UPPER BOUNDS ON THE DIAMETER FOR FINSLER MANIFOLDS WITH WEIGHTED RICCI CURVATURE

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Received 05 January, 2018

Abstract. In this paper we obtain some Cheeger-Gromov-Taylor type compactness theorems for a forward complete and connected Finsler manifold of dimensional $n \geq 2$ via weighted Ricci curvatures. The proofs are based on the index form of a minimal unit speed geodesic segment, Bochner-Weitzenböck formula and Hessian comparison theorem.

2010 Mathematics Subject Classification: 53C60; 53B40

Keywords: diameter estimate, distortion, Finsler manifold, $S$-curvature, weighted Ricci curvature

1. INTRODUCTION AND MAIN THEOREMS

In [8], Myers obtained a compactness theorem in Riemannian manifolds. The theorem of Myers concludes that if $\text{Ric} \geq (n - 1)K > 0$, then $\text{diam}(M) \leq \pi / \sqrt{K}$. Later, Cheeger-Gromov-Taylor [3] proved that if there exist $p \in M$ and $r_0, v > 0$ such that

$$\text{Ric} \geq (n - 1)\left(\frac{\frac{1}{4} + v^2}{r^2}\right)$$

holds for all $r(x) \geq r_0 > 0$ where $r$ is distance function defined with respect to a fixed point $p \in M$, i.e., $r(x) = d(x, p)$, then $M$ is compact and the diameter is bounded from above by $\text{diam}_p(M) < r_0 \pi / v$. By using Bakry-Emery Ricci tensor, $\text{Ric}_f = \text{Ric} + \text{Hess} f$. Soylu [12] attained a generalization of Cheeger-Gromov-Taylor’s compactness theorem.

For $m$-Bakry-Emery Ricci tensor, Wang [14] proved that, if the following inequality

$$\text{Ric}_{f,m} = \text{Ric} + \text{Hess} f - \frac{df \otimes df}{m-n} \geq -(m - 1)\frac{K_0}{(1+r)^2}$$

holds for all $x \in M$, where $K_0 < -\frac{1}{4}$ and $r$ is distance function defined with respect to a fixed point $p \in M$, then $M$ is compact and the diameter has the upper bound $\text{diam}(M) < 2(e^{2\pi / K} - 1)$, where $K = \sqrt{-K_0 - \frac{1}{4}}$.

We can find various kinds of generalizations of the Myers theorem in [4, 6, 7, 13, 15].

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Finsler geometry is a natural generalization of Riemannian geometry. The validity of the Myers compactness theorem for Finsler manifolds was shown by Shen [11] without any modification. Later, using the weighted Ricci curvature $\text{Ric}_N := \text{Ric} + \tilde{S} - \frac{S^2}{N-n} \geq K > 0$, $N \in (n, \infty)$, Ohta [9] obtained a compactness theorem and gave an upper bound for the diameter of $n$-dimensional Finsler manifolds as $\text{diam}(M) \leq \pi \sqrt{(N-1)/K}$. In [16], Wu establish a generalized Myers theorem under line integral curvature bound for Finsler manifolds. In [2], Anastasiei extended to Finsler manifolds the compactness theorems of Ambrose and Galloway (see [1] and [5], respectively). Yin [18] acquired two Myers-type compactness theorems for a Finsler manifold with a positive weighted Ricci curvature bound and an advisable condition on the distortion or the $S$-curvature.

Throughout this paper, $(M, F)$ is a connected forward complete $n$-dimensional smooth Finsler manifold, $r(x) = d(x, p)$ is the forward distance function from $p \in M$ and $d\mu$ is an arbitrary positive $C^\infty$-measure on $M$. Here, there is no canonical measure like the volume measure in Riemannian geometry. Thus we begin with an arbitrary measure on $M$.

We are now ready to give our main results.

