Abstract. The aim of this paper is to describe the topological $K$-ring, in terms of generators and relations, of a Springer variety $\mathcal{F}_\lambda$ of type $A$ associated to a nilpotent operator having Jordan canonical form whose block sizes form a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$. Our description parallels the description of the integral cohomology ring of $\mathcal{F}_\lambda$ due to Tanisaki and also the equivariant analogue due to Abe and Horiguchi.

1. Introduction

Fix a positive integer $n$ and consider the complete flag variety $\mathcal{F}(\mathbb{C}^n)$ (or more briefly $\mathcal{F}$) defined as

$$\mathcal{F}(\mathbb{C}^n) := \{ V := (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim V_i = i \text{ for all } i \}. $$

Let $N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ denote a nilpotent linear transformation of $\mathbb{C}^n$. The Springer variety of type $A$ associated to $N$ denoted by $\mathcal{F}_N$ is the closed subvariety of $\mathcal{F}$ defined as

$$\{ V \in \mathcal{F} \mid NV_i \subset V_{i-1} \text{ for all } 1 \leq i \leq n \}. $$

The Springer variety $\mathcal{F}_N$ is seen to be the subvariety of $\mathcal{F}$ fixed by the action of the infinite cyclic group generated by the unipotent element $U = I_n + N \in SL(n, \mathbb{C})$. Moreover, denoting by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ the partition of $n$ where the $\lambda_j$ are the sizes of the diagonal blocks of the Jordan canonical form of $N$, the variety $\mathcal{F}_N$ depends, up to isomorphism, only on the partition $\lambda$. This is so, because two different choices of nilpotent transformations corresponding to the same partition $\lambda$ are conjugates in $GL(n, \mathbb{C})$. For this reason, we assume that $N$ itself is in the Jordan canonical form: $N = J_\lambda := \text{diag}(J_{\lambda_1}, \ldots, J_{\lambda_l})$ with $\lambda_1 \geq \cdots \geq \lambda_l$ and denote the Springer variety $\mathcal{F}_N$ by $\mathcal{F}_\lambda$. (Here $J_p = (a_{i,j}) \in M_p(\mathbb{C})$ is the matrix where $a_{i,i+1} = 1$, $1 \leq i < p$, and all other entries are zero.) If $\lambda = (1, \ldots, 1)$, then $N = 0$ and we have

2020 Mathematics Subject Classification. Primary 55N15; Secondary 14M15, 19L19.

Key words and phrases. Springer varieties, flag varieties, $K$-theory, Chern character.
\( F_\lambda = \mathcal{F}(\mathbb{C}^n) = \mathcal{F} \). At the other extreme, when \( \lambda = (n) \), \( N \) is a regular nilpotent element and we see that \( \mathcal{F}(n) \) is the one-point variety consisting only of the standard flag \( 0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n \) where \( E_j \) is spanned by the standard basis vectors \( e_1, \ldots, e_j \) for \( 1 \leq j \leq n \).

Note that \( F_\lambda \) is stable by the action of the algebraic torus \( T^l_\mathbb{C} \cong (\mathbb{C}^*)^l \) contained in \( GL(n, \mathbb{C}) \) consisting of all diagonal matrices which commute with \( N \). We shall denote by \( T^l = (\mathbb{S}^1)^l \) the compact torus contained in \( T^l_\mathbb{C} \). Denoting the diagonal subgroup of \( GL(n, \mathbb{C}) \) by \( T^l_\mathbb{C} \), we have \( (t_1, \ldots, t_n) \in T^l_\mathbb{C} \) belongs to \( T^l_\mathbb{C} \) if and only if \( t_{a_j+i} = t_{a_j+1} \) for \( 1 \leq i \leq \lambda_{j+1} \) where \( a_j := \lambda_1 + \cdots + \lambda_j \), \( 1 \leq j \leq l-1 \) and \( a_0 = 0 \).

The variety \( F_\lambda \) was first studied by Springer (see [14], [15] and also [6]). In particular, Springer showed that there is a natural action of the symmetric group \( S_n \) on the rational cohomology \( H^*(F_\lambda; \mathbb{Q}) \) which is compatible with the standard action of \( S_n \) on \( H^*(\mathcal{F}; \mathbb{Q}) \). Moreover, the restriction homomorphism \( H^*(\mathcal{F}; \mathbb{Z}) \longrightarrow H^*(F_\lambda; \mathbb{Z}) \) induced by the inclusion \( F_\lambda \hookrightarrow \mathcal{F} \), is surjective (see [6]). The variety \( F_\lambda \) is not irreducible in general, but it is equidimensional. The irreducible components of \( F_\lambda \) are naturally labelled by the set of standard tableaux of shape \( \lambda \). See [13]. Under the \( S_n \)-action, the \( S_n \)-module, \( H^*(F_\lambda; \mathbb{Q}) \) is isomorphic to the representation \( M_\lambda \) of \( S_n \) induced from the identity representation of the subgroup \( \mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_l} \subset \mathfrak{S}_n \). See [9] and [5, p. 204].

De Concini and Procesi [5] gave a description of \( H^*(F_\lambda; \mathbb{C}) \) as the coordinate ring of an (unreduced) variety over \( \mathbb{C} \) which we now describe. Let \( \lambda^\vee \) denote the partition dual to \( \lambda \). Consider the coordinate ring \( \mathbb{C}[t_C \cap \mathcal{O}_{\lambda^\vee}] \) of the (non-reduced) scheme \( t_C \cap \mathcal{O}_{\lambda^\vee} \) (scheme theoretic intersection), where \( t_C = \text{Lie}(T^l_\mathbb{C}) \subset \mathfrak{g}(n, \mathbb{C}) = M_n(\mathbb{C}) \) and \( \mathcal{O}_{\lambda^\vee} \subset M_n(\mathbb{C}) \) denotes the closure of the orbit of \( J_{\lambda^\vee} \) under the adjoint action of \( GL(n, \mathbb{C}) \). De Concini and Procesi showed that \( H^*(F_\lambda; \mathbb{C}) \) is isomorphic to the algebra \( \mathbb{C}[t_C \cap \mathcal{O}_{\lambda^\vee}] \).

