The reduction on the linear stability of elliptic Euler-Moulton solutions of the \( n \)-body problem to those of 3-body problems

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Abstract

In this paper, we consider the elliptic collinear solutions of the classical \( n \)-body problem, where the \( n \) bodies always stay on a straight line, and each of them moves on its own elliptic orbit with the same eccentricity. Such a motion is called an elliptic Euler-Moulton collinear solution. Here we prove that the corresponding linearized Hamiltonian system at such an elliptic Euler-Moulton collinear solution of \( n \)-bodies splits into \((n-1)\) independent linear Hamiltonian systems, the first one is the linearized Hamiltonian system of the Kepler 2-body problem at Kepler elliptic orbit, and each of the other \((n-2)\) systems is the essential part of the linearized Hamiltonian system at an elliptic Euler collinear solution of a 3-body problem whose mass parameter is modified. Then the linear stability of such a solution in the \( n \)-body problem is reduced to those of the corresponding elliptic Euler collinear solutions of the 3-body problems, which for example then can be further understood using numerical results of Martinéz, Samà and Simó in [13] and [14] on 3-body Euler solutions in 2004-2006. As an example, we carry out the detailed derivation of the linear stability for an elliptic Euler-Moulton solution of the 4-body problem with two small masses in the middle.

Keywords: \( n \)-body problem, elliptic Euler-Moulton collinear solution, reduction, linear stability.

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1 Introduction and main results

When one considers a system of \( n \) bodies including the Earth, the Moon and \((n-2)\) space stations in the middle, one tries to find places for these space stations so that they can be easily put there and easily taken away. When \( n = 3 \), by the linear stability study it is well-known that such a middle place should be the Euler point, because at such a point the essential part of the linearized Hamiltonian system possesses two pairs of Floquet multipliers with suitable masses and eccentricity, one of which is elliptic and the other is hyperbolic. This paper is devoted to study the problem for general \( n \geq 3 \), and in fact here we prove that the study on such an \( n \)-body problem can be reduced to those of \((n-2)\) related 3-body problems.

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Recall that in the classical 3-body problem with three positive masses, a special solution was found by L. Euler in [3] of 1767. In this motion, the 3 bodies form always a collinear central configuration at any time in a fixed plane and each body runs along a special Keplerian elliptic orbit about the center of mass of the 3 bodies with the same eccentricity $e \in [0, 1)$. Then F. Moulton in [20] of 1910 proved that for every ordering of $n$ positive masses, there exists a unique collinear central configuration of $n$-bodies. After them in general, for the classical $n$-body problem we call a solution elliptic Euler-Moulton homographic motion of $n$-bodies (EEM for short below), if the $n$ bodies always form a collinear central configuration and each body travels along a specific Keplerian elliptic orbit about the center of mass of the system with the same eccentricity. Specially when $e = 0$, the $n$ bodies run circularly around the center of mass with the same angular velocity, which are called Euler-Moulton relative equilibria traditionally.

Given $n$ positive masses $m = (m_1, \ldots, m_n) \in (\mathbb{R}^+)^n$ on $n$ points $q = (q_1, \ldots, q_n) \in (\mathbb{R}^2)^n$ respectively. According to Newton’s gravitation law, their motion is governed by the system,

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad \text{for } i = 1, 2, \ldots, n,$$

(1.1)

where $U(q) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|q_i - q_j|^2}$ is the potential function and $| \cdot |$ denotes the norm of vectors in $\mathbb{R}^2$.

Let

$$\dot{\mathcal{X}} := \left\{ q = (q_1, q_2, \ldots, q_n) \in (\mathbb{R}^2)^n \left| \sum_{i=1}^n m_i q_i = 0, q_i \neq q_j, \forall i \neq j \right. \right\}.$$

Then critical points of the action functional

$$\mathcal{A}(q) = \int_0^{2\pi} \left[ \sum_{i=1}^n \frac{m_i |\dot{q}_i(t)|^2}{2} + U(q(t)) \right] dt$$

defined on the space $W^{1, 2}(\mathbb{R}/2\pi\mathbb{Z}, \dot{\mathcal{X}})$ correspond to $2\pi$-periodic solutions of the system (1.1) one-to-one. To transform (1.1) to a Hamiltonian system, we let $p = (p_1, \ldots, p_n)$ with $p_i = m_i \dot{q}_i \in \mathbb{R}^2$ for $1 \leq i \leq n$ and obtain

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{for } i = 1, 2, \ldots, n,$$

(1.2)

where the Hamiltonian function is given by

$$H(p, q) = \sum_{i=1}^n \frac{|p_i|^2}{2m_i} - U(q).$$

(1.3)

It is well-known (cf. [13], [14]) that the linear stability of an EEM solution of the 3-body problem with masses $m = (m_1, m_2, m_3) \in (\mathbb{R}^+)^3$ is determined by the eccentricity $e \in [0, 1)$ and the mass parameter

$$\beta = \frac{m_1(3x^2 + 3x + 1) + m_3x^2(x^2 + 3x + 3)}{x^2 + m_2[(x + 1)^2(x^2 + 1) - x^2]},$$

(1.4)

where $x$ is the unique positive solution of the Euler quintic polynomial equation

$$(m_3 + m_2)x^5 + (3m_3 + 2m_2)x^4 + (3m_3 + m_2)x^3 - (3m_1 + m_2)x^2 - (3m_1 + 2m_2)x - (m_1 + m_2) = 0,$$

(1.5)

and the three bodies form a central configuration of $m$, which are denoted by $q_1 = 0$, $q_2 = (x\alpha, 0)^T$ and $q_3 = ((1 + x\alpha, 0)^T$ with $\alpha = |q_2 - q_3| > 0$, $x\alpha = |q_1 - q_2|$. 

In this paper we prove that the linear stability problem of the EEM in the $n$-body case for every integer $n \geq 3$ can be in fact reduced to the linear stabilities of $(n - 2)$ related EEM of 3-body cases. More precisely,
based on the central configuration coordinate method of K. Meyer and D. Schmidt in [16], we reduce the linear stability of the $n$-body EEM to two parts symplectically, one of which is the same as that of the Kepler solutions, and the other is a $4(n - 2)$-dimensional Hamiltonian system whose fundamental solution is the essential part for the linear stability of the EEM of $n$-bodies. Then we prove that this essential part is the sum of $(n - 2)$ independent linear Hamiltonian systems, each of which is the essential part of the linearized Hamiltonian system of some EEM of a related 3-body problem.

To describe our main reduction result more precisely, given positive masses $m = (m_1, m_2, \ldots, m_n) \in (\mathbb{R}^+)^n$, let $a = (a_1, \ldots, a_n)$ be the unique $n$-body collinear central configuration of $m$ with $a_i = (a_{ix}, 0)^T$ for $1 \leq i \leq n$ which satisfies $a_{ix} < a_{jx}$ if $i < j$. Without loss of generality, we normalize the masses by

$$\sum_{i=1}^{n} m_i = 1,$$  \hspace{1cm} (1.6)

and normalize the positions $a_i$ with $1 \leq i \leq n$ by

$$\sum_{i=1}^{n} m_i a_i = 0, \quad \text{and} \quad \sum_{i=1}^{n} m_i a_i^2 = 2I(a) = 1. \hspace{1cm} (1.7)$$

Moreover, we define

$$\mu = U(a) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|a_i - a_j|},$$  \hspace{1cm} (1.8)

and

$$\tilde{M} = \text{diag}(m_1, \ldots, m_n). \hspace{1cm} (1.9)$$

Let $B = (B_{ij}) = \frac{1}{2}U''(a)$ be the Hessian of $U(q)$ at the collinear central configuration $q = a$ which is an $n \times n$ symmetric matrix given by

$$B_{ij} = \begin{cases} \frac{m_i m_j}{|a_i - a_j|^3}, & \text{if } i \neq j, \ 1 \leq i, j \leq n, \\ -\sum_{1 \leq j \leq n \atop j \neq i} \frac{m_i m_j}{|a_i - a_j|^3}, & \text{if } i = j, \ 1 \leq i \leq n. \end{cases} \hspace{1cm} (1.10, 1.11)$$

We let

$$D = \mu I_n + \tilde{M}^{-1} B. \hspace{1cm} (1.12)$$

Then the following lemma is crucial for our study, whose proof is due to C. Conley according to F. Pacella ([21], 1987) and R. Moeckel ([17] of 1990 as well as [18] of 1994). For reader’s conveniences, a sketch of this proof will be given in the Appendix of this paper below following [21], [17] and [18].

**Lemma 1.1** The $n \times n$ matrix $D$ possesses a simple eigenvalue $\lambda_1 = \mu > 0$ and a second eigenvalue $\lambda_2 = 0$. The other $n - 2$ eigenvalues of $D$ besides $\mu$ and this $0$ are non-positive. Consequently they satisfy

$$\lambda_1 > \lambda_2 = 0 \geq \lambda_3 \geq \cdots \geq \lambda_n. \hspace{1cm} (1.13)$$

Then we define

$$\beta_i = -\frac{\lambda_{i+2}}{\mu} \geq 0, \quad \forall \ 1 \leq i \leq n - 2. \hspace{1cm} (1.14)$$

Based on these $\beta_i$s, our main result of this paper is the following
Theorem 1.2 In the planar n-body problem with given masses \( m = (m_1, m_2, \ldots, m_n) \in (\mathbb{R}^+)^n \), denote the EEM with eccentricity \( e \in [0, 1) \) for \( m \) by \( q_{m,e}(t) = (q_1(t), q_2(t), \ldots, q_n(t)) \). Then the linearized Hamiltonian system at \( q_{m,e} \) is reduced into the sum of \((n-1)\) independent Hamiltonian systems, the first one is the linearized system of the Kepler 2-body problem at the corresponding Kepler orbit, and the \( i \)-th part of the other \((n-2)\) parts with \( 1 \leq i \leq n-2 \) is the essential part of the linearized Hamiltonian system of some EEM of a 3-body problem with the original eccentricity \( e \) and the mass parameter \( \beta_i \) given by (1.14) instead of that \( \beta \) given by (1.4).

Remark 1.3 (i) J. Liouville first observed in [10] of 1842 that the Moon stays always on the straight line passing through the centers of the Sun and the Earth and on the opposite side of the Sun with respect to the Earth, i.e., the Moon always enlightens the Earth during the nights, is impossible due to the instability of such a configuration. According to R. Moeckel (cf. p.300, [18]) of 1994, the stability analysis of collinear relative equilibria can be attributed to M. Andoyer [1] in 1906 and M. Meyer [15] in 1933. Subsequent studies on the linear stability of EEMs can be found in [16] of K. Meyer and D. Schmidt in 2005. [12], [13] and [14] of R. Martínez, A. Samà and C. Simó in 2004-2006, and the recent preprints [26] of Q. Zhou and Y. Long, and [6] of X. Hu and Y. Ou. Researches on Lagrangian equilateral triangle elliptic solutions (cf. [2]) and related topics were done by M. Gascheau (4], 1843), E. Routh ([23], 1875), J. Danby ([2], 1964), R. Moeckel ([19], 1995), G. Roberts ([22], 2002), X. Hu and S. Sun ([7], 2010), and X. Hu, Y. Long and S. Sun ([5], 2014).

(ii) Based on our above reduction theorems, the numerical results obtained by R. Martínez, A. Samà and C. Simó in [13] and [14] for 3-body Euler solutions can be applied to get the linear stability of the \( n \)-body elliptic Euler-Moulton collinear solutions using our formula of \( \beta_i \)s in (1.14) for any positive integer \( n \geq 3 \). The theoretical linear stability results on 3-body EEM obtained in papers [26] and [6] can also be applied too.

(iii) It may be worth to point out that the proof of our reduction Theorem 1.2 is based upon the results of [16] of 2005, and is independent of the results and their proofs in papers [13], [14], [26] and [6] for the 3-body case.

In the Section 2 of this paper we focus on the proof of Theorem 1.2. In Section 3, we study a special example of a collinear 4-body problem with two small masses in the middle. The two corresponding mass parameters \( \beta_1 \) and \( \beta_2 \) in (1.14) are calculated explicitly there, and hence their linear stability can be determined numerically using results in [13] and [14] of 2004-2006 for example. It is interesting to see that when the masses of the two middle particles tend to 0, the effect of both of them does not disappear. In the Appendix, a sketch of the proof of Lemma 1.1 is given.

2 Reduction from the collinear n-body problem to \((n-2)\) collinear 3-body problems

In their paper [16] of 2005, K. Meyer and D. Schmidt introduced the central configuration coordinates for a class of periodic solutions of the n-body problem. Our study on the EEM solutions of \( n \)-bodies is based upon their method. Here the key point is that we found the reduction of the linear stability of the \( n \)-body EEM problem to those of \((n-2)\) three body problems. This reduction needs more techniques for the \( n \) body case.

