The Harmonic Lagrange Top
and the Confluent Heun Equation

Sean R. Dawson1*, Holger R. Dullin1**, and Diana M. H. Nguyen1***

1School of Mathematics and Statistics, University of Sydney,
2006 New South Wales, Australia

Received November 01, 2021; revised May 25, 2022; accepted June 13, 2022

Abstract—The harmonic Lagrange top is the Lagrange top plus a quadratic (harmonic) potential term. We describe the top in the space-fixed frame using a global description with a Poisson structure on $T^*S^3$. This global description naturally leads to a rational parametrisation of the set of critical values of the energy-momentum map. We show that there are 4 different topological types for generic parameter values. The quantum mechanics of the harmonic Lagrange top is described by the most general confluent Heun equation (also known as the generalised spheroidal wave equation). We derive formulas for an infinite pentadiagonal symmetric matrix representing the Hamiltonian from which the spectrum is computed.

MSC2010 numbers: 70E17, 81Q99

DOI: 10.1134/S1560354722040049

Keywords: symmetric rigid body, Lagrange top, Hamiltonian Hopf bifurcation, quantisation, confluent Heun equation

Dedicated to the memory of Alexey Borisov

1. INTRODUCTION

The Lagrange top is a prime example of classical mechanics. Over centuries, it has been studied starting with Euler and Lagrange, and interest in its various features is blossoming again and again. Almost every modern development in mechanics has led to new insights about the Lagrange top. Before we attempt to describe the place of the Lagrange top in mechanics in the remainder of this introduction, let us formulate our main observation: the quantum mechanics of the harmonic Lagrange top is described by the most general confluent Heun equation (also known as the generalised spheroidal wave equation). By harmonic Lagrange top we mean the Lagrange top with an added harmonic (i.e., quadratic) potential. It provides an example of the subcritical and the supercritical Hamiltonian Hopf bifurcation and hence the quantisation of these bifurcations. The bulk of the paper is devoted to the description of the classical integrable system.

Rigid body dynamics is treated in most mechanics textbooks, e.g., [1, 26, 33, 37, 52]. Of the books devoted specifically to rigid body dynamics we highlight the monumental volumes of Klein & Sommerfeld [30] and the recent addition by Borisov & Mamaev [10]. Many special cases of rigid body dynamics including the Lagrange top are completely integrable Hamiltonian systems, and as such have been studied in detail in Bolsinov & Fomenko [7] and Cushman & Bates [13]. For all the references we inevitably missed in this introduction we refer to the extensive bibliography in [10].

In modern mechanics the (energy)-momentum map plays a central role. Singularity theory’s swallowtail was found as the set of critical values of the energy-momentum map of the Lagrange top in [15], also see [13]. The meaning of the swallowtail from the point of view of bifurcation theory, specifically the supercritical Hamiltonian Hopf bifurcation in the Lagrange top, was described...
The fact that the swallowtail may make the set of regular values in the image of the energy-momentum map non-simply connected is the essential observation that explains why the Lagrange top does not possess global action variables [14, 18]. Hamiltonian monodromy of the Lagrange top is described in [51]. Integrable discretisations of the integrable Lagrange top were found in [5]. The complex algebraic geometry of the Lagrange top was described in [25], and its bi-Hamiltonian structure in [50]. In KAM theory perturbations of the Lagrange top give a beautiful example worked out in detail in [29].

The quantisation of the symmetric top was first done in the early days of quantum mechanics [40] and leads to a hypergeometric equation, also see [32]. The study of polar molecules in an electric field leads to a Hamiltonian that is equivalent to the Lagrange top. In the physics literature this is referred to as the Stark effect, and was first studied in [45]. Matrix elements for the numerical computation of the spectrum were given in [47], and nearly 30 years later again in [27]. The discovery of quantum monodromy [18] was made in the smaller brother of the Lagrange top, the spherical pendulum, in [14]. The quantum monodromy in the Lagrange top itself has been studied in [31].

While so-called semi-toric systems with two degrees of freedom (somewhat like the spherical pendulum) are now in a precise sense completely understood classically [38] and quantum mechanically [24], the Lagrange top is still out of reach from this point of view. We should mention that many generalisations of the spherical pendulum have been studied, in particular, the magnetic spherical pendulum [12, 13], also see [44], and the quadratic spherical pendulum [21, 53]. The combination of both is the harmonic Lagrange top, which is the object of this paper. To our knowledge, it has not been considered in the literature in full generality. The so-called Kirchhoff top which has only quadratic terms in the potential has been studied in [3, 6]. A general potential with linear and quadratic terms was considered in [28] from the point of view of perturbation theory of the Euler top. The harmonic Lagrange top can also be considered as an example of the general idea described in [20], where a semi-toric system is deformed, preserving integrability. In particular, we find that the harmonic Lagrange top exhibits the subcritical and the supercritical Hamiltonian Hopf bifurcations.

As mentioned in the beginning, we want to draw attention to the fact that the quantisation of the Lagrange top leads to the confluent Heun equation. The Heun equation is a Fuchsian equation with 4 regular singular points, thus generalising the hypergeometric equation by one singularity, see, e.g., [2, 17, 41, 48]. An important physical application of the confluent Heun equation appears in the perturbation theory of a rotating black hole in general relativity [34, 39, 49]. In this context, expansions in terms of Jacobi polynomials have been given in [23], and series expansions for small potential are given in [46]. As we show below, the harmonic Lagrange top leads to the most general confluent Heun equation, unlike the above application in general relativity, which does not have enough parameters.

After this work was completed we learned that a physical interpretation for the additional quadratic (“harmonic”) term in the potential is provided by considering the Lagrange top on a vibrating suspension [35, 36]. In this context the focus-focus points in the model have been analysed in [11], also see [8]. Some of our results about the threads of focus-focus points in the bifurcation diagram overlap with [11], also see [42].

The structure of this paper is as follows. We give an introduction to the Lagrange top in the next section, where we emphasise the description in the spatial frame using quaternions and the corresponding Poisson structure. The various periodic flows and their differences when considering $T^*SO(3)$ or $T^*S^3$ (the quaternions) are discussed in Section 3, and the reductions to two degrees of freedom in Section 4. The traditional description in Euler angles is recalled in Section 5, which is needed for the quantisation. The main classical results are the description of the critical points in phase space and the corresponding critical values in the image of the energy-momentum map. There are 4 different cases, with one thread (the original Lagrange top), with two threads, with a triangular tube instead of the thread, and a triangular tube shrinking to a thread. In the final section we show that the quantum harmonic Lagrange top leads to the most general confluent Heun equation and compute the spectrum, which is displayed overlayed with (slices) of the classical energy-momentum map. A new method for the computation of the spectrum is presented.
2. HEAVY SYMMETRIC TOP

Consider a general rigid body with a fixed point. Assume that the symmetric inertia tensor $I$ with respect to that point has three distinct eigenvalues $I_1$, $I_2$, $I_3$, the moments of inertia, and assume that a body frame has been chosen in which the tensor of inertia is diagonal. For the symmetric top with $I_1 = I_2$ the location of the corresponding basis vectors is only defined up to a rotation about the symmetry axis (or figure axis) of the body. In the spatial coordinate frame, the $z$-axis is parallel to the direction of gravity. Let $V$ be the coordinate vector of a point in the body frame. The orthogonal matrix $R \in SO(3)$ describes how this point is moving in time when viewed in the spatial frame, $v = RV$.

For the free rigid body (Euler top), the fixed point of the body is the centre of gravity of the body. For the Lagrange top, the centre of gravity is on the figure axis but does not coincide with the fixed point of the body, which also lies on the said axis. Denote the unit vector along the figure axis by $a$ (in the spatial frame), then the potential energy in the field of gravity is $V = c_1 a_z$. In this paper, we are going to study the more general case

$$V(a_z) = c_1 a_z + c_2 a_z^2.$$  

The angular velocity $\Omega$ in the body frame is defined through $R$ by $R^t \dot{R}V = \Omega \times V$ for any vector $V$, or, equivalently, by $\Omega = R^t \dot{R}$. The kinetic energy of the rigid body is

$$T = \frac{1}{2} \Omega \cdot I \Omega,$$

where $I$ is the diagonal tensor of inertia and $\cdot$ denotes the Euclidean scalar product.

