Bell’s theorem without inequalities and only two distant observers

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Abstract

A proof of Bell’s theorem without inequalities and involving only two observers is given by suitably extending a proof of the Bell-Kochen-Specker theorem due to Mermin. This proof is generalized to obtain an inequality-free proof of Bell’s theorem for a set of $n$ Bell states (with $n$ odd) shared between two distant observers. A generalized CHSH inequality is formulated for $n$ Bell states shared symmetrically between two observers and it is shown that quantum mechanics violates this inequality by an amount that grows exponentially with increasing $n$.

In two recent papers[1,2], Cabello gave a proof of Bell’s theorem without inequalities by using a special state of four qubits shared between two distant observers. This improved upon the classic proof of Greenberger, Horne and Zeilinger[3] and Mermin[4] by reducing the number of distant observers from three to two. The purpose of this paper is to describe a variant of Cabello’s proof that avoids one of its shortcomings and can also be generalized to apply to a suitable entangled state of $2n$ qubits (namely, $n$ identical Bell states) shared symmetrically between two observers. The present proof, like Cabello’s[2] and several others before it[5], uses a common framework to prove both the Bell-Kochen-Specker (BKS)[6] and Bell[7] theorems. However, while Cabello proceeds backwards from the stronger (Bell) to the weaker (BKS) theorem, we proceed in the opposite direction. Our approach has the advantage over Cabello’s that it makes no use of either entanglement or communication between observers in proving the BKS theorem, and invokes these additional elements only in passing from the BKS to the Bell theorem.

The present proof is similar in overall structure to an earlier proof by Heywood and Redhead[8], although it differs in several specific respects. Both proofs exploit EPR type correlations to derive non-contextuality from locality, and then use a Kochen-Specker argument to establish the inevitability of non-locality. However while Heywood and Redhead use a singlet state of two spin-1 particles and the original Kochen-Specker argument in carrying out their proof, we use $n$ Bell states shared symmetrically between two observers and a more
transparent variant of the Kochen-Specker argument due to Mermin[9] in carrying out our proof. We believe the latter route offers both a gain in simplicity relative to the Heywood-Redhead argument and also has a more direct contact with experiment, since only qubits are involved in the present scheme and the question of imperfect correlations and/or detector efficiencies is also addressed.

Figure 1 shows a 3 x 3 array of observables pertaining to a pair of qubits used by Mermin[9] to prove the BKS theorem. Mermin’s proof is based on the elementary observations that: (a) each observable has only the eigenvalues ±1, (b) the observables in any row or column of the array form a mutually commuting set, and (c) the product of the observables (and hence their eigenvalues) in any row or column is +1, with the exception of the last column for which this product is −1. Armed with these facts, Mermin’s argument proceeds as follows:

Suppose an experimenter, Alice, who has two qubits in her possession carries out the measurements corresponding to the commuting observables in one of the rows or columns of Mermin’s square. The result will be a set of +1s and −1s for the measured eigenvalues satisfying the product constraint mentioned earlier. The product constraint can be restated as the “sum” constraint that the total number of -1s for any triad of commuting observables is always even, except for the last column for which it is odd. Now if Alice is a “realist” and believes that the eigenvalues she measures merely reflect preexisting properties of the qubits, she would be tempted to assign the value +1 or −1 to each of the nine observables in Mermin’s square in such a way that all the sum constraints on their values are met. However this is easily seen to be impossible by counting the total number of -1s in the square in two different ways: firstly, by summing over the rows (which leads to an even number) and, secondly, by summing over the columns (which leads to an odd number). This contradiction shows the impossibility of assigning preexisting values (or “elements of reality”[10]) to the concerned observables and constitutes Mermin’s proof of the BKS theorem.

