Geometrical construction of quantum groups
representations

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Abstract: We describe geometrically the classical and quantum inhomogeneous groups $G_0 = (\text{SL}(2, \mathbb{C}) \rtimes \mathbb{C}^2)$ and $G_1 = (\text{SL}(2, \mathbb{C}) \rtimes \mathbb{C}^2) \rtimes \mathbb{C}$ by studying explicitly their shape algebras as spaces of polynomial functions with a quadratic relations.

1 Introduction

The problem of describing quantum inhomogeneous group $G$, semi-direct product of a semi simple group by an abelian or more generally solvable normal subgroup is still not totally solved [5]. The main difficulty is the incompleteness of the family of irreducible finite dimensional representations for these groups.

In the order to overpass this problem, we have either to consider infinite dimensional irreducible representations or finite dimensional indecomposable representations.

This last family of representations is very large and hard to describe (generally we don’t have a classification for such representations). But in the case where the inhomogeneous group is a subgroup of a semi simple one $S$, we can try to restrict ourselves to the family of restrictions to $G$ of irreducible finite dimensional representations of $S$.

On the other hand, C. Ohn gave a geometrical description of the quantum group $G = \text{SL}(n, \mathbb{C})$ [1], by defining its shape algebra as the space of regular sections of the line bundles over some sheme in the flag manifold [4], or as the space of regular functions on a manifold $G/U$.

In this paper, we intend to describe explicitly the simplest examples of shape algebra for an inhomogeneous quantum group by using the geometric approach of C. Ohn.

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More precisely, we are looking at two subgroups $G_1$ and $G_0$ of $G = SL(3, \mathbb{C})$, which are inhomogeneous of the form $(SL(2, \mathbb{C}) \triangleright \mathbb{C}^2) \triangleright \mathbb{C}$ and $(SL(2, \mathbb{C}) \triangleright \mathbb{C}^2)$. In the Drinfeld-Jimbo quantum universal enveloping algebra $U_q(sl(3, \mathbb{C}))$, there is a sub-bigebra which is the quantum version $U_q(g_1)$ of the classical enveloping algebra of $g_1$. Unfortunately, there is no quantum subalgebra associated to $G_0$ in $U_q(sl(3, \mathbb{C}))$.

We intend to study, geometrically, this phenomena, thus we describe the classical shape algebras for $G_1$ and $G_0$ as regular functions under the complex manifolds $G_1/U_1$ and $G_0/U_0$. The first one is dense inside $G/U$ but for the second one $G_0/U_0$ is a closed submanifold of $G/U$ of smaller dimension.

In our opinion this is the geometrical presentation of the fact that $G_0$ is not a quantum subgroup of $G$.

We can, finally, quantify the shape algebra of $G_1$ by following the same computation as for $SL(3)$. The paper is organized as follows, after recalling our notations and the presentation of $U_q(g_1)$ in part 2, we describe explicitly the classical shape algebra for $G_1 = SL(3, \mathbb{C})$ as a space of polynomial functions or as a quadratic associative algebra generated by particular functions $p_i$ and $q_j$ with explicit quadratic relations.

In part 4, we recall the C. Ohn construction for the quantum shape algebra of $G$. As a result, this algebra is still quadratic, generated by the $p_i$ and $q_j$ with explicit deformed relations.

In the two last part, we describe first the classical shape algebras for $G_0$ and $G_1$ as algebras of polynomial functions, then the quantum shape algebra for $G_1$, as an explicit quadratic associative algebra generated by $p_i$, $q_j$ and $1/q_3$.

## 2 Notations and preliminaries

In this paper, we shall consider the following Lie groups:

$$G = SL(3, \mathbb{C}) = \{ g = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}, \quad \det g = 1 \}$$

$$G_1 = (SL(2, \mathbb{C}) \triangleright \mathbb{C}^2) \triangleright \mathbb{C} = \{ g_1 = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & z_3 \end{pmatrix}, \quad \det g_1 = 1 \}$$

and

$$G_0 = SL(2, \mathbb{C}) \triangleright \mathbb{C}^2 = \{ g_0 = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 1 \end{pmatrix}, \quad \det g_0 = 1 \}.$$
\( g = \mathfrak{sl}(3, \mathbb{C}) \), it is defined by its generators \( K_1^\pm, K_2^\pm, X_1, X_2, Y_1, Y_2 \) and the relations:

\[
\begin{align*}
K_1 K_1^{-1} &= K_1^{-1} K_1 = 1 \\
K_2 K_2^{-1} &= K_2^{-1} K_2 = 1 \\
K_1 Y_1 K_1^{-1} &= q^{-2} Y_1 \\
K_1 Y_2 K_1^{-1} &= q Y_2 \\
K_2 X_2 K_2^{-1} &= q Y_2 \\
X_1 Y_2 - Y_2 X_1 &= 0 \\
\end{align*}
\]

and

\[
\begin{align*}
X_1^2 X_2 - (q + q^{-1}) X_1 X_2 X_1 + X_2 X_1^2 &= 0 \\
X_2^2 X_1 - (q + q^{-1}) X_2 X_1 X_2 + X_1 X_2^2 &= 0 \\
Y_1^2 Y_2 - (q + q^{-1}) Y_1 Y_2 Y_1 + Y_2 Y_1^2 &= 0 \\
Y_2^2 Y_1 - (q + q^{-1}) Y_2 Y_1 Y_2 + Y_1 Y_2^2 &= 0 \\
\end{align*}
\]

