On Multivariate Singular Spectrum Analysis

Anish Agarwal∗
MIT
anish90@mit.edu

Abdullah Alomar
MIT
aalomar@mit.edu

Devavrat Shah
MIT
devavrat@mit.edu

Abstract

We analyze a variant of multivariate singular spectrum analysis (mSSA), a widely used multivariate time series method, which we find to perform competitively with respect to the state-of-art neural network time series methods (LSTM, DeepAR [16]). Its restriction for single time series, singular spectrum analysis (SSA), has been analyzed recently [10, 2]. Despite its popularity, theoretical understanding of mSSA is absent. Towards this, we introduce a natural spatio-temporal factor model to analyze mSSA. We establish the in-sample prediction error for imputation and forecasting under mSSA scales as \(1/\sqrt{NT}\), for \(N\) time series with \(T\) observations per time series. In contrast, for SSA the error scales as \(1/\sqrt{T}\) and for matrix factorization based time series methods (c.f. [25, 15]), the error scales as \(1/\min(N,T)\). We utilize an online learning framework to analyze the one-step-ahead prediction error of mSSA and establish it has a regret of \(1/(\sqrt{NT}0.04)\) with respect to in-sample forecasting error. By applying mSSA on the square of the time series observations, we furnish an algorithm to estimate the time-varying variance of a time series and establish it has in-sample imputation / forecasting error scaling as \(1/\sqrt{NT}\). To establish our results, we make three technical contributions. First, we establish that the “stacked” Page Matrix time series representation, the core data structure in mSSA, has an approximate low-rank structure for a large class of time series models used in practice under the spatio-temporal factor model. Second, we extend the theory of online convex optimization to address the variant when the constraints are time-varying. Third, we extend the analysis prediction error analysis of Principle Component Regression beyond the recent work of [3] to when the covariate matrix is approximately low-rank.
Contents

1 Introduction 4

1.1 Multivariate Singular Spectrum Analysis 4
1.2 Our Contributions 4
1.3 Literature Review 6

2 Spatio-Temporal Factor Model For Multivariate Time Series Analysis 7

2.1 Spatio-Temporal Factor Model 7

2.2 Examples Of \((G,}\epsilon)-Hankel Representable Time Series Dynamics 8

2.2.1 \((G,}\epsilon)-Hankel Time Series “Algebra” 8
2.2.2 \((G,}\epsilon)-LRF Time Series 8
2.2.3 “Smooth”, Periodic Time Series - \(C^0_{T,\text{per}}\) 8
2.2.4 Time Series with Latent Variable Model Structure 9

2.3 Stacked Hankel Matrix of mSSA is (Approximately) Low-Rank 9

3 mSSA Algorithm 10

3.1 mSSA Imputation and Forecasting: Mean and Variance Estimation 10
3.2 mSSA: An Online Convex Optimization Lens 11

4 Theoretical Results 12

4.1 Error Metrics for Evaluating Imputation and Forecasting Prediction Error 12
4.2 mSSA Mean Estimation - Finite Sample Analysis 12
4.3 mSSA Variance Estimation: Finite Sample Analysis 13
4.4 mSSA Forecasting (Online Variant) - Regret Analysis 13

5 Conclusion 15

A Experiments 17

A.1 Mean Estimation 17
A.1.1 Datasets 17
A.1.2 Mean Imputation 18
A.1.3 Mean Forecasting 18

A.2 Variance Estimation 19
A.2.1 Datasets 19
A.2.2 Variance Imputation and Forecasting 19
A.3 Algorithms Parameters and Settings 20

B Proofs - \((G,}\epsilon)-Hankel Representability of Different Time Series Dynamics 21

B.1 Proof of Proposition 4
B.2 Proof of Proposition 2.2 21
B.3 Proof of Proposition 2.4 21
B.3.1 Helper Lemmas for Proposition 2.4 21
B.3.2 Completing Proof of Proposition 2.4 ............................................. 22
B.4 Proof of Proposition 2.5 ............................................................. 22

C Proofs - Stacked Hankel Matrix is Approximately Low-Rank 24
C.1 Helper Lemmas ........................................................................... 24
C.2 Proof of Proposition 2.6 ............................................................. 24

D Imputation and Forecasting Error Analysis - Proof Notation 25
D.1 Induced Linear Operator .............................................................. 25

E Concentration Inequalities Lemmas 26
E.0.1 Classic Results ........................................................................ 26
E.0.2 High Probability Events for Imputation and Forecasting ........... 26

F HSVT Error 28

G Proofs - Imputation Analysis 32
G.1 Proof of Theorem 4.1 ................................................................. 32
G.2 Proof of Theorem 4.3 ................................................................. 32

H Proofs - Forecasting Analysis 33
H.1 Forecasting - Helper Lemmas ...................................................... 33
H.2 Proof of Theorem 4.2 ................................................................. 35
H.3 Proof of Theorem 4.4 ................................................................. 36

I Proofs - Regret Analysis 37
I.1 Regret - Helper Lemmas ............................................................. 37
I.1.1 Bounding $\Delta_{(T+1L,n)}$ ......................................................... 39
I.2 Proof of Theorem 4.5 ................................................................. 40
1 Introduction

Multivariate time series data is ubiquitous and is of great interest across many application areas, including cyber-physical systems, finance, retail, healthcare to name a few. The goal across these domains can be summarized as accurate imputation and forecasting of a multivariate time series in the presence of noisy and/or missing data along with providing meaningful uncertainty estimates.

Setup. We consider a discrete time setting with time indexed as $t \in \mathbb{Z}$. Let $f_n : \mathbb{Z} \to \mathbb{R}$, $n \in [N] := \{1, \ldots, N\}$ be the $N \in \mathbb{N}$ latent time series of interest. For $t \in [T]$ and $n \in [N]$, with probability $\rho \in (0,1]$, we observe the random variable $X_{nt}(t)$, where $X_{nt}(t) = f_{nt}(t) + \eta_{nt}(t)$. While the underlying time series $f_{nt}$ is of course strongly correlated, we assume the per-step noise $\eta_{nt}(t)$, are independent mean-zero random variables, with time-varying variance, $\mathbb{E}[\eta_{nt}^2(t)] := \sigma_{nt}^2(t)$. Note $\sigma_{nt}^2(t)$ is a key parameter of interest in time series analysis to do uncertainty quantification, especially in applications where the time series exhibits time-varying volatility (see [3]). We consider a time series model where $f_{nt}(t)$ and $\sigma_{nt}^2(t)$ satisfy a spatio-temporal factor model as described in detail in Section 2.

Goal. The objective is two-folds, for $n \in [N]$: (i) imputation – estimating $f_{nt}(t)$, $\sigma_{nt}^2(t)$ for all $t \in [T]$; (ii) forecasting – learning a model to forecast $f_{nt}(t), \sigma_{nt}^2(t)$ for $t > T$.

1.1 Multivariate Singular Spectrum Analysis

Singular Spectrum Analysis (SSA). We begin by describing SSA, a popular method in the literature for both imputation and forecasting (see [10, 2]) of a univariate time series. For the purposes of describing SSA, we assume access to only one time series $X_1(t)$. The variant of SSA considered in [2] performs three key steps: (a) transform $X_1(t)$, $t \in [T]$ into an $L \times T$ dimensional Page matrix by placing non-overlapping contiguous segments of size $L > 1$ (an algorithmic hyper-parameter) as columns (b) perform Principal Component Analysis (PCA), or equivalently Hard Singular Value Thresholding (HSV), on the resulting Page matrix (the number of principal components retained is an algorithmic hyper-parameter) to impute the time series; and (c) perform Ordinary Least Squares (OLS) on the retained principal components and the last row of the Page matrix to learn a linear forecasting model.

Multivariate Singular Spectrum Analysis (mSSA). Often, multiple related time series are available, e.g. prices of different stocks or sensor readings from cyber-physical systems. In this setting, the challenge is that one wants a method that both exploits the temporal structure (i.e., the structure that exists within a time series) along with the ‘spatial’ structure (i.e., the structure that exists across the time series). Towards this goal, mSSA is a natural extension of the SSA method where in Step 1 of the algorithm, a “stacked” Page matrix of dimension $L \times (NT/L)$ is constructed from the multivariate time series data by concatenating (column-wise) the various Page matrices induced by each time series. The subsequent two steps of mSSA remain equivalent to the SSA method introduced above [2].

Why Study mSSA? First, mSSA is a very well established and used in practice algorithm, yet lacks theoretical understanding. Second, we show that the variant of mSSA we propose has rather impressive empirical effectiveness. As seen in Table 2 despite mSSA’s simplicity, for both imputation and forecasting, it performs competitively with the current best neural network based time series libraries on various time series datasets [3]. In Section A we include details on the experimental setup and the datasets used. Third, mSSA significantly outperforms SSA – thus somehow the simple additional step of concatenating various Page matrices allows mSSA to exploit the additional “spatial” structure in the data that SSA cannot. This begs the question of when and why does mSSA work?

1.2 Our Contributions

Theoretical Analysis of mSSA. In Section 2 we establish both the imputation and (in-sample) forecasting prediction error scale as $(NT)^{-1/2}$. In particular, by first doing the core data transformation in mSSA of constructing the “stacked” Page matrix, we see the error scales as the product of $N$ and $T$. Our analysis allows one to reason about the sample complexity gains by exploiting both the low-rank spatial structure amongst the multivariate time series and the temporal structure within each time series. For example, in

Footnote 2 Formally, for a time series $f(t)$ with $T$ observations, the Page Matrix induced by it $M^f \in \mathbb{R}^{L \times (T/L)}$ is defined as $M^f_{ij} := f((i+j-1)L)$; the Hankel Matrix induced by it $H^f \in \mathbb{R}^{L \times T}$ is defined as $H^f_{ij} := f(i+j-1)$.

Footnote 3 The mSSA algorithm described is a variant of the standard SSA/mSSA method, (cf. [10]). However, the two core subroutines of performing PCA and subsequently OLS on the matrix induced from the time series remain the same. For a detailed comparison, please refer to our literature review.

Footnote 4 We note in companion work [11], we build a scalable, open-source time series prediction system (in line with the growing field of AutoML) using an incremental variant of mSSA. Via comprehensive empirical testing, we show mSSA outperforms all listed methods in Table 2 both in terms of statistical and just as importantly, computational performance. We include Table 2 in this work just as evidence of mSSA’s surprisingly high empirical effectiveness.
Algorithm for Multivariate Time-Varying Variance Estimation. Without making restrictive parametric assumptions, the challenge is the time-varying variance is not directly observed, not even a noisy version of it. Hence we believe an important algorithmic and theoretical contribution is the extension of mSSA we propose (see Section 3.1). We show it performs consistent imputation/forecasting of the time-varying variance parameter for a broad class of time series models, at rate \((NT)^{-1/2}\) (see Section 4.3). Further as empirical evidence, on real time series datasets, we find the stacked Page matrix does indeed have low-rank structure (see Table 3), and hence fits within our proposed model. The spatio-temporal factor model which brings together two heavily-studied models from well-established yet disparate literatures - the factor model used for panel data (i.e. multivariate time series data) in econometrics, and the low-rank Hankel model in the SSA literature. Under this model, in Proposition 2.6 we show that the stacked Page (and Hankel) matrix, the key data representation in mSSA, is indeed (approximately) low-rank.

Table 1: Comparison of finite-sample results with relevant algorithms in the literature.

| Method | Functionality | Mean Estimation | Variance Estimation | Regret |
|--------|---------------|-----------------|---------------------|--------|
|        |               | Imputation       | Forecasting         |        |
| This Work | Yes | Yes | (NT)^{-1/2} | (NT)^{-1/2} | (NT)^{-1/2} | (NT)^{-1/2} | N^{-1/2}T^{-0.014} |
| mSSA - Literature | Yes | No | – | – | – | – | – |
| SSA [10] | No | No | T^{-1/4} | – | – | – | – |
| Neural Network [16] | Yes | Yes | – | – | – | – | – |
| TRMF [25, 15] | Yes | No | (min(N,T))^{-1} | – | – | – | – |

Table 2: mSSA statistically outperforms SSA, other state-of-the-art algorithms, including LSTMs and DeepAR across many datasets. We use the average normalized root mean squared error (NRMSE) as our metric.

| Method | Electricity | Traffic | Synthetic | Financial | Mean Imputation (NRMSE) | Mean Forecasting (NRMSE) | Variance imputation Forecasting (NRMSE) |
|--------|-------------|---------|-----------|-----------|------------------------|------------------------|----------------------------------|
| mSSA   | 0.391       | 0.494   | 0.253     | 0.283     | 0.483                  | 0.525                  | 0.196                            |
| SSA    | 0.519       | 0.608   | 0.626     | 0.466     | 0.552                  | 0.704                  | 0.522                            |
| LSTM   | NA          | NA      | NA        | NA        | 0.551                  | 0.473                  | 1.203                            |
| DeepAR | NA          | NA      | NA        | NA        | 0.484                  | 0.474                  | 0.331                            |
| TRMF   | 0.694       | 0.512   | 0.335     | 0.513     | 0.534                  | 0.570                  | 0.267                            |
| Prophet| NA          | NA      | NA        | NA        | 0.582                  | 0.617                  | 1.005                            |

SSA we do not get this additional scaling given by the number of time series, \(N\), which might help explain mSSA’s vastly superior empirical performance, as seen in Table 2. Further, recent results from the low-rank matrix factorization time series literature (see [25, 15]), show imputation error scales as \(1/\min(N,T)\) (see Theorem 2 of [15]): it does not lead to consistency if \(N\) and \(T\) are not growing, e.g., having access to a single \((N=1)\) time series. See Table 1 for a summary of our theoretical results.

Approximate Low-Rank Hankel with Spatio-Temporal Factor Model. In Section 2.1 we propose a spatio-temporal factor model which brings together two heavily-studied models from well-established yet disparate literatures - the factor model used for panel data (i.e. multivariate time series data) in econometrics, and the low-rank Hankel model in the SSA literature. Under this model, in Proposition 2.6 we show that the stacked Page (and Hankel) matrix, the key data representation in mSSA, is indeed (approximately) low-rank.

In Section 2.2, we show many important time series models (e.g. any differentiable, periodic function) have this approximate low-rank structure. Towards that, we develop a representation “calculus” of this model class by establishing that it is closed under component wise addition and multiplication (cf. Proposition 2.1). Further as empirical evidence, on real time series datasets, we find the stacked Page matrix does indeed have low-rank structure (see Table 3), and hence fits within our proposed model. The spatio-temporal factor model is in line with that introduced in [25]. However, as stated above, it is the additional stacked Page matrix data transformation in mSSA which allows us prove the prediction error scales as the product of \(N\) and \(T\) rather than \(\min(N,T)\) as in [25].

Algorithm for Multivariate Time-Varying Variance Estimation. Estimating the time-varying variance parameter (a la GARCH-like model) is a key problem in time series analysis. Yet to the best of our knowledge, there does not exist a theoretically grounded method which accurately estimates this variance parameter without making restrictive parametric assumptions. The challenge is the time-varying variance is not directly observed, not even a noisy version of it. Hence we believe an important algorithmic and theoretical contribution is the extension of mSSA we propose (see Section 3.1). We show it performs consistent imputation/forecasting of the time-varying variance parameter for a broad class of time series models, at rate \((NT)^{-1/2}\) (see Section 4.3). Technically, we reduce the problem of estimating the time-varying variance parameter to that of accurately estimating the heteroscedastic variance parameter of a matrix, and furnish a simple two-step matrix estimation algorithm to do so. We note the simple “meta” variance estimation algorithm we propose allows for variance estimation using other time series methods that do imputation, e.g., in Appendix A.2 we use the method in [25] to do variance estimation via this approach. It could also be of independent interest in the matrix estimation literature to produce prediction intervals.

Online Variant of mSSA. Traditionally in time series analysis, the metric of evaluation is either parameter estimation (where explicit parametric assumptions are about the generating process) or the in-sample

---

4There seems to be a typo in Corollary 2 of [25] in applying Theorem 2: square in Frobenius-norm error is missing.
prediction error, which we provide bounds for. To study how mSSA “generalizes” (i.e. the quality of one-step ahead forecasts), we propose an online variant of mSSA and establish it has regret scaling as $N^{-\frac{1}{4}}T^{-0.04}$ (see Corollary 4.1). To do so, we establish an online variant of Principle Component Regression, a key step in the mSSA, has sub-linear regret in an error-in-variable regression setting. This required extending the theory of online convex optimization (OCO) to deal with time-varying constraints, a variation not addressed by existing results on OCO. We believe that our regret bound is not tight (w.r.t to scaling of $T$) and is an important direction for future research.

1.3 Literature Review

Given the ubiquity of multivariate time series analysis, we cannot possibly do justice to the entire literature. Hence we focus on a few techniques, most relevant to compare against, either theoretically or empirically.

**SSA.** For a detailed analysis of the standard SSA method, please refer to [10]. The main steps of SSA are given by: Step 1 - create a Hankel matrix from the time series data; Step 2 - do a Singular Value Decomposition (SVD) of it; Step 3 - group the singular values based on user belief of the model that generated the process; Step 4 - perform diagonal averaging to Hankelize” the grouped rank-1 matrices outputted from the SVD to create a set of time series; Step 5 - learn a linear model for each Hankelized” time series for the purpose of forecasting.

The theoretical analysis of this original SSA method has been on proving that many univariate time series have a low-rank Hankel representation, and secondly on defining sufficient (asymptotic) conditions for when the singular values of the various time series components are separable (thereby justifying Step 3 of the method).

In [2], the authors extend the class of time series considered to those with an approximate low-rank Hankel structure. Some subtle but core alterations to the SSA procedure were made. In Step 1, the Page Matrix rather than the Hankel was used, which allowed for independent noise assumptions to hold. In Steps 2-3, instead of doing the SVD of the matrix and grouping the singular values, only a single threshold is picked for the singular values (i.e., simply doing HSVT on the induced Page matrix). In Step 4, subsequently, a single linear forecasting model is learnt rather than a separate linear forecaster for each grouped time series. Under such a setting, they perform a finite-sample analysis of the variant of SSA. However, the analysis in [2] does fails short, even for the univariate case, as they do not explicitly show consistency of the SSA estimator for forecasting.

**mSSA.** mSSA is the natural extension of SSA for multivariate time series data and has been employed in a variety of applications with some empirical success (see [13][12][14]). However, theoretical analysis (and even practical guidance) regarding when and why it works is severely limited.

**Matrix Based Multivariate Time Series Methods.** There is a recent line of work in time series analysis (see [22][25]), where multiple time series are viewed collectively as a matrix, and some form of matrix factorization is done. Most such methods make strong prior model assumptions on the underlying time series and the algorithm changes based on assumptions of the model that generated the data. Further finite sample analysis of such methods is usually lacking. We highlight one method, Temporal Regularized Matrix Factorization (TRMF) (see [24]), which we directly compare against due to its popularity, and as it performs both imputation and forecasting. In [25], authors provide finite sample imputation analysis for an instance of the model considered in this work. In addition to the model being restrictive, the imputation error scales as $1/\min(N,T)$ which is weaker compared to our imputation error of $1/\sqrt{NT}$. For example, for $N=\Theta(1)$, their error bound remains $\Theta(1)$ for any $T$, while ours would decay with $T$ as $1/\sqrt{T}$.

**Other Relevant Time Series Methods.** Recently, with the advent of deep learning, neural network (NN) based approaches have been the most popular, and empirically effective. Some industry standard neural network methods include LSTMs, from the Keras library (a standard NN library, see [6]) and DeepAR (an industry leading NN library for time series analysis, see [16]).

**Time-Varying Variance Estimation.** The time-varying variance is a key input parameter in many sequential prediction/decision-making algorithms themselves. For example in control systems, the widely used Kalman Filter uses an estimate of the per step variance for both filtering and smoothing. Similarly in finance, the time-varying variance of each financial instrument in a portfolio is necessary for risk-adjustment. The key challenge in estimating the variance of a time series (which itself might very well be time-varying) is that unlike the actual time series itself, we do not get to directly observe it (nor even a noisy version of it). Despite the vast time series literature, existing algorithms to estimate time-varying variance are mostly heuristics and/or make restrictive, parametric assumptions about how the variance (and the underlying mean) evolves (e.g. ARCH/GARCH models). See [5]. Hence, provable finite-sample guarantees of these previous methods are highly restricted.