However the above BKS proof has the objectionable feature that an observable is assigned the same value whether it is measured as part of a row or column of observables. This assumption of “noncontextuality” has no empirical basis and, in the opinion of many physicists (including Bell himself[6]), considerably diminishes the force of the BKS theorem[11]. We now show how to rectify this defect and thereby promote Mermin’s BKS proof into a proof of Bell’s theorem. To do this we enlist the help of a second experimenter, Bob, give him two qubits of his own, and allow him to do everything Alice can at a location far removed from hers. The trick to ensuring that Alice and Bob can jointly prove Bell’s theorem is that the four qubits given to them are in the entangled state

$$\Psi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{13} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{24} = \frac{1}{2} (|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle),$$

(1)

where 0 and 1 denote basis states of a qubit and the subscripts 1, ..., 4 in the middle expression indicate the relative positions of these qubits in the last, expanded form of the state $|\Psi\rangle$. In other words, $|\Psi\rangle$ consists of a pair of identical
Bell states, with one member of each pair (qubits 1 and 2) going to Alice and the others (qubits 3 and 4) going to Bob. It is also assumed that the first and second members of each two particle observable in Fig.1 refer to qubits 1 and 2 for Alice and qubits 3 and 4 for Bob.

The state (1) possesses the interesting property that $O_i^A O_i^B |\Psi\rangle = |\Psi\rangle$ for $i = 1, \ldots, 9$, where $O_i^A$ is any one of Alice’s nine observables and $O_i^B$ is the same observable for Bob. This property implies that if Alice and Bob measure identical observables on their qubits, they always obtain the same eigenvalues, even if their measurements are carried out at spacelike separations. These correlated outcomes suffice to establish that all of Alice’s observables are "elements of reality”, and that the same is true of Bob’s. For Bob (or Alice) can use his (or her) measurement of a particular observable to instantly predict the outcome of the other person’s measurement of the same observable at a distant location without disturbing that person’s qubits in any way. Note further that either person’s ability to predict the value of the other’s observable is independent of whether the latter is measured by itself or as part of any commuting triad it happens to be a member of[12]. But this last statement is just the assumption of noncontextuality, now justified on the basis of the correlations in state (1) and the principle of locality, and serves to promote Mermin’s earlier BKS proof into a full fledged proof of Bell’s theorem.

Cabello’s proof[2] differs from ours in that state (1) is replaced by a direct product of singlets and the nine observables measured by Alice and Bob are not identical. However a more significant difference is that, in Cabello’s scheme, Alice and Bob are required to collaborate in measuring five non-local observables each made up of their separate observables. While the measurement of these non-local observables poses no problems for a Bell test, it imposes the unnecessary burden on a BKS test of requiring communication between the observers to achieve its goals.

The present BKS-Bell proof suggests a joint laboratory experiment for verifying the BKS and Bell theorems. However its practical realization is complicated by the fact that each observer needs to be able to measure a sequence of three commuting two-particle observables on his/her qubits. Such "non-demolition" measurements are possible to envision in principle[13], but they are rather challenging to carry out in practice.

The above Bell proof can be generalized to a set of $n$ Bell states (with $n$ odd) shared symmetrically between two observers. Consider the following $(n+2)$

sets of mutually commuting $n$-qubit observables, where each commuting set is shown on a separate line and the superscripts on the Pauli operators refer to the different qubits:
\[
\begin{align*}
\sigma_1^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \cdots, \\
\sigma_1^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \cdots, \\
\sigma_2^2 \sigma_3^3 \sigma_4^4 \cdots, \\
\sigma_1^1 \sigma_2^2 \sigma_3^3 \sigma_4^4, \\
\ldots \ldots \ldots \ldots \ldots \ldots, \\
\sigma_n^1 \sigma_1^1 \sigma_2^2 \sigma_3^3 \cdot \cdot \cdot \sigma_n^1, \\
\end{align*}
\] (2)

Each line after the first consists of one of the observables in the first line together with all the single particle observables of which it is made up. There are \((n + 1) + 2n = 3n + 1\) distinct observables in all, each of which occurs in exactly two commuting sets. The observables have the further properties that: (a) each has only the eigenvalues \(\pm 1\), and (b) the product of the observables (and hence their eigenvalues) in any commuting set is \(+1\), with the exception of the first set for which this product is \(-1\).

A BKS proof can now be constructed as follows. Suppose Alice is given \(n\) qubits and allowed to measure the above observables on them. If Alice is a "realist" and believes that the eigenvalues she measures already preexist in the qubits, she would be tempted to assign the value \(+1\) or \(-1\) to each of the observables in such a way that the products of the values corresponding to each of the rows in (2)-(7) is \(+1\), with the exception of the first row for which it is \(-1\). However this assignment is easily seen to be impossible by taking the product of all the value equations, for one then finds that the product of all the left sides is \(+1\) (because each observable value occurs exactly twice) whereas the product of all the right sides is \(-1\) (because of the first equation in this chain). This contradiction shows the impossibility of assigning preexisting values to the observables and proves the BKS theorem for a system of \(n\) qubits. We now show how to justify the assumption of noncontextuality made in this argument and thus promote it into a proof of Bell’s theorem.

