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \text{Tr} \left[ OV_h \rho V_h^\dagger \right] \right)^2 = \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{j=0_h}^{2h} \sum_{i=0_h}^{2h} (\cos 2\theta)^{|i| - |j|} (\sin 2\theta)^{|j|} \text{Tr} \left[ O_i \rho \right] \right)^2 \tag{31}
\]

\[
= \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{i=1_h}^{2h} (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right)^2 \tag{32}
\]

\[
\geq \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{i=1_h}^{2h} (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right)^2 \tag{33}
\]

Equation (33) is obtained by discarding the square of the latter term in the bracket of Eq. (32). We remark that if Eqs. (34) and (35) hold, we can prove Eq. (22) by using Eqs. (31-33).

\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{i=1_h}^{2h} (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right)^2 - \left( \text{Tr} \left[ OV_h (0) \rho V_h (0) \right] \right)^2 \geq -(6h - 2) \gamma^2 \|c\|_2^2 \|O\|_2^2. \tag{34}
\]

\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{j=0_h}^{2h} \sum_{i=0_h}^{2h} (\cos 2\theta)^{|i| - |j|} (\sin 2\theta)^{|j|} \text{Tr} \left[ O_i \rho \right] \sum_{i'=1_h}^{2h} (\cos 2\theta)^{|i'|} \text{Tr} \left[ O_{i'}^0 \rho \right] \geq -(6h^2 - 9h + 3) \gamma^2 \|c\|_2^2 \|O\|_2^2. \tag{35}
\]

In the following proof, we would derive Eqs. (34) and (35). We focus on the Eq. (34) first. In fact, the left side of Eq. (34) is bounded by

\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{i=1_h}^{2h} \left[ 1 - (\cos 2\theta)^{|i|} - 1 \text{Tr} \left[ O_i^0 \rho \right] \right]^2 \right) \geq 2 \left( \text{Tr} \left[ O_i^0 \rho \right] \right)^2 \tag{36}
\]

\[
= \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{i=1_h}^{2h} \left[ 1 - (\cos 2\theta)^{|i|} - 1 \text{Tr} \left[ O_i^0 \rho \right] \right]^2 \right) \tag{37}
\]

\[
\geq 2 \left[ \text{Tr} \left[ O_i^0 \rho \right] \right] \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{i=1_h}^{2h} \left[ 1 - (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right] \tag{38}
\]

\[
= -2 \left[ \text{Tr} \left[ O_i^0 \rho \right] \right] \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{i=1_h}^{2h} \left[ 1 - (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right] \tag{39}
\]

\[
\geq -2 \|c\|_2 \|O\|_2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{i=1_h}^{2h} \left[ 1 - (\cos 2\theta)^{|i|} \text{Tr} \left[ O_i^0 \rho \right] \right] \tag{40}
\]

\[
\geq -2 \|c\|_2^2 \|O\|_2^2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( 2 - (\cos 2\theta)^h - 1 \right). \tag{41}
\]
Equation (36) is obtained by using the definition (24). Equation (37) is derived by using Eq. (26). Equation (38) is obtained by using \((a - b)^2 - b^2 \geq -2|a| \cdot |b|\). Equation (39) yields from Eq. (26). Equation (40) is derived by using

\[
\left| \text{Tr} \left[ O_{i}^{j} \rho \right] \right| = \sum_{k} c_{k} \text{Tr} \left[ O_{i}^{j} \rho_{k} \right] \leq \sum_{k} |c_{k}| \left| \text{Tr} \left[ O_{i}^{j} \rho_{k} \right] \right| \leq \sum_{k} |c_{k}| \left\| O_{i}^{j} \right\|_{2} \leq \left\| c \right\|_{1} \| O \|_{2}. \tag{42}
\]

Equation (41) is obtained by using the \(h' = h\) case of Eq. (43), i.e.,

\[
\left| \sum_{j' = j + 1}^{j + k} 1 - (\cos 2\theta)^{1/1-1-j'-1-k} \right| \text{Tr} \left[ O_{i}^{j} \rho \right] \leq \left( (2 - \cos 2\theta)^{j'-1} - 1 \right) \left\| c \right\|_{1} \| O \|_{2} \tag{43}
\]

for all \(j' \in \{0, 1, \ldots, h\}, j \in \{0, 1, 2\}^{h-h'}, \) and \(j \in \{0, 1\}^{h-h'}\).

Equation (43) can be proved inductively. First, for the case \(h' = 0\), Eq. (43) holds trivially. Next we assume that Eq. (43) holds for the case \(h' = k\). Then for all \(i \in \{0, 1, 2\}^{h-h-1}\) and \(j \in \{0, 1\}^{h-h-1}\), we have

\[
\left| \sum_{j' = j + 1}^{2k+1} 1 - (\cos 2\theta)^{1/1-1-j'-1-k} \right| \text{Tr} \left[ O_{i}^{j} \rho \right] \leq \left( (2 - \cos 2\theta)^{h-k} - 1 \right) \left\| c \right\|_{1} \| O \|_{2}. \tag{45}
\]

For the case \(j' = 1\),

\[
\left| \sum_{j' = j + 1}^{2k} 1 - (\cos 2\theta)^{1/1-1-j'-1-k} \right| \text{Tr} \left[ O_{i}^{j} \rho \right] \leq \left( (2 - \cos 2\theta)^{h-k} - 1 \right) \left\| c \right\|_{1} \| O \|_{2}. \tag{46}
\]

Equation (45) is derived by using the \(h' = k\) case of Eq. (43). Equation (46) is derived by using \(j' = 1\). For the case \(j' = 0\),

\[
\left| \sum_{j' = j + 1}^{2k} 1 - (\cos 2\theta)^{1/1-1-j'-1-k} \right| \text{Tr} \left[ O_{i}^{j} \rho \right] \leq \left( (2 - \cos 2\theta)^{h-k} - 1 \right) \left\| c \right\|_{1} \| O \|_{2}. \tag{47}
\]
Equation (52) is derived by calculating expectation terms. Equation (53) yields from Eq. (25). Thus we have proved Eq. (43) since Eqs. (46) and (50) match the case of Eq. (43) and Eq. (26). Equation (50) is derived by using Eq. (25). Thus we have proved Eq. (43) since Eqs. (46) and (50) match the $h' = k + 1$ case.

Since $\cos 2\theta \geq 1 - 2\theta^2$, we could further bound Eq. (41) by

$$\geq -2\|\mathbf{c}\|^2_2\|O\|^2_2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} [(1 + 2\theta^2)^h - 1]$$ \hspace{1cm} (51)

$$= -2\|\mathbf{c}\|^2_2\|O\|^2_2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{t=1}^{h} \binom{h}{t} (2\theta^2)^t$$

$$= -2\|\mathbf{c}\|^2_2\|O\|^2_2 \sum_{t=1}^{h} \binom{h}{t} (2t - 1)!!(2\gamma^2)^t$$ \hspace{1cm} (52)

$$\geq -2\|\mathbf{c}\|^2_2\|O\|^2_2 \sum_{t=1}^{h} h(h - 1)t^{-1}2^{t-1} \left(\frac{1}{6h^2}\right)^{t-1} (2\gamma^2)$$

$$= -2\|\mathbf{c}\|^2_2\|O\|^2_2 2h\gamma^2 \left[1 + \frac{h - 1}{3h^2} \sum_{t=0}^{h-2} \left(\frac{h - 1}{3h^2}\right)^t\right]$$

$$\geq - (6h - 2) \|\mathbf{c}\|^2_2\|O\|^2_2 2\gamma^2.$$ \hspace{1cm} (54)

Equation (52) is derived by calculating expectation terms. Equation (53) yields from $\frac{(2t-1)!!}{t!} \leq 2^{t-1}$ and the condition $\gamma^2 \leq \frac{1}{12h^2}$. Thus, we have proved Eq. (34).

