NUMERICAL STUDIES OF DYNAMO ACTION IN A TURBULENT SHEAR FLOW. I.

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ABSTRACT

We perform numerical experiments to study the shear dynamo problem where we look for the growth of a large-scale magnetic field due to non-helical stirring at small scales in a background linear shear flow in previously unexplored parameter regimes. We demonstrate the large-scale dynamo action in the limit where the fluid Reynolds number (Re) is below unity while the magnetic Reynolds number (Rm) is above unity; the exponential growth rate scales linearly with shear, which is consistent with earlier numerical works. The limit of low Re is particularly interesting, as seeing the dynamo action in this limit would provide enough motivation for further theoretical investigations, which may focus attention on this analytically more tractable limit of Re < 1 compared to the more formidable limit of Re > 1. We also perform simulations in the regimes where (i) both (Re, Rm) < 1, and (ii) Re > 1 and Rm < 1, and compute all of the components of the turbulent transport coefficients (αij and ηij) using the test-field method. A reasonably good agreement is observed between our results and the results of earlier analytical works in similar parameter regimes.

Key words: dynamo – magnetic fields – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Magnetic fields observed in various astrophysical systems, such as the Earth, the Sun, disk galaxies, accretion disks, etc., possess large-scale magnetic fields in addition to a fluctuating component. The magnetic field survives for timescales much larger than the diffusion timescales in those systems, and therefore are thought to be self-sustained by turbulent dynamo action. The standard model of such a turbulent dynamo producing a large-scale magnetic field involves the amplification of seed magnetic fields due to the usual α effect, where α is a measure of the net kinetic helicity in the flow (see, e.g., Moffatt 1978; Parker 1979; Krause & Rädler 1980; Brandenburg & Subramanian 2005; Brandenburg et al. 2012). Since it is not necessary for the turbulent flow always to be helical, it is interesting to study the dynamo action in non-helically forced shear flows. Dynamo action due to shear and turbulence in the absence of the α effect received some attention in the astrophysical contexts of accretion disks (Vishniac & Brandenburg 1997) and galactic disks (Blackman 1998; Sur & Subramanian 2009). The presence of large-scale shear in turbulent flows is expected to have significant effects on transport properties (Rädler & Stepanov 2006; Rüdiger & Kitchatinov 2006; Leprovost & Kim 2009; Sridhar & Singh 2010; Singh & Sridhar 2011). It has also been demonstrated that the mean shear in conjunction with rotating turbulent convection gives rise to the growth of large-scale magnetic fields (Käpylä et al. 2008; Hughes & Proctor 2009). The problem we are interested in may be stated as follows: in the absence of the α effect, will it be possible to generate a large-scale magnetic field solely through the action of non-helical turbulence in the background shear flow on the seed magnetic field? This question was recently numerically studied by Brandenburg et al. (2008), Yousef et al. (2008a, 2008b). These works clearly demonstrated the growth of large-scale magnetic fields due to non-helical stirring at small scales in the background linear shear flow.

Although various mechanisms have been proposed to resolve the shear dynamo problem, it is still not clear what really drives the dynamo action in such systems. The presence of the magnetic helicity flux could potentially cause the growth of the large-scale magnetic field (Vishniac & Cho 2001; Brandenburg & Subramanian 2005; Shapovalov & Vishniac 2011). Yet another possibility that has been suggested is the shear-current effect (Rogachevskii & Kleeorin 2003, 2004, 2008) where the shear-current term in the expression for the mean electromotive force (EMF) is thought to generate the cross-shear component of the mean magnetic field from the shearwise component. However, some analytic calculations (Rädler & Stepanov 2006; Rüdiger & Kitchatinov 2006; Sridhar & Subramanian 2009a, 2009b; Sridhar & Singh 2010; Singh & Sridhar 2011) and numerical experiments (Brandenburg et al. 2008) find that the sign of the shear-current term is unfavorable for the dynamo action. The quasilinear kinematic theories of Sridhar & Subramanian (2009a, 2009b), and the low magnetic Reynolds number (Rm) theories of Sridhar & Singh (2010) and Singh & Sridhar (2011) found no evidence of dynamo action; in these works, a Galilean-invariant formulation of the shear dynamo problem was developed, in which the α effect was strictly zero, and unlike earlier works the shear was treated non-perturbatively. It has been suggested that the mean magnetic field could grow due to a process known as the incoherent α shear mechanism, in which the fluctuations in α with no net value, together with the mean shear, might drive the large-scale dynamo action (Sokolov 1997; Vishniac & Brandenburg 1997; Silant’ev 2000; Proctor 2007; Brandenburg et al. 2008; Kleeorin & Rogachevskii 2008; Sur & Subramanian 2009; Richardson & Proctor 2012; Sridhar & Singh 2014). Recent analytical works by Heinemann et al. (2011), McWilliams (2012), and Mitra & Brandenburg (2012) predict the growth of a mean-squared
magnetic field by considering a fluctuating $\alpha$ in the background shear in the limit of the small Reynolds numbers. Sridhar & Singh (2014) discuss the possibility of the growth of the mean magnetic field in a shearing background by considering zero-mean temporal fluctuations in $\alpha$, which have finite correlation times.

It should be noted that all previous numerical experiments performed thus far have been carried out for both the fluid Reynolds number ($Re$) and the magnetic Reynolds number ($Rm$) above unity, the limit at which a rigorous theory explaining the origin of the shear dynamo is yet to come. In order to achieve step-by-step progress analytically, it seems necessary to explore the regime $Re < 1$ and $Rm > 1$ before synthesizing a theory which is valid for both ($Re, Rm$) > 1.

