Slow pressure modes in thin accretion discs

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23 November 2009

ABSTRACT
Thin accretion discs around massive compact objects can support slow pressure modes of oscillations in the linear regime that have azimuthal wavenumber $m = 1$. We consider finite, flat discs composed of barotropic fluid for various surface density profiles and demonstrate through WKB analysis and numerical solution of the eigenvalue problem that these modes are stable and have spatial scales comparable to the size of the disc. We show that the eigenvalue equation can be mapped to a Schrödinger-like equation. Analysis of this equation shows that all eigenmodes have discrete spectra. We find that all the models we have considered support negative frequency eigenmodes; however, the positive eigenfrequency modes are only present in power law discs, albeit for physically uninteresting values of the power law index $\beta$ and barotropic index $\gamma$.

Key words: accretion discs; hydrodynamics; waves; methods: analytical

1 INTRODUCTION
Low-mass discs orbiting massive compact bodies are a feature of many astronomical systems. When the dynamics of a disc is dominated by the Newtonian gravitational force of the central body, the disc may be considered nearly Keplerian. In a purely Keplerian potential eccentric orbits do not precess because the orbital frequency is equal to the epicyclic frequency. In a nearly Keplerian disc there is a small difference between the orbital and epicyclic frequencies. This could be due to the self-gravity of the disc, thermal pressure in a gas disc, and random motions in a collisionless disc. This difference in frequencies manifests as a precession of eccentric orbits at rates that are small compared to the orbital and epicyclic frequencies. Then the disc may be able to support large-scale, slow, lopsided modes (Kato 1983; Sridhar, Syer & Touma 1999; Lee & Goodman 1999; Sridhar & Touma 1999). In the linear regime, these modes have azimuthal frequency, $m = 1$, whose first systematic investigation is due to Tremaine (2001). He studied slow modes in various types of discs (fluid, collisionless and softened gravity), with the focus largely on the effect of the self-gravity of the disc. In particular, a WKB analysis was used to show that the fluid disc can support large-scale slow modes when the Mach number, $\mathcal{M}$, is much larger than the Toomre $Q$ parameter (both parameters are defined in § 2). The assumption behind this analysis is that the self-gravity of the disc dominates fluid pressure. However, such is not the case for thin accretion discs around white dwarfs and neutron stars. Indeed, for a disc around a white dwarf (Frank, King & Raine 2002), we can estimate $\mathcal{M} \sim 50$ and $Q \sim 10^{10}$; hence the analysis of Tremaine (2001) is not directly applicable. The goal of this paper is to study large-scale $m = 1$ slow modes in thin accretion discs, where $Q \gg \mathcal{M} \gg 1$.

In § 2 we use the WKB approximation to establish that pressure, in the absence of self-gravity, can enable slow, $m = 1$ modes in thin accretion discs. The linear eigenvalue problem for slow pressure modes (or “p-modes”) is formulated in § 3 for a flat barotropic disc, which is axisymmetric in its unperturbed state. When appropriate boundary conditions are chosen, the eigenvalue equation reduces to a Sturm-Liouville problem. Since the differential operator is self-adjoint, the eigenfrequencies are real: therefore all $p$-modes are stable, and the eigenfunctions form a complete set of orthogonal functions. We also map the eigenvalue equation into a Schrödinger-like equation, which is useful in the interpretation of our numerical results. In § 4 we present numerical results for a variety of discs, namely (i) an approximation to the Shakura-Sunyaev thin disc, (ii) the Kuzmin disc which is more centrally concentrated, (iii) power-law discs. Of particular interest is the nature of the eigenfrequency; whether it is positive or negative. This has bearing on the excitation of these modes, because they are stable and will not grow spontaneously through a non-viscous instability. Comparison with earlier work, summary and conclusions follow in § 5.
2 SLOW PRESSURE MODES

Consider a flat thin disc of fluid orbiting a central mass $M$. The fluid is assumed to be barotropic and the disc is described by a surface density profile $\Sigma$. In our analysis we ignore viscous forces, assuming that they adjust to maintain a quasi-stationary flow with a small radial velocity, and have little effect on the perturbed flow. Thus, we start with the continuity and Euler equations in cylindrical polar coordinates

\[
\begin{align*}
\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R) + \frac{1}{R} \frac{\partial}{\partial \phi} (\Sigma v_\phi) &= 0, \\
\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + v_\phi \frac{\partial v_R}{\partial \phi} - \frac{\partial^2}{\partial R^2} &= -\frac{GM}{R^2} - \frac{\partial}{\partial R} (\Phi + h), \\
\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + v_\phi \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_R v_\phi}{R} &= -\frac{1}{R} \frac{\partial}{\partial R} (\Phi + h),
\end{align*}
\]

where $v_R$ and $v_\phi$ are the radial and azimuthal components of the fluid velocity, $h$ is the enthalpy per unit mass, and $\Phi$ is the gravitational potential due to the disc. For a barotropic fluid with an equation of state given by $p = \Sigma h$, the isentropic sound speed and enthalpy are given by

\[
c_s^2 = \gamma D \Sigma^{-1}, \\
h = \frac{D}{\gamma - 1} \Sigma^{-1} = \frac{c_s^2}{\gamma - 1},
\]

