A UNIVERSAL TORELLI THEOREM
FOR ELLIPTIC SURFACES

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Abstract. Given two semistable elliptic surfaces over a curve $C$ defined over a field of characteristic zero or finitely generated over its prime field, we show that any compatible family of effective isometries of the Néron-Severi lattices of the base changed elliptic surfaces for all finite separable maps $B \to C$ arises from an isomorphism of the elliptic surfaces. Without the effectivity hypothesis, we show that the two elliptic surfaces are isomorphic.

We also determine the group of universal automorphisms of a semistable elliptic surface. In particular, this includes showing that the Picard-Lefschetz transformations corresponding to an irreducible component of a singular fibre, can be extended as universal isometries. In the process, we get a family of homomorphisms of the affine Weyl group associated to $\tilde{A}_{n-1}$ to that of $\tilde{A}_{dn-1}$, indexed by natural numbers $d$, which are closed under composition.

1. Introduction

Let $X$ be a compact, connected, oriented Kähler manifold of dimension $d$ with an integral Kähler form $\omega \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$. The intersection product induces a graded algebra structure on the cohomology algebra

$$H(X) = \bigoplus_{i=0}^{2d} H^i(X, \mathbb{Z}),$$

where $H^i(X, \mathbb{Z})$ are the singular cohomology groups of $X$. Hodge theory provides a filtration of the complex cohomology groups $H^i(X, \mathbb{C})$. The Kähler form induces a polarization of the Hodge structure. The classical Torelli question is whether the space $X$ can be recovered from the polarized Hodge structure of the cohomology algebra equipped with the intersection product.

When $X$ is a compact, connected Riemann surface the Torelli question has an affirmative answer: the Riemann surface is determined by its associated polarized Hodge structure.

Now, let $\pi : X \to C$ be an elliptic surface over a smooth, projective curve $C$ over $\mathbb{C}$. Different aspects of the Torelli problem have been well studied for elliptic surfaces. It is known, for instance, that Torelli does not hold for elliptic surfaces (see Example 2.5).

One of the problems that arises with the Torelli question for elliptic surfaces is that the Néron-Severi group of the surface $X$ is not sufficiently large enough to distinguish

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between the surfaces. To rectify this problem, we argue in analogy with Tate’s isogeny conjecture, or with Grothendieck’s use of base changes. This leads us to base change the elliptic surfaces by finite maps $B \to C$, so that the Néron-Severi group of the base changed elliptic surfaces becomes larger.

We can consider all base changes of the elliptic surfaces by finite separable morphisms of the base curve, and a family of compatible isometries between the Néron-Severi groups of the base changed surfaces. For the existence of compatible isometries, we need to work with semistable elliptic surfaces.

The resulting object will carry an action of the absolute Galois group of the generic point of $C$, and the isometries will have to be equivariant with respect to the action of the Galois group. Considering the whole family of Néron-Severi lattices carries the risk that the collection of effective isometries of the Néron-Severi lattices can become larger corresponding to the growth of the Néron-Severi group. Miraculously, this does not happen. Working over a field $k$, which is of characteristic zero or finitely generated over its prime field, we show that compatible, effective isometries of the Néron-Severi lattices of the base changes for all base changes of the base curve, arises from an isomorphism of the elliptic surfaces. The introduction of the effectivity hypothesis is critical, enlarging the scope of the theorem, making it more natural and compelling, and follows the use of the effectivity hypothesis for the Torelli theorem for $K3$-surfaces proved by Piatetskii-Shapiro and Shafarevich.

We next consider describing the group of universal isometries of a semistable elliptic surface dropping the effectivity assumption on the isometries. We observe that the Picard-Lefschetz transformations based on the irreducible components of the singular fibres can be extended to give compatible, isometries of the Néron-Severi lattices of the base changes for all base changes of the base curve. The Picard-Lefschetz reflections of a singular fibre of type $I_n$ generates the affine Weyl group of $\tilde{A}_{n-1}$. The universality of Picard-Lefschetz reflections defines a family of representations of the affine Weyl group $\tilde{A}_{n-1}$ to $\tilde{A}_{d(n-1)}$ for any natural number $d$ which are closed under composition. Such maps correspond to the base change by a map of degree $d$ of the base curve, totally ramified of degree $d$ at the point corresponding to the singular fibre. The study of these representations allow us to determine the group of universal automorphisms of the Néron-Severi lattices attached to base changes of a semistable elliptic surface.

1.0.1. Outline of the paper. We now give an outline of the paper. In section 2, we introduce the notion of universal Néron-Severi groups and state the main theorems. In section 3, using base changes, we first show that universal isometries preserve fibres. We then show that univeral isometry determines the Kodaira-Néron type of the singular fibres. In section 4, appealing to theorems proving the Tate isogeny conjecture, we establish that the generic fibres are isogenous. Using the extra information coming from our hypothesis, it is shown in section 5 that the elliptic surfaces are indeed isomorphic.
Using the effectivity hypothesis, one concludes that sections are mapped to sections and the irreducible components of singular fibres are preserved by the universal isometry (see section 6). The fact that the elliptic surfaces are isomorphic allows us to compose the universal isometry with itself. Making use of the fact that torsion elements are determined by their intersection with the components of singular fibres, allows us to conclude in section 7 that the action of the universal isometry on torsion and the fibral divisors is geometric. The final proof of the effective universal Torelli theorem carried out in section 8 rests on the use of a Galois theoretic argument together with the geometry and arithmetic around the narrow Mordell-Weil group of the generic fibre.

The second half of the paper studies the representation theoretic and geometric aspects of the Picard-Lefschetz reflections based on the irreducible components of the singular fibres. We first determine the base change map on fibral divisors in section 9. This description allows us to arrive in an inductive manner the definition of the lifts of Picard-Lefschetz reflections to universal isometries. However it is more convenient to represent these reflections in terms of usual permutation notation, allowing us to come up with an alternate definition of the universal Picard-Lefschetz isometries. These facts and the representation theoretic aspects of the affine Weyl group of type $\tilde{A}$ that arise are studied in section 10.

In section 11 we show that the lifts of Picard-Lefschetz reflections we have defined indeed define isometries of the universal Néron-Severi group. In the last section 12 we determine the group of isometries of the universal Néron-Severi group of a semistable elliptic surface.

2. Elliptic surfaces and the main theorems

For a variety $Z$ defined over $k$, let $\bar{Z} = Z \times_k \bar{k}$, where $\bar{k}$ is a fixed separable closure of $k$. If $f : Z \to Y$ is a morphism of schemes defined over $k$, let $\bar{f} : \bar{Z} \to \bar{Y}$ denote the base change of the morphism $f$ to $\bar{k}$. The structure sheaf of a variety $X$ is denoted by $\mathcal{O}_X$.

Let $C$ be a connected, smooth, projective curve over a field $k$. An elliptic surface $\mathcal{E} : X \to C$ is a non-singular, projective surface $X$ defined over $k$ together with a surjective morphism $\pi : X \to C$ such that the following conditions are satisfied:

- The generic fibre $E$ over the function field $K = k(C)$ of $C$ is a smooth, irreducible curve of genus 1.
- The map $\pi : X \to C$ has a section.
- The curve $C$ has a $k$-rational point.
- $\bar{\pi}$ is relatively minimal, i.e., there is no irreducible, rational curve $D$ on $\bar{X}$ with self-intersection $D^2 = -1$ contained in a fibre of $\bar{\pi}$.
- The $j$-invariant of the generic fibre is not algebraic over $k$. Equivalently the elliptic surface is not potentially iso-trivial.
We refer to the excellent surveys ([M, SS] and [Si2, Chapter III]) for information about elliptic surfaces (primarily over algebraically closed ground fields).

2.1. Base change. Let $\mathcal{B}_C$ be the collection of triples $(l, B, b)$ consisting of the following

- $l$ is a finite separable extension of $k$ contained inside $\bar{k}$.
- $B$ is a geometrically integral, regular projective curve defined over $l$.
- $b : B \to C \times_k \text{Spec}(l)$ is a finite, separable morphism.

When the situation is clear, we drop the use of the subscript $C$, and also simply refer the morphism $b \in \mathcal{B}$.

For $b \in \mathcal{B}_C$, let $X_b$ be the relatively minimal regular model in the birational equivalence class of the base change surface $X^b := (X \times_k \text{Spec}(l)) \times_{C \times_k \text{Spec}(l)} B \to B$. The elliptic surface $\mathcal{E}_b : X_b \to B$ can be considered as the unique relatively minimal regular elliptic surface over $B$ with generic fibre the curve $E$ considered as an elliptic curve over $l(B)$.

2.2. Semistable elliptic surfaces. For a place $t$ of $k(C)$, let $\mathcal{O}_{C,t}$ denote the local ring at $t$. The elliptic surface defines an elliptic curve $E_t$ over $\mathcal{O}_{C,t}$. Define an elliptic surface $E$ to have semistable reduction at $t$, if the elliptic curve $E_t$ has either good or split multiplicative reduction modulo the maximal ideal in $\mathcal{O}_{C,t}$.

This amounts to saying that the fibre at $t$, is either an elliptic curve, or is of type $I_n$, $n \geq 1$ in the Kodaira-Néron classification of singular fibres. At a place having singular reduction of type $I_n$, $n \geq 3$, the special fibre $X_t$ of $X$ at $t$, is a reduced cycle consisting of $n$ smooth, rational curves, with self-intersection $-2$, and each curve intersecting its neighbours with multiplicity one.

Define an elliptic surface $\mathcal{E} : X \to C$ to be semistable, if it has semistable reduction at all places $v$ of $C$.

Notation. The ramification locus of $\pi$ is usually denoted by $S$. For $t \in S \subset C(k)$, the Kodaira-Néron type of the singular fibre is denoted by $I_{n_t}$. The irreducible components of the fibre at $t$ are denoted by $v^t_i$, $i \in \mathbb{Z}/n_t \mathbb{Z}$, where the component $v^0_0$ is the component intersecting the zero section. If $n_t \geq 3$, then

$$(v^t_i)^2 = -2, \quad v^t_i v^t_{i+1} = 1 \quad \text{for} \quad i \in \mathbb{Z}/n_t \mathbb{Z}.$$ 

At times the superscript $t$ is dropped. When base changes are involved, $w$ is used instead of $v$ to denote the components of the base changed surface.

The following well known theorems ensuring the existence of semistable base change and properties of semistable surfaces under further base change are crucial to the formulation and proof of the results of this paper:

Theorem 1. (i) Given an elliptic surface $X \to C$, there is a triple $(l, B, b) \in \mathcal{B}$, such that the base changed surface $\mathcal{E}_b : X_b \to B$ is semistable.
(ii) Suppose $X \to C$ is a semistable elliptic surface, and $(l, B, b) \in \mathcal{B}$. Then $X_b$ is the minimal desingularization of $X^b := (X \times_k \text{Spec}(l)) \times_C \text{Spec}(l)$ $B$. The surface $\mathcal{E}_b : X_b \to B$ is semistable and there is a finite, proper map $p_b : X_b \to X$, compatible with the map $b : B \to C$.

For the proof see ([Liu, Chapter 10]). We make the following observation about Part (ii). Let $b \in \mathcal{B}$, and $w$ be a place of $B$. Suppose $t'$ maps to $t$, and the local ring $\mathcal{O}_{B, t'}$ has ramification degree $d$ over $\mathcal{O}_{C, t}$. The base changed surface $X^b$ is normal with $A_d$-singularities at the points on the special fibre at $t'$ which maps to the singular points of the fibre of $X$ at $t$. The completed local ring at this singularity is of the form $\mathcal{O}_v[[x, y]]/(xy - z^d)$, and the singularity is resolved with $[d/2]$-blowups ([Liu, Chapter 10, Lemma 3.21], [HN, Section 2.1.7]). The surface $X_b$, the minimal regular model, is the minimal desingularization of the base changed surface $X^b$. In particular, this yields a morphism $X_b \to X$, compatible with the map $b : B \to C$.

Suppose $L$ is Galois over $K = k(C)$. The Galois group $G(L/K)$ sits inside a short exact sequence of the form,

$$1 \to G(L/lK) \to G(L/K) \to G(l/k) \to 1,$$

where the Galois group $G(L/lK)$ can be identified with the automorphism group $\text{Aut}(B/C \times_k \text{Spec}(l))$. Since $X_b$ is the minimal desingularization of $X^b$, the action of $G(L/K)$ on $X^b$ extends to yield an action of $G(L/K)$ on $X_b$.

2.3. Néron-Severi group. The Néron-Severi group of $\bar{X}$, is the group of divisors on $\bar{X}$ taken modulo algebraic equivalence. The Néron-Severi group $NS(X)$ of $X$ is defined to be the image of $\text{Pic}_{X/k}(k)$ in $NS(\bar{X})$, where $\text{Pic}_{X/k}$ is the Picard group scheme of $X$ over $k$. The intersection product of divisors on $\bar{X}$ induces a bilinear pairing $<, >$ on $NS(\bar{X})$, and hence on $NS(X)$. The intersection product of two divisors $D_1, D_2$ will be denoted by $< D_1, D_2 >$ or just $D_1.D_2$.

Suppose $X \to C$ is a semistable elliptic surface, and $(l, B, b) \in \mathcal{B}$. By Part (ii) of Theorem [1] we obtain a well-defined pullback map $p_b^* : NS(X) \to NS(X_b)$ of the Néron-Severi lattices, satisfying

$$< p_b^* x, p_b^* y > = \deg(b) < x, y > \quad x, y \in NS(X),$$

where $\deg(b)$ is the geometric degree of the maps $b$ and $p_b$.

2.4. Universal Néron-Severi group. We consider the Néron-Severi group of an elliptic surface $X \to C$, functorially with regard to arbitrary base changes given by finite, separable maps $B \to C$. The aim is to show that an effective natural transformation between two such Néron-Severi functors, arises from an isomorphism of the elliptic surfaces.

Definition 2.1. Let $\mathcal{E} : X \to C$ be a semistable elliptic surface defined over a field $k$. Define the universal Néron-Severi group $UNS(\mathcal{E})$ of $\mathcal{E}$ to be the collection of $(NS(X_b), <, >)$, where $b \in \mathcal{B}_C$, equipped with the pull back maps $p_a^* : NS(X_b) \to NS(X_{boa})$,.
for any pair of finite morphisms \( A \xrightarrow{a} B \xrightarrow{b} C, \ a \in B_B \).

**Definition 2.2.** A (universal) isometry \( \Phi : UNS(\mathcal{E}) \to UNS(\mathcal{E}') \) between universal Néron-Severi groups of two semistable elliptic surfaces \( \mathcal{E} : X \xrightarrow{\pi} C \) and \( \mathcal{E}' : X' \xrightarrow{\pi'} C \) is defined to be a collection of isometries 
\[
\phi_b : NS(X_b) \to NS(X'_b),
\]
indexed by \( b \in B_C \), such that for any sequence of finite maps \( A \xrightarrow{a} B \xrightarrow{b} C \) with \( b \circ a \in B \), \( \phi_{b \circ a} \circ p_a^* = p_a^* \circ \phi_b \), i.e., the following diagram is commutative:
\[
\begin{array}{ccc}
NS(X_{b\circ a}) & \xrightarrow{\phi_{b\circ a}} & NS(X_{b\circ a}) \\
p_a^* & & p_a^* \\
| & & | \\
| & & |
\end{array}
\]
\[
\begin{array}{ccc}
NS(X_b) & \xrightarrow{\phi_b} & NS(X_b) \\
\end{array}
\]

We also denote \( \phi_b \) simply by \( \Phi \) if the context is clear.

2.4.1. **Effective isometries.** We recall that a prime divisor on \( X \) ([H. Chapter II, Section 6]) is a closed integral subscheme of codimension one. A divisor \( D \) on \( X \) is said to be effective, if it can be written as a finite, non-negative integral linear combination of prime divisors. Given two elliptic surfaces \( X, X' \) over \( C \), a map \( \phi : NS(X) \to NS(X') \) is said to be effective, if it takes the class of an effective divisor on \( X \) to the class of an effective divisor on \( X' \).

A (universal) isometry \( \Phi \) of universal Néron-Severi groups of two semistable elliptic surfaces is said to be effective if for any \( b \in B_C \), the isometry \( \phi_b \) is effective, i.e., it takes the cone of effective divisors in \( NS(X_b) \) to the cone of effective divisors in \( NS(X'_b) \).

