SOME NEW NEAR-NORMAL SEQUENCES

DRAGOMIR Ž. DOKOVIĆ

Abstract. The normal sequences $NS(n)$ and near-normal sequences $NN(n)$ play an important role in the construction of orthogonal designs and Hadamard matrices. They can be identified with certain base sequences $(A; B; C; D)$, where $A$ and $B$ have length $n + 1$ and $C$ and $D$ length $n$. C.H. Yang conjectured that near-normal sequences exist for all even $n$. While this has been confirmed for $n \leq 30$, so far nothing else was known for $n > 30$. Our main result is that $NN(32)$ consists of 8 equivalence classes and we exhibit their representatives. We also construct representatives for two equivalence classes of $NN(34)$. On the other hand, we have shown by exhaustive computer searches that $NS(31)$ and $NS(33)$ are void.

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1. Introduction

We deal with quadruples $(A; B; C; D)$ of binary sequences, i.e., sequences with entries $\pm 1$. Base sequences are such quadruples, with $A$ and $B$ of length $m$ and $C$ and $D$ of length $n$, such that the sum of their nonperiodic autocorrelation functions is a $\delta$-function. The collection of such sequences is denoted by $BS(m, n)$.

In section 2 we recall the definition of base sequences and how they can be used to construct Hadamard matrices. In section 3 we define normal sequences, $NS(n)$, and near-normal sequences, $NN(n)$, as some special classes of base sequences $BS(n+1, n)$. We also recall some basic facts about these sequences: their use to construct Yang multiplications, their importance for the construction of $T$-sequences, and what is known about their existence. The $T$-sequences are not binary; they are quadruples of ternary sequences with entries from $\{0, \pm 1\}$, all of the same length. See the main text for the complete definition.

In section 4 we describe the new results that we have obtained. We have classified the near-normal sequences $NN(32)$ and constructed two...
non-equivalent near-normal sequences in $NN(34)$. Let us mention that in our recent paper [3] we have classified the near-normal sequences $NN(n)$ for all even $n \leq 30$. We have introduced there two equivalence relations in $NN(n)$: $BS$- and $NN$-equivalence. In this note we use only the $NN$-equivalence. We also report that our exhaustive searches have shown that $NS(31) = NS(33) = \emptyset$. As a consequence of these facts, we deduce that 63 and 67 are not Yang numbers while 69 is such a number. By definition, a Yang number is an odd integer $2s + 1$ such that $NS(s)$ or $NN(s)$ is not empty.

2. Base sequences

We denote finite sequences of integers by capital letters. If, say, $A$ is such a sequence of length $n$ then we denote its elements by the corresponding lower case letters. Thus

$$A = a_1, a_2, \ldots, a_n.$$ 

To this sequence we associate the polynomial

$$A(x) = a_1 + a_2 x + \cdots + a_n x^{n-1},$$ 

which we view as an element of the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. (As usual, $\mathbb{Z}$ denotes the ring of integers.) The nonperiodic autocorrelation function $N_A$ of $A$ is defined by:

$$N_A(i) = \sum_{j \in \mathbb{Z}} a_j a_{i+j}, \quad i \in \mathbb{Z},$$ 

where $a_k = 0$ for $k < 1$ and for $k > n$. Note that $N_A(-i) = N_A(i)$ for all $i \in \mathbb{Z}$ and $N_A(i) = 0$ for $i \geq n$. The norm of $A$ is the Laurent polynomial $N(A) = A(x)A(x^{-1})$. We have

$$N(A) = \sum_{i \in \mathbb{Z}} N_A(i)x^i.$$ 

To the sequence $A$ we associate two other sequences of the same length: the negation

$$-A = -a_1, -a_2, \ldots, -a_n$$ 

and the alternation

$$A^* = a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1}a_n.$$ 

By $A, B$ we denote the concatenation of the sequences $A$ and $B$.

The base sequences consist of four $\{\pm 1\}$-sequences $(A; B; C; D)$, with $A$ and $B$ of length $m$ and $C$ and $D$ of length $n$, such that

$$(2.1) \quad N(A) + N(B) + N(C) + N(D) = 2(m + n).$$
We denote by $BS(m, n)$ the set of such base sequences with $m$ and $n$ fixed.

It is known that $BS(n + 1, n) \neq \emptyset$ for $0 \leq n \leq 35$ (see [5, 7]) and that $BS(2n - 1, n) \neq \emptyset$ for all even $n = 2, 4, \ldots, 36$ (see [5, 7, 4]).

