CANONICAL CONNECTION ON CONTACT MANIFOLDS

YONG-GEUN OH, RUI WANG

Abstract. We introduce a canonical affine connection on the contact manifold $(Q, \xi)$, which is associated to each contact triad $(Q, \lambda, J)$ where $\lambda$ is a contact form and $J : \xi \to \xi$ is an endomorphism with $J^2 = -id$ compatible to $d\lambda$. We call it the contact triad connection of $(Q, \lambda, J)$ and prove its existence and uniqueness. The connection is canonical in that the pull-back connection $\phi^*\nabla$ of a triad connection $\nabla$ becomes the triad connection of the pull-back triad $(Q, \phi^*\lambda, \phi^*J)$ for any diffeomorphism $\phi : Q \to Q$ satisfying $\phi^*\lambda = \lambda$ (sometimes called a strict contact diffeomorphism). It also preserves both the triad metric $g(\lambda, J) = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda$ and $J$ regarded as an endomorphism on $TQ = \mathbb{R}\{X_{\lambda}\} \oplus \xi$, and is characterized by its torsion properties and the requirement that the contact form $\lambda$ be holomorphic in the CR-sense. In particular, the connection restricts to a Hermitian connection $\nabla^\pi$ on the Hermitian vector bundle $(\xi, J, g_\xi)$ with $g_\xi = d\lambda(\cdot, J\cdot)|_\xi$, which we call the contact Hermitian connection of $(\xi, J, g_\xi)$. These connections greatly simplify tensorial calculations in the sequel [OW1], [OW2] performed in the authors’ analytic study of the map $w$, called contact instantons, which satisfy the nonlinear elliptic system of equations

\[ \bar{D}_w w = 0, \ d(w^*\lambda \circ j) = 0 \]

in the contact triad $(Q, \lambda, J)$.


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1. Introduction

A contact manifold \((Q, \xi)\) is a \(2n+1\) dimensional manifold equipped with a completely non-integrable distribution of rank \(2n\), called a contact structure. Complete non-integrability of \(\xi\) can be expressed by the non-vanishing property
\[
\lambda \wedge (d\lambda)^n \neq 0
\]
for a one-form \(\lambda\) which defines the distribution, i.e., \(\text{ker} \lambda = \xi\). Such a one-form \(\lambda\) is called a contact form associated to \(\xi\). Associated to the given contact form \(\lambda\), we have the unique vector field \(X_\lambda\) determined by
\[
X_\lambda|\lambda \equiv 1, \quad X_\lambda|d\lambda \equiv 0.
\]
In relation to the study of pseudoholomorphic curves, one considers an endomorphism \(J : \xi \to \xi\) with \(J^2 = -\text{id}\) and regard \((\xi, J)\) a complex vector bundle. In the presence of the contact form \(\lambda\), one usually considers the set of \(J\) that is compatible to \(d\lambda\) in the sense that the bilinear form \(g_\xi = d\lambda(\cdot, J\cdot)\) defines a Hermitian vector bundle \((\xi, J, g_\xi)\) on \(Q\). We call the triple \((Q, \lambda, J)\) a contact triad.

For each given contact triad, we equip \(Q\) with the triad metric
\[
g = d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda.
\]
For a given Riemann surface \((\Sigma, j)\) and a smooth map \(w : \Sigma \to Q\), we decompose the derivative \(dw : T\Sigma \to TQ\) into
\[
dw = \pi dw + w^* \lambda X_\lambda
\]
as a \(w^* TQ\)-valued one-form and then
\[
d^\ast w := \pi dw = \partial^\ast w + \overline{\partial^\ast} w
\]
as a \(w^* \xi\)-valued one-form on \(\Sigma\) where \(\partial^\ast w\) (resp. \(\overline{\partial^\ast} w\)) is the \(J\)-linear part (resp. anti-\(J\)-linear part) of \(d^\ast w\).

Definition 1.1 (Contact Cauchy-Riemann map). Let \((\Sigma, j)\) be a Riemann surface with a finite number of marked points and denote by \(\tilde{\Sigma}\) the associated punctured Riemann surfaces. We call a map \(w : \tilde{\Sigma} \to Q\) a contact Cauchy-Riemann map if \(\overline{\partial^\ast} w = 0\).

It turns out that to establish the geometric analysis necessary for the study of associated moduli space, one needs to augment the equation \(\overline{\partial^\ast} w = 0\) by
\[
d(w^* \lambda \circ j) = 0. \quad (1.1)
\]
We refer to [H2] for the origin of such equation in contact geometry and to [OW1], [OW2] for the detailed analytic study of priori \(W^{k,2}\)-estimates and asymptotic convergence on punctured Riemann surfaces.
**Definition 1.2** (Contact instanton). Let \( \Sigma \) be as above. We call a pair of \((j, w)\) of a complex structure \(j\) on \(\Sigma\) and a map \(w: \dot{\Sigma} \to Q\) a **contact instanton** if it satisfies

\[
\bar{\mathcal{D}} w = 0, \quad d(w^* \lambda \circ j) = 0.
\]

In the course of studying the geometric analysis of such a map, we need to simplify the tensorial calculations by choosing a special connection as in (almost) complex geometry.

The main purpose of the present paper is to introduce the notion of the **contact triad connection** of the triad \((Q, \lambda, J)\) which is the contact analog to Ehresmann-Libermann’s notion of **canonical connection** on the almost Kähler manifold \((M, \omega, J)\). (See [EL], [L1], [L2], [Ko] for general exposition on the canonical connection.) We recall that the leaf space of Reeb foliations of the contact triad \((Q, \lambda, J)\) canonically carries a (non-Hausdorff) almost Kähler structure which we denote by \((\hat{Q}, \hat{d}\lambda, \hat{J})\).

**Theorem 1.3** (Contact triad connection). Let \((Q, \lambda, J)\) be any contact triad of contact manifold \((Q, \xi)\). Denote by \(g_\xi + \lambda \otimes \lambda =: g\) the natural Riemannian metric on \(Q\) induced by \((\lambda, J)\), which we call a contact triad metric. Then there exists a unique affine connection \(\nabla\) that has the following properties:

1. \(\nabla\) is a Riemannian connection of the triad metric.
2. The torsion tensor of \(\nabla\) satisfies \(T(X_\lambda, Y) = 0\) for all \(Y \in TQ\).
3. \(\nabla X_\lambda X_\lambda = 0\) and \(\nabla Y X_\lambda \in \xi\), for \(Y \in \xi\).
4. \(\nabla^\pi := \pi \nabla|_\xi\) defines a Hermitian connection of the vector bundle \(\xi \to Q\) with Hermitian structure \((d\lambda, J)\).
5. The \(\xi\) projection, denoted by \(T^\pi := \pi T\), of the torsion \(T\) is of \((0, 2)\)-type in its complexification, i.e., satisfies the following properties:

\[
T^\pi(JY, Y) = 0
\]

for all \(Y\) tangent to \(\xi\).

6. For \(Y \in \xi\),

\[
\nabla JY X_\lambda + J \nabla Y X_\lambda = 0.
\]

We call \(\nabla\) the contact triad connection.

We would like to note that Axioms (4) and (5) are nothing but properties of the canonical connection on the tangent bundle of the (non-Hausdorff) almost Kähler manifold \((\hat{Q}, \hat{d}\lambda, \hat{J}_\xi)\) lifted to \(\xi\). On the other hand, Axioms (1), (2), (3) indicate this connection behaves like the Levi-Civita connection when the Reeb direction \(X_\lambda\) get involved. Axiom (6) is an extra requirement to connect the information in \(\xi\) part and \(X_\lambda\) part, which is used to dramatically simplify our calculation in [OW1], [OW2].

The contact triad connection \(\nabla\) canonically induces a Hermitian connection for the Hermitian vector bundle \((\xi, J, g_\xi)\) with \(g_\xi = d\lambda(\cdot, J\cdot)|_\xi\). We call this vector bundle connection the **contact Hermitian connection**.

In fact, one can explicitly write the formula for the contact triad connection in terms of the Levi-Civita connection.

**Theorem 1.4.** Let \((Q, \lambda, J)\) be a contact triad and \(\Pi: TQ \to TQ\) be the idempotent with \(\text{Im} \Pi = \xi\) and \(\ker \Pi = \mathbb{R}\{X_\lambda\}\). Extend \(J\) to \(TQ\) setting \(J(X_\lambda) = 0\) and still
denote by the same $J$. Let $\nabla$ be the contact triad connection and $\nabla^{LC}$ be the Levi-Civita connection of the triad metric. Then we have
\[ \nabla = \nabla^{LC} + B \]
where $B$ is the tensor of type $(1,2)$ defined by
\[ B = B_1 + B_2 \]
where
\[ B_1(Z_1, Z_2) = -\frac{1}{2} J(\nabla^{LC}_{Z_1}, JZ_2), \]
\[ B_2(Z_1, Z_2) = \frac{1}{2} (-\langle Z_2, X_\lambda \rangle JZ_1 - \langle Z_1, X_\lambda \rangle JZ_2 + \langle JZ_1, Z_2 \rangle X_\lambda). \] (1.3)

In fact, the contact triad connection is one of the $\mathbb{R}$-family of affine connections satisfying Axioms (1) - (5) with (6) replaced by
\[ \nabla JY X_\lambda + J \nabla Y \alpha = c Y, \quad c \in \mathbb{R}. \]
Contact triad connection corresponds to $c = 0$ and the connection $\nabla + B_1$ corresponds to $c = -1$.

One immediate consequence of the uniqueness of the triad connection is the following naturality of the contact triad connection.

**Corollary 1.5 (Naturality).** Let $\nabla$ be the contact triad connection of the triad $(Q, \lambda, J)$. Consider any strict contact diffeomorphism $\phi : Q \to Q$, i.e., a diffeomorphism $\phi$ satisfying $\phi^* \lambda = \lambda$. Then the pull-back connection $\phi^* \nabla$ is the contact triad connection associated to the triad $(Q, \lambda, \phi^* J)$. In particular, this applies to any diffeomorphism arising from the Reeb flow $\phi^t$ of $\dot{x} = X_\lambda(x)$.

While our introduction of Axiom (6) is motivated by our attempt to simplify the tensor calculations [OW1], it has a nice geometric interpretation in terms of CR-geometry. We call a one-form $\alpha$ is CR-holomorphic if $\alpha$ satisfies
\[ \nabla X_\lambda \alpha = 0, \]
\[ \nabla Y \alpha + J \nabla J Y \alpha = 0 \quad \text{for} \quad Y \in \xi. \]

We refer to Definition 3.3 for the definition for arbitrary $k$-forms. With this definition, we prove the following whose proof we refer to the proof of Proposition 3.4.

**Proposition 1.6.** In the presence of other defining properties of contact triad connection, Axiom (6) is equivalent to the statement that $\lambda$ is holomorphic in the CR-sense.

An immediate consequence of this proposition is the following important property of the pull-back form $w^* \lambda$ for any contact Cauchy-Riemann map.

**Corollary 1.7.** Let $w : (\Sigma, j) \to (Q, J)$ be any Cauchy-Riemann map. Then $w^* \lambda$ is a holomorphic one-form in that
\[ \nabla^w (w^* \lambda) := \frac{\nabla w^* \lambda + J \nabla j(w^* \lambda)}{2} = 0. \]
Or equivalently, $w^* \lambda$ is the imaginary part of holomorphic section of the complex line bundle of $L^w_\lambda \to \Sigma$ where $L^w_\lambda = w^* L_\lambda$ is the pull-back bundle of the canonical complex line bundle $C \otimes \mathbb{R} \{X_\lambda\} \to \mathcal{Q}$.

Some further elaboration of this point of view in relation to the study of geometric analysis of contact instantons will be given in [Oh2].

