Abstract. We investigate the conditions that are sufficient to make the Ext-algebra of an object in a (triangulated) category into a Frobenius algebra, and compute the corresponding Nakayama automorphism. As an application, we prove the conjecture that $\text{hdet}(\mu_A) = 1$ for any noetherian Artin-Schelter regular (hence skew Calabi-Yau) algebra $A$.

1. Introduction

Let $k$ be an algebraically closed field. The goal of this paper is to study some categories with nice duality theories that arise from Artin-Schelter (or AS for short) Gorenstein $k$-algebras. By studying the Ext-algebras of objects in these categories we will obtain several interesting consequences for such Gorenstein algebras. This paper is a sequel to [RRZ]. In particular, we make significant progress on a conjecture in [RRZ] about the homological determinant of a Nakayama automorphism (see Theorem 1.3 below).

An especially pleasing form of duality is encapsulated in the definition of a Calabi-Yau triangulated category [Ke]. This is a Hom-finite $k$-linear triangulated category with a duality of the form
\[(E1.0.1)\quad \text{Hom}(X,Y)^* \cong \text{Hom}(Y, \Sigma^d X),\]
where $(-)^*$ denotes the $k$-linear dual and $\Sigma$ is the translation functor. Of course, the theory of Serre duality is the model for this definition, and the bounded derived category of coherent sheaves over a smooth projective variety with trivial dualizing sheaf is an important example. Other interesting examples include the bounded derived categories of finite-dimensional modules over Calabi-Yau algebras [Gi].

We are especially interested in derived categories of graded modules for those graded algebras that are important in noncommutative algebraic geometry, such as AS regular or Gorenstein algebras [AS], in which case the duality theory above does not capture all examples of interest. For this reason we study more general dualities of the form
\[(E1.0.2)\quad \text{Hom}(X,Y)^* \cong \text{Hom}(Y, \Sigma^d T^i \Phi(X)),\]
where $T$ is an automorphism of a triangulated category (typically coming from a shift of grading on modules), $\Phi$ is another automorphism called the Nakayama functor, and $\Sigma, T,$ and $\Phi$ all commute. Under some additional technical conditions, we call a $k$-linear Hom-finite triangulated category satisfying a duality of this form a skew Calabi-Yau triangulated category (see Section 2.3 for the precise definition).

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It is well-known that the duality condition (E1.0.1) endows each Ext-algebra $E(X) = \bigoplus_{i=0}^{d} \text{Hom}(X, \Sigma^i X)$ with the structure of a graded-symmetric Frobenius algebra. Our first theorem shows that this extends in some cases to our more general kind of duality.

**Theorem 1.1** (Theorem 2.7). If $C$ is a skew Calabi-Yau triangulated category with Nakayama functor $\Phi$, and $X$ is an object such that there is an isomorphism $\phi : X \rightarrow \Phi(X)$, then the Ext-algebra $E(X) = \bigoplus_{i,j} \text{Hom}(X, \Sigma^i T^j X)$ has the structure of a bigraded Frobenius algebra.

Furthermore, there is an explicit formula for the Nakayama automorphism $\mu$ of the Frobenius algebra $E(X)$ in terms of the given isomorphism $\phi$.

The main examples we have in mind are certain subcategories of derived categories of modules over Gorenstein rings. Recall that an N-graded k-algebra $A = \bigoplus_{i \geq 0} A_i$ is called AS Gorenstein if it is connected ($A_0 = k$) and locally finite ($\dim_k A_i < \infty$ for all $i \geq 0$), $A$ has finite injective dimension $d$ as an $A$-module, and $\text{Ext}^d_A(k, A) \cong k(1)$ (as graded modules), while $\text{Ext}^i_A(k, A) = 0$ if $i \neq d$. Here, $k = A/A_1$ is the trivial module and $k(1)$ is its graded shift by 1. If $A$ has finite global dimension in addition, then $A$ is called AS regular.

Let $A$ be noetherian AS Gorenstein, and let $\mathcal{D}_\epsilon(A)$ be the subcategory of the bounded derived category of finitely generated $\mathbb{Z}$-graded $A$-modules consisting of perfect complexes with finite-dimensional cohomology. Let $\text{Aut}_\mathbb{Z}(A)$ be the group of all graded algebra automorphisms of $A$. The algebra $A$ has an associated Nakayama automorphism $\mu_A \in \text{Aut}_\mathbb{Z}(A)$ (see Definition 3.1). The operation $M \mapsto \mu_M$ which twists the action on a graded left module $M$ by this automorphism induces a functor $\Phi : \mathcal{D}_\epsilon(A) \rightarrow \mathcal{D}_\epsilon(A)$. An object $X \in \mathcal{D}_\epsilon(A)$ is called $\Phi$-plain if $\phi(X) \cong X$, and we let $\mathcal{D}_{\epsilon \text{pl}}(A)$ be the thick subcategory of $\mathcal{D}_\epsilon(A)$ generated by $\Phi$-plain complexes. The complex shift functor $\Sigma : X \mapsto X[1]$ and the functor $T$ induced by the graded shift of modules $M \mapsto M(1)$ restrict to this category $\mathcal{D}_{\epsilon \text{pl}}(A)$. We prove the following.

**Proposition 1.2** (Proposition 3.3 and Theorem 3.5). Let $A$ be noetherian AS Gorenstein. Then $\mathcal{D}_{\epsilon \text{pl}}(A)$, with its functors $\Sigma$, $T$, and $\Phi$ as above, is a skew Calabi-Yau triangulated category.

In fact, we prove the proposition for a more general class of $\mathbb{N}$-graded algebras called generalized AS Gorenstein (Definition 3.1) which are locally finite but not necessarily connected.

If $A$ is noetherian (connected) AS Gorenstein, then each graded algebra automorphism of $A$ has a corresponding homological determinant, which is a nonzero scalar, and where $\text{hdet} : \text{Aut}_\mathbb{Z}(A) \rightarrow k$ is multiplicative. The homological determinant is fundamental to the study of group actions on noncommutative graded algebras (see Section 4 for more details). One of our motivating goals is to prove that the homological identity $\text{hdet}(\mu_A) = 1$ holds in wide generality. This identity has several significant applications; see [RRZ Corollaries 0.5, 0.6, 0.7].

In this paper, as an application of Theorem 1.1, we prove

**Theorem 1.3** (Theorem 5.3). If $A$ is noetherian connected AS Gorenstein and $\mathcal{D}_\epsilon(A) \neq 0$, then $\text{hdet}(\mu_A) = 1$.

The homological identity $\text{hdet}(\mu_A) = 1$ was proved in [RRZ Theorem 6.3] in the special case that $A$ is noetherian Koszul AS regular, and was conjectured to hold for all noetherian AS Gorenstein algebras in [RRZ Conjecture 6.4]. The hypothesis $\mathcal{D}_\epsilon(A) \neq 0$ of Theorem 1.3 seems very weak and we conjecture that it holds automatically. In any case, in Section 5
below we show that $D_\epsilon(A) \neq 0$ holds under very general conditions which cover most known
AS Gorenstein algebras. In particular, $D_\epsilon(A) \neq 0$ holds when $A$ is noetherian AS regular,
and so the conjecture is proved for all such regular algebras (Corollary 5.4).

As a second application of Theorem 1.1 we recover and generalize a result of Berger and
Marconnet [BM], as follows. We recall that connected graded noetherian AS regular algebras
$A$ are examples of skew Calabi-Yau algebras (see [RRZ Definition 0.1], [RRZ Lemma 1.2])
and are Calabi-Yau algebras when $\mu_A = 1$, a special case of particular interest.

**Theorem 1.4 (Theorem 4.2).** Let $A$ be connected graded noetherian AS regular of di-
mension $d$ with Nakayama automorphism $\mu_A$, where $A$ is generated in degree 1, and let $E := \bigoplus_i \text{Ext}_A^i(k,k)$ be the Ext-algebra of $A$. Let $\mu_E$ denote the Nakayama automorphism
of the Frobenius algebra $E$. Then identifying $(A_1)^* \quad \text{with} \quad E^1 = \text{Ext}_A^1(k,k), \text{we have} \quad \mu_E|_{E^1} = (-1)^{d+1}(\mu_A|A_1)^*$. In particular, $\mu_A = 1$ (that is, $A$ is a Calabi-Yau algebra) if and only if $E$
is a graded-symmetric Frobenius algebra.

Portions of this theorem have been proved in the case of (not necessarily noetherian)
$N$-Koszul algebras; see [BM, Theorem 6.3] for the description of $\mu_E|_{E^1}$ and [HVOZ, Proposition 3.3]
for the characterization of when $A$ is Calabi-Yau. But many regular algebras of
dimension 4 and higher are not $N$-Koszul. In fact, Theorem 1.2 describes $\mu_E$ entirely, not
just in degree 1, provided one can calculate an explicit isomorphism of complexes between
the minimal free resolution $X$ of $k$ and its twist $\Phi(X)$ by $\mu_A$. In addition, in Theorem 1.2 we
give a more general result which applies to certain not necessarily connected algebras. The
theorem also has further applications in noncommutative invariant theory; see Section 4.2.

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2. CALABI-YAU CATEGORIES AND SERRE FUNCTORS

2.1. (Bi)-graded categories. This section follows Van den Bergh’s treatment of Serre
duality and Calabi-Yau triangulated categories given in [Bo Appendix A]. Another standard reference
on the subject is [Ke].

Throughout the paper we work over an algebraically closed field $k$. Recall that a category
$\mathcal{C}$ is $k$-linear if each $\text{Hom}_\mathcal{C}(X,Y)$ is a $k$-vector space, composition of morphisms is $k$-bilinear,
and $\mathcal{C}$ has finite biproducts. For convenience, we assume that all categories in this paper are
$k$-linear, although some definitions make sense in greater generality.

A graded category is a pair $(\mathcal{C}, \Sigma)$, where $\mathcal{C}$ is a $k$-linear category, and $\Sigma$ is a $k$-linear
automorphism of $\mathcal{C}$. Given such a category with objects $X, Y \in \mathcal{C}$, we define

$$\text{Hom}^i(X,Y) = \text{Hom}(X, \Sigma^iY),$$

and then define the graded Hom-set

$$\text{Hom}^{gt}(X,Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(X,Y).$$
Graded composition is given, for $g \in \text{Hom}^i(Y, Z)$ and $f \in \text{Hom}^i(X, Y)$, by $g \ast f = \Sigma^i(g) \circ f$. It is easy to check that graded composition is associative. This allows one to define the category $\mathcal{C}^{gr}$ with the same objects as $\mathcal{C}$, but using graded Hom-sets and composition as above. In particular, for any object $X \in \mathcal{C}$, this endows $E(X) := \text{Hom}^{gr}(X, X) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(X, \Sigma^i X)$ with the structure of a $\mathbb{Z}$-graded $k$-algebra.

A graded functor between graded categories is a pair $(U, \eta^U): (\mathcal{C}, \Sigma_\mathcal{C}) \to (\mathcal{D}, \Sigma_\mathcal{D})$, where $U: \mathcal{C} \to \mathcal{D}$ is a $k$-linear functor and $\eta^U: U \circ \Sigma_\mathcal{C} \to \Sigma_\mathcal{D} \circ U$ is a natural isomorphism. To simplify notation, we write $\eta^U = \eta$ and write both $\Sigma_\mathcal{C}$ and $\Sigma_\mathcal{D}$ as $\Sigma$, when there is no chance of confusion. Given a graded functor $(U, \eta)$, for every integer $i \geq 1$ there is an associated natural isomorphism $\eta^i: U \circ \Sigma^i \to \Sigma^i \circ U$, defined on an object $X$ to be the composite

$$\eta^i_X: U \circ \Sigma^i(X) \xrightarrow{\eta_{\Sigma^i-1(X)}} \Sigma \circ U \circ \Sigma^{i-1}(X) \to \cdots \to \Sigma^{i-1} \circ U \circ \Sigma(X) \xrightarrow{\eta_X} \Sigma^i \circ U(X).$$

Of course, the use of the symbol $\eta^i$ is a (harmless) abuse of notation. We also get an induced natural isomorphism $\eta^{-1}: U \circ \Sigma^{-1} \to \Sigma^{-1} \circ U$ defined on an object $X$ by $(\eta^{-1})_X = \Sigma^{-1}(\eta_{\Sigma^{-1}(X)})$. Then the powers $\eta^{-i} = (\eta^{-1})^i$ are determined as above for $i \geq 1$. Finally, let $\eta^0: U \to U$ be the identity natural isomorphism by convention. It is routine to see that with these definitions, the following formula holds for all $i, j \in \mathbb{Z}$:

$$(E2.0.1) \quad \eta^{i+j}_X = \Sigma^i(\eta^j_X) \circ \eta^j_{\Sigma^{i}(X)}.$$

Now using (E2.0.1), one may easily check that the graded functor $U$ induces a functor $U^{gr}: \mathcal{C}^{gr} \to \mathcal{D}^{gr}$, which agrees with $U$ on objects and is defined on a homogeneous element $f \in \text{Hom}^i_c(X, Y) = \text{Hom}_{\mathcal{C}}(X, \Sigma^i Y)$ by

$$U^{gr}(f) = \eta^i_Y \circ U(f).$$

Suppose that $(U, \eta^U), (V, \eta^V): (\mathcal{C}, \Sigma_\mathcal{C}) \to (\mathcal{D}, \Sigma_\mathcal{D})$ are graded functors. A morphism of graded functors is a natural transformation $h: U \to V$ such that, for every object $X$ of $\mathcal{C}$, we have $\Sigma_\mathcal{D}(h_X) \circ \eta^U_X = \eta^V_X \circ h_{\Sigma_\mathcal{C}X}$. Of course, $h$ is called an isomorphism of graded functors if it is a natural isomorphism (whose inverse will necessarily be graded), and in this case we write $(U, \eta^U) \cong (V, \eta^V)$.

One can also compose graded functors in the obvious way. If $(U, \eta^U): (\mathcal{C}_1, \Sigma_1) \to (\mathcal{C}_2, \Sigma_2)$ and $(V, \eta^V): (\mathcal{C}_2, \Sigma_2) \to (\mathcal{C}_3, \Sigma_3)$ are graded functors, then the composite $(V, \eta^V) \circ (U, \eta^U)$ is defined to be $(V \circ U, \eta^{V \circ U})$: $(\mathcal{C}_1, \Sigma_1) \to (\mathcal{C}_3, \Sigma_3)$, where $\eta^{V \circ U}$ is defined for an object $X \in \mathcal{C}_1$ by $\eta^{V \circ U}_X = \eta^V_{\Sigma^i(X)} \circ V(\eta^i_X)$.

In this paper, we also need a bigraded (or multi-graded) version of the above constructions, which is a routine generalization. We only define the bigraded case. The extension to the case of more than two automorphisms to get multi-graded categories is similar, and is left to the reader. Suppose that the $k$-linear category $\mathcal{C}$ has two automorphisms $\Sigma$ and $T$, which commute in the sense that $\Sigma T = T \Sigma$ as functors. We say that $(\mathcal{C}, \Sigma, T)$ is a bigraded category. We can define bigraded Hom sets $\text{Hom}^{gr}(X, Y) = \bigoplus_{i,j} \text{Hom}^{ij}(X, Y)$, where $\text{Hom}^{ij}(X, Y) = \text{Hom}(X, \Sigma^i T^j Y)$. Composition is done in the obvious way: if $g \in \text{Hom}^{kl}(Y, Z)$ and $f \in \text{Hom}^{ij}(X, Y)$, then

$$g \ast f = \Sigma^i T^j(g) \circ f,$$

which is in $\text{Hom}(X, \Sigma^i T^j \Sigma^k T^l Z) = \text{Hom}(X, \Sigma^{i+k} T^{j+l} Z) = \text{Hom}^{i+k,j+l}(X, Z)$. Note that since we assume that $\Sigma$ and $T$ strictly commute, rather than $\Sigma T$ and $T \Sigma$ only being naturally
isomorphic, we silently commute $\Sigma$ and $T$ in formulas like this. Similarly as in the singly graded case, it is routine to check that graded composition is associative. Thus we may define a category $\mathcal{C}^{gr}$ with the same objects as $\mathcal{C}$, but with morphisms given by bigraded Hom-sets and composition as above. In particular, the endomorphism ring $\text{Hom}^{gr}(X, X)$ is a bigraded $k$-algebra, for any object $X$.

