Existence and Uniqueness of Traces on Discrete Quantum Groups

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Abstract

We find quantum group dynamic characterizations of the existence and uniqueness of tracial states on the reduced $C^*$-algebra $C_r(\hat{G})$ of an arbitrary discrete quantum group $G$. We prove that $C_r(\hat{G})$ admits a tracial state if and only there exists a $G$-invariant state if and only if the cokernel of the Furstenberg boundary of $G$, $\hat{H}_F$, is unimodular. We prove that there is a unique tracial state if and only if $\hat{H}_F$ coincides with the canonical Kac quotient of $\hat{G}$ and a certain quantum group $C^*$-algebra is exotic. Along the way, we obtain that $C_r(\hat{G})$ is nuclear and has a tracial state if and only if $G$ is amenable, which resolves an open problem due to C.-K Ng and Viselter, and Crann, in the discrete case. As applications of our work, we find that $C^*$-simplicity implies the Haar state is the only possible tracial state for arbitrary $G$, and we answer two questions of Kalantar, Kasprzak, Skalski, and Vergnioux in the case where $G$ is unimodular.

1 Introduction

The structure of tracial states and simplicity of $C^*$-algebras are properties of interest to operator algebraists (e.g., for classification theory). A particularly rich class of examples are group $C^*$-algebras, especially the reduced ones coming from discrete groups. In [6] and [21], group dynamical characterizations of the unique trace property and simplicity of reduced group $C^*$-algebras were achieved, where they showed that the reduced $C^*$-algebra $C_r(\hat{G})$ of a discrete group $G$ has a unique tracial state if and only if the action of $G$ on its Furstenberg boundary $\partial_F(G)$ is faithful and in turn that simplicity of $C_r(\hat{G})$ implies it has a unique trace (which is the canonical trace $1_{\{e\}} \in C_r(\hat{G})^* \subseteq \ell^\infty(G)$). Here, we remark that if there is a unique trace, then the structure of the tracial states is easily and fully understood. Given that $R_a(G) = \ker(G \curvearrowright \partial_F(G))$, where $R_a(G)$ is the amenable radical, this shows that the unique trace property is equivalent to having $R_a(G) = \{e\}$. The key point of this result is that in determining the tracial structure of $C_r(\hat{G})$, the tracial states of the form $1_N \in C_r(\hat{G})^*$ where $N$ is an amenable normal subgroup of $G$ are the ones to consider. In particular, the unique trace property is determined by a comparison of $1_{R_a(G)}$ with $1_{\{e\}}$.

Discrete quantum groups give rise to many examples of $C^*$-algebras that hold the reduced group $C^*$-algebras under their umbrella, namely, the reduced quantum group $C^*$-algebras. Serving as a stepping stone towards establishing quantum group dynamic machinery for
quantizing the unique trace property and simplicity of \( C_r(\hat{G}) \) where \( G \) is a discrete quantum group, Kalantar et al. \([16]\) constructed the Furstenberg boundary \( \partial_F(G) \) (see Section 4.1). Moreover, the cokernel of \( \ker(G \curvearrowright \partial_F(G)) \) was shown in \([16]\) to be equal to \( \ell^\infty(\mathbb{H}_F) \) where \( \mathbb{H}_F \) is a quantum subgroup of \( \hat{G} \) that is minimal as an object where \( \ell^\infty(\mathbb{H}_F) \) is relatively amenable (see Section 4.1). In particular, we might call \( \mathbb{H}_F \) the ‘relatively amenable coradical’ of \( G \). There is a catch, though. At the quantum level, the canonical trace is replaced with the Haar state \( h_{\hat{G}} \in C_r(\hat{G})^* \), however, \( h_{\hat{G}} \) may not be tracial and there are known examples where \( C_r(\hat{G}) \) has no trace (e.g., see \([3]\)).

**Definition 1.1.** We say a discrete quantum group \( G \) is \( C^* \)-simple if the reduced \( C^* \)-algebra \( C_r(\hat{G}) \) is simple. We say a unimodular discrete quantum group \( G \) has the unique trace property if the Haar state \( h_{\hat{G}} \) of \( \hat{G} \) is the unique tracial state.

**Theorem 1.2.** \([1,16]\) Let \( G \) be a unimodular discrete quantum group. We have that the action of \( G \) on \( \partial_F(G) \) is faithful if and only if \( G \) has the unique trace property.

Achieving a bonafide generalization of the classical group case, \( C^* \)-simplicity was also found to imply the unique trace property for unimodular discrete quantum groups.

**Theorem 1.3.** \([7]\) Let \( G \) be a discrete quantum group. If \( G \) is \( C^* \)-simple then the action of \( G \) on \( \partial_F(G) \) is faithful.

This completes the picture for the unimodular discrete quantum groups. In this note, we move to arbitrary \( G \), and we study the existence and uniqueness of traces. The position of \( \mathbb{H}_F \) as a closed quantum subgroup of \( \hat{G} \) in relation to the canonical Kac quotient \( \mathbb{H}_{Kac} \) (see Section 4.2) turns out to be the fundamental property that governs the existence and uniqueness of traces.

Our result, which informs the position of \( \mathbb{H}_F \) as a closed quantum subgroup of \( \hat{G} \) is the following, which generalizes the fact that every \( G \)-invariant state (invariant with respect to the conjugation action) in \( C_r(\hat{G}) \) concentrates on the amenable radical.

**Theorem 4.22.** Let \( G \) be a DQG. Every \( G \)-invariant state \( \tau \in C_r(\hat{G})^* \) concentrates on \( \hat{G}/\mathbb{H}_F \).

We must note that most of the above theorem was proven in the proof of \([16\) Theorem 5.3].

For classical groups, it turns the \( G \)-invariant states are the tracial states on \( C_r(\hat{G}) \) and one immediately sees that faithfulness of the Furstenberg boundary gives the unique trace property. This fact remains true for unimodular discrete quantum groups but fails in the arbitrary case, where it was shown in \([16]\) that the \( G \)-invariant tracial states are the KMS states of the scaling automorphism group of \( G \) (see \([16]\)). They also prove that faithfulness of \( G \curvearrowright \partial_F(G) \) implies \( C_r(\hat{G}) \) has no \( G \)-invariant states. In particular, from above, simplicity of \( C_r(\hat{G}) \) proves it has no \( G \)-invariant states.

Despite the apparent disparity between traces and \( G \)-invariant states in general, it turns out that Haar idempotents (see Section 2.4) are \( G \)-invariant if and only if they are tracial (see Proposition 4.13), generalizing the well-known fact that the Haar state is \( G \)-invariant if and only if it is tracial (see Section 4.3). In particular, using the work of Crann \([9]\), we
can prove that $C_r(\hat{G})$ is nuclear if and only if $G$ is amenable and $C_r(\hat{G})$ admits tracial state (Corollary 4.20). Thus we achieve the following.

Corollary 4.24 Let $G$ be a DQG. Then $H_F$ is a closed quantum subgroup of every Kac closed quantum subgroup $H$ of $\hat{G}$, where $\hat{G}/H$ is coamenable. In particular, the following hold:

1. $H_F$ is Kac if and only if $C_r(\hat{G})$ has a tracial state;
2. $H_F = H_{Kac}$ and $C_{rKac}(H_F) = C_r(\hat{G})$ if and only if $C_r(\hat{G})$ has a unique tracial state;
3. $H_F = H_{Kac}$ if and only if $C_r(\hat{G})$ has a unique idempotent tracial state.

See Section 3.2 for coamenability of quotients and coideals.

In particular, we have that the Haar idempotent $\omega_F$ induced by $H_F$ is tracial if and only if $C_r(\hat{G})$ admits a tracial state. So, it is the Haar state coming from the cokernel of $G \curvearrowright \partial_F(\hat{G})$ that governs the existence and uniqueness of traces of discrete quantum groups. With these observations, we are able to deduce the following as well.

Corollary 4.25 Let $G$ be a $C^*$-simple DQG. The only possible tracial state is the Haar state.

The amenable radical of a DQG was constructed in [10], which generalizes the notion of an amenable radical $R_a(G)$ for a discrete group $G$. Kalantar et al. asked for the dual notion of an amenable radical, a so called “coamenable coradical” of a compact quantum group [16, Question 8.2]. As a consequence of Theorem 4.22, we find that $H_F$ is the Kac coamenable coradical of $\hat{G}$. In particular, if $\hat{G}$ is Kac, then $H_F$ is the coamenable coradical (see Section 4.5), and so we resolve [10, Question 8.2] for unimodular discrete quantum groups.

From [1, Theorem 1.10] it follows that given a closed quantum subgroup $H$ of $\hat{G}$, if $\hat{G}/H$ is coamenable, then $\ell^\infty(H)$ is a relatively amenable coideal in $\ell^\infty(G)$ (see Section 4.1 for the definition of relative amenability). This partially answered [10, Question 8.1]. We fully resolve [10, Question 8.1] in the case of unimodular $G$.

Corollary 4.28 Let $G$ be DQG. If $H$ is a Kac closed quantum subgroup of $\hat{G}$, then $\hat{G}/H$ is a coamenable quotient if and only if $\ell^\infty(H)$ is relatively amenable in $\ell^\infty(G)$.

We discuss now the organization of our paper. Section 2 is reserved for preliminary concepts. We discuss locally compact quantum groups, $G$-boundaries, closed quantum subgroups, and coideals and idempotent states. In Section 3 we discuss coamenability of coideals. We introduce a $C^*$-algebraic framework for coideals, and prove basic facts that generalize known characterizations of coamenability of compact quantum groups for coideals. We also touch on the lattice structure of reduced idempotent states. We reserve Section 4 for our main theorems. We recall the construction of the Furstenberg boundary, the Kac and unimodularity properties, the construction of the canonical Kac quotient, and the basics of $G$-invariant states. We spend the remainder of the paper proving our main theorems highlighted above.
2 Preliminaries

2.1 Discrete Quantum Groups

The notion of a quantum group we will be using is the von Neumann algebraic one developed by Kustermans and Vaes [24]. A locally compact quantum group (LCQG) $\hat{G}$ is a quadruple $(L^\infty(\hat{G}), \Delta, h_L, h_R)$ where $L^\infty(\hat{G})$ is a von Neumann algebra; $\Delta : L^\infty(\hat{G}) \to L^\infty(\hat{G}) \otimes L^\infty(\hat{G})$ a normal unital *-homomorphism satisfying $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ (coassociativity); and $h_L$ and $h_R$ are normal semifinite faithful weights on $L^\infty(\hat{G})$ satisfying

$$h_L(f \otimes \text{id})\Delta(x) = f(1)h_L(x), \quad f \in L^1(\hat{G}), \quad x \in M_{h_L} \quad \text{(left invariance)}$$

and

$$h_R(\text{id} \otimes f)\Delta(x) = f(1)h_R(x), \quad f \in L^1(\hat{G}), \quad x \in M_{h_R} \quad \text{(right invariance)},$$

where $M_{h_L}$ and $M_{h_R}$ are the set of integrable elements of $L^\infty(\hat{G})$ with respect to $h_L$ and $h_R$ respectively. We call $\Delta$ the coproduct and $h_L$ and $h_R$ the left and right Haar weights respectively, of $\hat{G}$. The predual $L^1(\hat{G}) := L^\infty(\hat{G})^*$ is a Banach algebra with respect to the product $f \ast g := (f \otimes g)\Delta$ known as convolution.

Using $h_L$, we can build a GNS Hilbert space $L^2(\hat{G})$ in which $L^\infty(\hat{G})$ is standardly represented. There is a unitary $W_\hat{G} \in L^\infty(\hat{G}) \overline{\otimes} B(L^2(\hat{G}))$ such that $\Delta_G(x) = W_{\hat{G}}^*(1 \otimes x)W_{\hat{G}}$. The unitary $W_{\hat{G}}$ is known as the left fundamental unitary of $\hat{G}$ respectively. The left regular representation is the representation

$$\lambda_G : L^1(\hat{G}) \to B(L^2(\hat{G})), \quad f \mapsto (f \otimes \text{id})W_{\hat{G}}.$$

There is a dense involutive subalgebra $L^1_\#(\hat{G}) \subseteq L^1(\hat{G})$ that makes $\lambda_G|L^1_\#(\hat{G})$ a *-representation. We denote the von Neumann algebra $L^\infty(\hat{G}) = \lambda_G(L^1(\hat{G}))'$. There exists a LCQG $\hat{G} = (L^\infty(\hat{G}), \Delta, \hat{h}_L, \hat{h}_R)$, where $\Delta$ is implemented by $W_{\hat{G}} = \Sigma(W_G)^*$, where $\Sigma : a \otimes b \mapsto b \otimes a$ is the flip map. Pontryagin duality holds: $\hat{\hat{G}} = G$.

A discrete quantum group (DQG) is a LCQG $G$ where $L^1(G) (= L^1(G))$ is unital (cf. [29], [37]). Note also that we write $L^\infty(G) = L^\infty(G)$. We denote the unit by $e_G$.

