Lie group theory for nonlinear fractional K(m,n) type equation with variable coefficients

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Abstract

We investigated the analytical solution of fractional order K(m,n) type equation with variable coefficient which is an extended type of KdV equations into a genuinely nonlinear dispersion regime. By using the Lie symmetry analysis, we obtain the Lie point symmetries for this type of time-fractional partial differential equations (PDE). Also we present the corresponding reduced fractional differential equations (FDEs) corresponding to the time-fractional K(m,n) type equation.

Keywords: fractional differential equation, Lie symmetry analysis method, reduced equation, fractional order K(m,n) type equation.

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1. Introduction

Most problems in engineering, biology, applied mathematics and physics might be better modeled by using ordinary/partial differential equations with fractional (arbitrary) order. The method of group analysis for ordinary/partial differential equations, originally advocated by the Norwegian mathematician Sophus Lie during 1870s. The tangent structural equations under transformation groups is the fundamental idea of symmetry analysis. Numerous methods developed to solve differential equations based on Lie symmetry analysis.

In last few decades, many researcher studied different class of the fractional partial differential equations (FPDE). These equations arise in Various branches of sciences such as physics, biology, viscoelastic materials, electrochemistry, signal processing, fluid mechanics \[18, 25, 32, 1, 27\]. Integrals and derivatives are of any order in the fractional calculus \[18, 25\]. In the recent years, finding exact solutions of FDEs has gained much attention.

Many researchers have presented various techniques and methods for obtaining the numerical and analytical solutions of FDE, such as variational iteration method \[11\], the separating variables method \[7\], operational matrices \[15\], the fractional complex transform \[12\], the first integral method \[20\], and so on. In many years ago, there are many articles to obtain the analytical solution/s of
nonlinear PDE using Lie group theory. It is important to know, however, that few of them involve FDEs\[5\],\[9\],\[14\],\[17\],\[19\],\[30\],\[31\]. Furthermore, already few articles done in symmetries of variable coefficients FDEs such as [10, 21]. Our purpose is to study the time-fractional K(m,n) equation:

\[\frac{\partial^\alpha u}{\partial t^\alpha} + \zeta (u^m)_x + g(t)(u^n)_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (1)\]

or equivalently

\[D_t^\alpha u + \zeta (u^m)_x + g(t)(u^n)_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1. \quad (2)\]

Here \(m \neq 0\) are arbitrary constants, \(\zeta = \pm 1\) and \(g(t)\) is an arbitrary nonvanishing function of the variable \(t\). This equation for \(\alpha = 1\) and also with constant coefficient for \(0 < \alpha < 1\) has been discussed in [6] and [29].

In the follow, we study the above equation with \(m = 2, n = 3\). Eq. (1) is called the K(m,n) equation, when \(\zeta = g(t) = 1\). Rosenau introduced this equation in 1998 [26, 33] which is described the process of interpretation the role nonlinear dispersion in the formation of structures in liquid drops.

This article is organized as follows. In the next Section, it is given the analysis of Lie Symmetry group for a FPDE. Then in section 3, using Lie group, the Lie point symmetries of equation (1) are obtained. In Section 4, we perform Lie group on the equation (1) for obtaining invariant solutions and reduced fractional ODEs. Conclusions are given in the Section 5.

2. Lie symmetry analysis method for FPDEs

According to the importance of FPDEs in mathematics and physics, finding the exact solutions for these equations is very important. Although nonlinear FPDEs are difficult to solve, but many papers have been presented by scientists. Studying differential equations using the fundamental method of the Lie symmetries is interesting for many researchers. In the past century, many researchers have studied in the field of the Lie groups. Some of them are Ovsiannikov [24], Olver [23], Ibragimov [13], Baumann [2] and Bluman [3]. In this section, finding infinitesimal functions of FPDEs are given. Let us consider the below form of FPDEs:

\[D_t^\alpha u = F(x, t, u, u_{(1)}, \ldots), \quad \alpha > 0. \quad (3)\]

Where \(D_t^\alpha\) fractional derivative in the sense of Riemann-Liouville [14] and \(u\) is depend to \(x, t\).