**Theorem 1.** Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension $n$ with arbitrary volume form and let $r$ be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature

$$\text{Ric}_\infty := \text{Ric} + \tilde{S} \geq (n-1) \frac{H}{r^2},$$

and the distortion $|\tau| \leq (n-1)k$ for all $x \in M$ such that $r(x) \geq r_0 > 0$, where the constants $k$ and $H$ satisfy the inequalities $k \geq 0$ and $H > 1/4$. Then $M$ is compact and the diameter from the point $p \in M$ satisfies

$$\text{diam}_p(M) \leq r_0 \exp \left( \frac{2}{4H-1} \sqrt{32k^2 + (4H-1)^2} + 16k \sqrt{4k^2 + (4H-1)^2} \right).$$

The distortion $\tau$ is a smooth function on $M$ when $M$ is a Riemannian manifold. Therefore the diameter estimate (1.4) of Theorem 1 coincides with the diameter estimate of Theorem 1.1 in [12].

**Theorem 2.** Let $(M, F, d\mu)$ be a forward complete and connected Finsler manifold of dimension $n$ with arbitrary volume form and let $r$ be the distance function $r(x) = d(x, p)$ with respect to a fixed point $p \in M$. Assume that the weighted Ricci curvature

$$\text{Ric}_N := \text{Ric}_\infty - \frac{S^2}{N-n} \geq (n-1) \frac{H}{r^2}$$

for all $N \in (n, \infty)$ and $r(x) \geq r_0 > 0$, where $H > 1/4$. Then $M$ is compact and the diameter from the point $p \in M$ satisfies

$$\text{diam}_p(M) \leq r_0 e^{2\pi/\sqrt{4H-1}}.$$
The diameter estimate (1.6) obtained in the above theorem coincides with the result of Cheeger-Gromov-Taylor in [3] obtained for the original Ricci tensor in the Riemannian manifolds.

Theorem 3. Let \((M, F, d \mu)\) be a forward complete and connected Finsler manifold of dimension \(n\) with arbitrary volume form and let \(r\) be the distance function \(r(x) = d(x, p)\) with respect to a fixed point \(p \in M\). Suppose that the weighted Ricci curvature

\[
\text{Ric}_N := \text{Ric}_\infty - \frac{S^2}{N - n} \geq (N - 1) \frac{H}{(1 + r)^2}
\]

(1.7)

for all \(x \in M\) and \(N \in (n, \infty)\), where \(H > 1/4\). Then \(M\) is compact and the diameter satisfies

\[
\text{diam}(M) \leq (1 + \lambda)(e^{2\pi/\sqrt{4H-1}} - 1),
\]

(1.8)

where \(\lambda\) is the reversibility.

We review below some basic informations about the Finsler manifolds to be used in the proofs of main theorems.

2. A brief review of Finsler geometry

Let \((M, F)\) be a Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\). Let \(\pi : TM \to M\) be the natural projection and \((x, y)\) be a point of \(TM\) such that \(x \in M\) and \(y \in T_xM\). A Finsler metric is a \(C^\infty\)-Finsler structure of \(M\) with the following properties:

1. \(F\) is \(C^\infty\) on \(TM \setminus 0\) (Regularity),
2. \(F(x, \lambda y) = \lambda F(x, y)\) for all \(\lambda > 0\) (Positive homogeneity),
3. The \(n \times n\) Hessian matrix

\[
\begin{align*}
g_{ij} &:= \frac{1}{2} [F^2]_{y^i y^j} 
\end{align*}
\]

is positive-definite at every point of \(TM \setminus 0\) (Strong convexity).

The Chern curvature \(R^V\) for vectors fields \(X, Y, Z \in T_xM \setminus 0\) is defined by

\[
R^V(X, Y)Z := \nabla_X^V \nabla_Y^V Z - \nabla_Y^V \nabla_X^V Z - \nabla^V_{[X,Y]} Z,
\]

(2.1)

and the flag curvature is defined as follows:

\[
K(V, W) := \frac{g_V(R^V(W, V)W, V)}{g_V(W, V)g_V(W, W) - g_V(V, W)^2},
\]

(2.2)

where \(V, W \in T_xM \setminus 0\) are linearly independent vectors. Then the Ricci curvature of \(V\) (as the trace of the flag curvature) is defined by

\[
\text{Ric}(V) := \sum_{i=1}^{n-1} K(V, E_i),
\]

(2.3)
where \( \{E_1, E_2, \ldots, E_{n-1}, V/F(V) \} \) is an orthonormal basis of \( T_xM \) with respect to \( g \).

