On the conjectures regarding the 4-point Atiyah determinant

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Abstract

For the case of 4 points in Euclidean space, we present a computer aided proof of Conjectures II and III made by Atiyah and Sutcliffe regarding Atiyah’s determinant along with an elegant factorization of the square of the imaginary part of Atiyah’s determinant.

1 Introduction

The Atiyah determinant is a complex-valued determinant function $At(P_1, \ldots, P_n)$ associated with $n$ distinct points $P_1, \ldots, P_n$ of $\mathbb{R}^3$. It was constructed by M. F. Atiyah in [1] in his attempt at answering a natural geometric question posed in [3] and arising from the study of the spin statistics theorem using classical quantum theory. The original conjecture of Atiyah was that $At$ does not vanish for all configurations of distinct points $P_1, \ldots, P_n \in \mathbb{R}^3$. The conjecture was verified in the linear case (all points lie on a straight line) and in the case $n = 3$ by Atiyah in [1]. However, the case $n \geq 4$ turned out to be notoriously difficult. In a subsequent paper [2], Atiyah and Sutcliffe studied the function $At$ and added two new conjectures (after normalizing $At$) which imply the original conjecture of Atiyah. They provided compelling numerical evidence of the validity of all three conjectures. The three conjectures can be stated as follows: For all distinct points $P_1, \ldots, P_n$ of $\mathbb{R}^3$ (and all $n \geq 1$) we have:

(I) $At(P_1, \ldots, P_n) \neq 0$.

(II) $|At(P_1, \ldots, P_n)| \geq \prod_{i<j}(2r_{ij})$, where $r_{ij} = ||P_i - P_j||$.

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(III) $|At(P_1, \ldots, P_n)|^{n-2} \geq \prod_{k=1}^{n} |At(P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_n)|.$

From the statement of these conjectures we can see that (III) $\implies$ (II) $\implies$ (I).

The three conjectures have been very resistant since their inauguration time and only a few attempts on some special configurations were proved successfully. For example, Eastwood and Norbury [5] were able to prove the case $n = 4$ only for the first conjecture. Other proofs on special configurations include [4] and [6]. In this paper, we build on the work of Eastwood and Norbury by presenting a computer aided proof of conjectures (II) and (III) in the case $n = 4$ and we also give an elegant factorization of the square of the imaginary part of the Atiyah determinant.

The construction of the determinant is as follows: One starts with $n$ distinct points $P_1, \ldots, P_n \in \mathbb{R}^3$. By considering $P_j$ as an observer of the other $n - 1$ points we obtain $n-1$ vectors $\overrightarrow{P_jP_1}, \ldots, \overrightarrow{P_jP_{j-1}}, \overrightarrow{P_jP_{j+1}}, \ldots, \overrightarrow{P_jP_n}$ in $\mathbb{R}^3$. We lift each of these vectors from $\mathbb{R}^3$ to $\mathbb{C}^2$ using the Hopf map $h : \mathbb{C}^2 \to \mathbb{R}^3$ given by $h(z, w) = ((|z|^2 - |w|^2)/2, z\bar{w})$ to obtain $n - 1$ points of $\mathbb{C}^2$. Note that the lifts are not unique and are defined up to phase because $h(\lambda z, \lambda w) = |\lambda|^2 h(z, w)$. Consequently, our lifts can be considered as points of $\mathbb{C}P^1$. Taking the symmetric product of these lifts gives a vector $V_j$ in $\mathbb{C}^n$ because $\circ_n \mathbb{C}P^1 = \mathbb{C}^n$. Atiyah’s first conjecture was that $\{V_1, \ldots, V_n\}$ is a linearly independent set. In other words, the determinant of the matrix having the vector $V_j$ as its $j$th column is nonzero. This determinant (when properly normalized) is called the Atiyah determinant and is denoted by $At$.

It is immediate from the above construction that $At$ is coordinate free and is independent of solid motion. In other words, the determinant function $At$ is invariant under translations and rotations in $\mathbb{R}^3$. Moreover, $At$ gets conjugated under a plane reflection (see [1]). One consequence of this last property is that $At$ must be real-valued when the points are planar since a reflection in their plane leaves them fixed. In addition to these properties, the Atiyah determinant is built so that it is independent of the order of the points. In other words, if $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$ then $At(P_{j_1}, \ldots, P_{j_n}) = At(P_1, \ldots, P_n)$.

