Frustrated Synchronization in Competing Drive-Response Coupled Chaotic Systems

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Abstract

Chaotic systems can be synchronized by linking them to a common signal, subject to certain conditions. However, the presence of multiple driving signals coming from different systems, give rise to novel behavior. The particular case of Lorenz systems, with two independent systems driving another system through drive-response coupling has been studied in this paper. This is the simplest arrangement which shows the effect of “frustrated synchronization” due to competition between the two driver systems. The resulting response system attractor deviates significantly from the conventional Lorenz attractor. A new measure of desynchronization is proposed, which shows a power-law scaling relation with the competition parameter.

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Keywords: Chaos, Synchronization, Frustration, Competition, Lorenz System.
1 Introduction

The synchronization of chaotic systems is a difficult problem owing to their extremely sensitive dependence on initial conditions. Any initial correlation present between identical systems, starting from very close initial conditions, exponentially decrease to zero with time. Thus, for all practical purposes, any initial synchronization between the systems is bound to disappear rapidly. In recent times, however, some methods of achieving synchronized behavior between chaotic systems have been proposed. Pioneering work in this respect has been done by Pecora and Carroll [1], who used the concept of a response system locking on to a driver system. So far, such studies have been limited to driving a response system by a single driver system. However, the knowledge gained from studying such simple systems may not be adequate to give us an idea as to how systems consisting of multiple independent driver systems, competing with each other to synchronize the same response system, will behave. The Pecora-Carroll driving mechanism can be seen as the “strong-coupling” limit of a general scheme of directionally-oriented couplings in a network of chaotic elements.

The synchronization of bidirectionally coupled chaotic systems is stable provided the coupling strength is at least half the Lyapunov exponent of the system [2]. One-way coupling (or, driving one chaotic system by another) can also lead to synchronization, provided certain conditions are satisfied [1], [3], [4]. The drive-response method consists of the following steps. First an $n$-dimensional autonomous system

\[
\frac{dx}{dt} = F(x),
\]

is divided into two parts, driving ($x_d$) and responding ($x_r$):

\[
\frac{dx_d}{dt} = g(x_d, x_r), \quad \frac{dx_r}{dt} = h(x_d, x_r),
\]

where, $x_d = (x_1, \ldots, x_m)$, $g = [f_1(x), \ldots, f_m(x)]$, $x_r = (x_{m+1}, \ldots, x_n)$ and $h = [f_{m+1}(x), \ldots, f_n(x)]$.

A replica subsystem $x'_r$ identical to $x_r$ is then created and driven with the $x_d$ variables of the original system. Therefore, the replica subsystem equations are

\[
\frac{dx'_r}{dt} = h(x_d, x'_r).
\]

The responding subsystems $x_r$ and $x'_r$ will synchronize only if $\delta x_r = |x_r - x'_r| \to 0$. According to Pecora and Carroll, this occurs if and only if the conditional Lyapunov exponents of the $x_r$ subsystem are all negative.

Drive-response synchronization has been realized in various electrical circuit experiments. It has also been used in experiments of secure communication where a chaotic masking signal is added to the transmitted signal. It is then recovered at the receiving end by subtracting the chaotic signal regenerated by synchronization [5].
Besides the Pecora-Carroll method, other synchronization procedures have also been proposed. Of these, the Variable Control Feedback (VCF) method is of particular interest, as it can be used for both control and synchronization of chaos [3]. In fact, the Pecora-Carroll method turns out to be a special limiting case of this method. VCF consists of adding a feedback term to a dynamical system to guide it into some prescribed state. If \( \frac{dx}{dt} = F(x) \) be an \( n \)-dimensional dynamical system and \( x^* \) be the desired state to which the system has to be brought, then VCF involves modifying the system dynamics to:

\[
\frac{dx}{dt} = F(x) - \lambda(x - x^*)
\]

where \( \lambda \) is the set of \( n \) feedback multipliers. If \( x^* \) be the output of a chaotic system \( F'(x) \), then the system synchronizes with \( F(x) \). In the large-\( \lambda \) limit, VCF reduces to the Pecora-Carroll method. Specifically, the feedback parameters for the driving subsystem variables, \( \lambda_d \to \infty \), while the remaining \( \lambda \)s are set to zero.

In this paper some observations have been reported on the attractor structure of a chaotic system which has been subjected to simultaneous synchronization by two other identical chaotic systems competing with each other. Section 2 introduces the model used for studying competition among synchronizing chaotic systems and includes a short analysis of the fixed points and their stability. Section 3 contains the results of computer simulations of the system. Finally, possible directions of future research and the relevance of this type of research to the theory of neural computation are discussed.

