Total vertex product irregularity strength of graphs

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Abstract

Consider a simple graph $G$. A labeling $w : E(G) \cup V(G) \to \{1, 2, \ldots, m\}$ is called total vertex product-irregular, if all product degrees $pd_G(v) = w(v) \times \prod_{e \ni v} w(e)$ are distinct. The goal is to obtain a total vertex product-irregular labeling that minimizes the maximum label. This minimum value is called the total vertex product irregularity strength and denoted $tvps(G)$. In this paper we provide some general lower and upper bounds, as well as exact values for chosen families of graphs.

Keywords: product-irregular labeling, total vertex product irregularity strength, vertex-distinguishing labeling.

1 Introduction

Let $G = (V(G), E(G))$ a simple undirected graph with no loops. Let us assign a label (positive integer) to every edge $e$ and every vertex $v$ in $G$ and denote it by $w(e)$ and $w(v)$ respectively. For every vertex $v \in V(G)$ we define the product degree as

$$pd_G(v) = w(v) \times \prod_{e \ni v} w(e),$$

where $d_G(v)$ denotes the degree of vertex $v$ in $G$ and $e \ni v$ means that the vertex $v$ is incident to $e$. In particular, in the case of an isolated vertex $v$ we have $pd_G(v) = w(v).$
We call $w$ product-irregular if for every pair of distinct vertices $u, v \in V(G)$, $pd_G(u) \neq pd_G(v)$. The (total vertex product) strength of the labeling $w$ is defined as

$$tvps_w(G) = \max\{w(x) | x \in E(G) \cup V(G)\},$$

while the total vertex product irregularity strength of $G$ as

$$tvps(G) = \min\{tvps_w(G) | w \text{ is product-irregular}\}.$$

This concept is a variant of the product irregularity strength, introduced by Anholcer [3] and studied in [4] and [9]. This time the graphs under consideration have no isolated edges and at most one isolated vertex and the product degree is defined as

$$pd_G(v) = \begin{cases} \prod_{e \ni v} w(e) & \text{if } d_G(v) > 0, \\ 0 & \text{if } d_G(v) = 0. \end{cases}$$

Similarly as in the definition above, one defines the (product) strength of the labeling $ps_w(G)$ as the maximum label used and the goal is to find a labeling with minimum strength. This minimum value is denoted $ps(G)$ and called product irregularity strength of $G$.

The motivation to study this kind of problems were the well-known irregularity strength and total vertex irregularity strength, where the vertex weighted degrees are defined as the sums of labels (instead of products). The irregularity strength was defined by Chartrand et al. in [8] and investigated by numerous authors (see e.g. [1, 2, 6, 11, 12, 13, 15, 18]. Best published general result due to Kalkowski et al. (see [14]) is $s(G) \leq 6n/\delta$ for graphs with $\delta \geq 6$. It was improved by Majerski and Przybyło ([16]) for dense graphs of sufficiently large order ($s(G) \leq (4+o(1))n/\delta+4$ in this case).

The total vertex irregularity strength was in turn introduced by Baca et al. in [7] and then also attracted some attention (see e.g. [19, 6]). The best general result $tvs(G) \leq 3n/\delta$ is due to Anholcer et al. [5]. It was improved by Majerski and Przybyło [17] for dense graphs of sufficiently large order (in this case $tvs(G) \leq (2 + o(1))n/\delta + 4$).

Let us focus again on the product version of the problem. Anholcer [3] proved in particular the following lower bounds for general and regular graphs.

**Proposition 1.1 ([3])** For every graph $G$

$$ps(G) \geq \max_{\delta(G) \leq \delta \leq \Delta(G)} \left\{ \left[ \frac{d}{e} \cdot \frac{1}{n^d} - d + 1 \right] \right\}.$$
Proposition 1.2 ([3]) For every $r$-regular graph $G$ on $n$ vertices

$$ps(G) \geq \left\lceil \frac{r}{e} n^{1/r} - r + 1 \right\rceil.$$  

Also, the following upper bounds were given.

Proposition 1.3 ([3]) For every graph $G$ without isolated edges and with at most one isolated vertex, $ps(G) \leq p(|E(G)|)$, where $p(n)$ denotes the $n^{th}$ prime number.

The next bound follows from the work of Pikhurko [20].

Proposition 1.4 ([20]) For every sufficiently large graph $G$ without isolated edges and with at most one isolated vertex, $ps(G) \leq |E(G)|$.

Darda and Hujdurović in [9] improved these bounds for most graphs.

Proposition 1.5 ([9]) Let $X$ be a graph of order at least 4 with at most one isolated vertex and without isolated edges. Then

$$ps(X) \leq |V(X)| - 1.$$  

In the case of cycles the above results can be improved. In particular, the exact values for short cycles have been found. Also, two lower bounds for general cycles are given below.

