Zero forcing number of graphs with a power law degree distribution

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The zero forcing number is the minimum number of black vertices that can turn a white graph black following a single neighbour colour forcing rule. The zero forcing number provides topological information about linear algebra on graphs, with applications to the controllability of linear dynamical systems and quantum walks on graphs among other problems. Here, I investigate the zero forcing number of undirected simple graphs with a power law degree distribution $p_k \sim k^{-\gamma}$. For graphs generated by the preferential attachment model, with a diameter scaling logarithmically with the graph size, the zero forcing number approaches the graph size when $\gamma \rightarrow 2$. In contrast, for graphs generated by the deactivation model, with a diameter scaling linearly with the graph size, the zero forcing number is smaller than the graph size independently of $\gamma$. Therefore the scaling of the graph diameter with the graph size is another factor determining the controllability of dynamical systems.

I. INTRODUCTION

The mathematical formulation of complex systems made of heterogeneous components are plagued with unknown parameters. Yet, we can make general conclusions based on their topology. Topological analyses has been particularly successful in the investigation of complex dynamical systems, including the characterization of their phase space \cite{1} and their controllability \cite{2, 3}. Directed graphs are the natural backbone of dynamical systems with asymmetries between the forward and backward interactions between components. In contrast, undirected graphs are the choice of topology for classical and quantum walks on graphs and other quantum systems \cite{4, 5}.

In the control theory of classical linear dynamical systems, the topological analyses have been mapped to finding zero forcing sets. The zero forcing set is equivalent to finding the maximum matching of that directed graph \cite{6}. Since the running time of the Hopcroft-Karp algorithm is faster than $O(N^3)$, computing the matching number of a directed graph can be done in polynomial time ($P$ class of computational complexity).

In the context of the control theory of quantum linear dynamical systems, the topological analyses have been mapped to finding zero forcing sets. The zero forcing set of an undirected graph is defined as a subset of vertices that can change a graph coloured white to black according to the following dynamical rules. Start with a set of vertices coloured black and all other vertices white. At each step, all black vertices with exactly one white neighbour, colour that neighbour black. If the process ends colouring the whole graph black then the starting set of black vertices is a zero forcing set. The zero forcing number of a graph $G$, denoted by $Z(G)$, is defined as the size of the zero forcing set with minimum size. Burgarth et al demonstrated that finding the minimal input set to control quantum linear systems is equivalent to finding the zero forcing number of the associated undirected graph \cite{7}.

The zero forcing number has been studied extensively in the context of linear algebra because it allows to obtain bounds to the minimum rank of a graph \cite{2, 8, 9}. The minimum rank of a graph $G$, denoted by $R(G)$, is defined as the minimum rank of all real matrices with non-zero elements on the graph edges \cite{8}. It has been mathematically proven that \cite{9}.

\begin{equation}
N - Z \leq R
\end{equation}

where the equality is warranty when $G$ is an undirected or directed tree. Equation (2) holds true for graphs without loops (simple), with loops, undirected and directed, after small changes in the forcing rule \cite{8}. As demonstrated by Liu, Slotine and Barabási \cite{6}, we can use the Hopcroft-Karp algorithm to find the minimal set of inputs to control simple directed graphs in polynomial time. However,
finding the zero forcing sets of directed graphs with loops [10] or of undirected graphs [2] belongs to the class of NP-hard problems.

Many graphs representing real systems are characterized by a power low degree distribution \( p_k \sim k^{-\gamma} \), in-degree distribution \( p_i \sim i^{-\gamma_{in}} \) or out-degree distribution \( p_o \sim o^{-\gamma_{out}} \) [11]. Liu, Slotine and Barabási have shown that [6], for simple directed graphs with \( \gamma_{in} = \gamma_{out} = \gamma \), the minimum driving number is given by

\[
D(G) \approx N e^{-\frac{1}{2} \left( \frac{2}{\gamma} \right)^{(k)}}
\]

which approaches \( N \) when \( \gamma \to 2 \). Given the computational complexity differences between the zero forcing problems on undirected and directed graphs, it is worth asking whether \( Z(G) \to N(G) \) for undirected graphs with \( \gamma \to 2 \). Furthermore, we would like to know whether \( \gamma \to 2 \) is a sufficient condition.

II. LEAF REMOVAL ALGORITHM

Leaf-removal algorithms have been used to tackle NP-hard problems on graphs with a power law degree distribution [12] [13]. In the context of zero forcing, we can take advantage of the fact that, for a vertex with \( L \) leafs, we have no other choice but including including taking advantage of the fact that, for a vertex with

\[
\text{leaf-removal rule does not create new leafs and, therefore, the algorithm will stop after all vertices with two or more leafs have been pruned. To re-start the process I will remove the vertex with the largest degree and add it to the zero forcing set. Putting all together the algorithm proceed as follows.}
\]

1. Start with all vertices coloured white and an empty zero forcing set.
2. Create a list of all leafs in the current graph.
3. For each leaf in the list,
   - If the leaf is black, remove the leaf and colour the neighbour black. In the event that the neighbour becomes itself a leaf proceed recursively until no new leaf is created.
   - Otherwise, find all adjacent leafs, colour them black, remove them and add them to the zero forcing set.
4. If the graph is not empty, find a vertex with the current largest degree, colour the vertex black, remove the vertex, add the vertex to the zero forcing set, and move all neighbour vertices that become a leaf to the leaf list.
5. If the graph is empty stop, otherwise go back to step 2.