The angular momentum vector is defined by $L = I \Omega$. For the free rigid body $l = RL$ is a constant vector. For the Lagrange top instead there are only two conserved quantities given by

$$l_z = l \cdot e_z, \quad L_3 = L \cdot e_3 = R^t l \cdot e_3 = l \cdot Re_3 = l \cdot a.$$

In the spatial frame we have $e_z = (0, 0, 1)^t$ and in the body frame we have $e_3 = (0, 0, 1)^t$.

A beautiful global description of the dynamics of rigid bodies uses quaternions $x = (x_0, x_1, x_2, x_3)$ which are coordinates on the double cover of $SO(3)$ which is $S^3 \in \mathbb{R}^4$ given by $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$. Define

$$x_{\pm} = \begin{pmatrix} x_1 & -x_0 & \mp x_3 & \pm x_2 \\ x_2 & \pm x_3 & -x_0 & \pm x_1 \\ x_3 & \mp x_2 & \pm x_1 & -x_0 \end{pmatrix},$$

which satisfy $x_+ x_+^t = id, x_- x_-^t = id, x_+^t x_+ x_- = x_+, \text{ and } x_+^t x_- x_+ = x_-$. Then an orthogonal $3 \times 3$ matrix is given by $R = x_+ x_-^t$ and the last two identities in the previous sentence become $x_+^t R = x_+$ and $x_-^t R = x_-$. The matrices $x_{\pm}$ relate the angular velocities to the tangent vector of the sphere $\dot{x}$ by $\Omega = 2x_- \dot{x}$ and $\omega = 2x_+ \dot{x}$, see, e.g., [52, Section 16]. To see this, differentiate $R$ with respect to time, observe that $\dot{x}_+ x_- = x_+ \dot{x}_-$, and use $R^t x_+ = x_-$. Substituting $\Omega = 2x_- \dot{x}$ into the expression for $T$ gives

$$T = 2\dot{x}^t (x_-^t I x_-) \dot{x}.$$  

Differentiating with respect to $\dot{x}$ gives the conjugate momenta $p = 4(x_-^t I x_-) \dot{x}$ on $T^* S^3$. Using $L = I \Omega = 2I x_- \dot{x}$, we see that

$$L = 2I x_- \dot{x} = 2(x_- x_-^t) I x_- \dot{x} = \frac{1}{2} x_- p.$$  

Similarly, we have $l = \frac{1}{2} x_+ p$. It is valid to use the canonical bracket between $x$ and $p$ because the resulting Hamiltonian automatically preserves $|x| = 1$ and $x \cdot p = 0$. 

---

REGULAR AND CHAOTIC DYNAMICS Vol. 27 No. 4 2022
Now changing from canonical variables \((x, p)\) to non-canonical variables \((x, L)\) gives the Lie–Poisson structure in the body frame as [9, 10]

\[
B_- = \begin{pmatrix} 0 & \frac{1}{2}x_1^t \\ -\frac{1}{2}x_2 & -L \end{pmatrix}, \quad \dot{x} = \frac{1}{2}x_1^t \nabla_L H, \quad \dot{L} = -\frac{1}{2}x_2 \nabla_2 H + L \times \nabla_L H.
\]

Similarly, the Lie–Poisson structure in the space fixed frame is

\[
B_+ = \begin{pmatrix} 0 & \frac{1}{2}x_2^t \\ -\frac{1}{2}x_1 & -I \end{pmatrix}, \quad \dot{x} = \frac{1}{2}x_2^t \nabla_I H, \quad \dot{I} = -\frac{1}{2}x_1 \nabla_2 H - I \times \nabla_I H.
\]

Both Poisson structures have the Casimir \(x_0^2 + x_1^2 + x_2^2 + x_3^2\). The Poisson structure \(B_+\) is found by sandwiching the symplectic structure of the \((x, \mathbf{p})\) variables by the Jacobian of the transformation of \((x, I)\) and its transpose.

For the Euler top the usual Hamiltonian in the body frame is \(H = \frac{1}{2}l \cdot I^{-1} l\), and the complicated integrals are \(Rl\) (which imply the simple integral \(|l|^2\)). In the space fixed frame instead we have the complicated Hamiltonian \(H = \frac{1}{2}l \cdot RI^{-1}R^t l\) with the simple integrals \(I\). We mention the Euler top here to make the point that, for general moments of inertia, the description in the body frame is simpler. However, for a round rigid body with \(I_1 = I_2 = I_3\) both Hamiltonians are equally simple. Also, for a symmetric rigid body with say \(I_1 = I_2\), the spatial frame is useful because

\[
2T = l \cdot RI^{-1}R^t l = l \cdot \frac{1}{I_1} R(id + \delta e_3 e_3^t)R^t l = \frac{1}{I_1} (l^2 + \delta L_3^2),
\]

where \(\delta = I_1/I_3 - 1\). The important point is that \(L_3 = e_3 \cdot L = e_3 \cdot Rl = Re_3 \cdot l = l \cdot a\) is the angular momentum about the body’s symmetry axis \(e_3\) and hence a constant of motion for the symmetric top.

**Theorem 1.** The Lagrange top (symmetric heavy rigid body with a fixed point on the symmetry axis) in coordinates \(x \in S^3 \subset \mathbb{R}^4\) and angular momenta \(l\) in the space fixed frame has Hamiltonian

\[
H = \frac{1}{2I_1}(l_x^2 + l_y^2 + l_z^2 + \delta L_3^2) + V(x_0^2 + x_3^2 - x_1^2 - x_2^2)
\]

and Poisson structure \(B_+\), with integrals \(l_z\) and

\[
L_3 = 2l_z(-x_0 x_2 + x_1 x_3) + 2l_y(x_0 x_1 + x_2 x_3) + l_z(x_0^2 + x_3^2 - x_1^2 - x_2^2).
\]

The vector fields of \(l_z\) and \(L_3\) generate a \(T^2\) action with isotropies. The vector field of the Hamiltonian is

\[
X_H = \frac{1}{2I_1} X_{l^2} + \frac{\delta L_3}{I_1} X_{L_3} - \frac{1}{2}(0, 0, 0, 0, x_+ \nabla x V)^t. \quad (2.1)
\]

The functions \(H, l_z, L_3\) have pairwise vanishing Poisson bracket. The vector fields \(X_H, X_{L_3}\) and \(X_{l_z}\) are independent almost everywhere.

This theorem is well known for the case of a linear potential, and when using Euler angles it is part of most mechanics textbooks. Instead we offer a global description in the spatial frame with a Poisson structure. In addition, in order to make the connection with the general confluent Heun equation, we consider not just a linear potential (gravity), but in addition a quadratic term. After some preparations in the next sections discussing the torus action, the reduction, and briefly recalling Euler angles, the main technical part is the description of the set of critical values of the energy-momentum map in Theorem 2.
The vector field generated by $L_3$ in the space fixed coordinate system is
\[ X_{L_3} = B_+ \nabla L_3 = \frac{1}{2} (x^t R e_3, 0, 0, 0)^t = \left( \frac{1}{2} x^t e_3, 0, 0, 0 \right)^t \tag{3.1} \]
where we used the identity $x^t R = x^t$. This vector field can be easily integrated (two harmonic oscillators) to give the flow $\Phi^\varphi_{L_3}$. This flow rotates $(x_0, x_3)$ and $(x_1, x_2)$ by $\varphi/2$ clockwise. However, when the flow acts on $R$ it acts by multiplication by a counterclockwise rotation about the $z$-axis through $\varphi$ (not $\varphi/2$!) from the right. Thus, $L_3$ has $2\pi$-periodic flow on $T^*SO(3)$ and hence is an action variable.

