On primitive elements of algebraic function fields and models of $X_0(N)$

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Abstract

This paper is a continuation of our previous works where we study maps from $X_0(N)$, $N \geq 1$, into $\mathbb{P}^2$ constructed via modular forms of the same weight and criteria that such a map is birational (see Muić in Monatsh Math 180(3):607–629, 2016). In the present paper, our approach is based on the theory of primitive elements in finite separable field extensions. We prove that in most of the cases the constructed maps are birational. We consider particular cases and their equations in $\mathbb{P}^2$.

Keywords  Modular forms · Modular curves · Birational equivalence · Primitive elements

Mathematics Subject Classification  11F11 · 11F23

1 Introduction

Let $\mathbb{H}$ be the complex upper half-plane with the $SL_2(\mathbb{R})$-invariant hyperbolic measure defined by $dx
dy/y^2$, where the coordinates on $\mathbb{H}$ are written as $z = x + \sqrt{-1}y$, $y > 0$. Let $\Gamma$ be a Fuchsian group of the first kind [8, Sect. 1.7, p. 28]. By a theorem of Siegel [8, Theorem 1.9.1], a discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$ is a Fuchsian group of the first kind if and only if the hyperbolic volume of the quotient $\Gamma \setminus \mathbb{H}$ is finite: $\int\int_{\Gamma \setminus \mathbb{H}} \frac{dx
dy}{y^2} < \infty$. Examples of such groups are the important modular groups such as $SL_2(\mathbb{Z})$ and its congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, and $\Gamma(N)$ [8, Sect. 4.2].

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The quotient \( \Gamma \backslash \mathbb{H} \) can be compactified by adding a finite number of \( \Gamma \)-orbits of points in \( \mathbb{R} \cup \{\infty\} \) called cusps of \( \Gamma \) and we obtain a compact Riemann surface which will be denoted by \( \mathcal{R}_\Gamma \). As it is an irreducible complete smooth algebraic curve, we are interested in finding its plane realizations. Various aspects of modular curves has been studied in [1–4, 9, 12, 16] and [19]. We continue the approach presented in [10, 11], and [5], and start by introducing the notation and briefly repeat some results from [11]. In the present paper our approach is based on the theory of primitive elements in finite separable field extensions (see [20], and [18, Sect. 6.10]). We especially rely on the theory and criteria for birationality developed in [11].

Assume that \( \Gamma \) has at least one cusp. Let \( g(\Gamma) \) be the genus of \( \mathcal{R}_\Gamma \). For \( m \geq 2 \) an even integer, let \( M_m(\Gamma) \) (resp., \( S_m(\Gamma) \)) be the space of (resp. cuspidal) modular forms of weight \( m \) for \( \Gamma \). Assume \( \dim M_m(\Gamma) \geq 3 \). Let \( f, g, h \) be three linearly independent modular forms in \( M_m(\Gamma) \). Then, we define a holomorphic map \( \mathcal{R}_\Gamma \to \mathbb{P}^2 \) by

\[
\alpha_z \mapsto (f(z) : g(z) : h(z)). \tag{1.1}
\]

Since \( \mathcal{R}_\Gamma \) has a canonical structure of complex projective irreducible algebraic curve, this map can be regarded as a regular map between projective varieties. Consequently, the image is an irreducible projective curve which we denote by \( C(f, g, h) \).

The degree \( d(f, g, h) \) of the map (1.1) is by definition the degree of the field extension of the fields of rational functions:

\[
\mathbb{C}(C(f, g, h)) \subset \mathbb{C}(\mathcal{R}_\Gamma).
\]

This number is studied in [11] in great detail. The main result of [11] and its proof give a fairly detailed description of \( d(f, g, h) \) [11, Theorem 1-4]. In order to recall that result we introduce more notation.

We recall the notion of the divisor of \( f \in M_m(\Gamma), f \neq 0 \) (see [8, Sect. 2.3], or Sect. 2 in this paper). For each \( \alpha \in \mathcal{R}_\Gamma \), we may define the multiplicity \( v_\alpha(f) \) of \( f \) at \( \alpha \). The multiplicity \( v_\alpha(f) \) is a non-negative rational number, and, for all but finitely many points \( \alpha \in \mathcal{R}_\Gamma \), \( v_\alpha(f) = 0 \). We may define the divisor of \( f \) as follows:

\[
\text{div}(f) = \sum_{\alpha \in \mathcal{R}_\Gamma} v_\alpha(f) \alpha.
\]

The degree of this divisor is given by

\[
\text{deg}(\text{div}(f)) \overset{\text{def}}{=} \sum_{\alpha \in \mathcal{R}_\Gamma} v_\alpha(f) = \frac{m}{4\pi} \int_{\Gamma \backslash \mathbb{H}} \frac{dx\,dy}{y^2}.
\]

If \(-1 \in \Gamma \) and \( \Gamma \) is a subgroup of finite index in \( SL_2(\mathbb{Z}) \), then the right-hand side is given by the following well-known expression: \( \frac{m}{12}[SL_2(\mathbb{Z}) : \Gamma] \).

Now, [11, Theorem 1-4] gives the following equality:

\[
d(f, g, h) \cdot \text{deg} C(f, g, h) = \frac{m}{4\pi} \int_{\Gamma \backslash \mathbb{H}} \frac{dx\,dy}{y^2} - \sum_{\alpha \in \mathcal{R}_\Gamma} \min(v_\alpha(f), v_\alpha(g), v_\alpha(h)).
\]

Here, \( \text{deg} C(f, g, h) \) is the degree of the reduced homogeneous equation defining \( C(f, g, h) \) in \( \mathbb{P}^2 \).
In [11, Corollary 1.5], this was further refined as follows (recall that $m \geq 2$ is even):

$$
d(f, g, h) \cdot \deg C(f, g, h) = \begin{cases} 
\dim M_m(\Gamma) + g(\Gamma) - 1 - \sum_{a \in \mathfrak{R}_\Gamma} \min \left( \epsilon'_f(a), \epsilon'_g(a), \epsilon'_h(a) \right), \\
\dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m - \sum_{a \in \mathfrak{R}_\Gamma} \min \left( \epsilon_f(a), \epsilon_g(a), \epsilon_h(a) \right), \\
\quad \text{if } f, g, h \in S_m(\Gamma),
\end{cases} \quad (1.2)
$$

where $\epsilon_2 = 1$ and $\epsilon_m = 0$ for $m$ even, $m \geq 4$. Here, for example, $\epsilon'_f$ denotes the integral effective divisor on $\mathfrak{R}_\Gamma$ obtained from $\text{div}(f)$ by subtracting necessary contributions at elliptic points, and, in addition, if $f \in S_m(\Gamma)$, then we subtract necessary contribution from $\epsilon'_f$ at cusps, to get a divisor $\epsilon_f$. Details are standard, and they can be found in (see [11, Lemma 2.2], or Lemma 2.2 in this paper).

We need the following definition before we state the main result of the paper:

**Definition 1.3** Let $W \subset M_m(\Gamma)$ be a non-zero linear subspace. Then, we say that $W$ determines the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ if $\dim W \geq 2$, and there exists a basis $f_0, \ldots, f_{s-1}$ of $W$, such that $\mathbb{C}(\mathfrak{R}_\Gamma)$ is generated over $\mathbb{C}$ by the quotients $f_i/f_0$, $1 \leq i \leq s-1$.

Clearly, this notion does not depend on the choice of the basis used. Also, it is equivalent to the fact that the holomorphic map $\mathfrak{R}_\Gamma \rightarrow \mathbb{P}^{s-1}$ given by $a_z \mapsto (f_0(z) : \cdots : f_{s-1}(z))$ is birational onto its image in $\mathbb{P}^{s-1}$.

For example, if $\dim S_m(\Gamma) = \max (g(\Gamma) + 2, 3)$, then we can take $W = S_m(\Gamma)$ by general theory of algebraic curves [10, Corollary 3.4]. We recall that $\mathfrak{R}_\Gamma$ is hyperelliptic if $g(\Gamma) \geq 2$, and there is a degree two map onto $\mathbb{P}^1$. By general theory [7, Chapter VII, Proposition 1.10], if $g(\Gamma) = 2$, then $\mathfrak{R}_\Gamma$ is hyperelliptic. If $\mathfrak{R}_\Gamma$ is not hyperelliptic, then $\dim S_2(\Gamma) = g(\Gamma) \geq 3$, and we can take $W = S_2(\Gamma)$ using the fact that the holomorphic map $\mathfrak{R}_\Gamma \rightarrow \mathbb{P}^{g(\Gamma)-1}$ attached to a canonical divisor is an isomorphism (and in particular birational equivalence) onto its image [7, Chapter VII, Proposition 2.1], and the possibility to interpret cuspidal forms in $S_2(\Gamma)$ as holomorphic 1-forms on $\mathfrak{R}_\Gamma$ [8, Theorem 2.3.2].

Now, the first main result of the present paper is the following theorem:

**Theorem 1.4** Assume that $m \geq 2$ is an even integer. Let $W \subset M_m(\Gamma)$, $\dim W \geq 3$, be a subspace which determines the field of rational functions $\mathbb{C}(\mathfrak{R}_\Gamma)$ (see Definition 1.3). Let $f, g \in W$ be linearly independent. Then, there exists a non-empty Zariski open set $U \subset W$ such that for any $h \in U$ we have the following:

(i) $f, g, h$ are linearly independent;

(ii) $\mathfrak{R}_\Gamma$ is birationally equivalent to $\mathbb{C}(f, g, h)$ via the map (1.1).

Moreover, for each $h \in U$, the degree of $\mathbb{C}(f, g, h)$ is given by (1.2) with $d(f, g, h) = 1$. 

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We prove Theorem 1.4 in Sect. 3. For “generic” modular form $h$ stated in Theorem 1.4, the degree is always given by

$$\deg C(f, g, h) = \begin{cases} 
\dim M_m(\Gamma) + g(\Gamma) - 1, \\
\dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m \text{ if } f, g, h \in S_m(\Gamma).
\end{cases}$$

This was proved in Corollary 3.7. But as the referee suggested, the degree can be significantly lowered as we demonstrated by various results and examples in Sect. 5.

In Sect. 4 we prove the following corollary of Theorem 1.4. We recall that $g(\Gamma_0(N)) \geq 2$ unless

$$N \in \{1 - 10, 12, 13, 16, 18, 25\} \text{ when } g(\Gamma_0(N)) = 0,$$

$$N \in \{11, 14, 15, 17, 19 - 21, 24, 27, 32, 36, 49\} \text{ when } g(\Gamma_0(N)) = 1.$$ Let $g(\Gamma_0(N)) \geq 2$. Then, we remark that Ogg [13] has determined all $X_0(N)$ which are hyperelliptic curves. In view of Ogg’s paper, we see that $X_0(N)$ is not hyperelliptic for $N \in \{34, 38, 42, 43, 44, 45, 51 - 58, 60 - 70\}$ or $N \geq 72$. This implies $g(\Gamma_0(N)) \geq 3$.

