A KINETIC EQUATION FOR REPULSIVE COALESCING RANDOM JUMPS IN CONTINUUM

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Abstract. A continuum individual-based model of hopping and coalescing particles is introduced and studied. Its microscopic dynamics are described by a hierarchy of evolution equations obtained in the paper. Then the passage from the micro- to mesoscopic dynamics is performed by means of a Vlasov-type scaling. The existence and uniqueness of the solutions of the corresponding kinetic equation are proved.

1. Introduction

In this paper, we introduce and study the dynamics of an infinite system of particles located in \( \mathbb{R}^d \), which jump and merge (coalesce). Both jumping and coalescing are repulsive. The proposed model is individual based, which means that the description of its Markov dynamics are performed in terms of random changes of states of individual particles. In the proposed model, such changes include: (a) the particle located at a given \( x \in \mathbb{R}^d \) changes its position to \( y \in \mathbb{R}^d \) (jumping); (b) two particles, located at \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^d \), merge into a single particle located at \( z \in \mathbb{R}^d \) (coalescing).

The state space of the system considered is the configuration space, that is, the set of all locally finite subsets of \( \mathbb{R}^d \)

\[ \Gamma = \Gamma(\mathbb{R}^d) = \left\{ \gamma \subset \mathbb{R}^d : \gamma \cap \Lambda \text{ is finite for every compact } \Lambda \subset \mathbb{R}^d \right\}. \]

It can be given a measurability structure which turns \( \Gamma \) into a standard Borel space. This allows one to consider probability measures on \( \Gamma \) as states of the
system. To characterize them one uses observables, which are appropriate functions $F : \Gamma \to \mathbb{R}$. For an observable $F$ and a state $\mu$, the number

$$\int_{\Gamma} F d\mu$$

is the $\mu$-expected value of $F$. Then the evolution of states $\mu_0 \mapsto \mu_t$ can be described via the dual evolution $F_0 \mapsto F_t$ based on the following duality relation

$$\int_{\Gamma} F_0 d\mu_0 = \int_{\Gamma} F_t d\mu_0, \quad t > 0.$$

The evolution of observables is obtained in turn from the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \quad F_t|_{t=0} = F_0,$$

in which the ‘operator’ $L$ characterizes the model, see [6, 7, 8, 9] for more detail. In the proposed model, it has the following form

$$LF(\gamma) = \sum_{\{x,y\} \subset \gamma} \int_{\mathbb{R}^d} \tilde{c}_1(x, y; z; \gamma) \left( F(\gamma \setminus \{x, y\} \cup z) - F(\gamma) \right) dz$$

$$+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} \tilde{c}_2(x; y; \gamma) \left( F(\gamma \setminus x \cup y) - F(\gamma) \right) dy. \quad (1.1)$$

The first term of $L$ describes the coalescence occurring with intensity $\tilde{c}_1(x, y; z; \gamma)$. The particles located at $x$ and $y$ merge into a new particle located at a point $z$. The second term describes the jump of the particle located at $x$ to a point $y$ with intensity $\tilde{c}_2(x; y; \gamma)$. The model with $L$ consisting of the second term only is the Kawasaki model studied in [5]. The kernels $\tilde{c}_1$ and $\tilde{c}_2$ take into account also the influence of the whole configuration, which is supposed to be repulsive, see below.

In the present research, we follow the statistical approach, see, e.g., [5, 6, 7, 8, 9], in which the dynamics of the model are described by means of that of the corresponding correlation functions obtained from the following Cauchy problem

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_{t=0} = k_0,$$

in which the ‘operator’ $L^\Delta$ is related to $L$ in a certain way. In section 3 we calculate $L^\Delta$ following the scheme developed in [9]. Usually, equations for $k_t$ are studied in scales of the corresponding Banach spaces. However, as the structure of $L^\Delta$ obtained below is too complicated, in this work we do not study this equation, which is supposed to be done in a separate work. Instead, in section 4 we consider a simplified version obtained by means of a Vlasov-type scaling procedure developed in, e.g., [7], which is equivalent to passing to the so-called mesoscopic description. In particular, we informally obtain the kinetic equation and study its local solutions in an appropriate
Banach space, showing their existence and uniqueness. In the next section, we introduce necessary notions and technical tools.

2. Basic notions and tools

In this section we introduce basic notions and tools used for proving the results in the following sections. We give only a short description with references to the corresponding sources.

Note that each element $\gamma \in \Gamma$ is at most countable without finite limiting points. $\Gamma$ is endowed with the vague topology, which is the weakest topology that makes continuous the mappings

$$\gamma \to \sum_{x \in \gamma} f(x)$$

for all continuous compactly supported functions $f : \mathbb{R}^d \to \mathbb{R}$. For the general discussion on configuration spaces we recommend [6, 8].

One can also consider the space of all finite configurations

$$\Gamma_0 = \Gamma_0(\mathbb{R}^d) = \{ \eta \in \Gamma : \eta \text{ is finite} \},$$

which can be written down as

$$\Gamma_0 = \bigcup_{n=0}^{\infty} \Gamma^{(n)}$$

where $\Gamma^{(n)}(\mathbb{R}^d) = \left\{ \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d, x_i \neq x_j \text{ for } i \neq j \right\}$, $n \in \mathbb{N}_0$ and each $\Gamma^{(n)}$ is equipped with the topology based on the Euclidean topology of $\mathbb{R}^d$. Note that $\Gamma^{(0)} = \{\emptyset\}$. Therefore $\Gamma_0$ can be considered either with the topology induced from the vague topology of $\Gamma$ or with the topology of the disjoint union. These topologies are different but the corresponding Borel $\sigma$–algebras are equal. Both $(\Gamma, \mathcal{B}(\Gamma))$ and $(\Gamma_0, \mathcal{B}(\Gamma_0))$ are standard Borel spaces. Additionally $\Gamma_0 \in \mathcal{B}(\Gamma)$, which implies that $\mathcal{B}(\Gamma_0)$ is a sub-$\sigma$-field of $\mathcal{B}(\Gamma)$.

For each bounded Borel set $\Lambda \subset \mathbb{R}^d$, we define $p_{\Lambda}(\gamma) = \gamma_{\Lambda} = \gamma \cap \Lambda$ and denote $\Gamma_{\Lambda} = p_{\Lambda}(\Gamma)$. Note that $\Gamma_{\Lambda}$ is a subset of $\Gamma_0$, that is

$$\Gamma_{\Lambda} = \bigcup_{n=0}^{\infty} \Gamma^{(n)}_{\Lambda}, \quad \text{where} \quad \Gamma^{(n)}_{\Lambda} = \Gamma_{\Lambda} \cap \Gamma^{(n)}.$$

We say that a Borel set $A \subset \Gamma_0$ is bounded, if for some $N \in \mathbb{N}$ and a bounded $\Lambda \subset \mathbb{R}^d$

$$A \subset \bigcup_{n=0}^{N} \Gamma^{(n)}_{\Lambda}.$$

It can be shown that $G : \Gamma_0 \to \mathbb{R}$ is measurable if and only if, for any $n \in \mathbb{N}_0$, there exists a symmetric measurable function $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ such that $G^{(n)}(x_1, \ldots, x_n) = G(\{x_1, \ldots, x_n\})$, whenever $x_i \neq x_j$ for $i \neq j$. Here we use the convention that $G^{(0)} = G(\emptyset) \in \mathbb{R}$ is just a constant function.
By $B_{bs}(\Gamma_0)$ we denote the set of all bounded measurable functions $G : \Gamma_0 \to \mathbb{R}$ having bounded supports. Then the $K$-transform is defined as follows. For any $G \in B_{bs}(\Gamma_0)$, $KG : \Gamma \to \mathbb{R}$ is

$$ (KG)(\gamma) = \sum_{\eta \in \gamma} G(\eta), \quad (2.1) $$

where $\eta \in \gamma$ means that $\eta$ is a finite sub-configuration of $\gamma$, see [11]. Obviously, the $K$-transform is linear. It acts to $\mathcal{F}_{cyl}(\Gamma)$, i.e., to the set of all measurable cylinder functions $F : \Gamma \to \mathbb{R}$. It is also invertible with the inverse given by

$$ (K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta\setminus\xi|} F(\xi). $$

For $G_1, G_2 \in B_{bs}(\Gamma_0)$, it is known that

$$ (KG_1) \cdot (KG_2) = K(G_1 \star G_2), \quad (2.2) $$

where $G_1 \star G_2$ is the 'convolution' given by the formula

$$ (G_1 \star G_2)(\eta) = \sum_{\xi \subset \eta} G_1(\xi) \sum_{\zeta \subset \xi} G_2(\eta \setminus \xi \cup \zeta) \in B_{bs}(\Gamma_0). \quad (2.3) $$

