HILBERT DESINGULARIZATIONS FOR THREE DIMENSIONAL CANONICAL CYCLIC QUOTIENT SINGULARITIES

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Abstract. In this paper, we shall discuss Hilbert property of Hilb\(^G\)(\(\mathbb{C}^3\)), Fujiki-Oka resolutions and iterated Fujiki-Oka resolutions for three dimensional canonical cyclic quotient singularities by using the classification shown by Ishida and Iwashita[11]. Finally, we shall prove that there exists a Hilbert desingularization for any three dimensional canonical cyclic quotient singularity.

1. Introduction

Ishida and Iwashita classified three dimensional canonical cyclic quotient singularities[11]. Let \(G\) be a finite cyclic group of \(GL(3, \mathbb{C})\). Then \(\mathbb{C}^3/G\) has a canonical singularity if

(i) \(G \subset SL(3, \mathbb{C})\),

(ii) \(G = \frac{1}{r}(a_1, a_2, a_3)\) with \(gcd(a_i, a_j) = 1\) and \(a_i + a_j = r\) for some \(1 \leq i < j \leq 3\),

(iii) \(G = \frac{1}{4k}(1, 2k + 1, 4k - 2)\) with \(k \geq 2\),

(iv) \(G = \frac{1}{5}(1, 4, 7)\) or \(\frac{1}{14}(1, 9, 11)\).

The class (ii) has the two cases

(a) \(\frac{1}{r}(1, a, r - a)\) with \(gcd(r, a) = 1\) and \(r > a\),

(b) \(\frac{1}{r}(1, r - 1, a)\) with \(gcd(r, a) > 1\) and \(r > a\).

The form (ii)-(a) is a symbolic form as a three dimensional toric terminal quotient singularity.

Theorem 1.1[19], (5.2) Theorem. A 3-fold cyclic quotient singularity is terminal if and only if it is of type \(\frac{1}{r}(1, a, r - a)\) with a coprime to \(r\).

We note that (ii)-(b) can be expressed as

\[ G = \frac{1}{dr}(1, dr - 1, ad) \]

where \(gcd(r, a) = 1\), \(r > a\) and \(d > 1\).

We shall compute an irreducible component Hilb\(^G\)(\(\mathbb{C}^3\)) of \(G\)-Hilbert scheme \(G\)-Hilb(\(\mathbb{C}^3\)) dominating \(\mathbb{C}^3/G\) (see Section 2), Fujiki-Oka resolutions (see Section 3) and iterated Fujiki-Oka resolutions (see Section 4) for these quotient singularities in each case. These resolutions are toric varieties.

Key words and phrases. Hilbert desingularizations, Fujiki-Oka resolutions, iterated Fujiki-Oka resolution, \(G\)-Hilbert schemes, Quotient singularities, Canonical singularities, Toric varieties, Dimension three.
Let \( N \cong \mathbb{Z}^n \) be a free \( \mathbb{Z} \)-module of rank \( n \geq 1 \), and let \( N_\mathbb{R} := N \otimes \mathbb{Z} \cong \mathbb{R}^n \). The age of an element \( \nu \) is defined as the sum of all components of \( \nu \in N \). Let \( \mathcal{H} \) be the affine hyperplane of level 1,
\[
\mathcal{H} := \{(x_1, x_2, \ldots, x_n) \in N_\mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 1\}.
\]
Let \( \mathfrak{s} \) be the \textit{junior simplex} for a rational strongly convex polyhedral cone \( \sigma \subset N_\mathbb{R} \),
\[
\mathfrak{s} := \sigma \cap \mathcal{H}.
\]
Moreover, the \textit{junior simplex} \( \mathfrak{s} \) for a fan \( \Delta \) is
\[
\mathfrak{s} := \bigcup_{\sigma \in \Delta} (\sigma \cap \mathcal{H}).
\]
Every \( \sigma \subset N_\mathbb{R} \) has the \textit{Hilbert basis} of \( \sigma \) with reference to \( N \)
\[
\text{Hilb}_N(\sigma) := \left\{ \nu \in \sigma \cap (N \setminus \{0\}) \mid \nu \text{ can not be expressed as the sum of two other vectors belonging to } \sigma \cap (N \setminus \{0\}) \right\}.
\]
Let \( \Delta \) be a fan in which all cones are rational strongly convex polyhedral, and let \( \Delta(r) \) be the set of all \( r \)-dimensional cones in \( \Delta \). If \( \rho \in \Delta(1) \), then there exists a primitive element \( P(\rho) \in N \cap \rho \) with \( \rho = \mathbb{R}_{\geq 0} P(\rho) \). Therefore, we have the set of minimal generators of \( \sigma \in \Delta \)
\[
\text{Gen}(\sigma) := \{P(\rho) \mid \rho \in \Delta(1), \rho \prec \sigma\}
\]
where the symbol \( \rho \prec \sigma \) means that the face \( \rho \) is contained in \( \sigma \) as its face. As a natural extension, we also define the set of minimal generators for a fan \( \Delta \)
\[
\text{Gen}(\Delta) := \bigcup_{\sigma \in \Delta} \text{Gen}(\sigma).
\]

