NLO evolution of 3-quark Wilson loop operator

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ABSTRACT: It is well known that high-energy scattering of a meson from some hadronic target can be described by the interaction of that target with a color dipole formed by two Wilson lines corresponding to fast quark-antiquark pair. Moreover, the energy dependence of the scattering amplitude is governed by the evolution equation of this color dipole with respect to rapidity. Similarly, the energy dependence of scattering of a baryon can be described in terms of evolution of a three-Wilson-lines operator with respect to the rapidity of the Wilson lines. We calculate the evolution of the 3-quark Wilson loop operator in the next-to-leading order (NLO) and present a quasi-conformal evolution equation for a composite 3-Wilson-lines operator. We also obtain the linearized version of that evolution equation describing the amplitude of the odderon exchange at high energies.
1 Introduction

The description of the proton scattering in the framework of $k_T$-factorization can be addressed within the high energy operator expansion developed in [1]. In that paper this method was applied to derive the full leading order (LO) hierarchy of the low-x evolution equations for Wilson lines with arbitrary indices and to the most important case of the color dipole. In the next to leading order (NLO) the evolution equation for the color dipole was derived in [2], [3], [4] and the connected evolution of the 3 Wilson lines was found in [5]. Finally the full NLO hierarchy of the low-x evolution equations was written in [6] and the JIMWLK hamiltonian equivalent to it in [7].

In this framework the amplitude in the Regge limit can be written as a convolution of the impact factors and the matrix elements of the Wilson line operators. The impact factors consist of the wavefunctions of the incoming and outgoing particles, which describe
their splitting into the quarks and gluons propagating through the shockwave formed by the other particle. It is well known that the propagation of the fast particle is described by a Wilson line - infinite gauge link ordered along the classical trajectory of the fast particle. For the virtual photon or meson scattering the relevant two-Wilson-lines operator is a color dipole. In the proton case assuming SU(3) symmetry it is the baryon or 3-quark Wilson loop (3QWL) $\varepsilon^{ijh} \varepsilon_{ijh} U_{1i} U_{2j} U_{3h}$. Its leading order linear evolution equation was studied in the C-odd case within the JIMWLK formalism and proved equivalent to the C-odd BKP equation [8]-[9] in [10] and its nonlinear evolution equation was derived within Wilson line approach [1] in [11]. The connected contribution to the NLO kernel of the equation was calculated in [5]. In the momentum representation the evolution of this operator was first studied in [12], and the nonlinear equation was worked out in [13]. In the C-odd case the linear NLO evolution equation for the odderon Green function was obtained in [14].

Here the NLO evolution equation for the 3QWL operator is presented. Then as in [3], we construct the composite 3QWL operator obeying the quasi-conformal evolution equation and after that give its linearized kernel in the 3-gluon approximation. In this approximation we also linearize the BK equation and show that it contains the nondipole 3QWL operators.

After completion of this paper we were informed about JIMWLK calculation of the NLO evolution of 3-Wilson-line operator [15]. Both evolution kernels reproduce NLO BK in the dipole limit $\vec{r}_1 \rightarrow \vec{r}_2$ and survive other checks but the detailed comparison of these two results is beyond the scope of present paper. We also wish to compare our results with the results of S. Caron-Huot [16].

The paper is organized as follows. In Section 2 we remind the general logic of high-energy OPE. Sections 3 and 4 present the derivation of the NLO kernel for 3QWL operator and Section 5 describes the calculation of the quasi-conformal kernel for the composite 3QWL operator. The linearized kernel is given in Section 6. The main results of the paper are listed in Sect. 7 and conclusions in Sect. 8. Appendices comprise the necessary technical details.

2 Rapidity factorization and evolution of Wilson lines

Let us consider the proton scattering off a hadron target like another proton or nucleus. First, we assume that due to saturation the characteristic transverse momenta of exchanged and produced gluons are relatively high ($Q_s \sim 2 - 3$ GeV for $pA$ scattering at LHC) so the application of perturbation theory is justified. Alternatively, one may think about high-energy scattering of a charmed baryon.

If pQCD is applicable, in accordance with general logic of high-energy OPE we factorize all amplitudes in rapidity. First, we integrate over gluons with rapidity $Y$ close to that of the projectile proton $Y_p$. To this end we introduce the rapidity divide $\eta \leq Y_p$ which separates the “fast” gluons from the “slow” ones.

It is convenient to use the background field formalism: we integrate over gluons with $\alpha > \sigma = e^\eta$ and leave gluons with $\alpha < \sigma$ as a background field, to be integrated over later. Since the rapidities of the background gluons are very different from the rapidities of gluons
Figure 1. Rapidity factorization. The impact factors with $Y > \eta$ are given by diagrams in the shock-wave background. Wilson-line operators with $Y < \eta$ are denoted by dotted lines.

in our Feynman diagrams, the background field can be taken in the form of a shock wave due to the Lorentz contraction. To derive the expression of a quark (or gluon) propagator in this shock-wave background we represent the propagator as a path integral over various trajectories, each of them weighed with the gauge factor $P\exp(ig \int dx_\mu A^\mu)$ ordered along the propagation path. Now, since the shock wave is very thin, quarks (or gluons) do not have time to deviate in transverse direction so their trajectory inside the shock wave can be approximated by a segment of the straight line. Moreover, since there is no external field outside the shock wave, the integral over the segment of straight line can be formally extended to $\pm \infty$ limits yielding the Wilson-line gauge factor

$$U^\eta_x = P\exp \left[ ig \int_{-\infty}^{\infty} du \ p_1^\mu A^\mu_\eta \ (u p_1 + x_\perp) \right],$$

$$A^\eta_\mu (x) = \int d^4k \ \theta(e^\eta - |\alpha_k|) e^{ik \cdot x} A_\mu (k) \quad (2.1)$$

$$U (r', \eta) = P e^{ig \int_{-\infty}^{\infty} A^-_\eta (r^+ ) dr^+},$$

where $A^-_\eta$ is the external shock wave field built from only slow gluons

$$A^-_\eta = \int \frac{d^4 p}{(2\pi)^4} e^{-ipz} A^- (p) \ \theta(e^\eta - p^+). \quad (2.3)$$

(Our light-cone conventions are listed in the Appendix A). The propagation of a quark (or gluon) in the shock-wave background is then described by free propagation to a point of interaction with the shock wave, Wilson line $U$ at the point of interaction, and free propagation to the final point.

Thus, the result of the integration over rapidities $Y > \eta$ gives the proton impact factor proportional to product of two proton wavefunctions integrated over longitudinal

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momenta. This impact factor is multiplied by a “color tripole” - 3QWL operator made of three (light-like) Wilson lines with rapidities up to \( \eta \):

\[
B_{123} \equiv \varepsilon^{ijh} \varepsilon_{ijh} U_{1i} U_{2j} U_{3h} \equiv U_1 \cdot U_2 \cdot U_3,
\]

(2.4)

where \( U_i \equiv U(\vec{r}, \eta) \). (As discussed in Refs. [1, 6], these Wilson lines should be connected by appropriate gauge links at infinity making the operator (2.4) and similar many-Wilson-lines operators below gauge invariant.) It should be noted that the proton impact factor is non-perturbative, so at this point it can be calculated only using some models of proton wavefunctions.

At the second step the integrals over gluons with rapidity \( Y < \eta \) give matrix element of triple Wilson-line operator \( B_{123} \) between target states. The “rapidity cutoff” \( \eta \) is arbitrary (between rapidities of projectile and target) but it is convenient to choose it in such a way that the impact factor does not scale with energy so all energy dependence is shifted to the matrix element of triple Wilson-line operator (see the discussion in Ref. [17]). In the leading order the rapidity evolution of this operator was calculated in Ref. [11], [5] while in this paper we present the result for the NLO evolution.

3 NLO evolution of triple-Wilson-line operator

In this Section we outline the calculation of the NLO kernel for the rapidity evolution of 3QWL operator (2.4). In accordance with general logic of high-energy OPE in order to find the evolution of the Wilson-line operators with respect to rapidity cutoff we consider matrix element of operators with rapidities up to \( \eta_1 \) and we integrate over the region of rapidities \( \Delta \eta = \eta_1 - \eta_2 \) (where \( \eta_1 > \eta_2 > Y_{\text{target}} \)). Since particles with different rapidities perceive each other as Wilson lines, the result of integration will be \( \Delta \eta \) times kernel of the rapidity evolution times Wilson lines with rapidities up to \( \eta_2 \). As we discussed in Sect. 2, it is convenient to use background-field formalism where gluons with rapidities \( Y < \eta_2 \) form a narrow shockwave.

![Figure 2. Leading-order diagrams.](image)

The typical leading-order diagrams are shown in Fig. 2 and it is clear that at this order the evolution equation for 3-line operator can be restored from the evolution of two-line operators since all interactions are either pairwise (Fig. 2b) or self-interactions (Fig. 2a).
The result for the LO evolution has the form [11]

\[
\frac{\partial B_{123}}{\partial \eta} = \frac{\alpha_s^3}{4\pi^2} \int d\vec{r}_0 \left[ \frac{\vec{r}_{12}^2}{\vec{r}_{01} \cdot \vec{r}_{02}} \left( -B_{123} + \frac{1}{6}(B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210}) \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right].
\]

(3.1)

where \( \vec{r}_1, \vec{r}_2, \vec{r}_3 \) are the coordinates of the quark Wilson lines within the 3QWL and \( \vec{r}_0 \) is the coordinate of gluon Wilson line coming from the intersection with the shock wave. (In this paper we set \( N_c = 3 \) explicitly).

The typical diagrams in the next-to-leading order are shown in Fig. 3 where \( \vec{r}_0 \) and \( \vec{r}_4 \) are the coordinates of intersections with the shock wave. It is clear that at the NLO in addition to self-interaction (Fig.3a) and pairwise-interaction (Fig.3b) diagrams we have also the triple-interaction diagrams of Fig.3c type. It should be emphasized that for the self-interaction and pairwise diagrams one can use the results of Ref. [6] while the triple-interaction diagrams were already calculated in Ref. [5]. In this paper we will combine these results to get a concise expression for the evolution of three-Wilson-line operator (2.4).

Let us start with self- and pairwise-interactions of the type shown in Fig. 3 a-d. At \( N_c = 3 \) one can use the \( SU(3) \) identities

\[
U_4^{hn} = 2tr(t^b U_4 t^a U_4^\dagger), \quad (t^a)^i_j (t^a)^i_k = \frac{1}{2} \delta^i_j \delta^i_k - \frac{1}{6} \delta^j_i \delta^k_i
\]

(3.2)
to rewrite the result of [6] only through the Wilson lines in the fundamental representation. For the contribution of the the states with 2 gluons crossing the shockwave it reads

\[
\langle K_{NLO} \otimes (U_1)^{i_1}_{j_1} (U_2)^{i_2}_{j_2} \rangle_{2g} = -\frac{\alpha_s^2}{8\pi^2} \int d\vec{r}_0 d\vec{r}_4 \mathbf{G}_{12},
\]

(3.3)
\[ G_{12} = G_1 \left\{ (U_1 U_0^\dagger U_4)^{i_1}_{j_1} (U_0 U_1^\dagger U_2)^{j_3}_{j_1} + (U_2 U_0^\dagger U_1)^{i_3}_{i_1} (U_2 U_0^\dagger U_0)^{j_3}_{j_1} \right\} \\
+ G_2 (U_1)^{i_3}_{i_1} \left\{ (U_0 U_1^\dagger U_2 U_0^\dagger U_4)^{j_3}_{j_1} + (U_4 U_0^\dagger U_2 U_0^\dagger U_0)^{j_3}_{j_1} \right\} \\
+ G_3 (U_2)^{j_3}_{j_1} \left\{ (U_0 U_1^\dagger U_1 U_0^\dagger U_4)^{i_3}_{i_1} + (U_4 U_1^\dagger U_1 U_0^\dagger U_0)^{i_3}_{i_1} \right\} \\
+ G_4 \left\{ (U_4)^{i_3}_{i_1} (U_1 U_0^\dagger U_2)^{j_3}_{j_1} + (U_4)^{j_3}_{j_1} (U_2 U_0^\dagger U_1)^{i_3}_{i_1} \right\} \text{tr}(U_0 U_1^\dagger) \\
+ G_5 (U_2)^{j_3}_{j_1} (U_4)^{i_3}_{i_1} \text{tr}(U_0 U_1^\dagger) \text{tr}(U_0 U_1) \\
+ G_6 \left\{ (U_0)^{i_3}_{i_1} \left\{ (U_4 U_1^\dagger U_1 U_0^\dagger U_2)^{j_3}_{j_1} + (U_2 U_4 U_1 U_0^\dagger U_0)^{j_3}_{j_1} \right\} \\
- \left\{ (U_4)^{j_3}_{j_1} \left\{ (U_0 U_1^\dagger U_2)^{i_3}_{i_1} + (U_4)^{i_3}_{i_1} \left\{ (U_2 U_1^\dagger U_0)^{j_3}_{j_1} \right\} \text{tr}(U_0 U_1) \right\} \\
+ G_7 (U_1)^{j_3}_{j_1} (U_4)^{i_3}_{i_1} \text{tr}(U_0 U_1^\dagger) \text{tr}(U_0 U_2) \\
+ G_8 \left\{ (U_4)^{i_3}_{i_1} \left\{ (U_0 U_1^\dagger U_2 U_0^\dagger U_1)^{j_3}_{j_1} + (U_1 U_0^\dagger U_2 U_0^\dagger U_0)^{j_3}_{j_1} \right\} \\
- \left\{ (U_0)^{j_3}_{j_1} \left\{ (U_1 U_0^\dagger U_4)^{i_3}_{i_1} + (U_0)^{i_3}_{i_1} \left\{ (U_2 U_0^\dagger U_4)^{j_3}_{j_1} \right\} \text{tr}(U_0 U_2) \right\} \right\} . \tag{3.4} \]

Note that as discussed in Ref. [6], it is convenient to present some of the terms with one intersection in the two-intersection form with an additional integration over \( \vec{r}_4 \). In doing so, some \( U_4 \) and \( U_0 \) factors in Eq. (3.4) are replaced by \( U_4 - U_i \) and \( U_4 - U_i \) \( (i = 1, 2 \) or \( 3) \). We do not write such subtraction terms here since it is easier to make the subtraction after the color convolution. The functions have the form

\[ G_1 = - \frac{\vec{r}_{01}^2 - 2\vec{r}_{02}^2}{2\vec{r}_{02}^2\vec{r}_{04}^2 (\vec{r}_{24}^2 - \vec{r}_{02}^2)} - \frac{\vec{r}_{04}^2 \vec{r}_{12}^2 + \vec{r}_{02}^2 (\vec{r}_{14}^2 - \vec{r}_{12}^2) + (\vec{r}_{01}^2 + \vec{r}_{02}^2 - \vec{r}_{12}^2) \vec{r}_{24}^2}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{04}^2\vec{r}_{24}^2} \]

\[ + \frac{1}{\vec{r}_{01}^2\vec{r}_{24}^2 - \vec{r}_{02}^2\vec{r}_{14}^2} \left[ \frac{2\vec{r}_{12}^4 - \vec{r}_{04}^2 \vec{r}_{02}^2}{\vec{r}_{04}^2} - \frac{\vec{r}_{14}^2}{\vec{r}_{04}^2} - \frac{(\vec{r}_{02}^2 - \vec{r}_{04}^2)(\vec{r}_{14}^2 - \vec{r}_{01}^2)\vec{r}_{24}^2}{\vec{r}_{04}^2(\vec{r}_{24}^2 - \vec{r}_{02}^2)} \right] \]

\[ \times \ln \left( \frac{\vec{r}_{02}^2}{\vec{r}_{24}^2} \right) - \frac{1}{\vec{r}_{04}^2} + (0 \leftrightarrow 4, 1 \leftrightarrow 2). \tag{3.5} \]

\[ G_2 = \left( \frac{1}{\vec{r}_{02}^2 - \vec{r}_{24}^2} \right) \left[ \left( \frac{1}{\vec{r}_{04}^2} + \frac{1}{2\vec{r}_{02}^2\vec{r}_{24}^2} \right) \left( \frac{\vec{r}_{02}^2 + \vec{r}_{24}^2}{2} - \frac{2}{\vec{r}_{04}^2} \right) - \frac{\vec{r}_{02}^2 - \vec{r}_{24}^2}{4\vec{r}_{02}^2\vec{r}_{04}^2\vec{r}_{24}^2} \right] \]

\[ \times \ln \left( \frac{\vec{r}_{02}^2}{\vec{r}_{24}^2} \right) - \frac{1}{\vec{r}_{04}^2}. \tag{3.6} \]

\[ G_3 = G_2_{1+2}. \tag{3.7} \]

\[ G_4 = \left( \frac{\vec{r}_{02}^2 - \vec{r}_{24}^2}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{14}^2\vec{r}_{24}^2} \right) \left( \frac{\vec{r}_{02}^2\vec{r}_{14}^2 - \vec{r}_{01}^2\vec{r}_{24}^2}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{14}^2\vec{r}_{24}^2} \right) - \frac{\vec{r}_{02}^2 + \vec{r}_{24}^2}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{14}^2\vec{r}_{24}^2} + \frac{1}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{14}^2\vec{r}_{24}^2} + \frac{\vec{r}_{12}^4}{2\vec{r}_{01}^2\vec{r}_{02}^2\vec{r}_{14}^2\vec{r}_{24}^2} \right] 

\[ - 6 - \]
\[ \frac{1}{2} \ln \left( \frac{\bar{r}_{01}^2}{\bar{r}_{14}^2} \right) - G_1. \]  

\[ G_5 = \frac{2}{r_{04}^4} + \left( \frac{1}{2} \frac{1}{r_{01}^2 - r_{14}^2} \left[ \frac{4}{r_{04}^4} - \frac{1}{r_{01}^2} - \frac{1}{r_{14}^2} \right] \right) \ln \left( \frac{\bar{r}_{01}^2}{\bar{r}_{14}^2} \right). \]  

\[ G_6 = \frac{1}{2} \left( \frac{\bar{r}_{12}^2 - \bar{r}_{24}^2}{r_{01}^2 r_{14}^2 r_{24}^2} + \frac{1}{2} \frac{\bar{r}_{12}^2 + \bar{r}_{24}^2}{r_{01}^2 r_{24}^2} \right) \ln \left( \frac{\bar{r}_{01}^2}{\bar{r}_{14}^2} \right). \]  

\[ G_7 = G_5 |_{1=2}, \quad G_8 = G_6 |_{1=2,3=4}. \]  

After the convolution with \( \epsilon_{ij} \epsilon_{jk} (U_3)^{h'}_h \), (3.4) gives the contribution of the 2-gluon states to the evolution of the 3QWL operator \( U_1 \cdot U_2 \cdot U_3 \) describing the total interaction of Wilson lines 1 and 2, leaving Wilson line 3 intact.

\[ G_{12} \epsilon_{ij} \epsilon_{jk} (U_3)^{h'}_h = G_1 \left( U_0 U_4 U_1 \right) \left( U_1 U_1^U U_4 \right) + \left( U_2 U_4 U_0 \right) \left( U_4 U_0 U_1 \right) \cdot U_3 \]

\[ + \left( G_2 \left( U_0 U_4 U_3 U_2 U_3 U_4 + U_4 U_0 U_3 U_2 U_3 U_0 \right) \cdot U_1 \cdot U_3 + (1 \leftrightarrow 2) \right) \]

\[ - G_4 \left( U_0 U_1 \right) \left( U_1 U_1^U U_3 + U_2 U_0 U_1 \right) \cdot U_3 \cdot U_4 \]

\[ + \left( G_5 \ U_2 \cdot U_3 \cdot U_4 \right) \left( U_2 U_4 U_1 U_3 U_4 + U_4 U_0 U_1 U_3 U_4 \right) \cdot U_0 \cdot U_3 + (1 \leftrightarrow 2, 2 \leftrightarrow 4) \].

