IRREDUCIBILITY AND \( p \)-ADIC MONODROMIES ON THE SIEGEL MODULI SPACES

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ABSTRACT. We generalize the surjectivity result of the \( p \)-adic monodromy for the ordinary locus of a Siegel moduli space by Faltings and Chai (independently by Ekedahl) to that for any \( p \)-rank stratum. We discuss irreducibility and connectedness of some \( p \)-rank strata of the moduli spaces with parahoric level structure. Finer results are obtained on the Siegel 3-fold with Iwahori level structure.

1. Introduction

The present paper is a continuation of the author’s work [21]. In loc. cit. we have determined the number of irreducible components of a mod \( p \) Siegel moduli space with Iwahori level structure. The main ingredients are a result of Ngô and Genestier [15] that the ordinary locus is dense in the moduli space, and the surjectivity of a \( p \)-adic monodromy due to Faltings and Chai [7], also due to Ekedahl [5]. The goal of this paper is to investigate the same problem for the non-ordinary locus and smaller strata.

Let \( p \) be a rational prime number. Let \( N \geq 3 \) be a prime-to-\( p \) positive integer. We choose a primitive \( N \)-th root of unity \( \zeta_N \) in \( \overline{\mathbb{Q}} \subset \mathbb{C} \) and an embedding \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \). Let \( A_{g,1,N} \) denote the moduli space over \( \mathbb{Z}(\bar{\mathbb{F}}_p) \) of \( g \)-dimensional principally polarized abelian varieties with a full symplectic level-\( N \) structure with respect to \( \zeta_N \). The moduli scheme \( A_{g,1,N} \) has irreducible geometric fibers. Let \( A \) be the reduction \( A_{g,1,N} \otimes \mathbb{F}_p \) modulo \( p \). For each integer \( 0 \leq f \leq g \), let \( A^f \subset A \) be the locally closed reduced subscheme that classifies the objects \((A,\lambda,\eta)\) whose \( p \)-rank is \( f \). The \( p \)-rank of an abelian variety \( A \) is the dimension of \( A[p](\bar{\mathbb{k}}) \) over \( \mathbb{F}_p \). It is known due to Koblitz [12] that each stratum \( A^f \) is equi-dimensional of co-dimension \( f \) and the closure of the stratum \( A^f \) contains \( A^{f-1} \) for all \( f \). This result is generalized to the moduli spaces of arbitrary polarized abelian varieties by Norman and Oort [16].

Let \((\mathcal{X},\lambda,\eta)\to A^f\) be the universal family. The maximal etale quotient \( \mathcal{X}[p^\infty]^{et} \) of the \( p \)-divisible group \( \mathcal{X}[p^\infty] \) gives rise to a \( p \)-adic monodromy

\[
\rho^f : \pi_1(A^f,\bar{x}) \to GL_f(\mathbb{Z}_p),
\]

where \( \bar{x} \) is a geometric point of \( A^f \).

In this paper, we prove

**Theorem 1.1.** The homomorphism \( \rho^f \) is surjective.

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The case where \( f = g \) is a well-known result proved by Faltings and Chai [7] and independently by Ekedahl [5]. Theorem 1.1 answers a question raised in Tilouine [20, Remark below Theorem 2, p. 792].

A direct consequence of Theorem 1.1 is that the associated Igusa tower over each stratum \( A^f \) is irreducible except when \( f = 0 \) and \( g \leq 2 \) (this is the case where the stratum is supersingular, that is, it is entirely contained in the supersingular locus); see Section 3. We apply Theorem 1.1 to the almost ordinary locus of the moduli spaces with parahoric level structure and determine the number of irreducible components; see Section 4. In the special case of Iwahori level structure, we have the following result:

Let \( A_{g,1,N} \) denote the moduli space over \( \mathbb{Z}[[\xi]] \) which parametrizes equivalence classes of objects \((A,\lambda,\eta,H_{\bullet})_S\), where \( S \) is a \( \mathbb{Z}[[\xi]] \)-scheme, \((A,\lambda,\eta)\) is in \( A_{g,1,N}(S) \), and \( H_{\bullet} \) is a flag of finite flat subgroup schemes of \( A \): 

\[
H_{\bullet} : 0 \subset H_1 \subset H_2 \subset \cdots \subset H_g \subset A \]

such that each \( H_i \) is of rank \( p^i \) and \( H_g \) is isotropic for the Weil pairing \( e_{\lambda} \) induced by \( \lambda \). Let \( A_{g,0}(p,N) := A_{g,1,N} \otimes \overline{\mathbb{F}}_p \) be the reduction modulo \( p \) and let \( A_{g,0}^{g-1}(p) := A_{g,0}(p) \times_A A^{g-1} \), the almost ordinary locus of \( A_{g,0}(p) \). We prove (Corollary 4.3)

**Theorem 1.2.** For \( g \geq 2 \), the almost ordinary locus \( A_{g,0}^{g-1}(p) \) has \( g^2 - 1 \) irreducible components.

One might expect that the almost ordinary locus \( A_{g,0}^{g-1}(p) \) is dense in the non-ordinary locus \( A_{g,0}^{\text{non-ord}}(p) \). If this is true, then it would imply the same for the moduli spaces with any parahoric level structure as well (see an argument in [21] for the ordinary case), and then we could determine the number of irreducible components of these non-ordinary loci. However, it is false in general. We examine an example in Section 6 (see Proposition 6.3).

In Section 5 we show how to use Theorem 1.1 to determine the numbers of connected components of the \( p \)-rank strata.

In Sections 7 and 8 we give a geometric characterization of Kottwitz-Rapoport strata for the case \( g = 2 \). The characterization requires some knowledge on the supersingular locus. Therefore, a description of the supersingular locus is included. On the other hand, the characterization also gives us more information on the supersingular locus through the Kottwitz-Rapoport stratification. This information enables us to determine the number of irreducible components of each Kottwitz-Rapoport stratum.

2. Proof of Theorem 1.1

We may assume that \( 1 \leq f < g \) because the case \( f = g \) is done in [7] and there is nothing to show for \( f = 0 \). Since the stratum \( A^f \) is irreducible (Proposition 5.2), it suffices to show the statement for a specific geometric base point. Choose a point \( x_0 = \mathbf{A}_0 \) in \( A_{g-f,1,N} \otimes \overline{\mathbb{F}}_p \) whose \( p \)-rank is zero. Consider the morphism

\[
\alpha : A^\text{ord} \to A^f, \quad A \mapsto A_0 \times A,
\]

where \( A^\text{ord} \) is the ordinary locus of the reduction \( A_{f,1,N} \otimes \overline{\mathbb{F}}_p \) mod \( p \). Choose a geometric point \( \bar{x}_1 \) of \( A^\text{ord}_f \). We have the following commutative diagram for the
\[ \rho_f^{\text{ord}} : \pi_1(A_f^{\text{ord}}, \bar{x}_1) \to \rho^f : \pi_1(A_f, \bar{x}_0 \times \bar{x}_1) \to \GL_f(\mathbb{Z}_p). \]

Since \( \rho_{\text{ord}}^f \) is surjective \[7\], \( \rho^f \) is also surjective. This completes the proof of Theorem 1.1.

3. Irreducibility of the Igusa towers

3.1. Let \( f \) be an integer with \( 0 \leq f \leq g \). For each integer \( m \geq 0 \), let \( \mathcal{I}_f^m \) be the cover of \( A_f \) over \( \mathbb{F}_p \) which parametrizes equivalence classes of objects \((A, \lambda, \eta, \xi)\) where \( S \) is an \( \mathbb{F}_p \)-scheme, \((A, \lambda, \eta)_S \) is in \( A_f(S) \) and \( \xi \) is an isomorphism form \( \mu_{p^m} \) to the multiplicative part \( A[p^m]^{\text{mul}} \) of \( A[p^m] \) over \( S \). Let

\[ \mathcal{I}_f := \{ \mathcal{I}_f^m \}_{m \geq 0} \]

be the Igusa tower over the stratum \( A_f \).

**Theorem 3.1** (Oort). *Every non-supersingular Newton polygon stratum of \( A \) is irreducible.*

This is yet a unpublished result of Oort. See the sketch of his proof in \[18\].

**Proposition 3.2.** The stratum \( A_f \) is irreducible except when \( f = 0 \) and \( g \leq 2 \).

**Proof.** From all possible symmetric Newton polygons, we know that

(a) the stratum \( A_f \) is supersingular (means that every maximal point of \( A_f \) is supersingular) if and only if \( f = 0 \) and \( g \leq 2 \), and

(b) the stratum \( A_f \) contains a unique maximal Newton polygon stratum as an open dense subset.

Then the proposition follows from Theorem 3.1.

**Proposition 3.3.** The Igusa tower \( \mathcal{I}_f \) is irreducible except when \( f = 0 \) and \( g \leq 2 \).

**Proof.** The cover \( \mathcal{I}_f^m \) is etale over \( A_f \) and it represents the etale sheaf

\[ \mathcal{I}_{\text{sm}}(\mu_{p^m}^{\text{et}}, \mathcal{X}[p^m]^{\text{mul}}), \]

where \((\mathcal{X}, \lambda, \eta) \to A_f \) is the universal family. Therefore, the cover gives rise to the \( p \)-adic monodromy \( \rho_{\text{et}}^f : \pi_1(A_f, \bar{x}) \to \GL_f(\mathbb{Z}/p^m\mathbb{Z}) \). By Theorem 1.1, the homomorphism \( \rho_{\text{et}}^f \) is surjective for all \( m \geq 0 \). Therefore, each \( \mathcal{I}_f^m \) is irreducible if the base \( A_f \) is irreducible. Then the proposition follows from Proposition 3.2.

When \( f = 0 \), each member \( \mathcal{I}_f^m \) of \( \mathcal{I}_f \) is \( A^0 \); this is the trivial case. For the non-trivial cases \( f \geq 1 \), the Igusa towers are all irreducible.
3.2. We consider a variant of the Igusa towers. Let $f$ be an integer with $1 \leq f \leq g$. For each integer $m \geq 0$, let $H^f_{m} := (\mathbb{Z}/p^m \mathbb{Z} \times \mu_{p^m})^\otimes f$ and let $\varphi^f_m : H^f_{m} \times H^f_{m} \to \mu_{p^m}$ be the alternating pairing defined by

$$\varphi^f_m((m_i, \zeta), (n_i, \eta)) = \prod_{i=1}^{f} m_i^{m_i} \zeta^{-n_i}, \quad \forall m_i, n_i \in \mathbb{Z}/p^m \mathbb{Z}, \zeta, \eta \in \mu_{p^m}.$$

Let $J^f_m$ be the cover of $A^f$ over $\mathbb{F}_p$, which parametrizes equivalence classes of objects $(A, \lambda, \eta, \xi)_S$ where $S$ is an $\mathbb{F}_p$-scheme, $(A, \lambda, \eta)_S$ is in $A^f(S)$ and

$$\xi : (H^f_m)_S \to A[p^m]$$

is a monomorphism (both homomorphism and closed immersion) over $S$ such that

$$\varphi^f_m(x, y) = e_{A}(\xi(x, \xi(y)), \forall x, y \in H^f_{m},$$

where $e_{A}$ is the Weil pairing induced by $\lambda$. Let

$$J^f := \{J^f_m\}_{m \geq 0}$$

be the associated tower over the stratum $A^f$.

**Theorem 3.4.** The tower $J^f$ is irreducible, that is, each member $J^f_m$ is irreducible.

**Proof.** Let $(\mathcal{X}, \lambda, \iota) \to A^f$ be the universal family. Consider the canonical filtration

$$0 \subset \mathcal{X}[p^m]_{\text{mul}} \subset \mathcal{X}[p^m]_{0} \subset \mathcal{X}[p^m],$$

where $\mathcal{X}[p^m]_{0}$ is the neutral connected component of $\mathcal{X}[p^m]$. So we have two canonical short exact sequences

$$0 \to \mathcal{X}[p^m]_{\text{mul}} \to \mathcal{X}[p^m]_{0} \to \mathcal{X}[p^m]_{\text{loc,loc}} \to 0,$$

$$0 \to \mathcal{X}[p^m]_{0} \to \mathcal{X}[p^m] \to \mathcal{X}[p^m]_{\text{et}} \to 0.$$

Since these short exact sequences split over a perfect affine base in characteristic $p$, we can find a finite radical surjective morphism $\pi : \mathcal{A}' \to A^f$ such that the base change $\mathcal{X}[p^m]_{\mathcal{A}'}$ admits the canonical decomposition

$$(3.1) \quad \mathcal{X}[p^m]_{\mathcal{A}'} = (\mathcal{X}[p^m]_{\mathcal{A}'} \oplus \mathcal{X}[p^m]_{\mathcal{A}'_{\text{et}}} \oplus \mathcal{X}[p^m]_{\mathcal{A}'_{\text{loc,loc}}},$$

where $\mathcal{X}[p^m]_{\mathcal{A}'}$ is the base change $\mathcal{X}[p^m]_{\mathcal{A}'} \times_{A^f} \mathcal{A}'$, the middle part $\mathcal{X}[p^m]_{\mathcal{A}'}$ is the maximal etale subgroup scheme of $\mathcal{X}[p^m]_{\mathcal{A}'}$, and $\mathcal{X}[p^m]_{\mathcal{A}'_{\text{loc,loc}}}$ is the maximal locally-local subgroup scheme of $\mathcal{X}[p^m]_{\mathcal{A}'}$. Furthermore, we may choose $\mathcal{A}'$ to be irreducible. To see this, let $\mathcal{A}_0$ be a scheme over $\mathbb{F}_q$ such that $A^f \simeq \mathcal{A}_0 \otimes_{\mathbb{F}_q} \mathbb{F}_p$. Let $\{U\}$ be a finite open covering of affine subschemes of $\mathcal{A}_0$. Choose a positive integer $n$ large enough such that $\mathcal{X}[p^m]_{\mathcal{A}_0}$ admits the canonical decomposition as (3.1) over $U^{(q^n)}$ for each $U$, where $F^n : U^{(q^n)} \to U$ is the iterated relative Frobenius morphism over $\mathbb{F}_q$. The subgroup schemes

$$\{(\mathcal{X}[p^m]_{U^{(q^n)}})_{\text{sub}}(U^{(q^n)})\}_{U^{(q^n)}}, \quad \text{(resp. } \{(\mathcal{X}[p^m]_{U^{(q^n)}})_{\text{loc,loc}}(U^{(q^n)})\}_{U^{(q^n)}}\}$$

glue to a subgroup scheme $\mathcal{X}[p^m]_{\text{sub}}$ (resp. $\mathcal{X}[p^m]_{\text{loc,loc}}$) of $\mathcal{A}_0^{(q^n)}$. Clearly $\mathcal{A}_0^{(q^n)}$ is irreducible. We may take $\mathcal{A}' := \mathcal{A}_0^{(q^n)} \otimes_{\mathbb{F}_q} \mathbb{F}_p$ and let $\pi = F^n$, then $\mathcal{X}[p^m]_{\mathcal{A}'}$ admits the canonical decomposition.

