Global gradient estimates for very singular nonlinear elliptic equations with measure data

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Abstract

This paper continues the development of regularity results for nonlinear measure data problems

\[
\begin{aligned}
-\text{div}(A(x, \nabla u)) &= \mu \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

in Lorentz and Lorentz-Morrey spaces, where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), \( \mu \) is a finite Radon measure on \( \Omega \), and \( A \) is a monotone Carathéodory vector valued operator acting between \( W^{1,p}_0(\Omega) \) and its dual \( W^{-1,p'}(\Omega) \). It emphasizes that this paper covers the ‘very singular’ case of \( 1 < p \leq \frac{3n-2}{2n-1} \) and the problem is considered under the weak assumption that the \( p \)-capacity uniform thickness condition is imposed on the complement of domain \( \Omega \). There are two main results obtained in our study pertaining to the global gradient estimates of solutions (renormalized solutions), involving the use of maximal and fractional maximal operators in Lorentz and Lorentz-Morrey spaces, respectively. The idea for writing this working paper comes directly from the results of others for the same research topic, where estimates for gradient of solutions cannot be obtained for the ‘very singular’ case, still remains a challenge. Our main goal here is to develop results and improve methods in \[64, 66\] to the case when \( 1 < p \leq \frac{3n-2}{2n-1} \). In order to derive the gradient estimates in Lorentz and Lorentz-Morrey spaces, our approach is based on the good-\( \lambda \) technique proposed early by Q.-H. Nguyen et al. in \[55, 56\] and our previous works in \[64, 66\].

Keywords: Nonlinear elliptic equations; measure data; gradient estimates; Lorentz spaces; Lorentz-Morrey spaces; good-\( \lambda \) techniques.

Contents

1 Motivation and main results

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1 Motivation and main results

The presence of quasilinear elliptic equations with measure data $-\text{div}(A(x, \nabla u)) = \mu$ and their regularity results were early studied by P. Benilan et al. in [3], L. Boccardo et al. in [8, 9, 10]. In recent years, many different models arising in physics, chemistry, biology and various scientific fields that involve the measure data problems were also the subject of many researchers. Later, in this class of elliptic problems, there has been a growing interest in the research about existence, uniqueness of solutions and/or some regularity results. For instance, many notions of solutions (very weak solutions, entropy solutions, renormalized solutions and SOLA-Solutions Obtained by Limits of Approximations) were introduced in [63, 43, 54, 45], and theory of regularity was early studied by F. Murat et al. in [53, 54, 45], M. F. Betta et al. in [5, 4, 6] and L. Boccardo et al. in [8, 9, 10] etc. Afterwards, the development of solution estimates in weak Lebesgue and Sobolev spaces has been extensively studied in recent years, therein Lorentz spaces, Lorentz-Morrey and Orlicz spaces (weighted or non-weighted) could be listed. Lately, there have been many regularity results concerning to nonlinear elliptic equations with measure data have been extensively obtained. We refer the interested reader to [48, 49, 51, 50, 40, 41, 42] for potential estimates, to [34, 19, 21] for some estimates in Orlicz spaces and to [56, 64, 60, 67, 68, 69] for gradient estimates in Lorentz and Lorentz-Morrey spaces under various assumptions on domain $\Omega$ and the range of $p$.

1.1. Elliptic equations with measure data. Before diving into the motivation of our study, it is important to start with the problem description and some of its assumptions. Let us now consider the quasilinear elliptic equations in the presence of measure as following

$$-\text{div}(A(x, \nabla u)) = \mu \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$ (1.1)
where $\Omega$ is a bounded open subset of $\mathbb{R}^n$ ($n \geq 2$); the datum measure $\mu$ is defined in $\mathcal{M}_b(\Omega)$—the space of all Radon measures on $\Omega$ with bounded total variation; the nonlinearity $A$ here is a Carathéodory vector valued function defined on $W_0^{1,p}(\Omega)$ only satisfying the growth and monotonicity conditions:
\[
|A(x, \xi)| \leq \Lambda |\xi|^{p-1},
\]
\[
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \Lambda^{-1} (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2,
\]
for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$, $\Lambda$ is a positive constant. This operator and its properties are described in Section 2.

In the study of solution regularity estimates, we restrict ourselves to the very ‘singular’ case for which $1 < p \leq \frac{2n}{n+2}$, and we discuss how results in extending to this case. The solution to our problem (1.1) is set in the context of renormalized solutions, that will be addressed the precise definition in the next section. Moreover, in the present work, domain assumptions specify the domain $\Omega \subset \mathbb{R}^n$ has its complement $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-capacity thick. One notices that this $p$-capacity density condition is stronger than the Weiner criterion described in [50] as:
\[
\int_0^1 \left( \frac{\text{cap}_p((\mathbb{R}^n \setminus \Omega) \cap B_r(x), B_{2r}(x))}{\text{cap}_p(B_r(x), B_{2r}(x))} \right)^{\frac{1}{p-1}} dr = \infty,
\]
which characterizes regular boundary points for the $p$-Laplace Dirichlet problem, where one measures the thickness of complement of $\Omega$ near boundary by capacity densities. This class of domains is relatively large (including those with Lipschitz boundaries), and its definition will be highlighted in Section 2. Otherwise, this condition is weaker than the Reifenberg flatness assumption on $\Omega$, that was discussed in various studies, see [62, 24, 35, 47].

1.2. Previous research and Motivation. There have been long-standing contributions in the research of regularity of solutions to the class of nonlinear elliptic equations with measure data. For instance, firstly by L. Boccardo et al. in [8, 9, 10], under the assumption that $\Omega$ is bounded, when $\mu \in L^m(\Omega)$ for $1 \leq m < \frac{np}{n(p-1)+p}$, authors proved that the unique solution $u$ belongs to $W_0^{1,nm/(n-p+1)}(\Omega)$. Later, the borderline case when $m = \frac{np}{n(p-1)+p}$ ($p < n$) was derived locally by G. Mingione et al. in [49]. Since then, whenever $2 - \frac{1}{n} < p \leq 2$, several local Calderón-Zygmund type estimates for the elliptic problems (1.1) were developed in different works [41, 42, 48, 49, 50] via potential estimates. Particularly, authors therein proposed the local estimates of solution at least for the case $2 \leq p \leq n$, and the extension to global estimates has also been obtained by using maximal function.

More recently, under the $p$-capacity assumption on the complement of $\Omega$ (boundary is thick enough to satisfy a uniform density condition) for the regular case $2 - \frac{1}{n} < p \leq n$, the global gradient estimates of solutions to problem (1.1) have been obtained in the Lorentz setting due to [60]; and it enables author further to treat the global regularity estimates in Morrey spaces in supercritical case, see [59]. According to the range value of $p$, our previous works in [64, 67] studied the gradient regularity to equation (1.1) in the framework of Lorentz spaces $L^{s,t}(\Omega)$, and in Lorentz-Morrey spaces $L^{s,t;\infty}(\Omega)$ respectively, for $0 < s < p + \varepsilon$ remains bounded, $0 < t \leq \infty$ and $0 \leq \kappa < n$. 

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In order to achieve better results where \( L^{s,t} \) and \( L^{s,t,n} (\Omega) \) estimates are obtained for all \( 0 < s < \infty \), it requires some additional structural assumptions of the problem setting. For instance, domain \( \Omega \) is assumed under Reifenberg flatness condition (Lipschitz domains with small Lipschitz constants, with fractal boundaries), and more information to operator \( A \) that \( A(x, \xi) \) is continuously differentiable in \( \xi \) away from the origin and satisfies the smallness condition of BMO type. According to these assumptions, Q.-H. Nguyen et al. very recently have got better Lorentz \( L^{s,t} (\Omega) \) estimates to (1.7) in [56] for singular case (that is, \( \frac{2n}{2n-1} < p \leq 2 - \frac{1}{n} \)). Further on, there have been numerous studies over Reifenberg domain for divergence type elliptic equations (This class of domains include all \( C^1 \)-domains, Lipschitz domains with small Lipschitz constants, and domains with fractal boundaries). For related results, we refer to [18, 15, 16, 17, 22, 1, 23, 11, 14, 39, 57, 58, 69] and the reader can consult also the references therein.

Our study is motivated by the question raised in some recent advances by above cited papers, that how to treat such regularity results for the ‘very singular’ problem when \( 1 < p \leq \frac{3n-2}{2n-1} \). There are two main results obtained in this work, where we prove the Lorentz and Lorentz-Morrey gradient norm estimates of solutions in term of fractional maximal functions. To our knowledge, the technique using maximal functions has been a lot of works developed. The main tool of our work is that we rely on the method introduced in [56] and those of recent papers in [64, 66, 67, 68, 69] to obtain gradient estimates for the ‘very singular’ problem. We face with two difficulties in this research paper. On the one hand, how we construct and prove the comparison estimates when \( 1 < p \leq \frac{3n-2}{2n-1} \); another hand that a weak assumption of the boundary is considered (\( p \)-capacity assumption on domain) instead of the stronger Reifenberg condition.

1.3. Main results. Let us now state here for convenience the main results, via four important theorems presented as below. Throughout the paper, we always assume that complement of the bounded domain \( \Omega \) satisfies the \( \mu \) setting. For instance, domain \( \Omega \) is assumed under Reifenberg flatness condition with positive constants \( c_0 \) and \( r_0 \). Moreover, for simplicity and conciseness of notations, we introduce

\[
m^* = \frac{n}{2n(p - 1) + 2 - p} \quad \text{and} \quad m^{**} = \frac{np}{n(p - 1) + p}
\]  

in main theorems and their proofs.

**Theorem A** Let \( 1 < p \leq \frac{3n-2}{2n-1} \), \( \mu \in \mathcal{M}_b (\Omega) \) and \( Q = B_{\text{diam}(\Omega)} (x_0) \) with \( x_0 \) is fixed in \( \Omega \). Assume in addition that \( \mu \in L^m (\Omega) \) for some \( m \in (m^*, m^{**}) \). Then, for any renormalized solution \( u \) to (1.1) and \( \frac{n}{2n-1} < q < \frac{nm(p-1)}{n-m} \), there exist some constants

\[
a = a(n, p, \Lambda, c_0) \in (0, 1), \quad b = b(n, p, q, \Lambda, c_0) \in \mathbb{R}, \quad \varepsilon_0 = \varepsilon_0(n, p, q, \Lambda) > 0 \quad \text{and} \quad C = C(n, p, q, \Lambda, c_0, \text{diam}(\Omega)/r_0) > 0
\]

such that the following inequality

\[
\mathcal{L}^n \left( \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \varepsilon^{-a} \lambda, (M_{\mu}(|\mu|^{m}))^{\frac{1}{m(p-1)}} < \varepsilon^b \lambda \right\} \right) \leq C \varepsilon \mathcal{L}^n \left( \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \right),
\]

holds for all \( \lambda > 0 \) and \( \varepsilon \in (0, \varepsilon_0) \).
Throughout this paper, we use $\mathcal{L}^n(E)$ for the $n$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$. Moreover, for the sake of convenience, in Theorem A, Theorem B and in what follows, the set $\{x \in \Omega : |g(x)| > \lambda\}$ is simply denoted by $\{|g| > \lambda\}$.

As we mentioned above, the value of $m^{**}$ is prescribed to guarantee the existence of renormalized solution to equation (1.1). In addition, to conclude the Lorentz and Lorentz-Morrey gradient estimates, it is critically important for us to construct the good-$\lambda$ inequalities, in which some comparison estimates in the interior and on the boundary of domain have been effectively employed. In our analysis, results obtained in this paper are comparable with those in [56] Lemma 2.2] when $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. However, in our work, an additional assumption on the datum $\mu$ is imposed, for which $\mu$ is a function belonging to $L^m(\Omega)$ for $m > 1$. More specifically, we will focus our attention on datum $\mu$ assumed to be a function belonging to $L^m(\Omega)$, for $m > m^*$ specified above in Theorem A. Indeed, we notice that

$$p \leq \frac{3n-2}{2n-1} \iff m^* \geq 1,$$

and in this case, generally speaking, based on methods given in [2, 56], we cannot expect to obtain comparison results for a general measure datum $\mu$ (as a function in $L^1$) and the range of $p$: $p < \frac{3n-2}{2n-1}$. Otherwise, in [56], authors dealt with the case $p > \frac{3n-2}{2n-1}$ (implies further that $m^* < 1$), and this suffices to prove both interior and boundary estimates with $\mu \in L^1(\Omega)$ (generally a measure). Therefore, it is natural to expect a more appropriate method or to require that one or more assumptions on our initial data. For the reader’s convenience, in Section 3 we explain in details how to treat in particular the case of $1 < p < \frac{3n-2}{2n-1}$.

The next theorem B constructs the improved version of good-$\lambda$ inequality presented in Theorem A that can be applied to prove Lorentz-Morrey gradient estimates in Theorem D later. However, the proof of Theorem B is rather similar to that of Theorem A except for some estimates.

**Theorem B** Let $1 < p \leq \frac{3n-2}{2n-1}$, $\mu \in \mathcal{M}_b(\Omega)$ and $Q_1 = B_\rho(x_0)$, $Q_2 = B_{10\rho}(x_0)$, with $0 < \rho < \text{diam}(\Omega)$ and $x_0$ is fixed in $\Omega$. Assume in addition that $\mu \in L^m(\Omega)$ for some $m \in (m^*, m^{**})$. Then, for any renormalized solution $u$ to (1.1) and for $\frac{n}{n-1} < q < \frac{nm(m-1)}{n-m}$, there exist some constants $a = a(n,p,\Lambda,c_0) \in (0,1)$, $b = b(n,p,q,\Lambda,c_0) \in \mathbb{R}$ and $C = C(n,p,q,\Lambda,c_0,\text{diam}(\Omega)/\rho_0) > 0$ such that the following inequality

$$\mathcal{L}^n \left( \left\{ (M(\chi_{Q_2} \nabla u)^q)^{\frac{1}{q}} > \varepsilon^{-a} \lambda, (M_m(\chi_{Q_2} |\mu|^m)^{\frac{1}{m-q}} \leq \varepsilon^b \lambda \right\} \cap Q_1 \right)$$

$$\leq C \mathcal{L}^n \left( \left\{ (M(\chi_{Q_2} \nabla u)^q)^{\frac{1}{q}} > \lambda \right\} \cap Q_1 \right)$$  (1.4)

holds for any $\lambda > \varepsilon^{-b} p^{-\frac{n}{q}} \|\nabla u\|_{L^q(Q_2 \cap \Omega)}$ with $\varepsilon \in (0,1)$ small enough.

