A Uniform Bound of the Operator Norm of Random Element Matrices and Operator Norm Minimizing Estimation

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Abstract

In this paper, we derive a uniform stochastic bound of the operator norm (or equivalently, the largest singular value) of random matrices whose elements are indexed by parameters. As an application, we propose a new estimator that minimizes the operator norm of the matrix that consists of the moment functions. We show the consistency of the estimator.

Keywords: Random Matrix Theory, Operator Norm, Uniform Bound, Operator Norm Minimizing Estimator

1 Introduction

Since its introduction in nuclear physics (Widner (1955)) and mathematical statistics (Wishart (1928)), random matrix theory has been developed to understand the properties of the

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spectra of large dimensional random matrices generated by various distributions. These include the asymptotic theory of the empirical distribution of the eigenvalues of large dimensional random matrices and bounds on the extreme eigenvalues. For detailed results on these topics, readers can refer to recent surveys like Bai (2008), Edelman and Rao (2005), Bai and Silverstein (2010), and Tao (2012), among others.

In random matrix theory the study of the asymptotics of the largest eigenvalue of large dimensional random matrices goes back to Geman (1980). Suppose that $X$ is an $N \times T$ matrix consisting of random variables $x_{it}$. Many researchers have derived the limit of the largest eigenvalue of the sample covariance matrix, $\lambda_1(X'X)$, under various distributional conditions on the random matrix $X$. For example, when $X_{it}$ are iid $N(0, 1)$ and $\kappa := \lim \frac{N}{T}$, Geman (1980) showed that $\frac{1}{N}\lambda_1(X'X) \rightarrow a.s. \ (1 + \kappa^{1/2})^2$. Later Johnstone (2001) showed that the properly normalized $\lambda_1(X'X)$, $\frac{\lambda_1(X'X) - \mu_{NT}}{\sigma_{NT}}$ with $\mu_{NT} = (\sqrt{N - 1} + \sqrt{T})^2$ and $\sigma_{NT} = (\sqrt{N - 1} + \sqrt{T})(1/\sqrt{N - 1} + 1/\sqrt{T})^{1/3}$, converges to the Tracy-Widom law.

These results imply that $\lambda_1(X'X)$ is stochastically bounded by an order of $\text{max}(N, T)$, or equivalently, the operator norm of the random matrix $X$, $\|X\| := \sqrt{\lambda_1(X'X)}$ is stochastically bounded by an order $\sqrt{\text{max}(N, T)}$. In fact, the order of the bound does not require that the distribution of the random matrix is Gaussian and can be derived under much weaker conditions. For example, Latała (2005) showed that if $x_{it}$ is independent across $i, t$ with mean zero and uniformly bounded fourth moments, then $\|X\|$ is stochastically bounded by $\sqrt{\text{max}(N, T)}$. Moon and Weidner (2017) extended the result for the cases where $x_{it}$ are weakly correlated in $i$ or $t$. Other papers that have established similar bounds of $E(\|X\|)$ include Bandeira and Van Handel (2016) and Guédon, Hinrichs, Litvak, and Prochno (2017).

In this paper we extend the existing random matrix theory on a stochastic bound of the largest eigenvalue of a high dimensional matrix that consists of random elements. Suppose that $x_{it}(\beta)$ are stochastic processes indexed by parameter $\beta$ and let $X(\beta)$ be the $N \times T$ matrix consisting of $x_{it}(\beta)$. The first contribution of the paper is to derive a uniform stochastic bound of the largest singular value (or equivalently the operator norm) of $X(\beta)$. Sup-
pose that the parameter set \( B \) is equipped with (pseudo) metric \( d_\beta \) and \( N_\beta(B, d_\beta, \nu) \) is the covering number of \( B \) with diameter \( \nu \) in the metric \( d_\beta \). Under the regularity conditions including sub-Gaussianity of \( x_{it}(\beta) \), we show that 
\[
E\left( \sup_{\beta \in B} \|X(\beta)\| \right) \text{ is an order of } 
\log(\max(N,T)) \sqrt{\max(N,T)} + \int_0^{\operatorname{diam}(B)} \log N_\beta(B, d_\beta, \nu) d\nu + \sqrt{\max(N,T)} \int_0^{\operatorname{diam}(B)} \sqrt{\log N_\beta(B, d_\beta, \nu)} d\nu,
\]
where \( \operatorname{diam}(B) \) is the diameter of \( B \) in the metric \( d_\beta \).

As an application of the uniform stochastic bound, in Section 3 we propose a new estimator that minimizes the operator norm of the matrix that consists of the moment functions. We show the consistency of the estimator using the uniform bound of the operator norm of the moment function matrix.

Section 4 concludes the paper. The appendix contains all the technical proofs of the results in the main paper.

Notation: We use notation \( \max_i, \max_t, \max_{i,t} \) to denote \( \max_{1 \leq i \leq N}, \max_{1 \leq t \leq T}, \max_{1 \leq i \leq N, 1 \leq t \leq T}, \) respectively. We denote \( a \preceq b \) if there exist a universal constant \( C \) such that \( a \leq Cb \). For two random elements \( A \) and \( B \), \( A =_d B \) denotes that the distributions of \( A \) and \( B \) are identical.

