Spectral properties of non-selfadjoint Sturm-Liouville operator equation on the real axis

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Abstract
In this paper, we analyze the non-selfadjoint Sturm-Liouville operator $L$ defined in the Hilbert space $L_2(\mathbb{R}, H)$ of vector-valued functions which are strongly-measurable and square-integrable in $\mathbb{R}$. $L$ is defined

$$L(y) = -y'' + Q(x)y, \quad x \in \mathbb{R},$$

for every $y \in L_2(\mathbb{R}, H)$ where the potential $Q(x)$ is a non-selfadjoint, completely continuous operator in a separable Hilbert space $H$ for each $x \in \mathbb{R}$. We obtain the Jost solutions of this operator and examine the analytic and asymptotic properties. Moreover, we find the point spectrum and the spectral singularities of $L$ and also obtain the sufficient condition which assures the finiteness of the eigenvalues and spectral singularities of $L$.

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1. Introduction
Non-selfadjoint operators are seen in physical systems which do not involve the conservation of energy law. Some selfadjoint problems also give us non-selfadjoint operators after separation of variables. The theory of non-selfadjoint operators has initially begun to analyze ordinary differential equations. M.V. Keldysh played a significant role to develop a general theory for non-selfadjoint operators by inventing a new method for establishing the resolvent of an arbitrary completely continuous, non-selfadjoint operator of finite order [16, 17].

Spectral analysis of non-selfadjoint differential operators has been studied by M.A. Naimark [24, 25]. In particular, he analyzed the non-selfadjoint Sturm-Liouville operator defined by

$$l(y) = -y'' + p(x)y, \quad 0 < x < \infty,$$

$$y'(0) - hy(0) = 0,$$  \hspace{1cm} (1.2)

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Consider the operator \( l \) defined by Equations (1.1) and (1.2) in detail \([22–25, 28]\). The results of Naimark \([24, 25]\) have been generalized in \([7, 8]\) to the operator \( l_0 \) generated in \( L_2(\mathbb{R}) \) which is defined by

\[
l_0(y) = -y'' + q(x)y, \quad x \in \mathbb{R},
\]

where the potential \( q \) is a complex-valued function. The authors generalized the results of \([24]\) and applied to the non-selfadjoint Schrödinger operator in the three-dimensional space \([13]\).

Non-selfadjoint Hamiltonians and complex extensions of Quantum Mechanics have been studied by many mathematicians, recently. Moreover, spectral properties of the selfadjoint matrix differential and difference equations have been examined \([9, 10, 15]\). For the non-selfadjoint case, discrete spectrum and the spectral singularities of the matrix Sturm-Liouville operator were investigated \([4, 11, 26, 27]\). Further, the authors examined a system of non-selfadjoint Sturm-Liouville equations \([2, 5, 6]\).

B. M. Levitan et al. have studied the point spectrum of the following Sturm-Liouville operator equation in detail \([14, 19–21]\). Let \( H \) be a separable Hilbert space and \( L_2(\mathbb{R}_+, H) \) denote the space of vector-valued functions \( f(x) \) defined on \((0, \infty)\) which are strongly-integrable and also square-integrable on \((0, \infty)\) i.e.,

\[
\int_0^\infty \|f(x)\|^2 \, dx < \infty.
\]

Consider the operator \( l_1 \) defined on \( L_2(\mathbb{R}_+, H) \) by

\[
l_1(Y) = -Y'' + Q(x)Y, \quad 0 < x < \infty,
\]

and the boundary condition \( Y(0) = 0 \) where \( Q(x) \) is a completely continuous, selfadjoint operator defined on \( H \) for every \( x \in (0, \infty) \). Equation (1.4) is called Sturm-Liouville operator equation.

