Capacitated Vehicle Routing with Target Geometric Constraints

Kai Gao
Jingjin Yu

Abstract—We investigate the capacitated vehicle routing problem (CVRP) under a robotics context, where a vehicle with limited payload must complete delivery (or pickup) tasks to serve a set of geographically distributed customers with varying demands. In classical CVRP, a customer location is modeled as a point. In many robotics applications, however, it is more appropriate to model such “customer locations” as 2D regions. For example, in aerial delivery, a drone may drop a package anywhere in a customer’s lot. This yields the problem of CVRG (Capacitated Vehicle Routing with Target Geometric Constraints). Computationally, CVRP is already strongly NP-hard; CVRG is therefore more challenging. Nevertheless, we develop fast algorithms for CVRG, capable of computing high quality solutions for hundreds of regions. Our algorithmic solution is guaranteed to be optimal when customer regions are convex. Numerical evaluations show that our proposed methods significantly outperform greedy best-first approaches. Comprehensive simulation studies confirm the effectiveness of our methods.

I. INTRODUCTION

The Capacitated Vehicle Routing Problem (CVRP) naturally arises in many robotics applications, e.g., autonomous truck routing on road networks [1], aerial delivery [2], [3], clutter removal [4], and so on. In CVRP, a vehicle starting at a depot with limited load capacity is tasked to deliver (resp., pick up) goods to (resp., from) geographically scattered customers. The goal is to minimize the total distance that the vehicle must travel to deliver all delivery/pickup tasks.

CVRP is closely related to the vehicle routing problem (VRP) [5] and the traveling salesperson problem (TSP) [6], both of which are NP-hard problems that have been studied extensively [7], [8]. In the literature, effective solutions to CVRP have mainly been based on branch-and-bound, branch-and-cut, and related methods [9]–[13].

In this work, we examine a variant of CVRP when customer locations have non-trivial geometry and study the resulting optimality structure. In this CVRP variant, instead of having a point location, each customer now occupies a contiguous 2D region. We denote the problem as Capacitated Vehicle Routing with (non-trivial) Target Geometric Constraints (CVRG). CVRG accurately models multiple real-world application scenarios. For example, in aerial delivery (or pickup), a parcel may be dropped (picked up) anywhere inside a given region (Fig. 1). Similarly, optimal algorithms for CVRG can provide the most desirable solutions for helicopter-based rescuing missions where people are isolated on “islands” during floods or other natural disasters.

Our algorithmic attack on the CVRG problem seeks to decouple the CVRP element and the additional constraint induced by non-trivial customer location geometry. On the CVRP side, we examine existing state-of-the-art linear programming based solvers and develop a combinatorial algorithm that uses dynamic programming (DP) to chain together individual tours (a tour is a round trip of the robot starting from the depot; generally, multiple tours are required to solve a CVRP). On the geometric constraint side, we prove that optimizing the delivery/pickup locations over a tour with fixed customer sequence induces an optimization problem that can be solved efficiently when customer regions are convex, allowing us to construct efficient subroutines for computing the corresponding optimal tour. A hierarchical combination of the CVRP subroutines and the geometric optimization subroutines prove to be highly effective when compared with greedy best-first approaches, yielding solutions with much higher quality. At the same time, we note that CVRP is strongly NP-hard even when customer demand is lower bounded, which we prove via a reduction from 3-PARTITION [14]. The effective algorithmic solutions developed for CVRG, in particular an optimal solution method for convex customer regions and a fast, high-quality finite horizon heuristic algorithm based on DP, form the main contributions of this work. The algorithms developed in this paper are thoroughly evaluated in simulation studies, including a physics engine based drone delivery scenario, which confirm their effectiveness.

Beside its relevance to CVRP, CVRG is closely related to research on object rearrangement in robotics. A diverse set of methods have been applied to tackle object rearrangement tasks, including search based approaches [15] and symbolic
reasoning based approaches [16]. In contrast, in [17]–[20], more focus is put on taming the combinatorial explosion caused by the large number of objects. There, even the seemingly simple problem of rearranging unlabeled objects turns out to be NP-hard if an optimal solution is sought after [18], which echoes the computational challenge we face in the current study. From an application perspective, our work applies to scenarios including aerial delivery [2], [3], disaster response [4], [21], [22], among others. CVRG can also model the truck behavior in the truck-drone collaborative delivery problems [23] [24] [25]. For each delivery, the truck only needs to reach the neighborhood of the customer and let the drone do the last-mile delivery.

**Organization.** The rest of the paper is organized as follows. In Section II, CVRG is stated, followed by a preliminary structural analysis and a hardness proof. In Section III, we describe in detail our proposed algorithmic solutions for CVRG, which are thoroughly evaluated and compared in Section IV. We conclude in Section V.