The vector field generated by the integral $l_z$ is
\[ X_{l_z} = B_+ \nabla l_z = (\frac{1}{2} x^t e_z, -l \times e_z)^t. \tag{3.2} \]
Again, this vector field is easily integrated (three harmonic oscillators), giving the flow $\Phi^\varphi_{l_z}$. The action on $R$ is by multiplication with a counterclockwise rotation about the $z$-axis through $\varphi$ from the left. In addition, the momentum vector $l$ is rotated by the same rotation matrix. Thus, $l_z$ has $2\pi$-periodic flow on $T^*SO(3)$ and hence is an action variable.

The vector fields $X_{L_3}$ and $X_{l_z}$ are parallel when $l_x = l_y = 0$ and either $x_0 = x_3 = 0$ or $x_1 = x_2 = 0$. These critical points have $l \| a \| e_z$ and are called sleeping tops. In the first case $x_z = -1$ (hanging sleeping top), while in the second case $x_z = +1$ (upright sleeping top). The torus action is not free at these points because the rotations coincide. Since $L_3 = l \cdot a$ we see that $L_3 = -l_z$ for the hanging sleeping top and $L_3 = l_z$ for the upright sleeping top. The corresponding critical points of $H$ are two parabolas above $l_z \pm L_3 = 0$.

The vector fields $X_{l_x}$ and $X_{L_3}$ both have $2\pi$ periodic flows on $T^*SO(3)$, i.e., they map $x$ to $-x$ after time $2\pi$. When considered as flows on $S^3$, both flows have period $4\pi$. Now consider the vector fields generated by $l \pm L_3$. These are both $2\pi$ periodic vector fields on $T^*S^3$. Points with $l_x = l_y = 0$ and either $x_0 = x_3 = 0$ or $x_1 = x_2 = 0$, respectively, are fixed points of these flows. Nevertheless, they are action variables on $T^*S^3$. Notice that as flows on $T^*SO(3)$ the orbits of $l_z \pm L_3$ do not all have the same minimal period, since points with $l_z = l_y = 0 = 0$ and either $x_0 = x_3 = 0 = 0$ or $x_1 = x_2$ have minimal period $\pi$, while all other non-fixed points have minimal period $2\pi$. The $T^2$ action on $T^*S^3$ is, of course, still not free, the difference is that now the exceptional sets of points are found as those where one of the vector fields vanishes.

The vector field of the spherical Euler top is that of $l^2 = l_x^2 + l_y^2 + l_z^2$. The vector fields of $l_x$ and $l_y$ are permutations to that of $l_z$ given in (3.2). Combining these gives
\[ X_{l_z} = (x^t l, 0, 0, 0)^t. \]
Here the components of $l$ are all constant, and the flow of this vector field is a rotation about the axis $l$. This is also a periodic flow, but the period is not constant. To obtain constant period, we consider the flow generated by $l = \sqrt{l^2}$, which we denote by $X_l$. This flow commutes with the flows of $l_z$ and $L_3$, but not with that of $H$. The flow of $l^2$ leaves $l$ constant and so
\[
\Phi^\varphi_l = \exp \left( \frac{\alpha}{2l} \begin{pmatrix}
0 & l_x & l_y & l_z \\
-l_x & 0 & -l_z & l_y \\
-l_y & l_z & 0 & -l_x \\
-l_z & -l_y & l_x & 0
\end{pmatrix} \right) = \begin{pmatrix}
\cos \frac{\alpha}{2} & l_x/\sin \frac{\alpha}{2} & l_y/\sin \frac{\alpha}{2} & l_z/\sin \frac{\alpha}{2} \\
-l_x/\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} & -l_z/\sin \frac{\alpha}{2} & l_y/\sin \frac{\alpha}{2} \\
-l_y/\sin \frac{\alpha}{2} & l_z/\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} & -l_x/\sin \frac{\alpha}{2} \\
-l_z/\sin \frac{\alpha}{2} & -l_y/\sin \frac{\alpha}{2} & l_x/\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2}
\end{pmatrix}.
\]
Acting with this flow on the rotation matrix $R$ with initial condition $x = (1, 0, 0, 0)$ gives Rodrigues’ parametrisation of $SO(3)$ with rotation axis $l/l$ and rotation angle $\alpha$. Thus, Rodrigues’ formula gives the geodesics of the spherical top. When acting with this flow on $S^3$, it is periodic with period $4\pi$.

The reason we are including this flow is that there is an interesting difference between $SO(3)$ and $S^3$. On $T^*SO(3)$ the singular $T^3$ torus action generated by the commuting flows of $l_z$, $L_3$,...
and \(l\) is faithful. This means that outside the singularity where \(l_z + L_3 = 0\) the action on each \(T^3\) obtained by fixing the values of the generators is faithful. By contrast, when considering the \(T^3\) torus action generated by \(l_z + L_3\), \(l_z - L_3\), and \(l + l\) on \(T^*S^3\), the action is not faithful on regular tori. The reason is that, when flowing each flow only for angle \(\pi\), the first two flows together achieve \(x \rightarrow -x\), and this is cancelled by the flow \(\Phi^\pi_{2l} = \Phi^\pi_l\).

4. REDUCTIONS

The flows of \(l_z\) and \(L_3\) are global \(S^1\) actions, and hence allow for regular reduction. It is straightforward to obtain the reduced system from the global system with Poisson structure \(B_\pm\). The \(l_z\)-reduced system gives the well known Euler–Poisson equations, while the \(L_3\)-reduced equations are somewhat less well known in classical mechanics (see, e.g., [5, 13, 19]). The full reduction is singular because the \(T^2\) action of \(l_z\) and \(L_3\) is not free. The standard description of reduction uses \(zxz\)-Euler angles, the singular reduction using invariants is in [13]. A peculiar property of Euler angles is that the \(\psi\)-rotation leaves the figure axis invariant (it acts on the right), while the \(\phi\)-rotation leaves the direction of gravity invariant (it acts on the left), and hence Euler angles are neither space-fixed nor body-fixed. The quantisation of the top (see below) starts out with Euler angles [32], but in the end, writing the Hamiltonian using \(I^2\) and \(L_3^2\) shows that for the quantum mechanical description the spatial frame is also useful.

The reduction by the symmetry \(\Phi^\psi_{L_3}\) introduces the coordinates of the axis of the top \(a = R e_3\) as new coordinates. This is, in fact, reduction by invariants, since the third column of \(R\) is given by \((2(x_0 x_2 + x_1 x_3), -2x_0 x_1 + 2 x_2 x_3, x_0^2 + x_2^2 - x_1^2 - x_3^2)\) and these are all invariant under the two-oscillator flow \(\Phi^\psi_{L_3}\). We already noted that \(\Phi^\psi_{L_3}\) acts on \(R\) by multiplication by \(R_z(\psi)\) from the right, where \(\hat{R_e}(\psi)\) denotes a counterclockwise rotation about the \(z\)-axis by \(\psi\). Hence, \(RR_z(\psi)e_3 = Re_3 = a\) is invariant. The resulting reduced system has Poisson structure

\[
B^r_+ = \begin{pmatrix} 0 & -\hat{a} \\ -\hat{a} & -\hat{l} \end{pmatrix}, \quad \hat{a} = -a \times \nabla_l H, \quad \hat{l} = -a \times \nabla_a H - l \times \nabla_l H.
\]

Denote the Jacobian of the transformation from \((x, l)\) to \((a, l)\) by \(A\). Then \(B^r_+ = A^tB_+A\) when expressed in the new variables. The main identity in the reduction from \(B_+\) to \(B^r_+\) is \(\frac{1}{2} \frac{\partial a}{\partial x} x^t_+ = \hat{a}\). The Poisson structure \(B^r_+\) has Casimirs \(a^2 = 1\) and \(a \cdot l\) and the reduced Hamiltonian is

\[
H = \frac{1}{2I_1}(l^2 + \delta(a \cdot l)^2) + V(a_z).
\]

Since \(a \cdot l\) is a Casimir (equal in value to the generator of the symmetry \(L_3\)), it does not contribute to the dynamics but merely changes the value of the Hamiltonian.