## 2 Competition among synchronizing Lorenz systems

The investigation of competition among synchronizing chaotic systems was carried out using the Lorenz system of equations [4], [5]. This well-known paradigm of chaos is defined by the following set of equations:

\[
\begin{align*}
\frac{dx}{dt} &= \sigma (y - x), \quad (1) \\
\frac{dy}{dt} &= rx - y - xz, \quad (2) \\
\frac{dz}{dt} &= xy - bz, \quad (3)
\end{align*}
\]

where, \( \sigma \), \( r \) and \( b \) are real, positive parameters. There are three fixed points for this system: \( F_1 = (0,0,0) \), \( F_2 = (\sqrt{b(r-1)},\sqrt{b(r-1)}, r-1) \), and \( F_3 = (-\sqrt{b(r-1)},-\sqrt{b(r-1)}, r-1) \). The local stability of the fixed point \((x_f, y_f, z_f)\) is determined by the eigenvalues of the Jacobian

\[
J = \begin{vmatrix}
-\sigma & \sigma & 0 \\
(r - x_f) & -1 & -x_f \\
y_f & x_f & -b
\end{vmatrix}.
\]
Evaluation of the matrix shows that for $0 < r < 1$, $F_1$ is the only stable fixed point. For $r > 1$, $F_1$ becomes unstable and the phase-space trajectory of the system converges to either $F_2$ or $F_3$. For $r > r_c = \sigma(\sigma+b+3)/(\sigma-b-1)$ the system’s trajectory perpetually wanders along the extremely complicated structure of the stable and unstable manifolds of the fixed points, exhibiting chaotic behavior.

For the present work the effect of two driving systems, designated as driving systems 1 ($x_1, y_1, z_1$) and 2 ($x_2, y_2, z_2$), competing to synchronize a responding system ($x_3, y_3, z_3$) was studied. The responding system was driven using the $y$ variable. A competition parameter $a$ was defined to indicate the strength of the driving systems relative to each other. The maximum value of $a$ was normalized to unity. Therefore, the $y$ variable of the responding system was defined in terms of the two driving systems as:

$$y_3 = ay_1 + (1-a)y_2.$$  \hspace{1cm} (5)

We consider first the case where the two driving systems have the same $r$-parameter value, and then, the more general case, where the two $r$-values are different ($r_1$ and $r_2$, say). The $\sigma$ and $b$-parameter values are considered to be the same in all cases.

**Case I: $r_1 = r_2 = r$**

It is obvious that for $a = 1$ the responding system synchronizes with driver system 1, whereas for $a = 0$, it synchronizes with system 2. The attractor of the response system, is identical to that of the conventional Lorenz system (fig. 1(a)). For $0 < a < 1$, the responding system ($x_3, y_3, z_3$) has nine fixed points:

$\begin{align*}
F_1 & = (0,0,0), \\
F_2 & = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \\
F_3 & = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1), \\
F_4 & = (a\sqrt{b(r-1)}, a\sqrt{b(r-1)}, a^2(r-1)), \\
F_5 & = ((1-a)\sqrt{b(r-1)}, (1-a)\sqrt{b(r-1)}, (1-a)^2(r-1)), \\
F_6 & = (-a\sqrt{b(r-1)}, -a\sqrt{b(r-1)}, a^2(r-1)), \\
F_7 & = -(1-a)\sqrt{b(r-1)}, -(1-a)\sqrt{b(r-1)}, (1-a)^2(r-1)), \\
F_8 & = ((2a-1)\sqrt{b(r-1)}, (2a-1)\sqrt{b(r-1)}, (2a-1)^2(r-1)), \\
F_9 & = -(2a-1)\sqrt{b(r-1)}, -(2a-1)\sqrt{b(r-1)}, (2a-1)^2(r-1)).
\end{align*}$

Note that the first three fixed points are those of the uncoupled Lorenz system. To find out about the stability of these fixed points we need to calculate the eigenvalues of the corresponding Jacobian, $J'$. The partially block-diagonal form of the matrix makes the calculation easy:

$$J' = \begin{bmatrix} J & 0_{3\times 3} & 0_{3\times 2} \\
0_{3\times 3} & J & 0_{3\times 2} \\
A & B & J_R \end{bmatrix}, \hspace{1cm} (6)$$

where, $J$ is the Jacobian (eqn. 4) of the unperturbed Lorenz system of equations, $0_{m\times n}$ is a null
matrix having \( m \) rows and \( n \) columns, and the other matrices are defined as,

\[
A = \begin{bmatrix}
0 & a & \sigma & 0 \\
0 & a & x_{f_3} & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & (1 - a) & \sigma & 0 \\
0 & (1 - a) & x_{f_4} & 0
\end{bmatrix},
\]

and,

\[
J_R = \begin{bmatrix}
-\sigma & 0 \\
a y_{f_1} + (1 - a) y_{f_2} - b
\end{bmatrix}.
\]

Here \( f_k \) refers to the fixed point of the \( k \)th Lorenz system.

