ON LARGE DEVIATIONS OF TRAJECTORIES OF RANDOM WALKS UNDER THE CRAMÉR MOMENT ASSUMPTION

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Abstract. In 2013 A.A. Borovkov and A.A. Mogulskii proved a non-standard large deviations principle (LDP) for the trajectories of a random walk in $\mathbb{R}^d$ whose increments have the Laplace transform finite in a neighbourhood of zero. The rate function in this LDP has non-compact sub-level sets. In this note we give two ways to transform this result into a standard LDP. We also give an explicit integral representation of the rate function, obtained using methods of the calculus of variations. As an application of our results, we obtain standard LDPs for the perimeter and the area of the convex hull of the first $n$ steps of the random walk on the plane.

1. The Introduction

The study of large deviations of trajectories of random walks was initiated by A.A. Borovkov in the 1960’s. In 1976 A.A. Mogulskii [18] proved a large deviations result for the trajectories of a multidimensional random walk under the assumption that the Laplace transform of its increments is finite. In [18] Mogulskii also studied the large deviations under the weaker Cramér moment assumption when the Laplace transform is finite only in a neighbourhood of zero, but the practical use of his results appears to be significantly limited. It was not until the 2010s when A.A. Borovkov and A.A. Mogulskii [4, 5] obtained an accessible non-standard large deviations principle (LDP, in short) for the trajectories, which they call an extended LDP (Theorem A below). A significant issue encountered in [4, 5] is that the sub-level sets of the rate function are not compact. This explains why the upper bound in the extended LDP of [4, 5], where the infimum of the rate function is taken over shrinking neighbourhoods, is worse than the conventional one, where the infimum is taken over the closure; compare (3) with (4).

The difficulty in working under the Cramér moment assumption is that one has to consider trajectories of the random walk as elements of the space of functions of bounded variation in order to allow discontinuities. Essentially, this is needed because the rate function of the increments is not super-linear at infinity unless the Laplace transform of the increments is finite; cf. (17) and (18). This is in sharp contrast with the case of finite Laplace transform of the increments, where it suffices to work with absolutely continuous functions.

The current paper presents an attempt to bring the extended LDP of [4, 5] to a standard form. First, we use the insights from the calculus of variations, which offers well-developed...
methods for working with (integral) action functionals on the space of functions of bounded variation. We work with the weak-* and related topologies on this space instead of Skorokhod topologies $M_1$ and $M_2$ used in [4, 5]. This allows us to give an explicit integral representation (Theorem 1) for the rate function of Borovkov and Mogulskii [4], who gave it only in the one-dimensional case. This also lets us find a few classes of sets where the upper bound in the extended LDP of [4] coincides with the standard one (Proposition 2 and examples in Remark 3).

Our second take on standardization of the extended LDP of [4] explores the useful way to generate subsets of a functional space is by taking pre-images of a functional on this space. In the framework of LDP’s this corresponds to contraction principles. We present a contraction principle (Theorem 2) which transfers the extended LDP into a standard one. Our main application, which was the original motivation for this paper, concerns the perimeter and the area of the convex hull of a random walk on the plane (Proposition 3). Large deviations of these functionals were studied in detail by Akopyan and Vysotsky [1] in the case of finite Laplace transform of increments. For certain types of distributions of increments of the walk, we find the rate functions of the perimeter and the area explicitly (Proposition 4) using the integral representation of Theorem 1.

This paper is organized as follows. In Section 2 we give necessary definitions and state the results of Borovkov and Mogulskii. In Section 3 we define the weak-* and related topologies on the space of functions of bounded variation, and compare them with the closely connected Jakubowki topology $S$ and the Skorokhod topologies $M_1$ and $M_2$. Sections 4 and 5 contain our main results and their proofs, including applications for the perimeter and the area of convex hulls of planar random walks.

2. Notation and the extended LDP for trajectories

In this section we give the necessary definitions and provide a brief compact summary of the results of Borovkov and Mogulskii [4, 5].

2.1. Skorokhod topologies. We will write $x = (x^{(1)}, \ldots, x^{(d)})$ for the coordinates of $x \in \mathbb{R}^d$, $|x|$ for the Euclidean norm, and the dot ‘,’ for the scalar product on $\mathbb{R}^d$.

Denote by $D[0,1] = D([0,1];\mathbb{R}^d)$ the set of càdlàg functions on $[0,1]$, that is right-continuous $\mathbb{R}^d$-valued functions without discontinuities of the second kind. Denote by $D_0[0,1]$ the subspace of $D[0,1]$ of functions $h$ such that $h(0) = 0$. We will also consider the subspace $BV[0,1]$ of $D[0,1]$ of functions of bounded variation, and $BV_0[0,1] := D_0[0,1] \cap BV[0,1]$. The completed graph $\Gamma h$ of a function $h \in D[0,1]$ is a closed subset of $[0,1] \times \mathbb{R}^d$ defined as $\Gamma h := \bigcup_{t \in [0,1]} \{t\} \times [h(t-), h(t+)]$, where $h(0-) := h(0)$, $h(1+) := h(1)$ and $[u_1, u_2]$ denotes the line segment with the endpoints $u_1, u_2 \in \mathbb{R}^d$. We equip completed graphs with the topology induced from $[0,1] \times \mathbb{R}^d$.

We will consider several metrics and topologies on $D[0,1]$ and its subspaces. For the first one, we regard the completed graphs of càdlàg functions as the images of continuous curves in $\mathbb{R}^d$. Consider a set of parametrizations of the completed graph of a function $h \in D[0,1]$:

$$\Pi(h) := \{ \gamma \mid \gamma : [0,1] \to \Gamma h \text{ is bijective, continuous, and satisfying } \gamma(0) = (0, h(0)) \}.$$
The metric $\rho_1$ on $D[0, 1]$ is then defined as the least uniform distance between parametrized completed graphs, i.e.

$$\rho_1(h_1, h_2) := \inf_{\gamma_1 \in \Pi(h_1), \gamma_2 \in \Pi(h_2), t \in [0, 1]} \sup_{s \in [0, 1], x \in \Pi(h_i(s), h_i(x))} |\gamma_1(t) - \gamma_2(t)|, \quad h_1, h_2 \in D[0, 1]. \quad (1)$$

The topology generated by $\rho_1$ is the Skorokhod topology $M_1$; see Remarks 12.3.4 and 12.5.2 in the book by Whitt [21], where Chapter 12 gives a comprehensive treatise of the Skorokhod topologies $M_1$ and $M_2$.

The metric $\rho_2$ on $D[0, 1]$ is defined as the Hausdorff distance between completed graphs: for any $h_1, h_2 \in D[0, 1]$, put

$$\rho_2(h_1, h_2) := \max_{i=1, 2} \max_{s \in [0, 1], x \in \Pi(h_i(s), h_i(x))} \min_{t \in [0, 1], y \in \Pi(h_3(t), h_3(y))} |(s, x) - (t, y)|. \quad (2)$$

The topology generated by $\rho_2$ is the Skorokhod topology $M_2$. Finally, denote by $\rho$ the most common Skorokhod metric of time-changed uniform distance, which generates the topology $J_1$. We will use it only as a reference and will not really work with it.

From (1) and (2) we see that $\rho_2 \leq \rho_1$, hence $M_2 \subseteq M_1$. On the other hand, $M_1 \subseteq J_1$ (Billingsley [21, Theorem 12.3.2]). All the three topologies on $D[0, 1]$ are separable since so is $J_1$, and so are their induced versions on $BV_0[0, 1]$ (Billingsley [2 Section 14]). Finally, let us mention that the space $D[0, 1]$ is complete under neither of the three metrics $\rho, \rho_1, \rho_2$. However, the topologies $J_1$ and $M_1$ are completely metrizable, i.e. they are generated by some complete metrics $\rho'$ and $\rho'_1$ equivalent respectively to $\rho$ and $\rho_1$. These complete metrics are explicit (Billingsley [2, Section 14] and [21, Section 12.8]).

