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Pulse vaccination of an epidemic model with two parallel infectious stages and time delays

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Abstract
An epidemic model with two parallel infectious stages and time delays and pulse vaccination is proposed. We introduce four thresholds and further obtain the conditions that the disease will be extinct or not. Corollaries show that under condition that θ > max{θ1∗, θ2∗}, the disease will fade out, and if θ < min{θ1∗, θ2∗}, the disease will be endemic. Our results indicate that a larger pulse vaccination rate will lead to the eradication of a disease. Furthermore, two thresholds ℜ1∗ and ℜ2∗ show that the diversity of the contagiousness affects the basic properties of these models. In addition, numerical results indicate that the probability for an infected individual to enter different infective compartments greatly affects two infective compartments.

Keywords: Parallel infectious stages; Time delay; Pulse vaccination; Stability; Permanence

1. Introduction
Infectious diseases have tremendous influence on human life. Every year, millions of people die of various infectious diseases. Controlling infectious diseases has been an increasingly complex issue in recent years. Vaccination is an important strategy for the elimination of infectious diseases [7,11,16,18]. Recently, pulse vaccination epidemic models have been the subject of intense theoretical analysis [1,10,15,19]. Theoretical results show that the pulse vaccination strategy can be distinguished from the conventional strategies in leading to disease eradication at relatively low values of vaccination [1].

For many infectious diseases, and in particular for the diseases with a prolonged infectious period, the infection has different consequences. For instance, some infected hosts can be properly detected and treated, isolated or removed, whereas others remain undetected and untreated. It is obvious that in such circumstances the length of the infectious periods and the levels of contagiousness for these detected and undetected hosts can significantly differ. Moreover, the behavior of a virus within a host may be different. Some infections, such as Hepatitis B, can have either a highly contagious and comparatively short acute infectious stage, or can persist within a host for a long time without apparent

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Furthermore, if the transmission of the infection is governed by the nonlinear incidence of the infective compartments and after a constant infectious period [8] then into the recovered compartment.

In the study of infectious disease, a problem of interest is the coexistence of multiple strains. Examples of persistent co-circulating multistrain diseases include influenza [2], malaria [9], and dengue fever [6]. More recently, the avian flu viruses, including H5N1, were reported to coexist in several genotypes until 2004 [21]. Other examples of viruses possessing multiple strains may be seen in corona viruses, such as severe acute respiratory syndrome (SARS) [22]. The presence of multiple strains adds greater complication to models of disease dynamics due to an increasing number of stages through different infection–recovery combinatorics. So in this paper, we consider the coexistence of two strains.

Standard epidemiological models use a bilinear incidence rate based on the law of mass action [5,1]. However, as the number of susceptible populations is large, it is unreasonable to consider the bilinear incidence rate because the number of susceptible populations with every infective contact within a certain time is limited. If the population is saturated with infective population, there are kinds of incidence forms that are used in epidemiological model: the proportionate mixing incidence $\beta SI/N$ [4,20], nonlinear incidence $\beta S^p I^q$ [13,14] or $\beta I^h S(1 + \alpha I^k)$ [17].

However, epidemic models with nonlinear incidence, constant infectious period and vaccination have never been seen by now. Time delay and vaccination are introduced into epidemic disease models, which greatly enriches biologic background. Therefore, in the present paper, we create a new two-delay epidemic model with nonlinear incidence and study its dynamic behavior.

The main purpose of this paper is to establish sufficient condition for the disease to die out, and we obtain that a large pulse vaccination rate will lead to eradication of the disease. The second purpose of this paper is to establish a sufficient condition for the permanence of the epidemic model. The organization of this paper is as follows. In the next section, we propose a two-delay disease Model with pulse vaccination and infectious Period. To prove the main results, we also give two lemmas. In Section 3, using the discrete dynamical system determined by the stroboscopic map, we establish sufficient conditions for the global attractivity of infection-free periodic solution. The sufficient conditions for the permanence of the model are obtained in Section 4. In particular, some results are obtained in Section 5 if there is only one infective compartment. Finally, we give a brief conclusion.

2. Model formulation and preliminaries

Motivated by Andrei [3] and Gao [8], in this paper, we assume that there are two alternative infectious pathways and two noninteracting infective subclasses. We denote the fractions of the susceptible individuals, the infectious individuals that belong to different subclass, and the recovered individuals in the population by $S(t)$, $I_1(t)$, $I_2(t)$ and $R(t)$, respectively, that is, $S(t) + I_1(t) + I_2(t) + R(t) = N(t)$. Here $N(t)$ is the size of the total variable population at time $t$. We assume that after infection an individual immediately moves from the susceptibles compartment into one of the infective compartments and after a constant infectious period [8] then into the recovered compartment. Furthermore, if the transmission of the infection is governed by the nonlinear incidence $\beta I S^p$ and pulse vaccination is incorporated into this model, then the corresponding equations of the SIR model are:

