The “minimal” viscosity and elliptic flow at RHIC

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We show from covariant transport theory that, for a massless ideal gas equation of state, even a small shear viscosity to entropy density ratio \( \eta/s \approx s/(4\pi) \) generates significant 15 – 30% dissipative corrections to elliptic flow for conditions expected in mid-peripheral \((b = 8 \text{ fm})\) Au + Au collisions at \( \sqrt{s_{NN}} \approx 200 \text{ GeV} \) at RHIC.

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Introduction. The goal of the heavy-ion physics is to study the properties of nuclear matter at extreme energy densities and temperatures. Spectacular features of the data from the Relativistic Heavy Ion Collider (RHIC) for gold-gold reactions at center-of-mass energies \( \sqrt{s_{NN}} \sim 100 – 200 \text{ GeV} \) lead to the suggestion that the hot and dense matter created in the collision exhibits perfect fluid behavior. (A perfect fluid has vanishing shear and bulk viscosities and heat conductivity.) The cornerstone of this conclusion was the success of ideal (Euler) fluid dynamics in explaining the large second Fourier moment of the azimuthal momentum distribution for fixed \( p_T \), the so-called “elliptic flow” \( v_2(p_T) \equiv \langle \cos 2\phi \rangle_{p_T} \), observed in non-central collisions.

General considerations based on quantum mechanics, on the other hand, indicate a nonzero lower bound for shear viscosity. For example, \( \varepsilon \), combining the time-energy uncertainty principle with kinetic theory gives \( \eta/s \geq 1/15 \), where \( s \) is the entropy density. This result is also supported by calculations for \( N = 4 \) supersymmetric Yang-Mills theory via the gauge-theory - gravity AdS-CFT duality conjecture. Pioneering results predicted \( \eta/s \geq 1/(4\pi) \) \( \rho \), while recent calculations for an extended class of theories find a somewhat lower bound \( \eta/s \geq 16/25 \times 1/(4\pi) \). It is an open question whether either of these limits applies to QCD. But the possibility is intriguing because even a small \( \eta/s \sim O(1)/(4\pi) \) has significant dynamical effects in heavy-ion collisions.

Ideal hydrodynamics assumes local equilibrium throughout the evolution. For nonzero transport coefficients, on the other hand, the system departs from local equilibrium, leading to dissipative corrections. If the system stays sufficiently close to local equilibrium, dissipation can be investigated via causal dissipative hydrodynamics, for example, Israel-Stewart theory \( \rho \). Solution techniques for viscous hydrodynamic equations, in the minimally required 2+1 dimensions necessary for elliptic flow studies, have been recently developed and applied \( \rho \). However, Israel-Stewart theory comes from a truncation procedure \( \rho \), with uncontrolled errors (it lacks a small expansion parameter), and therefore its region of validity is not known. Moreover, the solutions are causal only in a region of hydrodynamic parameters, and their stability is not guaranteed \( \rho \).

Here we utilize instead the fully causal and stable covariant parton transport theory \( \rho \), for which covariant algorithms have been available in full 3+1D for quite some time \( \rho \). A forerunner of this analysis \( \rho \) considered large, constant elastic 2 \( \rightarrow \) 2 gluon-gluon cross sections \( \sigma \sim 45 \text{ mb} \), and found a significant 20 – 30% reduction of elliptic flow due to dissipation. Such dynamics gives an increasing \( \eta/s \) with time \( \rho \). In contrast, here we study dissipation for a “minimal” \( \eta/s \approx 1/(4\pi) \) that is constant in time. Preliminary results have been reported already at \( \rho \), and are confirmed here to be accurate.

Covariant transport theory near the hydrodynamic limit. We consider here, as in Refs. \( \rho \), the simplest but nonlinear form of Lorentz-covariant Boltzmann transport theory in which the on-shell phase space density \( f(x, p) \), evolves with an elastic 2 \( \rightarrow \) 2 rate as

\[
p^\mu \partial_\mu f_1 = S(x, p_1) + \frac{1}{2\pi} \int_{34} (f_3 f_4 - f_1 f_2) W_{12\rightarrow 34} \times \delta^4(p_1 + p_2 - p_3 - p_4)
\]

where the integrals are shorthands for \( \int \equiv \int d^3p/(2E_i) \). For dilute systems, \( f \) would be the phase-space distribution of quasi-particles, while the transition probability \( W = s(s - 4m^2) d\sigma/dt \) would be given by the scattering matrix element. Our interest here, on the other hand, is to study the theory near its hydrodynamic limit.

