A complete Riemann zeta distribution and the Riemann hypothesis

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Let \( \sigma, t \in \mathbb{R} \), \( s = \sigma + it \), \( \Gamma(s) \) be the Gamma function, \( \xi(s) \) be the Riemann zeta function and \( \xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\xi(s) \) be the complete Riemann zeta function. We show that \( \Xi_{\sigma}(t) := \frac{\xi(\sigma - it)}{\xi(\sigma)} \) is a characteristic function for any \( \sigma \in \mathbb{R} \) by giving the probability density function. Next we prove that the Riemann hypothesis is true if and only if each \( \Xi_{\sigma}(t) \) is a pretended-infinitely divisible characteristic function, which is defined in this paper, for each \( 1/2 < \sigma < 1 \). Moreover, we show that \( \Xi_{\sigma}(t) \) is a pretended-infinitely divisible characteristic function when \( \sigma = 1 \). Finally we prove that the characteristic function \( \Xi_{\sigma}(t) \) is not infinitely divisible but quasi-infinitely divisible for any \( \sigma > 1 \).

Keywords: characteristic function; Lévy–Khintchine representation; Riemann hypothesis; zeta distribution

1. Introduction and main results

1.1. Riemann zeta function and distribution

The famous Riemann zeta function \( \xi(s) \) is a function of a complex variable \( s = \sigma + it \), for \( \sigma > 1 \) defined by

\[
\xi(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},
\]

where the letter \( p \) is a prime number, and the product of \( \prod_p \) is taken over all primes. The Dirichlet series \( \sum_{n=1}^{\infty} n^{-s} \) and the Euler product \( \prod_p (1 - p^{-s})^{-1} \) converges absolutely in the half-plane \( \sigma > 1 \) and uniformly in each compact subset of this half-plane. The Riemann zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at \( s = 1 \) with residue 1. Denote the Gamma function by \( \Gamma(s) \). We have the following functional equation of the complete Riemann zeta function \( \xi(s) \) (see, for example, Titchmarsh [15], (2.1.13))

\[
\xi(s) = \xi(1-s), \quad \xi(s) := s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\xi(s). \tag{1.1}
\]

In view of the Euler product, it is seen easily that \( \xi(s) \) has no zeros in the half-plane \( \sigma > 1 \). It follows from the functional equation (1.1) and basic properties of the Gamma-function that \( \xi(s) \) vanishes in \( \sigma < 0 \) exactly at the so-called trivial zeros \( s = -2m, m \in \mathbb{N} \). In 1859, Riemann
stated that it seems likely that all nontrivial zeros lie on the so-called critical line \( \sigma = 1/2 \). This is the famous, yet unproved Riemann hypothesis. In 1896, Hadamard and de la Vallée-Poussin independently proved that \( \zeta(1 + it) \neq 0 \) for any \( t \in \mathbb{R} \) (see Titchmarsh [15], page 45). Hence, we can also see that no zeros of \( \zeta(s) \) lie on the line \( \Re(s) = 0 \) by (1.1). Therefore, the Riemann hypothesis is rewritten equivalently as

\[
\text{Riemann hypothesis} \quad \zeta(s) \neq 0 \quad \text{for } 1/2 < \sigma < 1.
\]

Put \( Z_{\sigma}(t) := \frac{\zeta(\sigma - it)}{\zeta(\sigma)} \), \( t \in \mathbb{R} \), then \( Z_{\sigma}(t) \) is known to be a characteristic function when \( \sigma > 1 \) (see Khintchine [5] or Gnedenko and Kolmogorov [3], page 75). A distribution \( \mu_{\sigma} \) on \( \mathbb{R} \) is said to be a Riemann zeta distribution with parameter \( \sigma \) if it has \( Z_{\sigma}(t) \) as its characteristic function. Recently, the Riemann zeta distribution is investigated by Lin and Hu [7], and Gut [4]. On the other hand, in Aoyama and Nakamura [1], Remark 1.13, it is showed that \( Z_{\sigma}(t) \) is not a characteristic function for any \( 1/2 \leq \sigma \leq 1 \). Afterwards, Nakamura [9] showed that \( F_{\sigma}(t) \), where \( F_{\sigma}(t) := \frac{f_{\sigma}(t)}{f_{\sigma}(0)} \) and \( f_{\sigma}(t) := \frac{\zeta(\sigma - it)}{(\sigma - it)} \), is a characteristic function for any \( 0 < \sigma \neq 1 \).

Note that there are some other papers connected to Riemann zeta function in probabilistic view. Biane Pitman and Yor [2] reviewed known results about \( \xi(s) \) which are related to one-dimensional Brownian motion and to higher dimensional Bessel processes. Lagarias and Rains [6] treated \( \pi^{-s/2} \Gamma(s/2) \zeta(s) \) and its generalizations and gave results connected to infinite divisibility.

### 1.2. Infinitely divisible and quasi-infinitely divisible distributions

A probability measure \( \mu \) on \( \mathbb{R} \) is infinitely divisible if, for any positive integer \( n \), there is a probability measure \( \mu_n \) on \( \mathbb{R} \) such that \( \mu = \mu_n^n \), where \( \mu_n^n \) is the \( n \)-fold convolution of \( \mu_n \). For instance, normal, degenerate, Poisson and compound Poisson distributions are infinitely divisible.