In our previous paper \([3]\), we considered the non-selfadjoint analogue of the above problem and investigated the spectral properties of the non-selfadjoint Sturm-Liouville operator equation on the half line on the contrary to \([14, 19–21]\). We also generalized the results in \([2, 4, 11, 26, 27]\) by considering the coefficients as operators not only finite dimensional matrices. In this study, we extend these results to the whole real axis. Explicitly, we focus on the following non-selfadjoint operator.

Assume \( H \) is a separable Hilbert space and \( H_1 := L_2(\mathbb{R}, H) \) denotes the space of vector-valued functions \( f(x) \) defined on \( \mathbb{R} \) which are strongly-integrable and square-integrable. Note that \( H_1 \) is a Hilbert space with the inner product (see \([29]\)):

\[
(f, g)_1 = \int_{-\infty}^{\infty} (f(x), g(x))_H \, dx.
\]

Let us consider the non-selfadjoint operator \( L \) defined in \( H_1 \):

\[
L(y) = -y'' + Q(x)y, \quad x \in \mathbb{R}, \tag{1.5}
\]

where the potential \( Q(x) \) is a non-selfadjoint, completely continuous operator in \( H \) for each \( x \in \mathbb{R} \). In this paper, we specify the domain of \( L \) and express the Jost solutions. Then, we find the discrete spectrum and the set of spectral singularities of \( L \) by using the properties of the Jost solutions. Finally, we prove that \( L \) has a finite number of eigenvalues and spectral singularities.
The domain $D(L)$ of $L$ is the subspace consisting of all $y \in H_1$ which satisfies the following conditions:

(i) $y$ is twice strongly-differentiable,
(ii) $L(y) \in H_1$.

Let us consider the eigenvalue equation:

$$-y'' + Q(x)y = \lambda^2 y, \quad x \in \mathbb{R}. \tag{1.6}$$

2. The Jost solutions of (1.6)

We shall also focus on the equation

$$-Y'' + Q(x)Y = \lambda^2 Y, \quad x \in \mathbb{R}, \tag{2.1}$$

where $Y(x)$ is an operator-valued function i.e., $Y(x)$ is an operator in $H$ for each $x \in \mathbb{R}$.

Lemma 2.1. Every sequence of solutions of Equation (1.6) can be represented as an operator-valued function which satisfies Equation (2.1). Conversely, one can construct a sequence of vector-valued functions which satisfy Equation (1.6) for a given operator-valued solution of Equation (2.1).

Proof. Since $H$ is a separable Hilbert space, there exists an orthonormal basis $(u_n)_{n \in \mathbb{N}}$. Suppose vector-valued functions $(y_n(x))_{n \in \mathbb{N}}$ satisfy Equation (1.6). We can construct an operator-valued function $Y(x)$ such that $Y(x)u_n = y_n(x)$ for every $n \in \mathbb{N}$. It is clear that $Y(x)$ satisfies Equation (2.1).

Conversely, suppose operator-valued function $Y(x)$ satisfies Equation (2.1). Let $y_n(x) = Y(x)u_n$ for every $n \in \mathbb{N}$. Then, it is clear that $(y_n(x))$ satisfies Equation (1.6) for every $n \in \mathbb{N}$.

As a result of this one to one correspondence, we can focus on the solutions of only one of the Equations (1.6)-(2.1).

We shall use the notations:

$$\sigma^+(x) = \int_x^\infty \|Q(t)\| \, dt, \quad \sigma_1^+(x) = \int_x^\infty \sigma^+(t) \, dt,$$

$$\sigma^-(x) = \int_{-\infty}^x \|Q(t)\| \, dt, \quad \sigma_1^-(x) = \int_{-\infty}^x \sigma^-(t) \, dt.$$

