DIFFUSION LIMIT FOR THE RADIATIVE TRANSFER EQUATION
PERTURBED BY A MARKOVIAN PROCESS

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Abstract

We study the stochastic diffusive limit of a kinetic radiative transfer equation, which is non-linear, involving a small parameter and perturbed by a smooth random term. Under an appropriate scaling for the small parameter, using a generalization of the perturbed test-functions method, we show the convergence in law to a stochastic non-linear fluid limit.

Keywords: Kinetic equations, non-linear, diffusion limit, stochastic partial differential equations, perturbed test functions, Rosseland approximation, radiative transfer.

1 Introduction

In this paper, we are interested in the following non-linear equation

\[
\begin{aligned}
\partial_t f_\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f_\varepsilon &= \frac{1}{\varepsilon^2} \sigma(f_\varepsilon) L(f_\varepsilon) + \frac{1}{\varepsilon} f_\varepsilon m_\varepsilon, \\
\int_{\mathbb{T}^N} \int_{\mathbb{R}} f_\varepsilon(0) = f_0, & t \in [0,T], x \in \mathbb{T}^N, v \in V.
\end{aligned}
\]  

(1.1)

where \((V, \mu)\) is a measured space, \(a : V \to \mathbb{R}^N, \sigma : \mathbb{R} \to \mathbb{R}\). The notation \(\overline{f}\) stands for the average over the velocity space \(V\) of the function \(f\), that is

\[
\overline{f} = \int_V f \, d\mu(v).
\]

The operator \(L\) is a linear operator of relaxation which acts on the velocity variable \(v \in V\) only. It is given by

\[
L(f) := \overline{F} F - f,
\]

(1.2)

where \(v \mapsto F(v)\) is a velocity equilibrium function such that

\[
F > 0 \text{ a.s., } \overline{F} = 1, \quad \sup_{v \in V} F(v) < \infty.
\]

(1.3)

The term \(m_\varepsilon\) is a random process depending on \((t,x) \in \mathbb{R}^+ \times \mathbb{R}^N\) (see section 2.2). The precise description of the problem setting will be given in the next section. In this paper, we study the behaviour in the limit \(\varepsilon \to 0\) of the solution \(f_\varepsilon\) of (1.1).

Concerning the physical background in the deterministic case \((m_\varepsilon \equiv 0)\), equation (1.1) describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion. The unknown \(f_\varepsilon(t,x,v)\) then stands for a distribution function of photons having position \(x\) and velocity \(v\) at time \(t\). The

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function $\sigma$ is the opacity of the matter. When the surrounding medium becomes very large compared to the mean free paths $\varepsilon$ of photons, the solution $f^\varepsilon$ to (1.1) is known to behave like $\rho F$ where $\rho$ is the solution of the Rosseland equation

$$
\partial_t \rho - \text{div}_x (\sigma(\rho)^{-1}K \nabla_x \rho) = 0, \quad (t,x) \in [0,T] \times \mathbb{T}^N,
$$

and $F$ is the velocity equilibrium defined above. This is what we call the Rosseland approximation. In this paper, we investigate such an approximation where we have perturbed the deterministic equation by a smooth multiplicative random noise. To do so, we use the method of perturbed test-functions. This method provides an elegant way of deriving stochastic diffusive limit from random kinetic systems; it was first introduced by Papanicolaou, Stroock and Varadhan [11]. The book of Fouque, Garnier, Papanicolaou and Solna [9] presents many applications to this method. A generalization in infinite dimension of the perturbed test-functions method arose in recent papers of Debussche and Vovelle [7] and de Bouard and Gazeau [6].

In the deterministic case (that is when $m^\varepsilon \equiv 0$), the Rosseland approximation has been widely studied. In the paper of Bardos, Golse and Perthame [1], they derive the Rosseland approximation on a slightly more general equation of radiative transfer type than (1.1) where the solution also depends on the frequency variable $\nu$. Using the so-called Hilbert’s expansion method, they prove a strong convergence of the solution of the radiative transfer equation to the solution of the Rosseland equation. In [2], the Rosseland approximation is proved in a weaker sense with weakened hypothesis on the various parameters of the radiative transfer equation, in particular on the opacity function $\sigma$.

In the stochastic setting, the case where $\sigma \equiv \sigma_0$ is constant has been studied in the paper of Debussche and Vovelle [7] where they prove the convergence in law of the solution of (1.1) to a limit stochastic fluid equation by mean of a generalization of the perturbed test-functions method. Thus the radiative transfer equation (1.1) is a first step in studying approximation diffusion on non-linear stochastic kinetic equations since the operator $\sigma(f)Lf$ stands for a simple non-linear perturbation of the classical linear relaxation operator $L$.

As expected, we have to handle some difficulties caused by this non-linearity. In the paper of Debussche and Vovelle [7] is proved the tightness of the family of processes $(\rho^\varepsilon)_{\varepsilon > 0}$ in the space of time-continuous function with values in some negative Sobolev space $H^{-\eta}(\mathbb{T}^N)$. In our non-linear setting, this is not any more sufficient to succeed in passing to the limit as $\varepsilon$ goes to 0. As a consequence, the main step to overcome this difficulty is to prove the tightness of the family of processes $(\rho^\varepsilon)_{\varepsilon > 0}$ in the space $L^2(0,T;L^2(\mathbb{T}^N))$. This is made using averaging lemmas in the $L^2$ setting with a slight adaptation to our stochastic context. The main results about deterministic averaging lemmas that we will use in the sequel can be found in the paper of Jabin [10]. We point out that, thanks to this additional tightness result, we could handle the case of a more general and non-linear noise term in (1.1) of the form $\frac{1}{\varepsilon}m^\varepsilon \lambda(f^\varepsilon)$ where $\lambda : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function. In particular, this remains valid in the linear case $\sigma \equiv 1$ studied in the paper [7] of Debussche and Vovelle so that this paper can provide some improvements to their result.

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2 Preliminaries and main result

2.1 Notations and hypothesis

Let us now introduce the precise setting of equation (1.1). We work on a finite-time interval \([0, T]\) where \(T > 0\) and consider periodic boundary conditions for the space variable: \(x \in \mathbb{T}^N\) where \(\mathbb{T}^N\) is the \(N\)-dimensional torus. Regarding the velocity space \(V\), we assume that \((V, \mu)\) is a measured space.

In the sequel, \(L^2_{p-1}\) denotes the \(F^{-1}\) weighted \(L^2(\mathbb{T}^N \times V)\) space equipped with the norm
\[
\|f\|^2 := \int_{\mathbb{T}^N} \int_V \frac{|f(x, v)|^2}{F(v)} \, d\mu(v) \, dx.
\]

We denote its scalar product by \((.,.)\). We also need to work in the space \(L^2(\mathbb{T}^N)\), which will be often written \(L^2\) for short when the context is clear. In what follows, we will often use the inequality
\[
\|f\|_{L^2} \leq \|\overline{f}\|,
\]
which is just Cauchy-Schwarz inequality and the fact that \(\overline{F} = 1\). We also introduce the Sobolev spaces on the torus \(H^\gamma(\mathbb{T}^N)\), or \(H^\gamma\) for short. For \(\gamma \in \mathbb{N}\), they consist of periodic functions which are in \(L^2(\mathbb{T}^N)\) as well as their derivatives up to order \(\gamma\). For general \(\gamma \geq 0\), they are easily defined by Fourier series. For \(\gamma < 0\), \(H^{-\gamma}(\mathbb{T}^N)\) is the dual of \(H^{-\gamma}(\mathbb{T}^N)\).

Concerning the velocity mapping \(a : V \to \mathbb{R}^N\), we shall assume that it is bounded, that is
\[
\sup_{v \in V} |a(v)| < \infty. \tag{2.1}
\]

Furthermore, we suppose that the following null flux hypothesis holds
\[
\int_V a(v) F(v) \, d\mu(v) = 0, \tag{2.2}
\]
and that the following matrix
\[
K := \int_V a(v) \otimes a(v) F(v) \, d\mu(v)
\]
is definite positive. Finally, to obtain some compactness in the space variable by means of averaging lemmas, we also assume the following standard condition:
\[
\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \mu(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}) \leq \varepsilon^\theta, \tag{2.3}
\]
for some \(\theta \in (0, 1]\).

Let us now give several hypothesis on the opacity function \(\sigma : \mathbb{R} \to \mathbb{R}\). We assume that

(H1) There exist two positive constants \(\sigma_*\), \(\sigma^* > 0\) such that for almost all \(x \in \mathbb{R}\), we have
\[
\sigma_* \leq \sigma(x) \leq \sigma^*;
\]

(H2) the function \(\sigma\) is Lipschitz continuous.

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Similarly as in the deterministic case, we expect with (1.1) that \(\sigma(\mathcal{F})L(f)\) tends to zero with \(\varepsilon\), so that we should determine the equilibrium of the operator \(\sigma(\cdot)L(\cdot)\). In this case, since \(\sigma > 0\), they are clearly constituted by the functions of the form \(\rho F\) with \(\rho\) being independent of \(v \in V\). Note that it can easily be seen that \(\sigma(\cdot)L(\cdot)\) is a bounded operator from \(L^2_{F-1}\) to \(L^2_{F-1}\) and that it is dissipative; precisely, for \(f \in L^2_{F-1}\),

\[
(\sigma(\mathcal{F})Lf, f) = -\|\sigma(\mathcal{F})Lf\|^2 \leq 0. \tag{2.4}
\]

In the sequel, we denote by \(g(t, \cdot)\) the semi-group generated by the operator \(\sigma(\cdot)L(\cdot)\) on \(L^2_{F-1}\). It verifies, for \(f \in L^2_{F-1}\),

\[
\begin{aligned}
\frac{d}{dt}g(t, f) &= \sigma(g(t, f))Lg(t, f), \\
g(0, f) &= f,
\end{aligned}
\]

and we can show that it is given by

\[
g(t, f) = \mathcal{F}F + (f - \mathcal{F}F)e^{-t\sigma(\mathcal{F})}, \quad t \geq 0, \ f \in L^2_{F-1}.
\]

With the hypothesis (H1) made on \(\sigma\), we deduce the following relaxation property of the operator \(\sigma(\cdot)L(\cdot)\)

\[
g(t, f) \to \mathcal{F}F, \quad t \to \infty, \quad \text{in } L^2_{F-1}. \tag{2.5}
\]

2.2 The random perturbation

The random term \(m^\varepsilon\) is defined by

\[
m^\varepsilon(t, x) := m\left(\frac{t}{\varepsilon^2}, x\right),
\]

where \(m\) is a stationary process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and is adapted to a filtration \((\mathcal{F}_t)_{t \geq 0}\). Note that \(m^\varepsilon\) is adapted to the filtration \((\mathcal{F}_{t \varepsilon^2})_{t \geq 0} = (\mathcal{F}_{t-2t})_{t \geq 0}\).