Note that reduction by the symmetry generated by the integral \(l_z\) is more complicated in the spatial frame since the flow is a rotation in \(x\) and in \(l_x, l_y\). However, when switching to the body frame, the flow of \(l_z\) (written in terms of \(L\)) is simpler. Reduction is achieved by introducing the invariant of the left action generated by \(l_z\), which is \(e^t_3R_z(\phi)R = e^t_3R = \Gamma t\) with Poisson structure

\[
B^r_+ = \begin{pmatrix} 0 & \hat{\Gamma} \\ \hat{\Gamma} & \hat{L} \end{pmatrix}, \quad \hat{\Gamma} = \Gamma \times \nabla L H, \quad \hat{L} = \Gamma \times \nabla_l H + L \times \nabla_l H.
\]

The reduction leads to the more familiar Hamiltonian of the Lagrange top given by

\[
H = \frac{1}{2I_1}(L_1^2 + L_2^2) + \frac{1}{2I_3}L_3^2 + V(\Gamma_3),
\]

where \(\Gamma\) is \(e_3\) viewed from the body frame. The Poisson structure is \(B^r_+\) with the opposite sign than \(B^r_+\). These are the equations usually called Euler–Poisson equations. Their advantage is that this reduction remains valid for an arbitrary rigid body with a fixed point, and this family for
appropriate moments of inertia and position of the centre of mass contains the Kovalevskaya top, the Euler top, and all other (non-integrable) tops.

The Hamiltonian Hopf bifurcation in the sleeping top with $a|l|e_2$ respectively $\Gamma||L||e_3$ is best described in the reduced system(s), because the corresponding periodic orbit becomes a relative equilibrium after reduction. It is easy to check that indeed these are equilibria, and linearising the Hamiltonian vector field about these equilibria yields a $6 \times 6$ matrix with 2 eigenvalues zero corresponding to the two Casimirs. The characteristic polynomial for the remaining non-trivial eigenvalues is

$$P_+(\lambda) = \lambda^4 + \lambda^2(\kappa^2 - 2f) + f^2 = 0, \quad \kappa = l_z/I_1 = \omega I_3/I_1, \quad f = a_z V'(a_z)/I_1, \quad a_z = \pm 1,$$

in the spatial frame and

$$P_-(\lambda) = P_+(\lambda) + \omega(\omega - \kappa)(2\lambda^2 + 2f + \omega(\omega - \kappa)), \quad \omega = l_z/I_3$$

in the body frame. The eigenvalues given by the roots of $P_+$ in the spatial frame are not the same as the eigenvalues given by the roots of $P_-$ in the body frame because in the latter case the system is described in a frame rotating with angular velocity $\omega$. However, they differ only by $\pm i\omega$. More precisely, let $\lambda_1, \lambda_1, \lambda_2, \lambda_2$ be the roots of $P_+$, then the roots of $P_-$ are $\lambda_1 + i\omega, \lambda_1 - i\omega, \lambda_2 + i\omega, \lambda_2 - i\omega$ such that the Floquet multipliers $\mu = \exp(\lambda T)$ of the periodic orbit with period $T = 2\pi/\omega$ are the same. The description in the spatial frame gives simpler formulas.

At the Hamiltonian Hopf bifurcation the eigenvalues change from all purely imaginary via a collision on the imaginary axis to a quadruple of complex eigenvalues. This occurs when the discriminant of $P_+(\lambda)$ considered as a quadratic equation in $\lambda^2$ changes from positive to negative. The discriminant is given by $\kappa^2(\kappa^2 - 4f)$. When $f$ is negative the eigenvalues are purely imaginary for any $\kappa$. When $f$ is positive the eigenvalues are purely imaginary when $\kappa^2 > 4f$, while the top is unstable with non-zero real parts of the eigenvalues when $\kappa^2 < 4f$. This is the classical stability condition for the Lagrange top, here obtained for arbitrary potential. In the critical case $\kappa^2 = 4f$ the eigenvalues collide and $\lambda^2 = -\kappa^2/4$.

5. EULER ANGLES

The Poisson structures $B_{\pm}$ allow for a global description of rigid body dynamics free of coordinate singularities. However, often explicit canonical coordinates are more convenient, and even essential for the quantisation of the problem. Such a coordinate system adapted to the symmetries is given by $zxz$-Euler angles such that

$$R = R_z(\phi)R_x(\theta)R_z(\psi).$$

The canonically conjugate momenta are denoted by $p_\phi, p_\theta, p_\psi$, respectively. Then we have that $l_z = p_\phi$ and $L_3 = p_\psi$. The Hamiltonian in these coordinates is

$$H = \frac{1}{2I_1}(2T_{\text{round}} + \delta p_\psi^2) + V(\cos \theta),$$

where $T_{\text{round}}$ is the kinetic energy of the spherical top with moment of inertia 1:

$$T_{\text{round}} = \frac{1}{2} \left( p_\theta^2 + \frac{1}{\sin^2 \theta} \left( p_\phi^2 + p_\psi^2 - 2p_\phi p_\psi \cos \theta \right) \right) = \frac{1}{2} l^2.$$

Notice that this round metric on $SO(3)$ is a metric of constant curvature and hence up to a covering equivalent to the metric of the round sphere $S^3$.

Away from the coordinate singularity where the torus action is not free, Euler angles are a smooth local coordinate system. Equilibrium points in $\theta$ are determined by $\partial H/\partial \theta = 0$. For later use, we denote this function by $H_{\theta}$, and similarly the 2nd derivative by $H_{\theta \theta}$. 

REGULAR AND CHAOTIC DYNAMICS Vol. 27 No. 4 2022
6. BIFURCATION DIAGRAM

The energy-momentum map from $T^*SO(3)$ to $\mathbb{R}^3$ is given by $(l_z, L_3, H)$ where $L_3$ is given in terms of $x$ and $l$ as in Theorem 1. The bifurcation diagram of this integrable system is the set of critical values of the energy-momentum map. Hence, we are interested in the rank of $(X_{l_z}, X_{L_3}, X_H)$. To determine where the rank drops we consider
\[ \alpha X_{L_3} + \beta X_{l_z} + \gamma X_H = 0. \] (6.1)

Theorem 2. The rank 1 points of the energy-momentum map are given by two parabolas of sleeping tops
\[ (l_z, L_3, H) = \left( m, \pm m, \frac{m^2}{2l_1}(1 + \delta) + V(\pm 1) \right). \] (6.2)
The rank 2 points have a rational parametrisation determined by \( l(\beta, a_z) = \frac{1}{2} I_1 V'(a_z) a + \beta e_z \) such that for \( a_z \in [-1, 1] \) and \( \beta \in \mathbb{R} \) the critical values of the energy-momentum map are

\[
\begin{align*}
    l_z(\beta, a_z) &= \frac{1}{2} I_1 V'(a_z) a + \beta, \\
    L_3(\beta, a_z) &= \frac{1}{2} I_1 V'(a_z) + \beta a_z,
\end{align*}
\]

\[
H(\beta, a_z) = \frac{1}{2 I_1}(l(\beta, a_z)^2 + \delta L_3(\beta, a_z)^2) + V(a_z).
\]

Proof. Notice that the last 3 components of \( X_V \) can be written as

\[
-\frac{1}{2} x_x + \nabla_z V(a_z(x)) = -a \times \nabla_a V(a_z) = -a \times e_z V'(a_z).
\]

Using \( x_+ e_3 = x_+ R e_3 = x_+ a \) in the flow of \( L_3(6.1) \) becomes

\[
\left( \frac{1}{2} x_+ ((a + \gamma \delta L_3)a + \beta e_z + \gamma l) \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

where \( \mu \) is an extra arbitrary parameter. This means that critical points of the momentum map occur when

\[
\begin{align*}
    (a + \gamma \delta L_3)a + \beta e_z + \gamma l &= 0 \\
    \gamma L_3' a + \mu e_z + \beta l &= 0
\end{align*}
\]

for \( \alpha, \beta, \gamma \) not all zero. Hence, the three vectors \( a, l, e_z \) are co-planar. Since \( a \) and \( e_z \) never vanish, there is no rank 0 point. We have the following four cases.

