Analysis of a class of globally divergence-free HDG methods for stationary Navier-Stokes equations

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Abstract

This paper analyzes a class of globally divergence-free (and therefore pressure-robust) hybridizable discontinuous Galerkin (HDG) finite element methods for stationary Navier-Stokes equations. The methods use the $P_k/P_{k-1}$ ($k \geq 1$) discontinuous finite element combination for the velocity and pressure approximations in the interior of elements, and piecewise $P_k/P_k$ for the trace approximations of the velocity and pressure on the inter-element boundaries. It is shown that the uniqueness condition for the discrete solution is guaranteed by that for the continuous solution together with a sufficiently small mesh size. Based on the derived discrete HDG Sobolev embedding properties, optimal error estimates are obtained. Numerical experiments are performed to verify the theoretical analysis.

Keywords: Navier-Stokes equations; HDG methods; Divergence-free; Uniqueness condition; Error estimates.
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1. Introduction

Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a Lipschitz polygonal/polyhedral domain. We consider the following stationary Navier-Stokes equations: seek the velocity \( u \) and the pressure \( p \) such that

\[
\begin{aligned}
-\nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\n\nabla \cdot u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Here \( \nu = \text{Re}^{-1} > 0 \) is the fluid viscosity coefficient with \( \text{Re} \) denoting the Reynolds number, and \( f \in [L^2(\Omega)]^d \) is the given body force.

A Galerkin mixed method for the problem (1) (or even Stokes equations) requires the pair of finite element spaces for the velocity and pressure to satisfy an inf-sup stability condition (see, e.g. \([4, 66, 64, 13, 6, 48]\) and books \([15, 44, 34, 46, 33]\)). To circumvent the inf-sup difficulty, many methods of stabilization have been developed for the Stokes or Navier-Stokes equations (cf. \([38, 7, 9, 19, 10, 21, 17, 16, 32, 8, 56, 41, 22, 68, 48]\) and the references therein).

It is very important to preserve the mass conservation in the numerical solution of the incompressible fluid flows, since the finite element methods with poor mass conservation may lead to instabilities for more complex problems (e.g. unsteady Naiver-Stokes equations) (cf. \([5, 11, 53, 42, 60, 52]\)). The exactly divergence free discretizations automatically lead to pressure-robustness. As pointed out in \([54, 47]\), classical mixed methods for the Stokes equations, constructed by satisfying the discrete inf-sup condition, usually lead to the lack of pressure-robustness, i.e. the velocity error of the methods depends on the best approximation error of the pressure scaled with the inverse of the viscosity. Therefore, it is desirable to design globally divergence-free methods for the Stokes equations to ensure mass conservation and pressure-robustness.

In recent years the DG framework \([29, 3]\) has become increasingly popular due to its attractive features like local conservation of physical quantities and flexibility in meshing. In \([28, 55, 27]\) DG methods were applied to solve the Navier-Stokes model (1). A local discontinuous Galerkin (LDG) method for (1) was proposed in \([25]\), where a globally divergence-free approximation for the velocity is obtained by post-processing. The hybridizable discontinuous Galerkin (HDG) framework, presented in \([24]\) for diffusion problems,
provides a unifying strategy for hybridization of finite element methods. The resultant HDG methods preserve the advantages of standard DG methods and lead to discrete systems of significantly reduced sizes. We refer to \cite{23, 58, 30, 59, 62, 61, 57, 51, 50, 31, 39, 63} for some HDG methods for the Stokes equations, the Navier-Stokes equations and Stokes-like equations. In particular, in \cite{63} a technique that introduces the pressure trace on the inter-element boundaries as a Lagrangian multiplier so as to derive a divergence-free velocity approximation, as same as our earlier work in \cite{20}, was used to construct a globally divergence-free HDG method for the unsteady Navier-Stokes equations.

It is well-known the existence and uniqueness of the weak solution to \eqref{eq:1} is under the smallness condition

\begin{equation}
\frac{N}{\nu^2} \| f \|_* < 1, 
\end{equation}

where the constant $N$ and the norm $\| \cdot \|_*$ are defined in Section 2. As for numerical schemes, the corresponding discrete smallness condition is

\begin{equation}
\frac{N_h}{\nu^2} \| f \|_{*, h} < 1,
\end{equation}

where $N_h$ and $\| \cdot \|_{*, h}$ are defined similarly to the continuous case but with specific finite element spaces (cf. Section 2). For standard finite element approaches, it has been shown in \cite{43} that

$$
\lim_{h \to 0} \| f \|_{*, h} = \| f \|_*, \quad \lim_{h \to 0} N_h = N.
$$

Thus, the continuous condition \eqref{eq:2}, together with a sufficiently small mesh size $h$, guarantees the discrete smallness condition \eqref{eq:3} for the conforming methods. In \cite{26, 67, 18} a fixed point argument was used to establish uniqueness for discontinuous Galerkin (DG) or HDG methods under the condition

\begin{equation}
(C_0/\nu^2) \| f \|_0 < 1,
\end{equation}

where the constant $C_0$ depends on the underlying numerical method. Note that \eqref{eq:4} is equivalent to

$$
\frac{N}{\nu^2} \| f \|_* < \frac{N \| f \|_*}{C_0 \| f \|_0}.
$$

3
when $f \neq 0$, which means that the condition (4) can be guaranteed by (2) if

$$\frac{\mathcal{N}\|f\|_*}{C_0\|f\|_0} \geq 1.$$  \hspace{1cm} (5)

However, the condition (5) does not hold in general.

So far, to our best knowledge, there is no proof of (3) under the condition (2) for nonstandard approaches. We emphasize that it is true, like in [26, 67, 18], that if $\nu^{-2}\|f\|_0$ is small enough, then (4) holds true and therefore the underlying discretization has a unique solution, but there is still a chance that (2) holds true but (4) does not! So it is possible that the continuous problem has a unique solution, while the corresponding discrete scheme does not! This theoretical gap will be filled by our analysis.

In this paper, we shall analyze a class of HDG methods for the Navier-Stokes problem (1). The methods use the $P_k/P_{k-1}$ ($k \geq 1$) discontinuous finite element combination for the velocity and pressure approximations in the interior of elements, and piecewise $P_k/P_k$ for the trace approximations of the velocity and pressure on the inter-element boundaries. We note that the finite element combinations used in our methods are inherited from our previous paper [20] for the Stokes equations, and later appears in [63] for the unsteady Navier-Stokes equations. Our analysis is of the following features.

- The discrete smallness condition (3) for the proposed HDG discretization is shown to be guaranteed by the continuous smallness condition (2) together with a sufficiently small mesh size. To the authors’ knowledge, this is the first proof of such a conclusion for HDG methods, and it is expectable to extend the proof to other nonstandard methods. The key to the proof is that we define two switch operators, one from the continuous spaces to the HDG spaces and the other one from the HDG spaces to the continuous spaces, which satisfy certain strict inequalities (cf. (56) and (65)).

- The methods are shown to yield globally divergence-free velocity approximations, and therefore are pressure-robust.

- Our treatment of the nonlinear term is different from that in [63]. In our discretization, we design the nonlinear term as an antisymmetry form which makes the numerical scheme to be of stability independent of the choice of the stabilization parameter. In [63] the nonlinear
term is discretized directly, and the stabilization parameter needs to be sufficiently large to ensure the stability.

- Optimal error estimates are established based on the derived discrete HDG Sobolev embedding properties. We note that [63] does not provide convergence analysis for the methods therein.

The rest of this paper is organized as follows. Section 2 introduces notations and the HDG formulations. Section 3 gives some interpolation properties. Section 4 discusses stability and continuous conditions. Section 5 is devoted to the a priori error analysis. Section 6 derives $L^2$ error estimates for the velocity. Section 7 describes the local elimination property of the HDG methods. Finally, Section 8 provides numerical experiments.

2. HDG formulations

2.1. Notation

For any bounded domain $\Omega \subset \mathbb{R}^s$ ($s = d, d-1$), let $H^m(\Omega)$ and $H^m_0(\Omega)$ denote the usual $m^{th}$-order Sobolev spaces on $\Omega$, and $\| \cdot \|_{m,\Omega}$, $| \cdot |_{m,\Omega}$ denote the norm and semi-norm on these spaces, respectively. Let $(\cdot, \cdot)_\Lambda$ be the inner product of $H^m(\Omega)$, with $(\cdot, \cdot)_\Omega := (\cdot, \cdot)_{0,\Omega}$. When $\Omega = \Omega$, we set $\| \cdot \|_m := \| \cdot \|_{m,\Omega}$, $| \cdot |_m := | \cdot |_{m,\Omega}$, $(\cdot, \cdot) := (\cdot, \cdot)_\Omega$. In particular, when $\Lambda \in \mathbb{R}^{d-1}$, we use $(\cdot, \cdot)_\Lambda$ to replace $(\cdot, \cdot)_\Omega$. We note that bold face fonts will be used for vector analogues of the Sobolev spaces along with vector-valued functions. For an integer $k \geq 0$, $P_k(\Lambda)$ denotes the set of all polynomials defined on $\Lambda$ with degree not greater than $k$. We also need the following spaces:

$$L^2_0(\Omega) := \{ q \in L^2(\Omega) : (q, 1) = 0 \},$$

$$H(\text{div}; \Lambda) := \{ v \in [L^2(\Lambda)]^s : \nabla \cdot v \in L^2(\Lambda) \}.$$

Let $T_h = \bigcup \{ T \}$ be a shape-regular simplical decomposition of the domain $\Omega$ with mesh size $h = \max_{T \in T_h} h_T$, where $h_T$ is the diameter of $T$. Let $E_h = \bigcup \{ E \}$ be the union of all edges (faces) of $T \in T_h$. For any simplex $T \in T_h$ and $E \in E_h$, we denote by $n_T$ and $n_E$ the outward unit normal vectors along $\partial T$ and $E$, respectively. Let $h_E$ denote the diameter of $E$.

We use $\nabla_h$ and $\nabla_h \cdot$ to denote the piecewise-defined gradient and divergence with respect to the decomposition $T_h$. We also introduce the following mesh-dependent inner product and mesh-dependent norm:

$$\langle u, v \rangle_{\partial T_h} := \sum_{T \in T_h} \langle u, v \rangle_{\partial T}, \quad \| u \|^2_{\partial T_h} := \sum_{T \in T_h} \| u \|^2_{\partial T}.$$
For simplicity, throughout this paper we use $a \lesssim b$ ($a \gtrsim b$) to denote $a \leq C b$ ($a \geq C b$), where $C$ is a positive constant independent of mesh sizes $h$, $h_T$, $h_E$ and the fluid viscosity coefficient $\nu$. In addition, $a \sim b$ simplifies $a \lesssim b \lesssim a$.

2.2. Basic results for Naiver-Stokes equations

Introduce the spaces

$$
V := [H^1_0(\Omega)]^d, \quad Q := L^2_0(\Omega), \quad W := \{v \in V : \nabla \cdot v = 0\},
$$

and define the following trilinear form: for any $(w, u, v) \in [H^1(\Omega)]^d \times [H^1(\Omega)]^d \times [H^1(\Omega)]^d$,

$$
b(w; u, v) := \frac{1}{2} (\nabla \cdot (u \otimes w), v) - \frac{1}{2} (\nabla \cdot (v \otimes w), u).
$$

By integration by parts, it is easy to see

$$
b(w; u, v) = (w \cdot \nabla u, v) \quad \forall w \in W, u, v \in V.
$$

We set

$$
\mathcal{N} := \sup_{0 \neq u, v, w \in W} \frac{(w \cdot \nabla u, v)}{|w|_1 |u|_1 |v|_1}, \quad \|f\|_* := \sup_{0 \neq v \in W} \frac{(f, v)}{|v|_1}. \tag{6} \tag{7}
$$

**Theorem 2.1** (cf. [44]). Let $\Omega$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$. Given $f \in [H^{-1}(\Omega)]^d$, the problem [1] admits at least one weak solution $(u, p) \in W \times Q$.

**Theorem 2.2** (cf. [44]). Under the hypotheses of Theorem 2.1, let $(u, p) \in W \times Q$ be a weak solution of [1], then $u$ satisfies

$$
\|\nabla u\|_0 \leq \nu^{-1} \|f\|_* . \tag{8}
$$

In addition, if

$$
(\mathcal{N}/\nu^2) \|f\|_* < 1 \tag{9}
$$

holds, then the problem [1] admits a unique solution $(u, p) \in W \times Q$.

In the rest of this paper, the solution $(u, p)$ is supposed to be unique and, more precisely, we assume that

$$
(\mathcal{N}/\nu^2) \|f\|_* \leq 1 - \delta \text{ for some } \delta > 0. \tag{10}
$$
2.3. HDG scheme

For any integer $k \geq 1$, and integer $m \in \{k - 1, k\}$, we introduce the following finite element spaces:

\[ \mathbb{K}_h := \{ \tau_h : \tau_h|_T \in [P_m(T)]^{d \times d}, \forall T \in \mathcal{T}_h \}, \]
\[ V_h := \{ \mathbf{v}_h : \mathbf{v}_h|_T \in [P_k(T)]^d, \forall T \in \mathcal{T}_h \}, \]
\[ \tilde{V}_h := \{ \mathbf{v}_h : \mathbf{v}_h|_E \in [P_k(E)]^d, \forall E \in \mathcal{E}_h, \text{ and } \mathbf{v}_h|_{\partial \Omega} = 0 \}, \]
\[ Q_h := \{ q_h \in L^2_0(\Omega) : q_h|_T \in P_{k-1}(T), \forall T \in \mathcal{T}_h \}, \]
\[ \tilde{Q}_h := \{ \tilde{q}_h \in P_k(E), \forall E \in \mathcal{E}_h \}. \]

Introducing $\mathbb{L} = \nu \nabla \mathbf{u}$ in (11), we can rewrite it as

\[
\begin{cases}
\nu^{-1} \mathbb{L} - \nabla \mathbf{u} = 0 & \text{in } \Omega, \\
-\nabla \cdot \mathbb{L} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{u} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then our HDG finite element scheme for (11) is given as follows: find $(\mathbb{L}_h, \mathbf{u}_h, \tilde{\mathbf{u}}_h, p_h, \tilde{p}_h) \in \mathbb{K}_h \times V_h \times \tilde{V}_h \times Q_h \times \tilde{Q}_h$ such that, for all $(\mathbb{G}_h, \mathbf{v}_h, \tilde{\mathbf{v}}_h, q_h, \tilde{q}_h) \in \mathbb{K}_h \times V_h \times \tilde{V}_h \times Q_h \times \tilde{Q}_h$,

\[
\nu^{-1} (\mathbb{L}_h, \mathbb{G}_h) + (\mathbf{u}_h, \nabla \cdot \mathbb{G}_h) - (\tilde{\mathbf{u}}_h, \mathbb{G}_h \mathbf{n})_{\partial \mathcal{T}_h} = 0, \tag{12a}
\]
\[
(\mathbf{v}_h, \nabla \cdot \mathbb{L}_h) - (\tilde{\mathbf{v}}_h, \mathbb{L}_h \mathbf{n})_{\partial \mathcal{T}_h} + (\nabla \cdot \mathbf{v}_h, p_h) + (\mathbf{v}_h \cdot \mathbf{n}, \tilde{p}_h)_{\partial \mathcal{T}_h} - \frac{1}{2} (\mathbf{v}_h \otimes \mathbf{u}_h, \nabla \mathbf{u}_h) + \frac{1}{2} (\tilde{\mathbf{v}}_h \cdot \tilde{\mathbf{u}}_h, \mathbf{u}_h) \]
\[
+ \frac{1}{2} (\mathbf{u}_h \otimes \mathbf{u}_h, \nabla \mathbf{u}_h) - \frac{1}{2} (\tilde{\mathbf{u}}_h \cdot \tilde{\mathbf{u}}_h, \mathbf{u}_h) \]
\[
- \nu (\tau (\mathbf{u}_h - \tilde{\mathbf{u}}_h), \mathbf{v}_h - \tilde{\mathbf{v}}_h)_{\partial \mathcal{T}_h} = - (f, \mathbf{v}_h), \tag{12b}
\]
\[
(\nabla \cdot \mathbf{u}_h, q_h) - (\mathbf{u}_h \cdot \mathbf{n}, \tilde{q}_h)_{\partial \mathcal{T}_h} = 0, \tag{12c}
\]

where $\tau|_E = h^{-1}_E$ for all $E \in \mathcal{E}_h$. To simplify notation, we set

\[ \mathcal{W}_h := (w_h, \tilde{w}_h), \quad \mathcal{U}_h := (u_h, \tilde{u}_h), \quad \mathcal{V}_h = (v_h, \tilde{v}_h), \]
\[ \mathcal{P}_h := (p_h, \tilde{p}_h), \quad \mathcal{Q}_h = (q_h, \tilde{q}_h), \]

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and define

\[ a_h(\mathbb{L}_h, \mathbb{G}_h) := \nu^{-1}(\mathbb{L}_h, \mathbb{G}_h), \]
\[ c_h(\mathbb{U}_h, \mathbb{G}_h) := (u_h, \nabla_h \cdot \mathbb{G}_h) - \langle \tilde{u}_h, \mathbb{G}_h n \rangle_{\partial \mathcal{T}_h}, \]
\[ d_h(\mathbb{V}_h, \mathbb{P}_h) := (\nabla_h \cdot v_h, p_h) + \langle v_h \cdot n, \tilde{p}_h \rangle_{\partial \mathcal{T}_h}, \]
\[ s_h(\mathbb{U}_h, \mathbb{V}_h) := \nu\langle (u_h - \tilde{u}_h), v_h - \tilde{v}_h \rangle_{\partial \mathcal{T}_h}, \]
\[ b_h(\mathcal{W}_h; \mathbb{U}_h, \mathbb{V}_h) := \frac{1}{2}(v_h \otimes w_h, \nabla_h u_h) - \frac{1}{2}(\nabla_v \cdot \tilde{w}_h n, u_h) \]
\[ - \frac{1}{2}(u_h \otimes w_h, \nabla_v v_h) + \frac{1}{2}(\tilde{u}_h \cdot \tilde{w}_h n, v_h). \]

