Invariant measures for the nonlinear stochastic heat equation with no drift term

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Abstract

This paper deals with the long term behavior of the solution to the nonlinear stochastic heat equation \( \partial u / \partial t - \frac{1}{2} \Delta u = b(u) \dot{W} \), where \( b \) is assumed to be a globally Lipschitz continuous function and the noise \( \dot{W} \) is a centered and spatially homogeneous Gaussian noise that is white in time. Using the moment formulas obtained in [9, 10], we identify a set of conditions on the initial data, the correlation measure and the weight function \( \rho \), which will together guarantee the existence of an invariant measure in the weighted space \( L^2_\rho(\mathbb{R}^d) \). In particular, our result includes the parabolic Anderson model (i.e., the case when \( b(u) = \lambda u \)) starting from the Dirac delta measure.

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Contents

1 Introduction .................................................. 2
  1.1 Main results .................................................. 4
  1.2 Outline and notation ....................................... 6

2 Moment estimates – Proof of Theorem 1.2 ............. 7

3 A factorization lemma ........................................ 9

4 Tightness and construction – Proof of Theorem 1.4 .... 14
  4.1 Proof of part (a) of Theorem 1.4 ......................... 14
  4.2 Proof of part (b) of Theorem 1.4 ......................... 16

5 Discussion and examples ..................................... 17
  5.1 Invariant measures for SHE with a drift term .......... 17
  5.2 The conditions for the spectral measures by Tessitore and Zabczyk ... 17
  5.3 Various initial conditions .................................. 20
  5.4 Bessel and other related kernels .......................... 22
  5.5 Examples of admissible weight functions ............... 26

References .................................................. 28
1 Introduction

In this paper, we study the following nonlinear stochastic heat equation (SHE):
\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) - \frac{1}{2} \Delta u(t, x) &= b(x, u(t, x))\dot{W}(t, x) \quad x \in \mathbb{R}^d, \ t > 0, \\
\ u(0, \cdot) &= \mu(\cdot).
\end{align*}
\] (1.1)

The noise, $\dot{W}(t, x)$, is a centered Gaussian noise that is white in time and homogeneously colored in space defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by the noise. Its covariance structure is given by

\[ J(\psi, \phi) := \mathbb{E}[W(\psi)W(\phi)] = \int_0^\infty ds \int_{\mathbb{R}^d} \Gamma(dx)(\psi(s, \cdot)*\tilde{\phi}(s, \cdot))(x), \] (1.2)

where $\psi$ and $\phi$ are continuous and rapidly decreasing functions, $\tilde{\phi}(x) := \phi(-x)$, “*” refers to the convolution in the spatial variable, and $\Gamma$ is a nonnegative and nonnegative definite tempered measure on $\mathbb{R}^d$ that is commonly referred to as the correlation measure. The Fourier transform of $\Gamma$ (in the generalized sense) is also a nonnegative and nonnegative definite tempered measure, which is usually called the spectral measure and is denoted by $\hat{f}(d\xi)$ (see (1.23) for the convention of Fourier transform). Moreover, in the case where $\Gamma$ has a density $f$, namely, $\Gamma(dx) = f(x)dx$, we write $\hat{f}(d\xi)$ as $f(\xi)d\xi$.

The initial condition, $\mu$, is a deterministic, locally finite, regular, signed Borel measure that satisfies the following integrability condition at infinity:

\[ \int_{\mathbb{R}^d} \exp(-a|x|^2) |\mu|(dx) < \infty \quad \text{for all } a > 0, \] (1.3)

where $|\mu| = \mu_+ + \mu_-$ and $\mu = \mu_+ - \mu_-$ refers to the Hahn decomposition of the measure $\mu$. Initial conditions of this type, introduced in [8] and further explored in [10, 9], are called rough initial conditions.

The function $b(x, u)$ is uniformly bounded in the first variable and Lipschitz continuous in the second variable, i.e., for some constants $L_b > 0$ and $L_0 \geq 0$,

\[ |b(x, u) - b(x, v)| \leq L_b|u - v| \quad \text{and} \quad |b(x, 0)| \leq L_0 \quad \text{for all } u, v \in \mathbb{R} \text{ and } x \in \mathbb{R}^d. \] (1.4)

In particular, our assumption allows the linear case $b(x, u) = \lambda u$, which is usually referred to as the parabolic Anderson model (PAM) [5].

The SPDE (1.1) is understood in its mild form:

\[ u(t, x) = J_0(t, x; \mu) + \int_0^t \int_{\mathbb{R}^d} b(y, u(s, y))G(t - s, x - y)W(ds, dy), \] (1.5)

where $G(t, x) = (2\pi)^{-d/2} \exp \left(-\frac{1}{2t} |x|^2 \right)$ is the heat kernel,

\[ J_0(t, x) = J_0(t, x; \mu) := (G(t, \cdot)*\mu)(x) = \int_{\mathbb{R}^d} G(t, x - y)\mu(dy) \] (1.6)

is the solution to the homogeneous equation, and the stochastic integral is the Walsh integral. We refer the interested readers to [10, 14, 15, 27] for more details of this setup.

The aim of this paper is to investigate the conditions required to guarantee the existence of an invariant measure for the solution to (1.1), which is a crucial step for the study of the
ergodicity of the system that requires the corresponding uniqueness. We direct the interested readers to [6, 12, 13] for more details about the invariant measure, its existence/uniqueness, and the ergodicity of the system. The general procedure for finding the invariant measure, especially in the setting of (1.1), has been laid out by Tessitore and Zabczyk [26], which involves two parts: first one needs to show that the laws of the solution to (1.1) form a family of \textit{Markov transition functions} on some Hilbert space, \( H \), and the corresponding \textit{Markovian semigroup} is Feller; and second one needs to establish that the moments of solution are bounded in time (see (1.8) below). For the second point, it requires some substantial work (see Theorems 1.2 and 1.4 below). On the other hand, the first point has been shown to be true for our case of interest (see, e.g., [13, Chapter 9]) with the following weighted \( L^2(\mathbb{R}^d) \) space as our underlying Hilbert space as in [26]:

\textbf{Definition 1.1 ([26])}. A function \( \rho : \mathbb{R}^d \mapsto \mathbb{R} \) is called an \textit{admissible weight function} if it is a strictly positive, bounded, continuous, and \( L^1(\mathbb{R}^d) \)-integrable function such that for all \( T > 0 \), there exists a constant \( C_\rho(T) \) such that

\[
(G(t, \cdot) * \rho(\cdot))(x) \leq C_\rho(T)\rho(x) \quad \text{for all } t \in [0, T] \text{ and } x \in \mathbb{R}^d.
\]  

(1.7)

Moreover, we denote by \( L^2_\rho(\mathbb{R}^d) \) the corresponding Hilbert space of \( \rho \)-weighted square integrable functions, and we use \( \langle \cdot, \cdot \rangle_\rho \) and \( \| \cdot \|_\rho \) to denote the inner product and norm on \( L^2_\rho(\mathbb{R}^d) \):

\[
\langle f, g \rangle_\rho := \int_{\mathbb{R}^d} f(x)g(x)\rho(x)dx \quad \text{and} \quad \| f \|_\rho := \int_{\mathbb{R}^d} |f(x)|^2\rho(x)dx.
\]

Accordingly, we will prove the existence of the invariant measure following the same strategy as in [26]. Let \( \mathcal{L}(u(t, \cdot; \mu)) \) denote the law of \( u(t, \cdot; \mu) \) starting from \( \mu \) at \( t = 0 \). We will first establish the tightness of \( \{ \mathcal{L}(u(t, \cdot; \mu)) \}_{t > t_0} \) for some \( t_0 \geq 0 \). A critical step in obtaining this tightness result is to show that the following moment is uniformly bounded in time (see Theorem 1.4):

\[
\sup_{t>0} \mathbb{E} \left( \| u(t, \cdot) \|_\rho^2 \right) < \infty. \tag{1.8}
\]

Then we will apply the \textit{Krylov-Bogoliubov theorem} (see, e.g., [13, Theorem 11.7]) to construct an invariant measure via

\[
\eta(A) = \lim_{n \to \infty} \frac{1}{T_n} \int_{T_n + t_0}^{T_n} \mathcal{L}(u(t, \cdot; \mu))(A)dt,
\]  

(1.9)

for some sequence \( \{T_n\}_{n \geq 1} \) with \( T_n \uparrow \infty \).

In the literature, the existence of invariant measure of the stochastic heat equation is more commonly studied with a drift term; we will postpone a brief review of this case to Section 5.1. In contrast, the existence of an invariant measure under the settings of equation (1.1) has rarely been studied. To the best of our knowledge, this current article and the one by Tessitore and Zabczyk [26] are the only papers that consider the case where the spatial domain is the whole space \( \mathbb{R}^d \), the diffusion term, \( b(x, u) \), is globally Lipschitz in the second variable, uniformly bounded in the first variable (see (1.4)) and there is no additional negative drift term to help. The major challenge is to identify the right conditions so that the probability moments of the solution are bounded in time (see (1.8)). The solution to (1.1) is usually \textit{intermittent}, namely, its moments possess a certain exponential growth in \( t \); see, e.g., [5, 18]. For that reason, one has to impose some additional assumptions either on the initial conditions, or the noise, or the coefficients of (1.1), or all of them in order to control the growth of the moments. The moment formulas obtained in [9, 10] play an important role in this context.
Here we emphasize that we study the invariant measure using the Walsh random field approach \cite{27}, whereas such studies are mostly carried out under the framework of the stochastic evolution in Hilbert spaces \cite{13}. Even though both theories are equivalent (see \cite{16}), the differences in many technical aspects are still substantial. As the random field approach often produces results that are more explicit, we try to use this approach to obtain more precise conditions for the existence of an invariant measure. For the initial conditions, the results in \cite{26} allow for bounded $L^2(\mathbb{R}^d)$ functions, although the authors proved their main result—Theorem 3.3 ibid.—only for the constant one initial condition. Here we give the precise conditions on the initial condition (see (1.20) below) which allows a wider class of data, including unbounded functions and measures such as the Dirac delta measure (see Examples 5.7 and 5.8). Note that the Dirac delta initial measure plays a prominent role in the study of the stochastic heat equation; see, e.g., \cite{1}. Regarding the noise, we give an explicit and easily verifiable condition—(1.10a)—on the spectral density $\hat{f}$ and present a few concrete examples (see Section 5.4). The comparisons of our conditions with those obtained by Tessitore and Zabczyk \cite{26} are given in Section 5.2.

Our proof relies on a factorization representation for the solution $u(t,x)$ to (1.1) (see Lemma 3.4), which is obtained under the random field framework, whereas such factorization lemma has been widely used in the framework of the stochastic evolution equation in Hilbert spaces; see Section 3 for more details. Finally, we point out that there is a miscellany of results in Section 5, which may have independent interest.

Now we are ready to motivate the conditions that we use and present the main results.

1.1 Main results

As mentioned earlier, in order to have moments bounded in time as in (1.8), one should better first identify the sharp conditions under which the second moments as a function of $t$, namely $t \mapsto \mathbb{E} \left( (u(t,x))^2 \right)$, with $x$ fixed, are bounded. This question has been answered in \cite{10, Theorem 1.3 and Lemma 2.5}, where necessary and sufficient conditions are given. More precisely, to have the second moment bounded in time with $x$ fixed, one needs to have the spatial dimension $d \geq 3$, and in addition, the spectral measure $\hat{f}$ and Lipschitz constant $L_b$ of $b(\cdot)$ need to satisfy the following two conditions:

\[
\Upsilon(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{|\xi|^2} < \infty \quad (1.10a)
\]

and

\[
64L_b^2 < \frac{1}{2\Upsilon(0)} \quad (1.10b).
\]

These two conditions will guarantee the existence of the following non-empty open interval:

\[
(2^7L_b^2\Upsilon(0), 1) \neq \emptyset. \quad (1.11)
\]

Note that condition (1.10a) is a strengthened version of Dalang’s condition:

\[
\Upsilon(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{\beta + |\xi|^2} < \infty, \quad \text{for some (and hence) all } \beta > 0. \quad (1.12)
\]

Recall that in order to obtain the Hölder continuity of the solution, one needs to strengthen (1.12) in a different way. Indeed, what is required is that for some $\alpha \in (0, 1]$,

\[
\Upsilon_\alpha(\beta) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{(\beta + |\xi|^2)^{1-\alpha}} < \infty \quad \text{for some (hence all) } \beta > 0; \quad (1.13)
\]
see [9, Theorem 1.8] or [24]. Likewise, one can further strengthen condition (1.13) to
\[ \Upsilon_\alpha(0) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{|\xi|^{2(1-\alpha)}} < \infty \text{ for some } \alpha \in (0, 1]. \] (1.14)
We use the convention that when \( \alpha = 0 \), we simply drop it from the expression \( \Upsilon_\alpha(\beta) \), i.e., \( \Upsilon(\beta) = \Upsilon_0(\beta) \). The relations of these conditions are illustrated in Figure 1.

\[
\begin{array}{ccc}
\Upsilon(\beta) < \infty & \Leftarrow & \Upsilon_\alpha(\beta) < \infty \\
\Upsilon(0) < \infty & \Uparrow & \Upsilon_\alpha(0) < \infty
\end{array}
\]

Figure 1: Relations among conditions (1.12), (1.13), (1.14) and (1.10a).

