Random Walks and Lévy Processes Conditioned Not to Overshoot

Sergey G. Foss∗
Heriot-Watt University, Edinburgh, UK and
Institute of Mathematics, Novosibirsk, Russia
and
Anatolii A. Puhalskii†
University of Colorado Denver, Denver, U.S.A. and
Institute for Problems in Information Transmission, Moscow, Russia

September 3, 2009

Abstract
Let ξ1, ξ2, . . . be i.i.d. random variables with negative mean. Suppose that E exp(λξ1) < ∞ for some λ > 0 and that there exists γ > 0 with E exp(γξ1) = 1. It is known that if, in addition, E ξ1 exp(γξ1) < ∞, then the most likely way for the random walk S_k = ∑_{i=1}^k ξ_i to reach a high level is to follow a straight line with a positive slope. We study the case where E ξ1 exp(γξ1) = ∞. Assuming that the distribution F(dx) = exp(γx)P(ξ1 ∈ dx) belongs to the domain of attraction of a spectrally positive stable law, we obtain a weak convergence limit theorem as r → ∞ for the conditional distribution of the process (r−1 ∑_{i=1}^{[t/F(r,∞)]} ξ_i, t ≥ 0) stopped at the time when it reaches level 1 given that the latter event occurs. The limit is an increasing jump process. It is shown to be distributed as an increasing stable Lévy process stopped at the time when it reaches level 1 conditioned on the event this level is not overshot. Some properties of this process are studied.

1 Introduction
Let ξ1, ξ2, . . . be i.i.d. random variables on a probability space (Ω, F, P) with Eξ1 < 0 and P(ξ1 > 0) > 0. Then the random walk S_n = ∑_{i=1}^n ξ_i tends to −∞ with probability 1 and the event that it exceeds a high level has a small albeit positive probability. The asymptotics of this probability have been studied extensively. Suppose E exp(λξ1) < ∞ for some λ > 0 and denote
γ = sup{λ : E exp(λξ1) ≤ 1}. (1.1)
Clearly, γ > 0 and E exp(γξ1) ≤ 1.
In the “classical” case where E exp(γξ1) = 1 and β = E ξ1 exp(γξ1) < ∞, the celebrated Cramer-Lundberg theorem asserts that, for a certain constant C_1,
P(\sup_{n} S_n > r) \sim C_1 e^{-γr} \quad \text{as } r \to \infty, (1.2)
where \sim stands for asymptotic equivalence. The limit is taken along all r if ξ1 has a non-lattice distribution and along multiples of the lattice span if ξ1 has a lattice distribution, see, for example, Asmussen [2, XIII.5], Borovkov [7, §22], or Feller [14, XII].

∗Email address: S.Foss@ma.hw.ac.uk
†Email address: anatolii.puhalskii@ucdenver.edu
If \( \mathbb{E} \exp(\gamma \xi_1) < 1 \) so that \( \mathbb{E} \exp(\lambda \xi_1) = \infty \) for all \( \lambda > \gamma \), then, under certain regularity assumptions on the distribution of \( \xi_1 \) (more specifically, provided it belongs to class \( S_\gamma \), see Tengels [23]),

\[
P(\sup_n S_n > r) \sim C_2 \mathbb{P}(\xi_1 > r) \quad \text{as } r \to \infty,
\]

where \( C_2 = \mathbb{E} \exp(\gamma \sup_n S_n)/(1 - \mathbb{E} \xi_1 \exp(\gamma \xi_1)) \), see, for example, Bertoin and Doney [5] and references therein. For earlier results, see Borovkov [7, §22]; recent developments can be found in Borovkov and Borovkov [8] and Zachary and Foss [25].

The borderline case where \( \mathbb{E} \exp(\gamma \xi_1) = 1 \) and \( \beta = \infty \) was first addressed in Borovkov’s monograph, Borovkov [7, §22]. More complete results have been obtained by Korshunov [18] who showed that if the distribution of \( \xi_1 \) is nonlattice and the distribution \( F(dx) = \exp(\gamma x) \mathbb{P}(\xi_1 \in dx) \) has a regularly varying righthand tail with index \(-\alpha\), where \( \alpha \in (1/2,1) \), then

\[
P(\sup_n S_n > r) \sim C_3 \frac{e^{-\gamma r}}{\gamma m(r)} \quad \text{as } r \to \infty,
\]

for a certain universal constant \( C_3 \) and for \( m(r) = \int_0^r F((u, \infty)) \, du \). A similar asymptotic relationship was shown to be valid for lattice distributions too, with a different constant \( C_3' \). The restriction on \( \alpha \) to be greater than 1/2 is due to the fact that a key element in the proof of (1.3) is a local renewal theorem with infinite (or non-existing) mean which has been established for \( \alpha \in (1/2,1) \) only, see Erickson [10].

One can also clarify in what way the event of attaining a high level \( r \) is most likely to occur. In the classical case, the trajectory \((S_{\lfloor rt \rfloor}/r, t \in \mathbb{R}_+)\), where \( \lfloor \cdot \rfloor \) stands for the integer part, stays with a probability close to one in a neighbourhood of the straight line with slope \( \beta \); see (1.9) below for the exact formulation. Thus, the event of reaching level \( r \) is realised typically via multiple (of order \( r \)) jumps of size of order 1. If \( \mathbb{E} \exp(\gamma \xi_1) < 1 \), then the high level is most likely to be reached at the very beginning of the random walk, which occurs due to a single big jump of order \( r \), see Borovkov and Borovkov [8] for the case of regular exponential distribution tails and Zachary and Foss [25] for the more general \( S_\gamma \) distributions. In fact, as it follows from the results of Zachary and Foss [25], the conditional distribution of the jump time converges weakly (no scaling is involved) to the geometric distribution with parameter \( p = 1 - \mathbb{E} \exp(\gamma \xi_1) \).

The purpose of this paper is to study the most likely way for the random walk to attain a high level \( r \) in the setting considered by Korshunov [18]. Not unexpectedly, the results and intuitive explanations in the borderline situation are essentially more intriguing and complicated. On the one hand, as \( \beta \uparrow \infty \) in the classical case, the trajectories \( S_{\lfloor rt \rfloor}/r \) for large values of \( r \) increasingly look like a jump at time 0. On the other hand, as \( \mathbb{E} \exp(\gamma \xi_1) \uparrow 1 \), the time for the big jump that reaches (or almost reaches) \( r \) to occur tends to infinity. Therefore, the typical jump sizes in the case we concern ourselves with here should, on the one hand, grow to infinity as \( r \to \infty \), but, on the other hand, be of a smaller order of magnitude than \( r \). The correct order of magnitude is captured by considering the random walk \( S_n \) with \( n \) growing to infinity at a slower rate than in the classical case. More specifically, under appropriate hypotheses, one should let \( n = [r^{1-\alpha}] \), which corresponds to the typical jump sizes of the random walk reaching level \( r \) being of order \( r^{1-\alpha} \), whereas the number of such jumps being of order \( r^{\alpha} \).

To provide better insight into the kind of results we obtain, let us recall the argument underlying the asymptotics in the classical case. Its main points can be found in Asmussen [14], who refers to Iglehart [16] and von Bahr [3]. Let \( \mathcal{F}_n \) denote the \( \sigma \)-algebra on \( \Omega \) generated by the \( \xi_i, i = 1,2,\ldots,n \). We may and will assume that the \( \sigma \)-algebra \( \mathcal{F} \) is generated by the \( \sigma \)-algebras \( \mathcal{F}_n, n = 1,2,\ldots \). Let measure \( \mathbb{P}^* \) on \((\Omega, \mathcal{F})\) be defined by

\[
\mathbb{P}^*(\Gamma) = \mathbb{E} \exp(\gamma S_n)1_\Gamma \quad \text{for } \Gamma \in \mathcal{F}_n.
\]

(1.4)
where $1_{\Gamma}$ denotes the indicator function of event $\Gamma$. It is a probability measure by the assumption that $E \exp(\gamma \xi_1) = 1$. The probability measures $P$ and $P^*$ are locally equivalent and $dP / dP^* \big|_{F_n} = \exp(-\gamma S_n)$. We also note that under $P^*$ the $\xi_t$ are i.i.d. with mean $\beta$.

For $r > 0$, let $\tau^{(r)}$ be the first time the random walk $S_n$ attains level $r$, i.e.,
\[
\tau^{(r)} = \min \{ n : S_n \geq r \}. \tag{1.5}
\]

Because $\{ \tau^{(r)} = n \} \in \mathcal{F}_n$,
\[
P(\tau^{(r)} < \infty) = \sum_{n=1}^{\infty} P(\tau^{(r)} = n) = \sum_{n=1}^{\infty} E^* e^{-\gamma S_n} 1_{\{ \tau^{(r)} = n \}} = E^* e^{-\gamma S^{(r)}_n} 1_{\{ \tau^{(r)} < \infty \}},
\]
where $E^*$ denotes expectation with respect to $P^*$. On noting that $P^*(\tau^{(r)} < \infty) = 1$ as $E^* \xi_1 > 0$, we conclude that
\[
P(\tau^{(r)} < \infty) = E^* \exp(-\gamma S^{(r)}_n). \tag{1.6}
\]

More generally, if $\Gamma \in \mathcal{F}_{\tau^{(r)}}$, $\mathcal{F}_{\tau^{(r)}}$ being the $\sigma$-algebra associated with the stopping time $\tau^{(r)}$, then by the fact that $\{ \tau^{(r)} = n \} \cap \Gamma \in \mathcal{F}_n$
\[
P(\Gamma \cap \{ \tau^{(r)} < \infty \}) = \sum_{n=1}^{\infty} P(\Gamma \cap \{ \tau^{(r)} = n \}) = \sum_{n=1}^{\infty} E^* e^{-\gamma S_n} 1_{\Gamma \cap \{ \tau^{(r)} = n \}} = E^* e^{-\gamma S^{(r)}_n} 1_{\Gamma},
\]
so
\[
P(\Gamma | \tau^{(r)} < \infty) = \frac{E^* e^{-\gamma S^{(r)}_n} 1_{\Gamma}}{E^* e^{-\gamma S^{(r)}_n} 1_{\Gamma}} = E^* e^{-\gamma \chi^{(r)}} 1_{\Gamma}, \tag{1.7}
\]
where
\[
\chi^{(r)} = S^{(r)}_\tau - r \tag{1.8}
\]
is the overshoot of the random walk $S_n$ over level $r$.

Suppose now that $\Gamma$ is the event $\{ \sup_{n \leq \tau^{(r)}} |S_n - \beta n| < \varepsilon r \}$, where $\varepsilon > 0$ is given. Since $\beta < \infty$, by the strong law of large numbers the $P^*$-probability of this event tends to 1 as $r \to \infty$. Also, the condition $\beta < \infty$ implies, provided the distribution of $\xi_1$ is nonlattice, that the $\chi^{(r)}$ under $P^*$ tend in distribution, as $r \to \infty$, to a proper random variable, say $\chi^{(\infty)}$, see, e.g., Asmussen [2, VIII.2], Gut [15, III.10], or Feller [14, XI.4]. Therefore, the $E^* \exp(-\gamma \chi^{(r)})$ converge to a positive limit as $r \to \infty$ and by (1.7)
\[
\lim_{r \to \infty} P( \sup_{n \leq \tau^{(r)}} |S_n - \beta n| < \varepsilon r | \tau^{(r)} < \infty ) = 1. \tag{1.9}
\]
(In particular, (1.2) with $C_1 = E e^{-\gamma \chi^{(\infty)}}$ follows by (1.6) and (1.8).) This argument breaks down in two places if $\beta = \infty$: we can no longer rely on the law of large numbers for the random walk and the $\chi^{(r)}$ might converge to infinity as $r \to \infty$. We assume, following Korshunov [18], that the distribution $F$ has a regularly varying righthand tail with index $-\alpha$, where $\alpha \in (1/2, 1)$. A sufficient (but not necessary) condition for this to hold is for the function $e^{\gamma x} P(\xi_1 > x)$ to be regularly varying with index $-\alpha - 1$, see Korshunov [18] for further comments. Note also that due to the fact that the lefthand tail of $F$ decays exponentially fast, its righthand tail is regularly varying with index $-\alpha$ if and only if $F$ belongs to the domain of attraction of the spectrally positive stable law with index $\alpha$, cf., Gnedenko and Kolmogorov [12] or Feller [14].

"Time" is scaled by $(1 - F(r))^{-1}$, where $F$ denotes the cumulative distribution function associated with $F$, so that the processes $(S_{t(1-F(r))^{-1} t}/r, t \in \mathbb{R}_+)$ under $P^*$ converge in distribution as $r \to \infty$ to an increasing pure-jump Lévy process $X = (X(t), t \in \mathbb{R}_+)$ with Lévy measure
\(\alpha x^{-\alpha -1} dx\), see Resnick \[20\] or Lemma A.1. If the righthand tail of \(F\) decays as \(x^{-\alpha}\), then the scaled time is \([r^\alpha t]\), as was discussed earlier.

Under the stated assumptions, the random variables \(\chi^{(r)} / r\) converge in distribution to a proper random variable \(\chi\), which assumes values in \((0, 1)\) and has density \(p_\alpha(x) = (\sin \pi \alpha / \pi)x^{-\alpha}(1+x)^{-1}\). (See Dynkin \[9\] Theorem 2, or Feller \[13\] XIV.3, for the case of renewal processes, Sinai \[22\] for the case of sums of random variables with a stable distribution, the case in question follows by an application of Lemma 2 in Korshunov \[18\]. A different proof of this result is given in appendix A.1. It is then plausible that in (1.7) one should be able to replace \(\chi^{(r)}\) with \(r \chi\) so that \(\exp(-\gamma \chi^{(r)})\) can be replaced with \(\exp(-r \gamma \chi)\). For large values of \(r\), the bulk of the contribution to \(E^\prime \exp(-r \gamma \chi)\) comes from the small values of \(\chi\), so the righthand side of (1.7) should be asymptotically equivalent to \(P^\prime(\Gamma | \chi = 0)\). If \(\Gamma\) is an event associated with the process \((S_{[(1-F^{(r))^{-1}]}(r, t \in \mathbb{R}_+)\), then it should translate in the limit into a similar event associated with \(X\). Besides, since \(\chi^{(r)}\) is the overshoot over level \(r\) by \(S_n\), we have that \(\chi^{(r)}/r\) is the overshoot over level 1 of the process \((S_{[(1-F^{(r))^{-1}]}(r, t \in \mathbb{R}_+)\). That the latter process converges to \(X\) suggests the conjecture that \(\chi\) should be the overshoot of \(X\) above level 1.

One is thus led to the conjecture that the conditional distribution of the process \((S_{[(1-F^{(r))^{-1}]}(r, t \in \mathbb{R}_+)\) given that \(\tau^{(r)} < \infty\) should converge to the conditional distribution of the process \((X(t \wedge \tau), t \in \mathbb{R}_+)\) given the event \(X(\tau) = 1\), where \(\tau = \inf\{t: X(t) \geq 1\}\). The main result of the paper confirms this conjecture. As a consequence, we have that if the distribution function \(F\) decays as \(x^{-\alpha}\), then, assuming \(X\) is defined on a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\), for \(B > 0\) and \(\varepsilon > 0\),

\[
\lim_{r \to \infty} P\left( \sup_{n \leq \tau^{(r)}}|S_n - B n^{1/\alpha}| < \varepsilon r|\tau^{(r)} < \infty\right) = P'\left(\sup_{t \leq \tau} X(t) - B t^{1/\alpha} | < \varepsilon |X(\tau) = 1\right),
\]

which can be regarded as a counterpart of (1.9).

Certainly, the above argument is by no means rigorous. To begin with, the process \(X\) “does not creep up”, i.e., it overshoots every level with probability one, see Bertoin \[4\], so the event \(\{X(\tau) = 1\}\) has zero probability, and the probability law of \((X(t \wedge \tau), t \in \mathbb{R}_+)\) conditioned on this event needs defining. We define it as the limit of the distributions of \((X(t \wedge \tau), t \in \mathbb{R}_+)\) given \(X(\tau) \leq 1 + \varepsilon\) as \(\varepsilon \to 0\). There still remains the issue of justifying the existence of the limit.

