On linear relations between cohomological invariants of compact complex manifolds

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Introduction

In [23], Kotschick and Schreieder solve the following three problems for compact Kähler manifolds:

1. Which linear relations between Hodge and Chern numbers are valid for all compact Kähler manifolds?
2. Which linear combinations of Hodge and Chern numbers are bimeromorphic invariants of all compact Kähler manifolds?
3. Which linear combinations of Hodge and Chern numbers are topological invariants of all compact Kähler manifolds?

Roughly speaking, the answer for all three questions is: ‘Only the expected ones’. The third question was famously asked by Hirzebruch [16]. The version of this question concerning the Chern numbers only had previously been solved by Kotschick [21], [22], using the complex bordism ring. A key new ingredient in [23] which allowed the systematic treatment of Hodge numbers was the consideration of the Hodge ring of Kähler manifolds. That is, the graded ring of \( \mathbb{Z} \)-linear combinations of compact complex Kähler manifolds, where two manifolds are identified if they have the same Hodge numbers. Somewhat surprisingly, it is finitely generated.

The method of [23] was recently applied to treat linear relations between Hodge numbers of varieties in positive characteristic in [10]. By a different method, it was shown that there are no unexpected polynomial relations between the Hodge numbers of compact Kähler manifolds [30] or varieties in positive characteristic [11].

In complex dimension \( \geq 3 \), there are many open questions concerning the relation of general compact complex manifolds to Kähler manifolds on the one side and merely almost complex manifolds on the other side. For example, there is no known topological obstruction distinguishing complex from almost complex manifolds. On the other hand, there exist manifolds admitting both Kähler and non-Kähler structures, but there is no known example of a manifold for which every complex structure is Kähler (not even \( \mathbb{C} \mathbb{P}^n \), \( n \geq 3 \)). In dimension 2, the
situation is better understood, and Hodge and Chern numbers play a key role in distinguishing between almost complex, complex and Kähler manifolds.

A systematic study of the spread of cohomological invariants for general complex manifolds therefore appears desirable. In this article, we take a first step in this direction by tackling questions 1 – 3 for arbitrary compact complex manifolds. We do so both in a narrower and a wider sense:

First, in sections [1, 2] and [3], we answer the immediate analogues of the above questions when asked for all compact complex manifolds instead of just Kähler ones. The main results are the following:

**Theorem A.**

1. All universal \( \mathbb{Z} \)-linear relations between the Hodge, Betti and Chern numbers of \( n \)-dimensional compact complex manifolds are combinations of the following:
   
   (a) Serre duality: \( h^{p,q} = h^{n-p,n-q} \)
   
   (b) Poincaré duality: \( b_k = b_{2n-k} \).
   
   (c) Degeneration of the Frölicher spectral sequence:
      
      If \( n \leq 2 \), \( b_k = \sum_{p+q=k} h^{p,q} \).
   
   (d) Hirzebruch Riemann Roch: \( \chi_p = td_p \)
   
   (e) Euler characteristics: \( \sum_p (-1)^p \chi_p = \chi \).
   
   (f) Connected components: \( h^{0,0} = b_0 \)

2. The only universal congruences between the Hodge, Betti and Chern numbers of \( n \)-dimensional compact complex manifolds are combinations of the ones given in the first point, \( b_n(X) \equiv 0 \, (2) \) in case \( n \) odd and the congruences involving Chern numbers alone.

**Theorem B.** A \( \mathbb{Z} \)-linear combination or congruence of Hodge and Chern numbers is a bimeromorphic invariant for all compact complex manifolds if and only if it is a linear combination or congruence of the \( h^{p,0} \) and \( h^{0,p} \) only.

**Theorem C.** For compact complex manifolds of dimension \( \geq 3 \), a rational linear combination of Hodge and Chern numbers is a universal ...

1. ... homeomorphism invariant if and only if it is a universal diffeomorphism invariant if and only if it is a linear combination of the Betti numbers if and only if it is a linear combination of the Euler characteristic and the number of connected components \( h^{0,0} \).

2. ... oriented homeomorphism invariant if and only if it is a universal oriented diffeomorphism invariant if and only if it is a linear combination of the Betti numbers and the Pontryagin numbers if and only if it is a linear combination of the Euler characteristic, the number of connected components \( h^{0,0} \) and the Pontryagin numbers.

In all of these statements, one has to read equality of linear combinations as equality modulo the relations listed in Theorem A. The restriction to \( n \geq 3 \) in
Theorem C is for simplicity only; the situation in low dimensions is well understood.

The proofs of these results follow the ones in the Kähler case treated in [23]. A central tool is the Hodge de Rham ring of compact complex manifolds $H^\text{de Rham}$, defined as formal $\mathbb{Z}$-linear combinations of compact complex manifolds where two such linear combinations are identified if they have the same Hodge and Betti numbers. Like the Hodge ring of Kähler manifolds, this ring is finitely generated. The analogous ring in the positive characteristic setting has been considered in [10] and for the formal part of the arguments we can make use of calculations from that article. Additional complications arise from the degeneration of the Frölicher spectral sequence for surfaces. Unlike in the Kähler or positive characteristic case, this forces us to find manifolds with prescribed cohomological properties up to dimension 4, which is in part done using SageMath.

The second part of the article starts from the observation that for general compact complex manifolds, Hodge and Betti numbers are just the tip of an iceberg of cohomology theories that are not visible on compact Kähler manifolds (where they are determined by Dolbeault cohomology). The most classical ones are the higher pages of the Frölicher spectral sequence [14], Bott Chern cohomology [8] and Aeppli cohomology [2]. More recent examples include the Varouchas groups [39], the cohomologies of the Schweitzer complex [34], the refined Betti numbers [37] and the higher-page analogues of Bott-Chern, Aeppli and Varouchas cohomologies [32].

In [37], an abstract definition of cohomological functors was given, encompassing all of the previously mentioned examples. It appears natural to pose questions 1 – 3 taking into account the dimensions of all cohomological functors at once. One is led to consider the following ring, which we discuss in Section 7.

**Definition D.** The universal ring of cohomological invariants $\mathcal{U}$ is the ring of formal $\mathbb{Z}$-linear combinations of biholomorphism classes of compact connected complex manifolds modulo the relation

$$X \sim Y :\iff H(X) \cong H(Y) \text{ for all cohomological functors } H.$$  

We give a diagrammatic description of elements in $\mathcal{U}$ which allows to write down an algebraic candidate for $\mathcal{U}$ defined by a few simple diagrammatic conditions. In contrast to the rings occurring in the first part of the article, this ring is not finitely generated. We find manifolds realizing most generators in small degrees and certain infinite sequences of generators in Section 8.

In sections 5, 6, we discuss the two natural quotients $\mathcal{FS}$ and $\mathcal{RB}$ of $\mathcal{U}$ arising by considering, instead of all cohomological invariants, only the higher pages of the Frölicher spectral sequence or only the refined Betti numbers (which are recalled in Section 4). Unlike most other cohomological invariants, both of these satisfy a Künneth formula, making $\mathcal{FS}$ and $\mathcal{RB}$ amenable to a more elementary treatment by embedding them into polynomial rings. On the other hand, using the results of [37], one obtains an injection

$$\mathcal{U} \hookrightarrow \mathcal{FS} \times \mathcal{RB}.$$
With the constructions present so far, we obtain the following results.

**Corollary E.** For \( n \leq 3 \), there are no unexpected universal \( \mathbb{Z} \)-linear relations or congruences between refined Betti numbers of \( n \)-dimensional compact complex manifolds or, more generally, between all cohomological invariants, except possibly \( e_{2,1}^0 = e_{0,1}^3 \) on threefolds.

**Corollary F.** Let \( K \) be a field. For \( n \leq 4 \) (resp. \( n \leq 5 \)), there are no unexpected \( K \)-linear combinations of cohomological invariants (resp. refined Betti numbers) that are bimeromorphic invariants of \( n \)-dimensional compact complex manifolds.

What the expected relations or linear combinations in each case are is recalled in the main body of the text. The dimension restrictions could be immediately improved by constructing more generators for \( RB_* \) and \( U_* \). This leads to challenging construction problems. These, and some related open questions, are surveyed in Section 10.

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**Notations and conventions:** All complex manifolds we consider will be assumed compact. Denote by \( CM_* \) the graded ring which in rank \( n \) consists of formal \( \mathbb{Z} \)-linear combinations of isomorphism classes of compact complex manifolds modulo the relation \([X] + [Y] = [X \sqcup Y]\), and with multiplication given by \([X] \cdot [Y] := [X \times Y]\). We will be interested in quotients of this ring defined by cohomological conditions. We sometimes omit the brackets and simply write \( X \) for the isomorphism class \([X]\).

## 1 Hodge, Betti and Chern numbers

The goal of this section is to prove Theorem A. The formal work necessary for this parallels that carried out by van Dobben de Bruyn in [10] for the purpose of studying analogous questions for varieties in positive characteristic, so we keep our notation close to that article.

**Definition 1.1.** Consider \( \mathbb{Z}[x,y,z] \) as a graded ring with \(|x| = |y| = 0\) and \(|z| = 1\). The **Hodge ring** \( H_* \) is the image of the map of graded rings

\[
 h : CM_* \longrightarrow \mathbb{Z}[x,y,z]
\]

induced by sending a complex manifold to its Hodge polynomial (augmented by a dimension counting variable as in [23]):

\[
 [X] \longmapsto h(X) := \sum_{p,q=0}^{\dim X} h_{p,q}(X)x^py^qz^{\dim X}.
\]

That \( h \) is a map of rings is a consequence of the Künneth formula for Dolbeault cohomology (see e.g. [13]). The subring \( H^K_* \) generated by images of compact Kähler manifolds is the Hodge ring of Kähler manifolds studied in [23].
Definition 1.2. Denote by \( \mathcal{H}_n^{\text{form}} := \bigoplus_{n \geq 0} \mathcal{H}_n^{\text{form}} \) the graded subring of \( \mathbb{Z}[x, y, z] \) s.t. \( \mathcal{H}_n^{\text{form}} \) is the space of formal Hodge polynomials of degree \( n \), i.e. those of the form

\[
\sum_{n \geq p, q \geq 0} h^{p, q} x^p y^q z^n \mbox{ s.t. } h^{p, q} = h^{n-p, n-q}.
\]

While the polynomial description of \( \mathcal{H}_* \) and \( \mathcal{H}_*^{\text{form}} \) is convenient to take the multiplicative structure into account, the reader may prefer to picture an element \( P = (\sum_{p, q} h^{p, q} x^p y^q)z^n \in \mathcal{H}_n^{\text{form}} \) of a fixed degree as a (formal) Hodge diamond, e.g. for \( n = 2 \):

\[
P \cong \begin{array}{ccc}
  & h^{2,0} & \\
 h^{2,2} & & h^{1,2} \\
 h^{1,1} & & h^{0,2} \\
 h^{1,0} & & h^{0,1} \\
 h^{0,0} & & \\
\end{array}
\]

As to the algebraic structure of \( \mathcal{H}_*^{\text{form}} \), we have:

**Theorem 1.3** (van Dobben de Bruyn). The map

\[
\phi : \mathbb{Z}[A, B, C, D] \longrightarrow \mathcal{H}_*^{\text{form}} \\
A \mapsto (1 + xy)z \\
B \mapsto (x + y)z \\
C \mapsto xyz^2 \\
D \mapsto (x + xy^2)z^2
\]

is a surjection of graded rings, where \( |A| = |B| = 1 \) and \( |C| = |D| = 2 \). The kernel \( I \) of \( \phi \) is the principal ideal generated by

\[
G := D^2 - ABD + C(A^2 + B^2 - 4C).
\]

As in [23], we will from now on write \( A, B, C, D \) also for their images under \( \phi \).

**Theorem 1.4.** There is an equality \( \mathcal{H}_* = \mathcal{H}_*^{\text{form}} \).

One may take as generators (the images of) \( \mathbb{CP}^1 \), an elliptic curve \( E \), any Kähler surface \( S_K \) of signature \( \pm 1 \) (e.g. \( \mathbb{CP}^2 \)) and any non-Kähler surface \( S_{NK} \).

**Proof.** As noted in [23, Cor. 3], \( h(\mathbb{CP}^1) = A, h(E - \mathbb{CP}^1) = B, h(S_K) \equiv \pm C \) mod \( (A^2, B^2, AB) \) and \( A, B, C \) generate the Hodge ring of Kähler manifolds, \( \mathcal{H}_*^K \), i.e. the subring \( \mathcal{H}_*^{\text{form}} \) generated by polynomials \( (\sum h^{p, q} x^p y^q)z^n \) satisfying the additional relation \( h^{p, q} = h^{q, p} \). Since a non-Kähler surface satisfies \( h^{0,1}(S_{NK}) = h^{1,0}(S_{NK}) + 1 \) and \( h^{0,2}(S_{NK}) = h^{2,0}(S_{NK}) \) (see e.g. [3]), its Hodge polynomial is congruent to \( -D \) modulo \( A^2, B^2, AB \) and \( C \).
Remark 1.5. For concreteness, one may pick $S_K$ to be $\mathbb{CP}^2$ and $S_{NK} = H$ to be a Hopf surface. Then $C = b(\mathbb{CP}^1 \times \mathbb{CP}^1 - \mathbb{CP}^2)$ and $D + 2C = h(E \times \mathbb{CP}^1 - H)$.

Remark 1.6. Theorem 1.4 was already known to D. Kotschick and S. Schreieder (unpublished).

Next, we consider the Betti numbers:

Definition 1.7. Let $\mathbb{Z}[t, z]$ be graded ring via $|t| = 0$, $|z| = 1$. The de Rham ring (of compact complex manifolds) is the image of the natural map of graded rings

$$dR : \mathcal{CM} \longrightarrow \mathbb{Z}[t, z]$$

induced sending a manifold to its (augmented) Poincaré polynomial

$$[X] \mapsto dR(X) := \sum_{k \geq 0} b_k(X) t^k z^{\dim_{\mathbb{C}} X}.$$

Definition 1.8. Let $\mathcal{DR}_n^\text{form} \subseteq \mathbb{Z}[t, z]$ be the space of formal Poincaré polynomials, i.e. those of the form

$$\sum_{k=0}^{n} b_k t^k z^n \quad \text{s.t.} \quad b_k = b_{2n-k} \quad \text{and} \quad 2|b_n \text{ if } n \text{ is odd.}$$

The direct sum $\mathcal{DR}_*^\text{form} = \bigoplus \mathcal{DR}_n^\text{form}$ is a graded subring of $\mathbb{Z}[t, z]$.

There is a natural map $s : \mathcal{H}_*^\text{form} \rightarrow \mathcal{DR}_*^\text{form}$ defined by $x, y \mapsto t, z \mapsto z$.

However, as remarked in [10], one has $s \circ h(X) = dR(X)$ if and only if the Frölicher spectral sequence degenerates at the first page.

Theorem 1.9 (van Dobben de Bruyn). Let $\mathbb{Z}[A, B, C, D]$ be graded as in Theorem 1.3. The map

$$\psi : \mathbb{Z}[A, B, C, D] \longrightarrow \mathcal{DR}_*^\text{form}$$

with

$$A \mapsto (1 + t^2) z$$

$$B \mapsto 2t z$$

$$C \mapsto t^2 z^2$$

$$D \mapsto (t + t^3) z^2$$

is a surjection of graded rings with $\psi = s \circ \phi$. The kernel of $\psi$ is the ideal

$$J = (A^2 C - D^2, AB - 2D, B^2 - 4C, BD - 2AC).$$

Since $s : \mathcal{H}_*^\text{form} \rightarrow \mathcal{DR}_*^\text{form}$ is surjective, we get:

Corollary 1.10. There is an equality $\mathcal{DR}_* = \mathcal{DR}_*^\text{form}$.

Next, we will consider Hodge and Betti numbers simultaneously.

