On the existence of helical invariant solutions to steady Navier-Stokes equations

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Abstract

In this paper, we investigate the nonhomogeneous boundary value problem for the steady Navier-Stokes equations in a helically symmetric spatial domain. When data is assumed to be helical invariant and satisfies the compatibility condition, we prove this problem has at least one helical invariant solution.

Keyword: Steady Navier-Stokes equations, helically symmetric flow

1 Introduction

Let $\Omega = \Omega_0 \cup \bigcup_{j=1}^{N} \bar{\Omega}_j$ be a bounded multiply connected domain in $\mathbb{R}^3$ with $C^2$-smooth boundary $\partial \Omega = \bigcup_{j=0}^{N} \Gamma_j$ consisting of $N+1$ disjoint components $\Gamma_j = \partial \Omega_j$, $j = 0, ..., N$. Consider the nonhomogeneous boundary value problem for the steady Navier-Stokes equations

$$
\begin{aligned}
(u \cdot \nabla) u + \nabla p &= \Delta u + f \quad \text{in} \ \Omega, \\
\text{div} \ u &= 0 \quad \text{in} \ \Omega, \\
\ u &= a \quad \text{on} \ \partial \Omega,
\end{aligned}
$$

(1.1)

where $u$ and $p$ are unknown velocity and pressure, $a$ and $f$ are given boundary value and body force, we assume the viscous coefficient $\nu = 1$ for simplicity. The boundary data should satisfy the compatibility condition

$$
\int_{\partial \Omega} a \cdot n dS = \sum_{j=0}^{N} \int_{\Gamma_j} a \cdot n dS = \sum_{j=0}^{N} F_j = 0,
$$

(1.2)

where $n$ is a unit outward normal vector to $\partial \Omega$.

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In his remarkable article [27], Leray proved the solvability of \((1.1)\) when the flux of the velocity across each connected component \(\Gamma_j\) of the boundary vanishes:

\[
\int_{\Gamma_j} \mathbf{a} \cdot \mathbf{n} \, dS = 0, \quad j = 0, \ldots, N.
\]

It remained an open problem as to whether the necessary condition \((1.2)\) is sufficient for \((1.1)\) to be solvable. This is also called Leray’s problem because it actually goes back to his paper [27].

Then, Leray’s problem has been studied in many papers [25, 40, 12, 13, 15, 3, 35, 36, 24, 37, 32, 33, 34], only recently its solvability was proved in bounded 2D domains and for 3D axially symmetric case under the sole necessary condition \((1.2)\) by M.Korobkov, K.Pileckas and R.Russo [21]. As far as we know, this problem for general 3D bounded domain remains open.

Given a positive number \(\sigma\), we define the action of the helical group of transformations \(G_\sigma\) on \(\mathbb{R}^3\) by

\[
S_{\theta,\sigma}(x) = \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \\ x_3 + \frac{\sigma}{2\pi} \theta \end{pmatrix}, \quad \theta \in \mathbb{R},
\]

that is, a rotation around the \(x_3\) axis with simultaneous translation along the \(x_3\) axis. \(G_\sigma\) is uniquely determined by \(\sigma\), which we will call the step. We say that the smooth function \(f(x)\) is helically symmetric or simply helical, if \(f\) is invariant under the action of \(G_\sigma\), i.e., \(f(S_{\theta,\sigma}(x)) = f(x)\), \(\forall \theta \in \mathbb{R}\). Similarly, we say that the smooth vector field \(\mathbf{u}(x)\) is helically symmetric, or simply helical, if it is covariant with respect to the action of \(G_\sigma\), i.e., \(M(\theta)\mathbf{u}(x) = \mathbf{u}(S_{\theta,\sigma}(x))\) for all \(\theta \in \mathbb{R}\), where

\[
M(\theta) := \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\] (1.3)

We learn from the definitions of helical function and vector field that they are \(\sigma\)-periodic in \(x_3\) variable. A domain \(\Omega \subset \mathbb{R}^2 \times T_\sigma\) is called a helical domain if for each point \(x \in \Omega\), \(S_{\theta,\sigma}(x) \in \Omega\) for any \(\theta \in \mathbb{R}\). Here we denote by \(T_\sigma = \mathbb{R}/\sigma\mathbb{Z}\) the corresponding 1-dimensional torus. In other words, the domain \(\Omega\) is evolved from a two dimensional multiply connected domain \(D \subset \mathbb{R}^2\), hence

\[
\Omega = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \exists (y_1, y_2) \in D \text{ and } \theta \in \mathbb{R} \text{ such that } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \cos \theta + y_2 \sin \theta \\ -y_1 \sin \theta + y_2 \cos \theta \\ \frac{\sigma}{2\pi} \theta \end{pmatrix} \right\}.
\]

There is an alternative definition of helical symmetry as follows. We rewrite a vector field \(\mathbf{u}(x) = (u_1, u_2, u_3)(x_1, x_2, x_3)\) with respect to the moving orthonormal frame associated to standard cylindrical coordinates \((r, \theta, z)\),

\[
\mathbf{e}_r = (\cos \theta, \sin \theta, 0), \quad \mathbf{e}_\theta = (-\sin \theta, \cos \theta, 0), \quad \mathbf{e}_z = (0, 0, 1),
\] (1.4)

as \(\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z\), where \(u_r, u_\theta, u_z\) are functions of \((r, \theta, z)\). We introduce two new independent variables in place of \(\theta\) and \(z\):

\[
\eta := \frac{\sigma}{2\pi} \theta + z, \quad \xi := \frac{\sigma}{2\pi} \theta - z.
\] (1.5)
A smooth function $p = p(r, \theta, z)$ is a helical function if and only if, when expressed in the $(r, \xi, \eta)$ variables, it is independent of $\xi$: $p = q(r, \frac{\sigma}{2\pi} \theta + z)$, for some $q = q(r, \eta)$. Indeed, by definition $f$ is helical if and only if $f(S_{\rho, \sigma}(x))$ is actually independent of $\rho$, being equal to $f(x)$. Then by the Chain Rule $\frac{d}{d\rho} f(S_{\rho, \sigma}(x)) = 0$ is equivalent to $\frac{d}{d\xi} \tilde{f}(r, \xi, \eta) = 0$, where

$$\tilde{f}(r, \xi, \eta) = f\left(r \cos \left(\frac{\pi}{\sigma}(\eta + \xi)\right), r \sin \left(\frac{\pi}{\sigma}(\eta + \xi)\right), \frac{\eta - \xi}{2}\right).$$

Similarly, a smooth vector field $u$ is helical if and only if there exist $v_r, v_\theta, v_z$ functions of $(r, \eta)$ such that $u_r(r, \theta, z) = v_r(r, \frac{\sigma}{2\pi} \theta + z)$, $u_\theta(r, \theta, z) = v_\theta(r, \frac{\sigma}{2\pi} \theta + z)$ and $u_z(r, \theta, z) = v_z(r, \frac{\sigma}{2\pi} \theta + z)$. Since $u_r(r, \theta, z) = u_r(r, \theta + 2\pi, z)$, then $v_r(r, \frac{\sigma}{2\pi} \theta + z) = v_r(r, \frac{\sigma}{2\pi} \theta + z + \sigma)$, that is to say, $v_r(r, \eta)$ is periodic in $\eta$ with period $\sigma$. Clearly, as $\sigma \to \infty$ a helical flow becomes 2D flow and when $\sigma = 0$ it becomes axisymmetric flow (the coefficients $u_r, u_\theta, u_z$ depend only on $r$ and $z$). Hence, a helical flow can be regarded as an interpolation between 2D and 3D axisymmetric flows.

It is well-known that the global regularity of unsteady Navier-Stokes equations is a longstanding open problem which is one of the seven Millennium Prize problems [11]. But if the initial data is two-dimensional or axisymmetric without swirl ([5, 6, 19]). However, if the initial data is helical invariant, then the global regularity of the Leray-Hopf weak solution was proved by A.Mahalov, E.Titi and S.Leibovich [30]. In particular, a helical invariant function $u$ satisfies the 2D Ladyzhenskaya inequality:

$$\|u\|_{L^2(\Omega)} \leq C\sigma^{\frac{1}{2}}\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}, \quad (1.6)$$

where $\Omega = \mathbb{R}^2 \times T_\sigma$. This coefficient $\sigma^{\frac{1}{2}}$ also reflects that the helical flow is an interpolation between 2D and 3D axisymmetric flows. There are more interesting iteratures concerning helical flows, see [2, 9, 38, 28, 17].

This note is devoted to solve the Leray’s problem in a helical domain with helical invariant data. More precisely, we prove the following theorem.

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^2 \times T_\sigma$ is a helical domain with $C^2$ boundary. If $a \in W^{3/2,2}(\partial\Omega)$ and $f \in L^2(\Omega)$ are helical invariant functions, condition (1.2) is fulfilled, then problem (1.1) admits at least one helical invariant weak solution.

**Remark 1.2.** Under the hypothesis of Theorem 1.1, every weak solution $u$ is indeed a strong solution, i.e. $u \in W^{2,2}(\Omega)$ (see, e.g., [16]). We need such regularity to apply Morse-Sard type theorem for regular level sets of Sobolev functions (see Theorem 2.4 and the commentary after the formula (3.48)), so $a \in W^{1,2}(\partial\Omega)$ is necessary.