For \( 0 < r < 1 \), the only stable fixed point is \( F_1 \). For \( r > 1 \), \( F_1 \) loses its stability, and there are four new stable fixed points: \( F_2, F_3, F_8 \) and \( F_9 \). For \( r > r_c = \sigma(\sigma + b + 3)/(\sigma - b - 1) \), these fixed points lose their stability and the system shows only chaotic behavior. The most interesting instance is that of \( a = 0.5 \), where maximal competition occurs. In this case, \( F_8 = F_9 = F_1, F_4 = F_5 \) and \( F_6 = F_7 \) (fig. 2). The attractor of the responding system is found to be stretched over its 3-dimensional phase space showing an extremely tangled structure (fig. 1(b)). This is due to the extremely complicated motion of the response system trajectory along the stable and unstable manifolds of the fixed points \( F_1, F_2, F_3, F_4 \) and \( F_6 \). The coupling with driver system 1 tries to force the response system into synchronization with it, but at the same time, the coupling with driver system 2 desynchronizes the trajectory. The synchronization is therefore ‘frustrated’ by the competition between the two driver systems. The “frustrated” response system attractor reduces to the conventional Lorenz attractor if \( a \to 0 \) or \( 1 \), when competition is absent.

The attractor structure is found to be quite robust. If we start from two different initial conditions for the responding system, \((x, y, z)\) and \((x', y', z')\), say, then for stable synchronization, the two respective trajectories should converge rapidly. However, whereas in the Pecora-Carroll case, convergence occurs to the standard Lorenz attractor, in this case, both the trajectories converge to the “frustrated” attractor.

The stability of synchronization can be demonstrated analytically by linear stability analysis of the error dynamics. Defining the dynamical error between two response system trajectories \((x, x')\) which have different initial conditions, as \( e = x - x' \), the error equations can be written as:

\[
\frac{de_x}{dt} = -\sigma e_x, \tag{10}
\]

\[
e_y = 0, \tag{11}
\]

\[
\frac{de_z}{dt} = (ay_1 + (1 - a)y_2)e_x - be_z. \tag{12}
\]

Here we have assumed that the equation parameters for the two systems are identical. The error system of equations has an equilibrium point at \( e = (0, 0, 0) \), which corresponds to perfect
synchronization. The local stability of synchronization can then be checked by looking at the eigenvalues of the Jacobian of the error equations:

\[
J_R = \begin{vmatrix}
\quad -\sigma \\
\quad a y_1 + (1 - a) y_2 - b
\end{vmatrix}
\]

The eigenvalues are \(-\sigma\) and \(-b\), which are the conditional Lyapunov exponents of the response system. As both eigenvalues are negative, the synchronization is locally stable, and any difference in initial conditions rapidly goes to zero. Note that, this does not prove the global stability of the synchronized state. However, simulations have verified that even in the presence of large deviations in initial conditions, synchronization with the “frustrated” trajectory is achieved. This indicates that, although exact synchronization with the driver system cannot be achieved, the “frustrated” system can still be used for secure communication through chaotic masking. This has been established through simulations reported below.

**Case II: \(r_1 \neq r_2\)**

When the value of the \(r\)-parameter of the two driving systems is not the same, the fixed points are given by:

