STATE SPACE REALIZATION OF EVEN GENERALIZED
POSITIVE AND ODD RATIONAL FUNCTION.
APPLICATIONS TO STATIC OUTPUT FEEDBACK

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Abstract. We here specialize the well known Positive Real Lemma (also known as the Kalman-Yakubovich-Popov Lemma) to complex matrix-valued rational functions, (i) generalized positive even and (ii) odd. On the way we characterize the (non) minimality of realization of arbitrary systems through (i) the corresponding state matrix and (ii) moving the poles by applying static output feedback. We then explore the application of static output feedback to both generalized positive even and to odd functions.

1. Introduction
For a half of a century, the Positive Real Lemma (also known as the Kalman-Yakubovich-Popov Lemma) has been recognized as a fundamental result in System Theory. We here exploit it to study two classes of rational functions, (i) generalized positive even and (ii) odd.

Let $\mathbb{C}_+$ and $\mathbb{C}_-$ be the open right and left halves of the complex plane respectively, and $\mathbb{P}_k$, $(\mathbb{P}_k)$ be the set of all $k \times k$ positive definite (semidefinite) matrices. Recall that a $p \times p$-valued function $F(s)$, analytic in $\mathbb{C}_+$ is said to be positive if

$$F(s) + F^*(s) \in \mathbb{P}_p, \quad s \in \mathbb{C}_+. \quad (1.1)$$

The study of rational positive functions, denoted by $\mathbb{P}$, has been motivated from the 1920’s by (lumped) electrical networks theory, see e.g. [6], [9]. From the 1960’s positive functions also appeared in books on absolute stability theory, see e.g. [37], [39].

A $p \times p$-valued function of bounded type in $\mathbb{C}_+$ (i.e. a quotient of two functions analytic and bounded in $\mathbb{C}_+$) is called generalized positive $\mathcal{GP}$ if

$$F(i\omega) + F^*(i\omega) \in \mathbb{P}_p, \quad a.e. \quad \omega \in \mathbb{R}, \quad (1.2)$$

where $F(i\omega)$ denotes the non-tangential limit of $F$ at the point $i\omega$.
It is interesting to note that both the function set $\mathcal{GP}$ and its subset $\mathcal{P}$, are closed under positive scaling, sum and inversion (when the given function has a non-identically vanishing determinant). We emphasize that for simplicity we adhere hereafter to the rational case. We shall find it convenient to use the notation

$$F^\#(s) := F^*(-s^*).$$

Recall that a matrix valued function $F(s)$ is called even if

$$F(s) = F^\#(s).$$

We shall denote by $\mathcal{Even}$ the set of even functions. Having $F \in \mathcal{Even}$ in particular implies that its zeroes and poles are symmetric with respect to the imaginary axis. Moreover, on the imaginary axis this $F$ is Hermitian, i.e.

$$F(i\omega) = F^*(\omega) \quad a.e. \quad \omega \in \mathbb{R}.$$

We shall say that a $p \times p$-valued function $\Psi(s)$ is even generalized positive, denoted by $\mathcal{GPE}$, if it satisfies both (1.2) and (1.3), i.e.

$$\Psi(i\omega) \in \mathbb{F}_p \quad a.e. \quad \omega \in \mathbb{R}.$$

Namely, $\mathcal{GPE} = \mathcal{GP} \cap \mathcal{Even}$. Note having that $\Psi \in \mathcal{GP}$ is equivalent to $(\Psi + \Psi^\#) \in \mathcal{GPE}$.

Scalar rational $\mathcal{GPE}$ functions were recently studied and then applied to Nevanlinna-Pick interpolation in [4, Section 5].

In analogy to $\mathcal{Even}$ functions in (1.3), we shall say that a function $F(s)$ is odd, denoted by $F \in \mathcal{Odd}$, if

$$F(s) = -F^\#(s).$$

This implies that on the imaginary axis $Odd$ functions are skew-Hermitian,

$$F(i\omega) = -F^*(i\omega) \quad a.e. \quad \omega \in \mathbb{R}.$$

Recall also that a matrix valued function $F(s)$ can always be partitioned to its even and odd parts, i.e.

$$F(s) = F_{\text{even}}(s) + F_{\text{odd}}(s) \quad F_{\text{even}} := \frac{1}{2}(F + F^\#) \quad F_{\text{odd}} := \frac{1}{2}(F - F^\#).$$

Clearly, $F_{\text{even}} \in \mathcal{Even}$ and $F_{\text{odd}} \in \mathcal{Odd}$. From (1.2) and (1.6) it follows that in fact $\mathcal{Odd} \subset \mathcal{GP}$. Scalar rational $\mathcal{Odd}$ functions were recently studied in [4, Section 4]. To further motivate the study of $\mathcal{Odd}$ functions, recall that the classical Nevanlinna-Pick framework it was shown in [45] that if there exists an interpolating function within $\mathcal{P}$, without loss of generality (but compromising the minimal degree) there exists an interpolating function within $\mathcal{PO} := \mathcal{P} \cap \mathcal{Odd}$. For an application of this observation see [12, Corollary 5.2.2].

Specializing the above even-odd partitioning to $\Psi \in \mathcal{GP}$ one obtains

$$\Psi(s) = \Psi_{\text{even}}(s) + \Psi_{\text{odd}}(s) \quad \Psi_{\text{even}} \in \mathcal{GPE} \quad \Psi_{\text{odd}} \in \mathcal{Odd}.$$

As already mentioned, scalar, rational $\mathcal{GPE}$ and $\mathcal{Odd}$ functions were recently studied in [4, Section 5] and [4, Section 4], respectively.

\footnote{To ease reading, hereafter $F(s)$ denotes an arbitrary rational function and $\Psi(s)$ is from the subset of $\mathcal{GP}$.}
We now recall in the concept of state space realization. Let $F(s)$ be a $p \times p$-valued rational function analytic at infinity, i.e. $\lim_{s \to \infty} F(s)$ exists. Denote by $q$ the McMillan degree of $F$. Namely, $F$ admits a state space realization

\begin{equation}
F(s) = C(sI - A)^{-1}B + D \quad L := \left(\begin{array}{cc}
A & B \\
C & D 
\end{array}\right)
\end{equation}

with $A \in \mathbb{C}^{n \times n}$, $n \geq q$, $B, C^* \in \mathbb{C}^{p \times n}$ and $D \in \mathbb{C}^{p \times p}$, namely, $L \in \mathbb{C}^{(n+p) \times (n+p)}$. If the McMillan degree of $F(s)$ satisfies $q = n$, the realization is called minimal.

The aim of this work is to characterize the realization of rational $\mathcal{GPE}$ and $\textit{Odd}$ functions and to give an application to static output feedback.

Related works on realization for the non-rational case exist, see e.g. [7] and more recently [10]. Restricting the discussion to rational functions, enables to offer simple explicit formulas. On realization, our main result is as follows.

**Theorem 1.1.** Let $\Psi(s)$ be a $p \times p$-valued rational function so that $\lim_{s \to \infty} \Psi(s)$ exists and let $L$ be the associated realization matrix,

\[
\Psi(s) = C(sI - A)^{-1}B + D \quad L := \left(\begin{array}{cc}
A & B \\
C & D 
\end{array}\right),
\]

see [15]. Let $q$ denotes the McMillan degree.

A. Let $L \in \mathbb{C}^{(2n+p) \times (2n+p)}$ with $q$ even, $2n \geq q$. The following are equivalent.

(i) $\Psi \in \mathcal{GPE}$.

(ii) There exist $2n \times 2n$ Hermitian non-singular matrices $H_1$ and $H_2$ so that,

\[
HL + L^*H \in \mathbb{F}_{2n+p} \quad ML + (ML)^* = 0_{2n+p},
\]

where

\[
H := \text{diag}\{H_1, \ I_p\} \quad M := \text{diag}\{H_2, \ iI_p\}.
\]

(iii) There exist realization matrices $L$ whose sub-blocks are

\begin{equation}
A = \begin{pmatrix}
\hat{A} & \hat{B}^* \\
0 & -\hat{C}^*
\end{pmatrix} \quad B = \begin{pmatrix}
\hat{B}D^* \\
-C^*\end{pmatrix} \quad C = \begin{pmatrix}
\hat{C} & \hat{D}\hat{B}^*
\end{pmatrix} \quad D = \hat{D}\hat{D}^*
\end{equation}

where $\hat{A}$ is $n \times n$, $\hat{B}, \hat{C}^*$ are $n \times p$ and $\hat{D}$ is $p \times p$. Furtheremore without loss of generality one can take spectrum $\hat{A}$ to be in $\mathbb{C}_+$. Moreover, if the realization is minimal, i.e. $2n = q$, up to similarity, $L$ is of the above form.

