Color Grosse-Wulkenhaar models: One-loop $\beta$-functions\footnote{1}

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Abstract

The $\beta$-functions of $O(N)$ and $U(N)$ invariant Grosse-Wulkenhaar models are computed at one loop using the matrix basis. In particular, for “parallel interactions”, the model is proved asymptotically free in the UV limit for $N > 1$, and has a triviality problem or Landau ghost for $N < 1$. The vanishing $\beta$-function is recovered solely at $N = 1$. We discuss various possible consequences of these results.

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1 Introduction

Noncommutative (NC) quantum field theory (NCQFT) [1] receives an increasing attention of the theorist community since the advent of a class of renormalizable theories built around the Grosse and Wulkenhaar model (GWm) [2, 3]. This model is a NC $\phi^4$ scalar field theory on the Moyal-Weyl Euclidean space with a particular modification of the propagator. The GWm is dual in the sense of Langmann-Szabo (LS) [4], i.e. considers in a dual manner positions and momenta. This requires the inclusion of an harmonic term in the “naive” NC $\phi^4$ theory which allows to circumvent the deadlock of UV/IR mixing and to insure the renormalizability at all orders of perturbations [2, 5].

As a corollary, a series of fascinating facts about the GWm have been highlighted [3, 5]-[11] using different field theory techniques, and other models have been proved renormalizable using the same ideas [12]. More specifically, the study of the renormalization group (RG) flow has been investigated in detail [10][11]. The subsumption of a vanishing $\beta$-function, obtained at one loop in [6], has been finally achieved at all orders in [11] (with $\Omega = 1$) with as major consequence that the GWm has no Landau ghost (Lg) or triviality problem. The Lg means that, if the renormalized coupling constant is kept fixed and small, the bare constant increases without apparent bound as the UV cutoff is removed. Equivalently, triviality means that the renormalized coupling constant vanishes if the bare constant is kept bounded and small as the UV cutoff is removed. These twin diseases affect all quantum field theories except non Abelian gauge theories, and it was completely unexpected that they do not occur in the GWm. Moreover, the theory is not asymptotically free either, but asymptotically safe: both bare and renormalized coupling constants remain bounded after removing the UV cutoff. A nice argument in order to understand the absence of the usual “charge screening” phenomenon is the following [10]. At one loop, the wave function renormalization (wfr) does not vanish in contradistinction with the wfr of the commutative theory. Taking into account this wfr, the $\beta$-function hitherto positive for $\Omega < 1$, tends to zero as $\Omega$ tends to 1. In a way, the RG flow grinds to a halt at $\Omega = 1$ because the LS-duality at that point renders perfectly indistinguishable positions and momenta. The above scenario called “death of Landau ghost” has been shown to hold to all orders of perturbations by a combination of Ward identities and Dyson-Schwinger equations [11]. Thus, quantum field theories on NC geometry are, at least in this sense, better behaved than ordinary ones and the GWm is a promising candidate for the constructive program [13].

$O(N)$ and $U(N)$ invariant NC models with a $N$-valued color index have been considered in [14], and the vacuum and symmetry breaking of GW models of this type has been investigated in [15]. In this paper, we investigate the $\beta$-functions of such models. It should be emphasized that the limits $N \to 0$ and $N \to \infty$ could be both of special interest, for they are related, respectively, to polymers with non-local self-avoiding in-

\footnote{Indeed, the commutative tadpole is local and then induces a null contribution to the wfr.}
interactions \[16\] and to a kind of solvable spherical \[17\] GWm. We focus, hereunder, on the one-loop computations recalling that the behavior of the coupling constant flow is really determined by the sign of the first non vanishing coefficient of the $\beta$-function. In contrast, with $N = 1$, we find non zero coefficients at one loop order, so without further computations, we can reach conclusions on the UV behavior of these rotation invariant GW models. For the class of $O(N)$ models with “parallel interaction”, we find that such models are asymptotically free in the UV regime for $N > 1$, and hence also susceptible of a full constructive analysis. The model at $N = 0$ is not asymptotically free. The $U(N)$ invariant complex NC $\phi_4^4$ theory is also discussed and has similar features according to the value of $N$. Finally let us mention that asymptotically safe models of the GWm type with vanishing $\beta$-function at all orders can be obtained by adding a magnetic field \[18\].

