ON FOURIER ORTHOGONAL PROJECTIONS
IN THE ROTATION ALGEBRA

S. WALTERS
The University of Northern British Columbia

November 30, 2000

Abstract. Projections are constructed in the rotation algebra that are orthogonal to their Fourier transform and which are fixed under the flip automorphism. Such projections are expected in a construction of an inductive limit structure for the irrational rotation algebra that is invariant under the Fourier transform. (Namely, as two circle algebras of the same dimension, that are swapped by the Fourier transform, plus a few points.) The calculation is based on Rieffel’s construction of the Schwartz space as an equivalence bimodule of rotation algebras as well as on the theory of Theta functions.

§ 1. Introduction

Let $A_\theta$ denote the rotation C*-algebra (where $0 < \theta < 1$). It is generated by unitaries $U, V$ satisfying $VU = \lambda UV$ where $\lambda = e^{2\pi i \theta}$. The Fourier transform is the automorphism $\sigma$ defined by $\sigma(U) = V$, $\sigma(V) = U^{-1}$. (Its square is the usual flip automorphism.) A hitherto open problem of Elliott, is whether $A_\theta$ is the inductive limit of Fourier invariant building blocks consisting of two circle algebras and a few points, and further, whether the fixed point subalgebra $A_\theta^\sigma$ is an AF-algebra. In [8], a clue is given by constructing a model of an inductive limit automorphism of order four that agrees with $\sigma$ on $K_1$ and such that the fixed point subalgebra is approximately finite dimensional. The model predicts the existence of projections that are orthogonal to their Fourier transform and fixed under the flip. The study of such projections would seem crucial in attempting to unravel the structure of the Fourier transform. Therefore, in this paper we show how one can explicitly construct such projections for $0 < \theta < 0.2345$, rational or irrational. (In more recent work [9], the results of this paper are used in an attempt to directly study Elliott’s problem.) The construction here is based on Rieffel’s realization [5] of the Schwartz space as an equivalence bimodule of rotation algebras associated with lattices $D$ in locally compact Abelian groups. The projection to be constructed has the C*-algebra-valued inner product form $\langle h, h \rangle_D$ for an appropriate Schwartz function $h$ that involves Theta functions. Our main result is thus the following.

1991 Mathematics Subject Classification. 46L80, 46L40.
Key words and phrases. C*-algebras, irrational rotation algebras, automorphisms, inductive limits, K-groups, AF-algebras, Theta functions.
Research partly supported by NSERC grant OGP0169928

(TeXFile: orthogonal.tex)

Typeset by Ams-Tex
Theorem 1.1. Let $0 < \theta < 0.2345$. There exists a projection $e$ of trace $\theta$ in the rotation algebra $A_\theta$ such that
\[ e\sigma(e) = 0, \quad \sigma^2(e) = e. \]

In particular, $e + \sigma(e)$ is a Fourier invariant projection of trace $2\theta$.

This has the following immediate consequence.

Corollary 1.2. Let $0 < \theta < 1$. Each number in the set
\[ \{(m^2 + n^2)\theta + k : m, n, k \in \mathbb{Z}\} \cap (0, 0.2345) \]
is the trace of a projection $e$ in $A_\theta$ satisfying $e\sigma(e) = 0$ and $\sigma^2(e) = e$. Further, $e + \sigma(e)$ is a Fourier invariant projection.

Proof. Let $\alpha = (m^2 + n^2)\theta + k$ for some integer $k$ be in the above set. Let $U_1 = \lambda^{mn/2}U^nV^m$ and $V_1 = \sigma(U_1) = \lambda^{mn/2}V^nU^{-m}$. Then
\[ V_1U_1 = \lambda^{mn}V^nU^{-m}U^nV^m = \lambda^{mn/2}\lambda^{n(n-m)}U^{n-m}V^{n+m} = \lambda^{m^2+n^2}U_1V_1, \]
and it is clear that $\sigma(V_1) = U_1^{-1}$. Therefore $\sigma$ leaves invariant the rotation subalgebra $A_\alpha$ generated by $U_1, V_1$, and induces the Fourier transform on it. By the Theorem there is a projection $e$ in $A_\alpha$ such that $e\sigma(e) = 0$, $\sigma^2(e) = e$, and $\tau(e) = \alpha$. From this one gets the $\sigma$-invariant projection $e + \sigma(e)$ of twice the trace. \( \square \)

We recall that for any $\theta$ in $(0, 1)$ Boca [1] (Corollary 1.5) constructed a Fourier invariant projection in $A_\theta$ of trace $\theta$. Presumably, the approach presented here combined with those of Boca (esp. Section 2 of [1]) can be used to show that if $0 < \theta - \frac{p}{q} < \frac{C}{q^2}$, where $C$ is a positive constant akin to Boca’s 0.948 (see his Proposition 2.1), or perhaps a little smaller, and if $p$ is a quadratic residue mod $q$, then there exists a projection of trace $q\theta - p$ satisfying the properties of Theorem 1.1 above. (Similarly, for rational pairs satisfying $0 < \frac{p}{q} - \theta < \frac{C}{q^2}$, where $q - p$ is a quadratic residue mod $q$, one gets such a projection of trace $p - q\theta$.)

Throughout, we adopt the notation $e(t) := e^{2\pi it}$.

The technical part of Theorem 1.1 follows from the following more general result.

Theorem 1.3. Let $A$ be a unital $C^\ast$-algebra and $u, v$ be any pair of unitaries in $A$, where the spectrum of $u$ is the unit circle. Let $\alpha$ and $\beta$ be positive numbers satisfying $\beta^2 = 4(\alpha^2 + 1)$. Then for $\alpha > 0.2568$ the following element
\[ X = \sum_{m, n \in \mathbb{Z}} e(\beta^2mn/2)e^{-\pi\alpha\beta^2n^2/2}e^{-\pi\beta^2m^2/(2\alpha)}\Theta(m, n)u^mv^n \]
is invertible in $A$, where
\[ \Theta(m, n) = \vartheta_2(\frac{\pi}{2}\beta^2n, 2i\alpha)\vartheta_3(i\frac{\pi}{2\alpha}\beta^2m, it_\alpha) + \vartheta_3(\frac{\pi}{2}\beta^2n, 2i\alpha)\vartheta_2(i\frac{\pi}{2\alpha}\beta^2m, it_\alpha) \]
and $t_\alpha = 4\alpha + \frac{2}{\alpha}$.

The Theta functions appearing here are recalled briefly in the next section.

