Analysis of pressure-robust embedded-hybridized discontinuous Galerkin methods for the Stokes problem under minimal regularity

Aaron Baier-Reinio · Sander Rhebergen · Garth N. Wells

Received: date / Accepted: date

Abstract We present analysis of two lowest-order hybridizable discontinuous Galerkin methods for the Stokes problem, while making only minimal regularity assumptions on the exact solution. The methods under consideration have previously been shown to produce $H(\text{div})$-conforming and divergence-free approximate velocities. Using these properties, we derive a priori error estimates for the velocity that are independent of the pressure. These error estimates, which assume only $H^{1+s}$-regularity of the exact velocity fields for any $s \in [0,1]$, are optimal in a discrete energy norm. Error estimates for the velocity and pressure in the $L^2$-norm are also derived in this minimal regularity setting. Our theoretical findings are supported by numerical computations.

Keywords Stokes equations · Minimal regularity · Embedded · Hybridized · Discontinuous Galerkin finite element methods · Pressure-robust

Mathematics Subject Classification (2010) 65N12 · 65N15 · 65N30 · 76D07

1 Introduction

Finite element methods for incompressible flows that are pressure-robust have become increasingly popular. Such methods produce approximate velocity fields...
for which the *a priori* velocity error estimates are independent of the pressure approximation. Numerous classical inf-sup stable finite elements, such as the MINI element [1] and Bernardi–Raugel elements [3], are not pressure-robust [20], with the velocity error polluted by the pressure approximation error scaled by the inverse of the viscosity, which can be large if the pressure is complicated or the viscosity is small.

One way of achieving pressure-robustness is by stable mixed methods with \( H(\text{div}) \)-conforming and divergence-free approximate velocities [20]. Methods with these properties may relax \( H^1 \)-conformity and use discontinuous velocity approximations, as constructing \( H^1 \)-conforming and inf-sup stable schemes that are also divergence-free is difficult [20]. For this reason, discontinuous Galerkin (DG) methods seem to be a natural candidate for the construction of pressure-robust schemes. Several classes of pressure-robust DG methods that produce \( H(\text{div}) \)-conforming and divergence-free approximate velocities were introduced in [8,38].

A drawback of DG methods is that they are, on a given mesh, typically computationally more expensive than standard conforming methods. Hybridized discontinuous Galerkin (HDG) methods were introduced to improve upon the computational efficiency of DG methods while retaining their desirable properties [4]. This is accomplished by introducing extra degrees of freedom on cell facets which allow for local cell-wise variables to be eliminated by static condensation. Examples of \( H(\text{div}) \)-conforming and divergence-free HDG methods are given in [9,30,31] for the Stokes problem and in [10,20,31] for the Navier–Stokes problem.

In this paper we study two closely related lowest-order hybridizable DG methods for the velocity-pressure formulation of the Stokes problem. Both methods produce \( H(\text{div}) \)-conforming and divergence-free approximate velocities, and are therefore pressure-robust. The first method is the lowest-order HDG method analyzed in [30,32]. The velocity finite element space for this method consists of discontinuous piecewise linear functions on cells and facets. As discussed in [32], the computational cost of this method can be reduced, while maintaining pressure-robustness, by using a continuous basis for the velocity facet space. Such an approach is reminiscent of embedded discontinuous Galerkin (EDG) methods [17]. This leads to the EDG–HDG method of [32], the lowest-order formulation of which is the second method considered in this paper.

In [32], optimal pressure-robust velocity error estimates were obtained for the HDG and EDG–HDG methods, assuming \( H^2 \)-regularity of the exact velocity solution. Moreover, numerical experiments in [32] suggest that even when \( H^2 \)-regularity of the exact velocity solution fails to hold, the methods remain convergent. Because of their computational efficiency, the lowest-order HDG and EDG–HDG methods are appealing for problems with minimal regularity. However, error analysis for this minimal regularity case has not been developed. This paper closes this gap by presenting analysis of these lowest-order methods under \( H^{1+s} \)-regularity of the exact velocity solution for any real number \( s \in [0,1] \).

To put the error analysis of this paper into a broader context, we review briefly the literature relevant to our analysis. Numerous classes of non-pressure-robust nonconforming methods for the Stokes problem were analyzed under minimal regularity assumptions in [2,27]. Key to the analysis of [2,27] is a so-called enrichment operator that maps nonconforming discrete functions to \( H^1 \)-conforming functions. More recently, by using enrichment operators that map discretely divergence-free functions to exactly divergence-free ones, this minimal regularity analysis has been

---

Aaron Baier-Reinio et al.
Analysis of EDG–HDG methods for the Stokes problem under minimal regularity

extended to pressure-robust schemes. This is done in the works of [23, 25, 29, 37], which establish quasi-optimal and pressure-robust error estimates for various finite element methods that achieve pressure-robustness by modifying the source term in the discrete formulation. In [29, 37] modified Crouzeix–Raviart methods are considered, while [25] focuses on modified conforming methods and [23] on modified DG methods. Finally, a variety of conforming and nonconforming pressure-robust methods based on an augmented Lagrangian formulation have been proposed and analyzed under minimal regularity assumptions in [24].

The main contributions of this paper are as follows. First, we derive a bound on the consistency error of the lowest-order HDG and EDG–HDG methods, by means of an enrichment operator of the type considered in [23]. A consequence of the hybridized formulation is that our consistency error bound contains a new term not found in previous works. However, we show that it is still possible to obtain optimal pressure-robust velocity error estimates in a discrete energy norm. Pressure-robust velocity error estimates in the $L^2$-norm are also derived, and we conclude our analysis by deriving $L^2$-error bounds for the pressure.

The remainder of this paper is organized as follows. In Section 2 we introduce the Stokes problem and the methods to be analyzed, and discuss some preliminary results. The main analysis is carried out in Section 3, where we derive our error estimates for the velocity and the pressure. In Section 4 our theoretical findings are illustrated by numerical examples, and the paper ends with conclusions in Section 5.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a connected and bounded domain with polyhedral boundary $\partial \Omega$. The codimension of $\partial \Omega$ is assumed to be one, but we do not require that $\Omega$ be a Lipschitz domain. In particular, domains with cracks are allowed. On a given set $D \subset \Omega$, we let $(\cdot, \cdot)_D$ denote the $L^2$-inner-product on $D$ and $\|\cdot\|_D$ the $L^2$-norm on $D$. Given an integer $k \geq 0$, we let $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ denote the usual $H^k$-norm and $H^k$-semi-norm on $D$, respectively. If $k > 0$ is not an integer, we let $\|\cdot\|_{k,D}$ denote the fractional order $H^k$-norm on $D$ as defined in [10, 11]. In the following we omit the subscript $D$ in the case of $D = \Omega$.

2.1 Stokes problem

Let $f \in L^2(\Omega)^d$ be a prescribed body force and $\nu > 0$ a given constant kinematic viscosity. The Stokes problem seeks a velocity field $u \in H^1_0(\Omega)^d$ and pressure field $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}$ such that

\[
\begin{align*}
\nu a(u, v) + b(v, p) &= (f, v) \quad \forall v \in H^1_0(\Omega)^d, \quad (2.1a) \\
b(u, q) &= 0 \quad \forall q \in L^2_0(\Omega), \quad (2.1b)
\end{align*}
\]

where $a : H^1_0(\Omega)^d \times H^1_0(\Omega)^d \to \mathbb{R}$ and $b : H^1_0(\Omega)^d \times L^2_0(\Omega) \to \mathbb{R}$ are the bilinear forms

\[
a(w, v) := (\nabla w, \nabla v), \quad b(v, q) := -(\nabla \cdot v, q).
\]
It is known that eq. (2.1) is well-posed, see e.g. [14, Chapter 4]. Furthermore, within the reduced space \( V := \{ v \in H^1_0(\Omega)^d : \nabla \cdot v = 0 \} \) of divergence-free functions, the velocity \( u \in V \) equivalently satisfies the reduced problem

\[
\nu a(u, v) = (f, v) \quad \forall v \in V.
\]  

(2.2)

We introduce the space of weakly divergence-free vector fields with vanishing normal component on \( \partial \Omega \),

\[
L^2_\sigma(\Omega) := \{ w \in L^2(\Omega)^d : (w, \nabla \psi) = 0 \ \forall \psi \in H^1(\Omega) \},
\]  

(2.3)

and note that every vector field \( f \in L^2(\Omega)^d \) admits a unique Helmholtz decomposition \[29, Theorem 2.1\]

\[
f = \nabla \phi + \mathbb{P} f,
\]

where \( \phi \in H^1(\Omega)/\mathbb{R} \) and \( \mathbb{P} f \in L^2_\sigma(\Omega) \).

The vector field \( \mathbb{P} f \) is called the Helmholtz projection of \( f \), see e.g. \[20, Section 2\]. We note that the reduced problem in eq. (2.2) is equivalent to

\[
\nu a(u, v) = (\mathbb{P} f, v) \quad \forall v \in V,
\]  

(2.4)

since for all \( v \in V \) it holds that \((f, v) = (\mathbb{P} f, v)\). In particular, the velocity solution \( u \) is determined only by the Helmholtz projection \( \mathbb{P} f \) of the body force. The presence of \( \mathbb{P} f \) in eq. (2.4) will turn out to play an important role in the pressure-robustness of our error estimates in Section 3 and we discuss why this is the case in Remark 3.1.

2.2 Mesh-related notation

Let \( \mathcal{T} = \{K\} \) be a conforming triangulation of \( \Omega \) into simplices \( \{K\} \). Let \( K \in \mathcal{T} \). We use \( \mathcal{F}_K \) to indicate the collection of \((d-1)\)-dimensional faces of \( K \). We set \( h_K = \text{diam}(K) \) and let \( n_K \) denote the outward unit normal on \( \partial K \). The mesh size is defined as \( h := \max_{K \in \mathcal{T}} h_K \) and the mesh skeleton is defined as \( \Gamma_0 = \bigcup_{K \in \mathcal{T}} \partial K \).

