Microfractured media with a scale and Mumford-Shah energies

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Abstract

We want to understand the concentration of damage in microfractured elastic media. Due to the different scalings of the volume and area (or area and length in two dimensions) the traditional method of homogenization using periodic arrays of cells seems to fail when applied to the Mumford-Shah functional and to periodically fractured domains.

In the present paper we are departing from traditional homogenization. The main result implies the use of Mumford-Shah energies and leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

1 Introduction

A new direction of research in brittle fracture mechanics begins with the article of Mumford & Shah [15] regarding the problem of image segmentation. This problem, which consists in finding the set of edges of a picture and constructing a smoothed version of that picture, turns to be intimately related to the problem of brittle crack evolution. In the before mentioned article Mumford and Shah propose the following variational approach to the problem of image segmentation: let \( g : \Omega \subset \mathbb{R}^2 \to [0,1] \) be the original picture, given as a distribution of grey levels (1 is white and 0 is black), let \( u : \Omega \to \mathbb{R} \) be the smoothed picture and \( K \) be the set of edges. \( K \) represents the set where \( u \) has jumps, i.e. \( u \in C^1(\Omega \setminus K, R) \). The pair formed by the smoothed picture \( u \) and the set of edges \( K \) minimizes then the functional:

\[
I(u, K) = \int_\Omega \alpha \, | \nabla u |^2 \, dx + \int_\Omega \beta \, | u - g |^2 \, dx + \gamma \mathcal{H}^1(K) .
\]

The parameter \( \alpha \) controls the smoothness of the new picture \( u \), \( \beta \) controls the \( L^2 \) distance between the smoothed picture and the original one and \( \gamma \) controls the total length of the edges given by this variational method. The authors remark that for \( \beta = 0 \) the functional \( I \) might be useful for an energetic treatment of fracture mechanics.

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An energetic approach to fracture mechanics is naturally suited to explain brittle crack appearance under imposed boundary displacements. The idea is presented in the followings.

The state of a brittle body is described by a pair displacement-crack. \((u, K)\) is such a pair if \(K\) is a crack — seen as a surface — which appears in the body and \(u\) is a displacement of the broken body under the imposed boundary displacement, i.e. \(u\) is continuous in the exterior of the surface \(K\) and \(u\) equals the imposed displacement \(u_0\) on the exterior boundary of the body.

Let us suppose that the total energy of the body is a Mumford-Shah functional of the form:

\[
E(u, K) = \int_{\Omega} w(\nabla u) \, dx + F(u_0, K)
\]

The first term of the functional \(E\) represents the elastic energy of the body with the displacement \(u\). The second term represents the energy consumed to produce the crack \(K\) in the body, with the boundary displacement \(u_0\) as parameter. Then the crack that appears is supposed to be the second term of the pair \((u, K)\) which minimizes the total energy \(E\).

Models for brittle damage, based on functionals of the Mumford-Shah type have have been proposed by Francfort-Marigo [11], Buliga [6], among others. Such models have been studied intensively from the mathematical point of view, especially by the Italian school of geometric measure theory, to name a few: De Giorgi, Ambrosio, Dal Maso, Buttazzo.

The first homogenization result, concerning the Mumford-Shah functional, seems to be Braides, Defranceschi, Vitali [5]. In this paper it is done the homogenization of a Mumford-Shah functional of the form:

\[
\int \Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) \, dx + \int_{S_u} g\left(\frac{x}{\varepsilon}, (u^+ - u^-) \otimes \nu_u\right) \, d\mathcal{H}^{n-1}
\]

The paper Focardi, Gelli [14] (and the references therein) are part of another line of research which might be relevant for this paper: homogenization of perforated domains.

In the present paper we are departing from traditional homogenization. The line of research concerning perforated domains is close to our problem, but for various reasons the results from perforated domains don’t apply here.

We want to understand he concentration of damage in microfractured elastic media. Due to the different scalings of the volume and area (or area and length in two dimensions) the traditional method of homogenization using periodic arrays of cells seems to fail when applied to the Mumford-Shah functional and to periodically fractured domains.