In following Theorems C and D some improved results of Lorentz and Morrey-Lorentz gradient estimates are given. This work extends our earlier works in [64, 67] when $\Omega$ satisfies the $p$-capacity condition and $p$ is singular. It also remarks that for specific case when $m \equiv 1$, the statements in Theorem C and D still hold, but they only make sense for a certain range of $p$, i.e. $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. A detailed explanation will be discussed in Section 3.
Theorem C Let $1 < p \leq \frac{3n-2}{2n-1}$ and $\mu \in \mathcal{M}_{b}(\Omega)$. Assume that the given measure data $\mu \in L^{n}(\Omega)$ for some $m \in (m^{*}, m^{**})$. Then there exists a constant $\Theta_{0} = \Theta_{0}(n, p, \Lambda, c_{0}) > p$ such that for any renormalized solution $u$ to (1.1), there holds
\begin{equation}
\|\nabla u\|_{L^{n}(\Omega)} \leq C \left\{ M_{m}(|\mu|^{m}) \right\}^{\frac{1}{m(n-1)}} \|\mu\|_{L^{n}(\Omega)},
\end{equation}
for any $0 < s < \Theta_{0}$ and $0 < t \leq \infty$. The constant $C$ depends on $n$, $p$, $\Lambda$, $s$, $t$, $c_{0}$ and $D_{0}/r_{0}$.

Theorem D Let $1 < p \leq \frac{3n-2}{2n-1}$ and $\mu \in \mathcal{M}_{b}(\Omega) \cap L^{m}(\Omega)$ for some $m \in (m^{*}, m^{**})$. Then, there exist $\Theta_{0} = \Theta_{0}(n, p, \Lambda, c_{0}) > p$, $\beta_{0} = \beta_{0}(n, p, \Lambda) \in (0, 1/2)$ such that for any renormalized solution $u$ to (1.1) with given data $\mu \in L^{m_{s}, m_{t}, \kappa}(\Omega)$ satisfying $0 < \kappa < \min \left\{ \frac{(n-m)^{\beta_{0}}}{m(p-1)}, n \right\}$, $0 < t \leq \infty$ and
\begin{equation}
\frac{\kappa}{n} \leq s < \min \left\{ \frac{\kappa}{m + m(p-1)(1 - \beta_{0})}, \frac{\kappa \Theta_{0}}{m \Theta_{0} + m(p-1)\kappa} \right\},
\end{equation}
there holds
\begin{equation}
\left\| |\nabla u|^{p-1} \right\|_{L_{\kappa}^{\frac{\infty}{\kappa}, \kappa} \cap \kappa \frac{\infty}{\kappa} m_{t}, \kappa}(\Omega) \leq C \left\{ |\mu|_{m} \right\}^{\frac{1}{m(n-1)}} \|\mu\|_{L^{n}(\Omega)}^{\frac{1}{m(n-1)}}.
\end{equation}

Here the positive constant $C$ depends on $n$, $m$, $p$, $\Lambda$, $s$, $t$, $\kappa$, $c_{0}$ and $\text{diam}(\Omega)/r_{0}$.

1.4. Contents and highlights of the paper. Let us briefly summarize the contents of the paper as follows. We begin with a few preliminaries about notation, definitions and some assumptions of the problem in Section 2. In Section 3 we prove some preparatory lemmas including some comparison estimates that are very important to prove the main theorems. It is worth emphasizing that most of comparison results have been formulated and proved regarding to (1.1) up to the boundary. As we mentioned above, these proofs are highlighted with specific data $p$, $\mu$ and the idea behind the proofs is the choice of an appropriate test function. It is important to realize that the comparison estimates for the case $1 < p \leq \frac{3n-2}{2n-1}$ is not new, Q.-H. Nguyen and N.C. Phuc have already stated and proved in their unpublished work earlier. Here, for the convenience of reader, we only present another route to the proof. We also thank and refer to the works by G. Maso et al. in [14] Section 4 and Q.-H. Nguyen et al. in [60] Lemma 2.2 in which the authors have shown some examples of test functions, that inspired us in these proofs.

For $1 < p \leq \frac{3n-2}{2n-1}$, we indicate in the last section some new results that we planned to develop in this paper. Section 4 is devoted to proofs of gradient norm regularity in both Lorentz and Lorentz-Morrey spaces applying useful results of the previous section. This part has two main features: on one hand including the proofs of proposed method so-called “good-$\Lambda$” that is very important and useful in our work, on the other hand the proofs of Theorems C and D are obtained.

2 Preliminaries

This section will introduce some convenient notations, assumptions on the problem (1.1): review a number of the most important definitions and collect some additional useful results that related to our study. For further details, the interested reader can also consult the literature through the mentioned citations therein.
2.1 Assumptions on domain, measure and operator

Capacity and Measures. In this paper, our results are obtained under a weak condition on the domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), the uniform capacity density condition. To our knowledge, this condition is very important for the existence of a solution to our problem, for a higher integrability property of the gradient. We first recall the definition of $p$-capacity of a set.

**Definition 2.1** Let $K \subset \Omega$ be a compact set. For any $p > 1$, the $p$-capacity of $K$ is defined as:

$$
cap_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p dx : \varphi \in C^\infty_c, \varphi \geq \chi_K \right\},
$$

where $\chi_K$ is the characteristic function of $K$.

This definition can be extended to capacity of any open set $U \subseteq \Omega$ by taking the supremum over the capacities of the compact sets $K \subseteq U$ as follows

$$
cap_p(U, \Omega) = \sup \left\{ \cap_p(K, \Omega), \ K \text{ compact, } K \subseteq U \right\}.
$$

Consequently, the $p$-capacity of any subset $B \subseteq \Omega$ is defined by:

$$
cap_p(B, \Omega) = \inf \left\{ \cap_p(U, \Omega), \ U \text{ open, } B \subseteq U \right\}.
$$

**Definition 2.2** ($p$-capacity uniform thickness) The set $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick condition if there exist two constants $c_0, r_0 > 0$ such that

$$
\cap_p((\mathbb{R}^n \setminus \Omega) \cap \overline{B}_r(x), B_{2r}(x)) \geq c_0 \cap_p(\overline{B}_r(x), B_{2r}(x)), \tag{2.1}
$$

for every $x \in \mathbb{R}^n \setminus \Omega$ and $0 < r \leq r_0$.

Based on what we know, domains satisfy definition of $p$-capacity uniform thickness include those with Lipschitz boundaries or even those that satisfy a uniform corkscrew condition (see [33]) and (2.1) still remains valid for balls centered outside a uniformly $p$-thick domain. Moreover, the uniform $p$-capacity is necessary for the validity of Poincaré’s and Sobolev’s inequalities, that are very helpful in our proofs later.

Every nonempty $\mathbb{R}^n \setminus \Omega$ is uniform $p$-thick for $p > n$ and this condition is nontrivial only when $p \leq n$. It is also clear that if $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick then it is uniformly $q$-thick for every $q > p$. For further properties of the $p$-capacity condition we refer to the books [33, Chater 2] and [33, Chapter 2].

Otherwise, in the problem setting, we define $\mathcal{M}_b(\Omega)$ is the space of all Radon measures on $\Omega$ with bounded total variation and $C_b(\Omega)$ as the space of all bounded, continuous functions on $\Omega$, so that $\int_{\Omega} \varphi d\mu < +\infty$ for all $\varphi \in C_b(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$. The positive part, negative part and the total variation of a measure $\mu$ in $\mathcal{M}_b(\Omega)$ are denoted by $\mu^+$, $\mu^-$ and $|\mu|$, respectively.
Definition 2.3 We say that a sequence \((\mu_n)\) converges to \(\mu\) in \(\mathcal{M}_b(\Omega)\) in a narrow topology if
\[
\lim_{n \to +\infty} \int_{\Omega} \varphi d\mu_n = \int_{\Omega} \varphi d\mu,
\] (2.2)
for every \(\varphi \in C_b(\Omega)\). Moreover, if (2.2) holds for all continuous functions \(\varphi\) with compact support in \(\Omega\), then we have the weak convergence in \(\mathcal{M}_b(\Omega)\).

Remark 2.4 It has been noted in [10] that a measure \(\mu \in \mathcal{M}_0(\Omega)\) if and only if \(\mu\) belongs to \(L^1(\Omega) + W^{-1,p'}(\Omega)\).

Remark 2.5 For every measure \(\mu\) in \(\mathcal{M}_b(\Omega)\), there exists a unique pair of measures \((\mu_0, \mu_s)\), with \(\mu_0 \in \mathcal{M}_0(\Omega)\) and \(\mu_s \in \mathcal{M}_s(\Omega)\) (the set of all measures \(\mu\) in \(\mathcal{M}_b(\Omega)\) for which there exists a Borel set \(E \subset \Omega\) with \(\text{cap}_p(E, \Omega) = 0\)), such that \(\mu = \mu_0 + \mu_s\), is \(\mu\) is nonnegative, so are \(\mu_0\) and \(\mu_s\). Therefore, the measures \(\mu_0\) and \(\mu_s\) will be called the absolutely continuous and the singular part of \(\mu\) with respect to the \(p\)-capacity.

Operator. In the setting of equation 
\[-\text{div}(A(x, \nabla u)) = \mu,\]
the nonlinear operator \(A : \Omega \times \mathbb{R}^n \to \mathbb{R}\) satisfies the following conditions:

(i) \(A\) is a Carathéodory vector valued function. That is, \(A(\cdot, \xi)\) is measurable on \(\Omega\) for every \(\xi \in \mathbb{R}^n\), and \(A(x, \cdot)\) is continuous on \(\mathbb{R}^n\) for almost every \(x\) in \(\Omega\).

(ii) Growth and Monotonicity. That is, for some \(1 < p \leq n\) there exist two positive constants \(\Lambda_1\) and \(\Lambda_2\) such that
\[
|A(x, \xi)| \leq \Lambda_1 |\xi|^{p-1},
\] (A1)
and
\[
\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \Lambda_2 (|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2
\] (A2)
holds for almost every \(x\) in \(\Omega\) and every \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}\).

From above assumptions, one sees that the operator \(A\) is defined on \(W^{1,p}_0(\Omega)\) with values in its dual space \(W^{-1,p'}(\Omega)\) (\(p'\) is the Hölder conjugate of \(p\)).

2.2 Renormalized solutions
The problem (1.1) does not admit a weak solution under these above assumptions. However, in general, one can expect to establish a notion of weak solution such that we can prove the existence and uniqueness of such solution. The concept of renormalized solution, was first introduced by R.J. DiPerna et al. in [25] when studying of the Boltzmann equation, and then adapted to nonlinear elliptic problems with Dirichlet boundary conditions by Boccardo et al. in [8]. An equivalent notion of solutions, called entropy solution, was then introduced by Bénilan et al. in [2].
In this part, let us also recall and reproduce the definition of renormalized solution, that was early studied in \cite{7, 10, 45}. To do this, we first introduce the truncature operator. For a given constant \( k > 0 \), we follow the notation of truncation operator at level \( \pm k \), that is \( T_k : \mathbb{R} \to \mathbb{R} \) defined by

\[
T_k(s) = \max \{-k, \min\{k, s\}\}, \quad k \in \mathbb{R}^+, \ s \in \mathbb{R},
\]

that belongs to \( W^{1,p}_0(\Omega) \), which satisfies \(-\text{div}A(x, \nabla T_k(u)) = \mu_k \) in the sense of distribution in \( \Omega \) for a finite measure \( \mu_k \) in \( \Omega \).

**Definition 2.6** Let \( u \) be a measurable function defined on \( \Omega \) which is finite almost everywhere, and satisfies \( T_k(u) \in W^{1,1}_0(\Omega) \) for every \( k > 0 \). Then, there exists a unique measurable function \( v : \Omega \to \mathbb{R}^n \) such that:

\[
\nabla T_k(u) = \chi_{\{|u| \leq k\}}v, \quad \text{almost everywhere in} \ \Omega, \ \text{for every} \ k > 0.
\]

Moreover, the function \( v \) is so-called “generalized distributional gradient” of \( u \), still denoted by \( \nabla u \). If \( u \in L^1_{\text{loc}}(\Omega) \), this differs from the distributional gradient of \( u \), and it coincides with the usual gradient for every \( u \in W^{1,1}(\Omega) \).

There are several equivalent definitions of renormalized solutions (see \cite{44, 45} and related references), here we will use the following Definition 2.7 throughout this paper.

**Definition 2.7** (Renormalized solution) Let \( \mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega) \), where \( \mu_0 \in \mathcal{M}_0(\Omega) \) and \( \mu_s \in \mathcal{M}_s(\Omega) \). A measurable function \( u \) defined in \( \Omega \) and finite almost everywhere is called a renormalized solution of (1.1) if \( T_k(u) \in W^{1,p}_0(\Omega) \) for any \( k > 0 \), \( |\nabla u|^{p-1} \in L^r(\Omega) \) for any \( 0 < r < \frac{n}{n-1} \), and \( u \) has the following additional property. For any \( k > 0 \) there exist nonnegative Radon measures \( \lambda^+_k, \lambda^-_k \in \mathcal{M}_0(\Omega) \) concentrated on the sets \( u = k \) and \( u = -k \), respectively, such that \( \mu^+_k \to \mu^+_s \), \( \mu^-_k \to \mu^-_s \) in the narrow topology of measures and that

\[
\int_{\{|u| < k\}} (A(x, \nabla u), \nabla \varphi)dx = \int_{\{|u| < k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda^+_k - \int_{\Omega} \varphi d\lambda^-_k,
\]

for every \( \varphi \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega) \).

The following Lemmas 2.8 were given in \cite{45} Theorem 4.1], that characterize the classical global Lebesgue gradient estimate for solution \( u \) to (1.1) with given measure data \( \mu \in L^1(\Omega) \) and the convergence result. And we refer the reader to \cite{2, 9, 10} for the proofs.