Notice that, if cracks are present in the domain, it is possible for two distinct elements \( K_1, K_2 \in \mathcal{T} \) to share a face \( \sigma \in \mathcal{F}_{K_1} \cap \mathcal{F}_{K_2} \) that lies on the boundary, i.e. \( \sigma \subset \partial \Omega \). In this case, it will not be convenient to view \( \sigma \) as a single mesh face, as is typically done for interior faces. Following \[35\], we therefore define the collection of mesh faces as the quotient set

\[
\mathcal{F}_h := \{ [(\sigma, K) : K \in \mathcal{T}, \sigma \in \mathcal{F}_K] \}/\sim,
\]

\((\sigma_1, K_1) \sim (\sigma_2, K_2) \iff [(\sigma_1, K_1) = (\sigma_2, K_2)] \) or \( \sigma_1 = \sigma_2 \) and \( \sigma_1 \notin \partial \Omega \).

For \( F = [(\sigma, K)] \in \mathcal{F}_h \) we set \( h_F := \text{diam}(\sigma) \). Also, surface integration on \( F \) is well-defined, with the understanding that \( \int_F v \, ds := \int_\sigma v \, ds \) for all \( v \in L^1(\sigma) \). The boundary faces \( \mathcal{F}_b \) and interior faces \( \mathcal{F}_i \) are naturally defined as

\[
\mathcal{F}_b := \{ [(\sigma, K)] \in \mathcal{F}_h : \sigma \subset \partial \Omega \}, \quad \mathcal{F}_i := \mathcal{F}_h \setminus \mathcal{F}_b,
\]

and we note that \( \partial \Omega = \bigcup_{[(\sigma, K)] \in \mathcal{F}_b} \sigma \) since the codimension of \( \partial \Omega \) is one.
If $F \in \mathcal{F}_i$ is an interior face belonging to two distinct elements $K_1, K_2 \in \mathcal{T}$, we let $n_F$ denote the unit normal on $F$ pointing from $K_1$ to $K_2$, and we define on $F$ the jump operator $[\cdot]$ and average operator $\{\cdot\}$ in the usual way:

$$[\phi]|_F := \phi|_{K_1} - \phi|_{K_2}, \quad (2.5)$$

$$\{\phi\}|_F := \frac{1}{2}(\phi|_{K_1} + \phi|_{K_2}), \quad (2.6)$$

where $\phi$ is any function defined piecewise on $K_1 \cup K_2$. The ambiguity in the ordering of $K_1, K_2$ will be unimportant. If $F \in \mathcal{F}_b$ is a boundary face belonging to $K \in \mathcal{T}$, we let $n_F$ denote the unit normal on $F$ outward to $K$, and we define on $F$ the jump and average operators as

$$[\phi]|_F = \{\phi\}|_F := \phi|_K, \quad (2.7)$$

where $\phi$ is any function defined on $K$.

Finally, the following definition will be used in Appendix A. Let $K \in \mathcal{T}$. Observe that we do not have $\mathcal{F}_K \subset \mathcal{F}_h$, because of how $\mathcal{F}_h$ is defined using equivalence classes. We therefore define $\mathcal{F}_{K,h} := \{(\sigma, K) \mid \sigma \in \mathcal{F}_K\}$ so that $\mathcal{F}_{K,h} \subset \mathcal{F}_h$ holds. The sets $\mathcal{F}_K$ and $\mathcal{F}_{K,h}$ intuitively encode the same information; they both contain exactly $(d+1)$ elements which describe the faces of $K$. The only difference is that $\mathcal{F}_{K,h}$ is defined using equivalence classes in $\mathcal{F}_h$.

2.3 Discrete finite element spaces and norms

We introduce the following low-order discontinuous finite element spaces on $\Omega$:

$$X_h := \{v_h \in L^2(\Omega)^d : v_h|_K \in [P_1(K)]^d \forall K \in \mathcal{T}\},$$

$$Q_h := \{q_h \in L^2_0(\Omega) : q_h|_K \in P_0(K) \forall K \in \mathcal{T}\},$$

where $P_k(D)$ is the space of polynomials with degree at most $k$ on $D$. Also, let $P_1(\mathcal{F}_h) := \prod_{F \in \mathcal{F}_h} P_1(F)$. We introduce the low-order discontinuous facet finite element spaces

$$\bar{X}_h := \{\bar{v}_h \in [P_1(\mathcal{F}_h)]^d : \bar{v}_h|_F = 0 \forall F \in \mathcal{F}_b\},$$

$$\bar{Q}_h := P_1(\mathcal{F}_h).$$

Notice that $\bar{X}_h$ can be viewed as the space of discontinuous piecewise-linear vector functions on $\Gamma_0$ that vanish on $\partial \Omega$. Likewise, $\bar{Q}_h$ can be viewed as the space of discontinuous piecewise-linear scalar functions on $\Gamma_0$, with the caveat that these functions are double-valued on boundary faces shared by two distinct cells.

It will also be convenient to introduce the extended velocity spaces

$$X(h) := X_h + H^{1/2}_0(\Gamma_0)^d, \quad (2.8a)$$

$$\bar{X}(h) := \bar{X}_h + H^{1/2}_0(\Gamma_0)^d, \quad (2.8b)$$

where $H^{1/2}_0(\Gamma_0)^d$ is the trace space of functions in $H^1_0(\Omega)^d$ restricted to $\Gamma_0$. We use boldface notation for function pairs in $X(h) \times \bar{X}(h)$ and $Q_h \times \bar{Q}_h$, i.e.

$$v = (v, \bar{v}) \in X(h) \times \bar{X}(h) \quad \text{and} \quad q_h = (q_h, \bar{q}_h) \in Q_h \times \bar{Q}_h.$$
Throughout this paper $\nabla_h : X(h) \to [L^2(\Omega)]^{d \times d}$ denotes the broken gradient $(\nabla_h v)|_K := \nabla(v|_K)$. On the space $X(h)$ we introduce the discrete $H^1$-norm

$$\|v\|_{d_g}^2 := \|\nabla_h v\|^2 + |v|^2_J,$$

where $|\cdot|^J$ is the following jump semi-norm on $X(h)$:

$$|v|^2_J := \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|v\|_F^2.$$

Similarly, on the product space $X(h) \times \bar{X}(h)$ we introduce the discrete $H^1$-norm

$$\|\|v\|\|_v^2 := \|\nabla_h v\|^2 + |v|^2_F,$$

where $|\cdot|^F$ is the following facet semi-norm on $X(h) \times \bar{X}(h)$:

$$|v|^2_F := \sum_{K \in \mathcal{T}_h} \frac{1}{h_K} \|v - \bar{v}\|_{\partial K}^2.$$

Finally, on the space $Q_h \times \bar{Q}_h$ we introduce the norm

$$\|\|q\|\|_p^2 := \|q_h\|^2 + \|\bar{q}_h\|^2_p,$$

where $\|\cdot\|_p$ is the following norm on $\bar{Q}_h$:

$$\|\bar{q}_h\|_p^2 := \sum_{K \in \mathcal{T}_h} \frac{h_K}{h} \|\bar{q}_h\|_{\partial K}^2.$$

We use $a \lesssim b$ to indicate $a \leq Cb$ where $C$ is a positive constant depending only on $d, \Omega$ and the shape-regularity of $T$. On occasion we will use inequalities of the form $a \leq C(s)b$, where $C(s)$ is a positive constant depending only on $d, \Omega$, shape-regularity of $T$ and $s$, where $s \in [0, 1]$ corresponds to the order of the fractional Sobolev space $H^{1+s}(\Omega)^d$. In these cases, we will use the notation $a \lesssim_s b$.

We conclude this subsection with the observation that

$$|v|^J \lesssim |v|^F,$$

which follows from the triangle inequality. Note that eq. (2.9) implies

$$\|v\|_{d_g} \lesssim \|v\|_v.$$

These inequalities will be used frequently in Section 3.
The hybridized and embedded–hybridized discontinuous Galerkin methods

The lowest-order HDG and EDG–HDG methods analyzed in [32] utilize the following finite element spaces:

\[ X^v_h := \begin{cases} X_h \times X_h & \text{(HDG method),} \\
X_h \times \left( X_h \cap C^0(\Gamma_0)^d \right) & \text{(EDG–HDG method),} \end{cases} \]

\[ Q^p_h := Q_h \times \tilde{Q}_h. \]

The HDG and HDG–EDG methods differ only in their choice of velocity facet space, which is discontinuous for the HDG method and continuous for the EDG–HDG method. The remainder of the analysis is agnostic as to whether the HDG or EDG–HDG method is considered, with the presented analysis holding for both methods.

The discrete formulation of eq. (2.1) seeks \((u_h, p_h) \in X^v_h \times Q^p_h\) such that

\[ \nu a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h) \quad \forall v_h \in X^v_h, \] (2.11a)

\[ b_h(u_h, q_h) = 0 \quad \forall q_h \in Q^p_h, \] (2.11b)

where \(a_h : X^v_h \times X^v_h \to \mathbb{R}\) and \(b_h : X_h \times Q^p_h \to \mathbb{R}\) are the bilinear forms

\[ a_h(v, w) := \sum_{K \in T} \int_K \nabla v : \nabla w \, dx + \sum_{K \in T} \int_{\partial K} (v - \bar{v}) \cdot (w - \bar{w}) \, ds 
- \sum_{K \in T} \int_{\partial K} \left[ (v - \bar{v}) \cdot \frac{\partial w}{\partial n_K} + (w - \bar{w}) \cdot \frac{\partial v}{\partial n_K} \right] ds, \]

(2.12)

\[ b_h(v, q) := - \sum_{K \in T} \int_K (\nabla \cdot v) q \, dx + \sum_{K \in T} \int_{\partial K} (v \cdot n_K) \bar{q} \, ds, \]

(2.13)

and \(\alpha > 0\) is a penalty parameter. It was shown in [30, Lemma 4.2] that for sufficiently large \(\alpha\) the following coercivity result holds:

\[ \|v_h\|^2 \lesssim a_h(v_h, v_h) \quad \forall v_h \in X^v_h. \] (2.14)

Let us also mention that inf-sup stability of \(b_h\) was established in [32, Lemma 8]:

\[ \|q_h\|_p \lesssim \sup_{v_h \in X^v_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_v} \quad \forall q_h \in Q^p_h. \] (2.15)