The main result, theorem 4.2, implies the use of Mumford-Shah energies and leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

Instead of performing a homogenization of the total energy of the microfractured body and then study the minimizers of the homogenized energy, we proceed along a different path. We study sequences of problems on fractured elastic bodies, indexed by
a scale parameter $\varepsilon$. Each such problem has (at least approximative) solutions. We find estimates of the area of the damaged region in terms of the scale $\varepsilon$.

2 Notations

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^2$, with locally Lipschitz boundary. We denote by $Y = [0, 1]^2$ the unit closed square in $\mathbb{R}^2$.

For a given $\varepsilon > 0$ let $Z_\varepsilon \subset \mathbb{R}^2$ be the lattice of points in $\mathbb{R}^2$ with coordinates of the form $(\varepsilon m, \varepsilon n)$, for all $m, n \in \mathbb{Z}$.

We denote by $Z(\varepsilon, \Omega) \subset Z_\varepsilon$ the set of all $z \in Z_\varepsilon$ such that $z + \varepsilon Y \subset \Omega$.

To any $z \in Z(\varepsilon, \Omega)$ we associate the cell $D_z = z + \varepsilon Y \subset \Omega$.

The set $Z(\varepsilon, \Omega)$ is finite for any $\varepsilon > 0$. We denote the cardinal of this set by $N(\varepsilon)$ and we notice that as $\varepsilon$ goes to 0 we have

$$\lim_{\varepsilon \to 0} \frac{N(\varepsilon)\varepsilon^2}{A(\Omega)} = 1,$$

where $A(\Omega)$ denotes the area of $\Omega$. Thus for small $\varepsilon$ the number of cells $N(\varepsilon)$ is approximately equal to $A(\Omega)/\varepsilon^2$.

3 The model

We take $\Omega$ to be the configuration set of a microfractured linear elastic body. We explain further what we mean by this.

The elastic properties of the body are described by an elastic potential

$$w : M_{\text{sym}}^2(\mathbb{R}) \to \mathbb{R}.$$  

We suppose that the function $w$ is quadratic and strictly positive definite.

For a given displacement $u : \Omega \to \mathbb{R}^2$, the elastic energy of the body is given by

$$\int_\Omega w(e(u)) \, dx,$$

where $e(u)$ is the deformation of the displacement $u$, that is the symmetric part of the gradient of $u$: for any $x \in \Omega$

$$e(u)(x) = \frac{1}{2} \left( \nabla u(x) + (\nabla u)^T(x) \right).$$

For a fixed $\varepsilon > 0$ we suppose that the body contains a distribution of micro-fractures at the scale $\varepsilon$, seen as a union of (Lipschitz) curves

$$F_\varepsilon = \bigcup_{z \in Z(\varepsilon, \Omega)} (z + \varepsilon F_z),$$
where for each \( z \in \mathbb{Z}(\varepsilon, \Omega) \) the (Lipschitz) curve \( F_z \) lies inside the unit cell \( Y \):

\[
F_z \subset (0,1)^2.
\]

We explain further what we mean by an imposed boundary displacement \( u_0 \), and what we mean by \( u = u_0 \) on the boundary of \( \Omega \).

We consider, for simplicity, that \( u_0 : \partial \Omega \rightarrow \mathbb{R}^n \) is a continuous and therefore bounded function. Then, for any \( u \in \text{SBD}(\Omega) \), \( u = u_0 \) if the approximate limit of \( u \) equals \( u_0 \) in any point of \( \partial \Omega \) where the first exists, i.e.: for all \( x \in \partial \Omega \), if there exists \( v(x) \) such that

\[
\lim_{\rho \to 0^+} \frac{\int_{B_\rho(x) \cap \Omega} | u(y) - v(x) | \, dy}{| B_\rho(x) \cap \Omega |} = 0
\]

then \( v(x) = u_0(x) \).

**Definition 3.1** The class of admissible displacements with respect to the distribution of cracks \( F_\varepsilon \) and with respect to the imposed displacement \( u_0 \) is defined as the collection of all \( u \in \text{SBD}(\Omega) \) such that

(a) \( u = u_0 \) on \( \partial \Omega \),

(b) \( F_\varepsilon \subset S_u \).

This class of admissible displacements is denoted by \( \text{Adm}(F_\varepsilon, u_0) \).