Let $J^f_m$ be the etale cover of $\mathcal{A}'$ that represents the etale sheaf

$$\mathcal{P}_m := \text{Isom}(H^f_{m}, \varphi^f_m, (\mathcal{X}[p^m]_{\mathcal{A}'} \oplus \mathcal{X}[p^m]_{\mathcal{A}'_{\text{et}}}, e_{\lambda})).$$
Since any section $\xi$ of $P_m$ is determined by its restriction $\xi$ on $(\mathbb{Z}/p^m)\oplus f$, the restriction map $\xi \mapsto \xi|_{(\mathbb{Z}/p^m)\oplus f}$ gives an isomorphism

$$P_m \simeq Isom((\mathbb{Z}/p^m)\oplus f, X[p^m]_{et}).$$

Therefore, $J_m'$ corresponds to the $p$-adic monodromy $\rho_m' : \pi_1(A', \bar{x}') \to GL_f(\mathbb{Z}/p^m\mathbb{Z})$ and we have the commutative diagram

$$\pi_1(A', \bar{x}') \xrightarrow{\pi_*} \pi_1(A', \bar{x}) \xrightarrow{\rho_m'} GL_f(\mathbb{Z}/p^m\mathbb{Z}).$$

Since $\pi_*$ is an isomorphism and $\rho_m'$ is surjective, $J_m'$ is irreducible. Let $J_m''$ be the base change of $J_m'$ over $\mathbb{Z}/p^m\mathbb{Z}$ via $\rho_m'$. Then the family $(\mathcal{X}', \lambda', \eta')$ gives rise to a morphism $\alpha : J_m' \to J_m''$. Clearly this map is surjective, hence $J_m''$ is irreducible.

**Remark 3.5.** The irreducibility of $J_1^g$ is studied in [21] (that was denoted $A_{1,1,N}^{ord}(\mathbb{C})$ there). The lines 1-2 of p. 2593 in loc. cit. are incorrect. The moduli scheme $A_{1,1,N}^{ord}(\mathbb{C})$ is not etale over $A_{1,1,N}^{ord}(\mathbb{C})$ because the extension

$$0 \to \mathcal{X}[p] \to \mathcal{X}[p] \to \mathcal{X}[p]_{et} \to 0$$

does not split over any finite etale base change. However, this does not effect the conclusion on irreducibility of $A_{1,1,N}^{ord}(\mathbb{C})$; we just need a modified argument as in the proof above.

4. **The almost ordinary locus of the moduli spaces with parahoric level structure**

4.1. We keep the notation as before. Let $\mathbf{k} = (k_1, \ldots, k_r)$ be a tuple of positive integers $k_i \geq 1$ with $\sum_{i=1}^{r} k_i \leq g$. Set $h(i) := \sum_{j=1}^{i} k_j$ for $1 \leq i \leq r$ and $h(0) = 0$. Let $A_{g,\mathbf{k},N}$ denote the moduli space over $\mathbb{Z}[\zeta_N]$ that parametrizes equivalence classes of objects $(A, \lambda, \eta, H_\bullet)_S$, where $S$ is a $\mathbb{Z}[\zeta_N]$-scheme, $(A, \lambda, \eta)$ is in $A_{g,1,N}(S)$, and $H_\bullet$ is a flag of finite subgroup schemes of $A[p]$

$$H_\bullet \quad 0 = H_{h(0)} \subset H_{h(1)} \subset \cdots \subset H_{h(r)} \subset A[p]$$

such that $H_{h(i)}$ is locally free of rank $p^{h(i)}$ and $H_{h(r)}$ is isotropic for the Weil pairing $e_\lambda$ induced by the polarization $\lambda$. When $r = g$, the moduli scheme $A_{g,\mathbf{k},N}$ is $A_{g,1,N}(\mathbb{C})$ defined in Section 1.

Let $A_{\mathbf{k}} := A_{g,\mathbf{k},N} \otimes \overline{\mathbb{F}}_p$ be the reduction modulo $p$. For $0 \leq f \leq g$, let $A_{\mathbf{k}}^f := A_{\mathbf{k}} \times_A A^f$, the $p$-rank $f$ stratum of the moduli space $A_{\mathbf{k}}$. For an $\mathbb{F}_p$-scheme $S$, the $S$-valued set $A_{\mathbf{k}}^f(S)$ consists of objects $(A, \lambda, \eta, H_\bullet)_S$ in $A_{\mathbf{k}}(S)$ such that the canonical morphism $S \to A$ given by the family $(A, \lambda, \eta)_S$ factors through the subscheme $A^f$. Note that from the definition one can not determine whether $A_{\mathbf{k}}^f$ is reduced. We will compute the number of irreducible components of $A_{\mathbf{k}}^f$ in the case where $f = g - 1$, the almost ordinary locus.
We first seek discrete invariants for geometric points on \( A^g_{\mathbb{K}} \). Let \( k \) be an algebraically closed field of characteristic \( p \). Fix a supersingular elliptic curve \( E_0 \) over \( \overline{\mathbb{F}}_p \). Let \((A, \lambda, \eta, H_*)\) be a point of \( A^g_{\mathbb{K}}(k) \). We have

\[
A[p] \simeq (\mathbb{Z}/p\mathbb{Z} \times \mu_p)^g \times E_0[p].
\]

There are two cases:

(a) \( H_{h(r)} \) does not have local-local part. This occurs only when \( h(r) < g \).

For each \( 1 \leq i \leq r \), the finite group scheme \( H_h(i)/H_h(i-1) \) has the form

\[
\mu_{p^{\tau(i)}} \times (\mathbb{Z}/p\mathbb{Z})^{h_1-\tau(i)}
\]

for a non-negative integer \( 0 \leq \tau(i) \leq k_i \).

(b) \( H_{h(r)} \) has non-trivial local-local part. There is a unique integer \( 1 \leq j \leq r \) such that \( H_{h(j)} \) has non-trivial local-local part and \( H_{h(j-1)} \) has no local-local part. For each \( 1 \leq i \leq r \), the finite group scheme \( H_h(i)/H_h(i-1) \) has the form

\[
\left\{
\begin{array}{ll}
\mu_{p^{\tau(i)}} \times (\mathbb{Z}/p\mathbb{Z})^{h_1-\tau(i)} & \text{for some integer } 0 \leq \tau(i) \leq k_i \\
\mu_{p^{\tau(i-1)}} \times (\mathbb{Z}/p\mathbb{Z})^{h_1-\tau(i)} \times \alpha_p & \text{for some integer } 0 \leq \tau(i) \leq k_i - 1
\end{array}
\right.
\]

if \( i \neq j \).

4.2. For each \( m \geq 0 \), define \( I(m) := [0, m] \cap \mathbb{Z} \). Set

\[
I^0_{\mathbb{K}} := \begin{cases} 
\emptyset & \text{if } h(r) = g; \\
\{0\} \times \prod_i I(k_i) & \text{if } h(r) < g.
\end{cases}
\]

For each \( 1 \leq j \leq r \), set

\[
I^j_{\mathbb{K}} := \{ \underline{\tau} = (j, \tau(1), \ldots, \tau(r)) : \tau(i) \in I(k_i) \text{ for } i \neq j \text{ and } \tau(j) \in I(k_j - 1) \}.
\]

The finite set \( I^0_{\mathbb{K}} \) (resp. \( I^j_{\mathbb{K}} \) for \( j > 0 \)) will be used to parameterize discrete invariants in the case (a) (resp. the case (b)).

For \( \underline{\tau} = (0, \tau(1), \ldots, \tau(r)) \in I^0_{\mathbb{K}} \), we say a geometric point \( \underline{A} \) in \( A^g_{\mathbb{K}} \) is of type \( \underline{\tau} \) if

- \( H_{h(r)} \) has no local-local part, and
- the multiplicative part of \( H_{h(i)}/H_{h(i-1)} \) is of rank \( p^{\tau(i)} \) for all \( 1 \leq i \leq r \).

For \( \underline{\tau} = (j, \tau(1), \ldots, \tau(r)) \in I^j_{\mathbb{K}} \), where \( 1 \leq j \leq r \), we say a geometric point \( \underline{A} \) in \( A^g_{\mathbb{K}} \) is of type \( \underline{\tau} \) if

- \( H_{h(j)} \) has local-local part and \( H_{h(j-1)} \) has no local-local part, and
- the multiplicative part of \( H_{h(i)}/H_{h(i-1)} \) is of rank \( p^{\tau(i)} \) for all \( 1 \leq i \leq r \).

Then we have an assignment \( \underline{A} \mapsto \underline{\tau}(\underline{A}) \), which gives a surjective map

\[
\tau : A^g_{\mathbb{K}} \to \prod_{0 \leq j \leq r} I^j_{\mathbb{K}}.
\]

It is easy to see that this map is locally constant for the Zariski topology (use the argument in the proof of Theorem 3.4). For a fixed type \( \underline{\tau} \), let \( A^g_{\mathbb{K},\underline{\tau}} \) be the union of the connected components of \( A^g_{\mathbb{K}} \) whose objects are of type \( \underline{\tau} \). We write \( A^g_{\mathbb{K}} \) into a disjoint union of open subschemes

\[
A^g_{\mathbb{K}} = \prod_{0 \leq j \leq r} \prod_{\underline{\tau} \in I^j_{\mathbb{K}}} A^g_{\mathbb{K},\underline{\tau}}.
\]
Theorem 4.1. For each integer \(0 \leq j \leq r\) and each type \(\tau \in I_k^j\), there is a finite surjective morphism \(\pi_{\tau} : J_{1}^{\tau-1} \to A_{k-1}^{g-1}\). Consequently, each stratum \(A_{k-1}^{g-1}\) is irreducible for \(g \geq 2\).

Proof. Let \((X, \lambda, \eta, \xi) \to J_{1}^{\tau-1}\) be the universal family. The image \(\xi(H_{1}^{\tau-1})\) is the etale-multiplicative part \(X[p]_{\text{em}}\) of \(X[p]\), and the orthogonal complement of \(\xi(H_{1}^{\tau-1})\) for the Weil pairing \(e_\lambda\) is the local-local part \(X[p]_{\text{loc,loc}}\) of \(X[p]\). Namely, we have
\[
X[p] = X[p]_{\text{em}} \times X[p]_{\text{loc,loc}}, \quad \xi : (\mu_p \times \mathbb{Z}/p\mathbb{Z})^{g-1} \cong X[p]_{\text{em}}.
\]

Let \(C\) be the kernel of the relative Frobenius morphism
\[
F_{X/J_{1}^{\tau-1}} : X[p]_{\text{loc,loc}} \to (X[p]_{\text{loc,loc}})(p).
\]

(a) If \(\tau \in I_k^0\), let \(K_i := \mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-\tau(i)}\) for \(1 \leq i \leq r\). For \(1 \leq m \leq r\), set \(H_{m} := \xi(\prod_{i=1}^{m} K_i)\). Then we define a family \((X, \lambda, \eta, H_{\bullet}) \to J_{1}^{\tau-1}\) and this family induces a natural morphism \(\pi_{\tau} : J_{1}^{\tau-1} \to A_{k-1}\).

(b) If \(\tau \in I_k^j\) for some \(1 \leq j \leq r\), let
\[
K_i := \begin{cases} 
\mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-\tau(i)} & \text{if } i \neq j; \\
\mu_p^{\tau(i)} \times (\mathbb{Z}/p\mathbb{Z})^{k_i-1-\tau(i)} & \text{if } i = j.
\end{cases}
\]

For \(1 \leq m \leq r\), set
\[
H_{m} := \begin{cases} 
\xi(\prod_{i=1}^{m} K_i) & \text{if } m < j; \\
\xi(\prod_{i=1}^{m} K_i) \times C & \text{if } m \geq j.
\end{cases}
\]

Then we define a family \((X, \lambda, \eta, H_{\bullet}) \to J_{1}^{\tau-1}\) and this family induces a natural morphism \(\pi_{\tau} : J_{1}^{\tau-1} \to A_{k-1}\).

It is clear that \(\pi_{\tau}\) factors through the almost ordinary locus \(A_{k-1}^{g-1}\). Moreover, the image lands in the open subscheme \(A_{k-1}^{g-1}\) by the construction. So we get the morphism \(\pi_{\tau} : J_{1}^{\tau-1} \to A_{k-1}^{g-1}\). One checks easily that \(\pi_{\tau}\) is surjective. Since the composition \(J_{1}^{\tau-1} \to A_{k-1}^{g-1} \to A^{g-1}\) is finite, \(\pi_{\tau}\) is finite. This completes the proof.

Note that in the proof we use the universal family to define the morphism \(\pi_{\tau}\) instead of defining \(\pi_{\tau}(x)\) pointwisely. The reason is that \(J_{1}^{\tau-1}\) or \(A_{k-1}^{g-1}\) (defined by the fiber product) could be non-reduced.

Corollary 4.2. For \(g \geq 2\), the almost ordinary stratum \(A_{k-1}^{g-1}\) has
\[
\sum_{j=0}^{r} |I_k^{j}| = (k_1 + 1) \cdots (k_r + 1) \left[\epsilon + \frac{k_1}{k_1+1} + \cdots + \frac{k_r}{k_r+1}\right]
\]
irreducible components, where \(\epsilon = 1\) if \(h(r) < g\) and \(\epsilon = 0\) if \(h(r) = g\).

For the Iwahori case, \(r = g\) and \(k_i = 1\) for all \(i\), so Corollary 4.2 gives

Corollary 4.3. For \(g \geq 2\), the almost ordinary locus \(A_{k-1}^{g-1}\) has \(g2^{g-1}\) irreducible components.
5. Connected components of \( p \)-rank strata

In the previous section we study irreducible components of the almost ordinary locus. In this section we consider lower \( p \)-rank strata. We know that when \( g \geq 2 \) and \( 0 \leq f \leq g - 2 \), the natural morphism \( \mathcal{A}_k^f \to \mathcal{A}_k^f \) is not finite in general. This limits the method of using \( p \)-adic monodromy to the irreducibility problem in the present case. The obstacle results from the fibration “moving \( \alpha_p \)-subgroups”. If one contracts the fibration, then one obtains a finite morphism for which the \( p \)-adic monodromy results can be applied. Proceeding this approach, we obtain information on connected components instead.

Keep the notation in the previous sections. Assume that \( g \geq 2 \).

