On the non-efficient PAC learnability of acyclic conjunctive queries

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Abstract

This note serves three purposes: (i) we provide a self-contained exposition of the fact that conjunctive queries are not efficiently learnable in the Probably-Approximately-Correct (PAC) model, paying clear attention to the complicating fact that this concept class lacks the polynomial-size fitting property, a property that is tacitly assumed in much of the computational learning theory literature; (ii) we establish a strong negative PAC learnability result that applies to many restricted classes of conjunctive queries (CQs), including acyclic CQs for a wide range of notions of "acyclicity"; (iii) we show that CQs are efficiently PAC learnable with membership queries.

Keywords: Computational Learning Theory, Conjunctive Queries, Inductive Logic Programming

1. Introduction

 Conjunctive queries (CQs) are an extensively studied database query language that plays a very prominent role in database theory. They correspond precisely to Datalog programs with a single non-recursive rule and to the positive-existential-conjunctive fragment of first-order logic. Since the evaluation problem for conjunctive queries is NP-complete, various tractable subclasses have been introduced and studied. These include different variants of acyclicity, in particular α-acyclicity, β-acyclicity, γ-acyclicity, and Berge-acyclicity, which form a strict hierarchy with Berge-acyclicity being most restrictive [9].

In this note, we address the question of the learnability of acyclic CQs from labeled examples, in Valiant’s well-known Probably Approximately Correct (PAC) learning model [23]. We give a self-contained proof that the class of all CQs as well as various classes of acyclic CQs are not efficiently PAC learnable. While the general idea of our proof is due to [16, 13], we strengthen the result and present it in a form that is easily accessible to modern-day database theorists.

The result q(I) of evaluating a k-ary CQ q on a database instance I, is a set of k-tuples of values from the active domain of I. An example, then, is most naturally viewed as a pair (I, a) where I is a database instance, and a a k-tuple of values from the active domain of I. Such a pair is said to be a positive (resp., negative) example for q if a ∈ q(I) (resp., a /∈ q(I)).

An efficient PAC algorithm is a (possibly randomized) polynomial-time algorithm that takes as input a set of examples drawn from an unknown probability distribution D and labeled as positive/negative according to an unknown target CQ q* to be learned, and that outputs a CQ q, such that, if the input sample is sufficiently large, then with probability at least 1 − δ, q has expected error at most ε, meaning that if we draw an example e from D, then with probability 1 − ε, q and q* assign the same label to e (cf. Figure 1). The required number of examples must furthermore be bounded by a polynomial function in |q*|, 1/δ, 1/ε, and the example size. We give a precise definition in Section 2.

Note that a PAC algorithm must perform well regardless of the choice of example distribution D. Furthermore, the polynomial bound is not allowed to explicitly depend on D (other than on the size of the examples). In this sense, the PAC model captures a strong form of distribution-independent learning.

We say that an example distribution D is single-instance if there exists a single database instance I such that all examples to which D assigns non-zero probability mass, are of the form (I, a) for some tuple a from the active domain of I. We will restrict our attention to single-instance example distributions. This means that the input of the PAC algorithm is a set of examples that all share the same database instance, and can be equivalently represented as (I, S+, S−) where I is a database instance and S+, S− are sets of tuples from the active domain of I (corresponding to the positive and negative examples). By restricting to single-instance distributions, we capture the natural scenario of learning CQs from positive and negative examples, where the examples are values from a single given database instance. Clearly, efficient PAC learnability w.r.t. all example distributions implies efficient PAC learnability w.r.t. single-instance example distributions.

Our main result is the following, stated, for simplicity, for unary CQs:

**Theorem 1.1.** (assuming RP ≠ NP) Let C be any class of unary CQs over a fixed schema S that contains at least a binary relation symbol and a unary relation symbol. If C

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includes all path-CQs, then C is not efficiently PAC learnable, even w.r.t. single-instance example distributions.

Here, by RP we mean the class of problems solvable by a randomized algorithm with one-sided error that runs in polynomial time, and by a path-CQ we mean a unary CQ of the form

$$q(x_1) := \exists x_2 \ldots x_n( R(x_1, x_2) \wedge R(x_2, x_3) \ldots \wedge R(x_{n-1}, x_n) \wedge P(x_{j_1}) \wedge \ldots \wedge P(x_{j_m}))$$

where $R$ is a binary relation symbol and $P$ is a unary relation symbol. That is, a path-CQ is a very simple type of CQ that describes an outgoing directed path of some given length, together with additional constraints involving a single unary relation symbol.

Note that efficient PAC learnability is not a monotone property of concept classes, and Theorem 1.1 says more than just that path-CQs are not efficiently PAC learnable.

In particular, Theorem 1.1 implies that the class of all CQs is not efficiently PAC learnable. As we mentioned earlier, various notions of acyclicity have been proposed for CQs. Since path-CQs are acyclic under each of these definitions, we obtain that none of the corresponding classes of CQs is efficiently PAC learnable. Finally, Theorem 1.1 also implies non-efficient PAC learnability of concept expressions in the description logics $\mathcal{EL}$ and $\mathcal{ELI}$ (even in the absence of a TBox) [3].

It is worth comparing the notion of a PAC learning algorithm to that of a fitting algorithm. Both types of algorithms take as input a set of labeled examples. A fitting algorithm decides the existence of a CQ that agrees with the labels of the input examples. The fitting problem is coNExpTime-complete for CQs [24, 19] and, in fact, is known to be hard already for some more restricted classes of acyclic CQs [19, 10, 11]. A PAC algorithm, on the other hand, produces a CQ that, with high probability, has a low expected error, but is not required to fit the input examples. Furthermore, a PAC algorithm operates on inputs for which a fitting CQ is promised to exist. There are also other subtle differences between the two problems (e.g., the running time of a PAC algorithm is allowed to depend on the size of the target CQ, while the complexity of fitting problems is traditionally measured as a function of the size of the input examples only). In spite of these differences, the two problems are closely related to each other (cf. Proposition 2.6 below). Indeed, the proof of Theorem 1.1 is based on a reduction from a suitable fitting problem.

We complement this with the following positive result:

**Theorem 1.2.** Fix any schema $S$ and $k \geq 0$. The class of all $k$-ary CQs over $S$ is efficiently PAC learnable with membership queries.