2.2. The extended LDP for the trajectories of random walks. We start with general definitions. Let $\mathcal{X}$ be a topological space equipped with the Borel $\sigma$-algebra. Let $\mathcal{I} : \mathcal{X} \rightarrow [0, \infty]$ be a lower semi-continuous function. By definition, this means that the sub-level sets $\{x \in \mathcal{X} : \mathcal{I}(x) \leq \alpha\}_{\alpha \in [0, \infty]}$ are closed. If $\mathcal{X}$ is a metric space (or a sequential space, introduced in Section 3.1 below), this reduces to $\mathcal{I}(x) \leq \lim \inf_{n \rightarrow \infty} \mathcal{I}(x_n)$ for any sequence $(x_n)_{n \in \mathbb{N}}$ converging to an $x \in \mathcal{X}$. We say that $\mathcal{I}$ is tight if its sub-level sets are compact.

We say that a sequence $(Z_n)_{n \geq 1}$ of random elements of $\mathcal{X}$ satisfies a large deviations principle (LDP) in $\mathcal{X}$ with the rate function $\mathcal{I}$ (and speed $n$) if for every Borel set $B \subset \mathcal{X}$,

$$\inf_{x \in \mathcal{X}} \mathcal{I}(x) \leq \lim \inf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq \lim \sup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq \inf_{x \in B} \mathcal{I}(x), \quad (3)$$

where, as usual, we agree that $\inf_{\varnothing} = +\infty$. If $\mathcal{X}$ is a metric space, we say, following Borovkov and Mogulskii [3], that $(Z_n)_{n \geq 1}$ satisfies an extended LDP in $\mathcal{X}$ if

$$\inf_{x \in \mathcal{I} \mathcal{B}} \mathcal{I}(x) \leq \lim \inf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq \lim \sup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(Z_n \in B) \leq \lim \inf_{\epsilon \rightarrow 0+} \inf_{x \in B^\epsilon} \mathcal{I}(x), \quad (4)$$

where $B^\epsilon$ denotes the open $\epsilon$-neighbourhood of $B$. The second definition is natural because if $\mathcal{I}$ is tight, then (3) and (4) are the same since (see [3 Lemma 1.1] or Dembo and Zeitouni [8, Lemma 4.1.6(b)])

$$\inf_{x \in \mathcal{I} \mathcal{B}} \mathcal{I}(x) = \lim_{\epsilon \rightarrow 0+} \inf_{x \in B^\epsilon} \mathcal{I}(x). \quad (5)$$

Note that for tight $\mathcal{I}$, the last infimum in (3) is always attained on some $x$. 


Now, let \((S_k)_{k \geq 1}\), where \(S_k = X_1 + \ldots + X_k\), be a random walk with independent identically distributed increments \(X_1, X_2, \ldots\) in \(\mathbb{R}^d\), where \(d \geq 1\). For any \(n \in \mathbb{N}\), let \(S_n(\cdot)\) be the piece-wise linear function on \([0, 1]\) defined by linear interpolation between its values at the points \(k/n, 0 \leq k \leq n\), that are given by \(S_n(k/n) = S_k\), where \(S_0 := 0\). These are time-rescaled trajectories of the random walk \(S\). We will regard them as random elements of the spaces \(D[0, 1]\) and \(BV[0, 1]\) equipped with the Borel \(\sigma\)-algebras generated by \(\rho_1\) or \(\rho_2\).

Let \(\mathcal{L}(u) := \mathbb{E}e^{u \cdot X_1}\), where \(u \in \mathbb{R}^d\), be the Laplace transform of the random vector \(X_1\) in \(\mathbb{R}^d\). Denote by \(D_{\mathcal{L}} := \{u \in \mathbb{R}^d : \mathcal{L}(u) < \infty\}\) the effective domain of \(\mathcal{L}\). We say that \(X_1\) satisfies the Cramér moment assumption if \(\mathcal{L}\) is finite in an open neighbourhood of 0, that is \(0 \in \text{int} \ D_{\mathcal{L}}\) in short. Denote by \(I\) the Legendre–Fenchel transform of \(\mathcal{L}\). This is a convex lower semi-continuous function from \(\mathbb{R}^d\) to \([0, \infty]\). The classical Cramér theorem states that under \(0 \in \text{int} \ D_{\mathcal{L}}\), the sequence \((S_n/n)_{n \geq 1}\) satisfies the LDP in the Euclidean space \(\mathbb{R}^d\) with the tight rate function \(I\). This justifies saying that \(I\) is the rate function of \(X_1\).

Recall that \(BV[0, 1] = BV([0, 1]; \mathbb{R}^d)\) denotes the set of right-continuous functions of bounded variation. Define the total variation \(\text{Var}(h)\) of an \(h \in BV[0, 1]\) as

\[
\text{Var}(h) := |h(0)| + \sup_{t \in [0, 1]: \#t < \infty} \int_0^1 |(h^t)'(s)| ds,
\]

where \(h^t\) denotes the continuous function on \([0, 1]\) defined by linear interpolation between its values at \(t \in \mathbb{R}^d\) that are given by \(h^t(t) := h(t)\). We can regard \(\text{Var}(h)\) as the length of \(h\) considered as a curve on \([0, 1]\) plus \(|h(0)|\). Similarly, define the non-negative functional \(I_D\) on \(D[0, 1]\) as \(I_D(h) := +\infty\) for \(h \not\in BV[0, 1]\) and

\[
I_D(h) := \sup_{t \in [0, 1]: \#t < \infty} \int_0^1 I((h^t)'(s)) ds, \quad h \in BV[0, 1].
\]

Recall that a subset of a metric space is totally bounded if it has a finite \(\varepsilon\)-net for every \(\varepsilon > 0\). Any totally bounded closed subset of a complete metric space is compact. On some occasions, we will use the subscript to indicate the metric (or, more generally, the topology) with respect to which the closure of a set is taken, e.g. write \(\text{cl}_{\rho_1}(B)\) instead of \(\text{cl} B\).

We now present the extended LDP for trajectories of random walks.

**Theorem A** (Borovkov and Mogulskii [1, 5]). Assume that \(X_1\) is a random vector in \(\mathbb{R}^d\), where \(d \geq 1\), such that \(0 \in \text{int} D_{\mathcal{L}}\). Then the sequence \((S_n(\cdot)/n)_{n \geq 1}\) satisfies an extended LDP \(\square\) in the metric spaces \((D[0, 1], \rho_i)\) for \(i = 1, 2\) with the (lower semi-continuous) rate function \(I_D\), which is convex and has totally bounded sub-level sets.

Moreover, \(I_D(h) \geq c_1 \text{Var}(h) - c_2\) for some constants \(c_1, c_2 > 0\) and any \(h \in D[0, 1]\).

Let us comment. Although \(I_D\) is not tight, the equality \(\square\) still holds for \(I_D\) in a number of cases discussed in Remark 3 below. Note that it is not the incompleteness of \((D[0, 1], \rho_1)\) which prevents \(I_D\) from being tight. In fact, the topology \(M_1\) generated by \(\rho_1\) is completely metrizable (Whitt [21, Section 12.8]) but the sub-level sets of \(I_D\), contained in those of \(\text{Var}\), are not compact in \(M_1\). To see this, consider the sequence \((\mathbb{1}_{[1/3, 1/2]}(n))_{n \geq 1}\), which cannot converge in \(M_1\) to an element of \(D[0, 1]\).
Remark 1. The assertions of Theorem A on \(S_n(\cdot)\) as well as all the other results of this paper remain valid for the processes \(S_{[n]}\) on \([0,1]\). This follows from \(\rho_i(S_{[n]}(\cdot)/n,S_{[n]}(\cdot)/n) \leq 2/n\). Moreover, we can replace \(D[0,1]\) by \(BV[0,1]\).