## II. Preliminaries

### A. CVRP with Target Geometric Constraints (CVRG)

In a standard **capacitated vehicle routing problem** (CVRP) [26], a robot (vehicle) with a fixed load capacity is tasked to transport goods to multiple customers from a depot. More formally, let \( d \in \mathbb{R}^2 \) be the location of a depot where a robot may carry a maximum load of \( W \) and deliver different loads to a set of \( n \) customers located at \( c_1, \ldots, c_n \), respectively, with \( c_i \in \mathbb{R}^2 \) for \( 1 \leq i \leq n \). Each customer has a demand “weight” of \( w_i \) that must be satisfied through delivery by the robot.\(^1\) The robot returns to the depot after all deliveries are finished. As a problem that is always feasible, the goal in solving CVRP is to minimize the total distance traveled by the robot. The problem of picking up goods is symmetrical to the problem of delivering goods. In presenting this work, we mainly use the delivery setup but note that the pickup setup is also used when appropriate and is equivalent.

CVRP generally requires a customer’s demand to be met by a single delivery. This makes sense as each delivery operation itself will incur a non-trivial overhead (for both parties). In robotics operations, the overhead can be dropping of packages from air or grasping an object. We mention that the results that we develop also apply if the environment is modeled as a graph, e.g., as a road network. When it comes to robotics applications, e.g., in aerial delivery, the customer location could have non-trivial geometry. We model such constraints by treating each delivery location as a simply-connected polygon.

**Problem 1 (CVRP with Target Geometric Constraints (CVRG)).** Let \( d \in \mathbb{R}^2 \) be the location of a depot with unlimited supply. There are \( n \) customers located in \( P_1, \ldots, P_n \), where \( P_i \subseteq \mathbb{R}^2 \) is a simply-connected polygon. Customer \( i \) has a demand of \( w_i \in (0, 1] \), which is satisfied as a single supply of \( w_i \) is delivered to any point \( p \in P_i \). Find the minimum total distance required for a robot with unit capacity to complete all deliveries, starting from and ending at the depot.

In CVRG, it is not required that for \( 1 \leq i, j \leq n \), \( P_i \cap P_j = \emptyset \), i.e., \( P_i \) and \( P_j \) may overlap. Whereas the non-overlapping case is suitable for applications like aerial delivery, the overlapping setup can be more suitable for applications like picking up objects which may fall on one another. We explicitly address the case where elements of \( \{P_i\} \) overlap.

### B. Strong NP-Hardness of CVRP/CVRG

The classical VRP problem is computationally intractable for large instances as the NP-hard TSP readily reduces to it [7]. The same can be shown for CVRP using similar arguments. However, solvers exist that can solve very large instances of TSP near optimally very fast. CVRP, on the other hand, proves much more challenging. We observe that CVRP is in fact strongly NP-hard, even when customer demand is lower bounded, i.e., the customer’s demand is no smaller than \( 1/k \) for some integer \( k \). Setting \( k = 4 \) is sufficient to show strong NP-hardness. To show this, we first introduce the strongly NP-hard 3-\textsc{Partition} problem [14].

**Problem: 3-\textsc{Partition}**

**INSTANCE:** A finite set \( A \) of \( 3m \) elements, a bound \( B \in \mathbb{Z}^+ \), and a “size” \( s(a) \in \mathbb{Z}^+ \) for each \( a \in A \), such that each \( s(a) \) satisfies \( B/4 < s(a) < B/2 \) and \( \sum_{a \in S} s(a) = mB \).

**QUESTION:** Is there a partition of \( S \) into \( m \) disjoint subsets \( S_1, \ldots, S_m \) such that for \( 1 \leq i \leq m \), \( \sum_{a \in S_i} s(a) = B \)?

**Theorem II.1.** CVRP is strongly NP-hard.

**Proof.** Given a 3-\textsc{Partition} instance with \( 3m \) elements, we construct a CVRP instance as follows. In \( \mathbb{R}^2 \), let \( d = (0, 0) \) be the depot and let there be \( 3m \) customers all located close to \((1, 0)\). That is, for any customer \( i \), \( 1 \leq i \leq 3m \), \( c_i \in B_{\epsilon}(1, 0) \), which is the \( \epsilon \) ball around the point \((1, 0)\) for some positive \( \epsilon \ll 1/m \). Let the robot have a unit capacity and let customer \( i \) have a demand of \( s(a_i)/B \). Since \( s(a_i) > B/4 \), each delivery can supply at most 3 customers. To show that CVRP is NP-hard, we will show that the 3-\textsc{Partition} problem admits an optimal partition if and only if the CVRP problem admits a total travel distance of no more than \( 2m + 6m\epsilon \).