Denote

$$ e(f, \gamma) = \prod_{x \in \gamma} f(x). $$

For a measurable compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$, the following holds

$$ K(e(f, \cdot))(\gamma) = e(1 + f, \gamma). \quad (2.4) $$

A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ is said to have finite local moments of all orders if for any $n \in \mathbb{N}$ and a bounded Borel $\Lambda \subset \mathbb{R}^d$,

$$ \int_{\Gamma_0} |\gamma\Lambda|^n \mu(d\gamma) < \infty, $$

where $|\eta|$ stands for the cardinality of $\eta \in \Gamma_0$. For such a measure $\mu$, one can define a correlation measure, $\rho_\mu$, on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ by

$$ \int_{\Gamma_0} G(\eta) \rho_\mu(d\eta) = \int_{\Gamma} (KG)(\gamma) \mu(d\gamma), \quad G \in B_{bs}(\Gamma_0). $$

By $\lambda$ we denote the Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$, that is, the correlation measure for the homogeneous Poisson measure with unit intensity. The Lebesgue-Poisson measure is uniquely defined by the following formula

$$ \int_{\Gamma_0} G(\eta) \lambda(d\eta) = G^{(0)} + \sum_{n=1}^\infty \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, x_2, \ldots, x_n) dx_1 dx_2 \cdots dx_n, \quad (2.5) $$
which has to hold for all $G \in B_{bs}(\Gamma_0)$. The Minlos lemma (see, e.g., [6, eq. (2.2)]), states that

$$\int_{\Gamma_0} \int_{\Gamma_0} \cdots \int_{\Gamma_0} G(\eta_1 \cup \eta_2 \cup \cdots \cup \eta_n) H(\eta_1, \eta_2, \ldots, \eta_n) \lambda(d\eta_1) \lambda(d\eta_2) \cdots \lambda(d\eta_n)$$

$$= \int_{\Gamma_0} G(\eta) \sum_{\eta \in \mathcal{A}} H(\eta_1, \eta_2, \ldots, \eta_n) \lambda(d\eta),$$

(2.6)

where $n$ is a positive integer, $G : \Gamma_0 \to \mathbb{R}$, $H : (\Gamma_0)^n \to \mathbb{R}$ are positive and measurable and the sum is taken over all $n$-part partitions $(\eta_1, \ldots, \eta_n)$ of $\eta$, where parts being empty configurations are also considered. For $n = 2$, we can rewrite (2.6) in the following form

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\eta, \xi) \lambda(d\eta) \lambda(d\xi) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi) \lambda(d\eta).$$

(2.7)

By taking

$$H(\eta_1, \eta_2) = \begin{cases} h(x, \eta_2), & \eta_1 = \{x\} \\ 0, & |\eta_1| \neq 1 \end{cases}$$

and using (2.5) we obtain the following special case of the Minlos lemma

$$\int_{\Gamma_0} \int_{\mathbb{R}^d} G(\eta \cup x) h(x, \eta) dx \lambda(d\eta) = \int_{\Gamma_0} \sum_{x \in \eta} G(\eta) h(x, \eta \setminus x) \lambda(d\eta).$$

(2.8)

Analogously, for

$$H(\eta_1, \eta_2, \eta_3) = \begin{cases} h(x, y, \eta_3), & \eta_1 = \{x\}, \eta_2 = \{y\} \\ 0, & |\eta_1| \neq 1 \text{ or } |\eta_2| \neq 1, \end{cases}$$

we have

$$\frac{1}{2} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(\eta \cup \{x, y\}) h(x, y, \eta) dx dy \lambda(d\eta)$$

$$= \int_{\Gamma_0} \sum_{\{x, y\} \subset \eta} G(\eta) h(x, y, \eta \setminus \{x, y\}) \lambda(d\eta)$$

(2.9)

3. Dynamics of the correlation functions

In this section, we follow the approach of [9] and obtain the operator $L^\Delta$ corresponding to (1.1). First, by using the $K$-transform (2.1) we pass to the quasi-observables and obtain the operator $\hat{L}$. Then by the Minlos lemma (2.6) we obtain the operator $L^\Delta$. Recall, that

$$L = L_1 + L_2,$$

where

$$L_1 F(\gamma) = \sum_{\{x, y\} \subset \gamma} \int_{\mathbb{R}^d} \tilde{c}_1(x, y; z; \gamma) \left( F(\gamma \setminus \{x, y\} \cup z) - F(\gamma) \right) dz.$$
and
\[ L_2 F(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} \tilde{c}_2(x; y; \gamma) \left( F(\gamma \setminus x \cup y) - F(\gamma) \right) dy. \]

We assume that
\[
\begin{align*}
\tilde{c}_1(x, y; z; \gamma) &= c_1(x, y; z)e(t_z^{(1)}, \gamma \setminus \{x, y\}), \\
\tilde{c}_2(x; y; \gamma) &= c_2(x; y)e(t_y^{(2)}, \gamma \setminus x),
\end{align*}
\]
with
\[
t_z^{(1)}(u) = e^{-\phi_1(z-u)}, \quad t_y^{(2)}(u) = e^{-\phi_2(y-u)}.
\]

The above \(c_1, c_2, \phi_1, \phi_2\) are positive real functions such that
\[
\begin{align*}
\tilde{c}_1(x, y; z; \gamma) &= (KC_{x,y,z}^1)(\gamma \setminus \{x, y\}), \\
\tilde{c}_2(x; y; \gamma) &= (KC_{x,y}^2)(\gamma \setminus x)
\end{align*} \tag{3.1}
\]
for some \(C_{x,y,z}^1\) and \(C_{x,y}^2\). We discuss the form of these functions later in this section. Additionally, we assume that
\[
\begin{align*}
c_1(x, y; z) &= c_1(y, x; z), \\
\int_{\mathbb{R}^d} c_1(x_1, x_2; x_3)dx_idx_j &= \langle c_1 \rangle < \infty, \quad i, j = 1, 2, 3, \quad i \neq j, \\
\int_{\mathbb{R}^d} c_2(x; y)dx &= \int_{\mathbb{R}^d} c_2(x; y)dy = \langle c_2 \rangle < \infty, \\
\int_{\mathbb{R}^d} \phi_1(x)dx &= \langle \phi_1 \rangle < \infty, \quad \int_{\mathbb{R}^d} \phi_2(x)dx = \langle \phi_2 \rangle < \infty.
\end{align*}
\]

Suppose that \(F = KG\), where \(G : \Gamma_0 \to \mathbb{R}\). Then by writing \(K \hat{L}G = LF\) we define
\[ \hat{L} = K^{-1}LK. \tag{3.2} \]

By the properties of the \(K\)–transform we derive an explicit formula for \(\hat{L}\).

**Proposition 3.1.** \(\hat{L}\) defined as above has the following form
\[
\hat{L}G(\eta) = \int_{\mathbb{R}^d} \sum_{\{x,y\} \subset \eta} \left[ C_{x,y,z}^1 \ast H_{x,y,z}^1(\eta \setminus \{x, y\}) \right]dz \\
+ \int_{\mathbb{R}^d} \sum_{x \in \eta} \left[ C_{x,y}^2 \ast H_{x,y}^2(\eta \setminus x) \right]dy,
\]
where
\[
\begin{align*}
H_{x,y,z}^1(\eta) &= G(\eta \cup z) - G(\eta \cup x) - G(\eta \cup y) - G(\eta \cup \{x, y\}), \\
H_{x,y}^2(\eta) &= G(\eta \cup y) - G(\eta \cup x). \tag{3.3}
\end{align*}
\]
The next step is to pass with the action of the operator $\hat{L}$ to the correlation functions, introducing $L^\Delta$. We can define the latter by the pairing $\langle \langle \hat{L}G, k \rangle \rangle = \langle \langle G, L^\Delta k \rangle \rangle$, that is

$$\int_{\Gamma_0} (\hat{L}G)(\eta)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} G(\eta)(L^\Delta k)(\eta)\lambda(d\eta) \quad (3.4)$$

Let us consider the integral from the left hand side of equation (3.4). Using the Minlos lemma (2.6) several times, we can transform it so that we can obtain $L^\Delta$.