Our proof is motivated by M.Korobkov, K.Pileckas and R.Russo’s approach which successfully solved the Leray’s problem in 2D and 3D axisymmetric domain in [21]. It is plausible that we can extend this result to the helical case. The proof in [21] was based on Leray’s contradiction arguments and on the integration using the coarea formula along the level lines of the total head pressure. Also they used some consequence of Bernoulli’s law for solutions $w$ to Euler equations with low regularity, namely, in [20, 22], the authors proved that for any compact connected set $K \subset \Omega$

$$\psi|_K \equiv \text{const} \Rightarrow \Phi|_K \equiv \text{const}, \quad (1.7)$$

3
where \( \psi \) is the corresponding stream function and \( \Phi = p + \frac{1}{2} |w|^2 \) is the total head pressure (the identity for \( \Phi \) is understood up to a negligible set of 1-dimensional measure zero).

The helical invariant functions form a closed subspace with respect to weak convergence, we can still use the contradiction arguments to derive a helical invariant weak solution \((w, p)\) to the Euler equations

\[
\begin{align*}
    w_r \frac{\partial w_r}{\partial r} + \left( \frac{\sigma}{2\pi r} w_\theta + w_z \right) \frac{\partial w_r}{\partial \eta} - \frac{w_\theta^2}{r} + \frac{\partial p}{\partial r} &= 0, \\
    w_r \frac{\partial w_\theta}{\partial r} + \left( \frac{\sigma}{2\pi r} w_\theta + w_z \right) \frac{\partial w_\theta}{\partial \eta} + \frac{w_r w_\theta}{r} + \frac{\sigma}{2\pi r} \frac{\partial p}{\partial \eta} &= 0, \\
    w_r \frac{\partial w_z}{\partial r} + \left( \frac{\sigma}{2\pi r} w_\theta + w_z \right) \frac{\partial w_z}{\partial \eta} + \frac{\partial p}{\partial \eta} &= 0, \\
    \frac{\partial}{\partial r} (rw_r) + \frac{\partial}{\partial \eta} \left( \frac{\sigma}{2\pi r} w_\theta + rw_z \right) &= 0, \\
    w|_{\partial \Omega} &= 0.
\end{align*}
\] (1.8)

Unfortunately, although an equation

\[
\left( w_r \frac{\partial}{\partial r} + \left( \frac{\sigma}{2\pi r} w_\theta + w_z \right) \frac{\partial}{\partial \eta} \right) \left( \frac{1}{2} |w|^2 + p \right) = 0.
\] (1.9)

is still valid, the Bernoulli’s law which appeared before is no longer true. To see this, let \( w_r = 0, w_\theta = r, w_z = -\frac{\sigma}{2\pi}, p = \frac{r^2}{2} \). This is an explicit solution to equations (1.8), the stream function is constant since we always have \( w_r = \frac{\sigma}{2\pi} w_\theta + w_z = 0 \), but the total head pressure \( \Phi = r^2 + \frac{\sigma^2}{8\pi} \) is not a constant. The similar problem appears in axially–symmetric case (see, e.g., [23, page 746]), nevertheless, the Bernoulli identity (1.7) will be satisfied here, due to zero boundary conditions (1.8) and since each straight line passing through a point of \( \Omega \) parallel to the axis of symmetry intersects \( \partial \Omega \).

But this is not true anymore for helical domains, therefore, the Bernoulli identity (1.7) fails in general for the solutions to (1.8) in helical case.

This is an obstacle in modifying the arguments in [22, 21, 23], since the Bernoulli’s law is needed to separate the boundary components on which the total head pressure attains its supremum from the others, this step played a crucial role in solving Leray’s problem in 3D axisymmetric domains.

On the other hand, if a function with a finite Dirichlet integral is helically symmetric, then its restriction to a two-dimensional hyperplane orthogonal to the axis of symmetry has a finite Dirichlet integral as well (see, e.g., the formula (3.17)), i.e., the corresponding plane function has no singularities. This circumstance allows us to simplify the arguments and carry out the proof without using identities of the form (1.7), which turned out to be so significant in the axisymmetric case (when the restriction of considered functions to the two-dimensional hyperplane have singularities near the symmetry axis).

This paper is organized as follows. In section 2, we deal with the nonhomogeneous boundary values and present some properties of the Sobolev functions. In section 3, we first use Leray’s contradiction arguments to derive a nontrivial solution to the helical Euler equations. Finally we construct a family of level sets of the total head pressure and deduce a contradiction via coarea formula.

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2 Preliminaries

In this section, we first find a solenoidal and helical invariant extension of the nonhomogeneous boundary values. Then we review the Morse-Sard property of Sobolev functions.

By a domain we mean an open connected set. We use standard notations for function spaces: $W^{k,q}(\Omega)$, $W^{a,q}(\partial \Omega)$, where $a \in (0, 1)$, $k \in \mathbb{N}_0$, $q \in [1, +\infty]$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces.

Denote by $H(\Omega)$ the closure of the set of all solenoidal smooth vector-functions having compact supports in $\Omega$ with respect to the norm $\|w\|_{H(\Omega)} = \left( \int_{\Omega} |\nabla w|^2 \right)^{\frac{1}{2}}$.

2.1 Extension of boundary values

We need to use the following symmetry assumptions:

(H0) $\Omega \subset \mathbb{R}^3$ is a helical domain with $C^2$ boundary and $O_{x_3}$ is a symmetry axis of $\Omega$.

(H1) The assumptions (H0) are fulfilled and both the boundary value $a \in W^{3/2, 2}(\partial \Omega)$ and $f = \text{curl} \ b \in L^2(\Omega)$ are helical invariant.

Lemma 2.1. If conditions (H1) and (1.2) are fulfilled, then there exists a helical solenoidal extension $A \in W^{2, 2}(\Omega)$ of $a$ with

$$\|A\|_{W^{2, 2}(\Omega)} \leq c \|a\|_{W^{3/2, 2}(\partial \Omega)}.$$ (2.1)

Proof. Let conditions (H1) and (1.2) be fulfilled. Define by $P$ the hyperplane $P = \{(y_1, y_2, 0) : y_1, y_2 \in \mathbb{R}\}$, with $\Omega \cap P = \Sigma = \partial \mathbb{D} = P \cap \partial \Omega$. Then by construction

$$\mathbb{D} = \Omega \cap \mathbb{D}.$$ (cf. with the Introduction). Note, that by our assumptions, $\mathbb{D}$ is a bounded two dimensional domain with $C^2$-smooth boundary $\Sigma = \partial \mathbb{D} = P \cap \partial \Omega$, 0 \not\in \mathbb{D}$, moreover,

every vector field $\tilde{A} : \mathbb{D} \to \mathbb{R}^3$ can uniquely be extended to a helical invariant vector field $A : \Omega \to \mathbb{R}^3$ such that $A|_\mathbb{D} = \tilde{A}$ on $\mathbb{D}$ and

$$\|A\|_{W^{2, 2}(\Omega)} \leq \epsilon \|\tilde{A}\|_{W^{2, 2}(\mathbb{D})}.$$ (2.2)

Indeed, if $\tilde{A}(r, \theta) = \tilde{A}_r(r, \theta) e_r + \tilde{A}_\theta(r, \theta) e_\theta + \tilde{A}_z(r, \theta) e_z$ with respect to the moving orthonormal frame associated to standard cylindrical coordinates $(r, \theta, z)$ (see (1.4)), then $A$ can be defined as $A = A_r e_r + A_\theta e_\theta + A_z e_z$ with

$$A_r(r, \theta, z) := \tilde{A}_r(r, \theta + \frac{2\pi}{\sigma} z), \quad A_\theta(r, \theta, z) := \tilde{A}_\theta(r, \theta + \frac{2\pi}{\sigma} z), \quad A_z(r, \theta, z) := \tilde{A}_z(r, \theta + \frac{2\pi}{\sigma} z)$$ (2.3)

(see the Introduction for the explanations). Using standard elementary formulas for divergence operator for plane vector fields in polar coordinate system $(r, \theta)$ on $\mathbb{D}$ and for spatial vector fields in
cylindrical coordinate system \((r, \theta, z)\) on \(\Omega\), it is easy to check that for above helical extension procedure the following property holds:

\[
\left( \text{div } A = 0 \text{ in } \Omega \right) \iff \left( \text{div } \tilde{B} = 0 \text{ in } \mathbb{D} \right), \tag{2.4}
\]

where we denote by \(\tilde{B}\) the plane vector field \(\tilde{B} = \tilde{A}_r e_r + (\tilde{A}_\theta + \frac{2 \pi}{\sigma} \tilde{A}_r) e_\theta\) on \(\mathbb{D}\). Moreover, using classical elementary formulas for surface integrals, we have

\[
\left( \int_{\partial \Omega} A \cdot n \, dS = 0 \right) \iff \left( \int_{\partial \Omega} \tilde{B} \cdot n \, ds = 0 \right),
\]

where \(dS\) and \(ds\) mean integration with respect to surface area and length respectively.