2.1 Precession rate in the unperturbed disc

We assume that the radial velocity of the unperturbed flow is much smaller than the azimuthal velocity and set it identically equal to zero; this assumption is justified below at the end of § 2.2. The unperturbed disc is assumed to be axisymmetric, therefore all $\phi$ derivatives are set to zero. Gas flows along circular orbits, with centrifugal balance maintained largely by the gravitational attraction of the central mass (with small but non trivial contributions from gas pressure and disc self-gravity). The azimuthal and radial frequencies, $\Omega > 0$ and $\kappa > 0$ respectively, associated with nearly circular orbits are given by

\[
\begin{align*}
\Omega^2 &= \frac{GM}{R^3} + \frac{1}{R} \frac{d}{dR} (\Phi_0 + h_0), \\
\kappa^2 &= \frac{GM}{R^3} + \frac{d^2}{dR^2} (\Phi_0 + h_0) + \frac{3}{R} \frac{d}{dR} (\Phi_0 + h_0),
\end{align*}
\]

where the subscript '0' indicates unperturbed quantities. The Mach number of the flow $\mathcal{M}(\Omega) = R \Omega(\Omega) / c_s(\Omega) \gg 1$. The dominant contribution to both $\Omega^2$ and $\kappa^2$ is due to the central mass, with small corrections coming from the disc self-gravity ($\Phi_0$) and enthalpy ($h_0$). Let us define the small parameter $\epsilon \ll 1$ as the larger of $(\Sigma \Omega R^2 / M)$ and $(h_0 R / GM)$. The apsidal angles of the nearly circular orbit of a fluid element precess at a rate given by

\[
\begin{align*}
\zeta &= \Omega - \kappa, \\
&= -\frac{1}{2\Omega} \left( \frac{d^2}{dR^2} + \frac{2}{R} \frac{d}{dR} \right) (\Phi_0 + h_0) + O(\epsilon^2).
\end{align*}
\]

Note that, in contrast to Tremaine (2001), we have retained the contribution from gas pressure (i.e. enthalpy). In fact, in thin accretion discs around compact stars, disc self-gravity is negligible and the contribution to $\zeta$ is almost entirely from gas pressure. The goal of this section is to establish that these discs have large-scale slow modes driven only by pressure. In the WKB analysis of linear modes given below we follow the presentation due to Tremaine (2001).

2.2 The WKB approximation

We consider linear perturbations (of the velocity, surface density etc) of the form

\[
A(R) \exp \left[ i \int_0^R k(R')dR' + m\phi - \omega t \right],
\]

where $k(R)$ and $m$ are the radial and azimuthal wavenumbers, respectively, and $\omega$ is the angular frequency of the mode. In the tight-folding limit where $|k(R)/R| \gg 1$, a dispersion relation between $\omega$ and $k(R)$ can be derived (Safronov 1962; Binney & Tremaine 2008):

\[
(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \Sigma_0 |k| + c_s^2 k^2.
\]

The disc is stable to axisymmetric ($m = 0$) perturbations if and only if

\[
Q \equiv \frac{c_s \kappa}{\pi G \Sigma_0} > 1.
\]

This readily satisfied in thin accretion discs around compact stars. Tremaine (2001) showed that the dispersion relation for $m = 1$ modes can be written as

\[
\omega = \zeta + \frac{\pi G \Sigma_0 |k|}{\Omega} - \frac{c_s^2 k^2}{2\Omega} + \frac{1}{\Omega} O(\zeta^2, \omega^2).
\]

In Tremaine (2001) it is argued that, when the pressure is negligible compared to disc self-gravity (i.e. $c_s \approx 0$), the WKB dispersion relation of equation (11) admits large-scale $|k(R)/R| \sim 1$ modes with frequencies $\omega \sim \zeta \sim (\Sigma \Omega R^2 / M)$. It may be verified that the condition of negligible pressure implies that the Mach number $\mathcal{M} \gg Q$. However, as we have argued in the introduction, this inequality is violated for thin accretion discs around compact stars where the opposite is true, i.e. $Q \gg \mathcal{M} > 1$. Hence we need to consider a situation that is complementary to the analysis of Tremaine (2001). Disc self-gravity being negligible in accretion discs, the precession rate is determined entirely by the gas pressure. Then equation (11) can be written as

\[
\zeta = \frac{1}{2\Omega} \left( \frac{d^2 h_0}{dR^2} + \frac{2}{R} \frac{d h_0}{dR} \right) + O(1/\mathcal{M}^4) \sim \frac{\Omega}{\mathcal{M}^2}.
\]

(see Katz 1983) and we can approximate equation (11) as

\[
\omega = \zeta - \frac{c_s^2 k^2}{2\Omega} + \frac{1}{\Omega} O(\zeta^2, \omega^2).
\]

For a disc with a non zero inner radius, eigenmodes must satisfy the Bohr-Sommerfeld quantization condition, given by

\[
\int_0^R k(R) dR = \left( n + \frac{3}{4} \right) 2\pi, \quad n = 0, 1, 2, \ldots
\]

Thus, there exists a \textit{prima facie} case for modes with frequencies $\omega \sim \zeta$, with radial wavenumbers,

\[
k(R) \sim \left| \frac{\Omega \zeta}{c_s^2} \right|^{1/2} \sim \frac{1}{R},
\]

comparable to the radial scale of the disc. However, this
tentative conclusion is based on a WKB analysis which may not be valid for modes with \( k(R)R \sim 1 \). This is the motivation for our studies of the eigenvalue problem for slow modes given below. Henceforth we ignore disc self-gravity altogether and consider only the effect of gas pressure.

