Direction sets, Lipschitz graphs and density

Alex Iosevich and Jonathan Pakianathan

October 17, 2018

Abstract

We consider the direction set determined by various subsets $E$ of Euclidean space and show that there is a trichotomy: Either (i) The subset is the graph of a Lipschitz function and the direction set is not dense in the sphere, (ii) The subset is the graph of a non-Lipschitz function and the direction set is dense but not everything, or (iii) The subset is not a graph (in a suitable sense) and every direction is determined by the set. We then explore a variety of results based on this trichotomy under additional assumptions on the set $E$.

1 Introduction

The purpose of this paper is study directions sets determined by subsets of the Euclidean space. Informally, direction sets consists of direction vectors determined by pairs of vectors from a given set. More precisely, we have the following definitions.

Definition 1.1. Fix a subset $E \subseteq \mathbb{R}^d$, $d \geq 2$. The (oriented) direction set $\hat{D}(E)$ determined by $E$ is the set

$$\hat{D}(E) = \left\{ \frac{y - x}{|y - x|} : x, y \in E, x \neq y \right\} \subseteq S^{d-1}.$$ 

Note that this consists of the unit direction vector of the ray from $x$ to $y$ as $x, y$ range over pairs of distinct elements of $E$. We say that $E$ determines all directions, or a dense subset of directions, respectively, if $\hat{D}(E) = S^{d-1}$ or $\hat{D}(E)$ is a dense subset of $S^{d-1}$, respectively. Note also that when $|E| \leq 1$ then $\hat{D}(E) = \emptyset$, but is nonempty otherwise.

It was shown by the first listed author, Mourgoglou and Senger ([1]; see also [2], Theorem 10.11) that if the Hausdorff dimension of $E \subseteq \mathbb{R}^d$, $d \geq 2$, is greater than $d - 1$, then the $(d - 1)$-dimensional Lebesgue measure of $\hat{D}(E)$, viewed as a subset of $S^{d-1}$, is positive. They also obtained a rather precise description of the distribution of these directions. In this paper we turn in a slightly different direction and obtain rather comprehensive qualitative information about the structure of subsets of $\mathbb{R}^d$ for which the direction set is not dense in the sphere.
It is often convenient to consider unoriented directions, which are elements of $S^{d-1}$ taken modulo the antipodal action $a(x) = -x$. It is well known that the quotient of the sphere $S^{d-1}$ when antipodal points are identified is the projective space $\mathbb{R}P^{d-1}$ whose elements can be thought of as either pairs of antipodal points on the sphere, $\pm \hat{u}$ or lines through the origin in $\mathbb{R}^d$. The natural map $\pi : S^{d-1} \to \mathbb{R}P^{d-1}$ is a double cover map such that for each line $L \in \mathbb{R}P^{d-1}$, $\pi^{-1}(L)$ is the pair of antipodal points where the line $L$ intersects the sphere.

**Definition 1.2.** Let $E \subseteq \mathbb{R}^d$, $d \geq 2$. The unoriented direction set $D(E)$ determined by $E$ is the image of the oriented direction set $\tilde{D}(E)$ under the double cover map

$$\pi : S^{d-1} \to \mathbb{R}P^{d-1}.$$ 

We may think of $D(E)$ as the set of parallel types of lines determined by $E$.

Note for every set $E$, the oriented direction set $\tilde{D}(E) \subseteq S^{d-1}$ can be seen to be invariant under the antipodal map $a$. This is because given a direction achieved by a pair $x, y \in E$, the antipodal direction is achieved by the same pair under reversal of the roles of $x$ and $y$. Due to this it is easily seen that $\tilde{D}(E) = S^{d-1}$ if and only if $D(E) = \mathbb{R}P^{d-1}$ and $\tilde{D}(E)$ is dense in $S^{d-1}$ if and only if $D(E)$ is dense in $\mathbb{R}P^{d-1}$.

In this paper when we refer to the graph set of a function as a subset of $\mathbb{R}^d$ we will always mean a set, that up to rotation is the graph of a scalar-valued function $f : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^{d-1}$ is arbitrary with respect to the standard axis-system. The set $U$ is arbitrary and can even be empty or a single point. The function in this graph is said to be Lipschitz if there is a positive constant $C$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in U$, and non-Lipschitz otherwise.

