COFIBRATION AND MODEL CATEGORY STRUCTURES FOR DISCRETE AND CONTINUOUS HOMOTOPY

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Abstract. We show that the categories $\text{PsTop}$ and $\text{Lim}$ of pseudotopological spaces and limit spaces, respectively, admit cofibration category structures, and that $\text{PsTop}$ admits a model category structure, giving several ways to simultaneously study the homotopy theory of classical topological spaces, combinatorial spaces such as graphs and matroids, and metric spaces endowed with a privileged scale, in addition to spaces of maps between them. In the process, we give a sufficient condition for a topological construct which contains compactly generated Hausdorff spaces as a subcategory to admit an $I$-category structure. We further show that, for a topological space $X \in \mathcal{C}$, the homotopy groups of $X$ constructed in the cofibration category on $\text{PsTop}$ are isomorphic to those constructed classically in $\text{Top}^\ast$.

1. Introduction

In [30], Steenrod proposed that the category of compactly generated Hausdorff spaces should be the appropriate setting in which to study the homotopy theory of topological spaces, arguing that this category contained the principal spaces of interest in algebraic topology, that it was closed under a number of standard operations, and that the operations in question satisfy a number of reasonable properties. In modern terminology, these last two conditions are encapsulated by the fact that the category of compactly generated Hausdorff spaces is Cartesian closed, and, over time, the category of compactly generated Hausdorff spaces did indeed become one of the standard categories in which to study the algebraic topology of (most) topological spaces.

In recent years, there has been an increasing interest in using tools from algebraic topology to study combinatorial objects such as graphs [11, 12, 13, 16, 18, 13, 14], as well as an interest in studying the algebraic topology of metric spaces which have been decorated with a privileged scale [23, 16, 29, 4, 28], which we call scaled metric spaces, following [28]. Furthermore, in topological data analysis [10], one encounters the problem of how to compare the topological invariants of a topological space with the invariants of an object in one or more of the above classes, making it important to work in a category which simultaneously contains both combinatorial objects as well as more classical topological spaces. We therefore find ourselves once again in search of a “convenient” category for the study of the algebraic topology of certain spaces of interest, this time combinatorial objects and scaled metric spaces, in addition to CW complexes. In [28], we proposed that this could be at least partially resolved by developing algebraic topology in the category $\text{Cl}$ of Cech closure spaces, since this category contains graphs, scaled metric spaces, and topological...
spaces as subcategories, and it also admits non-trivial maps between them. This perspective was further developed in [3] [9]. Unfortunately, however, the category of Čech closure spaces has a serious shortcoming: like the category of topological spaces, it is not Cartesian closed, and, in particular, there is no canonical closure structure on function spaces for which the evaluation map is continuous. We are also unaware of a subcategory of Cl which is Cartesian closed and which also contains compactly generated topological spaces, graphs, and scaled metric spaces as full subcategories, leaving us to investigate Cartesian closed categories which contain Cl itself as a subcategory.

The goal of the present article is to show that the category PsTop of pseudotopological spaces - the Cartesian closed hull of Cl [7] [24] - can serve as a convenient setting for the simultaneous study of homotopy theory on many types of combinatorial spaces as well as CW-complexes, and, in particular, that it is well-suited to studying homotopy theory on graphs, scaled metric spaces, topological spaces, and spaces of maps between them. We do this in a number of steps. First, we show that any Cartesian closed category C which contains compactly generated Hausdorff spaces as a subcategory and satisfies some reasonable additional conditions admits a cofibration category structure (in the sense of Baues [5] [6]). Second, we use the cofibration category structure to construct homotopy groups for pseudotopological spaces, and we show that the resulting homotopy groups are isomorphic to the classical homotopy groups when the pseudotopological space is also topological, i.e. that $\pi_n(X) = [S^n, X]$ as sets, and this can be given a group structure for $n \geq 1$. While the cofibration category structure is enough to build a rich homotopy theory on PsTop, we further improve on this in Section 5 by showing that the homotopy groups from the cofibration category may be used to define the build a model category structure on PsTop which extends the Quillen model category structure on Top.

2. PSEUDOTOPOLOGICAL SPACES

In this section, we introduce the category PsTop of pseudotopological spaces and continuous functions, and we recall a number of basic results about PsTop and several related categories for later reference. We also show that pseudotopological spaces contain as full subcategories the main categories of interest to combinatorics and applied topology: graphs, scaled metric spaces, matroids, topological spaces, and Čech closure spaces.

We begin by defining general convergence spaces, specializing this to pseudotopological spaces in Definition 2.6.

**Definition 2.1.** A *convergence space* is a pair $(X, \Lambda)$, where $X$ is a set and $\Lambda \subseteq \mathcal{F}(X) \times X$ is a relation between the set of filters on $X$, which we denote by $\mathcal{F}(X)$, and the elements of $X$, where the relation $\Lambda$ satisfies the following axioms.

1. If $(\lambda, x) \in \Lambda$ and $\lambda \subseteq \lambda'$, then $(\lambda', x) \in \Lambda$.
2. $\dot{x} \in \Lambda$ for every $x \in X$, where $\dot{x}$ denotes the filter generated by $x$.

We call such a relation $\Lambda$ a *convergence structure* or *convergence* on $X$.

If, in addition,

1. $(\lambda, x), (\lambda', x) \in \Lambda$, then $(\lambda \cap \lambda', x) \in \Lambda$,

then we say that $(X, \Lambda)$ is a *limit space*.

Given two convergence structures $\Lambda$ and $\Lambda'$ on a set $X$, we say that $\Lambda$ is *coarser* than $\Lambda'$, and $\Lambda'$ is *finer* than $\Lambda$, iff $\Lambda \subseteq \Lambda'$.
We will often write $\lambda \to x$ in $(X, \Lambda)$ for $(\lambda, x) \in \Lambda$, and we define
\[
\lim_{\Lambda} \lambda := \{ x \in X \mid (\lambda, x) \in \Lambda \}.
\]
When writing $\lim_{\Lambda} \lambda$ and $(X, \Lambda)$, we will sometimes omit the convergence structure $\Lambda$ when it is unambiguous. Finally, given a convergence structure $\Lambda$, we will abuse notation and write $\lambda \in \Lambda$ to indicate that there exists and element $x \in X$ such that $(\lambda, x) \in \Lambda$.

**Definition 2.2.** Let $(X, \Lambda_X)$ and $(Y, \Lambda_Y)$ be convergence spaces, and let $f : X \to Y$ be a map from $X$ to $Y$. We say that $f$ is **continuous** iff
\[
\lambda \to x \in \Lambda_X \Rightarrow f(\lambda) \to f(x) \in \Lambda_Y,
\]
where $f(\lambda)$ is the filter in $Y$ generated by the collection $\{ f(U) \subset Y \mid U \in \lambda \}$ of subsets of $Y$.

**Definition 2.3.** An **ultrafilter** $\gamma$ on a set $X$ is a filter such that, for every subset $A \subset X$, either $A \in \gamma$ or $X \setminus A \in \gamma$.

**Definition 2.4.** Given a filter $\lambda$ on a set $X$, we denote by $\beta(\lambda)$ the set of all ultrafilters containing $\lambda$.

With this definition, we recall the following.

**Proposition 2.5.** Any filter $\lambda$ on $X$ is equal to the intersection of all of the ultrafilters which contain it, i.e. $\lambda = \bigcap_{\gamma \in \beta(\lambda)} \gamma$.

In particular, Lemma 2.6 and the definition of a convergence space implies that $\lim_{\Lambda} \lambda \subset \bigcap_{\gamma \in \beta(\lambda)} \lim_{\Lambda} \gamma$ for every filter $\lambda$ on a convergence space. We define a pseudotopological space to be a convergence space in which the reverse inclusion also holds.

**Definition 2.6.** We say that a convergence space $(X, \Lambda)$ is a **pseudotopological space** iff
\[
\lim_{\Lambda} \lambda = \bigcap_{\gamma \in \beta(\lambda)} \lim_{\Lambda} \gamma,
\]
for every filter $\lambda$ on $X$, where $\beta(\lambda)$ is the collection of ultrafilters which contains $\lambda$.

Finally, we recall from [24] that pseudotopological spaces are also limit spaces.

**Proposition 2.7** ([24], Remark 2.3.1.2). The category $\text{PsTop}$ of pseudotopological spaces and continuous maps between them is a full subcategory of the category $\text{Lim}$ of limit spaces and continuous maps.

2.1. **Classes of Pseudotopological Spaces.** In this section, we give the principal examples of pseudotopological spaces which are of interest to discrete and continuous homotopy.

2.1.1. **Pseudotopologies from Cech Closure Spaces.** We recall from [12] that a Cech closure space (also called a pretopological space) is a pair $(X, c)$ where $X$ is a set and $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a function on the power set of $X$ which satisfies

1. $c(\emptyset) = \emptyset$,
2. $A \subset c(A) \forall A \subset X$, and
3. $c(A \cup B) = c(A) \cup c(B) \forall A, B \subset X$. 
A function \( f : (X, c_X) \to (Y, c_Y) \) between closure spaces is said to be continuous iff for all \( A \subseteq X \), 
\( f(c_X(A)) \subseteq c_Y(f(A)) \). Čech closure spaces (pretopologies) and continuous maps form a category, 
denoted \( \text{Cl} \) (or \( \text{PreTop} \)).

The fundamental fact about about pretopologies is that they are completely determined by the 
choice of a neighborhood filter at every point \( x \in X \), i.e.

**Theorem 2.8** ([12], Theorem 14.B.10). Let \( X \) be a set, and suppose that, for each element \( x \in X \), 
\( \mathcal{U}(x) \) is a collection of subsets of \( X \) satisfying

1. \( \mathcal{U}(x) \neq \emptyset \),
2. For each \( U \in \mathcal{U}(x) \), \( x \in U \), and 
3. For each \( U_1 \) and \( U_2 \) in \( \mathcal{U}(x) \), there exists a \( U \in \mathcal{U}(x) \) with \( U \subseteq U_1 \cap U_2 \).

Then there exists a unique closure operation \( c : \mathcal{P}(X) \to \mathcal{P}(X) \) such that, for each \( x \in X \), \( \mathcal{U}(x) \) is a local base for the neighborhood filter of \( x \in (X, c) \). Furthermore, \( c \) is given by the formula

\[
    c(A) := \{ x \mid x \in X, U \in \mathcal{U}(x) \Rightarrow U \cap X \neq \emptyset \}.
\]

**Definition 2.9.** Given a closure space \( (X, c) \), we define a convergence structure \( \Lambda_c \subseteq \mathcal{F}(X) \times X \) by: \( (\lambda, x) \in \Lambda_c \) iff \( \lambda \) contains the neighborhood filter of \( x \).

