Discontinuous phase transition in a core contact process on complex networks

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Abstract
To understand the effect of generalized infection processes, we suggest and study the core contact process (CCP) on complex networks. In CCP an uninfected node is infected when at least \( k \) different infected neighbors of the node select the node for the infection. The healing process is the same as that of the normal CP. It is analytically and numerically shown that discontinuous transitions occur in CCP on random networks and scale-free networks depending on infection rate and initial density of infected nodes. The discontinuous transitions include hybrid transitions with \( \beta = 1/2 \) and \( \beta = 1 \). The asymptotic behavior of the phase boundary related to the initial density is found analytically and numerically. The mapping between CCP with \( k \) and static \((k+1)\)-core percolation is supposed from the \((k+1)\)-core structure in the active phase and the hybrid transition with \( \beta = 1/2 \). From these properties of CCP one can see that CCP is one of the dynamical processes for the \( k \)-core structure on real networks.

1. Introduction
Epidemic spreading and information propagation on complex networks have been intensively studied to understand the mechanism of spreading over a network and to provide strategies to control the spread [1–10]. The typical models of spreading are the contact process (CP) and the susceptible-infected-susceptible (SIS) model. Both models undergo the absorbing phase transition [11]. The infection process in the CP is theoretically different from that in the SIS model. In the CP, an infected node on a network randomly selects an uninfected linked neighbor and infects only the selected neighbor [2]. In contrast, an infected node in the SIS model infects all uninfected neighbors [1].

This theoretical difference of the infection process makes the physical properties of the CP model on complex networks different from those of the SIS model. The CP model on random networks and scale-free networks (SFNs) undergoes the continuous transition. The continuous transition of the CP on SFNs with the degree exponent \( \gamma > 3 \) shows the mean-field behavior, whereas the critical phenomena of the CP on SFNs with \( \gamma < 3 \) depends on \( \gamma \) [3, 5]. In contrast, the SIS model on annealed SFNs either undergoes the continuous transition or has only the active phase [1, 3, 6–9]. On annealed SFNs with \( \gamma \geq 4 \), the transition of the SIS model shows the mean-field behavior [3, 8]. On the SFNs with \( 3 < \gamma < 4 \), the critical phenomena depends on \( \gamma \) [3, 8]. Furthermore, on the SFNs with \( \gamma \leq 3 \), the SIS model has only the active phase [1, 3, 4, 8]. Recently, the SIS model on quenched uncorrelated SFNs was argued to have only active phase regardless of \( \gamma \) [6]. This argument for the SIS model on quenched SFNs is still controversial [7, 9, 10]. Therefore, the phase transition property of epidemic spreading on complex networks intrinsically varies depending on the details of the infection process.

However, in real epidemic processes, it is very natural to consider generalized infection (or propagation) processes in which a node is infected by at least \( k \) simultaneous successful infection attempts from infected neighbors. Physically \( k \) means the infection or propagation threshold (IPT). The generalized infection (or propagation) processes are easily observed in real situations. For example, when multiple attackers cooperate with each other to launch polluted packets to a node in a wireless mesh network, the batch verification of the
unpolluted node, which holds for a single polluted packet, may not result in polluting the node [12]. In the opinion formation process, if a group of neighboring persons to a selected person unanimously shares an opinion, the group pressure causes the selected person to take the opinion of the group [13, 14]. In these processes the infected cluster is expected to show the k-core structure, defined as the maximal cluster in which each node has at least k neighbors [15–17]. The k-core has also been found in social networks, such as terrorist networks [18], the trust-aware recommender system dataset Epinion [19], etc [20]. These networks are so-called recommendation-based trust networks, in which simultaneous recommendations from members are necessary to accept a new member. For example, to prevent the spreading of dishonest information or to build a trust network, cross-checking of information was suggested [20], where the cross-check can also be described by simultaneous acceptances. This acceptance process should also be a kind of the generalized propagation process. Until now the k-core structure has mainly been studied by the k-core percolation. Unlike the static percolation models, the above-mentioned k-core structures, such as trust networks, are dynamically formed and changed over the time. For example, terrorist networks are not static but dynamic, because the groups become fortified by new members and/or shrunk by the processes in which members are removed, killed, captured or compromised [18]. One may ask why k-core structures are easily found in dynamical trust networks, or ask why k-core structures can provide the trust. In the above-mentioned examples the infection processes with the IPT k ≥ 2 are intrinsically based on the simultaneous selection of a common node by different infected nodes. Therefore, it is physically very important to study how the simultaneous CP-like infections dynamically form the k-core structure in order to understand the dynamical origin of formation and evolution of real networks.

