Feynman Identity: a special case. II.

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Abstract

In this paper, the results of part I regarding a special case of Feynman identity are extended. The sign rule for a path in terms of data encoded by its word and formulas for the numbers of distinct equivalence classes of nonperiodic paths of given length with positive or negative sign are obtained for this extended case. Also, a connection is found between these numbers and the generalized Witt formula for the dimension of certain graded Lie algebras. Convergence of the infinite product in the identity is proved.

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I. Introduction

This paper is the sequel to Ref. 1 where a special case of Feynman identity is investigated. This identity relates admissible graphs and classes of nonperiodic paths on a lattice and it is relevant in a combinatorial proof of Onsager’s closed formula for the partition function of the two dimensional free field Ising model\(^1\text{–}^6\). In the special case considered in\(^1\) and here the lattice \(G_R\) consists of \(R\) oriented loops and one site as shown in Figure 1. The case \(R = 1\) is trivial because only one non periodic path is possible. The case \(R = 2\) is the first nontrivial one and was investigated in Ref. 1.

![Diagram of the lattice \(G_R\).](image)

FIG. 1. The lattice \(G_R\).

In the present paper, the case with \(R > 2\) will be considered. In this case Feynman identity can be expressed as:

\[
\prod_{m_1,\ldots,m_R \geq 0} \left(1 + z_{i_1}^{m_1} \cdots z_{i_r}^{m_R}\right)^{N_+(m_1,\ldots,m_R)} \left(1 - z_{i_1}^{m_1} \cdots z_{i_r}^{m_R}\right)^{N_-(m_1,\ldots,m_R)} = \prod_{1 \leq j \leq R} (1 + z_j)
\]  

(1.1)

The exponents \(N_+(m_1,\ldots,m_R)\) and \(N_-(m_1,\ldots,m_R)\) are the numbers of distinct non-periodic paths with positive and negative signs, respectively, which traverse \(m_1\) times loop 1, \(m_2\) times loop 2, ..., \(m_R\) times loop \(R\). Sequences with \(m_i > 1\) and \(m_j \neq i = 0\) are excluded in (1.1) because they correspond to periodic paths.

On the left hand side of (1.1) there are sequences

\[S' = \{(m_1,\ldots,m_R) \mid m_i = 1, m_j \neq i = 0, i = 1,\ldots,R\}\]  

(1.2)

with \(N_+ = 1\) and \(N_- = 0\). Collecting these sequences we get exactly the right hand side of (1.1) and relation (1.1) can be equivalently expressed as

\[
\prod_{S \neq S'} \left(1 + z_{i_1}^{m_1} \cdots z_{i_r}^{m_R}\right)^{N_+} \left(1 - z_{i_1}^{m_1} \cdots z_{i_r}^{m_R}\right)^{N_-} = 1
\]  

(1.3)

Write the product on the left hand side of (1.3) in the following way, namely,

\[
\prod_{r=2}^{R} \prod_{G_r \in S(G)} \prod_{S(G_r)} \left(1 + z_{1_{i_1}}^{m_1} \cdots z_{i_r}^{m_R}\right)^{\theta_+(m_1,\ldots,m_r)} \left(1 - z_{1_{i_1}}^{m_1} \cdots z_{i_r}^{m_R}\right)^{\theta_-(m_1,\ldots,m_r)}
\]  

(1.4)
The second product runs over all subdiagrams $G_r$ of $G_R$ with $r$ loops $i_1, i_2, \ldots, i_r$, $2 \leq r \leq R$. The third product is over all sequences $S(G_r) = \{(m_{i_1}, \ldots, m_{i_r}), m_i > 0\}$ and $\theta_+(m_{i_1}, \ldots, m_{i_r})$ and $\theta_-(m_{i_1}, \ldots, m_{i_r})$ are the numbers of distinct classes of equivalence of nonperiodic paths with positive and negative signs, respectively, which traverse $G_r$.

Given any $G_r$, the paths over $G_r$ can be classified according to the number $N = m_{i_1} + \ldots + m_{i_r}$ where $m_i$ is the number of times loop $i$ is traversed by a path. Thus,

$$
\prod_{S(G_r)} (1 + z_{i_1}^{m_{i_1}} \ldots z_{i_r}^{m_{i_r}})^{\theta_+} (1 - z_{i_1}^{m_{i_1}} \ldots z_{i_r}^{m_{i_r}})^{\theta_-} = \prod_{N=r}^{\infty} \prod_{m_i > 0} (1 + z_{i_1}^{m_{i_1}} \ldots z_{i_r}^{m_{i_r}})^{\theta_+} (1 - z_{i_1}^{m_{i_1}} \ldots z_{i_r}^{m_{i_r}})^{\theta_-} \tag{1.6}
$$

It shall be proved that the infinite product given by relation (1.6) converges to 1 if $|z_i| < 1$. This is true for each $G_r$. The proof is carried out in Section 3 using certain relations satisfied by the numbers $\theta_\pm$, to be computed in the same section. In Section 4, using results from section 3, a statement is proved according to which the number of classes of nonperiodic paths of length $N$ which traverse $G_R$ counterclockwisely is given by Witt formula. Witt formula gives the dimension of certain vector spaces associated with finite Lie algebras. Then, based on recent results in this field a connection is found with the so called generalized Witt formula for the dimension of certain finite graded Lie algebras. In Section 2, a general formula for the sign of a path is obtained which is important in Section 3 to find the number of classes of nonperiodic paths.
II. Rule of signs

In this section, a general formula for the sign of any path $p$ over $G_R$ is computed. It is crucial in the calculation of the numbers $\theta_{\pm}$.

Following the methods of Ref. 1 a path $p$ over $G_R$ is given by a word. This is to be understood as an ordered sequence of letters $D^i_e$ where $i$ gives the loop of $G_R$ traversed by $p$ and $|e_i| = m_i$, how many times. The sign of $e_i$ indicates whether the loop is traversed following the direction assigned for it (in this case, the sign is positive) or the opposite direction (in this case, the sign is negative). A typical word is the following:

$$W(p) = D_{i_1}^{e_1} D_{i_2}^{e_2} ... D_{i_l}^{e_l}$$

where $l = r, r + 1, ..., N$. The order in which the letters appear in the word is important since it indicates the loops traversed by $p$ and in which order. The order is encoded in the sequence $S_l = (i_1, i_2, ..., i_l)$. The sequence $S_l$ is such that a loop $i$ appears at least once in the sequence, $i_k \neq i_{k+1}$ and $i_l \neq i_1$. The word representation seems to imply that line $i_1$ is special over the others as the one to be traversed first. This is a convention because the path is closed. After traversing $i_l$ the path joins $i_1$. Fix $i_1 = 1$, from now on.

