4\textsuperscript{th} order similarity renormalization of a model hamiltonian

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Abstract

We study the similarity renormalization scheme for hamiltonians to the fourth order in perturbation theory using a model hamiltonian for fermions coupled to bosons. We demonstrate that the free finite parts of counterterms can be chosen in such a way that the \( T \)-matrix is covariant up to the fourth order and the eigenvalue equation for the physical fermion reduces to the Dirac equation. Through this choice, the systematic renormalization scheme reproduces the model solution originally proposed by Glazek and Perry.

1 Introduction

Our study of the similarity renormalization scheme for hamiltonians is done in the model which consists of only two sectors in the Fock space. This great simplification of the space of states allows complete analysis of the renormalization scheme and still includes typical factors and divergences that appear in quantum field theory. Our model is based on Yukawa theory.

The hamiltonian of Yukawa theory truncated to one fermion and one fermion plus one boson Fock sectors leads to infinities in the fermion-boson \( T \)-matrix. Therefore, we introduce a cutoff \( \Lambda \) for the momentum transfer in the interaction part of the hamiltonian. The similarity transformation allows us to construct counterterms in the initial hamiltonian in such a way that the renormalized hamiltonian gives finite and cutoff independent results for the \( T \)-matrix. We construct renormalized hamiltonians using expansion in powers of the effective fermion-boson coupling constant and including terms up to the fourth order.

In the similarity renormalization scheme, one constructs effective hamiltonians \( H_\lambda \) which are functions of the width \( \lambda \). \( H_\lambda \) is obtained from the initial hamiltonian \( H_\Lambda \) with the imposed cutoff \( \Lambda \) and added counterterms by a unitary transformation. The transformation and counterterms are found order by order in perturbation theory using the requirement that matrix elements of \( H_\lambda \) are independent of the cutoff \( \Lambda \) when the cutoff goes to infinity.

To find the unknown finite parts of the counterterms we calculate the \( T \)-matrix for fermion - boson scattering. The condition that the \( T \)-matrix is covariant can be satisfied and it implies
relations between the finite parts of different counterterms. We also demand, that the physical fermion is described by the Dirac equation with the fermion mass equal to the fermion mass term in the fermion-boson sector. This demand also provides a relation between the finite parts of counterterms and it is called the threshold condition \[2\].

The model hamiltonian we study was originally considered by G\l{}azek and Perry \[3\]. They guessed the form of counterterms which remove divergences in $T$-matrix and they obtained covariant results for the $T$-matrix to all orders.

Our main question about the model was if the systematic similarity calculation, carried out in perturbation theory, produces the same solution to the hamiltonian renormalization problem as guessed by G\l{}azek and Perry. The cutoff in the model is limited by the triviality bound \[3\] but one can assume that the coupling constant is small enough for reliable use of the perturbation theory.

Section 2 presents the model. Sections 3 and 4 describe its renormalization to the fourth order. Section 5 explains connection with Ref. \[3\] We conclude in Section 6 and Appendix contains key details.

## 2 Model

The initial hamiltonian is a light-front hamiltonian for Yukawa theory projected on two Fock-space sectors: one with a fermion and one with a fermion and a boson.

$$H_\Lambda = H_{0f} + H_{0fb} + H_Y + H_+ + X_\Lambda,$$

where the free part is

$$H_{0f} = \sum_\sigma \int [p]|p\sigma\rangle\langle p\sigma| \frac{p^2 + m^2}{p^+},$$

$$H_{0fb} = \sum_\sigma \int [p,k]|p\sigma,k\rangle\langle p\sigma,k| \left(\frac{p^2 + m^2}{p^+} + \frac{k^2 + \mu^2}{k^+}\right),$$

boson creation and annihilation vertices are

$$H_Y = g \sum_{\sigma_1,\sigma_2} \int [p_1,p_2,k] \theta(\Lambda^2 - M_{p_2,k}^2) 2(2\pi)^3 \delta^3(p_1 - p_2 - k) \times$$

$$\times [p_2\sigma_2,k]\langle p_1\sigma_1|\bar{u}(p_2,\sigma_2)u(p_1,\sigma_1) + h.c.\rangle = H_{>} + H_{<},$$