Let us start computing $At$ in the cases $n = 2$ and $n = 3$. We will work throughout this paper with the normalization imposed by Atiyah on $At$ (to get rid of the phase factors) which requires that $(-\overrightarrow{w,z})$ must be the lift of $\overrightarrow{P_jP_i}$ whenever $i < j$ and $(z, w)$ is the chosen lift of $\overrightarrow{P_iP_j}$.

For the case $n = 2$, we have two distinct points $A$ and $B$. We can identify $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{C}$ and assume (possibly after a solid motion) that $A$ and $B$ have coordinates $(0, 0)$ and $(0, x)$ respectively, where $x > 0$ is the distance from $A$ to $B$. By choosing $(\sqrt{x}, \sqrt{x})$ as a lift of $\overrightarrow{AB}$, we are forced to take $(-\sqrt{x}, \sqrt{x})$ as a lift of $\overrightarrow{BA}$. Consequently, Atiyah’s determinant is:

$$At(A, B) = \left| \begin{array}{cc} \sqrt{x} & -\sqrt{x} \\ \sqrt{x} & \sqrt{x} \end{array} \right| = 2x,$$

where $x = ||\overrightarrow{AB}||$.

Let us now consider the case $n = 3$. Assume (possibly after a solid motion) that $A = (0, 0)$, $B = (0, x)$, and $C = (0, ze^{I\alpha})$ where $I$ denotes $\sqrt{-1}$, $y = ||\overrightarrow{BC}||$, $z = ||\overrightarrow{AC}||$, $|z| = ||\overrightarrow{AC}||$, and $|y| = ||\overrightarrow{BC}||$. Thus, $x = ||\overrightarrow{AB}||$.
is considered as a vision point we obtain $\overrightarrow{AB} = (0, x), \overrightarrow{AC} = (0, x^e\alpha)$ whose lifts under the Hopf map $h$ are $(\sqrt{x}, \sqrt{x})$ and $(\sqrt{z}, \sqrt{ze^{-\alpha}})$. Similarly, the lifts corresponding to the vision point $C$ are $(\sqrt{z}, \sqrt{z})$ and $(\sqrt{y}, \sqrt{ye^{-\beta}})$. The symmetric tensor product of the vectors are then $\sqrt{xyz}(1, 1+e^{-\alpha}, e^{-\alpha}), \sqrt{xyz}(1, 1+e^{\beta}, e^{\beta})$ and $\sqrt{xyz}(e^{(\alpha-\beta)}, e^{\alpha}, e^{-\beta})$, respectively. Consequently, we obtain the Atiyah determinant for three points as

$$At(A, B, C) = xyz \left| \begin{array}{ccc} 1 & 1 & e^{(\alpha-\beta)} \\ 1+e^{-\alpha} & -1-e^{\beta} & e^{(\alpha-\beta)} \\ e^{-\alpha} & e^{\beta} & -1 \end{array} \right|$$

This determinant expands to $xyz[6 + 2(\cos \alpha + \cos \beta + \cos \gamma)]$, which can be written as $xyz[8 + 8 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}]$. Using the identity $\sin \frac{\alpha}{2} = \frac{1}{2} \sqrt{\frac{(a+b-c)(a+c-b)}{bc}}$ and similar identities for $\sin \frac{\beta}{2}$ and $\sin \frac{\gamma}{2}$, we can rewrite the Atiyah determinant for three points as

$$At(A, B, C) = 8xyz + d_3(x, y, z), \quad (1.1)$$

where $d_3$ is the polynomial defined by $d_3(x, y, z) = (-x+y+z)(x+y-z)(x+y-z)$. From the triangle inequality it follows that $d_3(x, y, z)$ is nonnegative, and so Conjecture III is verified for three points.

## 2 The Case of Four Points

Given four points $A, B, C, D$ in $\mathbb{R}^3$, the vector $u = U(A, B, C, D)$ in $\mathbb{R}^6$, called the vector of pair-wise distances, is defined by $u = (a, b, c, x, y, z)$ where $a = ||\overrightarrow{AB}||$, $b = ||\overrightarrow{BD}||$, $c = ||\overrightarrow{CD}||$, $x = ||\overrightarrow{AB}||$, $y = ||\overrightarrow{BC}||$, $z = ||\overrightarrow{AC}||$ (see Figure 2). The function $U$, as defined above, maps $\mathbb{R}^{3 \times 4}$ into $\mathbb{R}^6$, and it is clear that $U$ is neither injective nor surjective. A vector $u \in \mathbb{R}^6$ is said to be feasible if it belongs to the range of $U$. For

$$x = ||\overrightarrow{AB}||$$ and $\alpha, \beta, \gamma$ are the angles indicated in figure 1. When the first point