Equivalently, $G$ is a compact quantum group (CQG), which is a LCQG where $h_G = \hat{h}_L = \hat{h}_R \in L^1(G)$ is a state, known as the Haar state of $\hat{G}$. When $G$ is discrete, the irreducible *-representations of $L^1(G)$, are finite dimensional, where a *-representation on locally compact $G$ is a representation that restricts to a *-representation on $L^1_\#(G)$. Given a representation $\pi : L^1(\hat{G}) \to B(\mathcal{H}_\pi) \cong M_{n_\pi}$, there exists an operator $U^\pi \in L^\infty(\hat{G}) \overline{\otimes} B(\mathcal{H}_\pi)$ such that

$$\pi(u) = (u \otimes \text{id})U^\pi, \quad u \in L^1(\hat{G}).$$

Equivalently, a representation $L^1(\hat{G}) \to B(\mathcal{H}_\pi)$ is a *-representation if $U^\pi$ is unitary. Representations $\pi$ and $\rho$ are unitarily equivalent if there exists a unitary $U \in M_{n_\pi}$ such that $(1 \otimes U^*)U^\pi(1 \otimes U) = U^\rho$. We let $\text{Irr}(\hat{G})$ denote the collection of equivalence classes of irreducibles. Note that we will abuse notation and simply write $\pi \in \text{Irr}(\hat{G})$ when we are choosing a representative from $[\pi]$. For each $\pi \in \text{Irr}(\hat{G})$ we let $\mathcal{H}_\pi$ denote the corresponding $n_\pi$-dimensional Hilbert space. In the instance where $\pi \in \text{Irr}(\hat{G})$, we write $U^\pi = [u^\pi_{i,j}]_{i,j}$, so $\pi(u) = [u(u^\pi_{i,j})]_{i,j}$ for $u \in L^1(\hat{G})$ and some orthonormal basis (ONB) $\{e^*_i\}$ of $\mathcal{H}_\pi$. We let $\mathcal{V} : L^1(\hat{G}) \to B(\mathcal{H}_\pi)$ denote the representation where $U^\mathcal{V} = [(u^\pi_{i,j})^*] = \overline{U^\pi}$. 

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It turns out that every $\ast$-representation decomposes into a direct sum of irreducibles and the left regular representation decomposes into a direct sum of elements from $Irr(\hat{G})$, each with multiplicity $n\pi$. Consequently, we have $W_{\pi} = \bigoplus_{\pi \in Irr(\hat{G})} U^\pi \otimes I_{\sigma\pi}$ using the identification $L^2(\hat{G}) = \bigoplus_{\pi \in Irr(\hat{G})} H_\pi \otimes H_\pi$.

We denote the $\ast$-algebra

$$\text{Pol}(\hat{G}) = \text{span}\{ u_{i,j}^\pi : 1 \leq i, j \leq n\pi, \pi \in Irr(\hat{G}) \} \subseteq L^\infty(\hat{G}).$$

It follows that $L^\infty(\hat{G}) = \text{Pol}(\hat{G})$ from Pontryagin duality.

Focusing on $G$, the above entails that we have $\ell^\infty(G) = \bigoplus_{\pi \in Irr(\hat{G})} M_{n\pi}$ as von Neumann algebras, which implies $\ell^1(G) = \bigoplus_{\pi \in Irr(\hat{G})} (M_{n\pi})_*$ spatially. We denote the $C^\ast$-algebra $c_0(G) = \bigoplus_{\pi \in Irr(\hat{G})} M_{n\pi}$, which we note satisfies $M(c_0(G)) = \ell^\infty(G)$.

For this discussion, we are required to discuss the $C^\ast$-algebraic formulation of quantum groups in the universal setting (cf. [23]). There exists a universal $C^\ast$-norm $|| \cdot ||_u$ on $\text{Pol}(\hat{G})$. Let $|| \cdot ||_r$ be the norm on $BL^2(\hat{G}))$. We define the unital $C^\ast$-algebras $C_u(\hat{G}) = \text{Pol}(\hat{G})_{|| \cdot ||_u}$ and $C_r(\hat{G}) = \text{Pol}(\hat{G})_{|| \cdot ||_r} \subseteq L^\infty(\hat{G})$. The universal property gives us a $C^\ast$-algebraic coproduct on $C_u(\hat{G})$: a unital $\ast$-homomorphism

$$\Delta_u : C_u(\hat{G}) \to C_u(\hat{G}) \otimes_{\min} C_u(\hat{G})$$

satisfying coassociativity. Likewise, $\Delta^\pi_u = \Delta_u|_{C_r(\hat{G})}$ gives us a $C^\ast$-algebraic coproduct on $C_r(\hat{G})$. The spaces $C_u(\hat{G})^\ast$ and $C_r(\hat{G})^\ast$, and are known as the universal and reduced measure algebras of $\hat{G}$ respectively. Similar to the von Neumann algebraic case, the coproduct on $C(\hat{G})$ induces a product on $C(\hat{G})^\ast$:

$$\mu \ast \nu(a) = (\mu \otimes \nu)(a), \ a \in C_u(\hat{G}), \mu, \nu \in C(\hat{G})^\ast,$$

making $C(\hat{G})^\ast$ a Banach algebra, where, above, $C(\hat{G})$ and $C(\hat{G})^\ast$ can be either the universal or reduced versions.

The universal property gives us a unital surjective $\ast$-homomorphism

$$\Gamma_G : C_u(\hat{G}) \to C_r(\hat{G})$$

satisfying $\Delta_u \circ \Gamma_G = (\Gamma_G \otimes \Gamma_G) \circ \Delta_u$. The adjoint of this map induces a completely isometric algebra homomorphism $C_r(\hat{G})^\ast \to C_u(\hat{G})^\ast$ such that $C_r(\hat{G})^\ast$ is realized as a weak$^\ast$ closed ideal in $C_u(\hat{G})^\ast$. A Hahn-Banach argument shows $L^1(\hat{G}) = C_r(\hat{G})^\ast$, so $L^1(\hat{G})$ is a closed ideal in $C_u(\hat{G})^\ast$ as well.

More generally, we say $C_\sigma(\hat{G})$ is a CQG $C^\ast$-algebra it is a $C^\ast$-algebraic completion of $\text{Pol}(\hat{G})$ and admits a coproduct $\Delta_\sigma$ such that $\Delta_\sigma|_{\text{Pol}(\hat{G})} = \Delta|_{\text{Pol}(\hat{G})}$. There always exists quotient maps $C_u(\hat{G}) \to C_\sigma(\hat{G}) \to C_r(\hat{G})$ that intertwine the coproducts such as $\Gamma_G$ does for the universal and reduced coproducts.

**Definition 2.1.** A CQG $C^\ast$-algebra $C_\sigma(\hat{G})$ is exotic if $C_u(\hat{G}) \neq C_\sigma(\hat{G}) \neq C_r(\hat{G})$.

The restriction of the coproduct is a unital $\ast$-homomorphism $\Delta_{\hat{G}} : \text{Pol}(\hat{G}) \to \text{Pol}(\hat{G}) \otimes \text{Pol}(\hat{G})$ that satisfies $\Delta_{\hat{G}}(u_{i,j}^\pi) = \sum_{t=1}^{n\pi} u_{i,t}^\pi \otimes u_{t,j}^\pi$. The unital $\ast$-homomorphism $\epsilon_{\hat{G}} : \text{Pol}(\hat{G}) \to \mathbb{C}$ is
Remark 2.2. The examples of DQGs where $\ell^\infty(G)$ is commutative are the discrete groups (cf. [32]), where if $G$ is a discrete group, then $G = (\ell^\infty(G), \Delta_G, m_L, m_R)$ where $m_L = m_R = h_L = h_R$ are the left and right Haar measures, and $\Delta_G(f)(s, t) = f(st)$.

The DQGs where $\ell^1(G)$ is commutative are the duals of compact groups, where if $G$ is a compact group, then $\hat{G} = (VN(G), \Delta_{\hat{G}}, \hat{h})$ where $\hat{h}$ is the Plancherel weight and $\Delta_{\hat{G}}(\lambda_G(s)) = \lambda_G(s) \otimes \lambda_G(s)$.

2.2 $\mathbb{G}$-Boundaries and Crossed Products

Let $\mathbb{G}$ be a DQG and $A$ be a unital $C^*$-algebra.

Definition 2.3. $A$ is a $\mathbb{G}$-$C^*$-algebra if there exists a unital injective $\ast$-homomorphism $\alpha: A \to M(c_0(\mathbb{G}) \otimes_{\min} A)$ satisfying

- $(id \otimes \alpha)(\Delta_A \otimes id)\alpha$;
- the closed linear span of $(c_0(\mathbb{G}) \otimes 1)\alpha(A)$ is norm dense in $c_0(\mathbb{G}) \otimes_{\min} A$.

We call $\alpha$ a (left) coaction of $\mathbb{G}$ on $A$.

Remark 2.4. Given a $\mathbb{G}$-$C^*$-algebra $A$, we may write $A = C(X)$, where $X$ is the underlying compact quantum space, and say that $\mathbb{G}$ acts on $X$.

For a $\mathbb{G}$-$C^*$-algebra $A$, we will use the notation $a \ast f = (f \otimes id)\alpha(a)$ for $a \in A$ and $f \in \ell^1(\mathbb{G})$. Given $\mathbb{G}$-$C^*$-algebras $A$ and $B$, we will say a ucp map $\phi: A \to B$ is $\mathbb{G}$-equivariant if for all $a \in A$ and $f \in \ell^1(\mathbb{G})$, we have $\phi(a) \ast f = \phi(a \ast f)$. For any $\mathbb{G}$-equivariant ucp map $\phi: \ell^1(\mathbb{G}) \to \ell^1(\mathbb{G})$, the space $\phi(\ell^1(\mathbb{G}))$ is a $\mathbb{G}$-$C^*$-algebra when considered as a $C^*$-algebra with the Choi-Effros product.

Example 2.5. The unital $C^*$-algebra $C_r(\mathbb{G})$ is a $\mathbb{G}$-$C^*$-algebra with coaction $\Delta_r(\hat{a}) = W_G(1 \otimes \hat{a})W_G$. Using the decomposition $W_G = \bigoplus_{\pi \in Irr(\mathbb{G})} \sum_{i,j} n_{ij} E_{i,j}^\pi \otimes (u_{i,j}^\pi)^*$, we obtain

$$\Delta_r(\hat{a}) = \sum_{\pi \in Irr(\mathbb{G})} \sum_{i,j,k,l} E_{i,j}^\pi E_{k,l}^\pi \otimes u_{i,j}^\pi \hat{a}(u_{k,l}^\pi)^*$$

$$= \sum_{\pi \in Irr(\mathbb{G})} \sum_{i,l=1}^{n_{ij}} E_{i,l}^\pi \otimes \sum_{k=1}^{n_{ij}} u_{i,k}^\pi \hat{a}(u_{i,k}^\pi)^*$$

$$= \sum_{\pi \in Irr(\mathbb{G})} \sum_{i,j=1}^{n_{ij}} E_{i,j}^\pi \otimes L_{i,j}^\pi(\hat{a})$$

where $L_{i,j}^\pi(\hat{a}) = \sum_{l=1}^{n_{ij}} u_{i,l}^\pi \hat{a}(u_{i,l}^\pi)^*$. The reduced crossed product of a $\mathbb{G}$-$C^*$-algebra $A$ and $\mathbb{G}$ is the closed linear span of $(C_r(\mathbb{G}) \otimes 1)\alpha(A)$ in $M(K(\ell^2(\mathbb{G})) \otimes_{\min} A)$. We denote the reduced crossed product of $A$ with $\mathbb{G}$ by $A \rtimes_r \mathbb{G}$. It turns out that $A \rtimes_r \mathbb{G}$ is a $\mathbb{G}$-$C^*$-algebra (see [10] Lemma 2.11 and the preceding sections), with coaction $\beta: A \rtimes_r \mathbb{G} \to M(c_0(\mathbb{G}) \otimes_{\min} A \times_r \mathbb{G})$ defined by setting $\beta(A) = W_{12}A_{23}W_{12}$. The coaction $\beta$ satisfies
This makes the canonical embeddings of $A$ and $C_r(\hat{G})$ into $A \rtimes_r G$ $G$-equivariant.

Given a $G$-$C^\ast$-algebra $A$ and $\mu \in A^\ast$, the Poisson transform of $\mu$ is the ucp $G$-equivariant map $P_\mu : A \to \ell^\infty(G)$ defined by $P_\mu(a) = (id \otimes \mu)(a)$ for $a \in A$.

**Definition 2.6.** A $G$-$C^\ast$-algebra $A$ is a $G$-boundary if the Poisson transform $P_\mu$ of every state $\mu \in A^\ast$ is completely isometric.

### 2.3 Closed Quantum Subgroups

We use [12] and [15] as our primary reference for closed quantum subgroups. We use the notion of a quantum subgroup in the sense of Woronowicz. Let $G$ and $\mathbb{H}$ be LCQGs. Then $\mathbb{H}$ is a closed quantum subgroup of $G$ if there exists a non-degenerate $*$-homomorphism $\pi^\mathbb{H}_G : C_u(G) \to M(C_u(\mathbb{H}))$ such that $(\pi^\mathbb{H}_G \otimes \pi^\mathbb{H}_G)\Delta^\mathbb{H}_G = \Delta^\mathbb{H}_G \circ \pi^\mathbb{H}_G$ and $\pi^\mathbb{H}_G(C_u(G)) = C_u(\mathbb{H})$.

Given a DQG $G$, the closed quantum subgroups $\mathbb{H}$ are realized as follows:

- $\mathbb{H}$ is a closed quantum subgroup of $G$ if there exists a normal, unital, surjective $*$-homomorphism $\sigma^\mathbb{H} : \ell^\infty(G) \to \ell^\infty(\mathbb{H})$ and $(\sigma^\mathbb{H} \otimes \sigma^\mathbb{H})\Delta_G = \Delta_\mathbb{H} \circ \sigma^\mathbb{H};$
- $\hat{\mathbb{H}}$ is a closed quantum subgroup of $\hat{G}$ if there exists a normal, unital, surjective $*$-homomorphism $\pi^\mathbb{H}_\hat{G} : \text{Pol}(\hat{G}) \to \text{Pol}(\hat{\mathbb{H}})$ and $(\pi^\mathbb{H}_\hat{G} \otimes \pi^\mathbb{H}_\hat{G})\Delta_\hat{G} = \Delta_\hat{\mathbb{H}} \circ \pi^\mathbb{H}_\hat{G}$.

