Gyrokinetic simulation studies for non-axisymmetric plasma confinement: turbulent transport and entropy transfer

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Abstract. Turbulent ion heat transport and entropy transfer in non-axisymmetric toroidal plasma confinement are investigated by means of gyrokinetic simulations. Elongation of fluctuation spectrum into the radial wavenumber space is observed in the ion temperature gradient (ITG) turbulence simulation for a helical plasma configuration reconstructed from experimental data, and is examined by the entropy transfer function. For studying turbulence and zonal flows in a poloidally-rotating helical plasma, the conventional flux tube simulation model is extended to a bundle of flux tubes, where a multi-scale model of zonal flows and turbulence is formulated and tested by simulations of the linear ITG instability and the zonal flow response.

1. Introduction

In fusion, space, and astrophysical plasmas with a high temperature, the conventional fluid approximation is considered to be invalid, because a mean-free-path of particles is often comparable to or longer than the size of interest. Kinetic approaches dealing with the particle distribution function in the phase space are indispensable to theoretical and numerical studies of the collisionless or weakly collisional plasmas. For studying a magnetized plasma turbulence, the nonlinear gyrokinetic equations \cite{1, 2} are derived from the Vlasov equation, where the phase space dimension is reduced by the gyro-phase average while preserving the spatial and time resolutions for drift waves.

Gyrokinetic simulations of drift wave turbulence in fusion plasmas have made a large progress in the last two decades \cite{3}, where interaction of ion temperature gradient (ITG) turbulence and zonal flows \cite{4} is one of the main issues. However, application to non-axisymmetric toroidal systems, such as the Large Helical Device (LHD) \cite{5}, still remains as one of the challenges, since the complicated equilibrium configuration demands careful numerical treatments and expensive computational costs. The ITG turbulent transport simulations with model LHD configurations have been carried out \cite{6, 7} by means of a gyrokinetic simulation code, GKV \cite{8}, where the zonal flow enhanced in an optimized field configuration with less effective helical ripple components leads to reduction of the turbulent ion heat transport. The gyrokinetic simulation for LHD plasmas has been extended so as to incorporate the equilibrium configuration reconstructed...
from the experimental data [9, 10, 11]. The GKV-X code successfully reproduces the ion heat transport observed in a high ion temperature discharge on LHD [11].

In the $\delta f$ approach employed in the GKV and GKV-X codes, time evolution of the perturbed distribution function $\delta f$ from the background is calculated in the three-dimensional real and the two-dimensional velocity spaces, where fluctuations of $\delta f$ are transferred into different spacial and velocity scales though linear and nonlinear advection terms. In the present study, we examine the nonlinear interactions of turbulence and zonal flows in the LHD configuration reconstructed from experimental data, by means of a transfer function of a quadratic integral of $\delta f$, that is, the entropy transfer function [12, 13].

It is known that the neoclassical particle transport in helical systems spontaneously generates an equilibrium scale radial electric field which drives the poloidal $E \times B$ rotation and improves collisionless orbits of helical-ripple-trapped particles. To incorporate the effect of poloidal plasma rotation, however, the field-line label dependence of the confinement field strength is necessary to be introduced, which demands extension of the conventional flux tube model [14]. In the later part of the paper, a novel idea for multi-scale simulations of zonal flows and turbulence will be discussed as generalization of the flux tube model.

This paper is organized as follows. The basic formulation of the $\delta f$ gyrokinetic equations is explained in the next section. In section 3, results of the flux tube simulations of the ITG turbulent transport in the LHD configuration are analyzed by means of the entropy transfer function. A new multi-scale simulation model using a bundle of flux tubes for non-axisymmetric turbulent transport in the LHD configuration are analyzed by means of the entropy transfer function [14]. In the later part of the paper, a novel idea for multi-scale simulations of zonal flows and turbulence will be discussed as generalization of the flux tube model.

This paper is organized as follows. The basic formulation of the $\delta f$ gyrokinetic equations is explained in the next section. In section 3, results of the flux tube simulations of the ITG turbulent transport in the LHD configuration are analyzed by means of the entropy transfer function. A new multi-scale simulation model using a bundle of flux tubes for non-axisymmetric systems with the equilibrium radial electric field will be discussed in section 4, where simulation results of the linear ITG instability growth and the zonal flow response are discussed. A summary is given in the last section.