### 2 A Uniform Bound of the Operator Norm of Random Element Matrices

Let \( x_{it}(\beta) \) be a sequence of stochastic processes indexed by \( \beta \in B \). We assume that \( x_{it}(\beta) \) are independent over \( i \) and \( t \) with mean zero and have bounded sample paths almost surely, that is, \( x_{it}(\beta) \in \ell^\infty(B) \). The index \( \beta \) can be a finite dimensional parameter, or a infinite dimensional element. We assume that the index set \( B \) is equipped with a pseudo metric \( d_\beta(\cdot, \cdot) \).

Let \( X(\beta) := [x_{it}(\beta)] \) the \( N \times T \) matrix consisting of \( x_{it}(\beta) \). Suppose that \( \|X(\beta)\| \) is the operator norm of random matrix \( X(\beta) \),

\[
\|X(\beta)\| := \sup_{\|u\|=1} \sup_{\|v\|=1} u'X(\beta)v.
\]
Define
\[
\|X(\beta)\|_B := \sup_{\beta} \|X(\beta)\|.
\]

The main goal of this section is to establish bounds of \(E\|X(\beta)\|_B\).

Let \(g_{it}\) be a sequence of iid \(N(0,1)\) random variables that are independent of \(x_{it}(\beta)\). Let \(G := [g_{it}]\) be the \(N \times T\) matrix consisting of \(g_{it}\). Let \(Z(\beta) := X(\beta) \circ G = [x_{it}(\beta)g_{it}]\) be the Hadamard product of the random element matrix \(X(\beta)\) and the Gaussian random matrix \(G\). The following lemma establishes the first upper bound of the expectation of the uniform matrix norm of \(X(\beta)\).

**Lemma 1.** There exists a finite constant \(C\) such that
\[
E\|X(\beta)\|_B \leq C E\|Z(\beta)\|_B.
\]

By definition we can express
\[
\|Z(\beta)\|_B := \sup_{\beta \in B} \sup_{\|u\| = 1} \sup_{\|v\| = 1} u'Z(\beta)v.
\]

Let \(z_{it}(\beta)\) denote the \((i,t)\)th component of \(Z(\beta)\). Let \(U := \{x \in \mathbb{R}^N : \|x\| = 1\}\) and \(V := \{x \in \mathbb{R}^T : \|x\| = 1\}\) be the unit spheres in \(\mathbb{R}^N\) and \(\mathbb{R}^T\), respectively. Let \(\theta = (\beta', u', v')' \in \Theta\), where \(\Theta := B \times U \times V\). Denote
\[
S(\theta) := u'Z(\beta)v = \sum_{i=1}^{N} \sum_{t=1}^{T} u_iz_{it}(\beta)v_t.
\]

Then, our problem becomes finding an upper bound of
\[
E [\|Z(\beta)\|_B] = E \left[ \sup_{\theta \in \Theta} S(\theta) \right]. \tag{1}
\]

To establish a bound of \(E [\sup_{\theta \in \Theta} S(\theta)]\), we need to define a pseudo-distance \(d(\theta_1, \theta_2)\) over the parameter set \(\Theta\) with which we define the entropy of the parameter set \(\Theta\) and control the continuity of \(S(\theta)\) as a stochastic process.

For this, suppose that \(d_{\beta}(\beta_1, \beta_2)\) is a distance defined on the index set \(B\). Let \(d_u(u_1, d_2) := \)
∥u_1 − u_2∥ and d_v(v_1, v_2) := ∥v_1 − v_2∥ be the Euclidean distances in the set U and V, respectively. We define 

\[ d(\theta_1, \theta_2) := d_\beta(\beta_1, \beta_2) + d_u(u_1, u_2) + d_v(v_1, v_2). \]

**Assumption 1.** Assume that the stochastic process \( x_{it}(\beta) \) satisfies the following conditions.

(i) The stochastic process \( x_{it}(\beta) \) is separable.

(ii) There exists a finite constant \( \sigma^2 \) such that

\[
\sup_{\beta \in B} \log \mathbb{E} (\exp(\lambda x_{it}(\beta))) \leq \lambda^2 \sigma^2 / 2
\]

for all \( \lambda > 0 \).

(iii) For all \((i, t)\) and \((\beta_1, \beta_2) \in B \times B\),

\[
P \{ |x_{it}(\beta_1) − x_{it}(\beta_2)| > x \} \leq C \exp \left( -\frac{x^2}{d_\beta(\beta_1, \beta_2)^2} \right).
\]

(iv) The index set \( B \) is totally bounded with respect to \( d_\beta(\cdot, \cdot) \).

