1. INTRODUCTION

A differentiable manifold \( M \) is said to be \textit{formal} if the algebra \( \bigwedge^\ast (M) \) of the differential forms on \( M \) is quasi isomorphic to its DeRham cohomology. (We recall that a morphism between Differential Graded Algebras is said to be a \textit{quasi isomorphism} if it induces an isomorphism in cohomology and that two DGA’s are said to be \textit{quasi isomorphic} if they are equivalent with respect to the equivalence relation generated by quasi isomorphisms (cf. also [2]).) It is well known that (cf. [6])

\[
(\bigwedge^\ast (M), \partial) \text{ is formal} \implies \text{from } H^\ast (\bigwedge^\ast (M), \partial) \text{ we can reconstruct (via its minimal model, Postnikov towers etc...)} \text{ the whole rational (i.e. all the cofinite) homotopy theory of } M.
\]

One (actually, almost the only effective) way to get formality is to be able to produce a suitable derivation \( \delta \) on \( \bigwedge^\ast (M) \), \( \delta : \bigwedge^k \to \bigwedge^{k+1} \) (for \( k = 0, ..., n \)), satisfying \( \delta^2 = 0 \) and such that \textit{d\delta-lemma} holds, i.e. \( (\ker \partial \cap \ker \delta) \cap (\im \partial + \im \delta) = \im \partial \delta \).

More precisely, the following general statements holds:

**Theorem 1.1** (cf. [6]). Let \( M \) be a smooth manifold with a derivation \( \delta : \bigwedge^k \to \bigwedge^{k+1} \) (for \( k = 0, ..., n \)), satisfying \( \delta^2 = 0 \) such that \textit{d\delta-lemma} holds. Then

\[
H(\bigwedge^\ast (M), \partial) = (\ker \partial \cap \ker \delta)/\im \partial \delta
\]

and so \( (\bigwedge^\ast (M), \partial) \) and \( (\bigwedge^\ast (M), \delta) \) are formal.

An example of such a situation is provided by Kähler manifolds: in this case, \( \delta = d^c := J^{-1} dJ \), where \( J \) is the complex structure (cf. again [6]).

We first show (Lemma 2.1, Remark 2.2) that the derivation \( \delta \) satisfying properties above must be of the form \( \delta = d_R := R \partial R^{-1} \), with \( R \in \text{End}(TM) \) (i.e., \( R \) is a field of non degenerate linear transformations of the tangent spaces).

Then, we prove (Lemma 2.3) that the supercommutation of \( \partial \) and \( \delta = d_R \) (which is a natural, essentially necessary condition to get a \textit{d\delta-lemma}) amounts to \( N_R \equiv 0 \), \( N_R \) being the Nijenhuis tensor of \( R \). Then, we are looking for sufficient conditions that ensure the \( \partial \partial R \)-lemma holds. For \( R \) self adjoint with respect to a Riemannian metric, it is done in Section 3. For \( R \) compatible with an almost symplectic structure this is done in Section 4. Finally, we show that, if \( t^R = -R \) and \( \det R \equiv 1 \), then

\[
N_R \equiv 0 \implies N_J \equiv 0
\]

where \( J \) is the orthogonal component of \( R \), in its polar decomposition and this also provides a new characterization of Kähler structures.

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2. Preliminary remarks: on the space of derivations

We begin with the following

**Lemma 2.1.** Let $M$ be a smooth compact manifold of dimension $n$ and let $\delta \in \text{End}(\wedge^*(M))$ such that:

a. $\delta(\alpha \wedge \beta) = \delta \alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \delta \beta$ and $|\delta| = 1$ , i.e. $\delta : \wedge^p(M) \rightarrow \wedge^{p+1}(M)$  

b. $\text{Ker} \delta \cap \wedge^0(M) = \mathbb{R}$

c. $\delta^2 = 0$;  

then $R : X \mapsto \delta_X$ belongs to $\text{End}(TM)$ and $\delta = d_R := RdR^{-1}$ (where, clearly, $\delta_X : f \mapsto \delta f(X)$)

**Proof.** The linearity of $R$ is evident. By (b), $R$ is nondegenerate. In order to prove $\delta = d_R$, let us note that for any $f \in \wedge^0(M)$, $X \in TM$, we have:

$$d_R f(X) = df(RX) = RXf = \delta_X f = (\delta f)(X)$$

i.e. $\delta$ coincides with $d_R$ on $\wedge^0(M)$; this, together with (a), (c), is sufficient to insure $\delta \equiv d_R$. □

**Remark 2.2.** Assume $\delta$ satisfies (a), (c) of lemma 2.1. Suppose

• the $d \delta$-lemma holds, i.e.:

$$(\text{Ker} d \cap \wedge^0(M)) \cap (\text{Im} d + \text{Im} \delta) = \text{Im} d \delta$$

• $d \delta + \delta d = 0$.