The paper is organized as follows. The next section is devoted to notations and general considerations for the $O(N)$ GWm. The complex case construction and treatment are deferred in Section 3 as a direct generalization of the real analysis. We assume the knowledge of renormalization and effective expansions as developed for instance in \[13\], and of NC field theories in the matrix basis as treated in \[2, 10, 11\]. Section 3 gives our main result, its proof and further discussions. The conclusion of this study is given in Section 4.

2 The $O(N)$ Grosse-Wulkenhaar model: Notations and considerations

Let us consider a real vector field $(\phi^a)_{a=1,...,N}$ theory, with $N \in \mathbb{N}$, defined by the GWm action in the NC Euclidean spacetime $\mathbb{R}^4$ \[2\]

$$S = \int d^4x \left\{ \sum_{a=1}^{N} \left( \frac{1}{2} \partial_\mu \phi^a \star \partial^\mu \phi^a + \frac{\mu}{2} (\phi^a)^2 + \frac{\Omega}{2} (\tilde{x} \phi^a)^2 \right) + \frac{\lambda_1}{2} \sum_{c,d=1}^{N} \phi^c \star \phi^c \star \phi^d \star \phi^d + \frac{\lambda_2}{4} \sum_{c,d=1}^{N} \phi^c \star \phi^d \star \phi^c \star \phi^d \right\}, \tag{2.1}$$

where $\tilde{x}_\nu = 2(\theta^{-1}_\nu x^\mu)$, $\theta^{-1}_\nu$ being the inverse of the anticommutative matrix associated with the Moyal $\star$-product. Two coupling constants $\lambda_1$ and $\lambda_2$ have been introduced for the two natural interactions in the quartic term. Now, we write the theory in the matrix basis and use the simpler normalizations of Ref.\[11\], with $\Omega = \frac{1}{3}$. The bare propagator is given by, for all $m, n, k, l \in \mathbb{N}^2$ and $\delta_{mn} := \delta_{m_1n_1} \delta_{m_2n_2},$

$$C_{mn;kl} = G_{mn} \delta_{ml} \delta_{nk}; \quad G_{mn} = (m + n + A)^{-1}; \quad m + n := m_1 + m_2 + n_1 + n_2, \tag{2.2}$$

\[4\]For any $N \geq 0$, we can check in (2.9) that indeed the RG flow leads to $\Omega = 1$ in the UV regime as in the $N = 1$ case.
with $A = 2 + \mu^2/4$, and the vertices take the form (henceforth, implicit sum from 1 to $N$ over repeated color indices is used)

$$V_1 = \frac{\lambda_1}{2} \sum_{m,n,k,l \in \mathbb{N}^2} \phi_{mn}^c \phi_{nk}^c \phi_{kl}^d \phi_{lm}^d, \quad V_2 = \frac{\lambda_2}{4} \sum_{m,n,k,l \in \mathbb{N}^2} \phi_{mn}^c \phi_{nk}^d \phi_{kl}^c \phi_{lm}^d.$$  \hfill (2.3)

We call $V_1$ the “parallel” vertex and $V_2$ the “crossed” vertex. After these manipulations, the action can be written as

$$S = \frac{1}{2} \sum_{m,n \in \mathbb{N}^2} \phi_{mn}^a G_{mn}^{-1} \phi_{nm}^a + V_1 + V_2.$$  \hfill (2.4)

**Renormalizability of the model.** At $\Omega = 1$, the propagator is diagonal in the color index and independent of its value and it is the same as in the ordinary GWm. Therefore, a slice decomposition of the propagator identical to the one of [5] leads to an identical power counting. One concludes that, as in the $N = 1$ case, the only divergent contributions come from graphs with two or four external legs with genus zero and exactly one broken external face. The techniques of subtraction of logarithmically divergent graphs and of mass and wave function renormalizations used in the GWm can be applied here. Hence, the $O(N)$ GWm as defined in (2.4) is renormalizable.