As a dynamical process for k-core, we suggest the core contact process (CCP) on complex networks, in which an uninfected node is infected when at least k different infected neighbors of the node select the node for infection. Each selected infection is exactly the same as the infection process of the CP. Thus, a CCP with k = 1 reduces to the CP exactly. The healing process of the CCP is the same as the normal CP. The k-core structure is manifested in the active state of the CCP. The CCP on annealed or quenched SFNs including random networks is proven to undergo discontinuous transition regardless of γ by the simple or heterogenous mean-field theory and by simulations. We first obtain the phase diagram of the CCP which depends on the infection rate λ and initial density ρo of infected nodes on the networks. The phase diagram has the same physical structure on any network. Thus we find from the obtained phase diagrams that the CCP undergoes discontinuous phase transitions depending on the infection rate λ and initial density ρo of infected nodes regardless of the kind of the network. Especially, the discontinuous phase transitions include hybrid transitions with order parameter exponent β = 1/2 and γ = 1. One of the peculiar physical facts in the CCP is that ρo is another relevant quantity to control the transition, because very small ρo cannot infect, even for large λ.

Another physical importance of the CCP is the possible mapping to the static k-core percolation. The theoretical importance of the mapping between the susceptible-infected-recovered (SIR) model and the random percolation proved by Grassberger [23] and the desirability of another such kind of mapping was pointed out by subsequent studies [11, 24]. From the same cluster structure and the same transition property the CCP is conjectured to be mapped into the static k-core percolation. We find that a CCP with k dynamically on SFNs with any γ, including random networks, forms the (k+1)-core structure in the active phase, which maps to the maximal cluster of the (k+1)-core percolation. The hybrid transition with β = 1/2 is exactly the same transition occurring in the k-core percolation [15–17]. From the relation of the CCP to the static k-core percolation, one can answer the question of why the k-core structures are easily formed by the generalized infection or propagation process in real networks.

Recently, the model in which the infection of a node occurs only by the k simultaneous SIS-type infection process was suggested and studied on complex networks [21]. This SIS-type model [21] with k = 1 exactly reduces to the SIS model. Therefore, the model was shown to have physical properties different from those of the CCP just like the CP has different physical properties from those of the SIS model [3, 8]. In particular, the SIS-type model [21] only has an active phase on SFNs with γ < 3 and undergoes a continuous transition on SFNs with γ = 3. In contrast, the CCP undergoes discontinuous transitions regardless of the network. Furthermore, the SIS-type model [21] never analyzed the k-core structure of active clusters except for the limit λ → ∞. Other important different properties of the CCP from those of the SIS-type model [21] on complex networks will be explained in the final section. Another variant model [22] with multiple SIS-type infection processes for a successful infection was suggested and studied by mean-field theories and by simulations on a two-dimensional square lattice. This variant model [22] with k = 1 does not reduce to the CP or the SIS model. Physical properties of the variant model on complex networks were never studied.
2. Mean-field theory