2.5. **An effective universal Torelli theorem.** Suppose \( \mathcal{E} : X \xrightarrow{\pi} C \) and \( \mathcal{E}' : X' \xrightarrow{\pi'} C \) are two semistable elliptic surfaces. By an isomorphism \( \theta : \mathcal{E} \to \mathcal{E}' \) of the elliptic surfaces, we mean an isomorphism \( \theta : X \to X' \) compatible with the projections, i.e., \( \pi' \circ \theta = \pi \). It is clear that for any \( b \in B_C \), \( \theta \) induces an effective isometry
\[
\theta_b : NS(X_b) \to NS(X'_b).
\]

Then \( \theta \) induces a universal effective isometry \( \Theta : UNS(\mathcal{E}) \to UNS(\mathcal{E}') \).

Our main theorem is the converse, that an effective Torelli holds in totality considering all base changes for elliptic surfaces:

**Theorem 2.** Let \( k \) be a field of characteristic zero or finitely generated over its prime field, and \( \mathcal{E} : X \xrightarrow{\pi} C, \ \mathcal{E}' : X' \xrightarrow{\pi'} C \) be semistable elliptic surfaces over \( k \).

Suppose \( \Phi : UNS(\mathcal{E}) \to UNS(\mathcal{E}') \) is an effective universal isometry between universal Néron-Severi datum attached to \( \mathcal{E} \) and \( \mathcal{E}' \) as defined above.

Then \( \Phi \) arises from an unique isomorphism \( \theta : \mathcal{E} \to \mathcal{E}' \) between the elliptic surfaces.
The isomorphism $\theta$ is defined over $k$ if $k$ is finitely generated over its prime field, and over a quadratic extension $l$ of $k$ in case $k$ is an arbitrary field of characteristic zero. If we assume further that the Kodaira types of the singular fibres of $X \to C$ are of type $I_n$ with $n \geq 3$, then the isomorphism $\theta$ can be defined over $k$.

This result is the analogue of the refined Torelli theorem for $K3$ surfaces ([BHPV, Theorem 11.1]), with the additional assumptions involving base changes.

**Example.** Suppose $X_s$ is a family of non-isotrivial elliptic surfaces over a curve $C$ parametrized by a irreducible variety $S$. We assume that the ground field $k$ is algebraically closed. Let $\eta$ be the generic point of $S$. For a general point $s \in S(k)$, i.e., outside of a countable union of proper closed subvarieties of $S$, the specialization map $i_s : NS(X_\eta) \to NS(X_s)$ is an isomorphism ([MP, Proposition 3.6]). By the continuity theorem for intersection products ([Fu, Theorem 10.2]), it follows that the specialization map $i_s$ is an isometry. This gives examples of non-isomorphic elliptic surfaces whose Néron-Severi groups are isometric.

Thus, in order to obtain Torelli type results, it is necessary to bring in extra inputs: for example, transcendental inputs like Hodge theory, or some kind of Galois or universal invariance like we do out here.

2.6. **A (non-effective) universal Torelli theorem.** We now give an analogue of the (weak) Torelli theorem for $K3$ surfaces ([BHPV, Corollary 11.2]), where we do not assume that the map $\Phi$ is effective, but with the additional assumptions involving base changes.

**Theorem 3.** Let $k$ be a field of characteristic zero or finitely generated over its prime field, and $E : X \to C$, $E' : X' \to C$ be semistable elliptic surfaces over $k$.

Suppose $\Phi : UNS(E) \to UNS(E')$ is an universal isometry between universal Néron-Severi datum attached to $E$ and $E'$ as defined above.

Then the surfaces $E$ and $E'$ are isomorphic. The isomorphism $\theta$ is defined over $k$ if $k$ is finitely generated over its prime field, and over a quadratic extension $l$ of $k$ in case $k$ is an arbitrary field of characteristic zero. If we assume further that the Kodaira types of the singular fibres of $X \to C$ are of type $I_n$ with $n \geq 3$, then the isomorphism $\theta$ can be defined over $k$.

**Remark.** H. Kisilevsky pointed out the relevance of these theorems to a conjecture of Y. Zarhin ([K]): suppose $E, E'$ are elliptic curves defined over a number field $k$: if the ranks of the Mordell-Weil groups of $E$ and $E'$ are equal over all finite extensions of $k$, are $E$ and $E'$ isogenous?
It would be interesting to know whether analogues of our theorems hold for elliptic curves defined over number fields.

Remark. It is possible to drop the semistability hypothesis, but instead require that an isometry of the universal Néron-Severi groups exists whenever both the elliptic surfaces acquire semistable reduction:

Theorem 4. Let \( k \) be a field of characteristic zero or finitely generated over its prime field, and \( \mathcal{E} : X \rightarrow C, \mathcal{E}' : X' \rightarrow C \) be elliptic surfaces over \( k \).

Suppose \( \Phi : \text{UNS}(\mathcal{E}) \rightarrow \text{UNS}(\mathcal{E}') \) is an universal isometry between universal Néron-Severi datum attached to \( \mathcal{E} \) and \( \mathcal{E}' \) as defined above.

Then the surfaces \( \mathcal{E} \) and \( \mathcal{E}' \) are isomorphic over \( k \) over a quadratic extension \( l \) of \( k \).

The theorem follows from Theorem 3, since the property of having semistable reduction is local. One can produce different curves \( B \rightarrow C \), whose function fields are disjoint and over which the elliptic surfaces become semistable. By descent, the isomorphism \( \theta \) defined over the various curves \( B \) will descend to an isomorphism between the two elliptic surfaces defined over \( C \).

2.7. A reformulation. Let \( \bar{K} \) denote a separable algebraic closure of \( K = k(C) \) containing \( \bar{k}_s \). Suppose \( L \) is a finite extension of \( K \) contained in \( \bar{K} \). Then \( L \) is of the form \( l(B) \), where \( l \) is the closure of \( k \) inside \( L \), and \( B \) is a geometrically integral, regular projective curve defined over \( l \). There is a bijective order reversing correspondence between finite extensions \( L \) of \( K \) contained in \( \bar{K} \) and finite, separable maps \( b : B \rightarrow C \times_k \text{Spec}(l) \), where \( B \) is an integral, normal projective curve defined over \( l \).

Suppose \( L/K \) is Galois and \( B \) is regular. The Galois group \( G(L/K) \) acts on \( \text{NS}(X_b) \). Assume now that \( X \rightarrow C \) is semistable. Consider the direct limit,

\[
\text{NS}(X)/C = \lim_{\rightarrow b \in B_C} \text{NS}(X_b).
\]

This acquires an action of the absolute Galois group \( G(\bar{K}_s/K) \) of \( K \). Given a sequence of finite maps \( A \xrightarrow{a} B \xrightarrow{b} C \) with \( b \circ a \in B \), we have \( \deg(b \circ a) = \deg(b)\deg(a) \). Define a normalized bilinear pairing on \( \text{NS}(X_b) \), by

\[
< x, y >_n = \frac{1}{\deg(b)} < x, y > \quad x, y \in \text{NS}(X_b).
\]

With this modified inner product, the map \( p_b^* \) gives an isometry of \( \text{NS}(X) \) into \( \text{NS}(X_b) \). Hence, the normalized inner products on \( \text{NS}(X_b) \), can be patched to give a symmetric, bilinear \( \mathbb{Q} \)-valued pairing, \( < \cdot, \cdot > : \text{NS}(X) \times \text{NS}(X) \rightarrow \mathbb{Q} \). Since \( p_b^* \) maps effective divisors to effective divisors, the cone of effective divisors in \( \text{NS}(X) \) can be defined, and it makes sense to define effective morphisms between the completed Néron-Severi groups. Theorem 2 can be reformulated as:
Theorem 5. Let $k$ be a field of characteristic zero or finitely generated over its prime field, and $\mathcal{E} : X \twoheadrightarrow C$, $\mathcal{E}' : X' \twoheadrightarrow C$ be semistable elliptic surfaces over $k$.

Suppose $\Phi : \overline{\text{NS}(X)/C} \rightarrow \overline{\text{NS}(X')/C}$ is an effective, $G(\bar{K}/k(C))$-equivariant isometry.

Then $\Phi$ arises from an isomorphism $\theta : \mathcal{E} \rightarrow \mathcal{E}'$ between the elliptic surfaces. The isomorphism $\theta$ is defined over $k$ if $k$ is finitely generated over its prime field, and over a quadratic extension $l$ of $k$ in case $k$ is an arbitrary field of characteristic zero. If we assume further that the Kodaira types of the singular fibre of $X \rightarrow C$ are of type $I_n$ with $n \geq 3$, then the isomorphism $\theta$ can be defined over $k$.

A similar reformulation can be given for Theorem 3.

2.8. Universal lifts of Picard-Lefschetz isometries. The question of finding examples of non-effective universal isometries, leads one to study Picard-Lefschetz reflections. Given an element $v \in \text{NS}(X)$ with $v^2 = -2$, the Picard-Lefschetz reflection $s_v$ based at $v$ is defined as,

$$s_v(D) = D + <D, v>v, \quad D \in \text{NS}(X).$$

The Picard-Lefschetz reflection $s_v$ is a reflection around the hyperplane orthogonal to $v$:

$$s_v^2 = Id \quad \text{and} \quad s_v(v) = -v.$$ 

The following theorem shows that in the semistable case, the Picard-Lefschetz reflections can be lifted to universal isometries:

Theorem 6. Let $\mathcal{E} : X \twoheadrightarrow C$ be a semistable elliptic surface over an field $k$. Suppose $x_0 \in C(k)$ is an element of the singular locus $S$ and that the fibre of $p$ over $x_0$ is of type $I_n$ for some $n \geq 3$. Let $v$ be an irreducible component of the fibre $p^{-1}(x_0)$. Then there exists a universal isometry $\Phi(v) : \text{UNS}(\mathcal{E}) \rightarrow \text{UNS}(\mathcal{E})$, lifting the Picard-Lefschetz reflections $s_v$:

$$\Phi(v) |_{\text{NS}(X)} = s_v.$$ 

2.9. Group of universal isometries. The proof of Theorem 6 allows us to determine the structure of the group of universal isometries of a semistable elliptic surface.

We recall that the affine Weyl group of type $\tilde{A}_{n-1}$, denoted here by $W_n$, is the group with the presentation,

$$\langle s_0, \cdots, s_{n-1} | (s_is_j)^{m_{ij}} = 1 \rangle,$$ (2.3)

where

$$m_{ii} = 1, \quad m_{i(i+1)} = 3 \quad \text{and} \quad m_{ij} = 2 \quad \text{for} \quad |i - j| \geq 2.$$ (2.4)

Here we are using the notation for a cyclic group of order $n$, and the obvious meaning for $|i - j|$.

Restricted to a fibre of Kodaira-Néron type $I_n$, the Picard-Lefschetz reflections based on the irreducible components of the fibre generates the affine Weyl group.
of type \( \tilde{A}_{n-1} \). Suppose that the irreducible components of the singular fibre are \( v_0, v_1, \cdots, v_{n-1}, v_n = v_0 \). The map \( s_{v_i} \mapsto s_i \), yields an identification of the group generated by the Picard-Lefschetz reflections to the affine Weyl group of \( \tilde{A}_{n-1} \).

**Theorem 7.** Let \( E : X \rightarrow C \) be a semistable elliptic surface over a field \( k \) of characteristic zero or finitely generated over its prime field. Assume that the singular locus \( S \) is contained in \( C(k) \) and let \( S = \{ s_1, \cdots, s_r \} \), and that the Kodaira-Néron type of the fibre over \( s_i \), \( i = 1, \cdots, r \) is \( I_{n_i} \), \( n_i \geq 3 \) respectively.

The group \( \text{EffAut}(\text{UNS}(X)) \) of universal effective isometries of \( E \) is a semidirect product \( E(k(C)) \rtimes \text{Aut}(E/k(C)) \), where \( E(k(C)) \) acts by translations of the section of \( \pi \) corresponding to an element of \( E(k(C)) \). The group \( \text{Aut}(E/k(C)) \) of automorphisms of the generic fibre is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

The group \( \text{Aut}(\text{UNS}(X)) \) of universal isometries of \( E \) is a semidirect product \( \{ \pm \text{Id} \} \times \prod_{i=1}^r W_{n_i} \rtimes \text{EffAut}(\text{UNS}(X)) \), where \( \{ \pm \text{Id} \} \) is central and is the isomorphism sending every divisor to its negative; the group \( \prod_{i=1}^r W_{n_i} \) is the group generated by the universal Picard-Lefschetz isometries corresponding to the irreducible components of the singular fibres of \( \pi \).

### 2.10. A class of representations of affine Weyl group of type \( \tilde{A}_{n-1} \).**

The process of showing that Picard-Lefschetz reflections based on the irreducible components of singular fibres of a semistable elliptic surfaces lift to an universal isomorphism yields an interesting class of representations of the affine Weyl group \( W_n \). Corresponding to a fibration with local ramification degree \( e \), we define a homomorphism, say \( R_e \), of \( W_n \) into \( W_{ne} \).

Let \( n \geq 3 \). Denote by \( V_n \) the vector space equipped with a symmetric bilinear form and basis \( v_i, i \in \mathbb{Z}/n\mathbb{Z} \) satisfying,

\[
v_i^2 = -2, \quad v_i v_{i+1} = 1, \quad v_i v_j = 0, \quad i, j \in \mathbb{Z}/n\mathbb{Z}, \; |i - j| \geq 2,
\]

where we are using the obvious meaning for \( |i - j| \geq 2 \).

**Definition 2.3.** Given a natural number \( n \), fix an orientation on \( \mathbb{Z}/n\mathbb{Z} \), for instance, by identifying \( \mathbb{Z}/n\mathbb{Z} \) with the \( n \)-th roots of unity. Given \( i, j \in \mathbb{Z}/n\mathbb{Z} \), define the vector \( v(i, j) \in V_n \), as

\[
v(i, j) = \sum_{k=i}^{j} v_k,
\]

where the indices occuring in the sum are taken with the positive orientation from \( i \) to \( j \). Another way of describing the indices occuring in the sum is that we take integral representatives for \( i \) and \( j \) (denoted by the same letter) such that \( i < j \) and then the sum goes from \( i \) to \( j \). It is assumed that the set of integers \( i \leq m \leq j \) maps injectively to \( \mathbb{Z}/n\mathbb{Z} \) upon reduction modulo \( n \).

The **support** of \( v(i, j) \), denoted by \( \text{spt}(v(i, j)) \) is defined to be the set of integers in the interval \([i, j] \). The **length** \( l(v(i, j)) \) of \( v(i, j) \) is the cardinality of the support of \( v(i, j) \).
It can be seen that \( v(i, j)^2 = -2 \), and that any two such distinct vectors are orthogonal provided their supports have a non-empty intersection (see Lemma 6). For \( n \geq 3 \), \( k \in \mathbb{Z}/n\mathbb{Z} \) and any natural number \( e \), define the set \( I(n, e, k) \) to be the collection of vectors of the form \( v(i, j) \in \mathbb{V}_{ne} \) satisfying the following properties:

- The support of \( v(i, j) \) contains \( ek \).
- The length of \( v(i, j) \) is \( e \).

2.11. Base change. For the vectors \( v_0, \cdots, v_{n-1} \) belonging to \( \mathbb{V}_n \), define the following vectors \( p^*_b(v_k) \in \mathbb{V}_{ne} \),

\[
p^*_b(v_k) = ew_{ek} + \sum_{0 < j < e} (e - j)(w_{ek-j} + w_{ek+j}),
\]

where \( w_i, i \in \mathbb{Z}/ne\mathbb{Z} \) forms the standard basis for \( \mathbb{V}_{ne} \). This defines a linear map \( p^*_b : \mathbb{V}_n \to \mathbb{V}_{ne} \). The significance of this map is given by Proposition 23 giving a description of the inverse images of the irreducible components of a singular fibre under a base change which is totally ramified of ramification degree \( e \) at the singular point under consideration.

Define a map \( R^e_n : W_n \to W_{ne} \) by defining on the generators \( s_0, \cdots, s_{n-1} \) of \( W_n \) as:

\[
R^e_n(s_k) = \prod_{v(i,j) \in I(n,e,k)} s_{v(i,j)}.
\]

The following theorem shows that the maps \( R^e_n \) form a system of representations of the affine Weyl groups, closed under composition:

**Theorem 8.** Let \( n \geq 3 \). With notation as above, the following holds:

1. For any \( v_i, i \in \mathbb{Z}/n\mathbb{Z} \),

\[
(R^e_n(s_{v_i}))(p^*_b v) = p^*_b(s_{v_i}(v)).
\]

2. For \( e \geq 1 \), \( R^e_n \) is a representation from \( W_n \) to \( W_{ne} \).

3. For any natural numbers \( e, f \),

\[
R^f_{ne} \circ R^e_n = R^{ef}_n.
\]

3. Action on fibral divisors

We first characterize fibres by an universal property involving divisibility, which allows us to show that an universal Torelli isomorphism preserves fibres up to a sign. Using this, it can be derived that an universal Torelli isomorphism preserves fibral divisors.
3.1. Structure of Néron-Severi group of an elliptic surface. We recall now some well known facts about the Néron-Severi group of an elliptic surface. Under our hypothesis on the \( j \)-invariant of \( E \), it is known by Néron’s theorem of the base ([LN, Sh, Theorem 1.2]), that \( NS(X) \) is a finitely generated abelian group. For elliptic surfaces, this can also be proved directly using the cycle class map and that algebraic and numerical equivalence coincides on an elliptic surface ([M Lecture VII, Sh Section 3]). From this last fact, it also follows that \( NS(X) \) is torsion-free. Further by Hodge index theorem, the intersection pairing is a non-degenerate pairing on \( NS(\tilde{X}) \otimes \mathbb{R} \) of signature \((1, \rho - 1)\) where \( \rho \) is the rank of \( NS(\tilde{X}) \).

