Towards harmonic superfield formulation of $\mathcal{N} = 4$, $USp(4)$ SYM theory with the central charge

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Abstract

We develop a superfield formulation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with the rigid central charge in $USp(4)$ harmonic superspace. The component formulation of this theory was given by Sohnius, Stelle and West [1], but its superfield formulation has not been constructed so far. We construct the superfield action, corresponding to this model, and show that it reproduces the component action from [1].

1 Introduction

Maximally extended $\mathcal{N} = 4$ SYM theory with R-symmetry group $SU(4)$ possesses many remarkable properties on classical and quantum levels and is widely explored in modern theoretical and mathematical physics. For the first time this theory was obtained by dimensional reduction of the ten dimensional $\mathcal{N} = 1$ SYM theory to the four dimensions [2]. Field content of this theory involves one vector, six real scalars and four Majorana spinors. By construction, such model is non-manifestly supersymmetric and the supersymmetry transformations are closed only on-shell. Its formulation in terms of the on-shell superfields was given in [3]. In many cases, especially to study the quantum aspects, it would be preferable to get an off-shell formulation of $\mathcal{N} = 4$ SYM theory. It is generally accepted that off-shell formulation of the supersymmetric theories is realized in terms of corresponding unconstrained superfields (see e.g. [4], [5]). However, in spite of the considerable efforts, superfield formulation of $\mathcal{N} = 4$ SYM theory in terms of unconstrained $\mathcal{N} = 4$ superfields is still unknown. The best, that has been obtained so far is its formulation in terms of $\mathcal{N} = 1$ superfields [4] and $\mathcal{N} = 2, 3$ harmonic superfields [6], [7]. Attempts to develop a consistent unconstrained $\mathcal{N} = 4$ harmonic superfield formulation for the $\mathcal{N} = 4$ SYM theory have not been successful so far [8].

We would like to draw attention to another $\mathcal{N} = 4$ supersymmetric model, namely $USp(4)$ SYM theory with central charge$^1$. Due to the central charge, the R-symmetry

$^1$Structure of the extended supersymmetry theories with the central charges is discussed in [9].
group of the corresponding superalgebra is a subgroup $USp(4)$ of the group $SU(4)$. Field content of such a gauge model involves one real vector, five real scalars, four Majorana spinors and auxiliary fields - axial vector and five real scalars. Furthermore, special constraints are imposed on the auxiliary and dynamical fields. After eliminating the auxiliary fields with the help of an additional scalar field, the conventional $SU(4), \mathcal{N} = 4$ SYM theory is restored. In Abelian case the constraint is solved for the auxiliary vector field in terms of an antisymmetric second rank field [1]. This field describes a propagating spin-0 mode which is known as a `notoph' [10]. As a result we get the conventional $\mathcal{N} = 4$ SYM theory where one of the scalars is replaced by an antisymmetric tensor field. In addition, with respect to central charge transformations the vector field transforms into the dual field strength of the antisymmetric tensor and the antisymmetric tensor transforms into the dual field strength of the vector. Therefore such a model can be treated as some kind of vector-tensor multiplet theory\(^2\).

In non-Abelian case this constraint is not solved in local form, however the constraint can be inserted into action with help of a scalar Lagrange multiplier\(^3\). It is evident that such a procedure is not supersymmetric, the authors of [1] expressed a hope that it can be made supersymmetric if it becomes possible to construct a superfield formulation.

In this paper we develop a superspace formulation of $\mathcal{N} = 4$ SYM theory with the rigid central charge [1] in terms of $\mathcal{N} = 4$ superfields. Some superspace aspects of the theory under consideration have been discussed in the earlier papers [16], [17], [18]. The paper [16] is devoted to construction of the above theory in terms of $\mathcal{N} = 1$ superfields where the component constraint from [1] has been written in $\mathcal{N} = 1$ superfield form. The paper [17] develops a gauge theory in the $\mathcal{N} = 4$, $USp(4)$ superspace. Here the superfield constraints, originating from the Bianchi identities and determining the correct $USp(4)$ vector multiplet with the central charge have been formulated and solved in terms of some superfield strengths. In the papers [18] the aspects of a $\mathcal{N} = 4$ SYM theory in harmonic superspace with the central charge were studied and applied for the construction of Abelian low-energy effective action. In the present paper we prove that the constraint for the auxiliary field, introduced in [1] for non-Abelian theory, automatically follows from the $\mathcal{N} = 4$ superfield constraints stipulated by the Bianchi identities, obtained in [17]. Also, we develop a $USp(4), \mathcal{N} = 4$ harmonic superspace formalism and propose a gauge invariant, $\mathcal{N} = 4$ supersymmetric action, which exactly reproduces the component action of [1] for non-Abelian theory.

The paper is organized as follows. In subsection 1.1 the theory of Sohnius, Stelle and West is briefly discussed. Section 2 is devoted to the structure of the constraints and their solutions in conventional $USp(4), \mathcal{N} = 4$ superspace. Here we show how the component constraint from [1] is derived from the constraints on superfield strengths. In section 3 we reformulate these constraints in $USp(4), \mathcal{N} = 4$ harmonic superspace, define the corresponding analytic subspace, propose the $\mathcal{N} = 4$ supersymmetric action and find its component form. Section 4 briefly summarizes the results. The notations, conventions

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\(^2\)The $\mathcal{N} = 2$ vector-tensor multiplet theories are discussed in [12], [13], [14], [15].

\(^3\)It was shown in [11] that in principle a local Lagrangian can be written but the resulting Lorentz and supersymmetry transformations are non-local.
and some details of computations are given in Appendices A, B and C.

1.1 \textit{USp}(4) SYM model with the central charge

\( \mathcal{N}=4, \text{USp}(4) \) SYM model with the central charge has been proposed by Sohnius, Stelle and West in the component formulation [1]. This model possesses the \textit{USp}(4) R-symmetry and is described by the action\(^4\)

\[
S = \text{tr} \int d^4x \left( -\frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} V_m V^m + \frac{1}{2} \nabla_m \phi_{ij} \nabla^m \phi^{ij} + \frac{1}{2} H_{ij} H^{ij} \right)
\]

\[
-\frac{i}{4} \bar{\lambda}^i \nabla \lambda_i - \bar{\lambda}^i [\lambda^j, \phi_{ij}] + \frac{1}{4} [\phi_{ij}, \phi_{kl}] [\phi^{ij}, \phi^{kl}] .
\]

Here the vector field \( A_m \), \textit{USp}(4)-Majorana spinor fields \( \lambda_{i\alpha} = \bar{\lambda}^{i\beta}(C^{-1})_{\beta\alpha} \Omega_{ji} \) and the antisymmetric, \( \Omega \)-traceless scalar fields \( \phi_{ij} \) are the propagating fields whereas the pseudovector \( V_m \) and antisymmetric, \( \Omega \)-traceless, scalar fields \( H_{ij} \) are the auxiliary fields. All fields take the values in the Lie algebra of a gauge group. \( \nabla_m \) are the conventional gauge covariant derivatives.

Supersymmetry transformations of the component fields are defined as follows

\[
\delta A_m = i \bar{\epsilon}^i \gamma_m \lambda_i, \quad \delta \phi_{ij} = -i (\bar{\epsilon}_{[i} \lambda_{j]} + \frac{1}{2} \Omega_{ij} \epsilon^k \lambda_k) , \quad \delta \lambda_i = -\frac{1}{2} \sigma_{mn} F^{mn} \epsilon_i + 2 \nabla \phi \cdot \epsilon_j + \gamma^5 \gamma^m V_m \epsilon_i + 2 \gamma^5 H_i \epsilon_j - 2i \phi_{ik} \phi^{kj} \epsilon_j , \quad \delta H_{ij} = i (\bar{\epsilon}_{[i} \gamma^5 \nabla \lambda_{j]} + \frac{1}{2} \Omega_{ij} \epsilon^k \gamma^5 \nabla \lambda_k) + \epsilon^k \gamma^5 [\gamma^k, \phi_{ij}] \left[ -2(\bar{\epsilon}_{[i} \gamma^5 [\lambda^k, \phi_{jk}]) + \frac{1}{2} \Omega_{ij} \epsilon^k \gamma^5 [\lambda^l, \phi_{kl}] \right] , \quad \delta V_m = i \bar{\epsilon}^i \gamma^5 \sigma_{mn} \nabla^n \lambda_i + 2 \epsilon^i \gamma^5 \sigma_{mn} [\lambda^k, \phi_{ik}] .
\]

The supersymmetry transformations (2) are closed off-shell, up to field dependent gauge transformations. The anticommutator of two supersymmetry transformations leads to coordinate translations, central charge transformations and gauge transformations on the gauge vector fields and the physical scalar fields. The transformations generated by the central charge are defined in the form

\[
\delta_z A_m = \omega V_m, \quad \delta_z \phi_{ij} = -\omega H_{ij} ,
\]

\[
\delta_z \lambda_i = -\omega (\gamma^5 \nabla \lambda_i - 2i \gamma^5 [\lambda^k, \phi_{ik}]) ,
\]

\[
\delta_z V_m = \omega \left( \nabla^n F_{nm} - \frac{1}{2} \{ \bar{\lambda}^k, \gamma_m \lambda_k \} + i [\phi^{kl}, \nabla_m \phi_{kl}] \right) .
\]

\(^4\)Here we follow the notations from [1]. Matrix \( \Omega \), associated with \textit{USp}(4) symmetry, is given in Appendix A.
\[ \delta_z H_{ij} = \omega(-\nabla^m \nabla_m \phi_{ij} + \frac{i}{4} \Omega_{ij} \{ \bar{\lambda}_k, \lambda^k \} - \{ \phi_{kl}, [\phi^{kl}, \phi_{ij}] \}) . \]

The action (1) for the USp(4) multiplet is invariant under the supersymmetry transformations if, and only if, the following additional non-linear constraint
\[ \nabla^m V_m + \frac{1}{2} \{ \bar{\lambda}_i, \gamma^5 \lambda_i \} - i[A_5, \phi_{ij}] = 0 , \] is satisfied. Moreover, the action (1) is invariant under central charge transformations (3) with the constant parameter \( \omega \).

