Gauge Zero-Modes on ALE Manifolds

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ABSTRACT
In this paper we find the general (i.e. valid for arbitrary values of the winding number) form of the gauge zero-modes, in the adjoint representation, for theories living on manifolds of the ALE type.

1 Introduction
In the past few years there has been a considerable progress in the understanding of non-perturbative effects in supersymmetric (SUSY) gauge field theories. In the case of the theory with global $N = 2$ SUSY, using a certain number of educated guesses, all the non-perturbative contributions to the holomorphic part of the action have been calculated [1]. Moreover, the results of [1] have been generalized to a certain number of curved manifolds,
to compute topological invariants of the Donaldson type [2]. A part of the computation in [1] has been checked by comparing with the results obtained by a saddle point approximation of the functional integral around a self-dual solution (with winding numbers one and two) of the equations of motion of the theory [3]. As of today, no checks have been performed on the results in [2]. While guess-work can be very powerful in certain occasions, the advantage of a direct computation of the functional integral lies in the ease with which it can be generalized to different situations. For example the breaking of SUSY in supergravity theories by non-perturbative effects, leads to an explanation of the generation of mass hierarchies, one of the most outstanding problems in today’s high-energy theoretical physics. The signature of these non-perturbative effects is the formation of fermionic or bosonic condensates which can be computed by the saddle point expansion we discussed before. While the calculations in [3] were performed in flat space, in the case of supergravity we need a generalization to curved manifolds. These manifolds have to obey certain requirements if we want the classical supergravity theory to be a low-energy description of a heterotic string theory (or, that is the same, if we want to satisfy, the low-energy equations of motion of the heterotic string). The latter, in its turn, can act as an ultraviolet cut-off of the otherwise non-renormalizable supergravity. It turns out that ALE manifolds and self-dual gauge connections (of winding numbers bigger than one) satisfy the necessary requirements [4]. Some preliminary computations of the above mentioned condensates, were performed in [5]. In doing the actual computations, an ingredient one can not do without is the explicit form of the zero-modes of the gauge fields in the adjoint representation: deducing this expression is the subject of this paper. The final result will be valid for arbitrary winding numbers. Gauge connections of arbitrary winding numbers,
were first constructed in [6], while ALE spaces (or gravitational instantons as they are also known) of arbitrary winding numbers were built in [7]. Finally gauge connections of arbitrary winding numbers on ALE manifolds were described in [8]. A short review of some of these results is given in the first two section of this work. In the middle sections of the work we find the form of the zero-modes in a way that strictly resembles [9, 10]. In the last section we check the general form in a particular case already studied in [11].

2 Review of Kronheimer Construction of ALE Spaces

Before facing the Kronheimer-Nakajima construction of all ADHM instantons on ALE surfaces, we need to review the fundamental points of the Kronheimer construction of ALE spaces [7]. From the mathematical point of view, ALE manifolds are obtained exploiting a procedure called hyper-Kähler quotient. This procedure is a little involved and calls for some explanations.

The starting point is the set

$$Y \equiv (\mathbb{H}^* \otimes_\mathbb{R} \text{End}(R))^\Gamma.$$  \hspace{1cm} (1)

End($R$) stands for the adjoint endomorphisms of the linear space $R$ of the regular (adjoint) representation of the discrete group $\Gamma \subset SU(2)$. $\mathbb{H}^*$ stands for the dual of the quaternion space $\mathbb{H}$. The action of $\Gamma$ on $\mathbb{H}^*$ is induced by the usual action of $Sp(1) \sim SU(2)$ on $\mathbb{H}$. The superscript $\Gamma$ in (1) means that we must choose $Y$ as the $\Gamma$-invariant subset of $(\mathbb{H}^* \otimes_\mathbb{R} \text{End}(R))$.

