The Hamiltonian in the unidirectional surface wave propagation

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Abstract. Unidirectional surface water waves used in this study were obtained from the Benjamin Bona Mahony (BBM) equation. The BBM wave research is carried out by observing the envelope of BBM wave groups that involve following the Nonlinear Schrodinger (NLS) Equation. Soliton on Finite Background (SFB) is an excellent NLS solution for describing the dynamics of wave propagation. It is known that there are modulation instability and phase singularity phenomena which can significantly affect the amplitude of the wave in its propagation. Hamiltonian contains potential energy which can explain the singularity and amplitude dynamics of the phase. Hamiltonian equations using the displaced phase-amplitude variables have been obtained and are presented in this article.

1. Introduction

The Benjamin Bona Mahony (BBM) equation is a mathematical model that describes the unidirectional of surface water waves in its propagation. In this study, the BBM equation was studied in detail to support the activity of generating high, unbroken waves known as extreme waves [1,2]. The generation of extreme waves in the laboratory is carried out to test the resistance of ship models and offshore structures to extreme waves that can suddenly appear offshore [3,4].

Research on BBM waves was carried out by observing the wave itself and also using the BBM wave group envelope. It is known that the BBM wave group involves the Schrodinger Nonlinear (NLS) equation and the Soliton on Finite Background (SFB) is one of the excellent NLS solutions to describe the dynamics of wave propagation. Halfiani et al [5] and Fadhiliani et al [6] have carried out research on the envelope of the BBM wave group. Halfiani et al [5] state that there was a modulation instability in its propagation and the plot of the envelope shows a phase singularity phenomenon. Fadhiliani et al [6] in his research also stated the same thing by using argand diagrams. Argand diagrams are obtained by transforming SFB variables with displaced phase-amplitude variables introduced by Karjanto [7] to study the dynamics of another one-way wave, namely the Kortweg de Vries (Kdv) wave.

The phase singularity phenomenon can significantly affect the amplitude of the wave [7]. The singularity and the dynamics of the phase-amplitude in the NLS equation can be described using the Hamiltonian because it contains potential energy [8]. As we know, the Hamiltonian is the sum of the kinetic energy and potential energy of the system.
The use of Hamiltonian formulas in hydrodynamic problems for water waves has been widely used, including by Groves et al [9] and Wahen et al [10]. Sultana et al [11] in his research also used the Hamilton principle to investigate the evolution of water waves and derive Hamilton’s canonical equation of motion. Craig et al [12] derives the Hamiltonian of the NLS equation and several other equations in the study of free surface water waves and continues by analyzing the stability of Benjamin-Feir [13]. Besides that, there is Kurnia et al [14] who applied the consistent Hamiltonian modeling method to obtain the high-order Hamiltonian equation with the Taylor expansion of the potential and vertical velocity around the water surface at rest based on the Hamiltonian formulation of water waves. Therefore, this study focuses on obtaining information about the distribution of energy by utilizing the Hamiltonian function and the dynamics of BBM waves in extreme waves generation in the laboratory.

2. The BBM Wave Group Envelope
The surface water wave propagation for normalized variables can be expressed in the form [15]:

$$\eta_t + \eta_x + \eta \eta_x - \eta_{xxt} = 0$$  \hspace{1cm} (1)

This equation is intended to model unidirectional wave propagation with large wavenumbers and small amplitudes. If the selected solution is in the wave group \( \eta(x,t) = a(x,t)e^{i\theta} + c.c \) where \( \theta = (kx - \omega t) \) dan \( k \) are the frequency and wavenumber, respectively. Halfiani shows that the envelope of the wave group propagates and involve the spatial NLS equation as follows:

$$A_\xi + i\beta A_{\tau\tau} + i\gamma |A|^2 A = 0$$  \hspace{1cm} (2)