Suppose that the condition

$$\int_{-\infty}^\infty (1 + |t|) \|Q(t)\| \, dt < \infty, \tag{2.2}$$

holds. Then, Equation (2.1) has operator solutions $E^+(x, \lambda)$ and $F^-(x, \lambda)$ satisfying the initial conditions:

$$\lim_{x \to \infty} e^{-i\lambda x} E^+(x, \lambda) = I, \quad \text{Im} \lambda \geq 0, \tag{2.3}$$

and

$$\lim_{x \to -\infty} e^{i\lambda x} F^-(x, \lambda) = I, \quad \text{Im} \lambda \geq 0, \tag{2.4}$$

respectively. Indeed, consider the integral equation

$$E^+(x, \lambda) = e^{i\lambda x} I + \frac{1}{\lambda} \int_x^\infty \sin (\lambda (t - x)) Q(t) E^+(t, \lambda) \, dt, \quad \text{Im} \lambda \geq 0,$$

which is easily seen to be a solution of Equation (2.1) satisfying (2.3). Similarly, if we define

$$F^-(x, \lambda) = E^+(-x, \lambda), \quad \text{Im} \lambda \geq 0,$$
it easily follows that $E^-(x, \lambda)$ satisfies (2.4). Under the condition (2.2), the solution $E^+(x, \lambda)$ can be represented (see [1]):

$$E^+(x, \lambda) = e^{i\lambda x} I + \int_x^\infty e^{i\lambda t} K^+(x, t) dt, \quad \text{Im}\lambda \geq 0.$$ (2.5)

Let us consider the equation;

$$- Z'' + ZQ(x) = \lambda^2 Z, \quad x \in \mathbb{R},$$ (2.6)

where $Z(x)$ is an operator-valued function. Similarly, Equation (2.6) has an operator solution $E^-(x, \lambda)$ which satisfies the initial condition;

$$\lim_{x \to -\infty} e^{i\lambda x} E^-(x, \lambda) = I, \quad \text{Im}\lambda \geq 0,$$

and has the representation

$$E^-(x, \lambda) = e^{-i\lambda x} I + \int_{-\infty}^x e^{-i\lambda t} K^-(x, t) dt, \quad \text{Im}\lambda \geq 0.$$

Further, the operator-valued kernels $K^+(x, t)$ are differentiable with respect to $x$ and $t$ and satisfy

$$\left\| K^+(x, t) \right\| \leq \frac{1}{2} \sigma^+(\frac{x+t}{2}) \exp \left[ \sigma^+_1(x) - \sigma^-_1(\frac{x+t}{2}) \right],$$ (2.7)

$$\left\| K^+_x(x, t) + \frac{1}{4} Q(\frac{x+t}{2}) \right\| \leq \frac{1}{2} \sigma^+_1(x) \sigma^-_1(\frac{x+t}{2}) \exp \sigma^+_1(x),$$ (2.8)

$$\left\| K^-_t(x, t) + \frac{1}{4} Q(\frac{x+t}{2}) \right\| \leq \frac{1}{2} \sigma^+_1(t) \sigma^-_1(\frac{x+t}{2}) \exp \sigma^+_1(t).$$ (2.9)

As a result, the solutions $E^+(x, \lambda)$ and $E^-(x, \lambda)$ are analytic for $\text{Im}\lambda > 0$ and continuous for $\text{Im}\lambda \geq 0$. $E^+(x, \lambda)$ and $E^-(x, \lambda)$ are called the Jost solutions of Equation (1.6). The proofs of above results are very similar to the matrix coefficient case which have been obtained in [1, 4]. In addition, we obtained analogous properties in our previous paper [3]. Hence, we omitted the proofs.

**Lemma 2.2.** Let $Y(x)$ be a solution of Equation (2.1) and $Z(x)$ be a solution of Equation (2.6). Then, the Wronskian $W[Y, Z](x) := Z'(x)Y(x) - Z(x)Y'(x)$ is independent of $x$.

