Topological defects as inhomogeneous condensates in Quantum Field Theory:
Kinks in $(1 + 1)$ dimensional $\lambda \psi^4$ theory

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We study topological defects as inhomogeneous (localized) condensates of particles in Quantum Field Theory. In the framework of the Closed–Time–Path formalism, we consider explicitly a $(1+1)$ dimensional $\lambda \psi^4$ model and construct the Heisenberg picture field operator $\psi$ in the presence of kinks. We show how the classical kink solutions emerge from the vacuum expectation value of such an operator in the Born approximation and/or $\lambda \to 0$ limit. The presented method is general in the sense that applies also to the case of finite temperature and to non–equilibrium; it also allows for the determination of Green’s functions in the presence of topological defects. We discuss the classical kink solutions at $T \neq 0$ in the high temperature limit. We conclude with some speculations on the possible relevance of our method for the description of the defect formation during symmetry–breaking phase transitions.

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I. INTRODUCTION

In recent years much attention has been devoted to the study of topological defects and to the issue of their formation in the course of symmetry–breaking phase transitions [1,2]. This interest arises essentially for two reasons. On one hand, topological defects are naturally inherent to many dynamical systems ranging from cosmology to condensed matter physics, and thus they provide a bridge between processes which manifest themselves at very different energies and time scales [3–5]. On the other hand, the study of the formation of defects during phase transitions offers an important way to understand better the underlying non–equilibrium dynamics of quantum fields [6,7].

The basic phenomenological picture of defect formation is presently understood via the Kibble–Zurek mechanism [8]. In many systems, when the phase transition associated with a spontaneous symmetry breaking takes place, some regions of space may remain trapped in the initial symmetric phase. These regions are called topological defects. In this way cosmic strings may have been formed in the early universe; vortices in superfluids and superconductors, and textures in some nematic liquid crystals are also created in a similar fashion. A full understanding of the process of defect formation in the context of Quantum Field Theory (QFT) is however still far to be reached, although recently there has been much progress in this direction [9,10]. There are two fundamental points which make difficult such a task: the emergence of the macroscopic (collective) behavior associated with the defects in question resulting out of the underlying microscopic (quantum) dynamics, and the intrinsic off–equilibrium character of the processes - realistic phase transitions - in which the defects are created.

In the present paper, we address the first question, i.e. we show how a self–consistent description of topological defects can be achieved in QFT in a way that is suitable for the study of their formation in non–equilibrium processes - task which we plan to tackle in our future work. We are inspired by an approach to the description of extended objects (solitons) as condensates of particles in QFT which was developed in 70’s by Umezawa, Semenoff et al. [11,12]. In their approach, known as the boson transformation method, the usual order of argumentation is reversed: solitons emerge naturally as macroscopic objects with the full quantum origin. The classical soliton solutions are then recovered in the $\hbar \to 0$ limit. Here we revisit this method and reformulate it in the framework of the Closed-Time-Path (CTP) formalism [13]. There are several advantages in doing this. First of all, we obtain a closed functional form for the Heisenberg picture field operator in the presence of defects. This is to be compared with the original approach in which such an expression is obtained recursively via the Yang–Feldman equation. The most important feature of the presented formulation relies however on the fact that the CTP formalism allows for a unified treatment of both the
zero temperature and the finite temperature cases. It is also suitable for the treatment of non-equilibrium situations: in this case, as it is well known, the use of CTP is essential \[16\]. Last but not least, our formulation allows to determine Green’s functions in the presence of defects, which can be useful for example for setting up self-consistently the Cauchy data for a non-equilibrium evolution \[17,18\].

In this paper, we consider a toy model system, namely the (1+1) dimensional $\lambda\psi^4$ theory, both at zero and finite temperature. We construct the Heisenberg field operator in the kink sector and recover the known (classical) kink solutions in the classical limit as vacuum expectation values of the field operator. We do this in order to set the mathematical framework and explain techniques which we are going to “loosely replicate” in our further work, where more realistic systems will be considered \[19\].

An interesting consequence of the present method are the recurrence relations we arrive at in Section III, which reveal an intimate connection between the (non-exact) solvability of the (1 + 1) $\lambda\psi^4$ theory and the solutions of Cauchy–Marley’s functional equations. In this paper we do not consider quantum corrections to the classical soliton solutions, although the method here presented allows for a systematic study of this aspect (see comments in Section III), which will be treated elsewhere.

We then go one step further and show how to extend the treatment for a system in thermal equilibrium saturated at some temperature $T$. We show that in the high $T$ limit it is indeed possible to get analytical solutions for the (inhomogeneous) order parameter. A delicate limit $\hbar \to 0$ at finite $T$ is discussed and compared with other existing treatments \[20\]. The results obtained are of general validity and can be applied, for instance, to more realistic higher dimensional cases for the study of restoration of symmetry.

The plan of the paper is as follow: In Section II we briefly review the boson transformation method and the Haag expansion which are the basic ingredients of our method. As we aim to study topological defects in both equilibrium and non-equilibrium media we formulate the whole approach in the CTP formalism.

In Section III we consider the zero-temperature (1+1) $\lambda\psi^4$ theory in the Goldstone phase. We construct the classical kink solutions from the vacuum expectation value of the Heisenberg picture field operator, in the Born approximation and/or the $\lambda \to 0$ limit. These limits are realized via residue calculations of certain functional relations.

In Section IV we extend our study to finite temperature. To outline the basic strategy we consider the case of large $T$: this allows us to obtain in a closed form the solution for the (thermal) order parameter. The high temperature analysis reveals that there cannot be any phase transition at high $T$ but, instead, the system is plagued by spontaneous fluctuations exemplified by the non-analytic behavior in the order parameter. We also show explicitly that one cannot talk about the loopwise expansion as being an expansion in $\hbar$ at finite temperature. This essentially distinguishes our approach from the ones where tree approximations at finite $T$ are considered as equivalent to classical limit.

We conclude in Section V with some comments and speculations on the possible relevance of the presented method for the description of defect formation during phase transition. We also discuss the rôle of the shift function (used for a construction of the inhomogeneous coherent states) as a possible alternative tool for the classification of topological defects.

In the Appendices are clarified some finer mathematical manipulations needed in the main body of the paper.

II. DESCRIPTION OF SOLITONS AS INHOMOGENEOUS PARTICLE CONDENSATES

It has been recognized since long time that due to the breakdown of von Neumann’s uniqueness theorem \[21\], the structure of Quantum Field Theory is extremely rich, allowing for a host of different (mutually unitarily inequivalent) Hilbert spaces for a given dynamics of Heisenberg fields. The choice of the Hilbert space is equivalent to fixing the boundary conditions for the (operatorial) Heisenberg equations. It should be born in mind that a particular choice of boundary conditions specifies the observational particle (or quasi-particle) content of the theory. In the perturbative Lehmann–Symanzik–Zimmermann (LSZ) approach, the boundary conditions are injected via the asymptotic fields (most commonly via the in-fields), and the functional relation between them and the Heisenberg fields is then known as \textit{Haag’s map}. One practically realizes such a map expanding the Heisenberg fields into a (functional) power series
of the asymptotic fields - the Haag’s expansion\[1\].

It is worth to stress that Haag’s expansion is not an operatorial relation, but holds true only in a weak sense, i.e. when matrix elements are considered. As we shall see in Section II.B, this is due to the fact that the Hilbert spaces for the Heisenberg fields and asymptotic fields are mutually unitarily inequivalent.

A. The “boson transformation” method

Starting from these basic observations, an approach to the description of extended objects (solitons) as condensates of particles in QFT was developed in the seventies by Umezawa et al.\[13,14\]. In this approach solitons emerge naturally as macroscopic objects (inhomogeneous condensate) with the full quantum origin, provided one chooses the “right” Hilbert space for the asymptotic fields. The classical soliton solutions are then recovered in the Born approximation.

Let us here briefly recall the main lines of the boson transformation method. To avoid unnecessary difficulties we will ignore, temporary, the renormalization problem. Let us consider the simple case of a dynamics involving one scalar field $\psi$ satisfying the equation of motion:

$$\Lambda(\partial) \psi(x) = J[\psi](x),$$

(1)

where $\Lambda(\partial)$ is a differential operator, $x \equiv (t, x)$ and $J$ is some functional of the $\psi$ field, describing the interaction. Let us now denote the asymptotic field by $\varphi(x)$, satisfying the equation

$$\Lambda(\partial) \varphi(x) = 0.

(2)

Equation (1) can be formally recast in the following integral form (Yang–Feldman equation):

$$\psi(x) = \varphi(x) + \Lambda^{-1}(\partial) * J[\psi](x),$$

(3)

where * denotes convolution. The symbol $\Lambda^{-1}(\partial)$ denotes formally the Green function for the $\varphi(x)$ field. The precise form of Green’s function is specified by the boundary conditions. Eq.(3) can be solved by iteration, thus giving an expression for the Heisenberg field $\psi(x)$ in terms of powers of the $\varphi(x)$ field; this is the Haag expansion (or “dynamical map” in the language of refs.\[13,14\]), which might be formally written as

$$\psi(x) = F[x; \varphi].$$

(4)

As already remarked, such an expression is valid only in a weak sense, i.e. for the matrix elements only. This implies that Eq.(4) is not unique, since different sets of asymptotic fields (and the corresponding Hilbert spaces) can be used in the construction. Let us indeed consider a c–number function $f(x)$, satisfying the same free equations of motion:

$$\Lambda(\partial) f(x) = 0,$$

(5)

then the corresponding Yang–Feldman equation takes the form

$$\psi^f(x) = \varphi + f + \Lambda^{-1}(\partial) * J[\psi^f](x).$$

(6)

The latter gives rise to a different Haag expansion for a field $\psi^f(x)$ still satisfying the Heisenberg equation (1):

$$\psi^f(x) = F[x; \varphi + f].$$

(7)

The difference between the two solutions $\psi$ and $\psi^f$ is only in the boundary conditions. An important point is that the expansion Eq.(7) is obtained from that in Eq.(4), by the space–time dependent translation

$$\varphi(x) \rightarrow \varphi(x) + f(x).$$

(8)

\[1\]The Haag expansion is often defined as a power series of a normal ordered product of the asymptotic fields\[2\]. The connection between the definition we use and the one in\[2\] can be established via the operatorial Wick’s theorem.
Eqs. (8) and (9) express the essence of the so called, *boson transformation theorem* [13]: the dynamics embodied in Eq. (1), contains an internal freedom, represented by the possible choices of the function \( f(x) \), satisfying the free field equation (8). Also observe that the transformation (8) is a canonical transformation since it leaves invariant the canonical form of commutation relations.

The vacuum expectation value of Eq. (3) gives \((\langle 0 \rangle \) denotes the vacuum for the free field \( \phi \))
\[
\phi^f(x) \equiv \langle 0 | \psi^f(x) | 0 \rangle = f + \langle 0 | \left[ \Lambda^{-1}(\partial) * \mathcal{J}[\phi^f](x) \right] | 0 \rangle .
\]  
(9)

Notice that the order parameter \( \phi^f(x) \) is of full quantum nature: the classical solution is obtained by means of the classical or Born approximation, which consists in taking \( \langle 0 | \mathcal{J}[\psi^f]| 0 \rangle = \mathcal{J}[\phi^f] \), i.e. neglecting all contractions of the physical fields. In this limit, \( \phi^f(x) = \lim_{\hbar \to 0} \phi^f(x) \) is the solution of the classical Euler–Lagrangian equation:
\[
\Lambda(\partial) \phi^f_{cl}(x) = \mathcal{J}[\phi^f_{cl}](x).
\]  
(10)

Beyond the classical level, in general, the form of this equation changes. The Yang–Feldman equation (1) describes not only the equations for the order parameter, i.e. Eq. (10), but also, at higher orders in \( \hbar \), the dynamics of one or more quantum physical particles in the potential generated by the macroscopic object \( \phi^f(x) \) [13]. Typical examples of interest include for instance a scattering of quasi–electrons on the Abrikosov vortices in type–II superconductors or scattering of second sound waves (thermal phonons) on the vortices in superfluid \( ^4 \)He.