Remark. The commutation relation between $u, v$ is not needed to obtain the invertibility. But it is not clear if in general $X$ is positive, which it in fact is when $u, v$ are the canonical unitary generators of the rotation algebra. Theorem 1.3 holds if $v^n$ is replaced by any sequence $v_n$ of unitaries in $A$ (while $u$ still has spectrum equal to the unit circle).
2. Preliminaries

Let us briefly recall Rieffel’s setup in [5]. Let $M$ be a locally compact Abelian group and let $G = M \times \hat{M}$, where $\hat{M}$ is the dual of $M$. Let $\mathfrak{h}$ denote the Heisenberg cocycle on the group $G = M \times \hat{M}$ given by $\mathfrak{h}((m, s), (m', s')) = \langle m, s' \rangle$, where $\langle m, s' \rangle$ is the canonical pairing $M \times \hat{M} \to \mathbb{T}$. The Heisenberg (unitary) representation $\pi : G \to \mathcal{U}(L^2(M))$ of $G$ is given by

$$\pi(m,s)f(n) = \langle n, s \rangle f(n+m)$$

where $m, n \in M$, $s \in \hat{M}$, and $f \in L^2(M)$. It has the properties

$$\pi_x \pi_y = \mathfrak{h}(x,y)\pi_{x+y} = \mathfrak{h}(x,y)\overline{\mathfrak{h}(y,x)}\pi_y \pi_x, \quad \pi_x^* = \mathfrak{h}(x,x)\pi_{-x}$$

for $x, y \in G$. If $D$ is a given lattice in $G$ (discrete cocompact subgroup), its covolume $|G/D|$ is the Haar measure of a fundamental domain for $D$. Its associated twisted group C*-algebra $C^*(D, \mathfrak{h})$ is the C*-subalgebra of the bounded operators on $L^2(M)$ generated by the unitaries $\pi_x$ for $x \in D$. The complementary lattice of $D$ is

$$D^\perp = \{ y \in G : \mathfrak{h}(x,y)\overline{\mathfrak{h}(y,x)} = 1, \forall x \in D \}.$$ 

The C*-algebra $C^*(D^\perp, \overline{\mathfrak{h}})$ can be viewed as the C*-subalgebra of bounded operators on $L^2(M)$ generated by the unitaries $\pi_y^*$ for $y \in D^\perp$. The Schwartz space on $M$, denoted $\mathcal{S}(M)$, is an equivalence bimodule with $C^*(D, \mathfrak{h})$ acting on the left and $C^*(D^\perp, \overline{\mathfrak{h}})$ acting on the right by

$$af = \int_D a(x)\pi_x(f)dx = |G/D| \sum_{x \in D} a(x)\pi_x(f)$$

$$fb = \int_{D^\perp} b(y)\pi_y^*(f)dy = \sum_{y \in D^\perp} b(y)\pi_y^*(f)$$

where $f \in \mathcal{S}(M)$, $a \in C^*(D, \mathfrak{h})$, $b \in C^*(D^\perp, \overline{\mathfrak{h}})$, and where the mass point measure $(dx)$ on $D$ is $|G/D|$ and on $D^\perp$ it is one. The inner products on $\mathcal{S}(M)$ with values in the algebras $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \overline{\mathfrak{h}})$ are given, respectively, by

$$\langle f, g \rangle_D = |G/D| \sum_{w \in D} \langle f, g \rangle_D(w) \pi_w, \quad \langle f, g \rangle_{D^\perp} = \sum_{z \in D^\perp} \langle f, g \rangle_{D^\perp}(z) \pi_z^*$$

where

$$\langle f, g \rangle_D(w_1, w_2) = \int_M f(x)g(x + w_1) \langle x, w_2 \rangle dx$$

$$\langle f, g \rangle_{D^\perp}(z_1, z_2) = \int_M \overline{f(x)}g(x + z_1)\langle x, z_2 \rangle dx$$

where $(w_1, w_2) \in D$ and $(z_1, z_2) \in D^\perp$. These satisfy the associativity condition

$$\langle f, g \rangle_D h = f \langle g, h \rangle_{D^\perp}.$$
(See [5], pages 266 and 269.) By Rieffel’s Theorem 2.15 in [5], the Schwartz space $S(M)$ is an equivalence $C^*(D, \mathfrak{h})$-$C^*(D^\perp, \mathfrak{h})$ bimodule. The canonical normalized traces are given by

$$\tau_D \left( \sum_{w \in D} a_w \pi_w \right) = a_0, \quad \tau_{D^\perp} \left( \sum_{z \in D^\perp} b_z \pi_z^* \right) = b_0,$$

$(a_w, b_w \in \mathbb{C})$ which satisfy the equation

$$\tau_D(\langle f, g \rangle_D) = |G/D| \tau_{D^\perp}(\langle g, f \rangle_{D^\perp}).$$

From this it follows that if $f$ is a Schwartz function such that $\langle f, f \rangle_{D^\perp} = 1$, then $e = \langle f, f \rangle_D$ is a projection in $C^*(D, \mathfrak{h})$ of trace $|G/D|$. In this case, one has the isomorphism (which may be called the “Morita transform”)

$$\mu : C^*(D^\perp, \mathfrak{h}) \rightarrow eC^*(D, \mathfrak{h})e, \quad \mu(x) = \langle fx, f \rangle_D, \quad \text{and} \quad \mu^{-1}(pyp) = \langle f, yf \rangle_{D^\perp}$$

where $y \in C^*(D, \mathfrak{h})$ and $x \in C^*(D^\perp, \mathfrak{h})$. Projections arising in this manner we shall call generalized Rieffel projections. This seems appropriate since in [3], Elliott and Lin showed that the classical Rieffel projections [4] in the rotation algebra can in fact be written in this manner. (And perhaps also because of Proposition 2.8 of [4].)

In the present paper, we shall take (as in [1]) $M = \mathbb{R}$ so that $\hat{M} = M$ in a natural fashion. This permits one to define the order four automorphism $R : G \rightarrow G$ by $R(u; v) = (-v; u)$, $u, v \in M$. If $D$ is a lattice subgroup of $G$ such that $R(D) = D$ (and hence $R(D^\perp) = D^\perp$), then (as in [1]) there are order four automorphisms $\sigma, \sigma'$ of $C^*(D, \mathfrak{h})$ and $C^*(D^\perp, \mathfrak{h})$, respectively, that satisfy

$$\sigma(\pi_w) = \mathfrak{h}(w, w) \pi_{Rw}, \quad \sigma'(\pi_z^*) = \mathfrak{h}(z, z) \pi_{Rz}^*, \quad (2.1)$$

for $w \in D$, $z \in D^\perp$, and

$$\sigma(\langle f, g \rangle_D) = \langle \hat{f}, \hat{g} \rangle_D, \quad \sigma'(\langle f, g \rangle_{D^\perp}) = \langle \hat{f}, \hat{g} \rangle_{D^\perp}, \quad (2.2)$$

where $\hat{f}$ is the Fourier transform of $f \in S(M)$ given by

$$\hat{f}(s) = \int_M f(x) \overline{\langle x, s \rangle} \, dx = \int_{-\infty}^{\infty} f(x) e(-sx) \, dx$$

for $s \in M$.

The main Theta functions used in this paper are

$$\vartheta_2(z, t) = \sum_{n} e^{\pi i t (n + 1/2)^2} e^{i 2z(n + 1/2)}, \quad \vartheta_3(z, t) = \sum_{n} e^{\pi i t n^2} e^{i 2zn}$$

for $z, t \in \mathbb{C}$ and $\text{Im}(t) > 0$, where all summations range over the integers. The zeros of $\vartheta_3$ are well-known to be given by

$$\left( \frac{\pi}{2} + m\pi + \left( \frac{\pi}{2} + n\pi \right) t, t \right),$$
where \( m, n \in \mathbb{Z} \) and \( \text{Im}(t) > 0 \) are arbitrary. We shall make use of the following identities for any integer \( k \):

\[
\vartheta_3(w + \frac{\pi}{2}kit, it) = e^{-ikw}e^{\pi tk^2/4} \vartheta_{3-k}(w, it) \tag{2.3}
\]
\[
\vartheta_2(w + \frac{\pi}{2}kit, it) = e^{-ikw}e^{\pi tk^2/4} \vartheta_{2+k}(w, it) \tag{2.4}
\]
\[
\vartheta(z + \pi itk, it) = e^{-2kiz}e^{\pi tk^2} \vartheta(z, it) \tag{2.5}
\]

where \( k = 0 \) if \( k \) is even and \( k = 1 \) if \( k \) is odd, and the third of these holds for \( \vartheta = \vartheta_2, \vartheta_3 \).