**Proof.** We have

$$-Y'' + Q(x)Y = \lambda^2 Y,$$

$$-Z'' + ZQ(x) = \lambda^2 Z.$$

If we multiply the first equality from the left with $Z$ and the second equality from the right with $Y$ and subtract them, we have

$$Z''(x)Y(x) - Z(x)Y''(x) = 0,$$

which implies $W[Y, Z](x)$ is constant and hence independent of the variable $x$. \qed

Let us define the function

$$D(\lambda) := W\left[ E^-(x, \lambda), E^+(x, \lambda) \right], \quad \text{Im}\lambda \geq 0.$$  

Since the Wronskian of $E^+(x, \lambda)$ and $E^-(x, \lambda)$ is independent of $x$, $D(\lambda)$ is a function of $\lambda$ which is also analytic for $\text{Im}\lambda > 0$ and continuous for $\text{Im}\lambda \geq 0$. The function $D(\lambda)$ is called the Jost function of Equation (1.6).
Theorem 2.3. The function $D(\lambda)$ satisfies
\[ D(\lambda) = 2i\lambda I - 2K^+(0,0) - 2K^-(0,0) + \int_0^\infty e^{i\lambda t} F(t) dt, \] (2.10)
where
\[ F(t) = K^+_x(0,t) - K^-_x(0,-t) - K^-(0,0)K^+_x(0,t) - K^+(0,0)K^-_x(0,-t) \]
\[ + K^-_x(0,t) \ast K^+_x(0,t) - K^-_x(0,-t) \ast K^+(0,t) + K^-_x(0,-t) \]
\[ - K^+_x(0,t), \] (2.11)
and $F \in L_1(\mathbb{R}, H)$ where "*" is the convolution operation.

Proof. Since the Wronskian of $E^+(x,\lambda)$ and $E^-(x,\lambda)$ is independent of $x$, we put $x = 0$ and obtain
\[ D(\lambda) = W \left[ E^-(x,\lambda), E^+(x,\lambda) \right] = E^+_x(\lambda)E^-_x(\lambda) - E^+_x(\lambda)E^-_x(\lambda). \]

By using the integral representations of $E^+(x,\lambda)$ and $E^-(x,\lambda)$ we get (2.10) and (2.11). From (2.7)-(2.9) we have $F \in L_1(\mathbb{R}, H)$. \hfill \Box

Theorem 2.4. The following asymptotic relations hold;
\[ D(\lambda) = 2i\lambda I - 2K^+(0,0) - 2K^-(0,0) + o(1), \quad \text{Im} \lambda \geq 0, \quad |\lambda| \to \infty, \] (2.12)
\[ D(\lambda) = 2i\lambda I + O(1), \quad \text{Im} \lambda \geq 0, \quad |\lambda| \to \infty. \] (2.13)

Proof. Let $\lambda \in \mathbb{R}$. By Riemann-Lebesgue Lemma for Fourier transforms [18] we have
\[ \int_0^\infty e^{i\lambda t} F(t) dt = o(1), \quad \lambda \in \mathbb{R}, \quad |\lambda| \to \infty. \] (2.14)
Now, let $\text{Im} \lambda > 0$. Lebesgue Theorem [18] implies
\[ \int_0^\infty e^{i\lambda t} F(t) dt = o(1), \quad \text{Im} \lambda > 0, \quad |\lambda| \to \infty. \] (2.15)
If we use (2.14), (2.15) we get (2.12). The proof is similar for (2.13). \hfill \Box

3. Point spectrum and spectral singularities of $L$

Now, we introduce the point spectrum and the set of spectral singularities of $L$ according to the definitions given in [22-24]
\[ \sigma_d(L) = \left\{ \lambda^2 : \text{Im} \lambda > 0, \ D(\lambda) \text{ is not invertible} \right\}, \]
\[ \sigma_{ss}(L) = \left\{ \lambda^2 : \lambda \in \mathbb{R} \setminus \{0\}, \ D(\lambda) \text{ is not invertible} \right\}. \]