The coalgebra structure on \( \mathcal{U}_q(\mathfrak{g}) \) is defined by the following coproduct:

\[
\begin{align*}
\Delta K_1^{\pm 1} &= K_1^{\pm 1} \otimes K_1^{\pm 1} \\
\Delta K_2^{\pm 1} &= K_2^{\pm 1} \otimes K_2^{\pm 1} \\
\Delta Y_1 &= Y_1 \otimes \mathbb{1} + \mathbb{1} \otimes Y_1 \\
\Delta Y_2 &= Y_2 \otimes \mathbb{1} + \mathbb{1} \otimes Y_2 \\
\end{align*}
\]

The generators \( K_i^{\pm 1} \) are formally identified with \( e^{\pm t H_i} \) if \( H_1 \) and \( H_2 \) are the usual basis for the Cartan subalgebra:

\[
H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

\( X_1 \) and \( Y_1 \) are the root vectors associated to simple roots:

\[
X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

and \( t \) and \( q \) are formally related by \( e^t = q \).

Since the Lie algebras \( \mathfrak{g}_0 \) and \( \mathfrak{g}_1 \) of \( G_0 \) and \( G_1 \) are subalgebras of \( g \), the classical universal enveloping algebras \( \mathcal{U}(\mathfrak{g}_0) \) and \( \mathcal{U}(\mathfrak{g}_1) \) are subalgebras of \( \mathcal{U}(g) \). Unfortunately, this does not hold at the quantum level.

**Lemma 1**

\( \mathcal{U}_q(\mathfrak{g}_1) \) is a sub-bigebra of \( \mathcal{U}_q(\mathfrak{g}) \) but \( \Delta(\mathcal{U}_q(\mathfrak{g}_0)) \) is not included in \( \mathcal{U}_q(\mathfrak{g}_0) \otimes \mathcal{U}_q(\mathfrak{g}_0) \).
Indeed, the bigebra $\mathcal{U}_q(\mathfrak{g}_1)$ can be written as:

$$\mathcal{U}_q(\mathfrak{g}_1) = \frac{T(K_1^{\pm 1}, K_2^{\pm 1}, X_1, Y_1, Y_2)}{T(K_1^{\pm 1}, K_2^{\pm 1}, X_1, Y_1, Y_2) \cap I}$$

where $T(K_1^{\pm 1}, K_2^{\pm 1}, X_1, Y_1, Y_2)$ is the tensor algebra generated by $K_1^{\pm 1}, K_2^{\pm 1}, X_1, Y_1$ and $Y_2$ and $I$ is the ideal given by relations (9) in which the generator $X_2$ does not arise. Moreover, the coproduct $\Delta$ of $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ can be restricted to $\mathcal{U}_q(\mathfrak{g}_1)$ as:

$$\Delta_{\mid \mathcal{U}_q(\mathfrak{g}_1)} : \mathcal{U}_q(\mathfrak{g}_1) \to \mathcal{U}_q(\mathfrak{g}_1) \otimes \mathcal{U}_q(\mathfrak{g}_1).$$

Nevertheless, the set of generators of $\mathcal{U}_q(\mathfrak{g}_0)$ does not contain $K_2$. Since $K_2$ arise in the expression of $\Delta Y_2$, then $\mathcal{U}_q(\mathfrak{g}_0)$ is not included in $\mathcal{U}_q(\mathfrak{g}_0) \otimes \mathcal{U}_q(\mathfrak{g}_0)$.

Due to this fact, many authors, attempting to describe inhomogeneous quantum groups like $SL(2, \mathbb{C}) \rtimes \mathbb{C}^2$, prefer to add a central dilatation (the $K_2$ element here) to build their model (see [5] for instance).

In this paper, we want to give another point of view more geometrical for this phenomena. Our starting point is the description, by C. Ohn [1], of quantum $SL(3, \mathbb{C})$ and its shape algebra.

### 3 Borel-Weil-Bott theorem for $SL(3, \mathbb{C})$

To describe $SL_q(3, \mathbb{C})$, we need an explicit realization for each irreducible unitary representation of $SL(3, \mathbb{C})$ and their tensor product. The unitary irreducible representation of $SL(3, \mathbb{C})$ are acting on the space of sections of line bundles over $G/B$, where $B$ is the Borel subgroup $\{ b = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}, \det b = 1 \}$

The characters $\chi_{n_1, n_2}$ of $B$ have the following form:

$$\chi_{n_1, n_2}(b) = a_1^{n_1} c_3^{n_2} = a_1^{(n_1 + n_2)} b_2^{n_2} \quad (n_1, n_2 \in \mathbb{Z}).$$

We denote the corresponding principal bundle by:

$$G \times_{\chi_{n_1, n_2}} \mathbb{C} \to G/B$$

(an element of $G \times_{\chi_{n_1, n_2}} \mathbb{C}$ is an equivalence class $[g, z] = [gb, \chi_{n_1, n_2}(b)^{-1}z]$ in $G \times \mathbb{C}$).

$G/B$ is the flag manifold $D$ of $\mathbb{C}^3$:

$$D = \{ \mathbb{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbb{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbb{C} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \} \simeq \{ [p], [q], p \cdot q = p_1q_1 + p_2q_2 + p_3q_3 = 0 \}$$
where the elements \([p]\) and \([q]\) of \(\mathbb{P}(\mathbb{C}^3)\) are respectively the line through
\[
p = \begin{bmatrix} p_1 = x_1 \\ p_2 = x_2 \\ p_3 = x_3 \end{bmatrix}
\]
and the line through \(q = \begin{bmatrix} q_1 = x_2 y_3 - x_3 y_2 \\ q_2 = x_3 y_1 - x_1 y_3 \\ q_3 = x_1 y_2 - x_2 y_1 \end{bmatrix}\) in \(\mathbb{C}^3\).