We now justify the assumption made at the beginning of § 2.1, that the radial velocity in the unperturbed disc is small, and may be ignored when studying slow modes. The time scale of radial spreading of the disc is, \( t_{\text{vis}} \sim R^2/\nu \sim 2M/(\alpha \Omega) \), where \( \alpha \ll 1 \) is the Shakura–Sunyaev viscosity parameter. The frequency of a slow mode is \( \omega = \omega_{\text{vis}} \sim \Omega R^{-2} \). Therefore \( \omega_{\text{vis}} \sim \alpha^{-1} \gg 1 \) implies that the radial spreading occurs over many slow mode periods.

3 FORMULATION OF THE EIGENVALUE PROBLEM

3.1 Eigenvalue equation

The linearized Euler, continuity and enthalpy equations that govern the perturbed flow are

\[
\begin{align*}
\frac{dv_{\phi 1}}{dt} & = -2\Omega(R)v_{\phi 1} + -\frac{\partial v_{\phi 1}}{\partial R}, \\
\frac{dv_{\phi 1}}{dt} & = -2B(R)v_{\phi 1} + -\frac{1}{R} \frac{\partial R}{\partial \phi}, \\
\frac{d\Sigma}{dt} & = -\frac{1}{R} \frac{\partial R}{\partial \phi} + \frac{1}{R} \frac{\partial R}{\partial R} (R \Sigma_0 v_{\phi 1}), \\
h_a & = c_s^2 \frac{\Sigma}{\Sigma_0}.
\end{align*}
\]

where the subscript ‘0’ stands for the unperturbed quantities and ‘1’ for the first order perturbed quantities. The Oort’s parameter \( B(R) \) is related to the epicyclic frequency through \( \kappa^2(R) = -4\Omega(R)B(R) \), and \( d/dt = (\partial/\partial t + \Omega \partial/\partial \phi) \) is the convective derivative with respect to the unperturbed flow.

We consider non-axisymmetric perturbations with azimuthal wave number \( m = 1 \), of the form \( T_1 = T_1(R) \exp[i(\phi - \omega t)] \), where \( T_1 \) stands for any perturbed quantity. Substituting this form in equations (14), (15), (16) and (17) yields

\[
\begin{align*}
i(\Omega - \omega)v_{Ra} - 2\Omega v_{\phi a} + \frac{dh_a}{dR} & = 0, \\
i(\Omega - \omega)v_{\phi a} - 2Bv_{Ra} + \frac{ih_a}{R} & = 0, \\
i(\Omega - \omega)\Sigma_a + \frac{\Sigma_0}{R} v_{\phi a} + \frac{1}{R} \frac{d}{dR} (R \Sigma_0 v_{Ra}) & = 0, \\
h_a & = c_s^2 \frac{\Sigma_a}{\Sigma_0}.
\end{align*}
\]

Solving equations (18) and (19) for the velocity amplitudes we obtain

\[
\begin{align*}
v_{Ra} & = \frac{i}{\Delta} \left[ (\Omega - \omega) \frac{dh_a}{dR} + 2\Omega \frac{R}{h_a} \right], \\
v_{\phi a} & = \frac{1}{\Delta} \left[ -2B \frac{dh_a}{dR} + \Omega - \omega \frac{R}{h_a} \right],
\end{align*}
\]

where \( \Delta = \kappa^2 - (\Omega - \omega)^2 \). These equations, along with equation (21), when substituted in (20) yields

\[
\begin{align*}
& \frac{d^2}{dR^2} + \left\{ \frac{d}{dR} \ln \left( \frac{R \Sigma_a}{\Delta} \right) \right\} \frac{d}{dR} + \\
& \frac{2\Omega}{\Omega - \omega} \left\{ \frac{d}{dR} \ln \left( \frac{\Sigma_0 \Omega}{\Delta} \right) \right\} - \frac{1}{R^2} h_a = \frac{h_a \Delta}{c_s^2}.
\end{align*}

which is the eigenvalue problem for undriven modes, with eigenvalue \( \omega \) and eigenfunction \( h_a \). This equation is a special case of equation (13) of Goldreich & Tremaine (1979); Tsang & Lai (2009), where \( m = 1 \), and the external and self gravity perturbations are set equal to zero. It can be noted that this equation becomes singular at \( \Omega = \omega \) and \( \Delta = 0 \). The former corresponds to the corotation resonance and the latter corresponds to the Lindblad Resonances (LR). Below we discuss the validity of equation (24) at LR; a similar analysis holds for the singularity at corotation radius but we do not discuss it in this paper since, as is argued later, for the slow modes the corotation radius has to lie outside the disc.