**Panel Data Models - Econometrics.** In a traditional panel data factor model in econometrics, for $n \in [N]$ and $t \in \mathbb{Z}$, $X_n(t) = \sum_{r=1}^{R} U_{nr} W_r + \eta_n$, where $U_{n,:}, W_r, \in \mathbb{R}^R$ for some fixed $R \geq 1$. The $R$-dimensional
vectors $U_{n,t}, W_{r,t}$, are referred to as the low-dimensional “factor loadings” associated with time and “space” respectively, and $\eta_{n,t}$ is the independent noise per measurement. Despite the ubiquity of such factor models for multivariate time series data, the time series dynamics associated with the latent factors $W_{r,t}$, are not explicitly modeled, a possible additional structure that can be further exploited. In the model proposed in this work, with interest towards analyzing mSSA, we extend this standard factor model by considering a time series model class for each latent time series factor, $W_{r,t} \in \mathbb{R}^T$ for $r \in [R]$. In particular, we consider time series model classes, where the Hankel matrix induced by each $W_{r,t}$ has an approximate low-rank structure (see Definition 2.1). This (approximate) low-rank Hankel model class is inspired and extended from the SSA literature (see [10][2]). We show that this approximate low-rank Hankel representation includes many important time series dynamics such as any finite sum and products of: harmonics; polynomials; any differentiable periodic function; any time series with a sufficiently “smooth” non-linear latent variable model representation (see Section 2.2).

In short, this spatio-temporal factor model we propose can be thought of as a synthesis of two standard time series models: (i) a low-rank factor model, traditionally used to analyze multivariate time series data in the econometrics literature; (ii) approximately low-rank Hankel models for “time factors”, a representation traditionally used to analyze univariate time series data in the SSA literature. Details of the model we propose can be found in Section 2.

2 Spatio-Temporal Factor Model For Multivariate Time Series Analysis

This section is organized into three parts: (i) Section 2.1 describes the spatio-temporal factor model we use; (ii) Section 2.2 describes the family of time series dynamics we consider specifically for the latent factors associated with time; (iii) Section 2.3 shows, theoretically and empirically, that the stacked Page (and Hankel) matrix can be found in Section 2.

2.1 Spatio-Temporal Factor Model

Spatio-Temporal Factor Model for Latent Multivariate Time-Varying Mean. Define $\mathcal{M}^f \in \mathbb{R}^{N \times T}$ as $\mathcal{M}^f_{\omega}(t) = f_n(t)$. We use the following spatio-temporal factor model,

**Property 2.1.** Let $\mathcal{M}^f$ satisfy $\mathcal{M}^f_{\omega}(t) = \sum_{r=1}^{R_1} U_{nr}^f \cdot W_{r,t}(t)$, where $W_{r,t}^f(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}, |U_{nr}^f| \leq \Gamma_1, |W_{r,t}^f(\cdot)| \leq \Gamma_2$.

**Interpretation.** In words, there are $R_1$ “fundamental” time series denoted by $W_{r,t}^f(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}, r \in [R_1]$ and for each $n \in [N], \mathcal{M}^f_{\omega}(\cdot)$ is obtained through weighted, by $U_{nr}^f$, combination of these $R_1$ times series. $\Gamma_1, \Gamma_2$ are standard boundedness assumptions for the underlying latent time-varying means. In the econometrics literature, particularly to model panel data, the $R$-dimensional vectors $U_{n,t}, W_{r,t}$ are referred to as the low-dimensional “factor loadings” associated with “space” and time respectively, and $\eta_{n,t}$ is the independent noise per measurement. Despite the ubiquity of such factor models in time series analysis, the time series dynamics associated with the latent factors $W_{r,t}$ are not explicitly modeled. This is exactly what a spatio-temporal model aims to circumvent. We specify and motivate the class of time series models we consider for $W_{r,t}^f(\cdot)$ below.

**Time Series Model Class Considered: $(G, \epsilon)$-Hankel Representable Time Series.** We now describe the approximately low-rank Hankel model we consider for the latent time series factors, $W_{r,t}^f(\cdot)$.

**Definition 2.1.** For any $T \geq 1$, let the Hankel matrix $H^f \in \mathbb{R}^{T \times T}$ induced by a time series $f$ be defined as $H^f_{i,j} := f(i+j-1), i,j \in [T]$. A time series $f$ is said to be $(G, \epsilon)$-Hankel representable if there exists $H^f_{(1)} \in \mathbb{R}^{T \times T}$, such that (i) $\text{rank}(H^f_{(1)}) \leq G$; (ii) $\|H^f - H^f_{(1)}\|_{\text{max}} \leq \epsilon$.

**Property 2.2.** For $r \in [R_1], W_{r,t}^f(\cdot)$ is $(G, \epsilon)$-Hankel, where $G_r \leq G_{\text{max}}, |\epsilon_r| \leq \epsilon_{\text{max}}$.

In Section 2.2 we establish that a large class of time series models admit such a $(G, \epsilon)$-Hankel representation.

Spatio-Temporal Model for Latent Multivariate Time-Varying Noise Variance. Let $\sigma^2_n : \mathbb{Z} \rightarrow \mathbb{R}$ represent the time-varying variances, i.e. $\sigma^2_n(t) \in \mathbb{E}[\eta^2_{n,t}(t)]$ for $(n,t) \in [N] \times \mathbb{Z}$. We denote the collection of time-varying variances as $\sigma^2$. Analogous to $\mathcal{M}^f$, the latent time-varying variance is described through, $\mathcal{M}^{\sigma^2} \in \mathbb{R}^{N \times T}$, which captures the spatial and temporal structure in the data.

**Property 2.3.** Let $\mathcal{M}^{\sigma^2}$ satisfy $\mathcal{M}^{\sigma^2}_{\omega}(t) = \sum_{r=1}^{R_2} U_{nr}^{\sigma^2} \cdot W_{r,t}^\sigma(\cdot)$, where $W_{r,t}^\sigma(\cdot) : \mathbb{Z} \rightarrow \mathbb{R}, |U_{nr}^{\sigma^2}| \leq \Gamma_3, |W_{r,t}^\sigma(\cdot)| \leq \Gamma_3$.

**Property 2.4.** For $r \in [R_2], W_{r,t}^\sigma(\cdot)$ is $(G_r^{\sigma^2}, \epsilon_r^{\sigma^2})$-Hankel, where $G_r^{\sigma^2} \leq G_{\text{max}}^{\sigma^2}, |\epsilon_r^{\sigma^2}| \leq \epsilon_{\text{max}}^{\sigma^2}$.

**Noisy, Sparse Observation Model.** We now state some assumptions on how the observations of $\mathcal{M}^f$ are corrupted by noise and missingness.
Property 2.5. We observe \( Z^X \in \mathbb{R}^{N \times T} \), where for \((n, t) \in [N] \times [T]\) and \(\rho \in (0, 1)\), we have \(Z^X_n(t) = X_n(t)\) with probability \(\rho\) and \(1-\rho\) without \(s\) with probability \(1-\rho\). Define \(Z^{X^2} \in \mathbb{R}^{N \times T}\) as \(Z^{X^2}_n(t) = (Z^X_n(t))^2\).

Property 2.6. \(\eta_n(t)\) are independent mean-zero sub-gaussian random variables such that \(\|\eta_n(t)\|_{\psi_2} \leq \gamma\).

2.2 Examples Of \((G,\epsilon)\)-Hankel Representable Time Series Dynamics

In this section, we show this approximate low-rank Hankel representation includes many important, standard time series dynamics considered in the literature, including: (i) any finite sum of harmonics, polynomials and exponentials (Proposition 2.3); (ii) any differentiable periodic function (Proposition 2.4) – we note, this is a heavily utilized time series model in signal processing; (iii) any time series with a Holder continuous non-linear latent variable model Hankel representation (Proposition 2.5). Furthermore, we show the class of \((G,\epsilon)\)-Hankel representable time series is closed under component wise addition and multiplication (Proposition 2.5). Importantly, we establish the more measurements we collect (i.e., as \(T\) grows), the approximation error for all these time series dynamics decays to zero.

2.2.1 \((G,\epsilon)\)-Hankel Time Series “Algebra”

Sums of Time Series. For two time series \(f_1\) and \(f_2\), let \(f_1 + f_2\) represent the time series induced by adding the values of the time series \(f_1\) and \(f_2\) component wise, i.e., \(f_1(t) + f_2(t) = f_1(t) + f_2(t)\) for all \(t \in \mathbb{Z}\).

Products of Time Series. For two time series \(f_1\) and \(f_2\), let \(f_1 \circ f_2\) represent the time series induced by multiplying the values of the time series \(f_1\) and \(f_2\) component wise, i.e., \(f_1(t) \times f_2(t) = f_1(t) \times f_2(t)\) for all \(t \in \mathbb{Z}\).

In the following proposition we establish that time series representable as \((G,\epsilon)\)-Hankel time series are “closed” under component wise sums and products.

Proposition 2.1. Let \(f_1\) and \(f_2\) be two time series which are \((G_1,\epsilon_1)\)-Hankel and \((G_2,\epsilon_2)\)-Hankel representable respectively. Then \(f_1 + f_2\) is a \((G_1 + G_2,\epsilon_1 + \epsilon_2)\)-Hankel. And, \(f_1 \circ f_2\) is a \((G_1G_2;3\max(\epsilon_1,\epsilon_2)\cdot\max(\|f_1\|_{\infty},\|f_2\|_{\infty}))\)-Hankel.

See Appendix [B.1] for proof of Proposition 2.1

2.2.2 \((G,\epsilon)\)-LRF Time Series

Definition 2.2 ((G,\epsilon)-LRF). A time series \(f\) is said to be a \((G,\epsilon)\)-Linear Recurrent Formula (LRF) if for all \(T \in \mathbb{Z}\) and \(t \in [T]\),

\[
f(t) = f'(t) + \epsilon(t),
\]

where: (i) \(f'(t) = \sum_{l=1}^{G} \alpha_l f'(t-l)\); (ii) \(|\epsilon(t)| \leq \epsilon\).

Now we establish \((G,\epsilon)\)-LRFs are \((G,\epsilon)\)-Hankel representable (see Appendix [B.2] for proof).

Proposition 2.2. If \(f\) is a time series that is \((G,\epsilon)\)-LRF representable, then it is \((G,\epsilon)\)-Hankel representable.

Finite Product of Harmonics, Polynomials and Exponentials. LRF’s cover a broad class of time series functions, including any finite sum of products of harmonics, polynomials and exponentials. The order \(G\) of the LRF induced by such a time series is quantified through the following proposition.

Proposition 2.3. (Proposition 5.2 in [2]) Let \(P_{m_a}\) denote a polynomial of degree \(m_a\). Then,

\[
f(t) = \sum_{a=1}^{A} \exp(\alpha_a t) \cdot \cos(2\pi \omega_a t + \phi_a) \cdot P_{m_a}(t)
\]

is \((G,0)\)-LRF representable, where \(G \leq A(m_{\max} + 1)(m_{\max} + 2)\), where \(m_{\max} = \max_{a \in A} m_a\).

Remark 2.1. We remark importantly, that given a time series of \(K\) harmonics, the order of the LRF induced by it is independent of the period of the \(K\) harmonics. Rather the order of the LRF scales only with the number of harmonics.

2.2.3 “Smooth”, Periodic Time Series - \(C^k_{T_{per}}\)

Here we establish, perhaps surprisingly, that a broad class of time series - essentially any periodic function that is sufficiently “smooth” (is in \(C^k_{T_{per}}\) as in Definition 2.3) has a \((G,\epsilon)\)-LRF representation. Intuitively, this is possible as by Fourier analysis, it is well-established that such functions are well-approximated by finite sums of harmonics, which themselves are LRFs (by Proposition 2.3).
Definition 2.3 \((C^k_{R,\text{per}}\text{-smoothness})\). A time series \(f\) is said to be in \(C^k_{R,\text{per}}\) if \(f\) is \(R\)-periodic (i.e. \(f(t+R) = f(t)\)) and the \(k\)-th derivative of \(f\), denoted \(f^{(k)}\), exists and is continuous.

Proposition 2.4. Let \(f\) be a time series such that \(f \in C^k_{R,\text{per}}\). Then for any \(G \in \mathbb{N}\), \(f\) is
\[
\left( 4G, C(k,R) \frac{\|f^{(k)}\|}{G^{k-0.5}} \right) - \text{Hankel representable}.
\]
Here \(C(k,R)\) is a term that depends only on \(k,R\); and \(\|f^{(k)}\|^2 = \frac{1}{R} \int_0^R (f^{(k)}(t))^2 dt\).

See Appendix B.3 for proof of Proposition 2.4.

2.2.4 Time Series with Latent Variable Model Structure

**Time Series with Latent Variable Model (LVM) Representations** We now show that if a time series has a Latent Variable Model (LVM) representation, and the latent function is Hölder continuous, then it has a \((G, \epsilon)\)-Hankel representation. We first define the Hölder class of functions. Note this class of functions is widely adopted in the non-parametric regression literature \([20]\). Given a function \(g : [0, 1)^K \rightarrow \mathbb{R}\), and a multi-index \(\kappa\), let the partial derivates of \(g\) at \(x \in [0, 1)^K\) (if it exists) be denoted as, \(\nabla_\kappa g(x) = \frac{\partial^{\kappa} g(x)}{(\partial_{x_1}^{\kappa_1}) \cdots (\partial_{x_n}^{\kappa_n})}\).

**Definition 2.4 ((\(\alpha, \mathcal{C}\))-Hölder Class).** Let \(\alpha, \mathcal{C}\) be two positive numbers. The Hölder class \(\mathcal{H}(\alpha, \mathcal{C})\) on \([0, 1)^K\) is defined as the set of functions \(g : [0, 1)^K \rightarrow \mathbb{R}\) whose partial derivatives satisfy, for all \(x, x' \in [0, 1)^K\),
\[
\sum_{\kappa : |\kappa| = |\alpha|} \frac{1}{|\kappa|!} |\nabla_\kappa g(x) - \nabla_\kappa g(x')| \leq \mathcal{C} \|x - x'\|_{\infty}^{\alpha - |\alpha|}.
\]
(1)

Here \(|\alpha|\) refers to the greatest integer strictly smaller than \(\alpha\).

**Remark 2.2.** Note if \(\alpha \in (0, 1]\), then \([1]\) is equivalent to the \((\alpha, \mathcal{C})\)-Lipschitz condition, for all \(x, x' \in [0, 1)^K\),
\[
|g(x) - g(x')| \leq \mathcal{C} \|x - x'\|_{\infty}^{\alpha - |\alpha|}.
\]
However for \(\alpha > 1\), \((\alpha, \mathcal{C})\)-Hölder smoothness no longer implies \((\alpha, \mathcal{C})\)-Lipschitz smoothness.

**Property 2.7 (Time Series with (\(\alpha, \mathcal{C}\))-Hölder Smooth LVM Representation).** Let \(H^f \in \mathbb{R}^{T \times T}\) be the Hankel matrix induced by a time series \(f\). Recall \(H^{f}_{t,s} := f(t+s-1)\). A time series \(f\) is said to have a \((\alpha, \mathcal{C})\)-Hölder Smooth LVM Representation if the Hankel matrix, \(H^f\), has the following representation
\[
H_{t,s}^f = g(\theta_t, \omega_s),
\]
where \(\theta_t, \omega_s \in [0, 1)^K\) are latent parameters. Moreover for all \(\omega_s\), \(g(\cdot, \omega_s) \in \mathcal{H}(\alpha, \mathcal{C})\) as defined in \([1]\).

As stated earlier, the domain of the latent parameters \(\theta_t, \omega_s\) in Property 2.7 is easily extended to any compact subset of \(\mathbb{R}^K\) by appropriate rescaling.

**Remark 2.3.** Note that a \((G, 0)\)-Hankel denoted by \(f\), has the following representation (see Proposition C.2),
\[
H_{i,j}^f = (a^{(i)})^T b^{(j)}
\]
for some latent vectors \(a^{(i)}, b^{(j)} \in \mathbb{R}^G\). Hence, a \((G, 0)\)-Hankel representable time series is an instance of a time series that satisfies Property 2.7 for all \(\alpha \in \mathbb{N}\), and \(\mathcal{C} = C\), for some absolute positive constant, \(C\). One can thus think of time series that satisfy Property 2.7 as generalizations to (sufficiently smooth) non-linear functions, instead of just linear products of the latent factors (as would have in a low-rank Hankel).

**Proposition 2.5.** Let a time series \(f\) satisfy Property 2.7 with parameters \(\alpha, \mathcal{C}\). Then for all \(\epsilon > 0\), \(f\) is
\[
(C(\alpha, \mathcal{K}) \left( \frac{1}{\epsilon} \right)^{K}) - \text{Hankel representable}.
\]
Here \(C(\alpha, \mathcal{K})\) is a term that depends only on \(\alpha\) and \(\mathcal{K}\).

See Appendix B.4 for proof of Proposition 2.5.

2.3 Stacked Hankel Matrix of mSSA is (Approximately) Low-Rank

**Stacked Page, Hankel Matrix Notation.** The notation below is a formal description of the stacked Page and Hankel matrices, one gets by performing the core mSSA data transformation (i.e., concatenating the induced matrices column-wise). The only difference is the stacked Hankel matrix contains overlapping columns (while the stacked Page matrix utilized in the proposed variant of mSSA does not). For a multivariate time series \(g\) with \(N\) time series and \(T\) observations, let \(L \in \mathbb{N}\) be a hyper-parameter and let \(P = \lfloor T/L \rfloor\). For simplicity,
throughout we shall assume that $T/L = |T|/|L|$, i.e., $L \times P = T$. Let $\tilde{P} := N \times P$. Let $M^g \in \mathbb{R}^{L \times P}$, $H^g \in \mathbb{R}^{L \times (N \times (T - L + 1))}$ be the induced stacked Page and Hankel matrices by $g$ (with hyper-parameter $L \geq 1$); for $l \in [L]$, $k_1 \in [P]$, $k_2 \in [T - L + 1]$, $n \in [N]$;
\[
M^g_{l[k_1 + P \times (n - 1)]} = g_n(l + (k_1 - 1)L), \quad H^g_{l[k_1 + P \times (n - 1)]} = g_n(l + k_2 - 1).
\]

**Theoretical Justification For Stacked Hankel Matrix Transformation.** Proposition 2.6 shows that the stacked Page and Hankel matrices, the core data structures of interest in mSSA, have an approximate low-rank representation under our proposed model. This approximate low-rank structure of $H^f$ (and $H^{\sigma^2}$) is what crucially allows us to connect the analysis mSSA to recent advances in matrix estimation and high-dimensional statistics. See Appendix C for a proof of Proposition 2.6.

**Proposition 2.6.** Let Properties 2.1 and 2.2 hold. Let $M^f, H^f$ be defined w.r.t $M^f$. Then, there exists a matrix $H^f_{(ir)}$ such that, rank($H^f_{(ir)}$) $\leq R_1 G_{\text{max}}^{(1)}$ and $\|H^f - H^f_{(ir)}\|_{\text{max}} \leq R_1 G_{\text{max}}^{(1)}$. As an immediate consequence, there exists $M^f_{(ir)}$ such that rank($M^f_{(ir)}$) $\leq R_1 G_{\text{max}}^{(1)}$ and $\|M^f - M^f_{(ir)}\|_{\text{max}} \leq R_1 G_{\text{max}}^{(1)}$.

**Corollary 2.1.** Let Properties 2.3 and 2.4 hold. Let $M^{\sigma^2}, H^{\sigma^2}$ be defined w.r.t $M^{\sigma^2}$. Then, there exists a matrix $H^{\sigma^2}_{(ir)}$ such that, rank($H^{\sigma^2}_{(ir)}$) $\leq R_2 G_{\text{max}}^{(2)}$ and $\|H^{\sigma^2} - H^{\sigma^2}_{(ir)}\|_{\text{max}} \leq R_2 G_{\text{max}}^{(2)}$. As an immediate consequence, there exists $M^{\sigma^2}_{(ir)}$ such that rank($M^{\sigma^2}_{(ir)}$) $\leq R_2 G_{\text{max}}^{(2)}$ and $\|M^{\sigma^2} - M^{\sigma^2}_{(ir)}\|_{\text{max}} \leq R_2 G_{\text{max}}^{(2)}$.

**Interpretation.** We see that the stacked Hankel matrix has (approximate) rank that scales no more than the product of the rank of the factor model $R_1$, and the maximum (approximate) rank, $c_{\text{rank}}(1)$, of the Hankel matrices induced by the latent factors $W^f(\cdot)$, for $r \in [R]$). Crucially, the rank does not scale with the number of time series, $N$ nor with the number of measurements, $T$. Indeed, we see in Table 3 that across standard benchmark datasets, the (approximate) rank of the stacked Hankel matrix indeed scales very slowly with the number of time series, $N$ and number of measurements, $T$. Hence these standard multivariate time series datasets seem to fit within our proposed model.

