Rapid Formation of Gas-giant Planets via Collisional Coagulation from Dust Grains to Planetary Cores

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Abstract

Gas-giant planets, such as Jupiter, Saturn, and massive exoplanets, were formed via the gas accretion onto the solid cores, each with a mass of roughly 10 Earth masses. However, rapid radial migration due to disk–planet interaction prevents the formation of such massive cores via planetesimal accretion. Comparably rapid core growth via pebble accretion requires very massive protoplanetary disks because most pebbles fall into the central star. Although planetesimal formation, planetary migration, and gas-giant core formation have been studied with a lot of effort, the full evolution path from dust to planets is still uncertain. Here we report the result of full simulations for collisional evolution from dust to planets in a whole disk. Dust growth with realistic porosity allows the formation of icy planetesimals in the inner disk (<10 au), while pebbles formed in the outer disk drift to the inner disk and there grow to planetesimals. The growth of those pebbles to planetesimals suppresses their radial drift and supplies small planetesimals sustainably in the vicinity of cores. This enables rapid formation of sufficiently massive planetary cores within 0.2–0.4 million years, prior to the planetary migration. Our models show that the first gas giants form at 2–7 au in rather common protoplanetary disks, in accordance with the exoplanet and solar systems.

Unified Astronomy Thesaurus concepts: Planet formation (1241); Protoplanetary disks (1300); Planetary system formation (1257); Solar system formation (1530)

1. Introduction

Gas-giant planets are formed via the rapid gas accretion of solid cores, each with about 10 M⊕ in protoplanetary disks (Ikoma et al. 2000), where M⊙ is the Earth mass. The formation of cores via the accretion of 10 km sized planetesimals is in the Jupiter–Saturn forming region estimated to be ∼10^7 yr (Kobayashi et al. 2011), which is longer than the disk lifetime (several million years; Haisch et al. 2001). In addition, the cores undergo the fast migration caused by the tidal interaction with the disk (called “Type I” migration; Ward 1997). They are lost prior to the gas accretion if the core formation timescale is longer than the migration timescale (∼10^7 yr; Tanaka et al. 2002). Recently, the rapid accretion of submeter-sized bodies (called “pebbles” in the context of planet formation) is argued (Ormel & Klahr 2010). Pebbles form via collisional coagulation in the outer disk and then drift to the core-growing inner disk. The accretion of such bodies may lead to the formation of massive cores in a timescale (∼10^7 yr) comparable to the migration timescale (Lambrechts & Johansen 2014). However, this process requires a massive disk because the pebble accretion is lossy. The capture rate of pebbles by a single planetary core is evaluated to be below 10% (Ormel & Klahr 2010; Lin et al. 2018; Okamura & Kobayashi 2021). Hence, the total pebble mass of a few hundred Earth masses is required for the formation of a core with 10 M⊕ (see a more detailed estimate in Section 3.2), while protoplanetary disks with such a large solid mass are very rare (Mulders et al. 2021).

As a process achieving a high conversion rate from dust or pebbles to kilometer-sized or larger bodies, planetesimal formation via collisional growth of icy pebbles is one of the most probable candidates. Recent models for collisional evolution of dust grains showed that pebbles grow to planetesimals in inner disks (∼10 au) in the realistic bulk density evolution model (Okuzumi et al. 2012). This process enhances the solid surface density in the inner disk, while 10 km sized or larger planetesimals slowly accrete onto cores. If the accretion of (sub)kilometer-sized planetesimals effectively occurs prior to planetesimal growth, cores are expected to grow in a short timescale (∼10^7 yr). In order to confirm rapid core formation, the treatment fully from dust to cores in a whole disk is required.

In this paper, we investigate the formation of solid cores of giant planets from dust grains in protoplanetary disks. In Section 2, we introduce the disk model that we apply. In Section 3, we analytically estimate the growth timescales of solid cores via planetesimal and pebble accretion, respectively. In addition, we also estimate the minimum disk masses required for the formation of single gas-giant cores via pebble accretion. In Section 4, we model a simulation for the collisional evolution of bodies from dust to planet (“dust-to-planet” simulation, hereafter DTPS), taking into account the bulk density evolution of dust aggregates. This model consistently includes planetesimal and pebble accretion. In Section 5, we show the result of a DTPS, where the rapid formation of solid cores via the accretion of planetesimals formed via drifting pebbles. In Section 6, we discuss the locations of giant planets in the solar system or for exoplanets, based on the results of DTPSs. In Section 7, we summarize our findings.

2. Disk Model

Planet formation occurs in a protoplanetary disk. We consider a power-law disk model for gas and solid surface
densities, $\Sigma_g$ and $\Sigma_s$, as
\[
\Sigma_g = \Sigma_g(r/1 \text{ au})^{-1},
\]
\[
\Sigma_s = \Sigma_s(r/1 \text{ au})^{-1},
\]
where $\Sigma_g,1 = 480 \text{ g cm}^{-2}$ and $\Sigma_s,1 = 8.5 \text{ g cm}^{-2}$ are the gas and solid surface densities at 1 au, respectively, and $r$ is the distance from the host star. The given solid/gas ratio is the same as that in the minimum-mass solar nebula (MMSN) model beyond the snow line (Hayashi 1981). However, we apply the shallower power-law index than the MMSN model according to the observation of protoplanetary disks (Andrews & Williams 2007). The surface densities $\Sigma_g$ and $\Sigma_s$ at $r = 12.5 \text{ au}$ correspond to those in the MMSN model, while $\Sigma_g$ and $\Sigma_s$ are smaller than those in the MMSN in the inner disk. The typical sizes of observed disks are $\approx 100 \text{ au}$ (Andrews et al. 2010). We set disk radii $\approx 108 \text{ au}$: disk masses correspond to $0.037 M_\odot$ ($\approx 220 M_\oplus$ in solid).

We set the temperature at the disk midplane as
\[
T = 200 \left( \frac{r}{1 \text{ au}} \right)^{-1/2} \text{ K}.
\]
The radial dependence of temperature is the same as the MMSN. However, we apply a low temperature according to Brauer et al. (2008) because of optically thick disks.

### 3. Analytic Estimate

#### 3.1. Core-growth and Migration Timescales

We here estimate the growth timescale of a solid core growing via collisions with planetesimals. Taking into account the gravitational focusing, the growth rate is given by $dM_p/dt \sim 2\pi G M_p \Sigma_g \Omega / v_{\text{rel}}$ (e.g., Goldreich et al. 2004), where $M_p$ and $R_p$ are the mass and radius of a planetary embryo, respectively; $\Omega$ is the Keplerian frequency; $v_{\text{rel}}$ is the relative velocity between the core and planetesimals; and $t$ is the time. For a solid core with $M_p \sim 10 M_\oplus$, planetary atmosphere enhances the collisional radius of the planet. The growth rate is estimated using $R_e$ instead of $R_p$, where $R_e$ is the enhancement radius via atmosphere (Inaba & Ikoma 2003). Assuming the relative velocity is determined by the equilibrium between gas drag and the stirring by the core, we estimate the growth timescale $t_{\text{grow}} = M_p / dM_p/dt$ (e.g., Kobayashi et al. 2010, 2011)

\[
t_{\text{grow}} \approx 8.4 \times 10^6 \left( \frac{\Sigma_s}{1.2 \text{ g cm}^{-2}} \right)^{-1}
\times \left( \frac{R_e / R_p}{3} \right)^{-1} \left( \frac{m}{10^9 \text{ g}} \right)^{2/15}
\times \left( \frac{\rho_b}{1.4 \text{ g cm}^{-2}} \right)^{-4/15} \left( \frac{M_p}{10 M_\oplus} \right)^{-1/3} \left( \frac{r}{7 \text{ au}} \right)^{13/10}
\times \left( \frac{T}{76 \text{ K}} \right)^{1/5} \left( \frac{\Sigma_g}{69 \text{ g cm}^{-2}} \right)^{2/5} \text{ yr},
\]

where $m$ and $\rho_b$ are the mass and bulk density of planetesimals, respectively, and $R_e = 3R_p$ is used from the previous estimate (see Figure 1 in Kobayashi et al. 2011) and the values related to the disk are chosen from those of the given disk at $r = 7 \text{ au}$.