We assume that, considered as a random process with values in a space of spatially dependent functions, \(m\) is a stationary homogeneous Markov process taking values in a subset \(W^{1,\infty}(\mathbb{T}^N)\). In the sequel, \(E\) will be endowed with the norm \(\| \cdot \|_\infty\) of \(L^\infty(\mathbb{T}^N)\). Besides, we denote by \(\mathcal{B}(E)\) the set of bounded functions from \(E\) to \(\mathbb{R}\) endowed with the norm \(\|g\|_\infty := \sup_{n \in E} |g(n)|\) for \(g \in \mathcal{B}(E)\).

We assume that \(m\) is stochastically continuous. Note that \(m\) is supposed not to depend on the variable \(v\). For all \(t \geq 0\), the law \(\nu\) of \(m_t\) is supposed to be centered

\[
\mathbb{E}m_t = \int_E n \, d\nu(n) = 0.
\]

We denote by \(e^{tM}\) a transition semi-group on \(E\) associated to \(m\) and by \(M\) its infinitesimal generator. \(D(M)\) stands for the domain of \(M\); it is defined as follows:

\[
D(M) := \left\{ u \in \mathcal{B}(E), \lim_{h \to 0} \frac{e^{hM}u - I}{h} u \text{ exists in } \mathcal{B}(E) \right\},
\]
and if \( u \in D(M) \), we have

\[
Mu := \lim_{h \to 0} \frac{e^{hM} - I}{h} u \text{ in } B(E).
\]

Moreover, we suppose that \( m \) is ergodic and satisfies some mixing properties in the sense that there exists a subspace \( \mathcal{P}_M \) of \( B(E) \) such that for any \( g \in \mathcal{P}_M \), the Poisson equation

\[
M\psi = g - \int_E g(n) \, d\nu(n) =: \tilde{g},
\]

has a unique solution \( \psi \in D(M) \) satisfying \( \int_E \psi(n) \, d\nu(n) = 0 \). We denote by \( M^{-1}\tilde{g} \) this unique solution, and assume that it is given by

\[
M^{-1}\tilde{g}(n) = -\int_0^\infty e^{tM}\tilde{g}(n)dt, \quad n \in E. \tag{2.6}
\]

In particular, we suppose that the above integral is well defined. We need that \( \mathcal{P}_M \) contains sufficiently many functions. Thus we assume that for all \( f, g \in L^2_{P-1} \), we have

\[
\psi^{(1)}_{f,g} : n \mapsto (fn, g) \in \mathcal{P}_M, \tag{2.7}
\]

and we then define \( M^{-1}I \) from \( E \) into \( W^{1,\infty}(\mathbb{T}^N) \) by

\[
(fM^{-1}I(n), g) := M^{-1}\psi^{(1)}_{f,g}(n), \quad \forall f, g \in L^2_{P-1}. \tag{2.8}
\]

Then, we also suppose that for all \( f, g, h \in L^2_{P-1} \) and all continuous operator \( B \) from \( L^2_{P-1} \) to the space of the continuous bilinear operators on \( L^2_{P-1} \times L^2_{P-1} \),

\[
\psi^{(2)}_{f,g} : n \mapsto (fnM^{-1}I(n), g), \quad \psi^{(3)}_{B,f,g,h} : n \mapsto B(f)(gn, hM^{-1}I(n)) \in \mathcal{P}_M. \tag{2.9}
\]

We need a uniform bound in \( W^{1,\infty}(\mathbb{T}^N) \) of all the functions of the variable \( n \in E \) introduced above. Namely, we assume, for all \( f, g \in L^2_{P-1} \) and all continuous operator \( B \) on \( L^2_{P-1} \),

\[
\|n\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*, \quad \|M^{-1}I(n)\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*,
\]

\[
\|M^{-1}\psi^{(2)}_{f,g}\| \leq C_*\|f\|\|g\|, \quad \|M^{-1}\psi^{(3)}_{B,f,g,h}\| \leq C_*\|B(f)\|\|f\|\|g\|, \tag{2.10}
\]

Finally, we suppose that for all \( f, g \in L^2_{P-1} \),

\[
n \mapsto (fM^{-1}I(n), g)^2 \in D(M) \text{ with } \|M[(fM^{-1}I(n), g)^2]\| \leq C_*\|f\|^2\|g\|^2. \tag{2.11}
\]

To describe the limiting stochastic partial differential equation, we then set

\[
k(x, y) = E \int_{\mathbb{R}} m_0(y)m_t(x) \, dt, \quad x, y \in \mathbb{T}^N.
\]

We can easily show that the kernel \( k \) belong to \( L^\infty(\mathbb{T}^N \times \mathbb{T}^N) \) and, \( m \) being stationary, that it is symmetric (see [7]). As a result, we introduce the operator \( Q \) on \( L^2(\mathbb{T}^N) \) associated to the kernel \( k \)

\[
Qf(x) = \int_{\mathbb{T}^N} k(x, y)f(y) \, dy,
\]

which is self-adjoint, compact and non-negative (see [7]). As a consequence, we can define the square root \( Q^{1/2} \) which is Hilbert-Schmidt on \( L^2(\mathbb{T}^N) \).

**Remark** The above assumptions on the process \( m \) are verified, for instance, when \( m \) is a Poisson process taking values in a bounded subset \( E \) of \( W^{1,\infty}(\mathbb{T}^N) \).
2.3 Resolution of the kinetic equation

In this section, we solve the linear evolution problem (1.1) thanks to a semi-group approach. We thus introduce the linear operator \( A := a(v) \cdot \nabla_x \) on \( L^2_{F^{-1}} \) with domain

\[
D(A) := \{ f \in L^2_{F^{-1}}, \nabla_x f \in L^2_{F^{-1}} \}.
\]

The operator \( A \) has dense domain and, since it is skew-adjoint, it is \( m \)-dissipative. Consequently \( A \) generates a contraction semigroup \((T(t))_{t \geq 0}\) (see [4]). We recall that \( D(A) \) is endowed with the norm \( \| \cdot \|_{D(A)} := \| \cdot \| + \| A \cdot \| \), and that it is a Banach space.

**Proposition 2.1.** Let \( T > 0 \) and \( f_0^\varepsilon \in L^2_{F^{-1}} \). Then there exists a unique mild solution of (1.1) on \([0, T]\) in \( L^\infty(\Omega) \), that is there exists a unique \( f^\varepsilon \in L^\infty(\Omega, C([0, T], L^2_{F^{-1}})) \) such that \( \mathbb{P} \)-a.s.

\[
f_t^\varepsilon = T \left( \frac{t}{\varepsilon} \right) f_0^\varepsilon + \int_0^t T \left( \frac{t - s}{\varepsilon} \right) \left( \frac{1}{\varepsilon^2} \sigma(f_s^\varepsilon) L f_s^\varepsilon + \frac{1}{\varepsilon} m_s^\varepsilon f_s^\varepsilon \right) ds, \quad t \in [0, T].
\]

Assume further that \( f_0^\varepsilon \in D(A) \), then there exists a unique strong solution \( f^\varepsilon \) which belongs to the spaces \( L^\infty(\Omega, C^1([0, T], L^2_{F^{-1}})) \) and \( L^\infty(\Omega, C([0, T], D(A))) \) of (1.1).

**Proof.** Subsections 4.3.1 and 4.3.3 in [4] gives that \( \mathbb{P} \)-a.s. there exists a unique mild solution \( f^\varepsilon \in C([0, T], L^2_{F^{-1}}) \) and it is not difficult to slightly modify the proof to obtain that in fact \( f^\varepsilon \in L^\infty(\Omega, C([0, T], L^2_{F^{-1}})) \) (we intensively use that for all \( t \geq 0 \) and \( \varepsilon > 0 \), \( \| m_t^\varepsilon \|_{W^{1, \infty}(\mathbb{T}^N)} \leq C_s \)).

Similarly, subsections 4.3.1 and 4.3.3 in [4] gives us \( \mathbb{P} \)-a.s. a strong solution \( f^\varepsilon \) in the spaces \( C^1([0, T], L^2_{F^{-1}}) \) and \( C([0, T], D(A)) \) of (1.1) and once again one can easily get that in fact \( f^\varepsilon \) belongs to the spaces \( L^\infty(\Omega, C^1([0, T], L^2_{F^{-1}})) \) and \( L^\infty(\Omega, C([0, T], D(A))) \). \( \square \)

**Remark** If \( f_0^\varepsilon \in D(A) \), we thus have, for \( \varepsilon > 0 \) fixed,

\[
\sup_{t \in [0, T]} \| f_t^\varepsilon \| + \sup_{t \in [0, T]} \| Af_t^\varepsilon \| \in L^\infty(\Omega).
\]

(2.12)

2.4 Main result

We are now ready to state our main result.

**Theorem 2.2.** Assume that \((f_0^\varepsilon)_{\varepsilon > 0}\) is bounded in \( L^2_{F^{-1}} \) and that

\[
\rho_0^\varepsilon := \int_{\mathbb{T}^N} f_0^\varepsilon \, d\mu(v) \xrightarrow{\varepsilon \to 0} \rho_0 \text{ in } L^2(\mathbb{T}^N).
\]

Then, for all \( \eta > 0 \) and \( T > 0 \), \( \rho^\varepsilon := \int f^\varepsilon \) converges in law in \( C([0, T], H^{-\eta}(\mathbb{T}^N)) \) and \( L^2(0, T; L^2(\mathbb{T}^N)) \) to the solution \( \rho \) to the non-linear stochastic diffusion equation

\[
d\rho - \text{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) \, dt = H \rho \, dt + \rho Q^\frac{1}{2} \, dW_t, \quad \text{in } [0, T] \times \mathbb{T}^N,
\]

(2.13)

with initial condition \( \rho(0) = \rho_0 \) in \( L^2(\mathbb{T}^N) \), and where \( W \) is a cylindrical Wiener process on \( L^2(\mathbb{T}^N) \),

\[
K := \int_{\mathbb{T}^N} a(v) \otimes a(v) F(v) \, d\mu(v)
\]

(2.14)
Proposition 3.1. Let $D\phi$ be a continuous and integrable \((L^2,\mathbb{P})\)-martingale problem. Furthermore, if \(\phi\) is differentiable with respect to \(f\), its differential at a point \(f\) and we identify the differential with the gradient.

**Remark** The limit equation (2.13) can also be written in Stratonovich form

\[
d\rho - \text{div}_x(\sigma(\rho)^{-1}K\nabla_x \rho) \, dt = \rho \circ Q^\frac{d}{dt} dW_t.
\]

**Notation** In the sequel, we denote by \(\leq\) the inequalities which are valid up to constants of the problem, namely \(C_*,N,\sup_{\varepsilon>0}\|f\|_1,\sup_{v \in V} |a(v)|,\sup_{v \in V} F(v),\sigma_*,\sigma^*,\|\sigma\|_{\text{Lip}}\) and real constants.