We now define the core contact process (CCP) with IPT $k$ on a complex network with size $N$, in which each node $i$ has degree $z_i$. The state of a node $i$ is either the uninfected state ($\phi_i = 0$) or the infected state ($\phi_i = 1$). At $t = 0$, randomly-selected $N_{\text{inf}}$ nodes are assigned to be infected. At each update, depending on the state $\phi_q(t)$ of a randomly chosen node $q$ and the states of its neighbors, $\{\phi_{q_1}(t), \phi_{q_2}(t), \ldots, \phi_{q_{\ell}}(t)\}$, $\phi_q(t + dt)$ is decided. With rate 1, the healing process, $\phi_q(t + dt) = 0$, is taken. For the infection process, the following process is taken. Each neighbor $q_i$ with $\phi_{q_i}(t) = 1$ selects $q$ with the probability $1/z_{q_i}$. Let $n_q$ be the number of such selections from neighbors. If $n_q \geq k$, $\phi_q(t + dt) = 1$ with rate $\lambda n_q$. If $n_q < k$, the infection process is rejected. When $k = 1$, CCP reduces to the normal CP [11].

The mean-field rate equation of CCP is first considered. This mean-field theory exactly holds on the annealed regular network in which every node has the same degree $z$ [25]. Let us start with the rate equation of the normal CP [11]

$$
\frac{d\rho}{dt} = \rho = -\rho + \lambda (1 - \rho) \sum_{i=1}^{z} \left( \frac{z}{\rho} \right)^{\rho} \left( 1 - \frac{\rho}{z} \right)^{z-\rho}.
$$

(1)

Each term in the sum of equation (1) means the rate that $\rho$ infected neighbors simultaneously select the center node for infection. Likewise, the mean-field rate equation for the CCP with IPT $k$ is

$$
\dot{\rho} = -\rho + \lambda (1 - \rho) \sum_{i=1}^{z} \left( \frac{z}{\rho} \right)^{\rho} \left( 1 - \frac{\rho}{z} \right)^{z-\rho}.
$$

(2)

Since the property of the transition is confirmed to physically be the same for the CCP with any $k (\geq 2)$, analysis of the CCP with $k = 2$ is mainly reported in this paper. For $k = 2$, the rate equation (2) becomes

$$
\dot{\rho} = -\rho + \lambda (1 - \rho) \left[ 1 - \left( 1 - \frac{\rho}{z} \right)^{z-1} \right].
$$

(3)

To describe the dynamical properties of the CCP in the mean-field level, $\dot{\rho}(\lambda)$ is plotted against $\rho$ for various $\lambda$ by using equation (3) in figure 1(a). As shown in figure 1(a), $\rho < 0$ for small $\lambda$, because $\rho(\rho = 1) = -1$ and $\rho(\rho = 0) = 0$. As $\lambda$ increases, the maximal $\dot{\rho}, \dot{\rho}_{\text{max}}$, also increases. At $\lambda = \lambda^*$, $\dot{\rho}_{\text{max}} = 0$ as shown in figure 1(a).

From the conditions $\dot{\rho}_{\text{max}} = \dot{\rho}(\lambda^*, \rho^*) = 0$ and $d\dot{\rho}(\lambda^*, \rho^*)/d\rho = 0$, $\lambda^*$ and $\rho^*$ should satisfy

$$
z^2 \left[ 1 - \frac{\rho^*}{z} \right]^{z-1} - (z - 1)^2 \left[ 1 - \frac{\rho^*}{z} \right]^{z-2} = z = 0
$$

(4)

and

$$
\frac{1}{\lambda^*} = \left( 1 - \rho^* \right) \left[ 1 - \left( 1 - \frac{\rho^*}{z} \right)^{z-1} \right].
$$

(5)

For example, $\lambda^* = 5.3591546 \ldots$ and $\rho^* = 0.4521022 \ldots$ are numerically obtained from equations (4) and (5) for $z = 10$ (see the inset of figure 1(a)). For $\lambda > \lambda^*$, the equation

$$
\lambda (1 - \rho) \left[ 1 - \left( 1 - \frac{\rho}{z} \right)^{z-1} \right] = 1
$$

(6)

has two non-trivial solutions $\rho_1$ and $\rho_2$ and $\rho > 0$ when $\rho \in (\rho_1, \rho_2)$. The dependence of $\dot{\rho}$ on $\lambda$ and $\rho$ is schematically shown in figure 1(b).

