The Lax operator of the Gaudin type models is a 1-form on the classical level. In virtue of the quantization scheme proposed in [Talalaev04] (hep-th/0404153) it is natural to treat the quantum Lax operator as a connection; this connection is a particular case of the Knizhnik-Zamolodchikov connection [ChervovTalalaev06] (hep-th/0604128). In this paper we find a gauge transformation which produces the "second normal form" or the "Drinfeld-Sokolov" form. Moreover the differential operator naturally corresponding to this form is given precisely by the quantum characteristic polynomial [Talalaev04] of the Lax operator (this operator is called the $G$-oper or Baxter equation). This observation allows to relate solutions of the KZ and Baxter equations in an obvious way, and to prove that the immanent KZ-equations has only meromorphic solutions. As a corollary we obtain the quantum Cayley-Hamilton identity for the Gaudin-type Lax operators (including the general $\mathfrak{g}l_n[t]$ case). The presented construction sheds a new light on a geometric Langlands correspondence. We also discuss the relation with the Harish-Chandra homomorphism.

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1 Introduction and main results

1.1 Simple fact from linear algebra

Let $L$ be a generic matrix over a field, then it can be conjugated to the "second normal form", i.e. there exists a matrix $C$, such that

$$
C L C^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
H_n & H_{n-1} & H_{n-2} & \cdots & H_2 & H_1
\end{pmatrix}
$$

where $H_i$ are the coefficients of the characteristic polynomial:

$$
det(L(z) - \lambda) = (-1)^n(\lambda^n - \sum_i H_{n-i}\lambda^i)
$$

i.e. $H_1 = Tr(L), H_n = (-1)^{n-1}det(L)$.

To prove this statement one just takes the matrix $C$ to be the matrix of the basis change to the basis consisting of vectors: $v, Lv, L^2v, \ldots, L^{n-1}v$, where $v$ is a generic vector.

Example 1 Let $L$ be a numerical $2 \times 2$ matrix

$$
L = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

with $c \neq 0$. This matrix can be transformed to the second normal form by the matrix

$$
C = \begin{pmatrix}
0 & 1 \\
c & d
\end{pmatrix}
$$

i.e.

$$
C L C^{-1} = \begin{pmatrix}
0 & 1 \\
-det(L) & tr(L)
\end{pmatrix}
$$

For the purposes of the present paper it is instructive to rewrite the formula above in the following more general way:

$$
C L = \begin{pmatrix}
0 & 1 \\
-det(L) & tr(L)
\end{pmatrix} C \Leftrightarrow C(L - \lambda) = \left( \begin{pmatrix}
0 & 1 \\
-det(L) & tr(L)
\end{pmatrix} - \lambda \right) C
$$

(2)
1.2 The quantization scheme

Here we recall the quantization scheme for the class of integrable systems with the rational Lax operator on the example of the Gaudin model.

**Lax operator** The Lax operator of this model is a rational function with distinct poles:

\[ L(z) = \sum_{i=1\ldots N} \frac{\Phi_i}{z - z_i}, \]

\( \Phi_i \in \text{Mat}_n \otimes \bigoplus \text{gl}_n \subset \text{Mat}_n \otimes U(\text{gl}_n)^{\otimes N} \) is defined by the formula:

\[ \Phi_i = \sum_{kl} E_{kl} \otimes e^{(i)}_{kl} \] (3)

where \( E_{kl} \) form the standard basis in \( \text{Mat}_n \) and \( e^{(i)}_{kl} \) form a basis in the \( i \)-th copy of \( \text{gl}_n \).

**Remark 1** All our considerations identically holds true for any other Lax operators satisfying the linear R-matrix commutation relation 24, in particular for the standard Lax operators for \( \text{gl}_n[t] \) given by the formula 22.