Let \( \widehat{\mu}(t) \) be the characteristic function of a probability measure \( \mu \) on \( \mathbb{R} \) and \( \text{ID}(\mathbb{R}) \) be the class of all infinitely divisible distributions on \( \mathbb{R} \). The following Lévy–Khintchine representation is well known (see Sato [14], Section 2). Put \( D_b := \{ x \in \mathbb{R} : -b \leq x \leq b \} \), where \( b > 0 \). If \( \mu \in \text{ID}(\mathbb{R}) \), then one has

\[
\widehat{\mu}(t) = \exp \left[ -\frac{a}{2} t^2 + i\lambda t + \int_{\mathbb{R}} \left( e^{itx} - 1 - itx 1_{D_b}(x) \right) \nu(dx) \right], \quad t \in \mathbb{R},
\]

(1.2)

where \( a \geq 0 \), \( \lambda \in \mathbb{R} \) and \( \nu \) is a measure on \( \mathbb{R} \) satisfies \( \nu([0]) = 0 \) and \( \int_{\mathbb{R}} (|x|^2 \wedge 1) \nu(dx) < \infty \). Moreover, the representation of \( \widehat{\mu} \) in (1.2) by \( a, \nu, \) and \( \lambda \) is unique. If the Lévy measure \( \nu \) in (1.2) satisfies \( \int_{|x| < 1} |x| \nu(dx) < \infty \), then (1.2) can be written by

\[
\widehat{\mu}(t) = \exp \left[ -\frac{a}{2} t^2 + i\lambda_0 t + \int_{\mathbb{R}} \left( e^{itx} - 1 \right) \nu(dx) \right], \quad \lambda_0 \in \mathbb{R}.
\]

(1.3)
For example, the Lévy measure of $Z_\sigma (t) := \zeta(\sigma - it) / \zeta(\sigma)$ can be given as in the following (see Gnedenko and Kolmogorov [3], page 75). Let $\delta_x$ be the delta measure at $x$. Then we have

$$\log Z_\sigma (t) = \int_0^\infty (e^{itx} - 1)N_\sigma (dx), \quad N_\sigma (dx) := \sum_{p=1}^\infty \sum_{r=1}^\infty \frac{p^{-r}\sigma}{r} \delta_{\log p}(dx). \quad (1.4)$$

On the other hand, there are non-infinitely divisible distributions whose characteristic functions are the quotients of two infinitely divisible characteristic functions. That class is called class of quasi-infinitely divisible distributions and is defined as follows.

**Quasi-infinitely divisible distribution.** A distribution $\mu$ on $\mathbb{R}$ is called quasi-infinitely divisible if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure $\nu$ is a signed measure on $\mathbb{R}$ with total variation measure $|\nu|$ satisfying $\nu(\{0\}) = 0$ and $\int_\mathbb{R} (|x|^2 \land 1) |\nu|(dx) < \infty$.

We have to mention that the triplet $(a, \nu, \lambda)$ in this case is also unique if each component exists and that infinitely divisible distributions on $\mathbb{R}$ are quasi-infinitely divisible if and only if $a \geq 0$ and the negative part of $\nu$ in the Jordan decomposition equals zero. The measure $\nu$ is called quasi-Lévy measure and has appeared in some books and papers, for example, Gnedenko and Kolmogorov [3], page 81, Lindner and Sato [8], Niedbalska-Rajba [10], and others (see also Sato [13], Section 2.4).

### 1.3. Main results

In the present paper, we give a complete Riemann zeta distribution by the normalized complete Riemann zeta function

$$\Xi_\sigma (t) := \frac{\xi(\sigma - it)}{\xi(\sigma)}, \quad \xi(\sigma - it) := (\sigma - it)(\sigma - 1 - it)\pi^{(\sigma - 1)/2} \Gamma\left(\frac{\sigma - it}{2}\right) \zeta(\sigma - it),$$

for any $\sigma \in \mathbb{R}$. It should be mentioned that $\Xi_\sigma (t)$ is symmetric about the vertical axis $\sigma = 1/2$ by the functional equation (1.1). Therefore, we only have to consider the case $\sigma \geq 1/2$. In order to state the main results, we introduce the following pretended-infinitely divisible distribution.

**Pretended-infinitely divisible distribution.** A distribution $\mu$ on $\mathbb{R}$ is called pretended-infinitely divisible if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure $\nu$ is a signed measure on $\mathbb{R}$ with $\nu(\{0\}) = 0$.

Namely, pretended-infinitely divisible distributions are infinitely divisible or quasi-infinitely divisible distributions without the condition $\int_\mathbb{R} (|x|^2 \land 1) |\nu|(dx) < \infty$.

The main results in this paper are following four theorems.
Theorem 1.1. The function $\Xi_\sigma(t)$ is a characteristic function for any $\sigma \in \mathbb{R}$. Moreover, the probability density function $P_\sigma(y)$ is given as follows:

$$P_\sigma(y) := \begin{cases} \frac{2}{\xi(\sigma)} \sum_{n=1}^{\infty} f(ne^{-\gamma}) e^{-\sigma y}, & y \leq 0, \\ \frac{2}{\xi(\sigma)} \sum_{n=1}^{\infty} f(ne^{\gamma}) e^{(1-\sigma)y}, & y > 0, \end{cases} \quad (1.5)$$

where $f(x) := 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2}$.