Definition 1.11. The Hodge de Rham ring $\mathcal{HDR}_*$ is the image of the diagonal map:

$$\text{hdR} : \mathcal{CM}_* \longrightarrow \mathcal{H}_* \times \mathcal{DR}_*$$

$$[X] \mapsto (h(X), dR(X))$$
As in [10], consider the following homomorphisms:

\[ \chi : \mathcal{H}_* \rightarrow \mathbb{Z}[z] \quad \chi : \mathcal{DR}_* \rightarrow \mathbb{Z}[z] \quad h^{0,0} : \mathcal{H}_* \rightarrow \mathbb{Z}[z] \quad b_0 : \mathcal{H}_* \rightarrow \mathbb{Z}[z] \]

\[ x, y \mapsto -1 \quad t \mapsto -1 \quad x, y \mapsto 0 \quad t \mapsto 0 \]

**Lemma 1.12** (van Dobben de Bruyn). *The kernel of \((b_0, \chi) : \mathcal{DR} \rightarrow \mathbb{Z}[z] \times \mathbb{Z}[z]\) is the ideal generated by the two elements*

\[ d := (t + 2t^2 + t^3)z^2, \]
\[ e := (t^2 + 2t^3 + t^4)z^3. \]

Further, define the Frölicher defect

\[ FD : \mathcal{H}_* \times \mathcal{DR}_* \rightarrow \mathcal{DR}_*, \]
\[ (a, b) \mapsto s(a) - b \]

and denote by \(FD\) the composition of \(FD\) and the truncation \(\mathcal{DR}_* \rightarrow \mathcal{DR}_{\leq 2}\).

Then a natural candidate for \(\mathcal{HDR}_*\) is the ring

\[ \mathcal{HDR}_*^{\text{form}} := \left\{ (a, b) \in \mathcal{H}_*^{\text{form}} \times \mathcal{DR}_*^{\text{form}} \mid \frac{\chi(a) = \chi(b)}{h^{0,0}(a) = b_0(b)} \right\}. \]

**Theorem 1.13.** *There is an equality \(\mathcal{HDR}_* = \mathcal{HDR}_*^{\text{form}}\). In fact, \(\mathcal{HDR}_*^{\text{form}}\) may be generated by the images of \(\mathbb{C}P^1, \mathbb{C}P^2\) and a finite number of complex nilmanifolds of complex dimension \(\leq 4\).*

A brief review of complex nilmanifolds and their cohomologies is given in Section 8.1

**Proof.** The ring \(\mathcal{H}_*\) may be generated by \(\mathbb{C}P^1, \mathbb{C}P^2\), an elliptic curve and the Kodaira-Thurston manifold. The latter two are nilmanifolds. For all of these the Frölicher spectral sequence degenerates, so given \((a, b) \in \mathcal{HDR}_*^{\text{form}}\), the element \((a, s(a))\) lies in \(\mathcal{HDR}_*\) and we may assume that \(a = 0\). Then \(b\) has to be concentrated in degrees \(\geq 3\) and satisfy \(\chi(b) = \chi(0) = 0\) and \(b_0(b) = h^{0,0}(b) = 0\). By Lemma 1.12 the kernel \(K\) of \((b_0, \chi) : \mathcal{DR}_* \rightarrow \mathbb{Z}[z] \times \mathbb{Z}[z]\) is the ideal generated by the two elements \(d\) and \(e\). Thus, \(K \cap \mathcal{HDR}_{\geq 3}\) is generated by \(e, \psi(A)d, \psi(B)d, \psi(C)d, \psi(D)d\). Now note that for any \(X \in \mathcal{CM}_*\), the element \((0, FD(X)) = (h(X), s(h(X)) - (h(X), dR(X)))\) lies also in \(\mathcal{HDR}_*\). Therefore, it suffices to find (linear combinations of) compact complex manifolds s.t. \(FD(X)\) equals \(e, \psi(A)d, \psi(B)d, \psi(C)d\) and \(\psi(D)d\). These may be generated as follows:

\[ e = FD(I) \text{ for } I \text{ a type ii-deformation of the Iwasawa-manifold, which has } \text{Betti numbers } (1, 4, 8, 10, 8, 4, 1) \text{ and Hodge numbers } (1, 4, 9, 12, 9, 4, 1), \text{ see [27] p. 96}. \]

\[ \psi(A)d + e = FD(I) \text{ for } I \text{ the Iwasawa manifold itself, which has } \text{Betti numbers } (1, 4, 8, 10, 8, 4, 1) \text{ and Hodge numbers } (1, 5, 11, 14, 11, 5, 1), \text{ see [27] p. 96}. \]

**Lemma 1.14.** *\(\psi(C)d = FD(X - Y)\) for \(X\) and \(Y\) nilmanifolds associated with the strongly non nilpotent Lie algebras of family I in [27, p. 96].*
Proof. These manifolds have real dimension 8. To avoid manual computations in the complex of left-invariant forms, we used SageMath code inspired by [5], yielding:

\[
(s \circ h)(X) = (1 + 5t + 11t^2 + 15t^3 + 16t^4 + 15t^5 + 11t^6 + 5t^7 + t^8)z^4 \\
(s \circ h)(Y) = (1 + 5t + 11t^2 + 14t^3 + 14t^4 + 14t^5 + 11t^6 + 5t^7 + t^8)z^4 \\
dR(X) = dR(Y) = (1 + 5t + 8t^2 + 11t^3 + 14t^4 + 11t^5 + 8t^6 + 5t^7 + t^8)z^4.
\]

The claim follows. \qed

Finally, \(\psi(B)d = 2e\) and \(\psi(A)e = \psi(D)d\), which finishes the proof of the theorem. \qed

Remark 1.15. If we care about rational generators only, one can omit the four-dimensional nilmanifolds, since \(2\psi(C)d = \psi(B)e\).

More generally, we can consider linear relations between Hodge, Betti and Chern-numbers. Let \(\Omega^U\) denote the complex bordism ring.

Definition 1.16. The Hodge de Rham Chern ring \(\mathcal{H}D \mathcal{R}_\ast\) is the image of the diagonal map \(CM_\ast \rightarrow \mathcal{H}D \mathcal{R}_\ast \times \Omega^U\). The Hodge Chern ring \(\mathcal{H}C_\ast\) is the image of the diagonal map \(CM_\ast \rightarrow \mathcal{H}C_\ast \times \Omega^U\).

The Todd genera define a homomorphism \(td_\ast : \Omega^U \rightarrow \mathbb{Z}[x,z]\) sending a class \([X] \in \Omega_n\) to the polynomial \(\sum_{p=0}^n td_p(X)x^p z^n\). Following [23], we interpret the Hirzebruch genus as the homomorphism

\[
\chi_\ast : \mathcal{H}_\ast \rightarrow \mathbb{Z}[x,z] \\
y \mapsto -1.
\]

Setting \(x = 1\), it specializes to the signature \(\sigma : \mathcal{H}_\ast \rightarrow \mathbb{Z}[z]\), and setting \(x = -1\), it specializes to the Euler characteristic \(\chi\). For any \(X \in \mathcal{H}_n\), write \(\chi_\ast(X) = \sum_{p} \chi_p(X)x^p z^n\). By precomposing with the projection to the first factor, we may also consider \(\chi_\ast\) as a map with source \(\mathcal{H}D \mathcal{R}_\ast\). Since the Chern-classes of all complex nilmanifolds vanish, Hirzebruch-Riemann-Roch together with Theorem 1.13 yields (c.f. [23] Thm. 8):

Lemma 1.17. The image \(\mathcal{H}ir_\ast\) of \(\chi_\ast\) is a polynomial ring, generated by the images of \(h(\mathbb{C}P^1)\) and \(h(\mathbb{C}P^2)\). The kernel of \(\chi_\ast\) may be generated by complex nilmanifolds.

Theorem 1.18. There is an equality

\[
\mathcal{H}D \mathcal{R}_\ast = \{(a,b) \in \mathcal{H}D \mathcal{R}_\ast \times \Omega^U \mid \chi_\ast(a) = td_\ast(b)\}.
\]

Proof. By Hirzebruch-Riemann-Roch, there is an inclusion from left hand side to right hand side. On the other hand, if \((hdR(X), [Y]) \in \mathcal{H}D \mathcal{R}_\ast \times \Omega^U\) s.t. \(\chi_\ast(X) = td_\ast(Y)\) for all \(p\), then \(\chi_\ast(X) = \chi_\ast(Y)\) for all \(p\). Hence, \(hdR([Y] - [X])\) lies in the kernel of \(\chi_\ast\). This kernel is generated by complex nilmanifolds, which have vanishing Chern classes, so in particular represent the zero element in \(\Omega^U\).

Thus, writing \(hdR(Y - X) = hdR(M)\), where \(M\) is a linear combination of nilmanifolds, we see that

\[
(hdR(X), [Y]) = (hdR(Y), [Y]) - (hdR(Y - X), 0) \\
= (hdR(Y), [Y]) - (hdR(M), [M])
\]

lies in the image of the diagonal map. \qed
Theorem \[\text{A}\] from the introduction is an immediate consequence.

Remark 1.19. Analogously, \(\mathcal{HC}_* = \{(a, b) \in \mathcal{H}_* \times \Omega_*^U \mid \chi_*(a) = td_*(b)\}\).

2 Bimeromorphic invariants

Consider the map \(p\) given by the composition

\[
\mathcal{H}_* \rightarrow \mathbb{Z}[x, y, z] \rightarrow \mathbb{Z}[x, y, z]/(xy)
\]

of inclusion and projection. Similarly, consider the map \(p_{\mathcal{HC}}\) given as the composition of \(p\) with the projection

\[
\mathcal{HC}_* \rightarrow \mathcal{H}_*.
\]

Consider the ideals \(B \subseteq \mathcal{H}_*, \text{ resp. } B_{\mathcal{HC}} \subseteq \mathcal{HC}_*\) generated by differences of bimeromorphic compact complex manifolds.

Theorem 2.1.

1. The degree \(n\) part of the image of \(p\) (and a fortiori of \(p_{\mathcal{HC}}\)) is free of rank \(2^n\), with a basis given by \(z^n, xz^n, yz^n, x^2z^n, y^2z^n, ..., x^n - 1 z^n, y^n - 1 z^n, (x^n + y^n)z^n\).

2. There are equalities \(\ker p = (C) = B\).

3. There is an equality \(\ker p_{\mathcal{HC}} = B_{\mathcal{HC}}\).

Proof. Parts 1 and 2 follow with exactly the same proof as in the positive characteristic setting \([10]\), using that the numbers \(h_{p,0}\) and \(h_{0,q}\) are bimeromorphic invariants. For this reason, also the inclusion \(B_{\mathcal{HC}} \subseteq \ker p_{\mathcal{HC}}\) in 3. is clear. For the other inclusion, we use the observation made in \([23, \text{Thm. 13}]\) that there exists a basis sequence \(\beta_1 = \mathbb{C}P^1, \beta_2 = \mathbb{C}P^2 - \mathbb{C}P^1 \times \mathbb{C}P^1 = -C, \beta_3, ...\) for \(\Omega_*^U\) with \(\beta_i \in B_{\mathcal{HC}}\) for \(i \geq 2\).

Since \(\mathcal{HC}_*\) is generated by \(E, H, \beta_1, \beta_2, ...,\) we get a composition of surjective maps

\[
\mathbb{Z}[E, H, \beta_1] \rightarrow \mathcal{HC}_*/B_{\mathcal{HC}} \xrightarrow{pr} \mathcal{H}_*/(C) = \mathbb{Z}[A, B, D]/(D^2 - ABD).
\]

Because in \(\mathcal{H}_*/(C)\) we have \(A = \beta_1, B = E - \beta_1\) and \(D = E \times \beta_1 - H\), the kernel of this composition is the principal ideal generated by the element \((E\beta_1 - H) - \beta_1(E - \beta_1)(E\beta_1 - H)\), which maps to zero in \(\mathcal{HC}_*/B_{\mathcal{HC}}\). Thus, the map \(\mathcal{HC}_*/B_{\mathcal{HC}} \rightarrow \mathcal{H}_*/B\) is an isomorphism. Since \(\ker p_{\mathcal{HC}} = pr^{-1}(\ker p) = \ker pr^{-1}(B)\), this completes the proof.

Proof of Theorem \[\text{B}\]. Any map \(\mathcal{HC}_* \rightarrow R\) for \(R = \mathbb{Z}\) or \(\mathbb{Z}/n\mathbb{Z}\) factors through \(\mathcal{HC}_*/B_{\mathcal{HC}}\). Since \(B_{\mathcal{HC}} = \ker p_{\mathcal{HC}}\), this means it depends on the coefficients of the monomials in part 1 of Theorem \[\text{2.1}\].

9
3 The rational Hirzebruch problem for general complex manifolds

The goal of this section is to prove Theorem C from the introduction. Consider the ideal $\mathcal{I} \subseteq H^{DR}_* \otimes \mathbb{Q}$ generated by differences of homeomorphic equidimensional complex manifolds.

Proposition 3.1. In degrees $n \geq 3$, $\mathcal{I}$ coincides with the kernel of the map

$$\text{pr} : H^{DR}_* \otimes \mathbb{Q} \rightarrow D^R_* \otimes \mathbb{Q}.$$  

Under the projection $H^{DR}_* \rightarrow H_*$ it is identified with the kernel of $(h^{0,0}, \chi) : H_* \otimes \mathbb{Q} \rightarrow \mathbb{Q}[z] \times \mathbb{Q}[z]$.

Proof. Since the Betti numbers are homeomorphism invariants, $\mathcal{I} \subseteq K := \ker \text{pr}$. On the other hand, $K$ consists of all elements $(a, 0) \in H^{DR}_* \subseteq H_* \times D^R_*$. They have to satisfy the relations $\chi(a) = \chi(0) = 0$, $h^{0,0}(a) = b_0(0) = 0$ and $FD_{\leq 2}(a, 0) = s(a)_{<2} = 0$. I.e. we have an identification $K = K' \cap \ker s_{\leq 2} \subseteq H_*$ where $K' := \ker(h^{0,0}, \chi) \subseteq H_*$. Since $FD_{\leq 2}(a, 0) = 0$ for all $a \in H_\geq 3$, this gives the identification $K = K'$ in degrees $\geq 3$. Since there is a commutative diagram

$$\xymatrix{ H_* \ar[r]^-{s} \ar[rd]_-{(h^{0,0}, \chi)} & D^R_* \ar[d]^-{(b_0, \chi)} \\ & \mathbb{Q}[z] \times \mathbb{Q}[z] }$$

using Lemma 1.12 we obtain that $K' = s^{-1}(d, e) = (\tilde{d}, \tilde{e}, \ker s)$, where $\tilde{d}, \tilde{e}$ are arbitrary preimages of $d$ and $e$. We will choose $\tilde{d} = (x + 2xy + xy^2)z^2$ and $\tilde{e} = (x^2 + x^2y + xy^2 + xy^3)z^3$. By Theorem 1.9 and Theorem 1.3, the kernel of $s$ is generated by $Q = A^2C - D^2$, $R := BD - 2AC$, $S := B^2 - 4C$ and $T := AB - 2D$. Thus, $K' \cap H_\geq 3$ has the following generators:

$$\text{A} \tilde{d}, \text{B} \tilde{d}, \tilde{e}, \text{AS}, \text{BS}, \text{AT}, \text{BT, R} \quad \text{in degree 3}$$
$$\text{C} \tilde{d}, \text{D} \tilde{d}, \text{CS}, \text{DS}, \text{CT}, \text{DT, Q} \quad \text{in degree 4}$$

We have to find differences of homeomorphic complex manifolds whose difference realizes these generators.

Degree 2: That $S$ can be realized as by a combination of differences of orientation reversingly homeomorphic complete intersections $X_1 - Y_1$ and $X_2 - Y_2$ has been shown in the proof of [23, Thm. 10], as a consequence of the results in [20]. By Theorem 1.4, we may therefore realize $\text{AS}, \text{BS, CS, DS}$ as differences of homeomorphic manifolds.

