On the Correlation Distribution for a Ternary Niho Decimation

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Abstract

In this paper, let \( n = 2^m \) and \( d = 3^{m+1} - 2 \) with \( m \geq 2 \) and \( \gcd(d, 3^n - 1) = 1 \). By studying the weight distribution of the ternary Zetterberg code and counting the numbers of solutions of some equations over the finite field \( \mathbb{F}_{3^n} \), the correlation distribution between a ternary \( m \)-sequence of period \( 3^n - 1 \) and its \( d \)-decimation sequence is completely determined. This is the first time that the correlation distribution for a non-binary Niho decimation has been determined since 1976.

Index Terms Niho decimation, correlation distribution, exponential sum, ternary Zetterberg code.

1 Introduction

Let \( p \) be a prime, \( n \) a positive integer and \( \{s(t)\} \) a \( p \)-ary \( m \)-sequence over the finite field \( \mathbb{F}_p \) with period \( p^n - 1 \). A \( d \)-decimation sequence of \( \{s(t)\} \) is given by \( \{s(dt)\} \) and the integer \( d \) is called a decimation. The cross-correlation function \( C_d(\tau) \) between \( \{s(t)\} \) and its \( d \)-decimation sequence \( \{s(dt)\} \) is defined by

\[
C_d(\tau) = \sum_{t=0}^{p^n-2} \omega_p^{s(t+\tau) - s(dt)},
\]

where \( \tau = 0, 1, \cdots, p^n - 2 \) and \( \omega_p = e^{\frac{2\pi\sqrt{-1}}{p}} \) is a primitive complex \( p \)-th root of unity. In the theory of sequence design, for a decimation \( d \) leading to low cross-correlation, it is interesting to determine the values of \( C_d(\tau) \) together with the number of occurrences of each value, which

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is known as the correlation distribution for the decimation \(d\). This problem has received a lot of attention since the 1960s, and many interesting theoretical results have been obtained \[8, 25, 21, 12, 13, 15, 9, 22, 7, 29, 26\]. For known results and some open problems in this direction, the reader is referred to \[22, \text{Section 2.2}\] and a recent survey paper \[14\].

A decimation \(d\) is called a Niho decimation over the finite field \(\mathbb{F}_{p^n}\) provided \(n = 2m\) for some positive integer \(m\) and

\[d \equiv p^i \pmod{p^n - 1}\]

for some \(i < n\). The Niho decimation was originally studied by Niho in his famous thesis \[21\], and it leads to at least four-valued cross-correlation \[1, 16\]. Some basic properties about Niho decimations can be found in \[4, 22\], and for more research problems involving them, the reader is referred to \[3, 1, 22, 17, 4, 23, 16, 6\]. In the binary case \(p = 2\), all the known Niho decimations for which the correlation distributions are completely determined can be found in the recent paper \[27\]. When \(p\) is odd, there are only two such Niho decimations below:

(i) \(d = 2p^m - 1\), where \(n = 2m\) and \(p^m \not\equiv 2 \pmod{3}\) \[12, \text{Theorem 4.3}\];

(ii) \(d = \frac{p^n - 1}{3} + p^s\), where \(n \equiv 2 \pmod{4}\), \(0 \leq s < n\), \(p \equiv 2 \pmod{3}\) and \(\frac{1}{3} p^{n-s} (p^n - 1) \not\equiv 2 \pmod{3}\) \[12, \text{Theorem 4.11}\].

Case (i) leads to four-valued cross-correlation, while Case (ii) leads to six-valued cross-correlation. Note that both (i) and (ii) were found by Helleseth in 1976, and since then no further results have been found.

Under the condition that \(\gcd(5, p^m + 1) = 1\), the positive integer

\[d = 3p^m - 2\]  \(\text{(2)}\)

is coprime to \(p^n - 1\) and is a typical Niho decimation over the finite field \(\mathbb{F}_{p^n}\), where \(n = 2m\). This decimation has been studied for a long time, and by the well-known Niho’s theorem \[21, 22, 6, 23\], it leads to at most six-valued cross-correlation. However, its correlation distribution is not completely determined yet. In an unpublished manuscript dated 1999 \[5\], Dobbertin, Helleseth and Martinsen proved that for \(p = 3\) the Niho decimation in \(2\) leads to five-valued cross-correlation, but they did not determine the correlation distribution. Later, in 2006, Dobbertin et al. published a result about the correlation distribution for the Niho decimation \(2\) in the binary case \[4\]. Concretely, when \(p = 2\) and \(m\) is odd, they expressed the
correlation distribution for the Niho decimation \([2]\) in terms of a class of exponential sums. Those exponential sums are involved in the Dickson polynomials and the Kloosterman sums, and are generally difficult to be evaluated in closed forms.

Recently, in \([27]\), for the binary case \(p = 2\), the problem of determining the correlation distribution for the Niho decimation \([2]\) was reduced to a combinatorial problem related to the unit circle of \(\mathbb{F}_{2^n}\). Further, inspired by the idea in \([2]\), the authors of \([27]\) established a connection between the combinatorial problem and the binary Zetterberg code, and then they determined the correlation distribution based on the weight distribution of the binary Zetterberg code in \([24, 20]\). In the present paper, by using similar techniques of \([27]\), we determine the correlation distribution for the Niho decimation \([2]\) in the ternary case. Compared with \([27]\), the procedure for establishing the connection here, however, is much more complicated since the connection in the ternary case is not so direct and obvious as that in the binary case. In addition, there is no weight formula available for the ternary Zetterberg code and thus we need to establish some of these formulas. This paper supplies a proof of a conjecture proposed by Dobbertin et al. in \([5]\). It is the first time that the correlation distribution for a non-binary Niho decimation has been determined since 1976.

The remainder of this paper is organized as follows. Section 2 states our main result on the correlation distribution for the Niho decimation \([2]\) in the ternary case. Section 3 introduces some notation and preliminaries. Section 4 is devoted to proving our main result, and the concluding remarks are given in Section 5.