In refs. [13,14], it is shown that the class of solutions of Eq. (3) which leads to non–trivial (i.e. carrying a non–zero topological charge) solutions of Eq. (1), are those which have some sort of singularity with respect to Fourier transform. These can be either *divergent singularities* or *topological singularities*. The first are associated to a divergence of \( f(x) \) for \( |x| = \infty \), at least in some direction. The second means that \( f(x) \) is not single–valued, i.e. it is path dependent. In both cases, the macroscopic object described by the order parameter, will carry a non–zero topological charge.

It is also interesting to consider the boson transformation at level of states [23]. For this purpose let us write the generator of the field shift (8), as
\[
\varphi^f(x) = e^{-iD} \varphi(x) e^{iD} = \varphi(x) + f(x) \quad \quad D = - \int f(y) \partial^\mu \varphi(y) d\sigma^\mu .
\]  
(11)

with \( d\sigma^\mu = n^\mu d\sigma \) where \( \sigma \) is a flat space–like surface and \( n^\mu \) its normal, both \( x \) and \( y \) belong to \( \sigma \). The action of \( D \) on the vacuum defines a coherent state \( |f \rangle \):
\[
\langle f | \varphi(x) | f \rangle = f(x), \quad |f \rangle = e^{iD} |0 \rangle.
\]  
(12)

The classical soliton solution is then obtained by taking the coherent–state expectation value of the Heisenberg field \( \psi \) in the \( \hbar \to 0 \) limit, i.e.
\[
\lim_{\hbar \to 0} \langle f | \psi(x) | f \rangle = \lim_{\hbar \to 0} \langle 0 | \psi^f(x) | 0 \rangle = \phi^f_{cl}(x).
\]  
(13)

Although \( D \) in (11) is a precise analog of the Weyl operator of the quantum mechanics defining the canonical coherent states [22], the states \( |f \rangle \) defined by (12) are not the usual coherent states. Indeed, due to the fact that \( f(x) \) is not Fourier transformable, \( |f \rangle \) are not eigenstates of the annihilation operator \( a(\mathbf{k}) \) albeit they still do saturate the Heisenberg uncertainty relations. A discussion of the coherent states corresponding to the soliton solutions can be found, for example, in refs. [25,26]. Connection of QFT coherent states with Haag’s theorem is discussed in [27].

### B. The Haag expansion in the Closed–Time Path formalism

Whilst the Yang–Feldman equations are quite involved and do not usually allow to proceed beyond few iterations, the alternative Heisenberg equations seems to be more versatile. This may be seen by considering the bare Heisenberg equations of the motion:
\[
\dot{\psi}_B(x) = i[H, \psi_B(x)]\]  
(14)
\[
\Pi_B(x) = i[H, \Pi_B(x)],
\]  
(15)
where $\Pi_B$ is the bare momentum conjugate to the bare field $\psi_B$ and $H$ is the full Hamiltonian in the Heisenberg picture ($H = \int d^3x \mathcal{H}(x)$). The formal solution of \textcolor{red}{[14]}--\textcolor{red}{[15]} is well known. Assuming that the Heisenberg and interaction pictures coincide at some time $t_i$, we have \textcolor{red}{[28]}:

\begin{align}
\psi_B(x) &= Z_\psi^{1/2} \Lambda^{-1}(t) \psi_{in}(x) \Lambda(t) \\
\Pi_B(x) &= Z_H^{1/2} \Lambda^{-1}(t) \Pi_{in}(x) \Lambda(t) \\
\Lambda(t) &= e^{i(t-t_i)H_{in}} e^{-i(t-t_i)H} \\
\Lambda(t_2) \Lambda^{-1}(t_1) &= U(t_2; t_1) = T \left[ \exp(-i \int_{t_i}^{t_2} d^4x \mathcal{H}_{in}(x)) \right].
\end{align}

$T$ is the usual time ordering and $Z_\psi$, $Z_H$ are the wave–function renormalizations (usually $\Pi \propto \psi$, and so $Z_\psi = Z_H$). Due to the fact that both $\psi_B$ and $\psi_{in}$ satisfy the canonical equal–time commutation relations, the solution $\psi_B$ in \textcolor{red}{[14]} must be understood in a weak sense, i.e. valid for each matrix element separately. If not, we would obtain the canonical commutator between $\psi_B$ and $\Pi_B$ being equal to $iZ_\psi \delta^3(x-y)$ and thus canonical quantization would require that $Z_\psi = 1$. On the other hand, non–perturbative considerations (e.g. the Källen–Lehmann representation) require $Z_\psi < 1$. The solution of this problem is well known \textcolor{red}{[21,29]}: the Hilbert spaces for $\psi_B$ and $\psi_{in}$ are different (unitarily non–equivalent), and the wave function renormalizations $Z_\psi$ and/or $Z_H$ are then “indicators” of how much the unitarity is violated. This conclusion is usually referred as Haag’s theorem \textcolor{red}{[21,31]}.

Let us also mention that the interaction picture evolution operator $U(t_2, t_1)$ alternatively reads \textcolor{red}{[32]}

\begin{align}
U(t_2, t_1) &= T^* \left[ \exp(i \int_{t_1}^{t_2} d^4x L_{in}(x)) \right],
\end{align}

where the symbol $T^*$ is called $T^*$ (or covariant) product \textcolor{red}{[30,32,33]} and $L_{in}$ is the interacting part of the density of the Lagrangian in the interaction picture. Eq.\textcolor{red}{(20)} is valid even when the derivatives of fields are present in $L^I$ (and thus $H^I \neq -L^I$).

Eq.\textcolor{red}{(16)} can be recast into a more useful form, namely

\begin{align}
\psi_B(x) &= Z_\psi^{1/2} U(t_i; t) \psi_{in}(x) U^{-1}(t_i; t) = Z_\psi^{1/2} U(t_i; t_f)U(t_f; t) \psi_{in}(x) U(t; t_i) \\
&= Z_\psi^{1/2} T_C \left[ \psi_{in}(x) \exp \left(-i \int_C d^4x \mathcal{H}_{in}(x) \right) \right] = Z_\psi^{1/2} T_C \left[ \psi_{in}(x) \exp \left(i \int_C d^4x L_{in}(x) \right) \right].
\end{align}

Here $C$ denotes a closed–time (Schwinger) contour, running from $t_i$ to a later time $t_f$; $t \leq t_f$ and back again (see Fig.1). Similarly, $T_C$ denotes the corresponding time–path ordering symbol (analogously for the $T^*_C$ ordering). In the limit $t_i \to -\infty$, we get that $\psi_{in}$ turns out to be the usual in–(or asymptotic) field. As the time $t_f$ is by construction arbitrary, it is useful, from a technical point of view, to set $t_f = +\infty$. Eq.\textcolor{red}{(21)} may be viewed as the Haag expansion of the Heisenberg field $\psi_B$.

Generalization of Eq.\textcolor{red}{(21)} to the case of more fields is straightforward. So, for instance, for the time ordered product of $n$ Heisenberg fields we may write

\begin{align}
T[\psi_B(x_1) \ldots \psi_B(x_n)] &= Z_\psi^{n/2} T_C^n \left[ \psi_{in}(x_1) \ldots \psi_{in}(x_n) \exp \left(i \int_C d^4x L_{in}(x) \right) \right].
\end{align}

When we consider \textit{vacuum expectation value} of Eq.\textcolor{red}{(22)}, we have at our disposal two equivalent representations for the $T = 0$ Green’s functions, namely

\begin{align}
2\text{The } T^* \text{ product is defined in such a way that for fields in the interaction picture it is simply the } T \text{ product with all the derivatives pulled out of the } T \text{-ordering symbol. Evidently, for free fields without derivatives: } T^*[\psi_{in}(x_1) \ldots \psi_{in}(x_n)] = T[\psi_{in}(x_1) \ldots \psi_{in}(x_n)].
\end{align}
\begin{align}
\langle 0 | T(\psi_B(x_1) \ldots \psi_B(x_n)) | 0 \rangle &= Z^{n/2}_\Psi \langle 0 | T_C^\star \left[ \psi_{in}(x_1) \ldots \psi_{in}(x_n) \exp \left( i \int_C d^4 x \mathcal{L}_{in}^I(x) \right) \right] | 0 \rangle \\
&= Z^{n/2}_\Psi \frac{\langle 0 | T_C^\star \left[ \exp \left( i \int_{-\infty}^{\infty} d^4 x \mathcal{L}_{in}^I(x) \right) \right] | 0 \rangle}{\langle 0 | T^\star \left[ \psi_{in}(x_1) \ldots \psi_{in}(x_n) \exp \left( i \int_{-\infty}^{\infty} d^4 x \mathcal{L}_{in}^I(x) \right) \right] | 0 \rangle}. 
\end{align}

Eq. (24) is the well known Gell-Mann–Low formula for Green’s functions. Note that the latter is true only for the vacuum expectation values, due to the stability of the vacuum $|0\rangle$:

$$T^\star \left[ \exp \left( i \int_{-\infty}^{\infty} d^4 x \mathcal{L}_{in}^I(x) \right) \right] | 0 \rangle = \alpha |0\rangle,$$

with $\alpha$ being an eigenvalue (basically a phase factor) of the interaction–picture evolution operator which corresponds to $|0\rangle$. In more general situations where expectation values are taken with respect to a state $|\psi\rangle \neq |0\rangle$, Green’s function cannot be recast in the form (24) and the $T_C$ prescription is obligatory. So for mixed states, $|\psi\rangle \rightarrow \rho$, $\langle \psi | \ldots | \psi \rangle \rightarrow \text{Tr} (\rho \ldots) = \langle \ldots \rangle$ and

$$\langle T(\mathcal{P}[\psi_B]) \rangle = \left\langle T_C^\star \left[ \mathcal{P}_{[\psi_{in}]} \exp \left( i \int_C d^4 x \mathcal{L}_{in}^I(x) \right) \right] \right\rangle,$$

where $\mathcal{P}_{[\ldots]}$ is an arbitrary (generally composite) polynomial in $\psi_B$, and the subscript $r$ suggests that the corresponding renormalization factors are included. Eq. (25) will be of a fundamental importance in our following considerations.

An important special case is that of a system in thermodynamical equilibrium. Then the statistical properties of the system are described by the canonical density matrix (for simplicity we omit from our consideration grand–canonical ensembles). As $\rho \propto e^{-\beta H}$, the density matrix is basically a generator of the (imaginary) time translations. Using Eq. (18) we may then write

$$e^{-\beta H} = e^{-\beta H_0} U(t_i - i\beta, t_i).$$

From this it is evident that one may substitute the full density matrix with the density matrix for the corresponding free system provided one adds to the path $C$ a vertical part running from $t_i - i\beta$ to $t_i$ (see Fig. 2). Advantage of this rather formal step is that the free density matrix is Gaussian and correspondingly $\langle T_C[\psi_{in}(x_1) \ldots \psi_{in}(x_n)] \rangle$ is 0 for $n$ odd and a symmetrized product of the (free) two–point Green’s function if $n$ is even. This is nothing but the thermal (or thermodynamical) Wick’s theorem - basis for a perturbation calculus in QFT at finite temperature. We shall elaborate more on this point in Section IV.

**C. Heisenberg operator in the presence of defects**

Having obtained a compact expression for the Heisenberg field in terms of the interacting fields, we can select the initial–time conditions corresponding to a particular physical situation. In other words (see Section II.A), we select...
the set of asymptotic fields describing the particle content of the theory in question. For instance, in the case of a type–II superconductor we must choose either quasi–electrons, if we are above the critical temperature (normal phase), or bogolons if we are below the critical temperature (superconducting phase) [34]. It is well known that the new vacuum state in the superconducting phase is an (homogeneous) condensate of Cooper’s pairs (BCS state).