In constrast, $T = (-x + y + x^2y - xy^2)z^2$ can never be realized. In fact, if two complex surfaces are homeomorphic, then their first Betti numbers have to coincide. But for surfaces, the Hodge numbers $h^{1,0}$ and $h^{0,1}$ (and therefore also $h^{2,1}$ and $h^{1,2}$) are determined by the first Betti number, so they would have to agree as well. Thus, we really have to realize all the other generators in degrees
3 and 4. We will do so in all cases by considering different left-invariant structures on a fixed nilmanifold:

**Degree 3:** \( \tilde{e} = h(I_{ii} - I_{iii}) \) is realized by the difference of a type (ii) and a type (iii) deformation of the Iwasawa manifold, see [27] p. 96.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 5 & 2 \\
0 & 1 & 0 & 2 & 5 & 2 \\
0 & 0 & 0 & 1 & 5 & 5 \\
0 & 1 & 0 & 1 & 4 & 4 \\
0 & 0 & 0 & 2 & 3 & 1 \\
0 & 1 & 0 & 2 & 3 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\( B\tilde{d} = h(X - Y) \) where the underlying manifold of both \( X \) and \( Y \) is a nilmanifold associated with the Lie algebra \( h_{15} \) and the complex structures correspond to parameters \( (\rho, B, c) = (1, 2, 0) \) and \( (1, B, c) \) with \( B \neq 0 \neq c \) in the notation of [9], see [9] proof of Thm. 4.1.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 4 & 2 \\
0 & 1 & 0 & 2 & 4 & 2 \\
0 & 0 & 0 & 1 & 5 & 5 \\
0 & 1 & 0 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\( A\tilde{d} - B\tilde{d} + 2\tilde{e} = h(I_i - I_{ii}) \) is realized by the difference of a type (i) and a type (ii) deformation of the Iwasawa manifold, hence \( Ad \in I \), see again [27].

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 4 & 2 \\
0 & 1 & 0 & 2 & 4 & 2 \\
0 & 0 & 0 & 1 & 5 & 5 \\
0 & 1 & 0 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{array} + 2 \cdot
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 4 & 2 \\
0 & 1 & 0 & 2 & 4 & 2 \\
0 & 0 & 0 & 1 & 5 & 5 \\
0 & 1 & 0 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{array} =
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 2 & 4 & 2 \\
0 & 1 & 0 & 2 & 4 & 2 \\
0 & 0 & 0 & 1 & 5 & 5 \\
0 & 1 & 0 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

\( BT = AS + 2B\tilde{d} - 4\tilde{e} \), so it is redundant.

\[
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} + 2 \cdot
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} =
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} - 4 \cdot
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\( AT + A\tilde{d} - 2B\tilde{d} + 3\tilde{e} + BT = h(X - Y) \) where, in the notation of [9], \( X \) and \( Y \) are nilmanifolds associated with \( h_{15} \) and the complex structures correspond to parameter values \( (\rho, B, c) = (0, 1, c) \) with \( c \neq 1 \) and \( (\rho, B, c) = (1, B, c) \) with...
$B \neq 0 \neq c$. Hence, $AT \in \mathcal{I}$.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0
\end{bmatrix}
= 1 3 3 1 - 1 3 3 1
\]

$R = 2\tilde{c} - B\tilde{d}$, so $R$ is redundant.

**Degree 4**: It remains to show that $C\tilde{d}, D\tilde{d}, CT, DT$ and $Q$ lie in $\mathcal{I}$. Since we work rationally, it will turn out we do not need new geometric generators.

Since $Q = A(BD - 2AC) - C(B^2 - 4C) = AR - CS$ in $H_*$ and $R, CS \in \mathcal{I}$, we have $Q \in \mathcal{I}$.

$DT = 2CS - AR \in \mathcal{I}$.

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0
\end{bmatrix}
= 2.
\]

$4CT = 2DS - ABS + B^2T$, so $CT \in \mathcal{I}$.

Finally, since $2Cd = Be$ and $2Dd = ABd$ in $D\mathcal{R}_*$, the elements $2C\tilde{d} - B\tilde{c}$ and $2D\tilde{d} - AB\tilde{d}$ lie in the kernel of $s$. As we have already found generators for all of $(\ker s)_{\geq 3}$ and $AB\tilde{d}, B\tilde{c} \in \mathcal{I}$, we obtain $C\tilde{d}, D\tilde{d} \in \mathcal{I}$. \qed
Remark 3.2. (Integral version) As the proof shows, the integral version of the theorem is true in dimension 3 and to make the same proof work for any dimension it would suffice to invert 2. The general case depends on the geometric realization of the integral generators for $K'$ in dimension 4. We are optimistic that further progress in the classification of left-invariant complex structures on fixed 8-dimensional nilpotent Lie-algebras, as begun in [24], combined with a systematic study of the Dolbeault cohomology stratification of their moduli space, will eventually settle this.

Remark 3.3. Some of the generators for $I$ may also be realized in other ways, e.g. using different complex structures on twistor spaces. We use nilmanifolds since they have vanishing Chern classes, which will be of advantage later on.

Adding what needs to be added, the rest of the proof of Theorem [C] proceeds almost literally as in the Kähler case. For the reader’s convenience, we give the full argument.

Proposition 3.4. In degrees $\geq 3$, $I$ coincides with the ideal generated by differences of diffeomorphic complex manifolds, and $I \cap \ker(\sigma)$ coincides with the ideal generated by differences of orientation preservingly homeomorphic (or diffeomorphic) manifolds.

Proof. As we have seen in the proof of Proposition 3.1, except for the multiples of $S$, all non-redundant generators of $I_{\geq 3}$ are given by differences of two copies of the same nilmanifold equipped with different left-invariant complex structures, which are trivially (even orientation-preservingly) isomorphic. For multiples of $S$, recall that $\mathcal{H}_n$ can be generated by $\mathbb{C}P^1, \mathbb{C}P^2$ an elliptic curve $E$ and a Hopf surface $H$. By a result of Wall [40], the manifolds $X_i$ and $Y_i$ are $h$-cobordant. Thus, by the $h$-cobordism theorem [36], $\mathbb{C}P^k \times X_i$ and $\mathbb{C}P^k \times Y_i$ are diffeomorphic. Products with $E \cong S^1 \times S^1$ and $H \cong S^1 \times S^3$ are handled by the following lemma from [18]:

Lemma 3.5. Let $M, N$ be $h$-cobordant manifolds of dimension $\geq 5$. Then $M \times S^1$ and $N \times S^1$ are diffeomorphic.

To take into account orientations, we first note that all generators of $I_{\geq 3}$, except products of the differences of orientation reversingly homeomorphic complete intersections $X_i - Y_i$ with pure powers of $\mathbb{C}P^2$, have vanishing signature, so that $I_{\geq 3} \cap \ker(\sigma)$ is generated by products of $X_i - Y_i$ with monomials containing at least one factor of $\mathbb{C}P^1$, $E$ and $H$ and those generators coming from differences of nilmanifolds. As already remarked above, the latter are differences of orientation preservingly diffeomorphic manifolds. On the other hand, since $\mathbb{C}P^1, E$ and $H$ admit orientation reversing self-diffeomorphisms, we may deduce that also products of $X_i$ and $Y_i$ with any of these three manifolds are orientation preservingly diffeomorphic.

Proposition 3.6. Let $J \subseteq \mathcal{HC}_* \otimes \mathbb{Q}$ be the ideal generated by differences of homeomorphic complex manifolds. In degrees $\geq 3$, it coincides with the ideal generated by differences of diffeomorphic complex manifolds and with the kernel $K$ of the composition

$$\mathcal{HC}_* \otimes \mathbb{Q} \xrightarrow{(x \mapsto \alpha(x))} \mathbb{Q}[z] \times \mathbb{Q}[z].$$
Proof. Let \( \beta_1 = \mathbb{CP}^1, \beta_2, ... \beta_i, ... \) be a generating sequence for \( \Omega^*_e \otimes \mathbb{Q} \) by compact complex manifolds. Given any element \((h(X), [X]) \in \mathcal{HC}_n\), write \([X] = [P]\) for some polynomial \(P\) in \(\beta_1, ..., \beta_n\). Then \((h(X) - h(P), 0) \in \mathcal{HC}_n\) and \(h(X) - P\) is in the kernel of the Hirzebruch genus, which may be generated by an elliptic curve \(E\) and a Hopf surface \(H\), which map to zero in \(\Omega^*_e\). From this, we see that \(\mathcal{HC}_e \otimes \mathbb{Q}\) is generated by an elliptic curve \(E\), a Hopf surface \(H\) and the \(\beta_i\), i.e. there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Q}[E, H, \beta_1, \beta_2, ...] & \rightarrow & \mathbb{Q}[z] \times \mathbb{Q}[z]. \\
\downarrow & & \\
\mathcal{HC}_e \otimes \mathbb{Q} \rightarrow & \mathcal{J} & \subseteq K.
\end{array}
\]

Clearly, \(\mathcal{J} \subseteq K\). On the other hand, by [22, Thm. 10], it is possible to choose a basis sequence \(\beta_1 = \mathbb{CP}^1, \beta_2, ... \) such that \(\beta_i \in \mathcal{J}\) for \(i \geq 2\). For such a basis sequence, the above diagram may be reduced to

\[
\begin{array}{ccc}
\mathbb{Q}[E, H, \beta_1] & \rightarrow & \mathbb{Q}[z] \times \mathbb{Q}[z]. \\
\downarrow & & \\
\mathcal{HC}_e \otimes \mathbb{Q}/(\beta_i)_{i \geq 2} & \rightarrow & \mathcal{J} \subseteq K.
\end{array}
\]

Write \(B = \beta_1 - E\) and \(H' = E^2 - H\). Then, since \(\chi(B) = 2, \ h^{0,0}(B) = 0\) and \(\chi(E) = 0, \ h^{0,0}(E) = 1, \ \chi(H') = 0 = h^{0,0}(H')\), the surjective map \(\mathbb{Q}[E, H', B] = \mathbb{Q}[E, H, \beta_1] \rightarrow \mathbb{Q}[z] \times \mathbb{Q}[z]\) has kernel generated by \(EB\) and \(H'\). Thus \(K = \langle EB, H', \beta_2, \beta_3, ... \rangle\). Since both \(EB\) and \(H'\) have vanishing Chern classes, their images in \(\mathcal{HC}_e \otimes \mathbb{Q}\) lie entirely in the ideal of elements of the form \((a, 0) \in \mathcal{HC}_e \subseteq H_* \times C_*\), which, by Remark 1.19, is identified with the kernel of the Hirzebruch genus \(\chi_1 : H_* \otimes \mathbb{Q} \rightarrow Hir_* \otimes \mathbb{Q}\). In particular, they lie in the kernel of \(\chi : H_+ \otimes \mathbb{Q} \rightarrow \Omega^*_e \otimes \mathbb{Q}\). Since they also lie in the kernel of \(h^{0,0}\), by Proposition 3.1, all multiples in degrees \(\geq 3\) lie in \(\mathcal{J}\). This completes the proof of \(K_{\geq 3} = J_{\geq 3}\).

To take into account diffeomorphisms instead of homeomorphisms, note that the generators \(\beta_i\) constructed in [22] are in fact differences of diffeomorphic projective varieties as soon as \(i \geq 3\), and the same holds for \(\beta_1 \cdot \beta_2\) and \(\beta_2 \cdot \beta_2\). Since \(\beta_2\) is a difference of orientation-reversingly homeomorphic simply connected manifolds, it follows as in the proof of Proposition 3.4 from the h-cobordism theorem and Lemma 3.3 that \(E \times \beta_2\) and \(H \times \beta_2\) are differences of diffeomorphic manifolds. As we have seen, all other generators for \(\mathcal{J}_{\geq 3}\) may be taken to be differences of diffeomorphic nilmanifolds with left invariant complex structures.

Proposition 3.7. Let \(\mathcal{JO} \subseteq \mathcal{HC}_e \otimes \mathbb{Q}\) be the ideal generated by differences orientation-preservingly homeomorphic manifolds. In degrees \(\geq 3\), it coincides with the ideal generated by differences of orientation-preservingly diffeomorphic manifolds and with the kernel \(K\) of the map

\[
\mathcal{HC}_e \otimes \mathbb{Q} \rightarrow \mathbb{Q}[z] \times \mathbb{Q}[z] \times \Omega^*_{\mathbb{C}^e} \otimes \mathbb{Q},
\]

given by \((\chi, h^{0,0})\) in the first two components and the forgetful map in the third.
Proof. Clearly, the ideal generated by differences of orientation preservingly
diffeomorphic manifolds is contained in $\mathcal{J}\mathcal{O}$ which is in turn contained in the
kernel. For the reverse inclusions, let us use the same basis sequence as in the
previous proof and consider the diagram

$$
\begin{array}{ccc}
\mathbb{Q}[E, H, \beta_1, \beta_2, \ldots] & \rightarrow & \mathbb{Q}[z] \times \mathbb{Q}[z] \times \Omega^{SO}_{\ast} \otimes \mathbb{Q}.
\end{array}
$$

The elements $E, \beta_1$ have trivial Pontryagin numbers and we see, as before, that
the subring of polynomials in these two elements surjects onto the first two factors. On the other hand, the polynomials in the $\beta_{2i}$ generate $\Omega^{SO}_{\ast}$ and have zero
image in the first two factors. Thus, the diagonal map is surjective and its kernel
is the ideal generated by the elements $\beta_i$ for $i$ odd, $E \cdot \beta_j, H \cdot \beta_j$ for $j$ even
and $EB, H'$ as above. Using [22 Thm. 7] for the $\beta_1 \cdot \beta_{i-1}, \beta_i \in \mathcal{J}\mathcal{O}$ for odd $i \geq 3,$
and, as before, that $E$ and $H$ admit orientation reversing self-diffeomorphisms,
all these elements are representable by differences of orientation preservingly
diffeomorphic manifolds.

Proof of Theorem C. It is known that all of the listed quantities are (un-) ori-
entropy homeomorphism / diffeomorphism invariants, so it remains to show the
converse. By Proposition 3.6, any linear map $h: \mathcal{H}C_n \rightarrow \mathbb{Q}, n \geq 3,$ which is
a (not necessarily orientation preserving) homeomorphism or diffeomorphism
invariant factors over $(h^{0,0}, \chi),$ i.e. it can be written as a linear combination
of Euler characteristic and connected components only. The proof for the ori-
entation preserving case is the same, using Proposition 3.7. The statements
involving Betti numbers follow from Theorem A.

4 Refined Betti numbers

We give a reminder and more elementary treatment of the refined Betti num-
bers. They were originally introduced in [37] from the point of view of the
structure theory of double complexes and we will pick up this connection in
Section 7. Until then, we keep our discussion independent of the results of [37].

Let $X$ be a compact complex manifold of dimension $n.$ The $k$-th complex de
Rham cohomology $H^k_{dR}(X)$ is naturally equipped with two filtrations, namely
the Hodge filtration and its conjugate:

$$
F^p := F^p H^k_{dR}(X) := \left\{ [\omega] \mid \omega \in \bigoplus_{r+s=k, r \geq p} A^r,s_X \right\}
$$

$$
F^q := F^q H^k_{dR}(X) := \left\{ [\omega] \mid \omega \in \bigoplus_{r+s=k, s \geq q} A^r,s_X \right\}
$$

15
If $X$ satisfies the $\partial\bar{\partial}$-Lemma, for example for $X$ Kähler, it is a standard result that $F$ and $\bar{F}$ induce a pure Hodge structure on the de Rham cohomology, i.e. that

$$H^k_{dR}(X) = \bigoplus_{p+q=k} F^p \cap \bar{F}^q.$$  

For general $X$, this is no longer true. This motivates the following definitions:

**Definition 4.1.** The total filtration on $H^k_{dR}(X)$ is defined by

$$F^l_{\text{tot}} := F^l_{\text{tot}} H^k_{dR}(X) := \sum_{p+q=l} F^p \cap \bar{F}^q.$$  

Denote by $(H^k_{dR}(X))_l := gr^l_F H^k_d(X) := F^l_{\text{tot}}/F^{l+1}_{\text{tot}}$ the $l$-th associated graded and write

$$H^{p,q} := \text{im} \left( F^p \cap \bar{F}^q \rightarrow (H^k_{dR}(X))_{p+q} \right) = \frac{F^p \cap \bar{F}^q}{F^{p+1} \cap \bar{F}^q + F^p \cap F^{q+1}}.$$  

**Remark 4.2.** The total filtration is invariant under conjugation and may therefore also be considered as a filtration on the real cohomology.