Table 3: Across standard benchmarks, effective rank of the stacked Hankel matrix scales slowly with the number of time series. Effective rank is defined as the number of singular values to capture $> 90\%$ of the spectral energy.

| Dataset     | N = 1 | N = 10 | N = 100 | N = 350 |
|-------------|-------|--------|---------|---------|
| Electricity | 19/24 | 37/43  | 44/60   | 31/52   |
| Financial   | 1/1   | 3/3    | 3/4     | 6/9     |
| Traffic     | 14/16 | 32/65  | 69/224  | 116/296 |

### 3 mSSA Algorithm

**Some Necessary Notation.** Recall from Section 2 we assume access to observations $Z^X_N(1 : T) \in \mathbb{R}^T$ for $n \in [N]$. Additionally, let $\tilde{\rho}$ denote the fraction of observed entries of $Z^X$, i.e.,
\[
\tilde{\rho} := \frac{1}{N^T \left( \sum_{n,t} I(Z^X(t)) \neq 0 \right)} > \frac{1}{N^T}.
\]

For a matrix $A \in \mathbb{R}^{L \times P}$, let $A_L$ refer to the last row of $A$ and let $A_{(L-1)\times P}$ refer to the sub-matrix induced by retaining its first $L - 1$ rows.

#### 3.1 mSSA Imputation and Forecasting: Mean and Variance Estimation

**Mean Estimation Algorithm.**

**Imputation.** The key steps of mean imputation are as follows.

1. **(Form Page Matrix)** Transform $X_1(1 : T), \ldots, X_N(1 : T)$ into a stacked Page matrix $Z^X \in \mathbb{R}^{L \times P}$ with $L \leq \tilde{P}$. Fill all missing entries in the matrix by 0.

2. **(Singular Value Thresholding)** Let SVD of $Z^X = USV^T$, where $U \in \mathbb{R}^{L \times L}, V \in \mathbb{R}^{P \times L}$ represent left, right singular vectors, $S = \text{diag}(s_1, \ldots, s_L)$ is diagonal matrix of singular values $s_1 \geq \cdots \geq s_L \geq 0$. Obtain $\hat{M}^{\text{Impute}}_{(l)} = US_kV^T$, where $S_k = \text{diag}(s_1, \ldots, s_k, 0, \ldots, 0)$ for some $k \in [L]$.

3. **(Output)** $\hat{f}_n(i + (j - 1)L) := \hat{M}^{\text{Impute}}_{l[i + P(n - 1)]}, i \in [L], j \in [P], n \in [N]$.

**Forecasting.** Forecasting includes an additional step of fitting a linear model on the de-noised matrix.

1. **(Form Sub-Matrices)** Let $Z^X_{L-1 \times P}$ be a sub-matrix of $Z^X$ obtained by removing its last row, $Z^X_{L}$. 
2. ( Singular Value Thresholding) Let SVD of $Z = USV^T$, where $U \in \mathbb{R}^{L-1 \times L-1}$, $V \in \mathbb{R}^{P \times L-1}$ represent left and right singular vectors and $S = \text{diag}(s_1, \ldots, s_{L-1})$ be diagonal matrix of singular values $s_1 \geq \ldots \geq s_{L-1} \geq 0$. Obtain $M^f = US_kV^T$ by setting all but top $k$ singular values to $0$, i.e. $S_k = \text{diag}(s_1, \ldots, s_k, 0, \ldots, 0)$ for some for some $k \in [L-1]$.

3. (Linear Regression) $\hat{\beta} \in \arg\min_{b \in \mathbb{R}^{L-1}} \| Z_L^\sigma - (\hat{M}^f)^Tb \|^2_2$.

4. (Output) For $s \in \{L, 2L, \ldots\}$, $n \in [N]$, $\hat{f}_n(s) := X_n(s-L+1:s-1)^T\hat{\beta}$.

### Variance Estimation Algorithm.

For variance estimation, the mean estimation algorithm is run twice, once on $Z^X$ and once on $Z^{X^2}$. To estimate the time-varying variance (for both forecasting and imputation), a simple post-processing step is done where the square of the estimate produced from running the algorithm on $Z^X$ is subtracted from the estimate produced from $Z^{X^2}$. These steps above can be viewed as a “meta”-algorithm to do variance estimation. Indeed in Appendix A.2, to compare the performance of mSSA for variance estimation with other algorithms, we utilize the method in [28] to do variance estimation via this “meta” algorithm.

### Imputation.

Below are the steps for variance imputation using mSSA.

1. (Impute $Z^X, Z^{X^2}$) Use the mean imputation algorithm on $X_1(1:T), \ldots, X_N(1:T)$ (i.e., $Z^X$) and $X_1^2(1:T), \ldots, X_N^2(1:T)$ (i.e., $Z^{X^2}$), to produce the de-noised Page matrices $\hat{M}^f$ and $\hat{M}^{f_2+\sigma^2}$, respectively.

2. (Output) Construct $\hat{M}^{f^2} \in \mathbb{R}^{L \times P}$, where $\hat{M}_{ij}^{f^2} := (\hat{M}^{f_2})^2$, and produce estimates, $\hat{\sigma}^2_{(i+(j-1)L)} := \hat{M}_{ij}^{f_2+\sigma^2} - \hat{M}_{ij}^{f_2}$, for $i \in [L], j \in [P]$.

### Forecasting.

Like mean forecasting, it involves an additional step after imputation.

1. (Forecast with $Z^X, Z^X_L, Z^{X^2}, Z^{X^2}_L$) Using the mean forecasting algorithm on $X_1(1:T), \ldots, X_N(1:T)$ and $X_1^2(1:T), \ldots, X_N^2(1:T)$, to produce forecast estimates $\hat{f}_n(T+1)$ and $\hat{f}^2_n(T+1)$, respectively.

2. (Output) Produce the variance estimate $\hat{\sigma}^2_{(T+1)} := \hat{f}^2_n(T+1) - (\hat{f}_n(T+1))^2$.

### Online Convex Optimization Lens

**Online Convex Optimization (OCO) Setup.** The aim of this section is to formally tie mSSA to an OCO setting. First, we briefly describe the setup/dynamics of OCO relevant to us. At each step, $t \in \mathbb{N}$ we receive a convex set $\Omega_t$ and choose an element in it, denoted as $\beta_t \in \Omega_t$. Subsequent to choosing $\beta_t$, we receive cost function $c_t$ and incur cost $c_t(\beta_t)$. Note the key difference from the traditional OCO setup is that the convex set $\Omega_t$ is varying over time, an area of limited study in the online learning literature.

### Notation for online-mSSA.

We begin by introducing some notation. Let $g$ be a multivariate time series, comprising of $N$ time series. For any $t \in \mathbb{N}$, let $Z^g_{(t)} \in \mathbb{R}^{N \times T}$ refer to the first $t$ (noisy, sparse) observations of $g$. For each time step $t \in \mathbb{N}$, we assume we get access to the observations of the various time series in the sequence given by this ordering. Let $(t, n) \in \mathbb{N} \times [N]$ be a double index denoting the current time step, $t$, and the number of time series $n \in [N]$ we have observed at this current time step thus far. Note, the total number of observations by index $(t, n)$ is equal to $(N(t-1)+n)$. Let the stacked Page matrix induced by the current observations be denoted as $Z^g_{(t, n)} \in \mathbb{R}^{L \times ((N(t-1)+n)/L)}$. Let $Z^g_{(t, n), L}$ refer to the last row of the stacked Page matrix and let $Z^g_{(t, n), L}$ be the sub-matrix of $Z^g_{(t, n)}$ obtained by removing the last row, $[Z^g_{(t, n), L}]$. For any $k \in [L-1]$, let $\hat{U}^{k}_{(t, n)} \in \mathbb{R}^{(L-1) \times k}$ refer to the first $k$ left singular vectors of $Z^g_{(t, n)}$. Let $\Omega_{(t)}^{k}$ refer to the linear subspace induced by $U^{k}_{(t)}$.

### mSSA (i.e., PCR) as Regularized Linear Regression.

For $(t, n) \in \mathbb{N} \times [N]$, if we ran the forecasting algorithm in Section 3.1 with all available data, define $\beta_{(t, n)}^* \in \mathbb{R}^{L-1}$ as the resulting linear model obtained (i.e., $\beta$ from Step 3 of the forecasting algorithm in Section 3.1). For any fixed $k \in [L-1]$, where $k$ is the number of singular vectors of $Z^g_{(t, n)}$ retained, it can be verified that the $\beta_{(t, n)}^*$ is the solution of $\beta_{(t, n)}^* = \arg\min_{\beta \in \Omega_{(t, n)}^{k}} \| Z^g_{(t, n), L} - (Z^g_{(t, n)})^T \beta \|^2_2$.

### Interpretation.

The forecasting algorithm in Section 3.1 essentially does PCR (see Proposition 3.1 of [3]). It is a standard result that PCR is a form of regularized linear regression where the linear model $\beta_{(t, n)}^*$ is constrained to lie in the subspace spanned by the first $k$ singular vectors (i.e., $\Omega_{(t, n)}^{k}$) of the covariate matrix (i.e., $Z^g_{(t, n)}$).
OCO Dynamics for mSSA Setting. We make the following assumptions: (i) we have access to $T$ data points for each of the $N$ time series before we start making one-step ahead forecasts using mSSA; (ii) for each time series $n \in [N]$, we make a forecast for $H$ time steps at points $\{T + L, T + 2L, \ldots, T + H \times L\}$; (iii) we assume $k$ the number of principal components we retain at each step is also specified beforehand (hence, $\Omega^k_{T + tL,n}$ for $t \in [H]$ is the induced convex set from which we pick the per step linear model). We can now specify the OCO framework for our mSSA setting. For notational simplicity, we drop dependence on $g$ below.

For $n \in [N]$ and for $t \in [H]$ (where $Z_n(T + tL)$ is the $T + tL$ observation of the $n$-th time series):

1. **Pick** $\hat{\beta}(T + tL,n)$ from $\Omega^k_{T + tL,n}$;
2. **Incur cost** $c(T + tL,n)(\hat{\beta}(T + tL,n)) := [Z_n(T + tL) - (Z_n(T + (t-1)L + 1 : T + tL - 1))T \hat{\beta}(T + tL,n)]^2$.

**offline-mSSA.** We shall utilize an online gradient descent variant of the mSSA algorithm: for $n \in [N],$

1. Initialize $\hat{\beta}(T + tL,n)$, compute cost $c(T + tL,n)(\hat{\beta}(T + tL,n))$.
2. **Update**, $t \in [1 : H]$ as:
   - (i) $\tilde{\beta}(T + (t + 1)L,n) = \hat{\beta}(T + tL,n) - \delta \nabla c(T + tL,n)(\hat{\beta}(T + tL,n))$;
   - (ii) $\hat{\beta}(T + (t + 1)L,n) = \text{argmin}_{w \in \Omega^k_{T + tL,n}} \|w - \tilde{\beta}(T + (t + 1)L,n)\|_2$.

**Algorithm Intuition.** In effect, we utilize the standard projected online gradient descent algorithm. As stated earlier, the key difference is in our setting, the domain $\Omega^k_{T,n}$ is changing at each time step. We give guidance on choice of $\delta$ when instatiating the regret bound in Theorem 4.1

4 **Theoretical Results**

From Section 2.3 recall definition of $L, P, P^*, M^g, M^g_L$ and $M^g$.

4.1 **Error Metrics for Evaluating Imputation and Forecasting Prediction Error**

**Imputation.** For an estimate $\hat{M}^{g(\text{Impute})}$ of $M^g$, our imputation error metric is $\text{MSE}(M^g, \hat{M}^{g(\text{Impute})}) := \frac{1}{N^2} \mathbb{E}[\|M^g - \hat{M}^{g(\text{Impute})}\|_F^2]$, where the expectation is over noise and $\|\cdot\|_F$ refers to the Frobenius norm.

**Forecasting.** For forecasting we evaluate mSSA through two error metrics, an “offline” benchmark (i.e., in-sample prediction error), and an online regret analysis (relative to this offline benchmark).

**Offline benchmark (in-sample error).** Recall $M^g_L \in \mathbb{R}^{P}$ refers to the last row of the induced stacked Page matrix. The “offline” benchmark corresponds to the in-sample prediction error with respect to $M^g_L$, i.e., for an estimate $\hat{M}^{g(\text{Forecast})}_L$, we define the error as $\text{MSE}(M^g, \hat{M}^{g(\text{Forecast})}) := \frac{1}{P} \mathbb{E}[\|M^g_L - \hat{M}^{g(\text{Forecast})}_L\|_2^2]$

**Regret Metric.** Recall the definition of $c_t(\cdot), \hat{\beta}^*_n(T + H \times L,N)$ and $\hat{\beta}(T + H \times L,N)$ from Section 3.2, $\hat{\beta}(T + tL,n)$ for $(n,t) \in [N] \times [H]$ are the estimates produced by online-mSSA. $\hat{\beta}(T + H \times L,N)$ is the estimate produced if one had access to all $(T + H \times L) \times N$ data points – it is exactly the in-sample prediction error, i.e., the offline benchmark denoted above. Regret is thus defined as,

\[
\text{regret} := \sum_{n=1}^{N} \sum_{t=1}^{H} [c_t(\hat{\beta}(T + tL,n)) - c_t(\hat{\beta}(T + H \times L,N))].
\]

4.2 mSSA Mean Estimation - Finite Sample Analysis.

**Rank, Singular Values of $M^f_{(lr)}, M^f_{(lr)}^{2 + \sigma^2}$.** Recall definition of $M^f_{(lr)}$ and $M^f_{(lr)}^{2 + \sigma^2}$ from Proposition 2.6 and Corollary 2.1 respectively. Denote rank of $M^f_{(lr)}$ as $r(l)$. For $i \in \{1, \ldots, r(l)\}$, let $\tau_i$ denote the $i$-th singular value of $M^f_{(lr)}$ ordered by magnitude. Define $M^f_{(lr)}^{2 + \sigma^2}$ as the entry-wise addition of $M^f_{(lr)}^2$ and $M^f_{(lr)}^{2}$.

\[\text{We make forecasts at multiples of } L. \text{ This is easily circumvented by having } L \text{ different models;} \text{ for ease of exposition, we keep it to only single model.}\]

\[\text{Note that } M^f_{(lr)} \text{ contains entries of the latent time series for multiples of } L, \text{ i.e., } \{L, 2L, \ldots, T\}. \text{ This can be addressed simply creating } L \text{ different forecasting models, for } L + i, \text{ for } i \in \{0, 1, \ldots, L - 1\}. \text{ The corresponding algorithm and theoretical results remain identical. However, this is likely a limitation of our analysis technique and is irrelevant in practice as evidenced in our experiments.}\]
where $M_{(r)}^{f_2}$ is the entry-wise squaring of $M_{(r)}^{f}$. Denote rank of $M_{(r)}^{f_2+\sigma^2}$ as $r'(2)$. For $i \in [r'(2)]$, let $\tau_i^{(2)}$ denote the $i$-th singular value of $M_{(r)}^{f_2+\sigma^2}$.

**Property 4.1.** The non-zero singular values, $\tau_i^{(1)}$, of $M_{(r)}^{f}$ are well-balanced, i.e., $(\tau_i^{(1)})^2 = \Theta(TN/r^{(1)})$. Similarly, the non-zero singular values, $\tau_i^{(2)}$, of $M_{(r)}^{f_2+\sigma^2}$ are well-balanced, i.e., $(\tau_i^{(2)})^2 = \Theta(TN/r^{(2)})$.

**Interpretation.** A natural setting in which Property 4.1 holds is the entries of $M_{(r)}^{f_2}$ satisfy $s^2 = \Theta(\zeta)$ for some $\zeta$. Then, $C r^{(1)} \zeta = \| M_{(r)}^{f_2} \|_F = \Theta(TN)$ for some constant $C$, i.e., $(\tau_i^{(1)})^2 = \Theta(TN/r^{(1)})$. An identical argument applies to $M_{(r)}^{f_2+\sigma^2}$. Further, see Proposition 4.2 of [3] for another canonical example of when Property 4.1 holds.

**Theorem 4.1** (mssa Mean Estimation: Imputation). Let Properties 2.1, 2.2, 2.5, 2.6 and 4.1 hold. Let $\rho \geq C \log((L^2)/\rho)$ for absolute constant $C > 0$, $P = L$ and rank($\hat{M}$) = $r^{(1)}$. Let $C_1$ be a term that depends polynomially on $\Gamma_1, \Gamma_2$. Then,

$$\text{MSE} (\hat{M}^{f}, \hat{M}^{f}_{\text{(Impu)}}) \leq C_1 \gamma^2 \frac{R_1^2 (G_{\text{max}}^{(1)})^3}{\rho^3} \left( \frac{1}{\sqrt{TN}} + \left( \frac{\epsilon_1^{(1)}}{\epsilon_{\text{max}}^{(1)}} \right)^2 \right) \log(P).$$

**Theorem 4.2** (mssa Mean Estimation: Forecasting). Let conditions of Theorem 4.1 hold. Then,

$$\text{MSE} (\hat{M}^{f}, \hat{M}^{f}_{\text{(Forecast)}}) \leq C_1 \gamma^4 \frac{R_1^2 (G_{\text{max}}^{(1)})^3}{\rho^3} \left( \frac{1}{\sqrt{TN}} + \left( \frac{\epsilon_1^{(1)}}{\epsilon_{\text{max}}^{(1)}} \right)^2 \right) \log(P).$$

**Interpretation.** The mssa imputation and (in-sample) forecasting error bounds in Theorems 4.1 and 4.2 scale as, $O((1/\sqrt{NT}) + (\epsilon_{\text{max}}^{(1)})^2)$. In both results the bound is dominated by the error introduced in Step 2 of the respective algorithms, i.e., the SVT step in Section 3.1. It is straightforward to verify that for the time series dynamics listed in Section 3.1 the approximation error term vanishes as we collect more data. The assumption rank($\hat{M}^{f}$) = $r^{(1)}$, requires that the number of principal components are chosen correctly, i.e., is equal to the (approximate) rank of the stacked Page matrix. In Theorem 11, in Appendix H we generalize the above results to the setting where the number of principal components is misspecified, i.e., rank($\hat{M}^{f}$) $\neq r^{(1)}$.

**Technical Innovations.** The key technical challenge in proving Theorem 4.1 is establishing when the covariate matrix is well-approximated in an entry-wise sense by a low-rank matrix (see Lemma 4.1). [3] establishes the effectiveness of PCR only when the covariate matrix is well-approximated in operator norm by a low-rank matrix; hence, the existing bound in [3] on operator norm error could diverge even if the entry-wise error is diminishing. The key technical challenge in proving Theorem 4.2 is establishing that a (approximate) linear relationship between $\hat{M}^{f}$ and $M_{(r)}^{f}$ exists (see Proposition H.1). This is what motivates learning a linear model in Step 3 of the forecasting algorithm in Section 3.1.

### 4.3 mssa Variance Estimation: Finite Sample Analysis

**Theorem 4.3** (mssa Variance Estimation: Imputation). Let the conditions of Theorem 4.1 hold. Let Properties 2.3 and 2.4 hold and rank($\hat{M}^{f_2+\sigma^2}$) = $r^{(2)}$. Then,

$$\text{MSE} (\hat{M}^{f_2+\sigma^2}, \hat{M}^{f_2+\sigma^2}_{\text{(Impu)}}) \leq C_1 \gamma^2 \frac{R_1^4 R_2^2 (G_{\text{max}}^{(1)})^2 (G_{\text{max}}^{(2)})^3}{\rho^4} \left( \frac{1}{\sqrt{TN}} + \left( \frac{\epsilon_1^{(1)}}{\epsilon_{\text{max}}^{(1)}} \right)^2 + \left( \frac{\epsilon_2^{(2)}}{\epsilon_{\text{max}}^{(2)}} \right)^2 \right) \log(P).$$

**Theorem 4.4** (mssa Variance Estimation: Forecasting). Let conditions of Theorem 4.3 hold. Then,

$$\text{MSE} (\hat{M}^{f_2+\sigma^2}, \hat{M}^{f_2+\sigma^2}_{\text{(Forecast)}}) \leq C_1 \gamma^4 \frac{R_1^4 R_2^6 (G_{\text{max}}^{(1)})^5 (G_{\text{max}}^{(2)})^6}{\rho^6} \left( \frac{1}{\sqrt{TN}} + \left( \frac{\epsilon_1^{(1)}}{\epsilon_{\text{max}}^{(1)}} \right)^2 + \left( \frac{\epsilon_2^{(2)}}{\epsilon_{\text{max}}^{(2)}} \right)^2 \right) \log(P).$$

**Interpretation.** The variance estimation bounds come from the bounds for mean estimation and an extra term that arises due to error in estimating $\hat{M}^{f_2+\sigma^2}$. This also leads to a higher polynomial dependence on the model parameters, $R_1, R_2, G_{\text{max}}^{(1)}, G_{\text{max}}^{(2)}$.