### 3 The generator

The process \(f^\varepsilon\) is not Markov (indeed, by (1.1), we need \(m^\varepsilon\) to know the increments of \(f^\varepsilon\)) but the couple \((f^\varepsilon,m^\varepsilon)\) is. From now on, we denote by \(\mathcal{L}^\varepsilon\) its infinitesimal generator, that is

\[
\mathcal{L}^\varepsilon \varphi(f,n) := \lim_{h \to 0} \frac{1}{h} \mathbb{E}[\varphi(f^\varepsilon_h,n^\varepsilon_h) - \varphi(f,n)|f^\varepsilon_0,m^\varepsilon_0 = (f,n)],
\]

where \(\varphi : L^2_{F^{-1}} \times E \to \mathbb{R}\) belongs to the domain of \(\mathcal{L}^\varepsilon\). Thus we begin this section by introducing a special set of functions which lie in the domain of \(\mathcal{L}^\varepsilon\) and satisfy the associated martingale problem.

In the following, if \(\varphi : L^2_{F^{-1}} \to \mathbb{R}\) is differentiable with respect to \(f \in L^2_{F^{-1}}\), we denote by \(D\varphi(f)\) its differential at a point \(f\) and we identify the differential with the gradient.

**Definition 3.1.** We say that \(\varphi : L^2_{F^{-1}} \times E \to \mathbb{R}\) is a good test function if

(i) \((f,n) \mapsto \varphi(f,n)\) is differentiable with respect to \(f\);

(ii) \((f,n) \mapsto D\varphi(f,n)\) is continuous from \(L^2_{F^{-1}} \times E\) to \(L^2_{F^{-1}}\) and maps bounded sets onto bounded sets;

(iii) for any \(f \in L^2_{F^{-1}}\), \(\varphi(f,\cdot) \in D_M\);

(iv) \((f,n) \mapsto M\varphi(f,n)\) is continuous from \(L^2_{F^{-1}} \times E\) to \(\mathbb{R}\) and maps bounded sets onto bounded sets.

**Proposition 3.1.** Let \(\varphi\) be a good test function. Then, for all \((f,n) \in D(A) \times E\),

\[
\mathcal{L}^\varepsilon \varphi(f,n) = -\frac{1}{\varepsilon}(Af,D\varphi(f)) + \frac{1}{\varepsilon^2}(\sigma(f)Lf,D\varphi(f)) + \frac{1}{\varepsilon}(fn,D\varphi(f)) + \frac{1}{\varepsilon^2}M\varphi(f,n).
\]

Furthermore, if \(f^\varepsilon_0 \in D(A)\),

\[
M^\varepsilon_\varphi(t) := \varphi(f^\varepsilon_t,m^\varepsilon_t) - \varphi(f^\varepsilon_0,m^\varepsilon_0) - \int_0^t \mathcal{L}^\varepsilon \varphi(f^\varepsilon_s,m^\varepsilon_s) \, ds
\]

is a continuous and integrable \((\mathcal{F}_t^\varepsilon)_{t \geq 0}\) martingale, and if \(|\varphi|^2\) is a good test function, its quadratic variation is given by

\[
\langle M^\varepsilon_\varphi \rangle_t = \int_0^t (\mathcal{L}^\varepsilon |\varphi|^2 - 2\varphi \mathcal{L}^\varepsilon \varphi)(f^\varepsilon_s,m^\varepsilon_s) \, ds.
\]
Proof. We compute the expression of the infinitesimal generator as follows:

\[ \mathcal{L}^\varepsilon \varphi(f, n) = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \varphi(f_h^n, m_h^n) - \varphi(f, n) \right] = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \varphi(f_h^n, m_h^n) - \varphi(f, m_h^n) \right] (f_0^n, m_0^n) = (f, n) \]

\[ = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \varphi(f_h^n, m_h^n) - \varphi(f, m_h^n) \right] (f_0^n, m_0^n) = (f, n) \]

\[ + \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \varphi(f, m_h^n) - \varphi(f, n) \right] m_0^n = n \]

Since \( \varphi \) verifies point (iii) of Definition 3.1, the second term of the last equality goes to \( \varepsilon^{-2} M \varphi(f, n) \) when \( h \to 0 \). We now focus on the first term. With points (i) – (ii) of Definition 3.1, we have that \( \varphi \) is continuously differentiable with respect to \( f \). Thus

\[ \varphi(f_h^n, m_h^n) - \varphi(f, m_h^n) = \int_0^1 D\varphi(f + s(f_h^n - f), m_h^n) (f_h^n - f) \, ds. \]

Besides, since \( f_0^n = f \in D(A) \), \( f^n \in C^1([0, T], L^2_{\mathbb{F}, \mathbb{P})} \) and we have

\[ f_h^n - f = h \int_0^1 \partial_t f_{\text{uh}} \, du. \]

Thus, we can rewrite the first term as

\[ = \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \varphi(f_h^n, m_h^n) - \varphi(f, m_h^n) \right] (f_0^n, m_0^n) = (f, n) \]

\[ = \lim_{h \to 0} \mathbb{E}_{(f, n)} \left[ \int_0^1 \int_0^1 a_h(w, s, u) \, du \, ds \right], \]

with \( a_h(w, s, u) := D\varphi(f + s(f_h^n - f), m_h^n) (\partial_t f_{\text{uh}}) \) and where \( \mathbb{E}_{(f, n)} \) denotes the expectation under the probability measure \( \mathbb{P}_{(f, n)} := \mathbb{P}(\cdot | (f_0^n, m_0^n) = (f, n)). \)

Recall that \( D\varphi \) is continuous with respect to \((f, n)\) thanks to point (ii) of Definition 3.1, that \( f^n \) is \( \mathbb{P}-\)a.s. in \( C^1([0, T], L^2_{\mathbb{F}, \mathbb{P}}) \) and that \( m^n \) is stochastically continuous to conclude that \( a_h \) converges in probability as \( h \to 0 \) to \( D\varphi(f, n) (\partial_t f_{\text{uh}}) \) in the probability space \( \Omega := (\Omega \times [0, 1] \times [0, 1], \mathbb{F}_{(f, n)} \otimes dx \otimes ds) \). Furthermore, we prove that \( (a_h)_{0 \leq h \leq 1} \) is uniformly integrable in \( \Omega \) since it is uniformly bounded with respect to \( 0 \leq h \leq 1 \) in \( L^\infty(\Omega) \). Indeed, with the fact that \( L \) is a bounded operator, with (H1) and the fact that \( \|n\|_{L^\infty(\Omega)} \leq 1 \) for all \( n \in E \), we get

\[ |a_h| \leq \|D\varphi(f + s(f_h^n - f), m_h^n)\| \|f_{\text{uh}}\| + \|A f_{\text{uh}}\|. \]

With (2.12), we set

\[ R := \sup_{t \in [0, T]} \|f^n\| + \sup_{t \in [0, T]} \|A f^n\| \in L^\infty(\Omega), \]

and define \( r := \|R\|_{L^\infty(\Omega)}. \) Then, since \( D\varphi \) maps bounded sets on bounded sets, we can bound the term \( \|D\varphi(f + s(f_h^n - f), m_h^n)\| \) by

\[ C := \sup \left\{ ||D\varphi(f, n)||, f \in B_{L^2_{\mathbb{F}, \mathbb{P}}}(0, \|f\| + r), n \in B_{E}(0, C_*) \right\}. \]

So we are led to

\[ \|a_h\|_{L^\infty(\Omega)} \leq C \cdot r, \]

which is what we announced. To prove the sequel of the proposition, we use the same kind of ideas and follow the proofs of [7, Proposition 6] and [9, Appendix 6.9].
4 The limit generator

In this section, we study the limit of the generator $\mathcal{L}^\varepsilon$ when $\varepsilon \to 0$. The limit generator $\mathcal{L}$ will characterize the limit stochastic fluid equation.

4.1 Formal derivation of the corrections

To derive the diffusive limiting equation, one has to study the limit as $\varepsilon$ goes to 0 of quantities of the form $\mathcal{L}^\varepsilon \varphi$ where $\varphi$ is a good test function. To do so, following the perturbed test-functions method, we have to correct $\varphi$ so as to obtain a non-singular limit. We search the correction $\varphi^\varepsilon$ of $\varphi$ under the classical form:

$$\varphi^\varepsilon := \varphi + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2.$$ 

In this decomposition, $\varphi_1$ and $\varphi_2$ are respectively the first and second order corrections and are to be defined in the sequel so that

$$\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L} \varphi + O(\varepsilon),$$

where $\mathcal{L}$ will be the limit generator. We restrict our study to smooth test-functions. Precisely, we introduce the set of spatial derivative operators up to order 3:

$$\mathcal{R} := \{\partial_{i_1}^\varepsilon \partial_{i_2}^\varepsilon \partial_{i_3}^\varepsilon, \varepsilon \in \{0,1\}^3, i \in \{1, ..., N\}^3, |i| \leq 3\}$$

and we suppose that the test-function $\varphi$ is a good test, that $\varphi \in C^3(L^2_{\rho-1})$ and that there exists a constant $C_\varphi > 0$ such that

$$\begin{align*}
|\varphi(f)| &\leq C_\varphi (1 + \|f\|^2), \\
\|A\partial^2 \varphi(f)\| &\leq C_\varphi (1 + \|f\|), \\
|D^2 \varphi(f)(\Lambda_1 h, \Lambda_2 k)| &\leq C_\varphi \|h\| \|k\|, \\
|D^3 \varphi(f)(\Lambda_1 h, \Lambda_2 k, \Lambda_3 l)| &\leq C_\varphi \|h\| \|k\| \|l\|, \\
\end{align*}$$

for any $f, h, k, l \in L^2_{\rho-1}$ and $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{R}$. Thanks to Proposition 3.1, and since $\varphi$ does not depend on $n \in E$, we can write

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) = -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2} (\sigma(f) Lf, D\varphi(f)) + \frac{1}{\varepsilon} (fn, D\varphi(f)) + \frac{1}{\varepsilon}(\sigma(f) Lf, D\varphi_1(f)) + \frac{1}{\varepsilon}(fn, D\varphi_1(f)) + \frac{1}{\varepsilon} M\varphi_1$$

$$- (Af, D\varphi_1(f)) + \frac{1}{\varepsilon}(\sigma(f) Lf, D\varphi_1(f)) + (fn, D\varphi_1(f)) + \frac{1}{\varepsilon}M\varphi_1 + \varepsilon(Af, D\varphi_2(f)) + (\sigma(f) Lf, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)) + M\varphi_2.$$ 

In the sequel, we do not care about the terms relative to the transport part $A$ of the equation since these terms will be handled as in the deterministic case (when $m^\varepsilon \equiv 0$). To be more precise, and as it will be shown in the sequel, the first term of (4.2) will give rise, as $\varepsilon$ goes to 0, to the deterministic term in the limit generator $\mathcal{L}$ and the first terms of (4.3) and (4.4) are respectively of orders $\varepsilon$ and $\varepsilon^2$. For the remaining terms, in a first step, we would like to cancel those who have a singular power of $\varepsilon$. Thus we should impose that the two following equations hold:

$$(\sigma(f) Lf, D\varphi(f)) = 0.$$
\[(\sigma(\bar{f})Lf, D\varphi_1(f)) + M\varphi_1 + (fn, D\varphi(f)) = 0.\]  
(4.6)

Let us say a word about the fact that we chose to handle the terms relative to the transport part of the equation separately. When trying to correct these terms thanks to the correctors \(\varphi_1\) and \(\varphi_2\), the non-linearity \(\sigma\) implies that the second corrector \(\varphi_2\), unless we can write it formally, does not behave properly any more.

