HAUSDORFF OPERATORS ON FOCK SPACES

GEORGIOS STYLOGIANNIS AND PETROS GALANOPoulos

ABSTRACT. Let \( \mu \) be a positive Borel measure on the positive real axis. We call Hausdorff operator \( H_\mu \) the integral operator

\[
H_\mu(f)(z) = \int_0^\infty \frac{1}{t} f\left(\frac{z}{t}\right) d\mu(t), \quad z \in \mathbb{C},
\]

where \( f \) is an analytic function in \( \mathbb{C} \). This is a more general form of the operator considered in [HuKyQu17]. Our objective is the study of the action of \( H_\mu \) on the Fock spaces \( F_p^\alpha \), where \( p \in [1, \infty) \) and \( \alpha > 0 \). It turns out that the boundedness is characterized through the behavior of the sequence

\[
\int_0^\infty \frac{1}{t^{2k+1}} d\mu(t), \quad k \in \mathbb{N}.
\]

We prove that for every \( f \in F_p^\alpha \), \( p \in [1, \infty] \),

\[
\|H_\mu\|_{F_p^\alpha \to F_p^\alpha} = \sup \left\{ \int_0^\infty \frac{1}{t^{2k+1}} d\mu(t) : k \in \mathbb{N} \right\}.
\]

As it is expected, we show that a little-\( o \) condition describes the compactness of the operator for every \( p \in (1, \infty) \). In addition, we study the Schatten class \( S_p(F_2^\alpha) \) membership of \( H_\mu \), for \( p \in (0, \infty) \). We prove that this is determined by the \( p \)-summability of the sequence

\[
\int_0^\infty \frac{1}{t^k} d\mu(t), \quad k \in \mathbb{N}^*.
\]

1. INTRODUCTION

Let \( \mu \) be a positive Borel measure on \((0, \infty)\). We formally consider the operator

\[
H_\mu(f)(z) := \int_0^\infty \frac{1}{t} f\left(\frac{z}{t}\right) d\mu(t), \quad z \in \mathbb{C},
\]

where \( f \) is a holomorphic function on the complex plane. We will refer to it as the Hausdorff operator induced by the measure \( \mu \).

In [HuKyQu17] the term Hausdorff operator is used for an operator of the type

\[
H_\phi(f)(z) := \int_0^\infty f\left(\frac{z}{t}\right) \frac{\phi(t)}{t} dt,
\]

where \( \phi = \phi(t) \) is a locally integrable function. There the authors consider its action on the Hardy spaces of the upper half plane and the question under discussion is to find the exact condition on \( \phi(t) \) that forces the operator to be bounded. As a consequence of the comparison between the two formulas above we are allowed to say

---

Date: July 23, 2020.

1991 Mathematics Subject Classification. Primary 47B38; Secondary 30H20, 46E15, 47B05, 47B10.

Key words and phrases. Hausdorff Operator, Fock Spaces, Entire Functions.
that the integral operator (1.1) is a more general version of (1.2). We also mention that this operator $H_\mu$ appears for the first time in \cite{St20} where it is studied on the holomorphic Bergman spaces of the upper half plane.

Hausdorff operators have a long standing history. Not only have they been considered in the frames of a complex variable setting but they naturally appear in the theory of spaces of functions defined on the real line. To prove our case we mention the connection of the real variable version of the Hausdorff operator to the Hilbert transform as it is indicated in \cite{HuKyQu17} and the references therein. On the other hand, one interesting aspect is the fact that, for specific choices of symbols $\phi$ in (1.2), we can get well known operators such as the Hardy and the Cesàro operator, among many others. This connection to classical operators of Mathematical Analysis is made clear through a series of papers. See \cite{LiMo00}, \cite{ChFaLi12}, \cite{ChFaZh12}, \cite{FaLi14}, \cite{GiMo95}, \cite{Ka01}. For more information on the subject we propose to the interested reader the review articles \cite{Li013} and \cite{ChFaWa14} and the references therein.

The aim of this work is to extend the study of the Hausdorff operator $H_\mu$, as defined in (1.1), in the case of the Fock spaces. In the following we briefly present all the basic properties of these spaces, needed to our study. Let $1 \leq p < \infty$ and $\alpha > 0$.