Base sequences can be used to construct Hadamard matrices. Recall that a Hadamard matrix of order $m$ is a $\{\pm 1\}$-matrix $H$ of order $m$ such that $HH^T = mI_m$, where $T$ denotes the transpose and $I_m$ the identity matrix. For instance if $(A; B; C; D) \in BS(n, n)$ then we can construct a Hadamard matrix $H$ of order $4n$ as follows. Let $A^c$ denote the circulant matrix having $A$ as its first row, and define similarly the circulants $B^c, C^c$ and $D^c$. The construction of $H$ is based on the Goethals-Seidel array

$$
\begin{bmatrix}
U & XR & YR & ZR \\
-XR & U & -Z^TR & Y^TR \\
-YR & Z^TR & U & -X^TR \\
-ZR & -Y^TR & X^TR & U
\end{bmatrix}.
$$

To obtain $H$ we just substitute the symbol $R$ with the $n \times n$ matrix having ones on the back-diagonal and all other entries zero, and substitute (in any order) the symbols $U, X, Y, Z$ with the four circulants $A^c, B^c, C^c, D^c$. The condition (2.1) guarantees that $H$ is indeed a Hadamard matrix.

In connection with this construction, observe that there is a map $BS(m, n) \rightarrow BS(m + n, m + n)$ sending

$$(A; B; C; D) \rightarrow (A, C; A, -C; B, D; B, -D).$$

3. Normal, near-normal and $T$-sequences

Normal resp. near-normal sequences, originally defined by C.H. Yang [9], can be viewed as a special type of base sequences $BS(n + 1, n)$ (see [5, 2]), namely such that $b_i = a_i$ resp. $b_i = (-1)^{i-1}a_i$ for $1 \leq i \leq n$. We denote by $NS(n)$ resp. $NN(n)$ the subset of $BS(n+1, n)$ consisting of normal resp. near-normal sequences.

Very little is known about the existence of normal sequences $NS(n)$. Golay sequences of length $n$ are two $\{\pm 1\}$-sequences $(A; B)$ of length $n$ such that $N(A) + N(B) = 2n$. If such sequences of length $n$ exist, we say that $n$ is a Golay number. The known Golay numbers are $n = 2^a10^b26^c$, where $a, b, c$ are arbitrary nonnegative integers. If $n$ is a Golay number, then $NS(n) \neq \emptyset$. Indeed, if $(A; B)$ are Golay sequences of length $n$, then $(A, +; A, -; B; B) \in NS(n)$. For $n \leq 30$ it is known (see [2, 1]) that $NS(n) = \emptyset$ iff

$$n \in \{6, 14, 17, 21, 22, 23, 24, 27, 28, 30\}.$$
The case of near-normal sequences $NN(n)$ is apparently more promising. We mention that if $n > 1$ and $NN(n) \neq \emptyset$, then $n$ must be even. The following question (now known as Yang’s conjecture) was raised about twenty years ago.

**Conjecture 3.1.** *(Yang [9])* $NN(n) \neq \emptyset$ for all positive even $n$’s.

It has been known since 1994 that near-normal sequences exist for even $n \leq 30$ (see [5]), but nothing else was known for larger values of $n$ (see [1]).

Some of the most powerful methods for constructing orthogonal designs and Hadamard matrices are based on $T$-sequences (see [1]). Let us recall that $T$-sequences are quadruples $(A; B; C; D)$ of $\{0, \pm 1\}$-sequences of the same length $n$ such that $N(A) + N(B) + N(C) + N(D) = n$ and, for each $i$, exactly one of $a_i, b_i, c_i, d_i$ is nonzero. We denote by $TS(n)$ the set of $T$-sequences of length $n$. It is known that $TS(n) \neq \emptyset$ for all odd $n < 100$ different from 73, 79 and 97. It has been conjectured that $TS(n) \neq \emptyset$ for all odd integers $n$.

Normal and near-normal sequences are important for the construction of $T$-sequences. If $NN(s)$ and $BS(m, n)$ are nonempty then there is a map, called *Yang multiplication* [9] [5]

$$NN(s) \times BS(m, n) \rightarrow TS((2s + 1)(m + n)),$$

and a similar statement is valid for $NS(s)$. For that reason it is customary to refer to the odd integer $2s + 1$ as a *Yang number* if $NS(s)$ or $NN(s)$ is nonempty.

**Remark 3.2.** While implementing in Maple the Theorems 1-4 of [9] we discovered two misprints: In the definition of $\tau_k$ on p. 770 one should replace the two $f_k^*$’s with $f_k$’s, and in the definition of $\beta_k$ on p. 773 one should replace $A$ with $A^*$. The asterisk is used in [9], and in this remark, to denote the reversed sequence. These errors were not easy to locate and correct, the same errors appear in [6].

