Disconnected stationary solutions for 3D Kolmogorov flow problem: preliminary results

Nikolay M. Evstigneev
Federal Research Center "Computer Science and Control", Institute for System Analysis, Russian Academy of Science, 117312, Moscow, pr. 60-letiya Oktyabrya, 9, Russia.
E-mail: evstigneevm@yandex.ru

Abstract. The extension of the classical A.N. Kolmogorov’s flow problem for the stationary 3D Navier-Stokes equations on a stretched torus for velocity vector function is considered. A spectral Fourier method with the Leray projection is used to solve the problem numerically. The resulting system of nonlinear equations is used to perform numerical bifurcation analysis. The problem is analyzed by constructing solution curves in the parameter-phase space using previously developed deflated pseudo arc-length continuation method. Disconnected solutions from the main solution branch are found. These results are preliminary and shall be generalized elsewhere.

1. Introduction

The classical A.N. Kolmogorov’s flow problem for the stationary 2D Navier-Stokes equations on a stretched torus [1] is generalized to the 3D geometry by simply extending additional spatial coordinate. The original 2D problem is extensively studied in many different papers, see [2–8]. Excellent literature overview is provided in [9] and interested reader can be referred to these papers.

The 3D generalization is considered in different variants: modification of aspect ratios, modification of forcing terms, different active groups of symmetry etc. A 3D Kolmogorov flow was studied in [10]. The problem was solved numerically on a periodic cuboid with forcing term in $y$ direction with variation in $x$ direction using pseudo-spectral method. Hyper-viscosity was used in order to perform simulation for high Reynolds numbers. It was found that the flow is highly turbulent and intermittent even at large scales. Also it was noticed that the flow is anisotropic at large scales but for large Reynolds number the flow is isotropic at small scales. It was demonstrated that anisotropic large-scale flow may be considered locally isotropic at scales which are approximately ten times smaller than the scale of the flow. In [11] authors considered 3D Kolmogorov flow with sinusoidal forcing term in $x$ direction with variation along $z$ direction. Asymptotic of high Reynolds numbers were obtained and compared with averaged experimental data. Statistical analysis revealed that the shear-driven turbulence studied in [11] has significant spectral anisotropy which increases with wave number. Some variations of the flow problem and mean properties of the 3D Kolmogorov flow problem setups is given in [12]. In the recent paper [13] authors studied the subcritical stability loss and transition to turbulence for the 3D Kolmogorov flow with canonical forcing in a cube. Another recent paper on 3D Kolmogorov flow with different forcing terms was considered in [14]. Recurrent flows were studied and two...
groups of characteristic solutions were found: obtained from the linear instabilities cascades from the main solution and those that were developed by the self sustained process of wave vertex interaction. This indicates high importance of multistability for the dynamical system. In the paper [15] authors considered 3D Kolmogorov flow using DNS calculations to study dependence of statistical turbulence properties on the aspect ratio of the computational domain. It was found that for the minimal computational domain, the velocity statistics exhibit symmetries that are directly imposed by the forcing properties. However, for larger domains, the translational invariance in the stream-wise direction appears to be broken and the turbulence statistics depend on the computational box aspect ratio. It was also concluded that the Kolmogorov flow is a very rich and complex test case that can be considered as an appropriate test for assessing large-eddy simulation of inhomogeneous, anisotropic, and sheared turbulent flow. There are more papers, for example [16], that are dedicated to the statistical properties of the 3D Kolmogorov flow used as a benchmark test for some LES models. The nonlinear analysis and transition to chaos for different aspect ratios and is considered in [17],[18].

In this paper we are interested in the bifurcation analysis of stationary solutions and existence of disconnected solutions. Let the solution function \( u \) depends on the parameter \( R \) and one can trace a continuous curve in the parameter-solution function space. The nontrivial solution curve \( u(R) \) that bifurcated from the trivial solution \( u_0(R) \) at point \( R^* \) is called connected solution curve and \( u_0(R^*) = u(R^*) \) at the bifurcation curve. All other curves that bifurcated from the connected curve are also called connected solution curves. Note, that the stability of these solutions is irrelevant and the connected curves are applicable to both subcritical and supercritical bifurcations. The disconnected solution curve \( u_d(R) \) is such a solution curve that \( u_d(R) \neq u(R) \) for any admissible value of \( R \), where \( u(R) \) is a connected curve. In order to study this problem we use the deflated pseudo arc-length continuation method [19], based on [20], with the eigenvalue solver [21, 22]. These methods can construct bifurcation diagrams of connected and disconnected solution branches alike and check linear stability of these branches. The application of these methods allowed us to find disconnected stationary solutions for the 2D Kolmogorov problem [23]. This time an attempt is made to find disconnected solutions in the 3D Kolmogorov flow with the original, 2D forcing which is trivially extended to the 3D vector.