From the behavior of $\dot{\rho}$, the dependence of $\rho(t \to \infty)$ on $\lambda$ and $\rho_{\text{c}}$ is automatically obtained as shown in figure 1(c). $\rho_{\text{c}} = 0$ for $\lambda < \lambda^*$ regardless of $\rho_{\text{c}}$. Furthermore, $\rho_{\text{c}} = 0$ for $\lambda > \lambda^*$ and $\rho_{\text{c}} < \rho_{\text{c}}(\lambda)$, because $\rho < 0$. In contrast, for $\lambda > \lambda^*$ and $\rho_{\text{c}} > \rho_{\text{c}}(\lambda), \rho_{\text{c}} = \rho_{\text{c}}(\lambda)$, because $\rho < 0$ for $\rho > \rho_{\text{c}}$ and $\rho > 0$ for $\rho_{\text{c}} < \rho < \rho_{\text{c}}$. The exact phase diagram for the CCP with $k = 2$ is shown in figure 1(c). The phase boundary consists of the vertical solid line, $\lambda = \lambda^*$, and the lower solid curve, $\rho_{\text{c}} = \rho_{\text{c}}(\lambda)$ with $\lambda > \lambda^*$. From equation (3) with $\dot{\rho} = 0$, the lower boundary $\rho_{\text{c}}(\lambda)$ in the limit $\lambda \to \infty$ behaves as
In the language of nonlinear dynamics, equation (3) exhibits a saddle-node bifurcation at \( \lambda^* \) [21]. From the saddle-node bifurcation, one expects the following phase transition nature depending on the control parameter \( \lambda \) and \( \rho \) as shown in figure 1(c) and 2. The main transition occurs while \( \lambda \) increases with \( \rho > \rho^* \) (the process (1) in the figure 1(c)). In the process (1) the discrete jump from \( \rho_s = 0 \) to \( \rho_s = \rho^* \) occurs at \( \lambda = \lambda^* \). To know the dependence of \( \rho \) on \( \lambda \) when \( \lambda \to \lambda^* \), let us put \( \rho_s = \rho_s(\lambda) = \rho^* + \delta \). Of course, \( \delta = 0 \) when \( \lambda = \lambda^* \). Then, from equations (4)–(6), the relation between \( \lambda \) and \( \delta \) to the order of \( \delta^2 \) is obtained as

\[
\lambda \approx \lambda^* + C_2 \lambda^* \delta^2,
\]

where

\[
C_2 = \frac{z-1}{2z^2} \left( 1 - \frac{\rho^*}{z} \right)^{z-3} (3z - z\rho^* - 2).
\]

Thus, one obtains

\[
\rho(\lambda) \approx \rho^* + \frac{1}{C_2^{1/2} \lambda^{\beta}} (\beta = 1/2)
\]

as shown in figure 2(1). Thus in the process (1), the discontinuous hybrid transition at \( \lambda = \lambda^* \) with \( \beta = 1/2 \) occurs. This main hybrid transition is exactly the same as that shown in the static \( k \)-core percolation model [15–17]. This correspondence between the CCP and static \( k \)-core percolation strongly supports the mapping between the two models as emphasized in introduction section. In the processes (2) and (3) in figure 1(c), transitions...
different from the main transition occur. Using a similar analytic method to that used in deriving equation (9), the transition in process (2) is shown to be a hybrid transition with $\beta = 1$ as in figure 2(2). The transition in process (3) is a simple jump. These two transitions physically come from the following property of the CCP. In the initial configuration with small $\rho_0$, the majority of the infected nodes are isolated. Thus, in the CCP, the isolated nodes cannot infect other nodes even for $\lambda > \lambda^*$.  