Next, we focus on the Eq. (35). The left side of Eq. (35) could be bounded by

$$= \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{j \geq 0, \gamma \|\mathbf{j}\|_1} \sum_{i = j + 1}^{2h} (\cos 2\theta)^{\|\mathbf{i}\|_1 - \|\mathbf{j}\|_1^h} (\sin 2\theta)^{\|\mathbf{j}\|_1} \mathbf{Tr} \left[O_i^\theta \rho \right] \sum_{i' = 1}^{2h} (\cos 2\theta)^{\|\mathbf{i'}\|_1} \mathbf{Tr} \left[O_i^\theta \rho \right]$$

$$\geq - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{j \geq 0, \gamma \|\mathbf{j}\|_1} \left[1 - (\cos 2\theta)^{\|\mathbf{j}\|_1 - \|\mathbf{j}\|_1^h} \mathbf{Tr} \left[O_i^\theta \rho \right] + \sum_{i = j + 1}^{2h} \mathbf{Tr} \left[O_i^\theta \rho \right]ight]$$

$$\cdot \left[1 - (\cos 2\theta)^{\|\mathbf{j}\|_1 - \|\mathbf{j}\|_1^h} \mathbf{Tr} \left[O_i^\theta \rho \right] + \sum_{i' = 1}^{2h} \mathbf{Tr} \left[O_i^\theta \rho \right] \right]$$ \hspace{1cm} (56)

$$\geq - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{j \geq 0, \gamma \|\mathbf{j}\|_1} \left[(2 - \cos 2\theta)^{h - \|\mathbf{j}\|_1} - 1 \right] \|\mathbf{c}\|_2 \|O_0^\theta\|_2 + \mathbf{Tr} \left[O_2^\theta \rho \right]$$

$$\cdot \left[(2 - \cos 2\theta)^{h - 1} \|\mathbf{c}\|_2 \|O_0^\theta\|_2 + \mathbf{Tr} \left[O_2^\theta \rho \right] \right]$$ \hspace{1cm} (57)
\[ \geq - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{j \geq 0, \gamma^2} (\sin 2\theta)^{\|j\|_1} (2 - \cos 2\theta)^{2\gamma - \|j\|_1} \|c\|^2_1 \|O\|^2_2 \]  
(58)

\[ \geq - \|c\|^2_1 \|O\|^2_2 \sum_{j \geq 0, \gamma^2} (\sin 2\theta)^{\|j\|_1} (2 - \cos 2\theta)^{2\gamma - \|j\|_1}. \]  
(59)

Equation (55) is obtained by noticing that the expectation of \( \sin^2 2\theta \cos^2 2\theta \) equals to zero, if \( a \) is odd. Equation (56) is obtained by using \( \sum a_i b_j \geq - (\sum |a_i|)(\sum |b_j|) \) and \( |a + b| \leq |a| + |b| \). Equation (57) is derived by using the \( h' = h \) case of Eq. (43) and Eq. (26). Equation (58) is obtained by using \( \|O_0\| = \|O\| \) and Eq. (42). Equation (59) is derived by using \( (\sin 2\theta)^2 \leq (2\theta)^2 \) and \( \cos 2\theta \geq 1 - 2\theta^2 \).

We proceed from Eq. (59), which could be further bounded by

\[ = - \|c\|^2_1 \|O\|^2_2 \sum_{t=1}^{\lceil h/2 \rceil} (\sin 2\theta)^{2t} \sum_{m=0}^{2h - 2t} \left( \begin{array}{c} 2h - 2t \\ m \end{array} \right) (2\theta)^m \]  
(60)

\[ = - \|c\|^2_1 \|O\|^2_2 \sum_{t=1}^{\lceil h/2 \rceil} (\sin 2\theta)^{2t} \sum_{m=0}^{2h - 2t} \left( \begin{array}{c} 2h - 2t \\ m \end{array} \right) (2\theta)^m (2t + 2m - 1)!! \]  
(61)

\[ \geq - \|c\|^2_1 \|O\|^2_2 \sum_{t=1}^{\lceil h/2 \rceil} \sum_{m=0}^{2h - 2t} \frac{h(h - 1)^{2t-1}}{2^{t-1}} \frac{(2h - 2)^m}{m!} 2^{2t + m}(2t + 2m - 1)!!(2t + 1)(2t + 3) \cdots (2t + 2m - 1) \gamma^{2t + 2m} \]  
(62)

\[ \geq - \|c\|^2_1 \|O\|^2_2 \sum_{t=1}^{\lceil h/2 \rceil} \sum_{m=0}^{2h - 2t} \frac{h(h - 1)^{2t-1}}{2^{t-1}} \frac{(2h - 2)^m}{m!} 2^{2t + m}(2h)^m \gamma^{2t + 2m} \]  
(63)

\[ \geq - \|c\|^2_1 \|O\|^2_2 \sum_{t=1}^{\lceil h/2 \rceil} \sum_{m=0}^{2h - 2t} \left( \begin{array}{c} 2h - 1 \\ m \end{array} \right) \frac{4h(h - 1)^{2t-1}}{2^{t-1}} \frac{(2h - 2)^m}{m!} 2^{2t + m}(2h)^m \gamma^{2t + 2m} \]  
(64)

\[ = - \|c\|^2_1 \|O\|^2_2 \left( \begin{array}{c} h/2 \\ 2t \end{array} \right) \left( \begin{array}{c} h/2 \\ 2t - 1 \end{array} \right) \gamma^2 \cdot \left( 1 + 2 \sum_{m=1}^{2h - 2t} \left[ 4h(h - 1)^2 \right]^m \right) \]  
(65)

\[ \geq - \|c\|^2_1 \|O\|^2_2 \frac{3h(h - 1)^2}{2}(1 + 12h(h - 1)^2) \]  
(66)

\[ \geq - (6h^2 - 9h + 3) \gamma^2 \|c\|^2_1 \|O\|^2_2. \]  
(67)

Equation (60) is obtained since the summation \( \sum_{j \geq 0, \gamma^2} \) contains \( \frac{h}{2t} \) terms such that \( \|j\|_1 = 2t \), for all \( t \in \{1, \ldots, \lfloor h/2 \rfloor \} \). Equation (61) is derived by calculating expectation terms. Equation (62) is obtained by using

\[ \left( \begin{array}{c} h/2 \\ 2t \end{array} \right) \leq \frac{h(h - 1)^{2t-1}}{(2t)!} = \frac{h(h - 1)^{2t-1}}{2^{t-1}(2t - 1)!!} \]  
and \( \left( \begin{array}{c} 2h - 2t \\ m \end{array} \right) \leq \frac{(2h - 2t)^m}{m!} \).