B. Let $L \in \mathbb{C}^{(n+p) \times (n+p)}$ with $n \geq q$ and let $\nu$ be so that $A$, the $n \times n$ part of $L$ has at most $\nu$ eigenvalues in $\mathbb{C}_-$ and at most $n - \nu$ eigenvalues in $\mathbb{C}_+$. Consider the following statements.

(i) $\Psi \in \textit{Odd}$

(ii) There exists a non-singular $n \times n$ Hermitian $\hat{H}$ s.t. $H := \text{diag}\{\hat{H}, \ I_p\}$ satisfies

\[
HL + L^*H = 0_{n+p}.
\]

(iii) Up to similarity the sub-blocks of the realization matrix $L$ are

\begin{equation}
A = \begin{pmatrix}
T_1 & \hat{A} \\
\hat{A}^* & T_2
\end{pmatrix} \quad B = \begin{pmatrix}
B_1 \\
B_2
\end{pmatrix} \quad C = \begin{pmatrix}
B_1^* - B_2^*
\end{pmatrix} \quad D = T_3
\end{equation}

with $T_1, T_2, T_3$ skew-Hermitian ($T_j = -T_j^*$) of dimensions $\nu \times \nu$, $(n - \nu) \times (n - \nu)$ and $p \times p$, respectively.
Then (ii) and (iii) are equivalent and in turn imply (i). If in addition the realization is minimal, i.e. \( q = n \), the the converse is true as well.

The result of part A can be compared with [8, Theorem 10.2] where a state space realization of \( GPE \) functions is given. While the development in [8] is self contained, we here rely on two classical results: Lemma 2.1 and Theorem 2.2 below. This allows us to obtain the above Theorem 1.1 A which is more general than [8, Theorem 10.2] in the following sense, (i) minimality of the realization is not assumed and (ii) \( D = \lim_{s \to \infty} \Psi(s) \) can be of any rank (including zero). Moreover, the explicit formulation in (1.8) and (1.9) enables us to directly address applications like static output feedback. For example, recall that a \( GPE \) function which is analytic on \( i\mathbb{R} \) admits a spectral factorization \( GG^\# \) with \( G(s) \) analytic in \( \mathbb{C}_+ \), see e.g. [8, Chapter 9], [20], [31, Section 19.3] and [40]. We here state a result, where some of the details will be clarified in the sequel.

**Proposition 1.2.** Let \( \Psi(s) \) be a \( p \times p \)-valued rational \( GPE \) function, which is not analytic on \( i\mathbb{R} \). Assume that \( \lim_{s \to \infty} \Psi(s) = 0 \) and let the state space realization be as in (1.8) with \( \hat{D} = 0 \).

There exists a static output feedback gain \( K, -K \in \mathbb{F}_p \) so that the closed loop system \( (I - \Psi K)^{-1} \Psi \) is analytic on \( i\mathbb{R} \), if and only if for all \( r \in \mathbb{R} \) the two following matrices

\[
\begin{pmatrix}
\hat{A} - irI_n & \hat{B} \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\hat{A} - irI_n & \hat{C}
\end{pmatrix}
\]

are of full rank.

Clearly, it is suffices to check the conditions only for all \( ir \in \text{spec}(\hat{A}) \).

The outline of the paper is as follows. Section 2 is devoted to providing a perspective on relevant existing literature and background to be used in the sequel. Part A of Theorem 1.1 is proved in Section 3. Aspects of the (non) minimality of the realization in (1.8) are addressed in Section 4. On the way, we introduce a test for the non-minimality of a state space realization of an arbitrary system (vanishing at infinity) by examining common eigenvalues between the \((n+p) \times (n+p)\) system matrix \( L \) and its \( n \times n \) submatrix \( \hat{A} \), see (1.7). Part B of Theorem 1.1 is proved in Section 5. In Section 6 we relate minimal realization of an arbitrary system (vanishing at infinity) with the ability of moving its poles through static output feedback. As a sample application of Theorem 1.1, in Section 7 we study the effect of static output feedback on \( GPE \) and \( Odd \) systems and then prove Proposition 1.2.

## 2. Background and perspective

In this section we state known results to be used in the sequel. Generalized positive functions were introduced in the context of the Positive Real Lemma (PRL), see 3 and references therein. Applications of \( GPE \) functions to electrical networks appeared in [26], and to control in [36], where they first casted in a Linear Matrix Inequality (LMI) framework, see e.g. [10] for more information on LMI. For more application of the generalized PRL, see [22].

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3The original formulation was real. The case we address is in fact generalized positive and complex; but we wish to adhere to the commonly used term: Positive Real Lemma.
We now recall in three characterizations of $\mathcal{GP}$ functions. We start with the Positive Real Lemma (PRL) as first presented in [14, Theorem 1] (up to substituting the real setting by a complex one). For sample of other versions the PRL see e.g. [8, Theorem 15.2], [22, Section 3A] and the discussions in [3, Section 2] and in [10, p. 34].

**Lemma 2.1.** Let $L$ be a $(n+p) \times (n+p)$ realization matrix of a $p \times p$-valued rational function $\Psi(s)$ of McMillan degree $q$, see (1.7).

(I) $\Psi \in \mathcal{GP}$ if there exists a Hermitian non-singular $\hat{H} \in \mathbb{C}^{n \times n}$ so that

\[ HL + L^* H \in \mathbb{P}_{n+p} \quad H = \text{diag}\{\hat{H}, I_p\}. \]

If in addition the realization is minimal, i.e. $q = n$, the converse is true as well: $\Psi \in \mathcal{GP}$ implies (2.1).

(II) If in part (I) $-\hat{H} \in \mathbb{P}_n$ then $\Psi \in \mathcal{P}$.

It is of interest to recall that in [3, Section 7] we pointed out that if $L$ satisfies (2.1), whenever non-singular, also $L^{-1}$ satisfies (2.1), with the same $H$.

As a second characterizations of $\mathcal{GP}$ functions recall that a $p \times p$-valued function $\Psi(s)$ belongs to $\mathcal{GP}$ if and only if it can be factored as

\[ \Psi(s) = G(s)P(s)G^#(s), \]

where $G, P$ are $p \times p$-valued, $G$ analytic in $\mathbb{C}_-$ and $P \in \mathcal{P}$. See for instance [21] for the corresponding result in the setting of functions meromorphic in the open unit disk rather than in the right open half-plane, see the discussion in [2, Section 1]. To present the third characterizations of $\mathcal{GP}$ functions we briefly mention that a rational function $\Psi$ is in $\mathcal{GP}$ if and only if the kernel

\[ \Psi(s) + \Psi(w)^* \]

has a finite number of negative squares in its domain of definition in $\mathbb{C}_+$. This equivalent characterization of (not necessarily rational) functions of the form (2.2) appeared for the scalar case in [13] and [15] and extended in [33], [35], [35]. The significance of (2.2) to scalar rational $\mathcal{GP}$ functions was recently treated in [2] and [4], where a more complete survey of the literature can also be found.

The following result is crucial to our construction.

**Theorem 2.2.** Let $\Psi(s)$ be a matrix valued rational function. The following are equivalent

(i) $\Psi \in \mathcal{GPE}$

(ii) $\Psi \in \mathcal{GP}$ and in addition $\Psi(i\omega) = \Psi(i\omega)^*$ a.e. $\omega \in \mathbb{R}$.

(iii) $\Psi(s) = G(s)G^#(s)$ and without loss of generality $G$ can be chosen so that $G$ and $G^{-1}$ are analytic in $\mathbb{C}_-$.

**Proof** The equivalence of (i) and (ii) follows from the definition of $\mathcal{GPE}$ functions, see (1.4).

The fact that (ii) $\implies$ (iii) is deep and was established in [44, Theorem 2].