Consider the tensor product of \(n\) Bell states \(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{11'} \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{22'} \otimes \ldots \otimes \frac{1}{\sqrt{2}}(00 + 11)_{nn'}\), and suppose that the unprimed member of each Bell state is given to Alice and the primed member to Bob. Suppose that Alice and Bob are each allowed to measure any of the \(3n + 1\) observables in (2)-(7) on their respective qubits. Then it is not difficult to verify that if Alice and Bob measure identical observables on their qubits, they always obtain the same eigenvalues. From this perfect correlation one can argue, as before, that either observer’s observables are elements of reality and hence that the BKS proof based on (2)-(7) can be promoted into a Bell proof.

The \(n = 3\) case of our BKS proof was given by Mermin[9], who arranged the ten relevant observables[14] at the vertices of a pentagram in such a way that each set of commuting observables lay along one of its edges. Mermin then converted this BKS proof into a Bell proof by assuming that the three qubits were in a GHZ state. The \(n = 5\) case of our BKS proof was given by DiVincenzo and Peres[15], who pointed out that it could be promoted into a Bell
proof if the five qubits were assumed to be in a state corresponding to one of the logical codewords of the five-qubit single error-correcting code discussed in [16]. The foregoing Bell proofs of Mermin and of DiVincenzo and Peres involve three and five separated observers, respectively, whereas our proofs involve only two observers who however share three or five Bell states symmetrically between themselves.

In an interesting paper dealing with the transition from quantum nonlocality to classical behavior in the limit of an infinite number of particles, Pagonis et al[17] introduce several alternative sets of commuting observables for $N$ qubits that allow closely related proofs of the BKS and Bell theorems to be given. Their Bell proofs require distributing a simultaneous eigenstate of the commuting observables to $N$ observers who then carry out measurements on their individual qubits in a generalization of the GHZ protocol. It is worth noting that the observables proposed by Pagonis et al can also be used by just two observers to carry out a joint BKS-Bell proof provided that they share $N$ Bell states symmetrically among themselves and exploit the identity of the eigenvalues of similar observables measured at their two ends. An interesting difference between the observables in (2)-(7) and those proposed by Pagonis et al is that our observables generally involve no more than three qubits whereas those of Pagonis et al always involve all $N$ qubits simultaneously. Another, more significant, difference is that in the scheme of Pagonis et al (and also that of Mermin-GHZ), the observables that are used to prove the BKS theorem are closely related to the entangled state used in the later Bell proof (the latter being a simultaneous eigenstate of the nontrivial commuting observables in the BKS proof). In contrast to this, the observables used in our BKS proofs bear no relation to the entangled (Bell) states used in the later Bell proofs.

Our Bell proofs assume that Alice and Bob share perfect Bell states and that their particle detectors are perfectly efficient. If these conditions are not met, our proofs lose their "all or nothing" character and can be rescued only by devising inequalities that are satisfied by local realism but violated by quantum mechanics (and experiment). We now exhibit one such inequality. Suppose Alice and Bob share $n$ EPR singlets, with Alice possessing one member of each pair and Bob the other. Consider the operator $B = B_1B_2...B_n$, where $B_i = \sigma^i \cdot \hat{a}(\sigma^i \cdot \hat{b} + \sigma^i \cdot \hat{b}') + \sigma^i \cdot \hat{a}'(\sigma^i \cdot \hat{b} - \sigma^i \cdot \hat{b}')$ is the usual CHSH operator[18] for the $i$-th singlet shared by Alice and Bob, with the unit vectors $\hat{a}, \hat{a}', \hat{b}, \hat{b}'$ being chosen so as to make the expectation value of $B_i$ in the singlet state achieve its maximal value of $2\sqrt{2}$. Then, on taking the expectation value of $B$ in a direct product of $n$ singlets one finds the value $(2\sqrt{2})^n$, which is to be contrasted with the maximal value of $2^n$ yielded by local realism. One therefore finds that the gulf between quantum mechanics and local realism grows exponentially with the number, $n$, of singlets considered, which parallels Mermin’s finding[19] for the $n$-particle GHZ state. It should be added that this Bell inequality does not rest upon a BKS proof, as was the case with our earlier inequality-free proofs. The same technique of "amplification" used here can be applied to qudits (i.e. higher spin particles) as well to produce a larger gulf between the predictions.
of local realism and quantum mechanics.