**Remarks**

(a) If Assumption II(ii) is satisfied, then it is well known (for example, see Lemma 5.1 of van Handel (2016)) that

\[
\sup_{\beta \in B} \mathbb{E} \left[ \max_{i, \bar{t}} |x_{i\bar{t}}(\beta)| \right] \leq \sqrt{2 \sigma^2 \log(NT)}.
\]

(b) Assumption II(iii) assumes that \( x_{it}(\beta) \) is a sub-Gaussian process with respect to a pseudo-metric \( d_\beta(\cdot, \cdot) \) of the index set \( B \) uniformly in \((i, t)\).

Let \( N_{\beta, \epsilon} \) be the \( \epsilon \)-net of \((B, d_\beta)\). Let \( N_{\beta}(B, d, \epsilon) \) be the covering number of \((B, d_\beta)\).
Theorem 1. Suppose that Assumption 1 holds. Then,

\[ \mathbb{E} \left[ \| Z(\beta) \|_B \right] \lesssim \log(\max(N, T)) \sqrt{\max(N, T)} \]

\[ + \int_0^{\text{diam}(B)} \log N_\beta(B, d_\beta, \nu) d\nu + \sqrt{\max(N, T)} \int_0^{\text{diam}(B)} \sqrt{\log N_\beta(B, d_\beta, \nu)} d\nu. \]

Combining the bounds of Lemma 1 and Theorem 1, we have the main result of the paper in the following corollary.

Corollary 1. Suppose that Assumption 1 holds. Then,

\[ \mathbb{E} \left[ \| X(\beta) \|_B \right] \lesssim \log(\max(N, T)) \sqrt{\max(N, T)} \]

\[ + \int_0^{\text{diam}(B)} \log N_\beta(B, d_\beta, \nu) d\nu + \sqrt{\max(N, T)} \int_0^{\text{diam}(B)} \sqrt{\log N_\beta(B, d_\beta, \nu)} d\nu. \]

3 Application: Estimator Minimizing Operator Norm

In this section, we investigate a new estimator that minimizes the operator norm of the moment function matrix. Suppose that \( \varepsilon_{it}(\beta) \in \mathbb{R}^L \) are \( L \) moment functions of \( \beta \in B \subset \mathbb{R}^K \) such that \( \mathbb{E}(\varepsilon_{it}(\beta_0)) = 0 \). For simplicity, assume that \( L = K = 1 \). Let \( \varepsilon(\beta) = [\varepsilon_{it}(\beta)] \), the \( N \times T \) matrix of moment functions.

The conventional method of moment estimator solves

\[ \tilde{\beta} = \arg \min_{\beta \in B} \left| \frac{1}{NT} \sum_{i,t} \varepsilon_{it}(\beta) \right| = \arg \min_{\beta \in B} \left| \frac{1}{\sqrt{N}} \left( \frac{\varepsilon(\beta)}{\sqrt{NT}} \right) \frac{1}{\sqrt{T}} \right|, \]

where \( 1_N \) is the \( N \)-vector of ones.

The new estimator we propose minimizes the operator norm of the moment function
matrix $\varepsilon(\beta)$,

$$\hat{\beta} := \arg\min_{\beta \in B} \frac{\|\varepsilon(\beta)\|}{\sqrt{NT}}$$

$$= \arg\min_{\beta \in B} \sup_{\|u\|=1,\|v\|=1} w' \left( \frac{\varepsilon(\beta)}{\sqrt{NT}} \right) v.$$ 

In this section we establish consistency of $\hat{\beta}$ using the random matrix theory in Corollary 1.

**Assumption 2.** We assume that (i) the parameter set of $\beta$, $B$, is totally bounded with respect to $d_\beta(\beta_1, \beta_2) = |\beta_1 - \beta_2|$, (ii) the centered moment function $\varepsilon_{it}(\beta) - E(\varepsilon_{it}(\beta))$ satisfies the sub-Gaussian conditions in Assumption 1(i)-(iii), and (iii) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|E(\varepsilon(\beta))\|}{\sqrt{NT}} > 2\delta$.

The conditions in Assumptions 2 (i)-(ii) assume that $\varepsilon_{it}(\beta) - E(\varepsilon_{it}(\beta))$ satisfies Assumption 1. The last condition (iii) corresponds to the identification condition of the extremum estimator.

For consistency of $\hat{\beta}$, it is enough to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|\varepsilon(\beta)\|}{\sqrt{NT}} - \frac{\|\varepsilon(\beta_0)\|}{\sqrt{NT}} > \delta \quad (3)$$

with probability approaching one. Suppose that we choose $\delta > 0$ is Assumption 2(iii). By the triangle inequality,

$$\inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|\varepsilon(\beta)\|}{\sqrt{NT}} \geq \inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|E(\varepsilon(\beta))\|}{\sqrt{NT}} - \sup_{\beta \in B} \frac{\|\varepsilon(\beta) - E(\varepsilon(\beta))\|}{\sqrt{NT}}.$$ 

Under Assumption 2(i) and (ii), which is equivalent to Assumption 1 from Corollary 1, we have

$$\sup_{\beta \in B} \frac{\|\varepsilon(\beta) - E(\varepsilon(\beta))\|}{\sqrt{NT}} = o_p(1).$$
Also, by Assumption 2(iii), we have

$$\inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|E(\epsilon(\beta))\|}{\sqrt{NT}} > 2\delta.$$ 

Therefore,

$$\inf_{|\beta - \beta_0| \geq \epsilon} \frac{\|\epsilon(\beta)\|}{\sqrt{NT}} \geq 2\delta + o_p(1) \geq \delta \text{ wp1.}$$

This shows that the $\hat{\beta}$ is consistent.