### 4.1.1 Equation on \(\varphi\)

Let us solve (4.5). We recall that \((g(t, f))_{t \geq 0}\) denotes the semigroup of the operator \(\sigma(\bar{f})L\).

Equation (4.5) gives immediately that the map \(t \mapsto \varphi(g(t, f))\) is constant. As a result, with (2.5),

\[\varphi(f) = \varphi(g(0, f)) = \varphi(g(\infty, f)) = \varphi(\bar{f}F),\]

so that \(\varphi\) only depends on \(\bar{F}F\). This implies, for all \(h \in L^2 F\),

\[(h, D\varphi(f)) = (\bar{h}F, D\varphi(\bar{f}F)).\]

(4.7)

### 4.1.2 Equation on \(\varphi_1\)

Next, we solve (4.6). We consider the Markov process \((g(t, f), m(t, n))_{t \geq 0}\). Its generator will be denoted by \(\mathcal{M}\). We observe that equation (4.6) rewrites:

\[\mathcal{M}\varphi_1(f, n) = -(fn, D\varphi(f)).\]

This Poisson equation will have a solution if the integral of \((f, n) \mapsto (fn, D\varphi(f))\) over \(L^2 F_{p-1} \times E\) equipped with the invariant measure of the process \((g(t, f), m(t, n))_{t \geq 0}\) is zero. So, we must verify that

\[\int_E (\bar{f}Fn, D\varphi(\bar{f}F)) d\nu(n) = 0,\]

and this relation does hold since \(m\) is centered. As a consequence, if we can prove the existence of the integral, we can write \(\varphi_1\) as

\[\varphi_1(f, n) = \int_0^\infty E(g(t, f)m(t, n), D\varphi(g(t, f))) dt.\]

Then, we use (4.7), \(g(t, f) = \bar{f}\) and (2.7) and (2.8) to obtain

\[\varphi_1(f, n) = \int_0^\infty E(\bar{f}Fm(t, n), D\varphi(\bar{f}F)) dt = -(\bar{f}FM^{-1}I(n), D\varphi(\bar{f}F)) = -(FM^{-1}I(n), D\varphi(f)).\]

We are now able to state the

**Proposition 4.1** (First corrector). Let \(\varphi \in C^3(L^2 F_{p-1})\) be a good test-function satisfying (4.1) and depending only on \(\bar{f}F\). For any \((f, n) \in L^2 F_{p-1} \times E\), we define the first corrector \(\varphi_1\) as

\[\varphi_1(f, n) := -(FM^{-1}I(n), D\varphi(f)).\]

Furthermore, it satisfies the bounds

\[(i) \ |\varphi_1(f, n)| \lesssim C_\varphi(1 + \|f\|)^2, \quad (ii) \ \|D\varphi_1(f, n)\| \lesssim C_\varphi(1 + \|f\|).\]

(4.8)

Note that the bounds (4.8) are consequences of (2.10) and (4.1).
4.1.3 Equation on $\varphi_2$

At this stage, we have

$$L^\varepsilon \varphi(f, n) = -\frac{1}{\varepsilon} (Af, D\varphi) + \mathcal{M} \varphi_2 + (fn, D\varphi_1(f)) \tag{4.9}$$

$$- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).$$

Note that the limit of $L^\varepsilon \varphi$ as $\varepsilon$ goes to 0 does depend on $n \in E$ with the term $(fn, D\varphi_1(f))$. Since the expected limit is $L \varphi$ where $\varphi$ does not depend on $n$, we have to correct this term to cancel the dependence with respect to $n$ of the limit. This is the aim of the second order correction $\varphi_2$. The right way to do so, given the mixing properties of the operator $\mathcal{M}$, is to subtract the mean value of this term under the invariant measure of the Markov process $(g(t, f), m(t, n))_{t \geq 0}$ governed by $\mathcal{M}$. We write

$$L^\varepsilon \varphi(f, n) = -\frac{1}{\varepsilon} (Af, D\varphi) + \int_E (\mathcal{F}Fn, D\varphi_1(\mathcal{F})) d\nu(n)$$

$$+ \mathcal{M} \varphi_2 + (fn, D\varphi_1(f)) - \int_E (\mathcal{F}Fn, D\varphi_1(\mathcal{F})) d\nu(n)$$

$$- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)),$$

and we can now define $\varphi_2$ as the solution of the well-posed Poisson equation

$$\mathcal{M} \varphi_2 = -(fn, D\varphi_1(f)) + \int_E (\mathcal{F}Fn, D\varphi_1(\mathcal{F})) d\nu(n).$$

Note that, thanks to the definition of $\varphi_1$ given above, we can compute

$$(\mathcal{F}Fn, D\varphi_1(\mathcal{F})) = -(fnM^{-1}I(n), D\varphi(f)) - D^2\varphi(f)(fM^{-1}I(n), fn) =: q(f, n)$$

As a result, we easily have the following proposition.

**Proposition 4.2** (Second corrector). Let $\varphi \in C^3(L^2_{p-1})$ be a good test-function satisfying (4.1) and depending only on $\mathcal{F}F$. For any $(f, n) \in L^2_{p-1} \times E$, we define the second corrector $\varphi_2$ as

$$\varphi_2(f, n) := \mathbb{E} \int_0^\infty \left( \int_E (q(\mathcal{F}F, n) d\nu(n) - q(g(t, f), m(t, n)) \right) dt,$$

which is well defined and satisfies the bounds

$$(i) \ |\varphi_2(f, n)| \lesssim C_\varphi(1 + \|f\|)^2, \quad (ii) \ \|AD\varphi_2(f, n)\| \lesssim C_\varphi(1 + \|f\|). \tag{4.10}$$

The existence of $\varphi_2$ is based on (2.9) and the bounds (4.10) are proved using (2.10) and (4.1).

4.1.4 Summary

The correctors $\varphi_1$ and $\varphi_2$ being defined as above in Propositions 4.1 and 4.2, we are finally led to

$$L^\varepsilon \varphi(f, n) = -\frac{1}{\varepsilon} (Af, D\varphi) + \int_E (\mathcal{F}Fn, D\varphi_1(\mathcal{F})) d\nu(n)$$

$$- (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).$$
We are now able to define the limit generator $\mathcal{L}$ as, for all $\rho \in L^2(\mathbb{T}^N)$,

$$\mathcal{L}\varphi(\rho) := (\text{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) F, D\varphi(\rho F)) - \int_E (\rho F_n M^{-1} I(n), D\varphi(\rho F)) d\nu(n)$$

$$- \int_E D^2 \varphi(\rho F)(\rho F M^{-1} I(n), \rho F) d\nu(n),$$

and we have shown the following equality

$$\mathcal{L}^\varepsilon \varphi(\varepsilon, n) = \mathcal{L} \varphi(F) - \frac{1}{\varepsilon}(Af, D\varphi(f)) - (\text{div}_x(\sigma(F)^{-1} K \nabla_x F) F, D\varphi(F))$$

$$- (Af, D\varphi_1(f)) - \varepsilon (Af, D\varphi_2(f)) + \varepsilon (f_n, D\varphi_2(f)).$$

### 5 Uniform bound in $L^2_{\mathbb{T}^{N-1}}$

In this section, we prove a uniform estimate of the $L^2_{\mathbb{T}^{N-1}}$ norm of the solution $f^\varepsilon$ with respect to $\varepsilon$. To do so, we will again use the perturbed test functions method. The result is the following:

**Proposition 5.1.** Let $p \geq 1$ and $f_0 \in D(A)$. We have the two following bounds

$$\mathbb{E} \sup_{t \in [0, T]} \| f_t^\varepsilon \|_p^p \lesssim 1,$$

$$\mathbb{E} \left( \int_0^T \| \sigma^\frac{1}{2}(F_t^\varepsilon) Lf_t^\varepsilon \|_2^2 \, ds \right)^p \lesssim \varepsilon^{2p}.$$

**Proof.** We set, for all $f \in L^2_{\mathbb{T}^{N-1}}$, $\varphi(f) := \frac{1}{2}\| f \|^2$, which is easily seen to be a good test function. Then, with Proposition 3.1, the fact that $A$ is skew-adjoint, (2.4), and the fact that $\varphi$ does not depend on $n \in E$, we get for $f \in D(A)$ and $n \in E$,

$$\mathcal{L}^\varepsilon \varphi(f, n) = -\frac{1}{\varepsilon}(Af, f) + \frac{1}{\varepsilon^2}(\sigma(F) Lf, f) + \frac{1}{\varepsilon}(fn, f) + \frac{1}{\varepsilon^2}M\varphi(f, n)$$

$$= -\frac{1}{\varepsilon^2} \| \sigma^\frac{1}{2}(F_t^\varepsilon) Lf \|_2^2 + \frac{1}{\varepsilon}(fn, f).$$

The first term has a favourable behaviour for our purpose. The second term is more difficult to control and we correct $\varphi$ thanks to the perturbed test-functions method to get rid of it: we recall the formal computations done in Section 4.1 and we set $\varphi_1(f, n) = -(f, M^{-1} I(n)f)$ and $\varphi^\varepsilon : = \varphi(f, n) + \varepsilon \varphi_1$. We can show that $\varphi_1$ is a good test function with, thanks to Proposition 3.1,

$$\varepsilon \mathcal{L}^\varepsilon \varphi_1(f, n) = \frac{2}{\varepsilon}(\sigma(F) Lf, M^{-1} I(n)f) - 2(Af, M^{-1} I(n)f)$$

$$- 2(fn, M^{-1} I(n)f) - \frac{1}{\varepsilon}(fn, f).$$

As a consequence, we are led to

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) = -\frac{1}{\varepsilon^2} \| \sigma^\frac{1}{2}(F_t^\varepsilon) Lf \|_2^2 - \frac{2}{\varepsilon}(\sigma(F) Lf, M^{-1} I(n)f) - 2(Af, M^{-1} I(n)f)$$

$$- 2(fn, M^{-1} I(n)f).$$
We use (2.10) and the hypothesis (H1) made on $\sigma$ to bound the second term:

$$
\frac{2}{\varepsilon} \langle \sigma(T) L f, M^{-1} I(n) f \rangle \leq 2 C_n (\sigma^*) \varepsilon^{-1} \| \sigma(T) L f \| f \|
\leq \frac{1}{2 \varepsilon^2} \| \sigma(T) L f \|^2 + 2 C_n^2 \| f \|^2.
$$