Tanisaki [16] described \( H^*(F_\lambda; \mathbb{C}) \) as a quotient of a polynomial ring over \( \mathbb{C} \) by an ideal, which has come to be known as the Tanisaki ideal. Tanisaki's description in fact yields the integral cohomology ring of \( F_\lambda \). Recently, the \( T^l \)-equivariant cohomology algebra \( H^*_T(F_\lambda; \mathbb{Z}) \) has been described by H. Abe and T. Horiguchi. It turns out that \( H^*_T(F_\lambda; \mathbb{Z}) \) is the quotient of a polynomial algebra over \( H^*_T(pt; \mathbb{Z}) = H^*(BT^l; \mathbb{Z}) \) modulo an ideal, which is a natural generalization of the Tanisaki ideal. This presentation recovers the presentation for the ordinary integral cohomology ring via the forgetful map \( H^*_T(F_\lambda; \mathbb{Z}) \longrightarrow H^*(F_\lambda; \mathbb{Z}) \).
Our aim in this paper is to describe the topological $K$-ring of the variety $\mathcal{F}_\lambda$, in terms of generators and relations, using Tanisaki’s description of the integral cohomology ring. The generators of the Tanisaki ideal admit topological interpretation in terms of the Chern classes of certain naturally defined line bundles over $\mathcal{F}_\lambda$. We interpret these relations as a consequence of a relation in the topological $K$-ring $K(\mathcal{F}_\lambda)$ among the line bundles over $\mathcal{F}_\lambda$.

Before stating the main result of this paper, we need the following notations.

A non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers where $\sum_{1 \leq j \leq l} \lambda_j = n$, will be identified with the partition $(\lambda_1, \ldots, \lambda_n)$ where $\lambda_j = 0$ for $j > l$. Let $1 \leq s \leq n$ and denote by $i = (i_1, \ldots, i_s)$ a strictly increasing sequence of integers between 1 and $n$. The set of all such sequences will be denoted $W_{n,s}$. Let $p_\lambda(s) := \lambda_{n-s+1} + \cdots + \lambda_n, 1 \leq s \leq n$.

We denote by $\mathcal{L}_i$ the canonical line bundle over $\mathcal{F}(\mathbb{C}^n)$ whose fibre over a flag $V$ is the vector space $V_i/V_{i-1}, 1 \leq i \leq n$. Let $L_i = \mathcal{L}_i|_{\mathcal{F}_\lambda}$.

We now state the main theorem. A more precise formulation as a quotient of a polynomial ring by the $K$-theoretic Tanisaki ideal will be given in §4.

**Theorem 1.1.** The ring $K^0(\mathcal{F}_\lambda)$ is generated by the classes $[L_i]$ for $1 \leq i \leq n$, subject only to the following (generating) relations:

$$\gamma^d([L_{i_1} \oplus \cdots \oplus L_{i_s}] - s) = 0$$

for $d \geq s + 1 - p_\lambda(s), \ i = (i_1, \ldots, i_s) \in W_{n,s}, 1 \leq s \leq n$. Moreover, $K^1(\mathcal{F}_\lambda) = 0$.

For the definition of $\gamma$-operations in $K$-theory, see [7, Chapter 12].

The structure of $K(\mathcal{F}_\lambda)$ as an abelian group is easily obtained from the fact that $\mathcal{F}_\lambda$ admits an algebraic cellular decomposition. The existence of such a decomposition was established by Spaltenstein [13], the total number of such cells being equal to the number of $T^l$-fixed points in $\mathcal{F}_\lambda$ which in turn equals $\binom{n}{l}$. Although these algebraic cells are topologically $\mathbb{C}^r$, in general they do not yield the structure of a CW complex. See [17] for an explicit example. However, the algebraic cell-structure allows one to compute the integral cohomology groups. In particular, $\text{rank}(H^*(\mathcal{F}_\lambda; \mathbb{Z})) = \binom{n}{l}$ and $H^k(\mathcal{F}_\lambda; \mathbb{Z}) = 0$ for $k$ odd. In particular, $K^1(\mathcal{F}_\lambda) = 0$ and $K^0(\mathcal{F}_\lambda)$ is a free abelian group of rank $\binom{n}{\lambda}$. 
The fact that $H^*(\mathcal{F}_\lambda; \mathbb{Z})$ is generated in degree 2 allows us to apply [11, Lemma 4.1] to $\mathcal{F}_\lambda$ to conclude that $K^0(\mathcal{F}_\lambda)$ is generated as a ring by the classes of line bundles.

Using a geometrical argument, we shall show that the ideal of relations among the $[L_j]$ contains the elements described in the theorem. To show that these are all the generating relations we use Tanisaki’s description of the ring $H^*(\mathcal{F}_\lambda; \mathbb{Z})$ and a purely algebraic argument exploiting the fact that the $K$-theoretic Tanisaki ideal admits a certain natural filtration.

Similar approaches were applied in our papers [11], [12] and [10], to obtain a presentation of the $K$-ring of a smooth projective toric variety, a quasitoric manifold, and a class of torus manifolds which include the class of smooth complete toric varieties.

2. Cohomology of Springer varieties

Let $V_j$ be the subbundle of the trivial vector bundle $\mathcal{F} \times \mathbb{C}^n$ whose fibre over the flag $V = (V_i) \in \mathcal{F}$ is just $V_j$. Denote by $L_i$ the line bundle $V_i/V_{i-1}$, $1 \leq i \leq n$, on $\mathcal{F}$. We denote the first Chern class of the line bundle $L_i$ by $x_i \in H^2(\mathcal{F}; \mathbb{Z})$, $1 \leq i \leq n$. One has an exact sequence of algebraic vector bundles $0 \to V_{s-1} \to V_s \to L_s \to 0$, which leads to an isomorphism of complex vector bundles for $1 \leq s \leq n$:

$$L_1 \oplus \cdots \oplus L_s \cong V_s. \tag{2.1}$$

Since, $V_n = n\epsilon$, the trivial vector bundle of rank $n$, we have

$$L_1 \oplus \cdots \oplus L_n \cong n\epsilon. \tag{2.2}$$

It follows that the Chern polynomial of $\bigoplus_{1 \leq i \leq n} L_j$ is trivial. That is,

$$\prod_{1 \leq i \leq n} (1 + x_it) = 1$$

and so the $j$th elementary symmetric polynomial $e_j(x) := e_j(x_1, \ldots, x_n)$ vanishes for $1 \leq i \leq n$. Borel has shown that $H^*(\mathcal{F}; \mathbb{Z})$ is generated by the $c_1(L_i) = x_i$, $1 \leq i \leq n$ and that the only generating relations among the Chern classes of $L_i$ are given by $e_j(x) = 0$, $1 \leq j \leq n$. (See [4].) Thus we have a presentation

$$H^*(\mathcal{F}; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/\langle e_j(x); 1 \leq j \leq n \rangle. \tag{2.3}$$