We need to give exact references since Theorem A is a combination of many results scattered through [4,5]. First of all, [4,5] work with the cylindrical \(\sigma\)-algebra on \(D[0,1]\), which equals the Borel \(\sigma\)-algebra of \(J_1\) by [2, Theorem 14.5]. By \(M_2 \subset M_1 \subset J_1\), this cylindrical \(\sigma\)-algebra contains the Borel \(\sigma\)-algebras generated by \(\rho_1\) and \(\rho_2\). Except for the lower semi-continuity, the results for \(\rho_1\) imply those for the shorter metric \(\rho_2\). However, the main focus in [4,5] is on \(\rho_2\) so we refer accordingly. The metric space \((D[0,1],\rho_2)\) is obtained by factorizing the larger space in [4] using the equivalence relation defined by the pseudometric considered there. This factorization is well-defined by the second statement of Theorem 5.1 in [4]. The extended LDP in Theorem A for \(\rho_2\) is then given by Theorem 5.5 combined with Remark 1. The lower semi-continuity of \(I_D\) w.r.t. \(\rho_2\) is by Theorem 5.2.(ii). The inequality for \(I_D\) holds by Theorem 5.2.(iii). Totally boundedness of the sub-level sets of \(I_D\) w.r.t. \(\rho_2\) follows from the inequality \(I_D(h) \geq c_1 \text{Var}(h) - c_2\) and Lemma 5.3, which states this property for the functional \(\text{Var}\) on \(D_0[0,1]\). Similarly, totally boundedness of the sub-level sets w.r.t. \(\rho_1\) follows from Lemma 6.2. Since \(I_D\) is lower semi-continuous on \(I_D\) w.r.t. \(\rho_2\), it is so w.r.t. the longer metric \(\rho_1\). Finally, the extended LDP for \((S_{[n]}(\cdot)/n)_{n \geq 1}\) holds in the space \((BV[0,1],\rho_1)\) (and hence \((BV[0,1],\rho_2)\) with the coarser topology) by Theorem 6.2 in [5]. This easily extends to \((D[0,1],\rho_1)\) using that \(I_D\) is lower semi-continuous on this space; cf. the proof of Lemma 4.1.5(a) in the book by Dembo and Zeitouni [8].

Note in passing that the inequality for \(I_D\) readily follows from \(I(v) \geq c_1 |v| - c_2\) for \(v \in \mathbb{R}^d\), which holds since \(I\) is a convex function which grows at least linearly at infinity; see (17) and (18) below.

### 3. The weak-* and related topologies on \(BV[0,1]\)

In this section we introduce the weak-* topology \(W_1\) on the space of functions of bounded variation, then present a convenient characterization of convergence in this topology, and conclude by comparing \(W_1\) with the Skorokhod topologies and the Jakubowski topology \(S_1\).

#### 3.1. The weak-* topology and a related metric.

Every \(h \in BV[0,1]\) is the distribution function of an \(\mathbb{R}^d\)-valued finite Borel measure on \([0,1]\), which we denote by \(dh\), i.e. it holds \(dh([0,x]) = h(x)\) for \(x \in [0,1]\). As in the case \(d = 1\), this correspondence is bijective (Folland [12, Theorem 3.29]). Note that [12] considers only complex-valued measures but all the cited results of [12] are actually valid for any \(d \geq 1\) since the consideration of \(\mathbb{R}^d\)-valued finite measures is coordinate-wise. For example, the integral of a measurable function \(f : [0,1] \to \mathbb{R}^d\) w.r.t. \(dh\) is defined as

\[
\int_0^1 f \cdot dh := \sum_{k=1}^d \int_{[0,1]} f^{(k)} dh^{(k)}, \quad h \in BV[0,1].
\]

(8)

Recall that \(\text{Var}(h)\) denotes the total variation of an \(h \in BV[0,1]\); see (6). This is a norm on \(BV[0,1]\), and it generates a topology. Both will be referred to as **strong**.
Denote by \( C[0, 1] = C([0, 1]; \mathbb{R}^d) \) the set of continuous functions on \([0, 1]\), and equip it with the supremum norm \( \| \cdot \|_\infty \). By the Riesz theorem (\([12\text{, Theorem 7.17}]\)), the dual of \((C[0, 1], \| \cdot \|_\infty)\) is isometrically isomorphic to \((BV[0, 1], \Var(\cdot))\) since we regard \(BV[0, 1]\) as the space of finite \(\mathbb{R}^d\)-valued Borel measures on \([0, 1]\). In particular, we have

\[
\Var(h) = \sup_{f \in C[0,1]; \|f\|_\infty \leq 1} \int_0^1 f \cdot dh, \quad h \in BV[0,1],
\]

i.e. the strong (total variation) norm is the operator norm. The weak-* topology (Högnäs \([13]\))

Theorem B

\[ \tilde{W} = \text{the space of finite} \BV \text{ which holds by the Banach–Alaoglu theorem. Hence every strongly bounded weak-* closed subset of} BV[0,1] \text{ is weak-* compact.} \]

Let us give a characterization of the weak-* convergence. Consider the norm

\[
\|h\|_* := \int_0^1 |h(s)|ds + |h(1)|, \quad h \in BV[0,1].
\]

on \(BV[0,1]\) and the metric \(\rho_*(g,h) := \|g - h\|_*\). It generates the topology, which we denote by \(\tilde{W}_*\) in view of the following result.

**Theorem B** (Högnäs \([13]\)). Suppose that \(\{g_\alpha\}_{\alpha \in A} \subset BV[0,1]\) is a strongly bounded net, i.e. \(\sup_{\alpha \in A} \Var(g_\alpha) < \infty\). Then the following are equivalent:
1) \(\lim_{\alpha \in A} \|g_\alpha\|_* = 0\);
2) \(\lim_{\alpha \in A} \int_0^1 f \cdot dg_\alpha = 0\) for any \(f \in C[0,1]\), i.e. \(\{g_\alpha\}_{\alpha \in A}\) converges weakly-* to zero on \([0,1]\).

**Corollary.** A strongly bounded subset of \(BV[0,1]\) is open/closed/compact in \(\tilde{W}_*\) whenever it is so in \(W_*\), and whenever it is so in \(\text{seq}(W)_*\), defined as the topology where a set is closed if and only if it is sequentially closed in \(W_*\).

Note that since the integral is defined in \([8]\) as the sum of coordinate integrals, Theorem \([13]\) fully reduces to the case \(d = 1\) as considered in \([13]\).

**Remark 2.** If the net \(\{g_\alpha\}_{\alpha \in A}\) is a sequence, i.e. \(A = \mathbb{N}\), then by the uniform boundedness principle and \([9]\), \(\sup_{\alpha \in A} \Var(g_\alpha) < \infty\) if and only if \(\sup_{\alpha \in A} \left| \int_0^1 f \cdot dg_\alpha \right| < \infty\) for any \(f \in C[0,1]\). Therefore a weakly-* convergent sequence is also convergent in the metric \(\rho_*\), but not vice versa.

The Corollary follows from the definition of the subspace topology, where a set is compact if and only if it is compact in the original topology, and fact that Theorem \([13]\) is equivalent to the assertion that \(\rho_*\) metrizes the (subspace) weak-* topology on strongly bounded subsets of \(BV[0,1]\) (for related references, see a general metrization result \([9\text{, Theorem V.5.1}]\) and \([14\text{, Lemma 2.9}]\)). The equivalence holds because convergence of nets fully defines any topology.
due to the general fact that in a topological space, a set is closed if and only if together with any converging net it contains all its limits (Engelking [10, Corollary 1.6.4]). In general, one cannot replace nets by sequences in this characterization of closed set. Whenever this is possible, i.e. any sequentially closed set is closed, the topology is called sequential. For example, any metrizable topology is sequential. Furthermore, it is known that \( W_\ast \) is not sequential, and thus \( W_\ast \subseteq \text{seq}(W_\ast) \). We also have \( \tilde{W}_\ast \subset \text{seq}(W_\ast) \) since by Remark 2 sequential convergence in \( W_\ast \) implies convergence in \( \tilde{W}_\ast \). Let us clarify that (semi-)continuity of a functional in the topology \( \text{seq}(W_\ast) \) is exactly its sequential (semi-)continuity in \( W_\ast \) ([6, Proposition 1.1.5(ii)] and [10, Proposition 1.6.15]).

Finally, note that the topologies \( \tilde{W}_\ast, W_\ast, \text{seq}(W_\ast) \) are separable and Hausdorff. Separability follows from the Corollary to Theorem [3] and the fact that for every \( n \in \mathbb{N} \), the metric space \( (B_n, \rho_\ast) \) is compact, hence totally bounded, and hence separable. The Hausdorff property follows similarly. Also note that the space \( (BV[0,1], \rho_\ast) \) is not complete.

