Localizable Entanglement in Antiferromagnetic Spin Chains

B.-Q. Jin and V.E. Korepin
C.N. Yang Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840
(Dated: November 1, 2018)

Spin chains play an important role in solid state physics. For example, inelastic neutron scattering on SrCuO$_2$ can be explained by spin 1/2 Heisenberg chains. Spin chains are also important for information theory. Recently, many-body systems attract lots of attention in the field of quantum information and quantum computation. There is a lot in common between quantum statistical mechanics and quantum information theory. The role of phase transitions for quantum information was emphasized in Ref. 12. Most direct relation between correlation functions and entanglement was discovered by F. Verstraete, M. Popp and J.I. Cirac [we shall use abbreviation VPC] in Ref. 13. They found that correlation function provides a bound for localizable entanglement (LE).

The LE of two spins is defined as the maximal amount of entanglement that can be localized on two marked spins on average by doing local measurements on the rest of the spins [assisting spins]. Here we assume that we consider a pure state $|\phi\rangle$ of all these spins. The LE has an operational meaning applicable to situations in which one would like to concentrate as much entanglement as possible between two particular particles out of multi-particle entangled state. Good examples are quantum repeaters and spintronics. Let us consider an example of $N$ qubits in GHZ state:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle + |11\ldots1\rangle).$$

We can measure assisting qubits in $|\pm\rangle$ basis. This will force two marked spins into a Bell state [maximally entangled state of two qubits].

Let us proceed to the formal definition for localizable entanglement $E_{ij}$ between two spins marked by $i$ and $j$. Consider a pure state of $N$ spins $|\phi\rangle$ [it is normalized $\langle\phi|\phi\rangle = 1$]. Every measurement basis specifies an ensemble of pure states $E = \{p_s, |\psi_s\rangle\}$. The index $s$ enumerates different measurement outcomes. It runs through $2^{(N-2)}$ values. Here $|\psi_s\rangle$ is a two-spin state after the measurement and $p_s$ is its probability. The LE is defined as

$$E_{ij} = \max_s \sum p_s E(|\psi_s\rangle).$$

Here $E(|\psi_s\rangle)$ is the entanglement of $|\psi_s\rangle$, characterized by concurrence. The concurrence $C$ was suggested by W.K. Wooters as a measure of entanglement. By definition it is $0 \leq C \leq 1$. VPC noticed that it is in particular convenient measure for LE. It is important for us that concurrence for two qubits state $|\phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ coincides with maximum correlation $C(|\phi\rangle) = 2|ad - bc|$. It is difficult to calculate LE. Instead, VPC found bounds for LE. The upper bound comes out of considering a global [joint] measurement on all assisting spins. It can be related to the entanglement of assistance, which is the maximum entanglement over all possible states of $N$ spins consistent with the density matrix of two marked spins. It was introduced by D.P. DiVincenzo, C.A. Fuchs, H.Mabuchi, J.A.Smolin, A.Thapliyal and A.Uhlmann. A simple formula for entanglement of assistance was found in Ref. 17. Let us denote the density matrix of two marked spins by $\rho_{ij}$. Matrix $X$ is a square root of the density matrix: $\rho_{ij} = XX^\dagger$. The entanglement of assistance measured by concurrence is given by the trace norm $tr[X^T(\sigma_y \otimes \sigma_y)X]$. Hence, the upper bound of LE is:

$$E_{ij} \leq \sqrt{\frac{s_{ij}^+ \cdot s_{ij}^-}{2}},$$

where

$$s_{ij}^\pm = \left(\frac{1 \pm \langle\phi|\sigma_z^i \sigma_z^j|\phi\rangle}{2} - \langle\phi|\sigma_z^i|\phi\rangle \pm \langle\phi|\sigma_z^j|\phi\rangle\right)^2.$$

The lower bound on LE is expressed in terms of correlation functions:

$$Q_{ij} = \langle\langle\phi|\sigma_z^i \sigma_z^j|\phi\rangle\rangle$$

$$\equiv \langle\phi|\sigma_z^i \sigma_z^j|\phi\rangle - \langle\phi|\sigma_z^i|\phi\rangle \langle\phi|\sigma_z^j|\phi\rangle.$$

$$E_{ij} \geq \frac{\sqrt{Q_{ij} \cdot Q_{ij}}}{2},$$

PACS numbers: 03.67.Mn, 03.67.-a, 73.43.Nq, 05.50.+q
The lower bound on LE is based on the following observation: Given a state of $N$ spins with fixed correlation functions $Q^{ij}_{\alpha\beta}$ between two spins (marked by $i$ and $j$) and directions $\alpha$ and $\beta$, there exist directions in which one can measure other spins [assisting spins], such that this correlation does not decrease. Using this observation, VPC found a lower bound for LE:

$$E_{ij} \geq \max_{\alpha} \left( |Q^{ij}_{\alpha\beta}| \right).$$

VPC explicitly evaluated these bounds for the ground state of the Ising model and showed that actual value of LE is close to the lower bound.

In this paper we consider the ground state of infinite anti-ferromagnetic XXX spin chain at zero temperature. We also consider anisotropic version: XXZ chain. We calculated the concurrence before the measurement [see Appendix A] and compare it to LE. Measurement raises the concurrence.