A bigraded functor between graded categories

$$(U, \eta^U, \theta^U) : (\mathcal{C}, \Sigma_\mathcal{C}, T_\mathcal{C}) \to (\mathcal{D}, \Sigma_\mathcal{D}, T_\mathcal{D})$$

is a $k$-linear functor $U : \mathcal{C} \to \mathcal{D}$ together with natural isomorphisms

$$\eta^U : U \circ \Sigma_\mathcal{C} \to \Sigma_\mathcal{D} \circ U \text{ and } \theta^U : U \circ T_\mathcal{C} \to T_\mathcal{D} \circ U,$$

satisfying the following compatibility condition: for every object $X \in \mathcal{C}$, there is a commutative diagram

$$(E2.0.2) \quad \begin{array}{ccc}
U \Sigma_\mathcal{C} T_\mathcal{C} X & \xrightarrow{\Sigma_\mathcal{D}(\theta_X) \circ \eta_{\Sigma_\mathcal{C}(X)}} & UT_\mathcal{C} \Sigma_\mathcal{C} X \\
\downarrow \Sigma_\mathcal{D}(\eta_X) \circ \theta_{\Sigma_\mathcal{C}(X)} & & \downarrow T_\mathcal{D}(\eta_X) \circ \theta_{\Sigma_\mathcal{C}(X)} \\
\Sigma_\mathcal{D} T_\mathcal{D} U X & \xrightarrow{T_\mathcal{D}(\eta_X) \circ \theta_{\Sigma_\mathcal{C}(X)}} & T_\mathcal{D} \Sigma_\mathcal{D} U X
\end{array}$$

where the horizontal arrows are induced from the respective equalities $\Sigma_\mathcal{C} T_\mathcal{C} = T_\mathcal{C} \Sigma_\mathcal{C}$ and $\Sigma_\mathcal{D} T_\mathcal{D} = T_\mathcal{D} \Sigma_\mathcal{D}$. Similarly as above, for simplicity from now on we will omit the superscripts on $\eta$ and $\theta$ and the subscripts on $\Sigma$ and $T$.

2.2. Triangulated categories and exact functors. The graded categories $(\mathcal{C}, \Sigma)$ we are interested in will typically be triangulated, and when this is the case $\Sigma$ will stand for the translation functor $X \mapsto X[1]$ unless otherwise indicated. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories with translation functors $\Sigma_\mathcal{C}$ and $\Sigma_\mathcal{D}$ respectively. A graded functor $(U, \eta, \theta) : \mathcal{C} \to \mathcal{D}$ is called exact if $U$ maps exact triangles in $\mathcal{C}$ to exact triangles in $\mathcal{D}$, that is, for any exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_\mathcal{C} X,$$

the triangle in $\mathcal{D}$

$$U(X) \xrightarrow{U(f)} U(Y) \xrightarrow{U(g)} U(Z) \xrightarrow{\eta_X \circ U(h)} \Sigma_\mathcal{D} U(X)$$

is exact.

Remark 2.1. Let $\mathcal{C}$ be a triangulated category, considered as a graded category $(\mathcal{C}, \Sigma)$ where $\Sigma$ is the translation functor. Then

$$\Sigma^+ := (\Sigma, \eta_X = \text{id}_{\Sigma(X)})$$
is a graded endofunctor of \((\mathcal{C}, \Sigma)\); however, \(\Sigma^+\) need not be exact. On the other hand,
\[
\Sigma^- := (\Sigma, \eta_X = -\text{id}_{\Sigma^2(X)})
\]
is a graded and exact functor of the triangulated category \((\mathcal{C}, \Sigma)\) \([\text{Bo}, \text{Example A.3.2}]\). Thus \(\Sigma^-\) is the natural way to make \(\Sigma\) into a graded functor in the triangulated setting.

2.3. **Categories with Serre functors.** Serre functors were introduced by Bondal and Kapranov in \([\text{BK}]\). A thorough treatment of Serre functors is given by Reiten and Van den Bergh in \([\text{RV}, \text{§I.1}]\), and from the perspective of graded categories in Van den Bergh’s appendix \([\text{Bo}, \text{Appendix A}]\). Let \(\text{Hom}\) denote \(\text{Hom}_\mathcal{C}\) for an abstract category \(\mathcal{C}\). The \(k\)-linear category \(\mathcal{C}\) is called **Hom-finite** if \(\text{Hom}(X,Y)\) is a finite-dimensional \(k\)-space for all \(X,Y \in \mathcal{C}\).

**Definition 2.2.** Given a \(k\)-linear Hom-finite category \(\mathcal{C}\), an autoequivalence \(F : \mathcal{C} \to \mathcal{C}\) is called a **(right) Serre functor** if for all \(X,Y \in \mathcal{C}\) there are isomorphisms \(\alpha_{X,Y} : \text{Hom}(X,Y) \to \text{Hom}(Y,FX)^*\) that are natural in both \(X\) and \(Y\).

For a given object \(X \in \mathcal{C}\), setting \(Y = X\) in the natural isomorphism above, there is a distinguished element \(\text{Tr}_X = \alpha_{X,X}(\text{id}_X) \in \text{Hom}(X,FX)^*\). This is the **trace map**
\[
\text{Tr}_X : \text{Hom}(X,FX) \to k.
\]
The trace map provides a nondegenerate bilinear pairing
\[
\text{Tr}_X(- \circ -) : \text{Hom}(Y,FX) \times \text{Hom}(X,Y) \to k,
\]
which satisfies the following identity for \((g,f) \in \text{Hom}(Y,FX) \times \text{Hom}(X,Y)\):
\[
\text{Tr}_X(g \circ f) = \text{Tr}_Y(F(f) \circ g),
\]
see \([\text{Bo}, \text{p. 29 (3)}]\). Conversely, the action of the functor \(F\) on objects along with the trace maps \(\text{Tr}_X\) are enough to determine the functor \(F\) and the isomorphisms \(\alpha_{X,Y}\) \([\text{RV}, \text{Prop. I.1.4}]\).

Suppose that \(F\) is a Serre functor on a Hom-finite, \(k\)-linear graded category \((\mathcal{C}, S)\). We claim that there is an induced commutation rule for the functors \(F\) and \(S\). Fix the isomorphisms \(\alpha_{X,Y}\) which give the Serre duality. Also, fix a nonzero scalar \(\epsilon \in k^\times := k \setminus \{0\}\). Consider the diagram
\[
\begin{align*}
\text{Hom}(X,Y) &\xrightarrow{\alpha_{X,Y}} \text{Hom}(Y,FX)^* \xrightarrow{(S^*)^{-1}} \text{Hom}(SY,SSFX)^* \\
\text{Hom}(SX,SY) &\xrightarrow{\alpha_{SX,SY}} \text{Hom}(SY,FSX)^*
\end{align*}
\]
in which all solid arrows are isomorphisms. Thus for every pair of objects there is an induced isomorphism \(\beta_{X,Y} : \text{Hom}(SY,SSFX)^* \to \text{Hom}(SY,FSX)^*\) such that the diagram commutes up to the scalar \(\epsilon\), in other words such that
\[
\beta_{X,Y} \circ (S^*)^{-1} \circ \alpha_{X,Y} = \epsilon(\alpha_{SX,SY} \circ S).
\]
The isomorphisms \(\beta_{X,Y}\) are clearly also natural in \(X\) and \(Y\). Fixing \(X\) and varying \(Y\), we get an isomorphism of functors \(\text{Hom}(\cdot,SSFX)^* \to \text{Hom}(\cdot,FSX)^*\), and thus by the Yoneda
embedding, an isomorphism of objects \( \eta_X : FSX \to SFX \). These \( \eta_X \) are natural in \( X \) and so define a natural isomorphism \( \eta = \eta_k(S) : FS \to SF \). Note that \( \beta_{XY} \) is then given by the \( k \)-linear dual of the map \( f \mapsto \eta_Y \circ f \) for \( f \in \text{Hom}(SY, FSX) \). One can also give \( \eta \) in terms of the trace functions, as is done in Van den Bergh’s appendix to [Bo]. Namely, specializing the diagram above to the case \( X = Y \), we see that \( \eta = \eta_k(S) \) satisfies the formula [Bo, p. 30 (4)]

\[
\text{Tr}_X(S^{-1}(\eta_X \circ f)) = \epsilon \text{Tr}_X(f),
\]

for all objects \( X \) and morphisms \( f : SX \to FSX \) of \( C \). One can check that the collection of trace identities in [2.2.2] for all objects \( X \) uniquely determines the natural isomorphism \( \eta \).

**Remark 2.3.** Suppose that \( C \) is a Hom-finite \( k \)-linear triangulated category with translation functor \( \Sigma \), such that \( C \) has a Serre functor \( F \).

1. We would like to also require that the natural transformation \( \eta_k(\Sigma) : F\Sigma \to \Sigma F \) makes \( F \) into an exact functor of \((C, \Sigma)\). This is always the case when one chooses \( \epsilon = -1 \) [Bo, Theorem A.4.4]. Thus \( \eta_{-1}(\Sigma) \) is the natural way of commuting \( F \) and \( \Sigma \) in a triangulated category.

2. On the other hand, for many other isomorphisms \( T \) of \( C \), it is most natural to take \( \epsilon = 1 \) and consider \( \eta_1(T) : FT \to TF \). This will be the case, for example, when we consider a subcategory \( D \) of the bounded derived category of complexes of graded modules over an algebra \( A \) with a \( \mathbb{Z} \)-grading, and \( T \) is induced by the shift of grading on a module.

We would also like to consider the bigraded case, for which the following lemma will be useful.

**Lemma 2.4.** Let \( C \) be a Hom-finite \( k \)-linear category with Serre functor \( F \) and fixed isomorphisms \( \alpha_{XY} \) implementing the Serre duality. Let \( S_1, S_2 \) be isomorphisms of \( C \) which commute, and consider the bigraded category \((C, S_1, S_2)\). Let \( \epsilon_1, \epsilon_2 \in k^\times \) and define \( \eta = \eta_{\epsilon_1}(S_1), \theta = \eta_{\epsilon_2}(S_2) \).

1. \( \rho = \eta_{\epsilon_1 \epsilon_2}(S_1 \circ S_2) \) satisfies \( \rho_X = S_1(\theta_X) \circ \eta_{S_2X} \) for all objects \( X \).
2. \((F, \eta, \theta)\) is a bigraded functor of \((C, S_1, S_2)\).
3. For all objects \( X \) and \( f \in \text{Hom}(S_1S_2^iX, FS_1S_2^jX) \), we have

\[
\text{Tr}_X(S^{-i}S_2^{-j}(\eta^i \theta^j)_X \circ f) = \epsilon_1 \epsilon_2 \text{Tr}_{S_1S_2^2X}(f).
\]

**Proof.**

1. This follows from a routine diagram chase, using the definition of the natural transformations \( \eta_k(\cdot) \).

2. Since \( S_1S_2 = S_2S_1 \) by assumption, applying part (1) to both \( S_1S_2 \) and \( S_2S_1 \) shows that the condition [2.0.2] holds.

3. Recalling the definition of the natural isomorphism \( \eta^i \theta^j \) from Section 2.1, this follows easily from (1), induction, and [2.2.2]. \( \square \)

The Serre functor \( F \) is known to be unique up to natural isomorphism of functors [RV Lemma I.1.3]. Once \( F \) is fixed, there is some choice in the isomorphisms \( \alpha_{XY} \) which implement the Serre duality. The natural transformation \( \eta_k(S) \) may depend on this choice and so is uniquely determined only once \( F \) and the \( \alpha_{XY} \) are fixed. To conclude this section, we discuss in more detail what effect changing the \( \alpha_{XY} \) has on \( \eta_k(S) \).
Fix the Serre functor $F$, and consider two families of isomorphisms satisfying Definition 2.2. say \( \{ \alpha_{X,Y} \} \) and \( \{ \alpha'_{X,Y} \} \). Then for all pairs $X, Y$, we get an isomorphism
\[
\gamma_{X,Y} = \alpha'_{X,Y} \circ \alpha^{-1}_{X,Y} : \text{Hom}(Y, FX)^* \rightarrow \text{Hom}(Y, FX)^*.
\]
Varying $Y$, by Yoneda’s lemma this gives a natural isomorphism $\Psi : F \rightarrow F$ such that $\gamma_{X,Y}$ is the $k$-linear dual of $f \mapsto \Psi \circ f$. Conversely, given a natural isomorphism $\Psi : F \rightarrow F$ and a family of isomorphisms $\{ \alpha_{X,Y} \}$ satisfying Definition 2.2, we can define $\gamma_{X,Y}$ as the $k$-linear dual of $f \mapsto \Psi \circ f$ and then define $\alpha'_{X,Y} = \gamma_{X,Y} \circ \alpha_{X,Y}$, giving another family $\{ \alpha'_{X,Y} \}$ of natural isomorphisms satisfying Definition 2.2. Thus the possible choices of $\{ \alpha_{X,Y} \}$ are in bijection with the group of natural isomorphisms from $F$ to itself, which is the group of units of the ring of endomorphisms of $F$ in the category of functors. Since $F$ is an autoequivalence of $C$, it is routine to check that the ring of endomorphisms of $F$ is isomorphic to the ring of endomorphisms of the identity functor. Moreover, the ring of endomorphisms of the identity functor is called the center of the category $C$. Thus the possible families of natural isomorphisms $\{ \alpha_{X,Y} \}$ satisfying Definition 2.2 are in bijective correspondence with the units of the center of $C$.

Now suppose we have fixed $F$ and a family of natural isomorphisms $\{ \alpha_{X,Y} \}$, and consider another choice $\{ \alpha'_{X,Y} \}$ corresponding to $\Psi : F \rightarrow F$ as above. Let $S$ be an isomorphism of $C$, and define the natural transformation $\eta = \eta_k(S) : SF \rightarrow FS$ for some fixed $\epsilon \in k^\times$, using the $\alpha_{X,Y}$. Let $\eta' = \eta'_k(S) : FS \rightarrow SF$ be defined using the $\alpha'_{X,Y}$ instead. A straightforward diagram chase shows that for any object $X$,
\[
(E2.4.1) \quad \eta'_X = S(\Psi_X)^{-1} \circ \eta_X \circ \Psi_{SX}.
\]
If one recalls the definitions from Section 2.1, another way to interpret this equation is to say that $(F, \eta)$ and $(F, \eta')$ are isomorphic as graded functors from the graded category $(C, S)$ to itself. Conversely, if an isomorphism of graded functors $(F, \eta_k(S)) \cong (F, \rho)$ is given, then one may easily see that there is a choice of natural isomorphisms $\{ \alpha'_{X,Y} \}$ satisfying Definition 2.2 such that $\rho = \eta'_k(S)$.

2.4. Frobenius endomorphism algebras. Our main application of the general theory of the past few sections will be to a much more specialized setting, which we describe now.

**Hypothesis 2.5.** Let $C$ be a Hom-finite $k$-linear triangulated category with translation $\Sigma$.

1. We assume that $C$ has a Serre functor $F$ and that isomorphisms $\alpha_{X,Y}$ implementing the Serre duality as in Definition 2.2 are fixed. We fix the natural isomorphism $\eta = \eta_{-1}(\Sigma) : FS \rightarrow \Sigma F$ which makes $F$ an exact functor of $(C, \Sigma)$.

2. We assume that $T$ is an automorphism of $C$ such that $T\Sigma = \Sigma T$ and that the identity natural isomorphism $1_{T\Sigma} : T\Sigma \rightarrow \Sigma T$ also makes $(T, 1_{T\Sigma})$ an exact functor of $(C, \Sigma)$. We fix the natural isomorphism $\theta = \eta_1(T) : FT \rightarrow TF$, so that $(F, \eta, \theta)$ is a bigraded functor of $(C, \Sigma, T)$, as in Lemma 2.4. We say that $(C, \Sigma, T)$ is a $\mathbb{Z}$-graded triangulated category.

3. We may work instead in the multi-graded setting and replace $T$ with a set $T_1, \ldots, T_w$ of pairwise commuting automorphisms of $C$, where each $T_i$ separately satisfies the properties of (2). In this case we say that $(C, \Sigma, T_1, \ldots, T_w)$ is a $\mathbb{Z}^w$-graded triangulated category. All of our results extend easily to this multi-graded case, but we will state them only in the case of a single automorphism $T$ for simplicity. Our results below
can also be specialized to the case \( w = 0 \), where there are no automorphisms \( T \), but we always assume that \( w = 1 \) unless otherwise stated.

(4) We assume that there is an additional autoequivalence \( \Phi \) of \( C \) which also commutes with both \( \Sigma \) and \( T \), such that \( (\Phi, 1_{\Phi \Sigma}) \) is an exact graded functor of \( (\mathcal{C}, \Sigma) \). We assume that the Serre functor \( F \) is of the form \( F = (\Sigma)^d \circ T^t \circ \Phi \). Since \( \Sigma, T \), and \( \Phi \) all commute, we also have \( F \Sigma = \Sigma F \) and \( FT = TF \) as functors.