### 4.4 mssa Forecasting (Online Variant) - Regret Analysis

**Additional Necessary Notation.** We follow the same notation as in Section 3.2; in particular for the latent time series $f$, and index $(t,n) \in \mathbb{N} \times [N]$, recall the definitions of $Z_{(t)}^f, Z_{(t,n)}^f, Z_{(n,t)}^f$, $[Z_{(t,n)}^f]_L$ and $\Omega_{(t,n)}^{(i)}$.
for \( k \in [L-1] \). Let \( v_{(t,n),1}^k, \ldots, v_{(t,n),\kappa}^k \) be an orthonormal basis of \( \Omega_t^{(k)} \) and \( v_{(t,n),1}^{k+1}, \ldots, v_{(t,n),\kappa}^{k-1} \) be additional orthonormal vectors to form a complete basis of \( \mathbb{R}^{\kappa-1} \). Let \( Z_{(t,n,j)} := [Z_{(t,n)}^j] \). Define \( a_{(t,n,j)} \in \mathbb{R}^{\kappa-1} \) as 
\[
Z_{(t,n,j)} = \sum_{i=1}^{\kappa-1} a_{(t,n,j),i} v_{(t,n),i}.
\]
Further we define \( M_{(t,n)}^f \), \( \tilde{M}_{(t,n)}^f \), \( [M_{(t,n)}^f]_L \) and \( \Omega_t^{(k)} \) for \( k \in [L-1] \), analogously to above, but now with respect to the underlying latent mean of \( f \), given by \( \mathcal{M}^f \). Let \( \tau_{(t,n)}^1, \ldots, \tau_{(t,n)}^k \) be the first \( k \) singular values of \( \tilde{M}_{(t,n)}^f \). For a linear subspace defined by \( \Omega_t \), let \( P_\Omega \) be the associated projection operator (or matrix).

**Theorem 4.5 (Online mSSA: Regret Analysis).** Recall \( T = L \times P \). Let Properties 2.1, 2.2, 2.5 and 2.6 hold. Let \( C_3 \) be a term that depends only polynomially on \( \Gamma_1, \Gamma_2, R_1, G^{(1)}_{\max} \) and \( \gamma \). Further, for \( n \in [N] \), \( t \in [H+1] \), for some \( \epsilon_1, \epsilon_2 \in (0,1) \) and some \( k \in [L-1] \), let the following hold:

\[
A.1 \quad P_{\Omega_{(T+t+L,n)}}^k = P_{\Omega_{(T+H+L,n)}}^k;
A.2 \quad \Omega_{(T+t+L,n)}^k = \Omega_{(T+H+L,n)}^{k+1}, \quad \Lambda_{(T+t+L,n)}^{k+1} = 0;
A.3 \quad L = P^{1-\epsilon_1}, \quad H = P^{1-\epsilon_2};
A.4 \quad \| \beta_{(T+t+L,n)} \|_2 \leq C_3 k, \quad \beta_{(T+t+L,n)} \in \Omega_{(T,n)}^k;
A.5 \quad \max_{i,j} |a_{(T+t+L,n,j)}^{(i)}| \leq C_3 \sqrt{\frac{T}{N}};
A.6 \quad |Z_{\epsilon} (T+t+L)| \leq C_3.
\]

Picking \( \delta = \frac{1}{\sqrt{3 \pi H}} + \frac{2 N}{H} \), we get that with probability at least \( 1 - 1/(NT) \),

\[
\frac{1}{NH} \text{regret} \leq C_3 k^{2.5} \sqrt{\frac{p^{1-2\epsilon_1+\epsilon_2}}{N} + p^{2-2\epsilon_1-0.5\epsilon_2}}.
\]

**Corollary 4.1.** Let conditions of Theorem 4.5 hold. Pick \( \epsilon_1 = 0.86, \epsilon_2 = 0.67 \). Then, with probability at least \( 1 - 1/(NT) \),

\[
\frac{1}{NH} \text{regret} \leq C_3 k^{2.5} \frac{1}{\sqrt{N} T^{0.04}}.
\]

**Justification of Assumptions.**

A.1 This is the key assumption made. In essence, it requires that \( T \) is large enough such that the latent \( k \)-dimensional subspaces of the stacked Page matrix induced by the time series, \( f \), i.e., \( \tilde{M}_{(t,n)}^f \) are no longer varying. An interpretation of this assumption is that for matrix-factorization based time series algorithms, such as mSSA, this serves as an analog of the i.i.d assumption of the underlying generating process that is made in classical generalization error analysis (e.g. Rademacher analysis).

A.2 This is analogous to Property 4.1 holding.

A.3 This requires that the number of steps, \( H \), that we do “online” forecasting grows sub-linearly in the number of observations during the “offline” phase of mSSA; the same goes for \( L \), which is intuitively the allowed model complexity.

A.4 This is a standard boundedness assumption i.e., the linear model should not scale larger than the dimension of the subspace we project onto.

A.5 This is to ensure consistency with Assumption (ii), i.e., the coefficients, \( a_{(\cdot)} \) we get after projection onto \( \Omega_{(T+t+L,n)}^k \) are well-balanced.

A.6 This can easily be verified to hold in high-probability using standard concentration inequalities for sub-Gaussian random variables.

**Technical Innovation.** Note the key technical difficulty of our setting is that the subspaces \( \Omega_{(T+t+L,n)}^k \) are varying over time (due to noise in the observations), a setting not studied closely in the online learning literature.
5 Conclusion

In this work, we provide theoretical justification of mSSA, a heavily used method in practice with limited theoretical understanding. Using a spatio-temporal factor model, we argue the finite sample error for imputation and forecasting scales as $\frac{1}{\sqrt{NT}}$ for $N$ time series with observations of $T$ time-steps. The key technical contributions are: (a) establishing the stacked Page matrix, the core data representation in mSSA, is approximately low-rank for a large model class; (b) advancing the finite-sample analysis of PCR to obtain tight results for mSSA; (c) furnishing a time varying variance estimation algorithm with provable guarantees; (d) introducing a framework of online learning for quantifying the ‘generalization’ error for time series analysis; (e) extending the theory of OCO to allow for a time varying constraint set. As an important direction for future research, it is worth exploring the sample complexity gains by using a variant of mSSA that utilizes higher-dimensional ‘tensor’ structure induced by the Page Matrices of multiple time series rather than “stacking” them up as a larger matrix.

References

[1] A. Agarwal, A. Alomar, and D. Shah. Tsps: A real-time time series prediction system. Working paper, 2020.
[2] A. Agarwal, M. J. Amjad, D. Shah, and D. Shen. Model agnostic time series analysis via matrix estimation. Proceedings of the ACM on Measurement and Analysis of Computing Systems, 2(3):40, 2018.
[3] A. Agarwal, D. Shah, D. Shen, and D. Song. On robustness of principal component regression. In Advances in Neural Information Processing Systems, pages 9889–9900, 2019.
[4] S. Bernstein. The Theory of Probabilities. Gasteizdat Publishing House, 1946.
[5] T. Bollerslev. Generalized autoregressive conditional heteroskedasticity. Journal of econometrics, 31(3):307–327, 1986.
[6] F. Chollet. keras. https://github.com/fchollet/keras, 2015.
[7] C. Davis and W. M. Kahan. The rotation of eigenvectors by a perturbation. iii. SIAM Journal on Numerical Analysis, 7(1):1–46, 1970.
[8] Facebook. Prophet. https://facebook.github.io/prophet/. Online; accessed 25 February 2020.
[9] M. Gavish and D. L. Donoho. The optimal hard threshold for singular values is $\frac{4}{\sqrt{3}}$. IEEE Transactions on Information Theory, 60(8):5040–5053, 2014.
[10] N. Golyandina, V. Nekrutkin, and A. A. Zhigljavsky. Analysis of time series structure: SSA and related techniques. Chapman and Hall/CRC, 2001.
[11] L. Grafakos. Classical fourier analysis, volume 2. Springer, 2008.
[12] H. Hassani, S. Heravi, and A. Zhigljavsky. Forecasting uk industrial production with multivariate singular spectrum analysis. Journal of Forecasting, 32(5):395–408, 2013.
[13] H. Hassani and R. Mahmoudvand. Multivariate singular spectrum analysis: A general view and new vector forecasting approach. International Journal of Energy and Statistics, 1(01):55–83, 2013.
[14] V. Oropeza and M. Sacchi. Simultaneous seismic data denoising and reconstruction via multichannel singular spectrum analysis. Geophysics, 76(3):V25–V32, 2011.
[15] N. Rao, H.-F. Yu, P. K. Ravikumar, and I. S. Dhillon. Collaborative filtering with graph information: Consistency and scalable methods. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 2107–2115. Curran Associates, Inc., 2015.
[16] D. Salinas, V. Flunkert, J. Gasthaus, and T. Januschowski. Deepar: Probabilistic forecasting with autoregressive recurrent networks. International Journal of Forecasting, 2019.
[17] R. Sen, H.-F. Yu, and I. S. Dhillon. Think globally, act locally: A deep neural network approach to high-dimensional time series forecasting. In Advances in Neural Information Processing Systems, pages 4838–4847, 2019.
[18] A. Trindade. UCI machine learning repository - individual household electric power consumption data set.
[19] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. arXiv preprint arXiv:1011.3027, 2010.
[20] L. Wasserman. All of nonparametric statistics. Springer, 2006.
[21] P.-A. Wedin. Perturbation bounds in connection with singular value decomposition. BIT Numerical Mathematics, 12(1):99–111, 1972.
[22] K. W. Wilson, B. Raj, and P. Smaragdis. Regularized non-negative matrix factorization with temporal dependencies for speech denoising. In *Ninth Annual Conference of the International Speech Communication Association*, 2008.

[23] WRDS. The trade and quote (taq) database.

[24] J. Xu. Rates of convergence of spectral methods for graphon estimation. *arXiv preprint arXiv:1709.03183*, 2017.

[25] H.-F. Yu, N. Rao, and I. S. Dhillon. Temporal regularized matrix factorization for high-dimensional time series prediction. In *Advances in neural information processing systems*, pages 847–855, 2016.
Overview of Appendix.

The supplementary material to the main body of this work primarily provides detailed technical proofs of the results stated in the main body.

Section A provides details of the experiments conducted to generate the empirical results reported in Table 2. Sections B and C provide supporting technical proofs for the fundamental representation result that forms the basis of our results.

Sections D, E and F provide supporting notations and results that are utilized in Section G to provide proof of imputation analysis and in Section H to provide proof of forecasting analysis (for both mean and variance). Finally, Section I provides proofs associated with regret analysis.

A Experiments

In this Section, we detail the experimental setup and the datasets used to produce the results in Table 2. Specifically, Section A.1 describes in detail the mean imputation and forecasting experiments. Section A.2 describes the variance imputation and forecasting experiments. Finally, Section A.3 provides details about the parameters and implementations used for all algorithms used in the experiments.

Note that in all experiments, Normalized Root Mean Squared Error (NRMSE) is used as an accuracy metric. In particular, time series is normalized to have zero mean and unit variance before calculating the RMSE. We use this metric as it weighs the error on each time series equally.

A.1 Mean Estimation

A.1.1 Datasets.

In the mean estimation experiments, we use three real-world datasets and one synthetic dataset. The description and preprocessing we do on these four datasets are as follows:

Electricity Dataset. This is a public dataset obtained from the UCI repository which shows the 15-minutes electricity load of 370 households (18). As was done in [25], [17], [16], we aggregate the data into hourly intervals and use the first 25968 time-points for training. The goal here is to do 24-hour ahead forecasts for the next seven days (i.e. 24-step ahead forecast).

Traffic Dataset. This public dataset obtained from the UCI repository shows the occupancy rate of traffic lanes in San Francisco (18). The data is sampled every 15 minutes but to be consistent with previous work in [25], [17], we aggregate the data into hourly data and use the first 10392 time-points for training. The goal here is to do 24-hour ahead forecasts for the next seven days (i.e. 24-step ahead forecast).

Financial Dataset. This dataset is obtained from Wharton Research Data Services (WRDS) and contains the average daily stock prices of 839 companies from October 2004 till November 2019 (23). The datasets were preprocessed to remove stocks with any null values, or those with an average price below 30$ across the aforementioned period. This was simply done to constrain the number of time series for ease of experimentation and we end up with 839 time series (i.e. stock prices of listed companies) each with 3993 readings of daily stock prices. For forecasting, we consider the predicting the 180 time-points ahead one point at a time, and we train on the first 3813 time points. The goal here is to do one-day ahead forecasts for the next 180 days (i.e. 1-step ahead forecast). We choose to do so as this is a standard goal in finance.

Synthetic Dataset. We generate the observation tensor \( X \in \mathbb{R}^{n \times m \times T} \) by first randomly generating the two vectors \( U \in \mathbb{R}^n = [u_1, ..., u_n] \) and \( V \in \mathbb{R}^m = [v_1, ..., v_m] \). Then, we generate \( r \) mixtures of harmonics where each mixture \( g_k(t) \), \( k \in [r] \), is generated as: \( g_k(t) = \sum_{h=1}^{4} \alpha_h \cos(\omega_h t / T) \) where the parameters are selected randomly such that \( \alpha_h \in [-1, 10] \) and \( \omega_h \in [1, 1000] \). Then each value in the observation tensor is constructed as follows:

\[
X_{i,j}(t) = \sum_{k=1}^{r} u_i v_j g_k(t)
\]

where \( r \) is the tensor rank, \( i \in [n], j \in [m] \). In our experiment, we select \( n = 20, m = 20, T = 15000 \), and \( r = 4 \). This gives us 400 time series each with 15000 observations. In the forecasting experiments, we use the first 14000 points for training. The goal here is to do 10-step ahead forecasts for the final 1000 points.
A.1.2 Mean Imputation.

**Setup.** We test the robustness of the imputation performance by adding two sources of corruption to the data - varying the percentage of observed values; and varying the amount of noise we perturb the observations by. We test imputation performance on how accurately we recover missing values. We compare the performance of mSSA with TRMF, a method which achieves state-of-the-art imputation performance. Further, we compare with the SSA variant introduced in [2]. See details about the implementation of, and parameters used for TRMF and SSA in Section A.3.

**Results.** Figures 1 (a)-(d) show the imputation error in the aforementioned datasets as we vary the fraction of missing values, while Figures 1 (e)-(h) show the imputation error as we vary $\sigma$, the standard deviation of the gaussian noise. We see that as we vary the fraction of missing values and noise levels, mSSA outperforms both TRMF and SSA in $\sim 80\%$ of experiments run. The average NRMSE across all experiments for each dataset is reported in Table 2, where mSSA outperforms every other method across all datasets.

A.1.3 Mean Forecasting.

**Setup.** We test the forecasting accuracy of the proposed mSSA against several state-of-the-art algorithms. For each dataset, we split the data into training and testing datasets as outlined in Section A.1.1. As was done in the imputation experiments, we vary the conditions of the datasets by varying the percentage of observed values and the noise levels.

We compare against several methods, namely: (i) SSA, specifically the variant introduced in [2]; (ii) LSTM, available through the Keras library (a standard deep learning library); (iii) DeepAR, a state-of-the-art, deep learning methods that deals with multivariate time series [16]. (iv) TRMF, a matrix factorization approach with temporal regularization hat has gained in popularity recently [25]; (v) Prophet, Facebook’s time series forecasting library [8]. Refer to Section A.3 for details about the implementations of these algorithms and the selected hyper-parameters.

**Results.** Figures 2 (a)-(d) show the forecasting accuracy of mSSA vs. these methods in the aforementioned datasets as we vary the fraction of missing values, while Figures 2 (e)-(h) shows the forecasting accuracy as we vary the standard deviation of the added gaussian noise. We see that as we vary the fraction of missing values and noise levels, mSSA is the best performing method in $\sim 40\%$ of experiments. In terms of the average NRMSE across all experiments, we find that mSSA outperforms every other method across all datasets except for the traffic dataset (see Table 2).
F + trend: we adapt the method by using the “meta”-algorithm described in Section 3.1. Refer to Section A.3 for details on AR processes with/outperforming industry standard methods as we vary the number of missing data and noise level. Figures varying variance estimation algorithms are limited in the literature), we restrict experiments in this section to A.2.2 Variance Imputation and Forecasting

A.2 Variance Estimation

A.2.1 Datasets

A.2.2 Variance Imputation and Forecasting

Results. In Table 4 we see that for imputation, the variance imputation version of the mSSA algorithm outperforms both TRMF and SSA across all (except for one) different time series dynamics, fraction of
observed values, and observation models. In particular, the ratio of TRMF’s NRMSE to mSSA’s ranges between [0.97, 2.15], and the ratio of SSA’s NRMSE to mSSA’s ranges between [1.04, 3.46].

Table 4: Variance imputation error in NRMSE

| Observation Model | mSSA | TRMF | SSA |
|-------------------|------|------|-----|
| p = 1.0           | 0.076| 0.099| 0.118|
| p = 0.8           | 0.122| 0.125| 0.141|
| p = 0.5           | 0.179| 0.181| 0.207|
| Har               | 0.075| 0.091| 0.103|
| Har + trend       | 0.133| 0.135| 0.142|
| Har + AR + trend  | 0.113| 0.138| 0.213|

In Table 5 we see that for forecasting, the variance forecasting version of the mSSA algorithm outperforms DeepAR and SSA in all cases. In particular, the ratio of DeepAR’s NRMSE to mSSA’s ranges between [1.01, 976.97], with significant improvements in the case of the Poisson observation model. Further, the ratio of SSA’s NRMSE to mSSA’s ranges between [1.03, 4.38].

Table 5: Variance forecasting error in NRMSE

| Observation Model | mSSA | DeepAR | SSA |
|-------------------|------|--------|-----|
| p = 1.0           | 0.106| 0.144| 0.156|
| p = 0.8           | 0.170| 0.184| 0.289|
| p = 0.5           | 0.189| 0.195| 0.249|
| Har               | 0.154| 0.155| 0.247|
| Har + trend       | 0.286| 0.232| 0.269|
| Har + AR + trend  | 0.189| 0.209| 0.455|
| Poisson           | 0.173| 0.263| 0.337|
| Har               | 0.143| 0.152| 0.157|
| Har + trend       | 1.320| 1.485| 0.920|
| Har + AR + trend  | 0.225| 0.239| 0.368|
| Bernoulli         | 0.163| 0.173| 0.175|
| Har               | 0.093| 0.182| 0.199|
| Har + trend       | 0.491| 1.403| 2.163|
| Har + AR + trend  | 0.255| 0.292| 0.507|

A.3 Algorithms Parameters and Settings

In this section, we describe the algorithms used throughout the experiments in more detail, as well as describing the hyper-parameters/implementation used for each method.

**mSSA & SSA.** Note that since the SSA’s variant described in [2] is a special case of our proposed mSSA algorithm, we use our mSSA’s implementation to perform SSA experiments, however without “stacking” the various Page matrices induced by each time series. For all experiments we choose \( k \), the number of retained singular values, in a data-driven manner. Specifically, we choose \( k \) based on the thresholding procedure outlined in [9], where the threshold is determined by the median of the singular values and the shape of the matrix. Finally, as guided by our theoretical results, we choose \( L = 1000 \) (100 for SSA) for all experiments, except for experiments on the financial dataset where we choose \( L = 500 \) (50 for SSA).

**DeepAR.** We use the default parameters of “DeepAREstimator” algorithm provided by the GluonTS package.

**LSTM.** Across all datasets, we use a LSTM network with three hidden layers each, with 45 neurons per layer, as is done in [17]. We use the Keras implementation of LSTM.

**Prophet.** We used Prophets Python library with the default parameters selected [8].

**TRMF.** We use the implementation provided by the authors in the Github repository associated with the paper ([25]). For the parameter \( k \), which represent the chosen rank for the \( T \times N \) Time series matrix, we use
We begin by stating some classic results from Fourier Analysis. To do so, we introduce some notation.

\[ (iii) \text{ produce the variance estimate } \sigma_n^2 = \hat{f}_n + \sigma_n^\text{TRMF} - \hat{f}_n^\text{TRMF}(t)^2. \]

B Proofs - \((G, \varepsilon)\)-Hankel Representability of Different Time Series Dynamics

B.1 Proof of Proposition 2.1

Proof. Noting that for any two matrices \(A\) and \(B\), it is the case that \(\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)\) and \(\text{rank}(A \circ B) \leq \text{rank}(A)\text{rank}(B)\) (where \(\circ\) denotes the Hadamard product), completes the proof. \(\square\)

B.2 Proof of Proposition 2.2

Proof. Proof is immediate from Definitions 2.1 and 2.2. \(\square\)

B.3 Proof of Proposition 2.4

B.3.1 Helper Lemmas for Proposition 2.4

We begin by stating some classic results from Fourier Analysis. To do so, we introduce some notation.

\(C[0,R]\) and \(L^2[0,R]\) functions. \(C[0,R]\) is the set of real-valued, continuous functions defined on \([0,R]\). \(L^2[0,R]\) is the set of square integrable functions, i.e., \(\int_0^R f^2(t)dt \leq \infty\).

**Inner Product of functions in \(L^2[0,R]\).** \(L^2[0,R]\) is an inner product space endowed with inner product defined as \(\langle f,g \rangle := \frac{1}{R} \int_0^R f(t)g(t)dt\), and associated norm as \(\|f\| := \sqrt{\int_0^R f^2(t)dt}\).