For the “only if” part, if the 3-\textsc{Partition} problem admits a partition of the \( 3m \) elements into \( m \) sets \( S_1, \ldots, S_m \) of three elements each such that \( \sum_{a \in S_i} s(a) = B \), then a robot can complete all deliveries using a total of \( m \) tours based on the partition. For the \( i \)-th tour, the robot may travel in a straight line to a customer in \( S_i \), which incurs a distance of no more than \( 1 + \epsilon \). Then, the robot will travel along straight line to the other two customers in \( S_i \). These tours will incur a cost of no more than \( 2\epsilon \) each. Finally, the robot can return to the depot with a distance of no more than \( 1 + \epsilon \). The total cost in then no more than \( 2m + 6\epsilon \).

For the “if” part, if the CVRP problem admits a solution with a cost of no more than \( 2m + 6m\epsilon \) Since \( \epsilon \ll 1/m, 2m + 6m\epsilon < 2m + 1 \) and the robot can only make no more than \( m \) tours from the depot and return. Since the robot can move at most a unit of supply to the customers per round tour, the
robot can move at most a total supply of \( m \). As the CVRP problem is solved, this means that the robot must complete delivery to exactly 3 customers since (1) partial deliveries are not allowed and (2) at most three customers’ demand can be fulfilled with a unit of supply.

With CVRP (and subsequently, CVRG) being strongly NP-hard, it does not admit an FPTAS (fully polynomial-time approximation scheme) unless \( P = NP \) [27]. That is, it is unlikely that efficient algorithms exist for solving CVRP/CVRP approximately optimally.

III. Optimally Solving CVRG

Our algorithmic solution for CVRG has two components: a dynamic programming (DP) based algorithm for CVRP (Section III-A) and a geometric optimization subroutine (Section III-B) that efficiently compute optimal single tours in the presence of the geometric constraint. Our DP algorithm for CVRG is guaranteed to be optimal when the customer regions are convex. We also examine the relevant scenario for CVRG is guaranteed to be optimal when the customer (Section III-B) that efficiently compute optimal single tours (Section III-A) and a geometric optimization subroutine (Section III-B).

A. Exact and Finite-Horizon DP Algorithms for CVRP

Exact combinatorial algorithm. Our exact algorithm for CVRP proceeds in two phases. In the first phase, we exhaustively compute the optimal cost \( c_s \) of each valid tour starting from the depot that visits a subset of customers \( s \). It is easy to see that there are at most \( 2^n \) tour combinations, many of which will be invalid as they will exceed the capacity limit. For each valid tour combination, the optimal cost is computed using an exact TSP solver [28]. The costs are then stored for the second phase of the computation.

In the second phase, to select the optimal set of disjoint tours, a straightforward application of dynamic programming (DP) is used. Let \( C \) be the set of all customers. Let \( I \subseteq C \) be the set of customers that have not been served and let \( S_I \) be the set of customer subsets of \( I \) that can be visited in a single tour without violating the capacity constraint. Let \( J_I \) be the optimal cost (i.e., the minimum total path length) to satisfy the demands of all of \( I \), the standard DP recursion is

\[
J_I = \min_{j \in S_I} (J_{I-j} + c_j). 
\]

Note that \( c_j \) is provided by the first phase through a direct look-up. The DP algorithm provides significant computational savings by storing the optimal cost of all possible \( I \)'s. For \( n \) customers, there are at most \( 2^n \) such \( I \)'s. The algorithm, denoted DP-CVRP, is straightforward to implement: we simply create a large enough table to hold the \( 2^n \) entries and then iteratively populate the entries.

Algorithm analysis. In the first phase, there are at most \( 2^n \) tours to examine. Since each tour visits no more than \( n \) elements, obtaining the optimal cost of a tour takes \( O(T(n)) \) time, where \( T(n) = O(n^22^n) \) with a dynamic programming TSP solver. The first phase then takes time \( O(n^24^n) \).

In the second phase, the incremental DP computation needs to go through \( 2^n \) possible \( I \)'s, starting from \( I = \emptyset \) and eventually reach \( I = C \). For each \( I \), \( S_I \) contains \( O(2^{|I|}) \) potential tours that need to be checked. The second phase then requires \( O(2^n2^n) \). Therefore, the overall computational complexity of DP-CVRP is then bounded by \( O(n^24^n) \).

With the application of DP cutting down the computation from a naive \( O(n!f(n)) \) (where \( f(n) \) is some polynomial function of \( n \) for computing the optimal cost of a single tour) to \( O(n^24^n) \), significantly larger CVRP instances can be solved optimally. As we will demonstrate, DP-CVRP is exact. More importantly, DP-CVRP readily allows the integration of additional geometric constraints in the CVRG formulation.

