MAGNETIC BRAIDING AND QUASI-SEPARATRIX LAYERS

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ABSTRACT

The squashing factor $Q$, a property of the magnetic field line mapping, has been suggested as an indicator for the formation of current sheets, and subsequently magnetic reconnection, in astrophysical plasmas. Here, we test this hypothesis for a particular class of braided magnetic fields which serve as a model for solar coronal loops. We explore the relationship between quasi-separatrix layers (QSLs), that is, layer-like structures with high $Q$ value, electric currents, and integrated parallel currents; the latter being a quantity closely related to the reconnection rate. It is found that as the degree of braiding of the magnetic field is increased, the maximum values of $Q$ increase exponentially. At the same time, the distribution of $Q$ becomes increasingly filamentary, with the width of the high-$Q$ layers exponentially decreasing. This is accompanied by an increase in the number of layers so that as the field is increasingly braided the volume becomes occupied by a myriad of thin QSLs. QSLs are not found to be good predictors of current features in this class of braided fields. Indeed, despite the presence of multiple QSLs, the current associated with the field remains smooth and large scale under ideal relaxation; the field dynamically adjusts to a smooth equilibrium. Regions of high $Q$ are found to be better related to regions of high integrated parallel current than to actual current sheets.

Key words: magnetic fields – MHD – Sun: corona

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1. INTRODUCTION

Several different mechanisms have been proposed as explanations for solar coronal heating and, indeed, it is likely that a number of different heating processes are responsible for the extreme temperature values found. Much of the corona is considered to be close to force-free, that is, in a state satisfying $\mathbf{j} \times \mathbf{B} \simeq 0$. This consideration led to the proposal of one of the earliest and most debated coronal heating theories, that of Parker’s original paper (Parker 1972). Parker theorized that the coronal magnetic field is unable to relax to a smooth force-free equilibrium following arbitrary perturbations of the field via footpoint motions and the consequence is a development of tangential discontinuities in the field. Magnetic reconnection may then occur across the resulting current sheets. Since Parker’s original paper a multitude of arguments for and against the hypothesis has been given but no unanimity presently exists (e.g., van Ballegooijen 1985; Galsgaard & Nordlund 1996; Ng & Bhattacharjee 1998; Craig & Sneyd 2005).

One technique presently employed for predicting current sheet formation in continuous fields is to examine a quantity known as the squashing factor, $Q$ (Titov et al. 2002; see the Appendix for further details of the function itself). This is a measure for continuous magnetic fields which describes the level of squashing of an infinitesimal flux tube; $Q$ is large in regions where the footpoint mapping of the field is highly distorted. Regions of very high $Q$ outline so-called quasi-separatrix layers, or QSLs (Priest & Démoulin 1995), can be considered as an analog of the separatrix surface in configurations with no null points. QSLs are thought to be sites of preferential current sheet formation (e.g., Démoulin et al. 1996; Titov et al. 2002), and a number of numerical experiments have supported this hypothesis (e.g., Longcope & Strauss 1994; Milano et al. 1999; Galsgaard et al. 2003; Aulanier et al. 2005). In addition, observational studies have attempted to examine the relation between QSLs and current sheets (e.g., Schmieder et al. 1997; Fletcher et al. 2001), although this is difficult due to the spatial resolution and quality of the data required for an accurate depiction of the QSLs.

In an earlier paper (Wilmot-Smith et al. 2009), we considered a class of braided magnetic fields. The braids were initially given in a closed form (and not in MHD equilibrium), then inserted into a numerical scheme and underwent magnetic relaxation toward a force-free equilibrium. In that paper, the current structure of the fields was looked at in some detail. The results are summarized for convenience and put into context in Section 3. Here, we return to these models for braided fields in order to examine the relationship between QSLs, currents, and a quantity which is important for magnetic reconnection in three dimensions, the parallel electric current integrated along magnetic field lines. As such, we begin by describing the braided fields under consideration.

2. MODEL FOR A CLASS OF BRAIDED FIELDS

In Wilmot-Smith et al. 2009, (hereafter Paper I), we introduced a class of braided fields designed to model magnetic fields in solar coronal loops. The fields are given as $E^n$, $n \in \mathbb{Z}^+$, where $E^n$ can be considered as a concatenation of $n$ times the “elementary” field $E$. The braided field $E^3$ is modeled on the pigtails as illustrated in Figure 1.