1) \( a \parallel e_z \). This means \( a_x = a_y = 0 \) and \( a_z = \pm 1 \). If \( l \neq 0 \) and \( l \) is not parallel to \( e_z \), then linear independence (6.3) implies \( \alpha = \beta = \gamma = 0 \). Hence, \( l \parallel e_z \parallel a \) (including \( l = 0 \)), and we can use \( l_z = m \) as a parameter and thus show the parametrisation of the sleeping tops (6.2). Recall that these are the points where the torus action is not free. All points along these parabolas have rank 1. Parts of these parabolas may be isolated threads of focus-focus type, while others form the edges of the surface of elliptic-elliptic type. The vertices of the parabolas where \( m = 0 \) and hence \( l = 0 \) are equilibrium points of \( X_H \) with \( a_z = \pm 1 \).

2) \( l \parallel e_z \). Set \( l = \lambda e_z \). If \( a \parallel e_z \), then this gives the sleeping top solution again. If \( a \) is not parallel to \( e_z \), then by linear independence, (6.3) implies \( \alpha + \gamma \delta L_3 = \beta + \gamma \lambda = 0 \) and \( I_1 V' = \mu + \beta \lambda = 0 \). If \( \gamma = 0 \), then this forces the trivial solution \( \alpha = \beta = 0 \) so \( \gamma \neq 0 \) and \( V' = 0 \). Since \( V' = c_1 + 2c_2 a_z \), this means \( a_z = \frac{-c_2}{c_1} =: a_{z0} \). Normalising \( \gamma = -1 \) gives \( \lambda = \beta \) and hence \( l = \beta e_z, a = (a_x, a_y, a_z0), L_3 = l \cdot a = \beta a_{z0} \) and hence using \( \beta = m \) as a parameter gives

\[
(l_z, L_3, H) = \left( m, m a_{z0}, \frac{m^2}{2 I_1} (1 + \delta a_{z0}^2) + V(a_{z0}) \right).
\]

Since \( |a_z| \leq 1 \), this parabola only exists when \( 2|c_2| > |c_1| \), while for \( 2|c_2| = \pm |c_1| \) it merges with the sleeping tops. The vertex of this parabola where \( m = 0 \) and hence \( l = 0 \) is an equilibrium point of \( X_H \) with \( |a_z| \leq 1 \). This vertex lies above or below the vertices of the parabolas of sleeping tops (6.2) described in case 1, depending on whether \( c_2 < -c_1/2 \) or \( c_2 > c_1/2 \).

3) \( l \parallel a \). This forces \( l = \lambda a = L_3 a \). If \( a \parallel e_z \), then this gives the sleeping top solution again. If \( a \) is not parallel to \( e_z \), then linear independence and (6.3) imply \( \alpha + (\delta + 1)L_3 = \beta = 0 \) and \( \gamma I_1 V' + \beta L_3 = \mu = 0 \). Again \( \gamma = 0 \) gives the trivial solution, so we can normalise \( \gamma = -1 \) and find \( \alpha = (1 + \delta)L_3 \) and \( V' = 0 \), as in case 2. Using \( L_3 = k \) as a parameter gives

\[
(l_z, L_3, H) = \left( k a_{z0}, k, \frac{k^2}{2 I_1} (\delta + 1) + V(a_{z0}) \right).
\]

Existence and limiting behaviour is as in case 2. The vertex of this parabola coincides with that of case 2.
4) General case where no pair of vectors is parallel. If \( \gamma = 0 \), then this gives \( \alpha = \beta = 0 \), while if \( \beta = 0 \), then this gives case 2. We now assume \( \beta \neq 0 \) and \( \gamma \neq 0 \). Eliminating \( l \) from (6.3) and using linear independence gives \( \mu = \frac{\beta^2}{\gamma} \) and \( \alpha + \gamma \delta L_3 = \frac{\alpha^2}{\gamma} I_1 V' \). Using this to eliminate \( L_3 \) in (6.3) gives \(-l = \frac{\gamma}{\beta} I_1 V' a + \beta e_z \). Normalising \( \gamma = -1 \), computing \( l_z = l \cdot e_3, L_3 = l \cdot a, \) and \( l^2 = l \cdot l \) gives the result. Notice that \( \beta \) is the angular velocity of the angle \( \phi \) conjugate to \( p_\phi \). □

![Fig. 2. Slices through the bifurcation diagram along \( l_z - L_3 = 0 \) (blue) and \( l_z + L_3 = 0 \) (red) for Figs. 1a, 1c, 1d using \( \hbar = (0.075, 0.15, 0.11) \) for the quantum spectrum, respectively.](image)

Note that in cases 2 and 3 the parabolas (6.4) and (6.5) are embedded in the surface of critical values described in case 4. Unlike the parabolas (6.2), the rank of these points is 2.

Since \( V' \) is linear in \( a_z \), we can eliminate \( a_z \) in favour of \( \tilde{\alpha} = I_1 V'(a_z)/\beta \). Notice that \( \alpha \) is the angular velocity of the angle \( \psi \), and \( \tilde{\alpha} \) is that angular velocity with \( \delta = 0 \). As a result, we obtain a polynomial parametrisation of the critical values of the energy-momentum map which after non-dimensionalisation is given by

\[
(\beta + a_z \tilde{\alpha}(1 - \tilde{\alpha} \beta), \tilde{\alpha} + a_z \beta(1 - \tilde{\alpha} \beta), \frac{1}{2}(\tilde{\alpha}^2 + \beta^2 + a_z(1 - \tilde{\alpha} \beta)(3\tilde{\alpha} \beta + 1) + \delta(\tilde{\alpha} + a_z \beta(1 - \tilde{\alpha} \beta)^2))
\]

with the constraint \(-1 \leq a_z(1 - \tilde{\alpha} \beta) \leq 1\) on the parameters \( \tilde{\alpha} \) and \( \beta \). When \( \delta = 0 \), any line determined by fixing \( \alpha \) and changing \( \beta \) or vice versa is a planar parabola. This means the surface is doubly foliated by (arcs of) planar parabolas. The two special parabolas (6.4) and (6.5) correspond to vanishing angular momentum \( \tilde{\alpha} = 0 \) and \( \beta = 0 \), respectively. Hence, for points on (6.4) the top does not rotate about its figure axis, while for points on (6.5) the figure axis does not rotate in space.
Both are extreme cases of resonant 2-tori where one frequency vanishes. Note that such solutions are impossible in the ordinary Lagrange top with $c_2 = 0$. The parabolas of rank 1 points (6.2) are not part of this foliation, instead they mark the endpoints of the parabolic arcs where $a_{z0}(1 - \bar{a}\beta) = \pm 1$.

The rational parametrisation from Theorem 2 is also useful when using the Euler angles. Inserting the parametrisation into the condition for an equilibrium point $H_0 = 0$ shows that it is identically satisfied. To determine the stability of the equilibrium, we evaluate the second derivative $H_{\theta\theta}$ on the rational parametrisation and find

$$H_{\theta\theta}(\beta, a_z) = \beta^2 I_1 - 2a_z V'(a_z) + (1 - a_z^2) V''(a_z) + \frac{1}{\beta^2 I_1} V'(a_z)^2.$$  \hspace{1cm} (6.6)

The transverse stability of a 2-torus is determined by the sign of $H_{\theta\theta}$ since it gives the curvature of the effective potential. Computing $H_{\theta\theta}$ on the parabolas of sleeping top (6.2) gives $m^2/(4I_1) \mp V'(\pm 1)$, reproducing the classical condition for the Hamiltonian Hopf bifurcation in Lagrange’s sleeping top found at the end of Section 4. Evaluating $H_{\theta\theta}$ as given in (6.6) on the parabola (6.4) gives $m^2/I_1 + 2c_2(1 - a_{z0}^2)$, and on the parabola (6.5) similarly gives $k^2/I_1 + 2c_2(1 - a_{z0}^2)$. When $c_2 < -c_1/2$, these are both negative for small $m$ or $k$, respectively, and hence unstable. These correspond to points on top of the triangular tube, which are hyperbolic. For sufficiently large angular momentum the sign flips, and they are points in the outer envelope surface of critical values. When $c_2 > c_1/2$ the 2nd derivative is always positive, hence in this case rank 2 points correspond to elliptic 2-tori.