It is well known (Boltzmann’s \( H \)-theorem) that \( \rho \) drives the system towards a fixed point, global equilibrium. In the hydrodynamic limit \( W \rightarrow \infty \), the transport solutions approach local equilibrium \( f(x, p) = g \exp[(\mu(x) - p_\mu u^\mu(x))/T(x)]/(2\pi)^3 \). A systematic expansion in small gradients around equilibrium via the Chapman-Enskog procedure \( \rho \) gives the viscous hydrodynamic equations by Navier and Stokes. However, this approximate theory is severely acausal. A causal formulation proposed by Mueller \( \rho \) and later generalized by Israel and Stuart (IS) \( \rho \) retains certain second-order derivative terms, resulting in relaxation equations. IS
theory can also be recovered from transport via the 14-moment expansion of Grad [9, 15]. The transport coefficients, and the microscopic relaxation times for the dissipative fluxes in IS, are all given by the differential cross section $d\sigma/dt$.

The key observation here is that one can use transport theory to solve causal viscous hydrodynamics provided one dials in the equation of state (EOS) and transport coefficients of interest. In this case, the “particles” and the specific “interaction” in the transport have no physical significance - they are only mathematical tools to reproduce the desired dynamical equations. Here we consider an ultrarelativistic gluon gas with $e = 3p$, applicable to the high-temperature plasma in the early stages at RHIC. In this case [9, 13], $\eta/\xi \approx 0.08/(5\sigma_{tr})$ and the shear stress relaxation time is $\tau_\eta = 6\lambda_{tr}/5$, where $\sigma_{tr}$ and $\lambda_{tr} \equiv 1/(n\sigma_{tr})$ are the transport cross section and transport mean free path, respectively. Note, for an isotropic cross section $\sigma_{tr} \approx 2\sigma_{tot}/3$.

Elliptic flow and $\eta/s$. For the above conditions,

$$\frac{\eta}{s} \approx \frac{\eta}{4n} \approx \frac{T\lambda_{tr}}{5} = \frac{T}{5n\sigma_{tr}}$$

(2)

Assuming the system stays close to local equilibrium, during the initial longitudinal ( Bjorken) expansion stage of the heavy-ion collision the density and temperature evolve as $n \sim 1/\tau, T(\tau) \sim \tau^{-1}/\sqrt{T}$ where $\tau \equiv \sqrt{T} = \frac{\lambda}{L}$ is the longitudinal ( Bjorken) proper time. For a constant cross section, $\eta/s \propto \tau^{2/3}$ then increases with time.

One might therefore think that Ref. [21] with constant $\sigma \sim 45$ mb overestimated dissipative effects in $Au+Au$ at RHIC energies. However, even though $\eta/s$ grew in that calculation, its initial value was really small (cf. Fig. 1). For the longitudinally boost invariant scenario assumed there, $\lambda_{tr} = \tau/(\sigma_{tr}dN/dydx_{z}^{2})$, and with the parameters of that calculation [31] $\lambda_{tr}(\tau_{0} = 0.1 fm) \approx 7.1 \times 10^{-3} fm^{3}$, i.e., $\eta/s \sim 1/(60\pi)$ at the very center of the collision. The ratio is way below the conjectured bounds even for the average density that is $2-3$ times smaller than the maximum.