The paper is laid out as follows. First, the governing system of equations is considered and numerical methods are discussed in short. Next, the obtained results are laid out. Finally, the discussion section is introduced. This is a short paper that contains preliminary, yet, important results.

2. Governing equations, analysis methods
The extension of the classical A.N. Kolmogorov’s flow problem for the 3D Navier-Stokes equations on a stretched torus for velocity vector function \( \mathbf{u} : T(\alpha)^3 \rightarrow \mathbb{R}^3 \) and pressure scalar function \( p : T(\alpha)^3 \rightarrow \mathbb{R} \) is considered as:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + \nabla p - R^{-1} \Delta \mathbf{u} - (\sin(\beta y); 0; 0)^T &= 0, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where \( R \) is the Reynolds number (bifurcation parameter), \( \alpha \) is the stretch factor for the domain \( T(\alpha)^3 := [0; 2\pi/\alpha] \times [0; 2\pi] \times [0; 2\pi] \) and \( \Delta \) is the Laplace operator. The force vector field depends only on the second spatial variable \( y \) and coefficient \( \beta \) is an integer.

The problem has a trivial solution:

\[
\mathbf{u}_0 = \frac{R}{\beta^2} (\sin(\beta y), 0, 0)^T.
\]

In this study we use \( \alpha = 1/2, \beta = 1 \) hence, the forcing term is taken from the original work [2]
with added zero to the $z$ - component. The main focus is on the stationary problem, hence $\partial u / \partial t = 0$ in (1).

We use the following standard series representation for the spatial periodic solutions with the mode vector $j = (j_x, j_y, j_z)^T$:

$$u(x, t) = \sum_{j \in \mathbb{Z}^3} \hat{u}_j(t) e^{i((\alpha j_x x) + (j_y y) + (j_z z))},$$

$$p(x, t) = \sum_{j \in \mathbb{Z}^3} \hat{p}_j(t) e^{i((\alpha j_x x) + (j_y y) + (j_z z))}.$$  

(4)

Inserting (4) into (1) and using Bubnov–Galerkin projection and divergence-free projection (Leray projector) we arrive at the following system:

$$P(B(\hat{u}, \hat{u})) - R^{-1}C\hat{u} - \hat{f} = 0,$$

(5)

where $P = E - GC^{-1}G^T$ is the discrete Leray projector, $E$ is the identity matrix, $G$ is the gradient operator matrix, its transpose is a divergence matrix and $C$ is the Laplace operator matrix. The convolution term is designated as $B(\hat{u}_1, \hat{u}_2)$ and $\hat{f}$ is the divergence free forcing term projection to the space, spanned by the trigonometric functions.

We apply numerical methods, so the system of equations is recast to finite dimensions by considering truncated series in (5) (hence, we can apply a computer to solve the problem) with the zero mean flow. We consider $N/\alpha \times N \times N$ Fourier harmonics. In these series we limit ourselves to $N = 128$ harmonics. The active number of degrees of freedom (DOF) is $DOF = 6N \times N \times (N/2 + 1) - 1 \approx 6.38 \cdot 10^6$ due to the reality condition. The resulting finite dimensional nonlinear and linear operators (the latter obtained analytically on a solution $u_0$) are used in the deflation continuation and stability analysis process and are designated as $F(U)$ and $A(U_s) := \partial F(U_s)/\partial U$, accordingly: the vector $U = (\hat{u}_x, \hat{u}_y, \hat{u}_z)^T$ is constructed of the finite dimensional expansion coefficients, see (4).