The nature of $E$ and the construction mechanism for $E^n$ may be seen from that figure. The field $E$ consists of two localized regions of twist in an otherwise uniform field. These two regions are of opposite sign but are of the same underlying form—the superposition onto a uniform vertical field, $b_0 \hat{z}$, of a toroidal flux ring given by

$$
\mathbf{B}_c = 2b_0 k \frac{r}{a} \exp \left( -\frac{r^2}{a^2} - \frac{z^2}{b^2} \right) \mathbf{e}_\phi.
$$
In the closed form; and hence an exact expression for the toroidal flux at the point \((x_c, y_c, z_c)\) is:

\[
B_c = 2 \frac{b_0 k}{a} \exp \left( \frac{-(x - x_c)^2 - (y - y_c)^2 - (z - z_c)^2}{a^2} \right) \\
\times \left( -y \hat{x} + (x - x_c) \hat{y} \right),
\]

(1)

where \(c = \{x, y, z, k, a, l\}\) and \(b_0 = 1\) is taken throughout. The closed form expression for \(E\) is given by

\[
\frac{b_0 \hat{z} + \sum_{i=1}^{2} B_i}{c_1}
\]

where \(c_1 = \{1, 0, -4, 1, \sqrt{2}, 2\}\) and \(c_2 = \{-1, 0, 4, -1, \sqrt{2}, 2\}\) and, similarly, that for \(E^n\) by

\[
\frac{b_0 \hat{z} + \sum_{i=1}^{2n} B_i}{c_1}
\]

with appropriate choices for the parameters \(c_i\). For each field \(E\), we are able to find the equations of the magnetic field lines in the closed form; and hence an exact expression for \(Q\) for each \(n\) can be easily calculated.

3. CURRENT STRUCTURES

In Paper I, a Lagrangian numerical scheme (Craig & Sneyd 1986) was used to carry out an ideal relaxation of \(E\) toward a force-free state with the field on the boundaries of the domain fixed. The maximum Lorentz force within the domain decreased by 2 orders of magnitude during the relaxation and the resultant field is close to force-free (for a precise definition of what is meant by “close” see Section 3.2 of Paper I). Beyond this point problems with the numerical accuracy of the scheme (Pontin et al. 2009) mean no further relaxation is possible until the numerical difficulties are resolved. In the relaxed state for \(E^3\), the current is smooth and of a large scale (see Figure 3 of Paper I)—two regions of current extend vertically throughout the domain and the maximum magnitude of the current, \((j_{\text{max}} = 1.47)\), is around half of that of the initial state. No evidence for current sheet formation in the ideal relaxation was found.

It is well known that continuous motions on the boundary of an initially smooth magnetic field cannot lead to truly singular current sheets under an ideal (line-tied) evolution (Van Ballegooijen 1985). However, it might seem that the lack of current sheets in our relaxed state contradicts previous results, for example those of Longbottom et al. (1998) and Galsgaard et al. (2003). These investigations found the formation current concentrations under shearing motions on the boundary of an initially current-free configuration. Since we can think of our braid as having been created by a sequence of such shearing motions applied to an initially homogeneous field, we might also expect to find similar current concentrations in the relaxed state.

A closer comparison, however, shows that the twist, measured by the ratio of the displacement distance \((l_d)\) to the length of the domain \((L)\) in which current concentration is observed, is comparatively high in the experiments of Longbottom et al. (1998) and Galsgaard et al. (2003). In particular, Longbottom et al. (1998) looked in detail into this question and found significant buildup of current for values above \(l_d/L \approx 0.6\) \((l_d = \text{“shear distance” in their terminology and } L = 1)\). If we were to create our elementary braid \(E\) (which comes closest to the configuration they considered) by a similar shearing pattern, we would obtain a ratio \(l_d/L < 0.18\) \((l_d \approx 2\sqrt{2}; L = 16)\). This is a value for which no noticeable current concentration has been observed (see Figure 2 of Longbottom et al. 1998) in agreement with our findings. A similar argument applies to the conditions under which Galsgaard et al. (2003) found current concentrations, although their experiment does not allow a direct comparison as they used a resistive code. In their case, values of \(l_d\) (“shear distance” in their terminology, \(L = 1)\) below 0.2 also showed no significant buildup of current.