Such considerations motivated us to look for a numerical experiment carried out in the regime where $Re < 1$ and $Rm > 1$.

In this paper, we present numerical simulations for the shear dynamo problem which can be broadly classified into the following three categories: (i) the regime where both $Re$ and $Rm$ are less than unity, which is done for comparison with earlier analytical work (Singh & Sridhar 2011); (ii) the regime where $Re > 1$ and $Rm < 1$; and (iii) the regime where $Re < 1$ and $Rm > 1$. We have used the Pencil Code$^6$ for all of the simulations presented in this paper and followed the method given in Brandenburg et al. (2008). In Section 2, we begin with the fundamental equations of magnetohydrodynamics in a background linear shear flow. We then consider the case where the mean-magnetic field is a function only of the spatial coordinate $x_3$ and time $t$. We briefly describe the transport coefficients and discuss the test field method. A few important details of the simulation are presented. In Section 3, we assemble all of the results in three parts, namely, part A, part B, and part C corresponding to the three categories discussed above. We also perform comparisons with the analytical work of Sridhar & Singh (2010) and Singh & Sridhar (2011). In Section 4, we present our conclusions.

2. THE MODEL AND NUMERICAL SET UP

Let $(e_1, e_2, e_3)$ be the unit basis vectors of a Cartesian coordinate system in the laboratory frame. Using the notation $x = (x_1, x_2, x_3)$ for the position vector and $t$ for time, we write the total fluid velocity as $(Sx_1e_1 + v)$, where $S$ is the rate of the shear parameter and $v(x, t)$ is the velocity deviation from the background shear flow. Let $B^{tot}$ be the total magnetic field which obeys the induction equation. We have performed numerical simulations using the Pencil Code which is a publicly available code suited for weakly compressible hydrodynamic flows with magnetic fields. We consider the velocity field $v$ to be compressible and write the momentum, continuity, and induction equations for a compressible fluid of mass density $\rho$:

$$
\left( \frac{\partial}{\partial t} + Sx_1 \frac{\partial}{\partial x_2} \right) v + (v \cdot \nabla) v = \frac{1}{\rho} \nabla P + J^{tot} \times B^{tot} \rho + F_{\text{visc}} + f
$$  

$$
\left( \frac{\partial}{\partial t} + Sx_1 \frac{\partial}{\partial x_2} \right) B^{tot} - SB^{tot}_1 e_2 = \nabla \times (v \times B^{tot}) + \eta \nabla^2 B^{tot},
$$  

where $F_{\text{visc}}$ denotes the viscous term, $f$ is the random stirring force per unit mass, and we write $J^{tot} = (\nabla \times B^{tot})$ for simplicity, instead of the usual definition $J^{tot} = (\nabla \times B^{tot})/\mu_0$, $\mu_0$ and $\eta$ represent the magnetic permeability and magnetic diffusivity, respectively. Our aim is to investigate the case of incompressible magnetohydrodynamics in a background linear shear flow with non-helical random forcing at small scales. In order to achieve this with Pencil Code, we limit ourselves to those cases for which the root mean squared velocity, $v_{rms}$, is small compared to the sound speed, making the Mach number ($Ma$) very small. In this case, the solutions of the compressible equations approximate the solutions of the incompressible equations. When the velocity field $v$ is incompressible (or weakly compressible), the viscous term in Equation (1) becomes $F_{\text{visc}} = \nu \nabla^2 v$ ($\nu$ denotes the coefficient of kinematic viscosity) and the right-hand side of the continuity equation (2) vanishes.

2.1. Mean-field Induction Equation

Various transport phenomena have traditionally been studied in the framework of mean-field theory (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005). Applying Reynolds averaging to the induction Equation (3) we find that the mean magnetic field, $\bar{B}(x, t)$, obeys the following (mean-field induction) equation:

$$
\left( \frac{\partial}{\partial t} + Sx_1 \frac{\partial}{\partial x_2} \right) \bar{B} - SB \bar{e}_2 = \nabla \times \bar{E} + \eta \nabla^2 \bar{B}
$$  

where $\eta$ is the microscopic resistivity, and $\bar{E}$ is the mean electromotive force (EMF), $\bar{E} = (v' \times b')$, where $v'$ and $b'$ are the fluctuations in the velocity and magnetic fields, respectively. We perform numerical simulations in a cubic domain with a size of $L \times L \times L$, where the mean-field $Q$ of some quantity $Q^{tot}$ is defined by

$$
Q(x_3, t) = \frac{1}{L^3} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} Q^{tot}(x_1, x_2, x_3, t) dx_1 dx_2 dx_3.
$$  

Thus, the mean-field quantities discussed here are functions of $x_3$ and time $t$. The mean EMF is generally a functional of the mean magnetic field, $B_1$. For a slowly varying mean magnetic field, the mean EMF can approximately be written as a function of $B_1$ and $B_{\text{rms}}$ (see Brandenburg et al. 2008; Singh & Sridhar 2011):

$$
\bar{E}_i = \alpha_{ii}(t) B_i(x, t) - \eta_{mii}(t) \frac{\partial B_i(x, t)}{\partial x_m},
$$  

where $\alpha_{ii}(t)$ and $\eta_{mii}(t)$ are the transport coefficients which evolve in time at the beginning and saturate at late times.