\(F_1 = (0, 0, 0)\),
\(F_2 = (a \sqrt{b(r_1 - 1)} + (1 - a) \sqrt{b(r_2 - 1)}, a \sqrt{b(r_1 - 1)} + (1 - a) \sqrt{b(r_2 - 1)}, a^2(r_1 - 1) + (1 - a)^2(r_2 - 1) + 2a(1 - a)\sqrt{r_1 - 1}(r_2 - 1))\),
\(F_3 = (-a \sqrt{b(r_1 - 1)} - (1 - a) \sqrt{b(r_2 - 1)}, -a \sqrt{b(r_1 - 1)} - (1 - a) \sqrt{b(r_2 - 1)}, a^2(r_1 - 1) + (1 - a)^2(r_2 - 1) + 2a(1 - a)\sqrt{r_1 - 1}(r_2 - 1))\),
\(F_4 = (a \sqrt{b(r_1 - 1)}, a \sqrt{b(r_1 - 1)}, a^2(r_1 - 1))\),
\(F_5 = ((1 - a) \sqrt{b(r_2 - 1)}, (1 - a) \sqrt{b(r_2 - 1)}, (1 - a)^2(r_2 - 1))\),
\(F_6 = (-a \sqrt{b(r_1 - 1)}, -a \sqrt{b(r_1 - 1)}, a^2(r_1 - 1))\),
\(F_7 = ((1 - a) \sqrt{b(r_2 - 1)}, (1 - a) \sqrt{b(r_2 - 1)}, (1 - a)^2(r_2 - 1))\),
\(F_8 = (a \sqrt{b(r_1 - 1)} - (1 - a) \sqrt{b(r_2 - 1)}, a \sqrt{b(r_1 - 1)} - (1 - a) \sqrt{b(r_2 - 1)}, a^2(r_1 - 1) + (1 - a)^2(r_2 - 1) - 2a(1 - a)\sqrt{r_1 - 1}(r_2 - 1))\),
\(F_9 = (-a \sqrt{b(r_1 - 1)} + (1 - a) \sqrt{b(r_2 - 1)}, -a \sqrt{b(r_1 - 1)} + (1 - a) \sqrt{b(r_2 - 1)}, a^2(r_1 - 1) + (1 - a)^2(r_2 - 1) - 2a(1 - a)\sqrt{r_1 - 1}(r_2 - 1))\).

Fig. 3 shows the \((r_1, r_2)\)-parameter space. The stable fixed points at different regions are indicated in the diagram. The dotted line corresponds to the special case \(r_1 = r_2\) which has been considered above. Note that, whereas in the general case all the fixed points are stable in some region or other, in the special case of \(r_1 = r_2\), four of the fixed points, viz., \(F_4, F_5, F_6\) and \(F_7\), are always unstable. When one of the \(r\)-values go over to the chaotic regime, while the other \(r\)-value remains fairly below it, asymptotic synchronization with the chaotic trajectory is observed. The time required to ultimately synchronize with the chaotic attractor is a function of both the \(r\)-parameter values. The synchronization is phase- synchronization rather than state- synchronization, as the response system chaotic attractor is a scaled replica of the driver system attractor. The scaling
factor is $a$ for synchronization with driving system 1, and $(1 - a)$, for driving system 2. When both the $r$-values are in the chaotic regime, the “frustrated synchronization” situation occurs.

3 Simulation Results

For conducting simulations, the parameter values chosen were $r_1 = r_2 = 28$, $\sigma = 10$ and $b = 8/3$. The trace of the Jacobian (which is equal to the sum of the Lyapunov exponents) for the total system, including the driver and response systems, is -40.0. So the overall system is diffusive and possesses an attractor. The competition parameter $a$ was varied in the interval $[0, 1]$. The differential equations were numerically solved using the fourth-order Runge-Kutta method with step-size = 0.025. The phase-space trajectory of the responding system ($x_3, y_3, z_3$) was observed with different values of $a$ from $t = 0$ to $t = 100$. At the limit $a = 0$ (or 1) the responding system trajectory is identical to that of an unperturbed Lorenz system (fig. 1(a)). However, as $a \rightarrow 0.5$ (where maximal competition occurs), the trajectory deviates more and more from the standard Lorenz form. At $a = 0.5$, the trajectory moves in a complicated path around the fixed points $F_1$, $F_2$ and $F_3$ (note that, at $a = 0.5$, $F_8 = F_0 = F_1$ (fig. 1(b)). It appears that for $a=0.5$, the $z$-variable time-series is much more correlated. This becomes clearer on taking a Fourier transform of the data. The power spectral density of the frustrated attractor time-series is low in the high-frequency end compared to the unperturbed system time-series.

The Lyapunov exponents were calculated using Gram-Schmidt technique to create an orthonormal basis every 0.5 seconds of simulation time (this time interval being roughly half the “period” of the Lorenz system) and then averaging over 100 iterations. As expected, of the eight exponents, six correspond to those for the two unperturbed driving Lorenz systems (0.84, 0, -14.51). The remaining two exponents are the conditional Lyapunov exponents of the responding system : -8/3 and -10. This implies the robustness of the “frustrated” attractor - as any deviation from the attractor rapidly diminishes.