Then also (b) of lemma 2.1 is fulfilled.

**Proof.** Indeed, if $f \in \text{Ker} \delta \cap \wedge^0(M)$, $f \neq \text{const}$, then

$$0 \neq df \in (\text{Ker} d \cap \text{Ker} \delta) \cap (\text{Im} d + \text{Im} \delta),$$

contradicting $df \notin \text{Im} d \delta$. □

For any $S \in \text{End}(TM)$, we define the *Nijenhuis tensor* of $S$ as the element

$$N_S \in \wedge^2(M) \otimes TM$$

given by

$$N_S(X, Y) := [SX, SY] + S^2[X, Y] - S[SX, Y] - S[X, SY];$$

It is known (and follows direct from definitions), that

• $N_{1+S} = N_S$ (where $I : TM \rightarrow TM$ is the identity)

• for any $\lambda \in C^\infty(M, \mathbb{R})$, $N_{\lambda I} = 0$

• if $R \in \text{End}(TM)$ then

$$N_{R^{-1}}(X, Y) = R^{-2}N_R(R^{-1}X, R^{-1}Y).$$

Let $V$ be a vector space. For any $L \in \text{End}(V)$ we consider

$$\tau(L) \in \text{End}(\wedge V^*)$$

defined as follows:

$$\tau(L)(\alpha)(v_1, ..., v_p) := \sum_{h=1}^p \alpha(v_1, ..., L(v_h), ..., v_p).$$

We recall that a *Differential Graded Lie Algebra* (DGLA) is a graded vector space

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ and a degree one graded derivation $d$ on $\mathfrak{g}$ in such a way that:

• $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$
• for homogeneous elements $a$, $b$, $c$, we have:

$$
[a, b] = -(-1)^{|a||b|}[b, a]
$$

$$
[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]] \quad \text{(Jacobi identity)}
$$

$$
d[a, b] = [da, b] + (-1)^{|a|}[a, db].
$$

• $d^2 = 0$

For example, there is a natural structure of DGLA on $\text{End}(\wedge^*(M))$:

the grading is obvious: $|P| = |P\alpha| - |\alpha|$, and the bracket $[,]$ and the derivation (we use the letter $\nabla$ for it) are given by

- $[P, Q] := P \circ Q - (-1)^{|P||Q|}Q \circ P$,
- $\nabla P := [d, P]$.

Let us recall the lemma (cf. e.g. [4]),

**Lemma 2.3.** Let $R \in \text{Aut}(TM)$. Then,

$$
d_R = d + [\tau(S), d] - r(R),
$$

where $\tau$ is defined by $[I, S := R - I$, and $r(R)$ is a zero order differential operator quadratic in $S$ defined as follows:

- $r(R) \equiv 0$ on $\wedge^0(M)$
- for $\alpha \in \wedge^1(M)$ $r(R)(\alpha)(X, Y)) := \alpha(R^{-1}N_R(X, Y))$
- extend to general $\alpha$ as skew-symmetric derivation.

**Proof.** It is enough to prove the lemma for $f \in \wedge^0(M)$ and for $\alpha \in \wedge^1(M)$. If $f \in \wedge^0(M)$, we have:

$$
d_R f(X) = df((I + S)X) = df(X) + ([\tau(S), d])f(X);
$$

if $\alpha \in \wedge^1(M)$ we have first:

$$
[[\tau(S), d]\alpha](X, Y) = SX\alpha(Y) - SX\alpha(X) +
\alpha(S[X, Y] - [SX, Y] - [X, SY]);
$$

then:

$$
(d_R\alpha)(X, Y) = (dR^{-1}\alpha)((I + S)X, (I + S)Y) =
$$

$$
(I + S)X\alpha(Y) - (I + S)Y\alpha(X) - \alpha((I + S)^{-1}[(I + S)X, (I + S)Y])
$$