2. Basic equations

We consider the electrostatic gyrokinetic equation [1], which is represented as an inhomogeneous advection-diffusion equation for perturbed gyrocenter distribution function, $\delta f$, on the five dimensional phase space,

$$
\left[ \frac{\partial}{\partial t} + v_\parallel \cdot \nabla + v_d \cdot \nabla - \frac{\mu}{m} (b \cdot \nabla B) \frac{\partial}{\partial v_\parallel} \right] \delta f + \frac{c}{B_0} \Phi \delta f = (v_\ast - v_d - v_\parallel b) \cdot \frac{e \nabla \Phi}{T_i} F_M + C \tag{1}
$$

where the parallel velocity, $v_\parallel$, and the magnetic moment, $\mu = mv_\perp^2/2B$, are chosen as the velocity coordinates. The real space (gyroceter) coordinates are defined by the minor radius (flux surface label), $r$, the field line label, $\alpha$, and the poloidal angle, $\theta$. Here, $\alpha = \zeta - q \theta$ for the toroidal angle, $\zeta$, and the safety factor, $q$. The equilibrium quantities and their local gradients are assumed to be constant in the vicinity of a flux surface at the minor radius $r = r_0$, where the local turbulence simulation is carried out. The abbreviations of $v_d$, $\Phi$, and $F_M$ denote the magnetic drift velocity, the gyro-averaged potential, and the Maxwellian distribution function, respectively. In the numerical simulation, it is convenient to use the coordinates $(x, y, z)$, such that $x = r - r_0$, $y = (r_0/q_0)[q(r) \theta - \zeta]$, and $z = \theta$ [14] with assumption of the constant magnetic shear, $s = (r_0/q_0) dq/dr$, in $r$. The diamagnetic drift velocity $v_\ast$ associated with density and temperature gradients is given as

$$
v_\ast \cdot \nabla = - \frac{cT_i}{e L_n B_0} \left[ 1 + \eta_i \left( \frac{m_i v_\ast^2}{2 T_i} - \frac{3}{2} \right) \right] \frac{\partial}{\partial y} . \tag{2}
$$

The linear model collision term is abbreviated as $C$. Scale lengths of the density and the ion temperature gradients are denoted by $L_n$ and $L_T$, respectively, providing $\eta_i = L_n/L_T$. Other notations are standard (see Ref.[6] for more details).
3. Flux tube simulations of gyrokinetic plasma turbulence

3.1. Local flux tube model

In general toroidal confinement systems, the field-line-label dependence of the confinement field strength appears in several terms of Eq. (1), such as the magnetic drift and mirror force terms etc. Assumption of the toroidal axisymmetry for the confinement field is valid in a core region of tokamaks, while the non-axisymmetric effect should be taken into account in helical plasmas, such as LHD. In either case, the scale length of the confinement field strength $B(\alpha, \theta)$ is assumed to be much longer than those of turbulence. The former is characterized by $R_0/M$ or $q_0R_0$ in the $\alpha$ and $\theta$ directions, respectively, where $M$ means the toroidal periodicity of the non-axisymmetric component of confinement field. The later is supposed to be of the order of thermal gyro-radius $\rho_i$. Indeed, the gyrokinetic ordering used in derivation of the gyrokinetic equations assumes the smallness parameter of $\rho_*/L \ll 1$ where $L$ means an equilibrium scale length, such as the minor radius of a torus. Therefore, it is natural to assume scale separation of the turbulence and the equilibrium. Fixing the field-line-label $\alpha$ in $B(\alpha, \theta)$ to a constant leads to the flux tube model [14] which is a standard gyrokinetic model for studying the local turbulent transport in tokamak and helical plasmas.

The local turbulence assumption in $r$ enables ones to assume the periodic boundary condition in the radial direction. In the local flux tube geometry, the modified periodic boundary condition for the $z$ ($\theta$) coordinate is applied to the turbulent fluctuations in case with the finite magnetic shear [14]. The normal periodic boundary condition is employed for the zonal flow components of which wavenumbers in the $y$ ($\alpha$) direction vanish. The locality assumption simplifies the numerical scheme for the flux tube model, where the spectral method is used for fluctuations in the $x$ and $y$ coordinates. Thus, the Fourier transformation is applied to Eq. (1) with a constant value of $\alpha$ in $B$, where the perpendicular wavenumber vector $k_\perp$ is defined for the $(x, y)$ plane.

