THE SAMUELSON’S MODEL AS A SINGULAR DISCRETE TIME SYSTEM

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Abstract. In this paper we revisit the famous classical Samuelson’s multiplier-accelerator model for national economy. We reform this model into a singular discrete time system and study its solutions. The advantage of this study gives a better understanding of the structure of the model and more deep and elegant results.

Keywords: Samuelson, macroeconomic, singular, system, difference equations

1 Introduction

Many authors have studied generalised discrete & continuous time systems, see [1-19], and their applications especially in cases where the memory effect is needed including generalised discrete & continuous time systems with delays, see [20-38]. Many of these results have already been extended to systems of differential & difference equations with fractional operators, see [43-49].

Keynesian macroeconomics inspired the seminal work of Samuelson (1939), who actually introduced the business cycle theory. Although primitive and using only the demand point of view, the Samuelson’s prospect still provides an excellent insight into the problem and justification of business cycles appearing in national economies. In the past decades, many more sophisticated models have been proposed by other researchers [20-38]. All these models use superior and more delicate mechanisms involving monetary aspects, inventory issues, business expectation, borrowing constraints, welfare gains and multi-country consumption correlations.

Some of the previous articles also contribute to the discussion for the inadequacies of Samuelson’s model. The basic shortcoming of the original model is: the incapability to produce a stable path for the national income when realistic values for the different parameters (multiplier and accelerator parameters) are entered into the system of equations. Of course, this statement contradicts with the empirical evidence which supports temporary or long-lasting business cycles.

In this article, we propose an alternative view of the model by reforming it into a singular discrete time system.

The paper is organized as follows. Section 2 provides a short review for the organization of the original model and in Section 3 we introduce the proposed reformulation into a system of difference equations. Section 4 investigates the solutions of the proposed system.
2 The original model

The original version of Samuelson’s multiplier-accelerator original model is based on the following assumptions:

Assumption 2.1. National income $T_k$ in year $k$, equals to the summation of three elements: consumption, $C_k$, private investment, $I_k$, and governmental expenditure $G_k$

$$T_k = C_k + I_k + G_k.$$  \hspace{1cm} (1)

Assumption 2.2. Consumption $C_k$ in year $k$, depends on past income (only on last year’s value) and on marginal tendency to consume, modeled with $a$, the multiplier parameter, where $0 < a < 1$,  

$$C_k = aT_{k-1}. \hspace{3cm} (2)$$

Assumption 2.3. Private investment $I_k$ in year $k$, depends on consumption changes and on the accelerator factor $b$, where $b > 0$. Consequently, $I_k$ depends on national income changes,  

$$I_k = b(C_k - C_{k-1}) = ab(T_{k-1} - T_{k-2}). \hspace{3cm} (3)$$

Assumption 2.4. Governmental expenditure $G_k$ in year $k$, remains constant  

$$G_k = \bar{G}.$$ 

Hence, the national income is determined via the following second-order linear difference equation  

$$T_{k+2} - a(1 + b)T_{k+1} + abT_k = \bar{G}. \hspace{3cm} \text{See} \ [39-42] \ \text{for the needed theory of difference equations that lead to the solution of the above equation.}$$

3 The reformulation - Singular Samuelson’s model

Let  

$$Y_k = \begin{bmatrix} T_k \\ C_k \\ I_k \end{bmatrix}$$

Then (1) can be written as  

$$0 = -T_k + C_k + I_k + G_k,$$

or, equivalently,  

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} Y_{k+1} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} Y_k + G_k.$$ 

The equation (2) can be written as  

$$C_{k+1} = aT_k.$$
or, equivalently,
\[
\begin{bmatrix}
0 & 1 & 0
\end{bmatrix} Y_{k+1} = \begin{bmatrix} a & 0 & 0 \end{bmatrix} Y_k.
\]
Finally, (3) can be written as
\[
I_{k+1} = b(C_{k+1} - C_k).
\]
or, equivalently,
\[
-bC_{k+1} + I_{k+1} = -bC_k.
\]
or, equivalently,
\[
\begin{bmatrix}
0 & -b & 1
\end{bmatrix} Y_{k+1} = \begin{bmatrix} 0 & -b & 0 \end{bmatrix} Y_k.
\]
Hence the above expressions can be written in the following matrix form
\[
FY_{k+1} = GY_k + V_k, \quad k = 2, 3, ..., \tag{4}
\]
where
\[
F = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & -b & 1
\end{bmatrix}, \quad G = \begin{bmatrix}
-1 & 1 & 1 \\
a & 0 & 0 \\
0 & -b & 0
\end{bmatrix}, \quad V_k = \begin{bmatrix} G_k \\
0 \\
0
\end{bmatrix}.
\]
Note that \( F \) is singular (\( \det F = 0 \)). Throughout the paper we will use in several parts matrix pencil theory to establish our results. A matrix pencil is a family of matrices \( sF - G \), parametrized by a complex number \( s \), see [46-53].

