The Milnor fibre signature is not semi-continuous

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Abstract. Consider the germ of an isolated surface singularity in \((\mathbb{C}^3, 0)\). The corresponding Milnor fibre possesses the homology lattice (the integral middle homology with a natural symmetric intersection form).

An old question of A.Durfee (1978) asks: is the signature of this form non-increasing under degenerations? The present article answers negatively: We give examples of Newton non-degenerate families where the signature increases under degeneration.

1. Introduction

Let \((X, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^3, 0)\) be the germ of a complex, locally analytic surface with isolated singularity at the origin. For a general introduction to the ‘Milnor package’ associated with \((X, 0)\), see [Milnor68], [Dimca92], [AGLV93] or [Seade2006].

Let \(B \subset \mathbb{C}^3\) be a small ball centered at the origin. Consider a small deformation: \(X_{\epsilon} = f^{-1}(\epsilon) \cap B\), where \(\epsilon \neq 0\) is sufficiently small with respect to the chosen radius of the ball. Then \(X_{\epsilon}\) is smooth and the diffeomorphism type of the pair \((B, X_{\epsilon})\) does not depend on \(\epsilon \neq 0\). \(X_{\epsilon}\) is called the Milnor fibre and it has the homotopy type of a bouquet \(\vee_\mu S^2\) of middle dimensional spheres, where \(\mu\) is the Milnor number, hence \(H_2(X_{\epsilon}, \mathbb{Z}) \approx \mathbb{Z}^\mu\). There is a natural symmetric intersection form on \(H_2(X_{\epsilon}, \mathbb{Z})\). Its extension to \(H_2(X_{\epsilon}, \mathbb{Z}) \otimes \mathbb{R}\) gives the triple \((\mu_+, \mu_0, \mu_-)\), the dimensions of maximal vector subspaces of \(H_2(X_{\epsilon}, \mathbb{Z}) \otimes \mathbb{R}\) on which the form is positive definite, zero or negative definite.

Let \((X_t, 0) \subset (\mathbb{C}^3, 0) \times (\mathbb{C}^1, 0)\) be a holomorphic family of surface germs, such that \((X_t, 0)\) has an isolated singularity for any \(t\) in the neighborhood of \(0 \in \mathbb{C}^1\). An old question of A.Durfee [Durfee, Conjecture 5.4], see also W.Neumann [N.T.R.82, Problems session, p.249], is about the behavior of signature \(\text{sign} = \mu_+ - \mu_-\) of the corresponding intersection forms under the degeneration:

\[\text{Is the signature non-increasing under the degeneration } t \to 0?\]

It seems that the question was not addressed previously. The common expectation was that the semi-continuity as above should hold, at least under some mild restrictions. And the counterexamples, if any, must be quite complicated.

Since any non–smooth singularity is realized as the degeneration of \((x, y, z) \mapsto x^2 + y^2 + z^2\) whose signature is \(-1\), a positive answer to the above question would imply that the signature of any non–smooth isolated hypersurface surface singularity is negative, which is the (weak) Durfee Conjecture [Durfee, Conjecture 5.2].

The present article shows that any tentative proof of the Durfee Conjecture which is based on semi–continuity of the signature in this direct way, will fail.

Note also that the signature can be ‘almost’ determined from the spectrum, which has a rather strong semi-continuity behavior, cf. [Varchenko83], [Steenbrink85]. The contribution in the signature, which is not covered by the spectrum, is the equivariant signature corresponding to the eigenvalue one of the monodromy, which depends on the structure of the Jordan blocks associated with this eigenvalue. The point is that exactly these blocks are responsible for the non–semi–continuity of the signature; and they are the key targets of the present article as well.

