Very ample line bundles, contextuality and quantum computation
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Abstract

I relate contextuality to line bundles. Line bundles are important in algebraic geometry, they determine through their global sections rational maps to projective spaces. I explain how such maps, if they exist, relate input and output of measurement based computation (MBQC) and show geometrically that, indeed, contextuality is a necessary resource for the computational advantage in MBQC.

1 Introduction

A line bundle is a rank one vector bundle, a familiar notion to theoretical physicists because of their occurrence in geometric Berry’s phase\footnote{Barry Simon, \textit{Holonomy, the quantum adiabatic theorem, and berry’s phase}, Phys. Rev. Lett. \textbf{51} (1983), 2167–2170.} for example. Here I explain how they also enter naturally into what we may call \textit{contextual MBQC}. Line bundles determine rational maps. I explain how such maps relate input and output of MBQC and show geometrically that, indeed, contextuality is a necessary resource for the computational advantage in MBQC.

2 Contextuality through poset maps

The convenient way for us to describe contextuality is using the \textit{spectral presheaf}, denoted by \(\Sigma\). Following\footnote{A. Döring and C. Isham, “What is a thing?”: topos theory in the foundations of physics, New structures for physics, Lecture Notes in Phys., vol. 813, Springer, Heidelberg, 2011, pp. 753–937. See also L. Loveridge, R. Dridi and R. Raussendorf, \textit{Non-classical logic of classically universal measurement-based quantum computation}, \url{http://arxiv.org/abs/1408.0745}}, one might think of \(\Sigma\) as the generalized phase space associated to the given quantum system. It is locally a set where locally refers to the different windows (contexts) from which one has access/interacts to/with quantum mechanics. I will present \(\Sigma\) as a poset\footnote{Partially ordered set.} map and mask the usual categorical definition. Contextuality of the quantum system translates to the following statement \textit{the set of global sections (also called global points) of \(\Sigma\) is empty}\footnote{This leads to a second and more concrete motivation for the use of such mathematics. Indeed, from object-oriented programming persecutive (that is, if one decides to simulate contextual MBQCs using Python for instance), a contextual MBQC is an}. A global
section of $\Sigma$ will be a map between the poset maps $\mathbf{1}$ and $\Sigma$.

It is enough to go through the different constructions through the well-known example of Mermin’s system\footnote{N. D. Mermin, *Simple unified form for the major no-hidden-variables theorems*, Phys. Rev. Lett. 65 (1990), no. 27, 3373–3376.}. The Hilbert space is $\mathcal{H} = \mathbb{C}^{2^3}$ and the problem is to assign spectral values to the ten operators $\sigma^1_x, \sigma^1_y, \sigma^1_z, \sigma^2_x, \sigma^2_y, \sigma^2_z, \sigma^3_x, \sigma^3_y, \sigma^3_z, \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3, \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_y^3$. The remaining local sections of $\Sigma$ will be a map between the poset maps $\mathbf{1}$ and $\Sigma$. This translates into two conditions. Firstly, for all $W \in \mathcal{W}$ one has the map $\lambda_W : \{\star\} \to \Sigma(W)$. What the map $\lambda_W$ does is to choose an element, $\lambda_W(\star)$, from the set $\Sigma(W)$ (thus the nature of the element inside the singleton set doesn’t matter!). Note that, $\lambda_W(\star)$ is a map itself i.e., a local valuation in $\Sigma(W)$. Secondly, the map $\lambda$ is compatible with the poset structure. That is, for all $W^r \in W$, the restriction of the local valuation $\lambda_W(\star)$ to $W^r$ is the local valuation $\lambda_{W^r}(\star)$.

I explain now how one can get the same inconsistent system for Mermin’s example. Assume that the spectral presheaf posses a global section $\lambda : \mathbf{1} \to \Sigma$. For each maximal context $\mathcal{W}_i$, this global section will pick out a local valuation $\lambda_{\mathcal{W}_i}(\star)$. Taking into account the multiplicativity of $\lambda_{\mathcal{W}_i}(\star)$ and the fact $\lambda$ satisfies the restriction condition above, one can safely drop the subscript $\mathcal{W}_i$ and gets the familiar inconsistent system

$$
\lambda(\star)(\sigma^1_x)(\lambda(\star))(\sigma^2_y)(\lambda(\star))(\sigma^3_z) = \lambda(\star)(\sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_z)
$$

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$$
\lambda(\star)(\sigma^1_x)(\lambda(\star))(\sigma^2_y)(\lambda(\star))(\sigma^3_z) = \lambda(\star)(\sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_z)
$$

with $\lambda(\star)(\sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_z)\lambda(\star)(\sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_z)\lambda(\star)(\sigma^1_x \otimes \sigma^2_y \otimes \sigma^3_z) = -1$.

