Radial Acceleration Relation from Ultra-light Scalar Dark matter

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We show that ultra-light scalar dark matter (fuzzy dark matter) in galaxies has a quantum mechanical typical acceleration scale about $10^{-10}$ m/s$^2$, which leads to the baryonic Tully-Fisher relation. Baryonic matter at central parts of galaxies acts as a boundary condition for dark matter wave equation and influences stellar rotation velocities in halos. This model also explains the radial acceleration relation and MOND-like behavior of gravitational acceleration found in various type of galaxies without any modification of gravity or mechanics. This analysis can be extended to the Faber-Jackson relation.

The baryonic Tully-Fisher relation (BTFR) [1] is a tight empirical correlation between the total baryonic mass ($M_b$) of a disk galaxy and its asymptotic rotation velocity $v_f$; $M_b \sim v_f^4$. Semi-analytic models for BTFR based on baryonic processes in a cold dark matter (CDM) cosmology predict significant scatter from individual galaxy formation history, but observed BTFR is largely independent of baryonic processes and has small scatter [2]. There is another strong relation called radial acceleration relation (RAR) between the radial gravitational acceleration traced by rotation curves (RCs) of galaxies and predicted acceleration by the observed baryon distributions [3]. There are models [4] based on CDM paradigm explaining RAR, but it is unclear whether this tight relation can survive chaotic processes of galaxy formation and merging. These relations are puzzling, because galactic halos seem to be dark matter (DM) dominated objects and RCs at outer parts of galaxies are believed to be governed mostly by DM not by baryons. There are other relations challenging conventional DM models such as Faber-Jackson relation or baryon-halo conspiracy [5].

On the other hand BTFR and RAR are consistent with Modified Newtonian dynamics (MOND) which was proposed to explain the flat RCs without introducing dark matter [6]. According to MOND Newtonian gravitational acceleration of baryonic matter $g_b$ should be replaced by

$$g_{obs} = \sqrt{g_b g^f},$$

(1)
when $g_b < g^f \sim 1.2 \times 10^{-10}$ m/s$^{-2}$. However, MOND also has its own difficulties in explaining the properties of galaxy clusters and cosmic background radiation [7].

In this letter, we show that ultra-light scalar dark matter (fuzzy dark matter) has a quantum mechanical typical acceleration scale $g^f$, which naturally leads to dynamically established BTFR. Without any modification of gravity or mechanics this model also explains the RAR and MOND-like behavior of gravitational acceleration.

Although the CDM model well explains observed large scale structures of the universe, it encounters many difficulties in explaining galactic structures. For example, numerical studies with CDM predict cuspy DM halos and many satellite galaxies, which are in tension with observational data [8–11]. Recently, there has been a renewed interest in scalar field dark matter [12–16] (SFDM, often also called fuzzy DM [17], ultra-light axion, BEC DM or wave DM) as a solution of these problems. In this model DM is a ultra-light scalar with mass $m \sim 10^{-22}$ eV in Bose-Einstein condensation (BEC). Its long Compton wavelength $\lambda_c = 2\pi \hbar/mc \approx 0.04$ pc suppresses the formation of structures smaller than a galaxy, while it plays the role of CDM at super-galactic scales. (See Refs. 18–25 for a review and references.) Since galaxies are non-relativistic objects, the typical length scale of a galaxy is the de Broglie length $\xi$ rather than $\lambda_c$, which helps in solving the problems of CDM.

In this model, galactic halos are self-gravitating giant boson stars where gravitational force of matter balances with quantum pressure from the uncertainty principle with spatial uncertainty about $\xi$. From the uncertainty principle $\xi m \sqrt{G M_c/\xi} \geq h$ one can estimate $\xi \approx h^2/G M_c m^2$, where $M_c$ is the mass scale and $v$ is a typical rotation velocity of a galaxy. If we identify $M_c \sim 10^8 M_\odot$ and $v \sim 300$ pc to be the typical mass and the size of the core of a dwarf galaxy, then $m \approx h/\sqrt{G M_c} \approx 10^{-12}$ eV. Note that $\xi$ is almost independent of other properties of the galaxy except for $M_c$. We suggest that $\xi$ and the uncertainty principle lead to a natural acceleration scale $g^f = G M_c/\xi^2 \approx h^2/m^2 \xi^3 \approx 10^{-10}$ m/s$^{-2}$ of SFDM. We will show that this acceleration scale $g^f$ from the uncertainty principle can explain the aforementioned relations of galaxies. In theoretical point of view, the value of $\xi$ for galaxy mass scale $M_c$ is almost the same as the crossover distance due to dark matter in quantum theory of gravity [26].