**Quantum characteristic polynomial** The quantum commutative family is constructed with the help of the quantum characteristic polynomial

\[ "\text{det}(L(z) - \partial_z) = Tr A_n(L_1(z) - \partial_z)\ldots(L_n(z) - \partial_z) = \sum_{k=0}^{n} C_n^k (-1)^{n-k} QI_k(z) \partial_z^{n-k} \] (4)

**Theorem** [Talalaev04] The coefficients \( QI_k(z) \) commute

\[ [QI_k(z), QI_m(u)] = 0 \]

and quantize the classical Gaudin hamiltonians.

In these formulas \( L_i(z) \) is an element of \( \text{Mat}_n^{\otimes n} \otimes U(\text{gl}_n)^{\otimes N} \) obtained via the inclusion \( \text{Mat}_n \hookrightarrow \text{Mat}_n^{\otimes n} \) as the \( i \)-th component. The element \( A_n \) is the normalized antisymmetrization operator in \( \mathbb{C}^{n^{\otimes n}} \).

**Remark 2** By the same formula in [ChervovTalalaev06] it was constructed a commutative subalgebra in \( U(\text{gl}_n)[t]/t^N \), \( U(\text{gl}_n)[t] \) and the center of the universal enveloping affine algebra on the critical level.
1.3 KZ-equation, G-opers and Baxter equation

The brief relation scheme is the following:

- **Langlands correspondence. G-opers.**
  The scalar differential operator given by
  \[ \chi(\text{"det"}(L(z) - \partial_z)) = \sum_{k=0}^{n} C_k^n (-1)^{n-k} \chi(QI_k(z)) \partial_z^{n-k} \tag{5} \]
  is called G-oper in the theory of the geometric Langlands correspondence, this defines a connection on the punctured disc (Galois side of the correspondence). Here \( \chi \) is a character of the commutative subalgebra generated by \( QI_k(z) \). It is related to the character of the center of \( U_{\text{crit}}(\hat{gl}_n) \) and hence to the representation of \( U_{\text{crit}}(\hat{gl}_n) \) (automorphic side of the correspondence).

- **Baxter equation. Baxter’s Q-operator.**
  Baxter equation and its solution (the Baxter’s Q-operator) are given by the formula:
  \[ \pi(\text{"det"}(L(z) - \partial_z))Q(z) = \sum_{k=0}^{n} C_k^n (-1)^{n-k} \pi(QI_k(z)) \partial_z^{n-k}Q(z) = 0. \tag{6} \]
  Here \((\pi, H)\) is a representation of the algebra \( U(\mathfrak{gl}_n)^{\otimes N} \) in a Hilbert space \( H \). \( U(\mathfrak{gl}_n)^{\otimes N} \) is the algebra of quantum observables of the Gaudin model, 
  \[ H = V_1 \otimes \ldots \otimes V_N \]
  is its Hilbert space. \( Q(z) \) is an \( \text{End}(H) \)-valued function.

- **Knizhnik-Zamolodchikov equation.**
  The standard KZ-equation [KnizhnikZamolodchikov84] for the particular value of the level is given by the formula:
  \[ \pi(L(z) - \partial_z)\Psi(z) = \left( \sum_{i=1}^{N} \frac{\sum_{kl} E_{kl} \otimes \pi_i(e_{kl})}{z - z_i} - \partial_z \right) \Psi(z) = 0. \tag{7} \]
  Here \((\pi, V_1 \otimes \ldots \otimes V_N)\) is a representation of \( U(\mathfrak{gl}_n)^{\otimes N} \) and \( \Psi(z) \) is a \( \mathbb{C}^n \otimes V_1 \otimes \ldots \otimes V_N \)-valued function. In [ChervovTalalaev04] it was shown that having a solution for the universal KZ equation \((L(z) - \partial_z)\Psi(z) = 0\) one obtains solutions for the universal G-oper just taking an arbitrary component \( \Psi(z)_i \) of the vector \( \Psi(z) \) \((\Psi(z)_i \text{ is a } V_1 \otimes \ldots \otimes V_N\text{-valued function.})\)

For more details we refer to our previous work [ChervovTalalaev06].