To be explicit, $(\mathbb{H}^* \otimes_\mathbb{R} \text{End}(R))$ is the set composed by the matrices

$$y = y^k \sigma_k, \hspace{0.5cm} k = 1, 2, 3, 4,$$  \hspace{1cm} (2)
where the $\bar{\sigma}_i$’s are, respectively, the standard $2 \times 2$ matrices $1, -i\sigma_3^P, -i\sigma_2^P, -i\sigma_1^P$, and the $y^k$’s are $|\Gamma| \times |\Gamma|$ adjoint matrices ($|\Gamma|$ is the dimension of $\Gamma$)

$$y = \begin{pmatrix} y^1 - iy^2 & -y^3 - iy^4 \\ y^3 - iy^4 & y^1 + iy^2 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta^\dagger \\ \beta & \alpha^\dagger \end{pmatrix},$$  \hspace{1cm} (3)

is the isomorphism

$$(\mathbb{H}^* \otimes_R \text{End}(R)) \sim \text{Hom}(S^+ \otimes R, Q \otimes R)_R, \hspace{1cm} (4)$$

where $S^+$ is isomorphic to $\mathbb{C}^2$ (in physicist’s language it is the space acted upon by right-handed spinors) and $Q$ is the linear space of the fundamental representation of $SU(2)$. Given a

$$\gamma = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in \Gamma \subset SU(2),$$

we can constrain $\alpha$ and $\beta$ imposing

$$R(\gamma^{-1})\alpha R(\gamma) = u\alpha + v\beta, \hspace{0.5cm} R(\gamma^{-1})\beta R(\gamma) = -\bar{v}\alpha + \bar{u}\beta, \hspace{1cm} (5)$$

where $R(\gamma)$ stands for the regular (adjoint) representation of $\gamma$.

The set $Y$, equipped with the Euclidean metric

$$ds^2 = \text{Tr}(dydy^\dagger),$$  \hspace{1cm} (6)

is a flat manifold with hyper-Kählerian structure in the sense of Calabi [14]. This means that we can define three covariantly constant endomorphisms of the tangent space $TY \equiv Y$, say $I, J, K$, respecting the quaternionic algebra

$$I^2 = J^2 = K^2 = -1; \hspace{0.5cm} IJ = -JI = K.$$  \hspace{1cm} (7)

Looking at the expression (2), it is easy to see that $I, J, K$ can be chosen as

$$I, J, K = \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3 \otimes 1_R.$$  \hspace{1cm} (8)
The hyper-Kähler manifold $Y$ plays the role of an immersion space in the Kronheimer construction. To see this we have to note that the metric (6) admits an isometry group $G \subset U(|\Gamma|)$ acting by the transformation law
\[
\alpha \mapsto g\alpha g^\dagger \quad \beta \mapsto g\beta g^\dagger ,
\]
where $g \in G$ is any element of the unitary group $U(|\Gamma|)$ commuting with the action of $\Gamma$ on $R$. Now, we are able to define the moment maps $\mu_i$, $i = 1, 2, 3$, as the elements of $G^*$ satisfying
\[
d(\mu_i \cdot \lambda) = \omega_i(V_\lambda) ,
\]
where $\lambda$ is any element of $G$, $(\cdot \cdot)$ is the internal product in $G$, $V_\lambda$ is the Killing vector corresponding to $\lambda$ and $\omega_1, \omega_2, \omega_3$ are the three closed Kähler 2-forms induced by the hyper-Kähler structure
\[
\omega_1 = \frac{1}{2} \text{Tr}(dyIdy^\dagger) \\
\omega_2 = \frac{1}{2} \text{Tr}(dyJdy^\dagger) \\
\omega_3 = \frac{1}{2} \text{Tr}(dyKdy^\dagger)
\]
(11)

Explicitly, one can see that $\mu_1, \mu_2, \mu_3$ are the three $|\Gamma| \times |\Gamma|$ traceless skew-adjoint matrices
\[
\mu_1 = \frac{1}{2} i ([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]) \\
\mu_2 = \frac{1}{2} i ([\alpha, \beta] + [\alpha^\dagger, \beta^\dagger]) \\
\mu_3 = \frac{1}{2} i ([\alpha, \beta] - [\alpha^\dagger, \beta^\dagger])
\]
(12)
Choosing a suitable linear combination we can write
\[
\mu_C = \frac{1}{2} [\alpha, \beta] \\
\mu_R = \frac{1}{2} i ([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger])
\]
(13)
Equating $\mu_C$ and $\mu_R$, respectively, to $\zeta_C$ and $\zeta_R$, where the $\zeta$’s are parameters laying in the center of $G^*$ (traceless matrices invariant under the action of $G$ given by (9)), one obtains the “level surfaces” $Y_\zeta$,
\[
\zeta_C = \frac{1}{2} [\alpha, \beta] \\
\zeta_R = \frac{1}{2} i ([\alpha, \alpha^\dagger] + [\beta, \beta^\dagger])
\]
(14)
The main result of [7] is that varying $\zeta_C$ and $\zeta_R$ and the group $\Gamma$ we can obtain all the hyper-Kähler four-manifolds with ALE structure as the manifold