Let $\mathcal{Z}$ and $\mathcal{Z}_+$ be the set of zeros of the Riemann zeta function which lie in the critical strip \{s $\in \mathbb{C}$: 0 $\leq \Re(s) < 1$, and the region \{s $\in \mathbb{C}$: 0 $< \Re(s) < 1$, $\Im(s) > 0$\}, respectively. If the Riemann hypothesis is true, then each $\rho \in \mathcal{Z}_+$ can be expressed by $\rho = 1/2 + i\gamma$, where $\gamma > 0$.

Theorem 1.2. The characteristic function $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function for any $1/2 < \sigma < 1$ if and only if the Riemann hypothesis is true. Furthermore, we have

$$\Xi_\sigma(t) = \exp\left[\int_{0}^{\infty} (e^{itx} - 1) v_\sigma(dx)\right],$$

$$v_\sigma(dx) := -\sum_{\gamma \in \mathcal{Z}_+} \frac{2\cos(\gamma x)}{xe^{(\sigma-1/2)x}} (dx) \quad (1.6)$$

under the Riemann hypothesis.

Let $\mathcal{Z}_+^R$ be the set of zeros of $\xi(s)$ which lie on the half line \{s $\in \mathbb{C}$: $\Re(s) = 1/2$, $\Im(s) > 0$\} and $\mathcal{Z}_+^N$ be the set of zeros of $\xi(s)$ which lie in the region \{s $\in \mathbb{C}$: $1/2 < \Re(s) < 1$, $\Im(s) > 0$\}. Note that $\mathcal{Z}_+^N = \emptyset$ if and only if the Riemann hypothesis is true. One has $\mathcal{Z} = \{\rho, 1 - \rho: \rho \in \mathcal{Z}_+^R \cup \{\rho, 1 - \rho, \overline{\rho}, 1 - \overline{\rho}: \rho \in \mathcal{Z}_+^N\}\}$ from $\xi(s) = \xi(1-s)$ and $\xi(\overline{s}) = \overline{\xi(s)}$.

Theorem 1.3. When $\sigma \geq 1$, we have

$$\Xi_\sigma(t) = \exp\left[\int_{0}^{\infty} (e^{itx} - 1) v_\sigma(dx)\right],$$

$$v_\sigma(dx) := -\sum_{1/2+i\gamma \in \mathcal{Z}_+^R} \frac{2\cos(\gamma x)}{xe^{(\sigma-1/2)x}} (dx) - \sum_{\beta+i\gamma \in \mathcal{Z}_+^N} \left(\frac{2\cos(\gamma x)}{xe^{(\sigma-\beta)x}} + \frac{2\cos(\gamma x)}{xe^{(\sigma-1+\beta)x}}\right) (dx). \quad (1.7)$$

Especially, $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function when $\sigma = 1$.

Theorem 1.4. When $\sigma > 1$, we have

$$\Xi_\sigma(t) = \exp\left[it\lambda_\sigma + \int_{0}^{\infty} (e^{itx} - 1 - itx 1_{D_{1/2}}(x)) v_\sigma(dx)\right],$$
\[ \lambda_\sigma := \frac{e^{-\sigma/2} - 1}{\sigma} + \frac{e^{(1-\sigma)/2} - 1}{\sigma - 1} + \frac{\log \pi}{2} + \frac{1}{2} \int_0^1 \left( \frac{e^{-\sigma x/2}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) \, dx - \frac{1}{2} \int_1^\infty e^{-x} \frac{dx}{x} \cdot \]

\[ \nu_\sigma(dx) := \frac{1(dx)}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1 + e^x}{xe^{\sigma x}} (dx) + \sum_{p} \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} - \delta_r \log p (dx). \]

Therefore, the characteristic function \( \Xi_\sigma(t) \) is not infinitely divisible but quasi-infinitely divisible when \( \sigma > 1 \).

We call the distribution defined by the characteristic function \( \Xi_\sigma(t) \) the completed Riemann zeta distribution. It is well known that \( \zeta(s) \) has zeros on \( \Re(s) = 1/2 \) (see Titchmarsh [15], Section 10). By the definition of pretended-infinitely divisible distribution and the fact that \( \exp(z) \neq 0 \) for any \( z \in \mathbb{C} \), the characteristic function does not have zeros. Thus, \( \Xi_\sigma(t) \) is not even a pretended-infinitely divisible characteristic function when \( \sigma = 1/2 \).

2. Proofs

2.1. Proof of Theorem 1.1

We quote the following fact from Patterson [12] (see also Biane Pitman and Yor [2], Section 2).

Lemma 2.1 (see Patterson [12], Section 2.10). Let \( f(x) := 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2} \). Then we have

\[ \xi(s) = 2 \int_1^\infty \sum_{n=1}^\infty f(nx) (x^{s-1/2} + x^{1/2-s}) x^{-1/2} \, dx. \] (2.1)

Note that the last integral is absolutely convergent for all values of \( s \).