Finally, we shall make use of the following equalities

\[
\int_{\mathbb{R}} e(Ax) e^{-\pi \alpha x^2} dx = \frac{1}{\sqrt{\alpha}} e^{-\pi A^2/\alpha} \tag{2.6}
\]

and

\[
\int_{\mathbb{R}^2} e(Ax + By) e^{-\pi \alpha(x^2+y^2)} e(-xy) dxdy = \frac{1}{\sqrt{\alpha^2+1}} \exp \left(-\frac{\pi}{\alpha^2+1} \left[ \alpha A^2 + \alpha B^2 - 2iAB \right] \right) \tag{2.7}
\]

where \( \alpha > 0 \) and \( A, B \in \mathbb{C} \).

§3. Schwartz Theta functions

Fix \( 0 < \theta < \frac{1}{2} \), take \( M = \mathbb{R} \), \( G = \mathbb{R} \times \mathbb{R} \), and consider the lattice

\[
D : \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\theta} & 0 \\ 0 & \sqrt{\theta} \end{bmatrix}.
\]

It’s covolume in \( G \) is \( |G/D| = \theta \) and the associated C*-algebra \( C^*(D, h) = A_\theta \) is generated by \( U_j = \pi_{\varepsilon_j} \), which satisfy \( U_1 U_2 = e(\theta) U_2 U_1 \). Letting \( \beta = 1/\sqrt{\theta} \), the complementary lattice is

\[
D^\perp : \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}.
\]

Letting \( V_j = \pi_{\delta_j}^* = \pi_{-\delta_j} \) one has \( V_1 V_2 = e(-1/\theta) V_2 V_1 = e(-\beta^2) V_2 V_1 \), and these unitaries generate the C*-algebra \( C^*(D^\perp, \bar{h}) = A_{1/\theta} \). The automorphism \( \sigma \) of \( A_\theta \) as in (2.1) is given by \( \sigma(U_1) = U_2 \), \( \sigma(U_2) = U_1^{-1} \), which is the Fourier transform. The primary task of this paper is to prove the following.

**Theorem 3.1.** For \( 0 < \theta < 0.2345 \), there exists an even Schwartz function \( h \) on \( \mathbb{R} \) such that

(i) \( \langle h, \bar{h} \rangle_{D^\perp} = 0 \),

(ii) \( \langle h, h \rangle_{D^\perp} \) is invertible,

(iii) \( \langle h, h \rangle_{D^\perp} \) is fixed under the flip \( \sigma^2 \).

Consequently, the element \( e = \langle ha, ha \rangle_{D^\perp} \) is a flip-invariant smooth projection in \( A_\theta \) of trace \( \theta \) such that \( e \sigma(e) = 0 \), where \( a = (\langle h, h \rangle_{D^\perp})^{-1/2} \). In particular, \( e + \sigma(e) \) is a Fourier invariant projection of trace \( 2\theta \).

(The latter follows from \( \langle ha, ha \rangle_{D^\perp} = 1 \).)
Remark. As commented below (see the remark before Proposition 4.4), one can in fact show, with a little extra effort, that for $0 < \theta \leq 0.2427$ the element $\langle h, h \rangle_{D\perp}$ is invertible for the Schwartz function $h$ constructed here. However, for $\theta > 0.2451$ the methods here would have to be modified.

We shall now examine the Schwartz functions on $\mathbb{R}$ to be used in the construction. These functions have the Gaussian-Theta form

$$g_{r,\gamma}(x) = e^{-\pi \alpha x^2} \sum_p a_p e((rp - \gamma)x) = e^{-\pi \alpha x^2} e(-\gamma x) \vartheta_3(-\frac{1}{2} \pi i \alpha + \pi r x, i \alpha)$$

where

$$a_p = e^{-\pi \alpha p^2} e^{\pi \alpha p}$$

and $\alpha > 0$ and $r, \gamma$ are real. It is not hard to check that $g = g_{r,\gamma} \in \mathbb{S}(\mathbb{R})$. We will choose these parameters so that

$$\langle g, \tilde{g} \rangle_{D\perp} = 0$$

and $\langle \tilde{g}, \tilde{g} \rangle_{D\perp} = 0$,

where $\tilde{g}(x) = g(-x) = \tilde{g}(x)$. One then puts $h = g + \tilde{g}$ so that $h$ is an even function satisfying $\langle h, \tilde{h} \rangle_{D\perp} = 0$ and $\langle h, h \rangle_{D}$ is fixed under the flip automorphism. For certain of these parameters it will be shown in the next section that $\langle h, h \rangle_{D\perp}$ is invertible.

For real $r, s, \gamma, \gamma'$, one has

$$\langle g_{r,\gamma}, \tilde{g}_{s,\gamma'} \rangle_{D\perp} (m \delta_1 + n \delta_2) = \int_{\mathbb{R}} g_{r,\gamma}(x) \tilde{g}_{s,\gamma'}(x + \beta n) e(\beta mx) \, dx$$

$$= \int_{\mathbb{R}^2} g_{r,\gamma}(x) g_{s,\gamma'}(y) e(-xy) e(\beta mx - \gamma y) \, dx \, dy$$

$$= \sum_{p,q} \bar{a}_p a_q \int_{\mathbb{R}^2} e((\beta m - (rp - \gamma))x + ((sq - \gamma') - \beta n)y) e^{-\pi \alpha (x^2 + y^2)} e(-xy) \, dx \, dy$$

$$= \frac{1}{\sqrt{\alpha^2 + 1}} \sum_{p,q} \bar{a}_p a_q \exp \left(-\frac{\pi}{\alpha^2 + 1} D_{pq} \right)$$

using (2.7), where

$$D_{pq} = \alpha [\beta m - rp + \gamma]^2 + \alpha [sq - \gamma' - \beta n]^2 - 2i [\beta m - rp + \gamma] [sq - \gamma' - \beta n]$$

$$= \alpha (\beta m - rp)^2 + 2\alpha (\beta m - rp) \gamma + \alpha \gamma'^2 + \alpha (sq - \beta n)^2 - 2\alpha (sq - \beta n) \gamma' + \alpha (\gamma')^2$$

$$- 2i (\beta m - rp) (sq - \beta n) + 2i (\beta m - rp) \gamma' - 2i \gamma (sq - \beta n) + 2i \gamma' \gamma'$$

$$= \alpha r^2 p^2 - 2r (\alpha \beta m + i \beta n + \alpha \gamma + i \gamma') p + \alpha s^2 q^2 - 2s (\alpha \beta n + i \beta m + \alpha \gamma' + i \gamma) q$$

$$+ 2irs pq + K'$$

where $K'$ consists of all terms not involving $p, q$. In order to make the cross term involving $2irs pq$ disappear in the summation, we shall assume that $|rs| = \alpha^2 + 1$. This makes the above summation separable and one gets

$$\langle g_{r,\gamma}, \tilde{g}_{s,\gamma'} \rangle_{D\perp} (m \delta_1 + n \delta_2) = C_{mn} E_1(m, n) E_2(m, n)$$
where $C_{mn}$ is an exponential constant (independent of $p,q$ but depending on $m,n$), and where

$$E_1(m,n) = \sum_p \bar{a}_p \exp \left( -\frac{\pi}{\alpha^2+1} \left[ \alpha r^2 p^2 - 2r(\alpha\beta m + i\beta n + \alpha\gamma + i\gamma') p \right] \right),$$

$$E_2(m,n) = \sum_q a_q \exp \left( -\frac{\pi}{\alpha^2+1} \left[ \alpha s^2 q^2 - 2s(\alpha\beta n + i\beta m + \alpha\gamma' + i\gamma) q \right] \right).$$