3. Heterogeneous mean-field theory

In the CCP on complex networks with the degree distribution $P(z)$, a node with higher degree can be infected more easily than a node with lower degree. To analyze the CCP on complex networks, the heterogeneous mean-field (HMF) theory of the CCP, which is believed to be exact on the annealed complex networks, [1, 6, 8], is now considered. The HMF rate equation of the normal CP [1, 2] is

$$\frac{d\rho_z}{dt} = -\rho_z + \lambda z (1 - \rho_z) \Theta = -\rho_z + \lambda (1 - \rho_z) \sum_{\ell=1}^{\infty} \left( \frac{z}{\rho_0} \right)^{\ell} \Theta^\ell (1 - \Theta)^{\ell-\ell}. \quad (10)$$

$\rho_z(t)$ is the density of the infected nodes with degree $z$, i.e., $\rho_z(t) \equiv \frac{n_z(t)}{NP(z)}$. Here $n_z(t) \equiv \sum_{t=1}^{N} \phi_z(t) \delta_{z,z'}$ and $\rho = \sum_{z=0}^{\infty} \rho_z P(z), \Theta \equiv \sum_{\ell=0}^{\infty} P(z'|z) \rho_z(z') = \rho/(z)$ on uncorrelated networks, because $P(z'|z) = z'P(z')/(z)$. Thus, the HMF rate equation for CCP with $k$ is $\rho_z = -\rho_z + \lambda (1 - \rho_z) \sum_{\ell=1}^{\infty} \left( \frac{z}{\rho_0} \right)^{\ell} \Theta^\ell (1 - \Theta)^{\ell-\ell}$. The HMF rate equation with $k = 2$ is reduced to be

$$\rho_z = -\rho_z + \lambda z (1 - \rho_z) \left[ 1 - \left(1 - \frac{\rho}{\rho_0}\right)^{z-1} \right]. \quad (11)$$

We first investigate the CCP on the annealed random networks (RNs) with $P(z) = e^{-\langle z \rangle} z^{\langle z \rangle}/\langle z \rangle!$. Since equation (11) cannot be solved analytically, a numerical integration is used. Since the distribution $P(z)$ rapidly decreases as $z$ increases for $z \gg \langle z \rangle$, the contribution of $[\rho_z(t)]_{z>z_{cut}}$ to $\rho$ is physically negligible if $z_{cut}$ is large enough. With $z_{cut} = 10^3, dt = 10^{-5}$ and $\langle z \rangle = 10$, the time evolution of $\rho(t)$ is enumerated as $\rho(t + dt) = \rho_z(t) + \dot{\rho}_z dt$.

To obtain the phase diagram on the annealed RNs, the dependence of $\rho$ on both $\lambda$ and $\rho$ is numerically studied. First, $\rho(\lambda)$ and $\lambda^*$ is obtained while $\lambda$ is increased with a sufficiently large $\rho_0$ (process (1)). $\lambda^*$ is estimated as $\lambda^* \approx 5.803521(1)$ and $\rho_0(\lambda)$ for $\lambda \gg \lambda^*$ is obtained as shown in figure 3(a), especially $\rho^* = \rho_0(\lambda^*) \approx 0.40668(1)$. 

![Figure 2. Discontinuous phase transitions for $z = 10$ in the processes (1) and (2) in figure 1(c).](image-url)
To determine the lower phase boundary $\rho_l(\lambda)$, the process (3) in which $\rho_\lambda$ is increased with a fixed $\lambda (> \lambda^*)$ is used. The obtained $\rho_l(\lambda)$ is shown in figure 3(a). In particular, from the relation $\rho_l(\lambda^{*+}) = \rho^*$, the result $\rho^* \approx 0.40668(1)$ is reconfirmed. Thus, the phase diagram of the CCP on the annealed RNs is obtained as figure 3(a). Furthermore, the transition natures in processes (1), (2), and (3) are numerically confirmed to be the same as those shown in figure 2. The numerical integrations with $dt = 10^{-6}$ and $z_{cut} = 10^4$ are also performed to confirm nearly the same results. All the results of the CCP on RNs from the HMF theory are exactly confirmed by simulations on annealed RNs.