Fix a ‘zero’ section \( O \) of \( \pi : X \rightarrow C \). Let \( T(\tilde{X}) \) denote the ‘trivial’ sublattice of \( NS(\tilde{X}) \), i.e., the subgroup generated by the zero section \( O \) and the irreducible components of the fibers of \( \pi \). By decomposing divisors into ‘horizontal’ sections and ‘vertical’ fibers, there is an exact sequence

\[
0 \rightarrow T(\tilde{X}) \rightarrow NS(\tilde{X}) \rightarrow \hat{Q} \rightarrow 1.
\]

Let \( K \) (resp. \( K' \)) be the function field of \( C \) over \( K \) (resp. \( \bar{k} \)). Let \( E \) be the generic fibre of \( \pi : X \rightarrow C \). This is an elliptic curve defined over \( K \), with origin \( 0 \in E(K) \) defined by the intersection of the section of \( O \) with \( E \).

Since \( \pi \) is proper, the group \( E(K) \) is canonically identified with the group of sections of \( \pi : X \rightarrow C \). Denote by \( (P) \) the image in \( X \) of the section of \( \pi \) corresponding to a rational element \( P \in E(K) \), and by \( D(P) \) the divisor \( (P) - (0) \) on \( X \). Let \( T(X) = NS(X) \cap T(\tilde{X}) \) be the trivial sublattice of \( NS(X) \). We have the following description of the Mordell-Weil lattice of the generic fibre due to Shioda and Tate, which for lack of a reference, we indicate the proof over arbitrary base fields:

**Proposition 9.** The section map \( sec : P \mapsto (P) \mod T(X) \) gives an identification of the Mordell-Weil group \( E(K) \) of the generic fibre \( E \) with the quotient group \( Q = NS(X)/T(X) \).

**Proof.** Over \( \bar{k} \), this is Theorem 1.3 in ([Sh]). Let \( G_k \) be the Galois group of \( \bar{k} \) over \( k \). Since \( Pic(X)(k) = Pic(X)(\bar{k})^{G_k} \), \( NS(X) \) defined as the image of \( Pic(X)(k) \) in \( NS(\bar{X}) \) is \( G_k \)-invariant. Given a divisor \( D \in NS(X) \), it can be written uniquely in \( NS(\bar{X}) \) as \( (P) + V \), for some \( P \in E(K') \) and \( V \in T(X) \). The Galois invariance of \( D \) by \( G_k \) implies that \( P \) is \( G_k \)-invariant and hence belongs to \( E(K) \). Hence the section map \( sec \) is surjective. Since \( sec \) is injective over \( \bar{k} \), it is injective (over \( k \)), and this proves the proposition. \( \square \)

3.2. Euler characteristics. We recall some facts about the Euler characteristics of (semistable) elliptic surfaces. Let \( \chi^t(X) \) denote the (topological) Euler characteristic of \( X \), given by the alternating sum of the \( \ell \)-adic betti numbers. Suppose that \( S \) is the ramification divisor of \( \pi \), and the singular fibre at \( t \in S \) is of type \( I_{n_t} \). It is known ([SS, Corollary 6.1]), that \( 12\chi^t(X) = \sum_{t \in S} n_t. \)

Let \( \mathcal{O}_X \) denote the structure sheaf of \( X \), and \( \chi(X) \) the (coherent) Euler characteristic \( \chi(X) = \sum_{i=0}^2 (-1)^i \dim(H^i(X, \mathcal{O}_X)) \). These two Euler characteristics are related
by the formula $\chi^t(X) = 12\chi(X)$. In particular, this shows that $\chi(X)$ is always positive.

Further, if $(P)$ is any section, then the self-intersection number $(P)^2 = -\chi(X)$ ([SS, Corollary 6.9]).

3.3. **Universal isometries preserve fibres.** In this section, our aim is to show that an universal isometry preserves the fibre up to a sign. Since $C$ has a point defined over $k$, the class of the fibre is in $NS(X)$.

**Definition 3.1.** Suppose $L$ is a lattice, a finitely generated free abelian group. An element $l \in L$ is said to be divisible by a natural number $d$ if $l \in dL$.

Equivalently, the coefficients with respect to any integral basis of $L$ are divisible by $d$.

Given a map $b : B \rightarrow C$ of degree $d$, the divisor $p_b^*(F) = dF_b$, where $F$ (resp. $F_b$) is the divisor corresponding to a fibre of $X \rightarrow C$ (resp. $X \rightarrow B$).

**Proposition 10.** Let $\Phi : UNS(\mathcal{E}) \rightarrow UNS(\mathcal{E}')$ be a universal isometry between universal Torelli datum corresponding to two semistable elliptic surfaces $\mathcal{E} : X \xrightarrow{\pi} C$ and $\mathcal{E}' : X' \xrightarrow{\pi'} C$ over a field $k$.

Let $F$ (resp. $F'$) denote the class in the Néron-Severi group $NS(X)$ (resp. $NS(X')$) corresponding to the fibre $\pi^{-1}(s)$ (resp. $\pi'^{-1}(s)$), for some $k$-rational point $s \in C(k)$. Then

$$\Phi(F) = \pm F'.$$

**Proof.** The subspace $[O, F]$ of $NS(X)$ generated by a section $O$ and the fibre $F$ is isomorphic to ([SS, Section 8.6]),

$$[O, F] = \begin{cases} [1] \oplus [-1] & \text{if } \chi(X) \text{ is odd} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \chi(X) \text{ is even} \end{cases}.$$

Since $[O, F]$ is unimodular, there is a direct sum decomposition,

$$NS(X) = [O, F] \oplus [O, F]^\perp.$$

From the Hodge index theorem, it follows that $[O, F]^\perp$ is negative definite.

Write $\Phi(F) = eO' + fF' + D'$, where $D'$ is orthogonal to both $O'$ and $F'$. Let $b : B \rightarrow C$ be a map of degree $d$. The divisor $p_b^*(F) = dF_b$, where $F$ (resp. $F_b$) is the divisor corresponding to a fibre of $X \rightarrow C$ (resp. $X \rightarrow B$). The pull-back divisor $p_b^*D'$ continues to be orthogonal to $O_b' = p_b^*O'$ and the fibre $F_b'$ of $X_b' \rightarrow B$, since $dF_b' = p_b^*F$.

We have $p_b^*F = dF_b$, where $F_b$ is a fibre of $X_b \rightarrow B$. Since $\Phi$ is universal, the divisor

$$\Phi(p_b^*F) = p_b^*(\Phi(F)) = eO_b' + f dF_b' + p_b^*D'.$$
is also divisible by $d$ in $NS(X')$. As $d$ can be arbitrarily chosen, it follows that $e = 0$. Then,

$$0 = F^2 = \Phi(F^2) = D'^2.$$  

The intersection pairing is definite on $[O', F']$. Hence $D' = 0$, and $\Phi(F) = fF$ for some integer $f$.

Since $\{O, F\}$ forms a basis of the unimodular subspace $[O, F]$, $F$ can be completed to a basis of $NS(X)$. The map $\Phi$ being an isometry, $f = \pm 1$ and this proves the proposition. 

3.4. Preservation of fibral divisors. Our aim now is to show that $\Phi$ preserves the space of fibral divisors.

**Proposition 11.** Let $\mathcal{E} : X \to C$ and $\mathcal{E} : X \to C$ be semistable elliptic surfaces over $k$, and $\Phi : UNS(\mathcal{E}) \to UNS(\mathcal{E}')$ be an universal isometry between the universal Torelli data of $\mathcal{E}$ and $\mathcal{E}'$.

Then $\Phi$ preserves the space of fibral divisors.

**Proof.** By the foregoing lemma, we assume after multiplying by the universal isometry $-1\text{Id}$ if required, that the universal isometry $\Phi$ preserves fibres: $\Phi(F_b) = F'_b$ for any $b \in \mathcal{B}$. Let $v$ be an irreducible component of a singular fibre. Write,

$$\Phi(v) = (P') - (O') + r(O') + V' + s F',$$

where $r$, $s \in \mathbb{Z}$, $V'$ is a fibral divisor whose support does not contain the zero component of any singular fibre i.e., the irreducible component of the singular fibre intersecting non-trivially the zero section. We have $(O')V' = FV' = 0$. Now,

$$\Phi(v)F' = ((P') - (O') + r(O') + V' + s F')F' = r.$$

Since

$$\Phi(v)F' = \Phi(v)\Phi(F) = \Phi(vF) = 0,$$

it follows that $r = 0$.

Let $S' \subset C(k)$ be the ramification locus of $\pi'$, and for a point $s \in S'$, let $V'_s$ be $s$-component of $V'$. We now claim that $V'^2 + 2(P')V' \leq 0$. For this, it is sufficient to show that $V'^2 + 2(P')V'_s \leq 0$ for any $s \in S'$.

Let the fibre of $\pi'$ at $s$ be of type $I_n$. Let $v'_0, \ldots, v'_{n-1}$ be the irreducible components of the fibre at $s$ written in the standard notation, where $v'_0$ is the component meeting the section $O'$. We use cyclic notation (congruence modulo $n$) and define $v'_{n} = v'_0$. Write $V'_s = a_1 v'_0 + \cdots + a_{n-1} v'_{n-1}$, for some $a_i \in \mathbb{Z}$. Here $a_0 = a_n = 0$ by assumption on $V'$. Suppose $(P')$ intersects the component $V'_s$ at the component $v_j$. Then $2(P')V'_j = 2a_j$.

We have,

$$V'^2 = -2 \sum_{i=0}^{n-1} a_i^2 + 2 \sum_{i=0}^{n-1} a_i a_{i+1} = - \sum_{i=0}^{n-1} (a_i - a_{i+1})^2.$$  


The integers \(a_1 - a_0, \ldots, a_j - a_{j-1}\) and \(a_j - a_{j-1}, \ldots, a_{n-1} - a_n\) give a partition of \(a_j\). Hence

\[
\sum_{i=0}^{j} (a_i - a_{i+1})^2 \geq \sum_{i=0}^{j} |a_i - a_{i+1}| \geq |a_j|.
\]

Similarly, \(\sum_{i=j}^{n} (a_i - a_{i+1})^2 \geq |a_j|\). Hence, we get that

\[
V_s^2 + 2(P')V' = -\sum_{i=0}^{n-1} (a_i - a_{i+1})^2 + 2a_j \leq 0.
\]

This proves that \(V^2 + 2(P)V' \leq 0\). Suppose now \(P' \neq O'\). Then,

\[
-2 = v^2 = \Phi(v)^2 = (P')^2 + (O')^2 - 2(P')(O') + V'^2 + 2(P)V' - 2(O)V'
\]

\[
\leq -2\chi(X').
\]

This yields a contradiction when \(\chi(X') > 1\), and proves the proposition in this case.

Suppose now that \(\chi(X') = 1\). Consider a base change \(b : B \to C\) such that \(\chi(X_b') > 1\). By the above argument, the base change of \(\Phi(v) = (P') - (O') + V' + sF'\) to \(X_b'\) is fibral. This implies that \((P') = (O')\), i.e., there is no sectional component and this proves the proposition. \(\square\)

3.5. Isomorphism of singular fibres. We now show that a universal Torelli isomorphism preserving fibral divisors yields an ‘identification’ of the singular fibres of \(X\) and \(X'\). Let \(\pi : X \to C\) be a (split) semistable elliptic surface over \(k\). Suppose \(t \in C(k)\) belongs to the singular locus of \(\pi\) and the Kodaira type of the fibre is \(I_n\). For \(n \geq 3\), the subgroup \(NS(X_t)\) of \(NS(X)\) generated by the irreducible components of the singular fibre \(X_t\) of \(\pi\) at \(t\), is isomorphic to the affine root lattice of type \(\tilde{A}_{n-1}\). When \(n = 1\), the fibre \(X_t\) is a nodal curve, and \(NS(X_t) \simeq \mathbb{Z}\) with trivial intersection pairing.

**Proposition 12.** Let \(\mathcal{E} : X \xrightarrow{\pi} C\) and \(\mathcal{E} : X \xrightarrow{\pi'} C\) be semistable elliptic surfaces over \(k\), and \(\Phi : UNS(\mathcal{E}) \to UNS(\mathcal{E}')\) be an universal isometry between the universal Torelli data of \(\mathcal{E}\) and \(\mathcal{E}'\).

Let \(S\) (resp. \(S'\)) be the places of \(C(k)\) such that the fibre \(X_t\) (resp. \(X'_t\)) for \(t \in S\) (resp. \(t \in S'\)) is singular. Then, \(S = S'\), and for each \(t \in S\) and \(\Phi\) restricts to an isomorphism \(NS(X_t) \to NS(X'_t)\).

**Proof.** Let \(N\) be the supremum over the natural numbers \(n\) such that the singular fibres of \(\pi\) and \(\pi'\) are of type \(I_n\). Suppose that the singular fibre of \(\pi\) (resp. \(\pi'\)) at \(t \in S \cup S'\) is of type \(I_m\) (resp. \(I_{m'}\)) with \(m, m' \geq 0\). Choose a degree \(d > N\) morphism \(B \to C\) which is totally ramified at \(v\) and unramified at all other points of \(S \cup S'\). Let \(z \in B(k)\) map to \(t\). The fibre at \(z\) of \(\pi\) (resp. \(\pi'\)) is of Kodaira type \(I_{md}\) (resp. \(I_{m'd}\)), and at other points of \(S \cup S'\) it remains unchanged.

By Proposition 11, \(\Phi\) sends fibral divisors to fibral divisors. Since there are no isometries between the root systems \(\tilde{A}_n\) and \(\tilde{A}_m\) for \(n \neq m\), and \(d > N\) it follows...
that \( m = m' \). Applying the same argument to all points \( t \in S \cup S' \) the proposition follows. \[ \square \]

4. ISOGENY OF GENERIC FIBRES

In this section, we show under the hypothesis of Theorem 3 that the generic fibres of \( \pi \) and \( \pi' \) are isogenous over a finite extension of \( k(C) \), by invoking the validity of the Tate isogeny conjecture for elliptic curves under our hypothesis on \( k \).

**Proposition 13.** Let \( k \) be a field of characteristic zero or finitely generated over its prime field, and \( E : X \xrightarrow{\pi} C, E' : X' \xrightarrow{\pi'} C \) be semistable elliptic surfaces over \( k \). Let \( \Phi : \text{UNS}(E) \to \text{UNS}(E') \) be a universal isometry.

For each rational prime \( \ell \) coprime to the characteristic of \( k \), there is an isogeny \( \psi_\ell : E \to E' \), defined over \( k(C) \) when \( k \) is finitely generated over its prime field and over \( \bar{k}(C) \) when \( k \) is an arbitrary field of characteristic zero, such that \( \psi_\ell \) is an isomorphism on the \( \ell \)-torsion of \( E \).

**Proof.** After multiplying by the \(-1\) map sending a divisor \( D \) on \( X \) to \(-D\), we can assume by Proposition 10, that \( \Phi \) maps the fibre of \( \pi \) to that of \( \pi' \). Choose a section \( O \) (resp. \( O' \)) of \( \pi \) (resp. \( \pi' \)). Write,

\[ \Phi(O) = (P') - (O') + r(O') + V', \]

where \( V' \) is a fibral divisor. Since \( 1 = OF = \Phi(O)F' \), we get \( r = 1 \). Thus the section \( O \) gets mapped to a section of \( \pi' \) modulo the lattice spanned by fibral divisors of \( X' \). Denote this section by \( O' \).