**Remark 4.3.** In the following, we will mainly care about the spaces $H^{p,q}$. The reader preferring a quicker but less symmetric definition may verify the existence of a natural isomorphism $H^{p,q} \cong gr^p_F \cdot gr^q_{\bar{F}} H^k_d(X)$.

By definition, we obtain:

**Lemma 4.4 (Weak Hodge decomposition).**

$$(H^k_{dR}(X))_l = \bigoplus_{p+q=l} H^{p,q}$$

The spaces occurring in this decomposition for $p+q \neq k$ therefore measure the defect of purity.

**Definition 4.5.** The numbers $b^{p,q}_k := b^{p,q}_k(X) := \dim H^{p,q}_k(X)$ are called the refined Betti numbers of $X$.

Since an associated graded of a filtered vector space has the same dimension as the original one, we obtain $b_k = \sum_{p+q=k} b^{p,q}_k$ and by construction $b_k = \sum_{p+q=k} b^{p,q}_k$ if and only if $F$ and $\bar{F}$ induce a pure Hodge structure. Outside certain triple degrees, the $b^{p,q}_k$ have to vanish (see Proposition 4.7 below). One may picture the $b^{p,q}_k$ as giving a three-dimensional analogue of the Hodge diamond, or equivalently a diamond for each $k$, e.g. for $n=2$:

$$\begin{array}{cccc}
& k_2^{2,2} & & \\
& k_2^{1,1} & k_2^{2,1} & k_2^{1,2} & k_2^{2,2} \\
k_1^{0,0} & k_1^{1,0} & k_1^{0,1} & k_1^{1,1} & k_1^{1,2} \\
k_1^{1,0} & k_1^{0,1} & k_1^{1,1} & k_1^{1,2} \\
k_2^{0,0} & & & \\
\end{array}$$
For Kähler (or \(\partial\bar{\partial}\)-)manifolds, there is an isomorphism \(H^{p,q}_\partial(X) \cong H^{p,q}_{\partial\bar{\partial}}(X)\), so the refined Betti numbers are a different generalisation of the Hodge numbers on Kähler manifolds.

For an explicit understanding of the spaces \(H^{p,q}_k\), the following Lemma is useful. It is the general version of the well-known fact that for \(X\) Kähler (or more generally carrying a pure Hodge structure), the spaces \(F^p \cap F^q \subseteq H^k_{\partial\bar{\partial}}(X)\) with \(p+q=k\) consist of classes representable by pure-type forms.

**Lemma 4.6.** Fix some \(k \in \mathbb{Z}_{\geq 0}\). The subspaces \(F^p \cap F^q \subseteq H^k_{\partial\bar{\partial}}(X)\) allow the following explicit description:

If \(p + q \geq k\):

\[
F^p \cap F^q = \left\{ \text{classes that admit a representative } \omega \in A^{r,s} \right\}
\]

For all \((r,s) \in \{(k-q,q), ..., (p,k-p)\}\).

If \(p + q \leq k\):

\[
F^p \cap F^q = \left\{ \text{classes that admit a representative } \omega \right\}
\]

Such that \(\omega = \sum_{j=p}^{k-q} \omega_{j,k-j} \) with \(\omega_{r,s} \in A^{r,s}\).

**Proof.** For an element \(\omega \in A_X\), let us denote by \(\omega^{r,s}\) its component in bidegree \((r,s)\). By definition, a class \(\epsilon \in H^{p,q}_\partial(X)\) is in \(F^p \cap F^q\) if it has a representative \(\omega = \sum_{r+s=k} \omega^{r,s}\) with \(\omega^{r,s} = 0\) for \(r < p\) and another one \(\omega' = \sum_{r+s=k} \omega'^{r,s}\) with \(\omega'^{r,s} = 0\) for \(s < q\). So the inclusions from right to left are immediate and it remains to show the converse.

Let \(\omega, \omega'\) be two representatives of a class \(\epsilon \in F^p \cap F^q\) as above and let \(\eta = \sum_{r,s \in \mathbb{Z}} \eta^{r,s}\) be of total degree \(k-1\) with \(\omega = \omega' + \eta\). This gives us a sequence of equations

\[
\omega^{r,s} = \omega'^{r,s} + \partial \eta^{r-1,s} + \bar{\partial} \eta^{r,s-1}.
\]

Set \(\bar{p} := \max\{p, k-q\}\) and \(\bar{q} := \max\{q, k-p\}\). Replacing \(\omega\) by the cohomologous \(\omega - \sum_{i \geq \bar{p}} d\eta^{i,k-i-1}\) and \(\omega'\) by \(\omega' - \sum_{i \geq \bar{q}} d\eta^{k-i-1,i}\), we may assume that \(\eta^{r,s} = 0\) for \(r \geq \bar{p}\) or \(s \geq \bar{q}\), \(\omega'^{r,s} = 0\) for \(r \not\in [p, \bar{p}]\) and \(\omega'^{r,s} = 0\) for \(r \not\in [q, \bar{q}]\).

Now we distinguish two cases. If \(\bar{p} = k - q\), i.e. \(k \geq p + q\), we are done. If \(\bar{p} = p > k - q\), the element \(\omega = \omega^{p,k-p}\) is pure of bidegree \((p, k-p)\) and \(\omega' = \omega^{k-q,q}\) pure of bidegree \((k-q,q)\). By (\(\ast\)), we obtain \(\omega = \partial \eta^{p-1,d-p}\) which is cohomologous to the pure element \(-\bar{\partial} \eta^{p-1,k-p}\). Applying the same reasoning over and over again, we obtain representatives for \(\epsilon\) that are pure in degrees \((k-q,q), ..., (p,k-p))\).

**Proposition 4.7.** The refined Betti numbers of an \(n\)-dimensional compact complex manifold \(X\) satisfy the following universal relations:

(B1) **Bounded support:** If \(k \leq n\), one has \(b^p_q(X) = 0\) unless \(0 \leq p, q \leq k\).

(B2) **Conjugation symmetry:** \(b^p_q(X) = b^q_p(X)\) for all \(p, q, k \in \mathbb{Z}\).

(B3) **Serre symmetry:** \(b^p_q(X) = b^{n-p,n-q}(X)\).
(B4) Boundary case vanishing: \( b_{1}^{1}(X) = 0 \) and \( b_{n}^{n,p}(X) = 0 \) for \( p > 0 \).

**Proof.** (B1) and (B2) follow directly from the definitions.

For (B3), consider the map of double complexes \( A_{X} \to DAX \), where \( DAX \) is the dual double complex, given by \( (DAX)^{p,q} := \text{Hom}(A_{X}^{n-p,n-q}, \mathbb{C}) \) with total differential

\[
delta^{p,q}_{DAX} := (\varphi \mapsto (-1)^{p+q+1} \varphi \circ d^{2n-p-q-1}).
\]

Noting that for the first page of the spectral sequence associated with the column filtration of \( DAX \) one has

\[
E^{p,q}_{1}(DAX) = (E^{n-p,n-q}_{1}(A_{X}))^\vee = (H^{n-p,n-q}_{A_{X}})^\vee,
\]

Serre duality [35] implies that the integration \( A_{X} \to DAX, \omega \mapsto \int_{X} \omega \wedge \cdot \) induces an isomorphism on the first pages of the spectral sequences. Therefore also

\[
E^{p,q}_{r}(A_{X}) \cong (E^{n-p,n-q}_{r}(A_{X}))^\vee
\]

for all \( r \geq 1 \). Thus:

\[
g^{p}_{F}H_{dR}^{k}(X) = E_{\infty}^{p-k,p}(A_{X}) \cong (E^{n-p,n-k+p}_{\infty}(A_{X}))^\vee = (g^{n-p}_{F}H_{dR}^{2n-k}(X))^{\vee}.
\]

As a consequence, integration induces isomorphisms

\[
F^{p}H_{dR}^{k}(X) \cong (F^{2n-p}H_{dR}^{2n-k}(X))^{\vee}
\]

for all \( p \). By conjugation, the same holds for the \( \bar{F} \). Thus, we obtain the desired isomorphisms \( H_{dR}^{p,q} \cong H^{p,n-q}_{dR} \).

(B4) follows from well-known arguments (see e.g. [6] Ch. IV Lem. 2.1-2.3]). We sketch the argument for the reader’s convenience: For the first part of (B4), pick an element \( \epsilon \in H_{dR}^{1,1} = F^{1} \cap F^{1} \) and two representatives \( \omega^{1,0} \in A_{X}^{1,0} \) and \( \omega^{0,1} \in A_{X}^{0,1} \). Thus, \( \omega^{0,1} - \omega^{1,0} = df \), i.e. \( \omega^{0,1} = \partial f \). Hence \( \partial \bar{\partial} f = d\omega^{0,1} = 0 \), i.e. \( f \) is pluriharmonic. But since \( X \) is compact, \( f \) has to be constant and therefore \( \epsilon = 0 \). Since \( \epsilon \) was arbitrary, this shows \( H_{dR}^{1,1} = 0 \).

As for the second part, let \( \epsilon \in F^{n} \cap F^{p} \subseteq H_{dR}^{0}(X) \) with \( p > 0 \). By Lemma 4.6 there exists a representative \( \omega \) of type \( (n,0) \) but also another one, say \( \eta \), of type \( (n-p,p) \). Writing out the equation \( \omega - \eta = d\theta \) by bidegrees, we see that \( \omega \) is a \( \partial \)-exact holomorphic \( n \)-form. Using Stokes’ theorem, this implies \( \omega = 0 \). Thus, in particular \( H_{dR}^{n,p} = 0 \) for \( p > 1 \).

**Proposition 4.8.** For a product \( X \times Y \) of compact complex manifolds, the Künneth formula is strictly compatible with the Hodge filtration and its conjugate. In particular, the following relation holds:

\[
b^{p,q}_{k}(X \times Y) = \sum_{k_{1} + k_{2} = k \atop p_{1} + p_{2} = p \atop q_{1} + q_{2} = q} b^{p_{1},q_{1}}_{k_{1}}(X) \cdot b^{p_{2},q_{2}}_{k_{2}}(Y)
\]

**Proof.** Recall that a map of filtered vector spaces \( f : (V,F^{\cdot}) \to (W,F^{\cdot}) \) is called strict if \( f(F^{p}) = \text{im} f \cap F^{p} \) for all \( p \).

The Künneth isomorphism in de Rham cohomology is induced by the map \( \pi_{X}^{\vee} \otimes \pi_{Y}^{\vee} : A_{X} \otimes A_{Y} \to A_{X \times Y} \) given by the two pullback maps. This is a
map of double complexes and the grading on $A_X \otimes A_Y$ is the tensor product grading. The Künneth formula in Dolbeault cohomology states that it induces an isomorphism in Dolbeault cohomology, i.e. the first page of the Fröhlicher spectral sequence. Hence, there are induced maps on all later pages which are isomorphisms as well. In particular this holds for the page $E_{\infty}$. But that page is the graded vector space associated with the Hodge filtration on $H_{dR}$ and a map of filtered vector spaces is a strict isomorphism if and only if the map of the associated graded is an isomorphism. The result for the conjugate filtration follows since $\pi^\ast X \otimes \pi^\ast Y$ is, as all maps of geometric origin, conjugation invariant.

5 The refined de Rham ring

By Proposition 4.8, we obtain a ring homomorphism

$$ rb : \mathcal{CM}_* \longrightarrow \mathbb{Z}[x, y, h, z] $$

$$ [X] \mapsto rb(X) := \sum_{p, q, k} b^{p, q}_k (X) x^p y^q h^k z \dim X. $$

We denote the image by $\mathcal{R}B_*$ and call it the refined de Rham ring.

**Definition 5.1.** $\mathcal{R}B^\text{form}_n$ and $\mathcal{R}B'_n$ are the graded subrings of $\mathbb{Z}[x, y, h, z]$ defined by:

$$ \mathcal{R}B^\text{form}_n := \left\{ \left( \sum_{p, q, k \geq 0} b^{p, q}_k x^p y^q h^k \right) \cdot z^n \in \mathbb{Z}[x, y, h, z] \left| \begin{array}{l} b^{p, q}_k \text{ subject to conditions (B1) – (B4) of Prop. 4.7} \end{array} \right. \right\} $$

and

$$ \mathcal{R}B'_n := \left\{ \left( \sum_{p, q, k \geq 0} b^{p, q}_k x^p y^q h^k \right) \cdot z^n \in \mathbb{Z}[x, y, h, z] \left| \begin{array}{l} b^{p, q}_k \text{ subject to conditions (B1) – (B3) of Prop. 4.7} \end{array} \right. \right\}. $$

**Theorem 5.2.** There is a surjective map of rings

$$ \Phi : \mathbb{Z}[A, B, C, L, M] \longrightarrow \mathcal{R}B'_* $$

$$ A \mapsto (1 + xyh^2)z $$

$$ B \mapsto (xh + yh)z $$

$$ C \mapsto xyh^2z^2 $$

$$ L \mapsto (h + x^2y^2h^3)z^2 $$

$$ M \mapsto (xyh + h)z. $$

Setting $|A| = |B| = |M| = 1$ and $|C| = |L| = 2$, it is compatible with the grading. The kernel is given by the principal ideal $I$ generated by

$$ AML - L^2 - CM^2 - CA^2 + 4C^2. $$

**Proof.** The degree $n$ part $\mathcal{R}B'_n$ is a free $\mathbb{Z}$ module. The proof of surjectivity works by writing down preimages for every element in a basis. The computation of the kernel will follow from a rank counting argument.
Fix some degree \( n \in \mathbb{Z}_{\geq 0} \). For \( k \in \{0, \ldots, 2n\} \) and \( p, q \in [\max(k - n, 0), \min(k, n)] \cap \mathbb{Z} \), define the polynomial
\[
\text{Sym}_{k}^{p,q}(n) := \alpha_{k,n}^{p,q} (x^{p}y^{q}h^{k} + x^{q}y^{p}h^{k} + x^{n-p}y^{n-q}h^{2n-k} + x^{n-q}y^{n-p}h^{2n-k})z^{n} \in \mathcal{RB}^{n}_{k},
\]
where \( \alpha_{k,n}^{p,q} \in \{1, \frac{1}{2}, \frac{1}{4}\} \) is the number of distinct monomials occurring divided by four. To explain the need for \( \alpha \), there are two involutions acting on the degree \( n \) part of \( \mathbb{Z}[x,y,h,z] \): the one flipping everything along the diagonal, i.e. \( x^{p}y^{q} \mapsto x^{q}y^{p} \), and the one flipping along the antidiagonal, i.e. \( x^{p}y^{q}h^{k}z^{n} \mapsto x^{n-p}y^{n-q}h^{2n-k}z^{n} \). These combine into an action of \((\mathbb{Z}/2\mathbb{Z})^{2}\). Conditions (B2) and (B3) ensure that elements in \( \mathcal{RB}^{n}_{k} \) are invariant with respect to this action. The polynomial \( \text{Sym}_{k}^{p,q}(n) \) is the sum of the elements in the orbit of \( x^{p}y^{q}h^{k}z^{n} \).

**Lemma 5.3.** The following polynomials constitute a basis for \( \mathcal{RB}^{n}_{k} \):
\[
\text{Sym}_{k}^{p,q}(n) \text{ for } 0 \leq k \leq n - 1 \text{ and } 0 \leq q \leq p \leq k
\]
\[
\text{Sym}_{k}^{p,q}(n) \text{ for } 0 \leq q \leq p \leq n \text{ and } p + q \leq n
\]
The rank of \( \mathcal{RB}^{n}_{k} \) is
\[
\text{rk } \mathcal{RB}^{n}_{k} = \left[ \frac{2n^{3} + 9n^{2} + 16n + 12}{12} \right].
\]

**Proof.** An arbitrary element \( P \in \mathcal{RB}^{n}_{k} \) is a sum of monomials subject to condition (B1). By the symmetry conditions (B2) and (B3), for every monomial \( a \cdot x^{p}y^{q}h^{k}z^{n} \) occurring in \( P \), every monomial in \( \text{Sym}_{k}^{p,q}(n) \) occurs with coefficient \( a \). Hence the \( \text{Sym}_{k}^{p,q}(n) \) with only condition (B1) as restriction on \( p, q, k \) form a generating set. It remains to reduce the redundancy caused by the symmetries \( \text{Sym}_{p}^{p,q}(n) = \text{Sym}_{k}^{p,q}(n) = \text{Sym}_{n-p,n-q}(n) \).

