Topological properties on the diameters of the integer simplex

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Abstract

Wide diameter $d_\omega(G)$ and fault-diameter $D_\omega(G)$ of an interconnection network $G$ have been recently studied by many authors. We determine the wide diameter and fault-diameter of the integer simplex $T_{m,n}^m$. Note that $d_1(T_{m,n}^m) = D_1(T_{m,n}^m) = d(T_{m,n}^m)$, where $d(T_{m,n}^m)$ is the diameter of $T_{m,n}^m$. We prove that $d_\omega(T_{m,n}^m) = D_\omega(T_{m,n}^m) = d(T_{m,n}^m) + 1$ when $2 \leq \omega \leq n$. Since a triangular pyramid $TP_L$ is $T_{3,L}^3$, we have $d_\omega(TP_L) = D_\omega(TP_L) = d(TP_L) + 1$ when $2 \leq \omega \leq 3$.

Keywords: integer simplex; triangular pyramid; wide diameter; fault-diameter.

1 Introduction

An interconnection network is conveniently represented by an undirected graph. The vertices(edges) of the graph represent the nodes(links) of the network. As a topology for an interconnection network of a multiprocessor system, the triangular pyramid (tripy for short), is proposed by Razavi and Sarbazi-Azad in [14]. Some basic properties such as Hamiltonian-connectivity, pancyclicity and a routing algorithm were investigated in the paper. We studied other properties such as symmetry, connectivity and fault-tolerant vertex-pancyclicity in [13].

Reliability and efficiency are important criteria in the design of interconnection networks. In graph theory and the study of fault-tolerance and transmission delay of networks, wide diameter and fault-diameter are two very important parameters and have been studied by many researchers. The diameters of Cartesian product graphs
were studied in [1, 2, 3, 6, 7, 16]. The parameters of some well-known networks such as hypercube, crossed cube etc. were studied in [4, 5, 9, 11, 12, 15, 18].

Wide diameter of a graph, which combines connectivity with diameter, is a parameter that measures simultaneously the fault-tolerance and efficiency of parallel processing computer networks. Let \( u \) and \( v \) be two vertices of a graph \( G \). The distance between \( u \) and \( v \), denoted by \( d_G(u, v) \), is the length of the shortest path between them. The diameter of \( G \), denoted by \( d(G) \), is the maximum distance between any two vertices. The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal results in a disconnected or trivial network. We say that \( G \) is \( k \)-connected for any \( 0 < k \leq \kappa(G) \). According to Menger’s theorem, there are \( k \) disjoint paths between any two vertices in a \( k \)-connected network. Let \( G \) be a \( k \)-connected graph with \( 1 \leq \omega \leq k \). The \( \omega \)-wide diameter \( d_\omega(G) \) of \( G \) is the minimum \( \ell \) such that there exist \( \omega \) internally vertex disjoint paths of length at most \( \ell \) from \( u \) to \( v \) for any two vertices \( u \) and \( v \). Throughout this paper, “disjoint paths” always means “internally vertex disjoint paths”. In particular, \( d_1(G) \) is just the diameter \( d(G) \) of \( G \). It is easy to see that

\[
d(G) = d_1(G) \leq d_2(G) \leq \cdots \leq d_{k-1}(G) \leq d_k(G).
\]

Failures are inevitable when a network is put in use. Therefore, it is significant to consider faulty networks. The fault-diameter can be used to estimate the effect on the diameter when faults occur. A small fault-diameter is desirable because the delay would be shorter when some nodes fail. The fault-diameter with faulty vertices was first studied by the authors in [10]. The \((\omega - 1)\)-fault-diameter of a graph \( G \) is defined as

\[
D_\omega(G) = \max\{d(G - F) : F \subseteq V(G), |F| < \omega\}
\]

where \( G - F \) denotes the subgraph induced by \( V(G) - F \). Note that \( D_\omega(G) < \infty \) if
and only if $G$ is $\omega$-connected. It is also clear that

$$d(G) = D_1(G) \leq D_2(G) \leq \cdots \leq D_{k-1}(G) \leq D_k(G).$$

It is well known (see [12]) that for any $k$-connected graph $G$ and any integer $\omega$, $1 \leq \omega \leq k$, we have $D_\omega(G) \leq d_\omega(G)$, where the equality holds for some well-known networks. However, it’s difficult to determine the wide diameter or fault-diameter of the tripy according to its definition in [14]. Fortunately, we find that the tripy $TP_L$ is a special integer simplex, and we determine the wide diameter and fault-diameter of the integer simplex. The two kinds of diameters of the tripy are deduced from the results of the integer simplex.

The rest of this paper is organized as follows. We give some definitions and notations in Section 2. The main results are given in Section 3.