Furthermore, for the last two terms, we write

$$
-2(Af, M^{-1} I(n) f) - 2(f n, M^{-1} I(n) f) = (f^2, AM^{-1} I(n)) - 2(f n, M^{-1} I(n) f)
\leq \| f \|^2 \| a \|_{L^\infty(V)} C_n + 2 C_n^2 \| f \|^2.
$$

To sum up, we have proved that

$$
\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) \lesssim \frac{1}{2 \varepsilon^2} \| \sigma(T) L f \|^2 + \| f \|^2.
$$

(5.3)

As in Proposition 3.1, since $\varphi^\varepsilon$ is a good test function, we now define

$$
M^\varepsilon(t) := \varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \varphi^\varepsilon(f^\varepsilon_0, m^\varepsilon_0) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s) \, ds,
$$

which is a continuous and integrable $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ martingale. By definition of $\varphi$, $\varphi^\varepsilon$ and $M^\varepsilon$, we obtain

$$
\frac{1}{2} \| f^\varepsilon_t \|^2 = \frac{1}{2} \| f^\varepsilon_0 \|^2 - \varepsilon(\varphi_1(f^\varepsilon_t, m^\varepsilon_t) - \varphi_1(f^\varepsilon_0, m^\varepsilon_0)) + \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s) \, ds + M^\varepsilon(t).
$$

Since we have obviously $|\varphi_1(f, n)| \lesssim \| f \|^2$, we can write, with (5.3),

$$
\| f^\varepsilon_t \|^2 \lesssim \| f^\varepsilon_0 \|^2 + \varepsilon \| f^\varepsilon_t \|^2 + \int_0^t \frac{1}{2 \varepsilon^2} \| \sigma(T) L f^\varepsilon_s \|^2 + \| f^\varepsilon_s \|^2 \, ds + \sup_{t \in [0,T]} |M^\varepsilon(t)|,
$$

i.e. for $\varepsilon$ sufficiently small,

$$
\int_0^t \frac{1}{2 \varepsilon^2} \| \sigma(T) L f^\varepsilon_s \|^2 ds + \| f^\varepsilon_t \|^2 \lesssim \| f^\varepsilon_0 \|^2 + \int_0^t \| f^\varepsilon_s \|^2 ds + \sup_{t \in [0,T]} |M^\varepsilon(t)|,
$$

and by Gronwall lemma,

$$
\int_0^t \frac{1}{2 \varepsilon^2} \| \sigma(T) L f^\varepsilon_s \|^2 ds + \| f^\varepsilon_t \|^2 \lesssim \| f^\varepsilon_0 \|^2 + \sup_{t \in [0,T]} |M^\varepsilon(t)|.
$$

(5.4)

Note that $|\varphi^\varepsilon|^2$ is a good test function with, thanks to (2.10) and (2.11),

$$
|\mathcal{L}^\varepsilon |\varphi^\varepsilon|^2 - 2 \varphi^\varepsilon \mathcal{L}^\varepsilon \varphi^\varepsilon| = |M| \varphi_1|^2 - 2 \varphi_1 M \varphi_1 \lesssim \| f \|^4,
$$

and that, with Proposition 3.1, the quadratic variation of $M^\varepsilon(t)$ is given by

$$
\langle M^\varepsilon \rangle_t = \int_0^t \langle \mathcal{L}^\varepsilon |\varphi^\varepsilon|^2 - 2 \varphi^\varepsilon \mathcal{L}^\varepsilon \varphi^\varepsilon \rangle(f^\varepsilon_s, m^\varepsilon_s) \, ds.
$$
As a result, with Burkholder-Davis-Gundy and Hölder inequalities, we get
\[ E \sup_{t \in [0,T]} |M^\varepsilon(t)|^p \lesssim E[(M^\varepsilon)_T]^\frac{p}{2} \lesssim \int_0^T E\|f^\varepsilon_s\|^{2p} \, ds. \tag{5.5} \]

Neglecting the first (positive) term of the left-hand side in (5.4), we have
\[ E\|f^\varepsilon_t\|^{2p} \lesssim E\|f^\varepsilon_0\|^{2p} + E \sup_{t \in [0,T]} |M^\varepsilon(t)|^p, \]
so that we get
\[ E\|f^\varepsilon_T\|^{2p} \lesssim E\|f^\varepsilon_0\|^{2p} + \int_0^T E\|f^\varepsilon_s\|^{2p} \, ds, \]
and, by Gronwall lemma,
\[ E\|f^\varepsilon_T\|^{2p} \lesssim E\|f^\varepsilon_0\|^{2p}. \tag{5.6} \]
This actually holds true for any \( t \in [0, T] \). Thus, using (5.5) and (5.6) in (5.4) finally gives the expected bounds. \( \square \)

**Remark** We define \( g^\varepsilon := f^\varepsilon - \rho^\varepsilon F = -LF^\varepsilon \). Since we have \( \sigma \geq \sigma_* \), the bound (5.2) gives that, for all \( p \geq 1 \),
\[ (\varepsilon^{-1}g^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{p-1})). \tag{5.7} \]

In the sequel, we must deal with the non-linear term. To do so, we need some compactness in the space variable of the process \((\rho^\varepsilon)_{\varepsilon>0}\). The following proposition is a first step to this purpose.

**Proposition 5.2.** We assume that hypothesis (2.3) is satisfied. Let \( p \geq 1 \) and \( s \in (0, \theta/2) \).

We have the bound
\[ E \left( \int_0^T \|\rho^\varepsilon_s\|_{H^s(\mathbb{T}^N)}^2 \, ds \right)^{p/2} \lesssim 1. \tag{5.8} \]

**Proof.** Note that with \( \sigma \leq \sigma_* \), the remark (5.7) and equation (1.1), we observe that
\[ (\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla x f^\varepsilon - f^\varepsilon m^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{p-1})). \]
Furthermore, \((f^\varepsilon)_{\varepsilon>0}\) is bounded in \( L^p(\Omega; L^2(0,T; L^2_{p-1}))\) with (5.1) and \( |m^\varepsilon| \leq C_* \) so that
\[ (\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla x f^\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(\Omega; L^2(0,T; L^2_{p-1})). \tag{5.9} \]

Then, thanks to (2.3), we apply an averaging lemma to conclude. Precisely, [10, Theorem 3.1] in the unstationary case applies a.s. with \( \beta = \gamma = 0, \, p_1 = q_1 = p_2 = q_2 = 2, \, a = 0, \, k = \theta \) and
\[ f = f^\varepsilon, \quad g = \varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla x f^\varepsilon, \]
and gives the bound
\[ \|\rho^\varepsilon\|_{H^s_{2,\infty}} \lesssim C \|f^\varepsilon\|^{\frac{2}{1}} \|\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla x f^\varepsilon\|^{\frac{2}{1}} \quad \text{a.s.} \]

Since, for any \( s < \theta/2, \, H^s \subset B^\frac{s}{2}_{2,\infty} \), it yields, for \( p \geq 1 \),
\[ E \left( \int_0^T \|\rho^\varepsilon_s\|_{H^s}^2 \, ds \right)^{p/2} \lesssim C E \left( \int_0^T \|f^\varepsilon_s\|^{2} \|\varepsilon \partial_t f^\varepsilon_s + a(v) \cdot \nabla x f^\varepsilon_s\| \, ds \right)^{p/2}, \]
so that the result follows with Cauchy Schwarz inequality and (5.1) and (5.9). This concludes the proof. \( \square \)
6 Tightness

We want to prove the convergence in law of the family \((\rho^\varepsilon)_{\varepsilon > 0}\); in this section, we study the tightness of the processes \((\rho^\varepsilon)_{\varepsilon > 0}\) in the space \(C([0, T], H^{-\eta}(\mathbb{T}^N))\) where \(\eta > 0\). In fact, this will not be sufficient to pass to the limit in the non-linear term. As a consequence, we also prove that \((\rho^\varepsilon)_{\varepsilon > 0}\) is tight in the space \(L^2(0, T; L^2(\mathbb{T}^N))\).

Proposition 6.1. Let \(\eta > 0\). Then the sequence \((\rho^\varepsilon)_{\varepsilon > 0}\) is tight in the spaces \(C([0, T], H^{-\eta}(\mathbb{T}^N))\) and \(L^2(0, T; L^2(\mathbb{T}^N))\).

Proof. Step 1: control of the modulus of continuity of \(\rho^\varepsilon\) in \(H^{-\eta}(\mathbb{T}^N)\). Let \(\eta > 0\) be fixed. For any \(\delta > 0\), we define

\[ w(\rho, \delta) := \sup_{|t-s| < \delta} \|\rho(t) - \rho(s)\|_{H^{-\eta}(\mathbb{T}^N)} \]

the modulus of continuity of a function \(\rho \in C([0, T], H^{-\eta}(\mathbb{T}^N))\). In this first step of the proof, we want to obtain the following bound

\[ Ew(\rho^\varepsilon, \delta) \lesssim \varepsilon + \delta^\tau, \tag{6.1} \]

for some positive \(\tau\). To do so, we use the perturbed test-functions method. Let \((p_j)_{j \in \mathbb{N}^N}\) the Fourier orthonormal basis of \(L^2(\mathbb{T}^N)\) and \(J\) the operator

\[ J := (I - \Delta_x)^{-\frac{1}{2}}. \]

Let \(j \in \mathbb{N}^N\). We set

\[ \varphi_j(f) := (f, p_j F), \quad f \in L^2_{F^{-1}}, \]

and we define the first order corrections by, see Section 4.1,

\[ \varphi_{1,j}(f, n) := -(f M^{-1} I(n), p_j F), \quad (f, n) \in L^2_{F^{-1}} \times E. \]

We finally define \(\varphi_j^\varepsilon := \varphi_j + \varepsilon \varphi_{1,j}\), which is easily seen to be a good test-function, so that, thanks to Proposition 3.1, we consider the continuous martingales

\[ M_j^\varepsilon(t) := \varphi_j^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \varphi_j^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) \, ds. \]

We also define,

\[ \theta_j^\varepsilon(t) := \varphi_j(f_0^\varepsilon) + \int_0^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) \, ds + M_j^\varepsilon(t). \]