Proposition 1.6 ([3]) For every $n > 2$

$$ps(C_n) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.$$  

Proposition 1.7 ([3]) For every $n > 17$

$$ps(C_n) \geq \left\lceil \left( \frac{n}{1 - \ln 2} \right)^{1/2} \right\rceil.$$  

Finally, some upper bounds for cycles have been proved.

Proposition 1.8 ([3])

$$ps(C_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \geq 6 \\ \lceil \frac{n}{3} \rceil & \text{if } n \geq 21 \end{cases}.$$  


Theorem 1.9 ([3]) For every \( \varepsilon > 0 \) there exists \( n_0 \) such that for every \( n \geq n_0 \)
\[
\text{ps}(C_n) \leq \lceil (1 + \varepsilon)\sqrt{2n \ln n} \rceil.
\]

The two last results hold also for paths \( P_n \) and all the Hamiltonian graphs of order \( n \). In the same paper upper bounds for grids and toroidal grids were given.

Theorem 1.10 ([3]) For every \( \varepsilon > 0 \) there exist \( n_j^{(0)}, j = 1, \ldots, k \) such that for every \( k \)-tuple \( (n_1, n_2, \ldots, n_k), n_j \geq n_j^{(0)}, j = 1, 2, \ldots, k \),
\[
1. \text{ps}(T_{n_1 \times n_2 \times \cdots \times n_k}) \leq \lceil (1 + \varepsilon)\sqrt{2(\sum_{j=1}^{k} \sqrt{n_j}) \ln (\sum_{j=1}^{k} n_j)} \rceil;
\]
\[
2. \text{ps}(G_{n_1 \times n_2 \times \cdots \times n_k}) \leq \lceil (1 + \varepsilon)\sqrt{2(\sum_{j=1}^{k} \sqrt{n_j}) \ln (\sum_{j=1}^{k} n_j)} \rceil.
\]

Skowronek-Kaziów considered the local versions of both problems (with vertex labels allowed or not). In both cases the goal was to distinguish only the neighboring vertices. In particular she proved that if the vertex labels are allowed, then 3 colors are enough to properly color any graph [22] and if one cannot label the vertices, then it is enough to use at most 4 colors [23]. Note that the respective upper bounds are 2 and 3, so the obtained results are almost optimal. In the case of complete graphs Skowronek-Kaziów obtains these lower bounds, so we can see that
\[
\text{ps}(K_n) = 3
\]
and
\[
\text{tvps}(K_n) = 2.
\]

In [4] Anholcer presented in particular results for bipartite graphs and forests.

Proposition 1.11 Let \( m \) and \( n \) be two integers such that \( 2 \leq m \leq n \). Then
\[
\text{ps}(K_{m,n}) = 3
\]
if and only if \( n \leq \binom{m+2}{2} \). Otherwise \( \text{ps}(K_{m,n}) \geq 4 \).

Let \( n_i \) denote the number of vertices of degree \( i \). Then, the following is true.

Theorem 1.12 Let \( D \geq 3 \) be arbitrary integer. For almost all forests \( F \) such that
1. \( \Delta(F) = D, \ n_2 = 0 \) and \( n_0 \leq 1 \),

2. if we remove all the pendant edges, then in the resulting forest \( F' \),
   \( n_2 = 0 \),

the product irregularity strength equals to

\[
ps(F) = n_1.
\]

Darda and Hujdurović [9] presented results for complete multipartite graphs. In particular, they proved the following

**Theorem 1.13** ([9]) Let \( m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_k \) be positive integers, and let \( K_{m_1,m_2,\ldots,m_k} \) be the complete multipartite graph such that \( m_k \leq m_1 + m_2 + \cdots + m_{k-1} \). Then \( ps(K_{m_1,m_2,\ldots,m_k}) = 3 \).

In this paper we present some lower and upper bounds on the total vertex product irregularity strength of graphs. In particular in section 2 some general results are presented, as well as the results for regular graphs. In section 3 lower and upper bounds for cycles and paths are given. Then we generalize these results for grids and toroidal grids in section 4. Finally some bounds for complete bipartite graphs are presented in section 5. We conclude the paper with some open problems.