One can also write

\[ \langle K_{NLO} \otimes (U_1)^{j_3}_{j_1} |_{2g} = -\frac{\alpha_s^2}{8\pi^4} \int d\bar{r}_0 d\bar{r}_4 \ G_{11}, \]

\[ G_{11} = G_3 \left( U_0 U_3 U_2 U_4 U_0 \right) + U_0 U_1 U_4 U_0 \]

\[ - \left( U_0 U_1 \right) \left( U_1 U_0 \right) \ U_0 \cdot U_0 - \left( U_0 U_4 \right) \left( U_4 U_0 \right) \ U_0 \cdot U_4 \]

\[ + G_9 \left( U_0 U_1 \right) \left( U_1 U_0 \right) \ U_0 \cdot U_0 \]

\[ \left( U_0 U_4 \right) \left( U_4 U_0 \right) \ U_0 \cdot U_4 \]

\[ G_9 = \frac{\bar{r}_{01}^2 - \bar{r}_{04}^2}{4 r_{01}^2 r_{04}^2 r_{14}^2} \ln \left( \frac{\bar{r}_{01}^2}{\bar{r}_{14}^2} \right). \]

And then take the convolution

\[ G_{(1)3} = G_{11} \left( U_3 \right)^{j_3}_{j_1} = G_3 \left( U_0 U_1 U_3 U_4 U_0 U_1 U_4 \right) + \left( U_0 U_4 U_1 U_0 U_4 U_3 \right) \]

\[ - \left( U_0 U_1 U_3 \right) \left( U_1 U_4 \right) \left( U_4 U_0 \right) - \left( U_0 U_4 \right) \left( U_1 U_0 \right) \left( U_4 U_3 \right) \]

\[ + G_9 \left( U_0 U_3 \right) \left( U_1 U_4 \right) \left( U_4 U_0 \right) \left( U_0 U_4 \right) \left( U_1 U_0 \right) \left( U_4 U_3 \right) \].
For the elements of $SU(3)$ group one has the identity

$$
\varepsilon^{ijh} \varepsilon_{i'j'h'} (U_1)_i^{i'} (U_1)_j^{j'} = 2(U_1)_k^{i'j'}, \quad U_1 \cdot U_3 = 2 \text{tr}(U_1^4U_3),
$$

(3.17)

Taking $\vec{r}_2 = \vec{r}_1$ in (3.12) one can check that it is related to (3.16) via this identity using the other $SU(3)$ identities (B.1) and (B.3). Taking the conjugate of $G_{11}$, one gets

$$
\langle K_{NLO} \otimes (U_1)_j^{j'} |_{2g} = \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \ G_1,
$$

(3.18)

$$
G_1 = G_3 \left( U_4U_0^1U_1U_3^1U_0 + U_0U_4^1U_1U_0^1U_4 \right)
- tr(U_1U_4^1)tr(U_4U_0^1)U_0 - tr(U_0U_4^1)tr(U_1U_0^1)U_4
+ G_9 \left( tr(U_1U_4^1)tr(U_4U_0^1)U_0 - tr(U_0U_4^1)tr(U_1U_0^1)U_4 \right),
$$

(3.19)

The contribution of the evolution of only one line $U_1$ to the evolution of the 3QWL reads

$$
G_{(1)23} = G_1 \varepsilon^{ijh} \varepsilon_{i'j'h'} \left(U_2\right)_i^{i'} \left(U_3\right)_h^{h'} = G_3 \left( U_4U_0^1U_1U_4^1U_0 + U_0U_4^1U_1U_0^1U_4 \right)
- tr(U_1U_4^1)tr(U_4U_0^1)U_0 - tr(U_0U_4^1)tr(U_1U_0^1)U_4
+ G_9 \left( tr(U_1U_4^1)tr(U_4U_0^1)U_0 - tr(U_0U_4^1)tr(U_1U_0^1)U_4 \right),
$$

(3.20)

Then the connected contribution of the evolution of lines 1 and 2 reads

$$
G_{(12)3} = \frac{1}{2} \left[ H_1 - (1 \leftrightarrow 2) \right]
\times \left[ (U_0U_4^1U_2) \cdot (U_1U_4^1U_4) \cdot U_3 - (U_0U_4^1U_1) \cdot (U_2U_4^1U_4) \cdot U_3 - (4 \leftrightarrow 0) \right]
+ H_2 \left[ tr(U_0U_1) \left( U_0U_4^1U_2 + U_2U_4^1U_0 \right) \cdot U_3 \cdot U_4 \right.
- \left( U_2U_0^1U_1U_4 U_0 + U_0U_4^1U_1U_0^1U_2 \right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0) \right]
+ H_3 \left[ tr(U_0U_1) \left( U_0U_4^1U_2 + U_2U_4^1U_0 \right) \cdot U_3 \cdot U_4 \right.
+ \left( U_2U_0^1U_1U_4 U_0 + U_0U_4^1U_1U_0^1U_2 \right) \cdot U_3 \cdot U_4 + (4 \leftrightarrow 0) \right]
+ H_4 \left[ tr(U_0U_4^1) \left( U_1U_0^1U_2 + U_2U_0^1U_1 \right) \cdot U_3 \cdot U_4 \right.
+ \left( U_0U_4^1U_1 \right) \cdot \left( U_2U_0^1U_3 \right) + \left( U_0U_4^1U_2 \right) \cdot \left( U_1U_0^1U_3 \right) \cdot U_3 + (4 \leftrightarrow 0) \right]
+ H_1 \left[ tr(U_4U_0^1) \right. 
\left. \left( U_1U_4^1U_2 + U_2U_4^1U_1 \right) \cdot U_0 \cdot U_3 - (4 \leftrightarrow 0) \right] + (1 \leftrightarrow 2).
$$

(3.21)
\begin{align}
H_2 &= \frac{1}{8} \left[ \frac{r_1^2}{r_{01}^2 r_{14}^2 r_{24}^2} + \frac{r_{12}^2 - r_{14}^2 - r_{24}^2}{r_{01}^2 r_{14}^2 r_{24}^2} \right] \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) \cdot (3.22) \\
H_3 &= \frac{1}{8} \left[ \frac{r_1^2}{r_{01}^2 r_{04}^2 r_{24}^2} - \frac{r_{12}^2}{r_{01}^2 r_{04}^2 r_{24}^2} - \frac{r_{12}^2 - r_{14}^2 - r_{24}^2}{r_{01}^2 r_{04}^2 r_{24}^2} \right] \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) \cdot (3.23) \\
H_4 &= \frac{-1}{4 r_{01}^4} - \frac{1}{8} \left[ \frac{r_1^2 (r_{12}^2 - r_{02}^2 + r_{24}^2)}{r_{01}^2 r_{02}^2 r_{14}^2 r_{24}^2} + \frac{r_{12}^2}{r_{01}^2 r_{14}^2 r_{24}^2} - \frac{r_{12}^2}{r_{01}^2 r_{02}^2 r_{14}^2 r_{24}^2} + \frac{r_{24}^2 + r_{02}^2 - r_{14}^2}{r_{01}^2 r_{04}^2 r_{14}^2 r_{24}^2} \right] \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) + \frac{1}{r_{01}^2 - r_{14}^2} \left( \frac{r_{12}^2 - r_{02}^2 + r_{24}^2}{r_{01}^2 r_{02}^2 r_{14}^2 r_{24}^2} - \frac{r_{12}^2 + r_{02}^2 - r_{14}^2}{r_{01}^2 r_{04}^2 r_{14}^2 r_{24}^2} - \frac{4 r_{14}^2}{r_{01}^2} + \frac{8}{r_{01}^2} \right) \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) \cdot (3.24) \\
H_5 &= \frac{1}{8} \left[ \frac{r_{13}^2 r_{02}^2}{r_{01}^2 r_{03}^2 r_{04}^2 r_{24}^2} - \frac{r_{12}^2 r_{13}^2}{r_{01}^2 r_{03}^2 r_{14}^2 r_{24}^2} + \frac{r_{12}^2 r_{34}^2}{r_{01}^2 r_{04}^2 r_{14}^2 r_{24}^2} - \frac{r_{13}^2 r_{24}^2}{r_{01}^2 r_{04}^2 r_{34}^2 r_{02}^2} \right] \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) \cdot (3.26) \\
H_6 &= \frac{1}{8} \left[ \frac{r_{13}^2 r_{02}^2}{r_{01}^2 r_{03}^2 r_{14}^2 r_{24}^2} - \frac{r_{12}^2 r_{13}^2}{r_{01}^2 r_{03}^2 r_{04}^2 r_{24}^2} + \frac{r_{12}^2 r_{34}^2}{r_{01}^2 r_{03}^2 r_{14}^2 r_{24}^2} - \frac{r_{13}^2 r_{24}^2}{r_{01}^2 r_{03}^2 r_{04}^2 r_{24}^2} \right] \ln \left( \frac{r_{01}^2}{r_{14}^2} \right) \cdot (3.27)
\end{align}

The fully connected “triple” contribution corresponding to the diagrams of Fig. 3 e,f can be taken from [6] or [5] and transformed to the form

\[ G_{(123)} = H_5 \left[ \left( U_0 U_4 \right)^2 \right] \cdot \left( U_1 U_0 \right)^2 \cdot U_4 - \left( U_0 U_4 \right)^2 \cdot \left( U_1 U_0 \right)^2 \cdot U_4 \\
+ \left( U_2 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_4 - \left( U_2 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_0 + (4 \leftrightarrow 0) \right] \\
+ H_6 \left[ \left( U_0 U_4 \right)^2 \right] \cdot \left( U_1 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_4 + \left( U_2 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_4 \\
+ \left( U_2 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_4 - \left( U_2 U_0 \right)^2 \cdot \left( U_3 U_4 \right)^2 \cdot U_0 - (4 \leftrightarrow 0) \right] \\
+ (1 \leftrightarrow 2) + (1 \leftrightarrow 3). \]
shockwave

write for the full contribution to the evolution of the 3QWL with 2-gluons intersecting the contributions of 2 (3.21) and 3 (3.26) Wilson lines from the previous section one can

Taking the contributions of the self-interaction on one Wilson line (3.20), the connected

4 Construction of the kernel: gluon part

The connection of our notations with the notations in \[6\] is given in the appendix A.

The connection of our notations with the notations in \[6\] is given in the appendix A.

\[\langle K_{NLO} \otimes B_{123}\rangle_{2g} = \langle K_{NLO} \otimes U_1 \cdot U_2 \cdot U_3\rangle_{2g} = \frac{-\alpha_s}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \mathbf{G}, \]

\[\mathbf{G} = \mathbf{G}_{(1)23} + \mathbf{G}_{(1)(2)3} + \mathbf{G}_{12(3)} + \mathbf{G}_{(1)(2)3} + \mathbf{G}_{1(2)3} + \mathbf{G}_{(1)(3)2} + \mathbf{G}_{(1)(2)3}, \] (4.2)

Here \(\langle \ldots \rangle\) stands for the connected contribution, i.e. \(\mathbf{G}_{(1)23}\) gives the contribution of the evolution of line 1 (3.20), with lines 2 and 3 being spectators, \(\mathbf{G}_{(1)(2)3}\) — the connected contribution of the evolution of lines 1 and 2 (3.21), with line 3 being intact, and \(\mathbf{G}_{(1)(2)3}\) — the fully connected contribution (3.26). All the rest can be obtained from them by \(1 \leftrightarrow 2 \leftrightarrow 3\) transformation.

There are several useful \(SU(3)\) identities, which help to reduce the number of color structures. They are listed in the appendix B. First we use (B.5) to get rid of the structure

\[\left(U_0 U_4^\dagger U_3 U_0^\dagger U_4\right) \cdot U_1 \cdot U_2 \] (4.3)

and the 2 ones it goes into after the \(1 \leftrightarrow 2 \leftrightarrow 3\) transformations with their symmetric counterparts w.r.t. \(0 \leftrightarrow 4\) exchange. Next we use (B.6) to eliminate 6 such contributions antisymmetric w.r.t. \(0 \leftrightarrow 4\) exchange as

\[\left(U_2 U_0^\dagger U_1 U_4^\dagger U_0 + U_0 U_4^\dagger U_1 U_0^\dagger U_2\right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0). \] (4.4)

After that we use (B.7) to express 6 structures like

\[\left(U_2 U_4^\dagger U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1 U_4^\dagger U_2\right) \cdot U_0 \cdot U_3 \]

(4.5)

and their symmetric counterparts w.r.t. \(0 \leftrightarrow 4\) exchange through other structures. Then via (B.8) we cancel 3 structures of the form

\[U_2 \cdot U_3 \cdot U_4 \text{ tr} \left(U_0^\dagger U_1\right) \text{ tr} \left(U_0 U_4^\dagger\right) - U_2 \cdot U_3 \cdot U_0 \text{ tr} \left(U_4^\dagger U_1\right) \text{ tr} \left(U_4 U_0^\dagger\right). \] (4.6)
Finally, by means of (B.9) we discard the 3 nonconformal terms proportional to

$$tr \left( U_0 U_1 U_2 + U_0 U_1 U_2 \right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0) \quad (4.7)$$

and the 2 structures they go into after the 1 \leftrightarrow 2 \leftrightarrow 3 transformations. Finally, we get

$$G = \left\{ (L_{12} + \tilde{L}_{12}) \left( U_0 U_4 \right) \cdot \left( U_1 U_0 U_2 \right) \cdot U_3 \cdot U_4 \right\} + \left\{ (M_{13} - M_{12} - M_{23} + M_2) \left[ \left( U_0 U_4 U_3 \right) \cdot \left( U_2 U_0 U_1 \right) \cdot U_3 + \left( U_1 U_0 U_2 \right) \cdot \left( U_3 U_4 U_0 \right) \cdot U_4 \right] 
+ \text{(all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4). \quad (4.8)$$

$$L_{12} = H_3 + H_4 - \frac{1}{2} G_3 + (1 \leftrightarrow 2) \quad (4.9)$$

$$= \frac{\vec{r}_{12}^2}{8} \left[ -\frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{24}^2} \vec{r}_{14}^2 \vec{r}_{24}^2 - \frac{1}{\vec{r}_{01}^2 \vec{r}_{04}^2 \vec{r}_{24}^2} - \frac{1}{\vec{r}_{02}^2 \vec{r}_{04}^2 \vec{r}_{14}^2} \right] \ln \left( \frac{\vec{r}_{12}^2 \vec{r}_{24}^2}{\vec{r}_{14}^2 \vec{r}_{02}^2} \right). \quad (4.10)$$

$$M_{12} = \frac{1}{2} \left\{ H_1 + H_2 - \frac{1}{2} G_9 + (1 \leftrightarrow 2) \right\} \quad (4.11)$$

$$= \frac{\vec{r}_{12}^2}{16} \left[ -\frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2 \vec{r}_{14}^2 \vec{r}_{24}^2} + \frac{1}{\vec{r}_{01}^2 \vec{r}_{02}^2 \vec{r}_{14}^2 \vec{r}_{24}^2} + \frac{1}{\vec{r}_{02}^2 \vec{r}_{01}^2 \vec{r}_{14}^2 \vec{r}_{24}^2} \right] \ln \left( \frac{\vec{r}_{12}^2 \vec{r}_{24}^2}{\vec{r}_{14}^2 \vec{r}_{02}^2} \right). \quad (4.12)$$

These functions obey the identities

$$M_2 |_{\vec{r}_1 \rightarrow \vec{r}_3} = \frac{\vec{r}_{23}^2}{4} \left[ -\frac{\vec{r}_{23}^2}{\vec{r}_{03}^2 \vec{r}_{02}^2 \vec{r}_{24}^2 \vec{r}_{34}^2} + \frac{1}{\vec{r}_{03}^2 \vec{r}_{02}^2 \vec{r}_{24}^2 \vec{r}_{34}^2} + \frac{1}{\vec{r}_{02}^2 \vec{r}_{03}^2 \vec{r}_{24}^2 \vec{r}_{34}^2} \right] \ln \left( \frac{\vec{r}_{24}^2}{\vec{r}_{23}^2} \right). \quad (4.13)$$

$$M_{13} - M_{12} - M_{23} + M_2 |_{\vec{r}_1 \rightarrow \vec{r}_3} = \tilde{L}_{23}. \quad (4.14)$$

$$M_{13} - M_{12} - M_{23} + M_2 |_{\vec{r}_1 \rightarrow \vec{r}_2} = M_{13} - M_{12} - M_{23} + M_2 |_{\vec{r}_3 \rightarrow \vec{r}_2} = 0. \quad (4.15)$$

Using these identities and (B.1) with $l = 3$, we get the dipole result

$$G |_{\vec{r}_1 \rightarrow \vec{r}_3} = 4(L_{32} + \tilde{L}_{32}) tr \left( U_0 U_4 \right) tr \left( U_3 U_0 \right) tr \left( U_4 U_2 \right)$$

$$- 4L_{32} tr \left( U_0 U_4 U_3 U_0 U_3 U_4 \right) + (0 \leftrightarrow 4). \quad (4.16)$$
This expression is twice the corresponding part of the BK kernel for $tr(U_2 U_1')$.

The only UV divergent term in (4.8) is the term proportional to $L_{12}$. This term has the same coordinate structure as the corresponding term in the dipole kernel. Therefore we can do the same subtraction as in the dipole case. Using (B.3), we get

$$
\left( U_0 U_4 \hat{U}_2 \right) \cdot \left( U_1 U_0 \hat{U}_4 \right) \cdot U_3 + tr \left( U_0 U_4 \right) \left( U_1 U_0 \hat{U}_2 \right) \cdot U_3 \cdot U_4 + \left( 2 \leftrightarrow 1 \right)|_{\tilde{r}_0 \rightarrow \tilde{r}_4}^{|r_0 \rightarrow r_4} = 3 [tr \left( U_1 U_4 \right) U_2 \cdot U_3 \cdot U_4 + tr \left( U_2 U_4 \right) U_1 \cdot U_3 \cdot U_4 - tr \left( U_3 U_4 \right) U_1 \cdot U_2 \cdot U_4] - U_1 \cdot U_2 \cdot U_3
$$

$$
= \frac{3}{2} [B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124}] - B_{123},
$$

(4.17)

$$
B_{123} = U_1 \cdot U_2 \cdot U_3 = e^{i j' k' h' c_{ijh} U_{1r} U_{2j'} U_{3k'}}.
$$

(4.18)

Therefore we can separate the result into the UV finite and divergent parts

$$
\langle K_{NLO} \otimes B_{123} \rangle|_{g} = -\frac{\alpha_s^2}{8\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \mathbf{G}_{finite} - \frac{\alpha_s^2}{8\pi^4} \int d\tilde{r}_0 \mathbf{G}_{UV},
$$

(4.19)

$$
\mathbf{G}_{finite} = \left\{ \tilde{L}_{12} \left( U_0 U_4 \hat{U}_2 \right) \cdot \left( U_1 U_0 \hat{U}_4 \right) \cdot U_3 + L_{12} \left( \left( U_0 U_4 \hat{U}_2 \right) \cdot \left( U_1 U_0 \hat{U}_4 \right) \cdot U_3 + tr \left( U_0 U_4 \right) \left( U_1 U_0 \hat{U}_2 \right) \cdot U_3 \cdot U_4 \right) \right\}
$$

$$
- \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] + \frac{1}{2} B_{123}
$$

$$
+ M_{13} - M_{12} - M_{23} + M_2 \left[ \left( U_0 U_4 \hat{U}_3 \right) \cdot \left( U_2 U_0 \hat{U}_1 \right) \cdot U_4 + \left( U_1 U_0 \hat{U}_2 \right) \cdot \left( U_3 U_4 \hat{U}_0 \right) \cdot U_4 \right] + \left( \text{all 5 permutations} 1 \leftrightarrow 2 \leftrightarrow 3 \right) + \left( 0 \leftrightarrow 4 \right).
$$

(4.20)

And $\mathbf{G}_{UV}$ is included into the term describing the contribution with one gluon crossing the shockwave in [6].

The contribution of the diagrams with 1 gluon intersecting the shockwave, which are not proportional to the $\beta$-function one can take from (5.27) in [5]

$$
\langle \tilde{K}_{NLO} \otimes B_{123} \rangle|_{g} = \frac{\alpha_s^2}{(2\pi)^3} \int d\tilde{r}_0 \left\{ \frac{r_{10} r_{20} r_{30} r_{40}}{r_{12} r_{24} r_{32} r_{40}} - \frac{r_{30} r_{20} r_{10} r_{40}}{r_{12} r_{24} r_{32} r_{40}} \right\} \ln \frac{r_{30}}{r_{31}} \ln \frac{r_{20}^2}{r_{31}^2} \left( B_{100} B_{320} - B_{300} B_{210} \right)
$$

$$
+ \frac{\alpha_s^2}{(2\pi)^3} \int d\tilde{r}_0 \left\{ \frac{1}{r_{10}^2} - \frac{r_{30}^2}{r_{30}^2 r_{10}^2} \right\} \ln \frac{r_{20}^2}{r_{31}^2} \ln \frac{r_{10}^2}{r_{31}^2} \left( B_{123} - \frac{1}{2} \left[ 3 B_{100} B_{320} + B_{300} B_{120} - B_{200} B_{130} \right] \right)
$$

$$
+ \alpha_s^2 \int d\tilde{r}_0 \left\{ \frac{1}{r_{10}^2} - \frac{r_{30}^2}{r_{30}^2 r_{10}^2} \right\} \ln \frac{r_{20}^2}{r_{31}^2} \ln \frac{r_{10}^2}{r_{31}^2} \left( \frac{1}{2} \left[ 3 B_{300} B_{120} + B_{100} B_{320} - B_{200} B_{130} \right] - B_{123} \right)
$$

$$
+ \left( 2 \leftrightarrow 1 \right) + \left( 2 \leftrightarrow 3 \right).
$$

(4.21)

This term has the correct dipole limit (see (5.28) in [5]).

The contribution proportional to $\beta$-function reads (from [6])

$$
\langle \tilde{K}_{NLO} \otimes B_{123} \rangle|_{g}^2 = \left\{ -\frac{\alpha_s^2}{(2\pi)^3} \frac{1}{2} \int d\tilde{r}_0 \left\{ \ln \left( \frac{r_{20}^2}{r_{02}^2} \right) \left( \frac{1}{r_{02}^2} - \frac{1}{r_{20}^2} \right) - \frac{r_{12}^2}{r_{01}^2 r_{02}^2} \ln \frac{r_{12}^2}{r_{01}^2} \right\} \right\}
$$

(12)
\[
+ \frac{1}{r_{02}^2} \ln \left( \frac{\vec{r}_{02}^2}{\mu_g^2} \right) + \frac{1}{r_{01}^2} \ln \left( \frac{\vec{r}_{01}^2}{\mu_g^2} \right) \\
\times \left( U_0 \cdot U_3 \cdot (U_2 U_0^\dagger U_1) + U_0 \cdot U_3 \cdot (U_1 U_0^\dagger U_2) + \frac{2}{3} U_1 \cdot U_2 \cdot U_3 \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \\
+ \left[ \frac{\alpha_s}{2\pi} \right]^3 \frac{11}{6} \int d\vec{r}_0 \ln \left( \frac{\vec{r}_{01}^2}{\mu_g^2} \right) \left( U_0 \cdot U_2 \cdot U_3 \operatorname{tr}(U_1 U_0^\dagger) - \frac{1}{3} U_1 \cdot U_2 \cdot U_3 \right) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right],
\]

\[\tag{4.22}\]

\[
\left( \frac{11}{3} \ln \frac{1}{\mu_g^2} = \frac{11}{3} \ln \left( \frac{\mu^2}{4\pi^2 \alpha_s(1)} \right) + \frac{67}{9} - \frac{\pi^2}{3}. \tag{4.23}\right.
\]

Or, after some algebra

\[
\left( \hat{K}_{NLO} \otimes B_{123} \right)_{1g}^3 = -\frac{\alpha_s^2}{(2\pi)^3} \frac{11}{6} \int d\vec{r}_0 \left[ \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) \left( \frac{1}{\vec{r}_{02}} - \frac{1}{\vec{r}_{01}} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu_g^2} \right) \right] \\
\times \left( \frac{3}{2} (B_{100} B_{230} + B_{200} B_{130} - B_{300} B_{210}) - B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3). \tag{4.24}\]

It also has the correct dipole limit

\[
\left( \hat{K}_{NLO} \otimes B_{122} \right)_{1g}^3 = -\frac{\alpha_s^2}{(2\pi)^3} \frac{11}{3} \int d\vec{r}_0 \left[ \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) \left( \frac{1}{\vec{r}_{02}} - \frac{1}{\vec{r}_{01}} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu_g^2} \right) \right] \\
\times \left( \frac{3}{2} (B_{100} B_{230} - B_{122}) \right), \tag{4.25}\]

and it matches the BFKL kernel [18]. Therefore the real part of the whole kernel reads

\[
(K_{NLO} \otimes B_{123})_{\text{real}} = -\frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 d\vec{r}_4 \textbf{G}_{\text{finite}} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \textbf{G}_{\text{real}}. \tag{4.26}\]

\[
\textbf{G}_{\text{real}} = -\frac{1}{2} \left[ \left( \frac{\vec{r}_{10} \vec{r}_{20}}{\vec{r}_{10}^2 \vec{r}_{20}^2} - \frac{\vec{r}_{30} \vec{r}_{20}}{\vec{r}_{30}^2 \vec{r}_{20}^2} \right) \ln \frac{\vec{r}_{30}^2}{\vec{r}_{31}^2} \ln \frac{\vec{r}_{10}^2}{\vec{r}_{31}^2} (B_{100} B_{320} - B_{300} B_{210}) \right. \\
- \left[ \frac{1}{\vec{r}_{10}^2} - \frac{\vec{r}_{30} \vec{r}_{10}}{\vec{r}_{30}^2 \vec{r}_{10}^2} \right] \ln \frac{\vec{r}_{30}^2}{\vec{r}_{31}^2} \ln \frac{\vec{r}_{10}^2}{\vec{r}_{31}^2} (B_{123} - \frac{1}{2} (3B_{100} B_{320} + B_{300} B_{120} - B_{200} B_{130}) \right) \\
+ \left[ \frac{11}{12} \left( \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) \left( \frac{1}{\vec{r}_{02}} - \frac{1}{\vec{r}_{01}} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu_g^2} \right) \right) \right] \\
\times \left( \frac{3}{2} (B_{100} B_{230} + B_{200} B_{130} - B_{300} B_{210}) - B_{123} \right) \\
+ \text{(all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3), \tag{4.27}\]

and \textbf{G}_{\text{finite}} is defined in (4.20). If we put \( \vec{r}_2 = \vec{r}_3 \) here, we get the dipole result (see (100) in [3])

\[
\textbf{G}_{\text{real}}|_{\vec{r}_2 = \vec{r}_3} = \left\{ \frac{11}{3} \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) \left( \frac{1}{\vec{r}_{02}} - \frac{1}{\vec{r}_{01}} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu_g^2} \right) \right\} + \frac{2}{3} \frac{\vec{r}_{02}^2}{\vec{r}_{20}^2 \vec{r}_{10}^2} \ln \frac{\vec{r}_{20}^2}{\vec{r}_{21}^2} \ln \frac{\vec{r}_{02}^2}{\vec{r}_{21}^2}. \tag{4.20}\]
Finally, from the condition that the kernel must vanish without the shockwave (if all the $B = 6$) and that the virtual contribution is proportional to $B_{123}$, we get the total kernel

$$
\langle K_{NLO} \otimes B_{123} \rangle = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{\text{finite}} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, \mathbf{G}',
$$

$$
\mathbf{G}' = \frac{1}{2} \left[ \frac{\vec{r}_{10}^2}{r_{10}^2 r_{30}^2} - \frac{\vec{r}_{22}^2}{r_{22}^2 r_{20}^2} \right] \ln \frac{\vec{r}_{20}^2}{\vec{r}_{21}^2} \ln \frac{\vec{r}_{10}^2}{\vec{r}_{21}^2} (B_{100} B_{320} - B_{200} B_{310})
$$

\begin{align*}
&- \frac{\vec{r}_{12}^2}{r_{10}^2 r_{20}^2} \ln \frac{\vec{r}_{12}^2}{\vec{r}_{10}^2} \ln \frac{\vec{r}_{12}^2}{\vec{r}_{12}^2} \left( 9B_{123} - \frac{1}{2} \left[ 2 (B_{100} B_{320} + B_{200} B_{130}) - B_{300} B_{120} \right] \right) \\
&+ \frac{11}{6} \left[ \ln \left( \frac{\vec{r}_{02}^2}{\vec{r}_{02}^2} - \frac{\vec{r}_{01}^2}{\vec{r}_{01}^2} \right) - \frac{\vec{r}_{12}^2}{r_{01}^2 r_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{r_{01}^2} \right) \right] \times \left( \frac{3}{2} (B_{100} B_{230} + B_{200} B_{130} - B_{300} B_{210}) - 9B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3). \tag{4.30}
\end{align*}

It differs from (4.27) in the coefficients of $B_{123}$’s which turn into 9’s; $\mathbf{G}_{\text{finite}}$ is defined in (4.20).