Therefore, the growth timescale is comparable to or longer than the lifetimes of protoplanetary disks $\approx 10^6 \text{ yr}$.

The gravitational interaction between a solid core and the protoplanetary disk induces radial migration of the core. The orbital decay timescale for the Type I migration is estimated to be (e.g., Tanaka et al. 2002)

\[
t_{\text{mig}} = \frac{1}{4} \left( \frac{\Sigma_g}{M_\oplus} \right)^{-1} \left( \frac{h_g}{r} \right)^2 \Omega^{-1},
\]

\[
= 1.6 \times 10^3 \left( \frac{1}{4} \right)^{-1} \left( \frac{M_p}{10 M_\oplus} \right)^{-1} \left( \frac{\Sigma_g}{69 \text{ g cm}^{-2}} \right)^{-1}
\times \left( \frac{h_g}{0.05 a} \right)^2 \left( \frac{r}{7 \text{ au}} \right)^{-1/2} \left( \frac{M_p}{M_\oplus} \right)^{3/2} \text{ yr},
\]

where $\Gamma$ is the dimensionless migration coefficient, $h_g$ is the scale height of the disk, the values of $\Sigma_g$ and $h_g$ are chosen from the given disk at $7 \text{ au}$. In the isothermal disk, $\Gamma \approx 4$ (Tanaka et al. 2002). The formation of planets prior to the orbital decay requires $t_{\text{grow}} \ll t_{\text{mig}}$. Once planetary embryos reaches $\sim 10 M_\oplus$, the rapid gas accretion of planetary embryos occurs (e.g., Mizuno 1980). Gas-giant planets formed by the gas accretion open up the gap around their orbits and the migration timescale is then much longer than the estimate in Equation (5) because of the onset of Type II migration. Therefore, the formation timescale of a massive core with $10 M_\oplus$ is required to be comparable to or shorter than the Type I migration timescale.

The collisional growth of dust grains forms pebbles, which drift inward. The growth rate of a planetary core via pebble accretion is given by

\[
\frac{dM_p}{dt} = \varepsilon \frac{dM_r}{dt},
\]

where $dM_r/dt$ is the mass flux of pebbles across the orbit of the core and $\varepsilon$ is the accretion efficiency of drifting pebbles. From $dM_p/dt$ given by Equation (14) of Lambrechts & Johansen (2014), the core-growth timescale via the accretion of drifting pebbles is estimated to be

\[
t_{\text{grow,pe}} = 2.0 \times 10^{5} \left( \frac{\varepsilon}{0.1} \right)^{-1} \left( \frac{\Sigma_{g,1}}{480 \text{ g cm}^{-2}} \right)^{2/5}
\times \left( \frac{T}{8.5 \text{ g cm}^{-2}} \right)^{-5/3} \left( \frac{t}{10^5 \text{ yr}} \right)^{-1/3} \text{ yr},
\]

where the value of $\varepsilon$ is used for typical pebble-sized bodies (Ormel & Klahr 2010; Okamura & Kobayashi 2021). Although the collisional cross sections for the pebble accretion are large thanks to strong gas drag for pebbles, the solid surface density of drifting pebbles is much lower because of their rapid drift. The growth timescale $t_{\text{grow,pe}}$ is therefore comparable to the migration timescale.

#### 3.2. Total Mass Required for Pebble Accretion

The required mass for the formation of a core with $M_p$ via pebble accretion is given by $M_0$ obtained from the integration of Equation (6). Integrating Equation (6) with the relation $\varepsilon \propto M_p^{2/3}$ (Ormel & Klahr 2010; Okamura & Kobayashi 2021),
we have

$$M_F = 3M_p/\varepsilon(M_p),$$

(8)

where we assume the initial core mass is much smaller than $M_p$. The pebble mass required for core formation, $M_F$, is inversely proportional to $\varepsilon(M_p)$ (see Equation (8)). Figure 1 shows $\varepsilon(10 M_\oplus)$ as a function of the dimensionless stopping time due to gas drag, $St$. For $St \approx 0.2 - 1$, $\varepsilon$ increases with decreasing $St$, because of slow drift for low $St$. However, for $St \leq 0.2$, $\varepsilon$ decreases with decreasing $St$. The horseshoe flow reduces the accretion band of pebbles for $St \approx 0.02 - 0.2$, while the outflow around the Bondi sphere disturbs the accretion of pebbles (Kuwahara & Kurokawa 2020; Okamura & Kobayashi 2021). These effects reduce $\varepsilon$ significantly. For $M_p \sim 10 M_\oplus$, $\varepsilon$ is estimated to be 0.1 or smaller for pebble-sized bodies (Figure 1). It should be noted that the estimate of $\varepsilon$ ignoring the realistic gas flow around a planet gives fatal overestimates for $St \lesssim 10^{-2}$ (see Figure 1 and compare the formulae by Okamura & Kobayashi 2021; Ormel & Liu 2018).

Although $\varepsilon \ll 0.1$ for $St \ll 0.1$, we consider $\varepsilon = 0.1$ for $St \approx 0.1$. Then the pebble mass required for core formation, $M_F$, is estimated to be about $300(\varepsilon/0.1)^{-1} M_\oplus$ from Equation (8). The pebble mass $M_p$ is limited by the total solid mass in a disk and thus the minimum disk mass required for a single-core formation is estimated as

$$M_{\text{disk,min}} = 0.05 \left( \frac{\varepsilon}{0.1} \right)^{-1} \left( \frac{M_p}{10 M_\oplus} \right)^{0.1} \left( \frac{\Sigma_{d,1}}{\Sigma_{g,1}} \right)^{-1} M_\oplus,$$

(9)

where $\Sigma_{d,1}/\Sigma_{g,1}$ is the metallicity in the disk. However, most protoplanetary disks are less massive than 0.1$M_\oplus$ (Andrews et al. 2010) and disks with the solid mass of 300$M_\oplus$ are very rare even in Class 0 objects ($\sim$1%; Mulders et al. 2021). Note that the required solid mass of 300$M_\oplus$ for pebble accretion is the minimum value. If smaller pebbles with $St \ll 0.1$ are considered, the required solid mass is much more than 300$M_\oplus$. Therefore, it seems difficult to explain giant exoplanets existing rather commonly ($\sim$10%; Mayor et al. 2011) with pebble accretion. To reconcile this issue, we need to increase $\varepsilon$ due to collisional growth of drifting pebbles (see $\varepsilon$ for $St \gg 1$ in Figure 1 and the discussion by Okamura & Kobayashi 2021).