The system of equations (18)–(21) describe an undriven, autonomous system. At the Lindblad resonance the algebraic equations (18) and (19) become indeterminate if no conditions are imposed on the enthalpy perturbations \( h_a \). It is easily seen that these equations become consistent if

\[
-\frac{i}{\Delta}(\Omega - \omega) = \frac{2\Omega}{\Omega - \omega} = \frac{dh_a}{dR} = \frac{ih_a}{R}.
\]

The first equality follows from \( \Delta = 0 \), Rearranging the second equality yields

\[
\left[ \frac{d}{dR} \left( R^2 h_a \right) - \frac{\omega}{\Omega} \left( R^2 \frac{dh_a}{dR} \right) \right]_{LR} = 0.
\]

This condition must be satisfied at the Lindblad resonances for all undriven modes. However, equation (20) may not be satisfied if the disc is driven by external forcing and may lead to curious dynamics around the LR and transport of angular momentum away from the LR due to external torquing (Goldreich & Tremaine 1979). In this work we confine our investigations to free modes of an undriven disc, therefore, as the above discussion shows, nothing special happens at the LR.

Slow Mode Approximation: We now make the ansatz that the perturbed flow supports frequencies that are small in comparison to the circular frequency, i.e., \( |\omega| \ll \Omega \). Therefore, when \( \omega \neq 0 \), the disc must be finite, with outer radius such that the orbital frequency at the outer edge is much greater than \( |\omega| \). Applying the slow mode approximation to equation (24) we obtain

\[
\frac{c_s^2 R^{3/2}}{\Sigma_0 \Omega} \frac{d}{dR} \left( \frac{\Sigma_0 \Omega}{\Delta} \frac{d}{dR} (R^2 h_a) \right) = R^2 h_a.
\]

Similar to equation (24), equation (27) too is singular at the Lindblad resonances, however the singularity at the corotation radius has gone away since this equation has been derived under the slow mode condition, \( |\omega| \ll \Omega \). The condition \( \Delta = 0 \) implies that at some radius, either \( \omega = \Omega - \kappa \) or \( \omega = \Omega + \kappa \). Since \( \kappa \simeq \Omega + O(\epsilon^2) \), we see that the second equality cannot be satisfied under the slow mode approximation. It is straightforward to see that the radius where this would be satisfied would be larger than the corotation radius due to the fact that the Keplerian circular frequency
falls off monotonically with radius. Therefore, there are no outer Lindblad resonance singularities for slow modes. However, the Inner Lindblad Radius (ILR), where $\omega = \Omega$, could very well lie inside the disc. Due to the fact that the disc surface density is completely arbitrary, there could in general be more than one ILRs. To make the problem well posed under the slow mode approximation, at the ILRs, the condition \( \frac{d}{dR}(R^2 h_a)|_{\text{ILR}} = 0 \).

We shall see later that the numerical solutions of equation (30) satisfy this condition and the velocity amplitude at the ILR remains finite; thus the linear approximation remains valid and nothing special happens at the ILR.

From equation (11), $\dot{x}/\Omega \simeq O(\mu^{-2})$. This allows us to approximate $\kappa^2 \simeq \Omega^2$, leading to $B \simeq -\Omega/4$. Using these in equation (23) we obtain

$$v_{\phi a} = \frac{\Omega}{2\Delta} \left[ \frac{dh_a}{dR} + \frac{2h_a}{R} \right],$$

$$= \frac{\Omega}{2\Delta} \frac{1}{R^2} \frac{d}{dR} \left( R^2 h_a \right).$$

Differentiating equation (27) and using equation (29) we obtain,

$$\frac{d}{dR} \left[ \left( \frac{c_s^2 R^{3/2}}{\Sigma_0 \Omega} \right) \frac{d\theta}{dR} \right] + \frac{2 R^{3/2}}{\Sigma_0} (\dot{x} - \omega) \theta = 0,$$

where we have used the variable $\theta = R^{1/2} \Sigma_0 v_{\phi a}$ and $\Delta \simeq 2\Omega(\omega - \dot{x})$, which is valid under the slow mode approximation.

### 3.2 Slow modes as a Sturm-Liouville problem

Before we proceed to specific examples, we have to choose the boundary conditions that we impose to solve equation (30). We first cast the equation in a dimensionless form by choosing a radius $R_\star$, at which we evaluate various quantities, $\Sigma_\star, c_s, \dot{x}_\star$, and $\Omega_\star$. We introduce the parameter $x = R/R_\star$, and similarly for a quantity $H'$, we use $H' = H/H_\star$, leading to the Sturm-Liouville form of the eigen equation

$$\frac{d}{dx} \left[ P(x) \frac{d\theta}{dx} \right] + (Q(x) + \lambda W(x)) \theta = 0,$$

where

$$P(x) = \frac{c_s^2 x^{3/2}}{\Sigma_0 \Omega'}, \quad W(x) = \frac{2x^{3/2}}{\Sigma_0},$$

$$Q(x) = \frac{2x^{3/2} \dot{x}_\star}{\Sigma_0}, \quad \theta = x^{1/2} v_{\phi a} \Sigma_0.$$

In equation (31), $\lambda = -\omega \mu^2/\Omega_\star$ is defined with a negative sign to make an explicit correspondence with the Schrödinger’s equation, to be introduced in § 3.3. Henceforth we reserve the term “eigenvalue” for $\lambda$, and use either “frequency” or “eigenfrequency” for $\omega$.

