A minimax principle to the injectivity of the Jacobian conjecture

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Abstract

The main result of this paper is to prove some type of Real Jacobian Conjecture. It is proved by the Minimax Principle and asserts if the eigenvalues of $F'(x)$ are bounded from zero and all the eigenvalues of $F'(x) + F'(x)^T$ are strictly same sign, where $F$ is $C^1$ mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, then $F$ is injective. Moreover $F$ has a $C^1$ mapping inverse.

1 Introduction

It’s well-known that Jacobian Conjecture is first proposed by Keller [Kel39].

Conjecture 1.1. (Jacobian Conjecture) Let $F(x)$ be $k^n \rightarrow k^n$ a polynomial map, where $k$ is a field of characteristic 0. If the determinant for its jacobian of the polynomial map is a non-zero constant, i.e., $\det JF(x) \equiv C \in k^*$, $\forall x \in k^n$. Then $F(x)$ has a polynomial inverse map.

For a long study history, it is still open, even for $n = 2$. Many results on it, see [BCW82].

A very important step, for example if $k = \mathbb{C}^n$, is the following result [CR91].

Theorem 1.2. Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a polynomial map. If $F$ is injective, then $F$ is bijective. Furthermore the inverse is also a polynomial map.

The theorem above applied to real case, however, is still open. Generally, the question is transformed to the injectivity of the map in real or complex case. Compared with the complex case, we have an analogous conjecture in real case. For $k^n = \mathbb{R}^n$, the conjecture is

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Conjecture 1.3. (Real Jacobian Conjecture) (RJC) If \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a polynomial map, \( \det F'(x) \) is not zero in \( \mathbb{R}^n \), then \( F \) is an injective map.

However, it is a pity that is false and Pinchuk\cite{Pin94} constructed a counter-example to (RJC) for \( n = 2 \).

The Pinchuk’s counter-example states sufficiently that the condition \( \det F'(x) \) is not zero in \( \mathbb{R}^n \) in (RJC) is much weaker to prove Conjecture 1.1.

In order to enhance a sufficient condition for injective, M. Chamberland and G. Meisters raise the following conjecture see \cite{CM98}, Conjecture 2.1:

Conjecture 1.4. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) map. Suppose there exists an \( \epsilon > 0 \) such that \( |\lambda| \geq \epsilon \) for all the eigenvalues \( \lambda \) of \( F'(x) \) for all \( x \in \mathbb{R}^n \). Then \( F \) is injective.

Remark 1.5. Pinchuk’s counter-example polynomial does not satisfy the hypotheses of the Conjecture 1.4.

They\cite{CM98} also obtain the weak result for the Conjecture 1.4.

Theorem 1.6. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^1 \) map. Suppose there exists an \( \epsilon > 0 \) such that \( |\mu| \geq \epsilon \) for all the eigenvalues \( \mu \) of \( F'(x)F'(x)^T \) for all \( x \in \mathbb{R}^n \). Then \( F \) is injective.

In this paper, we prove the conjecture 1.4 under an additional assumption and the main result is the following theorem.

Theorem 1.7. Suppose that \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) map. If there exists \( \epsilon > 0 \), such that \( |\lambda| \geq \epsilon, \mu \geq \epsilon \) or \( \mu \leq -\epsilon \) for all eigenvalues \( \lambda, \mu \) of \( F'(x), F'(x) + F'(x)^T \) respectively, for all \( x \in \mathbb{R}^n \). Then \( F \) is injective. Moreover \( F \) has a \( C^1 \) inverse map.

In order to prove the theorem 1.7, we need to give some definition and notation.

Notation:
- \( \mathbb{C} \): Complex field;
- \( \mathbb{R} \): Real field;
- \( A^T \): the transposition of matrix \( A \);
- \( \det A \): the determinant of matrix \( A \);
- \( F'(x) \): The Jacobian matrix of \( F(x) \);
- \( X' \): the dual space of \( X \);
- \( trA \): the trace of matrix \( A \);
- \( ||.|| \): the norm of \( \mathbb{R}^n \).
2 Minimax Principle

In this section, we will introduce some preparation for the Minimax Principle.