The planetesimal formation from icy pebbles would be a possible process, which achieve a high conversion rate from pebbles to kilometer-sized or larger bodies. To consider the collisional growth of pebbles into planetesimals, we need to review collisional fragmentation. The collisional simulations of icy dust aggregates shows the fragmentation velocity of aggregates, $v_f$, depends on the interaction of monomers determined by the surface energy of ice $\gamma_{\text{ice}}$. For $v_f = 80(\gamma_{\text{ice}}/0.1 \text{ J m}^{-2})^{1/5}/\text{m s}^{-1}$ for aggregates composed of submicron-sized monomers (Wada et al. 2013). The collisional velocities for pebble-sized bodies are mainly smaller than 50 m s$^{-1}$, so that collisional fragmentation is negligible if $\gamma_{\text{ice}} \sim 0.1 \text{ J m}^{-2}$. The surface energy of ice was estimated to be much lower than 0.1 J m$^{-2}$ from the measurement of the rolling friction force between 1.1 mm sized particles in laboratory experiments (Musiolik & Wurm 2019). However, the distinction between rolling and slide forces is difficult for such large particles so that Kimura et al. (2020a) explained the measurements including the temperature dependence as the slide forces given by the tribology theory with quasi-liquid layers without low $\gamma_{\text{sil}}$. In addition, the measurement in laboratory experiments showed the tensile strength of aggregates for ice is comparable to that for silicates, implying that $\gamma_{\text{ice}}$ is as small as the silicate surface energy, $\gamma_{\text{sil}}$ (Gundlach et al. 2018). Kimura et al. (2020b) explained the measured tensile strengths for ice and silicate by the Griffith theory using $\gamma_{\text{ice}} \sim \gamma_{\text{sil}} \sim 0.1 \text{ J m}^{-2}$. From a physical point of view, the surface energy should be greater than the surface tension, which is $\approx 0.08 \text{ J m}^{-2}$ even in the room temperature. Therefore, collisional fragmentation is negligible for pebble growth.

We additionally discuss the effect of collisions with large $m_1/m_2$, where $m_1$ and $m_2$ are the masses of colliding bodies ($m_1 > m_2$). Erosive collisions, large $m_1/m_2$ collisions with velocities higher than $v_f$, reduce the masses of the larger colliding bodies, which inhibit the growth via collisions with large $m_1/m_2$. Krijt et al. (2015) claimed the growth of pebbles were stalled by erosive collisions under the assumption that $v_f$ for $m_1/m_2 > 100$ is much smaller than that for $m_1 \sim m_2$. However, this assumption is inconsistent with the impact simulations of dust aggregates. Recent impact simulations with $m_1/m_2 \geq 100$ show $v_f$ for larger $m_1/m_2$ is higher than that for $m_1 \sim m_2$ (Hasegawa et al. 2021). Therefore, erosive collisions are insignificant for pebble growth.

4. Models for DTPS

We develop a simulation for the collisional evolution of bodies from dust to planets (DTPS). We introduce here the model for the DTPS.

4.1. Collisional Evolution

Collisions between bodies lead to planet formation. The surface number density $n_o$ of bodies with mass $m$ at the distance $r$ from the host star with mass $M_\ast$ evolves via collisions and...
radial drift. The governing equation is given by
\[
\frac{\partial}{\partial t} n_s(m, r) = \frac{1}{2} \int_0^\infty dm_1 \times \int_0^\infty dm_2 n(m_1, r)n(m_2, r) \\
\times K(m_1, m_2) \delta(m - m_1 - m_2) \\
- n(m, r) \int_0^\infty dm_2 n(m_2, r) K(m, m_2) \\
- \frac{1}{r} \frac{\partial}{\partial r} [rn(m, r)v_r],
\]
where \(K(m_1, m_2)\) is the collisional kernel between bodies with masses \(m_1\) and \(m_2\) and \(v_r\) is the radial drift velocity. We adopt
\[
v_r = v_{\text{drag}} + v_{\text{mig}},
\]
where \(v_{\text{drag}}\) and \(v_{\text{mig}}\) are the radial drift velocity due to gas drag and the Type I migration. We model as (see Appendix A for \(v_{\text{drag}}\) and Tanaka et al. 2002 for \(v_{\text{mig}}\))
\[
v_{\text{drag}} = -\frac{2r\Omega St}{1 + St^2} \left(0.34c_s^2 + \frac{4c_s^2}{\pi^2} + \eta^2\right)^{1/2},
\]
\[
v_{\text{mig}} = -\Gamma \left(\frac{\Sigma g^2}{M_g}\right) \left(\frac{m}{M_*}\right) \left(\frac{h_g}{a}\right) r\Omega,
\]
where \(\Gamma = 4\) is the dimensionless migration coefficient (Tanaka et al. 2002), \(e\) and \(i\) are the orbital eccentricity and inclination, \(St\) is the dimensionless stopping time due to gas drag, and
\[
\eta = -\frac{1}{2} \left(\frac{c_s}{r\Omega}\right)^2 \frac{\partial \ln(\rho_g c_s^2)}{\partial \ln r}.
\]
Here, \(c_s\) is the isothermal sound velocity. The dimensionless stopping time, \(St\), called the Stokes number, is given by (e.g., Adachi et al. 1976)
\[
St = \begin{cases} 
\frac{3m}{8\pi \Sigma_g} & \text{for } \frac{s}{\lambda_{\text{mfp}}} < \frac{9}{4}, \\
\frac{m}{6\pi \Sigma_g \lambda_{\text{mfp}}} & \text{for } \frac{9}{4} \leq \frac{s}{\lambda_{\text{mfp}}} < \frac{12h_g}{\eta\rho_g}, \\
\frac{4m}{\pi s \rho_g \eta r} & \text{for } \frac{s}{\lambda_{\text{mfp}}} \geq \frac{12h_g}{\eta\rho_g},
\end{cases}
\]
where \(\rho_g\) is the midplane gas density, \(h_g = c_s/\Omega\) is the gas scale height, and \(\lambda_{\text{mfp}}\) is the mean free path.

As discussed above, collisional fragmentation is negligible for \(St \lesssim 1\). For further collisional growth, fragmentation is unimportant until planetary embryo formation (Kobayashi et al. 2016; Kobayashi & Tanaka 2018). We ignore collisional fragmentation even after embryo formation because of the uncertainty of collisional outcome models. This crude assumption is good to compare with the studies for pebble accretion, in which collisional fragmentation is also ignored except for consideration of pebble sizes (Bitsch et al. 2018; Lambrechts et al. 2019; Johansen et al. 2019). In addition, collisional fragmentation for pebble formation works negatively for pebble accretion because of low \(\varepsilon\) for \(St \lesssim 10^{-2}\) (Figure 1). Therefore, we consider only the collisional merging.

For \(St \gtrsim 1\), the collisional kernel is scaled by the masses of bodies using the Hill radius \((h_{\text{H},1,2} = [m_1 + m_2]/3M_g r^{1/3})\):
\[
K(m_1, m_2) = \frac{r_{\text{H},1,2}^2 P(\Delta e, \Delta i)}{\Omega},
\]
where \(P\) is the collisional probability, and \(\Delta e, \Delta i\) are the Hill-scaled relative eccentricity and inclination between \(m_1\) and \(m_2\). Sufficient mutual interaction between planetesimals results in the uniform orbital phases and eccentricities and inclinations following Rayleigh distributions (Ida & Makino 1992). Therefore, \(\Delta \tilde{e}, \Delta \tilde{i} = (\tilde{e}_1^2 + \tilde{i}_1^2)^{1/2}/\tilde{h}_{\text{H},1,2}\), and \(\Delta \tilde{e}, \Delta \tilde{i} = (\tilde{e}_1^2 + \tilde{i}_1^2)^{1/2}/\tilde{h}_{\text{H},1,2}\), where \(\tilde{e}_1, \tilde{i}_1\) are the mean eccentricity and inclination of bodies with \(m_1, m_2\), respectively. The collisional probability \(P\) is given by the limiting solutions for \(\Delta \tilde{e}, \Delta \tilde{i} \ll 1\), \(\Delta \tilde{e}, \Delta \tilde{i} \approx 0.2 - 2\), and \(\Delta \tilde{e}, \Delta \tilde{i} \gg 1\) (Inaba et al. 2001). In addition, we consider the enhancement of \(P\) due to planetary atmospheres (Inaba & Ikoma 2003) and the strong gas drag around a massive planets (Ormel & Klahr 2010). The details of \(P\) are described in Appendices B and C.