## 2 Main result on the correlation distribution

Throughout the remainder of this paper, we always assume that \(p = 3\),

\[d = 3^{m+1} - 2,\]  \((3)\)

and \(n = 2m\) with \(m \geq 2\). Let \(\mathbb{F}_{3^n}\) denote the multiplicative group of the finite field \(\mathbb{F}_{3^n}\). Define

\[S(a, b) = \sum_{x \in \mathbb{F}_{3^n}^\times} \omega_3^{\text{Tr}_3^m(ax + bx^d)},\]  \((4)\)

where \((a, b) \in \mathbb{F}_{3^n} \times \mathbb{F}_{3^n}\), \(\omega_3 = e^{\frac{2\pi i}{3}}\) is a primitive complex 3-th root of unity, and \(\text{Tr}_3^m(\cdot)\) is the trace function from \(\mathbb{F}_{3^n}\) to \(\mathbb{F}_3\) \([19]\). The main results of this paper are given in Theorems 1 and 2 below.
Theorem 1  When \((a, b)\) runs through \(\mathbb{F}_{3^n} \times \mathbb{F}_{3^n}\), the value distribution of \(S(a, b)\) is given by

\[
\begin{align*}
    &3^n - 1, & 1 & \text{time}, \\
    &-3^m - 1, & \frac{(3^{2m} - 1)(11 \cdot 3^{2m} - 16 \cdot 3^m - (-1)^m 3^m + 6)}{30} & \text{times}, \\
    &-1, & \frac{(3^{2m} - 1)(3 \cdot 3^{2m} + (-1)^m 3^m + 2 \cdot 3^m + 2)}{8} & \text{times}, \\
    &3^m - 1, & \frac{(3^{2m} - 1)(3^{2m} - (-1)^m 3^m + 6)}{6} & \text{times}, \\
    &2 \cdot 3^m - 1, & \frac{(3^{2m} - 1)(3^{2m} + (-1)^m 3^m + 4 \cdot 3^m - 6)}{12} & \text{times}, \\
    &4 \cdot 3^m - 1, & \frac{(3^{2m} - 1)(3^{2m} - 6 \cdot 3^m - (-1)^m 3^m + 6)}{120} & \text{times}.
\end{align*}
\]

In order to make sure \(\gcd(3^{m+1} - 2, 3^m - 1) = 1\), it requires that \(\gcd(5, 3^n + 1) = 1\), which is equivalent to \(m \not\equiv 2 \pmod{4}\). As a consequence of Theorem 1, the correlation distribution for the ternary Niho decimation [3] can be derived immediately.

Theorem 2  Let \(n = 2m\) and \(d = 3^{m+1} - 2\) with \(m \not\equiv 2 \pmod{4}\). Then, the distribution of the cross-correlation function \(C_d(t)\) defined in [1] is given by

\[
\begin{align*}
    &-3^m - 1, & \frac{11 \cdot 3^{2m} - 16 \cdot 3^m - (-1)^m 3^m + 6}{30} & \text{times}, \\
    &-1, & \frac{3 \cdot 3^{2m} + (-1)^m 3^m + 2 \cdot 3^m - 14}{8} & \text{times}, \\
    &3^m - 1, & \frac{3^{2m} - (-1)^m 3^m + 6}{6} & \text{times}, \\
    &2 \cdot 3^m - 1, & \frac{3^{2m} + (-1)^m 3^m + 4 \cdot 3^m - 6}{12} & \text{times}, \\
    &4 \cdot 3^m - 1, & \frac{3^{2m} - 6 \cdot 3^m - (-1)^m 3^m + 6}{120} & \text{times}.
\end{align*}
\]

The proofs of Theorems 1 and 2 are presented in Section 4. In what follows, we provide two examples to verify our results.

Example 1  Let \(m = 3\), then \(n = 2m = 6\) and \(d = 3^{m+1} - 2 = 79\). Let \(\alpha\) be a primitive element of the finite field \(\mathbb{F}_{3^6}\) and \(\{s(t)\}\) be a ternary \(m\)-sequence given by \(s(t) = \text{Tr}_6^1(\alpha^t)\). By Magma, the value distribution of \(S(a, b)\) is given as follows

\[
\begin{align*}
    &728, & 1 & \text{time}, \\
    &-28, & 184912 & \text{times}, \\
    &-1, & 201656 & \text{times}, \\
    &26, & 92456 & \text{times}, \\
    &53, & 48776 & \text{times}, \\
    &107, & 3640 & \text{times}.
\end{align*}
\]
The distribution of the cross correlation function $C_d(\tau)$ between $\{\text{Tr}_1^6(\alpha^t)\}$ and its decimation sequence $\{\text{Tr}_1^6(\alpha^{dt})\}$ is given by

$$
\begin{cases}
-28, & 254 \text{ times,} \\
-1, & 275 \text{ times,} \\
26, & 127 \text{ times,} \\
53, & 67 \text{ times,} \\
107, & 5 \text{ times.}
\end{cases}
$$

**Example 2** Let $m = 4$, $n = 2m = 8$, $d = 3^{m+1} - 2 = 241$, and $\{s(t)\}$ be the ternary $m$-sequence given by $s(t) = \text{Tr}_1^8(\alpha^t)$, where $\alpha$ is a primitive element of the finite field $\mathbb{F}_{3^8}$. Then, by Magma, the value distribution of $S(a, b)$ is

$$
\begin{cases}
6560, & 1 \text{ time}, \\
-82, & 15481600 \text{ times}, \\
-1, & 16340960 \text{ times}, \\
80, & 7091360 \text{ times}, \\
161, & 3804800 \text{ times}, \\
323, & 328000 \text{ times},
\end{cases}
$$

and the distribution of the cross-correlation function $C_d(\tau)$ between $\{s(t)\}$ and $\{s(dt)\}$ is given by

$$
\begin{cases}
-82, & 2360 \text{ times,} \\
-1, & 2489 \text{ times,} \\
80, & 1081 \text{ times,} \\
161, & 580 \text{ times,} \\
323, & 50 \text{ times.}
\end{cases}
$$

The above numerical results are coincided with the results in Theorems 1 and 2.

