Gallai-Ramsey numbers for monochromatic $K_4^+$ or $K_3$

Xueli Su, Yan Liu
School of Mathematical Sciences, South China Normal University,
Guangzhou, 510631, P.R. China *

Abstract

A Gallai $k$-coloring is a $k$-edge coloring of a complete graph in which there are no rainbow triangles. For two given graphs $H, G$ and two positive integers $k, s$ with that $s \leq k$, the $k$-colored Gallai-Ramsey number $gr_k(K_3 : s \cdot H, (k - s) \cdot G)$ is the minimum integer $n$ such that every Gallai $k$-colored $K_n$ contains a monochromatic copy of $H$ colored by one of the first $s$ colors or a monochromatic copy of $G$ colored by one of the remaining $k-s$ colors. In this paper, we determine the value of Gallai-Ramsey number in the case that $H = K_4^+$ and $G = K_3$. Thus the Gallai-Ramsey number $gr_k(K_3 : K_4^+)$ is obtained.

Key words: Gallai coloring, Rainbow triangle, Gallai-Ramsey number, Gallai partition.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. For a graph $G$, we use $|G|$ to denote the number of vertices of $G$, say the order of $G$. The complete graph of

*This work is supported by the Scientific Research Fund of the Science and Technology Program of Guangzhou, China (authorized in 2019) and by the Qinghai Province Natural Science Foundation£¨No.2020-ZJ-924). Correspondence should be addressed to Yan Liu(e-mail:liuyan@scnu.edu.cn)
order $n$ is denoted by $K_n$. For a subset $S \subseteq V(G)$, let $G[S]$ be the subgraph of $G$ induced by $S$. For two disjoint subsets $A$ and $B$ of $V(G)$, $E_G(A, B) = \{ab \in E(G) \mid a \in A, b \in B\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The union of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The join of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by adding all edges joining each vertex of $G_1$ and each vertex of $G_2$. For any positive integer $k$, we write $[k]$ for the set $\{1, 2, \cdots, k\}$. An edge coloring of a graph is called monochromatic if all edges are colored by the same color. An edge-colored graph is called rainbow if no two edges are colored by the same color. 

Given graphs $H_1$ and $H_2$, the classical Ramsey number $R(H_1, H_2)$ is the smallest integer $n$ such that for any 2-edge coloring of $K_n$ with red and blue, there exists a red copy of $H_1$ or a blue copy of $H_2$. A sharpness example of the Ramsey number $R(H_1, H_2)$, denoted by $C_{(H_1,H_2)}$, is a 2-edge colored $K_{R(H_1,H_2)−1}$ with red and blue such that there are neither red copies of $H_1$ nor blue copies of $H_2$. Given graphs $H_1, H_2, \cdots, H_k$, the multicolor Ramsey number $R(H_1, H_2, \cdots, H_k)$ is the smallest positive integer $n$ such that for every $k$-edge colored $K_n$ with the color set $[k]$, there exists some $i \in [k]$ such that $K_n$ contains a monochromatic copy of $H_i$ colored by $i$. The multicolor Ramsey number is an obvious generalization of the classical Ramsey number. When $H = H_1 = \cdots = H_k$, we simply denote $R(H_1, \cdots, H_k)$ by $R_k(H)$. The problem about computing Ramsey numbers is notoriously difficult. For more information on classical Ramsey number, we refer the readers to [5, 9, 10]. In this paper, we study Ramsey number in Gallai-coloring. A Gallai-coloring is an edge coloring of a complete graph with no rainbow triangle. Gallai-coloring naturally arises in several areas including: information theory [14]; the study of partially ordered sets, as in Gallai’s original paper [9] (his result was restated in [12] in the terminology of graphs); and the study of perfect graphs [2]. More information on this topic can be found in [7, 8]. A Gallai $k$-coloring is a Gallai-coloring that uses at most $k$ colors. Given a positive integer $k$ and graphs $H_1, H_2, \cdots, H_k$, the Gallai-Ramsey number $gr_k(K_n : H_1, H_2, \cdots, H_k)$ is the smallest integer $n$ such that every Gallai $k$-colored $K_n$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in [k]$. Clearly, $gr_k(K_3 : H_1, H_2, \cdots, H_k) \leq R(H_1, H_2, \cdots, H_k)$ for any $k$ and $gr_2(K_3 : H_1, H_2) = R(H_1, H_2)$. When $H = H_1 = \cdots = H_k$, we simply denote $gr_k(K_3 : H_1, H_2, \cdots, H_k)$ by $gr_k(K_3 : H)$. 

2
When \( H = H_1 = \cdots = H_s(0 \leq s \leq k) \) and \( G = H_{s+1} = \cdots = H_k \), we use the following shorthand notation

\[
gr_k(K_3 : s \cdot H, (k - s) \cdot G) = gr_k(K_3 : H, \cdots, H, G, \cdots, G).
\]

The authors in [15] and [17] determined the Gallai-Ramsey number \( gr_k(K_3 : s \cdot K_4, (k - s) \cdot K_3) \) and \( gr_k(K_3 : s \cdot K_3, (k - s) \cdot C_4) \), respectively. In this paper, we investigate the Gallai-Ramsey number \( gr_k(K_3 : s \cdot K_4^+, (k - s) \cdot K_3) \), where \( K_4^+ = K_1 \vee (K_3 + K_1) \). We will prove the following result in Section 2.

**Theorem 1.** Let \( k \) be a positive integer and \( s \) an integer such that \( 0 \leq s \leq k \). Then

\[
gr_k(K_3 : s \cdot K_4^+, (k - s) \cdot K_3) = \begin{cases} 
17^{\frac{s}{2}} \cdot 5^{\frac{k-s}{2}} + 1, & \text{if } s \text{ and } (k - s) \text{ are both even}, \\
2 \cdot 17^{\frac{s}{2}} \cdot 5^{\frac{k-s-1}{2}} + 1, & \text{if } s \text{ is even and } (k - s) \text{ is odd}, \\
4 \cdot 17^{\frac{s}{2}} - 1 + 1, & \text{if } s = k \text{ and } k \text{ is odd}, \\
8 \cdot 17^{\frac{s}{2}} \cdot 5^{\frac{k-s-1}{2}} + 1, & \text{if } s \text{ and } (k - s) \text{ are both odd}, \\
16 \cdot 17^{\frac{s-1}{2}} \cdot 5^{\frac{k-s-2}{2}} + 1, & \text{if } s < k \text{ and } s \text{ is odd and } (k - s) \text{ is even}.
\end{cases}
\]

When \( s = k \) and \( s = 0 \), we can get the following Theorem 2 and Theorem 3 from Theorem 1 respectively. So Theorem 2 and Theorem 3 can be seen as two corollaries obtained from Theorem 1.