Thus, the problem of a helical solenoidal extension from the boundary of spatial domain \(\Omega\) can be reduced to the problem of solenoidal extensions for two dimensional vector fields in plane domains, which is well known classical fact (see, e.g., [16] Chapter III.3).

\[\square\]

Then by the existence results of Stokes system [16, Chapter 3], we can find a unique solution \(U \in W^{2,2}(\Omega)\) to the Stokes problem such that \(\mathbf{U} - A \in H(\Omega) \cap W^{2,2}(\Omega)\), and

\[
\int_\Omega \nabla \mathbf{U} \cdot \nabla \varphi \, dx = \int_\Omega \mathbf{f} \cdot \varphi \, dx, \quad \forall \varphi \in H(\Omega). \tag{2.5}
\]

Moreover,

\[
\| \mathbf{U} \|_{W^{2,2}(\Omega)} \leq c(\| a \|_{W^{1/2,2}(\partial \Omega)} + \| \mathbf{f} \|_{L^2(\Omega)}). \tag{2.6}
\]

Actually, uniqueness ensures the solution \(\mathbf{U}\) is also a helical invariant function, since \(A\) is helical invariant. Indeed, for any \(\xi_0 \in [0, \sigma]\) denote \(U_{\xi_0}(r, \eta, \xi) = \mathbf{U}(r, \eta, \xi - \xi_0)\), then \(U_{\xi_0} - A \in H(\Omega) \cap W^{2,2}(\Omega)\) since \(A\) is helical invariant. We learn from (2.5)

\[
\int_\Omega \nabla \mathbf{U}_{\xi_0} \cdot \nabla \varphi \, dx = \int_\Omega \nabla \mathbf{U} \cdot \nabla \varphi_{-\xi_0} \, dx = \int_\Omega \mathbf{f} \cdot \varphi_{-\xi_0} \, dx = \int_\Omega \mathbf{f}_{\xi_0} \cdot \varphi \, dx. \tag{2.7}
\]

By our assumption, \(\mathbf{f}\) is helical invariant, therefore, \(\mathbf{f}_{\xi_0} = \mathbf{f}\). We have \(\mathbf{U}_{\xi_0} = \mathbf{U}\), in another words, \(\mathbf{U}\) is independent of \(\xi\) and \(\mathbf{U}\) is helical invariant. Associated with the Stokes problem (2.5), there is a pressure term, which we denoted it to be \(p\).

Now we consider \(w = \mathbf{u} - \mathbf{U}, \tilde{p} = p - Q\), to simplify the notation, we still write \(\tilde{p}\) as \(p\), equation (1.1) is equivalent to

\[
\begin{cases}
-\Delta \mathbf{w} + (\mathbf{U} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{U} + (\mathbf{w} \cdot \nabla) \mathbf{U} = -\nabla p - (\mathbf{U} \cdot \nabla) \mathbf{U} & \text{in } \Omega, \\
\text{div } \mathbf{w} = 0 & \text{in } \Omega, \\
\mathbf{w} = 0 & \text{on } \partial \Omega.
\end{cases} \tag{2.8}
\]

By a weak solution to problem (1.1), we mean a function \(\mathbf{u} \in W^{1,2}(\Omega)\) such that \(\mathbf{w} = \mathbf{u} - \mathbf{U} \in H(\Omega)\) and for any \(\varphi \in H(\Omega),\)

\[
\langle \mathbf{w}, \varphi \rangle_{H(\Omega)} = -\int_\Omega (\mathbf{U} \cdot \nabla) \mathbf{U} \cdot \varphi \, dx - \int_\Omega (\mathbf{U} \cdot \nabla) \mathbf{w} \cdot \varphi \, dx \\
- \int_\Omega (\mathbf{w} \cdot \nabla) \mathbf{w} \cdot \varphi \, dx - \int_\Omega (\mathbf{w} \cdot \nabla) \mathbf{U} \cdot \varphi \, dx. \tag{2.9}
\]
By Riesz representation theorem, for any \( w \in H(\Omega) \) there exists a unique function \( Tw \in H(\Omega) \) such that the right hand side of (2.9) is equivalent to \( \langle Tw, \varphi \rangle_{H(\Omega)} \), for any \( \varphi \in H(\Omega) \). Moreover, \( T \) is a compact operator by the well-known result in [26].

**Lemma 2.2.** For any helical function \( w \in H(\Omega) \), \( Tw \in H(\Omega) \) is also a helical function.

*Proof.* The proof is the same as above. We omit the details here. \( \square \)

### 2.2 Morse-Sard property of Sobolev functions

The following lemma is concerned with the classical differentiability properties of Sobolev functions.

**Lemma 2.3** ([8]). If \( \psi \in W^{2,1}(\mathbb{R}^2) \), then \( \psi \) is continuous and there exists a set \( A_\psi \) such that \( H^1(A_\psi) = 0 \) and \( \psi \) is differentiable (in the classical sense) at each \( x \in \mathbb{R} \setminus A_\psi \). Furthermore, the classical derivative at these points \( x \) coincides with \( \nabla \psi(x) = \lim_{r \to 0} \frac{1}{r} \int_{B_r(x)} \nabla \psi(y) dy \), where \( \lim_{r \to 0} \frac{1}{r} \int_{B_r(x)} |\nabla \psi(y) - \nabla \psi(x)|^2 dy = 0 \).

Here and henceforth we denote by \( H^1 \) the one-dimensional Hausdorff measure, i.e., \( H^1(F) = \lim_{t \to 0^+} H^1 t(F) \), where \( H^1 t(F) = \inf \{ \sum_{i=1}^{\infty} \text{diam} F_i : \text{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \} \).

The following Morse-Sard Theorem for Sobolev function has been proved by J.Bourgain, M.Korobkov and J.Kristensen [4, 5].

**Theorem 2.4.** Let \( D \subset \mathbb{R}^2 \) be a bounded domain with Lipschitz boundary and \( \psi \in W^{2,1}(D) \). Then:

1. \( H^1(\{ \psi(x) : x \in \bar{D} \setminus A_\psi \ & \ \nabla \psi(x) = 0 \}) = 0 \);

2. for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every set \( U \subset \bar{D} \) with \( H^1 \infty(U) < \delta \) the inequality \( H^1(\psi(U)) < \epsilon \) holds;

3. for \( H^1 \)-almost all \( y \in \psi(\bar{D}) \subset \mathbb{R} \) the preimage \( \psi^{-1}(y) \) is a finite disjoint family of \( C^1 \)-curves \( S_j, j = 1, 2, ..., N(y) \). Each \( S_j \) is either a cycle in \( D \) (i.e., \( S_j \subset D \) is homemorphic to the unit circle \( S^1 \)) or a simple arc with endpoints on \( \partial D \) (in this case \( S_j \) is transversal to \( \partial D \)).

### 3 Proof of Theorem 1.1

#### 3.1 Contradiction argument

Without loss of generality, we may assume that \( f = \text{curl} b \in L^2(\Omega) \). In particular,

\[
\text{div} f = 0 \quad (3.1)
\]

in the sense of distributions.

It is obvious, (2.9) is equivalent to the operator equation \( w = Tw \) in \( H(\Omega) \). By Leray-Schauder theorem, to prove the existence of weak solution is sufficient to show that the solutions of equation

\[1^\text{By the Helmholtz–Weyl decomposition, } f \text{ can be represented as the sum } f = \text{curl} b + \nabla \varphi \text{ with } \text{curl} b \in L^2(\Omega), \text{ and the gradient part is included then into the pressure term (see, e.g., [26], [16]).} \]
\( w^{(l)} = \lambda^l w^{(l)} \) are uniformly bounded with respect to \( \lambda \in [0, 1] \). In another word, the solution \( w \in H(\Omega) \) satisfies for any \( \varphi \in H(\Omega) \),

\[
\int_{\Omega} \nabla w \cdot \nabla \varphi \, dx = -\lambda \int_{\Omega} (U \cdot \nabla) U \cdot \varphi \, dx - \lambda \int_{\Omega} (U \cdot \nabla) w \cdot \varphi \, dx
\]

are uniformly bounded in \( H(\Omega) \) with respect to \( \lambda \in [0, 1] \).

Assume this is false. Then there exist \( \{\lambda_n\}_{n \in \mathbb{N}} \subset [0, 1] \) and \( \{\hat{w}_n\}_{n \in \mathbb{N}} \subset H(\Omega) \) such that, for any \( \varphi \in H(\Omega) \),

\[
\int_{\Omega} \nabla \hat{w}_n \cdot \nabla \varphi \, dx = -\lambda_n \int_{\Omega} (U \cdot \nabla) U \cdot \varphi \, dx - \lambda_n \int_{\Omega} (U \cdot \nabla) \hat{w}_n \cdot \varphi \, dx
\]

and

\[
\lambda_n \to \lambda_0 \in [0, 1], J_n := \|\hat{w}_n\|_{H(\Omega)} \to \infty
\]

After integration by part, (3.2) is equivalent to

\[
\int_{\Omega} \nabla \hat{w}_n \cdot \nabla \varphi \, dx = \lambda_n \int_{\Omega} (U \cdot \nabla) \varphi \cdot U \, dx + \lambda_n \int_{\Omega} (U \cdot \nabla) \varphi \cdot \hat{w}_n \, dx
\]

there exists a subsequence \( w_{n_l} \) converging weakly in \( H(\Omega) \) to a function \( w \in H(\Omega) \). Then by compact embedding of Sobolev space, \( w_{n_l} \) converges strongly in \( L^r(\Omega) \), for any \( r \in [1, 6) \). Taking \( \varphi = J_n^{-1} \hat{w}_n \) in (3.3), we get

\[
\int_{\Omega} |\nabla w_n|^2 \, dx = \lambda_n \int_{\Omega} (w_n \cdot \nabla) w_n \cdot U \, dx + J_n^{-1} \lambda_n \int_{\Omega} (U \cdot \nabla) w_n \cdot U \, dx
\]

Therefore, taking into account (3.4) and passing to a limit as \( n_l \to \infty \) in (3.5), we obtain

\[
1 = \lambda_0 \int_{\Omega} (w \cdot \nabla) w \cdot U \, dx
\]

in particular, this implies \( \lambda_0 \in (0, 1) \).