Given the word (2.1) the sequence $(i_1, i_2, ..., i_l)$ can be decomposed into subsequences defined as follows. A subsequence is an ordered set of numbers formed in such a way that a) if $i, j$ are two elements inside the subsequence and $j$ comes after $i$ then $j > i$; b) a new subsequence begins whenever this ordering is broken in the sequence by two adjacent elements not satisfying a). For instance, a sequence with $l = 14$ for the case $r = 3$ is decomposed as follows:

$$(1213213232132) = (12)(123)(2)(13)(23)(23)(12)$$

Denote by $T$ the number of subsequences in a decomposition and let $t$ be the number of subsequences where the number 1 is not present. In the example above $T = 7$ and there are three subsequences where the number 1 is not present, namely, the subsequence (2) and (23) which appears twice. So, $t = 3$.

Let $n_i$ be the number of times line $i$ occur in the sequence $(i_1, ..., i_l)$ and $N_i$, the number of times line $i$ is covered by $p$. Define

$$n = n_2 + n_3 + ... + n_r$$

and

$$N = N_1 + N_2 + ... + N_r$$

In (2.3), the sum really begins with $n_2$. For instance, in the word $D_1^3D_2^5D_1^4D_3^2D_2^3$, $n_2 = 2$, $n_3 = 1$, $N_1 = 4$, $N_2 = 5$ and $N_3 = 2$.

The following result can now be stated:
Lemma 1: Given the word $W(p)$ for $p$, the sign of $p$ is given by:

$$(-1)^{N+n+s+t+1}$$  \hspace{1cm} (2.5)

Proof: Following Ref. 1, relation (2.5) can be derived from the representation of $p$ in terms of an appropriate closed normal plane curve compatible with $W(p)$ which winds the loops of $G_R$ according to some rules. The sign of $p$ comes from the number $V$ of selfcrossings of its normal curve. The rules fix the way we draw a curve. Then, just by looking at it we count the number $V$.

In Ref. 1, the case of a lattice with only two loops was considered. There the selfcrossings of a normal closed curve for $p$ were of two types:

1) Type-1 crossings. Crossings produced by the segments of the curve winding the loops of $G_R$ like the ones shown in Figs. 2 and 3;

2) Type-2 crossings. Crossings produced when the curve changes in direction in order to wind a loop in the opposite direction to that fixed for it like the ones shown in Fig. 4.

In the case of a lattice with more than two loops, a third type of crossings is possible to occur:

3) Type-3 crossings. Crossings produced when segments linking different loops of $G_R$ intersect.

Counting Type-1 crossings. According to the rules given in Ref. 1 the segment of curve $D_1^x$ is drawn making an inward spiral around loop $i$. The number of times loop $i$ is traversed by this segment is $|x|$. If $x > 0$, then the curve winds the loop following its orientation. If $x < 0$, then it winds $i$ with opposite orientation. At the end of the winding when the curve leaves loop $i$, it will cross itself a number of times given by $|x| - 1$.

Let’s first consider the case where all exponents are positive, $e_i > 0$. Denote by $e_{1,\alpha}$, $\alpha = 1, 2, ..., n_1$, the exponents of $D_1$ in $W(p)$ where $n_1$ is the number of times loop $i = 1$ appears in the sequence $(i_1, ..., i_l)$. Fig. 2 shows the sequence of segments

$$D_1^{e_{11}}, D_1^{e_{12}}, ..., D_1^{e_{1n_1}}$$  \hspace{1cm} (2.6)

The number of crossings in this case is (see Ref. 1, Section 2, for more details)

$$A_1 = (e_{11} - 1) + 2(e_{12} - 1) + ... + 2(n_1 - 1)(e_{1n_1} - 1)$$  \hspace{1cm} (2.7)
FIG. 2. How to count type-1 crossings .(I).

The other loops $i, i = 2, \ldots, r$, (see Fig. 3), contribute with

$$B_i = (e_{i1} - 1) + 3(e_{i2} - 1) + 5(e_{i3} - 1) + \ldots + (2n_i - 1)(e_{in_i} - 1) \quad (2.8)$$

After winding loop $i$, the curve goes to its point of “departure” at $e$ (see Fig. 2)
but to do so it has to cross all the segments which have already winded loop $i_1 = 1$.

The number of these crossings is given by

$$C = e_{12} + \ldots + e_{1n_1} \quad (2.9)$$

Therefore, the total number of crossings is

$$V = A_1 + \sum_{j=2}^{r} B_j + C \quad (2.10)$$

A simple calculation shows that in this case

$$(-1)^V = (-1)^{N+n+1} \quad (2.11)$$
Counting type-2 crossings. In order to find (2.11) we considered words with all the exponents $e_i$ positive. In general, however, they may be negative, too. So, now, we should consider this more general case.

Before winding a loop the curve approaches it from down the right. See Fig. 4. If $e_i > 0$ it goes on to wind the loop counterclockwisely $|e_i|$ times leaving in the end by the left side. When $e_i < 0$, the curve coming from down the right will first turn to the left and only then will go upward to wind the loop $|e_i|$ times clockwise crossing itself once on its way out. The second time the same loop is winded counterclockwisely, the curve will cross itself five times on its way out. See Fig. 5.
If there are $s_i$ occurrences of loop $i$ in the sequence $(i_1, ..., i_l)$ for $W$ with $e_i < 0$ the curve will cross itself

$$V_i = \sum_{x=1}^{s_i} (4x - 3)$$

(2.12)

times.

![Diagram](image)

FIG. 5. How to count type-2 crossings .(II).

The contribution of these crossings to the sign is, therefore,

$$(-1)^{V_i} = (-1)^{s_i}$$

(2.13)

The total contribution to the sign of $p$ coming from $s = \sum s_i$ negative exponents in $W$ is then given by

$$(-1)^s$$

(2.14)

and so we get

$$(-1)^{N+n+s+1}$$

(2.15)

**Counting type-3 crossings.** First, let’s consider the case of a sequence $(i_1, ..., i_l)$ that has a decomposition into $T$ subsequences all of them beginning with loop 1. Moreover, suppose that the subsequences up to the $(T-1)$-th subsequence are of the form $(1, 2, ..., r)$, that is, all loops of $G_r$ are present but there are gaps (the lack of one or more loops of $G_r$) inside the $T$-th subsequence.