I. XXX ANTI-FERROMAGNETIC SPIN CHAIN

The Hamiltonian for anti-ferromagnetic XXX spin chain can be written as

$$H_{XXX}^0 = \sum_m \left\{ \sigma_x^{(m)} \sigma_x^{(m+1)} + \sigma_y^{(m)} \sigma_y^{(m+1)} + \sigma_z^{(m)} \sigma_z^{(m+1)} \right\}.$$ (6)

Here $\sigma_x^{(m)}$, $\sigma_y^{(m)}$, $\sigma_z^{(m)}$ are Pauli matrix, which describe spin operators on $m$-th lattice site. Summation goes through the whole infinite lattice. The density of the Hamiltonian is a linear function of the swap gate.

Hans Bethe found eigenfunctions of the Hamiltonian of the model in 1931. The ground state $|\phi\rangle$ was found by Hulten in Ref. 20. We shall normalize it to 1. Correlations are defined as averages with respect to the ground state. They are isotropic

$$\langle \sigma_\alpha^{(i)} \sigma_\beta^{(j)} | \phi \rangle = \delta^{\alpha\beta} \langle \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle.$$ (7)

There is no magnetization $\sigma_\alpha$ ($\alpha = x, y$ or $z$)

$$\sigma_\alpha = \langle \sigma_\alpha^{(i)} | \phi \rangle = 0.$$ (8)

This simplifies the lower bound (5) of LE:

$$1 \geq E_{ij} \geq |\langle \sigma_\alpha^{(i)} \sigma_\alpha^{(j)} | \phi \rangle|.$$ (9)

Let us now use the explicit expression for correlations $\langle \sigma_\alpha^{(i)} \sigma_\alpha^{(j)} | \phi \rangle$ to calculate the lower bound:

$$\langle \sigma_z^{(m)} \sigma_z^{(m+1)} | \phi \rangle = \frac{1}{3} - \frac{4}{3} \ln 2 \approx -0.5908629072,$$ (10)

$$\langle \sigma_z^{(m)} \sigma_z^{(m+2)} | \phi \rangle = \frac{1}{3} - \frac{16}{3} \ln 2 + 3 \zeta(3) \approx 0.2427190798,$$ (11)

$$\langle \sigma_z^{(m)} \sigma_z^{(m+3)} | \phi \rangle = \frac{1}{3} - 12 \ln 2 + \frac{74}{3} \zeta(3) - \frac{56}{3} \zeta(3) \ln 2 \approx -6 \zeta(3)^2 - \frac{125}{6} \zeta(5) + \frac{100}{3} \zeta(5) \ln 2 \approx -0.2009945090.$$ (12)

It took a long time to evaluate correlations functions. Nearest neighbor correlation can be extracted from the ground state energy. Next to nearest neighbor correlation was calculated by M. Takahashi in 1977, see Ref. 21. Recently it was established that all correlations can be expressed as polynomials of $\ln 2$ and the values of Riemann zeta function at odd arguments. These polynomials have only rational coefficients. Third neighbor correlation was calculated by K. Sakai, M. Shiroishi, N. Nishiyama, M. Takahashi, see 22. These results give us the following bounds for localizable entanglement [LE]:

$$E_{j,j+1} \geq 0.5908629072,$$ (14)

$$E_{j,j+2} \geq 0.2427190798,$$ (15)

$$E_{j,j+3} \geq 0.2009945090.$$ (16)

$$E_{j,j+4} \geq 0.0346527769.$$ (17)

At large distances correlation functions exhibit critical behavior. Asymptotic was obtained in:

$$\langle \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle \rightarrow (-1)^{i-j} \left\{ \frac{2 \ln |i-j|}{\ln |i-j|} \right\}.$$ (18)

This helps us to estimate localizable entanglement asymptotically for two spins, which are far away, i.e. $|i-j| \rightarrow \infty$:

$$E_{ij} \geq \left\{ \frac{2 \ln |i-j|}{\ln |i-j|} \right\}.$$ (19)

Even better, but more complicating expression for lower bound of LE, can be extracted from the paper.
$E_{ij} \geq \sqrt{\frac{2}{\pi^2} \frac{1}{|i-j|^2}} \left\{ 1 + \frac{3}{8} \left( \frac{c}{2} - \frac{c^2}{8} \right) g^2 + \left( \frac{5}{128} - \frac{c}{16} - \frac{c^2}{8} \right) g^4 + \left( \frac{21}{1024} + \frac{7c}{256} + \frac{7c^2}{64} - \frac{c^3}{16} + \frac{13\zeta(3)}{32} \right) g^6 + O(g^8) \right\}$

Here the coupling constant $g$ depends on the distance $|i-j|$. It is defined by:

$$\sqrt{g}e^{\frac{ct}{\eta}} = 2\sqrt{2\pi}e^{\gamma E + c|j-i|}.$$  \hfill (21)

Here $\gamma_E = 0.5772$ is the Euler's constant and $c$ is a parameter [normalization point]. A good choice for $c$ is $c = -1$. This boundary for LE is suitable for the full range of distances. In Appendix A we calculated concurrence before the measurement. It is nonzero only for nearest neighbors, see Ref. 38 and (80). Clearly, the measurement raises the concurrence.