Similarly, for paramagnetic–ferromagnetic phase transition in the Heisenberg model we choose particles (atoms) with spin \( \frac{1}{2} \) above the Curie temperature (paramagnetic phase) whilst we choose magnons if we are below the Curie temperature (ferromagnetic phase). The vacuum in the ferromagnetic phase is then a state where all particles have spins aligned in the same direction - an (homogeneous) spin (or Bloch) coherent state.

Let us also mention the important case of the QCD phase transition. Here, in the confined phase (i.e. at energy scale \( \ll \Lambda_{QCD} \)) one usually chooses as asymptotic states hadrons and mesons (i.e. quark bound states). In the deconfined phase, i.e. at temperatures of order \( \sim 0.2 \text{GeV} \sim 10^{12} \text{K} \) where a quark gluon plasma starts to form and the chiral symmetry is restored, one could, in principle, directly work with quarks and gluons as relevant asymptotic fields. In this case, the structure of the confined phase vacuum is still far from being understood. The latter is sometimes identified with the (homogeneous) chiral condensate or with “color” BCS state, etc.

The choice of the initial–time data is therefore an important theoretical probe for the description of phase transitions and their underlying microscopic mechanism. If a phase transition is sufficiently rapid it may happen that stable topological defects are created; examples include vortices in rotating superfluid \( ^4\text{He} \) below the \( \lambda \) point, a rich variety of defects in \(^3\text{He} \), quantum magnetic flux tubes (Abrikosov vortices) in type–II superconductors, disclination lines and other defects in liquid nematic crystals, droplets of an unbroken phase in quark gluon plasma, etc. As we have already mentioned in Section II.A, such topological defects can be generated with a special choice of the initial–time data by means of the boson transformation with a Fourier non–transfomrable shift function \( f \). Loosely speaking, the emergence of topological defects may be seen in the corresponding vacuum state for the asymptotic fields which should be a “suitably chosen” inhomogeneous coherent state.

We can summarize our strategy as follows (see Fig.3): For a given dynamics, the CTP formulation gives a closed (functional) expression for the Heisenberg field operator(s) \( \psi \) in terms of the asymptotic (physical) fields. We then use the boson transformation to introduce the shift function \( f \) controlling the choice of the Hilbert space.

The next step is to consider the order parameter, i.e. \( \phi^f \equiv \langle 0| \psi^f |0 \rangle \). By taking the classical limit \( \phi^f_{cl} = \lim_{\hbar \to 0} \phi^f \), we determine the form of the shift function, say \( \tilde{f} \), corresponding to a particular (classical) soliton solution. This shift function is then used to obtain the Heisenberg field operator in the chosen soliton sector: \( \psi^\tilde{f} \). At this point there are various possibilities:

- calculate quantum corrections to the order parameter, by taking higher orders in the \( \hbar \) expansion of \( \langle 0| \psi^f |0 \rangle \);
- study finite temperature effects on the order parameter, by considering \( \langle \psi^f \rangle_\beta \), where \( \langle ... \rangle_\beta \) stands for thermal average;
- calculate Green’s functions in the presence of defects, both at zero and finite temperature, as \( \langle T[\psi^f \psi^f] \rangle \), etc.

A more ambitious task is the study of the non–equilibrium properties of QFT systems containing defects, e.g. consider phase transitions. This will be the object of our future work.
III. KINKS IN TWO–DIMENSIONAL $\lambda \psi^4$ THEORY AT $T = 0$

In this Section we apply the formal considerations developed above to a specific model, namely the two–dimensional $\lambda \psi^4$ theory at zero temperature. It is well known that Goldstone’s sector of the $\lambda \psi^4$ theory in $(1 + 1)$ dimensions possesses at a classical level a certain class of analytical extended solutions, namely the kink (antikink) solutions [35]. In the following, we derive the classical kink solutions at $T = 0$ and construct the Heisenberg operators in presence of kinks. As a further non–trivial step, we extend in the next Section this analysis to finite temperature.

A. The Heisenberg operator in the presence of a kink

Let us consider the case of a $1 + 1$ dimensional scalar system of the hermitian field $\psi$ with quartic interaction term. We have the following bare Lagrangian (throughout we adopt the Minkowski metric with signature $(+ -)$):

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \psi)^2 - \frac{1}{2} \mu_B^2 \psi^2 - \frac{\lambda_B}{4} \psi^4$$

$$= \frac{1}{2} (\partial_{\mu} \psi)^2 - \frac{1}{2} \mu^2 \psi^2 - \frac{\lambda}{4} \psi^4 + \mathcal{L}_{ct},$$

where $\mathcal{L}_{ct}$ is the counterterm Lagrangian:

$$\mathcal{L}_{ct} = \frac{1}{2} (Z_\psi - 1)(\partial_{\mu} \psi)^2 - \frac{1}{2} \delta \mu^2 \psi^2 - \frac{\lambda}{4} (Z_\lambda - 1) \psi^4.$$

Here we have introduced in the usual way a renormalized field, mass and coupling as: $\psi_B = Z_\psi^{1/2} \psi$, $Z_\psi \mu_B^2 = \mu^2 + \delta \mu^2$ and $\lambda_B = Z^{-2}_\psi Z_\lambda \lambda$. The corresponding renormalized Heisenberg equation of the motion for the field operator $\psi$ is (see e.g. [28])

$$(\partial^2 + \mu^2)\psi(x) = -\lambda [\psi^3(x)].$$

(27)

The squared bracket on the RHS of (27) denotes a renormalized composite operator (or Zimmerman normal ordering) corresponding to an unrenormalized operator $\psi^3$.

In the case when both $\mu^2 < 0$ and $\mu_0^2 < 0$ (i.e. in the Goldstone phase) it is well known that (27) admits at a classical level kink solutions [35,23,36]. We shall omit from the next discussion the Wigner phase (i.e. when $\mu_0^2 > 0$) since it does not enjoy the kink solutions. Let us define the following renormalized quantities
\[ \psi(x) = v + \rho(x) \quad \text{with} \quad v = \sqrt{-\frac{\mu^2}{\lambda}}, \quad -2\mu^2 = m^2, \quad g = \sqrt{2\lambda}. \]  

(28)

The parameter \( v \) represents a tree–level expectation value of \( \psi \) (i.e. the true classical minimum of the potential in (23)). As a consequence of the parameterization (28) we obtain

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \rho)^2 - \frac{1}{2} m^2 \rho^2 - g^2 \rho^4 - \frac{1}{2} m g \rho^3 + \frac{m^4}{8 g^2} + \mathcal{L}_ct, \]

where

\[ \mathcal{L}_ct = \frac{1}{2} (Z_N - 1) (\partial_{\mu} \rho)^2 - \frac{1}{4} \left( (Z_N - 1) \frac{3}{2} m^2 - \delta m^2 \right) \rho^2 - \frac{1}{2} g (Z_N - 1) m \rho^3 - \frac{g^2}{8} (Z_N - 1) \rho^4 \]

\[ - \frac{1}{2} \frac{m}{g} (\delta m^2 - (Z_N - 1) m^2) \rho + \frac{1}{4} \frac{m^2}{g^2} \left( \delta m^2 - \frac{1}{2} (Z_N - 1) m^2 \right). \]

Here we have used that \( \delta m^2 = -2\delta \mu^2 \). The form (29) of the Lagrangian is particularly important in doing perturbative theory. We may recall that both \( m \) and \( g \) in \( \mathcal{L} \) are clearly (by construction) arbitrary parameters and no physical observable should depend on their choice. But, in practice, when one deals with perturbative calculations, choice of one parameterization over another can save some labor and possibly cultivate on the intermediate stage a better physical intuition. Here, and in the work to follow, we explicitly choose the zero–momentum mass renormalization prescription: \( \Sigma_R(\rho^2 = 0) = -m \); with \( \Sigma_R \) being the renormalized self–energy. Similarly, we chose the zero–momenta coupling renormalization condition: \( \Gamma_R^{(4)}[0,0,0,0] = -\lambda \); with \( \Gamma_R^{(4)} \) being the renormalized 4–point proper vertex function. Both \( m \) and \( \lambda \) are chosen to be the tree level mass and coupling, respectively. The virtue of this prescription is that we keep from the very beginning the interpretation of \( v \) as the tree–level vacuum expectation value of \( \psi \). In addition, the zero–momentum renormalization prescription can be easily recast into conditions on the renormalized effective potential which is the most natural tool for a treatment of phase transitions (30,37) (this is particularly useful when comparing with existing results which are predominantly computed in the effective potential framework). The physical, but mathematically more involved on–shell renormalization may be obtained via a finite renormalization procedure (30). Let us also remark that in 2\( D \) case the \( \lambda \psi^4 \) theory is super–renormalizable and so both \( Z_N \) and \( Z_\psi \) are finite to each order in the perturbative expansion, while the divergence present in \( \delta m^2 \) is solely due to the tadpole diagram which is here only logarithmically divergent.

The renormalized Heisenberg equation for the field \( \rho \) is:

\[ (\partial^2 + m^2) \rho(x) = \frac{3}{2} g^2 \rho^2(x) - \frac{1}{2} g^2 \rho^3(x). \]

Note that if \( \mu^2 < 0 \) then \( m^2 > 0 \). The asymptotic \( (t \to -\infty) \) field can be then identified with the free massive field satisfying

\[ (\partial^2 + m^2) \rho_{ct}(x) = 0, \]

(31)

with \( m \) being the aforementioned tree level renormalized mass. Setting in Eq.(18) \( H_{ct}^0 = \frac{1}{2} \rho_{ct}^2 + \frac{1}{2} \rho^2 - m^2 \rho_{ct}^2 \) then the interaction Lagrangian \( \mathcal{L}' \) (entering Eq.(21)) is

\[ \mathcal{L}'[\rho] = \mathcal{L}_{ct}[\rho] - \frac{1}{8} g^2 \rho^4 - \frac{1}{2} m g \rho^3 + \frac{m^4}{8 g^2}, \]

and the dynamical map for the field \( \psi \) has the form

\[ \psi(x) = v + T_{\mathcal{E}}^* \left[ \rho_{ct}(x) \exp \left\{ i \int_C d^2 y \mathcal{L}'[\rho_{ct}] \right\} \right]. \]  

(32)

Note that the constant factor in \( \mathcal{L}' \) automatically cancels during the contour integration\(^3\). Let us now consider the boson transformation:

\(^3\)This fact, among others, shows that in the CTP formalism the UV divergence in the energy density of the vacuum is automatically cancelled.
\[ \rho_{in}(x) \to \rho_{in}(x) + f(x) \]  

(33)

\[ (\partial^2 + m^2)f(x) = 0. \]  

(34)

As a result we get the new Haag expansion for the field \( \psi^f(x) \)

\[ \psi^f(x) = v + T_C \left[ (\rho_{in}(x) + f(x)) \exp \left\{ i \int_C d^2 y \, L^f [\rho_{in}(y) + f(y)] \right\} \right] \]

\[ = v + \left[ \frac{\delta}{i\delta J(x)} + f(x) \right] \exp \left\{ i \int_C d^2 y \, L^f \left[ \frac{\delta}{i\delta J(y)} + f(y) \right] \right\} T_C \left( \exp i \int_C d^2 y \, J(y) \rho_{in}(y) \right) \bigg|_{J=0}, \]

(35)

where we have introduced the \( c \)-number source \( J \) in order to perform some formal manipulations. By use of the (operatorial) Wick’s theorem \[30\], we get

\[ \psi^f(x) = v + \left[ \frac{\delta}{i\delta J(x)} + f(x) \right] \exp \left\{ i \int_C d^2 y \, L^f \left[ \frac{\delta}{i\delta J(y)} + f(y) \right] \right\} \times \exp \left\{ -\frac{1}{2} \int_C d^2 y d^2 z \, J(y) \Delta_C(y; z) J(z) \right\} \bigg|_{J=0}, \]

(36)

where \( \Delta_C(y; z) = \langle 0 | T_C (\rho_{in}(x) \rho_{in}(y)) | 0 \rangle \) and \( | 0 \rangle \) is the vacuum for the \( \rho_{in} \) field. Once the function \( f \) is “properly” chosen (see below), Eq.\( (36) \) represents a convenient representation for the Heisenberg operator in presence of defects which can be used for further analysis \[30\].