Let \( d\mu = \sigma(x)dx^1dx^2\ldots dx^n \) be the volume form on \( M \). For a vector \( V \in T_xM \setminus 0 \),

\[
\tau(x, V) := \ln \frac{\sqrt{\det(g_{ij}(x, V))}}{\sigma(x)}
\]

(2.4)
is a scalar function on \( T_xM \setminus 0 \) which is called the distortion of \((M, F, d\mu)\). We say that the distortion \( \tau \) is a \( C^\infty \)-function, if \( M \) is a Riemannian manifold. Setting

\[
S(x, V) := \frac{d}{dt}(\tau(\gamma(t), \dot{\gamma}(t)))|_{t=0}.
\]

(2.5)

where \( \gamma \) is the geodesic with \( \gamma(0) = x \), \( \dot{\gamma}(0) = V \). \( S(x, \lambda V) = \lambda S(x, V) \) for all \( \lambda > 0 \). \( S \) is a scalar function on \( T_xM \setminus 0 \) which is called the S-curvature. From the definition, it seems that the S-curvature measures the rate of change in the distortion along geodesics in the direction \( V \in T_xM \).

For all \( N \in (n, \infty) \), we define the weighted Ricci curvature of \((M, F, d\mu)\) as follows (see [9]):

\[
\begin{align*}
\text{Ric}_N(V) &:= \text{Ric}(V) + \frac{S(V)}{N-n}, \\
\text{Ric}_\infty(V) &:= \text{Ric}(V) + S(V), \\
\text{Ric}_n(V) &:= \begin{cases} 
\text{Ric} + S(V), & \text{if } S(V) = 0 \\
-\infty, & \text{otherwise}
\end{cases}
\end{align*}
\]

Also \( \text{Ric}_N(c V) := c^2 \text{Ric}_N(V) \) for \( c > 0 \).

We say that \((M, F)\) is forward complete if each geodesic \( \gamma : [0, \ell] \to M \) is extended to a geodesic on \([0, \infty)\), in other words, if exponential map is defined on whole \( TM \). Then the Hopf-Rinow theorem gives that every pair of points in \( M \) can be joined by a minimal geodesic.

The Legendre transformation \( \mathcal{L} : TM \to T^*M \) is defined by

\[
\mathcal{L}(W) := \begin{cases} 
g_W(W), & W \neq 0, \\
0, & W = 0.
\end{cases}
\]

For a smooth function \( h : M \to \mathbb{R} \), the gradient vector of \( h \) at \( x \in M \) is defined as \( \nabla h(x) := \mathcal{L}^{-1}(dh) \).

Given a smooth vector field \( Z = Z^i \partial/\partial x^i \) on \( M \), the divergence of \( Z \) with respect to an arbitrary volume form \( d\mu = e^\varphi dx^1dx^2\ldots dx^n \) is defined by

\[
\text{div}Z := \sum_{i=1}^n \left( \frac{\partial Z^i}{\partial x^i} + Z^i \frac{\partial \varphi}{\partial x^i} \right).
\]

(2.6)

Then we define the Finsler-Laplacian of \( h \) by \( \Delta h := \text{div}(\nabla h) = \text{div}(\mathcal{L}^{-1}(dh)) \).

The following lemma is useful to prove Theorem 3 (see [17]).
Lemma 1. Let \((M, F, d\mu)\) be a Finsler \(n\)-manifold, and \(h : M \to \mathbb{R}\) a smooth function on \(M\). Then on \(U = \{x \in M : \nabla h(x) \neq 0\}\) we have
\[
\Delta h = \sum_i H(h)(E_i, E_i) - S(\nabla h) := \text{tr}_{\nabla h} H(h) - S(\nabla h),
\]
(2.7)
where \(E_1, E_2, \ldots, E_n\) is a local \(g_{\nabla h}\)-orthonormal frame on \(U\).

Finally, define reversibility \(\lambda := \lambda(M, F)\) as follows:
\[
\lambda := \sup_{x \in M, y \in TM \setminus 0} \frac{F(x, -y)}{F(x, y)}.
\]
(2.8)
Obviously, \(\lambda \in [1, \infty]\), and \(\lambda = 1\) if and only if \((M, F)\) is reversible.