Similar discussion of PDEs [4, 23], we can write

\[D_t^\alpha \tilde{u} = D_t^\alpha u + \varepsilon [\eta_t^{(\alpha)}(t, x, u_{(1)}, u_{(\alpha)}, \ldots)] + O(\varepsilon^2). \quad (4)\]

In view of by the prolongation formula, for \(\eta_t^{(\alpha)}\) we have [9]

\[\eta_t^{(\alpha)} = D_t^\alpha (\eta) + \xi^x D_t^\alpha (u_x) - D_t^\alpha (\xi^x u_x) + D_t^\alpha (D_t(\xi^t)u) - D_t^{\alpha + 1}(\xi^t u) + \xi^t D_t^{\alpha + 1} u, \quad (5)\]

and the total derivative operator \(D_t\) is defined by

\[D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_u x \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + u_{xx} x \frac{\partial}{\partial u_{xx}} + \cdots. \quad (6)\]
Simplifying (5) using the Leibniz formula [28]
\[
D_t^\alpha [f(t)g(t)] = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha) \Gamma(n+1)} D_t^{\alpha-n} f(t) D_t^n g(t), \quad \alpha > 0, \quad (7)
\]
we can write [16]:
\[
\eta^{(\alpha)} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta - \alpha D_t(\xi^t)) \frac{\partial^\alpha \eta}{\partial t^\alpha} - \sum_{m=1}^{\infty} \frac{\alpha}{m} \frac{\partial^m \eta}{\partial t^m} - \left( m + 1 \right) D_t^{m+1}(\xi^t) D_t^{\alpha-m}(u) - \sum_{m=1}^{\infty} \frac{\alpha}{m} D_t^{m}(u_x) D_t^{\alpha-m}(\xi^x). \quad (8)
\]
To obtain coefficients of \(X\), we must have:
\[
X^{(\alpha)}[D_t^\alpha u - F(t,x,u,u(1),\ldots)] D_t^\alpha u = F(t,x,u(1),\ldots) = 0, \quad (9)
\]
where
\[
X^{(\alpha)} = \xi^t(t,x,u) \frac{\partial}{\partial x} + \xi^i(t,x,u) \frac{\partial}{\partial t} + \eta(t,x,u) \frac{\partial}{\partial u} + \eta^{(1)}(t,x,u,u(1)) \frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}(t,x,u,u(1),\ldots,u(k)) \frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}} + \eta^{(\alpha)}(t,x,u,\ldots,u(\alpha)) \frac{\partial}{\partial u^{(\alpha)}_i}. \quad (10)
\]
Using these relations, we obtain the Lie symmetries.

3. Fractional Lie symmetries for time-fractional K(2,3)

Now we obtain the infinitesimal generator of the time-fractional K(2,3) equation
\[
\frac{\partial^\alpha u}{\partial t^\alpha} + (u^2)_x + g(t)(u^3)_{xxx} = 0, \quad t > 0, \quad 0 < \alpha < 1. \quad (11)
\]

**Theorem 1.** Lie symmetries for Eq. (11), which those are solutions of determining equations depend on the selection of the function \(g(t)\), are

**Case 1:** \(0 < \alpha < 1, \alpha \neq \frac{1}{2}, \frac{3}{4}, \frac{k}{b} \neq 0, b \neq 0.\)

**Case 1.1:** \(g(t)\) be an non-vanishing arbitrary function.

In this case, the infinitesimal generator is given by
\[
X_{1,1} = \frac{\partial}{\partial x}. \quad (12)
\]

For mentioned \(g(t)\) as follows, we have additional symmetries.

**Case 1.2:** \(g(t) = kt^b.\)

For this case, we have
\[
X_{1,2,1} = \frac{\partial}{\partial x}, \quad X_{1,2,2} = -t \frac{\partial}{\partial t} + (\alpha - b) x \frac{\partial}{\partial x} + (2 \alpha - b) u \frac{\partial}{\partial u}. \quad (13)
\]
Case 1.3: \( g(t) = k \).
In this case, the infinitesimal generators are as follows

\[
X_{1.3.1} = \frac{\partial}{\partial x}, \quad X_{1.3.2} = \alpha x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} + 2\alpha u \frac{\partial}{\partial u}.
\]  

(14)

Case 2: \( \alpha = \frac{1}{2} \), \( k, b \neq 0 \).
For \( \alpha = \frac{1}{2} \), functions of \( g(t) \) can be obtained as follows

\[ g(t) = ke^{bt}, \quad kt^b, \quad k. \]