We consider discs with an inner edge at $R_{\text{inner}}$, and an outer edge at $R_{\text{outer}}$, and we choose $R_\star = R_{\text{inner}}$. We now argue that the parameters of the disc and the central mass for astrophysically interesting discs are such that the slow mode condition can be easily satisfied everywhere inside the disc. In a Keplerian disc the slow mode condition $|\omega| \ll \Omega$ is satisfied everywhere in the disc if it is satisfied at the disc outer radius, this leads to

$$\left( \frac{R_{\text{outer}}}{R_\star} \right)^{3/2} \ll \frac{\mu^2}{|\lambda|},$$

where we have used $\lambda = -\omega \mu^2/\Omega_\star$, and $\Omega(R_{\text{outer}}) = \Omega_\star (R_\star/R_{\text{outer}})^{3/2}$. Typical expected values for $\mu$ are in the range $10^4-10^6$ [Frank, King & Raine 2002]. Most examples we consider have surface densities that decline by $R_{\text{outer}}/R_\star \approx 30-50$, therefore we see that for an eigenmode to be slow through out the disc, $|\lambda|$ has to be much smaller than $10^5-10^6$. We shall see in the examples that this condition is comfortably satisfied.

We integrate the eigen equation in the range $1 < x < R_{\text{outer}}$. We assume that the perturbations obey the boundary conditions,

$$\theta(1) = \Theta(x_{\text{outer}}) = 0.$$

We note that these boundary conditions make the differential operator in equation (31) self-adjoint: therefore the eigenvalues $\lambda$ are real, and all slow p-modes are stable. The complete set of eigenfunctions also form a complete basis; however, we note that not all eigenvalues are slow, and thus we do not expect this set to describe the evolution of arbitrary perturbations, but only the ones that obey the slow mode condition, $\Omega \gg |\omega|$.
3.3 Effective potential and WKB approximation

In the usual WKB approximation we substitute the trial solution
\[
\Theta(x) = A(x) \exp \left[ \frac{i}{\mu} \int x \, \kappa dx \right],
\]
in the following equation:
\[
\mu^2 \frac{d^2}{dx^2} \left( P(x) \frac{d\Theta}{dx} \right) + (Q(x) + \lambda W(x)) \Theta = 0.
\]
Here \( \mu \) is an ordering parameter which is finally set equal to unity. \( A(x) \) and \( \kappa(x) \) are the amplitude and the wavevector respectively. Collecting terms of zeroth order in \( \mu \) leads to the dispersion relation
\[
\kappa^2 = \frac{Q(x) + \lambda W(x)}{P(x)} = \frac{2\Omega}{c_s^2} (\dot{\omega} - \omega),
\]
which is identical to equation (31). However, we find that this dispersion relation, together with the Bohr-Sommerfeld quantization condition of equation (12) predicts eigenvalues that compare poorly with those obtained from numerical integration of the Sturm-Liouville equation. Hence we have reformulated equation (31), using new variables \( \eta(x) = \sqrt{P(x)} \) and \( \Psi = \sqrt{P(x)} \Theta \). Then equation (31) takes the Schrödinger-like form
\[
\Psi'' + K^2(x) \Psi = 0,
\]
where
\[
K^2(x) = \frac{1}{\eta^4(x)} \left[ Q(x) + \lambda W(x) - \eta(x) \eta''(x) \right],
\]
which on defining \( V(x) = (-Q(x) + \eta(x) \eta''(x))/W(x) \) can be written as
\[
K^2(x) = \frac{W(x)}{\eta^2(x)} [\lambda - V(x)].
\]
This dispersion relation differs from equation (31) and seems to better describe the numerical solutions, giving a match with the numerically obtained eigenvalues up to a few per cent, as can be seen in Table 1.

Note that \( K^2(x) \) in equation (39) differs from the standard form for the Schrödinger equation by the factor \( W(x)/\eta^4(x) \). However, it is very useful for discussions of the turning points, where \( K^2(x) = 0 \), separating classically accessible regions from the forbidden ones. The solution is oscillatory where \( K^2 > 0 \), implying \( \lambda > V(x) \), and non-oscillatory otherwise. Since the disc is finite the eigen spectrum is always discrete and there are two distinct types of spectra:

**Type I:** This occurs when there is at least one turning point within the disc. In the case of a single turning point we can have oscillatory solution on either side of the turning point, depending on the form of \( K^2(x) \). If there are more than one turning point then we could either have oscillatory behaviour confined between the turning points or outside, such as the case of the SS disc discussed below.

**Type II:** This occurs when there are no turning points within the disc and the discreteness of the spectra depends entirely on the size of the disc.

To obtain real eigenvalues we need to consider the possibility of satisfying \( K^2 > 0 \) in a bounded region, which could either be bounded by one or both of the disc boundaries or by turning points. Thus a useful first step is to plot this potential for the problem at hand. If the potential allows regions that can support bound states, we search for solutions numerically, and to verify our results we use the WKB approximation.