Some motivation for the study of the canonical connection is in order. For given triad $(Q, \lambda, J)$, one commonly considers its symplectization $W = \mathbb{R}_+ \times Q$ with the symplectic form $\omega = d(r\lambda)$ or $\mathbb{R} \times Q$ with $\omega = d(e^s\lambda)$ with $r = e^s$ in relation to the study of contact topology of contact manifold $(Q, \xi)$ in the literature starting from [H1]. A cylindrical almost complex structure on $\mathbb{R} \times Q$ is given by

$$J = J_0 \oplus J, \quad TW \cong \mathbb{R} \left\{ \frac{\partial}{\partial s} \right\} \oplus \mathbb{R} \{X_\lambda\} \oplus \xi.$$

Then a map $u : \Sigma \to \mathbb{R} \times Q =: W$ is pseudoholomorphic with respect to the cylindrical almost complex structure associated to the endomorphism $J : \xi \to \xi$ if and only if its components $a = s \circ u$ and $w = \pi \circ u$ satisfy the equation

$$\begin{cases}
\pi \left( \frac{\partial w}{\partial \tau} \right) + J(w) \pi \left( \frac{\partial w}{\partial t} \right) = 0 \\
w^* \lambda \circ j = da.
\end{cases} \quad (1.4)$$

(See [H1] for the relevant calculations.) Hofer-Wysocki-Zehnder [HWZ1, HWZ2] derived exponential decay estimates of proper pseudoholomorphic curves in symplectization by bruit force coordinate calculations which largely rely on a choice of some special coordinates around the given Reeb orbit and involve complicated coordinate calculations.

Our attempt to improve the presentation of this decay estimates using the tensorial language was the starting point of the research performed in the present paper. In this regard, it follows obviously from the equation that the function $a$ is uniquely determined by $w$ itself modulo the addition of constant on the surface $\Sigma$. Therefore one might hope to study the equation

$$\nabla^w \mathcal{J} = 0 \quad (1.5)$$

directly without involving the function $a$ on the contact manifold $Q$, not on the symplectization.

For the analytical study of (1.5), one needs to differentiate the equation several times. It turns out that for the purpose of taking the derivatives of the map $w$ several times, the contact triad connection on $Q$ (together with contact Hermitian connection on the complex vector bundle $\xi \to Q$), is much more convenient and easier to keep track of various terms arising from switching the order of derivatives than the commonly used Levi-Civita connection in the literature. This is partly because the contact Hermitian connection preserves both $J$ and $g_\xi$. In particular it preserves the type of the vector fields in the complexification $\xi \otimes \mathbb{C}$. It is also because the torsion $T$ of the triad connection we introduce in the present paper has various symmetry properties in terms of the given almost complex structure, in particular it satisfies

$$T(JY, Y) = T(Y, X_\lambda) = 0.$$
for all \( Y \in \xi \), and finally because the contact form \( \lambda \) is \( CR \)-holomorphic.

The advantage of these connections will become even more apparent in [OW2] when we study the Morse-Bott case, and in [Oh2] when we study the Fredholm theory and the corresponding index computations. Importance of the usage of the canonical connection in the computations of pseudoholomorphic curves on almost Kähler manifolds has been emphasized by the senior author’s book [Oh1, chapter 7]. This paper follows the same spirit this time on the contact manifolds by introducing the contact analog of the canonical connection introduced by Ehresman-Libermann [EL] for the almost Hermitian manifold which has been extensively studied by Kobayashi [Ko] in relation to his pursuit of the almost-complex version of Kobayashi hyperbolic space. While our discovery of this canonical connection was made largely by accident, which was mainly motivated by our attempt to simplify the tensor calculations performed in [OW1], [OW2], we anticipate that this connection will play the similar role in contact geometry of the triad as the canonical connection (resp. the Chern connection) does in almost complex geometry (resp. in complex geometry).

Some historical remarks concerning the study of special connections on contact manifolds are in order. There have been several literature that study special connections on contact manifolds in the context of metric contact geometry (see [N], [V] for example). Besides the different motivation of ours which stems from the applications to contact topology while others stem from their Riemannian geometric study of special connections, one essential difference between our contact triad connection and the ones considered in other literature seems to lie in the fact that others require that the Reeb vector field be parallel, while our contact triad connection does not. (See Definition 2.2 [V] and Definition 3.1 [N], for example.) It seems to the present authors that in the point of view of contact topology or of the relevant analysis of contact Cauchy-Riemann maps, there is no a priori reason why the Reeb vector field is required to be parallel with respect to a natural connection. In hindsight, we have found such a connection by replacing the parallelness of \( X_\lambda \) by a weaker condition of \( \lambda \)-being holomorphic in the \( CR \)-sense above, which turns out to exist uniquely for all contact triads \( (Q, \lambda, J) \).

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2. Review of the canonical connection of almost Kähler manifold

We first recall this construction specializing to the case of almost Kähler manifolds \((M, \omega, J)\).

Recall that an almost Hermitian manifold is a triple \((M, J, g)\) of an almost complex structure \(J\) and a metric that satisfies

\[ g(J\cdot, J\cdot) = g(\cdot, \cdot). \]
An affine connection $\nabla$ is called $J$-linear if $\nabla J = 0$. There always exists a $J$-linear connection for a given almost complex manifold. We denote by $T$ the torsion tensor of $\nabla$.

**Definition 2.1.** Let $(M, J, g)$ be an almost Hermitian manifold. A $J$-linear connection is called the canonical connection (or the Chern connection) if it satisfies

$$T(JY, Y) = 0,$$

for any vector field $Y$ on $M$.

Recall that any $J$-linear connection extended to the complexification $T_C M = TM \otimes_{\mathbb{R}} \mathbb{C}$ complex linearly preserves the splitting

$$T_C M = T^{(1,0)} M \oplus T^{(0,1)} M.$$

Similarly we can extend the torsion tensor $T$ complex linearly which we denote by $T_C$.

For the later purpose, we need to derive general properties of the torsion tensor. See [Ko], [Oh1, Section 7.1] for some basic properties of the canonical connection. However a nice exhaustive discussion on the general almost Hermitian connection is given by Gauduchon in [Ga] to which we refer readers for more details.

Following the notation of [Ko], we denote

$$\Theta = \Pi' T_C$$

which is a $T^{(1,0)} M$-valued two-form on $M$. Here $T_C = T \otimes \mathbb{C}$ is the complex linear extension of $T$ and $\Pi'$ is the projection to $T^{(1,0)} M$. We have the decomposition

$$\Theta = \Theta^{(2,0)} + \Theta^{(1,1)} + \Theta^{(0,2)}.$$

We can define the canonical connection in terms of the induced connection on the complex vector bundle $T^{(1,0)} M \to M$. The following lemma is easy to check by definition.

**Lemma 2.2.** An affine connection $\nabla$ on $M$ is the canonical connection if and only if the induced connection $\nabla$ on the complex vector bundle $T^{(1,0)} M$ has its complex torsion form $\Theta = \Pi' T_C$ satisfy $\Theta^{(1,1)} = 0$.

We particularly quote two theorems from Gauduchon [Ga], Kobayashi [Ko].

**Theorem 2.3.** On any almost Hermitian manifold $(M, J, g)$, there exists a unique Hermitian connection $\nabla$ on $TM$ leading to the canonical connection on $T^{(1,0)} M$. We call this connection the canonical Hermitian connection of $(M, J, g)$.

We recall that $(M, J, g)$ is called almost-Kähler if the fundamental two-form $\Phi = g(J\cdot, \cdot)$ is closed [KN].

**Theorem 2.4.** Let $(M, J, g)$ be almost Kähler and $\nabla$ be the canonical connection of $T^{(1,0)} M$. Then $\Theta^{(2,0)} = 0$ in addition, and hence $\Theta$ is of type $(0, 2)$.

The following properties can be derived from this latter theorem easily. (See [Ga], [Ko] for details.)

**Proposition 2.5.** Let $(M, J, g)$ be an almost Kähler manifold and $\nabla$ be the canonical connection. Denote by $T$ its torsion tensor. Then

$$T(JY, Z) = T(Y, JZ)$$

(2.2)
and
\[ JT(JY, Z) = T(Y, Z) \] (2.3)
for all vector fields \( Y, Z \) on \( W \).

Now we describe one way of constructing the canonical connection on an almost complex manifold described in [KN, Theorem 3.4] which will be useful for our purpose of constructing the contact analog thereof later. This connection has its torsion which satisfies
\[ N = 4T \]
where \( N \) is the Nijenhuis tensor of the almost complex structure \( J \) defined as
\[ N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y] \].
In particular, the complexification \( \Theta = \Pi T_C \) is of \((0, 2)\)-type.

We now describe the construction of this canonical connection by following [KN]. Let \( \nabla^{LC} \) be the Levi-Civita connection (in fact, we can do it for any torsion free affine connection) of an almost Hermitian manifold \((M, g, J)\) and consider the tensor field \( Q \) defined by
\[ 4Q(X, Y) = (\nabla^{LC}_J)X + J((\nabla^{LC}_J)X) + 2J((\nabla^{LC}_J)Y) \] (2.4)
for vector fields \( X, Y \) on \( M \). It turns out that when \((M, g, J)\) is almost Kähler, i.e., the two form \( g(\cdot, \cdot) \) is closed, the sum of the first two terms vanish.

**Lemma 2.6 (2.2.10) [Ga].** Assume \((M, g, J)\) is almost Kähler. Then
\[ \nabla^{LC}_J X + J(\nabla^{LC}_J Y) = 0 \] (2.5)
and so \( Q(X, Y) = \frac{1}{2}J(\nabla^{LC}_X J Y) \).

We now consider the standard averaged connection \( \nabla^{av} \) of multiplication \( J : TM \to TM \)
\[ \nabla^{av}_X Y := \nabla^{LC}_X Y + J^{-1}\nabla^{LC}_X (JY) = \nabla^{LC}_X Y - \frac{1}{2}J(\nabla^{LC}_X J Y). \]
We then have the following Proposition stating that this connection becomes the canonical connection. Its proof can be found in [KN, Theorem 3.4] or from section 2 [Ga] with a little more strengthened argument by using (2.5) for the metric property.

**Proposition 2.7.** Assume that \((M, g, J)\) is almost Kähler, i.e, the two-form \( \omega = g(J\cdot, \cdot) \) is closed. Then the average connection \( \nabla^{av} \) defines the canonical connection of \((M, g, J)\), i.e., the connection is \( J \)-linear, preserves the metric and its complexified torsion is of \((0, 2)\)-type.

In later sections, we will need a contact analog to this proposition.

3. Definition of the contact triad connection and its consequences

Let \((Q, \xi)\) be a contact manifold and a contact form \( \lambda \) be given. On \( Q \), the Reeb vector field \( X_\lambda \) associated to the contact form \( \lambda \) is the unique vector field satisfying
\[ X_\lambda |\lambda = 1, \quad X_\lambda |d\lambda = 0. \] (3.1)
Therefore the tangent bundle \( TQ \) has the splitting \( TQ = \mathbb{R}\{X_\lambda\} \oplus \xi \). We denote by
\[ \pi : TQ \to \xi \]
the corresponding projection. $J$ is a complex structure on $\xi$, and we extend it to $TQ$ by defining $J(X_\lambda) = 0$. We will use such $J : TQ \to TQ$ throughout the paper. If there is danger of confusion, we do not distinguish $J$ and $J|_\xi$.

**Definition 3.1** (Contact triad metric). Let $(Q, \lambda, J)$ be a contact triad. We call the metric defined by
\[
g(h, k) = g(\lambda, J)(h, k) := \lambda(h)\lambda(k) + d\lambda(\pi h, J\pi k).
\]
for any $h, k \in TQ$ the contact triad metric associated to the triad $(Q, \lambda, J)$.

In this section, we associate a particular type of affine connection on $Q$ to the given triad $(Q, \lambda, J)$ which we call the contact triad connection of the triple.