(5) Finally, we assume that the natural isomorphisms \( \eta \) and \( \theta \) are scalar multiples of the trivial commutations \( 1_{F \Sigma} : F \Sigma \to \Sigma F \) and \( 1_{F T} : FT \to TF \), in other words that \( \eta = s \cdot 1_{F \Sigma} \) and \( \theta = t \cdot 1_{F T} \) for some \( s, t \in k^x \).

In Section 3 we will give many examples satisfying the hypothesis above, which arise from considering subcategories of the bounded derived category of graded modules over an AS Gorenstein algebra \( A \) of dimension \( d \). In these examples \( \Sigma \) is the shift of complexes, \( T \) is induced from the shift of grading on modules, and \( \Phi \) is induced from twisting modules by some automorphism of \( A \). Moreover, in these examples we will calculate that \( s = (-1)^d \) and \( t = 1 \).

The main goal of the rest of this section is to show that in some cases the graded endomorphism algebra of an object in a category satisfying Hypothesis 2.5 is Frobenius, and to give a formula for its Nakayama automorphism. We begin by proving a bigraded version of the graded Serre duality result established by Van den Bergh in \[Bo\] Proposition A.4.3. As usual, a multi-graded version also holds. In the remaining results in this section, under Hypothesis 2.5 we freely commute the isomorphisms \( \Sigma \) to properly interpret the expression \( \text{Tr}_Y(\Phi(f) \ast g) \) in part (2) of the next result, one should identify \( \Sigma^{d-i}T^{l-j} \Phi \Sigma^i T^j \) with \( \Sigma^d T^l \Phi = F \).

**Lemma 2.6.** Assume Hypothesis 2.5 and its notation. Consider the bigraded category \( (\mathcal{C}, \Sigma, T) \) and its associated category \( \mathcal{C}^{gr} \) with bigraded Hom sets.

(1) Given objects \( X, Y \in \mathcal{C} \) and homogeneous elements \( f \in \text{Hom}^{i,j}(X, Y) \) and \( g \in \text{Hom}^{-i,-j}(Y, FX) \), one has

\[
\text{Tr}_X(g \ast f) = (-1)^i \text{Tr}_Y(F^{gr}(f) \ast g).
\]

(E2.6.1)

(2) The formula of part (1) can be reinterpreted as follows: if \( f \in \text{Hom}^{i,j}(X, Y) \) and \( g \in \text{Hom}^{d-i,l-j}(Y, \Phi(X)) \), then

\[
\text{Tr}_X(g \ast f) = (-s)^i t^j \text{Tr}_Y(\Phi(f) \ast g).
\]

**Proof.** (1) This is very similar to Van den Bergh’s proof of \[Bo\] Proposition A.4.3, but for completeness we include the details. Applying definitions we have

\[
\text{Tr}_Y(F^{gr}(f) \ast g) = \text{Tr}_Y(\Sigma^{-i}T^{-j}(F^{gr}(f)) \circ g)
\]

\[
= \text{Tr}_Y(\Sigma^{-i}T^{-j}((\eta^j \theta^i)_Y \circ F(f)) \circ g)
\]

\[
= \text{Tr}_Y(\Sigma^{-i}T^{-j}((\eta^j \theta^i)_Y \circ F(f) \circ \Sigma^i T^j(g)))
\]

\[
= (-1)^i \text{Tr}_{Y^{-T}Y}(F(f) \circ \Sigma^i T^j(g)) \quad \text{by Lemma 2.4.3}
\]

\[
= (-1)^i \text{Tr}_X(\Sigma^i T^j(g) \circ f) \quad \text{by (E2.2.1)}
\]

\[
= (-1)^i \text{Tr}_X(g \ast f),
\]

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as we needed.

(2) We can reinterpret

\[ g \in \text{Hom}^{d-i,i-j}(Y, \Phi(X)) = \text{Hom}(Y, \Sigma^{d-i} T^{i-j} \Phi(X)) \]

as the element

\[ \tilde{g} \in \text{Hom}^{-i,-j}(Y, \Sigma^{d} T \Phi(X)) = \text{Hom}^{-i,-j}(Y, FX). \]

Thus \( g \) and \( \tilde{g} \) are the same morphism, but as graded morphisms they have different degrees.

Now the graded Serre duality formula (E2.6.1) for \( f \) and \( \tilde{g} \) reads as follows:

\[
\text{Tr}_X(\Sigma^i T^j(\tilde{g}) \circ f) = \text{Tr}_X(\tilde{g} \ast f) = (-1)^i \text{Tr}_Y(F^{\text{gr}}(f) \ast \tilde{g}) = (-1)^i \text{Tr}_Y(\Sigma^{-i} T^{-j} (F^{\text{gr}}(f)) \circ \tilde{g}) = (-s)^{i} \text{Tr}_Y(\Sigma^{d-i} T^{i} \Phi(f) \circ \tilde{g}).
\]

Viewing \( g \) as an element of \( \text{Hom}^{d-i,i-j}(Y, \Phi(X)) \) again, this equation translates to the desired formula:

\[
\text{Tr}_X(g \ast f) = \text{Tr}_X(\Sigma^i T^j(g) \circ f) = \text{Tr}_X(\Sigma^j T^i(\tilde{g}) \circ f) = (-s)^{i} \text{Tr}_Y(\Sigma^{d-i} T^{i-j} \Phi(f) \circ \tilde{g}) = (-s)^{i} \text{Tr}_Y(\Phi(f) \ast g). \]

We now show that the bigraded endomorphism algebra

\[ \text{Hom}^{\text{gr}}(X, X) = \bigoplus_{i,j} \text{Hom}(X, \Sigma^i T^j X) \]

is a Frobenius algebra in certain cases. Since we do not yet have any assumptions to ensure that this algebra is finite dimensional, we rely on the infinite dimensional generalization of Frobenius algebra in [Ja]. Following Jans, a \( k \)-algebra \( A \) is Frobenius if there is an associative nondegenerate bilinear pairing \( (-,-) : A \times A \to k \), as well as an algebra automorphism \( \mu \) of \( A \) satisfying \( (x,y) = (\mu(y), x) \) for all \( x, y \in A \). This \( \mu \) is called a Nakayama automorphism, and in the classical case where \( A \) is finite dimensional, the existence of \( \mu \) follows directly from the existence of \( (-,-) \). For \( \text{Hom}^{\text{gr}}(X, X) \) to be Frobenius, one needs \( \Phi \) to preserve \( X \) in some sense. Having \( \Phi(X) = X \) would certainly be sufficient, but in practice one is much more likely to find \( X \cong \Phi(X) \) than equality on the nose. Under the latter assumption we can prove the following result. For any homogeneous element \( g \), the degree of \( g \) is denoted by \( |g| \) or \( \text{deg}(g) \) below.

**Theorem 2.7.** Assume Hypothesis [E2.6.1] and its notation, so \( (C, \Sigma, T) \) is a bigraded category. Let \( X \in C \) be an object such that there exists an isomorphism \( \phi : X \to \Phi(X) \). Then

\[ E(X) = \text{Hom}^{\text{gr}}(X, X) = \bigoplus_{i,j} \text{Hom}(X, \Sigma^i T^j X), \]
with multiplication given by composition $* \in C^\otimes$, is a bigraded Frobenius algebra. An associative nondegenerate bilinear form is defined on homogeneous elements $f, g \in E(X)$ with degrees $|g| = (i, j)$ and $|f| = (d - i, 1 - j)$ by

$$(f, g) = Tr_X(\phi \ast f \ast g) = Tr_X(\Sigma^d T^i(\phi) \circ \Sigma^i T^j(f) \circ g),$$

and by $(f, g) = 0$ if $|f| + |g| \neq (d, 1)$. The corresponding Nakayama automorphism is defined as follows: for any $g \in \text{Hom}^{i,j}(X, X)$,

$$g \mapsto (-s)^{t^i j^j} \phi^{-1} \ast \Phi(g) \ast \phi = (-s)^{t^i j^j} \Sigma^i T^j(\phi)^{-1} \circ \Phi(g) \circ \phi.$$

Proof. To check associativity of the pairing $(-, -)$, it suffices to assume that $a, b, c \in E(X)$ are homogeneous with $|a| + |b| + |c| = (d, 1)$ and note that

$$(a \ast b, c) = Tr_X(\phi \ast (a \ast b) \ast c) = Tr_X(\phi \ast a \ast (b \ast c)) = (a, b \ast c),$$

relying upon associativity of graded composition.

To see that the pairing is nondegenerate, it is again sufficient to work with homogeneous elements in $E(X)$. Suppose that $g \in \text{Hom}^{i,j}(X, X)$ is such that $(g, -): E(X) \to k$ is the zero functional; we will show that $g = 0$. In particular, for every $f \in \text{Hom}^{d-i,1-j}(X, X)$, we have

$$0 = (g, f) = Tr_X(\phi \ast g \ast f) = Tr_X(\Sigma^{d-i} T^{1-j}(\phi \ast g) \circ f),$$

where $\Sigma^{d-i} T^{1-j}(\phi \ast g) \in \text{Hom}(\Sigma^{d-i} T^{1-j} X, \Sigma^d T^i \Phi X) = \text{Hom}(\Sigma^{d-i} T^{1-j} X, F X)$. Because the pairing $Tr(-, -): \text{Hom}(\Sigma^{d-i} T^{1-j} X, F X) \times \text{Hom}(X, \Sigma^{d-i} T^{1-j} X) \to k$ is nondegenerate, it follows that $\Sigma^{d-i} T^{1-j}(\phi \ast g) = 0$. As $\Sigma$ and $T$ are automorphisms of $C$, we obtain $0 = \phi \ast g = \Sigma^{i} T^{j}(\phi) \circ g$. Now because $\phi$ is an isomorphism, we conclude that $g = 0$ as desired.

Finally, we compute the Nakayama automorphism, making use of the graded Serre duality formula from Lemma 2.6. Assume that $f, g \in E(X)$ are homogeneous, again with $|g| = (i, j)$ and $|f| = (d - i, 1 - j)$. Then $g \in \text{Hom}^{i,j}(X, X)$ and

$$\phi \ast f = \Sigma^{d-i} T^{1-j}(\phi) \circ f \in \text{Hom}(X, \Sigma^{d-i} T^{1-j} \Phi X) = \text{Hom}^{i,1-j}(X, F X).$$

Furthermore, as $\phi$ is an isomorphism in $C$ and $\phi \in \text{Hom}^{0}(X, \Phi X)$, it has an inverse in $C^\otimes$ which is also given by $\phi^{-1} \in \text{Hom}^{0}(\Phi X, X)$. So we have

$$(f, g) = Tr_X((\phi \ast f) \ast g)$$

$$= (-s)^{t^i j^j} Tr_X(\Phi(g) \ast (\phi \ast f)) \text{ by Lemma 2.6 (2)}$$

$$= (-s)^{t^i j^j} Tr_X(\phi \ast (\phi^{-1} \ast \Phi(g) \ast \phi) \ast f)$$

$$= (-s)^{t^i j^j} (\phi^{-1} \ast \Phi(g) \ast \phi, f).$$

Thus the homomorphism $g \mapsto (-s)^{t^i j^j} \Phi(g) \ast \phi$ (for $g \in \text{Hom}^{i,j}(X, X)$) satisfies the defining property of a Nakayama automorphism of $E(X)$. □

2.5. Skew Calabi-Yau triangulated categories. If $C$ is a Hom-finite $k$-linear triangulated category with Serre functor $F$, then as noted in Remark 2.3 we can make $F$ into an exact graded functor of $(C, \Sigma)$ using $\eta = \eta_{-1}(\Sigma)$, which satisfies the trace formula (E2.2.2) with $\epsilon = -1$. Then $C$ is called a Calabi-Yau triangulated category if $(F, \eta) \cong (\Sigma, -1_{\Sigma^2})^d = (\Sigma^d, (-1)^d 1_{\Sigma^{d+1}})$ as graded functors [Ke]. Recalling the discussion in Section 2.3, it is equivalent to say that $F = \Sigma^d$ is a Serre functor, and there is a choice of the isomorphisms $\{\alpha_{X,Y}\}$ implementing the Serre duality such that $\eta_{-1}(\Sigma) = (-1)^d 1_{\Sigma^{d+1}}$. Then Hypothesis 2.5 applies in this case with $w = 0$, $\Phi$ the identity functor, and with $s = (-1)^d$. Theorem 2.7 applies
to every object $X$ of $\mathcal{C}$ with $\phi = \text{id}_X$, so that the Ext-algebra $E(X)$ is always a graded-symmetric Frobenius algebra (that is, its Nakayama automorphism maps $g$ to $(-1)^{|g|(d+1)}g$). These results are well known in the context of Calabi-Yau triangulated categories; see [Bo, Proposition A.5.2] and [Kr, Proposition 2.2].

Recall that a full subcategory $\mathcal{D}$ of a triangulated category $(\mathcal{C}, \Sigma)$ is called a **triangulated subcategory** if $\mathcal{D}$ is closed under $\Sigma$ and if for all exact triangles, if two objects in the triangle are in $\mathcal{D}$, so is the third. The full triangulated subcategory $\mathcal{D}$ of $\mathcal{C}$ is called **thick** if it contains all direct summands of each of its objects. Given a class $\mathcal{S}$ of objects of $\mathcal{C}$, the triangulated (or thick) subcategory generated by $\mathcal{S}$ is the smallest triangulated (respectively, thick) subcategory of $\mathcal{C}$ containing $\mathcal{S}$. These categories have the following more explicit description. Let $\mathcal{S}_1$ be the class of objects of the form $\{\Sigma^i(X)|i \in \mathbb{Z}, X \in \mathcal{S}\}$. Define $\mathcal{S}_n$ inductively for $n \geq 2$ as the class of all objects $Y$ which occur in exact triangles $X \to Y \to \Sigma(X)$ with $X, Z \in \mathcal{S}_{n-1}$. Then the full subcategory with objects in $\mathcal{D} = \bigcup_{n \geq 1} \mathcal{S}_n$ is the triangulated subcategory generated by $\mathcal{S}$, and the full subcategory of all direct summands of objects in $\mathcal{D}$ is the thick subcategory generated by $\mathcal{S}$ [Kr, Section 3.3].

**Definition 2.8.** Let $(\mathcal{C}, \Sigma)$ be a $k$-linear triangulated category and $(\Phi, \eta)$ an exact autoequivalence of $\mathcal{C}$.

1. An object $X$ in $\mathcal{C}$ is called $\Phi$-**plain** if $\Phi(X) \cong X$.
2. Let $\Xi(\Phi)$ denote the class of $\Phi$-plain objects in $\mathcal{C}$, and let $\mathcal{C}^{\text{pl}}$ be the thick subcategory of $\mathcal{C}$ generated by $\Xi(\Phi)$. We say $\Phi$ is **plain** if $\mathcal{C}^{\text{pl}} = \mathcal{C}$.

It is obvious that the identity functor is plain. In general, $\Sigma^- = (\Sigma, -1_{\Sigma^2})$ is not plain. Similarly, in the $\mathbb{Z}$-graded triangulated categories $(\mathcal{C}, \Sigma, T)$ we study in the next section, usually $\Sigma$ and $T$ are not plain (see the proof of Lemma 3.6(1)). Intuitively, a plain functor must be close to being the identity functor. We now propose the following definition of **skew Calabi-Yau category** in the $\mathbb{Z}$-graded setting. Note that a modified version of the following definition can easily be given in both the multi-graded and the ungraded cases.

**Definition 2.9.** Let $(\mathcal{C}, \Sigma, T)$ be a $\mathbb{Z}$-graded $k$-linear Hom-finite triangulated category satisfying Hypothesis 2.3 so that in particular $\mathcal{C}$ has a Serre functor of the form $F = \Sigma^d \circ T^t \circ \Phi$.

1. We say that $\mathcal{C}$ is ($\mathbb{Z}$-graded) $\Phi$-**skew Calabi-Yau** if (for some choice of maps $\{\alpha_{X,Y}\}$ satisfying Definition 2.2) we have

\[
(F, \eta_{-1}(\Sigma)) = (\Sigma, -1_{\Sigma^2})^d \circ (T^t, 1_{\Sigma^t}) \circ (\Phi, 1_{\Sigma^\Phi}) = (-1)^d 1_{\Sigma^F} \quad \text{and} \quad
(F, \eta_1(T)) = (\Sigma, 1_{\Sigma^t})^d \circ (T^t, 1_{\Sigma^t+1}) \circ (\Phi, 1_{T \Phi}) = 1_{TF},
\]

as graded functors. Equivalently, $s = (-1)^d$ and $t = 1$ in Hypothesis 2.5.