**Theorem 2.** For a positive integer \( k \),

\[
gr_k(K_3 : K_4^+) = \begin{cases} 
17^{\frac{k}{2}} + 1, & \text{if } k \text{ is even}, \\
4 \cdot 17^{\frac{k-1}{2}} + 1, & \text{if } k \text{ is odd}.
\end{cases}
\]

**Theorem 3.** [3, 11] For a positive integer \( k \),

\[
gr_k(K_3 : K_3) = \begin{cases} 
5^{\frac{k}{2}} + 1, & \text{if } k \text{ is even}, \\
2 \cdot 5^{\frac{k-1}{2}} + 1, & \text{if } k \text{ is odd}.
\end{cases}
\]

To prove Theorem 1 the following theorem is useful.
Theorem 4. [9, 12, 11] (Gallai-partition) For any Gallai-coloring of a complete graph \( G \), there exists a partition of \( V(G) \) into at least two parts such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts. The partition is called a Gallai-partition.

Given a Gallai-partition \((V_1, V_2, \cdots, V_t)\) of a Gallai-colored complete graph \( G \), let \( H_i = G[V_i], h_i \in V_i \) for each \( i \in [t] \) and \( R = G[\{h_1, h_2, \cdots, h_t\}] \). Then \( R \) is said to be the reduced graph of \( G \) corresponding to the given Gallai-partition. By Theorem 4, all edges in the reduced graph \( R \) are colored at most two colors.

2 Proof of Theorem 1

First, recall some known classical Ramsey numbers of \( K_{4^+} + K_3 \) and \( K_3 \).

Lemma 2.1. [4, 13, 16]

\[
R(K_3, K_3) = 6, \quad R(K_{4^+}, K_3) = R(K_3, K_{4^+}) = 9, \quad R(K_{4^+}, K_{4^+}) = 18.
\]

For the sake of notation, now we define functions \( f_i \) for \( i \in [5] \) as follows.

\[
\begin{align*}
f_1(k, s) & = 17^s \cdot 5^\frac{k-s}{2}, \\
f_2(k, s) & = 2 \cdot 17^s \cdot 5^\frac{k-s-1}{2}, \\
f_3(k, s) & = 4 \cdot 17^{\frac{k-1}{2}}, \\
f_4(k, s) & = 8 \cdot 17^{\frac{k-1}{2}} \cdot 5^\frac{k-s-1}{2}, \\
f_5(k, s) & = 16 \cdot 17^{\frac{k-1}{2}} \cdot 5^\frac{k-s-2}{2}.
\end{align*}
\]

Let

\[
f(k, s) = \begin{cases} 
    f_1(k, s), & \text{if } s \text{ and } (k - s) \text{ are both even}, \\
    f_2(k, s), & \text{if } s \text{ is even and } (k - s) \text{ is odd}, \\
    f_3(k, s), & \text{if } s = k \text{ and } k \text{ is odd}, \\
    f_4(k, s), & \text{if } s \text{ and } (k - s) \text{ are both odd}, \\
    f_5(k, s), & \text{if } s < k, \text{ s is odd and } (k - s) \text{ is even.}
\end{cases}
\]

It is easy to check that
\[2f(k - 1, s) = \begin{cases} 
2f_2(k - 1, s), & \text{if } s \text{ and } (k - s) \text{ are both even,} \\
2f_1(k - 1, s), & \text{if } s \text{ is even and } (k - s) \text{ is odd,} \\
2f_3(k - 1, s), & \text{if } s = k - 1 \text{ and } k \text{ is even,} \\
2f_5(k - 1, s), & \text{if } s < k - 1, s \text{ and } (k - s) \text{ are both odd,} \\
2f_4(k - 1, s), & \text{if } s \text{ is odd and } (k - s) \text{ is even} 
\end{cases} \leq f(k, s), \tag{1}\]

\[5f(k - 2, s) = \begin{cases} 
5f_1(k - 2, s), & \text{if } s \text{ and } (k - s) \text{ are both even,} \\
5f_2(k - 2, s), & \text{if } s \text{ is even and } (k - s) \text{ is odd,} \\
5f_3(k - 2, s), & \text{if } s = k - 2 \text{ and } k \text{ is odd,} \\
5f_4(k - 2, s), & \text{if } s \text{ and } (k - s) \text{ are both odd,} \\
5f_5(k - 2, s), & \text{if } s < k - 2 \text{ and } s \text{ is odd and } (k - s) \text{ is even} 
\end{cases} \leq f(k, s), \tag{2}\]

and the following inequations hold.

\[f(k - 1, s - 1) \leq \frac{5}{16} f(k, s), \tag{3}\]

\[f(k - 2, s - 1) \leq \frac{1}{8} f(k, s), \tag{4}\]

\[f(k, s - 1) + f(k - 1, s - 1) \leq f(k, s). \tag{5}\]

Now we prove Theorem \[1]\.