**Remarks**

(i) If $\epsilon_{it}(\beta)$ are iid, then the identification condition Assumption 2 (iii) becomes the usual identification condition, that is, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{|\beta - \beta_0| \geq \epsilon} |E(\epsilon_{it}(\beta))| > 2\delta.$$  

This is because $\frac{\|E(\epsilon(\beta))\|}{\sqrt{NT}} = |E(\epsilon_{it}(\beta))| \frac{1}{\sqrt{N}} \frac{1}{\sqrt{T}} = |E(\epsilon_{it}(\beta))|$.

(ii) Suppose that $\epsilon_{it}(\beta) = (\epsilon_{1,1}(\beta), \ldots, \epsilon_{L,1}(\beta))' \in \mathbb{R}^L$. For the least operator norm objective function, we may consider

$$\sum_{l=1}^{L} \omega_l \frac{\|\epsilon_l(\beta)\|}{\sqrt{NT}},$$

where $\omega_l$ are weights.

(iii) We can also extend the objective function to be the sum of $R_{NT}$ largest singular values, where $R_{NT}$ is a sequence of positive increasing integers such that $R_{NT} \to \infty$ slowly satisfying $\frac{R_{NT}}{\sqrt{\min(N,T)}} \to 0$:

$$\frac{1}{\sqrt{NT}} \sum_{r=1}^{R_{NT}} s_r(\epsilon(\beta)),$$

where $s_r(A)$ is the $r^{th}$ largest singular value of matrix $A$. Since $\|\epsilon(\beta) - E(\epsilon(\beta))\| = \ldots$
$s_1(\varepsilon(\beta) - \mathbb{E}(\varepsilon(\beta)))$, we have

$$
\sup_{\beta \in B} \frac{1}{\sqrt{NT}} \sum_{r=1}^{R_{NT}} s_r(\varepsilon(\beta) - \mathbb{E}(\varepsilon(\beta)))
\leq R_{NT} \left\| \varepsilon(\beta) - \mathbb{E}(\varepsilon(\beta)) \right\| = O_p \left( \frac{R_{NT}}{\min(N, T)} \right) = o_p(1).
$$

4 Conclusion

In this paper, we derived a uniform stochastic bound of the operator norm of random element matrices. We apply it to derive the consistency of the new estimator that minimizes the operator norm of the moment functions. We want to leave it as a future research topic to derive the limiting distribution of the new estimator.
Appendix

**Proof of Lemma** The proof is similar to the arguments used in the proof of Theorem 2 of Latała (2005). Let $\tilde{X}(\beta)$ be the independent copy of $X(\beta)$. Let $\xi_{it}$ be a sequence of iid Rademacher random variables. Let $\Xi := [\xi_{it}]$ be the $N \times T$ matrix consisting of $\xi_{it}$. We assume that $X(\beta), \tilde{X}(\beta), G,$ and $\Xi$ are independent. For a random element $X$, we denote $E_X$ as a conditional expectation operator conditioning on $X$.

Notice that

$$
E\|X(\beta)\|_B = E\|X(\beta) - E_{X(\beta)}(\tilde{X}(\beta))\|_B = E\|E_{X(\beta)}(X(\beta) - \tilde{X}(\beta))\|_B \\
\leq E\|X(\beta) - \tilde{X}(\beta)\|_B \\
= E \left\| \left( X(\beta) - \tilde{X}(\beta) \right) \circ \Xi \right\|_B \\
\leq 2E \|X(\beta) \circ \Xi\|_B. \tag{4}
$$

Here the first equality holds because $\tilde{X}(\beta)$ is a copy of $X(\beta)$ whose elements have zero means, the second line holds by the Jensen’s inequality and the law of iterative expectation, and the third line holds because the distribution of $X(\beta) - \tilde{X}(\beta)$ is symmetric around zero and so $X(\beta) - \tilde{X}(\beta) = d (X(\beta) - \tilde{X}(\beta)) \circ \Xi$, and the last line holds by the triangle inequality.

From this inequality, w.l.o.g., we assume that the distribution of $x_{it}(\beta)$ is symmetric around zero, that is $x_{it}(\beta) = -x_{it}(\beta)$, so that $X(\beta) = d X(\beta) \circ \Xi$.