Equation (63) is derived by using \( t! \geq 2t-1 \) and

\[ (2t + 2k - 1)(2t + 2m - 2k + 1) \leq (2t + m)^2 \leq (2h)^2, \forall k \in \{1, \ldots, m - 1\}. \]

Equation (64) is obtained by splitting the summation \( \sum_{m=0}^{2h - 2t} \) and using \( m! \geq 2^{m-1}, \forall m \geq 1 \). Equation (66) is derived by calculating geometric sequences with the condition \( \gamma^2 \leq \frac{1}{12h^2} \). Equation (67) follows from the condition \( \gamma^2 \leq \frac{1}{12h^2} \). Thus, we have proved Eq. (35).
**Lemma A.4.** Let \( \rho \) be the density matrix of a quantum state. Let \( V_h = W_1 e^{-i\theta G_1} W_2 \cdots W_h e^{-i\theta G_h} \), where \( \{G_n\}_{n=1}^h \) is a list of hermitian unitaries and \( \{W_n\}_{n=1}^h \) is a list of unitary matrices. Denote by \( O \) an arbitrary hermitian quantum observable. Then

\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ O V_h \rho V_h^\dagger \right] \right)^2 \geq (1 - 4\gamma^2) \left( \frac{\partial}{\partial \theta} \text{Tr} \left[ O V_h \rho V_h^\dagger \right] \right)^2 - 96h^2(h-1)\gamma^2 \|O\|_2^2 - 20h^2(h-1)(h-2)\gamma^2 \|O\|_2^2,
\]

(68)

where \( \|O\|_2 \) denotes the spectral norm of \( O \) and the variance \( \gamma^2 \leq \frac{1}{16h^2} \).

**Proof.** For convenience, we follow the notation \( O^i \) in Eq. (24). We can obtain the detailed formulation of \( \frac{\partial}{\partial \theta} \text{Tr} \left[ O V_h \rho V_h^\dagger \right] \) by using the \( h' = h \) case of Eq. (27),

\[
\frac{\partial}{\partial \theta} \text{Tr} \left[ O V_h \rho V_h^\dagger \right] = \frac{\partial}{\partial \theta} \sum_{j=0}^{\frac{h}{2}} \sum_{i=j+\frac{h}{2}}^{2h} (\cos 2\theta) \|i\|_1 - \|j\|_1 - h \sin 2\theta \|j\|_1^1 \text{Tr} \left[ O^i \rho \right]
\]

(69)

\[
= 2 \sum_{j=0, i=j+1}^{\frac{h}{2}} (h + \|j\|_1 - \|i\|_1) (\cos 2\theta) \|i\|_1 - \|j\|_1 - h \sin 2\theta \|j\|_1^1 \text{Tr} \left[ O^i \rho \right]
\]

(70)

\[
+ 2 \sum_{j=0, i=j+1}^{\frac{h}{2}} \|j\|_1 (\cos 2\theta) \|i\|_1 - \|j\|_1 - h \sin 2\theta \|j\|_1 \text{Tr} \left[ O^i \rho \right]
\]

\[
+ 2 \sum_{\|j\|_1 = 1, i = j+1}^{\frac{h}{2}} (\cos 2\theta) \|i\|_1 - h \text{Tr} \left[ O^i \rho \right]
\]

(71)

Equation (69) follows from Eq. (27). Equation (70) is derived by calculating the gradient of sine and cosine terms. By discarding the square of the sum of the first and the third term in Eq. (71), we obtain

\[
\left( \frac{\partial}{\partial \theta} \text{Tr} \left[ O V_h \rho V_h^\dagger \right] \right)^2 \geq 4 \left( \sum_{\|j\|_1 = 1, i = j+1}^{\frac{h}{2}} (\cos 2\theta) \|i\|_1 - h \text{Tr} \left[ O^i \rho \right] \right)^2
\]

\[
+ 8 \left( \sum_{\|j\|_1 \geq 2, i = j+1}^{\frac{h}{2}} \|j\|_1 (\cos 2\theta) \|i\|_1 - \|j\|_1 - h \sin 2\theta \|j\|_1 \text{Tr} \left[ O^i \rho \right] \right)
\]

\[
\cdot \left( \sum_{\|j\|_1 = 1, i' = j'+1}^{\frac{h}{2}} (\cos 2\theta) \|i'\|_1 - h \text{Tr} \left[ O^{i'} \rho \right] \right)
\]

\[
+ 8 \left( \sum_{j=0, i=j+1}^{\frac{h}{2}} (h + \|j\|_1 - \|i\|_1) (\cos 2\theta) \|i\|_1 - \|j\|_1 - h \sin 2\theta \|j\|_1 \text{Tr} \left[ O^i \rho \right] \right)
\]

\[
\cdot \left( \sum_{\|j\|_1 = 1, i' = j'+1}^{\frac{h}{2}} (\cos 2\theta) \|i'\|_1 - h \text{Tr} \left[ O^{i'} \rho \right] \right)
\]

(72)
Let $\theta = 0$ in Eq. (71), we obtain
\[
\frac{\partial}{\partial \theta} \text{Tr} \left[ O V_\theta V_\theta^\dagger \right] \bigg|_{\theta = 0} = 2 \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \text{Tr} \left[ O_i^j \rho \right].
\]
(73)

Thus, we could obtain Eq. (68) if Eqs. (74-76) hold.
\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} (\cos 2\theta)^{||i||_1 - h} \text{Tr} \left[ O_i^j \rho \right] \right)^2 - (1 - 4\gamma^2) \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \text{Tr} \left[ O_i^j \rho \right] \right)^2
\geq - \frac{13}{3} h^2 (h - 1) \gamma^2 ||O||_2^2,
\]
(74)
\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} (h + ||j||_1 - ||i||_1) (\cos 2\theta)^{||i||_1 - ||j||_1 - h} (\sin 2\theta)^{||j||_1 + 1} \text{Tr} \left[ O_i^j \rho \right] \right)
\cdot \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} (\cos 2\theta)^{||i||_1 - h} \text{Tr} \left[ O_i^j \rho \right] \right)
\geq - \frac{5}{6} h^2 (h - 1) \gamma^2 ||O||_2^2,
\]
(75)
\[
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} (\cos 2\theta)^{||i||_1 - h} \text{Tr} \left[ O_i^j \rho \right] \right)
\cdot \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} (\cos 2\theta)^{||j||_1 - 1} \text{Tr} \left[ O_i^j \rho \right] \right)
\geq - \frac{5}{2} h^2 (h - 1) (h - 2) \gamma^2 ||O||_2^2.
\]
(76)

We begin by proving Eq. (74). The left side of Eq. (74) can be lower bounded as
\[
\geq \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \left[ \cos 2\theta - (\cos 2\theta)^{||i||_1 - h} - \cos 2\theta \right] \text{Tr} \left[ O_i^j \rho \right] \right)^2
\geq - 2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} (\cos 2\theta)^2 \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \left[ 1 - (\cos 2\theta)^{||i||_1 - h} \right] \text{Tr} \left[ O_i^j \rho \right] \cdot \sum_{||j'||_1 = 1}^{2h} \sum_{i' = j' + 1}^{2h} \text{Tr} \left[ O_{i'}^{j'} \rho \right]
\geq - 2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} (\cos 2\theta)^2 \sum_{||j||_1 = 1}^{2h} \left[ (2 - \cos 2\theta)^{h - 1} - 1 \right] ||O||_2 \cdot \sum_{||j||_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \text{Tr} \left[ O_i^j \rho \right]
\geq - 2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} (\cos 2\theta)^2 h \left[ (2 - \cos 2\theta)^{h - 1} - 1 \right] ||O||_2 \cdot h ||O||_2
\geq - 2 h^2 ||O||_2^2 \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( 1 + 2\theta^2 \right)^{h - 1} - 1 \right]
\geq - \frac{13}{3} h^2 (h - 1) \gamma^2 ||O||_2^2.
\]
(77)