**Definition 2.10.** Let \( (X, c) \) be a closure space. A set \( A \subseteq X \) is said to be closed iff \( c(A) = A \) and \( A \subseteq X \) is said to be open if \( X \setminus A \) is closed.

**Proposition 2.11** ([24], Remark 2.3.1.2). The maps \( (X, c) \mapsto (X, \Lambda_c) \) and \( f \mapsto f \), where \( f : (X, c) \to (Y, c') \) is a continuous map, defines a functor \( F : \text{Cl} \to \text{PsTop} \) which makes the category 
\( \text{Cl} \) of Čech closure spaces into a full subcategory of \( \text{PsTop} \).

Note that, in particular, the above proposition implies that for a closure structure \( c \), the induced 
convergence structure \( \Lambda_c \) is pseudotopological, and that a continuous function for the closure 
structure is continuous for the pseudotopological structure. We now make precise what it means for a 
pseudotopological space to be topological.

**Definition 2.12.** A pseudotopological space \( (X, \Lambda) \) is said to be topological iff there exists a 
topological closure space \( (X, c_\tau) \) for which \( \Lambda = \Lambda_{c_\tau} \), i.e. there is a topology \( \tau \) on \( X \) such that a filter \( \lambda \mapsto x \) in \( \Lambda \) iff \( \lambda \) contains the filter of open sets \( \mathcal{U}(x) \).

Closure structures are extremely common. Every topological space \( (X, \tau) \) admits a closure 
structure \( c_\tau \), also known as the Kuratowski closure structure, where \( c_\tau(A) = \bar{A} \), where \( \bar{A} \) denotes 
the topological closure of \( A \) (see [12] or [28] for further details). Closure structures on graphs and 
semi-pseudometric spaces were discussed in [12], Chapter 14, as well as in [28], and closure structures 
on scaled metric spaces may be constructed from an induced semi-pseudometric as described in [28].

Matroids, furthermore, may also be defined using a (topological) closure structure, which then also 
induces a pseudotopological structure.

A closure structure may be induced by a hypergraph in two ways. First, we may construct a 
closure structure from the induced graph, where two vertices are an edge in the graph if they are 
contained in an edge of the hypergraph. This closure structure is given by \( c(v) := \{ w \mid \exists e \in E \text{ such that } \{v, w\} \subseteq e \} \), and \( c(A) := \cup_{e \in E} c(v) \), where \( v \in V \) and \( A \subseteq V \). We denote the resulting 
pseudotopological space by \( (V, \Lambda_c) \). Alternately, given a hypergraph \( H = (V, E) \), we may create 
a topological closure structure on its induced simplicial complex \( \Sigma = \cup_{e \in E} (\mathcal{P}(v) \setminus \{\emptyset\}) \), the union of nonempty subsets of each edge, by taking the Alexandrov topology on \( \Sigma \). Note, however, that 
neither of these procedures of constructing closure structures on hypergraphs induces a fully faithful 
functor from hypergraphs to closure spaces.
2.1.2. Pseudotopologies on Spaces of Continuous Functions. Unlike in \textbf{Top} or \textbf{Cl}, spaces of functions between pseudotopological spaces have a canonical pseudotopological structure which makes the evaluation map continuous, called the \textit{continuous convergence structure}. We recall its definition from [7].

\textbf{Definition 2.13.} Given convergence spaces \((X, \Lambda_X)\) and \((Y, \Lambda_Y)\), we denote by \(\mathcal{C}(X, Y)\) the set of all continuous mappings from \(X\) to \(Y\), and we denote by

\[ \omega_{X,Y} : \mathcal{C}(X, Y) \times X \to Y \]

the evaluation mapping, defined by \(\omega_{X,Y}(f, x) := f(x)\).

The \textit{continuous convergence structure} \(\Lambda_C\) on \(\mathcal{C}(X, Y)\) is defined by the following:

\[ \mathcal{H} \to f \text{ in } \Lambda_C \text{ iff } \omega_{X,Y}(\mathcal{H} \times \lambda) \to f(x) \forall (\lambda, x) \in \Lambda_X. \]

We see that, by definition, the evaluation map \(\omega_{X,Y}\) is continuous. Furthermore, when the spaces \(X\) and \(Y\) are pseudotopological, then \((\mathcal{C}(X, Y), \Lambda_C)\) is also pseudotopological, as guaranteed by the following proposition.

\textbf{Proposition 2.14} ([7], Theorem 1.5.5). \((\mathcal{C}(X, Y), \Lambda_C)\) is a pseudotopological space iff \((Y, \Lambda_Y)\) is pseudotopological.

We note that the continuous convergence structure \(\Lambda_C\) is not in general pretopological [7, 17].

3. Convenient Categories

Pseudotopological spaces will be our prototype for a \textit{convenient category}, i.e. a category which contains the main spaces of interest and which is Cartesian closed. Before setting the definition of convenience, we quickly recall several facts about Cartesian closed categories. For more details, see the relevant chapters in [1, 26] on Cartesian closed categories and the books [24, 17] for more on the categorical properties of convergence spaces, limit spaces, and pseudotopological spaces.

3.1. Cartesian Closed Categories and Convenience.

\textbf{Definition 3.1.} A category \(\mathcal{C}\) is said to be \textit{Cartesian closed} if it has all finite products and for each \(\mathcal{C}\)-object \(C\), the functor \((X \times -) : \mathcal{C} \to \mathcal{C}\) is a left adjoint.

The following well-known proposition is now immediate from the definition.

\textbf{Proposition 3.2.} Let \(\mathcal{C}\) be a Cartesian closed category. Then for all \(\mathcal{C}\)-objects \(C\), the functor \((C \times -) : \mathcal{C} \to \mathcal{C}\) preserves colimits.

\textit{Proof.} Left adjoints preserve colimits ([26], Theorem 4.5.3). \(\square\)

\textbf{Remark 3.3.} Convergence spaces, limit spaces, and pseudotopological spaces are all Cartesian closed ([24], 3.1.9(5)(6), 3.3.4(1)). Topological spaces and closure spaces, however, are not Cartesian closed ([24], 3.1.9(1)).

For the purposes of the present article, we define a convenient category as follows.

\textbf{Definition 3.4.} A \textit{convenient category of spaces} \(\mathcal{C}\) is a Cartesian closed category which contains the categories of graphs, scaled metric spaces, and compactly generated Hausdorff spaces as subcategories.

\textbf{Example 3.5.} Generalized convergence spaces, limit spaces, and pseudotopological spaces are all convenient categories.
3.2. Homotopies and the Homotopy Gluing Property. We now discuss homotopies of maps and homotopy equivalence. In particular, we present a homotopy gluing criterion which we will see provides a sufficient condition for a convenient category to admit an $I$-category structure. We also show that pseudotopological spaces and limit spaces satisfy this criterion.

**Definition 3.6.** We say that a quadruple $(I, i_0, i_1, p)$ is a cylinder on a category $C$ iff $I : C \to C$ is a functor and $i_0, i_1 : \text{Id}_C \to I$ and $p : I \to \text{Id}_C$ are natural transformations such that, for all $C$-objects $X$, the compositions $p \circ i_k : X \to IX \to X, k \in \{0, 1\}$ are the identity. We call the $i_k$ the inclusions of the cylinder and $p$ the projection of the cylinder. When the $i_k$ and $p$ are understood, we will simply refer to the cylinder $(I, i_0, i_1, p)$ by the functor $I$.

A cylinder on a category $C$ gives rise to a notion of homotopy in the following way.

**Definition 3.7.** Let $f, g : X \to Y$ be morphisms in a category $C$ with a cylinder $(I, i_0, i_1, p)$. We say that a morphism $H : IX \to Y$ is a homotopy from $f$ to $g$ iff $H \circ i_0(X) = f$ and $H \circ i_1(X) = g$.

As we will see, it may occur that homotopies do not “glue” properly, that is, that homotopy as defined above is not necessarily a transitive relation. We will require our convenient categories to satisfy one additional requirement, which, as we will see in Proposition 3.8 below, gives a sufficient condition for the homotopy relation in Definition 3.7 to be transitive. This will be used in the construction of the $I$-category structure on $\text{Lim}$ and $\text{PsTop}$ in Section 4.2.

**Definition.** Let $(I, i_0, i_1, p)$ be a cylinder on a category $C$. Denote by $IA \cup_A IA$ the pushout

$$
\begin{array}{ccc}
A & \xrightarrow{i_0} & IA \\
\downarrow{i_1} & & \downarrow{q_1} \\
IA & \xrightarrow{q_0} & IA \cup_A IA
\end{array}
$$

We say that a cylinder $(I, i_0, i_1, p)$ has the homotopy gluing property iff, for any object $A \in C$, there exists an isomorphism $u : IA \cup_A IA \cong IA$ such that $u \circ q_k \circ i_k = i_k$ for $k = 0, 1$, i.e. such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i_k} & IA \\
\downarrow{i_k} & & \downarrow{q_k} \\
IA & \xleftarrow{u} & IA \cup_A IA
\end{array}
$$

commutes for $k = 0, 1$.

Note that the $i_k$, $k = 0, 1$, in the second diagram above are no longer the maps which are identified in the pushout, but the other ‘ends’ of the cylinder.

We now establish the transitivity of the homotopy relation for a cylinder which satisfies the homotopy gluing property.

