Weak nonlinearity for strong nonnormality

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Natural and forced oscillations in fluids

Helmholtz resonators

https://physics.case.edu

Turbulent shear layer

Brown and Roshko (1974)

Bluff body wakes

en.wikipedia.org/wiki/File:Vortex-street-1.jpg
Natural and forced oscillations in fluids

Sloshing

Bongarzone et al. (2022)

2D Backward facing step Re=500, Vorticity

Boujo et al. (2013), Mantic-Lugo et al. (2016)

Facing jets, Re=80, dye

Bertsch et al. (2020)
Intrinsic and self-sustained dynamics
Canonical example: von Karman vortex street for $\text{Re}>47$, facing jets, ...

(Natural) oscillators

Entry #V0036

Swinging Jets

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Intrinsic self-sustained dynamics
Linearly unstable
ill-posed linear response to harmonic forcing

Bongarzone et al. (2021)
Resonators

Linearly stable
Amplification of weakly damped eigenmodes
Examples: Helmholtz resonators, Mechanical oscillators, \textit{Sloshing}

Viola et al. (2017)
Resonators

Water: \( R = 2.5 \text{ cm}, h/R = 0.5, d/R = 0.2 \)

\[ f = \frac{\Omega}{2\pi} [\text{Hz}], \text{ forcing freq.} \]

\[ A/R, \text{ max resp. amp.} \]

\[ f_1,n = \frac{\omega_1,n}{2\pi} [\text{Hz}], \text{ natural freq.} \]

\[ \sigma_{1,n} [1/\text{s}], \text{ damping} \]

\[ \text{sloshing} \]

\[
X_0(t) = \bar{a}_x \sin(\Omega t) e_x
\]
Amplifiers

Gallaire et al. (2016)
Amplifiers

Linearly **stable**
Amplification of external disturbances, irrespective of the eigenfrequency spectrum
Selective but broad band response
Examples: Jets, Separated boundary layers,…

Backward facing step Re=500

Boujo et al. (2013), Mantic-Lugo et al. (2016)

Ducimetièrè et al. (2022)
Three categories of flows

Oscillators

Amplifiers

Resonators

Water: $R = 2.5$ cm, $h'/R = 0.5$, $d'/R = 0.2$
Outline

1. Natural and forced oscillations in flows

2. (Strong) non-normality in amplifiers

3. Weak non-linearity in resonators and oscillators

4. The odd alliance
   i. Harmonic response
   ii. Transient growth

5. Conclusions
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5. Conclusions

Trefethen (1993), Farrell and Ioannou (1996), Schmid and Heningson (1993)…
Non-normality: formal definition

A linear operator $L$ is normal if it commutes with its adjoint $LL^+ = L^+L$
In this case, the eigenmodes form an orthogonal basis

If $L$ is non-normal (and stable), the damping rates of the eigenvalues do not say everything about the dynamics, which possibly implies

1. Transient growth despite asymptotic decay at large time
2. Strong response to harmonic forcing, away from eigenfrequencies
3. Strong sensitivity to small operator perturbations

Trefethen (1993), Farrell and Ioannou (1996), Schmid and Heningson (1993)…
Non-normality and transient growth

\[ \frac{dq}{dt} = Lq \]

With solution in the form of matrix exponential

\[ q = \exp(tL)q_0; \quad q(t = 0) = q_0 \]

Maximum amplification (gain) at time \( t \) (horizon) over all initial conditions

\[ G(t) = \max_{q_0} \frac{||q||^2}{||q_0||^2} = \max_{q_0} \frac{||\exp(tL)q_0||^2}{||q_0||^2} \]
Non-normality and transient growth

\[ L = S \Lambda S^{-1} \]

**Eigenvalue decomposition**

\[ || \exp(tL) ||^2 = || \exp(tS\Lambda S^{-1}) ||^2 = || S \exp(t\Lambda) S^{-1} ||^2 \]

S: Column eigenvector
\( \Lambda \): Diagonal eigenvalues

**Traditional stability analysis:**

*Behavior deduced by system eigenvalues*
Bounds on matrix exponential norm

\[ e^{2t\lambda_{\text{max}}} \leq \|\exp(tL)\|^2 \leq \|S\|^2\|S^{-1}\|^2 e^{2t\lambda_{\text{max}}} \]

**Condition number:**

\[ \kappa(S) = \|S\|^2\|S^{-1}\|^2 \]

