Quantum radiation from spherical mirrors

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Abstract

We consider mirrors of the spherical shape, that can expand or contract. Due to the excitation of the vacuum around, some spherical waves radiated from vibrating mirrors are encountered. Using experience from well-known literature on studies of two-dimensional conformal models, we adopt a similar framework to investigate such quantum phenomena in four dimensions. We calculate quantum averages of the energy-momentum tensor for s-wave approximation.

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1 Introduction

According to Quantum Field Theory, a quantum vacuum state is the lowest eigenstate of the energy operator. In the language of the second quantization, it refers to the state with no particles. Sometimes, this simple basis state can become a quite complicated mathematical object if we try to describe it using observables related to quanta of elementary free fields. The fundamental observation is that all quantized fields have zero-point (vacuum) energies of size $\hbar\nu/2$, where eigenfrequencies can be subject to some complicated equations defining the actual physical system. In other words, the physical vacuum can yield a complicated state involving virtual photons, electrons and other elementary quanta. Vacuum field fluctuations lead to significant physical effects, including charge and mass renormalization, Lamb shifts, Casimir effects or Unruh-Davies effects. The importance of vacuum fluctuations is recognized immediately whenever some non-trivial boundary conditions are assumed in field theoretical models.

Among many interesting problems of this kind, some theoretical models are investigated where either a boundary of physical space or an interface between two different media changes its shape with time. A prototype example is a moving mirror (Unruh-Davies effect [1]). Induced by the motion of such objects, quanta of massless fields are created around by vacuum fluctuations and they can be radiated away. If a moving interface is neutral (as it usually occurs in physical applications), such radiation is weak. In practice, to get a significant radiation flux it is necessary to assume that interfaces or mirrors move with almost light-like velocities or undergo sudden or large accelerations. These theoretical ideas have been used to model the effects of strong gravitation on QFT or to derive Hawking radiation [2, 3], to describe the squeezing effects in quantum optics [4], or to exploit the Schwinger suggestion how to explain the phenomenon of sonoluminescence [5].

In this paper, we study the (3+1)-dimensional problem of a single spherically symmetric mirror. For the sake of simplicity, the case of massless scalar field is considered. The mirror acts as a perfectly reflecting infinite potential barrier, so this implies Dirichlet boundary conditions for quantum fields. Its radius depends on time, the mirror can expand or contract. We find the radiation flux. Our analysis is based on standard papers devoted to the two-dimensional moving mirrors [6, 7, 8], see also recent papers [4, 9, 10, 11, 12, 13, 14]. Maybe the most interesting physical situation involves a case when a mirror oscillates with the period related to the eigen-modes of the two-dimensional cavity. Under this resonance condition we get that the Casimir force is enhanced [15, 16]. The main feature of two-dimensional situation is that the conformal symmetry allows one to transform the problem of time-dependent boundaries to the problem with static boundaries. It is not possible in the four-dimensional case. The quantum radiation of scalar particles by a moving plane mirror in the four dimensions has been considered in [17] (for non-relativistic motions). This paper addresses itself to the calculation of vacuum energy-
momentum tensor in the presence of a moving spherical mirror in four dimensions. However, we restrict ourselves to take account only spherical waves and spherically symmetric quantum excitations.

2 Spherical mirror

The model, we are dealing with in this paper, is the quantum theory of a massless real scalar field $\phi$ in (3+1)-dimensional Minkowski space-time. The field is almost free, no self-interactions are present except of interactions with a perfectly reflecting, spherical mirror. By a perfect mirror we mean a surface $\Sigma$ on which the field is subjected to the Dirichlet boundary conditions,

$$\phi|_\Sigma = 0.$$  

Center of the mirror rests in the chosen inertial reference frame while its radius $R$ changes in time.

As we want to compute the energy flux carried by the particles produced due to the motion of the mirror, it is crucial to have a well defined notions of the vacuum state and of the particle both in the past and in the future. It is possible if the mirror is initially and finally static, so we assume that its radius changes in time according to the formula

$$R = \begin{cases} 
R_0 & \text{for } t \leq 0 \\
R(t) & 0 \leq t \leq t_1 \\
R_1 & t \geq t_1 
\end{cases} 
$$  

Due to spherical symmetry of the mirror, we are able to solve the d’Alambert equation,

$$\left(-\partial_t^2 + \Delta\right) \phi(t, \vec{r}) = 0, $$  

using the method of separation of the variables in the spherical coordinates. We get

$$\phi(t, \vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} \phi_l(t, r) Y_{lm}(\theta, \phi) + c.c., $$  

where $Y_{lm}(\theta, \varphi)$ are spherical harmonics, $C_{lm}$ are arbitrary complex numbers and the functions $\phi_l(t, r)$ are solutions of the equations

$$\left( \frac{\partial^2}{\partial t^2} - \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{l(l+1)}{r^2} \right) \phi_l(t, r) = 0, \quad \phi_l(t, R(t)) = 0, $$  

with $l$ being a positive integer.