While this shows that our results do not contradict previous findings of current sheet formation, it is perhaps still surprising that a smooth near force-free state for such a complex configuration exists. In order to understand this, one has to bear in mind that our equilibrium is not an exact force-free state but only close to a force-free state (for an exact definition how close see Section 3.2 of Paper I). If the residual forces in the relaxed state could be balanced by a pressure, our relaxed state would have a plasma beta of \(\beta \approx 0.009\); although this is well within the quality of “force-freeness” usually assumed for magnetic loops in the solar corona, it is not zero. Moreover, the field has comparatively low maximum force-free field parameter \(\alpha\) (when measured in a dimensionless way), and it is known that for sufficiently small values of this quantity, force-free equilibria exist for arbitrarily braided fields (Bineau 1972). Unfortunately, the proof given does not give an explicit upper bound but only provides the existence of such a number. These two properties, a low \(\alpha\) and an only approximately force-free state, appear to allow for a sufficiently large space of smooth solutions to encompass the braids we have investigated.
While in two dimensions the current itself is the crucial quantity for magnetic reconnection, in three dimensions the integrated parallel electric field along magnetic field lines plays an important role (Hesse & Schindler 1988). In resistive MHD, the parallel electric field is related to the parallel current by the relation \( \int E_i dl = n \int j_{||} dl \) (assuming a uniform resistivity \( n \)). Motivated by this consideration, in Paper I we examined the integrated parallel current, hereafter denoted by \( \mathcal{J}_\parallel \), in both the initial and relaxed states for \( E^3 \). In both cases, a highly filamentary structure containing multiple thin \( \mathcal{J}_\parallel \) layers was found. During the relaxation, the spatial structure of \( \mathcal{J}_\parallel \) was approximately conserved. More generally, by considering \( E^n \) for various \( n \), the thickness, \( d \), of the \( \mathcal{J}_\parallel \) layers was found to decrease exponentially with \( n \) (degree of braiding). These small-scale structures were found to be incompatible with the presence of a finite resistivity, as in the solar corona, and thus the result of the braiding process was shown to be the loss of equilibrium of the coronal field. The manner in which this occurs, as well as any resulting observational signatures, will require a full resistive MHD simulation and this is left for consideration at a later stage.

A quantity proposed as an indicator for the formation of current sheets (and hence for the occurrence of magnetic reconnection) in continuous fields is known as the squashing factor, \( Q \) (Titov et al. 2002, 2009). Layers in which \( Q \) is large, QSLs, can be understood as a generalization of separatrix surfaces for cases where no null points exist in the domain under consideration. Our previous findings motivate a consideration of the nature of \( Q \) in the class of braided magnetic fields \( E^m \). We relate these findings both to the current structure and the integrated parallel current structure of the fields.

4. SQUASHING FACTOR AND QSLs

A method for obtaining the squashing factor \( Q \) in the braided fields considered is outlined in the Appendix. Note that for \( E^0 \), the squashing factor \( Q \) and the newer definition of the slip-squashing factor (Titov 2007, Titov et al. 2009) give the same result since the magnetic field is normal to both \( z \)-boundaries.

We begin the discussion by considering the simplest field, \( E \). Under ideal relaxation toward a force-free equilibrium, the current, \( j \), in \( E \) remains smooth and large scale, with the maximum value being \( j_{\text{max}} = 1.07 \) in the relaxed state (compared with \( j_{\text{max}} = 2.83 \) initially). The squashing factor \( Q \) for \( E \) is shown in the left-hand image of Figure 2 as a function of field line positions on the lower boundary. High values of \( Q \) (regions where the field line mapping is strongly distorted) are found in two layer-like regions, and the maximum value of \( Q \) is \( Q_{1,\text{max}} = 241.5 \). Contours of the parallel current integrated along field lines, \( \mathcal{J}_\parallel \), are shown in the right-hand image of Figure 2. The characteristic scales of \( \mathcal{J}_\parallel \) are larger than those of \( Q \) and no relation between the two is evident.