\textbf{Definition 1.2.} The subdivision \( \Delta \) of \( \sigma \) is called a \textit{Hilbert desingularization} of \( \sigma \) if \( \Delta \) satisfies the following conditions:
\begin{itemize}
  \item \( \Delta \) is smooth,
  \item \( \text{Gen}(\Delta) = \text{Hilb}_N(\sigma) \).
\end{itemize}
Sometimes, this desingularization is simply written as \textbf{Hilb-desingularization}.

For the details of this definition, see Section 4 in [6]. The previous works on \textbf{Hilb}-desingularizations related to this paper are as follows.
\begin{itemize}
  \item For two dimensional toric singularity (i.e. cyclic quotient singularity \( \mathbb{C}^2/G \)), there exists a uniquely determined minimal resolution, and this resolution is a \textbf{Hilb}-desingularization.
  \item In dimension three, C. Bouvier and G. Sprinberg shows existence of \textbf{Hilb}-desingularization. However, it is not a unique [2].
  \item They give an example of non-existence of such desingularizations in dimension four [2].
  \item A crepant resolution is one of \textbf{Hilb}-desingularizations for toric quotient singularities in any dimension [7].
  \item For three dimensional terminal quotient singularities, Danilov [8] and Reid [19] introduce economic resolutions which is obtained by the sequence of weighted blow-ups. It coincides with a \textbf{Hilb}-desingularization.
\end{itemize}
Additionally, the goal of this paper is the following theorem.

**Theorem 4.3.** For any three dimensional canonical cyclic quotient singularity, there exists a Hilbert iterated Fujiki-Oka resolution.

2. **On G-Hilbert schemes $\text{Hilb}^G(\mathbb{C}^n)$**

We recall definition of $G$-graph. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ denote the coordinate ring of $\mathbb{C}^n$ and $M$ be the set of all monomials in $S$ and 1. Let $\rho_i$ be irreducible representation of $G$. The symbol $X^u$ denotes a monomial $x_1^{u_1} \cdots x_n^{u_n}$ in $M$ where $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0}$.

We write $\text{wt}(X^u) = \rho_i$ if $X^u(g \cdot p) = \rho_i(g)X^u(p)$ holds for any $g \in G$ and $p \in \mathbb{C}^n$. Since any monomial is contained in some $\rho_i$, we can define a map $\text{wt} : M \to \text{Irr}(G)$, where $\text{Irr}(G)$ is the set of irreducible representation of $G$. In this paper, we define $G$-graph by using the map "wt" for an ideal in $S$.

| $n=2$ | $\text{SL}(n, \mathbb{C})$ | $\text{GL}(n, \mathbb{C})$ |
|-------|-----------------|-----------------|
|       | a minimal resolution | (Ito, Nakamura[13]) | (Kidoh[16], Ishii[12]) |
| $n=3$ | a crepant resolutions | (Nakamura[17], Bridgeland, King, Reid[3]) | Singular in general |
|       | Not crepant in general | Example[10]: $G = \frac{1}{2}(1, 2, 3)$ | Example[10]: $G = \frac{1}{2}(1, 1, 1)$ |

Table 1: Properties of $\text{Hilb}^G(\mathbb{C}^n)$ for $\mathbb{C}^n/G$.