## 2 General Graphs

In order to obtain different product degrees, it is necessary to use different multisets of labels to label the edges incident to every vertex (although, of course, it does not need to be a sufficient condition). Let \( n_d \) denote the number of vertices of degree \( d \) (note that \( d + 1 \) labels are present in the product degree of such vertex) and let \( s \) be the largest integer used in the labeling. Then

\[
n_d \leq \binom{s+d}{s-1} = \binom{s+d}{d+1} < \left( \frac{e(s+d)}{d+1} \right)^{d+1},
\]

so for every \( \delta(G) \leq d \leq \Delta(G) \), where \( \delta(G) \) and \( \Delta(G) \) denote the minimum and maximum degree, respectively,

\[
s \geq \left\lceil \frac{d+1}{e} n_d^{1/(d+1)} - d \right\rceil.
\]

Thus, the following two observations easily follow.
Proposition 2.1 For every graph $G$

$$ps(G) \geq \max_{\delta(G) \leq d \leq \Delta(G)} \left\{ \left\lceil \frac{d + 1}{e} n^{1/(d+1)} - d \right\rceil \right\}.$$  

Proposition 2.2 It follows, that for every $r$-regular graph $G$ on $n$ vertices

$$tvps(G) \geq \left\lceil \frac{r + 1}{e} n^{1/(r+1)} - r \right\rceil.$$  

Now let us consider the upper bounds. Obviously, if one assigns the label 1 to every vertex, then the obtained product degrees are equal to the ones in the case when vertex labels are not allowed. It follows that for every graph $G$ without isolated edges and with at most one isolated vertex, $tvps(G) \leq ps(G)$. In particular, the results from propositions [13] [14] and [15] hold. However, if one considers a general graph, the best we can obtain is the order of $G$. In fact, after labeling all the edges in an arbitrary way, one can compute the temporary product degrees assuming that all the vertex labels equal to 1. Now the final labels of vertices, being distinct numbers from \{1, 2, \ldots, |V(G)|\}, are assigned in increasing order consistent with non-decreasing order of the temporary product degrees. Obviously the resulting sequence of product degrees is strictly increasing. So we obtain the following.

Proposition 2.3 For every graph $G$ of order $n$, $tvps(G) \leq n$.

This bound cannot be improved in the general case, as the example of the empty graph shows.

In the next section we provide some results for cycles and paths.

3 Cycles and Paths

Denote by $C_k$ a cycle of length $k$. Then [AZAM: write the correct values and the details of the proof; remember to change ps to tvps; last cycles that]:

Fact 3.1

$$tvps(C_3) = tvps(C_4) = 2$$
$$tvps(C_5) = tvps(C_6) = tvps(C_7) = tvps(C_8) = tvps(C_9) = tvps(C_{10}) = 3$$
$$tvps(C_{11}) = tvps(C_{12}) = tvps(C_{13}) = tvps(C_{14}) = tvps(C_{15}) = tvps(C_{16}) = 4$$
$$tvps(C_n) \geq 5 \text{ for all } n \geq 17$$
**Proof.** First we prove that the listed values of $tvps(C_n)$ are the smallest possible.

As it may be trivially seen, using a label 1 one can get one product: 1, so it is impossible to label $C_3$ with just this label. Thus, we have to use at least two labels to label $C_3$ and any longer cycle.

Similarly we can observe that when using labels 1 and 2, one can obtain four distinct product degrees: 1, 2, 4 and 8, so at least three labels are necessary for $C_5$ and any longer cycle.

Using labels 1, 2 and 3 it is possible to obtain ten products: 1, 2, 3, 4, 6, 8, 9, 12, 18 and 27, so we have to use at least four labels to obtain a product-irregular labeling of $C_5$ and any longer cycle.

Finally, using labels 1, 2, 3 and 4 allows us to obtain sixteen product degrees: 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48 and 64, so at least 5 colors are necessary to properly label $C_{17}$ and any longer cycle.

Now we are going to show that the values in the theorem are enough to obtain product-irregular labelings. For every cycle $C_n$, $3 \leq n \leq 16$, let the sequence of labels be

$$S_n = ([w(v_1)], [w(v_1 v_2)], [w(v_2)], \ldots, [w(v_{n-1} v_n)], [w(v_n)], [w(v_n v_1)])$$

(the labels of vertices are in the brackets). Sample sequences of labels minimizing the value of $tvps(C_n)$ are listed below:

