Admissible representations of simple affine vertex algebras

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Positive energy representations

- $\mathcal{V}$ – a $\mathbb{Z}$-graded vertex algebra
- $\mathcal{E}(\mathcal{V})$ – the category of graded $\mathcal{V}$-modules, $M \in \mathcal{E}(\mathcal{V})$ provided . . .
  - . . . is a $\mathcal{V}$-module
  - . . . is a $\mathbb{C}$-graded vector space
  - . . . $Y_M(a, z) = \sum_{n \in \mathbb{Z}} a^{M}_{(n)} z^{-n-1}$ has conformal dimension $m$ for $a \in \mathcal{V}_m$, i.e. $\deg a^{M}_{(n)} = -n + m - 1$
- $\mathcal{E}_+(\mathcal{V})$ – the category of positive energy $\mathcal{V}$-modules, $M \in \mathcal{E}_+(\mathcal{V})$ provided . . .
  - . . . $M$ belongs to $\mathcal{E}(\mathcal{V})$
  - . . . $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ with $M_{\lambda} \neq 0$
- the top degree component

\[
M = \bigoplus_{n=0}^{\infty} M_{\lambda+n} \rightarrow M_{\text{top}} = M_{\lambda}
\]
Zhu’s correspondence

- $A(\mathcal{V})$ – the Zhu’s algebra of $\mathcal{V}$
  - $A(\mathcal{V})$ is a unital associative algebra
  - $\pi_{\text{Zhu}} : \mathcal{V} \to A(\mathcal{V})$ a canonical surjective mapping
  - $M \in E_+(\mathcal{V}) \implies M_{\text{top}} \in A(\mathcal{V})$, the action of $\pi_{\text{Zhu}}(a) \in A(\mathcal{V})$ on $M_{\text{top}}$ is given through $a^M_{(\deg a - 1)}$ for $a \in \mathcal{V}$

- Zhu’s correspondence

\[
\begin{align*}
M & \mapsto M_{\text{top}} \\
\left\{ \text{simple positive energy} \right\} & \leftrightarrow \left\{ \text{simple } A(\mathcal{V})\text{-modules} \right\}
\end{align*}
\]
Affine vertex algebras

- $\mathfrak{g}$ – a simple Lie algebra
- $\mathfrak{b}$ – a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$
- $\kappa_\mathfrak{g}$ – the Cartan–Killing form on $\mathfrak{g}$, $\kappa_\mathfrak{g} = 2\mathfrak{h}^\vee \kappa_0$
- $\kappa$ – a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$, $\kappa = k\kappa_0$ for $k \in \mathbb{C}$
- $\hat{\mathfrak{g}}_\kappa$ – the affine Kac–Moody algebra associated to $\mathfrak{g}$ of level $\kappa$
  - $\mathfrak{g}_\kappa = \mathfrak{g}((t)) \oplus \mathbb{C}c$
  - $[a_m, b_n] = [a, b]_{m+n} + \kappa(a, b)\delta_{m,-n}c$, $a_n = a \otimes t^n$ for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$
  - $c$ is the cental element of $\hat{\mathfrak{g}}_\kappa$

- $\mathcal{V}_\kappa(\mathfrak{g})$ – the universal affine vertex algebra
  - $\mathcal{V}_\kappa(\mathfrak{g}) = U(\hat{\mathfrak{g}}_\kappa) \otimes U(\mathfrak{g}[[t]]) \oplus \mathbb{C}c \simeq U(\mathfrak{g} \otimes \mathbb{C} t^{-1} \mathbb{C}[t^{-1}]) \otimes \mathbb{C}c$
  - $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ for $a \in \mathfrak{g}$
  - $[a(z), b(w)] = [a, b](w)\delta(z-w) + \kappa(a, b)\partial_w \delta(z-w)$ for $a, b \in \mathfrak{g}$

- $\mathcal{L}_\kappa(\mathfrak{g})$ – the simple affine vertex algebra
- $\mathcal{V}_\kappa(\mathfrak{g})$ and $\mathcal{L}_\kappa(\mathfrak{g})$ are $\mathbb{N}_0$-graded vertex algebras
Zhu’s correspondence

- $\mathcal{V}_\kappa(g) \implies A(\mathcal{V}_\kappa(g)) \simeq U(g)$
  \begin{align*}
  \left\{ \text{simple positive energy } \mathcal{V}_\kappa(g)-\text{modules} \right\} & \leftrightarrow 1:1 \left\{ \text{simple } U(g)-\text{modules} \right\} \\
  \end{align*}