The state dependent proof is obtained by requiring that the global section restricts to a particular local section (for instance, restricts to $\lambda_{GHZ}$ on the fifth context).
3 Very ample line bundles

Line bundles are important in algebraic geometry, they determine rational maps to projective spaces. Very ample line bundles determine embeddings in projective spaces.

Line bundles we need here are the hyperplane bundles. The hyperplane bundle $H$ on a variety $X$ is defined as follow: The fiber $\pi^{-1}(p)$ over $p \in X \subset \mathbb{P}^n$ is the 1-dimensional vector space of linear functionals on the line $\ell \in \mathbb{P}^n$ that determines $p$. This line bundle has many global sections i.e., the homogenous coordinates on $X$.

Now, given a line bundle on $X$, let us choose a set $\{s_0, \ldots, s_n\}$ of linearly independent global sections. These sections define the rational map

$$X \rightarrow \mathbb{P}^n,$$

$$x \mapsto [s_0(x) : \cdots : s_n(x)].$$  \hfill (3.1)

The construction can be reversed: Every rational map $X \to \mathbb{P}^n$ is determined by some global sections of some line bundle over $X$. Indeed, the line bundle on $X$ will be the pull-back of the hyperplane bundle on $\mathbb{P}^n$.

The vector space spanned by these sections is called complete linear system if this vector space consists of all the global sections of the line bundle. A line bundle is called very ample if the rational map determined by its complete linear system is an everywhere defined morphism that defines an isomorphism onto its image.\footnote{4}

4 Spectral line bundle

I claim that the spectral presheaf $\Sigma$ can be represented as a line bundle. This uses the fact that, in algebraic geometry, a vector bundle and its sheaf of sections are two equivalent data.\footnote{5} Before going into this, let us first identify each set $\Sigma(W_i)$ (where $W_i$ are the maximal contexts) with the hyperplane bundle $H_i \rightarrow X_i$ whose global sections are the local sections in $\Sigma(W_i)$. The variety $X_i$ is defined by the different correlations between the local sections of $\Sigma(W_i)$ (for the $\ell_2$MBQC setting below, $X_i$ are linear vector spaces). The local sections in $\Sigma(W_i)$ represent now the coordinates functions on $X_i$ and their correlations represent the defining equations of $X_i$ expressed in this choice of coordinates. The

\footnote{6} For a very intuitive exposition about the basic facts used here, the reader might consider the excellent An Invitation to Algebraic Geometry by K., Smith et al.

\footnote{7} A rational function on a variety $X$ is a function of the form $F(T_1, \ldots, T_n)/G(T_1, \ldots, T_n)$ such that $F(T_1, \ldots, T_n)$ and $G(T_1, \ldots, T_n)$ are polynomials and $G(T_1, \ldots, T_n)$ is not a consequence of the defining equations of $X$. A rational map $\varphi : X \rightarrow Y$ to a variety $Y \subset \mathbb{P}^m$ is an $m$-tuple of rational functions $\varphi_1, \ldots, \varphi_m$. The image of $X$ under a rational map $\varphi$ is the set of points $\varphi(X) = \{\varphi(x) | x \in X$ and $\varphi$ defined at $x\}$.

\footnote{8} Let $X \rightarrow \mathbb{P}^n$ be the embedding of $X$ in projective space determined by a basis $s_0, \ldots, s_n$ of the global sections of a line bundle on $X$. Under this morphism, the sections $s_i$ become the coordinate functions. Thus, after embedding $X$ in $\mathbb{P}^n$ this way, the line bundle has become the hyperplane bundle. One may then think of a very ample line bundle as one that, for some embedding of $X$ in projective space, is the hyperplane bundle.

\footnote{9} Such connection wouldn’t be evident without the sheaf theoretic definition of contextuality.

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spectral presheaf is a line bundle which locally, with respect to the cover \( \{ X_i \} \), is the hyperplane \( H_i \to X_i \). The base variety is \( X := \bigcup X_i \). The different local sections of \( \Sigma \) are local sections for the newly constructed line bundle\(^{10}\). When the spectral line bundle is very ample, the spectral presheaf is a line bundle.

