In this work, we investigate a new warped five-dimensional, $Z_2$-symmetric thick brane solution in the presence of a real scalar field. We examine the different geometric aspects of the model. We discuss the stability of the solution under gravitational fluctuations and study both the graviton ground state and the continuum of Kaluza-Klein modes to find a correction to Newton’s law. Then, we study the quantum version of the model by deriving the Wheeler-DeWitt equation and looking for the corresponding solution. Finally, the cosmology of the brane is studied.

I. INTRODUCTION

Historically, String theory suggests that our universe lives in a higher dimensional space-time. Kaluza and Klein, in order to unify electromagnetism with Einstein gravity, proposed that space-time has more than three spatial dimensions [1]. Later, in order to explain some of the open questions in particle physics and cosmology such as the hierarchy problem (which refers to the difference in magnitude between the weak scale and the Planck scale) and cosmological constant, higher dimensional space-time with large extra dimensions were paid more attention. This line of thinking led to the braneworld scenarios which are submanifolds embedded in a higher dimensional space-time (bulk). In this theory, it is supposed that our universe is like a membrane and particles corresponding to electromagnetic, weak and strong interactions are confined within it. Only gravitation and some exotic matter (e.g., the dilaton field) could propagate in the bulk. Serious research in the field of extra dimensions came with the work of Arkani-Hamed, Dimopoulos, and Dvali who proposed the large extra dimensions model, which lowers the energy scale of quantum gravity to 1 TeV by localizing the standard model fields to a 4-brane so that the hierarchy problem can be addressed [2, 3]. But, subsequently, it was shown that propagation of gravity in the bulk is in contradiction with the observational fact that four-dimensional gravity satisfies an inverse-square Newtonian law. This problem was solved in a model proposed by Randall and Sundrum (RS) by relaxing assumptions that our four-dimensional universe is independent of the coordinates defining the extra dimensions [4, 5]. One can then show that (even when the extra dimensions are infinitely large) gravity can be localized near the 3-brane, and Newtonian gravity can be restored at long distances. Indeed, RS considered the 3-brane (the four-dimensional Minkowski space-time) embedded in the five-dimensional anti-de Sitter space-time ($\text{AdS}_5$). They found that there exists a massless graviton (0-mode) and massive gravitons (Kaluza-Klein modes). The massless graviton reproduces the Newtonian gravity on the 3-brane and Kaluza-Klein modes, which are the effect of the existence of the higher-dimension, give a correction to the Newtonian gravity [6, 7]. Generally, this model succeeds in the localization of gravity around the brane due to the warping of the extra-dimension, but, these kinds of models can be only treated as an approximation since any fundamental theory would have a minimal length scale. Because in RS braneworld scenarios, the brane is an infinitely thin object, the energy density of the brane is a delta-like function with respect to the fifth dimension coordinate. So this model is a very idealized braneworld model. Recently, more realistic thick brane models were investigated in higher dimensional space-time [8]-[12]. In thick brane scenarios, the coupling between gravity and scalars should be introduced. The presence of a scalar field makes the warp function to behave smoothly. In [13]-[23] some properties of brane models were investigated: localization of gravity, graviton ground state, stability.

This paper is organized as follows: In Sec. 2, the general formalism of the braneworld scenario will be reviewed. Section 3 is devoted to introducing a new model and in Sec. 4 the stability of the model is studied. In Sec. 5, the WD equation and its solution will be investigated. Finally, in Sec. 6, concluding remarks are presented.

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II. THICK BRANE FORMALISM

We consider a five-dimensional Einstein-scalar field theory with the following action

\[ S = \int d^5x \sqrt{g^{(5)}} \left( \frac{1}{4} R^{(5)} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right). \]  

(1)

By variation with respect to the metric and scalar field, we get

\[ R_{AB} - \frac{1}{2} g_{AB} R^{(5)} = T^{\phi}_{AB}, \quad \Box_5 \phi = \frac{dV}{d\phi}, \]  

(2)

where \( \Box_5 \phi = g^{AB} \partial_5 \phi \partial_B \phi \) and \( T^{\phi}_{AB} \) is the 5-D energy-momentum tensor of the scalar field which is given by

\[ T^{\phi}_{AB} = \partial_\mu \phi \partial_{\nu} \phi - g_{AB} \left( \frac{1}{2} \partial_C \phi \partial^C \phi + V(\phi) \right). \]  