By means of the quasi-neutrality condition, the electrostatic potential in the particle coordinates is calculated in the perpendicular wavenumber $k_\perp$ space,

$$\int J_0 f_{k_\perp} d^3v - \frac{e\phi_{k_\perp}}{T_i} n_0 (1 - \Gamma_0) = n_{e,k_\perp},$$  \hspace{1cm} (3)

where $f_{k_\perp}$ denotes the Fourier component of $\delta f$. The electron density is given by the adiabatic response model $n_{e,k_\perp}/n_0 = e\phi_{k_\perp}/T_e$ for turbulence components. For zonal flows, no electron density perturbation arises, where no parallel electric field exists. Thus, $n_{e,k_\perp}/n_0 = e(\phi_{k_\perp} - \langle \phi_{k_\perp} \rangle) / T_e$. The flux surface average is denoted by

$$\langle A \rangle = \int_{-N_\theta\pi}^{+N_\theta\pi} \sqrt{g_F} A_{k_\eta} dz / \int_{-N_\theta\pi}^{+N_\theta\pi} \sqrt{g_F} dz.$$ \hspace{1cm} (4)

The poloidal extent of the simulation domain is defined as $-N_\theta\pi \leq z \leq N_\theta\pi$ with a positive integer $N_\theta$. Usually, one assumes $N_\theta = 1$. The Jacobian of the flux tube coordinates, $(x, y, z)$, is denoted by $\sqrt{g_F}$.

The simplest flux tube coordinates are given by the circular co-centric flux surfaces with the assumption of a large aspect ratio torus, where the flux surface average is simply given with the Jacobian, $\sqrt{g_F} = 1/B(\alpha_0, z)$ (See Ref. [6, 8] for more detailed implementation of the conventional GKV code). In a recent version of our flux tube gyrokinetic code, GKV-X [9, 10], the Jacobian is obtained from the VMEC code [15] which is an equilibrium configuration solver for non-axisymmetric systems. Then, the LHD experimental configurations reconstructed by the VMEC code are introduced in the GKV-X simulation. In the GKV-X code, the derivative operators in Eq.(1) are also defined with metric coefficients given by the VMEC equilibrium calculation, while simpler expressions are used in the conventional GKV code, such as $\vec{b} \cdot \vec{\nabla} \delta f = (1/q_0R_0) (\partial \delta f / \partial z)$.

Hereafter, unless it is explicitly stated, we use the normalization units of $x = x' / \rho_i$, $t = t' v_{ti} / L_n, v = v' / v_{ti}, B = B' / B_0$, and $\phi = e\phi' L_n / T_i \rho_i$, where $T_i$, $v_{ti}$, and $B_0$ denote the...
ion temperature, the ion thermal speed, and the typical magnitude of the confinement field, respectively. Prime means dimensional quantities.

3.2. Entropy balance relations in a flux tube model

Magnitude of kinetic plasma fluctuations in the phase space can be characterized by a quadratic form of the perturbed distribution function, $|\delta f|^2$. The integral of $|\delta f|^2/2F_M$ on the phase space corresponds to the second-order difference of the micro- and macroscopic entropy, $-\int F_M \ln F_M dx dv$ and $-\int F_M \ln F_M dx dv$, respectively. The entropy variable $\delta S$ is defined by $\delta S = \sum_{k_\perp} \delta S_{k_\perp} = \sum_{k_\perp} \int (|f_{k_\perp}|^2/2F_M) d^3v$. A balance equation for $\delta S_{k_\perp}$ is derived from Eqs. (1) and (3) with a fixed field-line-label, $a = a_0$,

$$\frac{\partial}{\partial t} (\delta S_{k_\perp} + W_{k_\perp}) = \eta_k Q_{k_\perp} + T_{k_\perp} + D_{k_\perp}, \quad (5)$$

where $W_{k_\perp}$, $Q_{k_\perp}$, and $D_{k_\perp}$ mean the potential energy, the ion heat transport flux, and the collisional dissipation, respectively. The detailed definitions are found in Refs. [13] and [16]. The entropy transfer among different turbulent and zonal flow components is represented by $T_{k_\perp}$, which is derived from the $E \times B$ nonlinear term in Eq.(1) [12, 13], such that $T_{k_\perp} = \sum_{q_\perp} \sum_{p_\perp} \delta_{k_\perp + p_\perp + q_\perp} J [k_\perp | p_\perp, q_\perp]$ with

$$J [k_\perp | p_\perp, q_\perp] = \left\{ b \cdot (p_\perp \times q_\perp) \int d^3v \frac{1}{2F_M} \operatorname{Re} \left[ \Phi_{p_\perp} h_{q_\perp} h_{k_\perp} - \Phi_{q_\perp} h_{p_\perp} h_{k_\perp} \right] \right\}. \quad (6)$$