We will also need the following slightly different condition:
\[ \mathcal{H}_{\alpha/2}(t) < \infty \text{ for some } \alpha \in (0, 1] \text{ and for all } t > 0, \] (1.15)
where
\[ \mathcal{H}_\alpha(t) := \int_0^t dr r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp(-r|\xi|^2). \] (1.16)
The quantity \( \mathcal{H}_\alpha(t) \) will appear naturally in the proof of Lemma 3.2 below. As shown in Lemma 3.5 below, condition (1.14) will imply condition (1.15). However, if one assumes (1.10a), then these two conditions become equivalent.

We are now ready to state our two main results of the paper.

**Theorem 1.2.** Let \( u(t,x;\mu) \) be the solution to (1.1) starting from \( \mu \) which satisfies (1.3). Assume that

(i) \( \rho : \mathbb{R}^d \to \mathbb{R}_+ \) is a nonnegative \( L^1(\mathbb{R}^d) \) function;

(ii) for all \( t > 0 \), the initial condition \( \mu \) satisfies \( G_\rho(t;|\mu|) < \infty \) where
\[ G_\rho(t;\mu) := \int_{\mathbb{R}^d} J_0^2(t,x;\mu) \rho(x) \, dx; \] (1.17)

(iii) the spectral measure \( \hat{f} \) and the Lipschitz constant \( L_b \) satisfy the two conditions in (1.10).

Then there exists a unique \( L^2(\Omega) \)-continuous solution \( u(t,x) \) such that for some constant \( C > 0 \), which does not depend on \( t \), the following holds:
\[ \mathbb{E}\left( \|u(t,\cdot;\mu)\|_\rho^2 \right) \leq CG_\rho(t;\mu^*) < \infty, \quad \text{for any } t > 0, \] (1.18)
where \( \mu^* := 1 + |\mu| \).

This theorem will be proved in Section 2. We now state and prove a corollary which shows that the solution to (1.1) starting from an \( L^2(\mathbb{R}^d) \) initial condition will almost surely be in \( L^2_\rho(\mathbb{R}^d) \) for all \( t > 0 \).
Corollary 1.3. Under the same assumptions of Theorem 1.2, if in addition $\rho$ is admissible (see Definition 1.1), then the solution $u(t,\cdot;\zeta)$ to (1.1) almost surely exists in $L^2_{\rho}(\mathbb{R}^d)$ for all $t > 0$, whenever the initial condition is also in $L^2_{\rho}(\mathbb{R}^d)$, i.e., $\zeta \in L^2_{\rho}(\mathbb{R}^d)$.

Proof. Choose and fix an arbitrary $\zeta \in L^2_{\rho}(\mathbb{R}^d)$ and set $\zeta^* = 1 + |\zeta|$. It is clear that $\zeta^* \in L^2_{\rho}(\mathbb{R}^d)$.

By (ii) of Theorem 1.2, it suffices to show the finiteness of $\mathcal{G}_\rho(t,\cdot;\zeta^*)$ for all $t > 0$. Indeed, by Hölder’s inequality,

$$\mathcal{G}_\rho(t,\cdot;\zeta^*) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} G(t,x-y)\zeta^*(y)dy \right)^2 \rho(x)dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t,x-y)\zeta^*(y)^2 \rho(x)dydx.$$ 

Now for any $t > 0$, choose $T > t$ and let $C_\rho(T)$ be as in (1.7). Then,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t,x-y)\zeta^*(y)^2 \rho(x)dydx = \int_{\mathbb{R}^d} (G(t,\cdot)*\rho)(y)|\zeta(y)|^2 dy \leq C_\rho(T) ||\zeta^*||^2_{\rho} < \infty.$$ 

Since $t$ and $T$ are arbitrary, this completes the proof of the corollary.

Theorem 1.4. Let $u(t,x)$ be the solution to (1.1) starting from $\mu$ and let $\rho$ be an admissible weight function. Assume that

(i) there exists another admissible weight $\tilde{\rho}$ such that

$$\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} dx < \infty; \quad (1.19)$$

(ii) the weight function $\tilde{\rho}$ and the initial condition satisfy the following condition:

$$\limsup_{t \to 0} \mathcal{G}_{\rho}(t;|\mu|) < \infty; \quad (1.20)$$

(iii) the spectral measure $\hat{f}$ and the Lipschitz constant $L_b$ satisfy the two conditions in (1.10);

(iv) for some $\alpha \in (2^{-1}T(0)L^2_{b},1)$ (see (1.11)), the spectral measure $\hat{f}$ satisfies (1.14).

Then we have that

(a) for any $\tau > 0$, the sequence of laws of $\{L\mu(t,\cdot;\mu)\}_{t \geq \tau}$ is tight, i.e., for any $\epsilon \in (0,1)$, there exists a compact set $\mathcal{K} \subset L^2_{\rho}(\mathbb{R}^d)$ such that

$$L\mu(t,\cdot;\mu)(\mathcal{K}) \geq 1 - \epsilon, \quad \text{for all } t \geq \tau > 0; \quad (1.21)$$

(b) there exists an invariant measure for the laws $\{L\mu(t,\cdot;\mu)\}_{t > 0}$ in $L^2_{\rho}(\mathbb{R}^d)$.

This theorem will be proved in Section 4.

1.2 Outline and notation

The paper is organized as follows: we first prove Theorem 1.2 in Section 2. Then in Section 3, we study the factorization lemma. Then we proceed to prove Theorem 1.4 in Section 4. Finally, in Section 5 we make some further discussion on the main results and present various examples. In particular, in Section 5.1, we give a brief review of the problem of finding invariant measures for the SHE with a drift term; in Section 5.2, we compare our conditions on the spectral density with those obtained by Tessitore and Zabczyk [26]; in Section 5.3, we show that our results could
include a wider class of initial conditions; in Section 5.4, we carry out some explicit computations
for the Bessel and related kernels as the correlation functions; finally, in Section 5.5, we give a few examples of the admissible weight functions.

We conclude this Introduction by introducing some notation and formulas that we use throughout the paper. We will use $\|X\|_p$ to denote the $L^p(\Omega)$ norm, namely, $\|X\|_p = (E(|X|^p))^{1/p}$. We will also use the following factorization property of the heat kernel,

$$G(t, x)G(s, y) = G\left(\frac{ts}{t + s}, \frac{sy + ty}{t + s}\right) G(t + s, x - y), \quad (1.22)$$

which can be easily verified and has been used extensively and critically in [8, 10, 9]. Next, we will need the following spherical coordinate integration formula:

$$\int_{\mathbb{R}^d} f(|x|)dx = \sigma(S^{d-1}) \int_0^\infty f(r) r^{d-1}dr,$$

where $\sigma(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ and $\Gamma(x)$ denotes the Gamma function. We use $\sim$ to denote the standard asymptotic equivalent relation. Lastly, the convention of Fourier transform is given by (see Remark 5.1)

$$\hat{\phi}(\xi) = \mathcal{F}\phi(\xi) := \int_{\mathbb{R}^d} e^{-ix\xi} \phi(x)dx \quad \text{and} \quad \mathcal{F}^{-1}\psi(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \psi(\xi)d\xi. \quad (1.23)$$

2 Moment estimates – Proof of Theorem 1.2

We first state some known results and prove a moment bound in Corollary 2.3.

Theorem 2.1 (Theorem 1.2 of [9]). Suppose that

(i) the initial deterministic measure $\mu$ satisfies (1.3);

(ii) the spectral measure $\hat{f}$ satisfies Dalang’s condition (1.12).

Then (1.1) has a unique random field solution starting from $\mu$. Moreover, the solution is $L^2(\Omega)$ continuous and is adapted to the filtration $\{F_t\}_{t\geq 0}$.

Theorem 2.2 (Theorem 1.7 of [9]). Under the assumptions of Theorem 2.1, for any $t > 0$, $x \in \mathbb{R}^d$ and $p \geq 2$, the solution to (1.1), $u(t, x)$, given by (1.5) is in $L^p(\Omega)$ and

$$\|u(t, x)\|_p \leq \left[\tilde{c} + \sqrt{2}(G(t, \cdot) * |\mu|)(x)\right] H(t; \gamma_p)^{1/2}, \quad (2.1)$$

where $\tilde{c} = L_0/L_b$, $\gamma_p = 32pL_b^2$ (see (1.4) for $L_0$ and $L_b$) and the function $H(t; \gamma_p)$ is nondecreasing in $t$ (see [9] for the expression of the function $H$).

Corollary 2.3. Under the same setting as Theorem 2.2, if the two conditions in (1.10) hold (see also (1.11)), then

$$\|u(t, x)\|_p \leq C_p \left(1 + (G(t, \cdot) * |\mu|)(x)\right), \quad \text{for all } p \text{ such that } 1/p \in (64L_b^2 \Upsilon(0), 1), \quad (2.2)$$

where $C_p = (\sqrt{2} \vee \tilde{c}) \sup_{t \geq 0} H(t; \gamma_p)^{1/2} < \infty$. 

7
Proof. Lemma 2.5 of [10] gives one sufficient condition, namely $2\gamma_p \Upsilon(0) < 1$, under which the function $H((t; \gamma_p))$ is bounded in $t$. Therefore, by taking into account the expression of $\gamma_p$ in Theorem 2.2, we see that as a direct consequence of (2.1), whenever

$$32pL^2_0 < \frac{1}{2\Upsilon(0)},$$

we have the $p$-th moment bounded as given in (2.2). 

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Under condition (iii), we can apply Fubini’s Theorem and the moment bound (2.2) below to see that for some constant $C > 0$ independent of $t$, which may vary from line to line, that

$$E \left( \|u(t, \cdot; \mu)\|_p^2 \right) \leq C E \left[ \int_{\mathbb{R}^d} \left( 1 + (G(t, \cdot) * |\mu|)(x) \right)^2 \rho(x)dx \right]$$

$$= C \int_{\mathbb{R}^d} E \left[ \left( \left( G(t, \cdot) * (1 + |\mu|) \right)(x) \right)^2 \right] \rho(x)dx$$

$$= C G_p(t; \mu^*) < \infty,$$

where we recall that $\mu^* = 1 + |\mu|$. This proves Theorem 1.2.

Remark 2.4 (Restarted SHE). Recall that the Markov property of the solution to (1.1) implies that for any $t \geq t_0 > 0$,

$$u(t + t_0, x; \mu) \overset{\mathcal{L}}{=} u(t, x; u(t_0, \cdot; \mu)) =: v(t, x),$$

(2.4)

where $\mathcal{L}$ refers to the equality in law. Then $v$ satisfies the following restarted SPDE:

$$\begin{cases}
\frac{\partial v}{\partial t}(t, x) - \frac{1}{2} \Delta v(t, x) = b(x, v(t, x)) \hat{W}(t_0, x) & x \in \mathbb{R}^d, t > 0, \\
v(0, x) = u(t_0, x; \mu), & x \in \mathbb{R}^d,
\end{cases}$$

(2.5)

where $\hat{W}(t_0, x) := \hat{W}(t + t_0, x)$ denotes the time shifted noise, i.e.,

$$\int_0^t \int_{\mathbb{R}^d} W(s, dy) = \int_{t_0}^{t+t_0} \int_{\mathbb{R}^d} W(ds, dy).$$

(2.6)

Under the conditions in (1.10), Theorem 2.2 and (2.4) imply immediately that

$$\|v(t, x)\|_q = \|u(t + t_0, x; \mu)\|_q \leq C_q \left( 1 + (G(t + t_0, \cdot) * |\mu|)(x) \right) = C_q J_0(t + t_0, x; 1 + |\mu|),$$

for all $q \geq 2$ and $t > 0$, where the constant $C_q$ does not depend on $t$. Moreover, under the assumptions of Theorem 1.2, we have $v(0, \cdot) \in L^2_\rho(\mathbb{R}^d)$ a.s. and

$$E \left( \|v(t, x)\|_p^2 \right) = E \left( \|u(t + t_0, x; \mu)\|_p^2 \right) \leq C G_p(t + t_0; \mu^*) < \infty.$$
3 A factorization lemma

In this section, we establish a factorization lemma with corresponding moment estimates; see Lemmas 3.2 and 3.4 below. This factorization lemma appeared in [11]; check also Section 5.3.1 of [13]. For \( \alpha \in (0,1) \), \( t > 0 \) and \( x \in \mathbb{R}^d \), define formally

\[
(F_\alpha f)(t,x) := \int_0^t \int_{\mathbb{R}^d} (t-s)^{\alpha-1} G(t-s,x-y)f(s,y) \, ds \, dy \tag{3.1}
\]

and

\[
(Y_\alpha f)(t,x) := \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} G(t-s,x-y)f(s,y)W(ds,dy). \tag{3.2}
\]

For \( F_\alpha \), the first step of the proof of [26, Theorem 3.1] showed the following proposition:

**Proposition 3.1.** Let \( \rho \) and \( \tilde{\rho} \) be given as in condition (i) of Theorem 1.4 (see (1.19)). For any \( q > 2, t_0 > 0 \) and \( \alpha \in (q^{-1},2^{-1}) \), the operator \( F_\alpha \), as an operator from \( L^q((0,t_0); L^2_\rho(\mathbb{R}^d)) \) to \( L^2_\rho(\mathbb{R}^d) \), is compact.