(3)

The metric of a static five-dimensional space-time with the four-dimensional Poincare symmetry can be written as

\[ ds^2 = g_{AB} dx^A dx^B = a(w)^2 (-dt^2 + dx^2 + dy^2 + dz^2) + dw^2, \]  

(4)

in which \( a(w) \) is the warp function. Assuming that the scalar field \( \phi \) is a function of \( w \) only, the Einstein and scalar equations are explicitly given by:

\[ (\mu, \nu) : \quad 3 \frac{a''}{a} + 3 \frac{a'^2}{a^2} = -\frac{1}{2} \phi'^2 - V, \]  

(5)

\[ (w, w) : \quad 6 \frac{a'^2}{a} = \frac{1}{2} \phi'^2 - V, \]  

(6)

\[ (\text{scalar field}) : \quad \phi'' + 4 \frac{a'}{a} \phi' = \frac{dV}{d\phi}, \]  

(7)

where the prime denotes derivative with respect to \( w \). Only two of the above equations are independent from each other. In order to get first-order equations, sometimes, an auxiliary superpotential is introduced which is related to \( V \) according to \[ V = -6W(\phi)^2 + 9 \left( \frac{dW(\phi)}{d\phi} \right)^2. \]  

(8)

Using the above equations (5-7) we get

\[ \phi' = 3 \frac{dW}{d\phi}, \quad \frac{a'}{a} = -W(\phi). \]  

(9)

The energy density is then given by

\[ T_{00} = \rho(w) = a(w)^2 \left( \frac{1}{2} \left( \frac{d\phi}{dw} \right)^2 + V(\phi) \right) = \frac{d}{dw} \left( 3W(\phi)a(w)^2 \right). \]  

(10)

For a given metric function, one can determine the scalar field profile, potential and superpotential, using the above equations.

III. THE LAMBERTW (W) MODEL

We study the case of a (conformally) flat brane, with the line element

\[ ds^2 = a(w)^2 \eta_{\mu\nu} dx^\mu dx^\nu + dw^2, \quad a(w) = W \left( \frac{1}{1 + \alpha w^2} \right)^{1+\alpha w^2}, \]  

(11)
where $\eta_{\mu\nu}$ is the 4-dimensional Minkowski metric with signature $(-,+,+,+)$ and $\alpha$ is a positive constant parameter with dimension $1/[\text{length}]^2$ that controls the localization of the brane. Moreover, as one can see later in the definition of Einstein tensor, this quantity plays the role of cosmological constant on the brane. The warp factor has the $Z_2$ symmetry $a(w) = a(-w)$, and is plotted in Fig. 1. The asymptotic behavior of the metric is

$$ds^2 \approx \frac{0.02}{(\alpha^2 w^4)^{\alpha w^2 + 1}} \eta_{\mu\nu} dx^\mu dx^\nu + dw^2, \quad w \to \infty,$$

and near the brane

$$ds^2 \approx (0.3 - 0.8\alpha w^2) \eta_{\mu\nu} dx^\mu dx^\nu + dw^2, \quad w \to 0.$$  

Also, one can calculate the energy density from Eq. (10), which is plotted in Fig. 2. The asymptotic and near brane behavior of the energy density are

$$T_{00} \approx -\frac{0.44}{w^2 (\alpha^2 w^4)^{\alpha w^2}} (1 + \ln(\alpha w^2))^2, \quad w \to \infty,$$

$$T_{00} \approx 2.33\alpha - 12.3\alpha^2 w^2 + 3.34\alpha^3 w^4, \quad w \to 0.$$  

FIG. 1: The behavior of warp factor in terms of $w$ for $\alpha = 1, 1.1, 1.2, 1.3$ (downwards).
Considering Figs. 1 and 2 one finds that $\alpha$ is a parameter that controls both the localization width of the brane and the amplitude of the localization of energy density on the brane. As this parameter increase, the warp factor will be more localized and the amplitude of energy density decreases.
FIG. 4: \(aV\) as a function of the \(w\) at \(\alpha = 1, 1.1, 1.2, 1.3\).