Equation (77) follows from
\[
1 - 4\gamma^2 = \mathbb{E}_\theta [1 - 4\theta^2] \leq \mathbb{E}_\theta (1 - 2\theta^2)^2 \leq \mathbb{E}_\theta (\cos 2\theta)^2.
\]
27
Equation (78) is obtained by using $(a - b)^2 - b^2 \geq -2|a| \cdot |b|$. Equation (79) follows from the $h' = h$ and $\|j\| = 1$ case of Eq. (43). Equation (80) is derived by using Eq. (26). Equation (81) is obtained by using $\cos 2\theta \geq 1 - 2\theta^2$. Equation (82) follows from the derivation below.

$$
\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)}(1 + 2\theta^2)^{h-1} - 1 = \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \frac{h-1}{t} ((2\theta^2)^t
\sum_{t=1}^{h-1} \left( \begin{array}{c} h-1 \\ t \end{array} \right) (2t-1)!! (2\gamma^2)^t
\leq \sum_{t=1}^{h-1} (h-1)(h-2)^{t-1}2^{t-1}(2\gamma^2)^t
\leq 2(h-1)\gamma^2 \sum_{t=1}^{h-1} \left( \begin{array}{c} h \gamma^2 \\ t-1 \end{array} \right)
\leq \frac{13}{6}(h-1)\gamma^2.
$$

Equation (83) is obtained by calculating expectation terms. Equation (84) follows from $h^3 \geq 4(h-2)$ for integer $h$. Equation (85) is derived by calculating geometric sequences with the condition $\gamma^2 \leq \frac{1}{16\pi^2}$.

Next, we prove Eq. (75). The left side of Eq. (75) could be lower bounded by

$$
\begin{align*}
&\mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{\|j'\|_1 = 1}^{2h} \sum_{i' = j' + 1}^{2h} \left( \cos 2\theta \right)^{\|i'\|_1 - 1 - h} \text{Tr} \left[ O_{i'}^J \rho \right] \right) \\
&= \sum_{\|j\|_1 = 1}^{2h} \sum_{i = j + 1}^{2h} \left( \cos 2\theta \right)^{\|i\|_1 - 1 - h} \text{Tr} \left[ O_{i}^J \rho \right] \\
&\geq - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{\|j\|_1 = 1}^{2h} \left( \begin{array}{c} 2h \\ i' = j' + 1 \end{array} \right) \left( 1 - \cos 2\theta \right)^{\|i\|_1 - 1 - h} \text{Tr} \left[ O_{i'}^J \rho \right] + \sum_{\|j\|_1 = 1}^{2h} \left( \begin{array}{c} 2h \\ i = j + 1 \end{array} \right) \left( \sin 2\theta \right)^{\|i\|_1 + 1} \text{Tr} \left[ O_{i}^J \rho \right] \right) \\
&\geq - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{\|j^+\|_1 = 1}^{2h} \left( 2 - \cos 2\theta \right)^{h-1} \|O\|_2 \sum_{\|j\|_1 = 1}^{2h} \left( \begin{array}{c} 2h \\ i = j + 1 \end{array} \right) \left( \sin 2\theta \right)^{\|i\|_1 + 1} \sum_{i = j + 1}^{2h} \left( \|j\|_1 - h \right) \cos 2\theta \text{Tr} \left[ O_{i}^J \rho \right] \right) \\
\end{align*}
$$

Equation (86) is obtained by noticing that the expectation of $\sin^a 2\theta \cos^b 2\theta$ equals to zero, if $a$ is odd. Equation (87) is derived by using $|\sum_k a_k| \leq \sum_k |a_k|$. Equation (88) is obtained by using the $h' = h,$
\[ \|j\|_1 = 1 \text{ case of Eq. (43) and Eq. (26). Equation (89) follows from Eq. (26) and the } h' = h \text{ case of Eq. (90), i.e.} \]

\[
\left| \sum_{i' = j' + 1}^{2h'} (h' - \|j\|_1) - (\|i'\|_1 - \|j'\|_1 - h')(\cos 2\theta)\|i'\|_1 - \|j'\|_1 - h') \right| \leq g(h' - \|j\|_1) \|O\|_2
\]

for all \( h' \in \{0, 1, \ldots, h\} \), \( i \in \{0, 1, 2\}^{h-h'} \), and \( j \in \{0, 1\}^{h-h'} \), where

\[
g(x) = x \left( (3 - \cos 2\theta)(2 - \cos 2\theta)^{x-1} - 1 \right).
\]

Equation (90) can be proved inductively. First, Eq. (90) holds trivially when \( h' = 0 \). Next, we assume that Eq. (90) holds for the \( h' = k \) case. Then, for all \( i \in \{0, 1, 2\}^{h-k-1} \) and \( j \in \{0, 1\}^{h-k-1} \), we have

\[
\left| \sum_{i' = j' + 1}^{2h} \left[(k + 1 - \|j\|_1) - (\|i'\|_1 - \|j'\|_1 - k)(\cos 2\theta)\|i'\|_1 - \|j'\|_1 - k - 1\right] \right| \leq g(k + 1 - \|j\|_1) \|O\|_2.
\]

For the case \( j'_{k+1} = 1 \), we have

\[
\text{Eq. (92) } = \left| \sum_{i' = j' + 1}^{2h} \left[(k - \|j\|_1) - (\|i'\|_1 - \|j'\|_1 - k)(\cos 2\theta)\|i'\|_1 - \|j'\|_1 - k\right] \right| \leq g(k - \|j\|_1) \|O\|_2.
\]

Equation (93) follows from the \( h' = k \) case of Eq. (90). Equation (94) is obtained by using \( j'_{k+1} = 1 \). We remark that Eq. (94) matches the \( h' = k + 1 \) case of Eq. (90).