To construct this contact analog to the canonical connection of the case of almost Kähler manifolds, we note that the space of Reeb foliations of $(Q, \lambda)$ becomes naturally a (non-Hausdorff) almost Kähler manifold $(\tilde{Q}, \tilde{d}\lambda, \tilde{J}_\xi)$, which characterize this connection in the contact distribution $\xi$. At the same time, we expect this connection behaves in the Reeb vector field direction as the Levi-Civita connection, i.e., it is torsion free and metric. This intuition motivates the following definition of canonical connection of the contact triads $(Q, \lambda, J)$.

We recall $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$, and we denote by $\pi : TQ \to \xi$ the projection. Under this splitting, we may regard a section $Y$ of $\xi \to Q$ as a vector field $Y \oplus 0$. We will just denote the latter by $Y$ with slight abuse of notation.

Define $\nabla^\pi$ the connection of the bundle $\xi \to Q$ by
\[
\nabla^\pi Y = \pi \nabla Y.
\]
If there is no danger of confusion, we also use $\nabla^\pi = \Pi \nabla$ by abuse of notation.

**Definition 3.2** (Contact triad connection). We call an affine connection $\nabla$ on $Q$ the contact triad connection of the contact triad $(Q, \lambda, J)$, if it satisfies the following properties:

1. $\nabla^\pi$ is a Hermitian connection of the Hermitian bundle $\xi$ over the contact manifold $Q$ with Hermitian structure $(d\lambda, J)$.
2. The $\xi$ projection, denoted by $T^\pi := \pi T$, of the torsion $T$ satisfies the following properties:
   \[
   T^\pi(JY, Y) = 0 \tag{3.3}
   \]
   for all $Y$ tangent to $\xi$.
3. $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.
4. $\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.
5. For $Y \in \xi$,
   \[
   \nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = 0.
   \]
6. For any $Y, Z \in \xi$,
   \[
   \langle \nabla_Y X_\lambda, Z \rangle + \langle X_\lambda, \nabla_Y Z \rangle = 0.
   \]

It follows from the definition that the contact triad connection is a Riemannian connection of the triad metric. (The statements of this definition are equivalent to those given in the introduction. We state properties of contact triad connection here as above which are organized in the way how they are used in the proofs of uniqueness and existence.)
We would like to note that Axioms (1) and (2) in the direction of $\xi$ are nothing but the properties of canonical connection on the tangent bundle of the (non-Hausdorff) almost Kähler manifold $(\hat{Q}, d\hat{\lambda}, \hat{J})$ lifted to $\xi$. In addition to them, Axiom (1) also requires the property that $\nabla_{X_\lambda}$ preserve the Hermitian structure $(d\lambda, J)$. On the other hand, Axioms (3), (4), (6) indicate that this connection behaves like the Levi-Civita connection when the Reeb direction $X_\lambda$ get involved. Axiom (5) is an extra requirement to connect the information in $\xi$ part and $X_\lambda$ part, which is used to dramatically simplify our calculations of contact Cauchy-Riemann maps in [OW1], [OW2] (see Proposition 6.11). It turns out that this condition has the following elegant interpretation in terms of CR-geometry.

By the second part of Axiom (4), the covariant derivative $\nabla_{X_\lambda}$ restricted to $\xi$ can be decomposed into

$$
\nabla_{X_\lambda} = \partial^\nabla X_\lambda + \overline{\partial}^\nabla X_\lambda
$$

where $\partial^\nabla X_\lambda$ (respectively, $\overline{\partial}^\nabla X_\lambda$) is $J$-linear (respectively, $J$-anti-linear part). Axiom (6) then is nothing but requiring that $\partial^\nabla X_\lambda = 0$, i.e., $X_\lambda$ is anti $J$-holomorphic in the CR-sense. (It appears that this explains the reason why Axiom (5) gives rise to dramatic simplification in our tensor calculations performed in [OW1].)

One can also consider similar decompositions of one-form $\lambda$. For this, we need some digression. Define $J_\alpha$ for a $k$-form $\alpha$ by the formula

$$
J_\alpha(Y_1, \cdots, Y_k) = \alpha(JY_1, \cdots, JY_k).
$$

**Definition 3.3.** Let $(Q, \lambda, J)$ be a contact triad. We call a $k$-form is CR-holomorphic if $\alpha$ satisfies

$$
\nabla_{X_\lambda} \alpha = 0, \quad (3.4)
$$

$$
\nabla_Y \alpha + J\nabla_{JY} \alpha = 0 \quad \text{for } Y \in \xi. \quad (3.5)
$$

**Proposition 3.4.** Axiom (5) is equivalent to the statement that $\lambda$ is holomorphic in the CR-sense in the presence of other defining properties of contact triad connection.

**Proof.** We first prove $\nabla_{X_\lambda} \lambda = 0$ by evaluating it against vector fields on $Q$. For $X_\lambda$, the first half of Axiom (4) gives rise to

$$
\nabla_{X_\lambda} \lambda(X_\lambda) = -\lambda(\nabla_{X_\lambda} X_\lambda) = 0.
$$

For the vector field $Y \in \xi$, we compute

$$
\nabla_{X_\lambda} \lambda(Y) = -\lambda(\nabla_{X_\lambda} Y)
$$

$$
= -\lambda(\nabla_Y X_\lambda + [X_\lambda, Y] + T(X_\lambda, Y))
$$

$$
= -\lambda(\nabla_Y X_\lambda) - \lambda([X_\lambda, Y]) - \lambda(T(X_\lambda, Y)).
$$

Here the third term vanishes by Axiom (3), the first term by the second part of Axiom (4) and the second term vanishes since

$$
\lambda([X_\lambda, Y]) = \lambda(\mathcal{L}_{X_\lambda} Y) = X_\lambda[\lambda(Y)] - \mathcal{L}_{X_\lambda} \lambda(Y) = 0 - 0 = 0.
$$

Here the first vanishes since $Y \in \xi$ and the second because $\mathcal{L}_{X_\lambda} \lambda = 0$ by the definition of the Reeb vector field. This proves

$$
\nabla_{X_\lambda} \lambda = 0. \quad (3.6)
$$

We next compute $J\nabla_Y \lambda$ for $Y \in \xi$. For a vector field $Z \in \xi$,

$$
(J\nabla_Y \lambda)(Z) = (\nabla_Y \lambda)(JZ) = \nabla_Y(\lambda(JZ)) - \lambda(\nabla_Y(JZ)) = -\lambda(\nabla_Y(JZ))
$$
since $\lambda(JZ) = 0$ for the last equality. Then by the definitions of the Reeb vector field and the triad metric and the skew-symmetry of $J$, we derive

$$-\lambda(\nabla_Y(JZ)) = -\langle \nabla_Y(JZ), X_\lambda \rangle = \langle JZ, \nabla_Y X_\lambda \rangle = -\langle Z, J\nabla_Y X_\lambda \rangle.$$ 

Finally, applying (6), we obtain

$$-\langle Z, J\nabla_Y X_\lambda \rangle = -\langle X_\lambda, JZ \rangle = -\langle \nabla_Y Z, X_\lambda \rangle = -\lambda(\nabla_Y Z) = (\nabla_{JY})X_\lambda.$$ 

Combining the above, we have derived

$$J(\nabla_Y \lambda)(Z) = \nabla_{JY} \lambda(Z)$$

for all $Z \in \xi$. On the other hand, for $X_\lambda$, we evaluate

$$J(\nabla_Y \lambda)(X_\lambda) = \nabla_Y \lambda(JX_\lambda) = \nabla_Y \lambda(0) = 0.$$ 

We also compute

$$\nabla_{JY} \lambda(X_\lambda) = L_{JY}(\lambda(X_\lambda)) - \lambda(\nabla_{JY} X_\lambda).$$

The first term vanishes since $\lambda(X_\lambda) \equiv 1$ and the second vanishes since $\nabla_{JY} X_\lambda \in \xi$ by the second part of Axiom (4). Therefore we have derived

$$J(\nabla_Y \lambda) = \nabla_{JY} \lambda$$

which is equivalent to (3.7). Combining (3.6) and (3.7), we have proved that Axiom (6) implies $\lambda$ is holomorphic in the $CR$-sense. The converse can be proved by reading the above proof backwards. □

From now on, when we refer Axioms, we mean the properties in Definition 3.2. One very interesting consequence of this uniqueness is the following naturality result of the contact-triad connection.

**Theorem 3.5 (Naturality).** Let $\nabla$ be the contact triad connection of the triad $(\xi, \lambda, J)$ associated to any given constant $c \in \mathbb{R}$. Consider any strict contact diffeomorphism $\phi : Q \to Q$, i.e., a diffeomorphism $\phi$ satisfying $\phi^* \lambda = \lambda$. Then the pull-back connection $\phi^* \nabla$ is the contact triad connection associated to the triad $(\xi, \lambda, \phi^* J)$. In particular, this applies to any diffeomorphism arising from the Reeb flow $\phi^t$ of $\dot{x} = X_\lambda(x)$.

**Proof.** A straightforward computation shows that the pull-back connection $\phi^* \nabla$ satisfies all Axioms (1) – (6). Therefore by the uniqueness, $\phi^* \nabla$ is the canonical connection. □

**Remark 3.6.** We would like to remark that Axiom (1) includes the property

$$\nabla_{X_\lambda} d\lambda = 0$$

as a part of Hermitian property of the connection $\nabla^\pi$ on the Hermitian bundle $(\xi, d\lambda, J)$ over $Q$. However this is not a part of algebraic properties lifted from those of the canonical connection on $(\hat{Q}, \hat{d}\lambda, \hat{J})$ because the lifted properties do not say anything about the $X_\lambda$-direction. Because of this, it is not obvious whether the vanishing $\nabla_{X_\lambda} d\lambda = 0$ is consistent with other part of axioms. An examination of the uniqueness proof given in the section 4 though, shows that we never use the condition $\nabla_{X_\lambda} d\lambda = 0$ and so we can drop this requirement from Axiom (1) just by requiring $\nabla_Y d\lambda = 0$ for $Y \in \xi$. Furthermore the vanishing $\nabla_{X_\lambda} d\lambda = 0$ will automatically follow from Axioms (3) and (5c) (which is a generalization of Axiom (5), see (4.2) in next section) and does not lead to any contradiction with
other axioms: Existence proof given in section 6 will ensure that the connection satisfying all these axioms of the contract triad connection exist. Therefore Axiom (1) in the presence of other axioms is equivalent to the latter. We prefer to state Axiom (1) as it is because the statement is simpler and more natural to state.

In fact, the automatic vanishing $\nabla_{X_\lambda} d\lambda = 0$ derived from other parts of the axioms reflects nice interplay between the geometric structures of contact form $\lambda$ and of the endomorphism $J: \xi \to \xi$ via the compatibility requirement $g_{|\xi} = d\lambda(\cdot, J\cdot)$. The vanishing is a consequence of the symmetry of the operator $L_{X_\lambda} J: \xi \to \xi$ as stated in Lemma 5.2 in section 5, whose proof in turn follows from the invariance property $L_{X_\lambda} d\lambda = 0$ of $d\lambda$ under the Reeb flow.

4. Proof of the uniqueness of the contact triad connection

In this section, we give the uniqueness proof by analyzing the first structure equation and showing how every axiom determines the connection one forms. In the next two sections, we explicitly construct a connection by carefully examining properties of the Levi-Civita connection and modifying the constructions in [KN], [Ko] for the canonical connection, and then show it satisfies all the requirements and thus the unique contact triad connection.

We are going to prove the existence and uniqueness for a more general family of connections. First, we generalize the Axiom (5) to the following Axiom: For $Y \in \xi$,

$$\nabla_{JY} X_\lambda + J \nabla_Y X_\lambda \in \mathbb{R} \cdot Y,$$

and we denote Axiom $(5; c)$ by

$$(5; c) \text{ For a given } c \in \mathbb{R}, \quad \nabla_{JY} X_\lambda + J \nabla_Y X_\lambda = c Y, \quad Y \in \xi.$$  

(4.1)

In particular, Axiom (5) corresponds to Axiom $(5; 0)$.

