REGRESSION CONDITIONS THAT CHARACTERIZES FREE POISSON AND FREE KUMMER NON-COMMUTATIVE RANDOM VARIABLES.

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Abstract. We find asymptotic spectral distribution of matrix–Kummer eigenvalues. Then we formulate and prove the free analogue of HV independence property, which is known for classical Kummer and Gamma random variables. We also prove related characterization of free–Kummer and free–Poisson (Marchenko-Pastur) free random variables.

1. Introduction

This paper concerns some connections between classical probability and non-commutative probability. Especially, links between independence and freeness. We prove a free-analogue of a classical independence property of random variables with Kummer and Gamma distributions. Similar attempts have been successful for Lukacs’ characterization of Gamma distribution, [21], as well as for the Matsumoto-Yor characterization of GIG and Gamma distributions, [20]. Earlier, also Bernstein’s characterization of Gaussian r. v. ([2]) was proved for semicircle free r. v. ([14]). However it is not clear that any independence characterization, which is known to hold for commutative random variables, would also hold, in some formulations, for their non–commutative counterparts. An important property of Gaussian r. v., known as Cramer’s Theorem, is not true in free probability setting, [1].

Let us recall that random variable \( Y \) has Gamma distribution with parameters \( a, c > 0 \), we write \( Y \sim \mathcal{G}(a, c) \), if it has density

\[
f(y) \propto y^{a-1} e^{-cy} I_{(0,\infty)}(y).
\]

Random variable \( X \) has Kummer distribution with parameters \( a, c > 0, b \in \mathbb{R} \), we write \( X \sim \mathcal{K}(a, b, c) \), if it has density

\[
f(x) \propto x^{a-1} (1 + x)^{-(a+b)} e^{-cx} I_{(0,\infty)}(x).
\]

An interesting property noticed in [7] says, that if \( X \sim \mathcal{K}(a, b, c) \) and \( Y \sim \mathcal{G}(a + b, c) \) are independent, then random variables

\[
U := \frac{Y}{1+X} \quad \text{and} \quad V := X (1 + U)
\]

are also independent and \( U \sim \mathcal{K}(a + b, -b, c) \), \( V \sim \mathcal{G}(a, c) \). We call this the HV property for the sake of its authors names. In [15] under the assumption that densities of \( X \) and \( Y \) are locally integrable, the converse theorem was proved: if \( X \) and \( Y \) are independent and \( U \) and \( V \) defined in (1) are independent, then necessarily \( X \sim \mathcal{K}(a, b, c) \) and \( Y \sim \mathcal{G}(a + b, c) \) for some constants \( a, c > 0 \) and \( b > -a \). In [16] it was shown that it is enough to assume constant regression conditions: \( \mathbb{E}(V|U) = \alpha \) and \( \mathbb{E}(V^{-1}|U) = \beta \) instead of independence of \( U \) and \( V \) (in fact there

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are more possible conditions in the paper). Some of other properties and characterizations of that type (we mean regression conditions) have its counterparts in non-commutative probability:

- The basic example is Bernstein characterizations of Gaussian distribution, [2], and related free characterization of Wigner law, [14];
- Lukacs’ Theorem, [12], which characterizes Gamma distribution through independence of $X + Y$ and $X/Y$. In the non-commutative version Marchenko-Pastur (free-Poisson) distribution is characterized through conditions: $\varphi(\mathbb{E}(X + Y)) = \alpha(X + Y) + \alpha_0$, $\text{Var}(\mathbb{E}(X + Y)) = C(1 + a(X + Y) + b(X + Y)^2)$ in [3] or by constant conditional expectations of $X/(X + Y)$ and $(X/(X + Y))^{-1}$ given $X + Y$ in [19, 21];
- Matsumoto-Yor property and related characterization of GIG and Gamma, [11], also have their counterparts in free probability, cf. [20].

These are links between commutative and non–commutative probability spaces and between independence and freeness. In this paper we give one more link of similar type. Firstly we find a pair of distributions $\mu, \nu$ such that if $X \sim \mu$ and $Y \sim \nu$ are free, self-adjointed random variables from some $C^*$ algebra $\mathcal{A}$, then $\mathbb{U} = (I + X)^{-1/2} Y (I + X)^{-1/2}$ and $\mathbb{V} = (I + \mathbb{U})^{1/2} X (I + \mathbb{U})^{1/2}$ are free, compare with (1). Then we prove the main result of the paper, i.e. the constant regression characterization of the measures $\mu$ and $\nu$.

2. Preliminaries

A broad introduction to free probability can be found in [8] or [13]. We will introduce only that part of the theory that we need to present our results.

Let $\mathcal{A}$ be a $\star$-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ be a linear functional such that $\varphi(I) = 1$, where $I = 1_{\mathcal{A}}$. We assume that $\varphi$ is faithful, normal, tracial and positive. We call the pair $(\mathcal{A}, \varphi)$ a non–commutative $\star$-probability space and elements of $\mathcal{A}$ are called non-commutative random variables (and we will denote them with double letters, like $X$ or $Y$).

An important issue considered in the literature is $\mathcal{A} \subset B(H)$, where $H$ is a Hilbert space and $B(H)$ denotes the space of bounded linear operators from $H$ to $H$. If $\mathcal{A}$ is von Neumann algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is normal, faithful and tracial, then we say that $(\mathcal{A}, \varphi)$ is $W^*$-probability space. It is a special type of $C^*$-probability space.

Let $a_1, \ldots, a_n \in \mathcal{A}$. The numbers $\varphi(a_{i(1)} \cdots a_{i(n)})$, $i(j) \in \{1, \ldots, k\}$ for $j \in \{1, \ldots, n\}$ are called moments and by joint distribution of $(a_1, \ldots, a_k)$ we mean the collection of all moments. For a bounded, self–adjointed random variable $X \in \mathcal{A}$ we can define the $\star$-distribution of $X$ as a unique, real, compactly supported probability measure $\mu$. This measure $\mu$ is uniquely determined as such that for all $n \in \mathbb{N}$

$$\varphi(X^n) = \int_R t^n \, d\mu(t).$$

If support of $\mu$ is contained in $(0, \infty)$, then we say that $X$ is positive. For a family of random variables $(X_1, \ldots, X_n)$ we define its $\star$-distribution as probability measure $\mu_n$ on $\mathbb{R}^n$ such that for any polynomial $P \in \mathbb{C}(x_1, \ldots, x_n)$ in non-commuting variables

$$\varphi(P(X_1, \ldots, X_n)) = \int P(x_1, \ldots, x_n) \, d\mu_n(x).$$
Let \( (\mathcal{A}, \varphi) \) be a non-commutative probability space and let \( I \) be a finite index set. For each \( i \in I \) let \( \mathcal{A}_i \subset \mathcal{A} \) be a unital subalgebra. The subalgebras \( (\mathcal{A}_i)_{i \in I} \) are called free or freely independent, if \( \varphi(a_1 \cdots a_k) = 0 \) whenever the following four conditions hold:

1. \( k \) is a positive integer,
2. \( a_j \in \mathcal{A}_{i(j)} \) (\( i(j) \in I \)) for all \( j = 1, \ldots, k \),
3. \( \varphi(a_j) = 0 \) for all \( j = 1, \ldots, k \),
4. neighbouring elements are from different subalgebras, i.e., \( i(1) \neq i(2) \neq \ldots \neq i(k) \).

We say that the random variables \( (X_i)_{i \in I} \), \( X_i \in \mathcal{A} \) for each \( i \in I \), are free or freely independent, if their generated subalgebras are free, i.e. if \( (\mathcal{A}_i, 1)_{i \in I} \) are free, where \( \mathcal{A}_i \) is the unital subalgebra of \( \mathcal{A} \) generated by \( X_i \) for each \( i \in I \). Freeness of non-commutative random variables can also be expressed in terms of free-cumulants.