From Eq. (9), it will be easy to calculate:

\[
W = -2\alpha w \left( A - \frac{1}{1+A} \right), \tag{15}
\]

\[
\left( \frac{d\phi}{dw} \right)^2 = -6\alpha \ln A + \frac{12\alpha^2 w^2}{(1+\alpha w^2)(1+A)} + \frac{6\alpha}{1+A} + \frac{12\alpha^2 w^2 A}{(1+\alpha w^2)(1+A)^2}, \tag{16}
\]

where \(A = W \left( \frac{1}{1+\alpha w^2} \right)\). By using of ODE plot method, one can obtain the behavior of the scalar field from Eq. (16). As can be seen from Fig. 3, the scalar field has mirror symmetry \((\phi(w) = -\phi(-w))\). The potential of this system from Eq. (8) as a function of fifth dimension is given by:

\[
V(w) = -24\alpha^2 w^2 \left( \ln A - \frac{1}{1+A} \right)^2 - 3\alpha \left( \ln A - \frac{1}{1+A} \right) + \frac{6\alpha^2 w^2}{(1+A)(1+\alpha w^2)} \left( 1 + \frac{A}{(1+A)^2} \right). \tag{17}
\]

This potential is plotted in the Fig. 4. It can be seen that the potential is an even function of \(w\) \((V(w) = V(-w))\). In order to obtain \(V(\phi)\), we need to have \(\phi(w)\). Considering Eq. (16), it is obvious that one cannot obtain an analytic relation for \(\phi(w)\). However we have obtained approximation solutions for \(\phi\) for both small and large \(w\), as follows

\[
\phi \approx 7.23\alpha w + \mathcal{O}(w^3), \quad \text{as } w \to 0
\]

\[
\phi \approx 6\alpha w(1 + \ln(\alpha w^2)) + \mathcal{O} \left( \frac{1}{w} \right), \quad \text{as } w \to \infty \tag{18}
\]

Now, by obtaining the function \(w(\phi)\), one can get the scalar potential as follow

\[
V(\phi) \approx 3.61\alpha - 0.53\phi^2 + \mathcal{O}(\phi^4), \quad \text{as } w \to 0
\]

\[
V(\phi) \approx 6\alpha - 0.667\phi^2 + \mathcal{O} \left( \frac{1}{\phi^2} \right), \quad \text{as } w \to \infty \tag{19}
\]

in which confirm that the potential is an even function of scalar field \((V(\phi) = V(-\phi))\).
Here, we are going to investigate the curvature behavior of the spacetime. The Ricci and Kretschmann scalars are given as

\[
R = \frac{16\alpha \left[ (1 + \alpha w^2)(1 + A)^2(1 + A + 10\alpha Aw^2) \ln(A) + 5\alpha^2 w^4(1 + A) - \alpha w^2(3A^2 + 3A - 2) - (1 + A)^2 \right]}{(1 + A)^3(1 + \alpha w^2)},
\]

and

\[
K = R_{abcd}R^{abcd} = \\
+ \frac{128\alpha^2 \ln(A)(20\alpha^3 w^6 A^4 + 26\alpha^2 w^4 A^4 + 13\alpha w^2 A^3 + 7\alpha w^2 A^4 + 24\alpha w^2 A^3 - 18\alpha^2 w^4 A^2)}{(1 + A)^4(1 + \alpha w^2)} \\
+ \frac{128\alpha^2 \ln(A)(-8A\alpha^3 w^4 - 7A\alpha w^2 + 20\alpha^3 A^3 w^6 + A^4 + 3A^3 + 3A^2 + \alpha A w^2 + A - 2\alpha w^2)}{(1 + A)^4(1 + \alpha w^2)} \\
+ \frac{64\alpha^2(1 + 3\alpha^2 w^4 + 9\alpha^2 w^4 A^4 + 24\alpha w^2 A^3 + 6\alpha w^2 A + 32\alpha w^4 A^3 + 36\alpha w^4 A^2 + 12A^2 w^4)}{(1 + A)^6(1 + \alpha w^2)^2} \\
+ \frac{64\alpha^2(10\alpha A^2 w^6 + 20\alpha^4 w^6 A - 12\alpha^3 w^6 A^3 + A^4 + 4A^3 + 6A^2)}{(1 + A)^6(1 + \alpha w^2)^2}.
\]

In the limit of \( w \to \pm \infty \) and \( w \to 0 \), we find

\[
\lim_{w \to \pm \infty} R = -\infty,
\]

\[
\lim_{w \to 0} R = 19.3\alpha,
\]

\[
\lim_{w \to 0} K = 92.97\alpha^2,
\]

\[
\lim_{w \to \pm \infty} K = \infty.
\]

Notably, divergence values of curvature scalars for \( \omega \to \infty \) do not have a physical interpretation. In order to overcome such a problem, we applied a constraint on the free (positive) parameter \( \alpha \) in such a way that for \( \omega \to \infty \) we have finite value for \( \alpha w^2 \) (we set this finite value to one without loss of generality). Considering the mentioned limitation, one finds an asymptotically flat 5–dimensional spacetime (as \( \omega \to \infty \)).

