Eigenvalue estimates of the drifted Cheng-Yau operator on bounded domains in pinched Cartan-Hadamard manifolds

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Abstract. We show how a Bochner type formula can be used to establish universal inequalities for the eigenvalues of the drifted Cheng-Yau operator on a bounded domain in a pinched Cartan-Hadamard manifold with the Dirichlet boundary condition. In the first theorem, the hyperbolic space case is treated in an independent way. For the more general setting, we first establish a Rauch comparison theorem for the Cheng-Yau operator and two estimates associated with the Bochner type formula for this operator. Next, we get some integral estimates of independent interest. As an application, we compute our universal inequalities. In particular, we obtain the corresponding inequalities for both Cheng-Yau operator and drifted Laplacian cases, and we recover the known inequalities for the Laplacian case. We also obtain a rigidity result for a Cheng-Yau operator on a class of bounded annular domains in a pinched Cartan-Hadamard manifold. In particular, we can use, e.g., the potential function of the Gaussian shrinking soliton to obtain such a rigidity for the Euclidean space case. The fundamental gap conjecture is also addressed in this paper.

1. Introduction

In this paper \(M^n\), \(n \geq 2\), is an \(n\)-dimensional simply connected smooth manifold with a geodesically complete Riemannian metric \(\langle , \rangle\) of sectional curvatures satisfying \(-\kappa_1^2 \leq K \leq -\kappa_2^2\), where \(0 \leq \kappa_2 \leq \kappa_1\) are constants. It has been called a pinched Cartan-Hadamard manifold after the Cartan-Hadamard theorem, which says that a simply connected geodesically complete Riemannian manifold with non-positive sectional curvatures is diffeomorphic to a Euclidean space. The two key properties on Cartan-Hadamard manifolds are that any two points in \(M^n\) lie on a unique geodesic, and that distance functions are everywhere smooth and convex, see, e.g., Bishop and O’Neill [5] or the book by Petersen [27]. Consider a bounded domain \(\Omega \subset M^n\) with smooth boundary \(\partial \Omega\). Let \(T\) be a symmetric positive definite \((1,1)\)-tensor on \(M^n\), and \(\eta\) be a smooth function on \(M^n\). Due to the boundedness of \(\Omega\) there exist two positive constants \(\epsilon\) and \(\delta\) such that \(\epsilon \leq \langle TX, X \rangle \leq \delta\) for any unit vector field \(X\) on \(\Omega\).
Our aim here is to study the following eigenvalue problem with Dirichlet boundary condition:

\[
\begin{cases}
  -\mathcal{L} u = \lambda u & \text{in } \Omega, \\
  u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where

\[
\mathcal{L} u = \text{div}(T(\nabla u)) - \langle \nabla \eta, T(\nabla u) \rangle,
\]

Here, div stands for the divergence of smooth vector fields, and \( \nabla \) for the gradient of smooth functions.

Alencar, Neto and Zhou [1] showed the Bochner type formula (2.3) for the operator that has been introduced by Cheng and Yau [13] which can be understood as follows

\[
\Box f := \text{tr}(\nabla^2 f \circ T) = \langle \nabla^2 f, T \rangle,
\]

where \( \nabla^2 f \) is the Hessian of a smooth function \( f \) on \( M^n \). Shortly after that, the second author and Miranda [16] showed that Eq. (1.1) can be decomposed into the \( \eta \)-divergence tensor of \( T \) and the Cheng-Yau operator:

\[
\mathcal{L} f = \Box f + \langle \text{div}_\eta T, \nabla f \rangle,
\]

where \( \text{div}_\eta T := \text{div} T - \langle \nabla \eta, T(\cdot) \rangle \), and \( \text{div} T \) stands for the divergence tensor of \( T \), see Section 2.

For orientable compact Riemannian manifolds, it has been proved in [13] that the operator \( \Box \) is self-adjoint if and only if \( T \) is divergence free, i.e., \( \text{div} T = 0 \). In this case, Eq. (1.3) becomes

\[
\mathcal{L} f = \Box f - \langle \nabla \eta, T(\nabla f) \rangle,
\]

which is a first order perturbation of the Cheng-Yau operator. We call Eq. (1.4) a drifted Cheng-Yau operator with a drifting function \( \eta \). In particular, if \( \eta \) is constant, then \( \Box f \) is a Cheng-Yau operator with \( \text{div} T = 0 \).

For instance, if \( \text{Ric} \) stands for the Ricci tensor of \( \langle \cdot, \cdot \rangle \) and \( R = \text{tr}(\text{Ric}) \), then, it is known that \( \text{div} \text{Ric} = \frac{dR}{2} \) and \( \text{div}(\text{R}I) = dR \), so the Einstein tensor \( G := \text{Ric} - \frac{R}{2}I \) is divergence free, therefore \( \Box f = \langle \nabla^2 f, G \rangle \) is self-adjoint on compact Riemannian manifolds, and the drifted Cheng-Yau operator \( \mathcal{L} f = \langle \nabla^2 f, G \rangle - \langle \nabla \eta, G(\nabla f) \rangle \) is likely to have applications in physics, see, e.g., Serre [30]. We highlight that Serre’s work deals with divergence free positive definite symmetric tensors and fluid dynamics, there the reader can find examples and know where these tensors occur. In the last part of Section 2 we list some geometric examples of such tensors. We refer the reader to Proposition 5.1 in [1] or Navarro [26] for more related discussions.

The Bochner type formula in [1] has been extended in [16] for the more general expression of \( \mathcal{L} \), see Eq. (2.2). Moreover, it was observed that \( \mathcal{L} \) is self-adjoint in the Hilbert space \( \mathcal{H}_0^1(\Omega, e^{-\eta}dvol_\Omega) \), see Section 2. It is known that the Bochner technique gives many optimal bounds on the topology of compact Riemannian manifolds with nonnegative curvature. In contrast, we show here with how the Bochner type formula can be used to establish universal inequalities for the eigenvalues of the drifted Cheng-Yau operator on a bounded domain in a pinched Cartan-Hadamard manifold. This is a simple approach that has not been used yet for this operator. First, we establish a Rauch comparison theorem for the Cheng-Yau operator and two inequalities associated with the Bochner type formula for this operator, see Proposition 1. Next, we get some integral estimates of independent interest, see Propositions 2 and 3. As an application, we compute our main inequalities.
Let us consider the hyperbolic space \( \mathbb{H}^n(-1) \) with constant curvature \(-1\), namely, the open half space \( x_n > 0 \) with its standard metric \( g_{ij} = x_n^{-2} \delta_{ij} \). It is known that \( r(x_1, \ldots, x_n) = \ln x_n \) works as a distance function on \( \mathbb{H}^n(-1) \). This fact has been our motivation for the first theorem of the paper, in which we prove a universal quadratic inequality for the eigenvalues of \( \mathcal{L} \) on any bounded domain in hyperbolic space \( \mathbb{H}^n(-\kappa^2) \) with constant curvature \(-\kappa^2\) in the upper half space model. We observe that the hyperbolic space is treated in an independent way in this first result.

In what follows, a smooth function \( f \) is called radially constant if its radial derivative vanish: \( \partial_r f = 0 \), where \( r \) is a distance function.

**Theorem 1.** Let \( \lambda_i \) be the \( i \)-th eigenvalue of \( \mathcal{L} \) on a bounded domain \( \Omega \subset \mathbb{H}^n(-\kappa^2) \) with the Dirichlet boundary condition. If the drifting function \( \eta \) is radially constant and \( T(\nabla \ln x_n) = \psi \nabla \ln x_n \), for some radially constant function \( \psi \). Then, for any positive integer \( k \), we have

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \kappa^2 \right).
\]

Moreover, the first eigenvalue satisfies \( \lambda_1(\Omega) \geq \frac{\kappa}{4\delta^2} (n-1)^2 \varepsilon^2 \kappa^2 > 0 \), since \( n \geq 2 \) and \( \kappa > 0 \).

Notice that we can take \( T \) to be the Identity operator \( I \) on \( \mathfrak{X}(\mathbb{H}^n) \) in Theorem 1 to obtain the corresponding universal inequalities for both Laplacian and drifted Laplacian cases. In particular, Theorem 1 generalizes a result by Cheng and Yang [12] Theorem 1.2] obtained for the Laplacian on a bounded domain in a hyperbolic space \( \mathbb{H}^n(-1) \).

Now, we observe that if \( T \partial_r = \psi \partial_r \), then \( (\nabla \partial_r T) \partial_r = \nabla \partial_r T \partial_r = \partial_r \psi \partial_r \), since \( \nabla \partial_r \partial_r = 0 \). Hence, \( \psi \) is radially constant if and only if \( \nabla \partial_r T \partial_r = 0 \), i.e., \( T \partial_r \) is a parallel field for a smooth distance function \( r \). We say that \( T \) is radially parallel if \( \nabla \partial_r T = 0 \).

For our second result, it is convenient to consider the next constant, which depends on dimension and how the tensor \( T \) is bounded on the domain:

\[
a(n, \varepsilon, \delta) := -(n-1)^2 \varepsilon^2 + 2(n-1)\delta^2. \tag{1.5}
\]

Notice that, for the case of \( T = I \), we have \( \varepsilon = \delta = 1 \), then \( a(n, \varepsilon, \delta) > 0 \) for \( n = 2 \), and \( a(n, \varepsilon, \delta) \leq 0 \) for \( n \geq 3 \).

**Theorem 2.** Let \( \lambda_i \) be the \( i \)-th eigenvalue of the drifted Cheng-Yau operator with a drifting function \( \eta \) on a bounded domain \( \Omega \subset M^n \) with the Dirichlet boundary condition. Fix a origin \( o \in M^n \setminus \overline{\Omega} \), and let \( r(x) \) be the distance function from \( o \). If \( T \) is radially parallel and it has \( \partial_r \) as an eigenvector, then, for any positive integer \( k \), we have:

1. For \( a(n, \varepsilon, \delta) \leq 0 \),

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \kappa_2^2 \right.
\]

\[
+ 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) + 2C_0(n-1)(\kappa_1 + \frac{1}{\delta}) + C_1 \right).
\]
(2) For $a(n, \varepsilon, \delta) > 0$,
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \kappa_2^2 + 2(n-1)(\kappa_1^2 - \kappa_2^2) \right)
\]
\[
+ 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) + 2C_0(n-1)(\kappa_1 + \frac{1}{d}) + C_1 + \frac{a(n, \varepsilon, \delta)}{d^2}.
\]
Here $d = \text{dist}(\Omega, o)$, $C_0 = \delta^2 \max_{\Omega} |\eta|$ and $C_1 = \delta^2 \max_{\Omega}(2\eta - \eta^2)$, whereas $\eta$ and $\bar{\eta}$ stand for the first and second radial derivatives of $\eta(x)$, respectively.

Since $(M^n, \langle \cdot, \cdot \rangle)$ is a geodesically complete Riemannian manifold, there exists a sequence of compact subsets $K_j \subset M^n$ with $K_j \subset \text{int}K_{j+1}$ and $\bigcup_j K_j = M^n$ such that if $q_j \notin K_j$, then $r(x, q_j) \to \infty$. As $\Omega$ is bounded, we can assume $\Omega \subset K_j$ for some $j$, so that we can define $r(x)$ on $\Omega$ from $o = q_j \in M^n \setminus K_j$, for large enough $j$, such that both $a(n, \varepsilon, \delta)/d^2$ and $1/d$ are small enough. Moreover, we can take $C_0 = \delta^2 \max_{\Omega} |\nabla \eta|$ and $C_1 = \delta^2 \max_{\Omega}(2|\nabla \eta| + |\nabla \eta|^2)$, which do not depend on the distance function. In this context, the most convenient is to take $T = I$ in Theorem [2] to get the next result.

**Corollary 1.** Let $\lambda_i$ be the $i$-th eigenvalue of the drifted Laplacian with a drifting function $\eta$ on a bounded domain $\Omega \subset M^n$ with the Dirichlet boundary condition. For any positive integer $k$, we have
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( 4\lambda_i - (n-1)^2 \kappa_2^2 + 2(n-1)(\kappa_1^2 - \kappa_2^2) \right) + 2C_0(n-1)\kappa_1 + C_1,
\]
where $C_0 = \max_{\Omega} |\nabla \eta|$ and $C_1 = \max_{\Omega}(2|\nabla \eta| + |\nabla \eta|^2)$. Moreover, the first eigenvalue satisfies
\[
\lambda_1(\Omega) \geq \frac{1}{4} \left( (n-1)^2 \kappa_2^2 - 2(n-1)(\kappa_1^2 - \kappa_2^2) - 2C_0(n-1)\kappa_1 - C_1 \right).
\]

In Corollary [4] we prove the more general case of the previous corollary that we can obtain from Theorem [2].

Taking $\eta$ to be constant in the first part of Corollary [1], we immediately recover the result by Chen, Zheng and Lu [9, Theorem 1.1] obtained for the Laplacian on a bounded domain in a pinched Cartan-Hadamard manifold $M^n$, for $n \geq 3$. Observe that dimension 2 has not been an obstacle for us. The second consequence from this previous corollary, again for the Laplacian, we write in Corollary [2] for convenience.

