Scaling Properties of Scale-Free Networks in Degree-Thresholding Renormalization Flows

Dan Chen, Defu Cai, and Housheng Su

Abstract—We study the statistical properties of observables of scale-free networks in degree-thresholding renormalization (DTR) flows. For Barabási-Albert (BA) scale-free networks with different sizes, we find that their structural and dynamical observables have similar scaling behavior in the DTR flow. The finite-size scaling analysis confirms this view and reveals a scaling function with a single scaling exponent that collectively captures the changes of these observables. Furthermore, for the scale-free network with a single initial size, we use its DTR snapshots as the original networks in the DTR flows, then perform a similar finite-size scaling analysis. Interestingly, the initial network and its snapshots share the same scaling exponent as the BA synthetic network. Our findings have important guiding significance for analyzing the structure and dynamic behavior of large-scale networks. Such as, in large-scale simulation scenarios with high time complexity, the DTR snapshot could serve as a substitute or guide for the initial network and then quickly explore the scaling behavior of initial networks.

Index Terms—Degree-thresholding renormalization, finite-size scaling, scaling exponent, scale-free networks.

I. INTRODUCTION

T HE network widely exists in nature and human society and is a common language for describing and modeling complex systems. Its emergence contributes to a better understanding of the structure and functional attributes of the system [1], [2]. In this regard, the scale-free network [3] is undoubtedly a good example. Its degree distribution can usually be characterized by a power-law distribution of the form \( p(k) \sim k^{-\lambda} \). In addition, this characteristic effectively distinguishes scale-free networks from regular networks, Erdős-Rényi (ER) random networks [4], and Watts-Strogatz (WS) small-world networks [5]. As a result, scale-free networks have many peculiar properties, such as robustness under random attacks, vulnerability under targeted attacks [6], and faster spreading speed of viruses on scale-free networks [7].

In the following years, another important property of networks, self-similarity [8], has also been discovered by borrowing concepts related to the renormalization group [9] in statistical physics. Some major studies include: Song et al. [8] proposed a box-covering renormalization (BCR) method based on shortest path lengths between network nodes to reduce the size of the network, and they found that the degree distribution of real networks such as WWW remained approximately unchanged during the BCR iteration. Serrano et al. [10] presented a simple degree-thresholding renormalization (DTR) technique for mining the self-similar properties of real networks. Recently, García-Pérez et al. [11] offered a network geometric renormalization (GR) approach in the context of the hidden metric space model [10], [12], [13], [14], [15], which provides another insight for studying the structural symmetry of networks. Precisely, the BCR depends on the shortest path length between nodes, and the GR requires embedding the network into a hidden space [16]. The core idea behind the BCR and GR methods is to coarse-graining multiple nodes into a supernode, resulting in the formation of smaller-scale replica networks. In contrast, the DTR procedure induces a smaller-scale subgraph by extracting nodes larger than a given degree threshold in the initial network. Technically, the DTR procedure is easier to execute. In short, the significance of the network’s renormalization is finding smaller-scale replicas to replace the original network, then effectively exploring the self-similarity of the real-world network.

Renormalization is useful for transforming large-scale networks into smaller ones, and the DTR is especially simple and convenient in this regard. Therefore, in the context of DTR, we conduct a series of studies on synthetic and real scale-free networks. First, we find that the DTR procedure can approximately maintain the important structural properties of Barabási-Albert (BA) [3] and Chung-Lu (CL) [17], [18] scale-free networks (see Section II-A). Then, our results show that some observables of BA and CL scale-free networks follow the scaling properties in the DTR flow (see Section III-B). The finite-size scaling (FSS) [19] analysis confirms this view and reveals a scaling function to capture the observable’s variation (see Section III-C). Furthermore, the results of a CL or real scale-free network with a single initial size, using the DTR procedure to obtain the corresponding smaller-size subnetwork, the results show that the initial network and its subnetworks share the same scaling exponent as the BA network (see Section III-C). Finally, we present the conclusion of the paper and the prospect of future work (see Section IV).

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The main contributions of the paper are summarized as follows:

a) The statistical properties of representative observables of scale-free networks under the DTR procedure are studied and show that these observables obey the scaling law.

b) For BA scale-free networks with different initial sizes, the FSS confirmed this universal scaling law and revealed a scaling function with a single exponent (α = 1) to capture the behavior of observables in DTR flows. For CL scale-free networks with different initial sizes, our results show that the scaling exponent depends on its degree distribution exponent λ.

c) Finally, for the CL or real-world scale-free network with a single initial size, our results show that the initial network and its DTR snapshots share the same scaling exponent as the BA synthetic network.

d) From the perspective of the application, the scaling exponent obtained here can be used as the foundation for predicting the structure and dynamic characteristics of the large-scale network, which has important guiding significance for reducing the time complexity of the large-scale numerical simulation.