Let \( NC(n) \) denote the set of all non-crossing partitions of the set \( \{1, \ldots, n\} \). We define the free cumulants \( \kappa_n : \mathcal{A}^n \to \mathbb{C}, n \geq 1 \) as multi-linear functionals by the recursive moment-cumulant relation

\[
\varphi(a_1 \ldots a_n) = \sum_{\pi \in NC(n)} \kappa_{\pi}(a_1, \ldots, a_n),
\]

where \( \kappa_{\pi} \) is a product of cumulants over all blocks of \( \pi \) and the arguments are given by the elements corresponding to the respective blocks. For example, if \( \pi = \{\{1, 4\}, \{2, 3\}\} \) then \( \kappa_{\pi}(a, b, c, d) = \kappa_2(a, d) \kappa_2(b, c) \). So \( \kappa_1(a) = \varphi(a) \), \( \kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1) \varphi(a_2) \) and so on.

Random variables \( X \) and \( Y \) are free if and only if \( \kappa_n(a_1, \ldots, a_n) = 0 \) whenever: \( n \geq 2 \), \( a_i \in \{X, Y\} \) for all \( i \) and there are at least two indices \( i, j \) such that \( a_i = X \), \( a_j = Y \), cf. [18].

Throughout following sections, we will use well known formula, which connects free cumulants and moments:

\[
\varphi(X_1 \ldots X_n) = \sum_{k=1}^{n} \sum_{1 < i_2 < \ldots < i_k \leq n} \kappa_k(X_{i_1}, X_{i_1}, \ldots, X_{i_k}) \prod_{j=1}^{k} \varphi(X_{i_j}+1 \ldots X_{i_{j+1}-1}),
\]

where \( i_1 = 1 \), \( i_{n+1} = n + 1 \), cf. [4].

2.1. **Analytical tools.** Let \( (\mathcal{A}, \varphi) \) be a fixed non-commutative probability space. Let \( X \in \mathcal{A} \). Let us recall that \( r \)-transform of random variable \( X \) is:

\[
r_X(z) = \sum_{n=0}^{\infty} \kappa_{n+1}(X) z^n, \quad z \in \mathbb{C}^+,
\]

where \( \kappa_n(X) = \kappa_n(X, \ldots, X) \). It is known that if \( X \) has compact support, then its \( r \)-transform is analytic function in the neighborhood of \( 0 \) (as a function of complex variable). For a self-adjoint, bounded \( X \) from \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) with state \( \varphi \) and \( * \)-distribution \( \nu \) we can define **Cauchy transform** as

\[
G_X(z) = G_\nu(z) = \varphi \left( (z - X)^{-1} \right) = \int_{\mathbb{R}} (z - t)^{-1} d\nu(t)
\]

for \( z \in \mathbb{C} \setminus \mathbb{R} \). Let \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \Im(z) > 0 \} \) and \( \mathbb{C}^- = \{ z \in \mathbb{C} : \Im(z) < 0 \} \). Then \( G_X \) is an analytic function from \( \mathbb{C}^+ \) to \( \mathbb{C}^- \) (Lemma 3.1.2 in [13]). Cauchy transform characterizes distribution and measure \( \nu \) can be recovered form \( G_\nu \) by Stieltjes inverse formula:
2.2. Asymptotic freeness. Consider a family of random variables \( \left( X_1^{(n)}, \ldots, X_k^{(n)} \right) \) on a probability space \((A_n, \varphi_n), n = 1, 2, \ldots\). We say that \(\star\)-distribution \(\left( \mu_{\left( X_1^{(n)}, \ldots, X_k^{(n)} \right)} \right)_{n \geq 1}\) converges to \(\mu\) when \(n\) goes to infinity, if for any polynomial in non-commutative variables \(P \in \mathbb{C}<x_1, \ldots, x_k>\) the following holds:

\[
\int P(x_1, \ldots, x_k) \mu_{\left( X_1^{(n)}, \ldots, X_k^{(n)} \right)}(x) \xrightarrow{n \to \infty} \int P(x_1, \ldots, x_k) \mu(x).
\]

If \(\mu\) is a \(\star\)-distribution of \((X_1, \ldots, X_k)\), then we say that \(\left( X_1^{(n)}, \ldots, X_k^{(n)} \right)\) converges in distribution to \((X_1, \ldots, X_k)\). Then

\[
\varphi_n \left( P \left( X_1^{(n)}, \ldots, X_k^{(n)} \right) \right) \xrightarrow{n \to \infty} \varphi \left( P(X_1, \ldots, X_k) \right)
\]
for any $P \in \mathbb{C} \langle x_1, \ldots, x_k \rangle$. If $X_1, X_2, \ldots, X_k$ are free, then we say that $(X_1^{(n)}, X_2^{(n)}, \ldots, X_k^{(n)})_{n \geq 1}$ is asymptotically free.

**Empirical spectral distribution** of $N \times N$ random matrix $X_N$ is a random measure

$$P_N = \frac{1}{N} (\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_N}),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are eigenvalues of $X_N$.

Classical and powerful result on connection between random matrices and free random variables is due to Voiculescu and we cite it below.

**Theorem 2.4** ([23], Thm. 3.8). Consider $N \times N$ random matrices $X_N$ and $Y_N$ such that $X_N$ and $Y_N$ have almost surely an asymptotic spectral distribution when $N \to \infty$; $X_N$ and $Y_N$ are independent for $N \geq 1$; $Y_N$ is unitarily invariant ensemble. Then $X_N$ and $Y_N$ (as random variables in $(A_n, \varphi_N)$, where $A_n$ is algebra of symmetric $N \times N$ matrices and $\varphi_N = \frac{1}{N} \text{tr}$) are almost surely asymptotically free.

In [20] Szpojankowski used Theorem 2.4 to prove Matsumoto-Yor property in non-commutative setting. Here we will adapt this approach to some extent in order to prove non-commutative HV property. We start by recalling the random matrix version of HV property proved recently in [10] (Theorem 2.2.).

Let $M_N$ be the set of real symmetric $N \times N$ matrices. By $M_N^+$ we denote the cone of positive definite symmetric matrices. We say that random matrix $X$ has matrix–Kummer distribution with parameters $a > \frac{N-1}{2}$, $b \in \mathbb{R}$ and $\Sigma \in M_N^+$, if it has density:

$$MK_N(a, b, \Sigma)(dx) = \frac{1}{\Gamma_N(a)} \Psi(a, \frac{N+1}{2} + b; \Sigma) (\det x)^{a - \frac{N+1}{2}} (\det(e + x))^{-(a+b)} e^{-(\Sigma, x)} I_{M_N^+}(x) dx,$$

where $\Psi$ is the confluent hypergeometric function of the second kind with matrix argument (see for instance formula (2) in [9]) and $e$ is identity in $M_N^+$. The Wishart distribution with parameters $b > \frac{N-1}{2}$ and $\Sigma \in M_N^+$, $W(b, \Sigma)$, has density

$$WN(b, \Sigma)(dx) = \frac{(\det \Sigma)^b}{\Gamma_r(b)} (\det y)^{-\frac{N+1}{2}} e^{-(\Sigma, x)} I_{M_N^+}(x) dx.$$  

**Theorem 2.5** ([10]). Let $X$ and $Y$ be two independent random matrices valued in $M_N^+$. Assume that $X$ has matrix–Kummer distribution $MK(a, b, ce)$ and $Y$ has Wishart distribution $W(a + b, ce)$, where $a > \frac{N-1}{2}$, $b > \frac{N-1}{2} - a$, $c > 0$. Then the random matrices

$$U := [e + X]^{-1/2} Y [e + X]^{-1/2}, \quad V := [e + U]^{1/2} X [e + U]^{1/2}$$

are independent. Furthermore, $U \sim MK(a + b, -b, ce)$ and $V \sim W(a, ce)$.

It is known that if we take a sequence of real Wishart matrices $(y_n)$ with parameters $a_n$ and $\Sigma_n = \lambda_n e$, where $\lambda_n > 0$, $\frac{2a_n}{n} \to \lambda > 0$ and $\frac{2a_n}{n} \to 0$ then the free Poisson distribution $\nu(1/\lambda, \alpha)$ is an almost sure weak limit of the empirical spectral distribution.