Besides, the components of the Einstein tensor can be written as

\[
G_{\nu}^\mu = \frac{6\alpha \left[ (1 + \alpha w^2)(1 + A)^2(1 + A + 8\alpha Aw^2) \ln(A) + 4\alpha^2 w^4(1 + A) - \alpha w^2(3A^2 + 4A - 1) - (1 + A)^2 \right] \delta_{\nu}^\mu}{(1 + A)^4(1 + \alpha w^2)},
\]

and

\[
G_w = \frac{24\alpha^2 w^2((1 + A) \ln(A) - 1)^2}{(1 + A)^2},
\]

where in the limits \( w \to 0 \), one can find

\[
\lim_{w \to 0} G_{\nu}^\mu = -7.23\alpha \delta_{\nu}^\mu,
\]

\[
\lim_{w \to 0} G_w = 0.
\]

So, one can interpret the right hand side of Eq. (25) as the cosmological constant on the brane (immersed in 5–dimensional spacetime), i.e., \( \Lambda = 7.23\alpha \) and since \( \alpha \) is positive, the cosmological constant of the brane would be positive for this model. Indeed, regarding Eq. (11), it is clear that the four dimensional brane is conformally flat. However, the 5–dimensional spacetime is not flat at all. It is notable that for the limiting \( \omega \to 0 \), we investigated the geometrical properties of the brane immersed in 5–dimensional spacetime. In other words, although we use the limit \( \omega \to 0 \) to localize on the brane, it is notable that such a brane in submanifold of 5–dimensional spacetime. So after calculating the Einstein and Ricci tensors of 5–dimensional spacetime, we find the trace of cosmological constant near the brane and 5–dimensional spacetime behaves like a dS space for \( \omega \to 0 \).
IV. STABILITY

Another general feature concerns the stability of the gravity sector of the braneworld model. Here, we shall consider linear perturbations of the metric in the following form

$$ds^2 = a(w)^2(\eta_{\mu\nu} + \varepsilon h_{\mu\nu})dx^\mu dx^\nu + dw^2$$

(30)

where $\varepsilon h_{\mu\nu}$ is a small perturbation around the Minkowski metric. By considering the transverse and traceless gauge and re-defining the $w$-coordinate as $dw = a(w)dz$, the corresponding Schrödinger-like equation takes the following form

$$-\frac{d^2\Psi(z)}{dz^2} + U(z)\Psi(z) = \lambda^2\Psi(z),$$

(31)

where $\lambda^2$ accounts for the 4D mass of the excited Kaluza-Klein gravitational modes and $\Psi(z)$ is a wave function. The stability potential is given by

$$U(z) = \frac{3}{2} \frac{\ddot{a}(z)}{a(z)} + \frac{3}{4} \frac{\dot{a}(z)^2}{a(z)} = \frac{3}{4} \left( a''(w)a(w) + \frac{1}{4}a'(w)^2 \right).$$

(32)

The asymptotic behavior of the effective potential for our model is

$$U(w \gg 0) = \frac{0.27 \ln(\alpha w)}{(\alpha w^2)^\frac{1}{2}} + \mathcal{O}(\frac{1}{w^4}),$$

(33)

and near the center of potential

$$U(w \approx 0) = -0.59\alpha + 2.017\alpha^2w^2 + \mathcal{O}(w^3).$$

(34)

This potential is plotted in Fig. 5 for different values of $\alpha$. As one can see, this potential has only one minimum at $w = 0$ which implies stability. Also, the effective potential shows that gravitons are localized around the brane. We know that the solution of this Schrödinger equation at $w = 0$, represents the coupling of the massive modes with the matter on the brane. The zero mode is $\Psi_0 \propto a(z)^2$.