**Proof.** We first prove that \(gr_k(K_3 : s \cdot K^+_4, (k - s) \cdot K_3) \geq f(k, s) + 1\) by constructing a Gallai \(k\)-colored complete graph with order \(f(k, s)\) which contains neither monochromatic copy of \(K^+_4\) colored by one of the first \(s\) colors nor monochromatic copy of \(K_3\) colored by one of the remaining \(k - s\) colors. For this construction, we use the sharpness example of classical Ramsey results. Let \(Q_1 = C_{(K_3,K_3)}\), \(Q_2 = C_{(K^+_4,K^+_4)}\) and \(Q_3 = C_{(K^+_4,K_3)}\). Then by Lemma \[2,1]\, \(Q_1\) is a 2-edge colored \(K_5\), \(Q_2\) is a 2-edge colored \(K_{17}\) and \(Q_3\) is a 2-edge
colored $K_8$. We construct our sharpness example by taking blow-ups of these sharpness examples $Q_i \ (i \in \{1, 2, 3\})$. A blow-up of an edge-colored graph $G$ on a graph $H$ is a new graph obtained from $G$ by replacing each vertex of $G$ with $H$ and replacing each edge $e$ of $G$ with a monochromatic complete bipartite graph $(V(H), V(H))$ in the same color with $e$. By induction, suppose that we have constructed $G_0, G_1, \cdots, G_i$, where $G_0$ be a single vertex and $G_i$ is colored by the color set $[i]$ such that $G_i$ has no monochromatic copy of $K_4^+$ colored by one of the first $s$ colors and no monochromatic copy of $K_3$ colored by one of the remaining $i - s$ colors. If $i = k$, then the construction is completed. Otherwise, we consider the following cases.

**Case a.** If $i \leq s - 2$, then construct $G_{i+2}$ by a blow-up of $Q_2$ colored by the two colors $i + 1$ and $i + 2$ on $G_i$. Then $G_{i+2}$ has no monochromatic copy $K_4^+$ in the first $i + 2$ colors.

**Case b.** If $i = s - 1$ and $k = s$, then construct $G_{i+1}$ by a blow-up of $K_4$ colored by the color $i + 1$ on $G_i$. Then $G_{i+1}$ has no monochromatic copy $K_4^+$ in the first $i + 1$ colors.

**Case c.** If $i = s - 1$ and $k > s$, then construct $G_{i+2}$ by a blow-up of $Q_3$ colored by the two colors $i + 1$ and $i + 2$ on $G_i$. Then $G_{i+2}$ contains neither monochromatic copy of $K_4^+$ colored by one of the first $i + 1$ colors nor monochromatic copy of $K_3$ colored by the color $i + 2$.

**Case d.** If $i \geq s$ and $i = k - 1$, then construct $G_{i+1}$ by a blow-up of $K_2$ colored by the color $i + 1$ on $G_i$. Then $G_{i+1}$ contains neither monochromatic copy of $K_4^+$ colored by one of the first $s$ colors nor monochromatic copy of $K_3$ colored by one of the remaining $k - s$ colors.

**Case e.** If $i \geq s$ and $i \leq k - 2$, then construct $G_{i+2}$ by a blow-up of $Q_1$ colored by the two colors $i + 1$ and $i + 2$ on $G_i$. Then $G_{i+2}$ contains neither monochromatic copy of $K_4^+$ colored by one of the first $s$ colors nor monochromatic copy of $K_3$ colored by one of the remaining $i + 2 - s$ colors.

By the above construction, it is clear that $G_k$ is Gallai $k$-colored and contains neither monochromatic copy of $K_4^+$ in any of the first $s$ colors nor monochromatic copy of $K_3$ in any of the remaining $k - s$ colors. If $s$ and $k - s$ are both even, then by Case a, we can first construct $G_s$ of order $17^\frac{s}{2}$. Next by Case e, we continue to construct $G_{s+2}, G_{s+4}, \cdots, G_k$. So we can get $|G_k| = f_1(k, s)$. If $s$ is even and $k - s$ is odd, then by Case a, we can first construct $G_s$ of order $17^\frac{s}{2}$. Next by Case e, we continue to construct $G_{s+2}, G_{s+4}, \cdots, G_{k-1}$.
and we can get $|G_{k-1}| = 17^2 \cdot 5^{k-s-1}$. Finally by Case d, we can construct $G_k$ of order $f_2(k, s)$. Similarly, we can get that $|G_k| = f(k, s)$ in the remaining three cases. Therefore,

$$gr_k(K_3 : s \cdot K_4^+, (k - s) \cdot K_3) \geq f(k, s) + 1.$$ 

Now we prove that $gr_k(K_3 : s \cdot K_4^+, (k - s) \cdot K_3) \leq f(k, s) + 1$ by induction on $k + s$. Let $n = f(k, s) + 1$ and $G$ be a Gallai $k$-colored complete graph of order $n$. The statement is trivial in the case that $k = 1$. The statement holds in the case that $k = 2$ by Lemma 2.1. The statement holds in the case that $s = 0$ by Theorem 3. So we can assume that $s \geq 1$ and $k \geq 3$, and the statement holds for any $s'$ and $k'$ such that $s' + k' < s + k$. Then $f(k, s) \geq 16$ and $n \geq 17$. Suppose, to the contrary, that $G$ contains neither monochromatic copy of $K_4^+$ in any of the first $s$ colors nor monochromatic copy of $K_3$ in any of the remaining $k - s$ colors. By Theorem 4 there exists a Gallai-partition of $V(G)$. Choose a Gallai-partition with the smallest number of parts, say $(V_1, V_2, \ldots, V_t)$ and let $H_i = G[V_i]$ for $i \in [t]$. Then $t \geq 2$. Choose one vertex $h_i \in V_i$ and set $R = G[[h_1, h_2, \ldots, h_t]]$. Then $R$ is a reduced graph of $G$ colored by at most two colors.

We first consider the case that $t = 2$. W.L.O.G, suppose that the color on the edges between two parts is red. If red is in the last $k - s$ colors (so $s < k$), then $H_1$ and $H_2$ both have no red edges, otherwise, there exists a red $K_3$, a contradiction. Hence $|H_i| \leq gr_{k-1}(K_3 : s \cdot K_4^+, (k - s - 1) \cdot K_3) - 1$ for each $i \in [2]$. By the induction hypothesis and (1),

$$|G| = |H_1| + |H_2| \leq 2f(k - 1, s) \leq f(k, s) < n,$$

a contradiction. If red is in the first $s$ colors (W.L.O.G, suppose that red is the $s$th color), then $H_1$ has no red edges or $H_2$ has no red edges, otherwise, there exists a red $K_3^+$ in $G$, a contradiction. W.L.O.G, suppose that $H_1$ has no red edges. Then $|H_1| \leq gr_{k-1}(K_3 : (s - 1) \cdot K_4^+, (k - s) \cdot K_3) - 1$. If $H_2$ has a red $K_3$, then we can find a red $K_4^+$ in $G$, a contradiction. Hence $H_2$ contains no monochromatic copy of $K_4^+$ in any of the first $s - 1$ colors and no monochromatic copy of $K_3$ in any of the remaining $(k - s + 1)$ colors. Hence $|H_2| \leq gr_k(K_3 : (s - 1) \cdot K_4^+, (k - s + 1) \cdot K_3) - 1$. By the induction hypothesis and (5), we get that

$$|G| = |H_1| + |H_2| \leq f(k - 1, s - 1) + f(k, s - 1) \leq f(k, s) < n,$$
a contradiction.