**Remark 1.** For the Cheng-Yau operator case, the inequalities in Theorem [2] do not depend on the constants $C_0$ and $C_1$, since we can take $\eta$ to be constant. In the case of the drifted Cheng-Yau operator with the radial drifting function given by $\eta(x) = -2\ln(r(x))$, these inequalities do not depend on the constant $C_1$, since $\eta$ solves $2\eta - \eta^2 = 0$. Besides, $-\eta$ is a simple example of drifting function that we can take in Corollary 1 so that the estimate to be obtained do not depend on $C_1$.

McKean [24] proved that for a Cartan-Hadamard manifold $M^n$ of sectional curvatures satisfying $K \leq -\kappa^2$, for some positive constant $\kappa^2$, the spectrum of the Laplacian on $M^n$ lies in $\left\{ (n-1)^2\kappa^2/4, +\infty \right\}$, and this lower bound is sharp on the hyperbolic space $\mathbb{H}^2(-\kappa^2)$, see also Cheng [10]. In Pinsk [28], we found more
accurate bounds for this lower bound in the case of surfaces. As an application of our Theorem 1, we immediately obtain that the first eigenvalue of the Laplacian on $\Omega \subset \mathbb{H}^n(-\kappa^2)$ with the Dirichlet boundary condition satisfies $\lambda_1(\Omega) \geq (n-1)^2 \kappa^2 / 4$.

In the case of bounded domains in pinched Cartan-Hadamard manifolds, we ask the following question: How sharp is this lower bound? Here, as an application of Corollary 1, we immediately obtain a universal lower bound for the first eigenvalue of the Laplacian on bounded domains in a pinched Cartan-Hadamard manifold. More precisely:

**Corollary 2.** The first eigenvalue $\lambda_1$ of the Laplacian on a bounded domain $\Omega \subset M^n$ with the Dirichlet boundary condition satisfies

$$\lambda_1(\Omega) \geq \frac{(n-1)^2}{4} \kappa^2 - \frac{n-1}{2} (\kappa_1^2 - \kappa_2^2).$$

Moreover, this lower bound is positive for $0 < \kappa_2 \leq \kappa_1$ and $\sqrt{n+1} \kappa_2 > \sqrt{2} \kappa_1$.

Now, let us consider each eigenvalue $\lambda_i(\Omega)$ of the Laplacian on bounded domains $\Omega \subset \mathbb{H}^n(-\kappa^2)$ as a function of these domains with the Dirichlet boundary condition. In this setting, we show that $\lambda_i(\Omega) \to (n-1)^2 \kappa^2 / 4$ as $\Omega \to \mathbb{H}^n(-\kappa^2)$. This means that $\Omega$ includes an $n$-disk of radius $a > 0$ and we can make the radius $a \to +\infty$, since $\mathbb{H}^n(-\kappa^2)$ is a geodesically complete Riemannian manifold. We observe that this fact has been proved by Cheng and Yang [12] for the unit hyperbolic space. Here, we give a complete proof by combining our Theorem 1 with an appropriated approach by the second author and Marrocos [23] in the setting of the spectrum of warped metrics.

**Corollary 3.** The $i$-th eigenvalue $\lambda_i$ of the Laplacian on a bounded domain $\Omega \subset \mathbb{H}^n(-\kappa^2)$ with the Dirichlet boundary condition satisfies

$$\lim_{\Omega \to \mathbb{H}^n(-\kappa^2)} \lambda_i(\Omega) = \frac{(n-1)^2}{4} \kappa^2.$$

We identify the inequalities in Theorem 2 as the most appropriate tool for the applications of our results. Note that the appearance of the constants $C_0$ and $C_1$ are natural and motivate us to ask the following question:

*Under which conditions the estimates for the eigenvalues obtained from Theorem 2 do not depend on the constants $C_0$ and $C_1$ for a nontrivial drifting function $\eta$?*

We give an immediate answer to this question by considering a radially constant drifting function on any bounded domain $\Omega \subset M^n$. In other words, the next result shows that the behavior of estimates of the eigenvalues of the Cheng-Yau operator remains invariant by a particular first-order perturbation of this operator. Hence, we get a rigidity result for a Cheng-Yau operator on the class of radially constant drifting functions defined on any bounded domains in $M^n$. More precisely, we have:

**Corollary 4.** Let $\lambda_i$ be the $i$-th eigenvalue of the drifted Cheng-Yau operator with a radially constant drifting function on a bounded domain $\Omega \subset M^n$ with the Dirichlet boundary condition. Fix an origin $o \in M^n \setminus \Omega$, and let $r(x)$ be the distance function from $o$. If $T$ is radially parallel and it has $\partial_r$ as an eigenvector, then, all estimates for the sequence of eigenvalues $(\lambda_i)$ that we can obtain from Theorem 3 do not depend on this drifting function.
In the case of radial drifting functions, we again fix an origin $o \in M^n$ to define the distance function $r(x)$ from $o$, and let us consider the bounded annular domain

$$\Omega = \left\{ x \in M^n; \frac{2(n-1)(\kappa_1 + \alpha)R + 2}{c} < r(x)^2 < R^2 \right\},$$

(1.6)

where $\alpha = 1/d$, $cR > (n-1)(\kappa_1 + \alpha) + \sqrt{(n-1)^2(\kappa_1 + \alpha)^2 + 2c}$ and $c$ are positive constants. To obtain such a constant $R$, it is enough to consider the quadratic polynomial $\mathcal{P}(x) = cx^2 - 2(n-1)(\kappa_1 + \alpha)x - 2$. So, for the radial drifting function $\eta(x) = \frac{\sqrt{2}}{2}r(x)^2$ on $\Omega$, we have $C_0 = \delta^2cR$ and

$$\max_{\Omega} (2c - c^2r(x)^2) = 2c - c^2 \min_{\Omega} r(x)^2 = 2c - (2c(n-1)(\kappa_1 + \alpha)R + 2c).$$

Thus, it is null the expression $2C_0(n-1)(\kappa_1 + \frac{1}{2}) + C_1$ that appear in Theorem 2.

Hence, as in Corollary 3, we get a rigidity result for a Cheng-Yau operator on the class of bounded annular domains defined as above. More precisely, we immediately obtain:

**Corollary 5.** Let $\lambda_i$ be the $i$-th eigenvalue of the drifted Cheng-Yau operator with the drifting function $\eta(x) = \frac{\sqrt{2}}{2}r(x)^2$ on the bounded annular domain defined as in Eq. (1.6) with the Dirichlet boundary condition. Fix an origin $o \in M^n \setminus \Omega$, and let $r(x)$ be the distance function from $o$. If $T$ is radially parallel and it has $\partial_\nu$ as an eigenvector, then, all estimates for the sequence of eigenvalues $(\lambda_i)$ that we can obtain from Theorem 3 do not depend on the constants $C_0$ and $C_1$.

To obtain an application of Corollary 5, the reader can consider the drifted Cheng-Yau operator with the drifting function $\eta(x) = \frac{\sqrt{2}}{2}|x|^2$ on bounded annular domains defined as in (1.6) in Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{1}{2}|x|^2)$.

In Section 6, we address specifically the fundamental gap conjecture. There we are working on the behavior of the fundamental gap for a particular case of the operator $\mathcal{L}$ defined as in (1.1) on convex bounded domains in hyperbolic space $\mathbb{H}^2(-1)$. We prove that the fundamental gap for the operator $\operatorname{div}(\varphi \nabla u)$, with $\varepsilon \leq \varphi \leq \delta$ and $\varphi_r = 0$, on each set of a special family of convex domains in $\mathbb{H}^2(-1)$, it satisfies $(\lambda_2 - \lambda_1)D^2 < 3\pi^2\delta$, where $D$ stands for the diameter of each domain of this family, see Theorem 6. Observe that the quantity $(\lambda_2 - \lambda_1)D^2$ is invariant under the scaling of the metric, then, this same result also holds for any simply connected negative constant curvature space forms.

## 2. Preliminaries

In this section, we fix notation, comments about facts that will be used in our proofs, and list without proof all the main formulas which will be appropriated for us. The material we summarize here is known, Gomes and Miranda [16] is a reference for it.

Let $(M^n, \langle , \rangle)$ be an $n$-dimensional Riemannian manifold and $T$ be a $(0,2)$–tensor on $M^n$. Throughout the paper, we will be constantly using the identification of $T$ with its associated $(1,1)$–tensor by the equation

$$\langle T(X), Y \rangle = T(X, Y).$$

In particular, the metric tensor $\langle , \rangle$ will be identified with the identity $I$ in $\mathfrak{X}(M)$. 
Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis in \( T_pM \), and \( S \) be a \((1, 1)\)-tensor with adjoint \( S^* \). Recall that the Hilbert-Schmidt inner product is defined as

\[
\langle T, S \rangle := \text{tr}(TS^*) = \sum_{i=1}^{n} \langle T(e_i), S(e_i) \rangle.
\]

The divergence of a \((1, 1)\)-tensor \( T \) is defined as the \((0, 1)\)-tensor

\[
(\text{div} T)(v)(p) = \text{tr}(w \mapsto (\nabla_w T)(v)(p)),
\]

where \( p \in M^n, v, w \in T_pM, \nabla \) stands for the covariant derivative of \( T \) and \( \text{tr} \) is the trace operator calculated in the metric \( \langle , \rangle \). Note that we can use the identification \( (\text{div} T)(v) = \langle \text{div} T, v \rangle \).

In this paper, the manifold \((M^n, \langle , \rangle)\) is assumed to be complete and \( \Omega \subset M^n \) a bounded domain assumed to be connected and with smooth boundary \( \partial \Omega \). We are using the weighted measure \( \text{dm} = e^{-\eta}\text{dvol}_{\Omega}, \) for some smooth function \( \eta \). If \( \nu \) is the outward normal vector field on the boundary \( \partial \Omega \), then the divergence theorem remains true in the form

\[
\int_{\Omega} \mathcal{L} f \text{dm} = \int_{\partial \Omega} T(\nabla f, \nu) d\mu,
\]

where \( d\mu = e^{-\eta}\text{dvol}_{\partial \Omega} \) is the weighted measure on the boundary induced by \( \nu \). Thus, the “integration by parts” formula is

\[
\int_{\Omega} \ell \mathcal{L} f \text{dm} = -\int_{\Omega} T(\nabla \ell, \nabla f) d\mu + \int_{\partial \Omega} \ell T(\nabla f, \nu) d\mu,
\]

from which we conclude that \( \mathcal{L} \) is a self-adjoint operator in the Hilbert space \( \mathcal{H}^1_\Omega(\Omega, \text{dm}) \). Thus, the Dirichlet eigenvalue problem

\[
\begin{cases}
-\mathcal{L} u &= \lambda u \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega
\end{cases} \tag{2.1}
\]

has a real and discrete spectrum \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty \), where each \( \lambda_i \) is repeated according to its multiplicity. Eigenspaces belonging to distinct eigenvalues are orthogonal in \( L^2(\Omega, \text{dm}) \), which is the direct sum of all the eigenspaces. We refer to the dimension of each eigenspace as the multiplicity of the eigenvalue. Moreover, for any eigenfunction \( u_i \) we have

\[
\lambda_i = -\int_{\Omega} u_i \mathcal{L} u_i \text{dm} = \int_{\Omega} T(\nabla u_i, \nabla u_i) \text{dm}.
\]

The Bochner type formula for the more general expression of \( \mathcal{L} \) is given by

\[
\frac{1}{2} \mathcal{L}(|\nabla f|^2) = \langle \nabla (\mathcal{L} f), \nabla f \rangle + R_{\eta, T}(\nabla f, \nabla f) + \langle \nabla^2 f, \nabla^2 f \circ T \rangle - \langle \nabla^2 f, \nabla \nabla_f T \rangle \tag{2.2}
\]

where \( R_T(X,Y) = \text{tr}(Z \mapsto T \circ R(X,Z)Y), R_{\eta, T} := R_T - \nabla \text{(div}_\eta T) \) and \( R(X,Z)Y \) is the curvature tensor of the Riemannian metric \( \langle , \rangle \) on \( M^n \). Here, the map \( (\cdot)\circ T \) stands for the musical isomorphism.

In particular, by taking \( \eta \) to be constant and \( T \) divergence free in \( (2.2) \), we get the Bochner type formula for the Cheng-Yau operator as follows

\[
\frac{1}{2} \Box(|\nabla f|^2) = \langle \nabla \Box f, \nabla f \rangle + R_T(\nabla f, \nabla f) + \langle \nabla^2 f, \nabla^2 f \circ T \rangle - \langle \nabla^2 f, \nabla \nabla_f T \rangle \tag{2.3}
\]

Two special cases of divergence free positive definite symmetric tensors on \((H^n(-1), \langle , \rangle)\) are \( T = -\text{Ric} \) and \( T = -S \), where \( S = \frac{1}{n-2} \left( \text{Ric} - \frac{R}{2(n-1)} \langle , \rangle \right) \) is
the Schouten tensor of $\langle \cdot, \cdot \rangle$, for $n \geq 3$. More generally, these special tensors can be considered on pinched Cartan-Hadamard Einstein manifolds $(M^n, \langle \cdot, \cdot \rangle)$, since $\text{Ric} = \frac{R}{n} \langle \cdot, \cdot \rangle$ and by Schur’s lemma the scalar curvature of Einstein manifolds of dimension $n \geq 3$ must be constant. We highlight that geodesically complete noncompact Einstein manifolds with $\text{Ric} = -(n-1) \langle \cdot, \cdot \rangle$ have a very special behavior at infinity, see Gicquaud, Ji and Shi [15]. Another example is obtained from $\hat{S} = S - \text{tr}(S) \langle \cdot, \cdot \rangle$ on pinched Cartan-Hadamard manifolds $(M^n, \langle \cdot, \cdot \rangle)$, $n \geq 3$, in this case, we have $\text{div} S = 0$, since $\text{div} S = d\text{tr}(S)$.