However, one can guess at the predictable measure of jumps of the process \(X\) “conditioned not to overshoot level 1”, which we denote \(\bar{X}\) and call “a Lévy process conditioned not to overshoot”, or, in short, “a non-overshooting Lévy process”. It appears as though the intensity of jumps of size \(x\) of \(\bar{X}\) at a point \(\bar{X}(t) = u\), where \(u < u + x < 1\), can be obtained by “conditioning” the intensity of jumps of \(X\) on the event that the overshoot of \(X\) over 1 is not greater than \(\varepsilon\). In other words, it should be given approximately by the product of the intensity of jumps of \(X\) from \(u\) to \(u + x\), which is \(\alpha x^{-\alpha-1}\), with the probability for \(X\) not to exceed level 1 by more than \(\varepsilon\) when starting at \(u + x\) over the probability that \(X\) does not overshoot 1 by more than \(\varepsilon\) starting at \(u\). As follows by the results of Dynkin \[9\], the probability for the process \(X\) not to overshoot a level \(y > 0\) by more than \(\varepsilon > 0\) is asymptotically equivalent to \((\sin \pi \alpha / (\pi (1 - \alpha))) y^{\alpha - 1} \varepsilon^{1 - \alpha}\) as \(\varepsilon \to 0\) (see also Rogozin \[21\], and more details are given in the proof of Theorem 3.1 below). Therefore, the intensity of jumps of \(\bar{X}\) should be \(\alpha x^{-\alpha-1}(1 - x/(1 - u))^{\alpha - 1}\). It is thus akin to the arcsine law, which is not surprising given that we are concerned, in effect, with distributions of infinite mean here.

In order to substantiate the loose argument we have been indulging in so far and provide proofs, we use a less direct line of attack than the one suggested by the above discussion. Two limit theorems are established: we prove that both the conditional laws of \((X(t \wedge \tau), t \in \mathbb{R}_+)\) given \(X(\tau) \leq 1 + \varepsilon\) and the conditional laws of \((S_{[(1-F^{(r))^{-1}]}(r, t \in \mathbb{R}_+)\) given \(\tau^{(r)} < \infty\) converge, as \(\varepsilon \to 0\) and \(r \to \infty\), respectively, to the law of \(\bar{X}\). Proofs of both convergences are
similar. First, we compute the predictable measures of jumps of the processes in question under the “conditional” measures and then apply results on weak convergence of semimartingales. The actual argument is more involved for the partial-sum processes so much so that we have to introduce an additional requirement on the distribution function \( F \). Since the approaches of Korshunov [18] play an important part, we also have to treat the nonlattice and lattice cases separately.

As it happens, the exposition in the paper is reversed as compared with the order in which we have first arrived at the results. We begin with a study of the process \( \tilde{X} \), which we do in Section 2. We define this process by postulating its predictable measure of jumps, prove its existence and uniqueness, and study some of its properties. In particular, we compute the moments of the time it takes \( \tilde{X} \) to reach level one and show that certain exponential moments of this random variable are finite. In Section 3 we prove that \( \tilde{X} \) can be obtained as a limit in distribution of the processes \( (X(t \land \tau), t \in \mathbb{R}_+) \) conditioned on \( X(\tau) \leq 1 + \varepsilon \). In Section 4 we establish the main result of the paper on the convergence in distribution of the processes \( (S_{[(1- F(r))]^{-1} t \land \tau(r)}/r, t \in \mathbb{R}_+) \) conditioned on the event that \( \tau(r) < \infty \). The appendix consists of three subsections. Subsection A.1 contains a proof of the convergence in distribution of the processes \( (S_{[(1- F(r))]^{-1} t}/r, t \in \mathbb{R}_+) \) to \( X \) under measure \( \mathbb{P}^* \) based on the semimartingale weak convergence theory. As a byproduct, we extend Dynkin’s result [9] on the limit in distribution of \( \chi^{(r)}/r \) to the case of random walks. Then, in Subsection A.2 we present a more complete version of the proof of two theorems from Korshunov [18], and Subsection A.3 contains a proof of some useful properties of slowly and regularly varying functions. We hope that the results in the appendix are also of interest in themselves. All results except those of Section 4 and of Subsection A.2 actually hold for \( \alpha \in (0,1) \) and not just for \( \alpha \in (1/2,1) \).

We conclude the introduction with a list of notation and conventions adopted in the paper. \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{R} \) denotes the set of real numbers, \( \mathcal{B}(\mathbb{R}) \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R} \), and \( \mathbb{R}_+ \) denotes the subset of \( \mathbb{R} \) of nonnegative reals. For \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \), \( x \wedge y = \min(x,y) \), \( x \vee y = \max(x,y) \), and \( x^- = -x \wedge 0 \). Recall also that \( \lfloor x \rfloor \) denotes the integer part of \( x \) and \( 1_\Gamma \) denotes the indicator-function of event \( \Gamma \). Two positive functions \( f(x) \) and \( g(x) \) of a real-valued argument are said to be asymptotically equivalent as \( x \to \infty \), which is written as \( f(x) \sim g(x) \) if \( \lim_{x \to \infty} f(x)/g(x) = 1 \). We write \( f(x) = O(g(x)) \) if \( f(x) \leq C g(x) \) for some \( C > 0 \) and for all \( x \) great enough. Integrals of the form \( \int^b_a \) are understood as \( \int[a,b] \) unless otherwise indicated. For \( x > 0 \) and \( y > 0 \), \( \mathcal{B}(x,y) \) denotes Euler’s beta function defined by \( \mathcal{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} \, du \).

We denote by \( \mathbb{D} \) the space of \( \mathbb{R} \)-valued right-continuous functions on \( \mathbb{R}_+ \) with left-hand limits. Its elements are denoted with lower-case bold-face Roman characters, e.g., \( x = (x(t), t \in \mathbb{R}_+) \); \( x(t-) \) denotes the left-hand limit of \( x \) at \( t \), \( \Delta x(t) = x(t) - x(t-) \) denotes the size of the jump at \( t \). The space \( \mathbb{D} \) is endowed with the Skorohod \( J_1 \)-topology, is equipped with the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{D}) \), and is metrised by a complete separable metric, see Ethier and Kurtz [13], Jacod and Shiryaev [17], and Liptser and Shiryaev [19] for the definition and properties. \( \mathbb{D}_1 \) denotes the subset of \( \mathbb{D} \) of increasing functions starting at zero equipped with the subspace topology. All stochastic processes encountered in this paper have trajectories in \( \mathbb{D} \) and are considered as random elements of \( (\mathbb{D}, \mathcal{B}(\mathbb{D})) \). Weak convergence of probability measures on \( \mathbb{D} \) and convergence in distribution of stochastic processes are understood with respect to the Skorohod topology.

We recall that a filtered probability space, or a stochastic basis, \( (\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}) \) is defined as a probability space \( (\Omega, \mathcal{F}, \mathbf{P}) \) endowed with an increasing right-continuous flow \( \mathbf{F} = (\mathcal{F}(t), t \in \mathbb{R}_+) \) of sub-\( \sigma \)-algebras of \( \mathcal{F} \). Such a flow is also referred to as a filtration. We will assume without further mention that all \( \sigma \)-algebras we consider are complete with respect to the corresponding probability measure. For background on the general theory of stochastic processes, the reader is referred to
and let $\nu$. There exists a stochastic process $\tilde{X} = (\tilde{X}(t), t \in \mathbb{R}_+)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with the following properties:

1. $\tilde{X}$ is a pure-jump semimartingale with $\tilde{X}(0) = 0$,

2. $(\nu(\tilde{X}; dt, dx))$ is the $\tilde{\mathbb{F}}$-predictable measure of jumps of $\tilde{X}$.

The distribution of $\tilde{X}$ is specified uniquely. In addition, $\tilde{X}$ has increasing trajectories a.s., $\tilde{X}(t) \in [0, 1]$ a.s. for $t \in \mathbb{R}_+$, and $\tilde{X}(t) = 1$ for all $t$ large enough a.s.

Moreover, if $\bar{\tau} = \inf\{t \geq 0 : \tilde{X}(t) = 1\}$, then, for $n \in \mathbb{N}$,

$$\tilde{E}e^{-\bar{\tau}} = n! \prod_{k=1}^{n} \left( \int_{0}^{1} (1 - x^{\alpha k})^{\alpha x^{-\alpha - 1}}(1 - x)^{-\alpha - 1} dx \right)^{-1},$$

and $\tilde{E}e^{-c\bar{\tau}} \leq 1/(1 - c\tilde{E}\bar{\tau})$ when $c < 1/\tilde{E}\bar{\tau}$.

**Proof.** We start by showing that the process $\tilde{X}$ exists. Let $L = (L(t), t \in \mathbb{R}_+)$ be an increasing pure-jump Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy measure $\Pi(dx) = 1_{\{x < 0, 1\}}(1 - x)^{\alpha x^{-\alpha - 1}} dx$ and $L(0) = 0$. Assume also that $\mathcal{F}$ coincides with the $\sigma$-algebra generated by the $L(t), t \in \mathbb{R}_+$. Denote $\mathcal{F}_L(t)$ the $\sigma$-algebra on $\tilde{\Omega}$ generated by the $L(s), s \leq t$, and let $\mathbf{F}_L = (\mathcal{F}_L(t), t \in \mathbb{R}_+)$. The latter flow is right-continuous by Bertoin [4, Proposition I.1.2]. Let process $\tilde{X} = (\tilde{X}(t), t \in \mathbb{R}_+)$ solve the Doléans equation

$$\tilde{X}(t) = 1 - \int_{0}^{t} \tilde{X}(s-) dL(s). \quad (2.2)$$

The process $\tilde{X}$ is well defined (see Liptser and Shiryaev [19] for details), is a decreasing pure-jump process with $\tilde{X}(0) = 1$, and can be explicitly written as

$$\tilde{X}(t) = \prod_{0 < s \leq t} (1 - \Delta L(s)), \quad (2.3)$$
where the product is taken over the jumps of \( L \) and “the empty product” is taken to be equal to 1. Because \( \Delta L(t) \in (0, 1) \), \( \sum_{s \leq t} \Delta L(s) = L(t) < \infty \) and \( L(t) \to \infty \) as \( t \to \infty \) a.s., we have that \( \bar{X}(t) > 0 \) for all \( t \in \mathbb{R}_+ \) and \( \bar{X}(t) \to 0 \) as \( t \to \infty \), so we define \( \bar{X}(\infty) = 0 \). Let

\[
\sigma(t) = \inf\{s : \int_0^s \bar{X}(q)^\alpha dq > t\},
\]

(2.4)

where \( \sigma(t) = \infty \) if \( \int_0^\infty \bar{X}(q)^\alpha dq \leq t \). The latter integral is finite a.s. since

\[
\int_0^\infty \bar{X}(q)^\alpha dq \leq 1 + \sum_{n=1}^\infty \prod_{i=1}^n \psi_i,
\]

where \( \psi_i = \prod_{1<i<s} (1 - \Delta L(s))^\alpha \). (Note that the \( \psi_i \) are i.i.d., \( \psi_1 \leq 1 \) and \( \bar{P}(\psi_1 < 1) > 0 \) so that \( \bar{E}\psi_1 < 1 \). Further, \( \sigma(t) \) is an \( F_L \)-stopping time. Also, for \( t < \int_0^\infty \bar{X}(q)^\alpha dq \), it is an absolutely continuous and strictly increasing function of \( t \) with inverse \( \sigma^{-1}(t) = \int_0^t \bar{X}(q)^\alpha dq \) and

\[
\sigma(t) = \int_0^t \bar{X}(\sigma(s))^{-\alpha} ds.
\]

(2.5)

Also, \( \sigma(t) > t \) for \( t < \int_0^\infty \bar{X}(q)^\alpha dq \).

We define the process \( \bar{X} \) by

\[
\bar{X}(t) = 1 - \bar{X}(\sigma(t)).
\]

(2.6)

This is clearly an increasing pure-jump process with \( \bar{X}(0) = 0 \) and \( \lim_{t \to \infty} \bar{X}(t) = 1 \). We evaluate its predictable measure of jumps. By (2.2) and the form of the Lévy measure of \( L \), the process \( \bar{X} \) is \( F_L \)-adapted with predictable measure of jumps

\[
\tilde{\nu}([0,t], G) = \int_0^t \int_{G \setminus \{0\}} 1_{\{\bar{X}(s) < x < 0\}} \left( 1 + \frac{x}{\bar{X}(s)} \right)^{-\alpha-1} \bar{X}(s)^\alpha \alpha(-x)^{-\alpha-1} dx ds, G \in \mathcal{B}(\mathbb{R}).
\]

(2.7)

Since \( \sigma(t) \) is an \( F_L \)-stopping time, the \( \sigma \)-algebras \( \bar{F}(t) = F_L(\sigma(t)) \) are well defined and, by (2.6), the \( \bar{X}(t) \) are \( \bar{F}(t) \)-measurable for all \( t \in \mathbb{R}_+ \). Let \( \bar{F} = (\bar{F}(t), t \in \mathbb{R}_+) \). It is a right-continuous flow by the facts that the flow \( F_L \) is right-continuous and \( \sigma(t) \) is a right-continuous function of \( t \). Note also that \( \Delta \bar{X}(t) = 0 \) for \( t \geq \int_0^\infty \bar{X}(q)^\alpha dq \) because of the convergence \( \bar{X}(t) \to \bar{X}(\infty) \) as \( t \to \infty \) so that \( \sigma(t) < \infty \) if \( \Delta \bar{X}(t) > 0 \). Equations (2.5), (2.6), and (2.7) imply that the \( \bar{F} \)-predictable measure of jumps of \( \bar{X} \) is of the form

\[
\tilde{\nu}([0,t], G) = \int_0^t \int_{\mathbb{R}} 1_{\{x \in G\}} 1_{\{\sigma(s) < \infty\}} \tilde{\nu}(d\sigma(s), dx)
\]

(2.8)

\[
= \int_0^t \int_{\mathbb{R}} 1_{\{0 < x < 1 - \bar{X}(s)\}} \left( 1 - \frac{x}{1 - \bar{X}(s)} \right)^{\alpha-1} \alpha x^{-\alpha-1} dx ds, G \in \mathcal{B}(\mathbb{R}),
\]

(2.9)

as required.
We now assume that \( \hat{X} \) is a process as in the hypotheses of the theorem. Let \( \mu \) denote the measure of jumps of \( \hat{X} \), i.e.,
\[
\mu([0,t],G) = \sum_{0<s\leq t} 1_{\{\Delta \hat{X}(s) \in G\}}.
\]
Since \( \int_0^t \int_\mathbb{R} (1_{\{\hat{X}(s)\geq 1-x\}} + 1_{\{x\leq 0\}}) \nu(\hat{X};ds,dx) = 0 \), it follows that \( \int_0^t \int_\mathbb{R} (1_{\{\hat{X}(s-)\geq 1-x\}} + 1_{\{x\leq 0\}}) \mu(ds,dx) = 0 \) a.s., so \( 0 < \Delta \hat{X}(s) < 1 - \hat{X}(s-) \) a.s., in particular, \( \hat{X} \) is increasing and \( \hat{X}(t) \in [0,1] \) a.s. We also note that \( \lim_{t \to \infty} \hat{X}(t) = 1 \) a.s. To see the latter, denote, for \( \varepsilon \in (0,1) \), \( \tau_\varepsilon = \inf\{t : \hat{X}(t) \geq 1 - \varepsilon\} \). The \( \hat{F} \)-compensator of \( \hat{X} \) is the process \( (\int_0^t \int_\mathbb{R} x \nu(ds,dx), t \in \mathbb{R}_+) \).

Note that by (2.11) \( \int_0^t \int_\mathbb{R} x \nu(ds,dx) = \alpha B(\alpha, 1 - \alpha) \int_0^t (1 - \hat{X}(s))^{1-\alpha} ds \). Then, for \( t > 0 \), by the fact that \( \hat{X} \) is a bounded process,
\[
\hat{E} \hat{X}(t \wedge \tau_\varepsilon) = \alpha B(\alpha, 1 - \alpha) \hat{E} \int_0^{t \wedge \tau_\varepsilon} (1 - \hat{X}(s))^{1-\alpha} ds \geq \alpha B(\alpha, 1 - \alpha) \varepsilon^{1-\alpha} \hat{E}(t \wedge \tau_\varepsilon).
\]

It follows that \( \hat{E} \tau_\varepsilon < \infty \), so \( \tau_\varepsilon < \infty \) a.s., which proves the claim.

In order to prove that the distribution of \( \hat{X} \) is specified uniquely, we reverse the procedure we employed when establishing existence. In what follows, we reuse the earlier notation. Motivated by (2.9), according to which \( \hat{X}(t) = 1 - \hat{X}(\sigma^{-1}(t)) \), and noting that by (2.5) \( \sigma(t) = \int_0^t (1 - \hat{X}(s))^{-\alpha} ds \), we define \( \sigma^{-1}(t) \) in terms of \( \hat{X} \) as follows:
\[
\sigma^{-1}(t) = \inf\{s : \int_0^s (1 - \hat{X}(q))^{-\alpha} dq > t\}, \quad (2.10)
\]
where \( \tilde{\tau} = \inf\{s : \hat{X}(s) = 1\} \leq \infty \) and \( \sigma^{-1}(t) = \tilde{\tau} \) if \( \int_0^{\tilde{\tau}} (1 - \hat{X}(q))^{-\alpha} dq \leq t \). Note that \( \sigma^{-1}(t) \), as a function of \( t \), is increasing and right-continuous. Moreover, for \( t < \int_0^{\tilde{\tau}} (1 - \hat{X}(q))^{-\alpha} dq \), it is strictly increasing and absolutely continuous with respect to Lebesgue measure, and
\[
\sigma^{-1}(t) = \int_0^t (1 - \hat{X}(\sigma^{-1}(s)))^{\alpha} ds, \quad (2.11)
\]
where we define \( \hat{X}(\infty) = 1 \).