Then (12a)-(12c) can be rewritten as a compact form: find \((\mathbb{L}_h, \mathbb{U}_h, \mathbb{P}_h) \in \mathbb{K}_h \times [\mathbb{V}_h \times \tilde{\mathbb{V}}_h] \times [\mathbb{Q}_h \times Q_h]\) such that, for all \((\mathbb{G}_h, \mathbb{V}_h, \mathbb{Q}_h) \in \mathbb{K}_h \times [\mathbb{V}_h \times \tilde{\mathbb{V}}_h] \times [\mathbb{Q}_h \times Q_h]\),

\[ a_h(\mathbb{L}_h, \mathbb{G}_h) + c_h(\mathbb{U}_h, \mathbb{G}_h) = 0, \quad \text{(13a)} \]
\[ c_h(\mathbb{V}_h, \mathbb{L}_h) + d_h(\mathbb{V}_h, \mathbb{P}_h) - s_h(\mathbb{U}_h, \mathbb{V}_h) - b_h(\mathbb{U}_h; \mathbb{U}_h, \mathbb{V}_h) = - (f, \mathbb{V}_h), \quad \text{(13b)} \]
\[ d_h(\mathbb{U}_h, \mathbb{Q}_h) = 0. \quad \text{(13c)} \]

Introduce an operator \(K_h : \mathbb{V}_h \times \tilde{\mathbb{V}}_h \rightarrow \mathbb{K}_h\) defined by

\[ (K_h \mathbb{V}_h, \mathbb{G}_h) = -c_h(\mathbb{V}_h, \mathbb{G}_h) \quad \forall \mathbb{V}_h \in \mathbb{V}_h \times \tilde{\mathbb{V}}_h, \mathbb{G}_h \in \mathbb{K}_h. \quad \text{(14)} \]

It is easy to see that \(K_h\) is well defined. From (13a) we immediately have

\[ \mathbb{L}_h = \nu K_h \mathbb{U}_h. \]

Hence, by eliminating \(\mathbb{L}_h\) in (13a) and (13b), we can rewrite (13a)-(13c) as the following system:

Find \((\mathbb{L}_h, \mathbb{U}_h, \mathbb{P}_h) \in \mathbb{K}_h \times [\mathbb{V}_h \times \tilde{\mathbb{V}}_h] \times [\mathbb{Q}_h \times \tilde{Q}_h]\) such that

\[ \mathbb{L}_h - \nu K_h \mathbb{U}_h = 0, \quad \text{(15a)} \]
\[ \nu(K_h \mathbb{U}_h, K_h \mathbb{V}_h) - d_h(\mathbb{V}_h, \mathbb{P}_h) + s_h(\mathbb{U}_h, \mathbb{V}_h) + b_h(\mathbb{U}_h; \mathbb{U}_h, \mathbb{V}_h) = (f, \mathbb{V}_h), \quad \text{(15b)} \]
\[ d_h(\mathbb{U}_h, \mathbb{Q}_h) = 0, \quad \text{(15c)} \]

holds for all \((\mathbb{V}_h, \mathbb{Q}_h) \in [\mathbb{V}_h \times \tilde{\mathbb{V}}_h] \times [\mathbb{Q}_h \times \tilde{Q}_h]\)
3. Stability results

We devote this section to the analysis of stability of scheme (13a). To this end, we introduce the following two semi-norms:

\[ \| V_h \|_V := \| K_h(v_h, \hat{v}_h) \|_0^2 + \| \tau^{\frac{1}{2}}(v_h - \hat{v}_h) \|_{\partial T_h}^2, \]

\[ \| Q_h \|_Q := \| q_h \|_0^2 + \| \tau^{-\frac{1}{2}}(q_h - \hat{q}_h) \|_{\partial T_h}^2. \]

Here we recall that \( K_h \) is given by (14), and \( \tau |_E = h_E^{-1} \), for all \( E \in \mathcal{E}_h \).

It is easy to see that \( \| \cdot \|_V \) and \( \| \cdot \|_Q \) are norms on \( V_h \times \hat{V}_h \) and \( Q_h \times \hat{Q}_h \), respectively. In fact, if \( \| V_h \|_V = 0 \), then

\[ \nabla_h v_h = 0, \quad v_h - \hat{v}_h = 0, \]

which mean that \( v_h \) is piecewise constant with respect to \( T_h \), and \( v_h = \hat{v}_h \) on \( \mathcal{E}_h \). Since \( \hat{v}_h = 0 \) on \( \partial \Omega \), then \( v_h = \hat{v}_h = 0 \). Thus, \( V_h = 0 \). Similarly, we can show \( \| Q_h \|_Q = 0 \) leads to \( Q_h = 0 \).

3.1. Basic results

**Lemma 3.1.** For any \( V_h = (v_h, \hat{v}_h) \in V_h \times \hat{V}_h \), it holds

\[ \| V_h \|_V \sim \| \nabla_h v_h \|_0 + \| \tau^{\frac{1}{2}}(v_h - \hat{v}_h) \|_{\partial T_h}. \]  \( (16) \)

**Proof.** For \( V_h \in V_h \times \hat{V}_h \), by the definition of \( K_h \), it holds

\[ (K_h V_h, G_h) = (\nabla_h v_h, G_h) + \langle \hat{v}_h - v_h, G_h n \rangle_{T_h} \]

for all \( G_h \in K_h \). By taking \( G_h = K_h V_h \) and \( G_h = \nabla_h v_h \) in this equality, respectively, the estimate \( (16) \) follows from Holder’s inequality and the inverse inequality. \( \square \)

The following embedding relationships are standard (cf. [1, 65]).

Before proving this theorem, we begin with a well-known discrete Sobolev embedding inequality in [36, Theorem 2.1] and a continuous Sobolev embedding inequality.

**Lemma 3.2** (cf. [36, Theorem 2.1]). There is a constant \( C > 0 \), such that

\[ \| v_h \|_{0, \mu} \leq C \left[ \sum_{K \in T_h} \| \nabla v_h \|_{0,T}^2 + \sum_{F \in \mathcal{E}_h} h_E^{-1} \| [v_h] \|_{0,E}^2 \right]^{\frac{1}{2}}, \]
for $\mu$ satisfying

\[
\begin{cases}
1 \leq \mu < \infty, & \text{if } d = 2, \\
1 \leq \mu \leq 6, & \text{if } d = 3,
\end{cases}
\]

and for all $v_h \in V_h$.

Based on this lemma, we can obtain a discrete HDG Sobolev embedding relation, which will be used to derive the continuity results in Lemma 3.4.

**Lemma 3.3 (HDG Sobolev embedding).** It holds

\[
\|v_h\|_{0,\mu} \lesssim \|v_h\|_V \quad \forall \mathcal{V}_h = (v_h, \hat{v}_h) \in V_h \times \hat{V}_h
\]

for $\mu$ satisfying

\[
\begin{cases}
1 \leq \mu < \infty, & \text{if } d = 2, \\
1 \leq \mu \leq 6, & \text{if } d = 3.
\end{cases}
\]

**3.2. Stability conditions**

For the trilinear form $b_h(\cdot; \cdot; \cdot)$, we have the following boundedness results.

**Lemma 3.4.** For all $U_h, V_h, W_h \in V_h \times \hat{V}_h$ with $U_h = (u_h, \hat{u}_h), V_h = (v_h, \hat{v}_h), W_h = (w_h, \hat{w}_h)$, it holds

\[
|b_h(W_h; U_h, V_h)| \lesssim \|W_h\|_V \|U_h\|_V \|V_h\|_V, \quad (17)
\]

\[
|b_h(W_h; U_h, V_h)| \lesssim (\|w_h\|_{0,3} + h^{1-\frac{d}{2}} \|W_h\|_V) \|U_h\|_V \|V_h\|_V. \quad (18)
\]

Moreover, if $w_h \in H(\text{div}, \Omega) \cap V_h$ with $\nabla \cdot w_h = 0$, then

\[
|b_h(W_h; U_h, V_h)| \lesssim (\|u_h\|_{0,3} + h^{1-\frac{d}{2}} \|U_h\|_V) \|W_h\|_V \|V_h\|_V, \quad (19)
\]

\[
|b_h(W_h; U_h, V_h)| \lesssim (\|v_h\|_{0,3} + h^{1-\frac{d}{2}} \|V_h\|_V) \|W_h\|_V \|U_h\|_V. \quad (20)
\]

**Proof.** We first show (18). For all $W_h, U_h, V_h \in V_h \times \hat{V}_h$, by the definition of $b_h(\cdot; \cdot; \cdot)$ we have

\[
2b_h(W_h; U_h, V_h) = [(v_h \otimes w_h, \nabla_h u_h) - (u_h \otimes w_h, \nabla_h v_h)]
\]

\[
+ [(\hat{w}_h \cdot n, \hat{u}_h \cdot v_h) - (\hat{w}_h \cdot n, \hat{v}_h \cdot u_h)]
\]
By the Hölder’s inequality, the inverse inequality, and (115), we obtain
\[|R_1| \leq \sum_{T \in \mathcal{T}_h} \|u_h\|_{0,6,T} \|w_h\|_{0,3,T} \|\nabla v_h\|_{0,T} + \sum_{T \in \mathcal{T}_h} \|v_h\|_{0,6,T} \|w_h\|_{0,3,T} \|\nabla u_h\|_{0,T} \lesssim \|w_h\|_{0,3} \cdot \|\mathcal{U}_h\|_V \cdot \|\mathcal{V}_h\|_V. \]  
\tag{22}

From triangle inequality we have
\[|R_2| \leq \|(w_h - \hat{w}_h) n, (u_h - \hat{u}_h) v_h\|_{\partial T_h} + \|(w_h, (u_h - \hat{u}_h) v_h)_{\partial T_h}\|
+ \|(w_h - \hat{w}_h) n, (v_h - \hat{v}_h) u_h\|_{\partial T_h} + \|(w_h, (v_h - \hat{v}_h) u_h)_{\partial T_h}\|
= \sum_{i=1}^{4} T_i. \]
\tag{23}

By the Hölder’s inequality, the inverse inequality, and (115), we obtain
\[T_1 \leq \sum_{T \in \mathcal{T}_h} \|w_h - \hat{w}_h\|_{0,\partial T} \|u_h - \hat{u}_h\|_{0,3,\partial T} \|v_h\|_{0,6,\partial T} \lesssim \sum_{T \in \mathcal{T}_h} \|w_h - \hat{w}_h\|_{0,\partial T} h_T^{-(d-1)/6} \|u_h - \hat{u}_h\|_{0,\partial T} h_T^{-\frac{1}{6}} \|v_h\|_{0,6,T} \]
\[= \sum_{T \in \mathcal{T}_h} h_T^{-\frac{1}{2}} \|w_h - \hat{w}_h\|_{0,\partial T} h_T^{-\frac{1}{2}} \|u_h - \hat{u}_h\|_{0,\partial T} h_T^{-\frac{1}{6}} \|v_h\|_{0,6,T} \lesssim h^{1-d} \|\mathcal{W}_h\|_V \cdot \|\mathcal{U}_h\|_V \cdot \|\mathcal{V}_h\|_V, \]  
\tag{24}

\[T_2 \leq \sum_{T \in \mathcal{T}_h} \|w_h\|_{0,3,\partial T} \|u_h - \hat{u}_h\|_{0,\partial T} \|v_h\|_{0,6,\partial T} \lesssim \sum_{T \in \mathcal{T}_h} h_T^{-\frac{1}{2}} \|w_h\|_{0,3,T} \|u_h - \hat{u}_h\|_{0,\partial T} h_T^{-\frac{1}{6}} \|v_h\|_{0,6,T} \]
\[= \sum_{T \in \mathcal{T}_h} \|w_h\|_{0,3,T} h_T^{-\frac{1}{2}} \|u_h - \hat{u}_h\|_{0,\partial T} \|v_h\|_{0,6,T} \lesssim \|w_h\|_{0,3} \cdot \|\mathcal{U}_h\|_V \cdot \|\mathcal{V}_h\|_V. \]  
\tag{25}
Similarly, we have

\begin{align}
T_3 & \lesssim h^{-\frac{d}{2}} \|\mathcal{W}_h\|_V \cdot \|\mathcal{U}_h\|_V \cdot \|\mathcal{V}_h\|_V, \\
T_3 & \lesssim \|w_h\|_{0,3} \cdot \|\mathcal{U}_h\|_V \cdot \|\mathcal{V}_h\|_V.
\end{align}

As a result, the desired inequality (18) follows from (21)-(27).

In light of the facts that $h \lesssim 1$, $1 - \frac{d}{6} \geq \frac{1}{2}$, and $\|w_h\|_{0,3} \lesssim \|\mathcal{W}_h\|_V$, the inequality (18) indicates (17).

Now we turn to show (19). By integration by parts and the fact $\langle w_h \cdot n, \hat{u}_h \cdot \hat{v}_h \rangle_{\partial T_h} = 0$ we get

\begin{align}
2b_h(\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h) &= (v_h \otimes w_h, \nabla_h u_h) - (u_h \otimes w_h, \nabla_h v_h) \\
&\quad + \langle \hat{u}_h \otimes \hat{w}_h n, v_h \rangle_{\partial T_h} - \langle \hat{v}_h \otimes \hat{w}_h n, u_h \rangle_{\partial T_h} \\
&= -2(u_h \otimes w_h, \nabla_h v_h) + (\nabla \cdot w_h, u_h v_h) \\
&\quad + \langle w_h n, u_h v_h - \hat{u}_h \hat{v}_h \rangle_{\partial T_h} + \langle \hat{w}_h n, \hat{u}_h v_h - u_h \hat{v}_h \rangle_{\partial T_h} \\
&= (2 \langle w_h n, u_h (\hat{v}_h - \hat{v}_h) \rangle_{\partial T_h} \\
&\quad + \langle w_h n, (u_h - \hat{u}_h) \cdot (\hat{v}_h - \hat{v}_h) \rangle_{\partial T_h} \\
&\quad + \langle \hat{w}_h - w_h \rangle \cdot n, (\hat{u}_h - u_h) \cdot v_h \rangle_{\partial T_h} \\
&\quad + \langle \hat{w}_h - w_h \rangle \cdot n, u_h \cdot (\hat{v}_h - \hat{v}_h) \rangle_{\partial T_h} \\
&= \sum_{i=1}^5 S_i \tag{28}
\end{align}

In what follows we estimate $S_i$ term by term. It is easy to see

\begin{align}
S_1 & \leq 2\|u_h\|_{0,3} \|w_h\|_{0,6} \|\nabla_h v_h\|_0 + d \|u_h\|_{0,3} \|u_h\|_{0,6} \|\nabla_h w_h\|_0 \\
& \lesssim \|u\|_{0,3} \|\mathcal{W}_h\|_V \|\mathcal{V}_h\|_V, \tag{29}
\end{align}

\begin{align}
S_2 & \leq 2 \sum_{T \in \mathcal{T}_h} \|w_h\|_{0,6,T} \|u_h\|_{0,3,T} \|v_h - v_h\|_{0,T} \\
& \lesssim \sum_{T \in \mathcal{T}_h} \|w_h\|_{0,6,T} \|u_h\|_{0,3,T} h_T^{-\frac{1}{2}} \|v_h - v_h\|_{0,T} \\
& \lesssim \|u_h\|_{0,3} \|\mathcal{W}_h\|_V \|\mathcal{V}_h\|_V, \tag{30}
\end{align}

\begin{align}
S_3 & \leq \sum_{T \in \mathcal{T}_h} \|w_h\|_{0,6,T} \|u_h - \hat{u}_h\|_{0,3,T} \|v_h - \hat{v}_h\|_{0,T} \\
& \quad \|w_h\|_{0,6,T} \|u_h - \hat{u}_h\|_{0,3,T} \|v_h - \hat{v}_h\|_{0,T}
\end{align}
\[ \leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \| \mathbf{w}_h \|_{0,6,T} h_T^{(d-1)/6} \| \mathbf{u}_h - \hat{\mathbf{u}}_h \|_{0,\partial T} \| \mathbf{v}_h - \mathbf{v}_h \|_{0,\partial T} \]
\[ \lesssim h^{1-d/2} \| \mathcal{W}_h \|_V \| \mathcal{U}_h \|_V \| \mathcal{V}_h \|_V. \]  

(31)

Similarly, we have

\[ S_4 \lesssim h^{1-d/2} \| \mathcal{W}_h \|_V \| \mathcal{U}_h \|_V \| \mathcal{V}_h \|_V, \]  

(32)

\[ S_5 \lesssim h^{1-d/2} \| \mathcal{W}_h \|_V \| \mathcal{U}_h \|_V \| \mathcal{V}_h \|_V. \]  

(33)

Therefore, the inequality (19) follows from (28)-(33).

The inequality (20) follows similarly.

For the bilinear form \( d_h(\cdot, \cdot) \), we have the following inf-sup inequality.