![Figure 1: Three points](image-url)
then the 24 resultant permutations are not distinct, and Conjecture II (for four points) becomes

\[(II) \quad |At(A, B, C, D)| \geq 64abcxyz \quad \text{for all points} \ A, B, C, D \in \mathbb{R}^3.
\]

Atiyah’s determinant is designed to be invariant under permutations of the points. Each of the 24 possible permutations of the four points \(A, B, C, D\) results in a permutation of the pair-wise distances \(a, b, c, x, y, z\). Specifically, if \(u = (a, b, c, x, y, z) \in \mathbb{R}^6\), then the 24 resultant permutations are

\[
\begin{align*}
  u_0 &= (a, b, c, x, y, z) & u_1 &= (a, x, z, b, y, c) & u_2 &= (b, c, a, y, z, x) & u_3 &= (x, b, y, a, c, z) \\
  u_4 &= (c, a, b, z, x, y) & u_5 &= (z, y, c, x, b, a) & u_6 &= (y, z, c, x, a, b) & u_7 &= (c, b, a, y, x, z) \\
  u_8 &= (x, y, b, z, c, a) & u_9 &= (a, c, b, y, x, c) & u_{10} &= (z, x, a, y, b, c) & u_{11} &= (b, a, c, x, z, y) \\
  u_{12} &= (z, c, y, a, b, x) & u_{13} &= (x, z, a, y, c, b) & u_{14} &= (x, a, z, b, c, y) & u_{15} &= (y, x, b, z, a, c) \\
  u_{16} &= (y, b, x, c, a, z) & u_{17} &= (y, c, z, b, a, x) & u_{18} &= (c, y, z, b, a, x) & u_{19} &= (z, a, x, c, b, y) \\
  u_{20} &= (b, x, y, a, z, c) & u_{21} &= (c, z, y, a, x, b) & u_{22} &= (a, z, x, c, y, b) & u_{23} &= (b, y, x, c, z, a)
\end{align*}
\]

A function \(f : \mathbb{R}^6 \to \mathbb{R}\) is said to be **symmetric** if \(f(u) = f(u_i)\) for \(i = 0, 1, \ldots, 23\) and is **skew-symmetric** if \(f(u) = (-1)^i f(u_i)\) for \(i = 0, 1, \ldots, 23\). The **symmetric average** of \(f\) is the symmetric function \(av[f]\) defined by

\[
  av[f](u) = \frac{1}{24} \sum_{i=0}^{23} f(u_i).
\]

Using Maple, Eastwood and Norbury have found that the real part of \(At(A, B, C, D)\) can be expressed as \(\Re At(A, B, C, D) = d_4(u)\), where \(d_4\) is the homogeneous polynomial of degree 6 given by

\[
  d_4(u) = 60 p_4(u) + 4n_4(u) + 2z_4(u) + 12 av[a((b + c)^2 - y^2)d_3(x, y, z)], \quad (2.1)
\]

where \(p_4(u) = abcxyz\), \(d_3\) is defined in (1.1), \(n_4(u) = p_4(u) - d_3(xc, ay, bz)\) and

\[
  z_4(u) = a^2y^2(b^2 + c^2 + x^2 + z^2) + b^2z^2(a^2 + c^2 + x^2 + y^2) + c^2x^2(a^2 + b^2 + y^2 + z^2) \\
  - (a^4y^2 + a^2y^4 + b^4z^2 + b^2z^4 + c^4x^2 + c^2x^4) \\
  - (a^2b^2x^2 + a^2c^2z^2 + b^2c^2y^2 + x^2y^2z^2).
\]
Eastwood and Norbury use the notation $144V^2$ in place of $z_4(u)$. If $u = U(A, B, C, D)$, the value $z_4(u)$ equals $144V^2$, where $V$ denotes the volume of the tetrahedron formed by the points $A, B, C, D$, and it therefore follows that $z_4(u) \geq 0$. It would be erroneous to infer from this that the polynomial $z_4$ is nonnegative on all of $\mathbb{R}^6$; the above statement implies only that $z_4$ is nonnegative on feasible vectors.