II. PRELIMINARIES

A. Network Models

The BA scale-free network is a classic and highly significant model in network science. Its importance lies in its ability to provide insights into the scale-free properties and evolutionary mechanisms of networks. The formation of BA scale-free networks is primarily driven by two factors: growth and preferential attachment [3]. In this context, the network eventually develops into a scale-free network with a degree distribution exponent λ ≈ 3. The exact form of the degree distribution of the BA model is p(k) = [2m(m + 1)]/[k(k + 1)(k + 2)]. For large k, approximately satisfies p(k) ∼ k−3, where the parameter m is the number of links increased after adding a new node. The paper uses the Python-based NetworkX library [20] to generate a BA scale-free network.

In addition, compared with the real scale-free network, BA scale-free network is a special case. To this end, we employ a more general scale-free network model, CL scale-free network [17, 18], as an extension model, its degree distribution satisfies p(k) ∼ k−λ, and the exponent λ is used to control the heterogeneity of the network. We employ the recent algorithm presented by Fasino et al. [18] to create a series of CL networks with different sizes and degree distribution exponents. Specifically, the model needs to define a nonnegative real vector

\[ \mathbf{w} = (w_1, \ldots, w_{N_0})^T, w_i = c/(i + i_0)^p, \]

where \( i = 1, 2, \ldots, N_0, \ p = 1/(\lambda - 1), \ c = (1 - p)\langle k \rangle N_0^p, \ i_0 = \left( c/\kappa_{\text{max}} \right)^{1/p} - 1, \) \( \langle k \rangle \) and \( \kappa_{\text{max}} \) are the upper limits of the expected values of the average degree and the largest degree, respectively.

B. DTR

Serrano et al. [10] have shown that some basic properties of real scale-free networks and a class of geometric model, such as the complementary cumulative degree distribution, degree-dependent clustering coefficient, and degree-degree correlations, are self-similar in DTR flows. The specific steps of DTR flows are as follows: for an initial network \( G_0 \) with \( N_0 \) nodes, given degree threshold \( k_T = 0, 1, 2, \ldots \), then, nodes with degrees \( k > k_T \) are extracted from \( G_0 \) to obtain the subgraph \( G(k_T) \) (i.e., nodes with degrees less than or equal to \( k_T \) are deleted), we thereby obtain a series of downscaled subnetworks, and the number of nodes contained in the subnetwork \( G(k_T) \) is denoted by \( N_{k_T} \). Starting from a BA scale-free network with 100 nodes, we delete those nodes with degrees \( k \leq k_T = 2, 5 \) from the initial network, and obtain two subgraphs (see Fig. 1). The red node in the center is the hub node, which shows that the hub node always exists in the network during this process.

III. RESULTS AND DISCUSSION

A. Self-Similarity of Scale-Free Networks Along the DTR Flow

In this subsection, we examine the complementary cumulative degree distribution, degree-dependent clustering coefficient, and degree-degree correlations of synthetic scale-free networks. The results show that these fundamental structural properties are self-similar in DTR flows.

Figs. 2(a), (b) and (c) show the complementary cumulative degree distribution \( P_c(k) \), degree-dependent clustering coefficient \( c(k) \), and degree-degree correlations \( \kappa_{nn}(k) \) of a BA scale-free network and its corresponding subgraphs, respectively. Here \( c(k) \) is defined as the average of the local clustering coefficients of all nodes with degree k. The degree-degree correlation is measured by normalized average nearest-neighbor degree \( \kappa_{nn}(k) = \kappa_{nn}(k)/\langle k \rangle \). The results of different subgraphs can collapse on almost the same main curve, which indicates that these two characteristics of BA networks are self-similar in the DTR flow.
In this subsection, we study the statistical properties of observables of BA scale-free networks, CL scale-free networks, and real scale-free networks in the DTR flow. Our results show that these observables approximately obey the scaling law in the DTR flow. Specifically, we investigate some basic and important topological characteristics of the network, such as the largest node degree, average degree (edge density), average clustering coefficient, and the $i$th moment of the degree distribution.

Taking the largest degree $k_{T,max}$ as an example, in general, the probability that a node’s degree value is equal to or higher than $k_T$, which also seems to reflect the essential difference between the BA scale-free network and some self-similar synthetic scale-free networks (such as $S^3$ model [10]). We only show the result of $m = 5$, error bars show the standard deviations for different realizations, and all results are averaged over $100$ independent realizations.

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scarcity characteristics of the largest node degree of the initial BA scale-free network.

Fig. 3(b) shows the behavior of the subgraphs’ average degree \(\langle k \rangle_{k_T} \) in the DTR flow. Due to the finite size effect, the average degree of the BA network decreases slightly with the increase of \(k_T\). However, in the thermodynamic limit, it is obvious that the average degree of the subgraphs should be constant as a function of the degree threshold. The inset shows the dependence of the normalized average degree \(\langle k \rangle_{k_T} = \langle k \rangle_{k_T}/(N_{k_T} - 1)\) on \(N_{k_T}\), which approximately satisfies the power-law behavior, \(\langle k \rangle_{k_T} \sim N_{k_T}^{-\gamma}\). Fig. 3(c) shows the average clustering coefficient of the subgraph gradually increases along the direction of DTR flow. The results of the inset show that \(c_{k_T} \sim N_{k_T}^{-\mu}\), and the red line is the fitting result. Klemm et al. [23] have proved that the average clustering coefficient of the BA network meets \(c \sim (ln N)^2/N\), and the average clustering coefficient of subgraph \(G_{k_T}\) can be calculated based on this conclusion, as shown in the green circle, our results show that the green circle and the red square roughly overlap, which implies that the DTR procedure do not change the statistical law of the average clustering coefficient for the initial BA network.

