Global foliations of matter spacetimes with Gowdy symmetry

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Abstract
A global existence theorem, with respect to a geometrically defined
time, is shown for Gowdy symmetric globally hyperbolic solutions of
the Einstein-Vlasov system for arbitrary (in size) initial data. The
spacetimes being studied contain both matter and gravitational waves.

1 Introduction

An important problem in classical general relativity is the question of global
existence (in an appropriate sense) for globally hyperbolic solutions of the
vacuum-Einstein and matter-Einstein equations. The main motivation being
its relationship to the cosmic censorship conjectures. Strong cosmic censor-
ship has eg. by Eardley and Moncrief [EM] been formulated as a question
on global existence and asymptotic behaviour of solutions to the Einstein
equations, suggesting a definite method of analytical attack.

To begin studying the long-time behaviour of solutions to a complicated
partial differential equation system one might focus on families of solutions
with some prescribed symmetry. With the exception of the monumental
work on global nonlinear stability of the Minkowski space by Christodoulou
and Klainerman [CK], the practice in general relativity has for long been to
study “global existence” problems under symmetric assumptions.

One family of (cosmological) solutions which have been studied exten-
sively is the Gowdy spacetimes [G]. These spacetimes are vacuum but admit

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gravitational waves (in contrast to e.g. spherically symmetric spacetimes). Global existence has been shown for the Gowdy spacetimes [M], strong cosmic censorship is settled in the case of polarized Gowdy spacetimes [CIM], and much is known about the subset of the Gowdy spacetimes which admit an extension across a Cauchy horizon [CI].

In this paper we show global existence, with respect to a geometrically defined time, for matter spacetimes (Einstein-Vlasov) with Gowdy symmetry and thereby we extend Moncrief’s result [M] in the vacuum case. This is the first result which provides a global foliation of a spacetime containing both matter and gravitational waves. Moreover, for matter spacetimes there are only a few global results available all together. Let us briefly mention some of these results. First, by matter spacetimes we have in mind spacetimes where the matter consists of massive particles. One can also consider spacetimes which only contains radiation and important results have been obtained in this direction, e.g. Christodoulou has obtained strong results in the spherically symmetric case with a scalar field as matter model (see e.g. [Cu1], [Cu2] and the references therein). For spacetimes containing massive particles the main global results can be summarized as follows. Under a smallness condition on the initial data, Rein and Rendall have [RR] shown that solutions of the spherically symmetric Einstein-Vlasov system are geodesically complete. Some information on the large data problem was then obtained in [RRS]. Christodoulou has in a series of papers (see [Cu3] and the references therein) studied the Einstein-Euler equation in the spherically symmetric case for a special equation of state, adapted to understand the dynamics of a supernova explosion. He can globally control the solutions to the Cauchy problem and he finds solutions whose behaviour resembles qualitatively that of a supernova explosion. Finally, the most relevant results in the context of this paper are those on cosmological solutions by Rendall [Rl1-2] and Rein [Rn]. These are discussed in some detail in relation to our result below.

Our method of proof is inspired by a recent global foliation result for vacuum spacetimes admitting a $T^2$ isometry group, acting on $T^3$ spacelike surfaces [BCIM]. These spacetimes are more general than the Gowdy spacetimes: both families admit two commuting Killing vectors but in the Gowdy case there is the additional condition that the twists are zero. The twists are defined by

\[ c_1 = \epsilon_{\mu\nu\rho\delta}X^\mu Y^\nu \nabla^\rho X^\delta, \quad c_2 = \epsilon_{\mu\nu\rho\delta}X^\mu Y^\nu \nabla^\rho Y^\delta, \]

(1)

where $X, Y$ are Killing vectors associated with the isometry group. It follows
from the Einstein equations that in vacuum these quantities are constant throughout spacetime [G].

One difficulty in studying long-time existence problems in general relativity is the lack of having a fixed time measure. A solution which remains regular for an infinite range of one time scale may become singular within a finite range of another. In [BCIM] this problem is treated by choosing a coordinate system in which the time is fixed to the geometry of spacetime. In fact, the time is defined to be the area of the two dimensional spacelike orbits of the $T^2$ isometry group. These coordinates are called areal coordinates. The main theorem in [BCIM] shows that the entire maximal globally hyperbolic development of the initial hypersurface can be foliated by areal coordinates. These coordinates are however only used in a direct way in the future direction. To show that the past of the initial hypersurface is covered by areal coordinates the authors use conformal coordinates (the time is not fixed to the geometry of spacetime) in which the equations take a more suitable form for an analytical treatment. By a long chain of geometrical arguments it is then shown that the development in conformal coordinates admits a foliation by areal coordinates, and that it covers the past maximal globally hyperbolic development of the initial hypersurface.

We prove that $T^3 \times \mathbb{R}$–matter spacetimes with Gowdy symmetry admit global foliations by areal coordinates. The matter content is described by the Vlasov equation. This is a kinetic equation and gives a statistical description of a collection of collisionless “particles”. In the cosmological case the particles are galaxies or clusters of galaxies whereas in stellar dynamics they are stars. The Vlasov equation has shown to be suitable in general relativity for the study of the long-time behaviour of matter in gravitational fields. In particular it rules out the formation of shell-crossing singularities. For a discussion on the choice of matter model see [Rl4] and [Rl5].

To prove the existence of a global foliation we also work directly in areal coordinates in the expanding (future) direction, and in the contracting (past) direction, we first show a global existence theorem in conformal coordinates and then we invoke the geometrical arguments in [BCIM] to complete the proof. We point out that our result depends strongly on the exact structure of the Vlasov equation and do not hold for general matter models which are only restricted by certain inequalities on the components of the energy-momentum tensor.

A related and interesting result has recently been shown by Rendall [Rl1] (see also [Rl2]). He considers $T^2$ symmetric spacetimes for the Einstein-Vlasov and the Einstein-wave map equations and he shows that if such a spacetime admits at least one compact constant mean curvature (CMC) hy-
persurface then the past of that surface can be covered by a foliation of compact CMC hypersurfaces. The CMC- and the areal coordinate foliation are both geometrically based time foliations which provide frameworks for studying strong cosmic censorship and other global issues. The main motivation for developing techniques to obtain CMC foliations is that the definition of a CMC hypersurface does not depend on any symmetry assumptions and it is hence possible that CMC foliations will exist for rather general spacetimes. The areal coordinate foliation used here is less general since it is adapted to the symmetry, but leads in the Gowdy case (note that the results in [RII] apply to the more general $T^2$ symmetric spacetimes which we hope to explore in the future) to stronger results. Namely, the arguments in [RII] do not show that the entire future of the initial hypersurface can be covered, and the existence of the CMC foliation is only guaranteed under the hypothesis that spacetime admits at least one such hypersurface.

We also mention a result in this direction due to Rein [Rn]. He has studied cosmological Einstein-Vlasov spacetimes with stronger symmetry restrictions than in the Gowdy case (the spacetimes admit three Killing vectors). In these spacetimes gravitational waves cannot exist. For plane symmetry (the relevant case for us) he has shown existence back to the initial singularity for small initial data, and under the assumption that one of the field components is bounded, he obtains global existence for large data in the future direction. An interesting result in his work is that the initial singularity is shown to be a curvature singularity as well as a “crushing” singularity (see [ES]).

The outline of the paper follows in the large that of [BCIM]. In section 2 we describe Gowdy symmetry and give the equations for the Einstein-Vlasov system in areal and conformal coordinates. The main theorem is formulated in section 3 where we also describe the geometrical arguments in [BCIM] needed to complete the proof in the contracting direction. Section 4 is devoted to the analysis in the contracting direction. Estimates for the field components and the matter terms are derived in conformal coordinates, by using e.g. light-cone arguments and methods originally developed for the Vlasov-Maxwell equation. The analysis in the expanding direction is carried out in areal coordinates in section 5 where a number of estimates are derived. Light-cone arguments and an “energy” monotonicity lemma are important tools for obtaining bounds on the field components and their derivatives. The control of the matter terms and their derivatives rely on three lemmas. The first one is the “energy” monotonicity lemma just mentioned. Then, in the second lemma a careful analysis of the characteristic system associated with the Vlasov equation is carried out, which leads to a bound on the sup-
port of the momenta. The third lemma provides bounds on the derivatives of the matter terms and relies indirectly on the geodesic deviation equation. This equation relates the curvature tensor and the acceleration of nearby geodesics and has proved useful in previous studies of the Einstein-Vlasov system (see [RR], [Rn] and [Rl3]).

2 The Einstein-Vlasov system with Gowdy symmetry

Let us begin with a brief review of Gowdy symmetry. Consider a spacetime that can be foliated by a family of compact, connected, and orientable hypersurfaces. If the maximal isometry group of the spacetime is two dimensional, and if it acts invariantly and effectively on the foliation, then the isometry group must be $U(1) \times U(1)$. Moreover, the foliation surfaces must be homeomorphic to $T^3, S^1 \times S^2, S^3$ or $L(p, q)$ (the Lens space), and the action is unique up to equivalence. The Killing vector fields $X, Y$ associated with the isometry group have to commute in such a spacetime. We say that spacetimes satisfying the symmetry conditions above and in which both the twists $c_1, c_2$ (see §2) vanish have Gowdy symmetry. We remark that the term “Gowdy spacetime" is reserved for the vacuum case. For more background on Gowdy symmetry we refer to [G], [Cl].

As mentioned above there are several choices of spacetime manifolds compatible with Gowdy symmetry. In this paper we restrict our attention to the $T^3-$case. It is an interesting fact that in vacuum this is the only possibility if the condition of vanishing twists is relaxed.

The dynamics of the matter is governed by the Vlasov equation. This is a kinetic equation and models a collisionless system of particles, i.e. the particles follow the geodesics of spacetime. For a nice introduction to the Einstein-Vlasov system see [Rl3]. We also mention the survey of Ehlers [E] for more information on kinetic theory in general relativity, and the book by Binney and Tremaine [BT] for some applications of kinetic theory in stellar dynamics.

We will use two choices of coordinates, areal coordinates and conformal coordinates. It has been shown in [Cl] that, at least locally, any globally hyperbolic (non-flat) Gowdy spacetime on $T^3 \times \mathbb{R}$ admits each of these coordinates. Both sets of coordinates are chosen so that

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y},$$
are Killing vector fields ($a, b, c$ and $d$ are constants with $ad - bc \neq 0$), and in both cases $\theta \in S^1$ denotes the remaining spatial coordinate. Below the form of the metric and the Einstein-Vlasov system is given in areal and conformal coordinates. The functions $R, \alpha, U, A, \eta$ all depend on $t$ and $\theta$ and the function $f$ depends on $t, \theta$ and $v \in \mathbb{R}^3$.

**Areal Coordinates**

**Metric**

$$g = -e^{2(\eta - U)} \alpha dt^2 + e^{2(\eta - U)} d\theta^2 + e^{2U} (dx + A dy)^2 + e^{-2U} t^2 dy^2$$  \hspace{1cm} (2)

The Einstein-matter constraint equations

$$\frac{\eta_t}{t} = U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A_\theta^2) + e^{2(\eta - U)} \alpha \rho$$  \hspace{1cm} (3)

$$\frac{\eta_\theta}{t} = 2U_t U_\theta + \frac{e^{4U}}{2t^2} A_t A_\theta - \frac{\alpha \theta}{2t^2} - e^{2(\eta - U)} \sqrt{\alpha} J$$  \hspace{1cm} (4)

$$\alpha_t = 2t \alpha^2 e^{2(\eta - U)} (P_1 - \rho)$$  \hspace{1cm} (5)

The Einstein-matter evolution equations

$$\eta_{tt} - \alpha \eta_{\theta\theta} = \frac{\eta_t \alpha_\theta}{2} + \frac{\eta_\theta \alpha_t}{2} - \frac{\alpha_\theta^2}{4 \alpha} + \frac{\alpha \theta^2}{2} - U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 - \alpha A_\theta^2)$$

$$- \alpha e^{2(\eta - U)} P_3 - \frac{A_t^2}{t^2} \alpha e^{2(\eta + U)} P_2 - \frac{2A}{t} \alpha e^{2\eta} S_{23}$$  \hspace{1cm} (6)

$$U_{tt} - \alpha U_{\theta\theta} = -U_t + \frac{U_\theta \alpha_\theta}{2} + \frac{U_\theta \alpha_t}{2} + \frac{e^{4U}}{2t^2} (A_t^2 - \alpha A_\theta^2)$$

$$+ \frac{1}{2} e^{2(\eta - U)} \alpha (\rho - P_1 + P_2 - P_3)$$  \hspace{1cm} (7)

$$A_{tt} - \alpha A_{\theta\theta} = \frac{A_t}{t} + \frac{A_\theta \alpha_\theta}{2} + \frac{A_\theta A_t}{2\alpha} - 4A_t U_t + 4\alpha A_\theta U_\theta$$

$$+ 2t \alpha e^{2(\eta - 2U)} S_{23}$$  \hspace{1cm} (8)
The Vlasov equation

\[
\frac{\partial f}{\partial t} + \frac{\sqrt{\alpha}v^1}{v^0} \frac{\partial f}{\partial \theta} - \left[ (\eta_\theta - U_\theta + \alpha \eta + (U_t - U_i)v^1 - \sqrt{\alpha}e^{2U}A_\theta v^2v^3 \right] \frac{\partial f}{\partial v^0} \\
+ \frac{\sqrt{\alpha}U_\theta}{v^0} ((v^3)^2 - (v^2)^2) \frac{\partial f}{\partial v^1} - \left[ U_t v^2 + \sqrt{\alpha}U_\theta v^1v^2 \right] \frac{\partial f}{\partial v^2} \\
- \left[ \frac{1}{t} - U_t \right] v^3 - \sqrt{\alpha}U_\theta v^1v^3 + \frac{e^{2U}v^2}{t} (A_t + \sqrt{\alpha}A_\theta v^1) \right] \frac{\partial f}{\partial v^3} = 0 \tag{9}
\]