Case 2.1: \( g(t) = ke^{bt} \).
In this case, the infinitesimal generator is given by

\[ X_{2.1} = \frac{\partial}{\partial x}. \]  

(15)

Case 2.2: \( g(t) = kt^b \).
The infinitesimal generators in this case are

\[
X_{2.2.1} = \frac{\partial}{\partial x}, \quad X_{2.2.2} = (2b - 1)x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + 2(b - 1)u \frac{\partial}{\partial u}.
\]  

(16)

Case 2.3: \( g(t) = k \).
We obtain the infinitesimal generators as follows

\[
X_{2.3.1} = \frac{\partial}{\partial x}, \quad X_{2.3.2} = x \frac{\partial}{\partial x} - 2t \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial u}.
\]  

(17)

Case 3: \( \alpha = \frac{1}{3} \), \( k, b \neq 0 \).
For \( \alpha = \frac{1}{3} \), functions of \( g(t) \) can be obtained as follows

\[ g(t) = k\left(t - b\right)^{\frac{2}{3}}, \quad k\left(t^2 - b\right)^{\frac{1}{3}}, \quad ke^{bt}, \quad kt^b, \quad k. \]

Case 3.1: \( g(t) = k\left(t - b\right)^{\frac{2}{3}}, \quad k\left(t^2 - b\right)^{\frac{1}{3}}, \quad ke^{bt} \).
In these cases, the infinitesimal generator is given by

\[ X_{3.1} = \frac{\partial}{\partial x}. \]  

(18)

Case 3.2: \( g(t) = kt^b \).
The infinitesimal generators in this case are

\[
X_{3.2.1} = \frac{\partial}{\partial x}, \quad X_{3.2.2} = (3b - 1)x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} + (3b - 2)u \frac{\partial}{\partial u}.
\]  

(19)

Case 3.3: \( g(t) = k \).
In this case, we obtain the infinitesimal generators as follows

\[
X_{3.3.1} = \frac{\partial}{\partial x}, \quad X_{3.3.2} = -3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial u}.
\]  

(20)
**Proof.** The one-parameter Lie group of transformations in \(x, t, u\) with \(\varepsilon\) as the group parameter are given

\[
\begin{align*}
t^* &= t + \varepsilon \xi^i(t, x, u) + O(\varepsilon^2), \\
x^* &= x + \varepsilon \xi^x(t, x, u) + O(\varepsilon^2), \\
u^* &= u + \varepsilon \eta_u(t, x, u) + O(\varepsilon^2),
\end{align*}
\]

The Lie algebra of K(2,3) equation (Eq. (11)) is spanned by vector fields

\[
X = \xi^x(t, x, u) \frac{\partial}{\partial x} + \xi^t(t, x, u) \frac{\partial}{\partial t} + \eta_u(x, t, u) \frac{\partial}{\partial u},
\]

(21)

where

\[
\begin{align*}
\xi^x &= \frac{dx^*}{d\varepsilon} \big|_{\varepsilon=0}, & \xi^t &= \frac{dt^*}{d\varepsilon} \big|_{\varepsilon=0}, & \eta_u &= \frac{du^*}{d\varepsilon} \big|_{\varepsilon=0}.
\end{align*}
\]

(22)

Applying the \(X^{(\alpha)}\) to (11), leads

\[
X^{(\alpha)} \left[ \frac{\partial^2 u}{\partial t^2} + (u^2)_x + g(t)(u^3)_{xxx} \right] = 0.
\]

(23)

Expanding the (23), and solving this obtained set using the Maple, we can distinguish all selections of the function \(g(t)\). Finally, the Lie point symmetries for (11) can be obtained as follow.