It is quite a surprise (at least for us) that the semi-continuity is violated even for families of Newton-non-degenerate surfaces. (In fact, all our examples are degenerations of \(T_{p,q,r}\) singularities.)
Example 1.1. Consider the family of surfaces \( X_t = f_t^{-1}(0) \), where for any \( t \in \mathbb{C} \)
\[ f_t = txyz + xyz(x + y + z) + x^4y + y^4z + z^4x. \]

Note that \( X_t \) is Newton-non-degenerate for any \( t \). In fact, \( X_{t \neq 0} \) is equivalent with the
\( T_{13,13,13} \)-singularity \( \{xyz + x^{13} + y^{13} + z^{13} = 0\} \). Indeed, first by scaling of the coordinates,
one gets that \( f_{t \neq 0} \) is contact equivalent to \( xyz + x^4y + y^4z + z^4x \). Further, using the transformation
\[ x \rightarrow x + 4y^2z^3 - 6yz^6 + 4z^9 + 29y^{11}z^2, \]
\[ y \rightarrow x + 4x^2z^2 - 6x^6z^2 + 4x^9 + 29z^{11}x^2, \]
\[ z \rightarrow z + 4z^2y^3 - 6xy^6 + 4y^9 + 29x^{11}y^2 \]
and dividing by \( 1 + 82(x^8y^2 + 82y^8z^2 + 82z^8x^2) \) one gets \( xyz + x^{13} + y^{13} + z^{13} + \ldots \), where the higher order terms consist of monomials of total degree at least 15.

The relevant numerical data for the family is:

\[ \begin{array}{ccccc}
\mu & \mu_+ & \mu_0 & \mu_- & \mu_+ - \mu_- \\
X_0 & 45 & 5 & 3 & 37 & -32 \\
X_{t \neq 0} & 38 & 1 & 1 & 36 & -35 \\
\end{array} \]

Once a counterexample is found a natural question is: how significantly the signature can grow in degenerations?

Below we consider the families of Newton-non-degenerate surface singularities. We explain the idea behind example
1.1 and present additional interesting families. In particular, it appears that the signature can increase significantly.
We give an example when the change of \( \mu_+ - \mu_- \) is asymptotically \( \frac{\mu(X_0)}{12} \) (or \( \frac{\mu(X_{t \neq 0})}{9} \)).

Remark 1.2. If one considers the family \( X_t \) as a deformation of the central fibre and allows the singular point to split
into several ones then an immediate counterexample is obtained as the suspension of a Newton-non-degenerate curve singularity.

Let \( (C_0,0) \subset (\mathbb{C}^2,0) \) be a singularity of the topological type of \( x^3 = y^9 \), i.e. three smooth branches, pairwise tangent
with order 3. Consider the family \( C_t \) in which the initial singularity deforms to five singular points, of the types
\( (A_5, A_5, A_1, A_1, A_1) \). An example of such a family is: \( f_t(x,y) = y(y - x^3)(y + (x - t)^3) \). Here the \( A_5 \) singularities are
at the points \( (0,0) \) and \( (t,0) \), while the three nodes \( A_1 \) are the intersection points of the branches \( \{y - x^3 = 0\} \) and
\( \{y + (x - t)^3 = 0\} \).

Let \( X_t = \{z^2 + f_t(x,y) = 0\} \subset \mathbb{C}^3 \) be the stabilization of such a family. The relevant data for the signature is:

\[ \begin{array}{ccccc}
\mu & \mu_+ & \mu_0 & \mu_- & \mu_+ - \mu_- \\
X_0 & 16 & 2 & 0 & 14 & -12 \\
X_{t \neq 0} & 13 & 0 & 0 & 13 & -13 \\
\end{array} \]

(where, for each fixed \( t \), the invariants are summed up over all singular points of \( X_t \)). Hence \( \text{sign}(X_0) > \text{sign}(X_{t \neq 0}) \).

The original question [Durfee, Conjecture 5.6] and in [N.T.R.82] targets non-splitting families, hence from now on
we consider only this case: \( \text{Sing}(X_t) = \{0\} \) for all \( t \).

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2. Preliminaries

2.1. The Milnor number. For isolated Newton-non-degenerate singularities various topological invariants are
determined by the Newton diagram. It is defined as follows: first, one takes the Minkowski sum of the support of \( f \)
and the first octant \( \mathbb{R}^3_{\geq 0} \). Then the union of all the compact faces of the convex hull of this set is called the Newton
diagram, and it is denoted by \( \Gamma_f \).