**Normal stability problem:**

Orthogonal eigenvectors

*Eigenvalues capture the dynamics*

**Non-Normal stability problem:**

Non-orthogonal eigenvectors

*Eigenvalues capture the asymptotic dynamics, not the transient behavior*
Non-normality and transient growth

\[ \mathbf{x}(t) = a_1 e^{\lambda_1 t} \mathbf{e}_1 + a_2 e^{\lambda_2 t} \mathbf{e}_2 \]

\[ ||\mathbf{x}(t)||^2 = a_1^2 e^{2\lambda_1 t} ||\mathbf{e}_1||^2 + a_2^2 e^{2\lambda_2 t} ||\mathbf{e}_2||^2 + a_1 a_2 < \mathbf{e}_1, \mathbf{e}_2 > e^{(\lambda_1 + \lambda_2) t} \]

\[ \mathbf{B} = \begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix} \]

\[ \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \]

Also exponential decay is observed for large times, the non-orthogonal superposition of eigenvectors can lead to transient energy growth.
Non-normality and harmonic response
[non-normality is only a necessary condition for transient growth]
Non-normality, measles outbreak in the London tube network

Asilani & Carletti (2018)
Transient growth:

\[ \frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L \]

Backward facing step flow

Re=500
Backward facing step flow

\[
\frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L
\]

Transient growth:

Re=500

initial condition

amplified solution at t=58

Blackburn et al, 2008
Non-normality and harmonic response

\[ \frac{\partial u}{\partial t} = Lu + f \]

insert:

\[ f(x, t) = \hat{f}(x)e^{i\omega t} + c. c \]
\[ u(x, t) = \hat{u}(x)e^{i\omega t} + c. c \]

to obtain:

\[ \hat{u} = (i\omega I - L)^{-1} \hat{f} = R(i\omega)\hat{f} \]
Optimal response to harmonic forcing

\[
\frac{\partial u}{\partial t} = Lu + f
\]

insert:

\[
\begin{align*}
  f(x, t) &= \hat{f}(x)e^{i\omega_0 t} + c.c \\
  u(x, t) &= \hat{u}(x)e^{i\omega_0 t} + c.c
\end{align*}
\]

to obtain:

\[
\hat{u} = (i\omega_0 I - L)^{-1} \hat{f} = R(i\omega_0) \hat{f}
\]

optimize:

\[
G(i\omega_0) = \max_f \frac{\|\hat{u}\|}{\|\hat{f}\|} = \|R(i\omega_0)\| = \frac{1}{\epsilon_0}
\]

under the scalar product:

\[
\langle \hat{u}_a | \hat{u}_b \rangle = \int_{\Omega} \hat{u}_a^H \hat{u}_b d\Omega
\]
Optimal response to harmonic forcing

\[
\frac{\partial \mathbf{u}}{\partial t} = L \mathbf{u} + \mathbf{f}
\]

Insert:
\[
\begin{align*}
\mathbf{f}(\mathbf{x}, t) &= \hat{\mathbf{f}}(\mathbf{x}) e^{i\omega_0 t} + \mathbf{c}.\mathbf{c} \\
\mathbf{u}(\mathbf{x}, t) &= \hat{\mathbf{u}}(\mathbf{x}) e^{i\omega_0 t} + \mathbf{c}.\mathbf{c}
\end{align*}
\]

Optimize:
\[
G(i\omega_0) = \max_{\hat{\mathbf{f}}} \frac{\|\hat{\mathbf{u}}\|}{\|\hat{\mathbf{f}}\|} = \|R(i\omega_0)\| = \frac{1}{\epsilon_0}
\]
under the scalar product
\[
\langle \hat{\mathbf{u}}_a | \hat{\mathbf{u}}_b \rangle = \int_{\Omega} \hat{\mathbf{u}}_a^H \hat{\mathbf{u}}_b d\Omega
\]

Singular value decomposition:
\[
R(i\omega_0)^{-1} \hat{\mathbf{u}}_o = \epsilon_o \hat{\mathbf{f}}_o
\]

With normalization
\[
\langle \hat{\mathbf{u}}_o | \hat{\mathbf{u}}_o \rangle = \|\hat{\mathbf{u}}_o\|^2 = 1
\]
\[
\|\hat{\mathbf{f}}_o\| = 1
\]