**Corollary 1.5** Let $m \geq 2$ be an even integer. Assume that one of the following holds:

(A) $g(\Gamma_0(N)) \geq 1$, and $m \geq 4$ (if $N \neq 11$) or $m \geq 6$ (if $N = 11$);

(B) $X_0(N)$ is not hyperelliptic, and $m = 2$.

(In either case, $\dim S_m(\Gamma_0(N)) \geq 3$.) Let $f, g \in S_m(\Gamma_0(N))$ be linearly independent with integral $q$-expansions. Then, there exists infinitely many $h \in S_m(\Gamma_0(N))$ with integral $q$-expansion such that we have the following:

(i) $X_0(N)$ is birationally equivalent to $C(f, g, h)$ via the map (1.1), and

(ii) the reduced equation of $C(f, g, h)$ has integral coefficients up to a multiplication by a non-zero constant in $\mathbb{C}$.

Examples and improvements to Corollary 1.5 are included in Sect. 5 as we already mentioned above.

The rest of the paper is based on the other practical use of Theorem 1.4. The proof of Theorem 1.4 essentially is about the theoretical construction of primitive elements in the finite field extension $\mathbb{C}(g/f) \subset \mathbb{C}(\mathcal{O}_\Gamma)$. Methods used in the proofs of Theorem 1.4 and its Corollary 3.7 are great theoretical tools, but not fertile result-wise. Therefore, we looked for methods of determining primitive elements in a more direct way.

In Sect. 6 we discuss the special case of Theorem 1.4 when $\dim W = 4$. The approach is based on estimates based on the Primitive Element Theorem of finite separable field extensions in the form stated in [18, Sect. 6.10] adapted to our case via general Lemma 6.2, and estimates on absolute values of roots of polynomials (see Lemma 6.5), one of them is Mahler’s estimate [6]. The main results are Propositions 6.6 and 6.7. Proposition 6.6 is a general result, and Proposition 6.7 is a nice example for $W = S_4(\Gamma_0(14))$. The application of Proposition 6.7 is given by Corollary 6.8.

In Sect. 7 we adapt to our case the trial method, commonly used in the cases of algebraic number fields [20], where an element that is chosen from a certain subset of...
the field extension is tested for being primitive. We present a very efficient algorithm for computing model of $X_0(N)$ when $X_0(N)$ is not hyperelliptic and $g(Γ_0(N)) \geq 4$. We use $W = S_2(Γ_0(N))$. As an example, we consider the case $X_0(72)$.

We would like to thank the referee for suggestions on improvements of our results and methods.

2 Preliminaries

In this section, we recall necessary facts about modular forms and their divisors [8]. Let $H$ be the upper half-plane. Then, the group $SL_2(\mathbb{R})$ acts on $H$ as follows:

$$g.z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

We let $j(g, z) = cz + d$. The function $j$ satisfies the cocycle identity:

$$j(gg', z) = j(g, g'z)j(g', z). \quad (2.1)$$

Next, the $SL_2(\mathbb{R})$-invariant measure on $H$ is defined by $dx \, dy / y^2$, where the coordinates on $H$ are written in a usual way $z = x + \sqrt{-1}y$, $y > 0$. A discrete subgroup $Γ \subset SL_2(\mathbb{R})$ is called a Fuchsian group of the first kind if

$$\int\int_{Γ \backslash H} \frac{dx \, dy}{y^2} < \infty.$$

Then, adding a finite number of points in $\mathbb{R} \cup \{\infty\}$, called cusps, $\mathcal{F}_Γ$ can be compactified. In this way we obtain a compact Riemann surface $\mathfrak{R}_Γ$. One of the most important examples are the groups

$$Γ_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); \quad c \equiv 0 \pmod{N} \right\}, \quad N \geq 1.$$

We write $X_0(N)$ for $\mathfrak{R}_{Γ_0(N)}$.

Let $Γ$ be a Fuchsian group of the first kind. Let $m \geq 2$ be an even integer. We consider the space $M_m(Γ)$ (resp., $S_m(Γ)$) of all modular (resp., cuspidal) forms of weight $m$. We also need the following obvious property: for $f, g \in M_m(Γ)$, $g \neq 0$, the quotient $f/g$ is a meromorphic function on $\mathfrak{R}_Γ$.

Next, we recall from [8, Sect. 2.3] some notions related to the theory of divisors of modular forms of even weight $m \geq 2$ and state a preliminary result.

Let $m \geq 2$ be an even integer and $f \in M_m(Γ) - \{0\}$. Then, $v_{z-ξ}(f)$ denotes the order of the holomorphic function $f$ at $ξ$. For each $γ \in Γ$, the functional equation $f(γ. z) = j(γ, z)^m f(z)$, $z \in H$, shows that $v_{z-ξ}(f) = v_{z-ξ'}(f)$ where $ξ' = γ.ξ$.

Also, if we let

$$e_ξ = \#(Γ_ξ / Γ \cap \{±1\}),$$

where $Γ_ξ = \{ γ \in Γ \mid γ.ξ = ξ \}$. 

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then \( e_\xi = e_\xi' \), where \( \Gamma_\xi \) is the stabilizer of \( \xi \) in \( \Gamma \). The point \( \xi \in \mathbb{H} \) is elliptic if \( e_\xi > 1 \). Next, following [8, Sect. 2.3], we define

\[
\nu_\xi(f) = \nu_{z_-,\xi}(f)/e_\xi.
\]

Clearly, \( \nu_\xi = \nu_{\xi'} \), and we may let

\[
\nu_{a_\xi}(f) = \nu_\xi(f),
\]

where

\[
a_\xi \in \mathcal{R}_\Gamma \text{ is the projection of } \xi \text{ to } \mathcal{R}_\Gamma,
\]

a notation we use throughout this paper.

If \( x \in \mathbb{R} \cup \{ \infty \} \) is a cusp for \( \Gamma \), then we define \( \nu_x(f) \) as follows: Let \( \sigma \in SL_2(\mathbb{R}) \) such that \( \sigma \cdot x = \infty \). We write

\[
\{ \pm 1 \} \sigma \Gamma_x \sigma^{-1} = \{ \pm 1 \} \begin{pmatrix} 1 & lh' \\ 0 & 1 \end{pmatrix} ; \ l \in \mathbb{Z},
\]

where \( h' > 0 \). Then, we write the Fourier expansion of \( f \) at \( x \) as follows:

\[
(f|_m \sigma^{-1})(\sigma \cdot z) = \sum_{n=0}^{\infty} a_n e^{2\pi \sqrt{-1} \sigma \cdot z/h'}.
\]

We let

\[
\nu_x(f) = l \geq 0,
\]

where \( l \) is defined by \( a_0 = a_1 = \cdots = a_{l-1} = 0, a_l \neq 0 \). One easily see that this definition does not depend on \( \sigma \). In addition, if \( x' = \gamma \cdot x \), then \( \nu_{x'}(f) = \nu_x(f) \). Hence, if \( b_x \in \mathcal{R}_\Gamma \) is a cusp corresponding to \( x \), then we may define

\[
\nu_{b_x}(f) = \nu_x(f).
\]

Put

\[
\text{div}(f) = \sum_{a \in \mathcal{R}_\Gamma} \nu_a(f) a \in \mathbb{Q} \otimes \text{Div}(\mathcal{R}_\Gamma),
\]

where \( \text{Div}(\mathcal{R}_\Gamma) \) is the group of (integral) divisors on \( \mathcal{R}_\Gamma \).

Using [8, Sect. 2.3], this sum is finite i.e., \( \nu_a(f) \neq 0 \) for only a finitely many points. We let

\[
\deg(\text{div}(f)) = \sum_{a \in \mathcal{R}_\Gamma} \nu_a(f).
\]
Let $\vartheta_i \in \mathbb{Q} \otimes \text{Div}(\mathcal{R}_\Gamma)$, $i = 1, 2$. Then, we say that $\vartheta_1 \geq \vartheta_2$ if their difference $\vartheta_1 - \vartheta_2$ belongs to $\text{Div}(\mathcal{R}_\Gamma)$ and is non-negative in the usual sense.

**Lemma 2.2** Assume that $m \geq 2$ is an even integer. Assume that $f \in M_m(\Gamma)$, $f \neq 0$. Let $t$ be the number of inequivalent cusps for $\Gamma$. Then, we have the following:

(i) For $a \in \mathcal{R}_\Gamma$, we have $\nu_a(f) \geq 0$.

(ii) For a cusp $a \in \mathcal{R}_\Gamma$, we have that $\nu_a(f) \geq 0$ is an integer.

(iii) If $a \in \mathcal{R}_\Gamma$ is not an elliptic point or a cusp, then $\nu_a(f) \geq 0$ is an integer. If $a \in \mathcal{R}_\Gamma$ is an elliptic point, then $\nu_a(f) - \left\lfloor \frac{m}{2}(1 - 1/e_a) \right\rfloor$ is an integer.

(iv) Let $g(\Gamma)$ be the genus of $\mathcal{R}_\Gamma$. Then,

$$\deg(\text{div}(f)) = m(g(\Gamma) - 1) + \frac{m}{2} \left( t + \sum_{a \in \mathcal{R}_\Gamma, \text{ elliptic}} (1 - 1/e_a) \right)$$

$$= \frac{m}{4\pi} \int_\Gamma \frac{dx \, dy}{y^2}.$$

(v) Let $[x]$ denote the largest integer $\leq x$ for $x \in \mathbb{R}$. Then,

$$\dim S_m(\Gamma) = \begin{cases} 
(m - 1)(g(\Gamma) - 1) + \frac{m}{2}(1 - 1/e_a) & \text{if } m \geq 4, \\
\sum_{a \in \mathcal{R}_\Gamma, \text{ elliptic}} \left\lfloor \frac{m}{2}(1 - 1/e_a) \right\rfloor, & \text{if } m = 2,
\end{cases}$$

$$\dim M_m(\Gamma) = \begin{cases} 
\dim S_m(\Gamma) + t & \text{if } m \geq 4, \text{ or } m = 2 \text{ and } t = 0, \\
\dim S_m(\Gamma) + t - 1 = g(\Gamma) + t - 1 & \text{if } m = 2 \text{ and } t \geq 1.
\end{cases}$$

(vi) Let $c'_f$ be defined by

$$c'_f = \text{div}(f) - \sum_{a \in \mathcal{R}_\Gamma, \text{ elliptic}} \left( \frac{m}{2}(1 - 1/e_a) - \left\lfloor \frac{m}{2}(1 - 1/e_a) \right\rfloor \right) a.$$

Then, $c'_f$ is an integral effective divisor of degree

$$\begin{cases} 
\dim M_m(\Gamma) + g(\Gamma) - 1 & \text{if } m \geq 4, \text{ or } m = 2 \text{ and } t \geq 1, \\
2(g(\Gamma) - 1) & \text{if } m = 2 \text{ and } t = 0.
\end{cases}$$

(vii) Assume that $f \in S_m(\Gamma)$. Then, the integral divisor defined by $c_f = c'_f - \sum_{b \in \mathcal{R}_\Gamma, \text{ cusp}} b$ satisfies $c_f \geq 0$ and its degree is given by

$$\begin{cases} 
\dim S_m(\Gamma) + g(\Gamma) - 1 & \text{if } m \geq 4, \\
2(g(\Gamma) - 1) & \text{if } m = 2.
\end{cases}$$
The claims (i)–(v) are standard [8, Sects. 2.3, 2.4, 2.5]. The claim (vi) follows from (iii), (iv), and (v) (see [9, Lemma 4-1]). Finally, (vii) follows from (vi).