The matter quantities

\[
\rho(t, \theta) = \int_{\mathbb{R}^3} v^0 f(t, \theta, v) \, dv \tag{10}
\]
\[
P_k(t, \theta) = \int_{\mathbb{R}^3} \frac{(v^k)^2}{v^0} f(t, \theta, v) \, dv, \quad k = 1, 2, 3 \tag{11}
\]
\[
J(t, \theta) = \int_{\mathbb{R}^3} v^1 f(t, \theta, v) \, dv \tag{12}
\]
\[
S_{23}(t, \theta) = \int_{\mathbb{R}^3} \frac{v^2v^3}{v^0} f(t, \theta, v) \, dv \tag{13}
\]

Here the variables \(v\) are related to the canonical momenta \(p\) through

\[
v^0 = \sqrt{\alpha e^{\eta - U}} p^0, \quad v^1 = e^{(\eta - U)} p^1, \quad v^2 = e^U p^2 + A e^U p^3, \quad v^3 = t e^{-U} p^3, \tag{14}\]

and

\[
p^\mu := \frac{dx^\mu}{d\tau}, \quad x^\mu = (t, \theta, x, y),
\]

where \(\tau\) is proper time. It is assumed that all 'particles' have the same mass (normalized to one) and follow the geodesics of spacetime (collisionless particle system). Hence

\[
g_{\mu\nu} p^\mu p^\nu = -1,
\]

so that

\[
v^0 = \sqrt{1 + (v^1)^2 + (v^2)^2 + (v^3)^2}. \tag{15}\]

In conformal coordinates the function \(\alpha\) is removed, having the consequence that the orbital area function \(R\) now depends on both \(t\) and \(\theta\) (in areal coordinates \(R = t\)). In these coordinates the metric and the Einstein-Vlasov system take the following form.
Conformal coordinates

Metric

\[ g = e^{2(\eta - U)}(-dt^2 + d\theta^2) + e^{2U}(dx + Ay)^2 + e^{-2U}R^2dy^2 \] (16)

The Einstein-matter constraint equations

\[ U_t^2 + U_\theta^2 \left[ \frac{4U}{2} (A_t^2 + A_\theta^2) + \frac{R_{t\theta}}{R} - \frac{\eta_t R_t}{R} - \frac{\eta_\theta R_\theta}{R} \right] = -e^{2(\eta - U)} \rho \] (17)

\[ 2U_t U_\theta + \frac{4U}{2R^2} A_t A_\theta + \frac{R_{t\theta}}{R} - \frac{\eta_t R_t}{R} - \frac{\eta_\theta R_\theta}{R} = e^{2(\eta - U)} J \] (18)

(19)

The Einstein-matter evolution equations

\[ U_{tt} - U_{t\theta} = \frac{U_t R_\theta}{R} - \frac{U_\theta R_t}{R} + \frac{4U}{2R^2} (A_t^2 - A_\theta^2) \]
\[ + \frac{1}{2} e^{2(\eta - U)} (\rho - P_1 + P_2 - P_3) \] (20)

\[ A_{tt} - A_{t\theta} = \frac{R_t A_t}{R} - \frac{R_\theta A_\theta}{R} + 4(A_\theta U_\theta - A_t U_t) + 2Re^{2(\eta - U)} S_{23} \] (21)

\[ R_{tt} - R_{t\theta} = Re^{2(\eta - U)} (\rho - P_1) \]

\[ \eta_{tt} - \eta_{t\theta} = U_\theta^2 - U_t^2 + \frac{4U}{4R^2} (A_t^2 - A_\theta^2) - e^{2(\eta - U)} P_3 \]
\[ - \frac{A^2}{R^2} e^{2(\eta + U)} P_2 - \frac{2A}{R} e^{2\eta} S_{23} \] (23)

The Vlasov equation

\[ \frac{\partial f}{\partial t} + \frac{v^1}{v^0} \frac{\partial f}{\partial \theta} - \left[ (\eta_\theta - U_\theta)v^0 + (\eta_t - U_t)v^1 - U_\theta \frac{(v^2)^2}{v^0} \right] \]
\[ + U_\theta \frac{(v^3)^2}{v^0} - A_\theta \frac{2U_t v^2 v^3}{v^0} \frac{\partial f}{\partial v^1} - \left[ U_t v^2 + U_\theta \frac{v^1 v^2}{v^0} \right] \frac{\partial f}{\partial v^2} \]
\[ - \left[ (\frac{R_t}{R} - U_t)v^3 - (\frac{R_\theta}{R} - U_\theta) \frac{v^1 v^3}{v^0} + \frac{e^{2U} v^2}{R} (A_t + A_\theta \frac{v^1}{v^0}) \right] \frac{\partial f}{\partial v^3} = 0 \] (24)
The matter quantities $\rho, P_k, J$ and $S_{23}$ are given by (10)-(13), where in this case

$$v^0 = e^{\eta-U} p^0, \quad v^1 = e^{(\eta-U)} p^1, \quad v^2 = e^{U} p^2 + A e^{U} p^3, \quad v^3 = R e^{U} p^3,$$

and (15) holds here as well.

**Remark.** It might be instructive to relate the metric in (16) with that used by Rein [Rn] mentioned in the introduction. By letting $A = 0$ and $U = (1/2) \ln R$ in (16), we obtain a metric which admits three Killing vectors and which depends on two field components. The distribution function $f$ depends in this case on $p^1$ and $(p^2)^2 + (p^3)^2$.

### 3 The main theorem

Let $(h, k, f_0)$ be a Gowdy symmetric initial data set on $T^3$, by this we mean that $h$ is a Riemannian metric on $T^3$, invariant under an effective $T^2$ action; $k$ is a symmetric 2-tensor on $T^3$, also invariant under the same $T^2$ group action; the twists $c_1$ and $c_2$ are both zero; the initial distribution function $f_0$ is defined on $T^3$ and is invariant under the same $T^2$ group action and possesses the following additional symmetry, which reads, in coordinates that cast the metric in the forms (2) or (16),

$$f_0(\theta, p^1, p^2, p^3) = f_0(\theta, p^1, -p^2, p^3) = f_0(\theta, p^1, p^2, -p^3)$$

(this assumption is ncessary for the Einstein-Vlasov system to be compatible with the form of the metric); and that $(h, k, f_0)$ satisfy the Einstein-Vlasov constraint equations. We also assume that $(h, k)$ are $C^\infty$ on $T^3$ and that $f_0$ is a non-negative, not identically zero, $C^\infty$ function of compact support on the tangent bundle $T(T^3)$ of $T^3$.

**Remark.** The smoothness assumption on the initial data is not a necessary condition. It is included so that we can refer directly to the classical local existence theorems. However, the estimates in this paper provide the information needed for proving a local existence theorem for $C^2 \times C^1$ data $(h, k)$ and $C^1$ data $f_0$. Moreover, the assumption $f_0 \neq 0$ is here included for a technical reason and we refer to [M] or section 5 in this paper for the vacuum case. Indeed, it is in this case possible to work directly in areal coordinates and the estimates derived in section 5 are sufficient. See also the remark following lemma 1 in that section.

The results by Choquet-Bruhat [CB] and Choquet-Bruhat and Geroch [CBG], show that there exists a unique maximal globally hyperbolic development $(\Sigma \times \mathbb{R}, g, f)$ of a given initial data set on a three-dimensional manifold $\Sigma$ for the Einstein-Vlasov equation. Let us briefly comment upon
the initial conditions imposed. The relations between a given initial data set \((h, k)\) on a three-dimensional manifold \(\Sigma\) and the metric \(g\) on the spacetime manifold is that there exists an imbedding \(\psi\) of \(\Sigma\) into the spacetime such that the induced metric and second fundamental form of \(\psi(\Sigma)\) coincide with the result of transporting \((h, k)\) with \(\psi\). For the relation of the distribution functions \(f\) and \(f_0\) we have to note that \(f\) is defined on the mass shell (for \(m = 1\) it is the set of all future pointing unit timelike vectors). The initial condition imposed is that the restriction of \(f\) to the part of the mass shell over \(\psi(\Sigma)\) should be equal to \(f_0 \circ (\psi^{-1}, d(\psi)^{-1}) \circ \phi\) where \(\phi\) sends each point of the mass shell over \(\psi(\Sigma)\), to its orthogonal projection onto the tangent space to \(\psi(\Sigma)\).

Our main theorem now reads,

**Theorem 1** Let \((h, k, f_0)\) be a smooth Gowdy symmetric initial data set on \(T^3\). For some non-negative constant \(c\), there exists a globally hyperbolic spacetime \((M, g, f)\) such that

(i) \(M = (c, \infty) \times T^3\)

(ii) \(g\) and \(f\) satisfy the Einstein-Vlasov equation

(iii) \(M\) is covered by areal coordinates \((t, \theta, x, y)\), with \(t \in (c, \infty)\), so the metric globally takes the form (3).

(iv) \((M, g, f)\) is isometrically diffeomorphic to the maximal globally hyperbolic development of the initial data \((h, k, f_0)\).

As described in the introduction we prove global existence in conformal and areal coordinates for the past and future directions respectively. Then, in order to prove Theorem 1 in the past direction, we need to invoke substantial geometrical arguments from [BCIM]. For the future direction only a simple geometrical argument is needed for completing the proof. It should be pointed out that even if the geometrical results in [BCIM] concern the vacuum case they are true also for matter spacetimes as long as the Einstein-matter equations form a well-posed hyperbolic system, which of course is the case here.

In section 4 we show that the past maximal development of \((h, k, f_0)\) in terms of conformal coordinates, which we denote by \(D^-_{\text{conf}}(h, k, f_0)\), has \(t \to -\infty\) as long as \(R\) stays bounded away from zero. Starting from this result we briefly describe how the geometrical arguments in [BCIM] lead to a proof of Theorem 1 in the past direction.

First, in [BCIM] \(R\) is shown to be positive everywhere in the globally hyperbolic region of a \(T^2\) symmetric spacetime. Also, along any past inextendible timelike path in \(D^-_{\text{conf}}(h, k, f_0)\), \(R\) is shown to approach a limit \(R_0 \geq 0\) (to be identified with \(c\) in Theorem 1), which is independent of the
choice of path. Moreover, for any $\tilde{R} \in (R_0, R_1)$, where $R_1$ is the minimum value of $R$ on the initial hypersurface, the level set $R = \tilde{R}$ in $D_{\text{conf}}^- (h, k, f_0)$ is shown to be a Cauchy surface. From these facts it follows from arguments in [Cl] that $D_{\text{conf}}^- (h, k, f_0)$ admits areal coordinates to the past of the hypersurface $R = R_1$. Proposition 4 and 5 in [BCIM] then show that $D_{\text{conf}}^- (h, k, f_0)$ is also isometrically diffeomorphic to the maximal past development, $D^- (h, k, f_0)$ of $(h, k, f_0)$ on $T^3$.

In the future direction, global existence in areal coordinates is almost sufficient for proving Theorem 1. The only statement that remains to be proved in Theorem 1 is that the future maximal development is covered by $D^+_{\text{areal}} (h, k, f_0)$. This follows from a very short geometrical argument given in the proof of Proposition 5 in [BCIM].