The fact that (iii) $\implies$ (i) is straightforward. \[\square\]
The problem in item (iii) of finding $GG^\#$ out of $\Psi \in \mathcal{GPE}$ is known as spectral factorization whenever $\Psi$ and $\Psi^{-1}$ are analytic on the imaginary axis and if this restriction is relaxed, pseudo spectral factorization. As sample references on spectral factorization, see e.g. [8, Chapter 9] [20] [31, Section 19.3] and [40]. On pseudo-spectral factorization, see e.g. [8, Chapter 10], [41] and the earlier work of Youla on which we rely, [44]. The gap between spectral and pseudo spectral factorizations is addressed in [8, Chapters 9, 10]. Here, Proposition 1.2 is restated and proved in Proposition 7.4 below.

For completeness we mention that the spectral factorization problem has been extended to the case where $F \in \mathcal{E}$ even can be factored to

$$F(s) = G(s) \cdot \text{diag}\{I, -I\} \cdot G^\#(s)$$

see e.g. [29], [32] and [8, Part VII].

Note that $\Psi \in \mathcal{GPE}$ if and only if one substitutes in (2.2) $\Psi = \Psi^\#$ thus it is equivalent to having in (2.2)

$$P(s) \equiv P$$

for some constant positive semidefinite $P$. This conforms well with item (iii) in Theorem 2.2.

The following result from [1, Theorem 4.1], (see also [40, Proposition 2.1]) is adapted to our framework.

**Theorem 2.3.** Let $F(s)$ be a matrix valued rational function admitting a state space realization $L$ as in (1.7) of McMillan degree $q$.

If

$$(2.3)\quad ML + (ML)^* = 0_{n+p}$$

where

$$M := \text{diag}\{H, iI_p\},$$

for some Hermitian non-singular $H \in \mathbb{C}^{n \times n}$, then

$$(2.4)\quad F(i\omega) = F(i\omega)^* \text{ a.e. } \omega \in \mathbb{R}.$$  

Conversely, let $L$ be a minimal realization of $F(s)$ in (2.4), i.e. $q = n$. Then $L$ satisfies (2.3).

Note that $M$ is not Hermitian.

Clearly, if $F$ satisfies (2.4), so does $-F$ and also $F^{-1}$ (whenever the determinant is not identically zero). It is less obvious that some of this properties hold for the system matrix $L$. Namely, if $L$ satisfies (2.3) with $M = \text{diag}\{H, iI_p\}$, then this holds for $-L$ and whenever exist, $\pm L^{-1}$ satisfies (2.3) with $M = \text{diag}\{-H, iI_p\}$.

We conclude this section with recalling in a technical result on the state space realization of a composition of a pair of rational functions (series or cascade connection of systems in Electrical Engineering terminology). Namely $F_\alpha(z)$, $F_\beta(z)$ are of compatible dimensions and $F_\gamma(z)$ is obtained by

$$(2.5)\quad F_\gamma(z) = F_\alpha(z)F_\beta(z).$$

Assuming the state-space realization of each $F_\alpha(z)$ and $F_\beta(z)$ is known, one can construct a realization of the resulting $F_\gamma(z)$, see e.g. [27, Subsection 8.3.3], [43, Eq. (4.15)].

**Observation 2.4.** Given $l \times q$ and $q \times r$ valued rational functions $F_\alpha(z)$, $F_\beta(z)$, respectively, admitting state space realization

$$F_\alpha(z) = C_\alpha(zI - A_\alpha)^{-1}B_\alpha + D_\alpha \quad A_\alpha \in \mathbb{C}^{p_\alpha \times p_\alpha}$$
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\[ B_\alpha \in \mathbb{C}^{p_\alpha \times q} \quad C_\alpha \in \mathbb{C}^{l \times p_\alpha} \quad D_\alpha \in \mathbb{C}^{l \times q}. \]

\[ F_\beta(z) = C_\beta(zI - A_\beta)^{-1}B_\beta + D_\beta \quad A_\beta \in \mathbb{C}^{p_\beta \times p_\beta} \]

\[ B_\beta \in \mathbb{C}^{p_\beta \times r} \quad C_\beta \in \mathbb{C}^{q \times p_\beta} \quad D_\beta \in \mathbb{C}^{q \times r}. \]

The system matrix \( L_\gamma \) associated with the realization of \( F_\gamma(z) \) in (2.5) is given by,

\[
L_\gamma = \begin{pmatrix}
A_\alpha & B_\alpha C_\beta & B_\alpha D_\beta \\
0 & A_\beta & B_\beta \\
C_\alpha & D_\alpha C_\beta & D_\alpha D_\beta
\end{pmatrix}.
\]

3. PROOF OF THEOREM 1.1

(ii) \( \implies \) (i)
Substituting in Theorem 2.2 (ii) \( \implies \) (i) a combination of Lemma 2.1 (the PRL) and Theorem 2.3 establishes this part.

(i) \( \implies \) (iii)
First, substitute in (1.7) \( F(s) = G(s) \) so that its (not necessarily minimal) realization is

\[(3.1) \quad L_g = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\]

Then the corresponding realization of \( G(s)^\# \) is given by,

\[
\begin{pmatrix}
-\hat{A}^* & -\hat{C}^* \\
\hat{B}^* & \hat{D}^*
\end{pmatrix}.
\]

Now, substituting in Observation 2.4 \( F_\alpha = G \) and \( F_\beta = G^\# \) along with Proposition 2.2 establishes the structure in (1.8), i.e.

\[(3.2) \quad L = \begin{pmatrix}
\hat{A} & \hat{B}^* & \hat{B}\hat{D}^* \\
0 & -\hat{A}^* & -\hat{C}^* \\
C & DB^* & DD^*
\end{pmatrix}.
\]

Finally, we have shown above how to construct a realization of \( \mathcal{GPE} \) function of degree= 2n where n is arbitrary, see (3.2). Note now that if \( \Psi \in \mathcal{GPE} \) is realized by \( \hat{L} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \) of the form in (3.2), then the same \( \Psi \) can also realized by \( \hat{L} = \begin{pmatrix}
A & * & B \\
0 & * & 0 \\
C & * & D
\end{pmatrix} \), where * means an arbitrary block of suitable dimensions. It may be the case that one can not transform \( \hat{L} \) by change of coordinates to the form in (3.2) (in particular the dimension of the \( \hat{A} \) part in \( \hat{L} \) may be odd). However, as all minimal realizations are similar, see e.g. \[8, 29\], \[27\] Theorem 2.4-7, one of them is of the form (3.2).

(iii) \( \implies \) (ii)
We find it convenient to introduce the following four intermediate conditions:

\[(3.3) \quad \text{Condition (ii) in Theorem 1.1 is satisfied where } H_1 \text{ and } H_2 \text{ share the same inertia.}\]
(3.4) Condition (ii) in Theorem 1.1 A is satisfied with $H_2$ unitarily similar to $H_1$.

(3.5) Condition (ii) in Theorem 1.1 A is satisfied with $H_2$ unitarily similar to $H_1$, both involutions.

(3.6) Condition (ii) in Theorem 1.1 A is not satisfied with $H_1 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, $H_2 = i \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Trivially (3.6) $\Rightarrow$ (3.5) $\Rightarrow$ (3.4) $\Rightarrow$ (3.3) $\Rightarrow$ (ii), so all we need to show is that (iii) $\Rightarrow$ (3.6).

Indeed, a straightforward calculation reveals that the matrix $L$ in (3.2) satisfies the conditions in (3.6) where

$$H L + L^* H = \begin{pmatrix} 0 & \sqrt{2} B \\ \sqrt{2} B & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} D \\ \sqrt{2} D & 0 \end{pmatrix}^*$$. Verifying (2.3) is immediate so the proof is complete.

Note that the matrix $L$ in (3.2) has a special symmetry, i.e.

$$L = \begin{pmatrix} 0 & P_2 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & P_3 \end{pmatrix}$$

In particular, the $A$ matrix in (1.8) (the upper left block in $L$) has a Hamiltonian structure, see also (7.2) below. This will be exploited in the proof Proposition 7.4 below.

We now examine an aspect of Theorem 1.1 A: Above we have indirectly proved that $A$ (ii) $\Rightarrow$ (3.3) $\Rightarrow$ (3.4) $\Rightarrow$ (3.5) $\Rightarrow$ (iii), so all we need to show is that (iii) $\Rightarrow$ (3.6).