3. THE PROOFS OF THE THEOREMS

Let \((M, F, d\mu)\) be a Finsler manifold of dimensional \(n\) and \(r(x) = d(x, p)\) be a distance function with respect to a fixed point \(p \in M\). It is well known that \(r\) is only smooth on \(M - (C_p \cup \{p\})\) where \(C_p\) is the cut locus of the point \(p \in M\). We assume that \(r\) is a minimal unit speed geodesic segment. We have \(\nabla r = \dot{y}\) in the adapted coordinates with respect to the \(r\), and also have \(F(\nabla r) = 1\) (see [11]). On the other hand, using the Finsler metric we obtain a weighted Riemannian metric \(g_{\nabla r}\). Thus we can apply the Riemannian calculation for \(g_{\nabla r}\) (on \(M - (C_p \cup \{p\})\)).

In order to prove the Theorem 1 and Theorem 2, we use the index form of a minimal unit speed geodesic, and to prove Theorem 3, we use Bochner-Weitzenböck formula and Hessian comparison theorem in Finsler geometry.

Proof of Theorem 1. Let \(q \in M\) be a point and let \(\sigma\) be a minimal unit speed geodesic segment from \(p\) to \(q\) of length \(\ell\) such that \(\sigma(0) = p, \sigma(\ell) = q\) and \(\ell > r_0 > 0\). Since the inequality \(\ell > r_0\) holds, \(\ell\) can be parametrized by \(\mu > 0\) such that
\[
\ell = r_0 e^{\mu \pi} > r_0.
\]
(3.1)
By virtue of any subsegment of a minimal unit speed geodesic segment is also a minimal unit speed geodesic segment, we have the minimal unit speed geodesic segment \(\gamma\) defined by \(\gamma(t) = \sigma|_{[r_0, \ell]}(t)\) where \(\gamma : [r_0, \ell] \to M\) and \(\gamma(r_0) = \sigma(r_0) = \bar{q}\), \(\gamma(\ell) = \sigma(\ell) = q\). Let \(\{E_1 = \dot{\gamma}, E_2, \ldots, E_n\}\) be a parallel \(g_{\nabla r}\)-orthonormal frame along \(\gamma\) and let \(f \in C^\infty([r_0, \ell])\) be a real-valued smooth function such that \(f(r_0) = f(\ell) = 0\). Then we have
\[
I(f E_i, f E_i) = \int_{r_0}^{\ell} \left( g_{\nabla r}(\dot{f} E_i, \dot{f} E_i) - g_{\nabla r}(R_{\nabla r} f E_i, \nabla r) \right) dt.
\]
(3.2)
It is obvious that (3.2) yields, by $g \nabla_f (R^\nabla \nabla r, \nabla r) = 0$ and the assumption (1.3) given in Theorem 1,
\[
\sum_{i=2}^{n} I(f E_i, f E_i) = \int_{r_0}^{\ell} \left( (n-1) \dot{f}^2 - f^2 \text{Ric}(\nabla r) \right) dt
\]
\[
= \int_{r_0}^{\ell} \left( (n-1) \dot{f}^2 - f^2 \text{Ric}_\infty(\nabla r) + f^2 \dot{S}(\nabla r) \right) dt
\]
\[
\leq \int_{r_0}^{\ell} \left( (n-1) (f^2 - \frac{Hf^2}{r^2}) + f^2 \dot{S}(\nabla r) \right) dt. \tag{3.3}
\]
Here, the term $f^2 \dot{S}(\nabla r)$ equals to
\[
f^2 \dot{S}(\nabla r) = -2 f \dot{f} S(\nabla r) + \frac{d}{dt} (f^2 S(\nabla r)) = -2 f \dot{f} \frac{d}{dt} f + \frac{d}{dt} (f^2 S(\nabla r))
\]
\[
= 2r \frac{d}{dt} (f \dot{f}) - \frac{d}{dt} (\tau f \dot{f} + \frac{d}{dt} (f^2 S(\nabla r))). \tag{3.4}
\]
Integrating both sides of (3.4) and using the assumption $|\tau| \leq (n-1)k$, we obtain
\[
\int_{r_0}^{\ell} (f^2 \dot{S}(\nabla r)) dt = 2 \int_{r_0}^{\ell} \frac{d}{dt} (f \dot{f}) dt \leq 2(n-1)k \int_{r_0}^{\ell} \frac{d}{dt} (f \dot{f}) dt, \tag{3.5}
\]
because of $f(r_0) = f(\ell) = 0$. By use of (3.5), the inequality (3.3) becomes
\[
\sum_{i=2}^{n} I(f E_i, f E_i) \leq \int_{r_0}^{\ell} (n-1) \left( \dot{f}^2 - \frac{Hf^2}{r^2} \right) dt + 2(n-1)k \int_{r_0}^{\ell} \frac{d}{dt} (f \dot{f}) dt. \tag{3.6}
\]
Set
\[
f(t) = \mu r_0 \sqrt{r(y(t))} \sin(\frac{1}{\mu} \ln \frac{r(y(t))}{r_0}). \tag{3.7}
\]
Therefore we have
\[
\frac{1}{r_0^2 (n-1)} \sum_{i=2}^{n} I(f E_i, f E_i) \leq -\frac{1}{4} \int_{r_0}^{\ell} \frac{(4H - 1) \mu^2}{r} \sin^2(\frac{1}{\mu} \ln \frac{r}{r_0}) dr
\]
\[
+ \int_{r_0}^{\ell} \frac{1}{r} \left[ \cos^2(\frac{1}{\mu} \ln \frac{r}{r_0}) + \frac{2}{\mu} \sin(\frac{2}{\mu} \ln \frac{r}{r_0}) \right] dr
\]
\[
+ 2k \int_{r_0}^{\ell} \frac{1}{r} \left[ \frac{\mu}{2} \sin(\frac{2}{\mu} \ln \frac{r}{r_0}) + \cos(\frac{2}{\mu} \ln \frac{r}{r_0}) \right] dr. \tag{3.8}
\]
In (3.8), considering the change variable $u = \ln \frac{r}{r_0}$, by $\ell = r_0 e^{\mu \pi}$, we obtain
\[
\frac{1}{r_0^2 (n-1)} \sum_{i=2}^{n} I(f E_i, f E_i) \leq -\frac{1}{4} \int_{0}^{\mu \pi} (4H - 1) \mu^2 \sin^2(\frac{1}{\mu} u) du
\]
\[ + \int_0^{\mu \pi} \left( \cos^2 \left( \frac{1}{\mu} u \right) + \frac{\mu}{2} \sin \left( \frac{2}{\mu} u \right) \right) du \]
\[ + 2k \int_0^{\mu \pi} \left| \frac{\mu}{2} \sin \left( \frac{2}{\mu} u \right) + \cos \left( \frac{2}{\mu} u \right) \right| du. \]  
(3.9)