Strong nonnormality: \(\epsilon_o \ll 1\)
Optimal response to harmonic forcing

Bounds of resolvent norm

- **Diagonalize** the system matrix $L$

$$L = S \Lambda S^{-1}$$

*Eigenvalue decomposition*

$S$: Column eigenvector

$\Lambda$: Diagonal eigenvalues

$$\frac{1}{\text{dist}\{i\omega, \Lambda\}} \leq \| (i\omega - L)^{-1} \| = \| S(i\omega - \Lambda)^{-1} S^{-1} \| \leq \kappa(S) \frac{1}{\text{dist}\{i\omega, \Lambda\}}$$

Non-Normal system: upper and lower bound differ

we can have a pseudo-resonance

Strong amplification also far from system eigenfrequency

$$\kappa(S) \gg 1$$
Strong nonnormality

\[ \frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L \]

Harmonic forcing:

Transient growth:

Re=500

Mantic-Lugo & Gallaire, 2016

Blackburn et al, 2008
Transient growth:

\[ \frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L \]

Harmonic forcing:

- Nonnormal (linear)
- Normal (linear)
  \[ \propto 1/\sigma_{1,r} \]

Transient growth:

- Nonnormal (linear)

\[ \log(\text{amplification}) \]

\[ \sigma_r \] (growth rate)

\[ \sigma_i \] (frequency)
Effect of non-linearity?

\[ \frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L \]

Harmonic forcing

Transient growth

Bye pass transition (subcriticality)
Trefethen (1993)
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Canonical example
Bénard-von Karman street

Supercritical Hopf Bifurcation

\[ Re = 26 < Re_c \]

\[ Re = 140 > Re_c \]

\[ Re_c \approx 47 \]

Threshold
Non-linear saturation, oscillator

Unstable flow, $Re > Re_c \simeq 47$

Linear stability: $u = \hat{u}(x) \exp(\lambda t)$

$$\lambda \hat{u} + (\hat{u} \cdot \nabla) \hat{u} + \nabla \hat{p} - Re^{-1} \nabla^2 \hat{u} = 0$$

Linear analysis

Nonlinear analysis (DNS)

Manted-Lugo & Galleire, 2015
Frequency correction

- DNS
- Expe (Williamson ‘88)
- Linear base flow (Pier 2002)
Mean flow distortion
Recirculation length

Zielinska et al. (1997)
Mean flow distortion Reynolds stresses

Maurel et al. 1995
Zielinska et al. 1997
Noack et al. 2001
Farrell and Ioannou 1995, etc...

Fig. 4.—Governor and Throttle-Valve.

Watt regulator
Base flow

Mean flow

Re=100

Barkley (2006)
The mean flow is neutrally (marginally) stable

Barkley (2006), Pier (2002) weakly parallel hypothesis
Maurel et al. '94 "In 1921, Noether [17] noted in the wall-bounded shear flows an unstable mean-flow profile different from the undisturbed profile because of the presence of a steady wave and analysed this change with the Reynolds stress. He concludes that a self-consistent problem may be used to characterize the instability: the growth rate determines the unstable character of the system and the Reynolds tensor stabilizes the profile, so that the growth rate value diminishes until it reaches a zero value."

Outline of a theory of turbulent shear flow

By W. V. R. Malkus
Woods Hole Oceanographic Institution, Woods Hole, Massachusetts
(Received 4 June 1956)
Stuart-Landau amplitude equation

\[ \frac{dA}{dt} = \epsilon \lambda A - \epsilon (\mu + \nu)A|A|^2 \]

- \( \epsilon \) distance from threshold
- \( \lambda, \mu, \nu \) parameters to be computed (or fitted)

**Experimental**
Sreenivasan, Strykowsky & Olinger (1986)
Provansal, Mathis & Boyer (1987)

**Numeric**
Dusek, Le Gal & Fraunié (1994)

**Analytic**
Sipp & Lebedev (2007)
Linear stability theory

Linearized Navier Stokes

\[ \nabla \cdot u' = 0, \quad \partial_t u' + \nabla u' \cdot U + \nabla U \cdot u' + \nabla p' - Re^{-1} \nabla^2 u' = 0, \quad +bc \]

Base flow \( q_0 = (U, V, P)^T \), perturbation \( q_1 = (u', v', p')^T \)

Global Mode expansion: \( q_1(x, y, t) = \hat{q}_1(x, y) e^{(\sigma + i\omega)t} + c.c. \)

Equations at threshold \( (i\omega_e B + L_{**}) \hat{q}_1 = 0 \quad w_e \sim 0.74 \)

Jackson (1987), Zebib (1987), Ding & Kawahara (1999), Barkley (2006), Giannetti & Luchini (2003, 2007), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)…
Bifurcation theory