Proof of Theorem 1.1. By (2.1) and the change of variables \( x = e^{-y} \) and \( x = e^y \), we have

\[
\xi(\sigma - it) = 2 \int_1^\infty \sum_{n=1}^\infty f(nx)x^{\sigma-it-1} \, dx + 2 \int_1^\infty \sum_{n=1}^\infty f(nx)x^{it-\sigma} \, dx
\]

\[ = 2 \int_0^\infty \sum_{n=1}^\infty f(ne^{-y})e^{(1+it-\sigma)y}(-e^{-y}) \, dy + 2 \int_0^\infty \sum_{n=1}^\infty f(ne^y)e^{(it-\sigma)y}(e^y) \, dy
\]

\[ = 2 \int_0^\infty e^{it y} \sum_{n=1}^\infty f(ne^{-y})e^{-\sigma y} \, dy + 2 \int_0^\infty e^{it y} \sum_{n=1}^\infty f(ne^y)e^{(1-\sigma)y} \, dy. \]
Obviously, we have \( f(x) = 2\pi(2\pi x^4 - 3x^2)e^{-\pi x^2} > 0 \) for any \( x \geq 1 \). Hence, one has \( f(ne^{-y}) > 0 \) for any \( y \leq 0 \) and \( n \in \mathbb{N} \), and \( f(ne^y) > 0 \) for any \( y > 0 \) and \( n \in \mathbb{N} \). Thus it holds that

\[
\sum_{n=1}^{\infty} f(ne^{-y})e^{-\sigma y} > 0, \quad y \leq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} f(ne^y)e^{(1-\sigma)y} > 0, \quad y > 0.
\]

On the other hand, we have

\[
\xi(\sigma) = 2 \int_{0}^{\infty} \sum_{n=1}^{\infty} f(ne^{-y})e^{-\sigma y} dy + 2 \int_{0}^{\infty} \sum_{n=1}^{\infty} f(ne^y)e^{(1-\sigma)y} dy > 0
\]

from (2.1) and the argument above. Hence, \( P_\sigma(y) \) defined by (1.5) is nonnegative. Therefore, we have \( \Xi_\sigma(t) = \int_{\mathbb{R}} e^{iy} P_\sigma(y) dy \), where \( P_\sigma(y) \) is the probability density function. \( \square \)

**Remark 2.2.** It should be emphasised that \( \Xi_\sigma(t) \) is a characteristic function for any \( \sigma \in \mathbb{R} \). On the other hand, \( F_\sigma(t) := f_\sigma(t)/f_\sigma(0) \), where \( f_\sigma(t) := \xi(\sigma-it)/(\sigma-it) \), is not a characteristic function for \( \sigma = 0 \), \( 1 \) and \( \sigma < -1/2 \). This is proved as follows. When \( \sigma = 1 \), it is well known that \( \zeta(1+it) \neq 0 \), \( t \neq 0 \), and \( \zeta(\sigma) \) has an only one pole at \( s = 1 \). Hence, we have

\[
F_1(t) = \frac{1}{\xi(1)} \frac{\xi(1+it)}{1+it} = 0 \quad \text{for any} \quad t \neq 0,
\]

which contradicts the uniform continuity of characteristic function \( \hat{\mu}(t) \) and \( \hat{\mu}(0) = 1 \). A similar argument can be done when \( \sigma = 0 \) since \( \zeta(s)/s \) has a simple pole at \( s = 0 \). By (1.1) and Stirling’s formula, one has

\[
|\zeta(s)| = \pi^{\sigma-1/2}(|t/2| + 2)^{-\sigma+1/2}(1 + O((|t| + 2)^{-1}))|\zeta(1-s)|
\]

for \( \sigma < 0 \). On the other hand, for any \( \varepsilon > 0 \) there are arbitrarily large \( t \) which satisfy \( |\zeta(\sigma+it)| > (1-\varepsilon)|\zeta(\sigma)| \) when \( \sigma > 1 \) (see Titchmarsh [15], Theorem 8.4). Thus, we can find \( t \) which satisfies \( |\zeta(s)| > \pi^{\sigma-1/2}|t/2|^{-\sigma+1/2}\zeta(1-\sigma)/2 \). Hence, there exists \( t \in \mathbb{R} \) such that \( |F_\sigma(t)| > 1 \) when \( \sigma < -1/2 \) by the factor \( |t/2|^{-\sigma+1/2} \).

The absolute value of a characteristic function is not greater than 1 (see for instance Sato [14], Proposition 2.5). Hence, we have the following inequality by Theorem 1.1.

**Corollary 2.3 (see Patterson [12], Section 2.11).** For any \( t \in \mathbb{R} \) and \( 1/2 \leq \sigma \), we have

\[
|\zeta(s)| \leq \pi^{\sigma-1/2}(|t/2| + 2)^{-\sigma+1/2}(1 + O((|t| + 2)^{-1}))|\zeta(1-s)|
\]

**2.2. Proof of Theorem 1.2**

Recall that \( \mathcal{Z} \) is the set of zeros of the Riemann zeta function which lie in the critical strip \( \{s \in \mathbb{C}: 0 < \Re(s) < 1\} \) (see Section 1.3). Observe that by the functional equation and \( \zeta(s) = \zeta(\bar{s}) \)
if $\rho \in \mathbb{Z}$ then $1 - \rho, 1 - \overline{\rho} \in \mathbb{Z}$. There are no real elements of $\mathbb{Z}$ since $\xi(\sigma) < 0$ and $0 < \Gamma(\sigma/2)$ when $0 < \sigma < 1$ (see Section 1.1 and the proof of Theorem 1.1). Now we quote the following fact from Patterson [12].

**Lemma 2.4 (see Patterson [12], page 34).** Let $\mathbb{Z}_+ := \{ \rho \in \mathbb{Z} : \Im(\rho) > 0 \}$. Then $\sum_{\rho \in \mathbb{Z}_+} |\rho|^{-a}$ converges for all $a > 1$ and it holds that

$$\xi(s) = s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\xi(s) = \prod_{\rho \in \mathbb{Z}_+} \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{1 - \rho}\right)$$

(2.2)

the product being absolutely convergent for all $s \in \mathbb{C}$.