We also performed further research on CL scale-free networks and \(S^1\) geometric models, as shown in Figs. S3 and S4 of the Supplemental Material. For CL models, except for the average degree of subgraphs, the other results are similar to the BA network. For CL and \(S^1\) models, the average degree of the subnetwork increases with the increase of \(k_T\) [see Fig. S3(b) and Fig. S4(b)]. The subnetwork’s average degree depends on many factors, among which the structural characteristics of the network and the finite-size effect may be significant factors. In addition, from the results of BA and CL networks, it seems that the DTR cannot well retain the average clustering coefficient of initial networks [see Fig. 2(b), Fig. 3(c), Fig. S1(b), and Fig. S3(c)] but can well retain this feature of \(S^1\) models [see Fig. S2(b) and Fig. S4(c)].

Recently, Serafino et al. [16] have shown that finite-size effects usually hide the scale-free properties of real-world networks through finite-size scaling analysis and moment ratio tests. In this context, we employ the moment ratio test to verify the scale invariance of BA scale-free networks in DTR flows. The moment of degree distribution is helpful to understand the meaning of scale-free term [2]. Specifically, for a scale-free network, the \(i\)th moment of the degree distribution is

\[
\langle k^i \rangle = \int_{k_{min}}^{\infty} k^i p(k) dk.
\]

When \(i > \lambda\), the ratio of the \(i\)th moment of the degree distribution to the \((i - 1)\)th moment is independent of \(i\), satisfies \(\langle k^i \rangle / \langle k^{i-1} \rangle \sim N^d\), where \(d > 0\). For BA scale-free networks, the moments ratios for different \(i\) are parallel lines, as shown in Fig. 4(a). Additionally, we consider the BA scale-free network with the initial size of \(N = 10^5\) and perform DTR procedure on it to obtain a series of subnetworks. Then, applying the moment ratio experiment to these subnetworks, as shown in Fig. 4(b), we obtain a similar result, namely \(\langle k^i \rangle / \langle k^{i-1} \rangle \sim N^d\), where \(d \approx d_{k_T}\). We display the results of Fig. 4(a) and (b) on the same figure, as shown in Fig. 4(c), the result shows that the DTR procedure can roughly reverse the preferential attachment evolutionary growth process of the BA network, which means that the DTR can approximately return the BA network to an earlier state.

The results of Fig. 2 show that BA networks’ degree distribution and degree-degree correlation have scale invariance in the DTR flow. The largest degree, the normalized average degree, and the average clustering coefficient approximately obey the scaling behavior in the DTR flow. To a certain extent, the DTR process of BA networks can be approximately regarded as the opposite direction of its evolutionary growth process, which makes it possible to predict the structural characteristics of large-scale BA networks based on their small-scale subgraphs.

In fact, empirical studies show that the degree distribution of a large number of real-world networks approximately satisfies \(p(k) \sim k^{-\lambda}\), and in general, the exponent \(\lambda \in (2, 3)\). Next, we consider eight real-world scale-free networks, which belong
The basic topological properties of real scale-free networks and the values of some scaling exponents. From left to right, we report the network name, category, number of nodes \( N \), number of edges \( E \), the average degree \( \langle k \rangle \), the degree distribution exponent \( \lambda \), the largest degree exponent \( \beta \), the normalized average degree exponent \( \gamma \), and the scaling exponent \( \alpha^* \).

| Name            | Category   | \( N \)  | \( E \)  | \( \langle k \rangle \) | \( \lambda \) | \( \beta \) | \( \gamma \) | \( \alpha^* \) |
|-----------------|------------|---------|---------|----------------|-------------|----------|----------|-------------|
| Proteome        | Biological | 4100    | 13358   | 6.52           | 2.61        | 0.6781   | 1.3956    | 1.00(1)     |
| Internet        | Technological | 23748  | 58414   | 4.92           | 2.17        | 0.6752   | 1.3098    | 1.00(4)     |
| Caida20071105   | Technological | 26475  | 53381   | 4.03           | 2.17        | 0.7030   | 1.2677    | 1.00(5)     |
| Words           | Text       | 7377    | 44205   | 11.98          | 2.24        | 0.8302   | 1.5289    | 1.0(1)      |
| Frenchbookinter | Text       | 9424    | 23841   | 5.06           | 2.43        | 0.7948   | 1.3914    | 1.0(5)      |
| Japanesebookinter | Text     | 3177    | 7998    | 5.03           | 2.28        | 0.8028   | 1.4240    | 1.00(5)     |
| Youtube         | Affiliation | 124325 | 293342  | 4.72           | 2.49        | 0.7220   | 1.3300    | 1.00(5)     |
| Recordlabel     | Affiliation | 186758 | 232377  | 2.5            | 2.02        | 0.8798   | 0.9633    | 1.00(5)     |