The pseudo arc-length continuation method along with the eigenvalue solver were developed in [24-26] and are applied to the problem at hand. The used eigensolver is the Implicitly Restarted Arnoldi Method (IRAM) with the Cayley and the inexact complex exponent shift-inverse transformations. We only describe the main idea of the methods here, interested reader can be referred to the cited papers. The deflation process allows one to find all existing solutions at a particular value of the parameter $R_s$. For this one solves the problem:

$$G(||U||^K_{j=0}, U, R_s) = 0,$$

(6)

where the set $\{ U_j \}_{j=0}^K$ is the set of all found $K + 1$ solutions for the particular value of the parameter $R_s$ and $G(\cdot, \cdot, R)$ is the deflation operator which is build around the operator $F(U)$ that deflates all known solutions passed as the set in the first argument. In order to solve the problem (6) numerically, one applies the Newton’s method, the details on this are given in [26]. Once the solution is found, it is used to start the pseudo arc-length continuation process which is executed to trace the curve in the solution-parameter space. Once the curve is traced (either by looping on itself or by leaving the maximum defined parameter values) it’s stability is analyzed and bifurcation points are identified by inspecting the linear stability using the IRAM method and using the bisection method to locate a precise bifurcation point. The eigenvectors that span the unstable manifold are passed to the deflation process near the bifurcation points in order to find and perform the continuation of the connected bifurcation curves. However, the deflation is executed, also, with random initial guesses in the Newton’s method, hence some disconnected solutions, if there exist any, can be found. These obtained solutions also undergo the process of continuation and bifurcation analysis. This process is repeated until all solution curves (connected and disconnected) and bifurcations are identified. The system of equations is solved using multiple Graphics Processing Units (GPUs) the method is implemented on the C++. The implementation can be found in the author’s github.
3. Stability of the main solution

The stability of the main solution (3) is analyzed by introducing infinitesimal perturbations, designated by the apostrophe, as \( u = u_0 - u' \) and \( p = p_0 + p' \) in the classical manner, see [27], Chapter 3.

Inserting this ansatz into the system (1) and subtracting the original system yields the linearized system:

\[
\begin{align*}
\partial u'/\partial t + (u_0, \nabla)u' + (u', \nabla)u_0 + \nabla p' - R^{-1} \triangle u' &= 0, \\
\nabla \cdot u' &= 0.
\end{align*}
\]

The first equation in (7) is greatly simplified, since it contains only one component that depends only on \( y \) which we shall call, in accordance with the tradition, the shearwise direction. Let us introduce the designation \( \partial_x(\cdot) := \partial(\cdot)/\partial x \). In this case the system (7), (8) becomes:

\[
\begin{align*}
\partial_t(u'_x) + u_0x \partial_x(u'_x) + u'_y \partial_y(u'_x) + \partial_x p' - R^{-1} \triangle u'_x &= 0, \\
\partial_t(u'_y) + u_0x \partial_x(u'_y) + \partial_y p' - R^{-1} \triangle u'_y &= 0, \\
\partial_t(u'_z) + u_0x \partial_x(u'_z) + \partial_z p' - R^{-1} \triangle u'_z &= 0,.
\end{align*}
\]

One can derive the pressure equation from applying the divergence operator to (9), (10), (11). Using (12) one obtains:

\[
\Delta p' = -\partial_y(u_0x)\partial_x(u'_x).
\]

Applying the Laplace operator to (10) one eliminates the pressure term and obtains the Orr-Sommerfeld equation:

\[
((\partial_t + u_0x \partial_x) \triangle - \partial_{yy}(u_0x) \partial_x - R^{-1} \Delta^2) v = 0,
\]

where the designation \( v \) stands for the perturbation velocity component \( u'_y \). Application of the curl operator to the system (9), (10), (11) and usage of (12) to eliminate the terms like \( \partial_y(u_0x)\partial_x(u'_z) + \partial_y(u_0x)\partial_y(u'_y) \) yields the Squire equations, obtained by taking the shearwise direction of the curl, designated as \( \omega \):

\[
(\partial_t + u_0x \partial_x - R^{-1} \triangle) \omega = -\partial_y(u_0x)\partial_z(v).
\]