A non-commutative random variable is free-Poisson (or Marchenko-Pastur) with parameters $\lambda > 0$, $\gamma > 0$ if it has distribution $\nu(\lambda, \gamma)$ defined by

$$\nu = \max\{0, 1 - \lambda\} \delta_0 + \lambda \tilde{\nu},$$

where $\tilde{\nu}$ is the distribution of the limiting eigenvalue distribution of the empirical spectral distribution of $X_N$. Theorem 2.4 gives conditions for $X_N$ and $Y_N$ to be almost surely asymptotically free, and Theorem 2.5 gives conditions for $U$ and $V$ to be independent. The free Poisson distribution $\nu(\lambda, \gamma)$ is a special case of the free Poisson distribution, where $\lambda > 0$, $\gamma > 0$.
where the measure $\tilde{\nu}$ is supported on the interval $\left( \gamma (1 - \sqrt{\lambda})^2, \gamma (1 + \sqrt{\lambda})^2 \right)$ and has density

$$
\tilde{\nu}(dx) = \frac{1}{2\pi \gamma x} \sqrt{4\lambda \gamma^2 - (x - \gamma (1 + \lambda))^2} \, dx.
$$

The parameter $\lambda$ is called the rate and $\gamma$ – the jump size. If $X$ is free–Poisson with parameters $\gamma$ and $\lambda$, we denote it $X \sim f\text{Pois}(\gamma, \lambda)$.

Note that Theorem 2.5 holds for real Wishart and matrix–Kummer random matrices. Although its proof from [10] seems to be easily re writable for random matrices with complex entries, we do not want to do it here, as it is not necessary for our results.

3. ASYMPTOTIC EIGENVALUE DISTRIBUTION OF MATRIX–KUMMER RANDOM MATRIX

In order to prove or even state free HV property we need to find free counterpart of Kummer distribution. This section provides this. We begin with standard result in distribution of entries, we do not want to do it here, as it is not necessary for our results.

**Proposition 3.1.** Let $Z$ be symmetric $n \times n$ random matrix and let $g$ be its density with respect to Lebesgue measure on $\mathbb{R}^{n(n+1)/2}$. If there exist function $h : \mathbb{R}^n \to \mathbb{R}$ such, that $h(\lambda_1, \ldots, \lambda_n) = g(x_{ij}, 1 \leq i \leq j \leq n)$, where $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ are eigenvalues of $x = (x_{ij})_{i,j=1,\ldots,n}$ and $x_{ji} := x_{ij}$ for $n \geq j > i \geq 1$, then the density of the vector of eigenvalues of $Z$ has the following form

$$
\frac{\pi^{n(n+1)/4}}{\prod_{j=1}^n \Gamma(j/2)} h(\lambda_1, \ldots, \lambda_n) \prod_{i<j} |\lambda_i - \lambda_j| I(0 < \lambda_1 < \ldots < \lambda_n).
$$

**Definition 3.1.** If $g$ is a density on $\mathbb{R}^n$ with respect to Lebesgue measure and if $g$ can be written as

$$
g(x) = C \cdot \prod_{1 \leq i < j \leq n} |x_i - x_j| \frac{1}{\gamma} \exp \left\{-\frac{1}{\gamma} V_n(x) \right\} dx,
$$

where $x = (x_1, \ldots, x_n)$, then function $V_n : \mathbb{R} \to \mathbb{R}$ is called potential of $g$.

It follows from Prop. 3.1 that matrix–Kummer eigenvalues have joint density. Let $x_n \sim MK_n(a_n, b_n, \gamma_n \mathbf{e})$, where $a_n > (n - 1)/2$, $b_n \in \mathbb{R}$, $\gamma_n > 0$. Then

$$
h_{(a_n, b_n, \gamma_n \mathbf{e})}(\lambda) = C \cdot \prod_{i=1}^n \lambda_i^{a_n - \frac{n+1}{2}} \prod_{i=1}^n (1 + \lambda_i)^{-\frac{a_n + b_n}{2}} \exp \left\{-\gamma_n \sum_{i=1}^n \lambda_i \right\} I_{(0,\infty)^n}(\lambda),
$$

where $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$. From Prop. 3.1 it follows that eigenvalues of $x_n$ have joint density:

$$
D \cdot \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) e^{-\frac{n}{2} \sum_{i=1}^n \left( \frac{2\gamma_n}{n} \lambda_i + \frac{2\alpha_n}{n} \log(1+\lambda_i) - \frac{2\beta_n}{n} \log(\lambda_i) \right)} I(0 < \lambda_1 < \ldots < \lambda_n),
$$

where $D$ is normalizing constant, $\alpha_n = a_n - n + 1 > -1$ and $\beta_n = a_n + b_n \in \mathbb{R}$.

From now on let $\beta_n > 0$, $\alpha_n > -1$ and $\gamma_n > 0$. We also assume that $2\gamma_n/n \to \gamma > 0$, $2\beta_n/n \to \beta \in \mathbb{R}$, $(2\alpha_n - n)/n \to \alpha > 0$, so that $2\alpha_n/n \to \alpha + 1$. Moreover, let $V_n$ be the
potential related to the density (4), i.e.:
\[ V_n(x) = \frac{2\gamma_n}{n} x + \frac{2\beta_n}{n} \log(1 + x) - \frac{2\alpha_n}{n} \log(x), \quad x \in \mathbb{R}. \]

In the complex matrix case, one can set \( a_n = n\alpha, \beta_n = n\beta \) and \( \gamma_n = n\gamma \) and then potential \( V_n^{\text{complex}}(x) = V^{\text{complex}}(x) \) does not depend on \( n \) and one can use classical results to get the limit of empirical measure. Here we use result from [6]:

If \( \mu(x) \) define below:
\[ \gamma > 0 \] such that (5) can equivalently be written as system of algebraic equations:
\[ \beta \]
\[ \alpha = \sqrt{\frac{1}{n}} \sum_{i=1}^{n} \delta_{\lambda_i} \] almost surely converges weakly to probability measure \( \mu_V \) as \( n \to \infty \). The measure \( \mu_V \) minimizes energy of a field with external potential \( V \), which we define below:
\[ E_V(\mu) := \int \int_{\mathbb{R}^2} \log |s - t|^{-1} d\mu(s) d\mu(t) + \int V(s) d\mu(s). \]

Properties of measure \( \mu_V \), called the equilibrium measure, has been deeply analysed in general cases, see [17] for instance. Facts that we use below, can be found in [17] (see Theorems IV.1.11 and IV.3.1 there).

If \( V \) is convex on some closed interval \([A, B]\) or \( xV'(x) \) is increasing on \((A, B) \subset [0, \infty] \) then the support of the equilibrium measure \( \mu_V \) is a closed interval \([a, b] \subset [A, B]\), where \( a \) and \( b \) are such that
\[ \gamma > 0 \] such that (5) can equivalently be written as system of algebraic equations:
\[ \begin{aligned}
&\frac{1}{\pi} \int_a^b \frac{V'(x)}{\sqrt{(b-x)(x-a)}} dx = 0, \\
&\frac{1}{\pi} \int_a^b \frac{xV'(x)}{\sqrt{(b-x)(x-a)}} dx = 2.
\end{aligned} \]

Then
\[ \mu_V(x) = \frac{1}{2\pi} \sqrt{(x-a)(b-x)} \ PV \left( \frac{1}{\pi} \int_a^b \frac{V'(t)}{\sqrt{(t-a)(b-t)}} dt \right), \quad x \in [a, b]. \]

For \( V_n \) related to Kummer eigenvalues we have
\[ V(x) = \lim_{n \to \infty} V_n(x) = \gamma x + \beta \log(1 + x) - (\alpha - 1) \log(x), \]
which fulfills conditions (a), (b), (c). Note that if \( \beta \leq 0 \) then \( V \) is convex on \( \mathbb{R} \) and if \( \beta > 0 \) then \( xV'(x) \) is increasing on \( \mathbb{R} \). So in both cases \([A, B] = [0, \infty] \) and (5) holds. It can be calculated that (5) can equivalently be written as system of algebraic equations:
\[ \begin{aligned}
&\gamma + \frac{\beta}{\sqrt{(a+1)(b+1)}} - \frac{\alpha - 1}{\sqrt{ab}} = 0, \\
&\gamma \frac{a+b}{2} \alpha + 1 + \beta - \frac{\beta}{\sqrt{(a+1)(b+1)}} = 2,
\end{aligned} \]
a, b \in [0, \infty]. Moreover, in this case the equilibrium measure is
\[ \mu_V(x) = \frac{1}{2\pi} \sqrt{(x-a)(b-x)} \left( \frac{\alpha - 1}{x\sqrt{ab}} - \frac{\beta}{(1+x)\sqrt{(a+1)(b+1)}} \right) I(a,b)(x). \]
We will call this distribution free–Kummer distribution with parameters $\alpha$, $\beta$, $\gamma$. We will write $X \sim fK(\alpha, \beta, \gamma)$ if $\mu_X$ is $\star$-distribution of $X$.