**Theorem 3.1** (Inf-sup Stability). For all \( \mathcal{P}_h \in Q_h \times \widehat{Q}_h \), it holds

\[ \sup_{0 \neq \mathbf{v}_h \in \mathcal{V}_h \times \widehat{\mathcal{V}}_h} \frac{d_h(\mathcal{V}_h, \mathcal{P}_h)}{\| \mathcal{V}_h \|_V} \gtrsim \| \mathcal{P}_h \|_Q. \]

**Proof.** We use the Fortin technique to prove the discrete inf-sup condition.

**Step 1.** Since \( p_h \in Q \), by the continuous inf-sup condition [15, Chapter 1, Corollary 2.4], there exists a function \( \mathbf{v} \in H^1_0(\Omega) \) such that

\[ \nabla \cdot \mathbf{v} = -p_h, \quad |\mathbf{v}|_1 \lesssim \| p_h \|_0. \]  

(34)

Take

\[ \mathcal{R}_h := (\mathbf{r}_h, \widehat{\mathbf{r}}_h) = (\mathbf{P}_h^{BDM} \mathbf{v}, \Pi_k^\partial \mathbf{v}) \in \mathcal{V}_h \times \widehat{\mathcal{V}}_h, \]

then from the definition of \( d_h(\cdot, \cdot) \) and integration by parts it follows

\[ d_h(\mathcal{R}_h, \mathcal{P}_h) = - (p_h, \nabla \cdot \mathbf{P}_h^{BDM} \mathbf{v}) + \langle \widehat{\mathbf{p}}_h, \mathbf{P}_h^{BDM} \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \]

\[ = (\nabla p_h, \mathbf{P}_h^{BDM} \mathbf{v}) + \langle \widehat{\mathbf{p}}_h - p_h, \mathbf{P}_h^{BDM} \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}. \]

In view of the properties of \( \mathbf{P}_h^{BDM} \) in (118a) and (118b), integration by parts , the fact \( \langle \widehat{\mathbf{p}}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \), and (34), we have

\[ d_h(\mathcal{R}_h, \mathcal{P}_h) = (\nabla p_h, \mathbf{v}) + \langle \widehat{\mathbf{p}}_h - p_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}, \]

\[ = - (p_h, \nabla \cdot \mathbf{v}) + \langle \widehat{\mathbf{p}}_h, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} \]

\[ = - (p_h, \nabla \cdot \mathbf{v}) \]

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\[ = \|p_h\|_0^2.\]

Since \( r_h = P_k^{BDM} v, \tilde{r}_h = \Pi_k^0 v \), by (120), (34), and the approximation properties of \( P_k^{BDM} \) and \( \Pi_k^0 \), we get

\[ \|\nabla r_h\|_0 + \|\tau \frac{1}{2} (r_h - \tilde{r}_h)\|_{0, \partial T_h} = \|\nabla_h P_k^{BDM} v\|_0 + \|\tau \frac{1}{2} (P_k^{BDM} v - v)\|_{0, \partial T_h} \lesssim \|v\|_1 \lesssim \|p_h\|_0. \]

Thus, by Lemma 3.1 and the definitions of \( \|\cdot\|_V \) and \( \|\cdot\|_Q \), we obtain

\[ \|\mathcal{R}_h\|_V \lesssim \|p_h\|_0 \leq \|\mathcal{P}_h\|_Q. \]

**Step 2.** Let \( w_h \in V_h \), with \( w_h|_T \in [\mathcal{P}_k(T)]^d \), be determined by

\[
\begin{align*}
(w_h, \nabla \tau_{k-1})_T &= 0 & \forall \tau_{k-1} \in \mathcal{P}_{k-1}(T), \\
(w_h \cdot n, \tau_k)_E &= h_E \langle p_h - \tilde{p}_h, \tau_k \rangle_E & \forall \tau_k \in \mathcal{P}_k(E), E \in \partial T.
\end{align*}
\]

for any \( T \in T_h \). Standard scaling arguments show

\[ \|w_h\|_{0,T} \lesssim h_T^{3/2} \|p_h - \tilde{p}_h\|_{0, \partial T}, \]

which combines with an inverse inequality, yields

\[ \|W_h\|_V \lesssim \|\mathcal{P}_h\|_V \] (37)

with \( W_h := (w_h, 0) \). From (35)-(36) and the definition of \( d_h(\cdot, \cdot) \), it follows

\[
\begin{align*}
&d_h(W_h, P_h) = -(p_h, \nabla \cdot w_h) + \langle \tilde{p}_h, w_h \cdot n \rangle_{\partial T_h} \\
&\quad = (\nabla_h p_h, w_h) + \langle \tilde{p}_h - p_h, w_h \cdot n \rangle_{\partial T_h} \\
&\quad = \|\tau \frac{1}{2} (p_h - \tilde{p}_h)\|_{\partial T_h}^2. \quad (38)
\end{align*}
\]

**Step 3.** Take \( \mathcal{V}_h = \mathcal{R}_h + W_h \in V_h \times \tilde{V}^0 \), and (38), then we get

\[
\begin{align*}
d_h(\mathcal{V}_h, P_h) &= d_h(\mathcal{R}_h, P_h) + d_h(W_h, P_h) = \|\mathcal{P}_h\|_Q^2. \quad (39)
\end{align*}
\]

By (35) and (37) we obtain

\[ \|\mathcal{V}_h\|_V \leq \|\mathcal{R}_h\|_V + \|W_h\|_V \lesssim \|\mathcal{P}_h\|_Q. \]

This result, together with (39), implies

\[ d_h(\mathcal{V}_h, P_h) \gtrsim \|\mathcal{P}_h\|_Q \|\mathcal{V}_h\|_Q, \]

which finishes the proof.
4. Existence and uniqueness of discrete solution

4.1. Globally divergence-free velocity approximation

**Theorem 4.1.** If \( \mathbf{U}_h := (\mathbf{u}_h, \mathbf{\hat{u}}_h) \in \mathbf{V}_h \times \mathbf{\hat{V}}_h \) satisfies (13c), then we have
\[
\mathbf{u}_h \in H(\text{div}; \Omega), \quad \nabla \cdot \mathbf{u}_h = 0.
\]

**Proof.** Introduce a function \( \mathbf{\hat{r}}_h \in L^2(\mathcal{E}_h) \) with
\[
\mathbf{\hat{r}}_h|_E := \begin{cases} - (\mathbf{u}_h \cdot \mathbf{n})|_{\partial T_1 \cap E} - (\mathbf{u}_h \cdot \mathbf{n})|_{\partial T_2 \cap E}, & \text{if } E \in \mathcal{E}_h/\partial \Omega, \\ 0, & \text{if } E \in \partial \Omega \end{cases}
\]
for any \( E \in \mathcal{E}_h \), where \( T_1, T_2 \in \mathcal{T}_h \) are the adjacent elements sharing the common edge (face) \( E \). Then we take \( Q_h = (\nabla_h \cdot \mathbf{u}_h - c_0, \mathbf{\hat{r}}_h - c_0) \) in (13c), with \( c_0 = (\nabla_h \cdot \mathbf{u}_h, 1)/|\Omega| \), to get
\[
\|\nabla_h \cdot \mathbf{u}_h\|_0^2 + \sum_{E \in \mathcal{E}_h/\partial \Omega, E = \partial T_1 \cap \partial T_2} \| (\mathbf{u}_h \cdot \mathbf{n})|_{\partial T_1 \cap E} + (\mathbf{u}_h \cdot \mathbf{n})|_{\partial T_2 \cap E} \|^2_{0,E} = 0,
\]
which implies the desired conclusion. \( \square \)

4.2. Existence result

Define
\[
\mathbf{W}_h := \{ \mathcal{W}_h \in \mathbf{V}_h \times \mathbf{\hat{V}}_h : d_h(\mathcal{W}_h; Q_h) = 0, \forall Q_h \in Q_h \times \mathbf{\hat{Q}}_h \}.
\]
Notice that \( \mathbf{W}_h \) is non-trivial due to Theorem 3.1. From Theorem 4.1, for \( \mathbf{V}_h = (\mathbf{v}_h, \mathbf{\hat{v}}_h) \in \mathbf{W}_h \), we have \( \mathbf{v}_h \in H(\text{div}; \Omega) \) and \( \nabla \cdot \mathbf{v}_h = 0 \).

By Lemma 3.4, we can define
\[
\mathcal{N}_h := \sup_{0 \neq \mathcal{W}_h, \mathbf{U}_h, \mathbf{V}_h \in \mathbf{W}_h} \frac{b_h(\mathcal{W}_h; \mathbf{U}_h, \mathbf{V}_h)}{\|\mathcal{W}_h\|_V \|\mathbf{U}_h\|_V \|\mathbf{V}_h\|_V}.
\]
Similarly, since \( \|\mathbf{v}_h\|_0 \lesssim \|\mathbf{V}_h\|_V \), we can define a norm of \( \mathbf{f} \) by
\[
\|\mathbf{f}\|_{*,h} := \sup_{0 \neq \mathbf{v}_h \in \mathbf{W}_h} \frac{(\mathbf{f}, \mathbf{v}_h)}{\|\mathbf{V}_h\|_V}.
\]

**Theorem 4.2.** The HDG scheme (13a)-(13c) admits at least one solution \( (\mathbb{L}_h, \mathbf{U}_h, \mathbf{P}_h) \in \mathbb{K}_h \times [\mathbf{V}_h \times \mathbf{\hat{V}}_h] \times [Q_h \times \mathbf{\hat{Q}}_h] \) for a sufficiently small mesh size \( h \).
Proof. Introduce the trilinear form
\[ A_h(W_h; U_h, V_h) := \nu(K_h U_h; K_h V_h) + s_h(U_h; V_h) + b_h(W_h; U_h, V_h). \]
Then the following two results hold:
(i) \( A_h(\mathcal{V}_h; \mathcal{V}_h, \mathcal{V}_h) = \nu\|\mathcal{V}_h\|_V^2, \quad \forall \mathcal{V}_h \in W_h; \)
(ii) \( W_h \) is separable, and the relation \( \lim_{n \to \infty} U_h^n = U_h \) (weakly in \( W_h \)) implies
\[ \lim_{n \to \infty} A_h(U_h^n; U_h^n, V_h) = A_h(U_h; U_h, V_h), \quad \forall V_h \in W_h. \]
Notice that (i) is obvious. We only need to show (ii). Let \( \{U^n_h\}_{n=1}^\infty \) be a sequence in \( W_h \) such that
\[ U^n_h \to U_h \text{ weakly in } W_h \text{ as } n \to \infty. \] (44)
Since \( W_h \) is finite dimensional space, we know that \( W_h \) is separable and the weak convergence and strong convergence are equivalent on \( W_h \). Then we have
\[ \lim_{n \to \infty} \|U^n_h - U_h\|_V = 0. \] (45)
Hence,
\[
|A_h(U^n_h; U^n_h, V_h) - A_h(U_h; U_h, V_h)|
\]
\[
= |\nu(K_h U^n_h; K_h V_h) + s_h(U^n_h - U_h; V_h) + b_h(U^n_h - U_h; U^n_h - U_h, V_h)|
\]
\[
\leq \nu\|U^n_h - U_h\|_V \|V_h\|_V + \mathcal{N}_h\|U^n_h - U_h\|_V^2 \|V_h\|_V + 2\mathcal{N}_h\|U^n_h - U_h\|_V \|U_h\|_V \|V_h\|_V.
\]
which, together with (45) and the fact that \( \mathcal{N}_h \) can be bounded from above by a positive constant for a sufficiently small \( h \), yields
\[ \lim_{n \to \infty} A_h(U^n_h; U^n_h, V_h) = A_h(U_h; U_h, V_h), \quad \forall V_h \in W_h, \]
i.e. (ii) holds.

In light of [14, Page 280, Theorem 1.2], the results (i)-(ii) imply that there exists at least one \( U_h \in W_h \) satisfying
\[ A_h(U_h; U_h, V_h) = (f, \nu_h) \quad \forall V_h \in W_h. \]
Given such a \( U_h \in W_h \), by Theorem 3.1 there exists a unique \( P_h \in Q_h \times \hat{Q}_h \) satisfying
\[ d_h(\nu_h, P_h) = -(f, \nu_h) + \nu(K_h U_h; K_h V_h) + s_h(U_h; V_h) + b_h(W_h; U_h, V_h). \]
As a result, the triple \( (\mathbb{I}_h = \nu K_h U_h, \nu_h, P_h) \in K_h \times [V_h \times \hat{V}_h] \times [Q_h \times \hat{Q}_h] \) is a solution to the scheme (13a)-(13c). \( \square \)
4.3. Uniqueness result

**Theorem 4.3.** Let \((L_h, U_h, P_h) \in K_h \times [V_h \times \tilde{V}_h] \times [Q_h \times \tilde{Q}_h]\) be a solution to the problem \((13a)-(13c)\), then \(U_h\) satisfies

\[\|U_h\|_{V} \leq \nu^{-1}\|f\|_{*,h}.\]  

(46)

In addition, if

\[(N_h/\nu^2)\|f\|_{*,h} < 1,\]  

(47)

then the scheme \((13a)-(13c)\) admits a unique solution.

**Proof.** Take \(V_h = U_h, Q_h = P_h\) in \(15b\) and \(15c\), and add the two equations to get

\[\nu\|U_h\|^2_{V} = (f, u_h) \leq \|f\|_{*,h}\|U_h\|_{V},\]

which leads to (46).

Since \(L_h = \nu K_h U_h\), we only need to show the uniqueness of \(U_h\) and \(P_h\). Let \((U_h^1, P_h^1)\) and \((U_h^2, P_h^2)\) be two solutions to \((15b)-(15c)\), then we have, for \(i = 1, 2\) and \((V_h, Q_h) \in [V_h \times \tilde{V}_h] \times [Q_h \times Q_h]\),

\[\nu(K_hU_h^i, K_hV_h) - d_h(V_h, P_h^1) + s_h(U_h^i; V_h) + b_h(U_h^i; U_h^i, V_h) = (f; v_h),\]

\[d_h(U_h^i; Q_h) = 0.\]

These two equations imply

\[\nu(K_h(U_h^1 - U_h^2), K_hV_h) - d_h(V_h, P_h^1 - P_h^2)\]

\[+s_h(U_h^1 - U_h^2; V_h) = b_h(U_h^2; U_h^1, V_h) - b_h(U_h^1; U_h^1, V_h),\]

\[d_h(U_h^1 - U_h^2; Q_h) = 0.\]  

(48a)

(48b)

Take \(V_h = U_h^1 - U_h^2, Q_h = P_h^1 - P_h^2\) in the above two equations, and add them together to get

\[\nu\|U_h^1 - U_h^2\|^2_{V} = b_h(U_h^2; U_h^1, U_h^1 - U_h^2) - b_h(U_h^1; U_h^1, U_h^1 - U_h^2)\]

\[= b_h(U_h^2; U_h^1 - U_h^1, U_h^1 - U_h^2) + b_h(U_h^2 - U_h^1; U_h^1, U_h^1 - U_h^2)\]

\[= b_h(U_h^2 - U_h^1; U_h^1, U_h^1 - U_h^2)\]

\[\leq N_h\|U_h^1 - U_h^2\|^2_{V} \|U_h^1\|_{V}.\]

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where we have used the stability result (46). Thus, it holds

$$\nu (1 - (N_h/\nu^2)) ||f||_{*1} ||U_h^1 - U_h^2||_V^2 \leq 0.$$  \hspace{1cm} (49)

As a result, by (47) we get \(U_h^1 = U_h^2\). And \(P_h^1 = P_h^2\) follows from (48a) and Theorem 3.1. \(\square\)

4.4. Relationship between discrete and continuous conditions of uniqueness

In this subsection, we shall show that the uniqueness condition (47) for the discrete solution is consistent with the continuous condition (10).