3. Auxiliary results

The key to prove the two main theorems of this paper relies on Proposition 1 which is a slight modification of Proposition 1 in [16]. With this in mind, we establish the necessary tools to work with the operator defined in Eq. (1.1) which enable us to obtain more general results. We believe that such tools are of independent interest.

An important lemma that will be used in the proof of Proposition 1 is a known result in comparison geometry. Its proof can be found in [27].

Lemma 1. [Rauch Comparison] Assume that $(M^n, \langle \cdot, \cdot \rangle)$ satisfies $c \leq K \leq C$. If $\langle \cdot, \cdot \rangle = dr^2 + g_r$ represents the metric in the polar coordinates, then

$$\frac{sn'_C(r)}{sn_C(r)} g_r \leq \nabla^2 r \leq \frac{sn'_C(r)}{sn_C(r)} g_r,$$

where $sn_C(r)$ denotes the unique solution to

$$\ddot{x}(r) + \kappa \cdot x(r) = 0 \quad \text{with} \quad x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1.$$

In particular, $\frac{sn'_C(r)}{sn_C(r)} = \sqrt{-\kappa \cosh(\sqrt{-\kappa} r)}$ for $\kappa < 0$, and $\frac{sn'_C(r)}{sn_C(r)} = \frac{1}{\kappa}$ for $\kappa = 0$.

In what follows, $\Omega \subset M^n$, $n \geq 2$, is a bounded domain in an $n$-dimensional pinched Cartan-Hadamard manifold as described in our introduction.

Our first proposition establishes the Rauch comparison theorem for the Cheng-Yau operator and two estimates that will be used in the Bochner type formula for this operator.

Proposition 1. Fix an origin $o \in M^n \setminus \bar{\Omega}$, and let $r(x)$ be the distance function from $o$. Let $T$ be a symmetric positive definite $(1, 1)$–tensor on $M^n$ such that $\partial_r$ is an eigenvector of $T$. Then, for $C = -\kappa^2_2$ and $c = -\kappa^2_1$, the following holds on $\Omega$:

1. $(n-1)\varepsilon \frac{sn'_C(r)}{sn_C(r)} \leq \Box r \leq (n-1)\delta \frac{sn'_C(r)}{sn_C(r)}$.
2. $\langle \nabla^2 r, \nabla^2 r \circ T \rangle \leq (n-1)\delta \left( \frac{sn'_C(r)}{sn_C(r)} \right)^2$.
3. $R_T(\partial_r, \partial_r) \leq -\varepsilon (n-1)\kappa^2_2$.

Proof. Take $x \in \Omega$ and complete $\partial_r$ to an orthonormal basis $\{e_1, \ldots, e_n = \partial_r\}$ for $T_x M$ such that $T e_i = \psi_i e_i$. Note that $\varepsilon \leq \psi_i \leq \delta$ on $\Omega$, for $i = 1, \ldots, n$. Thus,

$$\Box r = \langle \nabla^2 r, T \rangle = \sum_{i=1}^{n} \langle \nabla^2 r(e_i), T(e_i) \rangle = \sum_{i=1}^{n-1} \psi_i \langle \nabla^2 r(e_i), e_i \rangle.$$

Since $r(x)$ is a convex function, i.e., its Hessian is positive semidefinite, one has

$$\varepsilon \Delta r = \varepsilon \sum_{i=1}^{n-1} \langle \nabla^2 r(e_i), e_i \rangle \leq \Box r \leq \delta \sum_{i=1}^{n-1} \langle \nabla^2 r(e_i), e_i \rangle = \delta \Delta r.$$
Hence, the first assertion follows from Lemma 1. To prove the second assertion, we compute
\[ (\nabla^2 r, \nabla^2 r \circ T) = \sum_{i=1}^{n} (\nabla^2 r(e_i), \nabla^2 r \circ T(e_i)) = \sum_{i=1}^{n-1} \psi_i (\nabla^2 r(e_i), \nabla^2 r(e_i)). \]
Thus, again we use that \( r(x) \) is a convex function and that \( \psi_i \leq \delta \) to obtain
\[ (\nabla^2 r, \nabla^2 r \circ T) \leq \delta |\nabla^2 r|^2. \]
Now, we use Lemma 1 to get the second assertion. For the third assertion, we have
\[ R_T(\partial_r, \partial_r) = \sum_{i=1}^{n-1} (R(e_i, \partial_r) \partial_r, T(e_i)) = \sum_{i=1}^{n-1} \psi_i (R(e_i, \partial_r) \partial_r, e_i) \]
\[ \leq -\sum_{i=1}^{n-1} -\psi_i \kappa^2_i \leq -(n-1)(\kappa^2_2) \]
This completes the proof of the proposition. \( \square \)

For the second proposition, it is convenient to consider the constant defined in Eq. (1.5).

**Proposition 2.** Fix an origin \( o \in M^n \setminus \Omega \), and let \( r(x) \) be the distance function from \( o \). Let \( T \) be a symmetric positive definite \((1,1)\)-tensor on \( M^n \) such that \( \partial_r \) is an eigenvector of \( T \), and \( u_i \) be an \( L^2(\Omega, dm) \)-normalized function. If \( T \) is radially parallel, then:

1. For \( a(n, \varepsilon, \delta) \leq 0 \), it is true that
\[ \int_{\Omega} u_i^2 \left( -\langle \Delta r \rangle^2 - 2\langle \nabla r, T \partial_r \rangle \right) dm \leq -(n-1)^2 \varepsilon^2 \kappa^2_2 + 2(n-1)(\delta^2 \kappa^2_1 - \varepsilon^2 \kappa^2_2). \]
2. For \( a(n, \varepsilon, \delta) > 0 \), we consider \( d = \text{dist}(\Omega, o) \) so that
\[ \int_{\Omega} u_i^2 \left( -\langle \Delta r \rangle^2 - 2\langle \nabla r, T \partial_r \rangle \right) dm \leq -(n-1)^2 \varepsilon^2 \kappa^2_2 + 2(n-1)(\delta^2 \kappa^2_1 - \varepsilon^2 \kappa^2_2) + \frac{a(n, \varepsilon, \delta)}{d^2}. \]

**PROOF.** We begin by estimating the expression \( -\langle \Delta r \rangle^2 - 2\langle \nabla r, T \partial_r \rangle \). For it, we use that \( T \) is radially parallel, i.e., \( \nabla_{\partial_r} T \) is null, so that, from Bochner type formula for the Cheng-Yau operator, we obtain
\[ -\langle \Delta r, \partial_r \rangle = R_T(\partial_r, \partial_r) + \langle \nabla^2 r, \nabla^2 r \circ T \rangle. \]
Since \( T \partial_r = \psi_n \partial_r \), from the second and third assertions of Proposition 1, we have
\[ -\langle \Delta r, \partial_r \rangle = -\psi_n \langle \nabla r, \partial_r \rangle \leq -(n-1) \varepsilon^2 \kappa^2_2 + (n-1) \delta^2 \left( \frac{\sinh^{-2}(\kappa_1 r)}{\sinh^{-2}(\kappa_2 r)} \right)^2. \] (3.1)
There are three cases to consider:

(a) \( 0 < \kappa_2 \leq \kappa_1 \) case: Inequality (3.1) becomes
\[ -\langle \Delta r, \partial_r \rangle \leq (n-1) \delta^2 \kappa^2_1 \frac{\cosh^2(\kappa_1 r)}{\sinh^2(\kappa_1 r)} - (n-1) \varepsilon^2 \kappa^2_2. \]
So, from the first assertion of Proposition $1$, we estimate the expression

$$-(\Box r)^2 - 2\langle \nabla \Box r, T \partial r \rangle \leq -(n-1)^2 \varepsilon^2 \sum_{j=1}^{2} \cot^2(\kappa_j r) + 2(n-1) \delta^2 \sum_{j=1}^{2} \cot^2(\kappa_1 r) - 2(n-1) \varepsilon^2 \sum_{j=1}^{2} \cot^2(\kappa_j r)$$

$$= -(n-1)^2 \varepsilon^2 \kappa_2^2 - \frac{(n-1)^2 \varepsilon^2 \kappa_2^2}{\sinh^2(\kappa_2 r)} + 2(n-1) \delta^2 \kappa_2^2 + \frac{2(n-1) \delta^2 \kappa_2^2}{\sinh^2(\kappa_1 r)} - 2(n-1) \varepsilon^2 \kappa_2^2$$

$$= -(n-1)^2 \varepsilon^2 \kappa_2^2 + 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) + \left[ -\frac{(n-1)^2 \varepsilon^2 \kappa_2^2}{\sinh^2(\kappa_2 r)} + \frac{2(n-1) \delta^2 \kappa_1^2}{\sinh^2(\kappa_1 r)} \right].$$

Since $0 < \kappa_2 \leq \kappa_1$ and $r > 0$, we get

$$\frac{\kappa_1^2}{\sinh^2(\kappa_1 r)} \leq \frac{\kappa_2^2}{\sinh^2(\kappa_2 r)}.$$

Thus,

$$-\frac{(n-1)^2 \varepsilon^2 \kappa_2^2}{\sinh^2(\kappa_2 r)} + \frac{2(n-1) \delta^2 \kappa_1^2}{\sinh^2(\kappa_1 r)} \leq -\frac{(n-1)^2 \varepsilon^2}{r^2} + 2(n-1) \delta^2 \frac{\kappa_1^2}{\sinh^2(\kappa_1 r)}. \quad (3.2)$$

(b) $0 = \kappa_2 < \kappa_1$ case: Again from the first assertion of Proposition $1$ and Inequality $(3.1)$, we estimate the expression

$$-(\Box r)^2 - 2\langle \nabla \Box r, T \partial r \rangle \leq -\frac{(n-1)^2 \varepsilon^2}{r^2} + 2(n-1) \delta^2 \frac{\kappa_1^2}{\sinh^2(\kappa_1 r)}$$

$$= 2(n-1) \delta^2 \kappa_1^2 + \left[ -\frac{(n-1)^2 \varepsilon^2}{r^2} + \frac{2(n-1) \delta^2 \kappa_1^2}{\sinh^2(\kappa_1 r)} \right].$$

Since $0 < \kappa_1$ and $r > 0$, we get

$$\frac{\kappa_1^2}{\sinh^2(\kappa_1 r)} \leq \frac{1}{r^2}. \quad (3.3)$$

Thus,

$$-\frac{(n-1)^2 \varepsilon^2}{r^2} + \frac{2(n-1) \delta^2 \kappa_1^2}{\sinh^2(\kappa_1 r)} \leq -\frac{(n-1)^2 \varepsilon^2}{r^2} + 2(n-1) \delta^2 \frac{\kappa_1^2}{\sinh^2(\kappa_1 r)}. \quad (3.4)$$

(c) $0 = \kappa_2 = \kappa_1$ case: Again from the first assertion of Proposition $1$ and Inequality $(3.1)$, we estimate the expression

$$-(\Box r)^2 - 2\langle \nabla \Box r, T \partial r \rangle \leq -\frac{(n-1)^2 \varepsilon^2}{r^2} + 2(n-1) \delta^2 \frac{1}{r^2}. \quad (3.5)$$

Now, with inequalities $(3.2), (3.4)$ and $(3.5)$ in mind, we immediately obtain the first integral estimate of the proposition for $a(n, \varepsilon, \delta) \leq 0$. For the case of $a(n, \varepsilon, \delta) > 0$, first note that

$$\int_{\Omega} u_i^2 \frac{a(n, \varepsilon, \delta)}{r^2} \, dm \leq \frac{a(n, \varepsilon, \delta)}{d^2}, \quad (3.6)$$

where $d = dist(\Omega, o)$, since that $0 < d \leq r(x)$, for all $x \in \Omega$. We now complete our proof immediately from inequalities $(3.2), (3.3), (3.4), (3.5)$ and $(3.6)$. □
Proposition 3. Fix an origin \( o \in M^n \setminus \bar{\Omega} \), and let \( r(x) \) be the distance function from \( o \). Let \( T \) be a symmetric positive definite \((1, 1)\)-tensor on \( M^n \) such that \( \partial_r \) is an eigenvector of \( T \), and \( u_i \) be an \( L^2(\Omega, dm) \)-normalized function. Then,
\[
\int_{\Omega} u_i^2 r g(T \partial_r, \nabla \eta) dm \leq C_0 (n-1) (\kappa_1 + \frac{1}{d}),
\]
where \( d = \text{dist}(\Omega, o) \) and \( C_0 = \delta^2 \max_{\Omega} |\eta| \).