and the seagull term is

$$H_+ = g^2 \sum_{\sigma_1,\sigma_2} \int [p_1,p_2,k_1,k_2] \theta(\Lambda^2 - M_1^2)\theta(\Lambda^2 - M_2^2) 2(2\pi)^3 \delta^3(p_2 + k_2 - p_1 - k_1) \times$$

$$\times [p_2\sigma_2,q_2]\langle p_1\sigma_1,k_1|\bar{u}(p_2,\sigma_2)\frac{\gamma^+}{2(p_1^+ + k_1^+)} u(p_1,\sigma_1).$$
$X_\Lambda$ in Eq.(1) is an unknown counterterm. We have introduced cutoffs on the invariant mass $\mathcal{M}^2 = (p + k)^2$ of the two particle sector in the interaction parts of the hamiltonian, $H_Y$ and $H_+$ (see also [4]). The integration measure is

$$[p] = \frac{d^2p^+ dp^+}{2(2\pi)^3 p^+},$$

$$|p\rangle = a_p^\dagger |0\rangle,$$

and $\delta^3(p) = \delta^2(p^\perp) \delta^1(p^+).$ Standard light-front commutation relations are

$$[a_p, a_k^\dagger] = 2(2\pi)^3 p^+ \delta^3(p - k).$$

### 3 Renormalization

The similarity transformation $S_\lambda$ transforms $H_\Lambda$ to a band-diagonal hamiltonian $H_\lambda$,

$$H_\lambda = S_\lambda^\dagger H_\Lambda S_\lambda.$$

Expressions for $S_\lambda$ and $H_\lambda$ are found in perturbation theory [4]. $X_\Lambda$ in $H_\Lambda$ is fitted order by order in $g$, so that $H_\lambda$ does not have $\Lambda$ dependent (i.e. divergent) matrix elements for $\Lambda \to \infty$. This can be guaranteed in any finite order in perturbation theory.

In the second order the transformation gives:

$$H_2^\lambda = f_\lambda \left( H_+ + X_2 - \frac{1}{2} \left[ (1 - f_\lambda) H_Y, (1 + f_\lambda) H_Y \right] \right).$$

The underlining denotes the energy denominator and $f_\lambda$ is the diagonal proximum operator (see Appendix).

In the fermion-boson – fermion-boson sector Eq. (10) reads

$$H^\lambda_{f\bar{f} - f\bar{f}} = f_\lambda \left( H_+ - \frac{1}{2} (1 - f_\lambda) H_\geq (1 + f_\lambda) H_\leq + \frac{1}{2} (1 + f_\lambda) H_\geq (1 - f_\lambda) H_\leq \right).$$

This expression is not divergent for $\Lambda \to \infty$, thus no counterterm is needed in this sector. However, in the fermion-fermion sector, one obtains

$$H^\lambda_{f\bar{f} - f\bar{f}} = - (1 - f_\lambda) H_\leq H_\geq + X_{2\Lambda}.$$

The loop integration in the first term is linearly divergent. The form of this divergence dictates the form of the second order counterterm. Explicitly, one has to choose

$$X_{2\Lambda} = \sum_\sigma \int [p] [p\sigma] \langle p\sigma | \frac{1}{p^+} \frac{g^2}{16\pi^2} \left[ \frac{1}{2} \Lambda^2 + (3m^2 - \mu^2) \log \frac{\Lambda^2}{m^2} + A \right],$$

where $A$ is an undetermined constant.
Higher order calculations lead to the following expressions for $X_{3\Lambda}$ and $X_{4f_b-f_b\Lambda}$:

\[
X_{3\Lambda} = X_{3Y} + X_{3+} = \frac{1}{4} g^2 \frac{\Lambda^2}{16\pi^2} \log \frac{\Lambda^2}{C} H_Y +
\]
\[
+ \sum_{\sigma_1, \sigma_2} \int \frac{\theta(\Lambda^2 - M^2)}{2(2\pi)^3} \delta^3(p_1 - p_2 - k) \times
\]
\[
\times \frac{3}{2} g^2 \frac{\Lambda^2}{16\pi^2} \log \frac{\Lambda^2}{D} \left[ p_2 \sigma_2, k \right] \langle p_1 \sigma_1 \mid \bar{u}(p_2, \sigma_2) \frac{\gamma^+}{2p_1^+} u(p_1, \sigma_1) + h.c. \right],
\]

\[
X_{4f_b-f_b\Lambda} = \frac{1}{2} \frac{g^2}{16\pi^2} \log \frac{\Lambda^2}{B} H_+ ,
\]

where $B$, $C$ and $D$ are finite unknown constants.