**Departure from threshold:**

\[
\frac{1}{Re} - \frac{1}{Re_*} = O(\epsilon^2) \equiv \epsilon^2 \delta
\]

*slow time scale* \( T = \epsilon^2 t \)

**Expansion:**

\[
q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \ldots
\]

- **\( q_0 \)** base flow,
- **\( q_1 \)** leading order perturbation \( q_1 = A(T)\hat{q}_1 A e^{i\omega_* t} + \text{c.c.} \)
- **\( q_2 \)** second order perturbation, no secular terms with frequency \( \omega_* \)

\[
q_2 = \delta \hat{q}_{2\delta} + |A|^2 \hat{q}_{2|A|^2} + (A^2 \hat{q}_{2A^2} e^{2i\omega_* t} + \text{c.c.})
\]
The Fredholm alternative

Orthogonal to the adjoint of $\hat{q}_{1A}$

$$\nabla \cdot \hat{u}^\dagger = 0, \quad \partial_t \hat{u}^\dagger + \nabla U^\text{T} \cdot \hat{u}^\dagger - \nabla \hat{u}^\dagger \cdot U + \nabla \hat{p}^\dagger - \Re^{-1} \nabla^2 \hat{u}^\dagger = 0,$$

Giannetti & Luchini (2003), Sipp & Lebedev (2007), Marquet, Sipp & Jacquin (2009)…
Normal form

• $A\varepsilon$ leading order determined by resonant terms at $\varepsilon^3$

\[
\frac{dA}{dT} = \lambda \delta A - \mu A|A|^2, \\
\lambda = S^{-1} \int_\Sigma \hat{q}^{\dagger} \cdot \hat{F}_{3A} \, dx\, dy, \\
\mu = -S^{-1} \int_\Sigma \hat{q}^{\dagger} \cdot \hat{F}_{3A|A|^2} \, dx\, dy, \\
S = \int_\Sigma \hat{q}^{\dagger} \cdot \hat{q}_A \, dx\, dy.
\]

Sipp & Lebedev (2007)
Normal form

\[
\frac{dA}{dT} = \lambda \delta A - \mu A |A|^2,
\]

\(\mu_r > 0 \Rightarrow\) predicts saturation \( |A|^2 = \frac{\lambda_r \delta}{\mu_r} \)

\(\Rightarrow\) nonlinear frequency correction \( \delta \omega = \lambda_i \delta - \mu_i |A|^2 \)

*Sipp & Lebedev (2007)*
Correct near threshold
Similar strategy for resonators
Spring softening in sloshing
Amplitude equation similar to Duffing oscillators

Ockendon & Ockendon (1973), ..., Bauerlein & Avilla (2021), Bongarzone et al. (2022), Marcotte et al. (2022)
Need for an “outstanding”, near neutral, eigenvalue

Spectrum of $L$

Cylinder flow $Re_c = 46.6$

Sipp and Lebedev, 2007

Sloshing Water: $R = 2.5$ cm, $h/R = 0.5$, $d/R = 0.2$

$\sigma_{1,n}$, damping

$\omega_{1,n}/2\pi$ [Hz], natural freq.

$\sigma_{1,n} = [1/s]$, damping

-> if needed, multimodal extension
Need for an “outstanding”, near neutral, eigenvalue

Spectrum of $L$

Backward facing flow Re=500

Ducimetière et al. (2022)
Need for an “outstanding”, near neutral, eigenvalue

Amplifiers

Oscillators

Resonators

Water: $R = 2.5 \text{ cm}$, $h/R = 0.5$, $d/R = 0.2$
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5. Conclusions
In nonnormal systems, the inverse resolvent (and inverse propagator) operator may lead to a notable mitigation of the response and are close to singular. A small perturbation is then sufficient to fill their kernel with the response and render them singular. This can be encompassed in a multiple-scale expansion, closed by means of classical solvability conditions. Weakly nonlinear amplitude equations can be derived for nonnormal systems, and the presence of a neutral or weakly damped mode is unnecessary.
Transient growth

Harmonic forcing

\[ \frac{\partial u}{\partial t} = Lu + f, \quad \begin{aligned} u(0) \neq 0 \quad \text{and} \quad LL^\dagger & \neq L^\dagger L \end{aligned} \]

Effect of nonlinearity?
DNS response

- Large Saturation Response
- Response position variation

Response forcing amplitude $\varepsilon=0.0001$

Optimal forcing $St=0.075$

Response forcing amplitude $\varepsilon=0.01$
DNS response

- Large Saturation Response
- Response position variation

Optimal forcing: $St=0.075$

$$G = \frac{\|u'\|}{\|f\|}$$
Saturating nonlinearities - Amplitude equation?