II. CRITICAL XXZ ANTIFERROMAGNET

Let us consider effects of anisotropy of interaction of spins. The Hamiltonian of the XXZ spin chain is:

$$H_{XXZ} = -\sum_m \left\{ g_x^{(m)} \sigma_x^{(m+1)} \sigma_x^{(m)} + g_y^{(m)} \sigma_y^{(m+1)} \sigma_y^{(m)} + \Delta (\sigma_z^{(m)} \sigma_z^{(m+1)} - 1) \right\}.$$  \hfill (22)

We shall consider critical regime ($-1 \leq \Delta < 1$) and we use parametrization:

$$\Delta = \cos(\pi \eta), \quad 0 < \eta < 1.$$  \hfill (23)

Let us remind that the case $\eta = 0$ corresponds to ferromagnetic XXX, which we are not considering here. Another case $\eta = 1$ corresponds to anti-ferromagnetic XXX, see previous section.

The case $\eta = 2/3$ corresponds to $\Delta = -1/2$. In this case the model admits much simpler solution than generic Bethe Anzats, see Ref. 34-35. In this case the model is super-symmetric, see Ref. 37.

Later we shall see that all these special cases $\eta = 0, 2/3, 1$ have high sensitivity to magnetic field interacting with spins.

For general values of $\eta$, there is no magnetization:

$$\sigma_\alpha = \langle \phi \sigma_\alpha^{(m)} | \phi \rangle = 0.$$  \hfill (24)

LE is bounded by maximal correlation function:

$$1 \geq E_{ij} \geq \max \left\{ |\langle \phi \sigma_x^{(i)} \sigma_x^{(j)} | \phi \rangle| \right\}. \hfill (25)$$

Correlation functions decays as power laws at large distances $|i-j| \gg 1/(2 - 2\eta)$. Leading terms of correlations are:

$$\langle \phi \sigma_x^{(i)} \sigma_x^{(j)} | \phi \rangle = \langle \phi \sigma_y^{(i)} \sigma_y^{(j)} | \phi \rangle = F |i-j|^{-\eta},$$

$$\langle \phi \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle = -\frac{1}{\pi^2 \eta} |i-j|^{-2} + A(1)^{-j} |i-j|^{-\frac{\eta}{2}}.$$  \hfill (26)

Many people worked on the subject. Important results are obtained in 29. A good collection of references can be found in the book 11, see pages 512, 549-553. Since $0 < \eta < 1$, it becomes clear that $\sigma_z$ correlations asymptotically dominate the lower bound:

$$|Q_{xz}^{ij}| = |Q_{xy}^{ij}| > |Q_{zz}^{ij}|.$$  \hfill (27)

Finally we got the following bound for LE:

$$E_{ij} \geq F |i-j|^{-\eta}.$$  \hfill (28)

This shows that anisotropy raises the lower bound for LE. The coefficient $F$ was calculated $^{29,32}$

$$F = \frac{1}{2(1-\eta)^2} \left[ \frac{\Gamma \left( \frac{\eta}{2(1-\eta)} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right)} \right]^{-\eta} \times \exp\left\{ -\int_0^\infty \frac{dt}{\tau} \left[ \frac{\sinh(\eta t)}{\sinh(\eta t) \cosh((1-\eta) t)} - \eta e^{-2t} \right] \right\}.$$  \hfill (29)

The plot is shown on Fig. 1. There is singularity in function $F$ for $\eta$ near 1:

$$F \sim (1-\eta)^{-\frac{1}{2}} \quad \text{when} \quad \eta \to 1 - 0.$$  \hfill (30)

Special case $\eta = 1$ corresponds to XXX antiferromagnet.

In Appendix A, we evaluated concurrence before the measurement. It vanishes at finite distance, see 30.
III. XXX ANTIFERROMAGNET IN A MAGNETIC FIELD

Let us come back to XXX model

\[ H_{XXX}^0 = \sum_m \sigma_x^{(m)} \sigma_x^{(m+1)} + \sigma_y^{(m)} \sigma_y^{(m+1)} + \sigma_z^{(m)} \sigma_z^{(m+1)} \]  

(30)

But now let us add magnetic field

\[ H_{XXX}^0 = H_{XXX}^0 - \sum_m h \sigma_z^{(m)}. \]  

(31)

This introduce anisotropy in a different way. In small magnetic field \( h \to 0 \), small magnetization develops: \( \sigma_z = h/\pi^2 \). For stronger magnetic field, magnetization increases. As magnetic field approaches its critical value \( h_c = 4 \), magnetization approaches 1 [ferromagnetic state with all spins up]:

\[ \sigma_z = 1 - \frac{2}{\pi^2} \sqrt{h_c - h}, \quad h \to h_c - 0. \]  

(32)

Exact expression for magnetization \( \sigma_z \) at arbitrary value of magnetic field is given on the pages 70-71 of the book [11]. Averages of other spin components over ground state are zero: \( \sigma_x = \sigma_y = 0 \).