There is also another term of order $g^4$ in the fermion-fermion part of $X_{\Lambda}$. We did not calculate it because our goal was to investigate the possibility of fitting finite parts of counterterms by requesting the $T$-matrix covariance in the fermion-boson channel (Section 4.1) and the emergence of the Dirac equation for physical fermions. As $X_{4f-f_A}$ does not contribute either to $T_4$ or the second order Dirac equation, it was irrelevant for our considerations. Also, $X_{4f-f_A}$ is more complicated to calculate than the terms we need to discuss here, because of two correlated loop integrations.

## 4 Finite parts of the counterterms

The renormalization procedure does not determine values of the finite parts of counterterms. To find them we need to introduce extra conditions. In principle, the constants should be fitted to match experiment. It is interesting to look for theoretical requirements of symmetries, which may constrain these constants. The $T$-matrix calculated with the general counterterms \([18],[19]\) is not automatically covariant. So, the covariance of the $T$-matrix provides useful conditions. Another condition will be provided by requiring that the full Hamiltonian eigenvalue equation could be reduced to a free Dirac equation.

### 4.1 $T$-matrix

We calculate our $T$-matrix using the formula

\[
T(E) = H_I + H_I \frac{1}{E - H_0 + i\epsilon} H_I + \cdots .
\]

The second order $T$-matrix has a covariant form and does not depend on the counterterms. $X_{\Lambda}$ starts contributing in the fourth order. The explicit $\Lambda$ dependence of counterterms cancels divergences in the loop integrations in other terms. So, $T_4$ is finite. However, it is not covariant automatically.

\[
\langle p_2 \sigma_2, k_2 | T_4 | p_1 \sigma_1, k_1 \rangle = \frac{g^4}{16\pi^2} \theta(\Lambda^2 - M^2_1) \theta(\Lambda^2 - M^2_2) 2(2\pi)^3 \delta^3(p_2 + k_2 - p_1 - k_1) \times
\]
To obtain a covariant result for $T_4$ we demand that the function $\Gamma_3(s)$ vanishes for arbitrary $s$. Its explicit form reads

$$\Gamma_3(s) = \frac{1}{s - m^2} \left[ (s - m^2) \frac{1}{2} \log \frac{C}{B} + 3m^2 \log \frac{m^2}{D} - A + 16\pi^2 \alpha_f(s)(s - m^2) + \gamma_f(s) \right],$$

where

$$s = (p_1 + k_1)^2 = M_1^2,$$

and functions $\alpha_f(s)$ and $\gamma_f(s)$ are given in Appendix. As $16\pi^2 \alpha_f(s)(s - m^2) + \gamma_f(s)$ turns out to be real and independent of $s$, the condition $\Gamma_3(s) = 0$ implies two relations:

$$B = C$$

and

$$A = -m^2 + \mu^2 \log \frac{\mu^2}{m^2} + 3m^2 \log \frac{m^2}{D}.$$  

### 4.2 Dirac equation

To describe a physical state in terms of free Fock states one considers the eigenvalue equation

$$H_\Lambda |P\sigma\rangle_{\text{physical}} = \frac{P_{+1}^2 + m^2}{P_+} |P\sigma\rangle_{\text{physical}}.$$  

The physical fermion state is a superposition of the bare fermion and fermion-boson states:

$$|P\sigma\rangle_{\text{physical}} = \sum_{\sigma_2} e^{\sigma_2}_{\sigma_1} |P\sigma_2\rangle + \sum_{\sigma_2} \int [p, k] 2(2\pi)^3 \delta^3(P - p - k) \phi^{\sigma_2}_{\sigma_1}(x, M^2) |p\sigma, k\rangle.$$  

By following steps from ref. [3] one can reduce Eq.(22) to

$$\left( \Xi_1 P_+ - \Xi_2 m + \Xi_3 \frac{\gamma^+}{2P_+} \right) \psi = 0,$$

for the one-body sector wavefunction $\psi$. Using our hamiltonian with counterterms restricted by conditions (20)-(21), one gets

$$\Xi_1 = 1 + \frac{g^2}{16\pi^2} \left[ \frac{3}{2} \log \frac{\Lambda}{\beta} - \beta(m^2) \right] + o(g^4),$$

$$\Xi_2 = 1 + \frac{g^2}{16\pi^2} \alpha(m^2) + o(g^4),$$

$$\Xi_3 = 0 + o(g^4).$$
Our earlier demand of the $T$-matrix covariance established the value of the mass counterterm $X_2$ in a way that also leads to the vanishing of $\Xi_3$ in order $g^2$.