Fig. 5. For real scale-free networks, shows the dependence of the largest degree \( k_{k_T,\max} \) of subnetwork \( G(k_T) \) on the size \( N_{k_T} \), where \( k_{k_T,\max} \sim N_{k_T}^\beta \), and the value of exponent \( \beta \) is shown in Table I.

to four different categories: Biological, Technological, Text, and Affiliation\(^3\). The detailed topological information is shown in Table I. After examining the statistical properties of these networks and their subnetworks, respectively, a conclusion similar to Fig. 3(a) is obtained, that is, the largest degree of the subnetwork satisfies \( k_{k_T,\max} \sim N_{k_T}^\beta \), as shown in Fig. 5. Furthermore, we also studied the dependence of the normalized average degree of subnetwork \( G(k_T) \) on \( N_{k_T} \) (detailed results as shown in Fig. S5), the results further show that \( \langle k \rangle_{k_T} \sim N_{k_T}^\gamma \). However, our results also show that too large degree thresholds \( k_T \) will cause \( \langle k \rangle_{k_T} \) to deviate from this power-law behavior significantly, which is obvious on Proteome, Words, Frenchbookinter, Japanesebookinter, and Youtube networks. For these real networks, \( \langle k \rangle_{k_T} \sim N_{k_T}^\gamma \) is also approximately true in the DTR flow, as shown in Fig. S6. Indeed, even if the scale-free network with a sufficient initial size is considered, a large degree threshold will eventually lead to the loss of their original scale invariance (or self-similarity). For this reason, the FSS analysis in the next section shows that the finite-size effect often masks the underlying scale invariance of many networks.

C. FSS

For complex systems, some observables deviating from thermodynamic limit behavior are often observed in actual numerical simulations due to the limit of infinite size cannot be reached. Here, we show that the potential scale invariance of network observables is often masked by finite-size effects. We employ the FSS analytical tool to prove this view and reveal a scaling function with a single exponent that jointly captures the scaling behavior of these observables. Using \( Z \) represents a particular observable, we find that under the FSS hypothesis,

\[
Z = f \left( n_{k_T} N_{0}^{1/\alpha} \right),
\]

where \( N_0 \) is the size of the initial network, \( n_{k_T} = N_{k_T}/N_0 \) is the relative size of the subnetwork \( G(k_T) \), and the behavior of the observable \( Z \) is completely determined by the exponent \( \alpha \) and the scaling function \( f \). Taking six observables

\(^3\)[Online]. Available: https://icon.colorado.edu/
in our recent work [24] as examples, we next investigate the FSS behavior of these observables in the DTR flow.

For BA scale-free networks with different initial sizes, Fig. 6(a), (c), and (e) show the dependence of structural observables $k_{T,max}$, $\langle k \rangle_T$, and $\langle c \rangle_T$ on the relative size $n_{k_T}$, respectively. The results show that the observable curves of networks with different sizes have very similar behavior in the DTR flow. Eq (3) shows that rescaling the horizontal axis $n_{k_T}$ of these curves to $n_{k_T} N_0^{1/\alpha}$ will cause them to collapse on the same master curve [see Fig. 6(b), (d), and (f)]. Notably, these observables share the same scaling exponent, $\alpha \approx 1$. Here, the optimal exponent $\alpha$ is obtained by measuring the quality $S$ of the collapse plot, where the $S$ is used to measure the mean square distance between the collapse data and the master curve, and its detailed definition is shown in Ref. [25]. Precisely, rescaling the horizontal axis $n_{k_T}$ of the original data points to $n_{k_T} N_0^{1/\alpha}$, and the $S$ between multiple rescaled curves is calculated, respectively, where the minimum point of $S$ corresponds to the optimal collapse exponent $\alpha$. Taking the normalized largest degree $k_{T,max}$ as an example [see Fig. 6(a)], Fig. 7(a) shows the dependence of the $S$ on exponent $\alpha$. When $\alpha = 0.99$, $S$ achieves the minimum value, which means that different curves have the best collapse [see Fig. 6(b)]. Figs. 7(b)–7(d) show the results of Fig. 6(b) at $\alpha = 1.0$, $\alpha = 0.5$, and $\alpha = 1.5$, respectively, and the results show that the overlap quality of Fig. 7(b) is significantly better than that of Fig. 7(c) and Fig. 7(d).