5.1. Fix a tuple \( k = (k_1, \ldots, k_r) \) of positive integers with \( \sum_{i=1}^r k_i \leq g \) and an integer \( f \) with \( 0 \leq f \leq g - 2 \) as before. Again, we first seek discrete invariants for geometric points in \( \mathcal{A}_k^f(k) \). Let \( (A, \lambda, \eta, H) \) be a point in \( \mathcal{A}_k^f(k) \). We have a decomposition

\[
H_{h(i)} = \left( H_{h(i)}^{\text{et}} \oplus H_{h(i)}^{\text{mul}} \right) \oplus H_{h(i)}^{\text{loc}, \text{loc}}
\]

into etale-multiplicative part and local-local part. Suppose that \( H_{h(i)}^{\text{et}} \oplus H_{h(i)}^{\text{mul}} \) has rank \( p^{a(i)} \) and \( H_{h(i)}^{\text{loc}, \text{loc}} \) has rank \( p^{b(i)} \). Put \( a(0) = 0 \) and \( m(i) := a(i) - a(i - 1) \) for each \( 1 \leq i \leq r \). It is easy to see that

\[
(5.1) \quad 0 \leq m(i) \leq k_i, \quad \text{and} \quad f - (g - h(r)) \leq \sum_{i=1}^r m(i) \leq f,
\]

where \( h(i) = \sum_{j=1}^i k_j \) as before. Let \( G_i := H_{h(i)}^{\text{et}} \oplus H_{h(i)}^{\text{mul}} \). Then the successive quotient \( G_i/G_{i-1} \) has rank \( p^{m(i)} \). Let

\[
\tau(i) := \log_p \text{rank}(G_i/G_{i-1})^{\text{mul}}, \quad \forall 1 \leq i \leq r.
\]

We have \( 0 \leq \tau(i) \leq m(i) \) for all \( i \). We call the pair of \( r \)-tuples

\[
\left( m, \tau \right) = [(m(1), \ldots, m(r)), (\tau(1), \ldots, \tau(r))]
\]

the graded etale-multiplicative type associated to the object \((A, \lambda, \eta, H)\), abbreviated as gem type.

Conversely, fix a tuple of integers \((m(1), \ldots, m(r))\) satisfying (5.1). Let \((A, \lambda, \eta)\) be an object in \( \mathcal{A}_k^f \) and \( G_\bullet \) a flag of finite flat subgroup schemes

\[
0 = G_0 \subset G_1 \subset \cdots \subset G_r \subset A[p]
\]

such that

1. the group scheme \( G_r \) is isotropic with respect to the Weil pairing \( e_\lambda \), and
2. each \( G_i \) has no local-local part and the quotient \( G_i/G_{i-1} \) has rank \( p^{m(i)} \).

Then one can lift to an object \((A, \lambda, \eta, H) \in \mathcal{A}_k^f(k)\) such that \( H_{h(i)}^{\text{et}} \oplus H_{h(i)}^{\text{mul}} = G_i \) for all \( i \).
5.2. Define
\[ \Sigma_0(k, f) := \{ \mathbf{m} = (m(1), \ldots, m(r)) \in \mathbb{Z}^r | \]
\[ 0 \leq m(i) \leq k_i, \forall i, \text{ and } f - (g - h(r)) \leq \sum_i m(i) \leq f \}, \]
(5.2)
\[ \Sigma(k, f) := \{ (\mathbf{m}, \mathbf{\tau}) = ((m(1), \ldots, m(r)), (\tau(1), \ldots, \tau(r))) \in \mathbb{Z}^r \times \mathbb{Z}^r | \]
\[ \mathbf{m} \in \Sigma_0(k, f), \text{ and } 0 \leq \tau(i) \leq m(i), \forall i \}. \]
(5.3)
We have a natural map from \( \Sigma(k, f) \) to \( \Sigma_0(k, f) \) sending \((\mathbf{m}, \mathbf{\tau})\) to \(\mathbf{m}\). The finite set \( \Sigma(k, f) \) is exactly that of all possible graded etale-multiplicative types of points in \( A^f_k \).

For any element \( \mathbf{m} \in \Sigma_0(k, f) \), we define a scheme \( T(\mathbf{m}) \) over \( \overline{\mathbb{F}_p} \) as follows. For any locally Noetherian \( \mathbb{F}_p \)-scheme \( S \), the \( S \)-valued set \( T(\mathbf{m})(S) \) classifies equivalence classes of objects \((A, \lambda, \eta, G_\bullet)_S\), where
- \((A, \lambda, \eta)_S\) is in \( A^f(S) \), and
- \( G_\bullet \) is a flag of finite flat subgroup schemes

\[ G_\bullet : \quad 0 = G_0 \subset G_1 \subset \cdots \subset G_r \subset A[p] \]
satisfying the conditions (1) and (2) above.

For any element \((\mathbf{m}, \mathbf{\tau}) \in \Sigma(k, f)\), let \( T(\mathbf{m}, \mathbf{\tau}) \) be the (open) subscheme of \( T(\mathbf{m}) \) that consists of objects \((A, \lambda, \eta, G_\bullet)_S\) such that the multiplicative part of the quotient \( G_i/G_{i-1} \) has rank \( \rho^{\tau(i)} \) for all \( 1 \leq i \leq r \). Put \( T^f_k := \coprod_{\mathbf{m} \in \Sigma_0(k, f)} T(\mathbf{m}) \). Clearly, we have
\[ T^f_k = \coprod_{(\mathbf{m}, \mathbf{\tau}) \in \Sigma(k, f)} T(\mathbf{m}, \mathbf{\tau}). \]

Let \((\mathcal{X}, \lambda, \eta, \tilde{H}_\bullet) \rightarrow A^f_k \) be the universal family over \( A^f_k \). Then there is a finite dominant homeomorphic morphism \( \pi : \mathcal{A}' \rightarrow A^f_k \) such that the base change \( \mathcal{A}'[p]_{/\mathcal{A}'} \) admits the canonical decomposition
\[ \mathcal{A}'[p]_{/\mathcal{A}'} = (\mathcal{A}'[p]_{/\mathcal{A}'})^{\text{mul}} \oplus (\mathcal{A}'[p]_{/\mathcal{A}'})^{\text{et}} \oplus (\mathcal{A}'[p]_{/\mathcal{A}'})^{\text{loc}, \text{loc}} \]
into etale-multiplicative part and local-local part; see the proof of Theorem 6.3. Accordingly, we have the same decomposition
\[ \tilde{H}_{h(i), \mathcal{A}'} = (\tilde{H}_{h(i)}^{\text{mul}} \oplus \tilde{H}_{h(i)}^{\text{et}}) \oplus \tilde{H}_{h(i)}^{\text{loc}, \text{loc}} \]
for all \( i \).

Since these subgroup schemes are locally free, their ranks are constant on each connected component of \( \mathcal{A}' \). Let \( \mathcal{A}'(\mathbf{m}, \mathbf{\tau}) \subset \mathcal{A}' \) be the union of the connected components whose objects are of gem type \((\mathbf{m}, \mathbf{\tau})\). Let \( \mathcal{A}(\mathbf{m}, \mathbf{\tau}) \subset A^f_k \) be the open subscheme \( \pi(\mathcal{A}'(\mathbf{m}, \mathbf{\tau})) \). In particular, \( \pi : \mathcal{A}'(\mathbf{m}, \mathbf{\tau}) \rightarrow A(\mathbf{m}, \mathbf{\tau}) \) is a homeomorphism. Again, we have
\[ A^f_k = \coprod_{(\mathbf{m}, \mathbf{\tau})} A(\mathbf{m}, \mathbf{\tau}), \quad A' = \coprod_{(\mathbf{m}, \mathbf{\tau})} A'(\mathbf{m}, \mathbf{\tau}), \]
(5.6)
where \((\mathbf{m}, \mathbf{\tau})\) runs through all elements in \( \Sigma(k, f) \).

Consider the universal family \((\mathcal{X}, \lambda, \eta, \tilde{H}_\bullet)_{\mathcal{A}'(\mathbf{m}, \mathbf{\tau})}\) restricted on the open subscheme \( \mathcal{A}'(\mathbf{m}, \mathbf{\tau}) \). Put
\[ \tilde{G}_i := \tilde{H}_{h(i)}^{\text{mul}} \oplus \tilde{H}_{h(i)}^{\text{et}}, \]
for all \( i \). We get a family \( (\mathcal{X}, \lambda, \eta, \tilde{G}_*)_{\mathcal{A}'(m, z)} \). This gives rise to a natural morphism
\[
(5.7) \quad c(m, z) : \mathcal{A}'(m, z) \to T(m, z),
\]
which is proper and surjective (see Subsection 5.1).

If \( f = 0 \), then the set \( \Sigma(k, f) \) consists of only one element \((0, 0)\). In this case, we have
\[
(5.8) \quad \mathcal{A}' = A^0_k, \quad T^0_k = A^0, \quad \text{and} \quad c(0, 0) : A^0_k \to A^0.
\]

**Proposition 5.1.**

1. The stratum \( \mathcal{A}^0 \) is connected and it is irreducible if \( g \geq 3 \).
2. Suppose that \( f \geq 1 \). Then there is a finite surjective morphism \( \mathcal{J}^f \to T(m, z) \).

Consequently, each scheme \( T(m, z) \) is irreducible.

**Proof.** (1) When \( g = 2 \), this is a special case of Proposition 7.3 in Oort [17]. When \( g \geq 3 \), this is obtained in the proof of Proposition 3.3 using Theorem 3.1, a result of Oort.

(2) The construction is similar to that as in Theorem 4.1. Therefore, we do not repeat it. 

**5.3.** Let \( \mathbf{n} = (n(1), \ldots , n(r')) \) be a tuple of positive integers with \( \sum_{i=1}^{r'} n(i) \leq g - f \). We may identify the moduli space \( \mathcal{A}_n \) with the moduli space that parametrizes equivalence classes of chains of isogenies
\[
\mathcal{A}_n : \quad A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{} \cdots \xrightarrow{} A_{r'-1} \xrightarrow{\alpha_{r'}} A_{r'},
\]
where

- each \( A_i \) is a polarized abelian scheme with a symplectic level-\( N \) structure,
- \( A_0 \) is an object in \( \mathcal{A}_n \), and
- each \( \alpha_i \) is an isogeny of degree \( p^{n(i)} \) that preserves the level structures and the polarizations except when \( i = 1 \), and in this case one has \( \alpha_1^* \lambda_1 = p \lambda_0 \).

Define \( W_n \subset \mathcal{A}_n \) to be the reduced subscheme consisting of objects \( A_i \) such that the kernel \( \ker \alpha_i \) is of local-local type for all \( i \).

Let \( (m, z) \) be an element in \( \Sigma(k, f) \). Let \( a(i) := \sum_{j=1}^{i} m(j) \) and \( b(i) := h(i) - a(i) \). Put \( a(0) = b(0) = 0 \). Write the set \( \{b(i) : 0 \leq i \leq r'\} \) as
\[
\{b'(0), b'(1), \ldots , b'(r')\} \quad \text{with} \quad 0 = b'(0) < b'(1) < \cdots < b'(r').
\]
We have \( r' \leq r, r' \leq b'(r') \) and \( h(r) - f \leq b'(r') \leq g - f \). Set \( n(i) := b'(i) - b'(i - 1) \) for \( 1 \leq i \leq r' \). So we define a tuple \( \mathbf{n} \) of integers from the pair \( (m, z) \), and have a scheme \( W_n \).

Let \( x = (A_x, \lambda_x, \eta_x, G_*) \) be a point in \( T(m, z)(k) \). The fiber \( c(m, z)^{-1}(x) \) consists of flags of finite flat subgroup schemes
\[
0 = K_{b(0)} \subset K_{b(1)} \subset \cdots \subset K_{b(r')} \subset A_x[p]
\]
such that \( K_{b(r')} \) is isotropic for the Weil pairing \( e_{\lambda_x} \) and each \( K_{b(i)} \) is local-local of rank \( p^{b(i)} \). This is the same as flags of finite flat subgroup schemes
\[
0 = K_{b'(0)} \subset K_{b'(1)} \subset \cdots \subset K_{b'(r')} \subset A_x[p]
\]
with the same properties. This proves \( c(m, z)^{-1}(x) = W_n(x) \), where
\[
W_n(x) := \{ A_i \in W_n : A_i = (A_x, \lambda_x, \eta_x) \}.
\]
It is easy to see that the reduced scheme $W_n(x)$ is connected. Indeed, for $1 \leq d \leq g - f$, let $1^d := (1, \ldots, 1)$ with length $d$. We see that the fiber of the natural morphism $W_1(x) \to W_1^{e_{d-1}}(x)$ is a projective space, as it is the family of $\alpha_p$-subgroups in $\ker \lambda_{e_{d-1}}$. This shows that the scheme $W_1^{e_{d-1}}(x)$ is connected. Since the forgetful morphism $W_1^{e_{d-1}}(x) \to W_1(x)$ is surjective, the scheme $W_1(x)$ is connected. In conclusion, we have proved

**Proposition 5.2.** Any fiber of the morphism $c(m, \tau)$ (5.7) is connected.

**Theorem 5.3.** Every open subscheme $A(m, \tau) \subset A^f$ is connected. Consequently, the $p$-rank $f$ stratum $A^f_2$ has $|\Sigma(k, f)|$ connected components.

**Proof.** It follows from Propositions 5.1 and 5.2 that $A'_2$ is connected. Since $A_2$ is homeomorphic to $A'_2$, it is connected. The second statement follows from (5.6).

**Remark 5.4.** The scheme $T^f_2$ is closely related to the Stein factorization of the natural morphism $A^f_2 \to A^f$. Indeed, let $T$ (resp. $T'$) be the Stein factorization of $A^f_2 \to A^f$ (resp. of $A' \to A^f$). Then there are natural finite morphisms $\pi_1 : T' \to T$ and $\pi_2 : T' \to T^f_2$. It is not hard to see that these morphisms are homeomorphic. In some sense, $T^f_2$ provides a “modular interpretation” of the scheme $T$.

6. The Siegel 3-fold with Iwahori level structure

In this section we describe the Kottwitz-Rapoport stratification on the Siegel 3-fold $A_{2, \Gamma_0(p)}$ with Iwahori level structure. Our references are de Jong [3], Kottwitz and Rapoport [13], T. Haines [8], and Ngô and Genestier [15]. The geometric part of Tilouine [20] is also helpful to us. Then we conclude the following results as consequences:

(a) The almost ordinary locus $A^f_{2, \Gamma_0(p)}$ is not dense in the non-ordinary locus $A^\text{non-ord}_{2, \Gamma_0(p)}$.

(b) The supersingular locus $S_{2, \Gamma_0(p)}$ of $A_{2, \Gamma_0(p)}$ is not equi-dimensional. It consists of both one-dimensional components and two-dimensional components.