In order to determine the function \( f \) leading to kink solutions, let us now consider the vacuum expectation value of the Heisenberg field \( \psi^f \). Here the normal ordered term drops, and we get \( \langle \ldots \rangle \equiv \langle 0 \ldots | 0 \rangle \)

\[ \langle \psi^f(x) \rangle = v + \left[ \frac{\delta}{i\delta J(x)} + f(x) \right] \exp \left\{ i \int_C d^2 y \, L^f \left[ \frac{\delta}{i\delta J(y)} + f(y) \right] \right\} \times \exp \left\{ -\frac{1}{2} \int_C d^2 y d^2 z \, J(y) \Delta_C(y; z) J(z) \right\} \bigg|_{J=0}. \]

(37)

By use of the relation \[30\]

\[ F \left[ \frac{\delta}{i\delta J} \right] G[J] = G \left[ \frac{\delta}{i\delta K} \right] F[K] e^{i \int K J} \bigg|_{K=0}, \]

(38)

we obtain

\[ \langle \psi^f(x) \rangle = v + \exp \left\{ -\frac{1}{2} \int_C d^2 y d^2 z \, \Delta_C(y; z) \frac{\delta}{i\delta K(y)} \frac{\delta}{i\delta K(z)} \right\} \times (K(x) + f(x)) \exp \left\{ i \int_C d^2 y \, [L^f [K(y) + f(y)] + K(y) J(y)] \right\} \bigg|_{K=J=0}. \]

(39)

We now perform a change of variables \( K(x) \to K(x) + f(x) \) and set to zero the term with \( J \) (there are no derivatives with respect to it). As a result we obtain

\[ \langle \psi^f(x) \rangle = v + \exp \left\{ -\frac{1}{2} \int_C d^2 y d^2 z \, \Delta_C(y; z) \frac{\delta}{i\delta K(y)} \frac{\delta}{i\delta K(z)} \right\} K(x) B[K] \bigg|_{K=f}, \]

(40)

with

\[ B[K] \equiv \exp \left\{ i \int_C d^2 y \, L^f [K(y)] \right\}. \]

(41)
We can thus express Eq. \((40)\) as a sum of three terms:

\[
\langle \psi^f(x) \rangle = v + C[K](x)|_{K=f} + D[K](x)|_{K=f}
\]  

\[
C[K](x) = \int_C d^2 y \delta \Delta_C(x; y) \frac{\delta}{\delta K(y)} \exp \left[ \frac{1}{2} \int_C d^2 z d^2 y \frac{\delta^2}{\delta K(z) \delta K(y)} B[K] \right],
\]

\[
D[K](x) = K(x) \exp \left\{ \frac{1}{2} \int_C d^2 x d^2 y \delta \Delta_C(x; y) \frac{\delta^2}{\delta K(x) \delta K(y)} \right\} B[K].
\]

Let us observe that \(C[K = 0](x)\) is independent on \(x\) since the inhomogeneity in the order parameter is controlled by the \(x\) dependence in \(K(x)\) (or precisely in \(f(x)\)). As a result we see that \(C[0]\) describes quantum corrections to the tree level homogeneous density of condensation \(v\) (note \(D[K = 0] = 0\)). Introducing the notation:

\[
C[K](x) \equiv C[K](x) - C[0], \quad \tilde{v} = v + C[0],
\]

with \(\tilde{v}\) being the renormalized order parameter, we arrive at the following alternative form for the order parameter:

\[
\langle \psi^f(x) \rangle = \tilde{v} + C[K](x)|_{K=f} + D[K](x)|_{K=f}.
\]

Note that both \(D[K]\) and \(C[K]\) now vanish when we set \(K = 0\), which is equivalent to considering the homogeneous condensation only. In the literature, \(C[0]\) is often denoted as \(\delta v\).

**B. The kink solution in the Born approximation**

So far, all result obtained were of a full quantum nature. We now deal with the Born, or classical, approximation of Eq.\((40)\) and for this purpose we reintroduce \(h\). Each propagator is then proportional to \(h\) whilst \(B\) has in the exponent the factor \(ih^{-1}\). The Born approximation means that only terms of order \(h^0\) in \(\langle 40 \rangle\) must be taken into account. In order to better understand the classical limit, let us view the field theory as a classical physicist does: one would then talk about a frequency \(\omega_0\) and a wave vector \(k\) of the field rather than about mass \(m\) and momentum \(p\). In QFT, the particle mass and momentum are obtained by multiplying \(\omega_0\) and \(k\) by \(\hbar\) (we do set \(c = 1\)). The units of \(\lambda = g^2/2\) are \([E^3 l^{-3}]\) where \(E\) means units of energy and \(l\) means units of length, so \([\lambda] = [h^{-1} l^{-2}]\). Note that the latter also implies a super-renormalizability of the theory. Similarly, the Fourier transform is expressed in terms of the product \(kx\) and not \(px\). These comments, even if trivial, will be of a particular importance in the finite temperature case where only a carefully performed classical limit can provide consistent results. For instance, in massless theories, a correct classical limit reveals the breakdown of classical perturbative expansion (infrared catastrophe) and then some resummation is required - the so called hard thermal loops (HTL) resummation \([38,39]\).

In rewriting some of the previous expressions making the \(h\) dependence explicit, we recall that the counterterm Lagrangian \(L_{ct}\) is of order \(O(h)\) and so in the expansion of \(C[f]\) and \(D[f]\) terms one always gets loop contributions. It is obvious that in doing classical limit such contributions must decouple and so in the following reasonings we may neglect \(L_{ct}\). We have \(\footnote{We use the identity}
\[
\exp \left[ \frac{1}{2} \sum_{ij} \Delta_{ij} \partial x_i \partial x_j \right] x_k B(x_k) = \left( x_k + \sum_j \Delta_{kj} \partial x_j \right) B(x_k + \sum_j \Delta_{kj} \partial x_j) 1,
\]

where \(x_k \rightarrow K(x)\) and \(B(x) \rightarrow B[K]\).

\(\footnote{After factorizing out \(h\), we shall in the following consider \(\Delta_C(\ldots) \propto h^0\).} \)
with
\[ a = \frac{1}{2} \int_C d^2 z d^2 y \frac{\Delta_C(z; y)}{\delta K(z) \delta K(y)} \delta^2 \]
\[ b = -\frac{i}{2} \int_C d^2 z \left[ \frac{g^2}{4} K^4(z) + \omega g K^3(z) \right]. \]

Keeping only the finite terms in the \( \hbar \to 0 \) limit we get\(^6\)
\[ C[K](x, h \to 0) = \int_C d^2 y \Delta_C(x; y) \frac{\delta}{\delta K(y)} \lim_{\hbar \to 0} \left( \exp \left[ \frac{\hbar a}{b} \right] \right) \]
\[ = \int_C d^2 y \Delta_C(x; y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^{n} b^{n+1} \]
\[ = i \int_{-\infty}^{\infty} d^2 y G_R(x, y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^{n} b^{n+1}, \]
where \( iG_R(x, y) = \theta(x_0 - y_0) \Delta(x; y) \) is the (position space) retarded Green’s function of the free theory and \( \Delta(x; y) \) is the Pauli–Jordan function:
\[ \Delta(x_0, x_1; 0, 0) = \langle 0 | \rho_{\infty}(x), \rho_{\infty}(0) | 0 \rangle = \int \frac{d^2 k}{(2\pi)} \delta(k^2 - \omega_0^2) \delta(k_0) e^{-ikx} = -\frac{i}{2} \theta(x_0 - |x_1|) J_0(\omega_0 \sqrt{x_0^2 - x_1^2}). \]

The \( D \) term gives
\[ D(x, h \to 0) = K(x) \lim_{\hbar \to 0} \left( \frac{1}{\hbar} \exp \left[ \frac{\hbar a}{b} \right] \right) = K(x) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^{n} b^{n}. \]

In Appendix A we show that \( D[K, h \to 0] = K \). The final result is then
\[ \langle \psi'_0(x) \rangle = v + f(x) + i \int_{-\infty}^{\infty} d^2 y G_R(x, y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^{n} b^{n+1} \bigg|_{K=f} \]
\[ = v + \sum_{n=1}^{\infty} P_n[K](x) \bigg|_{K=f}, \quad (54) \]
where
\[ P_1(x) = K(x) \]
\[ P_n(x) = \frac{1}{[(n-2)!]^2} \int_{-\infty}^{\infty} d^2 y G_R(x, y) a^{n-2} \left\{ \frac{3}{2} \omega_0 g K^2(y) + \frac{1}{2} g^2 K^3(y) \right\} b^{n-2}; \quad n \geq 2. \]

In Appendix C we show (using mathematical induction) that the following recurrence relation holds:
\[ P_n(x) = \int_{-\infty}^{\infty} d^2 y G_R(x, y) \left[ \frac{3}{2} \omega_0 g \sum_{i+j=n} P_i(y) P_j(y) + \frac{1}{2} g^2 \sum_{i+j+k=n+1} P_i(y) P_j(y) P_k(y) \right]; \quad n \geq 2, \quad (56) \]
where \( i, j, k = 1, 2, \ldots \). The recurrence condition (56) may be “diagonalized” in the following way

---

\(^6\)Throughout this paper \( \hbar \to 0 \) is a short-hand notation for \( \hbar \to 0_+ \).
\[ Q_1(x) = K(x) \]
\[ Q_2(x) = \int_{-\infty}^{\infty} d^2 y \frac{3}{2v} G_R(x, y) \sum_{i+j=2} Q_i(y) Q_j(y) \]
\[ Q_n(x) = \int_{-\infty}^{\infty} d^2 y G_R(x, y) \left[ \frac{3}{2} \omega_0 g \sum_{i+j=n} Q_i(y) Q_j(y) + \frac{1}{2} \sigma^2 \sum_{i+j+k=n} Q_i(y) Q_j(y) Q_k(y) \right] ; \quad n \geq 3. \]

with
\[ Q_n = P_1 P_{n+1} + P_n. \]

Note that both \( \sum_{i=1}^{\infty} P_i \) and \( \sum_{i=1}^{\infty} Q_i \) lead to the same result and so it does not really matter which recurrence equation will be solved. The formulation of the problem in terms of recurrence conditions (57) has however one crucial advantage, namely Eq. (57) belongs to the class of the so-called functional equations of Cauchy–Marley’s type \cite{40,41} where it is known that the fundamental solution cannot be expressed (apart from a very narrow class of kernels) in terms of elementary functions. This means that we cannot resolve (57) in terms of general \( K \) (or \( f \)), i.e. we cannot find the Bäcklund transformation between the solutions of Eq. (34) and those of Eq. (64) - the classical Euler–Lagrange equation of motion (see below). Nevertheless we can obtain all the analytical solutions of Eq. (64), namely the analytical kinks: This is done once we realize that the convolution of the 2D retarded Green’s function \( G_R(x) \) with an exponential is proportional to the very same exponential (actually the exponential is the only function having this property - see Appendix C). So we choose \( Q_n(x) \propto (Q_1(x))^n \) with \( Q_1(x) = f_0(x) \) being an exponential, Fourier non–transformable solution of the Klein–Gordon equation \cite{44}. We obtain:
\[ Q_n(x) = A_n f_0^n(x) = A_n e^{\pm \omega_0 n \gamma (x_1 - x_0 v)}, \]

where \( \gamma = (1 - u^2)^{-\frac{1}{2}} \) (\( u \) will be later interpreted as a velocity of a kink). Plugging this form into the recurrence relation (57) and using the result from Appendix C we arrive at the following equation for the factor \( A_n \)
\[ A_n = \frac{1}{(n^2 - 1)} \left\{ \frac{3}{2v} \sum_{i+j=n} A_i A_j + \frac{1}{2 \sigma^2} \sum_{i+j+k=n} A_i A_j A_k \right\}. \]