**Goals.** We want to compute at one loop the dynamics of the effective constant couplings, say

$$\lambda_{1,r} = -\frac{\Gamma_{4,||}(0,0,0,0)}{Z^2}, \quad \lambda_{2,r} = -\frac{\Gamma_{4,\times}(0,0,0,0)}{Z^2},$$  \hfill (2.5)

where the wave function renormalization is

$$Z = 1 - \partial_{m_1} \Sigma(m,n)|_{m=0=n}$$  \hfill (2.6)

and the self-energy $\Sigma(m,n)$ is the sum of the amputated one particle irreducible (1PI) amplitudes of the two point correlation function

$$\Sigma(m,n) = \langle \phi_{mn}^a \phi_{nm}^a \rangle_{1PI}.$$  \hfill (2.7)

The amputated 1PI four point functions in (2.5) are

$$\Gamma_{4,||}(m,n,k,l) = \langle \phi_{mn}^a \phi_{nk}^b \phi_{kl}^b \phi_{lm}^b \rangle_{1PI}, \quad \Gamma_{4,\times}(m,n,k,l) = \langle \phi_{mn}^a \phi_{nk}^b \phi_{kl}^a \phi_{lm}^b \rangle_{1PI}.$$  \hfill (2.8)

Note that $a$ and $b$ are fixed in equations (2.7) and (2.8). Furthermore, the derivative taken on $m_1$ (2.6) is actually a matter of choice since one obtains the same result by deriving by the external indices $m_2,n_1,n_2$ and putting the remaining to 0.

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*For $\Omega = 1$, the model is a pure matrix theory, in the sense that the probability distribution is independent for each matrix coefficient although it is not identically distributed (often called i.n.i.d.). As already remarked, for any fixed $N \geq 0$, the flow of $\Omega$ always goes rapidly to $\Omega = 1$ in the UV which is of course the small distance limit of interest [10].*
The flow of the $\Omega$ parameter, at one loop and up to a constant, is given as in [2] by

$$\Omega^2_r = \frac{1 - \Omega^2}{\Omega} \left(1 - \partial_{m_1} \Sigma(m, n)|_{m=0=n}\right) \quad (2.9)$$

and induces, for any $N \geq 0$, the UV fixed point $\Omega_{bare} = 1$ (see below Eq.(3.4)).

**Feynman rules.** In the scalar GWm, the Feynman rules are expressed in terms of ribbon graphs. Only graphs with genus $g = 0$ and one broken external face $B = 1$ with at most 4 external legs may diverge. They govern the RG flow of the parameters $\lambda$, $\mu$, $\Omega$ and the field strength $Z$ (or wfr) of the model. Considering the $O(N)$ model, it is convenient to add to the ribbon graphs an inner “thread” or “decoration” which represents the color index. The corresponding Feynman rules read off as

(i) To each ribbon line is associated a $G_{mn}$ propagator;

(ii) Each vertex of the first kind $V_1$ (see Figure 1) has a weight equals to $\lambda_1/2$;

(iii) Each vertex of the second kind $V_2$ (see Figure 1) has a weight equals to $\lambda_2/4$.

(iv) To each ribbon face is associated a sum over the $(m_1, m_2)$ corresponding integers. These sums together with the $G_{mn}$ propagators command to the usual power counting, which is the same as the one of the scalar model.

(v) The color sum produces an additional contribution of $N$ for each colored “bubble”. For instance, only $T_1$ (see Figure 2) will have a factor of $N$.

### 3 One loop $\beta$-functions

As was thoroughly argued in [10] and shall not be refrained here, the effective perturbative expansion is better in order to study the $\beta$-function for it just contains the necessary subtractions. We therefore use the effective expansions in the next developments. One notes that the sum involved in ensuing amplitude of graphs are divergent. It is only
after subtraction of the mass divergences and other subtractions that these sums appear now finite even after removing the cutoff. The following statement holds.

**Theorem 3.1** At $\Omega = 1$, we have

\[
\lambda_{1,r} = \lambda_1 - (\lambda_1^2 (1 - N) + \lambda_2^2) S^{(1)} + O(\lambda_{i=1,2}^2) + O(\lambda_1 \lambda_2),
\]

\[
\lambda_{2,r} = \lambda_2 + (2\lambda_1^2 - 2(1 - N)\lambda_1 \lambda_2) S^{(1)} + O(\lambda_{i=1,2}^2) + O(\lambda_1 \lambda_2),
\]

where $S^{(1)} := \sum_{p \in \mathbb{N}^2} 1/(p + A)^2$ which is logarithmically divergent when removing the UV cutoff.

The divergence of $S^{(1)}$ is logarithmic and corresponds both to the bubble four point function divergence and to the wave function renormalization of the tadpole after mass subtraction. The proof of this theorem involves the coming lemmas.