Assuming that the solutions to the equations (13), (14) can be expanded into the Fourier modes in the \( x \) and \( z \) directions, hence the following ansatz is introduced:

\[
\begin{align*}
v(t, x, y, z) &= \eta(y) \exp(\lambda t + i\alpha k_x x + i k_z z), \\
\omega(t, x, y, z) &= \nu(y) \exp(\lambda t + i\alpha k_x x + i k_z z),
\end{align*}
\]

with \( k_x \) and \( k_z \) are integer wavenumbers. The ansatz (15) is inserted into (13) and (14) to obtain the system of equations that controls the stability of the main flow:

\[
\begin{align*}
\left((\lambda + i\alpha k_x u_0x)(-\alpha^2 k_x^2 + \partial_{yy} - k_z^2) - i\alpha k_x \partial_{yy}(u_0x) - R^{-1}(-\alpha^2 k_x^2 + \partial_{yy} - k_z^2)^2\right) \eta(y) &= 0, \\
\left(\lambda + i\alpha k_x u_0x - R^{-1}(-\alpha^2 k_x^2 + \partial_{yy} - k_z^2)\right) \nu(y) &= -i k_z \partial_y(u_0x)\eta(y).
\end{align*}
\]

The main solution (3) is stable if \( \Re e(\lambda) < 0 \). The derivation of stability conditions follow [13] with little differences. Some cases can be considered:
• Assume that $\eta$ and $\nu$ are spatially uniform functions. Then the solution can be constructed as $u^* = (c_1 \exp(ik_y y), 0, c_2 \exp(ik_y y))^T$, hence $\lambda = -1/Rk_x^2$ and the main solution is stable, since the zero wavenumber is excluded by the assumption of the zero constant flow through the domain.

• If $\nu$ is not a spatially uniform function, then the equation (17) must be considered. The magnitude of perturbation is obtained by the multiplication of (17) to the conjugate function $\nu^*$, integration over the $y$ direction from 0 to $2\pi$ and considering the real part of the integral:

$$
\int_0^{2\pi} \Re(e^\lambda|\nu|^2\,dy) = -\int_0^{2\pi} \left( R^{-1}(\alpha^2 k_x^2 + k_y^2)|\nu|^2 + R^{-1}|\partial_y \nu|^2 \right)\,dy < 0.
$$

• For the case when the solution is assumed to be independent of $x$ and $z$ directions, i.e. $k_x = k_z = 0$ we, again, obtain $\lambda = -1/Rk_z^2$.

• For the nontrivial case one needs to consider the equations (16) and obtain the energy equation by multiplying it on the conjugate function, taking the real part and integrating over the shearwise direction. After some manipulations one arrives at the equation:

$$
\Re(e^\lambda)(2\alpha^2 k_x^2 + 2k_y^2 - 1) \int_0^{2\pi} |\partial_y(\eta)|^2\,dy + \Re(e^\lambda)(\alpha^2 k_x^2 + k_y^2 - 1) \int_0^{2\pi} (\alpha^2 k_x^2 + k_y^2)|\eta|^2\,dy + \\
+ \Re(e^\lambda) \int_0^{2\pi} (|\partial_y y(\eta)|^2\,dy = -R^{-1}(\alpha^2 k_x^2 + k_y^2 - 1) \int_0^{2\pi} (|\partial_y y - \alpha^2 k_x^2 - k_y^2)|\eta|^2\,dy - \\
- R^{-1} \int_0^{2\pi} |\partial_y((\partial_y y - \alpha^2 k_x^2 - k_y^2)\eta)|^2\,dy.
$$

The situation is more complicated than the one in the [13]. One can notice that all integrals are positive. Let us introduce some designations to these integrals in order simplify the equation: $A := \int_0^{2\pi} |\partial_y(\eta)|^2\,dy$, $B := \int_0^{2\pi} (\alpha^2 k_x^2 + k_y^2)|\eta|^2\,dy$, $C := \int_0^{2\pi} (|\partial_y y(\eta)|^2\,dy$, $D := \int_0^{2\pi} (|\partial_y y - \alpha^2 k_x^2 - k_y^2)|\eta|^2\,dy$, $F := \int_0^{2\pi} (|\partial_y((\partial_y y - \alpha^2 k_x^2 - k_y^2)\eta)|^2\,dy$. The most dangerous situation is $k_x = 0$, i.e., according to the Squire’s theorem for the shear flows the 2D perturbations are the most dangerous. Then:

$$
\Re(e^\lambda) = -R^{-1}(k_y^2/4 - 1)D + F \overline{(k_x^2/2 - 1)A + (k_y^2/4 - 1)B + C}.
$$

Now, one can see that the amplification of perturbations is possible for $k_x = \pm 1$ i.e. the most dangerous instabilities are of the lowest frequencies.