**Lemma 2.8** Let \( \Omega \) is an open bounded domain in \( \mathbb{R}^n \). Then, there exists \( C = C(n, p) > 0 \) such that for any the renormalized solution \( u \) to (1.1) with a given finite measure data \( \mu \) there holds:

\[
\|\nabla u\|_{L^{\frac{(p-1)n}{n-1}}(\Omega)} \leq C \|\mu(\Omega)\|^{\frac{1}{p-1}}.
\]

**Proposition 2.9** Let \( \mu \in L^1(\Omega) \) and a sequence \((u_k)_k\) be the renormalized solution of (1.1) with data \( \mu_k \in L^{\frac{n}{p}}(\Omega) \) such that \( \mu_k \to \mu \) weakly in \( L^1(\Omega) \). Then, there exists a subsequence \( \{u_{k'}\}_k' \) which converges to \( u \) in \( L^q(\Omega) \) for all \( q < \frac{(p-1)n}{n-1} \). Moreover, \( \nabla u_{k'} \to \nabla u \) in \( L^q(\Omega) \) for all \( q < \frac{(p-1)n}{n-1} \).
2.3 The Lorentz and Lorentz-Morrey spaces

Definition 2.10 (Lorentz spaces) For $0 < s < \infty$ and $0 < t \leq \infty$, we denote by $L^{s,t}(\Omega)$ (see [31]) the Lorentz spaces is the set of all Lebesgue measurable functions $g$ on $\Omega$ such that:

$$\|g\|_{L^{s,t}(\Omega)} = \left[ \int_0^\infty \lambda^s L^n (\{x \in \Omega : |g(x)| > \lambda\})^{\frac{t}{s}} \frac{d\lambda}{\lambda} \right]^\frac{1}{t} < +\infty. \quad (2.6)$$

If $t = \infty$, the space $L^{s,1}(\Omega)$ is the usual weak-$L^s$ or Marcinkiewicz spaces with the following quasinorm:

$$\|g\|_{L^{s,\infty}(\Omega)} = \sup_{\lambda > 0} \lambda L^n (\{x \in \Omega : |g(x)| > \lambda\})^{\frac{1}{s}}. \quad (2.7)$$

Cavalieri’s principle shows that when $s = t$, the Lorentz space $L^{s,s}(\Omega)$ becomes the Lebesgue space $L^s(\Omega)$. More precisely, the spaces are nested increasingly with respect to the second parameter $t$:

$$L^{s,1}(\Omega) \subset L^{s,t}(\Omega) \subset L^{s,\infty}(\Omega).$$

Definition 2.11 (Lorentz-Morrey spaces) A function $g \in L^{s,t}(\Omega)$ for $0 < s < \infty$, $0 < t \leq \infty$ is said to belong to the Lorentz-Morrey functional spaces $L^{s,t;\kappa}(\Omega)$ for some $0 < \kappa \leq n$ if

$$\|g\|_{L^{s,t;\kappa}(\Omega)} := \sup_{0 < \rho < \text{diam}(\Omega); x \in \Omega} \rho^{-\frac{n-\kappa}{s}} \|g\|_{L^{s,t}(B_{\rho}(x) \cap \Omega)} < +\infty. \quad (2.8)$$

When $\kappa = n$ the space $L^{s,t;n}(\Omega)$ is exactly the Lorentz space $L^{s,t}(\Omega)$.

2.4 Hardy-Littlewood Maximal functions

For the reader’s convenience, we abbreviate the open ball in $\mathbb{R}^n$ with center $x_0$ and radius $r$ by $B_r(x_0)$. And in this paper, we denote the average of a function $f \in L^1_{\text{loc}}(\Omega)$ over a ball $B_r(x) \subseteq \Omega$ by

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy.$$

Definition 2.12 (Fractional maximal functions, [37, 38]) Let $0 \leq \alpha \leq n$, the fractional maximal function $M_\alpha$ of a locally integrable function $g : \mathbb{R}^n \to [-\infty, \infty]$ is defined by:

$$M_\alpha g(x) = \sup_{\rho > 0} \rho^\alpha \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |g(y)|dy. \quad (2.9)$$
For the case $\alpha = 0$, one obtains the Hardy-Littlewood maximal function, $Mg = M_0g$, defined for each locally integrable function $g$ in $\mathbb{R}^n$ by:

$$Mg(x) = \sup_{\rho > 0} \int_{B_\rho(x)} |g(y)| dy, \quad \forall x \in \mathbb{R}^n. \quad (2.10)$$

Fractional maximal operators have many applications in partial differential equations, potential theory and harmonic analysis. Once we need to estimate some quantities of a function $g$, they can be shown to be dominated by $Mg$, or more generally by $M_\alpha g$.

The fundamental result of maximal operator is that the boundedness on $L^p(\mathbb{R}^n)$ when $1 < p \leq \infty$, that is there exists a constant $C = C(n, p) > 0$ such that:

$$\|Mg\|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad \forall g \in L^p(\mathbb{R}^n).$$

Moreover, $M$ is also said to be weak-type $(1,1)$, this means there is a constant $C(n) > 0$ such that for all $\lambda > 0$ and $g \in L^1(\mathbb{R}^n)$, it holds that

$$\mathcal{L}^n (\{M(g) > \lambda\}) \leq C(n)\frac{\|g\|_1}{\lambda}.$$

The standard and classical references can be found in many places such as [30, 31], and later also in [12]. Besides that, there are some well-known properties of maximal operators, that will be shown in following lemmas.

**Lemma 2.13** It refers to [31] that the operator $M$ is bounded from $L^s(\mathbb{R}^n)$ to $L^{s,\infty}(\mathbb{R}^n)$, for $s \geq 1$, this means,

$$\mathcal{L}^n (\{M(g) > \lambda\}) \leq \frac{C}{\lambda^s} \int_{\mathbb{R}^n} |g(x)|^s dx, \quad \text{for all } \lambda > 0. \quad (2.11)$$

**Lemma 2.14** In [31], it allows us to present a boundedness property of maximal function $M$ in the Lorentz space $L^{q,s}(\mathbb{R}^n)$, for $q > 1$ as follows:

$$\|Mg\|_{L^{q,s}(\Omega)} \leq C\|g\|_{L^{q,s}(\Omega)}. \quad (2.12)$$

### 3 Comparison results

The purpose of this section is to construct and prove some comparison estimates between solutions of (1.1) in $\Omega$ and the ones of homogeneous equations in arbitrary balls, that are very important to obtain our main results later.

In the remaining parts of this paper, we always mention $u$-the solution of our problem (1.1), the renormalized solution in which the existence and uniqueness always make sense. In addition, for the sake of simplicity, we assume in some lemmas below that $\Omega$ is the domain whose complement satisfies the $p$-capacity uniform thickness condition (2.1), $p$ is ‘very singular’ and two parameters $m^*$ and $m^{**}$ are clarified as in (1.2).
3.1 A priori global estimates for the gradient

Lemma 3.1 Let \( \mu \in L^m(\Omega) \) for some \( m \in (1, m^*) \) and \( u \) be the solution to (1.1). Then \( \nabla u \in L^{\frac{nm(p-1)}{n-m}}(\Omega) \) and there exists a positive constant \( C \) such that

\[
\|\nabla u\|_{L^{\frac{nm(p-1)}{n-m}}(\Omega)} \leq C \|\mu\|_{L^m(\Omega)}. \tag{3.1}
\]

Lemma 3.2 Let \( u \) be the renormalized solution to (1.1) with measure data \( \mu \in L^m(\Omega) \) for some \( m \in (1, m^*) \). Assume that a sequence \( (\mu_k)_k \) is the renormalized solution to (1.1) with data \( \mu_k \in L^\frac{m(p-1)}{n-m}(\Omega) \) such that \( \mu_k \) converges to \( \mu \) weakly in \( L^m(\Omega) \). Then, there exists a subsequence \( (\mu_{k'})_k' \) such that \( \mu_{k'} \) converges to \( u \) and \( \nabla \mu_{k'} \) converges to \( \nabla u \) in \( L^q(\Omega) \) for all \( 0 < q < \frac{nm(p-1)}{n-m} \).

These obtained results remain valid in the case of measure data \( \mu \in L^1(\Omega) \), i.e. \( m \equiv 1 \) and they are exact the ones concluded in Lemma 2.8 and Proposition 2.9, respectively. We next derive local interior and boundary estimates in terms of given data \( \mu \) in following lemmas, they are independent of the above ones.

3.2 Interior estimates

One firstly makes attention to the interior estimates. Let us fix a point \( x_0 \in \Omega \), for \( 0 < 2R \leq r_0 \) (\( r_0 \) is the constant given in (2.1)). For each ball \( B_R := B_R(x_0) \subset \Omega \) and \( u \in W^{1,p}_{\text{loc}}(\Omega) \), let \( w \in W^{1,p}_0(B_R) + u \) be the unique solution to the following equation

\[
-\text{div}(A(x, \nabla w)) = 0 \quad \text{in} \quad B_R, \quad w = u \quad \text{on} \quad \partial B_R. \tag{3.2}
\]

Here, we recall the following version of interior Gehring’s lemma applied to solution \( w \) of (3.2). This is also known as the “reverse” Hölder integral inequality with increasing supports (see [29], [28, Theorem 6.7]). Technique of using this inequality with small exponents to gradient estimates was proposed by G. Mingione et al. in [48] and along with it, many research approaches have since been developed.

Lemma 3.3 Let \( u \in W^{1,p}_{\text{loc}}(\Omega) \) be a solution to equation (1.1) and \( w \in W^{1,p}_0(B_R) + u \) be the unique solution to equation (3.2). There exists a constant \( \Theta_0 > p \) such that

\[
\left( \int_{B_{2r}(y)} |\nabla w|^\Theta_0 dx \right)^{\frac{1}{\Theta_0}} \leq C \left( \int_{B_{2r}(y)} |\nabla w|^{p-1} dx \right)^{\frac{p}{p-1}}, \tag{3.3}
\]

for all \( B_{2r}(y) \subset B_R \) and the positive constant \( C \) depends on \( n, p, \Lambda \).

We next perform a comparison gradient result for solutions to both problems (1.1) and (3.2). The conclusion of following result is a modified version of a result has been proved in [56] when \( \mu \in L^1(\Omega) \) and \( \frac{2n-2}{2n-1} < p \leq 2 - \frac{1}{n} \). Our approach is based on methods given in [2, 50], where the utilization of Hölder’s and Sobolev’s inequalities are very important to conclude some estimates. However, the method cannot be directly applied itself to obtain comparison results for a general measure datum \( \mu \) (as a function in \( L^1 \)) and the range of \( p; p < \frac{2n-2}{2n-1} \). In our discussion, one comes to expect a more
appropriate method or the requirement that one or more assumptions on initial data. And in this study, to overcome the difficulty of the very singular $p$, authors impose a stronger assumption on datum $\mu$, to be a function belonging to $L^m(\Omega)$, with $m > 1$ will be specified later in the proof. It is worth emphasizing that, the statement and proof of comparison estimates for the case $1 < p \leq \frac{3n-2}{m-1}$ are not new, Q.-H. Nguyen and N.C. Phuc have already written in their unpublished work earlier. Here, for the convenience of reader, we present another route to the proof.

Let us state that important result via following lemma whose proof is included in Appendix A at the very end of this paper. Here, it is noticeable that we derive the local estimates, in the ball $B_R$.

**Lemma 3.4** Let $1 < p \leq \frac{3n-2}{m-1}$ and $\mu \in L^m(B_R)$ for some $m \in (m^*, n)$. Assume that $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution to equation (1.1) and $w \in W^{1,p}_0(B_R) + u$ be the unique solution to equation (3.2). For any $q$ satisfying the following condition

$$\frac{n}{2n-1} < q < \frac{nm(p-1)}{n-m},$$

(3.4)

there exists a constant $C > 0$ only depending on $n$, $p$, $q$ and $m$ such that

$$\left( \int_{B_R} |\nabla u - \nabla w|^q dx \right)^{\frac{1}{q}} \leq CF_R(\mu)^{\frac{p}{m-1}} + CF_R(\mu) \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{q}},$$

(3.5)

where the function $F_R$ is defined by

$$F_R(\mu) = \left( R^n \int_{B_R} |\mu|^m dx \right)^{\frac{1}{m}}.$$  

(3.6)

The following result holds.

**Lemma 3.5** Let $w$ be solution to (3.2). Then, there exist two constants $\beta_0 = \beta_0(n, p, \Lambda) \in (0, 1/2]$ and $C_1 = C_1(n, p, \Lambda) > 0$ such that

$$\left( \int_{B_{\rho}(y)} |\nabla w|^p dx \right)^{\frac{1}{p}} \leq C_1 \left( \frac{\rho}{r} \right)^{\beta_0-1} \left( \int_{B_{r}(y)} |\nabla w|^p dx \right)^{\frac{1}{p}},$$

(3.7)

for any $B_{\rho}(y) \subset B_{r}(y) \subset B_R$. Moreover, for any $s \in (0, p]$ there exists a positive constant $C_2 = C_2(n, p, \Lambda, s)$ such that

$$\left( \int_{B_{\rho}(y)} |\nabla w|^s dx \right)^{\frac{1}{s}} \leq C_2 \left( \frac{\rho}{r} \right)^{\beta_0-1} \left( \int_{B_{r}(y)} |\nabla w|^s dx \right)^{\frac{1}{s}},$$

(3.8)

for any $B_{\rho}(y) \subset B_{r}(y) \subset B_R$.

**Proof.** The inequality (3.7) comes from the standard interior Hölder continuity of solutions, its proof can be found in [28, Theorem 7.7] and we do not write down all the details here. Applying the inequality (3.7) and Lemma 3.3 one obtains (3.8).