The space of holomorphic sections of the line bundle \(G \times_{\chi_{n_1,n_2}} \mathbb{C} \to G/B\) is non trivial if and only if \(n_1 \geq 0\) and \(n_2 \geq 0\). A section is a regular homogeneous function \(f\) from \(G\) to \(\mathbb{C}\) such that:
\[
f(gb) = \chi_{n_1,n_2}(b)^{-1} f(g), \quad \forall g \in G, \forall b \in B.
\]

Let \(U\) be the unipotent subgroup:
\[
U = \{ u = \begin{pmatrix} 1 & b_1 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \}.
\]

We can describe the homogeneous space \(G/U\) as the submanifold of \(\mathbb{C}^6\) defined by:
\[
G/U \simeq \{ p \in \mathbb{C}^3 \setminus \{0\}, \quad q \in \mathbb{C}^3 \setminus \{0\}, \quad p \cdot q = 0 \}.
\]

\(G/U\) is thus an affine submanifold of \(\mathbb{C}^6\), we shall consider its closure \(\overline{G/U}\) in the usual embedding of \(\mathbb{C}^6\) into \(\mathbb{P}(\mathbb{C}^7)\):
\[
\overline{G/U} = \{ \begin{bmatrix} p \\ q \\ 1 \end{bmatrix} \in \mathbb{P}(\mathbb{C}^7), \quad p \cdot q = 0 \}
\]

**Lemma 2 (Extension of sections)**

For each \(n_1, n_2\) in \(\mathbb{N}\), the sections of \(G \times_{\chi_{n_1,n_2}} \mathbb{C} \to G/B\) can be viewed as regular functions on \(\overline{G/U}\).

**Proof:** Since \(\chi_{n_1,n_2}\) is trivial on \(U\), \(f\) gives rise to a function, still denoted \(f\) on \(\overline{G/U}\).

If \(p_1 \neq 0\) and \(q_3 \neq 0\), then
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{y_1}{x_1} & \frac{y_1}{x_1} \\ 0 & 1 & \frac{y_2}{x_2} \\ 0 & 0 & 1 \end{bmatrix}.
\]

Let \(f\) be a section of our line bundle, then, after division by \(\det^k(g)\) with a well choice of \(k\), \(f\) is polynomial homogeneous in the \((x, y, z)\) variables with degree \(n_1 + n_2\) in \(x\), \(n_2\) in \(y\), 0 in \(z\) or:
\[
f(x, y, z) = \frac{P_1(p, q)}{p_1^{n_1}} \quad (p_1 \neq 0),
\]

5
$P_1$ being polynomial in $p$ and $q$, homogeneous with degree $n_1 + n_2$ in $p$ and homogeneous with degree $n_2$ in $q$.

Similarly, on the open subset $p_2 \neq 0$, $p_3 \neq 0$, we can write:

$$f(x, y, z) = \frac{P_2(p, q)}{p_2^{n_2}} \quad \text{and} \quad f(x, y, z) = \frac{P_3(p, q)}{p_3^{n_3}}$$

On the other hand, the ideal $(pq)$ generated by $p_1 q_1 + p_2 q_2 + p_3 q_3$ is prime. Indeed, if $P$ and $Q$ are polynomials such that:

$$P Q = (pq) R,$$

then, by division in $\mathbb{C}(q_1)[p_1, p_2, p_3, q_2, q_3]$, we get polynomial functions $T, S, T', S'$ such that:

$$q_1 P = (pq) T + S \quad \text{and} \quad q_1' Q = (pq) T' + S'$$

with $\deg_{p_1} S = \deg_{p_1} S' = 0$. Or:

$$q_1'' (pq) R = (pq)((pq) T T' + TS' + T'S) + SS'$$

Thus $SS' = (pq) R'$, this implies $R' = 0$ and $S$ or $S' = 0$. If for instance $S = 0$ then:

$$q_1' P = (pq) T$$

Then $\val_{q_1}(T) \geq l$ or $T = q_1' T'$, $P = (pq) T'$ is in $(pq)$.

Now by the Nullstellensatz, our equation:

$$f = \frac{P_1}{p_1^{n_2}} = \frac{P_2}{p_2^{n_2}} \quad p_1 \neq 0, \quad p_2 \neq 0 \quad p, q = 0$$

can be written as:

$$p_2^{n_2} P_1 - p_1^{n_2} P_2 = (pq) Q.$$ 

Thus $\val_{(p_1, p_2)} Q \geq n_2$ or $Q = \sum_{j=0}^{n_2} p_1^{n_2-j} p_2^j Q_j$. From that we get:

$$p_2^{n_2} (P_1 - (pq) Q_{n_2}) - p_1^{n_2} (P_2 - (pq) Q_0) = (pq)p_1 p_2 \sum_{j=1}^{n_2-1} p_1^{n_2-j} p_2^j-1 Q_j.$$ 

Thus

$$\val_{p_1} (P_1 - (pq) Q_{n_2}) \geq 1, \quad \val_{p_2} (P_2 - (pq) Q_0) \geq 1$$

And

$$P_1 - (pq) Q_{n_2} = p_1 P'_1, \quad P_2 - (pq) Q_0 = p_2 P'_2$$

then

$$f = \frac{P'_1}{p_1^{n_2-1}} = \frac{P'_2}{p_2^{n_2-1}}$$

and by induction there exists a polynomial function $P$ in the variables $p, q$ such that $f = P$ on $G/\mathcal{U}$. 