**Definition 3.1.** Given \( F, G \in \mathbb{R}^{r \times m} \) and an arbitrary \( s \in \mathbb{C} \), the matrix pencil \( sF - G \) is called:

1. Regular when \( r = m \) and \( \det(sF - G) \neq 0 \);
2. Singular when \( r \neq m \) or \( r = m \) and \( \det(sF - G) = 0 \).

**Corollary 3.1.** The system (4) has always a regular pencil \( \forall a, b \).

**Proof.** The determinant \( \det(sF - G) = s^2 - a(b + 1)s + ab \neq 0 \). Hence from Definition 2.1, the pencil is regular. The proof is completed.

The class of \( sF - G \) is characterized by a uniquely defined element, known as the Weierstrass canonical form, see [50-57], specified by the complete set of invariants of \( sF - G \). This is the set of elementary divisors of type \( (s - a_j)^{p_j} \), called **finite elementary divisors**, where \( a_j \) is a finite eigenvalue of algebraic multiplicity \( p_j \) \((1 \leq j \leq \nu)\), and the set of elementary divisors of type \( \frac{1}{s^q} \), called **infinite elementary divisors**, where \( q \) is the algebraic multiplicity of the infinite eigenvalue, \( \sum_{j=1}^\nu p_j = p \) and \( p + q = m \).

From the regularity of \( sF - G \), there exist non-singular matrices \( P, Q \in \mathbb{R}^{m \times m} \) such that
\[
P F Q = \begin{bmatrix}
I_p & 0_{p,q} \\
0_{q,p} & H_q
\end{bmatrix},
\]
\[
P G Q = \begin{bmatrix}
J_p & 0_{p,q} \\
0_{q,p} & I_q
\end{bmatrix}.
\]
$J_p, H_q$ are appropriate matrices with $H_q$ a nilpotent matrix with index $q$, $J_p$ a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix $Q$ can be written as

$$Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}.$$  \hfill (6)

$Q_p \in \mathbb{R}^{m \times p}$ and $Q_q \in \mathbb{R}^{m \times q}$. The matrix $P$ can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$  \hfill (7)

$P_1 \in \mathbb{R}^{p \times r}$ and $P_2 \in \mathbb{R}^{q \times r}$.

The solution of system (4) is given by the following Theorem:

**Theorem 3.1.** (See [1-19]) We consider the system (4). Since its pencil is always regular, its solution exists and for $k \geq 0$, is given by the formula

$$Y_k = Q_p J_p^k C + Q D_k.$$  

Where $D_k = \left[ \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i - \sum_{i=0}^{q-1} H_q^{i} P_2 V_{k+i} \right]$ and $C \in \mathbb{R}^p$ is a constant vector. The matrices $Q_p, Q_q, P_1, P_2, J_p, H_q$ are defined by (5), (6), (7).

## 4 Main Results

In this section we will present our main results. We will provide the solution to the system (4) and consequently we will derive the sequence for the national income, the consumption and the private investment.

**Theorem 4.1.** We consider the system (4). Then in year $k$, National Income $T_k$, Consumption $C_k$ and private Investment $I_k$ are given by:

$$T_k = s_1^{k+1} c_1 + s_2^{k+1} c_2 + a \sum_{i=0}^{k-1} [(s_1^{k-1} + s_2^{k-1})] G_i,$$

$$C_k = a(s_1^k c_1 + s_2^k c_2) + a^2 \sum_{i=0}^{k-1} [(s_1^{k-i-1} + s_2^{k-i-1})] G_i,$$

$$I_k = s_1^k (s_1 - a) c_1 + s_2^k (s_2 - a) c_2 + a \sum_{i=0}^{k-1} [(s_1 - a) s_1^{k-1} + (s_2 - a) s_2^{k-1})] G_i.$$

**Proof.** From Corollary 3.1, the pencil $sF - G$ is always regular. Furthermore the pencil has one infinite eigenvalue and two finite:

$$s_1 = \frac{a(1 + b) + \sqrt{a^2(1 + b)^2 - 4ab}}{2}, \quad s_2 = \frac{a(1 + b) - \sqrt{a^2(1 + b)^2 - 4ab}}{2}.$$  