In this section, we are considering moderate magnetic field \( 0 \leq h \leq h_c \). Asymptotic of correlation functions at large distances can be described as follows:

\[ \langle \phi | \sigma_x^{(i)} \sigma_x^{(j)} | \phi \rangle = A_1 \frac{1}{(i-j)^2} + A_2 \cos \left\{ \pi (1 - \sigma_z) |i-j| \right\} \]  

(33)

Here double bracket in the left hand side means that we subtracted \( \sigma_z^2 \), see (4). The coefficients \( A_1 \) and \( A_2 \) depend on magnetic field. For small magnetic field, critical index \( \theta \) is close to 1:

\[ \theta = 1 + [2 \ln(h_x/h)]^{-1} \quad h \to 0; \quad h_x = \sqrt{\frac{8s^3}{e}} \]  

(34)

and for the values of magnetic field close to critical point, \( \theta \) is close to 2:

\[ \theta = 2(1 - \frac{1}{\pi^2} \sqrt{h_c - h}), \quad h \to h_c; \quad h \leq h_c = 4. \]  

(35)

In Appendix B we discuss the dependence of \( \theta \) on magnetic field for intermediate fields. Fig. 4 for \( \eta = 1 \) shows that \( \theta \) is a monotonic function of the magnetic field. Asymptotic of other correlation functions are:

\[ \langle \phi | \sigma_x^{(i)} \sigma_x^{(j)} | \phi \rangle = \langle \phi | \sigma_y^{(i)} \sigma_y^{(j)} | \phi \rangle = A(h) |i-j|^{-1/\theta}. \]  

(36)

Coefficient \( A(h) \) vanishes as magnetic field approaches the critical value. Exact formula for \( \theta \) at any value of magnetic field can be found on the pages 73-76 of the book [11] and in [28]. It shows that \( 1/2 \leq 1/\theta \leq 1 \leq 2 \). This means that the lower bound of LE is dominated by \( \sigma_x \) correlations again

\[ E_{ij} \geq A(h) |i-j|^{-1/\theta}. \]  

(37)

Now let us discuss the upper bound (3) of LE. Because of translational invariance we have:

\[ s_{ij}^+ = \left( 1 + \langle \phi | \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle \right)^2 - 4 \sigma_z^2 \]

\[ s_{ij}^- = \left( 1 - \langle \phi | \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle \right)^2. \]  

(38)

At large space separations \( |i-j| \to \infty \), correlations can be simplified \( \langle \phi | \sigma_z^{(i)} \sigma_z^{(j)} | \phi \rangle \to \sigma_z^2 \). This means that both \( s_{ij}^+ \) approach \( (1 - \sigma_z^2)^2 \). Finally the bounds for LE for large \( |i-j| \) are

\[ A(h) |i-j|^{-1/\theta} \leq E_{ij} < \frac{4}{\pi^2} \sqrt{h_c - h}. \]  

(39)

In Appendix A we evaluated concurrence before the measurement. It vanishes at finite distance, see [23].

In the most general case of XXZ a magnetic field correlations can be described by the similar formulae, but parameters \( h_c, \sigma_z, \theta \) are different. We shall elaborate in the next section.

IV. XXZ ANTIFERROMAGNET IN A MAGNETIC FIELD

Let us add interaction with a magnetic field to XXZ spin chain:

\[ H_{XXZ}^h = H_{XXZ}^0 - \sum_m h \sigma_z^{(m)}. \]  

(41)

Small magnetic field leads to a small magnetization \( \sigma_z = \chi h \). The magnetic susceptibility \( \chi \) is:

\[ \chi = \frac{1 - \eta}{\pi \eta \sin \pi \eta}. \]  

(42)

Here we used parameter \( \eta \) related to anisotropy \( \Delta = \cos \pi \eta. \) The dependence of \( \chi \) on \( \eta \) is illustrated in the Fig 2. Let us discuss the plot. The meaning of a singularity at \( \eta = 0 \) is the following: The case \( \eta = 0 \) corresponds to ferromagnetic XXX. At zero magnetic field it has spontaneous magnetization pointed in an arbitrary direction. Weak magnetic field will align spins to the direction of the magnetic field. This makes susceptibility infinite. As \( \eta \) approaches 1, susceptibility approaches \( 1/\pi^2 \) [antiferromagnetic XXX case].

For stronger magnetic field, magnetization increases. As magnetic field approaches its critical value \( h_c = 2(1-\Delta) \), magnetization approaches 1:

\[ \sigma_z = 1 - \frac{2}{\pi^2} \sqrt{h_c - h}, \quad h \to h_c - 0. \]  

(43)
Averages of other spin components over ground state are zero: \( \sigma_x = \sigma_y = 0 \). Here we are considering moderate magnetic field \( 0 \leq h \leq h_c \). Asymptotic of correlation functions at large distances can be described by the formulae similar to XXX in a magnetic field case see [33], [36], but critical index \( \theta \) is different. A formula for \( \theta \) depends on anisotropy \( \Delta = \cos \pi \eta \). Let us first discuss small magnetic field \( h \to 0 \). Critical index is quadratic in magnetic field for \( 0 \leq \eta \leq 2/3 \):

\[
\theta = \frac{1}{\eta} \left( 1 + \alpha_1 h^2 \right)
\]

(44)