Lemma 4.1. For all \(w, u, v \in W\), it holds

$$|b_h(W_l; U_I, \mathcal{V}_I) - b(w; u, v)| \lesssim h^{-\frac{\theta}{2}} ||\nabla u||_0 ||\nabla w||_0 ||\nabla v||_0,$$ \hspace{1cm} (50)

where \(W_l, U_I\) and \(\mathcal{V}_I\) are defined as

\[ W_l := (P_k^R w, \Pi^0_k w), \quad U_I := (P_k^R u, \Pi^0_k u), \quad \mathcal{V}_I := (P_k^R v, \Pi^0_k v). \]

Proof. Set

$$R_1 := -(P_k^R u \otimes P_k^R w, \nabla_h P_k^R v) + \langle \Pi^0_k u \otimes \Pi^0_k w u, P_k^R v \rangle_{\partial T_h} - (\nabla \cdot (u \otimes w), v),$$

$$R_2 := (P_k^R v \otimes P_k^R w, \nabla_h P_k^R u) - \langle \Pi^0_k v \otimes \Pi^0_k w u, P_k^R u \rangle_{\partial T_h} + (\nabla \cdot (v \otimes w), u).$$

Then we have

$$b_h(W_l; U_I, \mathcal{V}_I) - b(w; u, v) = \frac{1}{2} (R_1 - R_2).$$ \hspace{1cm} (51)

Now we are going to estimate \(R_1\) and \(R_2\). From integration by parts and the fact \(\langle \Pi^0_k u \otimes \Pi^0_k w u, v \rangle_{\partial T_h} = 0\) it follows

$$R_1 = -((P_k^R u - u) \otimes P_k^R w, \nabla_h P_k^R v) - (u \otimes (P_k^R w - w), \nabla_h P_k^R v)$$

$$- (u \otimes w, \nabla_h (P_k^R v - v)) + \langle \Pi^0_k u \otimes \Pi^0_k w u, P_k^R v - v \rangle_{\partial T_h}$$

$$= -((P_k^R u - u) \otimes P_k^R w, \nabla_h P_k^R v) - (u \otimes (P_k^R w - w), \nabla_h P_k^R v)$$

$$+ (\nabla \cdot (u \otimes w), P_k^R v - v) + \langle \Pi^0_k u \otimes \Pi^0_k w u - u \otimes w u, P_k^R v - v \rangle_{\partial T_h}$$

$$= -((P_k^R u - u) \otimes P_k^R w, \nabla_h P_k^R v) - (u \otimes (P_k^R w - w), \nabla_h P_k^R v)$$
\[ + (\nabla \cdot (u \otimes w), P^RT_k v - v) + \langle (\Pi^0_k u - u) \otimes \Pi^0_k w n, P^RT_k v - v \rangle_{\partial T_h} \\
+ \langle u \otimes (\Pi^0_k w - w) n, P^RT_k v - v \rangle_{\partial T_h} \]
\[ =: \sum_{i=1}^{5} R_{1,i}. \]

Thanks to the Hölder’s inequality and the approximation properties of \( P^RT_k \), it holds
\[ |R_{1,1}| \leq \sum_{T \in T_h} \| P^RT_k u - u \|_{0,3,T}(\| P^RT_k w - w \|_{0,6,T} + \| w \|_{0,6,T}) \| \nabla_h P^RT_k v \|_{0,T} \]
\[ \leq (\| P^RT_k w - w \|_{0,6} + \| w \|_{0,6}) \sum_{T \in T_h} \| P^RT_k u - u \|_{0,3,T} \| \nabla_h P^RT_k v \|_{0,T} \]
\[ \lesssim (h^{1-d/3} + 1) \| \nabla w \|_0 \sum_{T \in T_h} h_T^{1-d/6} \| \nabla u \|_0 \| \nabla v \|_0, \]
\[ |R_{1,2}| \leq \sum_{T \in T_h} \| u \|_{0,6,T} \| P^RT_k w - w \|_{0,3,T} \| \nabla_h P^RT_k v \|_{0,T} \]
\[ \lesssim h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0, \]
\[ |R_{1,3}| \leq \sum_{T \in T_h} \| \nabla \cdot w \|_{0,T} \| u \|_{0,6,T} \| P^RT_k v - v \|_{0,3,T} \]
\[ + \sum_{T \in T_h} \| w \|_{0,6,T} \| \nabla u \|_{0,T} \| P^RT_k v - v \|_{0,3,T} \]
\[ \lesssim h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0, \]
\[ |R_{1,4}| \leq \sum_{T \in T_h} \| \Pi^0_k u - u \|_{0, \partial T} \| \Pi^0_k w \|_{0,6, \partial T} \| P^RT_k v - v \|_{0,3, \partial T} \]
\[ \lesssim \sum_{T \in T_h} h_T^{\frac{1}{2}} \| \nabla u \|_{0,T} h_T^{-\frac{1}{2}} (\| w \|_{1,T} + \| w \|_{0,6,T}) h_T^{2/3-d/6} \| \nabla v \|_{0,T} \]
\[ \leq (\| w \|_1 + \| w \|_{0,6}) \sum_{T \in T_h} h_T^{1-d/6} \| \nabla u \|_{0,T} \| \nabla v \|_{0,T} \]
\[ \lesssim h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0, \]
and
\[ |R_{1,5}| \leq \sum_{T \in T_h} \| u \|_{0,6, \partial T} \| \Pi^0_k w - w \|_{0, \partial T} \| P^RT_k v - v \|_{0,3, \partial T} \]

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\[ \sum_{T \in \mathcal{T}_h} T \left( \| u \|_{1,T} + \| u \|_{0,6,T} \right) h_T^{1-d/6} \| \nabla w \|_{0,T} \]
\begin{align*}
\leq & \left( \| u \|_1 + \| u \|_{0,6} \right) \sum_{T \in \mathcal{T}_h} h_T^{1-d/6} \| \nabla w \|_{0,T} \| \nabla v \|_{0,T} \\
\leq & h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0.
\end{align*}

As a result, we have
\[ |R_1| \lesssim h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0. \] (52)

Similarly, we can obtain
\[ |R_2| \lesssim h^{1-d/6} \| \nabla u \|_0 \| \nabla w \|_0 \| \nabla v \|_0, \]
which, together with (51) and (52), gives the desired result. \(\square\)

In view of the definitions (6), (7), (42), and (43), in what follows we shall show the relationships between \(\| f \|_{\ast,h} \) and \(\| f \|_{\ast} \) and between \(\mathcal{N}_h \) and \(\mathcal{N} \). To this end, for any \(\lambda \in (1, 2] \), let \( g \in [W^{0,\lambda}(\Omega)]^d \) be a given function, and let \((\Phi, \Psi) \in V \times Q \) be the solution of the auxiliary Stokes problem
\begin{align*}
-\Delta \Phi + \nabla \Psi &= g, \quad (53a) \\
\nabla \cdot \Phi &= 0 \quad (53b)
\end{align*}
with the regularity assumption
\[ \| \Phi \|_{1+s,\lambda} + \| \Psi \|_{s,\lambda} \lesssim \| g \|_{0,\lambda} \] (54)
for some \( s \in (\frac{1}{2}, 1] \).

**Remark 4.1.** As shown in [44, 49, 35], when \( \Omega \) is a bounded convex polygon in \( \mathbb{R}^2 \) or convex polyhedron in \( \mathbb{R}^3 \), the regularity estimate (54) holds true with \( s = 1 \).

**Lemma 4.2.** It holds that
\[ \lim_{h \to 0} \| f \|_{\ast,h} = \| f \|_{\ast}, \quad \lim_{h \to 0} \mathcal{N}_h = \mathcal{N}. \]
Proof. We divide the proof into three steps.

Step 1). Let $\Pi : V_h \times \hat{V}_h \to W$ be a switch operator from the discrete space to the continuous space defined as follows: for any $V_h = (v_h, \hat{v}_h) \in W_h$, $\Pi V_h \in W$ is determined by

$$(\nabla \Pi V_h, \nabla w) = (K_h V_h, \nabla w), \quad \forall w \in W. \quad (55)$$

Here we recall that the operator $K_h$ is given in (14). By testing (55) with $w = \Pi V_h \in W$ we have

$$\|\nabla \Pi V_h\|_0 \leq \|K_h V_h\|_0 \leq \|V_h\|_V. \quad (56)$$

Let $\mu \in [2, \infty)$ be the conjugate number of $\lambda$ with

$$\frac{1}{\mu} + \frac{1}{\lambda} = 1. \quad (57)$$

We consider the following problem: find $(\Phi, \Psi) \in V \times Q$ such that

$$-\Delta \Phi + \nabla \Psi = (\Pi V_h - v_h)^{\mu-1}, \quad (58a)$$
$$\nabla \cdot \Phi = 0. \quad (58b)$$

Then from the regularity (54) it follows

$$\|\Phi\|_{1+s, \lambda} \lesssim \|\Pi V_h - v_h\|_{0, \lambda} = \|\Pi V_h - v_h\|_{0, \mu}^{\mu-1}. \quad (59)$$

We use the equation (58a), integration by parts, the fact $\langle n \cdot \nabla \Phi, \hat{v}_h \rangle_{\partial T_h} = 0$, the definitions of $\Pi^{o}_m$, the equality (55), and the definition of $K_h$ to get

$$\|\Pi V_h - v_h\|_{0, \mu}^{\mu} = (-\Delta \Phi + \nabla \Psi, \Pi V_h - v_h)$$
$$= (\nabla \Phi, \nabla \Pi V_h) - (\nabla \Phi, \nabla_v v_h) + \langle n \cdot \nabla \Phi, v_h - \hat{v}_h \rangle_{\partial T_h}$$
$$= (\nabla \Phi, \nabla \Pi V_h) - (\Pi^{o}_m \nabla \Phi, \nabla_v v_h) + \langle n \cdot \nabla \Phi, v_h - \hat{v}_h \rangle_{\partial T_h}$$
$$= (\nabla \Phi, \nabla \Pi V_h - K_h V_h) + \langle n \cdot (\nabla \Phi - \Pi^{o}_m \nabla \Phi), v_h - \hat{v}_h \rangle_{\partial T_h}$$
$$= \langle n \cdot (\nabla \Phi - \Pi^{o}_m \nabla \Phi), v_h - \hat{v}_h \rangle_{\partial T_h}.$$ 

In light of the H"older’s inequality and the approximation properties of interpolations we have

$$\|\Pi V_h - v_h\|_{0, \mu}^{\mu} \lesssim \sum_{T \in T_h} \|n \cdot (\nabla \Phi - \Pi^{o}_k \nabla \Phi)\|_{0, \lambda, \partial T} \|v_h - \hat{v}_h\|_{0, \mu, \partial T}$$

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\[
\sum_{E \in \mathcal{E}_h} h_T^{s-\frac{1}{2}} \| \Phi \|_{1+s, \lambda, T^h}^{- (d-1) \left( \frac{1}{2} - \frac{1}{p} \right)} \left\| \mathbf{v}_h - \hat{\mathbf{v}}_h \right\|_{0, \partial T} \\
\lesssim h^{s-d(\frac{1}{2} - \frac{1}{p})} \| \Pi \mathbf{V}_h - \mathbf{v}_h \|_{0, \mu} \| \tau^\frac{1}{2} (\mathbf{v}_h - \hat{\mathbf{v}}_h) \|_{\partial T_h},
\]

which leads to
\[
\| \Pi \mathbf{V}_h - \mathbf{v}_h \|_{0, \mu} \lesssim h^{s-d(\frac{1}{2} - \frac{1}{p})} \| \tau^\frac{1}{2} (\mathbf{v}_h - \hat{\mathbf{v}}_h) \|_{\partial T_h}. \tag{60}
\]

With this estimate and (56) it is easy to get
\[
\| f \|_{*, h} = \sup_{\mathbf{v}_h \in \mathbf{V}_h \times \hat{\mathbf{V}}_h} \left( \frac{f, \mathbf{v}_h}{\| \mathbf{V}_h \|_V} \right) \\
= \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h \times \hat{\mathbf{V}}_h} \left( \frac{f, \Pi \mathbf{V}_h + (f, \mathbf{v}_h - \Pi \mathbf{V}_h)}{\| \mathbf{V}_h \|_V} \right) \\
\leq \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h \times \hat{\mathbf{V}}_h} \left( \frac{f, \Pi \mathbf{V}_h}{\| \mathbf{V}_h \|_V} \right) + C h^s \| f \|_0 \| \tau^\frac{1}{2} (\mathbf{v}_h - \hat{\mathbf{v}}_h) \|_{\partial T_h} \\
\leq \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h \times \hat{\mathbf{V}}_h} \left( \frac{f, \mathbf{v}_h}{\| \nabla \mathbf{V}_h \|_0} \right) + C h^s \| f \|_0 \\
\leq \| f \|_* + C h^s \| f \|_0. \tag{61}
\]

By the approximation properties of interpolations \( P^{RT}_k, \Pi^q_k \) and (56) we obtain
\[
\| (\Pi \mathbf{V}_h)_I \|_V \lesssim \| \mathbf{V}_h \|_V, \tag{62}
\]
where
\[
(\Pi \mathbf{V}_h)_I := (P^{RT}_k \Pi \mathbf{V}_h, \Pi^q_k \Pi \mathbf{V}_h).
\]

Using (60) and the approximation property of \( P^{RT}_k \) yields
\[
\| \mathbf{v}_h - P^{RT}_k \Pi \mathbf{V}_h \|_{0, 3} \lesssim \| \mathbf{v}_h - \Pi \mathbf{V}_h \|_{0, 3} + \| \Pi \mathbf{V}_h - P^{RT}_k \Pi \mathbf{V}_h \|_{0, 3} \\
\lesssim h^{s-d/6} \| \tau^\frac{1}{2} (\mathbf{v}_h - \hat{\mathbf{v}}_h) \|_{\partial T_h} + \| \nabla \Pi \mathbf{V}_h \|_0 \\
\lesssim h^{s-d/6} \| \mathbf{V}_h \|_V. \tag{63}
\]
Therefore, from (18), (19), (20), (62), and (63) it follows
\[ b_h(\mathcal{V}_h; \mathcal{U}_h, \mathcal{V}_h) - b_h((\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h); (\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h)) \]
\[ = b_h(\mathcal{W}_h - (\mathcal{W}_h); \mathcal{U}_h, \mathcal{V}_h) \]
\[ + b_h((\mathcal{W}_h; \mathcal{U}_h - (\mathcal{W}_h); \mathcal{V}_h) \]
\[ + b_h((\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h - (\mathcal{W}_h); \mathcal{V}_h) \]
\[ \lesssim h^{s-d/6}||\mathcal{W}_h||_V \cdot ||\mathcal{U}_h||_V \cdot ||\mathcal{V}_h||_V. \]

Hence, we get
\[ N_h = \sup_{0 \neq w_h, u_h, v_h \in W_h} \frac{b_h(\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h)}{||\mathcal{W}_h||_V \cdot ||\mathcal{U}_h||_V \cdot ||\mathcal{V}_h||_V} \]
\[ \leq \sup_{0 \neq w_h, u_h, v_h \in W_h} \frac{b(\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h)}{||\mathcal{W}_h||_V \cdot ||\mathcal{U}_h||_V \cdot ||\mathcal{V}_h||_V} \]
\[ + Ch^{s-d/6} \]
\[ \leq \sup_{0 \neq w_h, u_h, v_h \in W_h} \frac{b(\mathcal{W}_h; \mathcal{U}_h, \mathcal{V}_h)}{||\nabla \mathcal{W}_h||_0 ||\nabla \mathcal{U}_h||_0 ||\nabla \mathcal{V}_h||_0} \]
\[ + Ch^{s-d/6} \]
\[ = N + Ch^{s-d/6}. \] (64)

Step 2). Let \( \Pi_h : W \to W_h \) be the switch operator defined as follows: for any \( v \in W, \Pi_h v := (\pi_h v, \tilde{\pi}_h v) \in W_h \) is determined by
\[ \nu(K_h \Pi_h v; K_h \mathcal{W}_h) + s_h(\Pi_h v; \mathcal{W}_h) = \nu(\nabla v, K_h \mathcal{W}_h) \quad \forall \mathcal{W}_h \in W_h. \]
Taking $W_h = \Pi_h v$ in this equation, we have

$$\|\Pi_h v\|_V \leq \|\nabla v\|_0. \quad (65)$$

Let us consider the following problem: find $(\Phi, \Psi) \in V \times Q$ such that

$$-\Delta \Phi + \nabla \Psi = (v - \pi_h v)^{\mu - 1}, \quad (66a)$$
$$\nabla \cdot \Phi = 0. \quad (66b)$$

Then we apply the regularity (54) to get

$$\|\Phi\|_{2,\lambda} \lesssim \|(v - \pi_h v)^{\mu - 1}\|_{0,\lambda} = \|v - \pi_h v\|_{0,\mu}^{\mu - 1}. \quad (67)$$

From (66a), integration by parts, the Holder’s inequality, the approximation properties of interpolations it follows

$$\|v - \pi_h v\|_{0,\mu} = (-\Delta \Phi + \nabla \Psi, v - \pi_h v)$$
$$= (\nabla \Phi, \nabla v) - (\nabla \Phi, \nabla \pi_h v) + \langle n \cdot \nabla \Phi, \pi_h v - \hat{\pi}_h v \rangle_{\partial T_h}$$
$$= (\nabla \Phi, \nabla v) - (\Pi_m \nabla \Phi, \nabla \pi_h v) + \langle n \cdot \nabla \Phi, \pi_h v - \hat{\pi}_h v \rangle_{\partial T_h}$$
$$= (K_h(\Phi), \nabla v - K_h(\pi_h v, \hat{\pi}_h v)) + (\nabla \Phi - K_h(\Phi), \nabla v)$$
$$+ \langle n \cdot (\nabla \Phi - \Pi_m \nabla \Phi), \pi_h v - \hat{\pi}_h v \rangle_{\partial T_h}$$
$$+ \langle \tau(\Pi_m^\circ \Phi - \Pi_m^\circ \hat{\Phi}), \pi_h v - \hat{\pi}_h v \rangle_{\partial T_h}$$
$$\leq \sum_{T \in T_h} \|\nabla \Phi - \Pi_m^\circ \nabla \Phi\|_{0,T} \|\nabla v\|_{0,T}$$
$$+ \sum_{T \in T_h} \|\nabla \Phi - \Pi_m^\circ \nabla \Phi\|_{0,\lambda,\partial T} \|\pi_h v - \hat{\pi}_h v\|_{0,\mu,\partial T}$$
$$+ \sum_{T \in T_h} h_T^{-1} \|\Pi_m^\circ \Phi - \Phi\|_{0,\lambda,\partial T} \|\pi_h v - \hat{\pi}_h v\|_{0,\mu,\partial T}$$
$$\lesssim h^{s+d(\frac{1}{2} - \frac{1}{\mu})} \Phi_{1+s,\lambda}(\|\nabla v\|_0 + \|\tau^{\frac{1}{2}}(\pi_h v - \hat{\pi}_h v)\|_{\partial T_h})$$
$$\lesssim h^{s+d(\frac{1}{2} - \frac{1}{\mu})} \|v - \Pi_h v\|_{0,\mu}^{\mu - 1} \|\nabla v\|_0,$$

which leads to

$$\|v - \pi_h v\|_{0,\mu} \lesssim h^{s+d(\frac{1}{2} - \frac{1}{\mu})} \|\nabla v\|_0. \quad (68)$$
By (68) and (65) we have
\[
\| f \|_* = \sup_{0 \neq v \in W} \frac{(f, v)}{\| \nabla v \|_0}
\leq \sup_{0 \neq v \in W} \frac{(f, \pi_h v) + (f, v - \pi_h v)}{\| \nabla v \|_0}
\leq \sup_{0 \neq v \in W} \frac{(f, \pi_h v) + Ch \| f \|_0 \| \nabla v \|_0}{\| \nabla v \|_0}
\leq \sup_{0 \neq v \in W} \frac{(f, \pi_h v)}{\| \nabla v \|_0} + Ch \| f \|_0
\leq \sup_{0 \neq v \in W} \frac{(f, v_h)}{\| \nabla v \|_0} + Ch \| f \|_0
\leq \| f \|_* + Ch \| f \|_0.
\] (69)