Given a LCQG $G$, a closed quantum subgroup $\mathbb{H}$ has an associated left coaction $l^\mathbb{H} : L^\infty(G) \to L^\infty(\mathbb{H}) \otimes L^\infty(G)$ and right coaction $r^\mathbb{H}$. The right quotient space $G/\mathbb{H}$ is defined by setting

$$L^\infty(G/\mathbb{H}) = \{x \in L^\infty(G) : r^\mathbb{H}(x) = x \otimes 1\}.$$

The left quotient space $\mathbb{H}/G$ is defined analogously with $l^\mathbb{H}$. We say $\mathbb{H}$ is normal if $L^\infty(G/\mathbb{H}) = L^\infty(\mathbb{H}/G)$. Equivalently, $G/\mathbb{H}$ is a LCQG, with coproduct $\Delta_{G/\mathbb{H}} = \Delta_G|_{L^\infty(G/\mathbb{H})}$. It is straightforward to see that $L^\infty(G/\mathbb{H})$ is two-sided if and only if $\mathbb{H}$ is normal.

### 2.4 Coideals and Idempotent States

Let $G$ be a LCQG and $N$ a von Neumann algebra.

**Definition 2.7.** $N$ is a $G$-space if there exists a normal, unital, injective $*$-homomorphism $\alpha : N \to L^\infty(G) \otimes N$ satisfying $(id \otimes \alpha)\alpha = (\Delta_G \otimes id)\alpha$. We call $\alpha$ a (left) coaction of $G$ on $N$.

A right coaction is defined analogously by swapping $L^\infty(G)$ and $N$ with each other.

Given a right or left $G$-space $N$, we then have that $N_\ast$ is a left or right $L^1(G)$-module by setting

$$f \ast x = (id \otimes f)\alpha(x) \text{ or } x \ast f = (f \otimes id)\alpha(x), \quad f \in L^1(G), x \in N$$

respectively.

Given (right) $G$-spaces $N$ and $M$, we will say a map $\phi : N \to M$ is $G$-equivariant if it is a right $L^1(G)$-module map.

We consider $L^\infty(G)$ as a $G$-space with the coproduct.
Definition 2.8. A right coideal $N$ of a LCQG $G$ is a right $G$-space $N \subseteq L^\infty(G)$ with coaction $\Delta_G|_N$. We might also call $N$ a right $L^1(G)$-invariant on Neummann subalgebra of $L^\infty(G)$. The left coideals are the left $L^1(G)$-invariant ones. We say a coideal is two-sided if it is both left and right $L^1(G)$ invariant.

In particular, every right or left quotient $L^\infty(G/H)$ and $L^\infty(H\backslash G)$ is a right or left coideal respectively.

Example 2.9. If $G$ is a locally compact group, then the right coideals are of the form $L^\infty(G/H)$ where $H$ is a closed subgroup of $G$. The coideals of $VN(G)$ are necessarily two-sided, and are of the form $VN(H)$ where $H$ is a closed subgroup of $G$ (see [19]).

Let us fix a DQG $G$. In [2], it was shown that for every right coideal $N \subseteq L^\infty(\widehat{G})$, there exists a sequence $E = (E_\pi)_{\pi \in Irr(\widehat{G})}$, where each $E_\pi \subseteq \mathcal{H}_\pi$ is a subspace, such that $N = \text{Pol}(\widehat{E})$, where

$$\text{Pol}(\widehat{E}) = \{ u_{\xi,\eta}^\pi : \xi \in E_\pi, \eta \in \mathcal{H}_\pi, \pi \in \text{Irr}(\widehat{G}) \} \subseteq \text{Pol}(\widehat{G})$$

where $u_{\xi,\eta}^\pi = (\text{id} \otimes u_{\eta,\xi})U^\pi$. We will call $E$ the hull of $N$, and will denote $L^\infty(\widehat{E}) = N$.

Fundamental to the coideals of DQGs are the group-like projections.

Definition 2.10. For a LCQG $G$, an orthogonal projection $P \in L^\infty(G)$ is left or right group-like if

$$(1 \otimes P)\Delta_G(P) = P \otimes P \text{ or } (P \otimes 1)\Delta_G(P) = P \otimes P$$

respectively.

Given some sequence of subspace $E = (E_\pi)_{\pi \in \text{Irr}(\widehat{G})}$, let $P_E = \bigoplus_{\pi \in \text{Irr}(\widehat{G})} P_\pi \in \ell^\infty(G)$ be the orthogonal projection where each $P_\pi$ is the orthogonal projection onto $E_\pi$. It turns out that $P_E$ is group-like if and only if $E$ is the hull of a right coideal (see [19]). We identify the right coideal of $\ell^\infty(G)$,

$$\tilde{N}_P = \{ x \in \ell^\infty(G) : (1 \otimes P)\Delta_G(x) = x \otimes P \}.$$ 

We call $\tilde{N}_P$ the codual coideal of $L^\infty(\widehat{E})$ and vice versa. It turns out that every right coideal of $\ell^\infty(G)$ is of the form $\tilde{N}_P$ for some group-like projection $P$.

Of special interest are the coideals generated by idempotent states (see [13, 20, 21, 30, 31]). Given an idempotent state $\omega \in C_u(\widehat{G})^*$, the adjoint of the map $L^1(\widehat{G}) \ni f \mapsto f \ast \omega \in L^1(\widehat{G})$ is a right $L^1(\widehat{G})$-module ucp projection $R_\omega : L^\infty(\widehat{G}) \rightarrow L^\infty(\widehat{G})$.

Definition 2.11. We say a right coideal $N_\omega = L^\infty(\widehat{E}_\omega)$ is a (right) compact quasi-subgroup of $G$ if there exists an idempotent state $\omega \in M^u(G)$ such that $\lambda_G(\omega) = P_{E_\omega} = P_\omega$.

In this case we say $\omega$ generates $N_\omega$. Occasionally, we may call the hull $E_\omega$ compact instead.

In the case where $L^\infty(\widehat{E})$ is a compact quasi-subgroup, $P_E$ is both a right and left group-like projection.

Remark 2.12. Given a compact quasi-subgroup $L^\infty(\widehat{E}_\omega)$, $R_\omega$ restricts to a projection $R_\omega : \text{Pol}(\widehat{G}) \rightarrow \text{Pol}(\widehat{E}_\omega)$ and a right $C_c(\widehat{G})^*$-module ucp projection $R_\omega^u : C_c(\widehat{G}) \rightarrow C_c(\widehat{E}_\omega)$. There also exists a universal version $R_\omega^u : C_u(\widehat{G}) \rightarrow C_u(\widehat{E}_\omega)$, which satisfies $\Gamma_G \circ R_\omega^u = R_\omega^u \circ \Gamma_G$. 

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For a DQG $\hat{G}$, if $H$ is a closed quantum subgroup of $G$, then we have a natural embedding $L^\infty(\hat{H}) \subseteq L^\infty(\hat{G})$, that realizes $L^\infty(\hat{H})$ as a two-sided coideal of $\hat{G}$ for which $E_{\hat{H}} = \hat{H}$ is compact. We denote the corresponding idempotent state by $\omega_{\hat{H}}$. If $G$ is discrete, then the two-sided coideals of $G$ are exactly those of the form $\ell^\infty(H)$ where $H$ is a closed quantum subgroup of $\hat{G}$. Similarly, the two-sided coideals of $\hat{G}$ are those of the form $L^\infty(H)$ where $H$ is a closed quantum subgroup of $G$. Equivalently, the two-sided coideals are those where $\omega$ is central (see [11]).

The quotient spaces $L^\infty(\hat{G}/H)$ are compact quasi-subgroups and are generated by the idempotent states $\omega_{\hat{G}/H} = h_H \circ \pi_{\hat{H}}$.

**Definition 2.13.** Let $G$ be a DQG. An idempotent state of the form $\omega_{\hat{G}/H} = h_H \circ \pi_{\hat{H}}$, where $H$ is a closed quantum subgroup of $\hat{G}$, is known as a **Haar idempotent**.

The Haar idempotents can be distinguished with the following important theorem.

**Theorem 2.14.** ([14] Theorem 3.3) Let $G$ be a DQG and $\omega \in C_u(\hat{G})^*$ an idempotent state. Then $\omega$ is a Haar idempotent if and only if $I_\omega = \{a \in C_u(\hat{G}) : \omega(\ast a) = 0\}$ is a two-sided ideal (equivalently, $I_\omega$ is self-adjoint).

An important consequence of Theorem 2.14 for us will be that the tracial idempotent states are automatically Haar idempotents.

**Remark 2.15.** We should remark that [14] Theorem 3.3 did not precisely prove what we are claiming. In their proof, they assumed $\hat{G}$ was coamenable. This caveat is easily surpassed by experts, but we show why for convenience. If we assume $I_\omega$ is two-sided, then by going through their proof, we find that there exists a CQG $\hat{H}$ such that $C_u(\hat{G})/I_\omega \cong C_r(\hat{H})$ and $\omega$ factors into the Haar state on $\hat{H}$. If we let $\pi_{\hat{H}} : C_u(\hat{G}) \to C_r(\hat{H})$ be the quotient map, then it follows that $\pi_{\hat{H}}(Pol(\hat{G})) = Pol(\hat{H})$ since $\pi_{\hat{H}}$ preserves corepresentation operators and is surjective. Then the restriction $\pi_{\hat{H}}|_{Pol(\hat{G})} : Pol(\hat{G}) \to Pol(\hat{H})$ tells us $\hat{H} \leq \hat{G}$. The converse remains unchanged for general CQGs.

Similarly, if $\omega \in C_r(\hat{G})^*$, then $I'_\omega = \{a \in C_r(\hat{G}) : \omega(a^*a) = 0\}$ is a two-sided ideal if and only if there is a closed quantum subgroup $H \leq \hat{G}$ where $G/H$ is coamenable such that $\omega = h_H \circ \pi_{\hat{H}}^u$.

### 3 Coamenable Coideals

#### 3.1 Amenable and Coamenable Quantum Groups

**Definition 3.1.** A CQG $\hat{G}$ is **coamenable** if $L^1(\hat{G})$ has a bounded approximate identity. We say $G$ is **amenable** whenever $\hat{G}$ there exists a state $m \in L^\infty(\hat{G})^*$ satisfying $(m \otimes id)\Delta_{\hat{G}}(x) = m(x)$ for all $x \in L^\infty(\hat{G})$.

For discrete quantum groups we have the following characterization of coamenability which holds for LCQGs in general except for the equivalence between amenability of $G$ and coamenability of $\hat{G}$. This equivalence remains one of the most important problems in the theory of LCQGs.

**Theorem 3.2.** ([28] [35]) Let $G$ be a DQG. The following are equivalent:
1. $G$ is amenable;
2. $\hat{G}$ is coamenable;
3. $\Gamma_G : C_u(\hat{G}) \to C_r(\hat{G})$ is injective;
4. $\epsilon^*_G \in C_r(\hat{G})^*$.

The notion of a coamenable coideal that is discussed in the proceeding subsection will generalize most of this theorem to coideals.

The coproduct on $\ell^\infty(G)$ extends to a coproduct on $\mathcal{B}(\ell^2(G))$ by defining

$$\Delta^G_T = W^*_G (1 \otimes T) W^G.$$  

Then $\mathcal{B}(\ell^2(G))$ is a $\mathcal{T}(\ell^2(G))$-bimodule with respect to the actions

$$\mu \triangleleft T = (\text{id} \otimes \mu) \Delta^G_T (T) \text{ and } T \triangleleft \mu = (\mu \otimes \text{id}) \Delta^G_T (T), \quad \mu \in \mathcal{T}(\ell^2(G)), T \in \mathcal{B}(\ell^2(G)).$$

One last characterization of amenability of a DQG we will mention is the following (which holds for LCQGs in general).

**Theorem 3.3.** Let $G$ be a DQG. The following are equivalent:

1. $G$ is amenable;
2. there exists a right $\mathcal{T}(\ell^2(G))$-module conditional expectation $E : \mathcal{B}(\ell^2(G)) \to L^\infty(\hat{G})$ such that $E(\ell^\infty(G)) = C1$;
3. there exists a left $\mathcal{T}(\ell^2(G))$-module conditional expectation $E : \mathcal{B}(\ell^2(G)) \to L^\infty(\hat{G})'$ such that $E(\ell^\infty(G)) = C1$.

**Remark 3.4.** A similar result was independently achieved in [34].

### 3.2 $C^*$-algebraic Coideals

As discussed in the introduction, in the classical setting of a discrete group $G$, the unique trace property is detected by the normal amenable subgroups of $G$. More precisely, the unique trace property holds if and only if the amenable radical $R_a(G)$ is trivial. The underlying reason is that a tracial state that is not the Haar state exists if and only if an idempotent tracial state exists, which are of the form $1_N$ for an amenable normal subgroup $N \leq G$. The main results in [1] extended this picture to DQGs by studying coamenability of compact coideals (or compact quasi-subgroups) which are in correspondence with idempotent states on $C_r(\hat{G})$. Then, for unimodular DQGs, a tracial state on $C_r(\hat{G})$ that is not the Haar state exists if and only if an idempotent tracial state exists as well, which comes from the Haar state of a closed quantum subgroup $\mathcal{H} \leq \hat{G}$. So, we have a deep connection between the tracial states on $C_r(\hat{G})$ and coamenable coideals that are of quotient type.

In general, the coamenable compact coideals correspond to idempotent states on $C_r(\hat{G})$, but are not necessarily tracial. Coideals in general do not necessarily have corresponding idempotent states (e.g., the Podleš spheres [27] of $SU_q(2)$). This means an idempotent state formulation of coamenability does not exist. As we will explore in this section, it is still possible to define coamenability for coideals in general. We will spend this section building
a theory of coamenability of coideals and we will prove results that are analogous to the characterizations of amenability and coamenability described in the previous section.