This is a trivial version of Cauchy–Marley’s equation which has the only (non–zero) fundamental solution, namely \( A_n \propto (A_1)^n \). Using the standard identities:
\[ \sum_{i+j=n} 1 = n - 1 \quad ; \quad \sum_{i+j+k=n} 1 = \frac{1}{2} (n - 1) (n - 2), \]

it might be easily checked that a solution of the recurrence relation \( (57) \) reads
\[ A_n = 2v \left( \frac{s}{2v} \right)^n, \]

with \( c \) being a real constant. Thus, finally, we have
\[ \langle \psi_0^f(x) \rangle = v + 2v \sum_{n=1}^{\infty} \left( \frac{s f_0(x)}{2v} \right)^n = v \cosh \left[ \frac{1}{2} \ln \left( \frac{s f_0(x)}{2v} \right) \right]. \]

Here \( \ln(z) = \ln|z| + i \arg z \) is the usual principal value of the logarithm of \( z \). Note that \( K = f = s f_0 \). Thus, provided \( f(x) \) is an exponential solution of the linear equation \cite{44}, the solution \( (58) \) fulfils the (classical) Euler–Lagrange equation of motion:

\[ X(x) - f(x) = \int_{-\infty}^{\infty} d^2 y G_R(x, y) \left[ \frac{3}{2} \omega_0 g X^2(y) + \frac{1}{2} \sigma^2 X^3(y) \right]. \]
It is quite instructive to realize what happens if we choose

\[ s \]

the Heisenberg operator in a given kink sector. One can thus calculate Green’s functions and higher quantum corrections [45]. This asserts that a system behaves classically if its de Broglie wavelength \( \lambda_D \) is much smaller than the characteristic length scale \( r_0 \) of the theory: at this stage the wave properties of a system become unimportant. In our case the kink (antikink) de Broglie wavelength is \( \lambda_D = 2\pi \hbar /|p| = 2\pi \hbar /\gamma M_{cl}|u| \). On the other hand, \( r_0 \) in the theory can be identified with the kink (antikink) width (= \( 1/\gamma \omega_0 \)). So

\[ \lambda_D \ll r_0 \Rightarrow \hbar \lambda \ll \omega_0^2 |u| \iff \hbar g^2 \ll \omega_0^2 |u| \]  

(70)

The later is nothing but the expectation value of equation (27) in the Born approximation.

For instance, if we choose \( f_0(x) = e^{-\omega_0 \gamma(x_1-x_0u)} \) with \( s = -2\nu e^{\omega_0 \gamma a} \), we easily obtain the standard kink solution [12, 13]

\[ \langle \psi_0^f(x) \rangle = v \operatorname{th} \left[ \frac{\omega_0}{2} \gamma ((x_1-a) - x_0u) \right] \]  

(65)

describing a constantly moving kink of a permanent profile with a center localized at \( a + xu_0 \). Note that the former function \( f(x) \) is the solution of the homogeneous Klein–Gordon equation [57], is not Fourier transformable and fulfills the initial value condition: \( f(x_0 \to -\operatorname{sgn}(u)\infty, x_1 \to 0) \).

As the Lagrangian (26) is \( Z_2 \) invariant we could equally choose \( \langle 0|\psi(x)|0 \rangle = -v \). In this case we would get

\[ \langle \psi_0^f(x) \rangle = -v \operatorname{th} \left[ \frac{\omega_0}{2} \gamma ((x_1-a) - x_0u) \right] \]  

(66)

which is the antikink solution. Note that the antikink can be also alternatively obtained choosing \( f_0(x) = e^{+\omega_0 \gamma(x_1-x_0u)} \) and \( s = 2\nu e^{-\omega_0 \gamma a} \), provided we keep \( \langle 0|\psi(x)|0 \rangle = v \).

It is quite instructive to realize what happens if we choose \( s \) positive, i.e. \( s = 2\nu e^{\omega_0 \gamma a} \). In this case we obtain

\[ \langle \psi_0^f(x) \rangle = \pm v \operatorname{cth} \left[ \frac{\omega_0}{2} \gamma ((x_1-a) - x_0u) \right] \]  

(67)

(\( \pm \) sign depends on the choice of \( f_0(x) \)) which is also solution of Eq.(64). However, this solution is singular in \( x_1 = a = xu_0 \) and so it does not classify as a soliton even if, for example, the corresponding energy density

\[ \mathcal{H}_{cl}(x) = \frac{1}{2} \left( \langle \psi_0^f(x) \rangle \right)^2 + \frac{1}{2} \left( \langle \psi_0^f(x) \rangle \right)^2 - \frac{1}{4\omega_0^2} \langle \psi_0^f(x) \rangle^2 + \frac{\lambda}{4} \langle \psi_0^f(x) \rangle^4 + \frac{1}{8} \omega_0^2 v^2 \]

\[ = \frac{1}{4(1-u^2)} v^2 \omega_0^2 \operatorname{csch}^4 \left[ \frac{\omega_0}{2} \gamma ((x_1-a) - x_0u) \right] \]  

(68)

is localized near \( x_1 = xu_0 + a \), and the corresponding total energy

\[ E_{cl} = \int_{-\infty}^{\infty} dx_1 \mathcal{H}_{cl}(0, x_1) = \frac{\omega_0^3}{3\lambda} \gamma = M_{cl} \gamma , \]  

(69)

is finite (\( M_{cl} \) denotes the classical (static) kinkmass).

We thus see that, in the classical limit, the present method gives us a constructive way for finding classes of analytical solutions of non–linear equations. The fact that we could not find the fundamental solution of Eq.(54) is intimately connected with the theory of Cauchy–Marley’s functional equations. On the other hand the non–existence of the fundamental solution is fully consistent with the fact that \( (1 + 1) \) dimensional \( \lambda \psi^4 \) theory is not exactly solvable.

The above solutions for the shift function \( f \) corresponding to kink solutions can be now used in Eq.(36) to get the Heisenberg operator in a given kink sector. One can thus calculate Green’s functions and higher quantum corrections to \( \phi^f \). We show this elsewhere [55, 57].

C. The kink solution in the small coupling limit

We can obtain the classical limit of the order parameter [13] using a different standpoint, namely the semiclassical (or WKB) approximation [13]. This asserts that a system behaves classically if its de Broglie wavelength \( \lambda_D \) is much smaller than the characteristic length scale \( r_0 \) of the theory: at this stage the wave properties of a system become unimportant. In our case the kink (antikink) de Broglie wavelength is \( \lambda_D = 2\pi \hbar /|p| = 2\pi \hbar /\gamma M_{cl}|u| \). On the other hand, \( r_0 \) in the theory can be identified with the kink (antikink) width (= \( 1/\gamma \omega_0 \)). So

\[ \lambda_D \ll r_0 \Rightarrow \hbar \lambda \ll \omega_0^2 |u| \iff \hbar g^2 \ll \omega_0^2 |u| \]  

(70)
Note that the above condition can be fulfilled in all Lorentz frames of reference apart from the ones where the kink (antikink) is close to or exactly at rest. This has a simple physical meaning: the semiclassical approximation is based on the assumption that typical momentum and distance scales are much larger than the error scales saturating Heisenberg’s uncertainty relations. This assumption naturally fails when the typical momentum scale is close to zero and correspondingly for small $|u|$ the semiclassical approximation cannot be used.

Eq. (70) is fulfilled either if $\hbar \rightarrow 0$ (see previous subsection) or if $\lambda \rightarrow 0$, i.e. in the heavy mass approximation. We study the latter case in Appendix D: there we show that the small coupling limit, leads to the analytical kink (antikink) solutions of the classical Euler–Lagrange equation \([54]\). We stress once again that the classical soliton solutions obtained both in the previous and present subsections were determined as a classical limit of the full quantum mechanical expression for the order parameter \([10]\).

**IV. KINKS IN TWO–DIMENSIONAL $\lambda \psi^4$ THEORY AT $T \neq 0$**

So far we have dealt with QFT at zero temperature. However, in many physical situations, the zero temperature approximation is not appropriate. This happens for example in cosmology \([46–49]\), astrophysics \([50–52]\) and in the study of the quark–gluon plasma formation \([53–55]\) as well as in condensed matter. More in general, in all cases when one studies systems near the critical temperature (and so neither low or high temperature expansions properly describe the basic features) the finite–temperature treatment must be carefully taken into account.

It is important therefore to understand what happens with the topological defects when the QFT system is immersed in a heat bath (thermal reservoir) which is saturated at some temperature $T$. In this Section we aim to use our toy–model system to address this question and demonstrate the basic features of our approach in the finite–temperature case. Most of the results here obtained retain their validity when more realistic (higher dimensional) systems are considered.

As already discussed in Section II.B, in thermal equilibrium, the most convenient choice for the time path is the one in Fig.2, to which we refer as the thermal path. The crucial observation at finite temperature is that the operatorial Wick’s theorem still holds (see e.g. \([54]\)) and consequently Eq. (55) retains its validity provided the following substitutions are performed:

$$\Delta_C(x; y) = \langle 0 | T_C(\rho_{in}(x)\rho_{in}(y)) | 0 \rangle \rightarrow \Delta_C(x; y, T) = \langle T_C(\rho_{in}(x)\rho_{in}(y)) \rangle_{\beta}$$

$$\vdots:\vdots\vdots\vdots\vdots\rightarrow N(\ldots),$$

where $\langle \ldots \rangle_{\beta} = Tr(\ e^{-\beta H} \ldots \ ) / Tr(\ e^{-\beta H})$ and $\beta = 1/T$. The thermal normal ordering $N(\ldots)$ is defined is such a way \([54]\) that $\langle N(\ldots) \rangle_{\beta} = 0$, the dots stands for a product of $T = 0$ free fields. This is of a great importance as all the formal considerations developed in Section III.A go through also for finite $T$.

**A. The kink solution at finite $T$ in the classical approximation**

The classical approximation has proved to be a useful tool for study of low–energy and/or high–temperature properties of quantum fields at finite temperature. Examples include non–perturbative computations of the Chern–Simons diffusion rate \([22]\), sphaleron decay \([58]\) or plasmon properties of hot non–abelian gauge theories \([59]\); there is also a perturbative treatment via the classical Feynman diagrams \([60]\) which however will not be followed here.