**Lemma 3.1** At $\Omega = 1$, for the NC $(\phi^4)^4$ model at the first order in $\lambda_i$, $i = 1, 2$, the self-energy and the wave function normalization are

\[
\Sigma(m, n) = -(\lambda_1(N + 1) + \lambda_2) \sum_{r \in \mathbb{N}^2} (G_{mr} + G_{rn}),
\]

\[
Z = 1 - \partial_{m_1} \Sigma(m, n)|_{m=0=n} = 1 - (\lambda_1(N + 1) + \lambda_2) S^{(1)},
\]

respectively.

**Lemma 3.2** At $\Omega = 1$, for the NC $(\phi^4)^4$ model at one loop, the amputated 1PI four point functions are given by,

\[
\Gamma_{4,\parallel}(0, 0, 0, 0) = -\lambda_1 + (\lambda_1^2 (N + 1) + 2\lambda_1 \lambda_2 + \lambda_2^2) S^{(1)},
\]

\[
\Gamma_{4,\times}(0, 0, 0, 0) = -\lambda_2 + 4\lambda_1 \lambda_2 S^{(1)}.
\]

The remaining of this section is devoted to the proof of these lemmas and theorem.

**Proof of Lemma 3.1.** The self-energy is

\[
\Sigma(m, n) = \sum_{G_i} K_{G_i} S_{G_i}(m, n)
\]

where $G_i$ runs over two point 1PI graphs, $S_{G_i}(m, n)$ is its amplitude and $K_{G_i}$ is the corresponding combinatorial weight including color summation, that is the number of Wick contractions given rise to $G_i$ times the sum over color indices. At first order, only the tadpole graphs $T_1$, $T_2$ and $T_3$ (see Figure 2) contribute to (3.7) with the combinatorial factors

\[
K_{T_1} = 2N, \quad K_{T_2} = 2, \quad K_{T_3} = 4,
\]

respectively. We introduce $S^{(1)}(m, n) = \sum_{r \in \mathbb{N}^2} (G_{mr} + G_{rn})$ and get the amplitudes

\[
S_{T_1}(m, n) = -\frac{\lambda_1}{2} S^{(1)}(m, n), \quad S_{T_2}(m, n) = -\frac{\lambda_1}{2} S^{(1)}(m, n), \quad S_{T_3}(m, n) = -\frac{\lambda_2}{4} S^{(1)}(m, n)
\]

(3.9)
where the propagators $G_{mr}$ and $G_{rn}$ coincide with the up and down versions of the tadpole, respectively [10]. Equation (3.3) follows from (3.8) and (3.9). The wfr $Z$ is readily obtained by taking the correct derivative onto $\Sigma(m,n)$ and setting external indices to zero. □

**Proof of Lemma 3.2.** In order to determine the two four point functions driving the equations of the RG flows at one loop, we consider the “decorated” graphs of Figure 3. Given (3.11) and (3.12)-(3.14), the following expressions rest on a straightforward algebra

$$\Gamma_{4,\lambda}(m,n,k,l) = \sum_{g_i} K_{g_i} S_{g_i}(m,n,k,l),$$  \hspace{1cm} (3.10)

Here, $g_i$ are four point 1PI graphs with the required topology of amplitude $S_{g_i}(m,n,k,l)$ and of combinatorial weights $K_{g_i}$. The graphs $F_i$, $i = 1, 2, 3, 4, 5$, contribute to $\Gamma_{4\|}$ and $F_6$ to $\Gamma_{4,x}$ (Figure 3) and their combinatorial factors (including again color summation) are

$$K_{F_1} = 2^3 \cdot N, \quad K_{F_2} = 2^3, \quad K_{F_3} = 4^2 \cdot 2, \quad K_{F_4} = 2^3 \cdot 2, \quad K_{F_5} = (4 \cdot 2) \cdot 2, \quad K_{F_6} = (4 \cdot 2) \cdot 2.$$  \hspace{1cm} (3.11)