Proof. From the first assertion of Proposition 1 we estimate
\[
\int_{\Omega} u_i^2 r g(T \partial_r, \nabla \eta) dm \leq \left( \int_{\Omega} u_i^2 r (\nabla \eta)^2 dm \right)^{1/2} \left( \int_{\Omega} u_i^2 (T \partial_r, \nabla \eta)^2 dm \right)^{1/2}
\]
\[
\leq \delta \max_{\Omega} |\eta| \left( \int_{\Omega} u_i^2 (\nabla \eta)^2 dm \right)^{1/2}
\]
\[
\leq C_0 (n-1) \left( \int_{\Omega} u_i^2 \left( \frac{\sinh^{-1}(r)}{\sinh^{-1}(r)} \right)^2 dm \right)^{1/2}.
\]

There are two cases to consider:

(a) \( 0 < \kappa_2 \leq \kappa_1 \) and \( 0 = \kappa_2 < \kappa_1 \) cases:
\[
\int_{\Omega} u_i^2 r g(T \partial_r, \nabla \eta) dm \leq C_0 (n-1) \kappa_1 \left( \int_{\Omega} u_i^2 \cosh^{-1}(\kappa_1 r) \sinh^{-1}(\kappa_1 r) \right)^{1/2}
\]
\[
\leq C_0 (n-1) \kappa_1 + C_0 (n-1) \left( \int_{\Omega} u_i^2 \frac{\kappa_1}{\sinh^2(\kappa_1 r)} dm \right)^{1/2}.
\]

(b) \( \kappa_1 = \kappa_2 = 0 \) case:
\[
\int_{\Omega} u_i^2 r g(T \partial_r, \nabla \eta) dm \leq C_0 (n-1) \left( \int_{\Omega} u_i^2 \frac{1}{r^2} dm \right)^{1/2}.
\]

As we argued in the proof of Proposition 2 we obtain the required integral estimate of the present proposition.

\( \square \)

4. Proof of Theorems 1 and 2

The next step before giving the proof of the two main theorems is a universal quadratic inequality for the eigenvalues of Problem (2.1), which is an essential tool for us.

Proposition 4. Fix an origin \( o \in M^n \setminus \bar{\Omega} \), and let \( r(x) \) be the distance function from \( o \). Let \( \lambda_i \) be the \( i \)-th eigenvalue of Problem (2.1), and \( u_i \) be its corresponding \( L^2(\Omega, dm) \)-normalized eigenfunction. Then,
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{4 \delta^2}{\varepsilon} \lambda_i - \int_{\Omega} u_i^2 ((\nabla r)^2 + 2(T \partial_r, \nabla \nabla r)) dm \right).
\]

Proof. Proposition 1 in [16] says that
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} u_i^2 T(\nabla h, \nabla h) dm \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_{\Omega} \left( u_i \nabla h + 2 T(\nabla h, \nabla u_i) \right)^2 dm
\]
for any \( h \in C^3(\Omega) \cap C^2(\partial \Omega) \). By taking \( h = r \), and noting that \( \varepsilon \leq T(\partial_r, \partial_r) \leq \delta \), we obtain
\[
\varepsilon \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_{\Omega} \{ u_i \mathcal{L} r + 2T(\partial_r, \nabla u_i) \} \, dm. \tag{4.1}
\]
The integral in (4.1) is estimated as follows
\[
\int_{\Omega} \{ u_i^2 (\mathcal{L} r)^2 + 4u_i \mathcal{L} r T(\partial_r, \nabla u_i) + 4T(\partial_r, \nabla u_i)^2 \} \, dm.
\]

Moreover,
\[
\int_{\Omega} \{ u_i^2 (\mathcal{L} r)^2 + 2T(\partial_r, \nabla (u_i^2 \mathcal{L} r) - u_i^2 \nabla \mathcal{L} r) \} \, dm + 4 \int_{\Omega} |T(\nabla u_i)|^2 \, dm. \tag{4.2}
\]
Integration by part formula gives us
\[
\int_{\Omega} \{ u_i^2 (\mathcal{L} r)^2 + 2T(\partial_r, \nabla (u_i^2 \mathcal{L} r)) \} \, dm = - \int_{\Omega} u_i^2 (\mathcal{L} r)^2 \, dm. \tag{4.3}
\]
Moreover,
\[
\varepsilon \int_{\Omega} |\nabla u_i|^2 \, dm \leq \int_{\Omega} \langle \nabla u_i, T(\nabla u_i) \rangle \, dm = \lambda_i.
\]
So,
\[
\int_{\Omega} |T(\nabla u_i)|^2 \, dm \leq \delta^2 \int_{\Omega} |\nabla u_i|^2 \, dm \leq \frac{\delta^2 \lambda_i}{\varepsilon}. \tag{4.4}
\]
Using (4.2), (4.3) and (4.4), we get
\[
\int_{\Omega} \{ u_i^2 (\mathcal{L} r)^2 + 4u_i \mathcal{L} r T(\partial_r, \nabla u_i) + 4T(\partial_r, \nabla u_i)^2 \} \, dm.
\]

Substituting (4.5) into (4.1) we obtain the estimate of the proposition. \( \square \)

We are now ready to prove our main results.

4.1. Proof of Theorem 1.

PROOF. We begin by proving the result of the theorem for the hyperbolic space \( \mathbb{H}^n \) case with constant curvature \(-1\), i.e., the open half space \( x_n > 0 \) with its standard metric \( g_{ij} = x_n^{-2} \delta_{ij} \). For this case, we have
\[
\nabla \ln x_n = g^{ij} \partial_{ln} x_n \partial_j = x_n \partial_n \quad \text{and} \quad |\nabla \ln x_n| = 1.
\]
By hypotheses \( T(\nabla \ln x_n) = \psi \nabla \ln x_n = \psi x_n \partial_n \) and \( \partial_n \eta = 0 \), thus
\[
\langle \nabla \eta, T(\nabla \ln x_n) \rangle = \langle g^{ij} \partial_j \eta \partial_j, \psi x_n \partial_n \rangle = 0.
\]
Recall that \( \text{div}(a^i \partial_i) = \frac{1}{\sqrt{\text{det}(g_{ij})}} \partial_i (\sqrt{\text{det}(g_{ij})} a^i) \), and by assumption \( \partial_n \psi = 0 \), so that
\[
\mathcal{L}(\ln x_n) = \text{div}(T(\nabla \ln x_n)) - \langle \nabla \eta, T(\nabla \ln x_n) \rangle
\]
\[
= \text{div}(\psi x_n \partial_n) = x_n^2 \partial_n (\psi x_n^{1-n}) = (1-n)\psi.
\]
So,
\[
\langle T(\nabla \ln x_n), \nabla \mathcal{L}(\ln x_n) \rangle = \langle \psi x_n \partial_n, (1-n) \nabla \psi \rangle = (1-n)\psi x_n \partial_n \psi = 0.
\]
Since $\ln x_n$ works as a distance function on $\mathbb{H}^n(-1)$, we can take $r = \ln x_n$ in Proposition 4 to obtain
\[
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left(\frac{4\delta^2}{\varepsilon} \lambda_i - \int_{\Omega} u_i^2(1-n)^2 \psi^2 \, dm\right).
\]

Since $\varepsilon \leq \langle T(\nabla \ln x_n), \nabla \ln x_n \rangle = \psi$, we have $-\psi^2 \leq -\varepsilon^2$, which is enough to obtain the next inequality
\[
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left(\frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \right).
\]

Now, by rescaling the metric by a factor of $\kappa^{-2}$, the previous inequality for the case of $\mathbb{H}^n(-\kappa^2)$ becomes
\[
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left(\frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \kappa^2 \right).
\]

Moreover, quadratic estimate guarantees that
\[
\frac{4\delta^2}{\varepsilon} \lambda_i - (n-1)^2 \varepsilon^2 \kappa^2 \geq 0,
\]
for $i \in \{1, 2, \ldots\}$, but the interesting case is $\lambda_1 \geq \frac{\varepsilon}{4\delta^2} (n-1)^2 \varepsilon^2 \kappa^2 > 0$, since $n \geq 2$ and $\kappa > 0$.

4.2. Proof of Theorem 2.

PROOF. By Proposition 4 we just need to estimate the integral
\[
- \int_{\Omega} u_i^2 ((\mathcal{L}r)^2 + 2\langle T\partial r, \nabla \mathcal{L}r \rangle) \, dm,
\]
for the particular case of the drifted Cheng-Yau $\mathcal{L}r = \square r - \langle T\partial r, \nabla \eta \rangle$. First, we compute
\[
(\mathcal{L}r)^2 + 2\langle T\partial r, \nabla \mathcal{L}r \rangle = (\square r)^2 - 2\square r \langle T\partial r, \nabla \eta \rangle + \langle T\partial r, \nabla \eta \rangle^2 + 2\langle T\partial r, \nabla \square r \rangle - 2\langle T\partial r, \nabla \langle T\partial r, \nabla \eta \rangle \rangle = (\square r)^2 + 2\langle T\partial r, \nabla \square r \rangle - 2\square r \langle T\partial r, \nabla \eta \rangle - 2\langle T\partial r, \nabla \langle T\partial r, \nabla \eta \rangle \rangle + \langle T\partial r, \nabla \eta \rangle^2.
\]

Therefore
\[
- \int_{\Omega} u_i^2 ((\mathcal{L}r)^2 + 2\langle T\partial r, \nabla \mathcal{L}r \rangle) \, dm = \int_{\Omega} u_i^2 \left( (\square r)^2 - 2\langle T\partial r, \nabla \square r \rangle \right) \, dm + 2 \int_{\Omega} u_i^2 \triangle r \langle T\partial r, \nabla \eta \rangle \, dm + \int_{\Omega} u_i^2 \left( 2\langle T\partial r, \nabla \langle T\partial r, \nabla \eta \rangle \rangle - \langle T\partial r, \nabla \eta \rangle^2 \right) \, dm.
\]

Since $\nabla \partial r T = 0$ and $T\partial r = \psi \partial r$, we have that $\partial r \psi = 0$ and
\[
\langle \partial r, \nabla \langle T\partial r, \nabla \eta \rangle \rangle = \partial r \langle T\partial r, \nabla \eta \rangle = \psi \partial \eta.
\]

Thus,
\[
\int_{\Omega} u_i^2 \left( 2\langle T\partial r, \nabla \langle T\partial r, \nabla \eta \rangle \rangle - \langle T\partial r, \nabla \eta \rangle^2 \right) \, dm \leq \int_{\Omega} u_i^2 \psi^2 \left( 2\partial \eta - \eta \right)^2 d\mu \leq C_1.
\]
where \( C_1 = \delta^2 \max_\Omega (2\hat{y} - \hat{y}^2) \). The previous estimate together with Propositions 2 and 3 immediately implies the inequalities of Theorem 2.

\( \square \)

### 5. Applications of Theorem 1

As with the other proofs in this paper we need to proceed in stages. First, we recall a result by McKean [24], see alternatively Chavel [8]. Here, we give a proof by combining our Theorem 1 with an approach of the spectrum of warped metrics.

**Lemma 2.** Let \( B(a) \subset \mathbb{H}^n(-\kappa^2) \) be an \( n \)-disk of radius \( a > 0 \). The first eigenvalue \( \lambda_1 \) of the Laplacian on \( B(a) \) with the Dirichlet boundary condition satisfies

\[
\lim_{a \to +\infty} \lambda_1(B(a)) = \frac{(n-1)^2}{4}\kappa^2.
\]

**Proof.** We start by proving the result of the lemma for the unity case \( \mathbb{H}^n(-1) \). Recall that its metric in the polar coordinates is given by \((r,\phi) = dr^2 + \sinh^2 r d\Omega_{n-1}\) on \((0, +\infty) \times S^{n-1}\). Thus, we can address the eigenvalue problem as in [23, Section 2] from which we take \( \mu_0 = 0 \) and \( \psi \) to be constant, so that our eigenvalue problem for the Laplacian becomes

\[
\ddot{\phi}(r) + (n-1) \coth(r) \dot{\phi}(r) + \lambda \phi(r) = 0,
\]

for some \( \phi \in L^2\left(\left[\frac{1}{2}, a\right], \sinh^{2(n-1)}(r) dr^2\right) \). For our purpose, it is enough to consider \( \lambda = \frac{(n-1)^2}{4} \) and the function \( \coth(r) \) for large values of \( r \). So, from now on, we are considering the solution of the ODE on \((\frac{1}{2}, a)\) as follows

\[
\phi(r) + (n-1) \dot{\phi}(r) + \frac{(n-1)^2}{4} \phi(r) = 0.
\]

We define \( f \) on \( B(a) \subset \mathbb{H}^n(-1) \) given by \( f(r, \theta) = \psi(r) \), where

\[
\psi(r) = \begin{cases} 
\phi(r) & \text{if } r \in \left(\frac{1}{2}, a\right), \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( f \) is an admissible function for the Dirichlet eigenvalue problem on the space \( L^2(B(a), d\nu) \), where \( d\nu = \sinh^{n-1}(r) d\text{vol}_{B(a)} \). Besides, \( \psi \) satisfies

\[
\ddot{\psi}(r) + (n-1) \dot{\psi}(r) + \frac{(n-1)^2}{4} \psi = 0 \quad \text{on} \quad \left(\frac{1}{2}, a\right).
\]

Now, we use integration by parts to obtain

\[
\int_{B(a)} \psi^2 d\nu = \int_{B(a)} \left( (n-1) \psi - \coth(r) + \frac{(n-1)^2}{4} \psi^2 \right) d\nu
\]

which implies

\[
\int_{B(a)} \left( \psi^2 - \frac{(n-1)^2}{4} \psi^2 \right) d\nu \leq \int_{B(a)} (n-1) |\psi| |\dot{\psi}| \coth(r) - 1 |d\nu|
\]

