1. Introduction.

In the 90’s Klaus Altmann studied deformations of affine toric varieties in a series of papers (see [Al1]-[Al5]). His main construction in [Al2] combinatorially described the situation when one affine toric variety is embedded in a higher dimensional affine toric variety as a complete intersection. The embedding of an affine toric variety associated with a convex cone $\sigma$ corresponds to a Minkowski sum decomposition of a polyhedral slice of $\sigma$, which is obtained as an intersection of the cone $\sigma$ with a hyperplane. There is a natural grading on the coordinate ring of the ambient affine toric variety coming from a projection of one lattice onto another one, and the equations of the complete intersection were assumed to have the same degree with respect to this grading. This is why the deformations of these complete intersections were called “homogeneous” deformations of the affine toric variety in [Al2]. Altmann’s work with some insight from a question of Bernd Sturmfels was ingenious to realize that Minkowski sums of polyhedra are related to deformations of affine toric varieties. However, homogeneous deformations of an affine toric variety in [Al2] did not produce all the unobstructed directions of the infinitesimal space of deformations of the toric variety by Kodaira-Spencer map. A later construction in [Al5] was supposed (we assume) to produce deformations so that infinitesimally they will span all the unobstructed deformations, but that construction was “non-toric”.

In the present paper we first generalized Altmann’s construction of deformations of an affine toric variety as complete intersections in another affine toric variety using homogeneous coordinates due to David Cox (see [C]). We expect that by

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Kodaira-Spencer map they should span all the unobstructed directions of the infinitesimal space of deformations. Altmann’s “homogeneous” toric deformations are described by complete intersections where all the defining equations have the same degree using the usual definition of degrees of homogeneous coordinates from [C]. This allowed us to extend the construction of deformations to arbitrary toric varieties by introducing Minkowski sum decompositions of polyhedral complexes. Our previous construction of deformations of blow ups of Fano toric varieties in [M1, M2] is a particular case of the current one.

As it was outlined in our previous work [M2], we expect that all unobstructed deformations of Calabi-Yau complete intersections in toric varieties must be realized as complete intersections in higher dimensional toric varieties. Since for Calabi-Yau hypersurfaces these deformations were induced by deformations of the ambient toric varieties, which were also realized by complete intersections in higher dimensional toric varieties, this motivated our study of deformations of toric varieties.

When we first learned about Altmann’s work in 2004 from a referee to [M1], his work seemed like a completely different story and we did not study it for a while. But then we realized that we should combine deformations of affine toric varieties with the deformations arising from regluing of affine toric charts similar to the construction of [M1]. It is not easy if not possible to describe deformations of abstract algebraic varieties via regluing of affine open charts. So, a more common approach is to embed a given variety into another one as a complete intersection and then deform the defining equations as it was done in [M2] and [Al2]. This approach is also not easy as many such embeddings will only lead to trivial deformations, and a priori we do not know which embedding will lead to nontrivial ones. However, the similarity of the construction of [M2], which was obtained through working out some examples of deformations of Calabi-Yau hypersurfaces in weighted projective spaces similar to [COFKM], and Altmann’s construction in [Al2] has led us to finding some compatible language, which turned out to be via the homogeneous coordinates of toric varieties. This explains why we had to redo everything in [Al2] in terms of homogeneous coordinates.

It is not common to use homogeneous coordinates on affine toric varieties, but this approach produced more deformations of affine toric varieties than Altmann’s construction. Namely, we found deformations of an affine toric variety realized as complete intersections in another affine toric variety but the defining equations are in terms of homogeneous coordinates, and, in general, the embedded variety is not a complete intersection in terms of affine coordinate functions. This feature is very similar to the situation of a non-Cartier hypersurface in a weighted projective space. Globally it is given by one equation, and so it is a complete intersection in homogeneous coordinates even if we restrict to an affine open chart of the ambient space. However, since the degree is non-Cartier, the restriction of the hypersurface to some affine chart is not a complete intersection in terms of the coordinate functions of the chart. After the generalization of Altmann’s construction for affine toric varieties, we just needed to translate our fan construction (which made the embedding) from [M2] into the language of Minkowski sums in order to obtain a more general construction via Minkowski sum decompositions of polyhedral complexes.

This preprint is a work in progress.

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2. Deformations of affine toric varieties.

First, let us describe the construction of Altmann after some modification. Let $N$ be a lattice and $M$ be its dual lattice. An affine toric variety $X_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ corresponds to a rational convex polyhedral cone $\sigma \subset N_\mathbb{R}$ with apex at 0, where $\sigma^\vee$ is the dual cone. We will view elements of the semigroup ring as finite sums $\sum_{m \in \sigma^\vee \cap M} a_m \chi^m$ where $a_m \in \mathbb{C}$ are complex coefficients and $\chi^m$ are elements of the ring corresponding to lattice points $m$. Take $R \in \sigma^\vee \cap M$ and let

$$\sigma(R) = \sigma \cap \{n \in N_\mathbb{R}|(R, n) = 1\}$$

(the slice of $\sigma$ in the hyperplane). Let $\sigma(R)^c$ be the compact part of the polyhedra $\sigma(R)$. This is just the convex hull of the vertices of the possibly unbounded polyhedra. Then decompose $\sigma(R)^c = Q_0 + Q_1 + \cdots + Q_k$ into a Minkowski sum of polytopes such that the following two conditions are satisfied:

(*) the induced decomposition of a vertex of $\sigma(R)^c$

$$\text{vert}(\sigma(R)^c) = \text{vert}(Q_0) + \text{vert}(Q_1) + \cdots + \text{vert}(Q_k)$$

into the sum of vertices of $Q_0, Q_1, \ldots, Q_k$ has all but possibly one of the summands $\text{vert}(Q_0), \text{vert}(Q_1), \ldots, \text{vert}(Q_k)$ being lattice points,

(*** $Q_0 \subset \{n \in N_\mathbb{R}|(R, n) = 1\}$, $Q_i \subset \{n \in N_\mathbb{R}|(R, n) = 0\}$.

Consider the lattice $\tilde{N} = N \oplus \mathbb{Z}^k$ denoting by $\{e_1, \ldots, e_k\}$ the standard basis for the second component. Let $\{e_1, \ldots, e_k\}$ be the dual to $\{e_1, \ldots, e_k\}$ basis for the second component. Let the cone

$$\tilde{\sigma} = (\sigma, Q_0 - e_1 - \cdots - e_k, Q_1 + e_1, \ldots, Q_k + e_k)$$

in $\tilde{N}_\mathbb{R}$ be generated by the indicated sets.

The inclusion of cones $\sigma \subset \tilde{\sigma}$ induces a ring homomorphism $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \to \mathbb{C}[\sigma^\vee \cap M]$ corresponding to a morphism of toric varieties $X_\sigma \to X_{\tilde{\sigma}} = \text{Spec}(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}])$.

**Lemma 2.1.** The ring homomorphism $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \to \mathbb{C}[\sigma^\vee \cap M]$ is surjective.