- $\mathcal{L}_\kappa(g) \implies A(\mathcal{L}_\kappa(g)) \simeq U(g)/I_\kappa(g)$, $I_\kappa(g)$ is a two-sided ideal of $U(g)$
  \begin{align*}
  \left\{ \text{simple positive energy } \mathcal{L}_\kappa(g)-\text{modules} \right\} & \leftrightarrow 1:1 \left\{ \text{simple } U(g)/I_\kappa(g)-\text{modules} \right\} \\
  \end{align*}

- $E$ is a $U(g)/I_\kappa(g)$-module $\iff$ $E$ is a $g$-module, $I_\kappa(g) \subset \text{Ann}_{U(g)}E$
**Associated variety**

- $U(\mathfrak{g}) \implies \text{gr } U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$
- $I$ – a left ideal of $U(\mathfrak{g})$
- $\mathcal{V}(I)$ – the associated variety of $I$, $\mathcal{V}(I) \subset \mathfrak{g}^*$

$$\mathcal{V}(I) = \text{Specm}(S(\mathfrak{g})/\text{gr } I)$$

- $I$ is a two-sided ideal $\implies \mathcal{V}(I)$ is invariant under the adjoint action of $G$ (connected algebraic group with its Lie algebra $\mathfrak{g}$)
- $L^g_{\mathfrak{b}}(\lambda)$ – the simple highest weight module with highest weight $\lambda \in \mathfrak{h}^*$
- $I$ – a primitive ideal of $U(\mathfrak{g})$
  - there exists $\lambda \in \mathfrak{h}^*$ such that $I = \text{Ann}_{U(\mathfrak{g})} L^g_{\mathfrak{b}}(\lambda)$ [Duflo ’77]
  - $\mathcal{V}(I) = \overline{\mathcal{O}^*}$ for a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ [Joseph ’85]
- a simple $\mathfrak{g}$-module $E$ belongs to a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ if $\mathcal{V}(I) = \overline{\mathcal{O}^*}$
Zhu’s correspondence

- \( E \) is a simple \( U(\mathfrak{g})/I_\kappa(\mathfrak{g}) \)-module \( \iff \) \( E \) is a simple \( \mathfrak{g} \)-module, \( I_\kappa(\mathfrak{g}) \subset \text{Ann}_U(\mathfrak{g})E \)

- \( E \) is a simple \( \mathfrak{g} \)-module \( \implies \) \( \text{Ann}_U(\mathfrak{g})E = \text{Ann}_U(\mathfrak{g})L^g_b(\lambda) \) for some \( \lambda \in \mathfrak{h}^* \)

- The goal is to determine all \( \lambda \in \mathfrak{h}^* \) such that \( I_\kappa(\mathfrak{g}) \subset \text{Ann}_U(\mathfrak{g})L^g_b(\lambda) \)

- difficult problem in general

- known for generic and admissible levels
Admissible levels

- $\widehat{\mathfrak{g}}_\kappa$ – the affine Kac–Moody algebra
- $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes_\mathbb{C} \mathbb{C} \mathfrak{c} \oplus \mathbb{C} \mathfrak{c}$ – the Cartan subalgebra of $\widehat{\mathfrak{g}}_\kappa$
- $\widehat{\mathfrak{h}}^* \simeq \mathfrak{h}^* \oplus \mathbb{C} \Lambda_0$, $\Lambda_0|_{\mathfrak{h} \otimes_\mathbb{C} \mathfrak{c}} = 0$, $\Lambda_0(\mathfrak{c}) = 1$
- $\widehat{\Delta}(\lambda)$ – the integral root system of $\lambda \in \widehat{\mathfrak{h}}^*$

\[ \widehat{\Delta}(\lambda) = \{ \alpha \in \widehat{\Delta}^{\text{re}}; \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z} \}, \quad \widehat{\rho} = \rho + h^\vee \Lambda_0 \]

- a weight $\lambda \in \widehat{\mathfrak{h}}^*$ is admissible provided
  - i) $\lambda$ is regular dominant, that is $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}^{\text{re}}_+$
  - ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\Delta^{\text{re}}$

- a level $\kappa = k \kappa_0$ is admissible if $\lambda = k \Lambda_0$ is an admissible weight
- [Kac–Wakimoto ’89]

\[ k + h^\vee = \frac{p}{q} \text{ with } p, q \in \mathbb{N}, \ (p, q) = 1, \ p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases} \]
Admissible levels

Theorem (Arakawa ’15)

Let $\kappa = k\kappa_0$ be an admissible level for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then there exists a nilpotent orbit $\mathcal{O}_q$ of $\mathfrak{g}$ such that $\mathcal{V}(I_k) = \overline{\mathcal{O}_q^*}$.