**Proposition 3.8.** Let $f, g, h : X \to Y$ be morphisms in a category $C$ with cylinder $(I, i_0, i_1, p)$. Suppose that $F : IX \to Y$ is a homotopy from $f$ to $g$ and that $G : IX \to Y$ is a homotopy from $g$ to $h$. If $(I, i_0, i_1, p)$ has the homotopy gluing property, then the map $F \ast G := \alpha \circ u^{-1} : IX \to Y$ in
the commutative diagram below is a homotopy from $f$ to $h$.

\[
\begin{array}{ccc}
X & \xrightarrow{i_0} & IX \\
\downarrow{i_1} & & \downarrow{q_1} \\
IX & \xrightarrow{q_0} & IX \cup X
\end{array}
\]

(3.1)

\[
F \star G \circ i_0(X) = F \star G \circ u \circ q_0 \circ i_0(X) = \alpha \circ q_0 \circ i_0 = F \circ i_0 = f,
\]

\[
F \star G \circ i_1(X) = F \star G \circ u \circ q_1 \circ i_1(X) = \alpha \circ q_1 \circ i_1 = G \circ i_1 = h.
\]

Therefore, $F \star G$ is a homotopy from $f$ to $h$, as desired. \hfill \Box

3.2.1. Limit Spaces. The standard cylinder $IX := X \times [0, 1]$, If := $(f(x), t)$, where $(X, \Lambda)$ is a limit or pseudotopological space and $([0, 1], \tau)$ is the topological interval with its topological convergence structure $\tau$, satisfies the homotopy gluing property in both limit spaces and pseudotopological spaces. Note that $IX \cup_X IX \cong X \times ([0, 1] \cup_{t=0} [0, 1])$ since both limit spaces and pseudotopological spaces are Cartesian closed categories, so to check that $IX \cup_X IX \cong IX$, for any limit or pseudotopological space $X$, it suffices to show that $[0, 1] \cup_{t=0} [0, 1] \cong [0, 1]$ as limit spaces or pseudotopological spaces. Consider the isomorphism $\phi : [0, 1] \to [0, 1] \cup_{t=0} [0, 1]$ given by

\[
\phi(x, t) \mapsto \begin{cases} 
q_0(2t), & t \in [0, \frac{1}{2}) \\
q_1(2t-1), & t \in [\frac{1}{2}, 1]
\end{cases}
\]

where the maps $q_k : [0, 1] \to [0, 1] \cup_{t=0} [0, 1]$ are the maps from the pushout square of $[0, 1] \cup_{t=0} [0, 1]$. To check that $\phi$ is continuous, it suffices to check that $\phi$ is continuous at the point $\frac{1}{2} \in [0, 1]$. Suppose that a filter $\lambda \to \frac{1}{2}$ in the topological structure $([0, 1], \tau)$, and note that $\phi(\lambda)$ is generated by the sets of the form $(t_0, 1] \cup_{t=0} [0, t_1) \subset [0, 1] \cup_{t=0} [0, 1]$. Sets of the form $(t_0, 1]$ generate the neighborhood
filter $\mathcal{N}_1$ of the point 1 in $[0, 1]$, and, similarly, sets of the form $[0, t_1)$ generate the neighborhood filter $\mathcal{N}_0$ of 0 in $[0, 1]$. The filter $\phi(\lambda)$ therefore contains the filter $q_0(\mathcal{N}_1) \cap q_1(\mathcal{N}_1)$, which converges in quotient convergence structure of $[0, 1] \cup_{1 \sim 0} [0, 1]$. It follows that $\phi(\lambda) \to \frac{1}{2}$, and therefore $\phi$ is continuous.

Note that $\phi^{-1}$ is continuous since it is the dashed arrow in the pushout diagram

$$
\begin{array}{ccc}
I & \xrightarrow{i_0} & I \\
\downarrow{i_1} & & \downarrow{q_1} \\
I & \xrightarrow{q_0} & I \cup_{1 \sim 0} I
\end{array}
$$

where $\alpha_0, \alpha_1 : I \to I$ are given by $\alpha_k(t) = \frac{t+k}{2}$ and $I = [0, 1]$.

Since $\phi, \phi^{-1}$ are both bijective, we have that $IX \cong IX \cup_X IX$ in limit spaces, as desired. The proof for pseudotopological spaces is analogous.

3.2.2. A Non-Example: General Convergence Spaces. In general convergence spaces, it is no longer true that if the filters $\lambda_0 \to x$ and $\lambda_1 \to x$, then $\lambda_0 \cap \lambda_1 \to x$, and therefore the spaces $[0, 1]$ and $[0, 1] \cup_{1 \sim 0} [0, 1]$ are not isomorphic. See [24], Remark A.2.3 for a more complete discussion of this point in the context of Kent convergence spaces. General convergence spaces (and Kent convergence spaces) therefore do not satisfy the homotopy gluing property.

4. Convenient Cofibration Categories and Homotopy Groups

The general strategy for studying homotopy theory in a category is, first, to specify a class of morphisms, often called weak equivalences, at which one would like to localize the category, and, second, to endow the category with enough structure in order to develop tools to probe the resulting homotopy category. The structures which we will study in this article are model categories [25, 21, 20] and the cofibration category theory developed in [5, 6]. While not as modern as more recent approaches to homotopy theory through infinity categories [27, 15, 22], they nonetheless remain somewhat more accessible, and, once one has a model category, an infinity category structure may also be constructed if desired.

In this section, we describe the axioms for cofibration categories and I-categories from [5], and we briefly indicate how these structures may be used to construct homotopy groups. More details on these constructions and the consequences of this theory may be found in [5, 8]. After this brief introduction, we show that convenient categories admit an I-category structure if they admit a cylinder $(I, i_0, i_1, p)$ which satisfies the homotopy gluing property, and we deduce from this the existence of a cofibration category structure on $\text{Lim}$ and $\text{PsTop}$. The resulting homotopy groups from the cofibration category structure will then be used to construct a model category structure on $\text{PsTop}$ in Section 5.

4.1. Cofibration Categories and Homotopy. We begin with the definition of of a cofibration category.

**Definition 4.1** (Cofibration Category). A cofibration category is a tuple $(\mathcal{C}, \text{cof}, \text{we})$, where $\mathcal{C}$ is a category, and cof and we, the cofibrations and weak equivalences, respectively, are special classes of morphisms of $\mathcal{C}$ which satisfy the following axioms
(1) (Composition axiom) The isomorphisms in $\mathcal{C}$ are weak equivalences and also cofibrations. For two maps

\[ A \xrightarrow{f} B \xrightarrow{g} C, \]

if any two of $f, g$ and $g \circ f$ are weak equivalences, then so is the third. The composition of cofibrations is a cofibration.

(2) (Pushout axiom) For a cofibration $i : B \hookrightarrow A$ and a map $f : B \to Y$, there exists a pushout in $\mathcal{C}$

\[
\begin{array}{ccc}
B & \xrightarrow{f} & Y \\
\downarrow{i} & & \downarrow{i} \\
A & \xrightarrow{f} & A \cup_f Y
\end{array}
\]

and $\tilde{i}$ is a cofibration. Moreover,

(a) if $f$ is a weak equivalence, then so is $\tilde{f}$,

(b) if $i$ is a weak equivalence, then so is $\tilde{i}$.

(3) (Factorization axiom) For a map $f : B \to Y$ in $\mathcal{C}$ there exists a commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i} & Z \\
\downarrow{g} & & \downarrow{\sim} \\
A & \xrightarrow{\sim} & A
\end{array}
\]

where $i$ is a cofibration and $g$ is a weak equivalence.

(4) (Axiom on fibrant models) For each object $X$ in $\mathcal{C}$ there is a trivial cofibration $X \hookrightarrow RX$ (i.e. a cofibration which is also a weak equivalence) where $RX$ is fibrant in $\mathcal{C}$, i.e. each trivial cofibration $i : RX \hookrightarrow Q$ admits a retraction $r : Q \to RX, ri = 1_{RX}$.

We now summarize the construction of homotopy groups in a general cofibration category. We will discuss the specific cases of $\text{Lim}$ and $\text{PsTop}$ in Section 4.3.1. The following discussion is summarized from [5], II, Sections 1-6. We assume throughout that $\mathcal{C}$ is a cofibration category with an initial object $\ast$ such that $\text{Hom}(A, \ast)$ is non-empty for any $\mathcal{C}$-object $A$. Note that the latter condition is automatically satisfied if $\ast$ is also a terminal object.

**Definition 4.2.** Relative Cylinders. Let $i : B \hookrightarrow A$ be a cofibration. Then, by the Pushout Axiom (Definition 4.1 Item 2) there exists the pushout

\[
\begin{array}{ccc}
B & \xrightarrow{i} & A \\
\downarrow{1_A} & & \downarrow{1_A} \\
A & \xrightarrow{\phi} & A \cup_i BA
\end{array}
\]

We call $\phi = (1_A, 1_A)$ the folding map. By the Factorization Axiom (Definition 4.1 Item 3), there is a factorization of $\phi$

\[
A \cup_B A \xrightarrow{i} Z \xrightarrow{p} A
\]

where $i$ is a cofibration and $p$ is a weak equivalence. We call the triple $(Z, i, p)$ a relative cylinder on $B \hookrightarrow A$. We often write $I_B A$ or $Z$ for the relative cylinder $(Z, i, p)$.
Note the difference between relative cylinders and the cylinders of Definition 3.6.

**Definition 4.3.** Homotopy. Let $X$ be a fibrant object in a cofibration category $C$, and let $Y \subset X$ be a cofibration. We say that two maps $f, g : A \to X$ are *homotopy relative* $B$, denoted $f \simeq g \text{ rel } B$, iff there is a commutative diagram

$$
\begin{array}{ccc}
A \cup_B A & \xrightarrow{i} & Z \\
(f, g) \downarrow & & \downarrow H \\
X & \xrightarrow{H} & X
\end{array}
$$

where $Z$ is a relative cylinder on $B \to A$. We call $H : Z \to X$ a homotopy rel $B$ from $f$ to $g$.

**Definition 4.4.** Let $Y \subset X$ be a cofibration in a cofibration category $C$. Let $u : Y \to U$ be a map and let $\text{Hom}(X, U)^u$ denote the set of all extensions of $u$, i.e. the maps $f : X \to U$ in $C$ such that $f|_Y = u$.

**Proposition 4.5** ([5], Proposition II.2.2). Suppose that $U$ is a fibrant object in a cofibration category $C$, and let $Y \subset X$ be a cofibration. Then all of the cylinders on $Y \subset X$ define the same homotopy relation relative $Y$, denoted $\simeq \text{ rel } Y$, on the set $\text{Hom}(X, U)^u$, and, furthermore, the homotopy relation relative $Y$ is an equivalence relation.

**Definition 4.6.** Torus. We define the *torus* $\Sigma Y X$ for $Y \subset X$ by the pushout diagram

$$
\begin{array}{ccc}
X \cup_Y X & \xrightarrow{(1,1)} & X \\
\downarrow & & \downarrow 1 \\
I_Y X & \xrightarrow{r} & \Sigma Y X \\
\downarrow p & & \downarrow r \\
X & &
\end{array}
$$

where $p$ is the map in the definition of the cylinder $(I_Y X, i, p)$ and the existence of $r$ is guaranteed by the universal property of the pushout.

**Definition 4.7.** Based object. A *based object* in a cofibration category $C$ is a pair $(X, o_X)$, where $X$ is a cofibrant object (i.e. $\ast \to X$ is a cofibration) and where $o := o_X : X \to \ast$ is a map from $X$ to the initial object. We call $o = o_X$ the *trivial map* on $X$. We say that a map $f : (X, o_X) \to (Y, o_Y)$ between based objects is *based* iff $o_Y \circ f = o_X$. (Note that in general there may be many maps $X \to \ast$ to the initial object.)