**Definition 2.1.** Let $I \subset S$ be an ideal, we define a subset $\Gamma(I) \subset M$ such that $\{X \in M \mid X \notin I\}$. A Subset $\Gamma(I)$ is called a $G$-graph if the restriction map $\text{wt}: \Gamma(I) \to \text{Irr}(G)$ is a bijection.

In other words, the condition $\{X \in M \mid X \notin I\}$ is equivalent to $X \in \Gamma$ and $X$ is divided by $Y \in M$, then $Y \in \Gamma$. This ideal is called a definition ideal.

**Definition 2.2.** Let $A_{\Gamma}$ be a set of minimal generators of $I(\Gamma)$. We define the map $\text{wt}_{\Gamma} : M \to G$-graph as $\text{wt}_{\Gamma}(X^u) = \tilde{X}^u$ such that $\text{wt}(X^u) = \text{wt}(\tilde{X}^u)$. For a $G$-graph $\Gamma$, we define $S(\Gamma)$ to be the subsemigroup of $M$ generated by $\frac{x^m \cdot \text{wt}_{\Gamma}(X^u)}{\text{wt}_{\Gamma}(m \cdot x)}$ for all $m \in M$ and $x \in \Gamma$. In addition, we define the rational cone

$$\sigma(\Gamma) := \{w \in N_\mathbb{R} \mid w \cdot X^u > w \cdot \text{wt}_{\Gamma}(X^u) \text{ for all } X^u \in A_{\Gamma}\},$$

where $w \cdot X^u$ means standard inner product $w \cdot u$ in $\mathbb{R}^n$.

$\text{Fan}(G)$ denotes the fan defined by all $n$-dimensional closed cone $\sigma(\Gamma)$ and all their faces in $N_\mathbb{R}$. The following theorem says that we can calculate an irreducible component $\text{Hilb}^G(\mathbb{C}^n)$ of $G$-Hilb scheme $G$-$\text{Hilb}(\mathbb{C}^n)$ dominating $\mathbb{C}^n/G$ by using $G$-graph.

**Theorem 2.3.** ([17], Theorem 2.11) The following hold.

- $\text{Fan}(G)$ is a finite fan with its support $\Delta$.
- The normalization of $\text{Hilb}^G(\mathbb{C}^n)$ is isomorphic to the toric variety determined by $\text{Fan}(G)$.
For $G = \frac{1}{r}(1, a, b)$, let $v_1 = \frac{1}{r}(1, a, b) \in N$ and $v_i = \frac{1}{r}(i, \bar{a}i, \bar{b}i)$ where $\bar{x} \equiv x \pmod{r}$.

The case of $G = \frac{1}{4}(1, 4, 7)$. The fan $\text{Fan}(G)$ consists 21 pieces of three dimensional cones. Let $u_1 = e_1 + v_3$, $u_2 = e_2 + v_3$ and $u_3 = e_3 + v_3$. Then these three points are not in Hilbert basis. It follows that $\text{Gen}(\text{Fan}(G)) \neq \text{Hilb}_N(\sigma)$.

**Figure 1.** $\bullet$ of $\text{Fan}(G)$ for $\frac{1}{4}(1, 4, 7)$.

Since $G$-Hilb($\mathbb{C}^3$) is reducible in the case of $\frac{1}{14}(1, 9, 11)$ (see [4]), the toric variety $\text{Hilb}^G(\mathbb{C}^3)$ is different from $G$-Hilb($\mathbb{C}^3$).