Let us consider the case of the SS disc shown in Fig. 1. On the negative side the potential blows up at the inner edge at \( x = 1 \), where the perturbations are assumed to vanish. If \( \lambda < 0 \) is to be a valid eigenvalue, then it must satisfy the quantization condition
\[
\int_a^b K(x) \, dx = \left( n + \frac{3}{4} \right) \pi, \quad n = 0, 1, 2, \ldots
\]
where, \( x = a > 1 \) is a turning point which separates a classically accessible region (I) on the left from a forbidden region (II) on the right. The case of positive eigenvalues, \( \lambda > 0 \), is more interesting. If the eigenvalue is positive and smaller than the maximum value of \( V(x) \), then there are two turning points (say, \( x = a, b \) with \( b > a \)) separating three distinct regions. The classically forbidden region lies between \( x = a \) and \( x = b \), separating the two classically accessible regions \( (1, a) \) and \( (b, x_{outer}) \). If \( \lambda \) is greater than the maximum of \( V(x) \), then whole disc is classically allowed. Below we present numerical results on eigenvalues and eigenfunctions, and use WKB approximation to understand them. It turns out that WKB approximation is very useful even for the case of small quantum numbers.

![Figure 2. The effective potential for the Kuzmin disc. Positive values of \( \lambda \) lead to a discrete spectrum of both Type I and Type II eigenvalues. There are no negative eigenvalues.](image)

### 4 NUMERICAL RESULTS

As discussed in the last section, we consider finite disc and we expect the spectra of equation (31) to be discrete. These modes would have observational consequences since they...
would rotate at a definite frequency around the disc. The perturbations in the enthalpy would lead to azimuthal variations in the temperature and density across the disc, which might be observable depending on the amplitude of perturbations.

Since very little is known about the surface density profiles of the discs we carry out a simplistic calculation based on certain standard forms of the disc as test cases. We also consider the generic power law profile. Some of the profiles might be observable depending on the amplitude of perturbations.

Shakura-Sunyaev (SS) discs: We first consider the standard model of an accretion disc proposed by Shakura and Sunyaev (Shakura & Sunyaev 1973). The surface density and temperature of this disc are given by (see Frank, King & Raine (2002)):

\[ \Sigma_{SS} = 5.2 \alpha^{-4/5} \dot{M}_{16}^{7/10} m_1^{1/4} R_{10}^{-3/4} f^{14/5} \text{ g cm}^{-2}, \quad (41) \]

\[ T_{SS} = 1.4 \times 10^4 \alpha^{-1/5} \dot{M}_{16}^{3/10} m_1^{1/4} R_{10}^{-3/4} f^{6/5} \text{ K}, \quad (42) \]

where

\[ f = \left[ 1 - \left( \frac{R_*}{R_{10}} \right)^{1/2} \right]^{1/4}. \quad (43) \]

\( \dot{M}_{16} \) is the mass accretion rate in the units of \( 10^{16} \text{ g s}^{-1} \), \( m_1 \) is the mass of disc in solar mass units, \( R_{10} \) is radius in the units of \( 10^{10} \text{ cm} \), \( R_* \) is the radius of the central object in the units of \( 10^{10} \text{ cm} \), and \( \alpha \) is the Shakura-Sunyaev viscosity parameter.

Although the SS disc is not based on a barotropic model, we find that a barotropic disc with index \( \gamma = 2 \) serves as a reasonable approximation. We have considered two cases:

(i) Choosing \( \Sigma_0 = \Sigma_{SS} \) of equation (41), and deriving the temperature profile for \( \gamma = 2 \), we find that the effective potential is

\[ V_1(x) = \frac{344 - 590 \sqrt{\pi} + 225x}{800 x^{3/4} (1 - \sqrt{1/x})^{13/10}}. \quad (44) \]

(ii) Choosing \( T_0 = T_{SS} \) of equation (42), and deriving the surface density profile for \( \gamma = 2 \), we find that the effective potential is

\[ V_2(x) = \frac{12 - 22 \sqrt{\pi} + 9x}{32 x^{3/4} (1 - \sqrt{1/x})^{3/2}}. \quad (45) \]

where we have used the natural length scale given by the inner disc radius, which we have adopted for conversion of our eigen equation into a dimensionless form. As Fig (1) shows, \( V_1(x) \) and \( V_2(x) \) are quite similar to each other. The effective potential blows up at the inner edge of the disc and steadily climbs up above zero and then decreases asymptotically. In the immediate vicinity of the inner edge a negative \( \lambda \) gives an oscillatory solution, and to the right of it is a classical turning point. Beyond the turning point the solution is exponentially decaying. This suggests that discrete, Type I, negative eigenvalues might exist, however, both the numerical search and the WK3 quantitation condition (40), fail to find such discrete eigenvalues. For \( 0 < \lambda \lesssim 0.01 \), we find discrete, Type I eigenvalues for which the oscillatory behaviour is outside the region bounded by two turning points. For \( \lambda \gtrsim 0.01 \), we find discrete eigenvalues of Type II, where the separation between neighbouring eigenvalues decreases with increasing values of the outer radius of the disc.