Let
\[
\hat{X}(t) = 1 - \hat{X}(\sigma^{-1}(t)). \quad (2.12)
\]
As the random variable \( \sigma^{-1}(t) \) is an \( \hat{F} \)-stopping time, on the one hand, and a right-continuous function of \( t \), on the other hand, we have that the \( \sigma \)-algebras \( \hat{F}(t) = \hat{F}(\sigma^{-1}(t)) \) are well defined, the flow \( \hat{F} = (\hat{F}(t), t \in \mathbb{R}_+) \) is right-continuous, and the process \( \hat{X} \) is \( \hat{F} \)-adapted. Note also that \( \lim_{t \to \infty} \hat{X}(t) = \hat{X}(\tilde{\tau}) \). This has been proved for \( \tilde{\tau} = \infty \). If \( \tilde{\tau} < \infty \), then \( \hat{E} \Delta \hat{X}(\tilde{\tau}) = \hat{E} \sum_{s>0} \Delta \hat{X}(s) 1_{\{\hat{X}(s-) < 1, \Delta \hat{X}(s) = 1 - \hat{X}(s-)\}} = 0 \). It follows that the process \( \hat{X} \) is continuous at \( t = \int_0^{\tilde{\tau}} (1 - \hat{X}(q))^{-\alpha} dq \) if \( \tilde{\tau} < \infty \) and \( \hat{X}(t) \to 0 \) a.s. on the set where \( \tilde{\tau} = \infty \). Arguing in analogy with (2.3), we conclude that the predictable measure of jumps of \( \hat{X} \) is given by (2.7).

The process
\[
L(t) = -\int_0^t \frac{d\hat{X}(s)}{\hat{X}(s-)} = -\int_0^{t \wedge \tilde{\tau}} \frac{d\hat{X}(s)}{\hat{X}(s-)} = \int_0^{\sigma^{-1}(t) \wedge \tilde{\tau}} \frac{d\hat{X}(s)}{1 - \hat{X}(s-)} \quad (2.13)
\]
is well defined and finite-valued. To see the latter, note that if $\tilde{t} = \infty$, then $\sigma^{-1}(t) < \tilde{\tau}$, so the denominator on the rightmost side of (2.13) is bounded away from zero. Suppose, $\tilde{t} < \infty$. Since the $\mathcal{F}$-compensator of $X$ is the process $(\alpha \mathbb{B}(\alpha, 1 - \alpha) \int_0^t (1 - X(s))^{-\alpha} ds, t \in \mathbb{R}_+)$, the $\mathcal{F}$-compensator of $(\int_0^{\tau_n} d\tilde{X}(s)/(1 - \tilde{X}(s_-)), t \in \mathbb{R}_+)$ is the process $(\alpha \mathbb{B}(\alpha, 1 - \alpha) \int_0^{\tau_n} (1 - \tilde{X}(s))^{-\alpha} ds, t \in \mathbb{R}_+)$.

Since $\tau_n \to \tilde{\tau}$ as $\varepsilon \to 0$, by Liptser and Shiryaev [19, Theorem 2.6.1] $\tilde{\mathbb{P}}$-a.s.

$$\{ \int_0^{\tilde{\tau}} (1 - \tilde{X}(s))^{-\alpha} ds < \infty \} \subset \{ \int_0^{\tilde{\tau}} \frac{d\tilde{X}(s)}{1 - \tilde{X}(s_-)} < \infty \},$$

so $L(t) < \infty$ $\tilde{\mathbb{P}}$-a.s. on the set $\tilde{t} < \infty$.

By (2.7) and (2.13), the process $L$ is $\tilde{\mathbb{F}}$-adapted with predictable measure of jumps $\Pi(dx)ds$. Thus, $L$ is a Lévy process, so its distribution is specified uniquely. By (2.13) the process $\tilde{X}$ solves the Doléans equation (2.2), so its distribution is specified uniquely too (by (2.3)). As we have seen, (2.3) implies that $X(t) > 0$ for all $t$ and $\lim_{t \to \infty} X(t) = 0$ a.s. Hence, by (2.12) $\tilde{\tau} = \lim_{t \to \infty} \sigma^{-1}(t)$.

In addition, $\sigma^{-1}(t) < \tilde{\tau}$, so $\int_0^{\tilde{\tau}} (1 - X(q))^{-\alpha} dq = \infty$. Therefore by (2.11) and (2.12)

$$\tilde{\tau} = \int_0^{\infty} \tilde{X}(q)^{\alpha} dq.$$ (2.14)

In addition, (2.11) and (2.12) also imply that $\sigma(\sigma^{-1}(t)) = t$, where $\sigma(t)$ is defined by (2.4). Another application of (2.12) shows that equation (2.6) holds for $t < \tilde{\tau}$. By (2.11), $\sigma(\tilde{\tau}) = \infty$, so (2.6) holds for $t \geq \tilde{\tau}$. Thus, the distribution of $\tilde{X}$ is uniquely specified by the distribution of $\tilde{X}$. The proof of the uniqueness of $X$ is complete.

We now establish the formula for $\mathbb{E}\tau^n$ in the statement of the lemma. By (2.14),

$$\mathbb{E}\tau^n = \int_0^{\infty} \ldots \int_0^{\infty} \mathbb{E}\tilde{X}(q_1) \alpha \ldots \tilde{X}(q_n) \alpha dq_1 \ldots dq_n = n! \int_0^{\infty} \ldots \int_0^{\infty} \mathbb{E}\tilde{X}(q_1) \alpha \ldots \tilde{X}(q_n) \alpha dq_1 \ldots dq_n$$

$$\mathbb{E}\exp \left( \alpha \sum_{k=1}^n (n - k + 1)(\ln \tilde{X}(q_k) - \ln \tilde{X}(q_{k-1})) \right) dq_1 \ldots dq_n,$$ (2.15)

where $q_0 = 0$. By (2.3), $\ln \tilde{X}(t) = \sum_{0 < s \leq t} \ln(1 - \Delta L(s))$, so $\ln \tilde{X}$ is a pure-jump Lévy process with Lévy measure $1_{z \leq 0} \alpha e^{x\alpha} (1 - e^{-\alpha_1^\alpha - 1}) dx$. Therefore, for $u > 0$,

$$\mathbb{E}e^{u \ln \tilde{X}(q)} = \exp \left( q \int_{-\infty}^0 (e^{ux} - 1) \alpha e^{x\alpha} (1 - e^{-\alpha_1^\alpha - 1}) dx \right) = \exp \left( -q \int_0^1 (1 - x^u) \alpha x^{\alpha_1^\alpha - 1} (1 - x)^{-\alpha_1^\alpha - 1} dx \right),$$

so, by (2.15),

$$\mathbb{E}\tau^n = n! \int_0^{\infty} \ldots \int_0^{\infty} \prod_{k=1}^n \exp (-1 - q_{k-1}) \int_0^1 (1 - x^{\alpha(n-k+1)} \alpha x^{\alpha_1^\alpha - 1} (1 - x)^{-\alpha_1^\alpha - 1} dx) dq_1 \ldots dq_n$$

$$= n! \prod_{k=1}^n \int_0^{\infty} \exp (-q \int_0^1 (1 - x^{ak}) \alpha x^{\alpha_1^\alpha - 1} (1 - x)^{-\alpha_1^\alpha - 1} dx) dq = n! \prod_{k=1}^n \int_0^1 \left( (1 - x^{ak}) \alpha x^{\alpha_1^\alpha - 1} (1 - x)^{-\alpha_1^\alpha - 1} dx \right)^{-1},$$ (2.16)
which is the required result. In particular, $\bar{\tau} < \infty$ $\bar{\mathbb{P}}$-a.s.

We show that $\bar{\tau}$ has a light-tailed distribution. By (2.16),

$$\mathbb{E}^{\bar{\tau}_n} \leq n! \left( \int_0^1 (1 - x^\alpha)ax^{\alpha-1}(1-x)^{-\alpha-1} \, dx \right)^{-n} = n!(\mathbb{E}^{\bar{\tau}})^n.$$

The bound $\mathbb{E}\exp(c\bar{\tau}) \leq 1/(1 - c\mathbb{E}\bar{\tau})$ when $c\mathbb{E}\bar{\tau} < 1$ follows by the Taylor expansion for the exponential function.

Remark 2.1. A slightly more intricate argument shows that (2.16) implies that $\mathbb{E}e^{c_n\bar{\tau}} < \infty$ where $c_n = (\mathbb{E}^{\bar{\tau}_n}/n!)^{-1/n}$, for any $n$.

## 3 Convergence of conditioned Lévy processes

Recall that $X = (X(t), t \in \mathbb{R}_+)$ denotes an increasing pure-jump stable Lévy process starting at zero with Lévy measure $\alpha x^{-\alpha-1} \, dx$, where $\alpha \in (0, 1)$. We assume that $X$ is defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. We also denote, as in the introduction,

$$\tau = \inf\{t : X(t) \geq 1\}, \quad (3.1)$$

and let $\tilde{X}$ denote the process $X$ stopped at $\tau$: $\tilde{X}(t) = X(t \wedge \tau)$.

**Theorem 3.1.** The conditional laws of $\tilde{X}$ given the events $X(\tau) \leq 1 + \varepsilon$, considered as distributions on $\mathbb{D}$, weakly converge as $\varepsilon \downarrow 0$ to the law of $\tilde{X}$.

**Remark 3.1.** As a consequence, the distribution of $\tilde{X}$ can be interpreted as the distribution of $\tilde{X}$ conditioned on the event $X(\tau) = 1$, which justifies calling $\tilde{X}$ a Lévy process conditioned not to overshoot.

The proof of this theorem as well as the proof of Theorem 4.1 in Section 4 will be obtained by an application of the following result, which is a particular case of Theorem IX.3.21 in Jacod and Shiryaev [17].

**Lemma 3.1.** Consider a sequence $X^{(n)}$ of $\mathbb{R}$-valued pure-jump semimartingales with predictable measures of jumps $\nu^{(n)}(dt, dx)$ defined on filtered probability spaces $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)}, \mathbb{P}^{(n)})$. Suppose that the $X^{(n)}$ are of locally bounded variation, i.e., $\int_0^t \int_{\mathbb{R}} |x| \nu^{(n)}(ds, dx) < \infty$ for $t \in \mathbb{R}_+$. Let an $\mathbb{R}_+$-valued function $K(y; G)$, where $y \in \mathbb{R}$ and $G \in \mathcal{B}(\mathbb{R})$, be Borel-measurable in $y$ and be a $\sigma$-finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ in $G$ such that $K(y; \{0\}) = 0$. Suppose that the following conditions hold:

1. $\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |x| \, K(y; dx) < \infty$,

2. for an arbitrary $\mathbb{R}$-valued continuous function $g(x)$, $x \in \mathbb{R}$, such that $|g(x)| \leq M|x|$, $x \in \mathbb{R}$, with some $M > 0$, the function $\int_{\mathbb{R}} g(x) \, K(y; dx)$ is continuous in $y$,

3. for arbitrary $\delta > 0$, $t > 0$, and an $\mathbb{R}$-valued continuous function $g(x)$, $x \in \mathbb{R}$, such that $|g(x)| \leq M|x|$, $x \in \mathbb{R}$, with some $M > 0$,

$$\lim_{n \to \infty} \mathbb{P}^{(n)} \left( \left| \int_0^t \int_{\mathbb{R}} g(x) \nu^{(n)}(ds, dx) - \int_0^t \int_{\mathbb{R}} g(x) \, K(X^{(n)}(s); dx) \, ds \right| > \delta \right) = 0,$$
4. the $X^{(n)}(0)$ converge in distribution to a random variable $X_0$ as $n \to \infty$,

5. there exists at most one pure-jump semimartingale $X = (X(t), t \in \mathbb{R}_+)$ with initial condition $X_0$ and with predictable measure of jumps $\nu(dt, dx) = K(X(t); dx) dt$.

Then the $X^{(n)}$ converge in distribution to $X$.

Lemma 3.1 will be applied with

$$K(y; G) = \int_{G \setminus \{0\}} 1_{\{0 < x < 1 - y\}} \left(1 - \frac{x}{1 - y}\right)^{\alpha - 1} \alpha x^{-\alpha - 1} dx \tag{3.2}$$

when $y \in (0, 1)$ and $K(y; G) = 0$ otherwise.

**Lemma 3.2.** The function $K$ satisfies conditions 1 and 2 of Lemma 3.1.

**Proof.** We have for $g(x)$ as in condition 2 of Lemma 3.1 and for $y \in (0, 1),$

$$\int_{\mathbb{R}} g(x)K(y; dx) = (1 - y)^{-\alpha} \int_{0}^{1} g(x(1 - y))(1 - x)^{\alpha - 1} \alpha x^{-\alpha - 1} dx. \tag{3.3}$$

Thus, $\int_{\mathbb{R}} |x|K(y; dx) \leq \alpha B(\alpha, 1 - \alpha)$, so condition 1 of Lemma 3.1 holds. Condition 2 follows by the assumption that $|g(x)| \leq M|x|$, continuity of $g(x)$, and Lebesgue’s bounded convergence theorem.

**Proof of Theorem 3.1.** Let $\mathcal{F}'(t)$ denote the $\sigma$-algebra generated by the random variables $X(s), s \leq t$, $\hat{\mathcal{F}}(t) = \mathcal{F}'(t \wedge \tau)$, and $\hat{F} = (\hat{\mathcal{F}}(t), t \in \mathbb{R}_+)$. We introduce the change of measure on $\mathcal{F}'(\tau)$ by letting

$$Q'(\varepsilon)(\Gamma) = \frac{P'(\Gamma \cap \{X(\tau) \leq 1 + \varepsilon\})}{P'(X(\tau) \leq 1 + \varepsilon)}, \quad \Gamma \in \mathcal{F}'(\tau) \tag{3.4}$$

We need to prove that the distributions of $\hat{X}$ under $Q'(\varepsilon)$ weakly converge, as $\varepsilon \to 0$, to the distribution of $\hat{X}$. To this end, we will apply Lemma 3.1 with the function $K$ from (3.2). Condition 4 is obviously met. Conditions 1 and 2 of Lemma 3.1 hold by Lemma 3.2, condition 5 holds by Theorem 2.1. We thus need to verify condition 3 of Lemma 3.1.

General tools for calculating $\hat{\nu}'(\varepsilon)$ are provided in Jacod and Shiryaev [17] and Liptser and Shiryaev [19]. We refer the interested reader to these sources for substantiation of the subsequent argument. Let $\hat{Z} = (\hat{Z}(t), t \in \mathbb{R}_+)$ be the density process of $Q'(\varepsilon)$ with respect to $P'$ (we omit $\varepsilon$ in some of the notation to make for easier reading). It is defined by

$$\hat{Z}(t) = \mathbb{E}' \left( \frac{dQ'(\varepsilon)}{dP'} \bigg| \hat{\mathcal{F}}(t) \right) = \frac{P'(X(\tau) \leq 1 + \varepsilon| \mathcal{F}'(t \wedge \tau))}{P'(X(\tau) \leq 1 + \varepsilon)} \tag{3.5}$$

Introducing

$$\hat{u}'(\varepsilon)(y) = P'(X(\tau_y) \leq y + \varepsilon) \tag{3.5}$$

where $\tau_y = \inf\{t : X(t) \geq y\}$, we have, by the fact that $X$ has independent and stationary increments under $P'$,

$$\hat{Z}(t) = 1_{\{\tau \leq t\}} \frac{1_{\{X(\tau) \leq 1 + \varepsilon\}}}{\hat{u}'(\varepsilon)(1)} + 1_{\{\tau > t\}} \frac{\hat{u}'(\varepsilon)(1 - X(t))}{\hat{u}'(\varepsilon)(1)}. \tag{3.5}$$
There exists a version of $\tilde{\nu}^\varepsilon$ of the form
\begin{align}
\tilde{\nu}^\varepsilon(dt, dx) = \tilde{Y}(t, x) \tilde{\nu}'(dt, dx),
\end{align}
where $\tilde{\nu}'$ is the $\tilde{F}$-predictable measure of jumps of $\tilde{X}$ under $P'$ and
\begin{align}
\tilde{Y}(t, x) = \frac{1_{\{\tilde{Z}(t-)>0\}}}{\tilde{Z}(t-)} M^{p'}_{\mu} (\tilde{Z}|\tilde{P})(t, x).
\end{align}

We recall that $\tilde{P}$ denotes the $\sigma$-algebra on $\Omega' \times \mathbb{R}_+ \times \mathbb{R}$ which is the product of the predictable $\sigma$-algebra on $\Omega' \times \mathbb{R}_+$ associated with $\tilde{F}$ and the Borel $\sigma$-algebra on $\mathbb{R}$, and $M^{p'}_{\mu}$ denotes the measure on $\Omega' \times \mathbb{R}_+ \times \mathbb{R}$ defined by the equality
\begin{align}
M^{p'}_{\mu} f = E' \int_0^\infty \int_{\mathbb{R}} f(\omega, t, x) \tilde{\mu}(dt, dx)
\end{align}
for $f \geq 0$, where $\tilde{\mu}$ is the measure of jumps of $\tilde{X}$, i.e.,
\begin{align}
\tilde{\mu}([0, t], G) = \sum_{0<s \leq t \wedge \tau} 1_{\{\Delta X(s) \in G\}}.
\end{align}

Accordingly, $M^{p'}_{\mu} (\tilde{Z}|\tilde{P})(t, x)$ is the conditional expectation of $\tilde{Z}$ with respect to $\tilde{P}$, i.e., it is a $\tilde{P}$-measurable function $g(\omega, t, x)$ such that $M^{p'}_{\mu} h\tilde{Z} = M^{p'}_{\mu} hg$ for all nonnegative $\tilde{P}$-measurable $h$.