For fixed \( k \) and \( n \), one may picture the allowed monomials \( x^{p}y^{q}h^{k}z^{n} \) occurring as summands in elements of \( \mathcal{RB}^{n}_{k} \) in a ‘Hodge diamond’, e.g. for \( k = 2, n \geq 2 \), the following (up to multiplication with \( h^{2}z^{n} \)):
\[
\begin{array}{ccccccc}
  x^{2}y^{2} & x^{2}y^{1} & x^{1}y^{2} & x^{2} & xy & y^{2} & x \\
  & & &  & & & y \\
  & & & 1 & & & \\
\end{array}
\]

By the symmetry under exchange of \( x \) and \( y \), one only has to consider the left half (including the central column) of each diamond. Similarly, by the symmetry along \( x^{p}y^{q}h^{k}z^{n} \mapsto x^{n-p}y^{n-q}h^{2n-k}z^{n} \), one only has to consider the first \( n - 1 \) of the diamonds and the lower half (including central row) of the \( n \)-th diamond and there are no further redundancies. The \( \text{Sym}_{k}^{p,q}(n) \) in the statement of the lemma are exactly those such that \( x^{p}y^{q}h^{k}z^{n} \) lies in both of these regions.

It remains to show the statement of the rank. For the \( k \)-th diamond, where \( 0 \leq k \leq n - 1 \), one obtains \( T_{k+1} = 1 + 2 + \ldots + (k+1) = \frac{(k+2)(k+1)}{2} \) polynomials.
For the \( n \)-th diamond, one only has to count monomials in the lower left quarter. If \( n \) is odd (resp. even), this amounts to summing the even (resp. odd) numbers between 0 and \( n + 1 \), yielding \( \frac{(n+1)(n+3)}{2} \) (resp. \( \frac{(n+2)^2}{2} \)). Since the sum of the first \( r \) triangular numbers is \( \frac{r(r+1)(r+2)}{6} \), one obtains a total dimension of

\[
\text{rk } \mathcal{RB}'_n = \frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+3)}{4} = 2n^3 + 9n^2 + 16n + 9
\]

if \( n \) is odd and

\[
\text{rk } \mathcal{RB}'_n = \frac{n(n+1)(n+2)}{6} + \left(\frac{n+2}{2}\right)^2 = 2n^3 + 9n^2 + 16n + 12
\]

for \( n \) even.

Our goal is thus to show that all the basis elements \( \text{Sym}^{p,q}_k(n) \) lie in the subring generated by the images of \( A, B, C, M, L \) above. We do this by induction on \( n \).

For \( n = 1 \), one has \( A \mapsto \text{Sym}^{0,0}_0(1) \), \( B \mapsto \text{Sym}^{1,0}_1(1) \) and \( M \mapsto \text{Sym}^{0,0}_0(1) \) by definition.

From now on, let \( n \geq 2 \).

Since \( \Phi(C) = xyh^2z^2 = \text{Sym}^{1,1}_2(2) \) consists of one monomial, one has

\[ \text{Sym}^{p,q}_k(n) \cdot \Phi(C) = \text{Sym}^{p+1,q+1}_k(n+2) \]

Thus, if \( p, q \not\in \{0, k\} \) (and hence necessarily \( k \geq 2 \)) one has

\[ \text{Sym}^{p,q}_k(n) = \text{Sym}^{p-1,q-1}_{k-2}(n-2) \cdot \Phi(C) \]

and the left factor may assumed to be in the image of \( \Phi \) by induction.

If \( k \) is arbitrary, \( q = 0 \) and \( p = k \) one has, modulo \( \Phi(C) \):

\[
\text{Sym}^{k,0}_k(n) = \alpha^{k,q}_{k,n} \cdot (x^dh + y^kh + x^{-k}y^{n-k}h^{2n-k} + x^n y^{-k}h^{2n-k})z^n
\]

\[
\equiv [(xh+yh)]^k \cdot [(1 + xyh^2)z]^{n-k}
\]

\[
= \Phi(B)^k \Phi(A)^{n-k} \in \text{im } \Phi.
\]

Let now \( q = 0 \) and \( n > k > p \geq 0 \). One has

\[
\Phi(L) \cdot \text{Sym}^{p,0}_{k-1}(n) = \text{Sym}^{p,0}_k(n) + \text{Sym}^{p+2,2}_{k+2}(n)
\]

\[
\equiv \text{Sym}^{p,0}_k(n) \mod \Phi(C)
\]

and so \( \text{Sym}^{p,0}_k(n) \) is in the image of \( \Phi \) by induction.

Next, let \( n > p = k \geq q \geq 1 \). One computes

\[
\Phi(M) \cdot \text{Sym}^{k-1,q-1}_{k-1}(n-1) = \text{Sym}^{k,q}_k(n) + \text{Sym}^{k-1,q-1}_k(n)
\]

and so by induction \( \text{Sym}^{p,q}_k(n) \) is in the image of \( \Phi \) whenever \( \text{Sym}^{k-1,q-1}_k(n) \) is. But if \( q \geq 2 \), the latter is a multiple of \( \Phi(C) \) and for \( q = 1 \) it is \( \text{Sym}^{k-1,0}_k(n) \),
which is in the image of $\Phi$ by the previous paragraph.

It remains to treat the case $k = n$ and $p$ or $q$ in $\{0, n\}$. For $q = 0$ and $n - 1 > p \geq 0$, one has the equation

$$\Phi(M) \cdot \text{Sym}^{n-1,0}(n-1) = \text{Sym}^{n-1,0}(n) + \text{Sym}^{n+1,1}(n)$$

and the right summand is a multiple of $\Phi(C)$. For $p = n - 1, q = 0$, one has directly

$$\Phi(M) \cdot \text{Sym}^{n-1,0}(n-1) = \text{Sym}^{n-1,0}(n).$$

Since for $k = n$, one has the restriction $p + q \leq n$, this ends the verification of surjectivity.

**Lemma 5.4.** The degree $n$ part of $\mathbb{Z}[A, B, C, L, M]/I$ has rank

$$r_n := \text{rk}(\mathbb{Z}[A, B, C, L, M]/I)_n = \left\lceil \frac{2n^3 + 9n^2 + 16n + 12}{12} \right\rceil.$$

**Proof.** Recall that $A, B, M$ have degree 1 and $C, L$ degree 2. Hence, the generating function for the numbers $s_n := \text{rk}\mathbb{Z}[A, B, C, L, M]_n$ is given by

$$\sum_{n=0}^{\infty} s_nt^n = \left( \sum_{i=0}^{\infty} i^3 \right) \cdot \left( \sum_{i=0}^{\infty} i^{2i} \right)^2 = \left( \sum_{i=0}^{\infty} \left( \frac{i+2}{2} \right)^i \right) \cdot \left( \sum_{i=0}^{\infty} \left( \frac{i+1}{1} \right) i^{2i} \right) = \sum_{n=0}^{\infty} \left( \sum_{i+2j=n} \left( \frac{i+2}{2} \right) (j+1) \right) t^n.$$

Since $I$ is a principal ideal generated by an element of degree 4, one has $r_n = s_n - s_{n-4}$. In particular, $r_n = s_n$ for $n = 0, 1, 2, 3$ and in these cases the formula of the lemma is easily checked directly (the values being $r_0 = 1, r_1 = 3, r_2 = 8$ and $r_3 = 16$). The other cases follow by induction using

$$s_n = s_{n-2} + \sum_{i+2j=n} \left( \frac{i+2}{2} \right).$$

From this, we obtain the second part of the theorem, since $\Phi$ vanishes on the generator of $I$ and hence induces in each degree a surjective map

$$(\mathbb{Z}[A, B, C, M, L]/I)_n \twoheadrightarrow \mathbb{R}B'_n.$$

But both sides are free $\mathbb{Z}$-modules of the same rank, so the map has to be injective as well.
Remark 5.5. One may now check that $A, B, C, L$ can indeed be realized by complex manifolds as

$$
A = rb([\mathbb{CP}^1]) \\
B = rb([E] - [\mathbb{CP}^1]) \\
C = rb([\mathbb{CP}^1 \times\mathbb{CP}^1] - [\mathbb{CP}^2]) \\
L = rb([H] - 2[\mathbb{CP}^2] + [\mathbb{CP}^1 \times\mathbb{CP}^1]),
$$

where $E$ denotes an elliptic curve and $H$ a Hopf surface. In particular, we recover $\mathcal{H}^E = \mathbb{Z}[A, B, C] \subseteq \mathcal{RB}_*$. However, $M$ is not in the image of $rb$, since it violates condition (B4). For instance, it would correspond to a (formal linear combination of) curve(s) not satisfying the $\partial\bar{\partial}$-lemma. So we see that there is a single generator being responsible for the inclusion $\mathcal{RB}_* \subseteq \mathcal{RB}'_*$ failing to be an equality.

It remains possible (and plausible) that the ring $\mathcal{RB}^*_\text{form}$ equals $\mathcal{RB}_*$. To show this, one might proceed as before: write down generators for $\mathcal{RB}^*_\text{form}$ and show that they can all be realized by $\mathbb{Z}$-linear combinations of actual compact complex manifolds. Unlike in the previous cases, however, it turns out that $\mathcal{RB}^*_\text{form}$ is not finitely generated:

Theorem 5.6. Under the isomorphism

$$
\Phi : \mathbb{Z}[A, B, C, L, M]/I \rightarrow \mathcal{RB}_*,
$$

the subring $\mathcal{RB}^*_\text{form} \subseteq \mathcal{RB}_*$ corresponds to the subring generated by $A$, $B$, $C$, $L$, $ABM$, $CM$ and the collection $AM^{n+1}$, $M^nL$ for $n \in \mathbb{Z}_{\geq 1}$.

In terms of the polynomials $\text{Sym}^{p,q}_k(n)$, a set of generators for $\mathcal{RB}^*_\text{form}$ is given by

$$
\text{Sym}^{0,0}_0(1) = \Phi(A), \quad \text{Sym}^{1,0}_1(1) = \Phi(B), \quad C = \text{Sym}^{1,1}_2(2), \quad \text{Sym}^{0,0}_1(2) = \Phi(L),
$$

the polynomials

$$
\text{Sym}^{2,1}_2(3) = \Phi(ABM) - \Phi(BL) \\
\text{Sym}^{1,1}_3(3) = \Phi(CM)
$$

and the collection of polynomials

$$
L_n := \text{Sym}^{0,0}_{n-1}(n) \\
M_n := \text{Sym}^{n-1,n-1}_{n-1}(n)
$$

for $n > 2$.

Remark 5.7. The above theorem presents one infinite generating set. A priori it may still be possible to choose a finite collection of generators for $\mathcal{RB}^*_\text{form}$. However, such a collection would contain an element of maximal degree, and one may verify that $L_n$ and $M_n$ do not lie in the ideal generated by elements of degree $< n$.

As in Theorem 1.4, instead of $\mathbb{CP}^2$ and $H$, we may also use an arbitrary Kähler surface of signature $\pm 1$ and an arbitrary non-Kähler surface, although the exact formulas might differ.
Proof. Recall that $\mathcal{RB}_e^{form} \subseteq \mathcal{RB}_e'$ is defined by

$$\mathcal{RB}_e^{form} = \{ \sum b_k^{p,q}(n) \cdot \text{Sym}_k^{p,q}(n) \in \mathcal{RB}_n' \mid b_n^{p,0}(n) = b_1^{1,1}(n) = 0 \ \forall p < n \} \subseteq \mathcal{RB}_e'$$

and that $\Phi$ gives a surjective map

$$\Phi : \mathbb{Z}[A,B,C,M,L] \rightarrow \mathcal{RB}_e'.$$

We are going to compute the inverse image of $\mathcal{RB}_e^{form}$ under this map. Denote

$$S := \left\{ \sum_{a,b,c,d,e \in \mathbb{Z}_{\geq 0}} c_{a,b,c,d,e} A^a B^b C^d M^e L^f \mid c_{a,0,0,1,0} = c_{0,b,0,d,0} = 0 \ \forall a, b, d - 1 \geq 0 \right\}$$

and define $\overline{S}$ to be the reduction of $S$ modulo $I$. The set $S$ (and a fortiori $\overline{S}$) is a graded subring of $\mathbb{Z}[A,B,C,M,L]$ (resp. $\mathbb{Z}[A,B,C,M,L]/I$).

**Claim:** $\overline{S}$ is generated by $A, B, C, L, ABM, CM$ and the collection $AM^{n+1}, M^n L$ for $n \in \mathbb{Z}_{\geq 1}$.

Indeed, the (images of) the monomials $A^a B^b C^d M^e L^f$ which do not violate the defining conditions of $S$ generate $\overline{S}$. Fix such a monomial. If $d = 0$, it is a product of $A, B, C$ and $L$.

If $d \geq 1$ and $c \geq 1$, it is a product of $A, B, C, L$ and some $M^n L$.

If $c = 0$ and $d = 1$, either $c \geq 1$ or $a \neq 0 \neq b$. In the first case we have a product of $A, B, C, L$ and in the second case a product of $A, B$ and $ABM$.

If $c = 0$ and $d \geq 2$, either $c = 0$, in which case $a \geq 1$ and we have a product of $A, B$ and $AM^d$. Or $c \geq 1$ and, using that we work modulo $I$,

$$A^a B^b C^d M^e \equiv A^a B^b C^{d-1}(AM^{d-1} L - L^2 M^{d-2} - C A^2 M^{d-2} + 4 C^2 M^{d-2}).$$

The first three summands in the bracket are multiples of the claimed generators, so one may inductively reduce to the case $d \leq 1$, where the claim has been proven.

**Claim:** $\Phi(S) \subseteq \mathcal{RB}_e^{form}$.

Since $\Phi$ factors through reduction modulo $I$, this may be checked on the generators for $\overline{S}$, the nontrivial cases being $AM^{n+1}$ and $M^n L$. There we have

$$\Phi(AM^n) = \text{Sym}_0^{0,0}(1) \cdot [\text{Sym}_1^{0,0}(1)]^n$$

$$= (1 + xyh^2)z \cdot [(h + xyh)z]^n$$

$$= (1 + xyh^2)z \cdot \left( \sum_{i=0}^{n} \binom{n}{i} x^i y^i h^n z^i \right)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \text{Sym}_n^{i,i}(n + 1) \in \mathcal{RB}_{n+1}'$$
and

\[
\Phi(LM^n) = Sym_{1}^{0,0}(2) \cdot [Sym_{1}^{0,0}(1)]^n = (h + x^2y^2h^3)z^2 \cdot [(h + xyh)z]^n = (h + x^2y^2h^3)z^2 \cdot \left( \sum_{i=0}^{n} \binom{n}{i} x^iyh^nz^n \right) \\
= \sum_{i=0}^{n} \binom{n}{i} Sym_{n+1}^{i,i}(n + 2) \in RB_{n+2}^\prime.
\]

By staring for a moment at the defining conditions for \( S_n \) and \( RB_{n}^{\text{form}} \), one arrives at the following

**Observation:** For each \( n \), the modules \( R_n := \mathbb{Z}[A, B, C, M, L]_n / S_n \) and \( RB_{n}^{\prime}/RB_{n}^{\text{form}} \) are free of the same rank, which equals \( n \) for \( n = 0, 1 \) and \( n + 1 \) otherwise.

Since \( \Phi \) is surjects onto \( RB_{n}^{\prime} \), the cokernels \( R_n \sim = RB_{n}^{\prime} \) are isomorphic. Hence, applying the five-lemma to the map of short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & S_n & \rightarrow & \mathbb{Z}[A, B, C, M, L]_n & \rightarrow & R_n & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & RB_{n}^{\text{form}} & \rightarrow & RB_{n}^{\prime} & \rightarrow & RB_{n}^{\prime}/RB_{n}^{\text{form}} & \rightarrow & 0
\end{array}
\]

one obtains surjectivity of \( \Phi \) from \( S \) to \( RB_{n}^{\text{form}} \) and therefore an isomorphism \( S/(S \cap I) \cong RB_{n}^{\text{form}} \).

