On some projective unitary qutrit gates

Claire Levaillant

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Abstract. As part of a protocol, we braid in a certain way six anyons of topological charges 222211 in the Kauffman-Jones version of SU(2) Chern-Simons theory at level 4. The gate we obtain is a braid for the usual qutrit 222 but with respect to a different basis. With respect to that basis, the Freedman group of [6] is identical to the $D$-group $D(18, 1, 1; 2, 1, 1)$. We give a physical interpretation for each Blichfeld generator of the group $D(18, 1, 1; 2, 1, 1)$. Inspired by these new techniques for the qutrit, we are able to make new ancillas, namely $\frac{1}{\sqrt{2}}(|1 > + |3 >)$ and $\frac{1}{\sqrt{2}}(|1 > - |3 >)$, for the qubit 121.

1 Setting

Recently, there has been some interest in finding qutrit gates which are universal for quantum computation. When the group of qutrit gates in the projective unitaries $PU(3)$ acts irreducibly on $\mathbb{C}^3$, a result of [3] provides a sufficient condition named by the authors condition (⋆) for an $SU(3)$-subgroup of single projective unitary qutrit gates to form a dense set of $PU(3)$. This condition finds its origins in a 2002 work [4] by Michael Freedman, Alexei Kitaev and Jacob Lurie. An older result from Jean-Luc Brylinski and Ranee Brylinski [2] implies that such a dense set of 1-qutrit gates together with a 2-qutrit entangling gate is universal for quantum computation. Therefore, there have been some attempts and hopes, starting from a finite group of projective unitary qutrit gates obtained by anyonic braiding, to add an extra projective unitary gate which would this time be obtained by braiding and interferometric measurement and would make the group become infinite. We believe that such a group would then satisfy to the conditions mentioned above for density.

In [1], we study a finite subgroup of $SU(3)$ arising from anyonic braiding. This group has order 162 and is later enlarged to a group of order 648, the Freedman group, by a fusion operation (FFO for future reference) due to Mike Freedman, see [6]. Both groups, the one of order 162 and its extension of order 648 are isomorphic to $D$-groups in the 1916 classification of finite $SU(3)$-subgroups by Blichfeld (later augmented with two new groups), namely to $D(9, 1, 1; 2, 1, 1)$ and to $D(18, 1, 1; 2, 1, 1)$ respectively. In [6], it is shown further that the $D$-group $D(18, 1, 1; 2, 1, 1)$ is the Freedman group, with respect to a different basis, that is both groups are conjugate. Classically and originally, the group...
$D(18, 1, 1; 2, 1, 1)$ is defined by three matrix generators which first appeared in the 1916 book by Blichfeld as part of the three generic generators for the groups $D(n, a, b; d, r, s)$ from the series $(D)$. Our paper introduces a new set of four generators for the $D$-group $D(18, 1, 1; 2, 1, 1)$, but the group is only generated by three of them. These generators all arise from anyonic braiding and FFO. Both the Freedman group of order 648 and our physical interpretation of $D(18, 1, 1; 2, 1, 1)$ contain the center $Z_3$ of $SU(3)$, hence we note that the number of projective unitary qutrit gates available to us remains the same. In this first part of the paper, we consider the qutrit 2222 and a pair of 1’s, do some specific braids and fail to obtain a new gate. Of course the number of protocols available to us is extremely large, so our failure does not imply that by choosing such an ancilla we won’t ever obtain an interesting gate by braiding and measurement. Two fundamental facts are enlightened from this first part. First, when doing a full twist $\sigma_2$ on four particles 2211, it results in swapping the topological charges 0 and 2. Second, when doing a single braid $\sigma_2$ on four particles 2211, we obtain a qubit 2121 with the same proportion of $|1\rangle$ and $|3\rangle$. Since doing $\sigma_1$braids only introduces phases, we can thus make a qubit 1221 with equal norms of $|1\rangle$ and $|3\rangle$. This was unknown fact in [3] where in some protocols using braiding and interferometric measurement on the qubit 1221, we were missing such ancillas which play a crucial role for the no-leakage condition.

2 Result

We state below our result.