The function \( f \) is Newton-non-degenerate or non-degenerate with respect to its diagram if for any face \( \sigma \subset \Gamma_f \) the
restriction \( f|_\sigma \) is non-degenerate, i.e. the corresponding hypersurface has no singular points inside the maximal torus.

Let \( X = f^{-1}(0) \subset (\mathbb{C}^3,0) \) be a Newton-non-degenerate germ with Newton diagram \( \Gamma_f \). We can assume that \( \Gamma_f \) is
convenient (by adding to \( f \) the monomials \( x^N, y^N, z^N \) for \( N \gg 0 \)).

Let \( \mathbb{R}^3_{\geq 0} \supset \Gamma_{x,y} = \Gamma_f \cap \{z = 0\} \), similarly for \( \Gamma_y, \Gamma_z \). Note that all these diagrams are convex and
convenient. Let \( \mathbb{R}^1_{\geq 0} \supset \Gamma_x = \Gamma_f \cap \{z = 0 = y\} \), and similarly introduce \( \Gamma_y, \Gamma_z \).
The Milnor number is determined by the volumes under $\Gamma_f$ as follows [Kouchnirenko76]

\[ \mu = 3! Vol_3 - 2! Vol_2 + Vol_1 - 1, \]

where $Vol_3$ is the volume in $\mathbb{R}^3_{\geq 0}$ under the Newton diagram $\Gamma_f$, $Vol_2$ is the sum of volumes in $\mathbb{R}^2_{\geq 0}$ under the diagrams $\Gamma_{xy}$, $\Gamma_{yz}$, and $\Gamma_{zx}$, and $Vol_1$ is the sum of volumes in $\mathbb{R}^1_{\geq 0}$ under the diagrams $\Gamma_x$, $\Gamma_y$, and $\Gamma_z$. All the volumes are normalized so that the volume of the unit cube (in any dimension) is 1.

2.2. $(\mu_+, \mu_0, \mu_-)$. Given a symmetric intersection form let $M_+, M_-$, $M_0$ be the maximal subspaces of $H_2(X, \mathbb{R})$ on which the symmetric form is positive definite, zero, negative definite. So $\mu_- = \dim(M_-)$, and similarly for $\mu_0, \mu_+$.

Recall [Steenbrink76] that the spectrum of $(X, 0)$ is a set of rational numbers in $(-1, 2)$, with multiplicities. The spectrum is symmetric with respect to $\frac{1}{2}$. The values $Sp \cap (0, 1)$ correspond to $M_-$, those from $(Sp \cap (-1, 0)) \cup (Sp \cap (1, 2))$ correspond to $M_+$ and those from $Sp \cap \{0, 1\}$ correspond to $M_+ \cup M_0$. In this last case, the precise distribution into $M_+, M_0$ is determined by the Jordan block structure of the monodromy (which is not codified in the spectrum). Indeed, the cardinality of $Sp \cap \{0, 1\}$ is the dimension of the generalized eigen-space of the monodromy corresponding to eigenvalue one. Let $J_n$ denote the number of $n \times n$ Jordan blocks corresponding to the eigenvalue one. By the monodromy theorem $n$ is always $\leq 2$, see e.g. [DoornSteenbrink89] and the references therein. Then $J_1$ contributes 1 to $\mu_0$ (and provides either the spectral number 0 or the spectral number 1; by the symmetry of the spectrum, such blocks appear in ‘pair’, one of them having spectral number 0, the other 1), while $J_2$ contributes 1 to $\mu_0$ and 1 to $\mu_+$ (and provides two spectral numbers, namely 0 and 1).

The cardinalities of the spectral numbers in different intervals (as above) are computed from the Newton diagram as follows [Saito88]:

- $\#(Sp \cap (-1, 0))$ equals the number of $\mathbb{Z}^3_{>0}$ points strictly below $\Gamma_f$.
- $\#(Sp \cap \{0\})$ equals the number of $\mathbb{Z}^3_{>0}$ points on $\Gamma_f$; hence:
  - $\#(Sp \cap (-1, 0))$ equals the number of $\mathbb{Z}^3_{>0}$ points not above $\Gamma_f$. Hence, $\mu_0 + \mu_+$ is twice this number.