**Theorem 4.1.** For any $c \in \mathbb{R}$, there exists a unique connection satisfies Axiom (1)-(4), (6) and $(5; c)$.

In particular, there exists a unique contact triad connection $\nabla$ for the triad $(Q, \lambda, J)$.

**Proof.** (Uniqueness)

Choose a moving frame of $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ given by

$$X_\lambda, E_1, \ldots, E_n, JE_1, \ldots, JE_n$$

and denote its dual co-frame by

$$\lambda, \alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^n.$$  

We use the Einstein summation convention to denote the sum of upper indices and lower indices in this paper.

Assume the connection matrix is $(\Omega^i_j)$, $i, j = 0, 1, \ldots, 2n$, and we write the first structure equations as follows.

$$
\begin{align*}
  d\lambda & = -\Omega^0_0 \wedge \lambda - \Omega^0_k \wedge \alpha^k - \Omega^0_{n+k} \wedge \beta^k + T^0 \\
  d\alpha^j & = -\Omega^j_0 \wedge \lambda - \Omega^j_k \wedge \alpha^k - \Omega^j_{n+k} \wedge \beta^k + T^j \\
  d\beta^j & = -\Omega^{n+j}_0 \wedge \lambda - \Omega^{n+j}_k \wedge \alpha^k - \Omega^{n+j}_{n+k} \wedge \beta^k + T^{n+j}
\end{align*}
$$
Denote

\[
\Omega^i_j = \Gamma^i_{0,j} \lambda + \Gamma^i_{k,j} \alpha^k + \Gamma^i_{n+k,j} \beta^k
\]
\[
\Omega^{n+i}_{n+j} = \Gamma^{n+i}_{0,n+j} \lambda + \Gamma^{n+i}_{k,n+j} \alpha^k + \Gamma^{n+i}_{n+k,n+j} \beta^k
\]
\[
\Omega^j_i = \Gamma^j_{0,i} \lambda + \Gamma^j_{k,i} \alpha^k + \Gamma^j_{n+k,i} \beta^k
\]
\[
\Omega^i_{n+j} = \Gamma^i_{0,n+j} \lambda + \Gamma^i_{k,n+j} \alpha^k + \Gamma^i_{n+k,n+j} \beta^k
\]

Throughout the section, if not stated otherwise, we let \( i, j \) and \( k \) take values from 1 to \( n \).

We will analyze each axiom in Definition 3.2 and show how they set down the matrix of connection one forms.

We first state that Axioms (1) and (2) uniquely determine \((\Omega^i_j|\xi)_{i,j=1,\ldots,2n}\). This is exactly the same as Kobayashi’s proof for the uniqueness of Hermitian connection given in [Ko]. To be more specific, we can restrict the first structure equation to \( \Theta \) and get the following equations for \( \alpha \) given in [Ko, Theorem 2.3] and (4.5) and (4.6) uniquely settle down \( \Omega^0_0 \) and \( \Omega_0 \). In the rest of the proof, we will clarify how the Axioms (3), (4), (5; c), (6) uniquely determine \( \Omega^0_0 \), \( \Omega_0 \) and \((\Omega^j_i(X_\lambda))_{i,j=1,\ldots,2n}\). Compute the first equality in Axiom (4) and get

\[
\nabla_{X_\lambda} X_\lambda = \Gamma^0_{0,0} X_\lambda + \Gamma^k_{0,0} E_k + \Gamma^{n+k}_{0,0} J E_k = 0.
\]

Hence

\[
\Gamma^0_{0,0} = 0
\] (4.4)
\[
\Gamma^k_{0,0} = 0
\] (4.5)
\[
\Gamma^{n+k}_{0,0} = 0.
\] (4.6)

The second claim in Axiom (4) is equal to say

\[
\nabla_{E_k} X_\lambda \in \xi, \quad \nabla_{J E_k} X_\lambda \in \xi.
\] (4.7)

Similar calculation shows that

\[
\Gamma^0_{k,0} = 0
\] (4.8)
\[
\Gamma^0_{n+k,0} = 0.
\] (4.9)

Now (4.4), (4.5) and (4.6) uniquely settle down

\[
\Gamma^i_j - \Gamma^j_0 = -\langle [E_j, X_\lambda], E_k \rangle = -\langle L_{X_\lambda} E_j, E_k \rangle
\] (4.10)
\[
\Gamma^k_{n+j,0} - \Gamma^k_{0,n+j} = -\langle [J E_j, X_\lambda], E_k \rangle = -\langle L_{X_\lambda} (J E_j), E_k \rangle
\] (4.11)
and

\[ \Gamma_{j,0}^{n+k} - \Gamma_{0,j}^{n+k} = \langle [E_j, X_\lambda], JE_k \rangle = -\langle \mathcal{L}_{X_\lambda} E_j, JE_k \rangle \]  
\[ \Gamma_{n+j,0}^{n+k} - \Gamma_{0,0}^{n+k} = \langle [E_j, X_\lambda], JE_k \rangle = -\langle \mathcal{L}_{X_\lambda} (JE_j), JE_k \rangle. \]  

From Axiom (5; c), we have

\[ \Gamma_{j,0}^k + \Gamma_{n+j,0}^{n+k} = 0 \]  
\[ \Gamma_{j,0}^{n+k} - \Gamma_{n+j,0}^k = -c\delta_{j,k}. \]  

Now we show how to determine \( \Omega_j^0 \) for \( j = 1, \ldots, 2n \). For this purpose, we calculate \( \Gamma_{j,0}^k \). First, by using (4.14), we write

\[ \Gamma_{j,0}^k = \frac{1}{2} \Gamma_{j,0}^k - \frac{1}{2} \Gamma_{n+j,0}^{n+k}. \]  

Furthermore, using (4.10) and (4.13), we have

\[ \Gamma_{j,0}^k = \frac{1}{2} \Gamma_{j,0}^k - \frac{1}{2} \Gamma_{n+j,0}^{n+k} \]
\[ = \frac{1}{2} (\Gamma_{j,0}^k - (\langle X_\lambda, E_j \rangle, E_k) - \frac{1}{2} (\Gamma_{0,n+j}^{n+k} - \langle X_\lambda (JE_j), JE_k \rangle) \]
\[ = \frac{1}{2} (\Gamma_{j,0}^k - \Gamma_{0,n+j}^{n+k}) - \frac{1}{2} (\langle X_\lambda, E_j \rangle, E_k) - \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle \]
\[ = \frac{1}{2} (\Gamma_{j,0}^k - \Gamma_{0,n+j}^{n+k}) - \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle \]
\[ = \frac{1}{2} (\Gamma_{j,0}^k - \Gamma_{0,n+j}^{n+k}) + \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle \]

Notice the first term vanishes by Axiom (2). In particular, that is from \( \nabla_{X_\lambda} J = 0 \). Hence we get

\[ \Gamma_{j,0}^k = \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle. \]  

Following the same idea, we use (4.15) and calculate

\[ \Gamma_{j,0}^{n+k} = -\frac{1}{2} c\delta_{j,k} + \frac{1}{2} \Gamma_{j,0}^{n+k} + \frac{1}{2} \Gamma_{n+j,0}^k \]
\[ = -\frac{1}{2} c\delta_{j,k} + \frac{1}{2} (\Gamma_{j,0}^{n+k} - \langle X_\lambda (JE_j), JE_k \rangle) + \frac{1}{2} (\Gamma_{0,n+j}^k - \langle X_\lambda (JE_j), JE_k \rangle) \]
\[ = -\frac{1}{2} c\delta_{j,k} + \frac{1}{2} (\Gamma_{j,0}^{n+k} + \Gamma_{0,n+j}^k) - \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle + \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle \]
\[ = -\frac{1}{2} c\delta_{j,k} + \frac{1}{2} \langle X_\lambda (JE_j), JE_k \rangle \]
\[ = -\frac{1}{2} c\delta_{j,k} + \frac{1}{2} \langle (X_\lambda (JE_j), JE_k) \rangle. \]
Here the fourth equality is due to \( \nabla_{X\lambda} J = 0 \) as before. Then substituting this into (4.13) and (4.15), we get

\[
\Gamma_{n+j, 0}^k = \frac{1}{2} c\delta_{jk} + \frac{1}{2} \langle (L_{X\lambda} J) E_j, E_k \rangle
\]

and

\[
\Gamma_{n+j, 0}^n + k = \frac{1}{2} \langle (L_{X\lambda} J) E_j, E_k \rangle
\]

Together with (4.4), (4.5) and (4.6), \( \Omega_0 \) is uniquely determined by this way.

Furthermore (4.10), (4.11), (4.12) and (4.13), uniquely determine \( \Omega_i^{(X\lambda)} \) for \( i, j = 1, \ldots, 2n \).

Notice that for any \( Y \in \xi \), we derive

\[ \nabla_{X\lambda} Y \in \xi \]

from Axiom (3). This is because the axiom implies \( \nabla_{X\lambda} Y = \nabla_Y X\lambda + \mathcal{L}_{X\lambda} Y \) and the latter is contained in \( \xi \): the second part of Axiom (4) implies \( \nabla_Y X\lambda \in \xi \) and the Lie derivative along the Reeb vector field preserves the contact structure \( \xi \). It then follows that \( \Gamma_{0, l}^n = 0 \) for \( l = 1, \ldots, 2n \). At the same time, Axiom (6) implies

\[
\Gamma_{j,k}^0 = -\Gamma_{j,0}^k.
\]

for \( j, k = 1, \ldots, 2n \). Hence together with (4.8) and (4.9), \( \Omega_0 \) is uniquely determined.

We are done with the proof of uniqueness.

We end this section by giving a summary of the procedure we take in the proof of uniqueness which actually indicates a way how to construct this connection in later sections.

First, we use the Hermitian connection property, i.e., Axiom (1) and torsion property Axiom (2), i.e., \( T^\pi_{\xi} \) has vanishing \((1, 1)\) part, to uniquely fix the connection on \( \xi \) projection of \( \nabla \) when taking values on \( \xi \).

Then we use the metric property

\[ \langle X\lambda, \nabla_Y Z \rangle + \langle \nabla_Y X\lambda, Z \rangle = 0, \]

for any \( Y, Z \in \xi \), to determine the \( X\lambda \) component of \( \nabla \) when taking values in \( \xi \).

To do this, we need the information of \( \nabla_Y X\lambda \). As mentioned before the second part of Axiom (4) enables us to decompose

\[ \nabla X\lambda = \partial^\nabla X\lambda + \mathfrak{d}^\nabla X\lambda \]

with

\[
\partial^\nabla X\lambda = \frac{\nabla X\lambda - J\nabla_{J(L)} X\lambda}{2}, \quad \mathfrak{d}^\nabla X\lambda = \frac{\nabla X\lambda + J\nabla_{J(L)} X\lambda}{2}.
\]

The axiom \( \nabla_{X\lambda} J = 0 \) embedded into the Hermitian property of \( (\xi, g, J) \) is nothing but

\[ \nabla_{X\lambda} (JY) - J\nabla_{X\lambda} Y = 0 \]

Axiom (3), the torsion property \( T(X\lambda, Y) = 0 \), then interprets this one into

\[ \nabla_{JY} X\lambda - J\nabla_Y X\lambda = -(\mathcal{L}_{X\lambda} J) Y \]
which is also equivalent to saying
\[ J \mathcal{D}^X \lambda = \frac{1}{2} (\mathcal{L}_X, J) Y \quad \text{or} \quad \mathcal{D}^X \lambda = \frac{1}{2} (\mathcal{L}_X, J) JY. \] (4.17)

It turns out that we can vary Axiom (5) by replacing it to (5; c)
\[ \nabla_{JY} X = c Y, \quad \text{or equivalently} \quad \partial^X \nabla \lambda = \frac{c}{2} Y \] (4.18)

for any given real number c. This way we shall have one-parameter family of affine connections parameterized by \( \mathbb{R} \) each of which satisfies Axioms (1) - (4) and (6) with (5) replaced by (5; c).