**Figure 2.** $\bullet$ of $\text{Fan}(G)$ for $\frac{1}{14}(1, 9, 11)$.

The following six elements in $\text{Gen}(\text{Fan}(G))$ are not in $\text{Hilb}_N(\sigma)$.

- $v_6 = v_2 + v_4$, $v_{10} = v_2 + v_8$, $v_{12} = v_4 + v_8$,
- $u_1 = e_1 + v_4$, $u_2 = e_2 + v_2$, $u_3 = e_3 + v_8$.

This triangulation is not a Hilb-desingularization.

**Proposition 2.4.** Let $G = \frac{1}{4k}(1, 2k + 1, 4k - 2)$ with $k > 1$. Then $\text{Hilb}^G(\mathbb{C}^3)$ is singular.
Proof. For the ideal \( I = (x^2, y^2, xy, z^{2k}, xz^k, yz^k) \), the subset \( \Gamma(I) \) is
\[
\Gamma(I) = \{1, z, z^2, \ldots, z^{2k-1}, x, xz, xz^2, \ldots, xz^{k-1}, y, yz, yz^2, \ldots, yz^{k-1}\}.
\]
In this case, the map \( wt \) satisfies
\[
wt(z) = \rho_{4k-2}, \quad wt(xz) = \rho_{4k-1}, \quad wt(yz) = \rho_{2k-3}.
\]
Since we have the restriction map as
\[
\text{Irr}(G) = \{1, \rho_{4k-2}, \rho_{4k-4}, \ldots, \rho_2, \rho_1, \rho_{4k-1}, \ldots, \rho_{4k-2(k-2)-1}, \rho_{2k+1}, \rho_{2k-3}, \ldots, \rho_3\},
\]
\( \Gamma(I) \) is a \( G \)-graph.

For the above \( \Gamma(I) \), we have the subsemigroup
\[
S(\Gamma(I)) = \langle x^{ad}z, y^{rd-ad}z, yz^x, xz^{y^{2k}} \rangle,
\]
and the corresponding toric variety satisfies
\[
X(N, \sigma(\Gamma(I))) \cong \mathbb{C}[X, Y, Z, W]/(XW - YZ).
\]
This is singular. \( \square \)

**Proposition 2.5.** Let \( G = \frac{1}{rd}(1, rd-1, ad) \) with \( \text{gcd}(r, a) = 1 \). Then \( \text{Hilb}^G(\mathbb{C}^3) \) is singular.

Proof. We fix \( \Gamma \subset \mathcal{M} \) as
\[
\Gamma = \{1, x, \ldots, x^{ad-1}, z, y, y^2, \ldots, y^{rd-ad-1}\}.
\]
In this case, the subsemigroup \( S(\Gamma) \) is
\[
S(\Gamma) = \langle \frac{x^{ad}}{z}, \frac{y^{rd-ad}}{z}, \frac{yz}{x^{ad-1}}, \frac{xz}{y^{rd-ad-1}} \rangle.
\]
Therefore, the corresponding toric variety satisfies
\[
X(N, \sigma(\Gamma)) \cong \mathbb{C}[X, Y, Z, W]/(XZ - YW),
\]
and this is singular. \( \square \)

Therefore, we have the following table.

| \( \text{Hilb}^G(\mathbb{C}^3) \) ]| \( \text{(i)} \) | a crepant resolution = a \text{Hilb-desingularization} |
|----------------|----------------|--------------------------------------------------|
| \( \text{(ii)-(a)} \) | \( \beta \) | Singular (O. Kedzierski[15]) |
| \( \text{(iii)} \) | | Singular |
| \( \text{(iv)} \) | | Not \text{Hilb-desingularizations} |

Table 2: Hilbert property of \( \text{Hilb}^G(\mathbb{C}^3) \)
3. On Fujiki-Oka resolutions

The quotient singularity whose corresponding cone in \( N_\mathbb{R} \) has at least one smooth facets as an affine toric variety is said to be semi-isolated. In other words, a semi-isolated quotient singularity is the one containing at least one smooth lines \( \mathbb{C}^n \) as affine toric subvarieties. The Fujiki-Oka resolution is a canonical resolution for semi-isolated quotient singularities. For the details, see [9], [18]. This resolution can be constructed algorithmically by the computation of the Ashikaga’s continued fraction for some proper fractions.