### 1.1. Related work

Haussler [13] showed that the class of Boolean CQs over a schema that contains an unbounded number of unary relation symbols is not PAC-learnable (unless RP = NP). The essential part of the proof is to show that the fitting problem for the same concept class is NP-complete. Over a schema that consists of unary relation symbols only, every CQ is trivially Berge-acyclic. Therefore, this implies the non-PAC learnability of any CQs over a fixed finite schema that consists of unary relation symbols, however, is an important difference. Indeed, if one were to consider Boolean queries over a fixed finite schema that consists of unary relation symbols only, then the resulting concept class would be finite and trivially PAC learnable.

Kietz [16] proved that the class of unary CQs over a schema that contains a single binary relation symbol and an unbounded number of unary relation symbols is not PAC-learnable (unless RP = NP). Again the essential part of the proof is to show that the fitting problem is NP-complete. Kietz’s result already applies to path-CQs of length 1 with multiple unary relation symbols. This is only possible because of the infinite schema, as, otherwise, the concept class is again finite and trivially PAC learnable.

Hirata [14] showed that $\alpha$-acyclic conjunctive queries are not efficiently PAC learnable. The lower bound proof uses queries of unbounded arity. It is pointed out in [14] that their result only applies to the class of $\alpha$-acyclic CQs and the status of other classes of acyclic CQs was left
as an open problem. Thus, Theorem 1.1 improves over the results in [14] in two ways: (a) our hardness result holds already for unary CQs, and (b) it applies not only to the class of α-acyclic CQs but to a wide range of classes of acyclic CQs. On the other hand, the hardness result in [14] is stronger in that it pertains to the PAC prediction problem where the output of the algorithm is not required to be a concept from the concept class, but can be any polynomial-time evaluable concept.

We focus, in this note, on classes of CQs defined through acyclicity conditions. In the literature on inductive logic programming (ILP) various positive and negative PAC learnability results have been obtained for classes of CQs defined by different means (e.g., limitations on the use of existential variables, determinacy conditions pertaining to functional relations, and restricted variable depth). These are orthogonal to acyclicity. An overview can be found in [17, Chapter 18].

In [22], the authors study learnability of GAV schema mappings, which are closely related to Unions of Conjunctive Queries (UCQs). Specifically, it was proved in [22] that GAV schema mappings are not efficiently PAC learnable, assuming RP ≠ NP, on source schemas that contain at least one relation symbol of arity at least two, using a reduction of the non-PAC-learnability of propositional formulas in positive DNF. This result immediately implies that, for any schema S containing a relation symbol of arity at least two, and for each k ≥ 0, the class of k-ary UCQs over S is not efficiently PAC learnable, assuming RP ≠ NP. Additionally, in [22], the authors completely map out the (non-)learnability of restricted classes of UCQs definable by conditions on their Gaifman graph.

There is also another line of work on PAC learnability of conjunctive queries [7, 8, 5] that is somewhat different in nature: one fixes a schema S and an S-instance I and defines a concept class where the concepts are now all relations over the active domain of I definable by a k-ary CQ (as evaluated in I). PAC learning for various classes of Boolean formulas, such as 3-CNF, can be seen as a special case of this framework, for a specific choice of schema S and (two-element) instance I, where k then corresponds to the number of Boolean variables. Since, for a fixed choice of k, this yields a finite concept class, in this setting, one is interested in the complexity of PAC learning as a function of k. The mentioned papers establish effective dichotomies, showing that, depending on the choice of S and I, this concept class is either efficiently PAC learnable in k or is not even PAC predictable with membership queries (under suitable cryptographic assumptions). See also Remark 5.3 below.

2. Preliminaries

2.1. Conjunctive Queries

A schema S is a finite set of relation symbols with associated arity. An instance I over schema S is a finite set of facts over S, where a fact is an expression of the form R(a_1, . . . , a_n) where R ∈ S is an n-ary relation symbol and a_1, . . . , a_n are values. The active domain of an instance I, denoted by adom(I) is the (finite) set of values that occur in the facts of I.

A k-ary conjunctive query (CQ) over a schema S, for k ≥ 0, is an expression of the form

q(x) := ∃y(α_1 ∧ ⋯ ∧ α_n)

where x, y are tuples of variables, x has length k, and each conjunct α_i is an atomic formula using a relation symbol from S, such that each variable from x occurs in some conjunct. We denote by q(I) the set of all k-tuples a such that I |= q(a).

We will not define in depth the various notions of acyclicity that have been introduced and studied for CQs, but we mention that these include the notions mentioned in the introduction, that is, α-acyclicity, β-acyclicity, γ-acyclicity, and Berge-acyclicity. A classic result states that α-acyclic queries can be evaluated in polynomial time [25], by which we mean that, given an α-acyclic CQ q(x), an instance I and a tuple a of elements of the active domain of I, we can test in polynomial time whether a ∈ q(I).

The definition of path-CQs was already given in the introduction.

Example 2.1. An example of a path-CQ is the query

q(x) := ∃yzu(R(x, y) ∧ R(y, z) ∧ R(z, u) ∧ P(y) ∧ P(u))

Every path-CQ is Berge-acyclic, and hence, by the aforementioned result, path-CQs can be evaluated in polynomial time.

2.2. Computational Learning Theory

A concept class is a triple C = (Φ, Ex, ⊨), where Φ is a set of concepts, Ex is a set of examples, and ⊨ ⊆ Ex × Φ represents whether an example is a positive or a negative example for a given concept. We also denote by lab_φ(e) the label of e according to φ, that is, lab_φ(e) = + if e |= φ and lab_φ(e) = − otherwise. Two concepts φ, φ’ ∈ Φ are said to be equivalent if lab_φ(e) = lab_φ(e) for all e ∈ Ex.

A labeled example is a pair (e, s) with e ∈ Ex and s ∈ {+, −}. A concept φ ∈ Φ fits a set of labeled examples E if lab_φ(e) = s for all (e, s) ∈ E.

We will only consider countable concept classes in this paper. Concepts and examples are assumed to have an effective representation and a corresponding notion of size, which will be denoted by |φ| and |e|, respectively. We will also denote the set of all concepts (examples) of size at

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2This deviates slightly from the standard convention, which defines a concept class as a pair (Ex, C) where C ⊆ p(Ex) (and which defines, for e ∈ C, |e| as the size of the smallest representation of e). The difference in presentation is non-essential. We prefer this presentation as it makes it easier to spell out unambiguously the algorithmic problems that we consider (e.g., Definition 2.2).
most \( n \) by \( \Phi^{(n)} \) (respectively, \( E_x^{(n)} \)). For a finite set of (possibly labeled) examples \( E, ||E|| = \sum_{e \in E} |e| \).