When c fixed, i.e., under Axiom (5; c), we can uniquely determine \( \nabla_Y X \lambda \) to be
\[ \nabla_Y X \lambda = - \frac{1}{2} c JY + \frac{1}{2} (\mathcal{L}_X, J) JY. \]

Therefore, \( \nabla_Y, Y \in \xi \) is uniquely determined by this way, and we also get \( \nabla_Y X \lambda \) combined with the torsion property. Then the remaining property \( \nabla_{X \lambda} X \lambda = 0 \) now uniquely determines the connection.

5. Properties of the Levi-Civita Connection on contact manifolds

From the discussion in previous sections, the only thing left to do for the existence of the contact triad connection is to globally define a connection such that it can patch the \( \xi \) part of \( \nabla|_\xi \) and the \( X \lambda \) part of it. In particular, we seek for a connection that satisfies the following properties:

(1) it satisfies all the algebraic properties of the canonical connection of almost Kähler manifold [Ko] when restricted to \( \xi \).

(2) it satisfies metric property and has vanishing torsion in \( X \lambda \) direction.

With these into consideration, we construct such a connection by examining the properties of the Levi-Civita connection of the triad metric associated to the contact triad \( (Q, \lambda, J) \). Our interest in the special connection of contact triad is motivated by our attempt in [OW1] to optimally organize the complicated tensorial expressions in the tensor calculations that appear in the analytical study of the maps of pseudoholomorphic curves on contact manifold (or on its symplectization). Without this guidance, it would not have been possible for us to pinpoint the geometric properties laid out in our definition of the canonical connection in the way that the uniqueness and existence can be established. The presence of such a construction is a manifestation of delicate interplay between the geometric structures \( \xi, \lambda, \) and \( J \) in the geometry of contact triads \( (Q, \lambda, J) \). In this regard, the closeness of \( d\lambda \) and the definition of Reeb vector field \( X \lambda \) play important roles. In particular \( d\lambda \) plays the role similar to that of the fundamental two-form \( \Phi \) in the case of almost Kähler manifold [KN] (in a non-strict sense) in that it is closed.

This interplay turns out to be reflected already in several basic properties of the Levi-Civita connection of the contact triad metric exposed in this section. After we had derived these properties ourselves, we learned from Blair’s book [Bl] that the similar results had been previously derived in his book [Bl, Section 6.1, 6.2] in a much more general context of contact metric manifolds. However, partly because our sign convention of the compatible metric is different from the one in [Bl] and also because our derivation may be simpler, we include full derivation of these properties in this section for completeness’ sake and self-containedness unless the corresponding results can be directly quoted from [Bl].
Before we study the Levi-Civita connection, we would like to remind that we extended \( J \) to \( TQ \) by defining \( J(X_\lambda) = 0 \). We denote by \( \Pi : TQ \rightarrow TQ \) the idempotent associated to the projection \( \pi : TQ \rightarrow \xi \), i.e., the endomorphism satisfying

\[
\Pi^2 = \Pi, \quad \text{Im} \, \Pi = \xi, \quad \ker \, \Pi = \mathbb{R} \{ X_\lambda \}.
\]

We have now \( J^2 = -\Pi \). Moreover, for any connection \( \nabla \) on \( Q \),

\[
(\nabla J)J = - (\nabla \Pi) - J(\nabla J). \tag{5.1}
\]

Notice for \( Y \in \xi \), we have

\[
\Pi(\nabla \Pi)Y = 0 \tag{5.2}
\]

\[
(\nabla \Pi)X_\lambda = - \Pi \nabla X_\lambda. \tag{5.3}
\]

We denote the triad metric \( g = g(\lambda, J) \) as

\[
\langle X, Y \rangle := d\lambda(\Pi X, J \Pi Y) + \lambda(X)\lambda(Y)
\]

for any \( X, Y \in TQ \) for our computations hereafter.

We state the following obvious properties of \((g = : \langle \cdot, \cdot \rangle, \lambda, J)\).

**Lemma 5.1.**

\[
\langle X, Y \rangle = d\lambda(X, JY) + \lambda(X)\lambda(Y)
\]

\[
d\lambda(X, Y) = d\lambda(JX, JY).
\]

Therefore,

\[
\langle JX, JY \rangle = d\lambda(X, JY)
\]

\[
\langle X, JY \rangle = -d\lambda(X, Y)
\]

\[
\langle JX, Y \rangle = -\langle X, JY \rangle.
\]

However, we remark

\[
\langle JX, JY \rangle \neq \langle X, Y \rangle
\]

in general now, and hence there is no obvious analog of the fundamental 2 form \( \Phi \) defined as in [KN] for the contact case. This is the main reason that is responsible for the differences arising in the various relevant formulae between the contact case and the almost Hermitian case.

The following preparation lemma says that the linear operator \( \mathcal{L}_{X_\lambda} J \) is symmetric with respect to the metric \( g = : \langle \cdot, \cdot \rangle \). It is proved in [Bl] to which we refer readers for its proof.

**Lemma 5.2 (Lemma 6.2 [Bl]).** For \( Y, Z \in \xi \),

\[
\langle (\mathcal{L}_{X_\lambda} J)Y, Z \rangle = \langle Y, (\mathcal{L}_{X_\lambda} J)Z \rangle.
\]

The following properties of the Levi-Civita connection on contact manifolds can be also found in [D] Lemma 3 as well as in [Bl]. One amusing consequence of this lemma is that the Reeb foliation is a geodesic foliation for the Levi-Civita connection (and so for the contact triad connection) of the contact triad metric.

**Lemma 5.3.** For any vector field \( Z \) on \( Q \),

\[
\nabla^LC_Z X_\lambda \in \xi, \tag{5.4}
\]

and

\[
\nabla^LC_{X_\lambda} X_\lambda = 0. \tag{5.5}
\]
Next we derive the following lemma which is the contact analog to the Prop 4.2 in [KN] for the almost Hermitian case. This lemma can be also extracted from [Bl, Corollary 6.1].

**Lemma 5.4.**

\[
2 \langle (\nabla^L_X J)Y, Z \rangle = \langle N(Y, Z), JX \rangle - \langle JX, JY \rangle \lambda(Z) + \langle JX, JZ \rangle \lambda(Y)
\]

for \( X, Y, Z \in \mathcal{T}Q \), where \( N \) is the Nijenhuis tensor defined as

\[
N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].
\]

**Proof.** The left hand side

\[
2 \langle (\nabla^L_X J)Y, Z \rangle = \langle \nabla^L_X (JY), Z \rangle - \langle J\nabla^L_X Y, Z \rangle = \langle \nabla^L_X (JY), Z \rangle + \langle \nabla^L_X Y, JZ \rangle
\]

Since \( \nabla^L_X \) is the Levi-Civita connection with respect to the Riemannian manifold \((Q, g = \langle \cdot, \cdot \rangle)\), we have the following formula

\[
2 \langle \nabla^L_X Y, Z \rangle = X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle.
\]

Using Lemma 5.1 we derive

\[
2 \langle (\nabla^L_X J)Y, Z \rangle = 2 \langle \nabla^L_X (JY), Z \rangle + 2 \langle \nabla^L_X Y, JZ \rangle = R(X, Y, Z) + S(X, Y, Z),
\]

From it, where

\[
R(X, Y, Z) := (JY) d\lambda(X, JZ) - Y d\lambda(X, Z) + Z d\lambda(X, Y) - (JZ) d\lambda(X, JY) + d\lambda([X, JY], JZ) - d\lambda([X, Y], Z) - d\lambda([Z, X], JY) + d\lambda([Z, JY], JX) + d\lambda([JZ, Y], JX),
\]

and

\[
S(X, Y, Z) := (JY) (\lambda(X) \lambda(Z)) - (JZ) (\lambda(X) \lambda(Y)) + \lambda([X, JY]) \lambda(Z) + \lambda([JZ, X]) \lambda(Y) + \lambda([Z, JY]) \lambda(X) + \lambda([JZ, Y]) \lambda(X).
\]

The way to deal with \( R(X, Y, Z) \) is exactly the same as in the proof of [KN Proposition 4.2] if we replace \( d\lambda \) by the fundamental 2 form \( \Phi \) therein. For readers' convenience and for completeness' sake, we state and prove the following lemma.

**Lemma 5.5.**

\[
R(X, Y, Z) = \langle N(Y, Z), JX \rangle.
\]

**Proof.** The proof follows by a straightforward calculation using \( d(d\lambda) = 0 \) and organizing \( R(X, Y, Z) \) into

\[
R(X, Y, Z) = (d(d\lambda))(X, JY, JZ) - (d(d\lambda))(X, Y, Z) + \langle N(Y, Z), JX \rangle.
\]
Indeed, we can rewrite (5.6) into
\[
R(X, Y, Z) = -(JY)d\lambda(JZ, X) - (JZ)d\lambda(X, JY) - Xd\lambda(JY, JZ) \\
+ \lambda([X, JY], JZ) + \lambda([JZ, X], JY) + \lambda([JY, JZ], X) \\
+ X(d\lambda(Y, Z)) + Yd\lambda(Z, X) + Zd\lambda(X, Y) \\
- d\lambda([X, Y], Z) - d\lambda([Z, X], Y) - d\lambda([Y, Z], X) \\
+ d\lambda([Z, JY], JX) + d\lambda([JZ, Y], JX) \\
- d\lambda([JY, JZ], X) + d\lambda([Y, Z], X).
\]

The difference between this formula for \(R\) and (5.6) is as follows: Here beside rearranging the terms, we subtracted \(Xd\lambda(JY, JZ)\) in the first line and add back \(X(d\lambda(Y, Z)) = Xd\lambda(JY, JZ)\) to the third line, and add \(d\lambda([JY, JZ], X)\) to the second line and subtract it back in the fifth line.

Then the first two lines of this formula become \(-(d\lambda)(X, JY, JZ)\) and the second two lines become \((d\lambda)(X, Y, Z)\) and final two lines can be re-written into
\[
d\lambda([Z, JY], JX) + d\lambda([JZ, Y], JX) - d\lambda([JY, JZ], X) + d\lambda([Y, Z], X) \\
= -(J[JY, Z], JX) - \langle J[Y, JZ], JX \rangle - \langle ([Y, Z], JX) + \langle [JY, JZ], JX \rangle \\
= \langle N(Y, Z), JX \rangle.
\]

This finishes the proof.

For \(S(X, Y, Z)\), we use the formula
\[
d\lambda(X, Y) = X(\lambda(Y) - Y(\lambda(X))) - \lambda([X, Y]),
\]
to simplify it and get
\[
S(X, Y, Z) = -d\lambda(X, JY)\lambda(Z) + d\lambda(X, JZ)\lambda(Y) \\
= -\langle JX, JY \rangle\lambda(Z) + \langle JX, JZ \rangle\lambda(Y).
\]

Combining all these, we have derived the formula
\[
2\langle (\nabla^L_X J)Y, Z \rangle = \langle N(Y, Z), JX \rangle \\
- \langle JX, JY \rangle\lambda(Z) + \langle JX, JZ \rangle\lambda(Y).
\]

In particular, we obtain the following corollary.

**Corollary 5.6.** For \(Y, Z \in \xi\),
\[
\begin{align*}
2\langle (\nabla^L_Y J)X, Z \rangle & = \langle N(X, Z), JY \rangle + \langle Y, Z \rangle \\
& = -\langle (\mathcal{L}_X J)Z, Y \rangle + \langle Y, Z \rangle \\
2\langle (\nabla^L_Y J)Z, X \rangle & = \langle N(Z, X), JY \rangle - \langle Y, Z \rangle \\
& = \langle (\mathcal{L}_X J)Z, Y \rangle - \langle Y, Z \rangle \\
2\langle (\nabla^L_X J)Y, Z \rangle & = \langle N(Y, Z), JX \rangle.
\end{align*}
\]

**Proof.** This is a direct corollary from Lemma 5.4 except that we also use
\[
N(X, Z) = -J(\mathcal{L}_X J)Z \quad (5.7)
\]
and
\[
N(Z, X) = J(\mathcal{L}_X J)Z. \quad (5.8)
\]
for the first two conclusions.