**Proof.** This ring homomorphism is induced by the natural projection of lattices $\tilde{M} = M \oplus \mathbb{Z}^k \to M$ and it is sufficient to show that the map of the semigroups $\tilde{\sigma}^\vee \cap \tilde{M} \to \sigma^\vee \cap M$ is surjective. Let $s \in \sigma^\vee \cap M$ and denote by $\min(s, Q_i)$ the minimal value of vector $s$ on polytope $Q_i$ (this is the same as the value of the support function of polytope $Q_i$ on vector $s$ as defined in [F, p. 68]). Also, denote by $\lfloor \min(s, Q_i) \rfloor$ the greatest integer value of the corresponding real number. Then we claim that the lattice point

$$\tilde{s} = s - \sum_{i=1}^{k} \lfloor \min(s, Q_i) \rfloor e_i^*$$
is in $\hat{\sigma}^\vee \cap \hat{M}$, whence the statement follows. We can check this claim on the generators of $\hat{\sigma}$. If $n \in \sigma$, then $\langle \hat{s}, n \rangle = \langle s, n \rangle \geq 0$ since $s \in \sigma^\vee$. If $q_i \in Q_i$ for $i \geq 1$, then $\langle \hat{s}, q_i + e_i \rangle = \langle s, q_i \rangle - [\min(s, Q_i)] \geq 0$. If $q_0 \in Q_0$, then

$$\langle \hat{s}, q_0 - \sum_{i=1}^k c_i e_i \rangle = \langle s, q_0 + \sum_{i=1}^k [\min(s, Q_i)] \rangle = \sum_{i=0}^k [\min(s, Q_i)] = [\min(s, \sigma(R^c))] \geq 0.$$  

The last equality is true because all but possibly one of $\min(s, Q_i)$ are integers by the condition $(\ast)$ on the Minkowski decomposition and the sum of support functions of polytopes is the support function of the sum of polytopes.

$\Box$

**Remark 2.2.** An analog of this Lemma in a complicated setting was proved in (4.1.1) in [Al2, p. 162] but we found this elegant proof of it.

The above lemma tells us that we have an embedding of the toric varieties $X_\sigma \hookrightarrow X_{\hat{\sigma}}$. The next one shows the defining equations.

**Proposition 2.3.** The ideal $\ker(C[\hat{\sigma}^\vee \cap \hat{M}] \to C[\sigma^\vee \cap M])$ is a complete intersection generated by the regular sequence $\chi_{R^c+e_j^*} - \chi_R^R, \ldots, \chi_{R^c+e_k^*} - \chi_R^R$.

**Proof.** While our setting is slightly different from that of Altmann, the proof of this result is essentially the same as in [Al2, p. 162-163] and we will repeat it in Proposition 2.6 generalizing this result. The regularity of the sequence follows from the fact that this is a complete intersection given by “homogeneous” elements where degree is induced by the natural projection $\deg : \hat{M} \to M$, so that $\deg(\chi_{R^c+e_j^*}) = \cdots = \deg(\chi_{R^c+e_k^*}) = \deg(\chi_R^R) = R \in M$.

$\Box$

**Remark 2.4.** Condition $(\ast\ast)$ on the Minkowski sum decomposition was necessary only to fix the defining equations of $X_\sigma$ inside $X_{\hat{\sigma}}$. It was used to present the construction in [Al5]. If we drop this condition, we worked out that $R$ must be replaced by $R - \sum_{i=1}^k \langle R, Q_i \rangle e_i^* e_i$ in the equations. Note that in this case, for $i \neq 0$,

$$\langle R - \sum_{i=1}^k \langle R, Q_i \rangle e_i^*, Q_i + e_i \rangle = 0,$$

and

$$\langle R - \sum_{i=1}^k \langle R, Q_i \rangle e_i^*, Q_0 - e_1 - \cdots - e_k \rangle = \sum_{i=0}^k \langle R, Q_i \rangle = \langle R, \sigma(R^c) \rangle = 1.$$  

These are the same equalities if we left condition $(\ast\ast)$ and used $R$ instead of $R - \sum_{i=1}^k \langle R, Q_i \rangle e_i^*$. For the sake of simplicity we will keep condition $(\ast\ast)$ in this section and also in Section 4.

Thus we have our toric variety $X_\sigma$ embedded into another toric variety $X_{\hat{\sigma}}$ as a complete intersection given by $\chi_{R^c+e_j^*} - \chi_R^R$ for $j = 1, \ldots, k$. A $k$-parameter embedded deformation of $X_\sigma$ corresponds to the following natural diagram:

$$\begin{align*}
X_\sigma & \subset \mathcal{X} \subset X_{\hat{\sigma}} \times \mathbb{C}^k \\
\downarrow & \downarrow \quad \swarrow \\
\{0\} & \subset \mathbb{C}^k.
\end{align*}$$  

(2)
where the deformation family $X$ in $X_\sigma \times \mathbb{C}^k$ is given by $\chi^{R+e_i^*} - \chi^R - \lambda_j$, $j = 1, \ldots, k$, with $\lambda_j$ being coordinates on $\mathbb{C}^k$.

**Remark 2.5.** In [Al2, Al5], the deformation family of $X_\sigma$ was described differently by the map $X_\sigma \to \mathbb{C}^k$ given by the values of the functions $(\chi^{R+e_i^*} - \chi^R, \ldots, \chi^{R+e_i^*} - \chi^R)$ with the special fiber over the origin isomorphic to $X_\sigma$. But the total space $X_\sigma$ of this family is isomorphic to the above $X$.

By using homogeneous coordinates on the affine toric variety we found the following generalization of Altmann’s deformations of $X_\sigma$ using an arbitrary $R \in M$. As we described the above construction, notice that there is no problem to proceed with the construction of $\tilde{\sigma}$ even if $R \notin \sigma^\vee \cap M$. We still get the embedding $X_\sigma \hookrightarrow X_\tilde{\sigma}$. But the image will no longer be a complete intersection in terms of affine functions on $X_\tilde{\sigma}$ unless there is another $R' \in \sigma^\vee \cap M$ for which $\sigma(R') = \sigma(R)$, which is not possible for singular $X_\sigma$. We also note that $\chi^R, \chi^{R+e_i^*}, \ldots, \chi^{R+e_i^*}$ will not be functions on $X_\tilde{\sigma}$ if $R \notin \sigma^\vee \cap M$.

The crucial idea to generalize Altmann’s construction is to use the categorical quotient presentation of a toric variety from the work [C] of David Cox. For an affine toric variety we have presentation

$$\text{Spec}(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]) \simeq \mathbb{C}^l / G = \text{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_l]^G),$$

where $l$ is the number of edges of the cone $\tilde{\sigma}$ and $x_1, \ldots, x_l$ are the corresponding homogeneous coordinates of the the affine toric variety $X_\tilde{\sigma}$, and the group

$$G = \{(\mu_1, \ldots, \mu_l) \in \mathbb{C}^l | \prod_{j=1}^l \mu_j^{u_j} = 1 \ \forall \ u \in M\},$$

where $v_j$ is the primitive lattice generator of the edge of $\tilde{\sigma}$ corresponding to $x_j$, acts on $\mathbb{C}^l$ by multiplication. On the level of rings this corresponds to isomorphism

$$\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \simeq \mathbb{C}[x_1, x_2, \ldots, x_l]^G,$$

which sends $\chi^u$ to $\prod_{j=1}^l x_j^{u_j}$. In the future we will denote the last monomial by $x^u$.

**Proposition 2.6.** There is a natural ring isomorphism

$$\mathbb{C}[\sigma^\vee \cap M] \simeq (\mathbb{C}[x_1, \ldots, x_l]/I)^G,$$

where the ideal $I$ is generated by $(\chi^{R+e_i^*} - \chi^R) \prod_{(R,v_j)<0} x_j^{-(R,v_j)}$ for $i = 1, \ldots, k$.