**Theorem 1** The group $\tilde{G}$ generated by the four matrices

$$
\tilde{G}_1 = \begin{pmatrix}
-e^{\frac{7i\pi}{9}} & e^{\frac{7i\pi}{9}} \\
e^{-\frac{4i\pi}{9}} & -e^{\frac{4i\pi}{9}}
\end{pmatrix} \quad \tilde{G}_2 = \begin{pmatrix}
e^{\frac{7i\pi}{9}} & e^{\frac{7i\pi}{9}} \\
e^{-\frac{4i\pi}{9}} & -e^{\frac{4i\pi}{9}}
\end{pmatrix}
$$

$$
\tilde{FUM} = \begin{pmatrix}
-e^{\frac{2i\pi}{3}} & -e^{\frac{2i\pi}{3}} \\
e^{-\frac{i\pi}{3}} & e^{\frac{i\pi}{3}}
\end{pmatrix} \quad N = \begin{pmatrix}
-e^{-\frac{i\pi}{3}} & -e^{-\frac{i\pi}{3}} \\
e^{\frac{i\pi}{3}} & e^{\frac{i\pi}{3}}
\end{pmatrix}
$$

is a finite subgroup of $SU(3)$ of order 648. It is isomorphic to a semi-direct product $C_6 \times C_{18} \rtimes S_3$. The generators above are up to phase obtained by the following unitary operations in the Kauffman-Jones version of $SU(2)$ Chern-Simons theory at level 4.
The generator $N$ belongs to the subgroup generated by $\tilde{G}_2$. Moreover, we have

$$\tilde{G} = \langle \tilde{G}_1, \tilde{G}_2, FUM \rangle = D(18, 1, 1; 2, 1, 1)$$

### 3 Protocol

A starting point are braids on four anyons of topological charge 2 in the Jones-Kauffman version of $SU(2)$ Chern-Simons theory at level 4. We recall below the matrices $G_2$ for a $\sigma_2$-braid and $G_1$ for a $\sigma_1$-braid, also commonly called $R$-matrix, taken from [1]. All the matrices are defined in $SU(3)$, that is they are defined up to phase.

$$G_1 = \begin{pmatrix} e^{\frac{7i\pi}{9}} & 0 & 0 \\ 0 & e^{\frac{4i\pi}{9}} & 0 \\ 0 & 0 & e^{\frac{7i\pi}{9}} \end{pmatrix}, \quad G_2 = \begin{pmatrix} -\frac{1}{2}e^{\frac{4i\pi}{9}} & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & \frac{1}{2}e^{\frac{4i\pi}{9}} \\ \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & 0 & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} \\ \frac{1}{2}e^{\frac{4i\pi}{9}} & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & -\frac{1}{2}e^{\frac{4i\pi}{9}} \end{pmatrix}$$

On the matrices above, we notice the special roles played by the qutrits $|0\rangle$ and $|4\rangle$ on the one hand and $|2\rangle$ on the other hand. Explicitly, braiding anyons 1 and 2 maps the qutrit $|0\rangle$ to itself and the qutrit $|4\rangle$ to the qutrit $-|\bar{4}\rangle$, up to a common phase. Notice further that

$$G_2(|2\rangle) = e^{\frac{7i\pi}{9}} \frac{|0\rangle + |4\rangle}{\sqrt{2}}$$

and

$$G_2\left(\frac{|0\rangle + |4\rangle}{\sqrt{2}}\right) = e^{\frac{7i\pi}{9}} |2\rangle$$

and

$$G_2\left(\frac{|0\rangle - |4\rangle}{\sqrt{2}}\right) = -e^{\frac{4i\pi}{9}} \frac{|0\rangle - |4\rangle}{\sqrt{2}}$$
From now on, we will work in the new basis \((e_1, e_2, e_3)\) with
\[
e_1 = \frac{|0> + |4>}{\sqrt{2}}, \quad e_2 = |2>, \quad e_3 = \frac{|0> - |4>}{\sqrt{2}}
\]
The matrices of the \(\sigma_1\) and \(\sigma_2\) braids with respect to this new basis are the following. Again, in all what follows, we write the matrices involved with determinant 1, that is we drop a phase. And so we get:
\[
\begin{align*}
\tilde{G}_1 &= \begin{pmatrix}
e_{7/3} & e^{7\pi/9} \\
-e^{4\pi/3} & e^{7\pi/9}
\end{pmatrix} \\
\tilde{G}_2 &= \begin{pmatrix}
e_{7\pi/9} & e^{7\pi/9} \\
e^{4\pi/3} & -e^{4\pi/3}
\end{pmatrix}
\end{align*}
\]
With respect to our new basis, the FFO whose effect is to swap the qutrits \(|0>\) and \(|4>\)
is encoded as follows.
\[
\begin{pmatrix}
-1 & e^{2\pi/3} \\
e^{-2\pi/3} & -e^{2\pi/3}
\end{pmatrix}
\]
Note in \(PU(3) = SU(3)/Z_3\), this matrix is simply
\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]
We have an analogue for the qubit without need of fusing any particles but simply by using braids. Namely, a full twist like on the figure below has the effect of swapping the qubits \(|0>\) and \(|2>\). This is a fundamental observation in the protocol we will soon describe.
Proof. The fact that $|0>$ is mapped to $|2>$ essentially relies on the following two points.