Let $\Delta_I = \begin{pmatrix} 0 & P_2 & 0 \\ P_1 & 0 & 0 \\ 0 & 0 & P_3 \end{pmatrix}$ where $P_1, P_2 \in \mathbb{F}_n$ and $P_3 \in \mathbb{F}_l$ are arbitrary. Then $\Delta_I$ satisfies the conditions of (3.6) with the same $H$ and $M$ as in (3.6). This in turn implies that also $L + \Delta_I$ satisfies the same conditions. However, $L + \Delta_I$ is no longer of the form of (1.8). This is illustrated by the following example.

Let $L_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ of the of the form of (1.8) be a realization the rational $\mathcal{GPE}$ function $\psi_1(s) = \frac{1}{1-s^2} + 1$. Now in $\Delta_I$ substitute $P_1 = 1, P_2 = 2$ and $P_3 = 3$, i.e. $\Delta_I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Indeed, $\Delta_I$ satisfies condition (3.6) and so does

$$L_2 := L_1 + \Delta_I = \begin{pmatrix} -1 & 3 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$. Although $L_2$ is not of the form of (1.8), it is a state space realization of the rational $\mathcal{GPE}$ function $\psi_2(s) = \frac{1}{4-2s^2} + 4$. Indeed,
ψ_2(s) can also be realized by \( L'_2 = \begin{pmatrix} -2 & 4 \\ 0 & 2 \\ \sqrt{5} - 2 & 4 \\ 4 & \end{pmatrix} \) which is of the form of (1.8).

\[ \text{(3.4)} \implies \text{(3.5)}. \]

Consider the condition for the submatrix that \( H_1A + A^*H_1 \in \mathbb{F}_n \) and \( H_2A + A^*H_2 = 0_n \).

Take \( A = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix} \). \( H_1 = \text{diag}\{1, -5\} \) and \( H_2 = \begin{pmatrix} 0 & \sqrt{5} \\ \sqrt{5} & -4 \end{pmatrix} \). Then the conditions are satisfied with \( H_1 \) and \( H_2 \) unitarily similar, but neither is an involution.

\[ \text{(3.3)} \implies \text{(3.4)}. \]

Consider the condition for the submatrix that \( H_1A + A^*H_1 \in \mathbb{F}_n \) and \( H_2A + A^*H_2 = 0_n \).

Take \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). \( H_1 = \text{diag}\{1, -5\} \) and \( H_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). Then the conditions are satisfied with \( H_1 \) and \( H_2 \) sharing the same inertia, but they are not unitarily similar.

### 4. Minimality of the Realization

We now address the question of minimality of the obtained realization. We first resort to the following observation which goes beyond the scope of this work.

**Lemma 4.1.** Let \( L \in \mathbb{C}^{(n+p) \times (n+p)} \) be realization of a \( p \times p \)-valued rational function \( F(s) \) of McMillan degree \( n \). Namely,

\[
F(s) = C(sI - A)^{-1}B + D \quad \quad \quad \quad L := \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

see (1.7). If the realization is not minimal, there exists \( \lambda \in \text{spect}(A) \) so that \( \lambda \in \text{spect}(L) \) for all \( D \).

If \( \text{rank}(B) = \text{rank}(C) \) or the number of Jordan blocks in \( A \) is less or equal to \( \min(\text{rank}(B), \text{rank}(C)) \) then the converse is true as well.

**Proof** We first show that if the realization is not minimal there exists \( \lambda \in \text{spect}(A) \) so that \( \lambda \in \text{spect}(L) \) for all \( D \).

If the realization \( L \) is not controllable, there exists \( 0 \neq \hat{v} \in \mathbb{C}^n \) so that \( \hat{v}^*(A - \lambda I_n) = 0 \), for some \( \lambda \in \mathbb{C} \) and in addition \( \hat{v}^*B = 0 \), see e.g. [9 Theorem 3.3.1], [27 Theorem 2.4-8], [31 Theorem 4.3.3]. This implies that \( (n+p) \)-dimensional vector \( v := (\hat{v}) \) satisfies, \( v^*(L - \lambda I_{n+p}) = 0 \) with the same \( \lambda \). Namely, \( \lambda \in \text{spect}(A) \cap \text{spect}(L) \).

Similarly, if the realization \( L \) is not observable, there exists \( 0 \neq \hat{v} \in \mathbb{C}^n \) so that \( (A - \lambda I_n)\hat{v} = 0 \), for some \( \lambda \in \mathbb{C} \) and in addition \( C\hat{v} = 0 \), see e.g. [6 Theorem 3.3.6], [24 Theorem 2.4-8]. This implies that \( (n+p) \)-dimensional vector \( v := (\hat{v}) \) satisfies, \( (L - \lambda I_{n+p})v = 0 \) with the same \( \lambda \).

For the converse direction we now show that if the realization \( L \) is minimal one can construct \( D \) so that the matrix

\[
L - \lambda I_{n+p} = \begin{pmatrix} A - \lambda I_n & B \\ C & D - \lambda I_p \end{pmatrix}
\]
is nonsingular for each \( \lambda \in \text{spect}(A) \), i.e \( \text{spect}(A) \cap \text{spect}(L) = \emptyset \). To this end, take \( D \) so that \( \text{spect}(A) \cap \text{spect}(D) = \emptyset \). Hence, one can write

\[
L - \lambda I_{n+p} = \begin{pmatrix} I_n & B(D - \lambda I_p)^{-1} \\ 0 & I_p \end{pmatrix} \begin{pmatrix} A - \lambda I_n & B(\lambda I_p - D)^{-1}C \\ C & D - \lambda I_p \end{pmatrix}.
\]
In this case $\lambda \in \text{spect}(L)$ if and only if the matrix
\[ A - \lambda I_n + BKC \quad \text{with} \quad K := (D - \lambda I_p)^{-1} \]
is singular. The construction for a minimal realization of such $K \in \mathbb{C}^{n \times p}$ so that $A - \lambda I_n + BKC$ is nonsingular for all $\lambda \in \text{spect}(A)$ is established in Proposition 6.1 below, so the proof is complete.

Admittedly, we do not know whether or not the conditions on $\text{rank}(B)$ and $\text{rank}(C)$ in the above result are inherent to the problem or just a by-product of the technique we employed.

We now illustrate the fact that it may be the case that $\text{spect}(A) \cap \text{spect}(L) \neq \emptyset$ for some $D$, although a realization is minimal. Indeed, take $A = 0$ (scalar) $B^*, C \in \mathbb{C}^2$, both non-zero, and $D \in \mathbb{C}^{2 \times 2}$. Namely the following $3 \times 3$ system matrix $L = (\begin{smallmatrix} 0 & B^* \\ C & 0 \end{smallmatrix})$ is a minimal (degree=1) realization of a $2 \times 2$-valued rational function $F(s) = \frac{1}{s}CB + D$. Obviously, minimality is independent of $D$. However, if for example $F(s) = (\frac{1}{s} + \gamma)CB$, namely $D = \gamma CB$, where $\gamma \in \mathbb{C}$ is arbitrary, both $A$ and $L$ are singular, i.e. have a common eigenvalue (=zero).

The special structure of $D$ in the above example suggests that if a realization is minimal, the matrices $L$ and $A$ typically do not share common eigenvalues.

One can now exploit the special structure of the realization $\mathcal{GPE}$ functions in (3.2), (3.7), in applying Lemma 4.1 to obtain the following.

Observation 4.2. Let $L_g(s)$ in (3.1) be a realization of $p \times p$-valued rational function $G(s)$ and without loss of generality assume that $\text{spect}(\hat{A}) \subset \mathbb{C}_+$. Let $\hat{G}(s) = G(s) - \hat{D}$, i.e. $\hat{G}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$. The following is true.

A. Minimality of the realizations of $G(s)$ and of $\hat{G}(s)$ is equivalent.

B. The following are equivalent.

(i) The realization of $\hat{G}(s)$ is not minimal.

(ii) There exists $\lambda \in \text{spect}(\hat{A})$ so that for all $\hat{D}$, $\lambda \in \text{spect}(L_g)$ in (3.1).