where \( \beta = - \left( 2\omega p + \frac{\omega^2}{k} \right)/p^3 \) and \( \gamma = \omega \left( \frac{2}{p-1} - \frac{k}{8k^2\omega + 2k^2} \right)/p \) is a constant that depends on the frequency or wave number. Equation (2) is obtained by [5] for variable transformation to the space variable \( x \) and time \( t \) through the transformation \( \xi = \epsilon^2 x, \ \tau = \epsilon \left( t - \frac{\xi}{p} \right), a(x,t) = \epsilon A(\xi, \tau) \). Parameter \( p = (1 + 2k\omega)/(1 + k^2) \) are elements that appear in the variable transformation obtained to eliminate the resonance term in the process of determining the solution through the multiple scale method [5]. For problems involving the wave signal as the initial signal in the wavemaker to describe the propagation in space can using the NLS equation. The independent variables \( \xi \) and \( \tau \) have meaning depending on the problem described. In the dispersive wave problem, \( \xi \) and \( \tau \) represents the spatial and the time variable, respectively.

The solution of the spatial NLS equation which can describe the occurrence of extreme waves in a wave pool is SFB. SFB is a non linear extinction of a monochromatic signal with an amplitude of \( r_0 \), that is \( A_0(\xi) = r_0 e^{-i\tau_0^2} \xi \) which is interfered with by a modulating wave with small wavelength intervals resulting in an exponential increase in instability. These SFB waves reach their maximum amplitude at \( 0, \frac{2n\pi}{\nu} \), \( n \in \mathbb{Z} \) [16]. The spatial SFB solution can be written as:

$$A(\xi, \tau) = A_{SFB}(\xi, \tau) A_0(\xi)$$  \hspace{1cm} (3)

In the propagation of the BBM wave group, modulation stability and the phenomenon of phase singularity appears [5]. The phase singularity occurs when the real value of the amplitude \( |A_{SFB}| \) disappeared [16]. Phase singularity, here, is indicated by the presence of wave dislocation which is indicated by the merging of two waves (wave merging) or wave splitting [17].

3. Variational Formulation (Lagrangian and Hamiltonian Structures)
The variational formulation of the NLS equation to be used is in the form of a Lagrangian and Hamiltonian structure. As stated by Groesen et al [18] that the equation for the wave-ship interaction
is based on the Lagrangian principle of variational, which leads to the formulation of the combined system as the Hamiltonian system. Hamiltonian functions to regulate the dynamics of phase-amplitude because it contains potential energy [8].

Let $A$ be the solution to the NLS equation,

$$A_\xi + i\beta A_{\tau\tau} + i\gamma |A|^2 A = 0,$$

(4)

Then the Lagrangian $\mathcal{L}$ associated with equation (4) is:

$$\mathcal{L}(A) = \frac{1}{4} i \left( A^* \partial_\xi A - A \partial_\xi A^* \right) + \frac{1}{2} \beta |\partial_\tau A|^2 - \frac{1}{4} \gamma |A|^4,$$

(5)

with Euler-Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial A} = \frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}}{\partial A_{\tau}} \right) + \frac{\partial}{\partial \xi} \left( \frac{\partial \mathcal{L}}{\partial A_\xi} \right).$$

(6)

The Hamiltonian $H$ structure is derived based on the Lagrangian. The Hamiltonian density $\mathcal{H}$ is expressed as follows:

$$\mathcal{H}(A) = \frac{1}{4} i \left( A^* \partial_\xi A - A \partial_\xi A^* \right) - \mathcal{L}(A)$$

$$= - \frac{1}{2} \beta |\partial_\tau A|^2 + \frac{1}{4} \gamma |A|^4,$$

(7)

and Hamiltonian $H$ expressed as follows:

$$H(A) = \int_{-\infty}^{\infty} \mathcal{H}(A) \, d\tau.$$ 

(8)

The Hamiltonian function to prove the singularity for a finite time in the NLS equation. The NLS equation in the Lagrangian form is a dynamic system. It is integrable and as a consequence it has an infinite number of conserved quantities. Hamiltonian is one of the quantities in question, besides that there is also a form:

$$C_1 = \int_{-\infty}^{\infty} |A|^2 \, d\tau.$$ 

(9)

It is a simple physical meaning for the first three conservation integrals $C_1$. For the lowest order conserved quantity, i.e the wave energy’, ‘mass’, ‘wave action’, ‘plasmon number’, or ‘wave power’ in optics. For the second-order conserved quantity $C_2$ called the ‘(linear) momentum’ and can be written as

$$C_2 = \int_{-\infty}^{\infty} i \left( A^* \partial_\tau A - A \partial_\tau A^* \right) \, d\tau.$$ 