In order to prove the key lemma below, it will be necessary to refer to Lemma 3.6 in [28, Lemma 1.4] as follows, where its proof can be found therein.
Lemma 3.6 Let \( \eta \in (0, 1) \), \( 0 \leq \beta < \alpha \) and \( h : [0, R] \rightarrow [0, \infty) \) be a non-decreasing function. Suppose that

\[
h(\rho) \leq P \left[ \left( \frac{\rho}{R} \right)^\alpha + \varepsilon \right] h(r) + Q r^\beta,\]

for any \( 0 < \rho \leq \eta r < R \) and \( \varepsilon > 0 \), with positive constants \( P, Q \). Then, there exists a constant \( \varepsilon_0 = \varepsilon_0(P, \alpha, \beta, \eta) \) such that if \( \varepsilon \in (0, \varepsilon_0) \) then

\[
h(\rho) \leq C \left[ \left( \frac{\rho}{R} \right)^\alpha h(r) + Q \rho^\beta \right],\]

for all \( 0 < \rho \leq r \leq R \), where \( C \) is a positive constant depending on \( P, \alpha, \beta \).

Lemma 3.7 Let \( 1 < p \leq \frac{3n-2}{2n-1} \) and \( u \) be a solution to equation (1.1) with \( \mu \in L^m(\Omega) \) for some \( m \in (m^*, n) \). Let \( \beta_0 \in (0, 1/2] \) be as in Lemma 3.5. Then, for any \( q \in \left( \frac{n}{2n-1}, \frac{nm(p-1)}{n-m} \right) \) and

\[
m + m(p-1)(1 - \beta_0) < \sigma \leq n,
\]

there exists a constant \( C = C(n, p, \Lambda, c_0, \sigma) > 0 \) such that

\[
\left( \int_{B_{\rho}(y)} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \rho^{\frac{n-\delta}{\sigma}} \left[ M_\sigma^D \left( |\mu|^m \right)(y) \right]^{\frac{1}{m(p-1)}}, \quad (3.9)
\]

for all \( B_{\rho}(y) \subset \subset \Omega \), where \( \delta = \frac{\sigma-m}{m(p-1)} \) and \( D_0 = \text{diam}(\Omega) \).

**Proof.** Let us take \( B_{\rho}(y) \subset \subset \Omega \) and \( 0 < \rho \leq r/2 \). Applying Lemma 3.3 with \( B_{2\rho} = B_{2\rho}(y) \), one gives:

\[
\left( \int_{B_{\rho}(y)} |\nabla(u-w)|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B_{\rho}(y)} |\mu|^m dx \right)^{\frac{1}{m(p-1)}} \left( \int_{B_{\rho}(y)} |\nabla u|^q dx \right)^{\frac{2-n}{q}} + C \left( \int_{B_{\rho}(y)} |\mu|^m dx \right)^{\frac{1}{m}} \left( \int_{B_{\rho}(y)} |\nabla u|^q dx \right)^{\frac{2-n}{q}}. \quad (3.10)
\]

Thanks to Lemma 3.3 with \( B_{\rho}(y) \subset \subset B_{2\rho} \subset B_R \) and \( s = q \), we obtain that

\[
\left( \int_{B_{\rho}(y)} |\nabla w|^q dx \right)^{\frac{1}{q}} \leq C \left( \frac{\rho}{R} \right)^{\beta_0-1} \left( \int_{B_{2\rho/3}(y)} |\nabla w|^q dx \right)^{\frac{1}{q}}. \quad (3.11)
\]

Combining (3.10), (3.11) with the fact that

\[
\int_{B_{2\rho/3}(y)} |\nabla w|^q dx \leq C \int_{B_{\rho}(y)} |\nabla u|^q dx,
\]

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we obtain
\[
\left( \int_{B_p(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}} \leq \left( \int_{B_p(y)} |\nabla w|^q \, dx \right)^{\frac{1}{q}} + \left( \int_{B_p(y)} |\nabla u - \nabla w|^q \, dx \right)^{\frac{1}{q}}
\]
\[
\leq C \left( \frac{\rho}{r} \right)^{\frac{n}{q} + \beta_0 - 1} \left( \int_{B_r(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}}
\]
\[
+ C \rho^{\frac{n}{q}} \left( \int_{B_r(y)} |\mu|^m \, dx \right)^{\frac{1}{m}} + C \rho^{\frac{n(p-1)}{q}} \left( \int_{B_r(y)} |\mu|^m \, dx \right)^{\frac{m}{m(p-1)}} \left( \int_{B_r(y)} |\nabla u|^q \, dx \right)^{\frac{2-q}{q}}.
\]

(3.12)

For any \( \varepsilon \in (0, 1) \), using Young’s inequality for the last term on the right hand side of (3.12) and notice that \( \rho < r \), one finds
\[
\left( \int_{B_p(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \frac{\rho}{r} \right)^{\frac{n}{q} + \beta_0 - 1} \left( \int_{B_r(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}}
\]
\[
+ \varepsilon \left( \frac{\rho}{r} \right)^{\frac{n}{q}} \left( \int_{B_r(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}} + C \varepsilon \rho^{\frac{n}{q}} \left( \int_{B_r(y)} |\mu|^m \, dx \right)^{\frac{1}{m}}
\]
\[
\leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{n}{q} + \beta_0 - 1} + \varepsilon \right] \left( \int_{B_r(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}} + C \varepsilon \rho^{\frac{n}{q}} \left( \int_{B_r(y)} |\mu|^m \, dx \right)^{\frac{1}{m}}.
\]

(3.13)

For any \( m + m(p-1)(1 - \beta_0) < \sigma \leq n \), let us set \( \delta = \frac{\sigma - m}{m(p-1)} \), the inequality (3.13) can be rewritten as follows
\[
h(\rho) \leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{n}{q} + \beta_0 - 1} + \varepsilon \right] h(r) + C r^{\frac{n}{q} - \delta} \left( \int_{B_r(y)} |\mu|^m \, dx \right)^{\frac{1}{m(p-1)}}
\]
\[
\leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{n}{q} + \beta_0 - 1} + \varepsilon \right] h(r) + C r^{\frac{n}{q} - \delta} \left( M^D_\sigma(|\mu|^m) \right)^{\frac{1}{m(p-1)}}.
\]

where the function \( h : [0, D_0] \to [0, \infty) \) defined by
\[
h(\rho) = \left( \int_{B_{\rho\varepsilon}(y)} |\nabla u|^q \, dx \right)^{\frac{1}{q}}, \quad \rho > 0.
\]

(3.14)

Applying Lemma 3.3 with \( Q = C \left( M^D_\sigma(|\mu|^m) \right)^{\frac{1}{m(p-1)}} \), \( \alpha = \frac{n}{q} + \beta_0 - 1 \) and \( \beta = \frac{n}{q} - \delta \), there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( 0 < \rho < r \leq D_0 \) there holds
\[
h(\rho) \leq C \left[ \left( \frac{\rho}{r} \right)^{\frac{n}{q} - \delta} h(r) + \rho^{\frac{n}{q} - \delta} \left( M^D_\sigma(|\mu|^m) \right)^{\frac{1}{m(p-1)}} \right],
\]

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and we thus get
\[
\left( \int_{B_{\rho}(y)} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \rho^{\frac{n}{q} - \delta} \left[ \left( \frac{1}{D_0} \right)^{\frac{n}{q} - \delta} \left( \int_{\Omega} |\nabla u|^q dx \right)^{\frac{1}{q}} + \left( M_{D_0}(|\mu|^m) \right)^{\frac{1}{m(p-1)}} \right].
\]
(3.15)

According to Hölder’s inequality and (3.1) in Lemma 3.1, it gives that
\[
\left( \frac{1}{D_0} \right)^{\frac{n}{q} - \delta} \left( \int_{\Omega} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \left( \frac{1}{D_0} \right)^{\frac{n}{q} - \delta} D_0^{\frac{n}{q} - \delta} \left( \int_{\Omega} |\nabla u|^{\frac{n}{m(p-1)} m} dx \right)^{\frac{1}{m(p-1)}} 
\leq C \left( M_{D_0}(|\mu|^m) \right)^{\frac{1}{m(p-1)}}.
\]
(3.16)

From (3.15) and (3.16) we may conclude that (3.9) holds.

3.3 Boundary estimates

In the remaining part of this section, we are able to deal with the comparison results near the boundary of domain. Under the hypothesis that \( \mathbb{R}^n \setminus \Omega \) is uniformly \( p \)-thickness with constants \( c_0, r_0 \) as in (2.1), it is possible to prove estimates similar to what obtained in the interior of domain.

Let \( x_0 \in \partial \Omega \) be a boundary point and \( R \in (0, r_0/2) \), we set \( \Omega_{2R} = B_{2R}(x_0) \cap \Omega \). For any \( u \in W^{1,p}_{loc}(\Omega) \) being a solution to equation (1.1), let \( w \in W^{1,p}_{0}(\Omega_{2R}) + u \) be the unique solution to the following equation
\[-\text{div}(A(x, \nabla w)) = 0 \quad \text{in} \quad \Omega_{2R}, \quad w = u \quad \text{on} \quad \partial \Omega_{2R}.\]
(3.17)

Lemma 3.8 Let \( u \in W^{1,p}_{loc}(\Omega) \) be a solution to equation (1.1) and \( w \in W^{1,p}_{0}(\Omega_{2R}) + u \) be the unique solution to equation (3.17). There exists a constant \( \Theta_0 > p \) such that
\[
\left( \int_{B_{\rho}(y)} |\nabla w|^{\Theta_0} dx \right)^{\frac{1}{\Theta_0}} \leq C \left( \int_{B_{2\rho}(y)} |\nabla w|^{p-1} dx \right)^{\frac{1}{p-1}},
\]
(3.18)
for all \( B_{2\rho}(y) \subset B_{2R}(x_0) \) and the positive constant \( C \) depends on \( n, p, \Lambda \).

We remark that in several articles such as [56], the reverse Hölder are usually stated as the following form
\[
\left( \int_{B_{\rho/2}(y)} |\nabla w|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B_{3\rho/2}(y)} |\nabla w|^{p-1} dx \right)^{\frac{1}{p-1}},
\]
for all \( B_{3\rho}(y) \subset B_{2R}(x_0) \). However, we can prove (3.18) by the same technique as the proof of [64, Lemma 3.5]. We next state the counterpart of Lemmas 3.4 up to the boundary of the domain \( \Omega \).
Lemma 3.9 Let $1 < p \leq \frac{3n-2}{2n-1}$ and $\mu \in L^m(B_{2R})$ for some $m \in (m^*, n)$. Assume that $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a solution to equation (1.1) and $w \in W^{1,p}_0(\Omega_{2R}) + u$ be the unique solution to equation (3.17). For any $q$ satisfying the following condition

$$\frac{n}{2n-1} < q < \frac{nm(p-1)}{n-m},$$

there exists a constant $C > 0$ only depending on $n, p, q$ and $m$ such that

$$\left(\int_{B_{2R}(x_0)} |\nabla u - \nabla w|^q dx\right)^\frac{1}{q} \leq C \left(\int_{B_{2R}(x_0)} |\mu|^m dx\right)^\frac{1}{m(p-1)} + C \left(\int_{B_{2R}(x_0)} |\mu|^m dx\right)^{\frac{1}{m}} \left(\int_{B_{2R}(x_0)} |\nabla u|^q dx\right)^{\frac{2-p}{q}}.$$

Lemma 3.10 Let $w$ be solution to (3.17). Then, there exist two constants $\beta_0 = \beta_0(n, p, \Lambda) \in (0, 1/2]$ and $C_1 = C_1(n, p, \Lambda) > 0$ such that

$$\left(\int_{B_\rho(y)} |\nabla w|^p dx\right)^\frac{1}{p} \leq C_1 \left(\frac{\rho}{r}\right)^{\beta_0-1} \left(\int_{B_r(y)} |\nabla w|^p dx\right)^\frac{1}{p}$$

for any $B_\rho(y) \subset B_r(y) \subset B_{2R}(x_0)$. Moreover, for any $s \in (0, p]$ there exists a positive constant $C_2 = C_2(n, p, \Lambda, s)$ such that

$$\left(\int_{B_\rho(y)} |\nabla w|^s dx\right)^\frac{1}{s} \leq C_2 \left(\frac{\rho}{r}\right)^{\beta_0-1} \left(\int_{B_r(y)} |\nabla w|^s dx\right)^\frac{1}{s}$$

for any $B_\rho(y) \subset B_r(y) \subset B_{2R}(x_0)$.

We next state the selection Lemma 3.11 which establishes $L^q$-estimate for gradient of solution $u$ up to the boundary. The proof of such result is very similar and follows the argument of Lemma 3.10.

Lemma 3.11 Let $1 < p \leq \frac{3n-2}{2n-1}$ and $u$ be a solution to equation (1.1) with $\mu \in L^m(\Omega)$ for some $m \in (m^*, n)$. Let $\beta_0 \in (0, 1/2]$ be as in Lemma 3.10. Then, for any $m + m(p-1)(1-\beta_0) < \sigma \leq n$,

there exists a constant $C = C(n, p, \Lambda, c_0, \sigma) > 0$ such that

$$\left(\int_{B_\rho(y)} |\nabla u|^q dx\right)^\frac{1}{q} \leq C \rho^{\sigma-\delta} \left[ M_{\sigma}^{D_0}(|\mu|^m)(y)\right]^{\frac{1}{m(p-1)}},$$

for all $B_\rho(y) \cap \partial \Omega \neq \emptyset$, where $\delta = \frac{\sigma-m}{m(p-1)}$ and $D_0 = \text{diam}(\Omega)$. 

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4 Lorentz and Lorentz-Morrey global gradient estimates

In this section, the proofs of Theorem C and D are shown to give the global Lorentz and Lorentz-Morrey gradient estimates, respectively. To this end, we mainly use the good-\(\lambda\) method. It is worth mentioning that these results are connected to our previous ones in [64, 67], but for \(\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}\).

4.1 Good-\(\lambda\) type bounds

The good-\(\lambda\) method, roughly speaking, the good-\(\lambda\) inequality, allows to derive the local or point-wise estimates between operators, in terms of theoretical measure. It was first used by D. L. Burkholder et al. in [13]. As far as the authors know, gradient estimates of a class of general elliptic equations using this method were strongly applied by many authors. For instance, regularity results obtained in [52] for fractional integral operators, in [49] for fractional maximal operators. Later, several research papers also applied this method in their proofs of gradient estimates, such as [55, 56, 60, 61, 64, 66, 67, 68, 69], etc.