As for \( Y_\alpha \), we have the following two lemmas, which hold for both the non-restarted SHE \( (t_0 = 0) \) and the restarted SHE \( (t_0 > 0) \).

**Lemma 3.2.** Suppose that \( \mu \) —the initial condition for \( u \) —satisfies (1.3) and that \( \tilde{f} \) satisfies Dalang’s condition (1.12). Suppose that for some \( \alpha \in (0,1/2) \), \( \mathcal{H}_\alpha(t) \) defined in (1.16) is finite for all \( t > 0 \). Fix an arbitrary \( t_0 \geq 0 \). Let \( v(t,x) \) be the solution to the restarted SHE (2.5) and \( W_{t_0} \) be the time-shifted noise (see (2.6)) when \( t_0 > 0 \) and let \( v = u \) when \( t_0 = 0 \). Then

\[
Y_\alpha(s,y) := [Y_\alpha b(\cdot,v(\cdot,\cdot))](s,y) = \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(s-r,y-z)b(z,v(r,z))W_{t_0}(dr,dz) \tag{3.3}
\]

has the following properties:

1. for all \( q \geq 2, s > 0, y \in \mathbb{R}^d \),

\[
\|Y_\alpha(s,y)\|_q^q \leq H(s+t_0;32qL^2_\mu)J_0^2(s+t_0,y;\mu^*) \mathcal{H}_\alpha(s) < \infty, \tag{3.4}
\]

where we remind the reader that \( \mu^* := 1+|\mu| \), and we refer to Theorem 2.2 for the function \( H(t;\gamma) \);

2. under both conditions in (1.10), if \( \mathcal{H}_\alpha(t) \) is finite for some \( \alpha \in (64L^2_\rho \Upsilon(0),1/2) \), then for any \( q \) with \( 1/q \in (64L^2_\rho \Upsilon(0),\alpha) \), the function \( H(t;32qL^2_\mu) \) in (3.4) is uniformly bounded in \( t \geq 0 \), i.e., \( \sup_{t \geq 0} H(t;32qL^2_\mu) < \infty \);

3. under both conditions in (1.10), if \( \mathcal{H}_\alpha(t) \) is finite for some \( \alpha \in (64L^2_\rho \Upsilon(0),1/2) \), then for any \( q \) with \( 1/q \in (64L^2_\rho \Upsilon(0),\alpha) \) and for any nonnegative and \( L^1(\mathbb{R}^d) \)-function \( \rho \), there exists a constant \( \Theta = \Theta(q,L_\rho,L_0,\alpha) \), which does not depend on \( t \), such that for \( t > 0 \),

\[
\mathbb{E} \left( \int_0^t \|Y_\alpha(s,\cdot)\|_q^q \, ds \right) \leq \Theta \int_0^t [\mathcal{G}_\rho(s+t_0;\mu^*) \mathcal{H}_\alpha(s)]^{q/2} \, ds, \tag{3.5}
\]

which is finite provided that

\[
\int_0^t [\mathcal{G}_\rho(s+t_0;\mu) \mathcal{H}_\alpha(s)]^{q/2} \, ds < \infty. \tag{3.6}
\]

9
Remark 3.3. Condition (3.6) is true for $t_0 > 0$ because $G_\rho(t; \mu)$ is a continuous function for $t > 0$ and $H_\alpha(s)$ is continuous and bounded for $s \in [0, t]$ thanks to (1.16). However, when $t_0 = 0$, the situation is much trickier. For example, when the initial condition is the delta initial condition, we have that

$$G_\rho(t; \delta_0) = \int_{\mathbb{R}^d} G(t, x)^2 \rho(x) dx = G(2t, 0) \int_{\mathbb{R}^d} G(t/2, x) \rho(x) dx < \infty,$$

where one can obtain the second equality via (1.22). Hence, when $s \to 0$, $G_\rho(s; \delta_0)$ blows up with a rate $s^{-d/2}$. Considering that $H_\alpha(s)$ goes to zero with a different rate, one needs to combine these two rates to check if condition (3.6) holds. By introducing $t_0$ and restarting the heat equation, one can avoid this issue, that being the potential singularity of $G_\rho$ at $s = 0$.

Proof. In the proof, we use $C$ to denote a generic constant that may change its value at each appearance. We first prove (3.4). By the Burkholder-Davis-Gundy inequality and Minkowski’s integral inequality, we see that

$$\|Y_v(s, y)\|_q^2 \leq C \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^{2d}} d\mathbf{z}_1 d\mathbf{z}_2 \|b(z_1, v(r, z_1))\|_q \times \int \|b(z_1 - z_2)G(s - r, y - z_2)\|_q.$$

Note that for the Lipschitz condition in (1.4), we have that

$$|b(x, v)| \leq |b(x, v) - b(x, 0)| + |b(x, 0)| \leq L_b |v| + L_0 \leq C(1 + |v|), \quad C := L_b \vee L_0.$$

We apply this and the moment bound (2.1) to $\|b(z_i, v(r, z_i))\|_q$ above to see that

$$\|b(z_i, v(r, z_i))\|_q \leq C \left(1 + \|v(r, z_i)\|_q\right)$$

where the last step is due to the fact that $H(t; \gamma)$ is a nondecreasing function; see Lemma 2.6 of [10]. Therefore, by denoting $C_s := H\left(s + t_0; 32qL_b^2\right)$,

$$\|Y_v(s, y)\|_q^2 \leq CC_s \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^{2d}} d\mathbf{z}_1 d\mathbf{z}_2 \|G(s - r, y - z_1)\|_q \prod_{i=1}^2 \left(\|b(s, y - z_i)\|_q\right)$$

$$= CC_s \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^{2d}} \mu^*(d\mathbf{z}_1) \mu^*(d\mathbf{z}_2) \int_{\mathbb{R}^{2d}} d\mathbf{z}_1 d\mathbf{z}_2 \times \int \|b(z_1 - z_2)\|_q \prod_{i=1}^2 \left(\|G(r + t_0, z_i - \sigma_i)\|_q\right)$$

$$= CC_s \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^{2d}} \mu^*(d\mathbf{z}_1) \mu^*(d\mathbf{z}_2) \int_{\mathbb{R}^{2d}} d\mathbf{z}_1 d\mathbf{z}_2 \times \int \left(\|G(s + t_0, y - \sigma_1)\|_q\right) \prod_{i=1}^2 G\left(\frac{(r + t_0)(s - r)}{s + t_0}, z_i - \sigma_i, \frac{r + t_0}{s + t_0}, \frac{s - r}{s + t_0}, \frac{s - r}{s + t_0}y\right)$$

$$\leq CC_s (2\pi)^{-2d} \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^{2d}} \mu^*(d\mathbf{z}_1) \mu^*(d\mathbf{z}_2) \int_{\mathbb{R}^{2d}} d\mathbf{z}_1 d\mathbf{z}_2 \times \int \left(\|b(z_1 - z_2)\|_q\right) \prod_{i=1}^2 G\left(\frac{(r + t_0)(s - r)}{s + t_0}, z_i - \sigma_i, \frac{r + t_0}{s + t_0}, \frac{s - r}{s + t_0}, \frac{s - r}{s + t_0}y\right)$$
\[
\times \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -\frac{(r + t_0)(s - r)}{s + t_0} |\xi|^2 \right),
\]
where we have applied (1.22) and Plancherel’s theorem. Hence,
\[
\|Y_v(s,y)\|_q^2 \leq CC_s(2\pi)^{-2d} J_0^2(s + t_0, y; \mu^*) \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -\frac{(r + t_0)(s - r)}{s + t_0} |\xi|^2 \right).
\]
Because the function
\[
t_0 \mapsto \frac{r + t_0}{s + t_0} = 1 - \frac{s - r}{s + t_0}
\]
is nondecreasing in \(t_0\) whenever \(s > r > 0\), we can replace the two appearances of \(t_0\) in the exponent of the above inequality by zero to see that
\[
\|Y_v(s,y)\|_q^2 \leq CC_s(2\pi)^{-2d} J_0^2(s + t_0, y; \mu^*) \int_0^s dr (s - r)^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -\frac{r(s - r)}{s} |\xi|^2 \right). \tag{3.8}
\]
Furthermore, by symmetry of \(r(s - r)/s\) and the fact that \(r(s - r)/s \geq r/2\) for all \(r \in [0, s/2]\), we see that the above double integral is bounded by
\[
\leq 2 \int_0^{s/2} dr r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -\frac{r}{2} |\xi|^2 \right)
\]
\[
= 2^{2(1 - \alpha)} \int_0^{s/4} dr r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -r |\xi|^2 \right)
\]
\[
\leq 2^{2(1 - \alpha)} \int_0^s dr r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp\left( -r |\xi|^2 \right)
\]
\[
= 2^{2(1 - \alpha)} \mathcal{H}_\alpha(s).
\]
Plugging the above bound back to (3.8) proves (3.4).

Part (2) is a direct consequence of Theorem 2.2. It remains to prove (3.5). An application of Minkowski’s inequality shows that
\[
\mathbb{E} \left( \|Y_v(s, \cdot)\|_q^2 \right) = \left\| \int_{\mathbb{R}^d} Y_v(s, y) \rho(y) dy \right\|_{q/2}^2 \leq \left( \int_{\mathbb{R}^d} \|Y_v(s, y)\|_q^2 \rho(y) dy \right)^{q/2}. \tag{3.9}
\]
By the definition of \(\mathcal{G}_\rho(t; \mu)\) in (1.17) and by (3.4), we see that
\[
\int_{\mathbb{R}^d} \|Y_v(s, y)\|_q^2 \rho(y) dy \leq C \mathcal{G}_\rho(s + t_0; \mu^*) \mathcal{H}_\alpha(s).
\]
Plugging the above expression to the far right-hand side of (3.9) proves (3.5). Finally, the finiteness of the upper bound in (3.5) is guaranteed by condition (3.6). This completes the proof of Lemma 3.2. \(\square\)

**Lemma 3.4** (Factorization lemma). Suppose that \(\mu\) — the initial condition for \(u\) — satisfies (1.3) and \(\hat{f}\) satisfies Dalang’s condition (1.12). Assume that condition (1.16) is satisfied for some \(\alpha \in (0, 1/2)\). Fix an arbitrary \(t_0 \geq 0\). Let \(v(t, x)\) be the solution to the restarted SHE (2.5) and \(W_{t_0}\) be the time-shifted noise (see (2.6)) when \(t_0 > 0\) and let \(v = u\) when \(t_0 = 0\). Then the following factorization holds
\[
\frac{\sin(\alpha \pi)}{\pi} \int_0^t (t-s)^{\alpha-1} [G(t-s, \cdot) \ast Y_v(s, \cdot)] (x) ds = \int_0^t \int_{\mathbb{R}^d} G(t-r, x-z) b(z, v(r, z)) W_{t_0}(dr, dz),
\]
for all $t > 0$ and $x \in \mathbb{R}^d$. As a consequence,

$$v(t, x) = [G(t, \cdot) * u(t_0, \cdot; \mu)](x) + \frac{\sin(\alpha \pi)}{\pi} [F_\alpha Y_\alpha](t, x), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d. \quad (3.10)$$

**Proof.** The lemma is straightforward provided that one can switch the orders of stochastic and ordinary integrals:

$$\int_0^t (t-s)^{\alpha-1} [G(t-s, \cdot) * Y_\alpha(s, \cdot)](x)ds$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dy \, G(t-s, x-y) (s)$$

$$\times \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(s-r, y-z) b(z, v(r,z)) W_{t_0}(\text{dr, dz})$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dy \, G(t-s, x-y) (s)$$

$$\times \int_0^s \int_{\mathbb{R}^d} (s-r)^{-\alpha} G(t-r, y-z) b(z, v(r,z)) W_{t_0}(\text{dr, dz})$$

$$= \frac{\pi}{\sin(\alpha \pi)} \int_0^t \int_{\mathbb{R}^d} G(t-r, x-z) b(z, v(r,z)) W_{t_0}(\text{dr, dz}),$$

where the last step is the *Beta integral* which requires that $\alpha \in (0, 1)$. It remains to justify the two applications of the stochastic Fubini’s theorem (see Theorem 5.30 of Chapter one in [14], or also [27] or Theorem 4.33 of [13]) in (3.11) and (3.12) in the following two steps.