As a supplement, we also consider some dynamical observables that depend on the underlying structure of the network. Such as the normalized smallest nonzero eigenvalue of the Laplace matrix, $\lambda_{k_T,2}(L) = \Lambda_{k_T,2}(L) / N_{k_T}$, where $\Lambda_{k_T,2}(L)$ is the smallest nonzero eigenvalue of the Laplace matrix of the sub-network $G(k_T)$. Previous studies have shown that, in most cases, maximizing the convergence rate to the network-homogeneous state for undirected networks is equivalent to maximizing $\Lambda_{k_T,2}(L)$ [26]. We further consider the ratio of the largest eigenvalue $\Lambda_{k_T,max}(L)$ of the Laplace matrix to the smallest nonzero eigenvalue $\lambda_{k_T,2}(L)$, i.e., $Q_{k_T} = \Lambda_{k_T,max}(L) / \Lambda_{k_T,2}(L)$ [27], [28]. This quantity is related to the stability of the network synchronization process, and it gives the time that the system needs to recover to the stable synchronization state after disturbance [29], [30]. Finally, the largest eigenvalue of the adjacency matrix determines many dynamic behaviors of the network, and its normalized form is $\lambda_{k_T,max}(A) = \Lambda_{k_T,max}(A) / (N_{k_T} - 1)$, where $\Lambda_{k_T,max}(A)$ is the largest eigenvalue of the sub-network’s adjacency matrix. Castellano et al. [31] have shown that the thresholds of two highly correlated dynamic models depend on $\Lambda_{k_T,max}(A)$, one of which is the epidemic spreading, its threshold satisfies $\eta_c = 1 / \Lambda_{k_T,max}(A)$, and the synchronization threshold $\zeta_c = \zeta_0 / \Lambda_{k_T,max}(A)$ [32] for Kuramoto dynamics coupling parameter associated with $\Lambda_{k_T,max}(A)$. Therefore, these observables (called dynamical observables in this paper) have a crucial influence on the dynamical behavior of the underlying network structure, and investigating their scale

![Diagram](image-url)
invariance in the DTR flow is of great significance for predicting the dynamic behavior of large-scale networks.

As shown in Fig. S8, we observed that the scaling behavior of these dynamic observables is similar to that in Fig. 6. Under the rescaling of (3), these curves of BA networks with different sizes could collapse onto the same master curve. In particular, an same scaling exponent, $\alpha = 1$, is produced here. Our results have important guiding significance for studying the dynamic behavior of large-scale networks. For instance, in practice, one possible outcome is to predict the synchronization stability of the initial large-scale network by taking the subnetwork as the study object and combining this scaling exponent. Fig. 6 and Fig. S8 show the results of BA networks under the parameter $m = 5$, and a consistent conclusion can be obtained when the parameter $m$ is set to other values (see Figs. S9 and S10).

We perform the same study as Fig. 6 on CL networks with different heterogeneity exponent $\lambda$ (Figs. S11 and S12 give the results for $\lambda = 2.5$ and 3.5, respectively), examining the dependence of scaling exponent $\alpha$ on the heterogeneity exponent $\lambda$, as shown in Fig. 7 (red solid circle, blue solid square, and green solid triangle). Our results show that, for each observable, the exponent $\alpha$ decreases gradually with the increase of the $\lambda$, which means that the exponent $\alpha$ strongly depends on the heterogeneity of the network. Moreover, for different observables, the exponent $\alpha$ has a large difference. Particularly, when $\lambda = 3$, each observable corresponds to an approximately same exponent, i.e., $\alpha \approx 1$, which is consistent with the results of BA scale-free networks.

When evaluating the scaling exponent $\alpha$ of synthetic networks, we can obtain this result by generating a series of networks with different initial sizes. For real networks, however, there is usually a single initial size. A natural question is how to calculate the scaling exponent for a particular network with a single size. To address this issue, we conduct further research on CL scale-free networks with a single initial size. Specifically, we generate a CL scale-free network $G_0$ with $N_0 = 10^5$, and then obtain the subgraph $G(K_T)$ under the given degree threshold $K_T$. The $K_T$ used here is essentially the same as the $k_T$ used above, with the main difference being that $K_T$ is a degree threshold explicitly set for a single initial network, and its value range is usually limited to a very narrow interval. For example, as shown in Fig. S13, the size of the initial network $G_0$ is $N_0 = 10^5$, its three subgraphs under degree thresholds $K_T = 5, 10, 15$ are obtained, respectively, and the size of $G(K_T)$ is $N(K_T) = 51734, 19901,$ and 10664, respectively. Next, we take $G_0$ and its three subgraphs as new initial reference networks to investigate their FSS behavior in DTR flows (see Fig. S13 and Fig. S14). Therefore, the subgraph size is within a reasonable range in this context. In other words, the subgraph size $N(K_T)$ cannot be too close to $N_0$, nor too small. For simplicity, we chose the appropriate $K_T$ so that the size of the subgraph is roughly equal to $N_0/2, N_0/5,$ and $N_0/10$ (or $N_0/2, N_0/4,$ and $N_0/8$). However, the $k_T$ can take on a wide range of values. For instance, for the BA scale-free network with $N_0 = 10^4$, when $k_T = 100$, the size of the subgraph $G(k_T)$ is $N_{k_T} < 100$, which is more than a hundred times smaller than the original network (see Fig. 3).