**Proof.** Let $I'$ denote the $\ker(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \to \mathbb{C}[\sigma^\vee \cap M])$. Then

$$\mathbb{C}[\sigma^\vee \cap M] \simeq \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]/I'.$$

It is easy to find that the ideal $I'$ is generated by binomials:

$$I' = \langle \chi^r - \chi^s | r, s \in \tilde{\sigma}^\vee \cap \tilde{M}, r - s \in \ker(\tilde{M} \to M) \rangle.$$

Now consider a ring homomorphism $\phi : \mathbb{C}[x_1, \ldots, x_l]^G \to (\mathbb{C}[x_1, \ldots, x_l]/I)^G$, assigning $f \mapsto f + I$. It suffices to show that

$$\ker(\phi) = \langle x^r - x^s | r, s \in \tilde{\sigma}^\vee \cap \tilde{M}, r - s \in \ker(\tilde{M} \to M) \rangle.$$


Now we have to go along the steps of [Al2, p. 162-163]. Suppose \( r,s \in \tilde{\sigma} \cap \tilde{M} \) and \( r-s \in \ker(\tilde{M} \rightarrow M) \), then we can write \( r-s = \sum_{i=1}^{k} \alpha_i^+ e_i^* - \sum_{i=1}^{k} \alpha_i^- e_i^* \) with \( \alpha_i^+ \cdot \alpha_i^- = 0 \) and \( \alpha_i^+, \alpha_i^- \in \mathbb{Z}_{\geq 0} \) for all \( i \). Without loss of generality assume \( \sum_{i=1}^{k} \alpha_i^+ \geq \sum_{i=1}^{k} \alpha_i^- \).

Denote

\[
q = r - \sum_{i=1}^{k} \alpha_i^+ (R + e_i^*) = s - \sum_{i=1}^{k} \alpha_i^- (R + e_i^*) + \sum_{i=1}^{k} (\alpha_i^- - \alpha_i^+) R,
\]

which is in \( \tilde{\sigma} \cap \tilde{M} \) by an easy check restricting one of the presentations of \( q \) to the generators of the cone \( \tilde{\sigma} \). Hence,

\[
x^r - x^s = x^q \left( \sum_{i=1}^{k} \alpha_i^+ (R + e_i^*) \right) = x^q \left( \sum_{i=1}^{k} \alpha_i^- (R + e_i^*) \right) = \sum_{i=1}^{k} (\alpha_i^+ - \alpha_i^-) R.
\]

Now \( x^q \) is divisible by the monomial

\[
\prod_{(R,v_j) < 0} x_j^{-\sum_{i=1}^{k} \alpha_i^+ R,v_j} = \left( \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} \right)^{\sum_{i=1}^{k} \alpha_i^+}.
\]

Hence, we get \( x^r - x^s \in I \), showing that these lie in \( \ker(\phi) \). The other inclusion can be shown straightforward.

The above proposition means

**Corollary 2.7.** Affine toric variety \( X_\sigma \) is embedded into toric variety \( X_\phi \) as a complete intersection given by \( (x^{R+e_i^*} - x^R) \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} \) for \( i = 1, \ldots, k \).

**Remark 2.8.** If \( R \) is not in \( \sigma^* \cap M \), the ideal \( I' \) defining \( X_\sigma \) in \( X_\phi \) need not be a complete intersection, but using homogeneous coordinates we can still represent our toric variety \( X_\sigma \) as a complete intersection in \( X_\phi \), providing us more opportunities to deform the affine toric variety.

**Remark 2.9.** The binomial \( (x^{R+e_i^*} - x^R) \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} \) is actually independent of \( R \), i.e., for different \( R \)'s giving the same compact part \( \sigma(R)^c \) of the slice we get the same binomial. To see this note that the positive powers of \( x_j \) in this binomial appear for \( v_j \) that are multiples of vertices of \( Q_0 - e_1 - \cdots - e_k \) and \( Q_i + e_i \). In particular, we have

\[
(x^{R+e_i^*} - x^R) \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} = \prod_{(e_i^*,v_j) > 0} x_j^{(e_i^*,v_j)} - \prod_{(e_i^*,v_j) < 0} x_j^{-(e_i^*,v_j)},
\]

where the exponents are the “distances” of the lattice points \( v_j \) with respect to \( e_i^* \) from the linear subspace \( N \) inside \( \hat{N} \).

The \( k \)-parameter embedded deformation of \( X_\sigma \) in \( X_\phi \) corresponds to the same diagram as in (2) but given by the equations in homogeneous coordinates:

\[
\prod_{(e_i^*,v_j) > 0} x_j^{(e_i^*,v_j)} - \prod_{(e_i^*,v_j) < 0} x_j^{-(e_i^*,v_j)} + \lambda_i \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} = 0
\]
for \( i = 1, \ldots, k \). This deformation family we again denote by \( \mathcal{X} \) and since it depends on the lattice point \( R \in M \) and the Minkowski sum decomposition \( \sigma(R)^c = Q_0 + Q_1 + \cdots + Q_k \) of its slice we will often refer to this family as \( \mathcal{X}_{R,Q_0,Q_1,\ldots,Q_k} \).

**Remark 2.10.** There was a construction of deformations in (3.5) of [Al15] using an arbitrary \( R \in M \), but the ambient space of the deformation was not a toric variety.

Next we compute the Kodaira-Spencer map for the family \( \mathcal{X}_{R,Q_0,Q_1,\ldots,Q_k} \). The space \( T^1_{\mathcal{X}_\sigma} \) of infinitesimal deformations of \( X_\sigma \) was calculated by K. Altmann in [Al11]. It is \( M \)-graded:

\[
T^1_{\mathcal{X}_\sigma} = \bigoplus_{R \in M} T^1_{\mathcal{X}_\sigma}(-R),
\]

where the graded piece can be described as follows. Let \( a_1, \ldots, a_N \) be the primitive lattice generators of edges of \( \sigma \) and \( S = \{ s_1, \ldots, s_w \} \) be the set of generators of \( \sigma^\vee \cap M \). Denote \( S^R = \{ s \in S | \langle s, a_l \rangle < \langle R, a_l \rangle \} \) for \( l = 1, \ldots, N \). Then by Theorem in [Al11, p. 243]

\[
T^1_{\mathcal{X}_\sigma}(-R) = \left( L\left( \bigcup_{l=1}^N S^R_l \right) / \sum_{l=1}^N L(S^R_l) \right) ^* \otimes_{\mathbb{R}} \mathbb{C},
\]

where \( L(\ldots) \) denotes the \( \mathbb{R} \)-vector space of linear dependencies of the corresponding subset. All of the vector spaces \( L(\ldots) \) in the formula are naturally embedded into \( L(S) \), whose elements can be represented by the \( w \)-tuples of real coefficients of the relations.

**Theorem 2.11.** For the family \( \mathcal{X}_{R,Q_0,Q_1,\ldots,Q_k} \rightarrow \mathbb{C}^k \), the Kodaira-spencer map

\[
\rho_0 : T_{\mathbb{C}^k,0} \rightarrow T^1_{\mathcal{X}_\sigma}
\]

sends the basis vector \( \partial / \partial \lambda_i \) to the element in \( T^1_{\mathcal{X}_\sigma}(-R) \) induced by the map

\[
L\left( \bigcup_{l=1}^N S^R_l \right) \rightarrow \mathbb{R}, \quad (c_1, \ldots, c_w) \mapsto -\sum_{j=1}^w [\min(\langle s_j, Q_i \rangle)] c_j.
\]

**Proof.** Classification of infinitesimal deformations of an affine algebraic variety is well described in [E]. We will follow its procedure, however computation of the Kodaira-spencer map in the case \( R \in \sigma^\vee \cap M \) was done in [Al11, pp. 167-171] with a slightly different description.