3 The proof of Theorem 1.4 and a corollary

Proof We begin the proof of Theorem 1.4 with the following lemma:

Lemma 3.1 Assume that \( m \geq 2 \) is an integer. Let \( W \subset M_m(\Gamma) \), \( \dim W \geq 3 \), be a subspace which determines the field of rational functions \( \mathbb{C}(\mathcal{R}_\Gamma) \) (see Definition 1.3). Let \( f, g \in W \) be linearly independent. Then, there exists a non-empty Zariski open set \( U \subset W \) such that for \( h \in U \) we have the following:

(i) \( f, g, \) and \( h \) are linearly independent;
(ii) the field of rational functions \( \mathbb{C}(\mathcal{R}_\Gamma) \) is generated over \( \mathbb{C} \) by \( g/f \) and \( h/f \).

Proof We select a basis \( f_0, \ldots, f_{s-1} \) of \( W \) such that \( f = f_0 \) and \( g = f_1 \). By the assumption on \( W \), the field of rational functions \( \mathbb{C}(\mathcal{R}_\Gamma) \) is generated over \( \mathbb{C} \) by all \( f_i/f_0, 1 \leq i \leq s - 1 \). We let

\[
K = \mathbb{C}(f_1/f_0),
\]
and

\[
L = \mathbb{C}(\mathcal{R}_\Gamma) = \mathbb{C}(f_1/f_0, \ldots, f_{s-1}/f_0) = K(f_2/f_0, \ldots, f_{s-1}/f_0).
\]

Since \( L \) has transcendence degree 1 over \( \mathbb{C} \), \( f_2/f_0, \ldots, f_{s-1}/f_0 \) are all algebraic over \( K \). It is also obviously separable.

Lemma 3.2 There exists \( (\lambda_2, \ldots, \lambda_{s-1}) \in \mathbb{C}^{s-2} \) such that \( (\lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1})/f_0 \) generates \( L \) over \( K \), i.e.,

\[
L = K((\lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1})/f_0) = \mathbb{C}(f_1/f_0, (\lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1})/f_0).
\]

Proof Since \( \mathbb{C} \) is a subfield of \( K \), this follows using a variant of a proof of Primitive Element Theorem given by [18, Sect. 6.10] (see also the second paragraph in Sect. 6).

Now, we explain a systematic way to get them all. Let us fix an algebraic closure \( \overline{K} \) of \( K \) containing \( L \). We consider the polynomial ring \( K[X] \) in variable \( X \). For \( x \in L \), we define a \( K \)-linear endomorphism \( T_x(y) = xy \), and attach usual invariants from elementary Linear algebra: the minimal polynomial, say \( \mu(X, x) \in K[X] \), and characteristic polynomial \( k(X, x) = \det(X \cdot Id_L - T_x) \in K[X] \), where \( Id_L \) is identity on \( L \). The degree of \( k(X, x) \) is \( [L : K] \).

By elementary field theory, \( \mu(X, x) \) is also a unique monic irreducible polynomial of \( x \) over \( K \). Therefore, the roots in \( \overline{K} \) of \( \mu(X, x) \) are all simple. In addition, by elementary Linear algebra, \( \mu(X, x) \) and \( k(X, x) \) have the same set of roots in \( \overline{K} \), and the multiplicity of each root of \( \mu(X, x) \) is less than or equal to the multiplicity of the same root in \( k(X, x) \). This immediately implies

\[\mathbb{C}\]
Lemma 3.3 Let \( x \in L \). Then, \( L = K(x) \) if and only if all roots in \( \overline{K} \) of \( k(X, x) \) are simple.

Proof By elementary field theory, \( L = K(x) \) if and only if the degree of \( \mu(X, x) \) is \([L : K]\). By above discussion, this is equivalent to the fact that \( \mu(X, x) = k(X, x) \) since both polynomials are monic, and \( \mu(X, x) \) divides \( k(X, x) \). Again, by above considerations, this is equivalent to the fact that all roots in \( \overline{K} \) of \( k(X, x) \) are simple. \( \square \)

For \((\lambda_2, \ldots, \lambda_{s-1}) \in \mathbb{C}^{s-2}\), we consider the characteristic polynomial

\[
P(X, \lambda_2, \ldots, \lambda_{s-1}) \overset{\text{def}}{=} k(X, (\lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1})/f_0).
\]

The discriminant \( R \) of \( P(X, \lambda_2, \ldots, \lambda_{s-1}) \) with respect to \( X \) i.e., the resultant with respect to the variable \( X \) of the polynomial \( P(X, \lambda_2, \ldots, \lambda_{s-1}) \) and its derivative \( \frac{\partial}{\partial \lambda} P(X, \lambda_2, \ldots, \lambda_{s-1}) \) is a polynomial in \( \lambda_2, \ldots, \lambda_{s-1} \) with coefficients in \( K \). We remark that the degree of \( P(X, \lambda_2, \ldots, \lambda_{s-1}) \) is \([L : K] \geq 2\), and of \( \frac{\partial}{\partial \lambda} P(X, \lambda_2, \ldots, \lambda_{s-1}) \) is \([L : K] - 1 \geq 1\). Consequently, both depend on \( X \) as it is required in the definition of the resultant.

Lemma 3.4 The discriminant \( R \) is not identically equal to zero. Moreover, for \((\lambda_2, \ldots, \lambda_{s-1}) \in \mathbb{C}^{s-2}\), \( R(\lambda_2, \ldots, \lambda_{s-1}) \neq 0 \) if and only if \((\lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1})/f_0\) generates \( L \) over \( K \).

Proof The last claim follows from Lemma 3.3 and above definition of \( R \). The first claim follows from the last, and Lemma 3.2. \( \square \)

Still, the discriminant is a polynomial in variables \( \lambda_2, \ldots, \lambda_{s-1} \) with coefficients in \( K \). We recall that elements of the field \( K \) are rational functions on \( \mathcal{M} \). So, to obtain a polynomial with coefficients in \( \mathbb{C} \), we write \( \mathcal{P} \) for the (finite) set of all poles of all non-zero coefficients of \( R \). Then, for \( a \in \mathcal{M} \setminus \mathcal{P}, R(\lambda_2, \ldots, \lambda_{s-1})(a) \) is a polynomial in variables \( \lambda_2, \ldots, \lambda_{s-1} \) with coefficients in \( \mathbb{C} \). Obviously, for \((\lambda_2, \ldots, \lambda_{s-1}) \in \mathbb{C}^{s-2}, R(\lambda_2, \ldots, \lambda_{s-1}) \neq 0 \) is equivalent to the fact that there exists \( a \in \mathcal{M} \setminus \mathcal{P} \) such that \( R(\lambda_2, \ldots, \lambda_{s-1})(a) \neq 0 \). Thus, the condition

\[
R(\lambda_2, \ldots, \lambda_{s-1}) \neq 0 \tag{3.5}
\]

defines a Zariski open set in \( \mathbb{C}^{s-2} \).

Also, by Lemma 3.3, if \((\lambda_2, \ldots, \lambda_{s-1}) \) belongs to that Zariski open set, then

\[
h \overset{\text{def}}{=} \lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1} \in \mathbb{C} f_2 \oplus \cdots \oplus \mathbb{C} f_{s-1}
\]

generates \( L \) over \( K \). It does not affect the thing if we enlarge \( h \) to be

\[
h = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_{s-1} f_{s-1},
\]
where $\lambda_0, \lambda_1$ are arbitrary complex numbers. This means that $h$ can be selected from the Zariski open subset of $W$ given by (3.5) in coordinates

$$ W = \mathbb{C}f_0 \oplus \cdots \oplus \mathbb{C}f_{s-1}. $$

We consider the discriminant $R$ as a polynomial of all variables $\lambda_0, \ldots, \lambda_{s-1}$ but which does not depend on the first two variables. This completes the proof of the lemma. $\square$

Now, to complete the proof of Theorem 1.4, we need to prove the formula for the degree of $C(f, g, h)$. But, since by Lemma 3.1 the curve $C(f, g, h)$ is birational to $\mathcal{R}_\Gamma$, we have $d(f, g, h) = 1$ as explained in the introduction. Finally, the formula for the degree follows from [11] as we explained in the introduction before the statement of Theorem 1.4. $\square$

In the following corollary to Theorem 1.4 we need the next definition.

**Definition 3.6** Assume that $m \geq 2$ is an even integer. Let $W \subset M_m(\Gamma)$, $W \neq 0$, be a linear subspace. We say that $W$ is a base-point-free if one of the following holds:

(i) $W \nsubseteq S_m(\Gamma)$, and, for each $a \in \mathcal{R}_\Gamma$, there exists $f \in W$, $f \neq 0$, such that $c_f'(a) = 0$ (see Lemma 2.2 (vi) for notation), or

(ii) $W \subset S_m(\Gamma)$, and, for each $a \in \mathcal{R}_\Gamma$, there exists $f \in W$, $f \neq 0$, such that $c_f(a) = 0$ (see Lemma 2.2 (vii) for notation).

For example, $\dim S_m(\Gamma) \geq \max (g(\Gamma) + 2, 3)$, which implies $m \geq 4$, then we can take $W = S_m(\Gamma)$ by general theory of algebraic curves (see [10, Theorem 3.3]). Also, if $g(\Gamma) \geq 3$, then $W = S_2(\Gamma)$ is a base-point-free using isomorphism of $S_2(\Gamma)$ with the space of holomorphic differential forms on $\mathcal{R}_\Gamma$ [8, Theorems 2.3.2 and 2.3.3], and the fact that the corresponding canonical linear system is a base-point-free [7, Chapter VII, Lemma 1.14].

**Corollary 3.7** Assume that $m \geq 2$ is an even integer. Let $W \subset M_m(\Gamma)$, $\dim W \geq 3$, be a subspace which determines the field of rational functions $\mathbb{C}(\mathcal{R}_\Gamma)$ (see Definition 1.3), and is a base-point-free (see Definition 3.6). Let $f, g \in W$ be linearly independent. Then, there exists a non-empty Zariski open set $V \subset W$ such that for any $h \in V$ we have the following:

$$ \deg C(f, g, h) = \begin{cases} \dim M_m(\Gamma) + g(\Gamma) - 1 & \text{if } W \nsubseteq S_m(\Gamma), \\
\dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m & \text{if } W \subset S_m(\Gamma). \end{cases} \quad (3.8) $$

**Proof** We consider the case $W \subset S_m(\Gamma)$. The other one is analogous. Let us fix a basis $f_0, \ldots, f_{s-1}$ of $W$ such that $f = f_0$ and $g = f_1$. For $f, g \in W$, let us fix a Zariski open subset $U$ such that the conclusion of Theorem 1.4 holds. Since $W$ is a base-point-free, for each $a \in \text{supp}(c_{f_0})$, there exists $i_a \in \{1, \ldots, s - 1\}$ such that $a \notin \text{supp}(c_{f_{i_a}})$. Then, the rational functions $f_i / f_{i_a}$ are defined at $a$ since we have the following (see Lemma 2.2 (vi)):

$$ \text{div} \left( \frac{f_i}{f_{i_a}} \right) = \text{div}(f_i) - \text{div}(f_{i_a}) = c_{f_i} - c_{f_{i_a}}, $$

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where the rightmost is the difference of two effective divisors, so that the point \( a \) does not belong to the divisors of poles because of \( a \notin \text{supp}(c_i f_{ia}) \).