### 3.2. Comparison with the Skorokhod topologies

First compare \( \rho_\ast \) with the metric \( \rho_2 \) defined in Section [2.1]

**Lemma 1.** For any \( h \in BV[0,1] \), there exists an \( r = r(h) > 0 \) such that for any \( g \in D[0,1] \) satisfying \( \rho_2(g,h) < r \), we have

\[
\int_{0}^{1} |g(s) - h(s)| ds \leq 2d(\text{Var}(h) + 2)\rho_2(g,h).
\]

**Proof of Lemma 1.** By the inequalities \( |x| \leq |x(1)| + \ldots + |x(d)| \) for \( x \in \mathbb{R}^d \), and \( \text{Var}(h) \leq \text{Var}(h(k)) \) and \( \rho_2(g,(k), h(k)) \leq \rho_2(g,h) \) for \( k = 1, \ldots, d \), it suffices to prove the assertion only for \( d = 1 \). The last inequality readily follows from definition [2] of the metric \( \rho_2 \): first estimate \( |(s,x) - (t,y)| \geq |(s,x(k)) - (t,y(k))| \) and then, since the r.h.s. of this inequality does not depend on the remaining coordinates, eliminate them from the constraints under the maximum and the minimum in [2].

Now assume that \( d = 1 \) and consider the Borel set

\[
U := \{ (s,x) \in \mathbb{R}^2 : 0 \leq s \leq 1, g(s) \wedge h(s) \leq x \leq g(s) \vee h(s) \}.
\]

We claim that \( U \subset \text{cl}((\Gamma h)^{\rho_2(g,h)}) \), where \((\Gamma h)^{\rho_2(g,h)} \) is the Euclidean \( \rho_2(g,h) \)-neighbourhood of the completed graph of \( h \). In order to check this, pick an \( s \in [0,1] \). There is a point \((t,y) \in \Gamma h \) such that \( |(t,y) - (s,g(s))| \leq \rho_2(g,h) \). Hence a) \((s,x) \in \text{cl}((\Gamma h)^{\rho_2(g,h)}) \) for any \( x \in [g(s) \wedge y, g(s) \vee y] \), and b) using the definition of the completed graph, for any \( x \in [h(s) \wedge y, h(s) \vee y] \) there exists a \( u \in [s \wedge t, s \vee t] \) such that \((u,x) \in \Gamma h \), which by \( |s-t| \leq \rho_2(g,h) \) implies \((s,x) \in \text{cl}((\Gamma h)^{\rho_2(g,h)}) \). Combining a) and b) yields the claim by

\[
[g(s) \wedge h(s), g(s) \vee h(s)] \subset [g(s) \wedge y, g(s) \vee y] \cup [h(s) \wedge y, h(s) \vee y].
\]

Then, using Fubini’s theorem,

\[
\int_{0}^{1} |g(s) - h(s)| ds = \lambda(U) \leq \lambda(\text{cl}((\Gamma h)^{\rho_2(g,h)})) .
\]

where \( \lambda \) denote the Lebesgue measure on the plane. The limit \( \lim_{\varepsilon \to 0+} \lambda(\text{cl}((\Gamma h)^{\varepsilon}))/2\varepsilon \) is called the one-dimensional *Minkowski content* of the set \( \Gamma h \). As in the definition of the
metric $\rho_1$, we can regard this closed set as the image of a planar curve, and it is easy to check using the definition of the total variation of a function that the length of this curve does not exceed $\text{Var}(h) - |h(0)| + 1$. The Minkowski content of $\Gamma h$ is exactly the length of this curve; see Federer [11, Theorem 3.2.39]. Combining this result with (10) proves the lemma. □

Now we use Lemma [2] to clarify the relations between the introduced topologies on $BV[0,1]$. With no risk of confusion, in the following discussion we use the original notation $M_1$, $M_2$, $J_1$ to refer to the Skorokhod topologies induced on $BV[0,1]$. Then

$$\tilde{W}_* \subset M_1$$

(11)

by Lemma [1] since convergence of càdlàg functions in the metric $\rho_1$ (which generates $M_1$) implies convergence of their values at the endpoint 1. On the other hand, both $\tilde{W}_*$ and $W_*$ are incomparable with $M_2$. For example, for $g_n := \mathbb{1}_{[1-1/n,1]}$ and $g := \mathbb{1}_{[1]}$, we have $\rho_2(g_n, g) \to 0$ but $\rho_1(g_n, g) \not\to 0$ as $n \to \infty$. Note in passing that convergence in $\rho_2$ (which generates $M_2$) implies convergence in $\rho_1$ if the limit function is continuous at 1; this can be shown using that a point $(1, x)$ is a limit point for the completed graph $\Gamma h$ of a function $h \in D[0,1]$ if and only if $x \in \{h(1-), h(1)\}$.

Moreover, $W_*$ is incomparable with $M_1$. For example, for $g_n := \{n \cdot \}/\sqrt{n}$, where $\{\cdot\}$ denotes the fractional part, we have $\rho_1(g_n, 0) \to 0$ but $g_n$ does not converge weakly-* since its total variation explodes. Likewise, $W_*$ is incomparable with $J_1$. Both topologies are weaker than the strong topology: this holds for $W_*$ by definition, and for $J_1$, this follows from

$$\rho(g, h) \leq \|g - h\|_{\infty} \leq \text{Var}(g - h), \quad g, h \in BV[0,1],$$

(12)

where the second equality holds by (15) below.

3.3. Comparison with the Jakubowski topology. It is worth to compare the topologies $W_*$ and $\tilde{W}_*$ with the Jakubowski topology $S$, which appears to be quite useful; see Jakubowski [15] [16] for details and references. Essentially, this topology is defined by extending to $D[0,1]$ a modified version of $W_*$. By definition, a subset of $D[0,1]$ is closed in $S$ if and only if it is sequentially closed under the convergence $\to_S$, which in turn is defined as follows. We say that $f_n \to_S f_0$ as $n \to \infty$, where $f_0, f_1, \ldots \in D[0,1]$, if for any $\varepsilon > 0$ there exist $h_{0,\varepsilon}, h_{1,\varepsilon}, \ldots \in BV[0,1]$ such that $\|f_n - h_{n,\varepsilon}\|_{\infty} \leq \varepsilon$ for $n = 0, 1, \ldots$ and $h_{n,\varepsilon} \to h_{0,\varepsilon}$ weakly-* as $n \to \infty$. The topology $S$ is sequential due to the fact that every sequence that converges in $S$ contains an $\to_S$-convergent subsequence; see [16] Theorem 6.3. We will work, without further specification, with the restriction of $S$ to $BV[0,1]$, obtained from the convergence $\to_S$ on $BV[0,1]$ as above (it is the subspace topology induced by $S$).

The topology $S$ shares many properties with the topology $\tilde{W}_*$. Both are quite weak, for example, the functional $h \mapsto \sup_{0 \leq t \leq 1}(h(t) \cdot \ell)$ with a fixed non-zero $\ell \in \mathbb{R}^d$ is not continuous (as it is in $M_1$ or $M_2$) but only lower semi-continuous. Convergence in $M_1$ implies the $\to_S$-convergence, hence $S$ is weaker than $M_1$, while $S$ is incomparable with $M_2$ (16 Section 4). We claim that $\tilde{W}_*$ is weaker than $S$ (on $BV[0,1]$), hence

$$\tilde{W}_* \subset S \subset M_1.$$

(13)
In fact, every $\to_S$-convergent sequence in $BV[0,1]$ is bounded in the $\| \cdot \|_{\infty}$-norm by Theorem [2] and the second inequality in (12), and it converges pointwisely outside of a countable subset of $[0,1]$ ([15 Remark 2.4]). Then the sequence converges in $\rho_*$ by the dominated convergence theorem. This implies, since $S$ is sequential, that every set closed in $\widetilde{W}_*$ is also closed in $S$, and the claim follows.

The weak-* convergence in $BV[0,1]$ implies the $\to_S$-convergence, just take $h_{n,\varepsilon} := f_n$. Hence

$$\widetilde{W}_* \subset S \subset \text{seq}(W_*).$$

Although we cannot claim that $S \subset W_*$ (cf. Section 5.1), $S$ coincides with $W_*$ on strongly bounded sets, as follows from (14) and the Corollary to Theorem B. Finally, the topology $S$ is separable and Hausdorff because these properties hold for $W_*$ and $\text{seq}(W_*)$.

4. The integral representation of the rate function $I_D$

From this point on we work only with the space $BV_0[0,1]$. By exactly the same argument as above, the topologies on this space induced by $M_1$, $M_2$, $S$, seq$(W_*)$, $W_*$, and $\widetilde{W}_*$ are separable and Hausdorff.