Here, the non-adiabatic part of the perturbed gyrocenter distribution is given by $h_{k_\perp} = f_{k_\perp} + \Phi_{k_\perp} F_M$. The entropy transfer function $T_{k_\perp}$ is regarded as a kinetic extension of the Reynolds stress [12], and is useful for analyzing interactions among turbulence and zonal flows [13]. The total entropy balance relation in a single flux tube is given by the sum of Eq.(5) over $k_\perp$,

$$\frac{d}{dt} (\delta S + W) = \eta_k Q + D, \quad (7)$$

where each term stems from their counter part in Eq.(5) while $\sum_{k_\perp} T_{k_\perp}$ vanishes by construction. The entropy balance relations give us insights how fluctuations of $\delta f$ are transferred in the phase space. One of the simple examples is the collisionless damping of zonal flows and the geodesic acoustic mode (GAM) [17]. Suppose the initial condition given by an Maxwellian distribution with the poloidal and toroidal mode numbers of $m = n = 0$ but with a finite radial wavenumber so that the zonal flow perturbation is constant in $z$ with $k_x \neq 0$ and $k_y = 0$. As the right hand side of Eq. (7) vanishes ($Q = D = 0$), $\delta S + W$ should be conserved. The zonal flow with $m = n = 0$ couples with the $m = 1$ mode through the toroidicity, and starts to oscillate with the GAM frequency. The GAM oscillation with a $m = 1$ component is damped through the collisionless Landau damping by passing particles which generate fine-scale velocity-space structures [8]. As the GAM is damped, $W$ decreases while $\delta S$ increases. It means that the entropy is transferred from large to small velocity scales. In a long time limit, the zonal flow amplitude with $m = 0$ remains finite after the initial GAM damping [18], where a coherent structure of $f_{k_\perp}$ associated with the neoclassical polarization is formed in the trapped particle region of the velocity space [8].

In non-axisymmetric systems, the zonal flow damping process is influenced by radial drift motion of helical ripple trapped particles which lead to additional shielding of the zonal flow potential through mixing of the perturbed distribution function [19, 20, 21]. The gyrokinetic simulations of zonal flow damping in the LHD model configuration clearly captures the modulation of $f_{k_\perp}$ by the radial drift which also increases the entropy variable.
computational costs, we have reduced the resolution for \( z \) interaction condition, \( k \) found at
radial scales. \( (k \) contours. It is found that the fluctuation spectrum spreads into the high radial wavenumber
in Fig. 1 (right), where \( |k| \) found in Fig. 1 (left). Wavenumber spectrum of the electrostatic potential fluctuations is shown
with the results of Ref [11] within an error of 10% that is smaller than the fluctuation level of
\( n \) the resultant transport coefficient averaged for 60
\( L \)
\( q | \) is transferred to the higher radial wavenumber region, such as
\( q \) in the ITG turbulence, where the time average is taken for 40\( R_0/v_{ti} \) < \( t < 90R_0/v_{ti} \).

3.3. Turbulent entropy transfer in LHD plasma

A quantitative analysis of the nonlinear entropy transfer among zonal flows and turbulence was
first carried out for the tokamak ITG and ETG (electron temperature gradient) turbulence simulations [13], where the successive entropy transfer from low to high radial wavenumber region results in the elongated spectrum of ITG turbulence fluctuations. The same analysis has also been adopted to the model LHD configurations with a simple helical confinement field given by an analytic expression [16], where the conventional GKV code is employed. We have been developing a new simulation code package, GKV+, with a modern programming structure for higher flexibility in maintenance and extension, where the full geometrical computations used in the GKV-X are also incorporated. Then, the entropy transfer analysis can be applied to the ITG turbulence simulations for the LHD experimental configurations.