Now, we can form the following product of non-zero linear forms in \((\lambda_0, \ldots, \lambda_{s-1}) \in \mathbb{C}^s:\)

\[
\prod_{a \in \text{supp}(c_{f_0})} \left( \lambda_0 \frac{f_0}{f_{ia}}(a) + \lambda_1 \frac{f_1}{f_{ia}}(a) + \cdots + \lambda_{s-1} \frac{f_{s-1}}{f_{ia}}(a) \right).
\] (3.9)

Let \( \mathcal{U}' \subset W \) be the Zariski open subset \( \mathcal{U}' \subset W \) consisting of all \( \sum_{i=0}^{s-1} \lambda_i f_i \in W \) such that the product in (3.9) is not equal to zero. For \( \sum_{i=0}^{s-1} \lambda_i f_i \in \mathcal{U}' \), neither of \( a \in \text{supp}(c_{f_0}) \) belong to the divisor of zeros \( \text{div}_0 \left( \left( \sum_{i=0}^{s-1} \lambda_i f_i \right)/f_{ia} \right) \) of the corresponding rational function. Since \( a \notin \text{supp}(c_{f_ia}) \), and

\[
\text{div}_0 \left( \sum_{i=0}^{s-1} \lambda_i f_i \right) - \text{div}_\infty \left( \sum_{i=0}^{s-1} \lambda_i f_i \right) = \text{div} \left( \sum_{i=0}^{s-1} \lambda_i f_i \right) = c_{\sum_{i=0}^{s-1} \lambda_i f_i} - c_{f_{ia}},
\]

where the rightmost expression is the difference of two effective divisors, we obtain

\[
a \in \text{supp}(c_{f_0}) \implies a \notin \text{supp}(c_{\lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_{s-1} f_{s-1}}).
\] (3.10)

Finally, we define the Zariski open subset \( \mathcal{U} \) by \( \mathcal{V} = \mathcal{U} \cap \mathcal{U}' \). For \( h \) defined by \( h = \sum_{i=0}^{s-1} \lambda_i f_i \) in \( \mathcal{V} \), we have

\[
\deg C(f, g, h) = \dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m - \sum_{a \in \mathfrak{R}_F} \min \left( \epsilon_f(a), \epsilon_g(a), \epsilon_h(a) \right)
\]

by Theorem 1.4. Now, since \( f = f_0 \), using (3.10), we obtain

\[
\sum_{a \in \mathfrak{R}_F} \min \left( \epsilon_f(a), \epsilon_g(a), \epsilon_h(a) \right) = 0,
\]

proving the corollary.

\[\square\]

4 Proof of Corollary 1.5

**Proof** In the proof of Corollary 1.5 we use the following notation: \( \nu_{\infty}(\Gamma_0(N)) \) is the number of inequivalent cusps, \( \nu_2(\Gamma_0(N)) \) (resp., \( \nu_3(\Gamma_0(N)) \)) is the number of inequivalent elliptic points of order 2 (resp. 3) of the congruence subgroup \( \Gamma_0(N) \).

We begin with the proof of Corollary 1.5 assuming that (A) holds (see the statement of Corollary 1.5). First, we need to assure that \( W = S_m(\Gamma_0(N)) \) determines the field of rational functions \( \mathbb{C}(X_0(N)) \) (see Theorem 1.4). By general theory of algebraic curves [10, Corollary 3.7], it is enough to require

\[
\dim S_m(\Gamma_0(N)) \geq \max (g(\Gamma_0(N)) + 2, 3).
\]
Since we assume that
\[ g(\Gamma_0(N)) \geq 1, \]
We only need to require that
\[ \dim S_m(\Gamma_0(N)) \geq g(\Gamma_0(N)) + 2. \]
Using Lemma 2.2 (v), we obtain
\[ \dim S_m(\Gamma_0(N)) = (m - 1)(g(\Gamma_0(N)) - 1) + \left( \frac{m}{2} - 1 \right) v_{\infty}(\Gamma_0(N)) + \left\lceil \frac{m}{4} \right\rceil v_2(\Gamma_0(N)) + \left\lceil \frac{m}{3} \right\rceil v_3(\Gamma_0(N)). \]
By [8, Theorem 4.2.7], we have
\[ v_{\infty}(\Gamma_0(N)) = \sum_{d > 0, d|N} \phi((d, N/d)) \geq 3 \]
unless \( N \) is prime number in which case \( v_{\infty}(\Gamma_0(N)) = 2 \). Next, unless \( v_2(\Gamma_0(N)) = v_3(\Gamma_0(N)) = 0 \), above formula shows that for \( m = 4 \) we have
\[ \dim S_4(\Gamma_0(N)) \geq 3(g(\Gamma_0(N)) - 1) + 2 \left( \frac{4}{2} - 1 \right) + 1 = 3g(\Gamma_0(N)) \geq g(\Gamma_0(N)) + 2, \]
since we assume that \( g(\Gamma_0(N)) \geq 1 \). Then, for an even \( m \geq 6 \), we have
\[ \dim S_m(\Gamma_0(N)) \geq \dim (S_2(\Gamma_0(N)) \cdot S_{m-2}(\Gamma_0(N))) \geq \cdots \geq \dim (S_2(\Gamma_0(N)) \cdot S_4(\Gamma_0(N)) \geq \dim S_4(\Gamma_0(N)) \geq g(\Gamma_0(N)) + 2. \]
Similarly, the same inequality holds if \( v_2(\Gamma_0(N)) = v_3(\Gamma_0(N)) = 0 \) but \( N \) is not a prime. It remains to consider the case \( N \) is a prime and \( v_2(\Gamma_0(N)) = v_3(\Gamma_0(N)) = 0 \). In this case
\[ \dim S_4(\Gamma_0(N)) = 3g(\Gamma_0(N)) - 1 \geq g(\Gamma_0(N)) + 2 \]
if and only if \( g(\Gamma_0(N)) \geq 2 \). It remains to consider the case \( N \) is prime, \( v_2(\Gamma_0(N)) = v_3(\Gamma_0(N)) = 0 \), and \( g(\Gamma_0(N)) = 1 \). In this case [8, Theorem 4.2.11] gives us \([SL_2(\mathbb{Z}) : \Gamma_0(N)] = 12\). Applying [8, Theorem 4.2.5], we see that \( \psi(N) = N + 1 = 12 \) since \( N \) is prime. Hence, \( N = 11 \). In this case, we use [8, Theorem 4.2.5] to check that we indeed have \( v_2(\Gamma_0(11)) = v_3(\Gamma_0(11)) = 0 \) and \( g(\Gamma_0(11)) = 1 \). This gives us \( \dim S_4(\Gamma_0(11)) = 2 \) and
\[ \dim S_6(\Gamma_0(11)) = 4 > 3 = g(\Gamma_0(6)) + 2. \]
Again, for an even $m \geq 8$, we have
\[
\dim S_m(\Gamma_0(11)) \geq \dim (S_2(\Gamma_0(11) \cdot S_{m-2}(\Gamma_0(11))) \geq \cdots \\
\geq \dim (S_2(\Gamma_0(11) \cdot S_6(\Gamma_0(11))) \\
\geq \dim S_6(\Gamma_0(11)) \geq g(\Gamma_0(11)) + 2.
\]

Above considerations show that we can apply Theorem 1.4. Next, by Eichler–Shimura theory [15, Theorem 3.5.2], for each even integer $m \geq 2$ the space of cusp forms $S_m(\Gamma_0(N))$ has a basis as a complex vector space consisting of forms which have integral $q$-expansions. So, if we have $f, g \in S_m(\Gamma_0(N))$ with integral coefficients in their $q$-expansions, then we can select infinitely many $h$ which also have integral coefficients in their $q$-expansions in the set $U$ for $f$ and $g$ (see Theorem 1.4). This is because $\mathbb{Z}^l$ is Zariski dense in $\mathbb{C}^l$ for any $l \geq 1$. This proves (i).

To prove (ii), we write a reduced equation of $C(f, g, h)$ as follows:
\[
P = \sum_{\alpha=(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3_{\geq 0}, \quad |\alpha| \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \alpha_3 = l} a_{\alpha} x_0^{\alpha_1} x_1^{\alpha_2} x_2^{\alpha_3},
\]
where $x_0, x_1,$ and $x_2$ are variables, coefficients $a_{\alpha} \in \mathbb{C}$, and $l = \deg C(f, g, h)$.

Next, since $f, g, h \in S_m(\Gamma_0(N))$, we have that
\[
f^{\alpha_1} g^{\alpha_2} h^{\alpha_3} \in S_{lm}(\Gamma_0(N)),
\]
and those forms have integral $q$-expansions for all $\alpha \in \mathbb{Z}^3_{\geq 0}$ such that $|\alpha| = l$. Let us write their $q$-expansions as follows:
\[
f^{\alpha_1}(z) g^{\alpha_2}(z) h^{\alpha_3}(z) = \sum_{n=1}^{\infty} b_n^{\alpha} q^n.
\]
Then,
\[
P(f(z), g(z), h(z)) = 0, \quad \text{for all } z \in \mathbb{H}
\]
is equivalent to an infinite system of homogeneous equations for coefficients $a_{\alpha}$ given by
\[
\sum_{\alpha=(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3_{\geq 0}, \quad |\alpha| \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \alpha_3 = l} b_n^{\alpha} a_{\alpha} = 0 \quad n \geq 1.
\]
By elementary theory of algebraic curves, this homogeneous system has, up to a multiplication by a non-zero constant in $\mathbb{C}$, unique solution in complex numbers for coefficients $a_\alpha$. By the theory of homogeneous systems, this means that its rank is $l - 1$. But coefficients $b_\alpha^m$ are all integers, and the system has rank $l - 1$. This means that the homogeneous system has, up to multiplication by a non-zero constant in $\mathbb{Q}$, unique solution in integers for coefficients $a_\alpha$. This proves (ii). This completes the proof of Corollary 1.5.

The proof of Corollary 1.5 assuming (B) is very similar. First, we have

$$\dim S_2(\Gamma_0(N)) = g(\Gamma_0(N)) \geq 3.$$

Also, since $X_0(N)$ is not hyperelliptic, $W = S_2(\Gamma_0(N))$ determines the field of rational functions $\mathbb{C}(X_0(N))$ by general theory of algebraic curves [7, Chapter VII.2, Proposition 2.1]. Now, we complete the proof in the same way as we completed the proof of Corollary 1.5 assuming (A).