The coefficient

\[
\alpha_1 = \frac{(1 - \eta)^2}{4\pi \eta \tan \left( \frac{\pi \eta}{2(1 - \eta)} \right) \sin^2 \pi \eta}
\]

(45)

shows singularity at points \( \eta = 0 \) [ferromagnetic XXX] and \( \eta = 2/3 \) [\( \Delta = -1/2 \) case]:

\[
\alpha_1 \sim \frac{1}{\pi^4 \eta^2} \quad \text{for} \quad \eta \to 0 + 0
\]

(46)

\[
\alpha_1 \sim \frac{1}{8\pi^2 (\eta - 2/3)} \quad \text{for} \quad \eta \to \left( \frac{2}{3} \right) - 0
\]

(47)

The nature of dependence of the coefficient \( \alpha_1 \) is illustrated on Fig 3. In case \( 2/3 \leq \eta \leq 1 \) small \( h \) behavior is more complicated:

\[
\theta = \frac{1}{\eta} \left( 1 + \alpha_2 h^{4(\eta^{-1} - 1)} \right)
\]

(48)

Notice that the power of magnetic field changes monotonically from 2 at \( \eta = 2/3 \) to 0 at \( \eta = 1 \). An expression for the coefficient \( \alpha_2 \) is more complicated:

\[
\alpha_2 = 2\eta e^{\frac{2\pi}{\eta} h_0^{4(\eta^{-1})}} \tan \left( \frac{\pi}{\eta} \right) \frac{\Gamma^2 \left( 1 + \frac{1}{\eta} \right)}{\Gamma^2 \left( \frac{1}{3} + \frac{1}{\eta} \right)}
\]

(49)

Here

\[
\beta = (1 - \eta) \ln(1 - \eta) + \eta \ln \eta
\]

(50)

and

\[
h_0 = \frac{4\eta \sqrt{\pi} \sin \pi \eta}{(1 - \eta)} e^{\frac{\beta}{2(1 - \eta)}} \frac{\Gamma \left( \frac{3 - 2\eta}{2(1 - \eta)} \right)}{\Gamma \left( \frac{2 - \eta}{2(1 - \eta)} \right)}
\]

(51)

\( \alpha_2(\eta) \) shows singularity at point \( \eta = 2/3 \)

\[
\alpha_2 \sim \frac{1}{8\pi^2 (\eta - 2/3)} \quad \text{for} \quad \eta \to \left( \frac{2}{3} \right) + 0
\]

(52)

The nature of dependence of \( \alpha_2 \) on \( \eta \) is illustrated on Fig 3. We see that at \( \eta = 2/3 \) critical index \( \theta \) strongly depends on weak magnetic field. It also depends strongly on weak magnetic field for \( \eta = 1 \), which is antiferromagnetic XXX case.

For magnetic field close to critical, the index \( \theta \) approaches 2:

\[
\theta = 2 + \frac{4\sqrt{h_c - h}}{\pi \tan \left( \frac{\pi \eta}{2} \right) \tan \pi \eta}
\]

(53)

In Appendix B we discuss the dependence of \( \theta \) on magnetic field for intermediate fields. Fig. 4 shows the dependence of \( \theta \) on magnetic field for different values of \( \eta \). Note that the dependence is monotonic.

For XXZ in a magnetic field the lower bound of localizable entanglement is also given by \( \sigma_x \) correlations

\[
E_{ij} \geq \langle \phi | \sigma_x^{(i)} \sigma_x^{(j)} | \phi \rangle
\]

(54)

Asymptotic of the \( \sigma_x \) correlations is still given by the formula [36] with \( \theta \) described in this section.

V. SUMMARY

In this paper we showed that correlations in spin chains are important not only for condensed mater physics and
statistical mechanics but also for quantum information. We considered boundaries for localizable entanglement in the ground state of antiferromagnetic spin chains. We showed that anisotropy raises the localizable entanglement. We also calculated concurrence before the measurement to illustrate that the measurement raises the concurrence.

There are still two open problems left: i. to prove that localizable entanglement coincides with the lower bound $E_{ij} = \max_{\langle Q_{ij} \rangle}$. II. to calculate localizable entanglement for positive temperature.

Acknowledgments

We are grateful to I. Cirac and S. Lukyanov for discussions. The paper was supported by NSF Grant DMR-0302758.

Appendix A

In this Appendix, we calculate the concurrence between two marked spins before measurement. We show that the concurrence between $i$-th and $j$-th qubits vanishes at finite distance $|i - j|$ before the measurement. The density matrix $\rho$ of $i$-th and $j$-th spins can be represented as:

$$\rho = \frac{1}{4} \sum_{\mu} \sum_{\nu} (\sigma^{(i)}_{\mu} \otimes \sigma^{(j)}_{\nu})(\sigma^{(i)}_{\mu} \otimes \sigma^{(j)}_{\nu}) \quad (55)$$

To calculate concurrence we need

$$\hat{\rho} = (\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y). \quad (56)$$

Here $\rho^*$ is the complex conjugate of $\rho$. Subindex $\mu$ runs though all different values $\mu = 0, x, y, z$. Pauli matrix $\sigma_{\mu}$ are

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (57)$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (58)$$