This definition deserves an explanation. An admissible displacement \( u \) is a function which has to be equal to the imposed displacement on the boundary of \( \Omega \) (condition (a)). Any such function \( u \) is a special function with bounded deformation, that is a reasonably smooth function on the set \( \Omega \setminus S_u \) and the function \( u \) is allowed to have jumps along the set \( S_u \). For the technical details see the Appendix. We have to think about \( S_u \) as being a collection of curves, with finite length. Physically the set \( S_u \) represents the collection of all cracks in the body under the displacement \( u \). The condition (b) tells us that the collection of all cracks associated to an admissible displacement \( u \) contains \( F_\varepsilon \), at least.

**Definition 3.2** With the notations from definition 3.1, the total energy of an admissible displacement \( u \in \text{Adm}(F_\varepsilon, u_0) \) is given by

\[
E_\varepsilon(u) = \int_\Omega w(e(u)) \, dx + \text{GH}^1(S_u \setminus F_\varepsilon).
\]

The energy of an admissible displacement is of Mumford-Shah type. It contains two terms.

The first term measures the elastic energy of the body under the displacement \( u \). Notice that in the expression of the elastic energy we have integrated over the whole
domain $\Omega$. This is simply because the collection of cracks associated to $u$ (that is the set $S_u$) has Lebesgue measure 0, therefore we have

$$\int_{\Omega} w(e(u)) \, dx = \int_{\Omega \setminus S_u} w(e(u)) \, dx.$$ 

In physical terms, the right hand side expression would make more sense than the left hand side, but from the mathematical point of view they are the same. This is not meaning that the elastic energy neglects the fractures. Indeed, further we shall infimize the energy $E_\varepsilon$ over the whole set of admissible displacements. According to condition (b) of definition 3.1, this set is defined with respect to the collection of cracks $F_\varepsilon$, therefore the infimum of the energy $E_\varepsilon$ depends on the set of cracks $F_\varepsilon$.

The second term of the Mumford-Shah energy measures the surface energy caused by the apparition of new cracks. The collection of new cracks is the set $S_u \setminus F_\varepsilon$. The constant $G$ has the dimension of energy per unit area, and it is physically related to the Griffith constant.

In [4] has been proven that functionals like $E_\varepsilon$ are $L^1$ inferior semi-continuous and coercive, hence on closed subspaces $V$ of $\text{SBD}(\Omega)$ the functional $E_\varepsilon$ has a minimizer. Such a closed subspace of $\text{SBD}(\Omega)$ is the space of all admissible displacements $\text{Adm}(F_\varepsilon, u_0)$. Therefore we have:

**Theorem 3.3** On the space $\text{Adm}(F_\varepsilon, u_0)$ we consider the topology given by the convergence: $u_h \rightarrow u$ if

$$\begin{cases} u_h & \rightarrow L^2 \rightarrow u, \\ \mathcal{H}^{n-1}(S_u \Delta S_u) & \rightarrow 0. \end{cases}$$

Then there exists a minimizer of the functional $E_\varepsilon$ over the set $\text{Adm}(F_\varepsilon, u_0)$.

In the following section we shall use approximate minimizers.

**Definition 3.4** For a given $\delta > 0$, a function $u \in \text{Adm}(F_\varepsilon, u_0)$ is a $\delta$-approximate minimizer if

$$E_\varepsilon(u) \leq \delta + \inf \{ E_\varepsilon(v) : v \in \text{Adm}(F_\varepsilon, u_0) \}.$$ 

For fixed $\delta > 0$, we model an approximate displacement of a microfractured body as a sequence of displacements $u_\varepsilon$, with $\varepsilon$ converging to 0, such that for each $\varepsilon > 0$ the displacement $u_\varepsilon \in \text{Adm}(F_\varepsilon, u_0)$ is a $\delta$-approximate minimizer of the Mumford-Shah energy $E_\varepsilon$, over the set $\text{Adm}(F_\varepsilon, u_0)$.

Notice that in the model, at this stage, there is no relation between the crack sets $F_\varepsilon, F_{\varepsilon'}$, for two different scales $\varepsilon, \varepsilon'$.

**4 An estimate related to damage concentration**

For fixed $\varepsilon, \delta > 0$, given $F_\varepsilon$ and imposed boundary displacement $u_0$, let $u \in \text{Adm}(F_\varepsilon, u_0)$ be a $\delta$-approximate minimizer of the Mumford-Shah energy $E_\varepsilon$. 