**Lemma 3.2.** The Cauchy transform of free–Kummer $fK(\alpha, \beta, \gamma)$ is

$$G_{\mu}(z) = \frac{1}{2} \left( \gamma - \frac{a-1}{z} + \frac{\beta}{1+z} + \sqrt{(z-b)(z-a)} \left[ \frac{\beta}{(1+z)\sqrt{a+1)(b+1)} - \frac{a-1}{z\sqrt{ab}} \right] \right)$$

**Proof.** To prove Lemma it is enough to show that $\gamma > 0$. The proof goes as the proof of Theorem 2 in [5].

For our results we also need boundedness of the largest eigenvalue of Kummer matrix. For this reason we introduce large deviation principle (LDP). We say that LDP with rate function $I$ and speed $n$ holds for a sequence of measures $(\nu_n)$, if for any Borel set $\Gamma$:

$$-\inf\{I(x) : x \in \Gamma^0\} \leq \lim_{n \to \infty} \inf \nu_n \log \nu_n(\Gamma) \leq \lim_{n \to \infty} \sup \nu_n \log \nu_n(\Gamma) \leq -\inf\{I(x) : x \in \bar{\Gamma}\},$$

where $\Gamma^0$ is the interior and $\bar{\Gamma}$ is the closure of $\Gamma$. Function $I$ is a **good rate function**, if for any $\alpha \in \mathbb{R}$ the set $\{x : I(x) \leq \alpha\}$ is compact.

**Proposition 3.3.** The largest eigenvalue $\lambda_{\text{max}}$ of random matrix $MK(\alpha_n, \beta_n, \gamma_n)$ converges almost surely to $b$ (right end of the support) and satisfies a LDP on $\mathbb{R}^{+*}$ with speed $n$ and good rate function

$$I_{\alpha,\beta,\gamma}^*(t) = \begin{cases} \int_b^t \frac{1}{2} \sqrt{(x-a)(x-b)} \left( \frac{a-1}{x\sqrt{ab}} - \frac{\beta}{(1+x)\sqrt{(a+1)(b+1)}} \right) \, dx, & t > b \\ +\infty, & \text{otherwise} \end{cases}$$

**Proof.** The proof goes as the proof of Theorem 2 in [5].

Let

$$g(x) := \int_a^b \log |x-t|^{-1} \, \mu(dt) + \frac{1}{2} V(x) + \frac{1}{2} \int_a^b V(t) \, \mu(dt).$$

Notice that $I_{\alpha,\beta,\gamma}^*(t) = \int_b^t g'(x) \, dx$ for all $t > b$. Since

$$g'(x) = \frac{1}{2} V'(x) + \int_a^b \frac{1}{t-x} \, d\mu(t) = \frac{1}{2} \sqrt{(x-a)(x-b)} \left( \frac{a-1}{x\sqrt{ab}} - \frac{\beta}{(1+x)\sqrt{(a+1)(b+1)}} \right)$$

is positive ($\gamma > 0$), then $g$ is increasing on $(b, \infty)$. The statement follows from arguments in Section 4.2 in [5].
Corollary 3.4. The largest eigenvalue of matrix Kummer random matrix is asymptotically almost surely bounded.

4. Freeness property for free–Poisson and free–Kummer distributions

Theorem 4.1. Let \((A, \varphi)\) be a \(C^*\)-probability space. Let \(X\) have free Kummer distribution \(fK(a, a + b, c)\) and \(Y\) have free Poisson distribution \(f\text{Pois}(1/c, a + b)\), with \(a, c > 0\) and \(b \in \mathbb{R}\). If \(X\) and \(Y\) are free and if
\[
U := (I + X)^{-1/2} Y (I + X)^{-1/2} \quad \text{and} \quad V := (I + U)^{1/2} X (I + U)^{1/2}
\]
then \(U\) and \(V\) are free.

We provide two technical results to prove Theorem 4.1. The first one can be found in [20] (Lem. 3.2) in slightly different form.

Lemma 4.2. Let \((u_N)_{N \geq 1}, (z_N)_{N \geq 1}\) be two independent sequences of random matrices on probabilistic space \((\Omega, \mathcal{F}, P)\) and assume that \(u_N, z_N\) are \(N \times N\) matrices for every \(N\). Suppose that \(u_N\) and \(z_N\) have almost surely weak limits of their sequences of the empirical spectral distributions, \(\mu\) and \(\nu\), respectively. Also suppose that the smallest eigenvalue of \(z_N\) is asymptotically almost surely greater than a constant \(A > 0\) and that the largest eigenvalue of \(z_N\) is asymptotically almost surely smaller than a constant \(B > 0\). Let \((A, \varphi)\) be \(C^*\)-probability space. Assume that there exist \(U, Z \in A\) such that \(U\) and \(Z\) are free and \(U \sim \mu, Z \sim \nu\). Then for any complex polynomial \(Q \in \mathbb{C}(x) (x_1, x_2, x_3)\) in three non-commuting variables and for any \(\epsilon > 0\) we have
\[
\mathbb{P}
\left[
\left|\frac{1}{N} \text{tr} \left[ Q (u_N, z_N, z_N^{-1}) \right] - \varphi \left[ Q (U, Z, Z^{-1}) \right] \right| > \epsilon
\right] \xrightarrow{N \to \infty} 0.
\]

In [20] the lemma was proved for \(U\) with free GIG distribution, and \(Z\) which has free Poisson distribution (with certain parameters). Although our formulation is more general, the proof is the same.

Corollary 4.3. Let \((A, \varphi)\) be \(C^*\)-probability space. Assume that there exist \(X, Y \in A\) such that \(X\) and \(Y\) are free, \(X \sim fK(a, a + b, c)\) and \(Y \sim f\text{Pois}(1/c, a + b)\). Let \((x_N)_{N \geq 1}, (y_N)_{N \geq 1}\) be two independent sequences of random matrices, such that \(x_N \sim MK(a_N, b_N, c_N e)\), \(y_N \sim V(a_N + b_N, c_N e)\) and \(2a_N/N \to a, 2b_N/N \to b, 2c_N/N \to c\). Then for any complex polynomial \(Q \in \mathbb{C}(x) (x_1, x_2, x_3)\) in three non-commuting variables and for any \(\epsilon > 0\) we have
\[
\mathbb{P}
\left[
\left|\frac{1}{N} \text{tr} \left[ Q (e + x_N, (e + x_N)^{-1}, y_N) \right] - \varphi \left[ Q (I + X, (I + X)^{-1}, Y) \right] \right| > \epsilon
\right] \xrightarrow{N \to \infty} 0.
\]

Proof. The sequences of empirical spectral distribution of matrices \((x_N)_N\) and \((y_N)_N\) almost surely have their weak limits: \(fK(a, a + b, c)\) and \(f\text{Pois}(1/c, a + b)\). Eigenvalues of \(e + x_N\) are greater than 1. This implies that the support of the weak limit of empirical spectral measure of sequence \((e + x_N)_{N=1,2,\ldots}\) is separated from 0. Also the largest eigenvalue is asymptotically almost surely bounded as \(N \to \infty\) (Cor. 3.4). Then statement follows from Lemma 4.2. \(\Box\)

Now we are ready to prove the free HV property.