$$X_\zeta = Y_\zeta / G.$$  \hfill (15)

In particular, it is always possible to put $\zeta_R = 0$ and, for every choosing of $\zeta_C \neq 0$, we obtain an ALE manifold resembling $\mathbb{R}^4 / \Gamma$ at infinity.

More clearly, if we call $\xi$ the elements of $Y_\zeta$, the metric (6) induces on $Y_\zeta$ the metric

$$ds^2 = \text{Tr}(d\xi d\xi^\dagger).$$ \hfill (16)

Since the Kronheimer conditions (14) are invariant under the action of $G$ given by (9), the metric (16) still possesses the isometry $G$. According to (15) we can therefore obtain $X_\zeta$ by gauging the $G$-invariance in (16). The net effect of this procedure turns out to be the substitution of $d\xi$ with

$$d^R \xi = (d + [A^R_\xi, ]\xi)$$ \hfill (17)

in the metric of $Y_\zeta$, obtaining for $X_\zeta$ the metric

$$ds^2_{X_\zeta} = \text{Tr}(d^R \xi (d^R \xi)^\dagger),$$ \hfill (18)

where, mathematically speaking, $d^R$ is the covariant differentiation on the matrices $\xi$ saw as sections

$$\xi \in (\mathbb{H}^* \otimes_R \text{End}(\mathcal{R}))^\Gamma$$ \hfill (19)

of the so-called tautological bundle $\mathcal{R}$ obtained from the principal $G$-bundle $Y_\zeta$ as [8]

$$\mathcal{R} = Y_\zeta \times_G R.$$ \hfill (20)
It comes from [16] that $\mathcal{R}$ has a natural decomposition

$$\mathcal{R} = \bigoplus_{i=0}^{r-1} \mathcal{R}_i \otimes R_i,$$

(21)

where the $R_i$'s are all the irreducible linear spaces of the representation of $\Gamma$ ($R_0$ is the trivial representation), and that the structure group of $\mathcal{R}$ turns out to be

$$G = \bigotimes_{i \neq 0}^{r-1} U(|R_i|).$$

(22)

In this way the connection $A^\mathcal{R}_\xi$ can be deduced according to the decomposition (21) from the properties of the $A_i$ connections equipping the vector bundles $\mathcal{R}_i$. In particular [16] we have that $A_i$ possesses an antiself-dual curvature with finite action. As a connection in $G$, $A^\mathcal{R}_\xi$ is a skew-Hermitian connection. This means that, with the help of (2), we can write

$$ds^2_{\mathcal{X}_\xi} = 2 \sum_{k=1}^{4} \text{Tr}((d^\mathcal{R}_\xi)^k(d^\mathcal{R}_\xi)^k),$$

(23)

where we used the identity

$$\bar{\sigma}_k \sigma_i = i \bar{\eta}^a_{ki} \sigma_a^P + \delta_{ki} \mathbf{1}, \ a = 1, 2, 3.$$

(24)

The symbol $\bar{\eta}^a_{ki}$ is the skew-symmetric, antiself-dual 't Hooft symbol.

[18] will be useful later.

3 Kronheimer-Nakajima Construction

Now we are able to face the Kronheimer-Nakajima (KN) construction of all ADHM instantons with topological index $k/|\Gamma|$ on ALE manifolds.