Then we can assume that \( t \geq 3 \) and since \( t \) is smallest, \( R \) is colored by exactly two colors. Suppose that the two colors appeared in the Gallai-partition \((V_1, V_2, \cdots, V_t)\) are red and blue, that is, the reduced graph \( R \) is colored by red and blue. If the edge \( h_ih_j \) is red in \( R \), then \( h_j \) is said to be a red neighbor of \( h_i \). Let \( N_r(h_i) = \{ \text{all red neighbors of } h_i \} \), named the red neighborhood of \( h_i \), and \( N_b(h_i) \) be the blue neighborhood of \( h_i \) symmetrically. For each vertex \( h_i \in V(R) \) \((i \in [t])\), let \( d_r(h_i) = |N_r(h_i)| \), the red degree of \( h_i \) in \( R \). Then \( d_r(h_i) + d_b(h_i) = t - 1 \). If there exists one part (say \( V_1 \)) such that all edges joining \( V_1 \) to the other parts are colored by the same color, then we can find a new Gallai-partition with two parts \((V_1, V_2 \cup \cdots \cup V_t)\), which contradicts with that \( t \) is smallest. It follows that \( t \neq 3 \) and the following fact holds.

**Fact 1.** For any \( h_i \in V(R) \), we have that \( d_r(h_i) \geq 1 \) and \( d_b(h_i) \geq 1 \).

Now we can assume that \( t \geq 4 \). We consider the following three cases.

**Case 1.** Both red and blue are in the last \( k - s \) colors (so \( s \leq k - 2 \)).

This means that there is neither red \( K_3 \) nor blue \( K_3 \) in \( G \). Since \( R(K_3, K_3) = 6 \), we have \( t \leq 5 \). By Fact \( \text{II} \) \( H_i \) contains neither red edges nor blue edges for each \( i \). Then \( |H_i| \leq gr_{k-2}(K_3 : s \cdot K_4^+, (k-s-2) \cdot K_3) - 1 \) for \( 1 \leq i \leq t \). By the induction hypothesis and (2), we get that

\[
|G| = \sum_{i=1}^{t} |H_i| \leq t \cdot f(k-2, s) \leq 5f(k-2, s) \leq f(k, s) < n,
\]

a contradiction.

Now we only consider the case that red or blue are in the first \( s \) colors in the following.

Then we have the following Facts.

**Fact 2.** If red is in the first \( s \) colors, then for any \( p \in [t-1] \) and any \( p \) parts \( V_{j_1}, \cdots, V_{j_p} \), \( G[V_{j_1} \cup \cdots \cup V_{j_p}] \) has no red \( K_4 \), and the statement holds for blue symmetrically.

Otherwise, suppose that there is a red \( K_4 \) in \( G[V_{j_1} \cup \cdots \cup V_{j_p}] \). If there exists one red edge in \( E_G(V_{j_1} \cup \cdots \cup V_{j_p}, V(G) \setminus (V_{j_1} \cup \cdots \cup V_{j_p})) \), then \( G \) contains a red \( K_4^+ \), a
contradiction. It follows that all edges in \( E_G(V_{j_1} \cup \cdots \cup V_{j_p}, V(G) \setminus (V_{j_1} \cup \cdots \cup V_{j_p})) \) are blue. Then \( (V_{j_1} \cup \cdots \cup V_{j_p}, V(G) \setminus (V_{j_1} \cup \cdots \cup V_{j_p})) \) is a new Gallai-partition which contradicts with that \( t \geq 4 \) and \( t \) is smallest.

By Fact 1 and Fact 2 we have the following facts.

**Fact 3.** If red is in the first \( s \) colors, then every \( H_i \) has no red \( K_3 \), and the statement holds for blue symmetrically.

**Fact 4.** If red is in the first \( s \) colors and \( h_i \) is contained in a red \( K_3 \) of \( R \), then \( H_i \) has no red edge, and the statement holds for blue symmetrically.

**Fact 5.** If red is in the first \( s \) colors, and \( H_i \) and \( H_j \) both contain red edges, then the edges \( E_G(V_i, V_j) \) are blue, that is, edge \( h_ih_j \) is blue in \( R \), and the statement holds for blue symmetrically.

Let \( I_r = \{ h_i \in V(R) : H_i \text{ contains at least one red edge} \} \) and \( I_b = \{ h_i \in V(R) : H_i \text{ contains at least one blue edge} \} \). Clearly by Fact 5 if red is in the first \( s \) colors, then the induced subgraph of \( R \) by \( I_r \) is a blue complete graph and if blue is in the first \( s \) colors, then the induced subgraph of \( R \) by \( I_b \) is a red complete graph.

**Fact 6.** If red and blue are both in the first \( s \) colors, then \( |I_b \cap I_r| \leq 1 \).

Suppose, to the contrary, that \( h_i, h_j \in I_b \cap I_r \). We know that \( h_ih_j \) is either red or blue. Then we can find either a red \( K_4 \) or a blue \( K_4 \) in \( G[V_i \cup V_j] \), which contradicts with Fact 2.

Suppose that \( |H_i| \geq 2 \) for each \( i \in [l] \) and \( |H_j| = 1 \) for each \( j \) with \( l + 1 \leq j \leq t \). Then \( l \geq 1 \). Otherwise, \( R = G \). Then \( G \) is 2-edge colored, which contradicts with that \( k \geq 3 \). Let \( p_0 \) be the number of \( H_i(i \in [l]) \) which contains neither red nor blue edges, \( p_1 \) the number of \( H_i \) which contains either red or blue edges and \( p_2 \) the number of \( H_i \) which contains both red and blue edges. Then \( l = p_0 + p_1 + p_2, p_2 = |I_b \cap I_r| \) and \( p_1 + p_2 = |I_b \cup I_r| \).