Our main result is the following.

**Theorem 1.3.** Let $E \subseteq \mathbb{R}^d$. Then there is a trichotomy in that exactly one of the following statements holds:

- i) $E$ is the graph of a Lipschitz function $f : A \to \mathbb{R}$ for some $A \subseteq \mathbb{R}^{d-1}$ (up to rotation) and $D(E)$ is not dense in $\mathbb{R}P^{d-1}$.
- ii) $E$ is the graph of a non-Lipschitz function $f : A \to \mathbb{R}$ (up to rotation) and $D(E)$ is dense in $\mathbb{R}P^{d-1}$ but is not equal to all of $\mathbb{R}P^{d-1}$.
- iii) $E$ is not a graph of a scalar valued function and $D(E) = \mathbb{R}P^{d-1}$.

We can say a bit more with a few extra assumptions.

**Corollary 1.4.** Let $E$ be a compact subspace of $\mathbb{R}^d$. Then there is the following trichotomy:
• i) $E$ is the graph of a Lipschitz function $f : A \to \mathbb{R}$ for some compact $A \subseteq \mathbb{R}^{d-1}$, up to rotation, and $D(E)$ is not dense in $RP^{d-1}$.

• ii) $E$ is the graph of a continuous, non-Lipschitz function $f : A \to \mathbb{R}$, up to rotation, $A$ compact, and $D(E)$ is dense in $RP^{d-1}$ but is not equal to all of $RP^{d-1}$.

• iii) $E$ is not a graph of a scalar valued function and $D(E) = RP^{d-1}$.

Using the intermediate value theorem, one can further upgrade the result when $E$ is a compact, connected subset of $\mathbb{R}^2$.

**Corollary 1.5.** Let $E$ be a compact, connected subset of $\mathbb{R}^2$. Then there is the following trichotomy:

• i) $E$ is the graph of a Lipschitz function $f : [a,b] \to \mathbb{R}$, up to rotation, and $D(E)$ is not dense in $RP^1$.

• ii) $E$ is the graph of a continuous, non-Lipschitz function $f : [a,b] \to \mathbb{R}$ (up to rotation) and $D(E)$ misses exactly one point of $RP^1$.

• iii) $E$ is not a graph of a scalar valued function and $D(E) = RP^1$.

Note the last corollary shows that once a compact, connected subset of the plane misses two directions, it misses a nonempty open set of directions and up to rotation, it is the graph of a Lipschitz map over a closed interval $[a,b]$ (where $a = b$ is a possibility). Also note, that it is known that when the vector space of continuous real-valued functions on the interval $[a,b]$, $a < b$ is given the sup-norm (uniform convergence norm), the nowhere differentiable continuous functions form a co-meagre set (topological analog of full measure subset). Thus “most” continuous functions are nowhere differentiable (and hence not Lipschitz as Lipschitz functions are almost everywhere differentiable), and hence “most” continuous functions $f : [a,b] \to \mathbb{R}$ determine every possible secant slope in $\mathbb{R}$.

We also establish the following:

**Proposition 1.6.** Let $E \subseteq \mathbb{R}^d$ then $D(E)$ is countable if and only if $E$ is countable or $E$ is contained in a line, not necessarily through the origin.

Putting these results together it follows that if $E \subseteq \mathbb{R}^d, d \geq 2$ and $E$ has Hausdorff dimension $> d - 1$, then $D(E)$ is an uncountable dense subset of $\mathbb{R}P^{d-1}$. This is because the Hausdorff dimension of a graph set of a Lipschitz function $f : A \to \mathbb{R}, A \subseteq \mathbb{R}^{d-1}$ is at most $d - 1$. It also follows immediately that if $E$ is a compact connected subset of the plane of Hausdorff dimension $> 1$ then $D(E)$ misses at most one direction.
2 Proofs of the main results

Throughout the paper, \( x \) and \( y \) are \( d \)-dimensional vectors when we are in \( \mathbb{R}^d \). In two dimensions, \((x, y)\) denotes a 2-dimensional vector.