4 Analysis in the contracting direction

The local existence theorem of Choquet-Bruhat [CB] together with the result of Chrusciel (Lemma 4.2 in [Cl]) imply that for any Gowdy symmetric initial data set $(h, k, f_0)$ on $T^3$, we can find an interval $(\hat{t}_1, \hat{t}_2)$ and $C^\infty$ functions $R, U, \eta$ on $(\hat{t}_1, \hat{t}_2) \times T^3$, and a non-negative $C^\infty$ function $f$ on $(\hat{t}_1, \hat{t}_2) \times P$ ($P$ denotes the mass shell) such that: these functions satisfy the Einstein-Vlasov equations in conformal coordinate form and for some $t_0 \in (\hat{t}_1, \hat{t}_2)$, the metric $g$ induces initial data on the $t_0$-hypersurface which is smoothly spatially diffeomorphic to $(h, k)$, and the relation between $f$ and $f_0$ given above holds.

Now, in order to show that $D_{\text{conf}}^- (h, k, f_0)$ has $t \to -\infty$, as long as $R$ stays bounded away from zero, it is sufficient to prove that on any finite time interval $(\hat{t}, t_0]$, the functions $R, U, A, \eta, f$ and all their derivatives are uniformly bounded and that the supremum of the support of momenta at time $t$,

$$Q(t) := \sup \{|v| : \exists (s, \theta) \in [t, t_0] \times S^1 \text{ such that } f(s, \theta, v) \neq 0\},$$

(26)

is uniformly bounded. Note that the last condition implies that the matter quantities and their derivatives are uniformly bounded (if $|\partial f/\partial x^\mu| < C$).

Step 1. (Monotonicity of $R$ and bounds on its first derivatives.)

This is a key step and relies on Theorem 4.1 in [Cl] together with the arguments in [BCIM]. We have to check that the matter terms have the right signs so that these arguments still hold. The bounds on $R$ and its first derivatives will play a crucial role when we control the matter terms below.
First we show that $\nabla R$ is timelike. Let us introduce the null vector fields
\[ \partial_\xi = \frac{1}{\sqrt{2}}(\partial_t + \partial_\theta), \quad \partial_\lambda = \frac{1}{\sqrt{2}}(\partial_t - \partial_\theta), \]  
(27)
and let us set $F_\xi = \partial_\xi F$, $F_\lambda = \partial_\lambda F$ for a function $F$. After some algebra it follows that the constraint equations (17) and (18) can be written
\[ \partial_\theta R_\xi = \eta_\xi R_\xi - RU^2_\xi - Re^{2(\eta-U)}(\rho - J), \]  
(28)
\[ \partial_\theta R_\lambda = \eta_\lambda R_\lambda - RU^2_\lambda - Re^{2(\eta-U)}(\rho + J). \]  
(29)
Let $h_1$ and $h_2$ be defined by
\[ h_1 := RU^2_\xi + Re^{2(\eta-U)}(\rho - J), \]
and
\[ h_2 := RU^2_\lambda + Re^{2(\eta-U)}(\rho + J). \]
From (10) and (12) we have $\rho \geq |J|$, and since $R > 0$ it follows that both $h_1$ and $h_2$ are non-negative. Solving equation (28) gives for any $\theta_0 \in [0, 2\pi]$ (suppressing the $t$-dependence)
\[ R_\xi(\theta) = e^{\int_{\theta_0}^{\theta} \eta_\xi(\sigma) d\sigma} R_\xi(\theta_0) - \int_{\theta_0}^{\theta} e^{\int_{\tilde{\theta}_0}^{\tilde{\theta}} \eta_\xi(\sigma) d\sigma} h_1(\tilde{\theta}) d\tilde{\theta}. \]  
(30)
Since $R$ is $C^\infty$ on $S^1$ it can be identified with a periodic function on the real line. If now $R_\xi(\theta_0) = 0$ for any $\theta_0$ then $R_\xi(2\pi + \theta_0) = 0$, but from (30) this is only possible if $h_1$ vanishes identically. However, in the non-vacuum case (recall $f_0 \neq 0$) $h_1(t, \cdot)$ is strictly positive on some open set of $[0, 2\pi]$. Therefore $R_\xi$ is nonzero and has a definite sign. The same arguments apply to $R_\lambda$, and it follows that
\[ g^{\mu\nu} \partial_\mu R \partial_\nu R = e^{-2(\eta-U)} R_\xi R_\lambda \]
is strictly positive or strictly negative. The former possibility is ruled out since $\partial_\theta R = 0$ at some point on $S^1$. Thus $\nabla R$ is timelike. This means that $\partial_\theta R$ is nonzero everywhere. Our choice of time corresponds to contracting $T^2$ orbits so that $\partial_\theta R > 0$.

Next we show that $\partial_\theta R$ and $|\partial_\theta R|$ are bounded into the past. The evolution equation (22) can be written
\[ \partial_\lambda R_\xi = Re^{2(\eta-U)}(\rho - P_1), \]  
(31)
or equivalently,
\[ \partial_\xi R_\lambda = R e^{2(n-U)}(\rho - P_1). \]  
(32)
The right hand side is positive since \( \rho \geq P_1 \), see (10) and (11), and from (31) it follows that if we start at any point \((t_0, \theta_0)\) on the initial surface we obtain
\[ R_\xi(\theta_0 + s, t_0 - s) \leq R_\xi(t_0, \theta_0), \]  
(33)
and similarly from (32)
\[ R_\lambda(\theta_0 - s, t_0 - s) \leq R_\lambda(t_0, \theta_0). \]  
(34)
From these relations we get for any \( t \in (\tilde{t}, t_0) \) and any \( \theta \in S^1 \),
\[ R_\xi(t, \theta) \leq \sup_{\theta \in S^1} R_\xi(t_0, \theta), \]  
(35)
\[ R_\lambda(t, \theta) \leq \sup_{\theta \in S^1} R_\lambda(t_0, \theta). \]  
(36)
This yields
\[ R_t(t, \theta) \leq \sup_{\theta \in S^1} (R_\xi + R_\lambda)(t_0, \theta), \]  
(37)
and since \( \nabla R \) is timelike everywhere we have \( |R_t| > |R_\theta| \) and we find that both \( R_t \) and \( |R_\theta| \) are bounded into the past, so \( R \) is uniformly \( C^1 \) bounded to the past of the initial surface.

**Step 2.** (Bounds on \( U, A \) and \( \eta \) and their first derivatives.)
The bounds on \( U_t, A_t, U_\theta \) and \( A_\theta \) to the past of the initial surface are obtained by a light-cone estimate, which in this case, with one spatial dimension, is an application of the Gronwall method on two independent null paths. Then, by combining these results, one obtains the desired estimate.

The functions involved in the light-cone argument are quadratic functions in the first order derivatives of \( U \) and \( A \), defined by
\[ G = \frac{1}{2} R(U_t^2 + U_\theta^2) + \frac{e^{4U}}{8R}(A_t^2 + A_\theta^2), \]  
(38)
\[ H = RU_t U_\theta + \frac{e^{4U}}{4R} A_t A_\theta. \]  
(39)
We will see below that if we let the vector fields along the null paths act on \( G + H \) and \( G - H \) we obtain equations appropriate for applying a Gronwall argument. First, however, we motivate the definition of \( G \) and \( H \) by showing that they are components of an “energy-momentum tensor” of a wave map. So let us follow [BCIM] and give a brief discussion of such a map in this context. Consider a base Lorentzian manifold \((\mathcal{N}, \nu)\) with the two dimensional
manifold $\mathcal{N}$ corresponding to the past conformal coordinate development of $(h, k, f_0)$ with the metric

$$\nu := -dt^2 + d\theta^2,$$

and consider a family of target manifolds $(\mathbb{R}^2, d_{(t, \theta)})$ with

$$d_{(t, \theta)} := R(t, \theta)dU^2 + \frac{e^{4U}}{4R(t, \theta)}dA^2.$$

Define the map

$$\Phi : \mathcal{N} \rightarrow \mathbb{R}^2,$$

$$(t, \theta) \mapsto \Phi(t, \theta) = (U(t, \theta), A(t, \theta)), \quad (40)$$

with $d_{(t, \theta)}$ providing an inner product on their tangents, e.g.

$$<\Phi_t, \Phi_\theta> = d_{ab}\Phi^a_t\Phi^b_\theta = RU_tU_\theta + \frac{e^{4U}}{4R}A_tA_\theta.$$

There is a covariant derivative $D$ compatible with $\nu$ and semi-compatible with $d_{(t, \theta)}$. Using Greek indices for the base and Latin indices for the target the action $D$ can be expressed as

$$D^\mu\Phi^a_\lambda = \partial^\mu\Phi^a_\lambda + \Gamma^a_\lambda\Phi^b_\mu\Phi^c_\kappa - \Gamma^\kappa_\mu\Phi^a_\kappa, \quad (41)$$

with

$$\Gamma^U_{UU} = 0, \quad \Gamma^U_{UA} = 0, \quad \Gamma^U_{AA} = -\frac{e^{4U}}{2R^2},$$

$$\Gamma^A_{AA} = 0, \quad \Gamma^A_{AU} = 2, \quad \Gamma^A_{UU} = 0,$$

and the base Christoffel coefficients $\Gamma^\kappa_{\lambda\mu} = 0$ (in the expanding direction these will be nonzero). As pointed out above, $D$ is compatible with the flat metric $\nu$ but not with $d_{(t, \theta)}$, we have

$$D^\lambda d_{ab} = R_\lambda(\delta^U_{a}\delta^U_{b} - \frac{e^{4U}}{4R^2}\delta^A_{a}\delta^A_{b}).$$

By defining the wave operator $\Box := \nu^{\lambda\mu}D^\lambda D_\mu$ the evolution equations for $U$ and $A$ take the form

$$\Box U = \frac{U_tR_t}{R} - \frac{U_\theta R_\theta}{R} + \frac{1}{2}e^{2(\eta-U)}(p - P_1 + P_2 - P_3)$$

$$\Box A = -\frac{A_tR_t}{R} + \frac{A_\theta R_\theta}{R} + 2Re^{2(\eta-U)}S_{23}.$$
Now we define an “energy-momentum tensor” for the maps $\Phi$,

$$
T_{\lambda\mu} = <\Phi_\lambda, \Phi_\mu> - \frac{1}{2} \nu_{\lambda\mu} \nu_{\delta\gamma} <\Phi_\delta, \Phi_\gamma>
$$

$$
= RU_\lambda U_\mu + \frac{e^{4U}}{4R} A_\lambda A_\mu + \frac{1}{2} \nu_{\lambda\mu} [R(U_t^2 - U_\theta^2) + \frac{e^{4U}}{4R}(A_t^2 - A_\theta^2)].
$$

We find that $G = T_{tt}$ and $H = T_{t\theta}$.

Let us now derive bounds on $U$ and $A$ and their first order derivatives.

By using the evolution equation (20) and (21) we find

$$
\partial_\lambda (G + H) = \frac{-1}{2\sqrt{2}} R_\lambda \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2} (-A_t^2 + A_\theta^2) \right)
$$

$$
+ \frac{R}{2} U_\xi e^{2(\eta-U)} (\rho - P_1 + P_2 - P_3)
$$

$$
+ \frac{e^{2U}}{2R} A_\xi R e^{2(\eta-U)} S_{23},
$$

and

$$
\partial_\xi (G - H) = \frac{-1}{2\sqrt{2}} R_\lambda \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2} (-A_t^2 + A_\theta^2) \right)
$$

$$
+ \frac{R}{2} U_\lambda e^{2(\eta-U)} (\rho - P_1 + P_2 - P_3)
$$

$$
+ \frac{e^{2U}}{2R} A_\lambda R e^{2(\eta-U)} S_{23}.
$$

Now, integrating these equations along null paths starting at $(t_1, \theta)$ and ending at the initial $t_0$–surface, and adding the results we obtain

$$
G(t_1, \theta) = \frac{1}{2} [G + H](t_0, \theta) \pm (t_0 - t_1) + \frac{1}{2} [G + H](t_0, \theta + (t_0 - t_1))
$$

$$
- \frac{1}{2} \int_{t_1}^{t_0} K_1(s, \theta - (s - t_1)) + K_2(s, \theta + (s - t_1)) ds
$$

$$
- \frac{1}{2} \int_{t_1}^{t_0} [U_\xi T](s, \theta - (s - t_1)) + [U_\lambda T](s, \theta + (s - t_1)) ds
$$

$$
- \frac{1}{2} \int_{t_1}^{t_0} \left[ \frac{e^{2U}}{2R} A_\xi \tilde{T} \right](s, \theta - (s - t_1)) + \left[ \frac{e^{2U}}{2R} A_\lambda \tilde{T} \right](s, \theta + (s - t_1)) ds,
$$

where we have introduced the notations

$$
K_1 = \frac{-1}{2\sqrt{2}} R_\lambda \left( (U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2} (-A_t^2 + A_\theta^2) \right),
$$

$$
K_2 = \frac{-1}{2\sqrt{2}} R_\lambda \left( (U_t^2 - U_\theta^2 + \frac{e^{4U}}{R^2} (-A_t^2 + A_\theta^2) \right).
$$
\[ K_2 = \frac{-1}{2\sqrt{2}} R_\xi \left( (U_t^2 - U^2_{\theta}) + \frac{e^{4U}}{R^2} (-A_t^2 + A^2_{\theta}) \right), \quad (44) \]

\[ T = \frac{R}{2} e^{2(\eta-U)} (\rho - P_1 + P_2 - P_3) \quad (45) \]

\[ \tilde{T} = R e^{2(\eta-U)} S_{23}. \quad (46) \]