For the case \( j'_{k+1} = 0 \), the situation is more complicated. We have

\[
\text{Eq. (92) } = \left| \sum_{i' = j' + 1}^{2h} \left[(k + 1 - \|j'\|_1) - (\|i'\|_1 - \|j'\|_1 - k)(\cos 2\theta)\|i'\|_1 - \|j'\|_1 - k\right] \right| \leq g(k + 1 - \|j'\|_1) \|O\|_2.
\]

29
\[-(1 - \cos 2\theta) \sum_{j'+j''=1}^{2h} \left[ (k - \|j''\|) - (\|i''\|_1 - \|j''\|_1 - k) (\cos 2\theta)\|i''\|_1 - \|j''\|_1 - k \right] \text{Tr} \left[ O_{i''}^{j',\theta,j''} \rho \right] \]

\[
\leq \left\| O_{a_0,1,1}^{0,0,j} \right\|_2 + g(k - \|j''\|_1)\|O\|_2 + (1 - \cos 2\theta)(k + 1 - \|j''\|_1) \left\| O_{a_0,2,1}^{0,0,j} \right\|_2 \\
+ \cos 2\theta [ (2 - \cos 2\theta) k - \|j''\|_1 - 1 ] \left\| O_{a_0,2,1}^{0,0,j} \right\|_2 + (1 - \cos 2\theta)g(k - \|j''\|_1)\|O\|_2 \\
\leq \left[ (2 - \cos 2\theta)(k - \|j''\|_1) \left[ (3 - \cos 2\theta)(2 - \cos 2\theta) k - \|j''\|_1 - 1 \right] \\
+ (2 - \cos 2\theta) k - \|j''\|_1 + (1 - \cos 2\theta)(k + 1 - \|j''\|_1) \right] \|O\|_2 \\
\leq g(k + 1 - \|j''\| - j'_{k+1})\|O\|_2. 
\]

Equation (95) is obtained by using the \( h' = k \) case of Eq. (90). Equation (96) is obtained by using Eqs. (25) and (91). Equation (97) follows from Eq. (91) and the condition \( j'_{k+1} = 0 \). Since Eqs. (94) and (97) match the formulation of the \( h' = k + 1 \) case of Eq. (90), we have proved Eq. (90) for general cases.

We proceed from Eq. (89), which can be lower bounded by

\[
\begin{align*}
g &\geq -h(h - 1)\|O\|_2^2 \sum_{j=0}^{1} \sum_{\theta \sim N(0,\gamma^2)} (2\theta)^{\|j\|_1 + 1} 2(1 + 2\theta^2)^{2h-1-\|j\|_1} \\
&\geq -2h(h - 1)\|O\|_2^2 \sum_{\theta \sim N(0,\gamma^2)} \sum_{t=1}^{1} \left( \frac{h}{2t - 1} \right) (2\theta)^{2t} \sum_{m=0}^{2h-2t} \binom{2h - 2t}{m} (2\theta^2)^m \\
&= -2h(h - 1)\|O\|_2^2 \sum_{t=1}^{1} \left( \frac{h}{2t - 1} \right) \sum_{m=0}^{2h-2t} \binom{2h - 2t}{m} 2^{2t+m}(2t + 2m - 1)!! \gamma^{2t+2m} \\
&\geq -\frac{59}{6} h^2(h - 1)\gamma^2 \|O\|_2^2.
\end{align*}
\]

Equation (98) is obtained by using \( 1 \geq \cos 2\theta \geq 1 - 2\theta^2 \). Equation (99) is obtained since the summation \( \sum_{j=0}^{1} \) contains \( \binom{h}{2t - 1} \) terms such that \( \|j\|_1 = 2t - 1 \), for all \( t \in \{1, \ldots, \left\lfloor \frac{h+1}{2} \right\rfloor \} \). Equation (100) is derived by calculating expectation terms. Equation (101) is obtained by bounding the summation terms, i.e.

\[
\begin{align*}
&\sum_{t=1}^{1} \left( \frac{h}{2t - 1} \right) \sum_{m=0}^{2h-2t} \binom{2h - 2t}{m} 2^{2t+m}(2t + 2m - 1)!! \gamma^{2t+2m} \\
&\leq \sum_{t=1}^{1} \frac{(h(h - 1))^{2t-2}}{2^{t-1}(t-1)!(2t - 1)!!} \sum_{m=0}^{2h-2t} \frac{(2h - 2t)^m}{m!} 2^{2t+m}(2t + 1)!! (2t + 2m - 1)!! \gamma^{2t+2m} \\
&\leq \sum_{t=1}^{1} \frac{(h(h - 1))^{2t-2} \gamma^2}{t} \sum_{m=0}^{2h-2t} \frac{(2h - 2)^m}{m!} 2^{2t+m} \gamma^{2m} \\
&\leq 4h \gamma^2 \sum_{t=1}^{1} \frac{(h(h - 1)^2 \gamma^2)^{t-1}}{t!} \sum_{m=0}^{2h-2t} (2h^3 \gamma^2)^m
\end{align*}
\]
\[ \leq 4h\gamma^2 \frac{16}{15} \times \frac{8}{7} \leq \frac{59}{12}h\gamma^2. \]  

Equation (102) follows from \( t \geq 1 \) and \( (2t-1)! = 2^t t!(2t-1)! \). Equation (103) is obtained by using \( t \geq 1 \) and

\[ (2t + 2k - 1)(2t + 2m - 2k + 1) \leq (m + 2t)^2 \leq (2h)^2, \forall k \in \{1, \cdots, m\}. \]

Equation (104) is derived by using \( 8h(h - 1) \leq 2h^3 \) for the integer \( h \). Equation (105) is obtained by calculating geometric sequences with the condition \( \gamma^2 \leq \frac{1}{16h^2} \).

Finally, we prove Eq. (76). The left side of Eq. (76) could be lower bounded by

\[
\begin{align*}
= & - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \left( \sum_{||i'|| = 1}^{2h} (\cos 2\theta)||i'||1-h-1 \text{Tr} \left[ O'_{i'} \rho \right] \right) \\
& \cdot \left( \sum_{j=0}^{1} \sum_{i=j+1}^{2h} \frac{1}{||j||1} \frac{1}{||j||1} (\cos 2\theta)||i'||1-h-1 (\sin 2\theta)||i'||1-h-1 \text{Tr} \left[ O'_{i'} \rho \right] \right) \\
\geq & - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} \sum_{||j||1 = 1} (2 - \cos 2\theta)^h-1 ||O||_2 \sum_{j=0}^{1} \frac{1}{||j||1} \frac{1}{||j||1} (\cos 2\theta)^2 (2 \cos 2\theta)^h-1 ||O||_2 \\
& \left\{ \left| \sum_{i=j+1}^{2h} \frac{1}{||j||1} \frac{1}{||j||1} (\cos 2\theta)||i'||1-h \text{Tr} \left[ O'_{i'} \rho \right] \right| + \left| \sum_{i=j+1}^{2h} \frac{1}{||j||1} \frac{1}{||j||1} \text{Tr} \left[ O'_{i'} \rho \right] \right| \right\} \\
\geq & - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} h(2 - \cos 2\theta)^h-1 ||O||_2 \sum_{j=0}^{1} \frac{1}{||j||1} \frac{1}{||j||1} (\cos 2\theta)^2 (2 - \cos 2\theta)^h-1 ||O||_2 \\
& \left| \sum_{i=j+1}^{2h} \frac{1}{||j||1} \frac{1}{||j||1} (\cos 2\theta)||i'||1-h \text{Tr} \left[ O'_{i'} \rho \right] \right| \\
\geq & - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} h||O||_2 \sum_{j=0}^{1} \frac{2h}{j} \frac{2h}{j} (2t+1)(2t+1)(2t)^2 \left( 1 + \frac{2h}{2} \right)^2 \left( \frac{h}{2t+1} \right)^{2t+2t} \\
\geq & - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} h||O||_2 \sum_{t=1}^{\frac{h-1}{2}} \left( 2t+1 \right) \left( 2t+1 \right) \left( 2t \right)^2 \left( 1 + \frac{2h}{2} \right)^2 \left( \frac{h}{2t+1} \right)^{2t+2t} \\
\end{align*}
\]