Equating $H_{\theta\theta}$ to zero gives a relation between $\beta$ and $a_z$ which determines degenerate values in the bifurcation diagram. These are the cusp-shaped edges of the triangular tubes in Figs. 1a, 1b. The most degenerate situation occurs when simultaneously the 2nd and the 3rd $\theta$-derivative of $H$ vanish. This occurs for the special parameter values $a_z = -c_1/(2c_2)\), \beta^2 = -c_1^2/(8c_2 I_1)$ and $a_z = -c_1/(4c_2), \beta^2 = c_1^2/(2c_2 I_1) - 2c_2/I_1$. When these degenerate values for $a_z$ collide with $\pm 1$, the degenerate points disappear and the topological structure of the bifurcation diagram changes. This occurs for $c_1 = \pm 2c_2$ and $c_1 = -4c_2$. The plus sign yields imaginary $\beta$. The sign of $c_1$ can be made positive by the original choice of the body coordinate system. This can flip the sign of $c_1 a_z$ in the potential but leaves $c_2 a_z^2$ unchanged. Hence, there are 4 topologically distinct cases illustrated in Fig. 1:

A) $c_2/c_1 < -1/2$: triangular tube Fig. 1a;
B) $-1/2 < c_2/c_1 < -1/4$: triangular tube shrinking to a thread Fig. 1b;
C) $-1/4 < c_2/c_1 < 1/2$: one thread Fig. 1c;
D) $c_2/c_1 > 1/2$: two threads Fig. 1d.

To understand the figures corresponding to these 4 cases, it helps to consider how they bifurcate into each other. We stress again that we always consider $\delta = 0$, because adding the additional quadratic term in $L_3$ to the Hamiltonian deforms the bifurcation diagram, but does not essentially change it. Bifurcations similar to those found here have recently been described in [22, 43], in particular, also the related quantum monodromy in [43].

Let us start with the ordinary Lagrange top, $c_2 = 0$, $c_1 = 1$ by choice of a coordinate system and normalisation [15]. The bifurcation diagram for the harmonic Lagrange top is topologically the same for $-1/4 < c_2/c_1 < 1/2$. It is natural that a small enough quadratic term does not change the nature of the bifurcation diagram since the potential $V(a_z)$ is only changed a little since $|a_z| \leq 1$. The outer surface is a bowl that has at least two corners when cut at constant energy. For high energy there are four corners, while for low energy there are only two. The transitions are two supercritical Hamiltonian Hopf bifurcations where the sleeping top becomes stable. A thread of critical values detaches at these points of the surface. This thread is shown in blue in Fig. 1c. In Fig. 2 slices through the 3-dimensional bifurcation diagram are shown. Each blue curve is a slice with $l_z - L_3 = 0$ which contains the thread, while in the other slice $l_z + L_3 = 0$ the thread appears as a single isolated point. In these figures we also show the quantum spectrum, see the next section. This situation persists for $|c_2|$ not too large.
For $c_2/c_1 > 1/2$, a second thread emerges from the minimum of $H$, as shown in Fig. 1d and Fig. 2d. For low energies, the outer surface has no corners at all. For intermediate energy as visible at the top of Fig. 1d, there are 2 corners above where the red thread is attached, but the blue thread is not yet attached and the outer surface nearby is still smooth. For high energies, there are 4 corners.

A more dramatic change occurs when decreasing $c_2/c_1$ through $-1/4$. All attachment points of the threads in the two cases discussed so far are supercritical Hamiltonian Hopf bifurcations. When passing $-1/4$, the supercritical Hamiltonian Hopf bifurcation turns into a subcritical Hamiltonian Hopf bifurcation. The attachment point is replaced by a tube with triangular cross section that eventually contracts to a point and becomes a thread, as shown in Fig. 1b (zoomed in).

When decreasing $c_2/c_1$ further, the two subcritical Hamiltonian Hopf bifurcation values collide when $c_2/c_1 = -1/2$, and merge into a triangular tube shown in Fig. 1a and there is no bifurcation any more in the rank 1 points given by (6.2) with $L_3 = +m$. Instead this parabola marks the corner at the bottom of the triangular tube and for higher energies the corner in the outer surface. In this figure, the bounding box is chosen such that it cuts away parts of the surface facing the viewer so that the triangular tube becomes visible. The 0-slices are shown in Fig. 2a. The two bottom surfaces of the tube correspond to elliptic 2-tori, while the top surface of the tube corresponds to hyperbolic 2-tori. The top surface joins the bottom surfaces along a line of cusps where the triangular tube becomes visible. The 0-slices are shown in Fig. 2a. It is obtained by slicing the tube orthogonal to its long direction in the middle. When moving this slice away from the middle, the triangle moves up, but the bottom curve below it moves up faster, and eventually the corner of the triangle will pierce through that curve.

When viewing the critical values from below, the triangular tube pierces through the surface. The first bifurcation in the slice occurs when the top of the triangle (hyperbolic 2-tori) becomes tangent to the curve. This creates a pair of saddle-centre bifurcations of 2-tori and the corresponding critical values in the energy-momentum map are degenerate. In the rightmost slice the two cusps collide and annihilate and the slice becomes smooth. The reason that the two different slices in Fig. 3 appear somewhat similar is that, when $c_1 \to 0$, they actually become identical, see Fig. 4 below. In the left pane a perspective view looking down along the $H$-axis from above is shown, while in the right pane we are looking up along the $H$-axis from below the surface. This concludes the description of the four generic cases of the bifurcation diagram.

There are 3 degenerate cases separating A,B,C,D from each other. In addition, there are two non-generic cases that occur in the limit as $c_1 \to 0$. There are two different limiting cases depending on the sign of $c_2$. For positive $c_2$ we recover the case studied in [3], for which the two threads (6.2) intersect at their vertices. The case of negative $c_2$ is fundamentally different and was not considered in [3]. Again the two parabolas (6.2) intersect at their vertices, but they are now embedded in the surface of critical values and mark its edges, see Fig. 4. In addition, the triangular tube becomes symmetric in this limit, forming a kind of trampoline. The edge of the trampoline where $H_{\theta\theta} = 0$ is shown in orange. The cuspidal points of the trampoline touch the outer surface where the self-intersection of the surface stops, this is where the parabolas (6.4) and (6.5) (green and purple in Fig. 4) intersect the orange cusps. Viewed from below, this point is where the self-intersection of the surface stops and the parabolas become visible as embedded in the smooth outer surface (Fig. 4 right). Slices of the set of critical values for constant energy in this case have $D_4$ symmetry.

7. QUANTUM MECHANICS OF THE HARMONIC LAGRANGE TOP

The quantisation of the rigid body is textbook material, see, e.g., [32, §103]. The global action variables $l_z$ and $L_3$ become operators $\hat{l}_z = -i\partial/\partial\phi$ and $\hat{L}_3 = -i\partial/\partial\psi$, measured in units of $\hbar$. We denote the corresponding integer eigenvalues by $m$ and $k$ such that $\hat{l}_z \Psi = m \Psi$ and $\hat{L}_3 \Psi = k \Psi$ for a wave function $\Psi$.

