On Solvable Potentials for One Dimensional Schrödinger and Fokker-Planck Equations

George Krylov

Department of Physics, Belarusian State University,
Fr. Skariny av. 4, 220050 Minsk, Belarus
e–mail: krylov@dragon.bas-net.by

Abstract. One construction of exactly-solvable potentials for Fokker-Planck equation is considered based on supersymmetric quantum mechanics approach.

PACS numbers: 03.65.-w, 03.65.Ge, 03.65.Ca, 02.90.+p
Submitted to Journal of Physics A: Mathematical and General
1 Introduction

Exactly solvable models play an important role in the developing of all branches of physics as well as persist to be a genuine challenge to our understanding of nature beauty and simplicity. Comprehensive analysis of underlying reasons of solvability is still a dream today even in such developed field as quantum mechanics, whereas solvable models known in all other branches of physics, could be easily enumerate by hands.

Among the most developed approaches to solvability problem in quantum mechanics is supersymmetric quantum mechanics (SUSY QM) approach that has much in common with the early Infeld & Hull factorization [1]. It was developed by E. Witten [2] originally in the context of quantum field theory, but later gathered a lot of attention especially after L. Genderstein introduction of the concept of self-similar potential [3]. Today the SUSY QM approach has clearly outlined the region of its applicability as well as its connection with other analytical tools and methods. Some interesting new results on the topic were obtained in the way of generalization of SUSY approach on matrix models, higher order symmetries and can be found in [4]-[9]).

The another type of approaches of the solvability problem have to be mentioned, which were developed separately though been and becoming more and more closer to SUSY QM, are numerous algebraic approaches. In this context we would like to mention only a few papers, firstly relatively old classical papers [10],[11] with extensive reference lists therein and some recent ones [12]-[15] discussing different aspects of the problem.

There are also analytical ways of attacking the problem which are also very numerous and we only cite some recent papers [16]-[18], [19] demonstrating some new ideas but been away out of the topics discussed in the present work.

For Fokker-Planck equations solvable problems are still very rare events as well as developed approaches to this problem. Some recent results have been obtained for the generalization of SUSY QM approach on this situation [20] (see also the references therein).

The goal of the present paper is to establish close relations between the above mentioned equations based on the SUSY approach developed in [20]. As it will be demonstrated, the set of exactly solvable potentials for the Fokker-Planck equation is at least not poorer than that of Schrödinger one.

2 Diffusion equation and Superpotential

The diffusion (Fokker-Planck) equation for the distribution function $f(x, t)$ for a system in an external potential $U(x)$ has the following form

$$\frac{\partial f(x, t)}{\partial t} = \nabla \cdot \hat{D} \cdot \nabla f(x, t) + \nabla (f(x, t) \nabla U(x))$$

where $x$ are some space co-ordinates for the system, $\hat{D}$ is a diffusion tensor and we assume in subsequent that both diffusion tensor and external field potential do not depend on time explicitly.

In literature, it is commonly accepted that the only difference of diffusion equation (for spherically symmetric particles, when $\hat{D}_{ij} = D\delta_{ij}$) and Schrödinger’s one is imaginary time on respect
to real time (Vick’s rotation). Though it is evidently true for the case of a free particle, for the problem in an external field the only sight on the second equation

\[ i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(x, t) + U(x) \psi(x, t) \]  

immediately demonstrates that the external field is incorporated into the equation (2) in a different way on respect to that in the diffusion case (1).

The prominent feature of the eq. (1) is the existence of zero-mode (stationary or steady-state) solution \( f_s(x) \), which simply corresponds to generalization of known Boltzmann distribution \( f_s(x) = C \exp(-U(x)) \) (valid when diffusion tensor is spherical). Appropriate first order system of differential equations can be obtained from eq. (1) as

\[ \nabla \cdot \hat{D} \cdot \nabla f_s(x) + \nabla (f_s(x) \nabla U(x)) = \nabla \left( \hat{D} \cdot \nabla f_s(x) + f_s(x) \nabla U(x) \right) = 0 \]  

where \( \hat{D} \cdot \nabla f_s(x) + f_s(x) \nabla U(x) = \text{rot } b(x) \) (3)

In opposite, for the Schrödinger equation (2) the ground state is typically unknown and of most interest.

This, as we will see at least for one dimensional problems, is due to the fact that after transformation of the diffusion equation into the form of the Schrödinger one, we obtain the last in the supersymmetric quantum mechanics (SUSY) form directly and the proper partner Hamiltonian is just \( H_- \) [3].

Let us follow the way, similar to [20] and concentrate in the subsequent on 1D problems only (so that \( x \to x \)). We also assume the units’ choice such that \( \hbar = 1, m = 1, D = 1/2 \). It is worth to note that steady state solution reads \( f_s(x) = \exp(-2U(x)) \) with this units’ choice.