We cross-check this important finding with the transport opacity [20]

$$\chi = \frac{\sigma_{tr}}{\sigma_{tot}}(n_{coll}) = \left\langle \frac{dz}{\lambda_{tr}(x_{0} + 2n_{tr}, \tau = \tau_{0} + z)} \right\rangle$$

(3)

which is the number of collisions per particle weight by the transport cross section and averaged over initial coordinates and directions. $\chi$ is dominated by the early and densest longitudinal expansion stage, during which $\lambda_{tr} \propto \tau$, and thus

$$\chi \approx \frac{\tau_{0}}{(\lambda_{tr}(\tau_{0}))} \int_{0}^{L} \frac{dz}{z + \tau_{0}} \approx \frac{\tau_{0}}{(\lambda_{tr}(\tau_{0}))} \ln \frac{L}{\tau_{0}}$$

(4)

With $\chi \approx 21$ from the calculation and an estimate $L \sim 3-4$ fm for the size of the reaction zone, on average $\langle \lambda_{tr}(\tau_{0} = 0.1 fm) \rangle \approx 1.5 \times 10^{-2} fm$. This is about 2.5 times larger than the value estimated for the collision center, as expected.

From [21] we then find that for the situation in Ref. [21], $\eta/s$ evolves with time schematically as shown in Fig. 1. During the first few fermis of the evolution relevant for the buildup of elliptic flow [18, 21], the system stays closer to equilibrium than would be allowed by a “minimal” viscosity because the transport mean free path (or equivalently, the scattering rate) is not limited by any quantum bound. The situation only changes after $\tau \sim 1-3$ fm, when the transport mean free path becomes large enough to reach $\eta/s \approx 1/(4\pi)$.

To study elliptic flow for constant $\eta/s$, we therefore start with an initially modest transport cross section and increase it with time as $\sigma_{tr}(\tau) = \eta_{0}[\tau/(0.1 fm)]^{2/3}$. Such growth is already encoded in perturbative QCD, provided we ignore the running of the coupling: $\sigma_{tr} \approx (18\pi\alpha_{s}^{2}/s)(\ln(s/\mu_{D}^{2})) \sim (\pi\alpha_{s}^{2}/T^{2})(\ln(18/g^{2}))$, where $\mu_{D} \sim gT$ is the Debye mass. The initial conditions for $Au+Au$ at $\sqrt{s_{NN}} = 200$ GeV at impact parameter $b = 8$ fm (see footnote [31]) and the numerical solution technique MPC [22] are the same as in Ref. [21]. For numerical convenience we uniformly set $\sigma_{0} \approx 2.7$ mb, which ensures that on average $\eta/s \approx 1/(4\pi)$ in the system. In the center of the collision zone, (2) gives a lower value $\eta/s \approx 0.4/(4\pi)$, but that is compensated by the increase of $\eta/s$ with decreasing density as we go outward. With the average density $\langle n \rangle \sim n_{max}/2.5$ estimated earlier from [4], on average $4\pi\eta/s \approx 1$. A cross-check with the transport opacity $\chi \approx 16$ obtained from the growing-cross-section calculation $\sigma_{tr}(\tau) \propto \tau^{2/3}$ gives

$$\chi \approx \frac{3\tau_{0}}{2\langle \lambda_{tr}(\tau_{0}) \rangle} \left( \frac{L}{\tau_{0}} \right)^{2/3}$$

(5)

i.e., $\lambda_{tr}(\tau_{0}=0.1 fm) \approx 0.09-0.11$ fm and $4\pi\eta/s \sim 0.8-1$.

The above choice of $\sigma_{0}$ implies $\sim 5-10$ times higher two-body rates than perturbative QCD estimates. For our purposes this is not a problem, the rates are simply
adjusted to reproduce the desired $\eta/s$. With the inclusion of radiative $3 \leftrightarrow 2$ processes\textsuperscript{[29]}, even perturbative rates could generate a small $\eta/s \sim 0.1$.