The following two properties of concept classes will be important for us later on:

**Definition 2.2 (Polynomial-time evaluality).** A concept class is polynomial-time evaluable if there exists a polynomial-time algorithm that, given \( \phi \in \Phi \) and \( \epsilon \in Ex \), tests whether \( \epsilon \models \phi \).

**Definition 2.3 (Polynomial-size fitting property).** A concept class has the polynomial-size fitting property if for every finite set of labeled examples \( E \), the existence of a concept that fits \( E \) implies that there exists a fitting concept whose size is bounded by a polynomial in \( ||E|| \).

We now define the two algorithmic problems mentioned in the introduction, namely fitting and PAC learning.

**Definition 2.4 (Fitting problem).** The fitting problem (also known as consistency problem or separability problem) for a concept class \( C \) is the problem to decide, given a finite set of labeled examples \( E \), whether there exists a concept in \( C \) that fits \( E \).

In order to define PAC algorithms, we first need to introduce some terminology and notation. An example distribution for a concept class \( C = (\Phi, Ex, \models) \) is a probability distribution \( D \) over \( Ex \). Given concepts \( \phi, \phi^* \in \Phi \) and an example distribution \( D \),

\[
\text{error}_{\phi, \phi^*}(D) = \text{Pr}_{e \sim D}(\text{lab}_\phi(e) \neq \text{lab}_{\phi^*}(e))
\]

is the expected error of \( \phi \) relative to \( \phi^* \) and \( D \).

**Definition 2.5 (Efficient PAC learnability).** An efficient PAC algorithm for a concept class \( C \) is a pair \((A, f)\) where

- \( A \) is a randomized polynomial-time algorithm that takes a set of labeled examples and outputs a concept, and
- \( f(\cdot, \cdot, \cdot) \) is a polynomial function, such that, for all \( \delta, \epsilon \in \mathbb{R}^+ \), for all \( n, m \in \mathbb{N} \), for all example distributions \( D \) over \( Ex^{(n)} \), and for all \( \phi^* \in \Phi^{(n)} \), if the input consists of at least \( f(1/\delta, 1/\epsilon, n, m) \) examples drawn from \( D \) and labeled according to \( \phi^* \), then with probability at least \( 1 - \delta \), \( A \) outputs a concept \( \phi \) with error \( \phi, \phi^*, D(\phi) < \epsilon \).

4In general, a probability distribution over \( Ex \) is of the form \( D = (A, \mu) \) where \( A \subseteq 2^{Ex} \) is a \( \sigma \)-algebra, i.e., a collection of subsets (called “measurable events”) closed under complement and countable union, and where \( \mu: A \to [0, 1] \) such that \( \mu(\emptyset) = 0, \mu(Ex) = 1 \), and \( \mu(\bigcup_{i=1}^\infty X_i) = \sum_i \mu(X_i) \) for every countable sequence \( X_1, X_2, \ldots \) of pairwise disjoint events. Since in our case, \( Ex \) is always countable, we may assume that \( A = 2^{Ex} \), i.e., every event is measurable. In particular, this ensures that \( \text{error}_{\phi, \phi^*}(D) \) is well-defined.

If such an algorithm exists, we say that \( C \) is efficiently PAC learnable. If the function \( f \) depends only on \( \delta \) and \( \epsilon \) and not on \( n, m \), then we say that \((A, f)\) is a strongly efficient PAC algorithm, and that the concept class \( C \) is strongly efficiently PAC learnable.

This definition of efficient PAC algorithms is modeled after the one in the textbook [2] and is in line with the literature on inductive logic programming (cf., e.g., [17]). Alternatively, PAC learning algorithms can be defined as interactive algorithms which do not take examples as input, but rather, have access to a “sampling oracle” which provides randomly drawn labeled examples upon request. There are some subtle differences between the possible definitions, but our results hold regardless.4 We choose to work with the above definition, as it exhibits more clearly the relationship to fitting algorithms.

The following proposition relates the two algorithmic problems (fitting and PAC learning) to each other.

**Proposition 2.6 (Pitt and Valiant [18]).** Let \( C \) be a polynomial-time evaluable concept class with the polynomial-size fitting property. If \( C \) is efficiently PAC learnable, then the fitting problem for \( C \) is in \( RP \).

This is a well-known fact (cf. also [2, Thm 6.2.1]), although not in this precise formulation, as, usually, polynomial evaluable and the polynomial-size fitting property are tacitly assumed. To be self-contained, we outline the argument here.

4 Following the oracle-based presentation in, e.g., [15], one can define an efficient PAC learning algorithm for a concept class \( C \) to be a randomized polynomial-time algorithm that takes as input rational numbers \( \delta \) and \( \epsilon \) as well as a number \( n \) that bounds the size of the target concept \( \phi^* \), and has access to an oracle \( Ex^{\phi^*}_D \) which, when called, returns (in unit time) a random example drawn from \( D \) and labeled according to \( \phi^* \). For every choice of rational numbers \( \delta, \epsilon \in (0, 1) \), \( \phi^* \in \Phi \), \( n \geq |\phi^*| \), and for every example distribution \( D \), the algorithm must terminate in time polynomial in \( 1/\delta, 1/\epsilon, n \), and the size of the largest examples returned by the oracle during the given run of the algorithm. Furthermore, the algorithm must return a concept that with probability \( 1 - \delta \) satisfies \( \text{error}_{\phi, \phi^*, D}(\phi) < \epsilon \). This definition is used, e.g., in [22].

Note that, under this oracle-style definition of PAC learning algorithms, unlike in Definition 2.5, (i) \( D \) ranges over all distributions over \( Ex \), and (ii) not only the running time but also the number of examples drawn from the distribution is allowed to depend in a dynamic way on the size of examples drawn from the distribution. That is, if the learning algorithm encounters a large example \( e \), it may follow up by requesting a number of additional examples that is polynomial in the size of \( e \).

It is not hard to see that efficient PAC learnability in the above sense implies PAC learnability in the sense of Definition 2.5. Since our main result is a negative learnability result, it applies to both.

Note that a large part of the computational learning theory literature is concerned with learning domains where the examples space is either \( \mathbb{R}^d \) or \( \{0, 1\}^d (d \geq 1) \) and the analysis is parameterized by \( d \).