Let us return to (3.3), for any \( \zeta \in W^{1,2}_0(\Omega) \), consider the linear functional

\[
R_n(\zeta) = \int_{\Omega} \nabla \hat{w}_n \cdot \nabla \zeta \, dx - \lambda_n \int_{\Omega} (U \cdot \nabla) \zeta \cdot U \, dx - \lambda_n \int_{\Omega} (U \cdot \nabla) \zeta \cdot \hat{w}_n \, dx
\]

\[
- \lambda_n \int_{\Omega} (\hat{w}_n \cdot \nabla) \zeta \cdot \hat{w}_n \, dx - \lambda_n \int_{\Omega} (\hat{w}_n \cdot \nabla) \zeta \cdot U \, dx
\]
It is obvious that $R_n(\zeta)$ has the following estimate

$$|R_n(\zeta)| \leq C(||\hat{w}_n||_{H^2(\Omega)} + ||\hat{w}_n||^2_{H(\Omega)})$$

Again (3.3) implies

$$R_n(\varphi) = 0, \quad \forall \varphi \in H(\Omega).$$

Therefore, there exists a unique function $\hat{p}_n \in L^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q(x)dx = 0\}$ such that

$$R_n(\varphi) = \int_\Omega \hat{p}_n \varphi dx, \quad \forall \varphi \in H(\Omega).$$

actually $\hat{p}_n$ is a helical function, this can be established by uniqueness since the functions $\hat{w}_n$ and $U$ on the RHS of (3.7) are helical invariant. The proof of this part is similar to (2.7). Then we can learn from (3.8),

$$\|\hat{p}_n\|_{L^2(\Omega)} \leq C||R_n||_{H^{-1}(\Omega)}$$

$$\leq C(||\hat{w}_n||_{H^2(\Omega)} + ||\hat{w}_n||^2_{H(\Omega)}) + ||a||_{W^{3/2}(\partial\Omega)}^2 + ||f||_{L^2(\Omega)}^2$$

where we have used (2.6).

Combine (3.7) – (3.9), the pair ($\hat{w}_n, \hat{p}_n$) satisfies for any $\zeta \in W^{1,2}_0(\Omega)$

$$\int_\Omega \nabla \hat{w}_n \cdot \nabla \zeta dx - \lambda_n \int_\Omega (\nabla \cdot \nabla) \zeta \cdot \nabla \hat{p}_n dx \leq \int_\Omega (\nabla \cdot \nabla) \zeta \cdot \hat{w}_n dx$$

$$-\lambda_n \int_\Omega (\hat{w}_n \cdot \nabla) \zeta \cdot \hat{w}_n dx - \lambda_n \int_\Omega (\hat{w}_n \cdot \nabla) \zeta \cdot U dx = \int_\Omega \hat{p}_n \varphi dx \quad (3.11)$$

Let $\hat{u}_n = \hat{w}_n + U$, then (3.11) is equivalent to for any $\zeta \in W^{1,2}_0(\Omega)$

$$\int_\Omega \nabla \hat{u}_n \cdot \nabla \zeta dx - \int_\Omega \hat{p}_n \varphi dx \zeta dx = -\lambda_n \int_\Omega (\hat{u}_n \cdot \nabla) \hat{u}_n \cdot \zeta dx + \int_\Omega f \cdot \zeta dx$$

where we have used (2.5).

In another word, the pair ($\hat{u}_n, \hat{p}_n$) satisfies the following Navier-Stokes equations

$$\begin{cases}
-\Delta \hat{u}_n + \nabla \hat{p}_n = -\lambda_n (\hat{u}_n \cdot \nabla) \hat{u}_n + f & \text{in } \Omega, \\
\text{div } \hat{u}_n = 0 & \text{in } \Omega, \\
\hat{u}_n = a & \text{on } \partial\Omega.
\end{cases}$$

(3.12)

By the well-known $L^4$-estimates of Stokes systems (see, e.g., [16, P. 279, Theorem IV.4.1]), equations (3.12) imply

$$||\hat{u}_n||_{W^{3/2}(\Omega)} + ||\hat{p}_n||_{W^{1,3/2}(\Omega)} \leq C(||(\hat{u}_n \cdot \nabla) \hat{u}_n||_{L^{3/2}(\Omega)} + ||f||_{L^2(\Omega)} + ||a||_{W^{3/2}(\partial\Omega)})$$

$$\leq C(||\hat{u}_n||^2_{W^{1,2}(\Omega)} + ||f||_{L^2(\Omega)} + ||a||_{W^{3/2}(\partial\Omega)}) \quad (3.13)$$

note that we have used (3.10) in the last step.
Let \( u_n = J_n^{-1} \hat{u}_n \) and \( p_n = \lambda_n^{-1} J_n^{-2} \hat{p}_n \). It is obvious that \( \|u_n\|_{W^{1,2}(\Omega)} \) and \( \|p_n\|_{W^{1,3/2}(\Omega)} \) are uniformly bounded, and

\[
\begin{align*}
-\nu_n \Delta u_n + (u_n \cdot \nabla) u_n + \nabla p_n &= f_n \quad \text{in } \Omega, \\
\text{div } u_n &= 0 \quad \text{in } \Omega, \\
\text{div } u_n &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(3.14)

where \( \nu_n = \lambda_n^{-1} J_n^{-1} \), \( f_n = \lambda_n^{-1} J_n^{-2} f \), and \( a_n = J_n^{-1} a \). By the observation above, we can extract weakly convergent subsequences \( u_n \rightharpoonup w \) in \( W^{1,2}(\Omega) \) and \( p_n \rightharpoonup p \) in \( W^{1,3/2}_{\text{loc}}(\Omega) \). The pair \((w, p)\) satisfies the following Euler equations

\[
\begin{align*}
(w \cdot \nabla) w + \nabla p &= 0 \quad \text{in } \Omega, \\
\text{div } w &= 0 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(3.15)

We summarize the above results as follows.

**Lemma 3.1.** Assume that \( \Omega \subset \mathbb{R}^2 \times \mathbb{T}_r \) is a helical domain with \( C^2 \) boundary \( \partial \Omega \), \( f = \text{curl } b \), \( b \in W^{1,2}(\Omega) \) and \( a \in W^{3/2,2}(\partial \Omega) \) are both helical functions and (1.2) is valid. If the assertion of Theorem 1.1 is false, then there exist \( w, p \) with the following properties:

**E–H** The helical functions \( w \in W^{1,2}_0(\Omega), p \in W^{1,3/2}(\Omega) \) satisfy the Euler equations (3.15), moreover equation (3.6) holds.

**E–NS–H** Conditions **E–H** are satisfied, and there exist sequences of helical functions \( u_n \in W^{1,2}(\Omega), p_n \in W^{1,3/2}(\Omega) \) and numbers \( \nu_n \to 0^+, \lambda_n \to \lambda_0 > 0 \) such that the norms \( \|u_n\|_{W^{1,2}(\Omega)} \), \( \|p_n\|_{W^{1,3/2}(\Omega)} \) are uniformly bounded, the pair \((u_n, p_n)\) satisfies (3.14), and

\[
\|\nabla u_n\|_{L^{2}(\Omega)} \to 1, \quad \|\nabla u_n\|_{L^{2}(\Omega)} \to 1, \quad u_n \to w \text{ in } W^{1,2}(\Omega), \quad p_n \to p \text{ in } W^{1,3/2}(\Omega).
\]

Moreover,

\[
u_n \in W^{2,2}(\Omega) \quad \text{and} \quad p_n \in W^{2,1}_{\text{loc}}(\Omega).
\]

(3.16)

Note, that the last inclusion follows from (3.1) and from the fact that locally the laplacian \( \Delta p \) belongs to the Hardy space (see, e.g., [20, Section 5.2] for more detailed description).

### 3.2 Euler equations

In this subsection we discuss the properties satisfied by the helical invariant solution to Euler equation (3.15).