As the initial test, the same configuration as that used in the GKV-X simulation is employed, that is, the VMEC equilibrium reconstructed from the LHD experimental data #88343 [11]. Figure 1 (left) shows the time evolution of the ion heat transport coefficient normalized by the gyro-Bohm unit, \( \chi_{GB} = \rho_i^2v_{ti}/R_0 \), where the normalized minor radius \( \rho = 0.83 \). For saving the computational costs, we have reduced the resolution for \( z \), \( v_{ti} \), and \( \mu \) space by half. Nevertheless, the resultant transport coefficient averaged for 60\( L_n/v_{ti} \) < \( t < 100L_n/v_{ti} \) is 7.19\( \chi_{GB} \), and agrees with the results of Ref [11] within an error of 10% that is smaller than the fluctuation level of \( \chi_i \) found in Fig. 1 (left). Wavenumber spectrum of the electrostatic potential fluctuations is shown in Fig. 1 (right), where \( |\phi_{k_z}(z = 0)|^2 \) averaged from \( t = 40 \) to 90\( L_n/v_{ti} \) is plotted by color contours. It is found that the fluctuation spectrum spreads into the high radial wavenumber \( (k_z) \) region in comparison to the \( k_y \) direction. This is attributed to the strong interaction of zonal flows and turbulence, where the entropy fluctuations of turbulence are transferred to small radial scales.

The entropy transfer function \( \delta_{k_z} + p_{q_z} \) is plotted in Fig. 2 for \( p_{q_z} = (0.244, 0.127) \), where the time average \( \cdots \) is taken for 40\( L_n/v_{ti} < t < 90L_n/v_{ti} \). The entropy transfer to the mode with \( p_{q_z} = (0.244, 0.127) \) is represented by the red color which is typically found at \( q_{z} = (0, -0.127) \) and \((-0.244, 0) \). The wavenumbers of three modes satisfy the triad interaction condition, \( k_z + p_{q_z} + q_{z} = 0 \). In contrast, the entropy of \( p_{q_z} = (0.244, 0.127) \) mode is transferred to the higher radial wavenumber region, such as \( q_{z} = (-0.610, -0.127) \) or higher \( |q_z| \), where the transfer function has negative values colored by blue. The result confirms that the successive entropy transfer found in the previous works also acts in the LHD experimental

**Figure 1.** Results of ITG turbulence simulation for the LHD experiments #88343. (Left) Time history of the ion heat transport coefficient \( \chi_i \) normalized in the gyro-Bohm unit of \( \chi_{GB} = \rho_i^2v_{ti}/R_0 \). (Right) Wavenumber spectrum of the electrostatic potential fluctuations observed at \( z = 0 \) in the ITG turbulence, where the time average is taken for 40\( R_0/v_{ti} < t < 90R_0/v_{ti} \).
configuration analyzed here. An additional simulation carried out for $\rho = 0.46$ presents a weaker entropy transfer to high $k_x$ region, where the wavenumber spectrum of turbulence fluctuations looks more isotropic (not shown here) [11]. The obtained results suggest that the fluctuation spectrum or the radial correlation length, as well as the zonal flow and turbulence interactions, has a flux surface (or $\rho$) dependence in the LHD configuration.

4. Flux-tube bundle model for non-axisymmetric systems

In the previous section, we have discussed the entropy transfer process in the local flux tube model for the LHD configuration with no equilibrium-scale radial electric field $E_{r0}$. In the axisymmetric system, the rigid poloidal $E \times B$ rotation with a small poloidal Mach number causes only the Doppler shift of mode frequencies, but no influence on the turbulent transport. In the non-axisymmetric systems, however, the poloidal rotation by $E_{r0}$ plays a crucial role in determining the long-time response of zonal flows, which was pointed out by the gyrokinetic theory [22, 23, 24, 25]. Gyrokinetic simulations for non-axisymmetric systems have also confirmed enhancement of the residual zonal flow level due to the poloidal flow driven by the macro-scale (uniform and constant) $E_{r0}$ [25, 26, 27, 28]. In the helical systems, thus, the turbulent transport may be influenced by the radial electric field $E_{r0}$ spontaneously formed by the ambipolarity condition of the neoclassical particle transport, leading to coupling of the neoclassical and anomalous transport processes through zonal flows.