6
We shall call $\mathcal{O}(G/U)$ the shape algebra of the classical group $SL(3, \mathbb{C})$. The multiplication in this algebra:

$$\mathcal{O}(G/U) \otimes \mathcal{O}(G/U) \to \mathcal{O}(G/U)$$

is the dual form of the classical comultiplication $\Delta$ on $\mathcal{U}(\mathfrak{sl}(3))$. If $u$ belongs to $\mathcal{U}(\mathfrak{sl}(3))$ and if $f_1, f_2$ belongs to $\mathcal{O}(G/U)$, we can put:

$$\Delta u(f_1 \otimes f_2)(e, e) = u(f_1 f_2)(e)$$

(in fact $\Delta X = 1 \otimes X + X \otimes 1$ for any $X$ in $\mathfrak{sl}(3, \mathbb{C})$) and $\Delta$ is a morphism from $\mathcal{U}(\mathfrak{sl}(3))$ into $\mathcal{U}(\mathfrak{sl}(3)) \otimes \mathcal{U}(\mathfrak{sl}(3))$.

We can also see, as C. Ohn did in [1], the space $H^0(G/B, \chi_{n_1, n_2})$ of polynomial functions homogeneous with degree $n_1$ in $p$ and $n_2$ in $q$ as the dual $V_{n_1, n_2}$ of the “algebraic” space $V^{n_1, n_2}$. If $\varpi_1$ and $\varpi_2$ stands for the fundamental weights, $V^{n_1, n_2}$ is the carrying space of irreducible representation of $SL(3, \mathbb{C})$ with highest weight $\lambda = n_1 \varpi_1 + n_2 \varpi_2$. If $V^1$ is $\mathbb{C}^3$ with canonical basis $e_1, e_2, e_3$, then $V^{n_1, n_2}$ is explicitly realized as the submodule of $(V^1)^{\otimes n_1} \otimes (V^1 \wedge V^1)^{\otimes n_2}$ generated by the highest weight vector $v_\lambda = (e_1)^{\otimes n_1} \otimes (e_1 \wedge e_2)^{\otimes n_2}$. Denote $V^2$ the space $V^1 \wedge V^1$, we get an inclusion mapping:

$$V^{n_1, n_2} \subset (V^1)^{\otimes n_1} \otimes (V^2)^{\otimes n_2}.$$  

The natural identification between $H^0(G/B, \chi_{n_1, n_2})$ and $V_{n_1, n_2}$ being

$$\varphi \in V_{n_1, n_2} \mapsto f$$

such that $f(g) = \varphi(g, v_\lambda)$, for any $g$ in $G$.

The multiplication $m$ is thus the transposition of the family of injections:

$$V^{\lambda_1 + \lambda_2} \to V^{\lambda_1} \otimes V^{\lambda_2}.$$  

In fact as an algebra, $\mathcal{O}(G/U)$ is generated by $V_1$ and $V_2$ the duals of the fundamental representation of $SL(3, \mathbb{C})$ i.e. by the linear functions $p_1, p_2, p_3, q_1, q_2, q_3$ and sixteen quadratic relations:

$$p_i p_j = p_j p_i \quad (i < j) \quad q_j q_i = q_i q_j \quad (i < j) \quad p_i q_i = q_i p_i \quad p_i q_i = 0$$

We can replace the four last relations by:

$$p_1 q_1 = q_1 p_1 \quad p_3 q_3 = q_3 p_3 \quad p_2 q_2 + q_1 p_1 + q_3 p_3 = 0 \quad q_2 p_2 + p_1 q_1 + p_3 q_3 = 0$$

7
4 The C. Ohn construction

C. Ohn gave a more geometrical construction for the shape algebra. Let us recall quickly here, in the $SL(3, \mathbb{C})$ case, his construction.

The shape algebra is generated by $V_1 \oplus V_2$ (the linear polynomial functions in $p_1, p_2, p_3, q_1, q_2, q_3$). If $i$ and $j$ are in $\{1, 2\}$, we consider the tensor products $V^i \otimes V^j$.

Let $V^{ij}$ be the irreducible module with highest weight $\omega_i + \omega_j$. We define an explicit injective map:

$$V^{ij} \rightarrow V^i \otimes V^j$$

by constructing a system of vectors $e_{ij}^{ij}$ generating $V^{ij}$. We consider all the orthocell $C$ for $\mathfrak{sl}(3)$: $C$ is the right coset in the weyl group $W$ of $\mathfrak{sl}(3)$, for a subgroup $\Gamma$ generated by a set $A$ of pairwise commuting reflexions, ($A = \emptyset$ or $A = \{s_{\alpha_i}\} (i = 1, 2)$). Among all these orthocells, we select the small and $ij$-effective ones, (See [1] for explicit definition).