It is difficult to obtain the exact value of the critical Reynolds number for the stability loss but it is clear that the stability of the main solution is conditional. It will be investigated numerically in the next section.

### 4. Results

In order to represent the curves in solution-parameter space graphically one reduces the dimension of the solution by plotting those in some function that reduces dimension and is representative. Particular representation functions $g_j$ are shown on the axis labels in each graph. Let us designate $u$ as the pointwise absolute value of the physical velocity and designate the Fourier vector norm $||U||_2 := \sqrt{DOF^{-1} \sum_{j=0}^{DOF} \overline{U_j U_j}$. and are expressed as:

$$
g_1 := u \left( \begin{array}{ccc} 3\pi & 2\pi & 2\pi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \\
g_2 := u \left( \begin{array}{ccc} 3\pi & 2\pi & 2\pi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \\
g_3 := u \left( \begin{array}{ccc} 3\pi & 2\pi & 2\pi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \\
g_4 := u \left( \begin{array}{ccc} 3\pi & 2\pi & 2\pi \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{18}
$$
In this paper the problem is only investigated over the parameter segment $R \in [0, 5]$, where only stationary or laminar solutions reside.

4.1. All branches

The visualization of all solution branches for $R \in [0, 5]$ is presented in figure 1. The detection of bifurcation points is performed by the linear interpolation with the bisection algorithm and is executed up to $|\delta R| \leq 5 \cdot 10^{-5}$.

![Figure 1. Bifurcation diagrams of stationary solutions in the form "branch_number: branch_name".](image)

The main solution is depicted as the branch number zero. As at was shown in the previous section that it looses stability. In fact it suffers the pitchfork bifurcation at $R = 1.0125 \pm 0.0001$. All other solution curves are secondary. Curves 0–2 are connected curves (to the main branch number 0), curves 3–6 are disconnected curves. The connected curves are mostly bifurcated by the pitchfork bifurcation, while the disconnected curves bifurcated from the saddle-node bifurcation.

The unstable manifold dimension (the number of real eigenvalues with positive real parts plus the number of complex conjugate eigenvalues with positive real parts) for each curve is estimated and is shown in figure 2 in different projections. The bifurcation points are represented by the numbers, the value of the number corresponds to the increase or decrease of the unstable dimension caused by the bifurcation. Now let us analyze particular branches as well as their stability.

4.2. Connected branches

All connected curves, bifurcation points and unstable manifold dimensions are given in figure 3. This figure is used for reference and navigation.

We start the analysis from the curve 0 that bifurcates with the pitchfork at $R = 1.0125 \pm 0.0001$. The solution at the bifurcation point of curves 0 and 1 is shown at figure 4. The bifurcated solutions in the curve 1 are stable. The germ of the solution near the bifurcation point is shows in figure 5. One can notice that the germ perfectly matches the conclusion that was made during the stability analysis of the main solution, i.e. the germ $g$ is represented by $g \sim \exp(ik_xx)$ with $k_x = \pm 1$. The sign of the wavenumber defines the left or right tines of the pitchfork. The resulting solution on the curve 1 near the bifurcation point is stable, stationary and symmetric w.r.t the $z$-direction.
Figure 2. Bifurcation diagrams of stationary solutions with the unstable manifold dimension (represented by the color map) and bifurcation points (represented by numbers on curves).