In order to illustrate the implications of gaps for the counting of type-3 crossings we are going to consider the simplest case where there is only one gap, that between 1 and $i_a$, namely,

$$(1 \ i_a...r)$$

(2.16)
where $i_a \neq 2$ and lines $2, 3, \ldots, i_a - 1$ are not in this subsequence.

Having assumed that all subsequences up to the $(T-1)$-th have no gaps means that each of the $r$ loops of $G_r$ have been already winded before $(T-1)$ times. So, when the curve goes from line 1 to line $i_a$ it has to cross twice a bundle with $(T-1)$ segments of itself thus producing $2(T-1)$ crossings. See Fig. 6.

![Diagram](image)

FIG. 6. How to count type-3 crossings. (I).

(For convenience of presentation, in Figs. 6 and 7, the curve windings of loops have not been displayed like in Figs. 2 and 3.) For the same reason, if there were other gaps, whenever one gap is met by the curve in the same subsequence or in another one in the decomposition, the total number of type-3 crossings would be an even number. The number of type-3 crossings being an even number their contribution to the sign of $p$ is then $+1 \equiv (-1)^t$ with $t = 0$.

An odd number of type-3 crossings is of course possible but only when the sequence $(i_1, \ldots, i_T)$ is such that it has subsequences in its decomposition which does not initiate with loop 1. To see that, let’s consider a simple representative case, namely, the decomposition

$$(12\ldots r)(12\ldots r)(a\ldots r)$$

where $a \neq 1$ and the first $(T-1)$ subsequences before $(a\ldots r)$ have no gaps at all. When the curve goes over the first $(T-1)$ subsequences the loops of $G_r$ are winded $(T-1)$ times each. Then, it has to cross once a bundle with $(T-2)$ segments and then another bundle with $(T-1)$ segments before winding loop $a$. (See Fig. 7). The different numbers of segments in each bundle is solely due to the fact that loop 1 was not traversed by $p$. A total of $(2T-3)$ crossings is produced and their contribution to the sign is $-1 \equiv (-1)^t$ where $t = 1$. In a more general case where
there are more subsequences lacking loop 1 exactly \( t \) times one can show that the sign is \((-1)^t\). In this way we get the LHS of (2.5).

Corollary 1. The sign of \( p \) is given as well by

\[
(-1)^{N+l+s+T+1}
\] (2.18)

Proof: Add \( 2n_1 \) to the exponent of \((-1)\) in (2.5). Then,

\[
(-1)^{N+n+s+t+1} \equiv (-1)^{N+(n_1+n)+s+(t+n_1)+1}
\] (2.19)

But \( n_1 + n \equiv l \) and \( t + n_1 \equiv T \).

Corollary 2. The sign of a periodic word \( p \) equals the sign of its nonperiodic subword if \( p \) has odd period and it is \(-1\) if the period is an even number.

Proof: Suppose \( p \) given by the word

\[
D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} ... D_{i_l}^{e_{i_l}}
\] (2.20)

for a given \( l \leq N, N = \sum |e_i|, s \) negative exponents and \( T \) subsequences in the sequence \((i_1, ..., i_l)\). Suppose \( p \) is periodic with period \( g \). Then, \( p \) is the repetition of a non periodic subword \( g \) times, that is, \( p = (w)^g \) where

\[
w = D_{i_1}^{e_{i_1}} D_{i_2}^{e_{i_2}} ... D_{i_j}^{e_{i_j}}
\] (2.21)

with \( j = l/g, \) length \( L = N/g, s_0 = s/g \) negative exponents and \( T_0 = T/g \) subsequences in \((i_1, ..., i_j)\). The sign of \( p \) is

\[
(-1)^{N+l+s+T+1} = (-1)^{g(n+j+s_0+T_0)+1}
\] (2.22)

which equals \((-1)\) if \( g \) is an even number and equals the sign of \( w \) if \( g \) is an odd number. \( \square \)

FIG. 7. How to count type-3 crossings. (II).
III. The numbers $\theta_{\pm}$

In this section we compute explicit formulas for the weights $\theta_{\pm}$ and prove convergence of the infinite product (1.6).

Let’s consider the simpler case $z_1 = z_2 = \ldots = z_R = z$. In this case (1.6) becomes

\[
\prod_{N=r}^{\infty} (1 + z^N)^{\theta_+ (N, r)} (1 - z^N)^{\theta_- (N, r)}
\]

where

\[
\theta_{\pm} (N, r) = \sum_{m_i > 0 \atop m_{i_1} + \ldots + m_{i_r} = N} \theta_{\pm} (m_{i_1}, \ldots, m_{i_r})
\]

(3.2)

**Theorem 3.1.** Given $r, N \geq r$, the number of equivalence classes of nonperiodic paths of length $N$ and positive sign which traverses $r$ loops of $G_R$ is given by

\[
\theta_+ (N, r) = \sum_{\text{odd } g | N} \frac{\mu(g)}{g} \mathcal{F}_r \left( \frac{N}{g} \right)
\]

(3.3)

where the summation is over the odd divisors of $N$ only and

\[
2y \mathcal{F}_r (y) = \sum_{k=-1}^{r-1} (-1)^{r+k+1} \binom{r}{k+1} (2k+1)^y
\]

(3.4)

For the number $\theta_- (N, r)$ of equivalence classes of nonperiodic paths of length $N$ and negative sign which traverses $r$ loops of $G_R$, the following cases hold:

I) If $N$ is a) odd or prime or b) even but $N < 2r$, then

\[
\theta_- (N, r) = \theta_+ (N, r)
\]

(3.5)

II) If $N$ is even and $N \geq 2r$, then

\[
\theta_- (N, r) = \theta_+ (N, r) - \theta_+ \left( \frac{N}{2}, r \right)
\]

(3.6)

Furthermore, if $|z| < 1$ the product (3.1) converges to 1.