- $E$ is a simple $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module which belongs to $\mathcal{O}$ \implies $I_\kappa(\mathfrak{g}) \subset \text{Ann}_{U(\mathfrak{g})} E \implies \mathcal{O} \subset \overline{\mathcal{O}_q}$
**Principal admissible weights**

- A weight $\lambda \in \hat{h}^*$ is principal admissible of an admissible level $k$ if there is an element $y$ of the extended affine Weyl group $\hat{W}$ of $\mathfrak{g}$ such that $\hat{\Delta}(\lambda) = y(\hat{\Delta}(k\Lambda_0))$

$$\overline{Pr}_k = \{ \lambda \in h^* ; \lambda + k\Lambda_0 \in Pr_k \}$$

**Theorem (Arakawa ’16)**

Let $\kappa = k\kappa_0$ be an admissible level for $\mathfrak{g}$. Then the simple $\mathfrak{g}$-module $L^\mathfrak{g}_b(\lambda)$ with $\lambda \in h^*$ is a $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module if and only if $\lambda \in \overline{Pr}_k$.

- The goal is to describe the set $\overline{Pr}_k$
- For $\mathfrak{g} = \mathfrak{sl}_2$ all highest weights were classified by [Adamović–Milas ’95, Dong–Li–Mason ’97]
Principal admissible weights

- $\mathcal{O}$ – a nilpotent orbit of $\mathfrak{g}$

$$\overline{\text{Pr}_k^\mathcal{O}} = \{ \lambda \in \overline{\text{Pr}_k}; \mathcal{V}(\text{Ann}_{U(\mathfrak{g})} L^\mathfrak{g}_\mathfrak{b}(\lambda)) = \overline{\mathcal{O}^*} \}$$

- decomposition

$$\overline{\text{Pr}_k} = \bigsqcup_{\mathcal{O} \subseteq \overline{\mathcal{O}_q}} \overline{\text{Pr}_k^\mathcal{O}},$$
Principal admissible weights for $\mathfrak{sl}_{n+1}$

- $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $k \in \mathbb{Q}$
  \[ k + n + 1 = \frac{p}{q} \quad \text{with} \quad p, q \in \mathbb{N}, \ (p, q) = 1, \ p \geq n + 1. \]

- $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ – the set of simple roots

- $\mathcal{P}_{n+1}$ – the set of partitions of $n + 1$, it parameterizes nilpotent orbits of $\mathfrak{g}$

- $\mathcal{O}_q = \mathcal{O}[q^r, s]$, $n + 1 = qr + s$, $r, s \in \mathbb{N}_0$, $0 \leq s \leq q - 1$
  \[ \overline{\mathcal{P}}_{r_k} = \bigcup_{\mathcal{O} \subset \mathcal{O}_q} \overline{\mathcal{P}}_{r_k}^{\mathcal{O}}, \]

- $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r] \in \mathcal{P}_{n+1}$
  \[ \sum_\lambda = \{\alpha_1, \ldots, \alpha_{\lambda_1-1}, \alpha_{\lambda_1+1}, \ldots, \alpha_{\lambda_1+\lambda_2-1}, \ldots, \alpha_{\lambda_1+\ldots+\lambda_{r-1}+1}, \ldots, \alpha_{\lambda_1+\ldots+\lambda_r-1}\} \]

- $\mathfrak{p}_\lambda$ – the standard parabolic subalgebra of $\mathfrak{g}$ determined by $\sum_\lambda$
Parabolic subalgebras

- $\mathfrak{p}$ – a standard (contains $\mathfrak{h}$) parabolic subalgebra of $\mathfrak{g}$
  - $\Sigma_p = \{\alpha \in \Pi; \mathfrak{g}_\alpha \subset \mathfrak{p}\}$
  - $\mathcal{O}_p = (G.p^\perp)^{\text{reg}}$ – the Richardson orbit attached to $\mathfrak{p}$
  - $W_p$ – the subgroup of $W$ generated by $s_\alpha$ for $\alpha \in \Sigma_p$
  - $W^p$ – the set of minimal coset representatives of $W_p \backslash W$
- equivalence relation $\sim$

$$p_1 \sim p_2 \iff \text{there exists } w \in W \text{ such that } \Delta_{\Sigma_1} = w(\Delta_{\Sigma_2})$$