Using the approximation properties of interpolations and (56), we get
\[
\| \Pi_h v \|_V \lesssim \| \nabla v \|_0.
\] (70)

In light of (68) and the approximation property of $P^{RT}_k$, it holds
\[
\| \pi_h v - P^{RT}_k v \|_{0,3} \leq \| \pi_h v - v \|_{0,3} + \| v - P^{RT}_k v \|_{0,3} \lesssim h^{s-d/6} \| \nabla v \|_0.
\] (71)

Then, from (18), (19), (20), (70), and (71) it follows
\[
|b_h(\Pi_h w; \Pi_h u, \Pi_h v) - b_h(\mathcal{W}_I; \mathcal{U}_I, \mathcal{V}_I)| \leq |b_h(\Pi_h w - \mathcal{W}_I; \Pi_h u, \Pi_h v)| + |b_h(\mathcal{W}_I; \Pi_h u - \mathcal{U}_I, \Pi_h v)| + |b_h(\mathcal{W}_I; \mathcal{U}_I, \Pi_h v - \mathcal{V}_I)|
\lesssim h^{s-d/6} \| \nabla w \|_0 \| \nabla u \|_0 \| \nabla v \|_0.
\]

Note that (50) means
\[
|b_h(\mathcal{W}_I; \mathcal{U}_I, \mathcal{V}_I) - b(w; u, v)| \leq h^{1-\frac{d}{2}} \| \nabla w \|_0 \| \nabla u \|_0 \| \nabla v \|_0.
\]

Therefore,
\[
b(w; u, v) = b_h(\Pi_h w; \Pi_h u, \Pi_h v)
- (b_h(\Pi_h w; \Pi_h u, \Pi_h v) - b_h(\mathcal{W}_I; \mathcal{U}_I, \mathcal{V}_I))
- (b_h(\mathcal{W}_I; \mathcal{U}_I, \mathcal{V}_I) - b(w; u, v))
\]
\[\leq b_h(\Pi_h w; \Pi_h u, \Pi_h v) + Ch^{s-d/6}\|\nabla w\|_0\|\nabla u\|_0\|\nabla v\|_0,\]

which implies

\[N = \sup_{0 \neq w, u, v \in W} \frac{b(w; u, v)}{\|\nabla w\|_0\|\nabla u\|_0\|\nabla v\|_0} \leq \sup_{0 \neq w, u, v \in W} \frac{b_h(\Pi_h w; \Pi_h u, \Pi_h v) + Ch^{1-d}\|\nabla w\|_0\|\nabla u\|_0\|\nabla v\|_0}{\|\nabla w\|_0\|\nabla u\|_0\|\nabla v\|_0} \leq \sup_{0 \neq w, u, v \in W} \frac{b_h(\Pi_h w; \Pi_h u, \Pi_h v)}{\|\Pi_h w\|_V\|\Pi_h u\|_V\|\Pi_h v\|_V} + Ch^{1-d} \leq \sup_{0 \neq w_h, u_h, v_h \in W_h} \frac{b_h(W_h; U_h, V_h)}{\|W_h\|_V\|U_h\|_V\|V_h\|_V} + Ch^{1-d} = N_h + Ch^{s-d/6} \tag{72}\]

Step 3). By (61), (69), (64), and (72) we have

\[\|f\|_\ast - Ch^s\|f\|_0 \leq \|f\|_\ast,h \leq \|f\|_\ast + Ch^s\|f\|_0,\]

\[N - Ch^{s-d/6} \leq N_h \leq N + Ch^{s-d/6},\]

where we recall that \(s \in (\frac{1}{2}, 1]\) and \(d = 2, 3\). As a result, the desired results follow from the squeeze theorem immediately.

\[\square\]

**Remark 4.2.** In view of Lemma 4.2 and the condition (10), it holds

\[(N_h/\nu^2)\|f\|_\ast,h \leq 1 - \frac{\delta}{2}\] \tag{73}\]

when the mesh size \(h\) is sufficiently small.

5. **A priori error estimates**

By some simple calculations, we have the following lemma.

**Lemma 5.1.** For any \(u, w \in W\) and \(v_h \in V_h\), it holds

\[b_h(W_L; U_L, V_h) = (\nabla \cdot (u \otimes w), v_h) + E_N(w; u, v_h),\] \tag{74}\]
where
\[ E_N(w; u, V_h) = -\frac{1}{2}(P_k^{RT} u \otimes P_k^{RT} w - u \otimes w, \nabla_h v_h) + \frac{1}{2}(\Pi_k^0 u \otimes \Pi_k^0 wn - u \otimes wn, v_h)_{\partial T_h} \]

\[ -\frac{1}{2}(w \cdot \nabla u - P_k^{RT} w \cdot \nabla_h P_k^{RT} u, v_h) - \frac{1}{2}(\hat{v}_h \otimes \Pi_k^0 wn, P_k^{RT} u)_{\partial T_h}. \]  

(75)

**Proof.** By integration by parts, it arrives at
\[ -(P_k^{RT} u \otimes P_k^{RT} w, \nabla_h v_h) + (\Pi_k^0 u \otimes \Pi_k^0 wn, v_h)_{\partial T_h} = (\nabla \cdot (u \otimes w), v_h) - (P_k^{RT} u \otimes P_k^{RT} w - u \otimes w, \nabla_h v_h) + (\Pi_k^0 u \otimes \Pi_k^0 wn - u \otimes wn, v_h)_{\partial T_h}. \]

From integration by parts and the fact that \( \nabla \cdot w = 0 \), it follows
\[ -(v_h \otimes P_k^{RT} w, \nabla_h P_k^{RT} u) + (\hat{v}_h \otimes \Pi_k^0 wn, P_k^{RT} u)_{\partial T_h} = -(\nabla \cdot (u \otimes w), v_h) + (v_h \otimes w, \nabla u) - (v_h \otimes P_k^{RT} w, \nabla_h P_k^{RT} u) + (v_h \otimes \Pi_k^0 wn - u \otimes wn, v_h)_{\partial T_h} + (\hat{v}_h \otimes \Pi_k^0 wn, P_k^{RT} u)_{\partial T_h}. \]

Then the desired results follows from the definition of \( b_h \). \( \square \)

**Lemma 5.2.** Let \( (L, u, p) \) be the solution to (11). Then, for all \( (V_h, Q_h) \in [V_h \times \hat{V}_h] \times [Q_h \times \hat{Q}_h] \), it holds the equations
\[ \Pi_m^0 L - \nu K_h \mathcal{U}_I = 0, \]

(76a)
\[ \nu(K_h \mathcal{U}_I, K_h \mathcal{V}_h) - d_h(V_h, P_I) + s_h(U_I, V_h) + b_h(U_I; U_I, V_h) = (f, v_h) + E_L(u; V_h) + E_N(u, u; V_h) \]

(76b)
\[ d_h(U_I, Q_h) = 0, \]

(76c)

where \( E_N \) is defined in (75), and \( E_L \) is defined as
\[ E_L(u; V_h) := -\nu(v_h - \hat{v}_h, (\Pi_m^0 - 1)(\nabla u)n)_{\partial \mathcal{N}_h} - \nu(\tau(P_k^{RT} u - u), v_h - \hat{v}_h)_{\partial \mathcal{N}_h}. \]

In addition, it holds
\[ P_k^{RT} u |_T \in [P_k(T)]^d, \forall T \in \mathcal{T}_h. \]  

(77)
Proof. For any $T \in \mathcal{T}_h, \tau_k \in \mathcal{P}_k(T)$, by the property (113) we have
\[
(\nabla \cdot P^RT_k u, \tau_k)_T = (\nabla \cdot u, \tau_k)_T = 0,
\]
which implies that $\nabla \cdot P^RT_k u = 0$. Then the result (77) follows from Lemma A.6.

By the orthogonality of projections, integration by parts, and the fact $\nu \cdot \nu = 0$, we get
\[
a_h(\Pi^o_m \mathbb{L}, \mathbb{G}_h) + c_h(\mathcal{U}_I, \mathbb{G}_h)
= \nu^{-1}(\Pi^o_m \mathbb{L}, \mathbb{G}_h) + (P^RT_k u, \nabla_h \cdot \mathbb{G}_h) - (\Pi^o_k u, \mathbb{G}_h \nu)_{\partial T_h}
= \nu^{-1}(\mathbb{L}, \mathbb{G}_h) + (u, \nabla_h \cdot \mathbb{G}_h) - (u, \mathbb{G}_h \nu)_{\partial T_h}
= 0.
\]
Similarly, by the fact $-\nabla \cdot \mathbb{L} + \nabla p = f$ we obtain
\[
c_h(\mathcal{V}_h, \Pi^o_m \mathbb{L}) + d_h(\mathcal{V}_h, \mathcal{P}_I) - s_h(\mathcal{U}_I, \mathcal{V}_h)
= (v_h, \nabla_h \cdot \Pi^o_m \mathbb{L}) - (\tilde{v}_h, \Pi^o_m \mathbb{L} \nu)_{\partial T_h}
+ (\nabla_h \cdot v_h, \Pi^o_{k-1} p) + (v_h \nu, \Pi^o_k p)_{\partial T_h}
- \nu \langle \tau (\Pi^o_k P^RT_k u - \Pi^o_k u), \Pi^o_k v_h - \tilde{v}_h \rangle_{\partial T_h}
= (v_h, \nabla \cdot \mathbb{L}) - (\tilde{v}_h - v_h, (\Pi^o_m \mathbb{L} - \mathbb{L}) \nu)_{\partial T_h} - (v_h, \nabla p)
- \nu \langle \tau (P^RT_k u - u), \Pi^o_k v_h - \tilde{v}_h \rangle_{\partial T_h}
= -(f, v_h) + (\nabla \cdot (u \otimes u), v_h) + E_L(u; \mathcal{V}_h)
= -(f, v_h) + E_N(u; u, v_h) + E_L(u; \mathcal{V}_h) - b_h(\mathcal{U}_I; \mathcal{U}_I, \mathcal{V}_h).
\]
Thanks to $\nabla \cdot P^RT_k u = 0$ and the property (111a), we have
\[
d_h(\mathcal{U}_I, \mathcal{Q}_h) = (\nabla \cdot P^RT_k u, q_h) - (P^RT_k u \cdot \nu, \tilde{q}_h)_{\partial T_h} = 0.
\]
This completes the proof. 

Let us estimate the nonlinear error $E_N$ and linear error $E_L$ in the following lemma.

**Lemma 5.3.** Let $\mathbb{L} = \nu \nabla u, u \in [H^{r_u}(\Omega)]^d \cap \mathbb{W}, w \in [H^{r_w}(\Omega)]^d \cap \mathbb{W}$, with $r_u, r_w > \frac{3}{2}$, and $\mathcal{V}_h \in \mathcal{V}_h \times \tilde{\mathcal{V}}_h, \mathcal{V}_h \in \mathcal{V}_h \times \tilde{\mathcal{V}}_h$, then the following estimates holds
\[
|E_N(u, w; \mathcal{V}_h)| \lesssim \left( h^{1+u-\frac{d}{2}} |u_1|_W + h^{1+w-\frac{d}{2}} |w_1|_W |u|_{s_u} \right) ||\mathcal{V}_h||_V,
\]
\[
|E_L(u; \mathcal{V}_h)| \lesssim \nu h^{s_u-1} ||u||_{s_u} ||\mathcal{V}_h||_V,
\]
where $s_u := \min\{r_u, k + 1\}$ and $s_w := \min\{r_w, k + 1\}$. 

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Proof. We only prove the first estimate, since the second one follows similarly.

Set $E_N(u; u, v_h) = \sum_{i=1}^{3} E_i$, where

$$E_1 := -\frac{1}{2} \langle P^R_k u \otimes P^R_k w - u \otimes w, \nabla_h v_h \rangle,$$

$$E_2 := \frac{1}{2} \langle \Pi^0_k u \otimes \Pi^0_k w n - u \otimes w n, v_h \rangle_{\partial \Omega_h},$$

$$E_3 := -\frac{1}{2} (\omega \cdot \nabla u - P^R_k w \cdot \nabla_h P^R_k u, v_h) - \frac{1}{2} (\tilde{v}_h \otimes \Pi^0_k w n, P^R_k u)_{\partial \Omega_h}.$$

The orthogonality \[111b\] yields

$$|\langle P^R_k u \otimes (P^R_k w - w), \nabla_h v_h \rangle| = |\langle (P^R_k u - \Pi^0_k u) \otimes (P^R_k w - w), \nabla_h v_h \rangle| \lesssim h^{1+s_w} |\omega|_{1} |w|_{s_w} |\nabla_h|_{V}.$$ 

Similarity, we have

$$|\langle (P^R_k u - u) \otimes w, \nabla_h v_h \rangle| = |\langle (P^R_k u - u) \otimes (P^R_k w - \Pi^0_k w), \nabla_h v_h \rangle| \lesssim h^{1+s_u} |\omega|_{1} |u|_{s_u} |\nabla_h|_{V}.$$ 

Therefore, from the triangle inequality it follows

$$|E_1| \lesssim \left( h^{1+s_w} |\omega|_{1} |w|_{s_w} + h^{1+s_u} |\omega|_{1} |u|_{s_u} \right) |\nabla_h|_{V}.$$ 

Since

$$|\langle \Pi^0_k u \otimes (\Pi^0_k w - w) n, v_h - \tilde{v}_h \rangle_{\partial \Omega_h}| = |\langle (\Pi^0_k u - \Pi^0_k u) \otimes (\Pi^0_k w - w) n, v_h - \tilde{v}_h \rangle_{\partial \Omega_h}| \lesssim h^{1+s_w} |\omega|_{1} |w|_{s_w} |\nabla_h|_{V},$$

and

$$|\langle (\Pi^0_k u - u) \otimes w n, v_h - \tilde{v}_h \rangle_{\partial \Omega_h}| \lesssim h^{1+s_u} |\omega|_{1} |u|_{s_u} |\nabla_h|_{V},$$

by triangle inequality we obtain

$$|E_2| \lesssim \left( h^{1+s_w} |\omega|_{1} |w|_{s_w} + h^{1+s_u} |\omega|_{1} |u|_{s_u} \right) |\nabla_h|_{V}.$$ 

It is easy to see that

$$-2E_3 = \langle (u - P^R_k u) \cdot \nabla w, v_h \rangle + \langle P^R_k u \cdot \nabla_h (w - P^R_k w), v_h \rangle.$$
Subtracting (15a)-(15c) from (76a)-(76c), respectively, we get

\begin{align*}
\langle (\tilde{v}_h \otimes \Pi_k^0 w, P_k^{RT} u - u\rangle_{\partial \Omega_h} \\
= \langle (u - P_k^{RT} u) \cdot \nabla v, w_h \rangle + \langle (P_k^{RT} u - u) \cdot \nabla_h (w - P_k^{RT} w), v_h \rangle \\
+ \langle u \cdot \nabla_h (w - P_k^{RT} w), v_h \rangle + \langle (\tilde{v}_h - v_h) \otimes \Pi_k^0 w, P_k^{RT} u - u\rangle_{\partial \Omega_h} \\
+ \langle v_h \otimes \Pi_k^0 w, P_k^{RT} u - u\rangle_{\partial \Omega_h} \\
= - \langle (w - P_k^{RT} w) \cdot \nabla_h v, w_h - \Pi_k^0 u \rangle \\
+ \langle (w - P_k^{RT} w) \cdot n, (v_h - \tilde{v}_h) (u - \Pi_k^0 u)\rangle_{\partial \Omega_h} \\
+ \langle (P_k^{RT} w - w) \cdot \nabla_h (u - P_k^{RT} u), v_h \rangle \\
- \langle (w - \Pi_k^0 w) \cdot \nabla_h v, u - P_k^{RT} u \rangle \\
+ \langle (\tilde{v}_h - v_h) \otimes \Pi_k^0 w, P_k^{RT} u - u\rangle_{\partial \Omega_h} \\
+ \langle v_h \otimes (\Pi_k^0 w - w) n, P_k^{RT} u - u\rangle_{\partial \Omega_h}.
\end{align*}

So

\[ |E_3| \lesssim \left( h^{1+s_u-\frac{d}{2}} |u|_1 \|w\|_{s_u} + h^{1+s_u-\frac{d}{2}} |w|_1 |u|_{s_u,R} \right) \|V_h\|V. \]

Combining the above estimates of $E_1, E_2, E_3$ yields the desired conclusion.

\[ \square \]

**Theorem 5.1.** Let $(u, p) \in [H^{r_u}(\Omega)]^d \times H^r(\Omega)$, with $r_u > \frac{3}{2}$ and $r_p > \frac{1}{2}$, and $(\mathbb{I}_h u, \tilde{u}_h, p_h, \hat{p}_h) \in \mathbb{K}_h \times \mathbb{V}_h \times [Q_h^0 \times Q_h]$ be the solutions to (5a) and (15a)-(15c), respectively. Then it holds the following error estimates:

\begin{align}
\|\mathcal{U}_I - U_h\|_V & \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \|u\|_{s_u} |u|_1 + \nu \|u\|_{s_u} \right), \\
\|J_h p - p_h\|_Q & \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \|u\|_{s_u} |u|_1 + \nu \|u\|_{s_u} \right),
\end{align}

where $s_u = \min\{r_u, k+1\}$.