Then, we have

$$
E\|X(\beta) \circ G\|_B = E \| x_{it}(\beta) \xi_{it} \|_B \\
\geq E \left\| \left[ E_{x_{it}(\beta), \xi_{it}} (x_{it}(\beta) \xi_{it}) \|g_{it}\| \right] \right\|_B \\
= \sqrt{\frac{2}{\pi}} E \|X(\beta) \circ \Xi\|_B. \tag{5}
$$
From (4) and (5) we deduce the desired result of the lemma

\[ E \|X(\beta)\|_B \leq C E \|Z(\beta)\|_B, \tag{6} \]

where \( C \) is a finite constant.

\[ \square \]

**Proof of Theorem 1.** Let \( N_u \) and \( N_v \) be the \( \epsilon \)-nets of \((U, d_{u})\) and \((V, d_{v})\), respectively. Let \( N_u(U, d_{u}, \epsilon) \) and \( N_v(V, d_{v}, \epsilon) \) be the covering numbers of \((U, d_{u})\) and \((V, d_{v})\), respectively. Let \((\pi(u), \pi(v))\) be the element in the net product \( N_u \times N_v \) that is closest to \((u, v)\), so that \( \sup_{u \in U} \|u - \pi(u)\| \leq \epsilon \) and \( \sup_{v \in V} \|v - \pi(v)\| \leq \epsilon \).

Notice that

\[ \sup_{\beta \in B} \|Z(\beta)\| = \sup_{\beta \in B} \sup_{(u,v) \in U \times V} u' Z(\beta) v \]

\[ \leq \sup_{\beta \in B} \sup_{(u,v) \in U \times V} (u - \pi(u))' Z(\beta) v + \sup_{\beta \in B} \sup_{(u,v) \in U \times V} \pi(u)' Z(\beta)(v - \pi(v)) \]

\[ + \sup_{\beta \in B} \sup_{(u,v) \in U \times V} \pi(u)' Z(\beta) \pi(v) \]

\[ \leq \sup_{\beta \in B} \sup_{(u,v) \in U \times V} \|u - \pi(u)\| \|Z(\beta)\| \|v\| + \sup_{\beta \in B} \sup_{(u,v) \in U \times V} \|\pi(u)\| \|Z(\beta)\| \|v - \pi(v)\| \]

\[ + \sup_{\beta \in B} \sup_{(u,v) \in U \times V} \pi(u)' Z(\beta) \pi(v) \]

\[ \leq 2\epsilon \left( \sup_{\beta \in B} \|Z(\beta)\| \right) + \sup_{\beta \in B} \left( \max_{(u,v) \in N_u \times N_v} u' Z(\beta) v \right), \]

where the last line holds since \( \sup_{u \in U} \|u - \pi(u)\|, \sup_{v \in V} \|v - \pi(v)\| \leq \epsilon \). Therefore,

\[ E \left[ \sup_{\beta \in B} \|Z(\beta)\| \right] \leq \frac{1}{1 - 2\epsilon} E \left[ \max_{(u,v) \in N_u \times N_v} u' Z(\beta) v \right] \]

\[ \tag{7} \]

To find an upper bound of the right hand side of (7), we apply the chaining argument (to control \( \sup_{\beta \in B} \)) and the maximal inequality (to control \( \max_{(u,v) \in N_u \times N_v} \)).

For this, recall that \( d_{\beta}(\cdot, \cdot) \) is the distance defined on \( B \). We denote \( N_{\beta,k} \) as the \( 2^{-k} \)-net.
of \((B, d_β)\) such that \(N_{β,k} \subset N_{β,k+1}\), and \(N_β(B, d, 2^{-k})\) as the covering number. For each \(β \in N_{β,k}\), let \(π_k(β)\) be an element in \(β \in N_{β,k}\) such that

\[
d_β(β, π_k(β)) \leq 2^{-k}.
\]

Since the index set \(B\) is totally bounded with respect to the distance \(d_β(•, •)\), we can find the largest integer \(k_0\) such that \(2^{-k_0} \geq \text{diam}(B)\). By definition, any singleton, say \(\{β_0\} =: N_0\), is trivially a \(2^{-k_0}\)-net. Let \(π_0(β) := β_0\) for all \(β \in B\).

For \(k \geq k_0 + 1\), we have

\[
\sup_{β \in B} \left( \max_{(u,v) \in N_u \times N_v} u' Z(β) v \right)
\leq \sup_{β \in B} \left( \max_{(u,v) \in N_u \times N_v} u' Z(π_0(β)) v \right)
+ \sum_{k=k_0+1}^{n} \sup_{β \in B} \left( \max_{(u,v) \in N_u \times N_v} u' (Z(π_k(β)) - Z(π_{k-1}(β))) v \right)
+ \sup_{β \in B} \left( \max_{(u,v) \in N_u \times N_v} u' (Z(β) - Z(π_n(β))) v \right)
= \max_{(u,v) \in N_u \times N_v} u' Z(β_0) v
+ \sum_{k=k_0+1}^{n} \sup_{β \in N_{β,n}} \left( \max_{(u,v) \in N_u \times N_v} u' (Z(π_k(β)) - Z(π_{k-1}(β))) v \right)
+ \sup_{β \in B} \left( \max_{(u,v) \in N_u \times N_v} u' (Z(β) - Z(π_n(β))) v \right)
=: I + II + III,
\]

where the first equality holds since \(π_0(β) = β_0\) for all \(β \in B\) by definition.