As in Section 1, for the given masses \( m = (m_1, m_2, \ldots, m_n) \in (\mathbb{R}^+)^n \) satisfying (1.6), suppose the \( n \) particles are all on the \( x \)-axis with \( a_1 = (a_{1x}, 0)^T, a_2 = (a_{2x}, 0)^T, \ldots, a_n = (a_{nx}, 0)^T \) satisfying \( a_{ix} < a_{jx} \) if \( i < j \). In this section we always denote by \( a = (a_1, \ldots, a_n) \) the unique collinear central configuration for the
mass $m$ determined by [20]. Using normalization and notations (1.6)-(1.9), we have

$$
\sum_{j=1, j\neq i}^{n} \frac{m_j(a_{ij} - a_{ix})}{|a_{ij} - a_{ix}|^3} = \frac{U(a)}{2I(a)} a_{ix} = \mu a_{ix}.
$$

(2.1)

Based on the matrix $B$ of (1.10)-(1.11), besides $D$ we further define

$$\tilde{D} = \mu I_n + \tilde{M}^{-1/2} B \tilde{M}^{-1/2} = \tilde{M}^{1/2} D \tilde{M}^{-1/2}.
$$

(2.2)

where $\mu$ is given by (1.8).

Since $\tilde{D}$ is symmetric, all its eigenvalues are real, which are denoted by $\lambda_1 = \mu$, $\lambda_2 = 0$, $\lambda_3$, ..., $\lambda_n$ with corresponding eigenvectors $\tilde{v}_1 = \tilde{M}^{1/2} v_1$, $\tilde{v}_2 = \tilde{M}^{1/2} v_2$, $\tilde{v}_3$, ..., $\tilde{v}_n$. Moreover, we can suppose that $\tilde{v}_1$, $\tilde{v}_2$, ..., $\tilde{v}_n$ form an orthonormal basis of $\mathbb{R}^n$.

Letting $v_i = \tilde{M}^{-1/2} \tilde{v}_i$ for $3 \leq i \leq n$, we have

$$D v_i = \tilde{M}^{-1/2} \tilde{D} \tilde{M}^{1/2} (\tilde{M}^{-1/2} \tilde{v}_i) = \tilde{M}^{-1/2} \tilde{D} \tilde{v}_i = \tilde{M}^{-1/2} \lambda_i \tilde{v}_i = \lambda_i v_i.
$$

Thus $v_i$ is the eigenvector of $D$ belonging to its eigenvalue $\lambda_i$. Moreover, by the orthonormal basis property of $\tilde{v}_1$, $\tilde{v}_2$, ..., $\tilde{v}_n$, we have

$$v_i^T \tilde{M} v_j = v_i^T \tilde{v}_j = \delta_{ij}, \quad \forall \ 1 \leq i, j \leq n.
$$

(2.3)

Denote the eigenvector $v_i$ belonging to the eigenvalue $\lambda_i$ of the matrix $D$ by $v_i^T = (b_{1i}, b_{2i}, ..., b_{ni})$ for $3 \leq i \leq n$, i.e.,

$$D(b_{1k}, b_{2k}, ..., b_{nk})^T = \lambda_k (b_{1k}, b_{2k}, ..., b_{nk})^T, \quad 3 \leq k \leq n.
$$

(2.4)

Then it yields

$$\mu b_{ik} - \sum_{j=1, j\neq i}^{n} \frac{m_j(b_{jk} - b_{ik})}{|a_{ij} - a_{i}|^3} = \lambda_k b_{ik}, \quad 1 \leq i \leq n.
$$

(2.5)

Let

$$F_{ik} = \sum_{j=1, j\neq i}^{n} \frac{m_j(m_j b_{jk} - b_{ij})}{|a_{ij} - a_{i}|^3}, \quad 1 \leq i \leq n, 3 \leq k \leq n,
$$

(2.6)

then we have

$$F_{ik} = (\mu - \lambda_k) m_i b_{ik}.
$$

(2.7)

Moreover, we have

$$\sum_{i=1}^{n} F_{ik} b_{ik} = \sum_{i=1}^{n} (\mu - \lambda_k) m_i b_{ik}^2 = \mu - \lambda_k = \mu (1 + \beta_{k-2}),
$$

(2.8)

where in the last equality, we used (1.14).

Now as in p.263 of [16], we define

$$P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \quad Y = \begin{pmatrix} G \\ Z \\ W_1 \\ \vdots \\ W_{n-2} \end{pmatrix}, \quad X = \begin{pmatrix} g \\ z \\ w_1 \\ \vdots \\ w_{n-2} \end{pmatrix},
$$

(2.9)

where $p_i$, $q_i$ with $i = 1, 2, ..., n$, $G$, $Z$, $W_i$ with $1 \leq i \leq n - 2$, $g$, $z$, and $w_i$ with $1 \leq i \leq n - 2$ are all column vectors in $\mathbb{R}^2$. We make the symplectic coordinate change

$$P = A^{-T} Y, \quad Q = AX,$n

(2.10)
where the matrix $A$ is constructed as in the proof of Proposition 2.1 in [16]. More precisely, the matrix $A \in \text{GL}(\mathbb{R}^{2n})$ is given by

\[
A = \begin{pmatrix}
I & A_1 & B_{13} & \cdots & B_{1n} \\
I & A_2 & B_{23} & \cdots & B_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
I & A_n & B_{n3} & \cdots & B_{nn}
\end{pmatrix},
\] (2.11)

where each $A_i$ is a $2 \times 2$ matrix given by

\[
A_i = (a_i, ja_i) = \begin{pmatrix} a_{ix} & 0 \\
0 & a_{ix} \end{pmatrix} = a_{ix}I_2.
\] (2.12)

Let

\[
B_{ki} = \begin{pmatrix} b_{ki} & 0 \\
0 & b_{ki} \end{pmatrix} = b_{ki}I_2.
\] (2.13)

Then $A^TMA = I_{2n}$ holds (cf. (13) in p.263 of [16]).

As in Theorem 2.1 on pp.261-262, setting $G = g = 0$ to fix the center of mass at the origin as in p.271 of [16], after the transform (2.10) the Hamiltonian function of the $n$-body problem in the new variables becomes

\[
H(Z, W_1, \ldots, W_{n-2}, z, w_1, \ldots, w_{n-2}) = K(Z, W_1, \ldots, W_{n-2}) - U(z, w_1, \ldots, w_{n-2})
\] (2.14)

where the kinetic energy satisfies

\[
K = \frac{1}{2}(|Z|^2 + |W_1|^2 + \cdots + |W_{n-2}|^2),
\] (2.15)

and the potential function satisfies

\[
U(z, w_1, \ldots, w_{n-2}) = \sum_{1 \leq i < j \leq n} U_{ij}(z, w_1, \ldots, w_{n-2}),
\] (2.16)

with

\[
U_{ij}(z, w_1, \ldots, w_{n-2}) = \frac{m_im_j}{d_{ij}(z, w_1, \ldots, w_{n-2})},
\] (2.17)

\[
d_{ij}(z, w_1, \ldots, w_{n-2}) = |(A_i - A_j)z + \sum_{k=3}^{n}(B_{ik} - B_{jk})w_{k-2}|
\] (2.18)

where we have used (2.12) and (2.13). Recall that each $Z, W_i, z, w_i$ with $1 \leq i \leq n-2$ is a vector in $\mathbb{R}^2$. Here $z = z(t)$ is the Kepler elliptic orbit given through the true anomaly $\theta = \theta(t)$,

\[
r(\theta(t)) = |z(t)| = \frac{p}{1 + e \cos \theta(t)},
\] (2.19)

where $p = a(1 - e^2)$ and $a > 0$ is the latus rectum of the ellipse (2.19).

As in pp.271-273 of [16], we have the following proposition.

**Proposition 2.1** There exists a symplectic coordinate change

\[
\xi = (Z, W_1, \ldots, W_{n-2}, z, w_1, \ldots, w_{n-2})^T \mapsto \tilde{\xi} = (\tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2})^T,
\] (2.20)
such that using the true anomaly $\theta$ as the variable the resulting Hamiltonian function of the $n$-body problem is given by

$$H(\theta, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2})$$

$$= \frac{1}{2} (|\tilde{Z}|^2 + \sum_{k=1}^{n-2} |\tilde{W}_k|^2) + (\tilde{z} \cdot J \tilde{Z} + \sum_{k=1}^{n-2} \tilde{w}_k \cdot J \tilde{W}_k)$$

$$+ \frac{p - r(\theta)}{2p} (|\tilde{z}|^2 + \sum_{k=1}^{n-2} |\tilde{w}_k|^2) - \frac{r(\theta)}{\sigma} U(\tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}),$$

(2.21)

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $r(\theta) = \frac{p}{1 + e \cos \sigma}$, $\mu$ is given by (7.8), $\sigma = (\mu p)^{-1/4}$ and $p$ is given in (2.12).

**Remark 2.2** Proposition 2.1 is a modified version of Lemma 3.1 of [16] in our case of $n$-bodies. As pointed out in Section 11 of [11], in the 3-body case, the $\sigma$ in (2.23) given by $\sigma = p \beta^3$ in the original computation on line 9 of p.273 in [16] is incorrect, and should be corrected to $\sigma = (\mu p)^{-1/4}$. Note also that in the line 11 of p.273 in [16], the stationary solution $(0, 1, 0, 1, 0, 0, 0)^T$ is incorrect too and should be corrected to $(0, \sigma, 0, 0, \sigma, 0, 0)^T$ as in [11], and in general it may not be possible to have $\sigma = 1$.

**Proof of Proposition 2.1** Because of reasons mentioned in this remark, for reader’s conveniences, we give the complete details of the proof of this proposition below.

Following the proof of Lemma 3.1 of [16], we carry the coordinate changes in four steps.

**Step 1. Rotating coordinates via the matrix $R(\theta(t))$ in time $t$.**

We change first the coordinates $\xi$ to

$$\tilde{\xi} = (\tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2})^T \in (\mathbb{R}^2)^{n-1},$$

(2.22)

which rotates with the speed of the true anomaly. The transformation matrix is given by the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ The generating function of this transformation is given by

$$\hat{F}(t, Z, W_1, \ldots, W_{n-2}, z, w_1, \ldots, w_{n-2}) = -Z \cdot R(\theta) \tilde{z} - \sum_{i=1}^{n-2} W_i \cdot R(\theta) \tilde{w}_i,$$

(2.23)

and for $1 \leq i \leq n - 2$ the transformation is given by

$$z = -\frac{\partial F}{\partial Z} = R(\theta) \tilde{z}, \quad \dot{Z} = -\frac{\partial F}{\partial \tilde{z}} = R(\theta)^T Z,$$

(2.24)

$$w_i = -\frac{\partial F}{\partial W_i} = R(\theta) \tilde{w}_i, \quad \dot{W}_i = -\frac{\partial F}{\partial \tilde{w}_i} = R(\theta)^T W_i.$$ (2.25)

Writing $\hat{R}(\theta(t)) = \frac{d}{dt} R(\theta(t))$, and noting that $R(\theta)^T = R(\theta)^{-1}$ and $\hat{R}(\theta) = \hat{\theta} J R(\theta)$ we obtain the function

$$\dot{\hat{F}}_i = \frac{\partial F}{\partial t} = -Z \cdot \dot{R}(\theta) \tilde{z} - \sum_{i=1}^{n-2} W_i \cdot \dot{R}(\theta) \tilde{w}_i$$

$$= -\dot{Z} \cdot R(\theta)^T \dot{R}(\theta) \tilde{z} - \sum_{i=1}^{n-2} \dot{W}_i \cdot R(\theta)^T \dot{R}(\theta) \tilde{w}_i$$

$$= -\dot{\theta} \left( \dot{Z} \cdot R(\theta)^T J R(\theta) \tilde{z} + \sum_{i=1}^{n-2} \dot{W}_i \cdot R(\theta)^T J R(\theta) \tilde{w}_i \right)$$

$$= -\dot{\theta} \left( \dot{Z} \cdot J \tilde{z} + \sum_{i=1}^{n-2} \dot{W}_i \cdot J \tilde{w}_i \right).$$
Because by the definitions (2.12) of $A_i$s and (2.13) of $B_k$s, we obtain
\[ A_i R(\theta) = R(\theta)A_i, \quad B_k R(\theta) = R(\theta)B_k, \quad \forall \ 1 \leq i \leq n-2. \] (2.26)

By (2.16), this then implies
\[
U(z, w_1, \ldots, w_{n-2}) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|A_i - A_j| z + \sum_{i=1}^{n-2} (B_i - B_j) w_i} \\
= \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|A_i - A_j| R(\theta) z + \sum_{i=1}^{n-2} (B_i - B_j) R(\theta) w_i} \\
= \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|A_i - A_j| \hat{z} + \sum_{i=1}^{n-2} (B_i - B_j) \hat{w}_i} \\
= U(\hat{z}, \hat{w}_1, \ldots, \hat{w}_{n-2}),
\] (2.27)
by the orthogonality of $R(\theta)$. Because $\theta = \theta(t)$ depends on $t$, by adding the function $\frac{\partial \hat{F}}{\partial t}$ to the Hamiltonian function $H$ in (2.14), as in Line 5 in p.272 of [16], we obtain the Hamiltonian function $\hat{H}$ in the new coordinates:
\[
\hat{H}(t, \hat{Z}, \hat{W}_1, \ldots, \hat{W}_{n-2}, \hat{z}, \hat{w}_1, \ldots, \hat{w}_{n-2}) = H_0(Z, W_1, \ldots, W_{n-2}, z, w_1, \ldots, w_{n-2}) + \hat{F},
\]
\[
= \frac{1}{2}(|\hat{Z}|^2 + \sum_{i=1}^{n-2} |\hat{W}_i|^2) + (\hat{z} \cdot \hat{J} \hat{Z} + \sum_{i=1}^{n-2} \hat{w}_i \cdot J \hat{W}_i) \hat{\theta} - U(\hat{z}, \hat{w}_1, \ldots, \hat{w}_{n-2}),
\] (2.28)
where the variables of $H_0$ are functions of $\theta, \hat{Z}, \hat{W}_1, \ldots, \hat{W}_{n-2}, \hat{z}, \hat{w}_1, \ldots, \hat{w}_{n-2}$ given by (2.24)-(2.25).