**Proposition 3.2.** $L^\Delta$ defined as above is of the form

$$L^\Delta = L^\Delta_1 + L^\Delta_2,$$

where

$$L^\Delta_1 k(\eta) = \frac{1}{2} \int \left( \int \sum_{x \in \eta} c_1(x, y; z)k(\eta \setminus z \cup \xi + \{x, y\}) \right)$$

$$\times \ e(t^{(1)}_z - 1, \xi)e(t^{(1)}_y, \eta \setminus z)\lambda(d\xi)dx dy$$

$$- \frac{1}{2} \int \left( \int \sum_{y \in \eta} c_1(x, y; z)k(\eta \cup \xi \cup y) \right)$$

$$\times \ e(t^{(1)}_z - 1, \xi)e(t^{(1)}_y, \eta \setminus x)\lambda(d\xi)dy dz$$

and

$$L^\Delta_2 k(\eta) = \int \int \sum_{y \in \eta} k(\eta \setminus y \cup \xi \cup x)c_2(x; y)$$

$$\times \ e(t^{(2)}_y - 1, \xi)e(t^{(2)}_y - 1, \eta \setminus y)\lambda(d\xi)dx$$

$$- \int \int \sum_{x \in \eta} c_2(x; y)$$

$$\times \ \prod_{u \in \xi} e(t^{(2)}_y - 1, \xi)e(t^{(2)}_y - 1, \eta \setminus x)\lambda(d\xi)dy.$$

**Proof of Proposition 3.1.** First let us rewrite the operator $L$ in a more convenient form. Using (3.1) and recalling that any configuration treated as
a subset of $\mathbb{R}^d$ is Lebesgue measure-zero as it is countable, we have
\[
L_1 F(\gamma) = \sum_{\{x,y\} \subset \gamma} \int_{\mathbb{R}^d} \left( KC^1_{x,y;z}(\cdot) \left[ KG(\cdot \cup z) \right] - KG(\cdot \cup \{x,y\}) \right)(\gamma \setminus \{x,y\}) dz.
\]
Observe that for any $\xi \in \Gamma$, $x,y,z \notin \xi$ we have
\[
KG(\xi \cup z) = \sum_{\eta \subset \xi \cup z} \left[ G(\eta) + G(\eta \cup z) \right] = KG(\xi \cup \{x,y\})(\xi).
\]
and analogously
\[
KG(\xi \cup \{x,y\}) = KG(\xi \cup \{x,y\})(\xi).
\]
Using linearity of the $K-$transform and above observations, we obtain
\[
L_1 F(\gamma) = \sum_{\{x,y\} \subset \gamma} \int_{\mathbb{R}^d} \left( KC^1_{x,y;z}(\cdot) \left[ KG(\xi \cup z) - KG(\xi \cup \{x,y\}) \right] \right)(\gamma \setminus \{x,y\}) dz.
\]
Considering the second part of the operator, we have
\[
L_2 F(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} \left( KC^2_{x,y}(\gamma \setminus x) \left[ KG(\gamma \setminus x \cup y) - KG(\gamma) \right] dy
\]
\[
= \sum_{x \in \gamma} \int_{\mathbb{R}^d} \left( KC^2_{x,y}(\gamma \setminus x) \left[ KG(\gamma \setminus x \cup y) - KG(\gamma \setminus x \cup \{x,y\}) \right] dy.
\]
Using notion (3.3) and property (2.2) of the product of $K-$transforms, we derive
\[
L_1 F(\gamma) = \sum_{\{x,y\} \subset \gamma} \int_{\mathbb{R}^d} K\left[ C^1_{x,y;z} * H^1_{x,y;z} \right](\gamma \setminus \{x,y\}) dz,
\]
\[
L_2 F(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} K\left[ C^2_{x,y} * H^2_{x,y} \right](\gamma \setminus x) dy.
\]
Therefore
\[
L F(\gamma) = \sum_{\{x,y\} \subset \gamma} \int_{\mathbb{R}^d} K\left[ C^1_{x,y;z} * H^1_{x,y;z} \right](\gamma \setminus \{x,y\}) dz
\]
\[
+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} K\left[ C^2_{x,y} * H^2_{x,y} \right](\gamma \setminus x) dy.
\]
Recalling the definition (3.2) of the operator $\hat{L}$ and denoting
\[
\hat{L}_1 G(\eta) = K^{-1} L_1 F(\eta), \quad \hat{L}_2 G(\eta) = K^{-1} L_2 F(\eta)
\]
we obtain
\[
\hat{L}_1 G(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{\{x,y\} \subset \xi} \int_{\mathbb{R}^d} K\left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\xi \setminus \{x,y\}) dz
\]
\[
= \int \sum_{\{x,y\} \subset \eta} \sum_{\xi \subset \eta \setminus \{x,y\}} (-1)^{|\eta \setminus \{x,y\}|} K\left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\xi) dz
\]
\[
= \int \sum_{\{x,y\} \subset \eta} K^{-1} K\left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\eta \setminus \{x,y\}) dz
\]
\[
= \int \sum_{\{x,y\} \subset \eta} \left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\eta \setminus \{x,y\}) dz.
\]
and analogously
\[
\hat{L}_2 G(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{x \in \xi} \int_{\mathbb{R}^d} K\left[C_{x,y}^2 \ast H_{x,y}^2\right](\xi \setminus x) dy
\]
\[
= \int \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus x} (-1)^{|\eta \setminus \{x\}|} K\left[C_{x,y}^2 \ast H_{x,y}^2\right](\xi) dy
\]
\[
= \int \sum_{x \in \eta} K^{-1} K\left[C_{x,y}^2 \ast H_{x,y}^2\right](\eta \setminus x) dy
\]
\[
= \int \sum_{x \in \eta} \left[C_{x,y}^2 \ast H_{x,y}^2\right](\eta \setminus x) dy.
\]
Therefore
\[
\hat{L} G(\eta) = \int \sum_{\{x,y\} \subset \eta} \left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\eta \setminus \{x,y\}) dz
\]
\[
+ \int \sum_{x \in \eta} \left[C_{x,y}^2 \ast H_{x,y}^2\right](\eta \setminus x) dy.
\]

Proof of Proposition 3.2. Using the special case (2.9) of the Minlos lemma and proposition (3.1) we have
\[
\int_{\Gamma_0} (\hat{L}_1 G)(\eta) k(\eta) \lambda(d\eta)
\]
\[
= \int \int \sum_{\{x,y\} \subset \eta} \left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\eta \setminus \{x,y\}) k(\eta) dz \lambda(d\eta)
\]
\[
= \frac{1}{2} \int \int \left[C_{x,y;z}^1 \ast H_{x,y;z}^1\right](\eta) k(\eta \cup \{x,y\}) \lambda(d\eta) dx dy dz.
\]
Recalling the definition (2.3) of the convolution $\ast$ and using the Minlos lemma in the form (2.7) twice, we obtain
\[
\int_{\Gamma_0} (\hat{L}_1 G)(\eta) k(\eta) \lambda(d\eta) = \frac{1}{2} \int_{(R^d)^3} \int_{\Gamma_0} \sum_{\xi \subset \eta} C^1_{x,y;z}(\xi) \sum_{\zeta \subset \xi} H^1_{x,y;z}(\eta \setminus \xi \cup \zeta) \times k(\eta \cup \{x, y\}) \lambda(d\eta) dxdydz
\]
\[= \frac{1}{2} \int_{(R^d)^3} \int_{\Gamma_0} \sum_{\xi \subset \eta} C^1_{x,y;z}(\xi) \sum_{\zeta \subset \xi} H^1_{x,y;z}(\eta \cup \zeta) \times k(\eta \cup \{x, y\}) \lambda(d\eta) \lambda(d\xi) dxdydz
\]
\[= \frac{1}{2} \int_{(R^d)^3} \int_{\Gamma_0} \int_{(R^d)^3} \int_{\Gamma_0} \sum_{\xi \subset \eta} C^1_{x,y;z}(\xi) \sum_{\zeta \subset \xi} H^1_{x,y;z}(\eta \cup \zeta) \times k(\eta \cup \{x, y\}) \lambda(d\eta) \lambda(d\xi) \lambda(d\zeta) dxdydz.
\]

Using again the Minlos lemma (2.7), but in the opposite direction, we have
\[
\int_{\Gamma_0} (\hat{L}_1 G)(\eta) k(\eta) \lambda(d\eta) = \frac{1}{2} \int_{(R^d)^3} \int_{\Gamma_0} \sum_{\xi \subset \eta} C^1_{x,y;z}(\xi \cup \eta) C^1_{x,y;z}(\xi \cup \eta) \times k(\eta \cup \{x, y\}) \lambda(d\eta) \lambda(d\xi) \lambda(d\zeta) dxdydz.
\]