$$
\begin{align*}
S'(t) &= \lambda - \left(\beta_1 S^p(t) I_1(t) + \beta_2 S^p(t) I_2(t)\right) - \mu S(t), \\
I'_1(t) &= P_1\left(\beta_1 S^p(t) I_1(t) - \beta_1 e^{-\tau_1} S^p(t - \tau_1) I_1(t - \tau_1) + \beta_2 S^p(t) I_2(t)
- \beta_2 e^{-\tau_2} S^p(t - \tau_2) I_2(t - \tau_2)\right) - d_1 I_1(t), \\
I'_2(t) &= P_2\left(\beta_1 S^p(t) I_1(t) - \beta_1 e^{-\tau_1} S^p(t - \tau_1) I_1(t - \tau_1) + \beta_2 S^p(t) I_2(t)
- \beta_2 e^{-\tau_2} S^p(t - \tau_2) I_2(t - \tau_2)\right) - d_2 I_2(t), \\
R'(t) &= P_1\left(\beta_1 e^{-\tau_1} S^p(t - \tau_1) I_1(t - \tau_1) + \beta_2 e^{-\tau_2} S^p(t - \tau_2) I_2(t - \tau_2)\right)
+ P_2\left(\beta_1 e^{-\tau_1} S^p(t - \tau_1) I_1(t - \tau_1) + \beta_2 e^{-\tau_2} S^p(t - \tau_2) I_2(t - \tau_2)\right) - \sigma R(t), \\
S(t^+) &= (1 - \theta)S(t), \\
I_1(t^+) &= I_1(t), \\
I_2(t^+) &= I_2(t), \\
R(t^+) &= R(t) + \theta S(t),
\end{align*}
$$

$$t \neq nT, \quad n \in \mathbb{N},$$

$$t = nT, \quad n \in \mathbb{N}.$$
Here the constant $\lambda$ is the recruitment rate into the susceptible class (that includes immigrants and the newborns that are assumed to be susceptible); parameters $\mu, d_1, d_2$ and $\sigma$ are the susceptible, infectious and recovered natural mortalities, and $\beta_1$ and $\beta_2$ are the disease transmission coefficients; $\tau_i$ $(i = 1, 2)$ is the length of the infectious period. The term $\beta_i e^{-d_i \tau_i} S^p(t - \tau_i) I(t - \tau_i)$ reflects the fact that an individual has recovered from the $i$th infective compartments and is still alive after infectious period $\tau_i$ $(i = 1, 2)$.

Parameter $P_1$ $(i = 1, 2)$ is the probability for an infected individual to enter the $i$th infective compartment with $P_1 + P_2 = 1$. Pulse vaccination can be defined as the repeated application of vaccine across an age range. Assume the pulse scheme proposes to vaccinate a fraction $\theta S$ of the entire susceptible population in a single pulse, applied every $T$ years.

Adding all the equations in model (2.1), the total variable population size is given by the differential equation

$$ N'(t) = \lambda - \mu S(t) - d_1 I_1(t) - d_2 I_2(t) - \sigma R(t), $$

from which, we have

$$ N'(t) \leq \lambda - d N(t), $$

where $d = \min\{\mu, d_1, d_2, \sigma\}$. It follows that

$$ \lim_{t \to \infty} \sup N(t) \leq \frac{\lambda}{d}. $$

The first three equations in system (2.1) do not depend on the fourth and the fifth equation. Therefore, we restrict our attention to the following reduced system:

$$
\begin{align*}
S'(t) &= \lambda - \left(\beta_1 S^p(t) I_1(t) + \beta_2 S^p(t) I_2(t)\right) - \mu S(t), \\
I_1'(t) &= P_1 \left(\beta_1 S^p(t) I_1(t) + \beta_2 S^p(t) I_2(t) - \beta_1 e^{-d_1 \tau_1} S^p(t - \tau_1) I_1(t - \tau_1)\right) \\
&\quad - \beta_2 e^{-d_2 \tau_1} S^p(t - \tau_2) I_2(t - \tau_2) - d_1 I_1(t), \\
I_2'(t) &= P_2 \left(\beta_1 S^p(t) I_1(t) + \beta_2 S^p(t) I_2(t) - \beta_1 e^{-d_1 \tau_1} S^p(t - \tau_1) I_1(t - \tau_1)\right) \\
&\quad - \beta_2 e^{-d_2 \tau_1} S^p(t - \tau_2) I_2(t - \tau_2) - d_2 I_2(t), \\
S(t^+) &= (1 - \theta) S(t), \\
I_1(t^+) &= I_1(t), \\
I_2(t^+) &= I_2(t), \\
\end{align*}
$$

for $t \neq nT, n \in \mathbb{N}$.

The initial condition of (2.3) is given as

$$ S(t) = \phi_1(t), \quad I_1(t) = \phi_2(t), \quad I_2(t) = \phi_3(t), \quad t \in [-\tau, 0]. $$

Here $\tau = \max\{\tau_1, \tau_2\}, \phi = (\phi_1, \phi_2, \phi_3)^T \in \mathbb{P}_+ \cap \mathbb{P}_+$ is the space of all piecewise functions $\phi : [-\tau, 0] \to \mathbb{R}^3_+$ with points of discontinuity at $-nT(n \in \mathbb{N})$ of the first kind and which are continuous from the left, i.e., $\phi(-nT - 0) = \phi(-nT)$, where $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i \geq 0, i = 1, 2, 3\}$. Let $\Omega$ be the following subset of $\mathbb{R}^3_+$

$$\Omega = \left\{(S(t), I_1(t), I_2(t)) \in \mathbb{R}^3_+ | S(t) \geq 0, I_1(t) \geq 0, I_2(t) \geq 0, 0 \leq S(t) + I_1(t) + I_2(t) \leq \frac{\lambda}{d}\right\}. $$

From biological considerations, we discuss system (2.3) in the closed set $\Omega$. It can be verified that $\Omega$ is positively invariant with respect to system (2.3).