**Proof of Theorem 1.2.** If $\Xi(\sigma)(t)$ is a pretended-infinitely divisible characteristic function for any $1/2 < \sigma < 1$, then $\xi(s) \neq 0$ for any $1/2 < \sigma < 1$ by $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, $\Gamma(s) \neq 0$ for any $1/2 < \sigma < 1$ and the representation (1.2).

Next suppose that the Riemann hypothesis is true. Then we have $\rho = 1/2 + i\gamma$ and $1 - \rho = 1/2 - i\gamma$, where $\gamma > 0$ for $\rho \in \mathbb{Z}_+$. Note that the exponential distribution with parameter $a > 0$ is defined by $\mu(B) := a \int_{B \cap (0, \infty)} e^{-ax} \, dx$, where $B \in \mathcal{B}(\mathbb{R})$. The characteristic function is given by $\widehat{\mu}(t) = a/(a - it)$ (see, for example, Sato [14], page 13). Moreover, it is well known that

$$\frac{a}{a - iz} = \exp\left[\int_0^\infty (e^{ix} - 1)x^{-1}e^{-ax} \, dx\right], \quad a > 0, \, z \in \mathbb{R}$$

(2.3)

(see, for instance, Sato [14], page 45). The formula above holds if $a$ is replaced by $\alpha$ with $\Re(\alpha) > 0$. This is proved as follows. Put $\alpha = a + ib$, $a > 0$ and $b \in \mathbb{R}$. Then one has

$$\frac{\alpha}{\alpha - iz} = \frac{a + ib}{a} \frac{a}{a + ib - iz}
= \exp\left[\int_0^\infty (e^{ix} - 1)x^{-1}e^{-ax} \, dx - \int_0^\infty (e^{-ibx} - 1)x^{-1}e^{-ax} \, dx\right]
= \exp\left[\int_0^\infty (e^{ix} - 1)x^{-1}e^{-ax} \, dx\right], \quad \Re(\alpha) > 0,$$

(2.4)

by (2.3). Thus, it holds that

$$\left(1 - \frac{\sigma - it}{\rho}\right) \left(1 - \frac{\sigma}{\rho}\right)^{-1} = \frac{1/2 - \sigma + i(\gamma + t)}{1/2 + i\gamma} \frac{1/2 + i\gamma}{1/2 - \sigma + i\gamma} = \frac{\sigma - 1/2 - i\gamma - it}{\sigma - 1/2 - i\gamma}
= \exp\left[-\int_0^\infty (e^{ix} - 1)e^{(1/2 - \sigma + i\gamma)x} \, dx\right],$$
where $\sigma > 1/2$. It should be noted that we have $\sigma - it \neq \rho, 1 - \rho$ when $\sigma > 1/2$ under the Riemann hypothesis. Therefore, one has

$$\varphi_\rho(t) := \left(1 - \frac{\sigma - it}{\rho}\right)\left(1 - \frac{\rho}{\sigma}\right)^{-1}\left(1 - \frac{\sigma - it}{1 - \rho}\right)\left(1 - \frac{1 - \rho}{\sigma}\right)^{-1}$$

$$= \frac{\sigma - 1/2 - i\gamma - it}{\sigma - 1/2 - i\gamma}\frac{\sigma - 1/2 + iy + it}{\sigma - 1/2 + iy}$$

$$= \exp\left[-2\int_0^\infty \frac{\cos(\gamma x)}{xe^{(\sigma - 1/2)x}}\,dx\right].$$

(2.5)

We remark that $x^{-1}\cos(\gamma x)e^{(1/2 - \sigma)x}(dx)$ is not a measure but a signed measure since one has $-1 \leq \cos(\gamma x) \leq 1$ when $\gamma \in \mathbb{R}$. By (2.2) and the definition of $\Xi_\sigma(t)$, we have

$$\Xi_\sigma(t) = \prod_{\gamma \in \mathbb{N}} \frac{\sigma - 1/2 - i\gamma + it}{\sigma - 1/2 - i\gamma}\frac{\sigma - 1/2 + iy + it}{\sigma - 1/2 + iy}$$

$$= \exp\left[-2\sum_{1/2 + iy \in \mathbb{N}_{+}} \int_0^\infty \frac{(e^{it} - 1)\cos(\gamma x)}{xe^{(\sigma - 1/2)x}}\,dx\right].$$

This equality implies (1.6).

\[ \square \]

**Remark 2.5.** It should be mentioned that $\varphi_{1/2+i\gamma}(t)$ defined by (2.5) is not a characteristic function for any $\sigma > 1/2$. It is proved by as follows. Obviously, one has

$$|\varphi_{1/2+i\gamma}(t)|^2 = \frac{(\sigma - 1/2)^2 + \gamma^2 - t^2 + (2\sigma - 1)it}{(\sigma - 1/2)^2 + \gamma^2}.$$

If we take $t^2 = 2((\sigma - 1/2)^2 + \gamma^2)$, then $|\varphi_{1/2+i\gamma}(t)|^2 > 1$.

### 2.3. Proof of Theorem 1.3

Recall that $\mathbb{Z}$, $\mathbb{Z}_+^R$ and $\mathbb{Z}_+^N$ is the set of zeros of $\xi(s)$ which lie in $\{s \in \mathbb{C}: 0 < \Re(s) < 1\}$, $\{s \in \mathbb{C}: \Re(s) = 1/2, \Im(s) > 0\}$ and $\{s \in \mathbb{C}: 1/2 < \Re(s) < 1, \Im(s) > 0\}$, respectively. Then one has $\mathbb{Z} = \{\rho, 1 - \rho: \rho \in \mathbb{Z}_+^R\} \cup \{\rho, 1 - \rho, \bar{\rho}, 1 - \bar{\rho}: \rho \in \mathbb{Z}_+^N\}$. We have the following by Lemma 2.4.