2. If there is a **plain** exact autoequivalence $\Phi$ such that (E2.9.1) holds, then $\mathcal{C}$ is called ($\mathbb{Z}$-graded) $\Phi$-**skew Calabi-Yau of dimension** $d$. In this case, we say that $\Phi$ is a **Nakayama functor** for $\mathcal{C}$ and $t$ is the **AS index**.

The reason that 2.9(1) is not a sufficient definition for a skew Calabi-Yau category is that there is no guarantee of uniqueness of the data ($d, t, \Phi$) in the definition. The requirement that $\Phi$ is plain leads to uniqueness in the main case of interest studied in the next section; see Lemma 3.6(2) below. Presumably there may be some less stringent condition than plainness that could lead to uniqueness of ($d, t, \Phi$) for a $\Phi$-skew Calabi-Yau triangulated category.
The next lemma shows that there is a standard way of producing skew Calabi-Yau categories from Φ-Calabi-Yau categories. The lemma will be used in the next section.

**Lemma 2.10.** Let \( C \) be a \( \Phi \)-skew Calabi-Yau category. Then \((C^\text{pl}, \Sigma, T)\) is a skew Calabi-Yau category. That is, \( \Phi \) is plain on \( C^\text{pl} \).

**Proof.** Since \( T \) commutes with \( \Phi \), the functor \( T \) maps \( \Xi(\Phi) \) to itself. Since \((T, 1_{T\Sigma})\) is an exact functor of \((C, \Sigma)\), using the explicit description of the thick subcategory generated by a class of objects given earlier in this section, one easily sees that \( T \) restricts to the subcategory \( C^\text{pl} \). Similarly, \( T^{-1} \) restricts to \( C^\text{pl} \), so \( T \) restricts to an automorphism of \( C^\text{pl} \), and a straightforward argument can be used to show that \( T^{-1} \) is also exact (alternatively, one can apply [M, Lemma 49]). Similarly, \( \Sigma \) restricts to an automorphism of \( C^\text{pl} \), and \( \Phi \) restricts to an autoequivalence of \( C^\text{pl} \). Thus \( F \) restricts to an autoequivalence of \( C^\text{pl} \), and clearly Definition 2.9(1) still holds for the restricted category, that is, \((C^\text{pl}, \Sigma, T)\) is a \( \Phi \)-skew Calabi-Yau category. By the definition of \( C^\text{pl} \), \( \Phi \) is plain when restricted to \( C^\text{pl} \). The assertion follows.

\[ \square \]

3. **Skew Calabi-Yau categories related to AS Gorenstein algebras**

In this section, we show that a certain subcategory of the derived category of graded modules over an AS Gorenstein algebra is a skew Calabi-Yau triangulated category.

Throughout this section, \( A \) will be an algebra which is \( \mathbb{Z}^w \)-graded for some \( w \geq 1 \), and \( |x| \in \mathbb{Z}^w \) will indicate the degree of a homogeneous element in \( A \) or in a \( \mathbb{Z}^w \)-graded \( A \)-module. Any such algebra \( A \) is automatically also \( \mathbb{Z} \)-graded, using the homomorphism \( \mathbb{Z}^w \to \mathbb{Z} \) which adds the coordinates, and we use \(|x|\) to indicate the total degree of \( x \) under this \( \mathbb{Z} \)-grading. We are interested primarily in algebras which are \( \mathbb{N} \)-graded with respect to the \(|| \cdot |||-\)grading. Recall that an \( \mathbb{N} \)-graded algebra \( A = \bigoplus_{n \geq 0} A_n \) is connected if \( A_0 = k \), and locally finite if \( \dim_k A_n < \infty \) for all \( n \geq 0 \). Given an \( \mathbb{N} \)-graded algebra \( A \), let \( A\text{-Gr} \) be the category of \( \mathbb{Z} \)-graded left \( A \)-modules and let \( \Gamma = \Gamma_{mA} \) be the torsion functor \( A\text{-Gr} \to A\text{-Gr} \) which is defined as follows:

\[ \Gamma(M) = \{ x \in M | A_{\geq n} x = 0, \text{ some } n \geq 1 \} = \lim_{n \to \infty} \text{Hom}_A(A/A_{\geq n}, M). \]

If \( A \) is \( \mathbb{Z}^w \)-graded, then given a \( \mathbb{Z}^w \)-graded \( A \)-module \( M \), the notation \( M^* \) will indicate the graded dual, that is \( \bigoplus_{g \in \mathbb{Z}^w} \text{Hom}_k(M_g, k) \). If \( M \) is an \( (A, A) \)-bimodule and \( \sigma, \mu \) are automorphisms of \( A \), then \( \sigma M^\mu \) is the new bimodule with the same underlying vector space as \( M \), but with left and right actions twisted by the indicated automorphisms, that is with \( a * m * b = \sigma(a)m\mu(b) \) for \( m \in M, a, b \in A \). Similarly, we can define the twist \( \sigma M \) for a left module \( M \).

As in [RRZ], we would like our results to apply to \( \mathbb{N} \)-graded algebras which are locally finite but not necessarily connected. Following [RRZ] Definition 3.3, we generalize the notion of AS Gorenstein to this setting as follows.

**Definition 3.1.** Let \( A \) be a \( \mathbb{Z}^w \)-graded algebra, for some \( w \geq 1 \), such that it is locally finite and \( \mathbb{N} \)-graded with respect to the \(|| \cdot |||-\)grading. We say \( A \) is a **generalized AS Gorenstein algebra** if

1. \( A \) has injective dimension \( d \).
2. \( A \) is noetherian and satisfies the \( \chi \) condition [AZ] Definition 3.7, and the functor \( \Gamma_{mA} \) has finite cohomological dimension.
(3) There is an $A$-bimodule isomorphism $R^d\Gamma_{m_A}(A)^* \cong \mu A^1(-l)$, for some $l \in \mathbb{Z}$ (called the AS index) and for some graded algebra automorphism $\mu$ of $A$ (called the Nakayama automorphism).

If, in addition, $A$ has finite global dimension, then $A$ is called a generalized AS regular algebra.

An important consequence of the definition above is that a generalized AS Gorenstein algebra $A$ will have a rigid dualizing complex of the form $U := R^d\Gamma_{m_A}(A)^*[d] \cong \mu A^1(-l)[d]$ (see the proof of [RRZ, Lemma 3.5]).

Next, we define some important triangulated categories associated to a generalized AS Gorenstein algebra $A$. Let $\mathcal{D}(A)$ be the derived category of graded left $A$-modules, and let $\mathcal{D}^b_f(A)$ be the bounded derived category of finitely generated graded left $A$-modules, which can be viewed as a full triangulated subcategory of $\mathcal{D}(A)$. Let $\mathcal{E}(A)$ be the full triangulated subcategory of $\mathcal{D}^b_f(A)$ consisting of complexes with finite dimensional cohomologies. Recall that a complex is perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. Let $\mathcal{D}_{\text{perf}}(A)$ be the full triangulated subcategory of $\mathcal{D}(A)$ consisting of perfect complexes of graded left $A$-modules. Finally, let $\mathcal{D}_c(A)$ be the full triangulated subcategory of $\mathcal{D}(A)$ consisting of objects in both $\mathcal{D}_{\text{perf}}(A)$ and $\mathcal{E}(A)$. Note that if $A$ is generalized AS regular, then $\mathcal{D}^b_f(A) = \mathcal{D}_{\text{perf}}(A)$ and $\mathcal{D}_c(A) = \mathcal{E}(A)$. We let $\Sigma$ be the translation functor, in other words the shift $X \mapsto X[1]$ of complexes, in any of these triangulated categories.

Let $U$ be the rigid dualizing complex as above, and let $F$ be the functor $U \otimes_A^L \cdot$. Let $V = \mu A^1(-l)[d]$. Then it is obvious that $U \otimes_A^L V \cong V \otimes_A^L U \cong A$ as complexes of graded $A$-bimodules, so $G = V \otimes_A^L \cdot$ is an inverse of the automorphism $F$. Let $D = \text{RHom}_A(-, U)$ be the duality functor $\mathcal{D}^b_f(A) \to \mathcal{D}^b_f(A^{op})$, which has inverse $D^{op} = \text{RHom}_{A^{op}}(-, U) : \mathcal{D}^b_f(A^{op}) \to \mathcal{D}^b_f(A)$ [YZ Proposition 1.3]. Since $A$ is generalized AS Gorenstein, $\sum_i \text{dim}_k \text{Ext}_A^i(M, A)$ is finite for any finite dimensional graded $A$-module $M$. This implies that $D(X) \in \mathcal{E}(A^{op})$ whenever $X \in \mathcal{E}(A)$. Therefore the pair of functors $(D, D^{op})$ restrict to the subcategory $\mathcal{E}(A)$. The pair of functors $(D, D^{op})$ clearly restrict to $\mathcal{D}_{\text{perf}}(A)$, and thus also to $\mathcal{D}_c(A) = \mathcal{D}_{\text{perf}}(A) \cap \mathcal{E}(A)$. Similarly, $F$ and $G$ restrict to $\mathcal{D}_c(A)$.

We can now prove that the category $\mathcal{D}_c(A)$ satisfies Serre duality.

**Lemma 3.2.** Let $A$ be a generalized AS Gorenstein algebra, and maintain the notation introduced above.

1. [VdB] Theorem 5.1 and 6.3] $R \Gamma_{m_A}(-)^* \cong D(-)$, as functors from $\mathcal{D}^b_f(A)$ to $\mathcal{D}^b_f(A^{op})$.
2. If $Y$ is in $\mathcal{E}(A)$, then $Y^* \cong D(Y)$.
3. For any $X \in \mathcal{D}_{\text{perf}}(A)$ and $Y \in \mathcal{E}(A)$, there is an isomorphism
   
   $$(E3.2.1) \quad \alpha_{X,Y} : \text{Hom}_{\mathcal{D}(A)}(X, Y) \cong \text{Hom}_{\mathcal{D}(A)}(Y, F(X))^*,$$
   
   and these isomorphisms are natural in $X$ and $Y$.
4. The functor $F$ defined above is a Serre functor of the category $\mathcal{D}_c(A)$, and $D^{op}$ defines a Serre duality when restricted to $\mathcal{D}_c(A)$.

**Proof.** (1) Note that the indicated theorems from [VdB] do apply, since Van den Bergh’s theory on local duality works for locally finite noetherian algebras satisfying $\chi$ and having finite cohomological dimension (see the comments in [RRZ, Lemma 3.5]). Hence the assertion holds.
(2) This follows from part (1) and the fact that $R\Gamma_{m_A}(Y) = Y$ [VdB, Lemma 4.4].
(3) Since $X$ is a perfect complex, we have
\[ R\text{Hom}_A(X, Y) \cong R\text{Hom}_A(X, F(G(Y))) = R\text{Hom}_A(X, U \otimes_A^L G(Y)) \]
\[ = R\text{Hom}_A(X, U) \otimes_A^L G(Y) = D(X) \otimes_A^L G(Y). \]
Since $Y \in \mathcal{E}(A)$, it follows from induction on $\sum_n \dim_k H^n(Y[n])$ that $G(Y) \in \mathcal{E}(A)$. By part (2), $D(G(Y)) = G(Y)^* = R\text{Hom}_k(G(Y), k)$. By the above computation, adjointness, definition, and duality,
\[ R\text{Hom}_A(X, Y)^* \cong R\text{Hom}_k(D(X) \otimes_A^L G(Y), k) \]
\[ \cong R\text{Hom}_{A^e}(D(X), R\text{Hom}_k(G(Y), k)) \]
\[ \cong R\text{Hom}_{A^e}(D(X), D(G(Y))) \]
\[ \cong R\text{Hom}_A(G(Y), X) \cong R\text{Hom}_A(Y, F(X)). \]
This is equivalent to
\[ R\text{Hom}_A(X, Y) \cong R\text{Hom}_A(Y, F(X))^*. \]
The assertion follows from taking $H^0$ of the above.
(4) Since $F = U \otimes_A^L -$ is an autoequivalence of $\mathcal{D}_e(A)$, the assertion follows from part (3). \hfill \Box

Suppose that $A$ is $\mathbb{N}$-graded generalized AS Gorenstein. The shift of grading operation on a module $M \mapsto M(1)$, where $M(1)$ is defined by $M(1)_i = M_{i+1}$, gives an automorphism of the category $A\text{-Gr}$, which extends in an obvious way to complexes and thus gives an automorphism $T$ of $\mathcal{D}(A)$. Similarly, there is an automorphism of $A\text{-Gr}$ defined on objects by $M \mapsto \mu M$, where $\mu = \mu_A$ is the Nakayama automorphism, and as the identity on morphisms (using that $M$ and $\mu M$ have the same underlying $k$-space). It is easy to see that this functor may be identified with $\mu A^1 \otimes_A -$. This functor extends to complexes and thus gives an automorphism $\Phi$ of $\mathcal{D}(A)$, where in fact $\Phi = \mu A^1 \otimes_A^L -$. Both $T$ and $\Phi$ restrict to $\mathcal{D}_e(A)$. By the definition of $F$ and Lemma 3.2, $F := \mu A^k(-1)[d] \otimes_A^L -$ is the Serre functor of $\mathcal{D}_e(A)$, and clearly $F$ may also be written in the form $F = \Sigma^d \circ T^{-1} \circ \Phi$. It is also obvious that $\Sigma, T,$ and $\Phi$ pairwise commute.

We will see next that $(\mathcal{D}_e(A), \Sigma, T)$ is a $\Phi$-skew Calabi-Yau category in the sense of Definition 2.9.

**Proposition 3.3.** Let $A$ be $\mathbb{N}$-graded generalized AS Gorenstein and consider the bigraded category $(\mathcal{D}_e(A), \Sigma, T)$ defined above, where $\mathcal{D}_e(A)$ has Serre functor $F = \Sigma^d \circ T^{-1} \circ \Phi$. Fix the particular isomorphisms
\[ \alpha_{X,Y} : \text{Hom}_{\mathcal{D}_e(A)}(X, Y) \cong \text{Hom}_{\mathcal{D}_e(A)}(Y, F(X))^* \]
implementing the Serre duality which are determined in the course of the proof of Lemma 3.2 (3). Then

(a) $\eta = \eta_{-1}(\Sigma) = (-1)^d 1_{F\Sigma} : F\Sigma \to \Sigma F$, and
(b) $\theta = \eta_1(T) = 1_{FT} : FT \to TF$.

In particular, $(\mathcal{D}_e(A), \Sigma, T)$ satisfies Hypothesis 2.3 with $s = (-1)^d$ and $t = 1$, or equivalently $(\mathcal{D}_e(A), \Sigma, T)$ is a $\Phi$-skew Calabi-Yau triangulated category.
In order to prove Proposition 3.3, we found no alternative to a direct verification using the precise form of the isomorphisms $\alpha_{X,Y}$, and since these are defined by composing quite a few maps, the verification is rather long and technical. The reader willing to take the proof on faith may wish to skip ahead to Remark 3.4 at this point.

Since the proof below analyzes elements and morphisms concerning some Hom-sets, it is useful to recall some basic notation to be used in the proof. Recall that $A$-$Gr$ is the category of graded left $A$-modules with morphisms being the graded $A$-module homomorphisms of degree 0. For any graded left $A$-modules $M$ and $N$, let $\text{Hom}_A(M,N)$ denote the graded space $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_{-i}}(M,T^iN)$. Thus the degree 0 component of $\text{Hom}_A(M,N)$ is $\text{Hom}_A(M,N)_0 = \text{Hom}_{A_{-0}}(M,N)$. If $M$ is finitely generated, then $\text{Hom}_A(M,N)$, when forgetting the grading, agrees with the ungraded Hom set of all $A$-module homomorphisms from $M$ to $N$. Let $X$ and $Y$ be two complexes of graded left $A$-modules. Let $\text{Hom}_A(X,Y)$ be the complex of graded $k$-modules, defined by $\text{Hom}_A(X,Y)^i = \prod_{k \in \mathbb{Z}} \text{Hom}_A(X^k,Y^{i+k})$, with its standard differential. If $X$ is a perfect complex or if $Y$ is $K$-injective (or semi-injective), then

$$H^0 \text{Hom}_A(X,Y)_0 = \text{Hom}_{D(A)}(X,Y).$$

When $X$ and $Y$ are objects in a full triangulated subcategory $B$ of $D(A)$, then $\text{Hom}_B(X,Y) = \text{Hom}_{D(A)}(X,Y)$. In the next proof we work with morphisms at the level of complexes (or differential graded modules). Then by taking $H^0(-)_0$, we will then have morphisms at the level of derived categories.