6.1. **Local models.** The Kottwitz-Rapoport stratification is defined through local model diagrams. Let $O$ be a complete discrete valuation ring, $K$ its fraction field, $\pi$ an uniformizer of $O$, and $\kappa := O/\pi O$ the residue field. We require that $\text{char} \kappa = p > 0$. Set $V := K^{2n}$ and let $e_1, \ldots, e_{2n}$ be the standard basis. Denote by $\psi : V \times V \to K$ the non-degenerate alternating form whose non-zero pairings are

$\psi(e_i, e_{2n+1-i}) = 1, \quad 1 \leq i \leq n,$

$\psi(e_i, e_{2n+1-i}) = -1, \quad i \geq n + 1.$

The representing matrix for $\psi$ is

$$
\begin{pmatrix}
0 & \tilde{I} \\
\tilde{I} & 0
\end{pmatrix},
\quad \tilde{I} = \text{anti-diag}(1, \ldots, 1).
$$
Let $\text{GSp}_{2n}$ be the reductive algebraic group of symplectic similitudes with respect to $\psi$. Let

$$\pi L_0 = L_{-2n} \subset L_{-2n+1} \subset \cdots \subset L_{-1} \subset L_0 = \mathcal{O}^{2n}$$

be a chain of $\mathcal{O}$-lattices in $V$ where the lattice $L_{-i}$ is generated by $e_1, \ldots, e_{2n-i}, \pi e_{2n-i+1}, \ldots, \pi e_{2n}$. The $\mathcal{O}$-submodule in $V$ generated by $x_1, \ldots, x_k \in V$ is denoted by $<x_1, \ldots, x_k>$. Thus $L_{i-2n} = <e_1, \ldots, e_i, \pi e_{i+1}, \ldots, \pi e_{2n}>$.

For $0 \leq i \leq 2n$, let $\Lambda_{i-2n} = \mathcal{O}^{2n}$ and define $\beta_{i-2n}: \Lambda_{i-2n} \to \Lambda_{i-2n+1}$ for $i < 2n$ by

$$\beta_{i-2n}(e_{i+1}) = \pi e_{i+1}, \quad \text{and} \quad \beta_{i-2n}(e_j) = e_j \quad \text{for} \; j \neq i + 1.$$  

We have

$$\Lambda_{-2n} \xrightarrow{\beta_{-2n}} \Lambda_{-2n+1} \xrightarrow{\beta_{-3}} \cdots \xrightarrow{\beta_{-1}} \Lambda_{-1} \xrightarrow{\beta_{0}} \Lambda_0,$$

and there is a unique isomorphism $a_{-i}: \Lambda_{-i} \to L_{-i}$ with $a_0 = \text{id}$ such that the diagram

$$\begin{array}{c}
\Lambda_{-i} \xrightarrow{\beta_{-i}} \Lambda_{-i+1} \\
\downarrow a_{-i} \quad \quad \downarrow a_{-i+1} \\
L_{-i} \xrightarrow{\text{incl}} L_{-i+1}
\end{array}$$

commutes. Let $\tilde{\beta}_{-i}: \Lambda_{-i} \to \Lambda_0$ denote the composition of the morphisms

$$\Lambda_{-i} \xrightarrow{\beta_{-i}} \Lambda_{-i+1} \xrightarrow{\beta_{-i+1}} \cdots \xrightarrow{\beta_{-1}} \Lambda_{-1} \xrightarrow{\beta_{0}} \Lambda_0.$$  

Let $\psi_0 = \psi$ be on the form on $\Lambda_0 = L_0$. There is a perfect non-degenerate alternating form $\psi_n$ on $\Lambda_{-n}$ such that

$$\psi_0(\tilde{\beta}_{-n}(x), \tilde{\beta}_{-n}(y)) = \pi \psi_n(x, y), \quad \forall \; x, y \in \Lambda_{-n}.$$  

Let $\mathcal{M}^{\text{loc}}$ denote the projective $\mathcal{O}$-scheme that represents the functor which sends an $\mathcal{O}$-scheme $S$ to the set of the collections of locally free $\mathcal{O}_S$-submodules $\mathcal{F}_{-i} \subset \Lambda_{-i} \otimes \mathcal{O}_S$ of rank $n$ for $0 \leq i \leq n$ such that

(i) $\mathcal{F}_0$ and $\mathcal{F}_{-n}$ are isotropic with respect to $\psi_0$ and $\psi_{-n}$, respectively.

(ii) $\mathcal{F}_{-i}$ locally is a direct summand of $\Lambda_{-i} \otimes \mathcal{O}_S$ for all $i$.

(iii) $\beta_{-i}(\mathcal{F}_{-i}) \subset \mathcal{F}_{-i+1}$ for all $i$.

By an automorphism on $\Lambda_i \otimes \mathcal{O}_S$, where $S$ is an $\mathcal{O}$-scheme, we mean a collection of automorphisms $g_{-i}$ on $\Lambda_{-i} \otimes \mathcal{O}_S$ such that $g_{-i}$ commutes with the morphisms $\beta_{-i}$ for all $i$ and $g_0$ and $g_{-n}$ preserve the forms $\psi_0$ and $\psi_{-n}$, respectively, up to invertible scalars. We denote by $\text{Aut}(\Lambda_i \otimes \mathcal{O}_S, \psi_0, \psi_{-n})$ the group of automorphisms on $\Lambda_i \otimes \mathcal{O}_S$.

Let $\mathcal{G}$ be the group scheme over $\mathcal{O}$ that represents the functor

$$S \mapsto \text{Aut}(\Lambda_i \otimes \mathcal{O}_S, \psi_0, \psi_{-n}).$$

We know that $\mathcal{G}$ is an affine smooth group scheme over $\mathcal{O}$ whose generic fiber $\mathcal{G}_K$ is equal to $\text{GSp}_{2n}$. Furthermore, there is a left action of $\mathcal{G}$ on $\mathcal{M}^{\text{loc}}$. 
6.2. The Kottwitz-Rapoport stratification on $M_{\kappa}^{\text{loc}}$. Let $Fl$ be the space of chains of $O$-lattices in $V$ such that

\[ \pi L_0 = L_{-2n} \subset \cdots \subset L_{-1} \subset L_0 \]

such that

(i) $L_i/L_{i-1} \simeq \kappa$ for each $i$,
(ii) there is a non-degenerate alternating pairing $\psi'$ on $L_0$ with values in $O$ such that $\pi^m \psi' = \psi$ for some $m \in \mathbb{Z}$, and
(iii) set $\overline{L}_{-i} := L_{-i}/L_{-2n}$, we require that the orthogonal complement $\overline{L}_{-i}$ with respect to $\psi'$ is equal to $\overline{L}_{i-2n}$ for all $i$.

Note that a lattice chain $(L_i)$ in $Fl$ is determined by its members $L_{-i}$ for $0 \leq i \leq n$.

We regard $V$ as the space of column vectors. For $g \in \text{GSp}_{2n}(K)$, the map $g \mapsto (L_i) = (gL_i)$ gives a bijection $\text{GSp}_{2n}(K)/I \simeq Fl$, where $I$ is the stabilizer of the standard lattice chain.

Now we restrict to the equi-characteristic case $O = \kappa[[t]]$ and $\pi = t$. The space $Fl$ has a natural ind-scheme structure over $\kappa$ and is called the affine flag variety associated to $\text{GSp}_{2n}$ over $\kappa$. For any field extension $\kappa'$ of $\kappa$, we have a natural bijection $\text{GSp}_{2n}(\kappa'((t))) / I(\kappa') \simeq Fl(\kappa')$, where $I(\kappa')$ is the stabilizer of the standard lattice chain $(L_i \otimes \kappa'[[t]])$ base change over $\kappa'[[t]]$.

Let $Y$ be the closed subscheme of $Fl$ consisting of the lattice chains $L_i$ such that $tL_{-i} \subset L_{-i} \subset L_{-i}$, $0 \leq i \leq n$, such that $L_0/L_0 \simeq \kappa^n$.

The group $I$ acts on the ind-scheme $Fl$ by the left translation; it leaves the subscheme $Y$ invariant. Using the Bruhat-Iwahori decomposition

\[ \text{GSp}_{2n}(\kappa((t))) = \bigoplus_{x \in \tilde{W}} IxI, \]

the $I$-orbits are indexed by the extended affine Weyl group $\tilde{W}$ of $\text{GSp}_{2n}$:

\[ Fl = \bigoplus_{x \in \tilde{W}} Fl_x. \]

The extended affine Weyl group $\tilde{W}$ is the semi-direct product $X_*(T) \rtimes W$ of the Weyl group $W$ of $\text{GSp}_{2n}$ and the cocharacter groups $X_*(T)$, where $T$ is the group of diagonal matrices in $\text{GSp}_{2n}$. The cocharacter group $X_*(T)$ is

\[ \{(u_1, \ldots, u_{2n}) \in \mathbb{Z}^{2n} \mid u_1 + u_{2n} = \cdots = u_n + u_{n+1}, \}. \]

The Weyl group $W$ is a subgroup of the symmetric group $S_{2n} = W(\text{GL}_{2n})$ consisting of elements that commute with the permutation

\[ \theta = (1, 2n)(2, 2n-1) \cdots (n, n+1). \]

We identify the symmetric group $S_{2n}$ with the group of permutation matrices in $\text{GL}_{2n}$ in the way that
Let \( L \) be a lattice chain \( \mu \). Let \( \sigma \in S_{2n} \), the corresponding matrix \( w_\sigma \in \text{GL}_{2n} \) is given by \( w_\sigma(e_i) = e_{\sigma(i)}, \forall i \).

We may regard the group \( \tilde{W} \) as a subgroup of the group \( \mathbf{A}(\mathbb{R}^{2n}) \) of affine transformations on the space \( \mathbb{R}^{2n} \) of column vectors. For \( \nu \in X_*(T) \), we write \( t_\nu \) for the image of \( t \) under \( \nu \) in \( \text{GSp}_{2n}(\kappa(t)) \), also for the translation by \( \nu \) in \( \mathbf{A}(\mathbb{R}^{2n}) \). The element \( x = t_\nu w \) then is identified with the function \( x(v) = w \cdot v + \nu \) for \( v \in \mathbb{R}^{2n} \).

If \( x = (x_i) \in \mathbb{R}^{2n} \), we write \( |x| = \sum_i x_i \).

There is a partial order on the extended affine Weyl group \( \tilde{W} \) called the Bruhat order. According to the definition (see \( \text{[13, Subsection 3.2, Proposition 6.1.} \)), the Bruhat order on \( \tilde{W} \) is inherited from \( \tilde{W}_{\text{der}} \) as follows:

\[
x \leq y \text{ in } \tilde{W} \iff |x| = |y| \text{ in } \tilde{W}_{\text{der}} \setminus \tilde{W} \text{ and } 1 \leq yx^{-1} \text{ in } \tilde{W}_{\text{der}}.
\]

The choice of the Bruhat order on \( \tilde{W}_{\text{der}} \) depends on the choice of the Borel subgroup of \( \text{Sp}_{2n} \). We choose the Borel subgroup \( B \) to be the subgroup of upper triangular matrices in \( \text{Sp}_{2n} \). This also agrees with the choice (following Haines [8]) of the lattice chain \( L_{-2n} \). Let \( v_{-2n}, \ldots, v_0 \in \mathbb{Z}^{2n} \) be the integral vectors corresponding the lattices \( L_{-2n}, \ldots, L_0 \).

We have \( v_{-i} = (0^{2n-i}, 1^i) \). The sequence of integral vectors \( v_{-2n}, \ldots, v_0 \in \mathbb{Z}^{2n} \) forms an alcove in the sense of Kottwitz and Rapoport (see [13 Subsection 3.2, 4.2]). Let \( \mu = (1^n, 0^n) \in X_*(T) \), a dominant coweight. Following [13], we define the sets of \( \mu \)-permissible and \( \mu \)-admissible elements:

\[
\text{Perm}(\mu) := \{ x \in \tilde{W} \mid (0, \ldots, 0) \leq x(v_{-i}) - v_{-i} \leq (1, \ldots, 1) \text{ for all } i \text{ and } |x(0)| = n \}.
\]

\[
\text{Adm}(\mu) := \{ x \in \tilde{W} \mid \text{there is an element } w \in \text{W} \text{ such that } x \leq t_w(\mu) \}.
\]

**Proposition 6.1.** Notation as above.

1. The stratum \( \mathcal{F}_{l, x} \) is contained in \( \mathcal{Y} \) if and only if \( x \in \text{Perm}(\mu) \).
2. \( \text{Adm}(\mu) = \text{Perm}(\mu) \).

**Proof.** (1) Let \( x \in \tilde{W} \subseteq \text{GL}_{2n}(\kappa((t))) \). The lattice \( xL_{-i} \) corresponds to the element \( x(v_{-i}) \) in \( \mathbb{Z}^{2n} \). Then the condition \( tL_{-i} \subseteq xL_{-i} \subseteq L_{-i} \) is easily seen to be \( (0, \ldots, 0) \leq x(v_{-i}) - v_{-i} \leq (1, \ldots, 1) \). We also have \( \dim_n L_0/xL_0 = |x(0)| \).

Therefore, the statement follows.

(2) This is Theorem 4.5 (3) of Kottwitz and Rapoport [13].

**Remark 6.2.** The embedding \( \sigma \mapsto w_\sigma \) from \( S_{2n} \) to \( \text{GL}_{2n} \) in (6.1) does not send the Weyl group \( W \) into \( \text{GSp}_{2n} \). In fact for each \( \sigma \in W \), there is a unique element \( \epsilon_\sigma = \text{diag}(1, \ldots, 1, \epsilon_{\sigma, n+1}, \ldots, \epsilon_{\sigma, 2n}) \) with \( \epsilon_{\sigma, i} \in \{ \pm 1 \} \) such that \( w_\sigma' = w_\sigma \epsilon_\sigma \in \text{Sp}_{2n} \). However, since \( w_\sigma' t_\nu (w_\sigma')^{-1} = w_\sigma t_\nu w_\sigma^{-1} \) and \( t_\nu w_\sigma' L_{-i} = t_\nu w_\sigma L_{-i} \), it won’t effect any results if we choose the presentation of \( \tilde{W} \) in \( \text{GL}_{2n} \) either by \((\nu, \sigma) \mapsto t_\nu w_\sigma \) or by \((\nu, \sigma) \mapsto t_\nu w_\sigma' \). We make the first choice as it is easier not to deal with the signs. Another reason for this choice is that the lattice point in \( \mathbb{Z}^{2n} \) corresponding to \( t_\nu w_\sigma' L_{-i} = t_\nu w_\sigma L_{-i} \) is \( w_\sigma \cdot v_{-i} + \nu \) not \( w_\sigma' \cdot v_{-i} + \nu \).

To avoid confusing the standard lattice chain that defines the local model \( \mathbf{M}^{\text{loc}} \) and the lattice chain for the affine flag variety \( \mathcal{F}_{l} \), we use different notation to distinguish them. Let \( L_{-i} = \kappa[[t]]^{2n} \) for \( 0 \leq i \leq 2n \), and \( L_{i-2n} = \langle e_1, \ldots, e_i, te_{i+1}, \ldots, te_{2n} \rangle \).
Define \( \beta'_{-i}, \alpha'_{-i}, \psi'_0, \psi'_{-n}, G' \) as in Subsection 6.1. In particular, \( G' \) is a smooth affine group scheme over \( \kappa[[t]] \) and one has

\[
G'(S) = \text{Aut}(\Lambda'_i \otimes O_S, \psi'_0, \psi'_{-n})
\]

for any \( \kappa[[t]] \)-scheme \( S \). One also sees that the generic fiber of \( G' \) is \( GSp_{2n} \), \( G'(\kappa[[t]]) \)
is equal to \( I \), and that the special fiber \( G'_{\kappa} \) is canonically isomorphic to \( G_{\kappa} \).