It is well known that there are infinitely many Yang numbers. Indeed, if $s$ is a Golay number then $NS(2s + 1) \neq \emptyset$, and so $2s + 1$ is a Yang number. The known Yang numbers up to 100 are:

$$1, 3, 5, \ldots, 31, 33, 37, 39, 41, 45, 49, 51, 53, 57, 59, 61, 65, 81.$$  

It is also known that 35, 43, 47 and 55 are not Yang numbers.

4. **New results**

We show first that near-normal sequences $NN(32)$ and $NN(34)$ exist, and thereby confirm Yang’s conjecture for $n = 32$ and $n = 34.$
These sequences have been discovered by using the same algorithm as in our paper [2].

**Proposition 4.1.** The sets $NN(32)$ and $NN(34)$ are nonempty.

*Proof.* To prove this, it suffices to verify that $(A; B; C; D) \in NN(32)$, where

$$A = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$B = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$C = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$D = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

and $(P; Q; R; S) \in NN(34)$, where

$$P = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$Q = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$R = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -,$$

$$S = +, -, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, +, -.$$

The signs “+” and “−” stand for +1 and −1, respectively. It is tedious to verify by hand that $(A; B; C; D)$ and $(P; Q; R; S)$ are base sequences, but this can be easily done on a computer. The additional requirements for near-normality can be checked by inspection. \[
\]

**Proposition 4.2.** The number 69 is a (new) Yang number. The numbers 63 and 67 are not Yang numbers.

*Proof.* The first assertion holds since $NN(34) \neq \emptyset$. Our exhaustive computer searches showed that $NS(31) = NS(33) = \emptyset$. This implies the second assertion. \[
\]

As 32 is a Golay number, we know that $NS(32) \neq \emptyset$. Hence, the first unresolved case for the existence question of normal sequences $NS(n)$ is now $n = 34$.\]
In our paper [3], we have introduced two equivalence relations for near-normal sequences $NN(n)$: The $BS$-equivalence and the $NN$-equivalence. The former is finer than the latter. An $NN$-equivalence class may contain 1, 2 or 4 $BS$-equivalence classes. In this note we use only the $NN$-equivalence.

In the case $n = 32$ we have carried out an exhaustive search and found that $NN(32)$ consists of 8 $NN$-equivalence classes. In the case $n = 34$ our search was not complete and we constructed only two non-equivalent near-normal sequences. We list in Table 1 the representatives of these 10 $NN$-equivalence classes. The representatives are written in the compact encoded form. For the description of our encoding scheme see [2, 3]. The sequences $(A; B; C; D)$ and $(P; Q; R; S)$ displayed above are the first sequences in Table 1 for $n = 32$ and $n = 34$, respectively. The numbers $a, b, c, d$ resp. $a^*, b^*, c^*, d^*$ are the sums of the corresponding sequences $A, B, C, D$ resp. $A^*, B^*, C^*, D^*$. Note that

$$(A; B; C; D) \in NN(n) \Rightarrow (A^*; B^*; C^*; D^*) \in NN(n).$$

| $n = 32$ | $n = 34$ |
|-----------|-----------|
| $A & B$ | $C & D$ | $a, b, c, d$ | $a^*, b^*, c^*, d^*$ |
| 1 | 07656587173587123 | 1611375364252851 | $-5, 5, 8, 4$ | $7, -7, -4, 4$ |
| 2 | 07643217328262853 | 165722254564485 | $3, -7, 6, -6$ | $-5, 1, -10, 2$ |
| 3 | 07841512343414140 | 1663752642548557 | $9, 7, 0, 0$ | $9, 7, 0, 0$ |
| 4 | 07651732153537650 | 1767258654155337 | $3, 9, -2, 6$ | $11, 1, -2, 2$ |
| 5 | 07156434121787153 | 1867665578785216 | $3, 9, -6, 2$ | $11, 1, -2, 2$ |
| 6 | 0567146321465123 | 1166547238573585 | $11, 1, -2, 2$ | $3, 9, -6, -2$ |
| 7 | 051282658784653 | 1653815347277422 | $-1, -11, 2, 2$ | $-9, -3, -2, 6$ |
| 8 | 05126417143285123 | 1657686527418862 | $11, 1, -2, -2$ | $3, 9, 6, 2$ |

The $NN(32)$ above show that 65 is a Yang number. However this fact is already known since $NS(32) \neq \emptyset$. Nevertheless, each of the above ten near-normal sequences provides infinitely many (probably new) Hadamard matrices by using the Yang multiplication [3] and the infinite supply of known Williamson-type matrices (see e.g. [8]).

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NEAR-NORMAL SEQUENCES

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERTON, WATERLOO, ONTARIO, N2L 3G1, CANADA
E-mail address: djokovic@uwaterloo.ca