Next we prove the following lemma.
Lemma 5.7. For $Y, Z \in \xi$,

$$JN(Y, JZ) - \Pi N(Y, Z) = 0$$
$$\Pi N(Y, JZ) + \Pi N(Z, JY) = 0.$$

Proof. We compute for $Y, Z \in \xi$,

$$JN(Y, JZ) - \Pi N(Y, Z) = J ([JY, JJZ] - [Y, JZ] - J[Y, JJZ] - J[JY, JZ])$$
$$- \Pi ([JY, JZ] - [Y, Z] - J[Y, JZ] - J[JY, Z])$$
$$= J (-[JY, Z] + J[Y, Z] - J[JY, JZ])$$
$$- \Pi ([JY, JZ] - [Y, Z] - J[Y, JZ] - J[JY, Z])$$
$$= -J[JY, Z] - J[JY, JZ] - \Pi [Y, Z] + \Pi [JY, JZ]$$
$$- \Pi [JY, JZ] + J[JY, JZ] + J[JY, Z]$$
$$= 0.$$

For the second one, similarly, we compute

$$\Pi N(Y, JZ) + \Pi N(Z, JY) =$$
$$\Pi ([JY, JJZ] - [Y, JZ] - J[Y, JJZ] - J[JY, JZ])$$
$$+ \Pi ([JZ, JJY] - [Z, JY] - J[Z, JJY] - J[JZ, JY])$$
$$= -\Pi [JY, Z] - \Pi [Y, JZ] + J[Y, Z] - J[JY, JZ]$$
$$- \Pi [JZ, Y] - \Pi [Z, JY] + J[Z, Y] - J[JZ, JY]$$
$$= 0.$$

□

Together with the last equality in Corollary 5.6 and Lemma 5.7, we obtain the following lemma, which is the contact analog to Lemma 2.6.

Lemma 5.8.

$$\Pi (\nabla_{YJ}^{LC} J) X + J(\nabla_{YJ}^{LC} J) X = 0.$$  (5.9)

Proof. We look at for any $Z \in \xi$,

$$\langle \Pi (\nabla_{YJ}^{LC} J) X + J(\nabla_{YJ}^{LC} J) X, Z \rangle$$
$$= \langle \langle \nabla_{YJ}^{LC} J \rangle X, Z \rangle - \langle \langle \nabla_{YJ}^{LC} J \rangle X, JZ \rangle$$
$$= \frac{1}{2} \langle N(X, Z), JY \rangle - \frac{1}{2} \langle N(X, JZ), JY \rangle$$
$$= \frac{1}{2} \langle JN(X, JZ) - \Pi N(X, Z), Y \rangle = 0,$$

and then follows. □

The following result is an immediate consequence of Corollary 5.6 and the property $\nabla_{X_X} X_\lambda = 0$ of $X_\lambda$. We refer readers to [Bl, Corollary 6.1] for its proof.

Proposition 5.9 (Corollary 6.1 [Bl]).

$$\nabla_{X_X}^{LC} J = 0.$$
The following is equivalent to second part of Lemma 6.2 [Bl] after taking into consideration of different sign convention of the definition of compatibility of $J$ and $d\lambda$. For the sake of completeness and since this will be used in an important way later, we give its proof.

**Lemma 5.10 (Lemma 6.2 [Bl]).** For any $Y \in \xi$, we have

$$\nabla^L_Y X_\lambda = \frac{1}{2} JY + \frac{1}{2} (\mathcal{L}_X J)JY.$$  

**Proof.** Since the Levi-Civita connection is Riemannian, for any $Y, Z \in \xi$, we have

$$\langle \nabla^L_Y X_\lambda, Z \rangle = -\langle X_\lambda, \nabla^L_Y Z \rangle.$$  

Next we write

$$-\langle X_\lambda, \nabla^L_Y Z \rangle = \langle X_\lambda, (\nabla^L_Y J)(JZ) \rangle,$$

and then further by using the second equality in the Corollary 5.6, we have

$$\langle \nabla^L_Y X_\lambda, Z \rangle = -\langle X_\lambda, \nabla^L_Y Z \rangle = \langle X_\lambda, (\nabla^L_Y J)(JZ) \rangle = \frac{1}{2} \langle (\mathcal{L}_X J)(JZ), Y \rangle - \frac{1}{2} \langle Y, JZ \rangle = \frac{1}{2} \langle (\mathcal{L}_X J)Y, JZ \rangle - \frac{1}{2} \langle Y, JZ \rangle = -\frac{1}{2} \langle J(\mathcal{L}_X J)Y, Z \rangle + \frac{1}{2} \langle JY, Z \rangle = \frac{1}{2} JY + \frac{1}{2} (\mathcal{L}_X J)JY, Z \rangle$$

for any $Z \in \xi$. Here we use Lemma 5.2 for the forth equality. Since $\nabla^L_Y X_\lambda \in \xi$ for $Y \in \xi$, we are done with the proof. □

6. **Existence of the contact triad connection**

In this section, we establish the existence theorem of the contact triad connection in two stages. Recall that the space of affine connections on a given smooth manifold $M$ is an affine space and so for any given affine connection $\nabla$, the sum $\nabla + B$ defines a new affine connection for any given tensor $B$ of type $(\frac{1}{2})$ by the formula

$$\nabla^B_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + B(Z_1, Z_2).$$

Now consider the endomorphism $\mathcal{B}(Z_1)$ of $TM$ defined by

$$\langle \mathcal{B}(Z_1)(Z_2), Z_3 \rangle := \langle B(Z_1, Z_2), Z_3 \rangle.$$  

When $\nabla$ is Riemannian with respect to the given metric, the connection $\nabla^B$ remains Riemannin if $\mathcal{B}$ is skew-symmetric with respect to the associated inner product, i.e., it satisfies

$$\langle \mathcal{B}(Z_1)Z_2, Z_3 \rangle = -\langle \mathcal{B}(Z_1)Z_3, Z_2 \rangle. \quad (6.1)$$

First, we examine the relationship between the connections of two different $c$'s. Denote by $\nabla^{\lambda c}$ the unique connection associated to the constant $c$. The following proposition shows that $\nabla^{\lambda c}$ and $\nabla^{\lambda c'}$ for two different nonzero constants with the same parity are essentially the same in that it arises from the scale change of the contact form.
Proposition 6.1. Let \((Q, \lambda, J)\) be a contact triad and consider the triad \((Q, a\lambda, J)\) for a constant \(a > 0\). Then
\[\nabla^{a\lambda;1} = \nabla^{\lambda a}.
\]

Proof. By definition, \(\nabla^{a\lambda;1}\) is characterized by Axioms (1) - (4), (6) and (5;−1) for the triad \((Q, a\lambda, J)\). Since Axioms (1) - (4) and (6) are obviously scale-invariant of the contact form \(\lambda\), this connection satisfies all axioms for the triad \((Q, \lambda, J)\). The only thing left to check is Axiom (5;\(a\)). But this immediately follows from the relationship of the Reeb vector fields under the multiplication by a positive constant, which is
\[X_{a\lambda} = \frac{1}{a} X_\lambda.
\]
Therefore \(\nabla^{a\lambda;1}\) satisfies
\[\nabla^{a\lambda;1} J X_{a\lambda} + J \nabla^{a\lambda;1} X_{a\lambda} = Y
\]
which is equivalent to
\[\nabla^{a\lambda;1} X_\lambda + J \nabla^{a\lambda;1} X_\lambda = a Y.
\]
By the defining Axiom (5;\(a\)) for \(\nabla^{\lambda a}\), and the uniqueness thereof, we have proved
\[\nabla^{a\lambda;1} = \nabla^{\lambda a}.
\] This finishes the proof. □

In regard to this proposition, one could say that for each given contact structure \((Q, \xi)\), there are essentially two inequivalent \(\nabla^0\), \(\nabla^1\) (respectively three, \(\nabla^0\), \(\nabla^1\) and \(\nabla^{-1}\), if one fixes the orientation) choice of triad connections for each given projective equivalence class of the contact triad \((Q, \lambda, J)\). In this regard, the connection \(\nabla^0\) is essentially different from others in that this argument of scaling procedure of contact form \(\lambda\) does not apply to the case \(a = 0\) since it would lead to the zero form \(0 \cdot \lambda\). This proposition also reduces the construction essentially two connections of \(\nabla^{\lambda 0}\) and \(\nabla^{\lambda 1}\) (or \(\nabla^{\lambda -1}\)).

In the rest of this section, we will explicitly construct \(\nabla^{\lambda 1}\) and \(\nabla^{\lambda c}\) in two stages, by modifying the Levi-Civita connection by adding suitable tensors \(B\) as described above.

In the first stage, partially motivated by the construction of the Hermitian connection on almost Kähler manifold and exploiting the properties of the Levi-Civita connection we extracted in the previous section, we construct a connection named \(\nabla^{\text{tmp}1}\) and calculate its metric and torsion properties. This part itself provides a contact analog of Kobayashi’s construction of the Hermitian connection and the resulting connection \(\nabla^{\text{tmp}1}\) satisfies Axioms (1)-(4), (5;−1), (6).

In the second stage, we modify \(\nabla^{\text{tmp}1}\) to the final connection \(\nabla^{\text{tmp}2}\) mainly to deforming the property (5;−1) thereof to (5;c) leaving other properties of \(\nabla^{\text{tmp}1}\) intact for given constant \(c\). This \(\nabla^{\text{tmp}2}\) then satisfies all the axioms in Definition 3.2. The construction of \(\nabla^{\text{tmp}2}\) exhibits a way of combining the Levi-Civita connection and the canonical connection, so that the resulting connection behaves like the canonical connection on \(\xi\) and like the Levi-Civita connection on \(X_\lambda\). Indeed this is the original idea behind the choice of the axioms laid out in Definition 3.2.

Remark 6.2. We find that it is rather mysterious to see that the second stage of modification does not differentiate the case of \(c = 0\) from that of \(c \neq 0\), although the above mentioned scaling procedure of contact form clearly single out this case from others. It seems to say that even though the contact form degenerates as \(c \to 0\),
the associated contact triad connection itself converges to a smooth well-defined connection in the space of affine connections.

6.1. **Modification 1;** $\nabla^{tmp;1}$. We begin the first stage of constructing our first connection $\nabla^{tmp;1}$ motivated by the construction of canonical connection on almost Kähler manifold. This is the contact analog to this construction.

Define an affine connection $\nabla^{tmp;1}$ by the formula

$$\nabla^{tmp;1} Z_1 Z_2 = \nabla^{LC} Z_1 Z_2 - \Pi P(\Pi Z_1, \Pi Z_2)$$

where the bilinear map $P : \Gamma(TQ) \times \Gamma(TQ) \to \Gamma(TQ)$ over $C^\infty(Q)$ is defined by

$$4P(X, Y) = (\nabla^C_Y J)X + J((\nabla^C_Y J)X) + 2J((\nabla^C_Y J)Y) \quad (6.2)$$

for vector fields $X, Y$ in $Q$. (To avoid confusion with our notation $Q$ for the contact manifold and to highlight that $P$ is not the same tensor field as $Q$ but is the contact analog thereof, we use $P$ instead for its notation.) From (5.9), we have now

$$\Pi P(\Pi Z_1, \Pi Z_2) = \frac{1}{2} J((\nabla^L_{\Pi Z_1} J)\Pi Z_2).$$

According to the remark made in the beginning of the section, we choose $B$ to be

$$B_1(Z_1, Z_2) = -\Pi P(\Pi Z_1, \Pi Z_2) = -\frac{1}{2} J((\nabla^L_{\Pi Z_1} J)\Pi Z_2). \quad (6.3)$$

First we consider the induced vector bundle connection on the Hermitian bundle $\xi \to Q$, which we denote by $\nabla^{tmp;1, \pi}$; it is defined by

$$\nabla^{tmp;1, \pi} Y := \pi \nabla^{tmp;1} Y \quad (6.4)$$

for a vector field $Y$ tangent to $\xi$, i.e., a section of $\xi$ for arbitrary vector field $X$ on $Q$. We now prove the $J$ linearity of $\nabla^{tmp;1, \pi}$.