**Case (i).** We start with the case where \(B\) is a finite set. Then, there exists a finite \(n\) such that \(N_{β,n} = B\). In this case, \(III = 0\) because \(N_{β,n} = B\) for some finite \(n\) when \(B\) is a finite set.

(Term I) For \(I\), for \((u_j, v_j) \in N_u \times N_v\), write \(S_j := u'_j Z(β_0) v_j\) for \(j = 1, ..., N(U, d_u, ε) N(V, d_v, ε)\).
Notice that conditional on \( X(\cdot) \), \( S_j \) is a normal random variable with mean zero and variance \( \sum_{i=1}^{N} \sum_{t=1}^{T} u_{j,i}^2 v_{j,t}^2 x_{it}^2(\beta_0) \), which is bounded by \( \max_{i,t} x_{it}^2(\beta_0) \) because \( \sum_{i=1}^{N} u_{j,i}^2 = \sum_{t=1}^{T} v_{j,t}^2 = 1 \). Therefore, for any \( \tau > 0 \),

\[
\mathbb{E}_{X(\cdot)}(I) = \mathbb{E}_{X(\cdot)} \left[ \max_j S_j \right]
\]

\[
\leq \frac{1}{\tau} \log \mathbb{E}_{X} \left[ \exp(\tau \max_j S_j) \right] \leq \frac{1}{\tau} \log \left( \sum_{j=1}^{N} \mathbb{E}_{X(\cdot)} \left[ \exp(\tau S_j) \right] \right)
\]

\[
= \frac{1}{\tau} \log \left( \sum_{j=1}^{N} \exp \left( \frac{\tau^2}{2} \sum_{i=1}^{N} \sum_{t=1}^{T} u_{j,i}^2 v_{j,t}^2 x_{it}^2(\beta_0) \right) \right)
\]

\[
\leq \frac{1}{\tau} \log \left( \sum_{j=1}^{N} \exp \left( \frac{\tau^2}{2} \max_{i,t} x_{it}^2(\beta_0) \right) \right)
\]

\[
= \frac{1}{\tau} \log \left( \max_{\beta \in \mathbb{N}_{\beta,n}} \mathbb{E} \left[ \max_{i,t} |x_{it}(\beta)| \right] \right) \frac{1}{\tau} \log N_{u}(U, d_u, \epsilon) + \frac{\tau}{2} \max_{i,t} x_{it}(\beta_0)
\]

Since the above inequality holds for all \( \tau > 0 \), we have

\[
\mathbb{E}_{X(\cdot)}(I) \leq \inf_{\tau > 0} \left[ \frac{1}{\tau} \log N(u, d_u, \epsilon) + \frac{\tau}{2} \max_{i,t} x_{it}(\beta_0) \right]
\]

\[
= \sqrt{2} \log N_{u}(U, d_u, \epsilon) + \log N_{v}(V, d_v, \epsilon) \max_{i,t} x_{it}(\beta_0).
\]

Therefore, we have

\[
\mathbb{E}(I) \leq \max_{\beta \in \mathbb{N}_{\beta,n}} \mathbb{E} \left( \max_{i,t} |x_{it}(\beta)| \right) \sqrt{2} \log N_{u}(U, d_u, \epsilon) + \frac{\tau}{2} \max_{i,t} x_{it}(\beta_0)
\]

\[
\leq \sqrt{\sigma^2(\log N + \log T) \log N_{u}(U, d_u, \epsilon) + \log N_{v}(V, d_v, \epsilon)},
\]

where the last inequality holds by (2).
(Term II) For $II$, we first consider

$$
\sup_{\beta \in N_{\beta,n}} \max_{(u,v) \in N_u \times N_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v.
$$

For $\beta \in N_{\beta,n}$ and $(u_j, v_j) \in N_u \times N_v$, conditioning on $X(\cdot)$, the distribution of $u_j'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v_j$ is Gaussian with mean zero and variance

$$
\sum_{i=1}^N \sum_{t=1}^T u_{j,i}^2 v_{j,t}^2 (x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta)))^2.
$$

By similar arguments used in bounding $E_{X(\cdot)}(I)$, we have

$$
E_{X(\cdot)} \left[ \sup_{\beta \in N_{\beta,n}} \max_{(u,v) \in N_u \times N_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v \right]
\leq \frac{1}{\tau} \log \left( \exp \left( \tau \left[ \sup_{\beta \in N_{\beta,n}} \max_{(u,v) \in N_u \times N_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v \right] \right) \right)
= \frac{1}{\tau} \log \left( \sum_{(\pi_k(\beta),\pi_{k-1}(\beta))} \sum_{\beta \in N_{\beta,n}} \exp \left( \tau \left[ \sum_{i=1}^N \sum_{t=1}^T u_{j,i}^2 v_{j,t}^2 (x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta)))^2 \right] \right) \right).
$$