Having expressed $\Re At(A, B, C, D) = d_4(u)$ as in (2.1), Eastwood and Norbury then invoke the inequalities $z_4(u) \geq 0$, $(b + c)^2 \geq y^2$, $d_4(x, y, z) \geq 0$ and $abcxyz \geq d_3(xc, ay, bz)$ (ie $n_4(u) \geq 0$) to conclude that

$$|At(A, B, C, D)| \geq \Re At(A, B, C, D) = d_4(u) \geq 60 p_4(u)$$

which proves Conjecture I and comes close to proving Conjecture II.

Regarding the imaginary part of $At(A, B, C, D)$, Eastwood and Norbury have shown that its square can be written as $(\Im At(A, B, C, D))^2 = F_4(u)$, where $F_4$ is a symmetric homogeneous polynomial of degree 12. Whereas $d_4$ seems unwilling to be expressed in a simple manner, we have found that $F_4$ factors elegantly as

$$F_4 = w_4^2 z_4,$$

where $w_4$ is the skew-symmetric homogeneous polynomial of degree 3 given by

$$w_4(u) = (a^2 + y^2)(b - c - x + z) + (b^2 + z^2)(-a + c + x - y) + (c^2 + x^2)(a - b + y - z) + 2(cx + yz)(-a + b) + 2(ay + zx)(-b + c) + 2(bz + xy)(a - c).$$

Note that since $w_4$ is skew-symmetric it follows that $w_4^2$ is symmetric. As mentioned in the introduction, the imaginary part of $At(A, B, C, D)$ vanishes whenever the four points $A, B, C, D$ are coplanar and this is born out in the above factorization since $z_4(u) = 0$ when $A, B, C, D$ are coplanar. Being skew-symmetric, $w_4(u) = 0$ whenever $u = U(A, B, C, D)$ is invariant under an odd permutation of the four points $A, B, C, D$. For example, if $u = (a, b, c, x, y, z) = U(A, B, C, D)$ satisfies $b = x$ and $c = z$, then $\Re At(A, B, C, D) = U(D, B, C, A)$ and it follows that $w_4(u) = 0$. This line of reasoning yields the following.

**Corollary 1.** If $u = U(A, B, C, D)$ is invariant under an odd permutation of the four points $A, B, C, D$, then $\Im At(A, B, C, D) = 0$.

### 3 A linear program related to Conjecture II

Since $|At(A, B, C, D)| \geq \Re At(A, B, C, D) = d_4(u)$, in order to prove Conjecture II, it suffices to show that the polynomial $d_4$ satisfies

$$d_4(u) \geq 64 p_4(u) \text{ for all feasible vectors } u. \quad (3.1)$$

If one has in hand a collection $f_1, f_2, \ldots, f_k$ of symmetric homogeneous polynomials of degree 6 which are known to be nonnegative on feasible vectors, then one can 'have a go' by solving the linear program

Maximize $\alpha$

Subject to $d_4 = \alpha p_4 + \sum_{j=1}^k \lambda_j f_j$, with $\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0 \quad (3.2)$
If (3.2) is feasible and if the optimal objective value is \( \alpha = 64 \) (we will see later that \( \alpha > 64 \) is impossible), then we immediately obtain (3.1). The remaining difficulty is that of finding suitable polynomials \( \{f_j\} \). One means of generating a large collection of such polynomials, which we now describe, stems from the triangle inequality.

The four points \( A, B, C, D \) contain four (possibly degenerate) triangles and each triangle, by means of the triangle inequality, gives rise to three linear polynomials which are nonnegative when \( u = (a, b, c, x, y, z) \) is feasible. For example, the triangle \( A, B, C \) yields \(-x+y+z, -y+z+x, x+y-z\). In all, there are twelve such linear polynomials which we refer to as **triangular variables** and use the notation \( t = (t_1, t_2, \ldots, t_{12}) \), where

\[
\begin{align*}
t_1 &= -a + b + x & t_4 &= -b + c + y & t_7 &= -a + c + z & t_{10} &= -x + y + z \\
t_2 &= a - b + x & t_5 &= b - c + y & t_8 &= a - c + z & t_{11} &= x - y + z \\
t_3 &= a + b - x & t_6 &= b + c - y & t_9 &= a + c - z & t_{12} &= x + y - z
\end{align*}
\]