from which
\[ \frac{1}{r_0^2(n-1)} \sum_{i=2}^N \text{I}(f E_i, f E_i) &\leq \frac{\mu}{8} (4\pi - (4H-1)\pi \mu^2 + 16k \sqrt{\mu^2 + 4}). \]  
(3.10)

In the right hand side of (3.10), if the inequality
\[ 4\pi - (4H-1)\pi \mu^2 + 16k \sqrt{\mu^2 + 4} < 0 \]  
(3.11)
holds, then the index form I is not positive semi-definite. This is a contradiction. Hence, we must take
\[ 4\pi - (4H-1)\pi \mu^2 + 16k \sqrt{\mu^2 + 4} \geq 0. \]  
(3.12)
Thus
\[ \mu \leq \frac{2}{(4H-1)\pi} \sqrt{32k^2 + (4H-1)\pi^2 + 16k \sqrt{4k^2 + (4H-1)H \pi^2}}. \]  
(3.13)

Using the parametrization \( \ell = r_0 e^{\mu \pi} \) given in (3.1), we find
\[ \ell = r_0 e^{\mu \pi} \leq r_0 \exp \left( \frac{2}{4H-1} \sqrt{32k^2 + (4H-1)\pi^2 + 16k \sqrt{4k^2 + (4H-1)H \pi^2}} \right). \]  
(3.14)
Thus, \( M \) is compact and the diameter of \( M \) has the upper bound (1.4).