Substituting $a_p$, one has

$$E_1(m,n) = \sum_p e^{-\pi dp^2} \exp \left( \pi \alpha p + \frac{2\pi r}{\alpha^2+1} \left[ \alpha\beta m + i\beta n + \alpha\gamma + i\gamma' \right] p \right)$$

$$= \vartheta_3 \left( \frac{1}{2} i \pi \alpha + \frac{i \pi r}{\alpha^2+1} \left[ \alpha\beta m + i\beta n + \alpha\gamma + i\gamma' \right], id \right)$$

where $d = \alpha + \frac{\alpha r^2}{\alpha^2+1}$. Similarly, one can write $E_2$. In order to satisfy (3.1), one can choose either to make $E_1 \equiv 0$ or $E_2 \equiv 0$. For our purposes, it will be enough to make $E_1 \equiv 0$ for $\gamma' = \pm \gamma$. This will be arranged as follows. Since $\tilde{g}_{r,\gamma} = g_{-r,-\gamma}$, we want to have

$$\langle g_{r,\gamma}, \hat{g}_{r,\gamma} \rangle_{D^\perp} = 0, \quad \text{and} \quad \langle g_{r,\gamma}, \hat{g}_{-r,-\gamma} \rangle_{D^\perp} = 0,$$

and in both cases we have $s = \pm r$, $\gamma' = \pm \gamma$. Thus $r^2 = s^2 = \alpha^2 + 1$, and $d = 2\alpha$. Now $E_1 \equiv 0$ iff for each pair of integers $m,n$ there are integers $M,N$ such that

$$\frac{1}{2} i \alpha + \frac{i r}{\alpha^2+1} \left[ \alpha\beta m + i\beta n + \alpha\gamma + i\gamma' \right] = \frac{1}{2} M + \left( \frac{1}{2} + N \right) i 2\alpha.$$ 

Putting $m = n = 0$ gives integers $M_0, N_0$ such that

$$\frac{1}{2} i \alpha + \frac{i r}{\alpha^2+1} \left[ \alpha\gamma + i\gamma' \right] = \frac{1}{2} M_0 + \left( \frac{1}{2} + N_0 \right) i 2\alpha. \quad (3.2)$$

and upon subtracting one gets

$$\frac{r}{\alpha^2+1} [i\alpha m - \beta n] = M' + N' i 2\alpha,$$

where $M', N'$ are integers depending on $m,n$. Thus

$$-\frac{r\beta}{\alpha^2+1} n = M', \quad \frac{r\beta}{\alpha^2+1} m = 2N'.$$

The first of these implies that $\frac{|r|\beta}{\alpha^2+1} = v$ must be a positive integer. And from the second, one must have $v = 2L \geq 2$ for some positive integer $L$. Further, $r^2 = s^2 = \beta^2/v^2 = \alpha^2 + 1$. To get the greater range for $\theta = 1/\beta^2 \leq 1/v^2$, we shall henceforth choose $v = 2$ ($L = 1$). Therefore, we are lead to make the following choices

$$|r| = |s| = \frac{\beta}{2}, \quad \beta^2 = 4(\alpha^2 + 1).$$
So we take \( g := g_{\beta/2, \gamma} \). Now to satisfy \( \langle g_{\beta/2, \gamma}, \hat{g}_{\beta/2, \gamma} \rangle_{D^\perp} = 0 \), condition (3.2) must be satisfied with \( \gamma' = \gamma \) and becomes
\[
\frac{1}{2} i \alpha + \frac{\beta \gamma}{2(\alpha^2 + 1)} [i \alpha - 1] = \frac{1}{2} + M_0 + \left( \frac{1}{2} + N_0 \right) 2i \alpha \tag{3.3}
\]
and in order to satisfy \( \langle g_{\beta/2, \gamma}, \hat{g}_{-\beta/2, -\gamma} \rangle_{D^\perp} = 0 \), condition (3.2) must be satisfied with \( \gamma' = -\gamma \) and becomes
\[
\frac{1}{2} i \alpha + \frac{\beta \gamma}{2(\alpha^2 + 1)} [i \alpha + 1] = \frac{1}{2} + M'_0 + \left( \frac{1}{2} + N'_0 \right) 2i \alpha. \tag{3.4}
\]
Subtracting (3.3) from (3.4) implies that \( \frac{\beta \gamma}{\alpha^2 + 1} = \frac{4 \gamma}{\beta} \) must be an integer. We choose it to be 1. Hence we take
\[ \gamma = \frac{\beta}{4}. \]
(This choice will in fact turn out to facilitate our computations with Theta functions below and guarantees that \( \langle h, h \rangle_{D^\perp} \) is invertible for \( \beta \) a little greater than 2.) Now with \( r = \frac{\beta}{2} \) and \( \gamma = \frac{\beta}{4} \), (3.4) follows from (3.3) and the latter clearly holds with \( M_0 = -1, N_0 = 0 \).

We therefore conclude that
\[ \langle g_{\beta/2, \beta/4}, \hat{g}_{\beta/2, \beta/4} \rangle_{D^\perp} = 0, \quad \langle g_{\beta/2, \beta/4}, \hat{g}_{-\beta/2, -\beta/4} \rangle_{D^\perp} = 0, \]
as desired. For simplicity, we shall write \( g := g_{\beta/2, \beta/4} \) (where \( \beta \) is fixed). More specifically,
\[ g(x) = e^{-\pi \alpha x^2} \sum_p a_p e\left(\frac{\beta}{4} p - \frac{\beta}{4} x\right), \quad a_p = e^{-\pi \alpha p^2} e^{\pi \alpha p}. \]

**Remark.** If we drop the \( \frac{\beta}{4} \) term a calculation similar to that done in the next section shows that \( \langle h, h \rangle_{D^\perp} \) is singular.
§4. Invertibility of \( \langle h, h \rangle_{D^1} \)

This section is aimed at showing that the positive element \( \langle h, h \rangle_{D^1} \) is invertible, where \( h \) was constructed in the previous section. The calculation here also proves Theorem 1.3 (including the last assertion in the remark following it). We begin with a lemma.

**Lemma 4.1.** Let there be given two Gaussian-Theta Schwartz functions of the form

\[
f_1(x) = e^{-\pi \alpha_1 x^2} \sum_p a_p e([r_1 p - \gamma_1] x), \quad f_2(x) = e^{-\pi \alpha_2 x^2} \sum_q b_q e([r_2 q - \gamma_2] x),
\]

where \( \alpha_j > 0, \, r_j, \gamma_j \) are real, and \( a_p, b_q \) are rapidly decreasing sequences, and let \( \alpha = \alpha_1 + \alpha_2 \). For any real \( s, t \), one has

\[
\int_{\mathbb{R}} f_1(x) f_2(x + s) e(tx) \, dx = \frac{e(-\alpha_2 st/\alpha)}{\sqrt{\alpha}} e^{-\pi \alpha_1 \alpha_2 s^2/\alpha} e^{-\pi t^2/\alpha} e^{-\pi (\gamma_1 - \gamma_2)^2/\alpha} \cdot e^{2\pi (\gamma_2 - \gamma_1) t/\alpha} e(-(\alpha_2 \gamma_1 + \alpha_1 \gamma_2) \frac{\alpha}{\alpha}) \Theta(s, t)
\]

where

\[
\Theta(s, t) = \sum_{p, q} \bar{a}_p b_q \exp \left( \frac{2\pi}{\alpha} (\gamma_2 - \gamma_1) (r_2 q - r_1 p) \right) \\
\cdot \exp \left( -\frac{\alpha}{\alpha} [r_2 q - r_1 p]^2 + 2t(r_2 q - r_1 p) \right)
\]