Next consider the field \( E^1 \), which is based on the pigtail braid and on which most of the analysis of Paper I was based. Recall that a smooth equilibrium with only large-scale current structures was obtained in an ideal relaxation of this field.

The squashing factor \( Q \) on the lower boundary of the domain is shown in the left-hand image of Figure 3. Here, the maximum value of \( Q \) is \( Q_{1,\text{max}} = 7.17 \times 10^5 \) (with \( Q \) calculated at 13962 points on the lower boundary) and regions of enhanced \( Q \) occur in a multitude of thin layers in a large portion of the domain. Between these layers \( Q \) drops significantly, even to its minimum possible value of 2 in many locations. A typical half-width at half-maximum (HWHM) of the QSLs is \( \sim 10^{-3} \) or 0.01% of the domain width.

The integrated parallel current structure, \( \mathcal{J}_\parallel \), for \( E^1 \) is shown on the lower boundary of \( E^1 \) in the right-hand image of Figure 3. A resemblance between \( Q \) and \( \mathcal{J}_\parallel \) is shown, with both quantities having a similar global structure and containing thin layers of approximately equal width. However, \( Q \) is more filamentary than \( \mathcal{J}_\parallel \) and has a much greater range of values. In order to make a more precise comparison, the reader is referred to the upper panel of Figure 4 where both quantities are shown for only a subsection of the domain.

Finally, consider the field \( E^5 \). Both \( Q \) and \( \mathcal{J}_\parallel \) are shown over a subsection of the domain in the lower panel of Figure 4. The structures of the two quantities are highly filamentary, both containing extremely small scales (HWHM of \( \sim 10^{-5} \)). It is seen that \( Q \) and \( \mathcal{J}_\parallel \) here have a strong resemblance in terms of the global pattern of their thin layer-like structures. However, the locations of the regions of high (low) \( Q \) and high (low) \( \mathcal{J}_\parallel \) do not exactly coincide.
Considering now $Q$ alone for the various fields $E^n$, we seek to determine how the maximum value of $Q$ and the characteristic width (specifically HWHM), $d_Q$, of the layers of high $Q$ depend on $n$. It is found that the maximum of $Q$ for $E^n$, $Q_{n,\text{max}}$, in the domain increases exponentially with $n$—as shown in Figure 5(a)—for which a best fit line is

$$Q_{n,\text{max}} = 2.97 \times 10^{1.8n}.$$

In Paper I, a linear increase in the maximum of $\mathcal{J}_\parallel$ with $n$ was found, as expected, since the vertical length of the domain is also increasing linearly by construction of the fields. The typical width, $d_Q$, of the regions of enhanced $Q$ decreases with $n$. More precisely, there is an exponential decrease in $d_Q$ with $n$—as demonstrated in Figure 5(b)—for which a best fit line is

$$d_Q = 1.90 \times 10^{-0.93n}.$$

In Paper I, a similar exponential decrease in the width ($d_{\mathcal{J}_\parallel}$) of regions of high $\mathcal{J}_\parallel$ was found.

We now address the dependence of the maximum value of $Q$ and $Q_{n,\text{max}}$ on $n$. Recall that $Q_{n,\text{max}}$ was found to increase exponentially with $n$. Notice that by the construction of $E^n$ we may place an upper bound on the maximum value of $Q$ from the method of its calculation by composition of matrices. Letting $Q_i$ denote the maximum value of $Q$ for $E^i$, the submultiplicative property of the $p = 2$ norm tells us that
Galsgaard (2000) showed that the nature of the flow is important in determining whether currents accumulate at QSLs with a stagnation flow somewhere along the QSL necessary for obtaining strong current enhancement.

Figure 5. (a) Maximum value of $Q$ with $n$ in the full domain. An exponential increase in $Q$ with $n$ is shown. (b) HWHM of $Q$ layers with $n$. An exponential decrease in $Q$ with $n$ is found. Here, the width has been taken as the shortest such width along a cross section of the domain on the lower boundary.