For an arbitrary partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_l$ of $n$, the Springer variety $\mathcal{F}_\lambda$ is naturally imbedded in $\mathcal{F}$. The cohomology ring of the
Springer variety $F_{\lambda}$ has been described by Tanisaki [16] in terms of generators and relations, in a way that generalizes the above description of $H^*(F;\mathbb{Z})$. It turns out that, although Tanisaki considered cohomology with complex coefficients, his description, recalled below, is valid even when the coefficient ring is the integers (see [5]).

We need the following notation.

**Definition 2.1.** Let $[n] := \{1, 2, \ldots, n\}$. We define the function $p_{\lambda} : [n] \to [n] \cup \{0\}$ associated to a partition $\lambda$ of $n$ as follows:

\[(2.4) \quad p_{\lambda}(s) = \lambda_{n-s+1} + \cdots + \lambda_n, \quad 1 \leq s \leq n.\]

Thus $p_{\lambda}$ is a monotonically increasing function of $s$. The function $p_{\lambda^\vee}$ associated to the dual partition $\lambda^\vee$ is more relevant for us. Recall that the dual partition $\lambda^\vee$ is defined as $\lambda^\vee = (\eta_1, \ldots, \eta_n)$ where $\eta_j = \#\{i \mid \lambda_i \geq j\}$. Writing $\lambda$ as $1^{a_1} \cdot 2^{a_2} \cdots n^{a_n}$, where $a_j$ is the number of times $j$ occurs in $\lambda$, we have $\eta_j = a_j + \cdots + a_n$ for all $j \geq 1$.

We pause for an example.

**Example 2.2.** Let $n = 20$, and, $\lambda = (5, 4, 4, 2, 2, 2, 1)$ = $1^5 \cdot 2^4 \cdot 3^2 \cdot 4^2 \cdot 5^1$. Then $\lambda^\vee = (7, 6, 3, 3, 1)$ and $p_{\lambda^\vee}(s) = 0$ for $1 \leq s \leq 15$, $p_{\lambda^\vee}(16) = 1$, $p_{\lambda^\vee}(17) = 4$, $p_{\lambda^\vee}(18) = 7$, $p_{\lambda^\vee}(19) = 13$, $p_{\lambda^\vee}(20) = 20$.

**Definition 2.3.** Let $S = \mathbb{Z}[y_1, \ldots, y_n]$ be the polynomial ring in $n$-indeterminates $y_1, \ldots, y_n$. The Tanisaki ideal is the ideal $I_\lambda \subset S$ generated by the following elements:

\[e_d(y_{i_1}, \ldots, y_{i_s}), \quad \text{for } d \geq s + 1 - p_{\lambda^\vee}(s)\]

where $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq s \leq n$.

Recall that $L_j$ is the line bundle over $F_{\lambda}$ obtained as the restriction of $L_j$ on $F$.

**Theorem 2.4.** We keep the above notations. Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_l$ be a partition of $n$. Then one has an isomorphism of rings

\[H^*(F_{\lambda};\mathbb{Z}) \cong S/I_\lambda\]

where $c_1(L_j)$ corresponds to $y_j + I_\lambda$, $1 \leq j \leq n$. Moreover, the inclusion $\iota_\lambda : F_{\lambda} \to F$ induces a surjection $\iota_{\lambda}^* : H^*(F;\mathbb{Z}) \to H^*(F_{\lambda};\mathbb{Z})$. □

We shall abuse notation and denote by $y_j \in H^*(F_{\lambda};\mathbb{Z})$ the image of $y_j + I_\lambda$ under the isomorphism of the above theorem.

The rank of $H^*(F_{\lambda};\mathbb{Z})$ equals

\[\binom{n}{\lambda} = \frac{n!}{(\lambda_1! \cdots \lambda_l!)}\]
and \( \dim_{\mathbb{C}} F_{\lambda} = \sum_{1 \leq j \leq n} \eta_j \cdot (\eta_j - 1)/2 \) where \( \lambda^\vee = \eta_1 \geq \cdots \geq \eta_n \geq 0 \) is the partition of \( n \) that is dual to \( \lambda \).

3. Line bundles over \( F_{\lambda} \)

We begin by recalling the description of \( K(F) \) in terms of generators and relations. It turns out that \( K(F) \) is generated by the classes of the line bundles \( [\mathcal{L}_j], 1 \leq j \leq n \). (This can be seen using the Atiyah-Hirzebruch spectral sequence and the Chern character map \( \text{ch} : K(F) \to H^*(F; \mathbb{Q}) \).) In view of (2.2), we have an isomorphism

\[
(3.5) \quad e_k(\mathcal{L}_1, \ldots, \mathcal{L}_n) \cong \binom{n}{k} \epsilon
\]

for all \( k \geq 1 \). Indeed, we have an isomorphism

\[
(3.6) \quad \varphi : \frac{\mathbb{Z}[u_j; 1 \leq j \leq n]}{I} \to K(F)
\]

where \( \varphi(u_i) = [\mathcal{L}_i], 1 \leq i \leq n \), and the ideal \( I \) is generated by the elements \( e_k(u_1, \ldots, u_n) - \binom{n}{k}, 1 \leq k \leq n \). (See [8, \S3, Chapter IV].)