**Proof.** We have

\[
\int_{\mathbb{R}} f_1(x) f_2(x + s) e(tx) \, dx = \sum_{p, q} \bar{a}_p b_q \int_{\mathbb{R}} e^{-\pi \alpha_1 x^2} e(-[r_1 p - \gamma_1] x) e^{-\pi \alpha_2 (x+s)^2} e([r_2 q - \gamma_2] (x + s)) e(tx) \, dx
\]

\[
= \sum_{p, q} \bar{a}_p b_q e([r_2 q - \gamma_2] s) \int_{\mathbb{R}} e^{-\pi \alpha_1 (x^2 + \alpha_2 (x+s)^2)} e([r_2 q - r_1 p + \gamma_1 - \gamma_2 + t] x) \, dx
\]

\[
= \sum_{p, q} \bar{a}_p b_q \cdot \int_{\mathbb{R}} e([r_2 q - r_1 p + \gamma_1 - \gamma_2 + t] \frac{\alpha_{2s}}{\alpha}) e(-[r_2 q - r_1 p + \gamma_1 - \gamma_2 + t] x) \, dx
\]

\[
= \frac{1}{\sqrt{\alpha}} e^{-\pi \alpha_1 \alpha_2 s^2/\alpha} e(-[\gamma_1 - \gamma_2 + t] \frac{\alpha_{2s}}{\alpha}) \cdot \sum_{p, q} \bar{a}_p b_q \exp \left( -\frac{\alpha}{\alpha} [r_2 q - r_1 p + \gamma_1 - \gamma_2 + t]^2 \right)
\]
and since
\[ [r_2q - r_1p + \gamma_1 - \gamma_2 + t]^2 = (r_2q - r_1p)^2 + 2t(r_2q - r_1p) + t^2 \]
\[ + 2(\gamma_1 - \gamma_2)(r_2q - r_1p) + 2t(\gamma_1 - \gamma_2) + (\gamma_1 - \gamma_2)^2 \]

one gets
\[
\int_{\mathbb{R}} f_1(x)f_2(x+s)e(tx) \, dx \\
= \frac{1}{\sqrt{\alpha}} e^{-\pi \alpha s^2 / 2} e(-[\gamma_1 - \gamma_2 + t]^{\frac{\alpha s}{\alpha}}) \exp \left( -\frac{\pi}{\alpha} (\gamma_1 - \gamma_2)^2 \right) \exp \left( 2\frac{\pi}{\alpha} t(\gamma_2 - \gamma_1) \right) \\
\cdot e^{-\frac{\pi}{\alpha} s^2} \sum_{p,q} \bar{a}_p b_q e([r_2q - r_1p]s) e(-[r_2q - r_1p]^{\frac{\alpha s}{\alpha}}) \exp(\frac{2\pi}{\alpha} (\gamma_2 - \gamma_1)[r_2q - r_1p]) \\
\cdot \exp \left( -\frac{\pi}{\alpha} [(r_2q - r_1p)^2 + 2t(r_2q - r_1p)] \right)
\]

which gives the result in the statement of the lemma. □

Applying Lemma 4.1 to the functions \( g_\rho := g_{\rho \beta/2, \rho \beta/4} \) for \( \rho = \pm 1 \), one obtains (where \( \rho, \nu = \pm 1 \))
\[
\int_{\mathbb{R}} g_\rho(x)g_\nu(x+s)e(tx) \, dx \\
= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\alpha}} e^{-\pi \alpha s^2 / 2} e^{-\pi t^2 / (2\alpha)} \cdot \exp \left( -\frac{\pi \beta^2}{8\alpha} (\rho - \nu)^2 \right) \exp \left( \frac{\pi \beta t}{4\alpha} (\nu - \rho) \right) \exp \left( -\frac{\beta s}{\pi} (\rho + \nu) \right) \Theta_{\rho,\nu}
\]

where
\[
\Theta_{\rho,\nu} = \sum_{p,q} e^{-\pi \alpha (p^2 + q^2)} e^{\pi \alpha (p+q)} e\left( \frac{\beta}{4} [\rho p + \nu q] s \right) \exp \left( -\frac{\pi \beta^2}{8\alpha} (\rho - \nu)(\nu q - p \rho) \right) \\
\cdot \exp \left( -\frac{\beta\nu s}{8\alpha} [\beta(\nu q - p \rho)^2 + 4t(\nu q - p \rho)] \right).
\]

Making the replacements \( p \to \rho p \), \( q \to \nu q \) this becomes
\[
\Theta_{\rho,\nu} = \sum_{p,q} e^{-\pi \alpha (p^2 + q^2)} e^{\pi \alpha (p+q)} e\left( \frac{\beta}{4} (p + q) s \right) \exp \left( -\frac{\pi \beta^2}{8\alpha} (\rho - \nu)(q - p) \right) \\
\cdot \exp \left( -\frac{\beta\nu s}{8\alpha} [\beta(q - p)^2 + 4t(q - p)] \right).
\]