5
In this section we want to estimate the number of $\varepsilon$-cells $z + \varepsilon Y$, $z \in \mathbb{Z}(\varepsilon, \Omega)$, where the initial cracks $z + \varepsilon F_z$ propagated.

Let $l > 0$ be a given length.

**Definition 4.1** For any cell $D_z = z + \varepsilon Y$, $z \in \mathbb{Z}(\varepsilon, \Omega)$, and any $\delta$-approximate minimizer $u$ we define the emergent crack in the cell $D_z$ by

$$S_u(z) = (z + \varepsilon Y) \cap (S_u \setminus (z + \varepsilon F_z)) .$$

A cell $D_z$ is called active if the length of the emergent crack is greater than $\varepsilon l$, that is:

$$\mathcal{H}^1(S_u(z)) \geq \varepsilon l .$$

We denote by $M(\varepsilon, l)$ the number of active cells. (In this notation we don’t mention the dependence of $M(\varepsilon, l)$ on the $\delta$-approximate minimizer $u$.)

**Theorem 4.2** Suppose that for fixed $\delta > 0$, the crack sets $F_\varepsilon$ are chosen so that there exists an approximate displacement of a microfractured body $u_\varepsilon$, with $\varepsilon$ converging to 0, with the property that the sequence

$$\inf \{E_\varepsilon(v) : v \in \text{Adm}(F_\varepsilon, u_0)\}$$

is bounded.

Then the number of active cells $M(\varepsilon, l)$ is of order $1/\varepsilon$ and the area of the damaged region of the body

$$\text{Damaged}(\varepsilon, \Omega) = \bigcup_{D_z \text{ active}} D_z$$

is of order $\varepsilon$.

**Proof.** Let $M > 0$ such that for all $\varepsilon > 0$ we have

$$\inf \{E_\varepsilon(v) : v \in \text{Adm}(F_\varepsilon, u_0)\} \leq M .$$

According to definition 3.3 for any $\varepsilon > 0$ we have

$$E_\varepsilon(u_\varepsilon) = \int_\Omega w(e(u_\varepsilon)) \, dx + G\mathcal{H}^1(S_{u_\varepsilon} \setminus F_\varepsilon) \leq \delta + \inf \{E_\varepsilon(v) : v \in \text{Adm}(F_\varepsilon, u_0)\} \leq \delta + M .$$

From definition 4.1 we get the following estimate:

$$\mathcal{H}^1(S_{u_\varepsilon} \setminus F_\varepsilon) = \sum_{z \in \mathbb{Z}(\varepsilon, \Omega)} \mathcal{H}^1(S_u(z)) \geq M(\varepsilon, l) \varepsilon .$$

We have therefore

$$G M(\varepsilon, l) \varepsilon \leq G\mathcal{H}^1(S_{u_\varepsilon} \setminus F_\varepsilon) \leq E_\varepsilon(u_\varepsilon) \leq M + \delta .$$
All in all we have obtained the estimate:

\[ M(\varepsilon, l) \leq \frac{1}{\varepsilon} \frac{M + \delta}{Gl} . \]

The area of the damaged region of the body is

\[ \text{Area}(\text{Damaged}(\varepsilon, \Omega)) = \sum_{D_{\varepsilon} \text{ active}} \text{Area}(D_{\varepsilon}) = \varepsilon^2 M(\varepsilon, l) \leq \varepsilon \frac{M + \delta}{Gl} . \]

The proof is done. \( \square \)

5 Conclusions

The theorem implies that the area of the damaged region is much smaller than the total area of the body, as \( \varepsilon \) goes to zero. In this model the use of Mumford-Shah energies leads to an explanation of the observed concentration of damage in microfractured elastic bodies.

Notice that we need more precise estimates in order to prove that the damaged region (at the scale \( \varepsilon \)) converges, as \( \varepsilon \) goes to zero, to a curve with finite length. All we know at this moment is that the area of the damaged region goes to zero as the scale parameter \( \varepsilon \).

In experiments it has been observed that the damaged region is approximately straight. It is possible that Mumford-Shah energies might explain this, since geometries of the active crack set, that is \( S_u \setminus F_\varepsilon \), close to a straight line would be preferred by the energy \( E_\varepsilon \). See [7] for examples that in some situations the leading term of a Mumford-Shah energy is the one accounting for the length of the crack, and not the elastic energy part.