**Definition 2.1.** ((PS), sequence)
Let $X$ be a Banach space, $J \in C^1(X, \mathbb{R})$. If $\forall \{u_k\} \subset X$, $\exists c \in \mathbb{R}$, such that
\[ I(u_k) \to c, \quad I'(u_k) \to 0, \quad \text{in} \quad X', \]
as $k \to \infty$. Thus $\{u_k\}$ is a (PS)$_c$ sequence of $J$.

**Definition 2.2.** ((PS) condition)
If $\forall$ (PS)$_c$ sequence of $J$ has a convergent subsequence, then $J$ satisfies the (PS)$_c$ condition. If $\forall c$, it is said to satisfy the (PS) condition.

**Theorem 2.3.** (Minimax Principle) Let $X$ be a Banach space, and $J \in C^1(X, \mathbb{R})$. Suppose an open set $\Omega \subset X$, $u_0 \in \Omega$, and $u_1 \notin \Omega$. Set
\[ \Gamma = \{ l \in C([0,1], X) | l(i) = u_i, i = 0,1 \} \]
and
\[ c = \inf_{l \in \Gamma} \sup_{t \in [0,1]} J(l(t)). \]  

(1)

If
(a) $\alpha = \inf_{\partial \Omega} J(u) > \max\{J(u_0), J(u_1)\},$
(b) $J$ satisfies (PS)$_c$ condition.

Then $c$ is a critical value of $J$.

In this paper, we use $X = \mathbb{R}^n$ in Theorem 2.3.

3 The proof of Theorem 1.7

We have already introduced all the necessary ingredients for our theorem.

**Proof.** The proof is by contradiction. Suppose there exist $x_1, x_2 \in \mathbb{R}^n, x_1 \neq x_2$, such that $F(x_1) = F(x_2)$. Denote $I(x) = F(x + x_1) - F(x_2), \forall x \in \mathbb{R}^n$. So $I(x) \in \mathbb{R}^n$. We define
\[ J(x) = I(x)^T I(x), \quad \forall x \in \mathbb{R}^n. \]

Thus $J'(x) = 2I(x)^T I'(x)$. Let $x_0 = x_1 - x_2$, then $J(x_0) = J(0) = 0$.

Claim
\[ \det I'(x) \neq 0, \quad \forall x \in \mathbb{R}^n. \]  

(2)

If there exists $x' \in \mathbb{R}^n$, such that $det I'(x') = 0$. So $det F'(x' + x_1) = 0$. Thus, $\exists y \in \mathbb{R}^n, y \neq 0$, s.t.
\[ F'(x' + x_1)y = 0. \]  

(3)
That is $F'(x' + x_1)$ has a zero eigenvalue. It contradicts the eigenvalues of $F'(x)$ are bounded from zero.

Thus $I'(x)$ is an invertible matrix. If $J'(x) = 0$, $\forall x \in \mathbb{R}^n$, i.e. $2I(x)^T I'(x) = 0$, $\forall x \in \mathbb{R}^n$, thus $I'(x) I(x) = 0$, $\forall x \in \mathbb{R}^n$. $I(x) = 0$. So $J(x) = 0$.

Next, we prove $J$ satisfies the geometric condition (a) in Theorem 2.3. Since $J(0) = J(x_0) = 0$, it is sufficient to prove $\exists r$, such that

\[
J(u) > 0, \forall u \in \partial B_r(0). \tag{4}
\]

Claim: $x = 0$ is an isolated zero point of $J(x)$.

For each component $I_i(x)$ of $I(x)$, so $I_i(x) = I'(y_i)x$, here $y_i$ connects 0 to $x$, $i = 1, 2...n$. Define a continuous function $\beta(x)$ as

\[
\beta(x) = \begin{cases} 
(I_1'(y_1), I_2'(y_2)...I_n'(y_n))^T, & x \neq 0, \\
I'(0), & x = 0.
\end{cases}
\]

Thus $I(x) = \beta(x)x$, $\forall x \in \mathbb{R}^n$. Define

\[
\gamma(x_1, x_2..., x_n) = (I_1'(x_1), I_2'(x_2)...I_n'(x_n))^T.
\]

Thus $\gamma(x, x..., x) = I'(x)$ and $\gamma(y_1, y_2..., y_n) = \beta(x)$. Therefore

\[
\det \gamma(0, 0, ..., 0) = \det I'(0) \neq 0.
\]

By the continuity of $\gamma$, there exists a positive number $r > 0$, such that

\[
\det \gamma(x_1, x_2..., x_n) \neq 0, \text{ for } (x_1, x_2..., x_n) \in B_r(0).
\]

Thus $\det \beta(x) \neq 0, \forall x \in B_{r/\sqrt{n}}(0)$. Therefore 0 is an isolated zero point of $I(x)$.