With $2 \rightarrow 2$ scattering, particle number is conserved and thus $s = n(4 - \mu/T)$ where $\mu$ is the chemical potential. Therefore,\textsuperscript{[2]} acquires a small relative correction $\Delta s/s(\mu=0) = (1/4) \ln(n_{eq}/n) = (1/4) \ln(2T^{3}/(\pi^{2}n))$ logarithmic in density. For our gluon gas initial conditions ($g = 16$), it is about $\sim (-6)\%$ at the center of the collision, while $\sim 20\%$ for a low $n = n_{max}/3$. During the longitudinal expansion stage, the correction stays roughly constant because the dilution is largely compensated by cooling, $nT^{3} \approx const$. Dissipation of course still generates entropy because the temperature drops slightly slower\textsuperscript{[3]} than $T \propto \tau^{-1/3}$, but that effect cannot be very large for the system to stay near equilibrium. With $4\pi\eta/s \sim 0.8 - 1$ initially, we have a cushion for entropy production during later evolution. Therefore, we conclude that the average $\eta/s$ is set to the desired $\eta/s = 1/(4\pi)$ within about $20\%$ in the calculation.

Figure 2 shows differential elliptic flow $v_{2}(p_{T})$ results for $\tau_{0} = 0.1$ fm. Even for a “minimal” $\eta/s = 1/(4\pi)$ (filled squares), dissipation reduces elliptic flow at moderate $p_{T} \sim 2 - 3$ GeV by about $25\%$ relative to the ideal hydrodynamic limit (solid line). The relative change increases with decreasing $p_{T}$, and therefore dissipation also flattens the slope at low $p_{T}$\textsuperscript{[32]}. For comparison, we also show the result (open squares) for a constant cross section $\sigma_{\tau} = \sigma_{0}$, i.e., growing $\eta/s = (\tau/\tau_{0})^{2/3}/(4\pi)$, which of course generates much smaller elliptic flow.

![Figure 2](image_url)

**FIG. 2:** Differential elliptic flow $v_{2}(p_{T})$ as a function of $p_{T}$ in Au + Au at $\sqrt{s_{NN}} = 200$ GeV and $b = 8$ fm at RHIC, from ideal hydrodynamics (solid curve) using the codes in\textsuperscript{[2, 3]} and covariant transport (squares) using the MPC algorithm\textsuperscript{[22]}. An initial (thermalization) time of $\tau_{0} = 0.1$ fm/c was assumed. Transport results for a constant cross section (open squares) and for $\eta/s \approx 1/(4\pi)$ (filled squares) are shown, while the hydrodynamic curve is from\textsuperscript{[21]}.  

Figure 3 compares differential elliptic flow $v_{2}(p_{T})$ from the transport (squares) to the ideal hydrodynamic limit (solid line) for a later thermalization time $\tau_{0} = 0.6$ fm. In this case the initial temperature was adjusted to $T_{0} \approx 0.385$ GeV to account for cooling $T \sim \tau^{-1/3}$ because of $pdV$ work during longitudinal Bjorken expansion. With the rescaled temperature, results for both ideal hydrodynamics and transport with constant cross section (open squares) are essentially independent of $\tau_{0}$, as long as $\tau_{0}/R \ll 1$, as was also found in Ref.\textsuperscript{[21]} (cf. Fig. 2). However, with $\eta/s \approx 1/(4\pi)$, the dissipative reduction of elliptic flow (solid squares) relative to ideal hydrodynamics is more modest, $\sim 15\%$, for the larger $\tau_{0} = 0.6$ fm. This is similar in magnitude to recent results from 2+1D dissipative hydrodynamics\textsuperscript{[12, 14]}.

At fixed $\eta/s$, dissipative effects are weaker for larger $\tau_{0}$ because the initial value of the transport cross section is larger in that case. An equivalent explanation is that though $\eta/s$ is the same, initial velocity gradients in the system $\partial_{\mu}u_{\nu} \sim 1/\tau$ are smaller for larger $\tau_{0}$. Indeed, the Navier-Stokes correction to the stress tensor\textsuperscript{[1]} is

$$T^{\mu\nu} = T_{\text{ideal}}^{\mu\nu} + \eta(\nabla^{\mu}u^{\nu} + \nabla^{\nu}u^{\mu} - \frac{2}{3} \Delta^{\mu\nu}\partial_{\rho}u_{\rho})$$

(6)