A vector \( \alpha \in \mathbb{Z}^{12}_+ \) is called a **multi-index** with **order** \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{12} \). Employing the standard notation \( t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_{12}^{\alpha_{12}} \), we see that \( t^\alpha \) represents a homogeneous polynomial of degree \( |\alpha| \) in the variables \( (a, b, c, x, y, z) \). Applying the symmetric average, we conclude that \( \text{av}[t^\alpha] \) represents a symmetric homogeneous polynomial of degree \( |\alpha| \) which is nonnegative on feasible vectors. For integers \( \ell \geq 0 \), we define \( \mathbb{T}_\ell \) to be the set of all polynomials \( \text{av}[t^\alpha] \) with \( |\alpha| = \ell \):

\[
\mathbb{T}_\ell = \{ \text{av}[t^\alpha] : |\alpha| = \ell \} \quad (3.3)
\]

Numerically, we have found that if one chooses \( \{f_j\} \) equal to \( \mathbb{T}_6 \), then the linear program (3.2) is feasible and has optimal objective value \( \alpha = 32 \). The formulation (2.1) of Eastwood and Norbury can be understood in the context of (3.2) as the result of including, in addition to \( \mathbb{T}_6 \), the two symmetric polynomials \( z_4 \) and \( n_4 \) which are nonnegative on feasible vectors. Numerically solving (3.2) with \( \{f_j\} \) equal to \( \{z_4, n_4\} \cup \mathbb{T}_6 \), we have found that the optimal objective value is \( \alpha = 60 \), and (2.1) is indeed an optimal solution of (3.2) as the term \( \text{av}[a((b+c)^2 - y^2)d_3(x, y, z)] \) can be written as a nonnegative linear combination of polynomials in \( \mathbb{T}_6 \).

In order to further increase the optimal objective value \( \alpha \) in (3.2), we need other symmetric polynomials which are nonnegative on feasible vectors. In pursuit of this, we have identified the following twenty-one feasible vectors \( u \) where \( d_4(u) = 64 p_4(u) \) (all are obtained as \( u = U(A, B, C, D) \) with \( A, B, C, D \) collinear or non-distinct).

\[
\begin{align*}
(0, 1, 4, 1, 4, 4) &\quad (0, 4, 8, 4, 7, 8) &\quad (0, 6, 0, 6, 6, 0) \\
(0, 1, 1, 1, 2, 1) &\quad (0, 5, 5, 5, 5, 5) &\quad (0, 8, 8, 8, 1, 8) \\
(0, 1, 3, 1, 4, 3) &\quad (0, 6, 3, 6, 8, 3) &\quad (0, 6, 7, 6, 3, 7) \\
(0, 6, 6, 6, 9, 6) &\quad (0, 1, 1, 1, 0, 1) &\quad (0, 5, 3, 5, 3, 3) \\
(3, 3, 1, 0, 2, 2) &\quad (9, 9, 7, 0, 2, 2) &\quad (13, 13, 7, 0, 6, 6) \\
(19, 11, 7, 8, 4, 12) &\quad (17, 13, 4, 4, 9, 13) &\quad (15, 8, 7, 7, 1, 8) \\
(9, 8, 1, 1, 7, 8) &\quad (11, 9, 8, 2, 1, 3) &\quad (17, 9, 2, 8, 7, 15)
\end{align*}
\]

Both \( d_4 \) and \( p_4 \) vanish on the first fifteen of these vectors (counting horizontally), but are nonzero on the remaining six. In particular, since \( d_4(9, 8, 1, 1, 7, 8) = 64 p_4(9, 8, 1, 1, 7, 8) = \)
258048 > 0, it follows that there are no feasible solutions of (3.2) with \( \alpha > 64 \). On the other hand, if a feasible solution of (3.2) has been obtained with \( \alpha = 64 \), then it follows that \( f_j \) vanishes on all of the vectors in (3.4), whenever \( \lambda_j > 0 \). It has been verified that \( z_4 \) vanishes on all of these vectors, but \( n_4 \) does not. Therefore, the coefficient of \( n_4 \) will be 0 if (3.2) has been solved with \( \alpha = 64 \). We have considered numerous symmetric homogeneous polynomials of degree 6 which vanish on the vectors in (3.4), but only one of these has resulted in an improvement. Let \( v_4 \) denote the skew-symmetric homogeneous polynomial of degree 3 defined by

\[
v_4(u) = (b + z - c - x)(c + x - a - y)(a + y - b - z).
\]