Using \( O \) and \( O' \), define the \( G(\bar{K}_s/K) \)-invariant ‘trivial lattices’ \( T(X) = \lim_{b \in B} T(X_b) \) contained in \( NS(X) \) (and similarly for \( T(X') \)). From Proposition 6, we obtain an identification as \( G(\bar{K}_s/K) \)-modules,

\[ E(K_s) = \frac{NS(X)/T(X)}{T(X)}. \]  \[ (4.1) \]

For any natural number \( n \), let \( E[n] \) denote the group of \( n \)-torsion elements in \( E(\bar{K}_s) \). From the identification given by Equation 4.1 and by Proposition 9, there is a \( G(\bar{K}_s/K) \)-equivariant identification,

\[ E[n] = \{ D \in NS(X)/T(X) \mid nD \in T(X) \}/T(X). \]

Fixing a rational prime \( \ell \) coprime to the characteristic of \( k \), we get a compatible system of isomorphisms, \( \Phi[\ell^n] : E[\ell^n] \to E'[\ell^n] \). Taking the limit as \( n \to \infty \) yields an isomorphism,

\[ \Phi_{\ell^\infty} : T_\ell(E) \to T_\ell(E'), \]

of the Tate modules of the generic fibres of \( X \) and \( X' \).
Suppose $k$ is finitely generated over its prime field. By theorems of Tate, Serre, Zarhin and Faltings ([T, Se, Z, F, FW]), establishing the isogeny conjecture of Tate,

$$\text{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq \text{Hom}_{G_K}(T_\ell(E), T_\ell(E')),$$

there exists a morphism $\psi : E$ to $E'$ defined over $K$, and a scalar $a \in \mathbb{Z}_\ell$ such that $a \psi$ corresponds to $\Phi_{\ell \infty}$. Here we are using a theorem of Deuring ([Cl, Theorem 12]) that $\text{End}(E) = \mathbb{Z}$, since the $j$-invariant $j(E)$ of $E$ is not algebraic over the prime field contained in $k$.

The effect of $a \psi$ on $E[\ell^n]$ is given by that of an isogeny, $\psi_{\ell^n} : \bar{a}_n \psi : E \to E'$, where $\bar{a}$ denotes a lift to $\mathbb{Z}$ of the image in $\mathbb{Z}/\ell^n \mathbb{Z}$ of $a \in \mathbb{Z}_\ell$. Since the multiplication by $\ell$ maps from $E[\ell^{n+1}]$ to $E[\ell^n]$ are surjective and these groups are finite, the map $\psi_{\ell^n}$ coincides with $\Phi[\ell^n]$ on $E[\ell^n]$.

In particular, for each $\ell$ coprime to the characteristic of $k$, there is an isogeny $\psi_{\ell} : E \to E'$, which coincides with the action of $\Phi[\ell]$ on $E[\ell]$. Since $\Phi[\ell]$ is an isomorphism, this proves the proposition when $k$ is finitely generated over its prime field.

Now suppose $k$ is an arbitrary field of characteristic zero. We assume that $k$ is algebraically closed. The proposition follows now from the geometric analogue of the Tate isogeny theorem given by ([De, Corollaire 4.4.13]). Choose an embedding of $k$ into the complex numbers $\mathbb{C}$. Let $S$ be a finite subset of $C(k)$ containing the discriminant loci of $\pi$ and $\pi'$. The action of the absolute Galois group $G(K/K)$ acts via the algebraic fundamental group $\pi_1(C - S, p)$, where $p$ is some chosen basepoint.

By ([Sz, Theorem 4.6.10]), the fundamental groups are isomorphic for base change of algebraically closed fields of characteristic zero. Further if the elliptic curves $E$ and $E'$ are isogenous over $\mathbb{C}(C)$, then they are isogenous over some finite extension of $k(C)$ contained inside $\mathbb{C}(C)$, hence defined over $k(C)$ since $k = \bar{k}$. Hence it is enough to work over $k = \mathbb{C}$.

The $j$-invariants of $E$ and $E'$ are non-constant elements of $k(C)$. This property continues to hold for the base change of $E$ and $E'$ to $\mathbb{C}$. The maps $\pi$ defines an abelian scheme $X - \pi^{-1}(S) \to C - S$ (and similarly for $\pi'$). By ([De, Corollaire 4.4.13]),

$$\text{Hom}(X - \pi^{-1}(S), X' - \pi'^{-1}(S)) \simeq \text{Hom}(R_1\pi_*\mathbb{Z}, R_1\pi'_*\mathbb{Z}),$$

where the left hand side is as morphisms of abelian scheme over $C - S$, and the right hand side is as morphisms in the category of locally constant sheaves over $C - S$. By the universal coefficient theorem for homology, since $H^0$ is torsion-free, tensoring with $\mathbb{Z}_\ell$, we can identify $R_1\pi_*\mathbb{Z} \otimes \mathbb{Z}_\ell \simeq T_\ell(E)$ (and similarly for $E'$), as a module for the absolute Galois group $G_K$ of $K = \mathbb{C}(C)$. Since tensoring with $\mathbb{Z}_\ell$ is fully faithful, we obtain a $G_K$-equivariant isogeny $\psi : E \to E'$ for each rational prime $\ell$, defined over $\mathbb{C}(C)$. Arguing as above, proves the proposition. \(\square\)
5. Non effective universal Torelli

In this section we prove Theorem 3 but over \( \bar{k}(C) \) when \( k \) is an arbitrary field of characteristic zero.

**Proposition 14.** With the hypothesis of Theorem 3, the elliptic surfaces \( E \) and \( E' \) are isomorphic over \( k(C) \) when \( k \) is finitely generated over its prime field and over \( \bar{k}(C) \) when \( k \) is an arbitrary field of characteristic zero.

**Proof.** Let \( L = \bar{k}(C) \) when \( k \) is an arbitrary field of characteristic zero, and equal to \( k(C) \) when \( k \) is finitely generated over its prime field. Since the generic fibre uniquely determines the minimal regular model, it is enough to show that the generic fibres \( E \) and \( E' \) are isomorphic over \( L \).

By Proposition 13, for any rational prime, there is an isogeny \( \psi_\ell : E \to E' \), defined over \( L \), such that the order of the kernel of \( \psi_\ell \) is coprime to \( \ell \).

Suppose the kernel of \( \psi_\ell \) contains a group scheme of the form \( E[a] \) for some natural number \( a \). Since multiplication by \( a \) is an isomorphism of \( E \) to itself, quotienting by groups of the form \( E[a] \), we can assume that kernel of \( \psi_\ell \) is cyclic, in that it does not contain any subgroup scheme of the form \( E[a] \).

Choose some \( \ell \) coprime to the characteristic \( p \) of \( k \). Suppose for some \( \ell' \) coprime to \( p \), the \( \ell' \)-primary subgroup of \( \text{Ker}(\psi_\ell) \) is non-trivial. Consider the isogeny,

\[
A = \psi_{\ell'}^* \circ \psi_\ell : E \to E,
\]

where \( \psi_{\ell'}^* \) denotes the isogeny \( E' \to E \) dual to \( \psi_{\ell'} \).

Since \( \psi_{\ell'} \) has no element of order \( \ell' \) in its kernel, so does \( \psi_{\ell'}^* \). Since \( \text{End}(E) = \mathbb{Z} \) by Deuring’s theorem ([Cl, Theorem 12]), \( A \) is multiplication by some integer \( a \in \mathbb{Z} \). Thus \( \text{Ker}(A) = E[a] \).

On the other hand, the \( \ell' \)-primary part of \( \text{Ker}(A) \) is isomorphic to the \( \ell' \)-primary part of \( \text{Ker}(\psi_\ell) \), and this is not of the form \( E[\ell'^k] \) for any \( k \). This yields a contradiction and implies that the \( \ell' \)-primary part of \( \text{Ker}(\psi_\ell) \) is trivial for any \( \ell' \) coprime to the characteristic of \( k \). This also proves the proposition when characteristic of \( k \) is zero.

When the characteristic of \( k \) is \( p > 0 \), the foregoing argument implies that there is an isogeny \( \psi : E \to E' \), such that its kernel \( G \) is a finite group scheme of order \( p^k \), not containing any subgroup scheme of the form \( E[p] \). The group scheme \( E[p^k] \) is a semi-direct product of the cyclic étale group scheme \( \mathbb{Z}/p^k\mathbb{Z} \) by the connected group scheme \( E[p^k]^0 \). Suppose \( G \) has both an étale and connected component \( G^0 \). Both \( G^0 \) and \( G/G^0 \) will contain subgroup schemes or order \( p \). Then \( G \) will contain \( E[p] \), contradicting our assumption on \( G \). Hence we can assume that the kernel of both \( \psi \) and the dual isogeny \( \psi^* \) are either étale or a connected group scheme. If \( G \) is connected, then the kernel of the dual isogeny \( \psi^* \) is étale as together they make up \( E[p^k] \). Hence we can assume without loss of generality that \( G \) is étale, and is generated by a section \((P)\), given by a torsion-element of \( P \in E(k(C)) \) of order \( p^k \).
At a singular fibre, the group structure of the identity component is \( G_m \). Hence the section \((P)\) cannot pass through the identity component of any singular fibre. Suppose that the singular fibre at \( s \) of \( \pi \) is of Kodaira type \( I_n \) for some natural number \( n \). By (\cite{DoDo}, Theorem A.1), applied inductively, the Kodaira type of the singular fibre of \( \pi' \) is \( n/p^k \). But by Proposition 12 both \( X \) and \( X' \) have the same singular fibres. This implies that \( k = 0 \) and the proposition is proved. \( \square \)

Remark. The isomorphism allows us to consider the composition of the universal isometry \( \Phi \) with itself. This plays a crucial role in the proof of Theorem 2.

6. Effectivity

We now move towards the proof of Theorem 2. In this section, our aim is to prove the following proposition, giving the consequence of the effectivity hypothesis that is required for the proof of Theorem 2.

**Proposition 15.** Suppose \( \Phi : UNS(\mathcal{E}) \to UNS(\mathcal{E}') \) is an effective universal isomorphism between universal Torelli data of two semistable elliptic surfaces as in the hypothesis of Theorem 2. Then for any \( b \in B_C \), \( \phi_b : NS(X_b) \to NS(X'_b) \) sends the irreducible components of the singular fibres divisors of \( \pi \) to the irreducible components of the singular fibres divisors of \( \pi' \), and sections to sections.

We start with the following lemma characterising fibral divisors:

**Lemma 1.** Let \( D \) be an effective divisor on \( X \). Then \( D.F \geq 0 \). If moreover \( D.F = 0 \), then \( D \) is a fibral divisor.

**Proof.** It is sufficient to prove this over \( \bar{k} \), and we assume now that \( k = \bar{k} \). We can assume that \( D \) is an irreducible closed subvariety of \( X \) of codimension one. Suppose that \( \pi |_D : D \to C \) is dominant. Since \( D \) is a closed subvariety and \( \pi \) is proper, \( \pi |_D \) is surjective. Suppose for some \( x \in C(k) \), the fibre \( F_x \) at \( x \) is irreducible and \( D.F_x = D.F < 0 \). Then the fibre \( F_x \subset D \). Since this happens for almost all \( x \in C(k) \), the dimension of \( D \) will be 2, contradicting the fact that \( D \) is of dimension one. Hence \( D.F \geq 0 \).

If \( D.F = 0 \), then there exists a point \( x \in C(k) \) such that \( F_x \) and \( D \) are distinct effective divisors which are disjoint. Hence it follows that \( \pi |_D \) cannot be surjective. This implies that \( D \) is a fibral divisor. \( \square \)

**Lemma 2.** Suppose \( D \) is an irreducible subvariety of \( X \) and \( D.F = 1 \). Then \( D \) is a section, i.e., \( \pi |_D : D \to C \) is an isomorphism.

**Proof.** Since \( D.F = 1 \), the map \( \pi |_D : D \to C \) is surjective, and of degree 1. Hence the generic points, say \( D_0 \) of \( D \) and \( C_0 \) of \( C \) are isomorphic. Let \( s_0 : C_0 \to D_0 \) be an isomorphism. Since \( \pi \) is proper, this extends to a map \( S : C \to D \). The image \( s(C) \) is a proper closed subvariety of \( D \), and hence is equal to \( D \). This implies that \( D \) is a section. \( \square \)
6.1. Characterization of sections and irreducible fibre components.

**Definition 6.1.** An effective divisor \( D \in \text{NS}(X) \) is said to be *indecomposable*, if it cannot be written as a sum of two effective divisors in \( \text{NS}(X) \).

Note that if \( X \to C \) is an elliptic surface with a singular fibre of Kodaira type \( I_n \) with \( n \geq 2 \), then the fibre is not indecomposable in \( \text{NS}(X) \).

We now characterize sections and fibral divisors:

**Lemma 3.** Let \( \mathcal{E} : X \to C \) be a semistable elliptic surface over a field \( k \). Let \( D \in \text{NS}(X) \), and \( F \) denotes the divisor in \( \text{NS}(X) \) corresponding to a fibre of \( \pi \). Then the following holds:

1. Suppose \( n_t = 1 \) for all \( t \in S \). Then the fibre is indecomposable.
2. A divisor \( D \) is an irreducible component of a singular fibre of Kodaira type \( I_n \), \( n \geq 2 \) if and only if it is effective, indecomposable and \( D.F = 0 \).
3. A divisor \( D \) on \( X \) is a section, if and only if it is effective, indecomposable and \( D.F = 1 \).

**Proof.** (1) Suppose the fibre is not indecomposable, and is written as a non-negative linear combination of effective divisors. By Lemma 1, only fibral divisors occur non-trivially in such a sum. But since \( n_t = 1 \) for all \( t \in S \), the only fibral divisors contributing to \( \text{NS}(X) \) are multiples of \( F \).

(2) If \( D.F = 0 \), then \( D \) is a fibral divisor. Since any fibral effective divisor can be written as an integral linear combination of the irreducible components of singular fibres, if \( D \) is indecomposable then it has to be an irreducible component of a singular fibre of \( X \to C \).

Suppose \( v_0 \) is an irreducible component of the singular fibre of \( \pi \) at \( t_0 \) of Kodaira type \( I_{n_{t_0}} \), \( n_{t_0} \geq 2 \), and is not indecomposable. Then there is an expression of the form,

\[
v_0 = \sum_{i \in I} n_i (P_i) + \sum_{t \in S} l_{t,j} v_{t,j}, \quad n_i, \ l_{t,j} \geq 0 \quad \forall i \in I, \ t \in S, \ 0 \leq j \leq n_t - 1.
\]

where \( S \) is the ramification locus of \( \pi \) and for \( t \in S \), the fibre is of Kodaira type \( I_{n_t} \). Here \( I \) is a finite set and \((P_i)\) is a section. Intersecting with the fibre implies that \( n_i = 0 \) for all \( i \in I \).

Since the self-intersection of \( v_0 \) is \(-2\), this implies that \( l_{t_0,j_0} \geq 1 \), where \( v_0 = v_{t_0,j_0} \). This implies that there is a sum of the form \( \sum_{t \in S} l_{t,j} v_{t,j} \), \( l_{t,j} \geq 0 \) which is equivalent to \( 0 \) in \( \text{NS}(\bar{X}) \). Intersecting with the zero section \( O \) implies that \( l_{t,0} = 0 \) for all \( t \in S \), where \( v_{t,0} \) denotes the irreducible component in the fibre at \( t \) meeting the zero section. By the theorem of Tate-Shioda ([SS Corollary 6.13]), the rest of the components are linearly independent and hence \( l_{t,j} = 0 \) for all \( t \) and \( j \). This implies that the irreducible fibral divisor \( v_0 \) is indecomposable.
(3) If $D$ is an effective, indecomposable divisor on $X$, then $D$ is irreducible. By Lemma 2 if $D.F = 1$, then $D$ is a section.

Suppose the section $(P)$ where $P \in E(K)$, can be written as $(P) = \sum_{i \in I} n_i D_i$, where $D_i$ are irreducible subvarieties of $\overline{X}$ and $n_i \geq 0$ for $i \in I$. By Lemma 1 the intersection of any effective divisor with a fibre is non-negative. Upto reindexing, it can be assumed that there exists an index denoted $1 \in I, D_1.F = 1$ and $D_i.F = 0$ for $i \in I, i \neq 1$. By Lemma 2, $D_1$ is a section, say $(Q)$, for some $Q \in E(K)$. The function fields of the generic fibre $E$ and $X$ are isomorphic. Thus at the generic fibre $P$ and $Q$ are linearly equivalent. Being effective, this implies that $P = Q$. By Lemma 1, $\sum_{i \in I, i \neq 1} n_i D_i$ is a non-negative sum of irreducible components of fibres and is linearly equivalent to zero. By the theorem of Tate-Shioda, $n_i = 0$ for all $i \neq 1$, and this proves that $(P)$ is indecomposable.

\[\square\]

Proof of Proposition 15. By Proposition 10, an universal effective isometry preserves fibres. Hence Proposition 15 follows from the above lemmas.

6.2. Translations. Given a section corresponding to a rational element $P \in E(K)$, the translation map, $\tau_P : NS(X) \rightarrow NS(X)$, given by translating by the section $(P)$ is an isometry. Further it is effective.