**Proof:** The proof of (3.3) follows ideas from Ref. 1 and uses the general formula for the sign of a path computed in the previous section. The proof is lengthy and for this reason left to the Appendix. Let’s prove I) and II) and convergence. In Ref. 1 relations (3.5) and (3.6) were proved for the case $r = 2$ by direct, lengthy computation. It is possible to repeat that for the present case but simpler arguments can be used. They are as follows. A path traverses all $r$ lines of $G_r$ so its length is $N \geq r$. It is clear that if $N < 2r$ the path can not be periodic for to be periodic a
path must have a length which is a multiple of \( r \) and period \( g \geq 2 \) so its length should be \( N \geq 2r \). Since there is no periodic path with length \( N < 2r \) then in this case \( \theta_+ = \theta_- \). If \( N \) is any prime number, the only divisors of \( N \) are 1 and \( N \), hence, there can be no periodic paths with prime length and again \( \theta_+ = \theta_- \). Suppose now that \( N \) is an odd but nonprime number. Let’s prove that the numbers of periodic words with + and – signs are equal and therefore one must have \( \theta_+ = \theta_- \). Call \( N_0 \) the first odd, nonprime number greater than \( 2r \). If there are odd numbers in between \( 2r \) and \( N_0 \), then they must be prime and for these we have already proved that \( \theta_+ = \theta_- \). The divisors of \( N_0 \) are all odd. Denote them by \( 1, g_{0,1}, ..., g_{0,m}, N_0 \). Then, there are odd numbers \( n_{0,1}, ..., n_{0,m} \) such that \( N_0 = n_{0,1}g_{0,1} = ... = n_{0,m}g_{0,m} \). So, the number of periodic paths with period \( g_{0,i}, i = 1, 2, ..., m \), and positive (negative) sign is given by \( \theta_+(n_{0,i}) (\theta_-(n_{0,i})) \). Since \( n_{0,i} < N_0 \), \( n_{0,i} \) is prime and \( \theta_+(n_{0,i}) = \theta_-(n_{0,i}) \). Hence, one must have \( \theta_+(N_0) = \theta_-(N_0) \). Induction, now, proves that \( \theta_+(N) = \theta_-(N) \) for any odd and nonprime number \( N \).

Consider now the case \( N \geq 2r \), and \( N \) is an even number. Suppose, first, \( r \) odd and prime. From Corollary 2, section 2, the sign of a periodic word with even period is \(-1\) so in the case the period is \( g = 2 \), all the periodic words have sign \(-1\). The number of nonperiodic words with positive sign and length \( N/2 \) which make periodic words of length \( 2r \) is \( \theta_-(r) \), hence, subtracting the periodic words one gets \( \theta_-(2r) = \theta_+(2r) - \theta_-(r) \). Suppose now that \( r \) is odd but nonprime. In this case the divisors of \( N = 2r \) are \( 1, 2, a, 2r \) where \( a \) runs the divisors of \( r \), hence, \( a \) is odd. If the period is \( a \) then the length of nonperiodic subwords is \( 2r/a \), an even number. But one must have \( r < 2r/a < 2r \) which implies that \( 1 < a \) and \( a < 2 \), hence, the possible periodic words in this case have \( g = 2 \) as the only feasible case. For the case \( r \) even, the argument is the same. Now take \( N = 2(r + k) \). Then, \( \theta_-(N) \) for a lattice with \( r \) loops can be understood as being equivalent to the \( \theta_-(N) \) of paths for a lattice with \( r + k \) loops. Working on this lattice and using the previous arguments for the case \( k = 0 \) we obtain the desired result.

Convergence of the infinite product now follows easily using relations (3.5) and (3.6). Call \( P_{r,n} \) the partial product in (3.1) with \( N \) running from \( r \) up to \( n \). Using relations (3.5) and (3.6) it’s found that for \( n \geq 2r \)

\[
P_{r,n} = \prod_{j=[\frac{r}{2}]+1}^{n} (1 - z^{2j})^{\theta_+(j,r)}
\]

In the limit \( n \to \infty \) the infinite product only converges if \( |z| < 1 \) and it converges to 1. □

Relations (3.3-4) above reproduces precisely the results of Ref. 1.

Let’s consider now the case where the \( z_i \) are all distinct. For this case, a formula for \( \theta_+ \) is not yet available exception of the case \( r = 2 \) given in Ref. 1 but still one can prove the following.

**Theorem 3.2:** Given \( r \) and \( N \geq r \), the number of equivalence classes of nonperiodic
paths of length $N$ with + and − sign, $\theta_+(m_1, m_2, ..., m_r)$ and $\theta_-(m_1, m_2, ..., m_r)$, respectively, which traverse loops $i_1, ..., i_r$ a number of times given by $m_{i_1}, ..., m_{i_r}$, $m_i > 0$, respectively, $m_{i_1} + ... + m_{i_r} = N$ the following relations hold:

I) If $m_{i_1}, ..., m_{i_r}$ are 
   a) all odd, 
   b) coprime or just of distinct parity or 
   c) all even

and $N < 2r$, then

$$\theta_-(m_{i_1}, ..., m_{i_r}, N) = \theta_+(m_{i_1}, ..., m_{i_r}, N)$$  \hspace{1cm} (3.8)

II) If $m_{i_1}, ..., m_{i_r}$ are all even and $N \geq 2r$, then

$$\theta_-(m_{i_1}, ..., m_{i_r}, N) = \theta_+(m_{i_1}, ..., m_{i_r}, N) - \theta_+ \left( \frac{m_{i_1}}{2}, ..., \frac{m_{i_r}}{2}, \frac{N}{2} \right)$$ \hspace{1cm} (3.9)

Furthermore, for $|z_i| < 1$, $i = 1, 2, ..., r$, the product (1.6) converges to 1.

**Proof:** Similar to previous.
IV. Connection with Lie algebras

In this section a connection with finite dimensional Lie algebras is achieved. Firstly, the following is proved.

**Theorem 4.1.** The number of classes of nonperiodic paths of length \( N \) which traverse the lattice \( G_R \) counterclockwisely is given by:

\[
\theta(N) = \frac{1}{N} \sum_{g|N} \mu(g) R_{g}^N 
\]  

(4.1)

**Proof:** The proof below is a joint collaboration with A. L. Maciel.

A statement like the one above can be found in Ref. 4. The goal here is to prove it using formulas from section 3 and the Appendix.