- The quantum dimensions of particles of topological charge 1 and 3 are the same.

- The two diagonal coefficients of the squared $R$-matrix $R(2, 1)$ are opposite.

It then follows that $|2>$ is mapped to $|0>$ by unitarity of the matrix.

Let us justify the first point in more details. Acting on the qubit $|0>$, after doing an $F$-move with horizontal charge line 0 at the level of the second and the third anyon, followed by two $R$-moves, we obtain the diagram:

We then do an $F$-move again. When looking for the $|0>$ projection, the two unitary $6j$-symbols which are involved each contain a "0" which makes them be unitary theta symbols. Using the notations of [10] and [6], the two values $\theta^u(1, 2, 1)$ and $\theta^u(1, 2, 3)$ are identical since the quantum dimensions of particles of respective topological charge 1 and 3 are the same.

\[ \square \]

Protocol
(i) Take a qutrit $2222$ and a pair of 1’s out of the vacuum. Number the anyons 1, 2, 3, 4, 5, 6, those from the qutrit being numbered first.

(ii) Prepare the qutrit in one of the states $|2\rangle$ or $\frac{|0\rangle+|4\rangle}{\sqrt{2}}$ or $\frac{|0\rangle-|4\rangle}{\sqrt{2}}$.

(iii) Do a full twist on anyons 4 and 5 to "create" a 2 charge line in between the qutrit and the pair of ones.

(iv) Use this "extended" version of the qutrit to make braids in such a way that the outcome is a qutrit $2222$ on the first 3 anyons and a qubit $2211$ on the last 3.

(v) Go back to the original configuration of qutrit $2222$ and pair of 1’s by doing a full twist between anyons 4 and 5.

Step (ii) is summarized in the following figure and Step (iii) is represented in the figure below it.

The braids of step (iv) are now described below.
It is a consequence of the fusion rules that a "middle braid" on particles 4222 or 0222 or their respective vertical mirror images will map \( \mathbb{C}|2 \rangle \) into \( \mathbb{C}|2 \rangle \). Moreover, the braiding simply introduces the same phase \( e^{\frac{2\pi}{3}} \), whether dealing with 4222 or with 0222.

In light of this, it makes sense to do a full twist between anyons 2 and 3 on the figure above. It namely allows the charge line adjacent to the input (when going up the tree towards the root) to carry the charge 2 at the end of the braiding process in order for step \((v)\) to be successful independently from the input. After completing the whole protocol, we obtain a new matrix in \( SU(3) \), namely

\[
N = \begin{pmatrix}
  e^{\frac{8\pi}{9}} & e^{\frac{2\pi}{3}} \\
  e^{\frac{2\pi}{3}} & e^{\frac{8\pi}{9}} \\
  e^{\frac{2\pi}{3}} & e^{\frac{2\pi}{3}}
\end{pmatrix}
\]

4 Group structure

We will show the following result.

**Theorem 2** The group \( \tilde{G} \) generated by the matrices \( \tilde{G}_1, \tilde{G}_2, FUM \) and \( N \) has order 648 and is isomorphic to a semi-direct product \( C_6 \times C_{18} \rtimes S_3 \) with respect
to conjugation, for the action provided in Lemma 1 below. Moreover, it is the group $D(18, 1, 1; 2, 1, 1)$.