Now, we try to examine the eigenvalues of $L$ by employing the results in [17]. Let us recall;
\[ D(\lambda) = 2i\lambda I - 2K^+(0,0) - 2K^-(0,0) + \int_0^\infty e^{i\lambda t} F(t) dt, \quad \text{Im} \lambda \geq 0. \] (3.1)
Let
\[ A(\lambda) : = \frac{1}{2i\lambda} \left[ -2K^+(0,0) - 2K^-(0,0) + \int_0^\infty e^{i\lambda t} F(t) dt \right], \]
\[ G(\lambda) : = \frac{1}{2i\lambda} D(\lambda). \]
Then,
\[ G(\lambda) = I + A(\lambda), \quad \text{Im} \lambda \geq 0, \]
and for $\lambda \neq 0$, it follows $D(\lambda)$ is invertible iff $G(\lambda)$ is invertible. Hence
\[ \sigma_d(L) = \left\{ \lambda^2 : \text{Im} \lambda > 0, \ G(\lambda) \text{ is not invertible} \right\}, \]
\[ \sigma_{ss}(L) = \left\{ \lambda^2 : \lambda \in \mathbb{R} \setminus \{0\}, \ G(\lambda) \text{ is not invertible} \right\}. \]
Let us define $M_1 := \{ \lambda : \text{Im} \lambda > 0, \ G(\lambda) \text{ is not invertible} \}$.
It follows $\sigma_d(L) = \{ \lambda^2 : \lambda \in M_1 \}$.

Since $\int_{-\infty}^{\infty} (1 + |t|) \|Q(t)\| \, dt < \infty$ and $Q(x)$ is completely continuous operator for each $x \in \mathbb{R}$, it follows $F(t)$ is completely continuous operator for each $0 < t < \infty$ and as a result $A(\lambda)$ is completely continuous for $\text{Im} \lambda > 0$. Also, since $D(\lambda)$ is analytic for $\text{Im} \lambda > 0$, $A(\lambda)$ is also analytic in the same domain. Now, we can use the results in [17].

**Definition 3.1.** If

$$(I - R)(I + A) = I,$$

holds, then the operator $R$ is called the resolvent of the operator $A$, [17].

Let us denote the resolvent of $A(\lambda)$ by $R(\lambda)$. It follows

$$I - R(\lambda) = (I + A(\lambda))^{-1} = (G(\lambda))^{-1}.$$

If $I - R(\lambda)$ exists at least for one $\lambda$ which means $G(\lambda)$ is invertible, this implies $I - R(\lambda)$ exists on the domain $\mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} \lambda > 0 \}$ except for a set of isolated points, and also $I - R(\lambda)$ is a meromorphic operator function in the same domain [17]. It is obvious that $M_1 \neq \mathbb{C}_+$. This implies there exists at least one $\lambda$ such that $I - R(\lambda)$ is defined. [17] implies that $I - R(\lambda)$ should exist on the domain $\mathbb{C}_+$ except for a set of isolated points. These isolated points are obviously the eigenvalues of $L$. Moreover, $I - R(\lambda)$ is a meromorphic operator function on $\mathbb{C}_+$. Therefore, we can express $I - R(\lambda)$ as a ratio of two analytical functions in the domain $\mathbb{C}_+$ as:

$$(G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)},$$

where $S(\lambda)$ is an operator function and $d(\lambda)$ is a scalar function on $\mathbb{C}_+$. Moreover, the above isolated points are poles of the function $I - R(\lambda)$ and they are the zeros of the function $d(\lambda)$. As a result, it follows

$$M_1 = \{ \lambda : \text{Im} \lambda > 0, \ d(\lambda) = 0 \} .$$

**Theorem 3.2.** If the condition (2.2) holds, then $\sigma_d(L)$ is a bounded and countable set. Further, the limit points of $\sigma_d(L)$ should lie in a bounded interval of the real axis.