Collisionless interactions among bodies induce the evolution of their eccentricities and inclinations, which is sensitive to the mass spectrum of bodies. We calculate the \(e\) and \(i\) evolution with the mass evolution, taking into account the mutual interaction between bodies such as viscous stirring and dynamical friction, gas drag, and the perturbation from the turbulent density fluctuation (Kobayashi & Tanaka 2018). The detailed treatment of \(e\) and \(i\) evolution is described in Appendix D. We developed a simulation for planetesimal accretion (\(St \gg 1\)), which perfectly reproduces the result obtained from the direct N-body simulation (Kobayashi et al. 2010).

For \(St \lesssim 1\), we calculate \(P\) additionally using the scale height and the relative velocity. For \(St \ll 1\), the scale height for bodies with \(m_1\) and \(St_1\) is given by (Youdin & Lithwick 2007)
\[
h_{s,1} = h_g \left(1 + \frac{St_1}{\alpha_{\text{ef}}} \frac{1 + 2St_1}{1 + St_1}\right)^{-1/2},
\]
where \(\alpha_{\text{ef}}\) is the dimensionless turbulent parameter. We introduce the relative scale height between \(m_1\) and \(m_2\) as
\[
h_{s,1,2} = \left[\pi (h_{s,1}^2 + h_{s,2}^2)/2\right]^{1/2},
\]
The relative velocity is given by
\[
v_{\text{rel, gas}} = \Delta v_B^2 + \Delta v_r^2 + \Delta v_\theta^2 + \Delta v_\varphi^2 + \Delta v_z^2 + \Delta v_x^2,
\]
where \(\Delta v_B, \Delta v_r, \Delta v_\theta, \Delta v_\varphi, \Delta v_z, \Delta v_x\) are the relative velocities induced by the Brownian motion, radial and azimuthal drifts, vertical settling, and turbulence, respectively (detailed description in Appendix E). For \(St_1 \ll 1\) and \(St_2 \ll 1\), \(K\) is expressed using \(h_{s,1,2}\) and \(v_{\text{rel, gas}}\) as (Okuzumi et al. 2012)
\[
K = \frac{\pi (s_1 + s_2)^2 v_{\text{rel, gas}}}{2h_{s,1,2}^2}.
\]
We therefore expand Equation (16) to apply the case for \(St \lesssim 1\) using \(h_{s,1,2}\) and \(v_{\text{rel, gas}}\).

The collisional probability \(P\) is the function of \((\tilde{e}_1^2 + \tilde{i}_1^2)^{1/2}\) and \(\tilde{h}_{\text{H},1,2}\), which represent the relative velocity and the relative scale height, respectively. Therefore, we use the greater values of \((\tilde{e}_1^2 + \tilde{i}_1^2)^{1/2}\) or \(v_{\text{rel, gas}}/\tilde{h}_{\text{H},1,2}\) and \(\tilde{h}_{\text{H},1,2}\) or \(h_{s,1,2}/\tilde{h}_{\text{H},1,2}\) for the
function of \(\mathcal{P}\). We then calculate the collisional Kernel \(K\) for any \(St\). Using this method, we calculate \(K\) for \(St_1, St_2 \ll 1\), which corresponds to Equation (20). Therefore, we apply this method for bodies from dust grains to planets.

### 4.2. Bulk Density

The collisional growth of dust grains produces the fractal dust aggregates, whose \(\rho_b = (3m/4\pi s^3)\) is lower than the original material. The stopping time \(St\) depends on \(\rho_b\). The evolution of \(\rho_b\) is significantly important for collisional growth for \(St \ll 1\). We model \(\rho_b\) as

\[
\rho_b = \left[ \rho_{\text{mat}}^{-1} + (\rho_s + \rho_m + \rho_l)^{-1} \right]^{-1},
\]

where \(\rho_{\text{mat}} = 1.4 \text{ g cm}^{-3}\) is the material density, corresponding to the density of compact bodies or monomer grains in dust aggregates,

\[
\rho_s = \rho_{\text{mat}} \left( \frac{m}{m_{\text{mon}}} \right)^{-0.58},
\]

\[
\rho_m = 10^{-3} \text{ g/cm}^3,
\]

\[
\rho_l = \left( \frac{256\pi G^3 \rho_{\text{mat}}^9 s_{\text{mon}}^3 m_{\text{mon}}^2}{81E_{\text{roll}}^3} \right)^{1/5},
\]

where \(s_{\text{mon}} = 0.1 \mu\text{m}\) is the monomer radius, \(m_{\text{mon}} = 4\pi\rho_{\text{mat}} s_{\text{mon}}^3/3\) is the monomer mass, \(E_{\text{roll}} = 4.74 \times 10^{-9} \text{ erg}\) is the rolling energy between monomer grains, and \(G\) is the gravitational constant.

The densities \(\rho_s, \rho_m,\) and \(\rho_l\) almost correspond to the bulk density for small, intermediate, and large bodies, respectively. For small dust, collisional growth occurs without collisional compaction. Equation (22) is determined by the model given in the previous study (Okuzumi et al. 2012) under the assumption of the collisional evolution between same-mass bodies without collisional compaction, which is almost similar to the density evolution with the fractal dimension \(\sim 2\) (Okuzumi et al. 2012). For large bodies, the bulk density increases with increasing mass by self-gravity compaction until compact bodies with \(\rho_b = \rho_{\text{mat}}\) and the equilibrium density is given by Equation (24) (Kataoka et al. 2013).

For intermediate bodies, the bulk density is most important for \(St \sim 1\), which mostly determines the fate of bodies. The bulk density is determined by the compression due to ram pressure of the disk gas (Kataoka et al. 2013). We estimate \(\rho_b\) at \(St = 1\) under the assumption of the Epstein gas drag,

\[
\rho_b \sim 6.3 \times 10^{-4} \left( \frac{r}{10 \text{ au}} \right)^{-5/6} \text{ g cm}^{-3}.
\]

It should be noted that \(\rho_b\) for \(St = 1\) is smaller than that given by Equation (25) at \(r \lesssim 10\text{ au}\) because the Stokes gas drag is dominant for \(St \sim 1\) at the inner disk. Therefore, \(\rho_b\) for \(St \sim 1\) becomes up to \(\sim 10^{-3}\text{ g cm}^{-3}\) so that we simply choose the value of \(\rho_m = 10^{-3}\text{ g cm}^{-3}\) according to the estimate. Figure 2 shows the radii or \(St\) of bodies in the model as a function of mass.

### 5. Result

We perform a DTGS for the collisional evolution of bodies drifting due to gas drag and Type I migration in a protoplanetary disk. We set a disk with the inner and outer radii of \(\approx 3\text{ au}\) and \(\approx 108\text{ au}\), whose gas surface density is inversely proportional to \(r\) (see Equation (1)). The disk mass corresponds to 0.036\(M_\odot\), (total solid mass \(\approx 210M_\oplus\), which is smaller than the required mass for the pebble accretion (see Equation (9)). Solid bodies initially have a mass \(m = 5.9 \times 10^{-15}\text{ g}\) (corresponding to a radius of 0.1 \(\mu\text{m}\)). We set the turbulent strength to be \(a_{\text{Turb}} = 10^{-3}\).

Figure 3 shows the surface density of bodies whose masses are similar to \(m\) within about a factor of 2, as a function of \(m\) and the distance from the host star. Dust growth occurs around \(r \approx 5\text{ au}\) at \(t \approx 560\text{ yr}\) (Figure 3(a)). Largest bodies reach at \(m \sim 10^{13}\text{ g}\) at 3 au. The drift of bodies is controlled by the gas coupling parameter of bodies \(St\) (see Equation (15)). Bodies have the highest drift velocities at \(m \sim 10^9\), corresponding to \(St = 1\) (Figure 2). For low-density bodies, the collisional growth timescale is much shorter than the drift timescale so that large bodies with \(St \gtrsim 1\) are formed via collision growth (Okuzumi et al. 2012). Dust collisional growth propagates from the inner to outer disk (Figure 3(b); see also Ohashi et al. 2021). The dust growth front reaches 20, 50, and 90 au, and the outer boundary at \(t \approx 1.5 \times 10^4, 5.6 \times 10^4, 1.2 \times 10^5\), and \(2.1 \times 10^5\) yr, respectively (Figures 3(c)–(f)). Radial drift is more dominant than collisional growth for bodies with \(St \sim 1\) beyond 10 au. The drifting bodies grow to planetesimals in the disk inside 10 au.