**Definition 3.1.** Let \( n \) be an integer greater than or equal to 1. Let \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) and \( r \in \mathbb{N} \) which satisfies \( 0 \leq a_i \leq r - 1 \) for \( 1 \leq i \leq n \). We call the symbol

\[
\frac{a}{r} = \frac{(a_1, \ldots, a_n)}{r}
\]

an \( n \)-dimensional proper fraction. Moreover, the proper fraction such that at least one components of \( a \) are 1 is said to be semi-unimodular.

**Note 3.2.** For the cyclic groups \( G \) in the cases (ii)-(β), (iii), and (iv) in the classification of three dimensional canonical cyclic quotient singularities in Section 1, the generator \( g \) of \( G \) is semi-unimodular if and only if

(iii) \( g = \frac{1}{d_1}(1, 2k + 1, 4k - 2) \) or \( \frac{1}{d_1}(2k + 1, 1, 4k - 2) \),

(iv) \( g = \frac{1}{d_1}(1, 4, 7), \frac{1}{3}(7, 1, 4), \frac{1}{4}(4, 7, 1), \frac{1}{11}(1, 9, 11), \frac{1}{11}(11, 1, 9) \) or \( \frac{1}{11}(9, 11, 1) \)

where \( \gcd(r, a) = 1, r > a, d > 1 \) and \( k \geq 2 \).

The symbol \( \mathbb{Q}_n^{\text{prop}} \) means the set of \( n \)-dimensional proper fractions and the formal element \( \infty \). Similarly, \( \mathbb{Z}_n := \mathbb{Z}^n \cup \{\infty\} \).

Ashikaga’s continued fraction consists of a round down polynomial and a remainder polynomial, and these polynomials are obtained via round down maps and remainder maps for a semi-unimodular proper fraction. For the details, see [1]. In this paper, the round down polynomials play important roles especially.

**Definition 3.3.** Let \( \frac{(1, a_2, \ldots, a_n)}{r} \) be a semi-unimodular proper fraction. For \( 2 \leq i \leq n \), the \( i \)-th remainder map \( R_i : \mathbb{Q}_n^{\text{prop}} \rightarrow \mathbb{Q}_n^{\text{prop}} \) is defined by

\[
R_i \left( \frac{(1, a_2, \ldots, a_n)}{r} \right) := \begin{cases} 
\frac{(1, \overline{a}_2, \ldots, \overline{a}_i, \ldots, \overline{a}_n)}{\overline{a}_i} & \text{if } a_i \neq 0 \\
\infty & \text{if } a_i = 0
\end{cases}
\]

and \( R_i(\infty) = \infty \) where \( \overline{a}_j^{a_i} \) is an integer satisfying \( 0 \leq a_i < a_j \) and \( \overline{a}_j^{a_i} \equiv a_j \) modulo \( a_i \).

**Definition 3.4.** Let \( \frac{a}{r} \) be an \( n \)-dimensional semi-unimodular proper fraction. The remainder polynomial \( R_\ast \left( \frac{a}{r} \right) \in \mathbb{Q}_n^{\text{prop}}[x_2, \ldots, x_n] \) is defined by

\[
R_\ast \left( \frac{a}{r} \right) := \frac{a}{r} + \sum_{(i_1, i_2, \ldots, i_l) \in \mathbb{N}^l} (R_{i_1} \cdots R_{i_l} R_{i_{l+1}}) \left( \frac{a}{r} \right) \cdot x_{i_1} x_{i_2} \cdots x_{i_l}
\]

where we exclude terms with coefficients \( \infty \) or \( (0, 0, \ldots, 0) \).