Let us rewrite above using the definition (3.3) of $H^1_{x,y;z}(\eta)$.
\[
\int_{\Gamma_0} (\hat{L}_1 G)(\eta) k(\eta) \lambda(d\eta) = \frac{1}{2} \int_{(R^d)^3} \int_{\Gamma_0} \sum_{\xi \subset \eta} C^1_{x,y;z}(\xi \cup \eta) \times k(\eta \cup \{x, y\}) \lambda(d\eta) \lambda(d\xi) \lambda(d\zeta) dxdydz
\]

Using the special cases (2.8) and (2.9) of the Minlos lemma we obtain
\[
\int_{\Gamma_0} (\hat{L}_1 G)(\eta) k(\eta) \lambda(d\eta)
\]
\[= \int_{(R^d)^2} \int_{\Gamma_0} \sum_{\xi \subset \eta} k(\eta \cup \{x, y\}) \lambda(d\eta) \sum_{\zeta \subset \eta} C^1_{x,y;z}(\xi \cup \zeta) \lambda(d\xi) dxdydz
\]
\[= \int_{(R^d)^2} \int_{\Gamma_0} G(\eta) \left[ \frac{1}{2} \int_{(R^d)^2} \int_{\Gamma_0} \sum_{\xi \subset \eta} k(\eta \cup \{x, y\}) \sum_{\zeta \subset \eta} C^1_{x,y;z}(\xi \cup \zeta) \lambda(d\xi) dxdydz
\]
\[-\frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{x \in \eta} k(\eta \cup x \cup y) \sum_{\zeta \subset \eta \setminus x} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dydz \]

\[-\frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{y \in \eta} k(\eta \cup y \cup x) \sum_{\zeta \subset \eta \setminus y} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dxdz \]

\[-\int_{(\mathbb{R}^d)^2} \int_{\{x,y\} \subset \eta} \sum_{x \in \eta} k(\eta \cup x) \sum_{\zeta \subset \eta \setminus \{x,y\}} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dz \]

Employing the same technique to the second part of the operator \( \hat{L} \), we derive

\[\int_{\Gamma_0} (\hat{L}_2 G)(\eta) k(\eta) \lambda(d\eta) \]

\[= \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{\eta \in \eta} C_{x,y}^2(\xi \cup \zeta) H_{x,y}^2(\eta) k(\eta \cup x \cup y) \lambda(d\eta) \lambda(d\xi)dxdy \]

\[= \int_{\Gamma_0} G(\eta) \left[ \int_{(\mathbb{R}^d)^2} \sum_{y \in \eta} k(\eta \cup y \cup x) \sum_{\zeta \subset \eta \setminus y} C_{x,y,z}^2(\xi \cup \zeta) \lambda(d\xi)dx \right. \]

\[\left. \int_{(\mathbb{R}^d)^2} \int_{\{x,y\} \subset \eta} k(\eta \cup x) \sum_{x \in \eta} C_{x,y,z}^2(\xi \cup \zeta) \lambda(d\xi)dy \right] \lambda(d\eta) \]

Therefore, we obtain

\[L^\Delta k(\eta) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{x \in \eta} k(\eta \cup x \cup \{x, y\}) \sum_{\zeta \subset \eta \setminus z} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dxdy \]

\[-\frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{x \in \eta} k(\eta \cup x \cup y) \sum_{\zeta \subset \eta \setminus x} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dydz \]

\[-\frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\Gamma_0} \sum_{y \in \eta} k(\eta \cup y \cup x) \sum_{\zeta \subset \eta \setminus y} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dxdz \]

\[-\int_{(\mathbb{R}^d)^2} \int_{\{x,y\} \subset \eta} \sum_{x \in \eta} k(\eta \cup x) \sum_{\zeta \subset \eta \setminus \{x,y\}} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dz \]

\[+ \int_{(\mathbb{R}^d)^2} \int_{\eta \cup y \cup x \cup \zeta \subset \eta \setminus y} k(\eta \cup y \cup x) \sum_{\zeta \subset \eta \setminus y} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dx \]

\[-\int_{(\mathbb{R}^d)^2} \int_{\{x,y\} \subset \eta} \sum_{x \in \eta} k(\eta \cup x) \sum_{\zeta \subset \eta \setminus \{x,y\}} C_{x,y,z}^1(\xi \cup \zeta) \lambda(d\xi)dy \] (3.5)

Note that so far we have not used any assumption about coefficients \( \tilde{c}_1 \) and \( \tilde{c}_2 \) but that they can be written as results of action of the \( K \)–transform on
corresponding functions $C^1_{x,y;z}$ and $C^2_{x,y}$. Let us calculate explicit forms of these functions. Recall that
\[
\tilde{c}_1(x, y; z; \gamma) = c_1(x, y; z) \prod_{u \in \gamma \setminus \{x, y\}} e^{-\phi_1(z-u)},
\]
\[
\tilde{c}_2(x; y; \gamma) = c_2(x; y) \prod_{u \in \gamma \setminus x} e^{-\phi_2(y-u)}.
\]
We have
\[
K C^1_{x,y;z} = c_1(x, y; z)e(t^{(1)}_z - 1, \cdot),
\]
that is
\[
C^1_{x,y;z} = K^{-1} c_1(x, y; z)e(1 + t^{(1)}_z - 1, \cdot) = c_1(x, y; z)K^{-1} \sum_{\xi \subset \cdot} e(t^{(1)}_z - 1, \xi)
\]
\[
= c_1(x, y; z)K^{-1}e(t^{(1)}_z - 1, \cdot) = c_1(x, y; z)e(t^{(1)}_z - 1, \cdot).
\]
Therefore
\[
C^1_{x,y;z}(\eta) = c_1(x, y; z)e(t^{(1)}_z - 1, \eta). \tag{3.6}
\]
Analogously we can derive
\[
C^2_{x,y}(\eta) = c_2(x; y)e(t^{(2)}_y - 1, \eta). \tag{3.7}
\]
Using the above, we can rewrite the operator $L^\Lambda$. For convenience let us denote the part of it corresponding to the coalescence, that is the first four terms of (3.5), as $L^\Lambda_1$ and the part corresponding to the jumps, that is the last two terms of (3.5), as $L^\Lambda_2$. Substituting (3.6) we derive
\[
L^\Lambda_1 k(\eta) = \frac{1}{2} \int_{(R^d)^2} \int_{\Gamma_0} \sum_{x \in \eta} k(\eta \setminus \xi \cup \{x, y\}) \sum_{\zeta \subset \eta \setminus x} c_1(x, y; z) \times e(t^{(1)}_z - 1, \xi \cup \zeta)\lambda(d\xi)dx dy
\]
\[
- \frac{1}{2} \int_{(R^d)^2} \int_{\Gamma_0} \sum_{x \in \eta} k(\eta \cup \zeta \cup y) \sum_{\zeta \subset \eta \setminus y} c_1(x, y; z)e(t^{(1)}_z - 1, \xi \cup \zeta)\lambda(d\xi)dy dz
\]
\[
- \frac{1}{2} \int_{(R^d)^2} \int_{\Gamma_0} \sum_{y \in \eta} k(\eta \cup \zeta \cup x) \sum_{\zeta \subset \eta \setminus y} c_1(x, y; z)e(t^{(1)}_z - 1, \xi \cup \zeta)\lambda(d\xi)dx dz
\]
\[
- \int_{R^d} \int_{\Gamma_0} \sum_{\{x,y\} \subset \eta} k(\eta \cup \xi) \sum_{\zeta \subset \eta \setminus \{x,y\}} c_1(x, y; z)e(t^{(1)}_z - 1, \xi \cup \zeta)\lambda(d\xi)d\zeta
\]
and analogously using (3.7) we obtain
\[
L^\Lambda_2 k(\eta) = \int_{R^d} \int_{\Gamma_0} \sum_{y \in \eta} k(\eta \setminus y \cup \xi \cup x) \sum_{\zeta \subset \eta \setminus y} c_2(x; y)e(t^{(2)}_y - 1, \xi \cup \zeta)\lambda(d\xi)dx
\]
\[
- \int_{R^d} \int_{\Gamma_0} k(\eta \cup \zeta) \sum_{x \in \eta \setminus \zeta \cup \eta \setminus x} c_2(x; y)e(t^{(2)}_y - 1, \xi \cup \zeta)\lambda(d\xi)dy.
Consider the first component of $L^0_1$ and denote it as
\[
L^0_1 k(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} k(\eta \setminus z \cup \xi \cup \{x, y\}) \sum_{\zeta \subseteq \eta \setminus z} c_1(x, y; z)e(t_z^{(1)} - 1, \xi \cup \zeta)\lambda(\xi)\lambda(\zeta)dx dy.
\]