Let us first consider the matter terms. Note that for any \( t \in (\tilde{t}, t_0) \), the evolution equations (31) and (32) give

\[ R_\xi(t_0, \theta + (t_0 - t)) - R_\xi(t, \theta) = \sqrt{2} \int_t^{t_0} [Re^{2(\eta-U)}(\rho - P_1)](s, \theta + (s - t))ds, \quad (47) \]

and

\[ R_\lambda(t_0, \theta - (t_0 - t)) - R_\xi(t, \theta) = \sqrt{2} \int_t^{t_0} [Re^{2(\eta-U)}(\rho - P_1)](s, \theta - (s - t))ds. \quad (48) \]

Hence, since \( R \) is uniformly \( C^1 \) bounded to the past of the initial surface it follows that the right hand sides are uniformly bounded. From (10)-(11) we have

\[ \rho \geq P_1 + P_2 + P_3, \]

and thus

\[ 0 \leq (\rho - P_1 + P_2 - P_3) \leq 2(\rho - P_1), \]

and from (13) and the elementary inequality \( 2ab \leq a^2 + b^2 \), \( a, b \in \mathbb{R} \), we have

\[ 2|S_{23}| \leq P_2 + P_3 \leq \rho - P_1. \]

In view of (47) and (48) we therefore have that both

\[ \int_t^{t_0} T(s, \theta \pm (s - t))ds, \quad (49) \]

and

\[ \int_t^{t_0} |\tilde{T}(s, \theta \pm (s - t))|ds, \quad (50) \]

are uniformly bounded on \((\tilde{t}, t_0) \times S^1\), say by \( C(t) \). Now, by using the inequality \( 2ab \leq a^2 + b^2 \) again, we get

\[ |U_\xi| \leq \left( \frac{2G}{R} \right)^{1/2}, \]
\[
\frac{e^{2U}}{2R} |A_e| \leq \left( \frac{2G}{R} \right)^{1/2}.
\]

The same estimates also hold for \( U_\lambda \) and \( A_\lambda \). Since \( R_\xi \) and \( R_\lambda \) are uniformly bounded it clearly follows that

\[
|K_1| \leq \frac{CG}{R}, \quad |K_2| \leq \frac{CG}{R},
\]

for some constant \( C \). So the identity (42) now implies, as long as \( R \) stays bounded away from zero, that

\[
\sup_{\theta} G(t_1, \theta) \leq \sup_{\theta} 2G(t_0, \theta) + \sup_{\theta} 2G(t_0, \theta) + C \sup_{[t_1, t_0] \times S^1} (G, \cdot))^{1/2} + \int_{t_1}^{t_0} C \sup_{\theta} G(s, \cdot) ds.
\]

(51)

(52)

On a sub time interval \([\sigma_1, \sigma_0] \subset [t_1, t_0]\) on which \( \sup_{S^1} G(s, \cdot) \) is increasing as \( s \to \sigma_1 \), which is the case of interest for us since we wish to obtain a uniform upper bound on \( G \), we have

\[
\sup_{[\sigma_1, \sigma_0] \times S^1} (G, \cdot))^{1/2} \leq \sup_{S^1} (G(\sigma_1, \cdot))^{1/2}.
\]

Therefore, by dividing both sides by \( \sup_{S^1} (G(\sigma_1, \cdot))^{1/2} \) we obtain a Gronwall inequality on the time interval \([\sigma_1, \sigma_0] \) for \( \sup_{S^1} G^{1/2} \) leading to a upper bound on \( \sup_{S^1} G(\sigma_1, \cdot) \) as long as \( R \) stays bounded away from zero. By repeating this argument we conclude that \( \sup_{S^1} G \) is uniformly bounded on \([\sigma_1, \sigma_0] \times S^1 \), leading to bounds on \( U \) and its first order derivatives, and thus also on \( A \) and its first order derivatives, as long as \( R \) stays bounded away from zero. The bounds on \( |\eta|, |\eta_\rho| \) and \( |\eta_\theta| \) are obtained in a similar way since the evolution equation (23) can be written

\[
\partial_{\lambda} \eta_\xi = U_\theta - U_\xi + \frac{e^{2U}}{4R^2} (A_\theta^2 - A_\xi^2) - e^{2(\eta - U)} (P_3 + \frac{A^2}{R^2} e^{2U} P_2 + 2A e^{2U} S_3),
\]

(53)

or equivalently,

\[
\partial_{\xi} \eta_\lambda = U_\theta - U_\xi + \frac{e^{2U}}{4R^2} (A_\theta^2 - A_\xi^2) - e^{2(\eta - U)} (P_3 + \frac{A^2}{R^2} e^{2U} P_2 + 2A e^{2U} S_3).
\]

(54)

We found above that the integrals along null paths for the matter quantity \( Re^{2(\eta - U)}(\rho - P_1) \) were bounded to the past of the initial surface. Therefore,
since $0 \leq P_k \leq \rho - P_1$, $k = 2, 3$ and $|S_{23}| \leq \rho - P_1$ we have, as long as $R$ stays bounded away from zero, that the integrals along the null paths for the matter terms in the right hand sides above are bounded as well, since $U$ and $A$ are bounded. Now, since the first order derivatives of $U$ and $A$ are uniformly bounded we immediately obtain that $|\eta_\xi|$ and $|\eta_\lambda|$ are bounded by integrating the equations for $\eta$ along null paths. Since $\eta_t = \frac{1}{\sqrt{2}}(\eta_\xi + \eta_\lambda)$ and $\eta_\xi = \frac{1}{\sqrt{2}}(\eta_\xi - \eta_\lambda)$ we find that $\eta$ is uniformly $C^1$ bounded to the past of the initial surface as long as $R$ stays bounded away from zero.

Step 3. (Bound on the support of the momentum.)

Note that a solution $f$ to the Vlasov equation is given by

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)), \quad (55)$$

where $\Theta$ and $V$ are solutions to the characteristic system

$$\begin{align*}
\frac{d\Theta}{ds} &= \frac{V^1}{V^0}, \\
\frac{dV^1}{ds} &= -(\eta_\theta - U_\theta)V^0 - (\eta_t - U_t)V^1 + U_\theta \frac{(V^2)^2}{V^0} \\
&\quad - \left(\frac{R_\theta}{R} - \frac{R_t}{R}\right) \frac{(V^3)^2}{V^0} + \frac{A_\theta}{R} e^{2U} \frac{V^2 V^3}{V^0}, \\
\frac{dV^2}{ds} &= -U_t V^2 - U_\theta \frac{V^1 V^2}{V^0}, \\
\frac{dV^3}{ds} &= -\left(\frac{R_t}{R} - \frac{R_\theta}{R}\right) \frac{V^1 V^3}{V^0} \\
&\quad - \frac{e^{2U}}{R} (A_t + \frac{A_\theta V^1}{V^0}) V^2,
\end{align*}$$

and $\Theta(s, t, x, v), V(s, t, x, v)$ is the solution that goes through the point $(\theta, v)$ at time $t$. Let us recall the definition of

$$Q(t) := \sup\{|v| : \exists (s, \theta) \in [\tilde{t}, t_0] \times S^1 \text{ such that } f(s, \theta, v) \neq 0\}.$$ 

If $Q(t)$ can be controlled we obtain immediately from (11)-(12) bounds on $\rho, J, S_{23}$ and $P_k$, $k = 1, 2, 3$, since $||f||_\infty \leq ||f_0||_\infty$ from (55). Now, all of the field components and their first derivatives are known to be bounded on $(\tilde{t}, t_0)$, as long as $R$ stays bounded away from zero. Also, the distribution function has compact support on the initial surface and therefore $|V^k(t_0)| < C$. So by observing that $|V^k| < V^0$, $k = 1, 2, 3$, a simple Gronwall argument applied to the characteristic system gives uniform bounds on $|V^k(t)|$, $t \in (\tilde{t}, t_0)$, and it follows that $Q(t)$ is uniformly bounded on $(\tilde{t}, t_0)$.  

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**Remark.** By a Killing vector argument, bounds on $|V^2|$ and $|V^3|$ can be derived if merely $|U|$ and $|A|$ are bounded and $R > \epsilon > 0$. Such an argument will be used in the expanding direction.

**Step 4.** (Bounds on the second order derivatives of the field components and on the first order derivatives of $f$.) From the Einstein-matter constraint equations in conformal coordinates we can express $R_{\theta\theta}$ and $R_{\theta t}$ in terms of uniformly bounded quantities, as long as $R$ stays bounded away from zero. Therefore these functions are uniformly bounded and equation (22) then implies that $R_{tt}$ is uniformly bounded as well.

In the vacuum case one can take the derivative of the evolution equations and repeat the argument in step 2 to obtain bounds on second order derivatives of $U$ and $A$. Here we need another argument. First we write the evolution equations for $U$ and $A$ in the forms

$$U_{tt} - U_{\theta\theta} = \frac{(R_{\theta} - R_{t})}{2R}(U_{\theta} + U_{t}) - \frac{(R_{\theta} + R_{t})}{2R}(U_{t} - U_{\theta})$$

$$+ \frac{e^{4U}}{2R^{2}}(A_{t} - A_{\theta})(A_{t} + A_{\theta}) + \frac{1}{2}e^{2(\eta - U)}\kappa,$$  \hfill (56)

and

$$A_{tt} - A_{\theta\theta} = \frac{(R_{t} - R_{\theta})}{2R}(A_{\theta} + A_{t}) + \frac{(R_{\theta} + R_{t})}{2R}(A_{t} - A_{\theta})$$

$$- 2(A_{t} - A_{\theta})(U_{\theta} + U_{t}) - 2(A_{\theta} + A_{t})(U_{t} - U_{\theta})$$

$$+ 2Re^{2(\eta - 2U)}S_{23},$$ \hfill (57)

where $\kappa$ denotes $\rho - P_{1} + P_{2} - P_{3}$. Taking the $\theta$-derivative of these equations gives

$$\partial_{\lambda}\partial_{\xi}U_{\theta} = L + \frac{R_{\lambda}}{2R}\partial_{\xi}U_{\theta} + \frac{R_{\xi}}{2R}\partial_{\lambda}U_{\theta}$$

$$+ \frac{e^{4U}}{2R^{2}}(A_{\lambda}\partial_{\xi}A_{\lambda} + A_{\xi}\partial_{\lambda}A_{\theta}) + \frac{1}{4}e^{2(\eta - U)}\kappa_{\theta},$$ \hfill (58)

and

$$\partial_{\lambda}\partial_{\xi}A_{\theta} = L + \frac{R_{\lambda}}{2R}\partial_{\xi}A_{\theta} - \frac{R_{\xi}}{2R}\partial_{\lambda}A_{\theta} + 2U_{\xi}\partial_{\lambda}A_{\theta} + 2A_{\lambda}\partial_{\xi}U_{\theta}$$

$$+ 2U_{\lambda}\partial_{\xi}A_{\theta} + 2A_{\xi}\partial_{\lambda}U_{\theta} + 2Re^{2(\eta - 2U)}(S_{23})_{\theta},$$ \hfill (59)

Here, $L$ contains only $\kappa$ and $S_{23}$, first order derivatives of $U, A$ and $\eta$, and first and second order derivatives of $R$, which all are known to be bounded.
These equations can of course also be written in a form where the left hand 
sides read \( \partial_\xi \partial_\lambda U_\theta \) and \( \partial_\xi \partial_\lambda A_\theta \), respectively. By integrating these equations 
along null paths to the past of the initial surface, we get from a Gronwall 
argument a bound on 

\[
\sup_{\theta \in S^1} (|\partial_\xi U_\theta| + |\partial_\lambda U_\theta| + |\partial_\xi A_\theta| + |\partial_\lambda A_\theta|),
\]
as long as \( R \) is bounded away from zero, under the hypothesis that the 
integral of the differentiated matter terms \( \kappa_\theta \) and \( (S_{23})_\theta \) can be controlled. In 
order to bound these integrals we make use of a device introduced by Glassey 
and Strauss [GS] for treating the Vlasov-Maxwell equation. It is sufficient 
to show how one of the differentiated matter terms can be bounded since 
the arguments are similar in all cases. Let us consider the integral appearing 
by integrating (58) along the null path defined by \( \partial_\lambda = \partial_t - \partial_\theta \), 

\[
\frac{1}{4} \int^{t_0}_t \int_{\mathbb{R}^3} \left[ e^{2(\eta-U)} v^0 \partial_\theta f \right](s, \theta - (s-t), v) dv ds,
\]

where \( t \in (\tilde{t}, t_0] \). Next, define 

\[
W = \sqrt{2} \partial_\lambda = \partial_t - \partial_\theta, \quad S = \partial_t + \frac{v^1}{v^0} \partial_\theta.
\]