Recall that the Atiyah-Hirzebruch spectral sequence of a space \( X \) admitting the structure of a finite CW complex has \( E_2 - \) page defined as \( E_2^{p,q}(X) = H^p(X; K^q(pt)) \) and differential \( d_r \) has bidegree \( (r, 1 - r) \). The spectral sequence converges to \( K^*(X) \), that is, there exists a decreasing filtration \( \{K^p_{p+q}(X)\} \) such that \( E_{\infty}^{p,q} \cong G_p K_{p+q}(X) = K_{p+1}^{p+q}(X)/K_{p+1}^{p+2}(X) \) (see [3, p. 17]).

Recall that \( K^q(pt) \cong \mathbb{Z} \) for \( q \) even and \( K^q(pt) = 0 \) for \( q \) odd (see [3, p. 10]). Suppose that \( H^p(X; \mathbb{Z}) \) vanishes if \( p \) is odd and that it is a free abelian group when \( p \) is even. Then \( E_2^{p,q} = 0 \) unless both \( p, q \) are even and it follows that \( d_r = 0 \) for all \( r \). Thus the spectral sequence collapses and we have \( E_2^{p,q} = E_{\infty}^{p,q} \). Consequently, \( K^1(X) = 0 \) and \( K^0(X) \cong H^*(X; \mathbb{Z}) \) is a free abelian group.

Suppose that \( Y \) is another such space and that \( f : X \to Y \) induces a surjective homomorphism \( f^* : H^*(Y; \mathbb{Z}) \to H^*(X; \mathbb{Z}) \). The naturality of the spectral sequence yields a morphism \( \{E_r^{p,q}(Y), d_r\} \to \{E_r^{p,q}(X), d_r\} \). Since \( E_2^{p,q}(Y) = H^p(Y; K^q(pt)) \xrightarrow{f^*} H^p(X; K^q(pt)) = E_2^{p,q}(X) \) is surjective and since the differentials \( d_r \) vanish for both the sequences for \( r \geq 2 \), we see that \( f^* : E_\infty^{p,q}(Y) = E_2^{p,q}(Y) \to E_2^{p,q}(X) = E_\infty^{p,q}(X) \) is surjective. Moreover, both \( K^*(X) = K^0(X), K^*(Y) = \)
$K^0(Y)$ are free abelian groups. It follows that $f^!: K^*(Y) \to K^*(X)$ is surjective.

Setting $X = F_\lambda$, we obtain the following isomorphism of abelian groups

$$K^*(F_\lambda) = K^0(F_\lambda) \cong H^*(F_\lambda; \mathbb{Z}) \cong \mathbb{Z}(\lambda).$$

We next take $Y = F$ and $f$ to be the inclusion $\iota_\lambda : F_\lambda \hookrightarrow F$.

Let $y_i$ denote the image $\iota_\lambda^*(x_i) \in H^2(F_\lambda; \mathbb{Z})$ and $L_i$ denote the restriction $L_i|x_{F_\lambda}$ for $1 \leq i \leq n$. Thus $c_1(L_i) = y_i$. The surjectivity of $\iota_\lambda^*: H^*(F; \mathbb{Z}) \to H^*(F_\lambda; \mathbb{Z})$ implies that $H^*(F_\lambda; \mathbb{Z})$ is generated by $c_1(L_j) = y_j; 1 \leq j \leq n$.

By the above argument we conclude that the pull back map

$$\iota_\lambda^!: K(F) \to K(F_\lambda)$$

induced by the inclusion $\iota_\lambda : F_\lambda \to F$ is surjective. Furthermore, since $K(F)$ is generated, as a ring by the isomorphism classes of line bundles $[L_j], 1 \leq j \leq n$, it follows that $K(F_\lambda)$ is generated by $[L_j], 1 \leq j \leq n$.

Summarising the above discussion, we have the following proposition.

**Proposition 3.1.** The group $K(F_\lambda)$ is free abelian of rank $\binom{n}{\lambda}$. Also, $\iota_\lambda^!: K(F) \to K(F_\lambda)$ is a surjection. In particular, $K(F_\lambda)$ is generated as a ring by the classes of line bundles $[L_j] \in K(F_\lambda), 1 \leq j \leq n$. \hfill $\square$

### 3.1. The action of the symmetric group on $K(F_\lambda)$.

The symmetric group $S_n$ acts linearly on $\mathbb{C}^n$ by permuting the standard basis vectors $e_1, \ldots, e_n$. This action induces an action of $S_n$ on the flag variety $F$ and hence on $K(F)$ as well as on the singular cohomology algebra $H^*(F; \mathbb{Z})$. Explicitly, the $S_n$-action on $K(F)$ and on $H^*(F; \mathbb{Z})$ are given by permutation of the classes of line bundles $[L_j] \in K(F)$ and the Chern classes $x_j := c_1(L_j) \in H^*(F; \mathbb{Z}), 1 \leq j \leq n$, respectively. It is readily seen that the Chern character map

$$\text{ch}_F : K(F) \to H^*(F; \mathbb{Q})$$

which sends $[L_k]$ to \(\sum_{r \geq 0} \frac{x_k^r}{r!}\) is $S_n$-equivariant.

Springer [15] showed that the symmetric group $S_n$ acts on $H^*(F_\lambda; \mathbb{Q})$ and that $\iota_\lambda^*: H^*(F; \mathbb{Q}) \to H^*(F_\lambda; \mathbb{Q})$ is $S_n$-equivariant and surjective. Hotta and Springer [6] showed that in fact $\iota_\lambda^*: H^*(F; \mathbb{Z}) \to H^*(F_\lambda; \mathbb{Z})$ is $S_n$-equivariant and surjective (also see [5, p. 213]).
Proposition 3.2. The ring $K(\mathcal{F}_\lambda)$ admits an action of the symmetric group $\mathfrak{S}_n$ with respect to which the Chern character map

$$ch_{\mathcal{F}_\lambda} : K(\mathcal{F}_\lambda) \to H^*(\mathcal{F}_\lambda; \mathbb{Q})$$

as well as the pull back map $\iota_\lambda^! : K(\mathcal{F}) \to K(\mathcal{F}_\lambda)$ are both $\mathfrak{S}_n$-equivariant.