In the present paper we shall consider only the functions $\phi(t, \vec{r}) \equiv \phi(t, r)$ which are independent of the angular variables. In other words, in the partial wave expansion we confine
ourselves to the $s$–wave contribution with $l = 0$. This allows us to keep the technical details of the presented calculation at the relatively simple level. We plan to consider the more complicated case of the higher partial waves in the separate article.

A general classical solution of the d’Alambert equation, assumed to be independent of the angles and to vanish on the surface of the mirror, can be written in the form

$$
\phi(t, r) = \int_{-\infty}^{\infty} d\omega \beta(\omega)e^{-i\omega t}\psi_\omega(t, r) + c.c, \quad (5)
$$

where

$$
\psi_\omega(t, r) = \frac{1}{2ir} \left\{ e^{i\omega(\Delta r - \delta R)} - e^{-i\omega\Delta r} \right\}, \quad (6)
$$

and $\beta(\omega)$ is an arbitrary complex function. $\Delta r$ and $\delta R$ are defined by the formulae

$$
\Delta r = r - R_0, \quad \delta R = 2[R(t_R) - R_0]. \quad (7)
$$

$R(t_R)$ is a position of the mirror in a retarded time $t_R$, i.e. at the past moment when the wave incoming at the present time $t$ and position $r$ was reflected from the mirror,

$$
r - R(t_R) = t - t_R. \quad (8)
$$

From the eq. (8) it follows that $t_R$ and $R(t_R)$ are functions of $t - r$.

For $t \leq 0$ the mirror is at rest and we can perform the standard quantization procedure by imposing on the field $\hat{\phi}_{in}(t, r)$ (now promoted to the level of an operator) and its conjugated momentum,

$$
\hat{\pi}_{in}(t, r) = \partial_t \hat{\phi}_{in}(t, r),
$$

the canonical, equal time commutation relations written in the spherical coordinates,

$$
\left[ \hat{\pi}_{in}(t, r), \hat{\phi}_{in}(t, r) \right] = \frac{i}{r^2}\delta(r - r'). \quad (9)
$$

Subscript “in” serves to recall that the relations above are valid only for $t \leq 0$.

The field operator $\hat{\phi}_{in}(t, r)$ can be expanded in the basis of classical solutions of the d’Alambert equation,

$$
\hat{\phi}_{in}(t, r) = \frac{1}{2\pi} \int_{0}^{\infty} d\omega \sqrt{\omega} \left[ b_\omega e^{-i\omega t} + b_\omega^\dagger e^{i\omega t} \right] \psi_\omega^{(0)}(r), \quad (10)
$$

where

$$
\psi_\omega^{(0)}(r) = \frac{\sin (\omega\Delta r)}{r} = \psi_\omega(t, r) \quad \text{for} \quad t \leq 0.
$$

The operators $b_\omega, b_\omega^\dagger$ satisfy the standard commutation relations (which follow from (11)),

$$
\left[ b_\omega, b_\omega^\dagger \right] = \delta(\omega - \omega'), \\
\left[ b_\omega, b_{\omega'} \right] = \left[ b_\omega^\dagger, b_{\omega'}^\dagger \right] = 0,
$$

3
The above relation allow us to interprete corresponding operators as an annihilation and creation operators for the field quanta. Consequently, we can define the state $|0\rangle_{in}$ which contains no particles through the equation

$$b_\omega|0\rangle_{in} = 0 \quad \forall \omega.$$ (12)

From the moment $t = 0$, the mirror starts to move and finally, at $t = t_1$, it reaches the shape of a sphere with the radius $R = R_1$. In the period $0 \leq t \leq t_1$ the evolution of the field operator differs from the free one in (10) due to destorsion of the modes caused by the interaction with the moving boundary,

$$\hat{\phi}(t, r) = \frac{1}{2\pi} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \left[ b_\omega e^{-i\omega t}\psi_\omega(t, r) + b_\omega^\dagger e^{i\omega t}\psi_\omega^*(t, r) \right].$$ (13)

While the system remains in the state $|0\rangle_{in}$, it can no longer be viewed as containing no particles. This is due to the fact that for a generic motion of the mirror the functions $\psi_\omega(t, r)$ contain for $t \geq 0$ an admixture of the negative frequency modes $\sim e^{+\nu t}$, $\nu > 0$, and consequently $b_\omega$ cannot be interpreted as some annihilation operators for the “out” vacuum.