3. Global attractivity of infection-free periodic solution

To prove our main results, we state some notations and lemmas which will be essential to our proofs.

**Lemma 3.1** ([8]). Consider the following impulsive differential equations

$$
\begin{align*}
u'(t) &= a - bu(t), & t \neq nT, \\
u(t^+) &= (1 - \theta)u(t), & t = nT,
\end{align*}
$$
where \( a > 0, b > 0, 0 < \theta < 1 \). Then the above system exists a unique positive periodic solution given by
\[
    u^*(t) = \frac{a}{b} + \left( \bar{u} - \frac{a}{b} \right) e^{-b(t-nT)}, \quad nT < t \leq (n+1)T,
\]
which is globally asymptotically stable, where
\[
    \bar{u} = \frac{a (1 - \theta)(1 - e^{-bT})}{b (1 - (1 - \theta)e^{-bT})}.
\]

**Definition 3.1** ([12]). Let \( V : \mathbb{R}_+ \times \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \), then \( V \) is said to belong to class \( V_0 \) if

(i) \( V \) is continuous in \((nT, (n+1)T] \times \mathbb{R}_+^3\) and for each \( X \in \mathbb{R}_+^3 \), \( \lim_{t \rightarrow (nT)^+} V(t, y) = V(nT^+, X) \) exists.

(ii) \( V \) is locally Lipschitzian in \( X \).

**Lemma 3.2** ([12]). Let \( V \in V_0 \). Assume that
\[
\begin{align*}
    D^+ V(t, x) &\leq g(t, V(t, x)), \quad t \neq nT, \\
    V(t, x(t^+)) &\leq \phi_n(V(t, x)), \quad t = nT,
\end{align*}
\]
where \( g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is continuous in \((nT, (n+1)T] \times \mathbb{R}_+^3\) and for \( u \in \mathbb{R}_+^3 \), \( k \in \mathbb{N} \), \( \lim_{t \rightarrow (nT)^+} V(t, y) = V(nT^+, u) \) exists, \( \phi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is non-decreasing. Let \( r(t) \) be the maximal solution of the scalar impulsive differential equation
\[
\begin{align*}
    u'(t) &= g(t, u(t)), \quad t \neq nT, \\
    u(t^+) &= \phi_n(u(t)), \quad t = nT, \\
    u(0^+) &= u_0,
\end{align*}
\]
existing on \([0, \infty)\). Then \( V(0^+, x_0) \leq u_0 \) implies that \( V(t, x(t)) \leq r(t), \quad t \geq 0 \), where \( x(t) \) is any solution of (2.3).

Now we shall prove the disease-free periodic solution \((S^*(t), 0, 0)\) is the global attractivity. We first demonstrate the existence of the disease-free periodic solution, in which infectious individuals are entirely absent from the population permanently, i.e. \( I_1(t) \equiv 0, I_2(t) \equiv 0 \) for all \( t > 0 \). Under this condition, the growth of susceptible individuals must satisfy
\[
\begin{align*}
    \left\{ \begin{array}{l}
    S'(t) = \lambda - \mu S(t), \quad t \neq nT, \quad n \in \mathbb{N}, \\
    S(t^+) = (1 - \theta)S(t), \quad t = nT, \quad n \in \mathbb{N}.
    \end{array} \right.
\end{align*}
\]
(3.1)

According to Lemma 3.1, we know that the periodic solution of system (3.1)
\[
S^*(t) = \frac{\lambda}{\mu} \left( 1 - \frac{\theta e^{-\mu(t-nT)}}{1 - (1 - \theta)e^{-\muT}} \right), \quad nT < t \leq (n+1)T
\]
is globally asymptotically stable. Hence system (2.3) has a disease-free periodic solution \((S^*(t), 0, 0)\).

Denote
\[
\mathcal{R}_1^1 = \frac{2P_1 \beta_1}{d_1} \lambda \left( \frac{1 - e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}} \right)^p, \quad \mathcal{R}_1^2 = \frac{2P_2 \beta_2}{d_2} \frac{\lambda}{\mu} \left( \frac{1 - e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}} \right)^p.
\]

**Theorem 3.1.** If \( \mathcal{R}_1^1 < 1 \) and \( \mathcal{R}_1^2 < 1 \), then disease-free periodic solution \((S^*(t), 0, 0)\) of system (2.3) is globally attractive.

**Proof.** Since \( \mathcal{R}_1^1 < 1 \) and \( \mathcal{R}_1^2 < 1 \), we can choose \( \varepsilon_1, \varepsilon_2 > 0 \) sufficiently small such that
\[
2P_1 \beta_1 \left( \frac{\lambda}{\mu} \left( \frac{1 - e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}} \right)^p + \varepsilon_1 \right) < d_1, \quad 2P_2 \beta_2 \left( \frac{\lambda}{\mu} \left( \frac{1 - e^{-\mu T}}{1 - (1 - \theta)e^{-\mu T}} \right)^p + \varepsilon_2 \right) < d_2.
\]
(3.2)
From the first equation of system (2.3), we have $S'(t) < \lambda - \mu S(t)$, then we consider the following comparison system with pulse:

$$
\begin{align*}
    u'(t) &= \lambda - \mu u(t), \quad t \neq nT, \quad n \in N, \\
    u(t^+) &= (1 - \theta)u(t), \quad t = nT, \quad n \in N.
\end{align*}
$$

(3.3)