$$S_3 = ([1], [1], [2], [2], 2),$$
$$S_4 = ([1], [1], [1], [2], [2], [2], 2),$$
$$S_5 = ([1], [1], [2], [3], [3], [3], [3], [3], [3], [3], [2], 2),$$
$$S_6 = ([1], [1], [1], [2], [2], [3], [3], [2], 2),$$
$$S_7 = ([1], [1], [1], [2], [2], [2], [2], [2], [3], [3], [3], [2], 2, 1),$$
$$S_8 = ([1], [1], [3], [1], [1], [2], [2], [2], [2], [3], [3], [3], [2], 2, 1),$$
$$S_9 = ([1], [1], [1], [3], [2], [1], [1], [2], [2], [2], [2], [3], [3], [3], [2], 2, 1),$$
$$S_{10} = ([1], [1], [1], [3], [1], [3], [3], [2], 2, [2], [2], [2], [2], [3], [3], [2], [2], 1),$$
$$S_{11} = ([1], [1], [3], [1], [3], [1], [1], [1], [2], [2], [3], [3], [3], [4], [4], [4], [4], [3], [3], [2]),$$
$$S_{12} = ([1], [1], [3], [1], [3], [1], [4], [1], [1], [1], [2], [2], [3], [3], [3], [4], [4], [4], [4], [3], [3], [2]),$$
$$S_{13} = ([4], [1], [2], [1], [2], [3], [4], [2], [4], [2], [3], [3], [3], [4], [2], [4], [4], [4], [3], [3], [3], [3], [3], [2], 2),$$
$$S_{14} = ([4], [1], [2], [1], [4], [1], [2], [3], [4], [2], [4], [2], [3], [3], [4], [2], [4], [4], [4], [3], [3], [3], [3], [2], 2),$$
$$S_{15} = ([4], [1], [2], [1], [3], [1], [4], [1], [2], [3], [4], [2], [4], [2], [3], [3], [1], [3], [4], [2], [4], [4], [4], [4], [3], [3], [3], [3], [2], 2),$$
$$S_{16} = ([4], [1], [2], [1], [1], [3], [1], [4], [1], [2], [3], [4], [2], [4], [2], [3], [3], [1], [3], [4], [2], [4], [4], [4], [3], [3], [3], [3], [2], 2).$$
This completes the proof.

Now we are going to present the lower and upper bounds for an arbitrary cycle \( C_n \). Using similar argument as in Proposition 2.1 we can improve the general bound.

**Proposition 3.2** For every \( n > 2 \),

\[
tvps(C_n) \geq \lceil \sqrt[3]{6n - 1} \rceil.
\]

**Proof.** In the case of a cycle of length \( n \), the inequality

\[
n_d \leq \binom{s + d}{d + 1}
\]

takes the form

\[
s(s + 1)(s + 2) \geq 6n,
\]

which implies

\[
(s + 1)^3 \geq 6n,
\]

and finally

\[
s \geq \sqrt[3]{6n - 1}.
\]

Now we are going to present upper bounds on \( tvps(C_n) \). Recall that in the label sequences, the vertex labels are surrounded with square brackets.

**Lemma 3.3** For every \( s \geq 3 \) there exist product-irregular labelings of \( C_{3s-2} \), \( C_{3s-1} \) and \( C_{3s} \) with the label sequence containing the subsequence

\[
s - 1, [s - 1], s, [s], s, [s], s - 1.
\]

**Proof.** For \( C_7 \), \( C_8 \) and \( C_9 \), see the sequences given in the proof of Fact 3.1.

Now, assume that the statement is true for some \( s \). Take any \( n \), \( 3s - 2 \leq n \leq 3s \). Without loss of generality we can assume that \( w(v_{n-2}v_{n-1}) = s - 1 \), \( w(v_{n-1}) = s - 1 \), \( w(v_{n-1}v_n) = s \), \( w(v_n) = s \), \( w(v_nv_1) = s \), \( w(v_1) = s \), \( w(v_1v_2) = s - 1 \). As one can see, the three largest product degrees are \( w(v_n - 1) = s(s - 1)^2 \), \( w(v_n) = s^3 \) and \( w(v_1) = s^2(s - 1) \). We now extend the labeling to \( C_{n+3} \) adding only one label \( s + 1 \) in the following way (of course the edge \( v_nv_1 \) is removed together with its label: \( w(v_nv_{n+1}) = s \), \( w(v_{n+1}) = s \), \( w(v_{n+1}v_{n+2}) = s + 1 \), \( w(v_{n+2}) = s + 1 \), \( w(v_{n+2}v_{n+3}) = s + 1 \), \( w(v_{n+3}v_{n+4}) = s + 1 \), \( w(v_{n+4}v_{n+5}) = s + 1 \), etc.)
$w(v_{n+3}) = s + 1, w(v_{n+3}v_1) = s$. As one can easily check, the new labeling contains the sequence

$$s, [s], s + 1, [s + 1], s + 1, [s + 1], s.$$  

Moreover it preserves the product degrees for all $v_i$, $1 \leq i \leq n$, while for the three new vertices we have $pd(v_{n+1}) = s^2(s + 1)$, $pd(v_{n+2}) = (s + 1)^3$, $pd(v_{n+3}) = s(s + 1)^2$. Clearly, the new labeling satisfies the subsequence condition and is product-irregular, so the proof follows by induction.  