### 5 Construction of the kernel: quark part

One can take the quark contribution to the NLO evolution of 3QWL from [6]. The contribution with 2 quarks intersecting the shockwave without subtraction reads

$$
\langle K_{NLO} \otimes B_{123} \rangle_{\text{qy}} = -\frac{\alpha_s^2 n_f}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}^q,
$$

$$
\mathbf{G}^q = \left[ \left( (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_2 \cdot U_3 - 3U_2 \cdot U_3 \cdot U_4 \cdot tr(U_0^\dagger U_1) - \frac{1}{3} U_1 \cdot U_2 \cdot U_3 tr(U_0^\dagger U_4) \right) \right.
$$

\begin{align*}
&\times \frac{2}{3} \frac{1}{r_{01}^4} \left\{ \frac{(\vec{r}_{14} \vec{r}_{01})}{r_{14}^2 - r_{01}^2} \ln \left( \frac{\vec{r}_{14}^2}{r_{01}^2} \right) + 1 \right\} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \\
&+ \left[ \left( \frac{1}{3} (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} U_1 \cdot U_2 \cdot U_3 tr(U_0^\dagger U_4) \right) \right. \\
&\left. + (U_1 U_0^\dagger U_2) \cdot U_3 \cdot U_4 + (1 \leftrightarrow 2) \right) \left( \frac{1}{r_{01}^4} \left\{ \frac{(\vec{r}_{14} \vec{r}_{01})}{r_{14}^2 - r_{01}^2} \ln \left( \frac{\vec{r}_{14}^2}{r_{01}^2} \right) + 1 \right\} + \frac{L_{12}^q}{2} + (1 \leftrightarrow 2) \\
&\left. + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right) \\
&\left. + \frac{1}{2} \right\} \left( \frac{1}{3} (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} U_1 \cdot U_2 \cdot U_3 tr(U_0^\dagger U_4) \right)
\end{align*}

where

$$
L_{12}^q = \frac{1}{r_{01}^4} \left\{ \frac{\vec{r}_{02}^2 \vec{r}_{14}^2 + \vec{r}_{01}^2 \vec{r}_{24}^2 - \vec{r}_{02}^2 \vec{r}_{12}^2}{2(r_{02}^2 r_{14}^2 - r_{01}^2 r_{24}^2)} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{14}^2}{r_{01}^2 r_{24}^2} \right) - 1 \right\}. \tag{5.3}
$$

Using identity (B.15) one can see that this contribution is conformally invariant, indeed

$$
\mathbf{G}^q = \frac{1}{2} \left\{ \left( \frac{1}{3} (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} U_1 \cdot U_2 \cdot U_3 tr(U_0^\dagger U_4) \right) \right.
$$

\begin{align*}
&\times \left( \frac{3}{2} B_{100} B_{220} - B_{122} \right) \right\}
\end{align*}

\begin{align*}
\tag{4.28}
\end{align*}
Here as a result the full kernel in QCD reads

\[ \langle K_{NLO} \otimes B_{123} \rangle = - \frac{\alpha_s}{8\pi^4} \int d\vec{r}_0 d\vec{r}_1 \left( G_{finite} + n_f G_{finite}^q \right) - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, G^{q'}, \]  

(5.11)

\[
G^{q'} = \frac{1}{2} \left[ \frac{\vec{r}_{12}^2}{r_{12}^2} - \frac{\vec{r}_{10}^2}{r_{10}^2} \right] \ln \frac{\vec{r}_{20}^2}{\vec{r}_{21}^2} \ln \frac{\vec{r}_{10}^2}{\vec{r}_{12}^2} (B_{100}B_{320} - B_{200}B_{310}) \\
- \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \frac{\vec{r}_{10}^2}{r_{10}^2} \ln \frac{\vec{r}_{20}^2}{\vec{r}_{21}^2} \left( 9B_{123} - \frac{1}{2} \{2(B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \} \right) \\
+ \frac{\beta}{2} \left[ \ln \left( \frac{\vec{r}_{12}^2}{\vec{r}_{10}^2} \right) \left( \frac{1}{\vec{r}_{12}^2} - \frac{1}{\vec{r}_{10}^2} \right) - \frac{\vec{r}_{12}^2}{r_{01}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu^2} \right) \right] \\
\times \left( \frac{3}{2} \{B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210} \} - 9B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3). 
\]

(5.12)

Here \( G_{finite} \) is defined in (4.20) and \( G^q_{finite} \) is defined in (5.7).
6 Evolution equation for composite 3QWL operator

In this section we consider only the gluon part of the kernel since the quark one is quasi-conformal. To construct composite conformal operators we will use the prescription [4] (see also Ref. [19])

\[ O^{\text{conf}} = O + \frac{1}{2} \frac{\partial O}{\partial \eta} \left| \frac{r_{0a}^2}{r_{0m}^2} \rightarrow \frac{r_{0a}^2}{r_{0m}^2} \ln \left( \frac{r_{0a}^2}{r_{0m}^2} \right) \right|, \]  

(6.1)

where \( a \) is an arbitrary constant. For the conformal 3QWL operator we have the following ansatz

\[ B_{123}^{\text{conf}} = B_{123} + \frac{\alpha_s}{8\pi^2} \int d\vec{r}_4 \left[ \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \left( \frac{\alpha s r_{12}^2}{r_{12}^2} \right) \right] \times \left( -B_{123} + \frac{1}{6} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}) \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \]  

(6.2)

If we put \( \vec{r}_2 = \vec{r}_3 \), then

\[ B_{122}^{\text{conf}} = B_{122} + \frac{\alpha_s}{4\pi^2} \int d\vec{r}_4 \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \left( \frac{\alpha s r_{12}^2}{r_{12}^2} \right) \times \left( -B_{122} + \frac{1}{6} (B_{144}B_{224}) \right), \]

or

\[ \text{tr}(U_1 U_2^\dagger)^{\text{conf}} = \text{tr}(U_1 U_2^\dagger) + \frac{\alpha_s}{4\pi^2} \int d\vec{r}_4 \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \left( \frac{\alpha s r_{12}^2}{r_{12}^2} \right) \times \left( \text{tr}(U_1 U_2^\dagger) - 3\text{tr}(U_1 U_2^\dagger) \right), \]

(6.3)

which is exactly the composite dipole operator of [4]. Using SU(3) identity (B.3) one can rewrite (6.2) as

\[ B_{123}^{\text{conf}} = B_{123} + \frac{\alpha_s}{8\pi^2} \int d\vec{r}_4 \left[ \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \left( \frac{\alpha s r_{12}^2}{r_{12}^2} \right) \right] \times \left( -B_{123} + \frac{1}{6} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}) \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \]  

(6.4)

Then as in [4], for \( (-B_{123} + \frac{1}{6} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})) \)

\[ (-3B_{123} + \frac{1}{2} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))^{\text{conf}} \]

\[ = (-3B_{123} + \frac{1}{2} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))^{\text{conf}} \]

\[ + \frac{\alpha_s}{8\pi^2} \int d\vec{r}_0 \left( \frac{\alpha s}{r_{03}^2 r_{04}^2} \ln \left( \frac{\alpha s r_{03}^2}{r_{04}^2} \right) \right) \]

\[ + A_{14} \frac{\vec{r}_{14}^2}{r_{01}^2 r_{04}^2} \ln \left( \frac{\vec{r}_{14}^2}{r_{01}^2 r_{04}^2} \right) \]

\[ + A_{24} \frac{\vec{r}_{24}^2}{r_{02}^2 r_{04}^2} \ln \left( \frac{\vec{r}_{24}^2}{r_{02}^2 r_{04}^2} \right) \]

(6.5)

where the functions \( A \) are calculated in appendix C (C.4–C.8) according to model (6.1). Therefore the evolution equation for \( B_{123}^{\text{conf}} \) turns into

\[ \frac{\partial B_{123}^{\text{conf}}}{\partial \eta} = \frac{\alpha_s}{4\pi^2} \int d\vec{r}_4 \left[ \frac{\vec{r}_{12}^2}{r_{12}^2} \ln \left( \frac{\alpha s r_{12}^2}{r_{12}^2} \right) \right] \times \left( -B_{123} + \frac{1}{6} (B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}) \right)^{\text{conf}} \]
+ (1 \leftrightarrow 3) + (2 \leftrightarrow 3) - \frac{\alpha^2_s}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \tilde{G}_{\text{finite}} - \frac{\alpha^2_s}{8\pi^4} \int d\vec{r}_0 G' \\
- \frac{\alpha_s}{4\pi} \frac{\alpha_s}{8\pi^2} \int d\vec{r}_4 d\vec{r}_0 \left[ \frac{\vec{r}_2}{r_{14}^2} \left( A_{34} \frac{\vec{r}_2}{r_{03}^2} \ln \left( \frac{\vec{r}_2}{r_{03}^2} \right) + A_{13} \frac{\vec{r}_2}{r_{01}^2} \ln \left( \frac{\vec{r}_2}{r_{01}^2} \right) \right) + A_{23} \frac{\vec{r}_2}{r_{03}^2} \ln \left( \frac{\vec{r}_2}{r_{03}^2} \right) + A_{14} \frac{\vec{r}_2}{r_{01}^2} \ln \left( \frac{\vec{r}_2}{r_{01}^2} \right) + A_{24} \frac{\vec{r}_2}{r_{01}^2} \ln \left( \frac{\vec{r}_2}{r_{01}^2} \right) \right] \\
+ (A_{12} \frac{\vec{r}_2}{r_{01}^2} \ln \left( \frac{\vec{r}_2}{r_{01}^2} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right] \\
= \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \tilde{G}_{\text{finite}} \quad (6.6)
\end{align}

After simplification one has

\begin{align}
\frac{\partial B_{ij}^{\text{conf}}}{\partial \eta} &= \frac{\alpha_s}{4\pi^2} \int d\vec{r}_4 \left[ \frac{\vec{r}_2}{r_{14}^2} \left( -B_{123} + \frac{1}{6} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214}) \right) \right] \ln \left( \frac{\vec{r}_2}{r_{14}^2} \right) \\
&+ (1 \leftrightarrow 3) + (2 \leftrightarrow 3) + \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \tilde{G}_{\text{finite}} \quad (6.7)
\end{align}

Now we can symmetrize the last 3 lines of this expression w.r.t. 0 \leftrightarrow 4 transformation, i.e.

\begin{align}
A_{ij} F(\vec{r}_i) &\rightarrow \left[ A_{ij} F(\vec{r}_i) \right]^{\text{symm}} \\
&= \frac{[A_{ij} + A_{ij} (0 \leftrightarrow 4)] [F + F (0 \leftrightarrow 4)] + [A_{ij} - A_{ij} (0 \leftrightarrow 4)] [F - F (0 \leftrightarrow 4)]}{4} \\
&\quad (6.8)
\end{align}

After that one can use (B.9) to show that all the nonconformal terms have the SU(3) coefficients independent either of \vec{r}_i or of \vec{r}_0.

So first we add the symmetrized last 3 lines of the previous expression to the nonconformal part of \tilde{G}_{\text{finite}} (4.20). Taking into account (B.3), (B.9), and (B.13), we have

\begin{align}
- \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \tilde{G} &= \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left\{ \left( M_{13} - M_{12} + M_{23} + M_2 \right) \left( U_0 U_1 U_3 \right) \cdot \left( U_2 U_0 U_1 \right) \cdot U_4 \\
&+ (U_1 U_0 \cdot \left( U_3 U_4 \cdot U_0 \right) \cdot U_4) + (all \ permutations \, 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4)
\end{align}

\end{document}
Then, it is easier to transform (4.30) using integral (116) from [4]. We use it in the symmetric form

\[ A_{23} \begin{pmatrix} r_{12}^2 r_{13}^2 & r_{14}^2 r_{24}^2 \\ r_{34}^2 r_{14}^2 & r_{34}^2 r_{12}^2 \end{pmatrix} + A_{14} \begin{pmatrix} r_{14}^2 r_{24}^2 \\ r_{14}^2 r_{24}^2 \end{pmatrix} + A_{24} \begin{pmatrix} r_{24}^2 r_{12}^2 \\ r_{24}^2 r_{12}^2 \end{pmatrix} \]

\[ + A_{13} \begin{pmatrix} r_{13}^2 r_{23}^2 \\ r_{13}^2 r_{23}^2 \end{pmatrix} \] (1) + (2) \[
= -\frac{\alpha_s^2}{8\pi^4} \int d\mathbf{r}_0 d\mathbf{r}_4 \left\{ \left( \frac{\tilde{r}_{12}^4}{8\tilde{r}_{02}^2\tilde{r}_{03}^2\tilde{r}_{14}^2} \right) \left( \frac{\tilde{r}_{12}^4}{\tilde{r}_{14}^2\tilde{r}_{24}^2} \right) \left( \frac{1}{2} B_{003B012} - 2B_{001B023} \right) + \frac{\tilde{r}_{12}^2}{8\tilde{r}_{02}^2\tilde{r}_{04}^2\tilde{r}_{14}^2} \right\} \]

\[ \times \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \tilde{r}_{14}^2\tilde{r}_{24}^2 \right) \]

\[ + \ln \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{1}{2} B_{003B012} - 2B_{001B023} \right) \]

\[ + \ln \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{1}{2} B_{003B012} - 2B_{001B023} \right) \]

\[ + \ln \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{\tilde{r}_{04}^2\tilde{r}_{02}^2\tilde{r}_{34}^2}{\tilde{r}_{03}^2\tilde{r}_{04}^2\tilde{r}_{12}^2} \right) \left( \frac{1}{2} B_{003B012} - 2B_{001B023} \right) \]

Indeed, in this expression all the nonconformal terms have the $SU(3)$ coefficients independent either of $\tilde{r}_4$ or of $\tilde{r}_0$. Therefore one can integrate them w.r.t. $\tilde{r}_4$ or $\tilde{r}_0$. However, it is easier to transform (4.30) using integral (116) from [4]. We use it in the symmetric form

\[ \frac{\tilde{r}_{12}^2}{\tilde{r}_{02}^2\tilde{r}_{03}^2\tilde{r}_{14}^2} \ln \frac{\tilde{r}_{10}^2}{\tilde{r}_{12}^2} \ln \frac{\tilde{r}_{20}^2}{\tilde{r}_{12}^2} = \frac{2\pi \zeta (3)}{(3)} \left( \delta (\tilde{r}_{10}) + \delta (\tilde{r}_{20}) \right) \]

\[ + \frac{\tilde{r}_{12}^2}{\tilde{r}_{02}^2\tilde{r}_{03}^2\tilde{r}_{14}^2} \int d\mathbf{r}_4 \left( \frac{\tilde{r}_{20}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} + \frac{\tilde{r}_{10}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} - \frac{\tilde{r}_{12}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} \right) \ln \left( \frac{\tilde{r}_{10}^2\tilde{r}_{20}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} \right) \] (6.10)

Then,

\[ G' = \frac{1}{2} \left[ \frac{\tilde{r}_{13}^2\tilde{r}_{20}^2}{\tilde{r}_{30}^2\tilde{r}_{12}^2} + \frac{\tilde{r}_{32}^2\tilde{r}_{10}^2}{\tilde{r}_{30}^2\tilde{r}_{12}^2} \right] \int d\mathbf{r}_4 \left( \frac{\tilde{r}_{20}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} + \frac{\tilde{r}_{10}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} - \frac{\tilde{r}_{12}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} \right) \ln \left( \frac{\tilde{r}_{10}^2\tilde{r}_{20}^2}{\tilde{r}_{04}^2\tilde{r}_{14}^2} \right) \]
Next, we symmetrize the previous expression w.r.t. \(0\).

One can write it as

\[
\times (B_{100}B_{320} - B_{200}B_{310})
+ \left[ \frac{r_{12}^2}{r_{10}^2} \right] \int \frac{d^4 \tilde{r}}{2\pi} \left( \frac{r_{10}^2}{r_{01}^2} + \frac{r_{12}^2}{r_{14}^2} \right) \ln \left( \frac{r_{12}^2}{r_{14}^2} \right)
\]

\[
\times \left( 9B_{123} - \frac{1}{2} \left[ 2 (B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \right] \right)
- 2\pi \zeta (3) \left( \delta (\tilde{r}_{10}) + \delta (\tilde{r}_{20}) \right) \left( 9B_{123} - \frac{1}{2} \left[ 2 (B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \right] \right)
+ \delta (\tilde{r}_{12}) \left( 9B_{123} - \frac{1}{2} \left[ 2 (B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \right] \right)
\]

One can write it as

\[
G' = \left[ \frac{r_{12}^2}{r_{10}^2} \right] \int \frac{d^4 \tilde{r}}{2\pi} \left( \frac{r_{10}^2}{r_{01}^2} + \frac{r_{12}^2}{r_{14}^2} \right) \ln \left( \frac{r_{12}^2}{r_{14}^2} \right)
\]

\[
\times \left( 3B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210} \right) - \frac{1}{2} \left[ 2 (B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \right]
\]

\[
+ \frac{1}{2} \left[ \left( 9B_{123} - \frac{1}{2} \left[ 2 (B_{100}B_{320} + B_{200}B_{130}) - B_{300}B_{120} \right] \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right].
\]
it can be transformed to

\[
\langle K_{NLO} \otimes B_{123}^{(2)} \rangle = -\frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 d\vec{r}_4 \left( \left\{ \tilde{L}_{12}^C \left( U_0 U_1^\dagger U_2 \right) \cdot \left( U_1 U_2^\dagger U_3 \right) \cdot U_3 + \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] \right. \right.
\]

\[\left. \left. + M_{12}^C \left( U_0 U_1^\dagger U_3 \right) \cdot \left( U_2 U_3^\dagger U_4 \right) \cdot U_4 + \left( U_1 U_2^\dagger U_3 \right) \cdot \left( U_3 U_4^\dagger U_0 \right) \cdot U_4 \right\} + Z_{12} B_{003} B_{012} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right) \right] + (0 \leftrightarrow 4) \right) \right)
\]

\[\left( \frac{3}{2} \left( B_{100} B_{230} + B_{200} B_{130} - B_{300} B_{210} \right) - 9 B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right) \right) . \quad (6.15)
\]

Here

\[
L_{12}^C = L_{12} + \frac{\vec{r}_{\theta0}^2}{4\vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) + \frac{\vec{r}_{\theta2}^2}{4\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) , \quad (6.16)
\]

\[
\tilde{L}_{12}^C = \tilde{L}_{12} + \frac{\vec{r}_{\theta2}^2}{4\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) - \frac{\vec{r}_{\theta2}^2}{4\vec{r}_{\theta0}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) , \quad (6.17)
\]

\[
M_{12}^C = \frac{\vec{r}_{\theta2}^2}{16\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) + \frac{\vec{r}_{\theta2}^2}{16\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) + \frac{\vec{r}_{\theta2}^2}{16\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) + \frac{\vec{r}_{\theta2}^2}{16\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) + \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{8\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) + \frac{\vec{r}_{\theta2}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2}{8\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) + \frac{\vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{8\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) \right) . \quad (6.18)
\]

\[
Z_{12} = \frac{\vec{r}_{\theta2}^2}{8\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2} \left[ \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} - \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2} \right) \ln \left( \frac{\vec{r}_{\theta2}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) \right] + \frac{\vec{r}_{\theta0}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) + \frac{\vec{r}_{\theta1}^2}{\vec{r}_{\theta2} \vec{r}_{\theta4}^2} \ln \left( \frac{\vec{r}_{\theta0}^2 \vec{r}_{\theta2}^2 \vec{r}_{\theta4}^2}{\vec{r}_{\theta0}^2 \vec{r}_{\theta1}^2 \vec{r}_{\theta2}^2} \right) \right) - (1 \leftrightarrow 3) . \quad (6.19)
\]

and $L_{12}$ and $\tilde{L}_{12}$ are the elements of the nonconformal kernel defined in (4.9) and (4.10). Checking that $L_{12}^C$, $\tilde{L}_{12}^C$, $M_{12}^C$, and $Z_{12}$ have integrable singularities at $\vec{r}_2 = \vec{r}_0$ and that $L_{12}^C$, $\tilde{L}_{12}^C$, and $Z_{12}$ have integrable singularities at $\vec{r}_4 = \vec{r}_{1,2,3}$ is straightforward. To prove that all the terms with $M_{12}^C$ have safe behavior at $\vec{r}_4 = \vec{r}_{1,2,3}$ one has to use $SU(3)$ identity (B.14).
Now one can see that the NLO kernel for the evolution equation for the composite 3QWL operator $B_{123}^{conf}$ (6.2) is quasi-conformal if one expresses the LO kernel in terms of composite operator (6.5).