Following W.K. Wooters \textsuperscript{18} we define

$$R = \sqrt{\hat{\rho} \hat{\rho}} \sqrt{\rho} \quad (59)$$

We shall denote the eigenvalues of $R$ by $\lambda_k$ in a decreasing order

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4.$$

Then concurrence ($C$) can be expressed as

$$C = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}.$$

Let us start with the most general case $XXZ$ in a magnetic field:

$$\langle \sigma^{(i)}_x \rangle = \langle \sigma^{(j)}_x \rangle = 0; \quad \langle \sigma^{(i)}_z \rangle = \sigma_1 \quad 0 \leq \sigma < 1 \quad (60)$$

$$\langle \sigma^{(i)}_x \sigma^{(j)}_y \rangle = \langle \sigma^{(i)}_x \sigma^{(j)}_y \rangle = 0 \quad (61)$$

$$\langle \sigma^{(i)}_z \sigma^{(j)}_z \rangle = \langle \sigma^{(i)}_z \sigma^{(j)}_z \rangle = g_2(i - j) \quad (62)$$

$$\langle \sigma^{(i)}_z \sigma^{(j)}_x \rangle = G(i - j) = \sigma^2 + g_2(i - j) \quad (63)$$

$$0 < |g_2(i - j)| < |g_2(i - j)| < 1 \quad \text{for large } |i - j|. \quad (64)$$

Hence density matrix $\rho$ for $i$-th and $j$-th qubits (spins) can be expressed as

$$\rho = \frac{1}{4} I \otimes I + \frac{g_2(i - j)}{4} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$$

$$+ \frac{G(i - j)}{4} \sigma_z \otimes \sigma_z + \frac{\sigma}{4} (I \otimes \sigma_z + \sigma_z \otimes I). \quad (65)$$

All coefficients $\sigma$, $g_2(i - j)$, $G(i - j)$ are real, so

$$\rho^* = \rho \quad (66)$$

$$\hat{\rho} = (\sigma_y \otimes \sigma_y)\rho(\sigma_y \otimes \sigma_y) \quad (67)$$

First three terms in Eq. (65) commute with $\sigma_y \otimes \sigma_y$ and the last term anti-commute. Let us define

$$\rho_0 = \frac{I \otimes I + g_2(i - j)}{4} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$$

$$+ \frac{G(i - j)}{4} \sigma_z \otimes \sigma_z \quad \text{and} \quad (68)$$

$$m = \frac{\sigma}{4} (I \otimes \sigma_z + \sigma_z \otimes I). \quad (69)$$

Notice that $[I \otimes \sigma_z + \sigma_z \otimes I, \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y] = 0$. So we have

$$\rho = \rho_0 + m, \quad \hat{\rho} = \rho_0 - m, \quad [\rho_0, m] = 0, \quad (70)$$

$$[\rho, m] = 0, \quad [\hat{\rho}, m] = 0, \quad [\rho, \hat{\rho}] = 0. \quad (71)$$

Now we can simplify the expression for the matrix $R$:

$$R = \sqrt{\hat{\rho} \hat{\rho}} \sqrt{\rho} = \sqrt{\hat{\rho} \rho} = \sqrt{\rho_0^2 - m^2}. \quad (72)$$

Using this representation we can diagonalize $R$. Corresponding four eigenvalues $\{\lambda_k\}$ are:

$$\left\{ \frac{1}{4} \sqrt{(1 + G(i - j))^2 - 4g_2^2}, \frac{1}{4} \sqrt{(1 + G(i - j))^2 - 4g_2^2}, \frac{1 - G(i - j) + g_2(i - j)}{4}, \frac{1 - G(i - j) - g_2(i - j)}{4} \right\} \quad (73)$$

Now let us consider separately special cases:

I. $XXZ$ model with $h = 0$.

In this case, $\sigma = 0$ and

$$G(i - j) = g_2(i - j) = g.$$  

The set of eigenvalues of $R$ becomes

$$\{\lambda_k\} = \left\{ \frac{1 + g}{4}, \frac{1 + g}{4}, \frac{1 + g}{4}, \frac{1 - 3g}{4} \right\}. \quad (74)$$

To get an explicit expression for concurrence we need to consider separately two cases:
A: $g > 0$ (for $|i - j| =$ even)

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1 + g}{4}, \lambda_4 = \frac{1 - 3g}{4}. \tag{75}$$

We can calculate the concurrence $C$

$$C = \max\{0, -\frac{1 - g}{2}\} = 0, \tag{76}$$

since $|g| < 1$.

B: $g < 0$ (for $|i - j| =$ odd)

$$\lambda_1 = \frac{1 - 3g}{4}, \lambda_2 = \lambda_3 = \lambda_4 = \frac{1 + g}{4}. \tag{77}$$

The concurrence

$$C = \max\{0, -\frac{3g - 1}{4}\} \tag{78}$$

is non-zero only if $g < -\frac{1}{3}$. This happens only for $j = i \pm 1$ with

$$g = \langle \sigma^z_j \sigma^z_{j+1} \rangle = \frac{1 - 4n \ln 2}{3} \approx -0.591. \tag{79}$$

Hence we have

$$C_{j,j+1} = \ln 2 - \frac{1}{2} \approx 0.193,$$

$$C_{j,j+k} = 0 \text{ if } k > 1. \tag{80}$$

So, concurrence is non-zero only for nearest neighbors [for ground state of XXX model without magnetic field]. It was first discovered in Ref. 38.