**Lemma 3.4** For every $s \geq 4$ there exist product-irregular labelings of $C_{4s-3}$, $C_{4s-2}$, $C_{4s-1}$ and $C_{4s}$ with the label sequence containing the subsequence

$$s - 1, [s], s, [s - 2], s, [s], s, [s - 1], s - 1.$$  

**Proof.** For $C_{13}$, $C_{14}$, $C_{15}$ and $C_{16}$, see the sequences given in the proof of Fact 3.1. Now, assume that the statement is true for some $s$. Take any $n$, $4s - 3 \leq n \leq 4s$. Without loss of generality we can assume that $w(v_{n-2}v_{n-1}) = s - 1, w(v_{n-1}) = s, w(v_{n-1}v_n) = s, w(v_n) = s - 2, w(v_nv_1) = s, w(v_1) = s, w(v_1v_2) = s, w(v_2) = s - 1, w(v_2v_3) = s - 1$. As one can see, the four largest product degrees are $w(v_{n-1}) = s^2(s - 1)$, $w(v_n) = s^2(s - 2)$, $w(v_1) = s^3$ and $w(v_2) = s(s - 1)^2$. Similarly as in the proof of Lemma 3.3 we extend the labeling to $C_{n+4}$ adding only one label $s + 1$ by assigning $w(v_nv_{n+1}) = s, w(v_{n+1}) = s + 1, w(v_{n+1}v_{n+2}) = s + 1, w(v_{n+2}) = s - 1, w(v_{n+2}v_{n+3}) = s + 1, w(v_{n+3}) = s + 1, w(v_{n+3}v_{n+4}) = s + 1, w(v_{n+4}) = s, w(v_{n+4}v_1) = s$. As one can easily check, the new labeling contains the sequence

$$s, [s + 1], s + 1, [s - 1], s + 1, [s + 1], s + 1, [s], s.$$  

Moreover it preserves the product degrees for all $v_i$, $1 \leq i \leq n$, while for the four new vertices we have $pd(v_{n+1}) = s(s + 1)^2$, $pd(v_{n+2}) = (s - 1)(s + 1)^2$, $pd(v_{n+3}) = (s + 1)^3$, $pd(v_{n+4}) = s^2(s + 1)$. Clearly, the new labeling satisfies the subsequence condition and is product-irregular, so the proof follows by induction.  

The lemmas 3.3 and 3.4 imply the following.

**Proposition 3.5**

$$ps(C_n) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \geq 7 \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \geq 13 \end{cases}.$$  

Now we are going to consider the case of large cycles.
Theorem 3.6 For every $\varepsilon > 0$ there exists $n_0$ such that for every $n \geq n_0$

$$\text{ps}(C_n) \leq \lceil (1 + \varepsilon)3\sqrt{2}(1 + \varepsilon)n^{1/3}\ln n \rceil.$$  

Proof. Let $s = \lceil (1 + \varepsilon)3\sqrt{2}(1 + \varepsilon)n^{1/3}\ln n \rceil$ and let $p$ be the greatest prime such that $p \leq \lceil 2/3\pi(s) \rceil$. From Bertrand-Chebyshev theorem \cite{24}, $p \geq \lceil 2/3\pi(s) \rceil/2 + 1$.

We start with finding the edge labels.

For every $q$, $1 \leq q < p/2$ we define the sequence: $0$, $q \mod p$, $2q \mod p$, $\ldots$, $(p - 1)q \mod p$, $pq \mod p = 0$. We will refer to such a sequence as "the chain".

In any fixed chain, each number $a$, $0 \leq a \leq p - 1$, appears exactly once, as $q$ and $p$ are relatively prime, and so the order of $q$ in the additive group $\mathbb{Z}_p$ is equal to $p - 1$. Thus, if we join all those chains together and form a multichain (0’s being common members of every two neighboring chains), every pair of numbers from the considered set will appear at most once as a pair of consecutive numbers (here we use the assumption that $q < p/2$).

Next, we enumerate the primes: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on, and we assume also that $p_0 = 1$. Replace every number $a$ in the multichain constructed above by $p_a$. As no pair occurs twice, also each product occurs at most once. Moreover, as the neighboring numbers are always distinct, we do not obtain any square of a natural number. Now we join $r$ identical multichains together in the same way as we joined chains to obtain the multichain and close the cycle, where $1 \leq r \leq \lceil \pi(s) \rceil/3$ will be chosen later (in particular it may happen that $r = \lceil \pi(s) \rceil/3$. We are going to show that it is possible to label at least $n$ edges in this way.