2.1 Proof of Proposition 1.6

When \( E \subseteq \mathbb{R}^d \) is countable or contained in a line, it is clear that \( D(E) \) is countable. So let us just prove the converse. Suppose \( D(E) \) is countable and fix a point \( x \in E \). Then as \( D(E) \) is countable, \( E \) must be contained in a countable union of lines through \( x \). If \( E \) is countable we are done so assume it is uncountable. Then there must be a line \( L \) through \( x \) such that uncountably many elements of \( E \) lie in this line. If there was another line \( L' \) through \( x \) that contained an element \( e \in E \) besides \( x \), we would obtain uncountably many directions generated by \( E \) by noting that the lines through \( e \) and the various uncountable elements of \( E \cap L \) all have different directions. Thus it follows that if \( E \) is uncountable, it must be contained in a single line.

2.2 Proof of Theorem 1.3

First consider \( E \subset \mathbb{R}^d, d \geq 2 \) with \( D(E) \neq \mathbb{R}P^{d-1} \). As \( E \) fails to determine some direction, after a rotation, we may assume that \( E \) fails to determine the \( x_d \)-axis direction. The projection \( \pi : \mathbb{R}^d \to \mathbb{R}^{d-1} \) then determines a bijection \( E \to \pi(E) = A \subseteq \mathbb{R}^{d-1} \). Let \( f \) be the \( d \)-th coordinate function of the inverse of this bijection, it follows that \( E = Graph(f) = \{ (x, f(x)) | x \in A \subseteq \mathbb{R}^{d-1} \} \) is the graph set of a scalar valued-function. Conversely when \( E \) is the graph set of a scalar-valued function, by the vertical line test, it misses a direction.

Thus it follows that \( D(E) = \mathbb{R}P^{d-1} \) if and only if \( E \) is not a graph set of a scalar-valued function.

Thus it suffices for the remainder of the proof to only consider graph sets of functions

\[ f : A \to \mathbb{R} \text{ with } A \subseteq \mathbb{R}^{d-1}. \]

Let us first consider the case \( d = 2 \). Then \( \mathbb{R}P^1 \), the space of lines through the origin can be identified as the circle \([0, \pi]/(0 \sim \pi)\). This is because we can parameterize a line by the angle it makes with the \( x \)-axis, (angle \( \pi \) and angle 0 both give the same line, the \( x \)-axis). We can also use the slopes of these lines as the parametrization in which case \( \mathbb{R}P^1 \) is identified with the one-point-compactification of \( \mathbb{R} \) where the infinity slope corresponds to the \( y \)-axis. Under these identifications, dense subsets correspond to dense subsets so our conclusions are invariant no matter which picture we choose.

Under the slope identification of \( \mathbb{R}P^1 \), a graph set

\[ E = \text{Graph}(f) = \{ (x, f(x)) : x \in A \subseteq \mathbb{R}^1 \} \]
has $D(E)$ given by the set of secant slopes

$$\left\{ \frac{f(y) - f(x)}{y - x} : x, y \in A, x \neq y \right\}$$

of the graph.

Now suppose $E \subseteq \mathbb{R}^2$ does not have a dense set of directions, then $D(E)$ misses a nonempty open set of directions. We may rotate $E$ and assume this open set of directions includes the $y$-axis direction. Then as previously discussed it is the graph of a function $f : A \rightarrow \mathbb{R}$ whose secant slopes are bounded away from infinity as $D(E)$ avoids an open neighborhood of the infinite slope ($y$-axis).

In other words, there is a positive constant $C > 0$ such that

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C \text{ for all } x, y \in A, x \neq y.$$

It follows immediately then that $f$ is Lipschitz and $E$ is the graph set of a Lipschitz function. Conversely, the graph of a Lipschitz function clearly misses an open set about the infinite slope direction via the same picture.

Thus we have established in the case of subsets $E \subseteq \mathbb{R}^2$ that $D(E)$ is not dense in $\mathbb{R}P^1$ exactly when $E$ is the graph set of a Lipschitz function and $D(E) = \mathbb{R}P^1$ exactly when $E$ is not the graph set of a scalar valued function. Hence Theorem 1.3 is proven in the case $d = 2$.