Note that, \( \sup_{B(a)} |\coth(r) - 1| = |\coth(a/2) - 1| \) and \( |\dot{\psi}| = |\nabla f| \) on \( B(a) \). So,

\[
\|\nabla f\|^2 - \frac{(n-1)^2}{4} \|f\|^2 \leq (n-1) |\coth(a/2) - 1| \|f\| \|\nabla f\|.
\]

The previous inequality reads as

\[
\frac{\|\nabla f\|^2}{\|f\|^2} - (n-1) |\coth(a/2) - 1| \frac{\|\nabla f\|}{\|f\|} - \frac{(n-1)^2}{4} \leq 0.
\]
Moreover, it is clear that Theorem 1 to conclude that \( \lim \) 
\[
\frac{n-1}{2} \left| \coth(a/2) - 1 \right| + \frac{1}{2} \left( (n-1)^2 \right| \coth(a/2) - 1 \right|^2 + (n-1)^2 \right)^{\frac{1}{2}}.
\]
By Rayleigh’s theorem (see, e.g., Chavel [8])
\[
\sqrt{\lambda(B(a))} \leq \frac{n-1}{2} \left| \coth(a/2) - 1 \right| + \frac{1}{2} \left( (n-1)^2 \right| \coth(a/2) - 1 \right|^2 + (n-1)^2 \right)^{\frac{1}{2}}.
\]
Thus, \( \lim_{a \to 0} \lambda(B(a)) \leq \frac{(n-1)^2}{4}. \) From this latter result and Theorem 1 we get \( \lim_{a \to 0} \lambda(B(a)) = \frac{(n-1)^2}{4}. \) We now use the same argument as in the proof of Theorem 1 to conclude that \( \lim_{a \to \infty} \lambda(B(a)) = \frac{(n-1)^2}{4} \kappa^2 \) for the case of \( \mathbb{H}^n(-\kappa^2). \)

5.1. Proof of Corollary 3.

**Proof.** Here we are following the same steps of the proof given by Cheng-Yang [12 Corollary 1.3]. Let us consider into each bounded domain \( \Omega \subset \mathbb{H}^n(-\kappa^2) \) an \( n \)-disk \( B(a) \) of radius \( a > 0 \). Thus, from the domain monotonicity of eigenvalues (see, e.g., Chavel [8]), Lemma 2 and Theorem 1 we have
\[
\lambda_1(\Omega) \geq \frac{(n-1)^2}{4} \kappa^2 \quad \text{and} \quad \lim_{\Omega \to \mathbb{H}^n(-\kappa^2)} \lambda_1(\Omega) = \frac{(n-1)^2}{4} \kappa^2.
\]
Moreover, it is clear that
\[
\lambda_i(\Omega) \geq \frac{(n-1)^2}{4} \kappa^2 \quad \forall i > 1.
\]
Note that for the Laplacian case, we can work with \( v_i := 4\lambda_i - (n-1)^2\kappa^2 \), see Theorems 1 and 7. Since \( v_1 = 4\lambda_1 - (n-1)^2\kappa^2 \), then from (5.1) we obtain \( v_1(\Omega) \to 0 \) as \( \Omega \to \mathbb{H}^n(-\kappa^2) \). On the other hand, from (5.1), we get \( v_{i+1} \leq 5^2v_1 \forall i \geq 1 \). Consequently,
\[
0 = \lim_{\Omega \to \mathbb{H}^n(-\kappa^2)} v_{i+1}(\Omega) = \lim_{\Omega \to \mathbb{H}^n(-\kappa^2)} (4\lambda_{i+1}(\Omega) - (n-1)^2\kappa^2)
\]
which is enough to complete the proof of the corollary.

6. The fundamental gap for a class of operators on a class of convex domains in two-dimensional hyperbolic space

In this section, we address the fundamental gap for a certain class of operators \( \mathcal{L} \) defined as in (1.1), more precisely, we estimate the difference \( \lambda_2(\Omega) - \lambda_1(\Omega) \) between the first two eigenvalues of the eigenvalue problem for a special family of \( \mathcal{L} \) on a class of convex bounded domains \( \Omega \) with the Dirichlet boundary condition. In the first part, we are working with the class of the operators \( \mathcal{L} = \text{div}(\varphi \nabla u) \), for some radially constant function \( \varphi \), satisfying \( \varepsilon \leq \varphi \leq \delta \), for some positive constants \( \varepsilon \) and \( \delta \).

We begin with a brief historical background. The solution for the problem
\[
y''(t) + \lambda y(t) = 0 \quad n((0,t), \lambda > 0 \text{ and } y(0) = y(t) = 0, \text{ is given by } y(t) = \sum_{n} (A_n \sin(\sqrt{\lambda_n}t) + B_n \cos(\sqrt{\lambda_n}t)), \text{ where } \lambda_n = (n\pi)^2/\ell^2, \text{ hence, } \lambda_2 - \lambda_1 = 3\pi^2/\ell^2.
\]
This gap motivates us to think about the more general case of the Laplacian in convex bounded domains \( \Omega \subset \mathbb{R}^n \). For this case, it was observed in the 80’s by Michiel van den Berg [3] that for many convex domains, \( \lambda_2 - \lambda_1 \geq 3\pi^2/D^2 \), where \( D \) is the diameter of \( \Omega \). It was also independently suggested by Ashbaugh and
Benguria [3], and Yau [34] that this estimate of the gap remains true for any convex bounded domain in $\mathbb{R}^n$. It has been known as the fundamental gap conjecture. For the case of non-convex domains, it is known that the fundamental gap has no such a lower bound, and for non-connected cases, the gap may vanish.

In 2011, Andrews and Clutterbuck [2] proved such conjecture and suggested that it remains valid for the case of constant curvature spaces. In 2019, Seto, Wang and Wei [31] proved this new conjecture for convex domains with diameter $D \leq \pi/2$ in unit sphere $S^n$, $n \geq 3$. In 2020, He, Wei and Zhang [17] extended the Seto-Wang-Wei’s result to convex domains with diameter $D < \pi$ in unit sphere $S^n$, $n \geq 3$. In 2021, Dai, Seto and Wei [14] proved the conjecture for any convex bounded domains in $\mathbb{S}^2$.

However, the fundamental gap for the Laplacian behaves differently in negatively curved spaces. Indeed, Bourni et al. [6] constructed convex domains in $\mathbb{H}^2(-1)$, with diameter $D$, such that $(\lambda_2 - \lambda_1)D^2 < 3\pi^2$, and as the quantity $(\lambda_2 - \lambda_1)D^2$ is invariant under the scaling of the metric, this same result also holds for any simply connected negative constant curvature space forms. Very recently, for the more general case of convex domains in $\mathbb{H}^n(-1)$, $n \geq 2$, the same authors proved that $(\lambda_2 - \lambda_1)D^2$ can be arbitrarily small for domains of any diameter, see Bourni et al. [7, Theorem 1.1].

The essence of these results motivated us to work on the behavior of the fundamental gap for some case of our operator $\mathcal{L}$ on convex bounded domains in hyperbolic space. Here, we find a simple class of radially parallel $(1,1)$–tensors $T$ and drifting functions $\eta$ that define the operator $\mathcal{L}$ in [11], to answer positively the following two questions:

Is there some convex domain in hyperbolic space for which the fundamental gap for the operator $\text{div}(\varphi \nabla u)$ satisfies: $(\lambda_2 - \lambda_1)D^2 < 3\pi^2\delta$?

Is there some drifting function $\eta$ for which the fundamental gap for the operator $\text{div}(\varphi \nabla u) - (\nabla \eta, \varphi \nabla u)$ still satisfies the inequality $(\lambda_2 - \lambda_1)D^2 < 3\pi^2\delta$ on some convex domain in hyperbolic space?

As we mentioned before, the first question has been motivated by the work of Bourni et al. [6] that settled the case when $\varphi$ is constant. Here, to suit our case, we adapt the latter method, furthermore, we also provide some generalizations in comparison with the current literature.

Due to invariance of the quantity $(\lambda_2 - \lambda_1)D^2$, we can work, without loss of generality, on the hyperbolic space $\mathbb{H}^2(-1)$ with constant curvature $-1$, namely, the open half space $y > 0$ with the metric $g_{ij} = y^{-2}\delta_{ij}$. Such a metric in the coordinates $x = r \cos \theta$ and $y = r \sin \theta$, where $r > 0$ and $0 < \theta < \pi$, is written as follows

$$ds^2 = \frac{1}{\sin^2 \theta} \frac{dr^2}{r^2} + \frac{d\theta^2}{\sin^2 \theta} \quad (6.1)$$

so that $\{e_1 = r \sin \theta \frac{\partial}{\partial r}, e_2 = \sin \theta \frac{\partial}{\partial \theta}\}$ is an orthonormal frame, whose the nonzero Christoffel symbols are $\Gamma^1_{11} = -\Gamma^1_{12} = -\Gamma^1_{21} = \cos \theta$. Note that we can define an $(1,1)$–tensor $T$ on $\mathbb{H}^2(-1)$ by $T = \varphi I$, for $\varphi \in C^\infty(\mathbb{H}^2)$, with $\varepsilon \leq \varphi \leq \delta$, to be appropriately chosen.

First we compute $\text{div}(\varphi \nabla u)$ of a smooth function $u$ on $\mathbb{H}^2(-1)$. For this, we set $\nabla u = u_1 e_1 + u_2 e_2$, where $u_1 = e_1(u) = r \sin \theta u_r$ and $u_2 = e_2(u) = \sin \theta u_\theta$. Hence,

$$\langle \nabla_{e_1} \nabla u, e_1 \rangle = (u_{11} + u_2 \Gamma^1_{12}) = r^2 \sin^2 \theta u_{rr} + r \sin^2 \theta u_r - \sin \theta \cos \theta u_\theta,$$

$$\langle \nabla_{e_2} \nabla u, e_2 \rangle = u_{22} = \sin^2 \theta u_{\theta\theta} + \sin \theta \cos \theta u_\theta.$$
Thus,
\[
\Delta u = \langle \nabla_{e_1} \nabla u, e_1 \rangle + \langle \nabla_{e_2} \nabla u, e_2 \rangle = r^2 \sin^2 \theta u_{rr} + r \sin^2 \theta u_r - \sin \theta \cos \theta u_{\theta \theta} + \sin \theta \cos \theta u_{\theta} = r^2 \sin^2 \theta u_{rr} + r \sin^2 \theta u_r + \sin^2 \theta u_{\theta \theta}.
\]

Besides, \( \varphi_1 = e_1(\varphi) = r \sin \theta \varphi_r \) and \( \varphi_2 = e_2(\varphi) = \sin \theta \varphi_\theta \), thus
\[
\langle \nabla u, \nabla \varphi \rangle = u_1 \varphi_1 + u_2 \varphi_2 = r^2 \sin^2 \theta u_r + \sin^2 \theta \varphi_{\theta} u_{\theta},
\]
and recall that
\[
\text{div} (\varphi \nabla u) = \varphi \Delta u + \langle \nabla \varphi, \nabla u \rangle.
\]

Under the additional assumption of \( T \) be radially parallel, we get \( \varphi_r = 0 \), whence equation (6.3) becomes
\[
\text{div} (\varphi \nabla u) = \varphi (r^2 \sin^2 \theta u_{rr} + r \sin^2 \theta u_r) + \varphi \sin^2 \theta u_{\theta \theta} + \sin^2 \theta \varphi_{\theta} u_{\theta} + \lambda u = 0, \quad \text{in} \quad \Omega,
\]
which completes the first part.