In those cases, the size of examples is automatically fixed on ce and \( d \) is fixed. In contrast, the learning domains we consider here involve examples that can be of unbounded size, even for a fixed finite schema and/or a fixed target concept. The subtleties regarding example size described above are particularly relevant in our setting.
Proof. (of Proposition 2.6) Assume that there is an efficient PAC algorithm \((A, f)\) for \(C\). We use it to solve the fitting problem for \(C\) in randomized polynomial time. Assume that a set \(E\) of \(k\) labeled examples is given as the input. Let \(n = p(||Ex||)\), where \(p\) is the polynomial witnessing the fact that \(C\) has the polynomial-size fitting property. Let \(D\) be the uniform distribution on \(E\) (where each example in \(E\) gets probability mass \(1/k\)), and let \(m\) be the maximum size of an example in \(E\). Pick \(\delta < .5\) and \(\epsilon < 1/k\). We generate a new (polynomial-sized) collection of labeled examples \(E'\) by drawing \(f(1/\delta, 1/\epsilon, n, m)\) samples from distribution \(D\), and run algorithm \(A\) on it. Finally, we check that the output of \(A\) is a fitting concept for \(E\). If so, we answer Yes. Otherwise, we answer No.

Clearly, if there is no fitting concept, the output will be No. If, on the other hand, there is a fitting concept, then there is one of size at most \(n\), and hence, with probability \(1 - \delta\), the algorithm will output a concept with error less than \(\epsilon\). This in fact implies that the error is 0 (because if the query misclassifies an example to which \(D\) assigns non-zero mass, then it will have error at least \(1/k\)). Hence, with probability \(1 - \delta > 0.5\) the algorithm outputs Yes. \(\square\)

A variation on the same argument shows:

Proposition 2.7. If a concept class is strongly efficiently PAC learnable, then it has the polynomial-size fitting property.

Proof. The proof uses the same construction as before, except that the sample size now does not depend on \(n\). Furthermore, we omit the verification step where we confirm that the produced concept fits the input examples. Instead, we just output the result of the learning algorithm. In this way, we obtain a randomized polynomial-time algorithm that has a non-zero probability of outputting a fitting concept for given input labeled examples, whenever a fitting concept exists. The polynomial-size fitting property immediately follows from this (the run that outputs a fitting concept does so in polynomial time). \(\square\)

We will also make use of the following trivial fact:

Proposition 2.8. If a concept class \((\Phi, Ex, =)\) is efficiently PAC learnable, then, for every \(Ex' \subseteq Ex\), the concept class \((\Phi, Ex', =)\) is also efficiently PAC learnable.

Indeed, this follows from the fact that every example distribution over \(Ex'(n)\) is in particular also an example distribution over \(Ex(n)\) (that assigns no probability mass to any example in \(Ex \setminus Ex'\)).

Finally, we will also make use of a well known connection between PAC algorithms and Occam algorithms.

Definition 2.9 (Occam algorithm). An Occam algorithm for a concept class \(C = (\Phi, Ex, =)\), with parameters \(\alpha < 1\) and \(k \geq 1\), is an algorithm that takes as input a collection of examples \(E\) labeled according to some unknown concept \(\phi^* \in \Phi\) and produces a concept \(\phi \in \Phi\) fitting the input examples \(E\), such that \(|\phi| \leq |E|^\alpha |\phi^*|^k\). Furthermore, the running time is required to be bounded by a polynomial in \(|\phi^*|\) and \(||E||\).

Blumer et al. [4] proved that every Occam algorithm \(A\) yields an efficient PAC algorithm, namely \(A' = (A, f)\), where the sample-size polynomial \(f\) is chosen such that

\[ f(1/\delta, 1/\epsilon, n, m) = \left(\frac{n^k \ln 2 + \ln(2/\delta)}{\epsilon}\right)^{1/(1-\alpha)}. \]

Note that \(f\) does not depend on its fourth component \(m\) (i.e., the example size bound). Moreover, every Occam algorithm gives rise to an efficient PAC algorithm, not only in the sense of Definition 2.5 as explained above, but, by the same arguments, also when considering the oracle-based presentation of PAC algorithms (cf. footnote 4).

Theorem 2.10 ([4]). Every concept class for which there is an Occam algorithm is efficiently PAC learnable.

3. Classes of CQs as Concept Classes

Each class of CQs can be naturally viewed as a concept class. More precisely, fix a schema \(S\), an arity \(k \geq 0\), and a class \(C\) of \(k\)-ary CQs over \(S\). Each such choice of \(S, k,\) and \(C\) gives rise to a concept class \((C, Ex, =)\) where \(Ex\) is the class of all pairs \((I, a)\) with \(I\) an \(S\)-instance and \(a\) a \(k\)-tuple of elements of the active domain of \(I\), and \(=\) describes query answers, that is, \((I, a) = q(x)\) iff \(a \in q(I)\), for all \(q(x) \in C\) and \((I, a) \in Ez\). We will abuse notation and refer to this concept class \((C, Ex, =)\) simply as \(C\) when no ambiguity arises.

Theorem 3.1 ([19, 21]). Fix any schema \(S\) that contains at least one binary relation symbol, and some \(k \geq 0\).

1. The concept class of \(k\)-ary CQs over \(S\) is not polynomial-time evaluable (unless \(P = NP\)). Indeed, its evaluation problem is NP-complete.
2. The concept class of \(k\)-ary CQs over \(S\) lacks the polynomial-size fitting property. Indeed, the smallest fitting CQ for a given set of labeled examples is in general exponentially large.
3. The fitting problem for \(k\)-ary CQs over \(S\) is coNExpTime-complete.

By Proposition 2.7, this implies that CQs are not strongly efficiently PAC learnable. On the other hand, Proposition 2.6 does not immediately apply, and it thus does not follow from Theorem 3.1 that CQs are not efficiently PAC-learnable.

Let us now consider what happens when we shift from the class of all CQs to restricted classes of (unary) CQs that still include path-CQs. We will see in the next section that every such class of CQs has an NP-hard fitting problem (cf. Theorem 4.6). Furthermore, every such class of CQs lacks the polynomial-size fitting property:
Theorem 3.2. Fix a schema $S$ that contains at least a binary and a unary predicate, and let $C$ be any class of unary CQs over $S$ that includes all path-CQs. Then $C$ lacks the polynomial-size fitting property.

Proof. For $m \geq 1$, let $L_m$ denote the “lasso” instance, with active domain $a_0^m, \ldots, a_{2m-1}^m$ consisting of the facts $R(a_i^m, a_{i+1}^m)$ for all $i < 2m-1$ and $R(a_{2m-1}^m, a_0^m)$ and $P(a_i^m)$.

For $i \geq 1$, let $p_i$ be the $i$-th prime number (where $p_1 = 2$). Note that, by the prime number theorem, $p_i = O(i \log i)$.