Suppose $b : B \rightarrow C$ is in $B_C$. The rational element $P$ can be considered as an element in $E(L)$, where $L$ is the function field of $B$ and thus defines a translation isometry from $NS(X_b)$ to itself. It can be seen that $p_b^* \circ \tau_P = \tau_P \circ p_b^*$. Thus the collection of translations $\tau_P$ for $b \in B_C$ defines an effective isomorphism of the universal Néron-Severi group $UNS(X)$ of $X \rightarrow C$.

Proposition 14 gives an isomorphism of the elliptic surfaces $X$ and $X'$. Suppose that under this isomorphism the zero section $O'$ of $X' \rightarrow C$ maps to the section $O''$ of $X \rightarrow C$. The map $\Phi \circ \tau_{-O''} : UNS(X) \rightarrow UNS(X)$ gives an effective isomorphism of universal Torelli data from the elliptic surface $\mathcal{E}$ to itself, preserving the zero section.

From now on, we will assume that the universal isometry $\Phi$ as maps from $UNS(X)$ to itself preserving the zero section.

7. Revisiting action on fibral divisors

We state a special, refined version of the universal Torelli theorem, to take care of both Theorems 2 and 7.

Theorem 16. Let $\mathcal{E} : X \rightarrow C$ be a semistable elliptic surface over $k$. Let $\Phi : UNS(X) \rightarrow UNS(X)$ be an automorphism of the universal Néron-Severi group of $X$ satisfying the following:

- $\Phi$ preserves the fibre: $\Phi(F) = F$. 

• $\Phi$ preserves the zero section: $\Phi((O)) = (O)$.
• $\Phi$ maps the irreducible components of singular fibres to irreducible components of singular fibres.
• $\Phi$ sends sections to sections.

Then $\Phi$ arises from either the identity or the inverse map $\iota: P \mapsto -P$ of the generic fibre $E$ over $k(C)$.

From what has been done so far, under the hypothesis of Theorem 2, the hypothesis of Theorem 16 hold true. With a bit of descent, Theorem 2 will follow from Theorem 16.

The proof of Theorem will be given in Section 8. In this section, our aim is to show that $\Phi$ is partially geometric, in that it arises from an isomorphism of elliptic surfaces restricted to torsion and the fibral divisors.

7.1. Néron models, torsion elements and the narrow Mordell-Weil group. We recall some crucial facts that follow from the properties of Néron models. Given a point $t \in S$, let $O_t$ be the local ring of the curve $C$ at $t$, and $K_t$ be its quotient field. By localization, the elliptic surface defines an elliptic curve $E_t$ defined over $K_t$. Let $\mathcal{E}_t$ denote the Néron model of $E_t$. This is a group scheme defined over $O_t$, with the property that $E_t(K_t) = \mathcal{E}_t(O_t)$. The special fibre of the Néron model $\mathcal{E}_t$ can be identified with the complement of the singular locus in the fibre $X_t$. In particular, the collection of connected components $G_t$ of a singular fibre $X_t$ acquires a group structure, with the component intersecting the zero section as the identity element of the group law. When the fibre is of Kodaira-Néron type $I_n$, the group of connected components $G_t \simeq \mathbb{Z}/n\mathbb{Z}$. The specialization map yields a homomorphism $\psi: E_t(K_t) \to G_t$.

The main global ingredient in the proof of Theorem 16 is the following theorem ([SS, Corollary 7.5]), stating that a torsion section is determined by its intersections with the components of the singular fibres:

**Theorem 17.** The global specialization map yields an injective homomorphism,

$$\psi: E(K)_{\text{tors}} \to \prod_{t \in S} G_t,$$

where $E(K)$ is the torsion subgroup of $E(K)$.

Define the narrow Mordell-Weil group $E_O(K)$ to be the subspace of $E(K)$ consisting of the elements $P \in E(K)$ such that the section $(P)$ of $\pi: X \to C$ corresponding to $P$ intersects each singular fibre at the identity component. Equivalently, $E_O(K) = \text{Ker}(\psi)$. A consequence of Theorem 17 is that $E_O(K)$ is torsion-free.

7.2. $\Phi^2$ is partially geometric. For any $b \in \mathcal{B}$, let $\text{Tor}(X_b)$ denote the group of ‘torsion sections’ of $\pi_b: X_b \to B$, corresponding to the torsion elements in the generic fibre $E(k(X_b))$ of $\pi_b$ of order coprime to the characteristic of $k$. 
The fact that $\Phi$ can be considered as a self-map from $UNS(X)$ to itself, allows one to compose $\Phi$ with itself. We have,

**Proposition 18.** For any $b \in B$, the restriction of $\phi_b^2$ to $Tor(X_b)$ is the identity map.

*Proof.* The zero section $O$ of $\pi$ pulls back to the zero section $O_b$ of $\pi_b$. The zero section $O_b$ is fixed by $\phi_b$. Let $S_b$ be the ramification locus of $\pi_b$. Suppose the Kodaira type of the singular fibre at $t \in S_b$ is of type $I_{m_t}$ for some $m_t \in \mathbb{N}$. If $m_t \geq 3$, the singular fibre is a chain of rational curves each with self-intersection $-2$ and intersecting its neighbours with multiplicity one. Since the component intersecting the zero section is fixed by $\phi_b$, and $\phi_b$ is an isometry, it will either act as identity or act as an involution sending the divisor $v_j$ to $v_{-j}$, where we are using the notation as in section 2.9. If $m_t \leq 2$, then there at most two components. Thus $\phi_b^2$ acts as identity on each $G_t$ for each $t \in S_b$. The proposition now follows from Theorem [17]. \qed

7.3. **An application of Tate uniformization.** We now apply Tate’s uniformization of semistable elliptic curves to gain further control on $\Phi$. Fix a singular point $s \in S$ where the fibre is of type $I_n$ for some $n \in \mathbb{N}$. Corresponding to $s$, there is a non-trivial discrete valuation $\nu_s$ of the function field $K$ of $C$, and we let $\hat{K}$ be the completion of the $K$ with respect to the valuation $\nu_s$. Denote by $L$ the algebraic closure of $\hat{K}$.

Since the elliptic surface has semistable reduction at $s$, the $p$-adic uniformization theorem of Tate asserts the existence of $q \in \hat{K}^*$, such that there is a $G(L/\hat{K})$-equivariant isomorphism,

$$L^*/q^{\mathbb{Z}} \rightarrow E(L).$$

(7.1)

Fix a rational prime $\ell$ coprime to the characteristic of $k$. The $p$-adic uniformization theorem implies that the Tate module $T_\ell(E/\hat{K})$ of the elliptic curve $E$ considered over $\hat{K}$ sits in the following exact sequence of $G(L/\hat{K})$-modules,

$$1 \rightarrow T_\ell(G_m) \rightarrow T_\ell(E/\hat{K}) \rightarrow \mathbb{Z}_\ell \rightarrow 0.$$  \hspace{1cm} (7.2)

where $T_\ell(G_m) = \lim\limits_{\longleftarrow n \to \infty} \mu_{\ell^n}$, and $\mu_{\ell^n}$ is the group of $\ell^n$-th roots of unity in $L$.

Let $D_s \subset G(\hat{K}/K)$ denote the decomposition group at $s$, defined as the image under the restriction map to $\hat{K}_s$ of $G(L/\hat{K})$. Since $\Phi$ is $G(\hat{K}/K)$-equivariant, it is also $D_s$-equivariant. The Tate module $T_\ell(E)$ is isomorphic as abelian groups to $T_\ell(E/\hat{K})$. The decomposition group $D_s$ preserves the filtration given by Equation (7.2).

With respect to this filtration, it is known that the Zariski closure of the image of $D_s$ inside $Aut(V_\ell(E))$ contains the group of matrices which act as identity on the associated graded decomposition of the space $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell (\mathbb{S}^{[\ell]}).$ Choosing an appropriate basis, the Zariski closure of the image thus contains the subgroup $U$ of upper triangular unipotent matrices. Since $\Phi$ is equivariant with respect to the action of $D_s$, it is equivariant with respect to $U$. Hence we have,

**Lemma 4.** The map $\Phi$ acting on $V_\ell(E)$ is upper triangular with respect to the filtration given by Equation (7.2).
Since the only upper triangular matrices of order 2 are diagonal matrices with entries \( \{\pm 1\} \) along the diagonal, combining this with Proposition 18, we get

**Corollary 1.** Suppose \( n \geq 1 \). Let \( P \) be a generator for the group \( \mu_{\ell^n}(L) \subset E[\ell^n] \), and \( Q \) be a generator for the quotient group \( E[\ell^n]/\mu_{\ell^n}(L^*) \). Then
\[
\Phi((P)) = \pm(P) \quad \text{and} \quad \Phi((Q)) = \pm(Q).
\]

7.4. \( \Phi \) is geometric on torsion. Corollary 1 allows us to conclude that \( \Phi \) is geometric restricted to \( \ell^\infty \)-torsion:

**Proposition 19.** With hypothesis as in Theorem 16, \( \Phi \) restricted to \( E[\ell^n] \) is either identity or the inverse map \( P \mapsto -P \), where \( \ell \) is a rational prime coprime to the characteristic of \( k \).

**Proof.** Suppose for \( \ell^n \) and \( P, Q \in E[\ell^n] \) as in Corollary 1
\[
\Phi(P) = P \quad \text{and} \quad \Phi(Q) = -Q.
\]
Now \( P \) and \( Q + P \) also satisfy the hypothesis of Corollary 1. The foregoing equation yields, \( \Phi(Q + P) = -Q + P \). By Corollary 1, \( \Phi(Q + P) = -Q + P \) is equal to either \( Q + P \) or \( -(Q + P) \). This implies respectively, \( 2Q = 0 \) or \( 2P = 0 \). This yields a contradiction if \( \ell^n > 2 \). A similar argument works when \( \Phi(P) = -P \). \( \square \)

Hence restricted to \( E[\ell^n] \) and for any \( n \) uniformly, we have that \( \Phi \) is either identity or the additive inverse map. After multiplying the base change Torelli isomorphism \( \Phi \) by the morphism induced by the \(-Id\) isomorphism of the elliptic surface, we can assume that \( \Phi \) induces the identity map on \( E[\ell^n] \) and for any \( n \).

7.5. \( \Phi \) is geometric on fibres.

**Proposition 20.** With hypothesis as in Theorem 16, upto multiplication by an element of \( \text{Aut}(E/k(C)) \simeq \mathbb{Z}/2\mathbb{Z} \), \( \Phi_b \) acts as identity on the trivial lattice \( T(X_b) \) for any \( b \in B \).

**Proof.** Let \( \ell \) be a rational prime coprime to the characteristic of \( k \). By Proposition 19, we can assume that upto multiplying by an element of \( \text{Aut}(E/k(C)) \), \( \Phi \) acts trivially on \( E[\ell^\infty] \). We need to conclude that \( \Phi \) acts trivially on the fibral divisors.

For this, it is enough to prove it for some base change \( b \), since \( p_b^* \) is injective. The injectivity of \( p_b^* \) follows from the fact that the intersection pairing on the Néron-Severi group of an elliptic surface is non-degenerate taken in conjunction with Equation 2.2. Consider the base change over which the elements of \( E[\ell] \subset E[\ell(B)] \). It follows from the exact sequence (7.2), that for any singular fibre of \( \pi_b \), there will be a \( \ell \)-torsion section, not of order 2 in the group \( E[\ell]/\mu_{\ell} \), where \( \mu_{\ell} \) is sitting in \( E[\ell] \) by Tate uniformization. Since \( \Phi \) acts trivially on \( E[\ell] \), and \( \Phi \) respects the intersection product, it follows as in the proof of Proposition 18 that \( \Phi \) acts as identity on the irreducible components of any singular fibre. \( \square \)
8. Proof of Theorem 2

In this section, we give a proof of Theorem 16 (and Theorem 2). By Proposition 14 the elliptic surfaces \( X \to C \) and \( X' \to C \) become isomorphic (over a possibly quadratic extension of \( k(C) \) contained inside \( \bar{k}(C) \) in case the characteristic of \( k \) is zero). Assume now that the elliptic surfaces are isomorphic. By Corollary 13 the map \( \Phi \) preserves sections and the reducible components of the singular fibres divisors of \( \pi \). Translating by a section, we can assume that \( \Phi \) preserves the zero section of \( \pi \). Finally by Proposition 20 up to multiplication by an element of \( \text{Aut}(E/k(C)) \), we can assume that \( \Phi \) acts as identity on the trivial lattice. Thus the proof of Theorem 16 follows from the proof of the following theorem:

**Theorem 21.** With hypothesis as in Theorem 16 assume further that \( \phi_b \) acts as identity on the trivial lattice \( T(X_b) \) for any \( b : B \to C \). Then \( \Phi \) is the identity map.

**Proof.** For \((l, B, b) \in \mathcal{B}_C\), let \( K_b \) denote the function field \( l(B) \). Since \( \Phi \) fixes the trivial lattice, by passing to the quotient \( NS(X_b)/T(X_b) \), it yields a homomorphism, say \( \phi_b^0 : E(K) \to E(K) \) of the Mordell-Weil group \( E(K_b) \) of the generic fibre to itself. For \( P \in E(K_b) \), let \( u_b(P) = \phi_b^0(P) - P \). The map \( u_b : E(K_b) \to E(K_b) \) is a homomorphism, \( u_b(P + Q) = u_b(P) + u_b(Q) \). The universal property of \( \Phi \) implies that the maps \( u_b \) patch to give a map \( U : E(K) \to E(K) \).

**Lemma 5.** For any \( P \in E(K_b) \), \( u_b(P) \) lies in the narrow Mordell-Weil group \( E_O(K_b) \).

**Proof.** The group of components of the special fibre \( \mathcal{E}_{b,t} \) of the Néron model of the base changed elliptic curve \( E_b \) at a point \( t \) of ramification of \( \pi \) is indexed by the irreducible components of \( X_t \). Since by assumption, \( \Phi \) acts as identity on the set of irreducible components of the singular fibre, the sections \( \Phi((P)) \) and \( (P) \) pass through the same irreducible component of the singular fibre. By the group law on \( \mathcal{E}_{b,t} \), it follows that the element \( u_b(P) \) passes through the identity component of \( X_t \), and this proves the lemma.

Next, we observe the following Galois invariance property:

**Proposition 22.** Let \( L = k(B) \) be a finite Galois extension of \( K = k(C) \) for some \( b \in \mathcal{B}_C \). Suppose \( u \in E_O(K) \), \( u' \in E_O(L) \) and \( nu' = u \) for some \( n \) coprime to \( p \). Then \( u' \in E_O(K) \), i.e., \( u' \) is \( G(L/K) \)-invariant.

**Proof.** The section \((O)\) is defined over \( K \). Thus the identity components of the singular fibres of \( X_b \) are invariant by \( \text{Gal}(L/K) \). Now for any \( s \) in the ramification locus of \( \pi_b \),

\[
\sigma(u').x_0^0 = \sigma(u').\sigma(v_0^n) = u'.v_n^0 = 1.
\]

Hence we obtain that \( \sigma(u') \in E_O(L) \). Since \( n\sigma(u') = \sigma(nu') = \sigma(u) = u \), we have \( \sigma(u') = u' + t \), for some \( n \)-torsion element \( t \in E(L) \). Since \( u' \in E_O(L) \), the element \( t \in E_O(L) \). Since \( E_O(L) \) is torsion-free, this implies \( t = O \) and proves the proposition. \( \square \)
Corollary 2. Given any $0 \neq u \in E_0(K)$, there exists some $n$ sufficiently large such that any $u' \in E(\bar{K})$ with $nu' = u$, then $u'$ cannot lie in $E_0(K)$.

If $u' \in E_0(\bar{K})$, by the proposition, we have that $u' \in E(K)$, and hence in $E_0(K)$. By the theorems of Mordell-Weil ([31]), and Lang-Néron ([LN]), $E_0(K)$ is a finitely generated abelian group, and free by Theorem 17. Hence the corollary follows.

We can now finish the proof of Theorem 21. Given $P \in E(K)$ and $u(P) \in E_0(K)$, choose $n$ as in the above corollary, and $Q \in E(\bar{K})$ with $nQ = P$. Then $nU(Q) = U(P)$. Since $U(Q)$ belongs to $E_0(\bar{K})$, the corollary implies that $U(Q) = 0$, i.e., $\phi(0) = Q$. Since $nQ = P$, and sections are mapped to sections, this implies $\Phi((P)) = (P)$. □

8.1. Descent and proof of Theorems 2 and 3. The preceding arguments establish Theorem 2 and 3 except in the case when the characteristic of $k$ is zero and the Kodaira types of the singular fibres are of the form $I_n$ with $n \geq 3$. Here we have a priori that $\Phi$ arises from an isomorphism $\theta : \mathcal{E} \to \mathcal{E}'$ which is defined over a quadratic extension $l$ of $k$. Let $\sigma$ be the non-trivial element of $\text{Gal}(l/k)$. Denote by $\Phi^\sigma = \sigma \circ \Phi \circ \sigma$. These maps are equal on the Néron-Severi groups $N(X_t)$ of the singular fibres $X_t$, $t \in S$ of $X \to C$. It follows from Proposition 25 that the maps $\Phi$ and $\Phi^\sigma$ are equal. By Theorem 21, it follows that the maps $\theta$ and $\theta^\sigma$ are equal on $l^\infty$-torsion, and hence they are equal. This establishes the required descent property for the proofs of Theorems 2 and 3.