**Case 2.** Exactly one of red and blue is in the first \( s \) colors and the other is in the last \( k - s \) colors. (so \( s \leq k - 1 \)).

W.L.O.G., suppose that red appears in the first \( s \) colors, blue appears in the last \( k - s \) colors and red is the \( s \)th color. This means that there is neither red \( K_4^+ \) nor blue \( K_3 \) in
Then by Fact \( \text{II} \) we can get that every \( H_i \) contains no blue edge, which implies that \( p_2 = 0 \). Since \( R(K_4^+, K_3) = 9 \), we have that \( t \leq 8 \).

**Claim 1.** \( p_1 \leq 2 \).

Otherwise, suppose that \( p_1 \geq 3 \). Then W.L.O.G., suppose that \( H_1, H_2 \) and \( H_3 \) are three corresponding subgraphs which contain red edges. By Fact \( \text{V} \) \( R([h_1, h_2, h_3]) \) is a blue \( K_3 \). So \( G \) has a blue \( K_3 \), a contradiction.

**Claim 2.**

\[ |G| \leq \left( \frac{1}{8} p_0 + \frac{5}{16} p_1 \right) f(k, s) + (t - l). \]

**Proof.** First let \( i \in [l] \) and \( H_i \) contain no red edges. Then \( H_i \) is colored with exactly \( k - 2 \) colors and satisfies that it contains neither monochromatic \( K_4^+ \) in one of the first \( s - 1 \) color nor monochromatic \( K_3 \) in one of the remaining \( k - s - 1 \) colors. Hence by the induction hypothesis and (4),

\[ |H_i| \leq gr_{k-2}(K_3 : (s - 1) \cdot K_4^+, (k - s - 1) \cdot K_3) - 1 = f(k - 2, s - 1) \leq \frac{1}{8} f(k, s). \]

Next suppose that \( H_i \) contains a red edge. Then \( H_i \) is colored with exactly \( k - 1 \) colors and by Fact \( \text{III} \) \( H_i \) contains neither monochromatic \( K_4^+ \) in one of the first \( s - 1 \) colors nor monochromatic \( K_3 \) in one of the remaining \( k - s \) colors. Hence by the induction hypothesis and (3),

\[ |H_i| \leq gr_{k-1}(K_3 : (s - 1) \cdot K_4^+, (k - s) \cdot K_3) - 1 = f(k - 1, s - 1) \leq \frac{5}{16} f(k, s). \]

Therefore by the above inequalities, we get that

\[ |G| \leq \left( \frac{1}{8} p_0 + \frac{5}{16} p_1 \right) f(k, s) + (t - l). \]

\[ \square \]

We now consider subcases based on the values of \( l \) and \( t \).

**Subcase 2.1.** \( t \leq 5 \).
By Claim \([1]\) \(p_1 \leq 2\). Since \(p_0 + p_1 = l \leq t\) and by Claim \([2]\) we have that
\[
|G| \leq \left(\frac{1 \times 3}{8} + \frac{5 \times 2}{16}\right) f(k, s) < n,
\]
a contradiction.

**Subcase 2.2.** \(l \leq 2\) and \(6 \leq t \leq 8\).

Then by Claim \([1]\) and Claim \([2]\) we can get that
\[
|G| \leq \left(\frac{1 \times 0}{8} + \frac{5 \times 2}{16}\right) f(k, s) + (t - 2)
\leq \frac{5}{8} f(k, s) + 6
\leq f(k, s) < n,
\]
a contradiction.

**Subcase 2.3.** \(l \geq 3\) and \(6 \leq t \leq 8\).

**Claim 3.** In this case, \(p_1 \leq 1\). Further, if \(t \geq 7\), then \(p_1 = 0\).

**Proof.** First, to the contrary, we can assume that \(H_1\) and \(H_2\) both contain a red edge, and each other \(H_i\) contains no red edge by Claim \([1]\). Then by Fact \([5]\), we have that the edges in \(E_G(V_1, V_2)\) are blue. If there are two blue edges in \(\{h_1h_i| 3 \leq i \leq t\}\), then we can find a blue triangle \(h_1h_ph_q(2 \leq p < q \leq t)\) which contradicts with that \(G\) has no blue \(K_3\) or find a red \(K_3\) which is \(h_2h_ph_q(3 \leq p < q \leq t)\), which contradicts with Fact \([4]\). So we can assume that all edges in \(\{h_1h_i| 4 \leq i \leq t\}\) are red. It follows that either we can find a red triangle \(h_1h_ph_q(4 \leq p < q \leq t)\) which contradicts with Fact \([4]\) or we can find a blue \(K_3\) which is \(h_4h_5h_6\), a contradiction. Secondly, let \(t \geq 7\). To the contrary, we can assume that \(H_1\) contains a red edge and each other \(H_i\) contains no red edge. Then by Fact \([4]\) and since \(G\) has no blue \(K_3\), there are at most two red edges in \(\{h_1h_i| 2 \leq i \leq t\}\). So we can assume that all edges in \(\{h_1h_i| 4 \leq i \leq t\}\) are blue. It follows that either we can find a blue \(K_3\) which is \(h_1h_ph_q\), where \(4 \leq p < q \leq t\), a contradiction, or we can find a red \(K_4\) induced by \(\{h_4, h_5, h_6, h_7\}\), which contradicts with Fact \([2]\).

Thus, if \(t = 6\), then \(p_1 \leq 1\). By Claim \([2]\) we have that
\[
|G| \leq \left(\frac{5 \times 1}{16} + \frac{l - 1}{8}\right) f(k, s) + (6 - l)
\leq \frac{2l + 3}{16} f(k, s) + (6 - l) < f(k, s) < n,
\]
a contradiction.

If \( t \geq 7 \), then by Claim 3 and Claim 2, we have that \( p_1 = 0 \) and

\[
|G| \leq \frac{t}{8} + (t - l) \leq \frac{f(k, s)}{8} < n,
\]
a contradiction. The proof of Case 2 is completed.

**Case 3.** Red and blue are both in the first \( s \) colors (so \( s \geq 2 \)).

This means that there exists neither red \( K_4^+ \) nor blue \( K_4^+ \) in \( G \). W.L.O.G., suppose that red and blue are the \((s - 1)\)th and \(s\)th color, respectively. Since \( R(K_3^+, K_4^+) = 18 \), we know that \( t \leq 17 \). Since \( s \geq 2 \) and \( k \geq 3 \), \( f(k, s) \geq 35 \). First we prove some claims.