In order to obtain the good-\(\lambda\) type bounds, the main idea is to use the following lemma which is well-known as a substitution of the Calderón-Zygmund-Krylov-Safonov decomposition. The reader is referred to [20] for the proof of this lemma.

**Lemma 4.1** Let \(\varepsilon \in (0, 1), 0 < R_1 < R_2\) and the ball \(Q := B_{R_2}(x_0)\) for some \(x_0 \in \mathbb{R}^n\). Let \(V \subset W \subset Q\) be two measurable sets satisfying two following properties:

i) \(L^n(V) < \varepsilon L^n(B_{R_1})\);

ii) for all \(x \in Q\) and \(r \in (0, R_1]\), if \(L^n(V \cap B_r(x)) \geq \varepsilon L^n(B_r(x))\) then \(B_r(x) \cap Q \subset W\).

Then there exists a positive constant \(C\) depending on \(n\) such that \(L^n(V) \leq C\varepsilon L^n(W)\).

**Proof of Theorem A.** Let \(u\) be the renormalized solution to (1.1) and \(q < \frac{nm(p-1)}{n-m}\). We denote by \(Q\) the ball \(B_{D_0}(x_0)\) with \(x_0 \in \Omega\) fixed and \(D_0 = \text{diam}(\Omega)\). We need to find three constants \(a \in (0, 1), b \in \mathbb{R}\) and \(\varepsilon_0 > 0\) such that (1.3) holds for any \(\lambda > 0\) and \(\varepsilon \in (0, \varepsilon_0)\).

The main idea of the proof is to apply Lemma 4.1 for two measurable sets \(V_{\lambda, \varepsilon}\) and \(W_\lambda\) which are respectively defined by

\[
V_{\lambda, \varepsilon} = \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \varepsilon^{-a} \lambda, (M_m(|\mu|^m))^{\frac{1}{m(p-1)}} \leq \varepsilon^b \lambda \right\} \cap Q,
\]

and

\[
W_\lambda = \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap Q.
\]

for \(\lambda, \varepsilon > 0\). So we need to show that \(V_{\lambda, \varepsilon}\) and \(W_\lambda\) satisfy two hypotheses i) and ii) in Lemma 4.1. For the first one, we prove that

\[
L^n(V_{\lambda, \varepsilon}) \leq C\varepsilon L^n(B_{R_0}(0)),
\]

(4.1)
for all $\lambda > 0$, where $R_0 = \min\{D_0, r_0\}$. Without loss of generality, we may assume that $V_{\lambda, \varepsilon} \neq \emptyset$ which implies that there is $x_1 \in Q$ such that $(M_{\mu}(|\mu|^m)(x_1))^{\frac{1}{m(p-1)}} \leq \varepsilon b\lambda$, which implies from the definition of fractional maximal function $M_{\mu}$ that

$$
\|\mu\|_{L^m(\Omega)} \leq D_0^{\frac{n}{p-1}}(\varepsilon b\lambda)^{p-1}.
$$

(4.2)

Using the boundedness of the Hardy-Littlewood function $M$ from Lebesgue space $L^1(\mathbb{R}^n)$ into Marcinkiewicz space $L^{1,\infty}(\mathbb{R}^n)$ and Hölder’s inequality, one obtains that

$$
\mathcal{L}^n(V_{\lambda, \varepsilon}) \leq \frac{C}{(\varepsilon^{-a}\lambda)^q} \int_{\Omega} |\nabla u|^q dx
\leq \frac{C}{(\varepsilon^{-a}\lambda)^q} D_0^{-\frac{q(n-m)}{m(p-1)}} \left( \int_{\Omega} |\nabla u|^{\frac{nm(p-1)}{n-m}} \right)^{\frac{q}{n-m}},
$$

(4.3)

with notice that $q < \frac{nm(p-1)}{n-m}$. On the other hand, the gradient bound of $u$ in Lemma 3.1 states that

$$
\|\nabla u\|_{L^{\frac{nm(p-1)}{n-m}}(\Omega)} \leq C\|\mu\|_{L^m(\Omega)}^{\frac{1}{q}},
$$

which follows from (4.2) and (4.3) that

$$
\mathcal{L}^n(V_{\lambda, \varepsilon}) \leq \frac{C}{(\varepsilon^{-a}\lambda)^q} D_0^{-\frac{q(n-m)}{m(p-1)}} \left[ D_0^{\frac{n}{p-1}}(\varepsilon b\lambda)^{p-1} \right]^{\frac{q}{n-m}}
\leq C\varepsilon^{(a+b)q} \left( \frac{D_0}{R_0} \right)^{n} \mathcal{L}^n(B_{R_0}).
$$

(4.4)

It is very easy to see from (4.4) that (4.1) holds if provided $(a+b)q \geq 1$, where the constant $C$ depending on $(D_0/R_0)^n$.

In the next step, we must to verify that for all $x \in Q, r \in (0, R_0]$ and $\lambda > 0$, the following statement does hold:

$$
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \geq C\varepsilon \mathcal{L}^n(B_r(x)) \implies B_r(x) \cap Q \subset W_\lambda.
$$

Indeed, let $x \in Q$ and $0 < r \leq R_0$, and by contradiction, let us assume that $B_r(x) \cap Q \cap \hat{W}_\lambda \neq \emptyset$ and $V_{\lambda, \varepsilon} \cap B_r(x) \neq \emptyset$. Then, there exist $x_2, x_3 \in B_r(x) \cap Q$ such that

$$
|M(|\nabla u|^q)(x_2)| \leq \lambda,
$$

(4.5)

and

$$
(M_{\mu}(|\mu|^m)(x_3))^{\frac{1}{m(p-1)}} \leq \varepsilon b\lambda.
$$

(4.6)

The proof is completed by showing that the following estimate

$$
|V_{\lambda, \varepsilon} \cap B_r(x)| < C\varepsilon |B_r(x)|
$$

(4.7)

holds for a constant $C$ depending on $n, p, \Lambda, m, q, c_0$. 

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For every \( y \in B_r(x) \) and \( \rho \geq r \), we note that
\[
B_{\rho}(y) \subset B_{\rho+r}(x) \subset B_{\rho+2r}(x_2) \subset B_{3\rho}(x_2),
\]
which deduces to
\[
\sup_{\rho \geq r} \int_{B_{\rho}(y)} |\nabla u|^q \, dx \leq 3^n \sup_{\rho \geq 3r} \int_{B_{\rho}(x_2)} |\nabla u|^q \, dx \leq 3^n M(|\nabla u|^q)(x_2) \leq 3^n \lambda^q.
\]
Here the last inequality yields from (4.3). Thus,
\[
M(|\nabla u|^q)(y) \leq \max \left\{ \sup_{0 < \rho < r} \int_{B_{\rho}(y)} \chi_{B_{2r}(x)} |\nabla u|^q \, dx; \sup_{\rho \geq r} \int_{B_{\rho}(y)} |\nabla u|^q \, dx \right\}
\leq \max \left\{ M(\chi_{B_{2r}(x)}|\nabla u|^q)(y); 3^n \lambda^q \right\}, \quad \text{for all } y \in B_r(x),
\]
which implies that
\[
\left\{ (M(|\nabla u|^q))^\frac{1}{q} > 3^n \lambda \right\} \cap B_r(x) = \emptyset.
\]
Therefore, by choosing \( \varepsilon_0 \in (0, 1) \) such that \( \varepsilon_0^{-n} > 3^n \), we will get that
\[
V_{\lambda, \varepsilon} \cap B_r(x) = \left\{ (M(\chi_{B_{2r}(x)}|\nabla u|^q))^\frac{1}{q} > \varepsilon^{-n}\lambda, (M_m(|\mu|_m))^{\frac{1}{m-1}} \leq \varepsilon^p \lambda \right\} \cap Q \cap B_r(x)
\]  
(4.8)
for all \( \varepsilon \in (0, \varepsilon_0) \). Let \( u_k \in W^{1, q}_{0}(\Omega) \) be the unique solution to the following problem:
\[
\begin{cases}
-\text{div}(A(x, \nabla u_k)) = \mu_k & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial \Omega,
\end{cases}
\]  
(4.9)
where \( \mu_k = T_k(\mu) \). In order to prove (4.7), for the sake of clarity we will consider two cases: \( B_{4r}(x) \subset \subset \Omega \) and \( B_{4r}(x) \cap \Omega^c \neq \emptyset \).

Let us consider the first case \( B_{4r}(x) \subset \subset \Omega \). Applying Lemma 3.4 for \( w_k \) being the unique solution to:
\[
\begin{cases}
-\text{div}(A(x, \nabla w_k)) = 0, & \text{in } B_{4r}(x), \\
w_k = u_k, & \text{on } \partial B_{4r}(x),
\end{cases}
\]  
(4.10)
with \( \mu = \mu_k \) and \( B_R = B_{4r}(x) \), one has a constant \( C = C(n, p, \Lambda, q, m) > 0 \) such that:
\[
\left( \int_{B_{4r}(x)} |\nabla u_k - \nabla w_k|^q \, dx \right)^{\frac{1}{q}} \leq C [\mathcal{F}_{4r}(\mu_k)]^{\frac{1}{m-1}} + C \mathcal{F}_{4r}(\mu_k) \left( \int_{B_{4r}(x)} |\nabla w_k|^q \, dx \right)^{\frac{2-n}{m}},
\]  
(4.11)
where the function \( \mathcal{F}_{4r} \) is defined by
\[
\mathcal{F}_{4r}(\mu_k) = \left( (4r)^m \int_{B_{4r}(x)} |\mu_k|^m \, dx \right)^{\frac{1}{m}}.
\]
\[20\]
Moreover, applying the reverse Hölder’s inequality in Lemma 3.3 there exists a constant \( \Theta_0 > p \) such that

\[
\left( \frac{1}{B_{2r}(x)} \int |\nabla w_k|^{\Theta_0} dx \right)^{\frac{1}{\Theta_0}} \leq C \left( \frac{1}{B_{4r}(x)} \int |\nabla w_k|^{p-1} dx \right)^{\frac{1}{p-1}} \\
\leq C \left( \frac{1}{B_{4r}(x)} \int |\nabla u_k|^q dx \right)^{\frac{1}{q}} + C \left( \frac{1}{B_{4r}(x)} \int |\nabla u_k - \nabla w_k|^q dx \right)^{\frac{1}{q}},
\]

(4.12)

where the second inequality is obtained by using Hölder’s inequality with notice that \( q > p - 1 \).

On the other hand, it easy to see that the Lebesgue measure of \( V_{\lambda,\varepsilon} \cap B_r(x) \) given in (4.5) can be decomposed as follows

\[
\mathcal{L}^n(V_{\lambda,\varepsilon} \cap B_r(x)) \leq \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla (u - u_k)|^q \right) > 3^{-\frac{1}{q}} \varepsilon^{-a} \lambda \right\} \cap B_r(x) \right) \\
+ \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla (u_k - w_k)|^q \right) > 3^{-\frac{1}{q}} \varepsilon^{-a} \lambda \right\} \cap B_r(x) \right) \\
+ \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla w_k|^q \right) > 3^{-\frac{1}{q}} \varepsilon^{-a} \lambda \right\} \cap B_r(x) \right).
\]

(4.13)

Using the boundedness of the maximal function \( M \) from \( L^s(\mathbb{R}^n) \) into \( L^{s,\infty}(\mathbb{R}^n) \) with \( s = 1, s = \frac{2a}{q} > 1 \) for three terms on the right hand side of (4.13) respectively, one obtains that

\[
\mathcal{L}^n(V_{\lambda,\varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon^{-a} \lambda) q} \int_{B_{2r}(x)} |\nabla u - \nabla u_k|^q dx \\
+ \frac{C r^n}{(\varepsilon^{-a} \lambda) q} \int_{B_{2r}(x)} |\nabla u_k - \nabla w_k|^q dx \\
+ \frac{C r^n}{(\varepsilon^{-a} \lambda) q_0} \int_{B_{2r}(x)} |\nabla w_k|^{\Theta_0} dx.
\]

(4.14)

Combining both estimates (4.11) and (4.12) to (4.14) we get

\[
\mathcal{L}^n(V_{\lambda,\varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon^{-a} \lambda) q} \int_{B_{4r}(x)} |\nabla u - \nabla u_k|^q dx \\
+ \frac{C r^n}{(\varepsilon^{-a} \lambda) q} \left[ \mathcal{F}_{4r}(\mu_k)^{\frac{1}{p-1}} + \mathcal{F}_{4r}(\mu_k) \left( \int_{B_{4r}(x)} |\nabla u_k|^q dx \right)^{\frac{2-q}{q}} \right] \\
+ \frac{C r^n}{(\varepsilon^{-a} \lambda) q_0} \left[ \left( \int_{B_{4r}(x)} |\nabla u_k|^q dx \right)^{\frac{1}{q}} + \mathcal{F}_{4r}(\mu_k)^{\frac{1}{p-1}} \right] \\
+ \mathcal{F}_{4r}(\mu_k) \left( \int_{B_{4r}(x)} |\nabla u_k|^q dx \right)^{\frac{2-q}{q}} \Theta.
\]

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Passing $k \to \infty$ and applying Lemma 3.2 the above inequality becomes
\[
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon^{-a} \lambda)^q} \left[ \mathcal{F}_{4r}(\mu)^{\frac{1}{p-1}} + \mathcal{F}_{4r}(\mu) \left( \int_{B_{4r}(x)} |\nabla u|^q dx \right)^{\frac{2-p}{q}} \right]^q
\]
\[
+ \frac{C r^n}{(\varepsilon^{-a} \lambda)^{\Theta_0}} \left( \int_{B_{4r}(x)} |\nabla u|^q dx \right)^{\frac{1}{q}} + \mathcal{F}_{4r}(\mu) \left( \int_{B_{4r}(x)} |\nabla u|^q dx \right)^{\frac{2-p}{q}} \right]^\Theta_0 .
\] (4.15)

Since $|x - x_2| < r$, it follows $B_{4r}(x) \subset B_{5r}(x_2)$. Thus one obtains from (4.12) that
\[
\left( \int_{B_{4r}(x)} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq \left( \left( \frac{5}{4} \right)^n \int_{B_{5r}(x_2)} |\nabla u|^q dx \right)^{\frac{1}{q}}
\]
\[
\leq C \left( \mathcal{M}(|\nabla u|^q)(x_2) \right)^{\frac{1}{q}} \leq C \lambda .
\] (4.16)

Similarly, from $|x - x_3| < r$, we can get $B_{4r}(x) \subset B_{5r}(x_3)$, it gives from (4.10) that
\[
\mathcal{F}_{4r}(\mu) \leq C \left( \int_{B_{5r}(x_3)} |\mu|^m dx \right)^{\frac{1}{m}}
\]
\[
\leq C \left( \mathcal{M}_n(|\mu|^m)(x_3) \right)^{\frac{1}{m}} \leq C (\varepsilon^b \lambda)^{p-1}.
\] (4.17)

Taking into account (4.10) and (4.17) to (4.15), we may conclude that
\[
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon^{-a} \lambda)^q} \left[ (\varepsilon^b \lambda + (\varepsilon^b \lambda)^{p-1} \lambda^{2-p})^q \right]
\]
\[
+ \frac{C r^n}{(\varepsilon^{-a} \lambda)^{\Theta_0}} \left[ \lambda + (\varepsilon^b \lambda)^{p-1} \lambda^{2-p} \right]^{\Theta_0}
\]
\[
\leq C r^n \left( (\varepsilon^{a+b})^q + (\varepsilon^{a+b(p-1)})^q \right) + C r^n \varepsilon^{a\Theta_0} .
\] (4.18)

We are reduced to proving (4.17) by choosing $a, b$ in (4.26) such that $a\Theta_0 = 1$ and $(a + b(p - 1))q = 1$. Note that with this choice, we also have $a \in (0, 1)$ and $(a+b)q > 1$ which is assumed in (4.1).