9. Base change and fibral divisors

Let $\mathcal{E} : X \to C$ be a semistable elliptic surface over $k$. Fix a point $x_0$ of in the ramification locus of $\pi$ such that the fibre of $\pi$ over $x_0$ is of type $I_n$ for some $n \geq 2$. Let $v_i$ be the irreducible components of the singular fibre of $p$ at $x_0$ where the indexing is the group $\mathbb{Z}/n\mathbb{Z}$, and the intersection multiplicities are as follows for $i, j \in \mathbb{Z}/n\mathbb{Z}$:

$$v_i^2 = -2, \quad v_iv_{i+1} = 1 \quad \text{and} \quad v_iv_j = 0 \quad \text{for} \quad j \neq i, i-1, i+1.$$  

We assume that the zero section passes through $v_0$.

Let $b \in B_C$ be a separable, finite morphism of degree $d$. Fix a point $y_0 \in B(k)$ mapping to $x_0$. Assume that $y_0$ is totally ramified over $x_0$ of degree $d$. It is known that the fibre of $X_b \to B$ over the point $y_0$ is of type $I_{nd}$. Let $w_i, i \in \mathbb{Z}/nd\mathbb{Z}$ be the irreducible components of the singular fibre over $y_0$, with indexing similar to the one given above. We would like to describe the inverse image divisors $p_b^* (v_i)$.

Proposition 23. With notation as above,

$$p_b^* (v_i) = dw_{di} + \sum_{0 < j < d} (d-j)(w_{di-j} + w_{di+j}).$$

Proof. Since $b$ is totally ramified at $y_0$ of degree $d$, $p_b^* F_{x_0} = dF_{y_0}$, where $F_{x_0}$ (resp. $F_{y_0}$) denotes the fibre of $p$ at $x_0$ (resp. the fibre of $p_b$ at $y_0$).
For $i \in \mathbb{Z}/d\mathbb{Z}$, let $w_k$ denote the ‘inverse image divisor’, i.e., the strict transform in $X_b$ of the inverse image of $v_i$ in $X^b = X \times_C B$. The multiplicity of $w_k$ in $p^*_b(v_i)$ is $d$. Further $p^*_b(v_i)$ is an effective divisor, and
\[ dF_{y_0} = p^*_b F_{x_0} = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} p^*_b(v_i). \]

Hence,
\[ p^*_b(v_0) = dw_0 + \sum_{i \neq 0} a_i w_i, \quad \text{where} \quad 0 \leq a_i \leq d. \]

Since $w_{k_j}$ occurs in $p^*_b(v_j)$ with multiplicity $d$, it follows that $a_{k_j} = 0$ for $j \neq 0$.

The map $p_b : X_b \to X$ is a finite proper map. The divisors $w_j$ for $j \neq k_i$ for any $i$, map to a point under $p_b$ and are the exceptional divisors in $F_{y_0}$. By the projection formula of intersection theory ([H, Appendix A, p. 427]),
\[ p^*_b(w_j p^*_b(v_0)) = p^*_b(w_j)v_0 = 0. \]

Since $p_b$ is finite and proper, it follows that $w_j p^*_b(v_0) = 0$. Thus,
\[ 2a_j = a_{j-1} + a_{j+1}, \quad j \neq k_i \quad \text{for} \quad i \in \mathbb{Z}/n\mathbb{Z}. \quad (9.1) \]

Since the multiplicities $a_j$ are bounded by $d$, if the multiplicity $a_j = d$ for some $j$, then the neighbouring multiplicities $a_{j-1}$ and $a_{j+1}$ are also equal to $d$. But $a_{k_i} = 0$ for $i \neq 0$, and this implies that the multiplicity of any exceptional divisor $w_j$ occurring in $p^*_b(v_0)$ is strictly less than $d$.

Now, $(p^*_b v_0)^2 = dw_0^2 = -2d$. Since the exceptional divisors intersect trivially with $p^*_b(v_0)$, and the inverse image divisors $w_{k_j}$ have multiplicity zero in $p^*_b(v_0)$ for $j \neq 0$, we obtain
\[ d((w_0, dw_0) + a_1(w_0, w_1) + a_{-1}(w_0, w_{-1})) = -2d. \]

This implies, $a_1 + a_{-1} = 2d - 2$, and this is possible if and only if $a_1 = a_{-1} = d - 1$.

From Equation (9.1), it follows that for $0 < i \leq d$, $a_i = a_{-i} = (d - i)$.

Thus, $p^*_b v_0 = D_1 + D_2$, where $D_1 = dw_0 + \sum_{0<j<d} (d-j)(w_{-j} + w_j)$, and the multiplicity of the components $w_i$ for $|i| \leq d$ is zero in $D_2$. Hence the divisors $D_1$ and $D_2$ are orthogonal. The divisors $D_1$ and $D_2$ can be considered in the negative definite space spanned by the divisors $w_0, \ldots, w_{d-1}, w_{d+1}, \ldots, w_{nd-1}$. Since $D_1^2 = -2d = (p^*_b v_0)^2$, this implies $D_2 = 0$. Hence we obtain that
\[ p^*_b v_0 = dw_0 + \sum_{0<j<d} (d-j)(w_{-j} + w_j), \]

and a similar expression holds for each $p^*_b v_i$:
\[ p^*_b v_i = dw_{k_i} + \sum_{0<j<d} (d-j)(w_{k_i-j} + w_{k_i+j}). \]

Since the sum of these divisors is equal to $dF_{y_0}$, it follows that $k_i$ are multiples of $d$. Since $p^*_b v_i$ and $p^*_b v_{i+1}$ have non-zero intersection, we get $k_i = di$ for $i = 0, \ldots, d$, and this proves the proposition. □
10. A REPRESENTATION OF THE AFFINE WEYL GROUP
OF $A_{n-1}$: PROOF OF THEOREM 8

Our aim is to show that the Picard-Lefschetz isometries based at irreducible components of fibral divisors lift to universal isometries. We first work out the underlying representation theoretical aspect, which arise when we consider the action of the reflections on the subspace of the Néron-Severi group contributed by the components of a fibre.

Corresponding to a fibration with local ramification degree $e$, we define a homomorphism, say $R_e$, of $W_n$ into $W_{ne}$. We define this representation on the generators, and verify the braid relations are satisfied. We also need to check that this representation gives the lift of the Picard-Lefschetz reflections to an universal isometry. We first work out some of the linear algebra considering the vectors $v(i,j)$ defined as in Definition 2.3. We use the notation from Section 2.

**Lemma 6.** With notation as in Definition (2.3), the following holds:

1. $v(i,j)^2 = -2$. In particular, the transformation,
   $s_{v(i,j)}(v) = v + (v(i,j),v)v(i,j)$, $v \in V_n$,
   defines a reflection.

2. Suppose $v(i_1,j_1) \neq v(i_2,j_2)$ are distinct vectors of equal length such that the intersection of their supports is non-empty. Then they are orthogonal. Equivalently the reflections $s_{v(i_1,j_1)}$ and $s_{v(i_2,j_2)}$ commute.

3. Any vector $v$ with $v^2 = -2$ is of the form $v(i,j) + rF$, where $F = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} v_i$ is the 'fibre' and $r \in \mathbb{Z}$.

4. The following relation is satisfied by the reflections $s_{v(i,j)}$:
   $$s_{v(i,j)} = s_{v_j}s_{v(i,j-1)}s_{v_j}.$$ (10.1)

**Proof.** For the proof of (1),

$$v(i,j)^2 = \sum_{k=i}^{j} v_k^2 + v_i v_{i+1} + v_j v_{j-1} + \sum_{k=i+1}^{j-1} (v_k v_{k-1} + v_k v_{k+1})$$

$$= -2(j - i + 1) + 2 + 2(j - i - 1) = -2.$$

To prove (2), let $(i_1,j_1) = (i,j)$, and $i_2 = i + k$ with $k > 0$. The hypothesis imply that $k \leq (j - i)$, and $j_2 = j + k$. The inner product,

$$v(i,j)v(i+k,j+k) = v_{i+k-1}v_{i+k} + v(i+k,j)^2 + v_j v_{j+1}$$

$$= 1 - 2 + 1 = 0.$$

To prove (3), write $v = \sum_i a_i v_i + rF$, where we can assume by absorbing into the fibre component, that not all $a_i$ have the same sign. Write $v = v_+ - v_-$, where $v_+ = \sum_{a_i > 0} a_i v_i$ and $v_- = -\sum_{a_i \leq 0} a_i v_i$. Then,

$$v^2 = v_+^2 + v_-^2 - 2v_+ v_-.$$
If both $v_+$ and $v_-$ are non-zero, then $v^2 < -2$. Since $v(i, j) + v(j, i) = F$, by working with $-v$ if required, we can assume that $v_- = 0$. Assume now that the support of $v$, the set of indices $i$ such that $a_i > 0$ is an interval of the form $[k, l]$, $0 \leq k < l < n$. Write $v = v(k, l) + v'$, where $v' = \sum_{k \leq i \leq l} b_i v_i$, $b_i \geq 0$. Now,

$$v^2 = v(k, l)^2 + 2v(k, l)v' + v'^2 = -2 - 2 \sum_{k \leq i \leq l} b_i + b_k + b_l + 2 \sum_{k < i < l} b_i + v'^2 = -2 - b_k - b_l + v'^2.$$ 

Hence if $v^2 = -2$, then $v = v(k, l)$. This equation also implies that for $v$ as above $v^2 \leq -2$. Thus if $v^2 = -2$, then $v$ cannot be written as a sum of two vectors with disjoint support. This proves Part (3).

For (4), the vectors $v(i, j - 1)$ and $v_j$ generate a two dimensional non-degenerate subspace. On the orthogonal complement of this subspace, both the reflections act as identity, and hence it suffices to verify the formula

$$s_{v(i, j)} = s_{v_j} s_{v(i, j-1)} s_{v_j},$$

in the two dimensional situation. Since $v(i, j - 1)$ and $v_j$ are vectors with self-intersection $-2$ and $v(i, j - 1).v_j = 1$, this is classical. □

10.1. Cyclic permutations. Before we proceed with the proof of Theorem 8, it is convenient to represent the transformations in terms of standard permutation symbols $(j_1 \cdots j_k)$. We consider permutations on the cyclic set $\mathbb{Z}/n\mathbb{Z}$. Assign the permutation $s_{v_i} \mapsto (i, i + 1)$. This gives a representation of the affine Weyl group $W_n$ as permutations on the cyclic set $\mathbb{Z}/n\mathbb{Z}$. It is clear that the braid relations are satisfied.

We observe that dropping one of the generators, for example $s_0$ from the generators of the affine Weyl group, the remaining generators $s_1, \ldots, s_{n-1}$ satisfy the braid relations defining the symmetric group $S_{n+1}$ (the permutation group on $(n + 1)$-symbols) on $n$-generators:

$$m_{ii} = 1, m_{i(i+1)} = 3 \quad \text{and} \quad m_{ij} = 2 \quad \text{for} \quad |i - j| \geq 2, \quad i, j \in \{1, 2, \ldots, n\}.$$ 

The above permutation representation restricted to any of the symmetric groups obtained by omitting one of the generators $s_i$ is injective, since the group generated is not of the form $\mathbb{Z}/2\mathbb{Z}$ if the number of generators is at least two.

**Definition 10.1.** An expression (or an equation) in the free group on the generators of the affine Weyl group $W_n$ (or in $W_n$) is said to be local, if it does not involve all the generators in a non-trivial manner.

When the equation or expression is local, then to check its properties or the validity of the equation in the affine Weyl group, it is sufficient to work with the (local) symmetric group generated by the generating reflections of the affine Weyl group involved in the equality. In such a case, it is sufficient to work within the group of cyclic permutations.
We now recall the definition of the representation $R^e_n : W_n \rightarrow W_{ne}$. To keep track of the difference, the standard basis of $\mathbb{V}_n$ is given by $v_i$, $i \in \mathbb{Z}/n\mathbb{Z}$ and that of $\mathbb{V}_{ne}$ is given by $w_i$, $i \in \mathbb{Z}/ne\mathbb{Z}$. The representation $R^e_n$ is defined on the generators $s_{v_0}, \ldots, s_{v_{n-1}}$ of $W_n$ as:

$$R^e_n(s_{v_k}) = \prod_{w(i,j) \in I(n,e,k)} s_{w(i,j)},$$

(10.2)

where the set $I(n,e,k)$ is the collection of vectors of the form $w(i,j) \in \mathbb{V}_{ne}$ of length $e$ and support containing $ek$.

For the proof, especially of parts (2) and (3) of Theorem 8, we observe that the various statements are local in the above sense, that it is enough to work with the symmetric group and hence enough to work with the permutations. This is because if $e > 1$, then the elements $R^e_n(s_{v_k})$ are in some appropriate symmetric groups by Part (4) of Lemma 6.

### 10.2. An inductive definition

We first need to check that $R^e_n(s_{v_k})$ can be written in terms of the generators $s_{w_i}$. The following inductive definition of $R^e_n(s_{v_k})$ is arrived at trying to ensure that the lifts satisfy Part (1) of Theorem 8 of being compatible with the base change map on the Néron-Severi groups given by Proposition 23.

For $0 \leq j < n$, $1 \leq i \leq e$, define the following isometries of $\mathbb{V}_{ne}$:

$$T(v_j, 0) = U(v_j, 0) = s_{w_{je}}$$

(10.3)

$$U(v_j, i) = s_{w_{je-i}}s_{w_{je+i}}$$

(10.4)

$$T(v_j, i) = U(v_j, 0) \cdots U(v_j, i) = T(v_j, i - 1)U(v_j, i)$$

(10.5)

$$S(v_j, i) = T(v_j, i - 1) \cdots T(v_j, 0) = T(v_j, i - 1)S(v_j, i - 1).$$

(10.6)

Since $n \geq 3$ and $e \geq 1$, an inductive argument implies that for any given $j$ and $i$, the expressions $U(v_j, i), T(v_j, i)$ and $S(v_j, i)$ are all local.

**Lemma 7.** With the assignment $s_{w_i} \mapsto (i, i + 1)$,

$$T(v_j, k) = (je, je - 1, \ldots, je - k, 1, \ldots, je + k + 1)$$

(10.7)

$$S(v_j, k) = (je - k + 1, je + 1)(je - k + 2, je + 2) \cdots (je, je + k).$$

(10.8)

**Proof.** The proof is by induction on $k$, and we carry it out for $j = 0$. The transformation $U(v_0, k)$ is given in permutation notation as,

$$U(v_0, k) = (k, k + 1)(-k, -k + 1).$$

Assuming that the lemma has been proved for $k - 1$. Then

$$T(v_0, k) = T(v_0, k - 1)U(v_0, k)$$

$$= (0, -1, \ldots, -k + 1, 1, \ldots, k)(k, k + 1)(-k, -k + 1)$$

$$= (0, -1, \ldots, -k, 1, \ldots, k + 1).$$
Similarly, assuming that the proposition holds for $S(v_0, k - 1)$,
\[
S(v_0, k) = T(v_0, k - 1)S(v_0, k - 1)
= (0, -1, \cdots, -k + 1, 1, \cdots, k)(-k + 2, 1)(-k + 3, 2) \cdots (0, k - 1)
= (-k + 1, 1)(-k + 2, 2) \cdots (0, k).
\]
This proves the lemma.

**Lemma 8.** In $W_{ne}$, the reflection $s_{w(p,q)}$, $p < q$, $|q - p| \leq e$ corresponding to the vector $w(p,q) = w_p + w_{p+1} + \cdots + w_q$ is local. The cyclic permutation corresponding to $s_{w(p,q)}$ is the permutation $(p, q + 1)$.

**Proof.** The proof is by induction on $|q - p|$. The case $q = p$ follows from the definition.

By Part (4) of Lemma 6,
\[
s_{w(p,q)} = s_{w_q}s_{w(p,q-1)}s_{w_q}.
\]
This implies the locality of $s_{w(p,q)}$ in the given range. Translating to the permutation notation, we see that
\[
(q, q + 1)(p, q)(q, q + 1) = (p, q + 1).
\]
This proves the lemma.

