Deformations of Oka manifolds

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Abstract We investigate the behaviour of the Oka property with respect to deformations of compact complex manifolds. We show that in a family of compact complex manifolds, the set of Oka fibres corresponds to a $G_\delta$ subset of the base. We give a necessary and sufficient condition for the limit fibre of a sequence of Oka fibres to be Oka in terms of a new uniform Oka property. We show that if the fibres are tori, then the projection is an Oka map. Finally, we consider holomorphic submersions with noncompact fibres.

Keywords Oka manifold · Convex approximation property · Oka map · Deformation

Mathematics Subject Classification (2010) Primary 32G05; Secondary 32E10 · 32Q28

1 Introduction

The class of Oka manifolds has emerged from the modern theory of the Oka principle, initiated in 1989 in a seminal paper of Gromov [6]. They were first formally defined by Forstnerič in 2009 in the wake of his result that some dozen possible definitions are all equivalent [3]. The Oka property can be seen as an answer to the question: what should it mean for a complex manifold to be “anti-hyperbolic”? For more background, see the survey [5]. One of the many open problems in Oka theory is to clarify the place of Oka manifolds in the classification theory of compact complex manifolds, surfaces in particular. To address this problem, we need to understand how the Oka property behaves with respect to deformations of compact complex manifolds. In this paper, we take some first steps in this direction.

Let $\pi : X \rightarrow B$ be a family of compact complex manifolds, that is, a proper holomorphic submersion, and therefore a smooth fibre bundle, from a complex manifold $X$ onto a complex manifold $B$. Write $X_t$ for the compact complex manifold $\pi^{-1}(t)$, $t \in B$. We wish to say as much as possible about the structure of the set $O_\pi$ of $t \in B$ for which $X_t$ is Oka.
It is well known that the set of $t \in B$ for which $X_t$ is hyperbolic is open [7, Theorem 3.11.1]. Thus we might expect $O_{\pi}$ to be closed. It is not clear that this is a reasonable conjecture. We will prove two weaker results. We show that $O_{\pi}$ is $G_{\delta}$ (Corollary 8). We also prove (Corollary 14) that if $t_0 \to s$ in $B$ and $X_{t_0}$ is Oka for every $n \in \mathbb{N}$, then $X_s$ is Oka if (and in fact only if) the family $\{X_{t_n} : n \in \mathbb{N}\}$ is uniformly Oka in a sense introduced here (Definition 9).

As well as asking about Oka properties of the fibres of the projection $\pi$, we can ask about Oka properties of $\pi$ itself as a map. It is an open question whether $\pi$ is an Oka map if all its fibres are Oka. We show that the answer is affirmative if the fibres are tori (Theorem 16).

Finally, we indicate how our results in Sects. 2 and 3 can be extended to holomorphic submersions whose fibres are not necessarily compact. We point out that every $G_{\delta}$ subset of a complex manifold $B$ can be realised as $O_{\pi}$ for some holomorphic submersion $\pi : X \to B$ with noncompact fibres.

2 The convex approximation property for compact manifolds

We will formulate the Oka property of a complex manifold $Y$ as the convex approximation property: $Y$ is Oka if and only if for every $k \geq 1$, every holomorphic map to $Y$ from a compact convex subset $K$ of $\mathbb{C}^k$ can be uniformly approximated on $K$ by holomorphic maps $\mathbb{C}^k \to Y$. (By a holomorphic map on a compact set we mean a holomorphic map on an open neighbourhood of the set.)

Let $Y$ be compact. We fix a Hermitian metric $\omega$ on $Y$. We use it to filter sets of holomorphic maps by normal families. For our purposes, the choice of filtration seems immaterial. To get a quantitative handle on the Oka property of $Y$, we introduce the following definition.

**Definition 1** By a *quintuple* we shall mean a quintuple $(K, U, V, r, \epsilon)$, where $K$ is a non-empty compact subset of $\mathbb{C}^k$, $k \geq 1$, $U \subset V \subset \mathbb{C}^k$ are open neighbourhoods of $K$, $r > 0$, and $\epsilon > 0$. Note that $U$ and $V$ are assumed to be relatively compact in $\mathbb{C}^k$. For every quintuple $(K, U, V, r, \epsilon)$, let

$$
\sigma(K, U, V, r, \epsilon)(Y) = \sup_{f:U \to Y \text{ hol.}} \inf_{g:V \to Y \text{ hol.}} \|g^*\omega\|_V \in [0, \infty].
$$

Here, $\|\cdot\|$ denotes the supremum norm with respect to the Euclidean metric on $\mathbb{C}^k$, and the distance $d(f, g)$ is with respect to $\omega$. In the proof of Theorem 7 below, it is important to have a weak inequality in $\|f^*\omega\|_U \leq r$ and a strict inequality in $d(f, g) < \epsilon$. We take the infimum of the empty set to be $\infty$.

Clearly, $\sigma(K, U, V, r, \epsilon)(Y)$ increases as $r$ increases, $\epsilon$ decreases, $U$ shrinks, and $V$ expands. Also, $\sigma(K, U, V, r, \epsilon)(Y)$ is finite if and only if there is $R > 0$ such that every holomorphic map $f : U \to Y$ with $\|f^*\omega\|_U \leq r$ can be approximated to within $\epsilon$ on $K$ by a holomorphic map $g : V \to Y$ with $\|g^*\omega\|_V \leq R$. Since $Y$ is compact, whether $\sigma(K, U, V, r, \epsilon)(Y)$ is finite for all $r$ and $\epsilon$ with $K, U$, and $V$ fixed is independent of the choice of a Hermitian metric on $Y$.

**Proposition 2** The compact manifold $Y$ is Oka if and only if $\sigma(K, U, V, r, \epsilon)(Y)$ is finite for every quintuple $(K, U, V, r, \epsilon)$ such that $K$ is convex.

**Proof** $\Leftrightarrow$ This is easy (and does not require compactness of $Y$).

$\Rightarrow$ Suppose $\sigma(K, U, V, r, \epsilon)(Y) = \infty$ for some quintuple $(K, U, V, r, \epsilon)$ with $K$ convex. This means that for every $n \in \mathbb{N}$, there is a holomorphic map $f_n : U \to Y$ with $\|f_n^*\omega\|_U \leq r$. 