Finally, for $n \geq 1$, let $I_n$ be the disjoint union of $L_{p_i}$ for $i = 1, \ldots, n$, extended with the facts $R(b, b)$ for a fresh value $b$. We now construct our set of examples $E_n$ as follows:

- Positive example $(I_n, a_0^0)$ for $i = 1 \ldots n$.
- Negative example $(I_n, b)$.

It is easy to see that a fitting path-CQ for $E_n$ exists, namely the query

$q(x_1) := \exists x_2 \ldots x_k (R(x_1, x_2) \land \cdots \land R(x_k-1, x_k) \land P(x_k))$

where $k = \Pi_{i=1 \ldots n}(p_i)$.

We claim that every CQ that fits the examples must be of size at least $2^n$. Let $q(x)$ be any CQ that fits the examples. Since positive and negative examples are based on the same instance, we may assume that $q$ is connected. First of all, note that $q$ must contain a conjunct of the form $P(y)$ (otherwise it would fail to fit the negative example). Furthermore, $y$ is not the free variable $x$ and $q$ uses only the relation symbols $P$ and $R$ (otherwise it would fail to fit any positive example). Consider any path of minimal length from $x$ to $y$ in the directed graph where the vertices are the variables of $q$ and there is an edge from variable $z$ to variable $z'$ iff the atom $R(z, z')$ occurs in $q$. We can think of this path as an oriented path consisting of forward edges and backward edges. We define the net-length of this path as the number of forward edges minus the number of backward edges. We require the net-length to be divisible by $p_i$ for all $i = 1 \ldots n$ as otherwise $q$ would fail to fit the positive example $(I_i, a_0^i)$. Hence, the net-length must be at least $\Pi_{i=1 \ldots n}(p_i)$. It follows, then, that also the net-length (in the ordinary sense) of the path must be at least $\Pi_{i=1 \ldots n}(p_i)$. Therefore, every CQ that fits the above examples must have at least $\Pi_{i=1 \ldots n}(p_i)$ variables, which exceeds $2^n$. \hfill \square

The concept class of path-CQs is polynomial-time learnable, as follows from the fact that it forms a subclass of the class of $\alpha$-acyclic CQs, which is polynomial-time learnable [25]. We will make use of this in the next section.

Theorem 3.4 ([25]). Fix any schema $S$. The concept class of path-CQs over $S$ is polynomial-time learnable.

4. Non-Efficient PAC Learnability

The aim of this section is to improve on Corollary 3.3 by showing that no class of unary CQs that includes all path-CQs is efficiently PAC learnable (cf. Theorem 1.1 as stated in the introduction). For ease of exposition, in this section, we restrict attention to unary CQs.

Recall that we cannot use Proposition 2.6 directly to prove non-efficient PAC learnability, for two reasons. First, the polynomial-size fitting property does not hold for path-CQs. And second, the classes that we consider may contain CQs that are not path-CQs, and thus polynomial-time learnability also fails, despite Theorem 3.4. To circumvent the latter issue, we work with a restricted class of instances.

4.1. Tree-Shaped Instances

Definition 4.1 (Tree-Shaped Instances and CQs). Let $S$ be a schema that consists of a binary relation symbol $R$ and any number of unary relation symbols, and let $I$ be an $S$-instance. We say that $I$ is tree-shaped if the following two conditions hold:

1. There is a function level: adom($I$) $\rightarrow \mathbb{N}$ such that, for each fact $R(a, b)$ of $I$, level($b$) = level($a$) + 1.
2. $I$ does not contain any binary fact $R(a, b)$, $R(a', b)$ that agree on the second value but not on the first.

A CQ over $S$ is said to be tree-shaped if its canonical instance is tree-shaped.\footnote{The canonical instance of a CQ is simply the instance whose active domain consists of the variables of the query and whose facts are the conjuncts of the query.}

Lemma 4.2. Fix a schema $S$ that consists of one binary relation symbol and any number of unary relation symbols. Given a CQ $q$ over $S$,

1. we can test in polynomial time whether there exists a tree-shaped instance $I$ such that $q(I) \neq \emptyset$.
2. if the answer to the above question is positive, then we can construct in polynomial time a tree-shaped CQ $q'$ such that for all tree-shaped instances $I$, $q(I) = q'(I)$.

Proof. It suffices to prove the claim for connected CQs (the general case then follows by a component-wise analysis). Therefore, let $q$ be a connected CQ.
Let \( \sim \) be the smallest equivalence relation over the variables of \( q \) such that, whenever \( R(a, v) \) and \( R(u, v') \) are conjuncts of \( q \) and \( v \sim v' \) then also \( u \sim u' \). Let \( q' \) be the quotent of \( q \) w.r.t. \( \sim \). It is easy to see that, for all tree-shaped instances \( I \), \( a \in q(I) \) iff \( a \in q'(I) \) (here, the left-to-right direction uses the tree-shape of \( I \), while the right-to-left direction holds for every instance \( I \)).

If \( q' \) contains a directed cycle, then clearly, \( q'(I) = \emptyset \) for all tree-shaped instances \( I \). Hence, in this case, we are done.

Now assume that \( q' \) does not contain a directed cycle. Since \( q' \) is connected, there must then exist a (free or existentially quantified) variable \( y \) for which \( q' \) does not contain any conjunct of the form \( R(\cdot, y) \). Furthermore, any simple path from \( y \) to any other variable \( z \) must consist entirely of forward edges (otherwise, the path would be of the form \( y \overset{R}{\to} \cdots \overset{R}{\to} u \overset{R}{\leftarrow} v \overset{R}{\to} \cdots \overset{R}{\to} z \) and then \( u \) and \( w \) would have been identified when we constructed \( q' \)). It follows that \( q' \) is tree-shaped. Furthermore, let \( I_{q'} \) be the canonical instance of \( q' \). Then, clearly, \( q'(I_{q'}) \neq \emptyset \).

Since tree-shaped CQs are \( \alpha \)-acyclic, and hence can be evaluated in polynomial time (on arbitrary instances) \cite{DBLP:journals/jcss/LiP13}, Lemma 4.2 immediately implies:

**Proposition 4.3.** Fix a schema \( S \) consisting of one binary relation symbol and any number of unary relation symbols. For every class \( C \) of CQs over \( S \), the concept class \( (C, Ex_{\text{tree}}, =) \), where \( Ex_{\text{tree}} \) is the set of tree-shaped \( S \)-instances, is polynomial-time evaluable.

In what follows, we will therefore only work with tree-shaped instances.

### 4.2. A reduction from 3CNF satisfiability

Fix a schema \( S \) containing a binary relation symbol \( R \) and a unary relation symbol \( P \).