**Proof.** There is in $\tilde{G}$ a normal subgroup, say $\Delta$, generated by all the diagonal matrices. Moreover, there is a Klein group inside $\Delta$ generated by the two matrices $(\tilde{FUM})^3$ and its $G_1$-conjugate $\tilde{G}_1(\tilde{FUM})^3\tilde{G}_1^{-1}$. Indeed, we have

$$(\tilde{FUM})^3 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \tilde{G}_1(\tilde{FUM})^3\tilde{G}_1^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

**Lemma 1**

Our group $\tilde{G} = \langle \tilde{G}_1, \tilde{G}_2, FUM, N \rangle$ is isomorphic to

$$\langle N^2G_1^2 \times N \rangle \times (\tilde{FUM})^3, \tilde{G}_1(\tilde{FUM})^3\tilde{G}_1^{-1} \rangle \times S_3$$

with

$$S_3 = \{ \tilde{G}_1^9, \tilde{G}_2^9, \tilde{G}_1^9\tilde{G}_2^9, \tilde{G}_2^9\tilde{G}_1^9, \tilde{G}_2\tilde{G}_1\tilde{G}_2 \}$$

Denoting the latter set by $\{ t_3, t_1, c_1, c_2, t_2 \}$ and the generators from the direct product of two cyclic groups $C_6 \times C_{18}$ by

\[
\begin{align*}
    x_6 & = N^2G_1^2 (\tilde{FUM})^3 \\
    x_{18} & = N\tilde{G}_1(\tilde{FUM})^3\tilde{G}_1^{-1}
\end{align*}
\]

a presentation for this group is given by

\[
\left\langle x_6, x_{18}, t_1, t_2 \right| \begin{array}{ll}
    t_1^2 = t_2^2 & 1 = x_6^6 = x_{18}^1 = [x_6, x_{18}] = (t_1t_2)^3, \\
    t_1x_6t_1 & = x_6^{-1}, \quad t_2x_6t_2 = x_6^3x_{18}^3,
    t_1x_{18}t_1 & = x_6^3x_{18}^1,
    t_2x_{18}t_2 & = x_6^3x_{18}^3
\end{array}\right\}
\]

In the semi-direct product above, $N^2G_1^2$ is the matrix $B^2$ of $[11]$ with respect to the basis \( \left( e_1 = \frac{10 > +4 >}{\sqrt{2}}, e_2 = |2>, e_3 = \frac{10 > -4 >}{\sqrt{2}} \right) \).

The GAP ID for the presentation given above is

\[ [648, 259] \]

That is our group is the 259-th group of order 648 in the SmallGroups library by H. Besche, B. Eick and E. O'Brien dating from the beginning of the 2000 millenium. This is the same GAP ID as the one of $D(18, 1, 1; 2, 1, 1)$.

The group $\tilde{G}$ is precisely the group

$$D(18, 1, 1; 2, 1, 1)$$

defined by matrix generators by Blichfeld in 1916.
Proof of Lemma. We look for more cyclic groups generated by diagonal matrices and whose mutual intersections and intersection with the Klein group are trivial.

Begin obviously with the subgroup of $\sim G$ generated by the matrix $N$. Notice also $\sim G^2$ squared is a diagonal matrix. We have

$$\sim G^2 = \begin{pmatrix} e^{-\frac{4i\pi}{9}} & e^{\frac{8i\pi}{9}} \\ e^{-\frac{4i\pi}{9}} & e^{\frac{8i\pi}{9}} \end{pmatrix}, \quad N = \begin{pmatrix} e^{\frac{8i\pi}{9}} & e^{\frac{8i\pi}{9}} \\ e^{\frac{4i\pi}{9}} & e^{\frac{4i\pi}{9}} \end{pmatrix}$$

Now stare at these matrices. Both matrices have order 9. Because their diagonal phases in position (2,2) are identical, we see that the two subgroups $<\sim G^2>$ and $<N>$ intersect non-trivially only for the $k$-th powers of the generators with $k$ satisfying $1 \leq k \leq 8$ and

$$2k \equiv -4k \equiv 8k \pmod{18}$$

This implies that 3 must divide $k$. Then $k = 3$ or $k = 6$. In order to solve this unpleasant issue, we must "mix" the generators instead. We have

$$N^2 \sim G^2 = \begin{pmatrix} e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} \\ 1 \end{pmatrix}$$

and

$$\left(N^2 \sim G^2\right)^2 = \begin{pmatrix} e^{\frac{8i\pi}{3}} & e^{\frac{4i\pi}{3}} \\ 1 \end{pmatrix}$$

And so, we have

$$<N^2 \sim G^2> \cap <N> = \{I_3\}$$

In the Klein group, all the elements have order 2 and in a cyclic group of odd order, all the elements have an odd order. Hence $<N^2 \sim G^2>$ and $<N>$ don’t intersect with the Klein group.