II. $XXZ$ model at $h = 0$.

In this case $\sigma = 0$. At $|i - j| \to \infty$

$$g_x(i - j) > 0, \quad g_x(i - j) \geq |g_x(i - j)| \tag{81}$$

and both $g_x(i - j)$ and $g_x(i - j)$ decay as a function of the distance $|i - j|$. Eigenvalues of $R$ become

$$\lambda_1 = \frac{1 - g_x(i - j)}{4} + \frac{g_x(i - j)}{2}; \tag{82}$$

$$\lambda_2 = \lambda_3 = \frac{1 + g_x(i - j)}{4}; \tag{83}$$

$$\lambda_4 = \frac{1 - g_x(i - j)}{4} - \frac{g_x(i - j)}{2}. \tag{84}$$

Now we can calculate the concurrence:

$$C = \max\{0, \frac{1}{2} + g_x(i - j) - \frac{1}{2}g_x(i - j)\} \tag{85}$$

It vanishes at distance larger than $|i - j|_{min}$

$$\frac{1}{2} \simeq g_x(|i - j|_{min}) \simeq \frac{F}{|i - j|_{min}^2};$$

$$|i - j|_{min} \simeq (2F)^{\frac{1}{2}}. \tag{86}$$

III. XXX in a magnetic field.

$$G(i - j) = \sigma^2 + g_z(i - j) \tag{87}$$

For large $|i - j|$ both $g_x(i - j)$ and $g_x(i - j)$ become small. For magnetic field smaller then critical

$$g_z(i - j) \frac{1 + \sigma^2}{(1 - \sigma^2)^2} \ll 1$$

$$|g_z(i - j)| > |g_z(i - j)|. \tag{88}$$

Eigenvalues of $R$ in Eq. 73 become

$$\lambda_1 = \frac{1 - \sigma^2}{4} - \frac{g_z(i - j)}{2} + |g_x(i - j)|; \tag{89}$$

$$\lambda_2 = \lambda_3 = \frac{1 - \sigma^2}{4} + g_z(i - j) 1 + \sigma^2; \tag{90}$$

$$\lambda_4 = \frac{1 - \sigma^2}{4} - g_z(i - j) - |g_z(i - j)|, \tag{91}$$

and concurrence becomes

$$C = \max\{0, |g_z(i - j)| - \frac{1 - \sigma^2}{2} - \frac{g_z(i - j) 1 + \sigma^2}{2 1 - \sigma^2}\}. \tag{92}$$

Hence concurrence vanishes at distance larger than $|i - j|_{min}$

$$\frac{1 - \sigma^2}{2} = |g_x(|i - j|_{min})| = |A(h)||i - j|_{min}^{-1/\theta}$$

$$|i - j|_{min} = \frac{2A(h)}{1 - \sigma^2}^{\theta}. \tag{93}$$

Appendix B

In this Appendix, we discuss the dependence of critical exponent $\theta$ on the magnetic field $h$. We follow the book [11].

I. Let us start from $XXX$ model.

Energy of a magnon $\epsilon(\lambda)$ is defined by a set of equations:

$$\epsilon(\lambda) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} K(\lambda, \mu) \epsilon(\mu) \mu \mu = \epsilon_0(\lambda), \tag{94}$$

$$K(\lambda, \mu) = \frac{-2}{1 + (\lambda - \mu)^2}, \quad \epsilon_0(\lambda) = 2h - \frac{2}{4 + \lambda^2}. \tag{95}$$

With extra condition $\epsilon(\pm \Delta) = 0$. This set of equation determines the dependence of $\Lambda$ on magnetic field $h$. Here $\Lambda$ is a value of a spectral parameter at the Fermi edge. An important object is the fractional charge $Z(\lambda)$:

$$Z(\lambda) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} K(\lambda, \mu) Z(\mu) \mu \mu = 1. \tag{96}$$
The critical exponent is equal to:
\[ \theta = 2Z^2(\Lambda) \quad (97) \]

For XXX model, the critical field \( h_c = 4 \). For large magnetic field \( (|h| > h_c) \) \( \Lambda = 0 \). If magnetic field approaches the critical value from below, then
\[ \Lambda = \frac{1}{2}\sqrt{h_c - |h|} \quad \text{and} \quad \theta = 2 - \frac{4}{\pi}\Lambda \rightarrow 2. \quad (98) \]

If the magnetic field is small \( |h| \rightarrow 0 \) then
\[ \Lambda = \frac{1}{2\pi} \ln \left( \frac{(2\pi)^3}{eh^2} \right) \rightarrow \infty \quad \text{and} \quad \theta = 1 + \frac{1}{2\pi\Lambda} \rightarrow 1 \quad (99) \]

II. Now let us discuss XXZ model.

In this case we can use the same set of equations (94,95) with \( K(\lambda,\mu) \) and \( \epsilon_0(\lambda) \) replaced by:
\[ K(\lambda,\mu) = \frac{\sin(2\pi\eta)}{\sinh(\lambda - \mu + i\pi\eta)\sinh(\lambda - \mu - i\pi\eta)}, \]
\[ \epsilon_0(\lambda) = 2h - \frac{2\sin^2(\pi\eta)}{\cosh(\lambda + \frac{i\pi\eta}{2})\cosh(\lambda - \frac{i\pi\eta}{2})}. \quad (100) \]

For small magnetic field
\[ \Lambda = (1 - \eta)\ln \left( \frac{h_0}{h} \right) \rightarrow \infty \quad \text{when} \quad h \rightarrow 0 \quad (101) \]

Here \( h_0 \) is given by (91).