**Definition 4.8.** Suspension of a based object. For a based object $A$ in a cofibration category $C$ the suspension $\Sigma A$ is the based object which is defined by the pushout diagram

\[
\begin{array}{ccc}
A \vee A & \xrightarrow{(1,1)} & A \\
\downarrow & \xrightarrow{\text{push}} & \downarrow \text{push} \\
I_* A & \xrightarrow{\sigma} & \Sigma_* A \\
\downarrow & \xrightarrow{\text{push}} & \downarrow \text{push} \\
A & \xrightarrow{o_A} & \ast
\end{array}
\]
where $\Sigma A$ is the torus on $* \hookrightarrow A$. Note that the existence of the map $o_{*A}: \Sigma A \to *$ exists by the universal property of the pushout. Also note that, a priori, $\Sigma A$ depends on the choice of trivial map $o_A: A \to *$. However, in the cases of interest in this article, $*$ will be a terminal object as well as an initial object, so $\Sigma A$ will be canonical.

**Definition 4.9.** For a based object $A$ in a cofibration category $C$, we define $\pi^A_n(U) := [\Sigma^n A, U]$. These are groups for $n \geq 1$ by [5], II.5.13 (As noted in the discussion in [5], II.6.9-11.)

**Example 4.10.** Homotopy groups of topological spaces. Let $C = \text{Top}^*$, the category of pointed topological spaces (with initial object $*$), and let $A = S^0$, the 0-dimensional sphere. Then $\Sigma^n S^0$ is the $n$-th based suspension of $S^0$, which is homeomorphic to $S^n$, and $\pi_n(X) = \pi_n^S(X) = [\Sigma^n S^0, X]$, which are the usual homotopy groups in $\text{Top}^*$.

### 4.2. $I$-categories

We will not work directly with the axioms of a cofibration category, but instead with an auxiliary structure called an $I$-category, which is an axiomatization of the case when the category in question has a canonical cylinder, as is the case for $\text{Lim}$ and $\text{PsTop}$, as well as for convenient categories in general. An $I$-category is defined as follows. Note that we have slightly reorganized the axioms in [5], permuting the last two axioms and incorporating much of the cylinder axiom in the definition of the cylinder $(I, i_0, i_1, p)$.

**Definition 4.11.** An $I$-category is a tuple $(C, \text{cof}, (I, i_0, i_1, p), \emptyset)$ where $C$ is a category, $\text{cof}$ is a collection of $C$-morphisms called cofibrations, $(I, i_0, i_1, p)$ is a cylinder (Definition 3.6) on $C$, and $\emptyset$ is the initial object in $C$. The structure must satisfy the following axioms:

1. (Cylinder axiom) $I\emptyset = \emptyset$.
2. (Pushout axiom) For a cofibration $i: B \hookrightarrow A$ and a morphism $f: B \to X$ there exists the pushout
   \[
   \begin{array}{ccc}
   B & \xrightarrow{f} & X \\
   i \downarrow & & \downarrow i \\
   A & \longrightarrow & A \cup_B X
   \end{array}
   \]
   and $\bar{i}$ is a cofibration. Moreover, the functor $I$ carries pushouts to pushouts, i.e. $I(A \cup_B X) = IA \cup_I IB IX$.
3. (Cofibration axiom) The class $\text{cof}$ of cofibrations satisfy the following:
   a. Each isomorphism is a cofibration
   b. For every $C$-object $X$, the morphism $\emptyset \to X$ is a cofibration
   c. The composition of cofibrations is a cofibrations
   d. A cofibration $i: B \hookrightarrow A$ satisfies the following homotopy extension property. For each commutative solid arrow diagram
   \[
   \begin{array}{ccc}
   & A \\
   & \searrow^{i} \quad \swarrow_{i_k} \\
   B & \xleftarrow{i_k} & IA \\
   \downarrow^{i_k} & & \downarrow G \\
   IB & \longrightarrow & Y
   \end{array}
   \]
   in $C$, there exists an $H: IA \to Y$ which makes the entire diagram commute.
4. (Interchange axiom) For all $C$-objects $X$, there exists a morphism $T: IIX \to IIX$ with $Ti_k = Ii_k$ and $TI(i_k) = i_k$ for $k = 0, 1$. $T$ is called the interchange map.
(5) (Relative Cylinder axiom) For a cofibration \(i : B \xrightarrow{\imath} A\), the map \(j\) defined by the pushout diagram is a cofibration.

We will often abbreviate \((C, (I, i_0, i_1, p), \text{cof}, \emptyset)\) by \((C, I, \text{cof}, \emptyset)\) when the remainder of the cylinder is clear from context.

The main interest of an \(I\)-category structure is that, when a category \(C\) does have a canonical cylinder functor, it is often easier to verify the \(I\)-category axioms than it is to directly verify the cofibration category axioms. However, the following theorem from [6] shows that an \(I\)-category structure implies the existence of a cofibration category. We begin by defining homotopy equivalence in an \(I\)-category.

**Definition 4.12.** (Homotopy and homotopy equivalence in an \(I\)-category) Let \((C, I, \text{cof}, \emptyset)\) be an \(I\)-category. We say that two maps \(f, g : X \rightarrow Y\) are homotopic, written \(f \simeq g\), if there exists a morphism \(H : IX \rightarrow Y\) such that \(Hi_0 = f\) and \(Hi_1 = g\). We call \(H\) the homotopy from \(f\) to \(g\). We say that a map \(f : X \rightarrow Y\) is a homotopy equivalence if there is a map \(g : Y \rightarrow X\) such that \(fg \simeq 1_Y\) and \(gf \simeq 1_X\).

**Theorem 4.13** ([6], Theorem 3.3). Let \((C, \text{cof}, I, \emptyset)\) be an \(I\)-category. Then \(C\) is a cofibration category with the following structure. Cofibrations are those of the \(I\)-category structure, and weak equivalences are the homotopy equivalences from Definition 4.12.

Note that in this structure, all objects are fibrant and cofibrant in \(C\).

4.3. An \(I\)-category structure on convenient categories with the homotopy gluing property. We now show that any convenient category \(C\) admits a cylinder \((I, i_0, i_1, p)\), and then that if the cylinder satisfies the homotopy gluing property, then \(C\) admits a cofibration category structure.

**Proposition 4.14.** Let \(C\) be a category which contains the topological interval \([0, 1]\) and the one point set \(\{\ast\}\) as objects, such that the set maps \(\ast \rightarrow 0 \in [0, 1]\) and \(\ast \rightarrow 1 \in [0, 1]\) are elements of \(\text{Hom}_C(\{\ast\}, [0, 1])\). Then there exists a cylinder \((I, i_0, i_1, p)\) on \(C\).

**Proof.** We define the functor \(I - : C \rightarrow C\) to be the product functor \(- \times I\), and for each \(C\)-object \(X\), we define the \(i_{k,X}\) to be the unique dashed arrow in the diagram

\[
\begin{align*}
\text{Diagram (4.1)}
\end{align*}
\]

The unicity of the dashed arrow in the diagram

\[
\begin{align*}
\text{Diagram (4.1)}
\end{align*}
\]
implies the commutativity of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{i_{k,X}} & & \downarrow^{i_{k,Y}} \\
IX & \xrightarrow{I_f} & IY
\end{array}
\]

and therefore \(i_0, i_1 : \text{Id}_C \to I\) are natural transformations. Similarly, for each \(C\)-object \(X\) we define \(p_X : IX \to X\) to be the map \(IX \to X\) from the pullback diagram \((4.1)\) of \(IX\). We see from this diagram that the composition

\[
p \circ i_k(X) : X \to IX \to X
\]

is the identity of \(X\) by construction, for all \(X \in C\) and \(k = 0, 1\). Therefore, \((I, i_0, i_1, p)\) is a cylinder in \(C\). □

The following corollary is immediate.

**Corollary 4.15.** A convenient category \(C\) has a cylinder \((I, i_0, i_1, p)\) defined by

\[
\begin{align*}
IX & := X \times [0, 1] \\
I F(x,t) & := f \times 1_{[0,1]} \\
i_0 X & := 1_X \times \{0\} \\
i_1 X & := 1_X \times \{1\} \\
p(IX) & := p_X(IX)
\end{align*}
\]

where \(p_X : X \times [0, 1] \to X\) is the projection onto \(X\) in the universal property of the product.

We now define the cofibrations in a convenient category.

**Definition 4.16.** Let \(C\) be an convenient category. Then we define \(\text{cof}\) to be the collection of \(C\)-morphisms which satisfy the homotopy extension property in the cofibration axiom, i.e. Item 33d of Definition 4.11.

We now check the axioms of an \(I\)-category for a convenient category \(C\) whose cylinder \((I, i_0, i_1, p)\) satisfies the homotopy gluing property, with the cofibrations \(\text{cof}\) defined as in Definition 4.16 above. We take the initial object \(\emptyset\) to be the empty set.

**Proposition 4.17.** Let \(C\) be a convenient category, the cylinder \((I, i_0, i_1, p)\) be defined as in Proposition 4.15, and let \(\text{cof}\) be the class of cofibrations defined in Definition 4.16. Then \((C, I, \text{cof}, \emptyset)\) satisfies the Pushout Axiom.