$$
= X\alpha(Y) + SX\alpha(Y) - SY\alpha(X) - \alpha((I + S)^{-1}[(I + S)X, (I + S)Y])
$$

$$
d\alpha(X, Y) + \alpha([X, Y]) + ([\tau(S), d]\alpha)(X, Y) +
\alpha(S[X, Y] - [SX, Y] - [X, SY]) - \alpha((I + S)^{-1}[(I + S)X, (I + S)Y]) =
$$

$$
d\alpha(X, Y) + ([\tau(S), d]\alpha)(X, Y) +
\alpha((I + S)^{-1}[(I + S)[X, Y] - (I + S)S[X, Y] + (I + S)[SX, Y] + (I + S)[X, SY])
$$

$$
- \alpha((I + S)^{-1}[(I + S)X, (I + S)Y]) =
$$

$$
d\alpha(X, Y) + ([\tau(S), d]\alpha)(X, Y) - \alpha((I + S)^{-1}N_S(X, Y)
$$

\[\square\]

We will need the following

**Lemma 2.4.**

$$
[d_R, d] = 0 \iff N_R = 0 \iff d_R = d + [\tau(S), d] .
$$

**Proof.** Let us first show that $d$ commutes with $d + [\tau(S), d]$. Since $d^2 = 0$, $d$ commutes with itself. In order to show that $d$ commutes with $[\tau(S), d] = 0$, we use the Jacobi identity:

$$
[d, [\tau(S), d]] = [d, \tau(S)], d] = -[d, [\tau(S), d]] \implies [d, [\tau(S), d]] = 0.
$$

(In particular, the above observation shows the “$\implies$” direction of the lemma)
In order to show that $d$ commutes with $r(R)$ if and only if $N_R = 0$, we use that, for every $f \in \Lambda^0(M)$, $X, Y \in TM$, we have,

\[ [d, dR]f(X, Y) = (r(R)df)(X, Y) = df(R^{-1}N_R(X, Y)). \]

Clearly, the right hand side vanishes for all $X, Y$ if and only if $N_R = 0$.

\[ \square \]

**Remark 2.5.** The previous lemma says that, in $(\text{End}(\Lambda^\ast(M)), [\ , \ ], \nabla)$,

\[ \nabla d_R = 0 \iff N_R = 0 \iff d - d_R = \nabla \tau(S) \text{ i.e. } <d> = <d_R>. \]

Note also that:

1. $(\nabla d_R + \frac{1}{2}[d_R, d_R]) = 0$.

\[ \text{3. } dd_R\text{-lemma in the presence of a Riemannian metric} \]

Let $g$ be a Riemannian metric on $M$. We denote by $\ast$ the Hodge-star operation. The next two lemmas says that when certain (natural) conditions on $R$ are fulfilled, then the $dd_R$-lemma holds.

**Lemma 3.1.** Let $R \in Aut(TM)$ such that:

1. $N_R = 0$ (Lemma $[\ , \ ]d_R = d + [\tau(S), d]$ Lemma $[\ , \ ]d, d_R] = 0$)
2. There exists a Riemannian metric $g$ on $M$ such that
   a. $[d_R, d^{\ast}] = 0$
   b. $[d_R, d^{\ast}d] = 0$.

Then,

\[ \text{Ker } d \cap \text{Im } d_R = \text{Im } dd_R. \]

**Proof.** Set

\[ \Delta_R := [d_R, d^{\ast}_R]. \]

Clearly,

\[ [\Delta_R, d] = 0 = [\Delta, d_R]. \]

Note that $\Delta_R = R\tilde{\Delta} R^{-1}$, where $\tilde{\Delta}$ is the Laplacian operator with respect to $\tilde{g} = g(R\cdot, R\cdot)$. Consider the Hodge decomposition with respect to $\Delta$ and $\Delta_R$:

\[ I = H + \Delta G, \quad I = H_R + \Delta_R G_R. \]

Given $\alpha \in \text{Ker } d \cap \text{Im } d_R$ we have:

\[ \alpha = H(\alpha) + dd^\ast G(\alpha) \]
\[ \alpha = d_Rd^\ast_R G_R(\alpha). \]