Equation (106) is obtained by noticing that the expectation of \( \sin^a 2\theta \cos^b 2\theta \) equals to zero, if \( a \) is odd. Equation (107) follows from the derivation (86-88). Equation (108) is obtained by using the \( h' = h \), \( ||j||1 = 1 \) case of Eq. (43). Equation (109) follows from \( 1 \geq \cos 2\theta \geq 1 \) and \( (\sin 2\theta)^2 \leq (2\theta)^2 \). Equation (110) is obtained since the summation \( \sum_{j=0}^{1} \) contains \( \left( \frac{h}{2t+1} \right)^{2t} \) terms such that \( ||j||1 = 2t+1 \), for all \( t \in \{1, \cdots, \left\lfloor \frac{h}{2} \right\rfloor \} \). We further bound Eq. (110) by

\[
\begin{align*}
&= - \mathbb{E}_{\theta \sim \mathcal{N}(0, \gamma^2)} h||O||_2 \sum_{t=1}^{\frac{h-1}{2}} \left( \frac{h}{2t+1} \right) \left( 2t+1 \right) \left( 2t+1 \right) \left( 2t \right)^2 \left( 1 + \frac{2h}{2} \right)^2 \left( \frac{h}{2t+1} \right)^{2t+2t} \\
\geq & - h||O||_2 \sum_{t=1}^{\frac{h-1}{2}} \left( \frac{h}{2t+1} \right) \left( 2t+1 \right) \left( 2t+1 \right) \left( 2t \right)^2 \left( 1 + \frac{2h}{2} \right)^2 \left( \frac{h}{2t+1} \right)^{2t+2t} \\
\end{align*}
\]

31
Thus, we could obtain Eq. (4) if

\begin{align}
&\geq -h \|O\|_2^2 \sum_{t=1}^{\lfloor \frac{h-1}{2} \rfloor} h(h-1)(h-2)(h-3)^{2t-2} \frac{2^{2t}\gamma^{2t}}{(2t)!} \sum_{m=0}^{2h-2-2t} (2h-2-2t)^m 2^m (2t+2m-1)! \gamma^{2m} \\
&= -h^2 (h-1)(h-2) \|O\|_2^2 \sum_{t=1}^{\lfloor \frac{h-1}{2} \rfloor} \frac{(h-3)^{2t-2}}{2^t!(2t-1)!} 2^{2t}\gamma^{2t} \sum_{m=0}^{2h-2-2t} (2h-2-2t)^m 2^m (2t+2m-1)! \gamma^{2m} \\
&= -\frac{5}{2} h^2 (h-1)(h-2) \gamma^2 \|O\|_2^2.
\end{align}

Equation (111) is obtained by calculating expectation terms. Equation (112) is derived by using $t \geq 1$. Equation (113) follows from $(2t)! = 2^t!(2t-1)!$. Equation (114) is obtained by bounding the summation terms, i.e.

\begin{align}
&\sum_{t=1}^{\lfloor \frac{h-1}{2} \rfloor} \frac{(h-3)^{2t-2}}{2^t!(2t-1)!} 2^{2t}\gamma^{2t} \sum_{m=0}^{2h-2-2t} (2h-2-2t)^m 2^m (2t+2m-1)! \gamma^{2m} \\
&\leq \sum_{t=1}^{\lfloor \frac{h-1}{2} \rfloor} \frac{(h-3)^{2t-2}}{2^t-1} 2^t \gamma^{2t} \sum_{m=0}^{2h-2-2t} (2h-4)^m 2^m (2h-2)^m \gamma^{2m} \\
&\leq 2 \gamma^2 \sum_{t=1}^{\lfloor \frac{h-1}{2} \rfloor} [(h-3)^{2} \gamma^2]^{t-1} \sum_{m=0}^{2h-2-2t} (h^3 \gamma^2)^m \\
&\leq 2 \gamma^2 \left( \frac{16}{15} \right)^2 \leq \frac{5}{2} \gamma^2.
\end{align}

Equation (115) is obtained by using $t! \geq 2^{t-1}$, $\forall t \geq 1$ and

$$(2t+2k-1)(2t+2m-2k+1) \leq (m+2t)^2 \leq (2h-2)^2, \forall k \in \{1, \cdots, m\}.$$  

Equation (116) follows from $h^3 \geq 8(h-1)(h-2)$ for integer $h$. Equation (117) is obtained by calculating geometric sequences with the condition $\gamma^2 \leq \frac{1}{16h^2}$. Thus, we have proved Eq. (76).

\[\square\]

**B Proof of Theorem 4.1**

*Proof.* Denote by $I_S := \{m \mid m \neq 0, m \in [N]\}$ the set of qubits where the observable acts non-trivially. First, we notice that the norm of the whole gradient is lower bounded by that of particle derivatives summed over a part of parameters, i.e.

\begin{align}
&\mathbb{E}_\theta \|\nabla f\|^2 \geq \sum_{q=1}^{L} \sum_{n \in I_S} \mathbb{E}_\theta \left( \frac{\partial f}{\partial \theta_{q,n}} \right)^2.
\end{align}

Thus, we could obtain Eq. (4) if

\begin{align}
&\mathbb{E}_\theta \left( \frac{\partial f}{\partial \theta_{q,n}} \right)^2 \geq \frac{1}{S^{S+1}(L+2)^{S+1}} \text{Tr} [\sigma_j \rho_m]^2,
\end{align}

holds for any $q \in \{1, \cdots, L\}$ and $n \in I_S$.  

32
Now we begin to prove Eq. (119). Our main idea is to integrate the square of the partial derivative of $f$ with respect to $\theta = (\theta_1, \cdots, \theta_{L+2})$ by using Lemma A.1 and Lemma A.2.

We introduce several notations for convenience. Denote the variance $S_i$. We refer to Lemma A.1 for the $\rho$ is obtained by using the notation $p$ and $f$.

Now we begin to prove Eq. (119). Our main idea is to integrate the square of the partial derivative of $\theta$.

\begin{equation}
R_k(\theta_k) = e^{-i\theta_k G_{k+1}} \otimes e^{-i\theta_{k+2} G_{k+2}} \otimes \cdots \otimes e^{-i\theta_{k,N} G_{k,N}},
\end{equation}

and $G_{\ell,j}$ is the Hamiltonian corresponding to the parameter $\theta_{\ell,j}$. Denote by $\rho_k$ the state after the $k$-th layer, $\forall k \in \{0, 1, \cdots, 2L + 2\}$,

\begin{equation}
\rho_k := \begin{cases} 
\left( \prod_{i=k}^{k+1/2} CZ_i R_i(\theta_i) \right) \rho_{in} \left( \prod_{i=k+1/2}^{k+1} R_i(\theta_i) \right)^\dagger CZ_i \, & (k = 2\ell \leq 2L), \\
\rho_{k-1} R_{k+1/2} \left( \theta_{k+1/2} \right)^\dagger \, & (k = 2\ell + 1 \leq 2L + 1), \\
R_{k+2} \left( \theta_{L+2} \right) \rho_{k-1} R_{k+2} \left( \theta_{L+2} \right)^\dagger \, & (k = 2L + 2).
\end{cases}
\end{equation}

Thus, $\rho_k$ is parameterized by $\{\theta_1, \cdots, \theta_p\}$, where $p = \ell$ if $k = 2\ell \leq 2L$, $p = \ell + 1$ if $k = 2\ell + 1 \leq 2L + 1$, and $p = L + 2$ if $k = 2L + 2$.