**Claim 4.** For any \( h \in V(R) \), we have \( d_r(h) \leq 8 \) and \( d_b(h) \leq 8 \) in \( R \).

W.L.O.G., suppose, to the contrary, that \( d_r(h) \geq 9 \). Since \( R(K_3^+, K_4^+) = 9 \), then the subgraph of \( R \) induced by \( N_r(h) \) contains either a red \( K_3 \) or a blue \( K_4^- \). So \( R \) contains a red \( K_4^+ \) or blue \( K_4^- \). Thus \( G \) contains also, a contradiction.

**Claim 5.** If \( d_r(h) \geq 4 \), then \( h \notin I_r \), and if \( d_b(h) \geq 4 \), then \( h \notin I_b \).

Suppose that \( d_r(h) \geq 4 \). To the contrary, suppose that \( H \) contains a red edge. If the induced subgraph \( R[N_r(h)] \) contains a red edge, say \( h_p h_q \), then we find a red \( K_3 \) which is \( h_i h_p h_q \), which contradicts with Fact 3. Otherwise, \( R[N_r(h)] \) contains a blue \( K_4 \). So \( G[\bigcup_{h \in N_r(h)} V_p] \) contains a blue \( K_4 \), which contradicts with Fact 2. So we have that \( h \notin I_r \). The proof for blue is as same as the above one for red symmetrically.

**Claim 6.** \(|I_b| + |I_r| \leq 4\).

If there is a vertex \( h_i \in I_b \cap I_r \), then by Fact 4, \( h_i \) is contained in neither a red \( K_3 \) nor a blue \( K_3 \) in \( R \). By Fact 5, we know that the induced subgraph of \( R \) by \( I_r \) is a blue complete graph and the induced subgraph of \( R \) by \( I_b \) is a red complete graph. It follows that \(|I_b| \leq 2 \) and \(|I_r| \leq 2 \).

Now we can assume that \( I_b \cap I_r = \emptyset \) by Fact 6. To the contrary, suppose that \(|I_b| + |I_r| \geq 5 \). If \(|I_b| \geq 4 \), then by Fact 5, the subgraph \( R[I_b] \) contains a red \( K_4 \), which contradicts with Fact 2. Then \(|I_b| \leq 3 \). By the same reasons, we know that \(|I_r| \leq 3 \). Then
\(|I_b| = 3\) or \(|I_r| = 3\). W.L.O.G., suppose that \(|I_b| = 3\). Then \(|I_r| \geq 2\). Let \(I_b = \{h_1, h_2, h_3\}\) and \(h_4, h_5 \in I_r\). By Fact 5, \(h_1h_2h_3\) is a red triangle and \(h_4h_5\) is a blue edge in \(R\). It is easy to check that there exists a red triangle \(h_ph_ih_j\) such that \(p \in \{4, 5\}\) and \(i, j \in [3]\) or a blue \(K_3 = h_ih_4h_5\) such that \(i \in [3]\), which contradicts with Fact 4.

**Claim 7.**

\[|G| \leq \left( \frac{1}{17}p_0 + \frac{5}{2 \times 17}p_1 + \frac{5}{17}p_2\right) f(k, s) + (t - l).\]

**Proof.** First suppose that \(H_i(i \leq l)\) contains neither red nor blue edges. This means that \(H_i\) is colored with exactly \(k - 2\) colors and satisfies that it has neither monochromatic \(K_4^+\) in one of the first \(s - 2\) colors nor monochromatic \(K_3\) in one of the remaining \(k - s\) colors. It is easy to check that

\[\frac{f_j(k - 2, s - 2)}{f_j(k, s)} = \frac{1}{17},\]

for any \(1 \leq j \leq 5\). Hence by the induction hypothesis,

\[|H_i| \leq gr_{k-2}(K_3 : (s - 2) \cdot K_4^+, (k - s) \cdot K_3) - 1 = f(k - 2, s - 2) = \frac{1}{17}f(k, s).\]  \(\text{(6)}\)

Next suppose that \(H_i\) contains no red edges but contains blue edges. This means that \(H_i\) is colored with exactly \(k - 1\) colors and satisfies that it contains neither monochromatic \(K_4^+\) in one of the first \(s - 2\) colors nor monochromatic \(K_3\) in one of the remaining \(k - s + 1\) colors by Fact 3. It is easy to check that

\[\frac{f_2(k - 1, s - 2)}{f_1(k, s)} = \frac{f_3(k - 1, s - 2)}{f_3(k, s)} = \frac{f_5(k - 1, s - 2)}{f_4(k, s)} = \frac{2}{17},\]

\[\frac{f_1(k - 1, s - 2)}{f_2(k, s)} = \frac{f_4(k - 1, s - 2)}{f_5(k, s)} = \frac{5}{2 \times 17},\]

So by the induction hypothesis,

\[|H_i| \leq gr_{k-1}(K_3 : (s - 2) \cdot K_4^+, (k - s + 1) \cdot K_3) - 1 = f(k - 1, s - 2) \leq \frac{5}{2 \times 17} f(k, s).\]  \(\text{(7)}\)

The same inequality holds if \(H_i\) contains no blue edges but contains red edges.

Finally suppose that \(H_i\) contains both red and blue edges. This means that \(H_i\) is colored with all \(k\) colors and satisfies that it contains neither monochromatic \(K_4^+\) in one
of the first $s - 2$ colors nor monochromatic $K_3$ in one of the remaining $k - s + 2$ colors by Fact 3. It is easy to check that
\[ \frac{f(k,s-2)}{f(k,s)} \leq \frac{5}{17}, \]
so by the induction hypothesis,
\[ |H_i| \leq gr_k(K_3 : (s - 2) \cdot K_4^+, (k - s + 2) \cdot K_3) - 1 = f(k, s - 2) \leq \frac{5}{17} f(k, s). \quad (8) \]

Combining Inequalities (6)-(8), we have the following inequality
\[ |G| \leq \left( \frac{1}{17} p_0 + \frac{5}{2 \times 17} p_1 + \frac{5}{17} p_2 \right) f(k, s) + (t - l). \]

We now consider subcases based on the value of $l$ and $t$.