**Example 3.5.** Let \( X(N', \sigma) \) have a quotient singularity of \( \frac{1}{11}(1, 2, 8) \)-type, i.e., \( N' = \mathbb{Z}^3 + \mathbb{Z}_{11}(1, 2, 8) \) and \( \sigma = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0} e_3 \). Then, the cone \( \sigma \) is semi-unimodular over \( e_1 \),
and the Oka center is \( c = \frac{1}{11}(1, 2, 8) \), and the remainder polynomial of the proper fraction \( \frac{(1, 2, 8)}{11} \) is

\[
\mathcal{R}_* \left( \frac{(1, 2, 8)}{11} \right) = \frac{1}{11}(1, 2, 8) + \frac{1}{2}(1, 1, 0)x_2 + \frac{1}{8}(1, 2, 5)x_3 \\
+ \frac{1}{2}(1, 0, 1)x_3x_2 + \frac{1}{5}(1, 2, 2)x_3x_3 \\
+ \frac{1}{2}(1, 1, 0)x_3x_3x_2 + \frac{1}{2}(1, 0, 1)x_3x_3x_3.
\]

This expanding of Ashikaga’s continued fraction indicates that the toric variety after the blow-up with the Oka center \( \frac{1}{11}(1, 2, 8) \) has two semi-isolated quotient singularities of \( \frac{1}{2}(1, 1, 0) \)-type and \( \frac{1}{8}(1, 2, 5) \)-type. For these quotient singularities, the corresponding cones which appear in \( \sigma \) after the subdivision by \( \frac{1}{11}(1, 2, 8) \in \mathbb{N}' \) are

\[
\sigma_2 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}c + \mathbb{R}_{\geq 0}e_3 \quad \text{and} \quad \sigma_3 = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}c \quad \text{respectively}.
\]

\( \frac{1}{2}(1, 1, 0) \) and \( \frac{1}{8}(1, 2, 5) \) are the Oka center of semi-unimodular cones \( \sigma_2, \sigma_3 \) over \( e_1 \) respectively. Therefore, we can take blow-ups with the Oka centers again. The blow-up with Oka centers of \( X(\mathbb{N}', \sigma_3) \) consists a smooth toric variety and quotient singularities of \( \frac{1}{2}(1, 0, 1) \)-type and \( \frac{1}{8}(1, 2, 2) \)-type respectively. By repeating blow-ups with Oka centers, we have the smooth toric variety (see Fig. 3).

**Figure 3.** s of the basic triangulation by Fujiki-Oka resolution

We proved that all Fujiki-Oka resolutions are crepant for any three dimensional semi-isolated Gorenstein quotient singularity in [20].

**Corollary 3.6.** For all three dimensional semi-isolated Gorenstein quotient singularities, the Fujiki-Oka resolutions are crepant.

By Theorem 6.1 in [7], it follows that arbitrary crepant toric resolution for a Gorenstein abelian quotient singularity is a Hilb-desingularization. Therefore, arbitrary crepant Fujiki-Oka resolution is Hilb-desingularization for any three dimensional semi-isolated Gorenstein quotient singularity. This is the semi-isolated part of (i) in Section 1.

By Section 5 in [19], the economic resolution for the quotient singularity of \( \frac{1}{r}(1, a, r - a) \)-type where \( \gcd(r, a) = 1 \) (i.e., the three dimensional toric terminal quotient singularity) is a Hilb-desingularization. For that quotient singularity, the economic resolution coincides with the Fujiki-Oka resolution. Therefore, arbitrary Fujiki-Oka resolution is Hilb-desingularization even in the case of (ii)-(a).