### 3 Preliminaries

In this section, we introduce some notation and preliminaries. For convenience, sometimes we denote $3^m$ by $q$. Let $\alpha$ be a fixed primitive element of the finite field $\mathbb{F}_{3^m}$, then $\gamma = \alpha^{q+1}$ is a
primitive element of $\mathbb{F}_q$. The unit circle of $\mathbb{F}_{3^n}$ is defined by

$$U = \{ x \in \mathbb{F}_{3^n} \mid x \bar{x} = 1 \},$$

where $\bar{x} = x^q$. Note that $U$ is a cyclic subgroup of order $q + 1$ in the multiplicative group $\mathbb{F}_{3^n}^*$. Actually, $U = \{ \alpha^i \mid i = 0, 1, \cdots, q \}$ and $\alpha^{q-1}$ is a generator of $U$. Let

$$\Omega = \{ \alpha^i : i = 0, 1, \cdots, q \},$$

then every $x \in \mathbb{F}_{3^n}^*$ has a unique representation

$$x = \beta y$$

with $(\beta, y) \in \Omega \times \mathbb{F}_q^*$.

### 3.1 Ternary Melas codes and ternary Zetterberg codes

Let $m \geq 2$ and $\gamma = \alpha^{q+1}$ be the primitive element of $\mathbb{F}_q$. Denote the minimal polynomial of $\gamma^i$ over $\mathbb{F}_3$ by $m_i(x)$, where $i \in \{-1, 1\}$. The ternary Melas code $M(q)$ is the cyclic code over $\mathbb{F}_3$ of length $q - 1$ generated by $m_1(x)m_{-1}(x)$, and its dual code $M(q)^\perp$ is given by

$$\left\{ (\text{Tr}_1^m(ax + b/x)) : a, b \in \mathbb{F}_q \right\}.$$  

(8)

For each $i \in \{0, 1, \cdots, q\}$, let $A_i$ denote the number of codewords of weight $i$ in $M(q)$. In [11, Theorem 2.3], a formula for $A_i$ was derived, but the formula involves the traces of Hecke operators on certain spaces of cusp forms. Thus, for given $i$ and $m$, it is difficult to compute the value of $A_i$ explicitly. Later, in [10], the formula for $A_i$ was further illustrated and a table of formulas for $A_i$ with small $i$ was computed. In the following lemma, we give the first five formulas in the table, which are useful in this paper.

**Lemma 1** [10, Table 6.1] With the notation introduced above, we have

$$\begin{align*}
A_1 &= 0, \\
A_2 &= q - 1, \\
A_3 &= 0, \\
A_4 &= \frac{(q-1)(q-3)}{2}, \\
A_5 &= \frac{4(q-1)(q^2 + ((-1)^m - 14)q + 36)}{15}.
\end{align*}$$

(9)
Let $\delta$ be a generator of $U$. For $m \geq 2$, the ternary Zetterberg code $Z(q)$ is a cyclic code over $F_3$ of length $q + 1$ defined by

$$Z(q) = \left\{ (c_0, c_1, \cdots, c_q) \in F_3^{q+1} \mid \sum_{i=0}^q c_i \delta^i = 0 \right\}. \quad (10)$$

The dual code $Z(q)^\perp$ of $Z(q)$ has a very simple trace description [11]:

$$Z(q)^\perp = \{ (\text{Tr}_{q_1}^n(a_0 \delta^0), \text{Tr}_{q_1}^n(a_0 \delta), \cdots, \text{Tr}_{q_1}^n(a_0 \delta^q)) : a \in F_{q^n} \}. \quad (11)$$

Let $B_i$ denote the number of codewords with weight $i$ in $Z(q)$, $0 \leq i \leq q + 1$. Let

$$A_M(z) = \sum_{i=0}^{q-1} A_i z^i \text{ and } A_Z(z) = \sum_{i=0}^{q+1} B_i z^i$$

be the weight enumerators of $M(q)$ and $Z(q)$, respectively. In [11], by establishing a correspondence between the codewords of $M(q)^\perp$ and those of $Z(q)^\perp$, the following result is deduced.

**Lemma 2 [11, pp. 268]** With the notation above, let $q' = \frac{q}{3}$, then we have

$$\frac{q^2}{q+1}(1 + 2z)^{q' - 1} A_Z(z) = \frac{q^2}{q-1}(1 - z)^{q' + 1}(1 + z)^{q-1} A_M \left( \frac{-z}{z+1} \right) - 2(1 + 2z)^{2q'}(1 - z)^{2q'} - \frac{q}{q+1}(1 - z)^{4q'} + \frac{1}{q+1}(1 + 2z)^{4q'}. \quad (11)$$

Combining Lemmas 1 and 2, we can derive some weight formulas for $Z(q)$ as follows, which will play an important role in proving our main result in the sequel.

**Lemma 3** Let $B_i$ denote the number of codewords with weight $i$ in the ternary Zetterberg code $Z(q)$, then

$$\begin{cases}
B_0 = 1, \\
B_1 = 0, \\
B_2 = 3^m + 1, \\
B_3 = 0, \\
B_4 = \left\lfloor \frac{3^m - 1}{2} \right\rfloor, \\
B_5 = \frac{4(3^m+1)(3^{2m} - 6 \cdot 3^m - (-1)^m \cdot 3^{m+6})}{15}.
\end{cases}$$
Proof: For each \( i \) satisfying \( 0 \leq i \leq q + 1 \), by comparing the coefficients of \( z^i \) on both sides of (11), we have

\[
\frac{q^2}{q+1} \sum_{j=0}^{i} \binom{q'-1}{i-j} 2^{i-j} B_j = \frac{q^2}{q-1} \sum_{j=0}^{i} \left[ \binom{q'+1}{j} (-1)^j \sum_{k=0}^{i-j} A_k (-1)^k \binom{q-1-k}{k} \right] - 2 \sum_{j=0}^{i} \binom{2q'}{j} 2^j \binom{2q'}{i-j} (-1)^{i-j} - \frac{1}{q+1} \left( \binom{4q'}{i} (-1)^i + \frac{1}{q+1} \binom{4q'}{i} \right) 2^i,
\]

where \( \binom{u}{v} \) denotes the number of \( v \)-combinations of an \( u \)-set. In the above equation, let \( i \) take \( 0, 1, \ldots, 5 \), respectively, then a system of linear equations in variables \( B_0, B_1, \ldots, B_5 \) is established. Solving this system gives the desired result. \( \square \)

3.2 Some combinatorial problems related to the unit circle

Let \( U \) be the unit circle of \( \mathbb{F}_3^n \) defined in (5) and \( \delta \) be a generator of \( U \). Define a set

\[
\mathcal{T}_k = \{(t_1, t_2, \ldots, t_k) \mid \delta^{t_1} + \delta^{t_2} + \cdots + \delta^{t_k} = 0 \text{ with } 0 \leq t_1 < t_2 < \cdots < t_k \leq q\},
\]

where \( k \) is a positive integer and \( k \geq 2 \). Denote the cardinality of \( \mathcal{T}_k \) by \( | \mathcal{T}_k | \). For large \( k \), determining \( | \mathcal{T}_k | \) is a very complicated problem. Due to Lemma 3, \( | \mathcal{T}_k | \) can be computed for \( k = 3, 4, 5 \).