To complete the proof, we have to show that instead of \( \Phi(AM^n) \) and \( \Phi(LM^n) \), we might just as well take \( L_n = Sym_{n-1}^{0,0}(n) \) and \( M_n = Sym_{n-1}^{n-1,n-1}(n) \) as generators. In fact, by the formula for \( \Phi(LM^n) \) above, we have

\[
\Phi(LM^n) - L_{n+2} = \Phi(C) \cdot \sum_{i=1}^{n} \binom{n}{i} Sym_{n-1-i}^{i-i}(n),
\]

and the all summands in the sum on the right are again in \( RB_{n}^{\prime} \), so the claim follows by induction. Similarly,

\[
\Phi(AM^n) - L_{n+1} - M_{n+1} = \sum_{i=1}^{n} \binom{n}{i} Sym_{n-2-i}^{i-i}(n + 1) = \Phi(C) \cdot \sum_{i=1}^{n} \binom{n}{i} Sym_{n-2-i}^{i-i}(n - 1).
\]

**Remark 5.8.** We have already seen in Remark 5.5 that \( RB_{\leq 2} = RB_{\leq 2}^{\text{form}} \). We will also exhibit manifolds realizing all generators in degree 3, all \( L_n \), and the \( M_n \) for \( n \) odd. In particular, this shows that not only \( RB_{n}^{\text{form}} \) but also \( RB_{n}^{\prime} \) itself cannot be finitely generated. Since some of this discussion overlaps with construction problems appearing in the next sections, we delay it to Section 8.
6 Higher pages of the Frölicher spectral sequence

Instead of only the Dolbeault cohomology, we can also take into account higher pages of the Frölicher spectral sequence

$$E^p_q(X) = H^p_{\partial}(X) \Rightarrow H^{p+q}_{dR}(X).$$

**Definition 6.1.** Let $X$ be a compact complex manifold. The $r$-th Frölicher numbers are the integers

$$e^{p,q}_r(X) := \dim E^{p,q}_r(X).$$

The Künneth-formula for Dolbeault cohomology implies a Künneth formula for all higher pages of the spectral sequence (see proof of Proposition 4.8). Hence there is a homomorphism

$$e_r : \mathcal{CM}^* \longrightarrow \mathbb{Z}[x, y, z],$$

induced by sending an $n$-dimensional manifold $X$ to its $r$-th Frölicher polynomial

$$X \mapsto e_r(X) := \sum_{p+q=0}^n e^{p,q}_r(X)x^py^qz^n.$$

Denote the image of this map by $\mathcal{H}^r_\ast$. In Section 4 we have considered the case $\mathcal{H}^1_\ast = \mathcal{H}_\ast$. Because the higher pages of the Frölicher spectral sequence also satisfy Serre-duality (see the proof of Proposition 4.7 or [37], [31], [26]), $\mathcal{H}^r_\ast$ is contained in $\mathcal{H}^r_{form} = \mathcal{H}_\ast$. Furthermore, we have seen that $\mathcal{H}^r_{form}$ is generated by manifolds for which the Frölicher spectral sequence degenerates. For such manifolds, the Hodge polynomial coincides with the $r$-th Frölicher polynomial for any $r \geq 1$. Hence, we obtain the following immediate generalisation of Theorem A and B:

**Theorem 6.2.** For any $r \geq 1$, there is an equality $\mathcal{H}^r_\ast = \mathcal{H}_\ast$. In particular, for any fixed $r \geq 1$, there are no universal linear relations between the $r$-th Frölicher numbers of compact complex manifolds other than the ones induced by Serre duality.

**Corollary 6.3.** For any $r \geq 1$, the only linear combinations between $r$-th Frölicher numbers which are bimeromorphic invariants are (up to Serre duality) linear combinations of the $e^{p,0}_r$ and $e^{0,q}_r$ only.

Of course, this leaves open the question of what the universal linear relations between Frölicher numbers for different $r$ are. For instance, the following ones are well-known:

1. Connected components: $e^{0,0}_1 = e^{0,0}_r$ for all $r$.
2. Euler Characteristics: let

$$\chi^r_p(X) := \sum_{q \geq 0} (-1)^q e^{p-2(q-1)r, q}_r(X).$$

Then $\chi^r_p(X) = \chi^s_p(X)$ for all $r, s \geq 1$. 

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3. No differential ends in degree \((n,0)\), i.e. \(e^{n,0}_r(X) = e^{n,0}_r(X)\) for all \(r\).

4. No differential starts in degree \((0,0)\), i.e. \(e^{0,0}_r(X) = e^{0,0}_r(X)\) for all \(r\).

5. For \(\dim \mathbb{C} X = n\), the Frölicher spectral sequence degenerates at the \(n\)-th page, i.e. \(e^{p,q}_r(X) = e^{p,q}_{r+k}(X)\) for all \(k \geq 0\). (This is obtained from the trivial degeneration bound by dimension combined with 3.)

6. For \(\dim \mathbb{C} X \leq 2\), the Frölicher spectral sequence degenerates at the first page, i.e. \(e^{p,q}_1(X) = e^{p,q}_2(X)\) in addition to the above.

As before, one obtains a map

\[ E : \mathcal{M}_* \longrightarrow (\mathbb{Z}[x,y])^N[z] \]

induced by collecting all Frölicher polynomials:

\[ X \mapsto \left( \sum_{p,q=0}^n e^{p,q}_r(X)x^py^q \right) \cdot z^{\dim \mathbb{C} X} \]

Denote its image by \(FS_*\).

**Theorem 6.4.** The ring \(FS_*\) is not finitely generated.

**Proof.** If there was a finite generating set, say \(X_1, \ldots, X_k\), then by property 5 above, all coefficients \((P_1, \ldots, P_r, \ldots) \in (\mathbb{Z}[x,y])^N\) of elements in \(FS_*\) would satisfy \(P_r = P_{r'}\) whenever \(r, r' \geq N := \max\{\dim \mathbb{C} X_i\}\). In other words, there would be a number \(N\) such that the Frölicher spectral sequence of all compact complex manifolds, regardless of their dimension, degenerates at stage \(N\). This is known to be false, see [7].

The conditions 1 – 6, together with Serre duality on each page, describe a natural candidate \(FS^\text{form}_*\) for the image. One could now proceed, as in the previous section, to compute generators for \(FS^\text{form}_*\) and try to realize them by compact complex manifolds. We do not spell this out for two reasons: On the one hand, the conditions defining \(FS^\text{form}_*\) are a bit technical and we believe the description of the universal ring of cohomological invariants in the next section is the better framework to incorporate the Frölicher numbers. On the other hand, we are quite far at the moment from actually realizing the generators by manifolds, and, without that step, the formal work does not seem to be rewarding. See Section 10 for a summary of the open construction problems.

## 7 The universal ring of cohomological invariants

In this section we connect the previous results with the theory developed in [37] and define a ring encoding ‘all’ cohomological invariants. The following structure theorem will underlie much of the discussion:

**Theorem 7.1** ([19], [37]).

1. Any bounded double complex of \(\mathbb{C}\)-vector spaces is a direct sum of indecomposable double complexes.
2. Every indecomposable double complex is isomorphic to one of the following:

(a) squares

\[
\begin{array}{ccc}
C & \rightarrow & C \\
\downarrow & & \downarrow \\
C & \rightarrow & C
\end{array}
\]

(b) even length zigzags

\[
\begin{array}{ccc}
C & \rightarrow & C \\
\downarrow & & \downarrow \\
C & \rightarrow & C \\
\downarrow & & \downarrow \\
C & \rightarrow & C \\
\end{array}
\]

(c) odd length zigzags

\[
\begin{array}{ccc}
C & \rightarrow & C \\
\downarrow & & \downarrow \\
C & \rightarrow & C \\
\downarrow & & \downarrow \\
C & \rightarrow & C \\
\end{array}
\]

3. For \(X\) a compact complex manifold and \(A_X\) its double complex of \(C\)-valued forms, in any decomposition \(A_X = A_X^{sq} \oplus A_X^{zig}\) into a summand consisting of squares and one consisting of zigzags, the latter is finite dimensional.

Let us call complexes satisfying the property in point 3 of Theorem 7.1 essentially finite dimensional. Denote by \(\mathcal{D}\) the category of essentially finite dimensional bounded double complexes of \(C\)-vector spaces with linear maps compatible with bigrading and differentials. Let \(\mathcal{C}\) denote the category of compact complex manifolds. Sending a manifold \(X\) to the double complex \(A_X\) and a holomorphic map to the pullback of forms yields a functor \(A : \mathcal{C} \rightarrow \mathcal{D}\).

**Definition 7.2** (c.f. [37]). Let \(V\) denote the category of finite dimensional \(C\)-vector spaces.

1. A linear functor \(H : \mathcal{D} \rightarrow V\) is called cohomological if it sends squares to zero.

2. A functor \(H : \mathcal{C} \rightarrow V\) is called cohomological if it factors as \(H = H' \circ A\) for a cohomological functor \(H' : \mathcal{D} \rightarrow V\).

3. For any cohomological functor \(H\) on \(\mathcal{C}\) (resp. \(\mathcal{D}\)), the composition \(\dim \circ H\), considered as a map from isomorphism classes of objects to \(\mathbb{N}\), is called a cohomological invariant of compact complex manifolds (resp. of double complexes).

**Example 7.3.** As mentioned in the introduction, the functors \(H^k_{dR}, H^p,q_{\phi}, H^p,q_{\phi}, H^p,q_{BC}, H^p,q_A\), \(H^p,q_{r_BC}, H^p,q_{r_A}\), as well as their higher page analogues \(E^p,q_{r_B}, E^p,q_{r_A}, E^p,q_{r_BC}, E^p,q_{r_A}\), the cohomologies of the Schweitzer complexes, the Varouchas groups and the \(H^p,q_k\) are all cohomological, so their dimensions yield cohomological invariants.

**Remark 7.4.** The reader preferring a less phenomenological justification for the name cohomological functor may find comfort in the fact that direct sums of squares are exactly the projective (and injective) objects in \(\mathcal{D}\) (see [19]).
The following Lemma was essentially shown in [37]:

**Lemma 7.5.** Let $X$ and $Y$ be $n$-dimensional compact complex manifolds. The following conditions are equivalent:

1. $h(X) = h(Y)$ for all cohomological invariants $h$.
2. $e^{p,q}_k(X) = e^{p,q}_k(Y)$ and $b^{p,q}_k(X) = b^{p,q}_k(Y)$ for all $k, r, p, q \in \mathbb{Z}$.
3. Every isomorphism type of zigzags occurs with the same multiplicity in any decomposition into indecomposables of $A_X$ and $A_Y$.
4. $A_X$ and $A_Y$ are $E_1$-isomorphic, i.e. there exists a map of double complexes $f : A_X \rightarrow A_Y$ s.t. $H_\partial(f)$ and $H_\bar{\partial}(f)$ are isomorphisms.

**Proof.** Clearly, $1 \Rightarrow 2$. By [37, sect. 2], the multiplicities of even zigzags are counted by differentials on the Frölicher spectral sequence and the multiplicities of odd length zigzags are counted by the refined Betti-numbers. In particular condition 3 is equivalent to $b^{p,q}_k(X) = b^{p,q}_k(Y)$ and $\dim \text{im } d^{p,q}_r \cap E_r(X) = \dim \text{im } d^{p,q}_r \cap E_r(Y)$ for all $r, p, q \in \mathbb{Z}$, where $d^{p,q}_r$ denotes the differential on the $r$-th page of the Frölicher spectral sequence. Since the complexes appearing on each page of the Frölicher spectral sequence are bounded, and each page can be computed as the cohomology of the previous one, the dimensions of the images of the differentials are determined by the dimensions $e^{p,q}_r$ for all $r, p, q \in \mathbb{Z}$, hence $2 \Rightarrow 3$. Since a linear functor commutes with direct sums, it is determined by its values on indecomposable complexes. Thus $3 \Rightarrow 1$. The equivalence between 3 and 4 is [37, Prop. 12].

Let $I \subseteq \mathcal{CM}^*$ be the ideal generated by differences $[X] - [Y]$ of equidimensional manifolds such that $h(X) = h(Y)$ for all cohomological invariants $h$.

**Definition 7.6.** The quotient $U_* := \mathcal{CM}^*/I$ will be called the **universal ring of cohomological invariants**.

By point 2 of Lemma 7.5 we obtain:

**Corollary 7.7.** The universal ring of cohomological invariants $U_*$ is isomorphic to the image of the diagonal map $\mathcal{CM}^* \rightarrow \mathcal{FS}_* \times \mathcal{RB}_*$.

The results of either Section 5 or 6 imply:

**Corollary 7.8.** $U_*$ cannot be finitely generated.
The following is a well-defined map of rings (37 sect. 4) and by Lemma 7.5, \( U_* \) may be identified with the image:

\[
CM_* \longrightarrow \mathbb{R}[z]
\]

\[
[X] \longmapsto [X] := [A_X] \cdot z^{\dim C_X} = \sum_{Z \text{ zigzag}} \text{mult}_Z(A_X)[Z]z^{\dim C_X}.
\]

The isomorphism class of any indecomposable complex \( Z \) is uniquely defined by its support \( \text{supp}(Z) = \{(p, q) \mid Z^{p,q} \neq 0\} \). For any isomorphism class of an indecomposable complex \( Z \), there is the conjugate isomorphism class \( \tau Z \) with support \( \text{supp}(\tau Z) = \{(p, q) \mid (q, p) \in \text{supp}(Z)\} \) and, fixing a degree \( n \), the dual isomorphism class \( \sigma Z \) with support \( \text{supp}(\sigma Z) = \{(n - p, n - q) \mid (p, q) \in \text{supp}(Z)\} \).

**Definition 7.9.** We denote by \( U^* \subseteq \mathbb{R}[z] \) the graded subring consisting in degree \( n \) of those elements \( \sum_Z m_Z[Z]z^n \) which satisfy:

1. bounded support: The support of \( Z \) is contained in \( \{0, ..., n\}^2 \)
2. Serre and conjugation symmetry: \( m_Z = m_{\sigma Z} = m_{\tau Z} = m_{\sigma \tau Z} \).
3. only dots in the corners: If \( \{P\} \subset \text{supp}(Z) \) for \( P = (0, 0) \) or \( (n, 0) \), then \( m_Z = 0 \).
4. degenerate FSS: \( m_Z = 0 \) for even length zigzags if \( n \leq 2 \).

We have \( U_* \subseteq U^* \) (c.f. 37 sect. 4). By construction \( U^* \) is the free abelian group on certain elements \( \sum_Z m_Z[Z]z^n \) which satisfy:

\[
U^* = \mathbb{Z} \cdot \star \oplus \mathbb{Z} \cdot \star
\]

\[
U^* = \mathbb{Z} \cdot \star \oplus \mathbb{Z} \cdot \star \oplus \mathbb{Z} \cdot \star \oplus \mathbb{Z} \cdot \star
\]

Sometimes it will be convenient to also have a non-diagrammatic way of writing down a specific \( Sym_Z(n) \). Recall from 37 that the isomorphism class of an odd zigzag \( Z \) is labeled as \( S^{p,q}_k \) if \( b^k_{p,q} \) is the unique nonzero refined Betti number of \( Z \). Similarly, if \( Z \) is an even zigzag s.t. there is a nonzero differential \( d : E^p_{r,q}(Z) \rightarrow E^{p+r,q-r+1}_{r+1}(Z) \) in the ‘Frölicher’ (i.e. column) spectral sequence, we denote its isomorphism class by \( S^{p,q}_{1,r} \). E.g.:

\[
Sym_{S_{1,1}^1}(3) = \begin{array}{cc}
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}
\]

\[
Sym_{S_{2}^2}(3) = \begin{array}{cc}
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}
\]

The first 1 in the subscript stands for the use of the column spectral sequence as opposed to the row spectral sequence. Even zigzags not contributing to the column spectral sequence contribute to the row spectral sequence and are labeled as \( S^{p,q}_{k,r} \).
Remark 7.10. There is a section to the map $U^\text{form}_* \rightarrow \mathcal{RB}^\text{form}_*$ given by

$$s: \mathcal{RB}^\text{form}_* \longrightarrow U^\text{form}_*$$

$$\text{Sym}^{p,q}_k(n) \mapsto \text{Sym}^{p,q}_{k,v}(n),$$

allowing us to identify $\mathcal{RB}^\text{form}_*$ with a subring of $U^\text{form}_*$. Note however that $s \circ rb(X) = [X] \in U_*$ only for manifolds whose Frölicher spectral sequence degenerates at the first page. We encourage the reader to revisit Section 5 from this diagrammatic point of view.