Regarding the Jordan block structure on the eigenvalue 1 part of the monodromy one has ([DoornSteenbrink89], p. 227):

- $J_2$ is the number of points of $\mathbb{Z}^3_{>0} \cap \Gamma_f$ which do not lie in the interior of a two-dimensional (compact) face of $\Gamma_f$.

**Example 2.1.** Let $X = f^{-1}(0) \subset \mathbb{C}^3$ for $f = xyz + x^p + y^q + z^r$. There is only one point of $\mathbb{Z}^3_{>0}$ which does not lie above $\Gamma_f$, namely $(1, 1, 1)$; hence $\mu_+ + \mu_0 = 2$. This point does not lie in the interior of any face, hence it contributes to $J_2$ and $\mu_0 = 1 = \mu_+$.

Applying Kouchnirenko’s formula (or in any other way) we get: $\mu = p + q + r - 1$, $\mu_- = p + q + r - 3$ and the signature $\mu_+ - \mu_- = 4 - p - q - r$.

2.3. **Numerical computations.** The general procedures to compute the spectrum and the monodromy are realized in [GPS-Singular] (though they are often time consuming). For example, the code for the case above is:

```
LIB "sing.lib"; %%% Loading the basic library of singularities.
LIB "gmssing.lib"; %%% Loading the library of Gauss-Manin connection.
ring s = 0,(x,y,z),ds;
int p,q,r=4,5,6;
poly f = x * y + z + x^p + y^q + z^r;
milnor(f);
spectrum(f);
monodromy(f);
```

3. **The idea behind the examples**

As we consider the families of Newton-non-degenerate singularities, we prescribe the degeneration by erasing some vertices. For simplicity, we will only describe a ‘primitive’ degeneration: just one vertex is erased. By assumption, the vertex does not belong to the interior of a face or an edge. Hence, it is enough to consider just the relevant part of the Newton diagram.
Let \( a \in \mathbb{Z}^3_{\geq 0} \) be the apex of \( a \) (strictly convex, solid) cone \( C \). Erase the apex, get a new smaller body: \( C_{\text{new}} = \text{Conv}(C \setminus a) \cap \mathbb{Z}^3_{\geq 0} \). Let \( \partial C \) respectively \( \partial C_{\text{new}} \), denote the boundary of \( C \), respectively \( C_{\text{new}} \). An integral point of \( \partial C_{\text{new}} \setminus \partial C \) is called new.

We assume that there is no new point which lies in the interior of a 2–face of \( C_{\text{new}} \). Consider the boundary of \( \partial C_{\text{new}} \setminus \partial C \), it is a closed 1–dimensional piecewise linear loop. The interior points of its edges are called inner. Its vertices are called outer. Finally, let \( V \) denote the volume of \( C \setminus C_{\text{new}} \).

Using these notations and assumptions, the discussion of (2.2) implies the following:

**Proposition 3.1.** Let \( N_{\text{inner}}, N_{\text{outer}}, N_{\text{new}} \) be the number of inner, outer and new points. Then:

\[
\begin{align*}
\mu(X_0) - \mu(X_{\neq 0}) &= 6V \\
\mu_0(X_0) - \mu_0(X_{\neq 0}) &= -1 + N_{\text{inner}} + N_{\text{new}} \\
\mu_+(X_0) - \mu_+(X_{\neq 0}) &= 1 + N_{\text{inner}} + N_{\text{new}} \\
\mu_-(X_0) - \mu_-(X_{\neq 0}) &= 6V - 2N_{\text{new}} \\
\text{sign}(X_0) - \text{sign}(X_{\neq 0}) &= -6V + N_{\text{inner}} + 3N_{\text{new}} + 1.
\end{align*}
\]

Next, we estimate \( V \). Take an arbitrary elementary integral triangulation of the 2–faces of \( \partial C_{\text{new}} \setminus \partial C \), i.e. the vertices of each triangle belong to \( \mathbb{Z}^3_{\geq 0} \) and there are no integral points in the interior of the triangle or of its edges. Then \( C \setminus C_{\text{new}} \) is naturally subdivided into the pyramids with the triangles as bases and the apex \( a \in C \). Each pyramid is elementary, i.e. its only integral points are its corners.