- If \(0 < \alpha < 1\), \(\alpha \neq \frac{1}{2}, \frac{1}{3}\), \(k, b \neq 0\), and \(g(t)\) be an arbitrary nonvanishing function then we have:

\[
\xi^x = c_1, \quad \xi^t = 0, \quad \eta_u = 0.
\]

Thus, the infinitesimal generator is given by

\[
X_1 = \frac{\partial}{\partial x},
\]

- If \(0 < \alpha < 1\), \(\alpha \neq \frac{1}{2}, \frac{1}{3}\), \(k, b \neq 0\), and \(g(t) = kt^b\) then we have:

\[
\xi^x = c_1 + c_2(\alpha - b)x, \quad \xi^t = -c_2t, \quad \eta_u = c_2(2\alpha - b)u.
\]

So, the infinitesimal generators are

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = (\alpha - b)x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} + (2\alpha - b)u \frac{\partial}{\partial u}.
\]

- If \(0 < \alpha < 1\), \(\alpha \neq \frac{1}{2}, \frac{1}{3}\), \(k, b \neq 0\), and \(g(t) = k\) then we have:

\[
\xi^x = c_1 + c_2\alpha x, \quad \xi^t = -c_2t, \quad \eta_u = 2c_2\alpha u.
\]

Therefore, the infinitesimal generators are given by

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \alpha x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} + 2\alpha u \frac{\partial}{\partial u}.
\]

The proof for \(\alpha = \frac{1}{2}, \frac{1}{3}\) are similar. Therefore, proof is completed.
4. Reduced equations and invariant solution of (11)

Our purpose for (11) is to reduce it the coordinates \((x, t, u)\) using invariants \((r, z)\) to a new coordinates \([22]\).

Let us consider
\[
X = \xi^t(t, x, u) \frac{\partial}{\partial t} + \xi^x(t, x, u) \frac{\partial}{\partial x} + \eta_u(t, x, u) \frac{\partial}{\partial u},
\]
as a Lie point symmetry of the time-fractional K(2,3) equation
\[
\frac{\partial^\alpha u}{\partial t^\alpha} + (u^2)_x + g(t)(u^3)_{xxx} = 0, \quad 0 < \alpha < 1, \quad t > 0.
\]

We use two invariants \(z = \psi(x, t)\) and \(r = \varphi(x, t)\) which are linearly independent in the characteristic equations
\[
\frac{dt}{\xi^t(t, x, u)} = \frac{dx}{\xi^x(t, x, u)} = \frac{du}{\eta_u(t, x, u)},
\]
for obtaining the invariant solutions. After that, we assume one of those invariants is depend to another,
\[
z = h(r), \tag{24}
\]
then we solve (24) for \(u\). Finally, substituting \(u\) in Eq. (11) for the unknown function \(h\), a fractional ODE can be obtained. Now, we obtain corresponding reduced equations, invariants and group invariant solutions of (11) for different cases of \(g(t)\) and \(\alpha\) as follows.

Case 1:
- Case 1.1: \(0 < \alpha < 1, \alpha \neq \frac{1}{2}, \frac{1}{3}\) and \(g(t)\) is a nonvanishing arbitrary function.
- Case 1.2: \(\alpha = \frac{1}{2}, g(t) = \{k, kt^b, ke^{bt}\}\)
- Case 1.3: \(\alpha = \frac{1}{3}, g(t) = \{k, kt^b, ke^{bt}, k(t - b)^{\frac{1}{2}}, k(t^2 - b)^{\frac{1}{3}}\}\)

In these cases, according to the infinitesimal generator \(X = \frac{\partial}{\partial x}\), the similarity variables using the method of characteristics are as follows:
\[
z = u, \quad r = t, \tag{25}
\]
and a solution is
\[
z = h(r) \Rightarrow u = h(t). \tag{26}
\]
By substituting (26) into (11) we find the \(h(r)\). Thus \(h(r)\) must be satisfied:
\[
\frac{d^\alpha h(t)}{dt^\alpha} = 0. \tag{27}
\]
Then by solving the above equation by the Laplace transform[25], we have
\[
h(t) = \frac{\kappa t^{\alpha - 1}}{\Gamma(\alpha)}, \quad \kappa \text{ is a constant}. \tag{28}
\]
Case 2:

- **Case 2.1**: $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $g(t) = kt^b$.
  