**Proof.** Set

\[ e_h^p := \Pi_k^0 \mathbb{I}_h - \mathbb{I}_h, \]
\[ e_h^u := P_k^{RT} u - u_h, \quad e_h := \Pi_k^0 u - \tilde{u}_h, \quad e_h^{\tilde{u}} := (e_h^u, e_h^\tilde{u}), \]
\[ e_h^p := \Pi_k^{p-1} p - p_h, \quad e_h^p := \Pi_k^p p - p_h, \quad e_h^p := (e_h^p, e_h^p). \]

Subtracting (15a)-(15c) from (76a)-(76c), respectively, we get

\[ e_h^v - \nu K_h e_h^u = 0, \quad (80a) \]
\[ \nu(\mathcal{E}_h^u, K_h\mathcal{V}_h) + s_h(\mathcal{E}_h^u, \mathcal{V}_h) - d_h(\mathcal{V}_h, \mathcal{E}_h^p) \]
\[ + b_h(\mathcal{U}_I; \mathcal{U}_I, \mathcal{V}_h) - b_h(\mathcal{U}_h; \mathcal{U}_h, \mathcal{V}_h) = E_L(u; \mathcal{V}_h) + E_N(u, u, \mathcal{V}_h) \]
\[ d_h(\mathcal{E}_h^u, Q_h) = 0. \]

It is easy to verify that \((\mathcal{U}_I - \mathcal{U}_h, \mathcal{P}_I - \mathcal{P}_h) \in [\mathcal{V}_h \times \mathcal{V}_h] \times [Q_h^p \times \mathcal{Q}_h].\) Then we take \((\mathcal{V}_h, Q_h) = (\mathcal{U}_I - \mathcal{U}_h, \mathcal{P}_I - \mathcal{P}_h)\) in (80) to get
\[ \nu \| \mathcal{E}_h^u \|_V^2 = E_L(u; \mathcal{E}_h^u) + E_N(u, u; \mathcal{E}_h^u) - \{ b_h(\mathcal{U}_I; \mathcal{U}_I, \mathcal{E}_h^u) - b_h(\mathcal{U}_h; \mathcal{U}_h, \mathcal{E}_h^u) \}. \]

By the triangle inequality we have
\[ b_h(\mathcal{U}_I; \mathcal{U}_I, \mathcal{E}_h^u) - b_h(\mathcal{U}_h; \mathcal{U}_h, \mathcal{E}_h^u) = b_h(\mathcal{U}_I; \mathcal{E}_h^u, \mathcal{E}_h^u) + b_h(\mathcal{E}_h^u; \mathcal{U}_h, \mathcal{E}_h^u) \]
\[ \leq \mathcal{N}_h \| \mathcal{U}_h \|_V \| \mathcal{E}_h^u \|_V^2 \leq \mathcal{N}_h \nu^{-1} \| f \|_{*,h} \| \mathcal{E}_h^u \|_V^2. \]

Then from (73), we get
\[ \frac{\delta \nu}{2} \| \mathcal{E}_h^u \|_V^2 \leq (\nu - \mathcal{N}_h \nu^{-1} \| f \|_{*,h}) \| \mathcal{E}_h^u \|_V^2 \]
\[ \leq E_L(u; \mathcal{E}_h^u) + E_N(u, u; \mathcal{E}_h^u) \]
\[ \lesssim (h^{s_u + 1 - \frac{d}{2}} \| u \|_{s_u} \| u \|_1 + \nu h^{s_u - 1} \| u \|_{s_u}) \| \mathcal{E}_h^u \|_V, \]

which leads to
\[ \| \mathcal{E}_h^u \|_V \lesssim \delta^{-1} \nu^{-1} h^{s_u - 1} (h^{s_u + 1 - \frac{d}{2}} \| u \|_{s_u} \| u \|_1 + \nu \| u \|_{s_u}). \]

This yields (78).

From (15b) it follows
\[ d_h(\mathcal{V}_h, \mathcal{E}_h^p) = (K_h\mathcal{E}_h^u, K_h\mathcal{V}_h) + s_h(\mathcal{E}_h^u, \mathcal{V}_h) + b_h(\mathcal{U}_I; \mathcal{U}_I, \mathcal{V}_h) \]
\[ - b_h(\mathcal{U}_h; \mathcal{U}_h, \mathcal{V}_h) - E_L(u; \mathcal{V}_h) - E_N(u, u; \mathcal{V}_h) \]

Thanks to Theorem 3.1, it holds
\[ \| \mathcal{E}_h^p \|_Q \lesssim \sup_{0 \neq \nu_h \in \mathcal{V}_h \times \mathcal{V}_h} \frac{d_h(\mathcal{V}_h, \mathcal{E}_h^p)}{\| \mathcal{V}_h \|_V} \lesssim \delta^{-1} \nu^{-1} h^{s_u - 1} (h^{s_u + 1 - \frac{d}{2}} \| u \|_{s_u} \| u \|_1 + \nu \| u \|_{s_u}). \]

This yields (79).
Furthermore, Theorem 5.1 leads to the following a priori error estimates.

**Theorem 5.2.** Under the same conditions of Theorem 5.1, it holds the error estimates

\[
\| \nabla u - \nabla h u_h \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} | u |_1 + \nu \| u \|_{s_u} \right) + h^{s_u-1} \| u \|_{s_u},
\]

(81)

\[
\nu^{-1} \| L - L_h \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} | u |_1 + \nu \| u \|_{s_u} \right) + h^{s_u-1} \| u \|_{s_u},
\]

(82)

\[
\| p - p_h \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} | u |_1 + \nu \| u \|_{s_u} \right) + h^{s_p} \| p \|_{s_p},
\]

(83)

where \( L = \nu \nabla u \) and \( s_p := \min\{r_p, k\} \).

**Proof.** Using (16) and Theorem 5.1, we have

\[
\| \nabla h (P_k^R T u - u_h) \|_0 \lesssim \| U_I - U_h \|_V.
\]

Then, from the triangle inequality it follows

\[
\| \nabla u - \nabla h u_h \|_0 \lesssim \| \nabla h (P_k^R T u - u_h) \|_0 + \| \nabla h (P_k^R T u - u_h) \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1} \left( h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} | u |_1 + \nu \| u \|_{s_u} \right) + h^{s_u-1} \| u \|_{s_u},
\]

i.e. (81) holds.

The estimate (82) follows from the estimate (79) and the definition of the norm \( \| \cdot \|_V \). Similarly, Theorem 5.1 and a triangle inequality give (83).

**Remark 5.1.** From (81) we see that the velocity error is independent of the pressure. This means that our HDG scheme is pressure-robust.

### 6. \( L^2 \) error estimation for velocity by dual arguments

We follow standard dual arguments to derive an \( L^2 \) error estimate for velocity. To this end, introduce the following dual problem: find \((\Phi, \Psi)\) satisfying

\[
\begin{align*}
-\nu \Delta \Phi - u \cdot \nabla \Phi + (\nabla u)^T \Phi + \nabla \Psi &= \epsilon_h^u \quad \text{in } \Omega, \\
\nabla \cdot \Phi &= 0 \quad \text{in } \Omega, \\
\Phi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(84)
where $u$ and $u_h$ are the solutions of (1) and (13a), respectively. We assume the following regularity holds:

$$\|\Phi\|_{1+\alpha} + \|\Psi\|_{\alpha} \lesssim \|\epsilon_h^u\|_0,$$  \hspace{1cm} (85)

with $\alpha \in (\frac{1}{2}, 1]$ depending on $\Omega$.

**Lemma 6.1.** For any $\Phi$, $u \in V$ and $\mathcal{V}_h \in V_h \times \widehat{V}_h$ we have

$$b_h(\mathcal{V}_h; U_t, \Phi_t) = ((\nabla u)^T \Phi, \mathcal{V}_h) + E_N^*(\mathcal{V}_h; u, \Phi),$$  \hspace{1cm} (86)

where

$$E_N^*(\mathcal{V}_h; u, \Phi) = ((\nabla h P_k^{RT} u)^T P_k^{RT} \Phi - (\nabla u)^T \Phi, v_h) - \frac{1}{2} \langle \Pi_k \Phi \otimes \hat{v}_h n, P_k^{RT} u \rangle_{\partial \mathcal{T}_h}$$

$$+ \frac{1}{2} \langle \Pi_k \hat{u} \otimes \hat{v}_h n, P_k^{RT} \Phi \rangle_{\partial \mathcal{T}_h} - \frac{1}{2} \langle P_k^{RT} u \otimes v_h, P_k^{RT} \Phi \rangle_{\partial \mathcal{T}_h}.$$  

**Proof.** By direct calculations, we have

$$- (P_k^{RT} u \otimes v_h, \nabla h P_k^{RT} \Phi) + \langle \Pi_k \Phi \otimes \hat{v}_h n, P_k^{RT} \Phi \rangle_{\partial \mathcal{T}_h}$$

$$= ((\nabla u)^T \Phi, v_h) + ((\nabla h P_k^{RT} u)^T P_k^{RT} \Phi - (\nabla u)^T \Phi, v_h)$$

$$+ \langle \Pi_k \hat{u} \otimes \hat{v}_h n, P_k^{RT} \Phi \rangle_{\partial \mathcal{T}_h} - \langle P_k^{RT} u \otimes v_h, P_k^{RT} \Phi \rangle_{\partial \mathcal{T}_h},$$

Then the desired result follows immediately. \hfill \Box

**Lemma 6.2.** For all $u \in W \cap [H^{s_u}(\Omega)]^d$, $\Phi \in W \cap [H^{1+\alpha}(\Omega)]^d$ and $v_h \in V_h$, there holds

$$|E_N(u; u, \Psi)| \lesssim h^{1+s_u-\frac{4}{4+\alpha}}u_1 \|u\|_{s_u} \|\Psi\|_{1+\alpha},$$  \hspace{1cm} (87)

$$|E_N^*(\mathcal{V}_h; u, \Phi)| \lesssim h^\alpha \|u\|_{1+\alpha} \|\mathcal{V}_h\|_V,$$  \hspace{1cm} (88)

$$|E_L(u; \Phi)| + |E_L(\Phi; \epsilon_h^u)| \lesssim h^{s_u-1+\alpha} \|u\|_{s_u} \|\Phi\|_{1+\alpha}.$$  \hspace{1cm} (89)
Proof. It is easy to obtain
\[ E_N(u; v, \Psi_I) \lesssim h^{1+s_u} \frac{3}{2} |u|_1 \|u\|_{s_u} \|\Psi_I\|_V \lesssim h^{1+s_u} \frac{3}{2} A \|u\|_1 \|u\|_{s_u} \|\Psi\|_{1+\alpha} \]
and
\[ E_N^*(v_h; u, \Phi) = - (\nabla_h \cdot (P^{RT}_k \Phi \otimes v_h), P^{RT}_k u - u) \\
+ (P^{RT}_k \Phi u, v_h \cdot (P^{RT}_k u - u))_{\partial \Omega_h} \\
+ (P^{RT}_k \Phi - \Phi, v_h \cdot \nabla u) \\
- \frac{1}{2} \langle (v_h - v_h) \cdot \mathbf{n}, \Pi_m^\Phi \cdot (P^{RT}_k u - \Pi_m^\Phi u) \rangle_{\partial \Omega_h} \\
- \frac{1}{2} \langle (v_h - v_h) \cdot \mathbf{n}, (\Pi_m^\Phi - P^{RT}_k \Phi) \cdot \Pi_m^\Phi u \rangle_{\partial \Omega_h} \\
- \frac{1}{2} \langle v_h \cdot \mathbf{n}, P^{RT}_k \Phi \cdot (P^{RT}_k u - \Pi_m^\Phi u) \rangle_{\partial \Omega_h} \\
- \frac{1}{2} \langle v_h \cdot \mathbf{n}, \Pi_m^\Phi \cdot (P^{RT}_k u - \Pi_m^\Phi u) \rangle_{\partial \Omega_h} \\
\lesssim h^\alpha \|u\|_1 \|\Phi\|_{1+\alpha} \|v_h\|_V. \]

Then the thing left is to show (89). Since \( \Phi \in V \), \( u \in H^{1+s}(\Omega) \) \( s > \frac{1}{2} \) and \( m \leq k \), we have
\[ \langle n \cdot (\nabla u - \Pi_m^\Phi \nabla u), \Pi_m^\Phi \rangle_{\partial \Omega_h} = \langle n \cdot (-\Pi_m^\Phi \nabla u), \Pi_m^\Phi \rangle_{\partial \Omega_h} \\
= \langle n \cdot (-\Pi_m^\Phi \nabla u), \Phi \rangle_{\partial \Omega_h} \\
= \langle n \cdot (\nabla u - \Pi_m^\Phi \nabla u), \Phi \rangle_{\partial \Omega_h}. \]

By the approximation properties of \( P^{RT}_k \) and \( \Pi_m^\Phi \), the approximation property and stability of \( L^2 \) projection \( \Pi_k^\Phi \), we easily get
\[ E_L(u; \Phi_I) = \nu \langle n \cdot (\nabla u - \Pi_m^\Phi \nabla u), P^{RT}_k \Phi - \Pi_k^\Phi \rangle_{\partial \Omega_h} \\
+ \nu \langle \tau (P^{RT}_k u - u), P^{RT}_k \Phi - \Pi_k^\Phi \rangle_{\partial \Omega_h} \\
= \nu \langle n \cdot (\nabla u - \Pi_m^\Phi \nabla u), P^{RT}_k \Phi - \Phi \rangle_{\partial \Omega_h} \\
+ \nu \langle \tau (P^{RT}_k u - u), \Pi_k^\Phi (P^{RT}_k \Phi - \Phi) \rangle_{\partial \Omega_h} \\
\lesssim \nu h^{s_u-1+\alpha} \|u\|_{s_u} \|\Phi\|_{1+\alpha}, \]
and
\[ E_L(\Phi; \Phi_I) = \nu \langle n \cdot (\nabla \Phi - \Pi_m^\Phi \nabla \Phi), (P^{RT}_k u - u) - (\Pi_k^\Phi u - \hat{u}_h) \rangle_{\partial \Omega_h} \]

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Then there holds
\[ \nu \langle \tau (P_k^{RT} \Phi - \Phi), (P_k^{RT} u - u_h) - (\Pi_k^2 u - \hat{u}_h) \rangle_{\partial \Omega} \]
\[ \lesssim \nu h^{\alpha} |\Phi|_{1+\alpha} \| U_t - U_h \|_V \]
\[ \lesssim \nu h^{s_u-1+\alpha} \| u \|_{s_u} |\Phi|_{1+\alpha}, \]
which yield the desired result.

**Theorem 6.1.** Let \((u, p) \in [H^{r_u}(\Omega)]^d \times H^r(\Omega)\) and \((\mathbb{L}_h, u_h, \hat{u}_h, p_h, \hat{p}_h) \in \mathbb{K}_h \times [\mathbb{V}_h \times \hat{\mathbb{V}}_h] \times [Q_h^0 \times \hat{Q}_h]\) be the solutions to (1) and (15a)-(15c), respectively. Then it holds the error estimate
\[ \| u - u_h \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1+\alpha}(h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} |u|_1 + \nu \| u \|_{s_u}) + h^{s_u-1+\alpha} \| u \|_{s_u}, \]
(90)

where \(\alpha \in (\frac{1}{2}, 1)\).

**Proof.** Similar to Lemma 5.2 for all \((V_h, Q_h) \in [V_h \times \hat{V}_h] \times [Q_h \times \hat{Q}_h]\) it holds
\[ \nu (K_h \Phi_I, K_h \Phi_h) - d_h(\Phi_I, \Phi_h) + s_h(\Phi_I, \Phi_h) - b_h(\Phi_I, \Phi_h) \]
\[ = (e_h^u, v_h) + E_L(u, \Phi_h) + E_N(u, u, \Phi_h) + E_N(V_h, u, \Phi) \]
\[ d_h(\Phi_I, Q_h) = 0. \]

Then there holds
\[ \| e_h^u \|^2_0 = \nu (K_h \Phi_I, K_h \Phi_h) - d_h(\Phi_h, \Phi_h) + s_h(\Phi_I, \Phi_h) - b_h(\Phi_I, \Phi_h) \]
\[ + b_h(\Phi_h, \Phi_I) - E_L(\Phi, \Phi_h) - E_N(u, \Phi, \Phi_h) - E_N(\Phi_h, u, \Phi) \]
\[ = b_h(\Phi_h, \Phi_h) + E_L(u, \Phi_I) + E_N(u, u, \Phi_I) \]
\[ - E_N(\Phi, \Phi_h) - E_N(u, \Phi, \Phi_h) - E_N(\Phi_h, u, \Phi). \]

which, together with the regularity (85), indicate
\[ \| e_h^u \|_0 \lesssim \delta^{-1} \nu^{-1} h^{s_u-1+\alpha}(h^{s_u+1-\frac{d}{2}} \| u \|_{s_u} |u|_1 + \nu \| u \|_{s_u}). \]

This estimate, together with triangle inequality and the approximation property of the operator \(P_k^{RT}\), leads to the desired conclusion.