(10)

4. Displaced phase-amplitude variables

Transformation to displaced phase-amplitude variables is carried out to represent the phase-amplitude of the SFB wave with a finite background as a solution to the NLS equation. The form of a solution to the NLS equation using displaced phase-amplitude variables is given:

$$A(\xi, \tau) = A_0(\xi)F(\xi, \tau).$$ 

(11)
where $A_0(\xi) = r_0 e^{-iyr_0^2\xi}$ and $F$ will be determined later. The variables used are $\phi(\xi, \tau)$ as the displaced phase, $G(\xi, \tau)$ as the displaced amplitude parameter and for special cases $F$ is used as follows:

$$F(\xi, \tau) = G(\xi, \tau)e^{i\phi(\xi, \tau)} - 1, \quad G \text{ dan } \phi \in \mathbb{R},$$

so that the NLS equation solution will be observed with the following form:

$$A(\xi, \tau) = \left(G(\xi, \tau)e^{i\phi(\xi, \tau)} - 1\right)r_0 e^{-iyr_0^2\xi}. \tag{13}$$

Furthermore, equation (13) will be substituted to the Hamiltonian to obtain the transformed Hamiltonian (depending on $G$ and $\phi$) \[19\].

5. Results and Discussion

From equations (7) and (8), the form of the Hamiltonian equation $H$ is as follows:

$$H(A) = \int_{-\infty}^{\infty} \left(-\frac{1}{2} \beta |\partial_\tau A|^2 + \frac{1}{4} \gamma |A|^4\right) \, d\tau. \tag{14}$$

Next, the solution of SFB, $A(\xi, \tau)$ will be substituted which contains the displaced phase-amplitude variables in equation (14). The first derivative $A(\xi, \tau)$ with respect to $\tau$ will be obtained:

$$\partial_\tau A = (\partial_\tau G + iG\partial_\tau \phi)r_0 e^{i(\phi - yr_0^2\xi)}, \tag{15}$$

then the first part of equation (14) will be obtained:

$$\frac{1}{2} \beta |\partial_\tau A|^2 = \frac{1}{2} \beta \left|(\partial_\tau G + iG\partial_\tau \phi)r_0 e^{i(\phi - yr_0^2\xi)}\right|^2 = \frac{1}{2} \beta r_0^2 \left[(\partial_\tau G)^2 + G^2(\partial_\tau \phi)^2\right]. \tag{16}$$

The second part is related to potential energy, as follows:

$$\frac{1}{4} \gamma |A|^4 = \frac{1}{4} \gamma \left|G(\xi, \tau)e^{i\phi(\xi, \tau)} - 1\right|r_0 e^{-iyr_0^2\xi} \right|^4 = \frac{1}{4} \gamma r_0^4 (G(G - 2 \cos \phi) + 1)^2, \tag{17}$$

so that the result of the Hamiltonian $H(G, \phi)$ transformation is obtained as follows:

$$H(A) = \int_{-T/2}^{T/2} \left(-\frac{1}{2} \beta r_0^2 [(\partial_\tau G)^2 + G^2(\partial_\tau \phi)^2] + \frac{1}{4} \gamma r_0^4 (G(G - 2 \cos \phi) + 1)^2\right) \, d\tau. \tag{18}$$

Equation (18) resulted from the Hamiltonian transformation using the displaced phase-amplitude variables which will be studied in future studies. Meanwhile, the form $\phi(\xi, \tau)$ as the displaced phase and $G(\xi, \tau)$ as the displaced amplitude parameter has been previously described (see Fadhiliani et al \[20\]).

6. Conclusion

In the propagation of BBM waves, modulation instability occurs and the phenomenon of phase singularity appears, this is known from the analysis of the BBM wave envelope that involves following the NLS equation. The NLS solution, known as SFB, can describe the dynamics of the
BBM wave and to be presented, a transformation of the SFB variable is carried out with displaced phase-amplitude variables. The singularity and amplitude dynamics of the phase can be shown using the Hamiltonian because they contain potential energy. The Hamiltonian equation using the displaced phase-amplitude variables has been derived and is presented in this article.

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