Combining the two foregoing lemmas and the definition of $R_n^e(s_{v_j})$, we have the following corollary,

**Corollary 3.** For $0 \leq j < n$, the permutation realization of $R_n^e(s_{v_j})$ is
\[
((j - 1)e + 1, je + 1)((j - 1)e + 2, je + 2) \cdots (je, (j + 1)e).
\]
In particular, $R_n^e(s_{v_j}) \in W_{ne}$.

**Proof.** By definition,
\[
R_n^e(s_{v_j}) = s_{w((j-1)e+1,je)}s_{w((j-1)e+2,je+1)} \cdots s_{w(je,(j+1)e-1)}.
\]
The permutation realization of the right hand side is nothing more than
\[
S(v_j, e) = ((j - 1)e + 1, je + 1)((j - 1)e + 2, je + 2) \cdots (je, (j + 1)e).
\]

**Proof of Part (1) of Theorem 8** We want to show that the lift $R_n^e(s_{v_j})$ is compatible with the base change map $p_b^*: \mathbb{V}_n \to \mathbb{V}_{ne}$,
\[
R_n^e(s_{v_i})(p_b^*(v)) = p_b^*(s_{v_i}(v)), \quad i \in \mathbb{Z}/n\mathbb{Z}, \quad v \in \mathbb{V}_n, \quad (10.9)
\]
where $p_b^*$ is defined on the generators as,
\[
p_b^*(v_k) = ew_k + \sum_{i=1}^{e-1}(w_{ke-i} + w_{ke+i}), \quad (10.10)
\]
as dictated by Proposition 23.
Lemma 9. \hspace{1em} (1) For \( k \geq 1 \),
\begin{equation}
R_n^e(s_{v_j})(w_{j_e}) = -\sum_{i=(j-1)e+1}^{(j+1)e-1} w_i. \tag{10.11}
\end{equation}

(2) For \( e < |i - je| \),
\begin{equation}
R_n^e(s_{v_j})(w_i) = w_i. \tag{10.12}
\end{equation}

(3)
\begin{equation}
R_n^e(s_{v_j})(w_{(j+1)e}) = \sum_{i=0}^{e} w_{je+i} \hspace{1em} \text{and} \hspace{1em} R_n^e(s_{v_j})(w_{(j-1)e}) = \sum_{i=0}^{e} w_{je-i}. \tag{10.13}
\end{equation}

(4) For \( 0 < i < e \),
\begin{equation}
R_n^e(s_{v_j})(w_{je-i}) = w_{(j+1)e-i} \hspace{1em} \text{and} \hspace{1em} R_n^e(s_{v_j})(w_{je+i}) = w_{(j-1)e+i}. \tag{10.14}
\end{equation}

Proof. The vectors \( w(i, j) = w_i + \cdots + w_j \) are orthogonal to all the base vectors \( w_k \), except when \( k = i - 1, i, j, j + 1 \). In these cases,
\begin{align*}
(w(i, j), w_{i-1}) &= 1, \hspace{1em} (w(i, j), w_i) = -1, \hspace{1em} (w(i, j), w_j) = -1, \hspace{1em} (w(i, j), w_{j+1}) = 1.
\end{align*}
To simplify the indices, we prove the statement taking \( j = 0 \). We have,
\begin{equation}
R_n^e(s_{v_i}) = s_{w(-e+1,0)} s_{w(-e+2,1)} \cdots s_{w(0,e-1)}. \tag{10.15}
\end{equation}
For proving (1), all the reflections except \( s_{w(-e+1,0)} \) and \( s_{w(0,e-1)} \) fix the vector \( w_0 \). Hence,
\begin{align*}
R_n^e(s_{v_i})(w_0) &= s_{w(-e+1,0)} s_{w(0,e-1)}(w_0) \\
&= s_{w(-e+1,0)}(w_0 + (w(0, e - 1), w_0)w(0, e - 1)) \\
&= s_{w(-e+1,0)}(w_0 - w(0, e - 1)) = -s_{w(-e+1,0)}(w_1 + \cdots w_{e-1}) \\
&= -(w_2 + \cdots w_{e-1} + w_1 + (w_1, w(-e + 1, 0))w(-e + 1, 0)) \\
&= -\sum_{i=-e+1}^{e-1} w_i.
\end{align*}
For the proof of Part (2), we observe that the isometry \( R_n^e(s_{v_0}) \) involves only the Picard-Lefschetz reflections \( s_{w_l} \) for \( |l| < e \). Each one of these reflections fixes \( w_i \), since \( |l| + 1 \leq e < |i| \). Hence
\begin{equation}
R_n^e(s_{v_0})(w_i) = w_i \hspace{1em} |i| > e. \tag{10.16}
\end{equation}
For the proof of Part (3), reasoning as in the proof of Part (1),
\begin{align*}
R_n^e(s_{v_0})(w_e) &= s_{w(0,e-1)}(w_e) = w_e + (w_e, w(0, e - 1))w(0, e - 1) \\
&= \sum_{i=0}^{e} w_i.
\end{align*}
The proof of the other equality follows in a similar manner.
To prove Part (4), we observe that the only reflections occurring in $R^e_n(s_{v_0})$ not fixing $w_i$ are the reflections based on the vectors $w(i - e, i - 1)$ and $w(i - e + 1, i)$. These vectors are orthogonal. Thus,

$$R^e_n(s_{v_0})(w_i) = s_{w(i-e,i-1)}s_{w(i-e,i)}(w_i) = s_{w(i-e,i-1)}(w_i - w(i - e + 1, i))$$

$$= w_i + (w(i - e, i - 1), w_i)w(i - e, i - 1)$$

$$- w(i - e + 1, i) - (w(i - e, i - 1), w(i - e, i - 1))w(i - e, i + 1)$$

$$= w_i + w(i - e, i - 1) - w(i - e + 1, i)$$

$$= w_{i-e}.$$

We now establish Equation $[10.9]$. To do this, we do it for $j = 0$, and take $v$ to be one of the basis vectors. The Picard-Lefschetz isometries $s_{w_k}$ for $0 < |k| < e$ involved in the definition of $R^e_n(s_{v_0})$ correspond to exceptional divisors. As in the proof of Proposition $[23]$ the divisors $w_k$ for $0 < |k| < e$ are exceptional, and hence do not intersect the pullback divisors $p^*_b(v_i)$. Hence the reflections $s_{w_k}$ for $0 < |k| < e$ fix $p^*_b(v_i)$. The reflection $s_{v_0}$ fixes the pullback vectors $p^*_b(v_i)$ for $|i| > 1$. Hence $R^e_n(s_{v_0})$ fixes $p^*_b(v_i)$ when $|i| > 1$ and the theorem is proved for such basis vectors.

Hence we are reduced to checking the commutativity for the basis vectors $v_0$, $v_1$ and $v_{-1}$. Using various parts from Lemma $[9]$ we obtain

$$R^e_n(s_{v_0})(p^*_b(v_0)) = R^e_n(s_{v_0}) \left( ew_0 + \sum_{i=1}^{e-1} (e - i)(w_{-i} + w_i) \right) \quad \text{(by Equation $[10.10]$)}$$

$$= -e \left( \sum_{i=-(e-1)}^{e-1} w_i \right) + \sum_{i=1}^{e-1} (e - i)(w_{-e-i} + w_{-(e-i)}) \quad \text{(by Equations $[10.11]$ $[10.13]$)}$$

$$= - \left( ew_0 + \sum_{i=1}^{e-1} (e - i)(w_{-i} + w_i) \right) = -p^*_b(v_0).$$

This proves that

$$R^e_n(s_{v_0})(p^*_b(v_0)) = p^*_b(s_{v_0}(v_0)) = -p^*_b(v_0). \quad (10.15)$$

We now check the commutativity for the divisor $v_1$ (and the same proof works for $v_{-1}$). We can write,

$$p^*_b(v_1) = ew_1 + \sum_{i=1}^{e-1} (e - i)(w_{e-i} + w_{e+i}).$$
By Lemma 9

\[ R_n^e(s_{v0}) \left( e w_e + \sum_{i=1}^{e-1} (e-i)(w_{e-i} + w_{e+i}) \right) \]

\[ = e R_n^e(s_{v0})(w_e) + R_n^e(s_{v0}) \left( \sum_{i=1}^{e-1} (e-i)w_{e-i} \right) + R_n^e(s_{v0}) \left( \sum_{i=1}^{e-1} (e-i)w_{e+i} \right) \]

\[ = e \sum_{i=0}^e w_i + \sum_{i=1}^{e-1} (e-i)w_i + \sum_{i=1}^{e-1} (e-i)w_{e+i} \]

\[ = p_b^*(v_0) + p_b^*(v_1). \]

Hence we get,

\[ R_n^e(s_{v0})(p_b^*(v_1)) = p_b^*(s_{v0}(v_1)) = p_b^*(v_0 + v_1). \]  

(10.16)

This proves Part (1) of Theorem \ref{thm:main}

**Proof of Part (2) of Theorem \ref{thm:main}** We now show that \( R_n^e \) defines a representation of \( W_n \) to \( W_{ne} \). It follows from Part (2) of Lemma \ref{lem:local} that the transformations \( s_{v(i,j)} \) appearing in the definition of \( R_n^e(s_k) \) are reflections that commute with each other. Hence, \( R_n^e(s_k)^2 = 1 \).

We need to check the braid relations are satisfied by \( R_n^e(s_k) \). For this, it is convenient to work with the permutation realization of these isometries. Since \( n \geq 3 \), given any \( i, j \in \mathbb{Z}/n\mathbb{Z} \), the braid relations involving \( R_n^e(s_i) \) and \( R_n^e(s_j) \) are local. Hence we can work with the permutation representation of these expressions. We write down explicitly, the permutation realization of the transformations \( R_n^e(s_k) \) for \( k = 0, 1, m \):

\[ R_n^e(s_0) = (-e + 1, 1)(-e + 2, 2) \cdots (0, e) \]

\[ R_n^e(s_1) = (1, e + 1)(2, e + 2) \cdots (e, 2e) \]

\[ R_n^e(s_m) = ((m-1)e + 1, me + 1)((m-1)e + 2, me + 2) \cdots (me, (m+1)e), \]

where we have used the equality sign to denote the realization as permutations on the set \( \mathbb{Z}/ne\mathbb{Z} \). The transpositions \( (k, k+e) \) for \( k \leq 0 \) appearing in the realization of \( R_n^e(s_0) \) and the transposition \( (l, l+e) \) for \( (m-1)e + 1 \leq l \leq me \) appearing in the realization of \( R_n^e(s_m) \) for \( |m| \geq 2 \) commute with each other. Hence it follows that \( R_n^e(s_0) \) and \( R_n^e(s_m) \) for \( |m| \geq 2 \) commute.

It remains to show that \( (R_n^e(s_0)R_n^e(s_1))^3 = 1 \). The transposition \( (k, k+e) \) for \( k \leq 0 \) commutes with the transpositions \( v(l, l+e) \) for \( 1 \leq l \leq e \) except when \( l = k + e \). The product \( (k, k+e)(k+e, k+2e) \) is order 3. Hence it follows \( (R_n^e(s_0)R_n^e(s_1))^3 = 1 \).

A similar calculation applies by replacing the indices 0, 1 and \( m \), and this proves Part (2) of Theorem \ref{thm:main}

**Proof of Part (3) of Theorem \ref{thm:main}** We now want to prove that the family of representations of the affine Weyl groups we constructed satisfy the composition relation:

\[ R_{ne}^f \circ R_n^e = R_n^{ef}, \]
where $n, e, f$ are any natural numbers. This statement is the compatibility relation with respect to the composition of pullbacks that is required of the universal Picard-Lefschetz isometries. We first compute the lifts $R_n^e(s_{v(i,j)})$ of the reflection $s_{v(i,j)}$ based at the vector $v(i,j) = v_i + \cdots + v_j$:

**Lemma 10.** For $k \geq 1$, the permutation realization of $R_n^e(s_{v(j,j+k-1)})$ is given by

$$((j-1)e+1, (j+k)e+1) \cdots (je, (j+k)e).$$

**Proof.** The proof is by induction on $k$. We take $j = 0$ and for $k = 1$, the permutation realization of $R_n^e(s_{v(0,k)})$ is $(-e+1,1)(-e+2,2) \cdots (0,e)$. Assume that the lemma has been proved for $k - 1$. By Part (4) of Lemma 6, $s_{v(0,k)} = s_{v_k}s_{v(0,k-1)}s_{v_k}$. Hence the permutation realization of $R_n^e(s_{v(0,k)})$ is given by,

$$\{( (k-2)e+1, (k-1)e+1) \cdots ((k-1)e, ke) \}$$

$$\{(-e+1, (k-1)e+1) \cdots (0, (k-1)e) \}$$

$$\{((k-2)e+1, (k-1)e+1) \cdots ((k-1)e, ke) \}$$

$$= (-e+1, ke+1) \cdots (0, ke),$$

and this proves the lemma. \qed

From the definition of $R_n^e(s_{v(0)})$, we get

$$R_n^f(R_n^e(s_{v(0)})) = R_n^f(s_{v(-e+1,0)}) \cdots R_n^f(s_{v(0,e-1)}).$$

Upon substituting $n = ne$ and $e = f$, in the equation given by Lemma 10, the permutation realization (as permutations on $\mathbb{Z}/nef\mathbb{Z}$) of $R_n^f(s_{v(j,j+e)})$ is,

$$((j-1)f+1, (j+e)f+1) \cdots (je, (j+f)e).$$

Hence the permutation realization of $R_n^f(R_n^e(s_{v(0)}))$ is given by,

$$(-fe+1,1)(-fe+2,2) \cdots (0, fe),$$

which is equal to the permutation realization of $R_n^{ef}(s_{v(0)})$.

As all these expressions are local, the equality as permutations establishes Part (3) of Theorem 8 for the reflection $s_{v(0)}$. By symmetry it establishes for the other generators. Since we know that the collection of maps $R_n^e : W_n \to W_{ne}$ define homomorphisms as $n$ and $e$ varies, this establishes Part (3) of Theorem 8. \qed

**Corollary 4.** Let $n \geq 3$. For any $x \in W_n$, the collection of elements $R_n^e(x) \in W_{ne}$ are compatible isometries in the following sense: for any natural numbers $e, f$ and $w_j \in \mathbb{V}_{ne}$,

$$R_n^f(x)(p_b^*(w_j)) = p_b^*(R_n^e(x)(w_j)),$$

where

$$p_b^*(w_j) = f z_{j} + \sum_{l=1}^{f-1} (f-l)(z_{j-l} + z_{j+l}),$$

is the base change map defined from $\mathbb{V}_{ne} \to \mathbb{V}_{nef}$ as in Proposition 22 with standard bases $w_j, j \in \mathbb{Z}/ne\mathbb{Z}$ and $z_l, l \in \mathbb{Z}/nef\mathbb{Z}$ for $\mathbb{V}_{ne}$ and $\mathbb{V}_{nef}$ respectively.
11. Universal isometries: Proof of Theorem 6

We now show that the Picard-Lefschetz reflections define universal isometries of the family of Néron-Severi lattices $NS(X_b)$ as $b$ varies. Suppose $X \to C$ a semistable, elliptic surface and the Kodaira fibre type at a point $x_0 \in C(k)$ is of type $I_n$ with $n \geq 3$. Given an irreducible component $v$ of the singular fibre at $x_0$, the map

$$s_v(x) = x + < x, v > , \quad x \in NS(X),$$

defines the Picard-Lefschetz reflection based at $v$ of $NS(X)$. Let $v_0, \ldots, v_{n-1}$ be the irreducible components of the singular fibre $p^{-1}(x_0)$. The reflections $s_{v_i}$ generates the affine Weyl group $W_n(x_0)$ based on the fibre $x_0$, giving an action of $W_n(x_0)$ on the Néron-Severi group $NS(X)$ of $X$.

Suppose $b : B \to C$ is a finite, separable map in $\mathcal{B}_C$, and let $y_1, \ldots, y_r$ be the points of $B$ lying above $x_0$. We use the variable $y$ to denote one of the fibres. Suppose that the local ramification degree at $y$ is $e_y$. Let $w^y_0, \ldots, w^y_{ne_y-1}$ be the irreducible components of the singular fibre $p_b^{-1}(y)$. By the results of Section 10 there is a representation $R_y : W_n(x_0) \to Aut(NS(X))$ defined on the Picard-Lefschetz reflection based on the irreducible component $v_k$ of the fibre at $x_0$ as,

$$R_y(s_{v_k}) = \prod_{w(i,j) \in I(n,e_y,k)} s_{w^y(i,j)},$$

where the set $I(n, e_y, k)$ is the collection of vectors of the form $w^y(i, j) = \sum w^y_l$ of length $e_y$ and support containing $ke_y$. Define

$$PL_b(s_{v_k}) = \prod_{y \in b^{-1}(x_0)} R_y(s_{v_k}).$$

By construction, $\theta_b(s_{v_k})$ is an element of $Aut(NS(X))$.