We say that an entire function $f$ belongs to the Fock space $F^p_\alpha$ if

$$||f||_{p,\alpha} = \left(\int_\mathbb{C} |f(z)| e^{-\frac{\alpha}{2}|z|^2} \, dm(z)\right)^{\frac{1}{p}} < \infty,$$

where $dm(z)$ stands for the Lebesgue area measure. It is true that, for each $z \in \mathbb{C}$,

$$|f(z)| \leq ||f||_{p,\alpha} e^{\frac{\alpha}{2}|z|^2}, \quad \forall f \in F^p_\alpha.$$

In other words, on each $F^p_\alpha$ the point evaluation functionals are bounded. Based on these growth estimates, it is natural to define the space $F^\infty_\alpha$ that is, the space of all the entire functions $f$ such that

$$||f||_{\infty,\alpha} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

It is not difficult to verify that the Fock spaces form a chain of Banach spaces with the following containment property:

$$F^p_\alpha \subset F^q_\alpha \subset F^\infty_\alpha, \quad 1 \leq p < q < \infty.$$

All the inclusions above are strict. If $p = 2$, $F^2_\alpha$ is a Hilbert space. In this special case the norm of a function $f(z) = \sum_{n \geq 0} a_n z^n \in F^2_\alpha$ can be expressed in terms of its Taylor coefficients as

$$||f||_{F^2_\alpha}^2 = \sum_{n \geq 0} |a_n|^2 \frac{n!}{\alpha^n}.$$

The function

$$K_\alpha(w, z) = e^{\alpha w \overline{z}}, \quad z, w \in \mathbb{C}$$

is the reproducing kernel of $F^2_\alpha$ with norm evaluation

$$||K_\alpha(\cdot, z)||_{F^2_\alpha}^2 = \frac{\alpha}{\pi} \int_\mathbb{C} |e^{\alpha w \overline{z}}|^2 e^{-\alpha |w|^2} \, dm(z) = e^{\alpha |z|^2}.$$
The term reproducing is justified by the property according to which every \( f \in F_\alpha \) can be represented as
\[
(1.8) \quad f(z) = \langle f, \Phi_\alpha(z) \rangle_{F_\alpha^2} = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(w) e^{\alpha z \overline{w}} e^{-\alpha |w|^2} \, dm(w), \quad z \in \mathbb{C}.
\]

Observe that (1.8) remains true for every \( f \in F_\alpha^p, \ p \in [1, 2) \), due to the inclusion property (1.4). A standard reference for the theory of \( F_\alpha^p \) spaces is \( \text{Zh12} \).

Having in mind all the above, at first we will establish the well definition of the integral (1.1), equivalently of the Hausdorff operator, when acting on the Fock spaces. To do so, consider a \( z \in \mathbb{C} \) and an \( f \in F_\infty^\alpha \), which is the largest space as it is stated in (1.4). Taking into account the pointwise growth (1.3) we get that
\[
\int_0^\infty \frac{1}{t} \left| f \left( \frac{z}{t} \right) \right| \, d\mu(t) \leq \int_0^\infty \frac{1}{t} e^{\alpha |z|^2/2t} \, d\mu(t) \| f \|_{F_\infty^\alpha}
\]
\[
= \int_0^\infty \frac{1}{t} \left( \sum_{k \geq 0} \frac{1}{k!} \frac{\alpha^k |z|^{2k}}{2^k t^{2k}} \right) \, d\mu(t) \| f \|_{F_\infty^\alpha}
\]
\[
= \left[ \sum_{k \geq 0} \frac{1}{k!} \left( \int_0^\infty \frac{1}{t^{2k+1}} \, d\mu(t) \right) \frac{\alpha^k |z|^{2k}}{2^k} \right] \| f \|_{F_\infty^\alpha}
\]
\[
\leq \left( \sup_{k \in \mathbb{N}} \int_0^\infty \frac{1}{t^{2k+1}} \, d\mu(t) \right) \frac{\alpha^k |z|^{2k}}{2^k} \| f \|_{F_\infty^\alpha}.
\]
So, we conclude that under the condition
\[
(1.9) \quad \sup_{k \in \mathbb{N}} \int_0^\infty \frac{1}{t^{2k+1}} \, d\mu(t) < \infty
\]
the Hausdorff operator \( H_\mu \) is well defined on any Fock space \( F_\alpha^p \), \( p \in [1, \infty] \). In other words, it makes sense to consider positive Borel measures \( \mu \) on \((0, \infty)\) that fulfill property (1.9). Although the latter condition is equivalent to
\[
\sup_{m \in \mathbb{N}} \int_0^\infty \frac{1}{t^m} \, d\mu(t) < \infty.
\]
we refer to (1.9) due to the fact that it appears naturally in the proof.