($\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^{\mu}u^{\nu}$, $\nabla^{\mu} \equiv \Delta^{\mu\nu}\partial_{\nu}$, $u^{\mu}$ is the flow velocity), in case of a longitudinal boost-invariant expansion, implies viscous corrections to the transverse and longitudinal pressure $\Delta p_{T} = 2\eta/(3\tau)$, $\Delta p_{L} = -4\eta/(3\tau)$. Therefore, relative pressure corrections

$$\frac{\Delta p}{p} \sim \frac{\eta}{s} \frac{T_{s}}{pT\tau} = \frac{4\pi\eta}{s} \frac{1}{\pi T\tau}$$

(7)

decrease with time $\Delta p/p \propto \tau^{-2/3}$ if $\eta/s = const$; whereas $\Delta p/p \propto \tau^{0}$ if $\sigma_{\tau} = const$.

Based on the Navier-Stokes estimate\textsuperscript{[7]}, it is not a surprise that dissipative corrections are important for conditions expected at RHIC. $\Delta p/p \sim 20\%$ for $\tau_{0}T_{0} = 0.6$ fm $\times 0.385$ GeV and is almost $100\%$ for $\tau_{0}T_{0} = 0.1$ fm
× 0.7 GeV. In the former case, the correction is modest, and viscous hydrodynamics is likely applicable. In the latter case, however, a hydrodynamic approach seems questionable. It would be interesting to test this anticipated break-down of hydrodynamics against solutions of Israel-Stewart theory. We note that dissipation is expected to be relevant not only at RHIC but at the LHC as well.

Conclusions. We utilized covariant transport theory to study the effect of a “minimal” shear viscosity \( \eta = s/4\pi \) on differential elliptic flow \( v_2(p_T) \) in \( Au + Au \) collisions at \( \sqrt{s_{NN}} \sim 200 \) GeV at RHIC. The key ingredient is a transport cross section \( \sigma_{tr} \propto \tau^{2/3} \) that grows with time. We find significant reduction of elliptic flow relative to the ideal hydrodynamic limit, \( \sim 25\% \) reduction for a thermalization time \( \tau_0 = 0.1 \) fm, while \( \sim 15\% \) for \( \tau_0 = 0.6 \) fm. This indicates that even such a small shear viscosity cannot be ignored at RHIC, and thus the LHC as well, because gradients are large.

We note that this study set \( \eta/s \approx 1/(4\pi) \) only within 20\% and in an average sense, for numerical convenience. The evolution of the local density and temperature was approximated with analytic results for longitudinal Bjorken expansion. More accurate results could be obtained, in principle, via a transport cross section \( \sigma_{tr}(n, \tau) \) that depends explicitly on the local density. However, that is much more expensive numerically with the solution technique (cascade algorithm) utilized here.

In principle, radiative processes such as \( gg \leftrightarrow ggg \) can also be included\([28]\). However, near the hydrodynamic limit we do not expect large corrections to our results because the dynamics is determined solely by the equation of state and the viscosity \( \eta/s = 1/(4\pi) \). The main difference is that radiative processes allow for change of particle number, which should be only a modest refinement for the initial conditions considered in this study.

Finally, we emphasize that a simple ideal gas equation of state \( e = 3p \) has been considered, which also implies vanishing bulk viscosity. It would be important to repeat this study with an equation of state that is more realistic for quark gluon matter at moderate \( T \lesssim 300 \) MeV, and to investigate effects of bulk viscosity which is expected to rise sharply in the vicinity of \( T \sim 200 \) MeV\([30]\).

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[31] Au + Au at \( b = 8 \) fm impact parameter, binary collision transverse profile, \( dN(b=8fm)/dy \approx 250 \) (the corresponding maximum transverse density is \( dN/dydz_T^2 \approx 9.36 \text{ fm}^{-2} \), \( T_0 = 0.7 \) GeV, and Debye-screened cross section with \( \sigma_T \approx 15 \) mb.
[32] Even for the massless equation of state considered here, at very low \( p_T \lesssim 0.1 \) GeV, \( v_2(p_T) \propto p_T^2 \) as follows from continuity and differentiability of the phasespace density. But the quadratic behavior very near the origin is masked
by the linear rise in a wide $p_T \sim 0.2 - 1.2$ GeV window.