  In this case
  \[ X_{1,2.2} = -t \frac{\partial}{\partial t} + (\alpha - b)x \frac{\partial}{\partial x} + (2\alpha - b)u \frac{\partial}{\partial u}, \]  
  (29)
  
  so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:
  \[ r = tx^\frac{1}{\alpha - b}, \quad z = ux^\frac{b - 2\alpha}{\alpha - b}, \]  
  (30)
  
  and a solution for equation (11) is
  \[ z = h(r) \Rightarrow u = x^\frac{b - 2\alpha}{\alpha - b} h(tx^\frac{1}{\alpha - b}). \]  
  (31)

  We substitute (31) into (11) to find the $f(r)$ and $f(r)$ must be satisfied in the fractional ODE as follows:

  \[
  (b - \alpha)^3 \frac{\partial^\alpha h}{\partial r^\alpha} - 2(b - \alpha)^2 rh(r)h'(r) + 18(-1 + 2b - 5\alpha)kr^{2+b}h(r)h'(r)^2 \\
  -18kr^{3+b}h(r)h'(r)h''(r) + 3(2b^3 - 17b^2\alpha + 46b\alpha^2 - 40\alpha^3)kr^b h(r)^3 \\
  -6kr^{3+b}h'(r)^3 + (2b^3 - 8b^2\alpha + 10b\alpha^2 - 4\alpha^3)h(r)^2 - 3(11b^2 + 15\alpha + 74\alpha^2) \\
  -2b(29\alpha + 3) + 1)kr^{1+b}h'(r)h(r)^2 + 9(-1 + 2b - 5\alpha)kr^{2+b}h''(r)h(r)^2 \\
  -3kr^{3+b}h'''(r)h(r)^2 = 0.
  \]

  Where $\alpha \neq \frac{1}{2}, \frac{1}{3}$.

- **Case 2.2**: $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $g(t) = k$.
  
  For this case we have
  \[ X_{1,3.2} = \alpha x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} + 2\alpha u \frac{\partial}{\partial u}, \]  
  (32)
  
  so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:
  \[ r = tx^\frac{1}{\alpha}, \quad z = ux^{-2}, \]  
  (33)
  
  and a solution for (11) is
  \[ z = h(r) \Rightarrow u = x^2 h(tx^\frac{1}{\alpha}). \]  
  (34)

  Again we substitute (34) into (11) to obtain the $f(r)$. So $h(r)$ must satisfy in the fractional ODE as follows:

  \[
  \alpha^3 \frac{\partial^\alpha h}{\partial r^\alpha} + 2\alpha^2 rh(r)h'(r) + 18(5\alpha + 1)kr^2 h(r)h'(r)^2 \\
  +18kr^3 h(r)h'(r)h''(r) + 120k\alpha^3 h(r)^3 + 6kr^3 h'(r)^3 + 4\alpha^3 h(r)^2 \\
  +3(7\alpha^2 + 15\alpha + 1)kr h(r)^2 h'(r) + 9(5\alpha + 1)kr^2 h(r)^2 h''(r) \\
  +3kr^3 h(r)^2 h'''(r) = 0.
  \]

  Where $\alpha \neq \frac{1}{2}, \frac{1}{3}$.
Case 3:

- **Case 3.1:** $\alpha = \frac{1}{2}$, $g(t) = kt^b$.
  
  For this case we have
  
  $$X_{2.2.2} = (2b - 1)x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + 2(b - 1)u \frac{\partial}{\partial u}, \quad (35)$$
  
  so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:
  
  $$r = tx^{\frac{2}{1-2b}}, \quad z = ux^{\frac{2b-2}{1-2b}}, \quad (36)$$
  
  and in view of (26), a solution for (11) is
  
  $$u = x^{\frac{2b-2}{1-2b}} h \left( tx^{\frac{2}{1-2b}} \right). \quad (37)$$
  
  We substitute (40) into (11) to obtain $h(r)$. After that $h(r)$ must be satisfied in the FDE as follows:
  
  $$\frac{(2b - 1)^3}{4} \frac{\partial^6 h}{\partial r^6} - (1 - 2b)^2 rh(r)h'(r) + 18(4b - 7)kr^{2+b}h(r)h''(r)^2
  
  - 36kr^{3+b}h(r)h'(r)h''(r) + 3(4b^3 - 17b^2 + 23b - 10)kr^b h(r)^3
  
  - 12kr^{3+b}h'(r)^3 + (4b^3 - 8b^2 + 5b - 1)h(r)^2 - 6(11b^2 - 35b + 27)kr^{1+b} h'(r)h(r)^2
  
  + 9(4b - 7)kr^{2+b}h''(r)h(r)^2 - 6kr^{3+b}h'''(r)h(r)^2 = 0.$$  
  
  Where $\alpha = \frac{1}{2}$.