In the early growth (Figures 3(a) and (b)), the total solid surface densities are mainly determined by largest bodies. At \(t \approx 6 \times 10^4\text{ yr}\) (Figure 3(c)), the runaway growth of bodies with \(m = 10^{13} \sim 10^{16}\text{ g}\) occurs at \(r \lesssim 6\text{ au}\). The solid surface density of planetesimal-sized bodies (\(m \sim 10^{18}\text{ g}\)) becomes dominant. Planetary embryos with \(m \sim 10^{23}\text{ g}\) are formed at \(r \lesssim 10\text{ au}\) via the runaway growth (Figure 3(d)). The further growth of embryos occurs via collisions with planetesimals (Figure 3(e)).
The largest planetary embryos exceed 10 Earth masses even at $t \approx 2 \times 10^5$ yr (Figure 3(f)). Collisional growth successfully forms bodies with $m \approx 10^{10}$ g ($St \approx 1$) only at $r \lesssim 10$ au (Figures 3(d)–(f)). To overcome the drift barrier at $St \approx 1$, bodies with $St \approx 1$ should grow via collisions much faster than their radial drift. The requirement for this condition is that bodies with $St = 1$ feel gas drag in the Stokes regime (Okuzumi et al. 2012). Therefore, bodies for $St = 1$ have $s \approx 9 \lambda_{\text{mfp}}/4$ (see Equation (15)):

$$\frac{9 \Sigma}{9 \pi \rho_b \lambda_{\text{mfp}}} \approx 1.$$  \hspace{1cm} (26)

For bulk densities and disk conditions given in Sections 4.2 and 3.2, Equation (26) corresponds to $r_{\text{grow}} \ll 24$ au, where $r_{\text{grow}}$ is the radius inside which pebbles can grow to planetesimals. Therefore, collisional growth results in planetesimals with $St \gg 1$ for $r \lesssim 10$ au. The radial drift of planetesimals with $St \gg 1$ is much slower than that of pebbles $St \lesssim 1$; the pileup results in the enhancement of solid surface densities at $r \lesssim 10$ au (Figures 3(d)–(f) and 4(b)). Pebbles formed in the whole disk with the total solid mass $M_{\text{solid, disk}}$ finally drift inward across $r_{\text{grow}}$, so that the enhanced surface density is estimated to be $M_{\text{solid, disk}}/\pi r_{\text{grow}}^2 \approx 18(r_{\text{grow}}/10 \text{ au})^{-2} \text{ g cm}^{-2}$ (compare with Figure 4(b)).

Figure 4(a) shows the mass of the largest planetary embryos in each annulus of the disk. Planetary embryos acquire $10 M_\oplus$ around $r \approx 6$–7 au at $t \approx 2 \times 10^5$ yr. Such rapid formation of massive embryos is achieved via the pileup of bodies in

Figure 3. Solid surface density at $t = 5.6 \times 10^5$ (a), $2.1 \times 10^5$ (b), $1.5 \times 10^4$ (c), $5.6 \times 10^4$ (d), $1.2 \times 10^4$ (e), and $2.1 \times 10^5$ (f) yr, as a function of the mass of bodies and the distance from the host star. The values of the solid surface density are shown in the color bar.

Figure 4. Planet mass (a) and surface density of bodies (b) as a function of the distance from the host star, where the planet mass is given by the mass of the largest bodies in each annulus in the disk.
$r < 10 \text{ au}$ (Figure 4(b)). As explained above, bodies with $St < 1$ drift inward from the outer disks until the bodies grow to $St \gg 1$ in $r < 10 \text{ au}$. The solid surface density increases to 20 g cm$^{-2}$ at 7 au in $2 \times 10^5$ yr (see Figure 4(b)), the formation of cores with $10 M_\oplus$ requires the surface density of 3 g cm$^{-2}$. Therefore, only about 15% of bodies are needed for the core formation in the enhanced disk. The $\Sigma_a$ enhancement effectively accelerates the growth of cores. However, the growth rate depends on the mass spectrum of bodies accreting onto cores.

We additionally investigate which masses of bodies mostly contribute to the core growth. The cumulative accretion rate of bodies onto the largest core in the annulus at $r = 6.75$ au is shown in Figure 5(a). The contribution of pebbles ($St < 1$) to the accretion rate is minor, because the solid surface density of bodies with $St < 1$ is tiny (Figure 5(b)). Collisions with 100 m–10 km sized bodies of $m = 10^9–10^{19}$ g mainly contribute to the accretion rate, while the solid surface density is mainly determined by planetesimal-sized bodies of $m \sim 10^{19}$ g (Figure 5 and see also Figure 3). The atmospheric collisional enhancement promotes the accretion of subkilometer-sized bodies of $m \sim 10^{16}$ g, which are in the course of growing to planetesimals. The growth of planetary cores additionally increases their Hill radii so that collisions between planetary embryos occur. The embryo accretion therefore increases the total accretion rate by a factor of about 1.5 additionally (Chambers 2006; Kobayashi et al. 2010).

We again estimate the growth timescale of cores via planetesimal accretion in this condition using Equation (4) in Section 3.1. The solid surface density of planetesimals or planetesimal precursors increases to 15 g cm$^{-2}$ (Figure 5(b)). As mentioned above, the contribution of planetesimal precursors to the accretion is significant. The enhancement factor $R_e/R_p$ proportional to $m^{-1/2}$ is higher for small planetesimals (Kobayashi et al. 2011). The growth timescale is then estimated to be $t_{\text{grow}} \approx 1.5 \times 10^4 (\Sigma_a/15 \text{ g cm}^{-2}) (m/10^{16} \text{ g})^{1/23} \text{ yr}$, corresponding to the accretion rate of $4.0 \times 10^{24} \text{ g yr}^{-1}$ for $M_p = 10 M_\oplus$. This value is consistent with the accretion rate at $t \approx 1.9 \times 10^5$ yr (see Figure 5(a)).

The accretion of planetesimals with mass $m = 10^{15}–10^{19}$ g induces the rapid growth of planetary embryos. Such planetesimals are produced via collisional growth of planetesimal precursors with $m \sim 10^9–10^{15}$ g. The bulk density of such planetesimal precursors for $m \gtrsim 10^{13}$ g is given by $\rho_b \propto m^{2/5}$ (see Section 4.2). Their collisional timescale among planetesimal precursors with mass $m$ and radius $r_p$ is given by

$$t_{\text{col}} \approx \frac{m}{\pi r_p^2 \Sigma_a \Omega},$$

which is estimated to be $t_{\text{col}} \approx 4.5 \times 10^3$ yr at 7 au for $m = 10^{13}$ g and $\Sigma_a = 0.2 \text{ g cm}^{-2}$ according to Figure 5(b). On the other hand, the drift timescale of planetesimal precursors is given by

$$t_{\text{drift}} \approx \frac{St}{2 n h \Omega},$$

where $\eta$ is the dimensionless parameter depending on the pressure gradient. The drift timescale is estimated to be $t_{\text{drift}} \approx 3.5 \times 10^5$ yr for $m = 10^{13}$ g. Therefore, $t_{\text{col}}/t_{\text{drift}} \approx 1.3 \times 10^2 (\Sigma_a/0.2 \text{ g cm}^{-2})^{-1} \ll 1$ independent of $m$. For such planetesimal precursors, the collisional growth timescale is much shorter than the drift timescale.