![FIG. 5: The behavior of zero-mode wave function (dotted-dashed lines) and effective potential (solid lines) in terms of $w$ for $\alpha = 1, 1.1, 1.2, 1.3$ (downwards).](image)
In order to ensure stability, it is important to check the normalizability of $\Psi_0^2$. Normalizability is connected with the asymptotic behavior of the potential of the Schrödinger equation. If $U(z) > 0$ as $|z| \to \infty$, then $\Psi_0(z)$ is always normalizable. In the following, we want to obtain the correction from the massive modes of Kaluza-Klein for the four-dimensional gravitational coupling. First, we obtain Newtonian coupling by using of zero-modes as follows:

$$G_4 \sim \frac{1}{M^3} <\Psi_0^2|\Psi_0> \equiv \frac{1}{M^3} \Xi,$$

where $M$ is the five-dimensional fundamental scale and $\Xi \equiv \frac{\Psi_0^2(0)}{<\Psi_0|\Psi_0>}$. In order to obtain a correction for $G_4$ from massive Kaluza-Klein modes, we have used numerical method. In Fig. (6), we have plotted the behavior of $\Xi$ in terms of $\alpha$. Then, we have fitted the best function on the numeric data. So, the best function is as follows:

$$\Xi \sim \frac{0.08(\alpha + 14.07)(\alpha + 0.11)}{\alpha + 1.44} \sim 1 + 0.08\alpha + ... \quad \alpha \gg 1$$

FIG. 6: The behavior of $\Xi$ in terms of $\alpha$.

The effective potential between two point-like sources of mass $M_1$ and $M_2$ is from the contribution of the zero mode and the continuum KK modes that can be expressed as

$$U(r) = \frac{M_1M_2}{r}(G_4 + \delta G_4) = \frac{M_1M_2}{r}M^3(\Xi + \int_{\lambda_0}^{\infty} d\lambda e^{-\lambda r}|\Psi_\lambda(0)|^2),$$

in which the continuous spectrum starts at $\lambda_0$. In order to get the corrected Newtonian potential, we have obtained the wave function near $w = 0$ as follows:

$$\Psi_\lambda(w) \approx C_1 \sin(3.1\sqrt{(0.2\alpha + 0.3\lambda^2)}w) + C_2 \cos(3.1\sqrt{(0.2\alpha + 0.3\lambda^2)}w)$$

then $\Psi_\lambda(0) = C_2$ and by inserting it into Eq. (37), one gets

$$U(r) = \frac{M_1M_2}{r}M^3(\Xi + C_2^2 e^{-\lambda_0 r}/r),$$

in order to have a correction to the Newtonian potential $C_2 \neq 0$. From the above discussion, it can be seen that for large distance $r$ between two point-like sources, the correction of gravitational potential $e^{-\lambda_0 r}/r^2$ is very small.
compared with the Newtonian potential $\frac{1}{r}$, because the contribution of massive Kaluza-Klein modes is at large distances. However, when $r$ is the order of $\lambda_0^{-1}$ or smaller, the correction $\frac{e^{-\lambda_0 r}}{r^2} \sim \frac{1}{r^2}$ becomes important in the effective Newtonian potential \[23, 24\].

V. WHEELER-DEWITT EQUATION

The first approach to describe the universe based on the application of the quantum theory was presented in 1960 by Wheeler \[26\] and DeWitt \[27\]. They proposed a quantum gravity equation to describe the wave function of the universe, which is known as the Wheeler-DeWitt (WD) equation. This equation is analogous to a zero-energy Schrödinger equation in which the Hamiltonian could contain the gravitational field and scalar fields. But, one of the most important problems for solving the WD equation is the subject of initial conditions. Unlike a classical system, for cosmological models, there are no external initial conditions because there is no external time parameter to the universe. In order to solve this problem, two different approaches are used: the Hartle-Hawking no boundary \[28-31\] and the Vilenkin tunneling proposal \[32-35\]. The first proposal is that the wave function of the Universe is given by a path integral over compact Euclidean geometries so that this universe has no boundary in this space. The second one states that the universe spontaneously nucleates and then evolves along the lines of an inflationary scenario. The mathematical description of this approach is closely analogous to that of quantum tunneling through a potential barrier. In fact, only the outgoing modes of the wave function should be taken at the singular boundary of superspace \[36, 37\].