A consequence of the stability properties in eqs. (2.14) and (2.15) is that the discrete problem eq. (2.11) is well-posed, see e.g. [4, Chapter 4]. Furthermore, let us introduce the discrete reduced space

\[ V_h^s := \{ v_h \in X^v_h : b_h(v_h, q_h) = 0 \ \forall q_h \in Q^p_h \} \]

\[ = \{ v_h \in X^v_h : v_h \in X^{\text{BDM}}_h \text{ and } \nabla \cdot v_h = 0 \}, \]

where \(X^{\text{BDM}}_h\) is the lowest-order Brezzi–Douglas–Marini (BDM) space [4],

\[ X^{\text{BDM}}_h = \{ v_h \in X_h : [v_h]_F \cdot n_F = 0 \ \forall F \in \mathcal{F}_h \}. \]
Inf-sup stability of $b_h$ implies the best approximation result [5, Section 12.5]
\[
\inf_{\tilde{v}_h \in V_h} \| u - \tilde{v}_h \|_v \lesssim \inf_{v_h \in X_h} \| u - v_h \|_v,
\] (2.16)
where $u \in H^1_0(\Omega)^d$ is the velocity solution to eq. (2.3) and $\tilde{v}_h = (u, u) \in X(h) \times \tilde{X}(h)$. Also, the discrete velocity solution $u_h \in V_h$ to eq. (2.11) satisfies the discrete reduced problem
\[
\nu a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h^0.
\] (2.17)
However, for $v_h \in V_h^0$ it holds that $v_h \in L_2^0(\Omega)$ (recall that $L_2^0(\Omega)$ is defined in eq. (2.3)) and therefore $(f, v_h) = (\bar{P} f, v_h)$. Hence the reduced problem eq. (2.17) can equivalently be written as
\[
\nu a_h(u_h, v_h) = (\bar{P} f, v_h) \quad \forall v_h \in V_h^0.
\] (2.18)
Analogously to eq. (2.4), the presence of $\bar{P} f$ in eq. (2.18) will play an important role in the pressure-robustness of our error estimates in Section 3.

2.5 Enrichment and interpolation operators

Our minimal regularity error analysis of the lowest-order HDG and EDG–HDG methods will utilize an enrichment operator $E_h$ with the following properties.

**Lemma 2.1 (Enrichment operator)** There exists a linear operator $E_h : X_h^{\text{BDM}} \to H^1_0(\Omega)^d$ such that for all $v_h \in X_h^{\text{BDM}}$ we have

(i) $\int_F \| v_h \| ds = \int_F E_h v_h ds$ for all $F \in \mathcal{F}_i$.

(ii) $\nabla \cdot v_h = \nabla \cdot E_h v_h$.

(iii) $\sum_{K \in T} h^{2(k-1)}_K | v_h - E_h v_h |^2_{k, K} \lesssim | v_h |^2_{k}$ for all $k \in \{0, 1\}$.

(iv) $\| \nabla E_h v_h \|_{\text{dg}} \lesssim \| v_h \|_{\text{dg}}$.

An operator satisfying the statements in Lemma 2.1 was constructed in [23]. In [23] the construction is outlined in detail for the two-dimensional case, but sketched only briefly for the three-dimensional case. We present an alternative proof of Lemma 2.1 for the three-dimensional case in Appendix A. Our construction is based on the conforming and divergence-free finite element of [19] and is inspired by [29, Lemma 4.7], in which a similar result is established for Crouzeix–Raviart finite element functions. We mention in passing that our construction in Appendix A can also be adapted to the two-dimensional case by using the two-dimensional finite element of [18].

Let $X_h^2 := \{ v_h \in X_h \cap C^0(\Omega)^d : v_h|_{\partial \Omega} = 0 \}$ be the conforming analogue of $X_h$. Aside from $E_h$, we will also use the following quasi-interpolation operator $I_h$ to deduce optimal rates of convergence for the HDG and EDG–HDG methods.

**Lemma 2.2 (Quasi-interpolation operator)** There exists a linear operator $I_h : H^1_0(\Omega)^d \to X_h^2$ such that for all $s \in [0, 1]$ and $v \in H^1_0(\Omega)^d \cap H^{1+s}(\Omega)^d$ we have
\[
\| \nabla_h (v - I_h v) \| \lesssim_s h^s \| v \|_{1+s}.
\] (2.19)
Proof A proof of Lemma 2.2 can be found in [15, 24], although these works assume that \( \Omega \) is a Lipschitz domain. For the sake of completeness, we now show that the quasi-interpolation operator of [15] still satisfies eq. (2.10) when it is not assumed that \( \Omega \) is Lipschitz. The quasi-interpolation operator of [15] is given by \( I_h = A_h \circ \Pi_h \), where \( A_h : X_h \to X_h^2 \) is the lowest-order averaging operator introduced in [21, Theorem 2.2] and \( \Pi_h : H^1_0(\Omega)^d \to X_h \) the \( L^2 \)-orthogonal projector onto \( X_h \). Because we are assuming that \( \partial \Omega \) has codimension one, every boundary vertex of the mesh is contained in some boundary face of the mesh. As a result, by the arguments used in [21, Theorem 2.2]:

\[
\| \nabla_h (v_h - A_h v_h) \| \lesssim |v_h|_j. \tag{2.20}
\]

Let \( v \in H^1_0(\Omega)^d \cap H^{1+s}(\Omega)^d \). Using the triangle inequality, eq. (2.20), and a continuous trace inequality [12, Lemma 1.49], we have

\[
\| \nabla_h (v - I_h v) \| \lesssim \| \nabla_h (v - \Pi_h v) \| + \| \nabla_h (\Pi_h v - A_h \Pi_h v) \| \\
\lesssim \| \nabla_h (v - \Pi_h v) \| + \| \Pi_h v \|_j \\
= \| \nabla_h (v - \Pi_h v) \| + \| v - \Pi_h v \|_j \\
\leq \left( \sum_{K \in T} h_K^2 \| v - \Pi_h v \|_K^2 + \| v - \Pi_h v \|_{L^2(K)}^2 \right)^{1/2}. \tag{2.21}
\]

Finally, eq. (2.10) follows from eq. (2.21) and standard approximation properties of the \( L^2 \)-orthogonal projector \( \Pi_h \) (see e.g. [12, Section 1.4.4]). \( \Box \)

3 Pressure-robust error analysis under minimal regularity

In [22], optimal and pressure-robust error estimates for the HDG and EDG–HDG methods were derived assuming \( u \in H^1_0(\Omega)^d \cap H^{1+s}(\Omega)^d \). In this section, we carry out error analysis for the more general case of \( u \in H^1_0(\Omega)^d \cap H^{1+s}(\Omega)^d \) for \( s \in [0, 1] \).

3.1 Velocity error estimates

Thus far we have considered \( a_h \) on the finite element space \( X_h^2 \) (see eq. (2.10a)). The first step in our analysis is to extend \( a_h \) to the larger space \( X^+(h) := X(h) \times X(h) \) (see eq. (2.23)). The main difficulty is that for \( v \in X(h) \) and \( K \in T \) we have only \( \nabla v \in (L^2(K))^d \) and therefore \( \nabla v \) does not admit a well-defined trace on \( \partial K \). To deal with this problem, let \( \pi_K : (L^2(K))^d \to [P_0(K)]^d \) denote the \( L^2 \)-orthogonal projector onto \([P_0(K)]^d\). Hence \((G - \pi_K G, H)_K = 0\) for all \( G \in (L^2(K))^d \) and \( H \in [P_0(K)]^d \). For any \( v, w \in X^+(h) \) we now define

\[
a_h(v, w) := \sum_{K \in T} \int_K \nabla v : \nabla w \, dx + \sum_{K \in T} \frac{\alpha}{h_K} \int_{\partial K} (v - \bar{v}) \cdot (w - \bar{w}) \, ds \\
- \sum_{K \in T} \int_{\partial K} \left[ (v - \bar{v}) \cdot ([\pi_K \nabla w] n_K) + (w - \bar{w}) \cdot ([\pi_K \nabla v] n_K) \right] ds. \tag{3.1}
\]
We will use this bilinear form in the following analysis. Observe that eq. (3.1) reduces to the previous definition of $a_h$ (see eq. (2.12)) for $v, w \in X^v_h$. Moreover, the following boundedness result holds on the extended space $X^v(h)$.

**Lemma 3.1 (Boundedness of $a_h$)** For all $v, w \in X^v(h)$ there holds

$$a_h(v, w) \lesssim \|v\|_v \|w\|_v.$$  

**Proof** By definition we have that

$$a_h(v, w) = \sum_{K \in T} \int_K \nabla v : \nabla w \, dx + \sum_{K \in T} h_K \int_{\partial K} (v - \bar{v}) \cdot (w - \bar{w}) \, ds$$

$$+ \sum_{K \in T} -\int_{\partial K} (v - \bar{v}) \cdot (\{\pi_K \nabla w\} n_K) \, ds$$

$$+ \sum_{K \in T} -\int_{\partial K} (w - \bar{w}) \cdot (\{\pi_K \nabla v\} n_K) \, ds.$$  

An application of the Cauchy–Schwarz inequality yields $|I_1| + |I_2| \lesssim \|v\|_v \|w\|_v$. To bound $|I_3|$ we first apply Cauchy–Schwarz to get

$$|I_3| \leq |v|_F \left( \sum_{K \in T} h_K \|\pi_K \nabla w\|_{\partial K}^2 \right)^{1/2}$$

$$\lesssim |v|_F \left( \sum_{K \in T} \|\pi_K \nabla w\|_{K}^2 \right)^{1/2}$$

$$\leq |v|_F \left( \sum_{K \in T} \|\nabla w\|_{K}^2 \right)^{1/2}$$

$$\leq \|v\|_v \|w\|_v.$$  

(3.2)

For the second inequality in eq. (3.2) we used a discrete trace inequality, and for the third inequality we used stability of $\pi_K$. Similar reasoning shows that $|I_4| \lesssim \|v\|_v \|w\|_v$. This completes the proof.  