Let $\alpha = \inf_{\partial B_{r/\sqrt{n}}(0)} J(x)$. It is a positive number since $J(x)$ is continuous and is not zero on $\partial B_{r/\sqrt{n}}(0)$.

Thus $J(x)$ satisfies the condition (a) in theorem 2.3.

If $c$ is a critical value of $J$, that is $\exists x_c \in \mathbb{R}^n$ such that $J'(x_c) = 0$. Thus

\[
0 < \alpha \leq c = J(x_c) = 0.
\]

Obviously, it’s impossible.

By Theorem 2.3, the condition (b) i.e. $(PS)_c$ condition does not hold. There is a sequence $\{x_k\} \subset \mathbb{R}^n$, such that

\[
(i) J(x_k) \to c; \quad (ii) J'(x_k) \to 0; \quad (iii) \|x_k\| + \infty.
\]

If $\mu \leq -\varepsilon$ for all eigenvalues $\mu$ of $F'(x) + F'(x)^T$ for all $x \in \mathbb{R}^n$.

Let $\mu_0$ denote the maximum eigenvalue of a Hermitian matrix $A$. Define

\[
\mu_0 = \sup_{Y \neq 0} \frac{Y^T A Y}{Y^T Y}. \tag{5}
\]
Set $A = I'(x_k) + I'(x_k)^T$ and $Y = I(x_k)$. By (5), we obtain

$$\mu_0(x_k) \geq \frac{I(x_k)^TI'(x_k)I(x_k) + I(x_k)^T'I'(x_k)^T I(x_k)}{I(x_k)^T I(x_k)}$$

$$= \frac{2I(x_k)^TI'(x_k)I(x_k)}{I(x_k)^T I(x_k)}.$$  

(6)

By (i), one gets

$$I(x_k)^T I(x_k) = J(x_k) \rightarrow c > 0.$$  

(7)

By (ii) and (7), we obtain

$$2I(x_k)^T I'(x_k)I(x_k) \leq 2 \left\| I(x_k)^T I'(x_k) \right\| \| I(x_k) \|$$

$$= \left\| J'(x_k) \right\| \left( I(x_k)^T I(x_k) \right)^{\frac{1}{2}}$$

$$= \left\| J'(x_k) \right\| \sqrt{J(x_k)} \rightarrow 0.$$  

(8)

Combining (6), (7) with (8), as $k \rightarrow +\infty$, thus

$$\mu_0(x_k) \geq 0.$$  

(9)

By $\mu \leq -\epsilon$ for all the eigenvalue $\mu$ of $F'(x) + F'(x)^T$, thus

$$\mu_0(x_k) < 0.$$  

(10)

In (9) and (10), letting $k \rightarrow +\infty$, thus,

$$\mu_0(x_k) \rightarrow 0.$$  

It contradicts.

If $\mu \geq \epsilon$ for all eigenvalues $\mu$ of $F'(x) + F'(x)^T$ for all $x \in \mathbb{R}^n$.

Let $\mu_1 = \inf_{Y \neq 0} \frac{Y^T A Y}{Y^T Y}$. By the same method, we obtain contradiction.

Therefore $F$ is injective. Thus $F$ has an inverse map, denoted by $G$. Since $F \cdot G = \text{Id}$ and $F'$ exists, then if $\forall x \in \mathbb{R}^n$, such that $F \cdot G(x) = x$, then

$$F'(G(x)) \cdot G'(x) = 1.$$  

Since $F'$ exists and $\det F' \neq 0$, $G'(x)$ exists, and

$$G'(x) = F'(G(x))^{-1}.$$  

By $F \in C^1$, we obtain $G \in C^1$.

$$\square$$

**Remark 3.1.** A further consideration to the question, it is either to give an analogue to minimax principle in complex case by our theorem or to give an real analogue to theorem 1.2, which is great progress to the Jacobian conjecture.
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