Pebbles with $St \sim 0.1$ drift from the outer disk, and the collisional growth among the pebbles produces planetesimal precursors prior to their drift. Because $t_{\text{col}} \ll t_{\text{drift}}$ as estimated above, planetesimal precursors grow without significant drift in the inner disk ($<10$ au) until gravitational scatterings by planetary cores, which lead to the uniform distribution of planetesimal precursors around cores. The surface density of planetesimal precursors is much smaller than that of planetesimals (Figure 5(b)). Planetesimal precursors are maintained via the supply from the growth of pebbles drifting from the outer disk. This mechanism leads to the sustainable accretion of small planetesimals, resulting in the rapid growth within 0.2 Myr.

If the scattering of planetesimal precursors by a solid core is comparable to the drift, such solid cores may open gaps up in a planetesimal-precursor disk, which would reduce the accretion rate of small planetesimals (Levison et al. 2010). However, the collisional growth timescale of planetesimal precursors is much shorter than the gap-opening timescale comparable to $t_{\text{drifts}}$, so that planetesimals are then formed via collisional growth of precursors prior to gap opening. Therefore, the rapid growth is achieved without the gap opening in the solid disk.

### 6. Discussion

We show the rapid core formation at 6–7 au in $2 \times 10^5$ yr. We obtain a similar result for a weak turbulent level of $\alpha_D = 10^{-4}$, with which the simulation results in the core formation at $\approx 7$ au in $3 \times 10^5$ yr. A sufficient massive core starts gas accretion and Type II migration. The orbital semimajor axis of a gas giant with Jupiter mass resulting from the gas accretion and Type II migration is
about 0.9 times that of the original core (Tanaka et al. 2020). Therefore, the first gas giant is formed around 6 au. The giant planets in the solar system may experience migration. Outward migration of Neptune can explain the orbital eccentricities of planets in the solar system may experience migration. Outward migration timescales, resulting in the differentiation of gas-giant planets from protoplanetary disks. DTPPs show a disk with the solid mass of 210 $M_\oplus$ produces a gas giant orbiting at $\approx 5$ au, while inner gas-giant planets are formed in less massive disks, each with a solid mass of $>100 M_\oplus$. Thus, our model naturally explains the formation of gas-giant planets from protoplanetary disks, each with a solid mass of $\approx 100$–200 $M_\oplus$.

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### Appendix A

#### Radial Drift

For $St \gg 1$, the drift velocity induced by gas drag is given by (Adachi et al. 1976; Inaba et al. 2001)

$$v_{\text{drag}} = \frac{-2\pi\Omega}{St} \left\{ \left[ \frac{E(3/4) + K(3/4)}{3\pi} \right] e^2 + \frac{4t^2}{\pi^2} + \eta^2 \right\}^{1/2},$$

(A1)

where $K$ and $E$ are the complete elliptic integrals of the first and second kinds, and we ignore the higher-order terms of $e$ and $i$ because the terms are small enough (Kobayashi 2015). On the other hand, for $St \ll 1$ $v_{\text{drag}}$ is given by (Adachi et al. 1976)

$$v_{\text{drag}} = \frac{2\pi\Omega}{1 + St^2}.$$  

(A2)

We combine the both regimes as (Kobayashi et al. 2010),

$$v_{\text{drag}} = \frac{-2\pi\Omega}{1 + St^2} \times \left\{ \left[ \frac{E(3/4) + K(3/4)}{3\pi} \right] e^2 + \frac{4t^2}{\pi^2} + \eta^2 \right\}^{1/2}.$$  

(A3)

### Appendix B

#### Atmospheric Enhancement

The collisional radius is enhanced by an atmosphere. Inaba & Ikoma (2003) derived the enhancement factor for the radius,

$$\xi = \frac{3}{2} \frac{v_{\text{rel}}^2 + 2Gm/s}{v_{\text{rel}}^2 + 2Gm/r_H \rho_h},$$

(B1)

where $v_{\text{rel}}$ is the relative velocity, $\eta = (m/3M_\oplus)^{1/3}r$ is the Hill radius, and $\rho_h$ is the atmospheric density at $\xi s$ from the center of the body.
We derive $\rho_a$ according to Inaba & Ikoma (2003). The pressure $P_a$, temperature $T_a$, and density $\rho_a$ of the atmosphere at the distance $r_a$ from the body have the relations as follows:

$$\bar{P}_a = \rho_a \bar{T}_a, \quad \bar{T}_a^4 = 1 + W_0(\bar{P}_a - 1), \quad \bar{r}_a = 1 + \frac{1}{V_0} \left[4(\bar{T}_a - 1) + f(\bar{T}_a, w_0)\right],$$

where the dimensionless pressure $\bar{P}_a = P_a / P_o$, temperature $\bar{T}_a = T_a / T_o$, density $\bar{\rho}_a = \rho_a / \rho_o$, and distance $\bar{r}_a = r_a / r_o$ are scaled by the pressure, temperature, density, and distance at the outer boundary, respectively,

$$V_0 = \frac{Gm\rho_o}{r_o P_o}, \quad W_0 = \frac{3\kappa_a L_a P_o}{4\pi \alpha cGm^2 T_o^4}, \quad w_0 = |1 - W_0|^{1/4},$$

$$g(\bar{T}_a, w_0) = \begin{cases} w_0 \ln \left(\frac{\bar{T}_a - w_0 - 1}{\bar{T}_a - w_0} \frac{\bar{T}_a}{w_0} \frac{\bar{T}_a - 1}{\bar{T}_a - w_0} \right) \left(\arctan \frac{\bar{T}_a}{w_0} - \arctan \frac{1}{w_0} \right) & \text{for } W_0 < 1, \\ \frac{w_0}{\sqrt{2}} \ln \left(\frac{\bar{T}_a^2 + \sqrt{2} w_0 \bar{T}_a + w_0^2}{\bar{T}_a^2 - \sqrt{2} w_0 \bar{T}_a + w_0^2} \right) + 2 \left(\arctan \frac{\sqrt{2} w_0}{w_0^2 - \bar{T}_a} - \arctan \frac{\sqrt{2} w_0}{w_0^2 - 1} \right) & \text{for } W_0 \geq 1, \end{cases}$$

where $\alpha$ is the radiation density constant, $\kappa_a$ is the opacity, and $L_a$ is the luminosity.

The luminosity is given by

$$L_a = \text{MAX} \left(\frac{Gmn}{s}, 4\pi s^2 \sigma_{SB} T^4\right),$$

where $m$ is the accretion rate of the body, $\sigma_{SB}$ is the Stefan–Boltzmann constant, and the function $\text{MAX}(x, y)$ gives the larger $x$ and $y$.

We set the outer boundary values and opacity as follows:

$$T_o = T, \quad P_o = P, \quad r_o = \text{MIN} \left(\frac{Gm}{\kappa_a c_v}, r_{1\text{h}}\right), \quad \kappa_a = 4 \zeta + 0.01 \text{ cm}^2 \text{ g}^{-1}$$

for $T < T_o \leq 170K$, \quad \kappa_a = 2 \zeta + 0.01 \text{ cm}^2 \text{ g}^{-1}$

for $170K < T_o \leq 1700K$, \quad \kappa_a = 0.01 \text{ cm}^2 \text{ g}^{-1}$

for $T_o > 1700K$, \quad \kappa_a = 0.01 \text{ cm}^2 \text{ g}^{-1}$

where $P$ is the gas pressure at the disk midplane, $c_v$ is the isothermal sound velocity, $k_b$ is the heat capacity ratio, the function $\text{MIN}(x, y)$ gives the smaller $x$ and $y$, the subscripts of

170 K and 1700 K indicate the values at $T_a = 170$ K and 1700 K, respectively, and $\zeta$ is the reduction factor of the atmospheric opacity. A massive planetary body acquires an atmosphere. Small dust grains decrease until the formation of massive bodies. We therefore apply $\zeta = 10^{-4}$.