**Step 1.** In this step, we justify the change of orders in (3.11). Note that $t, x$ and $s$ are fixed. It suffices to prove the following condition:

$$I_1 := \int_{\mathbb{R}^d} dy \, G(t-s, x-y) \int_0^s \int_{\mathbb{R}^d} dr \, (s-r)^{-2\alpha} \int_{\mathbb{R}^{2d}} dz_1 dz_2$$

$$\times f(z_1 - z_2) \left( \prod_{i=1}^2 G(s-r, y-z_i) \right) \mathbb{E} \left( \prod_{i=1}^2 b(z_i, v(r, z_i)) \right)$$

$$= \int_{\mathbb{R}^d} dy \, G(t-s, x-y) \|Y_\alpha(s, y)\|_2^2 < +\infty,$$

which follows immediately from (3.4). Indeed,

$$\int_{\mathbb{R}^d} dy \, G(t-s, x-y) \|Y_\alpha(s, y)\|_2^2 \leq C \int_{\mathbb{R}^d} dy \, G(t-s, x-y) J_0^2(s + t_0, y; \mu^*) \mathcal{H}_\alpha(s)$$

$$= C \mathcal{H}_\alpha(s) \int_{\mathbb{R}^d} dy \, G(t-s, x-y) \int_{\mathbb{R}^{2d}} \mu^*(dz_1) \mu^*(dz_2)$$

$$\times G(s + t_0, y-z_1) G(s + t_0, y-z_2).$$

Now we bound the three heat kernels using (1.22) as follows:

$$G(t-s, x-y) \prod_{i=1}^2 G(s + t_0, y-z_i)$$

$$= \frac{G(2(t-s), x-y)^2}{G(4(t-s, 0)} \prod_{i=1}^2 G(s + t_0, y-z_i)$$

12
By the Cauchy Schwartz inequality, \((\ref{eq:cauchy_schwartz})\),

Therefore, \(I_1 \leq C_{t,s,t_0} H_\alpha(s) J_0^2 (2(t + t_0), x; \mu^*) < \infty\).

**Step 2.** Similarly, as for (\ref{eq:step_2}), we need to show that

\[
I_2 := \int_0^t ds \int_0^s dr (s-r)^{\alpha-1} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} \xi d\omega d\eta
\]

\[
\times f(z_1 - z_2) \left( \prod_{i=1}^2 G(t-r, x-z_i) \right) \mathbb{E} \left( \prod_{i=1}^2 b(z_i, v(r, z_i)) \right) < \infty.
\]

By the Cauchy Schwartz inequality, (\ref{eq:cauchy_schwartz_2}) and because \(\alpha \in (0, 1/2)\),

\[
I_2 \leq C \int_0^t ds \int_0^s dr (s-r)^{\alpha-1} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} \xi d\omega d\eta
\]

\[
\times \int_{\mathbb{R}^d} d\omega' d\eta' f(z_1 - z_2) \prod_{i=1}^2 G(t-r, x-z_i) J_0(r + t_0, z_i; \mu^*)
\]

\[
= C' \int_0^t dr (t-r)^{-\alpha} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} \xi d\omega d\eta
\]

\[
\times f(z_1 - z_2) \prod_{i=1}^2 G(t-r, x-z_i) J_0(r + t_0, z_i; \mu^*)
\]

Now by the same arguments as those leading to (\ref{eq:final_bound}) (with \(2\alpha\) there replaced by \(\alpha\)), we see that

\[
I_2 \leq C J_0^2 (t, x; \mu^*) \int_0^t dr r^{-\alpha} \int_{\mathbb{R}^d} f(\xi) \exp \left( - \frac{r(t-r)}{t} |\xi|^2 \right),
\]

which is finite by (\ref{eq:finite_condition}) where we replace \(\alpha\) with \(\alpha/2\) and repeat the same steps right after (\ref{eq:cauchy_schwartz_3}).

This completes the proof of Lemma 3.4. \(\square\)

Finally, we characterize conditions (\ref{eq:final_conditions}) and (\ref{eq:finite_condition}) in the following lemma:

**Lemma 3.5.** For all \(\alpha \in (0, 1/2]\), we have the following properties:

1. \((2\pi)^{-d} H_\alpha(t) \leq \Gamma (1 - 2\alpha) \Upsilon_{2\alpha}(0)\) for all \(t > 0\) and hence

\[
\Upsilon_{2\alpha}(0) < \infty \implies H_\alpha(t) < \infty \text{ for all } t > 0;
\]

2. \(\lim_{t \to \infty} (2\pi)^{-d} H_\alpha(t) = \Gamma (1 - 2\alpha) \Upsilon_{2\alpha}(0)\);
(3) if $\Upsilon(0) < \infty$, then the reverse implication of (3.13) holds.

Proof. We only need to consider the case when $\alpha > 0$. It is clear that the function $\mathcal{H}_\alpha(t)$ is nondecreasing. Hence, part (2) implies part (1). As for part (2), by Fubini’s theorem,

$$
\lim_{t \to \infty} \mathcal{H}_\alpha(t) = \int_0^\infty dr \, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) e^{-r|\xi|^2} = \Gamma(1-2\alpha) \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{|\xi|^{2(1-2\alpha)}} = C \Upsilon_{2\alpha}(0),
$$

with $C := \Gamma(1-2\alpha)(2\pi)^d$. Now for part (3), for any $t > 0$, by splitting the dr integral in (3.14) into two parts, we see that

$$
C \Upsilon_{2\alpha}(0) = \int_0^\infty dr \, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -r|\xi|^2 \right) = \mathcal{H}_\alpha(t) + I_\alpha(t),
$$

with

$$
I_\alpha(t) = \int_t^\infty dr \, r^{-2\alpha} \int_{\mathbb{R}^d} \hat{f}(d\xi) \exp \left( -r|\xi|^2 \right).
$$

Notice that

$$
I_\alpha(t) \leq t^{-2\alpha} \int_t^\infty dr \int_{\mathbb{R}^d} \hat{f}(d\xi) e^{-r|\xi|^2} = t^{-2\alpha} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{|\xi|^2} e^{-t|\xi|^2} \leq t^{-2\alpha} \int_{\mathbb{R}^d} \frac{\hat{f}(d\xi)}{|\xi|^2} = \frac{(2\pi)^d}{t^{2\alpha}} \Upsilon(0).
$$

Therefore,

$$
\Upsilon_{2\alpha}(0) \leq \frac{\mathcal{H}_\alpha(t)}{(2\pi)^d \Gamma(1-2\alpha)} + \frac{\Upsilon(0)}{\Gamma(1-2\alpha) t^{2\alpha}} < \infty, \quad \text{for all } t > 0,
$$

which proves part (3). □

4 Tightness and construction – Proof of Theorem 1.4

4.1 Proof of part (a) of Theorem 1.4

Before we start the proof of part (a) of Theorem 1.4, we first recall the following result:

Proposition 4.1 (Proposition 2.1 of [26]). For any admissible weight $\rho$, the operators on $L^2_\rho(\mathbb{R}^d)$ defined by $\varphi \mapsto \mathcal{G}(t, \cdot; \varphi(\cdot))(x)$ can be extended to a $C_0$ - semigroup on $L^2_\rho(\mathbb{R}^d)$. Moreover, if $\tilde{\rho}$ is another admissible weight such that

$$
\int_{\mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \, dx < \infty,
$$

then for any $t > 0$, the operators defined above are compact from $L^2_\rho(\mathbb{R}^d)$ to $L^2_\rho(\mathbb{R}^d)$.

Proof of Theorem 1.4 (a). In this proof, $u(t, x)$ refers to $u(t, x; \mu)$. Fix $\tau > 0$ and let $t_0 = \tau/2$. Throughout the proof, we have $t \geq \tau$. Let $v$ be the solution to (2.5) that is restarted from $t - t_0$. Then (see Figure 2 for an illustration)

$$
v_t(s, x) \equiv u(s, x; u(t - t_0, \cdot; \mu)) \quad \text{for } s \geq 0 \text{ and } t \geq \tau.
$$

(4.1)

According to Assumption (i), we can choose and fix some admissible weight function $\tilde{\rho}$ such that (1.19) is satisfied. Hence, by Proposition 4.1 below, the following set

$$
\mathcal{K}_1(\Lambda) := \left\{ (\mathcal{G}(t_0, \cdot; y(\cdot))(x) : \|y\|_{\tilde{\rho}} \leq \Lambda \right\} \quad \text{with } \Lambda > 0
$$

14
Figure 2: An illustration for the restarted SHE in (4.1).

is relatively compact in $L^2_b(\mathbb{R}^d)$.

Assumption (iii), i.e., (1.10), implies that the interval $(64L^2_b\Upsilon(0), 1/2)$ is not empty. Moreover, Assumption (iv), i.e., (1.14), guarantees that there exists a constant $\alpha$ in this interval, namely, $64L^2_b\Upsilon(0) < \alpha < 1/2$, such that (1.14) holds with $\alpha$ replaced by $2\alpha$, i.e., $\Upsilon_{2\alpha}(0) < \infty$. Now we can apply part (3) of Lemma 3.5, thanks to (1.10a), to see that $\Upsilon_{2\alpha}(0) < \infty$ if and only if (1.16) holds. Therefore, both Lemmas 3.2 and 3.4 (more precisely part (3) of Lemma 3.2) are applicable. In particular, Lemma 3.4 ensures that the following factorization is well-defined:

$$v(t_0, x) = \left(G(t_0, \cdot) \ast u(t_0, \cdot)\right)(x) + \frac{\sin(\alpha \pi)}{\pi} [F_\alpha Y_\alpha](t_0, x). \quad (4.2)$$

Part (3) of Lemma 3.2 shows that for any $q$ in the following range,

$$\frac{1}{q} < \alpha < \frac{1}{2} \quad \text{(or equivalently)} \quad 2 < \frac{1}{\alpha} < q < \frac{1}{64L^2_b\Upsilon(0)}, \quad (4.3)$$

we can apply Proposition 3.1 to see that the set

$$\mathcal{X}_2(\Lambda) := \left\{(F_\alpha h)(t_0, x) : \|h\|_{L^q((0,t_0); L^2_\rho(\mathbb{R}^d))} \leq \Lambda\right\}, \quad \text{with} \quad \Lambda > 0,$$

is relatively compact in $L^2_\rho(\mathbb{R}^d)$. Now for any $\Lambda > 0$, define the set $\mathcal{X}(\Lambda)$ as

$$\mathcal{X}(\Lambda) := \mathcal{X}_1(\Lambda) + \mathcal{X}_2(\Lambda)$$

$$= \left\{(G(t_0, \cdot) \ast g(\cdot))(x) + (F_\alpha h)(t_0, x) : \|g\|_{\tilde{\rho}} \leq \Lambda \quad \text{and} \quad \|h\|_{L^q((0,t_0); L^2_\rho(\mathbb{R}^d))} \leq \Lambda\right\}. \quad \text{Notice that from the factorization formula (4.2),}

$$\mathbb{P}[v(t_0, \cdot) \notin \mathcal{X}(\Lambda)] \leq \mathbb{P} \left[ \left(\int_0^{t_0} \|Y_\alpha(s, \cdot)\|_{\tilde{\rho}}^q \, ds\right)^{1/q} > \frac{\pi \Lambda}{\sin(\alpha \pi)} \right] + \mathbb{P} \left[\|u(t-t_0, \cdot)\|_{\tilde{\rho}} > \Lambda\right]$$

$$=: I_1 + I_2.$$

By Chebyshev’s inequality and (1.18), we see that

$$I_2 \leq \frac{1}{\Lambda^2} \mathbb{E} \left[\|u(t-t_0, \cdot)\|_{\tilde{\rho}}^2\right] \leq \frac{1}{\Lambda^2} G_{\tilde{\rho}}(t-t_0; \mu^*).$$
Because $G_\rho(t; \mu^*)$ is a continuous function for $t > 0$, and because it is also bounded at infinity, thanks to Assumption (ii) (see (1.20)), we have that

$$G_\rho(t - t_0; \mu^*) \leq \sup_{t \geq \tau} G_\rho(t - t_0; \mu^*) = \sup_{t \geq t_0} G_\rho(t; \mu^*) < \infty. \quad (4.4)$$