1.4 Main results

Conjugation of the quantum Lax operator to the Drinfeld-Sokolov form

Let \( L(z) \in \text{Mat}_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes \text{Fun}(z) \) be the quantum Lax operator of the Gaudin model, here \( \text{Fun}(z) \) is an appropriate space of functions on the formal parameter \( z \). Let
us denote by $L^{[i]}(z)$ the *quantum powers* of the Lax operator defined by the following formulas:

\[
L^{[0]} = Id \\
L^{[i]} = L^{[i-1]}L + \partial_z L^{[i-1]}
\]

**Theorem 1** The element $C(z)$ of $\text{Mat}_n \otimes U(\mathfrak{gl}_n)^{\otimes N} \otimes \text{Fun}(z)$ given by

\[
C(z) = \begin{pmatrix} v \\ vL \\ vL^2 \\ \vdots \\ vL^{[n-1]} \end{pmatrix}
\]

where $v \in \mathbb{C}^n$ provides the following gauge transformation

\[
C(z)(L(z) - \partial_z) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ QH_n & QH_{n-1} & QH_{n-2} & \cdots & QH_2 & QH_1 \end{pmatrix} - \partial_z C(z)
\]

where

\[
\text{"det"}(L(z) - \partial_z) = \text{Tr}A_n(L_1(z) - \partial_z)\ldots (L_n(z) - \partial_z) = (-1)^n(\partial_z^n - \sum_i QH_{n-i}\partial_z^i) \quad (10)
\]

One denotes the connection on the right hand side of the Drinfeld-Sokolov type.

**Corollary 1** The quantum powers of the Lax operator satisfy the quantum Cayley-Hamilton identity

\[
L^{[n]}(z) = \sum_{i=1}^{n} QH_i(z)L^{[n-i]}(z)
\]

**Proof** Considering the last line of the equation (9) one obtains

\[
vL^{[n-1]}(z)(L(z) - \partial_z) = \sum_{i=1}^{n} vQH_i(z)L^{[n-i]}(z) - \partial_z vL^{[n-1]}(z)
\]

which is equivalent to the corollary □

As another corollary we obtain the simple way to produce solutions of the universal $G$-oper from solutions for the KZ equation by a linear transformation. This method
is in agreement with the one considered in [ChervovTalalaev04].

**Historic remarks** The Cayley-Hamilton identity above is the first (to the best of our knowledge) case of such identity where the Lax operator with the spectral parameter is used. The unexpected appearance of the corrections to the powers of $L(z)$ differs it from the standard ones. For the Lax operator without spectral parameter the identities were discussed in [HC] and references therein (see also [Molev02] section 4.2 page 37). In [Kirillov00] (see section 2.4 ”Discussion”) the Cayley-Hamilton identity for $\mathfrak{gl}_n$ was treated (this paper was inspiring for us). Let us also mention that in [GelfandKrobLasconexeLecercleRetakhThibon94] (section 8.6 page 96) the generalization of the Hamilton-Cayley identity of somewhat different nature was found for matrices with coefficients in arbitrary noncommutative algebra.

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## 2 Main section

### 2.1 Comparison with the general KZ ↔ G-oper correspondence

In [ChervovTalalaev04] it was observed that there is a simple connection between solutions of the KZ equation

$$(L(z) - \partial_z)S(z) = 0$$

for $S(z)$ - a function which takes values in $\mathbb{C}^n \otimes V$ where $V$ is a representation of $U(\mathfrak{gl}_n) \otimes N$; and of the equation produced from the quantum characteristic polynomial

$$"\text{det}"(L(z) - \partial_z)\Psi(z) = 0 \quad (12)$$

for $\Psi(z)$ - a function with values in $V$. One has to take as $\Psi(z)$ the projection to the antisymmetric part of the expression $U(z) = v_1 \otimes \ldots \otimes v_{n-1} \otimes S(z)$ where $v_i$ are arbitrary vectors in $\mathbb{C}^n$. With special choice of vectors $v_i$ one shows that all components of $S(z)$ over $\mathbb{C}^n$ solve the equation (12).