The term with $Z$ can be integrated w.r.t. $\vec{r}_4$. The integral is calculated in the appendix D

$$\int \frac{d\vec{r}_4}{\pi} Z_{12} = \frac{\vec{r}_{32}^2}{8r_{02}^2r_{03}^2} \ln^2 \left( \frac{\vec{r}_{32} \cdot \vec{r}_{10}}{\vec{r}_{13} \cdot \vec{r}_{20}} \right) - \frac{\vec{r}_{12}^2}{8r_{01}^2r_{02}^2} \ln^2 \left( \frac{\vec{r}_{12} \cdot \vec{r}_{30}}{\vec{r}_{13} \cdot \vec{r}_{20}} \right).$$  \hspace{1cm} (6.20)

Finally, the kernel reads

$$\langle K_{NLO} \otimes B_{123}^{conf} \rangle = -\frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 d\vec{r}_4 \left\{ \left\{ \tilde{L}^C_{12} \left( U_0 U_4 \dagger U_2 \right) \cdot \left( U_1 U_0 \dagger U_4 \right) \cdot U_3 - \frac{3}{4} [B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124}] + \frac{1}{2} B_{123} \right\} + \left( \text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3 \right) \right\} + (0 \leftrightarrow 4)$$

$$-\frac{\alpha_s^2}{32\pi^3} \int d\vec{r}_0 \left( B_{003} B_{012} \left[ \frac{\vec{r}_{32}^2}{r_{03}^2r_{02}^2} \ln^2 \left( \frac{\vec{r}_{32} \cdot \vec{r}_{10}}{\vec{r}_{13} \cdot \vec{r}_{20}} \right) - \frac{\vec{r}_{12}^2}{r_{01}^2r_{02}^2} \ln^2 \left( \frac{\vec{r}_{12} \cdot \vec{r}_{30}}{\vec{r}_{13} \cdot \vec{r}_{20}} \right) \right] \right) + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3), \hspace{1cm} (6.21)$$

In the quark-diquark limit $\vec{r}_3 \to \vec{r}_2$ one has

$$\left\{ M^C_{12} \left[ \left( U_0 U_4 \dagger U_3 \right) \cdot \left( U_2 U_0 \dagger U_1 \right) \cdot U_4 + \left( U_1 U_0 \dagger U_2 \right) \cdot \left( U_3 U_4 \dagger U_0 \right) \cdot U_4 \right] + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4)$$

$$\to 2 \tilde{L}^C_{12} \left[ \text{tr} \left( U_0 \dagger U_4 \right) \text{tr} \left( U_2 \dagger U_0 U_4 \dagger U_1 \right) + \text{tr} \left( U_3 \dagger U_1 U_4 \dagger U_0 \right) \right] + 2 \text{tr} \left( U_0 \dagger U_1 \right) \text{tr} \left( U_2 \dagger U_4 \right) \text{tr} \left( U_4 \dagger U_0 \right) - (0 \leftrightarrow 4), \hspace{1cm} (6.22)$$

$$\left\{ \tilde{L}^C_{12} \left( U_0 U_4 \dagger U_2 \right) \cdot \left( U_1 U_0 \dagger U_4 \right) \cdot U_3 + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4)$$

$$\to 2 \tilde{L}^C_{12} \left[ \text{tr} \left( U_4 \dagger U_0 \right) \left( \text{tr} \left( U_0 \dagger U_1 U_2 \dagger U_4 \right) + \text{tr} \left( U_0 \dagger U_4 U_2 \dagger U_1 \right) \right) \right] - (0 \leftrightarrow 4), \hspace{1cm} (6.23)$$

$$\tilde{L}^C_{12} \left[ \left( U_0 U_4 \dagger U_2 \right) \cdot \left( U_1 U_0 \dagger U_4 \right) \cdot U_3 + \text{tr} \left( U_0 U_4 \dagger \right) \left( U_1 U_0 \dagger U_2 \right) \cdot U_3 \cdot U_4 + \frac{1}{2} B_{123}}$$
Using identity (B.3) and the fact that in the 3-gluon approximation

\[
- \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) + (0 \leftrightarrow 4)
\]

\[
\to 4L_{12}^C \left[ tr \left( U_{2}^\dagger U_{1} \right) - 3tr \left( U_{0}^\dagger U_{1} \right) tr \left( U_{2}^\dagger U_{0} \right) + tr \left( U_{0}^\dagger U_{1} \right) tr \left( U_{2}^\dagger U_{4} \right) tr \left( U_{4}^\dagger U_{0} \right) - tr \left( U_{0}^\dagger U_{1} U_{4}^\dagger U_{0} U_{2}^\dagger U_{4} \right) + (0 \leftrightarrow 4) \right].
\]

Therefore,

\[
\langle k_{NLO} \otimes B_{122}^{conf} \rangle = - \frac{\alpha_s^2}{2\pi^4} \int d\vec{r}_0 d\vec{r}_1 \left\{ \left\{ \left( \hat{L}_{12}^C + L_{12}^C \right) tr \left( U_{0}^\dagger U_{1} \right) tr \left( U_{2}^\dagger U_{4} \right) tr \left( U_{4}^\dagger U_{0} \right) \\
+ L_{12}^C \left[ tr \left( U_{2}^\dagger U_{1} \right) - 3tr \left( U_{0}^\dagger U_{1} \right) tr \left( U_{2}^\dagger U_{0} \right) - tr \left( U_{0}^\dagger U_{1} U_{4}^\dagger U_{0} U_{2}^\dagger U_{4} \right) \right] \right\} + (0 \leftrightarrow 4)
\]

\[
- \frac{3\alpha_s^2}{2\pi^4} \int d\vec{r}_0 \frac{11}{6} \left[ \ln \left( \frac{\vec{r}^2_{02}}{\vec{r}^2_{01}} \right) \left( \frac{1}{\vec{r}^2_{02}} - \frac{1}{\vec{r}^2_{01}} \right) - \frac{\vec{r}^2_{12}}{\vec{r}^2_{02}} \ln \left( \frac{\vec{r}^2_{12}}{\vec{r}^2_{02}} \right) \right] \times \left( tr \left( U_{0}^\dagger U_{1} \right) tr \left( U_{2}^\dagger U_{0} \right) - 3tr \left( U_{2}^\dagger U_{1} \right) \right) .
\]

This is twice the gluon part of the BK kernel (see (67) in [4]).

7 Linearization

In the 3-gluon approximation

\[
B_{003} B_{012} = 6B_{003} + 6B_{012} - 36.
\]

We use the following identity to linearize the color structures in (6.21).

\[
(U_0 U_1^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 + (1 \leftrightarrow 2, 0 \leftrightarrow 4) =
\]

\[
= (U_0 U_4^\dagger - E)(U_2 - U_4) \cdot (U_1 - U_0) U_0^\dagger U_4 \cdot U_3 + U_4 U_0^\dagger (U_1 - U_0) \cdot (U_2 - U_4) (U_1^\dagger U_0 - E) \cdot U_3
\]

\[
+ U_0 \cdot (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_3 + (U_0 U_1^\dagger U_2 + U_2 U_1^\dagger U_0) \cdot U_4 \cdot U_3
\]

\[
+ (U_2 - U_4) \cdot (U_1 - U_0) (U_0^\dagger U_4 - E) \cdot U_3 + (U_4 U_0^\dagger - E)(U_1 - U_0) \cdot (U_2 - U_4) \cdot U_3
\]

\[
+ 2(U_2 - U_4) \cdot (U_1 - U_0) \cdot U_3 - 2U_0 \cdot U_4 \cdot U_3.
\]

Here \(E\) is the identity matrix. In the 3-gluon approximation it reads

\[
(U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 + (1 \leftrightarrow 2, 0 \leftrightarrow 4)
\]

\[
= (U_0 - U_4)(U_2 - U_4) \cdot (U_1 - U_0) \cdot E + (U_1 - U_0)(U_2 - U_4)(U_0 - U_4) \cdot E
\]

\[
+ U_0 \cdot (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_3 + (U_0 U_1^\dagger U_2 + U_2 U_1^\dagger U_0) \cdot U_4 \cdot U_3
\]

\[
+ (U_2 - U_4) \cdot (U_1 - U_0)(U_4 - U_0) \cdot E + (U_4 - U_0)(U_1 - U_0) \cdot (U_2 - U_4) \cdot E
\]

\[
+ 2(U_2 - U_4) \cdot (U_1 - U_0) \cdot U_3 - 2U_0 \cdot U_4 \cdot U_3.
\]

Using identity (B.3) and the fact that in the 3-gluon approximation

\[
((U_0 - U_4)(U_2 - U_4) + (U_2 - U_4)(U_0 - U_4))(U_1 - U_0) \cdot E
\]
\[
3g \equiv -(U_2 - U_4) \cdot (U_0 - U_4) \cdot (U_1 - U_0), \quad (7.4)
\]
we get
\[
(U_0 U_4^\dagger U_2) \cdot (U_1 U_0^\dagger U_4) \cdot U_3 + (1 \leftrightarrow 2, 0 \leftrightarrow 4)
\]
\[
3g = -B_{134} + \frac{1}{2}(B_{100}B_{340} + B_{400}B_{130} - B_{300}B_{140})
- B_{023} + \frac{1}{2}(B_{044}B_{234} + B_{244}B_{034} - B_{344}B_{024})
+ 2(U_2 - U_4) \cdot (U_1 - U_0) \cdot U_3 - 2U_0 \cdot U_4 \cdot U_3
\]
\[
= B_{123} - 3B_{134} + \frac{1}{2}(B_{100}B_{340} + B_{400}B_{130} - B_{300}B_{140}) + (1 \leftrightarrow 2, 0 \leftrightarrow 4)
\]
\[
3g = B_{123} + 3(B_{100} + B_{340} + B_{400} + B_{130} - B_{300} - B_{140} - B_{134} - 6) + (1 \leftrightarrow 2, 0 \leftrightarrow 4). \quad (7.5)
\]
As a result the coefficient of $\tilde{L}_C^{12}$ in (6.21) reads
\[
\left(\left(U_0 U_4^\dagger U_2\right) \cdot \left(U_1 U_0^\dagger U_4\right) \cdot U_3 + (1 \leftrightarrow 2, 0 \leftrightarrow 4)\right) - (0 \leftrightarrow 4)
\]
\[
3g = (3B_{001} + 6B_{130} - (1 \leftrightarrow 2)) - (0 \leftrightarrow 4). \quad (7.6)
\]
Using integrals (114) and (125) from [4],
\[
\int d\tilde{r}_4 \tilde{L}_{12} = \frac{\pi^2}{2} \zeta (3) (\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})) ,
\]
and
\[
\int d\tilde{r}_4 \left[\frac{\tilde{r}_1^2}{4\tilde{r}_0^2\tilde{r}_4^2\tilde{r}_{24}^2} \ln \left(\frac{\tilde{r}_0^2\tilde{r}_{14}^2}{\tilde{r}_4^2\tilde{r}_{12}^2}\right) - \frac{\tilde{r}_2^2}{4\tilde{r}_0^2\tilde{r}_4^2\tilde{r}_{14}^2} \ln \left(\frac{\tilde{r}_0^2\tilde{r}_{24}^2}{\tilde{r}_4^2\tilde{r}_{12}^2}\right)\right]
\]
\[
= \pi^2 \zeta (3) (\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})) ,
\]
one has
\[
\int d\tilde{r}_4 \tilde{L}_C^{12} = \frac{3}{2} \pi^2 \zeta (3) (\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})), \quad (7.9)
\]
and
\[
- \frac{\alpha^2}{8\pi^4} \int d\tilde{r}_0 d\tilde{r}_1 \left\{ \left(\tilde{L}_C^{12} \left(U_0 U_4^\dagger U_2\right) \cdot \left(U_1 U_0^\dagger U_4\right) \cdot U_3\right)
+ \text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3 \right\} + (0 \leftrightarrow 4) \right) \]
\[
3g - \frac{\alpha^2}{8\pi^4} \int d\tilde{r}_0 (3B_{001} + 6B_{130} - (1 \leftrightarrow 2)) 3\pi^2 \zeta (3) (\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)
\]
\[
= - \frac{9\alpha^2}{8\pi^4} \zeta (3) (36 + B_{131} + B_{133} + B_{121} + B_{212} + B_{232} + B_{233} - 12B_{231}). \quad (7.10)
\]
The second structure reads
\[
\left(U_1 U_0^\dagger U_2 + U_2 U_0^\dagger U_1\right) \cdot U_3 \cdot U_4
\]
\[
= (U_1 - U_0) U_0^\dagger (U_2 - U_0) \cdot U_4 \cdot (U_4 - U_0) + (U_2 - U_0) U_0^\dagger (U_1 - U_0) \cdot U_3 \cdot (U_4 - U_0)
\]
\[
= -(U_2 - U_4) \cdot (U_0 - U_4) \cdot (U_1 - U_0), \quad (7.11)
\]
\[ + 2(U_2 + U_1-U_0) \cdot U_3 \cdot (U_4-U_0) + \left( U_1 U_0^\dagger U_2 + U_2 U_0^\dagger U_1 \right) \cdot U_3 \cdot U_0. \]  

(7.11)

Again, applying identity (B.3) and equality (7.4) one gets in the 3-gluon approximation

\[
\left( U_1 U_0^\dagger U_2 + U_2 U_0^\dagger U_1 \right) \cdot U_3 \cdot U_4 \equiv -(U_1 - U_0) \cdot (U_2 - U_0) \cdot (U_4 - U_0)
\]

\[ + 2(U_2 + U_1-U_0) \cdot U_3 \cdot (U_4-U_0) - B_{123} + \frac{1}{2} \left( B_{100} B_{230} + B_{200} B_{130} - B_{300} B_{120} \right). \]  

(7.12)

Finally, the coefficient of \( L_{12}^C \) in (6.21) reads

\[
\left[ \left( U_0 U_4^\dagger U_2 \right) - \left( U_1 U_0^\dagger U_4 \right) \right) \cdot U_3 + tr \left( U_0 U_4^\dagger \right) \left( U_1 U_0^\dagger U_2 \right) \cdot U_3 \cdot U_4
\]

\[- \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] + \frac{1}{2} B_{123} + (1 \leftrightarrow 2) \right] + (0 \leftrightarrow 4)
\]

\[ \equiv 9(B_{044} + B_{004} - 12). \]  

(7.13)

Therefore,

\[
- \frac{\alpha_s^2}{8\pi^2} \int d\vec{r}_0 d\vec{r}_4 \left\{ L_{12}^C \left[ \left( U_0 U_4^\dagger U_2 \right) - \left( U_1 U_0^\dagger U_4 \right) \right) \cdot U_3 + tr \left( U_0 U_4^\dagger \right) \left( U_1 U_0^\dagger U_2 \right) \cdot U_3 \cdot U_4
\]

\[- \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] + \frac{1}{2} B_{123} + (1 \leftrightarrow 2) \right] + (0 \leftrightarrow 4) \right] \]

\[ \equiv - \frac{9\alpha_s^2}{8\pi^2} \int d\vec{r}_0 d\vec{r}_4 (L_{12}^C + L_{13}^C + L_{23}^C)(B_{044} + B_{004} - 12). \]  

(7.14)

The third structure reads

\[
(U_2 U_0^\dagger U_1) \cdot U_4 \cdot (U_0 U_4^\dagger U_3) + (U_1 U_0^\dagger U_2) \cdot U_4 \cdot (U_3 U_4^\dagger U_0)
\]

\[= U_1 \cdot U_4 \cdot (U_3 U_4^\dagger U_0 + U_3 U_4^\dagger U_0) + (U_1 U_0^\dagger U_2 + U_2 U_0^\dagger U_1) \cdot U_4 \cdot U_0
\]

\[+ (U_2 - U_0) \cdot U_4 \cdot ((U_0 - U_4) U_1^\dagger (U_3 - U_4) + (U_3 - U_4) U_1^\dagger (U_0 - U_4))
\]

\[+ (U_1 - U_0) U_1^\dagger (U_2 - U_0) \cdot U_4 \cdot ((U_3 - U_4) U_1^\dagger U_0)
\]

\[+ (U_2 - U_0) U_1^\dagger (U_1 - U_0) \cdot U_4 \cdot (U_0 U_3^\dagger (U_3 - U_4))
\]

\[+ 2(U_2 - U_0) \cdot U_4 \cdot (U_3 - U_4) - 2U_1 \cdot U_4 \cdot U_0. \]  

(7.15)

Using (B.3) and (7.4),

\[
(U_2 U_0^\dagger U_1) \cdot U_4 \cdot (U_0 U_4^\dagger U_3) + (U_1 U_0^\dagger U_2) \cdot U_4 \cdot (U_3 U_4^\dagger U_0)
\]

\[\equiv - B_{013} + \frac{1}{2} \left( B_{344} B_{014} + B_{014} B_{134} - B_{144} B_{034} \right) - B_{124} + \frac{1}{2} \left( B_{100} B_{240} + B_{200} B_{140} - B_{004} B_{012} \right)
\]

\[- (U_2 - U_0) \cdot (U_1 - 3 U_4) \cdot (U_3 - U_4) - 2U_1 \cdot U_4 \cdot U_0
\]

\[\equiv 3(B_{010} - B_{441} + B_{020} - B_{442} - B_{040} + 2B_{440} - B_{120} + B_{140})
\]

\[\equiv - B_{013} + \frac{1}{2} \left( B_{344} B_{014} + B_{014} B_{134} - B_{144} B_{034} \right) - B_{124} + \frac{1}{2} \left( B_{100} B_{240} + B_{200} B_{140} - B_{004} B_{012} \right)
\]

\[- (U_2 - U_0) \cdot (U_1 - 3 U_4) \cdot (U_3 - U_4) - 2U_1 \cdot U_4 \cdot U_0
\]

\[\equiv 3(B_{010} - B_{441} + B_{020} - B_{442} - B_{040} + 2B_{440} - B_{120} + B_{140})
\]
\[ + B_{341} + B_{240} - 2B_{340} + B_{342} + B_{443} - B_{231} - 36. \]  

(7.16)

As a result

\[- \frac{\alpha^2}{8\pi^2} \int \! d\vec{r}_0 \! d\vec{r}_4 \left\{ \left( \mathcal{M}_{12}^C \left[ (U_2U_0') U_3' \right] \cdot (U_1U_0') U_4 \right) + \left( U_1U_0'U_2' \right) \cdot (U_3U_4') U_0 \right\} \]

\[ + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \bigg\} + (0 \leftrightarrow 4) \bigg\} \]

\[ \approx \frac{3\alpha^2}{32\pi^2} \int \! d\vec{r}_0 \! d\vec{r}_4 \left( \frac{3}{2} F_0(B_{040} - B_{044}) + \left\{ \frac{3}{2} F_{140} B_{140} + F_{100} B_{100} + F_{230} B_{230} \right\} \right) \]

\[ + (0 \leftrightarrow 4) \bigg\} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \bigg\). \]  

(7.17)

Here

\[ F_0 = \frac{\vec{r}_{12}^2}{2r_{14}^2r_{24}^2} - \frac{\vec{r}_{12}^2}{2r_{02}^2r_{04}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{02}^2 \vec{r}_{34}^2}{r_{14}^2r_{24}^2r_{03}^2} \right) - \frac{\vec{r}_{13}^2}{2r_{01}^2r_{03}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{13}^2 \vec{r}_{24}^2}{r_{03}^2r_{12}^2r_{14}^2} \right) \]

\[ + \frac{2\vec{r}_{34}^2}{r_{03}^2r_{04}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{02}^2 \vec{r}_{34}^2}{r_{04}^2r_{14}^2r_{24}^2} \right) \]  

(7.18)

\[ F_{140} = \frac{\vec{r}_{12}^2}{2r_{02}^2r_{04}^2r_{14}^2} - \frac{\vec{r}_{12}^2}{2r_{02}^2r_{04}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{04}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{04}^2r_{14}^2r_{24}^2} \right) \]

\[ - \frac{\vec{r}_{12}^2}{2r_{02}^2r_{03}^2r_{04}^2r_{14}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{24}^2 \vec{r}_{34}^2}{r_{04}^2r_{14}^2r_{23}^2} \right) - \frac{\vec{r}_{23}^2 \vec{r}_{12}^2}{2r_{02}^2r_{03}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{03}^2r_{14}^2r_{24}^2} \right) \]

\[ + \frac{\vec{r}_{23}^2}{r_{03}^2r_{04}^2r_{24}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{34}^2}{r_{04}^2r_{23}^2} \right) + \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{2r_{03}^2r_{04}^2r_{24}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{02}^2 \vec{r}_{34}^2}{r_{03}^2r_{14}^2r_{23}^2} \right). \]  

(7.19)

\[ F_{100} = \frac{\vec{r}_{23}^2}{2r_{03}^2r_{04}^2r_{24}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{04}^2 \vec{r}_{23}^2 \vec{r}_{14}^2r_{34}^2}{r_{02}^2r_{03}^2r_{04}^2r_{24}^2} \right) - \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{2r_{01}^2r_{02}^2r_{03}^2r_{04}^2r_{24}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{13}^2 \vec{r}_{24}^2}{r_{02}^2r_{03}^2r_{04}^2r_{14}^2} \right) \]

\[ - \frac{\vec{r}_{03}^2 \vec{r}_{04}^2}{2r_{02}^2r_{03}^2r_{04}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{04}^2 \vec{r}_{12}^2 \vec{r}_{64}^2}{r_{03}^2r_{04}^2r_{12}^2r_{14}^2} \right) - \frac{\vec{r}_{12}^2}{2r_{02}^2r_{04}^2r_{14}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{4}^2}{r_{03}^2r_{04}^2r_{12}^2r_{14}^2} \right) \]

\[ + \frac{\vec{r}_{23}^2 \vec{r}_{12}^2}{2r_{02}^2r_{03}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{03}^2r_{14}^2r_{24}^2} \right) + \frac{\vec{r}_{02}^2 \vec{r}_{12}^2}{2r_{02}^2r_{03}^2r_{04}^2r_{14}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{03}^2r_{04}^2r_{12}^2r_{14}^2} \right) \]  

(7.20)

\[ F_{230} = \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{2r_{01}^2r_{03}^2r_{04}^2r_{24}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{04}^2 \vec{r}_{13}^2 \vec{r}_{24}^2}{r_{02}^2r_{03}^2r_{04}^2r_{14}^2} \right) - \frac{\vec{r}_{23}^2}{2r_{03}^2r_{04}^2r_{14}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{04}^2 r_{13}^2 \vec{r}_{24}^2}{r_{02}^2r_{03}^2r_{14}^2} \right) \]

\[ + \frac{\vec{r}_{34}^2 \vec{r}_{12}^2}{2r_{03}^2r_{04}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{04}^2 \vec{r}_{12}^2 \vec{r}_{84}^2}{r_{03}^2r_{04}^2r_{12}^2r_{14}^2} \right) + \frac{\vec{r}_{12}^2}{2r_{02}^2r_{04}^2r_{14}^2} \ln \left( \frac{\vec{r}_{01}^2 \vec{r}_{04}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{03}^2 \vec{r}_{04}^2 \vec{r}_{12}^2r_{14}^2} \right) \]

\[ - \frac{\vec{r}_{23}^2 \vec{r}_{12}^2}{r_{02}^2r_{03}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{03}^2r_{14}^2r_{24}^2} \right) - \frac{\vec{r}_{12}^2}{2r_{01}^2r_{04}^2r_{14}^2r_{24}^2} \ln \left( \frac{\vec{r}_{02}^2 \vec{r}_{12}^2 \vec{r}_{34}^2}{r_{04}^2r_{14}^2r_{23}^2} \right). \]  

(7.21)
One can integrate $F_{100}$ and $F_{200}$ w.r.t. $\tilde{r}_4$. The integrals are given in appendix D (D.23) and (D.33).