Next, rewrite the formulation of Eq. (119) in detail:

\begin{align}
\mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{L+2}} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_i V(\theta) \rho_{in} V(\theta)^\dagger \right] \right)^2 \\
= \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{L+2}} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_i \rho_{2L+2} \right] \right)^2 \\
\geq \left[ 4\gamma^2 (1 - 4\gamma^2) \right] S_i \left( 1 - 4\gamma^2 \right) S_i+1 \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{L+1}} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} \rho_{2L+2} \right] \right)^2 \\
\geq \left[ 4\gamma^2 (1 - 4\gamma^2) \right] S_i \left( 1 - 4\gamma^2 \right) S_i+1 \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_L} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} \rho_{2L} \right] \right)^2 \\
\geq \left[ 4\gamma^2 (1 - 4\gamma^2) \right] S_i \left( 1 - 4\gamma^2 \right) S_i+1 \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_L} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} \rho_{2L} \right] \right)^2,
\end{align}

where $3|i$ denotes the index by replacing non-zero elements of $i = (i_1, \cdots, i_N)$ with 3 and $3|i; 1$ denotes the index by replacing non-zero elements of $i = (i_1, \cdots, i_N)$ with 3 if the original value is 1. We refer to $S_1$, $S_2$, and $S_3$ as the number of 1, 2, and 3 in the index $i$, respectively. Eq. (122) is obtained by using the notation $\rho_{2L+2}$ defined in (121). We obtain Eqs. (123) and (124) by using Lemma A.1 for the $R_Y$ and $R_X$ gate case, respectively. Equation (125) follows from $S = S_1 + S_2 + S_3$.

Then, we proceed from Eq. (125) and take the expectation for parameters in $(\theta_L, \cdots, \theta_{q+1})$.\n
\begin{align}
\text{Eq. (125)} &= \left[ 2\gamma (1 - 4\gamma^2) \right] 2^S \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_L} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} CZ_i R_L(\theta_L) \rho_{2L-2} R_L(\theta_L)^\dagger CZ_i^\dagger \right] \right)^2 \\
&= \left[ 2\gamma (1 - 4\gamma^2) \right] 2^S \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_L} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} R_L(\theta_L) \rho_{2L-2} R_L(\theta_L)^\dagger \right] \right)^2 \\
&\geq \left[ 2\gamma (1 - 4\gamma^2) \right] 2^S (1 - 4\gamma^2) S_{\theta_1} \cdots \mathbb{E}_{\theta_{L-1}} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} \rho_{2L-2} \right] \right)^2 \\
&\geq \left[ 2\gamma (1 - 4\gamma^2) \right] 2^S (1 - 4\gamma^2) (L-q)^S \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_q} \left( \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_{3|i} \rho_{2l} \right] \right)^2.
\end{align}
Equation (126) follows from the definition of $\rho_{2L}$ (121). Equation (127) is obtained since

$$\text{CZ}(\sigma_j \otimes \sigma_k)\text{CZ}^\dagger = \sigma_j \otimes \sigma_k, \forall j, k \in \{0, 3\}. $$

Equation (128) is derived by using the Lemma A.1. We repeat the derivation in Eqs. (126-128) inductively for parameters $(\theta_L, \cdots, \theta_{q+1})$, which yields Eq. (129).

Next, we consider the expectation with respect to $\theta_q$. We have

$$\text{Eq. (129)} = [2\gamma(1-4\gamma^2)]^{2S}(1-4\gamma^2)^{(L-q)S}\mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_q} \left[ \frac{\partial}{\partial \theta_{q,n}} \text{Tr} \left[ \sigma_3[i\rho_{2q-1}] \right] \right]^2 \geq [2\gamma(1-4\gamma^2)]^{2S}(1-4\gamma^2)^{(L-q)S}(1-4\gamma^2)^{S-1}[4\gamma^2(1-4\gamma^2)]^{4} \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{q-1}} \text{Tr}[\sigma_3[i\rho_{2q-2}]]^2, \quad (130)$$

where expectations with respect to parameters $\{\theta_{q,j}\}_{j \in \mathcal{I}_s, j \neq n}$ are calculated via Lemma A.1 and the expectation with respect to $\theta_{q,n}$ is calculated via Lemma A.2.

Finally we proceed from Eq. (130) and take the expectation for parameters in $(\theta_{q-1}, \cdots, \theta_1)$. We have

$$\mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{q-1}} \text{Tr}[\sigma_3[i\rho_{2q-2}]]^2 = \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{q-1}} \text{Tr}[\sigma_3[i\text{CZ}_{q-1} R_{q-1}(\theta_{q-1}) \rho_{2q-4} R_{q-1}(\theta_{q-1})^\dagger \text{CZ}_{q-1}^\dagger]]^2 \quad (131)$$

$$= \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{q-1}} \text{Tr}[\sigma_3[i R_{q-1}(\theta_{q-1}) \rho_{2q-4} R_{q-1}(\theta_{q-1})^\dagger]]^2 \quad (132)$$

$$\geq (1-4\gamma^2)^S \mathbb{E}_{\theta_{q-2}} \text{Tr}[\sigma_3[i\rho_{2q-4}]]^2 \quad (133)$$

$$\geq (1-4\gamma^2)^{(q-1)S} \text{Tr}[\sigma_3[i\rho_0]]^2. \quad (134)$$

Eq. (131) is derived by using the definition of $\rho_{2q-2}$. Eq. (132) is obtained since

$$\text{CZ}(\sigma_j \otimes \sigma_k)\text{CZ}^\dagger = \sigma_j \otimes \sigma_k, \forall j, k \in \{0, 3\}. $$

Eq. (133) is derived by using Lemma A.1. We repeat the derivation in Eqs. (131-133) inductively for parameters $(\theta_{q-1}, \cdots, \theta_1)$, which yields Eq. (134). Employing Eq. (134) to Eq. (130) yields

$$\text{Eq. (129)} \geq 4(4\gamma^2)^{S+1}(1-4\gamma^2)^{S(L+2)} \text{Tr} [\sigma_3[i\rho_0]]^2 \quad (135)$$

$$= 4 \left( \frac{1}{S(L+2)} \right)^{S+1} \left( 1 - \frac{1}{S(L+2)} \right)^{S(L+2)} \text{Tr} [\sigma_3[i\rho_0]]^2 \quad (136)$$

$$\geq 4 \left( \frac{1}{S(L+2)} \right)^{S+1} \left( 1 - \frac{1}{2} \right)^2 \text{Tr} [\sigma_3[i\rho_0]]^2 \quad (137)$$

$$= \frac{1}{S^{S+1}(L+2)^{(S+1)}} \text{Tr} [\sigma_3[i\rho_0]]^2. \quad (138)$$

Eq. (136) is derived by using the condition $\gamma^2 = \frac{1}{4S(L+2)}$. Eq. (137) is obtained by noticing that function $g(x) = (1 - \frac{1}{x})^x$ is monotonically increasing when $x \geq 2$. Thus, we have proved Eq. (119).
C Proof of Theorem 4.2