**Proof.** Since \(C\) is Cartesian closed by hypothesis, pushouts exist and \(I\) preserves pushouts. It remains to verify that \(\tilde{i}\) is a cofibration. That is, given the commutative diagram of solid arrows

we wish to show that there is a morphism \(H : I(A \cup_B Y) \to Z\) which makes the entire diagram commute.
First, we note that $I(A \cup_B Y) = IA \cup_I BY$. Second, $A \cup_B Y$ is a pushout, and therefore for any commutative diagram of solid arrows

$$
\begin{array}{c}
A \\
\downarrow \beta_B \\
Y
\end{array} \quad \begin{array}{c}
A \cup_B Y \\
\downarrow \beta_Y \\
Z
\end{array} \quad \begin{array}{c}
A \\
\downarrow i \\
B \\
\downarrow \beta_B \\
IB
\end{array}
$$

Since $IB \cup_I BY$ is also a pushout, we also have

$$
\begin{array}{c}
IA \\
\downarrow I_i \\
IB \\
\downarrow I_{\beta_Y} \\
IY
\end{array} \quad \begin{array}{c}
IA \cup_I BY \\
\downarrow I_i \\
IB
\end{array} \quad \begin{array}{c}
IA \\
\downarrow H \\
IB
\end{array}
$$

Since $i : B \to A$ is a cofibration, we know that, for any $C$-object $Z$

$$
\begin{array}{c}
A \\
\downarrow i \\
B \\
\downarrow i_{k,B} \\
IB
\end{array} \quad \begin{array}{c}
IA \\
\downarrow H \\
IB \quad I_{k,A} \quad I_{\beta_Y} \quad I_{\beta_Y}
\end{array}
$$

Putting these all together, we have the diagram

$$
(4.2)
$$

where the solid-arrow diagram commutes. Finally, since the $i_k$ are natural transformations, we have

$$
\begin{array}{c}
A \\
\downarrow q_A \\
IA
\end{array} \quad \begin{array}{c}
A \cup_B Y \\
\downarrow \beta_B \\
Y
\end{array} \quad \begin{array}{c}
IA \cup_{IA} BY \\
\downarrow i_k, A \cup_B Y \\
I(A \cup_B Y)
\end{array} \quad \begin{array}{c}
\quad \Downarrow f \\
\quad \Downarrow G
\end{array}
$$

$$
\begin{array}{c}
IA \\
\downarrow q_A \\
IA
\end{array} \quad \begin{array}{c}
IA \cup_{IA} BY \\
\downarrow i_k, A \cup_B Y \\
I(A \cup_B)
\end{array}
$$
From these last two diagrams, we conclude that
\[ f \circ q_A = H \circ i_{k,A} \]
\[ = F \circ Iq_A \circ i_{k,A} \]
\[ = F \circ i_{k,A \cup B} \circ q_A. \]

We also have
\[ f \circ \bar{i} = G \circ i_{k,Y} \]
\[ = F \circ I\bar{i} \circ i_{k,Y} \]
\[ = F \circ i_{k,A \cup B} \circ \bar{i}. \]

Since both \( f \) and \( F \circ i_{k,A \cup B} \) satisfy the pushout property of Diagram 4.2, we conclude that \( f = F \circ i_{k,A \cup B} \) by the unicity of \( f \). It now follows that \( F : I(A \cup B) \to Z \) makes the complete Diagram 4.2 commute, and since \( Z \) was an arbitrary \( C \)-object, we conclude that \( \bar{i} \) is a cofibration. □

**Proposition 4.18.** Let \( C \) be a convenient category, the cylinder \((I, i_0, i_1, p)\) be defined as in Proposition 4.15, and let \( \text{cof} \) be the class of cofibrations defined in Definition 4.16. Then \((C, I, \text{cof}, \varnothing)\) satisfies the Cofibration Axiom.

**Proof.** We prove Items 3a-3d in the cofibration axiom one by one.

a) Each isomorphism is a cofibration: Let \( i : B \to A \) be an isomorphism. Then \( Ii : IB \to IA \) is an isomorphism by construction, and for any diagram of solid arrows

we define \( H : IA \to Z \) by \( H := G \circ (Ii)^{-1} \), and \( H \) makes the full diagram commute. Since \( Z \) is arbitrary, \( i \) is a cofibration.

b) For every \( C \)-object \( X \), the morphism \( \varnothing \to X \) is a cofibration: Since \( I\varnothing = \varnothing \) by hypothesis, for each diagram of solid arrows

the map \( If : IX \to Z \) makes the full diagram commute.
c) The composition of cofibrations is a cofibration: Let \( i : B \hookrightarrow A \) and \( j : C \hookrightarrow B \) be cofibrations, and consider the map \( i \circ j \). We wish to show that, for any diagram of solid arrows

\[
\begin{array}{ccc}
A & \xrightarrow{f} & IA \\
\downarrow{i} & & \downarrow{H} \\
C & \xrightarrow{ik} & IB \\
\downarrow{i_k} & & \downarrow{G} \\
IC & & Z
\end{array}
\]

there exists a dashed arrow \( H : IA \to Z \) making the full diagram commute. We first consider the following diagram

\[
\begin{array}{ccc}
B & \xrightarrow{f \circ i} & Z \\
\downarrow{j} & & \downarrow{G'} \\
C & \xrightarrow{ik} & IB \\
\downarrow{i_k} & & \downarrow{G} \\
IC & & Z
\end{array}
\]

Since \( j : C \hookrightarrow B \) is a cofibration, there exists \( H' : IB \to Z \) making the above diagram commute. However, since \( i : B \hookrightarrow A \) is a cofibration, there exists \( H : IA \to Z \) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & IA \\
\downarrow{i} & & \downarrow{H} \\
B & \xrightarrow{ik} & IB \\
\downarrow{i_k} & & \downarrow{G'} \\
IC & & Z
\end{array}
\]

commute. Putting these together, we have

\[
\begin{array}{ccc}
A & \xrightarrow{f} & IA \\
\downarrow{i} & & \downarrow{H} \\
B & \xrightarrow{ik} & IB \\
\downarrow{i_k} & & \downarrow{G'} \\
C & \xrightarrow{i_k} & IB \\
\downarrow{i} & & \downarrow{G} \\
IC & & Z
\end{array}
\]

as desired.

d) A cofibration \( i : B \hookrightarrow A \) in \((C, I, \text{cof}, \emptyset)\) satisfies the homotopy extension property by definition.

\[\square\]

For the proof of the Relative Cylinder Axiom, we will require the following a preparatory lemma.
Lemma 4.19. A map \( f : B \to A \) is a cofibration as defined in Definition 4.16 iff there exists a retract \( r_k : IA \to IB \cup i_k A \) of the map \( j_k : IB \cup i_k A \to IA \) in the pushout diagram

\[
\begin{array}{c}
A \\
i_0 & \downarrow \searrow j_0 \\
\downarrow i & \downarrow \nearrow i_k \\
B & \downarrow i_0 \\
\downarrow i & IB & \downarrow i_k \\
\downarrow i_0 & \downarrow i_k \\
IA & \downarrow j_k & \downarrow IA \\
\end{array}
\]

Proof. First, suppose that \( i : B \to A \) is a cofibration. Then for any \( Y \) and maps \( f : X \to Y \) and \( G : IA \to Y \), there exists a map \( H : IX \to Y \) that makes the diagram

\[
\begin{array}{c}
A \\
i_0 & \downarrow \searrow f \\
\downarrow i & \downarrow \nearrow i_k \\
B & \downarrow i_0 \\
\downarrow i & IB & \downarrow i_k \\
\downarrow i_0 & \downarrow i_k \\
IA & \downarrow H & \downarrow Y \\
\end{array}
\]

commute. Let \( Y := A \cup i_k IB \), and let \( f : A \to A \cup i_k IB \) and \( G : IB \to A \cup i_k IB \) be the standard inclusions for the pushout. Then the map \( r = H : IA \to A \cup i_k IB \) is the desired retract, by the commutativity of the diagram.

Now suppose such a retract \( r : IA \to A \cup i_k IB \) of \( j_k \) exists. Then, for any \( C \)-object \( Y \) and maps \( f : A \to Y \) and \( G : IB \to Y \) such that the solid arrow diagram

\[
\begin{array}{c}
A \\
i_0 & \downarrow \searrow r \\
\downarrow i & \downarrow \nearrow h \downarrow \searrow \downarrow \uparrow i_k \\
\downarrow i_0 & IB & \downarrow i_k \\
\downarrow i_0 & IB & \downarrow i_k \\
IA & \downarrow h & \downarrow Y \\
\end{array}
\]

exists, then there exists a map \( h : A \cup i_k IB \to Y \) making the entire diagram commute, Defining \( H := h \circ r \), we see that \( i \) is a cofibration, as desired. \( \square \)

Proposition 4.20. Let \( C \) be a convenient category, the cylinder \((I, i_0, i_1, p)\) be defined as in Proposition 4.15, and let \( \text{cof} \) be the class of cofibrations defined in Definition 4.16. Suppose, in addition, that the cylinder \((I, i_0, i_1, p)\) satisfies the homotopy gluing property in \( C \). Then \((C, I, \text{cof}, \emptyset)\) satisfies the Relative Cylinder Axiom.
Proof. We follow the proof in \[6\] for topological spaces. For the map \( j : A \cup IB \cup A \to IA \) to be a cofibration, for any \( C \)-object \( X \) and any diagram of solid arrows

\[
\begin{array}{ccc}
A \cup IB \cup A & \xrightarrow{j} & IA \\
\downarrow & & \downarrow f \\
I(A \cup IB \cup A) & \xrightarrow{i_k} & IIA \\
\end{array}
\]

there must exist a dashed arrow \( H : IIA \to X \) making the full diagram commute. Let \( \gamma : I \times I \to \mathbb{R} \) be the map \( \gamma(x, t) := \max(2||x||, 2 - t) \), and define the homeomorphism \( \alpha : I \times I \to I \times I \) by \( \alpha(x, t) := (\gamma(x, y)^{-1}(1 + t)x, 2 - \gamma(x, t)) \). Consider the diagram

\[
\begin{array}{ccc}
(IA) \cup_{i_0} I(A \cup i_0B \cup i_1 B \ A) & \xrightarrow{j_0} & IIA \\
\downarrow & & \downarrow {\alpha \times 1_A} \\
I(A \cup IB) & \xrightarrow{T_{I_k}} & IIA \\
\end{array}
\]

where \( \beta \) is the restriction of \( \alpha \times 1_A \) which makes the diagram commute, \( \iota : A \cup IB \to IA \) is the inclusion, and \( T : IIA \to IIA \) is the map from the Interchange Axiom. Since \( C \) satisfies the Homotopy Gluing Property, \( \beta \) is a homeomorphism onto \( I(A \cup IB) \), and therefore the inverse map \( \beta^{-1} \) exists. (Note that without the homotopy gluing property, the inverse \( \beta^{-1} \) may not be continuous.) Defining \( \bar{r} := \beta^{-1}(Ir)(\alpha \times 1_A) \) where \( r : IA \to A \cup IB \) is the retract of \( \iota \), which exists because \( i : B \to A \) is a cofibration. We see that \( \bar{r} \) is the desired retract of \( j_0 \), and therefore \( j \) is a cofibration, as desired. \( \square \)

Proposition 4.21. Let \( C \) be a convenient category, the cylinder \( (I, i_0, i_1, p) \) be defined as in Proposition \[4,13\] and let \( \text{cof} \) be the class of cofibrations defined in Definition \[4,15\]. Then \( (C, I, \text{cof}, \varnothing) \) satisfies the Interchange Axiom.