Set $\gamma = d^\ast_R G_R(\alpha)$. Then,

\[ \gamma = H(\gamma) + dd^\ast G(\gamma) + d^\ast dG(\gamma) \]

and so

\[ d_R\gamma = d_RH(\gamma) + dd^\ast G(d_R\gamma), \]

i.e., $H(d_R\gamma) = d_RH(\gamma) + dd^\ast G(d_R\gamma)$.

and so:

\[ \alpha = dd^\ast G(\alpha) = d_Rd^\ast_R G_R(\alpha). \]

and finally

\[ \alpha = dd^\ast G(\alpha) = d_Rd^\ast_R G_R(dd^\ast G(\alpha)) = dd_R G_R(d^\ast d^\ast G(\alpha)), \]

\[ \square \]

**Corollary 3.2.** Let $R \in \text{End}(TM)$ such that:

1. $N_R = 0$ (and so $d_R = d + [\tau(S), d]$ and $[d, d_R] = 0$)
2. There exists a Riemannian metric $g$ on $M$ such that
a. \([d_R, dd^*] = 0\)
b. \([d_R, d^*d] = 0\)
c. \([d, d_Rd_R^*] = 0\)
d. \([d, d_R^*d_R] = 0\);
then the \(dd_R\)-lemma holds, i.e.
\[(\text{Ker } d \cap \text{Ker } d_R) \cap (\text{Im } d + \text{Im } d_R) = \text{Im } dd_R\]

4. \(dd_R\)-lemma in the almost symplectic setting

Let \((M, \kappa)\) be an almost symplectic, \(2n\)-dimensional compact manifold. We consider
\[\mathcal{M}_\kappa(M) := \left\{ g \in \text{Riem}(M) \mid d\mu(g) = \frac{\kappa^n}{n!} \right\}.\]

Recall (cf. [2]) that we can define the symplectic analog of the Hodge star
\[\star : \wedge^r(M) \rightarrow \wedge^{2n-r}(M)\]
by means of the relation
\[\alpha \wedge \star \beta = \kappa(\alpha, \beta) \frac{\kappa^n}{n!}\]
for \(\alpha, \beta \in \wedge^r(M)\).

Analog to the Riemannian case, we consider on \(\wedge^r(M)\)
\[d^\star := (-1)^r \star d \star.\]

For any \(g \in \mathcal{M}_\kappa(M)\)
there exists \(R \in \text{End}(TM)\) such that
\[g(X, Y) = \kappa(R^{-1}X, Y).\]
Clearly, \(R\) and \(R^{-1}\) are \(g\)-antisymmetric and \(det R \equiv 1\);
on \(\wedge^r(M)\) we have (cf. [3]):
\[\star R^{-1} = (-1)^r \star \text{i.e. } * R = (-1)^r \star \text{ and } R^{-1} \star = \star;\]
consequently:
\[d^\star_R = -R^{-1} \star d \star R = -d^\star.\]

We have the following

Lemma 4.1. Assume
- \(N_R \equiv 0\)
- \(d_R[d, d^\star] = 0 = [d, d^\star]d_R\)
- \(d_{R^{-1}}[d, d^\star] = 0 = [d, d^\star]d_{R^{-1}}\)
then, the \(dd_R\)-lemma holds.

Proof. We have:
\[\left[d, d_Rd_R^*\right] = dd_Rd_R^* - d_Rdd_R^* d = -d_R[d, d_R^*] = d_R[d, d^\star]\]
and, similarly:
\[\left[d, d_R^*d_R\right] = -[d, d^\star]d_R;\]
finally, we have:
\[\left[d_R, dd^*\right] = -d[d^*, d_R]\]
and
\[R^{-1}d[d^*, d_R]R = \pm d_{R^{-1}}[d^*, d].\]
Repeating this procedure with the other relation and applying Lemma 4.2 we obtain that \(dd_R\)-lemma holds. \(\square\)

Remark 4.2. If \(\kappa\) defines a symplectic structure, i.e. \(d\kappa = 0\), then \([d, d^\star] = 0\) (cf. e.g. [3]),
and so we only need \(N_R \equiv 0\):
5. Relaxing the condition $J^2 = -I$ in the definition of Kähler manifold.

