**t-STRUCTURES FOR RELATIVE \(\mathcal{D}\)-MODULES AND t-EXACTNESS OF THE DE RHAM FUNCTOR**

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**Abstract.** This paper is a contribution to the study of relative holonomic \(\mathcal{D}\)-modules. Contrary to the absolute case, the standard \(t\)-structure on holonomic \(\mathcal{D}\)-modules is not preserved by duality and hence the solution functor is no longer \(t\)-exact with respect to the canonical, resp. middle-perverse, \(t\)-structure. We provide an explicit description of these dual \(t\)-structures.

When the parameter space is 1-dimensional, we use this description to prove that the solution functor as well as the relative Riemann-Hilbert functor are \(t\)-exact with respect to the dual \(t\)-structure and to the middle-perverse one while the de Rham functor is \(t\)-exact for the canonical, resp. middle-perverse, \(t\)-structure and their duals.

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**Introduction.**

Let \(X\) and \(S\) be complex manifolds and let \(p_X\) denote the projection of \(X \times S \to S\). We shall denote by \(d_X\) and \(d_S\) their respective complex dimensions and will often write \(p\) instead of \(p_X\) whenever there is no ambiguity.

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An extensive study of holonomic and regular holonomic $\mathcal{D}_{X \times S/S}$-modules as well as of their derived categories was performed in \[17\] and \[18\]. Such modules are called for convenience respectively relative holonomic and regular holonomic modules. Relative holonomic modules are coherent modules whose characteristic variety, in the product $(T^*X) \times S$, is contained in $\Lambda \times S$ for some Lagrangian conic closed analytic subset $\Lambda$ of $T^*X$. Regular relative holonomic modules are holonomic modules whose restriction to the fibers of $p_X$ have regular holonomic $\mathcal{D}_X$-modules as cohomologies.

Another notion introduced in \[17\] was that of $\mathbb{C}$-constructibility over $p_X^{-1}\mathcal{O}_S$, conducting to the (bounded) derived category of sheaves of $p_X^{-1}\mathcal{O}_S$-modules with $\mathbb{C}$-constructible cohomology, the $\mathbb{S}$-$\mathbb{C}$-constructible complexes (this category is denoted by $\mathcal{D}^b_{c,c}(p_X^{-1}\mathcal{O}_S)$), where a natural notion of perversity was also introduced. In loc.cit. it was proved that the essential image of the de Rham functor $\text{DR}$ as well as of the solution functor $\text{Sol}$, when restricted to the bounded derived category of $\mathcal{D}_{X \times S/S}$-modules with holonomic cohomology, is $\mathcal{D}^b_{c,c}(p_X^{-1}\mathcal{O}_S)$. Recall that, denoting by $\text{pSol}(\mathcal{M})$ (resp. $\text{pDR}(\mathcal{M})$) the complex $\text{Sol}(\mathcal{M})[d_X]$ (resp. $\text{DR}(\mathcal{M})[d_X]$), these two functors satisfy a natural isomorphism of commutation with duality: $\mathcal{D} \text{pSol}(-) \simeq \text{pDR}(-)$.

Under the assumption $d_S = 1$, a right quasi-inverse functor to $\text{pSol}$, the functor $\text{RH}^S$, was introduced in \[18\], so naturally $\text{RH}^S$ is a functor from $\mathcal{D}^b_{c,c}(p_X^{-1}\mathcal{O}_S)$ to the bounded derived category $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ of $\mathcal{D}_{X \times S/S}$-modules with regular holonomic modules. $\text{RH}^S$ is the relative version of Kashiwara’s Riemann Hilbert functor $\text{RH}$ (cf. \[9\]) as explained in Section 3.a where we briefly recall its construction. Recall that the importance of this apparently restrictive assumption on $S$ is two-sided: for $d_S = 1$, $\mathcal{O}_S$-flatness and absence of $\mathcal{O}_S$ torsion are equivalent, so we can split proofs in the torsion case and in the torsion free case; on the other hand, although we will not enter into details here, the construction of $\text{RH}^S$ requires, locally on $S$, the existence of bases of the coverings of the subanalytic site $S_{\text{sa}}$ formed by $\mathcal{O}_S$-acyclic open subanalytic sets which is possible in the case $d_S = 1$.

The main goal of this paper is to prove the $t$-exactness of $\text{pSol}$, $\text{pDR}$ and $\text{RH}^S$ with respect to the $t$-structures involved (assuming $d_S = 1$). To be more precise, in the holonomic side we have the standard $t$-structure $P$ as well as its dual $\Pi$, which, contrary to the absolute case proved by Kashiwara in \[9\], do not coincide if $d_X \geq 1$, $d_S \geq 1$ which is not surprising due to the possible absence of $\mathcal{O}_S$-flatness. Similarly, on the $\mathbb{C}$-constructible side, we have the perverse $t$-structure $p$ introduced in \[17\] and its dual $\pi, \text{which do not coincide if } d_X, d_S \geq 1$ as well. Kashiwara’s paper \[10\] provides a wide setting for this kind of problems covering the case $d_X = 0$ (the $\mathcal{O}_S$-coherent case) as well as the standard $t$-structure on the $\mathbb{C}$-constructible case and the correspondent $t$-structure on $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$ via $\text{RH}$. We took there our inspiration, adapting the ideas of several proofs.

In Theorems 2.11 and 3.9 we completely describe $\Pi$ and $\pi$ for any $d_X$ and $d_S$. In particular, when $d_S = 1$, we prove in Proposition 2.6 that $\Pi$ is obtained by left tilting $P$ with respect to a natural torsion pair (respectively $P$ is obtained by...
right tilting II with respect to a natural torsion pair) and we conclude in Corollary 2.7 that the category of strict relative holonomic modules is quasi-abelian ([22]). Similar results are deduced for $\pi$ and $p$ in Proposition 5.7 leading to the conclusion that perverse $S$-C-constructible complexes with a perverse dual are the objects of a quasi-abelian category. Recall that the procedure of tilting a $t$-structure $(D^{\leq 0}, D^{>0})$ on a triangulated category $C$ with respect to a given torsion pair $(\mathcal{T}, \mathcal{F})$ on its heart has been introduced by Happel, Reiten and Smalø in their work [6]. Following the notation of Bridgeland ([2] and [3]) Polishuk proved in [21] that performing the left tilting procedure one gets all the $t$-structures $(D^{\leq 0}, D^{>0})$ satisfying the condition $D^{\leq 0} \subseteq D^{\leq 0} \subseteq D^{\leq 1}$. The relations between torsion pairs, tilted $t$-structures and quasi-abelian categories have been clarified in [1] and [4].

With these informations in hand we have the tools to prove, under the assumption $d_S = 1$, in Theorem 1.1 that $p_{DR}$ is exact with respect to $P$ and $p$ (so, by duality, with respect to $\Pi$ and $\pi$) which gives a precision to the behavior of $p_{DR}$ already studied in [18]. However, since it is not known if $RH_S$ provides an equivalence of categories for general $d_X$, we do not dispose of a morphism of functors $D RH_S(\cdot) \rightarrow RH_S(D(\cdot))$ allowing us to argue by duality as in the $C$-constructible framework. Nevertheless, by a direct proof, in Theorem 4.2 we prove that $RH_S$ is exact with respect to $p$ and $\Pi$ as well as to the dual structures $\pi$ and $P$.

1. TORSION PAIRS, QUASI-ABELIAN CATEGORIES AND T-STRUCTURES

A torsion pair in an abelian category $A$ is a pair $(\mathcal{T}, \mathcal{F})$ of strict (i.e. closed under isomorphisms) full subcategories of $A$ satisfying the following conditions: $\text{Hom}_A(T, F) = 0$ for every $T \in \mathcal{T}$ and every $F \in \mathcal{F}$; for any object $A \in A$ there exists a short exact sequence: $0 \rightarrow t(A) \rightarrow A \rightarrow f(A) \rightarrow 0$ in $A$ such that $t(A) \in \mathcal{T}$ and $f(A) \in \mathcal{F}$. The class $\mathcal{T}$ is called the torsion class and it is closed under extensions, direct sums and quotients, while $\mathcal{F}$ is the torsion-free class and it is closed under extensions, subobjects and direct products. In particular, $\mathcal{T}$ is a full subcategory of $A$ such that the inclusion functor $i_\mathcal{T} : \mathcal{T} \rightarrow A$ admits a right adjoint $t : A \rightarrow \mathcal{T}$ such that $ti_\mathcal{T} = \text{id}_\mathcal{T}$, and dually, the inclusion functor $i_\mathcal{F} : \mathcal{F} \rightarrow A$ admits a left adjoint $f : A \rightarrow \mathcal{F}$ such that $fi_\mathcal{F} = \text{id}_\mathcal{F}$.

In general, the categories $\mathcal{T}$ and $\mathcal{F}$ are not abelian categories but, as observed in [1, 5.4], they are quasi-abelian categories. Let us recall that an additive category $\mathcal{E}$ is called quasi-abelian if it admits kernels and cokernels, and the class of short exact sequences $0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0$ with $E_1 \cong \text{Ker} \beta$ and $E_3 \cong \text{Coker} \alpha$ is stable by pushouts and pullbacks. Both $\mathcal{T}$ and $\mathcal{F}$ admit kernels and cokernels such that: $\text{Ker}_\mathcal{T} = t \circ \text{Ker}_A$, $\text{Coker}_\mathcal{T} = \text{Coker}_A$, $\text{Ker}_\mathcal{F} = \text{Ker}_A$ and $\text{Coker}_\mathcal{F} = f \circ \text{Coker}_A$. Exact sequences in $\mathcal{T}$ (respectively in $\mathcal{F}$) coincide with short exact sequences in $A$ whose terms belong to $\mathcal{T}$ (respectively $\mathcal{F}$) and hence they are stable by pullbacks and push-outs thus proving that $\mathcal{T}$ and $\mathcal{F}$ are quasi-abelian categories. For more details on quasi-abelian categories we refer to Schneiders’ work [22].
Definition 1.1. ([6] Ch. I, Proposition 2.1, [2] Proposition 2.5). Let \( \mathcal{H}_D \) be the heart of a \( t \)-structure \( D = (D^<0, D^>0) \) on a triangulated category \( \mathcal{C} \) and let \( (\mathcal{F}, \mathcal{F}) \) be a torsion pair on \( \mathcal{H}_D \). Then the pair \( \mathcal{D}_{(\mathcal{F}, \mathcal{F})} := (D^<0_{(\mathcal{F}, \mathcal{F})}, D^>0_{(\mathcal{F}, \mathcal{F})}) \) of full subcategories of \( \mathcal{C} \)

\[
\begin{align*}
D^<0_{(\mathcal{F}, \mathcal{F})} & = \{ C \in \mathcal{C} | H_D^0(C) \in \mathcal{F}, \ H_D^0(C) = 0 \forall i > 1 \} \\
D^>0_{(\mathcal{F}, \mathcal{F})} & = \{ C \in \mathcal{C} | H_D^0(C) \in \mathcal{F}, \ H_D^0(C) = 0 \forall i < 0 \}
\end{align*}
\]

is a \( t \)-structure on \( \mathcal{C} \) whose heart is

\[
\mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} = \{ C \in \mathcal{C} | H_D^0(C) \in \mathcal{F}, \ H_D^1(C) = 0 \forall i \notin \{0, 1\} \}.
\]

Following [2] we say that \( D_{(\mathcal{F}, \mathcal{F})} \) is obtained by left tilting \( D \) with respect to the torsion pair \( (\mathcal{F}, \mathcal{F}) \) while the \( t \)-structure \( \widetilde{D}_{(\mathcal{F}, \mathcal{F})} := D_{(\mathcal{F}, \mathcal{F})}[1] \) is called the right-structure obtained by right tilting \( D \) with respect to the torsion pair \( (\mathcal{F}, \mathcal{F}) \) and in this case the right tilted heart is:

\[
\mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} = \{ C \in \mathcal{C} | H_D^0(C) \in \mathcal{F}, \ H_D^{-1}(C) \in \mathcal{F}, \ H_D^1(C) = 0 \forall i \notin \{0, -1\} \}.
\]

Remark 1.2. ([6]). Following the previous notations, whenever one performs a left tilting of \( D \) with respect to a given torsion pair \( (\mathcal{F}, \mathcal{F}) \) on \( \mathcal{H}_D \) one obtains the new heart \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \) and the starting torsion pair is “tilted” in the torsion pair \( (\mathcal{F}, \mathcal{F}[−1]) \) which is a torsion pair in \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \): the class \( \mathcal{F} \) placed in degree zero is the torsion class for this torsion pair while the old torsion class \( \mathcal{F} \) shifted by \( [−1] \) becomes the new torsion-free class and, for any \( M \in \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \), the sequence

\[
0 \to H^0(M) \to M \to H^1(M)[-1] \to 0
\]

is the short exact sequence associated to the torsion pair \( (\mathcal{F}, \mathcal{F}[−1]) \).