Now, we are working on the fundamental gap for \( \text{div} (\varphi \nabla u) \) in (6.4). We start with the following eigenvalue problem with the Dirichlet boundary condition:
\[
\varphi (r^2 \sin^2 \theta u_{rr} + r \sin^2 \theta u_r) + \varphi \sin^2 \theta u_{\theta \theta} + \sin^2 \theta \varphi_{\theta} u_{\theta} + \lambda u = 0, \quad \text{in} \quad \Omega,
\]
\( u = 0, \) on \( \partial \Omega. \)

For each \( \ell > 0, \theta_0 \in (0, \frac{\pi}{2}) \) and \( \theta_1 \in (\frac{\pi}{2}, \pi) \) consider the family of domains \( \Omega_{\ell, \theta_0, \theta_1} = \{(r, \theta) : 1 < r < e^\frac{\ell}{\pi}, \theta_0 < \theta < \theta_1\} \), see Figure 1. As the geodesics are the vertical lines \( x = \ell \) and semicircles centered on the axis \( x \), then the sets \( \Omega_{\ell, \theta_0, \theta_1} \) are convex domains in \( \mathbb{H}^2(-1) \). Since the metric in (6.1) is a warped product, we can use the method of separating variables (see, e.g., [8] page 41), from which we write \( u(r, \theta) = f(r)h(\theta) \), so that \( u_r = f_r h, u_{rr} = f_{rr} h, u_{\theta} = f h_\theta \) and \( u_{\theta \theta} = f h_{\theta \theta} \). Hence, from Problem (6.4), we get
\[
(r^2 f_{rr} + r f_r) h + (h_{\theta \theta} + \frac{\varphi_{\theta}}{\varphi} h_\theta + \frac{\lambda}{\varphi} \csc^2 \theta h)f = 0.
\]

Since \( f \) depends only on \( r \), and \( h \) depends only on \( \theta \), there exists a constant \( \mu \) such that we can interchange this problem by the following two eigenvalue problems
\[
r^2 f_{rr} + r f_r = -\mu f, \quad r \in (1, e^\frac{\ell}{\pi})
\]
\[
h_{\theta \theta} + \frac{\varphi_{\theta}}{\varphi} h_\theta + \frac{\lambda}{\varphi} \csc^2 \theta h = \mu h, \quad \theta \in (\theta_0, \theta_1),
\]
\[
\text{div} (\varphi \nabla u) = \varphi \Delta u + \langle \nabla \varphi, \nabla u \rangle.
\]
with the Dirichlet bounded conditions \( f(1) = f(e^{T}) = 0 \) and \( h(\theta_0) = h(\theta_1) = 0 \), respectively. Besides, since \( 1 < r < e^{T} \), we can make the change of variable \( f(r) = f(e^{t}) \), for \( 0 < t < \frac{\pi}{\ell} \). Whence, \( f_t = rf_r \) and \( f_{tt} = rf_r + r^2 f_{rr} \). So, from (6.7), we obtain
\[
f_{tt} = -\mu f, \quad t \in \left(0, \frac{\pi}{\ell}\right),
\]
and, thus, the Dirichlet bounded condition guarantees that \( f(t) = \sin(\sqrt{\mu}t) \), with \( \mu = (k\ell)^2 > 0 \), where \( k \) is a nonzero integer, i.e., \( f(t) = \sin(k\ell t) \). While Eq. (6.8) is rewritten as
\[
-(\phi h)_\theta + \mu \phi h = \lambda \csc^2 \theta h.
\]

6.1. Identifying the first two eigenvalues. For the sake of completeness, we start this section by stating the Courant results for nodal domains, the Sturm comparison theorem for Jacobi equations, and the Sturm-Liouville theorem.

**Theorem 3** (Courant results for nodal domains, see, e.g., 13, p. 14-15). The first eigenfunction \( \phi_1 \) of an second order elliptic differential operator with Dirichlet boundary condition is positive in \( \Omega \) when \( \Omega \subset \mathbb{R}^n \) is connected; the first eigenvalue \( \lambda_1 \) has multiplicity equal to 1; and the second eigenfunction \( \phi_2 \) has precisely 2 nodal domains. Moreover, \( \lambda_1 \) is characterized as being the only eigenvalue with eigenfunction of constant sign.

**Theorem 4** (Sturm comparison theorem, see 32, p. 104). Let \( f_i \) be non-trivial solutions of
\[
(p(x)f'_i(x))' + b_i(x)f_i(x) = 0, \quad \text{for} \quad i = 1, 2,
\]
where \( 0 < p \in C^1 \), \( b_1 \) and \( b_2 \) are continuous functions, and \( b_1 \geq b_2 \) for all \( x \). If \( x_1 < x_2 \) are two consecutive zeros of \( f_2 \), then \( f_1 \) has at least one zero in \((x_1, x_2)\), unless \( b_1(x) = b_2(x) \) and \( f_2(x) = kf_1(x), \ k \in \mathbb{R} \).

We now highlight the well-known eigenvalue problem of Sturm-Liouville.
\[
-(py')' + qy = \lambda \rho y \quad \text{on} \ [a, b],
\]
\[
y(a) = y(b) = 0,
\]
with the coefficients \( p \) and \( q \), and the weight function \( \rho \) satisfying:
\[
p \in C^1[a, b], \quad q, \ \rho \in C[a, b],
\]
\[
p(x) \geq d > 0 \quad \text{and} \quad q(x) + c_2 \rho(x) \geq 0 \quad \text{for all} \ x \in [a, b],
\]
\[
\rho(x) > 0 \quad \text{for all} \ x \in (a, b).
\]

**Theorem 5** (Sturm-Liouville, see, e.g., 21, p. 174-175). Under the hypotheses (6.12), the Sturm-Liouville eigenvalue problem (6.11) possesses infinitely many linear independent eigenfunctions \( u_n \in C^2[a, b] \) with eigenvalues \( \lambda_n \in \mathbb{R} \), which satisfy
\[
\int_a^b \rho u_n u_m = \delta_{nm} \quad \text{and} \quad \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty.
\]
Moreover, each of the eigenvalue \( \lambda_n \) not only have geometric multiplicity one, but by symmetry also algebraic multiplicity one and the \( n^{th} \) eigenfunction \( u_n \) of a Sturm-Liouville eigenvalue problem (6.11) has at most \( n - 1 \) simple zeros in \((a, b)\).
According to Theorem 3, the first eigenvalue $\lambda_1$ of Problem (6.5) on $\Omega_{\ell,\ell_0,\ell_1}$ is a strictly positive eigenfunction. Thus, $\mu = \ell^2$, because $f > 0$ on $(0, \pi)$, besides, $h > 0$ on $(\theta_0, \theta_1)$, and from Theorem 5 we have that $\lambda_1$ is the smallest $\lambda^2_{\ell,\ell_0}$ that solves the following problem:

$$-(\varphi h_\theta)_\theta + \ell^2 \varphi = \lambda \csc^2 \theta h, \quad \theta \in (\theta_0, \theta_1),$$

$$h(\theta_0) = h(\theta_1) = 0.$$  \hspace{1cm} (6.13)

Again from Theorem 3 we know that $\lambda_2$ is an eigenfunction that changes sign only once, so, $f$ or $h$ have to change of sign.

If $f$ change of sign, then $\mu = 4\ell^2$ and $f(t) = \sin(2\ell t)$ in Problem (6.9). In this case, $h > 0$, and thus, Theorem 5 guarantees that $\lambda_2$ is the smallest $\lambda^2_{\ell,\ell_0}$ that solves the problem

$$-(\varphi h_\theta)_\theta + 4\ell^2 \varphi = \lambda \csc^2 \theta h, \quad \theta \in (\theta_0, \theta_1)$$

$$h(\theta_0) = h(\theta_1) = 0.$$  \hspace{1cm} (6.14)

If $h$ change of sign, then $f$ is positive, and given by $\sin(\ell t)$, with $\mu = \ell^2$. In this case, $\lambda_2$ is given by $\lambda^2_{\ell,\ell_0}$ solving (6.13) with $h$ changing of sign exactly once, because Theorem 5. Hence, the second eigenvalue is:

$$\lambda_2 = \min\{\lambda^2_{\ell,\ell_0}, \lambda^2_{2,\ell_0}\}. \hspace{1cm} (6.15)$$

### 6.2. Estimates on the first and second eigenvalues.

In this section, we will compute estimates for the first two eigenvalues for the operator in (6.4) with the Dirichlet boundary condition on a domain to be appropriately chosen. We begin by defining the angle:

$$\theta_* = \min(\theta_0, \pi - \theta_1).$$

Note that $\csc^2 \theta$ is a decreasing function on $(0, \pi/2]$ and increasing on $[\pi/2, \pi)$, so that $1 \leq \csc^2 \theta \leq \csc^2(\theta_*)$, for all $\theta \in [\theta_0, \theta_1]$.

**Lemma 3.** The first eigenvalue $\lambda^\mu_{\ell,\ell_0}$ of Problem (6.8) satisfies

$$\varepsilon \sin^2(\theta_*) \left(\mu + \frac{\pi^2}{(\theta_1 - \theta_0)^2}\right) \leq \lambda^\mu_{\ell,\ell_0} \leq \delta \left(\mu + \frac{\pi^2}{(\theta_1 - \theta_0)^2}\right). \hspace{1cm} (6.16)$$

**Proof.** For the lower estimate, we consider a solution $h$ of (6.8). We multiply by $h$ both sides of (6.10), and we integrate from $\theta_0$ to $\theta_1$, to get

$$\lambda^\mu_{\ell,\ell_0} = \frac{\int_{\theta_0}^{\theta_1} \varphi ((\phi h)_\theta)^2 + \mu h^2 d\theta}{\int_{\theta_0}^{\theta_1} \csc^2 \theta h^2 d\theta} \geq \frac{\varepsilon}{\csc^2(\theta_*)} \left(\mu + \frac{\int_{\theta_0}^{\theta_1} (\phi h)_\theta^2 d\theta}{\int_{\theta_0}^{\theta_1} \phi^2 d\theta}\right).$$

By using the Wirtinger inequality $\int_0^D (h')^2 dx \geq \frac{\pi^2}{(\ell_1^D h^2 dx$, we obtain

$$\lambda^\mu_{\ell,\ell_0} \geq \varepsilon \sin^2(\theta_*) \left(\mu + \frac{\pi^2}{(\theta_1 - \theta_0)^2}\right).$$

For the upper estimate, we choose the test function $\phi = \sin(\frac{\theta - \theta_0}{\theta_1 - \theta_0})$. By Rayleigh’s theorem (see, e.g.,[21, p. 177]) and the fact that $\csc^2 \theta \geq 1$, we have

$$\lambda^\mu_{\ell,\ell_0} = \frac{\int_{\theta_0}^{\theta_1} \varphi ((\phi h)_\theta)^2 + \mu \phi^2 d\theta}{\int_{\theta_0}^{\theta_1} \csc^2 \theta \phi^2 d\theta} \leq \delta \left(\mu + \frac{\int_{\theta_0}^{\theta_1} (\phi h)_\theta^2 d\theta}{\int_{\theta_0}^{\theta_1} \phi^2 d\theta}\right) = \delta \left(\mu + \frac{\pi^2}{(\theta_1 - \theta_0)^2}\right).$$

$\square$
Lemma 4. The second eigenvalue $\lambda_2^\mu$ of Problem (6.8) satisfies

$$\varepsilon \sin^2 \theta_* \left( \mu + \frac{4\pi^2}{(\theta_1 - \theta_0)^2} \right) \leq \lambda_2^\mu \leq \delta \left( \mu + \frac{4\pi^2}{(\theta_1 - \theta_0)^2} \right).$$

(6.17)

Proof. Let $h_2^\mu$ be an eigenfunction corresponding to the second eigenvalue $\lambda_2^\mu$. From Theorem 5, there is only one $\theta_2 \in (\theta_0, \theta_1)$ such that $h_2^\mu(\theta_2) = 0$, besides, the eigenvalue $\lambda_2^\mu$ coincides with the first eigenvalue of $-(\varphi_0) + \mu \varphi h = \lambda \csc^2 \theta h$ with the Dirichlet bounded condition on any of the intervals $[\theta_0, \theta_2]$ or $[\theta_2, \theta_1]$. The lower and upper limits on (6.17) are obtained by Lemma 3 considering the intervals with the shortest and longest lengths among $[\theta_0, \theta_2]$ and $[\theta_2, \theta_1]$, respectively, and noticing that $\min\{\theta_2 - \theta_0, \theta_1 - \theta_2\} \leq (\theta_1 - \theta_0)/2$ and $\max\{\theta_2 - \theta_0, \theta_1 - \theta_2\} \geq (\theta_1 - \theta_0)/2$. 

Now, $\lambda_2$ will be appropriately chosen. For that, we fix $\theta_* > \frac{\pi}{6}$ and $\varepsilon > \delta/4$, so that $\frac{4\varepsilon \sin^2 \theta_* - \delta}{4\delta - \varepsilon \sin^2 \theta_*} > 0$, thus we can choose $\ell$ such that

$$\frac{\pi^2}{(\theta_1 - \theta_0)^2} \frac{4\varepsilon \sin^2 \theta_* - \delta}{4\delta - \varepsilon \sin^2 \theta_*} \geq \ell^2.$$ 

(6.18)

Taking $\mu = 4\ell^2$ in (6.16) and $\mu = \ell^2$ in (6.17), we get

$$\lambda_2^\mu - \lambda_1^\mu \geq \varepsilon \sin^2 \theta_* \left( \ell^2 + \frac{4\pi^2}{(\theta_1 - \theta_0)^2} \right) - \delta \left( \ell^2 + \frac{\pi^2}{(\theta_1 - \theta_0)^2} \right)$$

$$= \ell^2 \left( \varepsilon \sin^2 \theta_* - 4\delta \right) + \frac{\pi^2}{(\theta_1 - \theta_0)^2} \left( 4\varepsilon \sin^2 \theta_* - \delta \right)$$

$$= \ell^2 \left( \varepsilon \sin^2 \theta_* - 4\delta \right) + \frac{\pi^2}{(\theta_1 - \theta_0)^2} \left( 4\varepsilon \sin^2 \theta_* - \delta \right) \left( 4\delta - \varepsilon \sin^2 \theta_* \right).$$

Thus, by using (6.18), we conclude that

$$\lambda_2^\mu - \lambda_1^\mu \geq \ell^2 \left( \varepsilon \sin^2 \theta_* - 4\delta \right) + \ell^2 \left( 4\delta - \varepsilon \sin^2 \theta_* \right) = 0.$$ 

Hence, the second eigenvalue of Problem (6.5) on $\Omega_{\ell, \theta_0, \theta_1}$ must be $\lambda_2^\mu \ell^2$, see (6.15). Geometrically, this corresponds to a domain as in Figure 1 in which the opening angle is small compared to the vertical length.