We will work further on with the version of the predictable measure of jumps given identically by (3.6). By (3.5) and (3.8),
\begin{align}
M^{p'}_{\mu} h\tilde{Z} &= E' \int_0^\infty \int_{\mathbb{R}} h(\omega, t, x) \left( 1\{\tau \leq t\} \frac{1_{\{X(\tau) \leq 1+\varepsilon\}}}{\tilde{u}^{(\varepsilon)}(1)} + 1\{\tau > t\} \frac{\tilde{u}^{(\varepsilon)}(1-X(t))}{\tilde{u}^{(\varepsilon)}(1)} \right) \tilde{\mu}(dt, dx)
\end{align}
\begin{align}
= E' \int_0^\infty \int_{\mathbb{R}} h(\omega, t, x) \left( 1\{X(t-)<1\} 1\{X(t-)+x \leq 1+\varepsilon\} \frac{1_{\{X(t-)+x \leq 1+\varepsilon\}}}{\tilde{u}^{(\varepsilon)}(1)} 
\right.
\end{align}
\begin{align}
+ \left. 1\{X(t-)+x<1\} \frac{\tilde{u}^{(\varepsilon)}(1-X(t-)-x)}{\tilde{u}^{(\varepsilon)}(1)} \right) \tilde{\mu}(dt, dx).
\end{align}

Hence,
\begin{align}
M^{p'}_{\mu} (\tilde{Z}|\tilde{P})(t, x) = \frac{1}{\tilde{u}^{(\varepsilon)}(1)} \left( 1\{X(t-)<1\} 1\{1 \leq X(t-)+x \leq 1+\varepsilon\} + 1\{X(t-)+x<1\} \tilde{u}^{(\varepsilon)}(1-X(t-)-x) \right) 1\{x > 0\}.
\end{align}

On noting that
\begin{align}
\tilde{\nu}'([0, t], dx) = \alpha x^{-\alpha-1} dx \ t \wedge \tau,
\end{align}
we conclude by (3.5), (3.6), and (3.7), that
\begin{align}
\tilde{\nu}'(\varepsilon)(dt, dx) = \frac{1\{x > 0\}}{\tilde{u}^{(\varepsilon)}(1-X(t))} \left( 1\{X(t)<1\} 1\{1 \leq X(t)+x \leq 1+\varepsilon\} 
\right.
\end{align}
\begin{align}
+ \left. 1\{X(t)+x<1\} \tilde{u}^{(\varepsilon)}(1-X(t)-x) \alpha x^{-\alpha-1} dx dt \right). (3.9)
\end{align}
Recall, see Dynkin [9, Theorem 6] or Rogozin [21, Theorem 7], that
\[ \tilde{u}^\varepsilon(y) = \Phi_\alpha \left( \frac{\varepsilon}{y} \right), \] (3.10)
where
\[ \Phi_\alpha(y) = \frac{\sin \pi \alpha}{\pi} \int_0^y u^{-\alpha}(1 + u)^{-1} du. \]

We will need the easily verified properties that
\[ \lim_{y \to 0} y^{\alpha-1} \Phi_\alpha(y) = \frac{\sin \pi \alpha}{\pi(1 - \alpha)} \quad \text{and} \quad \sup_{y \geq 1} \Phi_\alpha(y) < \infty. \] (3.11)

Substituting (3.10) into (3.9) yields
\[ \tilde{\nu}^\varepsilon(dt, dx) = \frac{1\{x>0\}}{\Phi_\alpha(\varepsilon(1 - X(t))^{-1})} \left( 1\{X(t)<1\} 1\{1 \leq X(t) + x \leq 1 + \varepsilon\} + 1\{X(t)+x<1\} \Phi_\alpha(\varepsilon(1 - X(t) - x)^{-1}) \right) x^{-\alpha-1} \quad dx \quad dt. \] (3.12)

Condition 3 of Lemma 5.1 clearly holds if, for an arbitrary continuous function \( g(x) \) with \( |g(x)| \leq M|x| \),
\[ \lim_{\varepsilon \to 0} \sup_{\omega \in \Omega} \int_0^t \int_0^1 g(x) \tilde{\nu}^\varepsilon(ds, dx) - \int_0^t \int_0^1 g(x) K(\tilde{X}(t); dx) ds = 0. \] (3.13)

We proceed with a proof of (3.13). Let
\[ \tilde{\nu}_1^\varepsilon(dt, dx) = \frac{1\{x>0\}}{\Phi_\alpha(\varepsilon(1 - X(t))^{-1})} 1\{X(t)<1\} 1\{1 \leq X(t) + x \leq 1 + \varepsilon\} x^{-\alpha-1} \quad dx \quad dt, \] (3.14a)
\[ \tilde{\nu}_2^\varepsilon(dt, dx) = \frac{1\{x>0\}}{\Phi_\alpha(\varepsilon(1 - X(t))^{-1})} 1\{X(t)+x<1\} \Phi_\alpha(\varepsilon(1 - X(t) - x)^{-1}) x^{-\alpha-1} \quad dx \quad dt. \] (3.14b)

For \( z \in (0, 1) \), let
\[ R_1(z) = \frac{(1 - z)^{1-\alpha}}{\Phi_\alpha(\varepsilon(1 - z)^{-1})} \left[ \left( 1 + \frac{\varepsilon}{1 - z} \right)^{1-\alpha} - 1 \right]. \]

Due to the monotonicity of \( \Phi_\alpha \), if \( \varepsilon \geq 1 - z \), then
\[ R_1(z) \leq \frac{2^{1-\alpha}}{\Phi_\alpha(1)} \varepsilon^{1-\alpha}, \]
and if \( 0 < \varepsilon < 1 - z \), then
\[ R_1(z) \leq \frac{\tilde{C}_2}{\tilde{C}_1} \left( \frac{\varepsilon}{1 - z} \right)^{\alpha} (1 - z)^{1-\alpha} \leq \frac{\tilde{C}_2}{\tilde{C}_1} \varepsilon^{\alpha\wedge(1-\alpha)} \]
where \( \tilde{C}_1 = \inf_{0 < y \leq 1} y^{\alpha-1} \Phi_\alpha(y) > 0 \) and \( \tilde{C}_2 = \sup_{0 < y \leq 1} ((1 + y)^{1-\alpha} - 1)/y < \infty \). So, for any \( \varepsilon \in (0, 1) \),
\[ \int_0^t \int_0^\infty x \tilde{\nu}_1^\varepsilon(ds, dx) = \frac{\alpha}{1 - \alpha} \int_0^t 1\{X(s)<1\} R_1(X(s)) \; ds \leq \frac{\alpha}{1 - \alpha} \left[ \frac{2^{1-\alpha}}{\Phi_\alpha(1)} \varepsilon^{1-\alpha} + \frac{\tilde{C}_2}{\tilde{C}_1} \varepsilon^{\alpha\wedge(1-\alpha)} \right]. \] (3.15)
where the rightmost side in (3.15) tends to zero as \( \varepsilon \to 0 \). We conclude that
\[
\lim_{\varepsilon \to 0} \sup_{\omega \in \Omega'} \int_0^t \int_0^\infty x \tilde{\nu}_1^{(\varepsilon)}(ds, dx) = 0.
\] (3.16)

We now turn to the term \( \tilde{\nu}_2^{(\varepsilon)} \). For any continuous function \( g(x) \) with \( |g(x)| \leq M|x| \), we have
\[
M_1 := \left| \int_0^t \int_0^\infty g(x) \tilde{\nu}_2^{(\varepsilon)}(ds, dx) - \int_0^t \int_0^1 g(x) K(\hat{X}(s); dx) ds \right| \leq M \int_0^t R_2(X(s)) ds
\] (3.17)
where, for \( z \in [0, 1) \),
\[
R_2(z) = \int_0^{1/y_0} 1_{\{z+x<1\}} \left| \frac{\Phi_\alpha(\varepsilon(1-z-x)^{-1})}{\Phi_\alpha(\varepsilon(1-z)^{-1})} - \left( 1 - \frac{x}{1-z} \right)^{\alpha-1} \right| \alpha x^{-\alpha} dx.
\] (3.18)

Using the change of variables \( y = x/(1-z) \), we obtain
\[
R_2(z) = (1-z)^{1-\alpha} \int_0^{1/y_0} R_3(z,y) dy
\] (3.19)
where
\[
R_3(z,y) = \left| \frac{\Phi_\alpha(\varepsilon(1-z)^{-1}(1-y)^{-1})}{\Phi_\alpha(\varepsilon(1-z)^{-1})} - (1-y)^{\alpha-1} \right| \alpha y^{-\alpha}.
\]

From (3.11), for any \( y_0 \in (0, 1) \),
\[
\sup_{y_0<y<1} \sup_{u>0} \frac{\Phi_\alpha(u)}{\Phi_\alpha(u(1-y))} \cdot (1-y)^{1-\alpha} < \infty.
\]

So, for any \( \delta > 0 \), one can choose \( y_0 \in (0, 1) \) such that, for any positive \( \varepsilon \) and \( z \),
\[
\int_{y_0}^{1} R_3(z,y) dy < \delta.
\]

Fix \( \delta \in (0, 1/2) \). Now choose \( b > 0 \) such that, for any \( 0 < u < b \),
\[
(1-\delta) \frac{\sin \pi \alpha}{\pi (1-\alpha)} \leq \Phi_\alpha(u) u^{\alpha-1} \leq (1+\delta) \frac{\sin \pi \alpha}{\pi (1-\alpha)}.
\]
If \( \varepsilon \leq b(1-z)(1-y_0) \), then, for \( z \in [0, 1) \),
\[
\int_0^{y_0} R_3(z,y) dy \leq 4\alpha \delta B(1-\alpha, \alpha),
\]
and if \( \varepsilon > b(1-z)(1-y_0) \), then
\[
\int_0^{y_0} R_3(z,y) dy \leq \frac{\alpha}{1-\alpha} \cdot \frac{\Phi_\alpha(\infty)}{\Phi_\alpha(b(1-y_0))} + \alpha B(1-\alpha, \alpha)
\]
and, denoting by \( \hat{c}_0 \) the finite right hand side of the latter inequality,

\[
(1 - z)^{1-\alpha} \int_0^{y_0} R_3(z,y)dy \leq \hat{c}_0\varepsilon^{1-\alpha}b_\alpha^{-1}(1 - y_0)^{\alpha-1}.
\]

So, for any \( \delta \in (0,1/2) \) one can choose \( b > 0 \) such that, for any \( \varepsilon > 0 \),

\[
M_1 \leq Mt_\delta + 4Mt_\alpha\delta(1 - \alpha, \alpha) + Mt\hat{c}_0\varepsilon^{1-\alpha}b_\alpha^{-1}(1 - y_0)^{\alpha-1}.
\]

Letting first \( \varepsilon \) and then \( \delta \) tend to zero, we conclude that

\[
\limsup_{\varepsilon \to 0} \frac{M_1}{\varepsilon^\alpha} = 0.
\]

(3.20)

Convergence (3.13) follows by (3.16), (3.18), (3.19), (3.20) and the equality \( \tilde{\nu}'(\varepsilon) = \nu_1'(\varepsilon) + \nu_2'(\varepsilon) \)
(see (3.12), (3.14a) and (3.14b)). \( \square \)

4 Convergence of the conditioned random walk

In this section we state and prove the main result of the paper. We briefly recall the setting. We are concerned with the random walk \( S_n = \sum_{i=1}^n \xi_i \), where \( \xi_1, \xi_2, \ldots \) is an i.i.d. sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) such that \( E\xi_1 < 0 \) and, for some \( \gamma > 0 \), \( E \exp(\gamma \xi_1) = 1 \) and \( E \xi_1 \exp(\gamma \xi_1) = \infty \). The distribution \( \exp(\gamma x)P(\xi_1 \in dx) \) is denoted \( F \). We also denote by \( \tilde{F} \) the distribution of \( \xi_1 \) under \( P \) so that \( F(dx) = \exp(\gamma x)\tilde{F}(dx) \). We extend slightly the notation introduced in (1.5) and (1.8) by letting for \( r > 0 \)

\[
\tau(r) = \min\{n : S_n \geq r\}, \quad \chi(r) = S_{\tau(r)} - r.
\]

Let also

\[
X^{(r)}(t) = \frac{1}{r} \left[ \left(1/F(r)\right) \sum_{i=1}^{[t/(1-F(r))] - 1} \xi_i \right],
\]

\[
\hat{\tau}(r) = \inf\{t : X^{(r)}(t) \geq 1\}.
\]

(4.1)

(4.2)

Note that \( \hat{\tau}(r) = (1 - F(r))\tau(r) \). We denote by \( \tilde{X}^{(r)} = (\tilde{X}^{(r)}(t), t \in \mathbb{R}_+) \) the process \( X^{(r)} \) stopped at \( \hat{\tau}(r) \), i.e., \( \tilde{X}^{(r)}(t) = X^{(r)}(t \wedge \hat{\tau}(r)) \).

**Theorem 4.1.** Let the following conditions hold:

1. the righthand tail of the distribution function \( F \) is regularly varying at infinity with index \( -\alpha \),
   where \( \alpha \in (1/2,1) \),

2. there exist \( C > 0 \) and \( \rho \in (0,1) \) such that, for all \( y \) great enough and all \( x \in (\rho,1) \),

\[
\frac{1 - F(yx)}{1 - F(y)} \leq 1 + C(1 - x).
\]

If, in addition, \( F \) is a nonlattice distribution, then, as \( r \to \infty \), the conditional distributions of the \( \tilde{X}^{(r)} \) given \( \tau^{(r)} \leq \infty \) weakly converge to the distribution of the non-overshooting Lévy process \( \tilde{X} \). If, instead, \( F \) is a lattice distribution with span \( h \), then, as \( n \to \infty \), where \( n \in \mathbb{N} \), the conditional distributions of the \( \tilde{X}^{(nh)} \) given \( \tau^{(nh)} \leq \infty \) weakly converge to the distribution of the non-overshooting Lévy process \( \tilde{X} \).
Remark 4.1. Under condition 1, the function \( \ell(x) = x^\alpha (1 - F(x)) \) is slowly varying at infinity, i.e., \( \lim_{x \to \infty} \ell(yx)/\ell(x) = 1 \) for all \( y > 0 \). According to Karamata’s theorem, see Bingham, Goldie, and Teugels [6] or Feller [14], it admits the representation \( \ell(x) = c(x) \exp \left( \int_0^x \varepsilon(u)/u \, du \right) \), where \( c(x) \to c > 0 \) and \( \varepsilon(x) \to 0 \) as \( x \to \infty \). If \( c(x) \) in this representation is a constant, or converges to the limit quickly enough, then condition 2 of the theorem holds.

Remark 4.2. The full power of the requirements that \( \alpha \in (1/2, 1) \) and that \( r \) be taken as a multiple of the lattice span when \( F \) is a lattice distribution are only used in the proof of Lemma A.4. The rest of the proof applies to any distribution \( F \) from the domain of attraction of a stable law with index \( \alpha \in (0, 1) \).

Remark 4.3. Note that \( F \) is lattice if and only if \( \hat{F} \) is lattice with the same span.