Proof. To begin with, we define several notations for convenience. Denote by $\rho_j$ the state after the $j$-th parameterized operator, i.e.

$$
\rho_j(\theta_1, \cdots, \theta_j) = \left( \prod_{i=j}^{1} V_i(\theta_i) \right) \rho_{\text{in}} \left( \prod_{i=1}^{j} V_i(\theta_i)^\dagger \right).
$$

(139)

Denote by $O_j$ the observable, i.e.

$$
O_j = V_j(0)^\dagger \cdots V_L(0)^\dagger OV_L(0) \cdots V_j(0), \ \forall j \in \{1, \cdots, L\}.
$$

(140)

Now we begin to prove Eq. (7). First we remark that $\forall j \in [L]$, the $a_j \neq 1$ case can be converted to the $a_j = 1$ case by using the transformation

$$
\theta_j' = \frac{\theta_j}{a_j},
$$

where the variance of the new and the old parameter satisfies

$$
\operatorname{Var}[\theta_j'] = \frac{1}{a_j^2} \operatorname{Var}[\theta_j].
$$

In the following proof, we assume that $a_j = 1$, $\forall j \in [L]$. By using the parameter-shift rule, $\frac{\partial f}{\partial \theta_j}$ could be written as the linear sum of $2h$ expectations on the observable $O$ with coefficients $\pm 1$. Then for the case $\ell \leq L - 1$, we have

$$
\mathbb{E} \left( \frac{\partial f}{\partial \theta_{\ell}} \right)^2 = \mathbb{E} \cdots \mathbb{E}_{\theta_1} \left( \frac{\partial}{\partial \theta_{\ell}} \operatorname{Tr} \left[ OV_L(\theta_L) \rho_{\ell-1} V_L(\theta_L)^\dagger \right] \right)^2
$$

(141)

$$
\geq \mathbb{E} \cdots \mathbb{E}_{\theta_1} \left( \frac{\partial}{\partial \theta_{\ell}} \operatorname{Tr} \left[ OV(0) \rho_{\ell-1} V(0)^\dagger \right] \right)^2 - |12h_L(h_L - 1) + 4] 4h^2_L \gamma^2 \|O\|_2^2
$$

(142)

$$
= \mathbb{E} \cdots \mathbb{E}_{\theta_1} \left( \frac{\partial}{\partial \theta_{\ell}} \operatorname{Tr} [O\rho_{\ell-1}] \right)^2 - |12h_L(h_L - 1) + 4] 4h^2_L \gamma^2 \|O\|_2^2
$$

(143)

$$
= \mathbb{E} \cdots \mathbb{E}_{\theta_1} \left( \frac{\partial}{\partial \theta_{\ell}} (\theta_1, \cdots, \theta_{\ell-1}, 0) \right)^2 - |12h_L(h_L - 1) + 4] 4h^2_L \gamma^2 \|O\|_2^2
$$

(144)

where Eq. (141) follows from the definition of $\rho_j$ in Eq. (139). Equation (142) is obtained by using Lemma A.3, where $\|c\|_1 = 2h$. Equation (143) follows from the definition of $O_j$ in Eq. (140). Equation (144) follows from the formulation $f(\theta) = \operatorname{Tr}[O\rho(\theta)]$. By proceeding the derivation (141-144) for $L - \ell$ times, we have

$$
\mathbb{E} \left( \frac{\partial f}{\partial \theta_{\ell}} \right)^2 \geq \mathbb{E} \cdots \mathbb{E}_{\theta_1} \left( \frac{\partial f}{\partial \theta_{\ell}} (\theta_1, \cdots, \theta_{\ell}, 0, \cdots, 0) \right)^2 - \sum_{j=\ell+1}^{L} [12h_j(h_j - 1) + 4] 4h^2_j \gamma^2 \|O\|_2^2
$$

(145)
\[
\begin{align*}
\geq & \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{\ell-1}} (1 - 4\gamma_\ell^2) \left( \frac{\partial}{\partial \theta_\ell} \text{Tr} \left[ O_{\ell+1} V_\ell (\theta_\ell) \rho_{\ell-1} V_\ell (\theta_\ell) \right] \right)^2 \bigg|_{\theta_\ell=0} - 96h_\ell^2 (h_\ell - 1) \gamma_\ell^2 \|O\|^2_2 \\
- & 20h_\ell^2 (h_\ell - 1)(h_\ell - 2)\gamma_\ell^2 \|O\|^2_2 - \sum_{j=\ell+1}^L [12h_j (h_j - 1) + 4] 4h_j^2 \gamma_j^2 \|O\|^2_2 \\
\geq & \mathbb{E}_{\theta_1} \cdots \mathbb{E}_{\theta_{\ell-1}} \left( \frac{\partial f}{\partial \theta_\ell} (\theta_1, \cdots, \theta_{\ell-1}, 0, 0, \cdots, 0) \right)^2 - 4\gamma_\ell^2 (2h_\ell)^2 \|O\|^2_2 - 96h_\ell^2 (h_\ell - 1) \gamma_\ell^2 \|O\|^2_2 \\
- & 20h_\ell^2 (h_\ell - 1)(h_\ell - 2)\gamma_\ell^2 \|O\|^2_2 - \sum_{j=\ell+1}^L [12h_j (h_j - 1) + 4] 4h_j^2 \gamma_j^2 \|O\|^2_2,
\end{align*}
\] (146)

where Eq. (145) follows from definitions \(\rho_j\) (139) and \(O_j\) (140). Equation (146) is derived by using Lemma A.4. Equation (147) follows from the parameter-shift rule. We proceed from Eq. (147) by employing the derivation (141-144) for parameters \((\theta_{\ell-1}, \cdots, \theta_1)\), which yields

\[
\begin{align*}
\mathbb{E} \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 & \geq \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \bigg|_0 - \sum_{j=1}^{\ell-1} 16h_j^2 [3h_j (h_j - 1) + 1] \gamma_j^2 \|O\|^2_2 - \sum_{j=\ell+1}^L 16h_j^2 [3h_j (h_j - 1) + 1] \gamma_j^2 \|O\|^2_2 \\
- & 16h_\ell^2 \gamma_\ell^2 \|O\|^2_2 - 96h_\ell^2 (h_\ell - 1) \gamma_\ell^2 \|O\|^2_2 - 20h_\ell^2 (h_\ell - 1)(h_\ell - 2)\gamma_\ell^2 \|O\|^2_2 \\
\geq & \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \bigg|_0 - \sum_{j=1}^L 16h_j^2 [3h_j (h_j - 1) + 1] \gamma_j^2 \|O\|^2_2 \\
\geq & (1 - \epsilon) \left( \frac{\partial f}{\partial \theta_\ell} \right)^2 \bigg|_0.
\end{align*}
\] (149)

Equation (148) is obtained by using Lemma A.3, where \(\|c\|_1 = 2h\). Equation (149) follows from the condition \(\gamma_j^2 \leq \frac{a_j^2 \epsilon}{16h_j^2 (3h_j (h_j - 1) + 1) \|O\|^2_2} \left( \frac{\partial f}{\partial \theta_j} \right)^2 \bigg|_{\theta=0} \) and \(a_j = 1, \forall j \in \{L\}\). Thus, we have proved Eq. (7).