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### 5 Rational computations

The following definition is from\(^{11}\). An \( \ell\)-2 MBQC is the following data: a resource state \( \varphi \in \mathbb{C}^{2^n} \), classical input \( i \in \mathbb{Z}_2^m \) and classical output \( o : \mathbb{Z}_2^n \to \mathbb{Z}_2 \), \( i \mapsto o(i) \), a collection of local observables \( \{ O_k(q_k) \mid k \in \{ 1...n \} ; q_k \in \mathbb{Z}_2 \} \) for which the measurement outcomes of a given \( O_k(q_k) \) are labelled \( s_k(q_k) \in \{ 0,1 \} \) for each \( k \). The computed output \( o(i) = \sum_k s_k(q_k) \mod 2 \), and, with \( q = (q_1...q_n) \), the measured observable \( O_k(q_k) \) is related to the outcome \( s_k(q_k) \) by \( q = Q_i \) where \( Q : \mathbb{Z}_2^m \to \mathbb{Z}_2^n \).

This definition mimics Mermin’s example\(^{12}\) where the correlations between the local observables are multiplicative and which translates to linear correlations between the local sections. The argument presented below is valid for more general type of correlations and shows the role of contextuality for the general scheme of MBQC: if the MBQC is non contextual then there is a rational map relating the different correlations (technically, coordinate rings) to the output i.e., the value of the global section. The importance of this particular instance of MBQC, once it is supplemented with a classical controller, is that if it computes nonlinear function then one gets classical universal computation.

In the language and notations of the previous sections, we have \( n \) contexts \( \{ C_q := \{ O_1(q_1), ..., O_n(q_n) \} \}_{q=(q_1...q_n)\in\mathbb{Z}_2^n} \) labeled by \( \mathbb{Z}_2^n \) plus a special context \( C_{n+1} := \{ O_1(q_1) \otimes ... \otimes O_n(q_n) \}_{(q_1...q_n)\in\mathbb{Z}_2^n} \) which stabilizes the resource state \( \varphi \). Each context yields a maximal abelian subalgebras \( W_i := C_i'' \). The spectral presheaf evaluated at \( C_q \), \( q = (q_1...q_n) \) is the set \( \{ v_k : C_q \to \mathbb{Z}_2, v_k(O_k(q_k)) = s_k(q_k), k = 1..n \} \). The resource state defines a local section for the last context \( C_{n+1} \). The computation is quantum mechanical or contextual if the state dependent spectral presheaf (where we require the global section to restrict to \( \varphi \) on the last context \( C_{n+1} \), in the same we did with Mermin’s system in Section 2) has no global section. We reproduce the result of \(^{11}\): if the \( \ell_2\)MBQC is non contextual then the computed function is linear. Indeed, the fact that the spectral presheaf \( \Sigma \), and hence the spectral line bundle, posses a global section translates into a rational (in fact linear) map \( X \to \mathbb{P}^0 \) i.e., to a point, the output, which represents the value of the section at the different observables. This rational map will also map the different correlations (here linear correlations \( o(i) = \sum_k s_k(q_k) \)) between the local sections for each context (in particu-

\(^{10}\)The spectral presheaf is the sheaf of sections of the line bundle.

\(^{11}\)R. Raussendorf, Contextuality in measurement-based quantum computation, Phys. Rev. A 88 (2013), 022322.

\(^{12}\)Converted into \( \ell_2\)MBQC in J. Anders, D. Browne Computational Power of Correlations Phys. Rev. Lett. 102, 050502 (2009)
lar for the $n$ first contexts) to the coordinate values of the point. Thus, yielding a linear computation.

6 Conclusion

I have used the language of line bundles to describe the role of contextuality in MBQC. The spectral presheaf, which I define in Section 2, is identified with a line bundle in Section 4. Locally, the spectral presheaf is the hyperplane bundle $H_i \to X_i$. Each variety $X_i$ is coordinated with the local sections in $\Sigma(W_i)$. Their correlations define the coordinate ring associated to $X_i$. I have explained that if an MBQC, presented as a sub-functor of the spectral presheaf, is non contextual then there is a rational map $\bigcup X_i \to \mathbb{P}^0$. This map relates the different correlations between the local sections (i.e., the coordinate rings of the varieties $X_i$) and in particular the first $n$ contexts representing the input to the output i.e., the value of the global section. In short, contextual MBQCs are transcendental relations between input and output.

To conclude, it would be interesting to see if this framework can actually be used in constructing examples of what we may call contextuality based computation. An ideal paradigm of quantum computation defined within the framework of functors and line bundles which consumes contextuality whenever it is present.