Define by \( P \) the hyperplane \( P = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\} \). Then by construction

\[
\mathbb{D} = \Omega \cap P.
\]

It follows from the helical symmetry of the vector field \( w \) that \( \|w\|_{L^q(\mathbb{D})} = \sigma \|w(\cdot, 0)\|_{L^q(\mathbb{D})} \), hence we have

\[
\nabla w \in L^{2}(\mathbb{D}), \quad w \in L^{q}(\mathbb{D}) \quad \forall q < \infty,
\]

(3.17)
consequently, from Euler system we have

$$\nabla p \in L^q(\Omega) \quad \forall q < 2. \quad (3.18)$$

In this section and below for any set $S \subset \mathbb{R}^2$ we define $\bar{S} \subset \mathbb{R}^2 \times \mathbb{T}_\sigma$ to be the three dimensional set which is evolved from $S$, i.e

$$\bar{S} = \left\{ \begin{bmatrix} y_1 \cos \theta + y_2 \sin \theta \\ -y_1 \sin \theta + y_2 \cos \theta \end{bmatrix} : (y_1, y_2) \in S, \theta \in \mathbb{R} \right\}. \quad (3.19)$$

Denote $\Sigma_j := P \cap \Gamma_j$. Clearly, $\bar{\Sigma}_j = \Gamma_j, j = 0, \ldots, N$, and

$$\partial \Omega = \bigcup_{j=0}^N \Sigma_j. \quad (3.20)$$

The next statement was proved in [18, Lemma 4] and in [1, Theorem 2.2].

**Lemma 3.2.** If $(E-H)$ are satisfied, then

$$\exists \tilde{p}_j \in \mathbb{R} : p(x) \equiv \tilde{p}_j \text{ for } \mathcal{H}^2\text{-almost all } x \in \Gamma_j, j = 0, \ldots, N.$$  

In particular, by helical symmetry,

$$p(x) \equiv \tilde{p}_j \text{ for } \mathcal{H}^1\text{-almost all } x \in \Sigma_j, j = 0, \ldots, N.$$  

By simple calculation from (3.6), (3.15) and Corollary 3.2, it follows that

**Corollary 3.3.** If conditions $(E-NS-H)$ are satisfied, then

$$-\frac{1}{\lambda_0} = \sum_{j=0}^N \tilde{p}_j \int_{\Gamma_j} a \cdot n dS = \sum_{j=0}^N \tilde{p}_j \mathcal{F}_j. \quad (3.21)$$

Set $\Phi_n = p_n + \frac{1}{2} |u_n|^2$ and $\Phi = p + \frac{1}{2} |w|^2$. By the properties of Sobolev functions best representatives for $w, \Phi$ (see [10]), we get the following.

**Lemma 3.4.** If conditions $(E-H)$ hold, then there exists a set $A_w \subset \Omega$ such that

1. $\mathcal{H}^1(A_w) = 0$;
2. for all $x \in \Omega \setminus A_w$,

$$\lim_{\rho \to 0} \int_{B_\rho(x)} |w(y) - w(x)|^2 dy = \lim_{\rho \to 0} \int_{B_\rho(x)} |\Phi(y) - \Phi(x)|^2 dy = 0; \quad (3.22)$$
3. for every $\epsilon > 0$, there exists an open set set $U \subset \mathbb{R}^2$ with $\mathcal{H}_{\infty}^1(U) < \epsilon, A_w \subset U$ and such that the functions $w, \Phi$ are continuous on $\Omega \setminus U$.  

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3.3 Obtaining a contradiction

We consider two possible cases.

(a) The maximum of \( \Phi \) is attained on the boundary \( \partial \Omega \):

\[
\max_{j=0, \ldots, N} \tilde{p}_j = \text{ess sup}_{x \in \Omega} \Phi(x). \tag{3.23}
\]

(b) The maximum of \( \Phi \) is not attained on the boundary \( \partial \Omega \):

\[
\max_{j=0, \ldots, N} \tilde{p}_j < \text{ess sup}_{x \in \Omega} \Phi(x). \tag{3.24}
\]

3.3.1 If (a) happens.

Add to the pressure a constant such that \( \text{ess sup}_{x \in \Omega} \Phi(x) = 0 \).

Without loss of generality, we can reenumerate the boundary components in such a way that

\[
\tilde{p}_0 = \tilde{p}_1 = \cdots = \tilde{p}_M = 0, \quad 0 \leq M < N; \tag{3.25}
\]

\[
\tilde{p}_j < 0, \quad j = M + 1, \ldots, N. \tag{3.26}
\]

Denote \( p_* = \frac{1}{3} \max_{j=M+1, \ldots, N} \tilde{p}_j \). Then by construction

\[
0 > p_* > \tilde{p}_j, \quad \forall j = M + 1, \ldots, N. \tag{3.27}
\]

For sufficiently small parameter \( h > 0 \) and \( j \in \{0, \ldots, N\} \) denote \( \Sigma_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma_j) = h\} \), \( \mathbb{D}_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma_j) < h\} \), and

\[
\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_M, \quad \Sigma_h = \bigcup_{j=0}^M \Sigma_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma) = h\}, \quad \mathbb{D}_h = \bigcup_{j=0}^M \mathbb{D}_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma) < h\}, \tag{3.28}
\]

\[
\Sigma^- = \Sigma_{M+1} \cup \cdots \cup \Sigma_N, \quad \Sigma^-_h = \bigcup_{j=M+1}^N \Sigma_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma^-) = h\},
\]

\[
\mathbb{D}^-_h = \bigcup_{j=M+1}^N \mathbb{D}_{jh} = \{x \in \mathbb{D} : \text{dist}(x, \Sigma^-) < h\}. \tag{3.29}
\]

In particular, we have

\[
\{x \in \mathbb{D} : \text{dist}(x, \partial \mathbb{D}) = h\} = \Sigma_h \cup \Sigma^-_h. \tag{3.29}
\]

Since the distance function \( \text{dist}(x, \partial \mathbb{D}) \) is \( C^1 \)-regular and the norm of its gradient is equal to one in the neighborhood of \( \partial \Omega \), there is a constant \( \delta_0 > 0 \) such that for every positive \( h \leq \delta_0 \) the set \( \Sigma_{jh} \) is a \( C^1 \)-smooth curve homeomorphic to the circle, below we will call such curves as cycles. Respectively, \( \Sigma_h \) is a disjoint union of cycles.

By direct calculations, (3.15) implies

\[
\nabla \Phi = w \times \omega \quad \text{in} \ \Omega, \tag{3.30}
\]

\[\text{ess sup} \Phi = \infty \text{ is not excluded.}\]
where $\omega = \text{curl } w$, i.e.,

$$\omega = (\omega_r, \omega_\theta, \omega_z) = \left( -\frac{\partial v_\theta}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{v_\theta}{r}, \frac{\partial v_\theta}{\partial r} \right).$$

Set $\omega(x) = |\omega(x)|$.

From conditions (E–NS–H) and from helical symmetry by construction we obtain

$$u_n \rightharpoonup w \text{ in } W^{1,2}(\mathbb{D}), \quad (3.31)$$

$$p_n \rightharpoonup p \text{ in } W^{1,3/2}(\mathbb{D}). \quad (3.32)$$

Then from Theorem 3.2 in [1] (see also [20, Lemma 3.3]) we obtain

**Lemma 3.5.** There exists a subsequence $\Phi_{k_l}$ such that $\Phi_{k_l}|_{\Sigma_{jh}}$ converges to $\Phi|_{\Sigma_{jh}}$ uniformly for almost all $h \in (0, \delta_0)$ and for all $j = \{0, 1, \ldots, N\}$.

Below we assume (without loss of generality) that the subsequence $\Phi_{k_l}$ coincides with the whole sequence $\Phi_k$.

The value $h \in (0, \delta_0)$ will be called regular, if it satisfies the assertion of Lemma 3.5, i.e., if

$$\Phi_{k_l}|_{\Sigma_{jh}} \Rightarrow \Phi|_{\Sigma_{jh}} \quad \forall j = \{0, 1, \ldots, N\}. \quad (3.33)$$

**Lemma 3.6.** There exists a measurable set $\mathcal{H} \subset (0, \delta_0)$ such that

(i) each value $h \in \mathcal{H}$ is regular;

(ii) density of $\mathcal{H}$ at zero equals 1:

$$\lim_{h \to 0+} \frac{\text{meas}(\mathcal{H} \cap [0, h])}{h} = 1$$

(iii) $\lim_{\mathcal{H} \ni h \to 0+} \sup_{x \in \Sigma_{jh}} |\Phi(x) - \bar{p}_j| = 0 \quad \forall j = \{0, 1, \ldots, N\}$.

**Proof.** Fix $j \in \{0, 1, \ldots, N\}$. Since $w|_{\partial \mathbb{D}} \equiv 0$ and $\nabla w \in L^2(\mathbb{D})$, by Hardy inequality we have

$$\int_{\mathbb{D}_{jh}} |w|^2 = o(h^2).$$

Then by Hölder inequality,

$$\int_{\mathbb{D}_{jh}} |w| \cdot |\nabla w| = o(h).$$

From Euler equations we have

$$\int_{\mathbb{D}_{jh}} |\nabla \Phi| \leq 2 \int_{\mathbb{D}_{h}} |w| \cdot |\nabla w| = o(h).$$

Using Fubini theorems, we can rewrite the last estimate as

$$\int_{\mathbb{D}_{jh}} |\nabla \Phi| = \int_0^h \left( \int_{\Sigma_{jh}} |\nabla \Phi| \, ds \right) dt = o(h).$$
The last estimate implies easily that there exists a measurable set \( \mathcal{H} \subset (0, \delta_0) \) such that the density of \( \mathcal{H} \) at zero equals 1 (e.g., (3.34) holds), the restriction \( \Phi|_{\Sigma_jh} \) is an absolute continuous function of one variable for all \( h \in \mathcal{H} \), and

\[
\lim_{\mathcal{H} \ni h \to 0^+} \int_{\Sigma_jh} |\nabla \Phi| \, ds = 0. \tag{3.35}
\]

Denote by \( \Phi_h \) the mean value of \( \Phi \) over the curve \( \Sigma_ih \). Then from (3.35) we obtain immediately that

\[
\lim_{\mathcal{H} \ni h \to 0^+} \max_{x \in \Sigma_jh} |\Phi(x) - \Phi_h| = 0. \tag{3.36}
\]

Since by our assumptions \( \partial D \) is \( C^2 \)-smooth, the curves \( \Sigma_ih \) are uniformly \( C^1 \)-smooth for all sufficiently small \( h < \delta \). Then by well-known classical Sobolev theorems, the continuous trace operator

\[
T_{jh} : W^{1,3/2}(D) \ni f \mapsto f|_{\Sigma_jh} \in L^2(\Sigma_jh)
\]

is well defined. Moreover, for any \( f \in W^{1,3/2}(D) \) the real-valued function

\[
[0, \delta] \ni h \mapsto \int_{\Sigma_jh} |f| \, ds \tag{3.37}
\]

is continuous.