In the above context, $G_0$ and its subnetworks are taken as initial networks to study the FSS behavior of their observables in the DTR flow. By analogy with (3), under the FSS hypothesis, the observable $Z$ satisfies,

$$Z = f \left\{ n_{k_T} \left[ N \left( K_T \right) \right]^{1/\alpha} \right\}$$

(4)

where $N(K_T)$ is the size of the subnetwork $G(K_T)$, and $n_{k_T}$ is the relative size of the remaining network after removing nodes $k \leq k_T$ from the subnetwork $G(K_T)$. The behavior of the observable $Z$ is completely determined by the exponent $\alpha^*$ and the scaling function $f$. Interestingly, the scaling exponent of the initial network and its subnetwork is independent of the heterogeneity exponent $\lambda$ and shares the same scaling exponent with the BA network, namely, $\alpha^* \approx 1$, as shown in Fig. 7 (red hollow circle, blue hollow square, and green hollow triangle). In particular, these three topological observables considered here correspond to an approximately identical exponent $\alpha^*$.

Subsequently, we present a possible application: the exponent $\alpha^*$ is obtained by FSS analysis performed on the topological observables, then combined with this exponent to predict the dynamic characteristics of the initial network. The reason is that the pure calculation of the network’s topology observables (such as the largest degree of the network) is usually a task with low computational complexity. In contrast, calculating dynamic observables (such as the eigenvalues of the network) is usually a task with high computational complexity. The significance of the scaling exponent, $\alpha^*$, is that we can predict the dynamic characteristics of the initial large-scale network with the help of the subgraph without spending extra time to calculate them separately. For example, based on $\alpha^*$ and the subnetwork $G(K_T)$,
we can predict the behavior of the largest eigenvalue of the $G_0$ in the DTR flow, as shown in Fig. S15.

Finally, we confirm the universality of $\alpha^* = 1$ via real scale-free networks, and details of these real network datasets are shown in Table I. Fig. 8 shows the structural observables results of the Internet and its three subnetworks $G(K_T)$, where $K_T = 1, 2, 4$, and the results of dynamical observables are shown in Fig. S16. To ensure the subnetwork $G(K_T)$ has approximately the same topology properties as the original network (see Figs. S17-S19), the value of $K_T$ should not be selected too large. Then we take the Internet and its three subnetworks as the research object, and perform DTR on them, respectively. By rescaling the horizontal axis according to (4), these observable curves of subnetworks and the original Internet network can collapse onto the same master curve, where the exponent $\alpha^* \approx 1$.

For other real scale-free networks, approximately the same exponent $\alpha^*$ is obtained, as shown in Table I. The result confirms that scale-free networks and their subnetworks share the same universality class exponent.

D. Discussion

Figs. 6 and 7 show that the corresponding scaling exponent $\alpha$ of the BA and CL synthetic network is inconsistent. Next, we give the possible reasons: the CL network is a static model, and its connection mode differs from the preferential attachment growth mode of the BA network. For the BA model, the degree increases with the age of nodes, while the DTR tends to delete the most recent nodes. Therefore, DTR can approximately return the BA network to an earlier state, implying that the artificially generated smaller-size BA network can be approximately used as a nested subgraph of the larger-size BA network. Indeed, the degree distribution shows that the scale-free network of BA with different sizes is self-similar, as shown in Fig. S20(a). This result means that the scaling exponent obtained in the DTR flow is approximately the same ($\alpha \approx \alpha^* \approx 1$) whether BA networks with different sizes or the DTR snapshots are used as the original network. However, the degree distribution of CL models with different sizes has certain differences, especially in the tail [as shown in Fig. S20(b)]. This indicates that the artificially generated smaller-size CL network cannot be used as the nested subgraph of the larger-size CL network. In contrast, the subgraph extracted by DTR can be approximately used as the nested subgraph of the initial large-scale CL network [see Fig. S1(a)]. Therefore, exponents $\alpha$ and $\alpha^*$ are different for CL models. In conclusion, the results of the hollow symbol reported in Fig. 7 and the results of the real network produced in the same way in Fig. 8 describe a feature of finite-size self-similar networks in DTR flows from another perspective.

Furthermore, the BCR technique proposed by Song et al. [8] connects the representation of dimensionality with the definition of distance. They use the shortest path length between nodes to define the similar distance for the network’s renormalization and define the fractal dimension of the network. In this context, Radicchi et al. [33], [34] investigated the fixed point problem of networks in BCR flows. They found the behavior of some
variables, such as the largest degree, obeys simple scaling laws and is characterized by a critical exponent. Recently, considering the ultrasmall-world property of real-world networks, which cannot provide a wide range of distance values. Within the context of hidden metric spaces [10], the GR [11] technique comes into being, this method has achieved excellent results in most scenarios, which is further confirmed in a series of literature and publications [35], [36], [37], [38]. As a supplement, our recent study [24] shows that the $S^3$ geometric model and some real evolutionary networks also obey simple scaling laws in GR flows. The critical exponent is independent of the type of observables but depend only on the structural properties of the network. Technically speaking, DTR differs significantly from BCR and GR in its idea, and the DTR procedure is especially simple. Therefore, we systematically study the observables of synthetic and real scale-free networks and find that they also obey scaling laws in DTR flows. Notably, as a hierarchical subgraph extraction technique, DTR can perform FSS analysis on a network with a single initial size, which is the difference between this paper and previous studies [24], [33], [34].