**Lemma 3.2.** Set \( \delta := 6V - N_{\text{outer}} - N_{\text{inner}} - 2N_{\text{new}} + 2 \). Then \( \delta \geq 0 \). The equality occurs iff the volume of each of the pyramids above is \( \frac{1}{6} \).

**Proof:** By the construction above we get that \( 6V \geq \) the number of pyramids = the number of triangles in \( \partial C_{\text{new}} \setminus \partial C \). The later quantity is combinatorial, does not depend on a particular triangulation, and can be computed explicitly.

From the previous proposition we get \( \text{sign}(X_0) - \text{sign}(X_{\neq 0}) = 3 - \delta - N_{\text{outer}} + N_{\text{new}} \). Therefore, using this lemma in order to get an interesting example (when this expression is ‘large’, or at least it is positive), \( N_{\text{outer}} \) should be small, \( N_{\text{new}} \) should be large and the pyramids above should have the minimal volume \( \frac{1}{6} \) (or as small as possible).

### 4. A Degeneration with Big Increase of Signature

A simple generalization of the example 1.1 is the following.

**Proposition 4.1.** The invariants for the family of surfaces \( (X_t,0) = f_t^{-1}(0) \subset (C^3,0) \) with

\[
f_t = txyz + x^{3k+3}y + x^{k+1}yz + z^2x + y^2z
\]

are:

\[
\begin{array}{c|cccc}
\mu & \mu_+ & \mu_0 & \mu_- & \mu_+ - \mu_- \\
\hline
X_0 & 12k + 11 & 2k + 1 & 1 & 10k + 9 & -8k - 8 \\
X_{\neq 0} & 9k + 11 & 1 & 1 & 9k + 9 & -9k - 8 \\
\end{array}
\]

So the signature grows by \( k \), therefore asymptotically by \( \frac{\mu(X_0)}{12} \) (or \( \frac{\mu(X_{\neq 0})}{9} \)) when \( k \to \infty \).

Notice that \( X_{\neq 0} \) is equivalent to the singularity \( T_{6k+5,3k+4,3} \). This can be proved in many different ways, and, in fact, all these methods can be used in the similar statement of Example 1.1 as well. By the first method, one computes the resolution graph of the germ, e.g. by Oka’s algorithm [Oka87] (cf. also [BraunNemethi07]), and one gets a cyclic graph which characterizes the hypersurface \( T_{p,q,r} \) germs. Or, one can use the results [BraunNemethi07] on ‘equivalent Newton boundaries’ which identifies the triple \( (6k + 5, 3k + 4, 3) \) from the diagram. It is clear that \( f_{\neq 0} \) is equivalent to \( txyz + x^{3k+3}y + z^2x + y^2z \). Then consider the three 2–faces of the the diagram and notice that they intersect the three axes at \( (6k + 5, 0, 0), (0, 3k + 4, 0) \) and \( (0, 0, 3) \) respectively. Then use [BraunNemethi07, §3]. Finally, one can get the result by change of variables too.

**Proof:** For the computation of \( \mu_0, \mu_+ \) we consider the polynomial \( f_{\neq 0} \), then we look for the point of \( \mathbb{Z}^3_{\geq 0} \) lying under/not above \( \Gamma_{\neq 0} \). One sees that any such point is of the form \( (x,0,0) \). In addition no integral points lie in the interior of the faces or edges of \( \Gamma_f \) (or \( \Gamma_{\neq 0} \)). Hence:

- \( t \neq 0 \). There is only one point: \( (0,0,0) \in \Gamma_{\neq 0} \) it creates \( J_2 \), so contributes 1 to \( \mu_0 \) and 1 to \( \mu_+ \).
\[ t = 0. \] The points \((j, 0, 0)\) for \(0 \leq j < k\) lie under \(\Gamma_{\frac{t}{j}}\), they contribute \(2k\) to \(\mu_+\). The point \((k, 0, 0)\) in \(\Gamma_{\frac{t}{k}}\) contributes \(1\) to \(\mu_0\) and \(1\) to \(\mu_+\).

The Milnor number can be computed by Kouchnirenko formula or in any other way, then one obtains \(\mu_-\) too. ■

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