## 2 The Integer Simplex $T^m_n$

For graph-theoretical terminology and notation not defined here, we follow [17]. We first restate the definitions of triangular mesh and tripy originally proposed by Razavi and Sarbazi-Azad for completeness.

![Figure 1: The $T_3$.](image)

**Definition 1** A radix-$m$ triangular mesh network, denoted by $T_m$, consists of a set of vertices $V(T_m) = \{(x,y) | 0 \leq x + y \leq m\}$ where any two vertices $(x_1, y_1)$ and
$(x_2, y_2)$ are connected by an edge if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$, or $x_2 = x_1 + 1$ and $y_2 = y_1 - 1$, or $x_2 = x_1 - 1$ and $y_2 = y_1 + 1$.

Figure 1 shows a $T_3$ network.

A tripy is a hierarchy structure based on triangular meshes.

**Definition 2** An $L$-level tripy, denoted by $TP_L$, consists of a set of vertices $V(TP_L) = \{(k, (x, y))| 0 \leq k \leq L, 0 \leq x + y \leq k\}$. Vertex $(k, (x, y)) \in V(TP_L)$ is said to be a vertex at level $k$ with the coordinate $(x, y)$. The vertices at level $k$ form a network of $T_k$. Vertex $(k, (x, y))$ is also connected to vertices $(x, y)$, $(x + 1, y)$, and $(x, y + 1)$, in level $k + 1$, as child vertices, and to vertex $(x - 1, y)$ if $x > 0$, vertex $(x, y - 1)$ if $y > 0$, and vertex $(x, y)$ if $x + y < k$, as parents in level $k - 1$.

**Definition 3** The integer simplex with dimension $n$ and side-length $m$ is the graph $T^n_m$ whose vertices are the nonnegative integer $(n+1)$-tuples summing to $m$, with two vertices adjacent when they differ by 1 in two places and are equal in all other places.

In other word,

$$V(T^n_m) = \{v_n \cdots v_1 v_0 | \sum_{i=0}^{n} v_i = m, v_i (i \in \{0, 1, \ldots, n\}) \}$$

Two vertices $u = u_n \cdots u_1 u_0$ and $v = v_n \cdots v_1 v_0$ are adjacent if and only if $u_i = v_i - 1$, $u_j = v_j + 1$ and $u_k = v_k$ where $k \in \{0, 1, \ldots, n\} \setminus \{i, j\}$.

However, both the triangular mesh and the tripy are special kinds of integer simplex. Let $\sigma_1$ be a mapping from $V(T_m)$ to $V(T^2_m)$ defined by $\sigma_1((x, y)) = (m - (x + y), x, y)$, for any vertex $(x, y) \in V(T_m)$. Then, $\sigma_1$ is an isomorphism from $T_m$ to $T^2_m$. Let $\sigma_2$ be a mapping from $V(TP_L)$ to $V(T^3_L)$ defined by $\sigma_2((k, (x, y))) = (m - (k + x + y), k, x, y)$, for any vertex $(k, (x, y)) \in V(TP_L)$. Then, $\sigma_2$ is an isomorphism from $TP_L$ to $T^3_L$.

According to the definition, the number of vertices of $T^n_m$ is $\binom{n+m}{m}$, and the minimum degree of $T^n_m$ is $n$. The special cases of $T^n_m$ are the following. $T^1_1$ is $K_{d+1}$; $T^1_m$ is a path with $m + 1$ vertices; $T^2_m$ is the triangular mesh network $T_m$, and is also named as triangulated triangle with side length $m$ in [8]; $T^3_m$ is the triangular pyramid $TP_m$. 
which is studied in [13, 14]. Since $T_m^1$ is a path, its connectivity is 1. We assume $n \geq 2$ when we studied the wide diameter and fault-diameter of $T_m^n$.

## 3 Main results

We first construct disjoint paths of certain lengths joining two vertices in $T_m^n$. A path $P$ joining vertices $u$ and $v$ is also denoted by a $uv$-path. For any two vertices $u = u_n \cdots u_1 u_0$ and $v = v_n \cdots v_1 v_0$ of $T_m^n$, let $h(u, v) = \frac{1}{2} \sum_{i=0}^{n} |u_i - v_i|$. By the vertex definiton of $T_m^n$, $h(u, v) \leq m$.

**Lemma 1.** Let $u = u_n \cdots u_1 u_0$ and $v = v_n \cdots v_1 v_0$ be two vertices in $T_m^n$. If there are $p$ bit positions such that $u_i > v_i$, and $q$ bit positions such that $u_i < v_i$, then there are $n + 1 - (p + q) + pq$ disjoint $uv$-paths, such that $pq$ of them are of length $h(u, v)$, and the other $n + 1 - (p + q)$ of them are of length $h(u, v) + 1$.