5 Examples and improvements

In this section, we give examples of equations of the corresponding curves using SAGE and compute degrees. It is demonstrated how useful is the theory developed in [11] especially the test for birational equivalence stated in the Introduction of [11].

The following example came out from the remarks of the referee. The $q$-expansions are computed using SAGE.

**Proposition 5.1** Consider three linearly independent forms from the four dimensional space $S_4(\Gamma_0(14))$ of cusp forms of weight four for $\Gamma_0(14)$:

$$f = q^2 - 2q^5 - 2q^6 + q^7 - 6q^8 + 12q^{10} + 4q^{11} + 2q^{13} - 5q^{14} + 4q^{15} + 10q^{16} + \cdots,$$

$$g = q^3 - q^5 - 2q^6 - q^7 - 4q^8 + 6q^9 + 10q^{10} - 6q^{11} + 4q^{12} - 3q^{13} - 2q^{14} + \cdots,$$

$$h = q^4 - 2q^5 + q^7 + q^8 - 4q^{10} + 4q^{11} - 2q^{12} + 2q^{13} + 2q^{14} + 4q^{15} - 5q^{16} + \cdots.$$

Then, the map (1.1) is a birational equivalence of $X_0(14)$ and $\mathcal{C}(f, g, h)$. Moreover, $\deg \mathcal{C}(f, g, h) = 3$.

**Proof** Let $a_\infty$ be the $\Gamma_0(14)$-orbit of the cusp $\infty$. Since the forms have at least double zero at $a_\infty$, and $f$ has exactly double zero, we have

$$\sum_{a \in X_0(14)} \min(\epsilon_f(a), \epsilon_g(a), \epsilon_h(a)) \geq \min(\epsilon_f(a_\infty), \epsilon_g(a_\infty), \epsilon_h(a_\infty)) = 1.$$
Now, in view of (1.2), we have
\[ 1 \leq d(f, g, h) \cdot \deg C(f, g, h) \leq \dim S_4(\Gamma_0(14)) + g(\Gamma_0(14)) - 1 - \epsilon_4 - 1 = 3 \] (5.2)
since
\[ g(\Gamma_0(14)) = 1. \]

Using (5.2), we must have
\[ \deg C(f, g, h) \in \{1, 2, 3\}. \]

But \( \deg C(f, g, h) = 1 \) means that \( C(f, g, h) \) is a line which is clearly impossible since \( f, g, \) and \( h \) are linearly independent. The case \( \deg C(f, g, h) = 2 \) means that \( C(f, g, h) \) is an irreducible conic. Using
\[ 2d(f, g, h) = d(f, g, h) \cdot \deg C(f, g, h) \leq 3, \]
we must have
\[ d(f, g, h) = 1. \]

This means that \( X_0(14) \) is birationally equivalent to the conic \( C(f, g, h) \). But irreducible conic is non-singular. This means that \( X_0(14) \) isomorphic to a conic. This is a contradiction since conic has genus 0 while \( X_0(14) \) has genus 1.

Thus, \( \deg C(f, g, h) = 3. \) Consequently, \( d(f, g, h) = 1 \) proving the proposition. \(\Box\)

We were also informed by the referee that
\[ g^3 + 3h^3 + f^2h - fg^2 + gh^2 + fgh = 0. \]

We remark that Proposition 5.1 implies that the polynomial
\[ P = x_1^3 + 3x_2^2 + x_0x_2 - x_0x_1^2 + x_1x_2 + x_0x_1x_2 \] (5.3)
is irreducible. The equation has been computed using SAGE.

Using other three elements of the basis for \( S_4(\Gamma_0(14)) \) we obtain the following result:

**Proposition 5.4** Consider three linearly independent forms from the four dimensional space \( S_4(\Gamma_0(14)) \) of cusp forms of weight four for \( \Gamma_0(14) \):

\[ f = q - 2q^5 - 4q^6 - q^7 + 8q^8 - 11q^9 - 12q^{10} + 12q^{11} + 8q^{12} + 38q^{13} \ldots, \]
\[ g = q^2 - 2q^5 - 2q^6 + q^7 - 6q^8 + 12q^{10} + 4q^{11} + 2q^{13} - 5q^{14} + 4q^{15} + \ldots, \]
\[ h = q^3 - q^5 - 2q^6 - q^7 - 4q^8 + 6q^9 + 10q^{10} - 6q^{11} + 4q^{12} - 3q^{13} + \cdots. \]

Then, the map \((1.1)\) is a birational equivalence of \(X_0(14)\) and \(C(f, g, h)\). Moreover, \(\deg C(f, g, h) = 4\).

**Proof** This case is different than previous one since now \(f\) does not have a double zero at \(a_{\infty}\). Consequently, we do not know anything about
\[
\sum_{a \in X_0(14)} \min (c_f(a), c_g(a), c_h(a))
\]
besides it is \(\geq 0\). Now, in view of \((1.2)\), we have
\[
1 \leq d(f, g, h) \cdot \deg C(f, g, h) \leq \dim S_4(\Gamma_0(14)) + g(\Gamma_0(14)) - 1 - \epsilon_4 - 1 = 4.
\]
This implies that
\[
\deg C(f, g, h) \leq 4.
\]
Using SAGE we compute that \(P(f, g, h) = 0\), where
\[
P = -3x_0^2x_1^2 - 6x_0x_1^3 - 4x_1^4 + 3x_0^3x_2 + 6x_0^2x_1x_2 - 3x_0x_1^2x_2 - 2x_1^3x_2 \\
+ 10x_0^2x_2^2 + 2x_0x_1x_2^2 - 21x_1^2x_2^2 + 23x_0x_3^3 + 16x_1x_3^2 + 11x_2^4. \tag{5.5}
\]
In addition, using SAGE system, we checked that this polynomial is irreducible. Hence,
\[
\deg C(f, g, h) = 4.
\]
Consequently, we have
\[
4d(f, g, h) = d(f, g, h) \cdot \deg C(f, g, h) \leq 4.
\]
Hence, we have
\[
d(f, g, h) = 1.
\]
This proves the proposition. \(\Box\)

**Proposition 5.6** Let \(N \geq 1\) such that \(g(\Gamma_0(N)) \geq 3\). Then, there exists \(f, g,\) and \(h\) linearly independent in \(S_2(\Gamma_0(N))\) such that \(d(f, g, h) \cdot \deg C(f, g, h) \leq g(\Gamma_0(N)) + 1\). In particular, \(\deg C(f, g, h) \leq g(\Gamma_0(N)) + 1\). Moreover, if \((g(\Gamma_0(N)) + 1)/2 < \deg C(f, g, h)\), then \(X_0(N)\) is birational to \(C(f, g, h)\) via the map \((1.1)\).
Proof We use standard elementary argument (see for example [9, Lemma 4.3]). We let

$$W_i = \{ f \in S_2(\Gamma_0(N)); f = 0 \text{ or } \nu_{a_{\infty}}(f) \geq i \} \quad (5.7)$$

for all $i \geq 1$. Then, all $W_i$ are linear subspaces of $S_2(\Gamma_0(N))$. Moreover,

$$S_2(\Gamma_0(N)) = W_1 \supset W_2 \supset W_3 \supset \cdots,$$

where $W_i = W_{i+1}$, or $W_{i+1}$ is of codimension one in $W_i$ for all $i \geq 1$. Now, since $\dim S_2(\Gamma_0(N)) = g(\Gamma_0(N))$, by counting dimensions, we see that

$$\dim W_{g(\Gamma_0(N)) - 2} \geq 3.$$

Let us select linearly independent $f, g, h \in W_{g(\Gamma_0(N)) - 2}$. Then we obtain

$$\sum_{a \in X_0(N)} \min (\epsilon_f(a), \epsilon_g(a), \epsilon_h(a)) \geq \min (\epsilon_f(a_{\infty}), \epsilon_g(a_{\infty}), \epsilon_h(a_{\infty})) \geq g(\Gamma_0(N)) - 3.$$

Finally, we have (see (1.2))

$$d(f, g, h) \cdot \deg C(f, g, h) = 2g(\Gamma_0(N)) - 2 - \sum_{a \in X_0(N)} \min (\epsilon_f(a), \epsilon_g(a), \epsilon_h(a)) \leq 2g(\Gamma_0(N)) - 2 - (g(\Gamma_0(N)) - 3) = g(\Gamma_0(N)) + 1.$$

This proves the first claim of the proposition. Other claims follow from this immediately. \qed

We give the following corollaries to Proposition 5.6:

**Corollary 5.8** Let $N = 63$. Then, $g(\Gamma_0(63)) = 5$. A computation of the basis of $S_2(\Gamma_0(63))$ in SAGE implies that the basis of $W_3$ (see (5.7)) is given by

$$f \overset{\text{def}}{=} q^3 - q^6 + q^9 - q^{12} - 2q^{15} - q^{18} - q^{21} + 3q^{24} + \cdots,$$

$$g \overset{\text{def}}{=} q^4 + q^7 - 4q^{10} + 2q^{13} - 2q^{16} - 4q^{19} + 5q^{22} + \cdots,$$

$$h \overset{\text{def}}{=} 2q^5 - q^8 - 3q^{11} - q^{14} + 2q^{17} + q^{23} + \cdots.$$

The curve $X_0(63)$ is birational to the curve $C(f, g, h)$ via the map (1.1). The curve the curve $C(f, g, h)$ has degree $\deg C(f, g, h) = g(\Gamma_0(N)) + 1 = 6$, and it is defined via irreducible polynomial

$$-2x_0^4x_1^2 - x_0x_1^5 + x_0^5x_2 + 2x_0^2x_1^3x_2 + x_0^3x_1x_2^2 - x_1^4x_2^2 + 3x_0x_1^2x_2^3 - 3x_0^2x_2^4.$$

Proof The equation of the curve is computed in SAGE. Except indicated computations in SAGE, the claim follows from Proposition 5.6. \qed
Corollary 5.9 Let \( N = 93 \). Then, \( g(\Gamma_0(93)) = 9 \). A computation of the basis of \( S_2(\Gamma_0(93)) \) in SAGE implies that the basis of \( W_7 \) (see (5.7)) is given by
\[
\begin{align*}
f &\overset{\text{def}}{=} q^7 + q^8 + 2q^9 - 4q^{10} - q^{11} + 2q^{12} + 3q^{13} - \cdots, \\
g &\overset{\text{def}}{=} 2q^8 + 2q^9 - 6q^{10} + 3q^{12} + 5q^{13} - 4q^{14} - 6q^{15} - \cdots, \\
h &\overset{\text{def}}{=} 4q^9 - 4q^{10} - 3q^{11} - q^{12} + q^{13} - 3q^{15} + 2q^{16} + \cdots.
\end{align*}
\]
The curve \( X_0(93) \) is birational to the curve \( C(f, g, h) \) via the map (1.1). The curve \( C(f, g, h) \) has degree \( \deg C(f, g, h) = g(\Gamma_0(N)) + 1 = 10 \), and it is defined via irreducible polynomial
\[-30000x_0^8x_1^2 + 172400x_0^7x_1^3 - \cdots + 14065x_0^9x_2^3 + 355x_1x_2^9 - 1825x_2^{10}.\]
**Proof** The equation of the curve is computed in SAGE. Except indicated computations in SAGE, the claim follows from Proposition 5.6. \(\square\)