Proof. By the naturality of Chern character we have the following commuting diagram:

$$
\begin{array}{ccc}
K(\mathcal{F}) & \xrightarrow{\iota_\lambda^!} & K(\mathcal{F}_\lambda) \\
\downarrow ch_{\mathcal{F}} & & \downarrow ch_{\mathcal{F}_\lambda} \\
H^*(\mathcal{F}; \mathbb{Q}) & \xrightarrow{\iota_\lambda^*} & H^*(\mathcal{F}_\lambda; \mathbb{Q}).
\end{array}
$$

The vertical arrows are monomorphisms since the $K$-groups $K(\mathcal{F}), K(\mathcal{F}_\lambda)$ are free abelian groups and $ch \otimes \mathbb{Q}$ is an isomorphism.

Now, since $ch_{\mathcal{F}_\lambda}$ is injective, $\iota_\lambda^!$ is surjective and $\iota_\lambda^*_{\mathcal{F}}$ and $ch_{\mathcal{F}}$ are $\mathfrak{S}_n$-equivariant, the commutative diagram (3.8) implies that the image $ch_{\mathcal{F}_\lambda}(K(\mathcal{F}_\lambda))$ in $H^*(\mathcal{F}_\lambda; \mathbb{Q})$ is stable under the $\mathfrak{S}_n$-action. This further implies that $K(\mathcal{F}_\lambda)$ admits an action of $\mathfrak{S}_n$ such that $ch_{\mathcal{F}_\lambda}$ and $\iota_\lambda^!$ are equivariant. This can be seen more explicitly as follows.

Let $x \in K(\mathcal{F}_\lambda)$. Since $\iota_\lambda^!$ is surjective, $x = \iota_\lambda^!(y)$ for some $y \in K(\mathcal{F})$. We define the action of $\mathfrak{S}_n$ on (the right of) $K(\mathcal{F}_\lambda)$ as follows: $x \cdot \sigma := \iota_\lambda^!(y \cdot \sigma)$. Indeed from (3.8) we have

$$ch_{\mathcal{F}_\lambda} \circ \iota_\lambda^!(y \cdot \sigma) = \iota_\lambda^* \circ ch_{\mathcal{F}}(y \cdot \sigma).$$

Moreover, since $\iota_\lambda^*$ and $ch_{\mathcal{F}}$ are $\mathfrak{S}_n$-equivariant this implies

$$ch_{\mathcal{F}_\lambda} \circ \iota_\lambda^!(y \cdot \sigma) = (\iota_\lambda^* \circ ch_{\mathcal{F}}(y)) \cdot \sigma.$$

Again by the commutativity of (3.8) we get

$$ch_{\mathcal{F}_\lambda} \circ \iota_\lambda^!(y \cdot \sigma) = (ch_{\mathcal{F}_\lambda} \circ \iota_\lambda^!(y)) \cdot \sigma = ch_{\mathcal{F}_\lambda}(x) \cdot \sigma.$$

Thus by definition $ch_{\mathcal{F}_\lambda}(x \cdot \sigma) = ch_{\mathcal{F}_\lambda}(x) \cdot \sigma$. Since $ch_{\mathcal{F}_\lambda}$ is injective and the $\mathfrak{S}_n$ action on $H^*(\mathcal{F}_\lambda; \mathbb{Q})$ is well defined, it follows that the $\mathfrak{S}_n$ action on $K(\mathcal{F}_\lambda)$ is well defined. It also follows that $ch_{\mathcal{F}_\lambda}$ and $\iota_\lambda^!$ are $\mathfrak{S}_n$-equivariant. \qed

Since the action of $\mathfrak{S}_n$ on (the right of) $K(\mathcal{F})$ is obtained as $[L_j] \cdot \sigma = [L_{\sigma(j)}]$ for $1 \leq j \leq n$, and $\sigma \in \mathfrak{S}_n$, by Proposition 3.2 and Proposition 3.1, it follows that the $\mathfrak{S}_n$ action on $K(\mathcal{F}_\lambda)$ is given by $[L_j] \cdot \sigma = [L_{\sigma(j)}]$ for all $1 \leq j \leq n$, $\sigma \in \mathfrak{S}_n$.

Recall the function $p_{\lambda}$ defined in §2. The numbers $p_{\lambda}(s)$ are related to the nilpotent transformation $N = J_\lambda$ as follows.
Lemma 3.3. [16, Proposition 3]. With the above notations,
(i) $p_{\lambda V}(s) = \text{rank}(J_{\lambda}^{n-s})$, $1 \leq s \leq n$.
(ii) Let $U = U_1 \subset U_2 \subset \cdots \subset U_n = \mathbb{C}^n$ be a flag that refines the
partial flag $0 = \text{Im}(J_{\lambda}^{1}) \subset \text{Im}(J_{\lambda}^{n-1}) \subset \cdots \subset \text{Im}(J_{\lambda}^{2}) \subset \text{Im}(J_{\lambda}) \subset \mathbb{C}^n$. Then, for any $V \in \mathcal{F}_\lambda$ and any $s \geq 1$, we have $U_q \subset V_s$ where $q = p_{\lambda V}(s)$.

\[ \square \]

3.2. Sectioning canonical bundles over $\mathcal{F}_\lambda$. For $1 \leq s \leq n$ we let
\[ W_{n,s} := \{ i = (i_1, \ldots, i_s) \mid 1 \leq i_1 < \cdots < i_s \leq n \}. \]

Proposition 3.4. Let $1 \leq s \leq n$ and let $i \in W_{n,s}$. Then
\[ L_{i_1} \oplus \cdots \oplus L_{i_s} \cong \xi \oplus q\epsilon \]
for some complex vector bundle $\xi = \xi(i)$ over $\mathcal{F}_\lambda$ where $q := p_{\lambda V}(s)$.

Proof. Fix $s \leq n$. Since the action of $\mathfrak{S}_n$ on $K(\mathcal{F}_\lambda)$ permutes the
$L_j$, $1 \leq j \leq n$, we need only consider the case where $i_j = j$, $\forall$ $1 \leq j \leq s$.