**Step 2. Dilating coordinates via the polar radius $r = |z(t)|$.**

We change the coordinates $\hat{z}$ to $\tilde{z} = (Z, W_1, \ldots, W_{n-2}, \hat{z}, \hat{w}_1, \ldots, \hat{w}_{n-2})$ which dilate with $r = |z(t)|$ given by (2.19). The position coordinates are transformed by
\[
\hat{z} = r \tilde{z}, \quad \hat{w}_i = r \tilde{w}_i, \quad \forall \ 1 \leq i \leq n-2.
\] (2.29)
It is natural to scale the momenta by $1/r$ to get $\hat{Z} = Z/r$ and $\hat{W}_i = W_i/r$. But it turns out that the new transformation with $1 \leq i \leq n-2$
\[
\hat{Z} = \frac{1}{r} \tilde{Z} + \tilde{r}, \quad \hat{W}_i = \frac{1}{r} \tilde{W}_i + \tilde{r} \tilde{w}_i
\] (2.30)
makes the resulting Hamiltonian function simpler. This transformation is generated by the function
\[ F(t, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) = \frac{1}{r} (\tilde{Z} \cdot \tilde{z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot \tilde{w}_i) + \frac{r}{2r^2} (|\tilde{z}|^2 + \sum_{i=1}^{n-2} |\tilde{w}_i|^2),
\] (2.31)
and is given by
\[
\tilde{z} = \frac{\partial \tilde{F}}{\partial \tilde{Z}} = \frac{1}{r} \tilde{z}, \quad \tilde{Z} = \frac{\partial \tilde{F}}{\partial \tilde{z}} = \frac{1}{r} \tilde{Z} + \frac{\tilde{r}}{r} \tilde{z} = \frac{1}{r} \tilde{Z} + \tilde{r} \tilde{z},
\]
\[
\tilde{W}_i = \frac{\partial \tilde{F}}{\partial \tilde{W}_i} = \frac{1}{r} \tilde{w}_i, \quad \tilde{W}_i = \frac{\partial \tilde{F}}{\partial \tilde{w}_i} = \frac{1}{r} \tilde{W}_i + \frac{\tilde{r}}{r} \tilde{w}_i = \frac{1}{r} \tilde{W}_i + \tilde{r} \tilde{w}_i,
\]
with
\[
\frac{\partial \tilde{F}}{\partial t} = -\frac{\tilde{r}}{r^2} (\tilde{Z} \cdot \tilde{z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot \tilde{w}_i) + \frac{\dot{r} - \tilde{r}^2}{2r^2} (|\tilde{z}|^2 + \sum_{i=1}^{n-2} |\tilde{w}_i|^2)
\]
\[
= -\frac{\tilde{r}}{r} (\tilde{Z} \cdot \tilde{z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot \tilde{w}_i) + \frac{\dot{r} - \tilde{r}^2}{2} (|\tilde{z}|^2 + \sum_{i=1}^{n-2} |\tilde{w}_i|^2),
\] (2.32)
by (2.30).

In this case, as in the last two lines on p.272 of [16], the Hamiltonian function $\hat{H}$ in (2.28) becomes the new Hamiltonian function $\tilde{H}$ in the new coordinates:

\[
\tilde{H}(t, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) \equiv \hat{H}(t, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) + \tilde{F}_t
\]

\[
= \frac{1}{2r^2}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) + \frac{\dot{r}}{r}(\tilde{Z} \cdot \tilde{z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot \tilde{w}_i) + \frac{\dot{r}^2}{2} (|\tilde{z}|^2 + \sum_{i=1}^{n-2} |\tilde{w}_i|^2)
\]

\[
+ (\tilde{z} \cdot J \tilde{Z} + \sum_{i=1}^{n-2} \tilde{w}_i \cdot J \tilde{W}_i)\dot{\theta} - U(r\tilde{z}, r\tilde{w}_1, \ldots, r\tilde{w}_{n-2}) + F_t
\]

\[
= \frac{1}{2r^2}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) + \frac{r\dot{r}}{2} (|\tilde{z}|^2 + \sum_{i=1}^{n-2} |\tilde{w}_i|^2)
\]

\[
+ (\tilde{z} \cdot J \tilde{Z} + \sum_{i=1}^{n-2} \tilde{w}_i \cdot J \tilde{W}_i)\dot{\theta} - \frac{1}{r}U(r\tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}).
\]

(2.33)

**Step 3. Coordinates via the true anomaly $\theta$ as the independent variable.**

Here we want to use the true anomaly $\theta \in [0, 2\pi]$ as an independent variable instead of $t \in [0, T]$ to simplify the study. This is achieved by dividing the Hamiltonian function $\tilde{H}$ in (2.33) by $\dot{\theta}$. Assuming $\dot{\theta}(t) > 0$ for all $t \in [0, T]$, for $\xi \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^3)$ we consider the action functional corresponding to the Hamiltonian system:

\[
f(\xi) = \int_0^T \left( \frac{1}{2} \dot{\xi}(t) \cdot J\dot{\xi}(t) - \tilde{H}(t, \xi(t)) \right) dt
\]

\[
= \int_0^{2\pi} \left( \frac{1}{2} \dot{\xi}(t) \cdot J\dot{\xi}(t) - \frac{\tilde{H}(t, \xi(t))}{\dot{\theta}(t)} \right) \dot{\theta} dt
\]

\[
= \int_0^{2\pi} \left( \frac{1}{2} \dot{\xi}'(t) \cdot J\dot{\xi}(t) - \tilde{H}(\theta, \xi(t)) \right) \dot{\theta} dt.
\]

Here we used $\dot{\xi}'(t)$ to denote the derivative of $\dot{\xi}(t)$ with respect to the variable $\theta$. But in the following we shall still write $\xi'(t)$ for the derivative with respect to $t$ instead of $\dot{\xi}'(t)$ for notational simplicity.

It is well known that the elliptic Kepler orbit (2.19) satisfies

\[
r(t)^2 \dot{\theta}(t) = \sqrt{\mu \rho} = \sqrt{\mu a(1 - e^2)} = \sigma^2 \text{ with } \sigma = (\mu \rho)^{1/4}.
\]

Note that $a = \mu^{1/3}(T/2\pi)^{2/3}$ with $T$ being the minimal period of the orbit (2.19), we have

\[
\sigma = (\mu a(1 - e^2))^{1/4} = \mu^{1/3}(\frac{T}{2\pi})^{1/6}(1 - e^2)^{1/4} \in (0, \mu^{1/3}(\frac{T}{2\pi})^{1/6}]
\]

depending on $e$, when the mass $\mu$ and the period $T$ are fixed. Note that similarly we have $p = \sigma^4/\mu$ depends on $e$ too. Note that the function $r$ satisfies

\[
\dot{r} = \frac{\mu p}{r^3} - \frac{\mu}{r^2} = \mu \left( \frac{p}{r^3} - \frac{1}{r^2} \right).
\]

Therefore we get the Hamiltonian function $\tilde{H}$ in the new coordinates:

\[
\tilde{H}(\theta, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) \equiv \frac{1}{\dot{\theta}}\tilde{H}(t, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2})
\]

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in (2.34) becomes a new Hamiltonian function:

\[ H(\tilde{\theta}(t)) = \frac{1}{2r^2(t)\tilde{\theta}(t)}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) + \frac{r(t)\tilde{r}(t)}{2\tilde{\theta}(t)}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) \]

\[ + (\tilde{Z} \cdot J\tilde{Z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot J\tilde{W}_i) - \frac{1}{r(t)\tilde{\theta}(t)}U(\tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}) \]

\[ = \frac{1}{2\sigma^2}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) + (\tilde{Z} \cdot J\tilde{Z} + \sum_{i=1}^{n-2} \tilde{W}_i \cdot J\tilde{W}_i) \]

\[ + \frac{2\mu(p - r(\theta))}{2\sigma^2}(|\tilde{Z}|^2 + \sum_{i=1}^{n-2} |\tilde{W}_i|^2) - \frac{1}{\sigma^2}U(\tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}), \tag{2.34} \]

where \( r(\theta) = p/(1 + e \cos \theta) \). Note that now the minimal period \( T \) of the elliptic solution \( \tilde{z} = \tilde{z}(\theta) \) becomes \( 2\pi \) in the new coordinates in terms of true anomaly \( \tilde{\theta} \) as an independent variable.

**Step 4. Coordinates via the dilation of \( \sigma = (\mu p)^{1/4} \).**

The last transformation is the dilation

\[ (\tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) \mapsto (\sigma \tilde{Z}, \sigma \tilde{W}_1, \ldots, \sigma \tilde{W}_{n-2}, \sigma^{-1} \tilde{z}, \sigma^{-1} \tilde{w}_1, \ldots, \sigma^{-1} \tilde{w}_{n-2}). \tag{2.35} \]

This transformation is symplectic and independent of the true anomaly \( \theta \). Thus the Hamiltonian function \( \tilde{H} \) in (2.34) becomes a new Hamiltonian function:

\[ H(\theta, \tilde{Z}, \tilde{W}_1, \ldots, \tilde{W}_{n-2}, \tilde{z}, \tilde{w}_1, \ldots, \tilde{w}_{n-2}) = \tilde{H}(\theta, \sigma \tilde{Z}, \sigma \tilde{W}_1, \ldots, \sigma \tilde{W}_{n-2}, \sigma^{-1} \tilde{z}, \sigma^{-1} \tilde{w}_1, \ldots, \sigma^{-1} \tilde{w}_{n-2}) \]

\[ = \frac{1}{2}(|Z|^2 + \sum_{i=1}^{n-2} |W_i|^2) + (Z \cdot JW + \sum_{i=1}^{n-2} W_i \cdot JW_i) + \frac{2 - \mu}{2\sigma^2}(|Z|^2 + \sum_{i=1}^{n-2} |W_i|^2) - \frac{\mu}{\sigma^2}U(Z, W_1, \ldots, W_{n-2}), \tag{2.36} \]

where one \( \sigma \) is factored out from \( U(\sigma^{-1}z, \sigma^{-1}w_1, \ldots, \sigma^{-1}w_{n-2}) \).

The proof is complete. ■

Motivated by ideas in Sections 2 and 3 of [16], we now derive the linearized Hamiltonian system at such an EEM solution of \( n \)-bodies, where \( \sigma = (\mu p)^{1/4} \) is important.