In the case of $SL(3)$ we get fourteen small orthocells. Identifying the Weyl group as $S_3$, the six “trivials” are:

- $C_0^1 = \{[123]\}$
- $C_0^2 = \{[132]\}$
- $C_0^3 = \{[213]\}$

and the eight “non trivials” are:

- $C_1 = \{[123], [213]\}$
- $C_2 = \{[132], [231]\}$
- $C_3 = \{[312], [321]\}$
- $C_4 = \{[123], [132]\}$
- $C_5 = \{[213], [312]\}$
- $C_6 = \{[231], [321]\}$
- $C_7 = \{[132], [312]\}$
- $C_8 = \{[213], [231]\}$

For any $i, j$ the six trivial orthocells are $ij$-effective. Amid the others, we keep only $C_1, C_2, C_5, C_6, C_7$ for $i = j = 1$, $C_2, C_3, C_4, C_5, C_6$ for $i = j = 2$ and $C_2, C_3$ for $i = 1$ and $j = 2$ (or $i = 2$ and $j = 1$). To each orthocell $C$, we associate a vector $e_C^{ij}$ of $V^i \otimes V^j$ by the following rule:

First we realize the Weyl group $W$ as permutations matrices in $SL(3)$, then we put $w. e^{(i)} = e_w^{(i)}$ if $e^{(i)}$ is the highest weight vector for $V^i$ and finally:

$$e_C^{ij} = \sum_{L \subset A} e_{s_{\alpha_i}}^i \otimes e_{s_{\alpha_j}}^j$$

where $C = \{s_{\alpha_i}, \alpha_i \in A\}$, $\bar{L} = A \setminus L$ and $s_L$ is the product of $s_{\alpha_i}$ for $\alpha_i$ in $L$. 
In the $SL(3)$ case we get the following vectors:

\[
\begin{align*}
\epsilon_{C_1}^{11} &= \epsilon_{C_2}^{11} = e_1 \otimes e_1 & \epsilon_{C_5}^{11} &= \epsilon_{C_4}^{11} = e_2 \otimes e_2 \\
\epsilon_{C_1}^{11} &= \epsilon_{C_2}^{11} = e_3 \otimes e_3 & \epsilon_{C_5}^{11} &= \epsilon_{C_4}^{11} = e_1 \otimes e_2 + qe_2 \otimes e_1 \\
\epsilon_{C_1}^{11} &= \epsilon_{C_2}^{11} = e_2 \otimes e_3 + qe_3 \otimes e_2 & \epsilon_{C_5}^{11} &= e_1 \otimes e_3 + qe_3 \otimes e_1 \\
\epsilon_{C_1}^{22} &= \epsilon_{C_2}^{22} = (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) & \epsilon_{C_5}^{22} &= \epsilon_{C_4}^{22} = (e_1 \wedge e_3) \otimes (e_1 \wedge e_3) \\
\epsilon_{C_1}^{22} &= \epsilon_{C_2}^{22} = (e_2 \wedge e_3) \otimes (e_2 \wedge e_3) + q(e_2 \wedge e_3) \otimes (e_1 \wedge e_3) \\
\epsilon_{C_1}^{22} &= \epsilon_{C_2}^{22} = (e_1 \wedge e_2) \otimes (e_1 \wedge e_3) + q(e_1 \wedge e_3) \otimes (e_1 \wedge e_2) \\
\epsilon_{C_1}^{22} &= (e_2 \wedge e_1) \otimes (e_2 \wedge e_3) + q(e_2 \wedge e_3) \otimes (e_2 \wedge e_1)
\end{align*}
\]

With these notations $V^{ij}$ is linearly generated by the vectors $\epsilon_{C_i}^{ij}$ where $C$ is small and $ij$-effective [1].

**Remark 1** This construction can be compared with the Demazure’s one of a basis for an irreducible module for a simple Lie algebra [2]; in fact, the explicit C. Ohn construction gives a generating system only if the highest weight has the form $\lambda_1 + \lambda_2$ where $\lambda_1$ and $\lambda_2$ are fundamental.

For instance for $V^1 \otimes V^2$, the trivial small orthocells define in the dual of the Cartan subalgebras the 6 vectors, image of the weight $\varpi_1 + \varpi_2$ under the action of the Weyl group (the vertices of the hexagon) and the 2 nontrivial correspond to two representations of a subgroup $SL(2, \mathbb{C})$ inside $SL(3, \mathbb{C})$ (two of the diagonals of the hexagon), thus to 2 times the weight 0, the third diagonal corresponding...
to a non small orthocell is excluded.

Let us now choose invariant supplementary space for \( V_{ij} \) in \( V^i \otimes V^j \):

\[
V^1 \otimes V^1 = V^{11} \oplus \text{Vec}\{e_1 \otimes e_2 - e_2 \otimes e_1; e_2 \otimes e_3 - e_3 \otimes e_2; e_1 \otimes e_3 - e_3 \otimes e_1\}
\]

\[
V^2 \otimes V^2 = V^{22} \oplus \text{Vec}\{(e_1 \wedge e_3) \otimes (e_2 \wedge e_3) - (e_2 \wedge e_3) \otimes (e_1 \wedge e_3); (e_1 \wedge e_2) \otimes (e_1 \wedge e_3) - (e_1 \wedge e_3) \otimes (e_1 \wedge e_2); (e_1 \wedge e_2) \otimes (e_2 \wedge e_3) - (e_2 \wedge e_3) \otimes (e_1 \wedge e_2)\}
\]

\[
V^1 \otimes V^2 = V^{12} \oplus \text{Vec}\{e_1 \otimes (e_2 \wedge e_3) + e_2 \otimes (e_3 \wedge e_1) + e_3 \otimes (e_1 \wedge e_2)\}
\]

\[
V^2 \otimes V^1 = V^{21} \oplus \text{Vec}\{(e_2 \wedge e_3) \otimes e_1 + (e_3 \wedge e_1) \otimes e_2 + (e_1 \wedge e_2) \otimes e_3\}
\]

