The common treatment of time-dependent potentials, such as those used for radio frequency cavities, is to average a potential’s time component through the interval that the reference particle spends in the cavity. Such an approach, using the so-called transit-time factor, uses time as the independent variable in the Hamiltonian. In this paper, we instead propose a fully covariant Hamiltonian to treat the time component of the potential like any other space component. We show how to calculate the dynamics of the particles in a pill-box cavity using an explicit symplectic integrator. Finally, we compare the results with the simulator TraceWin.

I. INTRODUCTION

When a charged particle passes through an accelerator element which generates a time-dependent field, this yields a Hamiltonian which is no longer a constant of motion. This requires special techniques to handle the time component, such as the Transit-Time Factor for RF cavities, where the time component is averaged and removed from the Hamiltonian.

The time dependency of the Hamiltonian can also be removed if we consider the covariant Hamiltonian, in an 8-dimensional phase space, where together with the usual 6-dimensions ($x, P_x; y, P_y; z, P_z$) we also have time and its canonical conjugate ($t, P_t$). In this phase space, the Hamiltonian $H = H(x, y, z, t, P_x, P_y, P_z, P_t)$ is always a constant of motion and will satisfy four pairs of Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial P^{\mu}}$$

$$\frac{dP^{\mu}}{d\tau} = -\frac{\partial H}{\partial x^\mu}$$

where $\tau$ is the “proper” time of the particle—the time in a reference frame that moves with the particle; $x^\mu = ct, x, y, z$ when $\mu = 0, 1, 2, 3$; and $P^\mu$, defined rigorously in the next section, are the corresponding momenta.

The solutions of these equations of Hamilton can be obtained in many ways, the technique used to solve them depends on the form of the potential used. Expanding on preliminary studies\cite{1}, the potential considered in the example of this paper is a simple pill-box cavity. Such a potential can be split as explained in \cite{2}, so we will adopt that technique to have an explicit sympletic integrator.

The results obtained with this method will be compared with the standard simulator used at the European Spallation Source, TraceWin \cite{3}.

II. HAMILTONIAN

The Lorentz invariant

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = c^2 d\tau^2$$

leads to Lagrangian

$$L = \frac{m}{2} v^\mu v_\mu$$

where

$$v^\mu = \begin{pmatrix} c dt \\ dx \frac{dt}{d\tau} \\ dy \frac{dt}{d\tau} \\ dz \frac{dt}{d\tau} \end{pmatrix}$$

$$v_\mu = \begin{pmatrix} -c dt \\ dx \frac{dt}{d\tau} \\ dy \frac{dt}{d\tau} \\ dz \frac{dt}{d\tau} \end{pmatrix}.$$}

With an external electromagnetic quadri-potential $A^\mu = (\frac{\phi}{c}, A_x, A_y, A_z)$ the Lagrangian is \cite{4}

$$L = \frac{m}{2} v_\mu v^\mu + q v_\mu A^\mu$$

the corresponding Hamiltonian is calculated with the Legendre transform as

$$P_\mu = \frac{\partial L}{\partial v^\mu}$$

$$H = P_\mu v^\mu - L.$$ 

This Hamiltonian generates eight equations of motion with the constraint $v_\mu v^\mu = c^2$ so not all the variables are independent.

III. THE PILL-BOX CAVITY

The case discussed here is the simple pill-box cavity with the simplest accelerating mode (TM$_{01}$). The electric and magnetic fields, expressed in cylindrical coordinates plus time ($t, r, \theta, z$) are \cite{5}

$$E_z = E_0 J_0 \left( \frac{p_0}{a} r \right) \sin(\omega t + \phi_0)$$

$$B_\theta = E_0 \frac{c}{a} J_1 \left( \frac{p_0}{a} r \right) \cos(\omega t + \phi_0)$$

$$E_r = E_\theta = B_r = B_z = 0.$$
$J_0$ and $J_1$ are the Bessel functions of kind 0 and 1; $p_{01}$ is the first zero of $J_0$, that is, $J_0(p_{01}) = 0$; $a$ is the radius of the pill box cavity (this leaves the electric field as zero when $r = a$); $\omega$ is the oscillation frequency of the field in the cavity multiplied by 2\pi which is fixed by the cavity aperture through the relationship $\omega = c \frac{p_{01}}{a}$, and $\phi_0$ is the phase that the particle sees when it arrives to the entrance of the cavity.

These fields can be expressed as a potential:

$$A_r = A_\theta = 0$$  \hspace{1cm} (13)

$$A_z = \frac{E_0}{\omega} J_0\left(\frac{p_{01}}{a} r\right) \cos(\omega t + \phi_0).$$  \hspace{1cm} (14)

Because the potential is only in the $z$ direction, we do not need to transform the Lagrangian in cylindrical coordinates, but we can use the potential directly in the calculations recalling that $r = \sqrt{x^2 + y^2}$. The Lagrangian is

$$L = \frac{m}{2} \left( v_0 v^0 + v_1 v^1 + v_2 v^2 + v_3 v^3 \right) + q v^3 A_3.$$  \hspace{1cm} (15)

The conjugate momenta are then

$$P_0 = \frac{\partial L}{\partial v^0} = m v_0$$  \hspace{1cm} (16)

$$P_1 = \frac{\partial L}{\partial v^1} = m v_1$$  \hspace{1cm} (17)

$$P_2 = \frac{\partial L}{\partial v^2} = m v_2$$  \hspace{1cm} (18)

$$P_3 = \frac{\partial L}{\partial v^3} = m v_3 + q A_3$$  \hspace{1cm} (19)

so we can rewrite the 4-velocity as

$$v^\mu = \left( \frac{P^0}{m}, \frac{P^1}{m}, \frac{P^2}{m}, \frac{P^3 - q A^3}{m} \right).$$  \hspace{1cm} (20)

The Hamiltonian is

$$H = \frac{P_0 P^0}{2m} + \frac{P_1 P^1}{2m} + \frac{P_2 P^2}{2m} + \frac{(P_3 - q A^3)(P^3 - q A^3)}{2m} - L$$  \hspace{1cm} (21)

$$= \frac{P_0^2}{2m} + \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{(P_3 - q A^3)^2}{2m}.$$  \hspace{1cm} (22)

The Hamiltonian (22) can be treated with the explicit symplectic integrator first developed in [2] and discussed in [6] Chapter 12 Section 9.