**Remark 3** The role of the equation (12) is very important in the problem of solving the quantum model and in the Langlands correspondence, namely by restricting this equation to the common eigen-vector for the Gaudin hamiltonians one obtains the so-called G-oper.
the condition that this differential equation does not have monodromy is equivalent to the Bethe ansatz conditions for the eigenvalues of quantum hamiltonians.

The formula (2) of the main theorem also provides such a relation, namely: if $S(z)$ solves the KZ-equation

$$(L(z) - \partial_z)S(z) = 0$$

then the first vector component of $C(z)S(z)$ solves the Baxter-type (or G-oper) equation (12). Taking $C(z)$ in the form 8 with the vector $v = (0, ..., 1, ..., 0)$ we see that the first component of $C(z)S(z)$ is just $S(z)_i$.

**Corollary 2** The KZ equation

$$(\pi(\partial_z - L(z)))S(z) = 0$$

has only rational solutions (here $\pi$ is a finite-dimensional representation of $U(\mathfrak{g}_n^{\otimes N})$).

**Proof** It was conjectured in [ChervovTalalaev04] (on the basis of [Frenkel95] and the ideas of Baxter, Gaudin, Sklyanin) that the Baxter-type equation

$$\pi(\det(\partial_z - L(z)))\Psi(z) = 0$$

has only rational solutions. The conjecture was proved in [MukhinTarasovVarchenko05] (see theorem 4.1 page 12). On the other hand we know from the result of [ChervovTalalaev04] discussed above that any component $S(z)_i$ of the vector $S(z)$ solves Baxter equation, hence $S(z)$ is the vector-valued rational function as well. □

### 2.2 Conjugation

Let us firstly give some examples of quantum powers of the Lax operator (8):

\[
L^{[1]}(z) = L(z) \\
L^{[2]}(z) = L^2(z) + L'(z)
\]

Hence for the $n = 2$ case the matrix $C$ can be taken just in the classical form

$$C(z) = \begin{pmatrix} 0 & 1 \\ c(z) & d(z) \end{pmatrix}$$

(14)

where the quantum Lax operator is

$$L(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

(15)

**Proof of theorem**

For the general case let us consider a solution $\Psi(z)$ for the KZ equation

$$L(z)\Psi(z) = \partial_z\Psi(z)$$

6
Let $\Phi(z) = C(z)\Psi(z)$ where $C(z)$ is given by the formula (8). Then

\[
\begin{align*}
\Phi_1(z) &= < v, \Psi(z) > \\
\Phi_2(z) &= < vL(z), \Psi(z) >= < v, \partial_z \Psi(z) > \\
\vdots \\
\Phi_k(z) &= < v(L^{[k-1]}L(z) + \partial_z L^{[k-1]}), \Psi(z) > \\
&= < vL^{[k-1]}, \partial_z \Psi(z) > + < v\partial_z L^{[k-1]}, \Psi(z) > = \partial_z \Phi_{k-1}(z)
\end{align*}
\]

Hence such $C(z)$ transforms the KZ connection to a Drinfeld-Sokolov type connection

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
\partial_n(z) & \vartheta_{n-1}(z) & \vartheta_{n-2}(z) & \ldots & \vartheta_2(z) & \vartheta_1(z)
\end{pmatrix}
- \partial_z
\]

To prove that this is exactly the connection on the right-hand side of (9) one has to apply the fact due to [ChervovTalalaev04], that each component of the KZ solution $\Psi$ satisfy the $G$-oper. On the other hand by construction we obtain that $< v, \Psi(z) >$ satisfy the differential equation

\[
(\partial_z^n - \sum_{i=1}^n \vartheta_i(z)\partial_z^{n-i}) < v, \Psi(z) >= 0 \quad (16)
\]

Due to generality of the choice of the KZ solution $\Psi(z)$ and the vector $v \in \mathbb{C}^n$ we conclude that the differential operator (16) and the $G$-oper coincide up to multiplication by a function. They coincide because their leading terms do.