**Proof of Proposition 3.3.** For any perfect complexes $X, Y \in D_c(A)$, let

$$D(X) := \text{Hom}_A(X,U), \quad F(X) := U \otimes_A X, \quad \text{and} \quad G(Y) := V \otimes_A Y,$$

where $U = \mu^iA^{-1}(-1)[d]$ and $V = U^{-1} = A^\mu(1)[-d]$. All of these are perfect complexes over $A$ or over $A^{op}$. Note that since $U$ and $V$ are perfect, $U \otimes_A U$ and $V \otimes_A V$ can be computed by $U \otimes_A U$ and $V \otimes_A V$. Consider the following diagram, where $H_A$ (respectively, $H_k$) is an abbreviation for $\text{Hom}_A$ (respectively, $\text{Hom}_k$).

$$
\begin{array}{ccccccccc}
M_1 = H_A(X,Y)^* & \xrightarrow{\alpha_1} & M_2 = H_k(D(X) \otimes_A G(Y),k) & \xrightarrow{\alpha_2} & \ldots \\
| & \downarrow{\psi_1} & | & \downarrow{\psi_2} & | & \downarrow{\psi_3} & | & \downarrow{\psi_4} & | \\
M_1' = H_A(\Sigma X, \Sigma Y)^* & \xrightarrow{\alpha_1'} & M_2' = H_k(D(\Sigma X) \otimes_A G(\Sigma Y),k) & \xrightarrow{\alpha_2'} & \ldots \\
| & | & | & | & | & | & | & \\
\ldots M_3 = H_{A^{op}}(D(X),H_k(G(Y),k)) & \xrightarrow{\alpha_3} & M_4 = H_{A^{op}}(D(X),D(G(Y))) & \ldots \\
| & | & | & | & | & | & | & \\
| & | & | & | & | & | & | & \\
\ldots M_3' = H_{A^{op}}(D(\Sigma X),H_k(G(\Sigma Y),k)) & \xrightarrow{\alpha_3'} & M_4' = H_{A^{op}}(D(\Sigma X),D(G(\Sigma Y))) & \ldots \\
| & | & | & | & | & | & | & \\
| & | & | & | & | & | & | & \\
\ldots \xrightarrow{\alpha_4} M_5 = H_A(G(Y),X) & \xrightarrow{\alpha_5} & M_6 = H_A(Y,F(X)) & \ldots \\
| & | & | & | & | & | & | & \\
| & | & | & | & | & | & | & \\
\ldots \xrightarrow{\alpha_4'} M_5' = H_A(G(\Sigma Y),\Sigma X) & \xrightarrow{\alpha_4'} & M_6' = H_A(\Sigma Y,F(\Sigma X)) & \ldots \\
\end{array}
$$

Here, the upper horizontal maps $\alpha_i$ exist as isomorphisms in the derived categories, coming from the proof of Lemma 3.2(3), and the lower horizontal maps $\alpha_i'$ are those isomorphisms applied to the objects $\Sigma X$ and $\Sigma Y$; the vertical maps will be defined later in the proof. By
taking 0-th cohomology and the 0-th graded piece, we obtain a diagram that involves the
Hom sets in the derived category and isomorphisms denoted formally by \( H^0(\alpha_i)_0 \).

We now describe the \( \alpha_i \) more explicitly. The map \( \alpha_1 \) is the inverse of the \( k \)-linear dual of
the map \( \lambda_1 \), where \( \lambda_1 \) is the composition of the following isomorphisms
\[
H_A(X,Y) \to H_A(X,F(G(Y))) \to H_A(X,U \otimes_A G(Y)) \to H_A(X,U) \otimes_A G(Y).
\]
The map \( \alpha_2 \) is just the \( \text{Hom} - \otimes \) adjoint. The map
\[
\alpha_3 : H_{A^\op}(D(X), H_k(G(Y), k)) \to H_{A^\op}(D(X), D(G(Y))),
\]
which exists at the derived level, is induced by the isomorphism \( \lambda_3 : H_k(G(Y), k) \to D(G(Y)) \)
in the derived category provided by Lemma 3.2.2 since \( G(Y) \) is in \( D_*(A) \). Note that
\( H_k(G(Y), k) \cong H_A(G(Y), A^* \}\) by adjointness. Next we define a zig-zag of maps that become
isomorphisms after we take the 0-th cohomology. Let \( I(U) \) be a fixed \( A \)-bimodule injective
resolution of \( U \) and let \( I(U)^{\leq 0} \) be the truncation. Then we have quasi-isomorphisms
\[
U \xrightarrow{\lambda_{3,1}} I(U) \xleftarrow{\lambda_{3,2}} I(U)^{\leq 0}.
\]
We also have
\[
I(U)^{\leq 0} \xrightarrow{\lambda_{3,3}} \Gamma_m(I(U)^{\leq 0}) \xrightarrow{\lambda_{3,4}} H^0(\Gamma_m(I(U)^{\leq 0})) \cong A^*,
\]
where \( \lambda_{3,3} \) is the natural embedding (but not a quasi-isomorphism) and \( \lambda_{3,4} \) is a quasi-
isomorphism (following from the AS Gorenstein property). Since \( G(Y) \) is in \( D_*(A) \),
\[
H_A(G(Y), \lambda_{3,3}) : H_A(G(Y), \Gamma_m(I(U)^{\leq 0})) \to H_A(G(Y), I(U)^{\leq 0})
\]
is a quasi-isomorphism. Therefore each \( H_A(G(Y), \lambda_{3,i}) \) is a quasi-isomorphism for \( i = 1, 2, 3, 4 \).
Note that \( \lambda_3 \) is a composition of the maps \( H_A(G(Y), \lambda_{3,i}) \) (or their inverses), which
becomes a well-defined isomorphism in the derived category. Since \( \alpha_3 \) is the composition
\[
H_{A^\op}(D(X), H_k(G(Y), k)) \to H_{A^\op}(D(X), H_A(G(Y), A^*))
\]
\[
H_{A^\op}(D(X), H_A(G(Y), U)) \to H_{A^\op}(D(X), D(G(Y))),
\]
it exists and is an isomorphism at the derived level. Finally, \( \alpha_4 \) is the isomorphism given by
the duality \( D \), and \( \alpha_5 \) is another adjoint isomorphism.

For any \( X \) in \( D_*(A) \), let \( s : X \to \Sigma X \) be the standard shift map of degree 1, which is the
identity on elements of \( X \). This is denoted by \( \sigma \) in the notes AFH Section 1.13]. For any
element \( x \in X \), let \( s(x) \) be the corresponding element in \( \Sigma X \).

Now we fix some standard isomorphisms for any perfect complexes \( X \) and \( Y \) (over \( A \) or
over \( A^\op \)), which come with important sign conventions. First, we have an isomorphism
\[
\Sigma : \text{Hom}_A(X,Y) \to \text{Hom}_A(\Sigma X, \Sigma Y), \quad f \mapsto (-1)^{|f|} s \circ f \circ s^{-1}.
\]
Here, we think of \( f \) as some homogeneous element of degree \( j \) in \( \text{Hom}_A(X,Y)^j = \prod_i \text{Hom}(X^i, Y^{i+j}) \).
The sign, and all signs below, come from the Koszul sign convention.
Similarly, we fix the standard isomorphisms, where \( \otimes \) could be either \( \otimes_A \) or \( \otimes_k \) and where \( \text{Hom} \) could be either \( \text{Hom}_A \), or \( \text{Hom}_{A^{op}} \), or \( \text{Hom}_k \).

(E3.3.2) \[ t_1 : (\Sigma X \otimes Y) \to \Sigma(X \otimes Y), \quad (sx \otimes y) \mapsto s(x \otimes y), \]

(E3.3.3) \[ t_2 : (X \otimes \Sigma Y) \to \Sigma(X \otimes Y), \quad (x \otimes sy) \mapsto (-1)^{|x|} s(x \otimes y), \]

(E3.3.4) \[ (\Sigma^{-1} \otimes \Sigma) : (X \otimes Y) \to (\Sigma^{-1}X \otimes \Sigma Y), \quad (x \otimes y) \mapsto (-1)^{|x|}(s^{-1}x \otimes sy), \]

(E3.3.5) \[ h_1 : \text{Hom}(\Sigma X, Y) \to \Sigma^{-1} \text{Hom}(X, Y), \quad f \mapsto (-1)^{|f|} s^{-1}(f \circ s), \quad \text{and} \]

(E3.3.6) \[ h_2 : \text{Hom}(X, \Sigma Y) \to \Sigma \text{Hom}(X, Y), \quad f \mapsto s(s^{-1} \circ f), \]

all of which are of course natural in each coordinate. Since \( D = \text{Hom}_A(-, U) \), we get an induced natural isomorphism of functors\[ \Omega : \Sigma^{-1}D \to D\Sigma \]
coming from \( h_1^{-1} \). Since \( G = V \otimes_A - \) and \( F = U \otimes_A - \), we have fixed natural isomorphisms of functors\[ \Lambda_G : G\Sigma \to \Sigma G, \quad \Lambda_F : F\Sigma \to \Sigma F \]
coming from \( t_2 \).

Now all vertical maps in the diagram above are defined by taking the obvious map using a combination of the fixed isomorphisms above. Most importantly, we have \( \psi_1 = (\Sigma^*)^{-1} \), and \( \psi_6 = (\Lambda_F)^{-1}_X \circ \Sigma(-) \). As another example, \( \psi_5 \) is the map induced by the isomorphism \( DX \otimes^L GY \to D\Sigma X \otimes^L G(\Sigma Y) \) given by \( (\Omega_X \otimes \Lambda_G^{-1}) \circ (\Sigma^{-1} \otimes \Sigma) \). We leave the similar obvious definitions of \( \psi_3, \psi_4, \psi_5 \) to the reader.

The main technical step of the proof is to check that with the definitions of the maps given above, then square I anticommutes (or \((-1)\)-commutes), while Squares II-V all commute. Suppose for the moment that this has been verified. Then taking \( H^0 \) and dualizing everything, we will have shown that \( \beta_{X,Y} \circ (\Sigma^*)^{-1} \circ \alpha_{X,Y} = - (\alpha_{\Sigma X, \Sigma Y} \circ \Sigma) \), where \( \beta_{X,Y} \) is the dual of the map \( \text{Hom}_{D((A))}(\Sigma Y, \Sigma FX) \to \text{Hom}_{D((A))}(\Sigma Y, F\Sigma X) \) given by \( f \mapsto (\Lambda_F)_X \circ f \). In other words, this shows that \( \eta_{-1}(\Sigma) = \Lambda_F \). However, because \( U \) is a complex concentrated entirely in degree \( d \), the sign in the isomorphism \( t_2 \) from which \( \Lambda_F \) is derived implies that \( \Lambda_F \) is equal to \( (-1)^d 1_{\Sigma F} \), where \( 1_{\Sigma F} : \Sigma F \to F\Sigma \) is the trivial commutation. This will verify the result for \( \eta_{-1}(\Sigma) \).

The verification that square I anticommutes, while each square II-V commutes, is tedious, and we do not give all of the details, but we briefly sketch some parts. Square II is easily seen to commute using the functoriality of the adjoint isomorphism. The commutation of square IV is mostly a matter of using that the duality \( D \) is a functor. Square V commutes again since \( F \) and \( G \) are adjoint (in fact inverse equivalences), and the natural isomorphisms \( \Lambda_F, \Lambda_G \) used to commute these with \( \Sigma \) are defined in a way compatible with this adjointness.

The more difficult squares are I and III. If necessary, each square can be analyzed by carefully writing out the definitions of the maps and applying them to a test element, and this is what we resorted to in order to verify that square I anticommutes. The commutation of square III depends on the maps \( \lambda_{3,i} \) introduced earlier.

For those readers who would like to see more details, we now give a full proof of the anticommutativity of square I. We leave the more detailed verification of the commutativity of square IV to the other squares to the reader. Note that the \( k \)-linear dual of square I
is the composition of the following three squares

\[
\begin{array}{c}
H_A(X, Y) \xrightarrow{\lambda_{1,1}} H_A(X, FG(Y)) \\
\downarrow \Sigma \quad \downarrow I_1 \quad \downarrow w_2 \\
H_A(\Sigma X, \Sigma Y) \xrightarrow{\lambda_{1,1}} H_A(\Sigma X, FG(\Sigma Y)),
\end{array}
\]

\[
\begin{array}{c}
H_A(X, FG(Y)) \xrightarrow{\lambda_{1,2}} H_A(X, U \otimes_A G(Y)) \\
\downarrow w_2 \quad \downarrow I_2 \quad \downarrow w_3 \\
H_A(\Sigma X, FG(\Sigma Y)) \xrightarrow{\lambda_{1,2}} H_A(\Sigma X, U \otimes_A G(\Sigma Y)),
\end{array}
\]

and

\[
\begin{array}{c}
H_A(X, U \otimes_A G(Y)) \xrightarrow{\lambda_{1,3}} H_A(X, U \otimes_A G(Y)) \\
\downarrow w_3 \quad \downarrow I_3 \quad \downarrow w_4 \\
H_A(\Sigma X, U \otimes_A G(\Sigma Y)) \xrightarrow{\lambda_{1,3}} H_A(\Sigma X, U \otimes_A G(\Sigma Y)).
\end{array}
\]

The map \(\Sigma\) is given in \([3.3.1]\). The map \(\lambda_{1,1}\) sends \(f \in H_A(X, Y)\) to \(\eta_Y \circ f\), where \(\eta_Y : Y \to FG(Y)\) is the natural isomorphism sending \(y \to u \otimes v \otimes y\), where \(u\) and \(v\) are the (nonzero) generators of \(U\) and \(V\). The map \(w_2\) is the the composition of \(\Sigma\) with \(H_A(\Sigma X, \delta_Y)\), where \(\delta_Y : \Sigma FG(Y) \to FG(\Sigma Y)\) sends \(s(u \otimes v \otimes y) \to u \otimes v \otimes s(y)\). Note that \(\delta_Y\) is a composition of two different \(t_2\)'s. For any \(f \in H_A(X, Y)\) and \(sx \in \Sigma X\), we have

\[
[(w_2 \circ \lambda_{11})(f)](sx) = w_2(\eta_Y \circ f)(sx) \\
= H_A(\Sigma X, \delta_Y)[(-1)^{|f|} s \circ (\eta_Y \circ f) \circ s^{-1}(sx)] \\
= H_A(\Sigma X, \delta_Y)[(-1)^{|f|} s \circ (\eta_Y \circ f)(x)] \\
= H_A(\Sigma X, \delta_Y)[(-1)^{|f|} s \circ (u \otimes v \otimes f(x))] \\
= (-1)^{|f|} u \otimes v \otimes sf(x),
\]

and

\[
[(\lambda_{11} \circ \Sigma)(f)](sx) = \lambda_{11}'((-1)^{|f|} s \circ f \circ s^{-1})(sx) \\
= (-1)^{|f|} \eta_{\Sigma Y}(sf(x)) \\
= (-1)^{|f|} u \otimes v \otimes sf(x).
\]

So \(w_2 \circ \lambda_{11} = \lambda_{11}' \circ \Sigma\) and square \(I_1\) is commutative. By definition, \(F(GY) = U \otimes_A G(Y)\) (so \(\lambda_{1,2}\) is the identity) and the map \(w_3\) (like \(w_2\)) is the composition of \(\Sigma\) with \(H_A(\Sigma X, \eta_Y)\), where \(\eta_Y\) is viewed as a natural isomorphism from \(\Sigma(U \otimes G(Y)) \to U \otimes G(\Sigma)\). It is clear that square \(I_2\) is commutative. Finally we show that square \(I_3\) is anticommutative. The map \(\lambda_{1,3}^{-1} : H_A(X, U) \otimes_A G(Y) \to H_A(X, U \otimes_A G(Y))\) is defined as follows: for any homogeneous element \(f \otimes (v \otimes y) \in H_A(X, U) \otimes_A G(Y)\), \(\lambda_{1,3}^{-1}(f \otimes (v \otimes y))\) in \(H_A(X, U \otimes_A G(Y))\) is denoted by \(\{f, v, y\}\) and for any homogeneous element \(x \in X\),

\[
\{f, v, y\}(x) = (-1)^{|x|(d+|y|)} f(x) \otimes (v \otimes y).
\]
We have already seen the definition of \( w_3 \). The map \( w_4 \) is the composition of \( \Sigma^{-1} \otimes \Sigma \) with \( h_1^{-1} \otimes t_2^{-1} \), so it sends

\[
f \otimes v \otimes y \mapsto (-1)^d (f \circ s^{-1}) \otimes v \otimes sy.
\]

Then

\[
[(\lambda_{1,3})^{-1} \circ w_4(f \otimes v \otimes y)](sx) = [(\lambda_{1,3})^{-1}((-1)^d (f \circ s^{-1}) \otimes v \otimes sy)](sx)
\]

\[
= (-1)^{d+(|x|-1)(d+|y|-1)} (f \circ s^{-1})(sx) \otimes v \otimes sy
\]

\[
= (-1)^{\omega_1} f(x) \otimes v \otimes sy
\]

where

\[
\omega_1 \equiv d + (|x| - 1)(d + |y| - 1) \equiv |x|(|d| + |y|) + |x| + |y| + 1 \quad \text{mod} \ 2.
\]

On the other hand,

\[
[w_3 \circ (\lambda_{1,3})^{-1}(f \otimes v \otimes y)](sx) = [w_3(\{f, v, y\})(sx)
\]

\[
= H_A(\Sigma X, \delta Y)((-1)^{|f|+d+|y|} s \circ \{f, v, y\} \circ s^{-1})(sx)
\]

\[
= (-1)^{|f|+d+|y|} H_A(\Sigma X, \delta Y)(s \circ \{f, v, y\}(x))
\]

\[
= (-1)^{|f|+d+|y|+|x|(d+|y|)} H_A(\Sigma X, \delta Y)(s(f(x) \otimes v \otimes y))
\]

\[
= (-1)^{\omega_2} f(x) \otimes v \otimes sy
\]

where \( \omega_2 = |f| + d + |y| + |x|(d + |y|) \). Since \( |f| = |f(x)| - |x| = |u| - |x| = -d - |x| \), we have

\[
\omega_2 = -|x| + |y| + |x|(d + |y|) = \omega_1 - 1 \quad \text{mod} \ 2.
\]

Therefore \((\lambda_{1,3})^{-1} \circ w_4 = -w_3 \circ (\lambda_{1,3})^{-1}\) and square \( I_3 \) is anticommutative. Combining square \( I_1, I_2 \) and \( I_3 \) and then taking the \( k \)-linear dual, one sees that square \( I \) is anticommutative. Therefore we verify square \( I \).