Finite-horizon heuristic. While DP-CVRP computes exact solutions with decent performance, the computation time is still exponential with respect to \( n \). To address this issue, we introduce a finite horizon heuristic which restricts the number of customers examined at any given time. That is, some fixed \( h (h \leq n) \) customers are selected on which DP-CVRP is run. Because high quality solutions to CVRP are generally clustered (see, e.g., Fig. 13 and Fig. 14), these \( h \) candidates are selected in a way to facilitate such clustering. From running DP-CVRP on the \( h \) candidates, we pick two most ”convenient” tours with the least average cost (path length divided by the number of served customers), serve these customers, and repeat the process on the remaining customers. The heuristic algorithm, which we denote as FH-DP-CVRP, has an apparent complexity of \( O(nh^24^h) \), which is polynomial.

B. Optimal Subtours Crossing Multiple Regions

For the first phase of DP-CVRP, the optimal cost for a single tour is computed by solving a TSP with DP, where a certain DP property holds: given five customers \( c_1 \) to \( c_5 \), and two paths \( c_1c_2c_3c_4c_5 \) and \( c_1c_2c_3c_5c_4 \), the optimal cost of \( c_1c_2c_3 \) is the same in both and the computation for this part only needs to be done once. The property breaks as we compute an optimal tour passing through a set of convex polygons, as is required by CVRG. As an example, consider the setup in Fig. 2 with six rectangles and assume that the robot starts from \( P_1 \) and ends at \( P_6 \). For the visiting sequence \( P_1P_2P_3P_4P_5P_6 \), the blue solid path is the optimal local path. On the other hand, for the visiting sequence \( P_1P_2P_3P_4P_6 \), the red dashed path is optimal. We observe that the partial path between \( P_1 \) and \( P_3 \) cannot be reused.

Fig. 2: Dynamic programming property existing in TSP tour computation fails to hold as the vertices become polygons.

Though there is a lack of partial ordering in optimal tour computation for CVRG, for a tour with \( k \) sites where \( k \) is not large, which is the case due to limited robot capacity, sifting...
and \( \ell \) if \( \ell \) overlaps with a line segment of \( \mathcal{T} \). The sub-routine \textsc{Local-Improv} as the process acts as fixing two endpoints of an elastic band and let a point in the middle “slide” in a restricted area (e.g. a convex polygon) to a low energy configuration.

\begin{algorithm} \[ \text{	extbf{Algorithm 1: ELASTIC-IMPROV (Iterative Elastic Improvement)}} \]
\begin{algorithmic[1]
\State \textbf{Input :} \( P_1, \ldots, P_k \): polygon sequence; \( d \): depot location
\State \textbf{Output :} \( \tau \): the optimal tour, initially empty
\For {each \( P_i \), \( 1 \leq i \leq k \)}
\State \( p_i \leftarrow \) a random point on \( P_i \);
\State \( \tau \leftarrow \tau + p_i \); \( \text{``...''} \) denotes appending
\EndFor
\State \( \tau' = \emptyset \);
\While {\( \tau \neq \tau' \)}
\State \( \tau' \leftarrow \tau \); \( \tau \leftarrow \emptyset \);
\For {\( 1 \leq i \leq k \)}
\State \( \tau \leftarrow \tau' + \) \textsc{Local-Improv} (\( i \), \( \tau' \), \( P_i \), \( d \));
\EndFor
\EndWhile
\State \Return \( \tau \).
\end{algorithmic}
\end{algorithm}

To see that Algorithm 1 is globally optimal when the polygons \( P_1, \ldots, P_k \) are convex, we first examine the case where each \( P_i \) is a line segment \( L_i \). We start with defining the terms refraction, reflection, crossing, and mirror-reflection. Without loss of generality, we assume that a tour \( T \) never overlaps with a line segment \( L_i \) as the probability of a shortest tour overlapping with a line segment is zero.

\begin{definition} [Refraction, Reflection, Crossing, and Mirror-Reflection] \end{definition}
Let a line segment \( L \) and a tour \( T \) intersect at \( p \). We say \( T \) is refracted by \( L \) at \( p \) if the two line segments \( \ell_1 \) and \( \ell_2 \) of \( T \) meeting at \( p \) lie on different sides of \( L \). If \( \ell_1 \) and \( \ell_2 \) are co-linear, then the refraction is a crossing. Otherwise, if \( \ell_1 \) and \( \ell_2 \) lie on the same side of \( L \), \( T \) is reflected by \( L \). A reflection is a mirror-reflection if the bisector of the angle between \( \ell_1 \) and \( \ell_2 \) is perpendicular to \( L \).

These definitions are illustrated in Fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig3.png}
\caption{Illustrations of refraction, reflection, crossing, and mirror-reflection. The bold horizontal line is a line segment \( L \) and the line segments with arrows are part of a tour \( T \).}
\end{figure}

\begin{definition} [Elastic Band] \end{definition}
Given a point \( d \in \mathbb{R}^2 \) and a set of line segments \( L_1, \ldots, L_k \), we say a tour \( T \) starting at \( d \) and passing through \( L_1, \ldots, L_k \) in that order, at locations \( p_1, \ldots, p_k \), is an elastic band if for any \( i, 1 \leq i \leq k \), one of the following holds: 1) \( T \) is mirror-reflected by \( L_i \) at \( p_i \); 2) \( T \) crosses \( L_i \) at \( p_i \), or 3) \( p_i \) is an end point of \( L_i \) and the angle formed by the two line segment of \( T \) meeting at \( p_i \) and enclosing \( L_i \) is no less than \( \pi \).