### 3 Factorization of QCP

#### 3.1 Miura form

**Corollary 3**  
Let the QCP be represented in a factorized form

\[
"\text{det}"(\partial_z - L(z)) = \text{Tr} A_n(\partial_z - L_1(z))\ldots(\partial_z - L_n(z)) = (\partial_z - \chi_n(z))\ldots(\partial_z - \chi_1(z)) \quad (17)
\]

then there exists $C^d(z)$ belonging to some algebraic extension of the quantum algebra such that:

\[
C^d(z)(\partial_z - L(z)) = \left( \partial_z - \begin{pmatrix}
\chi_1(z) & 1 & 0 & 0 & \ldots & 0 \\
0 & \chi_2(z) & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & \chi_{n-1}(z) & 1 \\
0 & 0 & 0 & \ldots & 0 & \chi_n(z)
\end{pmatrix} \right) C^d(z) \quad (18)
\]

Let us denote the connection on the right hand side by $\partial_z - L_M$.  

We provide a proof in the appendix.
Remark 4  Similar factorizations of Baxter-type equations often occur in mathematical physics literature, for the general Gaudin and XXX models the explicit factorization of scalar $G$-opers related to our constructions were given recently by E. Frenkel, E. Mukhin, V. Tarasov, and A. Varchenko.

Remark 5  In fact the differential operator $D = \sum_i H_i(z) \partial_z^i$ can be represented in a factorized form $D = (\partial_z - \chi_n(z)) \cdots (\partial_z - \chi_1(z))$ non-uniquely. Let $\Psi_1, \ldots, \Psi_n$ be a basis of solutions of the equation $D\Psi = 0$, let us pick out $\chi_1, \ldots, \chi_n$ recursively such that

$$ (\partial_z - \chi_i) \cdots (\partial_z - \chi_1) \Psi_i = 0 $$

For example $\chi_1 = \Psi'_1/\Psi_1$. Hence different factorizations are parametrized by the $n$-dimensional flag variety.

Remark 6  The matrix $C(z)$ is invertible only in the field of fractions of $U(\mathfrak{gl}_n)^{\otimes N}$, this means that $\pi(\partial_z - L(z))$ and $\pi(\partial_z - L_{DS}(z))$ are not fully gauge equivalent, where $\pi$ is a representation of $U(\mathfrak{gl}_n)^{\otimes N}$. One can produce a solution of $\pi(\det^{n}(\partial_z - L(z)))$ from any solution of $\pi(\partial_z - L(z))$, but this correspondence is not necessarily a bijection.

Remark 7  All the results of this paper can be generalized to semisimple Lie algebras and quantum groups following the ideas discussed in [ChervovTalalaev06]. For example for Lax operators related to $\mathfrak{gl}_n[t]$ the same theorems should hold (the Lax operator can be considered in any representation, not only fundamental), for Yangian: $e^{\partial_z} - T(z)$ can be conjugated to $DS$-form with the coefficients given by $\det^n(e^{\partial_z} - T(z))$, for $U_q(\mathfrak{g})$ the same is valid for $q^{\partial_z} - L^+(q^z)$.

3.2 Harish-Chandra map

In fact the factorization formula (17) for a particular case of the QCP provides an explicit realization of the Harish-Chandra homomorphism.

Let us consider the representation $\pi, V_\lambda$ of $\mathfrak{gl}_n$ with the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, (i.e. that there exists a vector $|0> \in V_\lambda$ such that $\pi(e_{ii})|0> = \lambda_i|0>$ and $\pi(e_{ij})|0> = 0$ for $i < j$).