The analogue of the $D$ matrix of the ADHM construction on flat spaces (see for example [10] and references therein) is

$$D = (A \otimes \mathbf{1}_R - \mathbf{1}_V \otimes \xi) \oplus (\Psi \otimes \mathbf{1}_R),$$

(25)
where
\[ A = \left( \begin{array}{cc} A & -B^\dagger \\ B & A^\dagger \end{array} \right), \quad A \in (\mathbb{H} \otimes \mathbb{R} \text{End}(V)), \]
and
\[ \Psi = \left( \begin{array}{c} s \\ t^\dagger \end{array} \right), \quad s, t^\dagger \in \text{Hom}(V, W). \]

We choose \( V \) and \( W \) as, respectively, \( \mathbb{C}^k \) and \( \mathbb{C}^n \) isomorphic \( \Gamma \)-equivariant linear spaces, that is
\[ V = \bigoplus_{i=0}^{r-1} R_i \otimes V_i, \quad W = \bigoplus_{i=0}^{r-1} R_i \otimes W_i, \]
where \( V_i \sim \mathbb{C}^{v_i} \) and \( W_i \sim \mathbb{C}^{w_i} \) are \( \Gamma \)-invariant spaces.

The matrix \( D \) defined in eq. (25), as an operator
\[ D : S^+ \otimes V \otimes \mathcal{R} \to (Q \otimes V \otimes \mathcal{R}) \oplus (W \otimes \mathcal{R}), \]
is a \((2k + n)|\Gamma| \times 2k|\Gamma|\) matrix. We can obtain a \((2k + n) \times 2k\) matrix, as in the ADHM construction on flat spaces, simply reducing\footnote{In principle \( D^\Gamma \) is a \((2k + n)|\Gamma| \times 2k|\Gamma|\) matrix like \( D \). The point is that starting from \( D^\Gamma \) it is always possible to cancel \(|\Gamma|\) rows and columns without affecting all the KN construction.} \( D \) to his \( \Gamma \)-invariant restriction \( D^\Gamma \)
\[ D^\Gamma : S^+ \otimes (V \otimes \mathcal{R})^\Gamma \to (Q \otimes V \otimes \mathcal{R})^\Gamma \oplus (W \otimes \mathcal{R})^\Gamma. \]

The matrix \( D^\Gamma \) must satisfy the ADHM conditions
\[ (D^\Gamma)^\dagger D^\Gamma = F^{-1} = f^{-1} \otimes 1_{S^+}. \]

(14) and (31) together give
\[ [A, B] + ts = \zeta_R \]
\[ [A, A^\dagger] + [B, B^\dagger] - s^\dagger s + tt^\dagger = \zeta_C. \]
where, this time, $\zeta_R$ and $\zeta_C$ are

$$\zeta = \bigoplus_{i=0}^{r-1} \zeta_i 1_{V_i}.$$  \hfill (33)

where $\sum_i \zeta_i = 0$. Finally, the $SU(n)$ gauge bundle $E$ with instanton connection $A^E_\mu$ and antiself-dual curvature $F^E_{\mu\nu}$, is given by

$$E \equiv \text{Ker}(D^\Gamma)^\dagger.$$  \hfill (34)

The instanton connection $A^E_\mu$ is

$$A^E_\mu = U^\dagger \nabla^R_\mu U,$$  \hfill (35)

where the matrix

$$U : E \to (Q \otimes V \otimes R)^\Gamma \oplus (W \otimes R)^\Gamma$$  \hfill (36)

is chosen in accordance with the conditions

$$\begin{align*}
(D^\Gamma)^\dagger U &= 0 \\
U^\dagger U &= 1
\end{align*}$$  \hfill (37)

and the derivation $\nabla^R_\mu$ contains the Levi-Civita connection and the connection $A^R_\mu$ of the bundle $R$ acting on $U$ according to (36).

4 Curvature in KN Construction

To show the close analogy between KN and ADHM formalism on $\mathbb{R}^4$, we will verify the antiself-duality of the curvature given by the instanton connection (35). Our proof will be basically the same given by [9] in the case of the ADHM construction on $\mathbb{R}^4$.