**Fourier Representation of functions in \(L^2[0,R]\).** For a function, \(f\), in \(L^2[0,R]\), define \(f_G\) as follows

\[ S_G(t) = a_0 + \sum_{n=1}^G (a_n \cos(2\pi nt/R) + b_n \cos(2\pi nt/R)) \tag{2} \]

where for \(n \in [N]\) \((a_0,a_n,b_n)\) are called the Fourier coefficients of \(f\).

\[ a_0 := \langle f, 1 \rangle = \frac{1}{R} \int_0^T f(t)dt, \]

\[ a_n := \langle f, \cos(2\pi nt/R) \rangle = \frac{1}{R} \int_0^T f(t)\cos(2\pi nt/R)dt, \]

\[ b_n := \langle f, \sin(2\pi nt/R) \rangle = \frac{1}{R} \int_0^T f(t)\sin(2\pi nt/R)dt. \]

We now state a classic result from Fourier analysis.

**Theorem B.1.** If \(f \in C^k_{\text{per}R}\), for \(k \geq 1\), then \(S_G(t)\) converges pointwise to \(f(t)\), i.e., for all \(t \in R\)

\[ \lim_{G \to \infty} S_G(t) \rightarrow f(t). \]

We next show that if \(f\) is \(k\)-times differentiable, then its Fourier coefficients decay rapidly. Precisely,

**Lemma B.1.** If \(f \in C^k_{\text{per}R}\), for \(k \geq 1\), then for \(k' \in [k]\), the Fourier coefficients of \(f^{(k')}\), the \(k'\)-th derivative of \(f\), are defined as

\[ a_0^{(k)} = a_0, \quad a_n^{(k)} = -\left(\frac{2\pi n}{R}\right) b_n^{(k-1)}, \quad b_n^{(k)} = \left(\frac{2\pi n}{R}\right) a_n^{(k-1)} \]
Proof. We show it for \( a_n^{(1)} \). Extension to \( a_n^{(k')} \) for \( k' \in [k] \) follows by induction in a straightforward manner.

\[
a_n^{(1)} = \langle f^{(1)}, \cos(2\pi nt/R) \rangle = \frac{1}{R} \int_0^R f^{(1)}(t) \cos(2\pi nt/R) dt
\]

\[
= \frac{1}{R} \left[ \left[ f(t) \cos(2\pi nt/R) \right]_0^R - \frac{2\pi n}{R} \left[ \int_0^R f(t) \sin(2\pi nt/R) dt \right]_0^R \right]
\]

\[
= -\left( \frac{2\pi n}{R} \right) b_n^{(0)}
\]

(a) follows by integration by parts. The identity for \( b_n^{(k')} \) follows in a similar fashion, as does it for \( a_0^{(k)} \).

B.3.2 Completing Proof of Proposition \( \text{2.4} \)

Proof. For \( G \in \mathbb{N} \), let \( S_G \) be defined as in (2). Then for \( t \in \mathbb{R} \)

\[
|f(t) - S_G(t)| \leq \sum_{n=G+1}^{\infty} \left| a_n \cos(2\pi nt/R) + b_n \cos(2\pi nt/R) \right|
\]

\[
\leq \sum_{n=G+1}^{\infty} |a_n| + |b_n|
\]

\[
\leq \left( \frac{R}{2\pi n} \right)^k \left( |a_n| + |b_n| \right)^k
\]

\[
\leq \sqrt{2} \left( \frac{R}{2\pi} \right)^k \left( \sum_{n=G+1}^{\infty} \left( \frac{1}{n} \right)^2 \right)^k \left( \sum_{n=G+1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) \right)^{k/2}
\]

\[
\leq \sqrt{2} \left( \frac{R}{2\pi} \right)^k \frac{1}{G^{k-0.5}} \left( \sum_{n=G+1}^{\infty} \left( |a_n|^2 + |b_n|^2 \right) \right)
\]

\[
\leq \sqrt{2} \left( \frac{R}{2\pi} \right)^k \frac{||f||}{G^{k-0.5}}
\]

\[
= C(k, R) \frac{||f||}{G^{k-0.5}}
\]

where \( C(k, R) \) is a constant that depends only on \( k \) and \( R \); (a) follows from Theorem \( \text{B.1} \); (b) follows from Lemma \( \text{B.1} \); (c) follows from Cauchy-Schwarz inequality; (d) follows from Bessel’s inequality.

Hence \( S_G \), a sum of \( 2G \) harmonics, gives an uniform approximation to \( f \) with error at most \( C(k, R) \frac{||f||}{G^{k-0.5}} \). Noting \( 2G \) harmonics can be represented by an order-4G LRF (by Proposition \( \text{2.3} \)) completes the proof.

B.4 Proof of Proposition \( \text{2.5} \)

This analysis is closely adapted from [24] and is stated for completeness.

Proof. Step 1: Partitioning the space \( [0,1]^K \). Let \( E \) denote a partition of the cube \( [0,1]^K \) into a finite number (denoted by \( |E| \)) of cubes \( \Delta \). Let \( \ell \in \mathbb{N} \). We say \( P_{E,\ell} : [0,1]^K \to \mathbb{R} \) is a piecewise polynomial of degree \( \ell \) if

\[
P_{E,\ell} (\theta) = \sum_{\Delta \in E} P_{\Delta,\ell} (\theta) \mathbf{1}_\Delta (\theta \in \Delta),
\]

where \( P_{\Delta,\ell} : [0,1]^K \to \mathbb{R} \) denotes a polynomial of degree at most \( \ell \).

It suffices to consider an equal partition of \( [0,1]^K \). More precisely, for any \( k \in \mathbb{N} \), we partition the the set \( [0,1] \) into \( 1/k \) half-open intervals of length \( 1/k \), i.e., \( [0,1] = \bigcup_{i=1}^k [(i-1)/k, i/k) \). It follows that \( [0,1]^K \) can be partitioned into \( k^K \) cubes of forms \( \otimes_{j=1}^K [(i_j-1)/k, i_j/k) \) with \( i_j \in [k] \). Let \( E_k \) be such a partition with \( I_1, I_2, ..., I_{k^K} \) denoting all such cubes and \( z_1, z_2, ..., z_{k^K} \in \mathbb{R}^K \) denoting the centers of those cubes.
Step 2: Taylor Expansion of $g(\cdot, \omega_s)$. For Step 2 of the proof, to reduce notational overload, we suppress dependence of $\omega_s$ on $g$, we abuse notation by using $g(\cdot) = g(\cdot, \omega_s)$.

For every $I_i$ with $1 \leq i \leq k^K$, define $P_{i, \ell}(x)$ as the degree-$\ell$ Taylor’s series expansion of $g(x)$ at point $z_i$:

$$P_{i, \ell}(x) = \sum_{\kappa:|\kappa| \leq \ell} \frac{1}{\kappa!} (x-z_i)^\kappa \nabla_g g(z_i), \tag{4}$$

where $\kappa = (\kappa_1, \ldots, \kappa_d)$ is a multi-index with $\kappa = \prod_{i=1}^K \kappa_i!$, and $\nabla_k g(z_i)$ is the partial derivative defined in Section 2.2.4. Note similar to $g$, $P_{i, \ell}(x)$ really refers to $P_{i, \ell}(x, \omega_s)$.

Now we define a degree-$\ell$ piecewise polynomial as in (3), i.e.,

$$P_{E_{k, \ell}}(x) = \sum_{i=1}^{k^K} P_{i, \ell}(x) 1(x \in I_i). \tag{5}$$

For the remainder of the proof, let $\ell = |\alpha|$ (recall $|\alpha|$ refers to the largest integer strictly larger than $\alpha$).

Since $f \in \mathcal{H}(\alpha, L)$, it follows that

$$\sup_{x \in X} |g(x) - P_{E_{k, \ell}}(x)| \leq \sup_{1 \leq i \leq k^K, x \in I_i} \left| \sum_{\kappa:|\kappa| \leq \ell} \frac{\nabla_k g(z_i)}{\kappa!} (x-z_i)^\kappa + \sum_{\kappa:|\kappa| = \ell} \frac{\nabla_k g(z_i)}{\kappa!} (x-z_i)^\ell - P_{E_{k, \ell}}(x) \right|$$

by Step 2, we have that for all $t \in [L], s \in [T]$.

$$|H_{ts} - [H_{(t)}]_{(s)}| \leq Lk^{-\alpha}$$

It remains to bound the rank of $H_{(t)}$. Note that since $P_{E_{k, \ell}}(\theta, \omega_s)$ is a piecewise polynomial of degree $\ell = |\alpha|$, it has a decomposition of the form

$$[H_{(t)}]_{(s)} = P_{E_{k, \ell}}(\theta, \omega_s) = \sum_{i=1}^{k^K} \langle \Phi(\theta), \beta_{I_i, s} \rangle 1(\theta \in I_i)$$
where the vector
\[ \Phi(\theta) = \left( 1, \theta_1, \ldots, \theta_K, \theta_1^\ell, \ldots, \theta_K^\ell \right)^T, \]
i.e., is the vector of all monomials of degree less than or equal to \( \ell \). The number of such monomials is easily shown to be equal to \( C(\alpha, K) := \sum_{i=0}^{\lfloor \alpha \rfloor} (i+K-1)_{K-1} \).

Thus the rank of \( H_{(lr)} \) is bounded by \( k^K C(\alpha, K) \). Setting \( k = 1/\epsilon \) completes the proof.

\[ \square \]

\section*{C \ Proofs - Stacked Hankel Matrix is Approximately Low-Rank}

\subsection*{C.1 Helper Lemmas}

\begin{lemma}
Let Property 2.1 hold. Then
\[ \|M_f\|_{\text{max}} \leq R_1 \Gamma_1 \Gamma_2 \]
\end{lemma}

\begin{proof}
Proof is immediate. \[ \square \]

\begin{lemma}
Let Property 2.3 hold. Then
\[ \|M_{\sigma^2}\|_{\text{max}} \leq R_2 \Gamma_3 \Gamma_4 \]
\end{lemma}

\begin{proof}
Proof is immediate. \[ \square \]

\begin{proposition}
Let \( f \) be a \((G,0)\)-LRF, then for \( s \in \{1, \ldots, L\} \), \( t \in \{0, P, \ldots, (P-1) L\} \), \( f \) admits the representation
\[ f(s+t) = \sum_{g=1}^{G} \alpha_g a_g(s) b_g(t) \] \hspace{1cm} (6)
for some scalars \( \alpha_g \), and functions \( a_g : [L] \to \mathbb{R} \) and \( b_g : [P] \to \mathbb{R} \).
\end{proposition}

\begin{proof}
Note the page matrix \( M_f \) corresponding to time series \( f \) has rank at most \( G \). Thus the singular value decomposition of \( M_f \) has the form, \( M_f = \sum_{g=1}^{G} \alpha_g a_g' b_g' \) where \( \alpha_g \) are the singular values, and \( a_g' \in \mathbb{R}^L \), \( b_g' \in \mathbb{R}^N \) are the left and right singular vectors of \( M_f \) respectively. Thus \( M_{f(s)} = f(i+(j-1)L) = \sum_{g=1}^{G} \alpha_g a_g'(i) b_g'(j) \). Identifying \( a_g, b_g \) as \( a_g', b_g' \) respectively completes the proof. \[ \square \]

\begin{proposition}
Let \( f \) be a \((G,0)\)-Hankel, then for \( s \in \{1, \ldots, L\} \), \( t \in \{0, P, \ldots, (P-1) L\} \), \( f \) admits the representation
\[ f(s+t) = \sum_{g=1}^{G} \alpha_g a_g(s) b_g(t) \] \hspace{1cm} (7)
for some scalars \( \alpha_g \), and functions \( a_g : [L] \to \mathbb{R} \) and \( b_g : [P] \to \mathbb{R} \).
\end{proposition}

\begin{proof}
Identical to that of Proposition C.1 \[ \square \]

\subsection*{C.2 Proof of Proposition 2.6}

\begin{proof}
To reduce notational complexity, we suppress the superscript \( f \) for the remainder of the proof. For \( l \in [L] \), \( k \in [T] \), \( n \in [N] \) we have,

\[ \square \]
where (a) follows directly from Property 2.2 and Proposition C.2. Here \( \epsilon \) completes the proof.

\[
H_{l, [k+T \times (n-1)]} = H_n(l+(k-1)L) = \sum_{r=1}^{R_1} U_{nr} W_r(l+(k-1)L)
\]

\[
\equiv \sum_{r=1}^{R_1} U_{nr} \left( \sum_{g=1}^{G_r} \alpha_g^r b_g^r (k-1)L + \epsilon_r (l+(k-1)L) \right)
\]

\[
= R_1 \sum_{r=1}^{R_1} U_{nr} \epsilon_r (l+(k-1)L) + \sum_{r=1}^{R_1} U_{nr} (k-1)L
\]

This completes the proof.

\[\Box\]

## D Imputation and Forecasting Error Analysis - Proof Notation

### D.1 Induced Linear Operator

Consider a matrix \( B \in \mathbb{R}^{N \times P} \) such that \( B = \sum_{i=1}^{N \wedge P} \sigma_i(B)x_i y_i^T \). Here, \( \sigma_i(B) \) are the singular vectors of \( B \) and \( x_i, y_i \) are the left and right singular vectors respectively.

**Hard Singular Value Thresholding.** To that end, given any \( \lambda > 0 \), we define the map \( \text{HSVT}_\lambda : \mathbb{R}^{N \times P} \to \mathbb{R}^{N \times P} \), which simply shaves off the input matrix's singular values that are below the threshold \( \lambda \). Precisely,

\[
\text{HSVT}_\lambda(B) = \sum_{i=1}^{N \wedge P} \sigma_i(B) \mathbb{1}(\sigma_i(B) \geq \lambda)x_i y_i^T.
\]  

**Induced Linear Operator.** With a specific choice of \( \lambda \geq 0 \), we can define a function \( \varphi^B_\lambda : \mathbb{R}^P \to \mathbb{R}^P \) as follows: for any vector row \( w \in \mathbb{R}^P \) (i.e. \( w \in \mathbb{R}^{1 \times P} \)),

\[
\varphi^B_\lambda(w) = \sum_{i=1}^{N \wedge P} \mathbb{1}(\sigma_i(B) \geq \lambda)y_i y_i^T w^T.
\]

Note that \( \varphi^B_\lambda \) is a linear operator and it depends on the tuple \((B, \lambda)\); more precisely, the singular values and the right singular vectors of \( B \), as well as the threshold \( \lambda \). If \( \lambda = 0 \), then we will adopt the shorthand notation: \( \varphi^B_\lambda = \varphi^B_0 \).

**Lemma D.1 (Lemma 35 of [3]).** Let \( B \in \mathbb{R}^{N \times P} \) and \( \lambda \geq 0 \) be given. Then for any \( j \in [N] \),

\[
\varphi^B_\lambda(B_{j, \cdot}) = \text{HSVTV}_\lambda(B)_{j, \cdot}^T
\]

where \( B_{j, \cdot} \in \mathbb{R}^{1 \times P} \) represents the \( j \)th row of \( B \), and \( \text{HSVTV}_\lambda(B)_{j, \cdot} \in \mathbb{R}^{1 \times P} \) represents the \( j \)th row of matrix obtained after applying HSVT over \( B \) with threshold \( \lambda \).

**Proof.** By [3], the orthonormality of the right singular vectors and \( B_{j, \cdot}^T = B^T e_j \) with \( e_j \in \mathbb{R}^P \) with \( j \)th entry 1 and everything else 0, we have
E.2. Norm of matrices with sub-gaussian entries. \[ \varphi_X^R(B_{ij}) = \sum_{i=1}^{N \wedge p} I(\sigma_i(B) \geq \lambda)y_i y_i^T B_{ij} = \sum_{i=1}^{N \wedge p} I(\sigma_i(B) \geq \lambda)y_i y_i^T B_{ij} e_j \]
\[ = \sum_{i=1}^{N \wedge p} I(\sigma_i(B) \geq \lambda)y_i y_i^T \sum_{i'=1}^{N \wedge p} \sigma_{i'}(B)x_{i'} x_{i'}^T e_j = \sum_{i=1}^{N \wedge p} \sigma_{i}(B)I(\sigma_i(B) \geq \lambda)y_i y_i^T x_{i'} x_{i'}^T e_j \]
\[ = \sum_{i=1}^{N \wedge p} \sigma_{i}(B)I(\sigma_i(B) \geq \lambda)y_i y_i^T x_{i'} x_{i'}^T e_j = \sum_{i=1}^{N \wedge p} \sigma_{i}(B)I(\sigma_i(B) \geq \lambda)y_i x_{i'} e_j \]
\[ = \text{HSVT}_\alpha(B)^T e_j = \text{HSVT}_\alpha(B)^T. \]

\[ \square \]

E. Concentration Inequalities Lemmas

E.0.1 Classic Results

Theorem E.1. Bernstein’s Inequality. \[ \text{[4]} \]
Suppose that \( X_1, \ldots, X_n \) are independent random variables with zero mean, and \( M \) is a constant such that \( |X_i| \leq M \) with probability one for each \( i \). Let \( S := \sum_{i=1}^{n} X_i \) and \( v := \text{Var}(S) \). Then for any \( t \geq 0 \),
\[ P(|S| \geq t) \leq 2\exp\left(-\frac{3t^2}{6v + 2Mt}\right). \]

Theorem E.2. Norm of matrices with sub-gaussian entries. \[ \text{[19]} \]
Let \( A \) be an \( m \times n \) random matrix whose entries \( A_{ij} \) are independent, mean zero, sub-gaussian random variables. Then, for any \( t > 0 \), we have
\[ \|A\| \leq CK\sqrt{m + \sqrt{n} + t} \]
with probability at least \( 1 - 2\exp(-t^2) \). Here, \( K = \max_{i,j} \|A_{ij}\|_{\psi_2} \).

Lemma E.1. Maximum of sequence of random variables. \[ \text{[19]} \]
Let \( X_1, X_2, \ldots, X_n \) be a sequence of random variables, which are not necessarily independent, and satisfy \( \mathbb{E}[X_i^2] \leq K_i \) for some \( K_i > 0 \) and all \( i \). Then, for every \( n \geq 2 \),
\[ \mathbb{E}\max_{i \leq n} |X_i| \leq CK_i \log \frac{2}{\alpha} (n). \tag{11} \]

Remark E.1. Lemma E.1 implies that if \( X_1, \ldots, X_n \) are \( \psi_\alpha \) random variables with \( \|X_i\|_{\psi_\alpha} \leq K_\alpha \) for all \( i \in [n] \), then
\[ \mathbb{E}\max_{i \leq n} |X_i| \leq CK_\alpha \log \frac{1}{\alpha} (n). \]

E.0.2 High Probability Events for Imputation and Forecasting

Setup. Let \( X \) be an \( L \times \tilde{P} \) random matrix (with \( L \leq \tilde{P} \)) whose entries \( X_{ij} \) are independent sub-gaussian entries where \( \mathbb{E}[X] = M \) and \( \|X_{ij}\|_{\psi_2} \leq \sigma \). Let \( Y \) denote the \( L \times \tilde{P} \) matrix whose entries \( Y_{ij} \) are defined as
\[ Y_{ij} = \begin{cases} X_{ij} & \text{w.p. } p, \\ 0 & \text{w.p. } 1-p, \end{cases} \]
for some \( p \in (0,1] \). Let \( \bar{p} = \max \left\{ \frac{1}{L \tilde{P}} \sum_{i=1}^{L} \sum_{j=1}^{\tilde{P}} 1_{X_{ij} \text{ observed}}, \frac{1}{L \tilde{P}} \right\} \).
High Probability Events. Define events $E_1$ to $E_5$, for some positive absolute constant $C$ as

$$E_1 := \{ \hat{y} - p \leq p/20 \},$$

$$E_2 := \{ \| Y - pM \| \leq C\sigma\sqrt{\hat{p}} \},$$

$$E_3 := \{ \| Y - pM \|^2_{\infty, 2} \leq C\sigma^2 \hat{p} \},$$

$$E_4 := \{ \max_{j \in B} \| \varphi^B_{\lambda_k} (Y_j, -pM_j) \|^2_{L^2} \leq C\sigma^2 k\log(\hat{p}) \},$$

$$E_5 := \left\{ \left( 1 - \frac{20\log(L\hat{p})}{LP\hat{p}} \right) p \leq \hat{p} \leq \frac{1}{1 + \frac{20\log(L\hat{p})}{LP\hat{p}}} p \right\}.$$  

(12) (13) (14) (15) (16)

Here, $B \in \mathbb{R}^{L \times \hat{p}}$ is an arbitrary matrix such that $B = \sum_{i=1}^{L} \lambda_i(B)x_iy_i^T$, where $\sigma_i(B)$ are the singular vectors of $B$ and $x_i, y_i$ are the left and right singular vectors respectively. Recall the definition of $\varphi^B_{\lambda_k}$ in (9).