Corollary 5.10 Let \( N = 110 \). Then, \( g(\Gamma_0(110)) = 15 \). A computation of the basis of \( S_2(\Gamma_0(110)) \) in SAGE implies that the basis of \( W_{13} \) (see (5.7)) is given by
\[
\begin{align*}
f &\overset{\text{def}}{=} q^{13} + q^{14} - 3q^{16} - 5q^{18} + 5q^{20} - 2q^{21} - q^{22} - \cdots, \\
g &\overset{\text{def}}{=} 2q^{14} - 3q^{16} - q^{17} - 6q^{18} + q^{19} + 6q^{20} - 3q^{21} - \cdots, \\
h &\overset{\text{def}}{=} 3q^{15} + 4q^{16} - 4q^{17} + 7q^{18} + 5q^{19} - 2q^{20} - q^{21} + \cdots.
\end{align*}
\]
The curve \( X_0(110) \) is birational to the curve \( C(f, g, h) \) via the map (1.1). The curve \( C(f, g, h) \) has degree \( \deg C(f, g, h) = 15 < g(\Gamma_0(N)) + 1 = 16 \), and it is defined via irreducible polynomial
\[
\begin{align*}
&- 198700267941x_0^{13}x_1^2 + 1714521491172x_0^{12}x_1^3 - \cdots + 48120x_0^9x_2^{13} \\
&- 91118x_0x_1x_2^{13} + 43558x_1^2x_2^{13} + 173x_0x_2^{14} - 138x_1x_2^{14} + x_2^{15}.
\end{align*}
\]
**Proof** The equation of the curve is computed in SAGE. Except indicated computations in SAGE, the claim follows from Proposition 5.6. \(\square\)

Let us explain conjectural generalization of above corollaries. We say that \( \Gamma_0(N) \)-orbit \( a_\infty = \Gamma_0(N)\cdot \infty \) is a Weierstrass point for \( X_0(N) \) if \( g(\Gamma_0(N)) \geq 2 \), and there exists a non-zero \( f \in S_2(\Gamma_0(N)) \) such that \( v_{a_\infty}(f) \geq g(\Gamma_0(N)) + 1 \). This is a particular case of the much more general definition of a Weierstrass point on a compact Riemann surface [14, Definition 6.1]. By the same reference, if \( a_\infty \) is not a Weierstrass point, then there exists a basis \( h_1, \ldots, h_g \) of \( S_2(\Gamma_0(N)) \) such that \( v_{a_\infty}(h_i) = i \) for \( 1 \leq i \leq g \). We have that \( W_g(\Gamma_0(N)) \) (see (5.7)) has a basis \( h_g-2, h_g-1, h_g \). Obviously, computing the base of \( S_2(\Gamma_0(N)) \) in SAGE system it is easy to check whether or not \( a_\infty \) is a Weierstrass point on \( X_0(N) \). Using this method, one check that \( a_\infty \) is not a Weierstrass point on \( X_0(63), X_0(93) \), and \( X_0(110) \) (see Corollaries 5.8, 5.9, and
Remark that the conditions \( g(\Gamma_0(N)) = 3 \), \( a_\infty \) is not a Wie- restrass point for \( X_0(N) \), and \( X_0(N) \) is not hyperelliptic imply \( N \in \{ 34, 43, 45 \} \). Let \( f = h_1, g = h_2 \), and \( h = h_3 \). Then, the map (1.1) is a canonical isomorphism of \( X_0(N) \) onto \( C(f, g, h) \).

By general theory, the degree of \( C(f, g, h) \) is \( 2g(\Gamma_0(N)) - 2 = 4 \) which is equal to \( g(\Gamma_0(N)) + 1 \). We have computed many more examples of above sort that indicate validity of the following conjecture:

**Conjecture 5.11** Let \( N \geq 1 \) be such that \( g(\Gamma_0(N)) \geq 3 \), \( a_\infty \) is not a Wir estrass point for \( X_0(N) \), and \( X_0(N) \) is not hyperelliptic. Let \( f, g, h \in S_2(\Gamma_0(N)) \) be such that \( \nu_{a_\infty}(f) = g(\Gamma_0(N)) - 2 \), \( \nu_{a_\infty}(g) = g(\Gamma_0(N)) - 1 \), and \( \nu_{a_\infty}(h) = g(\Gamma_0(N)) \).

Then, the map (1.1) is birational equivalence, and the curve \( C(f, g, h) \) has degree \( \deg C(f, g, h) \leq g(\Gamma_0(N)) + 1 \).

Let us show that the assumption that \( X_0(N) \) is not hyperelliptic in above conjecture is necessary. First, \( a_\infty \) is not a Wir estrass point for \( X_0(48) \) since \( g(\Gamma_0(48)) = 3 \), and the basis of \( S_2(\Gamma_0(48)) \) is given by

\[
\begin{align*}
 f &= q - 2q^5 + q^9 - 2q^{13} + 2q^{17} - q^{25} + 6q^{29} - 4q^{33} + 6q^{37} - 6q^{41} + \cdots, \\
g &= q^2 - q^6 - 2q^{10} + q^{18} + 4q^{22} - 2q^{26} + 2q^{30} + 2q^{34} - 4q^{38} - 8q^{46} + \cdots, \\
h &= q^3 - 4q^{11} - 2q^{15} + 4q^{19} + 8q^{23} + q^{27} - 8q^{31} - 2q^{39} - 4q^{43} + \cdots.
\end{align*}
\]

The corresponding reduced equation of \( C(f, g, h) \) is given by the irreducible polynomial \( -x_1^2 + x_0x_2 \). Thus, \( \deg C(f, g, h) = (g(\Gamma_0(48)) + 1)/2 = 2 \). By [13], \( X_0(48) \) is hyperelliptic. Then, since the map (1.1) is a canonical map, it has degree two by general theory [7, Chapter VII, Proposition 2.2]. Thus, it is not a birational equivalence.

Let us show that the assumption that \( a_\infty \) is not a Wir estrass point for \( X_0(N) \) in above conjecture is necessary. The curve \( X_0(72) \) is not hyperelliptic [13], \( g(\Gamma_0(72)) = 5 \), and \( a_\infty \) is a Wir estrass point for \( X_0(72) \) since the basis of \( W_3 \) is given by

\[
\begin{align*}
 f &= q^3 - q^9 - 2q^{15} + q^{27} + 4q^{33} - 2q^{39} + \cdots, \\
g &= q^5 - 2q^{11} - q^{17} + 4q^{23} - 3q^{29} + \cdots, \\
h &= q^7 - q^{13} - 3q^{19} + q^{25} + 3q^{31} + 4q^{37} + \cdots.
\end{align*}
\]

The reduced equation of \( C(f, g, h) \) is given by the irreducible polynomial \( -x_0x_1^2 + x_0^2x_2 - 2x_1x_2^2 \). Hence, \( \deg C(f, g, h) = 3 \). Now, since the proof of Proposition 5.6 implies that

\[ 3 \cdot d(f, g, h) = d(f, g, h) \cdot \deg C(f, g, h) \leq g(\Gamma_0(72)) + 1 = 6, \]

we obtain \( d(f, g, h) \leq 2 \). We have the following proposition.

**Proposition 5.12** Under above assumptions, we have \( d(f, g, h) = 2 \).

**Proof** If not, then, by above discussion, \( d(f, g, h) = 1 \). Thus, \( X_0(72) \) is birationally equivalent to the irreducible curve in \( \mathbb{P}^2 \) given by the irreducible polynomial \( -x_0x_1^2 + \)
\( x_0^2 x_2 - 2x_1 x_2^2 \). One checks easily that this curve is non-singular. This implies that this curve has genus equal to 1 and the birational equivalence is an isomorphism. This is a contradiction since \( X_0(72) \) has genus equal to 5.

\[ \square \]

**Remark 5.13** We discuss one more proof of Proposition 5.12. First, using the reduced equation, it easy to check that \((1 : 0 : 0)\) is a non-singular point on \( C(f, g, h) \). Then, the implicit function theorem implies that the local coordinate in that point is \( x_1 / x_0 \).

Next, \((1.1)\) maps \( a_\infty \) onto \((1 : 0 : 0)\). The local coordinate at \( a_\infty \) is \( q \). Thus, in terms of local coordinates, the map \((1.1)\) is given by

\[ q \longmapsto \frac{g}{f} = q^2 \frac{1 - 2q^6 - q^{12} + 4q^{18} - 3q^{24} + \cdots}{1 - q^6 - 2q^{12} + q^{24} + 4q^{30} - 2q^{36} + \cdots}. \tag{5.14} \]

Now, we use deeper properties of the proof of [11, Theorem 1.4]. Since \( a_\infty \) is mapped onto a non-singular point, the paragraph before the statement of [11, Lemma 3-4], shows that the multiplicity of the map \((1.1)\) at point \( a_\infty \) is well-defined. Using \((5.14)\), we see that the multiplicity at \( a_\infty \) is at least two. But since \( d(f, g, h) \leq 2 \), [11, Lemma 3-4] implies that the multiplicity at \( a_\infty \) is exactly two, and \( d(f, g, h) = 2 \).

On the other hand, \( X_0(54) \) is not hyperelliptic by [13], \( g(I_0(54)) = 4 \), and \( a_\infty \) is a Wierestrass point for \( X_0(54) \) since the basis of \( W_2 \) is given by

\[
\begin{align*}
f &= q^2 - 2q^8 - q^{14} + 5q^{26} + 4q^{32} - 7q^{38} + \cdots, \\
g &= q^4 - q^{10} - 3q^{13} - q^{16} + 3q^{19} + q^{22} + 3q^{25} - q^{28} + 3q^{31} - 3q^{37} + \cdots, \\
h &= q^5 - q^8 - q^{11} + q^{20} - 2q^{23} + 3q^{26} + 2q^{29} + q^{32} - q^{35} - 3q^{38} + \cdots.
\end{align*}
\]

The corresponding reduced equation of \( C(f, g, h) \) is given by the irreducible polynomial

\[-x_0^2 x_1^3 + 3x_0 x_1 x_2 + x_0^3 x_2^2 - 3x_1^3 x_2^2 - x_0 x_2^4 - 3x_2^5.\]

Thus, \( \deg C(f, g, h) = g(I_0(54)) + 1 \). Now, Proposition 5.6 implies that the map \((1.1)\) is a birational equivalence.

### 6 Estimates for primitive elements

In this Section, we will look for applications and improvements on Theorem 1.4 in the case when the subspace \( W \) has dimension 4 (see Theorem 1.4 for the notation and assumptions on \( W \)). We use the Primitive Element Theorem of finite separable field extensions in the form stated in [18, Sect. 6.10]. We start by recalling certain facts from [18, Sect. 6.10].