Now let \( k = q - p \):
\[
\Theta_{\rho,\nu} = \sum_{p,k} e^{-\pi \alpha (p^2 + (p+k)^2)} e^{\pi \alpha (\rho p + \nu (p+k))} e\left( \frac{\beta}{4} (k + 2p) s \right) \exp \left( -\frac{\beta^2}{8\alpha} (\rho - \nu) k \right) \\
\cdot \exp \left( -\frac{\beta\nu s}{8\alpha} [\beta k^2 + 4tk] \right).
\]
To calculate this more explicitly, we write
\[
\Theta_{\rho,\nu} = \sum_k e^{-\pi t k^2/4} \exp \left( \left[ \pi \alpha \nu - \frac{\pi \beta t}{2\alpha} - \frac{\pi \beta^2 (\rho - \nu)}{8\alpha} \right] k \right) e^{-\pi \alpha (\rho + \nu) k^2/2} H(k)
\]
where \( t = 4\alpha + \frac{2}{\alpha} = \frac{\beta^2}{2\alpha} + 2\alpha \) and
\[
H(k) = \sum_p e^{-2\pi \alpha (p + \frac{k}{2})^2} \exp \left( \left[ \pi i \beta s + \pi \alpha (\rho + \nu) \right] (p + \frac{k}{2}) \right)
\]
where
\[
L = L_{\rho,\nu} := \pi i \beta s + \pi \alpha (\rho + \nu).
\]
For \( k \) even, one has \( H(k) = \vartheta_3 \left( \frac{i}{2} L, 2i\alpha \right) \), and for \( k \) odd \( H(k) = \vartheta_2 \left( \frac{i}{2} L, 2i\alpha \right) \). Letting
\[
M = M_{\rho,\nu} = \pi \alpha \nu - \frac{\pi \beta t}{2\alpha} - \frac{\pi \beta^2 (\rho - \nu)}{8\alpha} = \frac{\pi}{2} \alpha (\nu - \rho) - \frac{\pi \beta t}{2\alpha} + \frac{\pi \beta^2 (\nu - \rho)}{8\alpha},
\]
which, from \( \beta^2 = 4(\alpha^2 + 1) \), becomes
\[
M = \frac{\pi}{4} t_\alpha (\nu - \rho) - \frac{\pi \beta t}{2\alpha},
\]
and one obtains
\[
\Theta_{\rho,\nu} = \sum_k e^{-\pi t_\alpha k^2/4} e^{M k} H(k)
\]
\[
= \vartheta_3 \left( \frac{i}{2} L, 2i\alpha \right) \sum_k e^{-\pi t_\alpha k^2} e^{2M k} + \vartheta_2 \left( \frac{i}{2} L, 2i\alpha \right) \sum_k e^{-\pi t_\alpha (k + \frac{1}{2})^2} e^{2M (k + \frac{1}{2})}
\]
\[
= \vartheta_3 \left( \frac{i}{2} L, 2i\alpha \right) \vartheta_3 (iM, it_\alpha) + \vartheta_2 \left( \frac{i}{2} L, 2i\alpha \right) \vartheta_2 (iM, it_\alpha).
\]
Let \( \ell = \frac{1}{2} (\nu + \rho) \) and \( d = \frac{1}{2} (\nu - \rho) \) which are equal to 0, ±1. We have thus obtained
\[
\int \frac{g_\rho(x) g_\nu(x + s) e(tx)}{\sqrt{2\alpha}} dx = \frac{e \left( \begin{array}{c} -\frac{1}{2} st \\ \end{array} \right)}{\sqrt{2\alpha}} e^{-\pi \alpha s^2/2} e^{-\pi t^2/2} e^{\frac{-\pi \beta^2}{8\alpha} d^2} e^{\frac{\pi \beta t}{2\alpha} d} e^{-i \pi \beta s \ell} e^{-i \pi \beta t \ell} \left[ \vartheta_3 \left( \frac{i}{2} L, 2i\alpha \right) \vartheta_3 (iM, it_\alpha) + \vartheta_2 \left( \frac{i}{2} L, 2i\alpha \right) \vartheta_2 (iM, it_\alpha) \right].
\]
Using the identities (2.3) and (2.4) each of these Theta functions can be simplified as follows
\[
\vartheta_3 \left( \frac{i}{2} L, 2i\alpha \right) = e^{\frac{i\pi}{2} \beta s t} e^{\frac{i\pi}{2} \alpha \ell^2} \vartheta_3 (\frac{i}{2} \beta s, 2i\alpha)
\]
\[
\vartheta_2 \left( \frac{i}{2} L, 2i\alpha \right) = e^{\frac{i\pi}{2} \beta s t} e^{\frac{i\pi}{2} \alpha \ell^2} \vartheta_2 (\frac{i}{2} \beta s, 2i\alpha)
\]
\[
\vartheta_3 (iM, it_\alpha) = e^{-\frac{i\pi}{2} M d \beta t} e^{\frac{i\pi}{4} d^2 t_\alpha} \vartheta_3 (\frac{i}{2} \beta s, it_\alpha)
\]
\[
\vartheta_2 (iM, it_\alpha) = e^{-\frac{i\pi}{2} M d \beta t} e^{\frac{i\pi}{4} d^2 t_\alpha} \vartheta_2 (\frac{i}{2} \beta s, it_\alpha)
\]
and since \( d^2 + \ell^2 = \frac{1}{2} (\nu^2 + \rho^2) = 1 \) we get
\[
\int \frac{g_\rho(x) g_\nu(x + s) e(tx)}{\sqrt{2\alpha}} dx = \frac{e \left( \begin{array}{c} -\frac{1}{2} st \\ \end{array} \right)}{\sqrt{2\alpha}} e^{-\pi \alpha s^2/2} e^{-\pi t^2/2} e^{\frac{-\pi \beta^2}{8\alpha} d^2} e^{\frac{\pi \beta t}{2\alpha} d} e^{-i \pi \beta s \ell} e^{-i \pi \beta t \ell} \left[ \vartheta_3 (\frac{i}{2} \beta s, 2i\alpha) \vartheta_3 (\frac{i}{2} \beta s, it_\alpha) + \vartheta_2 (\frac{i}{2} \beta s, 2i\alpha) \vartheta_2 (\frac{i}{2} \beta s, it_\alpha) \right].
\]
Since \( h = g_1 + g_{-1} \), summing these over \( \rho, \nu = \pm 1 \) yields the following.
Lemma 4.2. For real \(s, t\) one has

\[
\int_{\mathbb{R}} h(x) h(x+s) e(tx) \, dx = \frac{e(-\frac{1}{2} st)}{\sqrt{2\alpha}} e^{-\pi\alpha s^2/2} e^{-\pi t^2/(2\alpha)} \Gamma\left(\frac{t}{2\alpha}, \frac{s}{2\alpha}\right)
\]

where

\[
\Gamma(u, v) = 4 e^{\frac{\pi}{2} \alpha} \left[ \vartheta_2\left(\frac{\pi}{2} \beta^2 v, 2i\alpha\right) \vartheta_3\left(i \frac{\pi}{2\alpha} \beta^2 u, it\alpha\right) + \vartheta_3\left(\frac{\pi}{2} \beta^2 v, 2i\alpha\right) \vartheta_2\left(i \frac{\pi}{2\alpha} \beta^2 u, it\alpha\right) \right]
\]

for \(u, v \in \mathbb{R}\).

This lemma now gives

\[
\langle h, h \rangle_{D^\perp}(m\delta_1 + n\delta_2) = \frac{1}{\sqrt{2\alpha}} e(-\beta^2 mn/2) e^{-\pi\alpha\beta^2 n^2/2} e^{-\pi\beta^2 m^2/(2\alpha)} \Gamma(m, n)
\]

hence

\[
\langle h, h \rangle_{D^\perp} = \sum_{m,n} \langle h, h \rangle_{D^\perp}(m\delta_1 + n\delta_2) V_2^n V_1^m
\]

\[
= \sum_{m,n} e(\beta^2 mn) \langle h, h \rangle_{D^\perp}(m\delta_1 + n\delta_2) V_2^n V_1^m
\]

\[
= \frac{1}{\sqrt{2\alpha}} \sum_{m,n} e(\beta^2 mn/2) e^{-\pi\alpha\beta^2 n^2/2} e^{-\pi\beta^2 m^2/(2\alpha)} \Gamma(m, n) V_1^m V_2^n
\]

\[
= \frac{1}{\sqrt{2\alpha}} \sum_{n} e^{-\pi\alpha\beta^2 n^2/2} \psi_n V_2^n
\]

where \(\psi_n = \psi_n(V_1)\), viewed as a period 1 function on \(\mathbb{R}\), is given by

\[
\psi_n(t) := \sum_{m} e(\beta^2 mn/2) e^{-\pi\beta^2 m^2/(2\alpha)} \Gamma(m, n) e(mt)
\]

\[
= 4 e^{\frac{\pi}{2} \alpha} \left[ \vartheta_2\left(\frac{\pi}{2} \beta^2 n, 2i\alpha\right) F_n(t') + \vartheta_3\left(\frac{\pi}{2} \beta^2 n, 2i\alpha\right) G_n(t') \right]
\]

where \(t' = t + \frac{1}{2} \beta^2 n\) and

\[
F_n(t) = \sum_{m} e^{-\pi\beta^2 m^2/(2\alpha)} \vartheta_3\left(i \frac{\pi}{2\alpha} \beta^2 m, it\alpha\right) e(mt)
\]

and

\[
G_n(t) = \sum_{m} e^{-\pi\beta^2 m^2/(2\alpha)} \vartheta_2\left(i \frac{\pi}{2\alpha} \beta^2 m, it\alpha\right) e(mt).
\]