- $[p]$ – the equivalence class attached to $\mathfrak{p}$
- $W_+$ – the subgroup of $W$ generated by the element $w_1 = s_1 s_2 \ldots s_n$
  - $w_1^{n+1} = e$
  - $W_+ = \{e, w_1, \ldots, w_n\}$, $w_j = w_1^j$
- equivalence relation $\sim_+$

$$p_1 \sim_+ p_2 \iff \text{there exists } w \in W_+ \text{ such that } \Sigma_1 = w(\Sigma_2).$$

- $[p]_+$ – the equivalence class attached to $\mathfrak{p}$
- $W^p_+ = \{w \in W_+; w(\Sigma_p) = \Sigma_p\}$
Theorem (Futorny, K., Morales)

Let $\mathcal{O}_\lambda$ be the nilpotent orbit of $\mathfrak{g}$ given by a partition $\lambda \in \mathcal{P}_{n+1}$. Then we have

$$\overline{\text{Pr}}^{\mathcal{O}_\lambda}_k = \bigcup_{[p] + \in [p_{\lambda t}] / \sim} \bigcup_{[w] \in W^+_p \setminus W^p} w^{-1} \cdot \Lambda_k(p),$$

where $W^+_p \setminus W^p$ denotes the set of orbits of $W^+_p$ on $W^p$.

$$\text{Pr}_{k,\mathbb{Z}} = \{ \lambda \in \widehat{\mathfrak{h}}^*; \lambda(c) = k, \langle \lambda, \alpha^\vee \rangle \in \mathbb{N}_0 \text{ for } \alpha \in \Pi, \langle \lambda, \theta^\vee \rangle \leq p - n - 1 \}$$

$$\Lambda_k(p) = \{ \lambda - (k + n + 1)\eta; \lambda \in \overline{\text{Pr}}^{\mathcal{O}_\lambda}_k, \eta \in P^\vee_+, (\eta, \theta) \leq q - 1, \Pi^\eta = \Sigma_p \}$$
Example 1

- $\mathfrak{g} = \mathfrak{sl}_6$, $\mathfrak{O} = \mathfrak{O}_{[4,2]}
- p_{[2,2,1,1]} = \circ \bullet \circ \circ \circ \circ
- [p_{[2,2,1,1]}] = \{ \circ \bullet \circ \circ \circ \circ, \circ \bullet \circ \circ \circ \circ, \circ \circ \bullet \circ \circ \circ, \circ \circ \circ \bullet \circ \circ, \circ \circ \circ \circ \bullet \circ, \circ \circ \circ \circ \circ \bullet \}

Equivalence classes

- $p_1 = \circ \bullet \circ \circ \circ \circ$, $p_2 = \circ \bullet \circ \circ \circ \\
[p_1]_+ = \{ \circ \bullet \circ \circ \circ \circ, \circ \bullet \circ \circ \circ \circ, \circ \bullet \circ \circ \circ \circ, \circ \bullet \circ \circ \circ \circ \}

W_{p_1}^+ = \{ e \}

[p_2]_+ = \{ \circ \bullet \circ \circ \circ \circ, \circ \bullet \circ \circ \circ \circ \}

W_{p_2}^+ = \{ e, w_3 \}$
Admissible weights

\[ \overline{\text{Pr}}^O_k = \bigsqcup_{w \in W^{p_1}} w^{-1} \cdot \Lambda_k(p_1) \sqcup \bigsqcup_{[w] \in W^{p_2}_+ \setminus W^{p_2}} w^{-1} \cdot \Lambda_k(p_2) \]

Representative sets

\[ \Lambda_k(p_1) = \left\{ \lambda - \frac{p}{q}(a\omega_2 + b\omega_4 + c\omega_5); \lambda \in \overline{\text{Pr}}_k, z, a, b, c \in \mathbb{N}, a + b + c \leq q - 1 \right\} \]

\[ \Lambda_k(p_2) = \left\{ \lambda - \frac{p}{q}(a\omega_2 + b\omega_3 + c\omega_5); \lambda \in \overline{\text{Pr}}_k, z, a, b, c \in \mathbb{N}, a + b + c \leq q - 1 \right\} \]
Example 2

- $g = sl_3$, $\mathcal{O} = \mathcal{O}_{[3]} = \mathcal{O}_{\text{reg}}$
- $p_{[1,1,1]} = \leftrightarrow = b$
- $[p_{[1,1,1]}] = \{ \leftrightarrow \}$
- Equivalence classes

\[ w_1 \rightarrow \therefore, \]