The conventional on-shell SU(4) \( \mathcal{N} = 4 \) SYM theory [2], [3] is obtained by introducing the scalar Lagrange multiplier \( A_5 \) for the constraint and eliminating the auxiliary fields \( V_m \) and \( H_{ij} \) from the equations of motion
\[ V_m = -\nabla_m A_5, \quad H_{ij} = i[A_5, \phi_{ij}] . \] In this case, the scalar field \( A_5 \) is unified with 5 scalar fields \( \phi_{ij} \) and as a result one gets 6 scalar fields of conventional SU(4) \( \mathcal{N} = 4 \) SYM theory.

The aim of the paper is to develop a formulation of the model under consideration in terms of \( \mathcal{N} = 4 \) harmonic superfields and present the action in superfield form.

2 USp(4), \( \mathcal{N} = 4 \) superspace and SYM model with central charge

The \( \mathcal{N}=4 \) central charge superspace is represented by the coordinates \( Z^M = \{ x^a, z, \vartheta^\alpha, \bar{\vartheta}^\dot{\alpha} \} \) and the supercovariant derivatives \( D_M = (\partial_m, \partial_z, D_i, \bar{D}_\dot{i}) \) 5. These derivatives are used for the definitions of the gauge covariant derivatives \( \nabla_M = D_M + i\Gamma_M \) with superconnections \( \Gamma_M \) and gauge transformations \( \nabla'_M = e^{i\tau} \nabla_M e^{-i\tau} \), where \( \tau \) is a gauge superfield parameter. Then one introduces the curvature tensors or superfield strengths defined on the USp(4) \( \mathcal{N} = 4 \) central charge superspace with the help of algebra:
\[ \{ \nabla_{\alpha i}, \nabla_{\beta j} \} = 2i\epsilon_{\alpha\beta}^j \Omega_{ij} \nabla_z + 2i\epsilon_{\alpha\beta}^j W_{ij}, \quad \{ \nabla_{\dot{\alpha} i}, \nabla_{\dot{\beta} j} \} = 2i\epsilon_{\dot{\alpha}\dot{\beta}}^j \Omega_{ij} \nabla_z - 2i\epsilon_{\dot{\alpha}\dot{\beta}}^j W_{ij} , \]
\[ \{ \nabla_{\alpha i}, \nabla_{\dot{\alpha} j} \} = -2i\Omega_{ij} \nabla_{\alpha\dot{\alpha}} . \]
where we impose the reality conditions under the internal symmetry:
\[ \Omega^{ij} W_{ij} = 0, \quad (W_{ij}) = W^{ij} = \Omega^{ik} \Omega^{jk} W_{kl} = -\frac{1}{2} \epsilon^{ijkl} W_{kl} . \] 7

5We mainly use the notations and conventions from [5] and [6]. See Appendix A for details.
6These gauge supercovariant derivatives obey the usual constraints
\[ \{ \nabla_{\dot{\alpha}}^{(i}, \nabla_{\dot{\beta})j} \} = 0, \quad \{ \nabla_{\alpha}^{i}, \nabla_{\dot{\alpha}j} \} - \frac{1}{4} \delta^j_i \{ \nabla_{\alpha}^{k}, \nabla_{\dot{\alpha}k} \} = 0 . \]

Here and below we use the notation \( \dot{\alpha} \) for the set \( \{ \alpha, \dot{\alpha} \} \).
Here $\Omega_{ij}$ is the invariant tensor of the $USp(4)$ group. The other commutators of the gauge supercovariant derivatives look like

$$\begin{align*}
[\nabla_{\alpha i}, \nabla_{\beta j}] &= iG_{\alpha i}, & [\nabla_{\dot{\alpha} i}, \nabla_{\beta j}] &= -iG_{\dot{\alpha} i}, \\
[\nabla_{\alpha i}, \nabla_{\alpha j}] &= iF_{\alpha i}, & [\nabla_{\dot{\alpha} i}, \nabla_{\dot{\alpha} j}] &= -i\bar{F}_{\dot{\alpha} i}, \\
[\nabla_{\alpha i}, \nabla_{\beta j}] &= i\bar{F}_{\alpha i}, & [\nabla_{\dot{\alpha} i}, \nabla_{\dot{\alpha} j}] &= -i\bar{F}_{\dot{\alpha} i}.
\end{align*} \tag{8}$$

In this representation $G_{\alpha i}$ is the $USp(4)$ Majorana spinor with the reality condition $(G_{\alpha i}) = \bar{G}_{\dot{\alpha} i} = \Omega^{ij}G_{\alpha j}$. As a result the gauge theory in $USp(4)$, $\mathcal{N} = 4$ central charge superspace is described by the superfields $W_{ij}, G_{\alpha i}, \bar{F}_{\alpha i}, V_m, F_{mn}$. The superfield strengths satisfy some number of the constraints to reduce the number of fields to an irreducible multiplet stipulated by the Bianchi identities. The solution of these relations determines the field content of the theory as well as the transformation laws of the component fields.

### 2.1 Solving the Constraints

One can prove that the Bianchi identities are satisfied if, and only if, all superfield strengths are expressed in terms of a single real scalar superfield $W$. We only list the results of [17] in our conventions concerning the solution to the constraints of the dimension from $3/2$ to $3$:

- **Solution to the $\dim = \frac{3}{2}$ Bianchi identities:**

  $$\begin{align*}
  F_{\alpha i} &= -\sigma^m_{\alpha \dot{\alpha}} \bar{G}_{\dot{\alpha} i}, & \bar{F}_{\alpha i} &= G^m_{\alpha \dot{\alpha}} ,
  \end{align*} \tag{9}$$

  $$\nabla_{\alpha i} W_{ij} = i\Omega_{ij} G_{\alpha k} + 2i\Omega_{k[\alpha j]} G_{\alpha j],} \quad \nabla_{\dot{\alpha} i} W_{ij} = i\Omega_{ij} \bar{G}_{\dot{\alpha} k} + 2i\Omega_{k[\alpha j]} \bar{G}_{\dot{\alpha} j]} , \tag{10}$$

  $$5iG_{\alpha i} = \nabla_{\dot{\alpha}} W_{ki}, \quad \nabla_{\dot{\alpha}} W_{ij} \equiv H_{ij} . \tag{11}$$

- **Solution to the $\dim = 2$ Bianchi identities:**

  $$\nabla_{\alpha i} G_{\beta j} = -\varepsilon_{\alpha \beta} H_{ij} - \frac{1}{2} \Omega_{ij} F_{\alpha \beta} + \frac{1}{2} \varepsilon_{\alpha \beta} [W_{ik}, W_{jk}] , \tag{12}$$

  $$\nabla_{\dot{\alpha} i} \bar{G}_{\dot{\beta} j} = -\varepsilon_{\dot{\alpha} \dot{\beta}} \bar{H}_{ij} - \frac{1}{2} \bar{\Omega}_{ij} \bar{F}_{\dot{\alpha} \dot{\beta}} - \frac{1}{2} \varepsilon_{\dot{\alpha} \dot{\beta}} [\bar{W}_{ik}, W_{jk}] ,$$

  $$\nabla_{\alpha i} \bar{G}_{\dot{\alpha} j} = i\Omega_{ij} V_{\alpha \dot{\alpha}} - \nabla_{\alpha \dot{\alpha}} W_{ij}, \quad \nabla_{\dot{\alpha} i} \bar{G}_{\dot{\alpha} j} = i\Omega_{ij} V_{\alpha \dot{\alpha}} + \nabla_{\alpha \dot{\alpha}} W_{ij} . \tag{13}$$

- **Solution to the $\dim = \frac{5}{2}$ Bianchi identities:**

  $$\nabla_{\dot{\alpha} i} \bar{G}_{\dot{\alpha} i} = \bar{G}_{\dot{\alpha} i} G^\alpha_{\dot{\alpha} i} + [W_{ik}, \bar{G}^{\dot{\alpha} k}], \quad \nabla_{\dot{\alpha} i} \bar{G}^{\dot{\alpha} i} = \nabla^{\dot{\alpha} \dot{\beta}} \bar{G}_{\dot{\alpha} i} - [W_{ik}, G^{\dot{\alpha} k}] , \tag{14}$$

  $$\nabla_{\alpha i} V_{\alpha} = \sigma^m_{\alpha \dot{\alpha}} [W_{ik}, \bar{G}^{\dot{\alpha} k}] + i(\sigma_{mn})_{\alpha \dot{\alpha}} \nabla_n G_{\beta i} . \tag{15}$$

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7 Do not confuse the superfields $V_m, F_{mn}$ and others in relations (8) with fields $V_m, F_{mn}$ in the component action (1). Actually component fields $\phi_{ij}, \chi_{ik}, V_m, F_{mn}$ will be the lowest components of the superfields $W_{ij}, G_{\alpha i}, V_m, F_{mn}$.

8 Note that in the Abelian case the superfields $G_{\alpha i}, \bar{G}_{\dot{\alpha} i}$ satisfy the constraints $D^i G_{\beta i} = D^i \bar{G}_{\dot{\alpha} i} = 0$. 