In order to calculate the amount of produced energy and momentum we shall compute the expectation value of the energy–momentum tensor $T_{\mu\nu}$. In the considered spherically symmetric case it has only two non-vanishing components: the energy density,

$$T_{tt} = \frac{1}{2} \left\{ \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial r} \right)^2 \right\},$$ (14)

and the radial flux of the energy (or the momentum density),

$$T_{tr} = \frac{1}{2} \left\{ \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial t} \right\}.$$ (15)

The corresponding continuity equation takes the form,

$$\partial_t T_{tt} - \frac{1}{r} \partial_r T_{tr} = 0.$$ (16)

Our main aim is to calculate quantum average values of the energy-momentum tensor. To establish a correct ultraviolet behaviour, we make use of point-splitting regularization method ($\epsilon \to 0$),

$$\langle T_{tt} \rangle^\epsilon \equiv \frac{1}{2} \left\langle \langle 0 | \partial_t \phi(t, r) \partial_t \phi(t + i\epsilon, r) + \partial_r \phi(t, r) \partial_r \phi(t + i\epsilon, r) | 0 \rangle_{in} \right\rangle.$$ (17)

After straightforward calculations we obtain the following result for the energy density:

$$\langle T_{tt} \rangle^\epsilon = \frac{1}{32\pi^2 r^2} \left\{ \frac{4}{\epsilon^2} - \frac{1}{3} \left\{ u + 2\mathcal{R}(u), u \right\} S - \frac{2}{r} \left\{ 4(1 + \mathcal{R}(u))[r - \mathcal{R}(u)] + \frac{\mathcal{R}(u)}{1 + 2\mathcal{R}(u)} \right\} \right\} + \ldots$$
\[
+ \frac{1}{r^2} \left\{ \ln \left[ 1 + \frac{4(r - \mathcal{R}(u))^2}{\epsilon^2} \right] - \ln (1 + 2\mathcal{R}(u)) \right\} + \mathcal{O}(\epsilon),
\]

and for the radial momentum density:
\[
\langle T_{tr} \rangle^\epsilon = \frac{1}{32\pi^2 r^2} \left( -\frac{1}{3} \left\{ u + 2\mathcal{R}(u), u \right\}_S - \frac{2}{r} \left[ \frac{\dot{\mathcal{R}}(u)}{r - \mathcal{R}(u)} + \frac{\ddot{\mathcal{R}}(u)}{1 + 2\mathcal{R}(u)} \right] \right) + \mathcal{O}(\epsilon),
\]

where we have used an abbreviation \( \mathcal{R}(u) = R(t_R(u)), \ u = t - r. \) The Schwartz derivative which appears in (18,19) is defined as
\[
\{ u + 2\mathcal{R}(u), u \}_S \overset{def}{=} \frac{2\dot{\mathcal{R}}(u)}{1 + 2\mathcal{R}(u)} - \frac{3}{2} \left( \frac{2\ddot{\mathcal{R}}(u)}{1 + 2\mathcal{R}(u)} \right)^2.
\]

As the formula (18) contains products of coinciding field operators taken in the same (for \( \epsilon \to 0 \)) space-time point, it diverges in the limit of vanishing regulator. The appearance of this divergence reflects well-known ultraviolet problems in continuum quantum field theories. However, we should keep in mind here that the difference of energy densities between two physically realizable situations has a definite meaning and thus should be finite.

Following this outline, we subtract from (18) the (regulated with the same prescription) vacuum energy density of the spherical wall with constant radius \( R_1, \)
\[
\langle T_{tr} \rangle^\epsilon |_{R_1} = \frac{1}{32\pi^2 r^2} \left( -\frac{1}{3} \left\{ u + 2\mathcal{R}(u), u \right\}_S - \frac{2}{r} \left[ \frac{\dot{\mathcal{R}}(u)}{r - \mathcal{R}(u)} + \frac{\ddot{\mathcal{R}}(u)}{1 + 2\mathcal{R}(u)} \right] \right) + \mathcal{O}(\epsilon),
\]

and define
\[
\langle T_{tr} \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \left( \langle T_{tr} \rangle^\epsilon - \langle T_{tr} \rangle^\epsilon |_{R_1} \right) = \frac{1}{32\pi^2 r^2} \left( -\frac{1}{3} \left\{ u + 2\mathcal{R}(u), u \right\}_S - \frac{2}{r} \left[ \frac{\dot{\mathcal{R}}(u)(r - R_1) + \mathcal{R}(u) - R_1}{(r - R_1)(r - \mathcal{R}(u))} + \frac{\ddot{\mathcal{R}}(u)}{1 + 2\mathcal{R}(u)} \right] + \frac{1}{r^2} \ln \left[ \frac{r - \mathcal{R}(u)}{(r - R_1)(1 + 2\mathcal{R}(u))} \right] \right).
\]