Then, after substitution \( f(x) = \exp \{ -U(x) - Et \} \psi(x) \) into

\[ \frac{\partial f(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 f(x, t)}{\partial t^2} + \frac{\partial}{\partial x} (f(x, t)U'(x)) \]  

we get the Schrödinger equation in the form

\[ \frac{1}{2} \psi''(x) + (E - V_q(x)) \psi(x) = 0 \]  

with a "quantum potential" \( V_q(x) \) given

\[ V_q(x) = \frac{1}{2} U'(x)^2 - \frac{1}{2} U''(x) \]  

It is important to note that steady state solution of (4) reads \( f_0(x) = \exp(-2U(x)) \) in our case.

The last equation is just in the form of SUSY QM approach with the superpotential given by \( W(x) = U'(x) \) [20] and the Hamiltonian operator having the factorized form

\[ \hat{H}_- = \hat{A}^\dagger \hat{A} = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + U'(x) \right) \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + U'(x) \right) \]  

(7)
It is worth to remark that quantum potentials for partner Hamiltonians $\hat{H}_\pm$ now correspond simply to $X$-axis reflection images of the original diffusion potential $U(x)$.

It is immediately follows from the (3) and (7) that the state $E = 0$ is the eigenstate of $H_-$.

The last clarifies the principal difference between the Schrödinger and diffusion equations. Whereas it can be highly nontrivial problem to construct an explicit factorization of a given Hamiltonian and find as a result the ground state of a quantum system, for a diffusion equation (at least for one-dimensional one) this is not a problem at all, as it can be always done in the way been outlined.

To construct solvable cases for a 1-D diffusion equation, one can exploit the supersymmetric form directly, considering known shape-invariant partner potentials [20].

We will choose another way, namely we try to answer the following question:

\[ \text{what superpotential } W(x) = U'(x) \text{ should be that leads to exactly-solvable potentials for the eq. (5)} \]

provided we use most general form of a 1D quantum potential allowing polynomial anzatz for a wave function [19].

Denoting a solvable quantum potential in (3) by $V_s(x)$, we consider now (3) as the Ricatti equation for superpotential $W(x)$

\[ W'(x) - W(x)^2 = -2 V_s(x) \]  

(8)

Here it worth to point out the following. We could split the energy parameter $E$ in (3) as $E = E_1 + E_2$ that leads to the appearance of one term e.g., $E_2$ in the right side of (3), that will be used in subsequent.

Now, we perform the known trick. Based on the correspondence of Ricatti and Schrödinger equation we introduce substitution

\[ W(x) = -\Psi'(x)/\Psi(x) \]  

(9)

and rewrite (8) in the form of the Schrödinger equation for a function $\Psi(x)$

\[ \frac{1}{2} \Psi''(x) + (E_2 - V_s(x)) \Psi(x) = 0 \]  

(10)

The last simply means that every eigenstate $\Psi_n(x)$ of a quantum solvable potential $V_s(x)$ gives a superpotential through the relation (3) that after integration gives for the the diffusion equation potential $U(x)$ a simple formula

\[ U_n(x) = U_0 + \log |\Psi_n(x)| \]  

(11)
Most important fact here is that the set of $U_n(x)$ leads to the same Schrödinger equation (4) with different splitting of the constant $E$, but of course gives different solutions for original diffusion equation (1).

One more comment worth to be done is though representation in the form () looks very known from the point of view of I Inverse Scattering Transform theory (see, e.g. [21]) as well as the formulae (8,11) are valid in all cases, but only for solvable quantum potentials we obtain a closed form solution of the original problem. Indeed, logarithm of some $n$-th eigenfunction of a solvable potential $V_s(x)$ gives diffusion potential (eq.(11)) whereas eigenfunctions modified by exponential factor (written before eq.(4)) give the eigenfunctions of the diffusion problem itself.

Finally, we express the main result of the paper as follows. The $i$-th eigenstate for some exactly solvable diffusion potential $U_n(x)$ (given by above written formula ()) reads

$$f_i(x,t) = \Psi_n(x) \exp \left(-\left(E_i + \Delta E\right)t\right) \Psi_i(x) \quad i = 0, 1, ..., n... \quad (12)$$

where $E_i$ is eigenenergy of the appropriate quantum potential () and $\Delta E = -E_0$ is the constant energy shift introduce to give proper energy of the ground state ($E = 0$) for the diffusion equation.

## 3 Conclusion

Simple construction we discussed opens a new sight on interrelation of Fokker-Planck and as- sociate Schrödinger problems. Having a significantly large number of exactly solvable quantum potentials (see, e.g. list in [22]) one can construct at least measure similar list of solvable diffusion cases having possibility to choose one appropriate to approximate some practical problems.

## 4 Acknowledgement

Author would like to thank Prof. Dr. Marko Robnik for numerous and stimulating discussions of this and related problems. This work has been support in parts by the Fund of Fundamental Researches of the Republic of Belarus (Project F00-158), and Swiss Science Foundation (Project SCOPES 7BYPJ065731).

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