Using the isomorphism \( \alpha'_{-i} : \Lambda'_{-i} \simeq L'_{-i} \), we regard the lattice \( L'_{-i} \), where \( (L_{-i}) \) is a member in \( \mathcal{Y} \), as a \( \kappa[[t]] \)-submodule of \( \Lambda'_{-i} \) containing \( t\Lambda'_{-i} \). Then we have an isomorphism \( b : \mathcal{Y} \cong M^\text{loc}_{\kappa} \), which maps any \( \kappa' \)-point of \( \mathcal{Y} \) to \( M^\text{loc}_{\kappa}(\kappa') \) by

\[
(L_{-i}) \mapsto (T_{-i}), \quad T_{-i} := L_{-i}/t\Lambda'_{-i} \subset \Lambda'_{-i} \otimes \kappa'
\]

where \( \kappa' \) is any field extension of \( \kappa \).

The action of \( I \) on the scheme \( \mathcal{Y} \) factors through the quotient \( G'(\kappa) \). We also know that the isomorphism \( b \) is \( G_{\kappa} = G'_{\kappa} \)-equivariant. Therefore, the stratification

\[
\mathcal{Y} = \coprod_{x \in \text{Adm}(\mu)} \mathcal{F}_x
\]

induces a stratification, called the Kottwitz-Rapoport stratification, on \( M^\text{loc}_{\kappa} \)

\[
M^\text{loc}_{\kappa} = \coprod_{x \in \text{Adm}(\mu)} M^\text{loc}_x
\]

so that the \( G_{\kappa} \)-orbit \( M^\text{loc}_x \) corresponds to the \( I \)-orbit \( \mathcal{F}_x \).

6.3. **Local model diagrams.** Let \( A'_{g, \Gamma_0(p), N} \) denote the moduli space over \( \mathbb{Z}_p[\zeta_N] \)
that parametrizes equivalence classes of objects

\[
A_* : \quad A_0 \xrightarrow{\alpha_1} A_1 \longrightarrow \cdots \longrightarrow A_{g-1} \xrightarrow{\alpha_g} A_g,
\]

where

- each \( A_i = (A_i, \lambda_i, \eta_i) \) is a polarized abelian scheme with a symplectic level-\( N \) structure,
- \( A_0 \) and \( A_g \) are objects in \( A_{g,1,N} \), and
- each \( \alpha_i \) is an isogeny of degree \( p \) that preserves the level structures and the polarizations except when \( i = 1 \), and in this case one has \( \alpha_1 \Lambda_1 = p\Lambda_0 \).

There is a natural isomorphism from \( A_{g, \Gamma_0(p), N} \) to \( A'_{g, \Gamma_0(p), N} \) [Proposition 1.7]. We will identify \( A_{g, \Gamma_0(p), N} \) with \( A'_{g, \Gamma_0(p), N} \) via this natural isomorphism.

Put \( n = g \) and \( \mathcal{O} = \mathbb{Z}_p \) in Subsection 6.1. We get a projective \( \mathbb{Z}_p \)-scheme \( M^\text{loc} \).

Let \( S \) be a \( \mathbb{Z}_p[\zeta_N] \)-scheme and \( A_* \) is an object in \( A_{g, \Gamma_0(p), N}(S) \). A *trivialization* \( \gamma \) from the de Rham cohomologies \( H^3_{\text{DR}}(A_*/S) \) to \( \Lambda_* \otimes \mathcal{O}_S \) is a collection of isomorphisms \( \gamma_i : H^3_{\text{DR}}(A_i/S) \rightarrow \Lambda_{-i} \otimes \mathcal{O}_S \) of \( \mathcal{O}_S \)-modules such that

- the diagram

\[
\begin{array}{ccc}
H^3_{\text{DR}}(A_i/S) & \xrightarrow{\alpha_i^*} & H^3_{\text{DR}}(A_{i-1}/S) \\
\downarrow{\gamma_i} & & \downarrow{\gamma_{i-1}} \\
\Lambda_{-i} \otimes \mathcal{O}_S & \xrightarrow{\beta_{-i}} & \Lambda_{-i+1} \otimes \mathcal{O}_S
\end{array}
\]

commutes for \( 1 \leq i \leq g \),

...
• If $e_{\lambda_0}, e_{\lambda_g}$ are the non-degenerate symplectic pairings induced by the principal polarizations $\lambda_0, \lambda_g$, then $\gamma_i^* \psi_i$ is a scalar multiple of $e_{\lambda_i}$ by some element in $O_S^\times$ for $i = 0, g$.

With the terminology as above, let $\tilde{A}_{g, \Gamma_0(p), N}$ denote the moduli space over $\mathbb{Z}_p[\zeta_N]$ that parametrizes equivalence classes of objects $(A_\bullet, \gamma)_S$, where

- $A_\bullet$ is an object in $A_{g, \Gamma_0(p), N}(S)$, and
- $\gamma$ is a trivialization from $H^1_{DR}(A_\bullet/S)$ to $A_\bullet \otimes \mathcal{O}_S$.

The moduli scheme $\tilde{A}_{g, \Gamma_0(p), N}$ has two natural projections $\varphi^{\text{mod}}$ and $\varphi^{\text{loc}}$. The morphism

$$\varphi^{\text{mod}} : \tilde{A}_{g, \Gamma_0(p), N} \to A_{g, \Gamma_0(p), N} \otimes \mathbb{Z}_p[\zeta_N]$$

forgets the trivialization. The morphism

$$\varphi^{\text{loc}} : \tilde{A}_{g, \Gamma_0(p), N} \to M^{\text{loc}} \otimes \mathbb{Z}_p[\zeta_N]$$

sends an object $(A_\bullet, \gamma)$ to $(\gamma(\omega_\bullet))$, where $\omega_\bullet = (\omega_i)$ is a system of $\mathcal{O}_S$-submodules in the Hodge filtration

$$0 \to \omega_i \to H^1_{DR}(A_i/S) \to R^1 f_* (\mathcal{O}_{A_i}) \to 0,$$

and $f : A_i \to S$ is the structure morphism. Thus, we have the diagram:

$$\xymatrix{ & \tilde{A}_{g, \Gamma_0(p), N} \ar[dl]_{\varphi^{\text{mod}}} \ar[dr]^{\varphi^{\text{loc}}} & \\
A_{g, \Gamma_0(p), N} \otimes \mathbb{Z}_p[\zeta_N] & M^{\text{loc}} \otimes \mathbb{Z}_p[\zeta_N]. & \\
}$$

The moduli scheme $\tilde{A}_{g, \Gamma_0(p), N}$ also has a left action by the group scheme $G$. By the work of Rapoport-Zink [19], de Jong [3], and Genestier (cf. Remark below Theorem 1.3 of [13]), we know

(a) $\varphi^{\text{mod}}$ is a left $G$-torsor, and hence it is affine and smooth.

(b) $\varphi^{\text{loc}}$ is smooth, surjective, $G$-equivariant, and of relative dimension same as $\varphi^{\text{mod}}$.

Let

$$A_{\Gamma_0(p)} := A_{g, \Gamma_0(p), N} \otimes \mathbb{F}_p, \quad \tilde{A}_{\Gamma_0(p)} := \tilde{A}_{g, \Gamma_0(p), N} \otimes \mathbb{F}_p, \quad M^{\text{loc}}_p := M^{\text{loc}} \otimes \mathbb{F}_p$$

be the reduction modulo $p$, respectively. Let $\tilde{A}_{\Gamma_0(p), x}$ be the pre-image of a KR-stratum $M^{\text{loc}}_x$. By (b), $\tilde{A}_{\Gamma_0(p), x}$ is stable under the $G_{\mathbb{F}_p}$-action. Since $\varphi^{\text{mod}}$ is a $G_{\mathbb{F}_p}$-torsor, the stratification

$$\tilde{A}_{\Gamma_0(p)} = \prod_{x \in \text{Adm}(\mu)} \tilde{A}_{\Gamma_0(p), x}$$

descends to a stratification, called the Kottwitz-Rapoport stratification, on $A_{\Gamma_0(p)}$:

$$A_{\Gamma_0(p)} = \prod_{x \in \text{Adm}(\mu)} A_{\Gamma_0(p), x}.$$

Each stratum $A_{\Gamma_0(p), x}$ is smooth of dimension same as $\dim M^{\text{loc}}_x$, which is the length $\ell(x)$ of $x$. 
6.4. $g=2$. We describe the set $\text{Adm}(\mu)$ of $\mu$-admissible elements and the Bruhat order on this set, in the special case where $g=2$. The closure $\bar{a}$ of the base alcove $a$ is the set of points $u \in \mathbb{R}^4$ such that $u_1 + u_4 = u_2 + u_3$ and

$$1 + u_1 \geq u_4 \geq u_3 \geq u_2.$$ 

This is obtained from [13, 12.2] by applying the involution $\theta$ since our choice of the standard alcove $\{v_i\}$ differs from $\{w_i\}$ in [13, 4.2] by the involution $\theta$. The simple reflections corresponding to the faces

$$u_3 = u_4, \quad u_2 = u_3, \quad 1 + u_1 = u_4,$$

are $s_1 = (1 \ 2)(3 \ 4), s_2 = (2 \ 3)$ and

$$s_0 = ((-1, 0, 0, 1), (14)) : (u_1, u_2, u_3, u_4) \mapsto (u_4 - 1, u_2, u_3, u_1 + 1).$$

One checks that $\tau := ((0, 0, 1, 1), (13)(24)) \in \bar{W}$ is the element in $\text{Adm}(\mu)$ that fixes $\bar{a}$. It is not hard to compute the set $\text{Perm}(\mu)$ from the definition. From this and the fact $\text{Adm}(\mu) = \text{Perm}(\mu)$ (Proposition 6.1), we get

$$\text{Adm}(\mu) := \{ \tau, s_1 \tau, s_0 \tau, s_2 \tau, s_0 s_1 \tau, s_0 s_2 \tau, s_1 s_2 \tau, s_2 s_1 \tau, s_1 s_0 \tau, s_0 s_1 s_0 \tau, s_1 s_0 s_2 \tau, s_2 s_1 s_2 \tau, s_0 s_2 s_1 \tau \}.$$ 

We compute and express these elements $x$ as $(\nu, \sigma) \in X_*(T) \times W$:

$$\tau = [(0, 0, 1, 1), (13)(24)], \quad s_1 \tau = [(0, 0, 1, 1), (14)(23)],$$
$$s_0 \tau = [(0, 0, 1, 1), (1342)], \quad s_2 \tau = [(0, 1, 0, 1), (1243)],$$
$$s_0 s_1 \tau = [(0, 0, 1, 1), (23)], \quad s_0 s_2 \tau = [(0, 1, 0, 1), (12)(34)],$$
$$s_1 s_2 \tau = [(1, 0, 1, 0), (23)], \quad s_2 s_1 \tau = [(0, 1, 0, 1), (14)],$$
$$s_1 s_0 \tau = [(0, 0, 1, 1), (14)], \quad s_0 s_1 s_0 \tau = [(0, 0, 1, 1), (1)],$$
$$s_1 s_0 s_2 \tau = [(1, 0, 1, 0), (1)], \quad s_2 s_1 s_2 \tau = [(1, 1, 0, 0), (1)],$$
$$s_0 s_2 s_1 \tau = [(0, 1, 0, 1), (1)].$$

(6.2)

For a later use, we also express these elements as $t_\nu w_\sigma$ in $GL_4(\kappa((t)))$:

$$\tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}, \quad s_1 \tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix},$$
$$s_0 \tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}, \quad s_2 \tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}.$$
The Bruhat order on $\text{Adm}(\tau)$ is given by the formula

\[
\ell(x) = \frac{1}{2} \#\text{Fix}(\sigma), \quad \text{where} \quad \text{Fix}(\sigma) := \{ i ; \sigma(i) = i \}.
\]

Put $\text{Adm}^i(\mu) := \{ x \in \text{Adm}(\mu); p\text{-rank}(x) = i \}$. One easily computes the $p$-ranks of elements in $\text{Adm}(\mu)$ using (6.3) and gets

\[
\begin{align*}
\text{Adm}^2(\mu) &= \{ s_0 s_1 s_0 \tau, s_1 s_0 s_2 \tau, s_2 s_1 s_2 \tau, s_0 s_2 s_1 \tau \}, \\
\text{Adm}^1(\mu) &= \{ s_0 s_1 \tau, s_1 s_2 \tau, s_2 s_1 \tau, s_1 s_0 \tau \}, \\
\text{Adm}^0(\mu) &= \{ \tau, s_1 \tau, s_0 \tau, s_2 \tau, s_0 s_2 \tau \}.
\end{align*}
\]

The Bruhat order on $\text{Adm}(\mu)$ can be described by the following diagram

Here $x \to y$ means $\ell(y) = \ell(x) + 1$ and $y = sx$ for some reflection $s$ associated an affine root, and $x \leq y$ if and only if there exists a chain

\[ x = x_1 \to x_2 \cdots \to x_k = y, \]

(cf. [13 1.1.1.2]). This diagram is obtained from reading the picture in Haines [8] Figure 2 in Section 4 of admissible alcoves for $\text{GSp}_4$.

**Proposition 6.3.** (1) The supersingular locus $S_{2,\Gamma_0(\mu)}$ of $A_{2,\Gamma_0(\mu)}$ consists of both one-dimensional irreducible components and two-dimensional irreducible components.

(2) The almost ordinary locus $A^1_{2,\Gamma_0(\mu)}$ is not dense in the non-ordinary locus $A_{2,\Gamma_0(\mu)}^\text{non-ord}$.
Theorem 6.4 (1) It follows from Proposition 6.3 (2) that for \( g \geq 2 \) and \( 1 \leq f \leq g-1 \), the \( p \)-rank-\( f \) stratum \( A_{\Gamma_0(p),kr}(\mathbb{Q}_p) \) is not dense in the stratum \( A_{\Gamma_0(p)}^{\leq f} \subset A_{\Gamma_0(p)} \) consisting of points with \( p \)-rank \( \leq f \). This shows that the collection \( \{ A_{\Gamma_0(p)}^f \} \) of \( p \)-rank strata does not form a stratification.

(2) It follows from Proposition 6.3 (1) that for \( g \geq 2 \) and \( 0 \leq f \leq g-2 \), the natural morphism \( A_{\Gamma_0(p)}^f \to A^f \) is not finite.

(3) Tilouine [20, p.790] examines the intersections of four components of the special fiber of the local model. He also concludes the same results above.