Performing a right tilting of \( D_{(\mathcal{F}, \mathcal{F})} \) with respect to the torsion pair \( (\mathcal{F}, \mathcal{F}[−1]) \) on \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \) one re-obtains the starting \( t \)-structure \( D \) endowed with its torsion pair \( (\mathcal{F}, \mathcal{F}) \). In such a way the right tilting by \( (\mathcal{F}, \mathcal{F}[−1]) \) in \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \) is the inverse of the left tilting of \( D \) with respect to \( (\mathcal{F}, \mathcal{F}) \) on \( \mathcal{H}_D \).

Any \( t \)-structure \( D_{(\mathcal{F}, \mathcal{F})} \) obtained by left tilting \( D \) with respect to a torsion pair \( (\mathcal{F}, \mathcal{F}) \) in the heart \( \mathcal{H}_D \) of a \( t \)-structure \( D \) in \( \mathcal{C} \) satisfies

\[
D^<0 \subseteq D^<0_{(\mathcal{F}, \mathcal{F})} \subseteq D^<1 \quad \text{or equivalently} \quad D^>1 \subseteq D^>0_{(\mathcal{F}, \mathcal{F})} \subseteq D^>0
\]

and hence the heart \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \) of the \( t \)-structure \( D_{(\mathcal{F}, \mathcal{F})} \) satisfies \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \subseteq D^{[0:1]} := D^<1 \cap D^>0 \). Dually any \( t \)-structure \( \widetilde{D}_{(\mathcal{F}, \mathcal{F})} := D_{(\mathcal{F}, \mathcal{F})}[1] \) obtained by right tilting \( D \) with respect to a torsion pair \( (\mathcal{F}, \mathcal{F}) \) in the heart \( \mathcal{H}_D \) of a \( t \)-structure \( D \) in \( \mathcal{C} \) satisfies

\[
D^<1 \subseteq \widetilde{D}^<0_{(\mathcal{F}, \mathcal{F})} \subseteq D^<0 \quad \text{or equivalently} \quad D^>0 \subseteq \widetilde{D}^>0_{(\mathcal{F}, \mathcal{F})} \subseteq D^>0^{-1}
\]

and hence \( \mathcal{H}_{D_{(\mathcal{F}, \mathcal{F})}} \subseteq D^{[0:1]} := D^<0 \cap D^>0^{-1} \).

Polishchuk in [21] Lemma 1.2.2 proved the following:
Lemma 1.3. In any pair of $t$-structures $D, D'$ on a triangulated category $C$ verifying the condition $D^{≤0} \subseteq D'^{≤0} \subseteq D'^{≤1}$ (resp. $D^{≤−1} \subseteq D'^{≤0} \subseteq D'^{≤0}$), the $t$-structure $D'$ is obtained by left tilting (resp. right tilting) $D$ with respect to the torsion pair $(T, F) := (D^{≤−1} \cap \mathcal{H}_D, D'^{≥0} \cap \mathcal{H}_D)$ (resp. $(T, F) := (D'^{≤0} \cap \mathcal{H}_D, D'^{≥1} \cap \mathcal{H}_D)$) and in particular, for any $A \in \mathcal{H}_D$, the approximating triangle for the $t$-structure $D'$ is the short exact sequence for this torsion pair.

Remark 1.4. In the work [5] the authors propose a generalization of the previous result. In [5, Theorem 2.14 and 4.3] the authors proved that, under some technical hypotheses, given any pair of $t$-structures $D, D'$ satisfying the condition:

$$D^{≤0} \subseteq D'^{≤0} \subseteq D'^{≤1}$$

one can recover the $t$-structure $D'$ by an iterated procedure of left tilting of length $\ell$ starting with $D$. Equivalently the $t$-structure $D'$ can be obtained by an iterated procedure of right tilting of length $\ell$ starting with $D$.

In particular by [5, Lemma 2.10 (ii)] these hypotheses are joined whenever, following the definition of Keller and Vossieck [15] (cf. also [5, Definition 6.8]), the $t$-structure $D'$ is left $D$-compatible i.e. the class $D'^{≤0}$ is stable under the left truncations $\tau^{≤k}_D$ of $D$ for any $k \in \mathbb{Z}$.

2. $t$-structures on $D^b_{hol}(D_{X \times S/S})$

Let $X$ and $S$ denote complex manifolds of dimension $n := d_X$ and $\ell := d_S$ and let $p_X : X \times S \to S$ be the projection on $S$. We denote by $D_{X \times S/S}$ the subsheaf of $D_{X \times S}$ of relative differential operators with respect to $p_X$. We denote by $D^b_{hol}(D_{X \times S/S})$ the bounded derived category of left $D_{X \times S/S}$-modules with coherent cohomologies.

As in the absolute case (in which $S$ is a point) the triangulated category $D^b_{coh}(D_{X \times S/S})$ is endowed with a duality functor: given $M \in D^b_{coh}(D_{X \times S/S})$ we set

$$D(M) := R\mathcal{H}(om_{D_{X \times S/S}}(M, D_{X \times S/S} \otimes_{D_{X \times S/S}} \Omega^{<1}_{X \times S/S}))[n]$$

and hence $M \cong DDDM$ (since, as explained in [17] Proposition 3.2], any coherent $D_{X \times S/S}$-module locally admits a free resolution of length at most $2n + \ell$).

In [17] 3.4] the authors introduce the notion of holonomic $D_{X \times S/S}$-module (whose characteristic variety is contained $\Lambda \times S$ for some closed conic Lagrangian complex analytic subset $\Lambda$ of $T^*X$) and they proved that the dual of a holonomic $D_{X \times S/S}$-module is an object in $D^b_{hol}(D_{X \times S/S})$ ([17 Corollary 3.6]). Hence the previous duality restricts into a duality in $D^b_{hol}(D_{X \times S/S})$ (the bounded derived category of left $D_{X \times S/S}$-modules with holonomic cohomologies), but despite the absolute case it is no longer true that the dual of a holonomic $D_{X \times S/S}$-module is a holonomic $D_{X \times S/S}$-module.
Due to the previous considerations, we can endow the triangulated category $\mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S)$ with two $t$-structures $P$ and $\Pi$: we denote by $P$ the natural $t$-structure and by $\Pi$ its dual $t$-structure with respect to the functor $D$. Thus, by definition, complexes in $P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X \times S/S)$ (respectively $P \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X \times S/S)$) are isomorphic in $\mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S)$ to complexes of $\mathcal{D}_X \times S/S$-modules which have zero entries in positive (respectively negative) degrees and holonomic cohomologies. The dual $t$-structure $\Pi$ is by definition:

\[
\Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X \times S/S) = \{ M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S) \mid DM \in P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X \times S/S) \}
\]

\[
\Pi \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X \times S/S) = \{ M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S) \mid DM \in P \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X \times S/S) \}.
\]

**Remark 2.1.** We have the following statements:

1. If $S = \{pt\}$ then $\Pi = P$ (cf. [3, 4.11]).

2. If $X = \{pt\}$ then $P$ is nothing more than the natural $t$-structure in $\mathcal{D}_{\text{coh}}(\mathcal{O}_S)$ and $\Pi$ is its dual $t$-structure with respect to the functor $D(\cdot) := R\text{Hom}_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ described by Kashiwara in [10, §4, Proposition 4.3] which we shall denote by $\pi$:

\[
\pi \mathcal{D}_{\text{coh}}^{\leq 0}(\mathcal{O}_S) = \{ M \in \mathcal{D}_{\text{coh}}^b(\mathcal{O}_S) \mid \text{codim Supp}(\mathcal{H}^k(M)) \geq k \}
\]

\[
\pi \mathcal{D}_{\text{coh}}^{\geq 0}(\mathcal{O}_S) = \{ M \in \mathcal{D}_{\text{coh}}^b(\mathcal{O}_S) \mid \mathcal{H}^k[Z](M) = 0 \text{ for any analytic closed subset } Z \text{ and } k < \text{codim } Z \}.
\]

Recall that, following [14], for $s \in S$ on denotes by $\text{Li}_s^*$ the derived functor $p_X^{-1}(\mathcal{O}_S/m) \otimes_{p_X^{-1}\mathcal{O}_S} (\cdot)$ where $m$ is the maximal ideal of functions vanishing at $s$.

**Lemma 2.2.** Let consider the functors $\text{Li}_s^* : \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S) \rightarrow \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X)$ with $s$ varying in $S$. The following holds true:

1. The complex $M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S)$ is isomorphic to 0 if and only if $\text{Li}_s^* M = 0$ for any $s$ in $S$;

2. $\text{Li}_s^* M \in \mathcal{D}_{\text{hol}}^{\leq k}(\mathcal{D}_X)$ for each $s \in S$ if and only if $M \in P \mathcal{D}_{\text{hol}}^{\leq k}(\mathcal{D}_X \times S/S)$;

3. if $\text{Li}_s^* M \in \mathcal{D}_{\text{hol}}^{\leq k}(\mathcal{D}_X)$ for each $s \in S$ then $M \in P \mathcal{D}_{\text{hol}}^{\leq k}(\mathcal{D}_X \times S/S)$;

4. $\text{Li}_s^* M \in \mathcal{D}_{\text{hol}}^{\geq k}(\mathcal{D}_X)$ for each $s \in S$ if and only if $M \in \Pi \mathcal{D}_{\text{hol}}^{\geq k}(\mathcal{D}_X \times S/S)$.

**Proof.** These statements are a slight generalization of [18 Corollary 1.11], with exactly the same idea of proof. In particular (4) can be deduced by duality from (2) since we can characterize the objects in $\Pi \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X \times S/S)$ as follows:

\[
M \in \Pi \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X \times S/S) \iff DM \in P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X \times S/S)
\]

by (2) $M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S)$ and $\forall s \in S$, $\text{Li}_s^* DM \cong D\text{Li}_s^* M \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X)$

$\iff M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_X \times S/S)$ and $\forall s \in S$, $\text{Li}_s^* M \in \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X)$
where the last equivalence holds true since in the absolute case the functor $D$ on $Dhol^b(D_X)$ is exact with respect to the natural $t$-structure.

$q.e.d.$

**Lemma 2.3.** We have the double inclusion

$$
\Pi Dhol^{\leq t}(D_{X\times S/S}) \subseteq P Dhol^{\leq 0}(D_{X\times S/S}) \subseteq \Pi Dhol^{\leq 0}(D_{X\times S/S})
$$

hence, given $M$ a holonomic $D_{X\times S/S}$-module, its dual satisfies

$$
DM \in P Dhol^{[0, t]}(D_{X\times S/S}).
$$

Proof. In general, if $M \in P Dhol^{\leq 0}(D_{X\times S/S})$, for any $s \in S$, $Li_s^*M \in Dhol^{\leq 0}(D_X)$ and hence $Li_s^*DM \cong DLI_s^*M \in Dhol^{\geq 0}(D_X)$ thus, according to (3) of Lemma 2.2, $DM \in P Dhol^{\geq 0}(D_{X\times S/S})$ and so $M \in \Pi Dhol^{\leq 0}(D_{X\times S/S})$.

According to the definitions, Lemma 2.2 and by the $t$-exactness of the functor $D$ in the absolute case, we have the following chain:

$$
\begin{align*}
M \in \Pi Dhol^{\leq -t}(D_{X\times S/S}) & \iff DM \in P Dhol^{\geq t}(D_{X\times S/S}) \\
& \Rightarrow \forall s \in S, Li_s^*DM \cong DLI_s^*M \in Dhol^{\geq 0}(D_X) \iff \forall s \in S, Li_s^*M \in Dhol^{\leq 0}(D_X) \\
& \iff M \in P Dhol^{\leq 0}(D_{X\times S/S}).
\end{align*}
$$

$q.e.d.$

The following result will be useful in the sequel:

**Lemma 2.4.** Let $N \in P Dhol^{\leq 0}(D_{X\times S/S})$. Then $DN$ is quasi-isomorphic to a bounded complex $\mathfrak{F}^\bullet$ of coherent $D_{X\times S/S}$-modules whose terms in negative degrees are zero while the terms in positive degrees are strict coherent $D_{X\times S/S}$-modules. In particular $\mathcal{H}^{\geq 0}DN$ is torsion free.

Proof. Since any coherent $D_{X\times S/S}$-module locally admits a resolution of finite length by free $D_{X\times S/S}$-modules of finite rank, any complex $N \in P Dhol^{\leq 0}(D_{X\times S/S})$ locally admits a resolution $\mathcal{L}^\bullet$ by free $D_{X\times S/S}$-modules of finite rank such that $\mathcal{L}^i = 0$ for any $i > 0$ and for $i \ll 0$. Thus $DN$ can be represented locally by the complex $\mathcal{L}^{\bullet \bullet} := \mathcal{H}om_{D_{X\times S/S}}(\mathcal{L}^\bullet, D_{X\times S/S} \otimes_{O_{X\times S}} O_{X\times S}^{-1})[n]$ whose terms are free $D_{X\times S/S}$-modules of finite rank and whose cohomology in negative degrees is zero. By the assumption

$$
DN \simeq P_{\tau^{\geq 0}}(DN) \simeq \mathfrak{F}^\bullet \in \Pi Dhol^{\leq 0}(D_{X\times S/S}) \text{ (since } N \in P Dhol^{\leq 0}(D_{X\times S/S})\text{)}
$$

with

$$
\mathfrak{F}^\bullet := \cdots 0 \rightarrow 0 \rightarrow \text{Coker}(d_{\mathcal{L}^1}^{-1}) \rightarrow \mathcal{L}^{t1} \rightarrow \cdots
$$

where $\text{Coker}(d_{\mathcal{L}^1})$ is placed in degree 0. It remains to prove that $\text{Coker}(d_{\mathcal{L}^1})$ is a strict coherent $D_{X\times S/S}$-module.