**Corollary 6.1.** If \(\Omega\) is convex, \(\delta^{-1} \nu^{-1}(h^{s_u+1-\frac{d}{2}} |u|_1 + \nu) \lesssim 1\), and the true solution \((u, p)\) is smooth enough, we will have the following optimal estimates:
\[ \| u - u_h \|_0 \lesssim h^{k+1} \| u \|_{k+1}, \]
(91a)
\[ \| L - \mathbb{L}_h \|_0 \lesssim h^k \| u \|_{k+1}, \]
(91b)
\[ \| p - p_h \|_0 \lesssim h^k (\| u \|_{k+1} + \| p \|_k). \]
(91c)
7. Numerical experiments

In this section, we provide some numerical results to verify the performance of the proposed HDG finite element methods for the Navier-Stokes model (1) in two dimensional. All tests are programmed in C++ using the Eigen library.

We take \( \Omega = (0, 1) \times (0, 1) \) and the Reynolds number \( \text{Re} = 1 \) in the two examples below.

Example 7.1

The forcing term \( \mathbf{f} \) is chosen so that the analytical solution to (1), with the homogeneous Dirichlet boundary condition, is given by

\[
\begin{align*}
    u_1 &= -x^2(x - 1)^2 y(y - 1)(2y - 1), \\
    u_2 &= x(x - 1)(2x - 1) y^2(y - 1)^2, \\
    p &= 10 \left( \left( x - \frac{1}{2} \right)^3 y^2 + (1 - x)^3 \left( y - \frac{1}{2} \right)^3 \right). 
\end{align*}
\]

(92)

The computational mesh is a regular triangulation of \( \Omega \). The results of the relative errors of the velocity and pressure approximations for \( k = 1, 2, 3 \) are presented in Table 1. The obtained optimal convergence rates agree with our theoretical results (91a), (91b) and (91c). In addition, the velocities are truly divergence-free.

Example 7.2

This numerical example is from [47], forcing term \( \mathbf{f} = (0, 10^6(3y^2 - y + 1))^T \), and the analytical solution to (1), with the homogeneous Dirichlet boundary condition, is given by

\[
\mathbf{u} = 0, \quad p = 10^6(y^3 - y^2/2 + y + 7/12).
\]

In [47], it has been shown that the discrete velocity solved by Taylor-Hood space \( P_3 - P_1 \) is far from being equal to zero for Stokes equations. We present the errors of the velocity and pressure approximations by our HDG method with \( k = 1, 2, 3 \) and the Taylor-Hood \( P_2 - P_1 \) method in Tables 2,3. We can see that the discrete velocity obtained by our HDG method is very close to zero, while the velocity obtained by Taylor-Hood \( P_2 - P_1 \) method is not so accurate, which is influenced by the pressure approximation. This confirms that our proposed HDG method is indeed pressure-robust.

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### Table 1: History of convergence for Example 7.1

| $k$ | $\frac{\sqrt{2}}{h}$ | $\frac{\|u-u_h\|_0}{\|u\|_0}$ | Error | Rate | $\frac{\|\mathbf{L}-L_h\|_0}{\|\mathbf{L}\|_0}$ | Error | Rate | $\frac{\|p-p_h\|_0}{\|p\|_0}$ | Error | Rate | $\int_{\Omega} |\nabla \cdot \mathbf{u}_h| \, dx$ |
|-----|------------------|------------------|-------|------|------------------|-------|------|------------------|-------|------|------------------|
| 1   | 4                | 1.6593E-01       | 1.5900E-01 |      | 5.1047E-01       | 3.94E-16 |
| 8   | 1.1561E-02       | 5.2372E-02       | 2.7139E-01 | 0.91  |
| 32  | 2.9109E-03       | 9.1087E-03       | 6.9135E-02 | 0.99  |
| 64  | 7.2931E-04       | 4.4150E-03       | 3.4600E-02 | 1.00  |

| 2   | 4                | 2.8355E-02       | 2.9255E-02 |      |
| 8   | 3.6823E-03       | 5.1938E-03       | 2.8525E-02 | 1.94  |
| 16  | 4.6102E-04       | 1.0012E-03       | 2.0551E-03 | 1.98  |
| 32  | 7.1330E-06       | 2.1915E-04       | 1.8070E-03 | 2.00  |

| 3   | 4                | 3.8396E-03       | 4.4818E-03 |      |
| 8   | 2.6811E-04       | 4.4213E-04       | 1.7949E-03 | 2.96  |
| 16  | 1.7149E-05       | 5.096E-05        | 2.260E-04  | 2.99  |
| 32  | 1.0762E-06       | 6.3021E-06       | 2.830E-05  | 3.00  |
| 64  | 6.7274E-08       | 7.9099E-07       | 3.5393E-06 | 3.00  |

### Table 2: Results of $\|u-u_h\|_0$ for Example 7.2

| $\frac{\sqrt{2}}{h}$ | $\|u-u_h\|_0$ | $\|\mathbf{L}-L_h\|_0$ | $\|p-p_h\|_0$ | $\int_{\Omega} |\nabla \cdot \mathbf{u}_h| \, dx$ |
|------------------|------------------|------------------|------------------|------------------|
| k = 1            | HDG              | Taylor-Hood      | $p_2 - p_1$      |
| 10               | 3.1264E-10       | 1.695E-10        | 5.4894E-10       | 1.4948E+00      |
| 20               | 4.7678E-10       | 5.1265E-10       | 5.2989E-10       | 9.4093E-02      |
| 40               | 1.1007E-09       | 1.1615E-09       | 1.0066E-08       | 5.8943E-03      |
| 80               | 3.1464E-09       | 1.9250E-09       | 1.5895E-09       | 3.6885E-04      |

### Table 3: History of convergence for $\|p-p_h\|_0/\|p\|_0$ Example 7.2

| $\frac{\sqrt{2}}{h}$ | $\|p-p_h\|_0/\|p\|_0$ | $\|\mathbf{L}-L_h\|_0$ | $\|p-p_h\|_0$ | $\int_{\Omega} |\nabla \cdot \mathbf{u}_h| \, dx$ |
|------------------|------------------|------------------|------------------|------------------|
| k = 1            | HDG              | Taylor-Hood      | $p_2 - p_1$      |
| 10               | 9.2848E-02       | 1.8531E-03       | 3.4725E-05       | 2.4070E-03      |
| 20               | 4.6471E-02       | 4.6392E-04       | 4.3406E-06       | 5.9988E-04      |
| 40               | 2.3241E-02       | 1.1602E-04       | 5.4258E-07       | 1.4984E-04      |
| 80               | 1.1621E-02       | 2.9007E-05       | 6.7822E-08       | 3.7452E-05      |
Appendix A Interpolation properties

This section is devoted to some standard interpolation properties used in this paper.

A.1 $L^2$ projection

For any $T \in T_h$, $E \in E_h$ and any integer $r \geq 0$, let $\Pi_o^r : L^2(T) \to P_r(T)$ and $\Pi_o^\delta : L^2(E) \to P_r(E)$ be the usual $L^2$-projection operators. From [65] we have Lemmas A.1 and A.2.

**Lemma A.1.** Let $r$ be a nonnegative integer and $\rho \in [1, \infty]$. Assume that

$$2 \leq \frac{d\rho}{d - (r + 1)\rho} \quad \text{when} \quad (r + 1)\rho < d.$$  

Then, for $j \in \{0, 1, \ldots, r+1\}$, there exists a constant $C$ independent of $T$ such that the estimate

$$|v - \Pi_o^r v|_{j, \mu, T} \leq Ch^{r+1-j+\frac{d-\delta}{\mu}} |v|_{r+1, \rho, T}, \quad \forall v \in W^{r+1, \rho}(T)  \tag{93}$$

holds for $\mu$ satisfying

$$
\begin{cases}
\rho \leq \mu \leq \frac{d\rho}{d - (r + 1 - j)\rho}, & \text{if} \quad (r + 1 - j)\rho < d, \\
\rho \leq \mu < \infty, & \text{if} \quad (r + 1 - j)\rho = d, \\
\rho \leq \mu \leq \infty, & \text{if} \quad (r + 1 - j)\rho > d,
\end{cases}
$$

and the estimate

$$|
abla^j(v - \Pi_o^r v)|_{0, \mu, \partial T} \leq Ch^{r+1-j+\frac{d-\delta}{\mu}} |v|_{r+1, \rho, T}, \forall v \in W^{r+1, \rho}(T)  \tag{95}$$

holds for $\mu$ satisfying

$$
\begin{cases}
\rho \leq \mu \leq \frac{(d - 1)\rho}{d - (r + 1 - j)\rho}, & \text{if} \quad (r + 1 - j)\rho < d, \\
\rho \leq \mu < \infty, & \text{if} \quad (r + 1 - j)\rho = d, \\
\rho \leq \mu \leq \infty, & \text{if} \quad (r + 1 - j)\rho > d.
\end{cases}
$$

**Lemma A.2.** Let $r$ be a nonnegative integer and $\rho \in [1, \infty]$. Assume that

$$2 \leq \frac{(d-1)\rho}{d - (r + 1)\rho} \quad \text{when} \quad (r + 1)\rho < d.$$  

Then there exists a constant $C$ independent of $T$ such that

$$\|v - \Pi_o^\delta v\|_{0, \mu, \partial T} \leq Ch^{r+1+\frac{d-\delta}{\mu}} |v|_{r+1, \rho, T}, \forall v \in W^{r+1, \rho}(T)  \tag{97}$$

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holds for \( \mu \) satisfying

\[
\begin{cases}
\rho \leq \mu \leq \frac{(d-1)\rho}{d-(r+1)\rho}, & \text{if } (r+1)\rho < d, \\
\rho \leq \mu < \infty, & \text{if } (r+1)\rho = d, \\
\rho \leq \mu \leq \infty, & \text{if } (r+1)\rho > d.
\end{cases}
\] (98)

By following the same lines as in the proofs of the above two lemmas in [65], it is easy to get the following more general results, i.e. Lemmas A.3 and A.4.

**Lemma A.3** (Approximation properties for \( \Pi_r^o \)). Let \( r \) be a nonnegative integer and \( \rho \in [1, \infty] \). For \( j \in \{0, 1, \ldots, r+1\} \) and \( s \in \{j, j+1, \ldots, r+1\} \), assume that \( 2 \leq \frac{d\rho}{d-s\rho} \) when \( s\rho < d \). Then there exists a constant \( C \) independent of \( T \) such that

\[
|v - \Pi_r^o v|_{j, \mu, T} \leq Ch^{|s-j| + \frac{d}{d-s\rho}} |v|_{s, \rho, T}, \forall v \in W^{s, \rho}(T)
\] (99)

holds for \( \mu \) satisfying

\[
\begin{cases}
1 \leq \mu \leq \frac{d\rho}{d-(s-j)\rho}, & \text{if } (s-j)\rho < d, \\
1 \leq \mu < \infty, & \text{if } (s-j)\rho = d, \\
1 \leq \mu \leq \infty, & \text{if } (s-j)\rho > d.
\end{cases}
\] (100)

In addition, for \( j \in \{0, 1, \ldots, r+1\} \) and \( s \in \{j+1, j+2, \ldots, r+1\} \), assume that \( 2 \leq \frac{d\rho}{d-s\rho} \) when \( s\rho < d \). Then there exists a constant \( C \) independent of \( T \) such that

\[
|\nabla^j (v - \Pi_r^o v)|_{0, \mu, \partial T} \leq Ch^{s-j+\frac{d}{d-s\rho}} |v|_{s, \rho, T}, \forall v \in W^{s, \rho}(T)
\] (101)

holds for \( \mu \) satisfying

\[
\begin{cases}
1 \leq \mu \leq \frac{(d-1)\rho}{d-(s-j)\rho}, & \text{if } (s-j)\rho < d, \\
1 \leq \mu < \infty, & \text{if } (s-j)\rho = d, \\
1 \leq \mu \leq \infty, & \text{if } (s-j)\rho > d.
\end{cases}
\] (102)
Lemma A.4 (Approximation properties for $\Pi_r^\partial$). Let $r$ be a nonnegative integer and $\rho \in [1, \infty]$. For $s \in \{1, 2, \ldots, r+1\}$, assume that $2 \leq \frac{d \rho}{d - s \rho}$ when $s \rho < d$. Then there exists a constant $C$ independent of $T$ such that

$$\|v - \Pi_r^\partial v\|_{0, \mu, \partial T} \leq C h_T^{s + \frac{d-1}{d} - \frac{d}{\rho}} |v|_{s, \rho, T}, \forall v \in W^{s, \rho}(T)$$

holds for $\mu$ satisfying

$$1 \leq \mu \leq \frac{(d-1) \rho}{d - s \rho}, \quad \text{if } s \rho < d,$$

$$1 \leq \mu < \infty, \quad \text{if } s \rho = d,$$

$$1 \leq \mu \leq \infty, \quad \text{if } s \rho > d.$$  

We also need the following boundeness results for the projections.

Lemma A.5 (Boundedness properties for $\Pi_r^\circ$ and $\Pi_r^\partial$). Let $r$ be a nonnegative integer. Then, for any $\mu \in [2, \infty]$ and $j \in \{0, 1, \ldots, r+1\}$, it holds

$$|\Pi_r^\circ v|_{j, \mu, T} \lesssim |v|_{j, \mu, T}, \forall v \in L^2(T),$$  

$$\|\Pi_r^\circ v\|_{0, \mu, \partial T} \lesssim \|v\|_{0, \mu, \partial T}, \forall v \in L^2(\partial T),$$

$$\|\Pi_r^\circ v\|_{0, 6, \partial T} \lesssim h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 6, T}), \forall v \in H^1(T),$$

$$\|v\|_{0, 6, \partial T} \lesssim h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 6, T}), \forall v \in H^1(T),$$

$$\|\Pi_r^\circ v\|_{0, 6, \partial T} \lesssim h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 6, T}), \forall v \in H^1(T).$$

Proof. We first show (105). For $j = 0$, by an scaling argument we have

$$\|\Pi_r^\circ v\|_{0, \mu, T} \lesssim h_T^{-d(\frac{1}{2} - \frac{1}{\mu})} \|\Pi_r^\circ v\|_{0, T}$$

$$\lesssim h_T^{-d(\frac{1}{2} - \frac{1}{\mu})} \|v\|_{0, T}$$

$$\lesssim h_T^{-d(\frac{1}{2} - \frac{1}{\mu})} h_T^d \|\hat{v}\|_{0, T}$$

$$\lesssim h_T^{-d(\frac{1}{2} - \frac{1}{\mu})} h_T^d \|\hat{v}\|_{0, \mu, T}$$

$$\lesssim h_T^{-d(\frac{1}{2} - \frac{1}{\mu})} h_T^d \|v\|_{0, \mu, T} = \|v\|_{0, \mu, T},$$

For $j \geq 1$, we obtain

$$|\Pi_r^\circ v|_{j, \mu, T} = |\Pi_r^\circ v - \Pi_r^{j-1} v|_{j, \mu, T}.$$
\[ = |\Pi^o_r (v - \Pi^o_{j-1} v)|_{j,\mu, T} \]
\[ \lesssim h_T^{-j-d(\frac{1}{2} - \frac{1}{p})} ||\Pi^o_r (v - \Pi^o_{j-1} v)||_{0, T} \]
\[ \lesssim h_T^{-j-d(\frac{1}{2} - \frac{1}{p})} ||v - \Pi^o_{j-1} v||_{0, T} \]
\[ \lesssim |v|_{j, \mu, T}. \]

Combining the above two estimates yields (105).

Secondly, let’s prove (106). Similar to (110), it holds
\[ \|\Pi^o_r \partial_r v\|_{0, \mu, \partial T} \lesssim h_T^{-(d-1)(\frac{1}{2} - \frac{1}{p})} \|\Pi^o_r v\|_{0, \partial T} \]
\[ \lesssim h_T^{-(d-1)(\frac{1}{2} - \frac{1}{p})} \|v\|_{0, \partial T} \]
\[ \lesssim h_T^{-(d-1)(\frac{1}{2} - \frac{1}{p})} \frac{d-1}{h_T^2} \|\hat{v}\|_{0, \partial T} \]
\[ \lesssim h_T^{-(d-1)(\frac{1}{2} - \frac{1}{p})} \frac{d-1}{h_T^2} \|\hat{v}\|_{0, \mu, \partial T} \]
\[ \lesssim h_T^{-(d-1)(\frac{1}{2} - \frac{1}{p})} \frac{d-1}{h_T^2} h_T^{\frac{1-d}{p}} \|v\|_{0, \mu, \partial T} = \|v\|_{0, \mu, \partial T}, \]
which means (106).

Next we derive (107). We have
\[ \|\Pi^o_r v\|_{0, 0, \partial T} \leq \|\Pi^o_r (v - \Pi^o_r v)\|_{0, 0, \partial T} + \|\Pi^o_r v\|_{0, 0, \partial T} \]
\[ \lesssim \|v - \Pi^o_r v\|_{0, 0, \partial T} + h_T^{-\frac{1}{6}} \|\Pi^o_r v\|_{0, 0, T} \]
\[ \lesssim h_T^{\frac{5}{6} - d/3} |v|_{1, T} + h_T^{-\frac{1}{6}} \|\Pi^o_r v\|_{0, 0, T} \]
\[ \lesssim h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 0, T}), \]
i.e. (107) holds.

Now we turn to show (108). It is easy to see
\[ \|v\|_{0, 0, \partial T} \leq \|v - \Pi^o_r v\|_{0, 0, \partial T} + \|\Pi^o_r v\|_{0, 0, \partial T} \]
\[ \lesssim h_T^{\frac{5}{6} - d/3} |v|_{1, T} + h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 0, T}) \]
\[ \lesssim h_T^{-\frac{1}{6}} (|v|_{1, T} + \|v\|_{0, 0, T}), \]
which yields (108).