Forcing optimal at each frequency
Optimal response to harmonic forcing

\[ \frac{\partial \mathbf{u}}{\partial t} = L \mathbf{u} + \mathbf{f} \]

insert:

\[ \mathbf{f}(\mathbf{x}, t) = \widehat{\mathbf{f}}(\mathbf{x})e^{i\omega_0 t} + c. c \]

\[ \mathbf{u}(\mathbf{x}, t) = \widehat{\mathbf{u}}(\mathbf{x})e^{i\omega_0 t} + c. c \]

to obtain:

\[ \widehat{\mathbf{u}} = (i\omega_0 I - L)^{-1} \widehat{\mathbf{f}} = R(i\omega_0) \widehat{\mathbf{f}} \]

optimize:

\[ G(i\omega_0) = \max_{\widehat{\mathbf{f}}} \frac{\|\widehat{\mathbf{u}}\|}{\|\widehat{\mathbf{f}}\|} = \|R(i\omega_0)\| = \frac{1}{\epsilon_0} \]

under the scalar product

\[ \langle \widehat{\mathbf{u}}_a | \widehat{\mathbf{u}}_b \rangle = \int_\Omega \widehat{\mathbf{u}}_a^H \widehat{\mathbf{u}}_b d\Omega \]

Singular value decomposition:

\[ R(i\omega_0)^{-1} \widehat{\mathbf{u}}_o = \epsilon_o \widehat{\mathbf{f}}_o \]

With normalization

\[ \langle \widehat{\mathbf{u}}_o | \widehat{\mathbf{u}}_o \rangle = \|\widehat{\mathbf{u}}_o\|^2 = 1 \]

\[ \|\widehat{\mathbf{f}}_o\| = 1 \]

apply

\[ R(i\omega_0)^{-1} \]

Strong nonnormality:

\[ \epsilon_o \ll 1 \]
Operator “singularization”

\[ R(i\omega_o)^{-1}[\hat{u}_o] = \epsilon_o \hat{f}_o \]

can be re-written as:

\[ [R(i\omega_o)^{-1} - \epsilon_o \hat{f}_o \langle \hat{u}_o | . \rangle] \hat{u}_o = 0 \]
Operator “singularization”

\[ R(i\omega_o)^{-1} \hat{u}_o = \epsilon_o \hat{f}_o \]

can be re-written as:

\[ [R(i\omega_o)^{-1} - \epsilon_o \hat{f}_o \langle \hat{u}_o | . \rangle] \hat{u}_o = 0 \]

Let us define

\[ \Phi(i\omega_o) = R(i\omega_o)^{-1} - \epsilon_o P \]

with

\[ P = \hat{f}_o \langle \hat{u}_o | . \rangle \]

such that

\[ \Phi(i\omega_o)\hat{u}_o = 0 \]

\[ \Phi(i\omega_o)^\dagger \hat{f}_o = 0 \]

This minimal operator perturbation makes the inverse resolvent singular.
inject

\[ U(t,T) = U_e + \sqrt{\epsilon_0} u_1(t,T) + \epsilon_0 u_2(t,T) + \sqrt{\epsilon_0^3} u_3(t,T) + O(\epsilon_0^2) \]

in forced Navier-Stokes equations:

\[
\begin{align*}
\sqrt{\epsilon_0} \left( \partial_t u_1 - Lu_1 \right) \\
+ \epsilon_0 \left( \partial_t u_2 - Lu_2 + C(u_1,u_1) \right) \\
+ \sqrt{\epsilon_0^3} \left( \partial_t u_3 - Lu_3 + 2C(u_1,u_2) + \partial_T u_1 \right) + O(\epsilon_0^2) = \sqrt{\epsilon_0^3} \phi \hat{f} e^{i\omega_0 t} + c.c
\end{align*}
\]

with

\[ C(a, b) = \frac{1}{2} \left[ (a \cdot \nabla) b + (b \cdot \nabla)a \right] \]
Multiple scale asymptotic expansion