In general, one can expand both non-zero $\Xi$’s in a power series in $g$

$$\Xi = \Xi^{(0)} + \Xi^{(2)} g^2 + \Xi^{(4)} g^4 + \cdots$$  

(28)

and one can translate the requirement that $m$ is the mass of physical fermions,

$$\left( p_m - \frac{\Xi^{(0)}_2 + \Xi^{(2)}_2 g^2 + \Xi^{(4)}_2 g^4 + \cdots}{\Xi^{(0)}_1 + \Xi^{(2)}_1 g^2 + \Xi^{(4)}_1 g^4 + \cdots} m \right) \psi = 0 ,$$  

(29)

into the condition for all coefficients

$$\Xi^{(i)}_1 = \Xi^{(i)}_2 .$$  

(30)

This is the threshold condition which makes the $T$-matrix threshold to appear at $s = (m + \mu)^2$, where $m$ is the position of its fermion pole.

Let us investigate which terms of $H$ contribute to $\Xi^{(i)}$. If one puts $g = 0$ then, the only condition one gets is

$$|P\sigma\rangle_{\text{physical}} = |P\sigma\rangle .$$  

(31)

Technically, the zeroth order terms $\Xi^{(0)}_1$ and $\Xi^{(0)}_2$ come from the inversion of $\sum_{\sigma} u_{P\sigma m} u_{P\sigma m} = p_m + m$, which is a part of $H_{\text{<}} H_{\text{>}}$. Dirac equation results in this order automatically; $\Xi^{(0)}_1 = \Xi^{(0)}_2$.

One can easily see that the second order terms $\Xi^{(2)}_1$ and $\Xi^{(2)}_2$ partly come from the term $H_{\text{<}} X_3$. So, one needs third order vertex corrections, such as $X_4$, to know all second order contributions to the Dirac equation. There is an unknown finite parameter $D$ in $X_4$. The condition $\Xi^{(2)}_1 = \Xi^{(2)}_2$ and Eqs. (25)-(26) lead to

$$\log \frac{D}{m^2} = \frac{2}{3} \cdot 16\pi^2 \left[ \alpha_f(m^2) + \beta_f(m^2) \right] .$$  

(32)

The functions $\alpha_f(s)$ and $\beta_f(s)$ are given in Appendix.

We see that the requirement that $m$ is equal to the mass of physical fermions implies one more condition on the free parts of counterterms.

### 4.3 Discussion

Collecting conditions (21), (21) and (22) together, and looking at the structure of the counterterms, we can observe the following. $X_{3Y}$ can be accounted for by changing the coupling constant of $H_Y$

$$g \rightarrow g + \frac{g^3}{64\pi^2} \log \frac{\Lambda^2}{C}$$  

(33)

in the original hamiltonian, while $X_{4fb-fb\Lambda}$ shifts $g^2$ in the seagull term $H_{\text{+}}$:

$$g^2 \rightarrow g^2 + \frac{g^4}{32\pi^2} \log \frac{\Lambda^2}{C} .$$  

(34)

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So, these two counterterms can be absorbed in one, $\Lambda$-dependent coupling constant \( \frac{g^3}{16\pi^2} \log \frac{m^2}{C} \). We have to stress that, in physical results, $\Lambda$ dependent logarithms $\log \frac{\Lambda^2}{m^2}$ cancel out, leaving

\[
g + \frac{1}{4} \frac{g^3}{16\pi^2} \log \frac{m^2}{C}. \tag{35}
\]

Thus, $g$ and $C$ will never appear independently, and we have one parameter, combination \( \frac{g^3}{16\pi^2} \log \frac{m^2}{C} \), that can be fixed from experiment.