The curvature $F^E_{\mu\nu}$ is given by the formula

$$F^E_{\mu\nu} = [\nabla^E_\mu, \nabla^E_\nu],$$  \hfill (38)
where $\nabla_E^\mu$ is the covariant derivative with respect to the Levi-Civita and $A_E^\mu$ connection given by (33). From (33) it follows that, on any section $\phi$ of the bundle $E$, one has

$$\nabla_E^\mu \phi = U^\dagger \nabla_R^\mu (U \phi).$$  \hfill (39)

Substituting (39) into (38) we find

$$F_E^{\mu\nu} = U^\dagger \nabla_{[\mu}(U U^\dagger \nabla_{\nu]}U),$$  \hfill (40)

where, for the sake of simplicity, we omitted the superscript $R$ of the covariant derivatives.

Expanding (40), we find

$$F_E^{\mu\nu} = U^\dagger \nabla_{[\mu}P \nabla_{\nu]}U + U^\dagger F_R^{\mu\nu} U,$$  \hfill (41)

where we put $F_R^{\mu\nu} = [\nabla_{\mu}, \nabla_{\nu}]$ and $P = U^\dagger U$. Since

$$P = 1 - D^\Gamma (D^\Gamma)^\dagger,$$  \hfill (42)

we find

$$F_E^{\mu\nu} = U^\dagger \nabla_{[\mu}D^\Gamma F \nabla_{\nu]}(D^\Gamma)^\dagger U + U^\dagger F_R^{\mu\nu} U,$$  \hfill (43)

where we used the identity $(D^\Gamma)^\dagger P = 0$.

The covariant derivative $\nabla_\mu$ acts on $D^\Gamma$ according to (30). Looking at the definition of $D$ given in (25), one can see that

$$\nabla_\mu D^\Gamma = -b^\dagger \nabla_\mu \xi,$$  \hfill (44)

where, from now on, we abbreviate $1_V \otimes \xi$ with $\xi$ and we put $b$ for the projection to $(Q \otimes V \otimes R)^\Gamma$ in $(Q \otimes V \otimes R)^\Gamma \oplus (W \otimes R)^\Gamma$ ($b$ is the analogue of the matrix which multiplies the coordinate $x \in \mathbb{R}^4$ in the ADHM construction in flat space [10] ) [13] then becomes

$$F_E^{\mu\nu} = U^\dagger b^\dagger \nabla_{[\mu} \xi F \nabla_{\nu]} \xi^\dagger b U + U^\dagger F_R^{\mu\nu} U.$$  \hfill (45)
From the properties of the bundle $\mathcal{R}$ described at the end of Sec. 1 we see that $F_{\mu\nu}^R$ is an antiself-dual quantity. The remaining part of (45) is more easily written in terms of differential 2-forms as

$$U^\dagger b^\dagger d^R \xi \wedge F d^R \xi^\dagger bU = -U^\dagger b^\dagger (d^R \xi)^k \bar{\sigma}_k \wedge F (d^R \xi)^\dagger \sigma_i bU. \tag{46}$$

Looking at (23), we see that, reducing $d^R \xi$ to a $2 \times 2$ matrix, $(d^R \xi)^k$ can be normalized by a constant factor to a local orthonormal basis $e^k \in T^*X_\zeta$. The matrix $F$ satisfying the condition (31) commutes with $\bar{\sigma}_k$ and $\sigma_i$, so that, remembering (24), it is easy to see that (46) also is an antiself-dual quantity.

5 Bosonic Zero-Modes

The bosonic zero-modes of a YM theory with antiself-dual curvature are determined [13] by

$$\nabla^E_{ad \ [\mu} Z_{\nu]} = -\ast \nabla^E_{ad \ [\mu} Z_{\nu]} \tag{47}$$

and

$$\nabla^E_{ad \ [\mu} Z^{\mu} = 0, \tag{48}$$

where, since $Z_\mu$ is in the adjoint representation of the gauge group $SU(n)$, we have to use here $\nabla^E_{ad \ [\mu}$ instead of $\nabla^E_\mu$. The symbol $\ast$ stands for the duality operator.