As we use $p$ labels in each chain, and every chain is based on distinct natural number smaller than $p/2$, it is possible to label

$$m \geq r p(p - 1)/2 \geq \lceil \pi(s)/3 \rceil \left(\frac{2}{3}\pi(s)\right)^2/8 \geq \frac{1}{54} \left\lceil \frac{s}{\ln s} \right\rceil^3$$

edges. The choice of $s$ guarantees that, for sufficiently large $n$, the inequality $m > n$ holds.

We use the maximum number of multichains and then the maximum number of chains that allows us to label at most $n$ edges. Now, the number of labeled edges is between $n - (p - 1)r/2 + 1$ and $n$. Let $t$ denote the number of remaining edges. Of course, $0 \leq t < (p - 1)r/2 - 1$. We still use exactly $p$ labels and no squares have occurred so far. Now let us choose $t$ edges from the last chains of as many multichains as necessary (i.e., choose all the edges in the last chains of $\lceil t/(p - 1) \rceil$ multichains and $t - (p - 1)\lceil t/(p - 1) \rceil$ in any other
mutichain. Then, replace each of those edges with two consecutive edges, not changing the labels. This will not change any of the product degrees that have been obtained so far, as the sequences of the form $p_i, p_j, p_k$ become sequences of the form $p_i, p_j, p_j, p_k$. Adding new edges only adds new product degrees being squares of distinct primes that have not appeared so far. So we obtain edge labeling of the cycle of length $n$. Note that every product degree appears at most $r$ times, as in every multichain all the degrees are distinct.

Now for every set of vertices with equal product degree we use the primes greater than any label used so far (there are exactly $r$ of them) to distinguish the product degrees of those vertices. This results with a product-irregular labeling of $C_n$.

In this section we considered the total vertex product irregularity strength of a cycle $C_n$. It is straightforward to see that the given results can be used also to obtain upper bounds on the total vertex product irregularity strength of paths (it is enough to remove any edge with label 1, Hamiltonian graphs (one labels with 1 all the edges not belonging to the Hamiltonian cycle) and semi-Hamiltonian graphs (one labels with 1 all the edges not belonging to the Hamiltonian path).

4 Grids and Toroidal Grids

Assume we are given $k$ paths $P_j$ ($j = 1, 2, \ldots, k$) with vertex sets $V_j$ ($j = 1, 2, \ldots, k$), where $|V_j| = n_j$. We define the grid $G_{n_1 \times n_2 \times \cdots \times n_k}$ as the product of those paths. More exactly, the vertex set $V$ of $G_{n_1 \times n_2 \times \cdots \times n_k}$ is the Cartesian product of the vertex sets $V_j$ ($j = 1, 2, \ldots, k$) : $V = V_1 \times V_2 \times \cdots \times V_k$.

We can consider the elements of $V$ as points in the $k$-dimensional Euclidean space with $i^{th}$ coordinate equal to the position of vertex on path $P_k$. Two vertices $(u_1, u_2, \ldots, u_k), (v_1, v_2, \ldots, v_k) \in V$ are adjacent if and only if all their coordinates but one, say $j^{th}$, are the same, and $|u_j - v_j| = 1$.

If we consider cycles instead of paths (in such situation elements of $V$ may be considered as elements of Cartesian product of $k$ cyclic groups $Z_{n_j}$, $j = 1, 2, \ldots, k$, instead of Euclidean space), we obtain the toroidal grid $T_{n_1 \times n_2 \times \cdots \times n_k}$.

**Lemma 4.1** Let $n_1, n_2, \ldots, n_k$ be natural numbers, $n_j \geq 3$ for all $j = 1, 2, \ldots, k$.

1. Let $p_1, p_2, \ldots, p_k$ and $r_1, r_2, \ldots, r_k$ be $2k$ natural numbers (not necessarily distinct) such that $p_j$ primes as the edge labels and $r_k$ primes as vertex labels are enough to obtain a product-irregular labeling of $C_{n_j}$.
Then \( \sum_{j=1}^{k} (p_j + r_j) \) primes are enough to obtain a product-irregular labeling of the toroidal grid \( T_{n_1 \times n_2 \times \ldots \times n_k} \).

2. Let \( p_1, p_2, \ldots, p_k \) and \( r_1, r_2, \ldots, r_k \) be \( 2k \) natural numbers (not necessarily distinct) such that \( p_j \) primes as the edge labels and \( r_k \) primes as vertex labels are enough to obtain a product-irregular labeling of \( P_{n_j} \) 
\( (j = 1, 2, \ldots, k) \). Then \( \sum_{j=1}^{k} (p_j + r_j) \) primes are enough to obtain a product-irregular labeling of the grid \( G_{n_1 \times n_2 \times \ldots \times n_k} \).