We make use of a reduction from the satisfiability problem for 3CNF formulas, inspired by \cite{DBLP:journals/tcs/BaruttiCF12, DBLP:conf/lpar/ChenCF08}. Let \( \phi = \phi_1 \land \cdots \land \phi_k \) be any 3CNF formula over a propositional signature \( \text{PROP} = \{ X_1, \ldots, X_m \} \). We denote by \( \text{LIT} = \{ X_i, \overline{X_i} \mid i \leq m \} \) the set of all literals over \( \text{PROP} \).

We define an \( S \)-instance \( I_{\phi} \) as follows

- \( R(a_i, p_{i,1}) \) and \( R(a_i, n_{i,1}) \) for \( i \leq m \)
- \( R(p_{i,j}, p_{i,j+1}) \) and \( R(n_{i,j}, n_{i,j+1}) \) for \( i \leq m, j < 2m \)
- For every literal \( l \in \text{LIT} \setminus \{ X_i \} \), we add \( P(p_{i,j}) \) and for every literal \( l \in \text{LIT} \setminus \{ X_i \} \), we add \( P(n_{i,j}) \)
- \( R(b, b_{i,1}) \) for \( i \leq k \)
- \( R(b_{i,j}, b_{i,j+1}) \) for \( i \leq k \) and \( b \leq 2m \)
- For each \( l \in \text{LIT} \) and \( i \leq k \) with \( l \) not occurring in the clause \( \phi_i \), we add \( P(b_{i,j}) \)

where \( j_i \), for a literal \( l \), is defined as \( 2i \) if \( l \) is of the form \( X_i \), and \( 2i - 1 \) if \( l \) is of the form \( \overline{X_i} \).

Let \( E_{\phi} = \{ ( (I_0, a_i), +) \mid i \leq m \} \cup \{ ( (I_0, b), -) \} \).

**Example 4.4.** Let \( \text{PROP} = \{ X_1, X_2 \} \) and consider the formula \( \phi = X_1 \land X_2 \land (\overline{X_1} \lor X_2) \). Then, the corresponding \( S \)-instance \( I_{\phi} \) can be depicted as follows (where each edge represents a directed \( R \)-edge from the upper node to the lower node):

![Diagram](image)

**Lemma 4.5.** For all 3CNF formulas \( \phi \):

1. From a satisfying assignment for \( \phi \), one can construct in polynomial time a path-CQ that fits \( E_{\phi} \).
2. Conversely, if there is a CQ that fits \( E_{\phi} \), then \( \phi \) has a satisfying assignment.

In particular, whenever there is a CQ that fits \( E_{\phi} \), then there is a fitting path-CQ of size polynomial in \( |\text{PROP}| \).

**Proof.** 1. Let \( v \) be a satisfying assignment for \( \phi \). Let

\[
q(x_0) := \exists x_1, \ldots, x_{2m} (R(x_0, x_1) \land \cdots \land R(x_{2m-1}, x_{2m}) \land \bigwedge_{l \in \text{LIT}} \phi_l(x) )
\]

Clearly, each \( a_i \in q(I_{\phi}) \) and \( b \notin q(I_{\phi}) \).

2. Let \( q(x) \) be a unary CQ that fits \( E_{\phi} \). By Lemma 4.2, we may assume that \( q \) is a tree-shaped CQ. Furthermore, we may assume without loss of generality that \( q \) is connected. Let \( \text{level}_q : \text{Vars}(q) \to \mathbb{N} \) be as given by Definition 4.1. We may assume \( \text{level}_q(x) = 0 \) (if there was any \( y \in \text{Vars}(q) \) with \( \text{level}_q(y) < \text{level}_q(x) \), then \( q \) would not fit the positive examples of \( E_{\phi} \)).

Thus, \( q(x) \) is a connected tree-shaped CQ, where \( x \) is the root of the tree. Since \( q(x) \) fits the negative example \( (I_0, b) \), we have that \( b \notin q(I_{\phi}) \). This means that either (i) \( q \) contains a conjunct of the form \( P(x) \), or (ii) for some \( y \in \text{Vars}(q) \) with \( \text{level}_q(y) = 1 \), the subtree of \( q \) rooted at \( y \) does not admit a homomorphism to \( (I_0, b_{i,1}) \) for any \( i \leq n \). It is easy to see that (i) cannot happen, because it would imply that \( q \) does not fit the positive examples in \( E_{\phi} \). Therefore, case (ii) must apply. Let \( y \) be the variable in question, and let us denote by \( q'(y) \) the subtree of \( q \) rooted at \( y \) (with \( y \) as its free variable).

We know that \( q'(I_{\phi}) \) does not contain \( b_{i,1} \) for any \( i \leq n \). Furthermore, it is easy to see (from the fact that \( q \) fits the positive examples in \( E_{\phi} \)), that for each \( i \leq m \), either \( p_{i,1} \) or \( n_{i,1} \) belongs to \( q'(I_{\phi}) \).
Now, let $L_y$ be the set
\[ \{ l \in \text{LIT} \mid q' \text{ has a conjunct } P(z) \text{ with } \text{level}_q(z) = j_l + 1 \} \]

Claim 1: $L_y$ does not contain both $X_i, \overline{X}_i$ for any $i \leq m$.

Claim 1 follows immediately from the fact that $q(x)$ fits the positive examples.

Claim 2: $L_y$ contains a literal from each clause of $\phi$.

Suppose, for the sake of a contradiction, that $\phi$ has a clause $\phi_i$, such that no literal occurring in $\phi_i$ belongs to $L_y$. Then, $b_{i,1}$ belongs to $q'(I_\phi)$, as witnessed by the variable assignment that maps each variable $z$ to $b_{i,\text{level}_q(z)}^{-1}$. However, we know that $b_{i,1} \notin q'(I_\phi)$, a contradiction.

Claim 1 and 2 together imply that $\phi$ is satisfiable. Indeed, it suffices to take any truth assignment consistent with the literals in $L_y$. \hfill $\square$

From Lemma 4.5, together with the NP-hardness of 3CNF satisfiability, we immediately get:

**Theorem 4.6.** Fix any schema $S$ containing at least a binary relation symbol and a unary relation symbol, and let $C$ be any class of unary CQs over $S$ that includes all path-CQs. Then the fitting problem for $C$ is NP-hard.

Now, putting everything together, we can prove Theorem 1.1, restated here:

**Theorem 4.7.** (assuming $RP \neq NP$) Fix a schema $S$ containing at least a binary relation symbol and a unary relation symbol. Let $C$ be any class of unary CQs over $S$ that includes all path-CQs. Then $C$ is not efficiently PAC learnable, even w.r.t. single-instance distributions.