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\[ \nabla_{\bar{\alpha}} V_m = -\sigma^{m}_{\alpha\dot{\alpha}}[W_{ik}, G^{\alpha k}] + i\nabla_n \bar{G}^{\beta\dot{\alpha}}(\bar{\sigma}_{mn})^{\dot{\beta}} \dot{\alpha}, \]
\[ \nabla_{\alpha} H_{jk} = -i\Omega_{jk} \nabla_{\alpha\dot{\alpha}} \bar{G}_i^{\dot{\alpha}} - 2i\Omega_{ij} \nabla_{\alpha\dot{\alpha}} \bar{G}^{\dot{\alpha} k}_i - i[W_{jk}, G_{\alpha}] - i\Omega_{jk}[W_{i\dot{\alpha}}, G^{\alpha}_{\dot{\alpha}}] - 2i\Omega_{ij}[W_{k\dot{\alpha}}, G^{\alpha}_{\dot{\alpha}}], \]
\[ \bar{\nabla}_{\bar{\alpha}} H_{jk} = i\Omega_{jk} \nabla_{\alpha\dot{\alpha}} G_i^{\alpha i} + 2i\Omega_{ij} \nabla_{\alpha\dot{\alpha}} G_i^{\alpha i} + i[W_{jk}, \bar{G}^{\dot{\alpha} i}_{\dot{\alpha}}] + i\Omega_{jk}[W_{i\alpha}, \bar{G}^{\dot{\alpha} i}_{\dot{\alpha}}] + 2i\Omega_{ij}[W_{k\alpha}, \bar{G}^{\dot{\alpha} i}_{\dot{\alpha}}], \]
\[ \nabla_{\alpha} F_{\alpha\beta} = 2i\nabla_{(\alpha\beta)} \bar{G}_i^{\dot{\alpha} i}, \quad \nabla_{\gamma i} F_{\alpha\beta} = -2i\bar{\epsilon}_{\gamma(\alpha} \nabla_{\beta)\dot{\alpha}} \bar{G}_i^{\dot{\alpha} i}, \quad (16) \]
\[ \nabla_{\alpha} F_{\alpha\beta} = 2i\nabla_{(\alpha\beta)} \bar{G}_i^{\dot{\alpha} i}, \quad \nabla_{\dot{\gamma} i} F_{\alpha\beta} = -2i\bar{\epsilon}_{\dot{\gamma}(\alpha} \nabla_{\beta)\dot{\alpha}} \bar{G}_i^{\dot{\alpha} i}. \]

- Solution to the dim = 3 Bianchi identities:
\[ \nabla_m V_m = -\frac{i}{4}[W_{ik}, H^{ik}] + \frac{1}{2}\{G_{\alpha k}, G^{\alpha k}\} + \frac{1}{2}\{\bar{G}_{\dot{\alpha} k}, \bar{G}^{\dot{\alpha} k}\}, \quad (17) \]
\[ \nabla_z V_m = -\sigma^{m}_{\alpha\dot{\alpha}}(G^{\alpha i}, \bar{G}_i^{\dot{\alpha}}) - \frac{i}{4}[W_{ik}, \nabla_m W^{ik}] - \nabla_a F_{am}, \quad (18) \]
\[ \nabla_z H_{jk} = \Box W_{jk} - \frac{i}{2} \Omega_{jk}\{G_{\alpha i}, G^{\alpha i}\} - 2i\{G_{\alpha j}, G^{\alpha}_k\} + \frac{i}{2} \Omega_{jk}\{\bar{G}_{\dot{\alpha} i}, \bar{G}^{\dot{\alpha} i}\} + 2i\{\bar{G}_{\dot{\alpha} j}, \bar{G}^{\dot{\alpha}}_k\} \quad (19) \]
\[ + \frac{1}{8} \Omega_{jk}[W_{il}[W^{i}_{\beta p}, W^{lp}_{\gamma}]]. \]

Using the covariant derivatives and solving the Bianchi identities, we can immediately write down the supersymmetry transformations (2) of the component fields in the form \( \delta \Phi = -\epsilon_{\alpha i} \nabla_{\alpha i} \Phi \), where \( \Phi \) is any of the superfields under consideration. The central charge transformations (3) are realized on \( \nabla_M \) and superfields \( \Phi \) as \( \delta_z \nabla_M = [\omega \nabla_z, \nabla_M] \) and \( \delta_z \Phi = \omega \nabla_z \Phi \) with \( \omega \) being a constant parameter, corresponding to global transformation of the central charge coordinate \( \delta z = \omega \).

By analogy with the case of the \( N = 2 \) models with the intrinsic central charge [12]-[15], the equations (14, 18, 19) can be called the generalized Dirac equation, the Yang-Mills equation and the Klein-Gordon equation respectively. Thus the central charge plays the role of ‘fifth coordinate’. As a consequence of (9)-(19), we can construct the complete power expansion of the \( W^{ij} = \sum_{k=0}^{\infty} W^{ij}_{(k)} z^k \) in superspace from the first two coefficients \( W^{ij}_{(0)}, W^{ij}_{(1)} \). The set of constraints (9)-(19) shows that the component fields in [1] are completely determined by the lowest orders in the expansion \( W_{ij}(x, \theta, \bar{\theta}, z) \) over \( z, \theta, \bar{\theta} \). Then it is clear that we can completely fix the dependence of all quantities under consideration on the central-charge coordinate \( z \), as was done for the theories with \( N = 2 \) rigid supersymmetry with the central charge [12], [14], [15].

Now we clarify how the conditions (5) appear in the superfield approach. It is known from [1] that the equations (5) allow us to reduce the \( N = 4, USp(4) \) SYM theory to the conventional \( N = 4, SU(4) \) SYM theory. The relations (5) are the solutions of equations of motion for auxiliary fields \( V_m, H_{ij} \) with help of constraint (4). Therefore, if we want to get the analogous relation between the above two theories in the superfield approach, we also should use, besides the identities (9) - (17), some additional equations or restrictions. We take such restrictions in the form
\[ \partial_z \Gamma_{m,z} = 0, \quad \partial_z W_{ij} = 0 \quad (20) \]
and show that these conditions are equivalent to (5). Consider the superfields $V_m$ and $H_{ij}$ defined by the identities (8) and (11): $iV_m = [\nabla_m, \nabla_z], H_{ij} = [\nabla_z, W_{ij}]$. Let $V_m \sim V_m|\theta=0$, $\phi_{ij} \sim W_{ij}|\theta=0$, $A_5 \sim \Gamma_\alpha|\theta=0$ and use the constraints (20). Then we immediately get the relations (5). As a result, we see how the conditions (5) are obtained in the superfield approach. They are the consequences of the restrictions (20), which should be added to the identities (8), (11) if we want to reduce the $USp(4)$ SYM theory to the on-shell $SU(4)$ SYM theory. In this case the identities (14), (17), (18), (19) are converted into equations of motion for conventional $\mathcal{N} = 4$ SYM theory. If we do not impose the conditions (20), we have the $USp(4)$ superfield gauge theory satisfying the identities (9) – (19).

The constraints (10) allow us to derive the important consequences. Specifying concrete values of the indices $i, j, k$ and using explicit form of the matrix $\Omega$ one obtains the various constraints for the superfield $W_{ij}$. For example

$$\nabla_{\alpha i} W_{12} = 0, \quad \nabla_{\alpha 2} W_{12} = 0,$$

and the other analogous equations. These equations mean the very special dependence of the superfield $W_{ij}$ on anticommuting coordinates

$$W_{12} = W_{12}(\theta^3, \theta^4, \bar{\theta}_2, \bar{\theta}_1), \quad W_{13} = W_{13}(\theta^2, \theta^4, \bar{\theta}_3, \bar{\theta}_1),$$

$$W_{24} = W_{24}(\theta^1, \theta^3, \bar{\theta}_4, \bar{\theta}_2), \quad W_{34} = W_{34}(\theta^1, \theta^2, \bar{\theta}_4, \bar{\theta}_3),$$

$$W_{14}(\theta^1, \bar{\theta}_i) + W_{23}(\theta^1, \bar{\theta}_i) = 0.$$

The coordinates $x^m$ and $z$ are not written down. As a result we see that $W_{ij}$ with different values $i, j$ belong to different subspaces of the full superspace. A more transparent covariant solution of such constraints will take place in the framework of the harmonic superspace.

3 Gauge theory in $USp(4)$ harmonic superspace

3.1 Harmonic formalism

Following the general scheme of harmonic superspace construction [6] we extend the $\mathcal{N} = 4$ central charge superspace with coordinates $Z^M = (x^m, z, \theta_{\alpha i}, \bar{\theta}_i^\alpha)$, by the eight-dimensional coset space $USp(4)/U(1) \times U(1)$ parametrized by the harmonic variables $u_i^{(\pm, 0)}, u_i^{(0, \pm)}$, which are inert under supersymmetry and take the values in the fundamental representation of $USp(4)$ [19], [18] (see Appendix B for details). Using the harmonics $u_i^{(\pm, 0)}, u_i^{(0, \pm)}$, we define the harmonic derivatives $\partial^{(q_1, q_2)}$, which are left-invariant vector fields on $USp(4)$, by the rule

$$\partial^{(\pm, 0)} = u_i^{(\pm, 0)} \frac{\partial}{\partial u_i^{(\pm, 0)}}, \quad \partial^{(0, \pm)} = u_i^{(0, \pm)} \frac{\partial}{\partial u_i^{(0, \pm)}},$$

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9 Structure of the harmonic variables for $\mathcal{N} = 2, 3, 4$ harmonic superspaces with various R-symmetries is discussed in [19].

10 Alternative formalism for the superfield description of this theory can, in principle, be based on the $SO(5)/U(2)$ harmonic superspace or on the $SO(5)/U(1) \times U(1)$ harmonic superspace [20].
\[ \partial^{(\pm,\pm)} = u^{(\pm,0)}_{i} \frac{\partial}{\partial u^{(\pm,0)}_{i}} + u^{0,\pm}_{i} \frac{\partial}{\partial u^{0,\pm}_{i}}, \quad \partial^{(\pm,\mp)} = u^{(\pm,0)}_{i} \frac{\partial}{\partial u^{(\pm,0)}_{i}} - u^{(0,\mp)}_{i} \frac{\partial}{\partial u^{(0,\mp)}_{i}}. \]

The commutation relations for the step operators (23) together with the two Cartan generators of the algebra \( usp(4) \) are given by (B.11). The algebra \( usp(4) \) is ten-dimensional and its rank equals two. Therefore there are 4 positive roots, 2 of which are simple roots and similarly for the negative roots. Also, there is a special involution (special complex conjugation)

\[ u^{(\pm,0)}_{i} = u^{(0,\pm)}_{i}, \quad u^{(0,\pm)}_{i} = -u^{(\pm,0)}_{i}, \quad (24) \]

and so on, allowing us to define a reality condition in harmonic superspace [19], [18].

With the help of the harmonics \( u^{(\pm,0)}_{i} \), \( u^{(0,\pm)}_{i} \) one can convert the spinor covariant derivatives into the operators

\[ \nabla^{(\pm,0)}_{\dot{\alpha}} = u^{(\pm,0)}_{i} \nabla^{i}_{\dot{\alpha}}, \quad \nabla^{(0,\pm)}_{\dot{\alpha}} = u^{(0,\pm)}_{i} \nabla^{i}_{\dot{\alpha}}, \quad (25) \]

and reformulate the superalgebra (6) in another, clearer form. Now we can rewrite the constraints (21) as the constraints in harmonic superspace where they will have a form of analyticity conditions.