(iii) For all $\hat{D}$ the realization $L$ in (3.2), of $G(s)G(s)^\#$ is not minimal.

(iv) There exists $\lambda \in \text{spect}(\hat{A}) \cup \text{spect}(-\hat{A}^*)$ so that for all $\hat{D}$, $\lambda \in \text{spect}(L)$ in (3.2).

C. Assume that the realization of $\hat{G}(s)$ is minimal. If $\text{spect}(\hat{A}) \not\subseteq i\mathbb{R}$, one can always find $\hat{D}$ so that the realization of $G(s)G(s)^\#$ in (3.2), is not minimal.

Proof A. As minimality of realization is equivalent to controllability and observability, the claim is obvious.

B. First note that if $C(sI - A)^{-1}B + D$ is a realization of a $\mathcal{GPE}$ function of the form (3.2), then from (3.7) it in particular follows that $C^* = \begin{pmatrix} 0 \\ I \end{pmatrix}$, which implies $\text{rank}(B) = \text{rank}(C)$. Thus, one can apply Lemma 4.1 which now implies (i) $\iff$ (ii) and (iii) $\iff$ (iv).

(i) $\implies$ (iii)

If the realization of $\hat{G}(s)$ is not minimal, assume it is not observable. Namely, there exists $0 \neq v \in \mathbb{C}^n$ so that \begin{pmatrix} v \\ 0 \end{pmatrix} is an $(n + p)$-dimensional vector of $L_G$ in (3.1)
Let $\Psi$ be of the form $\Psi(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ with $\hat{A} = 0$, $\hat{B} = 1$ and $\hat{C} = 1$.

Let $\Psi_2 \in \mathcal{GP}_E$ be of the form $\Psi_2(s) = \Psi_1(s)\Psi_1^\#(s) = -\frac{1}{s^2}$. Its realization is of the form of (1.8), given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = (1, 0), \quad D = 0.$$ (4.1)

In particular, this realization is minimal.

Consider now the $\mathcal{GP}_E$ function $\Psi_2(s) + D$, where $D \geq 0$ is a parameter. For all $D$, the realization is minimal, i.e. of degree 2.

5. Odd systems

Proof of Theorem 1.1 B

(ii) $\implies$ (iii)

Let $\tilde{H}$ has $\nu$ eigenvalues in $\mathbb{C}_-$ and $n - \nu$ eigenvalues in $\mathbb{C}_+$. Then up to similarity on $L$ (and $A$) and congruence on $H$ (and $\tilde{H}$) one can take $\tilde{H} = \text{diag}\{-I_{\nu}, I_{n-\nu}\}$ (and thus $H = \text{diag}\{-I_{\nu}, I_{n-\nu}, I_p\}$). Substituting in $HL + L^*H = 0_{n+p}$ yields (1.9).

(iii) $\implies$ (i)

Odd systems

Odd systems

Odd systems

Odd systems
A straightforward calculation yields \( \Psi(s) = C(sI_n - A)^{-1}B + D \). We shall concentrate on \( \Psi(s)_{s=i\omega} \) for \( \omega \in \mathbb{R} \). We first conformally partition

\[
(i\omega I_n - A)^{-1} = \begin{pmatrix} i\omega I_{n_{p_{1}}} - T_{1} & -A \\ -A^* & i\omega I_{n_{p_{2}}} - T_{2} \end{pmatrix}^{-1} = \begin{pmatrix} W(i\omega) X(i\omega) \\ Y(i\omega) Z(i\omega) \end{pmatrix}.
\]

We thus formally have,

\[
\Psi(i\omega) = B_{11} W(i\omega) B_{1} + B_{12} X(i\omega) B_{2} - B_{12} Y(i\omega) B_{1} - B_{22} Z(i\omega) B_{2} + T_{3}.
\]

As for all \( \omega \in \mathbb{R} \) the matrix \( i \cdot \text{diag}\{I_{n_{p_{1}}}, -I_{n_{p_{2}}}\} (i\omega I_n - A) \) is Hermitian, so is its inverse

\[
(i\omega I_n - A)^{-1} \cdot i \cdot \text{diag}\{-I_{n_{p_{1}}}, I_{n_{p_{2}}}\} = \begin{pmatrix} -iW(i\omega) iX(i\omega) \\ -iY(i\omega) iZ(i\omega) \end{pmatrix}.
\]

Namely, \( W(i\omega) = -W(i\omega)^* := T_{4}(i\omega) \), \( Z(i\omega) = -Z(i\omega)^* := T_{5}(i\omega) \) and \( X(i\omega) = Y(i\omega)^* \). Substituting now in (5.1) yields

\[
\Psi(i\omega) = B_{11} T_{4}(i\omega) B_{1} + B_{12} X(i\omega) B_{2} - (B_{11} X(i\omega) B_{2})^* - B_{22} T_{5}(i\omega) B_{2} + T_{3}.
\]

Namely, (1.6) is obtained and this part of the claim is established.

(i) \( \Rightarrow \) (ii)

First consider the condition in (ii) and note that if one denotes \( N := \text{diag}\{I_{n_{p_{1}}}, iI_{p}\} \) then

\[
0 = HL + L^*H = HNN^*L + L^*NN^*H = ML^* M^* = M\hat{L} + (M\hat{L})^*,
\]

where \( M := \text{diag}\{\hat{H}, iI_{p}\} \) and \( \hat{L} = N^*L = \begin{pmatrix} A & B \\ -iC & -iD \end{pmatrix} \) i.e. if \( L \) realizes \( \Psi(s) = C(sI_n - A)^{-1}B + D \), \( \hat{L} \) realizes \( F(s) := -i\Psi(s) \). It now follows that on the imaginary axis \( F(s) \) is Hermitian, see (2.4) and \( \Psi(s) \) is skew-Hermitian, i.e. \( \Psi \in \mathcal{Odd} \).

Assuming the realization is minimal, i.e. \( q = n \), both directions of Theorem 2.6 holds, so in the proof is complete.

Note that as \( \mathcal{Odd} \subset \mathcal{GP} \), Lemma 2.4 (the PRL) is satisfied: Indeed (5.2) is a special case of (2.1).

Of a special interest is the set \( \mathcal{PO} \) of positive-odd functions (a.k.a Lossless or Foster), i.e. \( \mathcal{Odd} \cap \mathcal{P} \). In electrical networks theory they are associated with L-C circuits. For more details see e.g. [9] 2.20, 2.36, 2.39, 7.33, 8.35, 8.36, 8.37, [6] pp. 12, 221, Theorem 2.7.4, [12] subsections 4.2, 5.1. Recall also from (2.2) it follows that \( \Psi \in \mathcal{Odd} \) can always be factored as \( \Psi = GPG^\# \) with \( P \in \mathcal{PO} \). Thus in a sense, \( \mathcal{PO} \) functions generate all \( \mathcal{Odd} \) functions. The following is known, see e.g. [6] Eq. (5.2.6)]. It is immediate from part B of Theorem 1.1 upon substituting in (1.9) \( \nu = n \).

**Corollary 5.1.** Let \( \Psi(s) \) be a \( p \times p \)-valued rational function so that \( \lim_{s \to \infty} \Psi(s) \) exists. Let \( L \in \mathbb{C}^{(n+p) \times (n+p)} \) be the associated system matrix. If

\[
L = \begin{pmatrix} T_{n} & B \\ B^* & T_{p} \end{pmatrix} \quad T_{n} \in \mathbb{C}^{n \times n} \quad T_{n} = -T_{n}^* \quad T_{p} \in \mathbb{C}^{p \times p} \quad T_{p} = -T_{p}^*.
\]

Then, \( \Psi \in \mathcal{PO} \).

If the realization is minimal, up to similarity, the converse is true as well.
6. Static output feedback - arbitrary functions

Recall that applying a static output feedback to an input-output system \( y(s) = F(s)u(s) \) (\( u \) is \( m \)-dimensional input and \( y \) is \( p \)-dimensional output) means taking \( u = Ky + u' \) with \( u' \) an auxiliary input and \( K \) a \( m \times p \) constant matrix. The resulting closed loop system is \( y(s) = F_{c,1}(s)u'(s) \) with \( F_{c,1} = (I_p - FK)^{-1} F \).