Therefore, we can bound $I_2$ from above with a constant that does not depend on $t \geq \tau$, namely,

$$I_2 \leq \frac{1}{\Lambda^2} \sup_{t \geq t_0} G_\rho(t; \mu^*) < \infty.$$

As for $I_1$, with the choice of $\alpha$ and $q$ in (4.3), one can apply Chebyshev’s inequality and part (3) of Lemma 3.2 to see that

$$I_1 \leq \frac{\sin^q(\alpha \pi)}{\pi^q \Lambda^q} \mathbb{E} \int_0^{t_0} \|Y_v(s, \cdot)\|_\rho^q \, ds \leq \frac{\sin^q(\alpha \pi)}{\pi^q \Lambda^q} \Theta \int_0^{t_0} (G_\rho(s + t - t_0; \mu^*) \mathcal{H}_\alpha(s))^{q/2} \, ds,$$

where the constant $\Theta$ does not depend on $t$. As we have seen from above, since $\mathcal{Y}_{2\alpha}(0) < \infty$, we can apply Lemma 3.5 to bound $\mathcal{H}_\alpha(s)$ from above by the following finite bound: $(2\pi)^d \Gamma(1 - 2\alpha) \mathcal{Y}_{2\alpha}(0)$. Hence, together with (4.4), we obtain the following upper bound for $I_1$ that is uniform in $t \geq \tau$:

$$I_1 \leq \frac{\sin^q(\alpha \pi) \Theta (2\pi)^{dq/2} t_0}{\Gamma (1 - 2\alpha)^{q/2} \pi^q \Lambda^q} \left( \sup_{t \geq t_0} G_\rho(t; \mu^*) \right)^{q/2} \mathcal{Y}_{2\alpha}^{q/2}(0).$$

Combining these two upper bounds, we see that

$$\mathbb{P} \left[ v(t_0, \cdot) \notin \mathcal{X}(\Lambda) \right] \leq \frac{\sin^q(\alpha \pi) \Theta (2\pi)^{dq/2} t_0}{\Gamma (1 - 2\alpha)^{q/2} \pi^q \Lambda^q} \left( \sup_{t \geq t_0} G_\rho(t; \mu^*) \right)^{q/2} \mathcal{Y}_{2\alpha}^{q/2}(0) + \frac{1}{\Lambda^2} \sup_{t \geq t_0} G_\rho(t; \mu^*) < \epsilon,$$

with the upper bound holding uniformly for all $t \geq \tau$. Hence, for any $\epsilon > 0$, by choosing $\Lambda > 0$ big enough such that

$$\frac{\sin^q(\alpha \pi) \Theta (2\pi)^{dq/2} t_0}{\Gamma (1 - 2\alpha)^{q/2} \pi^q \Lambda^q} \left( \sup_{t \geq t_0} G_\rho(t; \mu^*) \right)^{q/2} \mathcal{Y}_{2\alpha}^{q/2}(0) + \frac{1}{\Lambda^2} \sup_{t \geq t_0} G_\rho(t; \mu^*) < \epsilon,$$

we can ensure that

$$\mathbb{P} \left[ (u(t, \cdot) \in \mathcal{X}(\Lambda)) \right] = \mathbb{P} \left[ v(t_0, \cdot) \in \mathcal{X}(\Lambda) \right] \geq 1 - \epsilon, \quad \text{for all } t \geq \tau,$$

which proves part (a) of Theorem 1.4. \qed

### 4.2 Proof of part (b) of Theorem 1.4

**Proof.** Fix an arbitrary $\tau > 0$ and denote

$$U(T) := \frac{1}{T} \int_{\tau}^{T + \tau} \mathcal{L}(u(t; \cdot; \mu)) \, dt, \quad T > 0.$$

We claim that the family of laws $U(T, \cdot)$ for $T > 0$ is tight in $L^2(\mathbb{R}^d)$. Indeed, for any $\epsilon \in (0, 1)$, by part (a), there exists a compact set $\mathcal{K} \in L^2(\mathbb{R}^d)$ such that (1.21) holds. This implies that

$$U(T)(\mathcal{K}) = \frac{1}{T} \int_{\tau}^{T + \tau} \mathcal{L}(u(t; \cdot; \mu)) \, dt \geq \frac{1}{T} \int_{\tau}^{T + \tau} (1 - \epsilon) \, dt = 1 - \epsilon, \quad \text{for all } T > 0.$$

Let $\{T_n\}_{n \in \mathbb{N}}$ be any deterministic sequence such that $T_n \uparrow \infty$. Since $\{U(T_n)\}_{n \geq 1}$ is a tight sequence of measures, then there exists a subsequence $\{U(T_{n_m})\}_{m \geq 1}$ that converges weakly to a measure, $\eta$, on $L^2(\mathbb{R}^d)$ (e.g. see [3, Theorem 5.1]). Then one can apply the Krylov-Bogoliubov existence theorem (see, e.g., [13, Theorem 11.7]) to conclude that the measure $\eta$ is an invariant measure for $\mathcal{L}(u(t; \cdot; \mu))$, $t \geq \tau$. Finally, since $\tau$ can be arbitrarily close to zero, one can conclude part (b) of Theorem 1.4. \qed
5 Discussion and examples

5.1 Invariant measures for SHE with a drift term

In this part, we give a brief account of the case when the SHE has a drift term which plays a crucial role in controlling the moments. The equation usually takes the following form:

\[ \left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta \right) u(t, x) = g(x, u(t, x)) + b(x, u(t, x)) \dot{W}(t, x) \quad x \in \mathcal{O}, \; t > 0. \]  

(5.1)

The references in this part are far from being complete. The interested readers can find more references from the references below.

The first case is when the drift term \( g(\cdot) \) in (5.1) satisfies certain dissipativity conditions, which push the solution toward zero; see, e.g., [2, 4, 6, 7, 17]. Such a “negative” drift term helps to cancel the growth of the moments. Here is one example of such drift term: for some \( m, k_1, c_1 > 0 \) as \( |u| \to \infty \):

\[
\begin{align*}
g(u) &\leq -k_1 |u|^m + k_2 & u > 0, \\
g(u) &\geq c_1 |u|^m - c_2 & u < 0.
\end{align*}
\]

(5.2)

In particular, Cerrai [6, 7] and Brzeźniak and Gątarek [4] considered the case of a bounded spatial domain, while Assing and Manthey [2] and Eckmann and Hairer [17] considered the whole space \( \mathbb{R}^d \). Note that Eckmann and Hairer [17] studied the additive noise case along with a bounded initial condition.

Several works which do not require an added drift term with dissipativity as in (5.2) include Misiats et al [21, 22]. In the Theorem 1.2 of [22], they provide a result guaranteeing the existence of an invariant measure for the stochastic heat equation on the whole space \( \mathbb{R}^d \). More precisely, they allow for a drift term, \( g(x, u) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \), such that for all \( x \in \mathbb{R}^d \) and \( u_1, u_2 \in \mathbb{R} \),

\[ |g(x, 0)| \leq \varphi(x) \quad \text{and} \quad |g(x, u_1) - g(x, u_2)| \leq L \varphi(x)|u_1 - u_2|, \]

for some \( L > 0 \) where \( \varphi(x) \) must decay fast enough such that \( \varphi/\sqrt{\rho} \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( \rho \) is the admissible weight. Thus, \( f \equiv 0 \) is allowed. However, they require the following condition on the diffusion term \( b \):

\[ |b(x, u_1) - b(x, u_2)| \leq L \varphi(x)|u_1 - u_2|, \]

which excludes the parabolic Anderson model. Lastly, Theorem 1.2 ibid. requires the initial condition to be in \( L^2(\mathbb{R}^d) \), which excludes the two important cases, \( u(0, x) = 1 \) and \( u(0, \cdot) = \delta_0(\cdot) \). Our Theorem 1.4 includes both of these initial conditions; see Section 5.3.

5.2 The conditions for the spectral measures by Tessitore and Zabczyk

Tessitore and Zabczyk [26] established the existence of an invariant measure for (1.1) in \( L^2_{\rho}(\mathbb{R}^d) \) under the assumptions that (1) there exists a \( \varphi \in L^2_{\rho}(\mathbb{R}^d) \cap L^2_{\bar{\rho}}(\mathbb{R}^d) \) where \( \rho/\bar{\rho} \in L^1(\mathbb{R}^d) \) and the solution starting from \( \varphi \) is bounded in probability in \( L^2_{\rho}(\mathbb{R}^d) \) and (2) that the spectral density \( \hat{f} \) satisfies

\[
\hat{f} \in L^p(\mathbb{R}^d) \quad \text{where} \quad \frac{d-2}{d} < \frac{1}{p}, \quad (5.3)
\]

see Hypothesis 2.1 ibid. However, as was illustrated in Theorem 3.3 ibid., in order to apply this theorem to a specific initial condition in \( L^2_{\rho} \) (or to have moments uniformly bounded in time), the following additional assumptions were imposed:

\[
d \geq 3 \quad \text{and} \quad L^2_{\rho} > \frac{\Gamma(d/2 - 1)2^{d/2-2}}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \left| \mathcal{F}(\sqrt{\hat{f}}) \right| \ast \left| \mathcal{F}(\sqrt{\hat{f}}) \right| \right)(\xi)|\xi|^{2-d}d\xi, \quad (5.4)
\]

17
where the convention of the Fourier transform is given in Remark 5.1. With these assumptions, they were able to prove that (1.1) starting from the constant one initial condition satisfies (1.8) and thus is bounded in probability, verifying the existence of an invariant measure via the construction (1.9).

Remark 5.1. The Fourier transform may be defined differently depending on how one handles the $2\pi$ constant. In this paper (as in [10, 9]), we use the convention given in (1.23). Hence, Plancherel’s theorem takes the form of $\int_{\mathbb{R}^d} \psi(x)\phi(x)dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\psi}(\xi)\hat{\phi}(\xi)d\xi$. The authors in [26] did not explicitly mention their convention of the Fourier transform. However, the proof of Theorem 3.3 ibid. suggests that the following convention has been used:

\[ \hat{\phi}(\xi) = \mathcal{F}\phi(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\phi(x)dx \quad \text{and} \quad \mathcal{F}^{-1}\psi(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\psi(\xi)d\xi. \]

Hence, Plancherel’s theorem takes the form $\int_{\mathbb{R}^d} \psi(x)\phi(x)dx = \int_{\mathbb{R}^d} \hat{\psi}(\xi)\hat{\phi}(\xi)d\xi$, without the additional factor $(2\pi)^{-d}$. In particular, the spectral density $\gamma$ ibid. corresponds to $(2\pi)^{-d/2} \hat{f}$ in this paper. Our equation (5.4), which is condition (3.4) ibid., takes into account this difference, therefore explaining the slightly different factor in front of the integral in (5.4) from that in (3.4) ibid.

In the following, we will focus on the second condition in (5.4), which corresponds to (1.10). We claim that the latter is much easier to check than that of the former. The square root and absolute value in (5.4) make their condition more restrictive. Indeed, if $\mathcal{F}(\sqrt{f})$ is nonnegative, then the absolute values in (5.4) can be removed without ambiguity, which will reduce to our condition (1.10) up to a constant factor. However, when $\hat{f}$ is only nonnegative and not strictly positive, finding the right square root of $\hat{f}$ so that $\mathcal{F}(\sqrt{\hat{f}})$ is nonnegative becomes tricky. This last point will be illustrated by the examples below. We first set up the notation in Example 5.2.

Example 5.2. Let $d = 1$ and $g(x) = \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$. Then we have that $\hat{g}(\xi) = \xi^{-1}\sin(\xi)$. Now set $f(x) = (g * g)(x) = 2^{-1}\max(2 - |x|, 0)$. It is clear that $f$ is nonnegative. It is also nonnegative-definite because $\hat{f}(\xi) = \hat{g}(\xi)^2 = \xi^{-2}\sin^2(\xi) \geq 0$.

Up to a constant, one may replace $\mathcal{F}$ by $\mathcal{F}^{-1}$. The following example shows that $\mathcal{F}^{-1}(\sqrt{\hat{f}})(x)$ is signed and hence, the absolute value make spoil the oscillatory structure.