**Appendix C**

**Collisional Probability**

Taking into account the relative velocity induced by gas, we use

$$\dot{\varepsilon} = \text{MAX}(v_{\text{rel, gas}} h_{1,2} r, \dot{\varepsilon}_{1,2}).$$

We then introduce

$$I = \begin{cases} \varepsilon_{1,2} / \dot{\varepsilon} & \text{for } v_{\text{rel, gas}} h_{1,2} r / \Omega < \dot{\varepsilon}_{1,2}, \\ 0.812 & \text{for } v_{\text{rel, gas}} h_{1,2} r / \Omega > \dot{\varepsilon}_{1,2}, \end{cases}$$

$$I_t = \text{MAX}(h_{1,2} / r_h, \dot{\varepsilon} / \dot{\varepsilon}^*),$$

where $h_{1,2} / r_h$ is the dimensionless mutual Hill radius.

The formulae of collisional probabilities for $S_t$, $S_2 \gg 1$ are mainly modeled by Inaba et al. (2001; see also Ormel & Kobayashi 2012). Using $I$ and $I_t$, we modify the formula for $m_1 > m_2$ as

$$P = \begin{cases} \text{MIN}(P_{\text{mid}}, (P_{\text{low}}^2 + P_{\text{high}}^2)^{-1/2}) & \text{for } \text{MIN}(P_{\text{high}}, P_{\text{low}}) > P_{\text{low}}, \\ P_{\text{t}} & \text{otherwise,} \end{cases}$$

where

$$P_{\text{low}} = 11.3 \sqrt{\frac{\varepsilon_{1,2}}{\dot{\varepsilon}}},$$

$$P_{\text{mid}} = \frac{\varepsilon_{1,2}}{4\pi \dot{\varepsilon}^2} \left(17.3 + \frac{232}{\varepsilon_{1,2}}\right),$$

$$P_{\text{high}} = \frac{\varepsilon_{1,2}^2}{2\pi} \left(F(I, I_t) + 6G(I, I_t) \frac{\varepsilon_{1,2}^2}{\dot{\varepsilon}_{1,2}}\right),$$

$$P_{\text{t}} = \begin{cases} 2b_{\text{set}} \left(\frac{2}{b_{\text{set}}} + \frac{v}{h_{1,2}}\right) & \text{for } S_2 < \text{MIN}(1, 12h_{1,2} / \eta^3), \\ P_{\text{low}} + \frac{64}{\varepsilon_{1,2}} & \text{for } S_2 > \text{MAX}(\frac{\eta}{h_{1,2}}, 1), \\ 2\text{MAX}(b_{\text{hyp}}, b_{\text{set}}) v_2 & \text{otherwise.} \end{cases}$$

The dimensionless colliding radius of bodies $s_{1,2}$ is given by $(\xi_{1,2} s_{1,2}) / h_{1,2}$ with the enhancement factor $\xi_1$ due to planetary atmospheres given in Appendix B. The formula $P_{\text{t}}$ is obtained by Ormel & Klahr (2010), where $b_{\text{set}}$, $b_{\text{hyp}}$, and $v_a$ are given by the solution of $b_{\text{hyp}}^2 (b_{\text{set}} + 2\eta / 3h_{1,2}) = S_2$ with a factor of $[\exp(-(S_2 h_{1,2}^3 / 12\eta^3)^{0.65}]$, $\varepsilon_{1,2}^2 (\eta + 6 / \varepsilon_{1,2})^2$, and $\eta\sqrt{1 + 4\varepsilon_{1,2}^2 / s_{1,2} h_{1,2}(1 + S_2)}$, respectively.

The functions $F$ and $G$ are originally formulated by Greenzweig & Lissauer (1992). Taking into account $I_t$, we
modify them as

\[
F(I, l_i) = \frac{\sqrt{3} \pi E(\tilde{\xi}_F)}{l_F} \left\{ 1 + \frac{\tilde{\xi}_E - \tilde{\xi}_F}{2\tilde{\xi}_F^2} \right\}
\times \left( \frac{K(\tilde{\xi}_E) - 1}{E(\tilde{\xi}_E)} \right) - \frac{1}{1 - \tilde{\xi}_F^2} \right\}, \quad (C8)
\]

\[
G(I, l_i) = \frac{2\sqrt{3} \pi K(\tilde{\xi}_G)}{l(1 + l) \tilde{\xi}_G} \left\{ 1 + \frac{\tilde{\xi}_G - \tilde{\xi}_E}{2\tilde{\xi}_G^2(1 - \tilde{\xi}_G^2)} \right\}
\times \left( 1 - 2\tilde{\xi}_G - \frac{E(\tilde{\xi}_G)}{K(\tilde{\xi}_G)} \frac{1 - 3\tilde{\xi}_G^2}{1 - \tilde{\xi}_G^2} \right) \right\}, \quad (C9)
\]

where \( K \) and \( E \) are the complete elliptic integrals of the first and second kinds, respectively, \( \tilde{\xi}_F = \sqrt{3} \pi / 4l(1 + \arctan \sqrt{1 - l^2} - 1/\sqrt{1 - l^2}) \) for \( l < 1 \), \( \sqrt{3} \pi / 4l \) for \( l = 1 \), and \( \sqrt{3} \pi / 4l(1 + \arctan \sqrt{1 - l^2} - 1/\sqrt{1 - l^2}) - 1 \) for \( l > 1 \); \( \tilde{\xi}_F = 3(1 - 2l^2)(1 - 4\tilde{\xi}_F^2/\sqrt{3} \pi )/4(1 - l^2) \) for \( l = 1 \) and \( 1/2 \) for \( l > 1 \); \( \tilde{\xi}_G = \sqrt{3} \pi \sqrt{1 - I}/4\sqrt{1 - I} \) arctan\((I - 1)^2 - 1 \) for \( I < 1 \), \( \sqrt{3} \pi / 4 \) for \( I = 1 \), and \( \sqrt{3} \pi / 4\sqrt{I - 1/\sqrt{I^2 - 1}} \) for \( I > 1 \); and \( \tilde{\xi}_G = 3(1 - 4\tilde{\xi}_G^2(1 + I)/\sqrt{3} \pi )/4(1 - I^2) \) for \( I = 1 \) and \( 1/2 \) for \( I > 1 \). Our modifications for \( F \) and \( G \) are only \( l_i \) in the denominators of the first terms in Equations (C23) and (C24).

If \( l_i = l \), the formulae of Equations (C18)–(C21) are the same as those in Inaba et al. (2001). In the limit of \( St_1 \), \( St_2 \ll 1 \), \( \tilde{e}^* = \nu_{\text{rel, gas}}/\nu_{h_1,2} \), \( \Omega \gg 1 \) and \( l_i = h_{1,2} \Omega / r_{h, \text{rel, gas}} \), and \( I = 0.812 \). Therefore, \( P \) reduces to

\[
P \approx 1.57\tilde{e}_{1,2}^2 \nu_{\text{rel, h_{1,2}}}. \quad (C10)
\]

This collisional probability corresponds to that between dust grains with \( St_1 \), \( St_2 \ll 1 \).