Proof. Let \( I_0 \) and \( I_1 \) be distinct copies of \( I \). Then the limits \( I_0 \times I_1 \) and \( I_1 \times I_0 \) exist and there exists a \( C \)-morphism \( T_I : I_0 \times I_1 \to I_0 \times I_1 \) from the properties of the limits \( I_0 \times I_1 \) and \( I_1 \times I_0 \). We let \( T : IIA \to IIA \) be the morphism \( T := T_I \times \text{Id}_A \). Then

\[
T(i_k(IA)) = T(\{k\} \times I \times A) = T_I \times \text{Id}_A(\{k\} \times I \times A) = (I \times \{k\} \times A) = I(i_kA)
\]

for \( k \in \{0, 1\} \). We also have

\[
T(I(i_k(A))) = T(I \times \{k\} \times A) = T_I \times \text{Id}_A(I \times \{k\} \times A) = \{k\} \times I \times A = i_k(IA)
\]

as desired. \( \square \)

Putting these together, we have shown

Theorem 4.22. An admissible category \( (C, I, \text{cof}, \varnothing) \) is an \( I \)-category, with \( I, \text{cof}, \) and \( \varnothing \) defined as above.

The following corollaries are immediate.

Corollary 4.23. \( (\text{PsTop}, I, \text{cof}, \varnothing) \) is an \( I \)-category.

Corollary 4.24. \( (\text{Lim}, I, \text{cof}, \varnothing) \) is an \( I \)-category.
Combining Corollaries 4.23 and 4.24 with Theorem 4.13 we have

**Corollary 4.25.** (\textbf{PsTop}, cof, we) and (\textbf{Lim}, cof, we) are cofibration categories, where cof is defined in Definition 4.16 and the weak equivalences are as in Theorem 4.13.

4.3.1. On the homotopy groups and suspensions of spheres in \textbf{PsTop} and \textbf{Lim}. We now show that a topological $n$-sphere can be obtained by the pseudotopological based suspension of a topological $(n-1)$-sphere. It follows that the homotopy groups constructed from the cofibration category structure on pseudotopological spaces are, in fact, the homotopy classes of maps from the $n$-spheres, just as in the topological case. We begin with the following lemma about ultrafilters.

**Lemma 4.26.** Let $p : X \to Y$ be a surjective map of sets. Then $\mathcal{X}$ is an ultrafilter in $Y$ iff there exists an ultrafilter $\lambda$ in $X$ such that $p(\lambda) = \mathcal{X}$.

**Proof.** Suppose that there exists an ultrafilter $\lambda$ in $X$ with $p(\lambda) = \mathcal{X}$. Let $V \subseteq Y$. If $p^{-1}(V) \in \lambda$, then $V \in p(\lambda) = \mathcal{X}$. Conversely, if $X - p^{-1}(V) \in \lambda$, then $p(X - p^{-1}(V)) = Y - V \in \mathcal{X}$, so $\mathcal{X}$ is an ultrafilter.

Now suppose that $\mathcal{X}$ is an ultrafilter, and consider

$$p^{-1}(\mathcal{X}) := \{p^{-1}(V) \mid V \in \lambda\}$$

Since $\mathcal{X}$ has the finite intersection property, so does $p^{-1}(\mathcal{X})$, and therefore there is an ultrafilter $\lambda$ on $X$ which contains $p^{-1}(\mathcal{X})$. We show that $p(\lambda) = \mathcal{X}$. First, since $p^{-1}(\mathcal{X}) \subseteq \lambda$, it follows that $\mathcal{X} \subseteq p(\lambda)$. Now suppose that $U \subseteq \lambda$. If $p(U) \notin \mathcal{X}$, then $Y - p(U) \in \mathcal{X}$, and therefore $p^{-1}(Y - p(U)) \subseteq p^{-1}(\mathcal{X}) \subseteq \lambda$. However, $p^{-1}(Y - p(U)) \cap p^{-1}(p(U)) = \emptyset$, which is a contradiction. Therefore $p(U) \subseteq \lambda$, as desired.

\hfill\Box

**Theorem 4.27.** Let $(S^n, \tau)$ be a topological $n$-sphere with base point $\ast \in X$ and pseudotopology $\tau$ (induced by the topological structure). Let $\Sigma S^n := S^n \times [0, 1]/(x, k) \sim (x', k) \sim (\ast, t), k \in \{0, 1\}, t \in [0, 1]$, denote the pseudotopological based suspension of $S^n$. Then, for every $n \geq 1$, $\Sigma S^{n-1} \cong \tau(\Sigma S^{n-1}) \cong (S^n, \tau)$, where $\tau(\Sigma S^{n-1})$ is the topological modification of $\Sigma S^{n-1}$.

**Proof.** We consider the quotient map $p : S^{n-1} \times [0, 1] \to \Sigma S^{n-1}$, where $S^{n-1}$ and $[0, 1]$ are endowed with the usual topological pseudotopologies, and the quotient is taken in \textbf{PsTop}. Note, however, that $p$ is a topological homeomorphism on $S^{n-1} \times [0, 1] - Q := \{(x, k) \mid k \in \{0, 1\}\} \cup \{(\ast, t) \mid t \in [0, 1]\}$, and that this induces a pseudotopological homeomorphism on the region as well, by the definition of the quotient structure on $\Sigma S^{n-1}$.

Now consider $[\ast, 0] \in \Sigma S^{n-1}$, the equivalence class of the point $(\ast, 0) \in S^{n-1} \times [0, 1]$, and let $V$ be the filter of open neighborhoods of the point $[\ast, 0] \in S^{n-1}$ in the standard quotient topology. We show that a filter $\mathcal{X}$ on $\Sigma S^{n-1}$ converges in the pseudotopological quotient to $[\ast, 0]$ iff it contains $V$.

First, suppose that a filter $\mathcal{X}$ in $\Sigma S^{n-1}$ does not contain $V$ but converges to $[\ast, 0]$. Then there exists an open set $V \subseteq S^{n-1}$ which is not in $\mathcal{X}$. However, by definition of the quotient pseudotopology, there exists a filter $\lambda$ in $S^{n-1} \times [0, 1]$, which converges to a point in $p^{-1}[\ast, 0]$ such that $p(\lambda) = \mathcal{X}$. By definition of the pseudotopological suspension, however, $\lambda$ must contain $p^{-1}(V)$, and therefore $\mathcal{X}$ contains $V$, a contradiction. Therefore every filter which converges to $[\ast, 0]$ must contain $V$.

We now show that the filter $[\mathcal{V}]$ generated by $V$ on $\Sigma S^{n-1}$ converges to $[\ast, 0]$, from which it will follow that any filter which contains $V$ converges to $[\ast, 0]$. Let $\mathcal{X}$ be an ultrafilter on $\Sigma S^{n-1}$ which contains $[\mathcal{V}]$. By Lemma 4.26 there exists an ultrafilter $\lambda$ on $S^{n-1} \times [0, 1]$ such that $p(\lambda) = \mathcal{X}$. Since $S^{n-1} \times [0, 1]$ is compact, $\lambda$ converges to a point $(x, t) \in S^{n-1} \times [0, 1]$. Suppose that $p(x, t) \neq [\ast, 0]$. Then $(x, t) \notin S^{n-1} \times \{0, 1\} \cup \{\ast\} \times [0, 1]$.

However, there are open neighborhoods $U$ of $(x, t)$ and $V$
of $S^{n-1} \times \{0, 1\} \cup \{\ast\} \times [0, 1]$ such that $U \cap V = \emptyset$, but both $U, V \in \lambda$ by construction, which is a contradiction. Therefore $(x, t) \in p^{-1}[\ast, 0]$ and $\lambda' = p(\lambda) \to p(x, t) = [\ast, 0]$. Since $\lambda'$ is an arbitrary ultrafilter which contains $[\lambda]$, it follows that $[\lambda] \to [\ast, 0]$.

In conclusion, we have shown that the quotient pseudotopology on $\Sigma S^{n-1}$ is the pseudotopology induced by the quotient topology on $\Sigma S^{n-1}$, and therefore $\Sigma S^{n-1} \cong S^n$, as desired. □

5. The Quillen Model Category Structure on PsTop

In order to construct the Quillen model category structure on PsTop, we generalize the construction in [20] from Top to PsTop. In order to do so, we only need to show that a critical compactness result ([20] Propositions 4.10) is also true for relative cell complexes in pseudotopological spaces. Once this is accomplished, the rest of the proof in [20] then applies verbatim to PsTop, with the change that all of the maps, diagrams, and operations take place in PsTop and not in Top. In the following, we begin with a discussion of compactness in PsTop, after which we proceed to show the necessary generalization of [20], Proposition 4.10, which completes the construction, following [20], of the Quillen model category in the pseudotopological case. We do not repeat the full argument of [20] here.

5.1. Compactness in PsTop. We state the definition and several critical results for compactness in PsTop.

**Definition 5.1.** A pseudotopological space $(X, \Lambda)$ is compact iff every ultrafilter on $X$ converges.

Our first observation is that the image of a compact convergence space by a continuous function is a compact subspace.

**Proposition 5.2 ([7], Proposition 1.4.7).** Let $(X, \Lambda_X)$ be a compact pseudotopological space and let $(Y, \Lambda_Y)$ be any pseudotopological space. If $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a continuous surjection, then $(Y, \Lambda_Y)$ is compact.

As in topological and closure spaces, there is a characterization of compactness in terms of coverings. The relevant notion of covering in PsTop is the following.

**Definition 5.3.** Let $(X, \mathcal{F})$ be a pseudotopological space, and suppose that $A \subset X$. We say that a collection $\mathcal{C}$ of sets is a *covering system* of $A$ iff for every filter $\lambda \to x \in A$, there exists a set $C \in \mathcal{C}$ such that $C \in \lambda$. If $A = \{x\}$, then we say that $\mathcal{C}$ is a local covering system of $(X, \mathcal{F})$ at the point $x$.

It is important to note, however, that covering systems are distinct from covers of sets. We recall the definition of a cover to emphasize this point.

**Definition 5.4.** We say that a collection of subsets $\mathcal{C}$ of a set $X$ is a cover of $X$ iff $\bigcup_{U \in \mathcal{C}} U = X$. If $C$ is a cover $\mathcal{C}$ of $X$ and a subcollection $\mathcal{C}' \subset \mathcal{C}$ is also covers $X$, then we call $\mathcal{C}'$ a *subcover* of $\mathcal{C}$.

We now define the interior of a collection of subsets of a pseudotopological space.