To study the degree of synchronization, $z$-coordinates of the responding system state ($z_3$) were plotted against the $z$-coordinates of each of the driver system states ($z_1, z_2$), for different values of $a$. If the two are synchronized, the plot gives a straight line. This suggests that the linear correlation coefficients, $r$, between the driver and response system time series, can be used to obtain a quantitative measure of synchronization. The linear correlation coefficient between two time series data $x(t)$ and $y(t)$ ($t = 1, \ldots, n$), is given by

$$r_{x,y} = \frac{1}{n} \sum_{i=1}^{n} (x(i) - \bar{x})(y(i) - \bar{y}),$$

where $\bar{x}$ and $\sigma_x$ are the mean and standard deviation respectively, for the time series $x(t)$. A
measure of desynchronization is defined as

$$\delta = 1 - r_{22,23}.$$  \hspace{1cm} (14)

At \(a=0\), where there is exact synchronization between driver system 2 and the response system, \(\delta = 0\). This is a particularly robust measure, as \(\delta \to 0\) for both state- and phase- synchronization. The variation of \(\delta\) with \(a\) is shown in a logarithmic plot (fig. 4). The linear nature of the curve over at least 3 orders of magnitude as \(a \to 0\), indicates the presence of a power-law scaling relation of the form:

$$\delta \sim a^\beta,$$  \hspace{1cm} (15)

where the scaling exponent, \(\beta \simeq 2.0\). The scaling exponent was also obtained for \(r = 50\) and 70. In both cases, \(\beta \simeq 2.0\) within simulation error.

Another interesting feature studied was the fractal correlation dimension of the frustrated attractor (fig. 5), calculated using the FD3 (ver. 0.3) software [10]. For the unperturbed Lorenz system, this is very close to 2, as the attractor is almost 2-dimensional. As \(a\) increases from 0 to 0.5, the attractor deviates from this two-dimensional shape, which can be quantitatively measured by the correlation dimension. As \(a \to 0.5\), the attractor structure stretches out more and more over the three-dimensional space. This type of enhanced diffusion in phase space seems to be a generic feature of frustration in chaotic systems, and has been reported previously in the case of Coupled Map Lattices [11].

The simulations also showed the robustness of the “frustrated” attractor. Starting from different initial conditions, the response system trajectory was found to converge to the same attractor structure. This indicates that even in the absence of exact synchronization with any of the driver systems, the response system trajectory can be used as a chaotic masking signal for secure communication [3]. This was verified by adding a small amplitude periodic signal (e.g., a sine wave of frequency \(\omega = 1/200\)) to the response system \(y\)-variable time series. The resultant time series appears to be devoid of any periodic component (fig. 6, top). It is then used to drive another Lorenz system, and the \(x\)-variable time series of the two systems are subtracted from each other to retrieve the original signal (fig. 6, bottom). The modulation of the competition parameter, \(a\), by a binary signal for chaotic switching, is another possibility of using the competitive scheme for secure communication.

4 Discussion

The competitive scheme described here for \(y\)-variable coupling was also implemented for \(x\)- and \(z\)-coupling of Lorenz systems. In the former, similar generalized attractor structure was observed, while in the latter, where the Pecora-Carroll synchronization does not work, no such structure
could be observed. The work done here on coupled Lorenz systems can be extended to other systems defined by autonomous set of differential equations as well as discrete maps. However, it might be interesting to consider the result of competition in synchronizing non-autonomous systems (e.g., the Duffing oscillator). As such systems already have a forcing term present, which brings about the onset of chaos, the introduction of additional forcing terms can lead to qualitatively new behavior.

Competitive synchronization in extended systems might also lead to interesting phenomena. Lattices of (globally or diffusively) coupled chaotic elements, where each element can be used both to drive other elements, as well as respond to driving signals from yet another set of elements, and hence by a series of feedbacks drive its own driving systems, will serve to illustrate interactions between multiple competing synchronizing feedback loops. The motivation for such a study is that, in the human brain, synchronization of activity among different neurons appear to have an important functional role in the proper performance of perceptual tasks. It is to be noted that, single neurons are capable of chaotic behavior. As the brain is composed of densely connected networks of neurons, there is bound to be competitive synchronizing interactions between neural assemblies \[2\]. A dynamic competition parameter, which causes synchronization-desynchronization transitions between various neural sub-assemblies, is a possible mechanism for information processing in biological systems. The resultant dynamics will be radically different from the one we are led to expect by observing the dynamics of single neurons or small groups of neurons.

The above work describes the simplest competitive scenario which can show a qualitatively different dynamics from that in the non-competitive situation. It is at present not known how the nature of synchronization and the attractor structure of the responding system might be altered by increasing the number of competing driver systems. In the brain, where each neuron is connected to $\sim 10^4$ other neurons, the competitive situation is bound to be far more complicated. The manner in which such an extremely competitive synchronization scenario might influence the way in which neural networks perform computations and process information is a very interesting problem for the future.

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Figure 1 (a)

Figure 1 (b)
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