The number of classes of nonperiodic paths of length \( N \) which traverse counterclockwisely a sublattice of \( G_R \) with \( r \) bonds, \( r = 1, 2, ..., R \), is given by

\[
\theta_r(N) = \sum_{g|N} \mu(g) \sum_{\alpha=1}^{N/g} \frac{1}{\alpha} \left( \frac{N}{\alpha} - 1 \right) r w_r(\alpha) 
\]  

(4.2)

where \( w_r(\alpha) \) is given by (4.11)

\[
rw_r(\alpha) = \sum_{j=1}^{r} (-1)^{r+j} \left( \binom{r}{j} (j-1)^{\alpha} + (-1)^{\alpha+r} \right) 
\]  

(4.3)

Thus, the number of classes of nonperiodic paths of length \( N \) which traverse counterclockwisely the lattice \( G_R \) is

\[
\theta(N) = \sum_{r=1}^{R} \binom{R}{r} \theta_r(N) 
\]  

(4.4)

When \( N = 1 \), \( \theta_r(1) = 0 \) if \( r > 1 \), hence, \( \theta(1) = R \) which is in (4.1). Consider now the case when \( N \geq 2 \). Upon substitution of (4.2) into (4.4) one gets:

\[
\theta(N) = \frac{1}{N} \sum_{g|N} \mu(g) \sum_{\alpha=1}^{N/g} \left( \frac{N}{\alpha} \right) A(R, \alpha) 
\]  

(4.5)

where

\[
A(R, \alpha) = (-1)^{\alpha} \sum_{r=2}^{R} (-1)^{r} \binom{R}{r} + \sum_{q=1}^{R} \sum_{p=q}^{R} (-1)^{p+q} \binom{R}{p} \binom{p}{q} (q-1)^{\alpha} 
\]  

(4.6)

The term with the double summation can be written as

\[
\sum_{q=1}^{R-1} (q-1)^{\alpha} \sum_{p=q}^{R} (-1)^{p+q} \binom{R}{p} \binom{p}{q} + (R-1)^{\alpha} 
\]  

(4.7)
The summation over $p$ is equal to zero for $q < R^9$, so that

$$A(R, \alpha) = (-1)^\alpha (R - 1) + (R - 1)^\alpha \quad (4.8)$$

and

$$\theta(N) = \frac{1}{N} \sum_{g|N} \mu(g)(-R + R^{\frac{N}{g}}) \quad (4.9)$$

Using that $\sum_{g|N} \mu(g) = 0$ for $N \geq 2^7$, the result follows. □

The right hand side in formula (4.1) is well known in Lie algebra theory where it is known as the Witt formula. It gives the dimensions of the subspaces $L_N$ of a Lie algebra with a $\mathbb{Z}_{>0}$-gradation $L = \bigoplus_{N=1}^{\infty} L_N$ generated by an $R$-dimensional vector space over $\mathbb{C}^{7-8}$. Witt formula satisfies the identity

$$\prod_{n=1}^{\infty} (1 - z^n)^{\dim L_n} = 1 - Rz \quad (4.10)$$

called the denominator identity for the algebra.

Recently, in Refs. 7-8, Witt formula has been generalized. Let $V = \bigoplus_{i=1}^{\infty} V_i$ be a $\mathbb{Z}_{>0}$-graded vector space over $\mathbb{C}$ with $\dim V_i = d(i) < \infty$, $\forall i \geq 1$, and let $L = \bigoplus_{N=1}^{\infty} L_N$ be the free Lie algebra generated by $V$ with a $\mathbb{Z}_{>0}$-gradation induced by that of $V$. Then, the dimensions of the subspaces $L_N$ are given by the generalized Witt formula

$$\dim L_N = \sum_{g|N} \frac{\mu(g)}{g} W\left(\frac{N}{g}\right) \quad (4.11)$$

where $W$ is the Witt partition function. $W$ and the $d(i)$'s are related. For its definition in terms of $d(i)$'s, see Refs. 7-8. The denominator identity in this case is

$$\prod_{N=1}^{\infty} (1 - z^{N})^{\dim L_N} = 1 - f(z) \quad (4.12)$$

where

$$f(z) = \sum_{i=1}^{\infty} d(i)z^i \quad (4.13)$$

is related to Witt partition function as follows. Define

$$g(z) = \sum_{n=1}^{\infty} W(n)z^n \quad (4.14)$$

Then,

$$e^{-g(z)} = 1 - f(z) \quad (4.15)$$

Comparing (4.11) and (4.2) it is clear that they have the same general structure. For each $r$ let’s interpret (4.2) as the dimension of some Lie algebra $L^{(r)}$. Then the dimensions of the associated vector spaces are given by the following theorem:
Theorem 4.2. For each \( r = 1, 2, 3, \ldots \), consider the free Lie algebra \( L^{(r)} = \bigoplus_{N=1}^{\infty} L_{N}^{(r)} \) with \( \dim L_{N}^{(r)} \) given by (4.11) (or (4.2)) with Witt partition function

\[
W^{(r)}(n) = \sum_{\alpha=1}^{n} \frac{1}{\alpha} \binom{n-1}{\alpha-1} rw_r(\alpha)
\] (4.16)

and \( rw_r \) given by (4.3). Then, for \(|z| < r^{-1}\), the dimensions \( d_r(i) = \dim V^{(r)}_i \) of the vector spaces in \( V^{(r)} = \bigoplus_{i=1}^{\infty} V^{(r)}_i \), the vector space that generates \( L^{(r)} \), is given by

\[
d_r(i) = \frac{1}{i!} \frac{d^i f_r}{dz^i}(0)
\] (4.17)

where

\[
f_r(z) = 1 - \prod_{j=1}^{r} (1 - jz)^{C_j(r)}
\] (4.18)

and

\[
C_j(r) = (-1)^{j+r} \binom{r}{j}
\] (4.19)

In particular, for \( r = 1 \), \( d(j) = 0 \) if \( j \neq 1 \) and \( d(1) = 1 \); for \( r = 2 \),

\[
d(0) = 0, \quad d(j) = j - 1, \quad j \geq 1;
\] (4.20)

for \( r = 3 \),

\[
d(0) = d(1) = d(2) = 0, \quad d(j) = 2^{j-2}(j - 2)(j + 5), \quad j \geq 3.
\] (4.21)