**Definition 5.5.** Let $(X, \Lambda)$ be a pseudotopological space, and let $\mathcal{U}$ be a collection of subsets of $X$. We define the interior of $\mathcal{U}$, $\text{int}(\mathcal{U})$, to be the set

$$\text{int}(\mathcal{U}) := \{x \in X \mid \mathcal{U} \text{ is a covering system of } (X, \Lambda) \text{ at } x\}$$

When $\mathcal{U} = \{U\}$, we will write $\text{int}(U)$ or $\overset{\circ}{U}$ for $\text{int}(\mathcal{U})$. 
Proposition 5.6. If \((X, \Lambda_X)\) is a pseudotopological space and \((Y, \Lambda_Y) \subset (X, \Lambda_X)\) is a topological subspace, then the topological interior of \(Y\) is equal to \(\text{int}_{\Lambda_X}(Y)\).

We also recall the generalization to \(\text{PsTop}\) of the property that closed subsets of a compact set are compact. We make this precise with the following.

Definition 5.7. Let \((X, \Lambda)\) be a pseudotopological space. For each subset \(A \subset X\), we define the adherence \(a_\Lambda(A) := \{x \in X \mid \exists (\lambda, x) \in \Lambda \text{ such that } A \in \lambda\}\). We say that \(A \subset X\) is closed iff \(a_\Lambda(A) = A\).

Proposition 5.8 (\([7]\), Proposition 1.4.6(i)). A closed subspace \(C \subset X\) of a compact pseudotopological space \((X, \Lambda)\) is compact.

Finally, we recall that a compact topological space remains compact in \(\text{PsTop}\).

Proposition 5.9. Let \((X, \tau)\) be a compact topological space. Then \((X, \Lambda_\tau)\) is a compact pseudotopological space.

Proposition 5.10 (\([7]\), Proposition 1.4.15). A pseudotopological space \((X, F)\) is compact iff every covering system \(C\) of \((X, F)\) contains a finite subcover.

The next proposition will be needed for the proof of Theorem 5.16.

Proposition 5.11. Let \((X, \Lambda)\) be a pseudotopological space, let \(U \subset X\) be a subspace of \(X\), and suppose that \(C\) is a local covering system of \(X\) at a point \(x \in U\). Then the set \(C' := \{C \cap U \mid C \in C\}\) is a local covering system of the subspace \((U, \Lambda_U)\) at \(x\).

Proof. Recall that \((\lambda, x) \in \Lambda_U\) iff \(([\lambda], x) \in \Lambda\), where \([\lambda]\) is the filter in \(X\) generated by the sets in \(\lambda\). Now suppose that \(C\) is a local covering system of \((X, \Lambda)\) at a point \(x \in U\), and define \(C'\) as in the statement of the proposition. Then for every \((\lambda, x) \in \Lambda_U\), \(C\) contains a set \(A \in [\lambda]\). Since \([\lambda]\) is generated by \(\lambda\), there exists \(B \in \lambda\) such that \(B \subset A\). However, \(B \subset U\) by construction, and therefore \(B \subset A \cap U\). It follows that \(A \cap U \in \lambda \cap C'\). Since \((\lambda, x)\) is an arbitrary element of \(\Lambda_U\) which converges to \(x \in U\), \(C'\) is a local covering system of \((U, \Lambda_U)\) at \(x \in U\), and the proof is complete. □

We now state and prove the compactness theorem necessary for the construction of the Quillen model category structure on \(\text{PsTop}\). We begin with the definition of a cell attachment and a relative cell complex.

Definition 5.12. If \(X\) is a subspace of \(Y\) and there is a pushout square

\[
\begin{array}{ccc}
S^{n-1} & \longrightarrow & X \\
\downarrow & & \uparrow h \\
D^n & \longrightarrow & Y
\end{array}
\]

for some \(n \geq 0\), then we say that \(Y\) is obtained from \(X\) by attaching a cell or a cell attachment. When \(n = 0\), we let \(S^{-1} = \emptyset\) and \(D^0 = \{\ast\}\), the one-point set.

Definition 5.13. We say that a continuous map \(f : X \rightarrow Y\) between pseudotopological spaces is a relative cell complex if \(f\) is an inclusion and \(Y\) can be constructed from \(X\) by a (possibly infinite, and even transfinite) sequence of cell attachments as in Definition 5.12. If \(Y\) may be constructed from \(X\) by attaching a finite number of cells, then we say that it is a finite relative cell complex, and if \(\emptyset \rightarrow Y\) is a (finite) relative cell complex, then we say that \(Y\) is a (finite) cell complex.
Definition 5.14. Given a relative cell complex \( f : X \to Y \), a (possibly transfinite) sequence \( (e_0 = f, e_1, \ldots) \) of cell attachments which constructs \( Y \) from \( X \) is called a presentation of \( f \). Given \( \mathcal{E} := (e_0 = f, e_1, \ldots) \) of a relative cell complex \( f : X \to Y \), we say that an ordinal number \( \gamma \) is the presentation ordinal of \( e \) in \( \mathcal{E} \) iff \( e \) is the \( \gamma \)-th cell attachment in \( \mathcal{E} \). In particular, the presentation ordinal of \( X \) is 0. We say that \( \gamma \) is the presentation ordinal of \( f : X \to Y \) iff \( \mathcal{E} \cong \gamma \).

Remark 5.15. Note that any two presentations \( \mathcal{E} \) and \( \mathcal{E}' \) of \( f : X \to Y \) have the same presentation ordinal. Otherwise, without loss of generality, there is a cell in \( \mathcal{E} \) which is not in \( \mathcal{E}' \), so they cannot be constructions of the same space \( Y \) from \( X \).

We now state and prove the generalization of [20], Proposition 4.10 to \( \text{PsTop} \).

Theorem 5.16. If \( X \to Y \) is a relative cell complex in \( \text{PsTop} \), then a compact subset of \( Y \) intersects the interiors of only finitely many cells of \( Y - X \).

Proof. We follow the general strategy of the proof of [20], Proposition 4.10.

Let \( C \) be a compact subset of \( Y \). We construct a subset \( P \) of \( C \) by choosing a point of \( C \cap \hat{D} \) for each cell \( D \) such that the intersection \( C \cap \hat{D} \) is non-empty. We will first show that \( P \) is closed in \( C \), and therefore compact by Proposition 5.8. Suppose that \( c \in C \). The first step to prove that \( P \) is closed in \( C \) will be to show that there is a local covering system \( \mathcal{U}_c \) of \( Y \) at \( c \) such that

\[
\bigcup_{U \in \mathcal{U}_c} (U \cap P) = \begin{cases} \emptyset & c \notin P, \\ c & c \in P. \end{cases}
\]

If \( c \in C \cap (Y - X) \), then let \( e_c \) be the unique cell of \( Y - X \) which contains \( c \) in its interior. Otherwise, if \( c \in C \cap X \), let \( e_c = X \). Since there is at most one point of \( P \) in the interior of any cell of \( Y - X \) and the cells are homeomorphic to the topological space \( D^n \), we may choose an local \((e_c, \Lambda_{e_c})\)-covering system (i.e. a local covering system in the subspace \((e_c, \Lambda_{e_c}) \subset (Y, \Lambda_Y)\)) \( V_c = \{V_c\} \) of \( c \) consisting of a single set subset of \( Y \) such that \( V_c \subset \text{int}(e_c) \) and which satisfies

\[ V_c \cap P = \begin{cases} c & c \in P, \\ \emptyset & c \notin P. \end{cases} \]

In particular, if \( e_c \) is a cell of \( Y - X \), then we may choose \( V_c \) to be a small open neighborhood of \( c \) which avoids any point \( p \in e_c \cap P \) where \( p \neq c \). If \( e_c = X \), then we may take \( V_c = X \). Since \( P \cap X = \emptyset \) and \( X = X \) by definition, the condition is satisfied.

While \( c \) is only in the interior of a single cell \( e_c \), \( c \) may be glued to the boundary of subsequent cells in the process of constructing the relative cell complex. We will now use Zorn’s lemma to show that we can extend \( \mathcal{V}_c \) to a local \((Y, \Lambda_Y)\)-covering system of \( c \), which we call \( \mathcal{U}_c \), such that

\[
\bigcup_{U \in \mathcal{U}_c} (U \cap P) = \begin{cases} c & c \in P, \\ \emptyset & c \notin P. \end{cases}
\]

Fix a presentation of \( X \to Y \), and let \( \alpha \) be the presentation ordinal of the cell \( e_c \) in the presentation, where, in particular, \( \alpha = 0 \) if \( e_c = X \). Suppose that the presentation ordinal of the relative cell complex \( X \to Y \) is \( \gamma \). Consider the set \( T \) of ordered pairs \((\beta, W)\), where \( \alpha \leq \beta \leq \gamma \) and \( W \) is a local covering system of \( c \) in \( Y^\beta \) such that

\[ W \cap Y^\alpha := \{Y^\alpha \cap W \mid W \in W\} = V_c, \]

where \( Y^\alpha \) is the result of attaching the first \( \alpha \) cells to \( X \) in the presentation of \( X \to Y \). We define a preorder on \( T \) by \((\beta_1, W_1) \prec (\beta_2, W_2)\) iff \( \beta_1 < \beta_2 \) and \( W_2 \cap Y^{\beta_1} = W_1 \). If \( \{(\beta_s, W_s)\}_{s \in S} \) is a
chain in $T$, then $(\cup_{s \in S} \beta_s, \cup_{s \in S} \omega_s)$ is an upper bound in $T$ of the chain, so by Zorn’s lemma, we have that $T$ has a maximal element, which we denote $(\alpha, \mathcal{A})$.

We now show that $\alpha = \gamma$. Suppose instead that $\alpha < \gamma$, and consider the cells of presentation ordinal $\alpha + 1$. Since $Y$ has the final pseudotopology determined by $X$ and the cells of $Y - X$, it is enough to enlarge each $A \in \mathcal{A}$ to an $A' \subset Y^{\alpha+1}$ so that the intersection of $A'$ with the cell of presentation ordinal $\alpha + 1$ is open in that cell, and so that $A \cap P = A' \cap P$ (which, in particular is equal to either $\emptyset$ or $c$). If $h : S^{n-1} \rightarrow Y^{\alpha+1}$ is the attaching map for the cell $e$ of presentation ordinal $\alpha + 1$, then $h^{-1}(A)$ contains an open set $B_{A,e} \subset S^{n-1}$ for at least one $A \in \mathcal{A}$. Let $B'_e$ be an open collar neighborhood of $B$, which, in the case that $e \cap P \neq \emptyset$, is chosen to avoid the unique point $p \in e \cap P$. For each $A \in \mathcal{A}$, let $A'$ be the union of $A$ with all of the $B'_{A,e}$. Then define $\mathcal{A}' := \{A' \mid A \in \mathcal{A}\}$. It follows that the pair $(\alpha + 1, \mathcal{A}')$ is an element of $T$ greater than $(\alpha, \mathcal{A})$, contradicting the maximality of $(\alpha, \mathcal{A})$. We therefore have that $\alpha = \gamma$, and we define $\mathcal{U}_e := \mathcal{A}$.