We choose for $Z_\mu$ the form

$$Z_\mu = U^\dagger z_\mu^\dagger - z_\mu U, \tag{49}$$

where

$$z_\mu = -U^\dagger \nabla_\mu D^\Gamma FC^\dagger. \tag{50}$$

$C^\dagger$ is a constant matrix,

$$C^\dagger : (Q \otimes V \otimes \mathcal{R})^\Gamma \oplus (W \otimes \mathcal{R})^\Gamma \rightarrow S^+ \otimes (V \otimes \mathcal{R})^\Gamma, \tag{51}$$
satisfying the condition
\[ \nabla_\mu C^\dagger = 0. \] (52)

This is equivalent to say that
\[ C^\dagger = (\mathcal{B} \otimes \Phi) \otimes 1_R^\Gamma, \] (53)

with
\[ \mathcal{B} \in \text{Hom}_R(S^+ \otimes V, Q \otimes V)_R \] (54)
and
\[ \Phi \in \text{Hom}_R(V, W). \] (55)

To demonstrate that \( Z_\mu \) given by (49) is a solution of (47) and (48), we need the explicit expression of \( \nabla^E_{ad} Z_\nu \). Since, using (35),
\[ \nabla^E_{ad} Z_\nu = U^\dagger \nabla_\mu (U Z_\nu U^\dagger) U, \] (56)
we can write
\[ \nabla^E_{ad} Z_\nu = \nabla^E_{ad} z_\mu + \nabla^E_{ad} U^\dagger U. \] (57)

Now, to simplify the proof of (47) and (48) it is useful to note that the quantity \( z_\mu \) is a solution of
\[ \nabla^E_\mu z_\nu = -\ast \nabla^E_\mu z_\nu \] (60)
and
\[ \nabla^E_\mu z_\mu = 0. \] (61)
(60) is easily checked by writing explicitly the derivative $\nabla_\mu^E z_\nu$ and proceeding as in Sec. 3. The proof of (61) is a little more involved. First of all we have to note that, setting $C^\dagger \equiv (C^\dagger)^\alpha \chi_\alpha$, where the index $\alpha = 1, 2$ spans the $S^+$ space, it is always possible to write

$$z_\mu = z_\mu^\alpha \chi_\alpha.$$  \hspace{1cm} (62)

From (62) we can construct the left-handed spinor

$$\bar{\chi}_\dot{\alpha} = z_\mu^\alpha \sigma^\mu \sigma_{\alpha \dot{\alpha}};$$  \hspace{1cm} (63)

which is a solution [8] of the Dirac equation

$$\tilde{\nabla}_\nu^E (\bar{\chi}_\dot{\alpha} \sigma^\mu \sigma_{\alpha \dot{\alpha}}) \sigma^\nu \sigma^\dot{\beta} = 0,$$  \hspace{1cm} (64)

where $\tilde{\nabla}_\nu^E$ is the covariant derivative with respect to the Levi-Civita, spin and instanton $A_E^\mu$ connection.

Since on a self-dual background, like ALE manifolds, we have

$$\tilde{\nabla}_\nu^E (\bar{\chi}_\dot{\alpha} \sigma^\mu \sigma_{\alpha \dot{\alpha}} \sigma^\nu \sigma^\dot{\beta}) = \nabla_\nu^E z_\mu^\alpha \sigma^\mu \sigma_{\alpha \dot{\alpha}} \sigma^\nu \sigma^\dot{\beta},$$  \hspace{1cm} (65)

we can write

$$\nabla_\nu^E z_\mu^\alpha \sigma^\mu \sigma^\nu = \tilde{\nabla}_\nu^E z_\mu^\alpha (i \eta^{\mu \nu} \sigma^k + g^{\mu \nu}) = 0,$$  \hspace{1cm} (66)

where $\eta^{\mu \nu}$ is the 't Hooft self-dual symbol and $g^{\mu \nu}$ is the metric.

From (60) and (66), it comes that

$$\nabla_\nu^E z_\mu g^{\mu \nu} = 0,$$  \hspace{1cm} (67)

which is (51).

Using (60) and (61), (47) and (48) reduce then to

$$K_{[\mu \nu]} = - * K_{[\mu \nu]}$$  \hspace{1cm} (68)
and

$$K_\mu^\mu = 0. \quad (69)$$

From (57) it comes that

$$K_{[\mu\nu]} = U^\dagger F b^\dagger \nabla_\mu \xi ((D^\Gamma)^\dagger C + C^\dagger D^\Gamma) \nabla_\nu \xi^\dagger b U - \text{h.c.} \quad (70)$$

Comparing this expression with (45), one sees that

$$(D^\Gamma)^\dagger C + C^\dagger D^\Gamma = G^{-1} = g^{-1} \otimes 1_{S^+}, \quad (71)$$

is a sufficient condition to assure the antiself-duality of $K_{[\mu\nu]}$.