The color structure in the quark part of the kernel can be linearized via (7.12)

$$\frac{1}{2} \left\{ \left( \frac{1}{3} (U_1 U_0) U_4 + U_4 U_0 U_1 \right) \cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0^\dagger U_4) + (U_1 U_0) U_2 \cdot U_3 \cdot U_4 \right. $$

$$+ \frac{1}{6} B_{123} - \frac{1}{4} (B_{013} B_{002} + B_{001} B_{023} - B_{012} B_{003}) + (1 \leftrightarrow 2) \right\} + (0 \leftrightarrow 4) \right\}

Then

$$\delta \frac{1}{6} (12 - B_{004} - B_{044} + 2 (2B_{014} - B_{001} - B_{144})$$

$$+ 2 (2B_{024} - B_{002} - B_{244}) - 4 (2B_{034} - B_{344} - B_{003}) \right) \right\}. \quad (7.22)$$

Therefore

$$\frac{-\alpha_s^2 n_f}{8\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \ G^q \equiv -\frac{\alpha_s^2 n_f}{48\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \ \{(12 - B_{004} - B_{044}) (L_{12}^q + L_{13}^q + L_{23}^q)$$

$$+ 2 (2B_{014} - B_{001} - B_{144}) (L_{12}^q + L_{13}^q + L_{23}^q) + 2 (2B_{024} - B_{002} - B_{244}) (L_{12}^q + L_{25}^q - 2L_{31}^q)$$

$$+ 2 (2B_{034} - B_{344} - B_{003}) (L_{32}^q + L_{13}^q - 2L_{12}^q) \right\}. \quad (7.23)$$

Putting things together we have for the linearized kernel

$$\langle K_{NLO} \otimes B_{123}^\text{conf} \rangle \equiv \frac{2\pi^2}{4\pi^2} \xi (3 - \delta_{23} - \delta_{13} - \delta_{21}) (B_{123} - 6)$$

$$- \frac{9\alpha_s^2}{8\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \left( L_{12}^C + L_{13}^C + L_{23}^C \frac{n_f}{54} (L_{12}^q + L_{13}^q + L_{23}^q) \right) (B_{044} + B_{004} - 12)$$

$$- \frac{\alpha_s^2 n_f}{24\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \ \{(2B_{014} - B_{001} - B_{144}) (L_{12}^q + L_{13}^q + L_{23}^q) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \}$$

$$- \frac{-9\alpha_s^2}{64\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \ \left( F_{100} (B_{040} - B_{044}) + \{ F_{140} + (0 \leftrightarrow 4) \} \right) B_{140} + (all\ 5\ \text{perm.} \ 1 \leftrightarrow 2 \leftrightarrow 3) + \left( B_{100} + B_{230} + B_{200} + B_{130} - B_{300} - B_{210} - B_{123} - 6 \right) \right.$$
\[
\frac{\bar{r}_{12}^2}{8r_{04}^2} \left( \frac{1}{r_{02}^2 r_{14}^2} - \frac{1}{r_{01}^2 r_{24}^2} \right) \ln \left( \frac{r_{01}^2 r_{24}^2}{r_{14}^2 r_{02}^2} \right) + \frac{1}{2r_{04}^2} \tag{7.25}
\]

and

\[
\bar{F}_{100} = \left( \frac{\bar{r}_{12}^2}{r_{01}^2 r_{02}^2} - \frac{\bar{r}_{14}^2}{r_{02}^2 r_{03}^2} - \frac{2\bar{r}_{23}^2}{r_{02}^2 r_{03}^2} \right) \ln^2 \left( \frac{r_{02}^2 r_{14}^2}{r_{01}^2 r_{23}^2} \right) + \frac{\bar{r}_{23}^2}{2r_{02}^2 r_{03}^2} \ln^2 \left( \frac{r_{03}^2 r_{12}^2}{r_{02}^2 r_{13}^2} \right) \\
+ \tilde{S}_{123} I \left( \frac{\bar{r}_{12}^2}{r_{01}^2 r_{02}^2}, \frac{\bar{r}_{14}^2}{r_{02}^2 r_{03}^2}, \frac{\bar{r}_{23}^2}{r_{02}^2 r_{03}^2} \right) + (2 \leftrightarrow 3), \tag{7.26}
\]

\[
\bar{F}_{230} = \left( \frac{2\bar{r}_{12}^2}{r_{01}^2 r_{02}^2} - \frac{\bar{r}_{23}^2}{r_{02}^2 r_{03}^2} \right) \ln^2 \left( \frac{r_{03}^2 r_{12}^2}{r_{01}^2 r_{23}^2} \right) + \left( \frac{\bar{r}_{12}^2}{r_{01}^2 r_{02}^2} - \frac{\bar{r}_{13}^2}{r_{01}^2 r_{03}^2} \right) \ln^2 \left( \frac{r_{02}^2 r_{13}^2}{r_{01}^2 r_{23}^2} \right) \\
- \tilde{S}_{123} I \left( \frac{\bar{r}_{12}^2}{r_{01}^2 r_{02}^2}, \frac{\bar{r}_{14}^2}{r_{02}^2 r_{03}^2}, \frac{\bar{r}_{23}^2}{r_{02}^2 r_{03}^2} \right) + (2 \leftrightarrow 3). \tag{7.27}
\]

The functions \( \tilde{S}_{123} \) and \( I \) are defined in appendix D (D.16) and (D.12). If we consider the dipole limit \( \bar{r}_3 = \bar{r}_2 \) and take into account that in this limit

\[
\bar{F}_{230} |_{\bar{r}_3=\bar{r}_2} = \bar{F}_{120} |_{\bar{r}_3=\bar{r}_2} = \bar{F}_{100} |_{\bar{r}_3=\bar{r}_2} = \bar{F}_{300} |_{\bar{r}_3=\bar{r}_2} = 0, \tag{7.28}
\]

\[
F_0 + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) |_{\bar{r}_3=\bar{r}_2} = -16L_{12}^C, \tag{7.29}
\]

\[
(F_{140} + (0 \leftrightarrow 4)) + (2 \leftrightarrow 3) |_{\bar{r}_3=\bar{r}_2} = 0,
\]

\[
(F_{240} + (0 \leftrightarrow 4)) + (1 \leftrightarrow 3) |_{\bar{r}_3=\bar{r}_2} = (F_{340} + (0 \leftrightarrow 4)) + (2 \leftrightarrow 1) |_{\bar{r}_3=\bar{r}_2} = 0, \tag{7.30}
\]

we have the linearized BK kernel in the 3-gluon approximation, whose C-even part is the BFKL kernel [18]

\[
\langle K_{NLO} \otimes B_{122}^{conf} \rangle \equiv -\frac{3g_6}{4\pi^3} \int d\bar{r}_0 d\bar{r}_1 \left( L_{12}^C - \frac{n_f}{54} L_{12}^q \right) (B_{044} + B_{004} - 12) \\
- \frac{\alpha_s^2 n_f}{12g^2} \int d\bar{r}_0 d\bar{r}_1 \left( (2B_{014} - B_{001} - B_{144}) - (2B_{024} - B_{002} - B_{244}) \right) L_{12}^q \\
+ \frac{27\alpha_s^2}{2\pi^2} \zeta(3) (B_{122} - 6) - \frac{9\alpha_s^2}{4\pi^2} \int d\bar{r}_0 d\bar{r}_1 \tilde{L}_{12}^C (B_{044} - B_{040}) \\
- \frac{9\alpha_s^2}{8\pi^3} \beta \int d\bar{r}_0 \left[ \ln \left( \frac{\bar{r}_{12}^2}{\bar{r}_{02}^2} \right) \left( \frac{1}{r_{01}^2} - \frac{1}{r_{02}^2} \right) - \frac{\bar{r}_{12}^2}{\bar{r}_{01}^2 r_{02}^2} \ln \left( \frac{\bar{r}_{12}^2}{\bar{r}_{01}^2 r_{02}^2} \right) \right] (B_{100} + B_{220} - B_{122} - 6). \tag{7.31}
\]

Let us compare this kernel for \( B_{122} = 2tr(U_1^a U_1^b) \) with the linearized BK kernel in the 2-gluon approximation from [4]. One can see that their C-even parts coincide as they are fixed by the BFKL kernel [18]. However the 2-gluon approximation is not enough to catch the correct C-odd part of the kernel. But the 3-gluon approximation (7.31) allows one to write it. At once we see that even for the color dipole the C-odd part of the kernel in the 3-gluon approximation cannot be expressed through dipoles only. One necessarily needs to introduce the 3QWL operators as is clear from the second line of this expression. One can check it by direct calculation via (7.12), indeed

\[
12 \left( tr(U_1^a t^a U_1^b) tr(U_1^a t^a U_0^b) - (0 \rightarrow 4) \right) + (0 \leftrightarrow 4)
\]
The NLO kernel for the C-even Green function in the 3-gluon approximation reads
\[
\frac{1}{3} \text{tr}(U_0 U_1^\dagger)\text{tr}(U_2^\dagger U_1) + 3\text{tr}(U_4^\dagger U_1)\text{tr}(U_2^\dagger U_0)
- \text{tr}(U_0 U_1^\dagger U_1 U_2^\dagger) - \text{tr}(U_0 U_2^\dagger U_1 U_4^\dagger) - (0 \rightarrow 4)
\]
\[
= \left\{ \frac{1}{12} B_{044} B_{122} + \frac{3}{4} B_{144} B_{022} - \frac{1}{2} \text{tr}(U_1 U_2^\dagger U_0 + U_0 U_2^\dagger U_1) \cdot U_4 \cdot U_4 - (0 \rightarrow 4) \right\} + (0 \leftrightarrow 4)
\]
\[
= \frac{1}{2} \left\{ 12 - B_{044} - B_{004} + 2(2B_{014} - B_{001} - B_{144}) - 2(2B_{024} - B_{002} - B_{214}) \right\}.
\]

As in [11] to separate the C-even and C-odd contributions we introduce C-even (pomeron) and C-odd (odderon) Green functions
\[
B_{123}^+ = B_{123} + B_{123} - 12,
\]
and
\[
B_{123}^- = B_{123} - B_{123},
\]
where \(B_{123}\) is the 3-antiquark Wilson loop operator
\[
B_{123} = U_1^\dagger \cdot U_2^\dagger \cdot U_3^\dagger.
\]

The NLO kernel for the C-even Green function in the 3-gluon approximation reads
\[
(K_{NLO} \otimes B_{123}^{+\text{conf}}) = -\frac{9\alpha_s^2}{4\pi^4} \int d\vec{r}_0 d\vec{r}_1 (L_{12}^C + L_{13}^C + L_{23}^C - \frac{n_f}{54} (L_{12}^q + L_{13}^q + L_{23}^q)) B_{044}^{4-}
+ \frac{27\alpha_s^2}{4\pi^2} \zeta(3)(3 - \delta_{23} - \delta_{13} - \delta_{21}) B_{123}^+
- \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \{ F_{140} + (0 \leftrightarrow 4) \} B_{140}^+ + \text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3 \right)
- \frac{9\alpha_s^2}{16\pi^4} \int d\vec{r}_0 \left( \tilde{F}_{1100} B_{1100}^- + \tilde{F}_{230} B_{230}^+ + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right)
- \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 \left( \beta \left[ \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{12}^2} \right) \left( \frac{1}{\vec{r}_{01}^2} - \frac{1}{\vec{r}_{12}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{12}^2} \ln \left( \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2} \right) \right] \right)
\times (B_{100}^+ + B_{230}^+ + B_{200}^+ + B_{130}^+ - B_{300}^+ - B_{210}^+ - B_{123}^+) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3).
\]

In the 3-gluon approximation [11]
\[
B_{123}^+ \equiv \frac{1}{2} (B_{133}^+ + B_{211}^+ + B_{322}^+),
\]
which kills all the terms in the third line in (7.24) in the C-even case. This equality also leads to the fact that for model (6.1)
\[
B_{123}^{+\text{conf}} \equiv \frac{1}{2} (B_{133}^{+\text{conf}} + B_{211}^{+\text{conf}} + B_{322}^{+\text{conf}})
\]
we have
\[
(K_{NLO} \otimes B_{123}^{+\text{conf}}) = \frac{1}{2} (K_{NLO} \otimes (B_{133}^{+\text{conf}} + B_{211}^{+\text{conf}} + B_{322}^{+\text{conf}})).
\]

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This equality imposes the following constraints

\[ 0 = \{ F_{140} + (0 \leftrightarrow 4) \} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3), \] (7.40)

\[ 0 = \int d\vec{r}_0 \tilde{F}_{230}, \] (7.41)

\[ 0 = \int d\vec{r}_4 \left( \{ F_{140} + (0 \leftrightarrow 4) \} + (2 \leftrightarrow 3) \right) + \tilde{F}_{100} + \frac{1}{2} \tilde{F}_{230|1\leftrightarrow 3} + \frac{1}{2} \tilde{F}_{230|1\leftrightarrow 2}. \] (7.42)

Constraint (7.40) follows from definition of \( F_{140} \) (7.19) directly. Constraint (7.41) holds since thanks to conformal invariance

\[ \int d\vec{r}_0 \tilde{F}_{230} = \int d\vec{r}_0 \tilde{F}_{230|2=3} = 0. \] (7.43)

Using (7.26) and (7.27) one can rewrite constraint (7.42) as

\[ \int d\vec{r}_4 \left( \{ F_{140} + (0 \leftrightarrow 4) \} + (2 \leftrightarrow 3) \right) = \frac{\vec{r}_{23}^2}{2r_{02}^2r_{03}^2} \left( \ln^2 \left( \frac{\vec{r}_{03}^2 r_{12}^2}{r_{01}^2 r_{23}^2} \right) + \ln^2 \left( \frac{\vec{r}_{02}^2 r_{13}^2}{r_{01}^2 r_{23}^2} \right) ight) \]

\[ - \frac{1}{2} \left( \frac{\vec{r}_{12}^2}{r_{02}^2 r_{03}^2} + \frac{\vec{r}_{13}^2}{r_{01}^2 r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{02}^2 r_{13}^2}{r_{01}^2 r_{23}^2} \right). \] (7.44)

The calculation of the integral and proof of this identity is given in appendix D.

The NLO kernel for the C-odd Green function in the 3-gluon approximation reads

\[ \langle K_{NLO} \otimes B_{123}^{\text{conf}} \rangle \equiv \frac{27\alpha_s^2}{4\pi^2} \zeta(3) (3 - \delta_{23} - \delta_{13} - \delta_{21}) B_{123} \]

\[ - \frac{\alpha_s^2 \mu_f}{24\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left\{ \left( 2B_{014}^- - B_{001}^- - B_{144}^- \right) \left( L_{12}^0 + L_{13}^0 - 2L_{32}^0 \right) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right\} \]

\[ - \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( 2F_0 B_{040}^- + \{ F_{140} + (0 \leftrightarrow 4) \} B_{140}^- + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right) \]

\[ - \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 \left( \vec{F}_{100} B_{100}^- + \tilde{F}_{230} B_{230}^- + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right) \]

\[ - \frac{9\alpha_s^2}{16\pi^4} \int d\vec{r}_0 \left( \beta \left[ \ln \left( \frac{r_{01}^2}{r_{02}^2} \right) \left( \frac{\vec{r}_{12}^2}{r_{01}^2 r_{02}^2} \right) + \frac{1}{r_{01}^2 r_{02}^2} \ln \left( \frac{r_{12}^2}{\mu^2} \right) \right] \right) \times \left( B_{100}^- + B_{230}^- + B_{200}^- + B_{130}^- - B_{300}^- - B_{210}^- - B_{123}^- \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3). \] (7.45)

8 Results

In this section we list the main results of the paper. Taking the LO equation (3.1) and using (4.30) we can write the NLO evolution equation for 3QWL operator as

\[ \frac{\partial B_{123}}{\partial \eta} = \frac{\alpha_s(\mu^2)}{8\pi^2} \int d\vec{r}_0 \left( (B_{100} B_{320} + B_{200} B_{310} - B_{300} B_{210} - 6B_{123}) \right) \]

\[ \times \left\{ \frac{\vec{r}_{12}^2}{r_{01}^2 r_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{r_{01}^2}{r_{02}^2} \right) \left( \frac{\vec{r}_{01}^2}{r_{01}^2 r_{02}^2} - \frac{1}{r_{02}^2} \right) - \frac{\vec{r}_{12}^2}{r_{01}^2 r_{02}^2} \ln \left( \frac{r_{12}^2}{\mu^2} \right) \right] \right\} \]
Here the functions $L_{12}, \tilde{L}_{12}, M_{12}, M_2$ are defined in (4.9-4.12), $L^9_{12}$ is defined in (5.3), the $\overline{\text{MS}}$ renormalization scale $\mu^2$ is related to scale $\tilde{\mu}^2$ through (5.10),

$$\beta = \left( \frac{11}{3} - \frac{2n_f}{3} \right).$$

As we mentioned, all the expressions in this paper are written in the $\overline{\text{MS}}$ renormalization scheme.

The evolution equation for the composite 3QWL operator $B_{123}^{\text{conf}}$ (6.2)

$$B_{123}^{\text{conf}} = B_{123} + \frac{\alpha_s}{8\pi^2} \int d\vec{r}_4 \left[ \frac{\vec{r}_{12}^2}{\vec{r}_{41}^2 r_{42}^2} \ln \left( \frac{\alpha_s}{\mu^2} \right) \right]$$

follows from (6.21)

$$\frac{\partial B_{123}^{\text{conf}}}{\partial \eta} = \frac{\alpha_s}{8\pi^2} \int d\vec{r}_0 \left[ \left( (B_{100}B_{320} + B_{200}B_{130} - B_{300}B_{210}) - 6B_{123} \right)^{\text{conf}} \right]$$

$$\times \left( \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 r_{02}^2} \right) - \frac{3\alpha_s}{4\pi} \left[ \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) \left( \frac{\vec{r}_{01}^2}{\vec{r}_{02}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 r_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu^2} \right) \right] + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)$$

$$- \frac{\alpha_s^2}{32\pi^2} \int d\vec{r}_0 \left( B_{003}B_{012} \left[ \frac{\vec{r}_{32}^2}{\vec{r}_{03}^2 r_{02}^2} \ln^2 \left( \frac{\vec{r}_{32}^2 r_{10}^2}{\vec{r}_{13}^2 r_{20}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 r_{02}^2} \ln^2 \left( \frac{\vec{r}_{12}^2 r_{30}^2}{\vec{r}_{13}^2 r_{20}^2} \right) \right] \right)$$

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\[
\begin{align*}
+ (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \\
- \frac{\alpha_s^2 n_f}{16 \pi^4} \int d\vec{r}_0 d\vec{r}_4 \left\{ \left\{ \left( \frac{1}{3} (U_1 U_0^\dagger U_4 + U_4 U_0^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0^\dagger U_4) \ight) + (U_1 U_0^\dagger U_2) \cdot U_3 \cdot U_4 + \frac{1}{6} B_{123} - \frac{1}{4} (B_{013} B_{002} + B_{001} B_{023} - B_{012} B_{003}) \ight\} L_{12}^0 + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \\
- \frac{\alpha_s^2}{8\pi} \int d\vec{r}_0 d\vec{r}_4 \left( \left\{ \tilde{L}_{12}^{C} \left( U_0 U_4^\dagger U_2 \right) \cdot \left( U_1 U_0^\dagger U_4 \right) \cdot U_3 \ight\} + L_{12}^{C} \left[ \left( U_0 U_4^\dagger U_2 \right) \cdot \left( U_1 U_0^\dagger U_4 \right) \cdot U_3 + tr \left( U_0 U_4^\dagger U_2 \right) \right) \cdot U_3 \cdot U_4 \ight) \\
- \frac{3}{4} \left[ B_{144} B_{234} + B_{244} B_{134} - B_{344} B_{124} \right] + \frac{1}{2} B_{123} \\
+ M_{12}^{C} \left[ \left( U_0 U_4^\dagger U_3 \right) \cdot \left( U_2 U_0^\dagger U_1 \right) \right) \cdot U_4 + \left( U_1 U_0^\dagger U_2 \right) \cdot \left( U_3 U_4^\dagger U_0 \right) \cdot U_4 \ight) + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right\} + (0 \leftrightarrow 4) \right]. \tag{8.4}
\end{align*}
\]

Here the composite operator \((B_{100} B_{320} + B_{200} B_{310} - B_{300} B_{210} - 6 B_{123})^{conf}\) is defined in (6.5) according to the prescription (6.1) and the functions \(L_{12}^{C}, \tilde{L}_{12}^{C}, M_{12}^{C}\) are defined in (6.16-6.18).