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6 See http://www.nordita.org/software/pencil-code.
2.2. Transport Coefficients

Previous studies have shown that the mean value of \( \alpha_j \) is zero so long as the stirring is non-helical (Brandenburg et al. 2008; Sridhar & Subramanian 2009a, 2009b; Sridhar & Singh 2010; Singh & Sridhar 2011), but it shows zero-mean temporal fluctuations in simulations. Using the definition of the mean-field given in Equation (5), we note that the mean magnetic field \( \mathbf{B} = \mathbf{B}(x_3, t) \). The condition \( \nabla \cdot \mathbf{B} = 0 \) implies that \( B_3 \) is uniform in space and that it can be set to zero without loss of generality; hence, we have \( \mathbf{B} = (B_1, B_2, 0) \). Thus, Equation (6) for the mean EMF gives \( \mathbf{E} = (\mathbf{E}_1, \mathbf{E}_2, 0) \), with

\[
\mathbf{E}_i = \alpha_j B_j - \eta_j \mathbf{J}_j; \quad \mathbf{J} = \nabla \times \mathbf{B} = \left( \begin{array}{c} -\partial B_2 \over \partial x_3, \quad \partial B_1 \over \partial x_3, \quad 0 \end{array} \right), \tag{7}
\]

where all components of \( \alpha_j \) show zero-mean temporal fluctuations as the forcing is non-helical and the two indexed magnetic diffusivity tensor \( \eta_j \) has four components, \( (\eta_1, \eta_2, \eta_3, \eta_4) \), which are defined in terms of the three indexed object \( \eta_{\text{fl}} \) by

\[
\eta_i = \epsilon_{i3} \eta_{33}; \quad \text{implying, } \eta_1 = -\eta_{32}; \quad \eta_2 = \eta_{31}. \tag{8}
\]

By substituting Equation (7) for \( \mathbf{E} \) in Equation (4), we obtain the evolution equation for the mean magnetic field. The diagonal components, \( \eta_{11} \) and \( \eta_{22} \), augment the microscopic resistivity, \( \eta \), whereas the off-diagonal components, \( \eta_{12} \) and \( \eta_{21} \), lead to the cross-coupling of \( B_1 \) and \( B_2 \). It was shown in Singh & Sridhar (2011) that each component of \( \eta_{\text{fl}} \) starts from zero at time \( t = 0 \) and saturates at some constant value \( (\eta_{i}^{\text{sat}}) \) at late times. Here, we aim to measure these saturated quantities.

2.3. Test-field Method

We use the test-field method to determine the transport coefficients \( \alpha_j \) and \( \eta_j \). This procedure has been described in detail in Brandenburg et al. (2008 and references therein). A brief description of the method is as follows. Assume that \( \mathbf{B}^\prime \) is a set of test-fields and \( \mathbf{E}^\prime \) is the EMF corresponding to the test-field \( \mathbf{B}^\prime \). By subtracting Equation (4) from Equation (3), we get the evolution equation for the fluctuating field \( \mathbf{b}^\prime \). With properly chosen \( \mathbf{B}^\prime \) and flow \( \mathbf{v} \), we can numerically solve for the fluctuating field \( \mathbf{b}^\prime \). This enables us to determine \( \mathbf{E}^\prime \), which can then be used to find \( \alpha_j \) and \( \eta_j \) using \( \mathbf{E}_i^\prime = \alpha_j B_j^\prime - \eta_j \mathbf{J}_j^\prime \), where \( \mathbf{J}^\prime = \nabla \times \mathbf{B}^\prime \).

There could be various choices for the number and form of the test fields which essentially depend on the problem being solved. For our purposes, we have chosen the test fields, denoted as \( \mathbf{B}^{\text{test}} \), defined by

\[
\mathbf{B}^{1x} = B_x (\cos[k_3 x_3], 0, 0); \quad \mathbf{B}^{2z} = B_z (0, \cos[k_3 x_3], 0), \tag{9}
\]

where \( B \) and \( k \) are assumed to be constant. Using Equation (9) in the expression \( \mathbf{E}_i^\prime = \alpha_j B_j^\prime - \eta_j \mathbf{J}_j^\prime \), we find the corresponding mean EMF denoted by \( \mathbf{E}^{\text{test}} \) as

\[
\mathbf{E}_i^{1x} = \alpha_j B \cos[k_3 x_3] + \eta_j B_k \sin[k_3 x_3] \tag{10}
\]

Here, we have four equations but eight unknowns \( (\eta_{11}, \ldots \eta_{42}, \alpha_{11}, \ldots \alpha_{22}) \). Therefore, we further consider the following set of test fields, denoted as \( \mathbf{B}^{\text{test}} \), defined by

\[
\mathbf{B}^{1x} = B (\sin[k_3 x_3], 0, 0); \quad \mathbf{B}^{2z} = B (0, \sin[k_3 x_3], 0), \tag{11}
\]

where \( B \) and \( k \) are assumed to be constant as before. Using Equation (11) in the expression \( \mathbf{E}_i^\prime = \alpha_j B_j^\prime - \eta_j \mathbf{J}_j^\prime \), we find the corresponding mean EMF denoted by \( \mathbf{E}^{\text{test}} \) as

\[
\mathbf{E}_i^{1x} = \alpha_j B \sin[k_3 x_3] - \eta_j B_k \cos[k_3 x_3] \tag{12}
\]

Using Equations (10) and (12), we can write

\[
\alpha_{11} = \frac{1}{B} (\mathbf{E}_i^{1x} \cos[k_3 x_3] + \mathbf{E}_i^{2z} \sin[k_3 x_3]) \tag{13}
\]

\[
\alpha_{12} = \frac{1}{B} (\mathbf{E}_i^{1x} \cos[k_3 x_3] + \mathbf{E}_i^{2z} \sin[k_3 x_3]); \quad i = 1, 2, \tag{14}
\]

\[
\eta_{11} = \frac{1}{B_k} (\mathbf{E}_i^{2z} \sin[k_3 x_3] - \mathbf{E}_i^{1x} \cos[k_3 x_3]) \tag{15}
\]

\[
\eta_{12} = \frac{1}{B_k} (\mathbf{E}_i^{1x} \sin[k_3 x_3] - \mathbf{E}_i^{1x} \cos[k_3 x_3]); \quad i = 1, 2. \tag{16}
\]