Inject

\[ U(t,T) = U_e + \sqrt{\epsilon_o} u_1(t,T) + \epsilon_o u_2(t,T) + \sqrt{\epsilon_o^3} u_3(t,T) + O(\epsilon_o^2) \]

\[ \sqrt{\epsilon_o} (\partial_t u_1 - Lu_1) + \epsilon_o (\partial_t u_2 - Lu_2 + C(u_1,u_1)) + \sqrt{\epsilon_o^3} (\partial_t u_3 - Lu_3 + 2C(u_1,u_2) + \partial_T u_1) + O(\epsilon_o^2) = \sqrt{\epsilon_o^3} \phi \hat{f}_h e^{i\omega t} + c.c \]

with \[ C(a,b) = \frac{1}{2} [(a \cdot V)b + (b \cdot V)a] \]

Fourier-expand velocity fields:

\[ u_j(t,T) = \bar{u}_{j0}(T) + \sum_m (\bar{u}_{j,m}(T)e^{mi\omega t} + c.c) \]

\[ m = 1, 2, 3, ... \]
Multiple scale asymptotic expansion

$$\sqrt{\epsilon_0} \left( (i\omega_o - L)\overline{u}_{1,1} e^{i\omega_o t} + c.c + s_1 \right)$$

$$+ \epsilon_0 \left( (i\omega_o - L)\overline{u}_{2,1} e^{i\omega_o t} + c.c + s_2 + C(u_1, u_1) \right)$$

$$+ \sqrt{\epsilon_0^{-3}} \left( (i\omega_o - L)\overline{u}_{3,1} e^{i\omega_o t} + c.c + s_3 + 2C(u_1, u_2) + \partial_T u_1 \right) + O(\epsilon_0^2) = \sqrt{\epsilon_0^{-3}} \Phi \hat{h} e^{i\omega_o t} + c.c$$

with

$$s_j = -L\overline{u}_{j,0}(T) + \sum_m \left( (im\omega_o - L)\overline{u}_{j,m}(T)e^{mi\omega_o t} + c.c \right)$$
Multiple scale asymptotic expansion

\[
\sqrt{\varepsilon_0} \left( (i\omega_o - L)\overline{u}_{1,1} e^{i\omega_o t} + \text{c. c} + s_1 \right) \\
+ \varepsilon_0 \left( (i\omega_o - L)\overline{u}_{2,1} e^{i\omega_o t} + \text{c. c} + s_2 + C(u_1, u_1) \right) \\
+ \sqrt{\varepsilon_0}^3 \left( (i\omega_o - L)\overline{u}_{3,1} e^{i\omega_o t} + \text{c. c} + s_3 + 2C(u_1, u_2) + \partial_T u_1 \right) + O(\varepsilon_0^2) = \sqrt{\varepsilon_0}^3 \Phi \hat{f}_h e^{i\omega_o t} + \text{c. c} \\
\]

with

\[
s_j = -L\overline{u}_{j,0}(T) + \sum_m \left( (im\omega_o - L)\overline{u}_{j,m}(T)e^{mi\omega_o t} + \text{c. c} \right) \\
\]

Perturb the operator:

\[
\Phi(i\omega_o) = R(i\omega_o)^{-1} - \varepsilon_o P \\
\]
Multiple scale asymptotic expansion

\[
\sqrt{\epsilon_0} \left( (i\omega_o - L)\bar{u}_{1,1} e^{i\omega_o t} + c.c + s_1 \right) \\
+ \epsilon_0 \left( (i\omega_o - L)\bar{u}_{2,1} e^{i\omega_o t} + c.c + s_2 + C(u_1, u_1) \right) \\
+ \sqrt{\epsilon_0}^3 \left( (i\omega_o - L)\bar{u}_{3,1} e^{i\omega_o t} + c.c + s_3 + 2C(u_1, u_2) + \partial_T u_1 \right) + O(\epsilon_0^2) = \sqrt{\epsilon_0}^3 \phi \hat{f}_h e^{i\omega_o t} + c.c
\]

with \( s_j = -L\bar{u}_{j,0}(T) + \sum_m \left( (im\omega_o - L)\bar{u}_{j,m}(T)e^{mi\omega_o t} + c.c \right) \)

leads to:

\[
\sqrt{\epsilon_0} \left( \Phi \bar{u}_{1,1} e^{i\omega_o t} + c.c + s_1 \right) \\
+ \epsilon_0 \left( \Phi \bar{u}_{2,1} e^{i\omega_o t} + c.c + s_2 + C(u_1, u_1) \right) \\
+ \sqrt{\epsilon_0}^3 \left( \Phi \bar{u}_{3,1} e^{i\omega_o t} + c.c + s_3 + 2C(u_1, u_2) + \partial_T u_1 + \left( P\bar{u}_{1,1} e^{i\omega_o t} + c.c \right) \right) + O(\epsilon_0^2) = \sqrt{\epsilon_0}^3 \phi \hat{f}_h e^{i\omega_o t} + c.c
\]
Collect at first-order: \[ \Phi(i\omega_0) \tilde{u}_{1,1} = 0 \]

leading to:

\[ u_1(t, T) = A(T) \tilde{u}_0 e^{i\omega_0 t} + c.c \]
Collect at first-order: \( \Phi(i\omega_o)\bar{u}_{1,1} = 0 \) leading to: \( u_1(t, T) = A(T) \hat{u}_0 e^{i\omega_o t} + c.c \)