**Proof.** The relation (2.12) implies

$$G(\lambda) = I + o(1), \quad \text{Im} \lambda \geq 0, \quad |\lambda| \to \infty,$$

which means for sufficiently large $\lambda \in \mathbb{C}_+$, $G(\lambda) \to I$ and thus $G(\lambda)$ is invertible. Therefore, $M_1$ is bounded. Since the function $d(\lambda)$ is analytic, its zeros are isolated. This implies $M_1$ is countable. Further, the limit points of the zeros of $d(\lambda)$ should lie in an interval of the real line [12]. The proof is complete since

$$\sigma_d(L) = \{ \lambda^2 : \lambda \in M_1 \} .$$

Now, let us assume that the condition

$$\int_{-\infty}^{\infty} e^{\epsilon|t|} \|Q(t)\| \, dt < \infty, \quad \epsilon > 0,$$

holds.

**Theorem 3.3.** $L$ has a finite number of eigenvalues.

**Proof.** From the equalities (2.7)-(2.9) and (3.4) we have

$$\|K^+(x, t)\|, \quad \|K_x^+(x, t)\|, \quad \|K_t^+(x, t)\| \leq c \exp \left( -\epsilon \frac{x + t}{2} \right),$$
and hence
\[ \|F(t)\| \leq c \exp \left(-\epsilon \left(\frac{t}{2}\right)\right), \ \forall t \in [0, \infty), \]
where \( c \) is a constant positive. Further,
\[ \|F(t)\| e^{\lambda t} \leq c \exp^{-t \left(\frac{\lambda}{2} + \text{Im}\lambda\right)}, \ \forall t \in [0, \infty), \]
and thus
\[ \left\| \int_0^\infty e^{\lambda t} F(t) dt \right\| \leq \int_0^\infty \left| e^{\lambda t}\right| \|F(t)\| dt \]
\[ \leq \int_0^\infty c \exp^{-t \left(\frac{\lambda}{2} + \text{Im}\lambda\right)} dt, \]
\[ \int_0^\infty c \exp^{-t \left(\frac{\lambda}{2} + \text{Im}\lambda\right)} dt < \infty \iff \text{Im}\lambda + \frac{\epsilon}{2} > 0. \]
The Uniform Convergence Test implies that the integral \( \int_0^\infty e^{\lambda t} F(t) dt \) is uniformly convergent in the domain \( \text{Im}\lambda > -\frac{\epsilon}{2} \). This implies \( D(\lambda) \) and also \( G(\lambda) \) have analytic continuations to the domain \( \text{Im}\lambda > -\frac{\epsilon}{2} \). Since the analytic continuation is unique, it follows
\[ (G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \ \text{Im}\lambda > -\frac{\epsilon}{2}. \]
Let us recall \( M_1 = \{ \lambda : \text{Im}\lambda > 0, \ d(\lambda) = 0 \} \), \( \sigma_d(L) = \{ \lambda^2 : \lambda \in M_1 \} \) and \( M_1 \) is bounded. Suppose that \( M_1 \) is not finite. Let us recall Bolzano-Weierstrass Theorem which states that each bounded sequence in \( \mathbb{R} \) has a convergent subsequence. Bolzano-Weierstrass Theorem implies that \( M_1 \) must have one limit point. Also, Theorem 3.2 states that the limit points of \( M_1 \) can only lie on the real axis. However, since \( d(\lambda) \) is analytic in the domain \( \text{Im}\lambda > -\frac{\epsilon}{2} \), the limit points of \( M_1 \) should be on the boundary of this domain [12]. This contradicts with the assumption \( \epsilon > 0 \). Thus, \( M_1 \) and \( \sigma_d(L) \) are finite. \( \square \)