**Lemma E.2.** For some positive constant $c_1$

$$\mathbb{P}(E_1) \geq 1 - 2e^{-c_1 L \hat{N} p} - (1 - p)^{L \hat{p}},$$

$$\mathbb{P}(E_2) \geq 1 - 2e^{-p},$$

$$\mathbb{P}(E_3) \geq 1 - 2e^{-p},$$

$$\mathbb{P}(E_4) \geq 1 - \frac{2}{L^{10}\hat{p}^{10}}.$$  

(17) (18) (19) (20) (21)

**Proof. Bounding $E_1$.** Let $\hat{p}_0 = \frac{1}{L \hat{N}} \sum_{j=1}^{L} \sum_{j=1}^{N} \mathbb{1}_{X_j}$ observed, which implies $\mathbb{E} [\hat{p}_0] = p$. We define the event $E_0 := \{ \hat{p}_0 = \hat{p} \}$. Thus, we have that

$$\mathbb{P}(E_1) = \mathbb{P}(E_1 \cap E_0) + \mathbb{P}(E_1 \cap \overline{E}_0)$$

$$= \mathbb{P}(\| \hat{p}_0 - p \| \geq p/20) + \mathbb{P}(E_1 \cap \overline{E}_0)$$

$$\leq \mathbb{P}(\| \hat{p}_0 - p \| \geq p/20) + \mathbb{P}(E_0)$$

$$= \mathbb{P}(\| \hat{p}_0 - p \| \geq p/20) + (1 - p)^L,$$

where the final equality follows by the independence of observations assumption and the fact that $\hat{p}_0 \neq \hat{p}$ only if we do not have any observations. By Bernstein’s Inequality, we have that

$$\mathbb{P}(\| \hat{p}_0 - p \| \geq p/20) \geq 1 - 2e^{-c_1 L \hat{N} p}.$$ 

Bounding $E_2$. Since $\mathbb{E} [Y_{ij}] = pM_{ij}$, Theorem E.2 yields

$$\mathbb{P}(E_2) \geq 1 - 2e^{-N}.$$  

Bounding $E_3$. Observing,

$$\| Y_j - pM_j \|_{\infty, 2} \leq \| Y_j - pM_j \|^2_{L^2}$$

and Theorem E.2 is sufficient to show (19).

Bounding $E_4$. Recall $y_i \in \bar{P}$ is the $i$-th right singular vector of $B = Y - pM$. Then,

$$\| \varphi^B_{\lambda_k} (Y_j, -pM_j) \|^2_{L^2} \leq \sum_{i=1}^{k} \| y_i \|_2^2 \| (\psi^B_{\lambda_k} (Y_j, -pM_j)) \|_2^2 \leq \sum_{i=1}^{k} \| y_i \|_2^2 \| (Y_j, -pM_j) \|_2 \| (Y_j, -pM_j) \|_2,$$

where $Z_i = \| y_i \|_2 \| (Y_j, -pM_j) \|_2$. By definition of $\psi_2$ norm of a random variable since $y_i$ is unit norm vector, it follows that

$$\| Z_i \|_{\psi_2} = \| y_i \|_2 \| (Y_j, -pM_j) \|_{\psi_2} \leq \| (Y_j, -pM_j) \|_{\psi_2}.$$
Since the coordinates of $Y_j, -pM_j$, are mean-zero and independent, with $\psi_2$ norm bounded by $\sqrt{C}\sigma$ for some absolute constant $C > 0$, using Lemma H.10 of [3], it follows that
\[
P \left( \sum_{i=1}^{2k} Z_i^2 > t \right) \leq 2k \exp \left( -\frac{t}{kC\sigma^2} \right). \tag{22}
\]
Therefore, for choice of $t = C\sigma^2 k \log P$ (with large enough constant $C > 0$ and since $L \leq \bar{P}$) and union bound, we have that
\[
P \left( E \right) \leq \frac{2}{L^{10} P^{10}}. \tag{23}
\]
Bounding $E_5$. Recall definition of $\hat{p}$. Then by the binomial Chernoff bound, for $\varepsilon > 1$,
\[
P \left( \hat{p} > c \varepsilon p \right) \leq \exp \left( -\frac{(\varepsilon - 1)^2}{\varepsilon + 1} LP \right), \quad \text{and}
\]
\[
P \left( \hat{p} < \frac{1}{\varepsilon} p \right) \leq \exp \left( -\frac{(\varepsilon - 1)^2}{2\varepsilon^2} LP \right).
\]
By the union bound,
\[
P \left( \frac{1}{\varepsilon} p \leq \hat{p} \leq \varepsilon p \right) \geq 1 - \left[ P \left( \hat{p} > c \varepsilon p \right) + P \left( \hat{p} < \frac{1}{\varepsilon} p \right) \right].
\]
Noticing $\varepsilon + 1 < 2\varepsilon < 2\varepsilon^2$ for all $\varepsilon > 1$, and substituting $\varepsilon = \left( 1 - \sqrt{\frac{20\log(LP)}{LP}} \right)^{-1}$ completes the proof.

\[
\]

Corollary E.1. Let $E := E_1 \cap E_2$. Then,
\[
P (E^c) \leq C_1 e^{-c_2 p}, \tag{24}
\]
where $C_1$ and $c_2$ are positive constants independent of $L$ and $P$.

Corollary E.2. Let $E := E_2 \cap E_3 \cap E_4 \cap E_5$. Then,
\[
P (E^c) \leq \frac{C_1}{L^{10} P^{10}}, \tag{25}
\]
where $C_1$ is an absolute positive constant, independent of $L$ and $P$.

F HSVT Error

Lemma F.1. Let $M_f = M^{(f)}_{\sigma r}$, $E^{(f)}_{\sigma r}$. Recall for $r \in [L - 1]$, let $\tau_r, \mu_r$ denote the $r$-th singular value and right singular vector of $M^{(f)}_{\sigma r}$ respectively. Suppose that
\[
1. \quad \|Z^X - \rho M_f\|_2 \leq \Delta \text{ for some } \Delta \geq 0
\]
\[
2. \quad \frac{1}{\varepsilon} \rho \leq \hat{p} \leq \varepsilon \rho \text{ for some } \varepsilon \geq 1,
\]
Let $M^{(f)}_{\sigma r} = HSVT_{\sigma r}(M^{(f)}_{\sigma r})$. Let $\tilde{M}^{(f)}_{\sigma r} = \frac{1}{\rho} HSVT_{\sigma r}(Z^X)$, where $s_k$ is the $k$-th singular value of $Z^X$. Then for any $j \in [L - 1],$
\[
\left\| \tilde{M}^{(f)}_{\sigma r} - M^{(f)}_{\sigma r} \right\|_2 \leq \frac{8\varepsilon^2}{\rho^4} \left( \frac{\Delta^2 + \|E^{(f)}_{\sigma r}\|_2^2}{(\tau_{k+1} - \tau_k)^2} \right) \left( \|Z^X - \rho M_{f_j}\|_2 + \|\tilde{M}^{(f)}_{\sigma r} - M^{(f)}_{\sigma r}\|_2 \right)
\]
\[
+ \frac{4\varepsilon^2}{\rho^2} \|E^{(f)}_{\sigma r}\|_2 \left( \|Z^X - \rho M_{f_j}\|_2 + 2(\varepsilon - 1)^2 \|M_{f_j}\|_2 \right)
\]
\[
+ 4\|E^{(f)}_{\sigma r}\|_2 \left( \left\| M^{(f)}_{\sigma r} \right\|_2^2 + 2 \|M_{f_j}\|_2 \right).
\]

Proof. For ease of exposition, for the remainder of the proof, let: $A = \tilde{M}^{(f)}_{\sigma r} - Z^X$; $\hat{A} = \tilde{M}^{(f)}_{\sigma r}$; $A^{(f)}_{\sigma r}\; A^{k}_{\sigma r} = \tilde{M}^{(k)}_{\sigma r}$, $E_{\sigma r} = M^{(f)}_{\sigma r} - \tilde{M}^{(f)}_{\sigma r}$; and $E_2 = M^{(f)}_{\sigma r} - \tilde{M}^{(f)}_{\sigma r}$.

We will use notation $\lambda^r = s_k$, the $k$-th singular value of $Z^X$ for simplicity. Further, recall that $s_k, \mu_k$ denote the $r$-th singular value and right singular vector of $Z^X$ respectively. We prove our Lemma in three steps.
Step 1. Fix a row index \( j \in [L-1] \). Observe that

\[
\hat{A}_{j_k} - A_{j_k} = (\hat{A}_{j_k} - \varphi_Z X_j) + (\varphi_Z A_{j_k} - A_{j_k}).
\]

By definition (see (39)), we have that \( \varphi_Z : \mathbb{R}^p \to \mathbb{R}^p \) is the projection operator onto the span of the top \( k \) right singular vectors of \( Z \), namely, span\(\{u_1, \ldots, u_k\} \). Therefore,

\[
\varphi_Z (A_{j_k}) - A_{j_k} \in \text{span}\{u_1, \ldots, u_k\}.
\]

By choice, \( \text{rank}(\hat{A}) = k \); hence, by using Lemma D.1

\[
\hat{A}_{j_k} - \varphi_Z (A_{j_k}) = \frac{1}{\hat{\rho}} \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \in \text{span}\{u_1, \ldots, u_k\}.
\]

Hence, \( \langle \hat{A}_{j_k}, -\varphi_Z (A_{j_k}) \rangle = 0 \) and

\[
\left\| \hat{A}_{j_k} - A_{j_k} \right\|_2^2 = \left\| \hat{A}_{j_k} - \varphi_Z (A_{j_k}) \right\|_2^2 + \left\| \varphi_Z (A_{j_k}) - A_{j_k} \right\|_2^2
\]

by the Pythagorean theorem.

Step 2. We begin by bounding the first term on the right hand side of (27). Again applying Lemma D.1 we can rewrite

\[
\hat{A}_{j_k} - \varphi_Z (A_{j_k}) = \frac{1}{\hat{\rho}} \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) = \frac{1}{\hat{\rho}} Z_j - A_{j_k}
\]

Using the Parallelogram Law (or, equivalently, combining Cauchy-Schwartz and AM-GM inequalities), we obtain

\[
\left\| \hat{A}_{j_k} - \varphi_Z (A_{j_k}) \right\|_2^2 = \left\| \frac{1}{\hat{\rho}} \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 \\
\leq 2 \left\| \frac{1}{\hat{\rho}} \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 + 2 \left\| \varphi_Z (A_{j_k}) - A_{j_k} \right\|_2^2 \\
\leq \frac{2}{\hat{\rho}^2} \left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 + 2 \left( \frac{\varphi - \hat{\rho}}{\hat{\rho}} \right)^2 \left\| A_{j_k} \right\|_2^2 \\
\leq \frac{2\varepsilon^2}{\hat{\rho}^2} \left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 + 2(\varepsilon - 1)^2 \left\| A_{j_k} \right\|_2^2
\]

because Condition 2 implies \( \frac{\hat{\rho}}{\rho} \leq \varepsilon \) and \( \left( \frac{\varphi - \hat{\rho}}{\hat{\rho}} \right)^2 \leq (\varepsilon - 1)^2 \).

Note that the first term of (28) can further be decomposed as,

\[
\left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 \leq 2 \left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2 + 2 \left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2^2
\]

We now bound the first term on the right hand side of (29) separately. First, we apply the Davis-Kahan sinθ Theorem (see [7,21]) to arrive at the following inequality:

\[
\left\| \mathcal{P}_{u_1, \ldots, u_k} - \mathcal{P}_{\mu_1, \ldots, \mu_k} \right\|_2 \leq \left\| \frac{Z - \mu A}{\hat{\rho} \tau_k - \hat{\rho} \tau_{k+1}} \right\| \left\| \frac{\mu A - \rho A}{\hat{\rho} \tau_k - \hat{\rho} \tau_{k+1}} \right\| + \frac{\Delta}{\hat{\rho} \tau_k - \hat{\rho} \tau_{k+1}} \left\| E_1 \right\| + \frac{\Delta}{\rho \tau_k - \tau_{k+1}} \left\| E_1 \right\|
\]

where \( \mathcal{P}_{u_1, \ldots, u_k} \) and \( \mathcal{P}_{\mu_1, \ldots, \mu_k} \) denote the projection operators onto the span of the top \( k \) right singular vectors of \( Z \) and \( A_{1(\hat{\rho})} \), respectively. Note, we utilized Condition 1 to bound \( \left\| Z - \rho A \right\|_2 \leq \Delta \).

Then it follows that

\[
\left\| \varphi_Z (Z_j) - \varphi_Z (A_{j_k}) \right\|_2 \leq \left\| \mathcal{P}_{u_1, \ldots, u_k} - \mathcal{P}_{\mu_1, \ldots, \mu_k} \right\|_2 \left\| Z_j - \rho A_{j_k} \right\|_2 + \frac{\Delta}{\hat{\rho} \tau_k - \tau_{k+1}} \left\| E_1 \right\| \left\| Z_j - \rho A_{j_k} \right\|_2
\]
Combining the inequalities together, we have
\[
\|\hat{A}_j - \varphi_{X'}(A_j)\|_2^2 \leq \frac{8 \varepsilon^2}{\rho^4} \left( \frac{\Delta^2}{(\tau_k - \tau_{k+1})^2} + \frac{\|E_1\|_2^2}{(\tau_k - \tau_{k+1})^2} \right) \|Z_j - \rho A_j\|_2^2
\]
\[
+ \frac{4 \varepsilon^2}{\rho^2} \left\| \varphi_{X'}(Z_j - \rho A_j) \right\|_2^2 + 2(\varepsilon - 1)^2 \|A_j\|_2^2.
\]
(33)

**Step 3.** We now bound the second term of (27). Recalling \( A = A_{(tr)}^k + E_2 \) and using (30)
\[
\|\varphi_{X'}^2(A_j)\|_2^2 = \|\varphi_{X'}(A_{(tr)}^k + E_2)\|_2^2 \leq 2\|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2
\]
\[
= 2\|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2
\]
\[
\leq 2\|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2
\]
\[
\leq 2\|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2
\]
\[
\leq 2 \|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2
\]
\[
\leq 4 \left( \frac{\Delta^2}{(\tau_k - \tau_{k+1})^2} + \frac{\|E_1\|_2^2}{(\tau_k - \tau_{k+1})^2} \right) \|\varphi_{X'}^2(A_{(tr)}^k)\|_2^2 + 2\|\varphi_{X'}(E_2)\|_2^2.
\]
(34)

Inserting (33) and (34) back to (27), and collecting terms completes the proof.

**Corollary F.1.** Let the conditions of Lemma F.1 hold. Then for any \( j \in [L - 1] \),
\[
\|\tilde{M}^{(k)}_{(tr)} - M^f_{(tr)}\|_2^2 \leq \frac{8 \varepsilon^2}{\rho^4} \left( \frac{\Delta^2 + LP(R_1 \varepsilon_{\max}\Gamma_1^2)}{(\tau_k - \tau_{k+1})^2} \right) \|Z_j - \rho M^f_{(tr)}\|_2^2
\]
\[
+ \frac{4 \varepsilon^2}{\rho^2} \left\| \varphi_{X'}^2(\hat{Z}_j - \rho M^f_{(tr)}) \right\|_2^2 + 2(\varepsilon - 1)^2 \|M^f_{(tr)} - \tilde{M}^{(k)}_{(tr)}\|_2^2.
\]
(35)

**Proof.** Immediate from Lemma F.1 and Proposition 2.6

**Proposition F.1.** Assume Properties 2.7 to 2.11 and 2.2 hold. Then,
\[
E \left\| M^f - \tilde{M}^{(k)}_{(tr)} \right\|_2^2 \leq C'' \left( \gamma^2 R_1^2 \Gamma_1^2 \Gamma_2^2 \right) \left( \frac{k}{(\tau_k - \tau_{k+1})^2} + \frac{L((\varepsilon_{\max})^2 + (\varepsilon_{\max})^4)}{(\tau_k - \tau_{k+1})^2} \right) \log(P) \bar{P}^2
\]
\[
+ 4 \max_{j \in [L - 1]} \|\tilde{M}^{(k)}_{(tr)} - M^f_{(tr)}\|_2^2,
\]
where \( C'' \) is a term that depends only on \( \gamma^2, R_1^2, \Gamma_1^2, \Gamma_2^2 \).

**Proof.** Notation. For ease of exposition, for the remainder of the proof, let: \( A = M^f; Z = Z^X; \)
\( \hat{A} = \tilde{M}^{(k)}_{(tr)}; A_{(tr)} = M^f_{(tr)}; A_{(tr)}^{(k)} = M^{(k)}_{(tr)}; \) and \( E = M^f_{(tr)} - \tilde{M}^{(k)}_{(tr)} \).

**High Probability Conditioning Event.** Let \( E := E_2 \cap E_3 \cap E_4 \cap E_5 \) where \( E_2 \) to \( E_5 \) are defined in (13) to (16) respectively. Then,
\[
E \left( \|\hat{A} - A\|_2 \right) = E \max_{j \in [L - 1]} \|\hat{A}_j - A_j\|_2
\]
\[
= E \left\| \max_{j \in [L - 1]} \left( \hat{A}_j - A_j \right) \right\|_2 + E \left[ \max_{j \in [L - 1]} \|\hat{A}_j - A_j\|_2 \cdot 1(E^c) \right].
\]
(37)

**Upper bound on the first term in (37).** First note \( \varepsilon^2 \leq 10 \) since \( \rho \geq \frac{6 \log(P_L)}{P_L} \); and that
\[
\|A_{(tr)}^{(k)}\|_2 \leq \|A_j\|_2 \leq (R_1 \Gamma_1^2) \Gamma_2^2 \leq (R_1 \Gamma_1 \Gamma_2 + (\varepsilon_{\max})^2)^2 \bar{P}
\]
where (a) follows by Lemma C.1 and Proposition 2.6

Then conditioned on event \( E \), and by Corollary F.1,
\[
\max_{j \in [L - 1]} \|\hat{A}_j - A_j\|_2 \leq C \rho^4 \left( \gamma^2 \bar{P} + LP(R_1 \varepsilon_{\max} \Gamma_1^2) \right) \left( \frac{\gamma^2 \bar{P} + (R_1 \Gamma_1 \Gamma_2 + (\varepsilon_{\max})^2)^2 \bar{P}}{(\tau_k - \tau_{k+1})^2} \right)
\]
\[
+ \frac{C}{\rho^4} \left( \gamma^2 \log(P) \right) + 2P(R_1 \varepsilon_{\max} \Gamma_1^2)^2 + 4 \max_{j \in [L - 1]} \|E_j\|_2^2.
\]
(38)
Simplifying (38) by collecting terms, we have
\[
\max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2^2 \leq C^* \left( \frac{\gamma^2 R_1^2 \Gamma_1^2 \Gamma_2^2}{\rho^2} \right) \left( \frac{1}{(\tau_k - \tau_{k-1})^2} + \frac{k}{P^2} + \frac{L((\epsilon_{(1)})^2 + (\epsilon_{(1)})^4)}{(\tau_k - \tau_{k-1})^2} \right) \log(P) P^2
\]  
\[+ 4 \max_{j \in [L-1]} \| E_j \|_2^2. \tag{39}
\]
where \( C^* \left( \gamma^2 R_1^2 \Gamma_1^2 \Gamma_2^2 \right) \) is a term that depends only on \( \gamma^2 R_1^2 \Gamma_1^2 \Gamma_2^2 \).