Note that

\[ \theta_j^\varepsilon(t) = \varphi_j(f_0^\varepsilon) + \varepsilon(\varphi_{1,j}(f_0^\varepsilon, m_0^\varepsilon) - \varphi_{1,j}(f_0^\varepsilon, m_0^\varepsilon)), \tag{6.2} \]

so that, with the definitions of \(\varphi_j\) and \(\varphi_{1,j}\), Cauchy-Schwarz inequality, we easily get

\[ |\theta_j^\varepsilon(t)| \lesssim \sup_{t \in [0, T]} \|f^\varepsilon(t)\|_{L^2} = \sup_{t \in [0, T]} \|f^\varepsilon(t)\|. \]

Hence, by the uniform \(L^2_{F^{-1}}\) bound (5.1),

\[ E \sup_{t \in [0, T]} |\theta_j^\varepsilon(t)| \lesssim 1. \tag{6.3} \]
With (6.2) and the uniform $L^2_{F_{1-1}}$ bound (5.1), we also deduce
\[ \mathbb{E} \sup_{t \in [0,T]} |\varphi_j(\rho^\varepsilon(t)) - \theta_j^\varepsilon(t)| \lesssim \varepsilon. \] (6.4)

From now on, we fix $\gamma > N/2 + 2$ and we remark that, by (6.3), a.s. and for all $t \in [0,T]$, the series defined by $u^\varepsilon_t := \sum_{j \in \mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j$ converges in $L^2(\mathbb{T}^N)$. We then set
\[ \theta^\varepsilon(t) := J^{-\gamma} \sum_{j \in \mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j, \]
which exists a.s. and for all $t \in [0,T]$ in $H^{-\gamma}(\mathbb{T}^N)$. And with (6.4), we obtain
\[ \mathbb{E} \sup_{t \in [0,T]} \|\rho^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-\gamma}(\mathbb{T}^N)} \lesssim \varepsilon. \] (6.5)

Actually, by interpolation, the continuous embedding $L^2(\mathbb{T}^N) \subset H^{-\eta}(\mathbb{T}^N)$ and the uniform $L^2_{F_{1-1}}$ bound (5.1), we have
\[ \mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta}} \leq \mathbb{E} \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta}^\varepsilon}, \]
for a certain $\eta > 0$ if $\eta' \eta'' > 0$. As a result, it is indeed sufficient to work with $\eta = \gamma$. In view of (6.5), we first want to obtain an estimate of the increments of $\theta^\varepsilon$. We have, for $j \in \mathbb{N}^N$ and $0 \leq s \leq t \leq T$,
\[ \theta_j^\varepsilon(t) - \theta_j^\varepsilon(s) = \int_s^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f^\varepsilon, m^\varepsilon) \, d\sigma + M_j^\varepsilon(t) - M_j^\varepsilon(s). \] (6.6)

We then control the two terms on the right-hand side of (6.6). Let us begin with the first one. Note that, since $D\varphi_j(f) \equiv p_j F$ and $D\varphi_{1,j}(f) \equiv -M^{-1}(m) p_j F$, we obtain thanks to (4.9) with $\varphi_2 \equiv 0$,
\[ \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f^\varepsilon, m^\varepsilon) = -\frac{1}{\varepsilon}(Af^\varepsilon, p_j F) + (Af^\varepsilon, M^{-1}(m^\varepsilon) p_j F) - (f^\varepsilon, m^\varepsilon, M^{-1}(m^\varepsilon) p_j F). \]

Since, with (2.2), we have $\overline{a(v)f_j^\varepsilon} = \overline{a(v)g_j^\varepsilon}$ where $g^\varepsilon$ has been defined previously as $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F$, we can write
\[ (Af^\varepsilon, p_j F) = \int_{\mathbb{T}^N} \text{div}_x(a(v)f_j^\varepsilon) \, p_j \, dx = \int_{\mathbb{T}^N} \text{div}_x(a(v)g_j^\varepsilon) \, p_j \, dx = (Ag^\varepsilon, p_j F) \]
and, as a consequence, since $a$ is bounded, we are led to
\[ \frac{1}{\varepsilon}(Af^\varepsilon, p_j F) \lesssim \|\varepsilon^{-1}g^\varepsilon\| \|\nabla_x p_j\|_{L^2}. \]

Similarly, we can show that
\[ (Af^\varepsilon, M^{-1}(m^\varepsilon) p_j F) \lesssim \|g^\varepsilon\|(1 + \|\nabla_x p_j\|_{L^2}). \]

Since we have obviously $(f^\varepsilon, m^\varepsilon, M^{-1}(m^\varepsilon) p_j F) \lesssim \|f^\varepsilon\|$, we can conclude that
\[ |\mathcal{L}^\varepsilon \varphi_j^\varepsilon(f^\varepsilon, m^\varepsilon)| \lesssim C_j \left[ \|\varepsilon^{-1}g^\varepsilon\| + \|g^\varepsilon\| + \|f^\varepsilon\| \right], \] (6.7)
where \( C_j := 1 + \|\nabla_x p_j\|_{L^2} \leq 1 + |j| \). Thanks to (5.1) and (5.7) with \( p = 4 \), we have that \((\varepsilon^{-1} g^\varepsilon)_{\varepsilon > 0}, (g^\varepsilon)_{\varepsilon > 0}\) and \((f^\varepsilon)_{\varepsilon > 0}\) are bounded in \( L^4(\Omega; L^2(0, T; L^2_{p-1})) \). As a consequence, (6.7) and an application of Hölder’s inequality gives

\[
E \left| \int_s^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon (f^{\varepsilon}_s, m^{\varepsilon}_s) \, d\sigma \right|^4 \lesssim C_j^4 |t-s|^2.
\]

Furthermore, using Burkholder-Davis-Gundy inequality, we can control the second term of the right-hand side of (6.6) as

\[
E|\mu_j^\varepsilon(t) - \mu_j^\varepsilon(s)|^4 \lesssim E\langle \mu_j^\varepsilon \rangle_t - \langle \mu_j^\varepsilon \rangle_s |t-s|^2,
\]

where the quadratic variation \( \langle \mu_j^\varepsilon \rangle_t \) is given by

\[
\langle \mu_j^\varepsilon \rangle_t = \int_0^t (\mu |\varphi_{1,j}|^2 - 2\varphi_{1,j} M \varphi_{1,j})(f^{\varepsilon}_s, m^{\varepsilon}_s) \, ds.
\]

With the definition of \( \varphi_{1,j}, (2.10), (2.11) \) and the uniform \( L^2_{p-1} \) bound (5.1), it is now easy to get

\[
E|\mu_j^\varepsilon(t) - \mu_j^\varepsilon(s)|^4 \lesssim |t-s|^2.
\]

Finally we have \( E|\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s)|^4 \lesssim (1 + |j|^4)|t-s|^2 \). Since we took \( \gamma > N/2 + 2 \), we can conclude that

\[
E||\theta^\varepsilon(t) - \theta^\varepsilon(s)||^4_{H^{-\gamma(TN)}} \lesssim |t-s|^2.
\]

It easily follows that, for \( \upsilon < 1/2 \),

\[
E ||\theta^\varepsilon||^{4}_{W^{\upsilon,4}(0, T; H^{-\gamma(TN)})} \lesssim 1
\]

and by the embedding

\[
W^{\upsilon,4}(0, T; H^{-\gamma(TN)}) \subseteq C^\gamma(0, T; H^{-\gamma(TN)}), \quad \tau < \upsilon - \frac{1}{4},
\]

we obtain that \( Ew(\theta^\varepsilon, \delta) \lesssim \delta^\tau \) for a certain positive \( \tau \). Finally, with (6.5), we can now conclude the first step of the proof since

\[
Ew(\rho^\varepsilon, \delta) \leq 2E \sup_{t \in [0, T]} ||\rho_t^\varepsilon - \theta_j^\varepsilon||_{H^{-\gamma(TN)}} + Ew(\theta^\varepsilon, \delta) \lesssim \varepsilon + \delta^\tau. \tag{6.8}
\]

**Step 2: tightness in \( C([0, T]; H^{-\eta(TN)}) \).** Since the embedding \( L^2(TN) \subseteq H^{-\eta(TN)} \) is compact, and by Ascoli’s Theorem, the set

\[
K_R := \left\{ \rho \in C([0, T], H^{-\eta(TN)}), \sup_{t \in [0, T]} ||\rho||_{L^2(TN)} \leq R, \ w(\rho, \delta) < \varepsilon(\delta) \right\},
\]

where \( R > 0 \) and \( \varepsilon(\delta) \to 0 \) when \( \delta \to 0 \), is compact in \( C([0, T], H^{-\eta(TN)}) \). By Prokohrov’s Theorem, the tightness of \( \langle \rho^\varepsilon \rangle_{\varepsilon > 0} \) in \( C([0, T], H^{-\eta(TN)}) \) will follow if we prove that for all \( \sigma > 0 \), there exists \( R > 0 \) such that

\[
\mathbb{P}( \sup_{t \in [0, T]} ||\rho^\varepsilon||_{L^2(TN)} > R ) < \sigma, \tag{6.9}
\]

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and
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) = 0. \tag{6.10}
\]

With Markov’s inequality and the uniform \(L^2_{\rho_{t-1}}\) bound (5.1), we have
\[
\mathbb{P}\left( \sup_{t \in [0,T]} \|\rho^\varepsilon\|_{L^2(\mathbb{T}^N)} > R \right) \leq \mathbb{P}\left( \sup_{t \in [0,T]} \|f^\varepsilon\| > R \right) \lesssim R^{-1},
\]
which gives (6.9). And we deduce (6.10) by Markov’s inequality and the bound (6.1) since
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) \leq \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sigma^{-1} \mathbb{E}w(\rho^\varepsilon, \delta) \\
\lesssim \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \sigma^{-1}(\varepsilon + \delta^\varepsilon) = 0.
\]

**Step 3: tightness in \(L^2(0,T; L^2(\mathbb{T}^N))\).** Similarly, due to \([12, \text{Theorem 5}]\), the set
\[
K_R := \left\{ \rho \in L^2(0,T; L^2(\mathbb{T}^N)), \int_0^T \|\rho_t\|^2_{H^s(\mathbb{T}^N)} dt \leq R, \, w(\rho, \delta) < \varepsilon(\delta) \right\},
\]
where \(R > 0, s > 0\) and \(\varepsilon(\delta) \to 0\) when \(\delta \to 0\), is compact in \(L^2(0,T; L^2(\mathbb{T}^N))\). By Prokhorov’s Theorem, the tightness of \((\rho^\varepsilon)_{\varepsilon > 0}\) in \(L^2(0,T; L^2(\mathbb{T}^N))\) will follow if we prove that for all \(\sigma > 0\), there exists \(R > 0\) such that
\[
\mathbb{P}\left( \int_0^T \|\rho_t\|^2_{H^s(\mathbb{T}^N)} dt > R \right) < \sigma, \tag{6.11}
\]
and
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) = 0. \tag{6.12}
\]
But (6.11) and (6.12) are consequences of Markov’s inequality and the bounds (5.8) with \(p = 1\) and (6.1) so that the proof is complete. \(\square\)

### 7 Convergence

We conclude here the proof of Theorem 2.2. The idea is now, by the tightness result and Prokhorov Theorem, to take a subsequence of \((\rho^\varepsilon)_{\varepsilon > 0}\) that converges in law to some probability measure. Then we show that this limiting probability is actually uniquely determined by the limit generator \(\mathcal{L}\) defined above.