**Theorem 2.3** Using notations in (2.9), the EEM solution \( (P(t), Q(t))^T \) in time \( t \) of the system (1.2) with

\[ Q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, \ldots, r(t)R(\theta(t))a_{n})^T, \quad P(t) = M\dot{Q}(t), \tag{2.37} \]

where we denote by \( M = \text{diag}(m_1, m_1, \ldots, m_n, m_n) \), is transformed to the new solution \( (Y(\theta), X(\theta))^T \) in the true anomaly \( \theta \) as the new variable with \( G = g = 0 \) for the original Hamiltonian function \( H \) of (2.27), which is given by

\[
Y(\theta) = \begin{pmatrix}
\tilde{Z}(\theta) \\
\tilde{W}_1(\theta) \\
\vdots \\
\tilde{W}_{n-2}(\theta)
\end{pmatrix}, \quad X(\theta) = \begin{pmatrix}
\tilde{z}(\theta) \\
\tilde{w}_1(\theta) \\
\vdots \\
\tilde{w}_{n-2}(\theta)
\end{pmatrix}, \tag{2.38}
\]

Moreover, the linearized Hamiltonian system at the EEM solution

\[ \xi_0 \equiv (Y(\theta), X(\theta))^T = (0, \sigma, 0, \ldots, 0, 0, \sigma, 0, \ldots, 0, 0)^T \in \mathbb{R}^{2(n-1)} \]
depending on the true anomaly $\theta$ with respect to the Hamiltonian function $H$ of (2.21) is given by

$$
\dot{\zeta}(\theta) = JB(\theta)\zeta(\theta),
$$

(2.39)

with

$$
B(\theta) = H''(\theta, \bar{Z}, \bar{W}_1, \ldots, \bar{W}_{n-2}, \bar{z}, \bar{w}_1, \ldots, \bar{w}_{n-2})|_{\bar{\zeta} = \bar{\xi}_0} = \begin{pmatrix}
I_2 & 0 & \ldots & 0 & -J & 0 & \ldots & 0 \\
O & I_2 & \ldots & 0 & O & -J & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \ldots \ldots \\
O & \ldots & O & I_2 & O & \ldots & 0 & -J \\
J & O & \ldots & O & H_{Z\zeta}(\theta, \xi_0) & O & \ldots & 0 \\
O & J & \ldots & O & 0 & H_{W_1\zeta}(\theta, \xi_0) & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \ldots \ldots \\
O & \ldots & O & J & 0 & \ldots & O & H_{W_{n-2}\zeta}(\theta, \xi_0)
\end{pmatrix},
$$

(2.40)

and

$$
H_{Z\zeta}(\theta, \xi_0) = \left(\begin{array}{cc}
-\frac{2r \cos \theta}{1 + e \cos \theta} & 0 \\
0 & 0
\end{array}\right),
$$

$$
H_{W_i\zeta}(\theta, \xi_0) = \left(\begin{array}{cc}
\frac{2 \beta_i + 2 r \cos \theta}{1 + e \cos \theta} & 0 \\
0 & \frac{\beta_i + 1 + r \cos \theta}{1 + e \cos \theta}
\end{array}\right),
$$

for $1 \leq i \leq n - 2$, (2.41)

where each $\beta_i$ with $1 \leq i \leq n - 2$ is given by (1.14), and $H''$ is the Hessian Matrix of $H$ with respect to its variables $\bar{Z}, \bar{W}_1, \ldots, \bar{W}_{n-2}, \bar{z}, \bar{w}_1, \ldots, \bar{w}_{n-2}$. The corresponding quadratic Hamiltonian function is given by

$$
\frac{1}{2}\bar{Z}^2 + \bar{Z} \cdot J \bar{Z} + \frac{1}{2}H_{Z\zeta}(\theta, \xi_0)|\bar{\zeta}|^2 + \sum_{i=1}^{n-2} \left(\frac{1}{2}U_{i\zeta}^2 + \bar{W}_i \cdot J \bar{w}_i + \frac{1}{2}H_{W_i\zeta}(\theta, \xi_0)|\bar{w}_i|^2\right).
$$

(2.42)

**Proof.** In this proof, we generalize the computations in [26] for the EEM of the 3-body case to the $n$-body case here. For reader’s conveniences, we give all details here. We only need to compute $H_{Z\zeta}(\theta, \xi_0)$, $H_{Z\zeta}(\theta, \xi_0)$ and $H_{W_i\zeta}(\theta, \xi_0)$ for $1 \leq i, j \leq n - 2$ respectively.

In this proof we omit all the upper bars on the variables of $H$ in (2.21). By (2.21), we have

$$
H_Z = JZ + \frac{p-r}{\sigma}z - \frac{r}{\sigma}U_Z(z, w_1, \ldots, w_{n-2}),
$$

$$
H_{W_i} = JW_i + \frac{p-r}{\sigma}w_i - \frac{r}{\sigma}U_{W_i}(z, w_1, \ldots, w_{n-2}),
$$

and

$$
\begin{align*}
H_{Z\zeta} &= \frac{p-r}{\sigma}I - \frac{r}{\sigma}U_{Z\zeta}(z, w_1, \ldots, w_{n-2}), \\
H_{Z\zeta} &= \frac{p-r}{\sigma}U_{Z\zeta}(z, w_1, \ldots, w_{n-2}), \quad \text{for } l = 1, \ldots, n - 2, \\
H_{W_i\zeta} &= \frac{p-r}{\sigma}I - \frac{r}{\sigma}U_{W_i\zeta}(z, w_1, \ldots, w_{n-2}), \quad \text{for } l = 1, \ldots, n - 2, \\
H_{W_i\zeta} &= \frac{p-r}{\sigma}U_{W_i\zeta}(z, w_1, \ldots, w_{n-2}), \quad \text{for } l, s = 1, \ldots, n - 2, l \neq s,
\end{align*}
$$

(2.43)

where all the items above are $2 \times 2$ matrices, and we denote by $H_x$ and $H_{xy}$ the derivative of $H$ with respect to $x$, and the second derivative of $H$ with respect to $x$ and then $y$ respectively for $x$ and $y \in \mathbf{R}$.

By (2.17) for $U_{ij}$ with $1 \leq i < j \leq n$ and $1 \leq l \leq n - 2$, we obtain

$$
\begin{align*}
\frac{\partial U_{ij}(z, w_1, \ldots, w_{n-2})}{\partial \zeta} &= -\frac{m m_j (a_{ix} - a_{jx})}{|(a_{ix} - a_{jx})z + \sum_{k=3}^{n} (b_{ik} - b_{jk})w_{k-2}|^3} \left[ (a_{ix} - a_{jx})z + \sum_{k=3}^{n} (b_{ik} - b_{jk})w_{k-2} \right], \\
\frac{\partial U_{ij}(z, w_1, \ldots, w_{n-2})}{\partial W_i} &= -\frac{m m_j (b_{ij} - b_{jx})}{|(a_{ix} - a_{jx})z + \sum_{k=3}^{n} (b_{ik} - b_{jk})w_{k-2}|^3} \left[ (a_{ix} - a_{jx})z + \sum_{k=3}^{n} (b_{ik} - b_{jk})w_{k-2} \right],
\end{align*}
$$

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\[
\frac{\partial^2 U_{ij}}{\partial z^2} (z, w_1, \ldots, w_{n-2}) = -\frac{m_{ij} (a_{ix} - a_{jx})^2}{(a_{ix} - a_{jx}) z + \sum_{k=3}^{n} (b_{ik} - b_{jk}) w_{k-2}} I \\
+ \frac{3m_{ij} (a_{ix} - a_{jx})^2}{(a_{ix} - a_{jx}) z + \sum_{k=3}^{n} (b_{ik} - b_{jk}) w_{k-2}} I \\
\cdot \left[ (a_{ix} - a_{jx}) z + \frac{n}{k=3} (b_{ik} - b_{jk}) w_{k-2} \right] \left[ (a_{ix} - a_{jx}) z + \frac{n}{k=3} (b_{ik} - b_{jk}) w_{k-2} \right]^T,
\]
\[
\frac{\partial^2 U_{ij}}{\partial w_l^2} (z, w_1, \ldots, w_{n-2}) = -\frac{m_{ij} (b_{il+2} - b_{jil+2})^2}{(a_{ix} - a_{jx}) z + \sum_{k=3}^{n} (b_{ik} - b_{jk}) w_{k-2}} I \\
+ \frac{3m_{ij} (b_{il+2} - b_{jil+2})^2}{(a_{ix} - a_{jx}) z + \sum_{k=3}^{n} (b_{ik} - b_{jk}) w_{k-2}} I \\
\cdot \left[ (a_{ix} - a_{jx}) z + \frac{n}{k=3} (b_{ik} - b_{jk}) w_{k-2} \right] \left[ (a_{ix} - a_{jx}) z + \frac{n}{k=3} (b_{ik} - b_{jk}) w_{k-2} \right]^T.
\]

Set
\[
K = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Now evaluating the corresponding functions at the special solution \((0, \sigma, \ldots, 0, 0, \sigma, 0, \ldots, 0, 0)^T \in \mathbb{R}^{4(n-1)}\) of \((\mathbf{2.38})\) with \(z = (\sigma, 0)^T, w_l = (0, 0)^T\) for \(1 \leq l \leq n - 2\), and summing them up, we obtain
\[
\frac{\partial^2 U}{\partial z^2} \bigg|_{z_0} = \sum_{1 \leq i < j \leq n} \frac{\partial^2 U_{ij}}{\partial z^2} \bigg|_{z_0} \\
= \sum_{1 \leq i < j \leq n} \left( -\frac{m_{ij} (a_{ix} - a_{jx})^2}{(a_{ix} - a_{jx}) \sigma^3} I + \frac{3m_{ij} (a_{ix} - a_{jx})^2}{(a_{ix} - a_{jx}) \sigma^5} (a_{ix} - a_{jx})^2 \sigma^2 K_1 \right) \\
= \frac{1}{\sigma^3} \sum_{1 \leq i < j \leq 4} m_{ij} \frac{m_{ij}}{|a_{ix} - a_{jx}|} K \\
= \frac{\mu}{\sigma^3} K, \tag{2.44}
\]
\[
\frac{\partial^2 U}{\partial w_l^2} \bigg|_{z_0} = \sum_{1 \leq i < j \leq n} \frac{\partial^2 U_{ij}}{\partial w_l^2} \bigg|_{z_0} \\
= \sum_{1 \leq i < j \leq n} \left( -\frac{m_{ij} (b_{il+2} - b_{jil+2})^2}{(a_{ix} - a_{jx}) \sigma^3} I + \frac{3m_{ij} (b_{il+2} - b_{jil+2})^2}{(a_{ix} - a_{jx}) \sigma^5} (a_{ix} - a_{jx})^2 \sigma^2 K_1 \right).
\]
follows from the definition (2.8). Similarly, we have

\[
= \frac{1}{\sigma^3} \left( \sum_{1 \leq i < j \leq n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2})^2}{|a_{ix} - a_{jx}|^3} \right) K
\]

\[
= \frac{1}{\sigma^3} \left( \sum_{1 \leq i < j \leq n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) b_{i,j+2}}{|a_{ix} - a_{jx}|^3} - \sum_{1 \leq i < j \leq n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) b_{j,i+2}}{|a_{ix} - a_{jx}|^3} \right) K
\]

\[
= \frac{1}{\sigma^3} \left( \sum_{1 \leq i < j \leq n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) b_{i,j+2}}{|a_{ix} - a_{jx}|^3} + \sum_{1 \leq i < j \leq n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) b_{j,i+2}}{|a_{ix} - a_{jx}|^3} \right) K
\]

\[
= \frac{1}{\sigma^3} \left( \sum_{i=1}^{n} b_{i,i+2} \sum_{j=1, j \neq i}^{n} \frac{m_i m_j (b_{i,j+2} - b_{j,i+2})}{|a_{ix} - a_{jx}|^3} \right) K
\]

\[
= \frac{1}{\sigma^3} \left( \sum_{i=1}^{n} b_{i,i+2} F_{i,i+2} \right) K
\]

\[
= \frac{\mu (1 + \beta_{0})}{\sigma^3} K,
\]

(2.45)

where the last equality of the first formula follows from (1.8), and the last equality of the second formula follows from the definition (2.8). Similarly, we have

\[
\frac{\partial^2 U}{\partial z \partial w_i} |_{z_0} = \sum_{1 \leq i < j \leq 4} \frac{\partial^2 U_{ij}}{\partial z \partial w_i} |_{z_0}
\]

\[
= \sum_{1 \leq i < j \leq 4} \left( \frac{m_i m_j (a_{ix} - a_{jx}) (b_{i,j+2} - b_{j,i+2})}{|a_{ix} - a_{jx}|^3 \sigma^3} + \frac{3 m_i m_j (a_{ix} - a_{jx}) (b_{i,j+2} - b_{j,i+2}) (a_{ix} - a_{jx})^2 \sigma^2 K_i}{|a_{ix} - a_{jx}|^5} \right)
\]

\[
= \sum_{1 \leq i < j \leq n} \left( \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) \cdot \text{sign}(a_{ix} - a_{jx})}{|a_{ix} - a_{jx}|^2} \right) K \frac{1}{\sigma^3}
\]

\[
= \sum_{1 \leq i < j \leq n} \left( \frac{m_i m_j (b_{i,j+2} - b_{j,i+2}) \cdot (-1)}{|a_{ix} - a_{jx}|^2} \right) K \frac{1}{\sigma^3}
\]

\[
= -\left( \sum_{i=1}^{n} b_{i,i+2} \sum_{j=i+1}^{n} \frac{m_i m_j}{|a_{ix} - a_{jx}|^2} + \sum_{i=1}^{n} b_{i,i+2} \sum_{j=1}^{i-1} \frac{m_i m_j}{|a_{ix} - a_{jx}|^2} \right) K \frac{1}{\sigma^3}
\]

\[
= -\left( \sum_{i=1}^{n} b_{i,i+2} \sum_{j=i+1}^{n} \frac{m_i m_j}{|a_{ix} - a_{jx}|^2} + \sum_{i=1}^{n} b_{i,i+2} \sum_{j=1}^{i-1} \frac{m_i m_j}{|a_{ix} - a_{jx}|^2} \right) K \frac{1}{\sigma^3}
\]

\[
= \left( \sum_{i=1}^{n} b_{i,i+2} \sum_{j=i+1}^{n} \frac{m_i m_j}{|a_{ix} - a_{jx}|^3} + \sum_{i=1}^{n} b_{i,i+2} \sum_{j=1}^{i-1} \frac{m_i m_j}{|a_{ix} - a_{jx}|^3} \right) K \frac{1}{\sigma^3}
\]