The next ingredient in our analysis is to establish an upper bound on the consistency error for the velocity solution of the method in eq. (2.11).

**Lemma 3.2 (Consistency error for $a_h$)** Let $u \in H^1_0(\Omega)^d$ be the velocity solution of eq. (2.1), let $u = (u, u)$, and let $u_h \in X^v_h$ be the discrete velocity solution of eq. (2.11). Then for all $v_h \in X^v_h$ and $w_h \in V^v_h$ it holds that

$$a_h(u - u_h, w_h) \lesssim \left\{ \|u - v_h\|_v + |v_h|_G + \frac{1}{\nu} \text{osc}(Pf) \right\} |w_h|_F,$$

(3.3)

where we have introduced the notation

$$\text{osc}(g) := \sum_{K \in T} h_K^2 \|g\|_{K}^2 \quad \forall g \in L^2(\Omega)^d,$$

(3.4)

$$|t_h|_G^2 := \sum_{F \in \mathcal{F}_h} h_F \|\nabla t_h \cdot n_F\|_{F}^2 \quad \forall t_h \in X_h.$$  

(3.5)
Proof Let \( v_h \in X_h^k \) and \( w_h \in V_h^k \). We set
\[
z_h = w_h - (E_h w_h, E_h w_h) = (w_h - E_h w_h, \bar{w}_h - E_h w_h).
\]
Then
\[
a_h(u - u_h, w_h) = \left[ a_h(u, (E_h w_h, E_h w_h)) - a_h(u_h, w_h) \right]
+ a_h(u - v_h, z_h) + a_h(v_h, z_h)
= \left[ a(u, E_h w_h) - a_h(u_h, w_h) \right]
+ a_h(u - v_h, z_h) + a_h(v_h, z_h).
\]
(3.6)

We first bound \( I_1 \). Since \( w_h \in V_h^k \) we have \( w_h \in X_h^{\text{RDM}} \) with \( \nabla \cdot w_h = 0 \). Thus \( E_h w_h \in V \) by Item \( \text{iii} \) of Lemma \( 2.1 \). Using the reduced problems eq. (2.4) and eq. (2.18), the Cauchy–Schwarz inequality, and Item \( \text{iii} \) of Lemma \( 2.1 \) with \( k = 0 \),
\[
I_1 = \frac{1}{\nu} \langle Pf, E_h w_h - w_h \rangle
\leq \frac{1}{\nu} \text{osc}(Pf) \overline{\sum_{K \in T} h_K^2 \| E_h w_h - w_h \|_K^2}^{1/2}
\leq \frac{1}{\nu} \text{osc}(Pf) |w_h|_f
\leq \frac{1}{\nu} \text{osc}(Pf) |w_h|_F.
\]
(3.7)

We now bound \( I_2 \). Using Item \( \text{iii} \) of Lemma \( 2.1 \) with \( k = 1 \) we have
\[
\| z_h \|^2 = \| \nabla_h (w_h - E_h w_h) \|^2 + |w_h|_F^2 \lesssim |w_h|_F^2
\]
so that \( \| z_h \|_v \lesssim |w_h|_F \). Hence by Lemma \( 3.1 \) we have
\[
I_2 \lesssim \| u - v_h \|_v \| z_h \|_v \lesssim \| u - v_h \|_v |w_h|_F.
\]
(3.8)

To bound \( I_3 \) we use the definition eq. (3.1) of \( a_h \) and integrate by parts element-wise. Using that \( (\nabla^2 v_h)_K = 0 \) as \( v_h \) is piecewise linear, this results in
\[
I_3 = \sum_{K \in T} \frac{\alpha}{h_K} \int_{\partial K} \left( v_h - \bar{v}_h \right) \cdot (z_h - \bar{z}_h) \, ds - \int_{\partial K} (v_h - \bar{v}_h) \cdot ([\pi_K \nabla z_h] n_K) \, ds
+ \sum_{K \in T} \int_{\partial K} \bar{z}_h \cdot ([\nabla v_h] n_K) \, ds.
\]
(3.9)

Using the same arguments from Lemma \( 3.1 \) (namely those used in eq. (3.2)) one sees that
\[
I_{3,1} \lesssim |v_h|_F \| z_h \|_v = |u - v_h|_F \| z_h \|_v \lesssim \| u - v_h \|_v |w_h|_F.
\]
(3.10)
Also, rewriting $I_{3,2}$ in terms of facet integrals, applying Item 3 of Lemma 2.1 and using the Cauchy-Schwarz inequality, we find

$$ I_{3,2} = \sum_{F \in F} \int_F \bar{z}_h \cdot \left( [\nabla v_h] n_F \right) ds $$

$$ = \sum_{F \in F} \int_F \left( \bar{w}_h - E_h w_h \right) \cdot \left( [\nabla v_h] n_F \right) ds $$

$$ = \sum_{F \in F} \int_F \left( \bar{w}_h - \langle w_h \rangle \right) \cdot \left( [\nabla v_h] n_F \right) ds $$

$$ \lesssim |w_h|_{\Gamma} |v_h|_G. \tag{3.11} $$

Using eq. (3.10) and eq. (3.11) in eq. (3.9) we obtain

$$ I_3 \lesssim \left( ||| u - v_h |||_v + |v_h|_G \right) |w_h|_{\Gamma}. \tag{3.12} $$

Finally, using the bounds eqs. (3.7), (3.8) and (3.12) in eq. (3.6) yields the desired result eq. (3.3). \qed

With boundedness and consistency results established for the method eq. (2.11), we can now derive our main error estimate.

**Theorem 3.1 (Velocity error)** Let $u \in H^1(\Omega)^d$ be the velocity solution of eq. (2.1), let $u = (u,u)$, and let $u_h \in X^h$ be the discrete velocity solution of eq. (2.11). Then

$$ ||| u - u_h |||_v \lesssim \inf_{v_h \in X^h} \left[ ||| u - v_h |||_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(P_f). \tag{3.13} $$

**Proof** Let $v_h \in X^h$. Owing to eq. (2.16) we can find $\tilde{v}_h \in V^h$ with $||| u - \tilde{v}_h |||_v \lesssim ||| u - v_h |||_v$. Let $w_h = (u_h - \tilde{v}_h) \in V^h$. Using discrete coercivity eq. (2.14) along with the boundedness and consistency results Lemmas 3.1 to 3.2,

$$ |||w_h|||_v^2 \lesssim a_h(w_h, w_h) $$

$$ = a_h(u_h - u, w_h) + a_h(u - \tilde{v}_h, w_h) $$

$$ \lesssim \left[ ||| u - \tilde{v}_h |||_v + |\tilde{v}_h|_G \right] |||w_h|||_v. \tag{3.14} $$

Therefore, dividing eq. (3.14) by $|||w_h|||_v$ we arrive at

$$ ||| u - u_h |||_v \lesssim \left[ ||| u - \tilde{v}_h |||_v + |\tilde{v}_h|_G \right] + \frac{1}{\nu} \text{osc}(P_f). \tag{3.15} $$

Using the triangle inequality and eq. (3.15) we obtain

$$ ||| u - u_h |||_v \lesssim \left[ ||| u - \tilde{v}_h |||_v + |\tilde{v}_h|_G \right] + \frac{1}{\nu} \text{osc}(P_f). \tag{3.16} $$
Also, by the triangle inequality and a discrete trace inequality we have

$$
|\tilde{v}_h|_G \leq |\tilde{v}_h - v_h|_G + |v_h|_G
\leq \|\nabla_h (\tilde{v}_h - v_h)\|_v + |v_h|_G
\leq \|\tilde{v}_h - v_h\|_v + \|\tilde{v}_h - v_h\|_v + |v_h|_G.
$$

(3.17)

Combining eqs. (3.16) to (3.17) and using that

$$
\|\|u - \tilde{v}_h\|_v \leq \|\|u - v_h\|_v + \frac{1}{\nu}\|\text{osc}(\tilde{P} f)\|_v.
$$

The desired result eq. (3.15) follows as $v_h \in X_h$ is arbitrary.

**Remark 3.1 (Pressure-robustness of the data oscillation term)** As discussed in [29, Remark 5.9], the function $\frac{1}{\nu}\tilde{P} f$ is independent of both the pressure $p$ and the viscosity $\nu$. This can be seen by extending the domain of the Helmholtz projector $\tilde{P}$ to $[H^{-1}(\Omega)]^d$, and then utilizing the fact that $-\nu\Delta u + \nabla p = f$ holds in the distributional sense. One finds that

$$
\frac{1}{\nu}\tilde{P} f = \frac{1}{\nu}\tilde{P}(-\nu\Delta u + \nabla p) = \tilde{P}(-\Delta u) \in L^2(\Omega)^d,
$$

(3.18)

since $\tilde{P}(\nabla p) = 0$. We refer the reader to [28, Section 3] for a more detailed discussion of these ideas. A consequence of eq. (3.18) is that the data oscillation term appearing in eq. (3.13) can equivalently be written as $\frac{1}{\nu}\text{osc}(\tilde{P} f) = \text{osc}(\tilde{P}(-\Delta u))$. Because this quantity depends only on the velocity, the error estimate eq. (3.13) is pressure-robust. We emphasize that eq. (3.13) would not be a pressure-robust error estimate if it contained $\frac{1}{\nu}\text{osc}(f)$ instead of $\frac{1}{\nu}\text{osc}(\tilde{P} f)$.

Our next step is to show that the interpolation error term appearing in Theorem 3.1 converges optimally with respect to the mesh size $h$.