A similar proof, but much easier, will show that \( \eta_1(T) = 1_{TF} \). In the corresponding diagram, there are no signs to worry about. Each square will easily be seen to commute. Thus we omit this simpler part of the proof. Note that it is immediate from the definitions that \( T \) and \( \Phi \) are exact functors of \( \mathcal{D}_e(A) \) using the trivial commutations. Thus Hypothesis \[2.5\] is verified, with values \( s = (-1)^d \) and \( t = 1 \). So \((\mathcal{D}_e(A), \Sigma, T)\) is a \( \Phi \)-skew Calabi-Yau triangulated category by definition.

**Remark 3.4.** The values of \( s \) and \( t \) we calculated in the preceding result are of course the natural and expected ones. As we saw in Section \[2.3\], \(-1_{\Sigma^2} : \Sigma^2 \to \Sigma^2 \) makes \( \Sigma \) into an exact functor of \((\mathcal{D}(A), \Sigma)\), whereas the natural commutations with \( \Sigma \) make \( T \) and \( \Phi \) into exact functors. Since \( F = \Sigma^d \circ T^{-1} \circ \Phi \), then \((-1)^d 1_{F\Sigma} : F\Sigma \to \Sigma F \) makes \( F \) exact.

However, although Van den Bergh proves in \[18\] Theorem A.4.4] that the natural transformation \( \eta_{-1}(\Sigma) \) always makes \( F \) into an exact functor, there is no apparent reason why there should be a unique natural transformation that does so in general. Thus \((-1)^d 1_{F\Sigma} = \eta_{-1}(\Sigma)\) is not obviously forced.

In addition, as we discussed in Section \[2.3\], the exact form of \( \eta_{-1}(\Sigma) \) may depend on the choice of the isomorphisms \( \alpha_{X,Y} \) in the Serre duality, if the category has a large center. The
proof of Proposition 3.3 suggests that the choices of these isomorphisms of AS index of Lemma 3.6. Let Lemma 3.2(3) are particularly natural ones, since they lead to the expected $\eta_{-1}(\Sigma)$.

In the remaining results in this section, we continue to maintain the notation introduced before Proposition 3.3. We now prove the main result of this section.

**Theorem 3.5.** Let $A$ be a generalized AS Gorenstein algebra. Then $(\mathcal{D}_c^d(A), \Sigma, T)$ is a skew Calabi-Yau triangulated category.

**Proof.** To avoid triviality, we assume that $\mathcal{D}_c^d(A)$ is not zero. By Proposition 3.3, $(\mathcal{D}_c(A), \Sigma, T)$ is a $\Phi$-skew Calabi-Yau triangulated category. The assertion follows from Lemma 2.10. □

Next, we see that for the skew Calabi-Yau categories which arise in this section, the dimension, AS index, and Nakayama functor are uniquely determined.

**Lemma 3.6.** Let $A$ be an $\mathbb{N}$-graded generalized AS Gorenstein algebra, of dimension $d$ and of AS index $l$, with Nakayama automorphism $\mu = \mu_A$.

1. Let $G = \Sigma^a \circ T^b \circ \Phi$ for some $a, b \in \mathbb{Z}$. If $X$ is a nonzero object in $\mathcal{D}_c(A)$ and $G(X) \cong X$, then $a = b = 0$.

2. If we require the existence of a nonzero $\Phi$-plain object in $\mathcal{D}_c(A)$, then the decomposition $D^2.9.1$ is unique, namely, $(d, l, \Phi)$ is uniquely determined by $(E^2.9.1)$.

**Proof.** (1) Note that $H^*(\Phi(X)) = H^*(X)$ as graded $k$-spaces. Since $G(X) \cong X$, we have $\Sigma^a T^b H^*(X) \cong H^*(X)$. Since $X$ is a bounded complex of graded finitely generated left $A$-modules, $H^*(X)$ is left and right bounded (using the complex degree) and bounded below (using the internal grading). If $a \neq 0$ or $b \neq 0$ and if $G(X) \cong X$, we have that $H^*(X) = 0$. Equivalently, $X = 0$ in $\mathcal{D}_c(A)$. The assertion follows.

(2) Suppose that there is another decomposition $F = \Sigma^{d'} \circ T^{l'} \circ \Phi'$ such that there is a nonzero $\Phi'$-plain object in $\mathcal{D}_c(A)$. Then $\Phi' = \Sigma^{d - d'} \circ T^{l - l'} \circ \Phi$ and there is an $X \neq 0$ such that $\Phi'(X) \cong X$. By part (a), $d' - d = 0 = l' - l$. Thus $\Phi' = \Phi$. □

Suppose that $A$ is a connected $\mathbb{N}$-graded algebra. It was shown in [LPWZ Corollary D] that $A$ is AS regular if and only if the Ext-algebra $E(k)$ is Frobenius; this generalized the well-known result of Smith in the Koszul case [Sm]. To close this section, we will show that if $A$ is generalized AS regular, then the Ext-algebra $E(A_0)$ is Frobenius, where $A_0$ denotes the left (and right) graded $A$-module $A/A_{\geq 1}$.

**Proposition 3.7.** Suppose that $A$ is an $\mathbb{N}$-graded generalized AS regular algebra of dimension $d$, with Nakayama automorphism $\mu = \mu_A$.

1. $A_0 = A/A_{\geq 1}$, considered as a complex concentrated in degree 0, is a nonzero $\Phi$-plain object in $\mathcal{D}_c(A)$.

2. If $A_0$ is semisimple, then $\mathcal{D}_c(A) = \mathcal{D}_c^d(A)$ is a skew Calabi-Yau triangulated category of dimension $d$ with Nakayama functor $\Phi = ^{\mu}A^1 \otimes -$.

3. $E(A_0) := \bigoplus_{i,j} \text{Ext}^i_A(A_0, A_0(j))$ is a (finite dimensional) $\mathbb{Z}$-graded Frobenius algebra.

**Proof.** (1) Since $\mu$ is a graded algebra automorphism, $\Phi(A_0) \cong A_0$. Since $A$ is generalized AS regular, in particular noetherian of finite global dimension, all complexes in $\mathcal{D}_c^d(A)$ are perfect. Thus $D_c(A)$ is the same as $E(A)$, the subcategory of $\mathcal{D}_c^d(A)$ consisting of complexes with finite-dimensional cohomologies. In particular, $A_0 \in D_c(A)$. 

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Thus we need \( \mu \) cases where this holds: when \( \text{about the action. In this paper we only study the case where } H \). In Section 5 we will show that \( B_k \) is semisimple, \( D \) semisimple, \( \text{Application one. 4.1. } \)

First, note that in the case that \( A \) does not have finite global dimension, it is conceivable that \( B \) may be the zero category, in which case it carries no interesting information. However, in all of the examples we know, \( B \) is nonzero and we conjecture that this is always the case.

In Section 5 we will show that \( B \) is nonzero in many important common cases.

Theorem 2.7 applies only to a \( \Phi \)-plain object \( X \). Recall that \( \Phi \) is induced by the twist \( M \mapsto \mu M \) on left modules in this case, where \( \mu = \mu_A \) is the Nakayama automorphism of \( A \). Thus we need \( B^{pl} \) to be nonzero. In the remainder of the section, we explore two different cases where this holds: when \( \mu_A \) is of a special form, or when \( A \) has finite global dimension.

4. Applications of the formula for the Nakayama automorphism of \( E \)

In the previous section, we showed that given any generalized AS Gorenstein algebra \( A \), we can associate a \( \Phi \)-skew Calabi-Yau triangulated category \( B = D_\nu(A) \), the full subcategory of the bounded derived category of finitely generated graded \( A \)-modules consisting of perfect complexes with finite-dimensional cohomology. In this section, we show how one can get interesting information about \( A \) by using Theorem 2.7 to study the \( \text{Ext-algebras associated to objects } X \) in this category.

First, note that in the case that \( A \) does not have finite global dimension, it is conceivable that \( B \) may be the zero category, in which case it carries no interesting information. However, in all of the examples we know, \( B \) is nonzero and we conjecture that this is always the case. In Section 5 we will show that \( B \) is nonzero in many important common cases.

Theorem 2.7 applies only to a \( \Phi \)-plain object \( X \). Recall that \( \Phi \) is induced by the twist \( M \mapsto \mu M \) on left modules in this case, where \( \mu = \mu_A \) is the Nakayama automorphism of \( A \). Thus we need \( B^{pl} \) to be nonzero. In the remainder of the section, we explore two different cases where this holds: when \( \mu_A \) is of a special form, or when \( A \) has finite global dimension.

4.1. Application one. Let \( A \) be noetherian connected graded AS Gorenstein. Any Hopf algebra \( H \) acting on \( A \) such that \( A \) is a left \( H \)-module algebra determines a map \( \text{hdet} : H \to k \), the homological determinant [KKZ, Definition 3.3], which gives important information about the action. In this paper we only study the case where \( H = kG \) is a group algebra for a group \( G \) of graded automorphisms of \( A \). Recall from the previous section that the rigid dualizing complex of \( A \) is of the form

\[
R = R^d\Gamma_{m_A}(A)^*[d] \cong \mu A^1(-1)[d],
\]

where \( \mu = \mu_A \). Now since \( A \) has a left \( kG \)-module structure, \( R^d\Gamma(A)^* \) also obtains a left \( kG \)-module structure; see [KKZ, Section 3] or [RRZ, Remark 3.8] for more details. Thus via the isomorphism above, \( \mu A^1(-1)[d] \) has a left \( kG \)-action. Each element of \( kG \) acts on the 1-dimensional degree 1 piece by a scalar, and this assignment defines the homological determinant \( \text{hdet} = \text{hdet}(A) : kG \to k \). We do not give the precise definition since it is not needed in the proofs below. We will, however, rely on some theorems about \( \text{hdet} \) which are established in [RRZ].

Recall that \( |x| \) indicates the degree of a homogeneous element \( x \). For any \( \mathbb{Z} \)-graded algebra \( R \) and \( c \in k^\times \), we may define a graded automorphism \( \xi_c \) of \( R \) where \( \xi_c(x) = cx^{\langle x \rangle} \) for homogeneous elements \( x \). Similarly, if \( R \) is \( \mathbb{Z}^2 \)-graded then \( \xi_{c,d} \) is the \( \mathbb{Z}^2 \)-graded automorphism with
\[ \xi_{c,d}(x) = c^d \Phi x \] for homogeneous elements \( x \) of degree \( |x| = (i,j) \). This definition extends in an obvious way to multi-gradings.

Now suppose that \( A \) is connected graded noetherian AS Gorenstein as above, and that \( \mu = \mu_A \) happens to have the form \( \xi_c \) for some \( c \in k^\times \). While this seems quite restrictive, we will see in the proof of Theorem 5.3 that any noetherian AS Gorenstein algebra is closely related to another one with a Nakayama automorphism of this simple form. For any \( \mathbb{Z} \)-graded \( A \)-module \( M \), the map \( \rho_M : M \to \Phi(M) = \mu M \) given by \( m \mapsto c^{|m|} m \) for homogeneous elements \( m \) is an isomorphism of graded left \( A \)-modules. This extends in an obvious way to complexes and thus to objects of the derived category \( \mathcal{D}(A) \) and its subcategory \( \mathcal{B} = \mathcal{D}_c(A) \).

Thus for any object \( X \in \mathcal{B} \) there is an isomorphism \( \rho_X : X \to \Phi(X) \), and so we have a natural isomorphism of functors \( \rho : 1 \to \Phi \). In particular, we can apply Theorem 2.7 to obtain the following result about the bigraded Ext-algebra of \( X \).

**Theorem 4.1.** Let \( A \) be noetherian connected graded AS Gorenstein, with \( \mu_A = \xi_c \) for some \( c \in k^\times \). Assume that \( \mathcal{B} := \mathcal{D}_c(A) \neq 0 \). Let \( d \) be the injective dimension of \( A \), and let \( l \) be the AS index of \( A \).

1. For any nonzero \( X \in \mathcal{B} \), the Ext-algebra \( E = \bigoplus_{i,j} \text{Hom}_B(X, \Sigma^iT^jX) \) is Frobenius with bigraded Nakayama automorphism \( \mu_E = \xi_{(-1)^{d+1},c^{-1}} \).
2. \( c^l = 1 \), and \( \text{hdet} \mu_A = c^l = 1 \).

**Proof.** (1) As we have seen, there is an isomorphism \( \rho_X : X \to \Phi(X) \). We also know by Proposition 5.3 that \( (\mathcal{B}, \Sigma, T) \) satisfies Hypothesis 2.5 with \( s = (-1)^d \) and \( t = 1 \), and so Theorem 2.7 applies, where the Serre functor \( F \) of \( \mathcal{B} \) has the form \( \Sigma^dT^{-\ell}F \).

It follows easily from the definitions that for any object \( X \),

\[ \Sigma(\rho_X) = \rho_{\Sigma(X)} \quad \text{and} \quad T(\rho_X) = c \rho_{T(X)}. \]

Now let \( X \) be any nonzero object in \( \mathcal{B} \) and let \( \phi = \rho_X : X \to \Phi(X) \). Since \( X \) is perfect, by replacing \( X \) with a quasi-isomorphic complex of projectives, we can think of \( \phi \) as well as elements in \( \text{Hom}_B(X, \Sigma^iT^jX) \) as actual maps of complexes. Let \( g \in E_{i,j} = \bigoplus_{i,j} \text{Hom}_B(X, \Sigma^iT^jX) \) be a homogeneous element of degree \( (i,j) \). Then

\[ \Sigma^iT^j(\phi)^{-1} \circ \Phi(g) \circ \phi = \Sigma^iT^j(\rho_X)^{-1} \circ \Phi(g) \circ \rho_X = c^{-j}(\rho_{\Sigma^iT^jX})^{-1} \circ \Phi(g) \circ \rho_X, \]

using (E4.1.1). Let \( X^n \) be the \( n \)th term in the complex \( X \) and let \( x \in (X^n)_m \) be a homogeneous element of degree \( m \) in this module. Then we calculate

\[ c^{-j}(\rho_{\Sigma^iT^jX})^{-1} \circ \Phi(g) \circ \rho_X(x) = c^{-j}c^{-m}g(c^m x) = c^{-j}g(x). \]

Since this is independent of \( n \) and \( m \), we see that \( \Sigma^iT^j(\phi)^{-1} \circ \Phi(g) \circ \phi = c^{-j}g \), as morphisms of complexes.