Collect at second-order: \( \ldots \) leading to: \( u_2(t, T) = |A|^2 u_{2,0} + (A^2 e^{2i\omega_o t} \hat{u}_{2,2} + c.c) \)
Collect at first-order: \( \Phi(i\omega_o)\bar{u}_{1,1} = 0 \) leading to: \( u_1(t, T) = A(T) \hat{u}_o e^{i\omega_o t} + c.c \)

Collect at second-order: ... leading to: \( u_2(t, T) = |A|^2 u_{2,0} + (A^2 e^{2i\omega_o t} \hat{u}_{2,2} + c.c) \)

Collect at third-order: \( \Phi(i\omega_o)\bar{u}_{3,1} = -A|A|^2 [2C(\hat{u}_o, u_{2,0}) + 2C(\hat{u}_o, \hat{u}_{2,2})] - \hat{u}_o \frac{dA}{dT} - A \hat{f}_o + \phi \hat{f}_h \)
Collect at first-order: \( \Phi(i\omega_0)\vec{u}_{1,1} = 0 \)  
leading to: \( u_1(t, T) = A(T)\hat{u}_0 e^{i\omega_0 t} + c. c \)  

Collect at second-order: ...  
leading to: \( u_2(t, T) = |A|^2u_{2,0} + (A^2e^{2i\omega_0 t}\hat{u}_{2,2} + c. c) \)  

Collect at third-order: \( \Phi(i\omega_0)\vec{u}_{3,1} = -A|A|^2[2C(\hat{u}_0, u_{2,0}) + 2C(\hat{u}_0, \hat{u}_{2,2})] - \hat{u}_0 \frac{dA}{dT} - A\hat{f}_o + \phi \hat{f}_h \)
Multiple scale asymptotic expansion

Collect at first-order:
\[ \Phi(i\omega_0)\bar{u}_{1,1} = 0 \]
leading to:
\[ u_1(t, T) = A(T) \hat{u}_o e^{i\omega_0 t} + c.c \]

Collect at second-order:
\[ ... \]
leading to:
\[ u_2(t, T) = |A|^2 u_{2,0} + (A^2 e^{2i\omega_0 t} \hat{u}_{2,2} + c.c) \]

Collect at third-order:
\[ \Phi(i\omega_0)\bar{u}_{3,1} = -A|A|^2 [2C(\hat{u}_o, u_{2,0}) + 2C(\hat{u}_o, \hat{u}_{2,2})] - \hat{u}_o \frac{dA}{dT} - A \hat{f}_o + \phi \hat{f}_h \]

Impose its orthogonality with the kernel of the adjoint (« Fredholm alternative »):
\[ \frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu}{\eta} A|A|^2 \]

\[ \gamma = \langle \hat{f}_o | \hat{f}_h \rangle \quad \eta = \frac{1}{\langle \hat{u}_o | \hat{f}_o \rangle} \quad \frac{\mu}{\eta} = \langle \hat{f}_o | 2C(\hat{u}_o, u_{2,0}) \rangle \quad \frac{\nu}{\eta} = \langle \hat{f}_o | 2C(\hat{u}_o^*, \hat{u}_{2,2}) \rangle \]
Amplitude equation

\[ \frac{1}{\eta} \frac{dA}{dT} = \phi \gamma - A - \frac{\mu + \nu}{\eta} A |A|^2 \]

\[ \gamma = \langle \hat{f}_o, \hat{f}_h \rangle, \quad \eta = \frac{1}{\langle \hat{f}_o, \hat{u}_o \rangle}, \quad \frac{\mu}{\eta} = \langle \hat{f}_o, 2C(\hat{u}_o, \hat{u}_{2,0}) \rangle, \quad \frac{\nu}{\eta} = \langle \hat{f}_o, 2C(\hat{u}^*_o, \hat{u}_{2,2}) \rangle \]
Saturating nonlinearities

\[
\frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu + \nu}{\eta} A |A|^2
\]