Given the complexity of our configuration, it seems unlikely that all of the motions which occur during the relaxation are of the type unfavorable for the formation of current sheets unless the nature of the deformation applied ($\mathbf{v} \sim \mathbf{j} \times \mathbf{B}$) excludes any such favorable deformation. The second possibility is that following the applied deformations the system is able to adjust to a neighboring smooth equilibrium. This possibility is supported by the arguments given in Section 3 that our final state is a low alpha, near force-free state.

Although current layers are not found in our experiment, it does show layer-like structures of $\mathcal{J}_\parallel$. Such layers have a broader definition in which only the integral of $\mathbf{j}$ and not $\mathbf{j}$ itself needs to show small scales. The following example, however, shows that there is no simple relation between either $Q$ and $\mathbf{j}$ or between $Q$ and $\mathcal{J}_\parallel$. We have chosen this example because it is locally similar to the braided fields discussed, in which in both cases $Q$ and $\mathcal{J}_\parallel$ each have a layer-like structure, with multiple layers in which the quantities are enhanced. Consider the force-free magnetic field

$$\mathbf{B} = (c_1 \sin(\lambda x)/\lambda) + c_2 \cos(\sin(\lambda x)/\lambda)\hat{y} + (c_1 \cos(\sin(\lambda x)/\lambda) - c_2 \sin(\lambda x)/\lambda)\hat{z}$$

for which $\mathbf{j} = \alpha \mathbf{B}$ with $\alpha = \cos \lambda x$, and take $z \in [0, s]$. On the lower boundary, $z = 0$, we have

$$Q = 2 + s^2 \cos^2(\lambda x)(c_1 \cos(\sin(\lambda x)/\lambda) - c_2 \sin(\lambda x)/\lambda)^2$$

$$\mathcal{J}_\parallel = \int_{s=0}^{s} \mathbf{j} \cdot \mathbf{B} \ ds' = s(c_1^2 + c_2^2) \cos \lambda x,$$

and

$$|\mathbf{j}|^2 = (c_1^2 + c_2^2) \cos^2(\lambda x).$$

Here, the relationship between $Q$ and currents crucially depends on the choice of parameters. Setting $c_2 = 0$, for example, maxima and minima of $Q$, $|\mathcal{J}_\parallel|$, and $|\mathbf{j}|$ coincide. Setting instead $c_1 = 0$, minima in $Q$ correspond to both maximal and minimal regions of $|\mathcal{J}_\parallel|$ or $|\mathbf{j}|$ (see Figure 6). Here, we see that the maxima of $Q$ increase quadratically with the vertical extent ($s$) of the domain under consideration, while the maxima of $\mathcal{J}_\parallel$ increase only linearly and the maxima of $|\mathbf{j}|$ are independent of $s$. 

$Q_{n,\text{max}} \approx (Q_{1,\text{max}})^n$. Evaluating, we obtain the upper bound as $(Q_{1,\text{max}})^n \sim 10^{4.7 n}/200$, while the true dependence is given by $Q_{n,\text{max}} \sim 2.97 \times 10^{1.8 n}$. The dependence on the smaller power then comes from the filling factor of the field $E$, i.e., the number of points on the lower boundary of $E$ with highest $Q$, that get mapped under $\mathbf{F}$ to points with high $Q_1$ for the next composition (for $E^2$ etc.)

In the next section, we seek to explain these findings and explore theoretically the relationship between $Q$ and the current structures.

5. RELATING QSLs AND CURRENT STRUCTURES

As mentioned above, the squashing factor has been suggested as an indicator for the formation of current sheets. Layers in which $Q$ is large outline QSLs. The link between QSLs and current sheets relies on the following hypothesis (referred to below as the QSL-hypothesis). Where $Q$ is high, there is a sensitive dependence of the endpoints of magnetic field lines on the starting points (or vice versa since $Q$ is symmetric). Hence, certain motions of the plasma on one boundary (particularly those crossing the QSLs) would lead to very high velocities at the other boundary, provided the evolution is ideal and the magnetic configuration is unaffected by this perturbation. For sufficiently high $Q$ this is inconsistent with $v < v_A$, and so the magnetic configuration has to change. Then a lack of neighboring smooth equilibria can lead to the formation of current concentrations or even current sheets.