Conjecture 1  Consider the 1-spin Lax operator $L(z) = \Phi/z$ where $\Phi \in \text{Mat}_n \otimes U(\mathfrak{gl}_n)$, $\Phi = \sum E_{kl} \otimes e_{kl}$

$$ \pi(\det^n(\partial_z - L(z)))|0> = (\partial_z - \lambda_1/z)(\partial_z - \lambda_2/z)\cdots(\partial_z - \lambda_n/z)|0>,$$

$$ \pi(\det^n(\partial_z + L(z)))|0> = (\partial_z + \lambda_n/z)(\partial_z + \lambda_{n-1}/z)\cdots(\partial_z + \lambda_1/z)|0> $$

Remark 8  Actually one can prove that the coefficients of the 1-spin QCP belong to the center of the universal enveloping algebra $U(\mathfrak{gl}_n)$ so it is true that:

$$ \pi(\det^n(\partial_z - L(z))) = (\partial_z - \lambda_1/z)(\partial_z - \lambda_2/z)\cdots(\partial_z - \lambda_n/z)\text{Id} $$

and the same for $\pi(\det^n(\partial_z + L(z)))$. These formulas obviously provide an explicit description for the Harish-Chandra map.
**Remark 9** This conjecture is related to the much more general results of E. Frenkel, E. Mukhin, V. Tarasov, and A. Varchenko.

By a straightforward calculation we obtain

**Proposition 1** The conjecture is true for the $\mathfrak{gl}_2$ case.

Let us remark how the shifted action of the Weyl group, well-known in Harish-Chandra homomorphism theory, can be represented by different factorizations of QCP. The images of Casimir elements under the Harish-Chandra homomorphism are symmetric functions with respect to the action of the Weyl group. The action is given not by the naive formula $(\lambda_i) \rightarrow (\lambda_{\sigma(i)})$, it is shifted. In the $\mathfrak{gl}_2$ case the Weyl group is $S_2 = \langle 1, p \rangle$ which acts by

$$p(\lambda_1, \lambda_2) = (\lambda_2 - 1, \lambda_1 + 1).$$

**Observation** The same action of the permutation group $S_2$ arises considering different factorizations of the QCP:

$$(\partial_z - \frac{\lambda_1}{z})(\partial_z - \frac{\lambda_2}{z}) = (\partial_z - \frac{\lambda_2 - 1}{z})(\partial_z - \frac{\lambda_1 + 1}{z})$$

(21)

4 Appendices

4.1 Standard Lax operator for $\mathfrak{gl}_n[t]$ 

Let us recall some definitions for the reader’s convenience.

**Definition 1** The standard Lax operator for $U(\mathfrak{gl}_n[t]/(t^{N} = 0)$ $(N$ can be finite or infinite) and the standard Lax operator for $U(\mathfrak{gl}_n[t]/(t^{N} = 0)$ with a constant term $K$ are given by the formulas:

$$L(z) = \sum_{i=0}^{N-1} \Phi_i z^{-(i+1)}, \quad L_K(z) = K + \sum_{i=0}^{N-1} \Phi_i z^{-(i+1)}.$$  

(22)

Here $K$ is an arbitrary constant matrix, $z$ - a formal parameter (not the same as $t$!),

$$\Phi_i \in \left(Mat_n \otimes U(\mathfrak{gl}_n[t])/(t^{N} = 0)\right) \cong \left(Mat_n[U(\mathfrak{gl}_n[t])/(t^{N} = 0)]\right)$$

are given by

$$\Phi_i = \sum_{kl} E_{kl} \otimes e_{kl} t^i \iff (\Phi_i)_{kl} = e_{kl} t^i$$

(23)

where $E_{kl} \in Mat_n$, $e_{kl} t^i \in U(\mathfrak{gl}_n[t])$. Both $E_{kl}$ and $e_{kl}$ are the matrices with the only $(k,l)$-th nontrivial matrix element equal to 1. We consider them as elements of different algebras: the first is an element of the associative algebra $Mat_n$, the second - of the universal enveloping algebra $U(\mathfrak{gl}_n[t])$.