In order to investigate the canonical quantization of the brane, one should promote the metric $g_{ij}$, the conjugate momenta $\Pi_{ij}$, the Hamiltonian density $H$ and the momentum density $H_i$ to quantum operators satisfying canonical commutation relations. We begin with the metric

$$ds^2 = a(w)^2 \left(-dt^2 + \frac{b(t)^2}{1 - \kappa r^2} dr^2 + b(t)^2 dr^2 + b(t)^2 r^2 sin(\theta)^2 d\phi^2\right) + dw^2,$$

where $b(t)$ is the brane scale factor. Ricci scalar for this metric is given by

$$R^{(5)} = -\frac{2(4a(w)b(t)^2a''(w) + 6b(t)^2a'(w)^2 - 3b(t)b'(t) - 3b(t)^2 - 3\kappa)}{a(w)^2 b(t)^2},$$

where dot and prim are derivatives with respect to $t$ and $w$, respectively. By using Eq. (1), the Lagrangian is

$$L = 2a(w)^2 b(t)(-3\dot{b}^2 - 6b(t)^2a'(w)^2 + 4a(w)b(t)^2a''(w)) + \frac{1}{2} a(w)^2 b(t)^3 \dot{\varphi}(w, t)^2 -
\frac{1}{2} a(w)^4 b(t)^3 \varphi'(w, t)^2 - a(w)^4 b(t)^3 V(\varphi) + 6a(w)^2 b(t)\kappa.$$  

The momenta conjugate to $b$ and $\varphi$ are

$$\Pi_b = \frac{\partial L}{\partial \dot{b}} = -12a(w)^2 b(t)\dot{b}, \quad \Pi_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = a(w)^2 b(t)^3 \dot{\varphi}.$$  

The Hamiltonian constraint is therefore

$$H = \Pi_b \dot{b} + \Pi_\varphi \dot{\varphi} - L = -\Pi_b^2 + \frac{12}{b(t)^2} \Pi_\varphi^2 + U(a, b, \varphi) = 0,$$

where $U(a, b, \varphi)$ is

$$U(a, b, \varphi) = 96a^4 b^4(3a'^2 + 2a'') + 24a^4 b^4(\frac{\varphi}{2} - 2V(\varphi)) - 144a^4 b^2 \kappa.$$  

Making the replacement $\Pi_b \rightarrow -i \frac{\partial}{\partial b}$ and $\Pi_\varphi \rightarrow -i \frac{\partial}{\partial \varphi}$ and imposing $H \Psi = 0$ results in the following WD equation

$$\left(\frac{\partial^2}{\partial b^2} - \frac{12}{b^2} \frac{\partial^2}{\partial \varphi^2} + U(a, b, \varphi)\right) \Psi = 0.$$  

In order to solve the WD equation we follow the separation of variable method. Using
\[ \Psi(b, \varphi) = \Phi(\varphi)B(b), \]  
(47)
the WD equation becomes
\[ \frac{1}{B} \frac{d^2 B}{db^2} - \frac{12}{\Phi \Phi''} \frac{d^2 \Phi}{d\varphi^2} + \nu b^4 - 0b^2 = 0, \]
(48)
where by using of Eq. (16), (17) and (45), \( \nu = (96a^4(3a'' + 2aa'') + 24a^6(1/2\varphi'^2 + V(\varphi))) |_{w=0} = -15.1\alpha \) and \( \rho = 144a(w = 0)^4\kappa = 14.89\kappa. \) Here, we assumed \( \varphi(w, t) = \varphi(t) \) and \( V(\varphi) = 0. \) We thus obtain the following equations
\[ \frac{d^2 \Phi}{d\varphi^2} - m \Phi = 0, \]
(49)
\[ \frac{d^2 B}{db^2} + (\nu b^4 - 0b^2 - 12m b^2)B = 0, \]
(50)
where \( m \) is a separation constant. By solving equation (49) one obtains
\[ \Phi(\varphi) = c_1 e^{\sqrt{m}\varphi} + c_2 e^{-\sqrt{m}\varphi}. \]
(51)
If in Eq. (50) \( b \to 0, \) one obtains
\[ \frac{d^2 B}{db^2} - \frac{12m}{b^2}B = 0, \]
(52)
which has the following solution
\[ B(b \to 0) \approx c_1 \sqrt{b^{1+\sqrt{48m+1}}} + c_2 \sqrt{b^{1-\sqrt{48m+1}}}, \]
(53)
in the limit \( b \to \infty \) Eq. (50) becomes
\[ \frac{d^2 B}{db^2} + (\nu b^4 - 0b^2)B = 0, \]
(54)
which has the following solution:
\[ B(b \to \infty) \approx c_1 e^{f(b, \kappa)} HenuT(\lambda, \delta, \eta, \zeta b) + c_2 e^{-f(b, \kappa)} HenuT(\lambda, \delta, \eta, -\zeta b), \]
(55)
where
\[ f(b, \kappa) = \frac{-0.33 + 0.11 \times 10^{-9} I(-1.5\varphi + \nu b^2)b}{\sqrt{-\varphi}}, \]
\[ \lambda = \frac{-(0.16 + 0.28I)\varphi^2}{(-\nu)^{\frac{3}{2}}}, \]
\[ \delta = 0, \]
\[ \eta = \frac{-(0.57 + 0.99I)\varphi}{(-\nu)^{\frac{3}{2}}}, \]
\[ \zeta = (0.44 + 0.76I)(-\nu)^{\frac{3}{2}}b. \]