The equation for the composite 3QWL operator \(B_{123}^{conf}\) (6.2) linearized in the 3-gluon approximation is the result of (7.24) and (C.16)

\[
\begin{align*}
\frac{\partial B_{123}^{conf}}{\partial \eta} &= \frac{3 \alpha_s (\mu^2)}{4 \pi^2} \int d\vec{r}_0 \left[ \left( B_{100}^{conf} + B_{320}^{conf} + B_{200}^{conf} + B_{310}^{conf} - B_{300}^{conf} - B_{210}^{conf} - B_{123}^{conf} - 6 \right) \ight] \\
&\times \left( \frac{\vec{r}_{12}}{\vec{r}_0^2 \vec{r}_{01} \vec{r}_{02}} - \frac{3 \alpha_s}{4 \pi} \beta \left[ \ln \left( \frac{\vec{r}_{01}^2}{\vec{r}_0^2} \right) \left( \frac{1}{\vec{r}_{01}^2} - \frac{1}{\vec{r}_0^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_0^2 \vec{r}_{01} \vec{r}_{02}} \ln \left( \frac{\vec{r}_{12}^2}{\vec{r}_0^2} \right) \right) \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \\
&- \frac{9 \alpha_s^2}{64 \pi^3} \int d\vec{r}_0 \left( \tilde{F}_{100} B_{100} + \tilde{F}_{230} B_{230} + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right) \\
&+ \frac{27 \alpha_s^2}{4 \pi^2} \zeta(3) (3 - \delta_{23} - \delta_{13} - \delta_{21})(B_{123} - 6) \\
&- \frac{9 \alpha_s^2}{8 \pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \left( L_{12}^C + L_{13}^C + L_{23}^C \right) - \frac{n_f}{54} (L_{12}^q + L_{13}^q + L_{23}^q) \right) B_{044} + B_{004} - 12 \\
&- \frac{\alpha_s^2 n_f}{24 \pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \left( 2 B_{014} - B_{001} - B_{144} \right) (L_{12}^q + L_{13}^q - 2 L_{32}^q) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right) \\
&- \frac{9 \alpha_s^2}{64 \pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \left\{ F_0 B_{040} + F_{140} B_{140} + (0 \leftrightarrow 4) \right\} + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3) \right). \tag{8.5}
\end{align*}
\]

Here \(\delta_{ij} = 1\), if \(\vec{r}_i = \vec{r}_j\) and \(\delta_{ij} = 0\) otherwise; the functions \(F_0\) and \(F_{140}\) are defined in (7.18) and (7.19); \(\tilde{F}_{100}\) and \(\tilde{F}_{230}\) are defined in (7.26-7.27).
The linearized equation for C-even composite 3QWL Green function is the consequence of (7.36) and (C.16)

\[
\frac{\partial B_{123}^{\text{conf}}}{\partial \eta} = \frac{3g}{4\pi^2} 3\alpha_s \frac{\mu^2}{\mu^2} \int d\vec{r}_0 \left[ \left( B_{100}^{\text{conf}} + B_{230}^{\text{conf}} + B_{200}^{\text{conf}} + B_{310}^{\text{conf}} \right) - B_{300}^{\text{conf}} - B_{210}^{\text{conf}} - B_{123}^{\text{conf}} \right] \\
\times \left( \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2\vec{r}_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{\vec{r}_{02}^2}{\vec{r}_{01}^2} \right) \left( \frac{1}{\vec{r}_{02}^2} - \frac{1}{\vec{r}_{01}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2\vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu^2} \right) \right] \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \\
- \frac{9\alpha_s^2}{4\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( L_{12}^C + L_{13}^C + L_{23}^C - \frac{n_f}{54} (L_{12}^q + L_{13}^q + L_{23}^q) \right) (B_{044} + B_{004} - 12) \\
- \frac{9\alpha_s^2}{64\pi^3} \int d\vec{r}_0 \left( \tilde{F}_{100} B_{100}^{\text{conf}} + \tilde{F}_{230} B_{230}^{\text{conf}} \right) (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \\
+ \frac{27\alpha_s^2}{4\pi^2} \zeta(3) (3 - \delta_{23} - \delta_{13} - \delta_{21}) B_{123}^{\text{conf}} \\
- \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \{ F_{140} + (0 \leftrightarrow 4) \} B_{140}^{\text{conf}} + (\text{all 5 permutations} 1 \leftrightarrow 2 \leftrightarrow 3) \right). \quad (8.6)
\]

The linearized equations for C-odd composite 3QWL Green function is the consequence of (7.45) and (C.16)

\[
\frac{\partial B_{123}^{-\text{conf}}}{\partial \eta} = \frac{3g}{4\pi^2} 3\alpha_s \frac{\mu^2}{\mu^2} \int d\vec{r}_0 \left[ \left( B_{100}^{-\text{conf}} + B_{230}^{-\text{conf}} + B_{200}^{-\text{conf}} + B_{310}^{-\text{conf}} \right) - B_{300}^{-\text{conf}} - B_{210}^{-\text{conf}} - B_{123}^{-\text{conf}} \right] \\
\times \left( \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2\vec{r}_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{\vec{r}_{02}^2}{\vec{r}_{01}^2} \right) \left( \frac{1}{\vec{r}_{02}^2} - \frac{1}{\vec{r}_{01}^2} \right) - \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2\vec{r}_{02}^2} \ln \left( \frac{\vec{r}_{12}^2}{\mu^2} \right) \right] \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \\
- \frac{\alpha_s^2 \eta}{24\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( \{ 2B_{101}^- - B_{001}^- - B_{144}^- \} (L_{12}^q + L_{13}^q - 2L_{32}^q) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \right) \\
- \frac{9\alpha_s^2}{64\pi^3} \int d\vec{r}_0 \left( \tilde{F}_{100} B_{100}^- + \tilde{F}_{230} B_{230}^- \right) (1 \leftrightarrow 3) + (1 \leftrightarrow 2) \\
+ \frac{27\alpha_s^2}{4\pi^2} \zeta(3) (3 - \delta_{23} - \delta_{13} - \delta_{21}) B_{123}^- \\
- \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left( 2F_{040}^- + \{ F_{140}^- + (0 \leftrightarrow 4) \} B_{140}^- \right) + (\text{all 5 permutations} 1 \leftrightarrow 2 \leftrightarrow 3). \quad (8.7)
\]

From these expressions one can see that terms with \( L_{ij}, L_{ij}^C \), which comprise the BFKL kernels contribute only to the evolution of C-even part of the Green function while terms with \( F_0, \hat{L}_{ij}, \hat{L}_{ij}^C \) contribute only to the evolution of the C-odd one.

The BK equation for the color dipole \( B_{122} = 2tr(U_1 U_2^*) \) in the 3-gluon approximation reads (see (7.31))

\[
\frac{\partial B_{122}^{\text{conf}}}{\partial \eta} = \frac{3g}{2\pi^2} 3\alpha_s \frac{\mu^2}{\mu^2} \int d\vec{r}_0 (B_{100}^{\text{conf}} + B_{220}^{\text{conf}} - B_{122}^{\text{conf}} - 6)
\]
\[
\times \left( \frac{r_{12}^2}{r_{01}^2 r_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{r_{01}^2}{r_{02}^2} \right) \left( \frac{1}{r_{02}^2} - \frac{1}{r_{01}^2} \right) - \frac{r_{12}^2}{r_{01}^2 r_{02}^2} \ln \left( \frac{r_{12}^2}{\mu^2} \right) \right] \right) + \frac{27\alpha_s^2}{2\pi^2} \zeta(3)(B_{122} - 6) \\
- \frac{9\alpha_s^2}{4\pi^2} \int d\tilde{r}_0 d\tilde{r}_4 \left( L_{12}^C - \frac{n_f}{54} L_{12}^q \right) (B_{044} + B_{004} - 12) - \frac{9\alpha_s^2}{4\pi^2} \int d\tilde{r}_0 d\tilde{r}_4 \cdot L_{12}^C (B_{044} - B_{004}) - \frac{\alpha_s^2 n_f}{12\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \left\{ (2B_{014} - B_{001} - B_{144}) - (2B_{024} - B_{002} - B_{244}) \right\} L_{12}^q.
\]

As is clear from the last line the evolution of the color dipole in the 3-gluon approximation depends on the 3QL operators which have non-dipole structure. The BK equation for the C-even part of the color dipole operator \( B_{122}^+ = 2 tr(U_1^2 U_2^2) + 2 tr(U_1^2 U_2^2 - 6 \) in the 3-gluon approximation is the same as in the 2-gluon one (BFKL)

\[
\frac{\partial B_{122}^{+} (\eta)}{\partial \eta} = \frac{3\alpha_s (\mu^2)}{2\pi^2} \int d\tilde{r}_0 (B_{100}^{+} + B_{220}^{+} - B_{122}^{+}) \\
\times \left( \frac{r_{12}^2}{r_{01}^2 r_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{r_{01}^2}{r_{02}^2} \right) \left( \frac{1}{r_{02}^2} - \frac{1}{r_{01}^2} \right) - \frac{r_{12}^2}{r_{01}^2 r_{02}^2} \ln \left( \frac{r_{12}^2}{\mu^2} \right) \right] \right) - \frac{9\alpha_s^2}{2\pi^2} \int d\tilde{r}_0 d\tilde{r}_4 \left( L_{12}^C - \frac{n_f}{54} L_{12}^q \right) B_{044}^{+} + \frac{27\alpha_s^2}{8\pi^2} \zeta(3) B_{122}^{+}.
\]

At the same time the BK equation for the C-odd part of the color dipole operator \( B_{122}^{-} = 2 tr(U_1^2 U_2^2 - 2 tr(U_1^2 U_2^2 - 6 \) in the 3-gluon approximation reads

\[
\frac{\partial B_{122}^{-} (\eta)}{\partial \eta} = \frac{3\alpha_s (\mu^2)}{2\pi^2} \int d\tilde{r}_0 (B_{100}^{-} + B_{220}^{-} - B_{122}^{-}) \\
\times \left( \frac{r_{12}^2}{r_{01}^2 r_{02}^2} - \frac{3\alpha_s}{4\pi} \beta \left[ \ln \left( \frac{r_{01}^2}{r_{02}^2} \right) \left( \frac{1}{r_{02}^2} - \frac{1}{r_{01}^2} \right) - \frac{r_{12}^2}{r_{01}^2 r_{02}^2} \ln \left( \frac{r_{12}^2}{\mu^2} \right) \right] \right) - \frac{9\alpha_s^2}{2\pi^2} \int d\tilde{r}_0 d\tilde{r}_4 \cdot L_{12}^C B_{044}^{-} - \frac{27\alpha_s^2}{2\pi^2} \zeta(3) B_{122}^{-} - \frac{\alpha_s^2 n_f}{12\pi^4} \int d\tilde{r}_0 d\tilde{r}_4 \left\{ (2B_{014}^{-} - B_{001}^{-} - B_{144}^{-}) - (2B_{024}^{-} - B_{002}^{-} - B_{244}^{-}) \right\} L_{12}^q.
\]

This equation contains the nondipole 3QL operators in its quark part.

\section{Conclusions}

In this paper we constructed the NLO evolution equation for the "color triple" - three-quark Wilson loop operator \( \epsilon^{i'j'k'} \epsilon_{ijk} U_{1i'} U_{2j'} U_{3k'} \). As in the case of color dipole evolution, for the "rigid cutoff" \( Y < \eta \) of Wilson line the kernel of this equation has non-conformal terms not related to renormalization. We have constructed the composite 3QL operator (6.2) obeying the NLO evolution equation with quasi-conformal kernel. We linearized the quasi-conformal equation in the 3-gluon approximation. It is worth noting that our results have correct dipole limit in the case when the coordinates of two lines coincide. We also constructed the 3-gluon approximation of the BK equation and showed that it contains non-dipole 3QL operators (8.8), (8.10).

The 3QL operator may have many phenomenological applications. First, it is a natural \( SU(3) \) model for a baryon Green function in the Regge limit. Also, it is the
irreducible operator describing C-odd (odderon) exchange. For example as shown in the appendix E, the odderon part of the quadrupole operator \( tr(U_1 U_2^\dagger U_3 U_4^\dagger) \) in the 3-gluon approximation in \( SU(3) \) can be decomposed into a sum of 3QWLs

\[
2tr(U_1 U_2^\dagger U_3 U_4^\dagger) - 2tr(U_4 U_3^\dagger U_2 U_1^\dagger) \approx B_{144}^- + B_{322}^- - B_{333}^- - B_{211}^- + B_{124}^- + B_{234}^- - B_{123}^- - B_{134}^-.
\]

Moreover, even the NLO evolution equation for the dipole C-odd Green function in the 3-gluon approximation (8.10) in QCD can not be written without the introduction of the 3QWL operator.

The evolution equation for the C-odd part of the 3QWL operator is the generalization of the BKP equation for odderon exchange to the saturation regime. However, it is valid for the colorless object, i.e. for the function \( B_{ijk}^- = B^- (\vec{r}_i, \vec{r}_j, \vec{r}_k) \), which vanishes as \( \vec{r}_i = \vec{r}_j = \vec{r}_k \). The linear approximation of the equation for the C-odd part of the 3QWL should be equivalent to the NLO BKP for odderon exchange acting in the space of such functions. One may try to restore the full NLO BKP kernel from our result via the technique similar to the one developed for the 2-point operators in [20].

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**A Notations**

First, let us describe our notations. We introduce the light cone vectors \( n_1 \) and \( n_2 \)

\[
n_1 = (1,0,0,1), \quad n_2 = \frac{1}{2} (1,0,0,-1), \quad n_1^+ = n_2^- = n_1 n_2 = 1 \quad (A.1)
\]

so that for any vector \( p \) we have

\[
p^+ = p_+ = pm_2 = \frac{1}{2} (p^0 + p^3), \quad p^- = p_+ = pm_1 = p^0 - p^3, \quad (A.2)
\]

\[
p = p^+ n_1 + p^- n_2 + p_{\perp}, \quad p^2 = 2p^+ p^- - \vec{p}^2, \quad (A.3)
\]

\[
p k = p^\mu k_\mu = p^+ k^- + p^- k^+ - \vec{p} \vec{k} = p_+ k_- + p_- k_+ - \vec{p} \vec{k}. \quad (A.4)
\]

The index convention is \( a_i l_j = (ab)^i_j \).

Second, let us present the connection of our notation with the notation of [6]. In that paper the quark and gluon coordinates were denoted \( z_i \). So we have to change

\[
z_{1,2,3,4} \leftrightarrow \vec{r}_{1,2,3,4}, \quad z_5 \leftrightarrow \vec{r}_0. \quad (A.5)
\]
Assuming this substitution for self-interaction we get

\[ G_3 = \frac{I_1}{2z_{14}^2} \ln \frac{z_{14}^2}{z_{15}^2} - \frac{1}{z_{15}}, \quad G_9 = \frac{- (z_{14}, z_{15})}{2z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2}. \]  

(A.6)

For pairwise interaction we obtain

\[ 8H_1 = 2J_{1245} \ln \frac{z_{14}^2}{z_{15}^2}, \quad 8H_2 = -2(J_{1245} + J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2}, \]  

(A.7)

\[ 8H_3 = 2(J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2}, \]  

(A.8)

\[ 8H_4 = \ln \frac{z_{14}^2}{z_{15}^2} (2L - 2J_{1245} + 2J_{1254}) - \frac{2}{z_{15}^2}, \]  

(A.9)

or

\[ 8(H_3 + H_4) = 2L \ln \frac{z_{14}^2}{z_{15}^2} - \frac{2}{z_{15}^2}. \]  

(A.10)

Here \( L \) is defined as

\[ -2K - \frac{8}{z_{15}^2} + \frac{2J_1}{z_{14}^2} + \frac{2J_2}{z_{25}^2} = 2L \ln \frac{z_{14}^2}{z_{15}^2} + (1 \leftrightarrow 2) - \frac{4}{z_{15}^2} = 8(H_3 + H_4 + 1 \leftrightarrow 2). \]  

(A.11)

For triple interaction we have

\[ H_7 \equiv H_5 + H_6 = \frac{1}{2} J_{32145} \ln \frac{z_{14}^2}{z_{15}^2}, \quad H_8 \equiv H_5 - H_6 = -\frac{1}{2} J_{32154} \ln \frac{z_{14}^2}{z_{15}^2}. \]  

(A.12)

\section*{B \ \ \textit{SU(3)} identities}

Here we present the list of \( SU(3) \) identities used in the paper.

\[ U_i \cdot U_j \cdot U_k = (U_i U_j^\dagger) \cdot (U_j U_k^\dagger) \cdot (U_k U_i^\dagger) = (U_i^\dagger U_i) \cdot (U_j^\dagger U_j) \cdot (U_k^\dagger U_k), \]  

(B.1)

\[ \varepsilon^{ijk} \varepsilon^{i'j'k'} (U_1)_{i}^{i'} (U_1)_{j}^{j'} = 2(U_1)_{k}^{k'}, \quad U_1 \cdot U_1 \cdot U_3 = 2 \text{tr}(U_1^\dagger U_3). \]  

(B.2)

These identities follow from the definition of the group, namely from unitarity and the fact that the determinant of \( U \) is 1.

\[ (U_2 U_4^\dagger U_1 + U_1 U_4^\dagger U_2) \cdot U_1 \cdot U_3 = -B_{123} + \frac{1}{2} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214}). \]  

(B.3)

This identity can be checked using (B.1) with \( l = 4 \) and then expanding the product of Levi-Civita symbols as

\[ \varepsilon_{ijk} \varepsilon^{i'j'k'} = \begin{vmatrix} \delta_{i'}^{i} & \delta_{j'}^{j} & \delta_{k'}^{k} \\ \delta_{i}^{i'} & \delta_{j}^{j'} & \delta_{k}^{k'} \\ \delta_{i'}^{k} & \delta_{j'}^{k} & \delta_{k'}^{k} \end{vmatrix}. \]  

(B.4)

\[ 0 = \left| \left( U_0 U_4^\dagger U_3 U_0^\dagger U_4 \right) \cdot U_1 \cdot U_2 - U_1 \cdot U_2 \cdot U_4 \text{tr} \left( U_0^\dagger U_3 \right) \text{tr} \left( U_4^\dagger U_0 \right) \right|. \]
This identity relates the color structures in $G_{12(3)}$, $G_{1(23)}$ and $G_{13(2)}$. By $1 \leftrightarrow 2 \leftrightarrow 3$ transformation one can obtain 2 more identities and totally eliminate 3 color structures from $G_{12(3)}$, $G_{1(23)}$, and $G_{1(23)}$.

\[
0 = tr \left( U_0 U_0 \right) \left( U_0 U_4 + U_3 U_4 U_0 \right) \cdot U_1 \cdot U_4
\]

\[
+ tr \left( U_0 U_4 \right) \left( U_2 U_0 U_3 + U_3 U_0 U_2 \right) \cdot U_1 \cdot U_4
\]

\[
+ \left( U_0 U_4 \right) \cdot \left( U_3 U_0 U_4 \right) \cdot U_1 + \left( U_0 U_4 \right) \cdot \left( U_2 U_0 U_4 \right) \cdot U_1 \cdot (1 \leftrightarrow 2) + (4 \leftrightarrow 0). \quad (B.5)
\]

This identity relates all color structures in $G_{1(23)}$ and two structures in $G_{(123)}$. It goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

\[
0 = tr \left( U_0 U_2 \right) \left( U_0 U_4 U_1 + U_1 U_4 U_0 \right) \cdot U_3 \cdot U_4
\]

\[
+ \left( U_0 U_4 \right) \cdot \left( U_2 U_0 U_4 \right) \cdot U_3 + \left( U_0 U_4 \right) \cdot \left( U_1 U_4 U_0 \right) \cdot U_4
\]

\[
- tr \left( U_0 U_4 \right) \left( U_1 U_4 U_2 + U_2 U_4 U_1 \right) \cdot U_3 \cdot U_4
\]

\[
- \left( U_0 U_4 \right) \cdot \left( U_2 U_4 U_1 \right) \cdot U_3 - \left( U_0 U_4 \right) \cdot \left( U_1 U_4 U_1 \right) \cdot U_3
\]

\[
- \left( U_1 U_0 U_4 \right) \cdot \left( U_3 U_4 U_2 \right) \cdot U_0 + \left( U_1 U_4 U_2 \right) \cdot \left( U_3 U_0 U_4 \right) \cdot U_0
\]

\[
+ \left( U_2 U_4 U_2 \right) \cdot U_0 - \left( U_2 U_4 U_3 \right) \cdot \left( U_4 U_0 U_4 \right) \cdot U_0 + (4 \leftrightarrow 0). \quad (B.6)
\]

This identity relates 2 color structures in $G_{12(3)}$ and a structure in $G_{(123)}$. It also goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

\[
0 = \left[ U_0 \cdot U_1 \cdot U_2 tr \left( U_0 U_4 \right) \right] \left( U_4 U_0 \right) \cdot U_3
\]

\[
- tr \left( U_4 U_0 \right) \left( U_1 U_4 U_3 + U_3 U_4 U_1 \right) \cdot U_0 \cdot U_2 + \left( U_0 U_4 U_1 \right) \cdot \left( U_3 U_0 U_4 \right) \cdot U_2
\]

\[
+ \left( U_1 U_4 U_0 \right) \cdot \left( U_3 U_0 U_3 \right) \cdot U_2 + (1 \leftrightarrow 2) - (4 \leftrightarrow 0). \quad (B.7)
\]

This identity relates 2 color structures in $G_{1(23)}$, 2 color structures in $G_{1(23)}$ and a structure in $G_{13(2)}$. It goes into 2 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 3 more structures.

\[
0 = 2 tr \left( U_4 U_0 \right) \left( U_2 U_4 U_3 + U_3 U_4 U_2 \right) \cdot U_0 \cdot U_4
\]
\[ + \left( U_0 U_1 U_2 + U_1 U_2 U_0 \right) \cdot \left( U_2 U_1 U_3 + U_3 U_1 U_2 \right) \cdot U_4 \]
\[ + \left( U_0 U_1 U_2 - U_2 U_1 U_0 \right) \cdot \left( U_3 U_1 U_0 - U_1 U_1 U_3 \right) \cdot U_4 \]
\[ + \left( U_0 U_1 U_3 - U_3 U_1 U_0 \right) \cdot \left( U_2 U_0 ^\dagger U_1 - U_1 U_0 ^\dagger U_2 \right) \cdot U_4 - (4 \leftrightarrow 0). \]  

This identity relates 3 color structures in \( G_{(132)} \) and a color structure in \( G_{1(23)} \). It goes into 2 different identities after \( 1 \leftrightarrow 2 \leftrightarrow 3 \) transformation, which allows one to get rid of 3 more structures.