From Theorem 3.1, the solution of (4) is given by

$$Y_k = Q_p J_p^k C + Q \left[ \sum_{i=0}^{k-1} J_p^{k-i-1} P_1 V_i - \sum_{i=0}^{q-1} H_q^{i} P_2 V_{k+i} \right].$$
Since we have one infinite eigenvalue we have
\[ H_q = 0 \]
and \( J_p \) is the Jordan matrix of the two finite eigenvalues:
\[
Y_k = Q_p \left[ \begin{array}{cc} s_1^k & 0 \\ 0 & s_2^k \end{array} \right] C + Q \left[ \sum_{i=0}^{k-1} J^k_{p,i-1} P_1 V_i \right].
\]
The matrix \( Q_p \) has the two eigenvectors of the two finite eigenvalues:
\[
Q_p = \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix},
\]
while \( Q_q \) is the eigenvector of the infinite eigenvalue:
\[
Q_q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]
Hence:
\[
Q = \begin{bmatrix} s_1 & s_2 & 1 \\ a & a & 0 \\ s_1 - a & s_2 - a & 0 \end{bmatrix}
\]
and the solution \( Y_k \) takes the form:
\[
Y_k = \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix} \begin{bmatrix} s_1^k & 0 \\ 0 & s_2^k \end{bmatrix} C + \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix} \left[ \sum_{i=0}^{k-1} J^k_{p,i-1} P_1 V_i \right].
\]
Finally, where \( P_1 \) is the matrix which contains the right eigenvectors of the finite eigenvalues
\[
P_1 = \begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}.
\]
Hence
\[
Y_k = \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix} \begin{bmatrix} s_1^k & 0 \\ 0 & s_2^k \end{bmatrix} C + \begin{bmatrix} s_1 & s_2 \\ a & a \\ s_1 - a & s_2 - a \end{bmatrix} \left[ \sum_{i=0}^{k-1} J^k_{p,i-1} P_1 V_i \right].
\]
or, equivalently,
\[
Y_k = \begin{bmatrix}
   s_1^{k+1}c_1 + s_2^{k+1}c_2 + a\sum_{i=0}^{k-1}((s_1^{k-i} + s_2^{k-i})G_i \\
   a(s_1^k + s_2^k) + a^2\sum_{i=0}^{k-1}((s_1^{k-i-1} + s_2^{k-i-1})G_i \\
   s_1^k(s_1-a)c_1 + s_2^k(s_2-a)c_2 + a\sum_{i=0}^{k-1}((s_1-a)s_1^{k-i-1} + (s_2-a)s_2^{k-i-1})G_i
\end{bmatrix},
\]
or, equivalently,
\[
\begin{bmatrix}
   T_k \\
   C_k \\
   I_k
\end{bmatrix} = \begin{bmatrix}
   s_1^{k+1}c_1 + s_2^{k+1}c_2 + a\sum_{i=0}^{k-1}((s_1^{k-i} + s_2^{k-i})G_i \\
   a(s_1^k + s_2^k) + a^2\sum_{i=0}^{k-1}((s_1^{k-i-1} + s_2^{k-i-1})G_i \\
   s_1^k(s_1-a)c_1 + s_2^k(s_2-a)c_2 + a\sum_{i=0}^{k-1}((s_1-a)s_1^{k-i-1} + (s_2-a)s_2^{k-i-1})G_i
\end{bmatrix}.
\]

The proof is completed.

**Initial Conditions**

We assume system (4) and the known initial conditions (IC): \(Y_2\).

**Definition 4.1.** Consider the system (4) with known IC. Then the IC are called consistent if there exists a solution for the system (4) which satisfies the given conditions.

**Proposition 4.2.** (See [1-19]) The IC of system (4) are consistent if and only if
\[Y_2 \in \text{colspan} Q_p + QD_2.\]

**Proposition 4.3.** (See [1-19]) Consider the system (4) with given IC. Then the solution for the initial value problem is unique if and only if the IC are consistent. Then, the unique solution is given by the formula
\[Y_k = Q_pJ_p^kZ_p^p + QD_k.\]

where \(D_k = -\sum_{i=0}^{k-1}J_p^{k-i-1}P_iV_i - \sum_{i=0}^{q-1}H_{q}^{i}P_2V_{k+i}\) and \(Z_p^p\) is the unique solution of the algebraic system
\[Y_2 = Q_pZ_p^p + D_2.\]

**Proposition 4.3.** The reformulation - Singular Samuelson’s model has always a unique solution for given initial conditions

**Proof.** The reformulation - Singular Samuelson’s model has always a unique solution for given initial conditions is a singular system given by (6). For \(k = 2\) we get:
\[Y_2 = \begin{bmatrix}
   T_2 \\
   C_2 \\
   I_2
\end{bmatrix},
\]
or, equivalently,
\[Y_2 = \begin{bmatrix}
   T_2 \\
   aT_1 \\
   ab(T_1 - T_0)
\end{bmatrix}.\]
or, equivalently,

\[
Y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} T_2 + \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix} aT_1 + \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} aT_0.
\]

However

\[
colspan Q_p + QD_2 = \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} > + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

and hence from Proposition 4.1, the IC of the reformulation - Singular Samuelson’s model are always consistent and from Proposition 4.2, reformulation - Singular Samuelson’s model has a unique solution for given IC. The proof is completed.

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