7. Geometric characterization \((g=2)\)

Let \( a = (A_0 \to A_0, A_1 \to A_1) \) be a point in \( A_{\Gamma_0(p)}(k) \), where \( k \) is an algebraically closed field of characteristic \( p \). Then \( a \) lies in a Kottwitz-Rapoport stratum \( A_{\Gamma_0(p)},KR(a) \) for a unique element \( KR(a) \) in \( \text{Adm}(\mu) \). We would like to describe \( KR(a) \) from geometric properties of the point \( a \).

Put \( M_i := H^i_{\text{DR}}(A_i/k) \) and \( \omega_i := \omega_{A_i} \). We have

\[
M_2 \xrightarrow{\alpha} M_1 \xrightarrow{\alpha} M_0.
\]

Set \( G_0 := \ker(\alpha : A_0 \to A_1) \) and \( G_1 := \ker(\alpha : A_1 \to A_2) \); they are finite flat group scheme of rank \( p \), which is isomorphic to \( \mathbb{Z}/p, \mu_p \), or \( \alpha_p \). From the Dieudonné theory we know that

\[
\omega_{G_i} = \omega_i/\alpha(\omega_{i+1}), \quad \text{and} \quad \text{Lie}(G'_i) = M_i/\langle \omega_i + \alpha(M_{i+1}) \rangle,
\]

where \( G'_i \) is the Cartier dual of \( G_i \). Following de Jong [3], we define

\[
\sigma_i(a) := \dim \omega_i/\alpha(\omega_{i+1}), \quad \tau_i(a) := \dim M_i/\langle \omega_i + \alpha(M_{i+1}) \rangle.
\]

Clearly we have the following characterization of \( G_i \):

| \((\sigma_i(a), \tau_i(a))\) | \((0, 1)\) | \((1, 0)\) | \((1, 1)\) |
|---------------------------|-----------------|-----------------|-----------------|
| \(G_i\)                  | \(\mathbb{Z}/p\mathbb{Z}\) | \(\mu_p\)       | \(\alpha_p\)    |

When the \( p \)-rank of \( a \) is \( \geq 1 \), the chain of the \( p \)-divisible groups of \( a \) is determined by the invariants \((\sigma_i(a), \tau_i(a))\) up to isomorphism. In particular, the element \( KR(a) \) is determined by the invariants \((\sigma_i(a), \tau_i(a))\). To describe the correspondence, it suffices to compute these invariants for the distinguished point \( x \) in the stratum \( M_{p,c} \).

Recall how to associate a member in \( M_{p,c} \) to an element \( x = t_0u_0 \) in \( \text{Adm}(\mu) \). We first apply \( x \) to the standard lattice chain and get a lattice chain \((xL'_{-i})_{0 \leq i \leq 2}\). It follows from the permissibility that \( tL'_{-i} \subset xL'_{-i} \subset L'_{-i} \). Then there is a lattice \( L_{-i} \) in \( \Lambda'_{-i} \) so that its image under the isomorphism \( \Lambda'_{-i} \cong L'_{-i} \) is \( xL'_{-i} \). This way we associate an element \((L_{-i}/\iota\Lambda'_{-i})\) in \( M_{p,c} \) (via the isomorphism \( b : \mathcal{Y} \cong M_{p,c} \)). We use this element to compute the invariants \((\sigma_i, \tau_i)\).
Below we write $[L_0] = (e_1, e_2, e_3, e_4)^t, [L_{-1}] = (e_1, e_2, e_3, te_4)^t$, and $[L_{-2}] = (e_1, e_2, te_3, te_4)^t$ and write $\mathcal{L}_{-i} = \mathcal{L}_{-i}/t\mathcal{L}_{-i}$ and $\mathcal{K}_{-i} = \mathcal{N}_{-i}/\mathcal{L}_{-i}$. Recall that $\beta'$s are the maps between the lattices $\mathcal{N}_{-i}$ which correspond to the maps $\alpha$ on $M_i$ under a trivialization map $\gamma$. Then the invariants $(\sigma_i, \tau_i)$ are given by

$$\sigma_i = \dim \mathcal{L}_{-i}/\beta(\mathcal{L}_{-i-1}), \quad \tau_i = \dim \mathcal{K}_{-i}/\beta(\mathcal{K}_{-i-1}).$$

(1) When $x = s_0s_1s_0\tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$, we compute

$$x[L_0] = \begin{pmatrix} e_1 \\ e_2 \\ te_3 \\ te_4 \end{pmatrix}, \quad x[L_{-1}] = \begin{pmatrix} e_1 \\ e_2 \\ te_3 \\ t^2e_4 \end{pmatrix}, \quad x[L_{-2}] = \begin{pmatrix} e_1 \\ e_2 \\ t^2e_3 \\ t^2e_4 \end{pmatrix}.$$}

It follows that $\mathcal{L}_{-2} = \mathcal{L}_{-1} = \mathcal{L}_0 =< e_1, e_2, te_3, te_4 >$, $\mathcal{K}_{-2} = \mathcal{K}_{-1} = \mathcal{K}_0 =< e_3, e_4 >$. It follows that $\beta(\mathcal{L}_{-2}) = \beta(\mathcal{L}_{-1}) = < e_1, e_2 >$, $\beta(\mathcal{K}_{-2}) = < e_1 >$, and $\beta(\mathcal{K}_{-1}) = < e_3 >$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (0, 1)$.

Note that if $x$ is diagonal, then $\mathcal{L}_{-2} = \mathcal{L}_{-1} = \mathcal{L}_0$. Therefore, it is enough to compute $x[L_0]$, which is done this way in (2)–(4).

(2) When $x = s_0s_2s_1\tau = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} e_1 \\ e_2 \\ te_3 \\ te_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< e_1, te_2, e_3, te_4 >, \quad \mathcal{L}_0 =< e_1, e_3 >, \quad \mathcal{K}_0 =< e_2, e_4 >.$$}

It follows that $\beta(\mathcal{L}_{-1}) = < e_1, e_3 >$, $\beta(\mathcal{L}_{-2}) = < e_1 >$, $\beta(\mathcal{K}_{-1}) = < e_2 >$, and $\beta(\mathcal{K}_{-2}) = < e_2, e_4 >$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (1, 0)$.

(3) When $x = s_1s_0s_2\tau = \begin{pmatrix} t & 1 \\ t & 1 \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} te_1 \\ e_2 \\ te_3 \\ e_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< te_1, e_2, te_3, e_4 >, \quad \mathcal{L}_0 =< e_2, e_4 >, \quad \mathcal{K}_0 =< e_1, e_3 >.$$}

It follows that $\beta(\mathcal{L}_{-1}) = < e_2 >$, $\beta(\mathcal{L}_{-2}) = < e_2, e_4 >$, $\beta(\mathcal{K}_{-1}) = < e_1, e_3 >$, and $\beta(\mathcal{K}_{-2}) = < e_1 >$. This gives $(\sigma_0, \tau_0) = (1, 0)$ and $(\sigma_1, \tau_1) = (0, 1)$.

(4) When $x = s_2s_1s_2\tau = \begin{pmatrix} t & 1 \\ t & 1 \end{pmatrix}$, we compute $x[L'_0] = \begin{pmatrix} te_1 \\ te_2 \\ e_3 \\ e_4 \end{pmatrix}$ and obtain

$$\mathcal{L}_0 =< te_1, te_2, e_3, e_4 >, \quad \mathcal{L}_0 =< e_3, e_4 >, \quad \mathcal{K}_0 =< e_1, e_2 >.$$

It follows that $\beta(L_{-1}) = \langle e_3, e_4 \rangle$, $\beta(L_{-2}) = \langle e_4, e_1, e_2 \rangle$, $\beta(L_{-3}) = \langle e_1, e_2 \rangle$, and $\beta(L_{-4}) = \langle e_1, e_3 \rangle$. This gives $(\sigma_0, \tau_0) = (1, 0)$ and $(\sigma_1, \tau_1) = (1, 0)$.

(5) When $x = s_0s_1 \tau = \begin{pmatrix} 1 & 1 \\ t & 1 \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} e_1 & e_1 \\ te_3 & te_3 \\ e_2 & te_2 \\ te_4 & t^2e_4 \end{pmatrix}$ and obtain

$L_0 = \langle e_1, e_2, te_3, te_4 \rangle$, $L_{-1} = \langle e_1, e_2, e_3, te_4 \rangle$, $L_{-2} = \langle e_1, e_2, e_3, e_4 \rangle$, $L_{-3} = \langle e_1, e_2 \rangle$.

It follows that $\beta(L_{-1}) = \langle e_1, e_2 \rangle$, $\beta(L_{-2}) = \langle e_1, e_3 \rangle$, and $\beta(L_{-3}) = \langle e_4 \rangle$. This gives $(\sigma_0, \tau_0) = (0, 1)$ and $(\sigma_1, \tau_1) = (1, 1)$.

(6) When $x = s_1s_2 \tau = \begin{pmatrix} t & 0 \\ t & 1 \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} te_1 & te_1 \\ te_3 & te_3 \\ e_2 & te_2 \\ e_4 & te_4 \end{pmatrix}$ and obtain

$L_0 = \langle te_3, te_4, e_1, e_2 \rangle$, $L_{-1} = \langle te_4, e_1, e_2, e_3 \rangle$, $L_{-2} = \langle te_3, e_1, e_2, e_4 \rangle$.

It follows that $\beta(L_{-1}) = \langle e_2 \rangle$, $\beta(L_{-2}) = \langle e_4 \rangle$, and $\beta(L_{-3}) = \langle e_1, e_3 \rangle$.

(7) When $x = s_2s_1 \tau = \begin{pmatrix} 1 \\ t \end{pmatrix}$, we compute $x([L'_0], [L'_{-1}], [L'_{-2}]) = \begin{pmatrix} te_4 & te_4 \\ te_2 & te_2 \\ e_3 & te_3 \\ e_1 & te_1 \end{pmatrix}$ and obtain

$L_0 = \langle e_1, e_2, e_3, e_4 \rangle$, $L_{-1} = \langle e_1, e_2, e_3, e_4 \rangle$, $L_{-2} = \langle e_1, e_2, e_3, e_4 \rangle$.

It follows that $\beta(L_{-1}) = \langle e_3 \rangle$, $\beta(L_{-2}) = \langle e_4 \rangle$, and $\beta(L_{-3}) = \langle e_1, e_2 \rangle$. This gives $(\sigma_0, \tau_0) = (1, 1)$ and $(\sigma_1, \tau_1) = (1, 0)$. 

(8) When \( x = s_1s_0\tau = \begin{pmatrix} 1 & 1 \\ t & \end{pmatrix} \), we compute \( x([L'_0], [L'_{-1}], [L'_{-2}]) = \)

\[
\begin{pmatrix}
  te_4 & te_4 & te_4 \\
  e_2 & e_2 & e_2 \\
  te_3 & te_3 & t^2e_3 \\
  e_1 & te_1 & te_1
\end{pmatrix}
\]

and obtain

\[
\mathcal{L}_0 = \langle e_1, e_2, te_3, te_4 \rangle, \quad \mathcal{L}_0 = \langle e_1, e_2 \rangle, \quad \mathcal{X}_0 = \langle e_3, e_4 \rangle,
\]

\[
\mathcal{L}_{-1} = \langle te_1, e_2, te_3, e_4 \rangle, \quad \mathcal{L}_{-1} = \langle e_2, e_4 \rangle, \quad \mathcal{X}_{-1} = \langle e_1, e_3 \rangle,
\]

\[
\mathcal{L}_{-2} = \langle te_1, e_2, te_3, e_4 \rangle, \quad \mathcal{L}_{-2} = \langle e_2, e_4 \rangle, \quad \mathcal{X}_{-2} = \langle e_1, e_3 \rangle.
\]

It follows that \( \beta(\mathcal{L}_{-1}) = \langle e_2, e_4 \rangle, \beta(\mathcal{L}_{-2}) = \langle e_2, e_4 \rangle, \beta(\mathcal{X}_{-1}) = \langle e_3 \rangle, \) and \( \beta(\mathcal{X}_{-2}) = \langle e_1 \rangle. \) This gives \( (\sigma_0, \tau_0) = (1, 1) \) and \( (\sigma_1, \tau_1) = (0, 1) \).

We conclude the characterization of \( KR(a) \) when \( p\text{-rank of } a \) is \( \geq 1 \) in the following table:

| \( p\text{-rank}(a) \) | 2       | 2       | 2       | 2       | 1     | 1    | 1     | 1     |
|------------------------|---------|---------|---------|---------|-------|------|-------|-------|
| \( (\sigma_0(a), \tau_0(a)) \) | (0, 1)  | (0, 1)  | (1, 0)  | (1, 0)  | (0, 1) | (1, 0) | (1, 1) | (1, 1) |
| \( (\sigma_1(a), \tau_1(a)) \) | (0, 1)  | (1, 0)  | (0, 1)  | (1, 0)  | (1, 1) | (1, 1) | (1, 0) | (0, 1) |
| \( KR(a) \)            | 8081807 | 8082817 | 8180827 | 8251827 | 80817 | 81527 | 82517 | 81807 |

When \( a \) is supersingular (i.e. any member \( A_i \) is a supersingular abelian variety), the element \( KR(a) \) is not determined by the invariants \( (\sigma_i(a), \tau_i(a)) \). In fact, they are all \( (1, 1) \), as the schemes \( G_i \) are isomorphic to \( \alpha_p \). We treat this case separately.

8. Geometric characterization \((g = 2)\): the supersingular case

8.1. We continue with a geometric point \( a \) in \( A_{2, \Gamma_0(p)}(k) \), and suppose that the point \( a \) has \( p\text{-rank } 0 \). We know that \( (\sigma_i(a), \tau_i(a)) = (1, 1) \) for \( i = 0, 1 \). We define a new invariant \( \sigma_{02}(a), \tau_{02}(a) \) by

\[
\sigma_{02}(a) := \dim \omega_0/\alpha^2(\omega_2), \quad \tau_{02}(a) := \dim M_0/\omega_0 + \alpha^2(M_2).
\]

As in the previous section, we associated a distinguished point in \( M_{x}^{loc} \) to each element \( x = t_{w_\sigma} \) in \( \text{Adm}(\mu) \). We shall use this point to calculate the invariant \( (\sigma_{02}(a), \tau_{02}(a)) \). Below \( [L'_0], [L'_{-1}], [L'_{-2}], \mathcal{L}_{-1}, \mathcal{X}_{-1} \) are as in the previous section.
(1) When \( x = s_0s_2\tau = \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} \), we compute \( x([L'_0], [L'_1], [L'_2]) = \begin{pmatrix} te_2 & te_2 & te_2 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2e_4 \end{pmatrix} \) and obtain

\[
\mathcal{L}_0 = <e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 = <e_1, e_3 >, \quad \overline{\mathcal{X}}_0 = <e_2, e_4 >, \\
\mathcal{L}_{-1} = <e_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} = <e_1, e_4 >, \quad \overline{\mathcal{X}}_{-1} = <e_2, e_3 >, \\
\mathcal{L}_{-2} = <e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_{-2} = <e_1, e_3 >, \quad \overline{\mathcal{X}}_{-2} = <e_2, e_4 >.
\]

It follows that \( \beta^2(\overline{\mathcal{L}}_{-2}) = < e_1 > \) and \( \beta^2(\overline{\mathcal{X}}_{-2}) = < e_2 > \). This gives \( (\sigma_{02}, \tau_{02}) = (1, 1) \).