All these identities \( (B.5-B.9) \) can be checked using \( (B.1) \) with \( l = 1 \) and then expanding the product of Levi-Civita symbols via \( (B.4) \).

\[ 0 = 2 \text{tr}(U_0 ^\dagger U_3) \left( U_1 U_4 ^\dagger U_2 \right) \cdot U_0 \cdot U_4 - \left( U_1 U_4 ^\dagger U_2 \right) \cdot \left( U_3 U_0 ^\dagger U_4 \right) \cdot U_0 \]
\[ - \left( U_1 U_4 ^\dagger U_2 \right) \cdot \left( U_4 U_0 ^\dagger U_3 \right) \cdot U_0 - 2 \left( U_1 U_4 ^\dagger U_2 \right) \cdot U_3 \cdot U_4 \]
\[ - \left( U_1 U_4 ^\dagger U_2 \right) \cdot \left( U_3 U_0 ^\dagger U_4 + U_4 U_0 ^\dagger U_3 \right) \cdot U_0 \]
\[ - \left( U_1 U_4 ^\dagger U_2 \right) \cdot U_0 \cdot U_3 \]
\[ 0 = 2 \text{tr}(U_0 ^\dagger U_4) \left( U_1 U_4 ^\dagger U_2 \right) \cdot U_0 \cdot U_4 + \left( U_2 U_0 ^\dagger U_3 \right) \cdot U_1 \cdot U_4 \]
\[ + \left( U_0 U_1 U_3 \right) \cdot U_2 \cdot U_4 + \left( U_0 U_4 ^\dagger U_1 \right) \cdot \left( U_2 U_0 ^\dagger U_4 \right) \cdot U_3 \]
\[ + \left( U_1 U_4 ^\dagger U_0 \right) \cdot \left( U_3 U_0 ^\dagger U_2 \right) \cdot U_4 + \left( U_1 U_4 ^\dagger U_0 \right) \cdot \left( U_4 U_0 ^\dagger U_2 \right) \cdot U_3 \]
\[ - \left( U_1 U_4 ^\dagger U_2 \right) \cdot \left( U_3 U_0 ^\dagger U_4 + U_4 U_0 ^\dagger U_3 \right) \cdot U_0 \]
\[ - \left( U_1 U_4 ^\dagger U_2 \right) \cdot U_0 \cdot U_3 \]
\[ 0 = \text{tr}(U_0 ^\dagger U_2) \left( U_0 U_4 ^\dagger U_1 + U_1 U_4 ^\dagger U_0 \right) \cdot U_3 \cdot U_4 - 2 \left( U_1 U_4 ^\dagger U_2 + U_2 U_4 ^\dagger U_1 \right) \cdot U_3 \cdot U_4 \]
\[ + U_0 \cdot U_3 \cdot U_4 \text{tr}(U_0 ^\dagger U_1 U_4 ^\dagger U_2) + U_0 \cdot U_3 \cdot U_4 \text{tr}(U_0 ^\dagger U_2 U_4 ^\dagger U_1) \]
\[ - \left( U_1 U_4 ^\dagger U_3 \right) \cdot U_0 \cdot U_4 \]
\[ - \left( U_1 U_4 ^\dagger U_2 U_0 ^\dagger U_3 \right) \cdot U_0 \cdot U_3 \]
\[ - \left( U_0 U_4 ^\dagger U_1 \right) \cdot \left( U_2 U_0 ^\dagger U_3 \right) \cdot U_4 \]
\[ - \left( U_0 U_4 ^\dagger U_1 \right) \cdot \left( U_2 U_0 ^\dagger U_4 \right) \cdot U_3 \]
\[ - \left( U_0 U_4 ^\dagger U_1 \right) \cdot \left( U_2 U_0 ^\dagger U_4 \right) \cdot U_3 - \left( U_1 U_4 ^\dagger U_0 \right) \cdot \left( U_4 U_0 ^\dagger U_2 \right) \cdot U_3. \]

\[ 0 = \text{tr}(U_4 ^\dagger U_1) \left( U_2 U_0 ^\dagger U_4 + U_4 U_0 ^\dagger U_2 \right) \cdot U_0 \cdot U_3 \]
\[ + \left( U_0 U_4 ^\dagger U_1 \right) \cdot \left( U_3 U_0 ^\dagger U_2 \right) \cdot U_4 + \left( U_1 U_4 ^\dagger U_0 \right) \cdot \left( U_2 U_0 ^\dagger U_3 \right) \cdot U_4 \]
\[ + \left( U_2 U_0 ^\dagger U_4 \right) \cdot U_3 \cdot U_4 - \left( U_4 U_0 ^\dagger U_2 U_4 ^\dagger U_1 \right) \cdot U_0 \cdot U_3 - (1 \leftrightarrow 2, 0 \leftrightarrow 4). \]
These identities (B.11–B.13) also can be checked using (B.1) with \( l = 3 \) and then expanding the product of Levi-Civita symbols via (B.4).

\[
0 = \left( U_2 U_3 U_0 - U_0 U_4 U_2 \right) \cdot \left( U_3 U_0 U_1 - U_1 U_0 U_3 \right) \cdot U_4 \\
+ \left( U_3 U_4 U_2 - U_2 U_4 U_3 \right) \cdot \left( U_4 U_0 U_1 - U_1 U_0 U_4 \right) \cdot U_0 \\
+ \left( U_1 U_4 U_2 - U_2 U_4 U_1 \right) \cdot \left( U_3 U_4 U_0 - U_0 U_4 U_3 \right) \cdot U_4 + (0 \leftrightarrow 4). \tag{B.14}
\]

It can be proved using (B.1) with \( l = 4 \) and (B.4).

\[
0 = 2 (U_3 U_0 U_4 + U_4 U_0 U_3) \cdot U_1 \cdot U_2 - U_1 \cdot U_2 \cdot U_3 \text{tr}(U_0 U_4) + (U_0 U_1 U_4 + U_4 U_0 U_1) \cdot U_2 \cdot U_3 \\
+ 3 (U_3 U_0 U_1 + U_1 U_0 U_3) \cdot U_2 \cdot U_4 - 3 U_1 \cdot U_2 \cdot U_4 \text{tr}(U_0 U_3) + (1 \leftrightarrow 2). \tag{B.15}
\]

This identity is necessary for calculation of the quark contribution. It can be proved using (B.1) with \( l = 2 \) and (B.4).

C Construction of conformal 4-point operator

Here we derive the evolution equation for the operator

\[
\left( U_2 U_4 U_1 + U_1 U_4 U_2 \right) \cdot U_4 \cdot U_3 - 2 B_{123} \right) = (-3 B_{123} + \frac{1}{2} (B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214})). \tag{C.1}
\]

So one has to find the evolution of the operator \((U_1 U_4 U_2) \cdot U_4 \cdot U_3\) first. It reads

\[
\frac{\partial (U_1 U_4 U_2) \cdot U_4 \cdot U_3}{\partial \eta} = (U_1 U_4 U_2) \cdot U_4 \cdot U_3 \left( -\frac{\alpha_s}{2 \pi^2} \frac{4 \pi^2}{3} \right) \int d\bar{z}_0 \left( \frac{1}{r_{10}^2} + \frac{1}{r_{20}^2} + \frac{1}{r_{30}^2} + \frac{2}{r_{40}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( \ell^c U_1 U_4 U_2 \right) \cdot U_4 \cdot U_3 + \left( U_1 t^c U_4 U_2 t^c \right) \cdot U_4 \cdot U_3 \right] \int d\bar{r}_0 \left( \frac{r_{10}^2 r_{20}^2}{r_{10}^2 r_{20}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( \ell^c U_1 U_4 U_2 \right) \cdot U_4 \cdot (t^c U_3) + \left( U_1 t^c U_4 U_2 \right) \cdot U_4 \cdot (t^c U_3) \right] \int d\bar{r}_0 \left( \frac{r_{10}^2 r_{30}^2}{r_{10}^2 r_{30}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( U_1 U_4 U_2 \right) \cdot U_4 \cdot (t^c U_3) + \left( U_1 U_4 U_2 t^c \right) \cdot U_4 \cdot (t^c U_3) \right] \int d\bar{r}_0 \left( \frac{r_{20}^2 r_{30}^2}{r_{20}^2 r_{30}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( \ell^c U_1 U_4 U_2 \right) \cdot (t^c U_4) + \left( U_1 t^c U_4 U_2 \right) \cdot (t^c U_4) \right] \int d\bar{r}_0 \left( \frac{r_{10}^2 r_{40}^2}{r_{10}^2 r_{40}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( U_1 U_4 U_2 \right) \cdot (t^c U_4) + \left( U_1 U_4 U_2 t^c \right) \cdot (t^c U_4) \right] \int d\bar{r}_0 \left( \frac{r_{20}^2 r_{40}^2}{r_{20}^2 r_{40}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( U_1 U_4 U_2 \right) \cdot (t^c U_4) + \left( U_1 U_4 U_2 t^c \right) \cdot (t^c U_3) \right] \int d\bar{r}_0 \left( \frac{r_{20}^2 r_{30}^2}{r_{20}^2 r_{30}^2} \right) \\
- \frac{\alpha_s}{\pi^2} \left[ \left( U_1 U_4 U_2 \right) \cdot (t^c U_4) + \left( U_1 U_4 U_2 t^c \right) \cdot (t^c U_3) \right] \int d\bar{r}_0 \left( \frac{r_{20}^2 r_{30}^2}{r_{20}^2 r_{30}^2} \right).
\]
\[
- \left( U_1 U_4^t t^t U_2 \right) \cdot U_4 \cdot (t^t U_3) - \left( U_1 t^t U_4^t U_2 \right) \cdot U_4 \cdot (U_3 t^t)
\]
\[
+ \frac{\alpha_s}{\pi^2} \left( U_1 U_4^t t^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 t^t U_4^t U_2 \right) \cdot (U_4 t^t) \cdot U_3
\]
\[
+ \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{\left( U_1 U_4^t U_2 \right) \cdot (t^t U_4 t^d) \cdot U_3}{\vec{r}_{01}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot U_3
\]
\[
\frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{\left( U_1 U_4^t U_2 \right) \cdot (t^t U_4 t^d) \cdot U_3}{\vec{r}_{02}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
\frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{\left( U_1 U_4^t U_2 \right) \cdot (t^t U_4 t^d) \cdot U_3}{\vec{r}_{03}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
+ \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{U_0^{cd} \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3}{\vec{r}_{01}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
+ \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{U_0^{cd} \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3}{\vec{r}_{02}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
+ \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{U_0^{cd} \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3}{\vec{r}_{03}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
- \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{U_0^{cd} \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3}{\vec{r}_{04}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
- \frac{\alpha_s}{\pi^2} \int d\vec{r}_0 U_0^{cd} \left( \frac{U_0^{cd} \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3 + \left( U_1 U_4^t U_2 \right) \cdot (t^t U_4) \cdot U_3}{\vec{r}_{01}^2} \right)
+ \left( U_1 U_4^t U_2 \right) \cdot U_4 \cdot (t^t U_3 t^d)
\]
\[
\frac{\partial}{\partial t^t} \left( U_2 U_1^t U_1 + U_1 U_2^t U_2 \right) \cdot U_4 \cdot U_3 - 2B_{123} = \frac{\alpha_s}{\pi^2} \int d\vec{r}_0
\]
\[
\times \left( A_{34} \frac{\vec{r}_{34}^2}{\vec{r}_{03}^2 \vec{r}_{04}^2} + A_{13} \frac{\vec{r}_{13}^2}{\vec{r}_{03}^2 \vec{r}_{01}^2} + A_{23} \frac{\vec{r}_{23}^2}{\vec{r}_{03}^2 \vec{r}_{02}^2} + A_{14} \frac{\vec{r}_{14}^2}{\vec{r}_{04}^2 \vec{r}_{01}^2} + A_{24} \frac{\vec{r}_{24}^2}{\vec{r}_{02}^2 \vec{r}_{04}^2} + A_{12} \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \right) .
\]

Using (B.3) and (B.10–B.12) after the convolution one gets

\[
\frac{\partial}{\partial t^t} \left( U_2 U_1^t U_1 + U_1 U_2^t U_2 \right) \cdot U_4 \cdot U_3 - 2B_{123} = \frac{\alpha_s}{\pi^2} \int d\vec{r}_0
\]
\[
\times \left( A_{34} \frac{\vec{r}_{34}^2}{\vec{r}_{03}^2 \vec{r}_{04}^2} + A_{13} \frac{\vec{r}_{13}^2}{\vec{r}_{03}^2 \vec{r}_{01}^2} + A_{23} \frac{\vec{r}_{23}^2}{\vec{r}_{03}^2 \vec{r}_{02}^2} + A_{14} \frac{\vec{r}_{14}^2}{\vec{r}_{04}^2 \vec{r}_{01}^2} + A_{24} \frac{\vec{r}_{24}^2}{\vec{r}_{02}^2 \vec{r}_{04}^2} + A_{12} \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} \right) .
\]

Here

\[
A_{34} = -2 \left( U_2 U_1^t U_1 + U_1 U_2^t U_2 \right) \cdot U_4 \cdot U_3 + \left( U_3 U_1^t U_1 + U_1 U_3^t U_3 \right) \cdot U_4 \cdot U_2
\]
\[
+ \left( U_3 U_2^t U_2 + U_2 U_3^t U_3 \right) \cdot U_4 \cdot U_1 + \left( U_2 U_4^t U_1 + U_1 U_4^t U_2 \right) \cdot \left( U_3 U_0^t U_4 + U_4 U_0^t U_3 \right) \cdot U_0
\]

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\[-\left(U_2 U_4 U_6\right) \cdot \left(U_3 U_0 U_1\right) \cdot U_4 - \left(U_0 U_4 U_2\right) \cdot \left(U_1 U_0 U_3\right) \cdot U_4 \]
\[-\left(U_0 U_4 U_1\right) \cdot \left(U_2 U_0 U_3\right) \cdot U_4 - \left(U_1 U_4 U_0\right) \cdot \left(U_3 U_0 U_2\right) \cdot U_4.\]  
(C.4)

\[A_{13} = \left(U_0 U_4 U_2\right) \cdot \left(U_1 U_0 U_3\right) \cdot U_4 + \left(U_2 U_4 U_0\right) \cdot \left(U_3 U_0 U_1\right) \cdot U_4\]
\[+ \left(U_3 U_0 U_4 U_2\right) \cdot U_0 \cdot U_4 + \left(U_2 U_4 U_1 U_0 U_3\right) \cdot U_0 \cdot U_4\]
\[-2 \left(U_1 U_0 U_3 + U_3 U_0 U_1\right) \cdot U_0 \cdot U_2 - \left(U_3 U_4 U_2 + U_2 U_4 U_3\right) \cdot U_1 \cdot U_4\]
\[-\left(U_1 U_4 U_2 + U_2 U_4 U_1\right) \cdot U_3 \cdot U_4 + 4 U_1 \cdot U_2 \cdot U_3.\]  
(C.5)

\[A_{14} = \text{tr} \left(U_0 U_1\right) \left(U_2 U_4 U_0 + U_0 U_4 U_2\right) \cdot U_3 \cdot U_4 + \text{tr} \left(U_4 U_0\right) \left(U_1 U_0 U_2 + U_2 U_0 U_1\right) \cdot U_3 \cdot U_4\]
\[+ \left(U_2 U_4 U_0\right) \cdot \left(U_4 U_0 U_1\right) \cdot U_3 + \left(U_0 U_4 U_2\right) \cdot \left(U_1 U_0 U_1\right) \cdot U_3\]
\[-2 U_4 \cdot U_2 \cdot U_3 \text{tr} \left(U_4 U_1\right) - 2 U_1 \cdot U_2 \cdot U_3 - 4 \left(U_1 U_4 U_2 + U_2 U_4 U_1\right) \cdot U_3 \cdot U_4\]
\[+ \left(U_4 U_0 U_1 U_4 U_2\right) \cdot U_0 \cdot U_3 + \left(U_2 U_4 U_1 U_0 U_4\right) \cdot U_0 \cdot U_3.\]  
(C.6)

\[A_{12} = -2 \left(U_1 U_0 U_2 + U_2 U_0 U_1\right) \cdot U_3 \cdot U_4 - \text{tr} \left(U_4 U_0\right) \left(U_1 U_0 U_2 + U_2 U_0 U_1\right) \cdot U_3 \cdot U_4\]
\[+ 4 U_1 \cdot U_2 \cdot U_3 + 2 U_4 \cdot U_2 \cdot U_3 \text{tr} \left(U_4 U_1\right) + 2 U_4 \cdot U_1 \cdot U_3 \text{tr} \left(U_4 U_2\right)\]
\[- U_0 \cdot U_3 \cdot U_4 \left(\text{tr} \left(U_0 U_1 U_4 U_2\right) + \text{tr} \left(U_0 U_1 U_3 U_4\right)\right) .\]  
(C.7)

\[A_{23} = A_{13} |_{\vec{r}_1 \leftrightarrow \vec{r}_2}, \quad A_{24} = A_{14} |_{\vec{r}_1 \leftrightarrow \vec{r}_2}.\]  
(C.8)

Our prescription for the composite conformal operators reads [4] (see also Ref. [19])

\[O^{\text{conf}} = O + \frac{1}{2} \frac{\partial O}{\partial \eta} \left| \frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}} \rightarrow \frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}} \ln \left(\frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}}\right) \right|,\]  
(C.9)

where \(a\) is an arbitrary constant. Thus

\[B_{123}^{\text{conf}} = B_{123} + \frac{1}{2} \frac{\partial B_{123}}{\partial \eta} \left| \frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}} \rightarrow \frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}} \ln \left(\frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}}\right) \right|,\]  
(C.10)

\[= B_{123} + \frac{\alpha_s^3}{8\pi^2} \int d\vec{r}_0 \left[ \frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}} \ln \left(\frac{\vec{r}_{\alpha_{mn}}}{\vec{r}_{\eta_{mn}}}\right) \right] \times \left(-B_{123} + \frac{1}{6} \left(B_{100} B_{320} + B_{200} B_{310} - B_{300} B_{210}\right)\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\]  
(C.11)

\[\left(-3B_{123} + \frac{1}{2} \left(B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214}\right)\right)^{\text{conf}}\]
\[= \left(-3B_{123} + \frac{1}{2} \left(B_{144} B_{324} + B_{244} B_{314} - B_{344} B_{214}\right)\right)\]
Therefore

Here we describe the calculation of integral (6.20). It reads

\[
\int d\vec{r}_1 Z_{12} = J_{12} - (1 \leftrightarrow 3). \tag{D.1}
\]

\[
J_{12} = \frac{\vec{r}_{12}^2}{8\vec{r}_{01}^2\vec{r}_{02}^2} \int d\vec{r}_4 \left[ \left( \frac{\vec{r}_{03}^2}{\vec{r}_{04}^2\vec{r}_{24}^2} - \frac{\vec{r}_{02}^2}{\vec{r}_{04}^2\vec{r}_{24}^2} \right) \ln \left( \frac{\vec{r}_{02}^2\vec{r}_{14}^2}{\vec{r}_{04}^2\vec{r}_{12}^2} \right) + \frac{\vec{r}_{01}^2}{\vec{r}_{04}^2\vec{r}_{12}^2} \ln \left( \frac{\vec{r}_{02}^2\vec{r}_{14}^2}{\vec{r}_{03}^2\vec{r}_{24}^2} \right) + \frac{\vec{r}_{13}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} \ln \left( \frac{\vec{r}_{03}^2\vec{r}_{12}^2}{\vec{r}_{02}^2\vec{r}_{13}^2} \right) \right]. \tag{D.2}
\]

Since the integral is conformally invariant, one can set \( \vec{r}_0 = 0 \) and make the inversion, then calculate the integral and then again make the inversion and restore \( \vec{r}_0 \).

\[
J_{12} \xrightarrow{\vec{r}_0 = 0} \frac{\vec{r}_{12}^2}{8\vec{r}_{14}^2\vec{r}_{24}^2} \int d\vec{r}_4 \left[ \left( \frac{\vec{r}_{13}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} - \frac{\vec{r}_{12}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} \right) \ln \left( \frac{\vec{r}_{12}^2\vec{r}_{14}^2}{\vec{r}_{13}^2\vec{r}_{34}^2} \right) + \frac{\vec{r}_{12}^2}{\vec{r}_{14}^2\vec{r}_{34}^2} \ln \left( \frac{\vec{r}_{13}^2\vec{r}_{14}^2}{\vec{r}_{12}^2\vec{r}_{13}^2} \right) \right]. \tag{D.3}
\]
Using the integrals from appendix A in [21] we have

\[ J_{12} \to -\pi \frac{r_{12}^2}{8} \ln^2 \left( \frac{r_{13}^2}{r_{12}^2} \right). \]  

(D.5)

After inversion and restoring of \( \vec{r}_0 \) we get

\[ J_{12} = -\pi \frac{r_{12}^2}{8 r_0^2} \ln^2 \left( \frac{r_{12}^2 r_{30}^2}{r_{13}^2 r_{20}^2} \right). \]

(D.6)

Therefore

\[ \int \frac{d\vec{r}_4}{\pi} Z_{12} = \frac{r_{13}^2}{8 r_0^2} \ln^2 \left( \frac{r_{12}^2 r_{10}^2}{r_{13}^2 r_{20}^2} \right) - \frac{r_{12}^2}{8 r_0^2} \ln^2 \left( \frac{r_{12}^2 r_{30}^2}{r_{13}^2 r_{20}^2} \right). \]  

(D.7)

Now we will integrate \( F_{100} \) (7.20) w.r.t. \( \vec{r}_4 \). Again we set \( \vec{r}_0 = 0 \), do inversion, and calculate the integral in the \( d = 2 + 2 \epsilon \) dimensional space using the integrals from appendix A in [21] and

\[ \int \frac{d^{2+2\epsilon} r_{14}}{\pi^{1+\epsilon} \Gamma (1 - \epsilon)} r_{12}^2 r_{24}^2 = \frac{r_{13}^2 + r_{23}^2 - r_{12}^2}{r_{12}^2} \left( \frac{1}{\epsilon} + \ln \left( r_{12}^2 \right) \right) + O(\epsilon). \]  

(D.8)

We get

\[ \int \frac{d\vec{r}_4}{\pi} F_{100} \to 0 \leftrightarrow 3 \to \int \frac{d^4 r_4}{\pi} \left( \frac{r_{12}^2 r_{24}^2}{r_{12}^2} \ln \left( \frac{r_{14}^2}{r_{12}^2} \right) + \frac{r_{23}^2 r_{12}^2}{2 r_{14}^2 r_{24}^2} \ln \left( \frac{r_{14}^2 r_{23}^2}{r_{12}^2} \right) \right) \]

\[ + \frac{r_{13}^2 r_{12}^2}{2 r_{14}^2 r_{24}^2} \ln \left( \frac{r_{13}^2 r_{24}^2}{r_{12}^2} \right) - \frac{r_{13}^2}{r_{24}^2} \ln \left( \frac{r_{13}^2 r_{24}^2}{r_{14}^2} \right) + \frac{r_{23}^2}{2 r_{24}^2} \ln \left( \frac{r_{23}^2 r_{24}^2}{r_{14}^2} \right) \]

\[ + \frac{r_{13}^2}{2 r_{14}^2} \ln \left( \frac{r_{13}^2 r_{12}^2}{r_{14}^2} \right) - \frac{r_{13}^2}{r_{12}^2 r_{24}^2} \ln \left( \frac{r_{12}^2 r_{34}^2}{r_{14}^2} \right) - \frac{r_{12}^2}{2 r_{14}^2} \ln \left( \frac{r_{12}^2 r_{14}^2}{r_{13}^2} \right) + (2 \leftrightarrow 3) \]  

(D.9)

\[ d \to 2 \left( \frac{3 (r_{13}^2 - r_{12}^2)}{4} - \frac{r_{23}^2}{2} \right) \ln^2 \left( \frac{r_{12}^2}{r_{23}^2} \right) + \left( \frac{3}{4} r_{23}^2 - r_{12}^2 - r_{13}^2 \right) \ln^2 \left( \frac{r_{12}^2}{r_{13}^2} \right) + \frac{3}{2} S_{123} I \left( r_{12}^2, r_{13}^2, r_{23}^2 \right). \]  

(D.10)

Here

\[ S_{123} = r_{12}^4 + r_{13}^4 + r_{23}^4 - 2 r_{13}^2 r_{12}^2 - 2 r_{23}^2 r_{12}^2 - 2 r_{13}^2 r_{23}^2 \]  

(D.11)

is the Cayley-Menger determinant proportional to the squared area of the triangle with the corners at \( r = r_{1,2,3} \), and

\[ I(a, b, c) = \int_0^1 \frac{dx}{a(1 - x) + bx - cx(1 - x)} \ln \left( \frac{a(1 - x) + bx}{cx(1 - x)} \right) \]  