**Lemma 6.3.** Let $\pi : TQ \to \xi$ be the projection. Then

$$\nabla^{tmp;1, \pi} (JY) = J\nabla^{tmp;1, \pi} Y$$

for $Y \in \xi$ and all $X \in TQ$.

**Proof.** For $X \in \xi$,

\[
\begin{align*}
\nabla^{tmp;1} (JY) &= \nabla^{LC} (JY) - \Pi P(X, JY) \\
&= (J\nabla^L Y + (\nabla^L J)Y) - \frac{1}{2} J((\nabla^L J)Y) \\
&= J\nabla^L Y + (\nabla^L J)Y - \frac{1}{2} J((\nabla^L J)Y) + \frac{1}{2} J((\nabla^L P)Y) \quad (6.5) \\
&= J\nabla^L Y + (\nabla^L J)Y - \frac{1}{2} J((\nabla^L J)Y)
\end{align*}
\]

where we use (5.1) to get the last two terms in the third equality and use (5.2) to see that the last term in (6.5) vanishes. Hence,

$$\pi \nabla^{tmp;1} (JY) = \pi \nabla^{tmp;1} (JY) = J\nabla^L Y + \frac{1}{2} \pi ((\nabla^L J)Y).$$
On the other hand, we compute
\[
J\pi \nabla^{\text{tmp};1} X = J \left( \nabla^{L_C} X - \frac{1}{2} J(\nabla^{L_C} J)X \right)
= J\nabla^{L_C} X + \frac{1}{2} \pi(\nabla^{L_C} J)X.
\]
Hence we have now
\[
\pi \nabla^{\text{tmp};1} X \left( JY \right) = J\pi \nabla^{\text{tmp};1} X Y
\]
for \(X, Y \in \xi\).

On the other hand, we notice that \(\nabla^{\text{tmp};1} X \lambda = \nabla^{L_C} X \lambda\). By using Proposition 5.9, the equality
\[
\pi \nabla^{\text{tmp};1} X \left( JY \right) = J\pi \nabla^{\text{tmp};1} X Y
\]
also holds for \(X = X_\lambda\), and we are done with the proof. □

Next we study the metric property of \(\nabla^{\text{tmp};1}\) by computing \(\langle \nabla^{\text{tmp};1} X Y, Z \rangle + \langle Y, \nabla^{\text{tmp};1} X Z \rangle\) for arbitrary \(X, Y, Z \in TQ\).

Using the metric property of the Levi-Civita connection, we derive
\[
\begin{align*}
\langle \nabla^{\text{tmp};1} Y, Z \rangle + \langle Y, \nabla^{\text{tmp};1} Z \rangle - X \langle Y, Z \rangle
= & \quad \langle \nabla^{L_C} Y, Z \rangle + \langle Y, \nabla^{L_C} Z \rangle - X \langle Y, Z \rangle - \langle \Pi P(\Pi X, \Pi Y), Z \rangle - \langle Y, \Pi P(\Pi X, \Pi Z) \rangle \\
= & \quad -\langle \Pi P(\Pi X, \Pi Y), Z \rangle - \langle Y, \Pi P(\Pi X, \Pi Z) \rangle,
\end{align*}
\]
(6.6)
The following lemma shows that when \(X, Y, Z \in \xi\) this last line vanishes. This is the contact analog to Proposition 2.7 whose proof is also similar thereto this time based on Lemma 5.7. Since we work in the contact case for which we cannot directly quote its proof here, we give complete proof for readers’ convenience.

**Lemma 6.4.** For \(X, Y, Z \in \xi\),
\[
\langle P(X, Y), Z \rangle + \langle Y, P(X, Z) \rangle = 0.
\]
Therefore,
\[
\langle \nabla^{\text{tmp};1} Y, Z \rangle + \langle Y, \nabla^{\text{tmp};1} Z \rangle = X \langle Y, Z \rangle.
\]

**Proof.** We compute for \(X, Y, Z \in \xi\),
\[
\begin{align*}
\langle P(X, Y), Z \rangle + \langle Y, P(X, Z) \rangle
= & \quad \frac{1}{2} \langle J(\nabla^{L_C} J)Y, Z \rangle + \frac{1}{2} \langle Y, J(\nabla^{L_C} J)Z \rangle \\
= & \quad -\frac{1}{2} \langle (\nabla^{L_C} J)Y, JZ \rangle - \frac{1}{2} \langle JY, (\nabla^{L_C} J)Z \rangle \\
= & \quad -\frac{1}{4} \langle N(Y, JZ), JX \rangle + \frac{1}{4} \langle N(Z, JY), JX \rangle \\
= & \quad -\frac{1}{4} \langle \Pi N(Y, JZ) + \Pi N(Z, JY), JX \rangle = 0,
\end{align*}
\]
(6.7)
where we use the third equality of Corollary 5.6 for (6.7) and use the second equality of Lemma 5.7 for the vanishing of (6.8). □

Now, we are ready to state the following proposition.

**Proposition 6.5.** The vector bundle connection \(\nabla^{\text{tmp};1, \pi} := \pi \nabla^{\text{tmp};1}\) is an Hermitian connection of the Hermitian bundle \(\xi \to Q\).
Proof. What is now left to show is that for any \( Y, Z \in \xi \),
\[
\langle \nabla_{X_\lambda}^{\text{tmp};1} Y, Z \rangle + \langle Y, \nabla_{X_\lambda}^{\text{tmp};1} Z \rangle = X_\lambda \langle Y, Z \rangle,
\]
which immediately follows from our construction of \( \nabla^{\text{tmp};1} \) since
\[
\nabla_{X_\lambda}^{\text{tmp};1} Y = \nabla_{X_\lambda}^{L_C} Y,
\]
\[
\nabla_{X_\lambda}^{\text{tmp};1} Z = \nabla_{X_\lambda}^{L_C} Z.
\]
□

Next, we look at the metric property when the Reeb direction gets involved.

Lemma 6.6. For \( Y, Z \in \xi \),
\[
\langle \nabla_{X_\lambda}^{\text{tmp};1} X_\lambda, Z \rangle + \langle X_\lambda, \nabla_{X_\lambda}^{\text{tmp};1} Z \rangle = 0.
\]

Proof. We compute for \( Y, Z \in \xi \),
\[
\langle \nabla_{Y}^{\text{tmp};1} X_\lambda, Z \rangle + \langle X_\lambda, \nabla_{Y}^{\text{tmp};1} Z \rangle = \langle \nabla_{Y}^{L_C} X_\lambda, Z \rangle + \langle X_\lambda, \nabla_{Y}^{L_C} Z \rangle - \langle X_\lambda, \Pi P(Y, Z) \rangle = 0.
\]
This finishes the proof. □

Now we study the torsion property of \( \nabla^{\text{tmp};1} \). Denote the torsion of \( \nabla^{\text{tmp};1} \) by \( T^{\text{tmp};1} \). We complexify the contact structure \( \xi \) and denote it by \( \xi_C = \xi \otimes \mathbb{C} \). Then \( \xi_C \) has the decomposition
\[
\xi_C = \xi^{(1,0)} \oplus \xi^{(0,1)}.
\]
Denote \( \Pi' \) the projection to \( \xi^{(1,0)} \) and \( T^{\text{tmp};1}_C \) is the complexification of \( T^{\text{tmp};1} \).
Define
\[
\Theta^\pi = \Pi'T^{\text{tmp};1,\pi}_C.
\]

The proof of the following lemma follows essentially the same strategy as that of the proof of [KN, Theorem 3.4]. But we would like to highlight two conventions we are using which are different therefrom:

1. We use the definition
\[
N(Y, Z) = [JY, JZ] - [Y, Z] - J[Y, JZ] - J[JY, Z]
\]
without the factor of 2 differently from [KN].

2. Our definition of the wedge product is the one from [S] but not the one from [KN]. More specifically, in our convention, we have
\[
d\lambda(X, Y) = X[\lambda(Y)] - Y[\lambda(X)] - \lambda([X, Y])
\]
while the one from [KN] gives rise to
\[
2d\lambda(X, Y) = X[\lambda(Y)] - Y[\lambda(X)] - \lambda([X, Y]).
\]

Besides these differences of the convention, since the current case deals with the contact case, whose statements are significantly different from the almost Hermitian case, we give the complete proof for readers’ convenience.

Lemma 6.7. For \( Y \in \xi \),
\[
T^{\text{tmp};1}(X_\lambda, Y) = 0.
\]
If we decompose
\[
T^{\text{tmp};1}|_\xi = \pi T^{\text{tmp};1}|_\xi + \lambda(T^{\text{tmp};1,\pi}|_\xi) X_\lambda
\]
and denote $T^{\text{tmp};1,\pi}|_{\xi} := \pi T^{\text{tmp};1,\pi}|_{\xi}$, then

$$
T^{\text{tmp};1,\pi}|_{\xi} = \frac{1}{4} N|_{\xi}
$$

$$
\lambda(T^{\text{tmp};1|_{\xi}}) = 0.
$$

In particular, $\Theta|_{\xi}$ is of $(0, 2)$ form or equivalently $JT^{\text{tmp};1}(Y, Z) = T^{\text{tmp};1}(Y, Z)$ for all $Y, Z \in \xi$.

**Proof.** Since $\nabla^{\text{tmp};1} = \nabla^{LC} - \Pi P(\Pi, \Pi)$ and $\nabla^{LC}$ is torsion free, we derive for $Y, Z \in \xi$,

$$
T^{\text{tmp};1}(Y, Z) = T^{LC}(Y, Z) - \Pi P(Y, Z) + \Pi P(Z, Y)
$$

from the general torsion formula.

Next we calculate $-\Pi P(\Pi Y, \Pi Z) + \Pi P(\Pi Z, \Pi Y)$ using the formula

$$
\frac{1}{2} J\nabla^{LC}_{Y} JZ - \frac{1}{2} J\nabla^{LC}_{Z} JY = \frac{1}{4} \pi([JY, JZ] - \pi[Y, Z] - J[JY, Z] - J[Y, JZ])
$$

$$
= \frac{1}{4} \pi N(Y, Z).
$$

This follows from the general formula

$$
- P(Y, Z) + P(Z, Y) = \frac{1}{4} ([JY, JZ] - \Pi[Y, Z] - J[JY, Z] - J[Y, JZ]), \quad (6.9)
$$

whose derivation we postpone till Appendix.

On the other hand, since the added terms to $\nabla^{LC}$ only involves $\xi$-directions, the $X_\lambda$-component of the torsion does not change and so

$$
\lambda(T^{\text{tmp};1}|_{\xi}) = \lambda(T^{LC}|_{\xi}) = 0.
$$

This finishes the proof. \qed

From the definition of $\nabla^{\text{tmp};1}$, we have the following lemma from the properties of the Levi-Civita connection in Proposition 5.3.

**Lemma 6.8.** $\nabla^{\text{tmp};1}_{X_\lambda} X_\lambda = 0$ and $\nabla^{\text{tmp};1}_{Y} X_\lambda \in \xi$ for any $Y \in \xi$.

We also get the following property by using Lemma 5.10 for Levi-Civita connection.

**Lemma 6.9.** For any $Y \in \xi$, we have

$$
\nabla_{Y}^{\text{tmp};1} X_\lambda = \frac{1}{2} JY + \frac{1}{2} (\mathcal{L}_{X_\lambda} J) JY.
$$

We end the construction of $\nabla^{\text{tmp};1}$ by summarizing that $\nabla^{\text{tmp};1}$ satisfies Axioms (1)-(4),(6) and (5;−1).