We refind the sixteen relations defining the classical shape algebra by considering the relations:

\[
(I)_{ij} = \ker m_{ij} V^i \oplus V^j
\]

\[
(II)_{12} = R_{12}(e_{ij}^{12}) = e_{ij}^{21}, \quad (II)_{11} \text{ and } (II)_{22} \text{ are trivial.}
\]

where \( R_{12} \) is the isomorphism \( R_{12} : V^1 \otimes V^2 \to V^2 \otimes V^1 \). Explicitly, we find:

\[
(I)_{11} = \{p_ip_j = p_jp_i, (i < j)\} \quad \quad (II)_{12} = \{q_iq_j = q_jq_i, (i < j)\}
\]

\[
(I)_{12} = \{p_1q_1 + p_2q_2 + p_3q_3 = 0\} \quad \quad (II)_{12} = \{p_iq_j = q_jp_i, \forall i, j\}
\]

To define the quantum \( SL(3, \mathbb{C}) \), we start with its representation theory, similar to the representation theory for classical \( SL(3, \mathbb{C}) \). For instance, \( V^1 \) and \( V^2 \) becomes \[3\]:

\[
K_\beta e^i_w = q^{<w, \beta>^i} e^i_w \quad \quad K_\beta^{-1} e^i_w = q^{-<w, \beta>} e^i_w
\]

and

\[
X_\beta e^i_w = 0, \quad Y_\beta e^i_w = e^i_{\beta w} \quad \text{if } <w, \beta> \geq 1
\]

\[
X_\beta e^i_w = 0, \quad Y_\beta e^i_w = 0 \quad \text{if } <w, \beta> = 0
\]

\[
X_\beta e^i_w = e^i_{\beta w}, \quad Y_\beta e^i_w = 0 \quad \text{if } <w, \beta> = -1
\]

for \((i, j) = (1, 2)\) and \( \beta = \alpha_1 \) or \( \alpha_2 \).

Now we define the quantum \( e_C^{ij} \) by:

\[
e_C^{ij} = \sum_{L \subseteq A} q_L^{ij} e_{s_L^i w} \otimes e_{s_L^j w}.
\]

\( V^{ij} \) is still generated by the \( e_C^{ij} \), irreducible, with the highest weight \( \varpi_i + \varpi_j \).

We choose supplementary spaces for \( V^{ij} \) in \( V^i \otimes V^j \) by:

\[
V^1 \otimes V^1 = V^{11} \oplus \text{Vec}\{(p_ip_j - qp_jp_i)^*, (i < j)\}
\]

\[
V^2 \otimes V^2 = V^{22} \oplus \text{Vec}\{(q_ip_j - qp_jq_i)^*, (i < j)\}
\]

\[
V^1 \otimes V^2 = V^{12} \oplus \text{Vec}\{(q^{2-1}p_1q_1 + q^{-1}p_2q_2 + p_3q_3)^*\}
\]

\[
V^2 \otimes V^1 = V^{21} \oplus \text{Vec}\{(q_1p_1 + q^{-1}q_2p_2 + q^{-2}q_3p_3)^*\}
\]
Now the sixteen relations defining the quantum shape algebra are:

\[ (I)_{11} \quad p_i p_j - q p_j p_i = 0, (i < j) \]
\[ (I)_{22} \quad q q_i q_j - q q_j q_i = 0, (i < j) \]
\[ (I)_{12} \quad p_2 q_2 + q^{-1} p_1 q_1 + q p_3 q_3 = 0 \]
\[ (I)_{21} \quad q_2 p_2 + q q_1 p_1 + q^{-1} q q_3 p_3 = 0 \]
\[ (II)_{12} = (II)_{21} \quad p_i p_j - q p_j p_i = 0 \quad i \neq j \]
   \[ p_1 p_1 = q q_1 p_1, \quad p_3 q_3 = q^{-1} q q_3 p_3. \]

Since the quantum shape algebra is quadratic, we get here a description of this algebra.

**Theorem 1** (Quantum shape algebra for \( SL(3, \mathbb{C}) \))

The quantum shape algebra is the quotient of the tensor associative algebra \( T(p, q) \) generated by \( p_1, p_2, p_3, q_1, q_2, q_3 \) by the two sided ideal generated by these sixteen relations.

**Remark 2** In [1] C. Ohn describe geometrically the preceding construction as a deformation of a scheme \( E \) canonically defined in \( G/B \).

### 5 Geometrical construction for \( G_1 \) and \( G_0 \)

In this part, we try to adapt the preceding construction for the cases of \( G_0 \) and \( G_1 \). We denote by \( B_0 \) and \( B_1 \) the “Borel subgroup” \( B \cap G_0, B \cap G_1 \) for \( G_0 \) and \( G_1 \). We associate to them the “flag” manifolds \( D_0 = G_0/B_0 \) and \( D_1 = G_1/B_1 \).