We next consider the second case $B_{4r}(x) \cap \Omega^c \neq \emptyset$. Let $x_4 \in \partial \Omega$ such that
\[
|x_4 - x| = \text{dist}(x, \partial \Omega) \leq 4r .
\]

It is easy to see that $B_{4r}(x) \subset B_{5r}(x_4)$. Applying Lemma 3.9 with $v_k$ being the solution to:
\[
\begin{align*}
-\text{div}(A(x, \nabla v_k)) &= 0, & \text{in } B_{5r}(x_4), \\
v_k &= u_k, & \text{on } \partial B_{5r}(x_4),
\end{align*}
\] (4.19)
for \( \mu = \mu_k \) and \( B_{2R} = B_{8r}(x_4) \) and \( u_k \in W^{1,n}_0(\Omega) \) being the solution to (4.9), one has a constant \( C = C(n, p, \Lambda) > 0 \) such that:

\[
\left( \int_{B_{8r}(x_4)} |\nabla u_k - \nabla v_k|^q dx \right)^{\frac{1}{q}} \leq C \left[ \tilde{F}_{8r}(\mu_k) \right]^{\frac{p-1}{p}} + C \tilde{F}_{8r}(\mu_k) \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^{\frac{2-p}{q}},
\]

(4.20)

where the function \( \tilde{F}_{8r} \) is defined by

\[
\tilde{F}_{8r}(\mu_k) = \left( (8r)^n \int_{B_{8r}(x_4)} |\mu_k|^m dx \right)^{\frac{1}{m}}.
\]

Moreover, following the reverse Hölder’s inequality in Lemma 3.9 with \( \rho = 4r \) and notice that \( B_{4r}(x) \subset B_{8r}(x_4) \), one has

\[
\left( \int_{B_{2r}(x)} |\nabla v_k|^{\Theta_0} dx \right)^{\frac{1}{\Theta_0}} \leq C \left( \int_{B_{4r}(x)} |\nabla v_k|^{p-1} dx \right)^{\frac{1}{p-1}} \leq C \left( \int_{B_{8r}(x_4)} |\nabla v_k|^{p-1} dx \right)^{\frac{1}{p-1}},
\]

which follows from Hölder’s inequality that

\[
\left( \int_{B_{2r}(x)} |\nabla v_k|^{\Theta_0} dx \right)^{\frac{1}{\Theta_0}} \leq C \left( \int_{B_{8r}(x_4)} |\nabla v_k|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^{\frac{1}{q}} + C \left( \int_{B_{8r}(x_4)} |\nabla u_k - \nabla v_k|^q dx \right)^{\frac{1}{q}}.
\]

(4.21)

As the proof in the first case, we first obtain the following estimate from (4.8)

\[
\mathcal{L}^n(V_{\lambda,\epsilon} \cap B_r(x)) \leq \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla (u - u_k)|^q \right) \right\} \cap B_r(x) \right)
\]

\[
+ \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla (u_k - v_k)|^q \right) \right\} \cap B_r(x) \right)
\]

\[
+ \mathcal{L}^n \left( \left\{ M \left( \chi_{B_{2r}(x)} |\nabla v_k|^q \right) \right\} \cap B_r(x) \right),
\]

and then using the boundedness of the maximal function \( M \) on the right hand side to yield that

\[
\mathcal{L}^n(V_{\lambda,\epsilon} \cap B_r(x)) \leq \frac{C r^n}{(\epsilon - a \lambda)^q} \int_{B_{2r}(x)} |\nabla u - \nabla u_k|^q dx + \int_{B_{2r}(x)} |\nabla u_k - \nabla v_k|^q dx
\]

\[
+ \frac{C r^n}{(\epsilon - a \lambda)^q} \int_{B_{2r}(x)} |\nabla v_k|^{\Theta_0} dx
\]

\[
\leq \frac{C r^n}{(\epsilon - a \lambda)^q} \int_{B_{8r}(x_4)} |\nabla u - \nabla u_k|^q dx + \int_{B_{8r}(x_4)} |\nabla u_k - \nabla v_k|^q dx
\]

\[
+ \frac{C r^n}{(\epsilon - a \lambda)^q} \int_{B_{2r}(x)} |\nabla v_k|^{\Theta_0} dx.
\]

(4.22)
Taking into account (4.20) and (4.21) to (4.22), there holds
\[
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon - a\lambda)^q} \int_{B_{8r}(x_4)} |\nabla u - \nabla u_k|^q dx \\
+ \frac{C r^n}{(\varepsilon - a\lambda)^q} \left[ \tilde{F}_{8r}(\mu_k)^\frac{p-1}{q} + \tilde{F}_{8r}(\mu_k) \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{2}{q-p} \right] \\
+ \frac{C r^n}{(\varepsilon - a\lambda)^q} \left[ \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{1}{q} + \tilde{F}_{8r}(\mu_k) \right] \\
+ \tilde{F}_{8r}(\mu_k) \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{2}{q-p} \Theta_0 .
\]

Sending \( k \to \infty \) and using Lemma 3.2, the above inequality becomes
\[
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \leq \frac{C r^n}{(\varepsilon - a\lambda)^q} \left[ \tilde{F}_{8r}(\mu_k)^\frac{p-1}{q} + \tilde{F}_{8r}(\mu_k) \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{2}{q-p} \right] \\
+ \frac{C r^n}{(\varepsilon - a\lambda)^q} \left[ \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{1}{q} + \tilde{F}_{8r}(\mu_k) \right] \\
+ \tilde{F}_{8r}(\mu_k) \left( \int_{B_{8r}(x_4)} |\nabla u_k|^q dx \right)^\frac{2}{q-p} \Theta_0 . \quad (4.23)
\]

It is similar to the previous case, we estimate the right hand side of (4.23) by using (4.12) and (4.6) with the fact that
\[
B_{8r}(x_4) \subset B_{13r}(x_2) \cap B_{13r}(x_3).
\]

With this notice, thanks to (4.12) and (4.6) one obtains that
\[
\left( \int_{B_{8r}(x_4)} |\nabla u|^q dx \right)^\frac{1}{q} \leq C \left( \int_{B_{13r}(x_2)} |\nabla u|^q dx \right)^\frac{1}{q} \\
\leq C(M(\|\nabla u\|)(x_2))^\frac{1}{q} \leq C\lambda, \quad (4.24)
\]
and
\[
\tilde{F}_{8r}(\mu) \leq C \left( (13r)^m \int_{B_{13r}(x_3)} |\mu|^m dx \right)^\frac{1}{m} \\
\leq C(M_m(\|\mu\|^m)(x_3))^\frac{1}{m} \leq C(\varepsilon^b\lambda)^{p-1} . \quad (4.25)
\]

Taking into account (4.24) and (4.25) to (4.23), we may conclude that
\[
\mathcal{L}^n(V_{\lambda, \varepsilon} \cap B_r(x)) \leq C r^n \left( \varepsilon^{(a+b)q} + \varepsilon^{(a+b(p-1))q} \right) + C r^n \varepsilon^{a} \Theta_0 , \quad (4.26)
\]
which guarantees (4.7) by the same value of $a$, $b$ as in the first case. This ends the proof of Theorem A.

**Proof of Theorem B.** For any $\lambda > \epsilon^b \cdot \rho^{-\frac{n}{p-1}} \|\nabla u\|_{L^q(Q_2 \cap \Omega)}$, we have the following estimate

\[
\mathcal{L}^n \left( \left\{ (M(\chi_{Q_2}|\nabla u|^q))^{\frac{1}{q}} > \epsilon^{-a} \lambda, (M_m(\chi_{Q_2}|\mu|^m))^{\frac{1}{m(p-1)}} \leq \epsilon^b \lambda \right\} \cap Q_1 \right) \\
\leq \frac{C}{(\epsilon^{-a} \lambda)^q} \int_\Omega \chi_{Q_2} |\nabla u|^q \, dx \\
\leq \frac{C}{(\epsilon^{-a} \epsilon^{-b} \rho^{-\frac{n}{p-1}} \|\nabla u\|_{L^q(Q_2 \cap \Omega)})^q} \int_\Omega \chi_{Q_2} |\nabla u|^q \, dx \\
\leq C \epsilon^{(a+b)q} \mathcal{L}^n(Q_2) \\
\leq C \epsilon \mathcal{L}^n(Q_2).
\]

As in the proof of Theorem A we recall that $a$, $b$ will be chosen such that $(a+b)q \geq 1$ which guarantees the last inequality. The other steps of the proof can be performed by the same way as in Theorem A.

Using aforementioned technique, we next give short proofs of results in two sections as below.

### 4.2 Gradient estimate in Lorentz spaces

**Proof of Theorem C.** In what follows we prove Theorem C only for the case $t \neq \infty$, and for $t = \infty$ the proof is similar. Let us fix $\frac{p}{p-1} < q < \frac{mn(p-1)}{m-m}$. Thanks to Theorem A there exist constants $\Theta_0 > p$, $a = \Theta_0^{-1}$, $b \in \mathbb{R}$, $C > 0$ and $0 < \epsilon_0 < 1$ such that the following inequality

\[
\mathcal{L}^n \left( \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \epsilon^{-a} \lambda, (M_m(|\mu|^m))^{\frac{1}{m(p-1)}} \leq \epsilon^b \lambda \right\} \cap \Omega \right) \\
\leq C \epsilon \mathcal{L}^n \left( \left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap \Omega \right),
\]

(4.27)

holds for any $\epsilon \in (0, \epsilon_0)$ and $\lambda > 0$. By changing of variables from $\lambda$ to $\epsilon^{-a} \lambda$ in the standard definition of Lorentz space, one has

\[
\| (M(|\nabla u|^q))^{\frac{1}{q}} \|_{L^t,1(\Omega)} = s \int_0^\infty \lambda^t \mathcal{L}^n(\left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap \Omega) \frac{d\lambda}{\lambda} \\
= \epsilon^{-at} s \int_0^\infty \lambda^t \mathcal{L}^n(\left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \epsilon^{-a} \lambda \right\} \cap \Omega) \frac{d\lambda}{\lambda}.
\]

(4.28)

Applying (4.27) to (4.28), we obtain that

\[
\| (M(|\nabla u|^q))^{\frac{1}{q}} \|_{L^t,1(\Omega)} \leq C \epsilon^{-at} \frac{s}{\lambda} \int_0^\infty \lambda^t \mathcal{L}^n(\left\{ (M(|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap \Omega) \frac{d\lambda}{\lambda} + C \epsilon^{-at} s \int_0^\infty \lambda^t \mathcal{L}^n(\left\{ (M_m(|\mu|^m))^{\frac{1}{m(p-1)}} > \epsilon^b \lambda \right\} \cap \Omega) \frac{d\lambda}{\lambda}.
\]

(4.29)
Performing change of variables in the second integral on right-hand side of (4.29), one gets

\[
\left\| (\mathcal{M}(|\nabla u|^q))^\frac{1}{q} \right\|_{L^{s,t}(\Omega)}^t \leq C\varepsilon^{-a+\frac{1}{q}}\left\| (\mathcal{M}(|\nabla u|^q))^\frac{1}{q} \right\|_{L^{s,t}(\Omega)}^t \\
+ C\varepsilon^{-a-b}\left\| (\mathcal{M}_m(|\mu|^\alpha))^\frac{1}{m-\beta} \right\|_{L^{s,t}(\Omega)},
\]

which deduces to

\[
\left\| (\mathcal{M}(|\nabla u|^q))^\frac{1}{q} \right\|_{L^{s,t}(\Omega)}^t \leq C\varepsilon^{-a+\frac{1}{q}}\left\| (\mathcal{M}(|\nabla u|^q))^\frac{1}{q} \right\|_{L^{s,t}(\Omega)}^t \\
+ C\varepsilon^{-a-b}\left\| (\mathcal{M}_m(|\mu|^\alpha))^\frac{1}{m-\beta} \right\|_{L^{s,t}(\Omega)}. \tag{4.30}
\]

For any \(0 < s < a^{-1} = \Theta_0\) and \(0 < t < \infty\), we may choose \(\varepsilon \in (0, \varepsilon_0)\) small enough such that \(C\varepsilon^{-a+\frac{1}{q}} \leq 1/2\) and in conclusion we have obtained (1.5).

\[\square\]

### 4.3 Gradient estimate in Lorentz-Morrey spaces

In this subsection, we prove the Lorentz-Morrey gradient estimate for renormalized solution to (1.1). The following standard lemma is useful for our proof.