4.1. Notation. Denote by $AC[0,1] = AC([0,1]; \mathbb{R}^d)$ the set of absolutely continuous functions on $[0,1]$, and by $AC_0[0,1]$ the subspace of $AC[0,1]$ of functions $h$ such that $h(0) = 0$. By Lebesgue’s decomposition theorem, any function $h \in BV_0[0,1]$ admits a unique representation $h = h_a + h_s$, where $h_a \in AC_0[0,1]$ and $h_s \in BV_0[0,1]$ are, respectively, the absolutely continuous and singular components of $h$.

Denote by $V^h$ the total variation function of $h$, defined by $V^h(t) := \text{Var}(h(\cdot \land t))$ for $t \in [0,1]$. By the definition, $V^h \in BV_0[0,1]$ and $\text{Var}(h) = V^h(1)$. By [12 Theorem 3.29], $dV^h$ is exactly the total variation measure of the vector-valued measure $dh$. We have $dh \ll dV^h$, hence there exists the Radon–Nykodim density $\hat{h} : [0,1] \to \mathbb{R}^d$ defined by $dh(t) = \hat{h}(t) dV^h(t)$. This measure satisfies $|\hat{h}| = 1 \ dV^h$-a.e. ([12 Proposition 3.13]), hence

$$\text{Var}(h) = \int_0^1 (\hat{h} \cdot \dot{\hat{h}}) dV^h = \sup_{f : \|f\|_{\infty} \leq 1} \int_0^1 f \cdot \dot{h}, \quad h \in BV_0[0,1],$$

where the supremum is taken over measurable functions $f : [0,1] \to \mathbb{R}^d$; cf. (9). Since $dh_a(t) = h'_a(t) dt$, where $dt$ stands for the Lebesgue measure on $[0,1]$ and $h'_a$ is the derivative of the absolutely continuous function $h_a$, we have $h_a = h'_a/|h'_a|$ $dt$-a.e. on the set where $h'_a$ is non-zero.

Define the directional total variation of $h$, denoted by $d\sigma^h$, as the push-forward measure $d\sigma^h := dV^h \circ (\hat{h})^{-1}$ on the unit sphere $S^{d-1}$. Then $\text{Var}(h) = \sigma^h(S^{d-1})$ is the total variation of $h$. For example, if $d = 1$, the Hahn–Jordan decomposition gives the unique representation $h_+ = h_+^+ - h_+^-$, where $h_+^+ \in BV_0[0,1]$ are non-decreasing functions, and so $d\sigma^h = h_+^+(1) \delta_1 + h_+^-(1) \delta_{-1}$. Finally, note that $d\sigma^{h_a} = dV^{h_a} \circ (\hat{h}_a)^{-1}$, where $(\hat{h})_s = \hat{h}_s$ by $V^{h_s} = (V^h)_s$. We also have $dh = h'_a \ dt + h_s \ dV^{h_s}$, hence

$$\text{Var}(h) = \int_0^1 |h'_a(t)| dt + \sigma^{h_s}(S^{d-1}).$$

(16)
Denote by
\[ I_\infty(v) := \sup\{u \cdot v : u \in D_L\}, \quad v \in \mathbb{R}^d \] (17)
the support function of the convex set \( D_L \). This notation reflects that fact (Rockafellar [20, Theorem 13.3]) that \( I_\infty \) equals the so-called recession function of \( I \), which is convex, lower semi-continuous and positively homogeneous on \( \mathbb{R}^d \), and has the property (20, Theorem 8.5)
\[ I_\infty(v) = \lim_{t \to \infty} I(u + vt)/t = \sup_{t > 0} [(I(u + vt) - I(u))/t], \quad u \in D_L, v \in \mathbb{R}^d. \] (18)

4.2. The integral representation of \( I_D \) and related results. We are ready to state our first main result. Recall that seq(\( W^* \)) denotes the topology on \( BV_0[0,1] \) where a set is closed if and only if it is sequentially weak-* closed, \( \tilde{W}^* \) is the topology generated by the metric \( \rho^* \), and the Jakubowski topology \( S \) was introduced in Section 3.3.

**Theorem 1.** Assume that \( X_1 \) is a random vector in \( \mathbb{R}^d \), where \( d \geq 1 \), such that \( 0 \in \text{int} \ D_L \). Then the functional \( I_D \) on \( BV_0[0,1] \), defined in (7), is lower semi-continuous and tight, both properties valid in the topologies \( \tilde{W}^*, S, \) and \( \text{seq}(W^*) \). Moreover, we have
\[ I_D(h) = \int_0^1 I(h'_a(t))dt + \int_{S_{d-1}} I_\infty(\ell) d\sigma^{h_s}(\ell), \quad h \in BV_0[0,1]. \] (19)

The advantage of integral representation (19) is its explicitness and the ease to work with it using the developed methods of variational calculus. Note that if the Laplace transform of \( X_1 \) is finite on the whole \( \mathbb{R}^d \), then \( D_L = \mathbb{R}^d \) and so \( I_\infty(v) = +\infty \) for \( v \neq 0 \), hence \( I_D(h) = +\infty \) for \( h \notin AC_0[0,1] \). In dimension one (19) reduces to the following formula matching the one in [4, Theorem 3.3]:
\[ I_D(h) = \int_0^1 I(h'_a(t))dt + h_+^+(1)I_\infty(1) + h_-^-(1)I_\infty(-1). \]

The following statement is a direct corollary to Theorems 1 and A.

**Proposition 1.** Assume that \( X_1 \) is a random vector in \( \mathbb{R}^d \) such that \( 0 \in \text{int} \ D_L \). Then the sequence \( (S_n(\cdot)/n)_{n \geq 1} \) satisfies the LDP in the separable metric space \( (BV_0[0,1], \rho_s) \) and in the separable Hausdorff space \( (BV_0[0,1], S) \) with the tight convex rate function \( I_D \).

**Proof.** The properties of the topologies stated were shown in Section 3. The function \( I_D \) is lower semi-continuous and tight on both spaces by Theorem 1. Then, by Theorem 1 and inclusions (13), the sequence \( (S_n(\cdot)/n)_{n \geq 1} \) satisfies the extended LDP with the rate function \( I_D \). By tightness of \( I_D \) and equality (5), the extended LDP reduces to the standard LDP, as required. \( \square \)

Proposition 1 should have a limited use due to weakness of the topologies \( S \) and \( \tilde{W}_s \), although the Jakubowski topology \( S \) proved to be quite popular; see [16, Section 1] for references. The next statement, which follows as above, is arguably more important.
Proposition 2. Assume that $X_1$ is a random vector in $\mathbb{R}^d$ such that $0 \in \text{int} \mathcal{D}_L$. Let $B \subset BV_0[0,1]$ be such that $\chi_{\rho_1}(B)$ is sequentially weakly-* closed. Then
\[
\lim_{\delta \to 0+} \inf_{h \in \chi_{\rho_1}(B), \rho(h,B) < \delta} I_D(h) = \min_{h \in \chi_{\rho_1}(B)} I_D(h).
\]

Thus, for sets $B$ that are closed both w.r.t. $\rho_1$ and sequentially weakly-*, the upper bound in the extended LDP of Theorem A matches the standard one in [3] and, moreover, the infimum is always attained, therefore being a minimum.

Remark 3. The restrictive assumption of Proposition 2 is satisfied when $B$ is a sub-level set of a functional on $BV_0[0,1]$ that is lower semi-continuous both in $\text{seq}(W_*)$ and $M_1$. Examples of such functionals (which are not continuous in $\text{seq}(W_*)$) include:

- The action functional defined by the r.h.s. of (19) with $I$ replaced by any non-constant convex lower semi-continuous function $J : \mathbb{R}^d \to [0, +\infty]$. It is sequentially weak-* lower semi-continuous by [6, Corollary 3.4.2] exactly as used in the proof of Theorem 1. In particular, from (16) we see that $J = | \cdot |$ corresponds to the action functional $\text{Var}$.

- The maximum functional $h \mapsto \sup_{0 \leq t \leq 1} h(t) \cdot \ell$ with a fixed direction $\ell \in \mathbb{S}^{d-1}$. It is continuous in $M_1$ (and $M_2$), and its sequential weak-* lower semi-continuity easily follows from the càdlàg property of $h$ and the fact that every weak-* convergent sequence in $BV[0,1]$ converges pointwise except, possibly, for a countable subset of $(0,1)$ ([7, Proposition 2.27]).