In this last part of the introduction we present the main results of the paper. In section 2, we deal with the boundedness of the Hausdorff operator. To be more precise, we are able to prove the following:

**Theorem 1.1.** Let \( 1 \leq p \leq \infty \), \( \alpha > 0 \) and \( \mu \) be a positive Borel measure on \((0, \infty)\) such that (1.9) is true. Then the Hausdorff operator \( H_\mu \) is bounded on \( F_\alpha^p \). Moreover
\[
\| H_\mu \|_{F_\alpha^p \to F_\alpha^p} = \sup \left\{ \int_0^\infty \frac{1}{t^{2k+1}} \, d\mu(t) : k \in \mathbb{N} \right\}, \quad 1 \leq p \leq \infty.
\]
In section 3 we consider the compactness. By definition, \( \mathcal{H}_\mu \) is compact in \( F_\alpha^p \), \( p \in (1, \infty) \), if the image of the unit ball is compact in the norm topology of the space. To derive the characterization of compactness, firstly, we handle the case of \( F_\alpha^2 \) space. Then the one-side compactness theorem, Theorem 3.2 in [CoKuSc92], plays the role of a bridge in order to pass our results from \( F_\alpha^2 \) to \( F_\alpha^p \) for \( p \in (1, \infty) \). The outcome of our approach is the following:

**Theorem 1.2.** Let \( p \in (1, \infty) \) and \( \alpha > 0 \). Assume that \( \mu \) is a positive Borel measure on \((0, \infty)\) such that (1.9) is true. Then the Hausdorff operator \( \mathcal{H}_\mu \) is compact on \( F_\alpha^p \) if and only if

\[
\lim_{k \to \infty} \int_0^\infty \frac{1}{t^k} d\mu(t) = 0.
\]

In the last section, we study the Schatten classes of \( \mathcal{H}_\mu \). Before proceeding to the definition we need to remember that any compact operator on a Hilbert space can be decomposed in a specific way. To be precise, let \( T \) be a compact operator on a separable Hilbert space \( H \), then there exist orthonormal sets \( \{e_n\}, \{\sigma_n\} \) in \( H \) such that the operator can be represented as

\[
T(x) = \sum_n \lambda_n \langle x, e_n \rangle_H \sigma_n, \quad x \in H
\]

for an appropriate sequence of numbers \( \{\lambda_n\} \), which is known as the sequence of the singular values of \( T \). The number \( \lambda_n \) is called the \( n \)-th singular value of \( T \).

Now, let \( p \in (0, \infty) \). By \( S_p(H) \) we denote the Schatten \( p \)-class of operators on \( H \). This consists of those compact operators \( T \) on \( H \) whose sequence of singular numbers \( \lambda_n \) belongs to the sequence space \( l^p \). It is also well known that

\[
\lambda_n = \lambda_n(T) = \inf \{ \|T - K\| : \text{rank } K \leq n \}.
\]

Thus finite rank operators belong to every \( S_p(H) \). In some sense, the membership of an operator in \( S_p(H) \) measures the size of the operator. In the case \( 1 \leq p < \infty \), \( S_p(H) \) is a Banach space with the norm

\[
\|T\|^p = \sum_n |\lambda_n|^p,
\]

while for \( 0 < p < 1 \) is a complete metric space. For more information on the subject, we refer to [Zh90] and [GoKr69]. Concluding, the membership of \( \mathcal{H}_\mu \) in \( S_p = S_p(F_\alpha^2) \) is described by the following:

**Theorem 1.3.** Let \( \mu \) be a positive Borel measure on \((0, \infty)\) such that (1.9) is true. Then the Hausdorff operator \( \mathcal{H}_\mu \) is in the Schatten class \( S_p \), \( 0 < p < \infty \) if and only if

\[
\sum_{k=0}^{\infty} \left( \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) \right)^p < \infty.
\]
2. Boundedness

In this section, we study the boundedness of the Hausdorff operator. In order to establish the proof of Theorem 1.1, it is of great importance to recall first the fact that Fock spaces can serve as interpolating spaces. Using the standard notation on the subject we can state this property as follows. Suppose that $1 \leq p_0 < p_1 \leq \infty$ and $0 \leq \theta \leq 1$. Then

$$[F_{p_0}^\alpha, F_{p_1}^\alpha]_\theta = F_{p}^\alpha,$$

where

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

In terms of operator theory, it can be interpreted as a boundedness property of an operator acting on the spaces under discussion. We state it for the operator of our interest. Assume that

$$\|H_\mu(f)\|_{F_{p_0}^\alpha \to F_{p_0}^\alpha} \leq M_0^{1 - \theta}$$

and that

$$\|H_\mu(f)\|_{F_{p_1}^\alpha \to F_{p_1}^\alpha} \leq M_1^\theta.$$

Then $H_\mu$ is bounded in

$$[F_{p_0}^\alpha, F_{p_1}^\alpha]_\theta = F_{p}^\alpha,$$

with norm

$$\|H_\mu(f)\|_{[F_{p_0}^\alpha, F_{p_1}^\alpha]_\theta} \leq M_0^{1 - \theta} M_1^\theta.$$

For a complete presentation of the complex interpolation theory for the Fock spaces see [Zh12].