Then \( v_4 \) vanishes on the vectors in (3.4), and numerically solving (3.2) with \( \{f_j\} \) equal to \( \{z_4, n_4, v_4^2\} \cup T_6 \), we have found that the optimal objective value is \( \alpha = 188/3 \). Our obtained identity, which has been verified in Maple, is the following:

\[
d_4(u) = \frac{188}{3} p_4(u) + \frac{10}{3} z_4(u) + \frac{4}{3} n_4(u) + \frac{2}{3} v_4^2(u) + \frac{1}{3} \sum_{|\alpha|=6} \lambda_\alpha u^\alpha,
\]

where the six nonzero coefficients \( \lambda_\alpha \) and corresponding multi-indices \( \alpha \) are given by

| \( \alpha \) | \( \lambda_\alpha \) | \( \alpha \) | \( \lambda_\alpha \) | \( \alpha \) | \( \lambda_\alpha \) |
|---|---|---|---|---|---|
| 000, 001, 010, 112 | 6 | 000, 001, 011, 111 | 18 | 000, 001, 110, 102 | 6 |
| 001, 001, 011, 111 | 14 | 001, 001, 010, 111 | 24 | 001, 011, 100, 110 | 24 |

4 Proof of Conjecture II for four points

Let \( m_4 \) be the symmetric homogeneous polynomial of degree 6 defined by \( m_4 = d_4 - (64p_4 + 4z_4 + v_4^2) \), so that

\[
d_4 = 64p_4 + 4z_4 + v_4^2 + m_4.
\]

We will show that \( m_4 \) is nonnegative on feasible vectors, but unfortunately, we have been unable to formulate a proof using only polynomials of degree 6. Rather, we have had to multiply \( m_4 \) by \( p_4 \) and then work with polynomials of degree 12.

**Theorem 1.** The product \( p_4m_4 \) can be written as a nonnegative linear combination of polynomials in \( T_{12} \).

**Proof.** Using Maple, we have verified that \( 64p_4m_4 \) can be written as

\[
64p_4(u)m_4(u) = \sum_{|\alpha|=12} \lambda_\alpha u^\alpha,
\]

where the sixty-four nonzero coefficients \( \{\lambda_\alpha\} \) are all positive integers as given in the
following table.

| \( \alpha \) | \( \lambda_\alpha \) | \( 011,021,201,112 \) | 6 | \( 011,111,220,102 \) | 6 |
|---|---|---|---|---|---|
| 001,012,211,112 | 12 | \( 011,021,201,121 \) | 6 | \( 011,112,102,210 \) | 12 |
| 001,012,211,121 | 12 | \( 011,021,201,211 \) | 18 | \( 011,112,120,210 \) | 39 |
| 001,112,112,012 | 12 | \( 011,021,221,110 \) | 54 | \( 011,112,210,210 \) | 42 |
| 001,121,121,012 | 21 | \( 011,022,011,121 \) | 6 | \( 011,120,011,212 \) | 27 |
| 002,110,111,212 | 9 | \( 011,022,111,102 \) | 84 | \( 011,120,011,221 \) | 24 |
| 002,110,111,221 | 9 | \( 011,022,111,210 \) | 18 | \( 011,120,012,112 \) | 3 |
| 011,011,021,221 | 30 | \( 011,022,211,110 \) | 6 | \( 011,121,021,201 \) | 24 |
| 011,011,101,222 | 56 | \( 011,101,021,212 \) | 54 | \( 011,121,102,210 \) | 6 |
| 011,011,110,222 | 48 | \( 011,102,012,211 \) | 6 | \( 011,121,120,210 \) | 24 |
| 011,011,112,220 | 6 | \( 011,102,022,111 \) | 36 | \( 011,201,012,211 \) | 3 |
| 011,011,120,212 | 18 | \( 011,102,112,102 \) | 18 | \( 011,201,021,211 \) | 45 |
| 011,011,122,102 | 57 | \( 011,102,112,201 \) | 24 | \( 012,012,012,111 \) | 24 |
| 011,012,201,221 | 36 | \( 011,102,201,121 \) | 33 | \( 012,012,102,111 \) | 8 |
| 011,011,222,011 | 6 | \( 011,102,210,112 \) | 75 | \( 012,111,012,021 \) | 36 |
| 011,012,102,112 | 120 | \( 011,102,210,121 \) | 12 | \( 011,220,210 \) | 6 |
| 011,012,112,210 | 18 | \( 011,110,021,221 \) | 48 | \( 011,120,012,202 \) | 21 |
| 011,012,120,121 | 12 | \( 011,111,012,202 \) | 21 | \( 011,120,021,202 \) | 24 |
| 011,012,211,102 | 84 | \( 011,111,021,202 \) | 18 | \( 011,201,012,202 \) | 18 |
| 011,012,211,201 | 72 | \( 011,111,021,220 \) | 54 | \( 011,210,021,202 \) | 3 |
| 011,021,011,122 | 3 | \( 011,111,022,102 \) | 75 | \( 012,012,120,102 \) | 3 |
| 011,021,112,012 | 69 | \( 011,111,202,210 \) | 12 |

\[ \square \]