At nonzero temperature the question of $\hbar$ appearance is more delicate than in the zero–temperature case. The whole complication is hidden in the thermal propagator $\Delta_C(x; y, T)$. Whilst at $T = 0$ the latter is directly proportional to $\hbar$, at finite $T$ the situation is very different. Although this generic observation appears simple, there has been in the past confusion in the literature about this point which was understood properly only recently \([58,59]\). To understand the complications involved let us make $\hbar$ explicit. The free thermal propagator in spectral or Mills’s representation \([22,23]\) then reads

$$\Delta_C(x; y, T) = \hbar \int \frac{d^2k}{(2\pi)^2} e^{-ik(x-y)} \rho(k)[\theta_C(x_0 - y_0) + f_{\hbar}(\hbar k_0/T)]$$

$$= \Delta_C(x; y) + \Delta_C^T(x; y),$$

(71)
with
\[ \Delta_C(x; y) = \hbar \int \frac{d^2 k}{(2\pi)^2} e^{-ik(x-y)} \rho(k) [\theta_C(x_0 - y_0) - \theta(-k_0)] , \quad (72) \]
where the spectral density \( \rho(k) = (2\pi)\varepsilon(k_0) \delta(k^2 - \omega_0^2) \) with \( \varepsilon(k_0) = \theta(k_0) - \theta(-k_0) \). The contour step function \( \theta_C(x_0 - y_0) \) is 1 if \( y_0 \) precedes \( x_0 \) along the contour \( C \). The Bose–Einstein distribution \( f_b(x) = (e^x - 1)^{-1} \). It might be directly checked that we obtain the usual elements of the (free) thermal propagator in the so called Keldysh–Schwinger formalism \([33,39]\), i.e.:
\[ \Delta_{11}(x; y, T) = \hbar \int \frac{d^2 k}{(2\pi)^2} e^{-ik(x-y)} \left\{ \frac{i}{k^2 - \omega_0^2 + i\varepsilon} + 2\pi \delta(k^2 - \omega_0^2) f_b(h|k_0|/T) \right\} \]
\[ \Delta_{21}(x; y, T) = \hbar \int \frac{d^2 k}{(2\pi)^2} e^{-ik(x-y)} \{ \theta(k_0) + f_b(h|k_0|/T) \} \delta(k^2 - \omega_0^2) \]
\[ \Delta_{22}(x; y, T) = (\Delta_{11}(x; y, T))^* \]
\[ \Delta_{12}(x; y, T) = (\Delta_{21}(x; y, T))^* , \]

where "*" denotes complex conjugation and the particle mass \( m = \hbar \omega_0 \). We remark that the thermal part of \( \Delta_C(x; T) \) is identical for all matrix elements and that in (73) it appears \( f_b(h|k_0|/T) \) and not \( f_b(h|k_0|/T) \). We should also emphasize that \( k \) in the integration is a wave vector - a reciprocal length - and not a momentum.

Now, due to the mentioned analogy with the \( T = 0 \) situation, we may immediately write for the order parameter
\[ \langle \psi^f(x) \rangle_\beta = v + C[K](x; T)|_{K=f} + D[K](x; T)|_{K=f} , \quad (74) \]
where we took the thermal average of the expression analogous to the one in Eq.(36), but with the normal ordering and the propagator replaced with their thermal counterparts provided one uses the thermal propagator instead of \( \Delta_C(x; y) \).

Let us consider the classical limit of Eq.(74). The \( D[f](x; T) \) term then gives:
\[ D[f](x; T, h \to 0) = f(x) Res_{h=0} \left[ \frac{1}{\hbar} \exp \left( \frac{\hbar}{2} \int_C d^2 z d^2 y \Delta_C(z; y, T) \frac{\delta^2}{\delta K(z)\delta K(y)} \right) \times \exp \left( -\frac{i}{2\hbar} \int_C d^2 z \left[ \frac{\hbar^2}{4} K^4(z) + \omega_0 g K^3(z) \right] \right) \right]_{K=f} \]
\[ = f(x) Res_{h=0} \sum_{n,m=0} \left[ \hbar^{n-m-1} n! m! \left( \frac{1}{2} \int_C d^2 z d^2 y \Delta_C(z; y, T) \frac{\delta^2}{\delta K(z)\delta K(y)} \right)^n b^m \right] \] \( k = f \). (75)

Note that if \( \Delta^T_C(x; y) = 0 \) we recover the result \([28]\) of the previous section. Using the result of Appendix A we may directly write
\[ D[f](x; T, h \to 0) = f(x) Res_{h=0} \sum_{n,m=0} \left[ \hbar^{n-m-1} n! m! \delta_{n0} \delta_{m0} \right] = f(x) . \quad (76) \]

For the \( C[f](x; T) \) term we get, in the classical limit:
\[ C[f](x; T, h \to 0) \]
\[ = \int_C d^2 y \Delta_C(x, y) \frac{\delta}{\delta K(y)} Res_{h=0} \sum_{n,m} \left[ \hbar^{n-m-1} n! m! \left( \frac{1}{2} \int_C d^2 z d^2 w \Delta_C(z; w, T) \frac{\delta^2}{\delta K(z)\delta K(w)} \right)^n b^m \right] \] \( k = f \). (77)

There is no simple way how to evaluate (77) without performing an explicit Laurent’s expansion of \( \Delta_C(x, T) \) around \( h = 0 \). Using the Bernoulli expansion:
\[ \frac{x}{e^x - 1} = \sum_{a=0}^{\infty} B_a \frac{x^a}{a!} , \quad |x| < 2\pi , \]

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(\(B_\alpha\) are Bernoulli’s numbers) we may Laurent expand \(f_b\) as

\[
f_b(\hbar k_0/T) + \frac{1}{2} = \frac{T}{\hbar k_0} + \frac{1}{12} \frac{\hbar k_0}{T} - \frac{1}{90} \left( \frac{\hbar k_0}{T} \right)^2 + \ldots.
\]

(78)

The key observation is that the foregoing series converges only for \(\hbar |k_0| < 2\pi T\). The leading term in (78) gives the classical thermal part of the propagator\(^8\), and the classical approximation is then equivalent to taking the leading term in the Laurent expansion (78). The higher quantum corrections are due to the higher terms in the expansion, but for large \(|k_0|\) the expansion does not work, i.e. an expansion in \(\hbar\) is unwarranted. Of course, for \(\hbar |k_0| \gg T\) the distribution \(f_b\) is exponentially small (Wien’s distribution law) and it is dropped in comparison to the (zero point) first term in the integral (71), which returns the usual \(T = 0\) approach from the previous Section.

In order to get some quantitative results let us concentrate on the high temperature case (low temperature case is more involved and will be treated elsewhere). By high temperature we mean the temperature at which the Rayleigh–Jeans’s sector of the thermal propagator (71) approximates by itself sufficiently well the thermal part of the propagator. This should be taken with a grain of salt as the high temperature we intend to use should be still bellow the temperature threshold above which the heat–bath quasiparticles become instable due to the thermal fluctuations.

Because we are in the high temperature regime we take cut off in the \(k_0\) integration: \(|k_0| \ll \Lambda_1 \approx T/\hbar\). Due to \(\delta\)–function in the thermal propagator we must also impose cut off on \(k_1\) integration: \(|k_1| \ll \Lambda_2 \approx T/\hbar\). We tacitly assume that the domain of integration from \(T/\hbar\) to \(\infty\) will contribute modestly due to the exponentially small contribution to the thermal propagator.

Using (54), (53) and (77) we can now evaluate \(C[f](x; T, \hbar \to 0)\). Indeed

\[
C[f](x; T, \hbar \to 0) = \exp[c] \left\{ i \int_{-\infty}^{\infty} d^2 y G_R(x, y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^n b^{n+1} \right\} \bigg|_{K=f}
\]

\[
= \exp[c] \left\{ -K(x) - 2v - \frac{4v^2}{K(x) - 2v} \right\} \bigg|_{K=f},
\]

(79)

where

\[
c = \frac{\hbar}{2} \int_{C} d^2 z d^2 y \Delta_{\delta}^E(z; y) \frac{\delta^2}{\delta K(z) \delta K(y)}.
\]

(80)

Note that there is no \(\hbar\) in \(c\) in the large \(T\) approximation. In other words, the high temperature properties are injected into the order parameter calculation in a purely classical way. This might be compared with the usual observation that at large \(T\) thermal fluctuations dominate over quantum ones. Thus in our case we get (we omit \(\hbar\) in \(\Delta_{\delta}^E\) as it cancels anyway in the large \(T\) limit\(^8\))

\[
C[f](x; T, \hbar \to 0) = \left\{ -K(x) - 2v - \frac{4v^2}{K(x) + \int_{C} d^2 y \Delta_{\delta}^E(x; y) \frac{\delta^2}{\pi K(y)} - 2v} \right\} \bigg|_{K=f}.
\]

(81)

Applying the expansion

\[
\frac{1}{A + B} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \ldots,
\]

and identifying \(A = K(x) - 2v\) and \(B = \int_{C} d^2 y \Delta_{\delta}^E(x; y) \frac{\delta^2}{\pi K(y)}\) we get

\[^8\]We can call it Rayleigh–Jeans’s sector of the thermal propagator as the corresponding distribution function \(f(\omega) = T/\omega\) is nothing but Rayleigh–Jeans’s distribution law.

\[^9\]To derive Eq. (81) we use the relation: \(\exp \left( \sum_{ij} A_{ij} \partial_{x_i} \partial_{x_j} \right) F(x_k) = F(x_k + 2 \sum_{ij} A_{kj} \partial_{x_j} 1).\)
The series \( \Sigma(z) = 1 + \sum_{n=1}^\infty z^n (2n - 1)!! \equiv \sum_{n=0}^\infty \sigma_n z^n \) is clearly divergent for all non-trivial values of \( z = \Delta_{11}(0)/(K(x) - 2v)^2 \). Thus \( \Sigma(z) \) can be at best understood as an asymptotic series. Question arises then which function has \( \Sigma(z) \) as its asymptotic expansion. To determine this, let us assume initially that \( z \in \mathbb{C} \) and perform the Borel transform \( B_\Sigma(z) \) of \( \Sigma(z) \) by [64]

\[
B_\Sigma(z) = \sum_{n=0}^\infty B_n z^n \equiv \sum_{n=0}^\infty \frac{\sigma_n}{n!} z^n = \frac{1}{\sqrt{1 - 2z}}
\]

Consequently the Borel sum of \( \Sigma(z) \) is

\[
\int_0^\infty dt \, e^{-t} B_\Sigma(tz) = -\sqrt{\frac{\pi}{-2z}} \, e^{-1/2z} \, \text{Erfc} \left( \frac{1}{\sqrt{-2z}} \right),
\]

(84)

where \( \text{Erfc}(\ldots) \) is the complementary error function (see e.g., [22]). Relation (84) is true for any \( z \in \mathbb{C} \) with \( \arg(z) \neq 0 \), and so in such a region \( \Sigma(z) \) is Borel summable with the unique asymptotic function

\[
-\sqrt{\frac{\pi}{-2z}} \, e^{-1/2z} \, \text{Erfc} \left( \frac{1}{\sqrt{-2z}} \right) \sim \Sigma(z), \quad \arg(z) \neq 0.
\]

(85)

In the marginal case when \( \arg(z) = 0_- \) or \( \arg(z) = 0_+ \) we have

\[
- \frac{\sqrt{\pi}}{2z} e^{-1/2z} \, \text{Erfc} \left( \frac{i}{\sqrt{2z}} \right) \sim \Sigma(z) \quad \text{or} \quad i \frac{\sqrt{2\pi}}{z} e^{-1/z} - i \frac{\sqrt{\pi}}{2z} e^{-1/2z} \, \text{Erfc} \left( \frac{i}{\sqrt{2z}} \right) \sim \Sigma(z),
\]

(86)

respectively, depending which value of \( \sqrt{-1} \) one accepts along the branch cut. From the uniqueness of the asymptotic expansion [83] follows that when \( z \) is real, then the functions in (86) are the only ones which have \( \Sigma(z) \) as their asymptotic expansion. This dichotomy is inherent in theory of asymptotic expansions [32] and fortunately is not counterproductive here. Both functions differ by an exponent which rapidly vanishes when \( z \to 0 \) and accordingly they belong to the same equivalence class of asymptotic functions (see e.g., Poincaré criterion [64]). A brief inspection of (84) reveals an important feature of the class functions, namely that they are complex even when \( z \) is real.