By Lemma 3.1, we obtain the periodic solution of system (3.3):

$$
    u^*(t) = \frac{\lambda}{\mu} \left(1 - \frac{\theta e^{-\mu(t-nT)}}{1 - (1-\theta)e^{-\mu T}}\right), \quad nT < t \leq (n+1)T.
$$

Furthermore, from the second and the third equation, we have, for

$$
    I(t) = \frac{1 - e^{-\mu T}}{1 - (1-\theta)e^{-\mu T}} + \epsilon := \tilde{S}, \quad nT < t \leq (n+1)T, \quad n > m_1.
$$

(3.4)

By the comparison theorem [12], and from the fact that $S(t) < u(t) < u^*(t) + \epsilon$, there exists $m_1 \in Z_+$ such that $S(t) < u^*(t) + \epsilon \leq \frac{\lambda}{\mu} \left(1 - \frac{1 - e^{-\mu T}}{1 - (1-\theta)e^{-\mu T}}\right) + \epsilon$.

Furthermore, from the second and the third equation, we have, for $t > nT$ and $n > m_1$,

$$(P_2 I_1(t) + P_1 I_2(t))' \leq P_2 \left(2P_1 \beta_1 \bar{S}^p - d_1\right) I_1(t) + P_1 \left(2P_2 \beta_2 \bar{S}^p - d_2\right) I_2(t).$$

According to (3.2) and (3.3), we have $(P_2 I_1(t) + P_1 I_2(t))' \leq 0$, then $\lim_{t \to \infty} (P_2 I_1(t) + P_1 I_2(t)) = 0$. Therefore, $\lim_{t \to \infty} I_1(t) = 0$ and $\lim_{t \to \infty} I_2(t) = 0$, i.e. for any sufficiently small $\epsilon_3, \epsilon_4 > 0$, there exists an integer $m_2 > m_1$ such that $I_1(t) < \epsilon_3, I_2(t) < \epsilon_4$ for all $t > m_2 T$. From the first equation of system (2.3), we have $S'(t) > \lambda - (2\beta_1 \bar{S}^p - 2\beta_2 \bar{S}^p) S(t)$, then we consider the following comparison system with pulse:

$$
\begin{align*}
    u_1'(t) &= \lambda - (2\beta_1 \bar{S}^p - 2\beta_2 \bar{S}^p) u_1(t), \quad t \neq nT, \quad n \in N, \\
    u_1(t^+) &= (1 - \theta)u_1(t), \quad t = nT, \quad n \in N.
\end{align*}
$$

(3.5)

From Lemma 3.1, we obtain the periodic solution of system (3.5):

$$
    u_1^*(t) = \frac{\lambda}{\beta_1 \bar{S}^p - 2\beta_2 \bar{S}^p - 2\beta_2 \bar{S}^p} \left(1 - \frac{\theta e^{-\mu(t-nT)}}{1 - (1-\theta)e^{-\mu T}}\right), \quad nT < t \leq (n+1)T.
$$

By the comparison theorem [12], there exists an integer $m_3 > m_2$ such that

$$
    S(t) \geq u_1(t) > u_1^*(t) - \epsilon, \quad nT < t \leq (n+1)T, \quad n > m_3.
$$

(3.6)

Since $\epsilon$ is sufficiently small, from (3.4) and (3.6), we know that $S(t) - \epsilon < S(t) < S^*(t) + \epsilon, \quad nT < t \leq (n+1)T$.

Hence, infection-free periodic solution $(S^*(t), 0, 0)$ of system (2.3) is globally attractive. The proof is completed.

Next, we give equivalent conditions of Theorem 3.1 for a well biological meaning. They are given in the following corollaries.

Set

$$
    \theta_1^* = (e^{\mu T} - 1) \left(\left(\frac{2P_1 \beta_1 \lambda}{d_1 \mu}\right)^{1/p} - 1\right), \quad \theta_2^* = (e^{\mu T} - 1) \left(\left(\frac{2P_2 \beta_2 \lambda}{d_2 \mu}\right)^{1/p} - 1\right),
$$

and

$$
    T_1^* = \frac{1}{\mu} \ln \frac{\theta}{1 - \frac{d_1 \mu}{\sum_1 P_1 \beta_1} \frac{1}{1/p}}, \quad T_2^* = \frac{1}{\mu} \ln \frac{1 + \theta}{\frac{d_2 \mu}{\sum_2 P_2 \beta_2} \frac{1}{1/p}}.
$$

**Corollary 3.1.** Suppose that $(\frac{2P_1 \beta_1 \lambda}{d_1 \mu})^{1/p} > 1$ and $(\frac{2P_2 \beta_2 \lambda}{d_2 \mu})^{1/p} > 1$. Then disease-free periodic solution $(S^*(t), 0, 0)$ of system (2.3) is globally attractive provided that $\theta > \max\{\theta_1^*, \theta_2^*\}$. 

Corollary 3.2. If \( \left( \frac{d_1 \mu}{2 \lambda \rho_1 \rho_2} \right)^{1/p} < 1 \) and \( \left( \frac{d_2 \mu}{2 \lambda \rho_3 \rho_4} \right)^{1/p} < 1 \), then the disease-free periodic solution \( (S^*(t), 0, 0) \) of system (2.3) is globally attractive provided that \( T < \min\{T^1_+, T^2_+\} \).

Remark 3.1. Theorem 3.1 determines the global attractivity of the disease-free periodic solution of system (2.3) in \( \Omega \) for the case \( \mathfrak{R}^*_1 < 1 \) and \( \mathfrak{R}^*_2 < 1 \). Its epidemiology implication is that the infectious population vanishes in time so the disease dies out. Corollaries 3.1 and 3.2 imply that a larger pulse vaccination rate or a shorter pulse period of immune vaccination will lead to the eradication of a disease.