Next, for a given $\eta$ let us introduce $C(\eta) = \{\xi \in \Gamma_0 : \xi \cap \eta \neq \emptyset\}$. Then, because any configuration treated as a measurable subset of $\mathbb{R}^d$ is of Lebesgue measure 0 and the empty configuration does not belong to $C(\eta)$ for any $\eta \in \Gamma_0$, we have $\lambda(C(\eta)) = 0$ for every $\eta \in \Gamma_0$. Indeed, using the characterization \((2.5)\) of the integral w.r.t. the Lebesgue-Poisson measure we obtain
\[
\lambda(C(\eta)) = \int_{\Gamma_0} I_{C(\eta)}(x) \lambda(dx) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} I_{C(\eta)}^{(n)}(x_1, ..., x_n)dx_1...dx_n.
\]

First, notice that $I_{C(\eta)}^{(0)} = 0$, as empty configuration cannot have common part with any configuration. Then, because
\[
I_{C(\eta)}^{(n)}(x_1, ..., x_n) \leq I_{C(\eta)}^{(1)}(x_1) + I_{C(\eta)}^{(1)}(x_2) + ... + I_{C(\eta)}^{(1)}(x_n)
\]
we have for every $n \in \mathbb{N}$
\[
\int_{(\mathbb{R}^d)^n} I_{C(\eta)}^{(n)}(x_1, ..., x_n)dx_1...dx_n \leq n \int_{(\mathbb{R}^d)^{n-1}} \left[ \int_{\mathbb{R}^d} I_{C(\eta)}^{(1)}(x)dx \right] dx_1...dx_{n-1}.
\]

Taking into account that
\[
\int_{\mathbb{R}^d} I_{C(\eta)}^{(1)}(x)dx = \int_{\mathbb{R}^d} I_\eta(x)dx = l(\eta) = 0,
\]
where $l$ denotes the Lebesgue measure, one can clearly see that $\lambda(C(\eta)) = 0$. Therefore, when integrating over $\Gamma_0 \setminus C(\eta)$ instead of $\Gamma_0$, the result is the same. However, all subconfigurations $\zeta$ of $\eta$ are disjoint with any $\xi \in \Gamma_0 \setminus C(\eta)$, which allows us to separate the product taken over $\xi \cup \zeta$ into one taken over $\xi$ and another taken over $\zeta$. Thus we can write
\[
L^0_1 k(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x, y; z)k(\eta \setminus z \cup \xi \cup \{x, y\}) \times e(t_z^{(1)} - 1, \xi) \sum_{\zeta \subseteq \eta \setminus z} e(t_z^{(1)} - 1, \zeta)\lambda(\xi)\lambda(\zeta)dx dy.
\]

Recalling the definition \((2.1)\) of the $K$-transform and its property \((2.4)\) we have
\[
\sum_{\zeta \subseteq \eta \setminus z} e(t_z^{(1)} - 1, \zeta) = K\left(e(t_z^{(1)} - 1, \cdot)\right)(\eta \setminus z) = e(t_z^{(1)}, \eta \setminus z).
\]
Therefore we can rewrite the action of $L_{11}^{\Delta}$ in the form

$$L_{11}^{\Delta}k(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x,y,z)k(\eta \setminus z \cup \xi \cup \{x,y\})$$

$$\times e(t_z^{(1)} - 1, \xi)e(t_z^{(1)}, \eta \setminus z)\lambda(d\xi) dx dy.$$

Applying the same method for the rest of the $L_{1}^{\Delta}$ and for the $L_{2}^{\Delta}$ we obtain the result. □

4. The Vlasov Scaling and the Kinetic Equation

4.1. The Vlasov Scaling.

We follow the scaling technique described in [7]. Let us introduce the scale parameter $\epsilon \in [0,1]$ with $\epsilon = 1$ corresponding to the unscaled and $\epsilon = 0$ to the fully rescaled case. We alter the operator $L^{\Delta}$ by scaling $c_1 \to \epsilon c_1$, $\phi_1 \to \epsilon \phi_2$ and $\phi_2 \to \epsilon \phi_2$ for $\epsilon \in (0,1]$, which can be interpreted as weakening the interactions between particles. Altered in such a way operator we denote by $L_{\epsilon}^{\Delta}$. Next, we renormalize it defining

$$L^{\text{ren}}_{\epsilon}k(\eta) = \epsilon |\eta| L_{\epsilon}^{\Delta}(\epsilon^{-|\eta|}k(\eta)).$$

Let us consider the first component of the operator $L^{\text{ren}}_{\epsilon}$. We have (cf Proposition 3.2)

$$L^{\text{ren}}_{11,\epsilon}k(\eta) = \frac{1}{2} \epsilon |\eta| \int \int \sum_{z \in \eta} \epsilon c_1(x,y,z)\epsilon^{-|\eta|}z \cup \xi \cup \{x,y\} k(\eta \setminus z \cup \xi \cup \{x,y\})$$

$$\times \prod_{u \in \xi} (e^{-\epsilon \phi_1(z-u)} - 1) \prod_{u \in \eta \setminus z} e^{-\epsilon \phi_1(z-u)} \lambda(d\xi) dx dy.$$

Note that integrating over $(\mathbb{R}^d)^2 \setminus (\eta \times \eta)$ instead of $(\mathbb{R}^d)^2$ and over $\Gamma_0 \setminus (\eta \cup \{x,y\})$ instead of $\Gamma_0$ does not influence the result, so we have

$$L^{\text{ren}}_{11,\epsilon}k(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x,y,z)k(\eta \setminus z \cup \xi \cup \{x,y\})$$

$$\times \prod_{u \in \xi} \frac{1}{\epsilon} (e^{-\epsilon \phi_1(z-u)} - 1) \prod_{u \in \eta \setminus z} e^{-\epsilon \phi_1(z-u)} \lambda(d\xi) dx dy.$$

Let us pass with $\epsilon$ to the limit. Noting that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} (e^{-\epsilon \phi_1(z-u)} - 1) = -\epsilon \phi_1(z - u),$$

we can write

$$\lim_{\epsilon \to 0} L^{\text{ren}}_{11,\epsilon}k(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x,y,z)k(\eta \setminus z \cup \xi \cup \{x,y\})$$

$$\times \prod_{u \in \xi} (- \phi_1(z - u)) \lambda(d\xi) dx dy.$$
Let us denote $V = \lim_{\epsilon \to 0} I_{\epsilon}^{\text{ren}}$. Calculating analogously as above, one derives

$$
V k(\eta) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\partial \Omega_0} \sum_{z \in \eta} c_1(x, y; z) k(\eta \setminus z \cup \xi \cup \{x, y\}) \times \prod_{u \in \xi} (-\phi_1(z - u)) \lambda(d\xi) dxdy
$$

$$
- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\partial \Omega_0} \sum_{x \in \eta} c_1(x, y; z) k(\eta \cup \xi \cup y) \prod_{u \in \xi} (-\phi_1(z - u)) \lambda(d\xi) dydz
$$

$$
- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \int_{\partial \Omega_0} \sum_{y \in \eta} c_1(x, y; z) k(\eta \cup \xi \cup x) \prod_{u \in \xi} (-\phi_1(z - u)) \lambda(d\xi) dx dz
$$

$$
+ \int_{(\mathbb{R}^d) \times \partial \Omega_0} \sum_{y \in \eta} c_2(x; y) \prod_{u \in \xi} (-\phi_2(y - u)) \lambda(d\xi) dx
datax
$$

$$
- \int_{(\mathbb{R}^d) \times \partial \Omega_0} \sum_{x \in \eta} c_2(x; y) \prod_{u \in \xi} (-\phi_2(y - u)) \lambda(d\xi) dy.
datax
$$

Consider the following problem

$$
\frac{d}{dt} r_t = V r_t, \quad r_{t=0} = r_0 \tag{4.1}
datax
$$

in the Banach space

$$
K_\theta = \{r : \Gamma_0 \to \mathbb{R}^d : ||r||_\theta < \infty\},
datax
$$

where

$$
||r||_\theta = \text{ess sup}_{\eta \in \Gamma_0} e^{\theta||r||_{L^\infty}}.
datax
$$