**Lemma 3.3 (Interpolation error)** Let $\psi \in H^1_0(\Omega)^d \cap H^{1+s}(\Omega)^d$ with $s \in [0, 1]$, and set $\psi = (\psi, \psi)$. Then

$$
\inf_{v_h \in X_h} \left[ \|\|\psi - v_h\|_v + |v_h|_G \right] \lesssim_s h^s \|\psi\|_{1+s}.
$$

(3.19)

**Proof** For each $F \in \mathcal{F}_i$ we introduce the patch $\omega_F$ of elements sharing $F$,

$$
\omega_F := \bigcup \{K \in \mathcal{T} : F \text{ is a face of } K\}.
$$
Note that $\omega_F$ is the union of exactly two elements. Now let $v_h \in X_h^v$ and $g_F \in [P_0(\omega_F)]^{d \times d}$ be arbitrary. By a discrete trace inequality and the triangle inequality,

$$
|v_h|_G^2 \leq \sum_{F \in F_h} h_F \|\nabla_h v_h\|_F^2 \\
= \sum_{F \in F_h} h_F \|\nabla_h v_h - g_F\|_F^2 \\
\lesssim \sum_{F \in F_h} \|\nabla_h v_h - g_F\|_{\omega_F}^2 \\
\lesssim \sum_{F \in F_h} \left[\|\nabla_h (\psi - v_h)\|_{\omega_F}^2 + \|\nabla \psi - g_F\|_{\omega_F}^2\right] \\
\lesssim \|\psi - v_h\|_v^2 + \sum_{F \in F_h} \|\nabla \psi - g_F\|_{\omega_F}^2.
$$

(3.20)

Since $v_h \in X_h^v$ and $g_F \in [P_0(\omega_F)]^{d \times d}$ are arbitrary, eq. (3.20) yields that

$$
\inf_{v_h \in X_h^v} \left[\|\psi - v_h\|_v + |v_h|_G\right] \lesssim \left(\sum_{F \in F_h} \inf_{g_F \in [P_0(\omega_F)]^{d \times d}} \|\nabla \psi - g_F\|_{\omega_F}^2\right)^{1/2} \\
+ \inf_{v_h \in X_h^v} \|\psi - v_h\|_v.
$$

(3.21)

A bound for $I_1$ follows from the fractional order Bramble–Hilbert lemma \[13\] Theorem 6.1] applied to the patches $\omega_F$:

$$
I_1 \lesssim_s \left(\sum_{F \in F_h} h_F^{2s} \|\nabla \psi\|_{s,\omega_F}^2\right)^{1/2} \lesssim h^s \|\psi\|_{1+s}.
$$

(3.22)

To bound $I_2$ we take $v_h = (I_h \psi, I_h \psi) \in X_h^v$ where $I_h$ is the quasi-interpolation operator introduced in Lemma 2.2. We find that

$$
I_2 \leq \|\psi - v_h\|_v = \|\nabla_h (\psi - I_h \psi)\| \lesssim_s h^s \|\psi\|_{1+s}.
$$

(3.23)

Using the bounds eqs. (3.22) to (3.23) in eq. (3.21) yields the desired result. \(\Box\)

An immediate consequence of Theorem 3.1 and Lemma 3.3 is the following error estimate, which is pressure-robust and optimal in the discrete energy norm.

Corollary 3.1 (Pressure-robust error estimate) In addition to the assumptions of Theorem 3.1, assume that $u \in H^{1+s}(\Omega)^d$ with $s \in [0,1]$. Then

$$
\|u - u_h\|_v \lesssim_s h^s \|u\|_{1+s} + \frac{1}{\nu} \text{osc}(Pf).
$$

Also, owing to eq. (3.4), the data oscillation term can be estimated as

$$
\frac{1}{\nu} \text{osc}(Pf) \leq h \left\|\frac{1}{\nu} Pf\right\|.
$$
Remark 3.2 (Convergence under $H^1$-regularity) In the case of $s = 0$, where only $H^1$-regularity of $u$ is assumed, Corollary 3.1 does not predict that $\|u - u_h\|_v \to 0$ as $h \to 0$. This can still be proven, however, using Theorem 3.1 and a density argument. Indeed, let $\epsilon > 0$. By definition, $H^1_0(\Omega)^d$ is the closure of $C_\infty^0(\Omega)^d$ under the $H^1$-norm. As $u \in H^1_0(\Omega)^d$ we can therefore find $\phi \in C_\infty^0(\Omega)^d$ with $|u - \phi|_1 < \epsilon$. Setting $\phi = (\phi, \phi)$, the triangle inequality and Lemma 3.3 then yield

$$\inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] \leq \|u - \phi\|_v + \inf_{v_h \in X_h} \left[ \|\phi - v_h\|_v + |v_h|_G \right]$$

$$= |u - \phi|_1 + \inf_{v_h \in X_h} \left[ \|\phi - v_h\|_v + |v_h|_G \right] \leq \epsilon + h\|\phi\|_2 \leq 2\epsilon,$$

where the last inequality in eq. (3.24) holds for $h$ sufficiently small. But $\epsilon > 0$ is arbitrary, and therefore eq. (3.24) implies that

$$\lim_{h \to 0} \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] \right\} = 0. \quad (3.25)$$

Using eq. (3.25) in Theorem 3.1 we see that $\|u - u_h\|_v \to 0$ as $h \to 0$.

We now investigate convergence of the velocity in the $L^2$-norm, by means of the Aubin–Nitsche trick. In order to proceed we assume the domain $\Omega$ is such that the following regularity holds (see e.g. [11]).

**Assumption 1** (Regularity of the reduced Stokes problem) Let $s_0 \in [0, 1]$ be fixed. We assume that for all $g \in L^2(\Omega)^d$ there holds $\phi_g \in H^{1+s_0}(\Omega)^d$ and

$$\|\phi_g\|_{1+s_0} \lesssim s_0 \|g\|$$

where $\phi_g \in V$ is the solution to the reduced Stokes problem

$$a(\phi_g, v) = (g, v) \quad \forall v \in V.$$

**Theorem 3.2** (Velocity error in the $L^2$-norm) In addition to the assumptions of Corollary 3.1 and under Assumption 1 we have

$$\|u - u_h\|_{s_0} \lesssim s_0 h^{s_0} \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(Pf) \right\}$$

$$\lesssim s_0 h^{s_0} \|u\|_{1+s_0} + h^{1+s_0} \left\| \frac{1}{\nu} Pf \right\|. \quad (3.26a)$$

**Proof** Let $\phi \in V, \phi_h \in V^u_h$ solve the reduced problems

$$a(\phi, v) = (u - u_h, v) \quad \forall v \in V, \quad (3.27a)$$

$$a_h(\phi_h, v_h) = (u - u_h, v_h) \quad \forall v_h \in V^u_h. \quad (3.27b)$$
Set $\phi = (\phi, \phi)$. By Assumption 1 we have $\phi \in H^{1+s_0}(\Omega)^d$ and $\|\phi\|_{1+s_0} \lesssim s_0 \|u - u_h\|$. Applying Corollary 3.1 to the reduced problems eq. (3.27) (for which the source term is $u - u_h$ and the viscosity is one), we find that

$$
\|\phi - \phi_h\|_v \lesssim s_0 h^{s_0} \|\phi\|_{1+s_0} + \text{osc}(P(u - u_h))
\leq h^{s_0} \|\phi\|_{1+s_0} + h \|u - u_h\|
\lesssim s_0 h^{s_0} \|u - u_h\|.
$$

(3.28)

Using eqs. (3.27a) to (3.27b) and some algebraic manipulations, we have

$$
\|u - u_h\|^2 = (u - u_h, u) - (u - u_h, u_h)
= a(\phi, u) - a_h(\phi_h, u_h)
= a_h(u - u_h, \phi - \phi_h) + a_h(\phi - \phi_h, u_h)
\leq h \|u - u_h\| I_1 + h \|u - u_h\| I_2 + h \|u - u_h\| I_3.
$$

(3.29)

To bound $I_1$ we use Lemma 3.1 and eq. (3.28):

$$
I_1 \leq \|u - u_h\| \|\phi - \phi_h\|_v \lesssim s_0 h^{s_0} \|u - u_h\| \|u - u_h\|_v.
$$

(3.30)

To bound $I_2$ we use Lemma 3.2, Lemma 3.3 and Assumption 1. This yields

$$
I_2 \lesssim \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \text{osc}(P(u - u_h)) \right\} \|u_h\|_F
\lesssim s_0 \left\{ h^{s_0} \|\phi\|_{1+s_0} + \text{osc}(P(u - u_h)) \right\} \|u_h\|_F
\lesssim s_0 h^{s_0} \|u - u_h\| \|u - u_h\|_v.
$$

(3.31)

To bound $I_3$ we again use Lemma 3.2 along with eq. (3.28). We find

$$
I_3 \lesssim \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(Pf) \right\} \|\phi_h\|_F
\leq \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(Pf) \right\} \|\phi - \phi_h\|_v
\lesssim s_0 h^{s_0} \|u - u_h\| \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(Pf) \right\}.
$$

(3.32)

Using the bounds eqs. (3.30) to (3.32) in eq. (3.29), and using Theorem 3.1 to bound $\|u - u_h\|_v$, we obtain

$$
\|u - u_h\|^2 \lesssim s_0 h^{s_0} \|u - u_h\| \left\{ \inf_{v_h \in X_h} \left[ \|u - v_h\|_v + |v_h|_G \right] + \frac{1}{\nu} \text{osc}(Pf) \right\}.
$$

(3.33)

Dividing eq. (3.33) by $\|u - u_h\|$ we obtain eq. (3.28a). Finally, eq. (3.28b) follows from eq. (3.28a) and Lemma 3.3. \qed
3.2 Pressure error estimate

Let \((u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)\) be the solution of eq. (2.1) and \((u_h, p_h) \in X_h^p \times Q_h^p\) the solution of eq. (2.11). Let \(\pi_h : L^2(\Omega) \to Q_h\) be the \(L^2\)-orthogonal projector onto \(Q_h\). Note that \((\pi_h p - p_h) \in Q_h\) and therefore \((p - \pi_h p, \pi_h p - p_h) = 0\). Hence by the Pythagorean theorem, the pressure error can be decomposed as

\[
\|p - p_h\|^2 = \|p - \pi_h p\|^2 + \|\pi_h p - p_h\|^2.
\]

The term \(\|p - \pi_h p\|\) is the best approximation error of \(p\) by functions in the discrete space \(Q_h\) under the \(L^2\)-norm, and is unavoidably pressure-dependent. However, the following theorem shows that the second term \(\|\pi_h p - p_h\|\) can be bounded above by an error that is dependent on the velocity only.