The Kuzmin Disc: In contrast to the SS discs, the Kuzmin disc has a centrally concentrated surface density, and hence offers a distinct case in which to study slow modes. The surface density profile in this case is given by

\[ \Sigma(R) = \frac{a M_D}{2 \pi (R^2 + a^2)^{3/2}}, \quad (46) \]

where \( M_D \) is the mass in the disc and \( a \) is the core radius. The surface density extends all the way to \( R = 0 \). If we take \( a \) as the size of the inner radius of the disc then we can rescale our equation by choosing \( R_* = a \), leading to the effective potential

\[ V(x) = \frac{3x^4(5 - 4 \gamma + 3 \gamma^2) + x^2(6 - 8 \gamma) + 1}{8x^{1/2}(1 + x^2)^{(3+\gamma)/2}}. \quad (47) \]

In Fig (2) we have plotted this potential for \( \gamma = 4/3 \). As may be seen, this case qualitatively resembles Fig (4), and the discussion in § 2.4 carries through. For negative values of
we can infer from Fig (2) that $K^2(x) < 0$, so wave like solutions are not possible. For positive values of $\lambda$ in the range $0 < \lambda \lesssim 0.3$, we can have Type I eigenvalues, but this region of $\lambda$ is further divided into two parts: for $0 < \lambda \lesssim V(1)$, there is only one turning point and oscillatory behaviour is possible for radius greater than the turning point, and for $V(1) < \lambda \lesssim 0.3$, there are two turning points and oscillatory behaviour is possible outside the region bounded by the two turning points. For $\lambda \gtrsim 0.3$ the eigenvalues are of Type II. This behaviour is confirmed by numerical integration of the eigenvalue equation. We can also admit values of $\gamma$ other than $4/3$. However, the effective potential in equation (47) retains the general shape of Fig (2), and the conclusions stated above remain valid.

**Power Law Discs:** Certain physical models (e.g. Narayan & Yi (1994)) naturally lead to scale invariant discs that follow a power law profile. Although in these models there is no associated length scale, we choose to truncate the disc at the inner disc radius thus leading to the form

$$\Sigma_0 = \Sigma_* x^\beta,$$  \hspace{1cm} (48)

where, $\Sigma_* = \Sigma(R_*)$ and $\beta$ is the power law index. The potential is given by

$$V(x) = C(\beta, \gamma) x^\nu,$$  \hspace{1cm} (49)

where

$$C(\beta, \gamma) = \frac{1}{8} (3 + \beta (-4 + \gamma (4 + \gamma \beta))),$$  \hspace{1cm} (50)

$$\nu(\beta, \gamma) = \gamma \beta - \beta - 1/2.$$  \hspace{1cm} (51)

There are four distinct possibilities:

(i) $C(\beta, \gamma) < 0, \nu(\beta, \gamma) < 0$: The region in $\beta$-$\gamma$ space
satisfying these conditions is plotted in Fig (3). The regions where these conditions are satisfied are those with (1) \( \beta \) positive and \( \gamma \) negative or (2) very small negative \( \beta \) and very large \( \gamma \). Both these cases are unphysical. Note that there is a region with large, positive \( \beta \), and small, positive \( \gamma \), and these are also physically uninteresting.

\( C(\beta, \gamma) < 0, \nu(\beta, \gamma) > 0: \) It can be verified that for \( \nu(\beta, \gamma) > 0 C(\beta, \gamma) \) is always positive and hence it is impossible to satisfy these conditions.

\( C(\beta, \gamma) > 0, \nu(\beta, \gamma) > 0: \) These two constraints give us a region in the \( \beta-\gamma \) space, displayed in Fig (3), that admits physically reasonable values. For \( V(x_{\text{inner}}) < \lambda < V(x_{\text{outer}}) \), we have one turning point admitting Type I eigenvalues. In Fig (3) we plot two examples of eigenmodes for the case \( \beta = \gamma = 2 \). The eigenfunctions for small quantum numbers are found to be centrally concentrated while they extend to larger radii for larger quantum numbers. It should be noted that our solutions are regular at the turning point. The first few eigenvalues for this case are tabulated in Table 1, where we find an excellent match with the WKB eigenvalues. Outside this range of positive \( \lambda \) there are no turning points and hence only Type II eigenvalues are possible.

\( C(\beta, \gamma) > 0, \nu(\beta, \gamma) < 0: \) The region in \( \beta-\gamma \) space satisfying these constraints is plotted in Fig (5). For \( 0 < \lambda < V(1) \), we have one turning point admitting Type I eigenvalues. Here the classically accessible region is bounded by the turning point on left and \( x_{\text{outer}} \) on right. Numerical solution for \( \beta = -2 \) and \( \gamma = 2 \) are plotted in Fig (7). Comparison with Fig (3) shows that the solutions are more radially extended, with well separated peaks. For \( \lambda > V(1) \) there is no turning point and the eigenvalues are of Type II.

5 DISCUSSION AND CONCLUSIONS

We have presented a theory of slow \( m = 1 \) linear pressure modes (or “p-modes”) in thin accretion discs around massive compact objects, such as white dwarfs and neutron stars. These modes are enabled by the small deviation from a purely Keplerian flow, due to fluid pressure rather than disc self-gravity. For simplicity we have taken the fluid to be barotropic. Our formulation largely follows that of Tremaine (2001), although there is a key difference: using the WKB approximation, Tremaine (2001) argued that fluid discs for which disc self-gravity dominates fluid pressure can support slow modes, if the Mach number \( M \) is much larger than the Toomre \( Q \) parameter. This condition may be satisfied in relatively cool discs, but not for thin accretion discs around white dwarfs or neutron stars. In these discs, \( Q \gg M \gg 1 \), and the analysis in Tremaine (2001) does not apply, because disc self-gravity is negligible when compared with fluid pressure in thin accretion discs. This implies that the precession rate of the apsides (\( \varpi \)) of a fluid element in a nearly circular orbit is determined by the fluid pressure; to order of magnitude, \( \varpi \sim (\Omega/M)^2 \).