Hence, \( \partial_\theta \) and \( \partial_t \) can be expressed in terms of \( W \) and \( S \) by 

\[
\partial_\theta = \frac{v^0}{v^0 + v^1} (S - W), \quad (61)
\]

\[
\partial_t = \frac{v^0}{v^0 + v^1} (S + \frac{v^1}{v^0} W). \quad (62)
\]

Now, 

\[
[Wf](s, \theta - (s-t), v) = \partial_s[f(s, \theta - (s-t), v)],
\]

and from the Vlasov equation we get 

\[
[Sf](s, \theta - (s-t), v) = [-K \cdot \nabla_v f](s, \theta - (s-t), v),
\]

where it is clear which terms have been denoted by \( K = (K_1, K_2, K_3) \). By 
using (61) we can now evaluate the integral above by integrating by parts 
in \( s \) for the \( W \)-term and in \( v \) for the \( S \)-term, so that the remaining terms 
only involve bounded quantities. Note in particular the \( v \)-integrals are easily 
controlled in view of the uniform bound on \( Q(t) \). Thus, the integrals
of the differentiated matter terms can be controlled and the Gronwall argument referred to above goes through. So we obtain uniform bounds on $|\partial_t U_\theta|, |\partial_\theta U_\theta|, |\partial_\lambda A_\theta|$, and $|\partial_\lambda A_\theta|$, and therefore also on $|U_{\theta\theta}|, |U_{\theta t}|, |A_{\theta\theta}|$ and $|A_{\theta t}|$, as long as $R$ is bounded away from zero. The evolution equations (20) and (21) then give uniform bounds on $|U_{tt}|$ and $|A_{tt}|$. By differentiating equation (23), it is now straightforward to obtain bounds on the second order derivatives of $\eta$, using similar arguments to those already discussed here, in particular the integrals involving matter quantities can be treated as above. Bounds on the first order derivatives of the distribution function $f$ may now be obtained from the known bounds on the field components from the formula

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)),$$

(63)

since $f_0$ is smooth and since $\partial \Theta$ and $\partial V$ (here $\partial$ denotes $\partial_t, \partial_\theta$ or $\partial_v$) can be controlled by a Gronwall argument in view of the characteristic system.

Step 5. (Bounds on higher order derivatives and completion of the proof.) It is clear that the method described above can be continued for obtaining bounds on higher derivatives as well. Hence, we have uniform bounds on the functions $R, U, A, \eta$ and $f$ and all their derivatives on the interval $(\tilde{t}, t_0]$ if $R > \epsilon > 0$. This implies that the solution extends to $t \to -\infty$ as long as $R$ stays bounded away from zero. In view of the discussion after the statement of Theorem 1, this completes the proof of Theorem 1 in the contracting direction.

\[\square\]

5 Analysis in the expanding direction

To begin the analysis in the expanding direction (increasing $R$) in areal coordinates we need to start with data on a $R$ =constant Cauchy surface (recall that in areal coordinates $R = t$). That this can be done follows from the geometrical arguments in [BCIM] (cf. the discussion following the statement of Theorem 1). There it is shown that if Gowdy symmetric (or more generally $T^2$ symmetric) data is given on $T^3$, and if $R_0$ is the past limit of $R$ along past inextendible paths in $D^-_{\text{conf}}$ and if $R_1 := \inf_{T^3} R$, then for every $d \in (R_0, R_1)$, the $R = d$ level set $\Sigma_d$ is a Cauchy surface, and these $\Sigma_d$ foliate the region $D^-_{\text{conf}} \cap I^-(\Sigma_{R_1})$. Here $I^-(S)$ is the chronological past of $S$ (see [HE]). The surfaces $\Sigma_d$ lie to the past of the initial surface. Let us
pick one of them, say $\Sigma_{d_2}$. The spacetime $D^{-}(h, k, f_0)$ induces initial data for the areal component fields $(U, A, \eta, \alpha)$ and the distribution function $f$ on $\Sigma_{t_2=d_2}$. By combining the local existence proof in harmonic coordinates [CB], and the arguments in [Cl] which show that the spacetime admits areal coordinates, we obtain local existence for the initial value problem in these coordinates. Now, in order to extend local existence to global existence in these coordinates, it is again sufficient to obtain uniform bounds on the field components and the distribution function and all their derivatives on a finite time interval $[t_2, t_3]$ on which the local solution exists.

Step 1. (Bounds on $\alpha, U, A$ and $\tilde{\eta}$.)

In this step we first show an “energy” monotonicity lemma and then we show how this result leads to bounds on $\tilde{\eta} := \eta + \ln \alpha/2$ and on $U$ and $A$.

Let $E(t)$ be defined by

$$E(t) = \int_{S^1} \left[ \alpha^{-\frac{1}{2}} U_t^2 + \sqrt{\alpha} U_\theta^2 + \frac{e^{4U}}{4t^2} \left( \alpha^{-\frac{1}{2}} A_t^2 + \sqrt{\alpha} A_\theta^2 \right) + \sqrt{\alpha} e^{2(\eta - U)} \rho \right] d\theta.$$ 

Lemma 1 $E(t)$ is a monotonically decreasing function in $t$, and satisfies

$$\frac{d}{dt} E(t) = -2t \int_{S^1} \left[ \alpha^{-1/2} U_t^2 + \frac{e^{4U}}{4t^2} \sqrt{\alpha} A_\theta^2 + \sqrt{\alpha} e^{2(\eta - U)} (\rho + P) \right] d\theta \leq 0. \quad (64)$$

Proof. This is a straightforward computation, using the evolution equation for $U$ and $A$, the constraint equations and the Vlasov equation. First, using the evolution equations for $U$ and $A$ we obtain after a short computation

$$\frac{d}{dt} \int_{S^1} \left[ \alpha^{-1/2} U_t^2 + \sqrt{\alpha} U_\theta^2 + \frac{e^{4U}}{4t^2} \left( \alpha^{-1/2} A_t^2 + \sqrt{\alpha} A_\theta^2 \right) \right] d\theta$$

$$= \int_{S^1} \frac{\alpha t}{2\alpha^{3/2}} \left( U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A_\theta^2) \right) - \frac{2t}{\alpha} \left( \frac{U_t^2}{\sqrt{\alpha}} + \frac{e^{4U}}{4t^2} \sqrt{\alpha} A_\theta^2 \right) + \int_{S^1} \sqrt{\alpha} U_t e^{2(\eta - U)} (\rho - P_1 + P_2 - P_3) + \frac{e^{2\eta}}{t} \sqrt{\alpha} A_t S_{23} d\theta. \quad (65)$$

Next we have

$$\frac{d}{dt} \int_{S^1} \sqrt{\alpha} e^{2(\eta - U)} \rho d\theta = \int_{S^1} \frac{\alpha t}{2\sqrt{\alpha}} e^{2(\eta - U)} \rho d\theta$$

$$+ \int_{S^1} 2\sqrt{\alpha}(\eta_t - U_t)e^{2(\eta - U)} \rho d\theta + \int_{S^1} \sqrt{\alpha} e^{2(\eta - U)} \rho_t d\theta. \quad (66)$$

$$+ \int_{S^1} 2\sqrt{\alpha}(\eta_t - U_t)e^{2(\eta - U)} \rho d\theta + \int_{S^1} \sqrt{\alpha} e^{2(\eta - U)} \rho_t d\theta. \quad (67)$$
Now we use the Vlasov equation and the definition of $\rho$ to write the last integral as

$$
\int_{S^1} \sqrt{\alpha} e^{2(\eta-U)} \rho t d\theta = \int_{S^1} \int_{\mathbb{R}^3} \sqrt{\alpha} e^{2(\eta-U)} \left[ -\sqrt{\alpha} v^1 \partial_{\theta} f \right] dv d\theta
$$

$$
+ \int_{S^1} \int_{\mathbb{R}^3} \sqrt{\alpha} e^{2(\eta-U)} \left[ \eta_\theta - U_\theta + \frac{\alpha}{2\alpha} \sqrt{\alpha} (v^0)^2 + (\eta_t - U_t) v^1 v^0ight.
$$

$$
+ \sqrt{\alpha} U_\theta [(v^1)^2 - (v^2)^2] - \frac{e^{2U}}{t} \sqrt{\alpha} A_\theta v^2 v^3] \frac{\partial f}{\partial v^1} d\theta d\theta
$$

$$
+ \int_{S^1} \int_{\mathbb{R}^3} \sqrt{\alpha} e^{2(\eta-U)} \left[ (1 - U_t) v^3 v^0 - \sqrt{\alpha} U_\theta v^1 v^3 \right]
$$

$$
+ \frac{e^{2U}}{t} (v^0 A_t + \sqrt{\alpha} A_\theta v^1) v^2 \right] \frac{\partial f}{\partial v^3} d\theta d\theta.
$$

Integrating by parts (using the periodicity in $\theta$ and the compact support in $v$) and rearranging the terms we get

$$
\int_{S^1} \sqrt{\alpha} e^{2(\eta-U)} \rho t d\theta = - \int_{S^1} \int_{\mathbb{R}^3} \sqrt{\alpha} e^{2(\eta-U)} \left[ (v^0 + \frac{(v^1)^2}{v^0}) \eta_t - (v^0 + \frac{(v^1)^2}{v^0} + \frac{(v^2)^2}{v^0} + \frac{(v^3)^2}{v^0}) U_t + \frac{e^{2U}}{t} A_t \frac{v^2 v^3}{v^0} \right] f dv d\theta
$$

$$
- \int_{S^1} \int_{\mathbb{R}^3} \sqrt{\alpha} \frac{e^{2(\eta-U)}}{t} \left( v^0 + \frac{(v^3)^2}{v^0} \right) f dv d\theta.
$$

By adding (65) and (67), using (69) for the last term in (67), we see that the terms involving $U_t$ and $A_t$ vanish and we obtain

$$
\frac{d}{dt} E(t) = \int_{S^1} \frac{\alpha_t}{2\alpha^{3/2}} [U_t^2 + \alpha U^2_\theta + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A^2_\theta) + \alpha e^{2(\eta-U)} \rho] d\theta
$$

$$
+ \int_{S^1} \sqrt{\alpha} e^{2(\eta-U)} (\rho - P_1) \eta_t d\theta
$$

$$
- \int_{S^1} \frac{2U_t^2}{t} + \frac{e^{4U}}{2t^2} \sqrt{\alpha} A^2_\theta + \frac{\sqrt{\alpha}}{t} e^{2(\eta-U)} (\rho + P_3) d\theta.
$$

By substituting $\alpha_t$ and $\eta_t$ using the constraint equations

$$
\frac{\eta_t}{t} = U_t^2 + \alpha U^2_\theta + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A^2_\theta) + e^{2(\eta-U)} \alpha \rho,
$$

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and
\[ \alpha_t = 2t\alpha^2 e^{2(\eta - U)}(P_1 - \rho), \tag{72} \]
the first two terms cancel, and \((64)\) follows.

\[ \square \]

**Remark.** It is clear from \((64)\) that a Gronwall argument leads to a bound on \(E(t)\) also on \((0, t_2]\). For \(T^2\) symmetry and vacuum, which is considered in \([BCIM]\), this bound is not available. A natural question is then why the areal coordinates in our case has to be discarded in the analysis for the past direction. However, the analysis of the characteristic system associated to the Vlasov equation in lemma 2 depends on the time direction.