\[
= -\left( \sum_{i=1}^{n} b_{i,i+2} \sum_{j=1, j \neq i}^{n} \frac{m_i m_j (a_{ix} - a_{jx})}{|a_{ix} - a_{jx}|^3} \right) K \frac{1}{\sigma^3}
\]
\[
\frac{\partial^2 U}{\partial w_l \partial w_s} \bigg|_{\xi_0} = \sum_{1 \leq i < j \leq n} \frac{\partial^2 U_{ij}}{\partial w_l \partial w_s} \bigg|_{\xi_0} \\
= \sum_{1 \leq i < j \leq n} \left( - \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} + \frac{3m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^5} (a_{ix} - a_{jx})^2 \sigma^2 K_1 \right) \\
= 1 \left( \sum_{1 \leq i < j \leq n} \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} \right) K \\
= 1 \left( \sum_{1 \leq i < j \leq n} \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} - \sum_{1 \leq i < j \leq n} \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^5} (a_{ix} - a_{jx})^2 \sigma^2 K_1 \right) K \\
= 1 \left( \sum_{1 \leq i < j \leq n} \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} + \sum_{1 \leq j < i \leq n} \frac{m \cdot m_j (b_{i,l+2} - b_{j,l+2}) (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} \right) K \\
= \frac{1}{\sigma^3} \left( \sum_{i=1}^{n} b_{i,l+2} \sum_{j=1, j \neq i}^{n} \frac{m \cdot m_j (b_{i,s+2} - b_{j,s+2})}{|a_{ix} - a_{jx}|^3} \right) K \\
= \frac{1}{\sigma^3} \left( \sum_{i=1}^{n} b_{i,l+2} F_i (s+2) \right) K \\
= \frac{1}{\sigma^3} \left( \sum_{i=1}^{n} b_{i,l+2} (\mu - \lambda_s + 2) m_i b_{i,s+2} \right) K \\
= O,
\]

where in the third last equality, we used (2.6), and in the last equality of (2.47), we used (2.3) and (2.4). By (2.44), (2.45), (2.46) and (2.43), we have

\[
H_{zz} |_{\xi_0} = \frac{p - r}{p} I - \frac{r \mu}{l + \mu} K = I - \frac{r}{p} I - \frac{r \mu}{p l} K = I - \frac{r}{p} (I + K) = \left( \begin{array}{cc} \frac{2 - 2e \cos \theta}{1 + e \cos \theta} & 0 \\ \frac{2e \cos \theta}{1 + e \cos \theta} & 1 \end{array} \right),
\]

\[
H_{zwl} |_{\xi_0} = -\frac{p}{l} \frac{\partial^2 U}{\partial z \partial w_l} |_{\xi_0} = O,
\]

for \(1 \leq l \leq n - 2\),

\[
H_{zwl} |_{\xi_0} = -\frac{r}{l} \frac{\partial^2 U}{\partial \xi \partial w_l} |_{\xi_0} = O,
\]

for \(1 \leq l, s \leq n - 2\), \(l \neq s\),

\[
H_{zwl} |_{\xi_0} = \frac{p - r}{p} I - \frac{r (1 + \beta)}{l + \beta} \mu K = I - \frac{r}{p} I - \frac{r (1 + \beta)}{p l} \mu K
\]

\[
= I - \frac{r}{p} (I + (1 + \beta) K) = \left( \begin{array}{cc} \frac{2 - 2e \cos \theta}{1 + e \cos \theta} & 0 \\ \frac{2e \cos \theta}{1 + e \cos \theta} & 1 \end{array} \right),
\]

for \(1 \leq l \leq n - 2\).  
(2.48)

Thus the proof is complete.
Remark 2.4  (i) When we set $n = 3$ in Theorem 2.3, then $\beta_1$ is precisely the mass parameter $\beta$ defined by (1.4), and the corresponding linearized Hamiltonian system at the EEM $q_{m,e}(t)$ is given by

$$z' = J \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & \frac{1}{1+e}\cos(t) \end{pmatrix} z.$$  \hfill (2.49)

Note that this system was derived in [13] and [26] too.

(ii) The Hamiltonian equation of the $i$-th part of the other $(n-2)$ parts with $1 \leq i \leq n-2$ is given by

$$z' = J \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & \frac{1}{1+e}\cos(t) \end{pmatrix} z.$$  \hfill (2.50)

Also, $\beta_1$ coincides with $\beta_c$ in Table 2 of [13] when $\alpha = 1$.

Now we can give Proof of Theorem 1.2. Note that by Theorem 2.3 specially (2.39)-(2.41), we obtain that the matrix $H_{\theta}(\theta, \xi_0)$ together with the first identity matrix $I_2$ in the diagonal of the matrix $B(\theta)$ in (2.40) yield a 4-dimensional Hamiltonian system corresponding to the Kepler 2-body problem, and each matrix $H_{\theta}(\theta, \xi_0)$ together with the $(i+1)$-th identity matrix $I_2$ in the diagonal of the matrix $B(\theta)$ in (2.40) yield a 4-dimensional Hamiltonian system (2.50) with $\beta_i$ given by (1.14), which corresponds to the linear system (2.49) of the Euler 3-body problem with $\beta$ replaced by $\beta_i$ for $1 \leq i \leq n-2$. Therefore Theorem 1.2 holds. \hfill \square

3  A collinear 4-body problem with two small masses in the middle

We now consider the linear stability of special collinear central configurations in the four body problem with two small masses in the middle. A typical example is the EEM orbit of the 4-bodies, the Earth, the Moon and two space stations in the middle as mentioned at the beginning of this paper with $n = 4$. We try to give an analytical way following which one can numerically find out the best elliptic-hyperbolic positions for the two space stations using results in [13] and [14]. Specially, for the four masses we fix $m_1 = m \in (0, 1)$, and let $m_2 = \epsilon, m_3 = \tau \epsilon, m_4 = 1 - m - (\tau + 1)\epsilon$ with $\tau > 0$ and $0 < \epsilon < \frac{1}{\tau+1}$. They satisfy

$$m_1 + m_2 + m_3 + m_4 = 1.$$  \hfill (3.1)

Suppose $q_1, q_2, q_3$ and $q_4$ are four points on the $x$-axis in $\mathbb{R}^2$, and form a central configuration. Using notations similar to those in [26], we set

$$q_1 = (0, 0)^T, \quad q_2 = (\alpha x, 0)^T, \quad q_3 = (\alpha y, 0)^T, \quad q_4 = (\alpha, 0)^T,$$  \hfill (3.2)

where $\alpha = \alpha_{x, y} = |q_4 - q_1|, x = x_{, y, \tau}, y = y_{, y, \tau}$ satisfy $0 < x < y < 1$. Then the center of mass of the four particles is

$$q_e = m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4$$
$$= (m_2 x + m_3 y + m_4 \alpha, 0)^T$$
$$= ((1 - m) - (1 + \tau - x - \tau y)\epsilon)\alpha, 0)^T,$$  \hfill (3.3)

where (3.1) is used to get the last equality.
Next we study properties of this central configuration.

Then we have

$$\begin{align*}
a_{1x} &= [m - 1 + (1 + \tau - x - \tau y)e]a, \\
a_{3x} &= [m + y - 1 + (1 + \tau - x - \tau y)e]a,
\end{align*}$$

and

$$a_{iy} = 0, \quad \text{for } i = 1, 2, 3, 4. \quad (3.6)$$

For $i = 1, 2, 3$ and 4, let $a_i = q_i - q_c$, and denote by $a_{ix}$ and $a_{iy}$ the $x$ and $y$-coordinates of $a_i$ respectively. Then we have

$$\begin{align*}
a_{1x} &= (m - 1 + (1 + \tau - x - \tau y)e)a, \\
a_{2x} &= (m + x - 1 + (1 + \tau - x - \tau y)e)a,
\end{align*}$$

and

$$\begin{align*}
a_{3x} &= [m + y - 1 + (1 + \tau - x - \tau y)e]a, \\
a_{4x} &= [m + (1 + \tau - x - \tau y)e]a,
\end{align*}$$

and

$$a_{iy} = 0, \quad \text{for } i = 1, 2, 3, 4. \quad (3.6)$$

Next we study properties of this central configuration.

**Step 1. Computations on $a$ and $x, y$.**

Scaling $a$ by setting $\sum_{i=1}^{4} |a_i|^2 = 1$, we obtain

$$\frac{1}{\alpha^2} = \frac{\sum_{i=1}^{4} |a_i|^2}{\alpha^2} = m[m - 1 + (1 + \tau - x - \tau y)e]^2 + e[m + x - 1 + (1 + \tau - x - \tau y)e]^2$$

$$+ \tau e[m + y - 1 + (1 + \tau - x - \tau y)e]^2 + (1 - m - (\tau + 1)e)[m + (1 + \tau - x - \tau y)e]^2$$

$$= m(1 - m^2 + 2m(1 - m)(x + \tau y - 1 - \tau)e + m(x + \tau y - 1 - \tau)^2 e^2$$

$$+ e[(x - 1)^2 + (m + (1 + \tau - x - \tau y)e)^2 + 2(x - 1)(m + (1 + \tau - x - \tau y)e)]$$

$$+ \tau e[(y - 1)^2 + (m + (1 + \tau - x - \tau y)e)^2 + 2(y - 1)(m + (1 + \tau - x - \tau y)e)]$$

$$+ (1 - m)[m^2 + 2m(1 + \tau - x - \tau y) + (1 + \tau - x - \tau y)^2 e^2] - (\tau + 1)e(m + (1 + \tau - x - \tau y)e)^2$$

$$= m(1 - m) + [(1 - x)^2 + \tau(1 - y)^2 + m^2(1 + \tau - 2m(1 + \tau - x - \tau y)]e$$

$$+ [2m^2(1 + \tau)(1 + \tau - x - \tau y) - (1 + \tau - x - \tau y)^2]e^2 + m(1 + \tau)(1 + \tau - x - \tau y)^2 e^3. \quad (3.7)$$

Moreover, let

$$\alpha_0 = \lim_{\epsilon \to 0} \alpha = [m(1 - m)]^{-\frac{1}{2}}, \quad (3.8)$$

and

$$q_c, 0 = \lim_{\epsilon \to 0} q_c = (1 - m)\alpha_0, \quad (3.9)$$

and hence

$$a_{1x, 0} = \lim_{\epsilon \to 0} a_{1x} = -(1 - m)\alpha_0, \quad (3.10)$$

$$a_{4x, 0} = \lim_{\epsilon \to 0} a_{4x} = m\alpha_0. \quad (3.11)$$

The potential $\mu$ is given by

$$\mu = \mu_{\epsilon, \tau} = \sum_{1 \leq i < j \leq 4} \frac{m_i m_j}{|a_i - a_j|}, \quad (3.12)$$

and by Lemma 3 of [8], we have

$$\mu_0 = \lim_{\epsilon \to 0} \mu = \frac{m(1 - m)}{\alpha_0^3} = \alpha_0^{-3}. \quad (3.13)$$

In the following, we will use the subscript 0 to denote the limit value of the parameters when $\epsilon \to 0$.

Motivated by Proposition 1 in [24], we have
Lemma 3.1 When $\epsilon \to 0$, $a_2$ and $a_3$ must converge to the same point $a^\ast$. Moreover, $a_{1,0}, a^\ast, a_{4,0}$ is the central configuration of the restricted 3-body problem with given masses $\tilde{m}_1 = m, \tilde{m}_2 = 0, \tilde{m}_3 = 1 - m$ which the small mass lies in the segment between the other two masses.

Proof. If $a_2$ and $a_3$ do not converge to the same point when $\epsilon \to 0$, there is a sequence $\{\epsilon_n\}_{n=0}^\infty$ convergent to 0 such that

$$a^\ast_{23x} = \lim_{n \to \infty} (a_{2x} - a_{3x}) \neq 0. \quad (3.14)$$

Up to a subsequence of $\{\epsilon_n\}_{n=0}^\infty$, and we denote it still by $\{\epsilon_n\}_{n=0}^\infty$, we have

$$\lim_{n \to \infty} a_{2x} = a^\ast_{2x}. \quad (3.15)$$

Then

$$\lim_{n \to \infty} a_{3x} = \lim_{n \to \infty} a_{2x} + \lim_{n \to \infty} (a_{2x} - a_{3x}) = a^\ast_{2x} + a^\ast_{23x}. \quad (3.16)$$

Because $a_{1,2,3,4}$ form a central configuration, for the two middle points we have

$$\frac{m_1(a_{2} - a_{1})}{|a_{2} - a_{1}|^3} + \frac{m_3(a_{2} - a_{3})}{|a_{2} - a_{3}|^3} + \frac{m_4(a_{2} - a_{4})}{|a_{2} - a_{4}|^3} = \mu a_{2}, \quad (3.17)$$

$$\frac{m_1(a_{3} - a_{1})}{|a_{3} - a_{1}|^3} + \frac{m_2(a_{3} - a_{2})}{|a_{3} - a_{2}|^3} + \frac{m_4(a_{3} - a_{4})}{|a_{3} - a_{4}|^3} = \mu a_{3}. \quad (3.18)$$

Let $\epsilon = \epsilon_n, n \in \mathbb{N}$, and $n \to \infty$, together with (3.5), (3.9), (3.10), (3.13) and (3.14), (3.17) and (3.18) become

$$\frac{m}{(a^\ast_{2x} - a_{1x,0})^2} - \frac{1 - m}{(a^\ast_{2x} - a_{4x,0})^2} = \mu_0 a^\ast_{2x}, \quad (3.19)$$

$$\frac{m}{(a^\ast_{2x} + a^\ast_{23x} - a_{1x,0})^2} - \frac{1 - m}{(a^\ast_{2x} + a^\ast_{23x} - a_{4x,0})^2} = \mu_0 (a^\ast_{2x} + a^\ast_{23x}). \quad (3.20)$$

We define

$$f(t) = \frac{m}{(t - a_{1x,0})^2} - \frac{1 - m}{(t - a_{4x,0})^2} - \mu_0 t, \quad \text{for } t \in (a_{1x,0}, a_{4x,0}). \quad (3.21)$$

Then $f$ is a strictly decreasing function satisfying

$$\lim_{t \to a_{1x,0}} f(t) = +\infty, \quad \text{and} \quad \lim_{t \to a_{4x,0}} f(t) = -\infty. \quad (3.22)$$

Thus there is a unique zero point of $f$ in $[a_{1x,0}, a_{4x,0}]$, which we denote by $a^\ast_{x}$. Here (3.22) yields $a_{1x,0} < a^\ast_{x} < a_{4x,0}$.