Proof of Theorem 2. By similar arguments given in the proof of Theorem 1, we have
\[ \sum_{i=2}^N \text{I}(f E_i, f E_i) = \int_0^\ell \left( (n-1) f^2 - f^2 \text{Ric}(\nabla r) \right) dt. \]  
(3.15)

Using the assumption (1.5) in the above integral expression, we get
\[ \sum_{i=2}^N \text{I}(f E_i, f E_i) \leq \int_0^\ell \left( (n-1) f^2 - (N-1) \frac{H f^2}{r^2} \right) dt \]
\[ + \int_0^\ell \left( f^2 \hat{S}(\nabla r) - \frac{f^2 (S(\nabla r))^2}{N-n} \right) dt. \]  
(3.16)
In the inequality (3.16), the term \( f^2 \hat{S}(\nabla r) \) equals to
\[ f^2 \hat{S}(\nabla r) = -2f \hat{f} S(\nabla r) + \frac{d}{dt} (f^2 S(\nabla r)). \]  
(3.17)
Integrating both sides of (3.17), we obtain
\[ \int_{r_0}^{\ell} f^2 \dot{S}(\nabla r) dt = \int_{r_0}^{\ell} -2 f \dot{f} S(\nabla r) dt, \] (3.18)
by \( f(r_0) = f(\ell) = 0 \). If we take \( P = -\dot{f} \) and \( T = f S(\nabla r) \), then the Cauchy-Schwarz inequality
\[ \int_{r_0}^{\ell} P T dt = \int_{r_0}^{\ell} -f \dot{f} S(\nabla r) dt \leq \left( \int_{r_0}^{\ell} \dot{f}^2 dt \right)^{1/2} \left( \int_{r_0}^{\ell} f^2 (S(\nabla r))^2 dt \right)^{1/2}. \] (3.19)
Because of the facts
\[ A = (N - n) \int_{r_0}^{\ell} \dot{f}^2 dt \geq 0 \quad \text{and} \quad B = \frac{1}{N-n} \int_{r_0}^{\ell} f^2 (S(\nabla r))^2 dt \geq 0, \] (3.20)
where \( N \in (n, \infty) \), we have the inequality \( \sqrt{AB} \leq \frac{1}{2} (A + B) \), i.e.,
\[ \left( \int_{r_0}^{\ell} \dot{f}^2 dt \right)^{1/2} \left( \int_{r_0}^{\ell} f^2 (S(\nabla r))^2 dt \right)^{1/2} \leq \int_{r_0}^{\ell} \frac{1}{2} (N-n) \dot{f}^2 dt + \int_{r_0}^{\ell} \frac{f^2 (S(\nabla r))^2}{2(N-n)} dt. \] (3.21)
Using (3.21) in (3.19), we find
\[ \int_{r_0}^{\ell} -f \dot{f} S(\nabla r) dt \leq \int_{r_0}^{\ell} \left( \frac{1}{2} (N-n) \dot{f}^2 + f^2 \frac{(S(\nabla r))^2}{2(N-n)} \right) dt. \] (3.22)
Therefore we have
\[ \int_{r_0}^{\ell} f^2 \dot{S}(\nabla r) dt = \int_{r_0}^{\ell} -2 f \dot{f} S(\nabla r) dt \leq \int_{r_0}^{\ell} \left( (N-n) \dot{f}^2 + f^2 \frac{(S(\nabla r))^2}{N-n} \right) dt. \] (3.23)
Inserting (3.23) into (3.16), we obtain
\[ \sum_{i=2}^{n} I(f E_i, f E_i) \leq (N - 1) \int_{r_0}^{\ell} \left( \dot{f}^2 - \frac{H f^2}{r^2} \right) dt. \] (3.24)
In the inequality (3.24), let us consider the choice
\[ f(t) = \mu r_0 \sqrt{r(\gamma(t))} \sin\left( \frac{1}{\mu} \ln \frac{r(\gamma(t))}{r_0} \right). \] (3.25)
Thereby the inequality (3.24) yields
\[ \frac{1}{N-1} \sum_{i=2}^{n} I(f E_i, f E_i) \leq \int_{r_0}^{\ell} \frac{r_0^2}{r} \left( \cos^2\left( \frac{1}{\mu} \ln \frac{r}{r_0} \right) + \mu \sin^2\left( \frac{2}{\mu} \ln \frac{r}{r_0} \right) \right) dr. \]
In the above inequality, considering the change variable \( u = \ln \frac{r}{r_0} \), by use of \( \ell = r_0 e^{\mu \pi} \), we get