Within the experiments described above we are in a position to test the QSL-hypothesis, given that we can think of the relaxed configuration for $E^2$ as an ideal deformation of the initial configuration. The initial configuration shows an abundance QSLs but no strong current concentrations. In the subsequent ideal relaxation, the QSLs remain but no current sheets (in the sense of a sheet-like structure of $\mathbf{j}$) form. This result appears to contradict the QSL-hypothesis. There are, however, two ways in which the system can escape the necessity of forming current concentrations.

Firstly, the deformation applied could be unfavorable for the formation of current sheets (Titov et al. 2003). For example, Galsgaard et al. (2003) considered a configuration containing two intersecting QSLs and applied two types of deformations on the boundary. They found that only one of the types of deformation leads to significant buildup of current. Similarly, Galsgaard (2000) showed that the nature of the flow is important in determining whether currents accumulate at QSLs with
Figure 6. $Q$ (solid line) and $|\mathcal{J}_f|$ (dashed line) for the illustrative example of a particular one-dimensional force-free field (described in Section 5). Here $|\mathcal{J}_f| = 4|\mathcal{J}|$.

In this example, $Q$ is not a good predictor of current features. (Parameters: $c_1 = 0$, $c_2 = 1$, $\lambda = 1$, and $s = 4$.)

Figure 7. Illustrative example demonstrating properties of the field line mapping $\mathbf{F}^n(x, y)$. For the function $\psi = \exp(-x^2/4 - y^2/4)$, the left-hand image shows $\psi(\mathbf{F}(x, y))$, the center image shows $\psi(\mathbf{F}^2(x, y))$, and the right-hand image shows $\psi(\mathbf{F}^3(x, y))$. As $\mathbf{F}$ is repeatedly applied to the function, the image develops small scales; for this reason since $Q$ and $\mathcal{J}_f$ take (by definition) one value on each field line, they too must develop small scales on the boundaries of $E^n$ for increasing $n$.

(A color version of this figure is available in the online journal.)

Figure 8. (Left) $\log_{10} Q$ as a function of field line position in the central plane ($z = 0$) for the field $E^1$, and (right) integrated parallel current along field lines, $|\mathcal{J}_f|$, for the same field, also in the central plane.

(A color version of this figure is available in the online journal.)

Note that this is a specific illustration of a more general case of a force-free field $\mathbf{B} = B_y(x)\hat{y} + B_z(x)\hat{z}$ with force-free parameter $\alpha(x)$. Taking $z \in [0, s]$, then on the lower boundary $Q = 2 + s^2\alpha^2B_z^2$ while $\mathbf{j} = \alpha(B_y^2 + B_z^2)^{1/2}$ and $\int j \, dl = s\alpha(B_y^2 + B_z^2)^{1/2}$.

The above described situation corresponds well to the braided fields $E^n$ in which both $Q$ and $\mathcal{J}_f$ have a layer-like structure. For $E^n$ we also find that there is no exact correspondence between the layers of high $Q$ and greatest $\mathcal{J}_f$. However, there is a similarity between $Q$ and $\mathcal{J}_f$ in terms of their global structure, particularly for moderate and large $n$ (see Figures 3 and 4). This similarity comes not from an inherent relation between $Q$ and $\mathcal{J}_f$, but rather from the fact that both quantities are functions of the field lines.

To see this, we prescribe a smooth function $(\psi(x_0, y_0))$ of field lines on the lower boundary of each of the braided fields $E^n$, $n = 1, 2, \ldots$. We then map this function $(\psi)$ along field lines of $E^n$ to the upper boundary, i.e., find $\psi(\mathbf{F}^n(x_0, y_0))$.

This is demonstrated for a specific example in Figure 7 for $E^i$, $i = 1, \ldots, 3$. As $\mathbf{F}$ is repeatedly applied to any function $\psi(x_0, y_0)$, then the characteristics of the map will result in stretching and contracting of $\psi$ in some regions of the domain according to the nature of $\mathbf{F}$ at those locations, specifically the Lyapunov exponents of the map. After repeated application of $\mathbf{F}$ any two functions will appear similar; the apparent structure of the two functions will be determined by the mapping $\mathbf{F}$.