Applying now the fact that in \((1+1)\) dimensions the thermal tadpole \( \Delta_{11}(0) = T/2\omega_0 = T/2gv \), we see that there is no high \( T \) solution for the thermal kink as the order parameter turns out to be a complex number. This result is fully compatible with the absence of phase transitions in \((1+1)\) dimensions [33][77]. In fact, the emergence of the complex valued order parameter can be attributed (similarly as in the case when the effective action machinery is in use) to spontaneous fluctuations in the order parameter. The latter behavior is well known, for instance, from Ising–type models [68] or from numerical simulation in \((1+1)\) dimensional \( \phi^4 \) theories [10].

It is interesting to compare the above result with the one contained in Ref. [20]: there a tree approximation at finite temperature is used to get a “classical” kink solution which is of the same form as the zero–temperature one but with thermal parameters \( m(T) \) and \( \lambda(T) \) in the place of zero–temperature ones. From our discussion it is clear that the tree approximation is not useful in extracting classical results at finite temperature as the loopwise expansion is not anymore an expansion in \( \hbar \). In fact, a resummation of infinitely many (thermal) loops is indeed necessary [1], the resummation in [22] takes neatly care of this. However, the qualitative result of Ref. [20] remains valid: at some “critical” temperature the kink solution disappears. This is not evident in the present case because of the dimensionality (reflecting in the temperature dependence of the propagator), nevertheless the above analysis till

\[10\] It should be noted that the paradigmatic example of this fact is HTL resummation which is used to cure the breakdown of the conventional perturbative theory for infrared momenta in thermal field theories with light bosons (especially in gauge theories where symmetries prevent perturbative radiative generation of masses) [32][61].
first line of Eq. (79) is general and can be applied, for example, to $\lambda\psi^4$ in $(3 + 1)$ dimensions: there, in the thin wall approximation, analytic kink solutions with spherical symmetry have essentially the same form as in $(1 + 1)$ dimensions [69]. It is definitely an intriguing question if in this case the temperature at which the kink disappears coincides or not with the critical temperature at which the symmetry is restored. Note that both of these temperatures can be estimated in the present approach: for instance, the one for symmetry restoration can be read from the thermal order parameter in the case of homogeneous condensation $(K = 0)$. Work in this direction is currently in progress [19].

V. CONCLUSIONS AND OUTLOOK

In this paper we have shown how to describe topological defects as inhomogeneous condensates in Quantum Field Theory, both at zero and at finite temperature. To this end, we have used the Closed–Time–Path formalism, allowing us to obtain a closed functional expansion for the Heisenberg field operator in presence of defects: the inhomogeneity of the vacuum condensate is introduced by means of a “shift function” $f(x)$ which is solution of the equations for the physical fields (quasiparticles).

We applied this general scheme to a specific simple model - $(1+1)$ dimensional $\lambda\psi^4$ theory. We constructed the Heisenberg operator $\psi^f$ in the kink sector, showing explicitly how the known classical kink solutions arise from the vacuum expectation value of such an operator in the classical limit. The knowledge of $\psi^f$ gives us the possibility of calculating quantum corrections to the soliton solutions as well as Green’s functions in presence of kinks.

We have also shown how to extend this treatment to finite temperature and considered explicitly the case of high temperature. In this limit, we are able to calculate the order parameter in the classical limit, showing how kink solutions depend on temperature. This is of particular interest in the case of higher dimensionality: then the analysis presented here goes through in a similar way and it is possible to study the behavior of topological defects near the critical temperature, as well as to calculate $T_c$. In fact, when the case of homogeneous condensation ($f = 0$) is considered, our formulation offers a way to the study of restoration of symmetry, alternative to the traditional analysis based on effective potentials.

Natural extensions of the present work includes the sine–Gordon model and the $(3+1)$ dimensional $\lambda\psi^4$ theory. For the sine–Gordon model, which is integrable, it is of interest to investigate the correspondence between the present method and the Backlund transformations. We have indeed shown that in the classical limit, the introduction of the shift function $f$ allows for a linearization of the equation for the order parameter. Then for integrable systems, linear superposition of solutions of the (linear) equation for $f$ give rise to multisoliton solutions for the non-linear Euler equation: such a correspondence could indeed offer an alternative possibility for classifying soliton solutions (i.e. by means of the shift function $f$). On the other hand, we have explicitly shown that, in the case of $(1+1)$ dimensional $\lambda\psi^4$ (which is non–integrable), it is not possible to solve in general the recurrence relation establishing the non–linear to linear mapping, except for a restricted class of solutions for $f$. Other interesting aspects of the sine–Gordon model for which the present approach could be useful are the finite temperature behavior [70] and the duality with the Thirring model both at zero [36] and finite temperature [71].

For the $(3+1)$ dimensional $\lambda\psi^4$ theory with complex field - relativistic Landau–Ginzburg theory, we already found an interesting relation between the Green’s functions for the system with vortices and the one without vortices [15], allowing us to calculate a “topological” contribution to the hydrostatic pressure [19]. Temperature–induced restoration of symmetry is an other important aspect which can be studied in the present framework.

Finally, it is definitely a challenging task to extend the above formulation to the description of phase transitions in which defects could be created. The main problem in doing this is in the change of the set of the asymptotic fields when crossing different phases, which is not taken into account in the present formulation. Study in this direction is in progress.

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APPENDIX A

We show here that $D[f, h \to 0] = f$. To prove this we start with the definition of $D[K, h \to 0]$. Let us remind that

$$ D[K, h \to 0] = K(x) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^n b^n, \quad (87) $$

with $a, b$ being defined in Section III.A. We shall now show that $a^k b^l[K = f] = 1$ if $k = l = 0$ and 0 otherwise (both $k$ and $l$ are positive integers). In the case when $k = l = 0$ or $l = 0, k \neq 0$ our statement obviously holds. If $k = 0$ but $l \neq 0$ our statement is true due to the fact that the contour integration of the function which is continuous across the real–time axis is zero (clearly $f$ cannot be discontinuous as it is a solution of the homogeneous second–order differential equation). So let us concentrate on the remaining cases and to prove that then $a^k b^l[K = f] = 0$.

To show this let us formulate the following conjecture:

$$ a^k b^l[K] = \int_C d^2 x d^2 y \Delta_C(x; y) \left\{ \delta(x - y) F^{(k,l)}[K](x) + B^{(k,l)}[K](x, y) \right\}, \quad (88) $$

where both $F^{(k,l)}[K](x)$ and $B^{(k,l)}[K](x, y)$ are continuous functions across the real–time axis when $K = f$, i.e.

$$ Disc_{x,y} F^{(k,l)}[f](x, y) = Disc_{x,y} B^{(k,l)}[f](x, y) = Disc_{x,y} B^{(k,l)}[f](x, y) = 0. $$

To prove the conjecture (88) we shall use the mathematical induction w.r.t. $k$. For $k = 1$ the conjecture is true as

$$ a b^l[K] = \int_C d^2 x d^2 y \Delta_C(x; y) \left\{ l(l - 1) b^{l-2} \frac{\delta b}{\delta K(x) \delta K(y)} + l b^{l-1} \frac{\delta^2 b}{\delta K(x) \delta K(y)} \right\} = \int_C d^2 x d^2 y \Delta_C(x; y) \left\{ \delta(x - y) F^{(1,l)}[K](x) + B^{(1,l)}[K](x, y) \right\}. \quad (89) $$

From the structure of the $b$ term is obvious that both $F^{(1,l)}[f]$ and $B^{(1,l)}[f]$ are continuous across the real–time axis.

In the next induction step we shall assume that the relation (88) is valid for $k = n$ and we should prove the validity for $k = n + 1$. In the latter case we may write

$$ a^{n+1} b^l[K] = a(a^n b^l)[K] = \int_C d^2 x d^2 y \Delta_C(x; y) \frac{\delta^2}{\delta K(x) \delta K(y)} \int_C d^2 z_1 d^2 z_2 \Delta_C(z_1; z_2) \left\{ B^{(n,l)}[K](z_1, z_2) + \delta(z_1 - z_2) F^{(n+1,l)}[K](z_1) \right\}. \quad (90) $$

Because of the assumed property of $B^{(n,l)}$ and $F^{(n,l)}$ they may by written as

$$ B^{(n,l)}[K](x, y) = \sum_{p,q} f_{pq}(x, y) K_p(x) K_q(y) P_{pq}[b] $$

$$ F^{(n,l)}[K](x) = \sum_p f_p(x) K_p(x) P_p[b], \quad (91) $$

Where both $f_{pq}$ and $f_p$ are continuous across the real–time axes, $P_{pq}[b]$ and $P_p[b]$ are some polynomials in $b$ (not in $K$). Plugging (91) into (90) we obtain after some manipulations that

$$ a^{n+1} b^l[K] = \int_C d^2 x d^2 y \Delta_C(x; y) \left\{ B^{(n+1,l)}[K](x, y) + \delta(x - y) F^{(n+1,l)}[K](x) \right\}, \quad (92) $$

with $B^{(n+1,l)}[f]$ and $F^{(n+1,l)}[f]$ fulfilling the required conditions. For example $F^{(n+1,l)}[K]$ can be found to be

$$ F^{(n+1,l)}[K](x) = \Delta_{11}(0) \sum_p f_p(x) p(p - 1) K^{p-2}(x) P_p[b] $$

$$ - \frac{i}{2} \Delta_{11}(0) [3y^2 K^2(x) + 6mgK(x)] \frac{dP_q[b]}{db} \int_C d^2 z_1 \sum_p f_p(z_1) K^p(z_1). $$

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During the previous derivations we have used the fact that
\[ \int_C d^2 y \Delta_C(x,y) f^p(y) = i \int_{-\infty}^{\infty} d^2 y G_R(x,y) f^p(y) = z(x), \]  
where \( z(x) \) is a continuous across \( x_0 \). This is obvious as \( z(x) \) solves equation \((\partial^2 + m^2)z(x) = f^p(x) \) with the boundary conditions \( z(0, x_1) = 0 \) and \( dz(0, x_1)/dx_0 = 0 \) (see for example [72]). The second important trick which we have used was that
\[ \int_C d^2 x d^2 y \Delta_C(x,y) \delta(x - y) \ldots = \Delta_{11}(0) \int_C d^2 x \ldots. \]  
This closes the proof of our conjecture (88). As a result we may write
\[ a^k b^j[f] = \Delta_{11}(0) \int_C d^2 x F^{(k,l)}[f](x) + \int_C d^2 x d^2 y \Delta_C(x,y) B^{(k,l)}[f](x,y) = 0. \]  
In the previous we have applied both the fact that the contour integral over the continuous function across the real–time axis is zero and the identity
\[ \Delta_{11} - \Delta_{12} - \Delta_{21} + \Delta_{22} = 0, \]  
which renders the double–contour integration zero\(^\footnote{As the identity (96) is only based on the fact that } \)\(^{11}\). As a byproduct we obtain that only the first term in (87) survives, i.e. \( D[f, \hbar \to 0] = f \). One of the main virtue of our derivation is that it immediately extends to finite temperatures because all the steps in the above proof may be repeated almost word by word. This mainly goes into account of equations (93), (94) and (95) (and the arguments mentioned therein) which retain their validity even at the finite temperature level.