Thus, from Equations (13) and (14), we can determine the unknown quantities \( \alpha_j \) and \( \eta_j \). For the homogeneous turbulence being considered here, the transport coefficients need to be independent of \( x_3 \), and therefore the apparent dependence on \( x_3 \) through the terms \( \sin[k_3 x_3] \) and \( \cos[k_3 x_3] \) in Equations (13) and (14) has to be compensated for by the \( x_3 \)-dependent \( \mathbf{E}_i^\prime \)s given by Equations (10) and (12).

We use the “shear-periodic” boundary conditions to solve Equations (1)–(3) in the same manner as given in Brandenburg et al. (2008). Shear-periodic boundary conditions have been widely used in numerical simulations in a variety of contexts: simulations of local patches of planetary rings (Wisdom & Tremaine 1988), the local dynamics of differentially rotating disks in astrophysical systems (Balbus & Hawley 1998; Binney & Tremaine 2008), the nonlinear evolution of perturbed shear flow in two-dimensions with the ultimate goal of understanding the dynamics of accretion disks (Lithwick 2007), the shear dynamo (Brandenburg et al. 2008; Yousef et al. 2008a, 2008b; Käpylä et al. 2008), etc., are a few examples.

The random forcing function \( f \) in Equation (1) is assumed to be non-helical, homogeneous, isotropic, and delta-correlated in time. Furthermore, we assume that the vector function \( \mathbf{J} \) is solenoidal and that the forcing is confined to a spherical shell of magnitude \( |k_f| = k_f \), where the wavevector \( k_f \) signifies the energy-injection scale \( l_f = 2\pi/k_f \) of the turbulence. This can be approximately achieved by following the method described in Brandenburg et al. (2008). We note that although the random forcing \( f \) is delta-correlated in time, the resulting fluctuating velocity field \( v \) will not be delta-correlated in time (this is due to the inertia, as pointed out in Brandenburg et al. 2008). This has been rigorously proved in Singh & Sridhar (2011) in the limit of a small fluid Reynolds number, the limit which we aim to explore in the present manuscript. Another important fact to note is that in the limit of small \( \text{Re} \), the non-helical forcing has been shown to give rise to a non-helical velocity field in Singh & Sridhar (2011); whether or not this is true even in the limit of high \( \text{Re} \) has not yet been proven. Thus, performing the simulation in the limit of \( \text{Re} < 1 \) with non-
helical forcing guarantees that the fluctuating velocity field is also non-helical.

3. RESULTS AND DISCUSSION

We have explored the following three parameter regimes: (i) \( \text{Re} < 1 \) and \( \text{Rm} < 1 \); (ii) \( \text{Re} > 1 \) and \( \text{Rm} < 1 \); and (iii) \( \text{Re} < 1 \) and \( \text{Rm} > 1 \). All of the results obtained in our numerical simulations for various parameter regimes are presented. As all of the transport coefficients show temporal fluctuations about some constant value, we take long time averages of the quantities and denote them by \( \eta^n \). The turbulent diffusivity, \( \eta_t \), is defined in terms of the components of the magnetic diffusivity tensor as follows:

\[
\eta_t = \frac{1}{2} \left( \eta_{11}^\infty + \eta_{22}^\infty \right); \quad \eta_T = \eta + \eta_t.
\]  

(15)

We note that the rate of the shear parameter, \( S < 0 \), and \( K \) is the smallest finite wavenumber in the \( x_3 \) direction. We now define various dimensionless quantities: the fluid Reynolds number, \( \text{Re} = \nu \mu/(\eta k_f) \); the magnetic Reynolds number, \( \text{Rm} = \nu \mu/(\eta k_f) \); the Prandtl number, \( \text{Pr} \equiv \nu/\eta \); and the dimensionless shear parameter, \( S_0 = S/(\nu k_f) \). The symbols used in these definitions have their usual meanings.

3.1. \( \text{Re} < 1 \) and \( \text{Rm} < 1 \)

It is necessary to compare the numerical results obtained in this parameter regime with the earlier analytical work in which the general functional form for the saturated values of magnetic diffusivities, \( \eta_{ij} \), was predicted (see Equation (60) and the related discussion in Singh & Sridhar 2011). It is useful to recall the following expression for the growth rate of the mean magnetic field, obtained from the mean-field theory (see, e.g., Brandenburg et al. 2008; Singh & Sridhar 2011):

\[
\frac{\lambda_n}{\eta_T K^2} = -1 \pm \frac{1}{\eta_T} \sqrt{\eta_T \left( \frac{S}{K^2} + \eta_{22}^\infty \right) + \epsilon^2},
\]  

(16)

where,

\[
\epsilon = \frac{1}{2} \left( \eta_{11}^\infty - \eta_{22}^\infty \right); \quad \text{and} \quad S < 0.
\]  

(17)