From this observation, we deduce that the structure of both $Q$ and $\mathcal{J}_f$ will depend crucially on the plane in which they are viewed and, by the symmetry of the braid, have the same scale on both the lower and upper boundaries. Accordingly, $\mathcal{J}_f$ will, for sufficiently high $n$ (about 3 in this case), have a filamentary distribution on the lower boundary that is reflective of the field line mapping rather than the “underlying” nature of the parallel currents. These properties are confirmed by illustrating $Q$ and
for $E^\parallel$ in a cross section in the middle of the domain ($z = 0$), thereby subtracting as much as possible the effect of the field line mapping. This is shown in Figure 8 where the quantities are seen to attain larger scales than on the domain boundaries, and less similarity between $Q$ and $\mathcal{J}_\parallel$ is apparent. Another property evident from Figure 8 is that several locations are present where the layers of high $Q$ intersect. These intersections of QSLs outline hyperbolic flux tubes (HFTs; Titov et al. 2002), thought to be particularly preferential sites for current sheet formation; it is not a lack of HFTs in our configuration leading to the smooth current profile.

6. CONCLUSIONS

In this paper, we have examined the link between the squashing factor $Q$ and current structures in a class of braided magnetic fields. The squashing factor (Titov et al. 2002) is a geometric measure for continuous fields that serves to identify regions where the mapping of magnetic field lines is highly distorted. It is used to identify QSLs (Priest & Démoulin 1995)—features of continuous fields which provide an analog to separatrix surfaces in fields containing null points. Previously, regions of high $Q$ have been considered to be a reliable indicator for the formation of current sheets under certain deformations on the boundaries.

Here, we are interested in both the relationship between $Q$ and the current $J$, as well as that between $Q$ and the parallel current integrated along field lines, $\mathcal{J}_\parallel$. The reason for this is that in three dimensions a key quantity for magnetic reconnection is the parallel electric field integrated along field lines. In resistive MHD with uniform resistivity, the parallel electric field and parallel current are related via

$$\int E_\parallel \, dl = \eta \int J_\parallel \, dl = \eta \mathcal{J}_\parallel.$$ 

We have considered a particular class of braided magnetic fields with no net twist, the fields being labeled $E^n$ ($n = 1, 2, \ldots$), where an increase in $n$ results in a field of increased complexity. In an earlier paper (Wilmot-Smith et al. 2009, Paper I), the field $E^1$ was taken as an initial condition in a magnetic relaxation simulation. The field on the boundaries was held fixed and a magnetofrictional code used to carry out an ideal relaxation toward a force-free equilibrium. In that process, the spatial scales associated with the current, $J$, were found to remain large, suggesting a smooth equilibrium corresponding to each $E^n$ can be found.

In Paper I, the structure of $\mathcal{J}_\parallel$ for $E^1$ was examined. This quantity was shown to display small spatial scales and to have an increasingly filamentary structure with increasing $n$. This structure was preserved in the ideal relaxation process. The consequence was shown to be that for a coronal field with any finite resistivity, after a certain degree of braiding via photospheric motions the field will undergo a loss of equilibrium due to the high gradients in $\mathcal{J}_\parallel$, regardless of the current structure itself.

Here, we have investigated the nature of the squashing factor $Q$ and associated QSLs for the fields $E^n$. A method was given (see the Appendix) to calculate $Q$ exactly, a property that becomes important in the analysis where extremely small scales in the quantity are found. It was shown that for the field with the least structure, $E$, $Q$ is enhanced in two layers of width $\sim 5\%$ of the domain width, i.e., there are two QSLs present. As the degree of braiding is increased, $Q$ develops an increasingly filamentary structure, with many narrow layers of enhanced $Q$ present. These are true layers in the sense that $Q$ drops to low values between them. The typical width of the layers decreases exponentially with $n$—for $E^3$ a typical layer width is $0.01\%$ of the domain width. Additionally, with increasing $n$ the maximum value of $Q$ exponentially increases. The result of the braiding process is shown to be that the domain becomes filled with a multitude of QSLs.

Despite these findings, we find that under ideal relaxation the braided fields $E^n$ do not develop any small-scale structures in the current $J$. This demonstrates an alternative to the view that in a configuration with QSLs (outlined by regions of high $Q$), plasma motion is highly likely to lead to the formation of current sheets along the QSLs—another possibility is that the magnetic field may adjust to a neighboring smooth equilibrium. This equilibrium may itself still contain QSLs.