Finally, it remains to prove (109). We easily know
\[ \|\Pi^o_r v\|_{0, 0, \partial T} \leq \|\Pi^o_r v - \Pi^o_r v\|_{0, 0, \partial T} + \|\Pi^o_r v\|_{0, 0, \partial T} \]
\[ \lesssim \| \Pi_0^2 v - v \|_{0,6,\partial T} + \| \Pi_0^2 v \|_{0,6,\partial T} \]
\[ \lesssim h_T^{5/6 - d/3} | v |_{1,T} + h_T^{1/6} (| v |_{1,T} + \| v \|_{0,6,T}) \]
\[ \lesssim h_T^{-\frac{1}{6}} (| v |_{1,T} + \| v \|_{0,6,T}), \]

which finishes the proof. \( \square \)

### A.2 RT interpolation

For any nonnegative integer \( r \), we introduce the local Raviart-Thomas (RT) element

\[ RT_r(T) = [\mathcal{P}_r(T)]^d + x \mathcal{P}_r(T). \]

Lemmas A.6-A.8 show some properties of the RT projection [37].

**Lemma A.6.** For any \( v_h \in RT_r(T) \), \( \nabla \cdot v_h |_{T} = 0 \) implies \( v_h \in [\mathcal{P}_r(T)]^d \).

**Lemma A.7.** For any \( T \in \mathcal{T}_h \) and \( v \in [H^1(T)]^d \), there exists a unique \( P_{r}^{RT} v \in RT_r(T) \) such that

\[
\langle P_{r}^{RT} v \cdot n_E, w_r \rangle_E = \langle v \cdot n_E, w_r \rangle_E \quad \forall w_r \in \mathcal{P}_r(E), E \subset \partial T, \tag{111a} \\
\langle P_{r}^{RT} v, w_{r-1} \rangle_T = \langle v, w_{r-1} \rangle_T \quad \forall w_{r-1} \in [\mathcal{P}_r(T)]^d. \tag{111b}
\]

If \( r = 0 \), then \( P_{r}^{RT} v \) is determined only by (111a). Moreover, for any nonnegative integer \( s \), the following interpolation approximation property holds:

\[ \| v - P_{r}^{RT} v \|_{0,T} \lesssim h_T^{s} | v |_{s,T} \quad \forall 1 \leq s \leq r + 1, v \in [H^s(T)]^d. \tag{112} \]

**Lemma A.8.** The operator \( P_{r}^{RT} \) defined in Lemma A.7 satisfies

\[ (\nabla \cdot P_{r}^{RT} v, q_h)_T = (\nabla \cdot v, q_h)_T \quad \forall q_h \in \mathcal{P}_r(T), T \in \mathcal{T}_h. \tag{113} \]

The following result is very useful to our analysis.

**Lemma A.9.** For all \( T \in \mathcal{T}_h \), \( w \in H^1(T) \), and \( \mu \in [1, \infty] \), it holds

\[ \| w \|_{0,\mu,\partial T} \lesssim h_T^{-1/\mu} \| w \|_{0,\mu,T} + h_T^{(\mu-1)/\mu} | w |_{1,\mu,T}. \tag{114} \]

In addition, for all \( w \in \mathcal{P}_k(T) \), it holds

\[ \| w \|_{0,\mu,\partial T} \lesssim h_T^{-1/\mu} \| w \|_{0,T}. \tag{115} \]

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Proof. For any \( w \in H^1(T) \), we use the trace theorem on the reference element \( \hat{T} \) to get

\[
\| w \|_{0, \mu, \partial T}^\mu = \int_{\partial \hat{T}} | \hat{w} |^\mu | \partial \hat{T} | d \hat{s} \lesssim h_T^{d-1} \| \hat{w} \|_{0, \mu, \partial \hat{T}}^\mu \lesssim h_T^{d-1} \| \hat{w} \|_{1, \mu, \hat{T}}^\mu
\]

which indicates (114). The result (115) follows from (114) and the inverse inequality. \( \Box \)

By Lemma A.9 we can get the following approximation properties of \( P_{RT} \).

**Lemma A.10.** For \( s \in \{1, 2, \cdots, r + 1\} \) and \( j \in \{0, 1, \cdots, s\} \), it holds

\[
| v - P_{RT} v |_{j, \mu, T} \lesssim h_T^{s-j-d(\frac{1}{2} - \frac{1}{\mu})} | v |_{s, T} \quad \forall v \in [H^s(T)]^d
\]  

(116)

for \( \mu \) satisfying

\[
\begin{cases}
2 \leq \mu \leq \frac{2d}{d-2(s-j)}, & \text{if } 2(s-j) < d, \\
2 \leq \mu < \infty, & \text{if } 2(s-j) = d, \\
2 \leq \mu \leq \infty, & \text{if } 2(s-j) > d,
\end{cases}
\]

and

\[
\| \nabla^j (v - P_{RT} v) \|_{0, \mu, \partial T} \lesssim h_T^{s-j-d(\frac{1}{2} - \frac{1}{\mu})} | v |_{s, T}, \forall v \in [H^s(T)]^d
\]  

(117)

for \( \mu \) satisfying

\[
\begin{cases}
2 \leq \mu \leq \frac{2(d-1)}{d-2(s-j)}, & \text{if } 2(s-j) < d, \\
2 \leq \mu < \infty, & \text{if } 2(s-j) = d, \\
2 \leq \mu \leq \infty, & \text{if } 2(s-j) > d.
\end{cases}
\]

Proof. By a triangle inequality, an inverse inequality, the approximation properties of \( \Pi_{s-1}^0 \), and the approximation properties of \( P_{RT} \) in (112), we get

\[
| v - P_{RT} v |_{j, \mu, T} \leq | v - \Pi_{s-1}^0 v |_{j, \mu, T} + | \Pi_{s-1}^0 v - P_{RT} v |_{j, \mu, T}
\]

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This completes the proof.

A.3 BDM interpolation

From \[37\] for 2D and \[14\] for 3D.

We recall the definition and properties of the BDM projection \(P^{BDM}_r\) from [37] for 2D and [14] for 3D.

**Lemma A.11.** For any \(T \in T_h\), \(v \in [H^1(T)]^d\), and integer \(r \geq 1\), there exists a unique \(P^{BDM}_r v \in [P_r(T)]^d\) such that

\[
\langle P^{BDM}_r v \cdot n_E, w_r \rangle_E = \langle v \cdot n_E, w_r \rangle_E, \quad \forall w_r \in P_r(E), E \subset \partial T,
\]

\[
(P^{BDM}_r v, \nabla p_{r-1})_T = (v, \nabla p_{r-1})_T, \quad \forall p_{r-1} \in P_{r-1}(T),
\]

\[
(P^{BDM}_r v, \text{curl}(b_T p_{j-2}))_T = (v, \text{curl}(b_T p_{j-2}))_T, \quad \forall p_{j-2} \in P_{j-2}(T),
\]

which yields (116).

It remains to show (117). By the triangle inequality, (115), the approximation properties of \(\Pi^o_{s-1}\), and the approximation property of \(P^{RT}_r\) in (112), we have

\[
\left\| \nabla^j (v - P^{RT}_r v) \right\|_{0,\mu,\partial T} \leq \left\| \nabla^j (v - \Pi^o_{s-1} v) \right\|_{0,\mu,\partial T} + \left\| \nabla^j (\Pi^o_{s-1} v - P^{RT}_r v) \right\|_{0,\mu,\partial T} \\
\lesssim \left\| \nabla^j (v - \Pi^o_{s-1} v) \right\|_{0,\mu,\partial T} + h_T^{-j - \frac{1}{p} - d(\frac{1}{2} - \frac{1}{p})} \left\| \Pi^o_{s-1} v - P^{RT}_r v \right\|_{0,\mu,\partial T} \\
\lesssim \left\| \nabla^j (v - \Pi^o_{s-1} v) \right\|_{0,\mu,\partial T} + h_T^{-j - \frac{1}{p} - d(\frac{1}{2} - \frac{1}{p})} \left\| \Pi^o_{s-1} v - v \right\|_{0,T} \\
\lesssim h_T^{s-j - \frac{1}{p} - d(\frac{1}{2} - \frac{1}{p})} \left\| v \right\|_{s,T}.
\]

This completes the proof. \(\square\)
when \( d = 2,3 \) and \( j \geq 2 \), and
\[
(P_r^{BDM} v, w)_T = (v, w)_T, \quad \forall w \in P_r^*(T), \tag{118d}
\]
when \( d = 3 \), where \( b_T \) in (118c) is the bubble function on \( T \), and
\[
P_r^*(T) := \{ v \in [P_r(T)]^d : \nabla \cdot v = 0, \ v \cdot n_E = 0, \ \forall E \subset \partial T \}.
\]
Moreover, for any integer \( s \) with \( 1 \leq s \leq r + 1 \), and \( v \in [H^r(T)]^d \), it holds the following interpolation approximation properties:
\[
\| v - P_r^{BDM} v \|_{0,T} \lesssim h_T^s |v|_{s,T}, \tag{119}
\]
\[
\| v - P_r^{BDM} v \|_{0,\partial T} \lesssim h_T^{s-\frac{1}{2}} |v|_{s,T}. \tag{120}
\]

A.4 Approximation properties extended to real number

In this section, we will extend the approximation properties in Lemmas A.2, A.3 and A.10 to real index \( s \).

First, we recall the following classical results.

**Theorem A.1** (cf. [2, Page 220, Theorem 7.23], [12, Page 373, Proposition 14.1.5]). Given \( 0 < \theta < 1 \) and \( 1 \leq p \leq \infty \), and given Banach spaces \( A_1 \hookrightarrow A_0, B_1 \hookrightarrow B_0 \). Let \( K \) be a bounded linear operator from \( A_0 + A_1 \) into \( B_0 + B_1 \) having the property that \( K \) is bounded from \( A_i \) into \( B_i \), with norm at most \( M_i, i = 0,1 \); that is
\[
\| K u_i \|_{B_i} \leq M_i \| u_i \|_{A_i} \quad \forall u_i \in A_i, i = 1,2.
\]
Then \( K : A_{\theta,p} \rightarrow B_{\theta,p} \) is a bounded linear operator and
\[
\| K \|_{A_{\theta,p} \rightarrow B_{\theta,p}} \leq \| K \|_{A_0 \rightarrow B_0}^{1-\theta} \| K \|_{A_1 \rightarrow B_1}^\theta,
\]
where \( A_{\theta,p} := [A_0, A_1]_{\theta,p}, B_{\theta,p} := [B_0, B_1]_{\theta,p} \), see [12, Page 372] for detailed definitions.

**Theorem A.2** (cf. [12 Page 375, Theorem 14.2.3]). Let \( 0 < s < 1, 1 \leq p \leq \infty \) and \( \ell \geq 0 \) be an integer, if \( \Omega \) has a Lipschitz boundary, then
\[
[W^{\ell,p}(\Omega), W^{\ell+1,p}(\Omega)]_{s,p} = W^{\ell+s,p}(\Omega).
\]
With the above results, we are ready to prove the following fractional approximation properties of the $L^2$-projection $\Pi_r^o$.

**Lemma A.12.** Let $r$ be a nonnegative integer and $\rho \in [1, \infty]$. For $j \in \{0, 1, \ldots, r+1\}$ and real number $s \in [j, r+1]$, assume that $2 \leq \frac{d\rho}{d-\lfloor s \rfloor \rho}$ when $\lfloor s \rfloor \rho < d$. Then there exists a constant $C$ independent of $T$ such that

$$|v - \Pi_r^o v|_{j,\mu,T} \leq C h_T^{s-j+\frac{d-\rho}{\mu}} \|v\|_{s,\rho,T}, \forall v \in W^{s,\rho}(T)$$

(121)

holds for $\mu$ satisfying

$$\begin{cases} 
1 \leq \mu \leq \frac{d\rho}{d-\lfloor s \rfloor \rho}, & \text{if } \lfloor s \rfloor \rho < d, \\
1 \leq \mu < \infty, & \text{if } \lfloor s \rfloor \rho = d, \\
1 \leq \mu \leq \infty, & \text{if } \lfloor s \rfloor \rho > d.
\end{cases}$$

(122)

where $\lfloor s \rfloor$ stands for the integer part of $s$. In addition, for $j \in \{0, 1, \ldots, r+1\}$ and $s \in [j+1, r+1]$, assume that $2 \leq \frac{d\rho}{d-\lfloor s \rfloor \rho}$ when $\lfloor s \rfloor \rho < d$. Then there exists a constant $C$ independent of $T$ such that

$$|\nabla^j(v - \Pi_r^o v)|_{0,\mu,\partial T} \leq C h_T^{s-j+\frac{d-1}{\mu} - \frac{d}{\rho}} \|v\|_{s,\rho,T}, \forall v \in W^{s,\rho}(T)$$

(123)

holds for $\mu$ satisfying

$$\begin{cases} 
1 \leq \mu \leq \frac{(d-1)\rho}{d-\lfloor s \rfloor \rho}, & \text{if } \lfloor s \rfloor \rho < d, \\
1 \leq \mu < \infty, & \text{if } \lfloor s \rfloor \rho = d, \\
1 \leq \mu \leq \infty, & \text{if } \lfloor s \rfloor \rho > d.
\end{cases}$$

(124)

Proof. We only give a proof for (121). When $s$ is an integer, the result is followed by immediately. Therefore, we assume $m-1 < s < m$, where $m \geq 1$ is an integer, and we take $A_0 = W^{m-1,\rho}(\Omega)$, $A_1 = W^{m,\rho}(\Omega)$, $B_0 = B_1 = W^{m-1,\rho}(\Omega)$, $\theta = s - (m-1)$. From Theorem [A.1] Theorem [A.2] (99) and the fact $\lfloor s \rfloor = m - 1$, we have

$$\frac{\|(Id - \Pi_r^o)v\|_{m-1,\rho,T}}{\|v\|_{s,\rho,T}} \leq \|(Id - \Pi_r^o)\|_{W^{s,\rho}(T) \rightarrow W^{m-1,\rho}(T)}$$

$$\leq \|(Id - \Pi_r^o)\|_{W^{m-1,\rho}(T) \rightarrow W^{m-1,\rho}(T)} \|(Id - \Pi_r^o)\|_{W^{m,\rho}(T) \rightarrow W^{m-1,\rho}(T)}$$

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\[ \lesssim h_T^\theta \]
\[ = h_T^{s-(m-1)}, \]

where \( Id \) is the identity operator. With the above estimate, (99), and the fact \( [s] = m - 1 \), we can get

\[ |v - \Pi_r^0 v|_{j,\mu,T} = |v - \Pi_r^0 v - \Pi_r^0 (v - \Pi_r^0 v)|_{j,\mu,T} \lesssim h_T^{m-1-j+\frac{d}{\rho} - \frac{d}{\rho} |v - \Pi_r^0 v|_{m-1,\rho,T} \lesssim h_T^{s-j+\frac{d}{\rho} - \frac{d}{\rho} \|v\|_{s,\rho,T}} \]

holds for all \( v \in W^{s,\rho}(T) \). Then we finish our proof. \( \square \)

The proofs for the following two lemmas can be done in the same way as above.

**Lemma A.13.** Let \( r \) be a nonnegative integer and \( \rho \in [1, \infty] \). For \( s \in \{1, 2, \ldots, r+1\} \), assume that \( 2 \leq \frac{d\rho}{d-[s]\rho} \) when \( [s]\rho < d \). Then there exists a constant \( C \) independent of \( T \) such that

\[ \|v - \Pi_r^\theta v\|_{0,\mu,0T} \leq Ch_T^{s+\frac{d-1}{\rho} - \frac{d}{\rho} \|v\|_{s,\rho,T}}, \forall v \in W^{s,\rho}(T) \] (125)

holds for \( \mu \) satisfying

\[
\begin{cases}
1 \leq \mu \leq \frac{(d-1)\rho}{d-[s]\rho}, & \text{if } [s]\rho < d, \\
1 \leq \mu < \infty, & \text{if } [s]\rho = d, \\
1 \leq \mu \leq \infty, & \text{if } [s]\rho > d.
\end{cases}
\] (126)

**Lemma A.14.** For any real number \( s \in [1, r+1] \) and integer \( j \in \{0, 1, \ldots, s\} \), it holds

\[ |v - P_r^{HT} v|_{j,\mu,T} \lesssim h_T^{s-j-d(\frac{1}{2} - \frac{1}{\rho})} \|v\|_{s,T}, \forall v \in [H^s(T)]^d \] (127)

for \( \mu \) satisfying

\[
\begin{cases}
2 \leq \mu \leq \frac{2d}{d-2(s-j)}, & \text{if } 2(s-j) < d, \\
2 \leq \mu < \infty, & \text{if } 2(s-j) = d, \\
2 \leq \mu \leq \infty, & \text{if } 2(s-j) > d,
\end{cases}
\]
and

\[ \| \nabla^j (v - P_r^{kT} v) \|_{0, \partial T} \lesssim h^{-j - \frac{1}{2} - d \left( \frac{1}{2} - \frac{1}{2} \right)} \| v \|_{s, T}, \forall v \in [H^s(T)]^d \]  

(128)

for \( \mu \) satisfying

\[
\begin{aligned}
2 \leq \mu &\leq \frac{2(d - 1)}{d - 2([s] - j)}, & \text{if } 2([s] - j) < d, \\
2 \leq \mu < \infty &\text{, if } 2([s] - j) = d, \\
2 \leq \mu &\leq \infty, & \text{if } 2([s] - j) > d.
\end{aligned}
\]

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