Proof. Let us begin with the assumption that the measure $\mu$ satisfies condition (1.9). First we deal with case $p = 1$. For an $f \in F_{\alpha}^1$,

$$\|H_\mu(f)\|_{F_{\alpha}^1} = \frac{a}{2\pi} \int_C |H_\mu(f)(z)| e^{-\frac{\alpha}{2}|z|^2} \, dm(z)$$

$$= \frac{a}{2\pi} \int_C \left| \int_0^\infty \frac{1}{t} f \left( \frac{z}{t} \right) \, d\mu(t) \right| e^{-\frac{\alpha}{2}|z|^2} \, dm(z)$$

$$\leq \frac{a}{2\pi} \int_0^\infty \frac{1}{t} \int_C \left| f \left( \frac{z}{t} \right) \right| e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \, d\mu(t).$$

Since (1.7) holds, the function $f$ belongs to $F_{\alpha}^2$. As a result we can use the representation (1.8). Thus
\[ \| \mathcal{H}_\mu(f) \|_{F^1} \leq \frac{a}{2\pi} \int_0^\infty \int_C \left| f \left( \frac{z}{t} \right) \right| e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \, d\mu(t) \]

\[ = \frac{a}{2\pi} \int_0^\infty \int_C \left| f(w) \right| e^{\frac{\alpha}{2} |w|^2} e^{-\alpha |w|^2} \, dm(w) \, e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \, d\mu(t) \]

\[ \leq \frac{a}{2\pi} \int_0^\infty \int_C \left| f(w) \right| \left| e^{\frac{\alpha}{2} |w|^2} \right| e^{-\alpha |w|^2} \, dm(w) \, e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \, d\mu(t) \]

\[ = \frac{a}{2\pi} \int_0^\infty \int_C \left| f(w) \right| \left| e^{\frac{\alpha}{2} |w|^2} \right|^2 e^{-\frac{\alpha}{2}|z|^2} \, dm(z) \, e^{-\alpha |w|^2} \, dm(w) \, d\mu(t) \, . \]

Bringing in mind (1.7) we get that

\[ \| \mathcal{H}_\mu(f) \|_{F^1} \leq \frac{a}{2\pi} \int \int_C \left| f(w) \right| e^{\frac{\alpha}{2} |w|^2} e^{-\alpha |w|^2} \, dm(w) \, d\mu(t) \]

\[ = \frac{a}{2\pi} \int \int_C \left| f(w) \right| \int_0^\infty \frac{1}{t} e^{\frac{\alpha}{2} |w|^2} \, d\mu(t) \, e^{-\alpha |w|^2} \, dm(w) \]

The inner integral above can be estimated as follows:

\[ \int_{(0, \infty)} \frac{1}{t} e^{\frac{\alpha}{2} |w|^2} \, d\mu(t) = \int_{(0, \infty)} \frac{1}{t} \left( \sum_{k \geq 0} \frac{1}{k!} \frac{\alpha^k |w|^{2k}}{t^{2k}} \right) \, d\mu(t) \]

\[ = \sum_{k \geq 0} \frac{1}{k!} \frac{\alpha^k}{t^{2k+1}} \left( \int_{(0, \infty)} \frac{1}{t^{2k+1}} \, d\mu(t) \right) \, |w|^{2k} \]

\[ \leq \sup_{k \in \mathbb{N}} \left\{ \int_{(0, \infty)} \frac{1}{t^{2k+1}} \, d\mu(t) \right\} \sum_{k \geq 0} \frac{1}{k!} \frac{\alpha^k}{t^{2k+1}} \, |w|^{2k} \]

\[ = \sup_{k \in \mathbb{N}} \left\{ \int_{(0, \infty)} \frac{1}{t^{2k+1}} \, d\mu(t) \right\} \, e^{\frac{\alpha}{2} |w|^2} \, . \]