In SFDM model, DM scalar field $\phi$ has an action

$$S = \int \sqrt{-g} d^4 x \left[ -\frac{R}{16 \pi G} - \frac{g^\mu\nu \phi^\mu \phi^\nu - U(\phi)}{2} \right],$$

(2)
where the typical potential is $U(\phi) = \frac{\lambda^2}{8} |\phi|^2 + \frac{1}{4} |\phi|^4$. For fuzzy DM $\lambda = 0$. In the Newtonian limit the Einstein equation and the Klein-Gordon equation from the action
can be reduced to the Schrödinger equation [27]
\[ i\hbar \partial_t \psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + m\Phi \psi(r, t) \quad (3) \]
and the Poisson equation
\[ \Delta \Phi(r) = 4\pi G (\rho_d(r) + \rho_b(r)) \quad (4) \]
with a self-gravitation potential \( \Phi \) and wavefunction \( \psi \equiv \sqrt{m\phi} \). Here, \( \rho_d \) is a DM density and \( \rho_b \) is a baryonic matter density, both of which contribute to \( \Phi \). Since galaxies are non-relativistic, in this model a galactic DM halo is well described with the macroscopic wavefunction \( \psi \) which is a solution of the Schrödinger equation.

For simplicity we consider a spherical fuzzy DM halos. Integrating the above equation gives magnitude of total gravitational acceleration
\[ g_{\text{obs}}(r) \equiv |\nabla \Phi| = \frac{4\pi G}{r^2} \int_0^r (\rho_d(r') + \rho_b(r')) r'^2 \, dr' \equiv g_d(r) + g_b(r), \quad \text{(5)} \]
where \( g_d(r) \) is the acceleration from dark matter and \( g_b(r) \) from baryonic matter at galactocentric radius \( r \).

The Madelung representation \[ |\psi(r, t)| = \sqrt{\frac{\rho_d(r, t)}{\rho_d(r, t) + \rho_b(r, t)}} \quad (6) \]
is useful to calculate \( g_{\text{obs}} \) in a fluid approach. Substituting Eq. (6) into the Schrödinger equation, one can obtain a modified Euler equation
\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla \Phi + \frac{\nabla p}{\rho_d} - \nabla Q \frac{m}{m} = 0, \quad \text{(7)} \]
where \( \mathbf{v} \equiv \nabla S/2m, \mathbf{p} \), and \( Q \equiv \frac{\hbar^2}{2m} \frac{\nabla \sqrt{\rho_d}}{\sqrt{\rho_d}} \) are a fluid velocity, the pressure from a self-interaction (if \( \lambda \neq 0 \)), and a quantum potential, respectively. The quantum pressure \( \nabla Q/m \) helps fuzzy dark matter to overcome the small scale problems of CDM and plays an important role in this paper.

By taking \( \mathbf{v} = 0 \) and \( \partial_t \mathbf{v} = 0 \), we find a stationary equilibrium condition
\[ g_{\text{obs}}(r) = g_d(r) + g_b(r) = \frac{\hbar^2}{2m^2} \left| \nabla \left( \frac{\Delta \sqrt{\rho_d}}{\sqrt{\rho_d}} \right) \right|, \quad \text{(8)} \]
which describes the dynamical balance between the gravitational attraction and the quantum pressure. This is the key equation to understand the origin of RAR in our model. It is interesting that the fuzzy DM density profile \( \rho_d \) and hence the wavefunction \( \psi \) trace the total gravitational acceleration not just \( g_d \). Using an approximation \( \partial_t \sim 1/\xi \) in \( \nabla Q/m \) one can define the characteristic acceleration for fuzzy DM halos more precisely
\[ g^\dagger \equiv \frac{\hbar^2}{2m^2 \xi^3} = 2.2 \times 10^{-10} \left( \frac{10^{-22} \text{eV}}{m} \right)^2 \left( \frac{300 \text{pc}}{\xi} \right)^3 \text{m/s}^2. \quad \text{(9)} \]