The proof of Theorem 4.1 is similar to the proof of Theorem 3.1 and involves a change of measure. We start with introducing the change of measure. Let the absolutely continuous with respect to \( P^* \) probability measure \( Q^{(r)} \) on \((\Omega, F_{\tau(r)})\) be defined by

\[
\frac{dQ^{(r)}}{dP^*} = \frac{e^{-\gamma \chi^{(r)}}}{E^{*} e^{-\gamma \chi^{(r)}}}.
\]

By (4.7), for \( \Gamma \in F_{\tau(r)} \),

\[
P(\Gamma|\tau^{(r)} < \infty) = \frac{E^{*} e^{-\gamma \chi^{(r)}} \mathbf{1}_\Gamma}{E^{*} e^{-\gamma \chi^{(r)}}},
\]

so, for a Borel subset \( U \) of \( \mathbb{D} \),

\[
P(\hat{X}^{(r)} \in U|\tau^{(r)} < \infty) = Q^{(r)}(\hat{X}^{(r)} \in U).
\]

In order to be able to use Lemma 3.1 to establish convergence in distribution of the conditioned process \( \hat{X}^{(r)} \), we need to calculate the predictable measure of jumps of this process relative to \( Q^{(r)} \). Let \( \mu^{(r)} \) denote the measure of jumps of \( X^{(r)} \) so that

\[
\mu^{(r)}([0, t], G) = \sum_{i=1}^{\lfloor t/(1 - F^{(r)}) \rfloor} \mathbf{1}_{\{\xi_i/r \in G \setminus \{0\}\}},
\]

where \( G \) is a Borel subset of \( \mathbb{R} \). Obviously,

\[
X^{(r)}(t) = \int_0^t \int_\mathbb{R} x \mu^{(r)}(ds, dx).
\]

Let \( F^{(r)} = (F^{(r)}(t), t \in \mathbb{R}_+) \), where the \( \sigma \)-algebras \( F^{(r)}(t) \) are generated by the random variables \( X^{(r)}(s), s \leq t \), be the filtration associated with \( X^{(r)} \) and let \( \bar{F}^{(r)} = (\bar{F}^{(r)}(t), t \in \mathbb{R}_+) \), with \( \bar{F}^{(r)}(t) = F^{(r)}(t \wedge \tau^{(r)}) \), be the filtration associated with \( \hat{X}^{(r)} \). The \( F^{(r)} \)-predictable measure of jumps of \( X^{(r)} \) under measure \( P^* \) is of the form

\[
\nu^{(r)}([0, t], G) = \left[ \frac{t}{1 - F^{(r)}} \right] F(r(G \setminus \{0\})).
\]

Since \( \hat{X}^{(r)}(t) = X^{(r)}(t \wedge \tau^{(r)}) \), the measure of jumps of \( \hat{X}^{(r)} \) is given by

\[
\hat{\mu}^{(r)}([0, t], G) = \sum_{i=1}^{\lfloor (t \wedge \tau^{(r)})/(1 - F^{(r)}) \rfloor} \mathbf{1}_{\{\xi_i/r \in G \setminus \{0\}\}},
\]
Lemma 4.1. There exists a version of the $\hat{\nu}^{(r)}$-predictable measure of jumps of $\hat{X}^{(r)}$ under measure $Q^{(r)}$ of the form

$$\hat{\nu}^{(r)}(dt, dx) = 1_{\{\hat{X}^{(r)}(t-) < 1\}} \left[ 1_{\{\hat{X}^{(r)}(t-)+x > 1\}} \frac{e^{-\gamma(\hat{X}^{(r)}(t-)+x-1)}}{u^{(r)}(1 - \hat{X}^{(r)}(t-))} + 1_{\{\hat{X}^{(r)}(t-)+x < 1\}} \frac{u^{(r)}(1 - \hat{X}^{(r)}(t-) - x)}{u^{(r)}(1 - \hat{X}^{(r)}(t-))} \right] \nu^{(r)}(dt, dx).$$

Proof. We argue in analogy with the derivation of $\nu^{(c)}$ in the proof of Theorem 3.1. Let us introduce

$$\kappa^{(r)}(y) = \inf\{t \in \mathbb{R}_+ : X^{(r)}(t) \geq y\}$$

and

$$\eta^{(r)}(y) = X^{(r)}(\kappa^{(r)}(y)) - y.$$ (4.12)

Note that $\hat{\tau}^{(r)} = \kappa^{(r)}(1)$ and $\chi^{(ry)} = r\eta^{(ry)}(1) = r\eta^{(r)}(y)$, so by (4.10)

$$u^{(r)}(y) = \mathbb{E}^*e^{-\gamma\eta^{(r)}(y)}.$$ (4.13)

Also, by (4.3) the density process of $Q^{(r)}$ with respect to $P^*$, which we denote $Z = (Z(t), t \in \mathbb{R}_+)$, can be written as

$$Z(t) = \mathbb{E}^*\left[ \frac{e^{-\gamma\eta^{(r)}(1)}}{\mathbb{E}^*e^{-\gamma\eta^{(r)}(1)}} \mathcal{F}^{(r)}(t \land \kappa^{(r)}(1)) \right].$$

Since $\eta^{(r)}(1)$ is an $\mathcal{F}^{(r)}(\kappa^{(r)}(1))$-measurable random variable and $X^{(r)}$ has independent stationary increments,

$$Z(t) = 1_{\{\kappa^{(r)}(1) \leq t\}} \mathbb{E}^*\left[ \frac{e^{-\gamma\eta^{(r)}(1)}}{\mathbb{E}^*e^{-\gamma\eta^{(r)}(1)}} \mathcal{F}^{(r)}(\kappa^{(r)}(1)) \right] + 1_{\{\kappa^{(r)}(1) > t\}} \mathbb{E}^*\left[ \frac{e^{-\gamma\eta^{(r)}(1)}}{\mathbb{E}^*e^{-\gamma\eta^{(r)}(1)}} \mathcal{F}^{(r)}(t) \right]$$

$$= 1_{\{\kappa^{(r)}(1) \leq t\}} \frac{e^{-\gamma\eta^{(r)}(1)}}{\mathbb{E}^*e^{-\gamma\eta^{(r)}(1)}} + 1_{\{\kappa^{(r)}(1) > t\}} \mathbb{E}^*e^{-\gamma\eta^{(r)}(y)} \big|_{y=1-X^{(r)}(t)}. (4.13)$$

Thus, on recalling (4.13) and taking into account that $\hat{X}^{(r)}(t) = X^{(r)}(t)$ for $t < \kappa^{(r)}(1)$,

$$Z(t) = 1_{\{\kappa^{(r)}(1) \leq t\}} \frac{e^{-\gamma\eta^{(r)}(1)}}{u^{(r)}(1)} + 1_{\{\kappa^{(r)}(1) > t\}} \frac{u^{(r)}(1 - \hat{X}^{(r)}(t))}{u^{(r)}(1)}. (4.14)$$

By Liptser and Shiryaev [19] or Jacod and Shiryaev [17], the $\hat{F}^{(r)}$-predictable measure of jumps of $\hat{X}^{(r)}$ under $Q^{(r)}$ admits a version of the form

$$\hat{\nu}^{(r)}(dt, dx) = Y(t, x)\hat{\nu}^{(r)}(dt, dx),$$ (4.15)

where

$$Y(t, x) = \frac{1_{\{Z(t-) > 0\}}}{Z(t-)} \mathbb{M}^{P^*}(Z|\mathbb{P})(t, x). (4.16)$$
In order to find \( \mathbf{M}_{\mu^{(r)}}^{\ast}(Z|\bar{\mathcal{P}})(t, x) \), we write by (4.8), (4.14), and the definition of \( \eta^{(r)}(1) \) in (4.12), for a \( \bar{\mathcal{P}} \)-measurable nonnegative function \( h \),

\[
\mathbf{M}_{\mu^{(r)}}^{\ast} h Z = E^{\ast} \int_{0}^{\infty} \int_{\mathbb{R}} h(t, x) 1_{\{\bar{X}^{(r)}(t) < 1\}} \left( 1_{\{\bar{X}^{(r)}(t) + x \geq 1\}} e^{-\gamma r(\bar{X}^{(r)}(t) + x - 1)} \frac{u^{(r)}(1)}{u^{(r)}(1)} + 1_{\{\bar{X}^{(r)}(t) + x < 1\}} \frac{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)}{u^{(r)}(1)} \right) \mu^{(r)}(dt, dx).
\]

The expression in parentheses is \( \bar{\mathcal{P}} \)-measurable. Hence,

\[
\mathbf{M}_{\mu^{(r)}}^{\ast} (Z|\bar{\mathcal{P}})(t, x) = \frac{1_{\{\bar{X}^{(r)}(t) < 1\}}}{u^{(r)}(1)} \left( 1_{\{\bar{X}^{(r)}(t) + x \geq 1\}} e^{-\gamma r(\bar{X}^{(r)}(t) + x - 1)} + 1_{\{\bar{X}^{(r)}(t) + x < 1\}} u^{(r)}(1 - \bar{X}^{(r)}(t) - x) \right) \mu^{(r)}(dt, dx).
\]

The expression for \( \bar{\nu}^{(r)}(dt, dx) \) in the statement of the lemma follows now by (4.9), (4.14), (4.15), and (4.16).

In what follows, we work with the version of the \( \bar{F}^{(r)} \)-predictable measure of jumps of \( \bar{X}^{(r)} \) given in the statement of Lemma 4.1. We study the properties of \( \bar{\nu}^{(r)} \). Let

\[
\begin{align*}
\bar{\nu}_{1}^{(r)}(dt, dx) &= 1_{\{\bar{X}^{(r)}(t) < 1\}} 1_{\{\bar{X}^{(r)}(t) + x \geq 1\}} e^{-\gamma r(\bar{X}^{(r)}(t) + x - 1)} \frac{u^{(r)}(1)}{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)} \nu^{(r)}(dt, dx), \\
\bar{\nu}_{2}^{(r)}(dt, dx) &= 1_{\{\bar{X}^{(r)}(t) < 1\}} 1_{\{x < 0\}} \frac{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)}{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)} \nu^{(r)}(dt, dx), \\
\bar{\nu}_{3}^{(r)}(dt, dx) &= 1_{\{x > 0\}} 1_{\{\bar{X}^{(r)}(t) + x < 1\}} \frac{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)}{u^{(r)}(1 - \bar{X}^{(r)}(t) - x)} \nu^{(r)}(dt, dx).
\end{align*}
\]

Note that

\[
\bar{\nu}^{(r)} = \bar{\nu}_{1}^{(r)} + \bar{\nu}_{2}^{(r)} + \bar{\nu}_{3}^{(r)}.
\]

One can see that \( \bar{\nu}_{1}^{(r)} \) describes the intensity of jumps of \( \bar{X}^{(r)} \) reaching level 1, \( \bar{\nu}_{2}^{(r)} \) is concerned with the intensity of downward jumps, and \( \bar{\nu}_{3}^{(r)} \) characterises the intensity of upward jumps. Not unexpectedly, the first two measures are inconsequential, as the next lemma shows.

**Lemma 4.2.** Let the hypotheses of Theorem 4.1 hold. Then, for \( i = 1, 2 \),

\[
\lim_{n \to \infty} \sup_{\omega \in \Omega} \int_{0}^{t} \int_{\mathbb{R}} |x| \bar{\nu}_{i}^{(r)}(ds, dx) = 0
\]

if \( F \) is a nonlattice distribution and

\[
\lim_{n \to \infty} \sup_{\omega \in \Omega} \int_{0}^{t} \int_{\mathbb{R}} |x| \bar{\nu}_{i}^{(nh)}(ds, dx) = 0
\]

if \( F \) is a lattice distribution with span \( h \).

18
Proof. Suppose that $F$ is a nonlattice distribution. We start with $i = 1$. Using (4.7) and integrating on $x$ yields on taking into account that $\tilde{X}^{(r)}(s)$ is constant on the intervals $[i(1 - F(r)), (i + 1)(1 - F(r))]$,

$$
\int_{1 - \tilde{X}^{(r)}(s)}^{t} \int_{0}^{\infty} |x| \tilde{\nu}_{1}^{(r)}(ds, dx) \leq \int_{0}^{\infty} \frac{1}{u^{(r)}(1 - \tilde{X}^{(r)}(s))} \frac{e^{-\gamma r(\tilde{X}^{(r)}(s)+x-1)}}{u^{(r)}(1 - \tilde{X}^{(r)}(s))} \left| x \right| \nu^{(r)}(ds, dx)
$$

$$
= \int_{0}^{t} \frac{1}{u^{(r)}(1 - \tilde{X}^{(r)}(s))} \int_{0}^{\infty} x e^{-\gamma r(\tilde{X}^{(r)}(s)+x-1)} F(r dx) d \left[ (1 - F(r))^{-1} s \right]
$$

$$
= \frac{1}{1 - F(r)} \int_{0}^{t} \frac{1}{u^{(r)}(1 - \tilde{X}^{(r)}(s))} \int_{0}^{\infty} x e^{-\gamma r(\tilde{X}^{(r)}(s)+x-1)} F(r dx) ds
$$

$$
\leq \frac{t}{1 - F(r)} \sup_{y \in [0, 1]} \frac{1}{u^{(r)}(y)} \int_{y}^{\infty} x e^{-\gamma r(x-y)} F(r dx). \quad (4.19)
$$

For $y \in [0, 1]$ and $A \in (0, r)$, on recalling (4.13), employing a change of variables and integrating by parts,

$$
\frac{1}{(1 - F(r))u^{(r)}(y)} \int_{y}^{\infty} x e^{-\gamma r(x-y)} F(r dx) = \frac{1}{r(1 - F(r))} \frac{\int_{r y}^{\infty} x e^{-\gamma z} F(dx)}{E^{a} e^{-\gamma (r y)}} \sup_{z \leq A} \int_{0}^{\infty} e^{-\gamma z} F(dx)
$$

$$
\leq 1_{\{r y \leq A\}} \frac{1}{r(1 - F(r))} \frac{\int_{r y}^{\infty} x e^{-\gamma z} F(dx)}{E^{a} e^{-\gamma (r y)}} \sup_{z \leq A} \int_{0}^{\infty} e^{-\gamma z} F(dx)
$$

$$
+ 1_{\{r y > A\}} \frac{1}{r y(1 - F(r y))} \frac{y(1 - F(r y))}{E^{a} e^{-\gamma (r y)}} \frac{\int_{r y}^{\infty} x e^{-\gamma z} F(dx)}{1 - F(r)} \int_{r y}^{\infty} e^{-\gamma z} x F(dx) dz. \quad (4.20)
$$

We first work with the second term on the rightmost side. By the first assertion of part 2 of Lemma $A.5$ given arbitrary $\varepsilon > 0$, $\sup_{y \in [A/r, 1]} y(1 - F(r y))/(1 - F(r)) \leq 1 + \varepsilon$ for all $A$ and $r$ large enough. By Lemma $A.4$ $(r y)(1 - F(r y))E^{a} \exp(-\gamma (r y))$ converges to a positive limit as $r y \to \infty$.

Next, for $u > 0$,

$$
\int_{u}^{u+z} x F(dx) \leq (z + u) (F(z + u) - F(u)). \quad (4.21)
$$

Condition 2 of Theorem $\ref{thm1}$ implies that, for all $a > 0$, $\lim_{a \to \infty} x(F(x + a) - F(x)) = 0$, so

$$
\lim_{u \to \infty} (z + u) (F(z + u) - F(u)) = 0.
$$

Also, for $u \geq 1$,

$$
e^{-\gamma z}(z + u)(F(z + u) - F(u)) \leq e^{-\gamma z} z + e^{-\gamma z} u(F(z + u) - F(u)) 1_{\{z \geq 2 \ln u / \gamma\}} + e^{-\gamma z} u 1_{\{z > 2 \ln u / \gamma\}}
$$

$$
\leq e^{-\gamma z} z + e^{-\gamma z} u(F(2 \ln u / \gamma + u) - F(u)) + e^{-\gamma z / 2}.
$$
Another application of condition 2 shows that, for all \( a > 0 \), \( \lim_{x \to \infty} x(F(x + a \ln x) - F(x)) = 0 \). Thus, the rightmost side of the latter display is bounded above by a function of \( z \) with a finite \((0, \infty)\)-integral with respect to Lebesgue measure. Hence, by the dominated convergence theorem,

\[
\lim_{u \to \infty} \gamma \int_{0}^{u+z} e^{-\gamma z} \int_{u}^{\infty} x F(dx) \, dz = 0.
\]

Thus, given arbitrary \( \varepsilon > 0 \), if \( A \) is great enough, then the second term on the rightmost side of (4.20) is less than \( \varepsilon \). Since the first term on the rightmost side of (4.20) converges to zero as \( r \to \infty \) for fixed \( A \), we conclude that

\[
\lim_{r \to \infty} \sup_{y \in [0,1]} \frac{1}{(1 - F(r))^u(r)(y)} \int_{y}^{\infty} x e^{-\gamma r(x-y)} F(r \, dx) = 0.
\]

By (4.19), \( \lim_{r \to \infty} \sup_{\omega \in \Omega} \int_{0}^{\infty} \int_{\mathbb{R}} |x| \, \nu_2^{(r)}(dx, ds) = 0 \).