The conditions specified in the definition of elastic bands are necessary conditions for a tour to be locally optimal (shortest). We note that a tour \( T \) may intersect an \( L_i \) at more than one point. However, since the visiting order of \( L_1, \ldots, L_k \) is fixed, \( p_i \)'s are uniquely defined. It is clear that a shortest tour \( T \) going through \( L_1, \ldots, L_k \) in that order, must be an elastic band: if at some \( p_i \) the elastic band properties are not satisfied, \( T \) can be readily shortened.

\begin{lemma} \[ \text{Lemma III.1. A shortest tour \( T \) starting at a depot \( d \) and going through line segments \( L_1, \ldots, L_k \), in that order, at points \( p_1, \ldots, p_k \), must be an elastic band.} \]
\end{lemma}

Next, we will show uniqueness of elastic bands, which yields global optimality. The main idea behind the proof of uniqueness is to establish that two different optimal elastic bands going through the same set of line segments cannot meet back at the starting point (depot \( d \)). Showing this requires detailed cases analysis. We start with some necessary terminologies, definitions, and intermediate lemmas.

Let \( T \) be a tour intersecting line segment \( L_i \) at \( p \). As shown in Fig. 4, \( pp' \) is part of \( L_i \), where \( p \) may be an endpoint of \( L_i \). If \( T \) is an elastic band, then we have \( \theta_1 \leq \theta_2 \) in the setup given in Fig. 4a and \( \theta_3 \leq \theta_4 \) in the setup given in Fig. 4b. Otherwise, \( p \) can be moved toward the middle of \( pp' \) to reduce the length of \( T \). Specifically, \( \theta_1 < \theta_2 \) and \( \theta_3 < \theta_4 \) only when \( p \) is an endpoint of \( L_i \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{Given an elastic band \( T \) and the solid line is part of the line segment \( L_i \), we have \( \theta_1 \leq \theta_2, \theta_3 \leq \theta_4 \).}
\end{figure}

Given an elastic band \( T_j \) starting at \( d \) and intersecting with line segments \( L_1, \ldots, L_k \) in the given order at \( p_{j,i}, \ldots, p_{j,k} \), respectively, we define a ray \( R_{j,i} \) as follows.

\[ R_{j,i} = \begin{cases} \text{ray} \, p_{j,i}, \quad & \text{for } i = 0 \\ \text{ray} \, p_{j,i}p_{j,i+1}, \quad & \text{for } 0 < i < k \\ \text{ray} \, p_{j,k}d, \quad & \text{for } i = k \end{cases} \]

For discussing the relationship between two elastic bands, we introduce a definition convergence angle \( \theta \) for rays.

\begin{definition} \[ \text{Definition III.3. Let } R_1 \text{ and } R_2 \text{ be two rays with direction vectors } \vec{n}_1 \text{ and } \vec{n}_2, \text{ respectively. If } R_1 \text{ and } R_2 \text{ intersect, then the convergence angle } \theta \text{ is defined to be the angle between } \vec{n}_1 \text{ and } \vec{n}_2, \text{ i.e. } \arccos(\vec{n}_1 \cdot \vec{n}_2), \text{ as } \theta_1 \text{ in Fig. 5a. On the other hand, if the two rays do not intersect, then the convergence angle } \theta \text{ is defined as } -\arccos(\vec{n}_1 \cdot \vec{n}_2) \text{ (e.g. } -\theta_2 \text{ in Fig. 5b).} \]
\end{definition}

We note that the later case in the definition of the convergence angle includes two sub cases: two rays intersecting on their extensions (along the negative directions of the rays) or one ray intersecting the extension of another ray. According to the definition, two different rays intersect if and only if
they have a positive convergence angle.

Fig. 5: Two possible arrangements of two rays. In the second case, the intersection may be between one ray and the extension of a second ray.

**Definition III.4.** Let $T_1$ and $T_2$ be two elastic bands going through a line segment $L_i$ containing rays $p_{1,i}p_{i+1}$ and $p_{2,i}p_{2,i+1}$, respectively. When the two outgoing rays emit from $L_i$ from the opposite sides and it holds that

$$\angle p_{1,i+1}p_{1,i}p_{2,i} + \angle p_{2,i+1}p_{2,i}p_{1,i} < \pi,$$

we say that the rays form a zigzag with respect to $L_i$.