Supposing that the initial state is the Poisson measure, $r_0$ can be factorized

$$
r_0(\eta) = \prod_{x \in \eta} \rho_0(x).
datax
$$

If $r_t$ can be written in the product form, that is

$$
r_t(\eta) = \prod_{x \in \eta} \rho_t(x),
datax
$$

then we can write

$$
\frac{d}{dt} r_t(\eta) = \frac{d}{dt} \prod_{x \in \eta} \rho_t(x) = \sum_{x \in \eta} \left( \prod_{y \in \eta \setminus x} \rho_t(y) \right) \frac{d}{dt} \rho_t(x).
datax
$$

Therefore, by expressing $V r_t$ in the form

$$
V r_t(\eta) = \sum_{x \in \eta} \left( \prod_{y \in \eta \setminus x} \rho_t(y) \right) v(\rho_t, x),
datax
$$

we can obtain a problem for $\rho_t$ corresponding to (4.1), namely a kinetic equation
\[
\frac{d}{dt} \rho_t(x) = v(\rho_t, x), \quad \rho_{t=0} = \rho_0,
\] (4.2)
where $r_0(x) = \prod_{x \in \eta} \rho_0(x)$. Indeed, if $\rho_t$ is a solution of (4.2), then we can easily check that
\[
k_t(\eta) = \prod_{x \in \eta} \rho_t(x)
\]
is a solution of (4.1).

Let us denote the first component of $V$ by $V_1$. We have
\[
V_1 r_t(\eta) = \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x, y; z) r_t(\eta \setminus z \cup \xi \cup \{x, y\})
\]
\[
\times \prod_{u \in \xi} (-\phi_1(z - u)) \lambda(d\xi) dxdy
\]
\[
= \frac{1}{2} \int \int \sum_{z \in \eta} c_1(x, y; z) \prod_{v \in \eta \setminus z \cup \xi \cup \{x, y\}} \rho_t(v)
\]
\[
\times \prod_{u \in \xi} (-\phi_1(z - u)) \lambda(d\xi) dxdy.
\]
Because $l(\eta) = 0$ and $\lambda(C(\eta \cup \{x, y\})) = 0$, one can rewrite above as
\[
V_1 r_t(\eta) = \sum_{z \in \eta} \left( \prod_{v \in \eta \setminus z} \rho_t(v) \right) \left( \frac{1}{2} \int \int c_1(x, y; z)
\]
\[
\times \prod_{u \in \xi} (-\rho_t(x)\rho_t(y)) \prod_{u \in \xi} (-\rho_t(u)\phi_1(z - u)) \lambda(d\xi) dxdy \right).
\]
Therefore
\[
V_1 r_t(\eta) = \sum_{z \in \eta} \left( \prod_{v \in \eta \setminus z} \rho_t(v) \right) v_1(\rho_t, z),
\]
where
\[
v_1(\rho_t, z) = \frac{1}{2} \int c_1(x, y; z) \rho_t(x) \rho_t(y) \int \prod_{u \in \xi} (-\rho_t(u)\phi_1(z - u)) \lambda(d\xi) dxdy.
\]
Noting that for $a(u) = -\rho_t(u)\phi_1(z - u)$
\[
\int_{\Gamma_0} \prod_{u \in \xi} a(u) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} a(x_1)...a(x_n) dx_1...dx_n
\]
\[
= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int_{\mathbb{R}^d} a(u) du \right)^n = \exp \left( \int_{\mathbb{R}^d} a(u) du \right),
\]
we can reformulate the above obtaining

\[ v_1(\rho_t, x) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(y, z; x) \rho_t(y) \rho_t(z) \exp \left( - \int_{\mathbb{R}^d} \phi_1(x - u) \rho_t(u) du \right) dydz. \]

Calculating analogously as above, one can obtain explicit form of \( v \) and thus the following kinetic equation

\[
\frac{d}{dt} \rho_t(x) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(y, z; x) \exp \left( - \int_{\mathbb{R}^d} \phi_1(x - u) \rho_t(u) du \right) \rho_t(y) \rho_t(z) dydz \\
- \frac{1}{2} \int_{(\mathbb{R}^d)^2} \left( c_1(x, y; z) + c_1(y, x; z) \right) \exp \left( - \int_{\mathbb{R}^d} \phi_1(z - u) \rho_t(u) du \right) \rho_t(x) dydz \\
+ \int_{\mathbb{R}^d} c_2(y; x) \exp \left( - \int_{\mathbb{R}^d} \phi_2(x - u) \rho_t(u) du \right) \rho_t(y) dy \\
- \int_{\mathbb{R}^d} c_2(x; y) \exp \left( - \int_{\mathbb{R}^d} \phi_2(y - u) \rho_t(u) du \right) \rho_t(x) dy,
\]

\[ \rho_{t=0} = \rho_0. \] (4.3)

4.2. The Kinetic Equation.

Let us rewrite the problem (4.3) as

\[
\frac{d}{dt} \rho_t(x) = R_1(\rho_t, x) + R_2(\rho_t, x), \quad \rho_{t=0}(x) = \rho_0(x), \quad (4.4)
\]

where

\[
R_1(\rho_t, x) = -\frac{1}{2} \rho_t(x) \int_{(\mathbb{R}^d)^2} \left( c_1(x, y; z) + c_1(y, x; z) \right) \rho_t(y) dydz \\
- h(\rho_t, x) \int_{\mathbb{R}^d} c_2(x; y) dy \\
= -\rho_t(x) h(\rho_t, x)
\]

for

\[
h(\rho_t, x) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} \left( c_1(x, y; z) + c_1(y, x; z) \right) \rho_t(y) dydz + \langle c_2 \rangle
\]
and

\[
R_2(\rho_t, x) = \frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(y, z; x) \exp \left( - \int_{\mathbb{R}^d} \phi_1(x - u) \rho_t(u) du \right) \rho_t(y) \rho_t(z) dydz
+ \frac{1}{2} \int_{(\mathbb{R}^d)^2} \left( c_1(x, y; z) + c_1(y, x; z) \right)
\times \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_1(z - u) \rho_t(u) du \right) \right] \rho_t(x) \rho_t(y) dydz
+ \int_{\mathbb{R}^d} c_2(y; x) \exp \left( - \int_{\mathbb{R}^d} \phi_2(x - u) \rho_t(u) du \right) \rho_t(y) dy
+ \int_{\mathbb{R}^d} c_2(x; y) \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_2(y - u) \rho_t(u) du \right) \right] \rho_t(x) dy.
\]

Note that from (4.4) we can obtain the equivalent integral equation

\[
\rho_t(x) = \rho_0(x) \exp \left( - \int_0^t h(\rho_s, x) ds \right) + \int_0^t R_2(\rho_s, x) \exp \left( - \int_s^t h(\rho_\sigma, x) d\sigma \right) ds.
\]  

\[\text{(4.5)}\]

**Theorem 4.1.** Problem (4.4) with the initial condition \( \rho_0 \in L^\infty(\mathbb{R}^d) \), \( \rho_0 \geq 0 \) has the unique local classical solution.

Consider \( X_T = C([0, T] \to L^\infty(\mathbb{R}^d)) \), \( T > 0 \) with the norm

\[ ||\rho||_{T, \gamma} = \sup_{t \in [0, T]} e^{-\gamma(\xi^T)t} ||\rho_t||_{L^\infty}.\]

Denote

\[ B_{T^*, \gamma}(r) = \{ \rho \in X_T : ||\rho||_{T^*, \gamma} \leq r, \rho_t \geq 0 \ \forall t \in [0, T] \}, \]
\[ B_{T, \gamma}(r, \rho_0) = \{ \psi \in B_{T^*, \gamma}(r) : \psi_0 = \rho_0 \}, \]
where \( \rho_0 \in L^\infty(\mathbb{R}^d), \rho_0 \geq 0, r \geq ||\rho_0||_{L^\infty} \) and \( T, \gamma > 0 \).

**Lemma 4.2.** Given \( r > 0 \), there exist \( \gamma, \tilde{T} > 0 \) such that \( F \) defined by the RHS of (4.5) with the domain \( B_{T^*, \gamma}(r) \subset X_{T^*} \) acts again to the \( B_{T^*, \gamma}(r) \) for any \( T^* \in [0, \tilde{T}] \).