Figures 1–3 display the plots of \( \eta_t, \eta_{11}^\infty, \) and \( \eta_{22}^\infty \) versus the dimensionless parameter \( -S_0 \text{ Re} \), which demonstrate the comparison of the results from a direct numerical simulation with \( 64^3 \) mesh points with the theoretical results obtained in Singh & Sridhar (2011). The scalings of the ordinates have been chosen for compatibility with the functional form of Equation (60) in Singh & Sridhar (2011). However, it should be noted that we have performed simulations for values of \( -S_0 \text{ Re} \) up to about 0.7, whereas Singh & Sridhar (2011) have been able to explore larger values of \( -S_0 \text{ Re} \). The plots in Figures 1(a)–(c) are for \( \text{Pr} = 1 \), but for two sets of values of the Reynolds numbers \( \text{Re} = \text{Rm} \approx 0.16 \) (the “bold” lines represent the theory and the symbols “o” represent the simulations) and \( \text{Re} = \text{Rm} \approx 0.46 \) (the “dashed” lines represent the theory and the symbols “x” represent the simulations). Figures 2(a)–(c) are for \( \text{Re} \approx 0.13 \) and \( \text{Rm} \approx 0.64 \), corresponding to \( \text{Pr} \approx 5 \) (the “bold” lines represent the theory and the symbols “o” represent the simulations). Figures 3(a)–(c) are for \( \text{Re} \approx 0.13 \) and \( \text{Rm} \approx 0.025 \), corresponding to \( \text{Pr} \approx 0.2 \) (the “bold” lines represent the theory and the symbols “o” represent the simulations). Some noteworthy properties are as follows:

(i) As may be seen from Figure 1, the symbols “o” and “x” (also the bold and dashed lines) lie very nearly on top of each other. This implies that \( \eta_t/(\eta_T \text{ Re}^2), \eta_{11}^\infty/(\eta_T \text{ Re}^2), \) and \( \eta_{22}^\infty/(\eta_T \text{ Re}^2) \) are (approximately) functions of \( -S_0 \text{ Re} \) and \( \text{Pr} \). Therefore, the magnitude of \( \chi \) in Equation (60) of Singh & Sridhar (2011) should be much smaller than unity. This was predicted in Singh & Sridhar (2011), and thus our numerical findings are in good agreement with the simulations.
agreement with the theoretical investigations of Singh & Sridhar (2011).

(ii) We see that \( \eta \) is always positive. For a fixed value of \((-S_h \text{ Re})\) the quantity \( \eta/\eta_T \text{ Re}^2 \) increases with \( \text{Pr} \), and for a fixed value of \( \text{Pr} \) it slowly increases with \((-S_h \text{ Re})\) (which is consistent with Brandenburg et al. 2008). An excellent agreement between our numerical findings and the theory presented in Singh & Sridhar (2011) may be seen from the top panels of Figures 1–3.

(iii) The quantity \( \eta_2^{\infty} \) approaches zero in the limit where \((-S_h \text{ Re})\) is nearly zero. In the numerical simulation, it is seen to increase with \((-S_h \text{ Re})\) for a fixed value of \( \text{Pr} \), and for a fixed value of \((-S_h \text{ Re})\) it increases with \( \text{Pr} \). \( \eta_2^{\infty} \) is expected to behave in a more complicated manner. Different signs of \( \eta_2^{\infty} \) are reported in Brandenburg et al. (2008) and Rüdiger & Kitchatinov (2006), whereas both signs have been predicted in calculations of Singh & Sridhar (2011). The differences between the theory and simulations may be inferred from panel (b) in Figures 1–3.

(iv) As may be seen from the bottom panels of Figures 1–3, \( \eta_2^{\infty} \) is always positive. This agrees with the results obtained in earlier works (Rädler & Stepanov 2006; Rüdiger & Kitchatinov 2006; Brandenburg et al. 2008). Once again, the agreement between our numerical findings and the theoretical investigations of Singh & Sridhar (2011) for this crucial component of the diffusivity tensor is remarkably good.\(^7\)

Furthermore, we show the time dependence of the root-mean-squared value of the total magnetic field \( B_{\text{rms}} \) in Figure 4, which explicitly demonstrates the decay of \( B_{\text{rms}} \) following three sets of values of control parameters: (i) \( \text{Re} \approx 0.128, \text{Rm} \approx 0.643 \) (corresponding to \( \text{Pr} \approx 5.0 \)); \( S_h \approx -1.545 \); (ii) \( \text{Re} \approx 0.16, \text{Rm} \approx 0.16 \) (corresponding to \( \text{Pr} \approx 1.0 \)); \( S_h \approx -1.237 \); and (iii) \( \text{Re} \approx 0.127, \text{Rm} \approx 0.025 \) (corresponding to \( \text{Pr} \approx 0.25 \)); \( S_h \approx -1.560 \). Results shown in Figure 4 are from a direct numerical simulation with \( 64^3 \) mesh points and \( k_f/K = 10.03 \) for all three cases.

\[ B_{\text{rms}} / B_{\text{eq}} \]

Figure 4. Time dependence of the root mean squared value of the total magnetic field (scaled with respect to \( B_{\text{eq}} \)) vs. the dimensionless parameter \( (r_{\text{rms}} k_f) \). The bold line is for \( \text{Re} \approx 0.128, \text{Rm} \approx 0.643 \) (i.e., \( \text{Pr} = 5.0 \)); \( S_h \approx -1.545 \); the dashed line is for \( \text{Re} \approx 0.16, \text{Rm} \approx 0.16 \) (i.e., \( \text{Pr} = 1.0 \)); \( S_h \approx -1.237 \); and the dashed-dotted line is for \( \text{Re} \approx 0.127, \text{Rm} \approx 0.025 \) (i.e., \( \text{Pr} = 0.25 \)), and \( S_h \approx -1.560 \). \( k_f/K = 10.03 \) for all three cases.