Without loss of generality, assume \( i = 1 \). By [H, p. 80] the tangent vector \( \partial / \partial \lambda_1 \) to \( \mathbb{C}^k \) at 0 corresponds to the ring homomorphism \( \mathbb{C}[\lambda_1, \ldots, \lambda_k] \rightarrow \mathbb{C}[\varepsilon] / (\varepsilon^2) \), assigning \( \lambda_1 \mapsto \varepsilon, \lambda_i \mapsto 0 \), for \( i \neq 1 \). By the base change the infinitesimal deformation of \( X_\sigma \) corresponding to \( \partial / \partial \lambda_1 \) is the family \( \mathcal{X}' = \mathcal{X} \times_{\mathbb{C}^k} \text{Spec}(\mathbb{C}[\varepsilon] / (\varepsilon^2)) \rightarrow \text{Spec}(\mathbb{C}[\varepsilon] / (\varepsilon^2)) \). The first step to classify this infinitesimal deformation is to present it as an embedded infinitesimal deformation in an affine space. For this reason embed \( X_\sigma \) in the affine space \( \mathbb{C}^w \) by the surjective ring homomorphism \( \mathbb{C}[z_1, \ldots, z_w] \rightarrow \mathbb{C}[\sigma^\vee \cap M] \), sending \( z_j \mapsto \chi^{s_j} \). Denote by \( J \) the kernel of this ring homomorphism, then \( \mathbb{C}[\sigma^\vee \cap M] \simeq \mathbb{C}[z_1, \ldots, z_w] / J \). The infinitesimal deformation \( \mathcal{X}' \) of \( X_\sigma \) corresponds to the ring \( \hat{A} = \left( (\mathbb{C}[x_1, \ldots, x_l] \otimes \mathbb{C}[\varepsilon] / (\varepsilon^2)) / I_\varepsilon \right)^G \), where ideal \( I_\varepsilon \) is generated by

\[
(x^{R+\varepsilon} - x^R) \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} + \varepsilon \prod_{(R,v_j) < 0} x_j^{-\langle R,v_j \rangle} = 0.
\]
and by \((x^{R+e_i^*} - x^{R}) \prod_{\langle R,e_i \rangle < 0} x_j^{-(R,e_i)}\) for \(i \neq 1\). Associated to the embedding 

\[ X_\sigma \hookrightarrow X' \]

we have a surjective ring homomorphism \(\tilde{A} \rightarrow \mathbb{C}[\sigma' \cap M].\) We lift generator \(\chi_{s_j}\) of \(\mathbb{C}[\sigma' \cap M]\) by this surjective map to the coset of \(x^{s_j}\), where \(s_j = s_j - \sum_{i=1}^{k} [\min(s_j, Q_i)] e_i\). From the proof of Lemma 2.1, \(s_j \in \delta' \cap M.\) Then by [E, p. 414], we get a surjective homomorphism

\[ \mathbb{C}[z_1, \ldots, z_w] \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \rightarrow \tilde{A}, \]

(4)

sending \(z_j\) to the coset of \(x^{s_j}\), whose kernel we denote by \(\tilde{J}.\) From [E, p. 178] we know that \(\tilde{J}/J\varepsilon\) corresponds to the graph of a homomorphism 

\[ \psi \in \text{Hom}(J, \mathbb{C}[\sigma' \cap M]) \simeq \text{Hom}(J/J^2, \mathbb{C}[\sigma' \cap M]) \]

representing an element of 

\[ T^1_{X_\sigma} = \text{Hom}(J/J^2, \mathbb{C}[\sigma' \cap M])/\text{Hom}(\mathbb{C}[\sigma' \cap M], \mathbb{C}[\sigma' \cap M]). \]

In practice, we take a binomial generator \(z^a - z^b\) of ideal \(J\), where \(a = (a_1, \ldots, a_w)\), \(b = (b_1, \ldots, b_w) \in \mathbb{Z}_{\geq 0}^w\) with \(\sum_{j=1}^{w} (a_j - b_j)s_j = 0\), lift it to \(\mathbb{C}[z_1, \ldots, z_w] \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2)\) and then map it by (4) to the coset of \(x^{\sum_{j=1}^{w} a_j s_j} - x^{\sum_{j=1}^{w} b_j s_j}\) in \(\tilde{A}\). Since \(\sum_{j=1}^{w} a_j s_j - \sum_{j=1}^{w} b_j s_j \in \ker(M \rightarrow M)\), we can write \(\sum_{j=1}^{w} a_j s_j - \sum_{j=1}^{w} b_j s_j = \sum_{i=1}^{k} \alpha_i e_i^* - \sum_{i=1}^{k} \alpha_i e_i^*\) with \(\alpha_i^* \cdot \alpha_i^* = 0\) and \(\alpha_i^*, \alpha_i^* \in \mathbb{Z}_{\geq 0}\) for all \(i\). Denote 

\[ q = \sum_{j=1}^{w} a_j s_j - \sum_{i=1}^{k} \alpha_i^* (R + e_i^*) = \sum_{j=1}^{w} b_j s_j - \sum_{i=1}^{k} \alpha_i^* (R + e_i^*) + \sum_{i=1}^{k} (\alpha_i - \alpha_i^*) R, \]

where without losing generality we assume \(\sum_{i=1}^{k} \alpha_i^* \geq \sum_{i=1}^{k} \alpha_i^*\) and \(\alpha_1^* = 0\). Then in the ring \(\tilde{A}\): 

\[ x^{\sum_{j=1}^{w} a_j s_j} - x^{\sum_{j=1}^{w} b_j s_j} = x^q(x^{\sum_{i=1}^{k} \alpha_i^* (R + e_i^*)} - x^{\sum_{i=1}^{k} \alpha_i^* (R + e_i^*)} x^{\sum_{i=1}^{k} (\alpha_i - \alpha_i^*) R}) = x^q(x^{\sum_{i=1}^{k} \alpha_i^* R} (x^{R} + \varepsilon)^{\alpha_i^*} - x^{\sum_{i=1}^{k} \alpha_i^* R}) = \varepsilon^{\alpha_1^*} x^q(x^{R} x^{\sum_{i=1}^{k} \alpha_i^*})^{-1}. \]

(5)

Dividing this element by \(\varepsilon\) and mapping by \(\tilde{A} \rightarrow \mathbb{C}[\sigma' \cap M]\) we get the image of \(z^a - z^b\) by \(\psi\) is 

\[ \alpha_1^* (\chi^R) (\sum_{i=1}^{k} \alpha_i^*)^{-1} = \varphi(a - b) \chi^{\sum_{j=1}^{w} a_j s_j - R}, \]

where \(\varphi(c_1, \ldots, c_w) = -\sum_{j=1}^{w} [\min(s_j, Q_i)] c_j.\) Now our result follows from the correspondence in Theorem (3.4) in [AI3].

**Remark 2.12.** The Kodaira-Spencer map \(\rho_0\) sends \(\sum_{i=1}^{k} \partial/\partial \lambda_i\) to the element in 

\[ T^1_{X_\sigma}(-R) \]

induced by the map

\[ L(\bigcup_{i=1}^{N} S_i^R) \rightarrow \mathbb{R}, \quad (c_1, \ldots, c_w) \mapsto -\sum_{j=1}^{w} [\min(s_j, Q_0)] c_j. \]
This follows from the equation
\[
\sum_{i=0}^{k} [[\min\langle s_j, Q_i \rangle]] = [[\min\langle s_j, \sigma(R)^c \rangle]],
\]
as in the proof of Lemma 2.1, and also from the fact that for \( c_j \neq 0 \) we have
\[
[[\min\langle s_j, \sigma(R)^c \rangle]] = 0
\]
because the minimum of \( s_j \) occurs at one of the vertices of \( \sigma(R)^c \) of the form \( a_i/\langle R, a_i \rangle \) for \( \langle R, a_i \rangle \neq 0 \), as noticed in [A15], and for which
\[
\langle s_j, a_i \rangle < \langle R, a_i \rangle
\]
by the definition of \( S_i^R \) containing \( s_j \).