By Theorem 2.7, the action of the Nakayama automorphism \( \mu_E \) on the \((i,j)\)-graded piece is multiplication by the scalar \((-1)^{(d+1)i}c^{-j}\). In other words, thinking of \( E \) as \( \mathbb{Z}^2 \)-graded, we have

\[ \mu_E = \xi_{(-1)^{d+1},c^{-1}}. \]

(2) Since \( E \) is Frobenius with Nakayama automorphism \( \mu = \mu_E \), there is a nondegenerate bilinear form \( \langle -, - \rangle \) on \( E \), such that

\[ \langle a, b \rangle = \langle \mu(b), a \rangle \]
for \( a, b \in E \). Let \( \epsilon = (1, -) \). Then under the natural action of \( G = \text{Aut}_\mathbb{Z}(E) \) on \( E^* = \text{Hom}_k(E, k) \), we have

\[
\mu(\epsilon)(b) = \epsilon(\mu^{-1}(b)) = \langle 1, \mu^{-1}(b) \rangle = \langle b, 1 \rangle = \langle 1, b \rangle = \epsilon(b)
\]

and thus \( \mu(\epsilon) = \epsilon \). Note that \( E^* \) is \( \mathbb{Z}^2 \) graded with \( (E^*)_i = \text{Hom}_k(E_{-i}, k) \). Since \( \mu_E = \xi_{(-1)^{d+1}, c^{-1}} \), it is easy to see that \( \mu \) also must act on \( E^* \) via the same rule \( x \mapsto (-1)^{(d+1)c^{-1}x} \) for \( |x| = (i, j) \). Since \( \epsilon \) has degree \((-d, 1)\) we also have \( \mu(\epsilon) = (-1)^{-(d+1)d} e^{-1}\epsilon = e^{c^1} \epsilon \). Thus \( c^1 = 1 \).

Finally, since \( \mu_A = \xi \epsilon \) and \( A \) has AS index \( 1 \), it also follows that \( \text{hdet}_A(\mu_A) = c^1 \) [RRZ, Lemma 5.3]. Thus \( \text{hdet}(\mu_A) = 1 \).

In Section 5, we will extend the result \( \text{hdet} \mu_A = 1 \) to all AS Gorenstein algebras with \( D_\epsilon(A) \neq 0 \) [Theorem 5.3].

4.2. Application two. Throughout this application, we assume that \( A \) is an \( \mathbb{N} \)-graded generalized AS regular algebra, as defined in Definition 3.1. We also assume that \( A_0 \) is a semisimple \( k \)-algebra. Let \( X \) be the trivial module \( A_0 := A/A_{\geq 1} \), which is viewed as a complex concentrated in degree 0. Let \( \mathcal{B} = D_\epsilon(A) \) be defined as above. As already proved in Proposition 3.7, the bigraded Ext-algebra

\[
E = \bigoplus_{i,j} \text{Hom}_k(X, \Sigma^i T^j X) \cong \bigoplus_{i} \text{Ext}_A^i(A_0, A_0)
\]

is a graded Frobenius algebra. Theorem 2.7 also gives a formula for the Nakayama automorphism \( \mu_E \) of \( E \). In this section we study the formula for \( \mu_E \) more closely and show that it recovers and generalizes some existing results in the literature on the Nakayama automorphisms of Ext-algebras of regular algebras.

We first set up some notation. Let

\[
P = 0 \to P(-d) \to \cdots \to P(-2) \xrightarrow{\partial(-2)} P(-1) \xrightarrow{\partial(-1)} P(0) \to 0
\]

be a minimal graded projective resolution of \( X = A_0 \), where each term is of the form \( P^{(-i)} = A \otimes A_0 V^{(-i)} \) for some finite-dimensional graded left \( A_0 \)-module \( V^{(-i)} \). Note that throughout this section, we will place parentheses around the superscripts indicating cohomological degrees, in order to avoid notational confusion with exponents. Clearly, \( V^{(0)} = A_0 \).

Explicitly, one may construct the \( P^{(-i)} \) and \( \partial^{(-i)} \) inductively by letting \( K^{(-i+1)} = \ker \partial^{(-i+1)} \) (or \( K^{(0)} = A_{\geq 1} \) when \( i = 1 \)), taking \( V^{(-i)} = K^{(-i+1)}/A_{\geq 1} K^{(-i+1)} \), and letting \( \partial^{(-i)} : P^{(-i)} = A \otimes A_0 V^{(-i)} \to K^{(-i+1)} \) be defined by choosing some \( A_0 \)-module map \( j : V^{(-i)} \to K^{(-i+1)} \) such that \( \pi \circ j = 1 \), where \( \pi : K^{(-i+1)} \to V^{(-i)} \) is the natural surjection.

Let \( \sigma \in \text{Aut}_\mathbb{Z}(A) \). This restricts to an algebra isomorphism \( \sigma : A_0 \to A_0 \), and one has an isomorphism of left \( A \)-modules \( A_0 \cong \sigma A_0 \) (with underlying set map given by \( a \mapsto \sigma(a) \)). This may be lifted to a graded isomorphism of complexes \( \phi : P \to \sigma P \). Identifying the underlying vector space of \( \sigma P^{(-i)} \) with \( P^{(-i)} \) for each \( i \), we may also think of \( \phi : P \to P \) as a \( \sigma \)-linear map of complexes; that is, each map \( \phi^{(-i)} : P^{(-i)} \to P^{(-i)} \) satisfies \( \phi^{(i)}(ax) = \sigma(a)x \) for \( x \in P^{(-i)}, a \in A \).

It is easy to calculate \( \phi^{(0)} \) and \( \phi^{(1)} \) explicitly. Identifying \( P^{(0)} = A \), clearly we may take \( \phi^{(0)} = \sigma : A \to A \). By the description of the formation of the minimal projective resolution given above, we may take \( V^{(-1)} = A_{\geq 1}/A_{\geq 1} A_{\geq 1} \) as a left \( A_0 \)-module. Then \( \sigma \) restricts to \( A_{\geq 1} \) and factors through to give a map \( \sigma : V^{(-1)} \to V^{(-1)} \), which induces the
map \( \phi^{(1)} : P^{(-1)} \to P^{(-1)} \) by applying \( A \otimes A_0 \). For example, if \( A \) is generated in degrees 0 and 1, then we may identify \( V^{(-1)} \) with \( A_{\geq 1}/A_{\geq 2} = A_1 \) and we simply have \( \sigma = \sigma_{|A_1} \). If one has an explicit graded presentation of \( A \), it is also straightforward to calculate \( \phi^{(2)} \) explicitly in terms of the action of \( \sigma \) on the relations, but it could be computationally difficult to find the entire \( \sigma \)-linear map of complexes \( \phi \) explicitly when the global dimension of \( A \) is large.

Next, we recall that each \( \sigma \in \text{Aut}_Z(A) \) induces a (bi)-graded automorphism \( f_{\sigma} \) of the Ext-algebra \( E = \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0) \), so that the group \( \text{Aut}_Z(A)^{op} \) acts on \( E \) naturally. (In the case when \( A \) is connected, this graded automorphism \( f_{\sigma} \) of the Ext-algebra \( E = \bigoplus_{i \geq 0} \text{Ext}_A^i(k, k) \) was defined in [JZ] Lemma 4.2; see also [KKZ] Remark 5.11(a)). The map \( f_{\sigma} \) is described in the following simple way. Fix a \( \sigma \)-linear isomorphism \( \phi : P \to P \) as above. Since \( P \) is a minimal graded projective resolution, \( \partial^{(i)}(P^{(i)}) \subseteq A_{\geq 1}P^{(i-1)} \) for each \( i \). Thus the differentials in the complex \( \text{Hom}_A(P, A_0) \) are 0, and hence we may identify \( \text{Ext}_A^i(A_0, A_0) \) with \( \text{Hom}_A(P^{(-i)}, A_0) \) for each \( i \). Then \( f_{\sigma} : \text{Hom}(P^{(-i)}, A_0) \to \text{Hom}(P^{(-i)}, A_0) \) is given by \( g \mapsto \sigma^{-1} \circ g \circ \phi^{(-i)} \). (If \( g \) is \( A \)-linear, then \( g \circ \phi^{(-i)} \) is \( \sigma \)-linear and \( \sigma^{-1} \circ g \circ \phi^{(-i)} \) is again \( A \)-linear.)

We use the notation \( M^\vee = \text{Hom}_{A_0}(\cdot, A_0) \) for the dual of a graded left \( A_0 \)-module \( M \). Of course, this is the usual vector space dual \( M^* \) when \( A \) is connected. Note that we also have canonical isomorphisms

\[
\text{Hom}_A(P^{(-i)}, A_0) \simeq \text{Hom}_A(P^{(-i)}/A_{\geq 1}P^{(-i)}, A_0) \simeq \text{Hom}_{A_0}(V^{(-i)}, A_0) = (V^{(-i)})^\vee.
\]

In particular, this identifies \( \text{Ext}_A^1(A_0, A_0) \) with \( (V^{(-1)})^\vee \). In the special case that \( A \) is generated in degrees 0 and 1, we saw above that we can also identify \( V^{(-1)} \) with \( A_1 \), and hence \( \text{Ext}_A^1(A_0, A_0) \) is identified with \( (A_1)^\vee \). Now combining the observations above with Theorem 2.7 we obtain the main result of this section.

**Theorem 4.2.** Let \( A \) be generalized AS regular of global dimension \( d \), where \( A_0 \) is semisimple. Let \( \text{Aut}_Z(A)^{op} \) act on the Ext-algebra \( E(A_0) = \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0) \) as defined above.

1. \( E \) is Frobenius and \( \mu_E = \xi_{(-1)^{d+1}, 1} \circ f_{\mu_A} \).
2. Suppose that \( A \) is generated as an algebra in degrees 0 and 1. Under the natural identification of \( E^1 = \text{Ext}_A^1(A_0, A_0) \) with \( (A_1)^\vee = \text{Hom}_{A_0}(A_1, A_0) \), the map \( \mu_E|_{E^1} \) is identified with \( g \mapsto (-1)^{d+1}(\mu_A|_{A_0})^{-1} \circ g \circ \mu_A|_{A_1} \).
3. If \( A \) is connected and generated in degree 1, then \( \mu_E|_{E^1} = (-1)^{d+1}(\mu_A|_{A_1})^* \).

Moreover, \( \mu_A = 1 \) if and only if \( E \) is graded-symmetric.

**Proof.** (1) We need to relate the action of \( \mu_E \) on the Ext-algebra described in Theorem 2.7 in terms of the derived category to the action of \( \mu_A \) defined above. Keep all of the above notation; in particular, fix the minimal projective resolution \( P \) of \( A_0 \). As mentioned earlier several times, by standard facts about the derived category, \( \text{Ext}_A^i(A_0, A_0) \) may be identified with \( \text{Hom}_{D(A)}(P, \Sigma^iT^jP) \). Recall that the explicit identification is made as follows. As we already saw above, an element of \( \text{Ext}_A^i(A_0, A_0) \) corresponds to an element in \( \text{Hom}_A(P^{(-i)}, T^j(A_0)) \). Any element \( G(i) \in \text{Hom}_A(P^{(-i)}, T^j(A_0)) \) extends trivially to a map of complexes

\[
G : P \to \Sigma^iT^j(A_0).
\]

Using the quasi-isomorphism \( \Sigma^iT^jP \to \Sigma^iT^jA_0 \), \( G \) lifts to a map of complexes \( g : P \to \Sigma^iT^jP \). The \( g \) so obtained is unique up to homotopy. Conversely, since \( P \) is perfect, an
element of $\text{Hom}_{D_i(A)}(P, \Sigma^i T^j P)$ is given by an actual map of complexes $P \to \Sigma^i T^j P$, which is determined (up to homotopy) by the degree $i$ map $G(i) : P^{(-i)} \to T^j P^{(0)} \to T^j A_0$. 

Now consider the action of $\mu_E$ on $g \in \text{Hom}_{D_i(A)}(P, \Sigma^i T^j P)$ as described by Theorem 2.7. In degree $(i, j)$, the formula is

$$g \mapsto (-1)^{(d+1)i} \Sigma^i T^j (\phi)^{-1} \circ \Theta(g) \circ \phi.$$ 

Thus we only need to calculate the degree $i$ map in the morphism of complexes $\mu_E(g)$. Here, $\phi^{-1} : P^{(-i)} \to \Phi(P)^{(-i)}$ is the map of position $-i$ in the isomorphism of complexes $\phi : P \to ^{\mu_A} P$ fixed above. The second map $\Phi(g)$ has the same underlying function as $g$, and we only consider the position $-i$, which is the map $P^{(-i)} \to T^j P^{(0)}$. The third map $(\Sigma^i T^j (\phi))^{-1}$ when restricted to $T^j P^{(0)}$ is given by the isomorphism $\mu_A^{-1} : T^j (\mu_A P^{(0)}) \to T^j (P^{(0)})$. Now descending elements in $\text{Hom}_A(P^{(-i)}, T^j P^{(0)})$ to $\text{Hom}_A(P^{(-i)}, T^j A_0)$, the third map becomes the isomorphism $\mu_A^{-1} : T^j (\mu_A A_0) \to T^j (A_0)$. Altogether, under the identification of $\text{Hom}_{D_i(A)}(P, \Sigma^i T^j P)$ with $\text{Hom}_A(P^{(-i)}, T^j A_0)$ given above, we see that $\mu_E(g)$ is given in complex degree $i$ precisely by $(-1)^{(d+1)i} \mu_A^{-1} \circ g \circ \phi^{-1}$. In other words, this is exactly the same as the action $f_{\mu_A}$ of $\mu_A$ on $\text{Ext}^i_A(A_0, A_0)$ described above, except for the additional sign.

(2) Recall our earlier calculation that $\phi^{-1} : P^{(-1)} \to P^{(-1)}$ is induced by applying $A \otimes A_0$ to the map $\sigma : (\nu^{-1}) \to (\nu^{-1})$, which when $A$ is generated in degrees 0 and 1 is the same as $\mu_A|_{A_1}$ under the identification of $\nu^{-1}$ with $A_1$. The statement follows from restricting part (1) to degree 1, once we identify $\text{Hom}_A(P^{(-1)}, A_0)$ with $(A_1)^* = \text{Hom}_{A_0}(A_1, A_0)$.

(3) The first statement is a special case of (2), since $\mu^{-1} : A_0 \to A_0$ is trivial when $A_0 = k$, and $(\mu_A|_{A_1})^*$ is by definition equal to $g \mapsto g \circ \mu_A|_{A_1}$ for $g \in (A_1)^* = \text{Hom}_{A_0}(A_1, k)$. Because $A$ and $E$ are connected, the only inner automorphism of either algebra is the trivial one. Thus both $\mu_A$ and $\mu_E$ are uniquely determined in this case. Note that $E$ is graded-symmetric if and only if $\mu_E = \xi_{(-1)^{d+1}1}$. If $\mu_A = 1$, then $f_{\mu_A} = 1$ and so by part (1) we certainly get $\mu_E = \xi_{(-1)^{d+1}1}$. Conversely, if $\mu_E = \xi_{(-1)^{d+1}1}$ then we get $(\mu_A|_{A_1})^* = 1$, and thus $\mu_A|_{A_1} = 1$. Since $A$ is generated in degree 1, $\mu_A = 1$. \hfill $\square$

Part (2) of the theorem recovers and generalizes known results about the action of $\mu_E$ in degree 1, which were proved in the connected $N$-Koszul case only in [BM]. Note that our result allows one to calculate $\mu_E$ in fact in any degree, if one can calculate an explicit minimal resolution $P$ of $A_0$ and an explicit $\mu_A$-linear isomorphism $\phi : P \to P$.

We note that a generalized AS regular algebra $A$ will be a skew Calabi-Yau algebra as studied in [RRZ, Definition 0.1], in cases where $A$ is homologically smooth in the sense of that definition (this is automatic, for instance, when $A$ is connected graded [RRZ, Lemma 1.2]). When $A$ is skew Calabi-Yau, part (1) of the theorem shows that $A$ is Calabi-Yau ($\mu_A = 1$) if and only if $E$ is graded-symmetric.

An immediate consequence is the following theorem which answers a conjecture in [CWZ, Remark 4.2] affirmatively. See also the discussion after [CWZ, Theorem 0.1]. The definitions of the Hopf-theoretic notions involved in the following result may also be found in [CWZ].

**Theorem 4.3.** Let $A$ be a noetherian connected graded AS regular algebra generated in degree one with Nakayama automorphism $\mu_A$. Let $K$ be a Hopf algebra with bijective antipode $S$ coacting on $A$ from the right. Suppose that the homological codeterminant $\text{det}_{1.5}(b)$ of the $K$-coaction on $A$ is the element $D \in K$ and that the $K$-coaction on $A$ is
inner-faithful. Then
\[ \eta_D \circ S^2 = \eta_{\mu_A^r}, \]
where \( \eta_D \) is the automorphism of \( K \) defined by \( \eta_D(a) = D^{-1}aD \) and \( \eta_{\mu_A^r} \) is the automorphism of \( K \) given by conjugating by the transpose of the corresponding matrix of \( \mu_A^{r} \).