Example 5.3. Suppose that $d = 1$, $f(x) = (2\pi)^{-1}x^{-2}\sin^2(x)$ and $\hat{f}(\xi) = 2^{-2}\max(2 - |\xi|, 0)$. Then as explained in Example 5.2, both $f$ and $\hat{f}$ are non-negative and non-negative definite. We claim that

\[ \mathcal{F}^{-1}(\sqrt{\hat{f}})(x) \] takes both positive and negative values. \hspace{1cm} (5.5)

Indeed, for all $x \in \mathbb{R}$,

\[ \mathcal{F}^{-1}(\sqrt{\hat{f}})(x) = \frac{2}{\pi} \int_0^2 2^{-1}\sqrt{2 - \xi^2}\cos(x\xi)\ d\xi = \frac{1}{\pi} \int_0^{\sqrt{2}} \xi^2 \cos\left(2 - \xi^2\right)\ d\xi \]

\[ = \frac{1}{\pi} \cos(2x) \int_0^{\sqrt{2}} \xi^2 \cos(x\xi^2)\ d\xi + \frac{1}{\pi} \sin(2x) \int_0^{\sqrt{2}} \xi^2 \sin(x\xi^2)\ d\xi \]

\[ = -\frac{1}{\pi} \cos(2x) \int_0^{\sqrt{2}} \sin(x\xi^2)\ d\xi + \frac{1}{\pi} \sin(2x) \int_0^{\sqrt{2}} \cos(x\xi^2)\ d\xi, \]
where we have applied the change of variables $\xi' = \sqrt{2 - \xi}$ and an integration by parts. By using the Fresnel integrals (see, e.g., [23, 7.2 (iii)])

$$S(z) = \int_0^z \sin \left( \frac{\pi t^2}{2} \right) dt \quad \text{and} \quad C(z) = \int_0^z \cos \left( \frac{\pi t^2}{2} \right) dt,$$

the above expression can be further simplified via the change of variables $\xi' = \frac{\sqrt{2} |x|}{\pi \xi}$ to

$$\mathcal{F}^{-1}(\sqrt{\hat{f}})(x) = (8\pi)^{-1/2} |x|^{-3/2} \left( -\cos(2|x|)S \left( \frac{2\sqrt{|x|}}{\sqrt{\pi}} \right) + \sin(2|x|)C \left( \frac{2\sqrt{|x|}}{\sqrt{\pi}} \right) \right), \quad x \in \mathbb{R}.$$

This proves the claim in (5.5); see also Figure 3 for some plots.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{One example that $\mathcal{F}^{-1}(\sqrt{\hat{f}})(x)$ assumes both positive and negative values.}
\end{figure}

On the other hand, the next example shows that the spectral density given in Example 5.3 can be easily handled by our condition $-\Upsilon(0) < \infty$, i.e., (1.14).

**Example 5.4.** From the one-dimension case in Example 5.3, one can construct a $d$-dimensional counterpart: Let $f_1$ and $\hat{f}_1$ be the $f$ and $\hat{f}$, respectively, in Example 5.3. Then define

$$f_d(x) := \prod_{i=1}^d f_1(x_i), \quad x \in \mathbb{R}^d,$$

and hence

$$\hat{f}_d(\xi) := \prod_{i=1}^d \hat{f}_1(\xi_i), \quad \xi \in \mathbb{R}^d.$$ 

It is straightforward to verify that

$$-\Upsilon(0) = C \int_{\mathbb{R}^d} \prod_{i=1}^d \max\left\{ \frac{2 - |\xi_i|}{|\xi|^2(1-\alpha)} \right\} d\xi \leq C \int_{|\xi| \leq 2\sqrt{d}} \frac{2^d}{|\xi|^2(1-\alpha)} d\xi = C \int_0^{2\sqrt{d}} \frac{r^{d-1}}{r^2(1-\alpha)} dr.$$

Hence, if $\alpha > 1 - d/2$, then $-\Upsilon(0) < \infty$.

The next example illustrates the delicacy of choosing the right branches for the square root in (5.4).

**Example 5.5.** Let $f$ and $g$ be given as Example 5.2. In this case, $\hat{f}(\cdot)$ is only nonnegative (not strictly positive) with infinitely many zeros. Hence, when taking the square root of $\hat{f}(\xi)$ as in (5.4), one needs to wisely select the correct positive and negative branches: (1) Clearly, the signed version $\sqrt{\hat{f}(\xi)} = \xi^{-1} \sin(\xi)$ is preferable since its inverse Fourier transform can be easily computed, which is equal to $g(x)$. Moreover, because this inverse Fourier transform $g(x)$
is nonnegative, the absolute value signs in (5.4) do not pose any additional restrictions. (2) However, if one chooses the positive branches, namely, \( \sqrt{f(\xi)} = |\xi^{-1}\sin(\xi)| \), then it is not clear how to compute its Fourier transform. In general, some bad choices of the positive/negative branches may make the conditions in (5.4) fail. For example, such choice may turn \( \sqrt{f(\xi)} \) into a distribution, and then taking the absolute value of a distribution (unless it is a measure) may be problematic. Another issue that may arise is when \( \sqrt{f(\xi)} \) is a well-defined function, taking on both positive and negative values and after taking the absolute value, the integral in (5.4) may blow up.

### 5.3 Various initial conditions

In this part, we give some concrete examples of initial conditions.

**Example 5.6** (\( L^\infty(\mathbb{R}^d) \) initial condition). We emphasize that if the initial condition, \( \mu \), is deterministic and is such that \( \mu(dx) = \varphi(x)dx \) with \( \varphi \in L^\infty(\mathbb{R}^d) \), then all conditions related to \( \mathcal{G}_\rho(\cdot) \) in both Theorems 1.2 and 1.4 are trivially satisfied. To be more precise, both Conditions (1.17) and (1.20) hold because

\[
\mathcal{G}_\rho(t; |\varphi|) \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)} \|\rho\|_{L^1(\mathbb{R}^d)} < \infty \quad \text{uniformly for all } t \geq 0.
\]

**Example 5.7** (Delta initial condition). In this example, we study the case when the initial condition, \( \mu \), is the Dirac delta measure at zero, namely, \( \delta_0 \). Let \( \rho \) be a nonnegative \( L^1(\mathbb{R}^d) \) function. Since

\[
\mathcal{G}_\rho(t; \delta_0) = \int_{\mathbb{R}^d} G(t,x)^2 \rho(x)dx \leq G(t,0)^2 \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0,
\]

we see that both conditions (1.17) and (1.20) are satisfied. In particular, \( \limsup_{t>0} \mathcal{G}_\rho(t; \delta_0) = 0. \)

**Example 5.8** (More initial conditions not in \( L^2(\mathbb{R}^d) \)). In this example, we study the case when \( \mu(dx) = |x|^{-\alpha}dx \) for some \( \alpha \in (0,d) \). It is clear that when \( \alpha \in (d/2, d) \), \( \mu \in L^2(\mathbb{R}^d) \). However, in this case, we have

\[
J_0(t,x) = \left( G(t,\cdot) * | \cdot |^{-\alpha} \right)(x) \leq \left( G(t,\cdot) * | \cdot |^{-\alpha} \right)(0).
\]

On the other hand,

\[
\left( G(t,\cdot) * | \cdot |^{-\alpha} \right)(0) = \frac{2\pi^{d/2}}{\Gamma(d/2)} (2\pi t)^{-d/2} \int_{0}^{\infty} e^{-\frac{r^2}{2d}} r^{-\alpha+d-1} dr = C_* t^{-\alpha/2},
\]

with \( C_* = 2^{-\alpha/2} \Gamma((d-\alpha)/2) / \Gamma(d/2) \), which implies that

\[
\mathcal{G}_\rho(t; | \cdot |^{-\alpha}) \leq \int_{\mathbb{R}^d} J_0^2(t,0) \rho(x)dx = C^2_* t^{-\alpha} \|\rho\|_{L^1(\mathbb{R}^d)}, \quad \text{for all } t > 0.
\]

Therefore, we see that both conditions (1.17) and (1.20) are satisfied.

The following proposition shows that for initial conditions with unbounded tails, condition (1.17) may hold while condition (1.20) may fail.

**Proposition 5.9.** Suppose that \( \rho(x) = \exp(-|x|) \), which is an admissible weight function. Let the initial condition \( \mu \) be given as \( \mu(dx) = |x|^{-\alpha}dx \) with \( \alpha > 0 \). Then for some constants \( C, C' > 0 \) that depend on \( d \) and \( \alpha \), it holds that

\[
C'(1 + t^\alpha) \leq \mathcal{G}_\rho(t; |\mu|) \leq C(1 + t^\alpha), \quad \text{for all } t > 0.
\]

In particular, this implies that condition (1.17) is satisfied, but condition (1.20) fails.
Proof. Notice that by scaling arguments, \( \mathcal{G}_\rho(t; |\mu|) \) is equal to
\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} G(t, x - y)|y|^\alpha d y \right) \frac{1}{2} e^{-|x|^2} dx = \int_{\mathbb{R}^d} t^{\alpha + d/2} \left( \int_{\mathbb{R}^d} G(1, \xi - z) |z|^\alpha d z \right) \frac{1}{2} e^{-\sqrt{\alpha} |\xi|} d \xi.
\]
In the following, let \( C_d, C_\alpha, C'_\alpha, C_{\alpha,d} \) and \( C'_{\alpha,d} \) be generic constants that may depend on \( \alpha \) and \( d \) and may change their value at each appearance.

**Upper bound:** Because
\[
\int_{\mathbb{R}^d} G(1, z) |\xi - z|^\alpha d z \leq C_\alpha \int_{\mathbb{R}^d} G(1, z) (|\xi|^\alpha + |z|^\alpha) d z \leq C'_\alpha (1 + |\xi|^\alpha),
\]
we see that
\[
\mathcal{G}_\rho(t; \mu) \leq C_\alpha \int_{\mathbb{R}^d} t^{\alpha + d/2} (1 + |\xi|^{2\alpha}) e^{-\sqrt{\alpha} |\xi|} d \xi = C_{\alpha,d} (t^\alpha \Gamma(d) + \Gamma(d + 2\alpha)) = C'_{\alpha,d}(1 + t^\alpha) < \infty,
\]
which proves the upper bound in (5.6).

**Lower bound:** Now we prove the lower bound in (5.6). Indeed,
\[
\int_{\mathbb{R}^d} G(1, z) |\xi - z|^\alpha d z \geq \int_{\mathbb{R}^d} G(1, z) ||\xi| - |z||^\alpha d z \geq C_d \int_0^\infty ||\xi| - x|^\alpha e^{-\frac{x^2}{2}} x^{d-1} d x \geq C_d \int_1^2 ||\xi| - x|^\alpha d x = C_d \frac{\psi(\alpha)}{1 + \alpha},
\]
where, by considering three cases, we have
\[
\psi(r) = \begin{cases} 
(2 - r)^{\alpha + 1} - (1 - r)^{\alpha + 1} & \text{if } 0 < r < 1, \\
(2 - r)^{\alpha + 1} + (r - 1)^{\alpha + 1} & \text{if } 1 \leq r \leq 2, \\
(r - 1)^{\alpha + 1} - (r - 2)^{\alpha + 1} & \text{if } r > 2,
\end{cases}
\]
which is equal to \( \text{sgn}(2 - r)|r - 2|^{\alpha + 1} + \text{sgn}(r - 1)|r - 1|^{\alpha + 1} \). We claim that
\[
\inf_{r \geq 0} \frac{\psi(r)}{\sqrt{1 + r^{2\alpha}}} > 0. \tag{5.7}
\]
With (5.7), we have that
\[
\int_{\mathbb{R}^d} G(1, z) |\xi - z|^\alpha d z \geq C_{\alpha,d} \sqrt{1 + |\xi|^{2\alpha}}.
\]
Then, by the same arguments as above for the upper bound, we obtain the lower bound in (5.6).

It remains to prove (5.7), which will be proved in three cases.