**Lemma 4.3.** Let \( \rho_0 \in L^\infty(\mathbb{R}^d), \rho_0 \geq 0 \) and \( r \geq ||\rho_0||_{L^\infty} \). Let \( \tilde{T}, \gamma \) satisfy lemma 4.2 for this \( r \). We can choose \( T^* \in [0, \tilde{T}] \) in such a way that for any \( \rho, \psi \in B_{T^*, \gamma}(r, \rho_0) \) the inequality \( ||F(\rho) - F(\psi)||_{T^*, \gamma} \leq C||\rho - \psi||_{T^*, \gamma} \) holds for some constant \( C < 1 \).

**Proof of Theorem 4.1.** Choose \( r > ||\rho_0||_{L^\infty} \) and take corresponding \( \gamma, \tilde{T} \) from lemma 4.2. Take \( T^* \) as in lemma 4.3. Define the sequence of Picard
iterations \((\rho^{(n)})_{n \in \mathbb{N}}\) in the following way

\[
\rho^{(0)}_t = \rho_0 \quad \forall t \in [0, T^*], \\
\rho^{(n)}_t = F(\rho^{(n-1)}), \quad n \in \mathbb{N}. \tag{4.6}
\]

Obviously, \(\rho^{(0)} \in B_{T^*, \gamma}(r)\). Therefore, by lemma 4.2, \(\rho^{(n)} \in B_{T^*, \gamma}(r)\) for all \(n \in \mathbb{N}\) and from lemma 4.3 we obtain

\[
||\rho^{(n+k)} - \rho^{(n)}||_{T^*, \gamma} \leq ||\rho^{(1)} - \rho^{(0)}||_{T^*, \gamma} \sum_{i=1}^{k} C^{n+i-1} \leq ||\rho^{(1)} - \rho^{(0)}||_{T^*, \gamma} \frac{C^n}{1-C},
\]

where \(C < 1\) is a positive constant. Therefore \((\rho^{(n)})_{n \in \mathbb{N}}\) defined by (4.6) is a Cauchy sequence. As \(B_{T^*, \gamma}(r)\) is a closed subset of a Banach space, there exists

\[
\lim_{n \to \infty} \rho^{(n)} = \rho \in B_{T^*, \gamma}(r).
\]

Clearly \(F(\rho) = \rho\) and therefore \(\rho_t\) satisfies the integral equation (4.5) for \(t \in [0, T^*]\). Thus it is a local classical solution of (4.4).

Now suppose there is another local classical solution of this equation \(\psi\). Then \(\psi_0 = \rho_0\) and for \(r, \gamma, T^*\) as above, there exists \(T \leq T^*\) such that \(\psi \in B_{T, \gamma}(r)\). However, from lemma 4.3 we have

\[
||\rho - \psi||_{T, \gamma} = ||F(\rho) - F(\psi)||_{T, \gamma} \leq C||\rho - \psi||_{T, \gamma}
\]

for \(C < 1\), which means that

\[
||\rho - \psi||_{T, \gamma} = 0
\]

and thus \(\rho\) is the unique local classical solution. \(\square\)

**Proof of Lemma 4.2**. Take arbitrary \(T, \gamma > 0\) and \(\rho \in B_{T, \gamma}(r)\). Note that

\[
\begin{align*}
h(\rho_t, x) & \geq \langle c_2 \rangle, \\
R_2(\rho_t, x) & \leq \frac{3}{2} ||\rho_t||_{L^\infty}^2 \langle c_1 \rangle + 2 ||\rho_t||_{L^\infty} \langle c_2 \rangle, \\
\rho_t(x) & \leq ||\rho_t||_{L^\infty} \leq e^{\gamma(c_2)t} ||\rho||_{T, \gamma}, \tag{4.7}
\end{align*}
\]
It is obvious that $F$ preserves positiveness of $\rho$. Furthermore, using above estimates and the definition of $B_{T,\gamma}(r)$ we derive

$$
(F(\rho))_t(x) = \rho_0(x) \exp \left( - \int_0^t h(\rho_s, x) \, ds \right)
$$

$$
+ \int_0^t R_2(\rho_s, x) \exp \left( - \int_s^t h(\rho_\sigma, x) \, d\sigma \right) \, ds
$$

$$
\leq ||\rho_0||_{L^\infty} e^{-t(c_2)} + \int_0^t R_2(\rho_s, x) e^{(s-t)(c_2)} \, ds
$$

$$
\leq e^{-t(c_2)} \left[ ||\rho||_{T,\gamma} + \int_0^t \left( \frac{3}{2} c_1 e^{(2\gamma+1)(c_2)s} ||\rho||_{T,\gamma}^2 \right) \, ds \right].
$$

Therefore we obtain

$$
\left| \left| \left( F(\rho) \right)_t \right| \right|_{L^\infty} \leq e^{-t(c_2)} r \left[ 1 + \frac{3(c_1)r}{2(2\gamma+1)(c_2)} \left( e^{(2\gamma+1)(c_2)t} - 1 \right) \right]
$$

$$
+ \frac{2}{\gamma + 1} \left( e^{(\gamma+1)(c_2)t} - 1 \right)\right].
$$

Thus

$$
\left| \left| \left( F(\rho) \right)_t \right| \right|_{T,\gamma} \leq r \sup_{t \in [0,T]} f(t),
$$

where

$$
f(t) = e^{-\gamma+1)(c_2)t} \left[ 1 + \frac{3(c_1)r}{2(2\gamma+1)(c_2)} \left( e^{(2\gamma+1)(c_2)t} - 1 \right) \right]
$$

$$
+ \frac{2}{\gamma + 1} \left( e^{(\gamma+1)(c_2)t} - 1 \right).\right]
$$

Note that $f(0) = 1$. Additionally

$$
f'(t) = -(\gamma+1)(c_2) e^{-(\gamma+1)(c_2)t} \left[ 1 + \frac{3(c_1)r}{2(2\gamma+1)(c_2)} \left( e^{(2\gamma+1)(c_2)t} - 1 \right) \right]
$$

$$
+ \frac{2}{(\gamma + 1)} \left( e^{(\gamma+1)(c_2)t} - 1 \right)\right]
$$

$$
+ e^{-(\gamma+1)(c_2)t} \left[ \frac{3}{2} (c_1)r e^{(2\gamma+1)(c_2)t} + 2(c_2) e^{(\gamma+1)(c_2)t} \right].
$$

and hence

$$
f'(0) = -(\gamma+1)(c_2) + \left( \frac{3}{2} (c_1)r + 2(c_2) \right).\right]
Choosing $\gamma > 1 + \frac{3(c_1)}{2(c_2)}$, we have $f'(0) < 0$, which guarantees the existence of $\tilde{T}$ such that $\sup_{t \in [0,T]} f(t) = 1$. Taking $T = T^*$ for $T^* \in [0,\tilde{T}]$ yields

$$\|F(\rho)\|_{T^*,\gamma} \leq r.$$ 

Therefore $F(\rho) \in B_{T^*,\gamma}(r)$ for $\rho \in B_{T^*,\gamma}(r)$.

\[\square\]

\textbf{Proof of lemma 4.3.} We have

\[
\left( F(\rho) - F(\psi) \right)_t(x) = \rho_0(x) \exp \left( -\int_0^t h(\rho_s, x) \, ds \right) + \int_0^t R_2(\rho_s, x) \exp \left( -\int_s^t h(\rho_\sigma, x) \, d\sigma \right) \, ds \\
- \rho_0(x) \exp \left( -\int_0^t h(\psi_s, x) \, ds \right) - \int_0^t R_2(\psi_s, x) \exp \left( -\int_s^t h(\psi_\sigma, x) \, d\sigma \right) \, ds \\
= D_1 + \int_0^t D_2 \, ds, \tag{4.8}
\]

where

\[
D_1 = \rho_0(x) \left[ \exp \left( -\int_0^t h(\rho_s, x) \, ds \right) - \exp \left( -\int_0^t h(\psi_s, x) \, ds \right) \right]
\]

and

\[
D_2 = \int_0^t \left[ R_2(\rho_s, x) \exp \left( -\int_s^t h(\rho_\sigma, x) \, d\sigma \right) \\
- R_2(\psi_s, x) \exp \left( -\int_s^t h(\psi_\sigma, x) \, d\sigma \right) \right] \, ds.
\]

Take an arbitrary $T^* \in [0,\tilde{T}]$. We have

\[
|D_1| \leq \|\rho_0\|_{L^\infty} (c_1) \int_0^t \|\rho_s - \psi_s\|_{L^\infty} \, ds \leq r(c_1) t e^{\gamma(c_2)t} \|\rho - \psi\|_{T^*,\gamma}. \tag{4.9}
\]