APPENDIX B

In this appendix we aim to prove the recursion relation \((56)\). Let us first observe that for \( n = 2 \) we immediately obtain
\[ P_2(x) = \int_{-\infty}^{\infty} d^2 y G_R(x,y) \left[ \frac{3}{2} m g P_1(y) P_1(y) + \frac{1}{2} g^2 P_1(y) P_1(y) \right]. \]  
with \( P_1 = K \). Similarly, for \( n = 3 \) we may write
\[ P_3(x) = i \int_{-\infty}^{\infty} d^2 y G_R(x,y) a \left\{ \left[ \frac{3}{2} m g K^2(y) + \frac{1}{2} g^2 K^3(y) \right] b \right\} \]
\[ = \int_{-\infty}^{\infty} d^2 y G_R(x,y) \left[ \frac{3}{2} m g 2 P_1(y) P_2(y) + \frac{g^2}{2} 3 P_1(y) P_1(y) P_2(y) \right]. \]  
During the previous derivation we have took advantage of the relation \( \Delta(x,y) = -\Delta(y,x) \). No assumption about the actual behavior of \( K(x) \) was made at this stage. As a result we may conjecture that for the general \( n \) we have
\[ P_n(x) = \int_{-\infty}^{\infty} d^2 y G_R(x,y) \left[ \frac{3}{2} m g \sum_{i+j=n} P_i(y) P_j(y) + \frac{1}{2} g^2 \sum_{i+j+k=n+1} P_i(y) P_j(y) P_k(y) \right], \quad n \geq 2. \]  

\(^\footnote{As the identity (96) is only based on the fact that } \)

\[ \int_C d^2 x d^2 y \Delta(x,y) \Delta(x,y) F(x,y) = 0 \] provided \( \text{Disc}_{x,y} F(x,y) = \text{Disc}_{y,x} F(x,y) = 0. \]
To prove this conjecture we shall use the mathematical induction. Let us assume that Eq. (99) holds for \( n \). So for \( P_{n+1}(x) \) we may directly write

\[
P_{n+1}(x) = a \left[ P_n(x) \right] b \frac{1}{n^2}
\]

By performing the change of variables \( f \) with

\[
 w = \ldots
\]

where we used \( w \).

This proves our conjecture.

**APPENDIX C**

We calculate here the following integral:

\[
 I_n = \int_C d^2 z \; \Delta_C(x - z) \; f^n(z)
\]

with \( f(z) = A e^{-\gamma(z_1 - uz_0)} \). We get

\[
 I_n = A^n \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{\infty} dz_1 \left\{ \Delta_{11}(x - z) - \Delta_{12}(x - z) \right\} \; e^{-\gamma(z_1 - uz_0)}
\]

By performing the change of variables \( x - z = y \) and using Eq. (12) for \( \Delta_{11}(x - z) - \Delta_{12}(x - z) = \theta(x_0 - z_0) [\phi(x), \phi(z)] \), we get

\[
 I_n = -\frac{i}{2} A^n e^{-\gamma(x_1 - uz_0)} \int_0^\infty dy_0 \int_{-y_0}^{y_0} dy_1 \; J_0 \left[ m \sqrt{y_0^2 - y_1^2} \right] \; e^{\gamma y_1 y_0}
\]

\[
 = -\frac{i}{2} A^n e^{-\gamma(x_1 - uz_0)} \int_0^\infty dy_0 e^{-\gamma y_0} \int_0^1 dw \; J_0 [m y_0 (1 - w^2)] \; e^{\gamma y_0 w}
\]

where we used \( w = y_1/y_0 \). We then have (see [73] formula 6.616 n.5 after analytic continuation):

\[
 I_n = -i A^n \frac{e^{-\gamma(x_1 - uz_0)}}{m \sqrt{n^2 \gamma^2 - 1}} \int_0^\infty dy_0 \; e^{-\gamma y_0} \sinh(m y_0 \sqrt{n^2 \gamma^2 - 1})
\]

\[
 = -i A^n \frac{e^{-\gamma(x_1 - uz_0)}}{m^2 (n^2 \gamma^2 - 1)} \int_0^\infty dy_0' \; e^{-\frac{\gamma y_0'}{\sqrt{n^2 \gamma^2 - 1}}} \sinh(y_0')
\]

The last integral is tabulated (see [73], 3.541 n.1) with the result

\[
 \int_0^\infty dx \; e^{-\mu x} \sinh(x) = \frac{1}{4} B \left( \frac{\mu}{2} - 1, \frac{1}{2} \right) = \frac{1}{4} \frac{\Gamma \left( \frac{\mu}{2} - \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{\mu}{2} + \frac{1}{2} \right)} = \frac{1}{\mu^2 - 1}
\]

where \( \mu = n^2 u / \sqrt{n^2 \gamma^2 - 1} \). The final result is then

\[
 I_n = -i A^n \frac{e^{-\gamma(x_1 - uz_0)}}{m^2 (n^2 \gamma^2 - 1)} \frac{1}{\mu^2 - 1} = i A^n \frac{e^{-\gamma(x_1 - uz_0)}}{m^2 (n^2 \gamma^2 - 1)} \frac{n^2 \gamma^2 - 1}{n^2 \gamma^2 (1 - u^2) - 1} = i A^n \frac{e^{-\gamma(x_1 - uz_0)}}{m^2 (n^2 - 1)}
\]
So \( \int_C d^2z \Delta_C(x-z)f^n(z) \propto f^n(x) \) provided \( f \) is an exponential solution of the Klein–Gordon equation (34). On the other hand an exponential function is the only one which has this property for all \( n \). To see this let us reverse the former integral equation, i.e.

\[
(\partial^2 + m^2)f^n = (\partial_+ \partial_- + m^2)f^n = \eta f^n, \tag{107}
\]

where \( \eta \) is a proportionality constant and \( \partial_\pm = \partial/\partial x^\pm \) with \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \). As a result we get for \( n > 1 \) the non–linear differential equation

\[
\partial_+ f \partial_- f = \xi^2 f^2,
\]

with \( \xi = \sqrt{m^2/(n-1) + \eta/n(n-1)} \) which can be equivalently written as

\[
(\partial_+ f + \xi f)(\partial_- f + \xi f) = 0.
\]

The latter has clearly only exponential solutions.

**APPENDIX D**

In this appendix we show how the classical kink solutions emerge in the small coupling (large mass) limit. Let us put \( K \to K/g \). Then, as already done in Section III.B, we may argue that since \( \mathcal{L}_{ct} \) is of order \( \mathcal{O}(g^2) \) it automatically decouples in the following reasonings. As a result we obtain:

\[
D[K](x) = \frac{K(x)}{g} \exp[\hbar g^2 a] \exp \left[ \frac{1}{\hbar g^2} B \right]
\]

\[
C[K](x) = \hbar g \int_C d^2y \Delta_C(x,y) \frac{\delta}{\delta K(y)} \exp[\hbar g^2 a] \exp \left[ \frac{1}{\hbar g^2} B \right], \tag{108}
\]

with \( B = -\frac{i}{2} \int_C d^2z \left[ K^4(z)/4 + \omega_0 K^3(z) \right] \). Keeping only the leading terms in the \( g \to 0 \) limit (i.e. terms of order \( g^{-1} \)) we obtain

\[
D[K](x,g \to 0) = \frac{K(x)}{g} \text{Res}_{g \to 0} \left( \frac{1}{g} \exp[\hbar g^2 a] \exp \left[ \frac{B}{\hbar g^2} \right] \right) = \frac{K(x)}{g} \sum_{n=0} \frac{1}{(n!)^2} a^n B^n = \frac{K(x)}{g}, \tag{109}
\]

where the passage between second and third identity was done by means of the result in Appendix A. Analogously, for the \( C[K] \) term we arrive at

\[
C[K](x,g \to 0) = \frac{\hbar}{g} \int d^2y \Delta_C(x,y) \frac{\delta}{\delta K(y)} \text{Res}_{g \to 0} \left( g \exp[\hbar g^2 a] \exp \left[ \frac{1}{\hbar g^2} B \right] \right)
\]

\[
= \frac{i}{g} \int_{-\infty}^{\infty} d^2y G_R(x,y) \frac{\delta}{\delta K(y)} \sum_{n=0} \frac{1}{n!(n+1)!} a^n B^{n+1}. \tag{110}
\]

Let us remark that \( \hbar \) completely disappeared both from (109) and (110).

Similarly as in the case of the \( \hbar \) limit we may write

\[
\langle \psi_f(x) \rangle_{g \to 0} = v + \frac{f}{g} + \frac{i}{g} \int_{-\infty}^{\infty} d^2y G_R(x,y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^n B^{n+1} |_{K=f}
\]

\[
= v + \frac{1}{g} \sum_{n=1}^{\infty} R_n[K](x)|_{K=f}, \tag{111}
\]

where

\[
R_1(x) = K(x)
\]

\[
R_n(x) = \frac{1}{|(n-2)!|^2} \int_{-\infty}^{\infty} d^2y G_R(x,y) a^{n-2} \left\{ \left[ \frac{3}{2} \omega_0 K^2(y) + \frac{1}{2} K^3(y) \right] B^{n-2} \right\}; \quad n \geq 2. \tag{112}
\]
It is seen comparing Eq.(112) with Eq.(54) that the following recurrence relation holds:

\[
R_n(x) = -\int_{-\infty}^{\infty} d^2y G_R(x, y) \left[ \frac{3\omega_0}{2} \sum_{i+j=n} R_i(y)R_j(y) + \frac{1}{2} \sum_{i+j+k=n+1} R_i(y)R_j(y)R_k(y) \right] ; \quad n \geq 2 .
\]  

(113)

The “diagonalized” recurrence relation reads

\[
\begin{align*}
S_1(x) &= K(x) \\
S_2(x) &= \int_{-\infty}^{\infty} d^2y G_R(x, y) \frac{3\omega_0}{2} \sum_{i+j=2} S_i(y)S_j(y) \\
S_n(x) &= \int_{-\infty}^{\infty} d^2y G_R(x, y) \left[ \frac{3\omega_0}{2} \sum_{i+j=n} S_i(y)S_j(y) + \frac{1}{2} \sum_{i+j+k=n} S_i(y)S_j(y)S_k(y) \right] ; \quad n \geq 3 .
\end{align*}
\]  

(114)

This is, as before, the functional equation of Cauchy–Marley’s type. The corresponding analytical solution has the form \( S_n(x) \propto (S_1(x))^n \), with \( S_1(x) = f(x) \) being an exponential, Fourier non–transformable and \( g \) independent solution of the dynamical equation \( (113) \). So

\[
S_n(x) = \tilde{A}_n f^n(x) = \tilde{A}_n e^{-\omega_0 n \gamma(x_1 - x_0 u)} .
\]

Incorporating this solution in the recurrence relation \( (114) \), we are led to the conclusion that

\[
\tilde{A}_n = \frac{1}{(n^2 - 1)} \left\{ \frac{3}{2\omega_0} \sum_{i+j=n} \tilde{A}_i \tilde{A}_j + \frac{1}{2\omega_0^2} \sum_{i+j+k=n} \tilde{A}_i \tilde{A}_j \tilde{A}_k \right\} .
\]

(115)

By analogy with the calculation of Section III.B it is evident that the fundamental solution of this recurrence equation has the form \( \tilde{A}_n = 2\omega_0 \left( \frac{s}{2\omega_0} \right)^n \) where \( s \) is a real constant. The final form for the order parameter in the small coupling limit may be written as

\[
\langle \psi^f(x) \rangle_{g \to 0} = v + \frac{2\omega_0}{g} \sum_{n=1}^{\infty} \left( \frac{s f(x)}{2\omega_0} \right)^n .
\]

(116)

Recalling that \( v = \pm \omega_0 / g \) and choosing \( s = \pm 2\omega_0 e^{\omega_0 \gamma a} \) we can resum the series \( (110) \) with the result:

\[
\langle \psi^f(x) \rangle_{g \to 0} = v \mbox{th} \left[ \frac{\omega_0}{2} \gamma((x_1 - a) - ux_0) \right] ,
\]

(117)

for \( v \) positive and \( s \) negative, and

\[
\langle \psi^f(x) \rangle_{g \to 0} = -v \mbox{th} \left[ \frac{\omega_0}{2} \gamma((x_1 - a) - ux_0) \right] ,
\]

(118)

for both \( v \) and \( s \) negative. If \( s \) were positive, the expression \( (110) \) for the order parameter becomes the non–solitonic one with \( \mbox{cth} \) instead of \( \mbox{th} \). An analogous analysis can be done for \( f = e^{\omega_0 \gamma(x_1 - x_0 u)} \).

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