Now we consider the general case when $d > 2$. If $D(E)$ is not dense, then after a rotation we can assume it misses an open set about the $x_d$-axis direction. In particular, there is an $\epsilon > 0$ such that it does not generate any line within angle $\epsilon$ of the $x_d$-axis. Again projection to the first $d - 1$ coordinates is an injection on $E$ and so $E = \text{Graph}(f)$ where $f : A \rightarrow \mathbb{R}$, where $A \subseteq \mathbb{R}^{d-1}$ is the projection of $E$ to $\mathbb{R}^{d-1}$.

Now fix an affine line $L$ (not necessarily through the origin) in $\mathbb{R}^{d-1}$ then the unique affine 2-plane in $\mathbb{R}^d$ which contains $L$ and a line parallel to the $x_d$-axis, intersects $E$ in the graph of the function $f$ restricted to $A \cap L$. This restriction of $f$ to the line $L$ is then Lipschitz with Lipschitz constant $\tan(\frac{\pi}{2} - \epsilon)$ by the argument in the $d = 2$ case as its secant slopes are bounded away from the infinite slope which now corresponds to the $x_d$-axis direction. It is then clear that $f : A \rightarrow \mathbb{R}$ is Lipschitz as it is Lipschitz on $A \cap L$ for any line $L$, with a uniform Lipschitz constant $\tan(\frac{\pi}{2} - \epsilon)$. The Lipschitz condition involves two elements at a time and any two elements lie on a single affine line.

Conversely if $E = \text{Graph}(f)$ for some $f : A \rightarrow \mathbb{R}$, $A \subseteq \mathbb{R}^{d-1}$, then if $f$ is Lipschitz, there is a uniform bound on the secant slopes of $f$ restricted to $A \cap L$ for all lines $L$ through the origin in $\mathbb{R}^{d-1}$. This easily translates to the existance of an angle $\epsilon > 0$ such that $E$ does not generate any lines within $\epsilon$ angle of the $x_d$-axis.

Thus in the general $\mathbb{R}^d$ case we have seen that $D(E) = \mathbb{R}P^{d-1}$ if and only if $E$ is not the graph set of a scalar-valued function and $D(E)$ is not dense if and only if it is the graph set of a Lipschitz function. Thus the theorem is proven.
2.3 Proof of Corollary 1.4 and Corollary 1.5

By the proof of Theorem 1.3, when \( D(E) \neq RP^{d-1} \), \( E \) is (up to rotation) the graph set of a function \( f : A \rightarrow \mathbb{R} \) where \( A \subseteq \mathbb{R}^{d-1} \) is the projection of \( E \) to \( \mathbb{R}^{d-1} \). When \( E \) is compact (connected), \( A \) is also compact (connected). In particular when \( d = 2 \), \( A \) is a closed subinterval of \( \mathbb{R} \) as every compact, connected subset of \( \mathbb{R} \) is a closed interval (to see this just apply the intermediate and extreme value theorems to the inclusion map \( i : A \rightarrow \mathbb{R} \)).

When this graph set \( E \) is compact, it follows from the topological closed graph theorem ([3]) that \( f \) is continuous. Finally when \( d = 2 \), and \( E \) is compact and connected, then (up to rotation) \( E \) is the graph set of \( f : [a, b] \rightarrow \mathbb{R} \). The secant formula

\[
\frac{f(x) - f(y)}{x - y}
\]

defines a continuous function on the open triangle

\[
\{(x, y) \in [a, b] \times [a, b] : x < y\}.
\]

Since this triangle is connected it follows that \( D(E) \) is connected in \( RP^1 \). When the function \( f : [a, b] \rightarrow \mathbb{R} \) is continuous but not Lipschitz, it follows from Theorem 1.3 that its secants are dense and not bounded, this combined with \( D(E) \) is connected is enough to conclude then that

\[
D(E) = RP^1 - \{ \text{y-axis direction} \}
\]

using the intermediate value theorem. Use the identification of \( RP^1 \) via slopes so that

\[
RP^1 - \{ \text{y-axis direction} \}
\]

is identified with \( \mathbb{R} \). Thus all directions are achieved except the y-axis direction in this case. This concludes the proofs of these corollaries.

References

[1] A. Iosevich, M. Mourgoglou and S. Senger, *On sets of directions determined by subsets of \( \mathbb{R}^d \)*. J. Anal. Math. **116** (2012), 355-369. 1

[2] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, (1995). 1

[3] J. Munkres, *General Topology*, Prentice Hall, (1975). 6