According to Eq. (56), the effective potential is
\[ U(b, w = 0) = \nu b^4 - \rho b^2. \]
(56)
In order to investigate the WD equation, we need to know the properties of the superpotential \( U(b, w). \) The superpotential may have a maximum, necessary for quantum tunneling. For \( w = 0 \) and \( b \gg 0 \) the superpotential
consists of two terms, a curvature term $gb^2$ and the term $\nu b^4$. Since $\nu < 0$ ($\alpha > 0$) and $g < 0$ or ($\kappa < 0$), we have quantum tunneling.

In the following, in order to more study the case of $\kappa = -1$, we study cosmology in brane. So, By using of our ansatz metric (40), the explicit form of Einstein equation and the equation of motion for $\phi$ resulting from the action (2) are [37]

$$ (t, t) : \frac{3}{a^2} \left( a^2 + aa'' - \frac{\dot{b}^2}{b^2} - \frac{\kappa}{b^2} \right) = \frac{1}{2} \ddot{\phi}^2 - \frac{1}{2} \phi'^2 - V, \quad (57) $$

$$ (i, j) : \frac{1}{a^2} \left( -2 \frac{\dot{b}}{b} - \frac{\ddot{b}^2}{b^2} - \frac{\kappa}{b^2} + 3a^2 + 3aa'' \right) = -\frac{1}{2} \ddot{\phi}^2 - \frac{1}{2} \phi'^2 - V, \quad (58) $$

$$ (w, w) : \frac{3}{a^2} \left( - \frac{\ddot{b}}{b} + 2a^2 - \frac{\dot{b}^2}{b^2} - \frac{\kappa}{b^2} \right) = -\frac{1}{2} \ddot{\phi}^2 + \frac{1}{2} \phi'^2 - V, \quad (59) $$

$$(scalar field) : \ddot{\phi} + 3 \frac{\dot{b}}{b} \dot{\phi} + a^2 \frac{\partial V}{\partial \phi} - a^2 \phi'' + 4a' a \phi' = 0, \quad (60)$$

in the case of $\phi(w,t) = \phi(w)$, the equations simplify as follow:

$$ (t, t) : \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = \frac{a^2}{3} \left[ \frac{a^2}{a^2 + a''} + \frac{1}{2} \phi'^2 + V \right], \quad (61) $$

$$ (i, j) : \frac{2 \dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = a^2 \left[ \frac{a^2}{a^2 + a''} + \frac{1}{2} \phi'^2 + V \right], \quad (62) $$

$$ (w, w) : \frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = \frac{a^2}{3} \left[ 6 \frac{a^2}{a^2} - \frac{1}{2} \phi'^2 + V \right], \quad (63) $$

$$(scalar field) : \phi'' - 4 \frac{a'}{a} \phi' - \frac{\partial V}{\partial \phi} = 0, \quad (64) $$

where the left(right) hand sides depend only on $t$ ($w$). We then obtain the following set of equations for $b(t)$:

$$ \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = C_t, \quad (65) $$

$$ \frac{2 \dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = C_x, \quad (66) $$

$$ \frac{\dot{b}}{b} + \frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} = C_w, \quad (67) $$

where $C_{t,x,w}$ are constants. It is easy to see that in order for the first two equations to be consistent with the third one it is necessary that