Fig. 6: A zigzag is the case when the two tours leaving a line segment from the opposite sides and $\angle p_{1,i+1}p_{1,i}p_{2,i} + \angle p_{2,i+1}p_{2,i}p_{1,i} < \pi$.

**Lemma III.2.** Given two elastic bands $T_1$ and $T_2$ going through line segment $L_i$ containing rays $p_{1,i}p_{2,i}^0$ and $p_{2,i}p_{2,i}^0$, respectively. If $L_i$ is the first line segment in the visiting order, then the rays $p_{1,i}p_{2,i}^0$ and $p_{2,i}p_{2,i}^0$ cannot form a zigzag with respect to $L_i$.

**Proof.** Since $L_i$ is the first object in the visiting order, both $T_1$ and $T_2$ come from the same point $d$ before going through $L_i$. We prove the lemma by showing that $d$ cannot exist if $p_{1,i}p_{2,i}^0$ and $p_{2,i}p_{2,i}^0$ form a zigzag with respect to $L_i$.

Denote the intersection of elastic band $T_i$ and line segment $L_j$ by $p_{i,j}$. If a zigzag forms about $L_i$, then the inequality (2) holds, i.e. $\angle p_{1,i}p_{2,i}p_{2,i}^0 + \angle p_{2,i}p_{2,i}^0p_{1,i} < \pi$. Without loss of generality, we may assume $\angle p_{1,i}p_{2,i}p_{2,i}^0 < \pi/2$. Due to the properties of elastic bands, $d$ must be in the shaded area as shown in Fig. 7a, where $p_{1,0}p_{2,1}p_{1,2}$ is a crossing and $p_{0,1}p_{1,2}p_{1,3}$ is a mirror-reflection. Once $\angle p_{2,i}p_{2,i}^0p_{1,i}^0$ is fixed, there will be another shaded area corresponding to $T_2$, and $d$ has to be in the intersection of the two areas. To allow overlapping between the areas, $\angle p_{2,i}p_{2,i}^0p_{1>i}$ must be greater than $\pi/2$ and the dash area is as shown in Fig. 7b. With the inequality (2), the aforementioned two shaded areas must be disjoint, which leads to the nonexistence of $d$.

**Lemma III.3.** Consider two elastic bands $T_1$ and $T_2$. $v_i$ is the convergence angle of the corresponding rays $R_{1,i}$ and $R_{2,i}$. It is impossible to have a zigzag for an object $L_j$ if $v_i \in (-\pi, 0]$ holds for all $i = 1, 2, \ldots, j - 1$.

**Proof.** Assuming the contrary, then (2) holds. Without loss of generality, we can assume $\angle p_{1,i+1}p_{1,i}p_{2,i} < \pi/2$.
For case C–1 (as Fig. 9), \( v_{j-1} = -(\pi - \theta_2 - \theta_3) = -\pi + \theta_2 + \theta_4 < 0 \) and \( v_j = \pi - (\pi - \theta_1) = -\pi + \theta_1 + \theta_3 \).

With the properties of elastic bands, we have \( 0 < \theta_1 \leq \theta_2 \), \( 0 < \theta_3 \leq \theta_4 \), therefore \( -\pi < v_j \leq v_{j-1} \leq 0 \).

Fig. 9: When \( -\pi < v_{j-1} < 0 \), if two elastic bands \( T_1 \) and \( T_2 \) approach object \( L_j \) from the same side and form refractions, then \( -\pi < v_j \leq 0 \).

With similar reasoning, we can verify that \( -\pi < v_j \leq 0 \), when \( -\pi < v_{j-1} \leq 0 \) in case C–2.

Fig. 10: Two sub-cases in case C–3.

As for case C–3, there are two sub-cases (Fig. 10a and Fig. 10b). For the first sub-case, assuming that \( -\pi < v_i \leq 0 \), for all \( i < j \), \( p_{1,j-1}p_{1,j} \) and \( p_{2,j-1}p_{2,j} \) must be from the opposite sides of object \( L_{j-1} \) as shown in Fig. 11.

Fig. 11: Geometric analysis shows that \( -\pi < v_j < 0 \) in case C–3.

To allow \( v_j > 0 \), \( \theta_4 < \theta_1 \). By the properties of the elastic band, \( \theta_1 \leq \theta_2 \), \( \theta_3 < \theta_4 \). Therefore, \( \theta_4 < \theta_1 \leq \theta_3 < \theta_4 \leq \theta_2 \). Since \( \theta_4 < \theta_3 < \theta_1 \leq \theta_3 \leq \theta_4 \), the object \( L_{j-1} \) is gone through with a zigzag, which is impossible by Lemma III.3.