Now (3.19) and (3.20) yield two zero points $a^\ast_{2x}$ and $a^\ast_{2x} + a^\ast_{23x}$ of $f$ in $[a_{1x,0}, a_{4x,0}]$ respectively, which then yields a contradiction. Therefore, we have proved $\lim_{\epsilon \to 0} (a_{2x} - a_{3x}) = 0$.

Now we want to prove $\lim_{\epsilon \to 0} a_{2x} = a^\ast_{x}$. If not, there is a sequence $\{\tilde{\epsilon}_n\}_{n=0}^\infty$ converges to 0, such that $\lim_{n \to \infty} a_{2x} = \tilde{a}^\ast \neq a^\ast$. Then $\lim_{n \to \infty} a_{3x} = \tilde{a}^\ast$.

Now adding $\frac{m_2}{m_{2x} + m_3}$ times (3.17) to $\frac{m_3}{m_{2x} + m_3}$ times (3.18) yields

$$m_1 \left( \frac{1}{\tau + 1} \frac{a_{2} - a_{1}}{|a_{2} - a_{1}|^3} + \frac{1 - m}{|a_{3} - a_{1}|^3} \right) + m_4 \left( \frac{1}{\tau + 1} \frac{a_{2} - a_{4}}{|a_{2} - a_{4}|^3} + \frac{1 - m}{|a_{3} - a_{4}|^3} \right) = \mu_0 \frac{a_{2} + \tau a_{3}}{1 + \tau}, \quad (3.23)$$

Let $\epsilon = \tilde{\epsilon}_n, n \in \mathbb{N}$, and $n \to \infty$, together with (3.6), (3.10), (3.11) and (3.14), (3.22) becomes

$$\frac{m}{(\tilde{a}^\ast - a_{1x,0})^2} - \frac{1 - m}{(\tilde{a}^\ast - a_{4x,0})^2} = \mu_0 \tilde{a}^\ast. \quad (3.24)$$
then using also the property of unique zero point of $f(x)$, we obtain a contradiction. Thus we must have
\[ \lim_{x \to 0}a_{2x} = \lim_{y \to 0}a_{3y} = a_x^s. \]

By direct computations, we can check that $a_{10} = (a_{1x,0}, 0)^T$, $a_x^s = (a_x^s, 0)^T$ and $a_{40} = (a_{4x,0}, 0)^T$ form a collinear central configuration with given masses $\tilde{m}_1 = m$, $\tilde{m}_2 = 0$ and $\tilde{m}_3 = 1 - m$. The uniqueness is obtained by these three given ordered masses as in [20].

By Lemma 3.1, we can suppose
\[ \lim_{x \to 0} = \lim_{y \to 0} = x_0, \]
and hence
\[ a_{2x,0} = \lim_{x \to 0} a_{2x} = (m + x_0 - 1)\alpha_0, \]
\[ a_{3x,0} = \lim_{x \to 0} a_{3x} = (m + x_0 - 1)\alpha_0. \]

Note that $a_{10}, a_{20}$ and $a_{40}$ form a central configuration with given masses $\tilde{m}_1 = m$, $\tilde{m}_2 = 0$ and $\tilde{m}_3 = 1 - m$. Then $\tilde{x} = \frac{m}{\tau x_0}$ is the unique positive root of Euler’s quintic polynomial equation (cf. p. 276 of [25] and p. 29 of [11]):
\[ (1 - m)\tilde{x}^5 + (3 - 3m)\tilde{x}^4 + (3 - 3m)\tilde{x}^3 - 3m\tilde{x}^2 - 3m\tilde{x} - m = 0. \]

Thus $x_0$ satisfies:
\[ x_0^5 - (3 - m)x_0^4 + (3 - 2m)x_0^3 - mx_0^2 + 2mx_0 - m = 0. \]

Next we derive the equations satisfied by $x = x(\epsilon)$ and $y = y(\epsilon)$. Because $a_1, a_2, a_3$ and $a_4$ form a central configuration, we have
\[ \frac{\epsilon}{x^2a^2} + \frac{\tau\epsilon}{y^2a^2} + \frac{1 - m - (1 + \tau)\epsilon}{a^2} = \mu[1 - m - (1 + \tau - x - \tau y)\epsilon]\alpha, \]
\[ -\frac{m}{x^2a^2} + \frac{\tau\epsilon}{(y - x)^2a^2} + \frac{1 - m - (1 + \tau)\epsilon}{(1 - x)^2a^2} = \mu[1 - m - x - (1 + \tau - x - \tau y)\epsilon]\alpha, \]
\[ -\frac{m}{y^2a^2} - \frac{\epsilon}{(y - x)^2a^2} + \frac{1 - m - (1 + \tau)\epsilon}{(1 - y)^2a^2} = \mu[1 - m - y - (1 + \tau - x - \tau y)\epsilon]\alpha, \]
\[ -\frac{m}{a^2} - \frac{\epsilon}{(1 - x)^2a^2} - \frac{\tau\epsilon}{(1 - y)^2a^2} = \mu[m + (1 + \tau - x - \tau y)\epsilon]\alpha. \]

From (3.30) and (3.31), we have
\[
0 = \left[ \frac{\epsilon}{x^2} + \frac{\tau\epsilon}{y^2} + 1 - m - (1 + \tau)\epsilon \right] \left[ 1 - m - x - (1 + \tau - x - \tau y)\epsilon \right] \\
- \left[ -\frac{m}{x^2} + \frac{\tau\epsilon}{(y - x)^2} + \frac{1 - m - (1 + \tau)\epsilon}{(1 - x)^2} \right] \left[ 1 - m - (1 + \tau - x - \tau y)\epsilon \right] \\
+ \epsilon \left[ (1 - m)(1 + \tau - x - \tau y) + (1 - m - x)(\frac{1 + \tau}{x^2} + \frac{\tau}{y^2} - 1 - \tau) \right] \\
- (1 - m)(\frac{\tau}{(y - x)^2} - \frac{1 + \tau}{(1 - x)^2}) + \left( -\frac{m}{x^2} + \frac{1 - m}{(1 - x)^2} \right) (1 + \tau - x - \tau y) \\
+ \epsilon^2 \left[ \frac{1 + \tau}{x^2} - \tau - 1 - \tau)(1 + \tau - x - \tau y) + (\frac{\tau}{(y - x)^2} - \frac{1 + \tau}{(1 - x)^2})(1 + \tau - x - \tau y) \right] \\
= -(1 - m)\frac{x^5 - (3 - m)x^4 + (3 - 2m)x^3 - mx^2 + 2mx - m}{x^2(1 - x)^2}.
\]
Therefore,

\[ \text{Step 2.} \]

Similarly, from (3.30) and (3.32), we have

\[
+ \epsilon \left[ \frac{m - 1 - \frac{m}{x^2} + \frac{1 - m}{(1 - x)^2}}{y^2} \right] (1 + \tau - x - \tau y) + (1 - m - x)(\frac{1}{x^2} + \frac{\tau}{y^2} - 1 - \tau) \\
-(1 - m)\left( \frac{\tau}{(y - x)^2} - \frac{1 + \tau}{(1 - x)^2} \right) + \epsilon^2 \left[ \frac{1}{x^2} + \frac{\tau}{y^2} \right] (1 + \tau - x - \tau y) - \frac{1}{y^2} - 1 - \tau \right] (1 + \tau - x - \tau y). 
\]

(3.34)

We denote the right hand side of (3.34) by \( g_\epsilon(x, y) \), then \( x^2(1 - x)^2 y^2(1 - y)^2 g_\epsilon(x, y) \) is a binary polynomial in \( x, y \). Similarly, from (3.30) and (3.32), we have

\[
h_\epsilon(x, y) = 0, 
\]

(3.35)

where

\[
h_\epsilon(x, y) = - (1 - m)^2 \frac{y^2 - (3 - m)y^4 + (3 - 2m)y^3 - my^2 + 2my - m}{y^2(1 - y)^2} \\
+ \epsilon \left[ (m - 1 - \frac{m}{y^2} + \frac{1 - m}{(1 - y)^2})(1 + \tau - x - \tau y) + (1 - m - y)(\frac{1}{x^2} + \frac{\tau}{y^2} - 1 - \tau) \\
+(1 - m)\left( \frac{1}{(y - x)^2} + \frac{1 + \tau}{(1 - y)^2} \right) + \epsilon^2 \left[ \frac{1}{x^2} + \frac{\tau}{y^2} - \frac{1}{(y - x)^2} - \frac{1 + \tau}{(1 - y)^2} \right] (1 + \tau - x - \tau y). 
\]

(3.36)

Therefore, \( x \) and \( y \) can be solved out from \( g_\epsilon(x, y) = 0 \) and \( h_\epsilon(x, y) = 0 \).

Now by the first conclusion of Lemma 3.1, letting \( \epsilon \to 0 \) in the equations \( g_\epsilon(x, y) = 0 \) and \( h_\epsilon(x, y) = 0 \), we obtain

\[
\lim_{\epsilon \to 0} x^2(1 - x)^2 y^2(1 - y)^2 g_\epsilon(x, y) \\
= -(1 - m)^2 y^2(1 - y)^2 (3 - m)x^4 + (3 - 2m)x^3 - mx^2 + 2mx - m, 
\]

(3.37)

\[
\lim_{\epsilon \to 0} y^2(1 - y)^2 x^2(1 - x)^2 h_\epsilon(x, y) \\
= -(1 - m)x^2(1 - x)^2 y^2(1 - y)^2 (3 - 2m)y^3 - my^2 + 2my - m. 
\]

(3.38)

Here in (3.37) and (3.38) we have the same polynomial again as that in the left hand side of (3.29).

**Step 2. Computations on \( \beta_i \).**

Now in our case, \( D \) is given by

\[
D = \left( \begin{array}{c}
\mu - \frac{1}{a^2} \frac{m}{x^2a^2}, \\
\frac{m}{x^2a^2}, \\
\frac{m}{(y - x)^2a^2}, \\
\frac{m}{a^2}, \\
\mu - \frac{1}{a^2} \frac{m}{x^2a^2}, \\
\mu - \frac{1}{a^2} \frac{m}{(y - x)^2a^2}, \\
\mu - \frac{1}{a^2} \frac{m}{a^2}, \\
\mu - \frac{1}{a^2} \frac{m}{a^2} + \frac{e}{(1 - x)^2a^2}, \frac{\tau e}{(1 - x)^2a^2}, \frac{1 - m - (1 + \tau)e}{a^2}, \frac{1 - m - (1 + \tau)e}{a^2}
\end{array} \right).
\]

(3.39)

Recall that the other two eigenvalues of \( D \) are \( \lambda_3 \) and \( \lambda_4 \), then we have

\[
det(D - \lambda_4 I_4) = -\lambda(\mu - \lambda)(\lambda_3 - \lambda)(\lambda_4 - \lambda) \\
= \lambda^4 - (\mu + \lambda_3 + \lambda_4)\lambda^3 + (\lambda_3\lambda_4 + \mu(\lambda_3 + \lambda_4))\lambda^2 - \lambda_3\lambda_4\mu \lambda 
\]

(3.40)

On the other hand

\[
det(D - \lambda_4 I_4) = \lambda^4 - (trD)\lambda^3 + \left( \sum_{i,j=1,i<j}^4 \det E_{ij} \right)\lambda^2 + \ldots, 
\]