For the quotient singularity of \( \frac{1}{4k}(1, 2k + 1, 4k - 2) \)-type where \( k \geq 2 \) (resp. \( \frac{1}{dr}(1, dr - 1, ad) \)-type where \( \gcd(r, a) = 1, r > a \) and \( d > 1 \), the number of the smooth maximal cones appearing in the toric fan of each Fujiki-Oka resolution is greater than or equal to \( 2|G| \) by the computation of the remainder polynomial of the proper fractions of the both types.
We note that this property does not depend on changing semi-unimodular generator of $G$ by Note 3.2. Let $E$ and $F$ be the numbers of the edges and faces in $s$ of the toric fan corresponding to the Fujiki-Oka resolution respectively. In these case, we have the equation

$$ E = \frac{1}{2}(3F + s + 2) $$

where $s = \gcd(4k, 4k + 2) = 2$ (resp. $s = d$). We note that $s > 1$. Let $V$ be the the number of the vertices in $s$. By Euler’s polyhedron formula, we have

$$ V = \frac{1}{2}(F + s + 4). $$

Since the above inequality $F \geq 2|G|$ and $s > 1$, we have

$$ V \geq |G| + 3. $$

On the other hand, the following inequality with respect to the number of the Hilbert basis $\text{Hilb}_N(\sigma)$

$$(1) \quad \#\text{Hilb}_N(\sigma) \leq |G| + 2$$

holds. Therefore, we have

$$ V \geq \#\text{Hilb}_N(\sigma) + 1. $$

This inequality says that the Fujiki-Oka resolution is not $\text{Hilb}$-desingularization in each case of (ii)-(β) and (iii).

For the quotient singularities of $\frac{1}{5}(1,4,7)$-type and $\frac{1}{11}(1,9,11)$-type, the numbers of the smooth maximal cones appearing in the toric fan of the Fujiki-Oka resolution are 23 and 43 respectively by the computation of the remainder polynomial of the proper fractions. The numbers of the smooth maximal cone are invariant under changing semi-unimodular generator of $G$ by Note 3.2. By Euler’s polyhedron formula and the relation of the number of faces and edges in the triangulation of simplex spanned by $e_1, e_2, e_3$ corresponding to the Fujiki-Oka resolution, the formula

$$ V = \frac{1}{2}(F + 5) $$

holds. Therefore, the number of the vertices $V$ in the simplex is 14 and 24 in each case. Clearly, the inequality (1) does not stand, and the Fujiki-Oka resolutions in the case (iv) are not $\text{Hilb}$-desingularization.

| Fujiki-Oka resolution |  
|-----------------------|
| s.i. part of (i) | a crepant resolution = a $\text{Hilb}$-singularization  
| (ii)-(α) (β) | a $\text{Hilb}$-desingularization (economic)  
| (iii) | Not $\text{Hilb}$-desingularizations  
| (iv) | Not $\text{Hilb}$-desingularizations  

Table 3: Hilbert property of Fujiki-Oka Resolutions
4. On iterated Fujiki-Oka resolutions

It is known that there exists an iterated Fujiki-Oka resolution for Gorenstein abelian quotient singularities in all dimensions by Lemma 4.2 in [20]. Naturally, the definition of this resolution can be extended to the cyclic quotient singularities. Let $G \subset GL(n, \mathbb{C})$ be a cyclic subgroup and $H$ be a component of a decomposition by cyclic subgroups of $G$. If the singularity $\mathbb{C}^n/H$ is semi-isolated, then we have the Fujiki-Oka resolution $(\widetilde{Y}_H, \text{FO}_1)$ and the toric partial resolution $(Y_G, \phi)$ satisfying the following diagram:

\[
\begin{array}{c}
\mathbb{C}^n \\
\downarrow \pi_H \\
\widetilde{Y}_H \\
\downarrow \pi_{G/H} \\
Y_G = \widetilde{Y}_H/(G/H) \\
\downarrow \phi \\
\mathbb{C}^n/G
\end{array}
\]

where $\pi_H$ (resp. $\pi_{G/H}$) is the quotient map by $H$ (resp. $G/H$). If all maximal cones in the toric fan of $Y_G$ are semi-unimodular, then we have a Fujiki-Oka resolutions $(\widetilde{Y}_G, \text{FO}_2)$ for the quotient singularities in $Y_G$.