Now consider the local covering system $\mathcal{U} := \cup_{e \in \mathcal{C}} \mathcal{U}_e$ of $C$ in $Y$. If a point $c \in C, c \notin P$ is in the adherence $a_C(P)$, then there exists a filter $\lambda \rightarrow c$ in $C \subset Y$ such that $P \subset \lambda$, however, this contradicts the construction of $\mathcal{U}_e$, since $P \cap U = \emptyset$ for all $U \in \mathcal{U}_e$. Therefore, $a_C(P) = P$, and $P$ is closed in $C \subset Y$. $P$ is therefore compact in $C$. We also see that, when $c \in P$, the above procedure produces a $\Lambda \alpha$-local covering system $\mathcal{U}_e$ of $c$ in $Y$ such that $\cup_{U \in \mathcal{U}_e} (U \cap P) = \{c\}$.

5.2. **The Quillen Model Category Structure.** We begin this section by recalling the definition of a model category structure and state the main theorem of this section. As in [20], we use the definition of a model category from [19], Definition 7.1.3, in which a model category is required to admit all small (co)limits instead of only finite ones, and we ask that the factorizations not only exist but are also functorial. We first recall the definition of a model category from [19].

**Definition 5.17.** A model category is a category $\mathcal{M}$ with three distinguished classes of morphisms, called weak equivalences, cofibrations, and fibrations, which satisfy the following axioms:

1. (Limits) The category $\mathcal{M}$ is complete and co complete.
2. (2-out-of-3) If $f$ and $g$ are maps in $\mathcal{M}$ such that $gf$ is defined and any two of $f, g, and $gf$ are weak equivalences, then so is the third.
3. (Retract) If $f$ and $g$ are maps in $\mathcal{M}$ such that $f$ is a retract of $g$ and $g$ is a weak equivalence, a vibrations, or a cofibration, then so is $f$.
4. (Lifting) Given the commutative solid arrow diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow i & & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

the dashed arrow exists if either

(a) $i$ is a cofibration and $p$ is a trivial fibration, or
(b) $i$ is a trivial fibration and $p$ is a fibration.
5. (Factorization) Every morphism $g$ in $\mathcal{M}$ has two functorial factorizations:

(a) $g = \rho(g) \circ \gamma(g)$, where $\rho(g)$ is a trivial fibration and $\gamma(g)$ is a cofibration, and
(b) $g = \beta(g) \circ \iota(g)$, where $\iota(g)$ is a trivial cofibration and $\beta(g)$ is a fibration.
We now define the notions on pseudotopological spaces with which we will construct the Quillen model category on \( \text{PsTop} \). We will typically refer to the pseudotopological space \((X, \Lambda)\) as \( X \) when the structure \( \Lambda \) is unambiguous.

**Definition 5.18.** A continuous function \( f : X \rightarrow Y \) between pseudotopological spaces is a *weak homotopy equivalence* iff either \( X \) and \( Y \) are both empty or \( X \) and \( Y \) are both non-empty and the induced map
\[
f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))
\]
is an isomorphism for every \( n \geq 0 \) and every choice of base-point \( x \in X \).

**Definition 5.19.** A map \( \alpha \) is said to be a *retract* of a map \( \beta \) iff there exists a commutative diagram
\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
B & \xrightarrow{\beta} & B \\
\downarrow{1_B} & & \\
B & & B
\end{array}
\]

**Definition 5.20.** We say that a function \( p : X \rightarrow Y \) is a *Serre fibration* iff any diagram of solid arrows
\[
\begin{array}{ccc}
D^n & \xrightarrow{h} & X \\
\downarrow{p} & & \\
D^n \times I & \xrightarrow{} & Y
\end{array}
\]
adopts a lift \( h : D^n \times I \rightarrow X \).

We now state and prove the main theorem of this section.

**Theorem 5.21.** There is a model category structure on the category \( \text{PsTop} \) of pseudotopological spaces where

1. the weak equivalences of the model category are weak homotopy equivalences,
2. the cofibrations of the model category are continuous functions which are either relative cell complexes or retracts of relative cell complexes, and
3. the fibrations of the model category are Serre fibrations.

**Proof.** The proof is identical to the proof of [20], Theorem 2.5, with the exception that Theorem 5.16 replaces [20], Proposition 4.10, and that all morphisms and constructions in the proof are in the category \( \text{PsTop} \) instead of in \( \text{Top} \). \( \square \)

5.3. **Fibrant and Cofibrant Objects.** In this final section, we show that all objects in \( \text{PsTop} \) are fibrant and weakly homotopy equivalent to either a cell complex or the retract of a cell complex.

**Definition 5.22.** A *fibrant* object in a model category \( \mathcal{C} \) with a terminal object \( \{*\} \) is a \( \mathcal{C} \)-object \( X \) such that the map \( X \rightarrow \{*\} \) is a fibration.

**Proposition 5.23.** Every object in \( \text{PsTop} \) is fibrant.

**Proof.** The constant map \( X \rightarrow * \) is a Serre fibration for any pseudotopological space \( X \).\( \square \)

**Definition 5.24.** A cofibrant object in a model category \( \mathcal{C} \) with an initial object \( \emptyset \) is a \( \mathcal{C} \)-object \( X \) such that the map \( \emptyset \rightarrow X \) is a cofibration.
Proposition 5.25. The cofibrant objects in the Quillen model category on $\text{PsTop}$ are the cell complexes.

Proof. Immediate from the definitions of cofibrations and cell complexes. $\square$

We end with the following surprising consequence of these constructions.

Theorem 5.26. Every pseudotopological space $X$ is weakly homotopy equivalent to a cell complex or the retract of a cell complex.

Proof. The existence of the Quillen model category structure on $\text{PsTop}$ implies that every continuous map $g : Z \to X$ between pseudotopological spaces may be factored as $g = p \circ i$, where $i$ is a cofibration and $p$ is a fibration and a weak equivalence. In particular, when $Z = \emptyset$, this implies that there exists a pseudotopological space $Y$ such that $i : \emptyset \to Y$ is a cofibration and $p : Y \to X$ is a weak equivalence. By definition, $i : \emptyset \to Y$ is a cofibration if $Y$ is a cell complex or the retract of a cell complex. $\square$

Remark 5.27. Note that although the cells in a cell complex are topological, the process of construction a cell complex in $\text{PsTop}$ may a priori result in a non-topological space, although we are unaware of an example where this occurs.

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References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. Abstract and concrete categories: the joy of cats. Number 17. 2006. Reprint of the 1990 original [Wiley, New York; MR1051419].
[2] Eric Babson, Hélène Barcelo, Mark de Longueville, and Reinhard Laubenbacher. Homotopy theory of graphs. J. Algebraic Combin., 24(1):31–44, 2006.
[3] Hélène Barcelo and Reinhard Laubenbacher. Perspectives on $A$-homotopy theory and its applications. Discrete Math., 298(1-3):39–61, 2005.
[4] Laurent Bartholdi, Thomas Schick, Nat Smale, and Steve Smale. Hodge theory on metric spaces. Found. Comput. Math., 12(1):1–48, 2012. Appendix B by Anthony W. Baker.
[5] Hans Joachim Baues. Algebraic homotopy, volume 15 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1989.
[6] Hans-Joachim Baues. Combinatorial foundation of homology and homotopy. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999. Applications to spaces, diagrams, transformation groups, compactifications, differential algebras, algebraic theories, simplicial objects, and resolutions.
[7] R. Beattie and H.-P. Butzmann. Convergence structures and applications to functional analysis. Kluwer Academic Publishers, Dordrecht, 2002.
[8] Peter Bubenik and Nikola Miličević. Eilenberg-steenrod homology and cohomology theories for Čech’s closure spaces. 2021.
[9] Peter Bubenik and Nikola Miličević. Homotopy, homology, and persistent homology using closure spaces and filtered closure spaces. 2021.
[10] Gunnar Carlsson. Topology and data. Bull. Amer. Math. Soc. (N.S.), 46(2):255–308, 2009.
[11] Daniel Carranza and Chris Kapulkin. Cubical setting for discrete homotopy theory, revisited. 2022.
[12] Eduard Čech. *Topological spaces*. Revised edition by Zdeněk Frolík and Miroslav Katětov. Scientific editor, Vlastimil Pták. Editor of the English translation, Charles O. Junge. Publishing House of the Czechoslovak Academy of Sciences, Prague; Interscience Publishers John Wiley & Sons, London-New York-Sydney, 1966.

[13] Tien Chih and Laura Scull. A homotopy category for graphs. *J. Algebraic Combin.*, 53(4):1231–1251, 2021.

[14] Tien Chih and Laura Scull. Fundamental groupoids for graphs. *Categ. Gen. Algebr. Struct. Appl.*, 16(1):221–248, 2022.

[15] Denis-Charles Cisinski. *Higher categories and homotopical algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019.

[16] Jim Conant, Victoria Curnutte, Corey Jones, Conrad Plaut, Kristen Pueschel, Maria Lusby, and Jay Wilkins. Discrete homotopy theory and critical values of metric space. *Fund. Math.*, 227(2):97–128, 2014.

[17] Szymon Dolecki and Frédéric Mynard. *Convergence foundations of topology*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

[18] Alexander Grigor’yan, Yong Lin, Yuri Muranov, and Shing-Tung Yau. Homotopy theory for digraphs. *Pure Appl. Math. Q.*, 10(4):619–674, 2014.

[19] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[20] Philip S. Hirschhorn. The quillen model category of topological spaces. *Expositiones Mathematicae*, 37(1):2–24, 2019.

[21] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[22] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[23] Conrad Plaut and Jay Wilkins. Discrete homotopies and the fundamental group. *Adv. Math.*, 232:271–294, 2013.

[24] Gerhard Preuss. *Foundations of topology*. Kluwer Academic Publishers, Dordrecht, 2002. An approach to convenient topology.

[25] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[26] Emily Riehl. *Category Theory in Context*. Dover Publications, Inc, 2016.

[27] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*, volume 194 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2022.

[28] Antonio Rieser. Čech closure spaces: A unified framework for discrete and continuous homotopy. *Topology and its Applications*, 296:107613, 2021.

[29] Nat Smale and Steve Smale. Abstract and classical Hodge–de Rham theory. *Anal. Appl. (Singap.)*, 10(1):91–111, 2012.

[30] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

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