For simplicity, we adopt the common assumption that \( \lim_{s \to \infty} F(s) = 0 \) (“strictly proper” in engineering jargon). Thus, in (1.7) the \((n + p) \times (n + m)\) realization matrix \( L \) is of the form \( L = \begin{pmatrix} A_c & B \\ C & 0 \end{pmatrix} \). After applying a static output feedback, the closed loop realization matrix \( L_{c,1} \) is

\[
L_{c,1} = \begin{pmatrix} A_c & B \\ C & 0 \end{pmatrix}
A_c := A + BKC.
\]

The simplicity of the static output feedback has made it very attractive. However, exploring its properties turned out to be challenging. The most common associated problem has been stabilization, namely guaranteeing (in the continuous time case) that \( \text{spec}(A_c) \subset \mathbb{C}_- \). This is illustrated in a basic way in [27, Section 3.1] and for a sample of more recent references see e.g. [18, 23, 30, 38, 42]. One can go beyond stability, and in [3, Proposition 8.1] we characterized systems which may turned to be \( GP \), and in particular \( P \), through static output feedback.

Our first result here goes beyond the scope of this work. To this end, we need the following notation. Let \( L \) be realization of a \( p \times m \)-valued rational function \( F(s) \) of McMillan degree \( n \) with \( \lim_{s \to \infty} F(s) = 0 \). Namely,

\[
\begin{align*}
F(s) &= C(sI - A)^{-1}B \\
L &= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix},
\end{align*}
\]

see (1.7). I shall find it convenient to denote \( \beta := \text{rank}(B), \gamma := \text{rank}(C) \).

Namely, there exist nonsingular matrices \( R_b \in \mathbb{C}^{m \times m} \) and \( R_c \in \mathbb{C}^{p \times p} \) so that \( R_c F(s) R_b = \begin{pmatrix} \tilde{F}(s) & 0 \\ 0 & (p, \gamma) \times (m, \beta) \end{pmatrix} \) where \( \tilde{F}(s) \) is \( \gamma \times \beta \)-valued.

We shall also denote by \( r \) the number of Jordan blocks in \( A \).

**Proposition 6.1.** Let \( L \) be realization of \( F(s) \) as in (6.2). If this realization is not minimal, there exists \( \lambda \in \text{spec}(A) \) so that for all \( K \in \mathbb{C}^{m \times p} \), \( \lambda \in \text{spec}(A_c) \), see (6.1).

If \( \min(\beta, \gamma) \geq r \) or \( \beta = \gamma \), then the converse is true as well.

**Proof.** We first show that if the realization is not minimal then there exists \( \lambda \in \text{spec}(A) \) so that for all \( K \), \( \lambda \in \text{spec}(A_c) \). If the realization is not controllable, there exists a left eigenvector \( 0 \neq v_L \in \mathbb{C}^n \) (the subscript stands for “left”) so that \( v_L^*(A - \lambda I_n) = 0 \), for some \( \lambda \in \mathbb{C} \) and in addition \( v_L^*B = 0 \), see e.g. [27, Theorem 2.4-8], [31, Theorem 4.3.3]. This implies that

\[
v_L^*(A_c - \lambda I_n) = v_L^*(A + BKC - \lambda I_n) = v_L^*(A - \lambda I_n) = 0.
\]
Similarly, if the realization is not observable, there exists \( 0 \neq v_R \in \mathbb{C}^n \) (the subscript stands for “right”) so that \( (A - \lambda I_n)v_R = 0 \), for some \( \lambda \in \mathbb{C} \) and in addition \( C v_R = 0 \), see e.g. [27, Theorem 2.4-8]. This implies that
\[
(A_3 - \lambda I_n)v_R = (A + BKC - \lambda I_n)v_R = (A - \lambda I_n)v_R = 0.
\]

For the converse direction, assuming the realization is minimal, we now construct \( K \in \mathbb{C}^{m \times p} \) so that \( \text{spec}(A) \nsubseteq \text{spec}(A_3) \). To this end, \( K \) should be so that
\[
0 \neq v_L \in \mathbb{C}^n \quad v_L^*(A - \lambda I_n) = 0 \quad \Rightarrow \quad v_L^* (A_3 - \lambda I_n) = v_L^* BKC \neq 0
\]
\[
0 \neq v_R \in \mathbb{C}^n \quad (A - \lambda I_n)v_R = 0 \quad \Rightarrow \quad (A_3 - \lambda I_n)v_R = BKCv_R \neq 0
\]
where the subscripts \( L \) and \( R \) stand for “left” and “right”. We shall find it convenient to denote \( K := \delta \hat{K} \) where \( 0 \neq \delta \in \mathbb{C} \) will be later determined. Thus, we actually look for \( \hat{K} \) (which will turn to be an isometry if \( m \geq p \), else coisometry) so that
\[
(6.3) \quad 0 \neq v_L \in \mathbb{C}^n \quad v_L^*(A - \lambda I_n) = 0 \quad \Rightarrow \quad v_L^* BKC \neq 0
\]
\[
0 \neq v_R \in \mathbb{C}^n \quad (A - \lambda I_n)v_R = 0 \quad \Rightarrow \quad BKCv_R \neq 0.
\]
Consider now the singular values decomposition, see e.g. [24, Theorem 7.3.5], of the matrices \( B \) and \( C \)
\[
B = U_b \Sigma_b W_b \quad C = U_c \Sigma_c W_c
\]
with \( U_b, W_b \in \mathbb{C}^{n \times n} \), \( W_b \in \mathbb{C}^{m \times m} \) and \( U_c, W_c \in \mathbb{C}^{p \times p} \) all unitary and
\[
\Sigma_b = \left( \begin{array}{cc} \hat{\Sigma}_b & 0 \\ 0 & 0_{(n - \beta) \times (m - \beta)} \end{array} \right) \quad \quad \Sigma_c = \left( \begin{array}{cc} \hat{\Sigma}_c & 0 \\ 0 & 0_{(p - \gamma) \times (n - \gamma)} \end{array} \right)
\]
with \( \hat{\Sigma}_b \) and \( \hat{\Sigma}_c \) positive diagonal of dimensions \( \beta \times \beta \) and \( \gamma \times \gamma \) respectively.

If \( \beta = \gamma \) then it is sufficient to take
\[
\hat{K} = \left( \begin{array}{c} \hat{\Sigma}_b \Sigma_c \\ 0_{(n - \beta) \times (n - \beta)} \end{array} \right) \quad \quad \Sigma_c = \left( \begin{array}{c} \hat{\Sigma}_b \Sigma_c \\ 0_{(n - \beta) \times (n - \beta)} \end{array} \right)
\]
so that
\[
BKC = \delta U_b \Sigma_c \left( \begin{array}{c} \hat{\Sigma}_b \Sigma_c \\ 0_{(n - \beta) \times (n - \beta)} \end{array} \right) W_b.
\]
Minimality guarantees that \( (6.3) \) is satisfied.

Next assuming that \( \min(\beta, \gamma) \geq r \) let
\[
\mathcal{V}_L := \{ B^* v_L \mid v_L^*(A - \lambda I_n) = 0 \} \quad \quad \mathcal{V}_R := \{ C v_R \mid (A - \lambda I_n)v_R = 0 \}.
\]
Namely, \( \mathcal{V}_L \subset \mathbb{C}^m \) and \( \mathcal{V}_R \subset \mathbb{C}^p \) can be spanned by \( r \) vectors and let \( K_L \in \mathbb{C}^{n \times r} \) and \( K_R \in \mathbb{C}^{p \times r} \) be isometries whose range is equal to the span of \( \mathcal{V}_L \) and \( \mathcal{V}_R \), respectively. Thus, if one takes
\[
\hat{K} = K_L K_R^* ,
\]
minimality guarantees that \( (6.3) \) is satisfied. Thus, there is only the value of the scalar \( \delta \) yet to consider.