But for magnetic field close to critical:
\[ \Lambda = \frac{\sqrt{h_c - |h|}}{2\tan(\eta/2)} \rightarrow 0 \quad \text{when} \quad h \rightarrow \pm h_c. \quad (102) \]

The critical value of the magnetic field is:
\[ h_c = 2(1 - \Delta) = 2(1 - \cos(\pi\eta)) \quad (103) \]

For general magnetic field \( h \), we solved these equations numerically and found that both \( \Lambda(h) \) and \( \theta(h) \) are monotonic functions of \( h \). The numerical solution for \( \theta(h) \) was shown in Fig 4.

FIG. 4: Critical exponent \( \theta \) (Eqs. 97 and 99) versus magnetic field \( h \) for different values of anisotropy \( \eta \).
17 T. Laustsen, F. Verstraete and S. J. van Enk, arXiv: quant-ph/0206192, Quantum Information and Computation 3, 64 (2003)
18 W.K.Wooters, Phys. Rev. Lett. 80, page 2245, 1998
19 H. Bethe, Zeitschrift für Physik, 76, 205 (1931)
20 L. Hulthén, Ark. Mat. Astron. Physik A26, 1 (1939)
21 M. Takahashi, J. Phys. C: Solid State Phys. 10 (1977) 1289
22 H.E.Boos, V.E.Korepin, arXiv: hep-th/0104008, Journal of Physics A: Math. Gen. V34, 5311 (2001)
23 H.E.Boos, V.E.Korepin, hep-th/0105144 published in the book Math.Phys Odyssey 2001 in Progress in Mathematics, dedicated to 60 birthday of Professor McCoy, Birkhauser
24 H.E.Boos, V.E.Korepin, Y.Nishiyama and M.Shiroishi, cond-mat/0202346, Journal of Physics A: Math. Gen. V35, 4443 (2002)
25 H.E.Boos, V.E.Korepin and F.A. Smirnov, Nucl.Phys. B658, 417 (2003)
26 H.E. Boos, V.E. Korepin, F.A. Smirnov, New formulae for solutions of quantum Knizhnik-Zamolodchikov equations on level -4 , J. Phys. A37 323-336 (2004), hep-th/0304077
27 H.E. Boos, V.E. Korepin, F.A. Smirnov, New formulae for solutions of quantum Knizhnik-Zamolodchikov equations on level -4 and correlation functions , hep-th/0305135
28 K. Sakai, M. Shiroishi, Y. Nishiyama, M. Takahashi, Third Neighbor Correlators of Spin-1/2 Heisenberg Antiferromagnet,Phys.Rev. B67 (2003) 065101, cond-mat/0302564
29 S. Lukyanov, Correlation amplitude for the XXZ spin chain in the disordered regime, cond-mat/9809254
30 cond-mat/9712314, Nucl. Phys. B522 (1998), 533-549
31 I.A. Zaliznyak, C. Broholm, M. Kibune, M. Nohara and H. Takagi, Phys. Rev. Lett. 83, 5370 (1999).
32 M. J. Bhave, F. H. L. Essler, A. Grage, cond-mat/0312055
33 I. Affleck, Exact Correlation Amplitude for the S=1/2 Heisenberg Antiferromagnetic Chain , J.Phys.A 31, 4573 (1998), cond-mat/9802045
34 S. Lukyanov, V. Terras, Long-distance asymptotic of spin-spin correlation functions for the XXZ spin chain , Nucl.Phys. B654, 323-356 (2003), hep-th/0206093
35 A. V. Razumov, Yu. G. Stroganov Spin chains and combinatorics: twisted boundary conditions, J.Phys. A34 (2001) 5335-5340 , cond-mat/0102247
36 A. V. Razumov, Yu. G. Stroganov Spin chains and combinatorics, J.Phys. A34 (2001) 3185, cond-mat/0012141
37 Yu. G. Stroganov The Importance of being Odd , J.Phys. A34 (2001) L179-L186, cond-mat/0102035
38 P. Fendley, B. Nienhuis, K. Schoutens Lattice fermion models with super-symmetry , cond-mat/0307338, J.Phys. A36 (2003) 12399-12424
39 S.-J. Gu, H.-Q. Li and Y.-Q. Li, Phys. Rev. A68, 042330(2003), quant-ph/0307131
40 Entropy of entanglement is equal to \( H(\{1 + \sqrt{1-C^2}\}/2) \), here \( H(x) \) is Shanon entropy.
41 In this case the upper bound of \( E_{ij} \) given by (9) is 1 (any concurrence is bounded by 1).