Proof of Thm. 4.1. We want to show that algebras generated by \(U\) and \(V\) are free.

Let us take a sequence \((y_n)_{n \geq 1}\) of \(n \times n\) Wishart matrices with parameters \(\alpha_n + \beta_n\) and \(\gamma_n e\), where \(\alpha_n/n \to \alpha > 0, \beta_n/n \to \beta\) and \(\gamma_n/n \to \gamma > 0\). Moreover let \((x_n)_{n \geq 1}\) be a sequence
of matrix–Kummer matrices with parameters \( \alpha_n, \beta_n \) and \( \gamma_n \). Assume that \( (x_n) \) and \( (y_n) \) are independent. Then since matrix–Kummer and Wishart matrices are unitarily invariant and both possess almost sure limiting eigenvalue distributions, they are almost surely asymptotically free (Thm. 2.4). That means that if we define \( \mathcal{A}_n \) to be the algebra of random matrices of size \( n \times n \) with integrable entries and consider the state \( \varphi_n(a) = 1/n \tr(a) \), then for any polynomial in two non–commuting variables \( P \in \mathbb{C} \langle x_1, x_2 \rangle \) we have almost surely

\[
\lim_{n \to \infty} \varphi_n(P(x_n, y_n)) = \varphi(P(\bar{X}, \bar{Y})),
\]

where \( \bar{X} \) and \( \bar{Y} \) are as in the statement of the theorem.

By the HV property for random matrices we have that

\[
\mathbf{u}_n := (e + x_n)^{-1/2} y_n (e + x_n)^{-1/2} \quad \text{and} \quad \mathbf{v}_n := (e + \mathbf{u}_n)^{1/2} x_n (e + \mathbf{u}_n)^{1/2}
\]

are independent, \( \mathbf{u}_n \sim \mathcal{MK}(a_n + b_n, -b_n, \gamma_n e) \) and \( \mathbf{v}_n \sim \mathcal{W}(a_n, \gamma_n e) \) for any \( n \in \mathbb{N}_+ \). They are also almost surely asymptotically free. Let \( \bar{U}, \bar{V} \) be the limiting pair of non–commuting free random variables.

Then for any polynomial \( P \in \mathbb{C} \langle x_1, x_2 \rangle \), there exists \( Q \in \mathbb{C} \langle x_1, x_2, x_3 \rangle \) such, that we have

\[
\lim_{n \to \infty} \varphi_n \left( Q \left( e + x_n, (e + x_n)^{-1}, y_n \right) \right) = \lim_{n \to \infty} \varphi_n(P(\mathbf{u}_n, \mathbf{v}_n)) = \varphi(P(\bar{U}, \bar{V})).
\]

From Corollary 4.3 we know that \( \varphi_n \left( Q \left( e + x_n, (e + x_n)^{-1}, y_n \right) \right) \) converges in probability to

\[
\varphi \left( Q \left( \mathbb{I} + \bar{X}, (\mathbb{I} + \bar{X})^{-1}, \bar{Y} \right) \right).
\]

However, by (11) the sequence \( \left( \varphi_n \left( Q \left( e + x_n, (e + x_n)^{-1}, y_n \right) \right) \right)_{n \geq 1} \) has almost sure limit and thus we have

\[
\lim_{n \to \infty} \varphi_n \left( Q \left( e + x_n, (e + x_n)^{-1}, y_n \right) \right) = \varphi \left( Q \left( \mathbb{I} + \bar{X}, (\mathbb{I} + \bar{X})^{-1}, \bar{Y} \right) \right) = \varphi \left( P \left( U, V \right) \right).
\]

These imply that all joint moments of \( (U, V) \) and \( (\bar{U}, \bar{V}) \) are the same. Since \( \bar{U} \) and \( \bar{V} \) are free, then \( U \) and \( V \) are also free (recall that freeness is defined my joint moments). \( \square \)

5. Free regression characterization of HV type

5.1. Conditional expectation in a free world. We will formulate free version of the following classical characterization theorem from [16].

**Theorem 5.1.** Let \( X \) and \( Y \) be independent, positive, non-degenerate (commutative) random variables, such that \( \mathbb{E}X < \infty \), \( \mathbb{E}Y < \infty \) and \( \mathbb{E}X^{-1} < \infty \). Let \( U := Y/(1 + X) \), \( V := X(1 + U) \) and assume that there exist real constants \( \alpha \) and \( \beta \) such that \( \mathbb{E}(V|U) = \alpha \) and \( \mathbb{E}(V^{-1}|U) = \beta \). Then \( \alpha \beta > 1 \) and there exists a constant \( c > 0 \) such that

\[
X \sim \mathcal{K} \left( c, \beta \alpha \beta^{-1}, \alpha \beta^{-1} \right) \quad \text{and} \quad Y \sim \mathcal{G} \left( c, \beta \alpha \beta^{-1} \right).
\]

To do this we need to recall definition of non-commutative conditional expectation following [3, 22]. Let \( (\mathcal{A}, \varphi) \) be a \( W^* \)–probability space. Let \( \mathcal{B} \subset \mathcal{A} \) be a von Neumann subalgebra of \( \mathcal{A} \). Then there exists a unique faithful, normal projection \( \varphi(\cdot|\mathcal{B}) : \mathcal{A} \to \mathcal{B} \) such that \( \varphi(\varphi(\cdot|\mathcal{B})) = \varphi(\cdot) \). We call it a non-commutative conditional expectation from \( \mathcal{A} \) to \( \mathcal{B} \) with respect to \( \varphi \) (see [22], Vol I p. 332). The conditional expectation of a self–adjoint element \( \bar{X} \in \mathcal{A} \) is a unique self–adjoint element of \( \mathcal{B} \).
There are several important properties of a non-commutative conditional expectation \( \varphi(\cdot | \mathcal{B}) \) which we will use throughout the following sections. For that reason we cite these properties from [3]:

I. If random variables \( U, V \in \mathcal{A} \) are free, then \( \varphi(U|V) = \varphi(U) I \);

II. If \( X \in \mathcal{A}, Y \in \mathcal{B} \), then \( \varphi(XY) = \varphi(X|\mathcal{B}) Y \).

### 5.2. The characterization theorem

Free analogue of Theorem 5.1 will now be stated.

**Theorem 5.2.** Let \((\mathcal{A}, \phi)\) be a non–commutative \( \mathbb{W}^* \)-probability space. Let \( X \in \mathcal{A} \) and \( Y \in \mathcal{A} \) be self–adjoint, positive, free, compactly supported and non-degenerate random variables. Define

\[
U := (I + X)^{-1/2} Y (I + X)^{-1/2} \quad \text{and} \quad V := (I + U)^{1/2} X (I + U)^{1/2}
\]

and assume that

\[
\varphi(V | U) = \bar{\alpha} I, \quad \varphi(V^{-1} | U) = \bar{\beta} I
\]

for some constants \( \bar{\alpha}, \bar{\beta} > 0 \). Then \( X \sim f\mathcal{K}(\bar{\alpha} \gamma, a \gamma, \gamma) \) and \( Y \sim f\text{Pois}(1/\gamma, a \gamma) \), where \( \gamma = \bar{\beta} / (\bar{\alpha}^2 - 1) \) and \( a > 0 \).