- **Case 3.2:** $\alpha = \frac{1}{2}$, $g(t) = k$.
  
  For this case we have
  
  $$X_{2.3.2} = x \frac{\partial}{\partial x} - 2t \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial u}, \quad (38)$$
  
  so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:
  
  $$r = tx^2, \quad z = u x^{-2}, \quad (39)$$
  
  and in view of (26), a solution for (11) is
  
  $$u = x^2 h \left( tx^2 \right). \quad (40)$$
  
  To obtain $f(r)$, we substitute (40) into (11). Then $f(r)$ must satisfy in the fractional ODE as follows:
  
  $$\frac{1}{4} \frac{\partial^6 f}{\partial r^6} + 30fk(r)^3 + 12kr^3 f'(r)^3 + rf(r)f'(r) + 126kr^2 f(r)f'(r)^2
  
  + 36kr^3 f(r)^2 f''(r) + f(r)^2 + 162kr f'(r)f(r)^2 + 63kr^2 f(r)^2 f''(r)
  
  + 6kr^3 f(r)^2 f'''(r) = 0.$$  
  
  Where $\alpha = \frac{1}{2}$.
Case 4:

**Case 4.1:** $\alpha = \frac{1}{3}$, $g(t) = kt^b$.

For this case we have

$$X_{3.2.2} = (3b-1)x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} + (3b-2)u \frac{\partial}{\partial u},$$

(41)

so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:

$$r = tx^{\frac{3}{3b-1}}, \quad z = ux^{\frac{3b-2}{3b-1}},$$

(42)

and a solution to our equation is

$$z = h(r) \Rightarrow u = x^{\frac{3b-2}{3b-1}} h \left( tx^{\frac{3}{3b-1}} \right).$$

(43)

We substitute (43) into (11) to determine the $h(r)$. Then $h(r)$ must satisfy in the fractional ODE as follows:

$$(3b-1)^3 \frac{\partial^3 h}{\partial r^3} - 6(1-3b)^2 rh(r)h'(r) + 324(3b-4)kr^{2+b}h(r)h'(r)^2$$

$$-486kr^3h(r)h'(r)h''(r) + 3(54b^3 - 153b^2 + 138b - 40)kr^b h(r)^3$$

$$-162kr^{1+b}h'(r)^3 + (54b^3 - 72b^2 + 30b - 4)h(r)^2 - 9(99b^2 - 228b + 128)kr^{1+b}h'(r)h(r)^2$$

$$+162(3b - 4)kr^{2+b}h''(r)h(r)^2 - 81kr^{3+b}h'''(r)h(r)^2 = 0.$$

Where $\alpha = \frac{1}{3}$.

**Case 4.2:** $\alpha = \frac{1}{3}$, $g(t) = k$.

For this case we have

$$X_{3.3.2} = x \frac{\partial}{\partial x} - 3t \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial u},$$

(44)

so the similarity variables for this Lie point symmetry using the method of characteristics are as follows:

$$r = tx^3, \quad z = ux^{-2},$$

(45)

and a solution to our equation is

$$z = h(r) \Rightarrow u = x^2 h \left( tx^3 \right).$$

(46)

We substitute (46) into (11) to determine the $h(r)$. Then $h(r)$ must satisfy in the fractional ODE as follows:

$$\frac{\partial^\alpha h}{\partial r^\alpha} + 120kh(r)^3 + 162kr^3h'(r)^3 + 6rh(r)h'(r) + 1296kr^2h(r)h'(r)^2$$

$$+486kr^3h(r)h'(r)h''(r) + 4h(r)^2 + 1152krh'(r)h(r)^2 + 648kr^2h(r)^2h''(r)$$

$$+81kr^3h(r)^2h'''(r) = 0.$$

Where $\alpha = \frac{1}{3}$.
5. Conclusion

The corresponding invariants and group invariant solutions has been obtained for the time-fractional K(m,n) equation when the the fractional derivative is in the Riemann-Liouville sense. Finally, we reduced this time-fractional equation into a nonlinear ODE of fractional order. For this propose, the Lie group method and the symmetry properties have been investigated for the governing equation. They have been used to reduced the give FPDE to a corresponding FDE which might be solved easily.

Conflict of interest

The author declare that they have no conflict of interest.

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Authors’ contributions

All authors have read and approved the final manuscript.

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