(71) is identical to the $U(n)$ version of the condition found in [13] for the bosonic zero-modes on $\mathbb{R}^4$.

For what (69) is concerned, we see that

$$K_\mu^\mu = U^\dagger b^\dagger \bar{\sigma}^k ((D^\Gamma)^\dagger C - C^\dagger D^\Gamma) \sigma_k b U. \quad (72)$$

Remembering that $d^R \xi$ can be written as $e^k \bar{\sigma}_k$ and that $e^k$ is a local vierbein basis in $T^* X_\zeta$, we obtain

$$K_\mu^\mu = U^\dagger b^\dagger f \bar{\sigma}^k ((D^\Gamma)^\dagger C - C^\dagger D^\Gamma) \sigma_k b U. \quad (73)$$

(71) means that $(D^\Gamma)^\dagger C$ must be a matrix of the form

$$\begin{pmatrix} \gamma & -\delta^\dagger \\ \delta & \gamma^\dagger \end{pmatrix}.$$ 

Since for this kind of matrix we have

$$\sum_{k=1}^{4} \left( \bar{\sigma}^k \begin{pmatrix} \gamma & -\delta^\dagger \\ \delta & \gamma^\dagger \end{pmatrix} \sigma_k \right) = 2(\gamma + \gamma^\dagger) \otimes 1_{S^+}, \quad (74)$$

it is easy to verify that

$$K_\mu^\mu = 0. \quad (75)$$
6 Gauge Zero-modes for $k = 1/2$ on the Eguchi-Hanson Manifold

As an example of (49) and (50) we compute the zero-modes of the self-dual gauge potential of topological index $k = 1/2$ corresponding to the choice $\Gamma \equiv \mathbb{Z}_2$.

As it is explained in [4], in this case the ADHM-KN construction gives the simplest instanton connection possible on the Eguchi-Hanson (EH) manifold. As a first step, it is necessary to determine the expression of $\xi$ and, consequently the EH metric from the Kronheimer construction described in Sec. 1. The choice $\Gamma \equiv \mathbb{Z}_2$ means that $R$ is isomorphic to $\mathbb{C}^2$. In the decomposition (21) only $R_0$ and $R_1$ survive. This means that $A^R$ can be written as

$$A^R = \begin{pmatrix} 0 & 0 \\ 0 & A^{U(1)} \end{pmatrix},$$

where $A^{U(1)}$ is a suitable abelian connection with antiself-dual curvature.

Acting on $\xi$, which is a section of $\mathbb{H}^* \otimes \text{End}(\mathcal{R})$, (76) becomes

$$A^R_\xi = \begin{pmatrix} A^R & 0 \\ 0 & A^R \end{pmatrix}.$$

Furthermore, one can see [4] that

$$\xi = \begin{pmatrix} 0 & v^1 & 0 & -\lambda \bar{v}^2 \\ \lambda v^1 & 0 & -\lambda v^2 & 0 \\ 0 & v^2 & 0 & \lambda \bar{b}^1 \\ \lambda v^2 & 0 & \lambda v^1 & 0 \end{pmatrix},$$

where $v^1, v^2 \in \mathbb{C}$ and $\lambda = 1 + a^2 / \sum_{i=1}^2 |v^i|^2$, so that $d^R \xi$ (reduced to a $2 \times 2$ matrix) can be written as

$$d^R \xi = (d + A^{U(1)}) \begin{pmatrix} v^1 & -\lambda \bar{v}^2 \\ v^2 & \lambda \bar{v}^1 \end{pmatrix}.$$
The vierbein basis $e^k$ defined at the end of Sec. 3 is then

$$e^k = (d + A^{U(1)})\xi^k, \quad (80)$$

where

$$\xi^1 = \frac{1}{2}(v^1 + \lambda \bar{v}^1) \quad \xi^3 = \frac{1}{2}(v^2 + \lambda \bar{v}^2)$$
$$\xi^2 = \frac{1}{2}(v^1 - \lambda \bar{v}^1) \quad \xi^4 = \frac{1}{2}(v^2 - \lambda \bar{v}^2). \quad (81)$$