### Appendix D

#### Random Velocity Evolution

For \( St \gg 1 \), collisional evolution depends on \( e \) and \( i \). We consider the \( e \) and \( i \) evolution due to gravitational interaction (Ohtsuki et al. 2002), gas drag (Adachi et al. 1976), and collisional damping (Ohtsuki 1992). On the other hand, a body has an orbit determined by the Kepler law for \( St \ll 1 \). The orbital elements of \( e \) and \( i \) do not indicate the motion of bodies. However, the collisional velocity is determined by \( \nu_{\text{rel, gas}} \) instead of \( e \) and \( i \). Therefore, we calculate the \( e \) and \( i \) evolution via the following equations:

\[
\frac{de}{dt} = \begin{cases} \frac{0}{dt} + \frac{de}{dt} & \text{for } St < 1, \\ \frac{0}{dt} + \frac{de}{dt} & \text{for } St \geq 1, \end{cases} \quad (D1)
\]

\[
\frac{di}{dt} = \begin{cases} \frac{0}{dt} + \frac{di}{dt} & \text{for } St < 1, \\ \frac{0}{dt} + \frac{di}{dt} & \text{for } St \geq 1, \end{cases} \quad (D2)
\]

where the subscripts “g,” “i,” and “c” indicate the gravitational interaction, the gas drag, and the collisional damping.

The \( e \) and \( i \) evolution due to gravitational interaction is given by (Ohtsuki et al. 2002)

\[
\frac{de}{dt} = \Omega^2 \int dm_2 n_s(m_2) \frac{h_{1,2}^2 m_2}{(m_1 + m_2)^2} \times \left( m_2 P_{\nu S} + 0.7 P_{DF} \frac{m_2 e^2 - m_i e^2}{e_r^2 + e_i^2} \right), \quad (D3)
\]

\[
\frac{di}{dt} = \Omega^2 \int dm_2 n_s(m_2) \frac{h_{1,2}^2 m_2}{(m_1 + m_2)^2} \times \left( m_2 Q_{\nu S} + 0.7 Q_{DF} \frac{m_2 i^2 - m_i i^2}{i_r^2 + i_i^2} \right), \quad (D4)
\]

where

\[
P_{\nu S} = 73 \frac{\ln(10\lambda^2/\tilde{e}_{1,2}^2 + 1)}{10\lambda^2 / \tilde{e}_{1,2}^2} + 72 \Psi_{P\nu S}(l) \frac{\ln(1 + \lambda^2)}{\pi \tilde{e}_{1,2} \tilde{h}_{1,2}}, \quad (D5)
\]

\[
Q_{\nu S} = (4\lambda^2 + 0.2\lambda^3) \frac{\ln(10\lambda^2 \tilde{e}_{1,2}^2 + 1)}{10\lambda^2 / \tilde{e}_{1,2}^2} + 72 \Psi_{Q\nu S}(l) \frac{\ln(1 + \lambda^2)}{\pi \tilde{e}_{1,2} \tilde{h}_{1,2}}, \quad (D6)
\]

\[
P_{DF} = 100^2 \frac{\ln(10\lambda^2 + 1)}{10\lambda^2} + 576 \Psi_{PDF}(l) \frac{\ln(1 + \lambda^2)}{\pi \tilde{e}_{1,2} \tilde{h}_{1,2}}, \quad (D7)
\]

\[
Q_{DF} = 100^2 \frac{\ln(10\lambda^2 + 1)}{10\lambda^2} + 576 \Psi_{QDF}(l) \frac{\ln(1 + \lambda^2)}{\pi \tilde{e}_{1,2} \tilde{h}_{1,2}}, \quad (D8)
\]

with

\[
\Lambda = \frac{1}{3} (\tilde{e}_{1,2}^2 + i_r^2) (\tilde{e}_{1,2}^2 + 1), \quad (D9)
\]

\[
\Psi_{P\nu S}(l) = \int_0^1 \frac{5K(\sqrt{\frac{3(1 - \lambda^2)}{2}}, \frac{-12\lambda^2 \lambda^2}{1 + 3\lambda^2} E(\sqrt{\frac{3(1 - \lambda^2)}{2}})}{l + (l^{-1} - l)(\lambda^2)} d\lambda, \quad (D10)
\]

\[
\Psi_{Q\nu S}(l) = \int_0^1 \frac{K(\sqrt{\frac{3(1 - X^2)}{2}}, \frac{-12X^2 \lambda^2}{1 + 3X^2} E(\sqrt{\frac{3(1 - X^2)}{2}})}{l + (l^{-1} - l)(X^2)} d\lambda, \quad (D11)
\]

\[
\Psi_{PDF}(l) = \int_0^1 \frac{\frac{1 - \lambda^2}{1 + 3\lambda^2} E(\sqrt{\frac{3(1 - \lambda^2)}{2}})}{l + (l^{-1} - l)(\lambda^2)} d\lambda, \quad (D12)
\]

\[
\Psi_{QDF}(l) = \int_0^1 \frac{\lambda^2 \frac{1}{1 + 3\lambda^2} E(\sqrt{\frac{3(1 - X^2)}{2}})}{l + (l^{-1} - l)(X^2)} d\lambda. \quad (D13)
\]

Since \( e \) and \( i \) follow Rayleigh distributions, the evolution of mean \( e \) and \( i \) due to gas drag is given by (Adachi et al. 1976;
\[ \frac{de^2}{dt} = -\frac{2}{\eta s t} \left( \frac{9E(3/4)}{4\pi} e^2 + \frac{r^2}{\pi} + \frac{9\eta^2}{4} \right)^{1/2}, \quad (D14) \]

\[ \frac{d\Delta^2}{dt} = -\frac{2}{\eta s t} \left( \frac{E(3/4)}{\pi} e^2 + \frac{2r^2}{\pi} + \frac{\eta^2}{\pi} \right)^{1/2}. \quad (D15) \]

We only consider the leading-order terms of \( e \) and \( i \), because the higher-order terms of \( e \) and \( i \) are negligible for \( e \) and \( i \) with which we are concerned (Kobayashi 2015).

The collisional damping terms, \( de^2/dt \) and \( d\Delta^2/dt \), are given via the random velocities of collisional outcomes according to Kobayashi et al. (2010).

### Appendix E

**Relative Velocity for Strong Coupling with Gas**

To determine \( \Delta v_{\text{rel}, \text{gas}} \), we use the vertical averaged values for \( \Delta v_B \), \( \Delta v_r \), \( \Delta v_B \), and \( \Delta v_r \), given by (Adachi et al. 1976; Okuzumi et al. 2012; Ormel & Cuzzi 2007)

\[
\Delta v_B = \sqrt{\frac{8k_B T (m_1 + m_2)}{\pi m_1 m_2}},
\]  

(E1)

\[
\Delta v_r = \left| \frac{S_1}{1 + S_1^2} - \frac{S_2}{1 + S_2^2} \right| 2\eta r \Omega,
\]  

(E2)

\[
\Delta v_B = \left| \frac{1}{1 + S_1^2} - \frac{1}{1 + S_2^2} \right| \eta r \Omega,
\]  

(E3)

\[
\Delta v_C = \left| \frac{S_1}{1 + S_1^2} - \frac{S_2}{1 + S_2^2} \right| \frac{h_{s,1} h_{s,2} \Omega}{h_{s,1,2}},
\]  

(E4)

\[
\Delta \delta = (\Delta v_1^2 + \Delta v_2^2)^{1/2},
\]  

(E5)

where

\[
\Delta v_1^2 = \alpha_D \Delta v_2^2 \left( \frac{S_1^2}{S_1 + S_2} \right) - \left( \frac{S_1^2}{S_1 + S_1^2 + S_2} + \frac{S_2^2}{1 + S_2} \right)
\]  

(E6)

\[
\Delta v_2^2 = \alpha_D \Delta v_2^2 \left( S_{1,2}^* - S_{\text{min}} \right) \left( \frac{S_{1,2}^* + S_{\text{min}}}{S_1 + S_2} \right) \left( \frac{S_{1,2}^* + S_{\text{min}}}{S_1 + S_2} \right)
\]  

(E7)

\[ S_{\text{min}} = \sqrt{\pi} \lambda_{\text{min}} / 4 \sqrt{2} \alpha_D h_g, \text{ and } S_{1,2}^* = \text{MAX}(\text{MIN}(1,6S_1, 1)) \]

for \( S_1 \geq S_2 \).