**Proof.** **Step 1.** First step is to show that \( Y \) is free–Poisson random variable.

For any \( k \geq 0 \) we multiply both sides of (12) by \( U^k \) and take expectation \( \varphi \). Due to the property (II) we have

\[
\varphi\left(U^k (1 + X) + U^{k+1} (1 + X) - U^k - U^{k+1}\right) = \bar{\alpha} \varphi(U^k).
\]

Eq. (13) can be modified in a similar manner and we obtain a system of recursive equations holding for any \( k \geq 1 \)

\[
\beta_{k-1} + \beta_k = (\bar{\alpha} + 1) \alpha_k + \alpha_{k+1},
\gamma_k = \beta(\alpha_k + \alpha_{k+1}),
\]

where for \( k \geq 0 \):

\[
\alpha_k = \varphi\left([I + X]^{-1} Y^k\right) = \varphi\left([Y (I + X)^{-1}]^k\right),
\beta_k = \varphi\left(Y [I + X]^{-1} Y^k\right) = \varphi\left(Y [Y (I + X)^{-1}]^k\right),
\gamma_k = \varphi\left(X^{-1} [Y (I + X)^{-1}] Y^k\right) = \varphi\left(X^{-1} Y [I + X]^{-1} Y^k\right),
\delta_k = \varphi\left([I + X]^{-1} Y^k\right).
\]

For instance, notice that \( \alpha_0 = 1 \) and \( \beta_0 = \varphi(Y) \). For \( z \) from neighbourhood of 0 we can define

\[
A(z) := \sum_{n=0}^{\infty} z^n \alpha_n, \quad B(z) := \sum_{n=0}^{\infty} z^n \beta_n, \quad C(z) := \sum_{n=0}^{\infty} z^n \gamma_n, \quad D(z) := \sum_{n=0}^{\infty} z^n \delta_n.
\]

From Eq. (14) we obtain

\[
\begin{aligned}
\text{(d)} \\
B(z) + \frac{B(z) - \beta_0}{z} &= \frac{\bar{\alpha} + 1}{z} (A(z) - 1) + \frac{A(z) - z \alpha_1 - 1}{z^2},
\end{aligned}
\]
(e) 

\[ C(z) = \bar{\beta} \left( A(z) + \frac{A(z) - 1}{z} \right). \]

Also if \( r := r_Y \) is a \( r \)-transform of \( Y \), then these three relations hold:

(a) \( A(z) = 1 + zD(z) r(zD(z)) \):

From (2) it follows that for any \( n \geq 1 \) we have

\[ \alpha_n = \varphi \left( \left[ (I + X)^{-1} Y \right]^n \right) = \sum_{k=1}^{n} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \delta_{i_1} \ldots \delta_{i_k}, \]

where \( \kappa_k := \kappa_k(Y) \) is \( k \)-th free cumulant of \( Y \).

So

\[ A(z) = \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \delta_{i_1} \ldots \delta_{i_k} + 1 \]
\[ = \sum_{k=1}^{\infty} \kappa_k z^k \sum_{n=k}^{\infty} z^{n-k} \sum_{i_1 + \ldots + i_k = n-k} \delta_{i_1} \ldots \delta_{i_k} + 1 \]
\[ = \sum_{k=1}^{\infty} \kappa_k z^k D^k(z) + 1 \]
\[ = 1 + z D(z) r(zD(z)). \]

(b) \( B(z) = A(z) r(zD(z)) \):

From (2) it follows that for any \( n \geq 0 \) we have

\[ \beta_n = \varphi \left( Y \left[ Y(I + X)^{-1} \right]^n \right) \]
\[ = \sum_{k=1}^{n+1} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \alpha_{i_1} \delta_{i_2} \ldots \delta_{i_k}. \]

Then

\[ B(z) = \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n+1} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \alpha_{i_1} \delta_{i_2} \ldots \delta_{i_k} \]
\[ = \sum_{k=1}^{\infty} \kappa_k z^k \sum_{n=k-1}^{\infty} z^{n-k} \sum_{i_1 + \ldots + i_k = n-k} \alpha_{i_1} \delta_{i_2} \ldots \delta_{i_k} \]
\[ = \sum_{k=1}^{\infty} \kappa_k z^k \alpha_n D^{k-1}(z) \]
\[ = z A(z) r(zD(z)). \]

(c) \( C(z) [z r(zD(z)) - 1] = A(z) - 1 - \gamma_0 \):

Note that for \( n \geq 1 \) we have

\[ \gamma_n = \varphi \left( X^{-1} \left[ (I + X)^{-1} Y \right]^n \right) = \varphi \left( X^{-1} Y \left[ (I + X)^{-1} Y \right]^{n-1} \right) - \varphi \left( \left[ (I + X)^{-1} Y \right]^n \right). \]
Again Eq. (2) implies
\[
\varphi \left( X^{-1} Y \left( (I + X)^{-1} Y \right)^{n-1} \right) = \kappa_1 \gamma_{n-1} + \kappa_2 (\delta_0 \gamma_{n-2} + \delta_1 \gamma_{n-3} + \ldots + \delta_{n-2} \gamma_0) + 
\ldots + \kappa_n \delta_0^{n-1} \gamma_0 
= \sum_{k=1}^{n} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \gamma_{i_1} \delta_{i_2} \ldots \delta_{i_k}.
\]

If we multiply \( \gamma_n \) by \( z^n \) and sum over \( n = 0, 1, \ldots \), we have:
\[
C(z) = \gamma_0 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \kappa_k \sum_{i_1 + \ldots + i_k = n-k} \gamma_{i_1} \delta_{i_2} \ldots \delta_{i_k} - \sum_{n=1}^{\infty} z^n \alpha_n 
= \gamma_0 + \sum_{k=1}^{\infty} \kappa_k z^k \sum_{n=k}^{\infty} z^{n-k} \sum_{i_1 + \ldots + i_k = n-k} \gamma_{i_1} \delta_{i_2} \ldots \delta_{i_k} - (A(z) - 1) 
= \gamma_0 + zC(z) \sum_{k=1}^{\infty} \kappa_k z^{k-1} D(z)^{k-1} - A(z) + 1 
= \gamma_0 + 1 + z C(z) r(zD(z)) - A(z).
\]

To determine distribution of \( Y \) we will solve the system of equations (a)-(e) with respect to \( r \).

We rewrite the system here:
(a) \( A(z) = 1 + z D(z) r(zD(z)) \),
(b) \( B(z) = A(z) r(zD(z)) \),
(c) \( C(z) [z r(zD(z)) - 1] = A(z) - 1 - \gamma_0 \),
(d) \( B(z) + \frac{B(z) - \beta_0}{z} = (\bar{\alpha} + 1) \frac{A(z) - 1}{z} + \frac{A(z) - \alpha_1 - 1}{z^2} \),
(e) \( C(z) = \bar{\beta} \left( A(z) + \frac{A(z) - 1}{z} \right) \).

Firstly, we multiply Eq. (e) by \( z r(zD(z)) - 1 \) and then we plug it into Eq. (c). We obtain
\[
\bar{\beta}^{-1} (A(z) - 1 - \gamma_0) = \left( A(z) + \frac{A(z) - 1}{z} \right) (z r(zD(z)) - 1).
\]

Let \( h(z) := z D(z) r(zD(z)) \). Then (a) can be written in terms of \( h \) as: \( A(z) = 1 + h(z) \). From this and above equation we get:
\[
\bar{\beta}^{-1} (h(z) - \gamma_0) = \left( 1 + h(z) + \frac{1}{z} h(z) \right) \left( \frac{h(z)}{D(z)} - 1 \right).
\]

Then after simplification
\[
\left( h(z) + \frac{1}{z} h(z) \right) \frac{h(z)}{D(z)} = \bar{\beta}^{-1} h(z) - \bar{\beta}^{-1} \gamma_0 - \frac{h(z)}{D(z)} + 1 + h(z) + \frac{h(z)}{z}.
\]

On the other hand we can plug \( B(z) \) from (b) into (d) and multiply the latter by \( z \). Now we have
\[
(1 + z) A(z) r(zD(z)) - \beta_0 - \bar{\alpha} (A(z) - 1) + \alpha_1 + 1 = A(z) + \frac{A(z) - 1}{z}.
\]
In terms of \( h(z) = zD(z) r(zD(z)) = A(z) - 1 \) it reads as

\[
(16) \quad \left( h(z) + \frac{1}{z} h(z) \right) \frac{h(z)}{D(z)} = h(z) + \frac{h(z)}{z} + \beta_0 + \bar{\alpha} h(z) - \alpha_1 - \frac{h(z)}{D(z)} - \frac{h(z)}{zD(z)}.
\]