In this work, we employ scale-free networks as research objects, and these networks typically have a wide distribution of degree values. However, other homogeneous networks, such as ER random networks, the national highway network with homogeneous topology, or the European power grid with exponential degree distribution, tend to have a relatively narrow degree value range. For these latter networks, a small degree threshold may induce a subgraph with few nodes, which may bring inconvenience related research. Moreover, since node degree is a local network feature, the information it provides is often limited or insufficient to some extent. As such, it may be beneficial to explore other network features as alternative measures to the degree threshold. In light of these considerations, it is important to continue efforts in the future to develop a subgraph extraction or coarse-graining method that is appropriate for a wider range of scenarios.

IV. CONCLUSION

In this study, we investigate the statistical properties of some representative observables of scale-free networks in the DTR flow. Our experimental results indicate that these observables exhibit scaling behavior. However, deviations from pure scaling behavior due to finite network size effects are also observed. In the context of FSS analysis, we identify a scaling function that leads to data collapse for different network sizes. For BA scale-free networks of varying initial sizes, the FSS confirms the universality of the scaling law and reveals a scaling function with a single exponent, approximately equal to 1, to capture the behavior of observables in the DTR flow. For CL scale-free networks with different initial sizes, the scaling exponent depends on the degree distribution exponent. Additionally, for CL or real-world scale-free networks with a single initial size, our results show that the observables of both the initial network and its subnetworks share the same scaling exponent as the BA network. These findings suggest that subnetworks could be utilized to perform FSS and the scaling exponent could be determined from a single snapshot of the network’s topology. From a practical standpoint, the obtained scaling exponent can be employed to predict the structure and dynamic characteristics of large-scale networks, providing valuable guidance for reducing the time complexity of large-scale numerical simulations.

REFERENCES

[1] V. Latora, V. Nicosia, and G. Russo, Complex Networks: Principles, Methods and Applications. Cambridge, U.K.: Cambridge Univ. Press, 2017.
[2] A.-L. Barabási, Network Science. Cambridge, U.K.: Cambridge Univ. Press, 2016.
[3] A.-L. Barabási and R. Albert, “Emergence of scaling in random networks,” Science, vol. 286, no. 5439, pp. 509–512, 1999.
[4] P. Erdős and A. Rényi, “On random graphs,” Pub. Mathematische Debrecen, vol. 6, no. 290, pp. 290–297, 1959.
[5] D. Watts and S. H. Strogatz, “Collective dynamics of ‘small-world’ networks,” Nature, vol. 393, no. 6684, pp. 440–442, 1998.
[6] R. Albert, H. Jeong, and A. Barabási, “Error and attack tolerance of complex networks,” Nature, vol. 406, no. 6794, pp. 378–382, 2000.
[7] R. Pastor-Satorras and A. Vespignani, “Epidemic spreading in scale-free networks,” Phys. Rev. Lett., vol. 86, pp. 3200–3204, 2001.
[8] S. Song, S. Havlin, and H. A. Makse, “Self-similarity of complex networks,” Nature, vol. 433, no. 7024, pp. 392–395, 2005.
[9] L. P. Kadanoff, Statistical Physics: Static, Dynamics and Renormalization. Singapore: World Scientific, 2000.
[10] M. Á. Serrano, D. Krioukov, and M. Boguñá, “Self-similarity of complex networks and hidden metric spaces,” Phys. Rev. Lett., vol. 100, Feb. 2008, pp. 078701.
[11] G. García-Pérez, M. Boguñá, and M. A. Serrano, “Multiscale unfolding of real networks by geometric renormalization,” Nature Phys., vol. 14, no. 6, pp. 583–589, 2018.
[12] S. Yi, H. Jiang, Y. Jiang, P. Zhou, and Q. Wang, “A hyperbolic embedding method for weighted networks,” IEEE Trans. Netw. Sci. Eng., vol. 8, no. 1, pp. 599–612, Jan.–Mar. 2021.
[13] K. Huang, Z. Wang, and M. Jusup, “Incorporating latent constraints to enhance inference of network structure,” IEEE Trans. Netw. Sci. Eng., vol. 7, no. 1, pp. 466–475, Jan.–Mar. 2020.
[14] P. Cui, X. Wang, J. Pei, and W. Zhu, “A survey on network embedding,” IEEE Trans. Knowl. Data Eng., vol. 31, no. 5, pp. 833–852, May 2019.
[15] D. Chen, H. Su, and Z. Zeng, “Geometric renormalization reveals the self-similarity of weighted networks,” IEEE Trans. Comput. Social Syst., vol. 10, no. 2, pp. 426–434, Apr. 2023.
[16] M. Serafin et al., “True scale-free networks hidden by finite size effects,” Proc. Nat. Acad. Sci. USA, vol. 118, no. 2, 2021, Art. no. e2013825118.
[17] F. Chung and L. Lu, “Connected components in random graphs with given expected degree sequences,” Ann. Combinatorics, vol. 6, no. 2, pp. 125–145, 2002. [Online]. Available: https://link.springer.com/content/pdf/10.1007/PL00012580.pdf
[18] D. Fasino, A. Tonetto, and F. Tudisco, “Generating large scale-free networks with the chung–lu random graph model,” Networks, vol. 78, no. 2, pp. 174–187, 2021.
[19] H. E. Stanley, “Scaling, universality, and renormalization: Three pillars of modern critical phenomena,” Rev. Mod. Phys., vol. 71, pp. S358–S366, Mar. 1999.
[20] A. Hagberg, P. Swart, and D. S. Chult, “Exploring network structure, dynamics, and function using networkx,” Los Alamos Nat. Lab., Los Alamos, NM, USA, Tech. Rep. LA-UR-08-05495, 2008.
[21] A. Clauset, C. R. Shalizi, and M. E. Newman, “Power-law distributions in empirical data,” SIAM Rev., vol. 51, no. 4, pp. 661–703, 2009.
[22] J. Alstott, E. Bullmore, and D. Plenz, “Powerlaw: A python package for analysis of heavy-tailed distributions,” PLoS One, vol. 9, no. 1, 2014, Art. no. e85777.
[23] B. Klemm and V. M. Eguíluz, “Growing scale-free networks with small-world behavior,” Phys. Rev. E, vol. 65, May 2002, Art. no. 057102.
[24] D. Chen, H. Su, X. Wang, G.-J. Pan, and G. Chen, “Finite-size scaling of geometric renormalization flows in complex networks,” Phys. Rev. E, vol. 104, no. 3, 2021, Art. no. 034304.
[25] J. Houdayer and A. K. Hartmann, “Low-temperature behavior of two-dimensional Gaussian Ising spin glasses,” Phys. Rev. B, vol. 70, Jul. 2004, Art. no. 014418.