**Corollary 2.** The polynomial \( m_4 \) is nonnegative on feasible vectors and consequently (3.1) holds, which proves Conjecture II for four points.

**Proof.** Let \( u = U(A, B, C, D) \) be a feasible vector. It follows from (4.2) that \( p_4(u)m_4(u) \geq 0 \). If the points \( A, B, C, D \) are distinct, then \( p_4(u) > 0 \) and hence \( m_4(u) \geq 0 \). On the other hand, if \( A, B, C, D \) are not distinct, then they can be approximated by distinct points \( A', B', C', D' \) and it will then follow from the continuity of \( m_4 \) that \( m_4(u) \geq 0 \). \[ \square \]

## 5 Proof of Conjecture III for four points

Let \( P_4 \) denote the symmetric homogeneous polynomial of deg 12 given by

\[
P_4(u) := (8xyz + d_3(x, y, z))(8abx + d_3(a, b, x))(8acz + d_3(a, c, z))(8bcy + d_3(b, c, y)),
\]

whereby \( \text{At}(A, B, C) \) \( \text{At}(A, B, D) \) \( \text{At}(A, C, D) \) \( \text{At}(B, C, D) = P_4(u) \) when \( u = U(A, B, C, D) \). Since \( |\text{At}(A, B, C, D)|^2 \geq (\Re \text{At}(A, B, C, D))^2 = d_4^2(u) \), in order to prove Conjecture III, it suffices to show that

\[
d_4^2(u) \geq P_4(u) \text{ for all feasible vectors } u.
\] \quad (5.1)
Recall from (4.1) that $d_4$ has been written as $d_4 = 64p_4 + m_4 + (4z_4 + v_4^2)$, so it follows that

$$d_4^2 = (4z_4 + v_4^2)d_4 + (64p_4 + m_4)d_4 = (4z_4 + v_4^2)d_4 + (64p_4 + m_4)^2 + (64p_4 + m_4)(4z_4 + v_4^2) = (4z_4 + v_4^2)(d_4 + 32p_4 + m_4) + (64p_4 + m_4)^2 + 32p_4(4z_4 + v_4^2) + (64p_4 + m_4)(4z_4 + v_4^2)$.

With $M_4$ denoting the symmetric homogeneous polynomial of degree 12 defined by $M_4 = (64p_4 + m_4)^2 + 32p_4(4z_4 + v_4^2) - P_4$, we then have

$$d_4^2 = P_4 + (4z_4 + v_4^2)(d_4 + 32p_4 + m_4) + M_4. \tag{5.2}$$

**Theorem 2.** The polynomial $M_4$ is nonnegative on feasible vectors, and consequently (5.1) holds, which proves Conjecture III for four points.

**Proof.** Using Maple, we have verified that $128M_4$ can be written as

$$128M_4(u) = (4z_4(u) + v_4^2(u)) \sum_{|\alpha|=6} \mu_\alpha av[t^\alpha] + \sum_{|\alpha|=12} \nu_\alpha av[t^\alpha], \tag{5.3}$$

where the coefficients $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ are nonnegative integers: The 6 nonzero coefficients $\mu_\alpha$ and corresponding monomials $\alpha$ are given in the following table.

| 000, 001, 111, 110 | 1236 | 000, 101, 101, 101 | 3594 | 001, 010, 011, 011 | 300 |
|-------------------|------|-------------------|------|-------------------|------|
| 000, 100, 101, 111 | 60   | 000, 101, 110, 101 | 114  | 001, 011, 101, 001 | 1014 |

The 114 nonzero coefficients $\nu_\alpha$ and corresponding monomials $\alpha$ are given in the fol-
It now follows from (5.3) that $M_4$ is nonnegative on feasible vectors and we obtain (5.1) as a consequence of (5.2).
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