We fix \(\eta > 0\). By Proposition 6.1 and Prokhorov’s Theorem, there is a subsequence of \((\rho^\varepsilon)_{\varepsilon > 0}\), still denoted \((\rho^\varepsilon)_{\varepsilon > 0}\), and a probability measure \(P\) on the spaces \(C([0,T], H^{-\eta})\) and \(L^2(0,T; L^2)\) such that
\[
P^\varepsilon \to P \text{ weakly in } C([0,T], H^{-\eta}) \text{ and } L^2(0,T; L^2),
\]
where \(P^\varepsilon\) stands for the law of \(\rho^\varepsilon\). We now identify the probability measure \(P\).

Since the spaces \(C([0,T], H^{-\eta})\) and \(L^2(0,T; L^2)\) are separable, we can apply Skohorod representation Theorem [3], so that there exists a new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) and random variables
\[
\tilde{\rho}^\varepsilon, \tilde{\rho} : \tilde{\Omega} \to C([0,T], H^{-\eta}) \cap L^2(0,T; L^2),
\]

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with respective law $P^\varepsilon$ and $P$ such that $\tilde{\rho}^\varepsilon \to \tilde{\rho}$ in $C([0, T], H^{-n})$ and $L^2(0, T; L^2) \bar{P}$-a.s. In the sequel, for the sake of clarity, we do not write any more the tildes.

Note that, with (5.7), we can also suppose that $\varepsilon^{-1} g^\varepsilon$ converges to some $g$ weakly in the space $L^2(\Omega; L^2(0, T; L^2_{\mathbb{F}^{-1}}))$. Similarly, with (2.10), we assume that $m^\varepsilon$ converges to $m$ weakly in $L^2(\Omega; L^2(0, T; L^2_{\mathbb{F}^{-1}}))$. Before going on the proof, we want to identify the weak limit $g$ of $\varepsilon^{-1} g^\varepsilon$.

**Lemma 7.1.** In $L^2(\Omega; L^2(0, T; L^2))$, we have the relation

$$ a(v) g = -\sigma(\rho)^{-1} K \nabla_x \rho. $$

**Proof.** We define $D_T := (0, T) \times \mathbb{T}^N$. Since $f^\varepsilon$ satisfies equation (1.1), we can write, for any $\psi \in C_c^\infty(D_T)$ and $\theta \in L^\infty(V \times \Omega; \mathbb{R}^N)$,

$$ E \int_{D_T \times V} f^\varepsilon F^{-1} (-\varepsilon \partial_t \psi - a \cdot \nabla_x \psi) \, \theta = E \int_{D_T \times V} \frac{1}{\varepsilon} \sigma(f^\varepsilon) L f^\varepsilon F^{-1} \psi \, \theta + E \int_{D_T \times V} m^\varepsilon f^\varepsilon F^{-1} \psi \, \theta. $$

We recall that we set $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F$ and that $Lf^\varepsilon = Lg^\varepsilon$ so that we have

$$ E \int_{D_T \times V} -\varepsilon f^\varepsilon F^{-1} \partial_t \psi \, \theta - \rho^\varepsilon a \cdot \nabla_x \psi \, \theta - g^\varepsilon F^{-1} a \cdot \nabla_x \psi \, \theta = E \int_{D_T \times V} \sigma(\rho^\varepsilon) L(\varepsilon^{-1} g^\varepsilon) F^{-1} \psi \, \theta + E \int_{D_T \times V} m^\varepsilon f^\varepsilon F^{-1} \psi \, \theta. $$

Since $(f^\varepsilon)_{\varepsilon > 0}$ and $(\varepsilon^{-1} g^\varepsilon)_{\varepsilon > 0}$ are bounded in $L^2(\Omega; L^2(0, T; L^2_{\mathbb{F}^{-1}}))$ by (5.1) and (5.7), and with the $\bar{P}$-a.s. convergence $\rho^\varepsilon \to \rho$ in $L^2(0, T; L^2_{\mathbb{F}^{-1}})$ coupled with the uniform integrability of the family $(\rho^\varepsilon)_{\varepsilon > 0}$ obtained with (5.1), we have that the left-hand side of the previous equality actually converges as $\varepsilon \to 0$ to

$$ E \int_{D_T \times V} -\rho a \cdot \nabla_x \psi \, \theta. $$

Note that, $\bar{P}$-a.s., we have the following convergences in $L^2(0, T; L^2_{\mathbb{F}^{-1}})$

$$ \sigma(\rho^\varepsilon) \to \sigma(\rho), \quad L(\varepsilon^{-1} g^\varepsilon) \to Lg, \quad f^\varepsilon \to \rho F, \quad m^\varepsilon \to m, $$

where the first convergence is justified by the Lipschitz continuity of $\sigma$. As a result, since all the quantities above are uniformly integrable with respect to $\varepsilon$ thanks to (5.1), (5.7) and (2.10), the right-hand side of the previous equality converges as $\varepsilon \to 0$ to

$$ E \int_{D_T \times V} \sigma(\rho) L(g) F^{-1} \psi \, \theta + E \int_{D_T \times V} m \rho \psi \, \theta. $$

Thus, we have

$$ E \int_{D_T \times V} -\rho a \cdot \nabla_x \psi \, \theta = E \int_{D_T \times V} \sigma(\rho) L(g) F^{-1} \psi \, \theta + E \int_{D_T \times V} m \rho \psi \, \theta. $$

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Let $\xi$ be an arbitrary bounded measurable function on $\Omega$. We now set $\theta(v, \omega) = a(v)F(v)\xi(\omega)$; note that we do have $\theta \in L^\infty(V \times \Omega, \mathbb{R}^N)$. With (2.2) and the relation $Lg = gF - g$, we obtain

$$-E \int_{D_T \times V} \rho a \cdot \nabla_x \psi a F = -E \int_{D_T \times V} \sigma(\rho)g a(v)\psi.$$ 

Since this relation holds for any $\xi \in L^\infty(\Omega)$ and $\psi \in C^\infty(D_T)$, we deduce that $\nabla_x \rho \in L^2(\Omega, L^2(D_T))$ and that

$$\overline{a(v)g} = -\sigma(\rho)^{-1}K\nabla_x \rho,$$

and this concludes the proof. \hfill \Box

Let $\varphi \in C^3(L^2_{\rho_{-\varepsilon}})$ a good test-function satisfying (4.1). We define $\varphi^\varepsilon$ as in Section 4.1. Since $\varphi^\varepsilon$ is a good test-function, we have that

$$\varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \varphi^\varepsilon(f^\varepsilon_0, m^\varepsilon_0) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s) \, ds, \quad t \in [0, T],$$

is a continuous martingale for the filtration generated by $(f^\varepsilon_s)_{s \in [0, T]}$. As a result, if $\Psi$ denotes a continuous and bounded function from $L^2(\mathbb{T}^N)^n$ to $\mathbb{R}$, we have

$$E \left[ \left( \varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \varphi^\varepsilon(f^\varepsilon_s, m^\varepsilon_s) - \int_s^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f^\varepsilon_u, m^\varepsilon_u) \, du \right) \Psi(\rho^\varepsilon_{s_1},...,\rho^\varepsilon_{s_n}) \right] = 0,$$  \hfill (7.1)

for any $0 \leq s_1 \leq ... \leq s_n \leq s \leq t$. Our final purpose is to pass to the limit $\varepsilon \to 0$ in (7.1). In the sequel, we assume that the function $\varphi$ and $\Psi$ are also continuous on the space $H^{-\eta}$, which is always possible with an approximation argument: it suffices to consider $\varphi_r := \varphi((1-r\Delta_x)^{-2} \cdot)$ and $\Psi_r := \Psi((1-r\Delta_x)^{-2} \cdot)$ as $r \to 0$. With (4.12), we divide the left-hand side of (7.1) in four parts. Precisely, we define, for $i \in \{1,...,4\}$

$$\tau^\varepsilon_1 := \varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) - \varphi^\varepsilon(f^\varepsilon_0, m^\varepsilon_0),$$

$$\tau^\varepsilon_2 := \int_0^t \mathcal{L}^\varepsilon \varphi(\rho^\varepsilon_u) \, du,$$

$$\tau^\varepsilon_3 := \int_0^t \frac{1}{\varepsilon}(Af^\varepsilon_u, D\varphi(f^\varepsilon_u)) - (\text{div}_x(\sigma(\rho^\varepsilon_u)^{-1}K\nabla_x \rho^\varepsilon_u)F, D\varphi(\rho^\varepsilon_u)F)) \, du,$$

$$\tau^\varepsilon_4 := \int_0^t -(Af^\varepsilon_u, D\varphi_1(f^\varepsilon_u)) - \varepsilon(Af^\varepsilon_u, D\varphi_2(f^\varepsilon_u)) + \varepsilon(f^\varepsilon_u m^\varepsilon_u, D\varphi_2(f^\varepsilon_u)) \, du.$$