Similar to Proposition in [AI2, p. 172] we get a criterion when the deformation family has a non-trivial Kodaira-Spencer map, which implies a non-trivial deformation.

**Corollary 2.13.** For the family \( \mathcal{X}_{R,Q_0,Q_1,\ldots,Q_k} \rightarrow \mathbb{C}^k \), the Kodaira-Spencer map
\[
\rho_0 : T_{\mathcal{X}_0} \rightarrow T_{\mathcal{X}_0}^1
\]
is trivial if and only if all polytopes \( Q_0, Q_1, \ldots, Q_k \) are homothetic to \( \sigma(R)^c \) and either \( \sigma(R)^c \) is a lattice polytope or \( k \) of the summands \( Q_0, Q_1, \ldots, Q_k \) are lattice points.

**Proof.** As in [S], polytopes \( P \) and \( Q \) in \( N_\mathbb{R} \) are homothetic if \( Q = \delta \cdot P + n \) for some \( \delta > 0 \) and \( n \in \mathbb{N}_\mathbb{R} \). Suppose \( Q_i = \delta_i \cdot \sigma(R)^c + n_i \) for some \( \delta_i > 0 \) and \( n_i \in \mathbb{N}_\mathbb{R} \). If \( \sigma(R)^c \) is a lattice polytope, then its every Minkowski summand is so by the condition (b) on the Minkowski sum decomposition. Hence,
\[
\sum_{j=1}^{w} [[\min\langle s_j, Q_i \rangle]] c_j = \sum_{j=1}^{w} \min\langle s_j, Q_i \rangle c_j = \sum_{j=1}^{w} \delta_i c_j \min\langle s_j, \sigma(R)^c \rangle + \sum_{j=1}^{w} (c_j s_j, n_i) = 0
\]
for every \( \sum_{j=1}^{w} c_j s_j = 0 \) because for \( c_j \neq 0 \) we have \( \min\langle s_j, \sigma(R)^c \rangle = 0 \) as in Remark 2.12. Thus, by Theorem 2.11, the Kodaira-Spencer map \( \rho_0 \) is trivial. If \( Q_i = n_i \) is a lattice point, we again get
\[
\sum_{j=1}^{w} [[\min\langle s_j, Q_i \rangle]] c_j = (\sum_{j=1}^{w} c_j s_j, n_i) = 0.
\]
Using Remark 2.12, we conclude that \( k \) of the vectors \( \partial/\partial \lambda_1, \ldots, \partial/\partial \lambda_k, - \sum_{i=1}^{k} \partial/\partial \lambda_i \) are sent to 0 by the Kodaira-Spencer map \( \rho_0 \) if \( k \) of the summands \( Q_0, Q_1, \ldots, Q_k \) are lattice points. Therefore, \( \rho_0 \) is trivial in this case as well.

Conversely, suppose that the Kodaira-Spencer map \( \rho_0 \) is trivial, then
\[
\sum_{j=1}^{w} [[\min\langle s_j, Q_i \rangle]] c_j = 0
\]
for every \( \sum_{j=1}^{w} c_j s_j = 0 \) where \( c_j = 0 \) if \( s_j \notin \bigcup_{i=1}^{N} S_i^R \). Hence, for \( s_j \in \bigcup_{i=1}^{N} S_i^R \), we get
\[
[[\min\langle s_j, Q_i \rangle]] = \langle s_j, n_i \rangle
\]
for some \( n_i \in \mathbb{N}_\mathbb{R} \). Then \( [[\min\langle s_j, Q_i - n_i \rangle]] = 0 \). The
facets of polytope $\sigma(R)^c$ in the hyperplane where $R$ has value 1 are determined by the conditions $\min \langle s_j, \sigma(R)^c \rangle = 0$ for a subset of $s_j$ in $\bigcup_{l=1}^{N} S^R_l$. But for such $j$,

$$0 = \min \langle s_j, \sigma(R)^c \rangle = \sum_{i=0}^{k} \min \langle s_j, Q_i \rangle,$$

whence $\min \langle s_j, Q_i \rangle$ are integers by the condition (b) on the Minkowski sum decomposition. Therefore, we get stricter condition $\min \langle s_j, Q_i - n_i \rangle = 0$ for $s_j$ that were normals to the facets of $\sigma(R)^c$. Since $Q_i + n_i$ is a Minkowski summand of $\sigma(R)^c$ in a parallel hyperplane, the normals to its facets are among those for $\sigma(R)^c$ and it follows that $Q_i - n_i$ must be a multiple of $\sigma(R)^c$. Thus, $Q_i = \delta_i \sigma(R)^c + n_i$. If $\sigma(R)^c$ is a lattice polytope, then we are done. Suppose $a_l/(R, a_l)$ is a non-lattice vertex of $\sigma(R)^c$, or equivalently $\langle R, a_l \rangle > 1$. Since $S$ is generating $\sigma^c \cap M$, there is always $s_j$ such that $\langle s_j, a_l \rangle = 1$. Then $s_j \in S^R_l$ and $\langle s_j, n_i \rangle$ is an integer by the above. Now the Minkowski sum decomposition $\sigma(R)^c = Q_0 + Q_1 + \cdots + Q_k$ induces decomposition of the vertex

$$\frac{a_l}{\langle R, a_l \rangle} = \left( \delta_0 \frac{a_l}{\langle R, a_l \rangle} + n_0 \right) + \cdots + \left( \delta_k \frac{a_l}{\langle R, a_l \rangle} + n_k \right)$$

into the sum of vertices of $Q_0, Q_1, \ldots, Q_k$. By the condition (b) on this decomposition only one of the summands can be non-lattice. But if $\delta_i \frac{a_l}{\langle R, a_l \rangle} + n_i$ is a lattice point then

$$\langle s_j, \delta_i \frac{a_l}{\langle R, a_l \rangle} + n_i \rangle - \langle s_j, n_i \rangle = \delta_i \langle s_j, a_l \rangle = \frac{\delta_i}{\langle R, a_l \rangle}$$

is an integer which can happen only if $\delta_i = 0$ implying $Q_i = n_i$ is a lattice point. \qed

3. Minkowski sum decompositions of polyhedral complexes.

In order to generalize the construction of deformations of affine toric varieties we were forced to introduce the notion of a Minkowski sum decomposition of a polyhedral complex. We discovered this notion in 2007 and to the best of our knowledge it was never explicitly defined in any previously published work. In the next section we will see that this notion is very important for deformations of arbitrary toric varieties.

A polyhedral complex in a real vector space is a collection of polyhedra such that the intersection of any two polyhedra in the complex must be a face of each. By a Minkowski sum decomposition of a polyhedral complex we mean a Minkowski sum decomposition of each polyhedra in the complex such that for any two different intersecting polyhedra $P$ and $S$ of the complex the decompositions $P = P_1 + \cdots + P_n$ and $S = S_1 + \cdots + S_n$ induce compatible decompositions on their common face $P \cap S$.