An example: the flow past a backward-facing step

Set frequency, vary $Re$
Saturating nonlinearities

\[ \frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu + \nu}{\eta} A|A|^2 \]

An example: the flow past a backward-facing step

Set frequency, vary $Re$

Set $Re$, different frequencies
De-saturating nonlinearities

\[
\frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu + \nu}{\eta} A|A|^2
\]

An example: Orr mechanism in Poiseuille flow

Optimal forcing

\[ R(i\omega_0) \]

Optimal response

\[ \omega_0 = 0.38, \quad Re = 3000 \]
De-saturating nonlinearities

\[
\frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu}{\eta} A |A|^2
\]

An example: Orr mechanism in Poiseuille flow
Outline

1. Natural and forced oscillations in flows

2. (Strong) non-normality in amplifiers

3. Weak non-linearity in resonators and oscillators

4. The odd alliance
   i. Harmonic response
   ii. Transient growth

5. Conclusions
Transient growth:

$$\frac{\partial u}{\partial t} = Lu + f, \quad u(0) \neq 0 \quad \text{and} \quad LL^\dagger \neq L^\dagger L$$

Backward facing step flow

Re=500

initial condition

amplified solution at t=58

Blackburn et al, 2008
Operator “singularization”

\[ e^{-Lt_o}v_o = \epsilon_0 u_o, \quad \left[(e^{Lt_o})^\dagger\right]^{-1} u_o = \epsilon_0 v_o \]

Trajectory \[ l(t) = \epsilon_o e^{Lt} u_o \]

Perturbed operator \[ \Phi(t) = e^{-Lt} - \epsilon_o P(t), \quad \text{where} \quad P(t) = H(t) \frac{u_o \langle l(t), \ast \rangle}{\|l(t)\|^2} \]

The non-trivial kernel of \( \Phi(t) \) is \( l(t) \) for all \( t > 0 \)

Expansion \[ U(t, T) = U_e + \epsilon_o u_1(t, T) + \epsilon_o^2 u_2(t, T) + O(\epsilon_o^3) \]

Inject in Navier-Stokes \[ \epsilon_o (\partial_t - L)u_1 + \epsilon_o^2 \left[ (\partial_t - L)u_2 + C(u_1, u_1) + \partial_T u_1 \right] + O(\epsilon_o^3) = 0 \]

Perturb \[ \epsilon_o e^{Lt} \partial_t (\Phi u_1) + \epsilon_o^2 \left[ e^{Lt} \partial_t (\Phi u_2) + C(u_1, u_1) + \partial_T u_1 + e^{Lt} \partial_t (P(t)u_1) \right] + O(\epsilon_o^3) = 0 \]
Operator “singularization”

\[ \partial_t (\Phi u_1) = 0, \text{ subject to } u_1(0) = 0 \]

\[ u_1(t, T) = A(T)H(t)l(t) \]

\[
\frac{dA}{dt} = \epsilon_0 A^2 \frac{d\mu_2(t)}{dt}, \quad \text{with } A(0) = \alpha
\]

**initial amplitude**

where

\[ \mu_2(t) = \frac{\langle \tilde{u}_2(t), l(t) \rangle}{\langle l(t), l(t) \rangle} \]

and

\[ \frac{d\tilde{u}_2}{dt} = L\tilde{u}_2 - C(l, l), \quad \tilde{u}_2(0) = 0 \]

\[ C(a, b) = \frac{1}{2}((a \cdot \nabla)b + (b \cdot \nabla)a) \]
Results of the amplitude equation

* [small] subtlety.... one needs to push the reasoning at third order
Outline

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Summary

In nonnormal systems, the inverse resolvent (and inverse propagator) operator may lead to a notable mitigation of the response and are close to singular.
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A small perturbation is then sufficient to fill their kernel with the response and render them singular.
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In nonnormal systems, the inverse resolvent (and inverse propagator) operator may lead to a notable mitigation of the response and are close to singular.

A small perturbation is then sufficient to fill their kernel with the response and render them singular.

This can be encompassed in a multiple-scale expansion, closed by means of classical solvability conditions.

Weakly nonlinear amplitude equations can be derived for nonnormal systems, and the presence of a neutral or weakly damped mode is unnecessary.

**Harmonic forcing**

\[
\frac{1}{\eta} \frac{dA}{dT} = \gamma \phi - A - \frac{\mu + \nu}{\eta} A|A|^2
\]