Let us now define the quantity \(\tilde{\eta}\) by
\[ \tilde{\eta} = \eta + \frac{1}{2} \ln \alpha. \tag{73} \]

From the constraint equation \((6)\) we get
\[ \tilde{\eta}_\theta = 2tU_\theta U_\theta + \frac{e^{4U}}{2t} A_\theta A_\theta - t\sqrt{\alpha} e^{2(\eta - U)} J. \tag{74} \]

Now, from the elementary inequality \(|ab| \leq \frac{1}{2} a^2 + 2cb^2\), for any \(a, b, c \in \mathbb{R}, c > 0\), and from the fact that \(|J| \leq \rho\), it follows from lemma 1 that for any \(t \in [t_2, t_3)\)
\[ \int_{S^1} |\tilde{\eta}_\theta| d\theta \leq tE(t) \leq tE(t_2). \tag{75} \]

Hence, for any \(\theta_1, \theta_2 \in S^1\) and for any \(t \in [t_2, t_3)\) we have
\[ |\tilde{\eta}(t, \theta_2) - \tilde{\eta}(t, \theta_1)| = \int_{\theta_1}^{\theta_2} |\tilde{\eta}_\theta| d\theta \leq \int_{S^1} |\tilde{\eta}_\theta| d\theta \leq tE(t_2). \tag{76} \]

Next, using the constraint equations \((3)\) and \((5)\), we find that the time derivative of \(\tilde{\eta}\) satisfies
\[ \tilde{\eta}_t = t[U_\theta^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_\theta^2 + \alpha A_\theta^2) + \alpha e^{2(\eta - U)} P_1] \geq 0. \tag{77} \]

This relation leads to a control of \(\int_{S^1} \tilde{\eta} d\theta\) from above, namely
\[ \int_{S^1} \tilde{\eta}(t, \theta) d\theta - \int_{S^1} \tilde{\eta}(t_2, \theta) d\theta = \int_{t_2}^t \frac{d}{dt} \left( \int_{S^1} \tilde{\eta}(s, \theta) d\theta \right) ds \]
\[
\begin{align*}
&= \int_{t_2}^{t} \int_{S^1} \sqrt{\alpha s} \left[ \frac{U^2}{\sqrt{\alpha}} + \sqrt{\alpha U^2} + \frac{e^{4U}}{4s^2} (\frac{A^2}{\sqrt{\alpha}} + \sqrt{\alpha A^2}) + \sqrt{\alpha e^{2(\alpha-U)} P_1} \right] d\theta ds \\
&\leq \sup_{S^1} \sqrt{\alpha(t_2, \cdot)} \int_{t_2}^{t} sE(s) ds \leq C_1 \int_{t_2}^{t} sE(t_2) ds = C_1 E(t_2)(t^2 - t_2^2)/2 .
\end{align*}
\] (78)

In the first inequality above we used that \( P_1 \leq \rho \) and in the second that \( \alpha \) is a monotonically decreasing function in \( t \) (see (5)) and \( C_1 := \sup_{S^1} \sqrt{\alpha(t_2, \cdot)} \).

We are now in a position to obtain a upper bound on \( \tilde{\eta} \) itself. By letting \( C_2 := \int_{S^1} \tilde{\eta}(t_2, \theta) d\theta \) we get from (78) the inequality

\[
\frac{1}{2} C_1 E(t_2)(t^2 - t_2^2) + C_2 \geq \int_{S^1} \tilde{\eta}(t, \theta) d\theta
\] (79)

\[= 2\pi \max_{S^1} \tilde{\eta} + \int_{S^1} (\tilde{\eta} - \max_{S^1} \tilde{\eta}) d\theta .\] (80)

By applying (78) to the last term we find

\[
\frac{1}{2} E(t_2)(t^2 - t_2^2) + C_2 \geq 2\pi \max_{S^1} \tilde{\eta} - 2\pi tE(t_2) .
\] (81)

Therefore, for some bounded function \( C(t) \), we have the upper bound

\[
\max_{S^1} \tilde{\eta} \leq C(t) ,
\] (82)

and since \( \tilde{\eta}_t \geq 0 \) we conclude that \( \tilde{\eta} \) is uniformly bounded on \( S^1 \times [t_2, t_3] \).

**Remark.** In the analysis below \( C(t) \) will always denote a uniformly bounded function on \( [t_2, t_3] \). Sometimes we introduce other functions with the same property only for the purpose of trying to make some estimates become more transparent.

Next we show that the boundedness of \( E(t) \), together with the constraint equation (5), lead to a bound on \( |U| \). For any \( \theta_1, \theta_2 \in S^1 \), and \( t \in [t_2, t_3] \) we get by Hölder’s inequality

\[
|U(t, \theta_2) - U(t, \theta_1)| = \left| \int_{\theta_1}^{\theta_2} U_\theta(t, \theta) d\theta \right| \\
\leq \left( \int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta \right)^{1/2} \left( \int_{\theta_1}^{\theta_2} \sqrt{\alpha U^2_\theta} d\theta \right)^{1/2} .
\] (83)
The second factor on the right hand side is clearly bounded by \((E(t_2))^{1/2}\). For the first factor we use the constraint equation (5). This equation can be written as

\[
\partial_t (\alpha^{-1/2}) = t \sqrt{\alpha e^{2(\eta - U)}} (\rho - P_1),
\]  

(84)

so that for \(t \in [t_2, t_3)\)

\[
\alpha^{-1/2}(t, \theta) = \int_{t_2}^{t} s \sqrt{\alpha e^{2(\eta - U)}} (\rho - P_1) ds + \alpha^{-1/2}(t_2, \theta).
\]  

(85)

Since \(\rho \geq P_1\), the integrand is positive and bounded by the last term in the integrand of \(E(t)\). Letting \(C\) denote the supremum of \(\alpha^{-1/2}(t_2, \cdot)\) over \(S^1\) we get

\[
\int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta \leq \int_{t_2}^{t} s \int_{S^1} \sqrt{\alpha e^{2(\eta - U)}} \rho d\theta ds + 2\pi C
\]

\[
\leq E(t_2)(t^2 - t_2^2)/2 + 2\pi C.
\]  

(86)

Hence, for any \(\theta_1, \theta_2 \in S^1\) we have

\[
|U(t, \theta_2) - U(t, \theta_1)| \leq C(t).
\]  

(87)

Next we estimate \(\int_{S^1} U(t, \theta) d\theta\). Let \(C := \int_{S^1} U(t_2, \theta) d\theta\), we get by Hölder’s inequality

\[
\left| \int_{S^1} U(t, \theta) d\theta \right| = \left| \int_{t_2}^{t} \int_{S^1} U_t(s, \theta) d\theta ds + C \right|
\]

\[
\leq \int_{t_2}^{t} \int_{S^1} |U_t(s, \theta)| d\theta ds + |C|
\]

\[
\leq \int_{t_2}^{t} \left( \int_{S^1} \sqrt{\alpha} d\theta \right)^{1/2} \left( \int_{S^1} \alpha^{-1/2} U_t^2 d\theta \right)^{1/2} ds + |C|.
\]  

(88)

The right hand side is easily seen to be bounded since (5) shows that \(\sqrt{\alpha}\) is monotonically decreasing and (64) gives a bound for the second factor. Therefore

\[
\left| \int_{S^1} U(t, \theta) d\theta \right| \leq C(t),
\]

for some uniformly bounded function \(C(t)\). To obtain a uniform bound on \(U\) we combine these results. Let \(U_+(t) := \max_{S^1} U(t, \cdot)\), and \(U_-(t) := \min_{S^1} U(t, \cdot)\). We have

\[
2\pi U_\pm(t) = \int_{S^1} U(t, \theta) d\theta + \int_{S^1} (U_+(t) - U(t, \theta)) d\theta,
\]  

(89)
and the right hand side is bounded from below and above so $U$ is uniformly bounded on $[t_2, t_3) \times S^1$. These arguments also apply to $A$ as well, since the factor $e^{4U}$ is controlled by the uniform bound on $U$.

**Remark.** In the case studied in [BCIM], i.e. vacuum and $T^2$ symmetry, a bound on $\ln \alpha$, and thus on $\eta$, is directly available. On the other hand, the method used here to bound $U$ and $A$ does not apply. This may lead to a difficulty in generalizing the result in [BCIM] to matter spacetimes, since the bounds on $U$ and $A$ are crucial in order to treat the matter terms when the derivatives of $U$ and $A$ are to be bounded.

**Step 2.** (Bounds on $U_t, U_\theta, A_t, A_\theta, \eta_t, \alpha_t$ and $Q(t)$.)

To bound the derivatives of $U$ we use light-cone estimates in a similar way as for the contracting direction. However, the matter terms must be treated differently and we need to carry out a careful analysis of the characteristic system associated with the Vlasov equation. Let us define

$$ G = \frac{1}{2} (U_t^2 + \alpha U_\theta^2) + \frac{e^{4U}}{8t^2} (A_t^2 + \alpha A_\theta^2), \quad (90) $$

$$ H = \sqrt{\alpha U_t U_\theta} + \frac{e^{4U}}{4t^2} A_t A_\theta, \quad (91) $$

and

$$ \partial_\chi = \frac{1}{\sqrt{2}} (\partial_t + \sqrt{\alpha} \partial_\theta) \quad (92) $$

$$ \partial_\zeta = \frac{1}{\sqrt{2}} (\partial_t - \sqrt{\alpha} \partial_\theta) \quad (93) $$

A motivation for the introduction of these quantities is based on similar arguments as those given in step 2, section 4. For details we refer to [BCIM].

**Remark.** We use the same notations, $G$ and $H$, as in the contracting direction, and below we continue to carry over the notations. The analysis in the respective direction is independent so there should be no risk of confusion.

By using the evolution equation (7), a short computation shows that

$$ \partial_\zeta (G + H) = \frac{\alpha_t}{2\sqrt{2\alpha}} (G + H) $$

$$ - \frac{1}{\sqrt{2}t} \left( U_t^2 + \sqrt{\alpha} U_t U_\theta + \frac{e^{4U}}{4t^2} (\alpha A_\theta^2 + \sqrt{\alpha} A_t A_\theta) \right) $$

$$ + (U_t + \sqrt{\alpha} U_\theta) \frac{\alpha}{2\sqrt{2}} e^{2(\eta - U)} \kappa + \frac{\alpha e^{2\eta}}{2\sqrt{2}t} (A_t + \sqrt{\alpha} A_\theta) S_{23}, $$

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\[ \partial_t (G - H) = \frac{\alpha t}{2\sqrt{2t}} (G - H) \]
\[ \quad - \frac{1}{\sqrt{2t}} \left( U_t^2 - \sqrt{\alpha} U_t U_\theta + \frac{\alpha^2 U}{4t^2} (\alpha A_\theta^2 - \sqrt{\alpha} A_t A_\theta) \right) \]
\[ \quad + (U_t - \sqrt{\alpha} U_\theta) \frac{\alpha e^{2(\eta - U)}}{2\sqrt{2}} \kappa + \frac{\alpha e^{2\eta}}{2\sqrt{2t}} (A_t - \sqrt{\alpha} A_\theta) S_{23}. \]

(95)

Here \( \kappa = \rho - P_1 + P_2 - P_3 \). Now we wish to integrate these equations along the integral curves of the vector fields \( \partial_x \) and \( \partial_z \) respectively (let us henceforth call these integral curves null curves, since they are null with respect to the two-dimensional “base spacetime”). Below we show that the quantity
\[ \Gamma(t) := \sup_{\theta \in S^1} G(t, \cdot) + Q^2(t), \]
(96)
is uniformly bounded on \( [t_2, t_3] \) by deriving the inequality
\[ \Gamma(t) \leq C + \int_{t_2}^{t} \Gamma(s) \ln \Gamma(s) ds. \]
(97)

We begin with two observations. Let \( \gamma \) and \( X \) be a geodesic and a Killing vector field respectively in any spacetime. Then \( g(\gamma', X) \) is conserved along the geodesic. Here \( \gamma' \) is the tangent vector to \( \gamma \). In our case we have the two Killing vector fields \( \partial_x \) and \( \partial_y \). The particles follow the geodesics of spacetime with tangent \( p^\mu \), so \( g_{\lambda \mu} p^\mu (\partial_x)^\nu \) and \( g_{\lambda \mu} p^\mu (\partial_y)^\nu \) are thus conserved. Expressing \( p^\mu \) in terms of \( v^\mu \) (see (14)) we find that
\[ V^2(t)e^{U(t, \Theta(t))} \]
and
\[ V^2(t)Ae^U + V^3(t)te^{-U(t, \Theta(t))}, \]
are conserved. Here \( V^2(t), V^3(t) \) and \( \Theta(t) \) are solutions to the characteristic system associated to the Vlasov equation. From step 2 we have that \( U \) and \( A \) are uniformly bounded on \( [t_2, t_3] \). Hence \( |V^2(t)| \) and \( |V^3(t)| \) are both uniformly bounded on \( [t_2, t_3] \), and since the initial distribution function \( f_0 \) has compact support we conclude that
\[ \sup\{|v|^2 + |v|^3| : \exists(s, \theta) \in [t_2, t] \times S^1 \text{ with } f(s, \theta, v) \neq 0\}, \]
(98)
is uniformly bounded on \([t_2, t_3]\). Therefore, in order to control \(Q(t)\) it is sufficient to control

\[
Q^1(t) := \sup\{|v^1| : \exists (s, \theta) \in [t_2, t] \times \mathbb{S}^1 \text{ such that } f(s, \theta, v) \neq 0\}.
\] (99)

Below we introduce the uniformly bounded function \(\gamma(t)\) to denote estimates regarding the variables \(v^2\) and \(v^3\). Next we observe that there is some cancellation to take advantage of in the matter term \((\rho - P_1)\) which appears in the equations for \(G + H\) and \(G - H\) above. This term can be estimated as follows

\[
0 \leq (\rho - P_1)(t, \theta) = \int_{\mathbb{R}^3} \left( v^0 - \frac{(v^1)^2}{v^0} \right) f(t, \theta, v) dv \\
= \int_{\mathbb{R}^3} \frac{1 + (v^2)^2 + (v^3)^2}{v^0} f(t, \theta, v) dv \\
\leq \int_{\mathbb{R}^3} \left[ 1 + (v^2)^2 + (v^3)^2 \right] |f| \frac{dv}{\sqrt{1 + (v^1)^2}} \\
\leq \|f_0\|_{\infty} \gamma(t) \int_{|v^1| \leq Q^1(t)} \frac{dv^1}{\sqrt{1 + (v^1)^2}} \\
\leq C\gamma(t) \ln Q^1(t).
\] (100)

In a similar fashion we can estimate \(P_2, P_3\) and \(S_{23}\). Indeed, for \(k = 1, 2\), we have

\[
0 \leq P_k(t, \theta) = \int_{\mathbb{R}^3} \frac{(v^k)^2}{v^0} f(t, \theta, v) dv \\
\leq \|f_0\|_{\infty} \gamma(t) \int_{|v^1| \leq Q^1(t)} \frac{dv^1}{\sqrt{1 + (v^1)^2}} \\
\leq C\gamma(t) \ln Q^1(t).
\] (101)

The argument is almost identical for \(S_{23}\).