- For $d = 2$, the perimeter (and the mean width in higher dimensions) of the convex hull of the image of a planar curve, considered in Section 5 (with no use of the property discussed here). This follows from the previous item by Cauchy’s formula ([11, Eq. (53)]) and Fatou’s lemma.

Further examples of sets $B$ satisfying the assumption of Proposition 2 include:

- $B$ is the pre-image of a closed subset of the real line under a weakly-* continuous functional on $BV_0[0,1]$. Such functionals are of the form $h \mapsto \int_0^1 f \cdot dh$ for $f \in C[0,1]$. For example, by Proposition 1 and the contraction principle, this yields the LDP for the sequence $\left( \frac{1}{n} \sum_{k=1}^n f \left( \frac{k}{n} \right) X_k \right)_{k \geq 1}$ of weighted sums of i.i.d. random vectors in $\mathbb{R}^d$.

- $B = \{ h \in BV_0[0,1] : g_1 \leq h \leq g_2 \}$ for some $g_1, g_2 \in C[0,1]$ satisfying $g_1(0) \leq 0 \leq g_2(0)$. The minimizers of $I_D$ over $B$ are called the taut strings. In a probabilistic setup, taut strings were considered by Lifshits and Setzerqvist [17]. The set $B$ is closed in both topologies $\text{seq}(W_*)$ and $M_1$, as can be shown using the càdlàg property and the fact that every weak-* convergent sequence in $BV_0[0,1]$ converges pointwise except, possibly, for a countable subset of $(0,1)$.

Finally, it is worth to mention two general results related to the assumption discussed. By the Krein–Smulian theorem, a convex subset of a separable Banach space is sequentially weak-* closed if and only if it is weak-* closed (see [9, Theorem V.5.7] or [14, Theorem 2.10]). Also, by Mazur’s theorem, convex subsets of normed spaces have the same closures in the strong and the weak topologies. Unfortunately, the second result has no use here since $(C[0,1], \| \cdot \|_\infty)$ is not reflexive, hence $W_*$, the weak-* topology on $BV[0,1]$, is strictly weaker than the weak topology.

We will use the results and methods of the calculus of variations, referring to the book by Buttazzo [6]. The action (integral) functionals over finite Borel vector-valued signed
measures are considered in Chapter 3 of this book, where the notation \( C_0([0,1];\mathbb{R}^d) \) stands for our \( C[0,1] \). The main idea, described in Section 1.3 in [6], is to consider the relaxed functional \( \text{cl}_{\text{seq}}(W_0(I_C)) \) on \( BV_0[0,1] \), defined as the function whose epigraph is the closure in the topological space \((BV_0[0,1], \text{seq}(W_*)) \times ((-\infty, +\infty], | \cdot |) \) of the epigraph of \( I_C \). Equivalently, \( \text{cl}_{\text{seq}}(W_0(I_C)) \) is the maximal sequentially weak-* lower semi-continuous functional on \( BV_0[0,1] \) dominated by \( I_C \) ([6] Propositions 1.1.2(ii) and 1.1.5(ii)).

**Proof of Theorem 1.** Define the function \( I_C \) on \( BV_0[0,1] \) by putting \( I_C(h) := \int_0^1 I(h'(t))dt \) for \( h \in AC_0[0,1] \) and \( I_C(h) := +\infty \) for \( h \notin AC_0[0,1] \). Then \( I_C = I_D \) on \( AC_0[0,1] \) by [4] Theorem 5.3], hence

\[
I_D(h) \leq I_C(h), \quad h \in BV_0[0,1].
\]

In the new notation, the definition (7) of \( I_C \) reads as

\[
I_D(h) = \sup_{t \in [0,1]: \# t < \infty} I_C(h^t), \quad h \in BV_0[0,1].
\]

Moreover, for any sequence \((t_n)_{n \geq 1}\) that is dense in \([0,1]\), for \( t := \{t_1, \ldots, t_n\} \) we have

\[
I_D(h) = \lim_{n \to \infty} I_C(h^{t_n}), \quad h \in BV_0[0,1].
\]

since \( I_D \) is lower semi-continuous w.r.t. \( \rho_2 \) by Theorem [A] and, clearly, \( \rho_2(h^{t_n}, h) \to 0 \). Later on in this proof we will use that we also have \( h^{t_n} \to h \) in \( \rho_* \) and weakly-*, where the \( \rho_* \)-convergence follows from Lemma [1] and the equality \( h^{t_n}(1) = h(1) \), and the weak-* convergence then holds by Theorem [B] which applies by \( \text{Var}(h^{t_n}) \leq \text{Var}(h) \).

Let us prove lower semi-continuity of \( I_D \) in the metric \( \rho_* \). This will imply lower semi-continuity of \( I_D \) in the topologies \( \text{seq}(W_*) \) and \( S \) since \( \text{W}_* \), the topology generated by \( \rho_* \), is by ([14]) coarser than \( S \) and \( \text{seq}(W_*) \). Use that lower semi-continuity in metric spaces is a sequential property. Assume that there are \( g, g_1, g_2, \ldots \in BV_0[0,1] \) such that \( \rho_*(g_n, g) \to 0 \) but \( I_D(g) > \lim \inf g_n I_D(g_n) \). Since \( \rho_*(g_n, g) \to 0 \) means convergence in \( L_1 \) and at the endpoint 1, hence by considering a subsequence, we can assume w.l.o.g. that the convergence is pointwise on a subset of \([0,1]\) of full Lebesgue measure. Pick a sequence \((s_n)_{n \geq 1}\) of (distinct) elements of this set, such that the sequence is dense in \([0,1]\) and \( s_0 = 0, s_1 = 1 \).

For any \( n \geq 1 \), put \( s_n := \{s_1, \ldots, s_n\} \), and let \( \sigma_n \) be the permutation of length \( n \) such that \( s_{\sigma_n(1)} < \ldots < s_{\sigma_n(n)} \). For any integer \( i \) and \( n \) satisfying \( 1 \leq i \leq n \), we have \( g_n^{s_n}(s_i) = g_n(s_i) \). Therefore \( g_n^{s_n}(s_i) \to g(s_i) \) as \( n \to \infty \) for any fixed \( i \) by the choice of the sequence \((s_n)_{n \geq 1}\). Then by lower semi-continuity of \( I \), for any \( k \geq 2 \),

\[
I_C(g_n^{s_k}) = \sum_{i=0}^{k-1} (s_{\sigma(k+i+1)} - s_{\sigma(k+i)}) I \left( \frac{g_n(s_{\sigma(k+i+1)}) - g_n(s_{\sigma(k+i)})}{s_{\sigma(k+i+1)} - s_{\sigma(k+i)}} \right) \\
\leq \lim \inf_{n \to \infty} \sum_{i=0}^{k-1} (s_{\sigma(k+i+1)} - s_{\sigma(k+i)}) I \left( \frac{g_n(s_{\sigma(k+i+1)}) - g_n(s_{\sigma(k+i)})}{s_{\sigma(k+i+1)} - s_{\sigma(k+i)}} \right) = \lim \inf_{n \to \infty} I_C(g_n^{s_k}).
\]

From (21) we have \( I_C(g_n^{s_k}) \leq I_D(g_n) \), hence \( I_C(g_n^{s_k}) \leq \lim \inf_{n \to \infty} I_D(g_n) \). It remains to take \( k \to \infty \) and use (22) to arrive at \( I_D(g) \leq \lim \inf_n I_D(g_n) \), which contradicts our assumption that the lower semi-continuity is violated.
Furthermore, sub-level sets of $I_D$ are strongly bounded, as follows from the bound on $I_D$ in Theorem A. They are closed in the metric $\rho_*$ since $I_D$ is lower semi-continuous in $\rho_*$ as we just shown. Therefore, by the Corollary to Theorem A and the Banach–Alaoglu theorem, they are compact in $\tilde{W}_*$ and in seq($W_*).$ They are also compact in $S$ by the second inclusion in (14). This means that $I_D$ is tight in the three topologies, as required.