**Proof.** Without loss of generality, we may assume $u_i > v_i$ where $i \in \{n, n-1, \ldots, n-p+1\}$, and $u_i < v_i$ where $i \in \{0, 1, \ldots, q-1\}$. (Note that $p \geq 1$, $q \geq 1$, and $p + q \leq n + 1$). Then, $h(u, v) = \sum_{i=n-p+1}^{n}(u_i - v_i) = \sum_{i=0}^{q-1}(v_i - u_i)$.

There are $q$ different ways to make $u_{q-1} \cdots u_0$ up to $v_{q-1} \cdots v_0$. We modify the bits $(q-1), (q-2), \ldots, 1, 0$ in a rotational way.

The first way is that we modify the $(q-1)$th bit $u_{q-1}$ up to $v_{q-1}$, and then modify the $(q-2)$th bit $u_{q-2}$ up to $v_{q-2}$ and so on. At last modify the 0th bit $u_0$ up to $v_0$. We use $(q-1) - (q-2) - \cdots - 1 - 0$ to denote this way. That is $u_{q-1} u_{q-2} \cdots u_0 \rightarrow (u_{q-1} + 1) u_{q-2} \cdots u_0 \rightarrow \cdots \rightarrow v_{q-1} u_{q-2} \cdots u_0 \rightarrow v_{q-1}(u_{q-2} + 1) \cdots u_0 \rightarrow \cdots \rightarrow v_{q-1} v_{q-2} \cdots v_1 u_0 \rightarrow v_{q-1} v_{q-2} \cdots v_1(u_0 + 1) \rightarrow \cdots \rightarrow v_{q-1} \cdots v_0$.

In general, the $i$th way is $(q-i) - (q-i-1) - \cdots - 1 - 0 - (q-1) - \cdots - (q-i+1)$ for any $1 \leq i \leq q$.

Similarly, there are $p$ different ways to make $u_n \cdots u_{n-p+1}$ down to $v_n \cdots v_{n-p+1}$. We modify the bits $n, (n-1), \ldots, (n+1-p)$ in a rotational way.
The first way is that we modify the \( n \)th bit \( u_n \) down to \( v_n \), and then modify the \((n - 1)\)th bit \( u_{n-1} \) down to \( v_{n-1} \) and so on. At last modify the \((n - p + 1)\)th bit \( u_{n-p+1} \) down to \( v_{n-p+1} \). We use \( n - (n - 1) - \cdots - (n - p + 1) \) to denote this way.

In general, the \( j \)th way is \((n + 1 - j) - (n - j) - \cdots - (n - p + 1) - n - (n - 1) - \cdots - (n + 2 - j)\) for any \( 1 \leq j \leq q \).

Combining an \( i \)th way to make \( u_{q-1} \cdots u_0 \) up to \( v_{q-1} \cdots v_0 \) and a \( j \)th way to make \( u_n \cdots u_{n-p+1} \) down to \( v_n \cdots v_{n-p+1} \), we obtain a \( uv \)-path of length \( h(u, v) \). There are \( pq \) different ways to combine them and these paths are disjoint.

If \( p + q = n + 1 \), the conclusion is true.

If \( p + q < n + 1 \), then \( u_k = v_k < m \) for any bit \( k \) where \( q \leq k \leq n - p \). We construct \( n + 1 - (p + q) \) disjoint \( uv \)-paths of length \( h(u, v) + 1 \) in the following.

For any \( k \) satisfies \( q \leq k \leq n - p \), let \( u' = (u_{n-1} \cdots (u_k + 1) \cdots u_1 u_0 \) and \( v' = v_n v_{n-1} \cdots (v_k + 1) \cdots v_1 (v_0 - 1) \). We have \( h(u', v') = h(u, v) - 1 \). We can construct a \( u'v' \)-path \( P_k' \) of length \( h(u', v') \) as above and the \( k \)th bit of every vertex on the path is \( v_k + 1 \). Then \( P_k = u + P_k' + v \) is a \( uv \)-path of length \( h(u, v) + 1 \).

From the construction, these \((n + 1 - (p + q))\) \( uv \)-paths are disjoint and they are also disjoint with the previous \( pq \) \( uv \)-paths.