**IV. THE ALGORITHM**

The only problematic term in the Hamiltonian (22) is the one in $z$ because it mixes the momentum $P_z$ with the position contained in $A_z$. The idea of our integrator is to create a new function $U_z(t, x, y, z)$ such that $A_z = \frac{\partial U_z}{\partial z}$. In our case

$$U_z = \frac{E_0}{\omega} \frac{J_0\left(\frac{p_{01}}{a} r\right)}{J_1\left(\frac{p_{01}}{a} r\right)} m (\cos(\omega t + \phi_0)) z.$$  \hspace{1cm} (23)

The Lie transform of $U_z$ has the property:

$$e^{-q U_z}: z = z$$  \hspace{1cm} (24)

$$e^{-q U_z}: P_y = P_y - q \frac{\partial U_z}{\partial y}$$  \hspace{1cm} (25)

and as consequence we have

$$e^{-q U_z}: e^{-\frac{h}{2m} P_y^2} e^{q U_z}: = e^{-\frac{h}{2m}(P_y - q A_3)^2}.$$  \hspace{1cm} (26)

This splitting technique is similar to the usual drift-kick-drift, but here the term $U_z$ can be seen as a gauge transformation instead of a kick. Calling $K_1 = -\frac{p_3^2}{2m}$ and $K_i = \frac{p_i^2}{2m}$ with $i = x, y, z$ we have the second order explicit symplectic integrator as

$$S_2 = e^{-\frac{h}{2m} K_x} e^{-\frac{h}{2m} K_y} e^{-q U_z} e^{-\frac{h}{2m} K_z} e^{q U_z} \times \nonumber$$

$$e^{-h: K_i} \times \nonumber$$

$$e^{-q U_z} e^{-\frac{h}{2m} K_x} e^{q U_z} e^{-\frac{h}{2m} K_y} e^{-\frac{h}{2m} K_z}.$$  \hspace{1cm} (27)

This integrator can be extended to higher order integrators applying the technique of Yoshida [7] or the one of Suzuki [8] both techniques are explored in details in [9].

Every step of the integrator has to be evaluated on coordinates and momenta. The only terms different from the identity map are:

$$e^{-\frac{h}{2m} K_i}: i = x, y, z$$

$$e^{-\frac{h}{2m} K_i}: t = t - \frac{h}{m} P_i$$

$$e^{q U_z}: P_x = P_x - \frac{q x z E_0 p_{01}}{a r \omega} J_1\left(\frac{p_{01}}{a} r\right) m \cos(\omega t + \phi_0)$$

$$e^{q U_z}: P_y = P_y - \frac{q y z E_0 p_{01}}{a r \omega} J_1\left(\frac{p_{01}}{a} r\right) m \cos(\omega t + \phi_0)$$

$$e^{q U_z}: P_z = P_z - \frac{q E_0}{\omega} \frac{J_0\left(\frac{p_{01}}{a} r\right) m}{J_1\left(\frac{p_{01}}{a} r\right)} \sin(\omega t + \phi_0).$$  \hspace{1cm} (28)

**V. NUMERICAL RESULTS**

In order to compare the results of this symplectic integrator with an ordinary simulator as, for example, TraceWin [3] we need results in the same reference frame. The first step is thus to find the transformation from the covariant Hamiltonian to the Hamiltonian in the laboratory frame that uses $t$ as the independent variable. This can be done using Eq. (3) noting that $\frac{d}{dt} = \gamma$. From the
Using the fact that \( v^2 = \left( 1 - \frac{1}{\gamma^2} \right) \), we can connect the spatial component with the time component

\[
c^P_i = \gamma mc^2 + q\phi
\]

and this is

\[
c^P_0 = c\sqrt{(\vec{P} - q\vec{A})^2 + m^2c^2} + q\phi
\]

where the vector notation refers to the three spatial coordinates and the square is the norm square of the vector.

Using the fact that \( v^2 = \left( 1 - \frac{1}{\gamma^2} \right) \), we can connect the spatial component with the time component

\[
c^P_i = \gamma mc^2 + q\phi
\]

where the vector notation refers to the three spatial coordinates and the square is the norm square of the vector.

VI. CONCLUSIONS

We presented an algorithm for performing symplectic integration of Hamiltonian when the external potential depends on time. The main idea is to use the full covariant Hamiltonian and to integrate the field in 8-dimensional phase space applying a Lie-operator method to the field component as was done for \( s \)-dependent fields in [2]. A comparison with TraceWin, a well-established particle simulator, shows that the method reproduces the correct dynamics for the simple case of a pill-box cavity. More complicated potentials are possible if they respect the separability expressed by the equation [26].
FIG. 2. Comparison of the phase space on the three axes starting from the same initial distribution for TraceWin and our Covariant Hamiltonian algorithm.

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