The reader may wonder whether the multi-observer GHZ proof can be reduced to a two-observer proof of the sort discussed here by giving one particle to one observer and the other \( n - 1 \) particles to a second observer, who is situated nonlocally with respect to the first. However this will not work for the following reason. Consider, for simplicity, a three-particle GHZ state and suppose that particle 1 is given to Alice and particles 2 and 3 to Bob. Following Mermin’s procedure in [4] Alice and Bob would each measure the \( x-\) and \( y-\) component of spin of their particles and they would also jointly measure the four nonlocal observables

\[
O_1 = \sigma_1^1 \sigma_2^2 \sigma_3^3 (+1), \quad O_2 = \sigma_1^4 \sigma_2^2 \sigma_3^3 (+1), \quad O_3 = \sigma_1^1 \sigma_2^2 \sigma_3^2 (+1) \quad \text{and} \quad O_4 = \sigma_1^1 \sigma_2^2 \sigma_3^2 (-1),
\]

(8)

where the eigenvalue of each is indicated after it in parentheses. The GHZ proof works by showing that the product of the values of the observables \( O_1, O_2, O_3 \) and \( O_4 \) must be +1 (in contradiction to what is implied by (8)) because each of the individual spin components is an element of reality and occurs twice in the product \( O_1 O_2 O_3 O_4 \). However, if particles 2 and 3 are possessed by a single observer, only the products of their spin components \( \sigma_2^2 \sigma_3^3, \sigma_2^2 \sigma_3^3, \sigma_2^2 \sigma_3^2 \) and \( \sigma_2^2 \sigma_3^2 \) are elements of reality and, because each occurs only once in the product \( O_1 O_2 O_3 O_4 \), one can no longer conclude that this product has to have the value +1. One thus sees that the contradiction on which the three-particle GHZ proof is based disappears if the particles are shared by just two observers.

To conclude, we have presented a hierarchy of joint BKS-Bell proofs based on \( n \) identical Bell states shared symmetrically between two observers. Our proofs illustrate the close relationship between the two foundational theorems of quantum mechanics and show particularly how the weaker (BKS) theorem can serve as a catalyst in the proof of the stronger (Bell) one. Finally, we have derived a generalized CHSH inequality for \( n \) Bell states shared symmetrically between two observers and shown that quantum mechanics violates this inequality by an amount that increases exponentially with increasing \( n \).

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[11] See, however, Mermin’s paper in ref.9 for a thought provoking theoretical argument in support of noncontextuality.

[12] A more detailed explanation of this assertion is as follows. When Alice measures a particular observable on her qubits, she collapses them into the two-dimensional subspace associated with a particular eigenvalue (+1 or −1) of that observable. The correlations in state (1) then dictate that Bob’s qubits collapse into the same two-dimensional subspace of their Hilbert space. If Bob subsequently measures the same observable as Alice, either alone or in combination with any other observables that commute with it, his qubits remain within the selected two-dimensional subspace and he definitely obtains the same eigenvalue as Alice for the common observable measured.

[13] A "non-demolition" measurement on a set of qubits can be carried out by coupling them to ancilliary qubits and carrying out the usual (destructive) measurements on the ancilliary qubits. A quantum circuit, consisting of a sequence of one- and two-qubit gates, can be designed to implement any non-demolition measurement. However the practical implementation of the basic two-qubit XOR (or "controlled-not") gate is still in its infancy, and so the ability to carry out the required non-demolition measurements is still a
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\[
\begin{array}{ccc}
1 \otimes \sigma_z & \sigma_z \otimes 1 & \sigma_x \otimes \sigma_z \\
\sigma_z \otimes 1 & 1 \otimes \sigma_x & \sigma_x \otimes \sigma_x \\
\sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y
\end{array}
\]

Fig.1. A 3 x 3 array of observables for a pair of qubits used by Mermin (ref.9) to prove the Bell-Kochen-Specker theorem.