For the gyrokinetic simulations of helical plasma turbulence in case with $E_{r0}$, the conventional local flux tube model should be extended so as to introduce the field-line-label dependence of $B(\alpha, \theta)$ which appears in the magnetic drift, the mirror force, and other terms in Eq.(1). A preliminary simulation for the ITG turbulent transport in case with $E_{r0}$ has been carried out by means of an extended GKV code [27], where two different scale lengths (that is, an equilibrium scale associated with helical ripples, $R_0/M$, and the turbulence scale of $\rho_i$) are simultaneously involved in a single equation. It is contrast to the conventional flux tube model, and is not preferable from a viewpoint of the gyrokinetic ordering assuming separation of equilibrium and turbulence scales. To overcome limitation of the conventional methods, we have developed a novel gyrokinetic simulation model with multiple flux tubes, that is, a “flux-tube bundle” model of which concept was first discussed in Ref. [29].
4.1. Formulation of flux-tube bundle model

Let us start with the gyrokinetic equation in Eq.(1) where the explicit $\alpha$ dependence of $B$ remains. We postulate the field-line-label dependence on slow ($\alpha$) and fast ($y$) coordinates, such that $\delta f = \tilde{f}(x, \alpha, z, v_\parallel, \mu) + \tilde{f}(x, y, z, v_\parallel, \mu; \alpha)$ and $\Phi = \tilde{\Phi}(x, \alpha, z, \mu) + \tilde{\Phi}(x, y, z, \mu; \alpha) + \Phi_0$, where $y = (r_0/q_0)(q\theta - \zeta) = -(r_0/q_0)\alpha$ in the conventional flux tube model. The zonal flow and turbulence components are represented by $c_\parallel$ and $\cdots$, respectively. The equilibrium potential is denoted by $\Phi_0$ where $E_{r0} = -\partial \Phi_0/\partial r$. Assuming $[\delta f]_\alpha = \tilde{f}$ and $[\tilde{f}]_\alpha = 0$, where $[\cdots]_\alpha$ means the average over the $y$ coordinate, one finds the gyrokinetic equation for the zonal flows,

$$
\left[ \frac{\partial}{\partial t} + v_\parallel b \cdot \nabla + v_{dx} \frac{\partial}{\partial x} - \frac{\mu}{m} (b \cdot \nabla B) \frac{\partial}{\partial v_\parallel} + \omega_{q0} \frac{\partial}{\partial \alpha} \right] \tilde{f} = \left( -v_{dx} \frac{\partial \tilde{\Phi}}{\partial x} - v_\parallel b \cdot \nabla \tilde{\Phi} \right) \frac{e}{T_i} F_M + C(\tilde{f}) + S^{ZF} \tag{8}
$$

where $S^{ZF}$ represents the source term driven by the turbulence,

$$
S^{ZF} = -\frac{e}{B_0} \{ [\Phi, \delta f] \}_\alpha \, . \tag{9}
$$

The magnetic and diamagnetic drift terms in the $\alpha$ direction is neglected by the ordering of $|\partial/\partial x| \gg |\partial/\partial \alpha|$. This is because the $\alpha$ dependence appears in the equilibrium scale of $\sim R_0/M$, while the radial scale length is characterized by $\rho_i$. The poloidal $E \times B$ rotation with $\omega_q = eE_{r0}/q_0B_0$ should be included as the fifth term in the square brackets on the left hand side of Eq.(8), since it is usually larger than the magnetic and diamagnetic drift terms in the $\alpha$ directions and modifies the collisionless orbit of ripple-trapped particles.

By subtracting Eq. (8) from Eq. (1) with $\delta f = \tilde{f} + \Phi$ and $\Phi = \Phi + \tilde{\Phi} + \Phi_0$, one finds the gyrokinetic equation for $\tilde{f}(x, y, z, v_\parallel, \mu, \alpha)$,

$$
\left[ \frac{\partial}{\partial t} + v_\parallel b \cdot \nabla + v_{dx} \cdot \nabla - \frac{\mu}{m} (b \cdot \nabla B) \frac{\partial}{\partial v_\parallel} - \omega_{q0} \frac{\partial}{\partial y} \right] \tilde{f} + \frac{e}{B_0} \{ \Phi, \delta f \} = (v_\perp - v_\parallel b) \cdot \frac{e \nabla \tilde{\Phi}}{T_i} F_M + C(\tilde{f}) - S^{ZF} \tag{10}
$$

where the slow coordinate, $\alpha$, is treated as a parameter. It means that the local flux tube model with a fixed values of $\alpha$ can be applied to numerically solving Eq.(10). Interactions between zonal flows and turbulence are formulated though the nonlinear terms of $\{ \Phi, \delta f \}$ and $S^{ZF}$.