The next bifurcation number 2 in figure 3 is the Andronov-Hopf bifurcation. It occurs at $R = 3.46726 \pm 0.0001$. The stationary solution at the curve 1 at that point is presented in figure 4. It is also symmetric w.r.t. the $z$-direction, hence, the curve of periodic solutions that bifurcates from the curve 1 at point 2 will also be symmetric in that direction. This scenario is close to the one in the 2D Kolmogorov flow problem, see [6]. The solution loses stability and is unstable until $R = 5$. Bifurcation number 3 in the reference figure results in the other pitchfork bifurcation at $R = 4 \pm 0.0001$. This bifurcation spawns the curve number 2. The solutions near 2-nd bifurcation point are unstable. Another series of Hopf bifurcations can be observed near the bifurcation point number 5 that occur at $R = 4.36035 \pm 0.0001$, $R = 4.48749 \pm 0.0001$ and $R = 4.52029 \pm 0.0001$. The solution at bifurcation point 5 is visualized in figure 4. It is more complicated but still persists the symmetry in the $z$-direction.

The curve number 2 that bifurcates in point 3 from the curve number 1 at $R = 3.46726 \pm 0.0001$ is formed by the pitchfork bifurcation. The resulting curve is unstable, it’s germ is shown in figure 5 on the left. The next inverse transcritical bifurcation occurs in point 4 at $R = 4.33057 \pm 0.0001$, where the solution loses symmetry in $z$-direction and becomes stable. This kind of behavior, when the symmetric solution is unstable, while another non-symmetric
solution is stable can also be checked with the 3D Kolmogorov problem with more complicated forcing term, see [18], figure 6. The curve number 2 remains stable until the parameter reaches maximum observable value.

The connected curves demonstrate classical super/sub-critical transition process. The subcriticality is observed in the curve number 2, where the co-existing periodic or quasi-periodic orbits (that bifurcated earlier in points 2 or 5) may transfer to the stationary non-symmetric solution. This will be investigated in the next papers.

4.3. Disconnected branches

All disconnected curves, bifurcation points and unstable manifold dimensions are given in figure 6. This figure is used for reference and navigation.

The first disconnected curve is the curve number 3 that is formed by two saddle-node bifurcations and is a closed loop in visualization functions $g_3$, see figures 6 and 2. This curve is formed by the pair of bifurcations in points 3 at $R = 4.248980.0001$ and 1 at $R = 4.71496±0.0001$. The visualization of the solution at points 1 and 3 is shown in figure 7. The solutions are not symmetric w.r.t. the $z$-direction and are not connected to the main branch. This curve possesses the set of Hopf bifurcations at $R_{Hopf} = \{4.27852, 4.28142, 4.69577, 4.65858, 4.65807, 4.67752\}$. For example, bifurcation at point 2 spawns the periodic trajectory, it was confirmed by the direct simulation with the bifurcation solution used as the initial conditions.

The other disconnected curves 4, 5 and 6 have no stable solution regions in the considered range of parameter values and most probably don’t influence the dynamics of the system directly. The largest dimension of the unstable manifold is 8 at the curve number 4. It contains multiple Hopf (possibly secondary) and transcritical bifurcations. The germ of the solution at Hopf bifurcation in point 4 at $R = 4.56179±0.0001$ is presented in figure 8 top left. The saddle-node bifurcation is located in point 5 at $R = 4.41823 ± 0.0001$.

Curve number 5 also contains multiple Hopf and transcritical bifurcations. The curve itself represents the switching of symmetry w.r.t. the central point $2\pi, \pi, \pi$ in the $xz$ plane. This can be noticed in figure 8 on the two lower figures. Curve number six has a similar behavior. The curves are generated by the saddle-node bifurcations in point 8 at $R = 4.73496 ± 0.0001$ and point 9 at $R = 4.51659 ± 0.0001$. 

Figure 3. Bifurcation diagrams of connected solutions with bifurcation points and unstable manifold dimension presented by the color scale.
Figure 4. Absolute value of the velocity vector at bifurcation points of connected curves: 1 on curve 0 (top, left), germ of the curve 1 at point 1 (top, right), 2 on curve 1 (bottom, left), 5 on curve 1 (bottom, right).

Figure 5. Absolute value of the velocity vector at bifurcation points of connected curve 2: germ of the curve 2 at point 3 (left), 4 on curve 2 (right).
5. Discussion

We extend the conjecture in [7] on the 3D Kolmogorov flow problem and confirm our results from [17, 18] in that there are many disconnected solutions for this problem; these solutions cannot be obtained without the deflation process. Multiple stationary stable solutions are found for $R \in [4.6957, 4.7149]$.