**Lemma 2.6.** The sums $\sum_{\rho \in \mathbb{Z}_+^R} |\rho|^{-a}$ and $\sum_{\rho \in \mathbb{Z}_+^N} |\rho|^{-a}$ converge for all $a > 1$ and it holds that

$$\xi(s) = \prod_{1/2 + iy \in \mathbb{Z}_+^R} \left(1 - \frac{s}{1/2 + i\gamma}\right)\left(1 - \frac{s}{1/2 - i\gamma}\right)$$

$$\times \prod_{\rho \in \mathbb{Z}_+^N} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1 - \rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right)\left(1 - \frac{s}{1 - \bar{\rho}}\right).$$

(2.6)
the products being absolutely convergent for all $s \in \mathbb{C}$.

**Proof of Theorem 1.3.** Put $\bar{s} = \sigma - it$. Then we have
\[
\left(1 - \frac{\bar{s}}{\rho}\right)\left(1 - \frac{\sigma}{\rho}\right)^{-1} \left(1 - \frac{\bar{s}}{1 - \rho}\right)\left(1 - \frac{\sigma}{1 - \rho}\right)^{-1}
\]
\[
= \frac{\sigma - \beta - iy - it}{\sigma - \beta - iy} \frac{\sigma - 1 + \beta - iy - it}{\sigma - 1 + \beta - iy}
\]
\[
= \exp\left[ -\int_0^\infty (e^{itx} - 1)e^{(\beta - \sigma + iy)x} \frac{dx}{x} - \int_0^\infty (e^{itx} - 1)e^{(1 - \beta - \sigma + iy)x} \frac{dx}{x} \right]
\]
from (2.4). By replacing $\rho$ by $\bar{\rho}$, we obtain
\[
\left(1 - \frac{\bar{s}}{\rho}\right)\left(1 - \frac{\sigma}{\rho}\right)^{-1} \left(1 - \frac{\bar{s}}{1 - \rho}\right)\left(1 - \frac{\sigma}{1 - \rho}\right)^{-1}
\]
\[
= \exp\left[ -\int_0^\infty (e^{itx} - 1)e^{(\beta - \sigma - iy)x} \frac{dx}{x} - \int_0^\infty (e^{itx} - 1)e^{(1 - \beta - \sigma - iy)x} \frac{dx}{x} \right]
\]
We have to mention that one has $\beta - \sigma < 0$ and $1 - \beta - \sigma < 0$ since $\zeta(s) \neq 0$ for $\sigma \geq 1$ (see Remark 2.7 below). Hence, one has
\[
\frac{(1 - \frac{\bar{s}}{\rho})(1 - \frac{\sigma}{\rho})(1 - \frac{\bar{s}}{1 - \rho})(1 - \frac{\sigma}{1 - \rho})}{(1 - \frac{\sigma}{\rho})(1 - \frac{\sigma}{1 - \rho})(1 - \sigma/(1 - \rho))(1 - \sigma/(1 - \rho))}
\]
\[
= \exp\left[ -2 \int_0^\infty (e^{itx} - 1)\cos(\gamma x)(e^{(\beta - \sigma)x} + e^{(1 - \beta - \sigma)x}) \frac{dx}{x} \right].
\]
Therefore, we have
\[
\Xi_{\sigma}(t) = \exp\left[ -2 \sum_{1/2 + iy \in \mathbb{Z}^+_R} \int_0^\infty (e^{itx} - 1)\cos(\gamma x)e^{(1/2 - \sigma)x} \frac{dx}{x} \right.
\]
\[
- 2 \sum_{\beta + iy \in \mathbb{Z}^+_N} \int_0^\infty (e^{itx} - 1)\cos(\gamma x)(e^{(\beta - \sigma)x} + e^{(1 - \beta - \sigma)x}) \frac{dx}{x} \right]
\]
by (2.6) and the definition of $\Xi_{\sigma}(t)$.

**Remark 2.7.** By modifying the proof above, we can see that one has (1.7) for any $\sigma \geq \sigma_0 > 1/2$ if $\zeta(s)$ does not vanish for $\sigma \geq \sigma_0$.

**2.4. Proof of Theorem 1.4**

In order to prove Theorem 1.4, we first prove the following lemma which is an analogue of Nikeghbali and Yor [11], Lemma 2.9.
**Lemma 2.8.** Let $G_\sigma(t) = \Gamma(\sigma-it)/\Gamma(\sigma)$ for $0 < \sigma$. Then $G_\sigma(t)$ is an infinitely divisible characteristic function for any $\sigma > 0$. Moreover, one has

$$\log G_\sigma(t) = \text{it} \lambda^\#_\sigma + \int_0^\infty \left( e^{itx} - 1 - itx1_{[0, 1]}(x) \right) v^\#_\sigma(dx),$$

where

$$\lambda^\#_\sigma := \int_0^1 \left( \frac{e^{-\sigma x}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - \int_1^\infty \frac{e^{-x}}{x} dx,$$

and

$$v^\#_\sigma(dx) := \frac{1}{\chi e^{\sigma x}(1 - e^{-x})}.$$ 

**Proof.** By the integral representation of $\Gamma(s)$ and the change of variables $x = e^{-y}$, we have

$$G_\sigma(t) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-x} x^{\sigma-1} dx = \frac{-1}{\Gamma(\sigma)} \int_\infty^{-\infty} e^{-y} e^{y(1-\sigma+it)y} e^{-y} dy$$

$$= \frac{1}{\Gamma(\sigma)} \int_{-\infty}^{\infty} e^{iy} \exp(-\sigma y - e^{-y}) dy, \quad \sigma > 0.$$

Therefore, the probability density function is given by $\exp(-\sigma y - e^{-y})/\Gamma(\sigma)$.