**Subcase 3.1.** $13 \leq t \leq 17$.

By Claim 4 $d_r(h_i) \leq 8$ and $d_b(h_i) \leq 8$ in $R$ for any $i \in [l]$. Since $d_r(h_i) + d_b(h_i) = t - 1$, we have that $d_b(h_i) \geq 4$ and $d_r(h_i) \geq 4$ in $R$. Then by Claim 5 every $H_i$ contains neither red nor blue edge. So $p_2 = p_1 = 0$. Thus $p_0 = l$. By Claim 7 we have that
\[ |G| \leq \frac{l}{17} f(k, s) + (t - l) < f(k, s) + 1 = n, \]
a contradiction.

Next, we consider the case that $4 \leq t \leq 12$. By Fact 6 we have that $p_2 \leq 1$.

**Subcase 2.** $l \leq 3$.

Then $p_1 \leq 2$ if $p_2 = 1$ and $p_1 \leq 3$ if $p_2 = 0$. Hence by Claim 7 we get
\[ |G| \leq \begin{cases} \frac{10}{17} f(k, s) + (t - 3), & \text{if } p_2 = 1 \text{ and } p_1 \leq 2, \\ \frac{15}{34} f(k, s) + (t - 3), & \text{if } p_2 = 0 \text{ and } p_1 \leq 3 \end{cases} \]
\[ \leq f(k, s) < n, \]
a contradiction.

**Subcase 3.3.** $4 \leq l \leq 10$.

First suppose that $p_2 = 1$. It follows that $t \leq 7$. Otherwise, for any $h_i \in V(R)$, we have that $d_r(h_i) \geq 4$ or $d_b(h_i) \geq 4$ in $R$ since $d_r(h_i) + d_b(h_i) = t - 1$. Then by Claim 5, every
$H_i$ contains no red edge or contains no blue edge, which contradicts with the assumption that $p_2 = 1$. By Claim 6 we have that $p_1 \leq 2$. Thus, by Claim 7 we have that

$$|G| \leq \frac{14}{17} f(k, s) < n,$$

a contradiction.

Now we assume that $p_2 = 0$. By Claim 6 we have that $p_1 \leq 4$. This means that

$$|G| \leq \left\lfloor \frac{5}{2 \times 17} p_1 + \frac{1}{17} (l - p_1) \right\rfloor f(k, s) + (t - l) \leq \frac{16}{17} f(k, s) + 2 \leq f(k, s) < n,$$

a contradiction.

**Subcase 3.4.** $l \geq 11$.

Then $11 \leq l \leq t \leq 12$. Hence $d_r(h_i) \geq 4$ or $d_b(h_i) \geq 4$ for each $i \in [t]$. It follows that $p_2 = 0$ by Claim 5. If $d_r(h_i) \geq 4$ and $d_b(h_i) \geq 4$ for each $i \in [l]$, then $p_1 = 0$ by Claim 5. So $p_0 = l$. By Claim 7 we have that

$$|G| \leq \frac{l}{17} f(k, s) + (t - l) \leq f(k, s) < n,$$

a contradiction. So, W.L.O.G., we can assume that $d_r(h_1) \geq 4$ and $d_b(h_1) \leq 3$. By Claim 4 we know that $(d_r(h_1), d_b(h_1), l, t) \in \{(7, 3, 11, 11), (8, 2, 11, 11), (8, 3, 11, 12), (8, 3, 12, 12)\}$. Let $\tilde{d}_r(h_1) = |N_r(h_1) \cap \{h_1, \ldots, h_l\}|$ and $\tilde{d}_b(h_1) = |N_b(h_1) \cap \{h_1, \ldots, h_l\}|$. Then $(\tilde{d}_r(h_1), \tilde{d}_b(h_1), l) \in \{(7, 3, 11), (8, 2, 11), (8, 3, 12)\}$. Let $F$ be the subgraph of $R$ induced by $N_r(h_1) \cap \{h_1, \ldots, h_l\}$. So $|F| \geq 7$. Clearly, $F$ has no red $K_3$. Otherwise we can find a red $K_4$ obtained by a red $K_3$ in $F$ and $h_1$, which contradicts with Fact 2.

**Claim 8.** $V(F) \cap I_b = \emptyset$, $V(F) \cap I_r = \emptyset$ and $h_1 \notin I_r$.

First we claim that the red degree of $h_i$ in $F$ is at most 3 for each $h_i \in V(F)$. Otherwise, suppose that $h_i$ has at least four red neighbors in $F$. Since $F$ has no red $K_3$, we can find a blue $K_4$ induced by the red neighbors of $h_i$ in $F$, which contradicts with Fact 2. Then the blue degree of $h_i$ in $F$ is at least 3 for each $h_i \in V(F)$. It follows that every vertex $h_i$ of $F$ is contained in a blue triangle of $F$. Thus, by Fact 4 for each $h_i \in V(F)$, we have that $h_i \notin I_b$. Secondly, we claim that the blue degree of $h_i$ in $F$ is at most 5 for each $h_i \in V(F)$. Otherwise, we find a blue $K_3$ induced by the blue neighbors of $h_i$ in
$F$ since $R(K_3, K_3) = 6$ and $F$ has no red $K_3$. Then there exists a blue $K_4$ in $R$, which contradicts with Fact 2. It follows that the red degree of $h_i$ in $F$ is at least 1 for each $h_i \in V(F)$. Hence $h_1$ and $h_i$ are contained in the same red triangle of $R$ for each vertex $h_i$ of $F$. Thus, by Fact 4, $h_1 \notin I_r$ and for each $h_i \in V(F)$, we have that $h_i \notin I_r$.

By Claim 8, $p_0 \geq |F|$. Now we consider the case that $(\tilde{d}_r(h_1), \tilde{d}_b(h_1, l) = (8, 2, 11)$. Then $p_0 \geq 8$. By Fact 6, $p_2 \leq 1$.