Note that this scale has a quantum mechanical origin which is a unique feature of fuzzy DM. \( g^\dagger \) defined this way is almost independent of \( \rho_d \). This fact might explain the universality of \( g^\dagger \). Quite interestingly, if we use the typical core size of the dwarf galaxies (\( \sim 300 \text{pc} \)) as \( \xi \), one can reproduce the observed value \( g^\dagger = 1.2 \times 10^{-10} \text{m/s}^2 \) from the left. The dashed line represents the observed value \( g^\dagger = 1.2 \times 10^{-10} \text{m/s}^2 \).

Let us see how \( g^\dagger \) affects galaxies. According to precise numerical studies with fuzzy DM [29] a massive galaxy has a soliton-like core with size \( \sim \xi \) surrounded by a virialized halo of granules (also with size \( \sim \xi \)) having a Navarro-Frenk-White (NFW) density profile. In the regions where \( g_{\text{obs}} \gg g^\dagger \) (as in a center of a galaxy) baryonic matter is usually more concentrated than fuzzy DM and the gravitational acceleration is mainly from baryon mass. On the other hand, a DM dominated region usually has \( g_{\text{obs}} \leq g^\dagger \). Therefore, for massive galaxies, \( g^\dagger \) acts as a parameter discriminating baryonic matter dominated regions \( (r < \xi) \) from DM dominated regions \( (r > \xi) \). For baryonic matter dominated regions such as central parts of massive galaxies \( g_b \gg g_d \), and obviously \( g_{\text{obs}} \approx g_b \gg g^\dagger \), which explains the 1:1 linear part of RAR graph in Fig. 2.

On the other hand, there are three regions where \( g_{\text{obs}} \) can be much smaller than \( g^\dagger \): I) Outermost edge of galaxies \( (r > O(10^2) \text{kpc}) \). II) Outer parts of massive galaxies with almost flat RCs \( (\text{kpc} < r < O(10) \text{kpc}) \). III) Small dwarf galaxies \( (r < \text{kpc}) \).

Unlike MOND, in our model if a galaxy is well isolated from others, the rotation velocity in the region I is expected to drop off because of lack of matter. For ex-
ample, the Milky way and earlier galaxies seem to have falling RCs [30, 31] in the outermost edge. However, observational data in this region is still rare and uncertain, so we ignore this region in this letter to understand the observed RAR.

Since the observational data points satisfying Eq. (1) mainly come from the region II, and BTFR also relies on the flat rotation velocity data in this region we will focus on the flat RCs for which \( g_{\text{obs}} \sim r^{-1} \). There are many attempts to obtain the flat RCs with SFDM using excited states [14, 15, 34] or specific potentials [35, 36]. To find the RAR in the region II in fuzzy DM models we need to know \( \rho_d \). Numerical studies with only fuzzy DM indicate that DM halos have a solitonic core with size about \( \xi \) surrounded by an NFW-like profile from virialized granules [29]. Thus, an average DM density over the granules for this quasi-stationary system can be roughly given by using a step function \( \Theta \) [37]:

\[
\rho_d(r) \simeq \Theta(r_e - r)\rho_{\text{tot}} + \Theta(r - r_e)\rho_{\text{NFW}}. \tag{10}
\]

Here \( \rho_{\text{tot}} \propto 1/(1 + (r/r_e)^2)^8 \) is a soliton density, and the NFW profile is \( \rho_{\text{NFW}} = \rho_{\text{d0}} r_0 / (r + r_0^2/r_e^2) \) with constants \( \rho_{\text{d0}}, r_0, r_e \) and \( r_e \). Quite interestingly, a recent numerical work [38] found that if we include baryon (stars) in the inner halo, the total matter density \( \rho_{\text{tot}} \equiv \rho_b + \rho_d \) follows an almost isothermal profile \( \rho_{\text{tot}} \sim r^{-2} \) and \( g_{\text{obs}} \sim r^{-1} \) near the half-light radius where soliton-NFW transition happens. The cases exhibit this features are when \( \rho_b \) is comparable to \( \rho_d \) at the half mass radius, which is consistent with the condition to derive \( g^\dagger \) below Eq. (9). Therefore, in fuzzy DM model the region where \( g_{\text{obs}} \ll g^\dagger \) in massive galaxies usually corresponds to the region with almost flat RCs and \( r < O(10)\text{kpc} \) as observed.