We will denote
\[ C_u(\hat{E}) = \overline{\text{Pol}(\hat{E}) / ||\cdot||_u} \subseteq C_u(\hat{G}) \]
and
\[ C_r(\hat{E}) = \overline{\text{Pol}(\hat{E}) / ||\cdot||_r} \subseteq C_r(\hat{G}). \]

Notice that \( \Gamma_E := \Gamma_G|_{C_u(\hat{E})} : C_u(\hat{E}) \to C_r(\hat{E}) \) is a surjective unital \(*\)-homomorphism satisfying
\[ (\Gamma_G \otimes \Gamma_E)\Delta^u = \Delta^r \circ \Gamma_E. \]

Let \( C_{\text{env}}(\hat{E}) \) be the closure of \( \text{Pol}(\hat{E}) \) with respect to the universal norm:
\[ ||a||^{\text{env}}_E = \sup \{ ||a|| : ||\cdot|| \text{ is a } C^*\text{-seminorm} \}. \]
The standard argument will show that there is an identification \( S(\text{Pol}(\hat{E})) \cong S(C_{\text{env}}(\hat{E})) \): for each \( \mu \in S(\text{Pol}(\hat{E})) \) there exists a unique \( \mu \in S(C_{\text{env}}(\hat{E})) \) such that \( \hat{\mu}|_{\text{Pol}(\hat{E})} = \mu \). Then for \( a \in \text{Pol}(\hat{E}) \),
\[ ||aa^*||_u = \sup_{\mu \in S(\text{Pol}(\hat{E}))} \{ |\mu(aa^*)| \} = \sup_{\mu \in S(C_{\text{env}}(\hat{E}))} \{ |\mu(aa^*)| \}. \]

In the compact case, it turns out \( C_u(\hat{E}_\omega) \) is \( C_{\text{env}}(\hat{E}_\omega) \).

**Proposition 3.5.** Let \( E = E_\omega \) be compact. Then \( C_u(\hat{E}_\omega) = C_{\text{env}}(\hat{E}_\omega) \). In other words, \( C_{\text{env}}(\hat{E}_\omega) \) isometrically injects into \( C_u(\hat{G}) \).

**Proof.** The proof is essentially the same as the proof that \( C^*(H) \) isometrically embeds into \( C^*(G) \) whenever \( G \) is a discrete group and \( H \) is a subgroup. The projection \( R_\omega : \text{Pol}(\hat{G}) \to \text{Pol}(\hat{E}_\omega) \) induces a linear inclusion \( S(\text{Pol}(\hat{E}_\omega)) \to S(\text{Pol}(\hat{G})) \), \( \mu \mapsto \mu \circ R_\omega \). Consider \( a \in \text{Pol}(\hat{E}_\omega) \). Then
\[ ||a||^2_u = ||aa^*||_u = \sup_{\mu \in S(\text{Pol}(\hat{G}))} \{ |\mu(aa^*)| \} = \sup_{\mu \in S(\text{Pol}(\hat{E}_\omega)))} \{ |\mu \circ R_\omega(aa^*)| \} = \sup_{\mu \in S(\text{Pol}(\hat{E}_\omega)))} \{ |\mu(aa^*)| \} = (||a||^{\text{env}}_E)^2. \]

With the above proposition in hand, it is now straightforward to deduce that for two-sided coideals, their reduced \( C^* \)-algebras are equal to the reduced \( C^* \)-algebras that comprise their underlying quantum group.

**Proposition 3.6.** We have that \( C_r(\hat{E}_{\mathcal{R}}) = C_r(\hat{H}) \).

**Proof.** Note that Proposition 3.5 says that \( C_u(\hat{E}_{\mathcal{R}}) = C_u(\hat{H}) \). Denote \( I^H_G = \{ a \in C_u(\hat{G}) : h_G(a^*a) = 0 \} \). Then \( I^H_G = I^H_G \cap C_u(\hat{H}) \) and so
\[ C_r(\hat{E}_{\mathcal{R}}) = C_u(\hat{E}_{\mathcal{R}})/I^H_G = C_u(\hat{H})/I^H_G = C_r(\hat{H}). \]
The functional
\[ \epsilon_E^u := \epsilon_G^u|_{C_u(\hat{E})} : C_u(\hat{E}) \to C \]
is a state that satisfies \( \epsilon_E^u(u_{i,j}^\pi) = \delta_{i,j} \) for all \( u_{i,j}^\pi \in \text{Pol}(\hat{E}) \). As was shown in [2], if \( E = E_\omega \) is compact, then \( \omega|_{C_u(\hat{E})} = \epsilon_E^u \).

**Definition 3.7.** Let \( E \) be a hull for a coideal. We say \( E \) is **coamenable** if there exists \( \epsilon_E^* \in C_r(\hat{E})^* \) such that \( \epsilon_E^* \circ \Gamma_E = \epsilon_E^u \).

This a direct extension of the notion of a coamenable quotient from [16] and a coamenable compact quasi-subgroup from [1]. Recall from the latter the following.

**Proposition 3.8.** [1] Corollary 1.7] Let \( G \) be a DQG and \( E_\omega \) be a hull of a compact quasi-subgroup. Then \( E_\omega \) is coamenable if and only if \( \omega \in C_r(\hat{G})^* \).

Recall that \( G \) is coamenable if and only if \( C_u(\hat{G}) \cong C_r(\hat{G}) \). An argument verbatim to the argument used for [1] Theorem 2.2 will prove the following.

**Proposition 3.9.** We have that \( E \) is coamenable if and only if \( \Gamma_E : C_u(\hat{E}) \to C_r(\hat{E}) \) is injective.

For quotients, we have the following characterization.

**Theorem 3.10.** [16] Theorem 3.11] Let \( G \) be a DQG. We have that \( \hat{G}/\hat{H} \) is a coamenable quotient if and only if there exists a \(*\)-homomorphism \( \pi_H : C_r(\hat{G}) \to C_r(\hat{H}) \) such that \( \Gamma_G \circ \pi_H^* = \Gamma_H \circ \pi_H^* \).

Given that \( \hat{H} \leq \hat{G} \), because of the identity \( \Gamma_G \circ \pi_H^* = \Gamma_H \circ \pi_H^* \), we will normally just write \( \pi_H \) for \( \pi_H^* \) and \( \pi_H^* \) unless there is risk of confusion.

For DQGs, coamenability of their duals is the dual property of amenability: \( G \) is amenable if and only if \( \hat{G} \) is coamenable. For coideals, the dual property is relative amenability / amenability, which were properties defined and studied in detail in [1][16].

**Definition 3.11.** Let \( G \) be a LCQG. A right coideal \( N \) is **relatively amenable** in \( L^\infty(G) \) if there exists a \( G \)-equivariant ucp map \( L^\infty(G) \to N \). We say \( N \) is **amenable** if there exists a \( G \)-equivariant ucp projection \( L^\infty(G) \to N \).

**Remark 3.12.** For discrete \( G \), a coideal \( \ell^\infty(G/H) \) is relatively amenable if and only if it is amenable if and only if \( H \) is amenable. This statement was extended to the case where \( G \) is a discrete quantum group and \( H \) is a closed quantum subgroup independently in [16] and [1]. For a locally compact group \( G \) in general, it is only known that amenability of \( H \) is equivalent to amenability of \( L^\infty(G/H) \), and the coincidence of relative amenability and amenability of \( L^\infty(G/H) \) remains open (see [7]).

Let \( G \) be discrete. It was shown in [1] that if \( \tilde{N}_{P_\omega} \) is relatively amenable then the compact quasi-subgroup \( N_\omega \) is coamenable. The converse remains open in general, however, we make progress on this later in this paper. The following characterization of relative amenability and amenability of coideals was achieved.

**Theorem 3.13.** [1, Theorem 3.15 and Corollary 3.16] Let \( G \) be a DQG and \( \tilde{N}_P \) be a right coideal where \( P = \lambda_G(\omega) \) for an idempotent state \( \omega \in C_u(\hat{G})^* \). The following is true:
1. \( \tilde{N}_P \) is relatively amenable if and only if there exists a state \( m \in L^\infty(G) \) such that \( P(m \otimes \text{id})\Delta_G(x) = m(x)P \) for all \( x \in L^\infty(G) \);

2. \( \tilde{N}_P \) is amenable if and only if there exists a state \( m \in L^\infty(G) \) such that \( m(P) = 1 \) and \( P(m \otimes \text{id})\Delta_G(x) = m(x)P \) for all \( x \in L^\infty(G) \).

Proof. In Theorem 3.13, what was shown is that \( \tilde{N}_P \) is relatively amenable if and only if there exists a state \( m \in L^\infty(G)^* \) satisfying \( (\text{id} \otimes m)\Delta_G(x)(P \otimes 1) = m(x)P \). Since \( P \) is preserved by the unitary antipode (which we will not define here) [20, Lemma 1.3], the standard trick of turning right invariant states into left invariant states with the unitary antipode will work here to show \( P(m \otimes \text{id})\Delta_G(x) = m(x)P \).

We call a state \( m \) satisfying 1. of Theorem 3.13 an \( \text{\( P \)}} \) \text{-invariant} state.

Before getting to our main result, we require more elaboration on some finer details of the interplay between codual coideals in \( L^\infty(G) \) and \( L^\infty(G) \). Given that \( N = L^\infty(E) \) is a right coideal in \( L^\infty(G) \), we denote

\[
N^L = \{ u_{\xi,\eta}^\ast : \xi \in H_\pi, \eta \in E_\pi, \pi \in \text{Irr}(G) \},
\]

which is a left coideal in \( L^\infty(G) \) that satisfies \( N^L = L^\infty(G) \cap (\tilde{N}_P)^\prime \) and \( \tilde{N}_P = L^\infty(G) \cap (N^L)^\prime \). Recall that we have an action of \( L^\infty(G) \) on \( L^1(G) \) denoted by setting

\[
f x(y) = f(xy) \quad \text{and} \quad fx(y) = f(xy), \quad f \in L^1(G), \quad x, y \in L^\infty(G).
\]

We will have use for the fact

\[
(\lambda_G(P\ell^1(G))_{wk^*}) = \{ (f \otimes \text{id})(W_G(P \otimes 1)) : f \in L^1(G) \} = N^L,
\]

(see [120, 36]). Finally, we define the space

\[
M_P = \{ x \in L^\infty(G) : (1 \otimes P)\Delta_G(x)(1 \otimes P) = x \otimes P \} \supseteq \tilde{N}_P
\]

as defined in [1].

The weak* closed right invariant subspace \( M_P \) was defined in [1]. Its relevance was giving a partial converse of [1, Theorem 1.10]: it was proved that coamenability of \( \tilde{N}_P \) implies amenability of \( M_P \). A full converse would be achieved if \( M_P = \tilde{N}_P \), which occurs when \( \tilde{N}_P \) is amenable, \( \tilde{N}_P \) is of quotient type, or when \( G = \hat{K} \) for a compact group \( K \). We currently have no example where \( M_P \neq \tilde{N}_P \) and no proof that \( M_P = \tilde{N}_P \) in general (see [1]).

As with Theorem 3.13, it can be proved that \( M_P \) is amenable if and only if there exists a state \( m \in L^\infty(G)^* \) such that \( P(m \otimes \text{id})\Delta_G(x)P = m(x)P \) for all \( x \in L^\infty(G) \).

Remark 3.14. Contrary to our use of the word in this work, \( N^L \) is usually referred to as the codual of \( \tilde{N}_P \).

Theorem 3.15. Let \( G \) be a DQG. Let \( \tilde{N}_P \) be a right coideal in \( L^\infty(G) \) such that \( P = \lambda_G(\omega) \) for an idempotent state \( \omega \in C_u(G)^\ast \).

1. We have that \( \tilde{N}_P \) is amenable if and only if there exists a completely contractive left \( T(\ell^2(G)) \)-module unital projection \( E : B(\ell^2(G)) \to P(N^L)^\prime \) such that \( E(L^\infty(G)) = \mathbb{C}P \) and \( E|_{L^\infty(G)} \) is positive.
2. We have that $M_P$ is amenable if and only if there exists a left $PT(\ell^2(G))P$-module ucp projection $E : B(\ell^2(G)) \rightarrow P(N^L)'P$ such that $E(\ell^\infty(G)) = CP$.