The normal CP on scale-free networks (SFNs) with the degree distribution $P(z) = Az^{-\gamma}$ ($\gamma \leq 3$) obeys HMF theory [2, 28] even though the finite-size effect. Therefore, the numerical integrations for $z_{cut} = 2 \times 10^4, 5 \times 10^4, 1 \times 10^5, 2 \times 10^5, 5 \times 10^5$, and $1 \times 10^6$ are performed. As shown in the insets of figure 3(b), both $\lambda^*$ and $\rho^*$ converge to $\lambda^*$ ($z_{cut} \to \infty$) $\approx 12.68(1)$ and $\rho^*$ ($z_{cut} \to \infty$) $\approx 0.30058(2)$ are also confirmed to converge to finite values. From these results, the phase diagram of the CCP on the annealed SFNs with $\gamma = 2.5$ in the limit $z_{cut} \to \infty$ is determined as figure 3(b). We also find that the transition natures from the HMT theory on SFNs are the same as those from the mean-field theory in figure 2.

The asymptotic behavior of the phase boundary $\rho_l(\lambda)$ for $\lambda \to \infty$ on the annealed SFNs (or in figure 3(b)) is different from that in the mean-field theory or $\rho_l(\lambda) \propto z^{-1}$. From $\rho_l(\lambda) = \sum_x P(x) \rho_{12}(\lambda)$, $\rho_{12}(\lambda)$ is one of the solutions of equation (13) with $\rho_{\lambda} = 0$. Thus, from equation (11),

$$
\rho_{12}(\lambda) = \frac{\lambda^2 \rho_\lambda(\lambda) \left[ 1 - \left( 1 - \frac{\rho_\lambda(\lambda)}{\rho_\lambda} \right)^{z-1} \right]}{\rho_\lambda + \lambda^2 \rho_\lambda(\lambda) \left[ 1 - \left( 1 - \frac{\rho_\lambda(\lambda)}{\rho_\lambda} \right)^{z-1} \right]].
$$

(12)

Since $\rho_\lambda(\lambda) \ll 1$ in the limit $\lambda \to \infty$,

$$
\rho_{12}(\lambda) \approx \frac{\lambda \rho_\lambda(\lambda) (z - 1)/\langle z \rangle^2}{1 + \left[ \lambda \rho_\lambda(\lambda) (z - 1)/\langle z \rangle^2 \right]}. \tag{13}
$$

From $\rho_l(\lambda) = \int_{z_{min}}^{\infty} dz P(z) \rho_{12}(\lambda)$, and $P(z) = (\gamma - 1) z_{min}^{-1} z^{-\gamma}$

$$
\rho_l(\lambda) \approx (\gamma - 1) z_{min}^{-1} \int_{z_{min}}^{\infty} dz \rho_{12}(\lambda). \tag{14}
$$
If $\lambda \rho \lambda - \langle \rangle \ll z^2$, the integration (14) is nearly equal to $\int dzz z (z - 1)/\langle z \rangle^2$. If $\lambda \rho \lambda - \langle \rangle \gg z^2$, the integral is reduced to $\int dzz z z^2$. Thus, the integration of equation (14) is approximately divided into two subintegrals as

$$
\rho_\lambda (\lambda) \approx \langle \gamma - 1 \rangle_x
\int_{z_{\min}}^{z} dz \frac{z^{-\gamma} \rho_\lambda^2 (\lambda) (z - 1)/\langle z \rangle^2}{\langle z \rangle^2} + \int_{z}^{\infty} dz z^{-\gamma},
$$

where $\lambda \rho_\lambda^2 (\lambda) z_{\gamma} (z - 1)/\langle z \rangle^2 = 1$. With $z_{\gamma} = 0.5[1 + \sqrt{1 + 4\langle z \rangle^2 / \rho_\lambda^2 (\lambda)}] \approx \langle z \rangle (\rho_\lambda (\lambda) \sqrt{\langle \rangle} )^{-1}$,

$$
\rho_\lambda (\lambda) \approx \begin{cases} 
\frac{\gamma - 1}{3 - \gamma} (\langle z \rangle^{\gamma - 1})^{\gamma - 1} & \text{for } 2 < \gamma < 3 \\
\frac{\gamma - 1}{\gamma - 3} \lambda \rho_\lambda^2 (\lambda) \left[ z_{\min}^{\gamma - 1} \lambda \rho_\lambda^2 (\lambda) z_{\gamma}^2 \gamma - 3 - \gamma^2 \right] & \text{for } \gamma > 3
\end{cases}
$$