4. Permanence

In this section, we state the disease is endemic if the infectious population persists above a certain positive level for sufficiently large time. The endemicity of the disease can be well captured and studied through the notation of uniform persistence.

Definition 4.1. System (2.3) is said to be uniformly persistent if there exist positive constants \( M_i \geq m_i > 0 \) (Both are independent of the initial values), such that every solution \( (S(t), I_1(t), I_2(t)) \) with positive initial conditions of system (2.3) satisfies

\[
S(t) \leq M_1, \quad m_2 \leq I_1(t) + I_2(t) \leq M_2.
\]

Denoted by

\[
\mathfrak{R}^*_1 = \frac{\beta_1}{d_1} \left( \frac{\lambda}{\mu} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta) e^{-\mu T}} \right)^p (1 - e^{-d_1 T_1}), \quad \mathfrak{R}^*_2 = \frac{\beta_2}{d_2} \left( \frac{\lambda}{\mu} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta) e^{-\mu T}} \right)^p (1 - e^{-d_2 T_2})
\]

we have

Theorem 4.1. If \( \mathfrak{R}^*_1 > 1 \) and \( \mathfrak{R}^*_2 > 1 \), then the disease is uniformly persistent.

Proof. Let \( (S(t), I_1(t), I_2(t)) \) be any solution with initial values of system (2.3), then it is obvious that \( S(t) \leq \frac{1}{\theta}, I_1(t) \leq \frac{1}{\theta}, I_2(t) \leq \frac{1}{\theta} \) for all \( t > 0 \). We are left to prove there exist positive constants \( m_S, m_I \) and \( t_0 \) \( (t_0 \) is sufficiently large) such that \( S(t) \geq m_S, I(t) \geq m_I \) for all \( t > t_0 \).

Firstly, from the first equation of system (2.3), we have

\[
S'(t) \geq \lambda - \left( \mu + \frac{\beta_1}{d_1} \frac{\lambda}{d} + \frac{\beta_2}{d_2} \frac{\lambda}{d} \right) S(t).
\]

Considering the following comparison equations:

\[
\begin{align*}
 u_2(t) &= \lambda - \left( \mu + \frac{\beta_1}{d_1} \frac{\lambda}{d} + \frac{\beta_2}{d_2} \frac{\lambda}{d} \right) u_2(t), \quad t \neq nT, n \in N, \\
 u_2(t^+) &= (1 - \theta) u_2(t), \quad t = nT, n \in N,
\end{align*}
\]

by Lemma 3.1 and the comparison theorem [3], we know that for any sufficiently small \( \tilde{\varepsilon} > 0 \), there exists a \( t_0 \) (\( t_0 \) is sufficiently large) such that

\[
S(t) \geq u_2(t) > u_2^*(t) - \tilde{\varepsilon} \geq \frac{\lambda}{\mu + \frac{\beta_1}{d_1} \frac{\lambda}{d} + \frac{\beta_2}{d_2} \frac{\lambda}{d}} \frac{(1 - \theta)(1 - e^{-\left(\mu + \frac{\beta_1}{d_1} \frac{\lambda}{d} + \frac{\beta_2}{d_2} \frac{\lambda}{d}\right) T})}{1 - (1 - \theta) e^{-\left(\mu + \frac{\beta_1}{d_1} \frac{\lambda}{d} + \frac{\beta_2}{d_2} \frac{\lambda}{d}\right) T}} \tilde{\varepsilon} := m_S > 0.
\]

Now, we shall prove there exists an \( m_I > 0 \) such that \( I_1(t) + I_2(t) \geq m_I \) for all sufficiently large \( t > t_0 \). For clarity, we prove it in the following two steps:

Step 1. Since \( \mathfrak{R}^*_1 > 1 \) and \( \mathfrak{R}^*_2 > 1 \) there exist sufficiently small \( m_i^* > 0 \) and \( \tilde{\varepsilon} > 0 \) such that

\[
\beta_1 \eta^p (1 - e^{-d_1 T_1}) - d_1 > 0, \quad \beta_2 \eta^p (1 - e^{-d_2 T_2}) - d_2 > 0.
\]
where
\[
\eta = \frac{\lambda}{\mu + \beta_1 m^*_T + \beta_2 m^*_T} \left( \frac{(1 - \theta)(1 - e^{-(\mu + \beta_1 m^*_T + \beta_2 m^*_T)T})}{1 - (1 - \theta)e^{-(\mu + \beta_1 m^*_T + \beta_2 m^*_T)T}} \right) - \bar{\varepsilon}.
\]

For \( m^*_T \), there must exist a \( t_1 > 0 \) such that \( I_1(t_1) + I_2(t_1) \geq m^*_T \). Otherwise, \( I_1(t) + I_2(t) < m^*_T \) for all \( t > 0 \). From the first equation of system (2.3), we have
\[
S'(t) > \lambda - (\mu + \beta_1 m^*_T + \beta_2 m^*_T)S(t).
\]

Consider the following comparison equations:
\[
\begin{align*}
\begin{cases}
\displaystyle u'_3(t) = \lambda - (\mu + \beta_1 m^*_T + \beta_2 m^*_T)u_3(t), & t \neq nT, \ n \in \mathbb{N}, \\
\displaystyle u_3(T^+) = (1 - \theta)u_3(T), & t = nT, \ n \in \mathbb{N}.
\end{cases}
\end{align*}
\]