Comparing the left-hand sides of (15) and (16), we arrive at

\[
\bar{\beta}^{-1} h(z) - \bar{\beta}^{-1} \gamma_0 + 1 = \beta_0 + \bar{\alpha} h(z) - \alpha_1 - \frac{h(z)}{zD(z)}.
\]

Which can equivalently be written as

\[
h(z) = (\beta_0 - \alpha_1 - 1 + \bar{\beta}^{-1} \gamma_0) \left( \bar{\beta}^{-1} - \bar{\alpha} + \frac{1}{zD(z)} \right)^{-1}.
\]

Since \( h(z) = zD(z) r(zD(z)) = A(z) - 1 \), \( r \)-transform of \( \mathbb{Y} \) is analytic near 0 and \( \lim_{z \to 0} zD(z) = 0 \) (see definition of \( D \)), then for \( y \) close to 0 we have:

\[
r(y) = (\beta_0 - \alpha_1 - 1 + \bar{\beta}^{-1} \gamma_0) \left( y (\bar{\beta}^{-1} - \bar{\alpha}) + 1 \right)^{-1},
\]

\[
r'(y) = \frac{\beta_0}{1 - y (\bar{\alpha} - \bar{\beta}^{-1})}.
\]

The last equality holds due to the fact, that \( \alpha_1 + 1 = \gamma_0 / \bar{\beta} \). Eventually, we have

\[
(17) \quad r(y) = \frac{a}{1 - by},
\]

where \( a = \beta_0 \) i \( b = \bar{\alpha} - 1 / \bar{\beta} \). It is an \( r \)-transform of free-Poisson with parameters \((\bar{\alpha} \bar{\beta} - 1) / \bar{\beta} \) and \( a \bar{\beta} / (\bar{\alpha} \bar{\beta} - 1) \).

**Step 2.** To recover Cauchy transform of \( \mathbb{U} \) we use (a), (b) and (d). From (a) and formula for \( r \) we can deduce

\[
zD(z) = \frac{A(z) - 1}{(A(z) - 1)b + a}.
\]

Then (b) read as \( B(z) = A(z)((A(z) - 1)b + a) \). We plug that last in (d), which gives quadratic equation for \( A \):

\[
A^2(z) [z(1 + z)(\bar{\alpha} \bar{\beta} - 1) + A(z) \left[ z(1 + z)(\bar{\beta} a - \bar{\beta} \bar{\alpha} + 1) - \bar{\beta}(1 + z + \bar{\alpha} z) \right] + \bar{\beta} + z \bar{c} = 0,
\]

where \( \bar{c} = \bar{\beta}(1 + \bar{\alpha} - a + \alpha_1) \). Solution has the following form:

\[
A(z) = \frac{\bar{\beta}}{2(-1 + \bar{\alpha} \bar{\beta}) z (1 + z)} \left( 1 + \left( -\bar{\beta}^{-1} + 1 - a + 2 \bar{\alpha} \right) z + \left( -\bar{\beta}^{-1} - a + \bar{\alpha} \right) z^2 + \bar{\beta}^{-1} \sqrt{-4 \bar{\beta}(-1 + \bar{\alpha} \bar{\beta}) z (1 + z) [1 + (1 - a + \alpha + \alpha_1) z] + [z^2(1 + \bar{\beta} (a - \alpha)) + z (1 + \bar{\beta} (a - 1 - 2 \bar{\alpha})) - \bar{\beta}^2]} \right).
\]

Since Cauchy transform \( G \) of \( \mathbb{U} = (\mathbb{I} + \mathbb{X})^{-1/2} \mathbb{Y} (\mathbb{I} + \mathbb{X})^{-1/2} \) satisfies

\[
G(z) = \frac{1}{z} A \left( \frac{1}{z} \right),
\]

then

\[
(18) \quad G(z) = \frac{\bar{\beta}}{2(-1 + \bar{\alpha} \bar{\beta})} \left( 1 + \frac{\bar{\alpha}}{1 + z} - \frac{\bar{\beta}^{-1} + a - \bar{\alpha}}{z} + \frac{\bar{\beta}^{-1}}{z (1 + z)} \sqrt{p_1(z) - p_2(z)} \right),
\]
Figure 1. Left: $p_2$ (orange) for $\bar{\alpha} = 4.5$, $\bar{\beta} = 0.4$, $a = 3$, $\alpha_1 = 1.5$. Right: $p_1$ (blue) and $p_2$ (orange) for $\bar{\alpha} = 1$, $\bar{\beta} = 1.02$, $a = -2$, $\alpha_1 = 3$.

where

$$p_1(z) = \left[-\bar{\beta} z^2 + z \bar{\beta} (\bar{\beta}^{-1} + a - 1 - 2\bar{\alpha}) + 1 + a\bar{\beta} - \bar{\alpha}\bar{\beta}\right]^2,$$

$$p_2(z) = 4\bar{\beta} (-1 + \alpha\bar{\beta}) z (1 + z) (1 - a + \bar{\alpha} + \alpha_1 + z).$$

We have to find an admissible set of parameters $\bar{\alpha}, \bar{\beta}, a, \alpha_1$, such that $G$ is Cauchy transform of the probabilistic measure associated with $U$. For that reason we analyse the polynomial $p(\cdot) := p_1(\cdot) - p_2(\cdot)$, which is under the square root in formula (18). It is known that $G$ is analytic on $\mathbb{C}^+$ and the image $G((\mathbb{C}^+)) = \mathbb{C}^-$. We assumed that $\star$-distribution of $U$ is supported in $(0, \infty)$. These facts imply, that $p$

- (\star) does not have roots in $\mathbb{C}^+$ and so it does not have complex roots at all; then $p$ has 4 real roots (possibly multiple);
- (\star\star) can not be negative for negative (real) arguments: this follows from Stieltjes formula (3).

The roots of $p_2$ are: $z_1 = 0$, $z_2 = -1$ and $z_3 = a - \bar{\alpha} - \alpha_1 - 1$. Since $\varphi(\mathbb{V}|U) = \varphi(X + \mathbb{Y} - U|U)$ and $\alpha_1 = \varphi(\mathbb{U})$ if $a = \beta_0 = \varphi(\mathbb{Y})$, then from (12):

$$a - \bar{\alpha} - \alpha_1 = -\varphi(X) < 0.$$  

This implies that $z_3 < -1$, which will be of importance later. Now we can sketch $p_2$ – see Fig. (1).

We will show that $p_1$ has two double real roots now. Let

$$f(z) := z^2 - z (d - 1 - \bar{\alpha}) - d,$$
where \( d = \bar{\beta}^{-1} + a - \bar{\alpha} \). Then \( p_1(z) = \bar{\beta}^2 f^2(z) \) and \( f \) has the same roots as \( p_1 \). Note, that \( f \)'s discriminant is positive. Indeed:

\[
\Delta_f = (d - 1 - \bar{\alpha})^2 + 4d \\
= d^2 + 1 + \bar{\alpha}^2 + 2d + 2\bar{\alpha} - 2d\bar{\alpha} \\
= (d + 1 - \bar{\alpha})^2 + 4\bar{\alpha} > 0,
\]

where the last equality holds because \( \bar{\alpha} > 0 \).

We denote roots of \( p_1 \): \( \zeta_1 < \zeta_2 \). If \( \zeta_1 \cdot \zeta_2 > 0 \), then Viete’s formula implies \( \bar{\alpha} - a - \bar{\beta}^{-1} > 0 \). Then if \( \zeta_1, \zeta_2 > 0 \), we have \(- (\bar{\alpha} - a - \bar{\beta}^{-1}) - 1 - \bar{\alpha} = \zeta_1 + \zeta_2 > 0 \). But this is contrary to the fact that \( \bar{\alpha} > 0 \). On the other hand, when \( \zeta_1, \zeta_2 < 0 \), then \( p_1 \) and \( p_2 \) do not intersect (which is contrary to (\( \star \))) or they intersect in the second quarter of the plane. It contradicts (\( \star \star \)). This case is presented in the second graph in Fig. 1.