Let \( i = 2 \). By arguing in analogy with (4.19) and (4.20), and using the bound \( u^{(r)}(y) \leq 1 \), we have that for \( A > 0 \),

\[
\int_{0}^{t} \int_{\mathbb{R}} |x| \, \nu_2^{(r)}(dx, ds) \leq \frac{1}{1 - F(r)} \int_{0}^{t} \int_{\mathbb{R}} 1 \{ \hat{X}^{(r)}(s) < 1 \} x^{-} \frac{u^{(r)}(1 - \hat{X}^{(r)}(s) - x)}{u^{(r)}(1 - \hat{X}^{(r)}(s))} F(r \, dx) \, ds
\]

\[
\leq \frac{1}{r(1 - F(r))} \inf_{z \geq A} E^{*} e^{-\gamma \chi^{(z)}} \int_{0}^{t} \int_{\mathbb{R}} 1 \{ \hat{X}^{(r)}(s) < 1 \} 1 \{ r(1 - \hat{X}^{(r)}(s)) \leq A \} x^{-} F(dx) \, ds
\]

\[
+ \frac{1}{r(1 - F(r))} \sup_{z \geq A} E^{*} \exp(-\gamma \chi^{(z)}) \int_{0}^{t} \int_{\mathbb{R}} 1 \{ \hat{X}^{(r)}(s) < 1 \} 1 \{ r(1 - \hat{X}^{(r)}(s)) > A \} x^{-} F(dx) \, ds.
\]

Thus, on recalling Lemma A.4, given \( \varepsilon > 0 \), for all \( A \) large enough and all \( r > A \),

\[
\int_{0}^{t} \int_{\mathbb{R}} |x| \, \nu_2^{(r)}(dx, ds) \leq \frac{t}{r(1 - F(r))} \left( \frac{1}{\inf_{z \leq A} E^{*} e^{-\gamma \chi^{(z)}}} + 1 + \varepsilon \right) \int_{\mathbb{R}} x^{-} F(dx).
\]

Since \( \int_{\mathbb{R}} x^{-} F(dx) = \int_{-\infty}^{0} (-x) \exp(\gamma x) \hat{F}(dx) < \infty \), the function \( r^{\alpha}(1 - F(r)) \) is slowly varying at infinity, and \( \alpha < 1 \), we conclude that \( \sup_{\omega \in \Omega} \int_{0}^{t} \int_{\mathbb{R}} |x| \, \nu_2^{(r)}(dx, ds) \) converges to 0 as \( r \to \infty \).

The assertion of the lemma for \( F \) being nonlattice has been proved. The proof for the case where \( F \) is a lattice distribution with span \( h \) proceeds analogously. The only changes consist in assuming that \( r = nh \), that \( y \) is of the form \( k/n \) for \( k = 1, 2, \ldots, n \), that \( z \) is of the form \( kh \) for \( k = 1, 2, \ldots, n \), and that \( n \to \infty \), and in using the part of Lemma A.4 that concerns the lattice case.

We are now in a position to verify condition 3 of Lemma 3.1.

**Lemma 4.3.** Let the hypotheses of Theorem 4.1 hold, let \( K \) be defined by (3.2), and let \( g(x) \) be a bounded continuous function such that \( |g(x)| \leq M |x| \) for some \( M > 0 \). Let \( t > 0 \) and \( \varepsilon > 0 \) be
otherwise arbitrary. If, in addition, $F$ is a nonlattice distribution, then

$$
\lim_{r \to \infty} Q^{(r)}\left(\left| \int_0^t \int \tilde{\nu}^{(r)}(ds, dx) - \int_0^t \int g(x) K(\tilde{X}^{(r)}(s); dx) ds \right| > \varepsilon \right) = 0.
$$

If, instead, $F$ is a lattice distribution with span $h$, then

$$
\lim_{n \to \infty} Q^{(nh)}\left(\left| \int_0^t \int \tilde{\nu}^{(nh)}(ds, dx) - \int_0^t \int g(x) K(\tilde{X}^{(nh)}(s); dx) ds \right| > \varepsilon \right) = 0.
$$

**Proof.** Suppose $F$ is nonlattice. We will prove that $\lim_{r \to \infty} \sup_{\omega \in \Omega} M_2 = 0$ where

$$
M_2 = \left| \int_0^t \int g(x) \tilde{\nu}^{(r)}(ds, dx) - \int_0^t \int g(x) K(\tilde{X}^{(r)}(s); dx) ds \right|.
$$

By Lemma 4.2,

$$
\lim_{r \to \infty} \sup_{\omega \in \Omega} \int_0^t \int g(x) \tilde{\nu}_i^{(r)}(ds, dx) = 0, \ i = 1, 2.
$$

We turn our attention to $\tilde{\nu}_3^{(r)}$. By (4.17c),

$$
\tilde{\nu}_3^{(r)}(dt, dx) = 1_{\{0 < x < 1 - \tilde{X}^{(r)}(t)\}} \frac{u^{(r)}(1 - \tilde{X}^{(r)}(s) - x)}{u^{(r)}(1 - \tilde{X}^{(r)}(t)-)} \nu^{(r)}(dt, dx).
$$

Let $h(t, r) = [t/(1 - F(r))](1 - F(r))$. By (4.7),

$$
\int_0^t \int g(x) \tilde{\nu}_3^{(r)}(ds, dx) = \int_0^{h(t, r)} \int V_1(\tilde{X}^{(r)}(s), x) \frac{F(r(1 - \tilde{X}^{(r)}(s))) \, dx}{1 - F(r)} \, ds
$$

where

$$
V_1(z, x) = g((1 - z)x) 1_{\{z < 1\}} \frac{u^{(r)}((1 - z)(1 - x))}{u^{(r)}(1 - z)}.
$$

Let also

$$
V_2(z, x) = g((1 - z)x) 1_{\{z < 1\}} (1 - z)^{-\alpha}(1 - x)^{\alpha - 1}ax^{\alpha - 1}
$$

and

$$
V_3(z, x) = g((1 - z)x) 1_{\{z < 1\}} (1 - x)^{\alpha - 1}.
$$
Then, for $\eta \in (0, 1)$, recalling (3.3),

$$
M_2 \leq \int_0^t \int_0^{1-\eta} |V_2(\hat{X}^{(r)}(s), x)| \, dx \, ds + \int_0^t \int_0^{1-\eta} |V_1(\hat{X}^{(r)}(s), x)| \frac{F(r(1 - \hat{X}^{(r)}(s))) \, dx}{1 - F(r)} \, ds
$$

$$
+ \int_0^{h(t,x)} \int_0^{1-\eta} |V_3(\hat{X}^{(r)}(s), x)| \frac{F(r(1 - \hat{X}^{(r)}(s))) \, dx}{1 - F(r)} \, ds
$$

$$
+ \int_0^t \int_0^{1-\eta} |V_3(\hat{X}^{(r)}(s), x) - V_3(\hat{X}^{(r)}(s), x)| \frac{F(r(1 - \hat{X}^{(r)}(s))) \, dx}{1 - F(r)} \, ds
$$

$$
+ \int_0^t \int_0^{1-\eta} V_3(\hat{X}^{(r)}(s), x) \frac{F(r(1 - \hat{X}^{(r)}(s))) \, dx}{1 - F(r)} \, dx \, ds - \int_0^t \int_0^{1-\eta} V_2(\hat{X}^{(r)}(s), x) \, dx \, ds.
$$

We denote the terms on the righthand side of the latter inequality as $I_1$, $I_2$, $I_3$, $I_4$, and $I_5$, respectively. We treat them successively. We have

$$
I_1 \leq M \int_0^t (1 - \hat{X}^{(r)}(s))^{1-\alpha} \chi(\hat{X}^{(r)}(s) < 1) \int_0^{1-\eta} (1 - x)^{\alpha-1} \alpha x^{-\alpha} \, dx \, ds \leq Mt \int_0^{1-\eta} (1 - x)^{\alpha-1} \alpha x^{-\alpha} \, dx,
$$

so

$$
\lim_{\eta \to 0} \sup_{r \to \infty} \sup_{\omega \in \Omega} I_1 = 0.
$$

(4.24)

We now prove that

$$
\lim_{\eta \to 0} \sup_{r \to \infty} \omega \in \Omega I_2 = 0.
$$

(4.25)

By Lemma [A.4] $ry(1 - F(ry))u^{(r)}(y) \to C_0 > 0$ as $ry \to \infty$ and $\sup_{y \geq 0} ry(1 - F(ry))u^{(r)}(y) < \infty$. On denoting the latter supremum as $N$, we have that, provided $ry$ is great enough, for $x \in (0, 1)$,

$$
\frac{u^{(r)}(y(1 - x))}{u^{(r)}(y)} \leq \frac{2N}{C_0} \frac{1 - F(ry)}{(1 - x)(1 - F(ry)(1 - x)))}.
$$

(4.26)

Since $u^{(r)}(y(1 - x)) \leq 1$, we can write, for large enough $A \in (0, r)$ and for $B \in (0, A)$,

$$
I_2 \leq Mt \sup_{y \in [0, 1]} \int_{1-\eta}^1 xy \frac{u^{(r)}(y(1 - x))}{u^{(r)}(y)} \frac{F(ry \, dx)}{1 - F(r)} \, ds
$$

$$
\leq Mt \sup_{y \in [0, 1]} \int_{1-\eta}^1 \frac{A}{r(1 - F(r))} \frac{F(A)}{\inf_{z \leq A} x \exp(-\gamma x^z)} + Mt \sup_{y \in [A/r, 1]} \int_{1-\eta}^1 \frac{1 - F(r)}{(1 - x)(1 - F(ry)(1 - x)))} \frac{F(ry \, dx)}{1 - F(r)}
$$

$$
+ Mt \sup_{y \in [A/r, 1]} \int_{1-\eta}^1 \frac{1 - F(r)}{(1 - x)(1 - F(ry)(1 - x)))} \frac{F(ry \, dx)}{1 - F(r)}.
$$

(4.27)
The first term on the rightmost side of (4.27) tends to zero as \( r \to \infty \). By the fact that \( 1 - x \leq B/(ry) \) when the integrand of the second term is positive and by (4.26), that term is not greater than

\[
Mt \frac{2}{C_0} \sup_{y \in (A/r, 1]} \frac{y(1 - F(ry))}{1 - F(r)} \sup_{v > A} v(F(v) - F(v - B)).
\]

By the first assertion of part 2 of Lemma A.5, the supremum in the middle of the latter expression is bounded above for all great enough \( A \) and \( r \). By condition 2 of Theorem 4.1, the other supremum tends to zero as \( A \to \infty \). Thus, the second term on the rightmost side of (4.27) tends to zero as \( A \to \infty \) and \( r \to \infty \).

Let us consider the third term on the rightmost side of (4.27). Pick arbitrary \( \varepsilon \in (0, \alpha \wedge (1 - \alpha)) \).

The function \( \ell(x) = x^\alpha (1 - F(x)) \) is slowly varying at infinity, so by part 1 of Lemma A.5, for \( B \) large enough, provided \( ry(1 - x) > B \),

\[
\frac{1 - F(ry)}{1 - F(ry(1 - x))} = (1 - x)^\alpha \frac{\ell(ry)}{\ell(ry(1 - x))} \leq (1 + \varepsilon)(1 - x)^{\alpha - \varepsilon}.
\]

We obtain,

\[
\sup_{y \in (A/r, 1]} \int_{1 - \eta}^{1} \mathbf{1}_{(ry(1-x)>B)} \frac{1 - F(ry)}{(1 - x)(1 - F(ry(1-x)))} \frac{F(ry) dx}{1 - F(r)} \\
\leq (1 + \varepsilon) \sup_{y \in (A/r, 1]} \frac{y(1 - F(ry))}{1 - F(r)} \sup_{z > A} \frac{1}{1 - F(z)} \int_{1 - \eta}^{1} (1 - x)^{\alpha - \varepsilon - 1} F(z) dx. \quad (4.28)
\]

By the first assertion of part 2 of Lemma A.5, the first supremum on the right of (4.28) is bounded above for all \( A \) (and \( r \)) large enough.

For the second supremum, integration by parts yields

\[
\frac{1}{1 - F(z)} \int_{1 - \eta}^{1} (1 - x)^{\alpha - \varepsilon - 1} F(z) dx = (1 + \varepsilon - \alpha) \int_{1 - \eta}^{1} (1 - x)^{\alpha - \varepsilon - 2} \frac{F(z) - F(zx)}{1 - F(z)} dx \\
+ \eta^{\alpha - \varepsilon - 1} \frac{F(z) - F(z(1 - \eta))}{1 - F(z)}.
\]

By condition 2 of Theorem 4.1 provided \( z \) is large enough and \( \eta \) is small enough, we have, for \( x \in [1 - \eta, 1] \),

\[
\frac{F(z) - F(zx)}{1 - F(z)} \leq C(1 - x).
\]

Therefore,

\[
\int_{1 - \eta}^{1} (1 - x)^{\alpha - 2\varepsilon} \frac{F(z) - F(zx)}{1 - F(z)} dx \leq C \int_{1 - \eta}^{1} (1 - x)^{\alpha - \varepsilon - 1} dx.
\]

It follows that

\[
\lim_{\eta \to 0} \lim_{A \to \infty} \sup_{z > A} \frac{1}{1 - F(z)} \int_{1 - \eta}^{1} (1 - x)^{\alpha - \varepsilon - 1} F(z) dx = 0.
\]
By (4.28), we conclude that the third term on the rightmost side of (4.27) tends to zero as \( A \to \infty \) and \( \eta \to 0 \). Letting successively \( r \to \infty, A \to \infty, \) and \( \eta \to 0 \) in (4.27) and picking large enough \( B \) obtains (4.25).

Let us consider \( I_3 \). We have, by a change of variables,

\[
I_3 \leq M \int_{h(t,r)}^{t} \int_{0}^{1-\eta} (1 - \hat{X}^{(r)}(s))x1_{\{\hat{X}^{(r)}(s) < 1\}} (1 - x)^{\alpha - 1} \frac{F(r(1 - \hat{X}^{(r)}(s)))}{1 - F(r)} \, ds \text{d}x.
\]

By the second assertion of part 2 of Lemma A.5,

\[
\limsup_{r \to \infty} \frac{1}{1 - F(r)} \int_{0}^{1-\eta} xF(r \, dx) \leq \frac{1}{1 - \alpha}. \tag{4.29}
\]

Since \( t - h(t,r) \to 0 \) as \( r \to \infty \), it follows that

\[
\lim_{r \to \infty, \omega \in \Omega} \sup_{r=0} I_3 = 0. \tag{4.30}
\]

For \( I_4 \), we have reusing earlier arguments, for \( A > 0 \),

\[
I_4 \leq Mt \frac{A}{r(1 - F(r))} \left( \frac{1}{\inf_{z \leq A} E^* \exp(-\gamma \chi^{(z)})} + \eta^{\alpha - 1} \right) F(A)
\]

\[
+ Mt \eta^{-1} \sup_{z > A} \sup_{x \in [0,1-\eta]} \left| \frac{(1 - x)E^* \exp(-\gamma \chi^{(z(1-x))})}{E^* \exp(-\gamma \chi^{(z)})} - (1 - x)^{\alpha} \right| \left\{ \sup_{y \in [0,1]} \frac{y}{1 - F(r)} \right\}^{1-\eta} \int_{0}^{1-\eta} xF(r \, dx). \tag{4.31}
\]

Pick arbitrary \( \varepsilon \in (0, C_0) \). Since by Lemma A.4 \( E^* \exp(-\gamma \chi^{(z)}) \sim C_0/(z(1 - F(z))) \) as \( z \to \infty \) we have for all \( z \) large enough uniformly over \( x \in [0,1-\eta] \)

\[
\left| \frac{(1 - x)E^* \exp(-\gamma \chi^{(z(1-x))})}{E^* \exp(-\gamma \chi^{(z)})} - (1 - x)^{\alpha} \right| \leq \varepsilon \frac{1 - F(z)}{1 - F(z(1 - x))} + \frac{1 - F(z)}{1 - F(z(1 - x))} - (1 - x)^{\alpha} \right| = 0.
\]

The first summand on the right is not greater than \( \varepsilon \). The second summand tends to zero as \( z \to \infty \) uniformly over \( x \in [0,1-\eta] \) by the facts that \( F \) is regularly varying with exponent \( -\alpha \), the ratio \( (1 - F(z))/(1 - F(z(1 - x))) \) is monotonic in \( x \), and \( (1 - x)^{\alpha} \) is a continuous function. Thus,

\[
\lim_{z \to \infty} \sup_{x \in [0,1-\eta]} \left| \frac{(1 - x)E^* \exp(-\gamma \chi^{(z(1-x))})}{E^* \exp(-\gamma \chi^{(z)})} - (1 - x)^{\alpha} \right| = 0.
\]