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[26] T. Nishikawa, J. Sun, and A. E. Motter, “Sensitive dependence of optimal network dynamics on network structure,” Phys. Rev. X, vol. 7, Nov. 2017, Art. no. 041044.

[27] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, “Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize,” Phys. Rev. Lett., vol. 91, Jul. 2003, Art. no. 014101.

[28] L. Donetti, P. I. Hurtado, and M. A. Muñoz, “Entangled networks, synchronization, and optimal network topology,” Phys. Rev. Lett., vol. 95, Oct. 2005, Art. no. 188701.

[29] M. Barahona and L. M. Pecora, “Synchronization in small-world systems,” Phys. Rev. Lett., vol. 89, Jul. 2002, Art. no. 054101.

[30] D. Shi, G. Chen, W. W. K. Thong, and X. Yan, “Searching for optimal network topology with best possible synchronizability,” IEEE Circuits Syst. Mag., vol. 13, no. 1, pp. 66–75, Jan.–Mar. 2013.

[31] C. Castellano and R. Pastor-Satorras, “Relating topological determinants of complex networks to their spectral properties: Structural and dynamical effects,” Phys. Rev. X, vol. 7, Oct. 2017, Art. no. 041024.

[32] J. G. Restrepo, E. Ott, and B. R. Hunt, “Onset of synchronization in large networks of coupled oscillators,” Phys. Rev. E, vol. 71, Mar. 2005, Art. no. 036151.

[33] F. Radicchi, J. J. Ramasco, A. Barrat, and S. Fortunato, “Complex networks renormalization: Flows and fixed points,” Phys. Rev. Lett., vol. 101, Oct. 2008, Art. no. 148701.

[34] F. Radicchi, A. Barrat, S. Fortunato, and J. J. Ramasco, “Renormalization flows in complex networks,” Phys. Rev. E, vol. 79, Feb. 2009, Art. no. 026104.

[35] M. Zheng, G. García-Pérez, M. Boguñá, and M. Á. Serrano, “Scaling up real networks by geometric branching growth,” Proc. Nat. Acad. Sci. USA, vol. 118, no. 21, 2021, Art. no. e2018994118.

[36] M. Boguñá, I. Bonamassa, M. De Domenico, S. Havlin, D. Kríoukov, and M. Á. Serrano, “Network geometry,” Nature Rev. Phys., vol. 3, no. 2, pp. 114–135, 2021.

[37] M. Á. Serrano and M. Boguñá, The Shortest Path to Network Geometry: A Practical Guide to Basic Models and Applications. Cambridge, U.K.: Cambridge Univ. Press, 2021.

[38] P. Almagro, M. Boguñá, and M. Á Serrano, “Detecting the ultra low dimensionality of real networks,” Nature Commun., vol. 13, no. 1, pp. 1–10, 2022.

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