For any two vertices of \( T_m^n \), we construct \( n + 1 - (p + q) + pq \) disjoint \( uv \)-paths. Since \( p \geq 1 \) and \( q \geq 1 \), then \( pq \geq (p + q) - 1 \), and \( n + 1 - (p + q) + pq \geq n \). Hence, we construct at least \( n \) disjoint \( uv \)-paths with length at most \( m + 1 \). By Menger’s Theorem, \( n \leq \kappa(T_m^n) \). Note that the minimum degree of \( T_m^n \) is \( n \). By the well-known inequality \( \kappa(G) \leq \lambda(G) \leq \delta(G) \), we have the following corollary.

**Corollary 1.** \( \kappa(T_m^n) = \lambda(T_m^n) = \delta(T_m^n) = n \) and \( d_n(T_m^n) \leq m + 1 \).

For short we use \( a^i \) to denote \( i \) identical bit \( a \). For example, \( m000 = m000 \). Let \( u = u_n \cdots u_1 u_0 \) and \( v = v_n \cdots v_1 v_0 \) be two vertices of \( T_m^n \). Assume \( u_i > v_i \) where \( i \in \{ n, n - 1, \ldots, n - p + 1 \} \), and \( u_i < v_i \) where \( i \in \{ 0, 1, \ldots, q - 1 \} \). Then from the
definition of $T^n_m$, the distance between $u$ and $v$ is at least $h(u,v)$. By Lemma 1, there is a path of length $h(u,v)$ joining them. Letting $u = m0^{n-1}$ and $v = 0^n m$, we have $h(u,v) = m$. The following corollary is obtained.

**Corollary 2.** For any two vertices $u$ and $v$ in $T^n_m$, $d(u,v) = h(u,v)$. Furthermore, $d(T^n_m) = m$.

**Lemma 2.** Let $u = m0^n$ and $v = 0^n m$ be two vertices of $T^n_m$. Any shortest $uv$-path must contain the vertex $(m-1)0^n-1$.

**Proof.** Since $h(u,v) = m$, the length of a shortest $uv$-path is $m$.

Let $u'$ be a neighbor of $u$. The leftmost bit of $u'$ is $m-1$, and the other bits are 0 except that one bit is 1. If $u'$ is not the vertex $(m-1)0^n-1$, then the distance between $u'$ and $v$ is $m$. Hence, the length of a $uv$-path containing $u'$ is at least $m+1$. In other word, any shortest $uv$-path must contain the vertex $(m-1)0^n-1$. 

**Lemma 3.** $D_2(T^n_m) \geq m + 1$.

**Proof.** Since the fault-diameter $D_2(T^n_m)$ is defined as the largest distance between any pair of vertices if a fault occurs and our goal is to give a lower bound of this. We may select the following three vertices as follows. Let $u = m0^n$ and $v = 0^n m$ be two vertices of $T^n_m$, and let $w = (m-1)0^n-21$ be the faulty vertex and $G = T^n_m - w$. By Lemma 2, we have $d_{T^n_m}(u,v) = m$ and any shortest path in $T^n_m$ must contain the vertex $w$. Since $G = T^n_m - w$, $d_G(u,v) > d_{T^n_m}(u,v)$. We have $d_G(u,v) \geq m + 1$. Hence, $D_2(T^n_m) \geq m + 1$.

We have shown that $\kappa(T^n_m) = n$. We determine the fault-diameter $D_\omega(T^n_m)$ and wide diameter $d_\omega(T^n_m)$ for any $2 \leq \omega \leq n$ in the following.

**Theorem 1.** $d_\omega(T^n_m) = D_\omega(T^n_m) = d(T^n_m) + 1$ for $2 \leq \omega \leq n$. 

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Proof. We have \( d_n(T_n^m) \leq m + 1 \) by Corollary 1 and \( D_2(T_n^m) \geq m + 1 \) by Lemma 3. Since \( D_2(T_n^m) \leq d_2(T_n^m), D_n(T_n^m) \leq d_n(T_n^m), d_2(T_n^m) \leq \ldots \leq d_n(T_n^m) \) and \( D_2(T_n^m) \leq D_3(T_n^m) \leq \ldots \leq D_n(T_n^m), \) we have \( d_\omega(T_n^m) = D_\omega(T_n^m) = m + 1 \) for \( 2 \leq \omega \leq n. \)

By Corollary 2, \( d(T_n^m) = m. \) The conclusion is true. \( \square \)

For the tripy, we have the conclusion.

**Corollary 3.** \( d_\omega(TP_L) = D_\omega(TP_L) = d(TP_L) + 1 \) for \( 2 \leq \omega \leq 3. \)

Theorem 1 shows that the wide diameter equals the fault-diameter for the integer simplex \( T_n^m. \) They are increased only by one over the traditional diameter. Since the triangular pyramid is a special integer simplex, the tripy enjoys the similar property of strong resilience like the hypercube. And, the results add to the attractiveness of the tripy as compared to the hypercube.

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