By the same method, there exists \( t^* > 0 \), for \( t > t^* \), we have
\[
S(t) \geq u_3(t) > u^*_3(t) - \bar{\varepsilon} \geq \frac{\lambda}{\mu + \beta_1 m^*_T + \beta_2 m^*_T} \left( \frac{(1 - \theta)(1 - e^{-(\mu + \beta_1 m^*_T + \beta_2 m^*_T)T})}{1 - (1 - \theta)e^{-(\mu + \beta_1 m^*_T + \beta_2 m^*_T)T}} \right) - \bar{\varepsilon} := \eta. \quad (4.3)
\]

The second and the third equations of system (2.3) can be translated into the following form:
\[
I'_1(t) + I'_2(t) = \left( \beta_1 S^p(t)(1 - e^{-d_1 t_1}) - d_1 \right) I_1(t) + \left( \beta_2 S^p(t)(1 - e^{-d_2 t_2}) - d_2 \right) I_2(t)
+ \beta_1 e^{-d_1 t_1} \int_{t_1}^{t} S^p(\xi) I_1(\xi) d\xi + \beta_2 e^{-d_2 t_2} \int_{t_2}^{t} S^p(\xi) I_2(\xi) d\xi.
\]

Define a function \( V(t) \) as follows:
\[
V(t) = I_1(t) + I_2(t) - \beta_1 e^{-d_1 t_1} \int_{t_1}^{t} S^p(\xi) I_1(\xi) d\xi - \beta_2 e^{-d_2 t_2} \int_{t_2}^{t} S^p(\xi) I_2(\xi) d\xi.
\]

For \( t > t^* \), the derivative of \( V(t) \) along the solution of system (2.3) is
\[
V'(t) = \left( \beta_1 S^p(t)(1 - e^{-d_1 t_1}) - d_1 \right) I_1(t) + \left( \beta_2 S^p(t)(1 - e^{-d_2 t_2}) - d_2 \right) I_2(t)
+ \left( \beta_1 \eta^p(1 - e^{-d_1 t_1}) - d_1 \right) I_1(t) + \left( \beta_2 \eta^p(1 - e^{-d_2 t_2}) - d_2 \right) I_2(t).
\]

From (4.3), we obtain \( V'(t) > 0 \) for \( t > t^* \), which implies that \( V(t) \to \infty \) as \( t \to \infty \). This contradicts the boundedness of \( V(t) \). Hence, there exists a \( t_1 > 0 \) such that \( I_1(t_1) + I_2(t_1) \geq m^*_T \).

**Step 2.** According to step 1, for any positive solution \( (S(t), I_1(t), I_2(t)) \) of system (2.3), we are left to consider two cases. First, if \( I_1(t) + I_2(t) > m^*_T \) for all \( t > t_1 \), then our aim is obtained. Second, if \( I_1(t) + I_2(t) < m^*_T \) for some \( t > t_1 \), we can choose \( h > 0 \) and \( t_0 > \max\{t^*, t_1 + \tau\} \) (\( t_0 \) is demanded be sufficiently large) such that \( I_1(t) + I_2(t) \leq m^*_T \), \( I_1(t_0) + I_2(t_0) = I_1(t_0 + h) + I_2(t_0 + h) = m^*_T \) and \( S(t) > \eta \) for \( t \in [t_0, t_0 + h] \). Therefore, there exists a \( g \) \( (0 < g < \tau) \) such that for \( t \in [t_0, t_0 + g] \)
\[
I_1(t) + I_2(t) \geq \frac{m^*_T}{2}. \quad (4.4)
\]

In this case, we shall discuss three possible cases in terms of the sizes of \( g, h, \) and \( \tau \):

**Case I.** If \( h \leq g < \tau \), then it is obvious that \( I_1(t) + I_2(t) \geq m^*_T \), for \( t \in [t_0, t_0 + h] \).

**Case II.** If \( g < h \leq \tau \), then from the second equation of system (2.3), we can deduce
\[
I_1(t) + I_2(t) = \beta_1 \int_{t_1 + \xi}^{t} S^p(\xi) I_1(\xi)e^{-d_1 (t - \xi)} d\xi + \beta_2 \int_{t_2 + \xi}^{t} S^p(\xi) I_2(\xi)e^{-d_2 (t - \xi)} d\xi. \quad (4.5)
\]

By (4.3)–(4.5), we have
\[
\begin{align*}
I_1(t) + I_2(t) > \beta_1 \int_{t_0}^{t_0 + g} S^p(\xi) I_1(\xi)e^{-d_1 (t - \xi)} d\xi + \beta_2 \int_{t_0}^{t_0 + g} S^p(\xi) I_2(\xi)e^{-d_2 (t - \xi)} d\xi
&> (\beta_1 e^{-d_1 \tau} - \beta_2 e^{-d_2 \tau}) \eta g \frac{m^*_T}{2} := m^{**}_T
\end{align*}
\]
for \( t \in [t_0, t_0 + h] \).