It is worthy to emphasize that each graph $F_i$, for $i = 4, 5$, has a symmetric partner with the same amplitude, so that their combinatorial factor $K_{F_i}$, $i = 4, 5$, have taken into account such a symmetry factor. Things appear differently for the graph $F_6$ which has four partners by rotating the graph by $\pi/2$ but the amplitudes generated by these rotations are not all equal. The combinatorial factor $K_{F_6}$ considers only the rotation by $\pi$. We get the amplitudes such that

$$S_{F_1}(m,n) = \tilde{\lambda}_1 S(m,k), \quad S_{F_2}(n,l) = \tilde{\lambda}_1 S(n,l), \quad S_{F_4}(m,n) = \tilde{\lambda}_1 S(m,k),$$  \hspace{1cm} (3.12)

$$S_{F_3}(m,k) = \tilde{\lambda}_2 S(m,k), \quad S_{F_5}(m,k) = \tilde{\lambda}_{12} S(m,k),$$  \hspace{1cm} (3.13)

$$S_{F_6}(m,n,k,l) = \tilde{\lambda}_{12} (S(m,k) + S(n,l)),$$  \hspace{1cm} (3.14)

$$\forall m,k \in \mathbb{N}^2, \quad S(m,k) := \sum_{r \in \mathbb{N}^2} \frac{1}{(m+r+A)} \cdot \frac{1}{(k+r+A)}.$$
The lemma is proved from (3.15) and (3.16), once the external indices \( m, n, k, l \) are put to 0.

**Proof of Theorem 3.1.** The proof of equations (3.1) and (3.2) come out from the quotients

\[
\lambda_{r,1} = -\frac{\Gamma_{4,||}(0,0,0,0)}{Z^2} = -\frac{-\lambda_1 + \left(\lambda_1^2(N+3) + 2\lambda_1\lambda_2 + \lambda_2^2\right) S^{(1)}}{(1 - (\lambda_1(N+1) + \lambda_2) S^{(1)})^2}
\]

\[
= \lambda_1 - \gamma_{1||}^\lambda_1^2 - \gamma_{12}^\lambda_1\lambda_2 - \gamma_{2||}^\lambda_2^2 + O(\lambda_{i=1,2}^2) + O(\lambda_1\lambda_2), \tag{3.17}
\]

\[
\lambda_{r,2} = -\frac{\Gamma_{4,\times}(0,0,0,0)}{Z^2} = -\frac{-\lambda_2 + 4\lambda_1\lambda_2 S^{(1)}}{(1 - (\lambda_1(N+1) + \lambda_2) S^{(1)})^2}
\]

\[
= \lambda_2 - \gamma_{1||}^\times_1^\lambda_1^2 - \gamma_{12}^\times_1\lambda_2 - \gamma_{2||}^\times_2^\lambda_2^2 + O(\lambda_{i=1,2}^2) + O(\lambda_1\lambda_2). \tag{3.18}
\]

Expanding the rational function \( 1/Z^2 \) up to the second order in the coupling constants, we get the coefficients \( \gamma_k^{||,\times} \) as in (3.1) and (3.2), i.e.

\[
\gamma^{||}_1 = (1 - N), \quad \gamma^{||}_1 = 0, \quad \gamma^{||}_2 = 1, \tag{3.19}
\]

\[
\gamma^{\times}_1 = 0, \quad \gamma^{\times}_1 = 2(1 - N), \quad \gamma^{\times}_2 = -2. \tag{3.20}
\]

We are then lead to the one loop \( \beta \)-coefficients

\[
\beta^{||} = (1 - N), \quad \beta^{\times} = -2, \tag{3.21}
\]

which achieve the proof of the statement. □.

**Discussion.** For \( N > 1 \), one notes that \( \beta^{||} < 0 \). Consequently, in the UV limit and setting \( \lambda_2 = 0 \), the model is asymptotically free, i.e. the bare coupling \( \lambda_1 \) is screened in this limit. The renormalon problem has to be considered but its contribution possesses an alternate sign which in principle could be handled by some summability principle.
such as Borel resummation [13]. For \( N < 1 \), and still \( \lambda_2 = 0 \), the ordinary issue of Lg arises. Besides, if we assume that second order terms in equations (3.17) and (3.18) vanish, a way to force the system to be such that \( \lambda_{r,i} = \lambda_i \), the algebraic system reached can be inverted with nontrivial (not free theory) solutions

\[
[N = 1, \lambda_2 = 0], \quad [N = 2; \lambda_1 = -\lambda_2].
\]

The left hand side parameters can be considered as the fixed point of the GWm whereas, albeit unstable due to one of its sector, the second set of parameters and couplings defines a new renormalizable color model which is asymptotically safe and does not suffer of Lg. Further, this \( N = 2 \) model is not equivalent to the complex GWm [10] as one can immediately check by expanding the action in real and imaginary parts of the complex field \( \phi = \phi_1 + i\phi_2 \). As a consequence, although being somewhat a vector theory, the complex renormalizable GWm cannot be interpreted as a \( O(N) \) color model in the sense that we have defined it here.