When \( r > 2 \), we see that
\[
\frac{\psi(r)}{\sqrt{1 + r^{2\alpha}}} \geq C_\alpha \frac{(r - 1)^{\alpha + 1} - (r - 2)^{\alpha + 1}}{(1 + r)^{\alpha}} \geq C_\alpha \frac{(r - 1)^{\alpha}(r - 1) - (r - 1)^{\alpha}(r - 2)}{(1 + r)^{\alpha}} = C_\alpha \left( \frac{r - 1}{1 + r} \right)^{\alpha} = C_\alpha \left( 1 - \frac{2}{1 + r} \right)^{\alpha} \geq C_\alpha \left( 1 - \frac{2}{3} \right)^{\alpha}.
\]
Note that in the first inequality above, we have considered two cases: \( 2\alpha \geq 1 \) and \( 2\alpha < 1 \). When \( 2\alpha < 1 \), we have used the concavity of \( x^{2\alpha} \), namely, \((1 + r^{2\alpha})/2 \leq ((1 + r)/2)^{2\alpha}\); when \( 2\alpha \geq 1 \),
we have used the super-additivity of \( x^{2\alpha} \): namely, that for all \( a, b > 0 \) that \((a + b)^{2\alpha} \geq a^{2\alpha} + b^{2\alpha}\).

Therefore, \( \inf_{r > 2} \frac{\psi(r)}{\sqrt{1 + r^{2\alpha}}} > 0 \).

When \( r \in (1, 2] \), elementary calculations show that the minimum of \( \psi(r) \) is achieved at \( r = 3/2 \). Hence, \( \inf_{r \in (1, 2]} \frac{\psi(r)}{\sqrt{1 + r^{2\alpha}}} \geq \frac{\psi(3/2)}{\sqrt{1 + (3/2)^{2\alpha}}} > 0 \).

Similarly, when \( r \in (0, 1] \), by differentiation, one finds that the function \( \psi(r) \) is nonincreasing. Hence, the minimum is achieved at \( r = 1 \), \( \inf_{r \in (0, 1]} \frac{\psi(r)}{\sqrt{1 + r^{2\alpha}}} \geq \frac{\psi(1)}{\sqrt{2}} > 0 \).

Combining the above three cases proves (5.7) and hence, Proposition 5.9.

\[ \square \]

### 5.4 Bessel and other related kernels

In this part, we will make some explicit computations for Bessel and related kernels.

**Example 5.10** (Bessel kernel). Let \( f_s \) denote the Bessel kernel with a strictly positive parameter \( s > 0 \). It is known that (see, e.g., Section 1.2.2 of [19])

1. \( f_s(x) > 0 \) for all \( x \in \mathbb{R}^d \) and \( \| f_s \|_{L^1(\mathbb{R}^d)} = 1 \);
2. there exists a constant \( C(s, d) > 0 \) such that \( f_s(x) \leq C(s, d) \exp(-|x|/2) \) for \( |x| \geq 2 \);
3. there exists a constant \( c(s, d) > 0 \) such that
   \[
   \frac{1}{c(s, d)} \leq \frac{f_s(x)}{H_s(x)} \leq c(s, d) \quad \text{for } |x| \leq 2,
   \]
   with
   \[
   H_s(x) = \begin{cases}
   |x|^{s-d} + O(|x|^{s-d+2}) & \text{for } 0 < s < d, \\
   \log \left( \frac{2}{|x|} \right) + 1 + O(|x|^d) & \text{for } s = d, \\
   1 + O(|x|^{s-d}) & \text{for } s > d;
   \end{cases}
   \]
4. the Fourier transform of \( f_s \) is strictly positive:
   \[
   \mathcal{F} f_s(\xi) = \frac{1}{(1 + |\xi|^2)^{s/2}}. \tag{5.8}
   \]

Note that one can use (5.8) as the definition of the Bessel kernel. Properties 1 and 4 ensure that \( f_s \) is a nonnegative and nonnegative-definite tempered measure for all \( s > 0 \).

**Example 5.11** (Matérn class of correlation functions). The Matérn class of correlation functions has been widely used in spatial statistics; one may check the recent work [20] for references. Following Section 2.10 of [25], this class of correlation functions is given by

\[
K(x) = \phi \cdot (\alpha|x|)^\nu \mathcal{K}_\nu (\alpha|x|), \quad \text{for } x \in \mathbb{R}^d \text{ with } \phi > 0, \alpha > 0, \nu > 0, \tag{5.9}
\]

where \( \mathcal{K}_\nu (\cdot) \) is the modified Bessel function of second type, and \( \alpha \) and \( \nu \) refer to the scaling and smoothness parameters, respectively. From the inversion formula (see p. 46 ibid.), one sees that

\[
\mathcal{F} K(\xi) = (2\pi)^d \mathcal{F}^{-1} K(\xi) = (2\pi)^d f(|\xi|) \quad \text{with } f(\xi) = \frac{2^{\nu-1} \phi \Gamma(\nu + d/2) \alpha^{2\nu}}{\pi^{d/2} (\alpha^2 + |\xi|^2)^{\nu + d/2}}, \quad \xi \in \mathbb{R}^d.
\]

Comparing the above expression with (5.8), we see that the class of Bessel kernels \( f_s \), with \( s > d - 2 \) and \( d \geq 3 \), includes the Matérn class (5.9) as a special case under the following choice of parameters:

\[
\alpha = 1, \quad \nu = (s - d)/2, \quad \text{and } \phi = 2^{(2-d-s)/2} \pi^{-d/2} \Gamma(s/2)^{-1}.
\]

Note that the requirement of the smoothness parameter \( \nu > 0 \) for the Matérn class corresponds to the case of the Bessel kernel with \( s > d \).
The following proposition shows what conditions (1.14), (1.10a), and (1.15) reduce to for the Bessel kernel as the correlation function in terms of its parameters.

**Proposition 5.12** (Bessel kernel as correlation function). If the correlation function \( f \) is given by the Bessel kernel \( f_s(\cdot) \) with \( s > 0 \) defined in Example 5.10, then

\[
\Upsilon_\alpha(0) = \frac{\Gamma \left( \frac{d}{2} - 1 + \alpha \right) \Gamma \left( \frac{s-d}{2} + 1 - \alpha \right)}{2^d \pi^{d/2} \Gamma(d/2) \Gamma(s/2)} \quad \text{for all } s > d - 2(1 - \alpha) > 0, \quad (5.10)
\]

and in particular when \( \alpha = 0 \), (5.10) simplifies to the following:

\[
\Upsilon(0) = \frac{\Gamma \left( \frac{2 + s - d}{2} \right)}{2^{d-1} \pi^{d/2} (d-2) \Gamma(s/2)} \quad \text{for all } s > d - 2 > 0. \quad (5.11)
\]

In addition,

\[
\mathcal{H}_\alpha(t) < \infty \quad \forall t > 0 \iff 0 < \alpha < \frac{1}{2} - \frac{(d-s)_+}{4} \quad \text{and} \quad s > d - 2 > 0, \quad (5.12)
\]

where \( a_+ := \max(a, 0) \). Moreover, for \( \alpha \in (0, 1/2) \), we have the following asymptotic behavior of \( \mathcal{H}_\alpha(t) \) at \( t \to 0 \):

\[
\mathcal{H}_\alpha(t) = \begin{cases} 
\frac{\pi^{d/2} \Gamma \left( (d-s)/2 \right)}{((s-d)/2 + 1 - 2\alpha) \Gamma(d/2)} \, t^{(s-d)/2 + 1 - 2\alpha} & d - 2 < s < d \\
\frac{-\pi^{d/2} \Gamma \left( (d-s)/2 \right)}{(1 - 2\alpha) \Gamma(s/2)} \, t^{1 - 2\alpha} + O \left( t^{(s-d)/2 + 2(1-\alpha)} \right) & s = d \\
\frac{-\pi^{d/2}}{(1 - 2\alpha) \Gamma(d/2)} \, t^{1 - 2\alpha} \log \left( \frac{1}{t} \right) + O \left( t^2 \log(t) \right) & d < s < d + 2 \\
\frac{\pi^{d/2} \Gamma \left( (s-d)/2 \right)}{(1 - 2\alpha) \Gamma(s/2)} \, t^{1 - 2\alpha} + O \left( (s-d)/2 + 1 - 2\alpha \right) & d < s < d + 2 \\
\frac{\pi^{d/2}}{(1 - 2\alpha) \Gamma(d/2 + 1)} \, t^{1 - 2\alpha} + O \left( t^{2(1-\alpha)} \log(t) \right) & s = d + 2 \\
\frac{\pi^{d/2}}{(1 - 2\alpha) \Gamma(d/2 + 1)} \, t^{1 - 2\alpha} + O \left( t^{2(1-\alpha)} \right) & s > d + 2
\end{cases} \quad (5.13)
\]

where \( \psi(x) = \frac{d}{dx} \log(\Gamma(x)) \) refers to the digamma function and \( \gamma \approx 0.57721 \) to Euler’s constant; see, e.g., 5.2.2 and 5.2.3 on p. 136 of [23].

**Proof.** By the spherical coordinate integration formula and (5.8), for all \( \alpha \in \mathbb{R} \),

\[
\Upsilon_\alpha(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^{2(1-\alpha)} (1 + |\xi|^2)^{s/2}} = (2\pi)^{-d} C_d \int_0^\infty \frac{r^{d-1}}{r^{2(1-\alpha)} (1 + r^2)^{s/2}} dr,
\]

where...
where \( C_d := \frac{2\pi^{d/2}}{\Gamma(d/2)} \). Now by the change of variables \( z = r^2/(1 + r^2) \), we can evaluate the above integral explicitly by transforming it to the Beta integral:

\[
\int_0^\infty \frac{r^{d-1}}{r^{2(1-\alpha)}(1 + r^2)^{s/2}}dr = \frac{1}{2} \int_0^1 z^{d/2+\alpha-2}(1-z)^{(s-d)/2-\alpha}dz = \frac{\Gamma(d/2 - 1 + \alpha) \Gamma((s-d)/2 + 1 - \alpha)}{2\Gamma(s/2)}
\]

which is finite provided that \( s > d - 2(1-\alpha) > 0 \). This proves (5.10) and from this, we easily deduce (5.11) by letting \( \alpha = 0 \) in (5.10) and by applying the formula \( \Gamma(z+1) = z\Gamma(z) \), which holds for \( z \in \mathbb{C} \) such that \( \Re(z) > 0 \).

It remains to prove (5.13), which then implies (5.12). From (1.16) and by the spherical coordinate integration formula, for all \( t > 0 \),

\[
\mathcal{H}_\alpha(t) = \int_0^t dr \ r^{-2\alpha} \int_{\mathbb{R}^d} d\xi \ \exp(-r|\xi|^2) \frac{\Gamma\left((s-d)/2\right)}{(1+|\xi|^2)^{s/2}} = \frac{C_d}{2} \int_0^t dr \ r^{-2\alpha} \int_0^\infty du \ \exp(-ru)(1+u)^{-s/2}u^{d/2-1}
\]

\[
= \frac{C_d\Gamma(d/2)}{2} \int_0^t dr \ r^{-2\alpha} \int_0^\infty du \ \exp(-ru)(1+u)^{-s/2}u^{d/2-1} = \frac{C_d\Gamma(d/2)}{2} \int_0^t dr \ r^{-2\alpha}I(r).
\]

By [23, 13.4.4 on p.326], \( I(r) \) is equal to the confluent hypergeometric function:

\[
I(r) = U \left( \frac{d}{2}, \frac{2+d-s}{2}, r \right).
\]

By 18.2.18 – 13.2.22 on p. 323 ibid., we see that

\[
I(r) = \begin{cases} 
\frac{\Gamma\left((d-s)/2\right)}{\Gamma(d/2)} r^{(s-d)/2} + \frac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r^{(s-d)/2+1}\right) & d - 2 < s < d \quad 18.2.18, \\
-\frac{1}{\Gamma(d/2)} \left(\log(r) + \psi(d/2) + 2\gamma\right) + O\left(r \log(r)\right) & s = d \quad 18.2.19, \\
\frac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r^{(s-d)/2}\right) & d < s < d + 2 \quad 18.2.20, \\
\frac{1}{\Gamma(d/2 + 1)} + O\left(r \log(r)\right) & s = d + 2 \quad 18.2.21, \\
\frac{\Gamma\left((s-d)/2\right)}{\Gamma(s/2)} + O\left(r\right) & s > d + 2 \quad 18.2.22.
\end{cases}
\]

Then integrating the right-hand side of the above expressions against \( \pi^{d/2}r^{-2\alpha}dr \) over \([0,t]\) gives the five cases in (5.13). This completes the proof of Proposition 5.12. \( \square \)

Similarly, one can use the Bessel kernel as the spectral density. In this case, we have the following proposition:

**Proposition 5.13** (Bessel kernel as spectral density). Suppose that the spectral density \( \hat{f} \) is given by the Bessel kernel \( f_s(\cdot) \) defined in Example 5.10, or equivalently (see (5.8)), suppose that \( f(x) = (1 + |x|)^{-s/2} \) for \( s > 0 \). Then

\[
\Upsilon_\alpha(0) = \frac{\Gamma\left(1-\alpha\right) \Gamma\left(\alpha - 1 + s/2\right)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma\left(s/2\right)} \quad \text{for all} \ s > 2(1-\alpha) > 0,
\]

and in particular when \( \alpha = 0 \), (5.15) simplifies to the following:

\[
\Upsilon(0) = \frac{2^{1-2d}\pi^{-3d/2}}{(s-2)\Gamma(d/2)} \quad \text{for all} \ s > 2.
\]