**Example 3.1.** Consider the polyhedral complex obtained by subdividing a line segment in $N_2$ by points $n_1, \ldots, n_k$:

```
  n1  n2  . . .  nk−1  nk
```

\[ n1 \quad n2 \quad \ldots \quad nk−1 \quad nk \]
A polyhedra in this complex is a line segment \([n_i, n_{i+1}]\), which is also the convex hull of its vertices. We decompose them as follows:

\[
\begin{align*}
[n_1, n_2] & \quad [n_2, n_3] & \cdots & \quad [n_{k-2}, n_{k-1}] & \quad [n_{k-1}, n_k] \\
\| & \| & \cdots & \| & \| \\
[n_1, n_2] & n_2 & \cdots & n_2 & n_2 \\
+ & + & \cdots & + & + \\
0 & [0, n_3 - n_2] & \cdots & n_3 - n_2 & n_3 - n_2 \\
+ & + & \cdots & + & + \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
+ & + & \cdots & + & + \\
0 & 0 & \cdots & [0, n_{k-1} - n_{k-2}] & n_{k-1} - n_{k-2} \\
+ & + & \cdots & + & + \\
0 & 0 & \cdots & 0 & [0, n_k - n_{k-1}],
\end{align*}
\]

where by 0 we denoted the origin point in \(\mathbb{R}_\mathbb{R}\). Note that each vertex \(n_i\) has the same induced decomposition from the nearby line segments. Thus, we have a Minkowski sum decomposition of the polyhedral complex.

**Example 3.2.** Another, less trivial example, of a Minkowski sum decomposition of a polyhedral complex comes from the skeleton of a reflexive polytope. Consider the convex hull of points \((-1, -1), (2, -1), (-1, 2)\) in \(\mathbb{R}^2\). This is the dual of of the reflexive polytope coming from the fan of \(\mathbb{P}^2\). Subdividing the skeleton of this reflexive polytope by the lattice points \(n_1, \ldots, n_7\) inside its edges we obtain the following polyhedral complex.

Here is a nontrivial Minkowski sum decomposition of this complex:

\[
\begin{align*}
[n_1, n_2] &= [n_1, n_2] + 0 + 0 \\
[n_2, n_3] &= n_2 + [0, n_3 - n_2] + 0 \\
[n_3, n_4] &= n_2 + n_3 - n_2 + [0, n_4 - n_3] \\
[n_4, n_5] &= [n_2, n_9] + n_3 - n_2 + n_4 - n_3 \\
[n_5, n_6] &= n_9 + [n_3 - n_2, n_6] + n_4 - n_3 \\
[n_6, n_7] &= n_9 + n_6 + [n_4 - n_3, n_6] \\
[n_7, n_8] &= [n_9, n_1] + n_6 + n_6 \\
[n_8, n_9] &= n_1 + [n_6, 0] + n_6 \\
[n_9, n_1] &= n_1 + 0 + [n_6, 0]
\end{align*}
\]

4. Generalization of deformations for arbitrary toric varieties.

In this section we generalize the construction from Section 2 to the case of an arbitrary toric variety \(X_\Sigma\) associated to a fan \(\Sigma\) in \(\mathbb{R}\).
First, we do a simple case starting with one $R \in M$. The intersection of cones of the fan $\Sigma$ with the hyperplane $\{ n \in N_\mathbb{R} | \langle R, n \rangle = 1 \}$ forms a polyhedral complex consisting of polyhedra $\sigma(R) = \sigma \cap \{ n \in N_\mathbb{R} | \langle R, n \rangle = 1 \}$ for $\sigma \in \Sigma$. By taking the compact parts $\sigma(R)^c$ of these polyhedra we again get a polyhedral complex, denoted $\Sigma(R)^c = \{ \sigma(R)^c | \sigma \in \Sigma \}$. Decompose each $\sigma(R)^c = Q_0^\sigma + Q_1^\sigma + \cdots + Q_k^\sigma$ so that this forms a Minkowski sum decomposition of the polyhedral complex $\Sigma(R)^c$ satisfying

\((*)\) the induced decomposition of a vertex $\text{vert}(\sigma(R)^c) = \text{vert}(Q_0^\sigma) + \text{vert}(Q_1^\sigma) + \cdots + \text{vert}(Q_k^\sigma)$ of $\sigma(R)^c$ into the sum of vertices of $Q_0^\sigma, Q_1^\sigma, \ldots, Q_k^\sigma$ has all but possibly one of the summands $\text{vert}(Q_0^\sigma), \text{vert}(Q_1^\sigma), \ldots, \text{vert}(Q_k^\sigma)$ being lattice points,  
\((**)\) $Q_0^\sigma \subset \{ n \in N_\mathbb{R} | \langle R, n \rangle = 1 \}$, $Q_i^\sigma \subset \{ n \in N_\mathbb{R} | \langle R, n \rangle = 0 \}$, for $i = 1, \ldots, k$.

**Remark 4.1.** Following Remark 2.4, we note that condition $(**)$ is not necessary for the fan construction and Theorem 4.2 below.

As before we have extended lattices $\tilde{N} = N \oplus \mathbb{Z}^k$, $\tilde{M} = M \oplus \mathbb{Z}^k$ and the cones 
\[ \tilde{\sigma} = \langle \sigma, Q_0^\sigma - e_1 - \cdots - e_k, Q_1^\sigma + e_1, \ldots, Q_k^\sigma + e_k \rangle \]
in $\tilde{N}_\mathbb{R}$ for each $\sigma \in \Sigma$. Condition $(c)$ on the Minkowski sum decompositions guarantees that these cones $\tilde{\sigma}$ are compatible on their intersections, and, therefore, form the fan $\tilde{\Sigma} = \{ \tilde{\sigma} | \sigma \in \Sigma \}$ in $\tilde{N}_\mathbb{R}$. The natural inclusion map $\tilde{N}_\mathbb{R} \subset N_\mathbb{R}$ induces a map of the fan $\Sigma$ to $\tilde{\Sigma}$ and we have

**Theorem 4.2.** Associated to the map of fans $\Sigma \to \tilde{\Sigma}$, the map between toric varieties $X_\Sigma \to X_{\tilde{\Sigma}}$ is an embedding, whose image is a complete intersection given by the equations

\[ \prod_{\langle c_i^\sigma, v_j \rangle > 0} x_j^{\langle c_i^\sigma, v_j \rangle} - \prod_{\langle c_i^\sigma, v_j \rangle < 0} x_j^{-\langle c_i^\sigma, v_j \rangle} = 0, \]

for $i = 1, \ldots, k$ where $x_1, \ldots, x_n$ are homogeneous coordinates of the toric variety $X_\Sigma$ and $v_1, \ldots, v_n$ are primitive lattice generators of the rays in $\tilde{\Sigma}$.

**Proof.** Since the embedding property of varieties can be checked locally, from Lemma 2.1 we get the embedding part of the statement.

To show the second part it is sufficient to show that locally for each affine toric variety $X_{\tilde{\sigma}} \subset X_{\tilde{\Sigma}}$, the given equations induce the same equations as in Corollary 2.7, which determine $X_{\sigma}$ in $X_{\tilde{\sigma}}$. From [C], we know that

\[ \mathbb{C}[\tilde{\sigma}^\vee \cap M] \simeq \mathbb{C}[x_1, x_2, \ldots, x_n]_{v_{\tilde{\sigma}}}^{\tilde{G}}, \]

where $\mathbb{C}[x_1, x_2, \ldots, x_n]_{v_{\tilde{\sigma}}}^{\tilde{G}}$ means localization of the coordinate ring by the monomial $x^\sigma = \prod_{v_j \notin \sigma} x_j$ and the group $\tilde{G}$ defined the same way as in (3) but with $v_j$ corresponding to all rays of $\tilde{\Sigma}$. Now the subvariety of $X_\Sigma$ defined by the ideal $I$ in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, homogeneous with respect to the group $\tilde{G}$ is locally in each affine chart $X_{\tilde{\sigma}}$ corresponds to the ring $(\mathbb{C}[x_1, x_2, \ldots, x_n]_{v_{\tilde{\sigma}}}^{(I_{\tilde{\sigma}})})^{\tilde{G}}$. Assuming $I$ is generated by