The same conclusion holds for the second sub-case under case C–3. Therefore, none of the three cases (C–1, C–2, C–3) allows \( v_j > 0 \). And we conclude that once \( T_1 \) and \( T_2 \) diverge from one point, they are unable to converge to the same destination. \( \square \)

When the visiting sequence consists of line segments, Theorem III.1 ensures the uniqueness of the elastic band for each instance. Since a shortest tour is an elastic band, the solution we get from Algorithm 1 is optimal. Optimality of Algorithm 1 extends when objects are convex.

Corollary III.1. For a tour starting from the depot passing through a set of convex regions in a fixed order, its length has a unique global minimum, realized by an elastic band.

Proof. Assuming the contrary, we can let \( T_1 \) and \( T_2 \) be two different elastic bands. Denote the intersection of elastic band \( T_i \) and convex region \( R_j \) by \( p_{i,j} \). Recall that the proof of Theorem III.1 is based on the fact that segment \( p_{1,j}p_{2,j} \) is part of line segment \( L_j \). For each convex region \( R_j \), segment \( p_{1,j}p_{2,j} \) is still part of it. Therefore, the conclusion holds for the convex case, i.e., elastic bands \( T_1 \) and \( T_2 \) should be the same tour. \( \square \)

The uniqueness property stated in Corollary III.1 breaks down without the convexity assumption. Fig. 12 shows such an instance with two different elastic bands.

Fig. 12: An instance that has two different elastic bands when some of the regions are non-convex. \( R_1 \), \( R_2 \) are two non-convex regions and \( d \) is the depot. When the visiting sequence is fixed as \( P_1 \), \( T_1 \) and \( T_2 \) are two tours represented with blue arrows. Both of the tours are elastic bands.

C. Handling Overlaps

Polygon in CVRG may overlap (e.g., in a clutter removal scenario). For arbitrary polygonal objects \( i \) and \( j \), if \( i \) is placed on \( j \) or the opposite, we say object \( i \) and \( j \) overlap with each other. When overlaps occur, it is necessary to consider the partial orders before we schedule the pickup sequence. When the objects form a stack, we should always first pick up the objects on the top. We developed the corresponding DP algorithm for the overlapping cases and compared it with a greedy approach. Given the partial order constraints, invalid routes and invalid ordering among routes can be eliminated. Even when there is only one overlap in the environment, it will cut off the number of task sequences by 50%. In terms of optimality, corollary III.1 continues to hold. Due to limited space, evaluation for the overlapping cases is not presented in Section IV. We mention that, besides less computation cost, the results show no difference from the non-overlapping setting.

IV. Evaluation

We carried out extensive simulation studies to evaluate the performance of the proposed methods. These efforts are described and discussed in this section. The proposed algorithms are implemented in Python and all the experiments are executed on an Intel® Xeon® CPU at 3.00GHz.

A. Environment Setup

We selected multiple customer location and object weight distributions for realistic evaluation. More specifically, problem instances are generated in two phases. First, object weights are uniformly randomly selected with a lower and upper weight bound. For a given integer \( k \), three ranges are used: (1) \([0,1]\), (2) \([1/k,1]\), and (3) \([1/k,2/k]\). This choice models the practical setting that a lower limit on pickup/delivery weight is placed, below which it becomes uneconomical to do so. In our experiments, \( k = 7 \). The \([0,1]\) case consumes the case of \( k \) being arbitrarily large.

After weights are picked, the orientation of the objects are uniformly selected in \([0,2\pi]\). The mass center of the objects in the workspace are selected according to two distributions:
1) **Uniform**: The locations of objects are uniformly selected in the bounded 2D workspace.

2) **Gaussian**: The locations of objects follow a two-dimensional Gaussian distribution where heavy objects are closer to the mean of the distribution (Fig. 13(a)).

![Fig. 13: Gaussian (a) and inverse-Gaussian (b) distributions of customer locations. The size of the disc signifies the objects’ size and/or weights.](image)

We selected the Gaussian setup to model the collapsing of a large object into multiple smaller pieces (e.g., after an explosion), where it is likely for heavy pieces to be close to the epicenter. During the evaluation, we further examined the “inverse” Gaussian setting where heavy objects are more likely to be away from the Gaussian mean (Fig. 13(b)). We omit the result for this setting as it demonstrates performance characteristics similar to the uniform setting.

### B. Performance Evaluation of CVRG

We evaluate the performance of four algorithms on CVRG: (1) DP-CVRG, an exact algorithm based on DP-CVRP, with tour costs computed with ELASTIC-IMPROV, (2) FH, the finite horizon version of DP-CVRG, with \( h = 10 \), (3) GD, a greedy best-first algorithm that picks the closest object satisfying capacity constraints, (4) BCP-CVRG, which uses a state-of-the-art branch-cut-and-price method BCP-CVRP [29] to compute a solution with centroid of the objects and then use ELASTIC-IMPROV to shorten the paths. We also include BCP-CVRP, to see ELASTIC-IMPROV’s contribution.