(3.41)
where $E_{ij}$ is the principal minor when deleting all the rows and columns except for $i$ and $j$. Then we have

$$\mu + \lambda_3 + \lambda_4 = trD$$

$$= \sum_{i=1}^{4} D_{ji}$$

$$= \sum_{i=1}^{4} \left( \mu - \sum_{j=1, j\neq i}^{4} D_{ij} \right)$$

$$= 4\mu - \sum_{i,j=1, i\neq j}^{4} D_{ij}$$

$$= 4\mu - \frac{1}{\alpha^3} \left[ \frac{m + \epsilon}{x^3} + \frac{m + \tau\epsilon}{y^3} + 1 - (1 + \tau)\epsilon + \frac{(1 + \tau)\epsilon}{(y-x)^3} + \frac{1 - m - \tau\epsilon}{(1-x)^3} + \frac{1 - m - \epsilon}{(1-y)^3} \right],$$

(3.42)

$$\lambda_3\lambda_4 + \mu(\lambda_3 + \lambda_4) = \sum_{i,j=1, i<j}^{4} \det E_{ij}$$

$$= \sum_{i,j=1, i<j}^{4} \left( D_{ii}D_{jj} - D_{ij}D_{ji} \right)$$

$$= \sum_{i,j=1, i<j}^{4} D_{ii}D_{jj} - \sum_{i,j=1, i<j}^{4} D_{ij}D_{ji}$$

$$= \frac{\left( \sum_{i=1}^{4} D_{ij} \right)^2 - \sum_{i=1}^{4} D_{ii}^2}{2} - \sum_{i,j=1, i<j}^{4} D_{ij}D_{ji}$$

$$= \frac{1}{2}(trD)^2 - \frac{1}{2} \sum_{i=1}^{4} \left( \mu - \sum_{s=1, s\neq i}^{4} D_{is} \right)^2 - \sum_{i,j=1, i<j}^{4} D_{ij}D_{ji}$$

$$= \frac{1}{2}(trD)^2 - 2\mu^2 - \mu \sum_{i,j=1, i\neq j}^{4} D_{ij}^2 - \frac{1}{2} \sum_{i=1}^{4} \left( \sum_{s=1, s\neq i}^{4} D_{is}^2 \right) - \sum_{i,j=1, i<j}^{4} D_{ij}D_{ji}$$

(3.43)

Let

$$\delta = \frac{1}{2\mu} \sum_{i,j=1, i\neq j}^{4} D_{ij}$$

$$= \frac{1}{2\mu\alpha^3} \left[ \frac{m + \epsilon}{x^3} + \frac{m + \tau\epsilon}{y^3} + 1 - (1 + \tau)\epsilon + \frac{(1 + \tau)\epsilon}{(y-x)^3} + \frac{1 - m - \tau\epsilon}{(1-x)^3} + \frac{1 - m - \epsilon}{(1-y)^3} \right],$$

(3.44)

then we have

$$\lambda_3 + \lambda_4 = trD - \mu = 4\mu - 2\delta\mu - \mu = -(2\delta - 3)\mu,$$

(3.45)

$$\lambda_3\lambda_4 = -\mu(\lambda_3 + \lambda_4) + \frac{1}{2}(trD)^2 - 2\mu^2 + \mu \sum_{i,j=1, i\neq j}^{4} D_{ij} - \frac{1}{2} \sum_{i=1}^{4} \left( \sum_{s=1, s\neq i}^{4} D_{is} \right)^2 - \sum_{i,j=1, i<j}^{4} D_{ij}D_{ji}$$

$$= (2\delta - 3)\mu^2 + \frac{1}{2}(4 - 2\delta)^2\mu^2 - 2\mu^2 + 2\delta\mu^2 - \frac{1}{2\alpha^3} \left[ \frac{\epsilon}{x^3} + \frac{\tau\epsilon}{y^3} + 1 - m - (1 + \tau)\epsilon \right]^2.$$
Therefore, we obtain the stability pattern of our four body problem.

\[
\begin{align*}
\beta &= \beta_0, \text{ and } \tilde{\beta} \\
\text{Step 3.} \text{ Computations on the limit case.} \\
\text{We need to compute the mass parameter of the restricted three-body problem of given masses } m_1 = m, \quad m_2 = 0, \quad \text{and } m_3 = 1 - m. \text{ By (A.3) of [14], } \beta \text{ (they use } \beta_1 \text{ there) is given by}
\end{align*}
\]
\[
\beta_c = -1 + \frac{\alpha}{a^{\alpha+2}[1 + (\rho + 1)^2]} \left[ (\rho + 2)(\rho + 1)m_1 + m_2 + (\rho + 1)(m_2 + m_3(\rho + 1)) + \frac{m_3 - m_4\rho}{(\rho + 1)^{\alpha+2}} \right].
\]
where $\rho$ and $\alpha$ is given by (A.2) of [14].

Note that, when in our case $\alpha = 1$, (A.2) of [14] is just the Euler’s quintic equation, then together with $\rho = \frac{x_0}{1-x_0}$ of (3.28), we have

$$\beta = -1 + \frac{(\rho + 1)^3}{1 + (\rho + 1)^2} \left[ (\rho + 2) \frac{(\rho + 1)m}{\rho^3} + (1-m)(\rho + 1)^2 + \frac{(1-m - m\rho)}{(\rho + 1)^3} \right]$$

$$= -1 + \left( \frac{x_0}{1-x_0} \right)^3 \left[ \left( \frac{x_0}{1-x_0} + 2 \right) \frac{x_0}{(1-x_0)^3} + (1-m)\left( \frac{x_0}{1-x_0} \right)^2 + \frac{(1-m - m\rho)}{(1-x_0)^3} \right]$$

$$= -1 + \frac{1}{(1-x_0)(1-x_0)^2 + 1} \left[ \frac{m(2-x_0)(1-x_0)}{x_0^3} + (1-m) \frac{1}{(1-x_0)^2} + (1-x_0)^2(1-x_0 - m) \right]$$

$$= -1 + \frac{1}{(1-x_0)(1-x_0)^2 + 1} \left[ \frac{m(2-x_0)(1-x_0)}{x_0^3} - m \frac{(1-x_0)^2}{x_0^2} \right]$$

$$= -1 + \frac{1}{(1-x_0)(1-x_0)^2 + 1} \left[ \frac{(1-m)\frac{1}{(1-x_0)^2}}{(1-x_0)^2 + 1} + (1-m) \right]$$

$$= -1 + \frac{1}{(1-x_0)(1-x_0)^2 + 1} \left[ \frac{(1-m)\frac{1}{(1-x_0)^2}}{(1-x_0)^2 + 1} - (1-m) + (1-x_0)^2(1-x_0 - m) \right]$$

$$= -1 + \frac{1}{(1-x_0)(1-x_0)^2 + 1} \left[ \frac{x_0^5 + (3-m)x_0^4 - (3-2m)x_0^3 + m^2 - 2mx_0 + m}{x_0^2} \right]$$

$$= -1 + \frac{1}{(1-x_0)^2 + 1} - \frac{m}{x_0^3} + \frac{1}{(1-x_0)^2 + 1} \left[ \frac{1}{x_0^3} \right] \left( 3.51 \right)$$

where in the last equality, we used (3.29).

Following pp.171 in [24], for $q = (q_1, q_2)^T \in \mathbb{R}^2$, we define

$$V_2(q) = \frac{m}{|a_{1,0} - q_1|} + \frac{1 - m}{|a_{4,0} - q_1|} + \frac{1}{2} \alpha_0^{-3} |q|^2 \left( 3.52 \right)$$

where $\alpha_0^{-3}$ is an extra parameter because Z. Xia fixed $\lambda = 1$ of (1) in [24], but here we have $\lambda = \alpha_0^{-3}$. Then we have

$$\frac{\partial^2 V_2}{\partial^2 q_x} = -\frac{m}{|a_{1,0} - q_1|^3} - \frac{1 - m}{|a_{4,0} - q_1|^3} + \frac{1}{\alpha_0^3} + 3 \left[ \frac{m(1-m)\alpha_0 - q_x)\alpha_0^3}{|a_{1,0} - q_1|^5} + \frac{(1-m)(m\alpha_0 - q_y)\alpha_0^3}{|a_{4,0} - q_1|^5} \right] \left( 3.53 \right)$$

$$\frac{\partial^2 V_2}{\partial q_x \partial q_y} = -3 \left[ \frac{m(1-m)\alpha_0 - q_x)\alpha_0^3}{|a_{1,0} - q_1|^5} + \frac{(1-m)(m\alpha_0 - q_y)\alpha_0^3}{|a_{4,0} - q_1|^5} \right] \left( 3.54 \right)$$

$$\frac{\partial^2 V_2}{\partial^2 q_y} = -\frac{m}{|a_{1,0} - q_1|^3} - \frac{1 - m}{|a_{4,0} - q_1|^3} + \frac{1}{\alpha_0^3} + 3 \left[ \frac{m\alpha_0^2}{|a_{1,0} - q_1|^5} + \frac{(1-m)q_y^2}{|a_{4,0} - q_1|^5} \right] \left( 3.55 \right)$$

Therefore

$$\left. \frac{\partial^2 V_2}{\partial^2 q_x} \right|_{q=(x_0\alpha_0, 0)^T} = -\frac{m}{x_0^3}\alpha_0^3 - \frac{1 - m}{(1-x_0)^3}\alpha_0^3 + \frac{1}{\alpha_0^3} + 3 \left[ \frac{m\alpha_0^2}{x_0^5}\alpha_0^3 + \frac{(1-m)(1-x_0)^3}{(1-x_0)^5}\alpha_0^3 \right]$$

$$= \frac{2}{\alpha_0^3} \left[ \frac{m}{x_0^3}\alpha_0^3 + \frac{1 - m}{(1-x_0)^3}\alpha_0^3 \right] + \frac{1}{\alpha_0^3} \left( 3.56 \right)$$
\[ \frac{\partial^2 V_2}{\partial q_x \partial q_y} \big|_{q=(x_0,0)^T} = 0, \]

\[ \frac{\partial^2 V_2}{\partial^2 q} \big|_{q=(x_0,0)^T} = \begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}, \quad \text{(3.57)} \]

and hence

\[ D^2 V_2(q) \big|_{q=(x_0,0)^T} = \begin{pmatrix} (2\beta + 3)\alpha_0^{-3} & 0 \\ 0 & -(2\beta + 3)\alpha_0^{-3} \end{pmatrix}. \]

By the Case (ii) in p.173 of [24], we have

\[ \lim_{\epsilon \to 0} \frac{a_3 - a_2}{(m_2 + m_3)^\frac{1}{2}} = \lim_{\epsilon \to 0} r_2' = \pm [(2\beta + 3)\alpha_0^{-3}]^{-\frac{1}{2}}, \]

and hence

\[ \lim_{\epsilon \to 0} \frac{m_2}{|a_2 - a_3|^3} = \frac{1}{1 + \tau} \lim_{\epsilon \to 0} \frac{m_2 + m_3}{|a_2 - a_3|^3} = \frac{(2\beta + 3)\alpha_0^{-3}}{1 + \tau}, \quad \text{(3.61)} \]

\[ \lim_{\epsilon \to 0} \frac{m_3}{|a_2 - a_3|^3} = \frac{\tau}{1 + \tau} \lim_{\epsilon \to 0} \frac{m_2 + m_3}{|a_2 - a_3|^3} = \frac{\tau(2\beta + 3)\alpha_0^{-3}}{1 + \tau}. \]

Note that \( m_2 = \epsilon, m_3 = \tau \epsilon \) and \( \lim_{\epsilon \to 0} |a_i - a_j| \neq 0 \) if \( i < j, (i, j) \neq (2, 3) \), from (3.39), we have

\[ D_0 = \lim_{\epsilon \to 0} D = \begin{pmatrix} \frac{m m_0}{\tau_0} & 0 & (1 - m)\mu_0 \\ \frac{m \mu_0}{\tau_0} & \frac{2\beta + 3}{1 + \tau} & \frac{2\beta + 3}{1 + \tau} \mu_0 \\ \frac{m \mu_0}{\tau_0} & 0 & \frac{2\beta + 3}{1 + \tau} \mu_0 \end{pmatrix}, \]

where we have used (3.13), (3.51), (3.61) and (3.62). Then the characteristic polynomial of \( D_0 \) is given by

\[ \det(D_0 - \lambda I) = -\lambda(\mu_0 - \lambda) \left| \begin{array}{ccc} \beta & \frac{2\beta + 3}{1 + \tau} & \frac{2\beta + 3}{1 + \tau} \\ \frac{\mu_0}{\tau_0} & \frac{2\beta + 3}{1 + \tau} & \frac{2\beta + 3}{1 + \tau} \\ \frac{\mu_0}{\tau_0} & 0 & \frac{2\beta + 3}{1 + \tau} \end{array} \right| \]

\[ = \lambda(\lambda - \mu_0)(\lambda + \beta\mu_0)(\lambda + 3(\beta + 1)\mu_0). \]

Then all eigenvalues of \( D_0 \) are given by

\[ \lambda_{1,0} = \mu_0, \quad \lambda_{2,0} = 0, \quad \lambda_{3,0} = -\beta\mu_0, \quad \lambda_{4,0} = -3(\beta + 1)\mu_0, \]

and hence by (2.4), we have

\[ \beta_{1,0} = -\frac{\lambda_{1,0}}{\mu_0} = \beta, \]

\[ \beta_{2,0} = -\frac{\lambda_{4,0}}{\mu_0} = 3(\beta + 1). \]

From (3.65), the four eigenvalues of \( D_0 \) are different, then for \( \epsilon > 0 \) small enough, we have

\[ \lim_{\epsilon \to 0} \lambda_i = \lambda_{i,0}, \quad 1 \leq i \leq 4. \]

Thus, we also have

\[ \lim_{\epsilon \to 0} \beta_i = \beta_{i,0}, \quad 1 \leq i \leq 2. \]
Therefore, the linear stability problem of the limiting case of our four-body problem when letting \(e \to 0\) is reduced to the linear stability problems of two restricted three-body problems, for which one has mass parameter \(\beta\), and the other has mass parameter \(3(\beta + 1)\). Then the numerical results obtained by R. Martínez, A. Samà and C. Simó in [13] and [14] can be used to obtain the linear stability pattern of the limiting case of our four-body problem. We will compute a concrete example at the end of this paper.