(D.12)

\[ = \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3 (1 - x_1 - x_2 - x_3)}{(ax_1 + bx_2 + cx_3)(x_1 x_2 + x_1 x_3 + x_2 x_3)} \]  

(D.13)
\[
\int_0^1 dx \int_0^1 dz \frac{1}{c x (1 - x) z + (b (1 - x) + a x) (1 - z)}.
\] (D.14)

is symmetric w.r.t. interchange of its arguments function defined in [22]. Performing inversion and restoring \( r_0 \), we get

\[
\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3) = \left( \frac{3 \vec{r}_{23}^2}{4 \vec{r}_0^2 \vec{r}_0^2} - \frac{\vec{r}_{12}^2}{\vec{r}_0^2 \vec{r}_0^2} - \frac{\vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2 \vec{r}_{12}^2}{\vec{r}_0^2 \vec{r}_0^2 \vec{s}_{123}^2} \right)
\]

\[
+ \left( \frac{3 \vec{r}_{12}^2}{4 \vec{r}_0^2 \vec{r}_0^2} - \frac{3 \vec{r}_{13}^2}{4 \vec{r}_0^2 \vec{r}_0^2} - \frac{\vec{r}_{23}^2}{2 \vec{r}_0^2 \vec{r}_0^2} \right) \ln^2 \left( \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2 \vec{r}_0^2} \right)
\]

\[
+ \left( \frac{3 \vec{r}_{12}^2}{4 \vec{r}_0^2 \vec{r}_0^2} - \frac{3 \vec{r}_{13}^2}{4 \vec{r}_0^2 \vec{r}_0^2} - \frac{\vec{r}_{23}^2}{2 \vec{r}_0^2 \vec{r}_0^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2 \vec{r}_{12}^2}{\vec{r}_0^2 \vec{r}_0^2 \vec{r}_0^2} \right)
\]

\[
+ \frac{3}{2} \hat{S}_{123} \frac{\vec{r}_{12}^2}{\vec{r}_0^2 \vec{r}_0^2} \frac{\vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2} \frac{\vec{r}_{23}^2}{\vec{r}_0^2 \vec{r}_0^2} \delta(\vec{r}_{20})+ X \left( \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2} \frac{\vec{r}_{02}^2 \vec{r}_{23}^2}{\vec{r}_0^2 \vec{r}_0^2} \right) \delta(\vec{r}_{30})
\]

\[
+ Y \left( \frac{\vec{r}_{01}^2 \vec{r}_{23}^2}{\vec{r}_0^2 \vec{r}_0^2} \frac{\vec{r}_{02}^2 \vec{r}_{23}^2}{\vec{r}_0^2 \vec{r}_0^2} \right) \delta(\vec{r}_{10}).
\] (D.15)

Here

\[
\hat{S}_{123} = \left( \frac{\vec{r}_{12}^4}{\vec{r}_0^4 \vec{r}_0^4} + \frac{\vec{r}_{13}^4}{\vec{r}_0^4 \vec{r}_0^4} + \frac{\vec{r}_{24}^4}{\vec{r}_0^4 \vec{r}_0^4} - \frac{2 \vec{r}_{13}^2 \vec{r}_{12}^2}{\vec{r}_0^4 \vec{r}_0^4} - \frac{2 \vec{r}_{23}^2 \vec{r}_{12}^2}{\vec{r}_0^4 \vec{r}_0^4} - \frac{2 \vec{r}_{13}^2 \vec{r}_{23}^2}{\vec{r}_0^4 \vec{r}_0^4} \right)
\] (D.16)

and we added the delta-functional contributions, which may be lost via inversion. Thanks to conformal invariance of the integral such contributions may depend only on conformally invariant ratios. We can find the values of the unknown functions \( X \) and \( Y \) at \( \vec{r}_2 = \vec{r}_3, \vec{r}_2 = \vec{r}_1, \vec{r}_1 = \vec{r}_3 \). Using (7.9) we have

\[
\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_2 = \vec{r}_3} = 16 \int \frac{d\vec{r}_4}{\pi} L_{12}^0 = 24 \pi \zeta(3)|\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})|
\]

\[
= 2 X (1, \infty) \delta(\vec{r}_{20}) + Y (0, 0) \delta(\vec{r}_{10}).
\] (D.17)

Therefore

\[
X (1, \infty) = -12 \pi \zeta(3), \quad Y (0, 0) = 24 \pi \zeta(3).
\] (D.18)

\[
\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_1 = \vec{r}_3} = -4 \int \frac{d\vec{r}_4}{\pi} L_{12}^0 = -6 \pi \zeta(3)|\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})|
\]

\[
= X (0, 0) \delta(\vec{r}_{20}) + (Y (1, \infty) + X (\infty, 1)) \delta(\vec{r}_{10}).
\] (D.19)

Here again we used (7.9). Therefore

\[
X (0, 0) = 6 \pi \zeta(3), \quad Y (1, \infty) + X (\infty, 1) = -6 \pi \zeta(3).
\] (D.20)

If \( \vec{r}_2 \neq \vec{r}_3, \vec{r}_2 \neq \vec{r}_1, \vec{r}_1 \neq \vec{r}_3 \) then the arguments of \( X \) and \( Y \) are fixed by the integration w.r.t. \( \vec{r}_0 \)

\[
X \left( \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2} \frac{\vec{r}_{02}^2 \vec{r}_{13}^2}{\vec{r}_0^2 \vec{r}_0^2} \right) \delta(\vec{r}_{20}) = X (0, 0) \delta(\vec{r}_{20}) = 6 \pi \zeta(3) \delta(\vec{r}_{20}).
\] (D.21)
\[ Y \left( \frac{\vec{r}_{10}^2}{r_{01}^2}, \frac{\vec{r}_{23}^2}{r_{02}^2}, \frac{\vec{r}_{02}^2}{r_{03}^2}, \frac{\vec{r}_{12}^2}{r_{01}^2}, \frac{\vec{r}_{13}^2}{r_{01}^2} \right) \delta (\vec{r}_{10}) = Y (0, 0) \delta (\vec{r}_{10}) = 24\pi \zeta (3) \delta (\vec{r}_{10}). \]  
\[ \text{(D.22)} \]

As a result, one can write

\[ \int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3) = \left( \frac{3\vec{r}_{12}^2}{4r_{01}^2r_{02}^2} - \frac{\vec{r}_{13}^2}{2r_{01}^2r_{03}^2} - \frac{\vec{r}_{12}^2}{r_{02}^2r_{13}^2} \right) \ln^2 \left( \frac{\vec{r}_{02}^2\vec{r}_{13}^2}{r_{02}^2r_{13}^2} \right) \]
\[ + \left( \frac{3\vec{r}_{12}^2}{4r_{01}^2r_{02}^2} - \frac{3\vec{r}_{13}^2}{4r_{01}^2r_{03}^2} - \frac{\vec{r}_{23}^2}{r_{02}^2r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{02}^2\vec{r}_{13}^2}{r_{02}^2r_{13}^2} \right) \]
\[ + \left( \frac{3\vec{r}_{12}^2}{4r_{01}^2r_{02}^2} - \frac{3\vec{r}_{13}^2}{4r_{01}^2r_{03}^2} - \frac{\vec{r}_{23}^2}{r_{02}^2r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2\vec{r}_{12}^2}{r_{03}^2r_{12}^2} \right) \]
\[ + \frac{3}{2} S_{123} I \left( \frac{\vec{r}_{12}^2}{r_{01}^2r_{02}^2}, \frac{\vec{r}_{13}^2}{r_{01}^2r_{03}^2}, \frac{\vec{r}_{23}^2}{r_{02}^2r_{03}^2} \right) \]
\[ + 6\pi \zeta (3) (\delta (\vec{r}_{20}) + \delta (\vec{r}_{30})) + 24\pi \zeta (3) \delta (\vec{r}_{10}). \]
\[ \text{(D.23)} \]

Here \( \delta_{ij} = 1 \), if \( \vec{r}_1 = \vec{r}_j \) and \( \delta_{ij} = 0 \) otherwise. The last term is added since the total contribution at \( \vec{r}_1 = \vec{r}_2 = \vec{r}_3 \) is 0.

Now we will integrate \( F_{230} \) (7.21) w.r.t. \( \vec{r}_4 \). Again we set \( \vec{r}_0 = 0 \), do inversion, and calculate the integral in the d-dimensional space using the integrals from appendix A in [21] and (D.8). We get

\[ \int \frac{d\vec{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) \to \int \frac{d^4r_4}{\pi} \left( \frac{r_{12}^2}{2r_{14}^2r_{24}^2} \ln \left( \frac{r_{12}^4r_{34}^6}{r_{24}^2r_{12}^6} \right) + \frac{r_{12}^2}{2r_{14}^2} \ln \left( \frac{r_{12}^4r_{34}^8}{r_{24}^2r_{12}^6} \right) \right. \]
\[ - \frac{r_{12}^2}{2r_{24}^2} \ln \left( \frac{r_{12}^4}{r_{12}^4} \right) - \frac{r_{23}^2}{r_{14}^2} \ln \left( \frac{r_{13}^2r_{24}^2}{r_{14}^2} \right) \]
\[ - \frac{r_{23}^2}{2r_{24}^2} \ln \left( \frac{r_{23}^2r_{34}^2}{r_{24}^2r_{23}^2} \right) + \frac{r_{12}^2}{2r_{24}^2} \ln \left( \frac{r_{13}^2r_{24}^2}{r_{14}^2r_{34}^2} \right) \]
\[ - \frac{r_{23}^2}{r_{14}^2} \ln \left( \frac{r_{12}^2r_{24}^2}{r_{24}^2r_{12}^2} \right) - \frac{r_{13}^2}{r_{14}^2} \ln \left( \frac{r_{13}^2r_{24}^2}{r_{24}^2r_{13}^2} \right) \]
\[ - \frac{3}{4} r_{23}^2 \ln^2 \left( \frac{r_{12}^2}{r_{13}^2} \right) - \frac{1}{2} S_{123} I \left( r_{12}^2, r_{13}^2, r_{23}^2 \right). \]
\[ \text{(D.24)} \]

Again, inverting and restoring \( r_0 \) we have

\[ \int \frac{d\vec{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) = \left( \frac{\vec{r}_{12}^2}{2r_{01}^2r_{02}^2} + \frac{\vec{r}_{13}^2}{2r_{01}^2r_{03}^2} - \frac{3\vec{r}_{23}^2}{4r_{01}^2r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2\vec{r}_{12}^2}{r_{02}^2r_{13}^2} \right) \]
\[ + \left( \frac{3\vec{r}_{13}^2}{4r_{01}^2r_{03}^2} - \frac{3\vec{r}_{12}^2}{4r_{01}^2r_{02}^2} - \frac{\vec{r}_{23}^2}{r_{02}^2r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{02}^2\vec{r}_{13}^2}{r_{03}^2r_{13}^2} \right) \]
\[ + \left( \frac{3\vec{r}_{12}^2}{4r_{01}^2r_{02}^2} - \frac{3\vec{r}_{13}^2}{4r_{01}^2r_{03}^2} + \frac{\vec{r}_{23}^2}{r_{02}^2r_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2\vec{r}_{12}^2}{r_{01}^2r_{23}^2} \right) \]
\[
- \frac{3}{2} \tilde{S}_{123} I \left( \frac{\tilde{r}_{12}^2}{\tilde{r}_{01}^2 \tilde{r}_{02}^2}, \frac{\tilde{r}_{13}^2}{\tilde{r}_{01}^2 \tilde{r}_{03}^2}, \frac{\tilde{r}_{23}^2}{\tilde{r}_{02}^2 \tilde{r}_{03}^2} \right) \\
+ \tilde{X} \left( \frac{\tilde{r}_{02}^2 \tilde{r}_{13}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2}, \frac{\tilde{r}_{02}^2 \tilde{r}_{13}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2} \right) \delta (\tilde{r}_{20}) + \tilde{X} \left( \frac{\tilde{r}_{03}^2 \tilde{r}_{12}^2}{\tilde{r}_{02}^2 \tilde{r}_{13}^2}, \frac{\tilde{r}_{03}^2 \tilde{r}_{12}^2}{\tilde{r}_{02}^2 \tilde{r}_{13}^2} \right) \delta (\tilde{r}_{30}) \\
+ \tilde{Y} \left( \frac{\tilde{r}_{01}^2 \tilde{r}_{23}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2}, \frac{\tilde{r}_{01}^2 \tilde{r}_{23}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2} \right) \delta (\tilde{r}_{10}).
\]

(D.26)

Again, we can find the values of \(\tilde{X}\) and \(\tilde{Y}\) putting \(\tilde{r}_2 = \tilde{r}_3, \tilde{r}_2 = \tilde{r}_1, \tilde{r}_1 = \tilde{r}_3\) in this equation. Indeed via (7.9) we have,

\[
\int \frac{d\tilde{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) |_{\tilde{r}_2 = \tilde{r}_3} = -8 \int \frac{d\tilde{r}_4}{\pi} L_{12}^C = -12\pi \zeta (3) [\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})]
\]

\[
= 2 \tilde{X} (1, \infty) \delta (\tilde{r}_{20}) + \tilde{Y} (0, 0) \delta (\tilde{r}_{10}).
\]

(D.27)

Therefore

\[
\tilde{X} (1, \infty) = 6\pi \zeta (3), \quad \tilde{Y} (0, 0) = -12\pi \zeta (3).
\]

(D.28)

Using (7.9) again, we get

\[
\int \frac{d\tilde{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) |_{\tilde{r}_1 = \tilde{r}_3} = 8 \int \frac{d\tilde{r}_4}{\pi} L_{12}^C = 12\pi \zeta (3) [\delta (\tilde{r}_{10}) - \delta (\tilde{r}_{20})]
\]

\[
= \tilde{X} (0, 0) \delta (\tilde{r}_{20}) + \left( \tilde{X} (\infty, 1) + \tilde{Y} (1, \infty) \right) \delta (\tilde{r}_{10}).
\]

(D.29)

Therefore

\[
\tilde{X} (0, 0) = -12\pi \zeta (3), \quad \tilde{Y} (1, \infty) + \tilde{X} (\infty, 1) = 12\pi \zeta (3).
\]

(D.30)

If \(\tilde{r}_2 \neq \tilde{r}_3, \tilde{r}_2 \neq \tilde{r}_1, \tilde{r}_1 \neq \tilde{r}_3\) then the arguments of \(\tilde{X}\) and \(\tilde{Y}\) are fixed by the integration w.r.t. \(\tilde{r}_0\)

\[
\tilde{X} \left( \frac{\tilde{r}_{02}^2 \tilde{r}_{13}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2}, \frac{\tilde{r}_{02}^2 \tilde{r}_{13}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2} \right) \delta (\tilde{r}_{20}) = \tilde{X} (0, 0) \delta (\tilde{r}_{20}) = -12\pi \zeta (3) \delta (\tilde{r}_{20}),
\]

(D.31)

\[
\tilde{Y} \left( \frac{\tilde{r}_{01}^2 \tilde{r}_{23}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2}, \frac{\tilde{r}_{01}^2 \tilde{r}_{23}^2}{\tilde{r}_{03}^2 \tilde{r}_{12}^2} \right) \delta (\tilde{r}_{10}) = \tilde{Y} (0, 0) \delta (\tilde{r}_{10}) = -12\pi \zeta (3) \delta (\tilde{r}_{10}).
\]

(D.32)

Finally,

\[
\int \frac{d\tilde{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) = \left( \frac{\tilde{r}_{12}^2}{2\tilde{r}_{01}^2 \tilde{r}_{02}^2} + \frac{\tilde{r}_{13}^2}{2\tilde{r}_{01}^2 \tilde{r}_{03}^2} - \frac{3\tilde{r}_{23}^2}{4\tilde{r}_{02}^2 \tilde{r}_{03}^2} \right) \ln^2 \left( \frac{\tilde{r}_{03}^2 \tilde{r}_{12}^2}{\tilde{r}_{02}^2 \tilde{r}_{13}^2} \right)
\]

\[
+ \left( \frac{3\tilde{r}_{13}^2}{4\tilde{r}_{01}^2 \tilde{r}_{03}^2} - \frac{3\tilde{r}_{12}^2}{4\tilde{r}_{01}^2 \tilde{r}_{02}^2} + \frac{\tilde{r}_{23}^2}{\tilde{r}_{02}^2 \tilde{r}_{03}^2} \right) \ln^2 \left( \frac{\tilde{r}_{02}^2 \tilde{r}_{13}^2}{\tilde{r}_{01}^2 \tilde{r}_{23}^2} \right)
\]

\[
+ \left( \frac{3\tilde{r}_{12}^2}{4\tilde{r}_{01}^2 \tilde{r}_{02}^2} - \frac{3\tilde{r}_{13}^2}{4\tilde{r}_{01}^2 \tilde{r}_{03}^2} + \frac{\tilde{r}_{23}^2}{\tilde{r}_{02}^2 \tilde{r}_{03}^2} \right) \ln^2 \left( \frac{\tilde{r}_{03}^2 \tilde{r}_{12}^2}{\tilde{r}_{02}^2 \tilde{r}_{23}^2} \right)
\]

\[
- \frac{3}{2} \tilde{S}_{123} I \left( \frac{\tilde{r}_{12}^2}{\tilde{r}_{01}^2 \tilde{r}_{02}^2}, \frac{\tilde{r}_{13}^2}{\tilde{r}_{01}^2 \tilde{r}_{03}^2}, \frac{\tilde{r}_{23}^2}{\tilde{r}_{02}^2 \tilde{r}_{03}^2} \right)
\]

\[
- 12\pi \zeta (3) (\delta (\tilde{r}_{20}) + \delta (\tilde{r}_{30}) + \delta (\tilde{r}_{10})).
\]
\[ + 36 \pi \zeta(3) \delta_{23}(\vec{r}_{20}) + 36 \pi \zeta(3)(\delta_{13} + \delta_{12})\delta(\vec{r}_{10}) - 72 \pi \zeta(3)\delta_{13}\delta_{12}\delta(\vec{r}_{10}) \, . \] (D.33)

Now we will integrate (7.19) and prove equality (7.44). Again we set \( \vec{r}_0 = 0 \), do inversion, and calculate the integral in the \( d \)-dimensional space using the integrals from appendix A in [21] and (D.8). We get

\[
\int \frac{d^d \vec{r}_4}{\pi} \left( \{ F_{140} + (0 \leftrightarrow 4) \} + (2 \leftrightarrow 3) \right) \rightarrow \int \frac{d^d \vec{r}_4}{\pi} \left( \frac{r_{12}^2}{r_{14}^2} \ln \left( \frac{r_{12}^2 r_{34}^2}{r_{14}^2 r_{24}^2} \right) \right.
\]

\[
+ \frac{r_{12}^2}{r_{24}^2} \ln \left( \frac{r_{24}^2 r_{12}^2}{r_{34}^2} \right) - \frac{r_{24}^2 r_{12}^2}{r_{14}^2 r_{24}^2} \ln \left( \frac{r_{14}^2 r_{24}^2}{r_{12}^2 r_{34}^2} \right) - \frac{r_{23}^2}{r_{14}^2} \ln \left( \frac{r_{14}^2 r_{23}^2}{r_{12}^2 r_{34}^2} \right) - \frac{r_{23}^2}{r_{34}^2} \ln \left( \frac{r_{23}^2}{r_{24}^2} \right) + \frac{r_{13}^2}{r_{14}^2} \ln \left( \frac{r_{13}^2}{r_{24}^2} \right) + \frac{r_{34}^2}{r_{14}^2} \ln \left( \frac{r_{14}^2}{r_{24}^2} \right) + \frac{r_{34}^2}{r_{24}^2} \ln \left( \frac{r_{34}^2}{r_{23}^2} \right) + (2 \leftrightarrow 3) \, . \] (D.34)

Inverting and restoring \( \vec{r}_0 \), we get

\[
\int \frac{d^d \vec{r}_4}{\pi} \left( \{ F_{140} + (0 \leftrightarrow 4) \} + (2 \leftrightarrow 3) \right) = - \frac{1}{2} \left( \frac{\vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{02}^2} + \frac{\vec{r}_{13}^2}{\vec{r}_{01}^2 \vec{r}_{03}^2} \right) \ln^2 \left( \frac{\vec{r}_{03}^2 \vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{13}^2} \right)
\]

\[
+ \frac{\vec{r}_{23}^2}{2 \vec{r}_{02}^2 \vec{r}_{03}^2} \ln^2 \left( \frac{\vec{r}_{03}^2 \vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{23}^2} \right) + \frac{\vec{r}_{23}^2}{2 \vec{r}_{02}^2 \vec{r}_{03}^2} \ln^2 \left( \frac{\vec{r}_{03}^2 \vec{r}_{12}^2}{\vec{r}_{01}^2 \vec{r}_{23}^2} \right) . \] (D.36)

This integral has no delta functional contributions since it equals 0 at \( \vec{r}_1 = \vec{r}_2, \vec{r}_1 = \vec{r}_3, \vec{r}_3 = \vec{r}_2 \).

### E Decomposition of C-odd quadrupole operator

Here we demonstrate that the C-odd part of the quadrupole operator \( tr(U_1 U_3^I U_3^I) \) in the 3-gluon approximation in \( SU(3) \) can be decomposed into a sum of 3QWLs. Indeed

\[
2 tr(U_1 U_3^I U_3^I) = \left( (U_1 - U_2)(U_2^I - U_3^I)(U_3^I - U_4) \right) \cdot U_4 \cdot U_4
\]

\[
- B_{133} + B_{233} + B_{144} - B_{244} + B_{344} + B_{122} - 6 \] (E.1)

\[
\Rightarrow -(U_1 - U_2)(U_2 - U_3)(U_3 - U_4) \cdot E \cdot E - B_{133} + B_{233} + B_{144} - B_{244} + B_{344} + B_{122} - 6 . \] (E.2)

Therefore

\[
2 tr(U_1 U_3^I U_3^I) - 2 tr(U_3 U_3^I U_3^I) \Rightarrow - B_{133} + B_{233} + B_{144} - B_{244} + B_{344} + B_{122}
\]

\[
- ((U_1 - U_2)(U_2 - U_3)(U_3 - U_4) + (U_3 - U_4)(U_2 - U_3)(U_1 - U_2)) \cdot E \cdot E
\]

\[
\Rightarrow - B_{133} + B_{233} + B_{144} - B_{244} + B_{344} + B_{122} - 2(U_1 - U_2) \cdot (U_2 - U_3) \cdot (U_3 - U_4)
\]

\[
\Rightarrow - B_{133} + B_{233} + B_{144} - B_{244} + B_{344} + B_{122}
\]

\[
- (U_1 - U_2) \cdot (U_2 - U_3) \cdot (U_3 - U_4) + (U_3 - U_4) \cdot (U_2 - U_3) \cdot (U_3 - U_4)
\]

\[
= B_{144} + B_{233} - B_{211} + B_{124} + B_{324} - B_{123} - B_{134} . \] (E.3)
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