Since \( P(E) \leq 1 \), we have
\[
\mathbb{E} \left[ \max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2^2 \cdot \mathbf{1}(E) \right] \leq C^* \left( \frac{\gamma^2 R_1^2 \Gamma_1^2 \Gamma_2^2}{\rho^2} \right) \left( \frac{1}{(\tau_k - \tau_{k-1})^2} + \frac{k}{P^2} + \frac{L((\epsilon_{(1)})^2 + (\epsilon_{(1)})^4)}{(\tau_k - \tau_{k-1})^2} \right) \log(P) P^2
\]  
\[+ 4 \max_{j \in [L-1]} \| E_j \|_2^2. \tag{40}
\]

**Upper bound on the second term in (37).** To begin with, we note that for any \( j \in [L-1] \),
\[
\| \widehat{A}_j - A_j \|_2 \leq \| \widehat{A}_j \|_2 + \| A_j \|_2
\]
by triangle inequality. By the model assumption, the covariates are bounded (Property 2.1) and \( \| A_j \|_2 \leq (R_1 \Gamma_1 \Gamma_2) \sqrt{P} \) for all \( j \in [L-1] \). For the remainder of the proof, for ease of notation, let \( \Gamma := (R_1 \Gamma_1 \Gamma_2) \). By definition, for any \( j \in [L-1] \),
\[
\widehat{A}_j = \frac{1}{\rho} \text{HSVT}_k(Z)_{:,j}
\]
for a given threshold \( \lambda = s_k \), the \( k \)th singular value of \( Z \). Therefore,
\[
\| \widehat{A}_j \|_2 \leq \frac{1}{\rho} \| \text{HSVT}_k(Z)_{:,j} \|_2 \leq P L \| \text{HSVT}_k(Z)_{:,j} \|_2 \leq P L \| Z_{:,j} \|_2.
\]

Here, (a) follows from \( \rho \geq \frac{1}{P} \).
\[
\max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2 \leq \max_{j \in [L-1]} \| \widehat{A}_j \|_2 + \max_{j \in [L-1]} \| A_j \|_2
\]
\[
\leq P L \max_{j \in [L-1]} \| Z_{:,j} \|_2 + \Gamma \sqrt{P}
\]
\[
\leq (P^2 L + \sqrt{P}) \Gamma + P^2 L \max_{ij} |\eta_{ij}|
\]
\[
\leq 2P^2 L (\Gamma + \max_{ij} |\eta_{ij}|).
\tag{41}
\]
because \( \max_{j \in [L]} \| Z_{:,j} \|_2 \leq \sqrt{P} \max_{ij} |\eta_{ij}| \leq \sqrt{P} \max_{ij} |\eta_{ij}| \leq \sqrt{P} (\Gamma + \max_{ij} |\eta_{ij}|) \). Now we apply

Cauchy-Schwarz inequality on \( \mathbb{E} \left[ \max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2 \cdot \mathbf{1}(E^c) \right] \) to obtain
\[
\mathbb{E} \left[ \max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2 \cdot \mathbf{1}(E^c) \right] \leq \mathbb{E} \left[ \max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2^2 \right] \cdot \mathbb{E} \left[ \mathbf{1}(E^c) \right]^\frac{1}{2}
\]
\[
= \mathbb{E} \left[ \max_{j \in [L-1]} \| \widehat{A}_j - A_j \|_2^2 \right] \cdot \mathbb{P}(E^c)^{\frac{1}{2}}
\]
\[
\leq \left[ 4P^2 L^2 \mathbb{E} \left[ (\Gamma + \max_{ij} |\eta_{ij}|)^4 \right] \right]^{\frac{1}{2}} \cdot \mathbb{P}(E^c)^{\frac{1}{2}}
\]
\[
\leq 8\sqrt{2}P^3 L^2 \mathbb{E} \left[ (\Gamma^4 + \max_{ij} |\eta_{ij}|^4) \right] \cdot \mathbb{P}(E^c)^{\frac{1}{2}}
\]
\[
\leq 8\sqrt{2}P^3 L^2 (\Gamma^4 + \max_{ij} |\eta_{ij}|^4) \cdot \mathbb{P}(E^c)^{\frac{1}{2}}.
\tag{42}
\]
Here, (a) follows from (41); and (b) follows from Jensen’s inequality:
\[
\mathbb{E} \left[ (\Gamma + \max_{ij} |\eta_{ij}|)^4 \right] = \mathbb{E} \left[ (\frac{1}{2} (2\Gamma + 2\max_{ij} |\eta_{ij}|)^4 \right] \leq \mathbb{E} \left[ \frac{1}{2} (2\Gamma)^4 + (2\max_{ij} |\eta_{ij}|)^4 \right]
\]
\[
= 8\mathbb{E} \left[ (\Gamma^4 + \max_{ij} |\eta_{ij}|^4) \right] = 8 (\Gamma^4 + \max_{ij} |\eta_{ij}|^4);
\]
and (c) follows from the trivial inequality: \( \sqrt{A + B} \leq \sqrt{A} + \sqrt{B} \) for any \( A, B \geq 0 \).
Now it remains to find an upper bound for $\mathbb{E}[\max_{ij} |\eta_{ij}|^4]$. Note that for $\theta \geq 1$, $\eta_{ij}$ being a $\psi_2$-random variable implies that $|\eta_{ij}|^\theta$ is a $\psi_2/\theta$-random variable. With the choice of $\theta = 4$, we have that

$$\mathbb{E}[\max_{ij} |\eta_{ij}|^4] \leq C' \gamma^4 \log^2(\bar{P}L)$$

(43)

for some $C' > 0$ by Lemma E.1 (also see Remark E.1). Inserting (43) to (42) yields

$$\mathbb{E} \left[ \max_{j \in [L-1]} \| \hat{A}_{j, \cdot} - A_{j, \cdot} \|_F^2 \cdot I(E) \right] \leq 8\sqrt{2} \bar{P}^3 L^2 \left( \Gamma^2 + C'(\gamma^2 \log^2(\bar{P}L)) \right) \mathbb{P}(E)^{\frac{3}{2}}$$

$$\leq 32 \left( \Gamma^2 + C'(\gamma^2 \log^2(\bar{P}L)) \right) \frac{1}{\bar{P}L^3},$$

(44)

where (a) follows from recalling that $\mathbb{P}(E) \leq \frac{8}{\bar{P}L^3}$.

**Concluding the Proof.** Thus, combining (40) and (44) in (37) and noticing that term in (44) is smaller order term than that in (40), by redefining $C''(\gamma^2, R^2, \Gamma^2, \gamma^2, \bar{P})$, appropriately we obtain the desired bound:

$$\mathbb{E}\left[ \| \hat{A}_{j, \cdot} - A_{j, \cdot} \|_{\infty,2}^2 \right] \leq \frac{C' \gamma^2 (R^2, \Gamma^2, \Gamma^2) \rho^4}{\theta^2 \left( \gamma_k - \gamma_{k+1} \right)^2} + \frac{k}{\bar{P}^2} + \frac{L((\epsilon^{(1)}_{ij})_{\max}^2 + (\epsilon^{(1)}_{ij})_{\max}^4)}{(\gamma_k - \gamma_{k+1})^2} \log(\bar{P}) \bar{P}^2$$

$$+ 4 \max_{j \in [L-1]} \| E_{j, \cdot} \|_F^2$$

(45)

□

**G Proofs - Imputation Analysis**

**G.1 Proof of Theorem 4.1**

**Proof.** Observe that for any matrix, $A \in \mathbb{R}^{m \times n}$,

$$\frac{1}{4} \| A \|_{\infty,2}^2 \geq \frac{1}{m \bar{n}} \| A \|_F^2.$$

Then using Property 4.1 and Proposition 4.1 and simplifying terms gives the desired result. □

**G.2 Proof of Theorem 4.3**

**Proof.** For the purposes of the proof let $N$ and pick an arbitrary ordering of the $N$ time series, denoted as $f_1, \ldots, f_N$. For $i \in [L]$ and $j \in [\bar{P}]$, define $[M^{f^2}]_{ij} := ([M^f]_{ij})^2$ and $[M^{f^2 + \sigma^2}]_{ij} := [M^f + M^\sigma]_{ij}$

$$\mathbb{E}[\| \hat{M} - M \|_F^2] = \mathbb{E}[\| \hat{M}^{f^2} - M^{f^2} \|_F^2] = \mathbb{E}[\| \hat{M}^{f^2 + \sigma^2} - M^{f^2 + \sigma^2} \|_F^2]$$

First Term: $\mathbb{E}[\| \hat{M} - M \|_F^2]$

$$\mathbb{E}[\| \hat{M} - M \|_F^2] = \mathbb{E}[\sum_{n=1}^N \sum_{t=1}^T (\hat{f}_n(t) - f_n(t))^2]$$

$$= \mathbb{E}[\sum_{n=1}^N \sum_{t=1}^T (f_n(t) - \hat{f}_n(t))^2 (f_n(t) + \hat{f}_n(t))^2]$$

$$\leq \max_{n \in [N], t \in [T]} (f_n(t) + \hat{f}_n(t))^2 \mathbb{E}[\sum_{n=1}^N \sum_{t=1}^T (f_n(t) - \hat{f}_n(t))^2]$$

$$\leq 4 \left( R_1 \Gamma_1 \Gamma_2 \right)^2 \mathbb{E}[\| \hat{M} - M \|_F^2]$$

$$= 4 \left( R_1 \Gamma_1 \Gamma_2 \right)^2 \mathbb{E}[\hat{M}^{f(\text{Impute})} - M^{f^2}]^2.$$
where (a) follows from Lemma C.1.

Second Term: \( E\|\hat{M}f^{2+\gamma^2} - Mf^{2+\gamma^2}\|_F^2 \).

Note,
\[
\|X_n^2(t)\|_{\psi_2} = \|f_n^2(t) + 2f_n(t)\eta_n(t) + \eta_n^2(t)\|_{\psi_2} \\
\leq 2\|f_n^2(t)\|_{\psi_2} + 2\|\eta_n^2(t)\|_{\psi_2} \\
\leq 2(R_1^2\Gamma_1\Gamma_2)^2 + 2\gamma
\]

Further, by Corollary 2.1 and Proposition 2.1, we immediately have there exists a matrix \( M_{(tr)}^{f^{2+\gamma^2}} \in \mathbb{R}^{L \times \bar{P}} \) such that
\[
\text{rank}(M_{(tr)}^{f^{2+\gamma^2}}) \leq (R_1G_{\text{max}}^{(1)})^2 + R_2G_{\text{max}}^{(2)}, \quad \|M_{(tr)}^{f^{2+\gamma^2}} - \hat{M}_{(tr)}^{f^{2+\gamma^2}}\|_{\text{max}} \leq (R_1G_{\text{max}}^{(1)}\Gamma_1\Gamma_2) + R_2\epsilon_{\text{max}}^{(2)}\Gamma_3
\]

Then, by a straightforward modification of the proof of Theorem 4.1, we have
\[
\text{MSE}(M_{(tr)}^{f^{2+\gamma^2}}, \hat{M}_{(tr)}^{f^{2+\gamma^2}}) \leq C_2R_1^2R_2^3(G_{\text{max}}^{(1)})^2(G_{\text{max}}^{(2)})^2 \left( \frac{1}{\sqrt{TN}} + (\epsilon_{\text{max}}^{(1)})^2 + (\epsilon_{\text{max}}^{(2)})^2 \right) \log(\bar{P}).
\]

Adding the bounds we have for the first and second term completes the proof. \( \square \)

### H Proofs - Forecasting Analysis

#### H.1 Forecasting - Helper Lemmas

**Proposition H.1.** Let Properties 2.1 and 2.2 hold. Then there exists \( \beta^* \in \mathbb{R}^{L-1} \), with \( \|\beta^*\|_1 \leq CR_1G_{\text{max}}^{(1)} \), such that
\[
\| (\hat{M}^T)^T\beta^* - M_L^T \|_\infty \leq C(R_1)^2(G_{\text{max}}^{(1)} + 1)\epsilon_{\text{max}}^{(1)}\Gamma_1. \tag{46}
\]

Here \( C \) is an absolute constant.

**Proof.** To reduce notational complexity, we suppress the superscript \( f \) for the remainder of the proof.

Let \( H \) and \( H_{(tr)} \) be defined as in Proposition 2.6. Since \( \text{rank}(H_{(tr)}) \leq R_1G_{\text{max}}^{(1)} \), it must be the case that within the last \( R_1G_{\text{max}}^{(1)} \) rows, there exists at least one row (which we denote as \( r^* \)) that can be written as a linear combination of at most \( R_1G_{\text{max}}^{(1)} \) rows above it (which we denote as \( r_1, \ldots, r_{R_1G_{\text{max}}^{(1)}} \)).

Solely for the purposes of the remainder of the proof (and without any loss of generality), we redefine \( M, H, H_{(tr)} \) assuming access to data \( t \in [-T:2T] \) (instead of \( [1:T] \)).

Specifically there exists \( \theta_l \in \mathbb{R} \) for \( l \in [R_1G_{\text{max}}^{(1)}] \), such that for all \( t \in [L:T] \)[10]
\[
[H_{(tr)}]_{(r^*, t)} = \sum_{l=1}^{R_1G_{\text{max}}^{(1)}} \theta_l[H_{(tr)}]_{(r_l, t)}.
\tag{47}
\]

[10] Here is where we use the fact that we redefined \( M, H, H_{(tr)} \) with respect to \( t \in [-T:2T] \). Otherwise, we could only claim the equality in (47) for \( t \in [L:T - R_1G_{\text{max}}^{(1)}] \).
where (a) follows from Proposition 2.6 and

\[
\text{Proof.}
\]

Observing that every entry of \( \mathbf{H} \) follows from Proposition 2.6 and \( \mathbf{H} \) is independent mean-zero sub-gaussian random variables such that \( \mathbb{E} \mathbf{H} = 0 \), the noisy version has additive noise added to the corresponding entry in the component of \( \mathbf{H} \). Hence for all \( t \in [0:T] \),

\[
\left| \mathbf{H}(r^*, t) - \sum_{l=1}^{R_l G^{(1)}_{\text{max}}} \theta_l \mathbf{H}(r_l, t) \right| = \left| \mathbf{H}(r^*, t) + \mathbf{H}(r_l, t) - \theta_l \mathbf{H}(r_l, t) \right| 
\]

\[
\leq \left| \mathbf{H}(r^*, t) - \mathbf{H}(r_l, t) \right| + \left| \sum_{l=1}^{R_l G^{(1)}_{\text{max}}} \theta_l \mathbf{H}(r_l, t) \right| 
\]

\[
+ \left| \mathbf{H}(r_l, t) - \sum_{l=1}^{R_l G^{(1)}_{\text{max}}} \theta_l \mathbf{H}(r_l, t) \right| 
\]

\[
= \left| \mathbf{H}(r^*, t) - \mathbf{H}(r_l, t) \right| + \left| \sum_{l=1}^{R_l G^{(1)}_{\text{max}}} \theta_l \mathbf{H}(r_l, t) \right| 
\]

\[
\leq C \left( R_l G^{(1)}_{\text{max}} \Gamma_1 + (R_l G^{(1)}_{\text{max}}) R_l G^{(1)}_{\text{max}} \right) 
\]

\[
\leq C(1) G^{(1)}_{\text{max}} \Gamma_1 + C(1) G^{(1)}_{\text{max}} \Gamma_1 
\]

where (a) follows from Proposition 2.6 and \( C \) is an absolute constant.

Observing that every entry of \( \mathbf{M}^f_L \) appears in \( \mathbf{H}(r^*, .) \) and letting \( \beta^* := (\theta_1, \ldots, \theta_{R_l G^{(1)}_{\text{max}}}) \) completes the proof by redefining constants appropriately.

**Proposition H.2.** Assume Properties 2.1, 2.3, 2.6 and 2.2 hold. Then for some absolute constants, \( C_1 \geq 0 \),

\[
\text{MSE}(\mathcal{M}^f, \widetilde{\mathcal{M}}^{f(\text{fore})}) \leq C_1 \left( (R_l) (C^{(1)}_{\text{max}} + 1) \right)^2 + C_1 \left( (R_l) (C^{(1)}_{\text{max}}) \right)^2 \frac{\| \mathcal{M}^f - \mathcal{M}^{f,k} \|_{\infty}^2}{P} + K \frac{k^2}{P},
\]

where \( \mathcal{M}^{f,k} \) is the estimation obtained in the first step of the forecasting algorithm in Section 3.1 using threshold \( k \geq 1 \) for number of singular values, \( K \) is function of \( \gamma^2, R_l \Gamma_1 \Gamma_2, \rho^{-1} \).

**Proof.** Let \( Q := \rho(\mathcal{M}^f)^T \) and \( \mathcal{Q} := \rho(\mathcal{M}^{f,k})^T \). Define \( \eta_L \in \mathbb{R}^P \), where \( \eta_L := Z_L^X - \rho \mathbf{M}_L^f \). For each entry in \( Z_L^X \), it is equal to the noisy version underlying time series with probability \( \rho \), and otherwise 0; and the noisy version has additive noise added to the corresponding entry in the component of \( \mathbf{M}_L^f \). Therefore, \( \mathbb{E}[\eta_L] = 0 \) and using Property 2.6 as well as Lemma G.2 in [3], it follows that the coordinates of \( \eta_L \) are independent mean-zero sub-gaussian random variables such that \( \| \eta_L(s) \|_2 \leq C'(\gamma^2 + R_1 \Gamma_1 \Gamma_2) \) for \( s \in \mathbb{R}^P \), where \( C' > 0 \) is an absolute constant. Define \( K(\gamma^2, R_1 \Gamma_1 \Gamma_2) := C'(\gamma^2 + R_1 \Gamma_1 \Gamma_2) \).

Let \( \beta^* \) below be defined as in Proposition H.1. Then note that by the definition of \( \hat{\beta} \) in the algorithm,

\[
\| Z_L^X - \hat{Q} \hat{\beta} \|_2^2 \leq \| Z_L^X - \hat{Q} \beta^* \|_2^2 + \| \eta_L \|_2^2 + 2\eta_L^T (\rho \mathbf{M}_L^f - \hat{Q} \beta^*).
\]

Moreover,

\[
\| Z_L^X - \hat{Q} \hat{\beta} \|_2^2 = \| \mathbf{M}_L^f - \hat{Q} \beta^* \|_2^2 + \| \eta_L \|_2^2 + 2\eta_L^T (\hat{Q} \beta - \rho \mathbf{M}_L^f). \]

Combining (48) and (49) and taking expectations, we have

\[
\mathbb{E}[\| \rho \mathbf{M}_L^f - \hat{Q} \hat{\beta} \|_2^2] \leq \mathbb{E}[\| \rho \mathbf{M}_L^f - \hat{Q} \beta^* \|_2^2] + 2 \mathbb{E}[\eta_L^T \hat{Q} (\beta - \beta^*)].
\]

Let us bound the final term on the right hand side of (50). Under our independence assumptions, observe that

\[
\mathbb{E}[\eta_L^T \hat{Q} (\beta - \beta^*)] = \mathbb{E}[\eta_L^T] \mathbb{E}[\hat{Q}] \beta^* = 0.
\]

(51)
Recall \( \hat{\beta} = \hat{Q}^\dagger Z^T = \rho \hat{Q}^\dagger M^f_L + \hat{Q}^\dagger \eta_L \). Using the cyclic and linearity properties of the trace operator (coupled with similar independence arguments), we further have
\[
\mathbb{E}[\eta_L^T \hat{Q} \hat{\beta}] = \mathbb{E}[\rho \eta_L^T \hat{Q} \hat{Q}^\dagger] M^f_L + \mathbb{E}[\eta_L^T \hat{Q} \eta_L^T]
\]
\[
= \mathbb{E}
\left[
\text{Tr}
\left(\eta_L^T \hat{Q} \hat{Q}^\dagger \eta_L^T
\right)
\right]
\]
\[
= \mathbb{E}
\left[
\text{Tr}
\left(\hat{Q} \hat{Q}^\dagger \eta_L \eta_L^T
\right)
\right]
\]
\[
= \text{Tr}
\left(\mathbb{E}[\hat{Q} \hat{Q}^\dagger] \cdot \mathbb{E}[\eta_L \eta_L^T]
\right)
\]
\[
\leq (a) \leq C_\gamma K(\rho, \gamma)^2 \mathbb{E}\left[\text{Tr} \left( \hat{Q} \hat{Q}^\dagger \right) \right],
\]
where (a) follows from Property 2.6. Here \( \tilde{C} \) is an absolute constant that may depend on \( \gamma \) as well as \( \rho \).

Let \( \tilde{Q} = \tilde{U} S V^T \) be the singular value decomposition of \( \hat{Q} \). Then
\[
\hat{Q} \hat{Q}^\dagger = \tilde{U} S V^T \tilde{V} \tilde{S} \tilde{U}^T
\]
\[
= \tilde{U} \tilde{I} \tilde{U}^T.
\]

Here, \( \tilde{I} \) is a block diagonal matrix where its nonzero entries on the diagonal take the value 1. Plugging in (53) into (52), and using the fact that the trace of a square matrix is equal to the sum of its eigenvalues,
\[
\mathbb{E}\left[\text{Tr}\left(\hat{Q} \hat{Q}^\dagger\right)\right] = \mathbb{E}[\text{rank}(\hat{Q})] = k.
\]

We now turn our attention to the first term on the right hand side of (50). We obtain
\[
\|\rho M^f_L - (Q \beta^*) \|^2_2 = \|\rho M^f_L - (Q - \hat{Q}) \beta^* \|^2_2
\]
\[
\leq 2 \|\rho M^f_L - \beta^* \|^2_2 + 2 \|Q - \hat{Q} \|^2_2
\]
\[
\leq (a) \leq 2 \rho^2 \left( (R_1)^2 (G_{\text{max}}^1 + 1) \gamma \right)^2 \tilde{P} + 2 \|Q - \hat{Q} \|^2_2,
\]
where (a) follows from Proposition [H.1]

We also have,
\[
\mathbb{E}\|Q - \hat{Q} \|^2_2 = \rho^2 \mathbb{E}\left\| M^f - \tilde{M}^{f,k} \right\|^T_2 \beta^* \|^2_2
\]
\[
\leq \rho^2 \|\beta^*\|^2 \mathbb{E}\left\| M^f - \tilde{M}^{f,k} \right\|^2_2
\]
\[
\leq (b) \leq \rho^2 \left( C R_1 G_{\text{max}}^1 \right)^2 \mathbb{E}\|M^f - \tilde{M}^{f,k}\|^2_2.
\]

Collecting all the terms together, dividing by \( \rho^2 \times \tilde{P} \), using \( \tilde{K}(\gamma^2, R_1, \Gamma_1, \Gamma_2, \rho^{-1}) = K(\gamma^2, R_1, \Gamma_1, \Gamma_2) / \rho \) and redefining the constants, we obtain our desired result.