\section{3.2. \text{Re} > 1 and \text{Rm} < 1}

We explored this parameter regime for completeness in order to investigate the dynamo action when \( \text{Rm} < 1 \) and \( \text{Re} > 1 \). The kinematic theory of the shear-dynamo problem was developed in Sridhar & Singh (2010), which is valid for a low magnetic Reynolds number but places no restriction on the fluid Reynolds number. We computed all of the components of \( \alpha \) and \( \eta \) using the test-field method and investigated the possibility of a dynamo action. We find that all of the components of \( \alpha \) show fluctuations in time with mean zero, and therefore we do not expect the generation of any net helicity in the flow in these parameter regimes. We summarize all of our results for \( \text{Re} > 1 \) and \( \text{Rm} < 1 \) in detail in Table 1.

We find no evidence of dynamo action in this particular parameter regime. This is shown clearly in Figure 5, in which we plot the time dependence of root-mean-squared value of the total magnetic field \( B_{\text{rms}} \) and demonstrate the absence of the dynamo action in this parameter regime. Figure 5 shows results from the direct simulation with \( 64^3 \) mesh points for the following four sets of parameter values: (i) \( \text{Re} \approx 24.57, \text{Rm} \approx 0.614, k_f/K = 5.09, S_h \approx -0.118 \) (shown by the bold line); (ii) \( \text{Re} \approx 22.40, \text{Rm} \approx 0.448, k_f/K = 5.09, S_h \approx -0.128 \) (shown by the dashed line); (iii) \( \text{Re} \approx 43.17, \text{Rm} \approx 0.863, k_f/K = 3.13, S_h \approx -0.177 \) (shown by the dashed-dotted line); and (iv) \( \text{Re} \approx 36.54, \text{Rm} \approx 0.365, k_f/K = 3.13, S_h \approx -0.209 \) (shown by the triple-dot-dashed line).

\section{3.3. \text{Re} < 1 and \text{Rm} > 1}

We now report our analysis concerning the growth of the mean magnetic field in a background linear shear flow with non-helical forcing at small scales for the case where \( \text{Re} < 1 \) and \( \text{Rm} > 1 \). This is a particularly interesting regime for the following reasons: (i) it is an important fact to note that in the

\[ \eta/\eta_T \text{ Re}^2 \]

\[ r_{\text{rms}} k_f \]

\[ B_{\text{rms}} / B_{\text{eq}} \]

\[ \text{Re} \approx 0.128, \text{Rm} \approx 0.643 \]
limit of small Re, the non-helical forcing has been shown to give rise to a non-helical velocity field (see the discussion below Equation (46) of Singh & Sridhar 2011); (ii) for low Re the Navier–Stokes Equation (1) can be linearized, and thus it becomes an analytically more tractable problem compared to the case of high Re. Such solutions have been rigorously obtained without the Lorentz forces and were presented in Singh & Sridhar (2011). Therefore, it appears more reasonable to develop a theoretical framework in the limit Re < 1 and Rm > 1 before one aims to develop a theory which is valid for both (Re, Rm) > 1. Such thoughts motivated us to perform numerical experiments in this limit to look for the dynamo action. Figures 6–8 display the time dependence of the root-mean-squared value of the total magnetic field \( B_{\text{rms}} \) and spacetime diagrams of mean fields \( B_1(x, t) \) and \( B_2(x, t) \) for three different combinations of Re and Rm. These simulations were performed with 128³ mesh points. We have scaled the magnetic fields in Figures 6–8 with respect to \( B_{\text{eq}} \), where \( B_{\text{eq}} = (\mu_0 \langle \nu_{\text{rms}}^2 \rangle)^{1/2} \). Scalings in these figures have been chosen for compatibility with Figures 7 and 8 of Brandenburg et al. (2008). Below, we list a few useful points related to the dynamo action when Re < 1 and Rm > 1 based on careful investigation of Figures 6–8.

(i) The top panels of Figures 6–8 clearly show the growth of \( B_{\text{rms}} \), demonstrating the shear dynamo due to non-helical forcing \( (B_{\text{rms}}^2 = \langle B^2 \rangle + \langle b^2 \rangle) \), where \( B \) and \( b \) are the magnitudes of the mean and fluctuating magnetic fields, respectively. Thus, the \( B_{\text{rms}} \) field may grow either due to \( B \) or \( b \), or due to both \( B \) and \( b \).

(ii) Denoting the magnetic diffusion timescale as \( \tau_b = (\eta k^2)^{-1} \) and the eddy turn over timescale as \( \tau_{\text{edd}} = (\nu_{\text{rms}} k_f) \), we write \( \tau_b = \text{Rm} \tau_{\text{edd}} \). The magnetic fields in these simulations survive for times, say \( t = 640 \tau_{\text{edd}} \), which for Rm ≈ 32 (corresponding to Figure 8) implies \( t \approx 20 \tau_b \), i.e., 20 times the diffusion timescale. This is a clear indication of the dynamo action as the magnetic fields survive much longer than the magnetic diffusion timescale.
starts growing at earlier times. The magnetic field dynamo action. As mentioned earlier, our interest is in the range of values considered in this work. For larger values of Rm, we refer the reader to Brandenburg et al. (2008) where the possibility of the relevant component, \( \eta_{21} \), becoming negative at much larger Rm (>100) was discussed; however, the error bars were quite large, and therefore no conclusion could be drawn regarding the mean-field dynamo action. Although the forcing is done at a single length scale, a typical kinetic energy spectrum has a peak at the stirring scale with significantly less power at other length scales (e.g., see the dashed lines in the various panels of Figure 9). In Figure 9, we display the energy spectra obtained in one of the three simulations (for different combinations of the control parameters, all with Re < 1), corresponding to that shown in Figure 8. Thus, Figures 8 and 9 show results obtained from one particular simulation with 128\(^3\) mesh points, Re ≈ 0.641, Rm ≈ 32.039, \( k_f/K = 5.09 \), and \( S_h \approx -0.60 \). A few noteworthy points are discussed below in detail.