One of the equivalent definitions of the Kähler manifold is the following one: A Kähler manifold is a symplectic manifold $(M, \kappa)$ equipped with $J \in \text{End}(TM)$ such that the bilinear form $g$ defined by the equality $g(X, Y) := \kappa(X, JY)$ is a Riemannian metric and such that

\[ I \quad J^2 = -I \quad 0 \]

\[ II \quad N_J = 0. \]

A lot of papers study the consequences of relaxing the second condition $N_J = 0$. In this case, the structure $J$ is called an almost complex structure, and many papers are dedicated to almost complex structures satisfying additional conditions, see for example [5].

What about relaxing the first condition?

**Theorem 5.1.** Let $(M, \kappa)$ be an almost symplectic, $2n$-dimensional connected manifold; let again

\[
M_\kappa(M) := \left\{ g \in \text{Riem}(M) \mid d\mu(g) = \frac{\kappa^n}{n!} \right\}.
\]

Assume there exists $g \in M_\kappa(M)$ such that, representing $g$ via $\kappa$ by $R \in \text{End}(TM)$, i.e. for $R$ satisfying

\[ g(X, Y) = \kappa(RX, Y), \]

we have

\[ N_R = 0. \]

Then, the orthogonal component $J$ of $R$ in its $g$-polar decomposition is $g$–skew-symmetric and satisfies

\[ N_J = 0. \]

Moreover, if $d\kappa = 0$, then $(M, g, J)$ is a Kähler manifold.

**Proof.** The proof is organized as follows: we will first show that the orthogonal component $J$ of $R$ in its $g$-polar decomposition is actually a polynomial of $R$ (we will also see that the polynomial is real and odd). The property $N_J = 0$ will then follow from $N_R = 0$ by [5]. The closedness of the form $g(J\cdot, \cdot)$ will require certain additional work.

We consider $-R^2 := -R \circ R$. It is clearly self adjoint and positively definite with respect to $g$; by (2) we have $\det(R^2) = \text{const}$. Then, it is semi-simple, and all its eigenvalues are positive by linear algebra.

We denote by $m(x)$ the number of different eigenvalues of $-R^2$ at $x \in M$ and by $\lambda_1(x)^2 > \ldots > \lambda_{m(x)}(x)^2 \ (\lambda_j > 0 \ , \ 1 \leq j \leq m(x))$ the eigenvalues of $-R^2$ at $x \in M$.

We say that a point $x \in M$ is stable if $m(x)$ is constant in a neighborhood of $x$. By [3] Lemma 4], the set of stable points is open and everywhere dense on $M$. Later, we will even show that all points are stable. We shall first work near a stable point $x$.

By [5] Lemma 6], the Nijenhuis tensor $N_{-R^2} = 0$. By [7], in the neighborhood of $x$ there exists a coordinate system $\bar{x} = (x_1^1, \ldots, x_1^{2k_1}), \ldots, \bar{x}_m = (x_m^1, \ldots, x_m^{2k_m})$ such that in this coordinate system the matrix of $-R^2$ is block diagonal, the dimensions of the blocks are $2k_1, \ldots, 2k_m$, and such that the $j$th block is $\lambda_j^2$ times the identity $2k_j \times 2k_j$-matrix:

\[
- R^2 = \begin{pmatrix}
\lambda_1^2 \cdot I_{2k_1} & & \\
& \ddots & \\
& & \lambda_m^2 \cdot I_{2k_m}
\end{pmatrix}.
\]

Moreover, the function $\lambda_j$ does not depend on the variables $x_i^j$ for $j \neq i$.

This in particular implies that all eigenvalues $\lambda_j$ are actually constant: indeed, from (3) we know that the determinant of $-R^2$ is the product $(\lambda_1)^{2k_1} \cdot \ldots \cdot (\lambda_m)^{2k_m}$. By assumption, the determinant is constant. Since the functions $\lambda_j$ depend on its own variables, all functions $\lambda_i$ must be constant. Then, all points must be stable as we claimed before.
Remark 5.2. For further use let us note that, since the eigenspaces of $R$ corresponding to different eigenvalues are orthogonal, in these coordinates the matrix of $g$ is also block-diagonal with the same as in (3) dimensions of the blocks; by construction, the components of $R$ are also orthogonal with the same dimensions of the blocks

$$
\begin{pmatrix}
g_1 & & \\
& \ddots & \\
g_m & & 
\end{pmatrix}, \quad R = 
\begin{pmatrix}
R_1 & & \\
& \ddots & \\
& & R_m
\end{pmatrix}.
$$