From now on, the operator $\operatorname{div}(\varphi \nabla u)$ in (6.4) will be considered on the family of domains

$$\Omega_{\ell, \theta_0, \theta_1} = \{(r, \theta) : 1 < r < e^{\pi/\ell} \text{ and } \theta_0 < \theta < \theta_1 \text{ satisfying } 6.18\}. \quad (6.19)$$

To estimate the fundamental gap for $\operatorname{div}(\varphi \nabla u)$, with $\varepsilon \leq \varphi \leq \delta$ and $\varphi_r = 0$, we will need the diameter estimate of $\Omega_{\ell, \theta_0, \theta_1}$, which has been calculated by Bourni et al. in the more general configuration of this domain, namely:

Lemma 5. (Bourni et al. [6] \[6\]) $\frac{\pi^2}{\ell^2 \delta D_{\ell, \theta_0, \theta_1}} \to 1$ as $\ell \to 0$ or $\theta_* \to \frac{\pi}{6}$.

6.3. Estimate of the fundamental gap.

Lemma 6. The fundamental gap for $\operatorname{div}(\varphi \nabla u)$, with $\varepsilon \leq \varphi \leq \delta$ and $\varphi_r = 0$, on each domain $\Omega_{\ell, \theta_0, \theta_1}$ defined by (6.19), satisfies

$$3\varepsilon \sin^2 \theta_* \ell^2 < \lambda_2 - \lambda_1 < 3\delta \ell^2.$$ 

(6.20)

In particular, when $\varepsilon = \delta$ and taking $\theta_* \to \frac{\pi}{6}$, then the fundamental gap approaches the constant $3\delta \ell^2$. 


In the assumptions of the present lemma, we have $\lambda_1 = \ell^2$ and $\lambda_2 = \lambda_1^2$. Let $h^{(1)}$ and $h^{(2)}$ be the corresponding eigenfunctions of this eigenvalues, respectively, it follows from (6.13) and (6.14) that

$$
(\varphi h_{\theta}^{(1)})_{\theta} + (\lambda_1 \csc^2 \theta - \ell^2 \varphi)h^{(1)} = 0,
$$

$$
(\varphi h_{\theta}^{(2)})_{\theta} + (\lambda_2 \csc^2 \theta - 4\ell^2 \varphi)h^{(2)} = 0.
$$

To obtain the lower estimate, recall that $\csc^2 \theta \leq \csc^2 \theta_*$, The proof is by contradiction. Suppose that $\lambda_2 \leq \lambda_1 + 3\varepsilon\ell^2 \sin^2 \theta_*$, then

$$
\lambda_2 \csc^2 \theta - 4\ell^2 \varphi \leq \lambda_1 \csc^2 \theta + 3\ell^2 \varepsilon \sin^2 \theta_* \csc^2 \theta - 4\ell^2 \varphi \leq \lambda_1 \csc^2 \theta - \ell^2 \varphi.
$$

Moreover, observe that $\lambda_1 \csc^2 \theta + 3\ell^2 \varepsilon \sin^2 \theta_* \csc^2 \theta - 4\ell^2 \varphi = \lambda_1 \csc^2 \theta - \ell^2 \varphi$ is equivalent to $\sin^2 \theta = \xi \sin^2 \theta_*$, and as $\xi \leq 1$, then $\sin \theta \leq \sin \theta_*$, so, $\theta \leq \theta_*$. Hence, $\theta \leq \theta_0$ or $\theta \geq \theta_1$. Consequently, the inequality is strict in $(\theta_0, \theta_1)$, which allows us to use Theorem 4 to conclude that $h^{(1)}$ has at least one zero in the interval $(\theta_0, \theta_1)$, which contradicts Theorem 5. The upper estimate is obtained in the same way by using that $\csc^2 \theta \geq 1$.

In order to obtain an estimate for the fundamental gap of the operator $\text{div}(\varphi \nabla u)$ in (6.4) on each domain of the family in (6.19), we are using variational arguments as in [22] Sec. 3 and [6] Sec. 5, as well as the Sturm-Liouville’s result. For it, we consider the one-parameter family of problems

$$
(\varphi h_{\theta})_{\theta} + \lambda \csc^2 \theta h = \mu(s) \varphi(\theta)h \quad \text{em} \quad (\theta_0, \theta_1)
$$

$$
h(\theta_0) = h(\theta_1) = 0,
$$

where $h(\theta) = h^s(\theta)$, $\lambda = \lambda(s)$ and $\mu(s)$ is a smooth curve such that $\mu(0) = \ell^2$ and $\mu(1) = 4\ell^2$, with $0 \leq s \leq 1$. For each $s$, let $\lambda(s)$ be the smallest eigenvalue that is smooth at $s$, and let $h^s(\theta)$ be its first eigenfunction given by Theorem 5 which satisfies $\int_{\theta_0}^{\theta_1} \csc^2 \theta (h^s(\theta))^2 d\theta = 1$ and $h^s(\theta) > 0$ on $(\theta_0, \theta_1)$. Denoting by $f$ the derivative with respect to $s$ of a function $f(s)$, we get

$$
(\varphi(\theta) \hat{h}_{\theta})_{\theta} + \lambda(s) \csc^2 \theta \hat{h} - \mu(s) \varphi(\theta) \hat{h} + \lambda(s) \csc^2 \theta h = \mu(s) \varphi(\theta)h \quad \text{on} \quad (\theta_0, \theta_1).
$$

Multiplying the previous equation by $h$, and by integrating from $\theta_0$ to $\theta_1$, we have

$$
\int_{\theta_0}^{\theta_1} \left( (\varphi(\theta) \hat{h})_{\theta} + \lambda(s) \csc^2 \theta \hat{h} - \mu(s) \varphi(\theta) \hat{h} \right) h d\theta = \int_{\theta_0}^{\theta_1} \hat{\mu}(s) \varphi(\theta) h^2 d\theta - \int_{\theta_0}^{\theta_1} \lambda(s) \csc^2 \theta h^2 d\theta.
$$

(6.22)

Using integration by parts and (6.21) to evaluate the term

$$
\int_{\theta_0}^{\theta_1} (\varphi(\theta) \hat{h})_{\theta} h d\theta = \int_{\theta_0}^{\theta_1} (\varphi(\theta) h_{\theta})_{\theta} d\theta = \int_{\theta_0}^{\theta_1} (\mu(s) \varphi(\theta) \hat{h} - \lambda(s) \csc^2 \theta \hat{h} \hat{h}) d\theta,
$$

equation (6.22) reduces to

$$
\hat{\mu}(s) \int_{\theta_0}^{\theta_1} \varphi(\theta) h^2 d\theta = \lambda(s) \int_{\theta_0}^{\theta_1} \csc^2 \theta h^2 d\theta = \lambda(s).
$$

Note that we can take the curve $\mu(s) = \ell^2 + 3\ell^2 s$, so that

$$
\lambda(s) = 3\ell^2 \int_{\theta_0}^{\theta_1} \varphi(\theta) (h^s(\theta))^2 d\theta.
$$
Integrating from 0 to 1, and as \( \lambda(0) = \lambda_1 \) and \( \lambda(1) = \lambda_2 \), we obtain

\[
\lambda_2 - \lambda_1 \leq 3\ell^2\delta \max_{s \in [0,1]} \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta.
\] (6.23)

Now, it is enough to estimate the right-hand side of (6.23).

**Proposition 5.** It is valid that

\[
\max_{s \in [0,1]} \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta < 1.
\] (6.24)

**Proof.** First, note that \( \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta \leq \int_{\theta_0}^{\theta_1} \csc^2 \theta (h^s(\theta))^2 \, d\theta = 1 \). We will show that \( \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta < 1 \) for all \( s \in [0,1] \). For this purpose, we consider the angle \( \alpha \) satisfying

\[
\theta_0 < \alpha < \frac{\pi}{2}.
\]

The proof is by contradiction. Assume the equality

\[
\int_{\theta_0}^{\theta_1} \csc^2 \theta (h^s(\theta))^2 \, d\theta = 1
\]

for some \( s \in [0,1] \). Recall that \( \theta_1 > \frac{\pi}{2} \), so that we can write

\[
\int_{\theta_0}^{\alpha} \csc^2 \theta (h^s(\theta))^2 \, d\theta + \int_{\alpha}^{\theta_1} \csc^2 \theta (h^s(\theta))^2 \, d\theta = \int_{\theta_0}^{\alpha} (h^s(\theta))^2 \, d\theta + \int_{\alpha}^{\theta_1} (h^s(\theta))^2 \, d\theta.
\]

The choice of \( \alpha \) implies \( \csc^2 \theta \geq \csc^2 \alpha \) on \( (\theta_0, \alpha) \), moreover, \( 1 \leq \csc^2 \theta \), then

\[
\csc^2 \alpha \int_{\theta_0}^{\alpha} (h^s(\theta))^2 \, d\theta + \int_{\alpha}^{\theta_1} (h^s(\theta))^2 \, d\theta \leq \int_{\theta_0}^{\alpha} (h^s(\theta))^2 \, d\theta + \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta.
\]

Hence,

\[
(\csc^2 \alpha - 1) \int_{\theta_0}^{\alpha} (h^s(\theta))^2 \, d\theta \leq 0,
\]

which is a contradiction, because \( \csc^2 \alpha > 1 \). So, \( \int_{\theta_0}^{\theta_1} (h^s(\theta))^2 \, d\theta < 1 \), for all \( s \in [0,1] \), which is enough to obtain the result of the proposition.

**Theorem 6.** The fundamental gap for the operator \( \text{div}(\varphi \nabla u) \), with \( \varepsilon \leq \varphi \leq \delta \) and \( \varphi_r = 0 \), on each set of the family of convex domains \( \lim_{\theta_1 \to \frac{\pi}{2}} \Omega_{\ell,\theta_0,\theta_1} \), where \( \Omega_{\ell,\theta_0,\theta_1} \) is defined by (6.19), satisfies

\[
(\lambda_2 - \lambda_1)D^2 < 3\pi^2\delta,
\]

where \( D = \lim_{\theta_1 \to \frac{\pi}{2}} D_{\ell,\theta_0,\theta_1} \) is the diameter of each set of the family \( \lim_{\theta_1 \to \frac{\pi}{2}} \Omega_{\ell,\theta_0,\theta_1} \).

**Proof.** By (6.23) and Proposition 5, we have that

\[
\lambda_2 - \lambda_1 < 3\ell^2\delta.
\]

On the other hand, from Lemma 5

\[
3\ell^2\delta = \lim_{\theta_1 \to \frac{\pi}{2}} \frac{3\pi^2\delta}{D_{\ell,\theta_0,\theta_1}^2}
\]

which is enough to complete the result of the theorem.
From Theorem 6 we prove the next result for the operator \( \mathcal{L} \) defined as in (1.1), with \( T = \varphi I \) and \( \varphi_r = 0 \).

**Corollary 6.** Consider the operator \( \text{div}(\varphi \nabla u) \), with \( \varepsilon \leq \varphi \leq \delta \) and \( \varphi_r = 0 \), on each set of the family of convex domains \( \lim_{\theta_{\ell,\theta_0,\delta_1}} D_{\ell,\theta_0,\delta_1} \), where \( D_{\ell,\theta_0,\delta_1} \) is defined by (6.19). There exists a drifting function \( \eta \) such that the fundamental gap for \( \text{div}(\varphi \nabla u) \) remains invariant by a first order perturbation of this operator. More precisely, for the operator \( \mathcal{L} u = \text{div}(\varphi \nabla u) - \langle \nabla \eta, \varphi \nabla u \rangle \) it is true that

\[
(\lambda_2 - \lambda_1) D^2 < 3\pi^2 \delta,
\]

where \( D = \lim_{\theta_{\ell,\theta_0,\delta_1}} D_{\ell,\theta_0,\delta_1} \) is the diameter of each set of the family \( \lim_{\theta_{\ell,\theta_0,\delta_1}} \Omega_{\ell,\theta_0,\delta_1} \).

**Proof.** We claim that, any solution \( \eta \) of the equation

\[
2\text{div}(\varphi \nabla \eta) - \langle \nabla \eta, \varphi \nabla \eta \rangle = 0
\]

can be taken as a drifting function so that the fundamental gap for the operator \( \mathcal{L} = \text{div}(\varphi \nabla \cdot - \langle \nabla \eta, \varphi \nabla \rangle) \), with \( \varepsilon \leq \varphi \leq \delta \) and \( \varphi_r = 0 \), on each set of the family of convex domains \( \lim_{\theta_{\ell,\theta_0,\delta_1}} \Omega_{\ell,\theta_0,\delta_1} \), where \( \Omega_{\ell,\theta_0,\delta_1} \) is defined by (6.19), satisfies the required estimate. Indeed, we take \( u \) such that \( -\lambda u = \text{div}(\varphi \nabla u) \), and the change of variable \( u = e^{-\frac{\varphi}{2}} v \) together with the property of divergence of a vector field, to obtain

\[
\text{div}(\varphi \nabla u) = e^{-\frac{\varphi}{2}} \left( \text{div}(\varphi \nabla u) - \langle \nabla \eta, \varphi \nabla \eta \rangle \right) - \frac{e^{-\frac{\varphi}{2}}}{4} \left( 2\text{div}(\varphi \nabla \eta) - \langle \nabla \eta, \varphi \nabla \eta \rangle \right).
\]

If \( \eta \) solves (6.25), then \( -\lambda e^{-\frac{\varphi}{2}} v = -\lambda u = \text{div}(\varphi \nabla u) = e^{-\frac{\varphi}{2}} \mathcal{L} v \). Hence, \( \mathcal{L} v = -\lambda v \), and thus the estimate of the fundamental gap for \( \mathcal{L} \) follows from Theorem 6. \( \square \)

The reader can see that is easy to obtain solutions for equation (6.25). Next, we give a simple example.