**Case III.** If \( g < \tau < h \), we shall consider the following two cases, respectively.

**Case IIIa.** For \( t \in [t_0, t_0 + \tau] \), it is easy to obtain \( I_1(t) + I_2(t) > m_I^{**} \) from (4.6).

**Case IIIb.** For \( t \in [t_0 + \tau, t_0 + h] \), we claim \( I_1(t) + I_2(t) > m_I^* \). Otherwise, we suppose there is a \( \tilde{t} \in [t_0 + \tau, t_0 + h] \), such that \( I_1(t) + I_2(t) > m_I^{**} \) for \( t \in [t_0, \tilde{t}] \), and \( I_1(\tilde{t}) + I_2(\tilde{t}) = m_I^{**} \). From (4.2), (4.3) and (4.5), we have

\[
I_1(\tilde{t}) + I_2(\tilde{t}) = \beta_1 \int_{-\tau}^{\tilde{t}} S^p(\xi) I_1(\xi) e^{-d_1(\tilde{t} - \xi)} d\xi + \beta_2 \int_{-\tau}^{\tilde{t}} S^p(\xi) I_2(\xi) e^{-d_2(\tilde{t} - \xi)} d\xi > \frac{\beta_1 p^p}{d_1} (1 - e^{-d_1 \tau}) m_I^{**} + \frac{\beta_2 p^p}{d_2} (1 - e^{-d_2 \tau}) m_I^{**} > m_I^{**},
\]

which is contradictory to \( I_1(\tilde{t}) + I_2(\tilde{t}) = m_I^{**} \). Therefore, the claim holds true.

Since \( t_0 \) is arbitrary, there exists \( m_I := \min\{m_I^*, m_I^{**}\} > 0 \), such that \( I_1(t) + I_2(t) \geq m_I \) for all \( t > t_0 \). The proof is completed.

Denoted by

\[
\theta_1^* = \frac{(1 - e^{-\mu T}) (1 - \frac{\mu}{\lambda} \frac{d_1}{\beta_1 (1 - e^{-d_1 \tau})})^{1/p}}{1 - e^{-\mu T} + e^{-\mu T} \frac{\mu}{\lambda} \frac{d_1}{\beta_1 (1 - e^{-d_1 \tau})} \tau_1^*}, \quad \theta_2^* = \frac{(1 - e^{-\mu T}) (1 - \frac{\mu}{\lambda} \frac{d_2}{\beta_2 (1 - e^{-d_2 \tau})})^{1/p}}{1 - e^{-\mu T} + e^{-\mu T} \frac{\mu}{\lambda} \frac{d_2}{\beta_2 (1 - e^{-d_2 \tau})} \tau_2^*},
\]

\[
\tau_1^* = \frac{1}{d_1} \ln \frac{1}{1 - \frac{d_1}{\beta_1 (\frac{1}{\lambda} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta)e^{-\mu T}})^p}}, \quad \tau_2^* = \frac{1}{d_2} \ln \frac{1}{1 - \frac{d_2}{\beta_2 (\frac{1}{\lambda} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta)e^{-\mu T}})^p}},
\]

by Theorem 4.1, Parallel conclusions are given in the following corollaries.

**Corollary 4.1.** If \( \theta < \min\{\theta_1^*, \theta_2^*\} \), then the disease is uniformly persistent in model (2.3).

**Corollary 4.2.** If \( \beta_1 \left( \frac{\lambda}{\mu} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta)e^{-\mu T}} \right)^p > d_1 \) and \( \beta_2 \left( \frac{\lambda}{\mu} \frac{(1 - \theta)(1 - e^{-\mu T})}{1 - (1 - \theta)e^{-\mu T}} \right)^p > d_2 \), then the disease is uniformly persistent in model (2.3) provided that \( \tau_1 > \tau_1^* \) and \( \tau_2 > \tau_2^* \).

**Remark 4.1.** Corollaries 4.1 and 4.2 indicate that a less pulse vaccination rate or a longer infectious period will not lead to the eradication of the two diseases at the same time.

Next, we present some numerical simulations to illustrate the effects of different probability on population. Consider the following choice of parametric values:

\( \lambda = 2, \beta_1 = 0.8, \beta_2 = 0.6, d = 0.2, p = 1.2, \mu = 0.1, d_1 = 0.2, d_2 = 0.1, \tau_1 = 0.1, \tau_2 = 0.2, T = 2, \theta = 0.1 \).

System (2.3) is numerically solved for the above choice of parameters and time series are displayed in Fig. 1(a)–(c) with initial value \( S(0) = 0.1, I_1(0) = 1, I_2(0) = 1 \) for 50 pulsing cycles. The last 10 pulsing cycles are plotted in Fig. 1(d). The solid line denotes time series of the variables when \( P_1 = 0.3, P_2 = 0.7 \). The dotted line denotes time series of the variables when \( P_1 = 0.2, P_2 = 0.8 \). The dashed line denotes time series of the variables when \( P_1 = 0.1, P_2 = 0.9 \). Fig. 1 displays the effects of the probability for an infected individual to enter different infective compartments. For the above cases, the levels of the total infective are displayed in Fig. 2(a).