**Lemma 4** With the notation above, we have \( | \mathcal{T}_3 | = 0 \).

Proof: We only need to show that \( \mathcal{T}_3 \) is an empty set. Suppose, on the contrary, that there exists an element \( (t_1, t_2, t_3) \) in \( \mathcal{T}_3 \). Then, we can define a vector \( \mathbf{c} = (c_0, c_1, \ldots, c_{3n}) \) with

\[
c_{t_1} = c_{t_2} = c_{t_3} \in \mathbb{F}_3^* \text{ and } c_i = 0 \text{ for } i \not\in \{t_1, t_2, t_3\}.
\]

From the definition of \( Z(q) \) in (10) and the fact that \( \delta^{t_1} + \delta^{t_2} + \delta^{t_3} = 0 \), it follows that the vector \( \mathbf{c} \) defined above is a codeword of weight three in \( Z(q) \), a contradiction to Lemma 3 which states that there is no codeword with weight three in \( Z(q) \). Thus, \( \mathcal{T}_3 \) is empty. \( \square \)
Lemma 5  With the notation introduced above, $T_4$ is exactly given by

$$\left\{(t_1, t_2, t_3, t_4) \mid 0 \leq t_1 < t_2 < \frac{3^m + 1}{2}, \ t_3 = t_1 + \frac{3^m + 1}{2} \text{ and } t_4 = t_2 + \frac{3^m + 1}{2}\right\} \quad (13)$$

and thus $| T_4 | = \frac{3^{2m} - 1}{8}$.

Proof: Note that $\delta^{\frac{3^m + 1}{2}} = -1$. If $(t_1, t_2, t_3, t_4)$ is an element of (13), it is easily seen that $\delta^{t_1} + \delta^{t_2} + \delta^{t_3} + \delta^{t_4} = 0$. Hence, every element of (13) is also an element of $T_4$. In the sequel, it suffices to show that except for the elements of (13), there is no other 4-tuple $(t_1, t_2, t_3, t_4)$ belonging to $T_4$.

For each given element $(t_1, t_2, t_3, t_4)$ of (13) (also an element in $T_4$), let $c = (c_0, c_1, \ldots, c_q)$ be a vector satisfying

$$c_{t_1} = c_{t_3} \in \mathbb{F}_3^*, \ c_{t_2} = c_{t_4} \in \mathbb{F}_3^* \text{ and } c_i = 0 \text{ for } i \not\in \{t_1, t_2, t_3, t_4\}. \quad (14)$$

Then $c$ is a codeword with weight four in $Z(q)$. Note that for each given element $(t_1, t_2, t_3, t_4)$ of (13), there are four different vectors $c$ satisfying (14). Moreover, for different elements $(t_1, t_2, t_3, t_4)$ of (13), the corresponding codewords defined by (14) are also different. Thus, corresponding to the elements in (13), there are

$$\frac{3^{2m} - 1}{8} \times 4 = \frac{3^{2m} - 1}{2}$$

different codewords with weight four in $Z(q)$ since the total number of elements $(t_1, t_2, t_3, t_4)$ in (13) is $\frac{(1 + \frac{3^m - 1}{2}) \cdot \frac{3^m - 1}{2}}{2} = \frac{3^{2m} - 1}{8}$.

Suppose that there exists an element $(t_1, t_2, t_3, t_4)$ in $T_4$ but not in (13). Then, such an element $(t_1, t_2, t_3, t_4)$ gives at least two codewords $c = (c_0, c_1, \ldots, c_{3^m})$ with weight four in $Z(q)$, which are given by

$$c_{t_1} = c_{t_2} = c_{t_3} = c_{t_4} \in \mathbb{F}_3^* \text{ and } c_i = 0 \text{ for } i \not\in \{t_1, t_2, t_3, t_4\}.$$

Thus, except for the elements $(t_1, t_2, t_3, t_4)$ of (13), if there exists other 4-tuples $(t_1, t_2, t_3, t_4)$ in $T_4$, the total number of codewords having weight four in $Z(q)$ will be greater than $\frac{3^{2m} - 1}{2}$. So we arrive at a contradiction since by Lemma 3 the total number of codewords with weight
four in $Z(q)$ is exactly $\frac{3^{2m} - 1}{2}$. Therefore, except for the elements of \(13\), there is no other 4-tuple \((t_1, t_2, t_3, t_4)\) in $T_4$.

The cardinality of $T_4$ is equal to the number of elements in \(13\), which is $\frac{3^{2m} - 1}{8}$. □

**Lemma 6** With the notation of Lemma \(3\) let $T_5$ be defined by \(12\), then $|T_5| = B_5/2^5$.

Proof: See Appendix A. □

### 4 Proof for the main result

In this section, we will give the proof for the main result in Section 2. Let $S(a, b)$ be the exponential sum defined in \(4\). The possible values of $S(a, b)$ given in Lemma \(7\) below can be found by the techniques used in Lemma 2 of \(18\), which originate from the proof of Niho’s Theorem \(21\).

**Lemma 7** Let $S(a, b)$ be the exponential sum defined in \(4\). Then, the value of $S(a, b)$ is given by

$$(N(a, b) - 1)3^m - 1,$$

where $N(a, b)$ is the number of $z \in U$ such that

$$bz^5 + az^3 + az^2 + b = 0.$$  \(15\)

Note that \(15\) has at most five roots in $U$ since its degree is at most five. Thus, the possible values of $S(a, b)$ are $3^n - 1, -3^m - 1, -1, 2 \cdot 3^m - 1, 3 \cdot 3^m - 1,$ and $4 \cdot 3^m - 1$. In particular, $S(a, b) = 3^n - 1$ if and only if $a = b = 0$. Moreover, the following lemma can exclude a redundant value from the possible values of $S(a, b)$. A similar result was already presented in [22, Theorem 3.7]. For the reader’s convenience, we include a proof here.