**Remark.** Since the matter of interest is large momenta we have here assumed that \(Q^1(t) \geq 2\) to avoid the introduction of some immaterial constants in the estimates.

Let us now derive (97). As in step 2 in section 4 we integrate the equations above for \(G + H\) and \(G - H\) along null paths. For \(t \geq t_2\), let

\[
A(t, \theta) = \int_{t_2}^t \sqrt{\alpha}(s, \theta) ds,
\]
and integrate along the two null paths defined by $\partial \chi$ and $\partial \zeta$, starting at $(t_2, \theta)$ and add the results. We get for $t \in [t_2, t_3)$,

$$G(t, \theta) = \frac{1}{2} \{G + H\}(t_2, \theta - (A(t) - t_2)) + \frac{1}{2} \{G + H\}(t_2, \theta + (A(t) - t_2))$$

$$+ \frac{1}{2} \int_{t_2}^{t} K_1(s, \theta - (A(s) - t_2)) + K_2(s, \theta + (A(s) - t_2)) \, ds$$

$$+ \frac{1}{2} \int_{t_2}^{t} L_1(s, \theta - (A(s) - t_2)) + L_2(s, \theta + (A(s) - t_2)) \, ds$$

$$+ \frac{1}{2} \int_{t_2}^{t} [U \chi M](s, \theta - (A(s) - t_2)) + [U \zeta M](s, \theta + (A(s) - t_2)) \, ds$$

$$+ \frac{1}{2} \int_{t_2}^{t} \frac{A \chi M}{2t}(s, \theta - (A(s) - t_2)) + \left[\frac{A \zeta M}{2t}\right](s, \theta + (A(s) - t_2)) \, ds,$$

(102)

where

$$K_1 = \frac{\alpha t}{2\sqrt{2\alpha}} (G + H), \quad K_2 = \frac{\alpha t}{2\sqrt{2\alpha}} (G - H),$$

(103)

$$L_1 = -\frac{1}{\sqrt{2t}} \left( U_1^2 + \sqrt{\alpha U_1 U_\theta} + \frac{e^{4U}}{4t^2} (\alpha A^2_{\theta} + \sqrt{\alpha A_{\theta} A_{\theta}}) \right),$$

(104)

$$L_2 = -\frac{1}{\sqrt{2t}} \left( U_1^2 - \sqrt{\alpha U_1 U_\theta} + \frac{e^{4U}}{4t^2} (\alpha A^2_{\theta} - \sqrt{\alpha A_{\theta} A_{\theta}}) \right),$$

(105)

$$M = \frac{1}{2} e^{2(\bar{\eta} - U)} K, \quad \tilde{M} = e^{2\tilde{\eta}} S_{23}.$$  

(106)

Note that in the expression for $M$ and $\tilde{M}$ we used $\alpha e^{2\eta} = e^{2\tilde{\eta}}$. It is easy to see that both $G + H$ and $G - H$ can be written as sums of two squares. From the constraint equation equation (5) we find that $\alpha t / \alpha \leq 0$ so that $K_1$ and $K_2$ are nonpositive. Using the elementary inequality $2ab \leq a^2 + b^2$ and the fact that $|\bar{\eta}|$ and $|U|$ are uniformly bounded we obtain from (102) the inequality

$$\sup_{\theta} G(t, \cdot) \leq \sup_{\theta} G(t_2, \cdot) + \sup_{\theta} H(t_2, \cdot) + C \int_{t_2}^{t} \frac{1}{s} \sup_{\theta} G(s, \cdot) \, ds$$

$$+ \int_{t_2}^{t} C(s) \sup_{\theta} \sqrt{G(s, \cdot)} ((\rho - P_1 + P_2 - P_3) + S_{23}) \, ds$$

$$\leq C + C(t) \int_{t_2}^{t} \left[ \sup_{\theta} G(s, \cdot) + \sup_{\theta} \sqrt{G(s, \cdot)} \ln Q^1(s) \right] \, ds,$$

(107)
where (100) and (101) were used in the last inequality.

**Remark.** The sign of $K_1$ and $K_2$ simplified the estimate above. This is not crucial since $|\alpha_t|/\alpha$ is bounded by $\ln Q^1(t)$ which is sufficient for obtaining a bound on $\Gamma(t)$.

Let us now derive an estimate for $Q^1$ in terms of $\sup_\theta G$.

**Lemma 2** Let $Q^1(t)$ and $G(t, \theta)$ be as above. Then

$$|Q^1(t)|^2 \leq C + D(t) \int_{t_2}^{t} [(Q^1(s))^2 + \sup_\theta G(s, \cdot)] ds,$$

(108)

where $C$ is a constant and $D(t)$ is a uniformly bounded function on $[t_2, t_3]$.

**Proof.** The characteristic equation for $V^1$ associated to the Vlasov equation reads

$$\frac{dV^1(s)}{ds} = -(\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha}V^0 - (\eta_t - U_t)V^1$$

$$- \frac{\sqrt{\alpha}U_\theta}{V^0}((V^2)^2 - (V^3)^2) + \frac{\sqrt{\alpha}A_\theta}{sv^0}s^{2U}e^{2U}v^2v^3.$$  (109)

We will now split the right hand side into three terms to be analyzed separately. Expressing $\eta_\theta$ and $\eta_t$ by using the constraint equations (3) and (4) we obtain

$$\frac{d}{ds} (V^1(s))^2 = 2V^1(s)\frac{d}{ds} V^1(s) = T_1 + T_2 + T_3,$$  (110)

where

$$T_1 = -2V^1(s)[s\alpha e^{2(\eta-U)}(JV^0 + \rho V^1)],$$

$$T_2 = -2V^1(s) \left[s(U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4s^2} (A_t^2 + \alpha A_\theta^2))V^1ight.$$

$$+ 2s\sqrt{\alpha}U_\theta U_t V^0 - \sqrt{\alpha}U_\theta V^0 - U_t V^1 + \frac{e^{4U}}{2s} \sqrt{\alpha}A_t A_\theta V^0 \right],$$

$$T_3 = -2V^1(s) \left[\frac{\sqrt{\alpha}U_\theta}{V^0}((V^2)^2 - (V^3)^2) - \frac{\sqrt{\alpha}A_\theta}{sv^0}e^{2U}v^2v^3 \right].$$

Let us first estimate $T_1$. We split it into two terms

$$T_1 = T_1^- + T_1^+ = -2sV^1(s)e^{2(\tilde{\eta}-U)}(I^- + I^+),$$  (111)
where

\[ I^- = \int_{\mathbb{R}^2} \int_{-\infty}^{0} (v^1V^0 + v^0V^1) f(s, \theta, v)dv^1 dv^2 dv^3, \]
\[ I^+ = \int_{\mathbb{R}^2} \int_{0}^{\infty} (v^1V^0 + v^0V^1) f(s, \theta, v)dv^1 dv^2 dv^3. \]

Let us now consider the two cases \( V^1(s) > 0 \) and \( V^1(s) < 0 \). On a time interval where \( V^1(s) > 0 \), \( I^+ \) is nonnegative and \( T_1^+ \) can therefore be discarded since it is nonpositive. The kernel in \( I^- \) can be estimated as follows

\[ v^1V^0 + v^0V^1 = \frac{(v^1)^2(V^0)^2 - (v^0)^2(V^1)^2}{v^1V^0 - v^0V^1} = \frac{(v^1)^2(1 + (V^2)^2 + (V^3)^2)}{v^1V^0 - v^0V^1} + \frac{(V^1)^2(1 + (v^2)^2 + (v^3)^2)}{v^0V^1 - v^1V^0}. \]

Of course, the cancellation of the terms \((v^1)^2(V^1)^2\) is essential in this computation. The second term is positive since \( V^1(s) > 0 \) and \( v^1 < 0 \), and contributes negatively to \( T_1^- \) and can be discarded. The first term is negative and the modulus can be estimated by

\[ \frac{(v^1)^2(1 + (V^2)^2 + (V^3)^2)}{|v^1V^0 + v^0V^1|} \leq \frac{|v^1|(1 + (V^2)^2 + (V^3)^2)}{V^1}. \] (112)

In the expression for \( T_1 \) we first note that \( 2s\alpha e^{2(\eta - U)} = 2s e^{2(\eta - U)} \leq C(s) \).

Hence, on the time interval where \( V^1(s) > 0 \) we can estimate \( T_1 \) by

\[ T_1 \leq T_1^- \leq \|f_0\|_{\infty} C(s) V^1(s) \int_{\mathbb{R}^2} \int_{0}^{Q^1} \frac{v^1(1 + (V^2)^2 + (V^3)^2)}{V^1(s)} dv^1 du \]
\[ \leq \|f_0\|_{C(s)} \gamma(s) \int_{0}^{Q^1} v^1 dv^1 \leq C(s)(Q^1(s))^2. \] (113)

On a time interval where \( V^1 < 0 \) we see that \( T_1^- \) is nonpositive and can be discarded. We can then estimate \( T_1^+ \) by using almost identical arguments as for \( T_1^- \) and we get also on such a time interval,

\[ T_1 \leq T_1^+ \leq C(s)(Q^1(s))^2. \] (114)

Let us now consider \( T_2 \). We again study the cases \( V^1(s) > 0 \) and \( V^1(s) < 0 \). Assume first that \( V^1(s) > 0 \) on some time interval. The expression for \( T_2 \) can be written \( T_2 = T_2^p + T_2^r \) (p=principal, r=rest) where

\[ T_2^p = -2(V^1(s))^2 \left( [s(U_t + \sqrt{\alpha}U_\theta)^2 - (U_t + \sqrt{\alpha}U_\theta)] + [\epsilon t^4(U_t + \sqrt{\alpha}U_\theta)^2] \right) \]

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\[ T'_2 = 2(V^0(s) - V^1(s))V^1(s)\sqrt{\alpha U_\theta} - 2s\sqrt{\alpha U_t U_\theta} - \frac{e^{4U}}{2s}\sqrt{\alpha A_t A_\theta}. \]

For \( T'_2 \) we have
\[ |T'_2| = \frac{2(1 + (V^2)^2 + (V^3)^2)V^{1(s)}}{V^0 + V^1} \sqrt{\alpha U_\theta} - 2sU_t U_\theta - \frac{e^{4U}}{2s}\sqrt{\alpha A_t A_\theta} \]
\[ \leq (s + 1)\gamma(s) \sup G(s, \cdot). \quad (115) \]

Since the matter of interest is large \( G \) we have here assumed that \( \sqrt{G} \leq G \).
This assumption will be used below without comment. To estimate \( T_2^p \) we observe that for \( s \geq t_2 \),
\[ sa^2 - a \geq -\frac{1}{4s} \geq -\frac{1}{4t_2}, \text{ for any } a \in \mathbb{R}. \]

The term involving \( A \) contributes negatively and can be discarded, thus
\[ T_2^p \leq \frac{1}{2t_2}(V^1(s))^2 \leq C(\tilde{Q}^1(s))^2. \quad (116) \]

On a time interval where \( V^1(s) < 0 \), the same estimates hold. Indeed, we only have to write \( T_2 = T_2^p + T_2' \) in the form
\[ T_2^p = -2(V^1(s))^2 \left[ s(U_t - \sqrt{\alpha U_\theta})^2 - (U_t - \sqrt{\alpha U_\theta}) \right] + \frac{e^{4U}}{4s}(\sqrt{\alpha A_t A_\theta})^2 \]
and
\[ T_2' = 2(V^0(s) + V^1(s))V^1(s)[\sqrt{\alpha U_\theta} - 2s\sqrt{\alpha U_t U_\theta} - \frac{e^{4U}}{2s}\sqrt{\alpha A_t A_\theta}], \]
and the same arguments apply. Therefore we have obtained
\[ T_2 \leq T_2^p + |T_2'| \leq C(\tilde{Q}^1(s))^2 + C(s) \sup G(s, \cdot). \quad (117) \]