It is well-known that the Gaudin Lax operator given by the formula 3 as well as $L(z), L_K(z)$ satisfy the same commutation relation:
Proposition 2

\[ [\dot{L}(z), \dot{L}(u)] = \left( \frac{P}{z-u}, \dot{L}(z) + \dot{L}(u) \right) \] (24)

Remark 10  The Lax operator \( L(z) = K + \Phi/z \) defines an integrable system on coadjoint orbits of \( \mathfrak{gl}_n \). This system is sometimes called the Mishenko-Fomenko system, and the method to obtain it - the "argument translation method".

4.2 Conjugation to Miura form

Proposition 3  Consider the differential operator:

\[ \partial^n z - \sum_{i=1}^n H_i(z) \partial^{n-i} z = (\partial_z - \chi(z)) \ldots (\partial_z - \chi_1(z)) \]

then the Drinfeld-Sokolov connection

\[ \partial_z - L_{DS} \partial_z = \partial_z - \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
H_n(z) & H_{n-1}(z) & H_{n-2}(z) & \ldots & H_2(z) & H_1(z)
\end{pmatrix} \] (25)

is gauge equivalent to the Miura connection

\[ (\partial_z - L_M(z)) = \partial_z - \begin{pmatrix}
\chi_1(z) & 1 & 0 & 0 & \ldots & 0 \\
0 & \chi_2(z) & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & \chi_{n-1}(z) & 1 \\
0 & 0 & 0 & \ldots & 0 & \chi_n(z)
\end{pmatrix} \] (26)

Moreover if the vector \( \Psi(z) \) satisfies the equation \( (\partial_z - L_M(z))\Psi(z) = 0 \), then its first component provides a solution of the equation \( (\partial_z - \chi(z)) \ldots (\partial_z - \chi_1(z))\Psi_1(z) = 0 \).

Proof  We proceed by showing that there exists such an element \( B \) which transform the connection \( \partial_z - L_{DS} \) to the connection of the Miura form \( \partial_z - L_M \). Let \( S \) be a solution of the equation

\[ \partial_z S = \begin{pmatrix}
\chi_1 & 1 & 0 & 0 & \ldots & 0 \\
0 & \chi_2 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \chi_{n-1} & 1 \\
0 & 0 & 0 & \ldots & 0 & \chi_n
\end{pmatrix} S \] (27)
(we omit the dependence on the parameter $z$ for simplicity). One has
\[
\partial_z S_1 = \chi_1 S_1 + S_2 \\
\partial^2_z S_1 = (\partial_z \chi_1 + \chi_1^2) S_1 + (\chi_1 + \chi_2) S_2 + S_3
\]
etc.

Hence by a linear change of basis with the lower triangular matrix $B$ of the form
\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
\chi_1 & 1 & 0 & 0 & \ldots & 0 \\
\partial_z \chi_1 + \chi_1^2 & \chi_1 + \chi_2 & 1 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & 1
\end{pmatrix}
\] (28)

one transforms the connection of the Miura type $\partial_z - L_M$ to some connection of the DS type. The only thing to prove is that this connection is exactly $\partial_z - L_{DS}$. To do this one has to realize that the condition $(\partial_z - L_{DS}) \tilde{S} = 0$ is equivalent to the condition
"det"$(\partial_z - L)S_1 = 0$. Let us show that $S_1$ solves the equation "det"$(\partial_z - L)S_1 = 0$. Indeed,
\[
\begin{align*}
S_2 &= (\partial_z - \chi_1)S_1 \\
S_3 &= (\partial_z - \chi_2)(\partial_z - \chi_1)S_1 \\
\ldots \\
0 &= (\partial_z - \chi_n) \ldots (\partial_z - \chi_1)S_1
\end{align*}
\]

□

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