Recall, that by Lemma 3.2 the trace identity \( \Phi|_{\Sigma_j} \equiv \hat{p}_j \) holds. Since pressure is defined up to an additive constant, we can assume, without loss of generality, that \( \hat{p}_j = 0 \), i.e.,

\[
\Phi|_{\Sigma_j} = 0 \tag{3.38}
\]

in the sense of trace of Sobolev spaces. Hence, from (3.37)–(3.38) we obtain

\[
\int_{\Sigma_jh} |\Phi| \, ds \to 0 \quad \text{as} \quad h \to 0^+. \tag{3.39}
\]

This implies \( \Phi_h \to 0 \) as \( h \to 0^+ \). The last convergence together with the formula (3.35) give us the desired asymptotic

\[
\lim_{\mathcal{H} \ni h \to 0^+} \max_{x \in \Sigma_jh} |\Phi(x)| = 0. \tag{3.40}
\]

The Lemma is proved. \( \square \)

By Lemmas 3.5–3.6, decreasing \( \mathcal{H} \) if necessary and taking sufficiently large \( k \), we can assume, without loss of generality, that

**Corollary 3.7.** For all \( h \in \mathcal{H} \) we have

\[
\Phi_k(x) < p_* \quad \forall x \in \Sigma_jh. \tag{3.41}
\]

Also, by Lemma 3.6 we obtain

**Corollary 3.8.** For all \( \varepsilon > 0 \) there exists \( \delta_\varepsilon > 0 \) and \( k_\varepsilon \in \mathbb{N} \) such that \( \forall h \in \mathcal{H} \cap (0, \delta_\varepsilon) \) and \( \forall k \geq k_\varepsilon \) we have

\[
\Phi_k(x) > -\varepsilon \quad \forall x \in \Sigma_jh. \tag{3.42}
\]

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Denote $t_\ast = -p_\ast$. Let $0 < t_0 < t_\ast$. The next geometrical object plays an important role in the estimates below: for $t \in (t_0, t_\ast)$ and for all sufficiently large $k$ we define the level set $S_k(t, t_0) \subset \{ x \in D: \Phi_k(x) = -t \}$ separating boundary components $\Sigma$ from $\Sigma^-$ as follows. Namely, take $\epsilon = \frac{1}{2} t_0$ and the corresponding parameters $\delta_\epsilon > 0$, $k_\epsilon = k_\epsilon \in \mathbb{N}$ such that (3.41)–(3.42) holds for all $k \geq k_\epsilon$ and $h \in \mathcal{H} \cap (0, \delta_\epsilon)$. Fix a number $h_\circ \in \mathcal{H} \cap (0, \delta_\epsilon)$. In particular, we have

$$\forall t \in (t_0, t_\ast) \quad \forall k \geq k_\circ \quad \left( \Phi_k|_{\Sigma_h} > -t, \quad \Phi_k|_{\Sigma_{h_\circ}} < -t \right). \quad (3.43)$$

For $k \geq k_\circ$, $j = 0, 1, \ldots, M$, and $t \in (t_0, t_\ast)$ denote by $W_j^k(t_0; t)$ the connected component of the open set $\{ x \in D : \text{dist}(x, \partial D) > h_\circ \text{ & } \Phi_k(x) > -t \}$ such that $\partial W_j^k(t_0; t) \supset \Sigma_{h_j}$ (see Fig.1),

$$W_k(t_0; t) = \bigcup_{j=0}^M W_j^k(t_0; t), \quad S_k(t_0; t) = (\partial W_k(t_0; t)) \setminus \Sigma_{h_\circ}. \quad (3.44)$$

By construction, (see Fig.1),

$$\partial W_k(t_0; t) = \Sigma_{h_\circ} \cup S_k(t_0; t) \quad (3.45)$$

and

$$\partial W_k(t_0; t) \subset \{ x \in \mathbb{D} : \text{dist}(x, \partial \mathbb{D}) > h_\circ \text{ & } \Phi_k(x) = -t \} \cup \{ x \in \mathbb{D} : \text{dist}(x, \partial \mathbb{D}) = h_\circ \text{ & } \Phi_k(x) \geq -t \}. \quad (3.46)$$

Figure 1: $M = 0, N = 1$.

Therefore, by definition of $S_k(t_0; t)$ and in virtue of the identity (3.29),

$$S_k(t_0; t) \subset \{ x \in \mathbb{D} : \text{dist}(x, \partial \mathbb{D}) > h_\circ \text{ & } \Phi_k(x) = -t \} \cup \{ x \in \Sigma_{h_\circ}^- : \Phi_k(x) \geq -t \}. \quad (3.47)$$

But the last set $\{ x \in \Sigma_{h_\circ}^- : \Phi_k(x) \geq -t \}$ is empty because of (3.43). Therefore,

$$S_k(t_0; t) \subset \{ x \in \mathbb{D} : \text{dist}(x, \partial \mathbb{D}) > h_\circ \text{ & } \Phi_k(x) = -t \}. \quad (3.48)$$
Since by (E–NS–H) each $\Phi_k$ belongs to $W^{2,1}_{\text{loc}}(\mathbb{D})$, by the Morse-Sard theorem for Sobolev functions (see assertion (iii) of Theorem 2.4) we have that for almost all $t \in (t_0, t_*)$ the level set $S_k(t_0; t)$ consists of finitely many $C^1$-cycles and $\Phi_k$ is differentiable (in classical sense) at every point $x \in S_k(t_0; t)$ with $\nabla \Phi_k(x) \neq 0$. The values $t \in (t_0, t_*)$ having the above property will be called $k$-regular. (Note that $W_k(t_0; t)$ and $S_k(t_0; t)$ are well defined for all $t \in (t_0, t_*)$ and $k \geq k_0 = k_0(t_0)$.)

Recall that for a set $A \subset \mathbb{D}$ we denote by $A$ the three dimensional set in $\mathbb{T}_r$ which is evolved from $A$ (see (3.19)). By construction, for every regular value $t \in (t_0, t_*)$ the set $\overline{S}_k(t_0; t)$ is a finite union of smooth surfaces (tori), and

$$\int_{\overline{S}_k(t_0; t)} \nabla \Phi_k \cdot n \, dS = -\int_{\overline{S}_k(t_0; t)} |\nabla \Phi_k| \, dS < 0,$$

(3.49)

where $n$ is the unit outward normal vector to $\partial \overline{W}_k(t_0; t)$.

Note that $W_k(t_0; t)$ and $S_k(t_0; t)$ are well defined for all $t \in (t_0, t_*)$ and $k \geq k_0 = k_0(t_0)$. Now we are ready to prove the key estimate (which is analog of Lemma 3.8 from [21]).

**Lemma 3.9.** Let $0 < t_0 < t_*$. Then there exists $k_\epsilon = k_\epsilon(t_0)$ such that for every $k \geq k_\epsilon$ and for almost all $t \in (t_0, t_*)$ the inequality

$$\int_{\overline{S}_k(t_0; t)} |\nabla \Phi_k| \, dS < F t,$$

(3.50)

holds with the constant $F$ independent of $t, t_0$, and $k$.

**Proof.** Fix positive $t_0 < t_*$, $\epsilon = \frac{1}{2} t_0$, and the corresponding parameters $\delta_\epsilon > 0$, $k_\delta := k_\delta \in \mathbb{N}$ such that (3.41)–(3.42) holds for all $k \geq k_\delta$ and $h \in \mathcal{H} \cap (0, \delta_\epsilon)$. Fix a number $h_0 \in \mathcal{H} \cap (0, \delta_\epsilon)$. Then for all $t \in (t_0, t_*)$ and all $k \geq k_\delta$ we can define the set $S_k(t_0; t)$ as above. Moreover, for almost all $t \in (t_0, t_*)$ the set $S_k(t_0; t)$ consists of finitely many pairwise disjoint $C^1$-cycles (= $C^1$-smooth curves homeomorphic to the circle) and $\Phi_k$ is differentiable (in classical sense) at every point $x \in S_k(t_0; t)$ with $\nabla \Phi_k(x) \neq 0$. The values $t \in (t_0, t_*)$ having the above property will be called $k$-regular.

Respectively, for every regular value $t \in (t_0, t_*)$ the set $\overline{S}_k(t_0; t)$ is a finite union of smooth surfaces (tori), and the inequality (3.49) holds.