**Proof.** Let $Ex' = \{(I,a) \in Ex \mid I = I_\phi$ for some 3CNF-formula $\phi\}$. If the concept class $C = (C, Ex, \models)$ is efficiently PAC learnable, then, by Proposition 2.8, so is the concept class $C' = (C, Ex', \models)$. By Lemma 4.5, $C'$ has the polynomial-size fitting property. Furthermore, $C'$ is polynomial-time evaluable (Proposition 4.3). Therefore, by Proposition 2.6, if $C'$ is efficiently PAC learnable, then the fitting problem for $C'$ is solvable in RP. By Theorem 4.6, this implies that $RP = NP$.

Indeed, if we carefully inspect the proof of Proposition 2.6 and the construction of our examples, we see that even efficient PAC learnability w.r.t. single-instance distributions already gives us, in this way, an RP-algorithm for the fitting problem for $C'$.

**Remark 4.8.** The above proof involves path-CQs of unbounded depth, over a fixed schema. It is easy to see that if we were to bound both the depth of the path-CQs and keep the schema fixed, we would end up with a finite concept class, trivializing the PAC learning problem.

**Remark 4.9.** The above non-learnability proof cannot be adapted to UCQs. In fact, we crucially use the fact that the fitting problem for path-CQs is NP-hard whereas the fitting problem for UCQs that are unions of path-CQs can be solved in polynomial time. On the other hand, as mentioned earlier, it follows from results in [22] that UCQs are not efficiently PAC learnable, assuming RP $\neq$ NP.

**Remark 4.10.** The fact that the above proof involved a reduction from the satisfiability problem for 3CNF formulas is remarkable, given that 3CNF formulas are themselves efficiently PAC learnable [15].

**Remark 4.11.** The definition of efficient PAC learnability given in Definition 2.5 is also known as strong PAC learnability. In contrast, weak PAC learnability merely requires the existence of a learner that works for some non-trivial choice of $\delta$ and $\epsilon$. A well-known result in computational learning theory states that, for polynomial-time evaluable concept classes, weak learnability implies strong learnability (cf. [15]). Since the concept class of CQs is not polynomial-time evaluable, Theorem 1.1, taken at face value, does not imply that the same result holds in the weak PAC model. Nevertheless, inspection of our proof immediately shows that it yields the same result also for the weak PAC model.

### 5. PAC Learnability with Membership Queries

The concept class of all CQs is known to be efficiently exactly learnable with membership and equivalence queries, in Angluin’s [1] model of exact learning (this is implicit in [22] and a self-contained explicit proof can be found in [20, full version available on arXiv]). This result may seem counter-intuitive given that the concept class in question is not polynomially evaluable, but note that membership queries give the algorithm a means to effectively perform a limited form of evaluation, namely to evaluate the target query on a given example.

As pointed out in [22], the fact that CQs are efficiently exactly learnable with membership and equivalence queries, implies PAC learnability with membership queries and an NP-oracle (cf. [1]), where the NP-oracle is used for evaluating hypotheses on examples. It was left open in [22] whether CQs are efficiently PAC learnable with membership queries without an NP-oracle. We give an affirmative answer to this question here.

Formally, a membership oracle $\text{MEMB}_\phi$, for a concept $\phi$, is an oracle that, given any unlabeled example $e$, returns (in unit time) its label according to $\phi$. PAC learning with access to a membership oracle for the target concept, can be viewed as a formal model of active learning.

**Theorem 5.1.** Fix any schema $S$ and $k \geq 0$. There is an algorithm that takes as input a set $E$ of examples labeled according to any $k$-ary CQ $q^*$ over $S$ and has access to a membership oracle for $q^*$, and that outputs a $k$-ary CQ $q$
over \( S \) fitting the input examples with \(|q| \leq |q^*|\). Moreover, the running time of the algorithm is polynomial in \(||E||\) and \(|q^*|\).

**Proof.** The proof uses very similar ideas to the ones used in the aforementioned proof that CQs are efficiently exactly learnable with membership and equivalence queries [22, 20]. Before we describe the algorithm, we introduce a number of basic concepts.

Let \( I, J \) be instances over the same schema. A mapping \( h: \text{dom}(I) \to \text{dom}(J) \) is called homomorphism from \( I \) to \( J \) if \( R(h(c)) \in J \) for every \( R(c) \in I \). Given tuples \( a \) and \( b \) of values from \( I \) and \( J \), respectively, we write \((I, a) \to (J, b)\) to denote the existence of a homomorphism \( h \) from \( I \) to \( J \) with \( h(a) = b \). Homomorphisms compose in the sense that \((I, a) \to (J, b)\) and \((J, b) \to (K, c)\) imply \((I, a) \to (K, c)\).

The direct product \( I \times J \) of two instances (over the same schema \( S \)), is the \( S \)-instance consisting of all facts of the form \( R((a_1, b_1), \ldots, (a_n, b_n)) \), where \( R(a_1, \ldots, a_n) \) is a fact of \( I \) and \( R(b_1, \ldots, b_n) \) is a fact of \( J \). Note that the active domain of \( I \times J \) consists of pairs from \( \text{dom}(I) \times \text{dom}(J) \).

The direct product \( I \times J \) of two instances (over the same schema \( S \)), is the \( S \)-instance consisting of all facts of the form \( R((a_1, b_1), \ldots, (a_n, b_n)) \), where \( R(a_1, \ldots, a_n) \) is a fact of \( I \) and \( R(b_1, \ldots, b_n) \) is a fact of \( J \). Note that the active domain of \( I \times J \) consists of pairs from \( \text{dom}(I) \times \text{dom}(J) \).

A critical positive example for a CQ \( q^* \) is a positive example \((I, a) \to (J, b)\) such that, for every proper instance \( I' \subseteq I \), \((I', a) \) is a negative example for \( q^* \).

The following two claims are easy to prove (cf. [20]):

**Claim 1:** Given a positive example \((I, a)\) for an unknown CQ \( q^* \), we can construct from it in linear time a critical positive example \((I', a)\) for \( q^* \), with \( I' \subseteq I \), given access to a membership oracle for \( q^* \).

**Claim 2:** If \((I, a)\) and \((J, b)\) are positive examples for a CQ \( q^* \), then \((I, a) \times (J, b)\) is a well-defined example, and it is a positive example for \( q^* \).