The size of the resulting zero forcing set will be denoted by \( Z_{LM}(G) \), where \( LM \) stands for leaf and maximum degree removal. Since the maximum degree removal, step 4, is not necessarily optimal, this algorithm overestimates the zero forcing number,

\[
Z_{LM} \geq Z
\]

As a comparison, I will also calculate the minimum vertex covering using an adaptation of the vertex covering leaf removal algorithm [12]. The algorithm proceed as follows. Start with all vertices uncovered and a list of leafs in the graph. If the leaf list is non-empty, extract a leaf, remove the leaf, add the leaf neighbour to the vertex covering set and remove the leaf neighbour. Otherwise, find a vertex with the current largest degree, add it to the vertex covering set and remove it. Continue until the graph is empty. The minimum vertex covering will be denoted by \( V(G) \). The vertex cover size estimated by the leaf-maximum degree removal will be denoted by \( V_{LM}(G) \). Once again, because of the maximum degree removal rule, this algorithm overestimates the minimum vertex covering,

\[
V_{LM} \geq V
\]

III. PREFERENTIAL-ATTACHMENT GRAPHS

First, I will consider graphs generated by the preferential attachment model with initial attractiveness [14] (\( G_{PA} \)). The graph is started with a fully connected graph of \( m+1 \) vertices. Then add new vertices one by one until the targeted graph size is reached. Each time a new vertex is added it is connected to \( m \) existing and non-overlapping vertices, each selected with probability

\[
\pi_i = \frac{(a-1)m + k_i}{\sum_j (a-1)m + k_j}
\]

where the indexes run over vertices in the current graph and \( k_i \) denotes the degree of vertex \( i \). The parameter \( a > 0 \) represents a vertex independent attractiveness named initial attractiveness [14]. This model generates graphs with a power law degree distribution \( p_k \sim k^{-\gamma} \) with exponent [14]

\[
\gamma = 2 + a
\]

By tuning the initial attractiveness \( a \) we can obtain exponents in the range \( 2 < \gamma < \infty \). The case \( a = 1 \) (\( \gamma = 3 \)) corresponds to the original Barabási-Albert model [15].
Using the LM algorithms, I have estimated the zero forcing number and minimal vertex covering of graphs generated by the preferential attachment model (Fig. 2). We can observe that the zero forcing number approaches the graph size while the minimum vertex covering approaches zero. Around $\gamma = 2$ we observe the scalings (Fig. 3)

$$1 - Z_{LM}(G_{PA}) \approx c_z N(\gamma - 2)$$  \hspace{1cm} (8)

$$V_{LM}(G_{PA}) \approx c_v N(\gamma - 2)$$  \hspace{1cm} (9)

This linear scaling coincides with the result for directed graphs [6], as it can be verified expanding equation (8) around $\gamma = 2$. Substituting the scaling in equation (10) into the lower bound to the minimum graph rank (2) we obtain the minimum rank scaling

$$R(G_{PA}) \geq c_z N(\gamma - 2)$$  \hspace{1cm} (10)

Therefore, when $\gamma \approx 2$ there is the possibility that the minimum rank is 0.

Since the LM algorithm yields upper bounds it is worth asking how tight are those bounds, specially for $\gamma \approx 2$. To address this question I have calculated the number of vertices that were forced using the maximum degree step, denoted by $\Delta Z(G)$ and $\Delta V(G)$ for the zero forcing and vertex covering algorithms, respectively. We observe that $\Delta V(G_{PA}) / N \approx 0$ for all values of $\gamma$ explored (Fig. 4b). Therefore

$$V_{LM}(G_{PA}) \approx V(G_{PA})$$  \hspace{1cm} (11)

In contrast, $\Delta Z_{LM}(G_{PA}) > 0$ for most values of $\gamma$ and we cannot tell how good are the zero forcing number estimates. Nevertheless, for $\gamma \rightarrow 2$ we observe that $\Delta Z_{LM}(G_{PA}) / N \rightarrow 0$. In the vicinity of $\gamma = 2$ the LM algorithm provides good estimates of the zero forcing num-

FIG. 2. Zero forcing fraction $z = Z_{LM} / N$ and minimum vertex cover fraction $v = V_{LM} / N$ of preferential attachment graphs, with $N = 10,000$ and averaged over 100 graphs.