6.2. **Modification 2;** $\nabla^{\text{tmp};2}$. Now we introduce another modification $\nabla^{\text{tmp};2}$ starting from $\nabla^{\text{tmp};1}$ to make it satisfy Axiom (5;c) and preserve other axioms for any given constant $c \in \mathbb{R}$. Recall that $\nabla^{\lambda,0}$ is our definition of the contact triad connection and that $\nabla^{\text{tmp};1}$ satisfies (5;−1) and so $\nabla^{\text{tmp};1} = \nabla^{\lambda;−1}$. 
We define
\[ \nabla_{Z_1}^{\text{tmp;2}} Z_2 = \nabla_{Z_1}^{\text{tmp;1}} Z_2 - \frac{1}{2} (1 + c) \langle Z_2, X_\lambda \rangle J Z_1 - \frac{1}{2} (1 + c) \langle Z_1, X_\lambda \rangle J Z_2 + \frac{1}{2} (1 + c) \langle J Z_1, Z_2 \rangle X_\lambda. \]

In other words, we define \( \nabla_{\text{tmp;2}} \) = \( \nabla_{\text{tmp;1}} + B_2 \) for the tensor \( B_2 \) defined by
\[ B_2(Z_1, Z_2) = \frac{1}{2} (1 + c) (- \langle Z_2, X_\lambda \rangle J Z_1 - \langle Z_1, X_\lambda \rangle J Z_2 + \langle J Z_1, Z_2 \rangle X_\lambda). \]

From its expression, it follows that \( A_{\text{xi}}(1) \) is satisfied.

Proposition 6.10. The connection \( \nabla_{\text{tmp;2}} \) satisfies all the properties of the canonical connection with constant \( c \). In particular \( \nabla := \nabla_{\text{tmp;2}} \) with \( c = 0 \) is the contact triad connection.

Proof. For \( Y, Z \in \xi \), (6.12) gives \( \nabla_{\text{Y}}^{\text{tmp;2}, \pi} Z = \nabla_{\text{Y}}^{\text{tmp;1}, \pi} Z \). Hence we have
\[
\langle \nabla_{\text{X}}^{\text{tmp;2}, \pi} X, Z \rangle + \langle Y, \nabla_{\text{X}}^{\text{tmp;2}, \pi} Z \rangle = X \langle Y, Z \rangle
\]
for \( X, Y, Z \in \xi \) from the properties of \( \nabla_{\text{tmp;1}} \).

For \( Z \in \xi \), \( \nabla_{\text{X}}^{\text{tmp;2}, \pi} (JZ) = J \nabla_{\text{X}}^{\text{tmp;2}, \pi} Z \) follows from (6.13) and \( \nabla_{\text{X}}^{\text{tmp;1}, \pi} (JZ) = J \nabla_{\text{X}}^{\text{tmp;1}, \pi} Z \).

The metric property,
\[
\langle \nabla_{\text{X}}^{\text{tmp;2}, \pi} Y_1, Y_2 \rangle + \langle Y_1, \nabla_{\text{X}}^{\text{tmp;2}, \pi} Y_2 \rangle = 0
\]
for \( Y_1, Y_2 \in \xi \) immediately follows from that of \( \nabla_{\text{tmp;1}} \) and (6.13). Hence we have checked that Axiom (1) is satisfied.

We also have \( T_{\text{tmp;2}, \pi}(Y_1, Y_2) = T_{\text{tmp;1}, \pi}(Y_1, Y_2) \) again by (6.12) and (6.13). Therefore, Axiom (2) is satisfied.

For Axiom (3), we calculate for \( Y \in \xi \),
\[
T_{\text{tmp;2}}(X_\lambda, Y) = \nabla_{\text{X}}^{\text{tmp;2}, \pi} Y - \nabla_{\text{X}}^{\text{tmp;2}, \pi} X_\lambda - [X_\lambda, Y]
= \nabla_{\text{X}}^{\text{tmp;2}, \pi} Y - \frac{1}{2} (1 + c) J Y - \nabla_{\text{X}}^{\text{tmp;1}, \pi} X_\lambda + \frac{1}{2} (1 + c) J Y - [X_\lambda, Y]
= T_{\text{tmp;1}}(X_\lambda, Y) = 0.
\]

Axiom (4) immediately follows from the definition.

For Axiom (5), we compute
\[
\nabla_{\text{Y}}^{\text{tmp;2}, \pi} X_\lambda + J \nabla_{\text{Y}}^{\text{tmp;2}, \pi} X_\lambda
= \nabla_{\text{Y}}^{\text{tmp;1}, \pi} X_\lambda - \frac{1}{2} (1 + c) J Y + J \nabla_{\text{Y}}^{\text{tmp;1}, \pi} X_\lambda - J \frac{1}{2} (1 + c) J Y
= -Y + (1 + c) Y = c Y.
\]
Then we can uniquely get the following property
\[ \nabla_{\text{tmp}} Y X_{\lambda} = -\frac{1}{2} c J Y + \frac{1}{2} (\mathcal{L}_{X_{\lambda}} J) J Y \]
as explained at the end of Section 3, and Axiom (6) is preserved by \( \nabla_{\text{tmp}}^2 \) after a short calculation by recalling Lemma 6.9 together with \( \nabla_{\text{tmp}}^2 X_{\lambda} = \nabla_{\text{tmp}}^1 X_{\lambda} \). □

This proposition finally completes our construction of the contact triad connection, which is \( \nabla^{\lambda;0} \).

We now summarize the modifications that we have performed in the previous section. Our connection \( \nabla^{\lambda;-1} \) is nothing but
\[ \nabla^{\lambda;-1} = \nabla^{LC} + B_1 \]
with the tensor \( B_1 \) of the type \( \frac{1}{2} \) given by \( B_1(Z_1, Z_2) = -\Pi P(\Pi Z_1, \Pi Z_2) \), and
\[ \nabla = \nabla^{\nabla;\lambda} \]
is
\[ \nabla = \nabla^{\lambda;-1} + B_2 = \nabla^{LC} + B_1 + B_2 \quad (6.14) \]
where
\[ B_2(Z_1, Z_2) = \frac{1}{2} \left( -\langle Z_2, X_{\lambda} \rangle J Z_1 - \langle Z_1, X_{\lambda} \rangle J Z_2 + \langle J Z_1, Z_2 \rangle X_{\lambda} \right). \quad (6.15) \]

Before ending this section, we restate the following properties which will be useful for calculations involving contact Cauchy-Riemann maps performed in [OW1], [OW2].

**Proposition 6.11.** Let \( \nabla \) be the connection satisfying Axiom (1)-(4),(6) and (5; c), then
\[ \nabla Y X_{\lambda} = -\frac{1}{2} c J Y + \frac{1}{2} (\mathcal{L}_{X_{\lambda}} J) J Y. \]
In particular, for the contact triad connection,
\[ \nabla Y X_{\lambda} = \frac{1}{2} (\mathcal{L}_{X_{\lambda}} J) J Y. \]

**Proof.** We already gave its proof in the last part of Section 3. □

**Proposition 6.12.** Decompose the torsion of \( \nabla \) into \( T = \pi T + \lambda(T) X_{\lambda} \). The triad connection \( \nabla \) has its torsion given by \( T(X_{\lambda}, Z) = 0 \) for all \( Z \in TQ \), and
\[ \pi T(Y, Z) = \frac{1}{4} \pi N(Y, Z) = \frac{1}{4} (\langle \mathcal{L}_{J Y} J \rangle Z + \langle \mathcal{L}_{J Y} J \rangle Z) \]
\[ \lambda(T(Y, Z)) = d\lambda(Y, Z) \]
for all \( Y, Z \in \xi \).

**Proof.** We have seen
\[ \pi T^{\text{tmp};2}|_{\xi} = \pi T^{\text{tmp};1}|_{\xi} = \frac{1}{4} N^\pi|_{\xi}. \]
On the other hand, a simple computation shows
\[ N^\pi(Y, Z) = (\mathcal{L}_{J Y} J) Z - J(\mathcal{L}_{Y} J) Z = (\mathcal{L}_{J Y} J) Z + (\mathcal{L}_{Y} J) J Z. \]
This proves the first equality.

For the second, a straightforward computation shows
\[ \lambda(T^{\text{tmp};2}(Y, Z)) = \lambda(T^{\text{tmp};1}(Y, Z)) + (1 + c) \langle J Y, Z \rangle \]
\[ = (1 + c) d\lambda(Y, Z) \]
for general \( c \). Substituting \( c = 0 \), we obtain the second equality. This finishes the proof. \( \square \)

7. Epilogue

In most existing literature of symplectic topology and contact topology studying pseudoholomorphic curves, Levi-Civita connection of the compatible metric associated to almost Kähler structure \((M, \omega, J)\) is commonly used, largely prompted by the experience from Riemannian geometric calculations. However the senior author always feels that for the purpose of studying pseudoholomorphic curves the Levi-Civita connection is not the best connection to use, but a good choice of \( J \)-linear connection should be although one will lose torsion-freeness of the connection. After he learned about the canonical connection (or Ehresman-Liebermann connection) in almost Hermitian manifolds from a paper [Ko], he predicted that usage of canonical connection should be useful for some deeper study of pseudoholomorphic curves. We would like to refer readers to the introduction of [Ko] for nice explanation on the naturality of considering canonical connection over the Levi-Civita connection, and refer to [Oh1, chapter 7]) for some illustration of the advantage of using the canonical connection in geometric calculations involving pseudoholomorphic curves. Complication of the explicit formula given in Theorem 1.4 of the contact triad connection relative to the Levi-Civita connection and the usefulness of the contact triad connection in our tensor calculations in [OW1, OW2] indeed even more supports this point of view that an optimal choice of connection greatly facilitates geometric and analytic study also in symplectic and contact geometry.

8. Appendix

In this appendix, we give the proof of (6.9). By the definition of \( P \),

\[
-P(Y, Z) + P(Z, Y) = -\frac{1}{4} \left( (\nabla^L_C J)Y + J((\nabla^L_C J)Y) + 2J((\nabla^L_C J)Z) \right) + \frac{1}{4} \left( (\nabla^L_C J)Z + J((\nabla^L_C J)Z) + 2J((\nabla^L_C J)Y) \right)
\]

\[
= -\frac{1}{4} \left( \nabla^L_C (JY) - J\nabla^L_C Y + J\nabla^L_C (JY) - J\nabla^L_C (JZ) - 2J\nabla^L_C (JY) - 2JJ\nabla^L_C Z \right) + \frac{1}{4} \left( \nabla^L_C (JZ) - J\nabla^L_C Z + J\nabla^L_C (JZ) - J\nabla^L_C (JY) - 2J\nabla^L_C (JY) - 2JJ\nabla^L_C Y \right)
\]

\[
= -\frac{1}{4} \left( \nabla^L_C (JY) - J\nabla^L_C Y + J\nabla^L_C (JY) + \Pi\nabla^L_C Y + 2J\nabla^L_C (JZ) + J\nabla^L_C (JY) + 2\Pi\nabla^L_C Z \right) + \frac{1}{4} \left( \nabla^L_C (JZ) - J\nabla^L_C Z + J\nabla^L_C (JZ) + \Pi\nabla^L_C Z + 2J\nabla^L_C (JY) + 2\Pi\nabla^L_C Y \right)
\]

\[
= -\frac{1}{4} \left( \nabla^L_C (JZ) - \nabla^L_C (JY) \right) - \frac{1}{4} \Pi(\nabla^L_C Z - \nabla^L_C Y) - \frac{1}{4} J(\nabla^L_C Z - \nabla^L_C (JY)) - \frac{1}{4} J(\nabla^L_C (JZ) - \nabla^L_C (JY))
\]

\[
= -\frac{1}{4} \left[ (JY, JZ) - \Pi[Y, Z] - J[JY, Z] - J[Y, JZ] \right].
\]
This finishes the proof.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI, 53706 & IBS CENTER FOR GEOMETRY AND PHYSICS, POHANG & DEPARTMENT OF MATHEMATICS, POSTECH, POHANG, KOREA

E-mail address: oh@math.wisc.edu

E-mail address: rwang@math.wisc.edu