Finally, in theorem 4.2 we obtained an estimate of the number of cells where cracks of length at least \( \varepsilon l \) appear. It would be interested to study the interplay between \( \varepsilon \) and \( l \) in this estimate.

6 Appendix. Functions with bounded variation or deformation

This section is dedicated to a brief voyage through the spaces \( \text{SBV} \) and \( \text{SBD} \).

The space \( \text{SBV}(\Omega, R^n) \) of special functions with bounded variation was introduced by De Giorgi and Ambrosio in the study of a class of free discontinuity problems ([9], [1], [2]). For any function \( u \in L^1(\Omega, R^n) \) let us denote by \( Du \) the distributional derivative of \( u \) seen as a vector measure. The variation of \( Du \) is a scalar measure defined like this: for any Borel measurable subset \( B \) of \( \Omega \) the variation of \( Du \) over \( B \) is

\[ |Du|(B) = \text{sup} \left\{ \sum_{i=1}^{\infty} |Du(A_i)| : \bigcup_{i=1}^{\infty} A_i \subset B, A_i \cap A_j = \emptyset \quad \forall i \neq j \right\} . \]
A function $u$ has bounded variation if the total variation of $Du$ is finite. We send the reader to the book of Evans & Gariepy \cite{evans} for basic properties of such functions.

The space $SBV(\Omega, R^n)$ is defined as follows:

$$SBV(\Omega, R^n) = \{ u \in L^1(\Omega, R^n) : |Du| (\Omega) < +\infty , |D^s u| (\Omega \setminus S_u) = 0 \} .$$

The Lebesgue set of $u$ is the set of points where $u$ has approximate limit. The complementary set is a $L^n$ negligible set denoted by $S_u$. If $u$ is a special function with bounded variation then $S_u$ is also $\sigma$ (i.e. countably) rectifiable.

From the Calderon & Zygmund \cite{calderon} decomposition theorem we obtain the following expression of $Du$, the distributional derivative of $u \in SBV(\Omega, R^n)$, seen as a measure:

$$Du = \nabla u(x) \, dx + [u] \otimes n \, dH^{n-1} .$$

We shall use further the notation $\mu \ll \lambda$ if the measure $\mu$ is absolutely continuous with respect to the measure $\lambda$.

Let us define the following Sobolev space associated to the crack set $K$ (see \cite{belletini}):

$$W^{1,2}_K = \left\{ u \in SBV(\Omega, R^n) : \int_\Omega |\nabla u|^2 \, dx + \int_K [u]^2 \, dH^{n-1} < +\infty , |D^s u| \ll H^{n-1}_K \right\} .$$

It has been proved in \cite{belletini} the following equality:

$$W^{1,2}(\Omega \setminus K, R^n) \cap L^\infty(\Omega, R^n) = W^{1,2}_K(\Omega, R^n) \cap L^\infty(\Omega, R^n) . \quad (6.0.1)$$

A similar description can be made for the space of special functions with bounded deformation $SBD(\Omega)$ can be found in \cite{belletini}. For any function $u \in L^1(\Omega, R^n)$ we denote by $Eu$ the symmetric part of the distributional derivative of $u$, seen as a vector measure. We denote also by $J_u$ the subset of $\Omega$ where $u$ has different approximate limits with respect to a point-dependent direction. The difference between $S_u$ and $J_u$ is subtle. Let us quote only the fact that for a function $u \in SBV(\Omega, R^n)$ the difference of these sets is $H^{n-1}$-negligible.

The definition of $SBD(\Omega)$ is the following:

$$SBD(\Omega, R^n) = \{ u \in L^1(\Omega, R^n) : |Eu| (\Omega) < +\infty , |E^s u| (\Omega \setminus J_u) = 0 \} .$$

If $u$ is a special function with bounded deformation then $J_u$ is countably rectifiable. We have a decomposition theorem for $SBD$ functions, similar to Calderon & Zygmund result applied for $SBV$ functions. The decomposition theorem is due to Belletini, Coscia & Dal Maso \cite{belletini} and asserts that

$$Eu = \epsilon(u)(x) \, dx + [u] \otimes n \, dH^{n-1} .$$

Here $\otimes$ means the symmetric part of tensor product and $\epsilon(u)$ is the approximate symmetric gradient, hence the approximate limit of the symmetric part of the gradient of $u$.