**Proof of Theorem 4.3.** Replacing [CWZ, Lemma 4.1] by Theorem 4.2(2), the proof of [CWZ, Theorem 0.1] can be copied without much change. \( \square \)

The following further consequence generalizes [CWZ, Theorem 0.6] to the non-N-Koszul case. The proof is given in [CWZ, Proof of Theorem 0.6] by using Theorem 4.3.

**Corollary 4.4.** Let \( A \) be as in Theorem 4.3 and suppose that \( \text{char } k = 0 \). Further assume that \( \mu_A = \xi_{r} \) for some \( r \in k^\times \) and that \( H \) is a finite dimensional Hopf algebra that acts on \( A \) satisfying [CWZ, Hypothesis 0.3]. If the \( H \)-action on \( A \) has trivial homological determinant, then \( H \) is semisimple.

5. The \( \epsilon \)-condition

Let \( A \) be a \( \mathbb{Z}^w \)-graded algebra for some \( w \geq 1 \). We say that \( A \) is \( \mathbb{N}^w \)-graded if \( A_{i_1,i_2,...,i_w} \neq 0 \) implies that \( i_j \geq 0 \) for all \( 1 \leq j \leq w \). The algebra \( A \) is connected if \( A_{0,0,...,0} = k \). We also allow by convention \( w = 0 \), in which case \( A \) is ungraded. We do not require that \( A \) be locally finite in general in this section. Let \( A\text{-Gr} \) be the category of \( \mathbb{Z}^w \)-graded left \( A \)-modules. We write this as \( (A, \mathbb{Z}^w) \)-Gr if we need to emphasize the grading group, for example if \( A \) has multiple gradings.

**Definition 5.1.** Let \( A \) be a \( \mathbb{Z}^w \)-graded algebra, and let \( D^b(A\text{-Gr}) \) be the bounded derived category of \( \mathbb{Z}^w \)-graded left \( A \)-modules.

1. An object \( X \in D^b(A\text{-Gr}) \) is called an \( \epsilon \)-object if it is a perfect complex and \( H^0(X[n]) \) is a finite dimensional graded left \( A \)-module for all \( n \in \mathbb{Z} \).
2. Recall that \( D_\epsilon(A) \) is the subcategory of \( D^b(A\text{-Gr}) \) consisting of all \( \epsilon \)-objects. We say \( A \) satisfies the \( \epsilon \)-condition if \( D_\epsilon(A) \neq 0 \). We say that \( (A, \mathbb{Z}^w) \) satisfies the \( \epsilon \)-condition if the grading group needs emphasis.

The \( \epsilon \)-condition is a quite mild condition and we conjecture that all reasonable noetherian AS Gorenstein algebras satisfy it. In the next result, we give some of its important properties. First, we briefly recall the notion of a twist of a graded algebra. Let \( A \) be \( \mathbb{Z}^w \)-graded, and let a sequence \( \tilde{\sigma} := \{ \sigma_1, \cdots, \sigma_w \} \subset \text{Aut}_{\mathbb{Z}^w}(A) \) of pairwise commuting \( \mathbb{Z}^w \)-graded automorphisms of \( A \) be given. We write \( \sigma^r = \sigma_1^{r_1} \cdots \sigma_w^{r_w} \), for \( r = (r_1, \cdots, r_w) \in \mathbb{Z}^w \). The (left) graded twist of \( A \) associated to \( \tilde{\sigma} \) is a new graded algebra, denoted by \( A^\tilde{\sigma} \), such that \( A^\tilde{\sigma} = A \) as a \( \mathbb{Z}^w \)-graded vector space, and where the new multiplication \( \ast \) of \( A^\tilde{\sigma} \) is given by \( a \ast b = \sigma^{|r|}(a)b \) for all homogeneous elements \( a, b \in A \). Similarly, given a \( \mathbb{Z}^w \)-graded left \( A \)-module \( M \), we may define a \( \mathbb{Z}^w \)-graded \( A^\tilde{\sigma} \)-module \( M^\tilde{\sigma} \) with the same graded vector space as \( M \), but new action \( a \ast m = \sigma^{|r|}(a)m \) for homogeneous \( a \in A^\tilde{\sigma} \), \( m \in M \). The functor \( A\text{-Gr} \to (A^\tilde{\sigma})\text{-Gr} \) which sends \( M \) to \( M^\tilde{\sigma} \) and is the identity map on morphisms is an equivalence of categories [Zh, Theorem 3.1].

Below, we say that a commutative \( k \)-algebra is **affine** if it is finitely generated as a \( k \)-algebra.
Lemma 5.2.  
(1) Suppose that $A$ is a noetherian $\mathbb{Z}^w$-graded algebra with finite global dimension. If there is a nonzero finite dimensional graded module over $A$, then $A$ satisfies the $\epsilon$-condition.  
As a consequence, every connected graded noetherian $AS$ regular algebra satisfies the $\epsilon$-condition.

(2) Let $A$ be $\mathbb{Z}^w$-graded. Suppose that $\phi : \mathbb{Z}^w \to \mathbb{Z}^n$ is a group homomorphism. Then $A$ is also $\mathbb{Z}^n$-graded, where $A_r = \bigoplus_{s | \phi(s) = r} A_s$ for $r \in \mathbb{Z}^n$. If $(A, \mathbb{Z}^w)$ satisfies the $\epsilon$-condition, then $(A, \mathbb{Z}^n)$ satisfies the $\epsilon$-condition.

(3) Suppose that $A$ is connected $\mathbb{N}^w$-graded with $w > 0$ and that $A$ satisfies the $\epsilon$-condition. 
If there is a $\mathbb{Z}^w$-graded algebra map $A \to B$ such that $B_A$ is finitely generated, then $(B, \mathbb{Z}^w)$ satisfies the $\epsilon$-condition.  
As a consequence, every affine connected graded commutative algebra satisfies the $\epsilon$-condition.

(4) If $B$ is locally finite $\mathbb{N}$-graded and finite over its affine center, then $B$ satisfies the $\epsilon$-condition.

(5) Let $A$ and $B$ be connected $\mathbb{N}$-graded, and suppose that $A \otimes_k B$ is also noetherian. 
Then the following are equivalent: 
(i) $(A \otimes_k B, \mathbb{Z}^2)$ satisfies the $\epsilon$-condition; 
(ii) $(A \otimes_k B, \mathbb{Z})$ satisfies the $\epsilon$-condition; 
(iii) both $(A, \mathbb{Z})$ and $(B, \mathbb{Z})$ satisfy the $\epsilon$-condition.

As a consequence, $A$ satisfies the $\epsilon$-condition if and only if $A[t]$ does (as either a $\mathbb{Z}^2$ or $\mathbb{Z}$-graded algebra).

(6) $A$ satisfies the $\epsilon$-condition if and only if a graded twist $A^\phi$ does.

Proof. (1,2) These are clear.

(3) Let $X$ be a nonzero $\epsilon$-object over $A$. Let $Y = B \otimes_A^L X$. We claim that $Y$ is a nonzero $\epsilon$-object over $B$. To see that $Y$ is nonzero in the derived category, we note since $X$ is perfect that $Y = B \otimes_A X$ as a tensor product of complexes. Then $H^i(Y) = B \otimes_A H^i(X)$, where $i$ is the integer such that $H^i(X) \neq 0$ and $H^j(X) = 0$ for all $j > i$. Let $m$ be the unique graded maximal ideal of $A$, and let $M = A/m = A_{(0,0,...,0)}$ be the unique simple graded $A$-module. Since $H^i(X) \neq 0$, there is a surjective $A$-module map $f : H^i(X) \to M$. Since $B$ is a nonzero finitely generated graded right $A$-module, the graded Nakayama’s Lemma implies that $B \otimes_A M \neq 0$. Thus $B \otimes_A H^i(X) \neq 0$ and therefore $H^i(Y) \neq 0$.

Next, since $X$ is a perfect complex of $A$-modules, $Y = B \otimes_A X$ is a perfect complex of $B$-modules. Last, to see that $Y$ is an $\epsilon$-object over $B$, we note that $H^0(Y[n]) = H^n(Y) = \text{Tor}^A_n(B, X)$. So it suffices to show the assertion that $\text{Tor}^n_i(B, X)$ is finite dimensional for all $i$. By exact sequences and induction on $\sum_n \dim_k H^0(X[n])$ (forgetting that $X$ is perfect), we may reduce to the case that $X = M$, where $M$ is the unique simple graded left $A$-module. Since $A$ is noetherian, and $B_A$ is finitely generated, we may replace $B$ in $D^{-}(A_{gr})$ by a graded projective resolution $P$ of $B_A$ such that each term in the projective resolution is finitely generated. Then $Y = P \otimes_A M$ has finite dimensional cohomology as required.

For the consequence, take $A$ to be a connected graded affine commutative polynomial ring. By part (1), $A$ satisfies the $\epsilon$-condition. By the above paragraph, every factor ring of $A$ satisfies the $\epsilon$-condition.

(4) Let $Z$ be the center of $B$ and let $A = k \oplus Z_{\geq 1}$. Since $Z$ is affine by assumption, it easily follows that $A$ is affine as well. By part (3), $A$ satisfies the $\epsilon$-condition. Since $B$ is finite over $Z$, and $Z$ is clearly finite over $A$, $B$ is finite over $A$. The assertion follows from part (3).
(5) The implication (i) ⇒ (ii) is immediate from part (2). For (ii) ⇒ (iii), using the \( \mathbb{Z} \)-graded homomorphism \( A \otimes_k B \to A \otimes_k B/\geq_1 \cong A \otimes_k A \), \( A \) is a finitely generated \( A \otimes_k B \)-module. Thus \( (A, \mathbb{Z}) \)-satisfies the \( \epsilon \)-condition by part (3), and similarly for \( (B, \mathbb{Z}) \). Now suppose that \( (A, \mathbb{Z}) \) and \( (B, \mathbb{Z}) \) satisfy the \( \epsilon \)-condition, and let \( X \) and \( Y \) be nonzero \( \epsilon \)-objects over \( A \) and \( B \) respectively. Then \( X \otimes_k Y \) is a nonzero \( \mathbb{Z}_2 \)-graded \( \epsilon \)-object over \( A \otimes_k B \), proving (iii) ⇒ (i). The consequence follows by taking \( B = k[t] \).

(6) Note that the equivalence between \( (A, \mathbb{Z}^w) \text{-Gr} \) and \( (A, \mathbb{Z}^w) \text{-Gr} \) is naturally extended to the derived level. Then one checks that a nonzero object in \( D_\epsilon(A) \) maps to a nonzero object in \( D_\epsilon(A) \) under this equivalence. □

We can now prove the final major theorem of the paper.

**Theorem 5.3.** Let \( A \) be connected graded noetherian AS Gorenstein. Suppose that \( D_\epsilon(A) \neq 0 \). Then \( \text{hdet}_\mu A = 1 \). As a consequence, if \( A \) is connected graded noetherian AS regular, then \( \text{hdet}_\mu A = 1 \).

**Proof.** The proof depends heavily on the proof of [RRZ, Lemma 6.2]. The basic idea is to find a closely related AS Gorenstein algebra \( B \) such that \( \text{hdet}_\mu B = \text{hdet}_\mu A \) and \( \mu_B = \xi_c \) for some \( c \), and then apply Theorem 4.1.

In more detail, following the proof of [RRZ, Lemma 6.2], first one replaces \( A \) by a polynomial extension \( A[s] \) if necessary to avoid the case of AS index \(-1\). Note that the hypothesis \( D_\epsilon(A) \neq 0 \) still holds, by Lemma 5.2(5).

Then one takes the algebra \( A[t] \), as a \( \mathbb{Z}_2 \)-graded algebra, and does a \( \mathbb{Z}_2 \)-graded twist. Note that \( (A[t], \mathbb{Z}^2) \) satisfies the \( \epsilon \)-condition by Lemma 5.2(5), and so will a \( \mathbb{Z}_2 \)-graded twist by Lemma 5.2(6). Then one considers \( A[t] \) as a \( \mathbb{Z} \)-graded algebra via the “total degree” homomorphism \( \mathbb{Z}^2 \to \mathbb{Z} \); \( (A[t], \mathbb{Z}) \) satisfies the \( \epsilon \)-condition still by Lemma 5.2(2). Finally, one does a \( \mathbb{Z} \)-graded twist, arriving at an algebra \( B \), where \( (B, \mathbb{Z}) \) satisfies the \( \epsilon \)-condition by Lemma 5.2(6). By the proof of [RRZ, Lemma 6.2], \( \mu_B = \xi_c \) and \( \text{hdet}_\mu B = \text{hdet}_\mu A \).

Now by Theorem 4.1(2), \( \text{hdet}_\mu B = c^4 = 1 \), where \( c \) is the AS index of \( B \). So \( \text{hdet}_\mu A = 1 \). The consequence follows from Lemma 5.2(1). □

**Corollary 5.4.** If \( A \) is connected \( \mathbb{N} \)-graded noetherian AS Gorenstein and has any of the following additional properties, then \( \text{hdet}_\mu A = 1 \):

(i) \( A \) is a graded twist of an algebra which is finite over its affine center; or
(ii) \( A \) is a surjective image of a noetherian AS regular algebra.

**Proof.** By Theorem 5.3 we just need to verify that \( A \) satisfies the \( \epsilon \)-condition. This holds by Lemma 5.2(4)(6) in case (i), and by Lemma 5.2(1)(3) in case (ii). □

The following conjecture was made in [RRZ, Conjecture 6.4].

**Conjecture 5.5.** Let \( A \) be a noetherian connected graded AS Gorenstein algebra. Then \( \text{hdet}(\mu_A) = 1 \).

Most known AS Gorenstein algebras satisfy one of the conditions in Corollary 5.4, and so we have now proved the conjecture for the most common kinds of examples. Of course, the conjecture would be completely solved if there were a positive answer to the following question.

**Question 5.6.** Let \( A \) be a noetherian connected graded AS Gorenstein algebra. Does \( A \) satisfy the \( \epsilon \)-condition?
References

[AS] M. Artin and W.F. Schelter, Graded algebras of global dimension 3, Adv. Math. 66 (1987), no. 2, 171–216.

[AZ] M. Artin and J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (1994), no. 2, 228–287.

[AFH] L.L. Avramov, H.-B. Foxby and S. Halperin, Differential graded homological algebra, preprint.

[BM] R. Berger and N. Marconnet, Koszul and Gorenstein properties for homogeneous algebras, Algebr. Represent. Theory 9 (2006), no. 1, 67-97.

[Bo] R. Bocklandt, Graded Calabi-Yau algebras of dimension 3, J. Pure Appl. Algebra 212 (2008), no. 1, 14–32.

[BK] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337.

[CWZ] K. Chan, C. Walton and J.J. Zhang, Hopf actions and Nakayama automorphisms, J. Algebra 409 (2014), 26–53.

[Gi] V. Ginzburg, Calabi-Yau algebras, arXiv:math/0612139 (2006).

[HVOZ] J.W. He, F. Van Oystaeyen, Y. Zhang, Hopf algebra actions on differential graded algebras and applications, Bull. Belg. Math. Soc. Simon Stevin 18 (2011), no. 1, 99–111.

[Ja] J. P. Jans, On Frobenius algebras, Ann. of Math. 69 (1959), 392–407.

[JZ] N. Jing and J.J. Zhang, Gorensteinness of invariant subrings of quantum algebras, J. Algebra 221 (1999), no. 2, 669–691

[Ke] B. Keller, Calabi-Yau triangulated categories, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 467–489.

[KKZ] E. Kirkman, J. Kuzmanovich and J.J. Zhang, Gorenstein subrings of invariants under Hopf algebra actions, J. Algebra 322 (2009), no. 10, 3640–3669

[Kr] Henning Krause, Derived categories, resolutions, and Brown representability, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 101–139.

[LPWZ] D.-M. Lu, J.H. Palmieri, Q.-S. Wu and J.J. Zhang, Koszul equivalences in $A_{\infty}$-algebras, New York J. Math 14 (2008), 325–378.

[M] D. Murfet, Triangulated Categories Part I, notes available online at http://therisingsea.org/notes/TriangulatedCategories.pdf

[RV] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2002), no. 2, 295–366.

[RRZ] M. Reyes, D. Rogalski, and J. J. Zhang, Skew Calabi-Yau algebras and homological identities, Adv. Math. 264 (2014), 308–354.

[Sm] S. Paul Smith, Some finite-dimensional algebras related to elliptic curves, Representation theory of algebras and related topics, CMS Conf. Proc. 19, Amer. Math. Soc., Providence, RI, 1996, pp. 315–348.

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra 195 (1997), no. 2, 662–679.

[YZ] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra 213 (1999), no. 1, 1–51.

[Zh] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London Math. Soc. (3) 72 (1996), no. 2, 281-311.
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