So we conclude that \( \zeta_1 < 0 < \zeta_2 \). Or equivalently

\[
(19) \quad \bar{\beta}^{-1} + a - \bar{\alpha} = -\zeta_1 \cdot \zeta_2 > 0.
\]

Now, we want to show that \( p \) has two different, positive roots and one negative, which is a double root. Suppose that \( p \) has four different roots. Since \( \zeta_1 < 0 < \zeta_2 \) and \( p_2(x) > 0 \) for \( x > 0 \), then two roots of \( p = p_1 - p_2 \) are negative. We denote them \( y_1 < y_2 \). So \( p \) is negative in the interval \((y_1, y_2)\) (see Fig. 2), which is contrary to (\( \star \star \)). This implies \( p \) has double root \( x_0 \) \( (p_1 \) and \( p_2 \) are tangents at \( x_0 \)). Since \( p_1 \) and \( p_2 \) cannot be tangent outside interval \((z_3, -1)\), then \( x_0 < -1 \). This is in the second chart in Fig. 2. The other points of intersection of \( p_1 \) and \( p_2 \) are positive \( x_1 < x_2 \).

To find out how \( x_0, x_1 \) and \( x_2 \) depend on \( \bar{\alpha}, \bar{\beta} \) and \( a \) we use Viete’s formula once again. We have

\[
p(z) = (p_1 - p_2)(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4,
\]

where

\[
a_0 = (1 + a \bar{\beta} - \bar{\alpha} \bar{\beta})^2, \\
a_1 = 2 \left[1 + (1 - \bar{\alpha} + 2a_1) \bar{\beta} + (a^2 - a (1 + \bar{\alpha}) - \bar{\alpha} (1 + 2a_1)) \bar{\beta}^2 \right], \\
a_2 = 1 + (4 - 2a + 4a_1) \bar{\beta} + (1 - 4a + a^2 - 2\bar{\alpha} - 4\bar{\alpha}a_1) \bar{\beta}^2, \\
a_3 = 2\bar{\beta}(1 + \bar{\beta} - a \bar{\beta}), \\
a_4 = \bar{\beta}^2.
\]

We have

\[
(20) \quad x_1 x_2 x_0^2 = \frac{a_0}{a_4} = \left(\frac{1 + a \bar{\beta} - \bar{\alpha} \bar{\beta}}{\bar{\beta}}\right)^2,
\]

\[
(21) \quad x_1 + x_2 + 2x_0 = -\frac{a_3}{a_4} = -2 (1 - \alpha + \bar{\beta}^{-1}).
\]

Since \( x_0 < -1 < 0 \) and (19) holds, then from Eq. (20)

\[
(22) \quad x_0 = \frac{|\bar{\beta}^{-1} + a - \bar{\alpha}|}{\sqrt{x_1 x_2}} = \frac{-\bar{\beta}^{-1} + a - \bar{\alpha}}{\sqrt{x_1 x_2}}.
\]
Let \( q(z) := p(z - 1) \). Then
\[
q(z) = a'_0 + a'_1 z + a'_2 z^2 + a'_3 z^3 + a'_4 z^4,
\]
where
\[
\begin{align*}
&a'_0 = \bar{\alpha}^2 \bar{\beta}^2, \\
&a'_1 = -2\bar{\beta} \left[ 2\alpha_1 + a(-2 + \bar{\alpha} \bar{\beta}) - \bar{\alpha}(-1 + \bar{\beta} + 2\alpha_1 \bar{\beta}) \right], \\
&a'_2 = 1 - 2\bar{\beta}(1 + a - 2\alpha_1) + (1 + 2a + a^2 - 2\bar{\alpha} - 4\bar{\alpha} \alpha_1) \beta^2, \\
&a'_3 = -2\bar{\beta} (-1 + \beta + a\bar{\beta}), \\
&a'_4 = \bar{\beta}^2.
\end{align*}
\]
Notice that if \( q(x') = 0 \), then \( p(x' - 1) = 0 \) and so \( x' = x + 1 \), where \( x \) is a root of \( p \). So roots of \( q \) are exactly \( x_0 + 1 \) (double root), \( x_1 + 1 \) and \( x_2 + 1 \). Again from Viete’s formula we have
\[
(x_1 + 1)(x_2 + 1)(x_0 + 1)^2 = \frac{a'_0}{a'_4} = \bar{\alpha}^2.
\] (23)

From Eq. (23) and again from \( x_0 < -1 \), we have
\[
x_0 + 1 = -\frac{\bar{\alpha}}{\sqrt{(x_1 + 1)(x_2 + 1)}}.
\] (24)
Let us recall that roots $x_1$ and $x_2$ are the boundaries of the support of $U = (\mathbb{I} + X)^{-1/2} Y (\mathbb{I} + X)^{-1/2}$. Combining (21) with (24), we have

$$x_1 + x_2 + 2 \left( 1 - a + \tilde{\beta}^{-1} \right) - 2 \frac{\tilde{\alpha}}{\sqrt{(x_1 + 1)(x_2 + 1)}} - 2 = 0,$$

(25)

$$\frac{x_1 + x_2}{2} - a + \tilde{\beta}^{-1} - \frac{\tilde{\alpha}}{\sqrt{(x_1 + 1)(x_2 + 1)}} = 0.$$

From (22) and (24) we have:

$$\frac{\tilde{\alpha}}{\sqrt{(x_1 + 1)(x_2 + 1)}} - \frac{\tilde{\beta} + a - \tilde{\alpha}}{\sqrt{x_1 x_2}} + 1 = 0.$$  (26)

These are exactly conditions from (7) for support of free Kummer distribution with parameters $\gamma = \tilde{\beta}/(\tilde{\alpha} \tilde{\beta} - 1)$, $\alpha = (1/\tilde{\beta} - \tilde{\alpha} + a) \gamma + 1$ and $\beta = \tilde{\alpha} \gamma$. So we have

$$p(z) = (z - x_1)(z - x_2)(z - x_0)^2$$

and from (18)

$$G_U(z) = \frac{1}{2} \left[ \gamma - \frac{\alpha - 1}{z} + \frac{\beta}{1 + z} + \sqrt{(z - x_1)(z - x_2)} \left( \frac{\gamma}{1 + z} - \frac{\alpha - 1}{\sqrt{x_1 x_2} z (1 + z)} \right) \right]$$

$$= \frac{1}{2} \left[ \gamma - \frac{\alpha - 1}{z} + \frac{\beta}{1 + z} + \sqrt{(z - x_1)(z - x_2)} \left( \frac{\beta}{(1 + z)\sqrt{(x_1 + 1)(x_2 + 1)}} - \frac{\alpha - 1}{z \sqrt{x_1 x_2}} \right) \right],$$

where $x_1$ and $x_2$ are such that (25) and (26) hold. Since the support of $U$ is bounded, we choose the main branch of square root (conf. also proof of Lem. 3.2 for further reasoning). We finally obtained Cauchy transform of free Kummer with parameters $\alpha$, $\beta$, $\gamma$.

**Step 3.** Knowing distributions of $Y$ and $U$, we can calculate $X$ by its $S$-transforms. Let $S_z$ denote the $S$-transform of random variable $z$. Then, given that $X$ and $Y$ are free, we have

$$S_Y(z) S_{(1+X)^{-1}}(z) = S_U(z)$$

and this equality uniquely determines distribution of $X$. Theorem 4.1 implies that $X \sim fK(\beta, \alpha, \gamma)$. □

We have proved that free Poisson and free Kummer are the only distributions which maintain freeness of non-commutative random variables when transformed by the following mapping

$$(x, y) \mapsto \left( (I + x)^{-1/2} y (I + x)^{-1/2}, \left[ (I + x)^{-1/2} y (I + x)^{-1/2} \right]^{1/2} x \left[ (I + x)^{-1/2} y (I + x)^{-1/2} \right]^{1/2} \right).$$

It is still an open question if every independence characterization of random matrices has its analogon in free probability. If yes, then what is it?

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