(2) When \( x = s_0\tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix} \), we compute \( x([L'_0], [L'_1], [L'_2]) = \begin{pmatrix} te_3 & te_3 & te_3 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2e_4 \end{pmatrix} \)
and obtain

\[
\mathcal{L}_0 = < e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 = < e_1, e_2 >, \quad \overline{\mathcal{X}}_0 = < e_3, e_4 >, \\
\mathcal{L}_{-1} = < e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} = < e_1, e_4 >, \quad \overline{\mathcal{X}}_{-1} = < e_2, e_3 >, \\
\mathcal{L}_{-2} = < e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_{-2} = < e_1, e_3 >, \quad \overline{\mathcal{X}}_{-2} = < e_2, e_4 >.
\]

It follows that \( \beta^2(\overline{\mathcal{L}}_{-2}) = < e_1 > \) and \( \beta^2(\overline{\mathcal{X}}_{-2}) = 0 \). This gives \( (\sigma_{02}, \tau_{02}) = (1, 2) \).

(3) When \( x = s_1\tau = \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix} \), we compute \( x([L'_0], [L'_1], [L'_2]) = \begin{pmatrix} te_3 & te_3 & te_3 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2e_4 \end{pmatrix} \) and obtain

\[
\mathcal{L}_0 = < e_1, e_2, te_3, te_4 >, \quad \overline{\mathcal{L}}_0 = < e_1, e_2 >, \quad \overline{\mathcal{X}}_0 = < e_3, e_4 >, \\
\mathcal{L}_{-1} = < te_1, e_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} = < e_2, e_4 >, \quad \overline{\mathcal{X}}_{-1} = < e_1, e_3 >, \\
\mathcal{L}_{-2} = < te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} = < e_3, e_4 >, \quad \overline{\mathcal{X}}_{-2} = < e_1, e_2 >.
\]

It follows that \( \beta^2(\overline{\mathcal{L}}_{-2}) = 0 \) and \( \beta^2(\overline{\mathcal{X}}_{-2}) = 0 \). This gives \( (\sigma_{02}, \tau_{02}) = (2, 2) \).

(4) When \( x = s_2\tau = \begin{pmatrix} t & 1 \\ t & t \end{pmatrix} \), we compute \( x([L'_0], [L'_1], [L'_2]) = \begin{pmatrix} te_3 & te_3 & te_3 \\ e_1 & e_1 & e_1 \\ te_4 & te_4 & t^2e_4 \end{pmatrix} \) and obtain

\[
\mathcal{L}_0 = < e_1, te_2, e_3, te_4 >, \quad \overline{\mathcal{L}}_0 = < e_1, e_3 >, \quad \overline{\mathcal{X}}_0 = < e_2, e_4 >, \\
\mathcal{L}_{-1} = < e_1, te_2, te_3, e_4 >, \quad \overline{\mathcal{L}}_{-1} = < e_1, e_4 >, \quad \overline{\mathcal{X}}_{-1} = < e_2, e_3 >, \\
\mathcal{L}_{-2} = < te_1, te_2, e_3, e_4 >, \quad \overline{\mathcal{L}}_{-2} = < e_3, e_4 >, \quad \overline{\mathcal{X}}_{-2} = < e_1, e_2 >.
\]
It follows that $\beta^2(\mathcal{L}_{-2}) = 0$ and $\beta^2(\mathcal{K}_{-2}) = <e_2>$. This gives $(\sigma_{02}, \tau_{02}) = (2, 1)$.

(5) When $x = \tau = \begin{pmatrix} 1 \\ t \\ 1 \\ t \end{pmatrix}$, we compute $x([L_0^'], [L_{-1}^'], [L_{-2}^']) = \begin{pmatrix} te_3 \\ te_4 \\ te_4 \\ e_1 e_1 te_1 \\ e_2 te_2 te_2 \end{pmatrix}$ and obtain

\[ \mathcal{L}_0 = <e_1, e_2, te_3, te_4>, \quad \mathcal{L}_0 = <e_1, e_2>, \quad \mathcal{K}_0 = <e_3, e_4>, \]
\[ \mathcal{L}_{-1} = <e_1, te_3, e_4>, \quad \mathcal{L}_{-1} = <e_1, e_4>, \quad \mathcal{K}_{-1} = <e_2, e_3>, \]
\[ \mathcal{L}_{-2} = <te_1, te_2, e_3, e_4>, \quad \mathcal{K}_{-2} = <e_3, e_4>, \quad \mathcal{K}_{-2} = <e_1, e_2>. \]

It follows that $\beta^2(\mathcal{K}_{-2}) = 0$ and $\beta^2(\mathcal{K}_{-2}) = 0$. This gives $(\sigma_{02}, \tau_{02}) = (2, 2)$.

We conclude the result of our computation for characterizing $KR(a)$ when $p$-rank of $a$ is 0 in the following table:

| $p$-rank($a$) | 0 | 0 | 0 | 0 |
|---------------|---|---|---|---|
| $(\sigma_0(a), \tau_0(a))$ | (1, 1) | (1, 1) | (1, 1) | (1, 1) |
| $(\sigma_1(a), \tau_1(a))$ | (1, 1) | (1, 1) | (1, 1) | (1, 1) |
| $(\sigma_{02}(a), \tau_{02}(a))$ | (1, 1) | (1, 2) | (2, 1) | (2, 2) |
| $KR(a)$ | $s_0 s_2 \tau$ | $s_0 \tau$ | $s_2 \tau$ | $s_1 \tau$ |

To distinguish the types $s_1 \tau$ and $\tau$, we need to know a global description of the supersingular locus.

8.2. Description of the supersingular locus. Let $A_{2,p,N}$ denote the moduli space of polarized abelian surfaces of degree $p^2$ with a level-$N$ structure with respect to $\zeta_N$. Let $A_{2,1,N} \subset A_{2,1,N} \otimes \overline{\mathbb{F}}_p$ denote the subset of superspecial (geometric) points. Let $A \subset A_{2,p,N} \otimes \overline{\mathbb{F}}_p$ be the subset of (geometric) points $(A, \lambda, \eta)$ such that $\ker \lambda = \alpha_p \times \alpha_p$. Any member $A_0$ of $A$ is superspecial, as $A$ contains $\ker \lambda = \alpha_0 \times \alpha_p$. The sets $A_{2,1,N}$ and $A$ are finite, and every member of them is defined over $\overline{\mathbb{F}}_p$.

Recall that $A_{2,\Gamma_0(p)}$ denotes the reduction modulo $p$ of the Siegel 3-fold with Iwahori level structure, which parametrizes equivalence classes of objects $(A_0 \dashrightarrow A_1 \rightarrow A_2)$ in characteristic $p$ with conditions as before (Subsection 6.3). Let $S_{2,\Gamma_0(p)} \subset A_{2,\Gamma_0(p)}$ denote the supersingular locus, the reduced closed subscheme consisting of supersingular points. Clearly, we have (6.5)

$$S_{2,\Gamma_0(p)} = \coprod_{x \in \text{Adm}^2(\mu)} A_{\Gamma_0(p), x} \quad (g = 2).$$

For each $\xi = (A_\xi, \lambda_\xi, \eta_\xi) \in A$, let $W_\xi \subset S_{2,\Gamma_0(p)}$ be the reduced closed subscheme consisting of points $(A_0 \rightarrow A_1 \rightarrow A_2)$ such that $A_1 \simeq \xi$. For each $\gamma = (A_\gamma, \lambda_\gamma, \eta_\gamma) \in A_{2,1,N}$, let $U_\gamma \subset S_{2,\Gamma_0(p)}$ be the locally closed reduced subscheme consisting of points $(A_0 \rightarrow A_1 \rightarrow A_2)$ such that $A_0 \simeq \gamma$ and $A_1 \not\in A$. Let $S_\gamma$ be the Zariski closure of $U_\gamma$ in $S_{2,\Gamma_0(p)}$. Clearly, $S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$ if $\gamma_1 \neq \gamma_2$ and $W_{\xi_1} \cap W_{\xi_2} = \emptyset$ if $\xi_1 \neq \xi_2$.

**Theorem 8.1.** Notation as above.
(1) One has
\[ S_{2, \Gamma_0(p)} = \left( \prod_{\xi \in \Lambda} W_\xi \right) \cup \left( \prod_{\gamma \in \Lambda_{2,1,N}} S_\gamma \right) \]
as the union of irreducible components. Consequently, the supersingular locus has \(|\Lambda| + |\Lambda_{2,1,N}|\) irreducible components.

(2) For each \(\xi \in \Lambda\), the subscheme \(W_\xi\) is isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) over \(\mathbb{F}_p\). For each \(\gamma \in \Lambda_{2,1,N}\), the subscheme \(S_\gamma\) is isomorphic to \(\mathbb{P}^1\) over \(\mathbb{F}_p\). Furthermore, \(W_\xi\) and \(S_\gamma\) intersect transversally at most one point. The singular locus \(S_{2, \Gamma_0(p)}^{\text{sing}}\) is the intersection
\[ \left( \prod_{\xi \in \Lambda} W_\xi \right) \cap \left( \prod_{\gamma \in \Lambda_{2,1,N}} S_\gamma \right). \]

(3) One has \(|S_{2, \Gamma_0(p)}^{\text{sing}}| = |\Lambda_{2,1,N}|(p + 1)\) and
\[
|\Lambda| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p^2 - 1),
\]
\[
|\Lambda_{2,1,N}| = |\text{Sp}_4(\mathbb{Z}/N\mathbb{Z})| \frac{(-1)\zeta(-1)\zeta(-3)}{4}(p^2 + 1),
\]
where \(\zeta(s)\) is the Riemann zeta function.

The proof will be given in Subsection 8.3.

8.3. We use the classical contravariant Dieudonné theory. We refer the reader to Demazure [4] for a basic account of this theory. For a perfect field \(k\) of characteristic \(p\), write \(W := W(k)\) for the ring of Witt vectors over \(k\), and \(B(k)\) for the fraction field of \(W(k)\). Let \(\sigma\) be the Frobenius map on \(B(k)\). A quasi-polarization on a Dieudonné module \(M\) over \(k\) here is a non-degenerate (meaning of non-zero discriminant) alternating pairing
\[ \langle \cdot, \cdot \rangle : M \times M \to B(k), \]
such that \(\langle Fx, y \rangle = (x, Vy)^\sigma\) for \(x, y \in M\) and \((M^t, M^t) \subset W\). Here the dual \(M^t\) of \(M\) is regarded as a Dieudonné submodule in \(M \otimes B(k)\) using the pairing. A quasi-polarization is called separable if \(M^t = M\). Any polarized abelian variety \((A, \lambda)\) over \(k\) naturally gives rise to a quasi-polarized Dieudonné module. The induced quasi-polarization is separable if and only if \((p, \deg \lambda) = 1\).

Assume that \(k\) is an algebraically closed field field of characteristic \(p\).

Lemma 8.2.

(1) Let \(M\) be a separably quasi-polarized superspecial Dieudonné module over \(k\) of rank 4. Then there exists a basis \(f_1, f_2, f_3, f_4\) for \(M\) over \(W := W(k)\) such that
\[ Ff_1 = f_3, Ff_3 = pf_1, \quad Ff_2 = f_4, Ff_4 = pf_2 \]
and the non-zero pairings are
\[ \langle f_1, f_3 \rangle = -\langle f_3, f_1 \rangle = \beta_1, \quad \langle f_2, f_4 \rangle = -\langle f_4, f_2 \rangle = \beta_1, \]
where \(\beta_1 \in W(\mathbb{F}_p)^\times\) with \(\beta_1^p = -\beta_1\).

(2) Let \(\xi\) be a point in \(\Lambda\), and let \(M_\xi\) be the Dieudonné module of \(\xi\). Then there is a \(W\)-basis \(e_1, e_2, e_3, e_4\) for \(M_\xi\) such that
\[ Fe_1 = e_3, \quad Fe_2 = e_4, \quad Fe_3 = pe_1, \quad Fe_4 = pe_2, \]
and the non-zero pairings are
\[ (e_1, e_2) = - (e_2, e_1) = \frac{1}{p}, \quad (e_3, e_4) = -(e_4, e_3) = 1. \]

Proof. This is Lemma 4.2 of [22]. Statement (1) is a special case of Proposition 6.1 of [13], and the statement (2) is deduced from that proposition. □

Lemma 8.3.

(1) Let \((M_0, \langle \cdot, \cdot \rangle_0)\) be a separably quasi-polarized supersingular Dieudonné module of rank 4 and suppose \(a(M_0) = 1\). Let \(M_1 := (F, V)M_0\) and \(N\) be the unique Dieudonné module containing \(M_0\) with \(N/M_0 = k\). Let \(\langle \cdot, \cdot \rangle_1 := \frac{1}{p^2} \langle \cdot, \cdot \rangle_0\) be the quasi-polarization for \(M_1\). Then one has \(a(N) = a(M_1) = 2\), \(VN = M_1\), and \(M_1/M_1^1 \cong k \oplus k\) as Dieudonné modules.

(2) Let \((M_1, \langle \cdot, \cdot \rangle_1)\) be a quasi-polarized supersingular Dieudonné module of rank 4. Suppose that \(M_1/M_1^1\) is of length 2, that is, the quasi-polarization has degree \(p^2\).

(i) If \(a(M_1) = 1\), then letting \(M_2 := (F, V)M_1\), one has that \(a(M_2) = 2\) and \(\langle \cdot, \cdot \rangle_1\) is a separable quasi-polarization on \(M_2\).

(ii) Suppose \((M_1, \langle \cdot, \cdot \rangle_1)\) decomposes as the product of two quasi-polarized Dieudonné submodules of rank 2. Then there are a unique Dieudonné submodule \(M_2\) of \(M_1\) with \(M_1/M_2 = k\) and a unique Dieudonné module \(M_0\) containing \(M_1\) with \(M_0/M_1 = k\) so that \(\langle \cdot, \cdot \rangle_1\) (resp. \(p\langle \cdot, \cdot \rangle_1\)) is a separable quasi-polarization on \(M_2\) (resp. \(M_0\)).

(iii) Suppose \(M_1/M_1^1 \cong k \oplus k\) as Dieudonné modules. Let \(M_2 \subset M_1\) be any Dieudonné submodule with \(M_1/M_2 = k\), and \(M_0 \supset M_1\) be any Dieudonné overmodule with \(M_0/M_1 = k\). Then \(\langle \cdot, \cdot \rangle_1\) (resp. \(p\langle \cdot, \cdot \rangle_1\)) is a separable quasi-polarization on \(M_2\) (resp. \(M_0\)).