**Lemma 4.2** Let \(f \in L^{s,\kappa}(\Omega)\) for \(0 < s < \infty\), \(0 < t \leq \infty\) and \(0 < \kappa \leq n\). For \(0 < \sigma \leq \frac{\sigma_0}{n}\), there exists a constant \(C = C(n, s, \kappa, \sigma) > 0\) such that

\[
M_{\sigma}(f)(y) \leq C [M(f)(y)]^{1-\frac{\sigma_0}{\sigma}} \left( \| f \|_{L^{s,\kappa}(\Omega)} \right)^{\frac{\sigma_0}{\sigma}}, \tag{4.31}
\]

for \(\rho > 0\) and \(y \in \Omega\). In particular, there holds

\[
\| M_{\sigma}(f) \|_{L^{\infty}(\Omega)} \leq C \| f \|_{L^{s,\kappa}(\Omega)}, \tag{4.32}
\]

where \(D_0 = \text{diam}(\Omega)\).

**Proof.** Let \(\rho > 0\) and \(y \in \Omega\). For any \(0 < \alpha \leq 1\), we have

\[
\rho^{\alpha-n} \int_{B_{\rho}(y)} f(x) dx = \left( \rho^{-n} \int_{B_{\rho}(y)} f(x) dx \right)^{1-\alpha} \left( \rho^{\alpha-n} \int_{B_{\rho}(y)} f(x) dx \right)^{\alpha} \\
\leq C [M(f)(y)]^{1-\alpha} \left( \rho^{\alpha-n} \rho^{-\alpha} \| f \|_{L^{s,\kappa}(B_{\rho}(y))} \right)^{\alpha} \\
\leq C [M(f)(y)]^{1-\alpha} \left( \rho^{\alpha-n} \| f \|_{L^{s,\kappa}(B_{\rho}(y))} \right)^{\alpha} \\
\leq C [M(f)(y)]^{1-\alpha} \left( \| f \|_{L^{s,\kappa}(\Omega)} \right)^{\alpha}.
\]

Let us take \(\alpha = \frac{\sigma}{n}\), we obtain (4.31). By choosing \(\alpha = 1\) and taking the supremum both sides of this inequality for all \(0 < \rho < D_0\) and \(y \in \Omega\), we obtain (4.32) which completes the proof.

**Proof of Theorem D.** For simplicity of notation, we denote \(B_\rho := B_\rho(x_0)\) and \(B_{10\rho} := B_{10\rho}(x_0)\) with \(0 < \rho < D_0 = \text{diam}(\Omega)\) and \(x_0 \in \Omega\). Thanks to Theorem B
with \( q \in \left(\frac{n}{m-1}, \frac{nm(p-1)}{n-m}\right) \), there exist \( a = a(n,p,\lambda,c_0) \in (0,1), \ b \in \mathbb{R} \) and \( C = C(n,p,\lambda,c_0, D_0/r_0) > 0 \) such that the following estimate holds

\[
L^n \left( \left\{ (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} > \varepsilon^{-a} \lambda \right\} \cap B_{\rho} \right) \leq C \varepsilon L^n \left( \left\{ (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap B_{\rho} \right),
\]

for any \( \varepsilon \in (0,1) \) small enough and \( \lambda > \lambda_0 \), where

\[
\lambda_0 = \varepsilon^{-b} \rho - \frac{\beta}{q} \| \nabla u \|_{L^q(B_{10\rho} \cap \Omega)}.
\]

Thus, for all \( \lambda > \lambda_0 \) it gives

\[
L^n \left( \left\{ (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} > \varepsilon^{-a} \lambda \right\} \cap B_{\rho} \right) \leq C \varepsilon L^n \left( \left\{ (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} > \lambda \right\} \cap B_{\rho} \right) + L^n \left( \left\{ (M_m(\chi_{B_{10\rho}}|\mu|^m))^{\frac{1}{m(p-1)}} > \varepsilon^{b} \lambda \right\} \cap B_{\rho} \right).
\]

For the convenience of the reader, let us denote

\[
\alpha := \frac{m\kappa(p-1)}{\kappa - ms} \quad \text{and} \quad \beta := \frac{m\kappa(p-1)}{\kappa - mt}.
\]

By changing of variables in the definition of Lorentz norm we obtain that

\[
\| (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\beta}(B_{\rho})} \leq C \varepsilon^{-a} \lambda_0^\beta \| \nabla u \|_{L^q(B_{10\rho} \cap \Omega)}^\beta
\]

and

\[
C \varepsilon^{-a+b} \rho_0^\beta + C \varepsilon^{-a+b} \| (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\beta}(B_{\rho})}^\beta
\]

Then, it gives us the estimate:

\[
\| (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\beta}(B_{\rho})} \leq C \varepsilon^{-a} \lambda_0^\beta + C \varepsilon^{-a+b} \| (M(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\beta}(B_{\rho})}^\beta + C \varepsilon^{-a+b} \| (M_m(\chi_{B_{10\rho}}|\mu|^m))^{\frac{1}{m(p-1)}} \|_{L^{\beta}(B_{\rho})}^\beta.
\]
The inequality (4.37) holds for any $\alpha > 0$ and $0 < \beta < \infty$. We remark that this inequality even holds for $\beta = \infty$ by the same method. For $0 < \alpha < \Theta_0 := a^{-1}$, we may choose $\varepsilon \in (0, 1)$ small enough such that $C\varepsilon^{-a+\frac{1}{\alpha}} < 1/2$ in (4.37), then one obtains from (4.33) that
\[
\| (\mathcal{M}(\chi_{B_{10\rho}}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\alpha,\beta}(B_{\rho})} \leq C \rho^{\frac{\mu q}{p} - \frac{\mu}{q}} \| \nabla u \|_{L^{\alpha}(B_{10\rho} \cap \Omega)} + C \| (\mathcal{M}(\chi_{B_{10\rho}}|\mu|^m)) \|_{L^{\alpha,\beta}(B_{\rho})}^{\frac{1}{m(p-1)}}
\] (4.38)
Applying Lemma 3.7 with $\sigma = \frac{\kappa}{s}$ and $\delta = \frac{s-m\kappa}{m\kappa(p-1)}$ satisfying
\[
m + m(p-1)(1 - \beta_0) < \frac{\kappa}{s} \leq n,
\]
which deduces from (4.38) that
\[
\rho^{\frac{\mu q}{p} - \frac{\mu}{q}} \| (\mathcal{M}(\chi_{B_{10\rho}(x_0)}|\nabla u|^q))^{\frac{1}{q}} \|_{L^{\alpha,\beta}(B_{\rho}(x_0))} \leq C \| \mathcal{M}_0^\alpha (|\mu|^m) \|_{L^{\infty}(\Omega)}^{\frac{1}{m(p-1)}}
\] (4.39)
Here it is very easy to check that $\delta \alpha = \kappa$ using the definition of $\alpha$ in (4.35). By taking the supremum both sides of (4.39) for $0 < \rho < D_0$ and $x_0 \in \Omega$, it guarantees that
\[
\| \nabla u \|_{L^{\alpha,\beta,\kappa}(\Omega)} \leq C(I_1 + I_2),
\] (4.40)
where $I_1$ and $I_2$ are defined by
\[
I_1 = \| \mathcal{M}_0^\alpha (|\mu|^m) \|_{L^{\infty}(\Omega)}^{\frac{1}{m(p-1)}},
\]
\[
I_2 = \sup_{0 < \rho < D_0, x_0 \in \Omega} \rho^{\frac{\mu q}{p} - \frac{\mu}{q}} \| (\mathcal{M}(\chi_{B_{10\rho}(x_0)}|\mu|^m)) \|_{L^{\alpha,\beta}(B_{\rho}(x_0))}^{\frac{1}{m(p-1)}}.
\] (4.41)
Applying (4.31) in Lemma 1.2, one easily estimates $I_1$ as
\[
I_1 \leq C \| |\mu|^m \|_{L^{s,t,\kappa}(\Omega)}^{\frac{1}{m(p-1)}}.
\] (4.42)
It is necessary to estimate $I_2$ by the same norm in $L^{s,t,\kappa}(\Omega)$. For any $y \in B_{\rho}(x_0)$, thanks to (4.32) in Lemma 1.2 we have
\[
(\mathcal{M}(\chi_{B_{10\rho}(x_0)}|\mu|^m))(y) \leq C \left[ (\mathcal{M}(\chi_{B_{10\rho}(x_0)}|\mu|^m))(y) \right]^{1 - \frac{\mu q}{p}} \left( \| |\mu|^m \|_{L^{s,t,\kappa}(B_{10\rho}(x_0))} \right)^{\frac{\mu q}{p}},
\]
which implies that
\[
\|M_m(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}} \leq C \|M(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
\[
\leq C \|M(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
\[
= C \|M(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
Using the boundedness of the Hardy-Littlewood maximal function \(M\), one obtains that
\[
\|M_m(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))} \leq C \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
By the definition of Lorentz-Morrey norm with remark that \(\frac{s}{\alpha} = \frac{\kappa-ms}{m(p-1)\kappa}\), we deduce from the above inequality that
\[
\|M_m(\chi_{B_{10^\rho}(x_0)}|\mu|^m)\|_{L^{\alpha,\beta}(B_\rho(x_0))} \leq C \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
Combining \((4.41)\) and \((4.43)\), we get that
\[
I_2 \leq C \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))} \|\mu\|_{L^{\alpha,\beta}(B_\rho(x_0))}^{\frac{1}{m(p-1)}}
\]
Taking into account \((A.20)\) and \((A.23)\), we may conclude \((1.7)\). Finally, we note that all hypotheses that we need on parameters \(s\), \(\kappa\) are
\[
m + m(p-1)(1-\beta) = \frac{\kappa}{s} < n, \quad 0 < \frac{m(p-1)s\kappa}{\kappa-ms} < \Theta_0,
\]
which are equivalent to \((1.6)\). Moreover, we remark that
\[
\frac{\kappa}{n} < \frac{\kappa\Theta_0}{m\Theta_0 + m(p-1)\kappa} \iff \frac{\kappa}{m(p-1)} < \frac{(n-m)\Theta_0}{m(p-1)}
\]
and
\[
\frac{\kappa}{n} < \frac{\kappa}{m + m(p-1)(1-\beta)} \iff m < \frac{n}{1 + (p-1)(1-\beta)},
\]
which is always true for \(m < m^{**}\). The proof is complete.
Appendix A  Proof of Lemma 3.4

**Proof.** Here and in the following, for simplicity, let us denote \( v = u - w \). Then, for any \( q \in (0,1) \) and \( s > 0 \), using the Hölder inequality, one has

\[
\int_{B_R} |\nabla v|^q dx = \int_{B_R} |\nabla v|^q |v|^{-\frac{q}{q-1}} |v|^s dx \\
\leq \left( \int_{B_R} |\nabla v||v|^{-\frac{q}{q-1}} dx \right)^q \left( \int_{B_R} |v|^{1-\frac{s}{q}} dx \right)^{1-q}.
\]

(A.1)

Moreover, if \( q \) and \( s \) satisfy two conditions

\[
0 < s < q < 1 \text{ and } 1 \leq \frac{qs}{(q-s)(1-q)} \leq \frac{n}{n-1},
\]

(A.2)

then using Sobolev’s inequality for the function \( |v|^{1-\frac{s}{q}} \), there holds

\[
\int_{B_R} |v|^{\frac{q}{(q-s)(1-q)}} dx = \int_{B_R} \left( |v|^{1-\frac{s}{q}} \right)^{\frac{q}{(q-s)(1-q)}} dx \\
\leq C \left( \int_{B_R} |\nabla \left( |v|^{1-\frac{s}{q}} \right)| dx \right)^{\frac{q}{(q-s)(1-q)}} \\
\leq C \left( \int_{B_R} |\nabla v||v|^{-\frac{s}{q}} dx \right)^{\frac{q}{(q-s)(1-q)}}.
\]

(A.3)

Taking into account the estimate (A.3), we deduce from (A.1) that

\[
\left( \int_{B_R} |\nabla v|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B_R} |\nabla v||v|^{-\frac{s}{q}} dx \right)^{\frac{q}{q-1}}.
\]

(A.4)

The main point of proof we highlight here is to bound the following quantity

\[
K = \int_{B_R} |\nabla v||v|^{-\frac{s}{q}} dx,
\]

under conditions (A.2). We first remark that for \( 1 < p < 2 \), there holds

\[
|\nabla v| \leq C \left( (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla v|^\frac{p}{2} + (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla v||\nabla u|^{\frac{2-p}{2}} \right),
\]

which implies that

\[
K \leq C \int_{B_R} \left( |v|^{-\frac{q}{2}} \mathcal{G}(u,w)^\frac{p}{2} + |v|^{-\frac{q}{2}} \mathcal{G}(u,w)^\frac{1}{2} |\nabla u|^{\frac{2-p}{2}} \right) dx,
\]

(A.5)

where the function \( \mathcal{G} \) is defined by

\[
\mathcal{G}(u,w) = (|\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} |\nabla u - \nabla w|^2.
\]

(A.6)
In order to estimate $K$, we apply the variational form of equations (1.1) and (3.2) with a consistent test function. By subtracting two variational formulas of (1.1) and (3.2), we obtain that the following equation

$$
\hat{B}_R (A(x, \nabla u) - A(x, \nabla w), \nabla \varphi) dx = \hat{B}_R \varphi \mu dx,
$$

(A.7)

holds for any $\varphi \in W^{1,p}_0(B_R)$. The main idea of proof is to apply an appropriate test function $\varphi$ in (A.7).