**Lemma 8** Let $S(a, b)$ be the exponential sum defined in \(4\). Then, $S(a, b) \neq 3 \cdot 3^m - 1$ for any $(a, b) \in \mathbb{F}_{3^n} \times \mathbb{F}_{3^n}$.

Proof: By Lemma 7 it suffices to prove that \(15\) cannot have four roots in $U$. Now assume that \(15\) has four roots in $U$. Then, $b \neq 0$, $b/b \in U$, and the fifth root of \(15\) is also in $U$. Thus, when \(15\) has four roots in $U$, it must have three roots with multiplicity 1 and one root with multiplicity 2. Therefore, the derivative

$$2bz^4 + 2az$$  \(16\)
of \( bz^5 + \bar{a}z^3 + az^2 + \bar{b} \) has a common root with (15). Then, \( a \neq 0 \) and the only nonzero root of (16) satisfies
\[ z^3 = -a/b. \] (17)

Substituting (17) into (15), we get
\[ \bar{a}/\bar{b} = b/a, \]
which further implies (15) has the following factorization
\[ b(z^3 + a/b)(z^2 + \bar{b}/a) = 0. \]

The above factorization shows (15) has at most three roots in \( \mathbb{F}_{3^n} \), a contradiction. Thus, (15) cannot have four roots in \( U \). The desired conclusion is obtained. \( \square \)

By Lemmas 7 and 8, the nontrivial values of \( S(a, b) \) are \((i - 1)3^m - 1, i = 0, 1, 2, 3, 5\). When \((a, b)\) runs through \( \mathbb{F}_{3^n}^2 \setminus \{(0, 0)\} \), let \( \mu_i \) denote the number of occurrences of \((i - 1)3^m - 1\), where \( i = 0, 1, 2, 3 \) and \( \mu_4 \) denote the number of occurrences of \( 4 \cdot 3^m - 1 \). Determining the value distribution of \( S(a, b) \) is exactly determining the values of \( \mu_i, i = 0, 1, 2, 3, 4 \). In order to determine \( \mu_i \), sufficient independent equations in terms of \( \mu_i \)'s should be obtained, and an efficient way to get these equations is computing the power sums of \( S(a, b) \). The following lemma is very useful for computing the power sums of \( S(a, b) \). Its proof is routine and is omitted here.

**Lemma 9** Let \( N_r \) denote the number of solutions of
\[
\begin{align*}
x_1 + x_2 + \cdots + x_r &= 0, \\
x_1^d + x_2^d + \cdots + x_r^d &= 0,
\end{align*}
\] (18)
in \( (\mathbb{F}_{3^n}^*)^r \), where \( d \) is given in (3), then we have
\[ \sum_{(a,b) \in \mathbb{F}_{3^n}^2} S(a, b)^r = 3^{2n}N_r. \] (19)

Note that \( S(a, b) \) has five nontrivial values. In order to get five independent equations in terms of \( \mu_i \)'s, \( i = 0, 1, \ldots, 4 \), we need to determine \( N_r \) for \( r = 1, 2, 3, 4 \). When \( r \geq 3 \), it is usually difficult to determine the value \( N_r \). In [28], the authors introduced an elegant method for counting the number of solutions of certain equation systems related to generalized
Niho decimations. Their main idea is transforming the equation system similar to (18) into a matrix equation based the polar coordinate representations of the elements in the finite fields. Combining this idea and Lemma 6, the values of $N_3$ and $N_4$ can be determined as follows.

**Proposition 1** With the notation above, we have

$$
\begin{align*}
N_1 &= 0, \\
N_2 &= 3^n - 1, \\
N_3 &= (3^n - 1)(3^m - 2), \\
N_4 &= (3^n - 1)(5 \cdot 3^n - 12 \cdot 3^m - (-1)^m \cdot 3^m + 9).
\end{align*}
$$

*Proof:* It is easy to see that $N_1 = 0$ and $N_2 = 3^n - 1$. Thus, we only consider the cases $r = 3$ and 4.

**Determining $N_3$.** Let $N'_3$ be the number of solutions of

$$
\begin{align*}
x_1 + x_2 &= -1, \\
x_1^d + x_2^d &= -1.
\end{align*}
$$

Then,

$$
N_3 = (3^n - 1)N'_3.
$$

Let $(x_1, x_2) \in (\mathbb{F}_{3^n}^*)^2$. By (7), each $x_j$ can be represented as $x_j = \beta_j y_j$ with $y_j \in \mathbb{F}_{3^m}^*$ and $\beta_j \in \Omega$, where $j = 1, 2$ and $\Omega$ is defined in (6). Then,

$$
x_j^d = \beta_j^d y_j = \beta_j^{3(3^m - 1) + 1} y_j = u_j^3 \beta_j y_j,
$$

where $u_j = \beta_j^{3^m - 1}$. By the substitution in (22), we can write (20) as a matrix equation

$$
\begin{pmatrix}
1 & 1 \\
u_1^3 & u_2^3
\end{pmatrix}
\begin{pmatrix}
\beta_1 y_1 \\
\beta_2 y_2
\end{pmatrix} =
\begin{pmatrix}
-1 \\
-1
\end{pmatrix}.
$$

Therefore, $N'_3$ is the number of $(\beta_1, \beta_2, y_1, y_2) \in \Omega^2 \times (\mathbb{F}_{3^m}^*)^2$ satisfying (23). Let

$$
A =
\begin{pmatrix}
1 & 1 \\
u_1^3 & u_2^3
\end{pmatrix}.
$$
Then,
\[ \det(A) = u_2^3 - u_1^3 = (u_2 - u_1)^3. \]

Now we determine the solutions of (23) according to the determinant of the matrix \( A \).

**Case 1:** \( \det(A) = 0 \). Then, \( u_1 = u_2 \), which implies \( \beta_1 = \beta_2 \), and (23) becomes
\[
\begin{cases}
\beta_1 = \beta_2, \\
\beta_1 y_1 + \beta_2 y_2 = -1, \\
u_1^3 (\beta_1 y_1 + \beta_2 y_2) = -1,
\end{cases}
\]
which is equivalent to
\[
\begin{cases}
\beta_1 = \beta_2 = 1, \\
y_1 + y_2 = -1.
\end{cases}
\] (24)
The number of \((\beta_1, \beta_2, y_1, y_2) \in \Omega^2 \times (\mathbb{F}_{3^m}^*)^2\) satisfying (24) is \(3^m - 2\).