\[ \text{Proof.} \quad \text{Let us start with toroidal grids. We are going to prove the lemma by induction on } k. \text{ For } k = 1 \text{ it is trivially true. Let us assume that we have managed to label } T_{n_1 \times n_2 \times \ldots \times n_{k-1}} \text{ using } \sum_{j=1}^{k-1} p_j \text{ primes. To construct } T_{n_1 \times n_2 \times \ldots \times n_k}, \text{ we join } n_k \text{ copies of } T_{n_1 \times n_2 \times \ldots \times n_{k-1}} \text{ in such a way that for every vertex } v \text{ of } T_{n_1 \times n_2 \times \ldots \times n_{k-1}}, \text{ all copies of } v \text{ are joined with one copy of } C_{n_k}. \text{ The set of labels of the edges incident to each copy of } v \text{ is distinct, as it consists of constant set of labels used to label } T_{n_1 \times n_2 \times \ldots \times n_{k-1}} \text{ and changing pair of labels on the edges of } C_{n_k} \text{ (always distinct to the previous ones). We define the vertex label as the product of the labels of the corresponding vertices in the factors (i.e., in } T_{n_1 \times n_2 \times \ldots \times n_{k-1}} \text{ and } C_{n_k} \text{). This way we obtain the correct labeling of } T_{n_1 \times n_2 \times \ldots \times n_k} \text{ using } p_k + r_k \text{ new prime numbers. It gives us the desired number of labels.}

\]

\[ \text{In the case of } G_{n_1 \times n_2 \times \ldots \times n_k} \text{ we use } C_{n_j} \text{ instead of } P_{n_j}. \]

\[ \text{Note that the maximum label used in the above labeling is not greater than } \max\{P, R\}, \text{ where } P \text{ is the maximum edge label among all cycles (paths) and } R \text{ the product of the maximum vertex labels of all cycles (paths). Now we can formulate the main result of this section.} \]

\[ \textbf{Theorem 4.2} \quad \text{For every } k \geq 2 \text{ and every } \varepsilon > 0 \text{ there exist } n_j^{(0)} \text{, } j = 1, \ldots, k \text{ such that for every } k\text{-tuple } (n_1, n_2, \ldots, n_k), n_j \geq n_j^{(0)}, \text{ } j = 1, 2, \ldots, k; \]

1. \( \text{ps}(T_{n_1 \times n_2 \times \ldots \times n_k}) \leq \left[ (1 + \varepsilon)k^{k/2}(\sum_{j=1}^{k} n_j^{k/(2k+1)}) \sum_{j=1}^{k} \ln n_j \right]; \)

2. \( \text{ps}(G_{n_1 \times n_2 \times \ldots \times n_k}) \leq \left[ (1 + \varepsilon)k^{k/2}(\sum_{j=1}^{k} n_j^{k/(2k+1)}) \sum_{j=1}^{k} \ln n_j \right]. \)

\[ \text{Proof.} \quad \text{In order to find an irregular labeling of a cycle } C_n \text{ it is enough to use } p_j \text{ labels (primes or 1) to label the edges and} \]

\[ r_j = \left\lceil \frac{2n}{p_j(p_j - 1)} \right\rceil \]
labels for vertices (see the proof of Theorem 3.6). In particular, if \( p_j = \lfloor n_j^{k/(2k+1)} \rfloor \), then \( r_j \leq \lceil n_j^{1/(2k+1)} \rceil \). This means that we need at most

\[
\sum_{j=1}^{k} (\lfloor n_j^{k/(2k+1)} \rfloor + \lceil n_j^{1/(2k+1)} \rceil)
\]

distinct labels (all of them being primes or 1). Let

\[
s = \lfloor (1 + \varepsilon)k^{k-2} \left( \sum_{j=1}^{k} n_j^{k/(2k+1)} \right) \sum_{j=1}^{k} \ln n_j \rfloor
\]

and

\[
\varepsilon' = (1 + \varepsilon)^{1/k} - 1.
\]

We assign

\[
\sum_{j=1}^{k} \lceil n_j^{1/(2k+1)} \rceil
\]

smallest primes to the vertices (1 will be used as edge label). This means that the maximum vertex label \( R_j \) in every cycle satisfies the condition

\[
R_j < (1 + \varepsilon') \left( \sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \ln \left( \sum_{j=1}^{k} n_j^{1/(2k+1)} \right)
\]

for \( s_j \) (i.e. \( n_j \)) sufficiently large, \( j = 1, \ldots, n \). Consequently, the maximum vertex label \( R \) in \( T_{n_1 \times n_2 \times \cdots \times n_k} \) is not greater than

\[
R = \prod_{j=1}^{k} R_j < (1 + \varepsilon) \left( \left( \sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \ln \left( \sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \right)^k
\]

for \( n_j \) sufficiently large, \( j = 1, \ldots, k \). For \( n_j \) sufficiently large we have

\[
\ln \left( \sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \leq \sum_{j=1}^{k} \ln \left( n_j^{1/(2k+1)} \right).
\]