24
In addition,
\[ \mathcal{H}_\alpha(t) < \infty \quad \forall t > 0 \quad \iff \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad s > 0. \] (5.17)

Moreover, for \( \alpha \in (0, 1/2) \), we have the following asymptotic
\[ \mathcal{H}_\alpha(t) \sim \frac{t^{1-2\alpha}}{1-2\alpha}, \quad \text{as} \ t \downarrow 0. \] (5.18)

**Proof.** By similar arguments as Proposition 5.12, we have that
\[
\Upsilon_\alpha(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{f_s(\xi)}{\xi^{2(1-\alpha)}} \, d\xi = (2\pi)^{-2d} \int_{\mathbb{R}^d} \frac{\hat{f}_s(\xi)}{\xi^{d-2(1-\alpha)}} \, d\xi
\]
\[
= (2\pi)^{-2d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty \frac{r^{d-1}}{(1 + r^2)^{s/2}} r^{d-2(1-\alpha)} \, dr = \frac{\Gamma(1-\alpha)\Gamma(\alpha-1+s/2)}{2^{2d}\pi^{3d/2}\Gamma(d/2)\Gamma(s/2)}
\]
which is finite provided \( s > 2(1-\alpha) \). This proves both (5.15) and (5.16). As for (5.17),
\[
\mathcal{H}_\alpha(t) = \int_0^t dr \, r^{-2\alpha} \int_{\mathbb{R}^d} d\xi \, f_s(\xi) \exp(-r|\xi|^2)
\]
\[
= (2\pi)^{-d}\pi^{d/2} \int_0^t dr \, r^{-2\alpha} \int_{\mathbb{R}^d} d\xi \, \hat{f}_s(\xi) \exp\left(-\frac{|\xi|^2}{4r}\right) r^{-d/2}
\]
\[
= (2\pi)^{-d}\pi^{d/2} \int_0^t dr \, r^{-2\alpha} \int_{\mathbb{R}^d} d\xi \, (1 + |\xi|^2)^{-s/2} \exp\left(-\frac{|\xi|^2}{4r}\right) r^{-d/2}
\]
\[
= (2\pi)^{-d}\pi^{d/2} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^t dr \, r^{-2\alpha-d/2} \int_0^\infty dz \, z^{d-1}(1 + z^2)^{-s/2} \exp\left(-\frac{z^2}{4r}\right),
\]
where we have used Plancherel’s theorem and the following identities:
\[
\mathcal{F}(\exp(-|\cdot|^2))(\xi) = \pi^{d/2} \exp\left(-4^{-1}|\xi|^2\right) \quad \text{and} \quad \mathcal{F}(f(a\cdot))(\xi) = a^{-d}\mathcal{F}f(\xi/a).
\]

Then, by the same arguments as Proposition 5.12,
\[
\mathcal{H}_\alpha(t) = 2^{-d} \int_0^t dr \, r^{-2\alpha-d/2} I \left( \frac{1}{4r} \right), \quad \text{with} \quad I(r) = U \left( \frac{d}{2}, \frac{2 + d - s}{2}, r \right),
\]
where \( U \) is given in (5.14). As \( r \to \infty \), \( I(r) \sim r^{-d/2} \) thus as \( r \to 0 \), \( I \left( \frac{1}{4r} \right) \sim (4r)^{d/2} \) (see 13.2.6 on p. 322 of [23]). Hence, the above integral behave as follows:
\[
\mathcal{H}_\alpha(t) \sim \int_0^t dr \, r^{-2\alpha-\frac{d}{2}}(4r)^{d/2} = \int_0^t dr \, r^{-2\alpha} = \frac{t^{1-2\alpha}}{1-2\alpha},
\]
provided \( \alpha \in (0, 1/2) \), which proves both (5.17) and (5.18).

The necessity of the finiteness of \( \Upsilon(0) \) excludes the Riesz kernel as a choice for the spectral density. However, we can still construct a Riesz-type kernel which has polynomial growth at the origin and polynomial decay at infinity, but with different rates, using Propositions 5.12 and 5.13. This Riesz-type kernel gives another example of a kernel that is easily verifiable to be permissible under the conditions of our Theorem 1.4 above, while being demanding to verify using (5.4); see Example 5.14 for more details.
Example 5.14 (Riesz-type kernel). For \( s_1, s_2 \in (0, d) \), let \( f_{s_1} \) and \( f_{s_2} \) be Bessel kernels as in Example 5.10. Define

\[
    r(x) := f_{s_1}(x) + f_{s_2}(x)
\]

or equivalently \( \hat{r}(\xi) := \hat{f}_{s_1}(x) + f_{s_2}(x) \).

It is easy to see that \( r(\cdot) \) is both non-negative and non-negative definite which follows immediately from the linearity of the Fourier transform and the fact that the Bessel kernel is both non-negative and non-negative definite. Also, we easily deduce from properties (2) - (4) in Example 5.10 that

\[
    r(x) \sim \begin{cases} 
        |x|^{s_1-d} & |x| \to 0, \\
        |x|^{-s_2} & |x| \to \infty,
    \end{cases} \quad \text{and} \quad \hat{r}(\xi) \sim \begin{cases} 
        |\xi|^{s_2-d} & |\xi| \to 0, \\
        |\xi|^{-s_1} & |\xi| \to \infty.
    \end{cases}
\]

Propositions 5.12 and 5.13 imply that

\[
    \Upsilon_\alpha(0) = \frac{\Gamma \left( \frac{d}{2} - 1 + \alpha \right) \Gamma \left( \frac{s_1-d}{2} + 1 - \alpha \right)}{2^d \pi^{d/2} \Gamma(d/2) \Gamma(s_1/2)} + \frac{\Gamma(1-\alpha) \Gamma(\alpha - 1 + \frac{s_2}{2})}{2^{2d} \pi^{3d/2} \Gamma(d/2) \Gamma(s_2/2)} < \infty,
\]

provided that

\[
    0 < d - 2(1-\alpha) < s_1 < d \quad \text{and} \quad 0 < 2(1-\alpha) < s_2 < d.
\]

In contrast, it is not clear how to compute \( \mathcal{F}(\sqrt{r}) \); see condition (5.4).

5.5 Examples of admissible weight functions

In this part, we give some examples of the admissible weight functions. As given in Section 2 of [26], the following functions are admissible functions:

\[
    \begin{cases}
    \rho(x) = \exp(-a|x|) & a > 0, \\
    \rho(x) = (1 + |x|^\alpha)^{-1} & a > d.
    \end{cases}
\]  

(5.19)

The smaller the weight function \( \rho(\cdot) \) (not necessarily admissible) is, the larger the space \( L^2_{\rho}(\mathbb{R}^d) \) is. For example, one may choose \( \rho \) to be either a nonnegative function with compact support or the heat kernel itself \( G(1, \cdot) \). In both cases, \( \rho \) is smaller than those in (5.19) (up to a constant). However, one can easily check that the admissible condition (1.7) excludes these two cases. However, the examples in the following Proposition 5.15 seem to be less obvious:

Proposition 5.15. \( \rho_b(\cdot) \) is admissible if and only if \( b \in (0, 1) \) where

\[
    \rho_b(x) := \exp \left( -|x|^b \right), \quad x \in \mathbb{R}^d, \quad \text{with} \quad b > 0.
\]

Proof. From Definition (1.1), we see that \( \rho_b \) is admissible if and only if for all \( T > 0 \),

\[
    \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t} + |x|^b + |y|^b} \, dy < \infty.
\]

Denote the above integral by \( I(t,x) \). We will use \( C \) to denote a generic constant that does not depend on \( (t,x) \), which value may change at each occurrence.

We first assume that \( b \in (0, 1] \). In this case,

\[
    |x|^b \leq |x - y + y|^b \leq (|x - y| + |y|)^b \leq |x - y|^b + |y|^b.
\]
Hence,
\[
I(t, x) \leq C \int_{\mathbb{R}^d} t^{-d/2} e^{-\frac{|x-y|^2}{2t} + |x-y|^b} dy = C \int_{\mathbb{R}^d} t^{-d/2} e^{-\frac{|y|^2}{2t} + |y|^b} dy = C \int_0^\infty t^{-d/2} e^{-\frac{r^2}{2t} + r^{d-1}} dr.
\]
Then by applying the change of variables \( v = t^{-1/2} r \),
\[
I(t, x) \leq C \int_0^\infty e^{-\frac{1}{2} + T h^{2/2} r^{d-1}} dr \leq C \int_0^\infty e^{-\frac{1}{2} + T h^{2/2} r^{d-1}} dr < \infty.
\]
Next we assume that \( b > 1 \). We need to show that \( \rho_b \) is not admissible. Without loss of generality, we assume that \( d \geq 2 \). The case when \( d = 1 \) is easier and can be proved similarly as the proof below. It suffices to show that
\[
\lim_{r \to \infty} I(1/2, x_r) = \infty, \quad \text{where } x_r := (r, 0, \ldots, 0) \in \mathbb{R}^d.
\]
Without loss of generality, we may assume below that \( r \gg 2 \). Denote \( y = (y_1, \ldots, y_d) = (y_1, y_*) \) with \( y_* \in \mathbb{R}^{d-1} \). Using the subadditivity (resp. convexity) of \((x + y)^{b/2}\) when \( b \in (1, 2] \) (resp. \( b > 2 \)), we see that
\[
|y|^b = (y_1^b + y_2^b + \cdots + y_d^b)^{b/2} \leq c|y_1|^b + (y_2^b + \cdots + y_d^b)^{b/2} = c|y_1|^b + c|y_*|^b, \quad c := 1 \wedge 2^{b/2-1}.
\]
Hence,
\[
I\left(1/2, x_r\right) = C \int_{\mathbb{R}^d} e^{-\sum_{i=1}^d y_i^2 + (r^b - |y|^b)} dy \\
\geq C \int_{d_{R_{y_*}}} dy_* \int_{\mathbb{R}} dy_1 e^{-|y_1|^2 - |y_1|^b - |y_1|^{b-2} + (r^b - |y|^b)} \\
= C \int_{d_{R_{y_*}}} dy_* \int_{\mathbb{R}} dy_1 e^{-|y_1|^2 - c|y_1|^b} \\
= C \int_{y_1}^{r} e^{-y_1^2 + r^b - c|y_1|^b} dy \geq C \int_0^r e^{-y^2 + r^b - c|y|^b} dy = CK(r).
\]
It suffices to show that \( \lim_{r \to \infty} K(r) = \infty \), which is true when \( b > 2 \) because
\[
K(r) \geq \int_{r/2}^r e^{-y^2 + r^b - c|y|^b} dy \geq \int_{r/2}^r e^{-r^2 + r^b - c(r/2)^b} dy = \frac{r}{2} \exp\left(\left(1 - 2^{1-b/2}\right) r^b - r^2\right),
\]
which blows up as \( r \to \infty \). Hence, we may assume that \( b \in (1, 2] \). In this case, \( c = 1 \) and
\[
K'(r) = e^{r^b - r^2} + b \int_0^r e^{-y^2 + r^b - (r-y)^b} \left(r^{b-1} - (r-y)^{b-1}\right) dy.
\]
By the intermediate value theorem, we see that \( r^{b-1} - (r-y)^{b-1} = (b-1)y^{b-2} \xi \) for some \( \xi \in [r-y, r] \). Since \( b-1 \in (0, 1] \), this implies that \( r^{b-1} - (r-y)^{b-1} \geq (b-1)y^{b-2} \). Hence,
\[
K'(r) \geq e^{r^b - r^2} + b(b-1) \int_1^r e^{-y^2 + r^b - (r-y)^b} y^{b-2} dy \geq b(b-1)r^{b-2} \int_1^r e^{-y^2} dy.
\]
Another application of the intermediate value theorem shows that \( r^b - (r-1)^b = b\xi^{b-1} \) with \( \xi \in [r-1, r] \). Hence, \( r^b - (r-1)^b \geq b(r-1)^{b-1} \) and then
\[
K'(r) \geq C r^{b-2} \exp\left(b(r-1)^{b-1}\right).
\]
Hence, for \( r \gg 2 \), \( K'(r) \) is positive and unbounded as \( r \to \infty \). Therefore, this implies that \( K(r) \) blows up as \( r \to \infty \), which completes the proof of Proposition 5.15.
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