\[
\frac{1}{N-1} \sum_{i=2}^{n} I(f E_i, f E_i) \leq \int_0^{\mu \pi} r_0^2 \left( \cos^2 \left( \frac{1}{\mu} u \right) + \frac{\mu}{2} \sin \left( \frac{1}{\mu} u \right) \right) du
\]

\[
-\frac{1}{4} \int_0^{\mu \pi} r_0^2 (4H - 1) \mu^2 \sin^2 \left( \frac{1}{\mu} u \right) du,
\]

which implies

\[
\frac{1}{N-1} \sum_{i=2}^{n} I(f E_i, f E_i) \leq \frac{r_0^{2\mu \pi}}{8} (4 - (4H - 1) \mu^2) .
\]

In the right hand side of (3.28), if the inequality

\[ 4 - (4H - 1) \mu^2 < 0 \]

holds, then we conclude that the index form \( I \) is not positive semi-definite. But, since \( \gamma \) is minimal geodesic, this is a contradiction. Hence, we must take

\[ 4 - (4H - 1) \mu^2 \geq 0 .
\]

Thus we obtain

\[ \mu \leq \frac{2}{\sqrt{4H - 1}} .
\]

Using the parametrization \( \ell = r_0 e^{\mu \pi} \), we find

\[ \ell = r_0 e^{\mu \pi} \leq r_0 e^{2\pi / \sqrt{4H - 1}} .
\]

Thus, \( M \) is compact and the diameter of \( M \) has the upper bound (1.6).

**Proof of Theorem 3.** We know that \( r(x) = d(x, p) \) is a distance function from a fixed point \( p \in M \) and it is smooth on \( M \) \( - (C_p \cup \{p\}) \). Also it satisfies \( F(\nabla r) = 1 \). In Finsler geometry, recall that the Bochner-Weitzenböck formula [10] for a smooth function \( u \in C^\infty(M) \)

\[
0 = \Delta u \left( \frac{F(\nabla u)^2}{2} \right) = \text{Ric}_\infty(\nabla u) + D(\Delta u)(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)}. \tag{3.33}
\]

From the Bochner formula applied to distance function \( r \) and by Lemma 1, we have, on \( M \) \( - (C_p \cup \{p\}) \),

\[
0 = \text{Ric}_\infty(\nabla r) + D(\Delta r)(\nabla r) + \|\nabla^2 r\|^2_{HS(\nabla r)} \\
geq \text{Ric}_\infty(\nabla r) + g_r(\nabla r \Delta r, \nabla r) + \frac{(\Delta r + S(\nabla r))^2}{n-1}. \tag{3.34}
\]
By virtue of the inequality \((a + b)^2 \geq \frac{1}{\beta + 1} a^2 - \frac{1}{\beta} b^2\) holding for all real numbers \(a, b\) and positive real number \(\beta\), we have

\[
\left( \frac{\Delta r + S(\nabla r)}{n-1} \right)^2 \geq \frac{(\Delta r)^2}{(n-1)(\beta + 1)} - \frac{(S(\nabla r))^2}{(n-1)\beta}.
\] (3.35)

In the case where \(N > n\), taking \(\beta = \frac{N-n}{n-1} > 0\), (3.34) yields

\[
0 \geq \text{Ric}\nabla r + g_{\nabla r}(\nabla r, \Delta r) + \frac{(\Delta r)^2}{N-1} - \frac{(S(\nabla r))^2}{N-n}.
\] (3.36)

Applying the assumption (1.7) given in Theorem 3 to (3.36), we find

\[
0 \geq \partial_r (\Delta r) + \frac{(\Delta r)^2}{N-1} + (N-1)\frac{H}{(1+r)^2}.
\] (3.37)

The above inequality can be rewritten as

\[
0 \geq \partial_r \left( \frac{\Delta r}{N-1} \right) + \left( \frac{\Delta r}{N-1} \right)^2 + \frac{H}{(1+r)^2}.
\] (3.38)