Let us consider the distinguished triangle induced by the following short exact sequence of complexes of coherent $D_{X\times S/S}$-modules:
\[ \mathcal{L}^{* \geq 1} \quad \cdots \quad 0 \quad \xrightarrow{0} \quad \mathcal{L}^{* 1} \quad \xrightarrow{\mathcal{L}^{* 2}} \cdots \]

\[ \xrightarrow{\text{Coker}(d_{\mathcal{L}}^{-1, *})[0]} \quad \cdots \quad 0 \quad \xrightarrow{\text{Coker}(d_{\mathcal{L}}^{-1, *})} \quad \mathcal{L}^{* 1} \quad \xrightarrow{\mathcal{L}^{* 2}} \cdots \]

The triangle \( \text{Li}_s^* \mathcal{L}^{* \geq 1} \rightarrow \text{Li}_s^* \mathcal{F}^* \rightarrow \text{Li}_s^* (\text{Coker}(d_{\mathcal{L}}^{-1, *})) \Downarrow \) is distinguished, since each \( \mathcal{L}^{* i} \) is strict, \( \text{Li}_s^* \mathcal{L}^{* \geq 1} \in \mathcal{D}^{>1}_{\text{coh}}(\mathcal{D}_X) \) while \( \text{Li}_s^* \mathcal{F}^* \in \mathcal{D}^{0}_{\text{coh}}(\mathcal{D}_X) \) in view of Lemma 2.2 (4). Hence, for any \( s \in S \), \( \text{Li}_s^* (\text{Coker}(d_{\mathcal{L}}^{-1, *})) \in \mathcal{D}^{>0}_{\text{hol}}(\mathcal{D}_X) \), so \( \mathcal{H}^0 \text{Li}_s^* (\text{Coker}(d_{\mathcal{L}}^{-1, *})) = 0, \forall j \neq 0 \) since \( \text{Li}_s^* (\text{Coker}(d_{\mathcal{L}}^{-1, *})) \in \mathcal{D}^{<0}_{\text{hol}}(\mathcal{D}_X) \). According to [18, Lemma 1.13] we conclude that \( \text{Coker}(d_{\mathcal{L}}^{-1, *}) \) is strict and so \( \mathcal{H}^0 DN \) is torsion free. q.e.d.

**Remark 2.5.** In accordance with Lemma 2.4, if \( M \) is a torsion module, \( \mathcal{H}^0 D(M) \), being torsion free and a torsion module, is zero.

Following [20], \( p_X^{-1}O_S \)-flat holonomic \( \mathcal{D}_{X \times S/S}\)-modules are called strict. When \( d_S = 1 \), it is well known that \( p_X^{-1}O_S \)-flatness is equivalent to absence of \( p_X^{-1}O_S \)-torsion, hence a holonomic \( \mathcal{D}_{X \times S/S}\)-module \( M \) is strict if and only if for any \( f \in O_S \) the morphism \( M \xrightarrow{f} M \) (multiplication by \( f \)) is a monomorphism.

In this case, for a given coherent \( \mathcal{D}_{X \times S/S}\)-module \( M \), we denote by \( M_t \) the coherent sub-module of sections locally annihilated by some \( f \in O_S \) and we denote by \( M_{t, f} \) the quotient \( M/M_t \). We denote by \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_t \) the full subcategory of holonomic \( \mathcal{D}_{X \times S/S}\)-modules satisfying \( M_t \simeq M \) and by \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_{t, f} \) the full subcategory of holonomic \( \mathcal{D}_{X \times S/S}\)-modules satisfying \( M \simeq M_{t, f} \). The properties of torsion pair in \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) are clearly satisfied by \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_t, \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_{t, f} \).

Moreover this torsion pair is hereditary i.e. the class of torsion modules (which coincides with the class of holonomic \( \mathcal{D}_{X \times S/S}\)-modules \( M \) satisfying \( \dim_{p_X}(\text{Supp}(M)) = 0 \) plus the zero module) is closed under sub-objects and so it forms an abelian category.

**Proposition 2.6.** If \( d_S = 1 \), \( \Pi \) is the \( t \)-structure obtained by left tilting \( P \) with respect to the torsion pair \( (\text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_t, \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_{t, f}) \) in \( \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) while \( P \) is the \( t \)-structure obtained by right tilting \( \Pi \) with respect to the torsion pair \( (\text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_{t, f}, \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})_{t}[-1]) \) in \( \mathcal{H}_\Pi \).

**Proof.** By Lemma 2.3 we have

\[ \Pi \mathcal{D}_{\text{hol}}^{\leq -1}(\mathcal{D}_{X \times S/S}) \subset P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S/S}) \subset \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S/S}) \subset P \mathcal{D}_{\text{hol}}^{\leq 1}(\mathcal{D}_{X \times S/S}) \]

(the last inclusion on the right is obtained by shifting by \([-1]\) the first one) and hence, by Polishchuk’s result (Lemma 1.3), the \( t \)-structure \( \Pi \) is obtained by left tilting \( P \) with respect to the torsion pair

\[ (\Pi \mathcal{D}_{\text{hol}}^{\leq -1}(\mathcal{D}_{X \times S/S}) \cap \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}), \Pi \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_{X \times S/S}) \cap \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})). \]
Also by Lemma 2.3, if $M$ is holonomic, then $DM \in \mathcal{D}_{\text{hol}}^{[0,1]}(\mathcal{D}_{X/S})$, that is, $DM$ is concentrated in degrees 0 and 1. The result will then be a consequence of the following statements:

- (i) $M$ is a strict holonomic module if and only if $D(M)$ is concentrated in degree zero and strict.
- (ii) If $M$ belongs to $\text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})$, then $D(M)$ is concentrated in degree 1 and $P \mathcal{H}^1(DM)$ belongs to $\text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})$.

Item (i) is contained in Proposition 2 of [18]. Therefore it remains to check item (ii). Let $M \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})$. First we remark that, by the functoriality of the action of $p_X^{-1}\mathcal{O}_S$, all cohomology groups $P \mathcal{H}^i(DM)$ belong to $\text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})$.

As a consequence, the heart of $\Pi$ can be described as

$$\mathcal{H}_{\Pi} = \{ M \in \mathcal{D}_{\text{hol}}^{[0,1]}(\mathcal{D}_{X/S}) | P \mathcal{H}^0(M) \text{ strict and } P \mathcal{H}^1(M) \text{ torsion} \}$$

and thus the $t$-structure $P$ is obtained by right tilting $\Pi$ with respect to $(\text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})|_{t}, \text{Mod}_{\text{hol}}(\mathcal{D}_{X/S})|_{t[-1]})$ in $\mathcal{H}_{\Pi}$ (cf. [6] and Remark [12]).

q.e.d.

**Corollary 2.7.** If $d_S = 1$ then the full subcategory of strict holonomic $\mathcal{D}_{X/S/S}$-modules (thus holonomic $\mathcal{D}_{X/S/S}$-modules with a strict holonomic dual) is quasi-abelian.

Therefore the problem of expliciting $\Pi$ only matters for $d_S \geq 2$ and $d_X \geq 1$. The following Lemmas permit to describe the $t$-structure $\Pi$ in terms of support conditions as done by Kashiwara in the case of $X = \{pt\}$ (cf. [10]).

**Lemma 2.8.** The sets

$$S^k \subseteq \{ Z \mid Z \text{ is a closed analytic subset of } X \times S \text{ such that } Z \subseteq X \times W \text{ with } W \text{ a closed analytic subset of } S \text{ such that } \text{codim}_S(W) \geq k \}$$

with $k \in \mathbb{Z}$ form a support datum in $X \times S$ and the following classes:

- $S^k \mathcal{D}_{X/S/S}^{[0]} = \{ M \in \mathcal{D}_b(\mathcal{D}_{X/S/S}) | \text{codim}_X(\text{Supp}(P \mathcal{H}^0(M))) \geq k \}$
- $S^k \mathcal{D}_{X/S/S}^{[\geq 0]} = \{ M \in \mathcal{D}_b(\mathcal{D}_{X/S/S}) | P \mathcal{H}^0_{\mathcal{X} \times W}(M) = 0 \text{ for any closed analytic subset } W \text{ of } S \text{ and } k < \text{codim}_S(W) \}$

(respectively analogs $S^k \mathcal{D}_b(\mathcal{O}_{X/S}), S^k \mathcal{D}_b(\mathcal{O}_{X/S}))$ define a $t$-structure on $\mathcal{D}_b(\mathcal{D}_{X/S/S})$.

**Proof.** Let us recall that (cf. [10] page 850-852) a family of supports in $X \times S$ is a set $\Phi^k$ of closed subsets of $X \times S$ closed by closed subsets and finite unions and a support datum is a decreasing sequence $\Phi = \{ \Phi^k \}_{k \in \mathbb{Z}}$ of families of supports such that for $k < 0$, $\Phi^k$ is the set of all closed subsets of $X \times S$ and for $k \geq 0$, $\Phi^k = \emptyset$. 
Thus $s\mathcal{E} := \{s\mathcal{E}^k\}_{k \in \mathbb{Z}}$ is a support datum in $X \times S$. By [10, Theorem 3.5] (with $\mathcal{A} = \mathcal{D}_{X \times S/S}$) the following classes:

\[
\begin{align*}
&s\mathcal{E}^0(\mathcal{D}_{X \times S/S}) = \{ M \in \mathcal{D}^b(\mathcal{D}_{X \times S/S}) : \text{Supp}(\mathcal{H}^k(M)) \subset s\mathcal{E}^k \}, \\
&s\mathcal{E}^{\geq 0}(\mathcal{D}_{X \times S/S}) = \{ M \in \mathcal{D}^b(\mathcal{D}_{X \times S/S}) : \text{RT}_{[Z]}(M) \in P \mathcal{D}^{\geq k}(\mathcal{D}_{X \times S/S}), \forall Z \in s\mathcal{E}^k \}
\end{align*}
\]

(resp. their analogs $(s\mathcal{E}^0(\mathcal{O}_{X \times S}), s\mathcal{E}^{\geq 0}(\mathcal{O}_{X \times S}))$) define a $t$-structure on $\mathcal{D}^b(\mathcal{D}_{X \times S/S})$ (resp. $\mathcal{D}^b(\mathcal{O}_{X \times S})$). The assumption $\text{Supp}(P\mathcal{H}^k(M)) \in s\mathcal{E}^k$ is equivalent to require that $\text{codim}_X(\text{Supp}(P\mathcal{H}^k(M))) \geq k$ and so

\[
\begin{align*}
&s\mathcal{E}^0(\mathcal{D}_{X \times S/S}) = \{ M \in \mathcal{D}^b(\mathcal{D}_{X \times S/S}) : \text{codim}_X(\text{Supp}(P\mathcal{H}^k(M))) \geq k \}.
\end{align*}
\]

Let $Z \in s\mathcal{E}^k$ such that $Z \subset X \times W$ with $W$ a closed analytic subset of $S$ such that $\text{codim}_S W \geq k$. If $M$ satisfies $\text{RT}_{[X \times W]}(M) \in P \mathcal{D}^{\geq k}(\mathcal{D}_{X \times S/S})$ according to [7] and the left exactness of $\Gamma_{[Z]}(\cdot)$

\[
\text{RT}_{[Z]}(M) \cong \text{RT}_{[Z]}(\text{RT}_{[X \times W]}(M)) \in P \mathcal{D}^{\geq k}(\mathcal{D}_{X \times S/S})
\]

thus

\[
\begin{align*}
&s\mathcal{E}^{\geq 0}(\mathcal{D}_{X \times S/S}) = \{ M \in P \mathcal{D}^{\geq 0}(\mathcal{D}_{X \times S/S}) : \text{RT}_{[X \times W]}(M) \in P \mathcal{D}^{\geq k}(\mathcal{D}_{X \times S/S}) \\
&\quad \text{for any closed analytic subset } W \text{ of } S \text{ and } \text{codim}_S W \geq k \}.
\end{align*}
\]

q.e.d.