We replace $N$ by a conjugate $gNg^{-1}$ so that $\text{Im}(gNg^{-1}) = U_{p_{\lambda V}(k)} = \mathbb{C}^{p_{\lambda V}(k)}$ for $k \geq 1$. We may then choose the refinement $U \in \mathcal{F}$ to be the standard flag $0 \subset \mathbb{C} \subset \cdots \subset \mathbb{C}^n$. Thus $C^q \subset V_s$ for any $V \in \mathcal{F}_{gNg^{-1}}$. Let $\iota_q : \mathcal{F} \rightarrow \mathcal{F}$ be the translation by $g$:
$\iota_q : V \mapsto gV = gV_0 = 0 \subset gV_1 \subset \cdots \subset gV_n = \mathbb{C}^n$. Since $GL(n, \mathbb{C})$ is connected,
the composition $\mathcal{F}_N \xrightarrow{\iota_q} \mathcal{F} \xrightarrow{\iota_q} \mathcal{F}$, denoted $\iota_{\lambda, g}$ is homotopic to $\iota_\lambda$
and maps $\mathcal{F}_N$ onto $\mathcal{F}_{gNg^{-1}} \subset \mathcal{F}$. It follows that $\iota_\lambda$ and $\iota_{\lambda, g}$ induce
the same map in $K$-theory and singular cohomology. In particular,
$\iota^*_{\lambda, g}(L_j) = L_j \forall 1 \leq j \leq n$.

Let $G_{n,s} = G_s(\mathbb{C}^n)$ denote the Grassmann variety of $s$-planes in $\mathbb{C}^n$.
One has a projection $\pi_s : \mathcal{F}_N \rightarrow G_{n,s}$ defined as $V \mapsto V_s$.

Let $Y_q \subset G_{n,s}$ denote the subvariety $\{ U \in G_{n,s} \mid U \supseteq U_q \}$, $1 \leq q < s$.
Then $Y_q$ is isomorphic to a Grassmann variety $G_{n-q,s-q}$. A specific
isomorphism $Y_q \cong G_{s-q}(\mathbb{C}^n/U_q)$ is obtained by sending $U \in Y_q$ to $U/U_q$. The tautological complex vector bundle $\gamma_{n,s}$ is of rank $s$, whose
fibre over $A \in G_{n,s}$ is the vector space $A$. When restricted to $Y_q$, $\gamma_{n,s}$
has a trivial subbundle $q\epsilon$ of rank $q$. Indeed we have a commuting diagram
\[
\begin{array}{ccc}
Y_q \times U_q & \rightarrow & E(\gamma_{n,s}|_{Y_q}) \\
\downarrow & & \downarrow \\
Y_q & \xrightarrow{id} & Y_q
\end{array}
\]
where the vertical arrows are bundle projections. Therefore
\begin{equation}
\gamma_{n,s}|_{Y_q} \cong \omega \oplus q\varepsilon
\end{equation}
where \(\omega\) is the complex vector bundle over \(Y_q\) whose fibre over \(A \in Y_q\) is the complex vector space \(A' := A/C\).

From Proposition 3.3, the image of the composition
\[ \mathcal{F}_N = \mathcal{F}_\lambda \xrightarrow{i_{\lambda,q}} \mathcal{F} \xrightarrow{\pi_\lambda} G_{n,s}, \]
denoted \(\pi_{\lambda,s}\), is contained in \(Y_q\). Therefore, we have a commuting diagram
\[ \begin{array}{c}
\mathcal{F}_\lambda \xrightarrow{i_{\lambda,q}} \mathcal{F} \\
\pi_{\lambda,s} \downarrow \downarrow \pi_s \\
Y_q \hookrightarrow G_{n,s}
\end{array} \]
\begin{equation}
\begin{array}{c}
\end{array}
\end{equation}
Now \(\pi_s^*(\gamma_{n,s}) = V_s = L_1 \oplus \cdots \oplus L_s\) by (2.1). Therefore \(L_1 \oplus \cdots \oplus L_s = i_{\lambda,q}(L_1 \oplus \cdots \oplus L_s) = i_{\lambda,q} \circ \pi_s^*(\gamma_{n,s}) = \pi_{\lambda,s}^*(\gamma_{n,s}|_{Y_q}) = \pi_{\lambda,s}^*(\omega) \oplus q\varepsilon\), from (3.10). \(\Box\)

**Remark:** Note here that \(Y_q\) is nothing but the Schubert variety \(X(\sigma) = \{U \in G_{n,s} | \dim(U \cap C^{\sigma_i}) \geq i, \ 1 \leq i \leq s\}\) where \(\sigma_i = \begin{cases} i, & \text{if } i \leq q, \\ n - s + i, & \text{if } q < i \leq s. \end{cases}\)

### 3.3. The \(\gamma\)-operations in K-theory.
We recall here the \(\gamma\)-operations in \(K\)-theory and their relation to the \(\lambda\)-operations. We refer the reader to [7, §3, Chapter 12] for further details. Let \(X\) be a finite CW complex. The \(\gamma\)-operations \(\gamma^d : K(X) \to K(X), d \geq 0\), are defined in terms of the exterior power operations \(\lambda^d\) as follows:

\begin{equation}
\gamma_t(x) = \lambda_{t/(1-t)}(x) \ \forall x \in K(X),
\end{equation}
where \(\gamma_t(x) = \sum_{k \geq 0} \gamma^k(x)t^k\) and \(\lambda_t(x) = \sum_{k \geq 0} \lambda^k(x)t^k\) are regarded as elements of the formal power series ring \(K(X)[[t]]\) in the indeterminate \(t\). Since \(\gamma^0(x) = \lambda^0(x) = 1\), we can express \(\lambda_t(x)\) in terms of \(\gamma(x)\). Indeed we have \(\lambda_t(x) = \gamma_{t/(1+t)}(x)\). It follows from the definition of \(\gamma^d(x)\) that

\begin{equation}
\gamma^d(x) = \sum_{0 \leq k \leq d} \lambda^k(x)\binom{k + d - 1}{k - 1} = \lambda^d(x + d - 1).
\end{equation}
When \( x = [\xi] \in K(X) \) is the class of a vector bundle \( \xi \) of rank \( k \), we have \( \lambda_t(x) \) is a polynomial of degree \( k \) since the exterior power
\[(3.14) \quad \lambda^d([\xi]) = 0 = \lambda^d([\xi] + d - 1 - k)\]
for \( d \geq k + 1 \).