Finally we estimate \( T_3 \). It follows immediately that
\[ |T_3| \leq \gamma(s)\frac{|V^1(s)|}{V^0} \sqrt{\alpha U_\theta} + \frac{e^{2U}}{s} A_\theta \leq C(s) \sup G(s, \cdot). \quad (118) \]

The lemma now follows by adding the estimates for \( T_k, k = 1, 2, 3 \).
Combining the estimate for $(Q^1(t))^2$ in the lemma and the estimate (107) for $\sup_\theta G(t, \cdot)$, we find that $\Gamma(t)$ satisfies the estimate (97) and is thus uniformly bounded. The constraint equation (3) now immediately shows that $|\eta_t|$ is bounded by

$$2tG + te^{2(\tilde{\eta} - U)} \rho \leq C(t)[\sup_\theta G(t, \cdot) + (Q(t))^3],$$

since

$$\rho = \int_{\mathbb{R}^3} f dv \leq \|f_0\|_\infty \int_{|v| \leq Q(t)} dv \leq C(Q(t))^3.$$  

Analogous arguments show that $|\alpha_t|$ is uniformly bounded. The uniform bound on $G$ provides bounds on $|U_t|$ and $|A_t|$, but to conclude that $|U_\theta|$ and $|A_\theta|$ are bounded we have to show that $\alpha$ stays uniformly bounded away from zero. Equation (3) is easily solved,

$$\alpha(t, \theta) = \alpha(t_2, \theta)e^{\int_{t_2}^t F(s, \theta) ds},$$

where

$$F(t, \theta) := -2te^{2(\tilde{\eta} - U)}(\rho - P_1),$$

which is uniformly bounded from below. Hence $|U_\theta|$ and $|A_\theta|$ are bounded and step 2 is complete.

**Step 3.** (Bounds on $\partial f$, $\alpha_\theta$ and $\eta_\theta$.)

The main goal in this step is to show that the first derivatives of the distribution function are bounded. In view of the bound on $Q(t)$ we then also obtain bounds on the first derivatives of the matter terms $\rho, J, S_{23}$ and $P_k$, $k = 1, 2, 3$. Such bounds almost immediately lead to bounds on $\alpha_\theta$ and $\eta_\theta$.

Recall that the solution $f$ can be written in the form

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)),$$

where $\Theta(s, t, \theta, v), V(s, t, \theta, v)$ is the solution to the characteristic system

\begin{align*}
\frac{d\Theta}{ds} &= \sqrt{\alpha} \frac{V^1}{V^0}, \\
\frac{dV^1}{ds} &= -(\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha}V^0 - (\eta_t - U_t)V^1 \\
&\quad - \sqrt{\alpha}U_\theta \frac{(V^3)^2 - (V^2)^2}{V^0} + \frac{e^{2U}}{s} \sqrt{\alpha}A_\theta \frac{V^2V^3}{V^0}.
\end{align*}
\[
\frac{dV^2}{ds} = -U_t V^2 - \sqrt{\alpha U_\theta} \frac{V^1 V^2}{V^0}, \quad (123)
\]
\[
\frac{dV^3}{ds} = -\left(\frac{1}{s} - U_t\right) V^3 + \sqrt{\alpha U_\theta} \frac{V^1 V^3}{V^0} - \frac{\epsilon^2 U_\theta}{s} (A_t + \sqrt{\alpha A_\theta} \frac{V^1}{V^0}) V^2, \quad (124)
\]

with the property \( \Theta(t, t, \theta, v) = \theta, V(t, t, \theta, v) = v \). Hence, in order to establish bounds on the first derivatives of \( f \) it is sufficient to bound \( \partial \Theta \) and \( \partial V \) since \( f_0 \) is smooth. Here \( \partial \) denotes the first order derivative with respect to \( t, \theta \) or \( v \). Evolution equations for \( \partial \Theta \) and \( \partial V \) are provided by the characteristic system above. However, the right hand sides will contain second order derivatives of the field components, but so far we have only obtained bounds on the first order derivatives (except for \( \eta_\theta, \alpha_\theta \)). Yet, certain combinations of second order derivatives can be controlled. Behind this observation lies a geometrical idea which plays a fundamental role in general relativity. An important property of curvature is its control over the relative behaviour of nearby geodesics. Let \( \gamma(u, \lambda) \) be a two-parameter family of geodesics, i.e. for each fixed \( \lambda \), the curve \( u \mapsto \gamma(u, \lambda) \) is a geodesic. Define the variation vector field \( \mathbf{Y} := \gamma_{\lambda}(u, 0) \). This vector field satisfies the geodesic deviation equation (or Jacobi equation) (see eg. \([\text{HE}]\))
\[
\frac{D^2 \mathbf{Y}}{Du^2} = R_{\mathbf{Y} \gamma'} \gamma', \quad (125)
\]
where \( D/Du \) is the covariant derivative, \( R \) the Riemann curvature tensor, and \( \gamma' := \gamma_u(u, 0) \). Now, the Einstein tensor is closely related to the curvature tensor and since the Einstein tensor is proportional to the energy momentum tensor which we can control from step 2, it is meaningful, in view of \([\text{25}]\) (with \( Y = \partial \Theta \)), to look for linear combinations of \( \partial \Theta \) and \( \partial V \) which satisfy an equation with bounded coefficients. More precisely, we want to substitute the twice differentiated field components which appear by taking the derivative of the characteristic system by using the Einstein equations. The geodesic deviation equation has previously played an important role in studies of the Einstein-Vlasov system ([RR], [Rn] and [RI3]).

**Lemma 3** Let \( \Theta(s) = \Theta(s, t, \theta, v) \) and \( V^k(s) = V^k(s, t, \theta, v) \), \( k = 1, 2, 3 \) be a solution to the characteristic system \([121], [122], [123]\). Let \( \partial \) denote \( \partial_t, \partial_\theta \) or \( \partial_v \), and define
\[
\Psi = \alpha^{-1/2} \partial \Theta, \quad (126)
\]
\[ Z^1 = \partial V^1 + \left( \frac{\eta V^0}{\sqrt{\alpha}} - \frac{U_t V^0}{\sqrt{\alpha}} \frac{(V^0)^2 - (V^1)^2 + (V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} \right) \]
\[ + U_\theta \frac{V^1 ((V^2)^2 - (V^3)^2)}{(V^0)^2 - (V^1)^2} - \frac{A_\theta e^{2U}}{\sqrt{\alpha}} \frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2} \]
\[ + A_\theta \frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2} \partial \Theta, \]  
(127)

\[ Z^2 = \partial V^2 + V^2 U_\theta \partial \Theta, \]  
(128)

\[ Z^3 = \partial V^3 - (V^3 U_\theta - \frac{e^{2U}}{s} V^2 A_\theta) \partial \Theta. \]  
(129)

Then there is a matrix \( A = \{a_{lm}\}, l, m = 0, 1, 2, 3, \) such that
\[ \Omega := (\Psi, Z^1, Z^2, Z^3)^T \]
satisfies
\[ \frac{d\Omega}{ds} = A\Omega, \]  
(130)
and the matrix elements \( a_{lm} = a_{lm}(s, \Theta(s), V^k(s)) \) are all uniformly bounded on \( [t_2, t_3) \).

**Sketch of proof.** Once the ansatz \((126)-(129)\) has been found this is only a lengthy calculation. To illustrate the type of calculations involved we show the easiest case, i.e. the \( Z^2 \) term.

\[ \frac{dZ^2}{ds} = \frac{d}{ds}(\partial V^2 + V^2 U_\theta \partial \Theta) \]
\[ = \partial (\frac{d}{ds} V^2) + \frac{dV^2}{ds} U_\theta \partial \Theta \]
\[ + V^2 (U_{t\theta} + U_{\theta\theta} \frac{d\Theta}{ds}) \partial \Theta + V^2 U_\theta \frac{d\Theta}{ds} \partial \Theta. \]  
(131)

Now we use \((121)\) and \((123)\) to substitute for \( d\Theta/ds \) and \( dV^2/ds \). We find that the right hand side equals
\[ \partial (-U_t V^2 - \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0}) + (-U_t V^2 - \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0}) U_\theta \partial \Theta \]
\[ + V^2 (U_{t\theta} + U_{\theta\theta} \sqrt{\alpha} \frac{V^1}{V^0}) \partial \Theta + V^2 U_\theta \left( \frac{\alpha V^1}{2 \sqrt{\alpha} V^0} \partial \Theta + \sqrt{\alpha} \partial (\frac{V^1}{V^0}) \right). \]

Taking the \( \partial \) derivative of the first term we find that all terms of second order derivatives and terms containing \( \alpha_\theta \) cancel. Next, since
\[ - \sqrt{\alpha} U_\theta \partial \left( \frac{V^1 V^2}{V^0} \right) + \sqrt{\alpha} U_\theta V^2 \partial \left( \frac{V^1}{V^0} \right) = - \sqrt{\alpha} U_\theta \frac{V^1}{V^0} \partial V^2, \]  
(132)
we are left with
\[ \frac{dZ^2}{ds} = -(U_t V^2 + \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0})U_\theta \partial \Theta - (U_t + \sqrt{\alpha} U_\theta \frac{V^1}{V^0}) \partial V^2. \]  

Finally we express this in terms of \( \Psi, Z^1, Z^2 \) and \( Z^3 \). Here this is easy and we immediately get
\[ \frac{dZ^2}{ds} = -(U_t + \sqrt{\alpha} U_\theta \frac{V^1}{V^0})Z^2. \]

Clearly, the map \((\partial \Theta, \partial V^k) \mapsto (\Psi, Z^k)\) is invertible so that this step is easy also in the other cases. It follows that the matrix elements \( a_{2m}, m = 0, 1, 2, 3 \), are uniformly bounded on \([t_2, t_3)\) (only \( a_{22} \) is nonzero here). The computations for the other terms are similar. For the \( Z^1 \) term we point out that the evolution equations (7) and (8) should be invoked and that the matrix element \( a_{10} \) contains \( \eta_\theta \) and \( \alpha_\theta/2\alpha \), but they combine and form \( \tilde{\eta} \).

\[ \square \]

From the lemma it now immediately follows that \( |\Omega| \) is uniformly bounded on \([t_2, t_3)\). Moreover, since the system \((126)-(129)\) is invertible with uniformly bounded coefficients we also have uniform bounds on \( |\partial \Theta| \) and \( |\partial V^k|, k = 1, 2, 3 \). In view of the discussion at the beginning of this section we see that the distribution function \( f \) and the matter quantities \( \rho, J, S_{23} \), and \( P_k \), are all uniformly \( C^1 \) bounded. From the constraint equation (5) we now obtain a uniform bound on \( \alpha U_\theta \) by a simple Gronwall argument using as usual the identity \( \alpha e^{2(\tilde{\eta} - U)} = e^{2(\tilde{\eta} - U)} \). Finally this yields a uniform bound on \( \eta_\theta \) since
\[ \eta_\theta = \tilde{\eta}_\theta - \frac{\alpha_\theta}{2\alpha} \]
and \( \alpha \) stays uniformly bounded away from zero.

**Step 4.** (Bounds on second and higher order derivatives.)

It is now easy to obtain bounds on second order derivatives on \( U \) and \( A \) by using light cone arguments. We define \( G \) and \( H \) by

\[ G = \frac{1}{2}(U_t^2 + \alpha U_{\theta \theta}) + \frac{e^{4U}}{8t^2}(A_t^2 + \alpha A_{\theta \theta}^2), \]
\[ H = \sqrt{\alpha} U_t U_{\theta \theta} + \frac{e^{4U}}{4t^2} A_t A_{\theta \theta}, \]

and use the differentiated (with respect to \( t \)) evolution equations for \( U \) and \( A \) to obtain equations similar to (94) and (95). In this case a straightforward
light cone argument applies since we have control of the differentiated matter terms. \( U_{\theta\theta} \) and \( A_{\theta\theta} \) are then uniformly bounded in view of the evolution equations (7) and (8). Bounds on second order derivatives on \( f \) then follows from (130) by studying the equation for \( \partial \Omega \). The only thing to notice is that \( \tilde{\eta}_{\theta\theta} \) is controlled by (4). It is clear that this reasoning can be continued to give uniform bounds on \( [t_2, t_3) \) for higher order derivatives as well. In view of the discussion after the statement of Theorem 1 in section 3, this completes the proof of Theorem 1 in the expanding direction.

\[ \square \]

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