(i) Initially, the magnetic power is very small compared to the kinetic power and it is mainly concentrated at large \( k \) (i.e., small length scales), as may be seen from panel (a) of Figure 9. Also, there is essentially no magnetic power at small \( k \) (i.e., large length scales) during the initial stage of the simulation.

(ii) The strength of the total magnetic field decreases up to a certain time due to dissipation (compare panels (a) and (b) of Figure 9), before it starts building up due to the dynamo action.

(iii) From the top panel of Figure 8, we see that the root-mean-squared value of the total magnetic field starts growing due to the dynamo action (\( B_{rms}^2 = \langle B^2 \rangle + \langle b^2 \rangle \)), where \( B \) and \( b \) are the magnitudes of the mean and fluctuating magnetic fields, respectively). As the \( B_{rms} \) field may grow either due to \( B \) or \( b \), or due to both \( B \) and \( b \), it seems necessary to understand this in more detail. From Figure 9, it may be seen that the magnetic energy, when it begins to grow, grows at all scales until it saturates at small length scales.

(iv) The small-scale field grows, which averages out to zero, and hence does not show up in the spacetime diagrams of Figure 8. This is generally referred to as the fluctuation dynamo. The growth rate changes and becomes smaller after the fluctuation dynamo saturates (which happens at \( t_{rms} k_f \approx 150 \) in Figure 8 and the corresponding power spectrum at that time is shown in panel (d) of Figure 9).
(v) Although there is non-zero magnetic energy at the large scales at time $t \approx 150$ (see panel (d) of Figure 9), we begin to see some features in the spacetime diagrams of the mean magnetic field (shown in Figure 8) only beyond $t \approx 150$. Thus, it is possible that $B = 0$ while $(B^2)$ is finite.

(vi) The mean magnetic field starts developing beyond $t \approx 150$ (which is about five times the magnetic diffusion timescale) and saturates at $t \approx 330$ (see Figure 8), after which time the magnetic energy essentially stops evolving at all length scales, as may be seen from Figure 9.

(vii) When the magnetic energy saturates at some value, we see significant magnetic power at the largest scale.

We recall that in the kinematic stage, the magnetic field at all length scales grows at the same rate, i.e., the magnetic spectrum remains shape invariant (Brandenburg & Subramanian 2005; Subramanian & Brandenburg 2014). From panels (b) to (d) of Figure 9, the magnetic spectrum evolves in a nearly shape invariant manner. During this kinematic stage, much of the magnetic power still lies at small scales, but the power at the largest scales also grows with time. This initial growth of magnetic energy occurs at turbulent (fast) timescales. Toward the end of the kinematic stage, the growth rates of large- and small-scale magnetic fields become different due to the back reaction from Lorentz forces. The small-scale fields saturate, whereas the large-scale fields continue to grow, thus dominating over small-scale fields at much later times; shown in panels (e) and (f) of Figure 9.

It may be seen from the top panels of Figures 6–8 that $B_{rm}$ shows exponential growth. We denote the initial exponential growth rate of $B_{rm}$ as $\gamma$. It is evident from Figure (10) that the dimensionless growth rate ($\gamma' = \gamma/(\nu \eta)$) appears to scale as $\gamma' \propto -S_h$ in the range of parameters explored in this work. This result is in agreement with (Brandenburg et al. 2008; Yousef et al. 2008b; Heinemann et al. 2011; Richardson & Proctor 2012).

In Table 3, we summarize the details of various simulations performed in different parameter regimes. We note that larger shear contributes positively to the mean-field dynamo action; compare Runs C1 and C2, where the shear in C2 is five times larger compared to that in C1, with the rest of the parameters being the same.

### 4. CONCLUSIONS

We performed a variety of numerical simulations exploring the different regimes of the control parameters for the shear dynamo problem. The simulations were performed for the three following parameter regimes: (i) both $(Re, Rm) < 1$; (ii) $Re > 1$ and $Rm < 1$; and (iii) $Re < 1$ and $Rm > 1$. These limits, which were never explored in any earlier works, appeared interesting to us for following reasons: first, to compare the analytical findings of Singh & Sridhar (2011) with the results of numerical simulations in the parameter regimes when both $(Re, Rm) < 1$; and second, to look for the growth of the mean magnetic field in the limit where $Re < 1$. Exploring the possibility of the dynamo action when $Re < 1$ seems particularly interesting because, in the limit of small $Re$, non-helical forcing has been shown to give rise to non-helical velocity fields (see the discussion below Equation (46) of Singh & Sridhar 2011); whether or not this is true even in the limit of high $Re$ has not yet been proven. Thus, performing the simulation in this limit $(i.e., Re < 1)$ with non-helical forcing guarantees that the fluctuating velocity field is also non-helical. Also, for low $Re$, the Navier–Stokes Equation (1) can be linearized, and thus becomes an analytically more tractable problem compared to the case with high $Re$. Such solutions have been rigorously obtained without the Lorentz forces, and have been presented in Singh & Sridhar (2011).

In the present paper, we successfully demonstrated that the dynamo action is possible in a background linear shear flow due to non-helical forcing when the magnetic Reynolds number is above unity while the fluid Reynolds number is below unity, i.e., when $Re < 1$ and $Rm > 1$ (see Figures 6–9). A few important conclusions may be given as follows.