In this region, we can find a relation between \( g_{\text{obs}} \) and \( g_b \) by a simple reasoning. As \( r \) increases beyond baryon dominated regions, \( M_b(r) \) slowly approaches a total baryon mass \( M_b = \text{const.} \), and \( g_b(r) \) decreases faster than \( g_{\text{obs}}(r) \) does. At a point \( r^\dagger \) the acceleration \( g_b \) becomes comparable to \( g_{\text{obs}} \), and \( g_{\text{obs}} \) approaches the typical value \( g_b \), which means \( g_b(r^\dagger) \approx g_b^\dagger/2 \) and RCs become flat. The numerical work [38] indicates \( r^\dagger \) is about the half-light radius, i.e., \( M_b(r^\dagger) \approx M_b/2 \). Therefore, \( g_b(r^\dagger) \approx GM_b/r^2 \approx g^\dagger \). From Eq. (9) it implies

\[
r^\dagger \approx \sqrt{GM_b/g_b^\dagger} = \sqrt{2GM^2\xi^3M_b/h}. \tag{11}
\]

Thereby, bigger \( M_b \) means larger \( r^\dagger \). Around this point \( g_{\text{obs}} = |\delta \Phi/\partial r| \) starts to be small and the rotation velocity graph \( v(r) \approx \sqrt{\Phi(r)} \) has a gentle slope, which means almost flat RCs, i.e., \( v(r) \approx v_f \) [38]. Using \( r^\dagger \) above one can estimate the constant rotation velocity

\[
v_f \equiv \sqrt{r^\dagger g^\dagger} = \sqrt{GM_b/r^\dagger} \approx (GM_b g_b^\dagger)^{1/4}, \tag{12}
\]

which is just BTFR, \( M_b = Av_f^4 \) with

\[
A = (Gg_b^\dagger)^{-1} = \frac{2m^2\xi^3}{G\hbar^2} = 34.16 \left(\frac{m}{10^{-22}\text{eV}}\right)^2 \left(\frac{\xi}{300\text{pc}}\right)^3 M_\odot/(\text{km/s})^4. \tag{13}
\]

Remarkably, with Eq. (9) it reproduces the observed value \( A = 47 \pm 6 \ M_\odot\text{km}^{-1}\text{s}^4 \) [39], if \( m = 1.173 \pm 0.07 \times 10^{-22}\text{eV} \) for \( \xi = 300\text{pc} \). One of the advantages of our approach is that \( g^\dagger \) and \( A \) can be derived from the model. In our model, BTFR has a quantum mechanical origin, although it is a relation among macroscopic quantities of baryonic matter. (A Tully-Fisher-like relation between the total DM and the circular velocity was suggested for fuzzy DM in Ref. 40.) Following Ref. 41 we can derive the asymptotic form of RAR from the BTFR \( M_b = v_f^4/Gg_b^\dagger \),

\[
g_b(r \gg r^\dagger) \approx \frac{GM_b}{r^2} = \frac{1}{g_b^\dagger} \left(\frac{v_f^2}{r}\right)^2 = \frac{g_{\text{obs}}^2}{g^\dagger}, \tag{14}
\]