In the \( \tau \)-frame we have the obvious anticommuting relations

\[ \{ \nabla^{(\pm,0)}_{\dot{\alpha}}, \nabla^{(\pm,0)}_{\dot{\beta}} \} = 0, \quad \{ \nabla^{(0,\pm)}_{\dot{\alpha}}, \nabla^{(0,\pm)}_{\dot{\beta}} \} = 0, \quad (26) \]

\[ \{ \nabla^{(\pm,0)}_{\dot{\alpha}}, \nabla^{(\mp,0)}_{\dot{\alpha}} \} = \mp 2i \nabla_{\dot{\alpha}}, \quad \{ \nabla^{(0,\pm)}_{\dot{\alpha}}, \nabla^{(0,\mp)}_{\dot{\alpha}} \} = \mp 2i \nabla_{\dot{\alpha}}, \quad (27) \]

and the other anticommuting relations that define five harmonic projection of the tensor \( W^{ij} \)\(^{11}\)

\[ \{ \nabla^{(\pm,0)}_{\dot{\alpha}}, \nabla^{(0,-)}_{\dot{\beta}} \} = 2i \varepsilon_{\dot{\alpha} \dot{\beta}} \nabla_{\dot{\gamma}} \pm 2i \varepsilon_{\dot{\alpha} \dot{\beta}} W^{(0,0)}_{1}, \quad \{ \nabla^{(0,\pm)}_{\dot{\alpha}}, \nabla^{(0,-)}_{\dot{\beta}} \} = 2i \varepsilon_{\dot{\alpha} \dot{\beta}} \nabla_{\dot{\gamma}} \pm 2i \varepsilon_{\dot{\alpha} \dot{\beta}} W^{(0,0)}_{2}, \quad (28) \]

where:

\[ u^{(+,0)}_{i} u^{(-,0)}_{j} W^{ij} = W^{(0,0)}_{1}, \quad u^{(+,0)}_{i} u^{(0,+)}_{j} W^{ij} = W^{(+,+)}_{1}, \quad u^{(+,0)}_{i} u^{(0,-)}_{j} W^{ij} = W^{(+,-)}_{1}, \quad (28) \]

These definitions simply mean that \( W^{(q_{1}q_{2})} \) in \( \tau \)-basis depend linearly on harmonics \( u^{(\pm,0)}_{i}, u^{(0,\pm)}_{i} \) and all five harmonic projections of \( W^{ij} \) transform through each other under the action of symmetry generators. In particular

\[ \partial^{(+,0)} W^{(+,+)} = 0, \quad \partial^{(+,0)} W^{(+,-)} = W^{(+,+)}, \quad \partial^{(+,-)} W^{(+,-)} = -2W^{(0,0)}_{1}, \quad (29) \]

\(^{11}\)The upper sign here corresponds to \( \alpha \) in \( \dot{\alpha} \) and the lower sign corresponds to \( \dot{\alpha} \) in \( \dot{\alpha} \).
\[ \partial^{(-0)} W^{(+,-)} = W^{(-,+)} , \quad \partial^{(0,-)} W^{(+,+)} = W^{(+,-)} , \quad \partial^{(-,-)} W^{(+,+) = 2W^{(0,0)}} , \quad \partial^{(+,-)} W^{(+,+)} = \partial^{(-,+)} W^{(+,+)} = 0 . \]

The \( USp(4) \), \( \mathcal{N} = 4 \) harmonic superspace with coordinates \( \{ x^a, z, \theta^{(\pm,0)}, \theta^{(0,\pm)}, u \} \) contains several analytic subspaces of the full superspace with eight anticommuting coordinates. It can be checked that each of following four superfields lives in its own analytic subspace:

- \( W^{(+,+)}(\theta^{(+,0)}, \theta^{(0,+)}, \bar{\theta}^{(+,0)}, \bar{\theta}^{(0,+)}) \),
- \( \nabla^{(+,0)} W^{(+,+)} = \nabla^{(0,+)} W^{(+,+)} = \nabla^{(0,0)} W^{(+,+)} = 0 \), \hspace{1cm} (30)
- \( W^{(+,-)}(\theta^{(+,0)}, \theta^{(0,-)}, \bar{\theta}^{(+,0)}, \bar{\theta}^{(0,-)}) \),
- \( \nabla^{(+,0)} W^{(+,-)} = \nabla^{(0,-)} W^{(+,-)} = \nabla^{(0,0)} W^{(+,-)} = 0 \), \hspace{1cm} (31)
- \( W^{(-,+)}(\theta^{(-,0)}, \theta^{(0,+)}, \bar{\theta}^{(-,0)}, \bar{\theta}^{(0,+)}) \),
- \( \nabla^{(-,0)} W^{(-,+)} = \nabla^{(-,0)} W^{(-,-)} = \nabla^{(0,-)} W^{(-,+)} = 0 \), \hspace{1cm} (32)
- \( W^{(-,-)}(\theta^{(-,0)}, \theta^{(0,-)}, \bar{\theta}^{(-,0)}, \bar{\theta}^{(0,-)}) \),
- \( \nabla^{(-,0)} W^{(-,-)} = \nabla^{(0,-)} W^{(-,-)} = \nabla^{(0,-)} W^{(-,-)} = 0 \). \hspace{1cm} (33)

The coordinates \( x^m, z, u \) are manifestly not written down here. The relations (30)-(33) form the full list of constraints stipulated by the relations (10). In each of the above analytic subspaces the corresponding spinor derivatives become short. However, the dependence of the superfield \( W^{(0,0)}(\theta^{(\pm,0)}, \theta^{(0,\pm)}, \bar{\theta}^{(\pm,0)}, \bar{\theta}^{(0,\pm)}) \) on anticommuting coordinates is not restricted.

Acting on \( W^{(q_1,q_2)} \) by one, two, three or four spinor derivatives \( D^{(q_1,q_2)}_\alpha \), one obtains a set of relations which allow one to define the superstrength components. We do not write down all such relations. The relations that will be of further use have the form

\[ G^{(+,0)}_\alpha = i\nabla^{(+,0)} W^{(0,0)}_1 = \frac{i}{2} \nabla^{(0,+)} W^{(+,-)} = -\frac{i}{2} \nabla^{(0,-)} W^{(+,+)} , \hspace{1cm} (34) \]
\[ G^{(0,+)}_\alpha = -i\nabla^{(0,+)} W^{(0,0)}_1 = -\frac{i}{2} \nabla^{(+,0)} W^{(+,-)} = \frac{i}{2} \nabla^{(+,-)} W^{(+,+)} , \]
\[ \nabla^{(+,0)} G^{(+,0)}_\beta = \pm \varepsilon_{\dot{\alpha} \beta} [W^{(+,+)} W^{(+,-)}] , \hspace{1cm} (35) \]
\[ \nabla^{(-,0)} G^{(+,0)}_\beta = \varepsilon_{\dot{\alpha} \beta} H^{(0,0)} \pm \frac{1}{2} F_{\dot{\alpha} \beta} \pm \frac{1}{2} \varepsilon_{\dot{\alpha} \beta} \{ [W^{(+,+)} W^{(+,-)}] + [W^{(+,+)} W^{(+,-)}] \} , \]
\[ \nabla^{(0,-)} G^{(+,0)}_\beta = \varepsilon_{\dot{\alpha} \beta} H^{(+,-)} \pm \varepsilon_{\dot{\alpha} \beta} [W^{(0,0)} W^{(+,-)}, \nabla^{(0,-)} G^{(+,0)}_\beta = \nabla^{(0,-)} W^{(+,-)} , \]
Proceeding this way, we can find, in principle, the components of all harmonic projections of the superfield $W_{ij}$.\(^{12}\)

Further, we take the superfield $W^{(+,+)}$ as the basic superfield strength and construct superfield action in its terms. This superfield is a function on the analytic subspace of the harmonic superspace parameterized by

$$\{\zeta^M, u\} = \{x^m_A, z_A, \theta^{(0,+)}_\alpha, \theta^{(0,-)}_\alpha, \bar{\theta}^{(0,+)}_\alpha, \bar{\theta}^{(0,-)}_\alpha, u^{(\pm)}_i, u^{(0,\pm)}_i\} , \quad (36)$$

where

$$x^m_A = x^m - i\theta^{(-0)} \sigma^m \bar{\theta}^{(+)} - i\bar{\theta}^{(0,+)} \sigma^m \theta^{(-)} - i\theta^{(0,-)} \sigma^m \bar{\theta}^{(+)} - i\bar{\theta}^{(0,+)} \sigma^m \theta^{(-)} , \quad (37)$$

$$z_A = z + i\theta^{(-0)} \alpha \theta^{(+)} + i\bar{\theta}^{(0,+)} \alpha \bar{\theta}^{(-)} - i\bar{\theta}^{(0,+)} \alpha \bar{\theta}^{(+)} - i\bar{\theta}^{(0,+)} \alpha \bar{\theta}^{(-)} .$$

One can prove that the above analytic subspace is closed under the supersymmetry transformations and is real with respect to 'tilde'-conjugation (24). Hence, the analytic superfield $W^{(+,+)}$ also can be chosen real.

In the $\lambda$-frame the covariant spinor derivatives are

$$D^{(+,0)}_\alpha = \frac{\partial}{\partial \theta^{(-0)} \alpha} , \quad D^{(0,+)}_\alpha = \frac{\partial}{\partial \theta^{(0,-)} \bar{\alpha}} , \quad (38)$$

$$D^{(-,0)}_\alpha = -\frac{\partial}{\partial \theta^{(+0)} \alpha} + 2i\bar{\theta}^{(0)} \alpha \partial^{A} \bar{\alpha} - 2i\theta^{(0)} \alpha \partial^{A} \bar{\alpha} , \quad D^{(0,-)}_\alpha = -\frac{\partial}{\partial \theta^{(0,+) \alpha}} + 2i\bar{\theta}^{(0,-)} \alpha \partial^{A} \bar{\alpha} - 2i\theta^{(0,-)} \alpha \partial^{A} \bar{\alpha} ,$$

$$\tilde{D}^{(-,0)}_\alpha = -\frac{\partial}{\partial \theta^{(0,-)} \bar{\alpha}} - 2i\theta^{(0,-)} \alpha \partial^{A} \bar{\alpha} - 2i\bar{\theta}^{(0,-)} \alpha \partial^{A} \bar{\alpha} , \quad \tilde{D}^{(0,-)}_\alpha = -\frac{\partial}{\partial \theta^{(0,-)} \bar{\alpha}} - 2i\theta^{(0,-)} \alpha \partial^{A} \bar{\alpha} - 2i\bar{\theta}^{(0,-)} \alpha \partial^{A} \bar{\alpha} .$$

The harmonic derivatives $D^{(q_1,q_2)}$ in the analytic basis are presented in Appendix (B.14). It is obvious that the property (29) in the $\tau$-frame is becoming a requirement of the harmonic analyticity $D^{++} W^{(+,+)} = D^{(\pm,\mp)} W^{(+,+)} = D^{(0,++)} W^{(+,+)} = 0$ in the $\lambda$-frame.