**Case 2:** \( \det(A) \neq 0 \). Then, \( u_1 \neq u_2 \). For each given \((\beta_1, \beta_2) \in \Omega^2 \) such that \( u_1 \neq u_2 \), from (23), we can solve a unique solution \((y_1, y_2)\) as follows
\[
\begin{cases}
\beta_1 y_1 = \frac{u_2^3 - 1}{u_1^3 - u_2^3}, \\
\beta_2 y_2 = -\frac{u_1^3 - 1}{u_1^3 - u_2^3}.
\end{cases}
\] (25)
To ensure \(y_i \in \mathbb{F}_{3^m}^*, i = 1, 2\), we must have
\[
\begin{cases}
u_1 = \left( \frac{u_2^3 - 1}{u_1^3 - u_2^3} \right)^{3^{m-1}}, \\
u_2 = \left( \frac{u_1^3 - 1}{u_1^3 - u_2^3} \right)^{3^{m-1}},
\end{cases}
\]
which implies \(u_1 = u_2\), a contradiction. Thus, when \( \det(A) \neq 0 \), (23) has no solution.

Combining Cases 1 and 2, we have \(N_3' = 3^m - 2\). From (21), the desired result follows.

**Determining \(N_4\).** Similarly, let \(N_4'\) be the number of solutions of
\[
\begin{cases}
x_1 + x_2 + x_3 = -1, \\
x_1^d + x_2^d + x_3^d = -1.
\end{cases}
\] (26)
Then,
\[ N_4 = (3^n - 1)N_4'. \] (27)
Note that (26) is equivalent to
\[
\begin{align*}
  x_1 + x_2 + x_3 &= -1, \\
x_1^3 + x_2^3 + x_3^3 &= -1, \\
x_1^d + x_2^d + x_3^d &= -1.
\end{align*}
\] (28)

Using the substitution in (22) and noting that \(x_j^m = u_j \beta_j y_j\), where \(u_j = \beta_j \beta_{j-1}\) and \(j = 1, 2, 3\), (28) can be written as
\[
\begin{pmatrix}
  1 & 1 & 1 \\
  u_1 & u_2 & u_3 \\
  u_1^3 & u_2^3 & u_3^3
\end{pmatrix}
\begin{pmatrix}
  \beta_1 y_1 \\
  \beta_2 y_2 \\
  \beta_3 y_3
\end{pmatrix}
= 
\begin{pmatrix}
  -1 \\
  -1 \\
  -1
\end{pmatrix}.
\] (29)

Then, \(N'_4\) is the number of \((\beta_1, \beta_2, \beta_3, y_1, y_2, y_3) \in \Omega^3 \times \left(\mathbb{F}_{3^m}\right)^3\) satisfying (29). Let
\[
B = 
\begin{pmatrix}
  1 & 1 & 1 \\
  u_1 & u_2 & u_3 \\
  u_1^3 & u_2^3 & u_3^3
\end{pmatrix}.
\]

Then,
\[
\det(B) = -(u_1 - u_2)(u_1 - u_3)(u_2 - u_3)(u_1 + u_2 + u_3).
\]

Case I: \(\det(B) = 0\). By Lemma 4, \(u_1 + u_2 + u_3 = 0\) if and only if \(u_1 = u_2 = u_3\). Thus, we consider the following cases.

Subcase (a): \(u_1 = u_2 \neq u_3\). Then, \(\beta_1 = \beta_2 \neq \beta_3\) and (29) becomes
\[
\begin{align*}
\beta_1 &= \beta_2 \neq \beta_3, \\
\beta_1(y_1 + y_2) + \beta_3 y_3 &= -1, \\
u_1^3 \beta_1(y_1 + y_2) + u_3^3 \beta_3 y_3 &= -1.
\end{align*}
\] (30)

If \(y_1 + y_2 = 0\), then (30) is transformed into
\[
\begin{align*}
\beta_1 &= \beta_2 \neq \beta_3, \\
\beta_3 &= 1, \\
y_1 + y_2 &= 0, \\
y_3 &= -1.
\end{align*}
\] (31)
The number of \((\beta_1, \beta_2, \beta_3, y_1, y_2, y_3) \in \Omega^3 \times (\mathbb{F}_{3^m}^*)^3\) satisfying (31) is \(3^m(3^m - 1)\). If \(y_1 + y_2 \neq 0\), by arguments similar to Case 2, we can conclude that (30) has no solution. Therefore, when \(u_1 = u_2 \neq u_3\), (29) has
\[
3^m(3^m - 1)
\]solutions.

Similarly, when \(u_1 = u_3 \neq u_2\), or \(u_2 = u_3 \neq u_1\), the number of solutions of (29) is also given by (32).

**Subcase (b):** \(u_1 = u_2 = u_3\). Then, (29) becomes
\[
\begin{align*}
\beta_1 = \beta_2 = \beta_3, \\
\beta_1(y_1 + y_2 + y_3) &= -1, \\
u_1^3 \beta_1^3(y_1 + y_2 + y_3) &= -1,
\end{align*}
\]which implies
\[
\begin{align*}
\beta_1 = \beta_2 = \beta_3 &= 1, \\
y_1 + y_2 + y_3 &= -1.
\end{align*}
\]
The number of \((\beta_1, \beta_2, \beta_3, y_1, y_2, y_3) \in \Omega^3 \times (\mathbb{F}_{3^m}^*)^3\) satisfying (31) is \((3^m - 1) + (3^m - 2)^2\).