If \( \xi \) is a line bundle, then \( \lambda_t(x) = 1 + xt \). In the case when \( \xi \) is a trivial bundle, we have \( [\xi] = k \in K(X) \) and \( \lambda_t(k) = (1 + t)^k \) and so \( \lambda_t(-k) = (1+t)^{-k} \). The last equality in (3.14) follows from the identity \( \lambda_t(x + y) = \lambda_t(x)\lambda_t(y) \).

These basic facts will be used below.

### 4. Proof of Theorem 4.2

We begin by establishing the following proposition, which is an immediate corollary of Proposition 3.4.

**Proposition 4.1.** For any \( i \in W_{n,s} \) and any \( d \geq s + 1 - p_{\lambda^\vee}(s) \), the following relation holds in \( K(F) \):
\[(4.15) \quad \gamma^d([L_{i_1}] + \cdots + [L_{i_s}] - s) = \lambda^d([L_{i_1}] + \cdots + [L_{i_s}] + d - s - 1) = 0 \]
for all \( d \geq s + 1 - p_{\lambda^\vee}(s) \). Equivalently,
\[(4.16) \quad \lambda^d(\sum_{1 \leq j \leq s} [L_{i_j}] - q) = 0 \quad \forall \ d \geq s + 1 - q \]
where \( q = p_{\lambda^\vee}(s) \).

**Proof.** Set \( E = \bigoplus_{1 \leq j \leq s} L_{i_j} \). Then, by Proposition 3.4, \( E \cong \xi + q\epsilon \) where rank of \( \xi \) equals \( s - q \). (Here \( q = p_{\lambda^\vee}(s) \) as in the Propositon.) It follows that \( \gamma^d([E] - s) = \gamma^d([\xi] - s + q) = \lambda^d([\xi] + d - s + q - 1) = 0 \) for all \( d \geq \text{rank}([\xi]) + 1 = s + 1 - q \). The last equation is equivalent to the vanishing of \( \lambda^d([E] - q) \) for all \( d \geq s + 1 - q \) (see (3.14)). \( \square \)

Let \( i \in W_{n,s} \). We have
\[(4.17) \quad \lambda_i(\sum_{1 \leq j \leq s} [L_{i_j}] - q) = \prod_{1 \leq j \leq s} (1 + [L_{i_j}] t) \cdot (1 + t)^{-q} \]
is a polynomial of degree \( s - q \) where \( q = p_{\lambda^\vee}(s) \). Comparing the coefficient of \( t^d \) where \( d \geq s + 1 - q \), we obtain the following equation
in $K(F_{\lambda})$ from Proposition 4.1:

\[(4.18)\]

\[
h_d([L_{i_1}], \ldots, [L_{i_s}]) := \sum_{0 \leq k \leq d} (-1)^{d-k} e_k([L_{i_1}], \ldots, [L_{i_s}]) \cdot \binom{q + d - k - 1}{q - 1} = 0.
\]

When $s = n$, we have $q = n$. Now, from Equation 2.2 we have

\[
\sum_{1 \leq j \leq n} [L_j] - n = 0.
\]

Using this in Equation (4.17), we get $(1 + t)^n = \prod_{1 \leq j \leq n} (1 + [L_j]t)$. Thus in this case (4.15) follows from (3.5).

Let $R := \mathbb{Z}[u_1, \ldots, u_n]$, the polynomial algebra in $n$ variables $u_1, \ldots, u_n$. Let $1 \leq s \leq n$ and let $i \in W_{n,s}$. The ring $R$ is graded by setting $\deg(u_j) = 1 \quad \forall \quad j$. One has the augmentation $\theta : R \to \mathbb{Z}$ defined by $u_j \mapsto 1 \quad \forall \quad j \leq n$. For $i = (i_1, \ldots, i_s) \in W_{n,s}$, we denote by $u_i$ the sequence $(u_{i_1}, \ldots, u_{i_s})$. We define the $K$-theoretic Tanisaki ideal $I_{\lambda} \subset R$ to be the ideal generated by the elements $h_d(u_i)$ defined as

\[
h_d(u_i) := \sum_{0 \leq k \leq d} (-1)^{d-k} e_k(u_i) \cdot \binom{q + d - k - 1}{q - 1},
\]

where $1 \leq i_1 < \ldots < i_s \leq s$, $1 \leq s \leq n$. Since

\[
\sum_{0 \leq k \leq d} (-1)^{d-k} \binom{s}{k} \binom{q + d - k - 1}{q - 1} = \binom{s-q}{d} = 0
\]

as $d \geq s + 1 - q$, we have

\[
h_d(u_i) = \sum_{0 \leq k \leq d} (-1)^{d-k} (e_k(u_i) - \binom{s}{k}) \cdot \binom{q + d - k - 1}{q - 1}.
\]

The highest degree term in $h_d(u_i)$ equals $e_d(u_i)$. We set

\[
\bar{h}_d(u_i) := e_d(u_i) - \binom{s}{d}
\]

so that $\theta(\bar{h}_d(u_i)) = 0 = \theta(h_d(u_i))$.