As in section 3, we get:

\[ D_0 \simeq \{ [p], [q] \in \mathbb{P}(\mathbb{C}^3), \quad q_3 = 1, \quad pq = 0 \} \subset D \]
\[ D_1 \simeq \{ [p], [q] \in \mathbb{P}(\mathbb{C}^3), \quad q_3 \neq 0, \quad pq = 0 \} \subset D. \]

Then \( D_0 = D_1 \) is a dense subset of \( D \). The characters for \( D_0 \) and \( D_1 \) have the form:

\[ \chi^0_n : \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto a_1^{-n} \quad (n \in \mathbb{Z}) \]
\[ \chi^{1}_{n_1, n_2} : \begin{pmatrix} a_1 & b_1 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \mapsto a_1^{-n_1} c_3^{n_2} \quad (n_1, n_2 \in \mathbb{Z}). \]

Thus \( \chi^{1}_{n_1, n_2} \big|_{B_0} = \chi^0_{n_1} \) and the line bundles \( G_1 \times \chi^{1}_{n_1, n_2} \mathbb{C} \to D_1 \) and \( G_0 \times \chi^0_{n_1} \mathbb{C} \to D_0 \) are isomorphic for any \( n_2 \).

Let us consider now the space of holomorphic sections for these bundles. We
put $U_i = G_i \cap U$ then :

$$G_1 / U_1 \simeq \{ (p, q) \in \mathbb{C}^6, \ p \neq 0 \ q_3 \neq 0, \ pq = 0 \}$$
$$G_0 / U_0 \simeq \{ (p, q) \in \mathbb{C}^6, \ p \neq 0 \ q_3 = 1, \ pq = 0 \}.$$

As in section 3 we embed these spaces in $\mathbb{P}(\mathbb{C}^7)$ and take their closure :

$$\overline{G_1 / U_1} = \overline{G / U} = \left\{ \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}, \ pq = 0 \right\}$$
$$\overline{G_0 / U_0} = \left\{ \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}, \ q_3 = 1, pq = 0 \right\} \subset \overline{G_1 / U_1}$$

but $\overline{G_0 / U_0} \neq \overline{G_1 / U_1}$

**Theorem 2 (Space of section)**

- If $n_1 < 0$, 
  $$H^0(G_1 / B_1, \chi_{n_1, n_2}) \simeq H^0(G_0 / B_0, \chi_{n_1}) = \{ 0 \}.$$ 

- If $n_1 \geq 0$, then $H^0(G_1 / B_1, \chi_{n_1, n_2})$ is infinitely dimensional.

  Moreover, the space of holomorphic sections vanishes if $n = n_1 < 0$ since their restriction to $SL(2, \mathbb{C})$ are sections of the usual line bundle $SL(2, \mathbb{C}) \times \chi_{n_1, n_2}$.

**Proof**: First, the space of holomorphic sections vanishes if $n = n_1 < 0$ since their restriction to $SL(2, \mathbb{C})$ are sections of the usual line bundle:

$$SL(2, \mathbb{C}) \times \chi_{n_1, n_2} \mathbb{C} \rightarrow SL(2, \mathbb{C}) / B \cap SL(2, \mathbb{C})$$

There are no restrictions on $n_2$.

As above, a section $f$ in $H^0(G_1 / B_1, \chi_{n_1, n_2})$ can be viewed as a homogeneous function in $x, y, z$ with degree $n_1 + n_2$ in $x$, $n_2$ in $y$ and 0 in $z_3$. 

12
$f \in H^0(G_1/B_1, \chi_{n_1,n_2})$ has the form $
frac{\varphi(x,y,z)}{(\det g)^k(q_3(g))^l}$.

We multiply $f$ by $(\det g)^k$ and we choose $l$ as small as possible.

We get:

$$f = \frac{\varphi(x,y,z)}{(q_3(x,y))^l}.$$ 

Moreover, the covariance relation $f(g_1b_1) = \chi_{n_1,n_2}(b_1)^{-1}f(g_1)$ implies that $\varphi$ is an homogeneous polynomial function with degree 0 in $z$, $l$ in $y$ and $n_1 + n_2$ in $x$. $\varphi$ being $U_1$-invariant, since:

$$\begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} p_1 & 0 & 0 \\ p_2 & \frac{q_3}{p_1} & 0 \\ p_3 & \frac{-q_2}{p_1} & \frac{1}{q_3} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{y_1}{x_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

If $x_1 \neq 0$, we can write:

$$f(x,y) = \frac{F_1(p, q)}{p_1^{n_1}q_3^{l+n_2}}$$

for $p_1 \neq 0$

$$= \frac{F_2(p, q)}{p_2^{n_2}q_3^l}$$

for $p_2 \neq 0$

As in the section 3, the function $q^l_3 f$ coincides in fact with a polynomial function in the variables $p$ and $q$. Thus:

$$(**) \quad f(x,y) = \frac{1}{q_3^l} F(p, q) \quad (\deg_p F = n_1, \deg_q F = l + n_2 \geq 0).$$

But now $1/q_3^l$ is in $O(G/U)$, we can not eliminate the denominator $q_3^l$.

By the preceeding discussion, $F$ can be viewed as an element of $H^0(G/B, \chi_{n_1,l})$ and $(**)$ proves that:

$$H^0(G_1/B_1, \chi_{n_1,n_2}) \simeq \bigcup_{l = \sup(0,-n_2)}^{\infty} \frac{1}{q_3^l} H^0(G/B, \chi_{n_1,l+n_2}).$$

We don’t have a direct sum since for instance:

$$1 = \frac{q_3}{q_3} = ... = \frac{q_3}{q_3} \in H^0(G/B, \chi_{0,0}) \cap \frac{1}{q_3} H^0(G/B, \chi_{0,1}) \cap ... \cap \frac{1}{q_3} H^0(G/B, \chi_{0,l}).$$

Finally, $\frac{G_1}{U_1} = \frac{G}{U}$, then a function $f$ belongs to $O(\frac{G_1}{U_1} \cap \{q_3 \neq 0\})$ if and only if $f$ can be written:

$$f(x,y) = \frac{1}{q_3^l} F(p, q) \quad (l \geq 0),$$
Similarly to the $G$-algebras for $G$ over space of functions $f$. At the classical level this algebra is generated by the space $G$ to presentation for $G$. The sum is clearly direct. Similarly, since the bundle $G \times \chi_{n_1} \mathbb{C}$ is a restriction to $G_0$ of the bundle $G_1 \times \chi_{n_1,0} \mathbb{C} (\mathbb{R})$, we have:

$$H^0(G_0/B_0, \chi_{n_1}^0) = H^0(G_1/B_1, \chi_{n_1,0}^1)_{q_3=1}$$

this gives the last assertions of our theorem.