The gyrokinetic equation for the zonal flows, Eq.(8), can be numerically solved on discretized grids, that is, $\alpha_j = j\Delta \alpha$, where $\Delta \alpha$ is the grid spacing, and $j$ is an integer. We locate a flux tube simulation box at each grid point $\alpha = \alpha_j$ for calculating the turbulence components from Eq.(10). The multi-scale simulation model, thus, consists of the zonal flow solver and multiple flux tubes for turbulence [where a total number of flux tubes is equal to the number of grid points ($N_\alpha$) for discretizig Eq.(8)], and is called a “flux-tube bundle” model. Here, the same boundary conditions and numerical methods as those for the local flux tube model are applied to each flux tube at $\alpha = \alpha_j$. Finally, we postulate a closure relation, which relates the $k_y = 0$ component of $\tilde{f}_{j,k_y}$ in the $j$th flux tube to the zonal flow, so that

$$
\tilde{f}_{j,k_y=0} = \tilde{f}(\alpha_j) \, . \tag{11}
$$

It greatly simplifies the numerical treatment, since Eq.(8) at $\alpha = \alpha_j$ is the same as that for the $k_y = 0$ components of Eq.(10) except for the poloidal rotation term.
4.2. Linear ITG instability and zonal flow response

As Eq. (10) for \( k_y \neq 0 \) modes has the same form as that of the conventional flux tube model, except for the poloidal rotation term causing the Doppler shift of the mode frequency, the linear growth rate of the ITG instability remains unchanged in the flux-tube bundle model. Linear growth rates \( \gamma \) of the ITG mode at different \( \alpha_j \) are plotted in Fig. 3, where physical parameters are the same as those used in the inward-shifted case discussed in Ref. [21]. The numerical simulation is carried out using the grid points of \((256, 64, 24)\) in the \((z, v_{ki}, \mu)\) directions, respectively, where \(-\pi \leq z \leq \pi\), \(-5v_{ti} \leq v_{\|} \leq 5v_{ti}\), and \(0 \leq \mu \leq 12.5m_i v_{ti}^2 / B_0\). Eight flux tubes are set in a range of \(-\pi / M \leq \alpha \leq +\pi / M\) for \( M = 10 \) of the LHD configuration, and hence, \( \Delta \alpha = \pi / 40 \). Weak \( \alpha \) dependence of \( \gamma \) is found only in the low \( k_y \) region, which is consistent with the previous linear stability analysis [30].

The zonal flow response to a given source term is also calculated by linear gyrokinetic simulations using the flux-tube bundle model. Figure 4 shows the time-history of the real part of the zonal flow potential for cases with and without \( E_{r0} \). Physical and numerical parameters are the same as those for the simulation of the ITG instability shown above. One can clearly find that the residual zonal flow level after the initial GAM damping is enhanced in the case of \( M_p = 0.3 \) where the poloidal Mach number is defined as \( M_p = |\omega_0| R_0 q_0 / v_{ti} = (R_0 q_0 / r_0) e E_{r0} / (B_0 v_{ti})\). The residual level after the GAM damping is consistent with the results in Ref. [27]. As the Lenard-Bernstein collision operator is introduced with the collision frequency of \( \nu = 2 \times 10^{-3} v_{ti} / L_n \), the residual zonal flow slowly decays in time.

5. Summary

The gyrokinetic simulations towards quantitative evaluation and prediction of the turbulent transport in magnetic fusion plasmas have been applied to a non-axisymmetric toroidal fusion system, such as LHD. The ITG turbulence simulation using the GKV-X code has demonstrated the capability for reproducing the ion heat flux in the high ion temperature discharge of LHD experiment [11]. In this paper, we have shown the application of a new code package, GKV+, to the entropy transfer function analysis [13, 16] for the ITG turbulence simulation on the LHD experimental configuration. Analysis of the nonlinear entropy transfer function \( \delta_{k_\perp + p_\perp + q_\perp, 0} [p_\perp | q_\perp, k_\perp] \) confirms that the successive entropy transfer process also works in the LHD equilibrium configuration for the high ion temperature discharge.

In the latter part of this paper, a novel idea for the multi-scale simulation of turbulence and zonal flows is formulated for the non-axisymmetric system with poloidal rotation, and is
implemented with the flux-tube bundle model. The linear ITG instability and the zonal flow response are tested by means of the new simulation model. Application of the multi-scale simulation to the turbulent transport and the zonal flow generation is currently in progress, and the results will be published elsewhere.

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