It remains to prove the integral representation (19) for $I_D.$ Denote by $\mathcal{I}(h)$ the r.h.s. of (19). By [6, Corollary 3.4.2], it is a sequentially weakly-* lower semi-continuous functional on $BV_0[0,1].$ Then, since $\mathcal{I} = I_C$ on $AC_0[0,1],$ we have $\mathcal{I} = \cl_{\text{seq}(W_*)}(I_C)$ by [6, Theorem 3.3.1] using the fact that if for a lower semi-continuous convex non-negative function $J$ on $\mathbb{R}^n$ we put $J_C(h) := \int_0^1 J(h')dt$ for $h \in AC_0[0,1],$ then $J_C \equiv I_C$ on $AC_0[0,1]$ if and only if $I \equiv J.$

It is easy to check, using continuity of addition, that $\mathcal{I}$ is convex since so is $I_C.$ Then by [6, Propositions 1.3.1(ii) and 1.3.4(ii)],

$$\mathcal{I}(h) = \min \left\{ \liminf_{\alpha \to A} I_C(h_\alpha) : \{h_\alpha\}_{\alpha \in A} \text{ is a net converging weakly-* to } h \right\}$$

(actually, the minimum over nets can be replaced by infimum over converging sequences, by [6, Proposition 1.3.5 and Remark 1.3.6]). This gives $\mathcal{I} \leq I_D \leq I_C$ by (20) and (22), where $h^n \to h$ weakly-* as $n \to \infty,$ as explained right after (22). Therefore the required equality $\mathcal{I} = I_D$ follows from the fact that $\mathcal{I} = \cl_{\text{seq}(W_*)}(I_C)$ is the maximal sequentially weakly-* lower semi-continuous functional dominated by $I_C$ (see [6, Proposition 1.1.2(ii)]) since $I_D$ is sequentially weakly-* lower semi-continuous, as we proved above. □

5. The contraction principle and its applications

5.1. Contraction principle. We first prove the following version of the contraction principle to bring the extended LDP of Theorem A into a standard form matching (3). The argument actually works for any sequence of random elements of a metric space satisfying an extended LDP.

**Theorem 2.** Assume that $X_1$ is a random vector in $\mathbb{R}^d,$ where $d \geq 1,$ such that $0 \in \text{int} \ M.$ Let $M$ be a complete metric space, and $F : BV_0[0,1] \to M$ be a mapping that is uniformly continuous on totally bounded subsets of $(BV_0[0,1], \rho_1).$ Then the sequence of random elements $(F(S_n(\cdot)/n))_{n \geq 1}$ satisfies an LDP in $M$ with the tight rate function $J$ given by $J = \cl \tilde{J},$ where $\tilde{J}(x) := \inf_{h \in f^{-1}(x)} I_D(h)$ for $x \in M.$

Recall that $\cl \tilde{J}$ denotes the (lower semi-continuous) function whose epigraph is the closure of the epigraph of $\tilde{J}$ in the topological space $M \times (-\infty, +\infty].$

**Proof.** We will use the following representation (see Rassoul-Agha and Seppäläinen [19, Lemma 2.8]) for lower semi-continuous regularization $\cl \tilde{J}$ of $\tilde{J}:

$$\tilde{J}(x) = \sup \left\{ \inf_{y \in U} \tilde{J}(y) : U \subset M \text{ is open, } x \in U \right\}, \quad x \in M. \tag{23}$$

In particular, this implies that $\mathcal{J} \leq \tilde{J}$ and

$$\{x : \mathcal{J}(x) \leq \alpha\} \subset \cl \{x : \tilde{J}(x) \leq \alpha + \varepsilon\}, \quad \alpha, \varepsilon > 0. \tag{24}$$
The image of a totally bounded subset of the metric space $BV_0[0,1]$ under the uniformly continuous function $F$ is totally bounded in $\mathcal{M}$. Therefore by Theorem $\text{A}^{\text{a}}$ and the equalities

$$\{x : \tilde{\mathcal{J}}(x) < \alpha\} = \{x \in \mathcal{M} : \inf_{h \in f^{-1}(x)} I_D(h) < \alpha\} = F(\{h \in BV_0[0,1] : I_D(h) < \alpha\}),$$

the sub-level sets of $\tilde{\mathcal{J}}$ are totally bounded in $\mathcal{M}$, hence their closures are compact since $\mathcal{M}$ is complete. Together with (24), this implies compactness of the sub-level sets of $\mathcal{J}$, which are closed since $\mathcal{J}$ is lower semi-continuous. Thus, $\mathcal{J}$ is tight.

We will use that for any Borel set $B \subset \mathcal{M}$,

$$\inf_{x \in \text{int } B} \mathcal{J}(x) = \inf_{x \in \text{int } B} \tilde{\mathcal{J}}(x), \quad \inf_{x \in \text{cl } B} \mathcal{J}(x) = \lim_{\varepsilon \to 0^+} \inf_{x \in B^\varepsilon} \tilde{\mathcal{J}}(x). \quad (25)$$

The former equality follows straightforwardly from (23); the latter one follows from the former and (5). Denote by $m$ the metric on $\mathcal{M}$. Then from the uniform continuity of $F$ on sub-level sets of $I_D$, for any $\varepsilon, R > 0$ there exists a $\delta > 0$ such that $m(F(g), F(h)) < \varepsilon$ whenever $\rho(g, h) < \delta$ and $g, h \in \{I_D \leq R\}$; recall that $\{I_D \leq R\} \subset BV_0[0,1]$. By $(F^{-1}(B))^\delta = (\text{cl}(F^{-1}(B)))^\delta$, this yields

$$\{I_D \leq R\} \cap (\text{cl}(F^{-1}(B)))^\delta \subset F^{-1}(B^\varepsilon) \cap \{I_D \leq R\},$$

which implies, by first taking $\delta \to 0^+$ and then $\varepsilon \to 0^+$ and $R \to \infty$, that

$$\lim_{\delta \to 0^+} \inf_{h \in (\text{cl}(F^{-1}(B)))^\delta} I_D(h) \geq \lim_{\varepsilon \to 0^+} \inf_{h \in F^{-1}(B^\varepsilon)} I_D(h).$$

Combining this estimate with the definition of $\tilde{\mathcal{J}}$ and (25), we get the required standard LDP (3) from the extended LDP for $(S_n(\cdot)/n)_{n \geq 1}$ on $(BV_0[0,1], \rho_1)$ (see Remark [3]) and continuity of $F$. The latter property of $F$ follows from the assumption that $F$ is uniformly continuous on totally bounded sets and the fact that every convergent sequence in $\mathcal{M}$ is totally bounded. \(\Box\)

5.2. Applications to convex hulls of planar random walks. For an application of Theorem 2 consider the perimeter $P_n := \text{Per}(\text{conv}(0, S_1, \ldots, S_n))$ and the area $A_n := \text{area}(\text{conv}(0, S_1, \ldots, S_n))$ of the convex hull of the first $n$ steps of the random walk $S$ on the plane. Akopyan and Vysotsky [1] gave a detailed study of large deviations properties of these quantities in the case when the Laplace transform of the increments of $S$ is finite in the plane. Define the perimeter and the area of the convex hull of a planar curve respectively by $P(h) := \text{Per}(\text{conv}(Th))$ and $A(h) := \text{area}(\text{conv}(Th))$ for $h \in BV_0([0,1]; \mathbb{R}^2)$. Let us agree that the perimeter of a line segment equals its doubled length.

**Proposition 3.** Assume that $X_1$ is a random vector in the plane such that $0 \in \text{int } D_L$. Then the sequences $(P_n/(2n))_{n \geq 1}$ and satisfy $(A_n/n^2)_{n \geq 1}$ the LDPs in $\mathbb{R}$ with the respective rate functions $\mathcal{J}_P$ and $\mathcal{J}_A$ given by

$$\mathcal{J}_P(x) := \text{cl} \inf_{h \in BV_0[0,1]; P(h) = 2x} I_D(h), \quad \mathcal{J}_A(x) := \text{cl} \inf_{h \in BV_0[0,1]; A(h) = x} I_D(h), \quad x \geq 0. \quad (26)$$

**Proof.** From the Cauchy formula for the perimeter of a convex set on the plane ([3 Eq. (53)]),

$$|P(g) - P(h)| \leq 2\pi \min(P(g), P(h)) \rho_2(g, h), \quad g, h \in BV_0[0,1].$$
From the Steiner formula for the area of a neighbourhood of a convex set on the plane,

\[ |A(g) - A(h)| \leq \min(P(g), P(h))\rho_2(g, h) + \pi\rho_2^2(g, h), \quad g, h \in BV_0[0, 1]. \]

If \( T \) is a totally bounded subset of \((BV_0[0, 1], \rho_1)\), then \( \sup_{h \in T} \rho_1(h, 0) < \infty \), hence \( \sup_{h \in T} P(h) \) is finite. Therefore the functionals \( P \) and \( A \) are uniformly continuous on \( T \) by \( \rho_2 \leq \rho_1 \).