- $[b]^+ = \{ \leftrightarrow \}$, $W^b_+ = \{ e, w_1, w_2 \} = W_+$
Examples

Admissible weights

\[
\Pr_k^O = \bigsqcup_{[w] \in W^b_+ \setminus W^b} w^{-1} \cdot \Lambda_k(b)
\]

Representative sets

\[
\Lambda_k(b) = \left\{ \lambda - \frac{p}{q}(a\omega_1 + b\omega_2); \lambda \in \Pr_{k,Z}, a, b \in \mathbb{N}, a + b \leq q - 1 \right\}
\]

Hasse diagram for \(W^b\)

\[
\Pr_k^O = \Lambda_k(b) \sqcup s_1 \cdot \Lambda_k(b)
\]
Example 3

- $g = \mathfrak{sl}_4$
- Hasse diagram of nilpotent orbits

| $\mathcal{O}$ | dim | $\mathfrak{p}$ |
|---------------|-----|----------------|
| $\mathcal{O}_{[4]}$ | 12  |
| $\mathcal{O}_{[3,1]}$ | 10  |
| $\mathcal{O}_{[2^2]}$ | 8   |
| $\mathcal{O}_{[2,1^2]}$ | 6   |
| $\mathcal{O}_{[1^4]}$ | 0   |
Admissible weights

\[ \overline{\Pr_k} = \overline{\Pr_k}^{O\text{zero}} \sqcup \overline{\Pr_k}^{O\text{min}} \sqcup \overline{\Pr_k}^{O\text{rect}} \sqcup \overline{\Pr_k}^{O\text{subreg}} \sqcup \overline{\Pr_k}^{O\text{reg}} \]

\[ \overline{\Pr_k}^{O\text{zero}} = \overline{\Pr_k, \mathbb{Z}} \]

\[ \overline{\Pr_k}^{O\text{min}} = \bigsqcup_{w \in W_{\text{min}}} w \cdot \Lambda_k(p_{\alpha_1}^{\text{max}}), \quad W_{\text{min}} = \{e, s_1, s_2 s_1, s_3 s_2 s_1\} \]

\[ \overline{\Pr_k}^{O\text{rect}} = \bigsqcup_{w \in W_{\text{rect}}} w \cdot \Lambda_k(p_{\alpha_2}^{\text{max}}), \quad W_{\text{rect}} = \{e, s_2, s_1 s_2\} \]

\[ \overline{\Pr_k}^{O\text{subreg}} = \bigsqcup_{w \in W_{\text{subreg}}} w \cdot \Lambda_k(p_{\alpha_3}^{\text{min}}) \]

\[ W_{\text{subreg}} = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_3 s_2, s_1 s_2 s_1, s_3 s_1 s_2, s_3 s_2 s_1, s_2 s_3 s_1 s_2, s_3 s_1 s_2 s_1, s_2 s_3 s_1 s_2 s_1\} \]

\[ \overline{\Pr_k}^{O\text{reg}} = \bigsqcup_{w \in W_{\text{reg}}} w \cdot \Lambda_k(b), \quad W_{\text{reg}} = \{e, s_1, s_2, s_3, s_1 s_2, s_1 s_3\} \]
Examples

- **Representative sets**

\[
\overline{Pr}_{k,Z} = \{ \lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3; \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}_0, \\
\lambda_1 + \lambda_2 + \lambda_3 \leq p - 4 \}
\]

\[
\Lambda_k(p_{\alpha_1}^{\text{max}}) = \left\{ \lambda - \frac{p}{q} a \omega_1; \ \lambda \in \overline{Pr}_{k,Z}, \ a \in \mathbb{N}, \ a \leq q - 1 \right\}
\]

\[
\Lambda_k(p_{\alpha_2}^{\text{max}}) = \left\{ \lambda - \frac{p}{q} a \omega_2; \ \lambda \in \overline{Pr}_{k,Z}, \ a \in \mathbb{N}, \ a \leq q - 1 \right\}
\]

\[
\Lambda_k(p_{\alpha_3}^{\text{min}}) = \left\{ \lambda - \frac{p}{q} (a \omega_1 + b \omega_2); \ \lambda \in \overline{Pr}_{k,Z}, \ a, b \in \mathbb{N}, \ a + b \leq q - 1 \right\}
\]

\[
\Lambda_k(b) = \left\{ \lambda - \frac{p}{q} (a \omega_1 + b \omega_2 + c \omega_3); \ \lambda \in \overline{Pr}_{k,Z}, \ a, b, c \in \mathbb{N}, \\
\quad a + b + c \leq q - 1 \right\}
\]
Thank you for your attention!