The quantum mechanical harmonic Lagrange top has the Hamiltonian operator

$$\hat{H} = \frac{1}{2I_1} \left( \hat{l}^2 + \delta \hat{L}_3^2 \right) + c_1 \cos \theta + c_2 \cos^2 \theta,$$

(7.1)
Fig. 3. Slices of constant $l_z - L_3$ and $l_z + L_3$ near the most degenerate values. Top: slices with constant $l_z - L_3 = (0.200, 0.488, 0.755, 0.888, 1.15)$. Bottom: slices with constant $l_z + L_3 = (1.00, 1.36, 2.00, 2.16, 2.33)$. Parameters are the same as those in Fig. 1.
Fig. 4. Top and bottom view (shown in the left and right pane, respectively) of the limiting case with $c_1 = 0$ and negative $c_2 = -1$. Four parabolas corresponding to (6.2) (red and blue), (6.4) (green), (6.5) (purple) are shown, in addition to the cuspidal edge of the “trampoline” (orange).

where $\hat{l}$ is the total angular momentum operator. Explicitly the first part of the Hamiltonian operator is found as the Laplace–Beltrami operator of the metric $T_{\text{round}}$ of the spherical top, hence

$$\hat{l}^2 = -\frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} (m^2 + k^2 - 2mk \cos\theta),$$

where we have already replaced the operators $\hat{l}_z$ and $\hat{L}_3$ by their respective eigenvalues.

The equation $\hat{l}^2 f = j(j+1) f$ is a self-adjoint form of the hypergeometric equation. Setting the eigenvalue of $\hat{l}^2$ to $j(j+1)$ for positive integer $j$, solutions are given by $\sin |m+k| \theta \cos |m-k| \theta P_{j-\max(|k|,|m|)}^{m,k} (\cos\theta)$ where $P_{n,m}^{m_1,m_2}$ are the Jacobi polynomials. Up to normalisation and phase factors these are the Wigner-$D$ functions [4]. The equation has regular singular points at $\theta = 0, \pi$ with indices $\pm (m - k)$ and $\pm (m + k)$, respectively. Note that the global quantum numbers $m$ and $k$ appear as indices of regular singular points.

Adding the potential terms, and transforming to $z = \cos\theta$ brings us to the following observation.

Theorem 3. The quantisation of the harmonic Lagrange top leads to the most general confluent Heun equation (aka generalised spheroidal wave equation) which has the self-adjoint form

$$-\partial_z ((1 - z^2) \partial_z + \frac{k^2 + m^2 - 2kmz}{1 - z^2}) + \tilde{c}_1 z + \tilde{c}_2 z^2 - \lambda = 0,$$

where $z = \cos\theta$ and $\lambda$ is the spectral parameter related to the energy eigenvalue $E$ of the Hamiltonian by $\lambda = 2I_1 E/\hbar^2 - \delta k^2$, $\tilde{c}_1 = c_1 2I_1 / \hbar^2$, $\tilde{c}_2 = c_2 2I_1 / \hbar^2$.

In the form (7.2) the indices at $z = \pm 1$ are $\pm (m - k)/2$ and $\pm (m + k)/2$. This equation has an irregular singular point at infinity, which is obtained by the confluence of two regular singular points of the Heun equation. The Heun equation is the most general Fuchsian equation with 4 regular singular points. The Heun equation (after normalisation by Möbius transformations) has 6 parameters, 1 position of a pole, 4 indices, and the so-called accessory parameter. The pole position is used for the confluence, after which only two regular singular points remain. Hence, 2 indices remain as parameters (given by $\pm (m \pm k)/2$). Two additional parameters describe the behaviour near the irregular singular point, and the accessory parameter remains, so that there is a total of 5 parameters.
To transform into the standard form of the confluent Heun equation, see, e.g., [17], first shift to the standard poles by \( z \to (z + 1)/2 \), and then scale the dependent variable with \( \exp(2\sqrt{c_2})z^{m+k}|(z - 1)^{m-k}| \).

When considering the confluent Heun equation, the usual reference to its application in physics is to Teukolsky’s master equation [49], which appears in the perturbation theory around a rotating black hole, i.e., the Kerr metric. However, that equation only has 4 parameters, and one index-parameter is more restricted because it represents the spin of a particle. In this context, eigenvalues \( \lambda \) of the equation have been computed using expansion in Jacobi polynomials in [23]. Their results are not applicable to our case because their equation only has 4 parameters. To compute the spectrum in our case we generalise the papers [47], [27] which treat the case of a symmetric molecule (i.e., top) in an electric field, hence the Lagrange top (without the harmonic field). To extend their method, which is also an expansion in Jacobi polynomials (or rather the related Wigner-D-functions), we need to compute the matrix elements of \( \cos^2 \theta \). This leads to our final result.

**Theorem 4.** The spectrum of the harmonic Lagrange top (7.1) which is equivalent to the most general confluent Heun equation (7.2) can be computed from a penta-diagonal symmetric matrix

\[
\hat{H} = \hat{H}_0 + c_1 \hat{H}_1 + c_2 \hat{H}_1^2.
\]

For given fixed \( m, k \) the operator \( \hat{H}_0 \) is the diagonal representation of the Hamiltonian without potential and \( \hat{H}_1 \) is the tri-diagonal representation of \( \cos \theta \) in terms of Wigner-D basis functions.

**Proof.** The formulas for \( \hat{H}_0 \) and \( \hat{H}_1 \) are given in [47]. We repeat them here for convenience. The diagonal entries of \( \hat{H}_0 \) are \( \frac{k^2}{2j_1}(j(j + 1) + \delta k^2) \). The diagonal entries of \( \hat{H}_1 \) are \( a_j = -km/(j(j + 1)) \) and the off-diagonal entries are \( b_j = -\sqrt{(j^2 - k^2)(j^2 - m^2)/(j^2(4j^2 - 1))} \). The first entries in the matrix representing the operators have \( j = \max(m, k) \). Note that for \( m = k = 0 \) the diverging terms in \( b_j \) cancel and \( b_0 \) is defined. It is easy to compute the matrix elements of \( \cos^2 \theta \). This can be done by noticing that \( D_{2,0,0} = \frac{3}{2}\cos^2 \theta - \frac{1}{2} \). The matrix representation of \( D_{2,0,0} \) can be expressed in terms of Clebsch–Gordan coefficients. However, it is more efficient to use the fact that since \( \hat{H}_1 \) represents \( \cos \theta \) the matrix \( \hat{H}_1^2 \) represents \( \cos^2 \theta \). So instead of computing matrix elements of \( \cos^2 \theta \) from scratch in terms of Clebsch–Gordan coefficients, we can simply compute the square of the matrix representation of \( \hat{H}_1 \). In particular, the entries in the 2nd off-diagonal are given by products \( b_{j-1}b_j \).

The numerical convergence of these expressions is good, and the spectra displayed in Fig. 2 were computed from these matrices truncated at twice the maximal needed quantum number \( j \). Even though the term \( \delta L^2 \) in the Hamiltonian is important for the classical dynamics, its effect on the quantum spectrum is rather trivial, it simply adds \( \delta k^2 \). It does change the spectrum, but the change is simple, and for this reason in the figures we restricted attention to \( \delta = 0 \), the spherical top. Moreover, from the point of view of the computation of the spectrum of the general confluent Heun equation the term \( \delta k^2 \) is irrelevant.

Why is there a correspondence between the harmonic Lagrange top and the confluent Heun equation? This question may not have a definite answer, but it is suggestive that the harmonic potential is the most general potential for which the classical dynamics can be linearised using the Jacobian of an elliptic curve. This fact appears to be related to the fact that the corresponding quantum system is described by the confluent Heun equation. After adding higher-order terms to the potential, the system remains integrable and separable in the same way, but the classical dynamics will involve hyperelliptic curves, and the quantum system will be described by higher-order confluent Fuchsian equations. It would be interesting to make this observation more precise.