From Hölder’s Inequality it follows that

\[
\sum_{j=1}^{k} n_j^{1/(2k+1)} \leq \left( \sum_{j=1}^{k} n_j^{k/(2k+1)} \right)^{1/k} k^{1-1/k}
\]
and
\[
\sum_{j=1}^{k} \ln \left( n_j^{1/(2k+1)} \right) \leq \left( \sum_{j=1}^{k} \ln \left( n_j^{1/(2k+1)} \right) \right)^{\frac{1}{k}} k^{1-\frac{1}{k}} < \left( \sum_{j=1}^{k} \ln \left( k n_j^{1/(2k+1)} \right) \right)^{\frac{1}{k}} k^{1-\frac{1}{k}} < \left( \sum_{j=1}^{k} \ln \left( n_j^{k/(2k+1)} \right) \right)^{\frac{1}{k}} k^{1-\frac{1}{k}}
\]

This implies
\[
R < (1 + \varepsilon) k^{2k-2} \left( \sum_{j=1}^{k} n_j^{k/(2k+1)} \right) \sum_{j=1}^{k} \ln n_j \leq s.
\]

On the other hand, for \( n_j \) sufficiently large, \( j = 1, \ldots, n \), the highest edge label is not greater than
\[
(1 + \varepsilon) \left( \sum_{j=1}^{k} \left( \left\lfloor n_j^{k/(2k+1)} \right\rfloor + \left\lceil n_j^{1/(2k+1)} \right\rceil \right) \right) \ln \left( \sum_{j=1}^{k} \left( \left\lfloor n_j^{k/(2k+1)} \right\rfloor + \left\lceil n_j^{1/(2k+1)} \right\rceil \right) \right) < (1 + \varepsilon) \left( 2 \sum_{j=1}^{k} n_j^{k/(2k+1)} \right) 2 \sum_{j=1}^{k} \ln n_j^{k/(2k+1)} < s.
\]

Since in every cycle 1 appears as an edge label, the same reasoning proves the theorem for the grid \( G_{n_1 \times n_2 \times \ldots \times n_k} \).

5 Complete bipartite graphs

In this section we will prove a simple result for complete bipartite graphs.

**Proposition 5.1** Let \( m \) and \( n \) be two integers such that \( 3 \leq m \leq n \leq \binom{m+2}{2} \). Then
\[
tvps(K_{m,n}) = 3
\]
The fact that the labels 1, 2 and 3 are enough follows from Proposition 1.11 (it is enough to label all vertices with 1 and we are done). On the other hand, if we use only labels 1 and 2, then the product degrees could take at most $n + 2$ values, namely $1, 2, 4, \ldots, 2^{n+1}$, while there are at least $n + 3$ vertices, so at least third label is necessary.

\section{Conclusion}

In the paper we introduced a new graph invariant, total vertex product irregularity strength. It is a parameter analogous to the total vertex irregularity strength for the case where the weighted degrees are computed as products. We proved several results for general and regular graphs, as well as for some specific families of graphs, like cycles, paths and grids.

Most of our results show that the total vertex product irregularity strength of any $d$-regular graph of order $n$ is between $c_1 n^{1/(d+1)}$ and $c_2 n^{1/(d+1)} \ln n$ for some constants $c_1$ and $c_2$. Erdős \cite{erdos} proved that in the case of integers not greater than $n$, the cardinality $s(n)$ of a subset resulting in distinct pairwise products (multiplicative Sidon set) cannot be much greater than the number of primes not greater than $n$:

$$s(n) = \pi(n) + \Theta\left(\frac{n^{3/4}}{\ln^{3/2} n}\right).$$

This makes us believe that the upper bounds presented in this paper are closer to the exact value of $\text{tvps}(G)$ than the lower and the effort of the further research should be focused on the improvement of the latter ones. For that reason we formulate the two following open problems.

\textbf{Problem 6.1} Is there a constant $c$ such that for every $d$-regular graph $G$ of order $n$

$$\text{tvps}(G) \geq cn \ln \ln n?$$

\textbf{Problem 6.2} Is there a constant $c$ such that for every $d$-regular graph $G$ of order $n$

$$\text{tvps}(G) \geq \frac{cn \ln n}{\ln \ln n}?$$

We also believe that the logarithmic factor is enough to guarantee the existence of a product-irregular labeling of any graph. In particular, we pose the following conjecture.
Conjecture 6.3 There exists a constant c such that for every d-regular graph $G$ of order $n$

$$tvps(G) \leq cn^{1/(d+1)} \ln n.$$ 

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