As a consequence
(2.1) \[ \| \mathcal{H}_\mu(f) \|_{F_1^\alpha} \leq \sup_{k \in \mathbb{N}} \left\{ \int_{(0, \infty)} \frac{1}{t^{2k+1}} \, d\mu(t) \right\} \| f \|_{F_1^\alpha}. \]

On the other hand, the argument for the well definition of the operator provides the following estimation

(2.2) \[ \| \mathcal{H}_\mu(f) \|_{F_\infty^\alpha} \leq \sup_{k \in \mathbb{N}} \left\{ \int_{0}^{\infty} \frac{1}{t^{2k+1}} \, d\mu(t) \right\} \| f \|_{F_\infty^\alpha}. \]

Combining (2.1) and (2.2) with the interpolation property we get that

(2.3) \[ \| \mathcal{H}_\mu \|_{F_p^\alpha \to F_p^\alpha} \leq \sup_{k \in \mathbb{N}} \left\{ \int_{0}^{\infty} \frac{1}{t^{2k+1}} \, d\mu(t) \right\}, \quad 1 \leq p \leq \infty. \]

Actually, the norm of \( \mathcal{H}_\mu \) is exactly the above quantity. This is an easy consequence of following observation: The monomials

\[ f_m(z) = z^m, \quad m \in \mathbb{N} \]

belong to \( F_1^\alpha \) and thus in any \( F_p^\alpha, \ p \in [1, \infty] \), and they have the property that

\[ \mathcal{H}_\mu(f_m)(z) = \int_{(0, \infty)} \frac{1}{t^{m+1}} \, d\mu(t) \, f_m(z), \quad m \in \mathbb{N}. \]

By assumption, (1.9) holds and we know that this condition implies that \( \mathcal{H}_\mu \) is a bounded operator on any \( F_p^\alpha, \ p \in [1, \infty] \). Therefore

\[ \| \mathcal{H}_\mu \|_{F_p^\alpha \to F_p^\alpha} \geq \sup_{m \in \mathbb{N}} \frac{\| \mathcal{H}_\mu(f_m) \|_{F_p^\alpha}}{\| f_m \|_{F_p^\alpha}} \]

\[ = \sup_{m \in \mathbb{N}} \int_{0}^{\infty} \frac{1}{t^{m+1}} \, d\mu(t) \]

\[ \geq \sup_{k \in \mathbb{N}} \int_{0}^{\infty} \frac{1}{t^{2k+1}} \, d\mu(t). \]

\[ \square \]

3. COMPACTNESS

In this section we confront the problem of compactness. This will be a two step procedure. First we consider the Hilbert space case. For the necessity we need the fact that compactness can be equivalently determined by the property

(3.1) \[ \| \mathcal{H}_\mu(x_n) \|_{F_2^\alpha} \to 0, \quad \text{whenever} \ x_n \to 0 \ \text{weakly in} \ F_2^\alpha. \]
On the other hand the sufficient condition comes out of a well known property according to which if \( \{ T_n \} \) is a sequence of compact operators on \( F^2_\alpha \) such that

\[
\| \mathcal{H}_\mu - T_n \| \to 0, \quad n \to \infty
\]

then \( \mathcal{H}_\mu \) is compact on \( F^2_\alpha \). Combining the above arguments the result is:

**Theorem 3.1.** Assume that \( \mu \) is a positive Borel measure on \( (0, \infty) \) such that (1.9) is true. The Hausdorff operator \( \mathcal{H}_\mu \) is compact on \( F^2_\alpha \), \( \alpha > 0 \), if and only if

\[
\lim_{k \to \infty} \int_0^\infty \frac{1}{t^k} d\mu(t) = 0.
\]

**Proof.** Suppose that \( \mathcal{H}_\mu \) is compact on \( F^2_\alpha \). We consider the orthonormal basis

\[
e_k(z) = \sqrt{\frac{\alpha^k}{k!}} z^k, \quad k \in \mathbb{N}.
\]

By a standard argument we can easily check that

\[
e_k \to 0, \quad k \to \infty,
\]

weakly in \( F^2_\alpha \). The assumption that \( \mathcal{H}_\mu \) is compact and the property (3.1) imply that

\[
\| \mathcal{H}_\mu(e_k) \|_{F^2_\alpha} \to 0, \quad k \to \infty.
\]

Since

\[
\mathcal{H}_\mu(e_k)(z) = \int_0^\infty \frac{1}{t^{k+1}} \mu(t) e_k(z),
\]

it is immediate that

\[
\lim_{k \to \infty} \int_0^\infty \frac{1}{t^{k+1}} d\mu(t) = 0.
\]