The most interesting behavior of the system is expected in the vicinity of the points 1 and 2 on the disconnected curve number 3. In this case the solution may be trapped by the basin of attraction from the curve 3 and then may start developing through the alternate cascades of bifurcations that are not related to the main connected solution branches.

Additionally, such stationary and periodic orbits with multiple scenarios (originated from different regions) can serve as the recurrent solutions embedded into the turbulent chaotic flow. This question is yet to be understood and investigated.
Figure 8. Absolute value of the velocity vector at bifurcation points of disconnected curves: 4 on curve 4 (top, left), 7 on curve 6 (top, right), 8 on curve 5 (bottom, left), 10 on curve 5 (bottom, right).

Acknowledgments
This work was supported by the Russian Foundation for Basic Research, grant No. 18-29-10008mk and 20-07-00066.

References
[1] Arnol’d V and Meshalkin L 1960 Uspekhi Mat. Nauk 15(1(91)) 247–250
[2] Meshalkin L and Sinai I 1961 Journal of Applied Mathematics and Mechanics 25 1700–1705
[3] Okamoto H and Shoji M 1993 Japan Journal of Industrial and Applied Mathematics 10 191–218
[4] Okamoto H 1998
[5] Matsuda M and Miyatake S 2002 Tohoku Mathematical Journal 54 329–365
[6] Evstigneev N M, Magnitskii N A and Silaev D A 2015 Differential Equations 51 1292–1305
[7] Kim S C and Okamoto H 2015
[8] Tithof J, Suri B, Pallantla R K, Grigoriev R O and Schatz M F 2017 Journal of Fluid Mechanics 828 837–866
[9] Lucas D and Kerswell R 2014 Journal of Fluid Mechanics 750 518–554
[10] Borue V and Orszag S A 1996 306 293–323 URL https://doi.org/10.1017%2Fs0022112096001310
[11] Shebalin J V and Woodruff S L 1997 9 164–170 URL https://doi.org/10.1063%2F1.869159
[12] ROLLIN B, DUBIEF Y and DOERING C R 2017 670 204–213 URL https://doi.org/10.1017%2Fs0022112017003194
[13] Hiruta Y and Toh S 2020 89 044402 URL https://doi.org/10.7566%2FJpsj.89.044402
[14] Lucas D and Kerswell R 2017 817 URL https://doi.org/10.1017%2Fjfm.2017.97
[15] Sarris I E, Jeannart H, Carati D and Winckelmans G 2007 19 095101 URL https://doi.org/10.1063%2F1.2760280
[16] SL W, JV S and Hussaini M 1999 Direct-numerical and large-eddy simulations of a non-equilibrium turbulent kolmogorov flow Tech. Rep. NASA/CR-1999-209727, 99-45 Institute for Computer Applications in Science and Engineering 209727

[17] Evstigneev N and Magnitskii N 2017 *Journal of Applied Nonlinear Dynamics* **6** 345–353

[18] Evstigneev N, Magnitskii N and Ryabkov O 2019 *Journal of Applied Nonlinear Dynamics* **8** 595–619

[19] Evstigneev N 2019 *Communications in Computer and Information Science* **1063** 122–138

[20] Farrell P E, Birkisson Å and Funke S W 2015 *SIAM Journal on Scientific Computing* **37** A2026–A2045

[21] Evstigneev N 2017 *Communications in Computer and Information Science* **753** 301–316

[22] Evstigneev N 2018 *Journal of Physics: Conference Series* **1141**

[23] Evstigneev N M 2021 *Journal of Physics: Conference Series* **1730** 012078 URL https://doi.org/10.1088/2090/1/2090/1/78

[24] Evstigneev N M 2017 *Communications in Computer and Information Science* (Springer International Publishing) pp 301–316

[25] Evstigneev N M 2018 *Journal of Physics: Conference Series* **1141** 012121

[26] Evstigneev N M 2019 *Communications in Computer and Information Science* (Springer International Publishing) pp 122–138

[27] Schmid P J and Henningson D S 2001 *Stability and Transition in Shear Flows* (Springer New York) URL https://doi.org/10.1007/978-1-4613-0185-1

[28] Rosa R 1998 *Nonlinear Analysis: Theory, Methods & Applications* **32** 71–85

[29] Shimizu M and Manneville P 2019 *Physical Review Fluids* **4**