Proof. We will prove 2. The assertion in 1. follows by replacing each conjugation $PXP$ in the proof with $PX$. Assume that $\tilde{N}_P$ is amenable with $P$-invariant state $m$. We use an adaptation of the proof for [5 Theorem 3.3] (see also [34 Theorem 2.1]). Define $E : B(\ell^2(G)) \rightarrow B(\ell^2(G))$ by setting

$$E(T) = (m \otimes \text{id}) W_G^* (1 \otimes PTP) W_G = (m \otimes \text{id}) \Delta_G^1 (PTP), \quad T \in B(\ell^2(G)).$$

This map is clearly ucp. From the proof of [1 Theorem 3.1], we find that the property

$$m(x) = m(Px) = m(xP), \quad x \in \ell^\infty(G)$$

may be arranged. Consequently, we have

$$E(T) = (m \otimes \text{id}) \Delta_G^1 (PTP) = (1 \otimes P)(m \otimes \text{id}) \Delta_G^1 (T)(1 \otimes P) = PE(T)P \quad (1)$$

where we used group-likeness of $P$. Given this, notice that

$$(1 \otimes P) \Delta_G^1 \circ E(T)(1 \otimes P) = (1 \otimes 1 \otimes P)(m \otimes \text{id})(\Delta_G \otimes \text{id}) \Delta_G^1 (PTP)(1 \otimes 1 \otimes P)$$

$$= (m \otimes \text{id} \otimes \text{id})(\Delta_G \otimes \text{id})(1 \otimes P) \Delta_G^1 (PTP)(1 \otimes P)$$

$$= (1 \otimes P \otimes P)(m \otimes \text{id} \otimes \text{id})(\Delta_G \otimes \text{id}) \Delta_G^1 (T)(1 \otimes P \otimes P) \quad (\text{and group-likeness})$$

$$= (P \otimes 1 \otimes P)(1 \otimes m \otimes \text{id})(1 \otimes \Delta_G^1 (T))(1 \otimes 1 \otimes P) \quad (P\text{-invariance})$$

$$= P \otimes E(T) \quad (1).$$

Then, since $m(P) = 1$ and $E(T) = PE(T)P$,

$$E \circ E(T) = (m \otimes \text{id}) \Delta_G^1 \circ E(T) = E(T).$$

We showed $\Delta_G^1 (E(T)) = P \otimes E(T)$. On the other hand, if $\Delta_G^1 (T) = P \otimes T$, then

$$E(T) = P(m \otimes \text{id}) \Delta_G^1 (T)P = PTP$$

so

$$E(B(\ell^2(G))) = \{ T \in PB(\ell^2(G)) : \Delta_G^1 (T) = P \otimes T \}.$$  

Now, notice that

$$W_G^*(1 \otimes T)W_G = P \otimes T \iff (1 \otimes T)W_G = W_G(P \otimes T)$$

$$\quad \quad \iff (1 \otimes T)W_G(P \otimes 1) = W_G(P \otimes T)$$

$$\quad \quad \iff T \hat{x} = \hat{x}T \text{ for all } \hat{x} \in N^L,$$

which implies $E(B(\ell^2(G)) \subseteq P(N^L)'P$, and for $\hat{x} \in (N^L)'$,

$$E(P \hat{x} P) = (m \otimes \text{id})(P \otimes P)W_G^*(1 \otimes P \hat{x} P)W_G(P \otimes 1)$$

$$= (1 \otimes P)(m \otimes \text{id})(W_G(P \otimes 1))^{*}(1 \otimes P \hat{x} P)W_G(P \otimes 1)$$

$$= (1 \otimes P \hat{x} P)(m \otimes \text{id})(W_G(P \otimes 1))^{*}W_G(P \otimes 1) \quad (\text{since } P \hat{x} P \in (N^L)')$$

$$= (1 \otimes P \hat{x} P)(m \otimes \text{id})(P \otimes 1)W_G^*(P \otimes 1)$$

$$= P \hat{x} P.$$
Therefore, $E(B(ℓ^2(G))) = P(N^L)P$.

For the claim that $E|_{ℓ^∞(G)} = CP$, let $x ∈ ℓ^∞(G)$. Then

$$E(x) = (m ⊗ id)(Δ_G(PxP)) = P(m ⊗ id)(Δ_G(x)P) = m(x)P.$$

Finally, we show $E$ is left $PT(ℓ^2(G))P$-module. For this, let $τ ∈ T(ℓ^2(G))$ and $T ∈ B(ℓ^2(G))$.

Then,

$$E(PTP ⊙ T) = (m ⊗ id ⊗ PτP)(Δ^l_G ⊗ id)(P ⊗ P)(Δ^l_G(T)(P ⊗ P)) = (m ⊗ id ⊗ PτP)(id ⊗ Δ^l_G)Δ^l_G(PTP) \quad \text{(group-likeness)}$$

Conversely, set $m = ε_G ∘ E|_{ℓ^∞(G)}$. Since $E$ is upc, $m$ is a state. If we let $E(x) = PC_x$, where $c_x ∈ ℓ^∞(G)$, then

$$m(x) = ε_G(PC_x) = c_x,$$

so $E(x) = Pm(x)$. Since $σ$ is a lifting of convolution from $ℓ^1(G)$ (see [18]), $E$ is left $Pℓ^1(G)$-module. Then, for $f ∈ ℓ^1(G)$,

$$m * (PfP)(x) = (ε_G ⊗ id)(E ⊗ PfP)Δ_G(x) = (ε_G ⊗ id)(id ⊗ PfP)Δ_G(E(x)) = f(PE(x)P) = f(P)m(x).$$

This shows $P(m ⊗ id)Δ_G(x) = m(x)P$ as desired. □

**Remark 3.16.** 1. We have that Theorem 3.14 is a bona fide generalization of Theorem 3.13 $\iff$ iii. Indeed, we have that $G$ is amenable if and only if $C1 = N_1$ is an amenable coideal and in this case, $N^L_G = (N_1)' ∩ L^∞(G) = L^∞(G)$.

2. According to [1 Corollary 44], if $N_P$ is amenable then $M_P$ is amenable. Hence, if $N_P$ is amenable, then we obtain a upc left $PT(ℓ^2(G))P$-module projection $E : B(ℓ^2(G)) → P(N^L)P$ such that $E(ℓ^∞(G)) = CP$.

3. Suppose $H ≤ G$ is a quantum subgroup and $N_P$ is a left $PT(ℓ^2(G))P$-module projection $E : B(ℓ^2(G)) → P(N^L)P$ such that $E(ℓ^∞(G)) = CP$.

A related result was achieved by Crann [8 Corollary 3.6] in a much more general setting, which has the following consequence for DQGs: if $H$ is amenable then there exists a completely contractive right $L^1(\hat{G})$-module projection $L^∞(\hat{G}) → L^∞(\hat{H})$.

4. Suppose $H ≤ \hat{G}$ and set $P_H = \lambda_\hat{G}(h_H ∘ π_H)$. Then $N_P^H = ℓ^∞(H)$ and $N^L_{H_H ∘ π_H} = L^∞(H \setminus \hat{G})$. 

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3.3 The Semi-Lattice of Reduced Idempotent States

As stated in the previous section, a compact coideal \( N_\omega \) is coamenable if and only if \( \omega \in C_r(\widehat{G})^* \). In the classical setting of a discrete group \( G \), the idempotent state \( 1_{R_G(\hat{G})} \in C_r(\hat{G})^* \) is, in a sense we will define shortly, is the smallest tracial idempotent state in \( C_r(\hat{G}) \) and the Haar state is the largest. More generally, the containment of subgroups \( H_1 \leq H_2 \) is captured with the comparison of idempotent states \( 1_{H_2} \leq 1_{H_1} \), since subgroups of amenable groups are amenable, we have the hereditary property: \( 1_{H_2} \in C_r(\hat{G})^* \) then \( 1_{H_1} \in C_r(\hat{G})^* \). These considerations make the semi-lattice structure of idempotent states important for our purposes of studying idempotent tracial states. In this section, we will make some general observations regarding the semi lattice structure of reduced idempotent states for DQGs.

We will use the notation \( \text{Idem}(\hat{G}) \subseteq C_u(\hat{G})^* \) to denote the idempotent states and \( \text{Idem}_r(\hat{G}) = \text{Idem}(\hat{G}) \cap C_r(\hat{G})^* \) to denote the reduced idempotent states. Similarly, we let \( \text{Idem}^H(\hat{G}) \subseteq \text{Idem}(\hat{G}) \) denote the Haar idempotents and \( \text{Idem}^H_r(\hat{G}) = \text{Idem}^H(\hat{G}) \cap C_r(\hat{G})^* \).

We let \( \text{ZIdem}(\hat{G}) \) etc denote the central idempotents in \( C_u(\hat{G})^* \). We equip \( \text{Idem}(\hat{G}) \) with the following poset structure:

\[
\mu \leq \nu \quad \text{if} \quad \mu * \nu = \nu.
\]

We have the following equivalent ways of realizing this poset structure.

**Lemma 3.17.** [22, Lemma 2.1] Let \( G \) be a LCQG. The following are equivalent for \( \mu, \nu \in \text{Idem}(\hat{G}) \):

1. \( \mu \leq \nu \);
2. \( E_\mu \circ E_\nu = E_\nu \);
3. \( N_\nu \subseteq N_\mu \).

**Remark 3.18.** If \( G \) is discrete and \( L^\infty(\widehat{E}_i) \subseteq L^\infty(\hat{G}) \) are right coideals for \( i = 1, 2 \) with coduals \( \widehat{N}_{P_i} \subseteq \ell^\infty(G) \), then we obtain the additional relation:

\[
L^\infty(\widehat{E}_1) \subseteq L^\infty(\widehat{E}_2) \iff \widehat{N}_{P_2} \subseteq \widehat{N}_{P_1}.
\]

In [22], a meet and join operation was also defined on the idempotent states of a LCQG:

\[
\mu \lor \nu = \inf \{ \omega \in \text{Irr}(G) : \omega \geq \mu, \omega \geq \nu \}
\]

and

\[
\mu \land \nu = \sup \{ \omega \in \text{Irr}(G) : \omega \leq \mu, \omega \leq \nu \}.
\]

With respect to this ordering for a DQG \( G \), in both \( \text{Idem}(\hat{G}) \) and \( \text{Idem}^H(\hat{G}) \), we automatically see that \( \epsilon^u \) is the unique smallest element and \( h_{\hat{G}} \) is the unique largest element in the sense that

\[
\epsilon^u \leq \omega \quad \text{and} \quad \omega \leq h_{\hat{G}} \quad \text{for all} \quad \omega \in \text{Idem}(\hat{G}).
\]

As a consequence, we always have that \( \mu \lor \nu, \mu \land \nu \in \text{Idem}(\hat{G}) \).

We also have that \( h_{\hat{G}} \) is the unique largest element in \( \text{ZIdem}^H(\hat{G}) \), and so here, for any \( \mu, \nu \in \text{ZIdem}^H_r(\hat{G}) \), we have that \( \mu \lor \nu \in \text{ZIdem}^H_r(\hat{G}) \). What is interesting is finding a unique smallest element of the various reduced idempotent state spaces. For instance, for unimodular \( G \), the smallest element of \( \text{Idem}_r(\hat{G}) \) is \( h_{\hat{G}} \) if and only if \( G \) has the unique trace property. It turns out that there always exists minimal elements.
Proposition 3.19. Let $\mathbb{G}$ be a DQG. The following hold:

1. $\text{Idem}_r(\hat{\mathbb{G}})$ has minimal elements;
2. $\text{Idem}^H_r(\hat{\mathbb{G}})$ has minimal elements;
3. $\ell \text{Idem}_r(\hat{\mathbb{G}})$ has minimal elements.

Proof. 1. We will first show that there exists minimal elements in $\text{Idem}_r(\hat{\mathbb{G}})$ using Zorn’s lemma. Consider a chain $\{\omega_i : i \in I\}$ in $\text{Idem}_r(\hat{\mathbb{G}})$, where $\omega_i \leq \omega_j$ if $i \geq j$. Let $\omega$ be a weak$^*$ cluster point of the bounded net $(\omega_i)_i$. It is straightforward showing $\omega$ is an idempotent state, and since $C_r(\hat{\mathbb{G}})^*$ is weak$^*$ closed, $\omega \in \text{Idem}_r(\hat{\mathbb{G}})$. What remains is showing $\omega \leq \omega_i$ for all $i$. Indeed, take $i_0 \in I$, and for $a \in C_r(\hat{\mathbb{G}})$, suppose that $E_{\omega_{i_0}}(a) = a$. Let $\epsilon > 0$. For $u \in C_u(\hat{\mathbb{G}})^*$, find $i \geq i_0$ such that $|u \circ E_\omega(a) - u \circ E_{\omega_i}(a)| < \epsilon$. Then,

$$|u \circ E_\omega(a) - u(a)| = |u \circ E_\omega(a) - u \circ E_{\omega_i}(a)| < \epsilon.$$ 

We have shown $N_{\omega_{i_0}} \subseteq N_\omega$, which is the desired outcome.

2. If, as above, we instead have $(\omega_i)_i \subseteq \text{Idem}^H_r(\hat{\mathbb{G}})$, then $\omega \in \text{Idem}^H_r(\hat{\mathbb{G}})$. Indeed, we can prove that $I_\omega$ is self-adjoint and apply Theorem 2.14. For this, if $a \in I_\omega$, then

$$\omega(aa^*) = \lim_i \omega_i(aa^*) = 0,$$

because $\omega \leq \omega_i$ and each $I_{\omega_i}$ is self-adjoint for all $i$.

3. This is a straightforward adjustment of the above Zorn’s lemma argument. \qed

Example 3.20. Consider the free group on 2 generators, $\mathbb{F}_2 = \langle s_1, s_2 \rangle$. The amenable subgroups $\mathbb{Z} \cong \langle s_1 \rangle$ are maximal amenable subgroups, so that the coideals $VN(\langle s_i \rangle) \subseteq VN(\mathbb{F}_2)$ are coamenable. This means that $1_{\langle s_i \rangle} \in M_r(\mathbb{F}_2)$ are minimal (non-Haar, central) idempotent states that are distinct from the Haar state, but are incomparable to one another. In particular, we have that $1_{\langle s_1 \rangle} \wedge 1_{\langle s_2 \rangle} = 1_{\mathbb{F}_2} \notin \text{Idem}_r(\mathbb{F}_2)$. So, this gives us an example of a discrete quantum group where the meet operation is well-defined in neither $Z\text{Idem}_r(\hat{\mathbb{G}})$ nor $\text{Idem}_r(\hat{\mathbb{G}})$.

It is well-known that $\mathbb{F}_2$ is $C^*$-simple, thus the Haar state is the minimal (and only) element of $\text{Idem}^H_r(\mathbb{F}_2) = Z\text{Idem}_r(\mathbb{F}_2)$. We do not know if $\text{Idem}^H_r(\mathbb{G})$ has a smallest element in general.

Given a closed quantum subgroup $\mathbb{H}$ of a DQG $\mathbb{G}$, it is clear from the definition that coamenability of $L^\infty(\mathbb{H})$ as a coideal is just coamenability of $\mathbb{H}$. So, we see that $L^\infty(\mathbb{H})$ is a coamenable coideal if and only if $L_\infty(\mathbb{G}/\mathbb{H})$ is relatively amenable.