Let us now cook with the help of $R$ the field of endomorphisms $J$ such that it is the orthogonal component $R$ in its $g$–polar decomposition. We take the polynomial $P(X) = a_{2m-1}X^{2m-1} + \ldots + a_0$ of degree at most $2m - 1$ such that its value at the points $X = i\lambda_1, ..., i\lambda_m$ is equal to $i$ and such that its value at the points $X = -i\lambda_1, ..., -i\lambda_m$ is equal to $-i$. From general theory it follows that such polynomial is unique (since the values in $2m$ points determine a unique polynomial of degree $2m - 1$, see [1 §2 Ch. 1]). Since $P(\bar{X}) = P(X)$ for $2m$ points $X = \pm i\lambda_1, ..., \pm i\lambda_m$, the coefficients of the polynomial are real. Since $P(-X) = -P(X)$ for $2m$ points $X = \pm i\lambda_1, ..., \pm i\lambda_m$, the polynomial is odd (i.e., all terms of even degree are zero).

We would like to point out that, since $\lambda_i$ are constant, the coefficients of the polynomial are constant.

We now consider $J := P(R) = a_{2m-1}R^{2m-1} + \ldots + a_1 R$ (we understand $R^r$ as $R \circ R \circ \ldots \circ R$). Let us show that $J$ is indeed the orthogonal component of $R$ in its $g$–polar decomposition.

Evidently, the eigenvalues of $J$ are $P(\pm i\lambda_i) = \pm i$, and the algebraic multiplicity of each eigenvalue coincides with its geometric multiplicity. Then, $J^2 = -I$.

Now, since the polynomial $P$ is even, the bilinear form $g(J\cdot, \cdot)$ is skew-symmetric. Indeed, all terms of the polynomial of even degree are zero, and for every term of odd degree we have

$$
g(a_{2\ell-1}R^{2\ell-1}(U), V) = -g(a_{2\ell-1}R^{2\ell-2}(U), R(V)) = g(a_{2\ell-1}R^{2\ell-3}(U), R^2(V)) = \ldots = -g(U, a_{2\ell-1}R^{2\ell-1}(V))
$$

(each time we transport one $R$ to the right hand side we change the sign; all together we make odd number the sign change). Then, each term $g(a_{2\ell-1}R^{2\ell-1}, \cdot)$ is skew-symmetric implying $g(J\cdot, \cdot)$ is skew-symmetric as well.

Then, $J$ is a $g$–orthogonal operator. Indeed,

$$
g(JV, JU) = -g(JU, V) = g(U, V) = g(V, U).
$$

Now, the operator $R \cdot J = R \cdot P(R)$ is $g$–symmetric (implying $R = SJ$ for a certain $g$–symmetric operator $S$). Indeed, arguing as above, we have

$$
g(a_{2\ell-1}R^{2\ell}(U), V) = -g(a_{2\ell-1}R^{2\ell-1}(U), R(V)) = g(a_{2\ell-1}R^{2\ell-3}(U), R^2(V)) = \ldots = g(U, a_{2\ell-1}R^{2\ell}(V))
$$

(this time we transport $2\ell$ $R$’s from left to right, so we change the sign even number of times). Finally, $J = P(R)$ satisfies the following properties:

- It is $g$–orthogonal,
- $R = SJ$ for a certain $g$–symmetric operator.

Thus, $J$ is the orthogonal component of $R$ in its $g$–polar decomposition.

Our goal is to show that $(g, J)$ is a Kähler structure on $M$ provided $\kappa$ is closed. We already have seen that $J$ is $g$–skew-symmetric. The property $N_j \equiv 0$ follows from [5 Lemma 6].