Proof: Take \(|z| < r^{-1}\). Using (4.3), relation (4.16) can be expressed as

\[
W^{(r)}(n) = \sum_{j=1}^{r} (-1)^{j+r} \binom{r}{j} \frac{j^n}{n}
\] (4.22)

and substituting in (4.14), one obtains

\[
g(z) = \sum_{j=1}^{r} (-1)^{j+r+1} \binom{r}{j} \ln(1 - jz) = \ln \prod_{j=1}^{r} (1 - jz)^{-C_j}
\] (4.23)

and from (4.15), follows (4.18). \( f_r(z) \) is analytic in \( z = 0 \), so (4.17) holds. For \( r = 1 \), \( f_1(z) = z \); for \( r = 2 \),

\[
f_2(z) = \frac{z^2}{(1 - z)^2}
\] (4.24)

and for \( r = 3 \),

\[
f_3(z) = \frac{2z^3 - 3z^4}{(1 - 2z)^3}
\] (4.25)

Expanding (4.24) and (4.25) one gets (4.20) and (4.21). □
From (4.12) the denominator identities are
\[
\prod_{N=1}^{\infty} (1 - z^N)\theta_1(N) = 1 - z \tag{4.26}
\]
and
\[
\prod_{N=1}^{\infty} (1 - z^N)\theta_2(N) = 1 - \frac{z^2}{(1 - z)^2} \tag{4.27}
\]
and
\[
\prod_{N=1}^{\infty} (1 - z^N)\theta_3(N) = 1 - \frac{2z^3 - 3z^4}{(1 - 2z)^3} \tag{4.28}
\]
for \(r = 1, r = 2\) and \(r = 3\), respectively. (4.26) is trivially true. One can check that \(\theta_1(1) = 1\) and \(\theta_1(N) = 0\) if \(N \neq 1\).

**Theorem 4.3.** For each \(r = 2, 3, \ldots\), consider the free Lie algebra \(L^{(r)} = \bigoplus_{N=1}^{\infty} L_N^{(r)}\) with \(\dim L_N^{(r)}\) given by (4.11) with Witt partition function \(W^r(n) = 2W^r(n)\) where
\[
W^r(n) = \sum_{l=1}^{n} \frac{1}{l!} \left( \begin{array}{c} n - 1 \\ l - 1 \end{array} \right) 2^{l-1}r w_r(l) \tag{4.29}
\]
Then, for \(|z| < (2r + 1)^{-1}\), the dimensions \(d_r(i) = \dim V_i^{(r)}\) of the vector spaces in \(V^{(r)} = \bigoplus_{i=1}^{\infty} V_i^{r}\), the vector space that generates \(L^{(r)}\), is given by
\[
d_r(i) = \frac{1}{i!} \frac{d^i f_r}{dz^i}(0) \tag{4.30}
\]
where
\[
f_r(z) = 1 - (1 + z)(-1)^r\prod_{k=0}^{r-1} [1 - (2k + 1)z]^{-b(k)} \tag{4.31}
\]
with
\[
b(k) = \frac{r(-1)^{r+k}}{(k + 1)} \begin{pmatrix} r - 1 \\ k \end{pmatrix} \tag{4.32}
\]
In particular, for \(r = 2\),
\[
d(0) = d(1) = 0, \quad d(j) = 4(j - 1) \quad j \geq 2 \tag{4.33}
\]
For \(r = 3\), \(d(0) = d(1) = d(2) = 0\) and, for \(j \geq 3\)
\[
d(j) = \frac{6}{8} (-1)^j - 1 + \frac{5}{16} 3^{j-3}[(4j + 39)^2 - 43] - \frac{3^{j-2}}{16}[(4j + 13)^2 - 43] \tag{4.34}
\]
**Proof:** For each \(r\) take \(|z| < (2r + 1)^{-1}\). Upon substitution of \(W(n)\) in (4.14) one gets
\[
g(z) = \ln \left\{ (1 + z)(-1)^{r+1}\prod_{k=0}^{r-1} [1 - (2k + 1)z]^{b(k)} \right\} \tag{4.35}
\]
and from (4.15), follows (4.31). In particular, for $r = 2$,

$$f(z) = 1 - \frac{(1 + z)(1 - 3z)}{(1 - z)^2} = \frac{4z^2}{(1 - z)^2}$$

(4.36)

and for $r = 3$:

$$f(z) = 1 - \frac{(1 - z)^3(1 - 5z)}{(1 + z)(1 - 3z)^3}$$

(4.37)

Expanding (4.36) and (4.37) one gets (4.33) and (4.34). □

The denominator identity for the algebra in this case is

$$\prod_{j=1}^{\infty} (1 - z^j)^{\theta(j,r)} = (1 + z)^{(-1)^r} \prod_{k=0}^{r-1} \left[ 1 - (2k + 1)z \right]^{-b(k)}$$

(4.38)
Appendix A

Paths of length \( N \geq r \) are described by words of the form
\[
D_{i_1}^{e_1} D_{i_2}^{e_2} \ldots D_{i_l}^{e_l}
\]
where \( l = r, r + 1, \ldots, N \), \( i_k \neq i_{k+1}, i_k = 1, \ldots, r \), and
\[
\sum_{k=1}^{l} |e_{i_k}| = N
\]
The number \( W_r(N) \) of such words is given by
\[
W_r(N) = \sum_{l=r}^{N} 2^l p_l(N) r \omega_r(l)
\]
where
\[
p_l(N) = \binom{N-1}{l-1}
\]
is the number of unrestricted partitions of \( N \) into \( l \) nonzero parts \( |e_{i_k}| \), \( k = 1, 2, \ldots, l \). Since each \( e_i \) is either positive or negative there are \( 2^l \) ways of assigning + and − signs to these numbers. Call \( \alpha_1, \alpha_2, \ldots, \alpha_r \), the loops of \( G_r \). Given \( i_1 \in \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \), denote by \( \omega_r(l) \) the number of sequences \( (i_1, i_2, \ldots, i_l) \) with \( i_1 \) fixed and \( i_k \in \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \), such that: a) each \( \alpha_k \) shows up at least once in the sequence; b) \( i_k \neq i_{k+1} \); c) \( i_l \neq i_1 \). Since there are \( r \) possibilities for \( i_1 \) we multiply \( \omega_l(N) \) by \( r \) to get all possible sequences.

In the case \( r = 1 \), \( \omega_1(1) = 1 \). In order to compute a formula for \( \omega_r(l), r > 1 \), let’s drop conditions a) and c) for while. Then, given \( i_1 \) there are \( r - 1 \) possibilities for \( i_2 \), \( r - 1 \) for \( i_3 \) and so on until \( i_l \). Thus, \( (r-1)^{l-1} \) sequences are formed in this way. Among these sequences there are \( I_r(l) \) sequences with \( i_l = i_1 \). Among the remaining sequences there are \( q_r(l) \) sequences with \( i_l \neq i_1 \) and corresponding to paths which do not traverse all the \( r \) lines of \( G_r \) and \( \omega_r(l) \) sequences which obey the conditions a), b) and c). It is clear that
\[
\omega_r(l) = (r-1)^{r-1} - q_r(l) - I_r(l)
\]
In the sequel we derive formulas for \( I_r \) and \( q_r \).