If \( A = \lambda I_n \) for some \( \lambda \in \mathbb{C} \), minimality of the realization implies that \( m \geq \beta = n \) and \( p \geq \gamma = n \). Thus, \( \hat{K} = W_b^* U_c^* \) so any non-zero \( \delta \) will do. Hence, assume hereafter \( A \neq \lambda I_n \).
In fact, we claim that for any $|\delta|$ sufficiently small, $K$ satisfies the requirements. To this end recall, see e.g. [24] Corollary 7.3.8, that if $M$ and $\Delta$ are two matrices of the same dimensions then for all $k$,

$$\|\Delta\|_2 \geq |\sigma_k(M + \Delta) - \sigma_k(M)|$$

where $\sigma_k(\cdot)$ are the respective singular values ($\sigma_1 \geq \sigma_2 \geq \cdots$). For $j = 1, 2, \ldots$ let us denote by $\eta_j$ the smallest positive singular value of $A - \lambda_j I_n$ where $\lambda_j \in \text{spec}(A)$. Next let $\eta := \min(\eta_1, \eta_2, \ldots)$. Taking,

$$\delta = (2\eta\|B\|_2\|C\|_2)^{-1}$$

guarantees that none of the positive singular values of $A - \lambda_j I_n$ with $j = 1, 2, \ldots$ was moved to zero. Namely, $A_{cl} - \lambda I_n$ is nonsingular for all $\lambda \in \text{spec}(A)$, so the construction is complete.

As already mentioned, we do not know whether or not the above conditions on $\beta, \gamma$ are inherent to the problem or just a by-product of the technique we employed.

7. Static output feedback - GPE functions

We start with the following question: Under what conditions the $G\mathcal{P}$ class, and its subsets of $G\mathcal{PE}$ and Odd, are invariant under static output feedback.

**Proposition 7.1.** Consider the system $y(s) = \Psi(s)u(s)$ where $\Psi \in G\mathcal{P}$ is $p \times p$-valued rational function s.t. $\lim_{s \to \infty} \Psi(s) = 0$. Consider the static output feedback $u = Ky + u'$ where $K$ is a constant $p \times p$ matrix with $u'$ auxiliary input.

**A.** for all $K$ so that $-(K + K^*) \in \mathbb{F}_p$ also the resulting closed loop system is in $G\mathcal{P}$.

**B.** If in addition $\Psi$ is even, i.e. $\psi \in G\mathcal{PE}$ then, for all $K$ so that $-K \in \mathbb{F}_p$, also the resulting closed loop system is in $G\mathcal{PE}$.

**C.** If $\Psi \in \text{Odd}$, then, for all $K$ so that $K + K^* = 0$ also the resulting closed loop system is in Odd.

**Proof** Item A is part of [3] Proposition 8.1 (iii).

B. Substituting (1.8) in (6.1) yields, for the open loop

$$(7.1)\quad A = \begin{pmatrix} \hat{A} & B \hat{B}^* \\ 0 & -\hat{A}^* \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -\hat{C}^* \end{pmatrix}, \quad C = (\hat{C} \ 0), \quad D = 0.$$ 

Thus, for the closed loop

$$(7.2)\quad A_{cl} = \begin{pmatrix} \hat{A} & B \hat{B}^* \\ -\hat{C}^* K \hat{C} & -\hat{A}^* \end{pmatrix}.$$ 

Hence, for all $-K \in \mathbb{F}_p$, this $L_{cl}$ satisfies the conditions in (3.6), so this part of the claim is established.

C. Substituting (1.9) in (6.1) yields,

$$A_{cl} = \begin{pmatrix} T_1 + B_1 K B_1^* & \hat{A} - B_1 K B_2^* \\ \hat{A}^* + B_2 K B_1^* & T_2 - B_2 K B_2^* \end{pmatrix}, \quad B = (B_1 \ B_2), \quad C = (B_1^* - B_2^*) \quad D = 0.$$ 

Thus, for all $K$ skew-Hermitian ($K = -K^*$) this $L_{cl}$ satisfies condition (ii) in Theorem 1.1B with $H = \text{diag}\{-I_{\nu}, I_{n-\nu}, I_p\}$, so the claim is established $\square$
Example 7.2. We here illustrate item B in Proposition 7.1. Consider again the \( \mathcal{GP} \) function \( \Psi_2(s) = \frac{1}{s^2 + 1} \) from Example 4.3. Applying to \( \Psi_2(s) \) a static output feedback yields \( \Psi_{2c.1}(s) = \frac{1}{s^2 + 1} \), so that \( \Psi_{2c.1} \in \mathcal{GP} \) for all \( 0 > k \). Indeed, \( \Psi_{2c.1}(i\omega) = \frac{1}{\omega^2 + 1} \) for all \( \omega \in \mathbb{R} \).

We next use Proposition 7.1B to specialize the result of Proposition 6.1 to the class of \( \mathcal{GP} \) systems.

Corollary 7.3. Let \( \Psi \in \mathcal{GP} \) be a \( p \times p \)-valued rational function s.t. \( \lim_{s \to \infty} \Psi(s) = 0 \). Its realization matrix \( L = \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} \) satisfies (2.1) and the closed loop is as in (6.1). This realization is not minimal, if and only if there exists \( \lambda \in \text{spec}(A_{cl}) \), see (6.1), for all \( K - (K + K^*) \in \mathbb{P}_p \).

Indeed, as already pointed out in the proof of part B of Observation 4.2, from (3.7) it follows that \( \beta = \text{rank}(B) \) is equal to \( \gamma = \text{rank}(C) \). Thus, both directions of Proposition 6.1 hold.

Recall that in Theorem 2.2 and the proceeding discussion we pointed out that \( \Psi \in \mathcal{GP} \) always admits pseudo spectral factorization, i.e. \( \Psi = GG^* \) with \( G(s) \) analytic in \( \mathbb{C} \). It is desired for \( \Psi \) to admit spectral factorization, i.e. where the factor \( G(s) \) is analytic in \( \mathbb{C} \). Recall also that spectral factorization has various applications in control and filtering theories, see e.g. [6, 8 Section 16.3, 19 Section 6, 20, 22, 23, 31, 40]. We now address the following question: Assuming \( \Psi \in \mathcal{GP} \), does not admit spectral factorization, under what conditions can one apply a static output feedback so that \( \Psi_{cl} = (I_p - \Psi K)^{-1}\Psi \) does admit spectral factorization.

Specifically, in Proposition 7.1 and Corollary 7.3 we examined the use of static output feedback for moving the poles of \( \Psi \in \mathcal{GP} \) while retaining \( \Psi_{cl} \) in the \( \mathcal{GP} \) class (and \( \mathcal{GP} \) in particular). We know that a \( \mathcal{GP} \) function admits spectral factorization if it is analytic on the imaginary axis. Thus, the problem at hand is actually about moving by static output feedback poles of a \( \mathcal{GP} \) function away from the imaginary axis.

To gain intuition we first look at the scalar case. Let \( \Psi \in \mathcal{GP} \) be written as \( \Psi = \frac{N}{D} \) with \( N(s) \), \( D(s) \) polynomials so that, \( \frac{N(i\omega)}{D(i\omega)} \geq 0 \) (including +\( \infty \)) for all \( \omega \in \mathbb{R} \). Assuming that \( N(s) \) and \( D(s) \) have no common roots on \( i\mathbb{R} \), this in fact is equivalent to having for all \( \omega \in \mathbb{R} : N(i\omega) \geq 0, D(i\omega) \geq 0 \) and \( D(i\omega) + N(i\omega) > 0 \). This in turn is equivalent to having, for all \( \omega \in \mathbb{R} : N(i\omega) \geq 0, D(i\omega) \geq 0 \) and \( D(i\omega) + \alpha N(i\omega) > 0 \) for all \( \alpha > 0 \). Next recall that \( \Psi_{cl} = (I_p - \Psi K)^{-1}\Psi = \frac{N}{D + \alpha N} \). In conform with part B of Proposition 7.1 for all \( -k > 0 \): For all \( \omega \in \mathbb{R} \) the closed loop denominator satisfies \( (D(i\omega) - kN(i\omega)) > 0 \). Thus in fact \( \alpha = -k \) and the closed loop denominator is positive on \( i\mathbb{R} \). To conclude, if \( N(s) \) and \( D(s) \) have no common imaginary root, \( \Psi_{cl} \) is analytic on \( i\mathbb{R} \).