Next, we quote Malmstén’s formula (see, for example, Whittaker and Watson [16], page 249)

$$\log \Gamma(s) = \int_0^\infty \left( \frac{e^{-sx} - e^{-x}}{1 - e^{-x}} + (s - 1)e^{-x} \right) \frac{dx}{x}, \quad \sigma > 0.$$

Hence, it holds that

$$\log G_\sigma(t) = \int_0^\infty \left( \frac{e^{-x(\sigma-it)x} - e^{-\sigma x}}{1 - e^{-x}} - ite^{-x} \right) \frac{dx}{x}$$

$$= \int_0^1 \left( \frac{e^{itx} - 1 - itx}{e^{\sigma x}(1 - e^{-x})} - ite^{-x} + \frac{ixe^{-\sigma x}}{e^{\sigma x}(1 - e^{-x})} \right) dx + \int_1^\infty \left( \frac{e^{itx} - 1}{e^{\sigma x}(1 - e^{-x})} - ite^{-x} \right) \frac{dx}{x}$$

$$= \int_0^\infty \frac{e^{itx} - 1 - itx1_{[0, 1]}(x)}{xe^{\sigma x}(1 - e^{-x})} dx + if \int_0^1 \left( \frac{e^{-\sigma x} - e^{-x}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) dx - it \int_1^\infty e^{-x} \frac{dx}{x}.$$ 

Therefore, we obtain Lemma 2.8. 

For the reader’s convenience, we give a proof of (1.4). By the Euler product of $\zeta(s)$ and the Taylor expansion of $\log(1 - x)$, $|x| < 1$, one has

$$\log \frac{\zeta(\sigma-it)}{\zeta(\sigma)} = \sum_p \log \frac{1 - p^{-\sigma}}{1 - p^{-\sigma+it}} = \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r}\sigma \left( p^{rit} - 1 \right)$$

$$= \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r}\sigma \left( e^{rit} \log p - 1 \right) = \int_{-\infty}^\infty \left( e^{itx} - 1 \right) \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r}\sigma \delta_r \log p(dx).$$
This equality implies (1.4).

**Proof of Theorem 1.4.** We have

$$\Xi_{\sigma}(t) = \pi^{\nu/2} G_{\sigma/2}(t/2) \frac{\sigma - it}{\sigma} \frac{\sigma - 1 - it}{\sigma - 1} \frac{\zeta(\sigma - it)}{\zeta(\sigma)}$$

by the definition of $\Xi_{\sigma}(t)$. It holds that

$$\log G_{\sigma/2}(t/2) = \frac{it}{2} C(\sigma/2) + \int_0^\infty \frac{e^{it/2} x - 1 - it x 1_{[0,1]}(x)}{x e^{\sigma x/2} (1 - e^{-x})} \, dx$$

$$= \frac{it}{2} C(\sigma/2) + \int_0^\infty \frac{e^{it x} - 1 - it x 1_{[0,1]}(x)}{x e^{\sigma x} (1 - e^{-2x})} \, dx$$

from Lemma 2.8. Obviously, one has $1/2 < r \log p$ for any integer $r$ and prime number $p$ since $\log 2 = 0.6931471806 \ldots$. Hence by using (1.4), we have

$$\log \frac{\zeta(\sigma - it)}{\zeta(\sigma)} = \int_{-\infty}^\infty (e^{it x} - 1 - it x 1_{[0,1/2]}(x)) \sum_{p} \sum_{r=1}^\infty \frac{1}{p^{r_\sigma}} \delta_r \log p (dx).$$

When $\sigma > 1$, one has

$$\frac{\sigma - it}{\sigma} \frac{\sigma - 1 - it}{\sigma - 1} \frac{1 - e^{-\sigma/2}}{\sigma}$$

by (2.3). Thus, it holds that

$$\frac{\sigma - it}{\sigma} \frac{\sigma - 1 - it}{\sigma - 1} = \exp \left[ - \int_0^\infty (e^{it x} - 1 - it x 1_{[0,1/2]}(x)) \frac{1 + e^{x}}{xe^{\sigma x}} \, dx - \frac{1 - e^{-\sigma/2}}{\sigma} \right].$$

If $x$ is sufficiently large, then we have

$$\frac{1}{xe^{\sigma x} (1 - e^{-2x})} - \frac{1 + e^{x}}{xe^{\sigma x}} < 0.$$

Thus $\nu_\sigma$ in Theorem 1.4 is not a measure but a signed measure.