$X_2$ shifts the mass in the one fermion free energy. Sum of $H_y$ and one of the third order counterterms, $X_3+$, reproduces the same $\bar{u}u$ coupling but with shifted mass of the spinor in the one-particle sector, according to the formula

\[
\left(1 + \frac{\gamma^+ \delta m}{2p^+}\right) u_m(p,\sigma) = u_{m+\delta m}(p,\sigma). \tag{36}
\]

## 5 Comparison with Ref. [3]

It was shown in Ref. [3] that, in this model, to get finite and covariant results for the $T$-matrix to all orders of perturbation theory, and to get the mass in the Dirac equation which is required by the threshold condition, it is enough to (1) add to the bare cut-off hamiltonian a term that shifts the mass of fermions in the free part $H_0f$, (2) correspondingly, change the spinor mass in the vertex, see Eq. \((36)\), and (3) allow the coupling to depend on $\Lambda$.

When one rewrites the hamiltonian of Ref. [3] using the invariant mass cutoff and expands it in powers of $\tilde{g}(m^2)$ up to the fourth order, one gets the same result as obtained in our similarity calculation with

\[
g + \frac{1}{4} \frac{g^3}{16\pi^2} \log \frac{m^2}{C}. \tag{37}
\]

replaced by

\[
\tilde{g}(m) - \frac{1}{2} g^3(m) \alpha_f(m^2). \tag{38}
\]

So, one can choose $C$ leading to the same result as in Ref. [3].

## 6 Conclusion

This work provides an example of application of the similarity renormalization scheme in its algebraical version. We have shown how this systematic procedure leads from a divergent hamiltonian to a finite one. The finite hamiltonian gives a covariant scattering matrix in perturbation theory.

The hamiltonian we used was known to lead to covariant results when one introduced special counterterms. The question was if a systematic procedure, the similarity renormalization scheme, would produce the same solution. The answer is yes.

On the other hand, Ref. [4] has recently suggested that the model may find applications in pion-nucleon physics when another Fock sector, with one fermion and two bosons, is included.
Therefore, our work also suggests that a systematic improvement in the light-front hamiltonian approach to relativistic nuclear physics may be achievable using the similarity renormalization group techniques.

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Appendix

For any operator $A$,

$$A = \int \langle 1 | 2 \rangle A_{12},$$

(39)

$A$ is defined as

$$A = \int \langle 1 | 2 \rangle \frac{1}{E_2 - E_1} A_{12},$$

(40)

where $E$’s are eigenvalue of $H_0$. $A$ is a solution of an equation:

$$[A, H_0] = A.$$  

(41)

Action of diagonal proximum operator $f_{\lambda}$ is defined as follows :

$$f_{\lambda} A = \int \langle 1 | 2 \rangle f_{\lambda}(1, 2) A_{12}.$$  

(42)

We have chosen

$$f_{\lambda}(1, 2) = \theta (\lambda^2 - |M_1^2 - M_2^2|).$$  

(43)

Functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are defined by

$$\alpha(s) = -\frac{1}{16\pi^2} \int dM^2 dx \theta (\Lambda^2 - M^2) \frac{x}{M^2 - s + i\epsilon},$$

$$\beta(s) = -\frac{1}{16\pi^2} \int dM^2 dx \theta (\Lambda^2 - M^2) \frac{1}{M^2 - s + i\epsilon},$$

$$\gamma(s) = \int dM^2 dx \theta (\Lambda^2 - M^2) \frac{(1 - x)M^2 - \mu^2 + (1 - x)m^2}{M^2 - s + i\epsilon},$$

where $x$ is integrated over the whole kinematically allowed region. Their finite parts are defined by

$$\alpha_f(s) = \lim_{\Lambda \to \infty} \left[ \alpha(s) + \frac{1}{2 \cdot 16\pi^2} \log \frac{\Lambda^2}{m^2} \right],$$
\[ \beta_f(s) = \lim_{\Lambda \to \infty} \left[ \beta(s) + \frac{1}{16\pi^2} \log \frac{\Lambda^2}{m^2} \right], \]

\[ \gamma_f(s) = \lim_{\Lambda \to \infty} \left[ \gamma(s) - \frac{1}{2} \Lambda^2 - \frac{1}{2} (s - m^2 - 2\mu^2) \log \frac{\Lambda^2}{m^2} \right]. \]

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