**Theorem 3.3 (Pressure error)** Let \((u, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)\) solve eq. (2.1) and set \(u = (u, u)\). Let \((u_h, p_h) \in X_h^p \times Q_h^p\) solve eq. (2.11). Then

\[
\|\pi_h p - p_h\| \lesssim \left\{ \nu \inf_{w_h \in X_h^p} \left( \|u - w_h\|_v + \|v_h\|_G \right) + \text{osc}(\mathcal{P} f) \right\}.
\]

**Proof** Set \(r_h := (\pi_h p - p_h) \in Q_h\). We utilize the auxiliary inf-sup condition established in [32] Lemma 5], which tells us that

\[
|r_h| \lesssim \sup_{w_h \in X_h^\text{BDM} \times (X_h \cap C^0(I_h)^d)} \frac{(r_h, \nabla \cdot w_h)}{\|w_h\|_v}.
\]

Consider \(w_h \in X_h^\text{BDM} \times (X_h \cap C^0(I_h)^d)\). Then \(\|w_h\|_F \cdot n_F = 0\) for all \(F \in \mathcal{F}_h\) so that

\[
-(p_h, \nabla \cdot w_h) = b_h(w_h, p_h) = (f, w_h) - \nu a_h(u_h, w_h).
\]

On the other hand, since \(\nabla \cdot w_h = \nabla \cdot E_h w_h \in Q_h\) we have

\[
(\pi_h p, \nabla \cdot w_h) = (p, \nabla \cdot E_h w_h)
= -b(E_h w_h, p)
= -(f, E_h w_h) + \nu a(u, E_h w_h)
= -(f, E_h w_h) + \nu a_h(u, (E_h w_h, E_h w_h)).
\]

Set \(z_h = w_h - (E_h w_h, E_h w_h)\). Combining eq. (3.37) and eq. (3.38) we obtain

\[
(r_h, \nabla \cdot w_h) = (f, w_h - E_h w_h)
- \nu a_h(u_h, w_h) + \nu a_h(u, (E_h w_h, E_h w_h))
= (f, w_h - E_h w_h)
\]

\[
+ \nu \underbrace{a_h(u - u_h, (E_h w_h, E_h w_h))}_{I_1}
\]

\[
- \nu \underbrace{a_h(u_h, z_h)}_{I_2}.
\]

Since \(w_h \in X_h^\text{BDM}\) and \(\nabla \cdot (w_h - E_h w_h) = 0\) we have that \((w_h - E_h w_h) \in L^2(\Omega)\) (recall eq. 2.3). As a result, \(I_1 = (\mathcal{P} f, w_h - E_h w_h)\). Applying the Cauchy–Schwarz inequality and Item 11 of Lemma 2.1 with \(k = 0\) therefore yields

\[
|I_1| \lesssim \text{osc}(\mathcal{P} f) \|w_h\|_v \lesssim \text{osc}(\mathcal{P} f) \|w_h\|_v.
\]

(3.40)
A bound for $|I_2|$ follows from Lemma 3.1 and Item iv of Lemma 2.1:

$$|I_2| \lesssim ||u - u_h||_v ||(E_h w_h, E_h w_h)||_v$$
$$= ||u - u_h||_v ||E_h w_h||_{1d}$$
$$\lesssim ||u - u_h||_v ||w_h||_v. \quad (3.41)$$

Also, the same arguments used in Lemma 3.2 show that

$$|I_3| \lesssim \left[ ||u - u_h||_v + |u_h|_G \right] ||w_h||_v. \quad (3.42)$$

But for any $v_h \in X_h^v$, the triangle inequality and a discrete trace inequality yields

$$|u_h|_G \leq |u_h - v_h|_G + |v_h|_G$$
$$\lesssim \| \nabla_h (u_h - v_h) \| + |v_h|_G$$
$$\leq ||u_h - v_h||_v + |v_h|_G$$
$$\leq ||u - u_h||_v + \left[ ||u - v_h||_v + |v_h|_G \right]. \quad (3.43)$$

Combining eq. (3.42) and eq. (3.43) gives

$$|I_3| \lesssim \left\{ ||u - u_h||_v + \inf_{v_h \in X_h^v} ||u - v_h||_v + |v_h|_G \right\} ||w_h||_v. \quad (3.44)$$

Inserting the bounds eqs. (3.40), (3.41) and (3.44) into eq. (3.39), and using Theorem 3.1 to bound $||u - u_h||_v$, we obtain

$$(r_h, \nabla \cdot w_h) \lesssim \left\{ \nu \inf_{v_h \in X_h^v} \left[ ||u - v_h||_v + |v_h|_G \right] + \text{osc}(\mathbb{P}f) \right\} ||w_h||_v. \quad (3.45)$$

Finally, combining eq. (3.45) and the inf-sup condition eq. (3.36) we get

$$\|r_h\| \lesssim \left\{ \nu \inf_{v_h \in X_h^v} \left[ ||u - v_h||_v + |v_h|_G \right] + \text{osc}(\mathbb{P}f) \right\}$$

which is the desired result. \hfill \square

**Corollary 3.2 (Pressure convergence rate)** In addition to the assumptions of Theorem 3.3, assume that $(u, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for some $s \in [0, 1]$. Then

$$\|p - p_h\| \lesssim \|h^s\| \|p\|_{s} + \nu h^s \|u\|_{1+s} + h \|\mathbb{P}f\|. \quad (3.46)$$

**Proof** By standard approximation properties of the $L^2$-orthogonal projector,

$$\|p - \pi_h p\| \lesssim \|h^s\| \|p\|_{s}. \quad (3.47)$$

On the other hand, combining Theorem 3.3 and Lemma 3.3 we find that

$$\|\pi_h p - p_h\| \lesssim \nu h^s \|u\|_{1+s} + h \|\mathbb{P}f\|. \quad (3.48)$$

Using the bounds eq. (3.47) and eq. (3.48) in the decomposition eq. (3.34) yields the desired result in eq. (3.46). \hfill \square
4 Numerical examples

In this section we support our theoretical findings with numerical examples. Strictly speaking, the examples that we consider are outside of the scope of our theory, because they involve inhomogeneous Dirichlet boundary conditions. Nevertheless, we will see that our numerical observations agree with the theoretical predictions of Section 3.

All numerical examples have been implemented in NGSolve [33]. The penalty parameter is taken as $\alpha = 6k^2$ where $k$ is the polynomial degree of the velocity finite element space. We discuss numerical results only for the EDG–HDG method; our findings for the HDG method are very similar in all cases.

4.1 Convergence under minimal regularity

We consider the Stokes problem on the unit square $\Omega = (0,1)^2$ with $f = 0$ and $\nu = 1$. We impose Dirichlet boundary conditions on the discrete solution by interpolating the exact solution. The exact solution is taken from [36, Example 4] and in polar coordinates is given by

$$u = \frac{3}{2} \sqrt{r} \begin{bmatrix} \cos \left( \frac{\theta}{2} \right) - \cos \left( \frac{3\theta}{2} \right) \\ 3 \sin \left( \frac{\theta}{2} \right) - \sin \left( \frac{3\theta}{2} \right) \end{bmatrix}, \quad p = -6r^{-1/2} \cos \left( \frac{\theta}{2} \right). \quad (4.1)$$

We note that $(u,p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for all $0 \leq s < 1/2$.

The computed velocity and pressure errors for the EDG–HDG method using the lowest-order $P^1 - P^0$ discretization and the $P^2 - P^1$ discretization are shown in Table 4.1. Both discretizations are seen to converge at the same rate. The velocity error in the discrete $H^1$-norm and the pressure error in the $L^2$-norm are observed to converge as roughly $h^{1/2}$. This is consistent with the regularity of the exact solution and the predictions of Corollary 3.1 and Corollary 3.2. Finally, the velocity error in the $L^2$-norm is observed to converge as roughly $h^{3/2}$. Because $\Omega$ is convex and therefore Assumption $\ref{assump:convex}$ holds with $s_0 = 1$ (see e.g. [22]), this observed convergence rate is consistent with Theorem 3.2.

4.2 Pressure-robust velocity approximation

To demonstrate pressure-robustness in the minimal regularity setting, we consider a Stokes problem, taken from [36, Example 3], on the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ and we vary the viscosity $\nu$. Consider

$$\psi(\theta) = \frac{1}{1+\lambda} \sin((1+\lambda)\theta) \cos(\lambda \omega) - \cos((1+\lambda)\theta)$$

$$- \frac{1}{1-\lambda} \sin((1-\lambda)\theta) \cos(\lambda \omega) + \cos((1-\lambda)\theta),$$
Table 4.1  Computed errors for the minimal regularity test case of Section 4.1 using the EDG–HDG method with different polynomial orders. In all cases, the discrete velocity solution is divergence-free up to machine precision.

| Degree | Cells | $\|u - u_h\|$ | Rate | $\|u - u_h\|_v$ | Rate | $\|p - p_h\|$ | Rate |
|--------|-------|----------------|------|----------------|------|----------------|------|
| $P^1 - P^0$ | 24 | 7.2e-02 | - | 1.5e+00 | - | 5.8e+00 | - |
| | 96 | 2.2e-02 | 1.7 | 8.1e-01 | 0.9 | 1.2e+00 | 2.3 |
| | 384 | 2.8e-03 | 1.9 | 4.2e-01 | 0.5 | 5.8e-01 | 0.5 |
| | 1536 | 9.8e-04 | 1.5 | 3.0e-01 | 0.5 | 4.1e-01 | 0.5 |
| $P^2 - P^1$ | 24 | 2.8e-02 | - | 8.4e-01 | - | 1.4e+00 | - |
| | 96 | 7.6e-03 | 1.9 | 4.0e-01 | 1.1 | 5.2e-01 | 1.4 |
| | 384 | 2.7e-03 | 1.5 | 2.9e-01 | 0.5 | 3.7e-01 | 0.5 |
| | 1536 | 9.5e-04 | 1.5 | 2.0e-01 | 0.5 | 2.6e-01 | 0.5 |
| | 6144 | 3.4e-04 | 1.5 | 1.4e-01 | 0.5 | 1.8e-01 | 0.5 |

and let $\lambda = 856399/1572864 \approx 0.54$ and $\omega = 3\pi/2$. Our exact solution $(u, p)$ is given in polar coordinates by

$$u = r^\lambda \left[ (1 + \lambda) \sin(\theta) \psi(\theta) + \cos(\theta) \psi'(\theta) \right] - (1 + \lambda) \cos(\theta) \psi(\theta) + \sin(\theta) \psi'(\theta),$$

$$p = \nu p_1 + p_2,$$

where

$$p_1 = r^{\lambda-1} ((1 + \lambda)^2 \psi'(\theta) + \psi'''(\theta))/(1 - \lambda), \quad p_2 = x^3 + y^3.$$  

Note that $-\nabla^2 u + \nabla p_1 = 0$ and therefore $-\nu \nabla^2 u + \nabla p = f$ where $f = \nabla p_2$. Also, there holds $(u, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for all $0 \leq s < \lambda$.