6 Shape algebra for $G_1$

Similarly to the $G$-case, we shall define the (classical and quantum) shape algebras for $G_1$ as the vector space $\mathcal{O}(\overline{G_1/U_1} \cap \{q_3 \neq 0\})$ of all the line bundles over $G_1/B_1$.

At the classical level this algebra is generated by the space $V_1 \oplus V_2 \oplus V_{-1}$ where:

$$V_1 = Vec(p_1, p_2, p_3), \quad V_2 = Vec(q_1, q_2, q_3), \quad V_{-1} = \mathbb{C} \frac{1}{q_3}$$

In order to define the multiplication law of our shape algebra, we need a large family of representations $(\pi_i)$ of $G_1$ such that the representation $\pi_i \otimes \pi_j$ is a finite sum of some $\pi_k$.

Let us recall that $G_1$ is a classical and quantum subgroup of $G$ (see section 2), then each irreducible finite dimensional representation $V^\lambda$ of $G$ is a representation, still denoted $V^\lambda$ of $G_1$.

Since $V^\lambda$ is generated by a highest weight vector (and its dual $V_\lambda$ by a lowest weight vector, the funtion $p_3^{n_1} q_3^{n_2} q_3^{n_3}$), then $V^\lambda$ is an indecomposable representation of $G_1$, generally it is not irreducible: the $SL(2)$ module, generated by the highest weight vector, is a $G_1$-submodule without any direct factor.

We select now, the family $((V^{-1})^{\otimes i} \otimes V^\lambda)_{n_1 \geq 0, n_2 \geq 0, l \geq 0}$ as our family of representation for $G_1$ the dual of such a representation appears naturally as the space of functions $f$ in $\mathcal{O}(\overline{G_1/U_1} \cap \{q_3 \neq 0\})$ of the form:

$$f(x,y) = \frac{1}{q_3^3} F(p,q) \quad deg_p F = n_1, \quad deg_q F = n_2.$$  

Especially it is generated by $V_{-1}$, $V_1$ and $V_2$, since $V_{-1} \otimes V_i$ and $V_i \otimes V_{-1}$ are elements of our family of representation, there is no supplementary spaces, thus no relation like $I_{-12}$ or $I_{-11}$. However, the shape algebra is no more a direct sum of our representation spaces. Thus we have to add a new relation $I_{0,1,2} = I_{2,-1}^0$ since

$$(V_{-1})^{\otimes 0} \otimes V_{0,0} \subset V_{-1} \otimes V_2$$
1 ⊗ 1 = \frac{1}{q_3} \otimes q_3 = q_3 \otimes \frac{1}{q_3}

Thus the quadratic relations for the classical shape algebra of $G_1$ are those of the classical $G$:

\[ I_{11}, \; I_{12} = I_{21}, \; I_{22}, \; II_{12} = II_{21} \]

and

\[
\begin{align*}
I^0_{-12} & \quad (1/q_3).q_3 = 1 \\
I^0_{2-1} & \quad q_3.(1/q_3) = 1 \\
II_{-11} & \quad (1/q_3).p_i = p_i.(1/q_3) \\
II_{-12} & \quad (1/q_3).q_i = q_i.(1/q_3)
\end{align*}
\]

But the relations $II_{-11}$ and $II_{-12}$ are consequences of $I^0_{-12}$, $I^0_{2-1}$ and $I_{ij}$, $II_{ij}$ ($i \geq 0$, $j \geq 0$).

Let us, now consider the quantum case, thus $V^1$, $V^2$ are the restriction to $G_1$ of the $G$ module $V^1$, $V^2$, we define $V^{-1}$ as the one dimensional space $Cv$ with:

\[ X_1v = Y_1v = Y_2v = 0, \quad K_1v = v, \quad K_2v = q^{-1}v. \]

Then the quadratic relations in the shape algebra are:

\[ I_{11}, \; I_{22}, \; I_{12} = I_{21}, \; II_{12} = II_{21} \]

and

\[
\begin{align*}
I^0_{-12} & \quad (1/q_3).q_3 = 1 \\
I^0_{2-1} & \quad q_3.(1/q_3) = q
\end{align*}
\]

**Theorem 3 (Quantum shape algebra for $G_1$)**

*The quantum shape algebra is the quotient of the tensor associative algebra $T(p, q, 1/q_3)$, generated by $p_1, p_2, p_3, q_1, q_2, q_3, 1/q_3, by the above relations.***

Indeed, the shape algebra is a quotient of the algebra defined in theorem 3 but if $q = 1$ this quotient has to coincide with $O(G_1/U_1 \cap \{q_3 \neq 0\})$ which is the classical shape algebra, thus our quotient is trivial.

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