We also considered the relationship between $Q$ and the integrated parallel current, $\mathcal{J}_\parallel$. In the simplest braid, $E$, the two quantities are unrelated. With increasing braid complexity ($n$), $Q$ and $\mathcal{J}_\parallel$ have some similar characteristics; for moderate and large $n$ both quantities are filamentary with layer-like regions in which high values are obtained. However, no simple relationship has been found between the locations of maxima in the two quantities.

APPENDIX

Here, we give a method for calculating the squashing factor in the braided fields $E^n$. For this we must obtain the mapping of field lines from the lower boundary of the domain to the upper boundary. We make use of a property of the fields $E^n$, specifically that $E^n$ is a concatenation of $n$ times the “basic” field $E$.

Consider first the field $B_c + b_0 \hat{x}$ (see Equation (1)), that is, a single toroidal flux ring imposed on the background uniform field. The equations $\mathbf{X}(x_0, s)$ of the field line passing through the point $x_0 = (x_0, y_0, z_0)$ are obtained by integrating

$$\frac{d \mathbf{X}(s)}{ds} = \mathbf{B}(\mathbf{X}(s)),$$

where the parameter $s$ is related to the distance $\lambda$ along field lines by $d \lambda = |\mathbf{B}| \, ds$. The components of $\mathbf{X}(x_0, s)$ are given by

$$X = (x_0 - x_c) \cos \zeta - (y_0 - y_c) \sin \zeta + x_c,$$

$$Y = (y_0 - y_c) \cos \zeta + (x_0 - x_c) \sin \zeta + y_c,$$

$$Z = b_0 s + z_0,$$

where

$$\zeta = k \sqrt{\frac{\pi}{a}} \exp \left( \frac{-(x_0 - x_c)^2 - (y_0 - y_c)^2}{a^2} \right) \times \left( \text{erf} \left( \frac{b_0 s + z_0 - z_c}{l} \right) - \text{erf} \left( \frac{z_0 - z_c}{l} \right) \right).$$

For the field $E$, the two toroidal flux rings have been placed sufficiently far apart (compared with their characteristic length scales) that, in a plane equidistant between the flux rings, the field can be approximated as vertical, $b_0 \hat{z}$. Accordingly, we may replace the error functions in Equations (A1) by $\pm 1$ and obtain an expression for the mapping of field lines from the point $(x, y)$ in a two-dimensional plane (perpendicular to the field) below...
the single region of twist in $B_c + b_0 \mathbf{z}$ to the point $f(x, y)$, and in similar a plane above it:

$$f_p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$f_p : (x, y) \rightarrow ((x - x_c) \cos \xi - (y - y_c) \sin \xi + x_c, \times (y - y_c) \cos \xi + (x - x_c) \sin \xi + y_c),$$

where $p = \{x_c, y_c, k, a, l\}$ and

$$\xi = 2\sqrt{\pi k} \frac{1}{a} \exp \left( \frac{-(x - x_c)^2 - (y - y_c)^2}{a^2} \right). \quad (A2)$$

The mapping of field lines from the lower to the upper boundary of the field $E$ may then be found by composition of $f_p$ with itself using with the relevant parameters for each of the mappings:

$$F(x, y) = f_{p_1} \circ f_{p_1},$$

where $p_1 = \{1, 0, 1, \sqrt{2}, 2\}$ and $p_2 = \{1, 0, -1, \sqrt{2}, 2\}$. A similar mapping, $F^n(x, y)$, of the field lines from the lower to the upper boundary of $E^n$ is obtained by repeated composition of $F$. For example, the field line mapping from the lower to the upper boundary of $E^3$ is $F^3(x, y) = F \circ F^2 = F \circ F \circ F$.

The squashing factor $Q$ for $E^n$ can be found directly from the Jacobian $(DF^n)$ of the mapping $F^n$:

$$Q = \frac{||DF^n||^2}{\det(DF^n)},$$

where $|| \cdot ||$ denotes the entrywise 2 norm. In other words, letting

$$DF^n = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

say, then

$$Q = \frac{a^2 + b^2 + c^2 + d^2}{ad - bc}.$$