This is well-known; the proof is elementary and omitted. We remark that Dieudonné lattices in a supersingular four-dimensional polarized isocrystal are also classified in C. Kaiser [10, Section 3].

8.4. Let \((A_0, \lambda_0)\) be a superspecial principally polarized abelian surface and \((M_0, \langle \cdot, \cdot \rangle_0)\) be the associated Dieudonné module. Let \(\varphi : (A_0, \lambda_0) \to (A, \lambda)\) be an isogeny of degree \(p\) with \(\varphi^* \lambda = p \lambda_0\). Write \((M, \langle \cdot, \cdot \rangle)\) for the Dieudonné module of \((A, \lambda)\). Choose a basis \(f_1, f_2, f_3, f_4\) for \(M_0\) as in Lemma 8.2. Put \(M_2 := (F, V)M_0 = VM_0\). We have the inclusions
\[
M_2 \subset M \subset M_0.
\]
Modulo \(M_2\), a module \(M\) corresponds to a one-dimensional subspace \(M/M_2\) in \(M_0/M_2\).

As \(M_0/M_2 = k \subset F_1, f_2 >\), the subspace \(M/M_2\) has the form
\[
k < a f_1 + b f_2, \quad [a : b] \in P^1(k).
\]

Let \(\overline{M}_0 := M_0/pM_0\), and let
\[
\langle \cdot, \cdot \rangle_0 : \overline{M}_0 \times \overline{M}_0 \to k.
\]
be the induced perfect pairing.

Lemma 8.4. Notation as above, the following conditions are equivalent

(a) \(\ker \lambda \cong \alpha_p \times \alpha_p\).
(b) \langle \overline{M}, F \overline{M} \rangle_0 = 0\), where \(\overline{M} := M/pM_0\).
(c) \langle \overline{M}, V \overline{M} \rangle_0 = 0\).
(d) The corresponding point \([a : b]\) satisfies \(a^{p+1} + b^{p+1} = 0\)

Proof. One has \(M = k < f_1, f_3, f_4 >\) with \(f_1' = a f_1 + b f_2\). It is easy to see that
\[
\langle M, FM \rangle_0 = 0 \iff \langle f_1', FF_1' \rangle_0 = 0 \iff a^{p+1} + b^{p+1} = 0,
\]
and
\[
\langle M, VM \rangle_0 = 0 \iff \langle f_1', VF_1' \rangle_0 = 0 \iff a^{p+1} + b^{p+1} = 0.
\]
This shows that the conditions (b), (c) and (d) are equivalent.

Since \(\varphi^* \lambda = p \lambda_0\), we have \(\langle \cdot, \cdot \rangle = \frac{1}{p} \langle \cdot, \cdot \rangle_0\). The Dieudonné module \(M(ker \lambda)\) of the subgroup \(ker \lambda\) is equal to \(M/M'\). Hence the condition (a) is equivalent to the condition \(F\) and \(V\) vanish on \(M(ker \lambda) = M/M'\). On the other hand, the subspace \(M' := M/pM_0\) is equal to \(M'\) with respect to \(\langle \cdot, \cdot \rangle_0\). It follows that (a) is equivalent to the conditions (b) and (c).

It follows from Lemma \(\S 3\) that there are \(p+1\) isogenies \(\varphi\) so that \(ker \lambda \simeq \alpha_p \times \alpha_p\). Conversely, fix a polarized superspecial abelian surface \((A, \lambda)\) such that \(ker \lambda \simeq \alpha_p \times \alpha_p\). Then there are \(p^2 + 1\) degree-\(p\) isogenies \(\varphi: (A_0, \lambda_0) \rightarrow (A, \lambda)\) such that \(A_0\) is superspecial and \(\varphi^* \lambda = p \lambda_0\). Indeed, each isogeny \(\varphi\) has the property \(\varphi^* \lambda = p \lambda_0\) for a principal polarization \(\lambda_0\) (Lemma \(\S 3.3\)(iii)), and there are \(P^1(F_p)\) isogenies with \(A_0\) superspecial.

8.5. Let \(A_p\) be the moduli space of isogenies \(\alpha: \underline{A}_0 \rightarrow \underline{A}_1\) of degree \(p\), where \(\underline{A}_0\) is an object in \(\mathcal{A}_{2,1,N}\) and \(\underline{A}_1\) is an object in \(\mathcal{A}_{2,p,N}\) such that \(\alpha^* \lambda_1 = p \lambda_0\) and \(\alpha^* \eta_0 = \eta_1\). Let \(S_p \subset A_p \otimes \overline{\mathbb{F}}_p\) be the supersingular locus, the reduced closed subscheme consisting of supersingular points. For each \(\xi = (A_{\xi}, \lambda_{\xi}, \eta_{\xi}) \in \Lambda\), let \(V_\xi \subset S_p\) be the closed subvariety consisting of the isogenies \(\alpha: \underline{A}_0 \rightarrow \underline{A}_1\) such that \(\underline{A}_1 = \xi\). For each \(\gamma = (A_{\gamma}, \lambda_{\gamma}, \eta_{\gamma}) \in \Lambda_{2,1,N}\), let \(S'_\gamma \subset S_p\) be the closed subvariety consisting of the isogenies \(\alpha: \underline{A}_0 \rightarrow \underline{A}_1\) such that \(\underline{A}_1 = \gamma\).

It is known that the varieties \(V_\xi\) and \(S'_\gamma\) are isomorphic to \(P^1\) over \(\overline{\mathbb{F}}_p\) (cf. \(\S 3\)). We also know (22 Proposition 4.5)) that
\[
S_p = \left( \bigsqcup_{\xi \in \Lambda} V_\xi \right) \cup \left( \bigsqcup_{\gamma \in \Lambda_{2,1,N}} S'_\gamma \right)
\]
as the union of irreducible components.

Let \(pr: S_{2,1,0}(p) \rightarrow S_p\) be the natural projection.

8.6. Proof of Theorem \(\S 8.1\) (1) It is easy to see that
\[
S_{2,1,0}(p) = \left( \prod_{\xi \in \Lambda} W_\xi \right) \prod_{\gamma \in \Lambda_{2,1,N}} U_\gamma.
\]
The statement follows from this.

(2) Clearly we have
\[
W_\xi \simeq V_\xi \times V_\xi' \simeq P^1 \times P^1 \quad \text{(over } \overline{\mathbb{F}}_p\text{)},
\]
where \(V_\xi'\) is the variety parameterizing isogenies \(\alpha: \xi \rightarrow \underline{A}_0\) of degree \(p\) with \(\underline{A}_2\) in \(\mathcal{A}_{2,1,N} \otimes \overline{\mathbb{F}}_p\) satisfying \(\alpha^* \lambda_2 = \lambda_\xi\) and \(\alpha^* \eta_2 = \eta_\xi\). This completes the first assertion.
Let $a = (A_0 \overset{\alpha}{\to} A_1 \overset{\alpha}{\to} A_2)$ be a point in $U_\gamma(k)$. Since $A_2$ is determined by $A_1$ (cf. Lemma 8.3 (i) (ii)), the projection $pr$ induces an isomorphism
\[ pr : U_\gamma \iso pr(U_\gamma) \subset S'_\gamma. \]
As $U_\gamma$ is dense in $S_\gamma$ and $S'_\gamma$ is proper, $pr(S_\gamma) \subset S'_\gamma$. Since $pr$ is proper and $S'_\gamma$ is a smooth curve, the section $s : pr(U_\gamma) \to U_\gamma$ extends uniquely to a section $s : S'_\gamma \to S_\gamma$. This shows $pr : S_\gamma \iso S'_\gamma$, and hence $S_\gamma \iso \mathbb{P}^1$ over $\mathbb{F}_p$.

A component $S_\gamma$ meets a component $W_\xi$ if and only if $S'_\gamma$ and $pr(W_\xi)$ meet. Since $S'_\gamma$ and $V_\xi$ meet transversally at most one point [22 Proposition 4.5], the components $S_\gamma$ and $W_\xi$ meet transversally at most one point. Since any irreducible component of $S_{2,\Gamma_0(p)}$ is smooth, the singularity occurs only at the intersection of components $S_\gamma$ and $W_\xi$.

(3) We know that $S'_\gamma$ contains $p + 1$ points with $A_1 \in \Lambda$ (Subsection 8.3). Each component $S_\gamma$ meets $p + 1$ components of the form $W_\xi$, and hence has $p + 1$ singular points. This proves the first part.

The result (8.2) is due to Katsura and Oort [11, Theorem 5.1, Theorem 5.3] in a slightly different form. For another proof (using a mass formula due to Ekedahl [6] and some others), see [22 Corollary 3.3, Corollary 4.6].

8.7. It follows from the description of the supersingular locus that

(i) The closure of the stratum $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$ is $\coprod_{\xi \in \Lambda} W_\xi$.

(ii) The stratum $\mathcal{A}_{\Gamma_0(p),s_1\tau}$ is $\coprod_{\gamma \in \Lambda_1,N} U_\gamma$, as this is the complement of the closure $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$ in the supersingular locus $S_{2,\Gamma_0(p)}$.

(iii) The minimal stratum $\mathcal{A}_{\Gamma_0(p),\tau}$ is
\[
\left( \coprod_{\xi \in \Lambda} W_\xi \right) \cap \left( \coprod_{\gamma \in \Lambda_1,N} S_\gamma \right) = S_{2,\Gamma_0(p)}^{\text{sing}}.
\]

We compute the locus $\mathcal{A}_{\Gamma_0(p),s_0s_2\tau}$, the closure of the stratum $\mathcal{A}_{\Gamma_0(p),s_0\tau}$. This is the disjoint union of the subvarieties in the component $W_\xi$, for $\xi \in \Lambda$, defined by the closed condition $\tau_{02} = 2$ (Table 3.).

Let $A_1 = \xi \in \Lambda$ and $(M_1, \langle, \rangle_1)$ be the Dieudonné module of $A_1$. Choose a basis $e_1, e_2, e_3, e_4$ for $M_1$ as in Lemma 8.2. Let $M_2 \subset M_1 \subset M_0$ be a chain of Dieudonné modules with $M_0/M_1 \simeq M_1/M_2 \simeq k$. As $M_1/VM_1 = k < e_1, e_2 >$, the subspace $M_2/VM_1$ has the form
\[ k < ae_1 + be_2 >, \quad [a : b] \in \mathbb{P}^1(k). \]

As $V^{-1}M_1/M_1 = k < \frac{1}{p}e_3, \frac{1}{p}e_4 >$, the subspace $M_0/M_1$ has the form
\[ k < c - e_3 + d - e_4 >, \quad [c : d] \in \mathbb{P}^1(k). \]

Use this as coordinates for $W_\xi$, we get and fix an isomorphism $\Phi : W_\xi \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $A, B, C, D$ be lifts in $W$ of $a, b, c, d$, respectively. We have
\[ M_2 = < Ae_1 + Be_2, pe_1, pe_2, e_3, e_4 >, \quad \text{and} \quad M_0 = < e_1, e_2, e_3, e_4, C\frac{1}{p}e_3 + D\frac{1}{p}e_4 >. \]

The condition $\tau_{02} = 2$ says that in $\mathcal{M}_0 := M_0/pM_0$, the subspaces $\mathcal{V}M_0$ and $\mathcal{M}_2$ generate a two-dimensional subspace. As both have dimension two, the condition
means $VM_0 = M_2$. One has

$$VM_0 = \langle C^{-1} e_1 + D^{-1} e_2, pe_1, pe_2, e_3, e_4 \rangle.$$ 

As both submodules contain $VM_1$, modulo $VM_1$, we get

$$\langle e_1^{D^{-1}} + d^{-1} e_4 \rangle = \langle ae_1 + be_2 \rangle.$$ 

This gives the equation $a^p d - b^p c = 0$. We have shown that

$$\Phi(\Gamma_{p_0(p), s_2} \cap W_\xi) = \{ ([a : b], [a^p : b^p]) : [a : b] \in P^1 \} \simeq P^1.$$ 

This is the graph of the relative Frobenius morphism $F_{P_1/g_p} : P^1 \to P^1$. We carry out the similar computation and get

$$\Phi(\Gamma_{p_0(p), s_2} \cap W_\xi) = \{ ([e^p : d^p], [c : d]) : [c : d] \in P^1 \} \simeq P^1.$$ 

This is the transpose of the graph of the relative Frobenius morphism.

We summarize the results as follows.

**Proposition 8.5.** We have

$$\Gamma_{p_0(p), s_0 s_2} = \prod_{\xi \in \Lambda} W_\xi, \quad \Gamma_{p_0(p), s_1} = \prod_{\gamma \in \Lambda_{2,1}, N} S_\gamma,$$

$$\Gamma_{p_0(p), s_0 s_2} \simeq \prod_{\xi \in \Lambda} P^1, \quad \Gamma_{p_0(p), s_2} \simeq \prod_{\xi \in \Lambda} P^1, \quad \Gamma_{p_0(p), \tau} = S_{2, p_0(p)}^{sing}.$$ 

Consequently, we have

(i) The stratum $\Gamma_{p_0(p), s_0 s_2}$ has $|\Lambda|$ irreducible components.

(ii) The stratum $\Gamma_{p_0(p), s_1}$ has $|\Lambda_{2,1}, N|$ irreducible components.

(iii) The stratum $\Gamma_{p_0(p), s_0}$ has $|\Lambda|$ irreducible components.

(iv) The stratum $\Gamma_{p_0(p), s_2}$ has $|\Lambda|$ irreducible components.

(v) The stratum $\Gamma_{p_0(p), \tau}$ consists of $|\Lambda_{2,1}, N|(p + 1)$ points.

Proposition 8.3, Theorem 4.1 and Proposition 2.1 of [21] answer the question on irreducible components of each Kottwitz-Rapoport stratum in the moduli space $\mathcal{A}_{2, p_0(p)}$.

We end this paper with the following criterion to distinguish the types $s_1\tau$ and $\tau$.

This finishes our geometric characterization of Kottwitz-Rapoport strata for $g = 2$.

**Lemma 8.6.** Let $a = (A_0 \rightarrow A_1 \rightarrow A_2)$ be a point in $\mathcal{A}_{p_0(p), s_1\tau}(k)$, and let $M_2 \rightarrow M_1 \rightarrow M_0$ be the chain of the associated de Rham cohomologies. Let $\omega_i := \omega_{A_i} \subset M_i$ be the Hodge subspace. Then the point $a$ lies in $\mathcal{A}_{p_0(p), \tau}$ if and only if the condition $\langle \alpha(M_1), \alpha(\omega_1) \rangle_0 = 0$ holds.

**Proof.** It follows from Theorem 8.1 that $a$ lies in $\mathcal{A}_{p_0(p), \tau}$ if and only if the object $A_1$ lies in $\Lambda$ (Subsection 8.2). The statement then follows from Lemma 8.3 as one has $\omega_1 = VM_1$.

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