We compare the lowest-order EDG–HDG method to the lowest-order EDG method of [32] (see also [25] on the EDG method). The EDG method, which uses a continuous facet finite element space for both the velocity and pressure, is not pressure-robust [32]. We set the viscosity to be either $\nu = 1$ or $\nu = 10^{-5}$. The computed velocity errors for this example are shown in Figure 4.1.

For the EDG–HDG method, the velocity error is observed to be independent of the viscosity, confirming pressure-robustness. The velocity error for this method converges in the discrete $H^1$-norm as roughly $h^{0.54}$. This is consistent with the regularity of $u$ and Corollary 3.1. Furthermore, according to [5] Section 5, on this domain Assumption 1 holds with $s_0 \approx 0.54$. Therefore, Theorem 3.2 predicts the velocity error in the $L^2$-norm to converge as roughly $(h^{0.54})^2 = h^{1.08}$, which is consistent with the empirical convergence rates displayed in Figure 4.1.

When $\nu = 1$ the velocity error for the EDG method is comparable to that of the EDG–HDG method. However, when $\nu = 10^{-5}$ the velocity error for the EDG method increases substantially, at least in the regime of large $h$. In this regime, we hypothesize that the velocity error for the EDG method is dominated by the pressure best approximation error scaled by the inverse viscosity. Recalling that $p = \nu p_1 + p_2$, we can estimate the pressure best approximation error as

$$\inf_{q_h \in Q_h} ||p - q_h|| \leq \nu \inf_{q_h \in Q_h} ||p_1 - q_h|| + \inf_{q_h \in Q_h} ||p_2 - q_h|| \lesssim s \nu h^s ||p_1||_s + h^s ||p_2||_1,$$

(4.2)
Fig. 4.1  Computed velocity errors for the pressure-robustness test case of Section 4.2. We compare the lowest-order EDG and EDG–HDG methods with $\nu = 1$ and $\nu = 10^{-5}$.

Table 4.2  Computed errors for the cracked domain test case of Section 4.3 using the lowest-order EDG–HDG method. In all cases, the discrete velocity solution is divergence-free up to machine precision.

| Cells  | $\| u - u_h \|_v$ | Rate | $\| u - u_h \|_p$ | Rate | $\| p - p_h \|_v$ | Rate |
|--------|------------------|------|------------------|------|------------------|------|
| 1680   | 2.0e-03          |      | 4.5e-01          |      | 5.8e-01          |      |
| 6720   | 1.0e-03          | 1.0  | 3.2e-01          | 0.5  | 3.6e-01          | 0.7  |
| 26880  | 5.0e-04          | 1.0  | 3.3e-01          | 0.5  | 2.4e-01          | 0.6  |
| 107520 | 2.5e-04          | 1.0  | 1.6e-01          | 0.5  | 1.6e-01          | 0.6  |
| 430080 | 1.2e-04          | 1.0  | 1.1e-01          | 0.5  | 1.1e-01          | 0.5  |

for any $0 \leq s < \lambda$. For $h$ sufficiently large eq. (4.2) converges pre-asymptotically at a rate of $h^3$, while the asymptotic convergence rate of eq. (4.2) is $h^\alpha$. This behavior appears to be reflected in Figure 4.1 where for $\nu = 10^{-5}$ the velocity error of the EDG method pre-asymptotically converges at a faster rate than the EDG–HDG method.

4.3 Domain with a crack

We consider the Stokes problem on $\Omega = (-1/10,1/10)^2 \setminus ([0,1/10] \times \{0\})$ with $f = 0$ and $\nu = 1$. Notice that $\Omega$ has a crack along the positive $x$-axis. We use the
same exact solution from eq. (4.1). The computed velocity and pressure errors for the lowest-order EDG-HDG method are shown in Table 4.2. The velocity error in the discrete $H^1$-norm and the pressure error in the $L^2$-norm eventually both converge as roughly $h^{1/2}$. This is consistent with the regularity of the exact solution and the predictions of Corollary 3.1 and Corollary 3.2. Furthermore, according to [6, Section 5], on this domain Assumption 1 holds for any $s_0 < 1/2$. Therefore, Theorem 3.2 predicts the velocity error in the $L^2$-norm to converge as roughly $h^1$, which is consistent with the empirical convergence rate seen in Table 4.2.

5 Conclusions

We have analyzed two lowest-order hybridizable DG methods for the Stokes problem, while assuming only $H^{1+s}$-regularity of the exact velocity solution for any $s \in [0, 1]$. A salient feature of our analysis is that it allows for the case of a domain with cracks. The key ingredient in our analysis is a suitable upper bound on the consistency error of the hybridizable DG methods, which we have derived by means of a divergence-preserving enrichment operator. Our resultant error estimates for the velocity are pressure-robust and optimal in the discrete energy norm. We also obtained an error bound for the pressure that is dependent on the velocity only. Our theoretical findings are supported by various numerical examples.

Statements and Declarations

SR gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada through the Discovery Grant program (RGPIN-05606-2015).

References

1. Arnold, D.N., Brezzi, F., Fortin, M.: A stable finite element for the Stokes equations. Calcolo 21(4), 337–344 (1984). https://doi.org/10.1007/BF02576171
2. Badia, S., Codina, R., Gudi, T., Guzmán, J.: Error analysis of discontinuous Galerkin methods for the Stokes problem under minimal regularity. IMA J. Numer. Anal. 34(2), 800–819 (2013). https://doi.org/10.1093/imanum/drt022
3. Bernardi, C., Raugel, G.: Analysis of some finite elements for the Stokes problem. Math. Comput. 44(169), 71–79 (1985). https://doi.org/10.2307/2007993
4. Boffi, D., Brezzi, F., Fortin, M.: Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol. 44. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-36519-5
5. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, Texts in Applied Mathematics, vol. 15, 3rd edn. Springer, New York (2008). https://doi.org/10.1007/978-0-387-75934-0
6. Chorfi, N.: Geometric singularities of the Stokes problem. Abstr. Appl. Anal. 2014, Article ID 491326 (2014). http://dx.doi.org/10.1155/2014/491326
7. Cockburn, B., Gopalakrishnan, J., Lazarov, R.: Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. SIAM J. Numer. Anal. 47(2), 1319–1365 (2009). https://doi.org/10.1137/070706616
8. Cockburn, B., Kanschat, G., Schötzau, D.: A note on discontinuous Galerkin divergence-free solutions of the Navier–Stokes equations. J. Sci. Comput. 31(1-2), 61–73 (2007). https://doi.org/10.1007/s10915-006-9107-7
Analysis of EDG–HDG methods for the Stokes problem under minimal regularity

9. Cockburn, B., Sayas, F.-J.: Divergence-conforming HDG methods for Stokes flows. Math. Comput. 83(288), 1571–1598 (2014). https://doi.org/10.1090/S0025-5718-2014-02802-0

10. Dauge, M.: Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988). https://doi.org/10.1007/BFb0086682

11. Dauge, M.: Stationary Stokes and Navier–Stokes systems on two- or three-dimensional domains with corners. Part I. Linearized equations. SIAM J. Math. Anal. 20(1), 74–97 (1989). https://doi.org/10.1137/0520006

12. Di Pietro, D.A., Ern, A.: Mathematical Aspects of Discontinuous Galerkin Methods. Mathématiques et Applications, vol. 69. Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-22980-0

13. Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. Math. Comput. 34(150), 441–463 (1980). https://doi.org/10.1090/S0025-5718-1980-0559195-7

14. Ern, A., Guermond, J.-L.: Theory and Practice of Finite Elements, Applied Mathematical Sciences, vol. 159. Springer, New York (2004). https://doi.org/10.1007/978-1-4757-4355-5

15. Fu, G.: An explicit divergence-free DG method for incompressible flow. Comput. Methods Appl. Mech. Eng. 345, 502–517 (2019). https://doi.org/10.1016/j.cma.2018.11.012

16. Guzmán, J., Neilan, M.: Conforming and divergence-free Stokes elements in three dimensions. IMA J. Numer. Anal. 34(4), 1489–1508 (2014). https://doi.org/10.1093/imanum/dru053

17. Kreuzer, C., Verfürth, R., Zanotti, P.: Quasi-optimal and pressure robust discretizations of the Stokes equations by moment- and divergence-preserving operators. Comput. Methods Appl. Math. 21(2), 423–443 (2021). https://doi.org/10.1515/cmam-2020-0023

18. Kreuzer, C., Zanotti, P.: Quasi-optimal and pressure-robust discretizations of the Stokes equations by new augmented Lagrangian formulations. IMA J. Numer. Anal. 40(4), 2553–2583 (2019). https://doi.org/10.1093/imanum/drz044

19. Leube, R.J., Wells, G.N.: Energy stable and momentum conserving hybrid finite element method for the incompressible Navier–Stokes equations. SIAM J. Sci. Comput. 34(2), A889–A913 (2012). http://dx.doi.org/10.1137/100818683

20. Li, M., Mao, S., Zhang, S.: New error estimates of nonconforming mixed finite element methods for the Stokes problem. Math. Methods Appl. Sci. 37(7), 937–951 (2014). https://doi.org/10.1002/mma.2849

21. Linke, A., Merdon, C., Neilan, M.: Pressure-robustness in quasi-optimal a priori estimates for the Stokes problem. Electron. Trans. Numer. Anal. 52, 281–294 (2020). https://doi.org/10.1553/etna_vol52s281

22. Leube, R.J., Wells, G.N.: Energy stable and momentum conserving hybrid finite element method for the incompressible Navier–Stokes equations. SIAM J. Sci. Comput. 34(2), A889–A913 (2012). http://dx.doi.org/10.1137/100818683

23. Linke, A., Merdon, C., Neilan, M., Neumann, F.: Quasi-optimality of a pressure-robust nonconforming finite element method for the Stokes problem. Math. Comput. 87(312), 1543–1560 (2018). https://doi.org/10.1090/mcom/3344
A Enrichment operator

We prove here Lemma 2.11 for the three-dimensional case \( (d = 3) \). We shall use the angled bracket notation \( (\cdot, \cdot)_D \); this is simply used to denote the \( L^2 \)-inner-product on a domain \( D \) with dimension strictly less than \( d = 3 \).