To estimate $|D_2|$, consider two cases. First, suppose

\[
\int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) \, d\sigma \geq 0.
\]
Then

\[ |D_2| \leq \left| R_2(\rho_s, x) \exp \left[ - \int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) d\sigma \right] \right| \]

\[ - \left| R_2(\psi_s, x) \exp \left[ - \int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) d\sigma \right] \right| \]

\[ + \left| R_2(\psi_s, x) \exp \left[ - \int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) d\sigma \right] - R_2(\psi_s, x) \right| \]

\[ \leq \left| R_2(\rho_s, x) - R_2(\psi_s, x) \right| \]

\[ + R_2(\psi_s, x) \left\{ 1 - \exp \left[ - \int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) d\sigma \right] \right\} . \]

In the other case, when \( \int_s^t \left( h(\rho_\sigma, x) - h(\psi_\sigma, x) \right) d\sigma < 0 \), we have analogously

\[ |D_2| \leq \left| R_2(\rho_s, x) - R_2(\psi_s, x) \right| \]

\[ + R_2(\psi_s, x) \left\{ 1 - \exp \left[ - \int_s^t \left( h(\psi_\sigma, x) - h(\rho_\sigma, x) \right) d\sigma \right] \right\} . \]

Note that both \( R_2(\rho_s, x) \) and \( R_2(\psi_s, x) \), as both belong to \( B_{T^*, \gamma}(r) \), undergo the same estimation (cf. (4.7))

\[ R_2(\rho_s, x), R_2(\psi_s, x) \leq 3 \left( c_1 e^{2\gamma(c_2)s_r^2} + 2(c_2) e^{\gamma(c_2)s_r} \right), \]

which allows us to write

\[ |D_2| \leq \left| R_2(\rho_s, x) - R_2(\psi_s, x) \right| + \left( \frac{3}{2} \left( c_1 e^{2\gamma(c_2)s_r^2} + 2(c_2) e^{\gamma(c_2)s_r} \right) \right) \]

\[ \times \left\{ 1 - \exp \left[ - \int_s^t \left| h(\psi_\sigma, x) - h(\rho_\sigma, x) \right| d\sigma \right] \right\} . \] (4.10)

We have

\[ 1 - \exp \left[ - \int_s^t \left| h(\psi_\sigma, x) - h(\rho_\sigma, x) \right| d\sigma \right] \leq \int_s^t \left| h(\psi_\sigma, x) - h(\rho_\sigma, x) \right| d\sigma \]

\[ = \frac{1}{2} \int_s^t \int_\mathbb{R}^2 \left( c_1(x, y; z) + c_1(y, x; z) \right) \left( \rho_\sigma(y) - \psi_\sigma(y) \right) dydz \, d\sigma \]
\[
\begin{align*}
&\leq \int_s^t \|\rho_\sigma - \psi_\sigma\|_{L^\infty} \, d\sigma \leq \int_s^t \langle c_1 \rangle e^{(c_2)\sigma} \|\rho - \psi\|_{T^*_{\mathcal{R},\gamma}} \, d\sigma \\
&\leq \langle c_1 \rangle e^{(c_2)t} \|\rho - \psi\|_{T^*_{\mathcal{R},\gamma}},
\end{align*}
\]
which yields
\[
1 - \exp \left[ - \int_s^t \left| h(\psi_\sigma, x) - h(\rho_\sigma, x) \right| \, d\sigma \right] \leq \langle c_1 \rangle t e^{(c_2)t} \|\rho - \psi\|_{T^*_{\mathcal{R},\gamma}} \quad (4.11)
\]
Let us estimate
\[
\left| R_2(\rho_s, x) - R_2(\psi_s, x) \right|
\leq \frac{1}{2} \int_{(\mathbb{R}^d)^2} c_1(y, z; x) \left| \exp \left( - \int_{\mathbb{R}^d} \phi_1(x - u) \rho_s(u) \, du \right) \rho_s(y) \rho_s(z) \\
- \exp \left( - \int_{\mathbb{R}^d} \phi_1(x - u) \psi_s(u) \, du \right) \psi_s(y) \psi_s(z) \right| \, dy \, dz \\
+ \frac{1}{2} \int_{(\mathbb{R}^d)^2} (c_1(x, y; z) + c_1(y, x; z)) \left| \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_1(z - u) \rho_s(u) \, du \right) \right] \rho_s(x) \rho_s(y) \\
\times \rho_s(x) \rho_s(y) - \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_1(z - u) \psi_s(u) \, du \right) \right] \psi_s(x) \psi_s(y) \right| \, dy \, dz \\
+ \int_{\mathbb{R}^d} c_2(y; x) \left| \exp \left( - \int_{\mathbb{R}^d} \phi_2(x - u) \rho_s(u) \, du \right) \rho_s(y) \\
- \exp \left( - \int_{\mathbb{R}^d} \phi_2(x - u) \psi_s(u) \, du \right) \psi_s(y) \right| \, dy \\
+ \int_{\mathbb{R}^d} c_2(x; y) \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_2(y - u) \rho_s(u) \, du \right) \right] \rho_s(x) \\
- \left[ 1 - \exp \left( - \int_{\mathbb{R}^d} \phi_2(y - u) \psi_s(u) \, du \right) \right] \psi_s(x) \right| \, dy.
\]
Denote by $I_i$ the $i$-th component of the RHS of the above inequality for $i = 1, 2, 3, 4$. Then estimating analogously as above we derive
\[
I_3, I_4 \leq \langle c_2 \rangle \left( e^{2\gamma(c_2)\sigma} \langle \phi_2 \rangle r + e^{\gamma(c_2)\sigma} \right) \|\rho - \psi\|_{T^*_{\mathcal{R},\gamma}}.
\]
Moreover, noting that
\[
\left| \rho_s(y) \rho_s(z) - \psi_s(y) \psi_s(z) \right| \\
\leq \frac{1}{2} \left( \rho_s(z) + \psi_s(z) \right) \left| \rho_s(y) - \psi_s(y) \right| + \frac{1}{2} \left( \rho_s(y) + \psi_s(y) \right) \left| \rho_s(z) - \psi_s(z) \right|,
\]
we obtain
\[
I_1 \leq \frac{1}{2} (c_1) \left( 2e^{2\gamma(c_2)sT} + e^{3\gamma(c_2)sT} \right) \left\| \rho - \psi \right\|_{T^*, \gamma},
\]
\[
I_2 \leq (c_1) \left( 2e^{2\gamma(c_2)sT} + e^{3\gamma(c_2)sT} \right) \left\| \rho - \psi \right\|_{T^*, \gamma}.
\]

Therefore
\[
\left| R_2(\rho_s, x) - R_2(\psi, x) \right| \leq \frac{3}{2} (c_1) e^{\gamma(c_2)sT} \left( e^{2\gamma(c_2)sT} + e^{3\gamma(c_2)sT} \right) \left\| \rho - \psi \right\|_{T^*, \gamma}.
\] 

Substituting (4.11) and (4.12) into (4.10) and using it together with (4.9), we obtain (cf. [L8])
\[
\left| \left( F(\rho) - F(\psi) \right)_t \right| (x) \leq e^{\gamma(c_2)t} f(t) \left\| \rho - \psi \right\|_{T^*, \gamma},
\]
where
\[
f(t) = t \left[ \frac{3}{2} T^2 (c_1) e^{2\gamma(c_2)t} \left( (c_1)t + \langle \phi_1 \rangle \right) + re^{\gamma(c_2)t} \left( 2(c_1) \langle c_2 \rangle t + 3(c_1) + 2(c_2) \langle \phi_2 \rangle + 2(c_2) \right) \right].
\]

Therefore
\[
\left| \left| F(\rho) - F(\psi) \right| \right|_{T^*, \gamma} \leq \sup_{t \in [0, T^*]} f(t) \left\| \rho - \psi \right\|_{T^*, \gamma}
\]
Note that \( f(t) \) is continuous, increasing function of \( t \) and \( f(0) = 0 \). Thus, there exists \( T^{**} > 0 \) such that \( f(T^{**}) < 1 \) and \( f(t) \in [0, f(T^{**})] \) for \( t \in [0, T^{**}] \). Choosing \( T^* = \min(T^{**}, T) \), we obtain
\[
\left| \left| F(\rho) - F(\psi) \right| \right|_{T^*, \gamma} \leq C \left\| \rho - \psi \right\|_{T^*, \gamma}
\]
with \( C = f(T^*) \leq f(T^{**}) < 1 \). □

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