The main idea of the proof of (3.50) is quite simple: we will integrate the equation

$$\Delta \Phi_k = \omega_k^2 + \frac{1}{v_k} \text{div}(\Phi_k u_k) - \frac{1}{v_k} f_k \cdot u_k$$

(3.51)

over the suitable domain $\Omega_k(t)$ with $\partial \Omega_k(t) \supset \overline{S}_k(t_0; t)$ such that the corresponding boundary integrals

$$\left| \int_{\partial \Omega_k(t)} \nabla \Phi_k \cdot n \, dS \right|$$

(3.52)

$$\frac{1}{v_k} \left| \int_{\partial \Omega_k(t)} \Phi_k u_k \cdot n \, dS \right|$$

(3.53)

are negligible. We split the construction of the domain $\Omega_k(t)$ into two steps.

First of all, define the open set

$$D_k(t) := W_k(t_0; t) \cup \overline{D_{h_0}} \setminus \Sigma.$$
Then by construction (see, e.g., (3.28), (3.45)) we have
\[ \partial D_k(t) = \Sigma \cup S_k(t_0; t). \] (3.54)

Denote further
\[ \Omega_k(t) := \overline{D_k(t)}. \]
Then \( \Omega_k(t) \) is the open set in the three dimensional space and
\[ \partial \Omega_k(t) = \Gamma \cup \overline{S_k(t_0; t)}, \] (3.55)
where we denote \( \Gamma := \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_M \).

By direct calculations, (3.14) implies
\[ \nabla \Phi_k = -\nu_k \text{curl } \omega_k + u_k \times \omega_k + f_k = -\nu_k \text{curl } \omega_k + u_k \times \omega_k + \lambda_k \nu_k^2 \text{curl } b. \]

By the Stokes theorem, for any \( C^1 \)-smooth closed surface \( S \subset \Omega \) and \( g \in W^{2,2}(\Omega) \) we have
\[ \int_S \text{curl } g \cdot n \, dS = 0. \]
So, in particular,
\[ \int_S \nabla \Phi_k \cdot n \, dS = \int_S (u_k \times \omega_k) \cdot n \, dS. \] (3.56)
Recall that by the pressure normalization condition,
\[ \Phi|_\Gamma = 0. \] (3.57)

Our purpose on the next step is as follows: for arbitrary \( \varepsilon > 0 \) and for sufficiently large \( k \) to prove the estimates
\[ \left| \int_\Gamma \nabla \Phi_k \cdot n \, dS \right| = \left| \int_\Gamma (u_k \times \omega_k) \cdot n \, dS \right| < \varepsilon, \] (3.58)
\[ \frac{1}{\nu_k} \int_\Gamma \Phi_k u_k \, dS \bigg| < \varepsilon. \] (3.59)

Recall that in our notation \( u_k = \lambda_k \nu_k U + w_k \), where \( w_k \in H(\Omega) \), \( \|w_k\|_{H(\Omega)} = 1 \), and \( U \) is a solution to the Stokes problem with boundary value \( a \) and forcing term \( f \) (see (2.5)). In particular, we have
\[ u_k(x) \equiv \lambda_k \nu_k U(x) \quad \forall x \in \Gamma. \] (3.60)

To establish (3.59), we use the uniform boundedness
\[ \|\Phi_k\|_{L^2(\Omega)} + \|\nabla \Phi_k\|_{L^2(\Omega)} \leq C. \] (3.61)
From (3.61) and the weak convergence \( \Phi_k \rightharpoonup \Phi \) in \( W^{1,3/2}(\Omega) \) we easily have
\[ \Phi_k \rightharpoonup \Phi \quad \text{in } L^q(\Gamma) \quad \forall q \in [1, 2). \] (3.62)
Thus by virtue of (3.57),
\[ \int_\Gamma |\Phi_k| \, dS \to 0 \quad \text{as } k \to \infty. \] (3.63)
Since

\[ \|U\|_{L^\infty(\Omega)} \leq C \|U\|_{W^{2,2}(\Omega)} < \infty, \]  

by identity (3.60) we have

\[ \frac{1}{\nu_k} \left| \int_{\Gamma} \Phi_k u_k \, dS \right| = \lambda_k \left| \int_{\Gamma} \Phi_k U \, dS \right| \leq C \int_{\Gamma} |\Phi_k| \, dS \xrightarrow{k \to \infty} 0, \]

that implies the required estimate (3.59) for sufficiently large \( k \).

To prove (3.58), we need also the uniform estimate

\[ \|\nu_k u_k\|_{W^{2,3/2}(\Omega)} \leq C, \]  

where \( C \) is independent of \( k \) (this inequality follows from the construction, see (3.13)). Thus by Sobolev imbedding theorems

\[ \|\nu_k \nabla u_k\|_{L^1(\Gamma_h)} \leq C \forall h \in [0, h_0] \forall k \in \mathbb{N}, \]

where \( C > 0 \) is independent of \( k, h \), here \( \Omega_h = \{x \in \Omega : \text{dist}(x, \Gamma) \leq h\}, \Gamma_h = \{x \in \Omega : \text{dist}(x, \Gamma) = h\}. \) Moreover, by elementary calculations (3.66) implies the uniform Hölder continuity of the function \([0, h_0] \ni h \mapsto \|\nu_k \nabla u_k\|_{L^1(\Gamma_h)}\), i.e., there exists a constant \( \sigma > 0 \) (independent of \( k \)) such that

\[ \left| \int_{\Gamma_h'} |\nu_k \nabla u_k| \, dS - \int_{\Gamma_{h''}} |\nu_k \nabla u_k| \, dS \right| \leq \sigma |h' - h''|^{1/2} \forall h', h'' \in [0, h_0] \forall k \in \mathbb{N}. \]

From the last property and from the uniform boundedness of the Dirichlet integral

\[ \|\nabla u_k\|_{L^2(\Omega)} \leq 1 + \lambda_k \nu_k \|\nabla U\|_{L^2(\Omega)} \leq 2 \]  

(for sufficiently large \( k \)) one can easily deduce that

\[ \sup_{h \in [0, h_0]} \int_{\Gamma_h} |\nu_k \nabla u_k| \, dS \to 0 \quad \text{as} \quad k \to \infty, \]

in particular,

\[ \int_{\Gamma_0} |\nu_k \nabla u_k| \, dS \to 0 \quad \text{as} \quad k \to \infty. \]

Then from the identity (3.60) and the estimate (3.64) we have

\[ \left| \int_{\Gamma} (u_k \times \omega_k) \cdot n \, dS \right| = \lambda_k \nu_k \left| \int_{\Gamma} (U \times \omega_k) \cdot n \, dS \right| \leq C \int_{\Gamma_0} |\nu_k \nabla u_k| \, dS \to 0 \quad \text{as} \quad k \to \infty. \]

Hence the required estimate (3.58) is proved.
Recall that by construction we have \( \partial \Omega_k(t) = \Gamma \cup \overline{S}_k(t_0; t) \) (see (3.55)). Integrating the equation

\[
\Delta \Phi_k = \omega_k^2 + \frac{1}{\nu_k} \text{div}(\Phi_k u_k) - \frac{1}{\nu_k} f_k \cdot u_k
\]

over the domain \( \Omega_k(t) \), we obtain

\[
\int_{\overline{S}_k(t_0; t)} \nabla \Phi_k \cdot n \, dS + \int_{\Gamma} \nabla \Phi_k \cdot n \, dS = \int_{\Omega_k(t)} \omega_k^2 \, dx - \frac{1}{\nu_k} \int_{\Omega_k(t)} f_k \cdot u_k \, dx
\]

\[
+ \frac{1}{\nu_k} \int_{\overline{S}_k(t_0; t)} \Phi_k u_k \cdot n \, dS + \frac{1}{\nu_k} \int_{\Gamma} \Phi_k u_k \cdot n \, dS
\]

\[
= \int_{\Omega_k(t)} \omega_k^2 \, dx - \frac{1}{\nu_k} \int_{\Omega_k(t)} f_k \cdot u_k \, dx + r F_k + \frac{1}{\nu_k} \int_{\Gamma} \Phi_k u_k \cdot n \, dS,
\]

(3.74)

where \( F_k = \lambda_k (F_0 + \cdots + F_M) \) and \( F_j = \int \mathbf{a} \cdot n \, dS \) is the corresponding boundary flux (here we use the identity \( \Phi_k \equiv -t \) on \( \overline{S}_k(t_0; t) \)). In view of (3.49) and (3.58)–(3.59) we can estimate

\[
\int_{\overline{S}_k(t_0; t)} |\nabla \Phi_k| \, dS \leq r F_k + 3 \epsilon + \frac{1}{\nu_k} \int_{\Omega_k(t)} f_k \cdot u_k \, dx - \int_{\Omega_k(t)} \omega_k^2 \, dx
\]

(3.75)

with \( F_k = |F_k| \). By definition, \( \frac{1}{\nu_k} \| f_k \|_{L^2(\Omega)} = \lambda_k \nu_k \| f \|_{L^2(\Omega)} \to 0 \) as \( k \to \infty \). Therefore, using the uniform estimate \( \| u_k \|_{L^6(\Omega)} \leq \text{const} \), we have

\[
\left| \frac{1}{\nu_k} \int_{\Omega_k(t)} f_k \cdot u_k \, dx \right| < \epsilon
\]

for sufficiently large \( k \). Then (3.75) yields

\[
\int_{\overline{S}_k(t_0; t)} |\nabla \Phi_k| \, dS < r F_k + 4 \epsilon - \int_{\Omega_k(t)} \omega_k^2 \, dx.
\]

(3.76)

Since \( F_k = \lambda_k (F_0 + \cdots + F_M) \) is uniformly bounded, and \( \epsilon \) could be taken arbitrary small, the last inequality implies the required estimate (3.50) for sufficiently large \( k \). \( \Box \)

**Lemma 3.10.** Assume that \( \Omega \subset \mathbb{R}^2 \times \mathbb{T}_r \) is a helical domain with \( C^2 \) boundary \( \partial \Omega \), \( \mathbf{f} = \text{curl} \mathbf{b} \), \( \mathbf{b} \in W^{1,2}(\Omega) \) and \( \mathbf{a} \in W^{3/2,2}(\partial \Omega) \) are both helical functions, and (1.2) is valid. Then assumptions (E–NS–H) with (2.33) lead to a contradiction.