$$ C_w = \frac{C_x + C_t}{2}. \quad (68) $$
On the other hand, from the right hand sides of equations one obtains for the \( w \)-dependent functions the following equations

\[
C_x = 3C_t
\]  
(69)

so that all the constants can be expressed in terms \( C_w \)

\[
C_t = \frac{1}{2} C_w = \frac{1}{2} \lambda, \quad C_x = \frac{3}{2} C_w = \frac{3}{2} \lambda
\]  
(70)

where \( \lambda \) is a constant. Then by combination of Eqs. (65) - (67), one finds the following differential equations

\[
\frac{\ddot{b}}{b} - \frac{\dot{b}^2}{b^2} - \frac{\kappa}{b^2} = 0,
\]

(71)

\[
\frac{\dot{b}^2}{b^2} + \frac{\kappa}{b^2} - \frac{\lambda}{2} = 0.
\]

(72)

By solving Eq. (71), one can obtain

\[
b(t) = \frac{c_1}{2} \left( \frac{t + c_2}{e^{-c_1} + \kappa e^{-c_1}} \right).
\]

(73)

Now, by inserting scale factor (73) into the conditional equation (72), one finds

\[
c_1 = \sqrt{\frac{2}{\lambda}}.
\]

(74)

and therefore, scale factor can be written as

\[
b(t) = \sqrt{\frac{2}{\lambda}} \left( e^{\sqrt{\frac{2}{\lambda}}(t + c_2)} + \frac{\kappa e^{-\sqrt{\frac{2}{\lambda}}(t + c_2)}}{2} \right),
\]

(75)

where \( c_2 \) is an integration constant and \( \kappa = 0, \pm 1 \).

In the case of \( \kappa = -1 \) and \( \lambda < 0 \), the scale factor becomes

\[
b(t) = \sqrt{\frac{2}{\lambda}} \sin \left( \sqrt{\frac{2}{\lambda}}(t + c_2) \right)
\]

(76)

which is correspondence to the cyclic universe.

It is also notable that for \( \kappa = 0 \) and \( \lambda > 0 \), the expansion of universe becomes an exponential form while in the case of \( \kappa = 1 \) and \( \lambda > 0 \), the scale factor can be simplified as

\[
b(t) = \sqrt{\frac{2}{\lambda}} \cosh \left( \sqrt{\frac{\lambda}{2}}(t + c_2) \right).
\]

(77)

VI. CONCLUSION

In this paper, we have presented five-dimensional thick brane solutions supported by a scalar field. By choosing a special form for the warp factor, we have obtained regular solutions with finite energy density. These solutions are non-singular in the whole space-time even at the location of the brane. Also, we have investigated the stability of our thick brane solutions. The effective potential of the gravitons shows that there are bound states which are localized around the brane. Such gravitons make the four-dimensional gravity on the brane Newtonian if we take the thin brane and low energy limit. On the other hand, as \( z \to \infty \), the effective potential of the graviton approaches zero, this means, there is no mass gap between the excited KK modes and the massless ground state, while the probability density of the KK states is a maximum at the brane location. This means that the lighter KK excitations
are closer to the brane than the heavier ones and, hence, can interact with the graviton with a greater probability. The interplay between the probability of interaction and mass of the KK states is what generates the effective mass gap. We have briefly addressed the formalism of canonical gravity and the WD equation as applied to the brane. We have seen that only in the case of \( \kappa = -1 \), tunneling occurs which means that the appropriate classical cosmology subject to quantization is the spatially spherical case.

Despite our warp function is not a linear function of extra dimension when \( w \to \infty \) (see eq. (12)), the scalar potential is unbounded from below, i.e., it has not any minimum. This means while the system rolls down, since there is not a ground state for the system, it goes to negative infinity. So, the scalar potential is not an appropriate potential applicable to quantum effective field theory like Goldeston potential [39]. Even though, in order to have a good solution for thick brane, we should have following criteria: having a warp function localized around the thick brane, kink like scalar field and a scalar potential with at least two minima. It is obvious that satisfaction of the above conditions, simultaneously, is not possible for many candidates of warp function [40].

It is interesting to calculate the amplitude of the bulk tensor metric perturbations, the amplitude of the bound state modes and tunneling rate. In the case of \( \phi(w, t) = \phi(t) \) in Sec. 5, one can go further by investigating the cosmological inflation of the model in depth. We leave these works for future work.

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