**Example 1.** Let \( \varphi : (1, e^\frac{1}{2}) \times (\pi/3, 2\pi/3) \subset \mathbb{H}^2 \rightarrow \mathbb{R} \) be a radially constant function, defined by \( \varphi(r, \theta) = \sin \theta \). Note that \( \varphi \) and \( \theta \) satisfy the conditions in (6.19), for \( \theta_0 = \pi/3, \theta_1 = 2\pi/3, \theta_2 = \pi/3, \varepsilon = \sqrt{3}/2, \delta = 1 \) and \( \ell \) to be appropriately chosen. Under these conditions, the functions \( \eta(r, \theta) = -2\ln(1 - \ln \tan \frac{\theta}{2}) \) and \( \eta(r, \theta) = -2\ln(\pi - \ell \ln r) \) solve equation (6.25).

Indeed, we need to prove that \( 2\text{div}(\varphi \nabla \eta) - \langle \nabla \eta, \varphi \nabla \eta \rangle = 0 \). For it, we can proceed as in (6.2) and (6.4), to get the next expression, which is valid for any \( \varphi \) radially constant and for any \( \eta \),

\[
2\text{div}(\varphi \nabla \eta) - \langle \nabla \eta, \varphi \nabla \eta \rangle = 2(\varphi(r^2 \sin^2 \theta \eta_r + r \sin^2 \theta \eta_r) + \varphi \sin^2 \theta \eta_{\theta \theta} + \sin^2 \theta \varphi \eta_{\theta \theta}) - \varphi(r^2 \sin^2 \theta \eta_r + \sin^2 \theta \eta_{\theta \theta}).
\]

For \( \eta(r, \theta) = -2\ln(1 - \ln \tan \frac{\theta}{2}) \), we have

\[
\eta_r = 0, \quad \eta_\theta = \frac{2 \csc \theta}{1 - \ln \tan \frac{\theta}{2}} \quad \text{and} \quad \eta_{\theta \theta} = \frac{-2 \csc \theta \cot \theta}{1 - \ln \tan \frac{\theta}{2}} + \frac{2 \csc^2 \theta}{(1 - \ln \tan \frac{\theta}{2})^2}.
\]

Hence,

\[
2\text{div}(\varphi \nabla \eta) - \langle \nabla \eta, \varphi \nabla \eta \rangle = 2\varphi \sin^2 \theta \eta_{\theta \theta} + 2\sin^2 \theta \varphi \eta_{\theta \theta} - \varphi \sin^2 \theta \eta_{\theta \theta}^2 = 0.
\]

For \( \eta(r, \theta) = -2\ln(\pi - \ell \ln r) \), the computation is simple as in the previous case.
Remark 2. For the Laplacian case with Dirichlet boundary conditions on convex domains in $\mathbb{R}^n$, $n \geq 2$, Bourin et al. proved that the product of the fundamental gap with the square of the diameter can be arbitrarily small for domains of any diameter, see [7] Theorem 1.1. Then, we can say that they also proved that the drifted Laplacian has the same property when its drifting function solves $2\Delta \eta - |\nabla \eta|^2 = 0$.

7. Concluding remarks

Here, we give some applications of Theorem 2 in a more general context.

Corollary 7. Let $\lambda_1$ be the first eigenvalue of the drifted Cheng-Yau operator with a drifting function $\eta$ on a bounded domain $\Omega \subset M^n$ with the Dirichlet boundary condition. Fix an origin $a \in M^n \setminus \overline{\Omega}$, and let $r(x)$ be the distance function from $a$. If $T$ is radially parallel and it has $\partial_r$ as an eigenvector, then:

1. For $a(n, \varepsilon, \delta) \leq 0$,
\[
\lambda_1(\Omega) \geq \frac{\varepsilon}{4\delta^2} \left( (n-1)^2 \varepsilon^2 \kappa_2^2 - 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) - 2C_0(n-1)(\kappa_1 + \frac{1}{d}) - C_1 \right).
\]

2. For $a(n, \varepsilon, \delta) > 0$,
\[
\lambda_1(\Omega) \geq \frac{\varepsilon}{4\delta^2} \left( (n-1)^2 \varepsilon^2 \kappa_2^2 - 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) - 2C_0(n-1)(\kappa_1 + \frac{1}{d}) - C_1 - \frac{a(n, \varepsilon, \delta)}{d^2} \right),
\]
where the constants $d$, $C_0$ and $C_1$ are as in Theorem 2.

Proof. For simplicity and further reference, we define the auxiliary sequences:

1. For $a(n, \varepsilon, \delta) \leq 0$,
\[
v_1 := \frac{4\delta^2}{\varepsilon} \lambda_1 - (n-1)^2 \varepsilon^2 \kappa_2^2 + 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) + 2C_0(n-1)(\kappa_1 + \frac{1}{d}) + C_1.
\]

2. For $a(n, \varepsilon, \delta) > 0$,
\[
v_i := \frac{4\delta^2}{\varepsilon} \lambda_1 - (n-1)^2 \varepsilon^2 \kappa_2^2 + 2(n-1)(\delta^2 \kappa_1^2 - \varepsilon^2 \kappa_2^2) + 2C_0(n-1)(\kappa_1 + \frac{1}{d}) + C_1 + \frac{a(n, \varepsilon, \delta)}{d^2},
\]
where the constants $d$, $C_0$ and $C_1$ are as in Theorem 2. From (4.1) and the inequalities in Theorem 2, we have
\[
0 \leq \|u_i \mathcal{L} \eta + 2T(\partial_r, \nabla u_i)\|_{L^2(\Omega, dm)}^2 \leq v_i.
\]
Take $i = 1$ to obtain the result of the corollary. \hfill \Box

Now, we prove some estimates of eigenvalues of the drifted Cheng-Yau operator. In particular, we obtain the corresponding estimates for the Cheng-Yau operator, the drifted Laplacian and the Laplacian. For more in-depth information on these estimates, we refer the reader to [10].

Theorem 7. Let $\lambda_i$ be the $i$-th eigenvalue of the drifted Cheng-Yau operator with a drifting function $\eta$ on a bounded domain $\Omega \subset M^n$ with the Dirichlet boundary condition. Fix an origin $a \in M^n \setminus \overline{\Omega}$, and let $r(x)$ be the distance function from $a$. Suppose $T$ is radially parallel and it has $\partial_r$ as an eigenvector. Then, the sequence of eigenvalues $(\lambda_i)$ satisfies the following estimates in terms of the auxiliary sequences $(v_i)$ defined in Corollary 7:
\[
v_{k+1} \leq \left( 1 + \frac{4\delta^2}{\varepsilon^2} \right) k^{\frac{2\varepsilon^2}{\delta^2}} v_1.
\]

(7.1)
\[ v_{k+1} \leq \frac{1}{k} \left( 1 + \frac{4\delta^2}{\varepsilon^2} \right) \sum_{i=1}^{k} v_i. \]

(7.2)

\[ v_{k+1} \leq \left( 1 + \frac{2\delta^2}{\varepsilon^2} \right) \frac{1}{k} \sum_{i=1}^{k} v_i + \left[ \left( \frac{2\delta^2}{\varepsilon^2} \frac{1}{k} \sum_{i=1}^{k} v_i \right)^2 - \left( 1 + \frac{4\delta^2}{\varepsilon^2} \right) \frac{1}{k} \sum_{j=1}^{k} \left( u_j - \frac{1}{k} \sum_{i=1}^{k} v_i \right)^2 \right]^\frac{1}{2}. \]

(7.3)

\[ v_{k+1} - v_k \leq 2 \left[ \left( \frac{2\delta^2}{\varepsilon^2} \frac{1}{k} \sum_{i=1}^{k} v_i \right)^2 - \left( 1 + \frac{4\delta^2}{\varepsilon^2} \right) \frac{1}{k} \sum_{j=1}^{k} \left( u_j - \frac{1}{k} \sum_{i=1}^{k} v_i \right)^2 \right]^\frac{1}{2}. \]

(7.4)

Proof. Note that we can write each case of the sequence \((v_i)\) as follows

\[ v_i = \frac{4\delta^2}{\varepsilon} \lambda_i + C, \]

(7.5)

for an appropriated constant \(C\). With this simplified notation, inequalities of Theorem 2 become

\[ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{1}{\varepsilon} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \frac{4\delta^2}{\varepsilon} \lambda_i + C \right). \]

By (7.5), we get

\[ \lambda_{k+1} - \lambda_i = \frac{\varepsilon}{4\delta^2} (v_{k+1} - v_i). \]

Then,

\[ \sum_{i=1}^{k} (v_{k+1} - v_i)^2 \leq \frac{4\delta^2}{\varepsilon^2} \sum_{i=1}^{k} (v_{k+1} - v_i) v_i. \]

(7.6)

Besides, note that \(v_1 \leq v_2 \leq \cdots \to \infty\), since \(\lambda_1 \leq \lambda_2 \leq \cdots \to \infty\). Hence, each case of the sequence \((v_i)\) satisfies the assumptions as in Cheng and Yang \[11, 12\], from which we get (7.1). For a complete proof, see Miranda \[25\], Lemma 2.4 and Corollary 2.1.

By using (7.6) we follow the same steps as in Theorem 3 of \[16\] to prove the estimates in (7.2), (7.3) and (7.4).

Finally, we make some important observations of independent interesting for the more general expression of \(L\). For this, let \(N(\lambda)\) be the eigenvalue distribution function given by the number of eigenvalues \(\lambda_k\) smaller than a given \(\lambda\). We observe that the principal symbol of \(L\) is given by \(\sigma_L(x,\xi)\) for \(x \in M^n\) and \(\xi \in T_xM\), see Eqs. (1.2) and (1.3). It determines the first term in the asymptotics of the eigenvalue distribution function, as follows:

\[ N(\lambda) = c_0 \lambda^{-\frac{n}{2}} + O(\lambda^{-\frac{n+1}{2}}) \quad \text{as} \quad \lambda \to \infty \]

(7.7)

where

\[ c_0 = (2\pi)^{-n} \text{vol}\{ (x, \xi) : T_x(\xi, \xi) \leq 1 \} = (2\pi)^{-n} \int_{\{T_x(\xi, \xi) \leq 1\}} dx d\xi. \]

For more details, see \[19, 20, 29, 33\]. From (7.7), we obtain

\[ \lambda_k = c_0^{-\frac{1}{2}} k^{\frac{n}{2}} + O(1) \quad \text{as} \quad k \to \infty. \]

(7.8)
We now deduce two identities that have been used as necessary tools on eigenvalue problems in particular cases of $\mathcal{L}$, namely:

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \lambda_i = \frac{n}{n+2} c_0^{-\frac{2}{n}} \quad \text{and} \quad \lim_{k \to \infty} \frac{1}{k^2} \sum_{i=1}^{k} \lambda_i^2 = \frac{n}{n+4} c_0^{-\frac{2}{n}}. \quad (7.9)$$

Indeed, from (7.8) we get

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i = c_0^{-\frac{2}{n}} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^{\frac{2}{n}} \frac{1}{k} + O\left( \frac{1}{k} \right) \quad (7.10)$$

Consider $f(t) = t^{\frac{2}{n}}$ on $[0,1]$, and the partition $0 < \frac{1}{k} < \frac{2}{k} < \cdots < \frac{i}{k} < \cdots < 1$ of $[0,1]$ so that

$$\lim_{k \to \infty} \sum_{i=1}^{k} f\left( \frac{i}{k} \right) \frac{1}{k} = \lim_{k \to \infty} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^{\frac{2}{n}} \frac{1}{k} = \int_{0}^{1} t^{\frac{2}{n}} dt = \frac{n}{n+2}. \quad (7.11)$$

Thus, the first identity in (7.9) follows from (7.10).

For the second identity in (7.9), we begin by noting that

$$\lambda_i^2 = c_0^{-\frac{2}{n}} k^{\frac{2}{n}} + 2c_0^{-\frac{2}{n}} k^{\frac{2}{n}} O(1) + O(1)^2 \quad \text{as} \quad k \to \infty.$$  

So,

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i^2 = c_0^{-\frac{2}{n}} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^{\frac{2}{n}} \frac{1}{k} + 2O(1)c_0^{-\frac{2}{n}} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^{\frac{2}{n}} \frac{1}{k} + O(1)^2. \quad (7.12)$$

As in the first case, we use the function $t \mapsto t^{\frac{2}{n}}$ to get

$$\lim_{k \to \infty} \sum_{i=1}^{k} \left( \frac{i}{k} \right)^{\frac{2}{n}} \frac{1}{k} = \int_{0}^{1} t^{\frac{2}{n}} dt = \frac{n}{n+4}. \quad (7.11)$$

Hence, the second identity in (7.9) follows from (7.11) and (7.12).

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