Let \( K \subset L \) be a finite algebraic field extension. We assume that \( L \) is generated over \( K \) by two elements \( \alpha \) and \( \beta \). We are interested in the field of characteristic zero, but we work in a greater generality. We assume that \( K \) is infinite and \( K \subset L \) is separable. By the general theory [18, Sect. 6.10], since \( K \) is infinite, there exists a primitive element
of the field extension $K \subset L$ of the form $\alpha + c\beta$ for some $c \in K$. We just need to take $c \in K$ different than all

$$\frac{\alpha_i - \alpha}{\beta - \beta_j}, \quad 1 \leq i \leq m, \ 2 \leq j \leq m$$

where $\alpha_1 = \alpha, \ldots, \alpha_m$, and $\beta_1 = \beta, \ldots, \beta_n$ are all conjugates of $\alpha$ and $\beta$ in some algebraic closure of $K$ containing $L$. Let us recall a simple argument [18, Sect. 6.10]. Let $P$ and $Q$ be irreducible polynomials of $\alpha$ and $\beta$ over $K$, respectively. We write them in the form:

$$P(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0 = a_m (X - \alpha_1) \cdots (X - \alpha_m)$$

$$Q(X) = b_n X^n + b_{n-1} X^{n-1} + \cdots + b_1 X + b_0 = b_n (X - \beta_1) \cdots (X - \beta_m).$$

(6.1)

Select $c$ as above and let $\gamma = \alpha + c\beta$. Then $\beta$ is a common root of $P(\gamma - cT)$ and $Q(T)$ which are the polynomials with coefficients in $K(\gamma)$. Since $\beta$ is separable, all roots of $Q$ are simple, and because of our assumption $\beta$ is the only common root. So, computing greatest common divisor, we conclude that $X - \beta$ has coefficients in $K(\gamma)$. Hence $\beta \in K(\gamma)$. So, $\alpha = \gamma - c\beta \in K(\gamma)$. The claim follows. In this classical argument $\alpha$ is not necessarily separable but we would like to explain how to compute such $c$ without assuming that we know all roots. For this we need the assumption that $\alpha$ is separable.

The following lemma is an improvement of above argument in the case of finite extensions of algebraic function fields. The case of number fields is considered in [20].

Lemma 6.2 Let $K = k(T)$, a field of rational functions in one variable $T$ over a field $k$. Consider a finite separable algebraic field extension $K \subset L$. We assume that $L$ is generated over $K$ by two elements $\alpha$ and $\beta$. Let $P$ and $Q$ be irreducible polynomials of $\alpha$ and $\beta$ over $K$, respectively. Clearing denominators, we can write them in the form:

$$P(X, T) = a_m (T) X^m + a_{m-1}(T) X^{m-1} + \cdots + a_1(T) X + a_0(T),$$

$$Q(X, T) = b_n (T) X^n + b_{n-1}(T) X^{n-1} + \cdots + b_1(T) X + b_0(T) \in k[X, T],$$

where $a_i, b_j \in k[T]$. Assume that $\lambda \in k$ is selected such that the following holds:

(i) $P(X, \lambda)$ and $Q(X, \lambda)$ have degrees $m$ and $n$ as polynomials in $k[X]$, respectively, and

(ii) $Q(X, \lambda)$ considered as a polynomial in $X$ with coefficients in $k$, has simple roots in some (hence, any) algebraic closure of $k$.

Then, if we write $\bar{\alpha}_1, \ldots, \bar{\alpha}_m$ (resp., $\bar{\beta}_1, \ldots, \bar{\beta}_n$) for all roots of $P(X, \lambda)$ (resp., $Q(X, \lambda)$) in some algebraic closure of $k$, then for $c \in k$ different than all

$$\frac{\bar{\alpha}_i - \bar{\alpha}_{i_1}}{\bar{\beta}_{j_1} - \bar{\beta}_{j}}, \quad 1 \leq i, i_1 \leq m, \ 1 \leq j, j_1 \leq m, \ j \neq j_1,$$

we have that $\alpha + c\beta$ is a primitive element for the extension $K \subset L$.
Proof Consider $k$-algebra $k[T]_\lambda$ of all $a/b, a, b \in k[T], b(\lambda) \neq 0$. Let $m_\lambda$ be the maximal ideal in $k[T]_\lambda$ consisting of all functions vanishing at $\lambda$. Let $\overline{k(T)}$ be the algebraic closure of $K = k(T)$ containing $L$. Let $\overline{k[T]}_\lambda$ be the integral closure of $k[T]_\lambda$ in $\overline{k(T)}$. Let $\mathfrak{M}$ be a maximal ideal in $\overline{k[T]}_\lambda$ lying above $m_\lambda$. Then, $\overline{k[T]}_\lambda/\mathfrak{M}$ is the algebraic closure of $k[T]_\lambda/m_\lambda$.

Let us write in $\overline{k(T)}[X]$

$$P(X, T) = a_m(T)(X - \alpha_1) \cdots (X - \alpha_m),$$

$$Q(X, T) = b_n(T)(X - \beta_1) \cdots (X - \beta_n).$$

(6.3)

Then, clearly $\alpha = \alpha_1, \ldots, \alpha_m, \beta = \beta_1, \ldots, \beta_n$ are integral over $k[T]_\lambda$. Hence, they belong to $\overline{k[T]}_\lambda$. Let $\Lambda$ be the reduction homomorphism

$$\overline{k[T]}_\lambda \longrightarrow \overline{k[T]}_\lambda/\mathfrak{M}.$$ 

Applying $\Lambda$ to (6.3), we may assume that

$$\Lambda(\alpha_i) = \overline{\alpha}_i, \quad 1 \leq i \leq m,$$

$$\Lambda(\beta_i) = \overline{\beta}_i, \quad 1 \leq i \leq n.$$ 

(6.4)

Now,

$$c(\overline{\beta}_j - \overline{\beta}_j) \neq \overline{\alpha}_i - \overline{\alpha}_{i_1}$$

implies that

$$c(\beta_{j_1} - \beta_{j}) \neq \alpha_i - \alpha_{i_1}$$

for all $1 \leq i, i_1 \leq m, 1 \leq j, j_1 \leq m, j \neq j_1$. By the results recalled in the beginning of this Section, we obtain that $\alpha + c\beta$ is primitive for the extension $K \subset L$. ☐

To apply Lemma 6.2, we need the following lemma:

Lemma 6.5 Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 = a_n \prod_{i=1}^{n}(X - \alpha_i) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 1$. Then we have the following:

$$|\alpha_i - \alpha_j| < 2L(f).$$
If, in addition, $f$ has no multiple roots i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, then we have the following:

$$|\alpha_i - \alpha_j| > \sqrt{3n} \frac{-(n+2)}{2} L(f)^{-(n-1)}.$$ 

Here $L(f) = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$. 

**Proof** The first bound is elementary and well-known. It follows from the Rouché theorem in Complex analysis. We sketch the argument. If $R > 0$ is selected such that 

$$|a_n| R^n > \sum_{i=0}^{n-1} |a_i| R^i,$$

then all roots of $f$ belong to $|z| < R$. We may select 

$$R = \max \left\{ 1, \frac{|a_0| + \cdots + |a_{n-1}|}{|a_n|} \right\} \leq \max \left\{ 1, \frac{L(f)}{|a_n|} \right\}.$$ 

Now, we apply that $f$ has integral coefficients: $|a_n| \geq 1$.

The second bound is more complicated. It can be found in [6].

Finally, the main result of the present section is the following proposition.

**Proposition 6.6** Assume that $m \geq 2$ is an even integer. Let $W \subset M_m(\Gamma)$, $\dim W = 4$, be a subspace which determines the field of rational functions $\mathbb{C}(\mathfrak{R}_{\Gamma})$ (see Definition 1.3). Select a basis $\{f = f_0, g = f_1, f_2, f_3\}$ of $W$. We assume that all $f_i$ has integral $q$-expansions. Then, there exists an explicitly computable $c_0 \in \mathbb{Z}$ such that for all $c \in \mathbb{Z}$, $|c| \geq c_0$, $\mathfrak{R}_{\Gamma}$ is birationally equivalent to $\mathbb{C}(f, g, h_c)$ via the map (1.1) with $h = h_c$, where $h_c \overset{\text{def}}{=} f_2 + cf_3$.

**Proof** We use the notation of Lemma 6.2. Put 

$$K \overset{\text{def}}{=} \mathbb{C}(g/f) \text{ and } L \overset{\text{def}}{=} \mathbb{C}(\mathfrak{R}_{\Gamma}) = \mathbb{C}(f_1/f_0, f_2/f_0, f_3/f_0) = \mathbb{C}(g/f, f_2/f, f_3/f).$$

In the notation of Lemma 6.2, $\alpha = f_2/f$ and $\beta = f_3/f$.

Next, by the argument used in the proof of Corollary 1.5, since we assume that $f_i$ has integral $q$-expansions, we may assume that curves $\mathcal{C}(f, g, f_2)$ and $\mathcal{C}(f, g, f_3)$ have their reduced equations with coefficients in $\mathbb{Z}$. Dehomogenizing the reduced equations, we obtain two polynomials $P(X, T)$ and $Q(X, T)$ in $\mathbb{Z}[X, T]$ such that 

$$P(f_2/f, g/f) = 0 \text{ and } Q(f_3/f, g/f) = 0.$$ 

They are both irreducible as polynomials in $\mathbb{Q}[X, T]$. 

We select $\lambda \in \mathbb{Z}$ as required by (i) and (ii) in Lemma 6.2. For (ii), one might compute the discriminant of $Q(X, T)$ with respect to $X$ i.e., the resultant of $Q(X, T)$ and its
derivative with respect to \( X \). Since \( Q(X, T) \) is irreducible in \( \mathbb{Q}[X, T] \), the resultant \( R(T) \) is a polynomial in \( \mathbb{Z}[T] \) not identically equal to zero. Now, in the notation used in Lemma 6.2, we select \( \lambda \in \mathbb{Z} \) such that \( a_m(\lambda)b_n(\lambda)R(\lambda) \neq 0 \).

Finally, one can apply Lemma 6.2 combined with bounds of Lemma 6.5 applied to polynomials \( P(X, \lambda) \) and \( Q(X, \lambda) \) in \( \mathbb{Z}[X] \). The details are left to the reader as an easy exercise. \( \square \)

The bound mentioned in the proof of Proposition 6.6 is not very optimal as we observed by various computations using SAGE. The problem is with the Mahler’s estimate (see the second inequality in Lemma 6.4). But in some cases we can obtain good results. We include the following example:

**Proposition 6.7** Consider the four dimensional space \( W \overset{def}{=} S_4(\Gamma_0(14)) \) of cusp forms of weight four for \( \Gamma_0(14) \). It has a basis:

\[
\begin{align*}
    f &= f_0 = q - 2q^5 - 4q^6 - q^7 + 8q^8 - 11q^9 - 12q^{10} + 12q^{11} + 8q^{12} + 38q^{13} + \cdots, \\
    g &= f_1 = q^2 - 2q^5 - 2q^6 + q^7 - 6q^8 + 12q^{10} + 4q^{11} + 2q^{13} - 5q^{14} + 4q^{15} + \cdots, \\
    f_2 &= q^3 - q^5 - 2q^6 - q^7 - 4q^8 + 6q^9 + 10q^{10} - 6q^{11} + 4q^{12} - 3q^{13} + \cdots, \\
    f_3 &= q^4 - 2q^5 + q^7 + q^8 - 4q^{10} + 4q^{11} - 2q^{12} + 2q^{13} + 2q^{14} + 4q^{15} - 5q^{16} + \cdots.
\end{align*}
\]

Put \( h_c \overset{def}{=} f_2 + cf_3, \ c \in \mathbb{Z}, \) as in the statement of Proposition 6.6. Then, \( X_0(14) \) is birationally equivalent to \( \mathcal{C}(f, g, h_c) \) via the map (1.1) with \( h = h_c \) for \( |c| \geq 7 \).