In order to prove Theorem 6 that $PL_b(s_{v_k})$ defines a universal isometry, it needs to be checked its compatibility with the base change map $p_b^* : NS(X_b) \to NS(X_{boa})$ for maps $A \xrightarrow{a} B \xrightarrow{b} C$. On the fibral divisors this compatibility is given by Corollary 4. We need to check it only on sections.

Suppose $(P)$ is a section of $\pi$ not passing through $v_0$. Then $s_{v_0}$ fixes $(P)$. The pullback section $p_b^*((P))$ intersects the fibre over $y$ at one of the components $w_{ke_y}, k \neq 0$. Since the definition of $PL_b(s_{v_k})$ involves onlythe reflections corresponding to exceptional divisors $w_i, 0 \leq |i| < e_y$ $PL_b(s_{v_0})$ fixes $p_b^*((P))$.

Now lets assume that $(P)$ be a section of $\pi$ passing through $v_0$. Let $w^y_0$ be the identity component at the fibre over $y_0$ of the pullback divisor $p_b^*(v_0)$. The pullback section $p_b^*(P)$, is a section of $X_b \to B$ passing through $w^y_0$ for $i = 1, \ldots, r$. Using the fact that the reflections appearing in the definition of $R_y(s_{v_0})$ are mutually orthogonal
we get,
\[ R_y(s_{v_0})(p_b^*(P)) = s_{w^y(-e_y+1,0)}s_{w^y(-e_y+2,1)} \cdots s_{w^y(0,e_y-1)}(p_b^*(P)) \]
\[ = p_b^*(P) + w^y(-e_y + 1, 0) + w^y(-e_y + 2, 1) + \cdots + w^y(0, e_y - 1) \]
\[ = p_b^*(P) + e_y w^y_0 + \sum_{j=1}^{e_y-1} (e - j)(w^y_{-j} + w^y_j). \]

Then,
\[ PL_b(s_{v_0})((P)) = R_{y_1}(s_{v_0}) \cdots R_{y_r}(s_{v_0})((P)) \]
\[ = p_b^*(P) + \sum_{i=1}^r \sum_{j=1}^{e_{y_i}} \left( e_{y_i}w^y_0 + \sum_{j=1}^{e_{y_i}-1} (k - j)(w^y_{-j} + w^y_j) \right) \]
\[ = p_b^*((P) + v_0) = p_b^*(s_{v_0}((P)). \]

This proves the compatibility of \( PL_b(s_{v_0}) \) with the pullback map on sections, thereby showing that it defines a universal isometry, and finishes the proof of Theorem 6.

12. PROOF OF THEOREM 7

Let \( S \) be the singular locus of \( \pi : X \to C \). For \( t \in S \), let the singular fibre be of Kodaira type \( I_n \). The space \( N(X_t) \), the subspace of \( N(X) \) generated by the components of the fibre of \( \pi \) based at \( t \), equipped with its intersection pairing is isomorphic to the root lattice of type \( \tilde{A}_{n-1} \). Let \( A(N(X_t)) \) be the automorphism group of \( N(X) \) generated by the Picard-Lefschetz transformations based on the irreducible components of the fibre \( X_t \). The group \( A(N(X_t)) \) is isomorphic to the affine Weyl group \( W_n \).

It follows from Theorems 8 and 6 that the maps \( PL_b \) can be extended multiplicatively to give a representation of the product of the affine Weyl groups over \( t \in S \) as universal isometries of the elliptic surface \( \mathcal{E} \):

\[ PL : \prod_{t \in S} A(N(X_t)) \to \text{Aut}(UNS(X)). \]

Let \( \Phi \in \text{Aut}(UNS(X)) \) be an isometry of \( UNS(X) \). By Proposition 10 after multiplying by the automorphism \(-1\) if required, we can assume that \( \Phi(F) = F \). By Proposition 11 \( \phi : NS(X) \to NS(X) \) restricts to an isometry \( N(X_t) \to N(X_t) \) for \( t \in S \).

The space \( N(X_t) \) can be identified with the root lattice of the affine root system \( \tilde{A}_{n-1} \). Let \( v_i^t, i \in \mathbb{Z}/n_t \mathbb{Z} \) be a standard basis for \( N(X_t) \). We have two bases for this affine root system: \( \{v_0^t, \ldots, v_{n_t-1}^t\} \) and \( \{\phi(v_0^t), \ldots, \phi(v_{n_t-1}^t)\} \).

By [Kac, Proposition 5.9], there exists an element \( x_t \in A(N(X_t)) \simeq W_{n_t} \), that maps the basis \( \{\phi(v_0^t), \ldots, \phi(v_{n_t-1}^t)\} \) to the standard basis \( \{v_0^t, \ldots, v_{n_t-1}^t\} \) or to its negative. Since any element of \( A(N(X_t)) \) is generated by the Picard-Lefschetz transformations,
which preserve the fibre \( F = v_0^t + \cdots + v_{n-1}^t \), so does \( \phi \). It follows that \( x_t \) takes the basis \( \{ \phi(v_0^t), \cdots, \phi(v_{n-1}^t) \} \) to the standard basis \( \{ v_0^t, \cdots, v_{n-1}^t \} \).

By Theorems 4 and 8 we can assume that \( x_t \) defines (universal) automorphisms of \( UNS(X) \). Define \( \Psi \in \text{Aut}(UNS(X)) \) by

\[
\Psi = \Phi \circ \prod_{t \in S} PL(x_t).
\]

Denote by \( \psi \) its restriction to \( NS(X) \). We have,

**Proposition 24.** With notation as above, \( \psi \) maps sections to sections.

**Proof.** The property of \( \psi \) that we require in the proof is that \( \psi \) preserves the standard basis for each singular fibre of \( X \to C \). In particular, this implies that \( \psi(F) = F \).

By renaming if required, it is enough to show that the zero section \((O)\) is mapped to a section by \( \psi \). Write,

\[
\psi((O)) = (P) + V + rF,
\]

where \( V \) is a fibral divisor. It is enough to show that after translation by \((-P)\), \( \psi((O)) \) is a section. Hence we can assume that \( (P) = (O) \). We need to show that \( V \) and \( r \) are zero. Write \( V = \sum_{t \in S} V_t \), where \( V_t \) is the contribution to \( V \) from \( N(X_t) \).

We argue fibrewise and first show that each \( V_t \) is zero, up to modifying \( r \).

Fix \( t \) and for notational ease, we drop the superscript \( t \). Suppose that \( \psi(v_0) = v_k \) for some \( k \neq 0 \), and \( \psi(v_j) = v_0 \). Write \( V = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} a_i v_i \). Modify \( r \), such that \( a_0 = 0, k = j \). For \( l \neq 0, k \), the equation

\[
0 = (O).\psi^{-1}(v_l) = \psi((O)).v_l = (O).v_l + V.v_l = -2a_l + a_{l+1} + a_{l-1},
\]

yields the equality \( a_{l+1} = 2a_l - a_{l-1} \). Going from 0 to \( k \) in the increasing order, we get \( a_l = la_1 \) for \( l \leq k \). Going from \( 0 = n \) to \( k \) in the reverse order, we get \( a_{-l} = la_{-1} \) for \( l \leq n - k \). Hence we get \( ka_1 = ak = a_{-(n-k)} = (n-k)a_{-1} \).

From the equation,

\[
0 = (O).v_j = \psi((O)).\psi(v_j) = (O).v_0 + V.v_0 = 1 + a_1 + a_{-1},
\]

we get \( a_1 = -(1 + a_{-1}) \). Combining these two equations gives,

\[
-k(1 + a_{-1}) = (n-k)a_{-1}, \quad \text{i.e.,} \quad -k = na_{-1}.
\]

Since \( 0 < k < n \), this implies \( a_{-1} \) is non-integral, contradicting the integrality of the coefficients \( a_j \) of \( V \).

Hence this implies that \( k = 0 \), i.e., \( \psi(v_0) = v_0 \), and hence \( \psi(v_i) = v_i \) or \( v_{-i} \). In either case, for \( i \neq 0 \),

\[
0 = (O).v_i = \psi((O)).\psi(v_i) = (O).\psi(v_i) + V.\psi(v_i) = V.\psi(v_i).
\]

As the space generated by the vectors \( v_i \) for \( i \neq 0 \) is negative definite, this implies \( V = 0 \) and \( \psi((O)) = (O) + rF \), for some integer \( r \). Considering self-intersections,

\[
-\chi(X) = (O)^2 = \psi((O))^2 = (O)^2 + 2r = -\chi(X) + 2r,
\]

we get that \( r = 0 \) and hence \( \psi((O)) \) is a section. This proves the proposition. \( \Box \)
As a consequence of this proposition, translating by a section if required, we can assume that \( \psi((O)) = (O) \).

Since \( \psi((O)) = (O) \), it follows that \( \psi(v_t^i) = v_t^i \) for \( t \in S \) (this was proved as part of the proof of the proposition). Hence for any \( t \) and \( 1 \leq i < n_t, \psi(v_t^i) = v_t^i \) or \( v_t^{−i} \). In particular, \( \psi \) restricts to an involution restricted to \( N(X_t) \) for each \( t \in S \). We would like to extend these properties to the universal isometry \( \Psi \):

**Proposition 25.** Let \( \mathcal{E} : X \to C \) be a semistable elliptic surface. Let \( S \) be the singular locus, and assume that the singular fibre at \( t \in S \) is of Kodaira-Néron type \( I_{n_t} \) with \( n_t \geq 3 \). Suppose \( \Psi \) is an universal isometry of \( UNS(\mathcal{E}) \) such that \( \psi = \Psi|_{NS(X)} \) satisfies the following property \((E)\):

\[(E): \text{For all } t \in S, \psi(v_t^i) = v_t^i \text{ or } v_t^{−i}, \text{ where } \psi = \Psi|_{NS(X)} \text{ and } \{v_t^i, i \in \mathbb{Z}/n_t\mathbb{Z}\} \text{ are the irreducible components of the singular fibre at } t, \text{ and } v_0^t \text{ is the component meeting the section } (O).\]

Then for every \( b \in B_C \), the map \( \psi_b \) satisfies Property \( E \). Further, \( \Psi \) is uniquely determined by \( \psi \).

**Proof.** Fix a point \( t_0 \in S \) and a point \( t_0^b \) of \( B \) lying above \( t_0 \). Suppose that the Kodaira type of the fibre over \( t_0 \) (resp. \( t_0^b \)) is \( I_n \) (resp. \( I_{ne} \)). We can assume \( e > 1 \). Denote the irreducible components of the fibre \( X_{t_0} \) by \( v_i, \) \( i \in \mathbb{Z}/n\mathbb{Z} \) and those over \( t_0^b \) by \( w_i, \) \( i \in \mathbb{Z}/ne\mathbb{Z} \), where \( v_0, w_0 \) are the components meeting the zero section.

For \( k \in \mathbb{Z}/ne\mathbb{Z}, \psi_b(w_k)^2 = w_k^2 = -2 \). By Part (3) of Lemma 6, \( \psi_b(w_k) = w(i_k, j_k) + r_kF \) for some integers \( i_k, j_k, r_k \). Suppose \( w(i, j) \) and \( w(k, l) \) are two vectors whose supports intersect. We have,

\[
w(i, j), w(k, l) = \begin{cases} 
-2 & \text{if } i = k \text{ and } j = l, \\
-1 & \text{if either } i = k \text{ or } j = l \text{ and supports not equal,} \\
0 & \text{if none of the endpoints are equal.}
\end{cases}
\]

Since \( \psi_b(w_k)\psi_b(w_{k+1}) = 1 \), it follows that the supports of \( w(i_k, j_k) \) and \( w(i_{k+1}, j_{k+1}) \) do not intersect, and the union \([i_k, j_k] \cup [i_{k+1}, j_{k+1}]\) forms a connected segment. If for some \( k \), the segment \([i_{k+2}, j_{k+2}]\) intersects the segment \([i_k, j_k]\), then at least one of their endpoints have to coincide. By the above calculation, \( \psi_b(w_k)\psi_b(w_{k+2}) \) is either \(-2 \) or \(-1 \), contradicting the fact that it is equal to \( w_kw_{k+2} = 0 \) (as \( e > 1, ne \geq 5 \)). Hence the disjoint segments \([i_k, j_k]\) as \( k \) varies join together to form a connected segment without any back tracking, and fill up \( \mathbb{Z}/ne\mathbb{Z} \). These conditions force for each \( k, i_k = j_k \). Now,

\[
\psi_b(w_0)\psi_b(p^*_b(v_0)) = w_0p^*_b(v_0) = -2e + 2(e - 1) = -2.
\]

The support of the pullback divisor \( p^*_b(v_0) \) is the set \( |j| < e \). The exceptional divisors \( w_j, 0 < |j| < e \) do not intersect \( p^*_b(v_0) \). These conditions force \( \psi_b(w_0) = w_0 + r_0F \). It follows that \( \psi_b(w_k) = w_{\pm k} + r_kF \) for \( k \in \mathbb{Z}/ne\mathbb{Z} \). Intersecting with the zero section, we get \( r_k = 0 \) for all \( k \).
Suppose $\psi(v_k) = v_{-k}$ for all $k \in \mathbb{Z}/n\mathbb{Z}$. Since $\Psi$ is a universal isometry,
\[ p_b^*(v_k) = p_b^*(\psi(v_{-k})) = \psi_b(p_b^*(v_{-k})). \]

Hence we have,
\[ e^{\psi_b}(w_{-ke}) + \sum_{i=1}^{e-1} (e - i)\psi_b(w_{-ke+i}) + \psi_b(w_{-ke-i}) = ew_{ke} + \sum_{i=1}^{e-1} (e - i)(w_{ke+i} + w_{ke-i}). \]

This forces $\psi_b(w_{ke}) = w_{-ke}$ for $k \in \mathbb{Z}/n\mathbb{Z}$. The hypothesis $n \geq 3$, together with the fact proved above forces $\psi_b(w_k) = w_{-k}$ for $k \in \mathbb{Z}/ne\mathbb{Z}$. A similar argument works if we had assumed that $\psi(v_k) = v_k$ for all $k \in \mathbb{Z}/n\mathbb{Z}$, forcing in this case $\psi_b$ to be identity on the fibres above $t_0$.

It is clear that not only have we proved that $\Psi$ is uniquely determined by $\psi$, but in fact that the behaviour of $\psi_b$ on a singular fibre at $s \in B(k)$ is similar to that of $\psi$ on $b(s) \in C(k)$, in whether it acts as the identity or flips around the origin according respectively to the behaviour of $\psi$.

\textbf{Proof of Theorem 7.} We are now in a position to describe the automorphism group of the universal Néron-Severi group. Given an universal isometry $\Phi$, by Proposition 10, we first multiply by $-1$ if required to ensure that $\Phi$ fixes the fibre. By Proposition 12, the resulting automorphism restricts to an automorphism of $N(X_t)$ for each point $t \in S$, the ramification locus of $\pi$. By the argument given before the statement of Proposition 24, modify $\Phi$ by an element of the form $PL(x_t)$ for some element $x_t \in A(N(X_t))$ to ensure that the base morphism $\phi$ takes the standard basis of any singular fibre of $\pi$ to the standard basis.

This ensures, by Proposition 24, that $\phi$ preserves sections of $\pi$. Now we modify $\phi$ by a translation to ensure that the zero section ($O$) of $\pi$ is fixed. By Proposition 25, each $\phi_b$ for $b \in B_C$ preserves the standard basis of each fibre. In particular $\Phi$ preserves the irreducible components of the singular fibres. By Proposition 24 applied to each $b \in B_C$, $\Phi$ maps sections to sections.

This ensures that the hypothesis of Theorem 16 hold. As a consequence, $\Phi$ is either induced by the inverse map on the generic fibre or is the identity map. Hence the automorphism group is generated by the above transformations.

The transformation sending $x \mapsto -x$, $x \in UNS(X)$ is central in the automorphism group. Given an universal automorphism $\Phi$ of $UNS(X)$, it fixes the trivial lattice, and hence gives a compatible family of automorphisms, as $b$ varies in $B_C$, of the Mordell-Weil groups of the generic fibre $E(l(B))$. Since the Picard-Lefscetz tranformations act trivially on the Mordell-Weil groups, the kernel of this homomorphism is the group $PL(\prod_{t \in S} A(N(X_t)))$. The group generated by the translations by sections, the automorphism of the generic fibre and the central $-1$ element, project isomorphically as automorphisms of the Mordell-Weil lattices. This proves the semi-direct property of the automorphism group.

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