**Lemma 2.9.** Let consider $F : \mathcal{C} \longrightarrow \mathcal{T}$ a triangulated functor between two triangulated categories $\mathcal{C}$ and $\mathcal{T}$. Let $P := (P \mathcal{D}^{\leq 0}, P \mathcal{D}^{\geq 0})$ be a bounded $t$-structure on $\mathcal{C}$ and $P \mathcal{D}^{\leq 0}$ (resp. $P \mathcal{D}^{\geq 0}$) a class on $\mathcal{T}$ closed under extensions and shift by $[1]$ (resp. closed under extensions and shift by $[-1]$). The following statements hold true:

1. the functor $F(P \mathcal{D}^{\leq 0}) \subseteq P \mathcal{D}^{\leq 0}$ if and only if $F(\mathcal{H}_P) \subseteq P \mathcal{D}^{\leq 0}$;
2. the functor $F(P \mathcal{D}^{\geq 0}) \subseteq P \mathcal{D}^{\geq 0}$ if and only if $F(\mathcal{H}_P) \subseteq P \mathcal{D}^{\geq 0}$;
3. the previous conditions are simultaneously satisfied if and only if $F(\mathcal{H}_P) \subseteq P \mathcal{T}$

**Proof.** Let us recall that by definition a $t$-structure $P := (P \mathcal{D}^{\leq 0}, P \mathcal{D}^{\geq 0})$ on $\mathcal{C}$ is bounded if for any $X \in \mathcal{C}$ there exist $m \leq n \in \mathbb{Z}$ such that $X \in P \mathcal{D}^{\leq n} \cap P \mathcal{D}^{\geq m}$ and as remarked by Bridgeland in [2, Lemma 2.3] these $t$-structures are completely determined by their hearts (via its Postnikov tower).

The left to right implication is clear since $\mathcal{H}_P \subseteq P \mathcal{D}^{\leq 0}$ so let us suppose that $F(\mathcal{H}_P) \subseteq P \mathcal{D}^{\leq 0}$ and let us prove that $F(P \mathcal{D}^{\leq 0}) \subseteq P \mathcal{D}^{\leq 0}$. Recall that for any $X \in P \mathcal{D}^{\leq 0}$ there exists a suitable $k \in \mathbb{N}$ such that $X \in P \mathcal{D}^{\leq k} \cap P \mathcal{D}^{\geq k}$. Let us proceed by induction on $k \in \mathbb{N}$. For $k = 0$ we get $X \in \mathcal{H}_P$ and thus $F(X) \in P \mathcal{D}^{\leq 0}$ by hypothesis. Let us suppose by inductive hypothesis that the first statement holds true for $k$ and let $X \in P \mathcal{D}^{\leq 0} \cap P \mathcal{D}^{\geq k-1}$. By applying the functor $F$ to the distinguished triangle $P\mathcal{H}^{-k-1}(X)[k + 1] \to X \to P\mathcal{H}^{-k}(X) \to P\mathcal{H}^{-k-1}(X)[k + 1]$ we obtain $F(P\mathcal{H}^{-k-1}(X))[k + 1] \to F(X) \to F(P\mathcal{H}^{-k}(X)) \to F(P\mathcal{H}^{-k-1}(X))[k + 1]$. By hypothesis
\[ F(PH^{-1}(X))[k+1] \in \mathcal{T}_{\mathcal{D}^{\leq 0}} \subseteq \mathcal{T}_{\mathcal{D}^{\leq 0}} \] (thanks to the fact that \( \mathcal{T}_{\mathcal{D}^{\leq 0}} \) is closed under \([1]\)) and by inductive hypothesis \( F(P_{\mathcal{T}^{\geq -k}}(X)) \in \mathcal{T}_{\mathcal{D}^{\leq 0}} \). Thus \( F(X) \in \mathcal{T}_{\mathcal{D}^{\leq 0}} \) since \( \mathcal{T}_{\mathcal{D}^{\leq 0}} \) is closed under extensions. The second statement follows similarly and the third is the consequence of the first and second ones.

q.e.d.

**Lemma 2.10.** Let \( N \) be a coherent \( \mathcal{D}_{X \times S/S} \)-module. Then, for each \( k \),

\[ \text{codim char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(N, \mathcal{D}_{X \times S/S})) \geq k, \]

in particular

\[ \text{codim char}(\mathcal{H}_{\mathcal{D}_{X \times S/S}}^k(N)) \geq k + d_X. \]

**Proof.** According to the faithfull flatness of \( \mathcal{D}_{X \times S} \) over \( \mathcal{D}_{X \times S/S} \) and to \([8, \text{Theorem 2.19 (2)}]\), we have, for each \( k \),

\[ \text{codim char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(\mathcal{D}_{X \times S} \otimes \mathcal{D}_{X \times S/S} N, \mathcal{D}_{X \times S})) \]

= \[ \text{codim char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(N, \mathcal{D}_{X \times S/S}) \otimes \mathcal{D}_{X \times S} \mathcal{D}_{X \times S}) \geq k \]

Since

\[ \text{char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(N, \mathcal{D}_{X \times S/S}) \otimes \mathcal{D}_{X \times S/S}) = \pi^{-1} \text{char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(N, \mathcal{D}_{X \times S/S})) \]

where \( \pi : T^*X \times T^*S \to T^*X \times S \) is the projection, we conclude that

\[ \text{codim char}(\mathcal{E}_{\mathcal{D}_{X \times S/S}}(N, \mathcal{D}_{X \times S/S})) \geq k \]

as desired.

q.e.d.

We have now the tools to obtain the description of \( \Pi \) for arbitrary \( ds \):

**Theorem 2.11.** The \( t \)-structure \( \Pi \) on \( \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) can be described in the following way:

\[ \Pi_{\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})}^0(\mathcal{D}_{X \times S/S}) = \{ M \in \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \mid \forall k, \text{codim char}(P\mathcal{E}_{\mathcal{D}_{X \times S/S}}^k(M)) \geq k + d_X \} \]

\[ \Pi_{\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})}^0(\mathcal{D}_{X \times S/S}) = \{ M \in P\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \mid P\mathcal{E}_{\mathcal{T}\mathcal{X}_{W}^k}(M) = 0 \text{ for any closed analytic subset } W \text{ of } S \text{ and } k < \text{codim}_{S} W \}. \]

**Proof.** Note that the statement is true in the absolute case since we get \( \Pi_{\mathcal{D}_{\text{hol}}(\mathcal{D}_{X})} = P\mathcal{D}_{\text{hol}}(\mathcal{D}_{X}) \) (an holonomic \( \mathcal{D}_{X} \)-module whose characteristic variety has codimension grater than \( d_X \) is necessarily zero). In the case \( X \{pt\} \) the statement is true since we recover the \( t \)-structure \( \pi \) on \( \mathcal{D}_{\text{hol}}(\mathcal{O}_{S}) \) (see Remark \([24]\)).

**Step 1.** Following the notation of Lemma \([23]\) let us prove the equality

\[ \Pi_{\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})}^0(\mathcal{D}_{X \times S/S}) = \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap s_{\mathcal{E}} \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}). \]

We start by proving the inclusion \( \Pi_{\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})}^0(\mathcal{D}_{X \times S/S}) \subseteq \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap s_{\mathcal{E}} \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}). \) Let \( W \) be a closed analytic subset of \( S \) such that \( \text{codim } W \geq k \).

Let us prove that \( R\Gamma_{[X \times W]}(\mathcal{D}N) \in P\mathcal{D}^k(\mathcal{D}_{X \times S/S}) \) for any complex \( N \in \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \).
$P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S})$. This will be a consequence of Lemma 2.3. Indeed, keeping the notation of the proof of this Lemma, we have

$$R\Gamma_{[X \times W]}(\mathcal{D}N) \cong R\Gamma_{[W]}(p_X^{-1}O_S) \otimes_{p_X^{-1}O_S} \mathcal{I}^* \in P \mathcal{D}_{\text{hol}}^{\geq k}(\mathcal{D}_{X \times S / S})$$

since $R\Gamma_{[W]}(p_X^{-1}O_S) \in \mathcal{D}^{\geq k}(p_X^{-1}O_S)$.

Let us now prove the inclusion $\mathcal{D}_{\text{hol}}^{b}(\mathcal{D}_{X \times S / S}) \cap s^\mathcal{E} \mathcal{D}^{\geq 0}(\mathcal{D}_{X \times S / S}) \subseteq \Pi \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_{X \times S / S})$. Let $M \in \mathcal{D}_{\text{hol}}^{b}(\mathcal{D}_{X \times S / S}) \cap s^\mathcal{E} \mathcal{D}^{\geq 0}(\mathcal{D}_{X \times S / S})$. In view of Lemma 2.2 (4) it suffices to check that, for each $s \in S$, $Li_s^*M \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_{X})$. We have

$$Li_s^*M := \frac{O_{X \times S}}{(\mathcal{I}^*)_{(s)}} \otimes_{O_{X \times S}} M \cong \cong R\text{Hom}_{O_{X \times S}} \left( R\text{Hom}_{O_{X \times S}} \left( \frac{O_{X \times S}}{(\mathcal{I}^*)_{(s)}}, O_{X \times S} \right), M \right) \cong \cong R\text{Hom}_{O_{X \times S}} \left( R\text{Hom}_{p_X^{-1}O_S} \left( \frac{O_S}{(\mathcal{I}^*)_{(s)}}, p_X^{-1}O_S \right) \otimes_{p_X^{-1}O_S} O_{X \times S} \right) \otimes_{O_{X \times S}} M.$$ 

Since $s^\mathcal{E} \mathcal{D}^{\geq 0}(\mathcal{D}_{X \times S / S}) \subseteq s^\mathcal{E} \mathcal{D}^{\leq 0}(O_{X \times S})$ and

$$R\text{Hom}_{p_X^{-1}O_S} \left( \frac{O_S}{(\mathcal{I}^*)_{(s)}}, p_X^{-1}O_S \right) \otimes_{p_X^{-1}O_S} O_{X \times S} \in s^\mathcal{E} \mathcal{D}^{\leq 0}(O_{X \times S})$$

we conclude that $R\text{Hom}_{O_{X \times S}} \left( R\text{Hom}_{O_{X \times S}} \left( \frac{O_{X \times S}}{(\mathcal{I}^*)_{(s)}}, O_{X \times S} \right), M \right)[h], M) = 0$ for any $h > 0$. This proves that $Li_s^*M \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_{X})$ as desired.

Step 2. Let us prove that

$$\{ M \in \mathcal{D}_{\text{hol}}^{b}(\mathcal{D}_{X \times S / S}) \mid \text{codim Char}(P \mathcal{H}^k(M)) \geq k + d_X \} = \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S}).$$

First we prove the inclusion:

$$\{ M \in \mathcal{D}_{\text{hol}}^{b}(\mathcal{D}_{X \times S / S}) \mid \text{codim Char}(P \mathcal{H}^k(M)) \geq k + d_X \} \subseteq \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S}).$$

Let us argue by induction on $m$ such that $M \in P \mathcal{D}_{\text{hol}}^{\leq m}(\mathcal{D}_{X \times S / S})$ and that codim Char$\left(P \mathcal{H}^k(M)\right) \geq k + d_X$. For $m = 0$ we have by Lemma 2.3 $P \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S}) \subseteq \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S})$. Let us suppose that any complex in $P \mathcal{D}_{\text{hol}}^{\leq m}(\mathcal{D}_{X \times S / S})$ satisfying codim Char$\left(P \mathcal{H}^k(M)\right) \geq k + d_X$ belongs to $\Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S})$ and let $M \in P \mathcal{D}_{\text{hol}}^{\leq m+1}(\mathcal{D}_{X \times S / S})$ satisfying codim Char$\left(P \mathcal{H}^k(M)\right) \geq k + d_X$. By inductive hypothesis we have that $P \tau \leq m M \in \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S})$ and the distinguished triangle

$$P \tau \leq m M \rightarrow M \rightarrow P \mathcal{H}^{m+1}(M)[-m-1] \rightarrow$$

proves that $M \in \Pi \mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S / S})$ if and only if $P \mathcal{H}^{m+1}(M) \in \Pi \mathcal{D}_{\text{hol}}^{\leq -m-1}(\mathcal{D}_{X \times S / S})$. This last condition is satisfied in view of the assumption on $M$ according to Theorem 2.19 (1) together with the faithfulness flatness of $\mathcal{D}_{X \times S}$ over $\mathcal{D}_{X \times S / S}$, which shows that $D(P \mathcal{H}^{m+1}(M)) \in P \mathcal{D}_{\text{hol}}^{\leq m+1}(\mathcal{D}_{X \times S / S})$. 

Let us now prove the inclusion
\[ \Pi D_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S/S}) \subseteq \{ M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \mid \text{codim } \text{Char}(\mathcal{P} \mathcal{H}^k(M)) \geq k + d_X \}. \]

Recalling that \( \Pi D_{\text{hol}}^{\leq 0} := D(\Pi D_{\text{hol}}^{\geq 0}(\mathcal{D}_{X \times S/S})) \) we can apply Lemma 2.9 with \( F = D \) and so we need only to prove that given \( N \) a holonomic \( \mathcal{D}_{X \times S/S} \)-module, \( D(N) \) satisfies
\[ \text{codim } \text{Char}(\mathcal{P} \mathcal{H}^k(D(N))) \geq k + d_X \]
and this holds true by Lemma 2.10. q.e.d.