Lemma A.1.
\[
I_r(l) = \frac{(r-1)^{l-1} + (-1)^{l-1}(r-1)}{r}
\]
Proof: It is assumed that condition b) is always satisfied, so the number \( I_r(l) \) is equal to the number of sequences with \( l - 1 \) elements with \( i_{l-1} \neq i_1 \). The latter is equal to the total number of sequences with \( l - 1 \) elements given by \( (r-1)^{l-2} \) minus
the number of those sequences which terminate with \( i_{l-1} = i_1 \) given by \( I_r(l - 1) \). Then,
\[
I_r(l) = (r - 1)^{l-2} - I_r(l - 1)
\]
whose solution is easily found to be relation \((A.6)\). \(\square\)

**Lemma A.2.** Let \( r \geq 2 \). Then,
\[
q_r(l) = \sum_{k=0}^{r-2} (-1)^{r+k} \binom{r-1}{k} \frac{k^l + (-1)^l k}{k+1}
\]

**Proof:** When \( r = 2 \), \( q_2(l) = 0 \). Suppose now \( r > 2 \). Let’s consider the sequences \((i_1, ..., i_l)\) where only \( k \) of its elements, \( k < r \), are distinct. The number of distinct sequences of this type which begin with the same \( i_1 \) is given by \( w_k(l) \). Given \( i_1 \), it remains \( r - 1 \) elements in the set \( \{1, ..., r\} \) where \( k - 1 \) elements can be chosen to make the sequence. Taking \( k = 2, 3, ..., r - 1 \), it follows that
\[
q_r(l) = \sum_{k=2}^{r-1} \binom{r-1}{k-1} w_k(l)
\]

Now, a recurrence relation for \( q_r(l) \) can be obtained using \((A.5)\) and \((A.6)\). The solution is \((A.8)\) starting with \( k = 1 \). However, the case \( r = 2 \) can be included allowing \( k = 0 \). \(\square\)

**Lemma A.3.** Given \( r \geq 2 \),
\[
w_r(l) = \sum_{k=1}^{r-1} (-1)^{r+k+1} \binom{r-1}{k} \frac{k^l + (-1)^l k}{k+1}
\]

**Proof:** Substitute \((A.8)\) into \((A.5)\) and use \((A.6)\). \(\square\)

From \((A.10)\) one gets
\[
r w_r(l) = \sum_{j=2}^{r} (-1)^{r+j} \binom{r}{j} (j-1)^l + (-1)^l + r
\]
valid for \( \forall l \geq r \). Notice that \((A.11)\) can be extended to allow \( j = 1 \) and the case \( w_1(1) = 1 \).

Also, one can extend \( w_r(l) \) to allow \( l < r \) but in this case:

**Corollary A.1.** Given \( r \) and \( l \leq r - 1 \),
\[
w_r(l) = 0
\]

**Proof:** The case \( l = 1 \) follows trivially from \((A.10)\). A simple calculation also shows that
\[
w_r(l + 1) + w_r(l) = \sum_{\beta=0}^{r-1} (-1)^\beta (r - 1 - \beta)^l \binom{r-1}{\beta} = (r - 1)! S(l, r - 1)
\]
where $S(l, r - 1)$ is the Stirling number of second kind. Using induction and that $S(a, b) = 0$ for $a < b$, the result follows. □

In the sequel a formula for $\theta_+$, relation (3.3) is computed. Using (A.12), let’s rewrite (A.3) as

$$W_r(N) = \sum_{l=1}^{N} 2^l p_l(N) r w_r(l) \quad (A.14)$$

Following the ideas of Ref. 1 denote by $W_r(l/g, N/g, s/g, T/g)$ the number of non-periodic words plus their circular permutations associated to the numbers $l/g, N/g, s/g, T/g$. Then,

$$W_r(l, N, s, T) = \sum_{g|(l,N,s,T)} W_r(l/g, N/g, s/g, T/g) \quad (A.15)$$

where the summation is over the common divisors $g$ of $l, N, s$ and $T$. The term in (A.15) with $g = 1$ counts precisely the number of distinct nonperiodic words whereas the $g \neq 1$ terms count the periodic ones.

Applying Mobius inversion formula it follows that

$$\frac{W_r(l, N, s, T)}{g(l,N,s,T)} = \sum_{g|(l,N,s,T)} \mu(g) W_r(l/g, N/g, s/g, T/g) \quad (A.16)$$

where $\mu$ is the Mobius function. A formula for $W_r(l, N, s, T)$ can be computed after the the following decompositions in (A.3) are made:

$$2^l = \sum_{s=0}^{l} \binom{l}{s} \quad (A.17)$$

and

$$w_r(l) = \sum_{T} f_r(T, l) \quad (A.18)$$

where $f_r(T, l)$ is the number of sequences with the same number $T$ of subsequences. Explicit knowledge of $f_r$ will not be necessary. From (A.14) it follows that

$$W_r(l, N, s, T) = \binom{l}{s} \binom{N-1}{l-1} r f_r(T, l) \quad (A.19)$$

Using the rule of signs from section II),

$$\theta_{\pm}(N, r) = \sum_{l,s,T} \frac{W_r(l, N, s, T)}{l} \quad (A.20)$$

where the sum runs over all $l, s$ and $T$ satisfying the condition that $l + N + s + T$ is odd for $\theta_+$ and $l + N + s + T$ is even for $\theta_-$. 

21
From relations (A.19), (A.16) and (A.20) it seems to be the case that an explicit formula for \( f_r(T, l) \) is necessary in order to find \( \theta_\pm \). The next calculations will show that it suffices to know that \( f_r \) satisfy (A.18). Only the particular case of \( \theta_+ \) with \( N \) even will be considered explicitly. The other cases follow similar logic of calculation.

Suppose \( N \) is an even number. The condition that \( N+l+s+T \) be odd requires that

a) \( l, s, T \) odd;
b) \( l, s \) even and \( T \) odd ;
c) \( l, T \) even and \( s \) odd ;
d) \( s, T \) even and \( l \) odd.