The claim follows by Theorem 2 and the equalities \( P_n/(2n) = P(S_n(\cdot)/n)/2 \) and \( A_n/n^2 = A(S_n(\cdot)/n) \).

Following the ideas developed in \([1]\), we can find the rate functions \( J_P \) and \( J_A \) explicitly in certain cases. Put \( \mu := \mathbb{E}X_1 \). Consider the radial minimum rate function \( I(x) := \inf_{t \in \mathbb{R}} I(t) \) defined for \( x \geq 0 \). Its properties are given in \([1]\) Lemma 1. In particular, if \( 0 \in \text{int} \mathcal{D}_C \), this function is convex and decreasing on \([0, |\mu|] \), increasing on \([|\mu|, \infty) \), and lower semi-continuous. It is convex (on its domain) if the distribution of \( X_1 \) is rotationally invariant, as well in some other cases given by \([1]\) Proposition 2.

**Proposition 4.** Assume that \( X_1 \) is a random vector in the plane such that \( 0 \in \text{int} \mathcal{D}_C \). Then \( J_P \) and \( J_A \) increase on \([|\mu|, \infty) \) and \([0, \infty) \), respectively, and \( J_P = I \) on \([0, |\mu|] \). Moreover, if \( I \) is convex, then \( J_P = I \). If, in addition, the distribution of \( X_1 \) is rotationally invariant, then \( J_A(x) = I(\sqrt{2\pi}x) \) for \( x \geq 0 \).

Thus, from the monotonicity properties of \( J_P \) and \( J_A \), taking the lower semi-continuous minorant \( \ell \) in \([26]\) may change the values of the infima only at the points of discontinuity.

**Proof.** Let us prove the inequality \( J_P \geq I \) on \([0, |\mu|] \). First note that

\[ \int_S I_\infty(\ell) d\sigma^{h_s}(\ell) \geq I_\infty(h_s(1)), \quad h \in BV_0[0, 1]. \tag{27} \]

This inequality is trivial if \( h_s \) is zero, otherwise it follows from Jensen’s inequality and positive homogeneity of \( I_\infty \):

\[ \int_S I_\infty(\ell) d\sigma^{h_s}(\ell) = \int_0^1 I_\infty(\hat{h}_s) dV^{h_s} = \text{Var}(h_s) \int_0^1 I_\infty(\hat{h}_s)(dV^{h_s}/\text{Var}(h_s)) \]

\[ \geq \text{Var}(h_s) \cdot I_\infty\left( \int_0^1 \hat{h}_s(dV^{h_s}/\text{Var}(h_s)) \right) = \text{Var}(h_s) \cdot I_\infty\left( \frac{1}{\text{Var}(h_s)} \int_0^1 dh_s \right) = I_\infty(h_s(1)). \]

For the absolutely continuous component of \( h \), by Jensen’s inequality we have

\[ \int_0^1 I(h_a(t)) dt \geq I(h_a(1)), \quad h \in BV_0[0, 1]. \tag{28} \]

Furthermore, from \([18]\) we obtain that for any \( v_1, v_2 \in \mathbb{R}^2 \) and any \( t > 0 \),

\[ I(v_1) + I_\infty(v_2) \geq I(v_1) + I(\mu + v_2t)/t = \frac{1 + t}{1 + t} I(v_1) + \frac{1}{1 + t} I(\mu + v_2t). \]

Hence, using convexity of \( I \), then taking \( t \to \infty \), and then using lower semi-continuity of \( I \),

\[ I(v_1) + I_\infty(v_2) \geq I(v_1 + v_2), \quad v_1, v_2 \in \mathbb{R}^2. \tag{29} \]
Combining (27), (28), (29) and using Theorem 1 then gives \( I_D(h) \geq I(h(1)) \) for any \( h \in BV_0[0,1] \). Therefore, using the trivial inequality \( P(h) \geq 2|h(1)| \) and the fact that \( I \) is decreasing on \([0, |\mu|] \), we get

\[
\inf_{h: P(h) = 2x} I_D(h) \geq \inf_{h: P(h) = 2x} I(h(1)) \geq \min_{h: |h(1)| \leq x} I(h(1)) = \min_{y \leq x} I(y) = I(x), \quad x \in [0, |\mu|].
\]

Since \( I \) is lower semi-continuous, this implies the required inequality \( \mathcal{J}_P \geq I \) on \([0, |\mu|] \).

We now show that this equality extends to \([|\mu|, \infty) \) if \( I \) is convex. Let \( \ell_\infty \) be a direction that minimizes the lower semi-continuous function \( I_\infty \) on the unit circle \( S \). Equivalently, \( \ell_\infty \) is the direction to a closest point of \( \partial D_C \) to the origin; see (17). Then

\[
\int_S I_\infty(\ell) d\sigma(h_\ell(\ell)) \geq I_\infty(\text{Var}(h_\ell)\ell_\infty), \quad h \in BV_0[0,1].
\]  

(30)

and by Jensen’s inequality applied for \( I \),

\[
\int_0^1 I(h'_a(t))dt \geq \int_0^1 I(|h'_a(t)|)dt \geq I(\int_0^1 |h'_a(t)|dt) = I(\text{Var}(h_\ell)), \quad h \in BV_0[0,1].
\]

(31)

Furthermore, from (18) we obtain that for any \( x_1, x_2 \geq 0 \) and any \( t > 0 \),

\[
I(x_1) + I_\infty(x_2\ell_\infty) \geq I(x_1) + I(\mu + x_2\ell_\infty)/t \geq \frac{1 + t}{t} I(x_1) + \frac{1}{1 + t} I(|\mu + x_2\ell_\infty|),
\]

which by convexity and lower semi-continuity of \( I \) implies, similarly to (29), that

\[
I(x_1) + I_\infty(x_2\ell_\infty) \geq I(x_1 + x_2), \quad x_1, x_2 \geq 0.
\]

(32)

Combining (30), (31), (32) and using Theorem 1 then gives

\[
I_D(h) \geq I(\text{Var}(h)), \quad h \in BV_0[0,1].
\]

(33)

Then we argue as in the proof of Theorem 1: using the geometric inequality \( \text{Var}(h) \geq \frac{1}{2} P(h) \) (Corollary 5) and the fact that \( I \) increases on \([|\mu|, \infty) \), we get

\[
\inf_{h: P(h) = 2x} I_D(h) \geq \inf_{h: \text{Var}(h) \geq x} I_D(h) \geq \inf_{h: \text{Var}(h) \geq x} I(\text{Var}(h)) \geq \inf_{y \geq x} I(y) = I(x), \quad x \geq |\mu|.
\]

Since \( I \) is lower semi-continuous, this implies the required inequality \( \mathcal{J}_P \geq I \) on \([|\mu|, \infty) \).

It remains to prove the opposite inequality on \([0, \infty) \). Since \( I \) is lower semi-continuous, for any \( x \geq 0 \) there exists a direction \( \ell_x \in S \) such that \( I(x) = I(x\ell_x) \). Define the function \( h_x(t) := tx\ell_x \) for \( t \in [0, 1] \). Since \( I_D(h_x) = I(x) \), we always have \( \mathcal{J}_P \leq I \), as follows from the definition of the lower semi-continuous minorant. The formulas for \( \mathcal{J}_P \) are now proved.

The formula for \( \mathcal{J}_A \) for rotationally invariant distributions of increments follows analogously: use (33) and the isoperimetric inequality for convex hulls \( A(h) \leq \text{Var}(h)^2/(2\pi) \) (Theorem 1, Eq. (39)) instead of \( \text{Var}(h) \geq \frac{1}{2} P(h) \); see the proof of Part 2 of Theorem 2 in [1] for details.

Finally, the monotonicity properties of \( \mathcal{J}_P \) and \( \mathcal{J}_A \) follow by repeating the respective simple arguments in the proofs of Parts 1 of Theorems 1 and 2 in [1]. We omit the details since the difference is really minimal – both Jensen’s inequalities (27) and (28) should be used instead of the second one solely used in [1]. Notice that we are not claiming strict monotonicity, which was the case in [1], since we do not claim that the infima in (26) are always attained.
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