**Remark 2.12.** In the course of the previous proof, following the notation of Lemma 2.8 we show that \( D_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \cap s^C \mathcal{D}_{\geq 0}(\mathcal{D}_{X \times S/S}) = \Pi D_{\text{hol}}^{\geq 0}(\mathcal{D}_{X \times S/S}) \) while we only have an inclusion
\[ D_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \cap s^C \mathcal{D}_{\leq 0}(\mathcal{D}_{X \times S/S}) \subseteq \Pi D_{\text{hol}}^{\leq 0}(\mathcal{D}_{X \times S/S}). \]

**Remark 2.13.** We conclude by the previous Theorem 2.11 that the \( t \)-structure \( \Pi \) is left \( P \)-compatible (cf. Remark 1.4) and so, according to Lemma 2.3 and to [5, Theorem 4.3], it can be recovered from \( P \) via an iterated right tilting procedure of length \( \ell \).

### 3. \( t \)-structures on \( D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \)

In \( D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \) the natural dualizing complex is \( p_X\mathcal{O}_S = p_X^{-1}\mathcal{O}_S[2d_X] \) and one defines the duality functor (cf. [17] for details) by setting
\[ D(F) = R5\text{com}_{p_X^{-1}\mathcal{O}_S}(F, p_X^{-1}\mathcal{O}_S)[2d_X]. \]

Hence the canonical morphism \( F \to DD(F) \) is an isomorphism for any \( F \in D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \).

**Definition 3.1.** [17, 2.7] The perverse \( t \)-structure \( p \) on the triangulated category \( D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \) is given by
\[ p D_{\mathcal{C}, c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) = \{ F \in D_{\mathcal{C}, c}(p^{-1}\mathcal{O}_S) \mid \forall \alpha, i^{-k}_F \in \mathcal{D}_{\text{coh}}^{\leq -d_X}(p_X^{-1}\mathcal{O}_S), \text{ for some adapted } \mu\text{-stratification } (X_{\alpha}) \} \]
\[ p D_{\mathcal{C}, c}^{\geq 0}(p_X^{-1}\mathcal{O}_S) = \{ F \in D_{\mathcal{C}, c}(p^{-1}\mathcal{O}_S) \mid \forall \alpha, i^{+k}_F \in \mathcal{D}_{\text{coh}}^{\geq -d_X}(p_X^{-1}\mathcal{O}_S), \text{ for some adapted } \mu\text{-stratification } (X_{\alpha}) \} \]

or equivalently
\[ p D_{\mathcal{C}, c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) = \{ F \in D_{\mathcal{C}, c}^b(p^{-1}\mathcal{O}_S) \mid \forall \alpha, i^{-k}_F \in \mathcal{D}_{\text{coh}}^{\leq -d_X}(\mathcal{O}_S), \text{ for any } x \in X_{\alpha} \}
\text{ and for some adapted } \mu\text{-stratification } (X_{\alpha}) \}
\[ p D_{\mathcal{C}, c}^{\geq 0}(p_X^{-1}\mathcal{O}_S) = \{ F \in D_{\mathcal{C}, c}^b(p^{-1}\mathcal{O}_S) \mid \forall \alpha, i^{+k}_F \in \mathcal{D}_{\text{coh}}^{\geq d_X}(\mathcal{O}_S), \text{ for any } x \in X_{\alpha} \}
\text{ and for some adapted } \mu\text{-stratification } (X_{\alpha}) \}. \]

(See [11, Definition 8.3.19] for the definition of adapted \( \mu \)-stratification.) Hence its dual \( \pi \) with respect to the functor \( D \) is
\[ p D_{\mathcal{C}, c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) = \{ M \in D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \mid DM \in p D_{\mathcal{C}, c}^{\geq 0}(p_X^{-1}\mathcal{O}_S) \}
\[ p D_{\mathcal{C}, c}^{\geq 0}(p_X^{-1}\mathcal{O}_S) = \{ M \in D_{\mathcal{C}, c}^b(p_X^{-1}\mathcal{O}_S) \mid DM \in p D_{\mathcal{C}, c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) \}. \]
Notation 3.2. We shall denote by \( \text{perv}(p_X^{-1}(\mathcal{O}_S)) \) the heart of the \( t \)-structure \( p \).

We have the following statements:

1. If \( S = \{ pt \} \) then \( p \) equals the middle-perversity \( t \)-structure (cf. [8, 4.11]).
2. If \( X = \{ pt \} \) then \( p \) is, as above, the standard \( t \)-structure in \( \mathcal{D}_{\text{coh}}^b(\mathcal{O}_S) \) and \( \pi \) is the dual \( t \)-structure in \( \mathcal{D}_{\text{coh}}^b(\mathcal{O}_S) \) described by Kashiwara in [10] (cf. Remark 2.1).

Therefore, the problem of explicating \( \pi \) only matters for \( d_S \geq 1 \) and \( d_X \geq 1 \).

Lemma 3.3. Let us consider the functors \( Li^*_s : \mathcal{D}_{\text{C},c}^b(p_X^{-1}\mathcal{O}_S) \to \mathcal{D}_{\text{C},c}^b(\mathcal{C}_X) \) with \( s \) varying in \( S \). The following holds true:

1. the complex \( F \in \mathcal{D}_{\text{C},c}^b(p_X^{-1}\mathcal{O}_S) \) is isomorphic to 0 if and only if \( Li^*_s F = 0 \) for any \( s \) in \( S \);
2. \( Li^*_s F \in \mathcal{D}_{\text{C},c}^k(\mathcal{C}_X) \) for each \( s \in S \) if and only if \( F \in \mathcal{D}_{\text{C},c}^{\leq k}(p_X^{-1}\mathcal{O}_S) \);
3. if \( Li^*_s F \in \mathcal{D}_{\text{C},c}^k(\mathcal{C}_X) \) for each \( s \in S \) then \( F \in \mathcal{D}_{\text{C},c}^{\leq k}(p_X^{-1}\mathcal{O}_S) \).

Proof. (1) is proved in [17, Proposition 2.2]. The other implications can also be deduced by the proof of Proposition 2.2 in [17]. q.e.d.

Statement (2) of the previous Lemma affirms that a complex \( F \) belongs to the aisle of the natural \( t \)-structure on \( \mathcal{D}_{\text{C},c}^b(p_X^{-1}\mathcal{O}_S) \) if and only if any \( Li^*_s F \) belongs to the aisle of the natural \( t \)-structure on \( \mathcal{D}_{\text{C},c}^b(\mathcal{C}_X) \). This result admits the following counterpart for the perverse \( t \)-structure thus obtaining an analog of Lemma 2.2.

Lemma 3.4. The following statements hold true:

1. \( Li^*_s F \in \mathcal{D}_{\text{C},c}^{\leq k}(\mathcal{C}_X) \) for each \( s \in S \) if and only if \( F \in \mathcal{D}_{\text{C},c}^{\leq k}(p_X^{-1}\mathcal{O}_S) \);
2. if \( Li^*_s F \in \mathcal{D}_{\text{C},c}^{\geq k}(\mathcal{C}_X) \) for each \( s \in S \) then \( F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \);
3. if \( F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \) then \( Li^*_s F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \) for each \( s \in S \).
4. if \( F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \) then \( Li^*_s F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \) for each \( s \in S \).

Proof. (1) Recall that \( F \in \mathcal{D}_{\text{C},c}^b(p_X^{-1}\mathcal{O}_S) \) belongs to \( \mathcal{D}_{\text{C},c}^{\leq k}(p_X^{-1}\mathcal{O}_S) \) if for some adapted \( \mu \)-stratification \( (X_\alpha)_{\alpha \in A} \) we have
\[
\forall \alpha, i_\alpha^{-1} F \in \mathcal{D}_{\text{C},c}^{\leq k-d_\alpha}(p_{X_\alpha}^{-1}(\mathcal{O}_S))
\]
or equivalently, by Lemma 3.3 (2),
\[
\forall \alpha, Li^*_s i_\alpha^{-1} F \cong i_\alpha^{-1} Li^*_s F \in \mathcal{D}_{\text{C},c}^{\leq k-d_\alpha}(\mathcal{C}_X) \quad \forall s \in S
\]
which is equivalent to
\[
Li^*_s F \in \mathcal{D}_{\text{C},c}^{\leq k}(\mathcal{C}_X).
\]

(2) If \( Li^*_s F \in \mathcal{D}_{\text{C},c}^{\geq k}(\mathcal{C}_X) \) for each \( s \in S \) we get:
\[
\forall \alpha, Li^*_s i_\alpha^{-1} F \cong i_\alpha^{-1} Li^*_s F \in \mathcal{D}_{\text{C},c}^{\geq k-d_\alpha}(\mathcal{C}_X) \quad \forall \alpha, s \in S
\]
and so by (3) of Lemma 3.3 we obtain \( F \in \mathcal{D}_{\text{C},c}^{\geq k}(p_X^{-1}\mathcal{O}_S) \).

(3) can be deduced by duality from (1) since \( Li^*_s DF \cong DLi^*_s F \) for any \( F \in \mathcal{D}_{\text{C},c}^b(p_X^{-1}\mathcal{O}_S) \) we have:
Lemma 3.5. \( F \in \pi D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) \iff DF \in p D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) \)

by (1) \( F \in D_{\mathbb{C}^c}^b(p_X^{-1}\mathcal{O}_S) \) and \( \forall s \in S, Li_s^*DF \equiv DLi_s^*F \in D_{\mathbb{C}^c}^{\leq 0}(\mathcal{C}_X) \)

\( \iff F \in D_{\mathbb{C}^c}^b(p_X^{-1}\mathcal{O}_S) \) and \( \forall s \in S, Li_s^*F \in D_{\mathbb{C}^c}^{\geq 0}(\mathcal{C}_X) \)

where the last equivalence holds true since in the absolute case the functor \( D \) on \( D_{\mathbb{C}^c}^b(\mathcal{C}_X) \) is \( t \)-exact with respect to the perverse \( t \)-structure.

Let us prove (4): we have

\[
F \in p D_{\mathbb{C}^c}^{\geq 0}(p_X^{-1}\mathcal{O}_S) \iff \\
R\Gamma X_\times S(F) \in D^{\geq -d_X}(p_X^{-1}\mathcal{O}_S) \forall \alpha, \text{ for some adapted } \mu \text{-\emph{stratification } } (X_\alpha) \Rightarrow \\
R\Gamma X_\times (Li_s^*F) \equiv Li_s^*R\Gamma X_\times S(F) \in D^{\geq -d_X-\ell}(X) \forall s \in S, \forall \alpha, \\
\text{for some adapted } \mu \text{-\emph{stratification } } (X_\alpha) \\
\iff Li_s^*F \in p D_{\mathbb{C}^c}^{\geq -\ell}(X).
\]

q.e.d.

Lemma 3.5. We have the double inclusion

\[
\pi D_{\mathbb{C}^c}^{\leq -\ell}(p_X^{-1}\mathcal{O}_S) \subseteq p D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}\mathcal{O}_S) \subseteq \pi D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}\mathcal{O}_S)
\]

hence, given a perverse \( p_X^{-1}(\mathcal{O}_S) \)-module \( F \), its dual satisfies

\[
DF \in p D_{\mathbb{C}^c}^{[0,\ell]}(p_X^{-1}(\mathcal{O}_S)).
\]

Proof. If \( F \in p D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}(\mathcal{O}_S)) \) by (1) of Lemma 3.4 we get for any \( s \in S, Li_s^*F \in p D_{\mathbb{C}^c}^{\leq 0}(\mathcal{C}_X) \) and hence \( Li_s^*DF \equiv DLi_s^*F \in p D_{\mathbb{C}^c}^{\geq 0}(\mathcal{C}_X) \). Thus, according to (2) of Lemma 3.4 \( DF \in p D_{\mathbb{C}^c}^{\geq 0}(p_X^{-1}(\mathcal{O}_S)) \) and so \( F \in \pi D_{\mathbb{C}^c}^{\leq 0}(p_X^{-1}(\mathcal{O}_S)) \).

According to the definitions, Lemma 3.4, and by the \( t \)-exactness of the functor \( D \) for the perverse \( t \)-structure in the absolute case, we have:

\[
F \in \pi D_{\mathbb{C}^c}^{\leq -\ell}(p_X^{-1}(\mathcal{O}_S)) \iff DF \in p D_{\mathbb{C}^c}^{\geq \ell}(p_X^{-1}(\mathcal{O}_S))
\]

\( \Rightarrow \forall s \in S, Li_s^*DF \equiv DLi_s^*F \in D_{\mathbb{C}^c}^{\geq 0}(\mathcal{C}_X) \Rightarrow \forall s \in S, Li_s^*F \in D_{\mathbb{C}^c}^{\geq 0}(\mathcal{C}_X) \)

\( \iff F \in p D_{\mathbb{C}^c}^{\geq 0}(p_X^{-1}(\mathcal{O}_S)). \)

q.e.d.