By Subcases (a) and (b), when \(\det(B) = 0\), the number of solutions of (29) is given by
\[
3 \cdot 3^m(3^m - 1) + (3^m - 1) + (3^m - 2)^2. \tag{35}
\]

**Case II:** \(\det(B) \neq 0\). Then, for each given \((\beta_1, \beta_2, \beta_3) \in \Omega^3\) such that \(\det(B) \neq 0\), from (29), we can solve a unique solution \((y_1, y_2, y_3)\) as follows
\[
\begin{align*}
\beta_1 y_1 &= -\frac{(u_2 - 1)(u_3 - 1)(u_2 + u_3 + 1)}{(u_1 - u_2)(u_1 - u_3)(u_1 + u_2 + u_3)}, \\
\beta_2 y_2 &= -\frac{(u_1 - 1)(u_3 - 1)(u_1 + u_2 + 1)}{(u_2 - u_1)(u_2 - u_3)(u_1 + u_2 + u_3)}, \\
\beta_3 y_3 &= -\frac{(u_3 - u_1)(u_3 - u_2)(u_1 + u_2 + u_3)}{(u_3 - u_1)(u_3 - u_2)(u_1 + u_2 + u_3)}.
\end{align*}
\]
To ensure \((y_1, y_2, y_3) \in (\mathbb{F}_{3^m}^*)^3\), we must have
\[
\begin{align*}
u_1 &= -\frac{(u_2 - 1)(u_3 - 1)(u_2 + u_3 + 1)}{(u_1 - u_2)(u_1 - u_3)(u_1 + u_2 + u_3)}^{3^m - 1}, \\
u_2 &= -\frac{(u_1 - 1)(u_3 - 1)(u_1 + u_2 + 1)}{(u_2 - u_1)(u_2 - u_3)(u_1 + u_2 + u_3)}^{3^m - 1}, \\
u_3 &= -\frac{(u_3 - u_1)(u_3 - u_2)(u_1 + u_2 + u_3)}{(u_3 - u_1)(u_3 - u_2)(u_1 + u_2 + u_3)}^{3^m - 1},
\end{align*}
\]
which implies
\[
\begin{align*}
(1 - u_1) (u_1 + u_2 + u_3 + 1)^{3m+1} &= 1 - u_1, \\
(1 - u_2) (u_1 + u_2 + u_3 + 1)^{3m+1} &= 1 - u_2, \\
(1 - u_3) (u_1 + u_2 + u_3 + 1)^{3m+1} &= 1 - u_3.
\end{align*}
\]
Thus, (36) is equivalent to
\[
\begin{align*}
u_i &\neq 1, \ i = 1, 2, 3, \\
u_1 &\neq u_2, \ u_2 \neq u_3, \ u_1 \neq u_3, \\
u_1 + u_2 + u_3 &\neq 0, \\
u_1 + u_2 + 1 &\neq 0, \\
u_1 + u_3 + 1 &\neq 0, \\
u_2 + u_3 + 1 &\neq 0, \\
u_1 + u_2 + u_3 + 1 &\in U.
\end{align*}
\]
(37)

Note that there is a one-to-one correspondence between $\beta \in \Omega$ and $u = \beta^{3m-1} \in U$. Therefore, the above analysis shows that when $\det(B) \neq 0$, the number of solutions of (29) is the number of $(u_1, u_2, u_3) \in U^3$ satisfying (37), which is exactly the number of $(u_1, u_2, u_3, u_4) \in U^4$ such that
\[
\begin{align*}
u_i, \ i = 1, 2, 3, 4, \text{ and } 1 \text{ are pairwise distinct,} \\
u_1 + u_2 + u_3 + u_4 + 1 &= 0.
\end{align*}
\]
(38)

By Lemma 6, the number of $(u_1, u_2, u_3, u_4) \in U^4$ satisfying (38) is
\[
\frac{5!B_5}{2^5(3^m+1)}.
\]
(39)

By (35), (39) and Lemma 8, we have
\[
N_4' = 5 \cdot 3^m - 12 \cdot 3^m - (-1)^m 3^m + 9.
\]
(40)

From (27) and (40), it follows the value of $N_4$.

**Remark 1** The proof of Proposition 1 is similar to that of Proposition 2 in [27], where similar problem was considered over the finite field of characteristic 2. It is interesting that for $p = 2$ or 3, the problem of determining $N_4$ over $\mathbb{F}_{p^n}$ can be reduced to the same combinatorial problem over $\mathbb{F}_{p^n}$ as stated in (38).
Remark 2  Compared with [27], the procedure for establishing the relation between the combinatorial problem (38) and the Zetterberg codes in this paper is more complicated. For \( p = 3 \), the relation is not so obvious as that for \( p = 2 \) and thus we have to carry out some analysis before obtaining relation (see Lemma 6). In addition, there are no explicit weight formulas available for the ternary Zetterberg codes and we need to establish some of these formulas.

With the above preparations, we can give the proofs of the main results now.

Proof of Theorem 1: By Lemma 9 and Proposition 1, the power sum \( \sum_{(a,b) \in \mathbb{F}_3^n} S(a,b)^r \) can be computed for \( r = 1, 2, 3, 4 \). Then,

\[
\begin{cases}
\sum_{i=0}^{4} \mu_i = 3^n - 1, \\
\sum_{i=0}^{3} ((i-1)3^m - 1)^r \mu_i + (4 \cdot 3^n - 1)^r \mu_4 = \sum_{(a,b) \in \mathbb{F}_3^n} S(a,b)^r - (3^n - 1)^r, \quad r = 1, 2, 3, 4
\end{cases}
\]

forms a system of five linear equations in five variables \( \mu_i, i = 0, 1, 2, 3, 4 \). The coefficient matrix of this system is a Vandermonde matrix. By solving this equation system, \( \mu_i, i = 0, 1, 2, 3, 4 \), are obtained. Thus, the value distribution of \( S(a,b) \) defined (4) is determined.

\( \square \)

Proof of Theorem 2: When \( m \not\equiv 2 \pmod{4} \), \( \gcd(d, 3^n - 1) = 1 \). Thus, \( S(0,b) = -1 \) for any \( b \in \mathbb{F}_3^n \). We also have \( S(a,0) = -1 \) for any \( a \in \mathbb{F}_3^* \), and \( S(0,0) = 3^n - 1 \). Then, by Theorem 1, the value distribution of \( S(a,b) \) as \( (a,b) \) runs through \( \mathbb{F}_3^n \times \mathbb{F}_3^n \) can be calculated. Let \( d^{-1} \) denote the inverse of \( d \) modulo \( 3^n - 1 \). We have \( S(a,b) = S(a/(-b)^{d^{-1}}, -1) \) for \( b \in \mathbb{F}_3^n \). Moreover, for each fixed \( b \in \mathbb{F}_3^n \), the value distribution of \( S(a/(-b)^{d^{-1}}, -1) \) as \( a \) runs through \( \mathbb{F}_3^n \) is the same as that of \( C_d(\tau) \) when \( \tau \) runs from 0 to \( 3^n - 2 \). Thus, the multiset \( \{S(a,b) \mid (a,b) \in \mathbb{F}_3^n \times \mathbb{F}_3^n\} \) is the multiset sum of \( 3^n - 1 \) identical multisets \( \{C_d(\tau) \mid \tau = 0, 1, \ldots, 3^n - 2\} \). By direct calculations, we get the desired result.