i.e., \( g_{\text{obs}} = \sqrt{g^\dagger g_b^\dagger} \). This is the MOND-like behavior of \( g_{\text{obs}} \) in the RAR graph at large radii where \( g_b \ll g^\dagger \) and \( v(r) \approx v_f \). Thus, in our model MOND is just an effective phenomenon of fuzzy DM. Therefore, fuzzy DM can explain the apparent successes of both of CDM and MOND, because it acts as CDM at super-galactic scales and as an effective MOND at galactic scales due to the finite length scale \( \xi \). The mass discrepancy-acceleration relation (MDAR) appears [42], because \( M_{\text{tot}}(r)/M_b(r) = g_{\text{obs}}(r)/g_b(r) \approx \sqrt{g^\dagger/g_b} \), where \( M_{\text{tot}}(r) \) is the total mass.
enclosed within $r$. We now understand how RAR behaves in our model in two extreme limits where $g_b \gg g^\dagger$, or $g_b \ll g^\dagger$. An approximated function linking the two limits for RAR is $g_{obs} = g_b + \sqrt{g^\dagger g_b}$, which is a simple sum of $g_b$ and $g_{ob}$ $\geq \sqrt{g^\dagger g_b}$ in Eq. (8) (See Fig. 2). BTFR and RAR in our model can have small scatter because these relations are from the dynamical equilibrium condition rather than from forming history of galaxies or from baryon physics.

Eq. (8) seems to explain some other mysteries in massive galaxies. First, for galaxies with flat RCs we can roughly approximate the total density with a core-isothermal one $\rho_{obs} \simeq \sigma^2/2\pi Gr(r^2 + r^{'2}) \equiv \rho_c/(1 + (r/r^{'})^2)$ up to a few $r^\dagger$ as an effective core size. This leads to an universal surface density of cored galaxies [43]

$$\Sigma \simeq \rho_c r^\dagger \simeq \frac{\sigma^2}{2\pi Gr^\dagger} \simeq \frac{g^\dagger}{2\pi G}.$$  (15)

Here $\sigma$ is the stellar velocity dispersion and $g^\dagger \simeq \sigma^2/r^\dagger$. With Eq. (9) this reproduces the observed value [44] $\Sigma = 141^{+82}_{-52} M_{\odot}/pc^{-2}$ for $m = 1.33^{+35}_{-27} \times 10^{-12} eV$ and $\xi = 300$ pc. Second, for the isothermal distribution where $g_{obs} \ll g^\dagger$ the wavefunction $\psi$ in the region II should be dynamically adjusted to satisfy Eq. (8) under the small variation of $\rho_b(r)$, which explains the baryon-halo conspiracy for flat RCs [45]. Finally, we observe that Eq. (8) can be rearranged to an integro-differential equation for $\rho_d(r)$;

$$g_b(r) = -g_d(\rho_d(r)) + \left| \frac{\nabla Q(\rho_d(r))}{m} \right|,$$  (16)

where $g_b$ plays a role of a source term or a boundary condition. A solution $\rho_d(r)$ of this wave equation at large $r$ should be such that the right hand side approaches $g_b(r) \simeq GM_b/r^2$. For this solution details of baryon distribution at central regions except for $M_b$ are not so much relevant. This explains why $g_d$ and hence $g_{obs}$ are so sensitive to $g_b$ in massive galaxies despite of variety of the galaxies and at the same time insensitive to other visible matter properties like luminosity.

We move to the region III. In small dwarf galaxies the spatial size of baryonic matter distribution is comparable to that of DM halos, and $M_b$ can not play a role of central boundary condition as in the region II. Thereby, the arguments related to flat RCs do not hold in this region. In fuzzy DM model these galaxies are similar to the ground state (soliton) of boson stars which has a minimum mass comparable to the quantum Jeans mass.

The mass ($M_{tot}$)-radius ($R$) relation of solitonic core from the boson star theory is $M_{tot}R = \beta h^2/Gm^2$, where, for example, the constant $\beta = 3.925$ for the half mass radius of DM [20]. Therefore, using the mass-radius relation the core of DM dominated dwarf galaxies has a typical acceleration

$$g_0 = \frac{GM_{tot}}{R^2} \simeq \frac{G^3 m^4 M^3_{tot}}{\beta^2 h^4} \geq \frac{Gm^4}{\beta^2 h^4},$$  (17)

which gives $4.1 \times 10^{-12} ms^{-2}$ for $m = 1.35 \times 10^{-22} eV$ and $M_{tot} = 10^8 M_{\odot}$. Here we identify $\gamma M_J$ to be the minimum galaxy mass from the quantum Jeans mass