**The model limit \( N = 0 \).** As previously claimed, the model limit \( N = 0 \) can be interpreted as a model of polymers with nonlocal repulsing interaction. Using the explicit form of the Moyal kernel in position space [7], we see that what is suppressed is no longer when the polymer chain crosses itself, as in the usual commutative case, but when four points in the polymer chain sit at the corners of a parallelogram, and the suppression factor is really an oscillation proportional to the area of that parallelogram. This may seem completely unphysical, but in dimension 2, the Moyal geometry is really the one induced by a constant magnetic field perpendicular to the plane (see [3] and references therein). Renormalizable Moyal interactions may be selected by a RG process, so that such models may be physically relevant to the growth of two dimensional charged polymers under strong magnetic field. This model can therefore be thought as some kind of “polymer version” of the quantum Hall effect [3].

Let us rapidly discuss the features of the \( \beta \)-function of this model at one loop with respect to the above calculations. Setting \( N = 0 \) in (3.1) and (3.2), we have the response

\[
\lambda_{1,r} = \lambda_1 - (\lambda_1^2 + \lambda_2^2) S^{(1)} + O(\lambda_{i=1,2}) + O(\lambda_1 \lambda_2),
\]

\[
\lambda_{2,r} = \lambda_2 + (2\lambda_2^2 - 2\lambda_1 \lambda_2) S^{(1)} + O(\lambda_{i=1,2}) + O(\lambda_1 \lambda_2).
\]

Still, the trivial problem and Lg arises for \( \beta || > 0 \) (\( \lambda_2 = 0 \)) but the model is asymptotically free in the infrared direction, a reminiscence behavior of the ordinary commutative \( \phi^4 \) theory.

**The model limit \( N = \infty \).** If the index color \( N \) tends to infinity independently of the matrix indices, we need to rescale the coupling \( \lambda_i \) into \( \lambda_i/N \) to get a non trivial limit. A further Wick-ordered interaction cancels the tadpoles such that only chain of bubble graphs survive and form an explicit computable geometric series. We recover thence the usual integrability of the so called spherical model. The \( \beta \)-function is obtained by
taking the $N \to \infty$ in equations (3.4) and (3.5) with adapted constant couplings. We find ($\lambda_2 = 0$)

$$-\lambda^\infty_{r,1} = -\lambda_1 - \lambda_1^2 + O(\lambda_1^3). \quad (3.25)$$

The model then is asymptotically free as expected since we had asymptotic freedom for $N > 1$.

Far more interesting would be to investigate “double limits” in which both the matrix indices $m, n$ and the color index $a$ are sent to infinity in a coupled way. This may completely change the UV behavior of the theory. For instance, we typically no longer have renormalizability in $D = 4$ (unless we also change the propagator dependence on the color index). Remark that the crossed vertex $\lambda_2$ is similar to the Barrett-Crane vertex of the 2+1 dimensional group field theory approach to quantum gravity [19]. Thus, models with such double limits clearly deserves further study.

**The $U(N)$ NC $\phi^4$ theory.** By an extension of the previous formulation, we can easily discuss some features of the $U(N)$ invariant complex NC $\phi^4$ model. Let us recall that the ordinary complex GWm has some peculiarities of cyclic orientation and restricted kinds of contractions between $\bar{\phi}$ and $\phi$ [10, 11]. Given $(\phi^a)_{a=1,...,N}$ a complex vector field with complex conjugated $(\bar{\phi}^a)_{a=1,...,N}$, compiling the complex constraints onto a color model, the next vertices are found (see Figure 4 for ribbon representations)

$$V^c_{||,1} = \frac{\lambda_1}{2} \sum_{m,n,k,l \in \mathbb{N}^2} \bar{\phi}^a_{mn} \phi^a_{nk} \phi^b_{kl} \phi^b_{lm}, \quad V^c_{||,2} = \frac{\lambda_2}{2} \sum_{m,n,k,l \in \mathbb{N}^2} \bar{\phi}^a_{mn} \phi^b_{nk} \phi^a_{kl} \phi^b_{lm}, \quad (3.26)$$

$$V^c_x = \frac{\lambda_x}{2} \sum_{m,n,k,l \in \mathbb{N}^2} \bar{\phi}^a_{mn} \phi^b_{nk} \bar{\phi}^a_{kl} \phi^b_{lm}. \quad (3.27)$$