We sum up the main facts about functions with bounded variation or deformation, in the following three theorems.
Theorem 6.1 Let \( u \in L^1(\Omega, \mathbb{R}^m) \). Then

- (De Giorgi) If \( u \in BV(\Omega, \mathbb{R}^m) \) then \( S_u \) is countably rectifiable, \( \mathcal{H}^{n-1}(S_u \setminus J_u) = 0 \) and in \( \mathcal{H}^{n-1} \)-almost every point \( x \in S_u \) exists the approximate limits of \( u \) in the directions \( v(x) \) and \( -v(x) \) where \( v(x) \) is the normal to \( S_u \) in \( x \).

- (Kohn, Ambrosio, Coscia, Dal Maso) Let \( m = n \) and \( u \in BD(\Omega) \). Let \( \Theta_u \) be the Kohn set:

\[
\Theta_u = \left\{ x \in \Omega : \limsup_{\rho \to 0^+} \frac{|E_u| (B_\rho(x))}{\rho^{n-1}} > 0 \right\}
\]

Then \( \Theta_u \) is countably rectifiable, \( J_u \subseteq \Theta_u \) and \( \mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0 \).

Theorem 6.2 Let \( u \in L^1(\Omega, \mathbb{R}^m) \). Then

- (Calderon, Zygmund) If \( u \in BV(\Omega, \mathbb{R}^m) \) then \( u \) is approximately differentiable \( \mathcal{L}^n \)-a.e. in \( \Omega \). The approximate differential map \( x \mapsto \nabla u(x) \) is integrable. \( Du \) splits into three mutually singular measures on \( \Omega \)

\[
Du = \nabla u \, dx + |u| \otimes \nu_{S_u} + Cu
\]

where \( |u| \) is the jump of \( u \) in respect with the normal direction on \( S_u \) \( \nu \). \( Cu \) is the Cantor part of \( Du \) defined by \( Cu(A) = D^u(A \setminus S_u) \) where \( D^u \) is the singular part of \( Du \) in respect to \( \mathcal{L}^n \).

- (Belletini, Coscia, Dal Maso) Let \( m = n \) and \( u \in BD(\Omega) \). Then \( u \) has symmetric approximate differential \( \epsilon(u) \) \( \mathcal{L}^n \)-a.e. in \( \Omega \) and \( Eu \) splits into three mutually singular measures on \( \Omega \)

\[
Eu = \epsilon(u) \, dx + |u| \otimes \nu_{J_u} + E^u
\]

Moreover \( u \) is approximately differentiable \( \mathcal{L}^n \)-a.e. in \( \Omega \).

Theorem 6.3 The following are true:

- \( W^{1,1}(\Omega, \mathbb{R}^m) \subset BV(\Omega, \mathbb{R}^m) \). The inclusion is continuous in respect with the Banach space topologies. If

\[
u \in SBV(\Omega, \mathbb{R}^m)
\]

then

\[
u \in W^{1,1}(\Omega \setminus S_u, \mathbb{R}^m)
\]

Moreover if \( \nu \in W^{1,1}(\Omega \setminus K, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m) \), where \( K \) is a closed, countably rectifiable set with \( \mathcal{H}^{n-1}(K) < +\infty \), then \( \nu \in SBV(\Omega, \mathbb{R}^m) \) and \( \mathcal{H}^{n-1}(K \setminus S_u) = 0 \).

- Let \( LE^1(\Omega) \) be the Banach space of \( L^1(\Omega, \mathbb{R}^m) \) functions with \( L^1 \) symmetric differential. If \( \nu \in SBD(\Omega) \) then \( \nu \in LE^1(\Omega \setminus J_u) \). Let \( K \) be a closed, countably rectifiable set with \( \mathcal{H}^{n-1}(K) < +\infty \). If \( \nu \in LE^1(\Omega \setminus K) \cap L^\infty(\Omega, \mathbb{R}^m) \) then \( \nu \in SBD(\Omega) \) and \( \mathcal{H}^{n-1}(K \setminus J_u) = 0 \).
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