**Theorem H.1.** Assume Properties 2.1, 2.5, 2.6 and 2.2 hold. Further, let (i) \( \tilde{P} \geq L \). Then,
\[
\text{MSE}(M^f, \tilde{M}^{\text{forecast}}) \leq \frac{C^* (\gamma^2, R_1^2, \Gamma_1^2, \Gamma_2^2, (G_{\text{max}}^1)^2)^2}{\rho^4} \left( \frac{P + PL \left( (\epsilon_1^1)^2 + (\epsilon_1^2)^2 \right)}{\gamma k + \tau_{k+1}} \right) \log(\tilde{P})
\]
\[
+ \frac{C \left( R_1 G_{\text{max}}^1 \right)^2 \tilde{P}}{L} \left\| M^{(l)} - \tilde{M}^{(l)} \right\|^2_2
\]
\[
\text{where } C^* (\gamma^2, R_1^2, \Gamma_1^2, \Gamma_2^2) \text{ is a term that depends only on } \gamma^2, R_1^2, \Gamma_1^2, \Gamma_2^2 \text{ and } C \geq 0 \text{ is an absolute constant.}
\]

**Proof.** Immediate from Propositions [H.2] and [F.1].

**H.2 Proof of Theorem F.2**

**Proof.** Immediate from Theorem [F.1] by simplifying terms and using Property 4.1.
H.3 Proof of Theorem 4.4

Proof. For the purposes of the proof let \( N \) and pick an arbitrary ordering of the \( N \) time series, denoted as \( f_1, \ldots, f_n \). Let \( \widehat{M}_L^{f_1(\text{Forecast})}, \widehat{M}_L^{f_2(\text{Forecast})}, \widehat{M}_L^{f_3(\sigma^2)(\text{Forecast})} \in \mathbb{R}^P \) be the induced vectors corresponding to the ordering of the \( N \) time series chosen from \( \widehat{M}_n^{f_1(\text{Forecast})}, \widehat{M}_n^{f_2(\text{Forecast})}, \widehat{M}_n^{f_3(\sigma^2)(\text{Forecast})} \) respectively.

Let \( M_L^{f_1^2}, M_L^{f_2^2}, M_L^{f_3^2+\sigma^2} \in \mathbb{R}^P \) be analogously defined but with respect to the latent time series \( \sigma^2, f^2 \) and \( f_3^2+\sigma^2 \). Here \( f^2 \) is a component wise squaring of the time series \( f \).

\[
\mathbb{E}\|\widehat{M}_L^{f_1(\text{Forecast})} - M_L^{f_1^2}\|_2 = \mathbb{E}\|\widehat{M}_L^{f_1(\text{Forecast})} - M_L^{f_2^2}\|_2
\]

where (a) follows from Lemma C.1.

**First Term:** \( \mathbb{E}\|\widehat{M}_L^{f_1(\text{Forecast})} - M_L^{f_1^2}\|_2 \)

\[
= \mathbb{E}\sum_{n=1}^{N} \left( f_n(tL) - \hat{f}_n(tL) \right)^2
\]

\[
\leq \max_{n \in [N], t \in [T]} \left( f_n(tL) + \hat{f}_n(tL) \right)^2 \mathbb{E}\sum_{n=1}^{N} \left( f_n(tL) - \hat{f}_n(tL) \right)^2
\]

\[
= 4 \left( R_1 \Gamma_1 \Gamma_2 \right)^2 \mathbb{E}\sum_{n=1}^{N} \left( f_n(tL) - \hat{f}_n(tL) \right)^2
\]

where (a) follows from Lemma C.1.

**Second Term:** \( \mathbb{E}\|\widehat{M}_L^{f_3(\sigma^2)(\text{Forecast})} - M_L^{f_3^2+\sigma^2}\|_2 \)

Note for all \( t \in [T] \) and \( n \in [N] \),

\[
\|X_n^2(tL)\|_{\psi_2} = \|f_n^2(t) + 2 f_n(t) \eta_n(t) + \eta_n^2(t)\|_{\psi_2} \leq 2 \|f_n^2(t)\|_{\psi_2} + 2 \|\eta_n(t)\|_{\psi_2} \\
\leq 2 \left( R_1 \Gamma_1 \Gamma_2 \right)^2 + 2 \gamma
\]

By Corollary 2.1 and Proposition 2.1, we immediately have there exists a matrix \( M_L^{f_3(\sigma^2)(\text{Forecast})} \in \mathbb{R}^{(L-1) \times P} \) such that

\[
\text{rank}(M_L^{f_3(\sigma^2)(\text{Forecast})}) \leq (R_3 \Gamma_{\max}^{(1)})^2 + R_2 \gamma \Gamma_{\max}^{(2)}.
\]

Recall \( M_L^{f_1^2}, M_L^{f_2^2}, M_L^{f_3^2+\sigma^2} \in \mathbb{R}^{(L-1) \times P} \) refer to the first \( L - 1 \) rows of the induced stacked Page matrices with respect to the latent time series \( \sigma^2, f^2 \) and \( f_3^2+\sigma^2 \). Note \( M_L^{f_1^2}, M_L^{f_2^2}, M_L^{f_3^2+\sigma^2} \in \mathbb{R}^P \) defined above are the last row. Let \( R^* = (R_3 \Gamma_{\max}^{(1)})^2 + R_2 \gamma \Gamma_{\max}^{(2)} \) and recall \( \epsilon^* = (R_3 \Gamma_{\max}^{(1)} \Gamma_{\max}^{(2)} + R_2 \gamma \Gamma_{\max}^{(2)} \Gamma_{\max}^{(3)} \). Then by a straightforward modification of the proof of Proposition H.1 we see that there exists a \( \beta^* \in \mathbb{R}^{L-1} \), with \( \|\beta^*\|_1 \leq CR^* \), such that

\[
\|M_L^{f_3^2+\sigma^2} \|_{\psi_2} - \|M_L^{f_3^2+\sigma^2} \|_{\psi_2} \leq (R^* + 1) \epsilon^*.
\]

(58)

Here \( C \) is an absolute constant (in short, the linear approximation error scales as the product of the approximate rank, \( R^* \), and the low-rank Hankel approximation error, \( \epsilon^* \).

By showing the existence of \( \beta^* \), we can now apply a straightforward modification of the proof of Theorem 4.2 to get

\[
\text{MSE}(\hat{M}_L^{f_3^2+\sigma^2}, \hat{M}_L^{f_3^2+\sigma^2(\text{Forecast})}) \leq C_1 \gamma \left( \frac{R_1^{a_1} R_2^{a_2} (G_{\text{max}}^{(1)})^3 (G_{\text{max}}^{(2)})^3}{\rho^2} \right) \left( \frac{1}{\sqrt{TN} + (\epsilon_{\text{max}}^{(1)})^2 + (\epsilon_{\text{max}}^{(2)})^2} \right) \log(P).
\]

36
Adding the bounds we have for the first and second term completes the proof.

\[ \text{regret} \leq C \frac{20}{20} \left( B^2 + NHB^2D + B^2 \sum_{n=1}^{N} \sum_{t=1}^{H} \Delta(t, tL, n) \right). \]  

(59)

\[ \text{Proof}. \text{ For simplicity, write } \beta^* := \beta^*_{(T+H \times L,N)} \text{ as in the definition of regret, let } \beta^*(T+tL, n) \text{ be the projection of } \beta^* \text{ onto } \Omega^k_{(T+tL, n)}. \text{ Then for } n \in [N], \text{ and } t \in [H], \]

\[ (c(T+tL, n) \beta^*(T+tL, n) - c(T+tL, n) \beta^*)) \leq \nabla c(T+tL, n) (\hat{\beta}(T+tL,n) - \beta^*), \]  
due to convexity of $c(T+tL,n)(\cdot)$

\[ = \left( \frac{\beta(T+tL,n) - \beta(T+(t+1)L,n)}{\delta} \right) (\hat{\beta}(T+tL,n) - \beta^*), \]  
due to update definition in online-mSSA

\[ = \frac{1}{20} \left( \| \beta^* - \hat{\beta}(T+tL,n) \|^2_2 - \| \beta^* - \hat{\beta}(T+(t+1)L,n) \|^2_2 + \| \hat{\beta}(T+tL,n) - \hat{\beta}(T+(t+1)L,n) \|^2_2 \right) \]

\[ = \frac{1}{20} \left( \| \beta^* - \hat{\beta}(T+tL,n) \|^2_2 - \| \beta^* - \hat{\beta}(T+(t+1)L,n) \|^2_2 + \delta^2 \| \nabla c(T+tL,n) (\hat{\beta}(T+tL,n)) \|^2_2 \right), \]

where we again use the definition of the online-mSSA. First,

\[ \| \beta^* - \hat{\beta}(T+tL,n) \|^2_2 = \| \beta^* - \beta^*_0(T+tL,n) + \beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n) \|^2_2 \]

\[ = \| \beta^* - \beta^*_0(T+tL,n) \|^2_2 + \| \beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n) \|^2_2 + 2(\beta^* - \beta^*_0(T+tL,n) \beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n)) \]

\[ = \| \beta^* - \beta^*_0(T+tL,n) \|^2_2 + \| \beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n) \|^2_2 \]

In above, $\langle \beta^* - \beta^*_0(T+tL,n), \beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n) \rangle = 0$ due to $\beta^* - \beta^*_0(T+tL,n)$ being orthogonal to any vector in $\Omega^k_{(T+tL,n)}$ including $\beta^*_0(T+tL,n) - \hat{\beta}(T+tL,n)$.

Next,

\[ \| \beta^* - \hat{\beta}(T+(t+1)L,n) \|^2_2 = \| \beta^* - \beta^*_0(T+(t+1)L,n) + \beta^*_0(T+(t+1)L,n) - \hat{\beta}(T+(t+1)L,n) \|^2_2 \]

\[ = \| \beta^* - \beta^*_0(T+(t+1)L,n) \|^2_2 + \| \beta^*_0(T+(t+1)L,n) - \hat{\beta}(T+(t+1)L,n) \|^2_2 \]

\[ + 2(\beta^* - \beta^*_0(T+(t+1)L,n) \beta^*_0(T+(t+1)L,n) - \hat{\beta}(T+(t+1)L,n)) \]

\[ \geq \| \beta^* - \beta^*_0(T+(t+1)L,n) \|^2_2 + \| \beta^*_0(T+(t+1)L,n) - \hat{\beta}(T+(t+1)L,n) \|^2_2 \]

\[ + 2(\beta^* - \beta^*_0(T+(t+1)L,n) \beta^*_0(T+(t+1)L,n) - \hat{\beta}(T+(t+1)L,n)) \]

In above, the inequality follows since $\hat{\beta}(T+(t+1)L,n)$ is projection of $\beta^*_0(T+(t+1)L,n)$ on linear sub-space $\Omega^k_{(T+tL,n)}$ which contains $\beta^*_0(T+(t+1)L,n)$ and hence the inequality for $\| \cdot \|^2_2$. Hence we have,
Then, \( \sum_{n \in \{N\}} c(t_{L,n}) (\hat{\beta}(T+tL,n) - \beta^*) \) \\
\leq \left\| \beta^* - \beta^*_{(T+tL,n)} \right\|_2^2 + \left\| \beta^*_{(T+tL,n)} - \hat{\beta}_{(T+tL,n)} \right\|_2^2 + 2(\beta^* - \beta^*_{(T+(t+1)L,n)\beta^*_{(T+(t+1)L,n)}} - \hat{\beta}_{(T+(t+1)L,n)}) + \delta^2 \left\| \nabla c(t_{L+n}) (\hat{\beta}(T+tL,n)) \right\|_2^2 \\
\leq \sum_{n \in \{N\}} H \left( \beta^*_{(T+(t+1)L,n)} - \beta^*_{(T+(t+1)L,n)} - \hat{\beta}_{(T+(t+1)L,n)} \right)
\leq 8B^2 + P\delta^2 D + \sum_{n=1}^N H \left( \beta^*_{(T+(t+1)L,n)} - \beta^*_{(T+(t+1)L,n)} - \hat{\beta}_{(T+(t+1)L,n)} \right)
\leq 8B^2 + P\delta^2 D + \sum_{n=1}^N H \left( \beta^*_{(T+(t+1)L,n)} - \beta^*_{(T+(t+1)L,n)} - \hat{\beta}_{(T+(t+1)L,n)} \right)
\leq B^2 \sum_{n=1}^N H \Delta_{(T+tL,n)}.

\]

**Proposition I.2 (Bounding \( D \)).** Let \( B, D \) be defined as in Proposition I.1. Let \( \alpha^i_{(T+tL,n,j)} \) and \( Z_n(T+tL) \) be defined as in Section 4A. Assume,

- \( \max_{t \in [L-1], n \in \{N\}, j \in [P+1]} |\alpha^i_{(T+tL,n,j)}| = D_1 \)
- \( \max_{t \in [H+1], n \in \{N\}} ||Z_n(T+tL)|| = D_2 \)

Then,
\[
D \leq C \cdot k^3 \cdot B^2 \cdot D_1^4 \cdot D_2^2 
\]

where \( C \) is an absolute constant.

**Proof:**
\[
\left\| \nabla c(t_{L+n}) (\hat{\beta}(T+tL,n)) \right\|_2^2 = \left\| \nabla \hat{\beta}_{(T+tL,n)} \left( Z_n(T+tL) - \left( Z_n(T+(t-1)L+1 : T+tL-1) \right)^T \hat{\beta}_{(T+tL,n)} \right) \right\|_2^2 \\
= \left\| \nabla \hat{\beta}_{(T+tL,n)} \left( Z_n(T+tL) - \sum_{i=1}^k \alpha^i_{(T+tL,n,j)} (v^i_{(t,n)})^T \hat{\beta}_{(T+tL,n)} \right) \right\|_2^2 \\
= \left\| 2 \left( Z_n(T+tL) - \sum_{i=1}^k \alpha^i_{(T+tL,n,j)} (v^i_{(t,n)})^T \hat{\beta}_{(T+tL,n)} \right) \left( \sum_{i=1}^k \alpha^i_{(T+tL,n,j)} v^i_{(t,n)} \right) \right\|_2^2 \\
\leq 4 \left( Z_n(T+tL) \right)^2 + (kBD_1)^2 \left( \sum_{i=1}^k \alpha^i_{(T+tL,n,j)} v^i_{(t,n)} \right)^2 \\
\leq 4 \left( (D_2)^2 + (kBD_1)^2 \right) k \left( D_1 \right)^2
\]
Hence,
\[ D \leq C \cdot k^3 \cdot B^2 \cdot D_1^4 \cdot D_2^2 \]
where \( C \) is an absolute constant.

### I.1.1 Bounding \( \Delta_{(T+t)L,n} \)

**Proposition I.3.** Recall \( \tau^1_{(t,n)}, \ldots, \tau^k_{(t,n)} \) be the first \( k \) singular values of \( \tilde{M}^f_{(t,n)} \). Assume, for \( n \in [N] \) and \( t \in [H+1] \),
- \( P_{\Omega^k_{(T+t)L,n}} = P_{\Omega^k_{(T+H+L,N)}} \)
- Assume Properties 2.1, 2.3, 2.6 and 2.7 hold.

Let \( C_3 \) be large enough constant defined as in Theorem 4.5. Then, with probability at least \( 1 - 1/(NT)^4 \),
\[ ||P_{\Omega^k_{(T+t)L,n}} - P_{\Omega^k_{(T+H+L,N)}}|| \leq C_3 \frac{\sqrt{LNH}}{\tau^k_{(T+t)L,n} - \tau^{k+1}_{(T+t)L,n} - (\sqrt{L} + \sqrt{N(P+H)})}. \]

**Proof.** Let \( \tilde{\tau}^1_{(T+t)L,n}, \ldots, \tilde{\tau}^k_{(T+t)L,n} \) be the first \( k \) (ordered by magnitude) singular values of \( \tilde{Z}^f_{(T+t)L,n} \).
Define \( \tilde{Z}^f_{(T+t)L,n} \in \mathbb{R}^{L \times (N(P+H)))} \), which is induced from \( \tilde{Z}^f_{(T+t)L,n} \) by stacking columns of all 0s to the right of it to make it of dimension \( L \times (N(P+H)) \). Let \( E_{(T+t)L,n} := \tilde{Z}^f_{(T+t)L,n} - \tilde{Z}^f_{(T+H+L,N)} \).

By the Davis-Kahan \( \sin\Theta \) theorem, we have,
\[ ||P_{\Omega^k_{(T+t)L,n}} - P_{\Omega^k_{(T+H+L,N)}}|| \leq C_3 \frac{\|E_{(T+t)L,n}\|}{\tilde{\tau}^k_{(T+t)L,n} - \tilde{\tau}^{k+1}_{(T+t)L,n}} \]
\[ \leq C_3 \frac{\sqrt{LNH}}{\tilde{\tau}^k_{(T+t)L,n} - \tilde{\tau}^{k+1}_{(T+t)L,n}} \]

By the Cauchy Interlacing Theorem and an application of Theorem E.2 on \( Z^f_{(T+t)L,n} - \tilde{M}^f_{(T+t)L,n} \), with appropriately chosen large enough \( C_3 \geq 0 \), it follows that with probability at least \( 1 - 1/(NT)^4 \), we have that for \( i \in [L-1] \)
\[ |\tilde{\tau}^i_{(T+t)L,n} - \tau^i_{(T+t)L,n}| \leq C_3(\sqrt{L} + \sqrt{N(P+H)}) . \]

Hence,
\[ C_3 \frac{\sqrt{LNH}}{\tilde{\tau}^k_{(T+t)L,n} - \tilde{\tau}^{k+1}_{(T+t)L,n}} \leq C_3 \frac{\sqrt{LNH}}{\tau^k_{(T+t)L,n} - \tau^{k+1}_{(T+t)L,n} - (\sqrt{L} + \sqrt{N(P+H)})}. \]

**Corollary I.1.** For \( n \in [N] \) and \( t \in [H+1] \), assume:
- Conditions of Proposition I.3 hold;
- \( \tau^k_{(T+t)L,n} = \Omega\left(\sqrt{\frac{LNP}{k}}\right) \), \( \tau^{k+1}_{(T+t)L,n} = 0 \); and
- \( H = o(P) \).

Then for \( n \in [N] \), \( t \in [H] \), we have \( ||P_{\Omega^k_{(T+t)L,n}} - P_{\Omega^k_{(T+H+L,N)}}|| \leq C_3 \frac{\sqrt{LNP}}{P} \).

**Proof.** Using, Proposition I.3, we have
\[ ||P_{\Omega^k_{(T+t)L,n}} - P_{\Omega^k_{(T+H+L,N)}}|| \leq C_3 \frac{\sqrt{LNH}}{\tau^k_{(T+t)L,n} - \tau^{k+1}_{(T+t)L,n} - (\sqrt{L} + \sqrt{N(P+H)})} \leq C_3 \frac{\sqrt{LNP}}{P} \]

\( \square \)
1.2 Proof of Theorem 4.5

Proof. From Propositions I.1, I.2 and Corollary I.1, we have

\[
\frac{1}{NH}\text{-regret} \leq C_3 k^{2.5} \left( \frac{1}{NH} + \sqrt{\frac{H}{P}} + \delta L^2 \right),
\]

with probability at least \(1 - 1/(NT)\) by Union Bound.

Recall \(L = P^{1-\epsilon_1}, H = P^{1-\epsilon_2}\). Then, \(\frac{1}{NH} + \sqrt{\frac{H}{P}} = \frac{1}{NP^{1-\epsilon_2}} + \frac{1}{P^{0.5\epsilon_2}}\).

Setting \(\delta = \frac{1}{L} \sqrt{\frac{1}{NH} + \sqrt{\frac{H}{P}}} = \frac{1}{L} \sqrt{\left( \frac{1}{NP^{1-\epsilon_2}} + \frac{1}{P^{0.5\epsilon_2}} \right)}\), we get

\[
\left( \frac{1}{\delta} \left( \frac{1}{NH} + \sqrt{\frac{H}{P}} \right) + \delta L^2 \right) = 2P^{1-\epsilon_1} \sqrt{\left( \frac{1}{NP^{1-\epsilon_2}} + \frac{1}{P^{0.5\epsilon_2}} \right)}
\]

\[
= 2 \sqrt{\left( \frac{P^{2-2\epsilon_1}}{NP^{1-\epsilon_2}} + \frac{P^{2-2\epsilon_1}}{P^{0.5\epsilon_2}} \right)} = 2 \sqrt{\left( \frac{P^{1-2\epsilon_1+\epsilon_2}}{N} + P^{2-2\epsilon_1-0.5\epsilon_2} \right)}
\]

\(\Box\)