FIG. 3. Scaling of the zero forcing fraction $z = Z_{LM} / N$ and minimum vertex cover fraction $v = V_{LM} / N$ of preferential attachment graphs, with $N = 10,000$ and averaged over 100 graphs.
For each isolated star, all but one leaf are subject to the leaf forcing rule in Fig. 1 while the last leaf is forced by the star hub, resulting in

\[ \frac{Z(G_{0*})}{N} = \sum_{k>1} p_k(k-1) = 2p_1 - 1 < 1 \]  

A similar result is obtained for graphs made by a string of stars \((G-\star)\). In this case the degree distribution satisfy the constraints

\[ \sum_{k>1} p_k(k-1) = p_1 \]  
\[ \sum_{k>1} p_k = p_2 \]

Applying the leaf removal rule (Fig. 1) to all leafs of the first star will yield a new string of stars where the first star has been removed. Applying this star removal recursively will end forcing the all vertices. Except for the first star, we need to force \(k-2\) leafs at each star removal, resulting in

\[ \frac{Z(G-\star)}{N} = \sum_{k>1} p_k(k-2) = p_1 - p_2 < 1 \]

Therefore, regardless of the shape of the degree distribution, \(Z(G_{0*}) < N\) and \(Z(G-\star) < N\). A power low degree distribution \(p_k \sim k^{-\gamma}\) with \(\gamma \rightarrow 2\) is not a sufficient condition for \(Z(G) = N(G)\).

The deactivation graphs of Klemm and Eguíluz [16] \((G_D)\) provide another counter example inspired on natural rules of network evolution. The basic idea of the deactivation graph model is that, with time, some vertices will no longer participate in the network evolution, becoming inactive or deactivated. That could model the retirement of a scientist in the context of co-authorship networks for example. The deactivation graphs studied here are generated as follows. Start with a fully connected graph of \(m+1\) active vertices. At each graph evolution step, add a new active vertex, connect the new vertex to the pre-existing \(m\) active vertices and set one of the active vertices \((i \in A)\) inactive with a probability

\[ \pi_i = \frac{\sum_{s \in A} ((a-1)m + k_s)^{-1}}{(a-1)m + k_i} \]

The deactivation model generate graphs with a power low degree distribution \(p_k \sim k^{-\gamma}\) with exponent \([16]\)

\[ \gamma = 2 + a \]

By tuning \(a\) we can thus generate power law exponents in the range \(2 \leq \gamma < \infty\).

Using the leaf-maximum degree removal algorithms, I have estimated the zero forcing number and minimal vertex covering of deactivation graphs (Fig. 5). For the
deactivation graphs the zero forcing number does not approach the graph size when $\gamma \to 2$ (Fig. 5a). In fact, $Z_{LM}(G_D)/N < 1$ and $V_{LM}(G_D)/N > 0$ for all values of $\gamma$ (Fig. 5a,b).

As shown before, there is a key difference between the preferential-attachment and deactivation graphs regarding the graph diameter, denoted by $d$. The preferential-attachment generate small-world graphs for $\gamma > 3$ ($d \sim \ln N$) and ultra-small graphs ($d \sim \ln \ln N$) for $2 < \gamma < 3$ [17]. In contrast, deactivation graphs are effectively one-dimensional ($d \sim N$) [18]. This, together with the analysis of star graphs, indicates that the small-world property is a requirement to obtain $Z(G) \approx N$ when $\gamma \to 2$.

V. CONCLUSIONS

By means of numerical simulations, I have demonstrated that, in the class of preferential-attachment graphs, the zero forcing number approaches the graph size ($Z \to N$) when the exponent of the power law degree distribution approaches 2 ($\gamma \to 2$). This extends the Liu-Slotine-Barabási result for directed graphs [6] to undirected graphs. Through the analysis of some counterexamples, I have also shown that the small-world property of preferential-attachment graphs is a necessary requirement for this result.

[1] R. Gilmore, Rev. Mod. Phys. 70, 1455 (1998).
[2] M. Trefois and J.-C. Delvenne, Linear Algebra Its Appl. 484, 199 (2015).
[3] Y.-Y. Liu and A.-L. Barabási, Rev. Mod. Phys. 88, 035006 (2016).
[4] C. Godsil and S. Severini, Phys. Rev. A 81, 052316 (2010).
[5] M. Faccin, T. Johnson, J. Biamonte, S. Kais, and P. Migdal, Phys. Rev. X 3, 041007 (2013).
[6] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, Nature 473, 167 (2011).
[7] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young, IEEE Transactions on Automatic Control 58, 2349 (2013).
[8] A. M. R. S. G. W. Group, Linear Algebra Its Appl. 428, 1628 (2008).
[9] L. Hogben, Linear Algebra and its Applications 432, 1961 special issue devoted to the 15th ILAS Conference at Cancun, Mexico, June 16-20, 2008.
[10] Aazami, A., “Hardness results and approximation algorithms for some graph problems,” (2008).
[11] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[12] A. Vázquez and M. Weigt, Phys. Rev. E 67, 027101 (2003).
[13] J.-H. Zhao, Y. Habibulla, and H.-J. Zhou, J. Stat. Phys. 159, 1154 (2015).
[14] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. 85, 4633 (2000).
[15] A. Barabási and A. Réka, Science 286, 509 (1999).
[16] K. Klemm and V. Eguíluz, Phys. Rev. E 65, 036123 (2002).
[17] R. Cohen and S. Havlin, Phys. Rev. Lett. 90, 058701 (2003).
[18] A. Vázquez, M. Boguñá, Y. Moreno, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. E 67, 046111 (2003).