Similarly, we can consider the solution of the Grassmann constraints (31)-(33) for other superfields $W^{(q_1,q_2)}$ by passing to the corresponding analytic coordinates.

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12It is easy to prove, analogously to [21], that $W^{(+,+)}$ in the absence of the central charge also satisfies the conditions of linearity with respect to each variable $\theta$: $D^{(-,0)}_\alpha D^{(+,0)}_\alpha W^{(+,+)} = 0$, since $D^{++} D^{(-,0)}_\alpha D^{(+,0)}_\alpha W^{(+,+)} = D^{(-,0)}_\alpha D^{(+,0)}_\alpha W^{(+,+)} = 0$.
3.2 Superfield action

In this subsection we formulate the superfield action. We show that such a superfield action is written in terms of superfield $W^{(+,+)}$ and exactly reproduces the component action (1).

The action under consideration must be gauge invariant, $\mathcal{N} = 4$ supersymmetric and invariant under central charge transformation. To find such action, one uses a prescription, which was formulated in the $\mathcal{N} = 2$ central charge harmonic superspace [15]. We will see that this prescription perfectly works in $USp(4)$, $\mathcal{N} = 4$ harmonic superspace.

We propose the superfield action for $USp(4)$ SYM theory in the form

$$ S \sim \text{tr} \int d\zeta d^4 \theta (\theta^{(0,+)} d(\bar{\theta}^{(0,+)} + (\bar{\theta}^{(0,+)} - (\theta^{(0,+)})(\bar{\theta}^{(0,+)} - (\theta^{(0,+)}))L^{(2,2)} ,$$

and show that it reproduces the component action from [1]. Here

$$ L^{(2,2)} = W^{(+,+)} W^{(+,+)} ,$$

is an analytic (30) and harmonically 'short' superfield (29). The analytic superspace dimensionless integration measure looks like

$$ d\zeta = d^4 x d^2 \theta d^2 \bar{\theta} ,$$

where $du$ denotes the left-right invariant measure on the $USp(4)/U(1) \times U(1)$ coset$^{13}$.

The action (39) is obviously gauge invariant. Also, this action is $\mathcal{N} = 4$ supersymmetric. The proof of this statement is analogous to one in $\mathcal{N} = 2$ theory with intrinsic central charges [15]. The $\mathcal{N} = 4$ supersymmetry coordinate transformations in the theory under consideration are

$$ \delta x^A_{\alpha \dot{\alpha}} = -2i(\epsilon^{\alpha \dot{\alpha}}(0,-) \bar{\theta}^{(0,+)} + \epsilon^{\alpha \dot{\alpha}}(0,+)) + \theta^{\alpha \dot{\alpha}}(0,+)) \Delta^{(0,-)}(0,-),$$

$$ \delta z_A = 2i(\epsilon^{\alpha \dot{\alpha}}(0,+)) + \epsilon^{\alpha \dot{\alpha}}(0,+)) \Delta^{(0,+)}(0,+) = 0 .$$

Under these transformations the action (39) transforms as follows

$$ \delta S |_{\epsilon,0} = \int d\zeta d^4 \theta ((\theta^{(0+,+)} - (\bar{\theta}^{(0+,+) + (\theta^{(0,+)})^2} - (\bar{\theta}^{(0+,+) + (\theta^{(0,+)})^2})^2 \frac{\partial L^{(2,2)}}{\partial z_A} \Delta^{(0,+)}(0+,+) + (\theta^{(0,+)})^2 - (\bar{\theta}^{(0+,+) + (\theta^{(0,+)})^2} ) \Delta^{(2,2)} ) \} .$$

After integrating by parts and using the identities

$$ D^{(0,+,+)} \delta z_A = 2i(\epsilon^{\alpha \dot{\alpha}}(0,+)) + \epsilon^{\alpha \dot{\alpha}}(0,+)) \Delta^{(0,+)}(0,+) = 0 ,$$

one gets

$$ \delta S |_{\epsilon,0} = - \int d\zeta d^4 \theta ((\theta^{(0+,+)} - (\bar{\theta}^{(0+,+) + (\theta^{(0,+)})^2} - (\bar{\theta}^{(0+,+) + (\theta^{(0,+)})^2})^2 \Delta^{(2,2)} ) \Delta^{(0,+)}(0+,+) = 0 .$$

$^{13}$See the details in Appendix B.
Analogously $\delta S|_{(0,-)} = 0$. As a result, the action (39) is $\mathcal{N} = 4$ supersymmetric.

Though the action (39) does not contain integration over $z$, it actually is $z$-independent due to the identities $D^{(++,0)}\mathcal{L}^{(2,2)} = D^{(0,++)}\mathcal{L}^{(2,2)} = 0$. Indeed, the $z$ derivative of (39) is

$$\frac{\partial}{\partial z} S \sim i \int d\zeta (-4,-4) du \left( (\dot{\theta}^{(0,+)})^2 - (\ddot{\theta}^{(0,+)})^2 \right) (\dot{\phi}^{(++,0)} - 2i\theta^{(++,0)} \sigma^m \bar{\varphi}^{(+,0)} \partial_m) \mathcal{L}^{(2,2)} = 0.$$

It is easy to see that the integrand in above relation is a total $x$ and $u$-derivative and disappears upon integration. Therefore this action is invariant under central charge transformation $\delta x^m_A = 0, \delta z_A = \omega$ with the rigid parameter $\omega$.

Now we rewrite the action (39) in component form. To do that we should integrate over harmonic and over all anticommuting coordinates. We take into account that on a gauge invariant quantities $D^2 = \nabla^2$ [4]. Also one uses the integration rule

$$\int d\zeta (-4,-4) du = \frac{1}{256} \int d^4x du^{(\pm,0)} du^{(0,\pm)} (\nabla^{(-,0)})^2 (\nabla^{(-,0)})^2 (\nabla^{(0,-)})^2 (\nabla^{(0,-)})^2.$$  \hspace{1cm} (43)

First, one integrates over anticommuting coordinates $\theta^{(0,+)}, \bar{\theta}^{(0,+)}$ and gets\(^{14}\)

$$S \sim \text{tr} \int d^4x du^{(\pm,0)} d^2\theta^{(+,0)} \mathcal{L}^{(++,0)} - \text{tr} \int du^{(\pm,0)} d^4x d^2\theta^{(+,0)} \mathcal{L}^{(++,0)},$$  \hspace{1cm} (44)

where

$$\mathcal{L}^{(++,0)} = \frac{1}{4} \int du^{(0,\pm)} ((\nabla^{(-,0)})^2 - (\nabla^{(-,0)})^2) \mathcal{L}^{(2,2)}$$  \hspace{1cm} (45)

$$= -\frac{1}{2} \int du^{(0,\pm)} (G^{(+,0)}{\bar{G}}^{(+,0)} + \bar{G}^{(+,0)} G^{(+,0)} - 2iH^{(++,0)} W^{(+,0)}).$$

Second, one integrates in (45) over harmonics $u_i^{(0,\pm)}$ and finds

$$\mathcal{L}^{(++,0)} = u^{(+,0)}(u^{(+,0)}) L_{ij},$$  \hspace{1cm} (46)

where

$$L_{ij} = -\frac{1}{2} (G^a_i G_{a \beta} + \bar{G}^a_i \bar{G}_{a \beta} + \frac{i}{2} H^i_k W_{jk}).$$

Third, using the table of integrals (B.18) over harmonic variables $u_i^{(\pm,0)}$ for the expression $\frac{1}{4}(\nabla^2)^2 \mathcal{L}^{(++,0)}$ one gets

$$S \sim -\frac{1}{20} \text{tr} \int d^4x (\nabla^a(i \nabla^j) - \nabla^a(i \nabla^j)) L_{ij}|_{\theta = 0} = \text{tr} \int d^4x \mathcal{L},$$  \hspace{1cm} (47)

where the integrand is as follows:

$$\mathcal{L} = -\frac{1}{40} (\nabla^3 i \nabla^j) G^a_i G_{a \beta} + \nabla^3 j G^{ai} \nabla^j G_{a \beta} - \nabla^3 G^a_i \nabla^j G_{a \beta}.$$  \hspace{1cm} (48)

\(^{14}\)Here the identities $\frac{1}{4}(D^{(0,-)})^2 (\theta^{(0,+)})^2 = 1, \frac{1}{4}(\bar{D}^{(0,-)})^2 (\bar{\theta}^{(0,+)})^2 = 1$ and the definitions (34), (35) are used.

12
component field theory [1], the supersymmetry transformations and the central charge. Also it was proved that the Lagrange multiple 
A transformation are the consequences of the Bianchi identities for the superfield strengths. The gauge theory in \( \text{USp} \) rigid central charge. Component formulation of this theory was given in [1]. We studied these conditions, some of Bianchi identities are converted into equations of motion for conventional \( \mathcal{N} = 4 \) superconnections \( \Gamma_{m,z} \) and superstrength \( W_{ij} \). Thus, all the transformation rules (2), (3) and the constraint (4) on the auxiliary fields \( V_m, H_{ij} \) are a consequence of the Bianchi identities. As a result, we finally derive the action (1) from the superfield action (39).