**Proof:** Fix positive \( t_0 < t_* \). Put \( t_i = 2^{-i} t_0 \), \( i = 1, 2, \ldots \). Take the corresponding parameters \( k_{si} = k_s(t_i) \) such that the inequality (3.50) holds for all \( k \geq k_{si} \) and for almost all \( t \in (t_i, t_*). \) In particular, (3.50) holds for all \( k \geq k_{si} \) and for almost all \( t \in (t_i, 2t_i). \)

For \( k \geq k_{si} \) put

\[
E_{ki} = \bigcup_{t \in [t_i, 2t_i]} \overline{S}_k(t; t).
\]
By the Coarea formula (see, e.g., [31]), for any integrable function \( g : E_{ki} \to \mathbb{R} \) the equality
\[
\int_{E_{ki}} g |\nabla \Phi_k| \, dx = \int_{t_i}^{2t_i} \int_{\tilde{S}_k(t_i; t)} g(x) \, d\tilde{S}_k^2(x) \, dt
\] holds. In particular, taking \( g = |\nabla \Phi_k| \) and using (3.50), we obtain
\[
\int_{E_{ki}} |\nabla \Phi_k|^2 \, dx = \int_{t_i}^{2t_i} \int_{\tilde{S}_k(t_i; t)} |\nabla \Phi_k| d\tilde{S}_k^2(x) \, dt \leq \int_{t_i}^{2t_i} \mathcal{F} t \, dt = \frac{\mathcal{F}}{2} ((2t_i)^2 - (t_i)^2) \leq 2\mathcal{F} t_i^2
\] (3.78)
Now, taking \( g = 1 \) in (3.77) and using the Hölder inequality we have
\[
\int_{t_i}^{2t_i} \tilde{S}_k^2(\tilde{S}_k(t_i; t)) \, dt = \int_{E_{ki}} |\nabla \Phi_k| \, dx \leq \left( \int_{E_{ki}} |\nabla \Phi_k|^2 \, dx \right)^{\frac{1}{2}} \left( \operatorname{meas}(E_{ki}) \right)^{\frac{1}{2}} \leq \sqrt{2\mathcal{F} t_i^2 \operatorname{meas}(E_{ki})} \leq t_i \sqrt{2\mathcal{F} \operatorname{meas}(E_{ki})}.
\] (3.79)
By construction (see the arguments after the formula (3.43)), for all \( k \)-regular values \( t \) the set \( \tilde{S}_k(t_i; t) \) is a finite union of \( C^1 \)-smooth surfaces (tori) separating \( \Gamma = \Gamma_0 \cup \ldots \Gamma_M \) from \( \Gamma_{M+1} \cup \ldots \cup \Gamma_N \). It implies, in particular, that \( \tilde{S}_k^2(\tilde{S}_k(t_i; t)) \geq C_* = C_*(\Omega) > 0 \). Therefore, by virtue of (3.79) we have
\[
C_* t_i \leq t_i \sqrt{2\mathcal{F} \operatorname{meas}(E_{ki})}, \tag{3.80}
\]
in other words,
\[
C_* \leq \sqrt{2\mathcal{F} \operatorname{meas}(E_{ki})} \tag{3.81}
\]
for all \( i \in \mathbb{N} \) and for all \( k \geq k_i \). By constructions, for all \( k \geq k_i \) the sets \( E_{ki}, E_{k(i-1)}, \ldots, E_{k2}, E_{k1} \) are pairwise disjoint. Therefore, the measure of some \( E_{ki} \) could be made arbitrary small (for sufficiently large \( t \) and \( k \)). This obviously contradicts the estimate (3.81). The Lemma is proved. \( \square \)

### 3.3.2 If (b) happens.

Suppose now that the maximum of \( \Phi \) is not attained on the boundary \( \partial \Omega^3 \):
\[
\max_{j=0, \ldots, N} \hat{p}_j < \operatorname{ess sup}_{x \in \Omega} \Phi(x). \tag{3.82}
\]
Adding a constant to the pressure, we can assume without loss of generality that
\[
\max_{j=0, \ldots, N} \hat{p}_j + p_* < 0 < \operatorname{ess sup}_{x \in \Omega} \Phi(x). \tag{3.83}
\]
The proof for this case can be carried out with the same arguments as in the previous subsection with obvious simplifications. Let us describe some details. We start from the following simple fact.

**Lemma 3.11.** Under assumptions (3.83) there exists a straight segment \( F \subset \mathbb{D} \) such that \( F \cap A_w = \emptyset \) (i.e., \( F \) consists of the regular points of \( \Phi \) and \( w \), see Lemma 3.4), and
\[
0 < \inf_{x \in F} \Phi(x),
\]
moreover, the uniform convergence
\[
\Phi_k|_F \Rightarrow \Phi|_F \tag{3.84}
\]
holds.

\(^{3}\text{ess sup} \Phi = \infty \) is not excluded.
This fact follows easily from the definition of Sobolev spaces and from the weak convergence $\Phi_k \rightharpoonup \Phi$ in $W^{1,3/2}(D)$ (see, e.g., the proof of Theorem 3.2 in [1] for details), so we omit its proof here.

Fix the segment $F$ from Lemma 3.11. Denote $t_* = -p_*$ and fix a positive $t_0 < t_*$. Using the arguments from the previous subsection, we can find a small parameter $h_0 > 0$ and a number $k_0 \in \mathbb{N}$ such that

$$\forall t \in (t_0, t_*) \forall k \geq k_0 \quad \left(\Phi_k|_F > 0, \quad \Phi_k|_{\Sigma_h} < -t_0^{(i)}\right),$$

(3.85)

where now by definition $\Sigma_h = \{x \in D : \text{dist}(x, \partial D) = h\}$ (see Fig. 3).

![Figure 3.](image)

Further, using the same arguments, for almost all $t \in (t_0, t_*)$ we can find a set $S_k(t_0; t)$ consisting of finite disjoint family of $C^1$ smooth closed curves (cycles) such that $\Phi_k \equiv -t$ on $S_k(t_0; t)$. Moreover, there is an open set $W_k(t_0; t) \subset D$ satisfying the relations

$$W_k(t_0; t) \supset F,$$

(3.86)

$$\partial W_k(t_0; t) = S_k(t_0; t)$$

(3.87)

(cf. with (3.45)), and

$$\int_{S_k(t_0; t)} \nabla \Phi_k \cdot n \, dS = -\int_{\tilde{S}_k(t_0; t)} |\nabla \Phi_k| \, dS < 0,$$

(3.88)

where $n$ is the unit outward normal vector to $\partial \tilde{W}_k(t_0; t)$. Here, by construction, $\tilde{S}_k(t_0; t)$, is a finite union of smooth disjoint surfaces (tori).

Further we can prove the estimate (3.50) for our case integrating the same identity (3.51) over the domain $\Omega_k(t) = \tilde{W}_k(t_0; t)$ with $\partial \Omega_k(t) = \tilde{S}_k(t_0; t)$. Note that now the proof is even much simpler since we have no boundary integrals over subsets of $\partial \Omega$ — in other words, now we do not need to prove estimates of type (3.52)–(3.53) and (3.58)–(3.59).

After the estimate (3.50) is obtained, we can derive a contradiction in exactly the same way as in Lemma 3.10 of the previous section.

Now we can summarize the results of the last two subsections in the following statement.

**Lemma 3.12.** Assume that $\Omega \subset \mathbb{R}^2 \times T_\sigma$ is a helical domain with $C^2$ boundary $\partial \Omega$, $f = \text{curl} \, b$, $b \in W^{1,2}(\Omega)$ and $a \in W^{3/2,2}(\partial \Omega)$ are both helical functions, and (1.2) is valid. Let $(\text{E–NS–H})$ be fulfilled. Then each assumptions (3.23) or (3.24) lead to a contradiction.
Proof of Theorem 1.1. Let the hypotheses of Theorem 1.1 be satisfied. Suppose that its assertion fails. Then, by Lemma 3.1, there exist \( w, p \) and a sequence \( (u_k, p_k) \) satisfying (E–NS–H), and by Lemma 3.12 these assumptions lead to a contradiction. \( \Box \)

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