(16)

Thus, in the limit $\lambda \to \infty$,

$$
\rho_\lambda (\lambda) \propto \lambda^{-\kappa}, \kappa = \begin{cases} 
\frac{\gamma - 1}{2(\gamma - 2)} & \text{for } 2 < \gamma < 3 \\
1 & \text{for } \gamma > 3
\end{cases}
$$

(17)

These asymptotic behaviors, i.e., $\kappa = 1.5$ on SFNs with $\gamma = 2.5$ and $\kappa = 1$ on RNs, are confirmed from the numerical integrations.

4. Core contact process on quenched complex networks

To understand the physical properties of the CCP on quenched complex networks, the transition natures of the CCP on quenched RNs and SFNs are studied by simulation. A quenched network with $N$ nodes and a given degree distribution $P(z)$ is constructed by the configuration model [26]. By similar processes taken to determine the phase diagram by numerical integrations, the phase diagrams on the quenched RNs and SFNs with $\gamma = 2.5$ are obtained by the simulations.

As shown in the inset of figure 4(a), the phase diagram of CCP on the quenched RNs is physically the same as that from the mean-field theory in figure 1(c). The phase diagram on RNs is confirmed to barely depend on the network size $N$ by the simulations on RNs with $N$ up to $N = 1.6 \times 10^5$. It is found from simulations that the macroscopic active clusters of the CCP with IPT $k = 2$ on the quenched RNs always have the 3-core structure as shown in figure 4(a). The hybrid transition in process (1) is physically the same transition as in the $k$-core...
percolation on RNs with \( k \geq 3 \) [15–17]. Therefore, the \((k+1)\)-core structure should be formed in the active phase of the CCP with IPT \( k \). This \((k+1)\)-core structure as well as the hybrid transition with \( \beta = 1/2 \) strongly supports the mapping between the CCP and the static \( k \)-core percolation emphasized in the introduction.

The phase diagram of the CCP on quenched SFNs with \( \gamma = 2.5 \) is also obtained by simulation as the inset of figure 4(b) shows. Due to the strong finite-size effect or hub effect, \( \lambda^* \) and \( \rho^* \) of the CCP on SFNs depend strongly on \( N \) as shown in figures 4(b) and (c). To analyze the finite-size effect systematically, the simulations are performed for \( N = 1 \times 10^4, 2 \times 10^4, 4 \times 10^4, 8 \times 10^4, \) and \( 1.6 \times 10^5 \). First, the simulations are performed following the process (1) with \( \rho_o = 1 \) to determine \( \lambda^* \) and \( \rho_1 (\lambda) \). From the fact that \( \rho_1 (\lambda) > 0 \) for \( \lambda > \lambda^* \), we estimate \( \lambda^* (N) \) and \( \rho_1 (\lambda, N) \) is also estimated by using the process (3). As shown in the phase diagram for \( N = 1.6 \times 10^5 \) in the inset of figure 4(b), \( \rho^* (N) \) is larger than \( \rho_1 (\lambda^*, N) \) due to the strong finite-size effect. However, as shown in figure 4(b), both \( \rho^* (N) \) and \( \rho_1 (\lambda^*, N) \) converge to the same value 0.205(5) in the limit \( N \to \infty \). Therefore, in the thermodynamic limit, the same phase diagram as for the mean-field theory with \( \lambda^* \approx 17.9 \) and \( \rho^* \approx 0.205 \) is expected on quenched SFNs.