Definition 3.6. Let \( d_S = 1 \). A perverse sheaf \( F \in \text{perv}(p_X^{-1}(\mathcal{O}_S)) \) is called \textit{torsion-free} if for any \( s \in S \) we have \( Li_s^*F \in \text{perv}(\mathcal{C}_X) \). We will denote by \( \text{perv}(p_X^{-1}(\mathcal{O}_S))_{\text{tf}} \) the full subcategory of perverse sheaves which are torsion-free.
In other words, for each $s_0 \in S$, given a local coordinate on $S$ vanishing on $s_0$, the morphism $F \to F$ is injective in the abelian category $\text{perv}(p^{-1}_X(\mathcal{O}_{S}))$.

**Proposition 3.7.** If $d_S = 1$, $\pi$ is the t-structure obtained by left tilting $p$ with respect to the torsion pair

$$(\pi D^{≤-1}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)), \pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)))$$

and $\pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)) = \text{perv}(p^{-1}_X(\mathcal{O}_S))_{tf}$.

**Proof.** By Lemma 3.5 $\pi D^{≤-1}_{cc}(p^{-1}_X\mathcal{O}_S) \subset p D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \subset \pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S)$ hence, by Polishchuk result (Lemma 1.3), the t-structure $\pi$ is obtained by left tilting $p$ with respect to the torsion pair

$$(\pi D^{≤-1}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)), \pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)))$$

By [13] Lemma 1.9 $\pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \cap \text{perv}(p^{-1}_X(\mathcal{O}_S)) = \text{perv}(p^{-1}_X(\mathcal{O}_S))_{tf}$. q.e.d.

**Corollary 3.8.** The full subcategory of perverse $S$-$\mathcal{C}$-constructible sheaves with a perverse dual is quasi-abelian.

We have the following description of $\pi$ for arbitrary $d_S$:

**Theorem 3.9.** The t-structure $\pi$ on $D^{b}_{cc}(p^{-1}_X\mathcal{O}_S)$ can be described in the following way:

- $\pi D^{≤0}_{cc}(p^{-1}_X\mathcal{O}_S) = \{ F \in D^{b}_{cc}(p^{-1}_X\mathcal{O}_S) | i^{-1}_x F \in \pi D^{≤-d\alpha}_{coh}(\mathcal{O}_S) \text{ for any } x \in X_\alpha \text{ and for some adapted } \mu \text{-stratification } (X_\alpha) \}$

- $\pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) = \{ F \in D^{b}_{cc}(p^{-1}_X\mathcal{O}_S) | i^{!}_x F \in \pi D^{>0}_{coh}(\mathcal{O}_S) \text{ for any } x \in X_\alpha \text{ and for some adapted } \mu \text{-stratification } (X_\alpha) \}$

where the t-structure $\pi$ on $D^{b}_{coh}(\mathcal{O}_S)$ is the dual of the canonical t-structure described in Remark 2.24.

**Proof.** Following the definition of the perverse t-structure and [17] Remark 2.24,

$$F \in \pi D^{≤0}_{cc}(p^{-1}_X\mathcal{O}_S) \iff D F \in \pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \iff$$

$$\forall \alpha, \forall x \in X_\alpha, i^{!}_x D F \cong D i^{-1}_x F \in \pi D^{≤-d\alpha}_{coh}(\mathcal{O}_S) \iff$$

$$\forall \alpha, \forall x \in X_\alpha, i^{-1}_x F \in \pi D^{≤-d\alpha}_{coh}(\mathcal{O}_S).$$

Dually

$$F \in \pi D^{>0}_{cc}(p^{-1}_X\mathcal{O}_S) \iff$$

$$D F \in \pi D^{≤0}_{cc}(p^{-1}_X\mathcal{O}_S) \iff$$

$$\forall \alpha, \forall x \in X_\alpha, i^{-1}_x D F \cong D i^{!}_x F \in \pi D^{≤-d\alpha}_{coh}(\mathcal{O}_S) \iff$$

$$\forall \alpha, \forall x \in X_\alpha, i^{!}_x F \in \pi D^{≤-d\alpha}_{coh}(\mathcal{O}_S).$$
q.e.d.

**Remark 3.10.** Let denote by \( p_{r,k} \) the truncation functor with respect to the \( t \)-structure \( p \) on \( D_{coh}(p_X^{-1}O_S) \). We observe that given \( F \in \pi D_{coh}^b(p_X^{-1}O_S) \) we get by the previous Theorem 3.9 that \( \tau_{r,k} F \in \pi D_{coh}^b(p_X^{-1}O_S) \) for any \( k \in \mathbb{Z} \) since the functors \( i_x^{-1} \) are exact and the \( t \)-structure \( \pi \) on \( D_{coh}^b(X) \) is stable by truncation with respect to the standard \( t \)-structure. So, in analogy with Remark 2.13 the \( t \)-structure \( \pi \) is left \( p \)-compatible and, according to Lemma 2.3 and to [5, Theorem 4.3], it can be recovered from \( p \) via an iterated right tilting procedure of length \( \ell \).

We can now explicitly describe the torsion class in the abelian category \( perv(p_X^{-1}(O_S)) \) as follows:

**Proposition 3.11.** Let \( d_S = 1 \). We have:

\[
perv(p_X^{-1}(O_S)) := \pi D_{coh}^b(p_X^{-1}O_S) \cap perv(p_X^{-1}(O_S)) = \{ F \in perv(p_X^{-1}(O_S)) | \text{codim} \, p_X(Supp \, F) \geq 1 \}
\]

**Proof.** We observe that \( perv(p_X^{-1}(O_S)) \) is \( \pi D_{coh}^b(p_X^{-1}O_S) \cap \pi D_{coh}^b(p_X^{-1}O_S) \) (since \( \pi D_{coh}^b(p_X^{-1}O_S) \subseteq \pi D_{coh}^b(p_X^{-1}O_S) \)). Let us recall that in the case \( d_S = 1 \) the dual \( t \)-structure on \( D_{coh}(O_S) \) described in Remark 2.1 reduces to:

\[
\begin{align*}
\pi D_{coh}^b(O_S) &= \{ M \in D_{coh}^b(O_S) | \text{codim} \, Supp(\mathcal{H}(M)) \geq 1 \} \\
\pi D_{coh}^b(O_S) &= \{ M \in D_{coh}^b(O_S) | \mathcal{H}(M) \text{ is strict} \}
\end{align*}
\]

where we recall that since \( d_S = 1 \) the condition \( \text{codim} \, Supp(\mathcal{H}(M)) \geq 1 \) is equivalent to \( d_{supp(\mathcal{H}(M))} = 0 \) or \( M = 0 \).

Accordingly to Proposition 4.8 an object \( F \) belongs to \( perv(p_X^{-1}(O_S)) \) if and only if it verifies the following two conditions where \( (X_\alpha) \) is a \( \mu \)-stratification of \( X \) adapted to \( F \):

\[
\begin{align*}
(\text{i}) & \quad \forall \alpha, i_{\alpha,x}^{-1} F \in D_{coh}^b(X_\alpha) \quad \text{and} \quad \text{codim} \, Supp(i_{\alpha,x}^{-1} \mathcal{H}^{-d_{x_\alpha}}(F)) \geq 1, \quad \forall x \in X_\alpha \\
(\text{ii}) & \quad \forall \alpha, i_{\alpha,x}^{-1} F \in D_{coh}^b(X_\alpha) \quad \text{and} \quad \text{codim} \, Supp(i_{\alpha,x}^{-1} \mathcal{H}^{-d_{x_\alpha}}(p_X^{-1}(O_S))) \geq 1.
\end{align*}
\]

Recall that, locally on \( X_\alpha \), \( F \simeq p_{X_\alpha}^{-1} G \), for some \( G \in D_{coh}(O_S) \) and so \( (\text{i}) \) is equivalent to the following

\[
(\text{i}^\prime) \quad i_{\alpha,x}^{-1} F \in D_{coh}^b(X_\alpha) \quad \text{and} \quad \text{codim} \, p_{X_\alpha}(Supp(i_{\alpha,x}^{-1} \mathcal{H}^{-d_{x_\alpha}}(F))) \geq 1.
\]

**Step 1.** Let us prove that, for any \( F \in perv(p_X^{-1}(O_S)) \), \( H^om_{perv(p_X^{-1}O_S)}(F,F) \simeq H^om_{D_{coh}(p_X^{-1}O_S)}(F,F) := H^0 R^\alpha H^om_{p_X^{-1}O_S}(F,F) \) satisfies:

\[
\text{codim} \, p_X(Supp(H^0 R^\alpha H^om_{p_X^{-1}O_S}(F,F))) \geq 1.
\]

We recall that \( R^\alpha H^om_{p_X^{-1}O_S}(F,F) \in D_{coh}^b(p_X^{-1}O_S) \) since \( F \in perv(p_X^{-1}(O_S)) \) (see [17, Proposition 2.26]). If \( \text{codim} \, p_X(Supp(H^0 R^\alpha H^om_{p_X^{-1}O_S}(F,F))) = 0 \) let \( X_\alpha \) be a stratum of maximal dimension such that

\[
\text{codim} \, p_{X_\alpha}(Supp(i_{\alpha,x}^{-1} H^0 R^\alpha H^om_{p_X^{-1}O_S}(F,F))) = 0.
\]
Let $V$ be an open neighbourhood of $X_α$ in $X$ such that $V \times X_α$ intersects only strata of dimension $> d_{X_α}$, and let $j_α : (V \times X_α) \times S \to V \times S$ be the inclusion. Then the complex $i^{α−1}_α Rj_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F) \in D_{coh}(p^{-1}_X(0_S))$ and $\mathcal{H}om^{−1}_α Rj_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F) \cong i^{α−1}_α j_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F)$ and so

$$\text{codim } p_{X_α}(\text{Supp}(i^{α−1}_α Rj_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F))) \geq 1.$$ 

By the conditions (i') and (ii) we deduce that

$$\mathcal{H}om^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F) \cong \mathcal{H}om^{−1}_α R\hom_{p^{→}_{X_α}O_S}(i^{α−1}_α F, i^{α}_α F)$$

$$\cong \mathcal{H}om_{p^→_{X_α}O_S}(\mathcal{H}om^{−1}_α (i^{α−1}_α F), \mathcal{H}om^{−1}_α (i^{α}_α F))$$

and since codim $p_{X_α}(\text{Supp}(i^{α−1}_α R\hom_{p^→_{X_α}O_S}(F, F))) \geq 1$ we obtain

$$\text{codim } p_{X_α}(\text{Supp}(\mathcal{H}om^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F))) \geq 1.$$ 

From the distinguished triangle

$$i^{α}_α R\hom_{p^{→}_{X}O_S}(F, F) \longrightarrow i^{α−1}_α R\hom_{p^→_{X_α}O_S}(F, F)$$

$$\longrightarrow i^{α−1}_α Rj_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F) \longrightarrow 1,$$

we obtain the short left exact sequence

$$0 \longrightarrow \mathcal{H}om^{−1}_α R\hom_{p^{→}_{X}O_S}(F, F) \longrightarrow \mathcal{H}om^{−1}_α R\hom_{p^{→}_{X_α}O_S}(F, F)$$

$$\longrightarrow \mathcal{H}om^{−1}_α Rj_α∗j^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F)$$

which proves that codim $p_{X_α}(\text{Supp}(i^{α−1}_α R\hom_{p^→_{X_α}O_S}(F, F))) \geq 1$ since both the first and the third term of the sequence satisfy this condition.

**Step 2.** Let now deduce from step 1 that, for any $F \in \text{perv}(p^{-1}_X(0_S))_t$, codim $p_X(\text{Supp } F) \geq 1$.

The previous condition implies $\dim(p_X(\text{Supp } \mathcal{H}om_{\text{perv}(p^{-1}_XO_S)(F, F)})) = 0$ for any $F \neq 0$ and hence $\forall (x_0, s_0) \in X \times S$, choosing a local coordinate $s$ in $S$ vanishing in $s_0$, by the $S$-constructibility of $\mathcal{H}om_{\text{perv}(p^{-1}_XO_S)(F, F)} \cong \mathcal{H}om^{−1}_α R\hom_{p^→_{X_α}O_S}(F, F)$ there exists a positive integer $N$ such that in a neighbourood of $(x_0, s_0)$, $(s − s_0)^N \text{Hom}_{\text{perv}(p^{-1}_XO_S)(F, F)} = 0$. Therefore $(s − s_0)^N \text{id}_F = 0$ and so $\text{id}_{(s−s_0)^N F} = 0$ which entails the result. q.e.d.