The $U(N)$-version of the action as set in [10, 11] can be written as, with the matrix operator $X_{mn} = m \delta_{mn}$,

$$S = \frac{1}{2} \sum_{n,m \in \mathbb{N}^2} (\bar{\phi}^a_{mn} X_{mn} \phi^a_{nm} + \phi^a_{mn} X_{mn} \bar{\phi}^a_{nm}) + V^c_{||,1} + V^c_{||,2} + V^c_x. \quad (3.28)$$

The Gaussian measure has the same covariance (propagator) as in the real theory. Arguments towards the effective expansions and computation of the $\beta$-functions ($\beta_{||,1}, \beta_{||,2},$
β_c) naturally follow. In a similar way that we treated the O(N) model, we compute, on one side, the self-energy Σ^c(m, n) associated with two point amputated 1PI and the wfr Z^c = 1 − ∂Σ^c(0, 0) and, on the other side, the four point 1PI amputated amplitude Γ^c_{4,σ,i}, σ = ||, ×, i = 1, 2, 0, in order to evaluate at first order the RG flows

\[ \lambda^c_{r,σ,i} = \frac{Γ^c_{4,σ,i}(0,0,0,0)}{(Z^c)^2}, \]  

where

\[ Σ^c(m, n) = ∑_{G_i} K^c_{G_i} S_{G_i}(m, n), \quad Γ^c_{4,σ,i}(m, n, k, l) = ∑_{G_{σ,i}} K^c_{G_{σ,i}} S_{G_{σ,i}}(m, n, k, l). \]  

The ribbon graphs in the complex case are very similar to the real case. However, one has to implement the “skeleton” with an oriented boundary so that, mainly the U(N) ribbons have typically the same structure as in the previous situation. The graphs T_i’s and F_i’s, according to complex vertices V^c_{σ,i}, have two different orientations. We will denote the complex tadpoles and four point 1PI graphs by T^c_i and F^c_i, the meaning of i remaining the same as above, but the index s = ± is fixed according to the orientation (sign + is affected to the counterclockwise rotation in the “bubble”, see for instance Figures 5 and 6). We find the combinatorial factors (including sum over colour indices) and their amplitude sums, with S(m) := ∑_{ρ∈N^2} G_{mp},

\[ K^c_{T_{1,+}} = 2N = K^c_{T_{1,-}}, \quad K^c_{T_{2,+}} = K^c_{T_{2,-}} = K^c_{T_{3,+}} = K^c_{T_{3,-}} = 2; \]  

\[ S_{T_{1,+}}(m) = −\frac{λ_2}{2} S(m), \quad S_{T_{1,-}}(n) = −\frac{λ_1}{2} S(n), \quad S_{T_{2,+}}(m) = −\frac{λ_1}{2} S(m), \]  

\[ S_{T_{2,-}}(n) = −\frac{λ_1}{2} S(n), \quad S_{T_{3,+}}(m) = −\frac{λ_1}{2} S(m), \quad S_{T_{3,-}}(n) = −\frac{λ_1}{2} S(n); \]  

\[ K^c_{F_{1,+}} = 2^3 N = K^c_{F_{1,-}}, \quad K^c_{F_{2,+}} = 2^3 = K^c_{F_{2,-}}, \quad K^c_{F_{3,+}} = 2^3 = K^c_{F_{3,-}}; \]  

\[ K^c_{F_{4,+}} = 2^3 = K^c_{F_{4,-}}, \quad K^c_{F_{5,+}} = 2^3 = K^c_{F_{5,-}}, \quad K^c_{F_{6,+}} = 2^3 = K^c_{F_{6,-}}; \]  

\[ S_{F_{1,+}} = \tilde{λ}_2 S(m, k), \quad S_{F_{1,-}} = \tilde{λ}_1 S(n, l), \quad S_{F_{2,+}} = \tilde{λ}_1 S(m, k), \]  