Finally, we investigate the effect of parameter \( p \) on the extinction of the disease. Let \( \lambda = 2, \beta_1 = 0.8, \beta_2 = 0.6, \mu = 0.1, d_1 = d_2 = 0.1, \tau_1 = \tau_2 = 0.1, T = 2, \theta = 0.1 \). The levels of the total infective are displayed in Fig. 2(b) as parameter \( p \) takes 0.6, 1 and 1.2, respectively.
5. An epidemic model with pulse vaccination and time delay

If there is only an infective compartment $I_2(t)$, then system (2.1) becomes

\[
\begin{align*}
S'(t) &= \lambda - \beta_2 S(t) I_2(t) - \mu S(t), \\
I_2'(t) &= \beta_2 S(t) I_2(t) - \beta_2 e^{-d_2 \tau_2} S(t - \tau_2) I_2(t - \tau_2) - d_2 I_2(t), \\
R'(t) &= \beta_2 e^{-d_2 \tau_2} S(t - \tau_2) I_2(t - \tau_2) - \sigma R(t), \\
S(t^+) &= (1 - \theta) S(t), \\
I_2(t^+) &= I_2(t), \\
R(t^+) &= R(t) + \theta S(t),
\end{align*}
\]

\(t \neq nT, \quad n \in \mathbb{N}.
\]

\(t = nT, \quad n \in \mathbb{N}.
\]

(5.1)

Denoted by

\[
\Re_a = \frac{\beta_2 \lambda}{d_2 \mu} \left( \frac{1 - e^{-\mu T}}{1 - (1 - \theta) e^{-\mu T}} \right)^p,
\]

\[
\Re^* = \frac{\beta_2}{d_2} \left( \frac{\lambda (1 - \theta)(1 - e^{-\mu T})}{\mu (1 - \theta) e^{-\mu T}} \right)^p (1 - e^{-d_2 \tau_2})
\]
Fig. 2. (a) The levels of the total infective under the different probability ratios of an infected individual to enter the two infective compartments. (b) The effect of parameter $p$ on whether the disease will be extinct or not.

and

$$\theta^* = (e^{\mu T} - 1) \left( \left( \frac{\beta_2 \lambda}{d_2 \mu} \right)^{1/p} - 1 \right)$$

by Theorems 3.1 and 4.1, we get the corollary as follows.

**Corollary 5.1.** If $\Re_1^* < 1$, then disease-free periodic solution $(S^*(t), 0, \frac{A}{d} - S^*(t))$ of system (5.1) is globally attractive, if $\Re_2^* > 1$, then the disease is permanent in model (5.1).

**Corollary 5.2.** Suppose that $(\frac{\beta_2 \lambda}{d_2 \mu})^{1/p} > 1$. If $\theta > \theta^*$, then disease-free periodic solution $(S^*(t), 0, \frac{A}{d} - S^*(t))$ of system (5.1) is globally attractive.

Denoted by

$$\theta^* = \frac{(1 - e^{-\mu T}) \left( 1 - \frac{\mu}{\lambda} \left( 1 - \frac{d_2}{\beta_2 (1 - e^{-d_2 T_2})} \right)^{1/p} \right)}{1 - e^{-\mu T} + e^{-\mu T} \frac{\mu}{\lambda} \left( 1 - \frac{d_2}{\beta_2 (1 - e^{-d_2 T_2})} \right)^{1/p}}$$

and

$$\tau_2^* = \frac{1}{d_2} \ln \frac{1}{1 - \frac{d_2}{\beta_2 \left( \frac{\lambda (1 - \theta)(1 - e^{-\mu T})}{\mu (1 - (1 - \theta) e^{-\mu T})} \right)^p}}$$

by Corollaries 4.1 and 4.2, we get the corollary as follows.

**Corollary 5.3.** If $\theta < \theta^*$ hold, then the disease is permanent in model (5.1).

**Corollary 5.4.** If $\beta_2 \left( \frac{\lambda (1 - \theta)(1 - e^{-\mu T})}{\mu (1 - (1 - \theta) e^{-\mu T})} \right)^p > d_2$, then the disease will be endemic and system (5.1) is permanent provided that $\tau_2 > \tau_2^*$.

6. Conclusions

Some infectious diseases possess multiple parallel infectious stages such as Hepatitis B. This leads to the diversity of the levels of contagiousness and the lengths of the infective period which affect the system dynamics. In view of these facts and based on pulse vaccination, an epidemic model with two parallel infectious stages and time delays and pulse vaccination is proposed. We introduce four thresholds $\Re_{1x}, \Re_{2x}, \Re_{1z}^*$ and $\Re_{2z}^*$ and further obtain that the disease will be extinct if $\Re_{1x}^* < 1$ and $\Re_{2x}^* < 1$, and persistent if $\Re_{1x}^* > 1$ and $\Re_{2x}^* > 1$. Corollaries 3.1 and 4.1 show that
under condition that $\theta > \max \{\theta_1^*, \theta_2^*\}$ the disease will fade out, and if $\theta < \min \{\theta_1^*, \theta_2^*\}$, the disease will be uniformly persistent. Our results indicate that a larger pulse vaccination rate will lead to the eradication of a disease. Furthermore, two thresholds $R_1^*$ and $R_2^*$ show that the diversity of the contagiousness affects the basic properties of these models.

Finally, we illustrate how probability for an infected individual to enter different infective compartments affects two infective compartments. (1) Corollaries 3.1 and 5.2 display the conditions of the eradication of the disease when there are two infective compartments and only one infective compartment, respectively. Obviously, the threshold $\max \{\theta_1^*, \theta_2^*\}$ is larger than $\theta_*$, which implies that a larger fraction of susceptibles should be vaccinated against the disease if multiple strains coexist. (2) Fig. 2(a) indicates that the number of the infected decreases as the probability ratio ($P_1 : P_2$) of an infected individual to enter the two infective compartments increases when the other parameters are the same as those in Fig. 1. (3) During the above investigation, we find that whether the disease will be extinct or not is dependent on the parameter $p$ in the nonlinear incidence $\beta IS^p$ which governs the transmission of the infection (see Fig. 2(b)).

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