\begin{align*}
&\leq \frac{1}{\tau} \log \left( N_{\beta}(N_{\beta,n}, d_{\beta}, 2^{-k})^2 \times N(U, d_u, \epsilon) \times N(V, d_v, \epsilon) \times \exp \left( \frac{\tau^2}{2} \sum_{i=1}^N \sum_{t=1}^T \max_{\beta \in N_{\beta,n}} (x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta)))^2 \right) \right) \\
&= \frac{1}{\tau} \left[ 2 \log N_{\beta}(N_{\beta,n}, d_{\beta}, 2^{-k}) + \log N(U, d_u, \epsilon) + \log N(V, d_v, \epsilon) \right] \\
&\quad + \frac{\tau}{2} \max_{\beta \in N_{\beta,n}} \left( \sum_{i=1}^N \sum_{t=1}^T (x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta)))^2 \right) \]}

where the last inequality holds since the number of $\{(\pi_k(\beta), \pi_{k-1}(\beta)) : \beta \in N_{\beta,n}\}$ is bounded by $N_{\beta}(N_{\beta,n}, d_{\beta}, 2^{-k}) \times N_{\beta}(N_{\beta,n}, d_{\beta}, 2^{-k+1}) \leq N_{\beta}(N_{\beta,n}, d_{\beta}, 2^{-k})^2$ and $\sum_{i=1}^N u_{j,i}^2 = \sum_{t=1}^T v_{j,t}^2 = 1.$
Notice that since \( x^2 \leq \exp(x^2) - 1 \) for all \( x \) and by Lemma \( \bullet \) under Assumption \( \blacksquare \) we can find a constant \( C \) such that

\[
\mathbb{E} \left( \max_{\beta \in \mathbb{N}_{\beta,n}} \max_{i,t} |x_{i,t}(\pi_k(\beta)) - x_{i,t}(\pi_{k-1}(\beta))| \right)^2 
\leq C' 2^{-2k} \left( 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N + \log T \right).
\]

Therefore, for all \( \tau > 0 \), we have

\[
\mathbb{E} \left[ \sup_{\beta \in \mathbb{N}_{\beta,n}} \max_{(u,v) \in \mathbb{N}_u \times \mathbb{N}_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v \right]
\leq \frac{1}{\tau} \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N(\mathbb{U}, d_u, \epsilon) + \log N(\mathbb{V}, d_v, \epsilon) \right]
\quad \quad \quad \quad \quad \quad \quad \quad + C' \tau 2^{-2k} \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N + \log T \right].
\]

By minimizing the above upper bound with respect to \( \tau > 0 \), we can bound

\[
\mathbb{E} \left[ \sup_{\beta \in \mathbb{N}_{\beta,n}} \max_{(u,v) \in \mathbb{N}_u \times \mathbb{N}_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v \right]
\leq C' 2^{-k} \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N(\mathbb{U}, d_u, \epsilon) + \log N(\mathbb{V}, d_v, \epsilon) \right]^{1/2}
\times \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N + \log T \right]^{1/2}.
\]

Therefore,

\[
\mathbb{E}(II) = \sum_{k=k_0+1}^{n} \mathbb{E} \left[ \sup_{\beta \in \mathbb{N}_{\beta,n}} \max_{(u,v) \in \mathbb{N}_u \times \mathbb{N}_v} u'(Z(\pi_k(\beta)) - Z(\pi_{k-1}(\beta)))v \right]
\leq \sum_{k=k_0+1}^{n} 2^{-k} \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N(\mathbb{U}, d_u, \epsilon) + \log N(\mathbb{V}, d_v, \epsilon) \right]^{1/2}
\times \left[ 2 \log N_{\beta}(\mathbb{N}_{\beta,n}, d_{\beta}, 2^{-k}) + \log N + \log T \right]^{1/2}. \quad (10)
\]
Combining the bounds of (9) and (10), when $B = N_{\beta,n}$ for some finite $n$, we have

$$
\mathbb{E} \left[ \sup_{\beta \in N_{\beta,n}} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) \right] \\
\preceq \sigma (\log N + \log T)^{1/2} (\log N_u(U, d_u, \epsilon) + \log N_v(V, d_v, \epsilon))^{1/2} \\
+ \sum_{k=k_0+1}^n 2^{-k} \left[ 2 \log N_\beta(N_{\beta,n}, d_\beta, 2^{-k}) + \log N(U, d_u, \epsilon) + \log N(V, d_v, \epsilon) \right]^{1/2} \\
\times \left[ 2 \log N_\beta(N_{\beta,n}, d_\beta, 2^{-k}) + \log N + \log T \right]^{1/2}.
$$

**Case (ii).** Suppose that the number of the elements of $B$ is infinity. Since the stochastic process is $x_{it}(\beta)$ is separable (Assumption 1(i)), there are finitely many elements in $Z(\beta)$, and the sets $N_u$ and $N_v$ are finite, we can find a countable set $B^*$ such that

$$
\sup_{\beta \in B} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) = \sup_{\beta \in B^*} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) \text{ a.s.}
$$