For \( K \in \mathcal{T} \) we recall from Section 2.2 that \( F_{K,h} \subset F_k \) denotes the four faces of \( K \). Also, let \( E_K \) denote the six edges of \( K \) and \( V_K \) the four vertices of \( K \). The collection of all mesh edges is written as \( \mathcal{E}_h := \bigcup_{K \in \mathcal{T}} E_K = \mathcal{E}_b \cup \mathcal{E}_i \) where \( \mathcal{E}_b \) denotes the boundary edges and \( \mathcal{E}_i \) the interior edges. Likewise, the collection of all mesh vertices is written as \( V_h := \bigcup_{K \in \mathcal{T}} V_K = V_b \cup V_i \) where \( V_b \) denotes the boundary vertices and \( V_i \) the interior vertices. For an interior edge \( e \in \mathcal{E}_i \), we define the average of a function \( v \) on \( e \) as

\[
\|v\|_e := \frac{1}{|T_e|} \sum_{K \in T_e} v_K |e|,
\]

where \( T_e := \{ K \in \mathcal{T} : e \in E_K \} \) denotes the collection of elements having \( e \) as an edge and \( v_K := v|K \). On boundary edges \( e \in \mathcal{E}_b \), it will be convenient to define \( \|v\|_e := 0 \). Similarly, for an interior vertex \( a \in V_i \), the average of a function \( v \) on \( a \) is defined as

\[
\|v\|_a := \frac{1}{|T_a|} \sum_{K \in T_a} v_K (a),
\]

where \( T_a := \{ K \in \mathcal{T} : a \in V_K \} \) denotes the collection of elements having \( a \) as a vertex. On boundary vertices \( a \in V_b \) it will be convenient to define \( \|v\|_a := 0 \). Finally, throughout this proof we continue to use the definition eqs. 2.10 to 2.14 of the average operator on faces.

For \( K \in \mathcal{T} \), let \( V(K) \) denote the local three-dimensional Guzmán–Neilan finite element space defined by [19] eq. (3.9). This space has the properties [19] Lemma 3.4

\[
\begin{align*}
\mathcal{P}_1(K)^3 \subset V(K), & \quad V(K) \subset [W^{1,\infty}(K) \cap C^0(\bar{K})]^3, \\
\nabla \cdot V(K) & \subset \mathcal{P}_0(K), \quad V(K)|_{\partial K} \subset [\mathcal{P}_1(\partial K)]^3.
\end{align*}
\]
Moreover, a set of unisolvent degrees of freedom for \( v \in V(K) \) is given by \cite{[3]} Theorem 3.5

\[
\begin{align*}
  v(a) & \quad \forall a \in V_K, \\
  \langle v, w \rangle_e & \quad \forall e \in E_K, w \in [P_1(e)]^3, \\
  \langle v, w \rangle_F & \quad \forall F \in F_{K,h}, w \in [P_0(F)]^3.
\end{align*}
\]

(A.1a) \hspace{1cm} (A.1b) \hspace{1cm} (A.1c)

For \( K \in \mathcal{T} \) we now define the local operator \( E_K : X_h^{\text{BDM}} \rightarrow V(K) \) as follows. For \( v_h \in X_h^{\text{BDM}} \) we require that

\[
\begin{align*}
  (E_K v_h)(a) & = \frac{1}{h} \sum_{a \in V_K} v_h(a), \\
  \langle E_K v_h - \frac{1}{h} v_h, w \rangle_e & = 0, \\
  \langle E_K v_h, w \rangle_F & = 0.
\end{align*}
\]

(A.2a) \hspace{1cm} (A.2b) \hspace{1cm} (A.2c)

The degrees of freedom eq. \( \text{A.1} \) imply that the operator \( E_K \) is well-defined. We then define \( E_h : X_h^{\text{BDM}} \rightarrow H^1_0(\Omega)^d \) by \( E_h v_h = E_K v_h \) for all \( v_h \in X_h^{\text{BDM}} \). Utilizing the inclusion \( V(K)|_{\partial F} \subset [P_3(\partial K)]^3 \) one can show that \( [E_h v_h]|_F = 0 \) for all \( F \in F_h \), and thus \( E_h v_h \in H^1_0(\Omega)^d \) holds. It remains to verify that \( E_h \) satisfies Items i to iv from Lemma 2.1.

That Item ii holds is an immediate consequence of eq. \( \text{A.2a} \). To prove Item iii consider the space \( P_{0,h} = \{ q_h \in L^2(\Omega) : q_h|_K \in P_0(\Omega) \forall K \in \mathcal{T} \} \) of piecewise constant functions. Let \( v_h \in X_h^{\text{BDM}} \) and \( q_h \in P_{0,h} \). Then element-wise integration by parts and eq. \( \text{A.2a} \) shows that

\[
\begin{align*}
\int_\Omega (\nabla \cdot E_h v_h) q_h \, dx &= \sum_{F \in F_h} \int_F \langle E_h v_h \cdot n_F \rangle [q_h] \, ds \\
&= \sum_{F \in F_h} \int_F \langle \frac{1}{h} v_h \cdot n_F \rangle [q_h] \, ds \\
&= \int_\Omega (\nabla \cdot v_h) q_h \, dx,
\end{align*}
\]

where the last equality in eq. \( \text{A.3} \) follows from the fact that \( \langle v_h \rangle|_F \cdot n_F = 0 \) for all \( F \in F_h \).

But Item iv now follows from eq. \( \text{A.3} \) as \( \nabla \cdot (v_h, v_h) \subset P_{0,h} \) and \( q_h \subset P_{0,h} \) is arbitrary.

To prove Item iv fix \( k \in \{0,1\} \) and \( v_h \in X_h^{\text{BDM}} \). Consider \( K \in \mathcal{T} \) and set \( w_K := E_K v_h, w_K = v_h|_K \) and \( z_K = w_K - v_K \). Since \( v_K \in [P_1(K)]^3 \subset V(K) \), there holds \( z_K \in V(K) \).

A scaling argument utilizing the degrees of freedom eq. \( \text{A.1} \) then shows that

\[
\begin{align*}
  h^{2(k-1)} K || z_K ||^2_{V(K)} & \leq \sum_{a \in V_K} h_K || z_K(a) ||^2 + \sum_{e \in E_K} \sup_{\| n_e \| = 1} ||(z_K, n_e) ||^2 \\
  & + \sum_{F \in F_{K,h}} \frac{1}{h_K} \sup_{\| n_F \| = 1} ||(z_K, n_F) ||^2 \\
  & \leq \sum_{a \in V_K} h_K || v_h(a) - v_K(a) ||^2 + \sum_{e \in E_K} \| v_h \|_e - v_K \|_e^2 \\
  & + \frac{1}{h_K} \sum_{F \in F_{K,h}} \| v_h \|_F - v_K \|_F^2.
\end{align*}
\]

(A.4)

Because \( \partial \Omega \) has codimension one, every boundary vertex of the mesh is contained in some boundary face of the mesh, and likewise for boundary edges. As a result, the same arguments...
used in \cite{29} Lemma 4.7] show that

\begin{equation}
I_1 \lesssim \sum_{a \in \mathcal{V}_K} \sum_{F \in \mathcal{F}_a} \frac{1}{h_F} \|v_h\|_F^2,
\end{equation}

\begin{equation}
I_2 \lesssim \sum_{e \in \mathcal{E}_K} \sum_{F \in \mathcal{F}_e} \frac{1}{h_F} \|v_h\|_F^2,
\end{equation}

where $\mathcal{F}_a \subset \mathcal{F}_h$ denotes the collection of all mesh faces having $a$ as a vertex, and $\mathcal{F}_e \subset \mathcal{F}_h$ denotes the collection of all mesh faces having $e$ as an edge. We note that, due to midpoint continuity of Crouzeix–Raviart elements, there is no term analogous to $I_3$ in \cite{29} Lemma 4.7.

Fortunately, it is easy to see that we can bound $I_3$ by means of

\begin{equation}
I_3 \lesssim \sum_{F \in \mathcal{F}_{K,h}} \frac{1}{h_F} \|v_h\|_F^2.
\end{equation}

Using the bounds eqs. (A.5) to (A.7) in eq. (A.4), and summing over $K \in \mathcal{T}$, one obtains

\[
\sum_{K \in \mathcal{T}} h_K^{2(k-1)} |E_h v_h - v_h|_{1,K}^2 \lesssim \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|v_h\|^2_F = |v_h|_{1,F}^2,
\]

so that Item III holds. Lastly, Item IV follows from Item III with $k = 1$ and the triangle inequality:

\[
\|\nabla E_h v_h\| \leq \left( \sum_{K \in \mathcal{T}} |v_h - E_h v_h|_{1,K}^2 \right)^{1/2} + \left( \sum_{K \in \mathcal{T}} |v_h|_{1,K}^2 \right)^{1/2}
\]

\[
\lesssim |v_h|_{1,F} + \left( \sum_{K \in \mathcal{T}} |v_h|_{1,K}^2 \right)^{1/2}
\]

\[
\lesssim \|v_h\|_{dg},
\]

This completes the proof of Lemma 2.1.