Recall that $\mathbb{H}$ is normal if and only if $\omega_\mathbb{H}$ is Haar. The amenable radical $R_a(\mathbb{G})$ is the largest normal amenable closed quantum subgroup of $\mathbb{G}$ (see [15] Proposition 3.15 and the preceding section). This makes $L_\infty(\mathbb{G}/R_a(\mathbb{G}))$ the smallest relatively amenable quotient by a normal closed quantum subgroup. Since $L_\infty(\mathbb{G}/\mathbb{H}_1) \subseteq L_\infty(\mathbb{G}/\mathbb{H}_2)$ if and only if $L_\infty(\mathbb{H}_2) \subseteq L_\infty(\mathbb{H}_1)$ if and only if $\omega_{\mathbb{G}/\mathbb{H}_1} \leq \omega_{\mathbb{G}/\mathbb{H}_2}$, we are able to deduce the following.

Proposition 3.21. Let $\mathbb{G}$ be a DQG. We have that $\omega_{\mathbb{G}/R_a(\mathbb{G})} = \min Z\text{Idem}_r^H(\hat{\mathbb{G}})$. 

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4 Traces on Quantum Groups

4.1 The Furstenberg Boundary

The Furstenburg boundary was constructed for DQGs in [16] with the purpose of obtaining quantum group dynamic results for DQGs. Its role was instrumental in the resolution of problems like characterization the unique trace property and $C^*$-simplicity of reduced $C^*$-algebras of groups, as well as clarifying the relationship between the unique trace property and $C^*$-simplicity (see [6,17]). It plays a similar role for unimodular DQGs [1,16] and will play such a role for us in our present paper.

We define a poset on the $G$-equivariant ucp maps $\ell^\infty(G) \to \ell^\infty(G)$ by setting
\[
\phi \leq \psi \text{ if } ||\phi(x)|| \leq ||\psi(x)|| \text{ for all } x \in \ell^\infty(G)
\]
for such $\phi, \psi : \ell^\infty(G) \to \ell^\infty(G)$. There exists minimal elements with respect to this poset.

The $G$-$C^*$-algebra $\phi(\ell^\infty(G))$, where $\phi$ is minimal with respect to the above poset, is a $G$-boundary that does not depend on the choice of minimal $\phi$.

**Definition 4.1.** The Furstenberg boundary is the $G$-boundary $C(\partial_F(G))$ that is isomorphic to any $\phi(\ell^\infty(G))$ where $\phi$ is a minimal $G$-equivariant ucp map.

In other words, the Furstenberg boundary is constructed out of minimal relatively amenable spaces. It turns out to be the universal boundary in the sense that for any $G$-boundary $A$, there is a completely isometric ucp $G$-equivariant embedding $A \to C(\partial_F(G))$ (see [16, Theorem 4.16]). It also satisfies the following three properties (see [16, Proposition 4.10] and [16, Proposition 4.13]).

**Definition 4.2.** Let $A$ be a $G$-boundary. We say $A$ is $G$-essential if any ucp $G$-equivariant ucp $A \to B$ where $B$ is a $G$-$C^*$-algebra, is completely isometric. We say $A$ is $G$-rigid if identity map is the unique $G$-equivariant ucp map $A \to A$. We say $A$ is $G$-injective if for any $G$-$C^*$-algebras $A$ and $B$ with ucp $G$-equivariant ucp maps $\psi : A \to C(\partial_F(G))$ and $\iota : A \to B$, where $\iota$ is completely isometric, there exists a ucp $G$-equivariant map $\phi : B \to C(\partial_F(G))$ such that $\psi = \phi \circ \iota$.

Of critical importance to us is the cokernel of the Furstenberg boundary.

**Definition 4.3.** The cokernel of $\partial_F(G)$ is the two-sided coideal [16, Proposition 2.9]
\[
N_F := \{P_\mu(a) : a \in C(\partial_F(G)), \mu \in C(\partial_F(G))^*\}'' \subseteq \ell^\infty(G).
\]

Let $\widehat{H}_F$ denote be closed quantum subgroup of $\widehat{G}$ such that $\ell^\infty(\widehat{H}_F) = N_F$. We also call $\widehat{H}_F$ the cokernel of $\partial_F(G)$. We say the action of $G$ on $\partial_F(G)$ is faithful when $\widehat{H}_F = G$.

**Remark 4.4.** For a discrete group $G$, the kernel of the action of $G$ on $\partial_F(G)$ is $R_a(G)$, the amenable radical of $G$. Then the cokernel is $G/R_a(G)$.

For DQGs, the cokernel of the Furstenberg boundary has a similar structure. The cokernel of $\partial_F(G)$ turns out to be the unique smallest relatively amenable two-sided coideal of $G$. Then we have $\ell^\infty(\widehat{H}_F) \subseteq \ell^\infty(G/R_a(G))$, where we recall that $R_a(G)$ is the amenable radical of $G$. Whether the reverse containment holds or not remains open (see [16, Question 8.3]).
4.2 The Kac Property and Canonical Kac Quotient

The Kac property is the fundamental property governing traciality of idempotent states, so we recall its important aspects here. Given an arbitrary DQG $\mathcal{G}$, for every $\pi \in \text{Irr}(\hat{\mathcal{G}})$, there exists a unique positive invertible matrix $F_\pi$ such that

$$(1 \otimes F_\pi)(U_\pi^* (1 \otimes F_\pi^{-1})) = ((U_\pi^*)^t)^{-1}$$

and $\text{tr}(F_\pi) = \text{tr}(F_\pi^{-1}) > 0$, where $U_\pi = [(u_{i,j}^\pi)^*]$. For a given $\pi \in \text{Irr}(\hat{\mathcal{G}})$, we call $F_\pi$ the $F$-matrix for $\pi$. Then we have the decompositions

$$h_L = \bigoplus_{\pi \in \text{Irr}(\hat{\mathcal{G}})} \text{tr}(F_\pi) \text{tr}(F_\pi^{-1})$$

and

$$h_R = \bigoplus_{\pi \in \text{Irr}(\hat{\mathcal{G}})} \text{tr}(F_\pi) \text{tr}(F_\pi^*)$$

where $\text{tr}(I_{n_\pi}) = n_\pi$ is the normalization of $\text{tr}|_{M_{n_\pi}}$.

It was shown by Daws [11] that a set of representatives in $\text{Irr}(\hat{\mathcal{G}})$ may be chosen so that the $F$-matrices are diagonal, which is something we will do. In this case, given $\pi \in \text{Irr}(\hat{\mathcal{G}})$, we call $F_\pi = \text{diag}(\lambda_1, \ldots, \lambda_{n_\pi})$, Schur’s orthogonality is realized as the formulas:

$$h_{\hat{\mathcal{G}}}((u_{i,j}^\pi)^* u_{k,l}^\pi) = \delta_{\pi,\sigma} \delta_{i,k} \delta_{j,l} \frac{\lambda_j^{-1}}{\text{tr}(F_\pi)}$$

and

$$h_{\hat{\mathcal{G}}}((u_{i,j}^\pi)^* u_{k,l}^\pi) = \delta_{\pi,\sigma} \delta_{i,k} \delta_{j,l} \frac{\lambda_j}{\text{tr}(F_\pi)}$$

Definition 4.5. A DQG $\mathcal{G}$ is unimodular if $h_L = h_R$. We say $\hat{\mathcal{G}}$ is Kac if $h_{\hat{\mathcal{G}}}$ is a tracial state.

Unimodularity of $\mathcal{G}$ is well-known to be equivalent to Kacness of $\hat{\mathcal{G}}$. There is the further well-known characterization.

Theorem 4.6. Let $\mathcal{G}$ be a DQG. The following are equivalent:

1. $\mathcal{G}$ is unimodular;
2. $\hat{\mathcal{G}}$ is Kac;
3. every $\pi \in \text{Irr}(\hat{\mathcal{G}})$ has $F_\pi = I_{n_\pi}$;
4. $((U_\pi^*)^t)^{-1} = U_\pi$ for every $\pi \in \text{Irr}(\hat{\mathcal{G}})$.

Now, let us recall the canonical Kac quotient constructed by Soltan [33]. Define the ideal

$$I_{Kac} = \{ a \in C_u(\hat{\mathcal{G}}) : \tau(aa^*) = 0 \text{ for every tracial state } \tau \in C_u(\hat{\mathcal{G}})^* \}.$$

Then, $\mathbb{H}_{Kac}$ is a Kac quantum subgroup of $\hat{\mathcal{G}}$ such that $C_u(\hat{\mathcal{G}})/I_{Kac} \cong C_u(\mathbb{H}_{Kac})$. We call $\mathbb{H}_{Kac}$ the canonical Kac quotient of $\hat{\mathcal{G}}$. Soltan showed that $\mathbb{H}_{Kac}$ is a closed quantum subgroup of $\hat{\mathcal{G}}$, and it follows more or less from the definitions that every Kac closed quantum subgroup of $\hat{\mathcal{G}}$ is a closed quantum subgroup of $\mathbb{H}_{Kac}$. We denote the corresponding Haar idempotent by $\omega_{Kac}$.

Remark 4.7. If we let

$$I_{r,Kac} = \{ a \in C_r(\hat{\mathcal{G}}) : \tau(aa^*) = 0 \text{ for every tracial state } \tau \in C_r(\hat{\mathcal{G}})^* \}$$
when tracial states exist, then Soltan’s construction yields a CQG $C^*$-algebra $C_{rKac}(\mathbb{H}_{Kac}) \cong C_r(\hat{G})/I_{Kac}^r$. Because of the quotient $C_r(\hat{G}) \to C_{rKac}(\mathbb{H}_{Kac})$, if $\hat{G}$ is not coamenable, then $C_{rKac}(\mathbb{H}_{Kac}) \neq C_u(\mathbb{H}_{rKac})$. On the other hand, we have been unable to determine whether or not $C_{rKac}(\mathbb{H}_{Kac}) \neq C_r(\mathbb{H}_{Kac})$, i.e., whether or not $C_{rKac}(\mathbb{H}_{Kac})$ is exotic, without the assumption that $\omega$ is the unique tracial state on $C_r(\hat{G})$.

Recall that because of Theorem 2.14, tracial idempotent states are automatically Haar. In fact, we see that $\mathbb{H}$ is a Kac closed quantum subgroup of $\hat{G}$ if and only if $\omega_{\mathbb{H}/\mathbb{H}} = h_{\mathbb{H}} \circ \pi_\mathbb{H}^r$ is tracial. Then if $\hat{G}$ is unimodular, we find that an idempotent state is Haar if and only if it is tracial.

**Remark 4.8.** If we have a tracial state $\tau \in C_u(\hat{G})^*$, then the idempotent state achieved by taking a weak cluster point of the Cesaro sums $\frac{1}{n} \sum_{k=1}^n \tau^* \tau$ is a tracial idempotent state (see [25]). In particular, if a tracial state exists, then a tracial Haar idempotent exists.

**Proposition 4.9.** The following hold:

1. $C_r(\hat{G})^*$ has a tracial state if and only if there exists a Kac closed quantum subgroup $\mathbb{H} \leq \hat{G}$ such that $\hat{G}/\mathbb{H}$ is co-amenable;

2. $C_r(\hat{G})^*$ has a unique idempotent tracial state if and only if $\mathbb{H}_{Kac}$ is the only Kac closed quantum subgroup such that $\hat{G}/\mathbb{H}_{Kac}$ is coamenable;

3. $C_r(\hat{G})^*$ has a unique tracial state if and only if $\mathbb{H}_{Kac}$ is the only Kac closed quantum subgroup such that $\hat{G}/\mathbb{H}_{Kac}$ is coamenable and $C_{rKac}(\mathbb{H}_{Kac}) = C_r(\mathbb{H}_{Kac})$.

**Proof.** 1. As discussed in Remark 4.10, a tracial Haar idempotent exists, and hence a Kac closed quantum subgroup with coamenable quotient. Conversely, $\omega_{\mathbb{H}/\mathbb{H}}$ is a tracial state.

2. Suppose $C_r(\hat{G})^*$ has a unique idempotent tracial state. Then, for any Kac closed quantum subgroup $\mathbb{H} \leq \hat{G}$ where $\hat{G}/\mathbb{H}$ is coamenable, we find that $\omega_{\mathbb{H}/\mathbb{H}} = \omega_{Kac}$ by uniqueness. Conversely, if $\omega$ is a tracial idempotent state, it is a tracial Haar idempotent and it has an associated Kac closed quantum subgroup $\mathbb{H} \leq \hat{G}$ such that $\hat{G}/\mathbb{H}$ is coamenable. Then $\mathbb{H} = \mathbb{H}_{Kac}$, which implies $\omega_{\mathbb{H}/\mathbb{H}} = \omega_{Kac}$ by uniqueness.

3. The forward direction is similar to the forward direction of 2. Conversely, let $\tau \in C_r(\hat{G})^*$ be a tracial state. The closed quantum subgroups of $\mathbb{H}_{Kac}$ are the Kac closed quantum subgroups of $\hat{G}$. Moreover, if $\mathbb{H}_{Kac}/\mathbb{H}$ were to be coamenable, then so would be $\hat{G}/\mathbb{H}$ because then we obtain the quotient map

$$\pi = \pi_{\mathbb{H}} \circ \pi_{\mathbb{H}_{Kac}} : C_r(\hat{G}) \to C_r(\mathbb{H}_{Kac}) \to C_r(\mathbb{H}),$$

where $\pi_{\mathbb{H}} : C_r(\mathbb{H}_{Kac}) \to C_r(\mathbb{H})$ and $\pi_{\mathbb{H}_{Kac}} : C_r(\hat{G}) \to C_r(\mathbb{H}_{Kac})$ are as in Theorem 3.10. It is straightforward to check that $\pi \circ \Gamma_{\mathbb{H}} = \Gamma_{\mathbb{H}} \circ \pi$ so we deduce that $\hat{G}/\mathbb{H}$ is coamenable.