If $p_2 = 0$, then by Claim 7 we get that

$$|G| \leq \left[ \frac{1}{11}p_0 + \frac{5}{2 \times 11}(11 - p_0) \right] f(k, s) + (t - l) \leq \frac{31}{34} f(k, s) + 1 < f(k, s) < n,$$

a contradiction. If $p_2 = 1$, we can assume that $h_p \in I_r \cap I_b$. Then by Claim 8, $h_p \in N_b(h_1)$. By Fact 2, $h_1 \notin I_b$. Then by Claim 8, $h_1 \notin I_r \cup I_b$. So $p_0 \geq |F| + 1 = 9$. Then by Claim 7, we get that

$$|G| \leq \left[ \frac{9}{11} + \frac{5}{2 \times 11} + \frac{5}{2 \times 11} \right] f(k, s) + (t - l) \leq \frac{33}{34} f(k, s) + 1 < f(k, s) + 1 = n,$$

a contradiction.

Now we consider the remaining cases that $(\tilde{d}_r(h_1), \tilde{d}_b(h_1, l) \in \{(7, 3, 11), (8, 3, 12)\}$. Then we can assume that $N_b(h_1) \cap \{h_1, \cdots, h_l\} = \{h_o, h_p, h_q\}$. First consider the case that there is a blue edge spanned by vertices in $\{h_o, h_p, h_q\}$, say $h_p h_q$. Then $h_1 h_p h_q$ is a blue $K_3$. By Fact 4, we have that $h_1, h_p, h_q \notin I_l$. Then by Claim 8, $h_1 \notin I_r \cup I_b$. So $p_0 \geq |F| + 1$. If $p_2 = 0$, then by Claim 7 we get that

$$|G| \leq \left[ \frac{1}{11}p_0 + \frac{5}{2 \times 11}(l - p_0) \right] f(k, s) + (t - l) \leq \frac{33}{34} f(k, s) + 1 < f(k, s) + 1 = n,$$

a contradiction. If $p_2 = 1$, then $h_o \in I_r \cap I_b$ and $h_o h_p, h_o h_q$ are red. Then by Fact 2, $h_p, h_q \notin I_r \cup I_b$. So $p_0 = |F| + 3$. Then by Claim 7, we get that

$$|G| \leq \left[ \frac{11}{17} + \frac{5}{17} \right] f(k, s) + t - l < f(k, s) + 1 = n,$$

a contradiction.

Secondly, we consider the case that $h_o h_p h_q$ is a red $K_3$. By Fact 4, $h_o, h_p, h_q \notin I_r$. Then $p_2 = 0$. By Fact 2, for each vertex $h_i$ in $F$, there is at least one blue edge in $E_R(h_i, \{h_o, h_p, h_q\})$. This means that there are at least 7 blue edges between $V(F)$ and $\{h_o, h_p, h_q\}$. By the pigeonhole principle, there is a vertex in $\{h_o, h_p, h_q\}$, say $h_o$, such that
$|N_b(h_o) \cap V(F)| \geq 3$. If the subgraph induced by $N_b(h_o) \cap V(F)$ contains no blue edge, then $N_b(h_o) \cap V(F)$ along with $h_1$ induce a red $K_4$ in $R$, which contradicts with Fact 2. Then the subgraph induced by $N_b(h_o) \cap V(F)$ contains a blue edge. So $h_o$ is contained in both a red $K_3$ and a blue $K_3$. Thus $h_o \not\in I_b \cup I_r$ by Fact 4. So $p_0 \geq |F| + 1$. By Claim 7, we have that

$$|G| \leq \left[ \frac{1}{17}p_0 + \frac{5}{2 \times 17}(l - p_0) \right] f(k, s) + (t - l) \leq \frac{33}{34} f(k, s) + 1 < f(k, s) + 1 = n,$$

a contradiction.

Complete the proof of Case 3 and then the proof of Theorem 1.

References

[1] K. Cameron and J. Edmonds. Lambda composition. *J. Graph Theory*, 26(1):9–16, 1997.

[2] K. Cameron, J. Edmonds, and L. Lovász. A note on perfect graphs. *Period. Math. Hungar.*, 17:173–175, 1986.

[3] F. R. K. Chung and R. L. Graham. Edge-colored complete graphs with precisely colored subgraphs. *Combinatorica*, 3:315–324, 1983.

[4] M. Clancy. Some small ramsey numbers. *J. Graph Theory*, pages 89–91, 1977.

[5] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp. The size ramsey number. *Period. Math. Hungar.*, 9 (1–2):145–161, 1978.

[6] R. Faudree and Schelp R. A survey of results on the size ramsey number. *in: Paul Erdős and his mathematics, II (Budapest, 1999), in: Bolyai Soc. Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest*, pages 291–309, 2002.

[7] J. Fox, A. Grinshpun, and Pach J. The erdos-hajnal conjecture for rainbow triangles. *J. Combin. Theory Ser. B*, 111:75–125, 2015.
[8] S. Fujita, C. Magnant, and K. Ozeki. Rainbow generalizations of ramsey theory: a survey. *Graphs and Combin.*, 26:1–30, 2010.

[9] T. Gallai. Transitiv orientierbare graphen. volume 18, pages 25–66. 1967.

[10] R. Graham, B. Rothschild, and J. Spencer. Ramsey theory. *Wiley, New York*, 1990.

[11] A. Gyárfás, G. Sárközy, A. Sebő, and S. Selkow. Ramsey-type results for gallai-colorings. *J. Graph Theory*, 64:233–243, 2010.

[12] A. Gyárfás and G. Simonyi. Edge colorings of complete graphs without tricolored triangles. *J. Graph Theory*, 46(3):211–216, 2004.

[13] H. Harborth and I. Mengersen. All ramsey number for five vertices and seven or eight edges. *Discrete Math.*, pages 91–98, 1988.

[14] J. Körner and G. Simonyi. Graph pairs and their entropies: modularity problems. *Combinatorica*, 20:227–240, 2000.

[15] H. Liu, Magnant C., Saito A., I. Schiermeyer, and Y. Shi. Gallai-ramsey number for $k_4$. *J. Graph Theory*, pages 1–14, 2019.

[16] S. P. Radziszowski. Small ramsey numbers. *Electron. J. Combin.*, 1994.

[17] H. Wu and Magnant C. Gallai-ramsey numbers for monochromatic triangles or 4-cycles. *Graphs Combin.*, 34:1315–1324, 2018.