The expression (19), which contains only products of distinct operators, is finite in the vanishing regulator limit, thus there is no need for any subtraction procedure,
\[
\langle T_{tr} \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \langle T_{tr} \rangle^\epsilon.
\]

The renormalized quantities (21,22) satisfy the same continuity equation as their classical counterparts (14,15),
\[
\partial_t \langle T_{tr} \rangle_{\text{ren}} - \frac{1}{r} \partial_r \langle T_{tr} \rangle_{\text{ren}} = 0.
\]

It is worth to stress here that there is no contribution to the total energy from the subtracted expression (20). The quadratic term produces an infinite constant, while the other two terms in (21) return zero when integrated out over the whole space outside the sphere.
Using (21) we are finally at the position to calculate (for \( t \geq t_1 \)) the total energy. We obtain surprisingly compact formula,

\[
E = \int_{t=\text{const}} dV \langle T_{tt} \rangle_{\text{ren}} = \frac{1}{12\pi} \int_{R_1}^{\infty} dr \left( \frac{\dot{R}(u)}{1 + 2\dot{R}(u)} \right)^2.
\]  

(24)

Let us note that respective boundary terms vanish here due to the assumption of static mirror asymptotical states and the symmetry of the considered physical system. We observe that the only term in Eq.(21) which contributes to the total energy is just the one containing the Schwartz derivative.

In the remainder of this paper, we consider an example being an explicitly solvable problem. It refers to the following time-dependence of the radius of the spherical mirror,

\[
R(t) = \begin{cases} 
R_0 & \text{for } t \leq 0 \\
t + R_0 - a + a \exp \left(-\frac{t}{a}\right) & 0 \leq t \leq \bar{t} \\
t + R_0 - b - t_1 + b \exp \left(\frac{(t_1 - t)}{b}\right) & \bar{t} \leq t \leq t_1 \\
R_1 & t \geq t_1
\end{cases}
\]  

(25)

where

\[
t_1 = \frac{(R_1 - R_0)\bar{t}}{\bar{t} - v_{\text{max}}} , \quad b = -\frac{a(t_1 - \bar{t})}{\bar{t}}.
\]  

(26)

Here \( a, \bar{t}, \) the initial and final radii \( R_0 \) and \( R_1 \) are the parameters defining the \( C^1 \)–class function \( R(t) \) and \( v_{\text{max}} \) is a maximal velocity of the mirror

\[
v_{\text{max}} = 1 - \exp \left(-\frac{\bar{t}}{a}\right).
\]  

(27)

The retarded time can be calculated by solving Eq.(8),

\[
t_R(u) = \begin{cases} 
u + R_0 & \text{for } u \leq -R_0 \\
-a \ln \left(\frac{a - u - R_0}{a}\right) & -R_0 \leq u \leq a v_{\text{max}} - R_0 \\
t_1 - b \ln \left(\frac{b + t_1 - u - R_1}{b}\right) & a v_{\text{max}} - R_0 \leq u \leq t_1 - R_1 \\
u + R_1 & u > t_1 - R_1
\end{cases}
\]  

(28)

The basic function \( \mathcal{R}(u) \) is obtained immediately,

\[
\mathcal{R}(u) = t_R(u) - u,
\]  

(29)

and allows to compute explicitly the total energy \([24]\):

\[
E = \frac{1}{24\pi a(R_1 - R_0 + a v_{\text{max}} - t)} \left[ \frac{v_{\text{max}}}{1 - v_{\text{max}}^2} + \text{artanh} \ v_{\text{max}} \right].
\]  

(30)
3 Conclusions

In this paper, the radiation emitted by the spherical mirror has been considered. We assumed the movement of the mirror lasts a finite period of time. This assumption helps to define and interpret asymptotic spaces of physical states uniquely. As the initial state it is considered the vacuum state (no particles present). The vacuum is perturbed by the moving mirror, and some flux of particles (radiation) is seen by an observer in the laboratory frame. The radiation is a pure quantum effect. We have calculated the vacuum expectation value of the energy-momentum tensor. To define finite results, we adopted the time-splitting regularization technique. To make calculations simpler, we restrict ourselves only to spherical waves and spherically symmetric quantum excitations. We obtained that the radiation flux depends only upon values of the mirror radius and its time derivatives evaluated along the intersection of the world history of the mirror with the observer’s past light cone.

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