1. We did not find any dynamo action in the limit where both $(Re, Rm) < 1$ (see Figure 4). We note that all of the simulations were performed in a fixed cubic domain with a size of $2\pi^3$, and the average outer scales of the turbulence in these models were always about 10 times smaller than the domain; see Section 3.1. This scale separation factor of 10 might not yet be sufficient, in principle, and growth at scales larger than the currently chosen $\lambda_3$ extent cannot be ruled out. We computed all of the transport coefficients using test-field simulations and compared with the theoretical work of Singh & Sridhar (2011); see Figures 1–3. A good agreement between the theory and the simulations was found for all of the components of the magnetic diffusivity tensor, $\eta_{ij}$, except for $\eta_{22}$, which is expected to behave in a complicated fashion (Rüdiger & Kitchatinov 2006; Brandenburg et al. 2008; Singh & Sridhar 2011).

2. $\eta_{22}$ was always found to be positive in all of the simulations performed in different parameter regimes. This is in agreement with earlier conclusions that the shear-current effect cannot be responsible for the dynamo action.
Figure 9. Panels (a)–(f) show magnetic (bold line) and kinetic (dashed line) energy spectra from the direct simulation presented in Figure 8 with $Re \approx 0.641$, $Rm \approx 32.039$, $k_f/K = 5.09$, and $S_0 \approx -0.60$ for different values of $(t \Upsilon_{rms} k_f)$. 
agreement with the results from numerical simulations. In a recent analytical study, Sridhar & Singh (2014) have shown that the growth of the mean magnetic field is possible due to fluctuating $\alpha$ with non-zero correlation times in a shearing background. They derive the dimensionless parameters controlling the nature of the dynamo (or otherwise) action. The numerical computation of these dynamo numbers in simulations of the shear dynamo will be the focus of a future investigation.

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REFERENCES

Balbus, S. A., & Hawley, J. F. 1998, RvMP, 70, 1
Binney, J., & Tremaine, S. 2008, Galactic Dynamics (2nd ed.; Princeton, NJ: Princeton Univ. Press)
Blackman, E. G. 1998, ApJL, 496, L17
Brandenburg, A., Rädler, K.-H., Rheinhardt, M., & Käpylä, P. J. 2008, ApJ, 676, 740
Brandenburg, A., Sokoloff, D., & Subramanian, K. 2012, SSRv, 169, 123
Brandenburg, A., & Subramanian, K. 2005, PhR, 417, 1
Heinemann, T., McWilliams, J. C., & Schekochihin, A. A. 2011, PhRvL, 107, 255004
Hughes, D. W., & Proctor, M. R. E. 2009, PhRvL, 102, 044501
Käpylä, P. J., Korpi, M. J., & Brandenburg, A., 2008, A&A, 491, 353
Kleeröni, N., & Rogachevskii, I. 2008, PhRvE, 77, 036307
Krause, F., & Rädler, K.-H. 1980, Mean-field Magnetohydrodynamics and Dynamo Theory (Oxford: Pergamon Press, Ltd.)
Leprovost, N., & Kim, E.-j 2009, ApJL, 696, L125
Lithwick, Y. 2007, ApJL, 670, 789
McWilliams, J. C. 2012, JFM, 699, 414
Mitra, D., & Brandenburg, A. 2012, MNRAS, 420, 2170
Moffatt, H. K. 1978, Magnetic Field Generation in Electrically Conducting Fluids (Cambridge: Cambridge Univ. Press)
Parker, E. N. 1979, Cosmical Magnetic Fields: Their Origin and their Activity (Oxford: Clarendon Press; New York: Oxford Univ. Press)
Proctor, M. R. E. 2007, MNRAS, 382, L39
Rädler, K.-H., & Stepanov, R. 2006, PhRvE, 73, 056311
Richardson, K. J., & Proctor, M. R. E. 2012, MNRAS, 422, L53
Rogachevskii, I., & Kleeröni, N. 2003, PhRvE, 68, 036301
Rogachevskii, I., & Kleeröni, N. 2004, PhRvE, 70, 046310
Rogachevskii, I., & Kleeröni, N. 2008, AN, 329, 732
Rüdiger, G., & Kitchatinov, L. L. 2006, AN, 327, 298
Shapovalov, D. S., & Vishniac, E. T. 2011, ApJ, 738, 66
Silant’ev, N. A. 2000, A&A, 364, 339
Singh, N. K., & Sridhar, S. 2011, PhRvE, 83, 056309
Sokolov, D. D. 1997, ARep, 41, 68
Sridhar, S., & Singh, N. K. 2010, JFM, 664, 265
Sridhar, S., & Singh, N. K. 2014, MNRAS, 445, 3770
Sridhar, S., & Subramanian, K. 2009a, PhRvE, 80, 066315
Sridhar, S., & Subramanian, K. 2009b, PhRvE, 79, 045305
Subramanian, K., & Brandenburg, A. 2014, MNRAS, 445, 2930
Sur, S., & Subramanian, K. 2009, MNRAS, 392, L6
Vishniac, E. T., & Brandenburg, A. 1997, ApJ, 475, 263
Vishniac, E. T., & Cho, J. 2001, ApJ, 550, 752
Wisdom, J., & Tremaine, S. 1988, AJ, 95, 925
Yousef, T. A., Heinemann, T., Rincon, F., et al. 2008a, AN, 329, 737
Yousef, T. A., Heinemann, T., Schekochihin, A. A., et al. 2008b, PhRvL, 100, 184501