Note that up to a multiplicative positive constant, \(\langle h, h \rangle_{D^\perp}\) is the element \(X\) of Theorem 1.3, whose proof is furnished by the following argument. We now calculate \(F_n\) and \(G_n\) explicitly as follows. For convenience, let

\[
\tau_\alpha := \frac{\beta^2}{2\alpha} = 2(\alpha + \frac{1}{\alpha}) = t_\alpha - 2\alpha.
\]
For each real \( t \) one has

\[
F_n(t) = \sum_m e^{-\pi \beta^2 m^2 / (2\alpha)} \sum_k e^{-\pi t \alpha k^2} e^{2 i k \pi \beta^2 m} e^{2 \pi i m t}
\]

\[
= \sum_k e^{-\pi t \alpha k^2} \sum_m e^{-\pi \beta^2 m^2 / (2\alpha)} e^{2 i m [\pi t + i \pi \tau \alpha / 2 \alpha \beta^2 k]}
\]

\[
= \sum_k e^{-\pi t \alpha k^2} \sum_m e^{-\pi \tau \alpha m^2} e^{2 i m [\pi t + i \pi \tau \alpha k]}
\]

\[
= \sum_k e^{-\pi t \alpha k^2} e^{\pi \tau \alpha k^2} \sum_m e^{-\pi \tau \alpha (m+k)^2} e^{2 i m \pi t}
\]

\[
= \sum_k e^{-2 \pi \tau \alpha k^2} e^{-2 i k \pi t} e^{-\pi \tau \alpha m^2} e^{2 i m \pi t}
\]

\[
= \vartheta_3(\pi t, 2i\alpha) \vartheta_3(\pi t, i\tau \alpha)
\]

A similar computation gives

\[
G_n(t) = \vartheta_2(\pi t, 2i\alpha) \vartheta_2(\pi t, i\tau \alpha).
\]

Thus we obtain

\[
\psi_n(t) = 4 e^{\pi \alpha / 2} \left[ \vartheta_2(v_n, 2i\alpha) \vartheta_3(\pi t', 2i\alpha) \vartheta_3(\pi t', i\tau \alpha) + \vartheta_3(v_n, 2i\alpha) \vartheta_2(\pi t', 2i\alpha) \vartheta_2(\pi t', i\tau \alpha) \right],
\]

where \( v_n := \frac{\pi}{\beta^2} \beta^2 n \) and \( t' := t + \frac{1}{\beta^2} \beta^2 n \). Write this in the form

\[
\psi_n(t) = 4 e^{\pi \alpha / 2} \vartheta_3(v_n, 2i\alpha) \vartheta_3(\pi t', 2i\alpha) \vartheta_3(\pi t', i\tau \alpha) \cdot Q_n(t)
\]

where

\[
Q_n(t) = \frac{\vartheta_2(v_n, 2i\alpha)}{\vartheta_3(v_n, 2i\alpha)} + \frac{\vartheta_2(\pi t', 2i\alpha)}{\vartheta_3(\pi t', 2i\alpha)} \frac{\vartheta_2(\pi t', i\tau \alpha)}{\vartheta_3(\pi t', i\tau \alpha)}.
\]

It is clear that \( \psi_n \) is a real function. Our next goal is to show that \( \psi_0 \) is positive. We have

\[
\psi_0(t) = 4 e^{\pi \alpha / 2} \vartheta_3(0, 2i\alpha) \vartheta_3(\pi t, 2i\alpha) \vartheta_3(\pi t, i\tau \alpha) \cdot Q_0(t),
\]

\[
Q_0(t) = \frac{\vartheta_2(0, 2i\alpha)}{\vartheta_3(0, 2i\alpha)} + \frac{\vartheta_2(\pi t, 2i\alpha)}{\vartheta_3(\pi t, 2i\alpha)} \frac{\vartheta_2(\pi t, i\tau \alpha)}{\vartheta_3(\pi t, i\tau \alpha)}
\]

and it is enough to show that \( Q_0(t) > 0 \). This is an immediate consequence of the following two facts:

(F1) For all real \( s, t \) and \( \alpha > 0 \), one has

\[
\frac{\vartheta_2(\pi t, i\tau \alpha)}{\vartheta_3(\pi s, i\tau \alpha)} < 0.09.
\]

(F2) For each real \( t \), one has

\[
\left| \frac{\vartheta_2(\pi t, 2i\alpha)}{\vartheta_3(\pi t, 2i\alpha)} \right| < \frac{\vartheta_2(0, 2i\alpha)}{\vartheta_3(0, 2i\alpha)}.
\]
We make an upper bound estimates for each of the quotients here as follows

\[
\frac{\vartheta_2(\pi t, i\tau_\alpha)}{\vartheta_3(\pi s, i\tau_\alpha)} \leq \frac{\vartheta_2(0, i\tau_\alpha)}{\vartheta_3(\pi s, i\tau_\alpha)} < \frac{\vartheta_2(0, 4i)}{1 - 2e^{-4\pi}} \approx 0.0864
\]

which proves (F1). (Note that here we have used the well-known fact that for each real \( t \) and \( x > 0 \) one has \( 0 < \vartheta_3(\pi t, ix) \leq \vartheta_3(t, ix) \leq \vartheta_3(0, ix) \).) To show (F2), one need only check that the function \( t \mapsto \frac{\vartheta_2(\pi t, 2i\alpha)}{\vartheta_3(\pi t, 2i\alpha)} \) has period 2, is decreasing over the interval \([0, 1]\), increasing over \([1, 2]\), and hence attains its maximum value when \( t = 0 \).

We have

\[
\left| \frac{\psi_n(t)}{\psi_0(s)} \right| = \frac{\vartheta_3(v_n, 2i\alpha)}{\vartheta_3(0, 2i\alpha)} \frac{\vartheta_3(\pi t', 2i\alpha)}{\vartheta_3(\pi s, 2i\alpha)} \frac{\vartheta_3(\pi t', i\tau_\alpha)}{\vartheta_3(\pi s, i\tau_\alpha)} \cdot Q_n(t) \cdot \frac{Q_0(t)}{Q_0(s)}.
\]

We make an upper bound estimates for each of the quotients here as follows

\[
0 < \frac{\vartheta_3(v_n, 2i\alpha)}{\vartheta_3(0, 2i\alpha)} \leq 1
\]

\[
0 < \frac{\vartheta_3(\pi t', 2i\alpha)}{\vartheta_3(\pi s, 2i\alpha)} \leq \frac{\vartheta_3(0, 2i\alpha)}{\vartheta_3(\pi s, 2i\alpha)}
\]

\[
0 < \frac{\vartheta_3(\pi t', i\tau_\alpha)}{\vartheta_3(\pi s, i\tau_\alpha)} \leq \frac{\vartheta_3(0, i\tau_\alpha)}{\vartheta_3(\pi s, i\tau_\alpha)} \leq \frac{\vartheta_3(0, 4i)}{\vartheta_3(\pi s, 4i)} \leq 1.00001.
\]

The first two inequalities are obvious. The third inequality follows from the fact that \( \vartheta_3(0, ix) \) is a decreasing function of \( x \) and \( \vartheta_3(\pi t, ix) \) is an increasing function for, say, \( x > 0.2 \), and since \( \tau_\alpha \geq 4 \) for all \( \alpha > 0 \). The latter assertion can be seen from the expansion

\[
\vartheta_3(\pi t, ix) = 1 - 2 \left( [e^{-\pi x} - e^{-\pi x^2}] + [e^{-\pi x^3} - e^{-\pi x^4}] + \ldots \right)
\]

and noting that each of the bracketed functions is a decreasing function for \( x > 0.2 \) as can be verified directly. One thus gets

\[
\left| \frac{\psi_n(t)}{\psi_0(s)} \right| \leq 1.00001 \frac{\vartheta_3(0, 2i\alpha)}{\vartheta_3(\pi s, 2i\alpha)} \frac{|Q_n(t)|}{Q_0(s)}.
\]

Using (F1) and (F2) one gets

\[
|Q_n(t)| \leq 1.09 \frac{\vartheta_2(0, 2i\alpha)}{\vartheta_3(0, 2i\alpha)}, \quad Q_0(t) > 0.91 \frac{\vartheta_2(0, 2i\alpha)}{\vartheta_3(0, 2i\alpha)}.
\]

This proves the following result.