In particular, $(\mathbb{H}_{Kac})_F = \mathbb{H}_{Kac}$, i.e., the action of $\mathbb{H}_{Kac}$ on its Furstenburg boundary is faithful, so $C_r(\mathbb{H}_{Kac})$ has a unique trace. Since $\tau$ is a tracial state, and $I_{Kac}^r \subseteq \ker(\tau)$ using a standard Cauchy-Schwarz argument, there exists a state $\hat{\tau} \in C_{rKac}(\mathbb{H}_{Kac})^*$ such that $\hat{\tau} \circ q = \tau$ where $q : C_r(\hat{G}) \to C_{rKac}(\mathbb{H}_{Kac}) = C_{rKac}(\mathbb{H}_{Kac})$ is the quotient map discussed in the above remark. Clearly $\hat{\tau}$ is tracial and since $C_{rKac}(\mathbb{H}_{Kac}) = C_r(\mathbb{H}_{Kac})$, $\hat{\tau} = h_{\mathbb{H}_{Kac}}$ by uniqueness. Therefore, $\tau = \omega_{Kac}$. □

**Remark 4.10.** Notice that we showed that $\hat{G}/\mathbb{H}_{Kac}$ is coamenable if there exists a tracial state in $C_r(\hat{G})$. 20
4.3 $\mathbb{G}$-Invariant States

An important feature of unimodular DQGs that was used to in [16] to prove that faithfulness of $\mathbb{G} \cong \partial_F(G)$ implies the unique trace property was that tracial states are $\mathbb{G}$-invariant, allowing one to extend tracial states on $C_r(\hat{\mathbb{G}})$ to $C(\partial_F(G)) \rtimes_r \mathbb{G}$. As mentioned in the introduction, there is no correspondence of tracial states with $\mathbb{G}$-invariant states in general.

In this section, however, we will establish such a correspondence for tracial states. In this introduction, there is no correspondence of tracial states with $\mathbb{G}$-invariant states in general. In this section, however, we will establish such a correspondence for tracial idempotent states, based on the fact that a Haar state is tracial if and only if it is $\mathbb{G}$-invariant [16, Lemma 5.2]. An immediate payoff is that we settle an open problem in [9, 26] in the discrete case regarding the equivalence between nuclearity of and existence of a tracial state on $C_r(\hat{\mathbb{G}})$ with amenability of $\mathbb{G}$.

For the moment, we consider a general LCQG $\mathbb{G}$.

**Definition 4.11.** A state $\mu \in C_r(\hat{\mathbb{G}})^*$ or $\mu \in L^\infty(\hat{\mathbb{G}})^*$ is $\mathbb{G}$-invariant if $(\text{id} \otimes \mu)\Delta_l^\mathbb{G} = \mu$.

**Remark 4.12.** Instead of $\mathbb{G}$-invariance, Crann [9] used the terminology inner invariance, and said that $\mathbb{G}$ is topologically inner amenable if there exists an inner invariant state in $C_r(\hat{\mathbb{G}})^*$ and is inner amenable if there exists an inner invariant state in $L^\infty(\hat{\mathbb{G}})^*$.

**Remark 4.13.** Let $\mathbb{G} = G$ be a discrete group. Then for $\hat{\alpha} \in C_r^*(G)$,

$$\Delta_l^\mathbb{G}(\hat{\alpha}) = \sum_{s \in G} \delta_s \otimes \lambda(s)\hat{\alpha}\lambda(s)^*.$$

We recover the conjugation action of $G$ on $C_r^*(G)$ as follows: for $s_0 \in G$,

$$(\delta_{s_0} \otimes \text{id})\Delta_l^\mathbb{G}(\hat{\alpha}) = \lambda(s_0)\hat{\alpha}\lambda(s_0)^* = s_0 \cdot \hat{\alpha}.$$

We find a noncommutative analogue of the conjugation action by the irreducibles of $\hat{\mathbb{G}}$. We define the convolution product:

$$f \triangleleft \mu = (f \otimes \mu)\Delta_l^\mathbb{G}, \quad f \in \ell^1(\mathbb{G}), \quad \mu \in C_r(\hat{\mathbb{G}})^*.$$

If we let $\delta_{i,j}^\pi$ be the dual basis element with respect to the matrix units $E_{i,j}^\pi$, then $\delta_{i,j}^\pi \triangleleft \mu = \mu \circ L_{i,j}^\pi$, where we recall that $L_{i,j}^\pi(\hat{\alpha}) = \sum_{t=1}^n u_{t,i}^\pi \hat{\alpha}(u_{t,j}^\pi)^*$. In particular, we immediately see the following.

**Proposition 4.14.** Let $\mathbb{G}$ be a DQG. We have that $\mu \in M^r(\mathbb{G})$ is $\mathbb{G}$-invariant if and only if $\mu \circ L_{i,j}^\pi = \delta_{i,j}^\pi \mu$ for every $\pi \in \text{Irr}(\hat{\mathbb{G}})$ and $i, j$.

**Remark 4.15.** The above characterization of $\mathbb{G}$-invariance allows us to easily see that that $h_{\mathbb{G}}$ is $\mathbb{G}$-invariant whenever it is tracial. Indeed, recall that the condition $(U^\pi)^t\overline{U^\pi} = I_n = \overline{U^\pi}(U^\pi)^t$ for every irreducible $\pi$ is equivalent to $\mathbb{G}$ being Kac. Then for $a \in \text{Pol}(\hat{\mathbb{G}})$,

$$h_{\mathbb{G}}(\sum_{t=1}^n u_{t,i}^\pi a(u_{t,j}^\pi)^*) = h_{\mathbb{G}}(a \sum_{t=1}^n (u_{j,t}^\pi)^* u_{i,t}^\pi) = \delta_{i,j} h_{\mathbb{G}}(a).$$

**Definition 4.16.** Let $\mathbb{G}$ be a DQG. The scaling group of $\hat{\mathbb{G}}$ is the one-parameter group of automorphisms $(\tau_t)_{t \in \mathbb{R}}$ of $L^\infty(\hat{\mathbb{G}})$, with analytic extension to $\mathbb{C}$, that satisfies

$$\tau_t(u_{i,j}^\pi) = \sum_{k,l=1}^n ((F^\pi)^t)_{i,k}((F^\pi)^{-t})_{l,j} u_{k,l}^\pi.$$
For a discrete group $G$, the $G$-invariant functionals on $C_r(G)$ are exactly the tracial states on $C_r(G)$. For unimodular discrete $G$, a state is tracial if and only if it is $G$-invariant \[16\] Lemma 5.2. This changes for non-unimodular $G$, however. We have that the $G$-invariant states are the KMS states of the scaling group (with inverse temperature 1) on $C_r(G)$ \[15\] Lemma 5.2. More precisely, a state $\tau \in C_r(G)^*$ is $G$-invariant if and only if $\tau(ab) = \tau(\tau_i(b)a)$ for every $a, b \in C_r(G)$.

**Remark 4.17.** Let $\tau \in C_r(G)^*$ be a $G$-invariant state. Since it is KMS, as with tracial states (see Remark \[LS\]), we find that a weak* limit $\omega$ of the Cesaro sums of $\tau$ is KMS, and hence $\omega$ is a $G$-invariant idempotent state. Indeed, it can be shown that $(\tau_z \otimes \tau_z) \Delta^*_\omega = \Delta^*_\omega \circ \tau_z$ for every $z \in \mathbb{C}$. Then, for $a, b \in C_r(G)$,

$$\tau \ast \tau(ab) = (\tau \otimes \tau) \left( (\Delta^*_\omega(a) \Delta^*_\omega(b)) \right) = (\tau \otimes \tau) \left( ((\tau_i \otimes \tau_i) \Delta^*_\omega(b)) \Delta^*_\omega(a) \right) = (\tau \otimes \tau) \Delta^*_\omega((\tau_i(b)) \Delta^*_\omega(a)) = \tau \ast \tau(\tau_i(b)a).$$

So, the convolution powers of a KMS state is still KMS, and hence so would be $\omega$.

A straightforward application of the Cauchy-Schwarz inequality informs us that $\omega$ is a Haar idempotent: indeed, if $a \in I_\omega \cap \text{Pol}(\widehat{G})$, then

$$|\omega(aa^*)|^2 = |\omega(\tau_i(a^*)a)|^2 \leq \omega(\tau_i(a^*)\tau_i(a^*))\omega(a^*a) = 0,$$

so $a^* \in I_\omega \cap \text{Pol}(\widehat{G})$.

The Haar state is $G$-invariant if and only if it is tracial. This was proven with \[16\] Lemma 5.2 and is also something that is relatively straightforward to deduce from \[21\] Corollary 3.20. It turns out this can be witnessed by Haar states realized as Haar idempotents on larger CQGs.

**Proposition 4.18.** Let $G$ be a DQG. A Haar idempotent state on $C_r(G)$ is $G$-invariant if and only if it is tracial.

Before completing the proof, we remind the reader of important features of closed quantum subgroups of CQGs (see \[36\] for the compact case and \[12\] for the locally compact case). If $H \subset \widehat{G}$ and $\widehat{G}/H$ is amenable (so $\pi_H : C_r(G) \rightarrow C_r(H)$), then there is an injective, normal unital $*$-homomorphism $\gamma_H : \ell^\infty(H) \rightarrow \ell^\infty(G)$ satisfying $\Delta \circ \gamma_H = (\gamma_H \otimes \gamma_H)\Delta_{\widehat{H}}$ so that $\gamma_H(\ell^\infty(H))$ is the corresponding two-sided coideal mentioned in Section 2.4. The pre-adjoint is a surjective algebra homomorphism $(\gamma_H^*) : \ell^1(G) \rightarrow \ell^1(H)$. It turns out that $(\gamma_H \otimes \text{id})W_{\widehat{H}} = (\text{id} \otimes \pi_H)W_{\widehat{G}}$.

**Proof.** Let $\omega = h_{\widehat{H}} \circ \pi_{\widehat{H}}$ be a tracial Haar idempotent. In particular, $H$ is Kac. We have that $U_{\widehat{H}} = (\pi_{\widehat{H}} \otimes \text{id})U_{\widehat{H}}^\ell = [\pi_{\widehat{H}}(u^\tau_{i,j})]$ is a unitary corepresentation matrix of $H$. Then, since $H$ is Kac and $*$-representations decompose into irreducibles, we have $U_{\widehat{H}}^\ell U_{\widehat{H}}^\ell = I_{N_H}$. Therefore, for $a \in C_r(G)$,

$$h_{\widehat{H}} \circ \pi_{\widehat{H}}(L^\tau_{i,j}(a)) = h_{\widehat{H}} \left( \sum_{t=1}^{N_H} \pi_{\widehat{H}}(u^\tau_{i,t})\pi_{\widehat{H}}(a)\pi_{\widehat{H}}((u^\tau_{j,t})^*) \right) = h_{\widehat{H}}(\pi_{\widehat{H}}(a) \sum_{t=1}^{N_H} \pi_{\widehat{H}}((u^\tau_{j,t})^*)\pi_{\widehat{H}}(u^\tau_{i,t})) = \delta_{i,j}h_{\widehat{H}} \circ \pi_{\widehat{H}}(a).$$
Conversely, assume that \( h_\mathbb{H} \circ \pi_\mathbb{H} \) is \( G \)-invariant. Given a state \( f \in \ell^1(\tilde{\mathbb{H}}) \), find a state \( \varphi \in \ell^1(G) \) such that \( \varphi \circ \gamma_\mathbb{H} = f \). Then, for \( a \in C_\tau(\tilde{G}) \)

\[
(f \otimes h_\mathbb{H}) \Delta^G_\mathbb{H}(\pi_\mathbb{H}(a)) = (\varphi \circ \gamma_\mathbb{H} \otimes h_\mathbb{H}) W^\ast G(1 \otimes \pi_\mathbb{H}(a)) W^G_\mathbb{H} \\
= (\varphi \circ h_\mathbb{H})(\gamma_\mathbb{H} \otimes id) W^G_\mathbb{H}(1 \otimes \pi_\mathbb{H}(a))(\gamma_\mathbb{H} \otimes id) W^G_\mathbb{H} \\
= (\varphi \circ h_\mathbb{H})(id \otimes \pi_\mathbb{H})(W^G_\mathbb{H})(1 \otimes \pi_\mathbb{H}(a))(id \otimes \pi_\mathbb{H})(W^G_\mathbb{H}) \\
= (\varphi \circ h_\mathbb{H} \circ \pi_\mathbb{H}) W^G_\mathbb{H}(1 \otimes a) W^G_\mathbb{H} \\
= h_\mathbb{H}(\pi_\mathbb{H}(a)).
\]

So, \( h_\mathbb{H} \) is \( \tilde{\mathbb{H}} \)-invariant which implies it is tracial, and we deduce \( h_\mathbb{H} \circ \pi_\mathbb{H} \) is tracial. \( \square \)

By taking Cesaro sums of convolution powers of traces or KMS states, we immediately deduce the following with Proposition 4.18 in hand, which, despite the apparent disparity between \( G \)-invariant states and tracial states, a relationship remains none-the-less.

**Corollary 4.19.** Let \( G \) be a DQG. Then \( C_\tau(\tilde{G}) \) has \( G \)-invariant state if and only if it has a tracial state.

This partially resolves an open problem from [9,20] at the level of CQGs, which generalizes the equivalence between nuclearity of \( C_\tau(G) \) with amenability of a discrete group \( G \). Recall that a \( C^* \)-algebra \( A \) is **nuclear** if for every \( C^* \)-algebra \( B \) we have \( A \otimes_{\min} B = A \otimes_{\max} B \).