Since also (4.29) holds, the second summand on the right of (4.31) tends to zero as \( A \) and \( r \) tend to infinity. Since the first summand tends to zero as \( r \to \infty \), we arrive at the convergence

\[
\limsup_{r \to \infty, \omega \in \Omega} I_4 = 0. \tag{4.32}
\]
We now consider \( I_5 \). We have for \( \delta \in (0, 1 - \eta) \), employing a change of variables in the inside integral of the first term on the lefthand side of the first inequality,

\[
I_5 \leq \int_0^t \int_0^{1-\eta} |V_3(\hat{X}^{(r)}(s), x)| \frac{F(r(1 - \hat{X}^{(r)}(s)) \, dx)}{1 - F(r)} \mathbf{1}_{\{\hat{X}^{(r)}(s) \in (1-\delta, 1]\}} \, ds \\
+ \int_0^t \int_0^{1-\eta} |V_2(\hat{X}^{(r)}(s), x)| \mathbf{1}_{\{\hat{X}^{(r)}(s) \in (1-\delta, 1]\}} \, dx \, ds \\
+ \left| \int_0^t \int_0^{1-\eta} V_3((\hat{X}^{(r)}(s), x) \frac{F(r(1 - \hat{X}^{(r)}(s)) \, dx)}{1 - F(r)} \mathbf{1}_{\{\hat{X}^{(r)}(s) \leq 1-\delta\}} \, ds \\
- \int_0^t \int_0^{1-\eta} V_2((\hat{X}^{(r)}(s), x) \mathbf{1}_{\{\hat{X}^{(r)}(s) \leq 1-\delta\}} \, dx \, ds \right| \\
\leq M \eta \alpha^{-1} \left( \int_0^{\delta(1-\eta)} x \frac{F(r \, dx)}{1 - F(r)} + \delta^{1-\alpha} \int_0^{1-\eta} \alpha x^{-\alpha} \, dx \right) \\
+ t \sup_{y \in [\delta, 1]} \int_0^{1-\eta} g(yx) (1-x)^{\alpha-1} \frac{F(ry \, dx)}{1 - F(r)} - \int_0^{1-\eta} g(yx) y^{-\alpha}(1-x)^{\alpha-1} \alpha x^{-\alpha-1} \, dx \right|. \tag{4.33}
\]

On recalling the second assertion of part 2 of Lemma [A.5], we conclude that the term in the parentheses on the rightmost side of (4.33) tends to zero as \( r \to \infty \) and \( \delta \to 0 \), i.e.,

\[
\lim_{\delta \to 0} \lim_{r \to \infty} M \eta \alpha^{-1} \left( \int_0^{\delta(1-\eta)} x \frac{F(r \, dx)}{1 - F(r)} + \delta^{1-\alpha} \int_0^{1-\eta} \alpha x^{-\alpha} \, dx \right) = 0. \tag{4.34}
\]

The other summand is not greater than

\[
t \sup_{y \in [\delta, 1]} \int_0^{1-\eta} |g(yx)| \frac{1 - F(ry)}{1 - F(r)} - y^{-\alpha} \left| (1-x)^{\alpha-1} \frac{F(ry \, dx)}{1 - F(r)} + t \sup_{z \geq r \delta} \sup_{y \in [\delta, 1]} W(0, y, z) \right| \\
\leq M \eta \alpha^{-1} \sup_{y \in [\delta, 1]} \frac{1 - F(ry)}{1 - F(r)} - y^{-\alpha} \sup_{y \in [\delta, 1]} \int_0^{1-\eta} yx \frac{F(ry \, dx)}{1 - F(r)} \\
+ M \eta \alpha^{-1} \sup_{z \geq r \delta} \int_0^{\delta} x \frac{F(z \, dx)}{1 - F(z)} + M \eta \alpha^{-1} \int_0^{\delta} \alpha x^{-\alpha} \, dx + t \sup_{z \geq r \delta} \sup_{y \in [\delta, 1]} W(\delta, y, z) \tag{4.35}
\]

where, for \( 0 \leq a \leq 1 \),

\[
W(a, y, z) = \int_0^{1-\eta} g(yx) y^{-\alpha}(1-x)^{\alpha-1} \frac{F(z \, dx)}{1 - F(z)} - \int_0^{1-\eta} g(yx) y^{-\alpha}(1-x)^{\alpha-1} \alpha x^{-\alpha-1} \, dx \right|. 
\]
Since $1 - F(y)$ is regularly varying with index $-\alpha$,
\[
\sup_{y \in [\delta, 1]} \left| \frac{1 - F(ry)}{1 - F(r)} - y^{-\alpha} \right| \to 0 \text{ as } r \to \infty.
\]
Also,
\[
\sup_{y \in [\delta, 1]} \left| \int_0^{1-\eta} yx \frac{F(ry \, dx)}{1 - F(ry)} - y^{-\alpha} \right| \to 0 \text{ as } r \to \infty.
\]
On taking into account the second assertion of part 2 of Lemma A.5, we obtain that
\[
\lim_{r \to \infty} M t \eta^{\alpha} (1 - \eta) \sup_{y \in [\delta, 1]} \left| \frac{1 - F(ry)}{1 - F(r)} - y^{-\alpha} \right| \sup_{y \in [\delta, 1]} \left| \int_0^{1-\eta} yx \frac{F(ry \, dx)}{1 - F(ry)} \right| = 0.
\]
Another application of the second assertion of part 2 of Lemma A.5 yields
\[
\lim_{z \to \infty} \sup_{y \in [\delta, 1]} \left| \frac{1 - F(zx)}{1 - F(z)} - y^{-\alpha} \right| \sup_{y \in [\delta, 1]} \left| \int_0^{1-\eta} x \frac{F(ry \, dx)}{1 - F(ry)} \right| = 0.
\]
Of course, also,
\[
\lim_{\delta \to 0} M t \eta^{\alpha} (1 - \eta) \int_0^{\delta} \frac{x}{1 - F(z)} = 0.
\]
Let us now consider the last term on the righthand side of (4.33). Since $1 - F$ is regularly varying at infinity with index $-\alpha$, the $F(zx)/(1 - F(z))$, considered as measures on $[\delta, 1 - \eta]$, weakly converge as $z \to \infty$ to $\alpha x^{-\alpha - 1} \, dx$. Since $g(x)$ is a continuous function, the functions $(g(yx)y^{-\alpha}(1 - x)^{\alpha - 1}, x \in [\delta, 1 - \eta])$ are uniformly bounded and equicontinuous over $y \in [\delta, 1]$. Therefore,
\[
\lim_{z \to \infty} \sup_{y \in [\delta, 1]} W(\delta, y, z) = 0.
\]
Thus, by (4.33), (4.34), (4.35), (4.36), (4.37), (4.38), and (4.39),
\[
\lim_{r \to \infty} \sup_{\omega \in \Omega} I_5 = 0.
\]
Putting together (4.24), (4.24), (4.25), (4.30), (4.32), and (4.40), we conclude that
\[
\lim_{r \to \infty} \sup_{\omega \in \Omega} \left| \int_0^t \int_0^t g(x) \tilde{\nu}^{(r)}(s, dx) - \int_0^t \int_0^t g(x) K(\tilde{X}^{(r)}(s); dx) \, ds \right| = 0.
\]
On recalling (4.18) and (4.23), we arrive at (4.22).

The assertion of the lemma for $F$ being a nonlattice distribution has been proved. If $F$ is a lattice distribution with span $h$, the proof proceeds analogously provided one assumes that $r = nh$, where $n \in \mathbb{N}$, that the $y$’s are of the form $k/n$ for $k = 1, 2, \ldots, n$, that the $z$’s are of the form $kh$ for $k \in \mathbb{N}$, and that $n \to \infty$. \hfill \Box
Proof of Theorem 4.1. We apply Lemma 3.1 to the processes \( \hat{X}^{(r)} \) under the measures \( Q^{(r)} \) if \( F \) is a nonlattice distribution and to the processes \( \hat{X}^{(nh)} \) under the measures \( Q^{(nh)} \) if \( F \) is a lattice distribution with span \( h \). Conditions 1 and 2 of the lemma follow by Lemma 3.2. Condition 3 holds by Lemma 4.3. Condition 4 is obviously met. Condition 5 holds by Theorem 2.1.

Acknowledgements. This research was initiated during the second author’s visit to the Heriot-Watt University. The warm welcome of the Department of the Actuarial Mathematics and Statistics at the Heriot-Watt University and the support of the European Commission under the Marie Curie International Incoming Fellowship Programme are gratefully acknowledged.

A Auxiliary results

A.1 Convergence of the unconstrained random walk

We recall that \( X \) is an increasing stable Lévy process starting at zero with Lévy measure \( \alpha x^{-\alpha-1} dx, x > 0 \), defined on a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\), and the processes \( X^{(r)} \) are defined by (4.1).

Lemma A.1. Let the righthand tail of the distribution \( F \) be regularly varying at infinity with index \(-\alpha\), where \( \alpha \in (0, 1) \). Then the \( X^{(r)} \) under measure \( P^* \) converge in distribution to \( X \).

The proof will use the following implication of Lemma 3.1.

Lemma A.2. Consider a sequence \( X^{(n)} = (X^{(n)}(t), t \in \mathbb{R}_+) \) of \( \mathbb{R} \)-valued pure-jump semimartingales of locally bounded variation with independent increments defined on filtered probability spaces \((\Omega^{(n)}, \mathcal{F}^{(n)}, F^{(n)}, P^{(n)})\). Let \( X \) be a Lévy process on a probability space \((\Omega, \mathcal{F}, P)\), with Lévy measure \( K \) such that \( \int_{\mathbb{R}} 1 \wedge |x| K(dx) < \infty \). Suppose that \( X^{(n)}(0) = 0 \) \( P^{(n)} \)-a.s. and \( X(0) = 0 \) \( P \)-a.s. If for an arbitrary \( \mathbb{R} \)-valued bounded continuous function \( g(x), x \in \mathbb{R}, \) such that \( |g(x)| \leq M|x|, x \in \mathbb{R}, \) in a neighbourhood of the origin with some \( M > 0 \),

\[
\lim_{n \to \infty} \frac{1}{t} \int_{0}^{t} \int_{\mathbb{R}} g(x) \nu^{(n)}(ds, dx) = \frac{1}{t} \int_{\mathbb{R}} g(x) K(dx),
\]

where \( \nu^{(n)}(dt, dx) \) denotes the \( F^{(n)} \)-predictable measure of jumps of \( X^{(n)} \), then the \( X^{(n)} \) converge in distribution to \( X \).

Proof. The process \( X \) is a well-defined Lévy process, see Bertoin [4] or Jacod and Shiryaev [17], so the assertion of the lemma follows by Lemma 3.1. One can also apply Theorem VII.3.4 in Jacod and Shiryaev [17] or Theorem 7.3.1 in Liptser and Shiryaev [19].

Proof of Lemma A.1. The processes \( X^{(r)} \) under \( P^* \) are pure-jump semimartingales of locally bounded variation with independent increments. Their \( F^{(r)} \)-predictable measures of jumps under \( P^* \) are given by (4.7), so for \( g(x) \) as in the hypotheses,

\[
\int_{0}^{t} \int_{\mathbb{R}} g(x) \nu^{(r)}(ds, dx) = \frac{1}{(1 - F(r))^{-1} t} \int_{\mathbb{R}} g(x) F(rdx). \tag{A.1}
\]
We may and will assume that for a suitable $M'$, $|g(x)| \leq M'|x|$ for all $x$. We have on writing $1 - F(x) = x^{-\alpha} \ell(x)$, where $\ell$ is slowly varying at infinity,

$$\left| (1 - F(r))^{-1} t \right| \int_{-\infty}^{0} g(x) F(rdx) \leq M' \frac{r^{\alpha - 1}}{\ell(r)} t \int_{-\infty}^{0} |x| F(dx).$$

Since $\int_{-\infty}^{0} |x| F(dx) = \int_{-\infty}^{0} |x| \exp(\gamma x) \hat{F}(dx) < \infty$ and $\alpha < 1$, we obtain that

$$\lim_{r \to \infty} \frac{t}{\ell(r)} \int_{-\infty}^{0} |g(x)| F(rdx) = 0. \quad (A.2)$$

For $\varepsilon > 0$,

$$\left| (1 - F(r))^{-1} t \right| \int_{0}^{\varepsilon} |g(x)| F(rdx) \leq tM' \int_{0}^{\varepsilon} r F(rdx).$$

Hence, by the second assertion of part 2 of Lemma A.5,

$$\lim_{r \to \infty} \limsup_{\varepsilon \to 0} \frac{t}{\ell(r)} \int_{0}^{\varepsilon} |g(x)| F(rdx) = 0. \quad (A.3)$$

The hypotheses on $F$ imply that, for $x > \varepsilon > 0$, $(F(rx) - F(r\varepsilon))/(1 - F(r)) \to \varepsilon^{-\alpha} - x^{-\alpha}$ as $r \to \infty$, so the $F(dx)/(1 - F(r))$, as measures on $[\varepsilon, \infty)$, weakly converge to the measure $\alpha x^{-\alpha - 1} dx$. On recalling that $g(x)$ is a bounded and continuous function, we conclude that

$$\lim_{r \to \infty} \frac{t}{\ell(r)} \int_{\varepsilon}^{\infty} g(x) F(dx) = t \int_{\varepsilon}^{\infty} g(x) \alpha x^{-\alpha - 1} dx. \quad (A.4)$$

By (A.1), (A.2), (A.3), and (A.4),

$$\lim_{r \to \infty} \int_{0}^{t} \int_{\mathbb{R}} g(x) \nu^{(r)}(ds, dx) = t \int_{0}^{\infty} g(x) \alpha x^{-\alpha - 1} dx,$$

which completes the proof by Lemma A.2.

As a byproduct, we derive an extension of Dynkin’s result, Dynkin [9, Theorem 2], on the asymptotic behaviour of the overshoot $\chi^{(r)}$ (see (1.8)). We recall that $\tau$ is defined by (3.1) and let $\chi$ denote the overshoot $X(\tau) - 1$. As we have mentioned earlier, by Dynkin [9, Theorem 6] or Rogozin [21, Theorem 7],

$$\mathbb{P}'(\chi \leq x) = \frac{\sin \pi \alpha}{\pi} \int_{0}^{x} u^{-\alpha}(1 + u)^{-1} du.$$

Thus, “the arcsine law” for $\chi^{(r)}/r$ cited in the introduction is a consequence of the following result.

**Lemma A.3.** Under the hypotheses of Lemma A.1 as $r \to \infty$, the $\chi^{(r)}/r$ converge in distribution on $\mathbb{R}_+$ to $\chi$. 

28
**Proof.** Let \( \mathbb{D}_1 \) denote the subset of \( \mathbb{D} \) of unbounded functions \( x \) with \( x(0) = 0 \). It is an easy consequence of Theorem 7.2 in Whitt [24] that the mapping \( j \) from \( \mathbb{D}_1 \) to \( \mathbb{R}_+ \) defined by \( j(x) = \inf\{t : x(t) \geq 1\} \) is continuous at every strictly increasing \( x \in \mathbb{D}_1 \). Thus, the assertion of the lemma follows by Lemma [A.1], the fact that \( X \) is strictly increasing \( \mathbb{P}^\ast \)-a.s., see Bertoin [4], and the continuous mapping theorem for convergence in distribution, see, e.g., Jacod and Shiryaev [17]. \( \square \)

### A.2 A proof of two results by Korshunov [18]

The result formulated in this Subsection is contained in Theorems 1 and 2 in Korshunov [18]. However, some details of the proof are omitted there (especially for the lattice case), so we fill in the gaps in our proof below. We recall that \( \mathbb{E}^\ast \) denotes expectation with respect to measure \( \mathbb{P}^\ast \) defined by (1.1).

**Lemma A.4.** Let condition 1 of Theorem [4.1] hold.

1. If, in addition, \( F \) is a nonlattice distribution, then, for some \( C_0 > 0 \), as \( r \to \infty \),

\[
\mathbb{E}^\ast e^{-\gamma \chi^{(r)}} \sim \frac{C_0}{r(1 - F(r))}.
\]

(A.5)

2. If, instead, \( F \) is a lattice distribution with span \( h \), then, for some \( C'_0 > 0 \), as \( n \to \infty \),

\[
\mathbb{E}^\ast e^{-\gamma \chi^{(nh)}} \sim \frac{C'_0}{nh(1 - F(nh))}.
\]

Remark A.1. Note that, by Karamata theorem, the coefficients \( C_3 \) in (1.3) and \( C_0 \) in (A.5) are related by \( C_0 = C_3(1 - \alpha)/\gamma \).