5. Summary and discussion

In summary, the absorbing transitions in the CCP with IPT \( k \geq 2 \) on RNs and SFNs with \( \gamma < 3 \) are investigated. The phase diagram of the CCP on the complex networks is numerically shown to be the same as that analytically obtained from mean-field theory. Discontinuous hybrid transitions with \( \beta = 1/2 \) and \( \beta = 1 \) are found in the CCP. In particular, the \((k+1)\)-core structure found in the active phase and the hybrid transition with \( \beta = 1/2 \) provide strong evidence for the mapping between the CCP with \( k \) and static \((k+1)\)-core percolation. We also find that the lower boundary of the phase diagram for \( \lambda \gg 1 \) asymptotically behaves as \( \rho_1 (\lambda) \sim \lambda^{-\kappa} \) with \( \kappa = (\gamma - 1)/(2(\gamma - 2)) \) for \( 2 < \gamma < 3 \) and \( \kappa = 1 \) for \( \gamma \geq 3 \). From these properties of the CCP, it can be seen that the CCP is the key dynamical process for the formation of the \( k \)-core structure in real networks. Next we comment on the network resilience under random damage, to which \( k \)-core percolation was argued to be relevant [27]. Since \( \lambda^* \) in the CCP is normally somewhat larger than the epidemic threshold \( \lambda_c \) in the normal CP and low \( \rho_0 \) cannot make the active phase, the network resilience in the CCP is dynamically enhanced compared to the normal CP.

Finally, we want to compare the physical difference between the CCP and SIS-type models [21, 22]. The physical difference also means a difference in physical properties. We also want to explain the difference of physical properties. Furthermore, we discuss the advancements that our study on the CCP on complex networks represent, compared to the SIS-type models [21, 22].

The physical difference is the infection process. For the infection of a node in the CCP, the node should be selected simultaneously by two or more different infected nodes. Thus, an infection in the CCP consists of multiple simultaneous CP-like infections [2]. In contrast, for the infection of a node in the SIS-type models [21, 22] the node should be the only simultaneously linked neighbors of two or more different infected nodes. An infection in the SIS-type models [21, 22] consists of multiple simultaneous SIS-type [1] infections. The different physical properties owing to the physical difference between the CCP and SIS-type model [21] are as follows. On SFNs with \( \gamma < 3 \), the SIS-type model [21] has only the active state and undergoes no transition as with the original SIS-model [1]. The SIS-type model [21] on SFNs with \( \gamma = 3 \) undergoes a continuous transition. In contrast, the CCP shows discontinuous transitions regardless of the underlying topology. In the study of a variant SIS type model [22], the physical properties on complex networks were not examined, and thus different physical properties of the model cannot be compared.

Furthermore, our study of the CCP makes advances with respect to the work on the SIS-type model [21]. We clearly find the complete phase diagrams of the CCP depending on \( \lambda \) and \( \rho_0 \) with the lower boundary as equation (17) on complex networks (see figures 3 and 4). The phase diagrams are not only obtained by analytical methods but also confirmed by simulations, in particular an elaborate simulation on quenched SFNs with \( \gamma < 3 \) (see figure 4(b)). In the study of the SIS-type model [21], the phase diagrams were found only on regular random and Erdős–Rényi graphs by simulations. The mapping between the CCP with \( k \) and the static \((k+1)\)-core structure in the active phase and the hybrid transition with \( \beta = 1/2 \). From this mapping, one can answer the question of why the \( k \)-core structures are easily formed by the
generalized infection or propagation process in real networks. In contrast, the cluster structure in an active state such as the $k$-core structure has not been studied in the SIS-type model [21] except for the statement that the SIS-type model with $\lambda \to \infty$ on RNs might be related to a bootstrap percolation.

Recently generalized SIR models on the complete graph (CG), in which the infection of a node depends on accumulated CP-type contacts $D(t)$ over the previous $T$ time steps, were suggested and studied [29]. In these models the infection occurs if $D(t) \geq d^*$. The generalized SIR models on CGs were shown to undergo continuous transitions and discontinuous transitions depending on the conditions, such as whether $R$ becomes $S$ with a certain probability or not. The models with certain conditions to show discontinuous transitions may have similar aspects to the CCP, because $k$ in the CCP might be physically related to $d^*$ in the generalized SIR models. The generalized SIR models have never been studied on complex networks and thus the direct comparison between two models on complex networks is a meaningful future study.

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