Let us now prove prove that the form $g(J\cdot, \cdot)$ is also closed. We will work locally, in a coordinate system $\bar{x}$ constructed above. Combining these with the form (3) of $-R^2$, we obtain that the matrix of $J$ is given by

$$
J = 
\begin{pmatrix}
\frac{1}{\lambda_1}R_1 & & \\
& \ddots & \\
& & \frac{1}{\lambda_m}R_m
\end{pmatrix}
$$

Combining (3) and (4) we see that the matrix of $\kappa(\cdot, \cdot) := g(R\cdot, \cdot)$ (in the coordinate system $\bar{x}$ above) is given by the matrix

$$
\begin{pmatrix}
\frac{1}{\lambda_1}R_1 & & \\
& \ddots & \\
& & \frac{1}{\lambda_m}R_m
\end{pmatrix}
$$
Then, by (6), the matrix of $g(J, \cdot)$ is

\[
\begin{pmatrix}
-Jg_1 \\
\vdots \\
-Jg_m
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{\lambda_1} R_1 g_1 \\
\vdots \\
-\frac{1}{\lambda_m} R_m g_m
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\lambda_1} \kappa \\
\vdots \\
\frac{1}{\lambda_m} \kappa
\end{pmatrix}.
\]

In what follows we will use the convention

\[\bar{x} = (\bar{x}_1 = (x_1^1, \ldots, x_{2k_1}^1), \ldots, \bar{x}_m = (x_1^m, \ldots, x_{2k_m}^m)) = (y_1^1, \ldots, y_{2n}^2),\]

i.e., $y^1 := x_1^1, \ldots, y_{2k_1}^1 := x_{2k_1}^1, \ldots, y_{2n}^2 := x_{2k_m}^m$.

Now we use that the differential of the form $\kappa$ is given by

\[
d\left( \sum_{p,q=1}^{2n} \kappa_{pq} dy^p \wedge dy^q \right) = \sum_{p,q,s=1}^{2n} \left( \frac{\partial}{\partial y^p} \kappa_{pq} \right) dy^p \wedge dy^q \wedge dy^s.
\]

If the matrix of the form $\kappa$ is as in (6), i.e., if

\[
\kappa = \sum_{\alpha, \beta = 1}^{2k_1} \kappa_{\alpha \beta} dx^\alpha_1 \wedge dx^\beta_1 + \cdots + \sum_{\alpha, \beta = 1}^{m} \kappa_{\alpha \beta} dx^\alpha_m \wedge dx^\beta_m,
\]

then, the differential of $\kappa$ is

\[
d\kappa = d \frac{1}{\lambda_1} \kappa + \cdots + d \frac{m}{\lambda_m} \kappa = \sum_{i=1}^{m} \left( \sum_{p=1}^{2n} \sum_{\alpha, \beta = 1}^{2k_i} \left( \frac{\partial}{\partial y^p} \kappa_{\alpha \beta} \right) \right) dy^p \wedge dx^\alpha_i \wedge dx^\beta_i.
\]

We see that the components of the differentials of $d \frac{i}{\lambda_i} \kappa$ and $d \frac{m}{\lambda_m} \kappa$ do not combine for $i \neq j$. Indeed, every component of $d \frac{i}{\lambda_i} \kappa$ is proportional to a certain $dy^p \wedge dx^\alpha_i \wedge dx^\beta_i$, and every component of $d \frac{j}{\lambda_j} \kappa$ is proportional to a certain $dy^p \wedge dx^\alpha_j \wedge dx^\beta_j$. Then, $d \kappa = 0$ implies $d \frac{i}{\lambda_i} \kappa = 0$ for all $i$.

Now, by (7), the form $g(J, \cdot)$ is given by

\[
g := \kappa R \cdot \cdot
\]

Since $\lambda_i$ are constants as we explained above, and $d \frac{i}{\lambda_i} \kappa = 0$, then the differential of (8) vanishes. Thus, $g(J, \cdot)$ is closed as we claim. Theorem 5.4 is proved.

**Definition 5.3.** Let $(M, \kappa)$ be a $2n$-dimensional (compact) symplectic manifold; $R \in End(TM)$ is said to be $\kappa$-calibrated if

\[
g := \kappa(R \cdot, \cdot)
\]

is a Riemannian metric such that $d\mu(g) = \frac{\kappa^n}{n!}$.

From Theorem 5.4, we immediately obtain

**Corollary 5.4.** Let $(M, \kappa)$ be a $2n$-dimensional connected symplectic manifold. Then the following statements are equivalent:

- $(M, \kappa)$ admits a Kähler structure $g, J$ such that $\kappa(\cdot, \cdot) = g(J(\cdot, \cdot)$.
there exists $R \in \text{End}(TM)$ such that it is $\kappa$-calibrated and such that $N_R \equiv 0$.

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