\[ \prod_{\langle c_i^\sigma, v_j \rangle > 0} x_j^{\langle c_i^\sigma, v_j \rangle} - \prod_{\langle c_i^\sigma, v_j \rangle < 0} x_j^{-\langle c_i^\sigma, v_j \rangle} \]

and that $v_1, \ldots, v_l$ are the primitive lattice generators of $\tilde{\sigma}$ as in Proposition 2.6, we get natural isomorphisms

\[ \mathbb{C}[\sigma^\vee \cap M] \simeq (\mathbb{C}[x_1, \ldots, x_l]/I)^{\tilde{G}} \simeq (\mathbb{C}[x_1, x_2, \ldots, x_n]_{v_{\tilde{\sigma}}}^{I_{\tilde{\sigma}}})^{\tilde{G}}, \]
where the last one is induced by the ring homomorphism

\[
(C[x_1, x_2, \ldots, x_n])[x^s] \xrightarrow{\tilde{G}} C[x_1, \ldots, x_l]^{G}, \quad \prod_{j=1}^{n} x_j^{(u, v_j)} \mapsto \prod_{j=1}^{l} x_j^{(u, v_j)},
\]

which is simply evaluation at \( x_{l+1} = x_{l+2} = \cdots = x_n = 1 \).

\[ \square \]

A \( k \)-parameter embedded deformation of \( X_{\Sigma} \) corresponds to the diagram

\[
\begin{array}{ccc}
X_{\Sigma} & \subset & \tilde{X} \\
\downarrow & & \downarrow \\
\{0\} & \subset & C^k
\end{array}
\]

where the deformation family \( \tilde{X} \) in \( X_{\Sigma} \times C^k \) is given by the equations in homogeneous coordinates:

\[
\prod_{(e_i^*, v_j) > 0} x_j^{(e_i^*, v_j)} - \prod_{(e_i^*, v_j) < 0} x_j^{-(e_i^*, v_j)} + \lambda_i \prod_{(R, v_j) < 0} x_j^{-(R, v_j)} = 0
\]

for \( i = 1, \ldots, k \).

**Remark 4.3.** Complete intersection and compatibility in affine charts of \( X_{\Sigma} \) guarantee the flatness of this family. To check that the monomials in the equation are of the same homogeneous degree consider values of the points \( R + e_i^* \) and \( R \) on all \( v_j \)'s. Following Remark 2.4, lattice point \( R \) must be replaced by \( R - \sum_{i=1}^{k} \langle R, Q_i \rangle e_i^* \) in the above equations if we drop condition \((**\)).

**Example 4.4.** Let \( X_{\Sigma} \) be the weighted projective space \( P(1, 1, 2) \). The primitive lattice generators of the rays of the fan can be chosen as

\[
\{n_1 = (1, 0), n_2 = (-1, -2), n_3 = (0, 1)\}
\]

and the fan looks like in the picture.

Here, the dotted line is where the values of \( R = (2, -2) \) are 1. The compact part of the polyhedral complex obtained by intersecting the cones of the fan of \( X_{\Sigma} \) with the hyperplane \( \{n \in \mathbb{N}_R | \langle R, n \rangle = 1 \} \) consists of just one line segment \([n_1, n_2]\) with its vertices. It admits a Minkowski sum decomposition satisfying the conditions (a)-(c):

\[
[n_1, n_2] = \frac{n_1}{2} + [0, \frac{n_2 - n_1}{2}].
\]

The intersections of the cones of the fan adjacent to this line segment are just vertices \( \frac{n_1}{2}, \frac{n_2}{2} \) and they admit the induced decompositions.
According to our construction the rays of the fan $\tilde{\Sigma}$ are spanned by $\frac{2}{3} - e_1$, $e_1$, $\frac{2}{3} + e_1$, and $e_2$ in $N_\mathbb{R}$. Identifying naturally $N_\mathbb{R} = N_\mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^3$, we get that these points correspond to $(\frac{2}{3}, 0, -1)$, $(0, 0, 1)$, $(-1, -1, 1)$ and $(0, 1, 0)$ in $\mathbb{R}^3$. And replacing the first point with $(1, 0, -2)$ we get the primitive lattice generators of the fan $\Sigma$. The maximal cones of the fan $\tilde{\Sigma}$ are spanned by all possible triplets among these lattice points except for the combination $(0, 1, 0)$, $(0, 0, 1), (0, 1, 0), (0, 0, 1), (0, 1, 0)$. This simply means that $X_\Sigma$ is not compact. If homogeneous coordinates $x_1, x_2, x_3, x_4$ correspond to $(1, 0, -2), (0, 0, 1), (0, 1, 0), (0, 1, 0)$, then the equation defining $X_\Sigma$ in $X_\Sigma$ is $x_1^3 = x_2 x_3$ (Exponents are given by the “distance” of lattice points from $N$.) By adding the missing maximal cone to $\tilde{\Sigma}$ we get the fan of $\mathbb{P}^3$ and the hypersurface $x_1^3 = x_2 x_3$ simply misses the point $x_2 = x_3 = x_4 = 0$ in $\mathbb{P}^3$. One can actually work out that the map $P(1, 1, 2) = X_\Sigma \to X_\Sigma \subset \mathbb{P}^3$ is given by

$$(y_1, y_2, y_3) \mapsto (y_1 y_2, y_1^2, y_2^2, y_3)$$

in homogeneous coordinates. The deformation that we constructed is given by $x_2 x_3 - x_1^3 + \lambda x_1^4$ which smoothes out the singularity.

**Example 4.5.** Our general construction also generalizes the results of [M1, M2]. To see this let $X_{\Sigma_1}$ be the toric blow up of a Fano toric variety corresponding to a reflexive polytope $\Delta$, such that the rays of $\Sigma$ pass through all lattice points $n_1, \ldots, n_k$ of an edge $\Gamma^*$ of the dual polytope $\Delta^*$.

Let $R \in M$ such that $-R$ is in the interior of $\Gamma = \{u \in \Delta \cap M | \langle u, \Gamma^* \rangle = -1\}$, the codimension 2 face of $\Delta$ dual to the edge $\Gamma^*$. Then the compact part of the polyhedral complex, obtained by intersecting the fan $\Sigma$ with the hyperplane where $R$ has value 1, is just the collection of line segments $[n_i, n_{i+1}]$ together with their vertices. If we use the Minkowski sum decomposition of the polyhedral complex as in Example 3.1 and apply our construction of $\tilde{\Sigma}$ we obtain the same fan as $\Sigma(\Gamma^*)$ in [M2, Section 4]. The minimal lattice generators of $\tilde{\Sigma}$ according to our construction are $n_1 - e_1 - \cdots - e_k$, $n_2 - e_1 - \cdots - e_k, e_j, n_2 - n_1 + e_j$ for $j = 1, \ldots, k$ which are up to notation the same as in that paper.

We expect that the constructed families of deformations of $X_\Sigma$ for different $R \in M$ and corresponding Minkowski sum decompositions of polyhedral complexes infinitesimally, by Kodaira-Spencer map, generate all unobstructed deformations of a toric variety.

5. Combining deformation families.

In this section we will combine the constructed deformation families of toric varieties corresponding to different $R$’s. This section will be appear later, but so far the reader may try the following.

**Exercise 5.1.** Consider the toric variety which corresponds to the fan which subdivides the reflexive polytope in Example 3.2. This toric variety is the crepant resolution of the Fano toric variety which is mirror of $\mathbb{P}^2$. Now generalize the fan
construction of the previous section for the Minkowski sum decomposition of the polyhedral complex in (7). The bigger fan should correspond to the toric variety which contains all unobstructed deformations of the crepant resolution of the Fano toric variety. Describe the deformation family. It will be clear how to construct the family if you look at the next section.