On the other hand, for \( N+l+s+T \) to be even requires

a) \( l, s, T \) even ;
b) \( l, s \) odd and \( T \) even ;
c) \( l, T \) odd and \( s \) even ;
d) \( s, T \) odd and \( l \) even.

In the case that \( N \) is odd just swift the words even and odd in items a) − d) to get the conditions for \( N+l+s+T \) to be odd and likewise in items e) − h) to get the conditions for \( N+l+s+T \) to be even.

For \( N \) even, the number \( \theta_+(N, r) \) is computed summing over \( l, s, T \) satisfying the conditions a), b), c) and d) above, that is,

\[
\theta_+(N, r) = \left[ \sum_{a} + \sum_{b} + \sum_{c} + \sum_{d} \right] \frac{1}{l} \sum_{g|(l,N,s,T)} \mu(g)W_r \left( \frac{l}{g}, \frac{N}{g}, \frac{s}{g}, \frac{T}{g} \right) \tag{A.21}
\]

Let’s consider the a)-sum. First, the sum over the odd numbers \( l, s, T \) multiples of a given \( g \) is performed, and then the sum over the \( g \)'s which divide \( N \). From (A.14) the possible values of \( l \) odd are \( l = 1, 3, \ldots, N-1 \). Among these, the values multiples of a given \( g \) are \( l = (2x+1)g \) for \( x = 0, 1, \ldots, \frac{N}{2g} - 1 \), hence the possible values of \( s \) are \( s = (2y+1)g \) for \( y = 0, 1, \ldots, x \) and \( T = (2z+1)g \). The values of \( z \) are not important here and in the calculations below.

Upon substitution of these into (A.19) and (A.21) one gets for the a)-sum:

\[
\sum_{odd \ g|N} \mu(g) \sum_{x=0}^{\frac{N}{2g}-1} \frac{1}{2x+1} \left( \frac{N}{g} - 1 \right) \sum_{y=0}^{x} \left( \frac{2x+1}{2y+1} \right) r \sum_{z} f_r(2z+1, 2x+1) \tag{A.22}
\]

The b)-sum in (A.21) is over the even values of \( l, s \) and odd values of \( T \). The common divisors are among the odd divisors \( g \) of \( N \) and \( l = g(2x) \), with \( x = 1, \ldots, N/2g \),
hence, $s = g(2y)$, with $y = 0, 1, \ldots, x$ and $T = g(2z) + 1$. For the b)-sum one gets:

$$\sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=1}^{\frac{N}{g}} \frac{1}{2x} \left( \frac{\frac{N}{g} - 1}{2x - 1} \right) \sum_{y=0}^{x} \left( \frac{2x}{2y} \right) r \sum_{z} f_r(2z + 1, 2x) \quad (A.23)$$

The $l, T$ even and $s$ odd are given by $l = g(2x)$, with $x = 1, 2, \ldots, N/2g$, $s = g(2y+1)$, with $y = 0, 1, \ldots, x$ and $T = g(2z)$. Then, for the c)-sum:

$$\sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=0}^{\frac{N}{2g}-1} \frac{1}{2x + 1} \left( \frac{\frac{N}{g} - 1}{2x} \right) \sum_{y=0}^{x} \left( \frac{2x + 1}{2y + 1} \right) r \sum_{z} f_r(2z, 2x + 1) \quad (A.24)$$

The last summation in (A.21) is over the even values of $s, T$ and the odd $l$, namely, $l = g(2x + 1)$, with $x = 0, 1, \ldots, \frac{N}{2g} - 1$; $s = g(2y)$, with $y = 0, 1, \ldots, x$ and $T = g(2z)$. For the d) sum:

$$\sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=0}^{\frac{N}{2g}-1} \frac{1}{2x + 1} \left( \frac{\frac{N}{g} - 1}{2x} \right) \sum_{y=0}^{x} \left( \frac{2x + 1}{2y + 1} \right) r \sum_{z} f_r(2z, 2x + 1) \quad (A.25)$$

Adding (A.22) to (A.25) and using that

$$\sum_{y=0}^{x} \left( \frac{2x + 1}{2y + 1} \right) = \sum_{y=0}^{x} \left( \frac{2x + 1}{2y} \right) = 2^x \quad (A.26)$$

yields

$$\sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=0}^{\frac{N}{2g}-1} \frac{1}{2x + 1} \left( \frac{\frac{N}{g} - 1}{2x} \right) 2^{2x} r \sum_{z} [f_r(2z + 1, 2x + 1) + f_r(2z, 2x + 1)]$$

$$= \sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=0}^{\frac{N}{2g}-1} \frac{1}{2x + 1} \left( \frac{\frac{N}{g} - 1}{2x} \right) 2^{2x} r \sum_{z} f_r(T, 2x + 1) \quad (A.27)$$

Adding (A.23) to (A.24) yields:

$$\sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{x=0}^{\frac{N}{2g}} \frac{1}{2x} \left( \frac{\frac{N}{g} - 1}{2x - 1} \right) 2^{2x-1} r \sum_{z} f_r(T, 2x) \quad (A.28)$$

At last, adding (A.28) to (A.27):

$$\theta_+(N, r) = \sum_{odd \, g|N} \frac{\mu(g)}{g} \sum_{T} \left[ \sum_{x=0}^{\frac{N}{2g}} \frac{1}{2x + 1} \left( \frac{\frac{N}{g} - 1}{2x} \right) 2^{2x} r f_r(T, 2x + 1) + \right.$$

23
\[ + \sum_{x=1}^{N-1} \frac{1}{2x} \left( \frac{N}{g} \left( \frac{N}{g} - 1 \right) \right) 2^{2x-1} r f_r(T, 2x) \]\(\text{A.29}\)

Now, put the \(x\)-sums together and use (A.18) to get:

\[ \theta_+(N, r) = \sum_{\text{odd } g|N} \mu(g) \sum_{l=1}^{N} \frac{1}{l} \left( \frac{N}{g} \left( \frac{N}{g} - 1 \right) \right) 2^{l-1} r w_r(l) \]\(\text{A.30}\)

Although obtained for \(N\) even, this formula holds for \(N\) odd as well. Using (A.11) and performing the summation over \(l\) yields result (3.4). Relations (3.5-6) can be obtained in the same fashion as above (see Ref. 1, too) or faster as in section 3.
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