Given a set \( E \) of labeled examples, the algorithm now proceeds as follows. Let \((I_1, a_1), \ldots, (I_n, a_n)\) be an enumeration of the positive examples in \( E \). We construct, by induction on \( n \), a critical positive example \((J, b)\) such that there is a homomorphism from \((J, b)\) to each \((I_i, a_i)\). This is done by applying Claim 1 and Claim 2 in an interleaved fashion. More precisely:

- **Set** \((J, b) = (J_n, b_n)\). **Note** that, by Claim 2 and the fact that homomorphisms compose, each \((J_i', b_i')\) is a well-defined example that has a homomorphism to all examples \((I_1, a_1), \ldots, (I_i, a_i)\). Thus, \((J, b)\) has a homomorphism to all positive examples. Let \( b = b_1, \ldots, b_k \) and let \( q \) be the canonical CQ of \((J, b)\), that is, the CQ \( q(x_{b_1}, \ldots, x_{b_k}) \) that has a conjunct for every fact of \( J \), where each element \( b \in \text{dom}(J) \) is replaced by a corresponding variable \( x_b \). Then it is easy to see that \( q \) fits the positive examples in \( E \) (this follows from the fact that \((J, b)\) has a homomorphism to each positive example), and \( q \) fits the negative examples of \( E \) (because we know that \((J, b)\) is a positive example for \( q^* \), and \( q \) failed to fit one of the negative examples, this would mean that \((J, b)\) has a homomorphism to the negative example in question, which, by composition of homomorphisms, would lead to a contradiction with the fact that \( q^* \) fits the negative example in question).

Furthermore, one can easily see that any critical positive example \((I, a)\) for \( q^* \) satisfies \(|I| \leq |q^*|\). Hence, each \( J_i \) satisfies \(|J_i| \leq |q^*|\). This implies, in particular, that \(|q| \leq |q^*| \) as required. Moreover, it implies that \(|J_i'| \in O(||E|| \cdot |q^*|)\), for all \( i \). Since \( J_i \) is obtained from \( J_i' \) in linear time by Claim 1, the running time of this algorithm is \( O(||E||^2 \cdot |q^*|)\).

The algorithm given in Theorem 5.1 is an Occam algorithm (with \( \alpha = 0 \) and \( k = 1 \)) in the sense of Definition 2.9, except for the fact that it uses a membership oracle. While Theorem 2.10 is stated for the case without membership queries, its proof applies also to Occam algorithms with membership queries, yielding efficient PAC learnability with membership queries (stated as Theorem 1.2 in the introduction):

**Corollary 5.2.** Fix any schema \( S \) and \( k \geq 0 \). The class of all \( k \)-ary CQs over \( S \) is efficiently PAC learnable with membership queries.

**Remark 5.3.** The proof of Theorem 5.1 establishes something stronger, namely that CQs are efficiently PAC learnable with membership queries even when the schema \( S \) and the arity \( k \) are not fixed but treated as part of the input of the learning task. This is remarkable, because it follows from results in [7] that CQs are not PAC predictable with membership queries, when the arity is treated as part of the input (under suitable cryptographic assumptions). However, note that efficient PAC learnability (with membership queries) implies PAC predictability (with membership queries) only for concept classes that are polynomial-time evaluable, which the class of CQs is not.

**Remark 5.4.** We expect that, with respect to each of the various notions of “acyclicity” mentioned in the introduction, acyclic CQs are efficiently PAC learnable with membership queries. However, since efficient PAC learnability (with or without membership queries) is not a mono-
tune property of concept classes, this presumably requires a case-by-case analysis.

**Remark 5.5.** The proof of Theorem 5.1 can easily be modified to apply to the concept class of Unions of Conjunctive Queries (UCQs), and, consequently, to the concept class of GAV schema mappings, showing that both are efficiently PAC learnable with membership queries. This resolves an open question in [22].

6. Conclusion

We established a strong negative PAC learnability result that pertains to any class of CQs that includes path-CQs. On the other hand, we showed that CQs are efficiently PAC learnable with membership queries.

We conclude by discussing two other ways in which one could try to overcome the main negative result.

6.1. PAC prediction

In the PAC prediction model, the output of an algorithm does not have to be from the concept class of the target concept, but instead must be from any polynomial-learnable concept class. Note that the PAC prediction model is not necessarily a weakening of PAC learning: it is so only when the concept class is itself polynomial-time learnable. We leave it as an open question whether a version of Theorem 1.1 can be proved for the PAC prediction model. Note that Cohen [6] showed that CQs are not efficiently PAC-predictable (under suitable cryptographic assumptions) when the arity of the query is unbounded.

6.2. Loosening the running time requirements

Another approach towards overcoming our negative result is to loosen the running time requirements on a PAC learning algorithm. In particular, we could allow the running time to depend in a superpolynomial way on the size of the target concept.

**Theorem 6.1 ([25]).** Fix a finite schema S, and let C be any class of α-acyclic CQs over S. The concept class C is polynomial-time learnable. Indeed, its evaluation problem can be solved by an algorithm that runs in time \( O(|q| \cdot |e|) \).

By brute-force enumeration, this implies:

**Proposition 6.2.** Fix a finite schema S, and let C be any class of α-acyclic CQs. The fitting problem can be solved by an algorithm that, on inputs \( E \) for which a fitting concept \( q \) exists, terminates in time \( 2^{O(|q|)} \cdot O(|E|) \) and produces a fitting concept of size at most \(|q|\).

Note that, in particular, the fitting algorithm described in Proposition 6.2 is guaranteed to produce a fitting concept of minimal size. That is (modulo the running time requirements), it is an Occam algorithm in the sense of Definition 2.9.

Now, recall that we defined an efficient PAC algorithm as a pair \((A, f)\), where \( A \) is a randomized polynomial-time algorithm that takes a set of labeled examples and produces a concept, and \( f \) is a polynomial function satisfying suitable conditions. We could weaken this, by allowing \( A \) to run within the time bounds described in Proposition 6.2. Let us refer to this modified definition as weakly efficient PAC algorithm. Then, it follows from Proposition 6.2 and the proof of Theorem 2.10 that:

**Proposition 6.3.** Fix a finite schema S, and let C be any class of α-acyclic CQs over S. The concept class C has a weakly efficient PAC algorithm.

This raises a number of questions: (1) is the \( 2^{O(|q|)} \cdot O(|E|) \) bound provided by Proposition 6.3 optimal? (2) does the class of all CQs admit a weakly efficient PAC algorithm? and if not, (3) can we characterize, in the style of Grohe [12], the classes of CQs that admit a weakly efficient PAC algorithm.

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