Before stating the result we need some preliminaries. For a pair \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times m} \) and a region \( \Omega \subseteq \mathbb{C} \), one can define the condition,

\[
\text{rank}(A - \lambda I_n : B) = n \quad \forall \lambda \in \Omega.
\]

In the spirit of Proposition 6.1 when restricted to the imaginary axis, the realization is minimal.
If \( \Omega = \mathbb{C} \) the pair \( A, B \) is said to be controllable. If \( \Omega \) is a subset of \( \mathbb{C} \), this is equivalent to the existence of a matrix \( R \in \mathbb{C}^{m \times n} \) so that \( \text{spec}(A + BR) \cap \Omega = \emptyset \) (in particular, for \( \Omega = \mathbb{C}_+ \) the pair is said to be stabilizable, see e.g. [27 p. 205] [31 sub-section 4.4]).

Similarly, for a pair \( A \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{p \times n} \) one can define the condition,

\[
\text{rank} \left( \frac{A - \lambda I_n}{C} \right) = n \quad \forall \lambda \in \Omega.
\]

For \( \Omega = \mathbb{C} \) the pair \( A, C \) is said to be observable, for \( \Omega = \mathbb{C}_+ \) the pair is detectable, see e.g. [31, sub-section 4.4]).

It is of interest to point here out that in Proposition 6.1 we required minimality of realization, i.e. \( \Omega = \mathbb{C} \). From [30, Theorem] it follows that having the pair \( A, B \) stabilizable and the pair \( A, C \) detectable (i.e. \( \Omega = \mathbb{C}_+ \)) is necessary for stabilizability of a system by static output feedback. Below, we can be “modest” by resorting to \( \Omega = i\mathbb{R} \) and reformulate Proposition 1.2.

**Proposition 7.4.** Let \( \Psi(s) \) be a \( p \times p \)-valued rational \( \mathcal{GPE} \) function, which does not admit spectral factorization. Assume that \( \lim_{s \to \infty} \Psi(s) = 0 \) and let the state space realization be as in (7.1).

There exists a static output feedback gain \( K, -K \in \mathbb{F}_p \) so that \( \Psi_{\text{cl}}(s) \) admits spectral factorization, if and only if for \( r \in \mathbb{R} \) the two following matrices

\[
\left( \hat{A} - irI_n \right) \quad \text{and} \quad \left( \hat{A} - irI_n \right)
\]

are of full rank.

**Proof.** First, recall that \( \Psi_{\text{cl}}(s) \) admits spectral factorization if and only if in the corresponding state space realization, the spectrum of \( A_{c.l.} \) avoids the imaginary axis. Following Proposition 7.4 this in turn is equivalent to finding conditions on \( \hat{A}, \hat{B} \) and \( \hat{C} \) so that there exists \( -K \in \mathbb{F}_p \) so that \( A_{c.l.} \) in (7.2), will have no eigenvalues on the imaginary axis, i.e. the matrix

\[
A_{c.l.} - irI_{2n} = \left( \begin{array}{cc} \hat{A} - irI_n & \hat{B}^* \\ -\hat{C}^* & -\left(\hat{A} - irI_n\right)^* \end{array} \right) \quad \forall r \in \mathbb{R}
\]

is nonsingular. Note now that the nonsingularity of \( A_{c.l.} - irI_{2n} \) in (7.3) is equivalent to that of

\[
M := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} (A_{c.l.} - irI_{2n}) = \begin{pmatrix} \hat{C}^* & \hat{B}^* \\ \hat{A} - irI_n \end{pmatrix}. \quad \forall r \in \mathbb{R}
\]

Namely, we search for \( -K \in \mathbb{F}_p \) so that for all \( r \in \mathbb{R} \) the Hermitian matrix \( M \) is nonsingular.

Next consider the Lyapunov equation \( Q = HM + M^*H \) with \( H = \text{diag}\{-I_n, I_n\} \) and

\[
Q = 2\text{diag}\{-\hat{C}^*K\hat{C}, \hat{B}\hat{B}^*\}.
\]

Now for all \( -K \in \mathbb{F}_p \), indeed \( Q \in \mathbb{F}_{2n} \). This implies that the matrix \( M \) has at most \( n \) eigenvalues in each open half plane, see e.g. [25 Lemma 2.4.5]. Furthermore, from the Generalized Inertia Theorem for the Lyapunov equation, see e.g. [25 Theorems 2.4.7, 2.4.10], it follows that there are exactly \( n \) eigenvalues in each open half plane, if and only if there exists \( K, -K \in \mathbb{F}_p \) so that the pair \( M, Q \) is observable.

\[ \text{An additional Riccati type condition makes it also sufficient.} \]
Recall now that the pair $M, Q$ is not observable, if and only if, there exists a (right) eigenvector of $M$ which lies in the null-space of $Q$. Namely, there exists $0 \neq v \in \mathbb{C}^{2n}$ so that $Mv = \lambda v$ for some $\lambda \in \mathbb{R}$ and $Qv = 0$. This in turn is equivalent to having, $(M + \frac{i}{Q})v = \lambda v$ and $Qv = 0$. Note that

$$\hat{M} := M + \frac{i}{Q}Q = \left( \begin{array}{cc} 0 & (\hat{A} - irI_n)^* \\ \hat{A} - irI_n & 0 \end{array} \right).$$

To summarize: The pair $\hat{M}, Q$ is observable if and only if the pair $M, Q$ is observable and this is equivalent to the nonsingularity of $\hat{M}$ (which in turn is equivalent to the nonsingularity of $A_{e,1} - irI_{2n}$).

Now, the pair $\hat{M}, Q$ is observable if and only if there exists $K, -K \in \mathbb{T}_p$ so that for all $r \in \mathbb{R}$, both matrices: $(\hat{A} - irI_n)$ and $(\hat{A} - irI_n)^*$ are of full rank.

Next, the matrix $(\hat{A} - irI_n)^*$ is of full rank, if and only if the matrix $(\hat{A} - irI_n : \hat{B})$ is of full rank, so the first part of the condition in the claim is established.

Note now that if the matrix $(\hat{A} - irI_n)$ is of full rank for some $K, -K \in \mathbb{T}_p$ it is of full rank for $K = -I_p$, i.e. the matrix $(\hat{A} - irI_n)$ is of full rank. This in turn is equivalent to having the matrix $(\hat{A} - irI_n)$ of full rank, so the second part of the condition in the claim is established and the proof is complete.

We now illustrate the result of Proposition 7.4.

**Example 7.5.** Consider the $\mathcal{GPE}$ function $\Psi_2(s) = \frac{1}{s - k}$ from Examples 4.3 and 7.2. As shown in Example 4.3 $\Psi_2$ admits only pseudo-spectral factorization. However, $\Psi_{2c.1}(s) = \frac{1}{s + k}$ with $0 > k$, see Example 7.2 can be factored to $\Psi_{2c.1} = G_2G_2^\#$ with $G_2(s) = \frac{1}{\sqrt{-k-s}}$. From Example 4.3 it follows that

$$A_{c.} = \begin{pmatrix} \hat{A} & \hat{B}^* \\ -k\hat{C}^* & -\hat{A} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \quad 0 > k.$$

Indeed, the realization $\hat{A}, \hat{B}, \hat{C}$ given in Example 4.3 is minimal and hence in particular the conditions in Proposition 7.4 are satisfied. Thus, indeed the matrix $A_{c.}$ has no imaginary eigenvalues.

From the proof of Proposition 7.4 it follows that if $\Psi \in \mathcal{GPE}$ and $K, -K \in \mathbb{T}_p$ are so that $\Psi_{c.} = (I_p - \Psi K)^{-1}\Psi$ admits spectral factorization, the same is true for $\alpha K$ where $\alpha > 0$ may be arbitrarily small. Namely, $\Psi_{c.}(s)$ may be a small perturbation of $\Psi(s)$.

We conclude this section by noting that upon comparing Propositions 6.1 and 7.4 one can make the following statement.

**Corollary 7.6.** Let $G(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ be a realization of a rational function so that both $\hat{B}, \hat{C} \in \mathbb{C}^{p×n}$ are of the same rank. The following are equivalent.

- There exists a static output feedback gain $K \in \mathbb{C}^{p×p}$ so that the closed loop system $(I_p - GK)^{-1}G$ is analytic on $i\mathbb{R}$.
- There exists a static output feedback with $-K \in \mathbb{T}_p$, so that the closed loop system $(I_p - GG^\# K)^{-1}GG^\#$ is analytic on $i\mathbb{R}$.
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