**Remark 3.12.** Let $d_S = 1$. In Definition 3.6 we denoted by $\text{perv}(p^{−1}_X(0_S))_t$ the full subcategory of perverse sheaves which are torsion-free (i.e. for any $s \in S$ $L_i^*F \in \text{perv}(\mathcal{C}_X)$) while in Proposition 3.11 we proved that $\text{perv}(p^{−1}_X(0_S))_t = \{ F \in \text{perv}(p^{-1}_X(0_S)) | \text{codim } p_X(\text{Supp } F) \geq 1 \}$. Hence (cf. Proposition 3.7) $\pi$ is the $t$-structure obtained by left tilting $p$ with respect to the torsion pair $(\text{perv}(p^{−1}_X(0_S))_t)$, $\text{perv}(p^{−1}_X(0_S))_t$ in $\text{perv}(p^{−1}_X(0_S))_t$ while $p$ is the $t$-structure obtained by right tilting $\pi$ with respect to the tilted torsion pair.
(perv(\(p_X^{-1}(\mathcal{O}_S)\))_\mathcal{H}, perv(\(p_X^{-1}(\mathcal{O}_S)\))_\mathcal{H}[-1]) in \(\mathcal{H}_\pi\). In particular we obtain that

\[ \mathcal{H}_\pi = \{ F \in p \mathcal{D}^{0,1}_{\text{c},c}(p_X^{-1}\mathcal{O}_S) | p^*\mathfrak{f}^0(F) \text{ torsion free and } p^*\mathfrak{f}^1(M) \text{ torsion} \}. \]

4. \textit{t-exactness of the} \(p\text{DR}\) \textit{functors for} \(d_S = 1\)

4.a. \textbf{Reminder on the construction of} \(\text{RH}^S\). For details on the relative subanalytic site and construction of relative subanalytic sheaves we refer to \([16]\). For details on the construction of \(\text{RH}^S\) we refer to \([18]\).

We shall denote by \(\text{Op}(Z)\) the family of open subsets of a subanalytic site \(Z\). One denotes by \(\rho\), without reference to \(X \times S\) unless otherwise specified, the natural functor of sites \(\rho: X \times S \rightarrow (X \times S)_{\text{sa}}\) associated to the inclusion \(\text{Op}((X \times S)_{\text{sa}}) \subset \text{Op}(X \times S)\). Accordingly, we shall consider the associated functors \(\rho, \rho^{-1}\) introduced in \([13]\) and studied in \([19]\).

One also denotes by \(\rho': X \times S \rightarrow X_{\text{sa}} \times S_{\text{sa}}\) the natural functor of sites. We have well defined functors \(\rho'_*\) and \(\rho'_!\) from \(\text{Mod}(\mathcal{C}_{X \times S})\) to \(\text{Mod}(\mathcal{C}_{X_{\text{sa}} \times S_{\text{sa}}})\).

Note that \(W \in \text{Op}(X_{\text{sa}} \times S_{\text{sa}})\) if and only if \(W\) is a locally finite union of relatively compact subanalytic open subsets \(W\) of the form \(U \times V, U \in \text{Op}(X_{\text{sa}}), V \in \text{Op}(S_{\text{sa}})\). Note that there is a natural morphism of sites \(\eta: (X \times S)_{\text{sa}} \rightarrow X_{\text{sa}} \times S_{\text{sa}}\) associated to the inclusion \(\text{Op}(X_{\text{sa}} \times S_{\text{sa}}) \rightarrow \text{Op}((X \times S)_{\text{sa}})\).

In the absolute case, the Riemann-Hilbert reconstruction functor \(\text{RH}^S\) introduced by Kashiwara in \([9]\) from \(\mathcal{D}^{b}_{\text{e},c}(\mathcal{C}_X)\) to \(\mathcal{D}^{b}((\mathcal{D}_X)\) was later denoted by \(\mathcal{T}\text{Hom}(\cdot, \mathcal{O}_X)\) in \([13]\) where it was extensively studied. In \([14]\) the authors showed that it can be recovered using the language of subanalytic sheaves as \(\rho^{-1}\mathcal{T}\text{Hom}(\cdot, \mathcal{O}_X^!\) where \(\mathcal{O}_X^!\) is the subanalytic complex of tempered holomorphic functions on \(X_{\text{sa}}\).

Let \(F\) be a subanalytic sheaf on \((X \times S)_{\text{sa}}\). Following \([16]\), one denotes by \(F^{S,2}\) the sheaf on \(X_{\text{sa}} \times S_{\text{sa}}\) associated to the presheaf

\[ \text{Op}(X_{\text{sa}} \times S_{\text{sa}}) \rightarrow \text{Mod}(\mathcal{C}) \]

\[ U \times V \mapsto \Gamma(X \times V; \rho^{-1}\Gamma_{U \times S}F) \simeq \text{Hom}(\mathcal{C}_U \boxtimes \rho_!\mathcal{C}_V, F) \]

\[ \simeq \lim_{W \in V} \Gamma(U \times W; F). \]

One also denotes by \((\ast)^{RS,2}\) the associated right derived functor.

Then \(\mathcal{O}^{S,2}_{X \times S} = (\mathcal{O}^!_{X \times S})^{RS,2}\) is an object of \(\mathcal{D}^b(\rho'_!p^{-1}\mathcal{O}_S)\) and we also have \(\mathcal{O}_{X \times S} \simeq \rho'^{-1}(\mathcal{O}^{S,2}_{X \times S})\) (cf. \([16]\) for details).

The functor \(\text{RH}^S: \mathcal{D}^{b}_{\text{e},c}(p_X^{-1}(\mathcal{O}_S)) \rightarrow \mathcal{D}^{b}((\mathcal{D}_X)_{X \times S/S})\) was then defined in \([18]\) by the expression

\[ \text{RH}^S(F) = \rho'^{-1} \mathcal{T}\text{Hom}(\rho'_!p_X^{-1}\mathcal{O}_S, p'_*F, \mathcal{O}^{S,2}_{X \times S}; [d_X]). \]

When \(F\) is \(S - \mathbb{C}\) constructible, then \(\text{RH}^S(F)\) has regular holonomic \(\mathcal{D}_{X \times S/S}\)-cohomologies (\([18]\) Th. 3)).
4.b. Main results and proofs. The main results of this section are Theorem 4.1 and Theorem 4.2 below.

**Theorem 4.1.** If $d_S = 1$ the functor $\text{pDR}$ is t-exact with respect to the t-structures $P$ and $p$ above and consequently, $\text{pDR}$ is also t-exact with respect to the dual t-structures $\Pi$ and $\pi$.

**Theorem 4.2.** If $d_S = 1$ the functor $\text{RH}^S$ is t-exact with respect to the t-structures $p$ and $\Pi$ as well as with respect to their dual t-structures $\pi$ and $P$.

We shall need the following results.

**Lemma 4.3.** Let $q$ denote the projection $X \times S \to X$. Let be given a perverse sheaf $F$ on $X$ (where we consider the middle perversity on $D^b_{C, c}(\mathbb{C}_X)$). Let $\mathcal{G}$ be a sheaf of $\mathbb{C}$-vector spaces on $S$. Assume that $F \boxtimes \mathcal{G} := q^{-1}F \otimes p^{-1}\mathcal{G}$ belongs to $D^b_{C, c}(p_X^{-1}\mathcal{O}_S)$. Then $F \boxtimes \mathcal{G}$ is perverse in $pD^b_{C, c}(p_X^{-1}\mathcal{O}_S)$.

**Proof.** The result is trivial for $d_S = 0$. Let us now consider $d_S \geq 1$. Let $(X_\alpha)_{\alpha \in A}$ be a stratification adapted to $F$. Then $(X_\alpha)_{\alpha \in A}$ is also adapted to $q^{-1}F \otimes p^{-1}\mathcal{G}$ as an object of $D^b_{C, c}(p_X^{-1}\mathcal{O}_S)$ because the microsupport of $q^{-1}F$, $\text{SS}(q^{-1}F)$, cuts $\text{SS}(p^{-1}\mathcal{G})$ along $T_X^s X \times T^s S$ (cf. [11] Propositions 5.4.5 and 5.4.14). Moreover, if $F \in pD^b_{C, c}(\mathbb{C}_X)$ it is clear that $F \boxtimes \mathcal{G} \in pD^b_{C, c}(p_X^{-1}\mathcal{O}_S)$.

Let us now prove that $F \boxtimes \mathcal{G} \in pD^b_{C, c}(p_X^{-1}\mathcal{O}_S)$. Let $i_\alpha$ denote either the inclusion $X_\alpha \subset X$ or $X_\alpha \times S \subset X \times S$, for each $\alpha \in A$. Let $q_\alpha$ (resp. $p_\alpha$) denote the restriction of $q$ (resp. of $p$) to $X_\alpha \times S$. We have the following commutative diagram

\[
\begin{array}{ccc}
X_\alpha \times S & \xrightarrow{q_\alpha} & X_\alpha \\
\downarrow{i_\alpha} & & \downarrow{i_\alpha} \\
X \times S & \xrightarrow{q} & X
\end{array}
\]

We get a sequence of isomorphisms in $D(\mathbb{C}_{X_\alpha \times S})$:

\[
i_\alpha^!(q^{-1}F \otimes p^{-1}\mathcal{G}) \simeq i_\alpha^!q^{-1}F \otimes i_\alpha^{-1}p^{-1}\mathcal{G}
\]

\[
\simeq i_\alpha^!\alpha^!F[-2d_s] \otimes p_\alpha^{-1}\mathcal{G}
\]

\[
\simeq (qi_\alpha)^!(F[-2d_S] \otimes p_\alpha^{-1}\mathcal{G})
\]

\[
\simeq (i_\alpha q_\alpha)^!(F[-2d_S] \otimes p_\alpha^{-1}\mathcal{G})
\]

\[
\simeq q_\alpha^{-1}i_\alpha^{-1}F[-2d_S] \otimes p_\alpha^{-1}\mathcal{G}
\]

\[
\simeq q_\alpha^{-1}i_\alpha^{-1}F[-2d_S + 2d_S] \otimes p_\alpha^{-1}\mathcal{G}
\]

Since $i_\alpha^!F \in D^{> - d_\alpha}(\mathbb{C}_{X_\alpha})$, the last isomorphism shows that $i_\alpha^!(q^{-1}F \otimes p^{-1}\mathcal{G}) \in D^{> - d_\alpha}(\mathbb{C}_{X_\alpha \times S})$ as desired.

q.e.d.
Proof of Theorem 4.1. The second statement follows obviously from the first thanks to the $t$-exactness of the duality functors (by definition of the dual $t$-structures and the commutation of $p\text{DR}$ with duality (cf. [17, Th. 3.11]). Let us now prove the first part of the statement. According to Lemma 2.9, it is sufficient to prove that if $\mathcal{M}$ is a holonomic relative module then $p\text{DR}(\mathcal{M})$ is perverse.

Noting that, for $d_S = 1$, strictness is equivalent to absence of $\mathcal{O}_S$-torsion, and that, according to Proposition 2 of [13], the statement is true assuming that $\mathcal{M}$ is strict, we are reduced to prove the statement assuming that $\mathcal{M}$ is a torsion module. In such a case, for any $(x_0, s_0) \in X \times S$, choosing a local coordinate $s$ in $S$ vanishing in $s_0$, by the coherency of $\mathcal{M}$ we can find a positive integer $N$ such that $(s - s_0)^N \mathcal{M} = 0$ in a neighborhood of $(x_0, s_0)$. Arguing by induction on $N$ we are led to assume $N = 1$, in particular we may assume $\mathcal{M} = \mathcal{M}/(s - s_0)\mathcal{M}$ hence $\mathcal{M}$ is naturally a holonomic $D_X$-module where we identify $X$ to $X \times \{s_0\}$.

Locally, we get a chain of natural isomorphisms in $D^b(\mathcal{O}_S)$

$$p\text{DR} \mathcal{M} = R\mathcal{H}om_{D_X\times_S}(\mathcal{O}_X\times_S, \mathcal{M})[d_X]$$

$$\simeq R\mathcal{H}om_{D_X\times_S/(s-s_0)}(\mathcal{O}_X\times_S/(s-s_0)\mathcal{O}_X\times_S, \mathcal{M}/(s-s_0)\mathcal{M})[d_X]$$

We conclude a local isomorphism in $D^b(C_X\times_S)$

$$p\text{DR} \mathcal{M} \simeq q^{-1} F \otimes p^{-1} C_{\{s_0\}}$$

where $F$ is the perverse sheaf $p\text{DR}(\mathcal{M}/(s - s_0)\mathcal{M})$ sur $X = X \times \{s_0\}$. The result then follows by Lemma 4.3 q.e.d.

Corollary 4.4. The functor $p\text{Sol}$ is $t$-exact with respect to the $t$-structures respectively $P$ on $D^b(\mathcal{O}_X\times_S)\mathcal{O}_S^{op}$ and $\pi$ on $D^b_c(p^{-1}\mathcal{O}_S)$.

Proof. The statement follows immediately from the relation $D^{p\text{DR}} = p\text{Sol}$ (cf. [17, Corollary 3.9]). q.e.d.