Conversely suppose that

\[
\lim_{k \to \infty} \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) = 0.
\]

For a given \( \varepsilon > 0 \), there is a \( k_0 \in \mathbb{N} \) such that, for every \( k > k_0 \), we have

\[
\int_0^\infty \frac{1}{t^{1+k}} d\mu(t) < \varepsilon.
\]

Now consider the finite rank operators

\[
\mathcal{H}_{\mu,k}(f)(z) = \sum_{n=0}^k \int_0^\infty \frac{1}{t^{1+n}} d\mu(t) a_n z^n,
\]
where \( f(z) = \sum a_n z^n \in F^2_\alpha \). If we take advantage of the expression of the norm (1.5) we get that for every \( k > k_0 \),

\[
\|H_\mu(f) - H_{\mu,k}(f)\|_{F^2_\alpha}^2 = \sum_{n=k+1}^{\infty} \left( \int_0^\infty \frac{1}{t^{1+n}} d\mu(t) \right)^2 |a_n|^2 \frac{\alpha^n}{n!}
\]

\[
< \varepsilon \sum_{n=k+1}^{\infty} |a_n|^2 \frac{\alpha^n}{n!}
\]

\[
< \varepsilon \|f\|_{F^2_\alpha}^2.
\]

As a consequence of the condition (3.2) we prove that \( H_\mu \) is compact.

\[\square\]

In order to complete the scheme, we have to pass from the case \( p = 2 \) to any \( p \in (1, \infty) \). For the sufficiency we will apply an appropriate argument of interpolation for the compactness as we did for the boundedness of the operator. The study of interpolation properties of compact operators is a classical problem in interpolation theory, with very natural applications to other branches in analysis. The behaviour of compact operators under the complex method is one of the main open problems in this subject. It turns out that if the two couples of spaces are formed by Banach lattices of functions plus minor assumptions, then an one-sided compactness result is true for the complex method. We recall here that a Banach lattice \( X \), is a vector lattice that is at the same time a Banach space with a norm which satisfies the monotonicity condition

\[|x| \leq |y| \Rightarrow \|x\| \leq \|y\|, \quad x, y \in X.\]

This procedure is established in [CoKuSc92]. We comment in brief how it works in our case. The spaces \( F^p_\alpha \), for \( 1 < p < \infty \) and \( \alpha > 0 \), are separable Banach lattices. As a result, see pp. 287 in [CoKuSc92], their norm is absolutely continuous. So the Fock spaces \( F^p_\alpha \) satisfy the assumptions of Theorem 3.2 [CoKuSc92]. Therefore, if \( 1 < p_1 < p_2 < \infty, \alpha > 0 \),

\[H_\mu : F^{p_i}_\alpha \to F^{p_i}_\alpha, \quad i = 1, 2\]

is bounded and

\[H_\mu : F^{p_i}_\alpha \to F^{p_i}_\alpha, \quad i = 1, 2\]

is compact, for at least one index \( i \) then

\[H_\mu : F^p_\alpha \to F^p_\alpha\]

is compact for every \( p \in (p_1, p_2) \). Now we are in position to present the
proof of Theorem 1.2: More or less, necessity it is proved as in the Hilbert space case. Suppose that $\mathcal{H}_\mu$ is compact on $F^p_\alpha$. We consider the functions

\begin{equation}
\tilde{e}_k(z) = \frac{(\alpha p)^{k/2}}{\Gamma\left(\frac{kp}{2} + 1\right)^{1/p}} z^k, \quad k \in \mathbb{N}.
\end{equation}

Then, it is easy to see ([CoMac95] [Corollary 1.3]) that

$$\tilde{e}_k \to 0 \quad \text{weakly in } F^p_\alpha$$

and thus, as we showed in Theorem 3.1,

$$\lim_{k \to \infty} \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) = 0.$$ 

Conversely suppose that

$$\lim_{k \to \infty} \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) = 0.$$ 

For sure the operator is bounded on any $F^p_\alpha$, $p \in [1, \infty]$ and $\alpha > 0$. In addition, by Theorem 3.1, $\mathcal{H}_\mu$ is compact on $F^2_\alpha$. Now, Theorem 3.2 of [CoKuSc92] comes into play and allows us to apply the interpolation argument presented above. This results to the compacteness of $\mathcal{H}_\mu$ on any $F^p_\alpha$, $p \in (1, \infty)$.

\[\square\]