**Lemma 4.3.** For all integers \( n \), one has

\[
\|\psi_0^{-1}\psi_n\| \leq 1.19782 \frac{\vartheta_3(0, 2i\alpha)}{\vartheta_3(\pi s, 2i\alpha)}.
\]
Therefore one gets
\[ \| \psi_0^{-1} \sqrt{2} \langle h, h \rangle_{D^\perp} - I \| = \left\| \sum_{n \neq 0} e^{-\pi \alpha \beta^2 n^2 / 2} \psi_0^{-1} \psi_n V_n \right\| \leq 1.19782 \frac{\vartheta_3(0, 2i\alpha)}{\vartheta_3(\frac{\pi}{2}, 2i\alpha)} \cdot [\vartheta_3(0, \frac{i}{2} \alpha \beta^2) - 1]. \]

The right side of this inequality, as a function of \( \alpha > 0 \) (recall \( \beta^2 = 4(\alpha^2 + 1) \)), is less than 1 for \( \alpha > 0.2568 \), i.e. for \( 0 < \theta < 0.2345 \) or \( \beta > 2.065 \), and therefore we conclude that for this \( \theta \)-range, the element \( \langle h, h \rangle_{D^\perp} \) is invertible. This completes the proof of \( (ii) \) in Theorem 3.1 and, in particular, yields Theorem 1.3.

Remark. One can in fact show, with a little extra effort, that for \( 0 < \theta \leq 0.2427 \) if one calculates the norms \( \| \psi_0^{-1} \psi_n \| \) more precisely (for the first few \( n \)'s, say) that the the element \( \langle h, h \rangle_{D^\perp} \) is invertible (since the norm of the preceding sum is still less than 1). However, if one does the same for \( \theta > 0.2451 \) the norm can be greater than 1. (Of course, this does not mean that \( \langle h, h \rangle_{D^\perp} \) is singular, but that the methods here would have to be modified.)

We conclude by obtaining bounds on the function \( \psi_0 \) and describe its asymptotic behaviour in \( \alpha \), a result that will be used in future work ([9]).

**Proposition 4.4.** For \( \alpha > \frac{1}{4} \) and all real \( t \) one has \( 4 < \psi_0(t) < 18 \). In addition, \( \psi_0(t) \to 8 \) uniformly in \( t \) as \( \alpha \to \infty \).

**Proof.** From above we had
\[ \psi_0(t) = 4e^{\pi \alpha / 2} \vartheta_3(0, 2i\alpha) \vartheta_3(\pi t, 2i\alpha) \vartheta_3(\pi t, i\tau_\alpha) \cdot Q_0(t). \]
Let
\[ E(t) = \frac{\vartheta_3(\pi t, i\tau_\alpha)}{\vartheta_3(\pi t, i\tau_\alpha)} \]
which goes to zero uniformly in \( t \) as \( \alpha \to \infty \). By (F1) one has \( |E(t)| < 0.09 \). By (F2) one has
\[ 4[1 - E(t)] \vartheta_3(0, 2i\alpha) < Q_0(t) < 4[1 + E(t)] \vartheta_3(0, 2i\alpha) \]
hence
\[ 4[1 - E(t)] R(t) < \psi_0(t) < 4[1 + E(t)] R(t) \]
where
\[ R(t) = e^{\pi \alpha / 2} \vartheta_3(0, 2i\alpha) \vartheta_3(\pi t, 2i\alpha) \vartheta_3(\pi t, i\tau_\alpha). \]
From the inequalities
\[ 2 < e^{\pi \alpha / 2} \vartheta_3(0, 2i\alpha) < 2 \vartheta_3(0, 2i\alpha) \]
\[ \vartheta_3(\frac{\pi}{2}, 2i\alpha) \leq \vartheta_3(\pi t, 2i\alpha) \leq \vartheta_3(0, 2i\alpha) \]
\[ \vartheta_3(\frac{\pi}{2}, i\tau_\alpha) \leq \vartheta_3(\pi t, i\tau_\alpha) \leq \vartheta_3(0, i\tau_\alpha) \]
one obtains
\[ 2 \vartheta_3(\frac{\pi}{2}, 2i\alpha) \vartheta_3(\frac{\pi}{2}, i\tau_\alpha) < R(t) < 2 \vartheta_3(0, 2i\alpha)^2 \vartheta_3(0, i\tau_\alpha). \]

\[ \text{(†)} \]

\[ \text{(*)} \]
Since it is not hard to see that the left side of (*) is an increasing function of \( \alpha \geq \frac{1}{4} \), it is greater that its value at \( \alpha = \frac{1}{4} \) which, when computed (and using \( \tau_\alpha \geq 4 \)), gives a value greater than 1.17. The right side of (*) is a decreasing function of \( \alpha \) (as is each factor), so is at most equal to its value when \( \alpha = \frac{1}{4} \), which is less than 4.03. Therefore, \( 1.17 < R(t) < 4.03 \) for all \( t \) and \( \alpha \geq \frac{1}{4} \), which yields the claimed inequality of the lemma for \( \psi_0 \). Finally, note that the left and right sides of (*) converge to 2 as \( \alpha \to \infty \), so that \( R(t) \to 2 \) uniformly in \( t \). Therefore from (†) one has \( |\psi_0(t) - 4R(t)| \leq |E(t)|R(t) \) and thereby obtains the claimed asymptotic behaviour of \( \psi_0 \). □

References

[1] F. P. Boca, *Projections in rotation algebras and theta functions*, Comm. Math. Phys. **202** (1999), 325–357.
[2] G. Elliott and D. Evans, *The structure of the irrational rotation C*-algebra*, Ann. Math. **138** (1993), 477–501.
[3] G. Elliott and Q. Lin, *Cut-down method in the inductive limit decomposition of non-commutative tori*, J. London Math. Soc. (2) **54** (1996), no. 1, 121–134.
[4] M. Rieffel, *C*-algebras associated with irrational rotations, Pacific J. Math. **93** (1981), no. 2, 415–429.
[5] ———, *Projective modules over higher-dimensional non-commutative tori*, Canad. J. Math **40** (1988), 257–338.
[6] S. G. Walters, *Chern characters of Fourier modules*, Canad. J. Math. **52** (2000), no. 3, 633–672.
[7] ———, *K-theory of non commutative spheres arising from the Fourier automorphism*, Canad. J. Math. (2000), 40 pages (to appear).
[8] ———, *On the inductive limit structure of order four automorphisms of the irrational rotation algebra*, Internat. J. Math. (2001), 8 pages (to appear).
[9] ———, *Partially approximating invariant subalgebras of the irrational rotation C*-algebra*, in preparation (2000), 22 pages.

Department of Mathematics and Computer Science, The University of Northern British Columbia, Prince George, B.C. V2N 4Z9 CANADA

E-mail address: walters@hilbert.unbc.ca or walters@unbc.ca

Website: http://hilbert.unbc.ca/walters