**Corollary 4.20.** Let \( G \) be a DQG. We have that \( C_\tau(\tilde{G}) \) is nuclear and has a tracial state if and only if \( G \) is amenable.

*Proof.* It is was proven in [9] (combined with [35] and building off the work in [26]) that \( C_\tau(\tilde{G}) \) is nuclear and has a \( G \)-invariant state if and only if \( G \) is amenable. The proof is complete with Corollary 4.19. \( \square \)

### 4.4 Existence and Uniqueness of Traces

**Definition 4.21.** Let \( G \) be a DQG and \( \mathbb{H} \) a closed quantum subgroup of \( \tilde{G} \). We say a state \( \omega \in C^*_u(\tilde{G}) \) **concentrates** on \( \tilde{G} / \mathbb{H} \) if \( \tau \circ R_\omega = \omega_{\mathbb{H}} \mid_{\text{Pol}(\tilde{G})} \).

The following was essentially shown in the proof of [16, Theorem 5.3]. We complete the proof here.

**Theorem 4.22.** Let \( G \) be a DQG. Every \( G \)-invariant state \( \tau \in C_\tau(\tilde{G})^* \) concentrates on \( \tilde{G} / \mathbb{H}_F \).

*Proof.* Let \( \tau \) be a \( G \)-invariant state, and using \( G \)-injectivity of \( C(\partial_F(G)) \), we obtain a ucp \( G \)-equivariant extension \( M_\tau : C(\partial_F(G)) \times_F G \rightarrow C(\partial_F(G)) \) of \( \tau \). It was shown in the proof of [16, Theorem 5.3] that for all \( y \in \bar{N}_F \), \( \lambda_G(\tau)y = \epsilon_G(y)x \). We remind the reader how this is done here.

By \( G \)-rigidity, the restriction of \( M_\tau \) to \( \alpha(C(\partial_F(G)) \) is equal to \( \alpha^{-1} \), so we conclude that \( \alpha(C(\partial_F(G)) \) lies in the multiplicative domain of \( M_\tau \). In particular,

\[
M_\tau(\alpha(x)(\hat{a} \otimes 1)) = \tau(\hat{a})x = M_\tau((\hat{a} \otimes 1)\alpha(x))
\]
for all \( x \in C(\partial_F(\mathbb{G})) \) and \( \hat{a} \in C_r(\hat{\mathbb{G}}) \). Let \( \beta \) be the coaction of \( \mathbb{G} \) on \( C(\partial_F(\mathbb{G})) \times_{\tau} \mathbb{G} \) (see Section 2.2). The equation \((\text{id} \otimes \beta)\beta = (\Delta_{\mathbb{G}} \otimes \text{id})\beta \) implies that

\[
(W_{\mathbb{G}}^* \otimes 1)(1 \otimes \alpha(x)) = \beta(\alpha(x))(W_{\mathbb{G}}^* \otimes 1), \quad x \in C(\partial_F(\mathbb{G})).
\]

By applying \text{id} \otimes M_\tau to both sides of the above equation and using \( \beta|_{\alpha(C(\partial_F(\mathbb{G})))} = \text{id} \otimes \alpha \), we obtain,

\[
\alpha(x)((\text{id} \otimes \tau)(W_{\mathbb{G}}^*) \otimes 1) = (\text{id} \otimes \tau)(W_{\mathbb{G}}^*) \otimes x = ((\text{id} \otimes \tau)(W_{\mathbb{G}}^*) \otimes 1)\alpha(x).
\]

Therefore \((\text{id} \otimes \tau)(W_{\mathbb{G}}^*)P_\mu(x) = \mu(x)(\text{id} \otimes \tau)(W_{\mathbb{G}}^*) = P_\mu(x)(\text{id} \otimes \tau)(W_{\mathbb{G}}^*) \) for every \( \mu \in C(\partial_F(\mathbb{G}))^* \) and \( x \in C(\partial_F(\mathbb{G})) \). Then, \( \mu(x) = \epsilon_C \circ P_\mu(x) \) and \( W_{\mathbb{G}} = \Sigma(W_{\mathbb{G}}^*) \), we deduce that

\[
\epsilon_C(\lambda)\lambda_C^\tau(\tau) = \lambda_C(\tau)y = y\lambda_C^\tau(\tau), \quad \text{for all } y \in N_F.
\]

In particular, for \( P_F = \lambda_C(\omega_F) \in \tilde{N}_F, \)

\[
\lambda_C^\tau(\tau \ast \omega_F) = \lambda_C^\tau(\tau)P_F = \lambda_C^\tau(\tau) = \lambda_C(\omega_F \ast \tau).
\]

Then, \( \tau \circ R_{\epsilon_C - \omega_F}|_{\text{Pol}(\hat{\mathbb{G}})} = \tau \ast (\epsilon_C - \omega_F)|_{\text{Pol}(\hat{\mathbb{G}})} = 0 = \tau \circ L_{\epsilon_C - \omega_F}|_{\text{Pol}(\hat{\mathbb{G}})} \).

**Remark 4.23.**

1. Arrange a set of representatives of the irreducibles of \( \hat{\mathbb{G}} \) so that \( \lambda_C(\omega_F) \) is diagonal (see [2] Lemma 4.7). Then Theorem 4.22 says that \( \tau(u_{i,j}^\tau) = 0 \) whenever \( u_{i,j}^\tau \notin \text{Pol}(\hat{\mathbb{G}}/\mathbb{H}_F) \) or \( u_{i,j}^\tau \notin \text{Pol}(\mathbb{H}_F \setminus \hat{\mathbb{G}}) \).

More specifically, let \( \mathbb{H} \) be a Kac closed quantum subgroup of \( \hat{\mathbb{G}} \) where \( \hat{\mathbb{G}}/\mathbb{H} \) is coamenable. From Theorem 4.22 and Proposition 4.18 we have that the associated (tracial) Haar idempotent \( \omega_{\hat{\mathbb{G}}/\mathbb{H}} \in C_r(\hat{\mathbb{G}}) \) satisfies \( \omega_{\hat{\mathbb{G}}/\mathbb{H}} \ast \omega_F = \omega_{\hat{\mathbb{G}}/\mathbb{H}} \). This means that \( L^\infty(\hat{\mathbb{G}}/\mathbb{H}) \subseteq L^\infty(\hat{\mathbb{G}}/\mathbb{H}_F) \) using Lemma 3.17 and so \( \mathbb{H}_F \) is a closed quantum subgroup of \( \mathbb{H} \).

2. The above observation makes Theorem 4.22 a noncommutative version of [6] Theorem 4.1, which states that every tracial state of \( C_r(G) \), where \( G \) is a discrete group, concentrates on the amenable radical of \( G \), i.e., \( \tau(\lambda(s)) = 0 \) for every \( s \in G \setminus R_a(G) \).

From here, we can completely settle the existence and uniqueness of traces for arbitrary discrete quantum groups.

**Corollary 4.24.** Let \( \mathbb{G} \) be a DQG. Then \( \mathbb{H}_F \) is a closed quantum subgroup of every Kac closed quantum subgroup \( \mathbb{H} \) of \( \hat{\mathbb{G}} \), where \( \hat{\mathbb{G}}/\mathbb{H} \) is coamenable. In particular, the following hold:

1. \( \mathbb{H}_F \) is Kac if and only if \( C_r(\hat{\mathbb{G}}) \) has a tracial state (equivalently, has a \( \mathbb{G} \)-invariant state);

2. \( \mathbb{H}_F = H_{Kac}^{\mathbb{G}} \) and \( C_r(Kac(\mathbb{H}_F)) = C_r(\mathbb{H}_F) \) if and only if \( C_r(\hat{\mathbb{G}}) \) has a unique tracial state.

3. \( \mathbb{H}_F = H_{Kac}^{\mathbb{G}} \) if and only if \( C_r(\hat{\mathbb{G}}) \) has a unique idempotent tracial state (equivalently, a unique idempotent \( \mathbb{G} \)-invariant state).

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Proof. The statements regarding $\mathbb{G}$-invariant states are due to Proposition 4.18.

1. If $C_r(\hat{\mathbb{G}})$ has a tracial state, then Theorem 4.9 implies that a closed quantum subgroup $\mathbb{H}$ such that $\hat{\mathbb{G}}/\mathbb{H}$ is coamenable exists, from which we have that $\mathbb{H}_F$ is Kac because it is a closed quantum subgroup of $\mathbb{H}$. Conversely, if $C_r(\hat{\mathbb{G}})$ has no tracial states, then $\mathbb{H}_F$ could not be Kac because then $\omega_F \in C_r(\hat{\mathbb{G}})^*$ would be tracial.

2. Recall from the proof of Theorem 4.9 that the existence of a tracial state implies $\hat{\mathbb{G}}/\mathbb{H}_{Kac}$ is coamenable. Then, as discussed in Remark 4.23, $\mathbb{H}_F$ is a closed quantum subgroup of $\mathbb{H}_{Kac}$. If $C_r(\hat{\mathbb{G}})$ has a unique trace, then $\omega_F = \omega_{Kac}$, so $\mathbb{H}_F = \mathbb{H}_{Kac}$. Moreover, since $\omega_F \in C_r(\hat{\mathbb{G}})^*$ is the unique trace,

$$I_{Kac}^r = \{ a \in C_r(\hat{\mathbb{G}}) : \omega(a^*a) = 0 \} =: I_{\omega_F}^r$$

and $C_r(\hat{\mathbb{G}})/I_{\omega_F}^r = C_r(\mathbb{H}_F)$.

Conversely, for every Kac closed quantum subgroup $\mathbb{H}$ of $\hat{\mathbb{G}}$ where $\hat{\mathbb{G}}/\mathbb{H}$ is coamenable, we have $L^\infty(\hat{\mathbb{G}}/\mathbb{H}_{Kac}) \subseteq L^\infty(\hat{\mathbb{G}}/\mathbb{H}) \subseteq L^\infty(\hat{\mathbb{G}}/\mathbb{H}_F)$. So, $\mathbb{H} = \mathbb{H}_F = \mathbb{H}_{Kac}$, which then Theorem 4.9 implies $C_r(\hat{\mathbb{G}})$ has a unique trace.

3. Similar to proof of 2.. \qed

We can package our results into the following dichotomy for $C^*$-simple discrete quantum groups.

Corollary 4.25. Let $\mathbb{G}$ be a $C^*$-simple DQG. The only possible tracial or $\mathbb{G}$-invariant state is the Haar state.

Proof. Assume $\mathbb{G}$ is unimodular. Then Corollary 1.3 tells us the Haar state is the unique trace and hence unique $\mathbb{G}$-invariant state given the identification between tracial states and $\mathbb{G}$-invariant states. Now suppose $\mathbb{G}$ is non-unimodular. If $C_r(\hat{\mathbb{G}})$ had a tracial state or $\mathbb{G}$-invariant state, then Corollary 4.24 would tell us $\mathbb{H}_F$ is Kac. Simplicity then implies $\omega_F$ is faithful so that $\omega_F = h_{\hat{\mathbb{G}}}$, which contradicts unimodularity of $\mathbb{G}$. \qed

There are many known examples $C^*$-simple DQGs, both unimodular and non-unimodular (e.g., the free unitary and orthogonal quantum groups [3]), thus giving many examples of reduced $C^*$-algebras where the Haar state is the unique trace and where there no traces. We are unaware of an example of a DQG with a unique trace that is not the Haar state, however.

Question 4.26. Is it possible for a DQG to have a unique trace that is not the Haar state?

If the answer is negative and examples could be found where $\mathbb{H}_F = \mathbb{H}_{Kac}$, then $C_rKac(\mathbb{H}_{Kac})$ would provide examples of exotic $C^*$-algebras.

4.5 The Coamenable Coradical and Duality of Relative Amenability and Coamenability

In terms of the lattice of idempotent states of a CQG, given the existence of tracial state on $C_r(\hat{\mathbb{G}})$, Theorem 4.22 says that $\hat{\mathbb{G}}/\mathbb{H}_F$ is the largest coamenable quotient where $\mathbb{H}_F$ is Kac. So, whenever $\mathbb{G}$ is unimodular, $\hat{\mathbb{G}}/\mathbb{H}_F$ is the largest coamenable quotient. For unimodular $\mathbb{G}$, this answers [16, Question 8.2].
Definition 4.27. Let $G$ be a DQG such that $C_r(\hat{G})$ has a tracial state. We call $\mathbb{H}_F$ the Kac coamenable coradical of $\hat{G}$. When $G$ is unimodular, we just call $\mathbb{H}_F$ the coamenable coradical.

It turns out that the Kac closed quantum subgroups of $\hat{G}$ satisfy duality between coamenability and relatively amenability.

Corollary 4.28. Let $G$ be DQG. If $H$ is a Kac closed quantum subgroup of $\hat{G}$, then $\hat{G}/H$ is a coamenable quotient if and only if $\ell^\infty(\hat{H})$ is relatively amenable in $\ell^\infty(G)$.

Proof. Suppose $\hat{G}/H$ is a coamenable quotient. From Theorem 4.22 we have $L^\infty(\hat{G}/H) \subseteq L^\infty(\hat{G}/\mathbb{H}_F)$. Then $\tilde{N}_F \subseteq \ell^\infty(\hat{H})$, which implies $\ell^\infty(\hat{H})$ is relatively amenable because $\tilde{N}_F$ is. The converse is due to [1, Theorem 1.10].

In particular, for unimodular DQGs, we have established a positive answer to [16, Question 8.1].

Remark 4.29. Suppose $G$ is a unimodular DQG. We have that $\omega_F = \min \text{Idem}^H(\hat{G})$.

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