4. Schatten Classes

The objective of this last part is the study of the membership of $\mathcal{H}_\mu$ in the Schatten Classes $S_p(F^p_\alpha)$. In order to do so we begin with the natural assumption that $\mathcal{H}_\mu$ is compact that is

$$\lim_{k \to \infty} \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) = 0.$$ 

Now recall relation (3.4). From one point of view, this property obviously implies that

$$\left\{ \int_0^\infty \frac{1}{t^{1+k}} d\mu(t), \ k = 0, 1, 2, ... \right\} \subset \sigma_p(\mathcal{H}_\mu),$$

where $\sigma_p(\mathcal{H}_\mu)$ is the point spectrum of the Hausdorff operator. On the other hand, it justifies that $\mathcal{H}_\mu$ is a diagonal operator with respect to the orthonormal basis

$$e_n(z) = \sqrt{\frac{\alpha^n}{n!}} z^n, \quad n \in \mathbb{N}.$$ 

The terms of the sequence

$$\left\{ \int_0^\infty \frac{1}{t^{1+k}} d\mu(t), \ k = 0, 1, 2, ... \right\}$$

are the diagonal entries. Being $\mathcal{H}_\mu$ a compact and diagonal operator, in its turn actually implies that the point spectrum is included in the closure of the set of the
diagonal entries. Therefore we are allowed to say that
\[ \sigma_p(\mathcal{H}_\mu) \subset \left\{ \int_0^\infty \frac{1}{t^{1+k}} d\mu(t), \ k = 0, 1, 2, \ldots \right\} \cup \{0\}. \]

As a reference for the spectral theory of diagonal operators we propose [Ha82][pp. 34].

All the above and the fact that a diagonal operator is normal allow us to employ one last argument according to which the membership of a compact and normal operator in the Schatten Class \( \mathcal{S}_p \) is equivalent to the \( p \)-summability of its eigenvalues. For example see [GoKr69][pp. 93] and [Zh90][pp. 22 ex. 5].

Now we are in position to state the main result of this section.

**Theorem 4.1.** Let \( \mu \) be a positive Borel measure on \((0, \infty)\) such that
\[ \lim_{k \to \infty} \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) = 0. \]

The Hausdorff operator \( \mathcal{H}_\mu \) belongs to the Schatten class \( \mathcal{S}_p(F_2^\alpha) \), \( p \in (0, \infty) \), if and only if
\[ \sum_{k=0}^\infty \left( \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) \right)^p < \infty. \]

Closing the section we would like to observe that in the Hilber-Schmidt case the above characterization can be realised through an integral condition. This is stated in the following:

**Corollary 4.2.** The Hausdorff operator \( \mathcal{H}_\mu \) is Hilbert-Schmidt if and only if
\[ \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{1}{st(1-st)} d\mu(t) d\mu(s) < \infty. \]

**Proof.** Firstly notice that
\[ \|\mathcal{H}_\mu(\bar{e}_k)\|_{F_2^\alpha} = \int_0^\infty \frac{1}{t^{1+k}} d\mu(t). \]

Thus
\[
\sum_{k=0}^\infty \|\mathcal{H}_\mu(\bar{e}_k)\|_{F_2^\alpha}^2 = \sum_{k=0}^\infty \left( \int_0^\infty \frac{1}{t^{1+k}} d\mu(t) \right)^2
\]
\[
= \sum_{k=0}^\infty \int_0^\infty \int_0^\infty \frac{1}{(st)^{1+k}} d\mu(t) d\mu(s)
\]
\[
= \sum_{k=0}^\infty \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{1}{(st)^{1+k}} d\mu(t) d\mu(s)
\]
\[
= \int_{\mathbb{R}^+ \times \mathbb{R}^+} \sum_{k=0}^\infty \frac{1}{(st)^{1+k}} d\mu(t) d\mu(s)
\]
\[
= \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{1}{st(1-st)} d\mu(t) d\mu(s).
\]

\[\square\]


References

[BaBoMiMi16] S. Ballamoole, J. O. Bonyo, T. L. Miller and V. G. Miller, Cesáro-like operators on the Hardy and Bergman spaces of the half plane. Complex Anal. Oper. Theory 10 (2016), no. 1, 187-203.

[ChFaLi12] J. Chen, D. Fan and J. Li, Hausdorff operators on function spaces, Chin. Ann. Math. (Ser. B), 33 (2012), 537-556.

[ChFaZh12] J. Chen, D. Fan and C. Zhang, Multilinear Hausdorff operators and their best constants, Acta Math. Sinica (Ser. B), 28 (2012), 1521-1530.