Our evaluation models aerial delivery applications (Fig.1), where each customer location is a polygon. A typical solution is illustrated in Fig. 14 (the left figure), where the blue lines are the tours. In generating the polygonal regions for evaluation, we made the diameter of the polygon bounded by \( \frac{1}{10} \) of the workspace side length. The result is given in Fig. 15. The solution produced by BCP-CVRG is not optimal and is about 5% worse than the solution produced by FH, which computes near optimal solution (DP-CVRG computes optimal solution when ELASTIC-IMPROV is optimal).

![Fig. 14: (left) A (convex polygon only) CVRG instance with an optimal solution generated by DP-CVRG. (right) A CVRG instance with non-convex polygons and an optimal solution.](image)

The second evaluation examines the Gaussian setup with all three weight distributions. The running time and solution quality results are given in Fig. 16 and Fig. 17 respectively. We also observe that FH runs much faster as we lower bound the object weight, as expected in practice. In terms of solution optimality, FH is near optimal and significantly outperforms the greedy method (by 50%) as well as BCP-CVRG (by 15% to 20% percent). This shows superiority of FH in solution quality.

![Fig. 15: Algorithm performance for CVRG under uniform customer location distribution with 10-40 customers: (left: time (s); right: optimality (unitless)). BCP-CVRP and BCP-CVRG are overlaid with the shorter bar in front. For computational time, BCP-CVRP is faster.](image)

![Fig. 16: Algorithm running time in seconds for CVRG under uniform distribution with 10-40 customers for different weight distributions: [left] \([0,1]\), [middle] \([1/k,1]\), [right] \([1/k, 2/k]\).](image)

![Fig. 17: Corresponding solution quality for cases in Fig. 16.](image)

We evaluated performance of DP-CVRG and FH for cases where customer regions are non-convex. For the case where there are 10 regions, we compute the optimal solution by splitting each non-convex region into multiple convex ones and then run a modified version of DP-CVRG. The result confirms that DP-CVRG and FH both compute the same as the exact optimal solution. A typical case is illustrated in Fig. 14 (the right figure). The evaluation empirically suggests that FH is expected to compute high-quality solution even when regions are non-convex.

In addition to comprehensive numerical studies, we carried out physics based simulations of the aerial delivery scenario in the Unreal Engine to obtain a more realistic estimate of the travel and delivery time. The setup is similar to that illustrated in Fig. 1 and Fig. 14, with 10 to 20 objects. Over 10 runs of different setups for 10 objects, the time cost ratio between DP-CVRG/FH, BCP-CVRG, and GD is 1.07/1.17 (for this case, recall that DP-CVRG and FH compute the same optimal solutions). For 20 objects, the ratio between FH, BCP-CVRG, and GD is 1.02/1.16.
Considering the time that is required for making deliveries, these results largely agree with the earlier numerical results. Our methods for CVRG have excellent scalability. In under 200 seconds, BCP-CVRG and FH readily scale to over 200 objects. Outcome of the computational experiments are given in Fig. 18 in which uniform object distribution with uniform weight distribution in \([0, 1]\) is used.

We observe that the trend agrees with earlier results. In particular, for CVRG, FH does considerably better than both BCP-CVRG (~10%) and the greedy method (~20%).

As a last evaluation, we attempted large scale instances for CVRG where the regions are chosen following the Gaussian distribution. As shown in Fig. 19, the trend of the solution quality agrees with previous experiments under Gaussian distribution. On the other hand, the running time for FH is comparatively large. This can be attributed to the regions with low weights tending to be the last delivery targets. When there are hundreds of regions in the instance, most of the candidates in the last rounds of DP process are with low weights. This causes the amount of candidates in the later tours to be disproportionately large, requiring more time to go through.

In this work, we examined CVRG as a CVRP variant where the target locations have non-trivial geometry, with applications toward a variety of robotics tasks including aerial delivery, rescue, clutter removal, and so on. Solving CVRG optimally requires the careful selection of exact customer locations for completing pickup or delivery tasks. We developed multiple efficient algorithms for tackling CVRG, and evaluated their performance as the number of customers, customer geometry, and customer distribution changes. In all cases, it was shown that our algorithms provide significant advantage in computing high quality solutions as compared with greedy approaches and CVRP solver based methods.

V. CONCLUSION

In this work, we examined CVRG as a CVRP variant where the target locations have non-trivial geometry, with applications toward a variety of robotics tasks including aerial delivery, rescue, clutter removal, and so on. Solving CVRG optimally requires the careful selection of exact customer locations for completing pickup or delivery tasks. We developed multiple efficient algorithms for tackling CVRG, and evaluated their performance as the number of customers, customer geometry, and customer distribution changes. In all cases, it was shown that our algorithms provide significant advantage in computing high quality solutions as compared with greedy approaches and CVRP solver based methods.

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