Lemma 7.11. The map

$$\mathcal{H}^K_* \longrightarrow U^\text{form}_*$$

$$x^py^qz^n \mapsto [S^{p,q}]_{z^n}$$

identifies the Hodge ring of Kähler manifold with the subring of $U^\text{form}_*$ consisting of linear combinations of zigzags of length one (‘dots’).

Under this identification, we have

$$A = [\mathbb{CP}^1] = \begin{array}{ccc} & \ast & \\ \ast & & \\ & & \end{array}$$

$$B = [E] - [\mathbb{CP}^1] = \begin{array}{ccc} & \ast & \\ \ast & & \\ & & \end{array}$$

and

$$C = [(\mathbb{CP}^1)^2] - [\mathbb{CP}^2] = \begin{array}{ccc} & & \\ & & \ast \\ & & \\ & & \end{array}$$

As before, we obtain a ring-theoretic reformulation of the question whether all linear relations between arbitrary cohomological invariants reduces to the ones listed above:

Question 7.12. Is $U_* = U^\text{form}_*$?

It will be often convenient to calculate modulo $\mathcal{H}^K_*$. E.g. for $S_{NK}$ any non-Kähler surface, we have

$$[S_{NK}] = \begin{array}{ccc} & & \\ & & \\ \ast & & \\ \ast & & \\ & & \end{array} \in U_2/\mathcal{H}^K_2,$$

which is a generator for $U^\text{form}_*/\mathcal{H}^K_2$.

Proposition 7.13.

1. There are equalities $\mathcal{RB}_{\leq 2} = U_{\leq 2} = U^\text{form}_{\leq 2}$

2. The quotient $U^\text{form}_*/U_3$ is either trivial or generated by

$$\text{Sym}^{1,3}_{1,0}(3) = \begin{array}{ccc} & & \\ & & \\ \ast & & \\ \ast & & \\ \ast & & \\ & & \end{array}$$
Proof. The first statement follows from the preceding discussion. For the second point, we write out a basis of $\mathcal{U}^\text{form}_3 / \mathcal{H}^K_3$:

\[
\frac{\mathcal{U}^\text{form}_3}{\mathcal{H}^K_3} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.
\]

We will see in the next section that all basis elements except possibly $\text{Sym}_{S^1,0}(3)$ can be realized by formal $\mathbb{Z}$-linear combinations of compact complex manifolds. For now, we note only that

\[
\mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z} + \mathbb{Z}
\]

so by part 1, we have $\text{Sym}_{S^1,0}(3) \in \mathcal{U}^\text{form}_3$ and it suffices to realize either $\text{Sym}_{S^1,0}(3)$ or $\text{Sym}_{S^1,1}(3)$.

\[\square\]

8 Construction of certain generators

We construct all remaining generators announced in Proposition 7.13 and Remark 5.8. In particular, this proves Corollary E.

8.1 Nilmanifolds

We briefly recall some elements of the theory of left-invariant complex structures on nilmanifolds. We refer for example to [33] and the references therein for further detail.

Let $\mathfrak{g}$ be a finite dimensional nilpotent real Lie algebra. Associated with it, there is a simply connected nilpotent Lie group $N$ and we may identify $\mathfrak{g}$ with the left invariant vector fields on $N$. An almost complex structure on $\mathfrak{g}$ is a direct sum decomposition of the complexified dual $\mathfrak{g}^\mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ such that $\mathfrak{g}^{1,0} = \mathfrak{g}^{0,1}$. It is said to be integrable, or a complex structure, if the exterior differential

\[
d : \mathfrak{g}^\mathbb{C} \longrightarrow \Lambda^2 \mathfrak{g}^\mathbb{C}
\]

\[
\omega \longrightarrow (\omega(X,Y) \mapsto -\omega([X,Y])
\]

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is of type $(1,0) + (0,1)$ with respect to the associated decomposition $\Lambda^\bullet g^c = \bigoplus_{p,q} \Lambda_{p,q}^g$, where $\Lambda_{p,q}^g = \Lambda^p g^{1,0} \otimes \Lambda^q g^{0,1}$. Hence, for any complex structure, there is a finite dimensional double complex $\Lambda_g := (\Lambda_0, \Lambda_g, \partial, \bar{\partial})$ which can be identified with the subcomplex of left-invariant forms on $N$. Conversely, the space $\Lambda_{\emptyset,0}^g = g^{1,0}$ together with the map $d : \Lambda_{\emptyset,0}^g \rightarrow \Lambda_{\emptyset,2}^g \oplus \Lambda_{\emptyset,1}^g$ determine $g$ and the complex structure.

By work of Mal’cev [25], the Lie group $N$ admits a (automatically cocompact) lattice $\Gamma$ if and only if $g$ admits a rational structure, i.e. if there is a rational nilpotent Lie algebra $g_\mathbb{Q}$ such that $g_\mathbb{Q} \otimes \mathbb{R} \cong g$. Given such $\Gamma$, the associated quotient $X = X(g, \Gamma) = \Gamma \backslash N$ is called a nilmanifold. Any complex structure on $g$ induces one on $X$ and we obtain an inclusion of double complexes $\Lambda_g \subseteq A_X$.

As an immediate consequence of [33, Prop. 3.16] we get:

**Theorem 8.1.** For any real nilpotent Lie algebra $g$ (admitting a rational structure) and a fixed complex structure, there exists a lattice $\Gamma$ in the associated nilpotent Lie group $N$ such that the natural map

$$H^{p,q}_\partial(\Lambda_g) \rightarrow H^{p,q}_\partial(\Gamma \backslash N)$$

is an isomorphism for all $p, q \in \mathbb{Z}$.

**Remark 8.2.** It is conjectured that this holds for all lattices. This is known to be true in many special cases, in particular for $\dim \mathbb{R} g \leq 6$.

If in the following we speak of a nilmanifold associated with a given nilpotent Lie algebra with complex structure, we always take the lattice to be chosen as above. By Lemma 7.5 we obtain:

**Corollary 8.3.** For any $g$, $\Gamma$, $N$ as above, the natural map

$$H(\Lambda_g) \rightarrow H(\Gamma \backslash N) := H(A_{\Gamma \backslash N})$$

is an isomorphism for any cohomological functor in the sense of Definition 7.2.

In particular, for fixed $g$ and complex structure, computation of any cohomological invariant of the corresponding complex nilmanifold is a matter of finite dimensional linear algebra and can therefore in principle be done by a computer.

### 8.2 The Iwasawa manifold and its deformations

The Iwasawa manifold $I$ is a nilmanifold with left invariant structure determined by the structure equations

$$d\varphi_1 = d\varphi_2 = 0 \text{ and } d\varphi^3 = \varphi^1 \wedge \varphi^2.$$  

It has a six-dimensional locally complete family of small deformations, which has been stratified by Nakamura into three classes $i, ii, iii$, according to the values of the Hodge numbers [27]. Later, D. Angella has further refined this stratification into five classes $i, ii.a, ii.b, iii.a$ and $iii.b$, taking into account the values of Bott-Chern cohomology as well [4]. The values of Dolbeault and
Bott-Chern cohomology on each stratum is (by definition) constant and listed in [4]. Since the dimension is small, using the formulae

\[2(h_{BC}(X) - h_{\partial}(X)) = \sum_{Z \text{ zigzag}} (l(Z) - 2) \text{ mult}_Z(A_X)\]

and some elementary combinatorics in the style of [31, Lemma 4.5], one verifies that the \(E_1\)-isomorphism type, i.e. the class in \(U_3\) is also constant along each stratum. We give here the result only\(^3\) for brevity ignoring summands consisting of dots, i.e. the formulae are to be read in \(U_3/H_3^K\):

\[
\begin{align*}
[I_i] &\equiv 1 + 2 \\
[I_{ii,a}] &\equiv 1 + 2 \\
[I_{ii,b}] &\equiv 1 + 2 \\
[I_{iii,a}] &\equiv 1 + 2 \\
[I_{iii,b}] &\equiv 1 + 2
\end{align*}
\]

Since we already know that \(Sym_{S_2^1,0}(3) \in B \cdot U_2^{form} \subseteq U_3^{form}\), we obtain

\(Sym_{S_2^1,1}(3), Sym_{S_2^1,1}(3), Sym_{S_1^1,0}(3) \in U_3\).

### 8.3 A threefold with \(b_{3,1}^1(X) = 1\)

Denote by \(N\) a complex 3-dimensional nilmanifold with holomorphic differentials \(\varphi_1, \varphi_2, \varphi_3\) and nonzero differential \(\varphi_3 \mapsto \varphi_1 \wedge \varphi_2 - \varphi_1 \wedge \overline{\varphi}_2\). A tedious but

\(^3\)See also [13] where the same result is obtained for examples in all classes except \(ii.b\), which is not treated.
elementary computation (or use of SageMath) yields:

\[
N = + 2 \cdot + + + + 4 \cdot + 3 \cdot + + + \equiv + + + \in \mathcal{U}/\mathcal{H}_3
\]

Since the first two summands \(Sym_{1,0}^3(3)\) and \(Sym_{2,1}^3(3)\) have already been shown to lie in \(\mathcal{RB}_s \subseteq \mathcal{U}_s\), we get that also \(Sym_{3,1}^3(3) \in \mathcal{RB}_3\), resp. \(Sym_{s+1}^3(3) \in \mathcal{U}_3\).

### 8.4 Oeljeklaus-Toma manifolds and the generators \(L_n\)

Let \(K\) be a number field with exactly \(s + 2t\) distinct embeddings \(\sigma_1, \ldots, \sigma_n\) into \(\mathbb{C}\), the first \(s\) of which have image contained in the reals, while \(\sigma_{s+i} = \bar{\sigma}_{s+t+i}\) are pairs of distinct conjugate complex embeddings. Let \(\mathcal{O}_K \subseteq K\) denote the ring of algebraic integers of \(K\) and \(\mathcal{O}_K^{*+} \subseteq \mathcal{O}_K\) the subgroup of those elements which are positive under all real embeddings. There is an action of \(\mathcal{O}_K \rtimes \mathcal{O}_K^{*+}\) on \(\mathbb{H}^s \times \mathbb{C}^t\) by translations and dilations, i.e.

\[(a, b). (z_1, \ldots, z_{s+t}) = (\sigma_1(b) \cdot z_1 + \sigma_1(a), \ldots, \sigma_{s+t}(b) \cdot z_{s+t} + \sigma_{s+t}(a)).\]

As a consequence of Dirichlet’s unit theorem, one can show that there always exist subgroups \(U \subseteq \mathcal{O}_K^{*+}\) such that the quotient by the above action restricted to \(\mathcal{O}_K \rtimes U\) is a compact manifold (of dimension \(s + t\)). We will be only interested in the case \(t = 1\), where one may take \(U = \mathcal{O}_K^{*+}\).

**Definition 8.4 (28).** An Oeljeklaus-Toma manifold (or OT-manifold) of type \((s, t)\) is any manifold \(X(K, U)\) associated to a pair \((K, U)\) as above.

Combining [29, Cor. 4.7., Rem. 4.8.] and [17, Prop. 6.4.], one obtains:

**Theorem 8.5.** Let \(n \in \mathbb{Z}_{>1}\). For an OT manifold \(X\) of type \((n - 1, 1)\), one has

\[
h^{p,q}(X) = \begin{cases} \binom{n-1}{q} & \text{if } p = 0 \\ \binom{n-1}{q-1} & \text{if } p = n \\ 0 & \text{else.} \end{cases}
\]

**Corollary 8.6.** Let \(n \in \mathbb{Z}_{>1}\) and \(X\) an OT manifold of type \((n - 1, 1)\). The refined Poincaré polynomial of \(X\) is given as:

\[
rb(X) = \sum_{k=0}^{n-1} \binom{n-1}{k} Sym_k^{k,k}(n)
\]
Proof. From Theorem 8.5, we see that $E_{\infty}^{p,q}(X) = 0$ unless $p = 0, n$. In particular, the projection

$$H^k_{dR}(X) \to \gr^0_F H^k_{dR}(X) = E_{\infty}^{0,k}$$

is an isomorphism for all $k \leq n$, i.e. $F^0 H_{dR}(X) = H^k_{dR}(X)$ and $F^1 H^k_{dR}(X) = 0$ and by conjugation also $F^0 H_{dR}(X) = H^k_{dR}(X)$ and $F^1 H^k_{dR}(X) = 0$. The result now follows from the definition of the $b^{p,q}_k(X)$.

Remark 8.7. Since the Frölicher spectral sequence for OT-manifolds degenerates, we could also make this calculation in $U_*^n$. Corollary 8.8. If $RB_{<n} = RB_{<n}^{form}$, then $L_n \in RB_n$.

Proof. By [28], OT manifolds of given type $(s,t) \in \mathbb{Z}_2^2$ always exist, so we may take an OT manifold $X$ of type $(n-1,1)$. Then by Corollary 8.6, we have

$$rb(X) = L_n + \sum_{k=0}^{n-2} \left( \binom{n-1}{k} \right) \text{Sym}^{k,k}(n).$$

Since $RB_{<n}^{form} = (W_{n-1}RB_{<n}^{form}) \oplus (M_n,L_n)$, where $W_n RB_{<n}^{form}$ is the subring generated in degrees $\leq k$ and no $M_n$-summand appears in $rb(X)$, we conclude that $L_n \in RB_n$.

Remark 8.9. In particular, we finish the proof of $RB_{<3} = RB_{<3}^{form}$.

8.5 The generators $M_n$ for $n$ odd

To construct the generators $M_n$, we search for a complex $n$-fold $X_n$ which satisfies $b_{n-1,n-1}^{n-1}(X_n) = 1$. Then, modulo the degree $n$-part of the ring generated by all other generators of degree $\leq n$, we have $rb(X_n) \equiv M_n$. By Lemma 4.6, $b_{n-1,n-1}^{n-1}(X_n)$ is the dimension of the subspace of $H_{dR}^{n-1}(X_n)$ consisting of classes which admit both a holomorphic and an antiholomorphic representative.

We have seen that for $n = 3$ such a manifold may be found among small deformations of the Iwasawa manifold. The following example, due to Daniele Angella, generalizes this and shows that such $X_n$ exist in every odd dimension.

Consider the $n = 2m+1$-dimensional nilpotent Lie algebra $m$ with complex structure defined by the structure equations:

$$d\varphi^\ell = 0 \quad \text{for} \quad \ell \in \{1, \ldots, 2m\},$$

$$d\varphi^{2m+1} = -\sum_{\ell=1}^{m} \varphi^{2\ell-1} \wedge \varphi^{2\ell} + \sum_{\ell=1}^{m} \sqrt{-1} \varphi^{2\ell-1} \wedge \varphi^{2\ell-1}.$$

For $i \leq j$ write $\varphi^{i,j}$ for the product of all $\varphi^\ell$ (in ascending order) except $\varphi^i$ and $\varphi^j$. These forms form a basis for $\Lambda^{2m-1,0}_m$. We have

$$\partial \varphi^{i,j} = \begin{cases} -\varphi^i \wedge \ldots \wedge \varphi^{2m} & \text{if } i = 2\ell - 1, j = 2\ell \text{ for some } \ell \in \{1, \ldots, m\} \\ 0 & \text{else,} \end{cases}$$
in particular, \( \dim \partial \cap \Lambda^{2m}_m = 1 \). Since a holomorphic form cohomologous to any other form would have to be \( \partial \)-exact, we have \( b_{n-1,n-1}^{n-1}(\Lambda_m) \leq 1 \). On the other hand, for any \( \ell \in \{1, \ldots, m\} \),

\[
\left[ \sum_{\ell=1}^{m} \varphi^{2\ell-1} \wedge \varphi^{2\ell} \right] = \left[ \sum_{\ell=1}^{m} \sqrt{-1} \varphi^{2\ell-1} \wedge \bar{\varphi}^{2\ell-1} \right] = \left[ \sum_{\ell=1}^{m} \varphi^{2\ell-1} \wedge \bar{\varphi}^{2\ell} \right] \in H^2_{dR}(\Lambda_m, \mathbb{C}),
\]

whence also

\[
\left[ \left( \sum_{\ell=1}^{m} \varphi^{2\ell-1} \wedge \varphi^{2\ell} \right)^m \right] = \left[ \left( \sum_{\ell=1}^{m} \sqrt{-1} \varphi^{2\ell-1} \wedge \bar{\varphi}^{2\ell-1} \right)^m \right] \in H^{2m}_{dR}(\Lambda_m, \mathbb{C}).
\]

The class above is represented by \( m! \varphi^1 \wedge \cdots \wedge \varphi^{2m} \neq 0 \), so \( b_{n-1,n-1}^{n-1}(\Lambda_m) = 1 \). By Corollary 8.3, we may now pick \( X_n = \Gamma \backslash \mathcal{N} \), where \( \mathcal{N} \) is the simply connected Lie group corresponding to \( \mathfrak{m} \) and \( \Gamma \) is any lattice such that the Dolbeault cohomology is computed by left invariant forms.