Finally, we show $\int_{\mathbb{R}} (|x|^2 + 1)|\nu_\sigma|(dx) < \infty$ when $\sigma > 1$. By using $(1 - e^{-2})x \leq 1 - e^{-2x}$ for $0 \leq x < 1$ and $1 - e^{-2} \leq 1 - e^{-2x}$ for $x \geq 1$, we have

$$\int_0^\infty \frac{(1 - e^{-2})(|x|^2 + 1)}{xe^{\sigma x} (1 - e^{-2})} \, dx \leq \int_0^1 \frac{dx}{e^{\sigma x}} + \int_1^\infty \frac{dx}{xe^{\sigma x}} < \int_0^\infty \frac{dx}{e^{\sigma x}} < \infty.$$
Obviously, it holds that
\[
\int_0^\infty \frac{(1 + e^x)(|x|^2 \wedge 1)}{xe^{\sigma x}} \, dx < 2 \int_0^\infty \frac{(|x|^2 \wedge 1)}{xe^{(\sigma - 1)x}} \, dx < 2 \int_0^\infty \frac{dx}{e^{(\sigma - 1)x}} < \infty.
\]
From \(\sum_p p^{-\sigma} < \sum_{n=2}^{\infty} n^{-\sigma} = \zeta(\sigma) - 1\), one has
\[
\int_0^\infty \sum_p \sum_{r=1}^{\infty} \frac{1}{r} p^{-r\sigma} \delta_{r \log p}(dx)
= \sum_p \sum_{r=1}^{\infty} \frac{1}{r} p^{-r\sigma} < \sum_p \sum_{r=1}^{\infty} p^{-r\sigma} < \sum_{n=1}^{\infty} n^{-\sigma} + \sum_p \sum_{r=2}^{\infty} p^{-r\sigma}
= \zeta(\sigma) + \sum_p \frac{p^{-2\sigma}}{1 - p^{-\sigma}} < \zeta(\sigma) + \sum_{n=2}^{\infty} \frac{n^{-2\sigma}}{1 - 2^{-\sigma}}
< \zeta(\sigma) + (1 - 2^{-\sigma})^{-1} \zeta(2\sigma) < \infty.
\]
Therefore the characteristic function \(\Xi_\sigma(t)\) is not infinitely divisible but quasi-infinitely divisible. \(\square\)

**Remark 2.9.** Suppose \(\sigma \neq 1\) and put
\[
\Xi^*_\sigma(t) := \frac{\sigma - 1}{\sigma - 1 - it} \Xi_\sigma(t).
\]
Then \(\Xi^*_\sigma(t)\) is a characteristic function for any \(\sigma \neq 1\) by the fact that the product of a finite number of characteristic functions is also a characteristic function. By modifying the proof above, we have
\[
\Xi^*_\sigma(t) = \exp \left[ it \lambda^*_\sigma + \int_0^\infty \left( e^{itx} - 1 - itx 1_{[0,1/2]}(x) \right) \nu^*_\sigma(dx) \right],
\]
\[
\lambda^*_\sigma := \frac{1 - e^{-\sigma/2}}{\sigma} + \log \pi \frac{2}{\sigma} + \frac{1}{2} \int_0^1 \left( \frac{e^{-\sigma x/2}}{1 - e^{-x}} - \frac{e^{-x}}{x} \right) \, dx - \frac{1}{2} \int_1^{\infty} \frac{e^{-x} \, dx}{x},
\]
\[
\nu^*_\sigma(dx) := \frac{1(dx)}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1(dx)}{xe^{\sigma x}} + \sum_p \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx)
\]
for \(\sigma > 1\). Therefore the characteristic function \(\Xi^*_\sigma(t)\) is infinitely divisible for any \(\sigma > 1\) since one has
\[
\frac{1}{xe^{\sigma x}(1 - e^{-2x})} - \frac{1}{xe^{\sigma x}} > 0, \quad x > 0.
\]
Moreover, we can see that every characteristic function \(\Xi^*_\sigma(t)\) is a pretended-infinitely divisible characteristic function for each \(1/2 < \sigma < 1\) if and only if the Riemann hypothesis is true by
an argument similar to that in the proof of Theorem 1.2. In addition, it holds that

\[\Xi^*_\sigma(t) = \exp\left[\int_{-\infty}^{\infty} (e^{itx} - 1) \nu^*_\sigma(dx)\right],\]

\[\nu^*_\sigma(dx) := \frac{1_{(-\infty,0)}(dx)}{-xe^{(\sigma-1)x}} - \sum_{1/2+i\gamma \in \mathbb{Z}} \frac{2\cos(\gamma x)}{xe^{(\sigma-1/2)x}} 1_{(0,\infty)}(dx),\]

for \(1/2 < \sigma < 1\), under the Riemann hypothesis. This is proved by (1.6) and

\[\frac{\sigma - 1}{\sigma - 1 - it} = \frac{1 - \sigma}{1 - \sigma + it} = \exp\left[\int_{0}^{\infty} \frac{(e^{-itx} - 1)e^{(\sigma-1)x} dx}{x}\right] = \exp\left[\int_{-\infty}^{0} \frac{(e^{itx} - 1) dx}{xe^{(\sigma-1)x}}\right] = \exp\left[\int_{0}^{-\infty} \frac{(e^{itx} - 1) dx}{xe^{(\sigma-1)x}}\right],\]

when \(1/2 < \sigma < 1\).

It is well known that convolving a density with a normal density to make distributions more well-behaved. In this case the exponential distribution is the one that makes things nicer since when \(\sigma > 1\), the complete Riemann zeta distribution defined by \(\Xi^*_\sigma(t)\) and the distribution defined by \(\Xi^*_\sigma(t)\) are quasi-infinitely divisible and infinitely divisible, respectively.

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