**Proof** We apply Proposition 6.6. First, \( W = S_4(\Gamma_0(14)) \) determines the field of rational functions on \( X_0(4) \) by Proposition 5.1. Next, we recall form the proof of Proposition 6.6 that \( \alpha = f_2/f \) and \( \beta = f_3/f \). Then, (5.5) imply that we have

\[
P(X, T) = 11X^4 + (23 + 16T)X^3 + (10 + 2T - 21T^2)X^2 + (3 + 6T - 3T^2 - 2T^3)X + (-3T^2 - 6T^3 - 4T^4),
\]

and using similar computation in SAGE we obtain:

\[
Q(X, T) = 11X^4 - (18 + 24T)X^3 - (5 + 9T + 3T^2)X^2 - (1 + 2T + 4T^2 + 6T^3)X + T^4.
\]

One sees that we can select \( \lambda = 0 \) to ensure that the assumptions (i) and (ii) of Lemma 6.2 hold.

Using SAGE, it is easy to compute roots of both polynomials \( P(X, 0) \) and \( Q(X, 0) \) with required precision. From that we obtain for the roots \( \alpha_i \) of \( P(X, 0) \) the bound

\[
|\alpha_i - \alpha_j| < 1.638,
\]

and for the roots \( \beta_i \) of \( Q(X, 0) \) we have

\[
|\beta_i - \beta_j| > 0.2595.
\]
Thus, we have

\[ \left| \frac{\alpha_i - \alpha_{i_1}}{\beta_{j_1} - \beta_j} \right| \leq 6.308 < 7, \]

for all \( 1 \leq i, i_1 \leq 4, \) and \( 1 \leq j, j_1 \leq 3, \) \( j \neq j_1. \) Thus, by Lemma 6.2, we can select \( |c| \geq 7, c \in \mathbb{Z}, \) to obtain the claim of the proposition.

The estimate given by Proposition 6.7 is quite good. They depend on our choice of \( \lambda = 0. \) It is also possible to select for example \( \lambda = 1 \) which results in a weaker estimate \( |c| \geq 9, \) or \( \lambda = -1 \) which results in a better estimate \( |c| \geq 4. \)

We remark that the methods used in the proof of Proposition 5.4 imply that

\[ d(f, g, h_c) \cdot \deg C(f, g, h_c) \leq 4, \quad c \in \mathbb{Z}. \]

As in Sect. 5, we use SAGE to compute the reduced equation of \( \deg C(f, g, h_c) \) for \( c = 0, \ldots, 6. \) The degree is always equal to 4. In particular, in view of above estimate, we see that \( X_0(14) \) is birational to \( C(f, g, h_c) \) via the map (1.1) with \( h = h_c \) for \( c \in \{0, \ldots, 6\}. \) Combining with Proposition 6.7, we obtain the following corollary:

\textbf{Corollary 6.8} \( X_0(14) \) is birationally equivalent to \( C(f, g, h_c) \) via the map (1.1) with \( h = h_c, \) for all integers \( c \geq 0. \)

It seems that this result can not be established by the methods of Sect. 5. This shows the usefulness of the methods of the present section.

\section{The trial method for primitive elements}

Let \( W \subset S_m(\Gamma), m \geq 2, \) be a non-zero subspace that determines the field of rational \( \mathbb{C}(\mathfrak{R}_\Gamma) \) (see Definition 1.3). Furthermore, we assume that \( \dim W = s \geq 4. \) Let \( f_0, \ldots, f_{s-1} \) be a basis of \( W. \) We let \( f = f_0 \) and \( g = f_1. \) Then, Theorem 1.4 guarantees that in \( W \) we can find infinitely many forms \( h \) such that \( \mathfrak{R}_\Gamma \) is birationally equivalent to \( C(f, g, h) \) via (1.1). But the proof does not provide a computable manner of determining at least one such \( h. \) In this section, we present a simple algorithm for this. We adapt to our case the trial method, commonly used in the cases of algebraic number fields, [20], where an element that is chosen from a certain subset of the field extension is tested for being primitive.

As in the proof of Lemma 3.1, we denote \( K \overset{\text{def}}{=} \mathbb{C}(f/g), \) and

\[ L \overset{\text{def}}{=} \mathbb{C}(\mathfrak{R}_\Gamma) = \mathbb{C}(f_1/f_0, f_2/f_0, \ldots, f_{s-1}/f_0) = \mathbb{C}(f/g, f_2/f, \ldots, f_{s-1}/f). \]

We observe \( L \) is a finite algebraic extension of \( K, \) and we have the following:

\[ L = K(f_2/f_0, \ldots, f_{s-1}/f_0). \]
We are interested in finding a primitive element of $L$ over $K$ which has the form of a linear combination of the generators $f_2/f_0, \ldots, f_{s-1}/f_0$. From the proof of Lemma 3.1, we know that the coefficients of this linear combination must be from a Zariski open set in $\mathbb{C}^{s-2}$, and since $\mathbb{Z}^{s-2}$ is Zariski dense in $\mathbb{C}^{s-2}$, we can find a $\mathbb{Z}$-linear combination which is primitive for $L$. The trial method consists of testing various $\mathbb{Z}$-linear combinations for the condition of being primitive element.

For $a \overset{\text{def}}{=} (a_2, a_3, \ldots, a_{s-1}) \in \mathbb{Z}^{s-2}$, we let

$$h \overset{\text{def}}{=} h_a \overset{\text{def}}{=} a_2 f_2/f_0 + \cdots + a_{s-1} f_{s-1}/f_0 \in L. \quad (7.1)$$

Since, by our assumption $W \subset S_m(\Gamma)$, we have

$$d(f, g, h) \cdot \deg C(f, g, h) \leq \dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m,$$

using (1.2). Thus, if

$$\deg C(f, g, h) > \frac{\dim S_m(\Gamma) + g(\Gamma) - 1 - \epsilon_m}{2}, \quad (7.2)$$

then we obtain

$$d(f, g, h) = 1.$$

This means that $\mathfrak{R}_\Gamma$ is birationally equivalent to $C(f, g, h)$ via (1.1).

We organize $(s-2)$-tuples in $\mathbb{Z}^{s-2}$ as follows:

$$S_M \overset{\text{def}}{=} \left\{ a_2 f_2/f_0 + \cdots + a_{s-1} f_{s-1}/f_0; \quad a_i \in \mathbb{Z}, \quad 2 \leq i \leq s-1, \quad \sum_{i=2}^{s-1} |a_i| = M \right\}$$

for all $M \in \mathbb{Z}_{\geq 1}$. For $M \geq 1$, we order elements of $S_M$ using the lexicographical order.

To apply this simple method, we perform the following algorithm which stops after finitely many steps:

1. Let $M = 1$. Repeat the following:
2. For $a \in S_M$, we repeat the following: compute $\deg C(f, g, h)$, and test (7.2) for $h = h_a$. If (7.2) holds, then the algorithm stops. OUTPUT: $h$ such that $h/f$ is a primitive element for the extension $K \subset L$.
3. Increase $M$ by one, and return to step (2).

Let $\Gamma = \Gamma_0(N)$ such that $g(\Gamma_0(N)) \geq 4$, and $X_0(N)$ is not hyperelliptic [13] (or Introduction). Then, it is well-known that $S_2(\Gamma_0(N))$ determines the field of rational functions on $X_0(N)$. Since also its dimension is equal to $g(\Gamma_0(N)) \geq 4$, we may select $W = S_2(\Gamma_0(N))$. In this case, the inequality (7.2) is

$$\deg C(f, g, h) > g(\Gamma_0(N)) - 1. \quad (7.3)$$
As an example, we consider the case \( N = 72 \). Then \( g(\Gamma_0(72)) = 5 \), and we may take

\[
\begin{align*}
  f &= f_0 = q^3 - q^9 - 2q^{15} + q^{27} + 4q^{33} - 2q^{39} + \cdots, \\
  g &= f_1 = q^5 - 2q^{11} - q^{17} + 4q^{23} - 3q^{29} + \cdots, \\
  f_2 &= q^7 - q^{13} - 3q^{19} + q^{25} + 3q^{31} + 4q^{37} + \cdots, \\
  f_3 &= q - 2q^{13} - 4q^{19} - q^{25} + 8q^{31} + 6q^{37} + \cdots, \\
  f_4 &= q^2 - 4q^{14} + 2q^{26} + 8q^{38} + \cdots.
\end{align*}
\]

The example \( N = 72 \) already appears in Sect. 5 in the text after Conjecture 5.11 (see Proposition 5.12). Applying above algorithm, we obtain the following:

1. For \( M = 1 \), we have three cases in their lexicographical order \( a = (0, 0, 1) \), \( (0, 1, 0) \), and \( (1, 0, 0) \). We have \( \deg C(f, g, h_a) = 3, 2, \) and \( 3 \), respectively. In any case, \( \deg C(f, g, h_a) \leq g(\Gamma_0(72)) - 1 = 4 \). So, we go to the next step.
2. For \( M = 2 \), in the lexicographical order, we have the following:

1. \( a = (0, 0, 2) \), \( \deg C(f, g, h_a) = 3 \leq g(\Gamma_0(72)) - 1 = 4 \);
2. \( a = (0, 1, 1) \), \( \deg C(f, g, h_a) = 3 \leq 4 \);
3. \( a = (0, 2, 0) \), \( \deg C(f, g, h_a) = 2 \leq 4 \);
4. \( a = (1, 0, 1) \), \( \deg C(f, g, h_a) = 7 > 4 \); STOP.

Hence, the map (1.1) with \( h = h_{(1,0,1)} \) is a birational equivalence of \( X_0(72) \) and \( C(f, g, h_{(1,0,1)}) \). The reduced equation of \( C(f, g, h_{(1,0,1)}) \) is given by the irreducible polynomial

\[
\begin{align*}
  x_0^7 - 4x_0^6x_1 - 3x_0^4x_1^3 - 8x_0^3x_1^4 - x_0^2x_1^5 - 4x_0x_1^6 - 4x_1^7 - 4x_0^5x_1x_2 \\
  + 2x_0^3x_1^3x_2 - 4x_0^2x_1^4x_2 - x_0^4x_1^2x_2^2 + 8x_0^3x_1^2x_2^2 - 4x_0x_1^4x_2^2 + 8x_1^5x_2^2 \\
  + 4x_0^2x_1^2x_2^3 - 4x_1^3x_2^4.
\end{align*}
\]

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