4 Summary

We have developed the harmonic superspace formulation of \( \mathcal{N} = 4 \) SYM theory with the rigid central charge. Component formulation of this theory was given in [1]. We studied the gauge theory in \( \text{USp}(4) \), \( \mathcal{N} = 4 \) superspace and showed that all the constraints on the component field theory [1], the supersymmetry transformations and the central charge transformation are the consequences of the Bianchi identities for the superfield strengths. Also it was proved that the Lagrange multiple \( A_5 \) and the expressions of auxiliary fields \( V_m \) and \( H_{ij} \) in terms of \( A_5 \), used in [1], have a natural origin as the conditions of central charge independence of gauge superconnections \( \Gamma_{m,z} \) and superstrength \( W_{ij} \). Under these conditions, some of Bianchi identities are converted into equations of motion for conventional \( \mathcal{N} = 4 \) SYM theory and thus the \( \text{USp}(4) \) SYM theory is transformed into conventional \( \text{SU}(4) \) SYM theory.

\[ L = -\frac{1}{4} F_{mn} F_{mn} - \frac{1}{2} V^m V_m + \frac{1}{8} H^i H_i + \frac{1}{8} \nabla^m W_{ij} \nabla_m W_{ij} + \frac{1}{16} [W_{ik}, W_{jl}] [W^i, W^{jl}] \] (49)

It is easy to see that each term in the action (49) has the corresponding analogus term in the action (1), that is eq. (49) coincides with eq. (1) up to the coefficients. But all such coefficients can be absorbed in the redefinition of the fields in (49). Besides, one points out that the relation (17) at \( z = 0 \) and switched off anticommuting variables exactly reproduces the constraint (4). Therefore the action (49) is automatically accompanied by the constraint (4). Thus, all the transformation rules (2), (3) and the constraint (4) on the auxiliary fields \( V_m, H_{ij} \) are a consequence of the Bianchi identities. As a result, we finally derive the action (1) from the superfield action (39).

\[ L = -\frac{1}{4} F_{mn} F_{mn} - \frac{1}{2} V^m V_m + \frac{1}{8} H^i H_i + \frac{1}{8} \nabla^m W_{ij} \nabla_m W_{ij} + \frac{1}{16} [W_{ik}, W_{jl}] [W^i, W^{jl}] \] (49)

\[ + i G^{\alpha i} \nabla_{\alpha \dot{\alpha}} \bar{G}_{\dot{\alpha}} + \frac{i}{2} [W_{ik}, G^{\alpha k}] G^{\alpha i} + \frac{i}{2} [W_{ik}, G^{k \dot{\alpha}}] \bar{G}_{\dot{\alpha}}^{\alpha i} \].

15 The results of calculations for each line in (48) are given in Appendix C.
We constructed the $USp(4)$, $\mathcal{N} = 4$ harmonic superspace and several corresponding analytic subspaces. It is proved that Bianchi identities in conventional and harmonic $\mathcal{N} = 4$ superspaces with central charge provide the superspace treatment of all components of the model [1]. The harmonic superspace under consideration allows us to introduce the analytic superfield strength $W^{(+,+)}$ (see Subsection 3.1), which is a basic object for a analytic superfield Lagrangian $\mathcal{L}^{(2,2)}$ (see Subsection 3.2). Gauge invariant, $\mathcal{N} = 4$ supersymmetric action, invariant under the central charge transformations is proposed and it is proven that this action reproduces the component action given in [1].

However, we should note that the superfield strength $W^{(+,+)}$ is not expressed yet in terms of unconstrained superfield prepotentials. The procedure, which allowed to construct harmonic superspace formulation of $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetric theories in terms of the unconstrained analytic superfields, does not work apparently, in its literal form, for $\mathcal{N} = 4$ SYM theory with the central charge. Finding the prepotentials requires developing the new approaches. Nevertheless we hope that the approach developed in this paper will be useful for understanding the possibilities to construct the unconstrained superfield formulation $\mathcal{N} = 4$ SYM theory with central charge.

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5 Appendix A: Notations and Conventions

In this Appendix we list the notations and conventions for spinor algebra and some useful formulas of the two-component spinor formalism [6], [5].

The spinor indices are raised and lowered by means of the antisymmetric symbols $\varepsilon_{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}$ which have the properties

$$\varepsilon_{\alpha\gamma} \varepsilon^{\gamma\beta} = \delta^\beta_\alpha, \quad \varepsilon_{\dot{\alpha}\dot{\gamma}} \varepsilon^{\dot{\gamma}\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \quad (A.1)$$

The $USp(4)$ indices are raised and lowered by means of the antisymmetric matrix $\Omega_{ij}$, $(\Omega_{ik} \Omega^{kj} = \delta^j_i)$ according to the rule

$$\psi^{\alpha i} = \varepsilon^{\alpha\beta} \Omega^{ij} \psi^{\beta j}, \quad \bar{\psi}^{\dot{\alpha} i} = \varepsilon^{\dot{\alpha}\dot{\beta}} \Omega^{ij} \bar{\psi}^{\dot{\beta} j}. \quad (A.2)$$
The matrix $\Omega_{ij}$ has the form

$$
\Omega_{ij} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

(A.3)

The products of anticommuting spinors are defined as follows:

$$
\theta^a \theta^b = -\frac{1}{2} \varepsilon^{a\beta} \theta^2, \quad \bar{\theta}^a \bar{\theta}_\beta = \frac{1}{2} \varepsilon_{a\beta} \bar{\theta}^2, \quad \bar{\theta}^a \bar{\theta}_\beta = -\frac{1}{2} \varepsilon_{a\beta} \bar{\theta}^2, \quad \bar{\theta}^a \bar{\theta}_\beta = \frac{1}{2} \varepsilon_{a\beta} \bar{\theta}^2,
$$

$$
\psi_{\alpha} \chi_{\beta} = \frac{1}{2} \varepsilon_{\alpha\beta} \psi_{\delta} \chi_{\delta} + \frac{1}{2} \psi_{(\alpha} \chi_{\beta)}, \quad \bar{\psi}_{\dot{\alpha}} \bar{\chi}_{\dot{\beta}} = -\frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\delta}} \bar{\chi}_{\dot{\delta}} + \frac{1}{2} \bar{\psi}_{(\dot{\alpha}} \bar{\chi}_{\dot{\beta})}.
$$

The sigma matrices $\sigma^m_{\alpha\dot{\alpha}}, \bar{\sigma}^m_{\dot{\alpha} \alpha} = \varepsilon^{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} \sigma^m_{\beta\dot{\beta}}, \sigma_a \bar{\sigma}_b = \eta_{ab} - i \sigma_{ab}$ have the basic properties

$$
\sigma^b_{\alpha\dot{\alpha}} \bar{\sigma}^\alpha_{\dot{\alpha}} = 2 \delta^b_a, \quad \sigma^m_{\alpha\dot{\alpha}} \sigma^\beta_{\dot{\beta} m} = 2 \delta^\beta_{\dot{\beta}} \delta^\alpha_{\dot{\alpha}},
$$

(A.4)

$$
\sigma_a \sigma_b \sigma_c = \eta_{ab} \sigma_c - \eta_{ac} \sigma_b + \eta_{bc} \sigma_a - i \varepsilon_{abcd} \sigma^d,
$$

$$
\sigma_a \sigma_b \sigma_c = \eta_{ab} \sigma_c - \eta_{ac} \sigma_b + \eta_{bc} \sigma_a + i \varepsilon_{abcd} \sigma^d,
$$

$$
\frac{i}{2} \varepsilon_{abcd} \sigma^{cd} = \sigma_{ab}, \quad \frac{i}{2} \varepsilon_{abcd} \sigma^{cd} = -\sigma_{ab},
$$

$$
\sigma^a_{\alpha\beta} \sigma^\gamma_{ab} = 4(\delta^\gamma_{\alpha\beta} + \delta^\beta_{\alpha\gamma}), \quad \sigma^\dot{\alpha}_{\dot{\alpha} \dot{\beta}} \sigma^\dot{\beta}_{\dot{\beta} \dot{\gamma}} = 4(\delta^\gamma_{\dot{\alpha}\dot{\beta}} + \delta^\dot{\beta}_{\dot{\alpha}\gamma}),
$$

$$
(\sigma_{mn})_{\alpha\beta} (\sigma_{ab})^\alpha_{\dot{\alpha}} = 2(\eta_{ma} \eta_{nb} + i \varepsilon_{mnab}), \quad (\bar{\sigma}_{mn})_{\dot{\alpha} \dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\alpha}}_{\alpha} = 2(\eta_{ma} \eta_{nb} - i \varepsilon_{mnab}).
$$

The relation between spinor and vector representations looks like

$$
x^\dot{\alpha}\alpha = x^a \sigma^a_{\alpha\dot{\alpha}} \quad x^a = \frac{1}{2} x^\dot{\alpha}\alpha \sigma^a_{\dot{\alpha} \alpha},
$$

(A.5)

$$
F^{\alpha\beta} = (\sigma_{mn})^{\alpha\beta} F^{mn}, \quad \bar{F}^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}} F^{mn}, \quad F_{mn} = \frac{1}{8} F^{\alpha\beta} (\sigma_{mn})_{\alpha\beta} + \frac{1}{8} \bar{F}^{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}}.
$$

The rigid covariant derivatives have the form

$$
D_m = \partial_m, \quad D^i_a = \frac{\partial}{\partial \theta^a_i} + i \bar{\theta}^i_{\dot{\alpha}} \partial_{\dot{\alpha} a} - i \theta^i_{\alpha} \partial_z, \quad \bar{D}_{\dot{i}} = -\frac{\partial}{\partial \theta^\alpha_{\dot{i}}} - i \theta^\alpha a \partial_{\dot{a} a} - i \bar{\theta}^\alpha_{\dot{a}} \partial_z, \quad (A.6)
$$

and satisfy the relations

$$
\{D^i_a, \bar{D}_{\dot{i} j} \} = -2i \delta^i_{\dot{j}} \partial_{\dot{a} a}, \quad \{D^i_a, D^j_{\dot{b}} \} = -2i \varepsilon_{\alpha\beta} \Omega^{ij}_{\alpha\beta} \partial_z, \quad \{\bar{D}_{\dot{i} a}, \bar{D}_{\dot{j} b} \} = 2i \varepsilon_{\dot{a} \dot{b}} \Omega_{ij} \partial_z. \quad (A.7)
$$

The expressions for the $N = 4$ supersymmetry generators are

$$
Q^i_{\alpha} = i \frac{\partial}{\partial \theta^a_i} + \bar{\theta}^i_{\dot{\alpha}} \partial_{\dot{a} a} - \theta^i \partial_z, \quad \bar{Q}^i_{\dot{\alpha} a} = -i \frac{\partial}{\partial \theta^a_i} - \theta^i \partial_{\dot{a} a} - \bar{\theta}^i_{\dot{a}} \partial_z. \quad (A.8)
$$
7 Appendix B. Harmonics for $USp(4)/U(1) \times U(1)$

Central charge breaks the $U(4)$ R-symmetry group of the $\mathcal{N} = 4$ superalgebra down to $USp(4)$ [9]. Therefore, to construct the corresponding harmonic superspace we should define the $USp(4)$ harmonic variable. The various cosets of the $USp(4)$ group were introduced and studied in [19], [18]. In the present work we are interested in the harmonics on the $USp(4)/U(1) \times U(1)$ coset. In this Appendix we describe the properties of such harmonics.