6. **Deformations of Fano toric varieties.**

In this section we will show how to combine the constructed deformation families in the case of a Fano toric variety.

A Fano toric variety corresponds to a reflexive polytope $\Delta$ and has a (normal) fan whose cones are generated by the proper faces of the dual polytope $\Delta^* = \{ n \in \mathbb{N}_R | \langle m, n \rangle \geq -1 \forall m \in \Delta \}$. 

We will denote this fan $\Sigma_\Delta$. Now for those $R \in M$ such that $u = -R$ is in the interior of a face $\Gamma$ of $\Delta$, the compact part $\Sigma_\Delta(R)^\circ$ of the polyhedral complex obtained by intersecting the fan $\Sigma_\Delta$ with the hyperplane $\{ n \in N_R | \langle R, n \rangle = 1 \}$ consists of the faces of the dual face $\Gamma^* = \{ n \in \mathbb{N}_R | \langle m, n \rangle = -1 \forall m \in \Gamma \}$ to $\Gamma$. Note that the union $\Sigma_\Delta(R)^\circ$ for different $R$'s forms a polyhedral complex consisting of the proper faces of $\Delta^*$. To obtain a Minkowski sum decomposition of this polyhedral complex we can simply use a Minkowski sum decomposition of the polytope $\Delta^* = \Delta_0^* + \Delta_1^* + \cdots + \Delta_k^*$ defined by lattice polytopes. This induces Minkowski sum decompositions for each face $\Gamma^* = \Gamma_0^* + \Gamma_1^* + \cdots + \Gamma_k^*$.

In the lattices $\tilde{N} = N \oplus \mathbb{Z}^k$, $\tilde{M} = M \oplus \mathbb{Z}^k$ we construct the cones $\tilde{\sigma}_{\Gamma} = \langle \Gamma_0^* - e_1 - \cdots - e_k, \Gamma_1^* + e_1, \ldots, \Gamma_k^* + e_k \rangle$ in $\tilde{N}_R$ for each $\sigma_{\Gamma} \in \Sigma_\Delta$, where $\sigma_{\Gamma}$ is the cone generated by $\Gamma^*$. These cones $\tilde{\sigma}_{\Gamma}$ form a fan which we denote $\tilde{\Sigma}_\Delta$.

Following the proof of Theorem 4.2 we get

**Theorem 6.1.** Associated to the map of fans $\Sigma_\Delta \to \tilde{\Sigma}_\Delta$, the map between toric varieties $X_{\Sigma_\Delta} \to X_{\tilde{\Sigma}_\Delta}$ is an embedding, whose image is a complete intersection given by the equations

$$\prod_{\langle e_i^*, v_j \rangle > 0} x_j^{\langle e_i^*, v_j \rangle} - \prod_{\langle e_i^*, v_j \rangle < 0} x_j^{-\langle e_i^*, v_j \rangle} = 0,$$

for $i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are homogeneous coordinates of the toric variety $X_{\tilde{\Sigma}_\Delta}$ and $v_1, \ldots, v_n$ are primitive lattice generators of the rays in $\tilde{\Sigma}_\Delta$.

**Remark 6.2.** The fan $\tilde{\Sigma}_\Delta$ consists of the cones over the faces of the reflexive polytope $\tilde{\Delta}^* = \text{Conv}(\Delta_0^* - e_1 - \cdots - e_k \cup \Delta_1^* + e_1 \cup \cdots \cup \Delta_k^* + e_k)$, whence the toric variety $X_{\tilde{\Sigma}_\Delta}$ is also Fano. Important applications of this construction will appear in [M3].
If \( l(\Delta) \) denotes the number of lattice points in the reflexive polytope \( \Delta \), then we construct \((kl(\Delta) - k)\)-parameter embedded deformations of \( X_{\Sigma_\Delta} \) corresponding to the diagram

\[
\begin{array}{ccc}
X_{\Sigma_\Delta} & \subseteq & X_\Delta \\
\downarrow & & \downarrow \\
\{0\} & \subseteq & \mathbb{C}^{kl(\Delta) - k},
\end{array}
\]

where the deformation family \( \mathcal{X}_\Delta \) in \( X_{\Sigma_\Delta} \times \mathbb{C}^{kl(\Delta) - k} \) is given by the equations in homogeneous coordinates:

\[
\prod_{(\ell^*, v) > 0} x_j^{(\ell^*, v)} - \prod_{(\ell^*, v) < 0} x_j^{-(\ell^*, v)} + \sum_{\Delta \subseteq \text{int}(\Gamma)} \sum_{i,u} \lambda_{i,u} \prod_{l \in \Delta} x_j^{(u - \sum_{i=1}^k (u, \Gamma_i) \ell_i^*)} = 0
\]

for \( i = 1, \ldots, k \).

**Remark 6.3.** The above construction of deformations of Fano toric varieties can be described without homogeneous coordinates. Indeed, let \( \sigma = \{(l\Delta^*, t) \mid t \in \mathbb{R}_{\geq 0}\} \subset \mathbb{N}_\mathbb{R} \oplus \mathbb{R} \)

be the reflexive Gorenstein cone and let \( R = (0, 1) \in M \oplus \mathbb{Z} \). Then the slice of \( \sigma \) in the hyperplane where \( R \) has value 1 is the polytope \((\Delta^*, 1)\). From (8) we get the induced decomposition

\[
(\Delta^*, 1) = (\Delta_0^*, 1) + (\Delta_1^*, 0) + \cdots + (\Delta_k^*, 0).
\]

Applying (1) we get the cone

\[
\tilde{\sigma} = \{(l\Delta_0^*, 1) - e_1 - \cdots - e_k, (\Delta_1^*, 0) + e_1, \ldots, (\Delta_k^*, 0) + e_k\}
\]

in \( \mathbb{N}_\mathbb{R} \oplus \mathbb{R}^{k+1} \), which is reflexive Gorenstein of index \( k + 1 \) by [BBo]. By Proposition 2.3 we have embedding \( X_\sigma \hookrightarrow X_{\tilde{\sigma}} \) given by \( \chi^{R + e_i^*} - \chi^R, \ldots, \chi^{R + e_k^*} - \chi^R \). Its deformation then corresponds to the quotient ring \( \mathbb{C}[\tilde{\var} \cap M] \otimes \mathbb{C}[\Delta]/\hat{I} \), where \( \hat{I} = M \oplus \mathbb{Z}^{k+1} \) and \( \hat{I} \) is generated by

\[
\chi^{R + e_i^*} - \chi^R + \sum_{\Gamma \subseteq \Delta} \sum_{u \in \text{int}(\Gamma)} \lambda_{i,u} \chi^{u - \sum_{i=1}^k (u, \Gamma_i) \ell_i^*},
\]

for \( i = 1, \ldots, k \). For the Fano toric variety \( X_{\Sigma_\Delta} = \text{Proj}(\mathbb{C}[\sigma^* \cap M]) \), we get its deformations inside \( X_{\Sigma_\Delta} = \text{Proj}(\mathbb{C}[\tilde{\sigma}^* \cap M]) \) as complete intersections \( \text{Proj}(\mathbb{C}[\tilde{\var}^* \cap M]/\hat{I}) \) for different values of \( \lambda \)'s. An important point of this remark is that the Gorenstein cone construction used in Mirror Symmetry (see [BBo]) is a particular case of Altmann’s construction in [Al2]!

More deformations of Fano and general toric varieties will be added later. The current preprint is a work in progress. Applications to deformations and classification of Calabi-Yau complete intersections and Mirror Symmetry will follow as well in a parallel paper [M3].

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