Switching to coordinates

$$v^1 = \sqrt{r^2 - a^2} \cos(\frac{\psi + \phi}{2})$$
$$v^2 = \sqrt{r^2 - a^2} \sin(\frac{\psi - \phi}{2}) \quad (82)$$

and choosing $A^{U(1)}$ (in the same coordinates) as the monopole potential [12]

$$A^{U(1)} = \frac{ia^2}{r^2} \left( d\psi + \cos \theta d\phi \right) = -i \frac{a^2}{r^2} \sigma_3, \quad (83)$$

we find that $ds^2 = \sum_k (e^k)^2$ gives the EH metric.

Incidentally, we note that in the limit $a^2 \to 0$ ($X_\zeta \to \mathbb{R}^4/\mathbb{Z}_2$) the basis $e^k \equiv (dR^k)\xi^k$ reduces to the canonical basis of differential 1-forms in $\mathbb{R}^4$, $dx^1, ..., dx^4$, with $x^1, ..., x^4 \equiv \xi^1, ..., \xi^4 \in \mathbb{R}^4$.

Knowing the expression of the matrix $\xi$ we are able to find $D$ and, consequently, $D^\Gamma$. One can see [4] that

$$D^\Gamma = \begin{pmatrix} v^1 & -\lambda \bar{v}^2 \\ v^2 & \lambda \bar{v}^1 \\ s^1 & -\mu \bar{s}^2 \\ s^2 & \mu \bar{s}^1 \end{pmatrix}, \quad (84)$$

with $s^1, s^2 \in \mathbb{C}$ and $\mu = 1 - \frac{a^2}{(\sum_{i=1}^2 |s^i|^2)}$. From [84], solving (84), we can find the expression of the matrix $C$ determining the bosonic zero-modes of the instanton potential.

It is easy to see that $C$ can be chosen as

$$C = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ s^1 & s^2/\mu & s^1/\mu & -\bar{s}^1/\mu \\ s^2 & -\bar{s}^2/\mu & -\bar{s}^1/\mu & \mu \end{pmatrix}, \quad (85)$$
so that
\[(D^\Gamma)^\dagger C + C^\dagger D^\Gamma = \sum_i |s_i|^2 \otimes 1_{s^+}. \tag{86}\]

From (84) we can calculate \(U\), which turns out to be

\[U = \frac{|s|}{|v|\sqrt{|v|^2 + |s|^2}} \begin{pmatrix} v^1 & \mu \bar{v}^2 \\ v^2 & -\mu \bar{v}^1 \\ -|v|^2 s^1 & -\lambda |v|^2 \bar{s}^2 \\ -|s|^2 s^2 & \lambda |s|^2 \bar{s}^1 \end{pmatrix}, \tag{87}\]

where we put \(|s|^2 = \sum_i |s_i|^2, |v|^2 = \sum_i |v_i|^2, \) and \(F\). As a consequence

\[F = (|v|^2 + |s|^2) \otimes 1_{s^+}. \tag{88}\]

Putting all the pieces of our construction together as in (49) and (50), we find for \(Z = Z_\mu dx^\mu\) in EH coordinates the expression

\[Z = 2i \frac{t^2 + a^2}{\sqrt{t^4 - a^4}} \begin{pmatrix} f^3 \sigma_z & f^1 (\sigma_x - i\sigma_y) \\ f^1 (\sigma_x + i\sigma_y) & -f^3 \sigma_z \end{pmatrix}, \tag{89}\]

where \(t^2 = 2|s|^2 + a^2\) and

\[f^1 = \frac{t^2 r^2 + a^4}{(r^2 + t^2)^2}, \quad f^3 = \frac{(r^4 - a^4)\sqrt{t^4 - a^4}}{r^2(r^2 + t^2)^2}. \tag{90}\]

(89) gives the bosonic zero-modes, already found in [4] with different methods.

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