\[ S_{F_{2,-}} = \tilde{λ}_2 S(n, l), \quad S_{F_{3,+}} = \tilde{λ}_x S(m, k), \quad S_{F_{3,-}} = \tilde{λ}_x S(n, l), \]  

\[ S_{F_{4,+}} = \tilde{λ}_{12} S(m, k), \quad S_{F_{4,-}} = \tilde{λ}_{12} S(n, l), \quad S_{F_{5,+}} = \tilde{λ}_{2x} S(m, k), \]  

\[ S_{F_{5,-}} = \tilde{λ}_{1x} S(n, l), \quad S_{F_{6,+}} = \tilde{λ}_{1x} S(m, k), \quad S_{F_{6,-}} = \tilde{λ}_{2x} S(n, l). \]
The below expressions are deduced by direct calculation at one loop

\[ Z^c = 1 - (\lambda_1 + \lambda_2 N + \lambda_x)S^{(1)}, \] (3.40)

\[ \Gamma^c_{4,||,1}(0, 0, 0, 0) = -\lambda_1 + (\lambda_2^2(1 + N) + 2\lambda_1 \lambda_2 + 2\lambda_2 \lambda_x + \lambda_x^2)S^{(1)}, \] (3.41)

\[ \Gamma^c_{4,||,2}(0, 0, 0, 0) = -\lambda_2 + (\lambda_1^2(1 + N) + 2\lambda_1 \lambda_2 + 2\lambda_1 \lambda_x + \lambda_x^2)S^{(1)}, \] (3.42)

\[ \Gamma^c_{4,\times}(0, 0, 0, 0) = -\lambda_x + 2(\lambda_1 + \lambda_2)\lambda_x S^{(1)}, \] (3.43)

so that renormalized coupling flows can be inferred

\[-\lambda^c_{\parallel,1} = -\lambda_1 - (2\lambda_1^2 + 2\lambda_1 \lambda_2(N - 1) + 2\lambda_1 \lambda_x) \] (3.44)

\[-\lambda^c_2 = -\lambda_2 - (2\lambda_2^2 N + 2\lambda_2 \lambda_x - \lambda_1^2(1 + N) - 2\lambda_1 \lambda_x - \lambda_x^2)S^{(1)}, \] (3.45)

\[-\lambda^c_x = -\lambda_x - (2\lambda_x^2 - 2\lambda_2 \lambda_x (1 - N)). \] (3.46)

Keeping \( \lambda_x \) fixed to zero, \( \lambda_1 = \lambda_2 \) and for \( N > 1 \), we get \( \lambda^c_{\parallel,1} > \lambda_{\text{bare},1} \) and \( \lambda^c_{\parallel,2} < \lambda_{\text{bare},2} \) and, consequently, the “parallel” vertex \( V_1 \) determines an UV asymptotic freedom while \( V_2 \) takes the opposite direction of the triviality issue. If we impose the equations \( \lambda^c_{\parallel,i} = \lambda_{\text{bare},i}, i = 1, 2, \times \), we find the solutions

\[ [\lambda_x = 0, \ N = 1, \ \lambda_1 = \pm \lambda_2], \quad [N = 2, \ \lambda_1 = \lambda_2 = -\lambda_x]. \] (3.48)

The first set of parameters hints the complex renormalizable GWm [11] only if \( \lambda_1 = +\lambda_2 \).

The second set of parameters implies an unstable model in the “crossed” vertex sector if we assume that \( \lambda_1 > 0 \), but it is actually a new color model without Lg and possessing a fixed point in the RG flow of all of its renormalized coupling constants. This solution can be seen as the complex counterpart of the previous solution for \( N = 2 \) found in the real situation.

4 Conclusion

This letter has considered and analyzed the \( O(N) \) and \( U(N) \) GWm and computed their \( \beta \)-functions at one loop of perturbation. The real and complex NC color models are characterized by two and three kinds of vertices, respectively. Their RG flows have been determined at one loop in order to find associated the \( \beta \)-functions. Real and complex GWm’s are encoded as a limit \( N = 1 \). This study has revealed the UV asymptotic freedom of a particular class of models with \( N > 1 \) and only “parallel” couplings. For \( N = 0 \), the triviality problem or Lg of the ordinary \( \phi^4_3 \) has shown up again.

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