We may restate Theorem 4.2 in view of Proposition 4.1 and Equations (4.17) and (4.18) as follows:

**Theorem 4.2.** We keep the above notations. Let

\[
\psi_{\lambda} : R = \mathbb{Z}[u_1, \ldots, u_n] \to K(F_{\lambda})
\]

be the ring homomorphism defined by $\psi_{\lambda}(u_j) = [L_j], \quad 1 \leq j \leq n$. Then $\psi_{\lambda}$ is surjective and $\ker(\psi_{\lambda}) = I_{\lambda}$. 
Proof. In view of (4.18), it is clear that $I_\lambda \subset \ker(\psi_\lambda)$. Also, by Proposition 3.1, $\psi_\lambda$ is surjective. Since $\text{rank}(\mathcal{K}(\mathcal{F}_\lambda)) = \text{rank}(H^*(\mathcal{F}_\lambda; \mathbb{Z})) = \binom{n}{d}$, it suffices to show that $R/I_\lambda$ is a free abelian group of rank $\binom{n}{d}$. We will show that $R/I_\lambda$ has a filtration such that the associated graded ring is isomorphic as an abelian group to $S/I_\lambda \cong H^*(\mathcal{F}_\lambda; \mathbb{Z})$.

Let $R^d \subset R$ denote the abelian group generated by elements of the form $P(u) - \theta(P(u)) \in \ker(\theta)$, where $P(u)$ is a polynomial in $u_1, \ldots, u_n$ of degree at most $d$. Note that $R^0 = 0$ and $R^d \subset R^{d+1}$ and $R^i \cdot R^j \subset R^{i+j} \quad \forall \ i, j \geq 1$. Clearly $h_d(u_i) - \overline{h_d}(u_i) \in R^{d-1}$.

Let $k \geq 1$, and let $s = (s_1, \ldots, s_k)$ where $1 \leq s_1 \leq \cdots \leq s_k \leq n$. We denote by $S_k$ the set of all such $k$-tuples $s$. For $s \in S_k$ we define the following sets

$$\mathcal{W}_s := W_{n,s_1} \times \cdots \times W_{n,s_k} \quad \text{and} \quad \mathcal{D}_s := \{ d = (d_1, \ldots, d_k) \mid s_j + 1 - p_{\lambda^\vee}(s_j) \leq d_j \leq s_j \quad \forall \ j \}. $$

For a given $d \in \mathcal{D}_s$ and an integer $d \geq \sum_{j=1}^k d_j$, we set

$$\mathcal{C}_{d,d} := \{ c = (c_1, \ldots, c_k) \mid c_j \geq 1, \sum_{1 \leq j \leq k} c_j \cdot d_j = d \}.$$

An element of $\mathcal{W}_s$ will be denoted $\iota = (i(1), \ldots, i(k))$.

Given $s \in S_k$, $\iota \in \mathcal{W}_s$, $d \in \mathcal{D}_s$, $c \in \mathcal{C}_{d,d}$, we set

$$h_{i,d,c}(u) := h_{d_1}(u_{i(1)})^{c_1} \cdots h_{d_k}(u_{i(k)})^{c_k} \in \mathcal{I}_\lambda^d$$

where

$$\mathcal{I}_\lambda^d = \mathcal{I}_\lambda \cap R^d.$$  

Let $\mathcal{H}_{d,k}$ be the set of all elements $h_{i,d,c}(u)$, as we vary $s \in S_k$, $\iota \in \mathcal{W}_s$, $d \in \mathcal{D}_s$, $c \in \mathcal{C}_{d,d}$. Let

$$\mathcal{H}_d = \bigcup_k \mathcal{H}_{d,k}.$$

Note that the leading term of $h_{i,d,c}(u)$, when expressed as a polynomial in $u_1, \ldots, u_n$, equals that of $e_{i,d,c}(u) := e_{d_1}(u_{i(1)})^{c_1} \cdots e_{d_k}(u_{i(k)})^{c_k}$. Since $\theta(e_{i,d,c}(u)) \neq 0$, the element $e_{i,d,c}(u)$ is not in $R^d$. But evidently $e_{i,d,c}(u) - \theta(e_{i,d,c}(u))$ belongs to $R^d$. Setting $v_j = u_j - 1, 1 \leq j \leq n$, we have $e_{i,d,c}(u) \equiv e_{i,d,c}(u) - \theta(e_{i,d,c}(u)) \equiv h_{i,d,c}(u) \mod R^{d-1}$.

Recall, from Definition 2.3, that $I_\lambda \subset S = \mathbb{Z}[y_1, \ldots, y_n]$ is the Tanisaki ideal generated by the homogeneous polynomials

$$e_d(y_i), \ i \in \mathcal{W}_{n,s}, \ d \geq s + 1 - p_{\lambda^\vee}(s), \ 1 \leq s \leq n.$$
(see Definition 2.3). Let $S_k$ denote the homogeneous polynomials in $S$ of degree $k$ where $\deg(y_j) = 1$ for all $j \leq n$ and $I_{\lambda,k} := I_{\lambda} \cap S_k$. Writing $S^k = \bigoplus_{1 \leq j \leq k} S_j$ and $I^k_{\lambda} = \bigoplus_{1 \leq j \leq k} I_{\lambda,j}$, we see that under the isomorphism $\alpha : R \to S$ defined by $u_j \mapsto y_j$ for $1 \leq j \leq n$, we have $\alpha(R^k) = S^k$ for all $k \geq 1$. Also $\alpha(I_k^k) \equiv I^k_{\lambda} \mod S^{k-1}$ for all $k \geq 0$. Therefore $\alpha$ induces an isomorphism of abelian groups

$$S_k/I_{\lambda,k} \cong S^k/(I^k_{\lambda} + S^{k-1}) \cong R^k/(I_k^k + R^{k-1}).$$

Since by Theorem 2.4, for every $k$ we have

$$S_k/I_{\lambda,k} \cong H^k(F_\lambda; \mathbb{Z}),$$

we conclude that, as abelian groups,

$$R/I_\lambda \cong H^*(F_\lambda; \mathbb{Z}).$$

This completes the proof. □

**Acknowledgments:** We thank Professor Hiraku Abe for informing us about the work of Spaltenstein [13] and also for sending us a copy of it. We thank the referee for a careful reading of the manuscript and for valuable comments and suggestions.

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CHENNAI MATHEMATICAL INSTITUTE H1, SIPCOT IT PARK, SIRUSERI KELAMBAKKAM 603103 INDIA

Email address: sankaran@cmi.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, MADRAS, CHENNAI 600036, INDIA

Email address: vuma@iitm.ac.in