**Proof.** We introduce strict ascending ladder indices \( T_1, T_2, \ldots \) by letting \( T_0 = 0 \) and \( T_n = \min\{k > T_{n-1} : S_k - S_{T_{n-1}} > 0\} \) for \( n \in \mathbb{N} \). Let \( \zeta_k = S_{T_k} - S_{T_{k-1}} \) for \( k \in \mathbb{N} \). Under \( \mathbb{P}^\ast \), the \( \zeta_k \) are a.s. finite and i.i.d., and \( \mathbb{E}^\ast T_1 < \infty \), see Asmussen [2, VIII.2]. We let \( F_+ \) denote the common distribution function of the \( \zeta_k \) (under \( \mathbb{P}^\ast \)). Adapting the argument of the proof of Lemma 2 in Korshunov [18], we write, for \( x \geq 0 \),

\[
\frac{1 - F_+(x)}{1 - F(x)} = \int_{-\infty}^{0} \frac{1 - F(x - u)}{1 - F(x)} H(du),
\]

where \( H(u) = 1_{\{u = 0\}} + \sum_{k=1}^{\infty} \mathbb{P}^\ast(S_1 \leq 0, S_2 \leq 0, \ldots, S_k \leq 0, S_k \leq u) \) for \( u \leq 0 \). Under condition 1 of Theorem [4.1] \( \lim_{x \to \infty}(1 - F(x - u))/(1 - F(x)) = 1 \), so by Lebesgue’s bounded convergence theorem,

\[
\lim_{x \to \infty} \frac{1 - F_+(x)}{1 - F(x)} = H(0).
\]

Since \( H(0) = 1 + \sum_{k=1}^{\infty} \mathbb{P}^\ast(T_1 > k) = \mathbb{E}^\ast T_1 \), we conclude that

\[
1 - F_+(x) \sim (1 - F(x)) \mathbb{E}^\ast T_1 \quad \text{as} \quad x \to \infty.
\]

(A.6)

Thus, \( 1 - F_+(x) \) is regularly varying at infinity with index \(-\alpha\).

Since \( \chi^{(r)} \) is the overshoot over level \( r \) of the random walk \( S_n \), it is also the overshoot over \( r \) of the random walk associated with the \( \zeta_k \). Denoting by \( H_+(x) \) the corresponding renewal function, i.e., \( H_+(x) = 1_{\{x \geq 0\}} + \sum_{n=1}^{\infty} \mathbb{P}^\ast(\sum_{k=1}^{n} \zeta_k \leq x) \), we have

\[
\mathbb{E}^\ast e^{-\gamma \chi^{(r)}} = \int_{[0,r]} \int_{[r-x,\infty)} e^{-\gamma(y-(r-x))} F_+(dy) H_+(dx).
\]
On introducing

\[ z(x) = \int_{[x, \infty]} e^{-\gamma(y-x)} F_+(dy), \]  
(A.7)

we obtain that

\[ \mathbb{E}^* e^{-\gamma \chi(r)} = \int_{[0, r]} z(r - x) H_+(dx) - z(0) \Delta H_+(r). \]  
(A.8)

Note that \( z(x) = O(1/x) \) as \( x \to \infty \), which follows from the following calculations:

\[
\begin{align*}
  z(x) &= \int_{[x, x+\ln x/\gamma]} + \int_{(x+\ln x/\gamma, \infty)} \leq (F_+(x + \frac{\ln x}{\gamma}) - F_+(x - 1)) + \frac{1}{x}, \\
  F_+(x + \frac{\ln x}{\gamma}) - F_+(x - 1) &\sim \frac{\alpha}{\gamma} \frac{(1 - F_+(x)) \ln x}{x},
\end{align*}
\]

where for the latter equivalence we used the fact that \((1 - F_+(x + \ln x/\gamma))/(1 - F_+(x - 1)) \sim (1+\ln x/(\gamma x))^{-\alpha}\) by the uniform convergence theorem for regularly varying functions, see Bingham, Goldie, and Teugels [6, Theorem 1.5.2].

Suppose now that \( F \) is nonlattice. Then \( F_+ \) is a nonlattice distribution too, Asmussen [2, VIII.1]. By (A.7), the function \( z(x) \) is directly Riemann integrable, as defined in Feller [14, XI.1], and

\[ \int_{0}^{\infty} z(x) \, dx = \int_{0}^{\infty} e^{-\gamma x} (1 - F_+(x)) \, dx. \]

Thus, by Theorem 3 of Erickson [10], as \( r \to \infty \),

\[ \int_{[0, r]} z(r - x) H_+(dx) \sim \left( \int_{0}^{r} (1 - F_+(x)) \, dx \right)^{-1} \frac{\sin \pi \alpha}{\pi(1 - \alpha)} \int_{0}^{\infty} e^{-\gamma x} (1 - F_+(x)) \, dx. \]

In addition, by Theorem 1 of Erickson [10], as \( r \to \infty \),

\[ \Delta H_+(r) \int_{0}^{r} (1 - F_+(x)) \, dx \to 0. \]

If we also recall (A.6) and the fact that, according to Karamata’s theorem (see Proposition 1.5.8 in Bingham, Goldie, and Teugels [6]), \( \int_{0}^{r} (1 - F_+(x)) \, dx \sim r (1 - F_+(r))/(1 - \alpha) \), we obtain the asymptotic equivalence asserted in part 1 with

\[ C_0 = \frac{1}{\mathbb{E}^* \mathbb{E}_1} \frac{\sin \pi \alpha}{\pi} \int_{0}^{\infty} e^{-\gamma x} (1 - F_+(x)) \, dx. \]

For lattice distributions, we haven’t been able to find in the literature an analogue of Erickson’s Theorem 3. Therefore, we, in effect, deduce it from the local renewal theorem of Garsia and Lamperti [11] for our particular case by using the approach of Erickson [10]. As a matter of fact, we improve on Erickson’s argument so that we can give a streamlined proof of his Theorem 3.
Let $F$ be lattice with span $h$. Then $F_+$ is also lattice with span $h$. We can write for $\theta \in (0,1)$ and suitable $A>0$, on recalling that $z(x) = O(1/x)$ as $x \to \infty$,

\[
\int_{[0,nh]} z(nh-x)H_+(dx) = \int_{[0,\theta nh]} z(nh-x)H_+(dx) + \int_{(\theta nh,nh]} z(nh-x)H_+(dx) \leq \frac{A}{(1-\theta)nh}H_+(\theta nh) + \int_{(\theta nh,nh]} z(nh-x)H_+(dx). \quad (A.9)
\]

By the fact that the tail of $F_+$ is regularly varying at infinity with index $-\alpha$, we have (see Feller [14, XIV.3] or Bingham, Goldie, and Teugels [6, 8.6]) that $H_+(x) \sim (\sin \pi \alpha/\pi \alpha)(1-F_+(x))^{-1}$ as $x \to \infty$, so

\[
\lim_{\theta \to 0} \limsup_{n \to \infty} nh(1-F_+(nh))\frac{A}{(1-\theta)nh}H_+(\theta nh) = 0. \quad (A.10)
\]

Next,

\[
nh(1-F_+(nh)) \int_{(\theta nh,nh]} z(nh-x)H_+(dx) = nh(1-F_+(nh)) \sum_{k=0}^{\infty} z((n-k)h)\Delta H_+(kh) = h \sum_{k=0}^{\infty} z(kh) \frac{n}{n-k} \frac{1-F_+(nh)}{1-F_+((n-k)h)} m(n-k) \mathbf{1}_{\{n-k \geq \theta n\}}, \quad (A.11)
\]

where we used the notation $m(k) = k(1-F_+(kh))\Delta H_+(kh)$ for $k > 0$ and $m(k) = 0$ for $k \leq 0$. Since the $\zeta_i$ assume values $kh$, $k \in \mathbb{N}$, it follows, by Garsia and Lamperti [11], that

\[
\lim_{k \to \infty} m(k) = \frac{\sin \pi \alpha}{\pi}. \quad (A.12)
\]

Hence,

\[
\lim_{n \to \infty} \frac{1}{n-k} \frac{1-F_+(nh)}{1-F_+((n-k)h)} m(n-k) \mathbf{1}_{\{n-k \geq \theta n\}} = \frac{\sin \pi \alpha}{\pi}. \quad (A.13)
\]

Also by the uniform convergence theorem for regularly varying functions,

\[
\lim_{n \to \infty} \sup_{k \leq n-\theta n} \left| \frac{1-F_+(nh)}{1-F_+((n-k)h)} - \frac{n^{-\alpha}}{(n-k)^{-\alpha}} \right| = 0.
\]

Thus,

\[
\limsup_{n \to \infty} \sup_{k=0,1,2,...} \frac{1-F_+(nh)}{1-F_+((n-k)h)} m(n-k) \mathbf{1}_{\{n-k \geq \theta n\}} < \infty,
\]

so by (A.13), Fatou’s lemma, and Lebesgue’s bounded convergence theorem,

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} z(kh) \frac{n}{n-k} \frac{1-F_+(nh)}{1-F_+((n-k)h)} m(n-k) \mathbf{1}_{\{n-k \geq \theta n\}} = \frac{\sin \pi \alpha}{\pi} \sum_{k=0}^{\infty} z(kh). \quad (A.14)
\]

Putting together (A.6), (A.9), (A.10), (A.11), and (A.14), we conclude that

\[
\lim_{n \to \infty} nh(1-F(nh)) \int_{[0,nh]} z(nh-x)H_+(dx) = \frac{1}{E^*T_1} \frac{\sin \pi \alpha}{\pi} \sum_{k=0}^{\infty} z(kh)h. \quad (A.15)
\]
By Garsia and Lamperti [11, A.6], and(A.12), $nh(1 - F(nh))z(0)\Delta H_+ (nh) \to h z(0)(E^* T_1)^{-1}\sin(\pi \alpha)/\pi$ as $n \to \infty$, so the second assertion of the lemma follows by (A.8) and (A.15) with

$$C_0' = \frac{1}{E^* T_1} \frac{\sin \pi \alpha}{\pi} \sum_{k=1}^{\infty} z(kh)h = \frac{1}{E^* T_1} \frac{\sin \pi \alpha}{\pi} \sum_{k=0}^{\infty} e^{-\gamma kh}(1 - F_+(kh))h.$$

\[\square\]

### A.3 Useful properties of slowly and regularly varying functions

The following lemma comes in useful in the proof of the convergence of the conditioned random walk. Note that the first part is Potter’s theorem (see Bingham, Goldie, and Teugels [6, Proposition 1.5.6]).

**Lemma A.5.**

1. Let $L(x)$ be a slowly varying at infinity function. Then, given an arbitrary $\varepsilon > 0$, there exists $x_0 > 0$ such that $L(x)/L(y) \leq (1 + \varepsilon)((x/y) \lor (y/x))^{\varepsilon}$ for all $x \geq x_0$ and $y \geq x_0$.

2. If $F$ is regularly varying at infinity with index $-\alpha$, where $\alpha \in (0, 1)$, then

$$\limsup_{r \to \infty} \sup_{A \to \infty} \frac{y(1 - F(ry))}{1 - F(r)} \leq 1$$

and, for $y \in [0, 1],

$$\limsup_{r \to \infty} \frac{1}{1 - F(r)} \int_0^y xF(r \, dx) \leq \frac{y^{1-\alpha}}{1 - \alpha}.$$

**Proof.** By Karamata’s representation theorem,

$$L(x) = c(x) \exp \left( \int_1^x \frac{\varepsilon(u)}{u} \, du \right),$$

where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$. Therefore, for all $x$ and $y$ large enough,

$$\frac{L(x)}{L(y)} = \frac{c(x)}{c(y)} \exp \left( \int_1^x \frac{\varepsilon(u)}{u} \, du \right) \leq (1 + \varepsilon) \exp \left( \varepsilon \int_1^y \frac{1}{u} \, du \right).$$

The inequality in part 1 of the statement now follows by a simple algebraic manipulation.

In order to prove the first inequality of part 2, note that the function $\ell(x) = x^\alpha(1 - F(x))$ is slowly varying at infinity. Hence, for given arbitrary $\varepsilon \in (0, 1 - \alpha)$, we have by part 1 for all $y \in (0, 1]$ and $r$ such that $ry$ is large enough

$$\frac{y(1 - F(ry))}{1 - F(r)} = \frac{y^{1-\alpha} \ell(ry)}{\ell(r)} \leq y^{1-\alpha}(1 + \varepsilon)y^{-\varepsilon} \leq 1 + \varepsilon.$$

We prove the second inequality. Integration by parts yields

$$\int_0^y xF(r \, dx) = \int_0^y (F(ry) - F(rx)) \, dx.$$
On picking $A \in (0, ry)$ and partitioning the integration interval $[0, y]$ into two pieces $[0, A/r]$ and $(A/r, y]$, we have

$$\frac{1}{1 - F(r)} \int_0^y x F(r \, dx) \leq \frac{A}{r(1 - F(r))} + \frac{1}{1 - F(r)} \int_{A/r}^y (1 - F(rx)) \, dx.$$ 

(A.16)

Let $\varepsilon \in (0, 1 - \alpha)$ be otherwise arbitrary. If $A$ is large enough, then by part 1, on recalling that the function $x^\alpha(1 - F(x))$ is slowly varying at infinity and $y \leq 1$, we have for all $x \in [A/r, y]$

$$\frac{1 - F(rx)}{1 - F(r)} \leq (1 + \varepsilon)x^{-\alpha - \varepsilon}.$$

Therefore, for these $A$ and $r$,

$$\frac{1}{1 - F(r)} \int_{A/r}^y (1 - F(rx)) \, dx \leq (1 + \varepsilon) \int_0^y x^{-\alpha - \varepsilon} \, dx = \frac{(1 + \varepsilon)y^{1 - \alpha - \varepsilon}}{1 - \alpha - \varepsilon}.$$

The required bound follows now by (A.16) and the fact that the first term on the right of (A.16) tends to zero as $r \to \infty$ (and as $A$ is kept fixed large enough).

**References**

[1] S. Asmussen. Conditioned limit theorems relating a random walk to its associate, with applications to risk reserve processes and the GI/G/1 queue. *Adv. in Appl. Probab.*, 14(1):143–170, 1982.

[2] S. Asmussen. *Applied Probability and Queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[3] B. von Bahr. Ruin probabilities expressed in terms of ladder height distributions. *Scand. Actuar. J.*, pp. 190–204, 1974.

[4] J. Bertoin. *Lévy Processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[5] J. Bertoin and R.A. Doney. Some asymptotic results for transient random walks. *Adv. Appl Prob.*: 28(2), 207–226, 1996.

[6] N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[7] A.A. Borovkov. *Stochastic Processes in Queueing Theory*. Nauka, 1972. (in Russian, English translation: Springer, 1976).

[8] A.A. Borovkov and K.A. Borovkov. On large deviation probabilities for random walks. II. Regular exponentially decreasing distributions. *Teor. Veroyatn. Primen.*, 49(2):209–230, 2004. English translation: *Theory of Probability and its Applications*, 49(2):189–206, 2004.
[9] E.B. Dynkin. Some limit theorems for sums of independent random variables with infinite mathematical expectations (in Russian). Izv. Akad. Nauk SSSR. Ser. Mat., 19:247–266, 1955. English translation: Select. Transl. Math. Statist. and Probability, Vol. 1, pp. 171–189, Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I., 1961.

[10] K.B. Erickson. Strong renewal theorems with infinite mean. Trans. Amer. Math. Soc., 151:263–291, 1970.

[11] A. Garsia and J. Lamperti. A discrete renewal theorem with infinite mean. Comment. Math. Helv., 37:221–234, 1962/1963.

[12] B. V. Gnedenko and A. N. Kolmogorov. Limit distributions for sums of independent random variables. Translated from the Russian, annotated, and revised by K. L. Chung. With appendices by J. L. Doob and P. L. Hsu. Revised edition. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills., Ont., 1968.

[13] S.N. Ethier and T.G. Kurtz. Markov Processes. Characterization and Convergence. Wiley, 1986.

[14] W. Feller. An Introduction to Probability Theory and its Applications. Vol. II. Second edition. John Wiley & Sons Inc., New York, 1971.

[15] A. Gut. Stopped Random Walks: Limit Theorems and Applications. Springer, New York, 1988.

[16] D.L. Iglehart. Extreme values in the GI/G/1 queue. Ann. Math. Statist., 43:627–635, 1972.

[17] J. Jacod and A.N. Shiryaev. Limit Theorems for Stochastic Processes. Springer, 1987.

[18] D.A. Korshunov. The critical case of the Cramér-Lundberg theorem on the asymptotics of the distribution of the maximum of a random walk with negative drift. Sibirsk. Mat. Zh., 46(6):1335–1340, 2005.

[19] R.Sh. Liptser and A.N. Shiryayev. Theory of Martingales. Kluwer, 1989.

[20] S.I. Resnick. Point processes, regular variation and weak convergence. Adv. in Appl. Probab., 18(1):66–138, 1986.

[21] B.A. Rogozin. The distribution of the first ladder moment and height, and fluctuation of a random walk (in Russian). Teor. Verojatnost. i Primenen., 16(4):593–613, 1971. English translation: Theory of Probability and its Applications, 16(4):575 – 595, 1971.

[22] Ya.G. Sinay. On the distribution of the first positive sum for the sequence of independent random variables. Teor. Verojatnost. i Primenen., 2:126–135, 1957.

[23] J.L. Teugels. The class of subexponential distributions. Ann. Probability, 3(6):1000–1011, 1975.

[24] W. Whitt. Some useful functions for functional limit theorems. Math. Oper. Res., 5(1):67–85, 1980.

[25] S. Zachary and S. Foss. On the exact distributional asymptotics for the supremum of a random walk with increments in a class of light-tailed distributions. Siberian Math. J., 47(6): 1265–1274, 2006.