**Remark 8.10.** The manifolds \( X_n \) constructed above may be taken to be (small) deformations of the complex manifolds \( \eta_2^{2m+1} \) considered by Alessandrini and Bassanelli [3]. The latter are defined as quotients of the nilpotent Lie group

\[
\left\{ \begin{pmatrix}
1 & z_1 & z_3 & \cdots & z_{2m-1} \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 1 \\
& 1 & 0 & z_{2m-2} \\
& & 1 & z_{2m} \\
& & & 1
\end{pmatrix} \in \text{GL}(m+2; \mathbb{C}) \right\}
\]

by the co-compact discrete subgroup generated by such matrices with entries in \( \mathbb{Z}[\sqrt{-1}] \). For \( m = 1 \), this yields the Iwasawa manifold. A basis for the left-invariant \((1,0)\)-forms is given by

\[
\omega^1 = dz_1, \ldots, \omega^{2m} = dz_{2m}, \quad \omega^{2m+1} = dz_{2m+1} - \sum_{\ell=1}^{m} z_{2\ell-1} dz_{2\ell},
\]

with structure equations

\[
d\omega^1 = \ldots = d\omega^{2m} = 0, \quad d\omega^{2m+1} = -\sum_{\ell=1}^{m} \omega^{2\ell-1} \wedge \omega^{2\ell}.
\]

We consider the small deformation corresponding to the direction

\[
s(t) = \left[ \sqrt{-1} t \sum_{\ell=1}^{m} \left( \frac{\partial}{\partial z_{2\ell}} + z_{2\ell-1} \frac{\partial}{\partial z_{2m+1}} \right) \otimes \bar{\omega}^{2\ell-1} \right] \in H^{0,1}_{\pi}(X, T^{1,0}X)
\]

for \( t \in \mathbb{R} \) with \(|t| < \epsilon\). It is straightforward to check that \( s(t) \) satisfies the obstruction given by the Maurer-Cartan equation, so it defines a family \( \{X_t\}_{|t|<\epsilon} \)
of small deformations of $X = X_0$.

The holomorphic coordinates on $X_t$ are given by the solutions $(\zeta_\ell(t))_{\ell \in \{1, \ldots, 2m+1\}}$ of the initial-value pde

$$\begin{cases}
\bar{\partial} \zeta_\ell(t) - s(t) \zeta_\ell(t) = 0 \\
\zeta_\ell(0) = z_\ell.
\end{cases}$$

It is straightforward to check that the following works:

$$\zeta_{2\ell-1}(t) = z_{2\ell-1}, \quad \zeta_{2\ell}(t) = z_{2\ell} + \sqrt{-1} t \bar{z}_{2\ell-1}, \quad \text{for } \ell \in \{1, \ldots, 2m\},$$

$$\zeta_{2m+1} = z_{2m+1} + \sum_{\ell=1}^{m} \sqrt{-1} t |z_{2\ell-1}|^2.$$

We can then use the following coframe of invariant $(1,0)$-form on $X_t$:

$$\varphi^\ell(t) = d\zeta_\ell(t) \quad \text{for } \ell \in \{1, \ldots, 2m\},$$

$$\varphi^{2m+1}(t) = d\zeta_{2m+1}(t) - \sum_{\ell=1}^{m} (z_{2\ell-1} d\zeta_{2\ell}(t) + \sqrt{-1} t \bar{z}_{2\ell-1} d\zeta_{2\ell-1}(t)).$$

The structure equations in this coframe are

$$d\varphi^\ell(t) = 0 \quad \text{for } \ell \in \{1, \ldots, 2m\},$$

$$d\varphi^{2m+1}(t) = - \sum_{\ell=1}^{m} \varphi^{2\ell-1}(t) \wedge \varphi^{2\ell}(t) + \sum_{\ell=1}^{m} \sqrt{-1} t \varphi^{2\ell-1}(t) \wedge \varphi^{2\ell-1}(t).$$

For $t = 1$, we recover the previous examples, but the same argument there applies for any $t \neq 0$.

### 8.6 Frölicher differentials in dimension 3

In [9], the Frölicher spectral sequence of left-invariant structures on 6-dimensional nilmanifolds is studied. Only three nilpotent Lie algebras admit left-invariant complex structures with $E_2 \neq E_3$, namely $h_{13}$, $h_{14}$ and $h_{15}$. The last example has a particularly rich behaviour: The left-invariant complex structures are parametrized by a triple $(\rho, B, C) \in \{0,1\} \times \mathbb{R}_{>0} \times \mathbb{C}$ (subject so some conditions), and depending on the parameter values, the behaviour of the Frölicher spectral sequence changes. In fact, inspection of the proof of [9, Thm. 4.1] yields the following equalities in $U_3/R^3$:

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\[ \rho = c = 0, B = 1 : \quad [X_{(\rho, c, B)}] \equiv \begin{array}{c|c|c|c|c|c|c|c} & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} + 2 \]

\[ \rho = 1, c = 0, B \neq 1 : \quad [X_{(\rho, c, B)}] \equiv \begin{array}{c|c|c|c|c|c|c|c} & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} + \delta_B \]

\[ \rho = 1, |B - 1| \neq c \neq 0 : \quad [X_{(\rho, c, B)}] \equiv \begin{array}{c|c|c|c|c|c|c|c} & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \]

\[ \rho = 0, |B| \neq c \neq 0 : \quad [X_{(\rho, c, B)}] \equiv \begin{array}{c|c|c|c|c|c|c|c} & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \]

Since we have already seen that \( \mathcal{RE}_3 = \mathcal{RE}_3^{\text{form}} \), we obtain that

\[
\text{Sym}_{S_{1,2}^0}(3), \text{Sym}_{S_{1,1}^0}(3), \text{Sym}_{S_{2,2}^0}(3) \in \mathcal{U}_3,
\]

leaving open only the question whether or not \( \text{Sym}_{S_{1,1}^0}(3) \in \mathcal{U}_3 \). An inspection of the other two possibilities of nilmanifolds with nonzero \( E_2 \)-differentials yields:

**Proposition 8.11.** For all left-invariant structures on 6-dimensional nilmanifolds, the differential \( d_0^{1,1} : E_2^{0,1} \to E_2^{2,0} \) vanishes. In particular, it is not possible to realize \( \text{Sym}_{S_{1,2}^0}(3) \) using complex nilmanifolds.

### 9 Bimeromorphic invariants, reprise

We prove Corollary [F]. Let \( \mathcal{BR} \subseteq \mathcal{RB}_* \) and \( \mathcal{B}^d \subseteq \mathcal{U}_* \) be the ideals generated by differences of pairs bimeromorphic manifolds.

**Lemma 9.1.** The ideals \( \mathcal{BR} \subseteq \mathcal{RB}_* \) and \( \mathcal{B}^d \subseteq \mathcal{U}_* \) are principal ideals generated by

\[
C = \text{Sym}_{S_2^{1,1}}(2) = \begin{array}{c|c|c|c|c|c|c|c} & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}
\]

**Proof.** If \( Z \subseteq X \) is a submanifold of codimension \( r \geq 2 \) and \( \tilde{X} \) the blowup of \( X \) at \( Z \), then \( [\tilde{X}] - [X] = \sum_{i=0}^{r-2} C^i \cdot [Z]_2^{r-1-i} \in \mathcal{U}_* \) by [15]. In particular, taking a point blowup shows that \( C \) is contained in the respective ideals. On the other hand, every bimeromorphic map is the composition of blow-ups and blowdowns by the weak factorisation theorem [1], [41], hence the other inclusion holds.

**Lemma 9.2.**
1. The degree $n$ part of $\mathcal{RB}^\text{form}_n / \mathcal{RB}_n^\text{form} \cdot C$ has a basis consisting of $\text{Sym}^{n,0}_n(n)$, $\text{Sym}^{2,2}_n$ and

\[
\text{Sym}^{p,0}_k(n) \text{ for } 0 \leq p \leq k \leq n - 1 \\
\text{Sym}^{k,p}_k(n) \text{ for } 0 \leq p \leq k \leq n - 1 \\
\text{Sym}^{n-1,p}_k(n) \text{ for } 2 \leq p \leq n - 1
\]

2. The degree $n$ part of $\mathcal{U}^\text{form}_n / \mathcal{U}_n^\text{form} \cdot C$ has a basis consisting of the $\text{Sym}_n Z(n) \in \mathcal{U}_n$ where $Z$ is a zigzag satisfying at least one of the following conditions:

(a) the support of $Z$ contains a point $(p,q)$ with $p$ or $q$ equal to 0 or $n$,
(b) the length of $Z$ is at least two and $(1,1) \in Z$,
(c) the length of $Z$ is at least two and $(n-1,1) \in Z$.
(d) $[Z] = S_{1,1}^{1,2}$ if $n = 4$.

Part 1 is a special case of part 2 which is readily seen by the diagrammatic description and the restrictions given in Definition 7.9 and we omit a formal proof. For example, a basis for the degree 3 part of $\mathcal{U}^\text{form}_3 / \mathcal{U}_3^\text{form} \cdot C$ is given by all $\text{Sym}_3 Z(3) \in \mathcal{U}_3$ except the following:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\hline
\end{array}
\quad \quad \quad
\begin{array}{c|c|c|c|c|c|c|c}
\hline
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\hline
\end{array}
\]

\textbf{Corollary 9.3.} Let $R$ be a ring. Any map $\mathcal{RB}^\text{form}_n/(C) \to R$ (resp. $\mathcal{U}^\text{form}_n/(C) \to R$) can be written as an $R$-linear combination of the refined Betti numbers (resp. multiplicities of zigzags) that are the coefficients of the basis elements in Lemma 9.2.

\textbf{Proof Corollary} \(\blacksquare\) By Lemma 9.1, there is a natural map $\eta : \mathcal{U} / \mathcal{U}_n \cdot \mathcal{B}^\text{ld} \to \mathcal{U}^\text{form} / \mathcal{U}_n^\text{form} \cdot C$. The map $\eta$ is injective in degree $n$ if and only if $\mathcal{U}^\text{form} / \mathcal{U}_n^\text{form} \cdot C \cap \mathcal{U}_n = \mathcal{U}_{n-2} \cdot C$. This holds for $n \leq 4$ since then $\mathcal{U}^\text{form}_{n-2} = \mathcal{U}_{n-2}$. Now let $\varphi : (\mathcal{RB}_n / \mathcal{RB} \otimes K)_n \to K$ be a map to some field $K$, i.e. a $K$-linear combination of refined Betti-numbers which is a bimeromorphic invariant. Since $\eta^\text{ld}_K$ is an injective map of vector spaces, we can find a $\varphi' : (\mathcal{U}^\text{form} / \mathcal{U}_n^\text{form} \cdot C)_n \to K$ such that $\varphi = \varphi' \circ \eta^\text{ld}$ and apply Corollary 9.3. The case of $\mathcal{RB}_n$ is analogous, using that $\mathcal{RB}_{\leq 3} = \mathcal{RB}^\text{form}_{\leq 3}$.

\textbf{10 Summary of some open problems}

Solving the following problem would show $\mathcal{RB}_n = \mathcal{RB}^\text{form}_n$:

\textbf{Problem 10.1.} For even $n \geq 4$, construct an $n$-dimensional compact complex manifold $X_n$ with $\eta^{n-1,n-1}_{n-1}(X_n) = 1$, i.e. supporting a nonzero $n - 1$ de Rham class, unique up to scalar, which can be represented by both a holomorphic and an antiholomorphic form.
Ideally, such examples would also allow to show the slightly stronger statement

\[
\text{Sym}_{\mathcal{S}_{n-1,n-1}}(n) = \mathcal{U}_n.
\]

Apart from this and maybe some sporadic generators in small degrees, the main problem in showing a hypothetical equality \(\mathcal{U}_* = \mathcal{U}_{form}^*\) should consist in finding differentials in the Frölicher spectral sequence in minimal dimension and extremal degree:

**Problem 10.2.** For every \(n \geq 3\), construct a \(n\)-dimensional compact complex manifold \(X_n\) with nonvanishing differentials on page \(E_{n-1}\) starting in degree \((0,n-1)\) or \((0,n-2)\). More precisely, where for readability we omit zigzags determined via the real structure, show that the following lie in \(\mathcal{U}_n\):

\[
\text{Sym}_{\mathcal{S}_{0,n-1},n-1}(n) = \mathcal{U}_n \quad \text{and} \quad \text{Sym}_{\mathcal{S}_{0,n-2},n-1}(n) = \mathcal{U}_n.
\]

A solution to this problem would in particular solve the following problem with \(r(n) = n-1\) for \(n \geq 3\):

**Problem 10.3.** Determine for every dimension \(n\), the maximal number \(r = r(n)\) such that the Frölicher spectral sequence of an \(n\)-manifold may have a nonzero differential at page \(r\).

It is believed that it is possible to solve problem 10.3 using nilmanifolds with possibly high nilpotency step, see [7]. By the examples in [2] used in Section 8.6 one has indeed \(r(3) = 2\). On the other hand, by Proposition 8.11 it is already in dimension 3 not possible to give a positive answer to the stronger problem 10.2 using nilmanifolds only. It would be interesting to see whether problem 10.1 can be solved for four-dimensional nilmanifolds.

Even stronger than the question \(\mathcal{U}_* = \mathcal{U}_{form}^*\), one may ask what universal linear relations between cohomological invariants and Chern numbers exist.

**Conjecture 10.4.** Modulo the relations involving only cohomological invariants or only Chern numbers, the only \(\mathbb{Z}\)-linear relations or congruences between cohomological invariants and Chern numbers of compact complex manifolds are the ones induced by Hirzebruch Riemann Roch.

**Remark 10.5.** If one finds a generating set for \(\mathcal{U}_*\) consisting of \(\mathbb{C}P^1, \mathbb{C}P^2\) and otherwise only manifolds with vanishing Chern classes (e.g. complex nil- or solvmanifolds), this would follow with the same proof as Theorem 1.18.
For completeness, note that to address the integral version of the Hirzebruch question for Hodge and Betti numbers, in Section 3 it remained to solve the following:

Problem 10.6. Find $\mathbb{Z}$-linear combinations $Z = \sum a_i (X_i - Y_i)$, where $X_i$ and $Y_i$ are homeomorphic, or even orientation preservingly diffeomorphic, complex 4-folds, such that each of the following can be realized as $h(Z)$:

\[
CT = (xy^2 - x^2y + x^3y^2 - x^2y^3)z^4 \\
C\tilde{d} = (x^2y + 2x^2y^2 + x^2y^3)z^4 \\
D\tilde{d} = (x^2 + 2x^2y + 2x^2y^2 + 2x^2y^3 + x^2y^4)z^4.
\]

Finally, it seems natural to ask the following generalization of Hirzebruch’s question:

Question 10.7. Which linear combinations or congruences of cohomological invariants and Chern numbers are topological invariants of compact complex manifolds?

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