Let \( M_2 := \{ \lambda : \lambda \in \mathbb{R}, \ G(\lambda) \text{ is not invertible} \} \). It is obvious that
\[ \sigma_{ss}(L) = \left\{ \lambda^2 : \lambda \in M_2 \right\} \sim \{0\}. \]
From the representation (3.2), we have
\[ \frac{G(\lambda)}{d(\lambda)} S(\lambda) = I, \ \lambda \in \mathbb{C}_+, \]
and this implies \( S(\lambda) \) is invertible iff \( G(\lambda) \) is invertible or equivalently \( d(\lambda) \neq 0 \) for \( \lambda \in \mathbb{C}_+ \). If \( d(\lambda) \neq 0 \) it follows
\[ (S(\lambda))^{-1} = \frac{G(\lambda)}{d(\lambda)}, \ \lambda \in \mathbb{C}_+, \]
and also
\[ G(\lambda) = d(\lambda) (S(\lambda))^{-1}, \ \lambda \in \mathbb{C}_+. \] (3.5)
Since \( G(\lambda) \) is continuous on the real line, Equation (3.5) suggests that the functions \( S(\lambda) \) and \( d(\lambda) \) are continuous on the real line. Hence, we can extend the representation (3.2) continuously to the real line and obtain
\[ (G(\lambda))^{-1} = \frac{S(\lambda)}{d(\lambda)}, \ \lambda \in \overline{\mathbb{C}_+}. \] (3.6)
(3.6) implies that \( G(\lambda) \) is invertible iff \( d(\lambda) \neq 0 \) for \( \lambda \in \mathbb{R} \). Thus, we have
\[ M_2 = \{ \lambda \in \mathbb{R} : d(\lambda) = 0 \}. \]
Theorem 3.4. $M_2$ is compact and has zero Lebesgue measure under the condition (2.2).

Proof. Theorem 3.2 implies $M_2$ is bounded. We only have to show that $M_2$ is closed. Let $\{\lambda_n\} \subset M_2$ such that $\lambda_n \to \lambda_0$. $\{\lambda_n\} \subset M_2$ implies $\lambda_n \in \mathbb{R}$ and $G(\lambda_n)^{-1}$ doesn’t exist for $n \in \mathbb{N}$. Further, $\lambda_n \to \lambda_0$ implies $\lambda_0 \in \mathbb{R}$. We have $G(\lambda)$ is a continuous operator function on the real line. Now, $\lambda_n \to \lambda_0$ suggests that $G(\lambda_n) \to G(\lambda_0)$ where the latter convergence is strong.

Let $GL(H) := \{ A : A \text{ is invertible, bounded, linear operator on } H \}$. It is well known that $GL(H)$ is an open subset of the space $B(H)$ of bounded, linear operators on $H$. It follows $B(H) \setminus GL(H)$ is a closed set. This implies $G(\lambda_0) \in B(H) \setminus GL(H)$ and $\lambda_0 \in M_2$. Finally, Privalov’s Theorem states that $M_2$ has zero Lebesgue measure [12].

Corollary 3.5. $\sigma_{ss}(L)$ is bounded and has zero Lebesgue measure, under the condition (2.2).

Theorem 3.6. $L$ has a finite number of spectral singularities, under the condition (3.4).

Proof. It can be shown similarly (see the proof of Theorem 3.3) that $G(\lambda)$ has an analytic continuation to the domain $\text{Im} \lambda > -\frac{\epsilon}{2}$ for arbitrary $\epsilon > 0$. Since this analytic continuation is unique, it follows

$$(G(\lambda))^{-1} = I - R(\lambda) = \frac{S(\lambda)}{d(\lambda)}, \quad \text{Im} \lambda > -\frac{\epsilon}{2}.$$ 

Suppose that $M_2$ is not finite. Since $M_2$ is bounded (see Theorem 3.2), Bolzano-Weierstrass Theorem implies that $M_2$ has a limit point. This limit point as a zero of the analytic function $d(\lambda)$ should lie on the boundary of the domain $\text{Im} \lambda > -\frac{\epsilon}{2}$ [12]. It contradicts with $\epsilon > 0$.

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