We know from the Hessian comparison theorem in [17], if there is a local vector field \(X\) on an open set \(U\) of \(p \in M\) with \(g_{\nabla r}(X, X) = 1\) and \(g_{\nabla r}(\nabla r, X) = 0\), then \(H(r)(X, X) \sim \frac{1}{r}\) as \(r \to 0^+\). Hence, using the Lemma 1, we have

\[
\lim_{r \to 0^+} r \left( \frac{1}{N-1} \Delta r \right) = \lim_{r \to 0^+} r \left( \frac{1}{N-1} \left( \text{tr}_{\nabla r} H(r) - S(\nabla r) \right) \right) = \frac{n-1}{N-1} < 1. \] (3.39)

where \(N > n\). By (3.38) and (3.39), we obtain, on \(M \setminus \{p\}\),

\[
\frac{1}{N-1} \Delta r \leq \frac{1}{2(1+r)} \left( 1 + \sqrt{4H - 1} \cot \left( \frac{\sqrt{4H - 1}}{2} \ln(1+r) \right) \right),
\] (3.40)

where \(H > 1/4\). Indeed, the function

\[
Y(r) = \frac{1}{2(1+r)} \left( 1 + \sqrt{4H - 1} \cot \left( \frac{\sqrt{4H - 1}}{2} \ln(1+r) \right) \right)
\] (3.41)

is a solution of the Riccati differential equation

\[
Y'(r) + (Y(r))^2 + \frac{H}{(1+r)^2} = 0.
\] (3.42)

Because of \(\lim_{r \to 0^+} rY(r) = 1\) and (3.39), we have

\[
\lim_{r \to 0^+} r \left( \frac{1}{N-1} \Delta r \right) \leq \lim_{r \to 0^+} rY(r).
\] (3.43)

Thus, for a sufficiently small positive constant \(\varepsilon \in (0, T)\) the inequality

\[
\frac{1}{N-1} \Delta r(e) \leq Y(e)
\] (3.44)
is ensured. In that case, the Riccati comparison theorem gives the inequality

\[
\frac{1}{N-1} \Delta r(t) \leq Y(t)
\]

(3.45)

for every \( t \in [\varepsilon, T) \).

Let \( q \in \mathcal{M} \) be any point, and let \( \sigma \) be a minimal unit speed geodesic segment from \( p \) to \( q \). Suppose that the inequality

\[
d(p, q) > e^{2\pi/\sqrt{4H-1}} - 1
\]

(3.46)
is satisfied. Then, since \( \sigma \) is a minimal unit speed geodesic segment from \( p \) to \( q \), we have the fact that the point \( \sigma(e^{2\pi/\sqrt{4H-1}} - 1) \) is outside the cut locus of \( p \in \mathcal{M} \), i.e.,

\[
\sigma(e^{2\pi/\sqrt{4H-1}} - 1) \in \mathcal{M} \setminus (C_p \cup \{p\}).
\]

(3.47)

Therefore the distance function \( r \) is smooth at this point. Namely, at this point, left hand side of (3.40) is a constant. However, the right side of (3.40) tends to \(-\infty\) as \( r \to (e^{2\pi/\sqrt{4H-1}} - 1)^- \), i.e.,

\[
\lim_{r \to (e^{2\pi/\sqrt{4H-1}} - 1)^-} \frac{1}{2(1+r)} \left( 1 + \sqrt{4H-1} \cot \left( \frac{\sqrt{4H-1}}{2} \ln(1+r) \right) \right) = -\infty.
\]

(3.48)

This is a contradiction. Hence, (3.46) does not hold. It must be

\[
d(p, q) \leq e^{2\pi/\sqrt{4H-1}} - 1.
\]

(3.49)

Therefore \( \mathcal{M} \) is compact. Let \( \lambda \) be the reversibility. For any points \( p', q' \in \mathcal{M} \), due to the triangle inequality and the inequality (3.49), we obtain

\[
d(p', q') \leq d(p', p) + d(p, q') \leq \lambda d(p, p') + d(p, q').
\]

(3.50)

and so

\[
d(p', q') \leq (1 + \lambda)(e^{2\pi/\sqrt{4H-1}} - 1).
\]

(3.51)

This completes the proof of theorem.

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