A WKB analysis was used first to argue that thin accretion discs can support large-scale (\( k(R)R \sim 1 \)), \( m = 1 \) p-modes with small angular frequencies, \( \omega \sim \varpi \sim (\Omega/M)^2 \). As noted by Tremaine (2001), these long wavelength modes may dominate the appearance of the disc, and are not expected to be damped by viscosity. We derived an eigen equation for slow linear modes and showed that it is identical to a Sturm-Liouville problem. The differential operator being self-adjoint implies that the eigenvalues are all real, so that all slow p-modes are stable. This corresponds to the result in Tremaine (2001) that all slow modes of the softened gravity disc are stable. We solved the Sturm-Liouville problem numerically and those obtained using the WKB approximation. The columns \( \lambda_{50} \) and \( \lambda_{200} \) are the eigenvalues corresponding to \( x_{\text{outer}} = 50 \) and 200 respectively. The match between numerical and WKB eigenvalues is within a few per cent, and remains good even for small quantum numbers. The eigenvalues are not very sensitive to precise value of \( x_{\text{outer}} \).

| \( n \) | WKB \( \lambda \) | Numerical \( \lambda_{50} \) | Numerical \( \lambda_{200} \) |
|------|----------|----------------|----------------|
| 0    | 13.42    | 13.86          | 13.86          |
| 1    | 30.15    | 30.73          | 30.73          |
| 2    | 52.34    | 53.06          | 53.06          |
| 3    | 79.99    | 80.92          | 80.91          |
| 4    | 113.22   | 114.32         | 114.31         |
| 5    | 151.99   | 153.29         | 153.26         |
| 6    | 196.31   | 197.83         | 197.75         |
| 7    | 246.18   | 247.96         | 247.80         |
| 8    | 301.60   | 303.72         | 303.40         |
| 9    | 362.57   | 365.16         | 364.54         |
| 10   | 429.09   | 432.31         | 431.24         |

Table 1. The eigenvalues for the power law model with \( \beta = \gamma = 2 \) are tabulated here for comparison between those obtained numerically and those obtained using the WKB approximation. The columns \( \lambda_{50} \) and \( \lambda_{200} \) are the eigenvalues corresponding to \( x_{\text{outer}} = 50 \) and 200 respectively. The match between numerical and WKB eigenvalues is within a few per cent, and remains good even for small quantum numbers. The eigenvalues are not very sensitive to precise value of \( x_{\text{outer}} \).

\(^1\) The positions of the ILRs, obtained from, \( \lambda = -Q(x)/W(x) \) are slightly different from the turning points obtained from the equation \( \lambda = -(Q(x) - \eta(x)\eta''(x))/W(x) \).
merically for a variety of unperturbed discs, and summarise our results below.

(i) The first corresponds to two different kinds of barotropic approximations to the Shakura-Sunyaev thin disc, which have modes with negative eigenfrequencies.

(ii) The second is the Kuzmin disc, which is more centrally concentrated. As earlier, this too supports only negative eigenfrequencies.

(iii) Power-law discs can support modes with negative eigenfrequencies for reasonable values of $\beta$ and $\gamma$. For certain combinations of these parameters power law discs can support positive eigenfrequencies as well; however the range of parameters turn out to be physically uninteresting.

If slow modes are stable, it is necessary to consider how they could be excited. Since they have azimuthal wavenumber $m = 1$, we need look for excitation mechanisms which possess the same symmetry, at least in the linear limit. One possibility is from the stream of matter from the secondary star that feeds the accretion disc. When viewed in the rest frame of the primary, the region where the stream meets the outer edge of the disc rotates in a prograde sense with angular frequency equal to the orbital frequency of the binary system. Since the orbital frequency of the binary can be much smaller than the orbital frequency of the gas in the accretion disc, there is the possibility of the resonant excitation of a slow mode if it has positive frequency. In the cases we have considered, we find only negative frequencies belonging to a discrete spectrum, allowing only for non–resonant driving.

There is, however, another alternative that does not rely on external sources of excitation. Zhang & Lovelace (2005) have studied linear waves in thin accretion discs and applied their theory to slow $m = 1$ modes around black holes. They used the pseudo-Newtonian gravitational potential of Paczynsky & Wiita (1980) to model the general relativistic effects due to a Schwarzschild black hole. In this case $\dot{\varpi}$ is due to the deviation of the pseudo-Newtonian potential from a Kepler potential, so their slow modes are not driven by pressure (as is true in all the cases we have considered); hence more detailed comparison with our work is not possible. What is interesting is that they find that their slow modes to have negative energy and angular momentum, and suggest that slow modes may be excited spontaneously through the action of viscous forces. This possibility should be examined in the context of the $p$-modes we have studied. However, this would require reformulating the eigenvalue problem taking into account viscous effects.

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