The $USp(4)$ harmonic variables are $4 \times 4$ unitary matrices $u^I_i$ with unit determinant preserving the antisymmetric tensor (symplectic metric) $\Omega^{ij}$ (A.3)

$$u^I_i u^I_j = \delta^I_j, \quad u^I_i \Omega^{ij} u^I_j = \Omega^{ij}, \quad (B.1)$$

These basic relations give us a possibility to convert $USp(4)$ tensors into $U(1)$ ones and vice versa, namely

$$u^{i(+),j(-)}_i u^{j(+),i(-)}_j = u^{i(0,+),j(0,-)}_i u^{j(0,-),i(0,+)}_j = 1, \quad (B.2)$$

and completeness conditions

$$\det(u) = \varepsilon^{ijkl} u^{i(j+,k,-)}_i u^{j(k+,l,-)}_j u^{k(l+,i,-)}_k u^{l(i+,j,-)}_l = 1, \quad (B.3)$$

These basic relations give us a possibility to convert $USp(4)$ indices into $U(1)$ ones and vice versa, namely

$$\psi^I_i = u^{i(+),j(-)}_i \psi^{j(+),i(-)}_j - u^{i(-),j(+)}_i \psi^{j(0,+),i(0,-)}_j + u^{i(0,+),j(0,-)}_i \psi^{j(0,-),i(0,+)}_j. \quad (B.4)$$

Thus, the harmonic variable may be treated as the vielbeins which will be used to transform $USp(4)$ tensors into $USp(4)$ singlets.

Besides the usual complex conjugation

$$\overline{(u^{(\pm),i}_i)} = \mp u^{(\mp,i)}_i, \quad \overline{(u^{(0,\pm)}_i)} = \mp u^{(0,\mp)}_i,$$

there is the following 'tilde'-conjugation for harmonics [19], [18]

$$\tilde{u}^{i(+),j(-)}_i = u^{i(0,\pm),j(-)}_i, \quad \tilde{u}^{i(-),j(+)}_i = u^{i(0,\pm),j(0,\pm)}_i, \quad \tilde{u}^{i(0,+),j(0,-)}_i = u^{i(0,\pm),j(0,\pm)}_i, \quad \tilde{u}^{i(0,-),j(0,+)}_i = \pm u^{i(0,\pm),j(0,\pm)}_i. \quad (B.5)$$
This operation involves a complex conjugate and one of the reflections in the Weyl algebra \( usp(4) \). It acts only on the tangent indices of harmonics and can be selected in several ways. Among the many possible definitions of combined conjugation, (B.5) is most suitable for our purposes. It is the conjugation which allows us to define real objects in harmonic superspace with \( USp(4) \) harmonics.

There are also important identities with harmonics

\[
\tilde{u}^{i(0,+)} \bar{u}^{j(0,-)} \varepsilon_{ijkl} = + u_{[k}^{(0,+)} u_{l]}^{(0,-)}, \quad \tilde{u}^{i(0,+)} \bar{u}^{j(0,-)} \varepsilon_{ijkl} = u_{[k}^{(0,+)} u_{l]}^{(0,-)}, \quad \text{(B.6)}
\]

\[
\tilde{u}^{i(0,+)} \bar{u}^{j(0,-)} \varepsilon_{ijkl} = - u_{[k}^{(0,+)} u_{l]}^{(0,-)}, \quad \tilde{u}^{i(0,+)} \bar{u}^{j(0,-)} \varepsilon_{ijkl} = - u_{[k}^{(0,+)} u_{l]}^{(0,-)}, \quad \text{and}
\]

\[
\tilde{u}^{i(0,+)} \bar{u}^{j(0,+)} \varepsilon_{ijkl} = - u_{[k}^{(0,+)} u_{l]}^{(0,+)}, \quad \tilde{u}^{i(0,+)} \bar{u}^{j(0,+)} \varepsilon_{ijkl} = - u_{[k}^{(0,+)} u_{l]}^{(0,+)}.
\]

For the anticommuting variables we define the harmonic projections

\[
\theta^I_{\alpha} = -\tilde{u}^{I\alpha} \theta_{\alpha}, \quad \bar{\theta}^I = u^I \theta^I_{\alpha} = -\bar{u}^{I\alpha} \bar{\theta}_{\alpha},
\]

and following the rules of conjugation

\[
\bar{\theta}^{(0,\pm)}_{\alpha} = \bar{\theta}^{(0,\pm)}_{\alpha}, \quad \bar{\theta}^{(0,\pm)}_{\alpha} = \bar{\theta}^{(0,\pm)}_{\alpha}, \quad \bar{\theta}^{(0,\pm)}_{\alpha} = -\theta^{(0,\pm)}_{\alpha}, \quad \bar{\theta}^{(0,\pm)}_{\alpha} = -\theta^{(0,\pm)}_{\alpha}.
\]

Analogously, we define the harmonic projections for covariant spinor derivatives

\[
D_{\alpha}^{(\pm,0)} = \pm \frac{\partial}{\partial \theta^{(\pm,0)\alpha}} + i \bar{\theta}^{(\pm,0)\dot{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(\pm,0)\alpha} \partial_{\alpha a} - i \bar{\theta}^{(0,\pm)\dot{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(0,\pm)\alpha} \partial_{\alpha a}, \quad D_{\alpha}^{(0,\pm)} = \pm \frac{\partial}{\partial \theta^{(0,\pm)\alpha}} + i \bar{\theta}^{(0,\pm)\dot{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(0,\pm)\alpha} \partial_{\alpha a} - i \bar{\theta}^{(0,\pm)\dot{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(0,\pm)\alpha} \partial_{\alpha a},
\]

\[
D_{\bar{\alpha}}^{\pm,0} = \pm \frac{\partial}{\partial \bar{\theta}^{(\pm,0)\bar{\alpha}}} - i \theta^{(\pm,0)\bar{\alpha}} \partial_{\alpha a} - i \bar{\theta}^{(\pm,0)\bar{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(0,\pm)\bar{\alpha}} \partial_{\alpha a} - i \bar{\theta}^{(0,\pm)\bar{\alpha}} \partial_{\dot{\alpha}a}, \quad D_{\bar{\alpha}}^{0,\pm} = \pm \frac{\partial}{\partial \bar{\theta}^{(0,\pm)\bar{\alpha}}} - i \theta^{(0,\pm)\bar{\alpha}} \partial_{\alpha a} - i \bar{\theta}^{(0,\pm)\bar{\alpha}} \partial_{\dot{\alpha}a} - i \theta^{(0,\pm)\bar{\alpha}} \partial_{\alpha a} - i \bar{\theta}^{(0,\pm)\bar{\alpha}} \partial_{\dot{\alpha}a}.
\]

They are also related by the conjugation as follows

\[
\bar{D}_{\alpha}^{(\pm,0)} = -\bar{D}_{\bar{\alpha}}^{(0,\pm)}, \quad \bar{D}_{\alpha}^{(0,\pm)} = -\bar{D}_{\bar{\alpha}}^{(\pm,0)}, \quad \bar{D}_{\alpha}^{(\pm,0)} = \bar{D}_{\bar{\alpha}}^{(\pm,0)}, \quad \bar{D}_{\alpha}^{(0,\pm)} = \bar{D}_{\bar{\alpha}}^{(0,\pm)}.
\]

The right action of \( usp(4) \) algebra on a space of harmonics \( u' \) is generated by the differential operators (23), (B.12). The commutation relations among these operators show that the \( \partial^{(++0)}, \partial^{(--),+}, \partial^{(+-),+}, \partial^{(0++)} \) are rising operators, \( \partial^{(--0)}, \partial^{(+,-)}, \partial^{(-,+)}, \partial^{(0--)} \) are lowering ones and \( j_1, j_2 \) are the Cartan generators in \( usp(4) \) algebra. There are also the following reality properties

\[
\bar{\partial}^{(++0)} = \partial^{(0++)}, \quad \bar{\partial}^{(--0)} = \partial^{(--0)}, \quad \bar{\partial}^{(+-+)} = \partial^{(+,-)}, \quad \bar{\partial}^{(++0)} = \partial^{(0++)}, \quad \bar{\partial}^{(--0)} = \partial^{(--0)}, \quad \bar{\partial}^{(+-+)} = \partial^{(+,-)}, \quad \bar{\partial}^{(0++)} = \partial^{(0++)}, \quad \bar{\partial}^{(--0)} = \partial^{(--0)}, \quad \bar{\partial}^{(+-+)} = \partial^{(+,-)}.
\]

It is easy to check that these derivatives satisfy the relations

\[
[j_1, \partial^{(q_1,q_2)}] = q_1 \partial^{(q_1,q_2)}, \quad [j_2, \partial^{(q_1,q_2)}] = q_2 \partial^{(q_1,q_2)}, \quad (B.11)
\]
where

$$\dot{j}_1 = u_i^{(+,0)} \frac{\partial}{\partial u_i^{(+,0)}} - u_i^{(-,0)} \frac{\partial}{\partial u_i^{(-,-)}}, \quad \dot{j}_2 = u_i^{(0,+)} \frac{\partial}{\partial u_i^{(0,+)}} - u_i^{(0,-)} \frac{\partial}{\partial u_i^{(0,-)}}.$$  \hspace{1cm} (B.12)

It is also useful to find the commutation relations of $R$-symmetry generators with harmonic projections of the spinor covariant derivatives

$$[\partial^{(\pm,\pm)}, D^{(\mp,0)}_\alpha] = D^{(\pm,0)}_\alpha, \quad [\partial^{(0,\pm)}, D^{(0,\mp)}_\alpha] = D^{(0,\pm)}_\alpha,$$

$$[\partial^{(\pm,\mp)}, D^{(\mp,\mp)}_\alpha] = D^{(\pm,0)}_\alpha, \quad [\partial^{(\pm,\mp)}, D^{(\mp,\pm)}_\alpha] = D^{(0,\pm)}_\alpha,$$

$$[\partial^{(-,+)}, D^{(+,-)}_\alpha] = D^{(+,0)}_\alpha, \quad [\partial^{(+,-)}, D^{(-,+)}_\alpha] = -D^{(0,-)}_\alpha,$$

$$[\partial^{(-,+)}, D^{(+,-)}_\alpha] = -D^{(0,+)}_\alpha, \quad [\partial^{(-,+)}, D^{(-,+)}_\alpha] = D^{(-,0)}_\alpha.$$

Besides, all derivatives are eigenvectors of the charge operators $j_1, j_2$.