**Example 3.2 Computations on the actual case of the Earth-Moon-two space stations system.**

We denote by ESSM system the short hand notation for the Earth-two space stations-Moon system. From https://en.wikipedia.org/wiki/Earth and https://en.wikipedia.org/wiki/Moon, one can find that the mass of Earth is \(E = 5.97237 \times 10^{24}\) kg, the mass of the Moon is \(M = 7.342 \times 10^{22}\) kg, the distance between the Earth and the Moon is \(d = 384405\) km, and the actual eccentricity of the orbit of Moon is \(e \approx 0.0549\). This eccentricity is viewed as that of the orbits in the ESSM system.

By the normalization of the masses, we have

\[
m = \frac{E}{E + M} \approx 0.9879.
\]

(3.70)

For two space stations in the line segment between the Earth and the Moon, as their masses tends to 0 their limit position \(x_0\) given by (3.25) is determined by (3.29) and \(m\). When \(m\) is given by (3.70), by a numerical computation, we have

\[
x_0 \approx 0.8493
\]

(3.71)

By the distance between the Earth and the Moon, the distance between the limit position of the two space stations and the Moon is \(d_{SM} = d \times (1 - x_0) \approx 57930\) km.

Via (3.51), the constant \(\beta\) for the EEM of the 3-body problem is given by

\[
\beta = -1 + \frac{m}{x_0^3} + \frac{1 - m}{(1 - x_0)^3} \approx 4.1481.
\]

(3.72)

Thus the linear stability property of the ESSM system is determined by the eccentricity \(e \approx 0.0549\) of their orbits and the following two mass parameters:

\[
\beta_1 = \beta \approx 4.1481, \quad \beta_2 = 3(\beta + 1) \approx 15.4442.
\]

(3.73)

On the other hand, by (1.5)-(1.8) of (25), we have

\[
\hat{\beta}_2 \approx 2.7122, \quad \hat{\beta}_2^{4} \approx 4.9437, \quad \hat{\beta}_4 \approx 14.6764, \quad \hat{\beta}_2 \approx 18.9243,
\]

(3.74)

where \(\hat{\beta}_n\) and \(\hat{\beta}_{n+1}, n \in \mathbb{N}\) are the parameter values when the resonances of the linearized system appear. Indeed, \(\hat{\beta}_n\) is the \(n\)-th value such that \(\gamma_{\beta,0}(2\pi)\) has eigenvalue 1, and \(\hat{\beta}_{n+1}\) is the \(n\)-th value such that \(\gamma_{\beta,0}(2\pi)\) has eigenvalue \(-1\). Here \(\gamma_{\beta,0}(2\pi)\) is the end matrix at time \(t = 2\pi\) of the fundamental solution of the linearized Hamiltonian system (2.49) at the Euler solution EEM \(q_{m,e}\) with \(e = 0\) of the 3-body problem. Hence in our case,

\[
\hat{\beta}_2 < \beta_1 < \hat{\beta}_2^{4}, \quad \hat{\beta}_4 < \beta_2 < \hat{\beta}_2.
\]

(3.75)

Since the eccentricity \(e \approx 0.0549\) is very small, numerical computations show that the linear stability property is the same as that of \(e = 0\). Then by Theorem 1.5 of (26), the linear stability pattern of the ESSM system is

\[
R(\theta_1) \circ D(2) \circ R(\theta_2) \circ D(2)
\]

(3.76)

for some \(\theta_1\) and \(\theta_2 \in (0, \pi)\). Here for \(\theta \in \mathbb{R}\) and \(\lambda \in \mathbb{R} \setminus \{0, \pm 1\}\) we denote the elliptic and hyperbolic matrices by

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\]

respectively.
4 Appendix: A sketch of the proof of Lemma 1.1

For reader’s conveniences, following [17] of R. Moeckel (cf. also [21], [18]), next we sketch the ideas of the proof of Lemma 1.1 due to C. Conley.

A sketch of the proof of Lemma 1.1. Note first that both the matrices $D$ in (1.12) and $\dot{D}$ in (2.1) possess the same eigenvalues by the definition of $\dot{D}$. Because $\dot{D}$ is symmetric, all its eigenvalues are real, and then so does $D$, although it may not be symmetric in general.

Note that $2B(a) = U''(a)$ is the Hessian of $U(q)$ at the collinear central configuration $q = a$, and $U''(a)+U(a)\dot{M}$ is the Hessian of $U|_S$ with $S$ being the hypersurface determined by (1.7). By the homogeneity of $U$, we obtain that $D$ has the first eigenvalue $\lambda_1 = \mu = U(a)$ with the eigenvector $v_1 = (1, 1, \ldots, 1)^T$, i.e., $(Dv_1)_i = \mu$ holds for $1 \leq i \leq n$.

From the definition (2.1) of $a$ as a central configuration, we obtain that $D$ has the second eigenvalue $\lambda_2 = 0$ with the eigenvector $v_2 = (a_{1x}, a_{2x}, \ldots, a_{nx})^T$. More precisely for $1 \leq i \leq n$ by (2.1) we have

$$(Dv_2)_i = (\mu - \sum_{j=1, j \neq i}^{n} \frac{m_j}{|a_i - a_j|^3}) a_{ix} + \sum_{j=1, j \neq i}^{n} \frac{m_j a_{jx} - a_{ix}}{|a_j - a_i|^3} = 0.$$ 

Note that by (1.6)-(1.7), the vectors $v_1$ and $v_2$ form an $\tilde{M}$-orthonormal sub-basis, i.e., they satisfy $v_1^T \tilde{M} v_1 = 1$, $v_1^T \tilde{M} v_2 = 0$, and $v_2^T \tilde{M} v_2 = 1$. Denote all the other eigenvalues of $D$ by $\lambda_3, \ldots, \lambda_n$. Next goal is to show that the other $(n-2)$ eigenvalues of $D$ are non-positive.

Following [17], this is equivalent to showing that all the eigenvalues of $D$ are non-positive when we restricted to the subspace spanned by vectors orthogonal to $\tilde{M} v_1$, and observing that this is equivalent to showing that in the flow on the space of lines through the origin determined by the following linear system on $u$,

$$\dot{u} = M^{-1} B(a) u, \quad (4.1)$$

the line determined by $v_2$ is an attractor.

Let

$$K = \left\{ u = (u_1, u_2, \ldots, u_n)^T \left| \sum_{i=1}^{n} m_i u_i = 0, \ u_1 \leq u_2 \leq \ldots \leq u_n \right. \right\}.$$

Then for any $u \in K$, we have $u \perp \tilde{M} v_1$. Moreover, we have $rv_2 \in K$ for any $r \in \mathbb{R}$. We will show that, around the line in $K$ which is carried strictly inside itself by the flow defined by (4.1) except for the origin.

Note that the boundary $\partial K$ of $K$ consists of points where one or more equalities hold. However, except for the origin, at least one strict inequality must hold, otherwise $u = k(1, 1, \ldots, 1)^T \in K$ and hence $k = 0$. Consider a boundary point with

$$u_i = u_{i+1} = \cdots = u_j < u_{j+1}, \quad 1 \leq i < j < n,$$

or

$$u_{i-1} = u_i = \cdots = u_j < u_{j+1}, \quad 1 < i < j \leq n.$$

The differential equation (4.1) becomes

$$\dot{u_i} = \sum_{k \neq i} \frac{m_k}{r_{ik}} (u_k - u_i), \quad \dot{u_j} = \sum_{k \neq j} \frac{m_k}{r_{jk}} (u_k - u_j),$$
where \( r_{mk} = |a_{mx} - a_{kx}| \) for \( m = i, j \). Since \( u_i = u_j \) we get

\[
\dot{u}_j - \dot{u}_i = \sum_{k \neq i, j} m_k (u_k - u_j) \left[ \frac{1}{r_{jk}^3} - \frac{1}{r_{ik}^3} \right].
\]

Every term in this sum is non-negative, since

(i) if \( k < i \), \( (u_k - u_i) \leq 0 \) and \( \frac{1}{r_{jk}} - \frac{1}{r_{ik}} < 0 \);

(ii) if \( i \leq k \leq j \), \( u_k - u_i = 0 \);

(iii) if \( k > j \), \( (u_k - u_i) \geq 0 \) and \( \frac{1}{r_{jk}} - \frac{1}{r_{ik}} > 0 \).

Moreover, at least one term is strictly positive since not all of \( u_i \) with \( 1 \leq i \leq n \) are equal. Thus \( \dot{u}_j - \dot{u}_i > 0 \)
and the boundary point moves into the interiors of the cone \( K \) as required.

Now we consider the central configurations in \( \mathbb{R}^3 \). Let

\[
S = \left\{ q = (q_1, q_2, \ldots, q_n)^T, q_i \in \mathbb{R}^3 \middle| \sum_{i=1}^{n} m_i q_i = 0, \sum_{i=1}^{n} m_i q_i^2 = 1, q_i \neq q_j \text{ if } i \neq j \right\},
\]

\[
C = \left\{ q = (q_1, q_2, \ldots, q_n)^T \in S \middle| q_i \in \mathbb{R} \times \{0\} \times \{0\}, \forall 1 \leq i \leq n \right\},
\]

\[
E = \left\{ q = (q_1, q_2, \ldots, q_n)^T \in S \middle| q_i \in \{0\} \times \mathbb{R}^2, \forall 1 \leq i \leq n \right\},
\]

\[
\tilde{C} = \{ q = (q_1, q_2, \ldots, q_n)^T \in S \mid q \text{ is collinear along some line} \}.
\]

Then \( C \subset \tilde{C} \) holds and \( \tilde{C} \) is the orbit of \( C \) under \( \text{SO}(3) \).

Now on \( S \), the central configuration equation is

\[
F(q) = \tilde{M}^{-1} U'(q) + U(q) q = 0,
\]

where \( U'(q) \) denotes the gradient of \( U \) with respect to \( q = (q_1, \ldots, q_n) \). Then when we consider the gradient flow of the system

\[
\dot{q} = F(q),
\]

a central configuration is a fixed point of this flow. Note that \( C, \tilde{C} \) and \( E \) are invariant sub-manifolds under the gradient flow of (4.2). For the central configuration \( q_0 = (q_{1,0}, q_{2,0}, \ldots, q_{n,0}) \) with \( q_{i,0} = (a_{ix}, 0, 0)^T \), we have

\[
F'(q_0)|_C = -2\tilde{M}^{-1} B + \mu I_n, \quad (4.3)
\]

\[
F'(q_0)|_E = \text{diag}(\tilde{M}^{-1} B, \tilde{M}^{-1} B) + \mu I_{2n}. \quad (4.4)
\]

Note that in the first Corollary on p.507 of [17], R. Moeckel proved that any orbits near \( \tilde{C} \) are attracted to \( \tilde{C} \) by the gradient flow of (4.2). Therefore it yields that \( F'(q_0)|_E \) in (4.4) is non-negative definite as required.

In fact, using notations in [17], an explicit neighborhood \( \mathcal{U} = \{ q \in S \mid \Theta(q) \leq \frac{\pi}{4} \} \) of \( \tilde{C} \) in \( S \) can be defined such that the orbits of the gradient flow of (4.2) in \( \mathcal{U} \) get more and more collinear.

Here following [17] the function \( \Theta(q, L) \) measures the approximate collinearity of a configuration \( q \in S \) and a line \( L \) in \( \mathbb{R}^3 \) is defined by

\[
\Theta(q, L) = \max_{i \neq j} \angle(L, q_i - q_j),
\]

where \( \angle(L, q_i - q_j) \) denotes the acute angle between \( L \) and \( q_i - q_j \). \( \Theta(q, L) \) vanishes if and only if \( q \) is collinear along a line parallel to \( L \). Then let

\[
\Theta(q) = \min_L \Theta(q, L),
\]
which vanishes if and only if \( q \) is collinear.

Note that in \( \mathcal{U} \), \( \Theta(q) \) is strictly decreasing along orbits \( q = q(t) \) of the gradient flow of (4.2), and it suffices to prove

\[
\Theta(q(t)) < \Theta(q(0)), \quad \forall \ t > 0.
\]

(4.5)

Now we refer readers to pp.504-505 of [17] on the details of the proof of (4.5).

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