We now exhibit a symmetric group $S_3$ inside $\sim G$. It suffices to notice that $\sim G_1^9$ and $\sim G_2^9$ are the two usual permutation matrices associated with the respective two cycles of $Sym(3)$. On the other hand, we have

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ -1 \end{pmatrix}$$
These matrices provide the additional matrices respectively associated with the three transpositions (13), (12) and (23) of $\text{Sym}(3)$.

It remains to show that each of the $\tilde{G}$-generators $N, F \tilde{U} M, \tilde{G}_1$ and $\tilde{G}_2$ can be written as a product of an element of the direct product and a group element of $S_3$. The result from the Lemma will then classically follow.

First and foremost, we are able to write, using the fact that $\tilde{G}_1$ has order 18,

$$\tilde{G}_1 = N^{-10} \left( N^2 \tilde{G}_1^2 \right)^5 \tilde{G}_1^9$$

Next, it suffices to notice that

$$(F \tilde{U} M)^4 = N^3$$

and so,

$$F \tilde{U} M = N^3 (F \tilde{U} M)^{-3}$$

In particular, we see that the matrix corresponding to the FFO is in the direct product. This was expected since it is a diagonal matrix. Further, we have

$$N^2 \tilde{G}_2^2 = (F \tilde{U} M)^2$$

We derive

$$\tilde{G}_2 = N^{-2} (F \tilde{U} M)^2$$

Now write

$$\tilde{G}_2 = \left( \tilde{G}_2^2 \right)^5 \tilde{G}_2^9$$

in order to conclude.

Finally, it is straightforward to see that $\tilde{G} = D(18, 1, 1; 2, 1, 1)$. Recall below the Blichfeld generators of $D(18, 1, 1; 2, 1, 1)$.

$$F(18, 1, 1) = \begin{pmatrix} e^{\frac{i\pi}{9}} & e^{\frac{i\pi}{3}} & e^{\frac{-2i\pi}{9}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

We see that $E = \tilde{G}_2 \tilde{G}_1$ and $B = \tilde{G}_2 \tilde{G}_1 \tilde{G}_2$, hence $E$ and $B$ both belong to $\tilde{G}$. Further, we have

$$F(18, 1, 1) = (F \tilde{U} M)^3 N^{-1}$$  (1)
Thus, we see that $F(18,1,1)$ also belongs to $\tilde{G}$ and the Blichfeld generator $F(18,1,1)$ can be expressed in terms of the FFO matrix and the $N$ gate. We conclude that the groups $\tilde{G}$ and $D(18,1,1;2,1,1)$ are identical since by [6] and the current work, they have the same order. We now state below a theorem about a physical interpretation of the original Blichfeld generators of $D(18,1,1;2,1,1)$.

**Theorem 3** The Blichfeld generators from $D(18,1,1;2,1,1)$ can be physically realized as follows.

The subgroup $< \tilde{G}_1, \tilde{G}_2, FUM >$ has order 648 since it is conjugate to the Freedman group $< G_1, G_2, FUM >$ of the same order. Hence it is actually
the whole group \( \tilde{G} \) since as part of our work we showed that \( \tilde{G} \) has order 648. Then the matrix \( N \) must be obtained by braiding and FFO. In fact, it is simply obtained by braiding as shown on the figure above. We summarize our results in the Theorem below.

**Theorem 4**

(i) \[ \tilde{G} = < \tilde{G}_1, \tilde{G}_2, \tilde{FUM} > \]

(ii) The matrix \( N \) is obtained by braiding in an adequate way 4 anyons 2222 with respect to the basis \((e_1, e_2, e_3)\). Explicitly, we have

\[ N = \tilde{G}_2^{-4} \]

**Proof.** Point (i) was already discussed. As for point (ii), simply notice that

\[ N = F(9, 1, 1)^4 \]

and

\[ F(9, 1, 1)^{-1} = \tilde{G}_2 t_1 \]

Recall

\[ t_1 = \tilde{G}_2^9 \]

Hence,

\[ F(9, 1, 1) = \tilde{G}_2^8 \]

\[ \square \]