We call the resolution $(\widetilde{Y}_G, \text{FO}_2 \circ \phi)$ an iterated Fujiki-Oka resolution for a cyclic quotient singularity $\mathbb{C}^n/G$. Clearly, an alternative Fujiki-Oka resolution can be seen an iterated Fujiki-Oka resolution.

We proved the existence of a crepant iterated Fujiki-Oka resolution for any three dimensional Gorenstein abelian quotient singularity [20].

**Corollary 4.1.** Assume that $G$ is a finite abelian subgroup of $SL(3, \mathbb{C})$. Then, a crepant iterated Fujiki-Oka resolution exists for $\mathbb{C}^3/G$.

For the case of (iii) and (iv), we have convenient subgroup to find Hilbert iterated Fujiki-Oka resolutions.

**Lemma 4.2.** Let $G$ be $\frac{1}{4k}(1, 2k + 1, 4k - 2)$ with $k > 1$, $\frac{1}{9}(1, 4, 7)$ or $\frac{1}{11}(1, 9, 11)$. Then, $G$ contains a cyclic subgroup in $SL(3, \mathbb{C})$.

Proof. If $G = \frac{1}{4k}(1, 2k + 1, 4k - 2)$ with $k > 1$, then $G$ contains

$$H_1 := \frac{1}{2k}(1, 1, 2k - 2).$$

If $G = \frac{1}{9}(1, 4, 7)$, then $G$ contains

$$H_2 := \frac{1}{3}(1, 1, 1).$$

If $G = \frac{1}{11}(1, 9, 11)$, then $G$ contains

$$H_3 := \frac{1}{7}(1, 2, 4).$$

$\square$
Since the generators $H_1$, $H_2$ and $H_3$ are semi-unimodular, we have Gorenstein cyclic quotient singularities as $\mathbb{C}^n/H$ in the above diagram.

**Theorem 4.3.** For any three dimensional canonical cyclic quotient singularity, there exists a Hilbert iterated Fujiki-Oka resolution.

Proof. Since a toric crepant resolution is a Hilb-desingularizations by the discussion in Section 3, arbitrary crepant iterated Fujiki-Oka resolution for Gorenstein cyclic quotient singularities is a Hilb-desingularization. This is in the case (i). The Hilbert property for the case of (ii)-(α) has been shown in Section 3. The remaining part is in the case of (iii) and (iv).

By Lemma 4.2, we have the Fujiki-Oka resolutions $(\tilde{Y}_{H_i}, FO_i)$ where $i = 1, 2, 3$. Then, $Y_G$ has a three dimensional toric terminal quotient singularity in each affine piece. By Theorem 1.1, arbitrary three dimensional toric terminal quotient singularity is $\frac{1}{r}(1, a, r-a)$-type for integers $r$ and $a$ where $0 < a < r$ and $\text{gcd}(r, a) = 1$. We note that the Hilbert basis in each affine piece is not necessary a part of the Hilbert basis globally. However, the age of elements which are not in Hilbert basis globally are greater than or equal to 2. Therefore, this discussion can be reduct to the case (i) in each affine piece, and we take Hilb-desingularizations as FO2. Then, we have an iterated Fujiki-Oka resolution $(\tilde{Y}_G, FO_2 \circ \phi)$ which is a Hilb-desingularization. □

By the above theorem, we have the following table.

| iterated Fujiki-Oka resolution |
|-----------------------------|
| (i) | a crepant resolution = a Hilb-desingularization |
| (ii)-(α) | a Hilb-desingularization (economic) |
| (β) | ³ a Hilb-desingularization |
| (iii) | ³ a Hilb-desingularization |
| (iv) | ³ a Hilb-desingularization |

Table 4: Hilbert property of iterated Fujiki-Oka Resolutions

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