Remark 4.5. However the functor $p\text{Sol} : D^b(\mathcal{O}_X\times_S)\mathcal{O}_S^{op} \to D^b_c(p^{-1}\mathcal{O}_S)$ is not $t$-exact with respect to the $t$-structures respectively $P$ on $D^b(\mathcal{O}_X\times_S)\mathcal{O}_S^{op}$ and $\pi$ on $D^b_c(p^{-1}\mathcal{O}_S)$ as shown by the following example:

Example 4.6. Let $X = \mathbb{C}^*$ and $S = \mathbb{C}$ with respective coordinates $x$ and $s$. Let $\mathcal{M}$ be the quotient of $\mathcal{D}_X\times_S$ by the left ideal generated by $\partial_x$ and $s$. Then $\mathcal{M}$ can be identified with $\mathcal{O}_X\times(0)$ with the $s$-action being zero and the standard $\partial_x$-action. We notice that $\mathcal{M}$ is holonomic, but not strict. As a $\mathcal{D}_X\times_S$-module, it has the following resolution:

$$0 \to \mathcal{D}_X\times_S \xrightarrow{p \mapsto (p\partial_x, Ps)} \mathcal{D}_X^2\times_S \xrightarrow{(Q,R) \mapsto -Q\partial_x - Qs} \mathcal{D}_X\times_S \to \mathcal{M} \to 0.$$
where $\phi(f) = (\partial_x f, sf)$ and $\psi(g, h) = sg - \partial_x h$. We know that $^p\text{Sol}(M)$ is constructible, and since we work on $\mathbb{C}^*$, we see that its cohomology is $S$-locally constant. We note that $\mathcal{H}^0(\text{Sol}(M)|_{X \times \{0\}}) \neq 0$, since $(g, h) = (0, 1)$ is a nonzero section of it. Therefore, $^p\text{Sol}(M)$ does not belong to $^p\mathcal{D}_{\mathbb{C}^0}^{-\leq 0}(p_X^{-1}O_S)$ and so $^p\text{Sol}(M)$ is not perverse.

However $^p\text{DR}M$ is a perverse object: it is realized by the complex

$$0 \rightarrow M \xrightarrow{-1} \frac{\partial_x}{0} M \rightarrow 0$$

and the surjectivity of $\partial_x$ on $O_{X \times \{0\}}$ entails that $\mathcal{H}^0^p\text{DR}(M) = 0$. Moreover $\mathcal{H}^j\mathcal{R}^j\Gamma_{X \times S}^p\text{DR}M = \mathcal{H}^j^p\text{DR}M = 0$, for $j < -1$.

**Proof of Theorem 4.2**

i) Let us prove the first $t$-exactness. By Lemma 2.4 we have to prove that $\mathcal{R}^h(\text{Perv}(p_X^{-1}O_S)) \subseteq \Pi^\oplus D_{\text{hol}}^\geq 0(\mathcal{D}_{X \times S/S}) \cap \Pi^\oplus D_{\text{hol}}^\leq 0(\mathcal{D}_{X \times S/S})$. Recall that $\mathcal{R}L_i^\ast(F) \cong L_i^\ast \mathcal{R}^h(F)$ by [18, Proposition 3.25]. According to Lemma 3.4 given $F \in \text{perv}(p_X^{-1}O_S)$ we have $L_i^\ast F \in \mathcal{D}_{\mathbb{C}^0}^\geq 0(C_X)$ for each $s \in S$ and hence $\mathcal{R}L_i^\ast(F) \cong L_i^\ast \mathcal{R}^h(F) \in \mathcal{P} D_{\text{hol}}^\leq 0(\mathcal{D}_{X})$ for each $s \in S$ (since the functor $R^h$ is $t$-exact in the absolute case) and so by Lemma 2.2 we obtain $\mathcal{R}^h_F(\Pi^\oplus D_{\text{hol}}^\geq 0(\mathcal{D}_{X \times S/S})).$

It remains to prove that $\mathcal{R}^h(\text{Perv}(p_X^{-1}O_S)) \subseteq \Pi^\oplus D_{\text{hol}}^\leq 0(\mathcal{D}_{X \times S/S})$. Let $F \in \text{perv}(p_X^{-1}O_S)$. According to Lemma 3.4 for any $s \in S$, $L_i^\ast F \in \mathcal{D}_{\mathbb{C}^0}^\leq 1(X)$. Hence $L_i^\ast(\mathcal{R}^h F) \cong \mathcal{R}H(L_i^\ast F) \in D_{\text{hol}}^\leq 1(\mathcal{D}_{X})$ and thus by (2) of Lemma 2.2 we obtain $(*) \mathcal{R}^h_F \in \mathcal{P} D_{\text{hol}}^\leq 1(\mathcal{D}_{X \times S/S})$. By Proposition 2.6 and Definition 1.1 it is sufficient to prove that $(**)^p\mathcal{H}^1(\mathcal{R}^h(F))$ is a torsion module.

We divide the question in two cases, the torsion case and the torsion free case. Let us first suppose that $F \in \text{perv}(p_X^{-1}(O_S))$. According to Proposition 3.11 we have $\text{codim}_{p_X}(\text{Sup}(F)) \geq 1$ and so also $\text{codim}_{p_X}(\text{Sup}(^p\mathcal{H}^1(\mathcal{R}^h(F)))) \geq 1$.

Let us now suppose that $F \in \text{perv}(p_X^{-1}(O_S))_t$. According to [18, Cor.4], $\mathcal{R}^h(F)$ is a regular strict holonomic $\mathcal{D}_{X \times S/S}$-module so it belongs to $\Pi^\oplus D_{\text{hol}}^\leq 0(\mathcal{D}_{X \times S/S}$ which achieves the proof of i).

ii) Let us now prove the second $t$-exactness. By Lemma 2.9 we have to prove that $\mathcal{R}^h(\mathcal{H}_\pi) \subseteq \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})$. Given $F \in \mathcal{H}_\pi$ we know, according to Remark 3.12 that $F \in \mathcal{P}_{[0,1]}(p_X^{-1}O_S)$ with $^p\mathcal{H}^0(F)$ strict where $^p\mathcal{H}^1(F)$ is a torsion module. So, by Proposition 3.11 we have $\text{codim}_{p_X}(\text{Sup}(^p\mathcal{H}^1(F))) \geq 1$. Let us consider the distinguished triangle $^p\mathcal{H}^0(F) \rightarrow F \rightarrow ^p\mathcal{H}^1(F)[-1] \rightarrow 1$ (which provides the short exact sequence of $F$ with respect to the torsion pair $(\text{perv}(p_X^{-1}(O_S))_t, \text{perv}(p_X^{-1}(O_S))_t[-1])$ in $\mathcal{H}_\pi$). According to [18, Cor.4] we conclude that $\mathcal{R}^h(\text{perv}(^p\mathcal{H}^0(F)))$ is a strict relative holonomic $\mathcal{D}_{X \times S/S}$-module while, by the previous $t$-exactness, $\mathcal{R}^h(\text{perv}(^p\mathcal{H}^1(F)[-1])) = \mathcal{R}^h(\text{perv}(^p\mathcal{H}^1(F)))[1] \in \mathcal{H}_\pi[1]$. Therefore, according to Proposition 2.6 we have

$$\mathcal{H}_\pi[1] = \{ \mathcal{M} \in \mathcal{P} D_{\text{hol}}^{-1,0}(\mathcal{D}_{X \times S/S}) \mid ^p\mathcal{H}^{-1}(\mathcal{M}) \text{ strict and } ^p\mathcal{H}^0(\mathcal{M}) \text{ torsion} \}. $$
On the other hand, since $\operatorname{codim}(p_X(S)) \geq 1$, the cohomology sheaves of $\operatorname{RH}^\ast(p\mathcal{H}^1(F)[1])$ are torsion $\mathcal{D}_{X \times S/S}$-modules. Therefore $p\mathcal{H}^{-1}(\operatorname{RH}^\ast(p\mathcal{H}^1(F))[1])$, being strict, must be equal to 0, in other words $\operatorname{RH}^S(p\mathcal{H}^1(F)[1]) \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})$ which ends the proof.

In general for $\dim S \geq 1$ we have the following result:

**Proposition 4.7.** The functor $p\mathcal{D}R$ satisfies the following conditions:

1. $p\mathcal{D}R(p\mathcal{D}R(\mathcal{D}_{\text{hol}}(X \times S/S)) \subseteq \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$;
2. $p\mathcal{D}R((\mathcal{D}_{\text{hol}}(X \times S/S) \cap s\mathcal{D})(\mathcal{D}_{\text{hol}}(X \times S/S)) \subseteq \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$.

**Proof.** The first item is contained in [15, Proposition 1.15 (1)].

We set for short $s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) := \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap s\mathcal{D}(\mathcal{D}_{X \times S/S})$ and let us prove that $p\mathcal{D}R(s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})) \subseteq \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$ (recall that $s\mathcal{D}_{X \times S/S} = M \in \mathcal{D}_{X \times S/S} | \operatorname{codim}(p_X(S)) \geq k$).

We denote by $p\tau_{\leq k}$ the truncation functor with respect to the $t$-structure $P$ on $\mathcal{D}_{\text{hol}}(X \times S/S)$. Given $M \in s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})$, for any $k \in \mathbb{Z}$ both $p\tau_{\leq k}M$ and $p\tau_{\geq k+1}M$ belong to $s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})$ (since $p\mathcal{H}^i(p\tau_{\leq k}M)) = p\mathcal{H}^i(M))$ for $i \leq k$ or zero otherwise.

Let us prove that:

$$M \in s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap p\mathcal{D}R(\mathcal{D}_{X \times S/S}) \Rightarrow p\mathcal{D}R(M) \in \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$$

by induction on $k \geq 0$.

Let $k = 0$ by Lemma 3.5 and Lemma 3.5 we get $p\mathcal{D}R(p\mathcal{D}R(\mathcal{D}_{X \times S/S})) \subseteq \mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S})$ and $p\mathcal{D}R(p_X^{-1}\mathcal{O}_S) \subseteq \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$ and so $(I_0)$ holds true by Lemma 3.5. Let us suppose that $(I_k)$ holds true and let us prove $(I_{k+1})$. Let consider $M \in s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap p\mathcal{D}R(p\mathcal{D}R(\mathcal{D}_{X \times S/S}))$. The distinguished triangle

$$p\tau_{\leq k}M \rightarrow M \rightarrow p\tau_{\geq k+1}(M)[-k-1] \rightarrow$$

induces the distinguished triangle

$$p\mathcal{D}R(p\tau_{\leq k}M) \rightarrow p\mathcal{D}R(M) \rightarrow p\mathcal{D}R(p\tau_{\geq k+1}(M)[-k-1]) \rightarrow$$

By inductive hypothesis $p\mathcal{D}R(p\tau_{\leq k}M) \in \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$ since $p\tau_{\leq k}M \in s\mathcal{D}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \cap p\mathcal{D}R(p\mathcal{D}R(\mathcal{D}_{X \times S/S}))$. In order to conclude it is enough to prove that $p\mathcal{D}R(p\tau_{\geq k+1}(M)[-k-1]) \in \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$.

By Proposition 3.4 we have to prove that

$$i_{x}^{-1}(p\mathcal{D}R(p\tau_{\geq k+1}(M)[-k-1])) \in \mathcal{D}_{\text{coh}}(\alpha)(\mathcal{O}_S)$$

for any $\alpha$ and any $x \in X_\alpha$, for some adapted $\mu$-stratification $(X_\alpha)$. By the first item we have $p\mathcal{D}R(p\tau_{\geq k+1}(M)) \in \mathcal{D}_{\text{hol}}(p_X^{-1}\mathcal{O}_S)$ and thus (see Definition 3.1)

$$i_{x}^{-1}(p\mathcal{D}R(p\tau_{\geq k+1}(M)[-k-1])) \in \mathcal{D}_{\text{coh}}(\alpha+k+1)(\mathcal{O}_S)$$

$q.e.d.$
for any $\alpha$ and any $x \in X_\alpha$, for some adapted $\mu$-stratification $(X_\alpha)$. Moreover codim $p_X(\text{Supp}(P^k\mathcal{H}^{k+1}(M))) \geq k+1$ since $M \in \mathcal{S}_\mu^{\leq 0}(\mathcal{D}X\times S/S)$ and thus codim $p_X(\text{Supp} p^{-1}_D(\mathcal{H}^{k+1}(M))[-k-1]) \geq k+1$ which proves (see Remark 2.1) that $p^{-1}_D(\mathcal{H}^{k+1}(M))[-k-1] \in \mathcal{H}^{\leq \text{coh}}_{d_S}(\mathcal{O}_S)$.

q.e.d.

However we don’t know if $\mathcal{S}_\mu^{\leq 0}(\mathcal{D}X\times S/S) = \mathcal{H}^{\leq \text{coh}}_{d_S}(\mathcal{O}_S)$. If that equality holds true then the previous Proposition would imply that the functor $P^D$ is $t$-exact with respect to the $t$-structures $P$ and $p$ above without restriction on $d_S$, and consequently, $P^D$ is also $t$-exact with respect to the dual $t$-structures $\Pi$ and $\pi$.

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