Last, we comment on the two groups \( Fr(162 \times 4) \) and \( \tilde{G} = D(18, 1, 1; 2, 1, 1) \). By [6],

\[ O^T Fr(162 \times 4) O = \tilde{G} \]

with

\[ O = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \]

where \( Fr(162 \times 4) \) denotes the Freedman group. We read that \( O \) is the transition matrix from

\[ (|0>, |2>, |4>) \]

to

\[ \left( \frac{|4> + |0>}{\sqrt{2}}, |2>, \frac{|4> - |0>}{\sqrt{2}} \right) \]
Thus, we see that $\tilde{G} = D(18, 1, 1; 2, 1, 1)$ encodes the $\sigma_1$ and $\sigma_2$- braids and FFO on 4 anyons of topological charge 2, with respect to either basis

$$\left(\frac{\ket{4}+\ket{0}}{\sqrt{2}}, \ket{2}, \frac{\ket{4}-\ket{0}}{\sqrt{2}}\right)$$

$$\left(\frac{\ket{0}+\ket{4}}{\sqrt{2}}, \ket{2}, \frac{\ket{0}-\ket{4}}{\sqrt{2}}\right)$$

In other words, we have

$$Fr(162 \times 4) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} Fr(162 \times 4) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

That is, if we swap the first row and third row and the first column and the third column of a Freedman matrix, we again obtain a Freedman matrix.

## 5 New ancilla for the qubit 1221

In [3], we seek ancillas of the form $x \ket{1} + y \ket{3}$ with $|x| = |y|$ for the qubit 1221.

The fact that the norms in $\ket{1}$ and $\ket{3}$ are equal is a necessary condition for no-leakage on some protocols we test which use a combination of braiding and interferometric measurements. Such an ancilla cannot be realized by a combination of $\sigma_1$- and $\sigma_2$-braids on the qubit 1221. Indeed, the matrix for a $\sigma_2$-braid is the following.

$$\begin{pmatrix} -\frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

And the matrix for a $\sigma_1$-braid is simply a diagonal matrix with phases on the diagonal. Thus, an idea to create such ancillas is to start with the qubit 2211 instead. We have seen when working on the qutrit that a full twist in the center has the effect of swapping $\ket{0}$ and $\ket{2}$. If instead we do a single braid in the
center, we obtain the following matrix

\[
\begin{pmatrix}
|0> & |2> \\
|1> & \left( \frac{1}{\sqrt{2}} e^{i \frac{2\pi}{3}} \right) \\
|3> & \left( \frac{1}{\sqrt{3}} e^{-i \frac{5\pi}{6}} - \frac{i}{\sqrt{3}} \right)
\end{pmatrix}
\]

for the action

Thus, by doing

we obtain
6 Discussion

In the current paper, we give a physical interpretation of the actual original $D(18, 1, 1; 2, 1, 1)$ as defined by generators in [7], while in [6] we give a physical interpretation of an isomorphic copy of that group.

It is disappointing but not surprising that we did not succeed to increase the number of qutrit gates by doing our protocol. Enlarging such a number is not an easy problem. In fact, even using protocols with both braiding and interferometric measurement does not easily lead to finding additional gates which are not issued from braids we already have (cf Bauer’s beautiful programming in [3] to test such protocols by brute computer force).

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References

[1] B. Bauer and C. Levaillant, A new set of generators and a physical interpretation for the $SU(3)$ finite subgroup $D(9, 1, 1; 2, 1, 1)$, Quantum Information Processing Vol. 12, Issue 7 (2013) 2509 – 2521

[2] J-L. Brylinski and R. Brylinski, Universal quantum gates, arXiv:quant-ph/0108062v1

[3] B. Bauer, P. Bonderson, M.H. Freedman, M. Hastings, C. Levaillant, Z. Wang, J. Yard, Anyonic gates beyond braiding, in preparation
[4] M.H. Freedman, A. Kitaev, J. Lurie, Diameters of homogeneous spaces, Math. Res. Letters 10 (2003) 11 – 20

[5] L. Kauffmann and S. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds* Ann. Math. Studies, Vol 134, Princeton, NJ:Princeton Univ. Press 1994

[6] C. Levaillant, The Freedman group: a physical interpretation for the SU(3)-subgroup $D(18, 1, 1; 1, 1)$ of order 648, preprint 2013, arXiv:1309.3580

[7] G.A. Miller, H.F. Blichfeldt and L.E. Dickson, *Theory and Applications of Finite Groups*, John Wiley and Sons, New York 1916

[8] P.O. Ludl, Comments on the classification of the finite subgroups of SU(3), J. Phys. A: Math. Theor. 44 (2011) 255204

[9] W. Grimus and P.O. Ludl, On a characterization of the SU(3) subgroups of type $C$ and $D$, preprint 2013, arXiv:1310.3746

[10] Z. Wang, *Topological quantum computation*, CBMS monograph, Vol 112, American Mathematical Society 2010

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106

E-mail address: claire@math.ucsb.edu