Denote $B_n$ as the first $n$ elements of $B^*$ in arbitrary order. Then, by the monotone convergence theorem, we have

$$
\mathbb{E} \left[ \sup_{\beta \in B} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) \right] = \mathbb{E} \left[ \sup_{\beta \in B^*} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) \right] \\
= \sup_{n \geq 1} \mathbb{E} \left[ \sup_{\beta \in B_n} \left( \max_{(u,v) \in N_u \times N_v} u'Z(\beta)v \right) \right].
$$

Then, applying the bound of Case (i) together with $N_\beta(B_n, d_\beta, \epsilon) \leq N_\beta(B, d_\beta, \epsilon)$ for all
$\epsilon > 0$, we can bound
\[
\mathbb{E} \left[ \sup_{\beta \in B_n} \left( \max_{(u,v) \in N_u \times N_v} u' Z(\beta) v \right) \right] \\
\lesssim \sigma (\log N + \log T)^{1/2} (\log N_u(U, d_u, \epsilon) + \log N_v(V, d_v, \epsilon))^{1/2} \\
+ \int_0^{\text{diam}(B)} [2 \log N_\beta(B, d, \nu) + \log N(U, d_u, \epsilon) + \log N(V, d_v, \epsilon)]^{1/2} \\
\times [2 \log N_\beta(B, d, \nu) + \log N + \log T]^{1/2} d\nu.
\]

According to Lemma 5.13 of van Handel (2016), the covering numbers $N_u(U, d_u, \epsilon)$ and $N_v(V, d_v, \epsilon)$ are bounded by

\[
N_u(U, d_u, \epsilon) \leq \left( \frac{3}{\epsilon} \right)^N, \quad N_v(V, d_v, \epsilon) \leq \left( \frac{3}{\epsilon} \right)^T.
\]

Then, we have
\[
\mathbb{E} \left[ \sup_{\beta \in B_n} \left( \max_{(u,v) \in N_u \times N_v} u' Z(\beta) v \right) \right] \\
\lesssim \log(\max(N, T)) \sqrt{\max(N, T)} \\
+ \int_0^{\text{diam}(B)} \log N_\beta(B, d, \nu) d\nu \\
+ \sqrt{\max(N, T)} \int_0^{\text{diam}(B)} \sqrt{\log N_\beta(B, d, \nu)} d\nu.
\]

Combining the above bound with (7), we deduce the desired bound for the theorem.

\[\square\]

Define $\psi_p(x) := \exp(x^p) - 1$. For random variable $X$, let $\|X\|_{\psi_p} := \inf \left\{ C > 0 : \mathbb{E} \left[ \psi_p \left( \frac{|X|}{C} \right) \right] \leq 1 \right\}$, the $L_{\psi_p}$-Orlicz norm of $X$.

**Lemma 2.** Suppose that Assumption 1 holds. Consider the $\pi_k(\beta)$ as defined in (8). Then,
there exists a finite constant $C$ such that

$$
\left\| \max_{\beta \in \mathcal{B}} \max_{1 \leq i \leq N, 1 \leq t \leq T} |x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta))| \right\|_{\psi_2} 
\leq C 2^{-k} \sqrt{2 \log N_{\beta}(B, d_\beta, 2^{-k}) + \log N + \log T}.
$$

Proof of Lemma 2. Notice that the number of $\{(\pi_k(\beta), \pi_{k-1}(\beta)) : \beta \in \mathcal{B}\}$ is bounded by $N_{\beta}(B, d_\beta, 2^{-k}) \times N_{\beta}(B, d_\beta, 2^{-k+1}) \leq N_{\beta}(B, d_\beta, 2^{-k})^2$. By Lemma 2.2.2 of Van Der Vaart and Wellner (1996), we have

$$
\left\| \max_{\beta \in \mathcal{B}} \max_{1 \leq i \leq N, 1 \leq t \leq T} |x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta))| \right\|_{\psi_2} 
\leq C \psi_2^{-1}(N_{\beta}(B, d_\beta, 2^{-k})^2 NT) \max_{\beta \in \mathcal{B}} \max_{1 \leq i \leq N, 1 \leq t \leq T} \|x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta))\|_{\psi_2}.
$$

By definition $\psi_2^{-1}(n) = \sqrt{\log n + 1} \leq 2 \sqrt{\log n}$ for $n \geq 2$. Also, since $x_{it}(\beta)$ is a sub-Gaussian process with respect to a pseudo-metric $d_\beta(\cdot, \cdot)$, by Lemma 2.2.1 of Van Der Vaart and Wellner (1996) we have

$$
\|x_{it}(\pi_k(\beta)) - x_{it}(\pi_{k-1}(\beta))\|_{\psi_2} \leq \sqrt{6} d_\beta(\pi_k(\beta), \pi_{k-1}(\beta)) 
\leq d_\beta(\pi_k(\beta), \beta) + d_\beta(\beta, \pi_{k-1}(\beta)) 
\leq 3 \times 2^{-k}
$$

Therefore, we have the required result for the lemma.

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