**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.
REFERENCES

1. Arnol'd, V.I., *Mathematical Methods of Classical Mechanics*, 2nd ed., Grad. Texts in Math., vol.60, New York: Springer, 1997.
2. Arscott, F.M., *Periodic Differential Equations: An Introduction to Mathieu, Lamé, and Allied Functions*, Internat. Ser. Monogr. Pure Appl.Math., vol.66, New York: Pergamon, 1964.
3. Bates, L. and Zou, M., Degeneration of Hamiltonian Monodromy Cycles, *Nonlinearity*, 1993, vol.6, no.2, pp.313–335.
4. Biedenharn, L.C. and Louck, J.D., *Angular Momentum in Quantum Physics: Theory and Application*, Encyclopedia Math. Appl., vol.8, Reading, Mass.: Addison-Wesley, 1981.
5. Bobenko, A.I. and Suris, Yu. B., Discrete Time Lagrangian Mechanics on Lie Groups, with an Application to the Lagrange Top, *Comm. Math. Phys.*, 1999, vol.204, no.1, pp.147–188.
6. Bogoyavlenskii, O.I., Euler Equations on Finite-Dimensional Lie Coalgebras, Arising in Problems of Mathematical Physics, *Russian Math. Surveys*, 1992, vol.47, no.1, pp.117–189; see also: *Uspekhi Mat. Nauk*, 1992, vol.47, no.1(283), pp.107–146.
7. Bolsinov, A. V. and Fomenko, A. T., *Integrable Hamiltonian Systems: Geometry, Topology, Classification*, Boca Raton, Fla.: Chapman & Hall/CRC, 2004.
8. Borisov, A. V. and Ivanov, A. P., A Top on a Vibrating Base: New Integrable Problem of Nonholonomic Mechanics, *Regul. Chaotic Dyn.*, 2004, vol.9, no.3, pp.255–264.
9. Dullin, H. R., Poisson Integral for Symmetric Rigid Bodies, *Regul. Chaotic Dyn.*, 2004, vol.9, no.3, pp.255–264.
10. Dullin, H. R. and Pelayo, Á., Generating Hyperbolic Singularities in Semitoric Systems via Hopf Bifurcations, *J. Nonlinear Sci.*, 2016, vol.26, no.3, pp.787–811.
11. Efstathiou, K., *Metamorphoses of Hamiltonian Systems with Symmetries*, Lect. Notes in Math., vol.1864, Berlin: Springer, 2005.
12. Le Floch, Y. and Vũ Ngọc, S., The Inverse Spectral Problem for Quantum Semitoric Systems, arXiv:2104.06704 (2021).
13. Gavrilov, L. and Zhivkov, A., The Complex Geometry of the Lagrange Top, *Enseign. Math.* 2, 1998, vol.44, nos.1–2, pp.133–170.
14. Goldstein, H., *Classical Mechanics*, 2nd ed., Reading, Mass.: Addison-Wesley, 1980.
15. Hajnal, J. V. and Opat, G. I., Stark Effect for a Rigid Symmetric Top Molecule: Exact Solution, *J. Phys. B*, 1991, vol.24, no.12, pp.2799–2805.
16. Hanßmann, H., Quasi-Periodic Motions of a Rigid Body: 1. Quadratic Hamiltonians on the Sphere with a Distinguished Parameter, *Regul. Chaotic Dyn.*, 1997, vol.2, no.2, pp.41–57.
29. Broer, H.W., Hanßmann, H., Hoo, J., and Naudot, V., Nearly-Integrable Perturbations of the Lagrange Top: Applications of KAM-Theory, in Dynamics & Stochastics: Festschrift in Honour of M.S. Keane, D.D. Denteneer, F. Hollander, E. Verbitskiy (Eds.), IMS Lecture Notes Monogr. Ser., vol. 48, Beachwood, OH: Inst. Math. Statist., 2006, pp. 286–303.
30. Klein, F. and Sommerfeld, A., Über die Theorie des Kreisels, Leipzig: Teubner, 1910.
31. Kozin, I.N. and Roberts, R.M., Monodromy in the Spectrum of a Rigid Symmetric Top Molecule in an Electric Field, J. Chem. Phys., 2003, vol. 118, no. 23, pp. 10523–10533.
32. Landau, L.D. and Lifshitz, E.M., Course of Theoretical Physics: In 10 Vols.: Vol. 3. Quantum Mechanics (Nonrelativistic Theory), 3rd ed., Oxford: Butterworth-Heinemann, 2003.
33. Landau, L.D. and Lifshitz, E.M., Course of Theoretical Physics: Vol. 1. Mechanics, 3rd ed., Oxford: Pergamon, 1976.
34. Leaver, E.W., Solutions to a Generalized Spheroidal Wave Equation: Teukolsky’s Equations in General Relativity, and the Two-Center Problem in Molecular Quantum Mechanics, J. Math. Phys., 1986, vol. 27, no. 5, pp. 1238–1265.
35. Markeev, A.P., On the Theory of Motion of a Rigid Body with a Vibrating Suspension, Dokl. Phys., 2009, vol. 54, no. 8, pp. 392–396; see also: Dokl. Akad. Nauk, 2009, vol. 427, no. 6, pp. 771–775.
36. Markeev, A.P., On the Motion of a Heavy Dynamically Symmetric Rigid Body with Vibrating Suspension Point, Mech. Solids, 2012, vol. 47, no. 4, pp. 373–379; see also: Izv. Akad. Nauk: Mekh. Tverd. Tela, 2012, no. 4, pp. 3–10.
37. Marsden, J.E. and Ratiu, T.S., Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems, Texts Appl. Math., vol. 17, New York: Springer, 1994.
38. Pelayo, A. and Vu Ngoc, S., Semitoric Integrable Systems on Symplectic 4-Manifolds, Invent. Math., 2009, vol. 177, no. 3, pp. 571–597.
39. Press, W.H. and Teukolsky, S.A., Perturbations of a Rotating Black Hole: 2. Dynamical Stability of the Kerr Metric, Astrophys. J., 1973, vol. 185, pp. 649–673.
40. Reiche, F., Die Quantelung des symmetrischen Kreisels nach Schrödingers Undulationsmechanik, Z. Physik, 1926, vol. 39, nos. 5–6, pp. 444–464.
41. Henr’s Differential Equations, F.M. Arscott, S. Yu. Slavyanov, D. Schmidt, G. Wolf, P. Maroni, A. Duval (Eds.), Oxford: Oxford Univ. Press, 1995.
42. Ryabov, P.E. and Sokolov, S.V., Bifurcation Diagram of One Model of a Lagrange Top with a Vibrating Suspension Point, in Presentation at the 4th Internat. Conf. “Topological Methods in Dynamics and Related Topics” (Nizhny Novgorod, Russia, Aug 2021).
43. Sadovskii, D.A. and Zhilinskii, B.I., Hamiltonian Systems with Detuned 1 : 1 : 2 Resonance: Manifestation of Bidromy, Ann. Physics, 2007, vol. 322, no. 1, pp. 164–200.
44. Saksida, P., Neumann System, Spherical Pendulum and Magnetic Fields, J. Phys. A, 2002, vol. 35, no. 25, pp. 5237–5253.
45. Schlier, Ch., Der Stark-Effekt des symmetrischen Kreiselmoleküls bei hohen Feldstärken, Z. Physik, 1955, vol. 141, nos. 1–2, pp. 16–18.
46. Seidel, E., A Comment on the Eigenvalues of Spin-Weighted Spheroidal Functions, Class. Quantum Gravity, 1989, vol. 6, no. 7, pp. 1057–1062.
47. Shirley, J.H., Stark Energy Levels of Symmetric-Top Molecules, J. Chem. Phys., 1963, vol. 38, no. 12, pp. 2896–2913.
48. Slavyanov, S. and Lay, W., Special Functions: A Unified Theory Based on Singularities, Oxford: Oxford Univ. Press, 2000.
49. Teukolsky, S.A., Perturbations of a Rotating Black Hole: 1. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations, Astrophys. J., 1973, vol. 185, pp. 635–648.
50. Tsiganov, A.V., On Bi-Hamiltonian Geometry of the Lagrange Top, J. Phys. A, 2008, vol. 41, no. 31, 315212, 12 pp.
51. Vivolo, O., The Monodromy of the Lagrange Top and the Picard–Lefschetz Formula, J. Geom. Phys., 2003, vol. 46, no. 2, pp. 99–124.
52. Whittaker, E.T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 4th ed., New York: Cambridge Univ. Press, 1989.
53. Zou, M., Kolmogorov’s Condition for the Square Potential Spherical Pendulum, Phys. Lett. A, 1992, vol. 166, nos. 5–6, pp. 321–329.