$$M_J(z) = \frac{\pi^{15/6}}{6} \left( \frac{\hbar}{G^2 m} \right)^{2} \tilde{\rho}(z)^{1/2},$$  (18)

where $\gamma \simeq 0.5$ is a numerical constant from numerical studies and $\tilde{\rho}(z)$ is the background matter density at redshift $z$. Since relevant mass here is the total mass $M_{tot} = M_b + M_d$, $g_{obs}$ is insensitive to the fraction of baryonic matter as long as $M_b \ll M_{tot}$. This explains the flattening and large scatter of the RAR curve for small dwarf galaxies where $g_b < 10^{-12} m/s^2$ (See Fig. 2). Note that $g_0$ has a minimum value from the quantum Jeans mass $M_J$.

Regarding galaxy formation, fuzzy DM has only two free parameters, the particle mass $m$ and the background matter density $\tilde{\rho}(z)$. If we represent $\xi$ with these parameters, we can fully determine $g^\dagger$ and $A$ from the model. From the boson star mass-radius relation [14, 46], a natural candidate for $\xi$ is suggested [47, 48] to be

$$\xi = \beta \frac{\hbar}{G M_{tot} m^2} = \frac{3 \beta h^{1/2}}{4 \pi^{13/4} (G m^2 \tilde{\rho}(z))^{1/4}},$$  (19)

which is about $2 kpc$ for $M_{tot} = 10^8 M_{\odot}$ and $m = 1.3 \times 10^{-22} eV$. This size is somewhat larger than the observed core size $r_c \sim O(10^2) pc$ for a massive galaxy, although the profile of the core is quite similar to the ground state of boson stars. According to numerical studies with fuzzy DM, the smallness of $r_c$ is attributed to the nonlocal uncertainty principle applied to $r_e$ and $\sigma$, i.e., $r_e, \sigma \sim \hbar/m$ [49]. More precisely, $r_c = 1.6a^{1/2}(10^{-22} eV/m)(10^9 M_{\odot}/M_h)^{1/3} kpc$, where $M_h$ is a halo mass [49] and $a$ is the scale factor of the universe. It gives $\xi \simeq r_c = 300 pc$ for typical halos with $M_h = 10^{11} M_{\odot}$ and $m = 1.15 \times 10^{-22} eV$ at present ($a = 1$). Since $r_c$ is a slow function of $M_h$, $\xi$ is almost independent of properties of massive galaxies such as luminosity. Therefore, fuzzy DM can reproduce the observed $g^\dagger$ from the first principle. From the $r_c$ formula we expect $g^\dagger \propto a^{-3/2} \propto (1+z)^{3/2}$.

Our analysis can be easily extended to the Faber-Jackson relation [50], which is an empirical relation $L \propto \sigma^4$ between the luminosity $L$ and the central stellar velocity dispersion $\sigma$ of elliptical galaxies. If we assume baryon mass to light ratio $\Upsilon_b \equiv M_b/L \simeq 3 M_{\odot}/L_{\odot}$ is almost constant for elliptical galaxies [51] and $\sigma \sim v_f$, BTFR in Eq. (13) implies

$$L = \frac{M_b}{\Upsilon_b} \simeq \frac{34.16 m^4}{\Upsilon_b} \left( \frac{m}{10^{-22} eV} \right)^2 \left( \frac{\xi}{300 pc} \right)^3 \frac{M_{\odot}}{(km/s)^4},$$  (20)
which is comparable to the observed value $L \simeq 10L_{\odot} \sigma^4/(\text{km/s})^4$ [50]. Due to differences in $\Upsilon_b$ for individual galaxies, we expect larger scatter in the Faber-Jackson relation than in BTFR as observed. There are many studies on the characteristic mass and length scale in SFDM models, however little attention has been given to the characteristic acceleration so far [52]. The acceleration scale of fuzzy DM related to the scaling laws such as BTFR and Faber-Jackson relations can play an important role in evolution of galaxies and deserves further studies. These relations and observed MOND-like phenomen in galaxies can add another support for fuzzy DM. This work will provide an avenue to understanding the nature of quantum gravity because the properties of characteristic length scale is related to those in emergent quantum gravity [26].

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