For $k > 0$ and $\varepsilon > 0$, let us consider the truncation function defined for every $t \in \mathbb{R}$ as follows

$$T_k(t) = \max \{-k, \min \{k, t\}\},
$$

and

$$T^\varepsilon_k(t) = (k + \varepsilon) \text{sign}(t) \max \left\{0, \min \left\{1, \frac{|t| - \varepsilon}{k}\right\}\right\}.
$$

For any $\alpha > 0$, one can see that $T^\varepsilon_k(|v|^\alpha - 1)v \in W^{1,p}_0(B_R)$. Applying techniques related to the method we use (see [45] or [56] for some examples of formally test functions), we may choose this function as a test function $\varphi$ in (A.7) and notice that

$$|T^\varepsilon_k(t)| \leq |T_k(t)|
$$

for all $t \in \mathbb{R}$, to obtain that

$$\hat{\{x \in B_R : |v|^\alpha - 1 G(u, v) \leq \lambda\}} |v|^\alpha - 1 G(u, v) dx \leq C \hat{B}_R |\mu| dx,
$$

(A.8)

Let $m \in (1, n)$ and $m' = \frac{m}{m-1}$ denotes the Hölder conjugate exponent to $m$. Thanks to Hölder inequality, it follows from (A.8) that

$$\int_{\{x \in B_R : |v| < k\}} |v|^\alpha - 1 G(u, v) dx \leq C \left( \int_{B_R} (T_k^\alpha(|v|^\alpha))^{m'} dx \right)^{\frac{1}{m}} \left( \int_{B_R} |\mu|^m dx \right)^{\frac{1}{m'}}.
$$

Moreover, using an interesting property of $T_k^\alpha$ that

$$T_k^\alpha(|v|^\alpha) \leq k^{\alpha - \gamma} |v|^\gamma,
$$

for any $\gamma \in (0, \alpha)$, it deduces to

$$\int_{\{x \in B_R : |v| < k\}} |v|^\alpha - 1 G(u, v) dx \leq C k^{\alpha - \gamma} \left( \int_{B_R} |v|^\gamma m' dx \right)^{\frac{1}{m'}} \left( \int_{B_R} |\mu|^m dx \right)^{\frac{1}{m}}.
$$

(A.9)

For $k, \lambda > 0$, let us now introduce the following function

$$\mathcal{H}(k, \lambda) = \mathcal{L}^n \left( \{x \in B_R : |v| > k; |v|^\alpha - 1 G(u, v) > \lambda\} \right),
$$

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which is non increasing in the second variable $\lambda$. This follows that
\[
\mathcal{H}(0, \lambda) \leq \frac{1}{\lambda} \int_0^\lambda \mathcal{H}(0, t) dt \leq \mathcal{H}(k, 0) + \frac{1}{\lambda} \int_0^\lambda (\mathcal{H}(0, t) - \mathcal{H}(k, t)) dt,
\]
which can be rewritten as follows
\[
\mathcal{L}^n \left( \{ x \in B_R : |v|^{\alpha-1} \mathcal{G}(u, w) > \lambda \} \right) \leq \mathcal{L}^n \left( \{ x \in B_R : |v| > k \} \right) + \frac{1}{\lambda} \int_0^\lambda \mathcal{L}^n \left( \{ x \in B_R : |v| \leq k, |v|^{\alpha-1} \mathcal{G}(u, w) > t \} \right) dt
\]
\[
\leq k^{-\beta} \|v\|_{L^\beta(B_R)}^\beta + \frac{1}{\lambda} \int_{\{x \in B_R : |v| \leq k\}} |v|^{\alpha-1} \mathcal{G}(u, w) dx,
\]
where $\beta = \gamma m'$. Combining between (A.9) and (A.10), one gets that
\[
\mathcal{L}^n \left( \{ x \in B_R : |v|^{\alpha-1} \mathcal{G}(u, w) > \lambda \} \right) \leq k^{-\beta} \|v\|_{L^\beta(B_R)}^\beta + \frac{1}{\lambda} C k^{\alpha-\gamma} \|v\|_{L^\gamma(B_R)}^\gamma \|\mu\|_{L^m(B_R)}. \quad (A.11)
\]
In order to balance contributions coming from two terms on the right hand side of (A.11), we may choose
\[
\begin{align*}
k = \left( \lambda \|v\|_{L^\beta(B_R)}^\beta \|\mu\|_{L^m(B_R)}^{-1} \right)^{1/\alpha+\beta-\gamma}.
\end{align*}
\]
Thus for any $\lambda > 0$, there holds
\[
\lambda \mathcal{L}^n \left( \{ x \in B_R : |v|^{\alpha-1} \mathcal{G}(u, w) > \lambda \} \right) \leq C \|\mu\|_{L^m(B_R)} \|v\|_{L^\beta(B_R)}^\alpha,
\]
which leads to
\[
\| |v|^{\alpha-1} \mathcal{G}(u, w) \|_{L^{\alpha+\beta-\gamma, \infty}(B_R)} \leq C \|\mu\|_{L^m(B_R)} \|v\|_{L^\beta(B_R)}^\alpha, \quad (A.12)
\]
for all $\alpha, \beta > 0$. For any $0 < \eta < \frac{\beta}{\alpha+\beta-\gamma}$, using the boundedness property from Marcinkiewicz space $L^{\frac{\eta(\alpha+\beta-\gamma)}{\alpha+\beta}}(B_R)$ to Lebesgue space $L^1(B_R)$ (see for instance [31] or [65, Lemma 2.1]) and (A.12), we obtain that
\[
\begin{align*}
\int_{B_R} |v|^{\eta(\alpha-1)} \mathcal{G}(u, w) \eta dx &\leq C \mathcal{L}^n(B_R)^{1-\frac{\eta(\alpha+\beta-\gamma)}{\beta}} \| |v|^{\eta(\alpha-1)} \mathcal{G}(u, w) \eta \|_{L^{\frac{\eta(\alpha+\beta-\gamma)}{\alpha+\beta}}(B_R)}^{\beta} \\
&\leq CR^{-\frac{\eta(\alpha+\beta-\gamma)}{\beta}} \| |v|^{\alpha-1} \mathcal{G}(u, w) \|_{L^{\frac{\eta(\alpha+\beta-\gamma)}{\alpha+\beta}}(B_R)}^{\beta} \\
&\leq CR^{-\frac{\eta(\alpha+\beta-\gamma)}{\beta}} \|\mu\|_{L^m(B_R)} \|v\|_{L^\beta(B_R)}^{\eta}.
\end{align*}
\]
Moreover, it is similar to estimate (A.13) using Sobolev’s inequality, one easily concludes
\[
\|v\|_{L^\beta(B_R)} \leq C \|\nabla \left( |v|^{1-\frac{\eta}{\beta}} \right) \|_{L^1(B_R)}^{\frac{\beta}{\eta+\beta}} \leq C \|\nabla \left( |v|^{1-\frac{\eta}{\beta}} \right) \|_{L^1(B_R)}^{\frac{\beta}{\eta+\beta}},
\]
which can be rewritten as follows
\[
\mathcal{L}^n \left( \{ x \in B_R : |v|^{\alpha-1} \mathcal{G}(u, w) > \lambda \} \right) \leq \mathcal{L}^n \left( \{ x \in B_R : |v| > k \} \right) + \frac{1}{\lambda} \int_0^\lambda \mathcal{L}^n \left( \{ x \in B_R : |v| \leq k, |v|^{\alpha-1} \mathcal{G}(u, w) > t \} \right) dt
\]
\[
\leq k^{-\beta} \|v\|_{L^\beta(B_R)}^\beta + \frac{1}{\lambda} \int_{\{x \in B_R : |v| \leq k\}} |v|^{\alpha-1} \mathcal{G}(u, w) dx,
\]
for any \( \beta \) satisfying the following constraint
\[
1 \leq \frac{q\beta}{q-s} \leq \frac{n}{n-1}.
\] (A.14)

We remark here \( \frac{1}{m} = \frac{\beta - \gamma}{\beta} \), it deduces from (A.13) and the above inequality that
\[
\left( \int_{B_R} |v|^\eta G(u, w)^q dx \right)^{\frac{1}{\eta}} \leq CR^n \left( \frac{\eta - \alpha + \frac{\beta - \gamma}{\beta}}{\eta - \alpha} \right)^{-1} \left\| \mu \right\|_{L^m(B_R)} \left\| \frac{u}{q-s} \right\|_{L^\infty} \left( \frac{\eta}{\eta - \alpha} \right)^{-1} F_R(\mu) K^\frac{q}{q-s},
\] (A.15)

where \( F_R(\mu) \) defined as in (3.6).

Let us now comeback to estimate \( K \). We can rewrite (A.5) as
\[
K \leq C(I_1 + I_2),
\] where \( I_1 \) and \( I_2 \) are defined by
\[
I_1 = \int_{B_R} |v|^{\frac{q}{\eta}} G(u, v)^{\frac{1}{p}} dx, \quad I_2 = \int_{B_R} |v|^{\frac{q}{\eta}} G(u, v)^{\frac{1}{p}} |\nabla u|^{\frac{2-p}{2}} dx.
\] (A.16)
The first term \( I_1 \) can be estimated by applying (A.15) with \( \eta = \frac{1}{p} \) and \( \alpha = 1 - \frac{sp}{q} \), under the following conditions
\[
s < \frac{q}{p} \quad \text{and} \quad \frac{1}{p} < 1 - \frac{2p}{q} + \beta - \gamma.
\] (A.17)

Proceeding in the calculations under these above conditions and with notation \( \delta_1 = n \left( p - \frac{1-2p}{p} \right) - 1 \), one has
\[
I_1 = \int_{B_R} |v|^{\frac{q}{\eta}} G(u, v)^{\frac{1}{p}} dx \leq C\varepsilon K^\frac{q}{p(\eta - s)} \left( R^\delta_1 F_R(\mu) \right)^{\frac{1}{p}}. \tag{A.18}
\]
Let us recall a type of Young inequality for two positive number \( a, b \) below. For \( r \in (0, 1) \) and \( \varepsilon > 0 \), there exists a positive constant \( c(\varepsilon, r) \) only depending on \( r \) and \( \varepsilon \) such that
\[
a^r b \leq c(\varepsilon, r) b^{1-r}.
\] (A.19)

Using this inequality with \( r = \frac{2 - sp}{p(q-s)} \in (0, 1) \), we deduce from (A.18) that
\[
I_1 \leq C\varepsilon K + c(\varepsilon) \left( R^\delta_1 F_R(\mu) \right)^\frac{1}{p(1-r)} = C\varepsilon K + c(\varepsilon) \left( R^\delta_1 F_R(\mu) \right)^\frac{q}{n(p-1)}. \tag{A.20}
\]

In order to estimate the second term \( I_2 \), we first use Hölder inequality with \( q > \frac{2-p}{2} \) as follows
\[
I_2 = \int_{B_R} |v|^{\frac{q}{\eta}} G(u, v)^{\frac{1}{p}} |\nabla u|^{\frac{2-p}{2}} dx
\leq \left( \int_{B_R} |v|^{\frac{2q}{q-s}} G(u, v)^\frac{q}{q-s} dx \right)^{\frac{2q-p}{2q}} \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{2q}}. \tag{A.21}
\]
Applying (A.15) with $\eta = \frac{q}{2q+p-2}$, $\alpha = 1 - \frac{2s}{q}$ and let us denote

$$\delta_2 = n \left( \frac{2q + p - 2}{q} - \frac{1 - \frac{2s}{q}}{\beta} \right) - 1,$$

we obtain from (A.21) that the following inequality

$$I_2 \leq C \kappa^{\frac{q-2s}{2(q-s)}} \left( R^{\delta_2} F_R(\mu) \right)^{\frac{1}{q}} \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{2q}}$$

holds under the conditions

$$s < \frac{q}{2} \quad \text{and} \quad \frac{q}{2q + p - 2} < \frac{1 - \frac{2s}{q}}{\beta - \gamma}. \quad (A.22)$$

Applying Young inequality (A.19) again with $r = \frac{q-2s}{2(q-s)} \in (0,1)$, one has

$$I_2 \leq C \varepsilon \kappa + c(\varepsilon) \left[ \left( R^{\delta_2} F_R(\mu) \right)^{\frac{1}{q}} \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{2q}} \right]. \quad (A.23)$$

Combining between (A.16) to (A.20) and (A.23) with a small enough $\varepsilon$, we obtain that

$$\kappa \leq C \left( R^{\delta_1} F_R(\mu) \right)^{\frac{q-s}{q}} + C \left( R^{\delta_2} F_R(\mu) \right) \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{q}}. \quad (A.24)$$

Thanks to (A.4) and (A.24), it follows that

$$\left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \left( R^{\delta_1} F_R(\mu) \right)^{\frac{1}{q}} + C \left( R^{\delta_2} F_R(\mu) \right) \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{q}}$$

which leads to

$$\left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \left( R^{\delta_1} F_R(\mu) \right)^{\frac{1}{q}} + R^{\delta_2} F_R(\mu) \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{q}}. \quad (A.25)$$

Combining all conditions on $\beta$, it is easy to see that the optimal choice of $\beta$ is the largest value in (A.14). So we can fix $\beta = \frac{q-s}{n-1}$. By taking $s = \frac{nq(1-q)}{n-1}$ we may obtain that

$$\frac{\delta_1}{p-1} - \frac{n}{q} = \frac{\delta_2}{q} - \frac{n}{q} + \frac{n(2-p)}{q} = 0,$$

which guarantees that

$$\left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{1}{q}} \leq C \left( F_R(\mu) \right)^{\frac{1}{q}} + F_R(\mu) \left( \int_{B_R} |\nabla u|^q dx \right)^{\frac{2-p}{q}}. \quad (A.25)$$

It is worth mentioning that in this case $\frac{q-s}{q} = \frac{n}{n-1}$ which yields the condition (A.2). The final task is to set a range of $q$ such that (A.17) and (A.22) satisfy.
By simple computation with notices that $\beta = \frac{nm}{n-q}$ and $s = (1-q)\beta$, one obtains that (A.25) holds under several conditions as follows

\[
s < \frac{q}{2} \quad \text{and} \quad q > \frac{2-p}{2} \iff q > \max \left\{ \frac{n}{2n-1}, \frac{2-p}{2} \right\} = \frac{n}{2n-1},
\]

\[
\frac{1}{p} < \frac{1}{1 - \frac{nm}{n} + \beta - \gamma} \iff q < \frac{nm(p-1)}{n-m},
\]

\[
\frac{q}{2q + p - 2} < \frac{1}{1 - \frac{2p}{q} + \beta - \gamma} \iff q < \frac{nm(p-1)}{n-m}.
\]

We remark moreover that

\[
\frac{n}{2n-1} < \frac{nm(p-1)}{n-m} \iff m > m^*.
\]

Hence, we may conclude the proof that all conditions in (A.17) and (A.22) hold for any $q$ satisfying (3.4). 

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