In the analytic basis (36) the operators of the algebra $usp(4)$ read as:

$$D^{(+,0)} = \theta^{(+,0)} \frac{\partial}{\partial \theta^{(-,0)}} + \bar{\theta}^{(+,0)} \frac{\partial}{\partial \bar{\theta}^{(-,0)}} - 2i\theta^{(0,+)} \frac{\partial}{\partial \theta^{(0,-)}} + i(\theta^{(0,+)} \theta^{(0,-)} - \theta^{(0,+)} \bar{\theta}^{(0,-)}) \partial_z,$$

$$D^{(0,+,0)} = \theta^{(0,+)} \frac{\partial}{\partial \theta^{(0,-)}} + \bar{\theta}^{(0,+)} \frac{\partial}{\partial \bar{\theta}^{(0,-)}} - 2i\theta^{(0,+)} \frac{\partial}{\partial \theta^{(0,-)}} + i(\theta^{(0,+)} \theta^{(0,-)} - \theta^{(0,+)} \bar{\theta}^{(0,-)}) \partial_z,$$

$$D^{(\pm,\pm)} = \theta^{(\pm,\pm)} \frac{\partial}{\partial \theta^{(\pm,\pm)}} + \bar{\theta}^{(\pm,\pm)} \frac{\partial}{\partial \bar{\theta}^{(\pm,\pm)}} - 2i\theta^{(\pm,0)} \frac{\partial}{\partial \theta^{(\pm,0)}} + i(\theta^{(\pm,0)} \theta^{(\pm,0)} - \theta^{(\pm,0)} \bar{\theta}^{(\pm,0)}) \partial_z,$$

$$D^{(\pm,\mp)} = \theta^{(\pm,\mp)} \frac{\partial}{\partial \theta^{(\pm,\mp)}} + \bar{\theta}^{(\pm,\mp)} \frac{\partial}{\partial \bar{\theta}^{(\pm,\mp)}} - 2i\theta^{(\pm,0)} \frac{\partial}{\partial \theta^{(\pm,0)}} + i(\theta^{(\pm,0)} \theta^{(\pm,0)} - \theta^{(\pm,0)} \bar{\theta}^{(\pm,0)}) \partial_z,$$

$$D^{(-,0)} = \theta^{(-,0)} \frac{\partial}{\partial \theta^{(+,0)}} + \bar{\theta}^{(-,0)} \frac{\partial}{\partial \bar{\theta}^{(+,0)}} - 2i\theta^{(-,0)} \frac{\partial}{\partial \theta^{(+,0)}} + i(\theta^{(-,0)} \theta^{(+,0)} - \theta^{(-,0)} \bar{\theta}^{(+,0)}) \partial_z,$$

$$D^{(0,-)} = \theta^{(0,-)} \frac{\partial}{\partial \theta^{(0,)}} + \bar{\theta}^{(0,-)} \frac{\partial}{\partial \bar{\theta}^{(0,)}} - 2i\theta^{(0,-)} \frac{\partial}{\partial \theta^{(0,)}} + i(\theta^{(0,-)} \theta^{(0,-)} - \theta^{(0,-)} \bar{\theta}^{(0,-)}) \partial_z,$$

$$D^{(0,-)} = \theta^{(0,-)} \frac{\partial}{\partial \theta^{(0,-)}} + \bar{\theta}^{(0,-)} \frac{\partial}{\partial \bar{\theta}^{(0,-)}} - 2i\theta^{(0,-)} \frac{\partial}{\partial \theta^{(0,-)}} + i(\theta^{(0,-)} \theta^{(0,-)} - \theta^{(0,-)} \bar{\theta}^{(0,-)}) \partial_z.$$
\[ J_1 = j_1 + \theta^{(+,0)} \frac{\partial}{\partial \theta^{(+,0)}} + \bar{\theta}^{(+,0)} \frac{\partial}{\partial \theta^{(+,0)}} - \theta^{(-,0)} \frac{\partial}{\partial \theta^{(-,0)}} - \bar{\theta}^{(-,0)} \frac{\partial}{\partial \theta^{(-,0)}} , \]

\[ J_2 = j_2 + \theta^{(0,+)} \frac{\partial}{\partial \theta^{(0,+)}} + \bar{\theta}^{(0,+)} \frac{\partial}{\partial \theta^{(0,+)}} - \theta^{(0,-)} \frac{\partial}{\partial \theta^{(0,-)}} - \bar{\theta}^{(0,-)} \frac{\partial}{\partial \theta^{(0,-)}} . \]

The Grassmann and harmonic measure of integration over the USp(4) analytic harmonic superspace is \( d\zeta^{(-4,-4)} = dz^A dz^B d\bar{z}^\alpha d\bar{z}^{\bar{\alpha}} d\bar{\theta}^{(0,+) d\bar{\theta}^{(0,+)} d\bar{\theta}^{(0,+)} du , \) where the superscript \((-4, -4)\) refers to the \((q_1, q_2)\) charges. The measure is normalized so that

\[ \int d\zeta^{(-4,-4)} (\theta^{(+,0)})^2 (\bar{\theta}^{(0,+)} )^2 (\theta^{(+,0)})^2 (\bar{\theta}^{(0,+)} )^2 = 1 . \] (B.15)

Functions of harmonics are defined by their harmonic expansion

\[ f^{(q_1, q_2)}(u) = \sum_{k_1, k_2=0}^\infty f^{(1_{i1} \ldots 1_{k_1+q_1})} (j_1 \ldots J_{k_2+q_2}) u^{(+,0)}_{(i_1)} \ldots u^{(+,0)}_{(i_{k_1+q_1})} u^{(-,0)}_{(i_{k_1+q_1+1})} \ldots u^{(-,0)}_{(i_{k_2+q_2})} \]

\[ \times u^{(0,+)}_{(j_1)} \ldots u^{(0,+)}_{(j_{k_2+q_2})} u^{(0,-)}_{(j_{k_2+q_2+1})} \ldots u^{(0,-)}_{(j_{k_2+q_2+1})} . \] (B.16)

The direct consequences of the above definitions are the following rules

\[ \partial^{(+,0)} f^{(q_1, q_2)}(u) = 0 \rightarrow f^{(q_1, q_2)}(u) = 0, q_1 < 0; \quad \partial^{(0,+)} f^{(q_1, q_2)}(u) = 0 \rightarrow f^{(q_1, q_2)}(u) = 0, q_2 < 0 . \]

The harmonic integral is defined to select the USp(4) singlet

\[ \int du = 1, \quad \int du f^{(q_1, q_2)}(u) = \delta^{(q_1, 0)} \delta^{(0, q_2)} f^{(q_1, q_2)}(0) . \] (B.17)

Using this definition and the reduction identities [6] one can derive the integral table

\[ \int du u^{(+,0) i} u^{(-,0)} = \frac{1}{4} \delta^{ij} \int du u^{(+,0) i} u^{(+,0) j} u^{(-,0)} = \frac{1}{4} 5 \delta^{ij} , \quad \text{etc.} \] (B.18)

9 Appendix C.

The results of the transformations of each line in (48) are expressed as follows

- \[ 20 i \nabla_{\alpha} \bar{G}_i^{\alpha} G^{\alpha} - 2 H^{ij} H_{ij} + 5 F^{\alpha \beta} F_{\alpha \beta} - [W_{ik}, W_{j}^k][W^i_l, W^j_l] , \]

- \[ 24 i [W_{ik}, \bar{G}^{\alpha k}] G^{\alpha} - 20 i \nabla_{\alpha} \bar{G}^{\alpha} i G^{\alpha} + 20 V^{\alpha \alpha} V_{aa} - \nabla^{\alpha} W^{ij} \nabla_{\alpha} W_{ij} , \]

- \[ -24 i [W_{ik}, G^{\alpha k}] G^{\alpha} + 20 i \nabla_{\alpha} G^{\alpha} i \bar{G}^{\alpha} + 20 V^{\alpha \alpha} V_{aa} - \nabla_{\alpha} W^{ij} \nabla_{\alpha} W_{ij} , \]

- \[ 20 i \nabla_{\alpha} G^{\alpha i} \bar{G}_i^{\alpha} - 2 H^{ij} H_{ij} + 5 F^{\alpha \beta} F_{\alpha \beta} - [W_{ik}, W_{j}^k][W^i_l, W^j_l] , \]
\[ 40i \nabla_{\alpha\dot{\alpha}} \bar{G}_{i}^\dot{\alpha} G^{\alpha i} - 8H_{ij} H_{ij} - 4[W_{ik}, W_{j}^{k}][W_{l}^{i}, W_{j}^{l}] - 8\nabla^{m}W_{ij} \nabla_{m} W_{ij} - 16i[W_{ik}, G_{\dot{\alpha}}^{k}]G^{\alpha i} + 40i[W_{ik}, G_{\dot{\alpha}}^{k}]G^{\alpha i} , \]

\[ -40i \nabla_{\alpha\dot{\alpha}} G_{i}^{\alpha} \bar{G}_{i}^\dot{\alpha} - 8H_{ij} H_{ij} - 4[W_{ik}, W_{j}^{k}][W_{l}^{i}, W_{j}^{l}] - 8\nabla^{m}W_{ij} \nabla_{m} W_{ij} - 16i[W_{ik}, G^{\alpha k}]G^{i}_{\alpha} - 40i[W_{ik}, \bar{G}_{\dot{\alpha}}^{k}]G^{\alpha i} . \]

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