[ChFaWa14] J. Chen, D. Fan and S. Wang, Hausdorff Operators on Euclidean Spaces, Appl. Math. J. Chinese Univ. (Ser. B) (4) 28 (2014), 548-564.

[ChFaZh16] J. Chen, D. Fan and X. Zhu, The Hausdorff operator on the Hardy space $H^1(\mathbb{R}^1)$. Acta Math. Hungar. 150 (2016), no. 1, 142-152.

[CoKuSc92] F. Cobos, T. Kázmér and T. Schonbek, One-sided compactness results for Aronszajn-Gagliardo functors. J. Funct. Anal. 106 (1992), no. 2, 274-313.

[CoMac95] C. C. Cowen and B. D. MacCluer, Composition operators on spaces of analytic functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[FaLi14] D. Fan and X. Lin, Hausdorff operator on real Hardy spaces, Analysis (Berlin), 34 (2014), 319-337.

[GiMo95] D. V. Giang and F. Móricz, The Cesáro operator is bounded on the Hardy space $H^1$. Acta Sci. Math. (Szeged) 61 (1995), no. 1-4, 535-544.

[Ha82] P. R. Halmos, A Hilbert space problem book. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. Springer-Verlag, New York-Berlin, 1982.

[HuKhZh15] Hu, Bingyang; Khoi, Le Hai; Zhu, Kehe Frames and operators in Schatten classes. Houston J. Math. 41 (2015), no. 4, 1191-1219.

[HuKyQu18] H. D. Hung, L. D. Ky and T. T. Quang, Norm of the Hausdorff operator on the real Hardy space $H^1(\mathbb{R})$. Complex Anal. Oper. Theory 12 (2018), no. 1, 235-245.

[HuKyQu17] H. D. Hung, L. D. Ky, and T. T. Quang, Hausdorff operators on holomorphic Hardy spaces and applications. Proceedings of the Royal Society of Edinburgh, 1-18.

[JaPeRo87] S. Janson, J. Peetre and R. Rochberg, Hankel forms and the Fock space. Rev. Mat. Iberoamericana 3 (1987), no. 1, 61-138.

[Ka01] Y. Kanjin, The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$. Studia Math. 148 (2001), no. 1, 37-45.

[Li013] E. Liflyand, Hausdorff operators on Hardy spaces. Eurasian Math. J. 4 (2013), no. 4, 101-141.

[LiMi09] E. Liflyand and A. Miyachi, Boundedness of the Hausdorff operators in $H^p$ spaces, $0 < p < 1$, Boundedness of the Hausdorff operators in $H^p$ spaces, $0 < p < 1$. Studia Math. 194
(2009), no. 3, 279-292.

[LiMo02] E. Liflyand, F. Móricz, Commuting relations for Hausdorff operators and Hilbert transforms on real Hardy spaces. Acta Math. Hungar. 97 (2002), no. 1-2, 133-143.

[LiMo01] E. Liflyand, F. Móricz, The multi-parameter Hausdorff operator is bounded on the product Hardy space $H^{11}(\mathbb{R} \times \mathbb{R})$. Analysis (Munich) 21 (2001), no. 2, 107-118.

[LiMo00] E. Liflyand and F. Móricz, The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$, Proc. Amer. Math. Soc. 128 (2000), no. 5, 1391-1396.

[Mo05] F. Móricz, Multivariate Hausdorff operators on the spaces $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Anal. Math. 31 (2005), no. 1, 31-41.

[RuFa16] J. Ruan and D. Fan, Hausdorff operators on the power weighted Hardy spaces. J. Math. Anal. Appl. 433 (2016), no. 1, 31-48.

[St20] G. Stylogiannis, Hausdorff operators on Bergman spaces of the upper half plane. Concr. Oper. 7 (2020), no. 1, 69-80.

[Tu06] J. Tung, Taylor coefficients of functions in Fock spaces. J. Math. Anal. Appl. 318 (2006), no. 2, 397-409.

[Zh12] K. Zhu, Analysis on Fock spaces. Graduate Texts in Mathematics, 263. Springer, New York, 2012.

[Zh90] K. Zhu, Operator theory in function spaces. Second Edition, Mathematical surveys and Monographs, Vol 138, AMS, 2007.

Department of Mathematics, University of Thessaloniki, Thessaloniki 54124, Greece
E-mail address: stylog@math.auth.gr & g.stylog@gmail.com

Department of Mathematics, University of Thessaloniki, Thessaloniki 54124, Greece
E-mail address: petros gala@math.auth.gr