**Study of $\tau^\varepsilon_1$.** We recall that $\varphi^\varepsilon(f^\varepsilon_t, m^\varepsilon_t) = \varphi(\rho^\varepsilon_t)F + \varepsilon \varphi_1(f^\varepsilon_t, m^\varepsilon_t) + \varepsilon^2 \varphi_2(f^\varepsilon_t, m^\varepsilon_t)$ so that, with the $\mathbb{P}$-a.s. convergence of $\rho^\varepsilon$ to $\rho$ in $C([0, T), H^{-\eta})$ and the bounds (i) of (4.8) and (4.10), we have that $\tau^\varepsilon_1$ converges $\mathbb{P}$-a.s. to $\varphi(\rho_t F) - \varphi(\rho_s F)$ as $\varepsilon$ goes to 0. Furthermore, with the continuity of $\Psi$ in $H^{-\eta}$, we also have that $\Psi(\rho^\varepsilon_{s_1},...,\rho^\varepsilon_{s_n})$ converges $\mathbb{P}$-a.s. to $\Psi(\rho_{s_1},...,\rho_{s_n})$. Finally, since the family $\tau^\varepsilon_1 \Psi(\rho^\varepsilon_{s_1},...,\rho^\varepsilon_{s_n})$ is uniformly integrable with respect to $\varepsilon$ thanks to (4.1), the bounds (i) of (4.8) and (4.10) and the uniform $L^2_{\rho_{-\varepsilon}}$ bound (5.1), we have that

$$E[\tau^\varepsilon_1 \Psi(\rho^\varepsilon_{s_1},...,\rho^\varepsilon_{s_n})] \to E[(\varphi(\rho_t F) - \varphi(\rho_s F)) \Psi(\rho_{s_1},...,\rho_{s_n})].$$
Study of $\tau_2^\varepsilon$. We recall, with (4.11), that
\[
\mathcal{L}_t \varphi(\rho_u^\varepsilon) = \langle \mathrm{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K_n \rho_u^\varepsilon F), D\varphi(\rho_u^\varepsilon F) \rangle - \int_E (\rho_u^\varepsilon FM^{-1}I(n), D\varphi(\rho_u^\varepsilon F)) \, dv(n) - \int_E D^2 \varphi(\rho_u^\varepsilon F)(\rho_u^\varepsilon FM^{-1}I(n), \rho_u^\varepsilon F_n) \, dv(n).
\]
Thanks to the $\mathbb{P}$–a.s. convergence of $\rho^\varepsilon$ to $\rho$ in $L^2(0,T;L^2)$ and with $\varphi \in C^3(L^2_{\mathbb{P}^{-1}})$, we can pass to the limit $\varepsilon \to 0$ in the term
\[
\int_s^t \int_E -(\rho_u^\varepsilon FM^{-1}I(n), D\varphi(\rho_u^\varepsilon F)) - D^2 \varphi(\rho_u^\varepsilon F)(\rho_u^\varepsilon FM^{-1}I(n), \rho_u^\varepsilon F_n) \, dv(n) \, du.
\]
Regarding the first term of $\mathcal{L}_t \varphi(\rho_u^\varepsilon)$, we introduce
\[
G(\rho) := \int_0^\rho \frac{dy}{\sigma(y)},
\]
which is, thanks to the hypothesis (H1) made on $\sigma$, Lipschitz continuous on $L^2(\mathbb{T}^N)$. Now the first term of $\mathcal{L}_t \varphi(\rho_u^\varepsilon)$ writes
\[
\langle \mathrm{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K_n \rho_u^\varepsilon F), D\varphi(\rho_u^\varepsilon F) \rangle = \langle \mathrm{div}_x \nabla_x G(\rho_u^\varepsilon F), D\varphi(\rho_u^\varepsilon F) \rangle.
\]
Furthermore, with (4.1), the mapping $\rho \mapsto \partial_{x_i,x_j} D\varphi(\rho F)$ is continuous on $L^2(\mathbb{T}^N)$. As a result, we can now pass to the limit in the term
\[
\int_s^t \langle \mathrm{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K_n \rho_u^\varepsilon F), D\varphi(\rho_u^\varepsilon F) \rangle \, du.
\]
To sum up, we obtain that $\tau_2^\varepsilon$ converges $\mathbb{P}$–a.s. to $\int_s^t \mathcal{L}_t \varphi(\rho_u) \, du$ as $\varepsilon$ goes to 0. Finally, since the family $\tau_2^\varepsilon \Psi(\rho_{s_1}^\varepsilon,\ldots,\rho_{s_n}^\varepsilon)$ is uniformly integrable with respect to $\varepsilon$ thanks to (4.1) and the uniform $L^2_{\mathbb{P}^{-1}}$ bound (5.1), we have that
\[
\mathbb{E}[\tau_2^\varepsilon \Psi(\rho_{s_1}^\varepsilon,\ldots,\rho_{s_n}^\varepsilon)] \to \mathbb{E} \left[ \left( \int_s^t \mathcal{L}_t \varphi(\rho_u) \, du \right) \Psi(\rho_{s_1},\ldots,\rho_{s_n}) \right].
\]

Study of $\tau_3^\varepsilon$. First of all, we observe that, with the decomposition $f^\varepsilon = \rho^\varepsilon F + g^\varepsilon$, (4.7) and (2.2),
\[
-\varepsilon^{-1}(Af_u^\varepsilon,F^\varepsilon) = -\varepsilon^{-1}(Ag_u^\varepsilon,F^\varepsilon),
\]
so that, with the $\mathbb{P}$–a.s. convergences in $L^2(0,T;L^2)$
\[
\varepsilon^{-1}g^\varepsilon \to g, \quad \rho^\varepsilon \to \rho,
\]
and the continuity of the mapping $\rho \mapsto AD\varphi(\rho F)$ thanks to (4.1), we obtain that the first term of $\tau_3^\varepsilon$ converges $\mathbb{P}$–a.s. to
\[
-\int_s^t (Ag_u F, D\varphi(\rho_u F)) \, du.
\]
We recall that this is valid for all $n$ to the opposite of (7.1). As a result, $\tau_{\bar{3}}$ converges $\mathbb{P}$-a.s. to $0$. Finally, since the family $\tau_{\bar{3}}\psi(\rho_{s_{1}}^{-},...,\rho_{s_{n}}^{-})$ is uniformly integrable with respect to $\varepsilon$ thanks to (4.1), the uniform $L_{\varepsilon}^{2}$ bound (5.1) and the bound (5.7) on $(\varepsilon^{-1}g^{\tau})_{\varepsilon>0}$, we have that

$$
\mathbb{E}[\tau_{\bar{3}}\psi(\rho_{s_{1}}^{-},...,\rho_{s_{n}}^{-})] \to 0.
$$

**Study of $\tau_{\bar{4}}$.** If we transform the two first terms of $\tau_{\bar{4}}$ exactly as we do for the first term of $\tau_{\bar{3}}$, it is then easy, using the uniform bounds (5.1) and (5.7) and the bounds (ii) of (4.8) and (4.10), to get

$$
\mathbb{E}[\tau_{\bar{4}}\psi(\rho_{s_{1}}^{-},...,\rho_{s_{n}}^{-})] = O(\varepsilon).
$$

To sum up, we can pass to the limit $\varepsilon \to 0$ in (7.1) to obtain

$$
\mathbb{E}\left[\left(\varphi(\rho_{t}F) - \varphi(\rho_{0}F) - \int_{s}^{t} \mathcal{L}\varphi(\rho_{u})\,du\right)\psi(\rho_{s_{1}}^{-},...,\rho_{s_{n}}^{-})\right] = 0. \quad (7.3)
$$

We recall that this is valid for all $n \in \mathbb{N}$, $0 \leq s_{1} \leq ... \leq s_{n} \leq s \leq t \in [0,T]$ and all $\varphi$ continuous and bounded function on $L^{2}(\mathbb{T}^{n})$. Now, let $\xi$ be a smooth function on $L^{2}(\mathbb{T}^{n})$. We choose $\varphi(f) = (f, \xi F)$. We can easily verify that $\varphi$ and $|\varphi|^{2}$ belong to $C^{3}(L_{\varepsilon}^{2})$ and that they are good test-function satisfying (4.1). Thus, we obtain that

$$
N_{t} := \varphi(\rho_{t}F) - \varphi(\rho_{0}F) - \int_{0}^{t} \mathcal{L}\varphi(\rho_{u})\,du, \quad t \in [0,T],
$$

$$
|\varphi|^{2}(\rho_{t}F) - |\varphi|^{2}(\rho_{0}F) - \int_{0}^{t} \mathcal{L}|\varphi|^{2}(\rho_{u})\,du, \quad t \in [0,T],
$$

are continuous martingales with respect to the filtration generated by $(\rho_{s})_{s \in [0,T]}$. It implies (see appendix 6.9 in [9]) that the quadratic variation of $N$ is given by

$$
\langle N \rangle_{t} = \int_{0}^{t} \left[\mathcal{L}|\varphi|^{2}(\rho_{u}) - 2 \varphi(\rho_{u})\mathcal{L}\varphi(\rho_{u})\right]\,du, \quad t \in [0,T].
$$

Furthermore, we have

$$
\mathcal{L}|\varphi|^{2}(\rho_{u}) - 2 \varphi(\rho_{u})\mathcal{L}\varphi(\rho_{u}) = -2 \int_{\mathbb{R}^{\infty}}(\rho_{u}F_{n}, \xi F)(\rho_{u}FM^{-1}I(n), \xi F)\,dv(n)
$$

$$
= 2\mathbb{E} \int_{0}^{\infty}(\rho_{u}F_{m_{0}}, \xi F)(\rho_{u}FM_{1}, \xi F)\,dt
$$

$$
= \mathbb{E} \int_{\mathbb{R}}(\rho_{u}F_{m_{0}}, \xi F)(\rho_{u}FM_{1}, \xi F)\,dt
$$

$$
= \int_{\mathbb{T}^{n}} \int_{\mathbb{T}^{n}} \rho_{u}(x)\xi(x)\rho_{u}(y)\xi(y)k(x,y)\,dx\,dy
$$

$$
= \|\rho_{u}Q_{\tau}^{2}\xi\|_{L_{2}^{2}}^{2}.
$$

22
This is valid for all smooth function $\xi$ of $L^2(T^N)$ so we deduce that

$$M_t := \rho_t - \rho_0 - \int_0^t \text{div}_x(\sigma(\rho_s)^{-1}K\nabla_x \rho_s) \, ds - \int_0^t \rho_s H \, ds, \quad t \in [0, T],$$

is a martingale with quadratic variation

$$\int_0^t \rho_s Q_s^\frac{1}{2} \left( \rho_s Q_s^\frac{1}{2} \right)^* \, ds.$$

Thanks to martingale representation Theorem, see [5, Theorem 8.2], up to a change of probability space, there exists a cylindrical Wiener process $W$ such that

$$\rho_t - \rho_0 - \int_0^t \text{div}_x(\sigma(\rho_s)^{-1}K\nabla_x \rho_s) \, ds - \int_0^t \rho_s H \, ds = \int_0^t \rho_s Q_s^\frac{1}{2} \, dW_s, \quad t \in [0, T].$$

This gives that $\rho$ has the law of a weak solution to the equation (2.13) with paths in $C([0, T], H^{-\eta}) \cap L^2(0, T; L^2)$. Since this equation has a unique solution with paths in the space $C([0, T], H^{-\eta}) \cap L^2(0, T; L^2)$, and since pathwise uniqueness implies uniqueness in law, we deduce that $P$ is the law of this solution and is uniquely determined. Finally, by the uniqueness of the limit, the whole sequence $(P^\varepsilon)_{\varepsilon > 0}$ converges to $P$ weakly in the spaces of probability measures on $C([0, T], H^{-\eta})$ and $L^2(0, T; L^2)$. This concludes the proof of Theorem 2.2.

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