\( \square \)

5 Conclusion

In this paper, for the ternary Niho decimation \( d \) given in (4), the distribution of the cross-correlation function \( C_d(\tau) \) defined in (1) is derived. It is the first time that the correlation distribution for a non-binary Niho decimation has been determined since 1976. The main idea used here is similar to that in [27] but the present paper involves more detailed computation.
techniques (see Remarks 1 and 2 at the end of Proposition 1). When \( p > 3 \), some conclusions presented in this paper no longer hold, and there is no result about the weight distribution of the \( p \)-ary Zetterberg code. Thus, when \( p > 3 \), maybe new techniques are required to derive the correlation distribution for the Niho decimation (3). The readers are invited to attack this problem.

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Appendix A

Before we prove Lemma 6 in detail, we mention the following properties of \( T_5 \), which will be employed in the sequel.

**Lemma 10** Let \( T_5 \) be defined by (12) and \( (t_1, t_2, t_3, t_4, t_5) \) be an element of \( T_5 \). Define \( t'_k = t_k + \varepsilon_k \frac{3^m+1}{2} \pmod{3^m+1} \), where \( \varepsilon_k \in \{0,1\} \) and \( k = 1,2,3,4,5 \). Then,

(i) the difference between any two of \( t_1, t_2, t_3, t_4 \) and \( t_5 \) modulo \( 3^m+1 \) cannot be \( \frac{3^m+1}{2} \);

(ii) \( \delta t_1 + \delta t_2 + \delta t_3 + \delta t_4 + \delta t_5 = 0 \) if and only if \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0 \) or 1.

**Proof:** (i) Suppose without loss of generality that \( t_2 \equiv t_1 + \frac{3^m+1}{2} \pmod{3^m+1} \). Then, we have

\[ \delta t_3 + \delta t_4 + \delta t_5 = 0, \]

which leads to a contradiction due to Lemma 4. Thus, the desired result follows.

(ii) Obviously, the sufficient condition is true. It remains to prove the necessary condition. Suppose that \( \varepsilon_k, \ k = 1,2,3,4,5 \), are not all equal. Then, assume without loss of generality that \( \varepsilon_1 = \varepsilon_2 = 1 \) and \( \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0 \), which implies that

\[ -\delta t_1 - \delta t_2 + \delta t_3 + \delta t_4 + \delta t_5 = 0. \]

The above equation together with \( \delta t_1 + \delta t_2 + \delta t_3 + \delta t_4 + \delta t_5 = 0 \) gives

\[ \delta t_3 + \delta t_4 + \delta t_5 = 0, \]
a contradiction to Lemma 4. In the same way, other cases will also lead to a contradiction. Thus, the necessary condition is also true. □

Proof of Lemma 4. The conclusion of this lemma follows from the following three claims:

(i) from each element \((t_1, t_2, t_3, t_4, t_5)\) of \(T_5\) we can construct \(2^5\) pairwise distinct codewords with weight five of \(Z(q)\);

(ii) in (i) the codewords with weight five constructed from different elements of \(T_5\) are also pairwise distinct;

(iii) each codeword with weight five in \(Z(q)\) can be constructed from an element of \(T_5\).

Proof of Claim (i). Let \((t_1, t_2, t_3, t_4, t_5)\) be an element in \(T_5\) and \((v_1, v_2, v_3, v_4, v_5)\) an arbitrary vector in \((\mathbb{F}_3^5)^5\). Define \(\varepsilon_k = 1\) if \(v_k = -1\) and \(\varepsilon_k = 0\) otherwise, and let

\[
t'_k \equiv t_k + \varepsilon_k \frac{3^m + 1}{2} \pmod{3^m + 1},
\]

where \(1 \leq k \leq 5\). Then, we have

\[
v_1\delta^{t'_1} + v_2\delta^{t'_2} + v_3\delta^{t'_3} + v_4\delta^{t'_4} + v_5\delta^{t'_5} = 0
\]

since \(\delta^{t_1} + \delta^{t_2} + \delta^{t_3} + \delta^{t_4} + \delta^{t_5} = 0\) and \(\delta^{3^m+1} = -1\). By Lemma 10 (i), \(t'_k, 1 \leq k \leq 5\), in (42) are also pairwise distinct. Let \(c = (c_0,c_1,\cdots,c_q)\) be a vector defined by

\[
c_{t'_k} = v_k, \ k = 1,2,\cdots,5 \text{ and } c_j = 0 \text{ for } j \notin \{t'_1,t'_2,t'_3,t'_4,t'_5\}.
\]

Then, from (42) and the definition of \(Z(q)\) in (11), one knows that the above vector \(c\) is a codeword of weight five in \(Z(q)\). Thus, for a given element \((t_1, t_2, t_3, t_4, t_5)\) in \(T_5\), from each vector \((v_1, v_2, v_3, v_4, v_5)\) \(\in (\mathbb{F}_3^5)^5\) we can construct a codeword \(c\) having weight five in \(Z(q)\) by (11) and (13).

Moreover, for a fixed element \((t_1, t_2, t_3, t_4, t_5)\) in \(T_5\), the codewords with weight five of \(Z(q)\) constructed from different vectors of \((\mathbb{F}_3^5)^5\) are distinct. The reason is given as follows. Assume that \((v_{1,1},v_{1,2},v_{1,3},v_{1,4},v_{1,5})\) and \((v_{2,1},v_{2,2},v_{2,3},v_{2,4},v_{2,5})\) are two different vectors in \((\mathbb{F}_3^5)^5\). Let \(\varepsilon_{i,k} = 1\) if \(v_{i,k} = -1\) and \(\varepsilon_{i,k} = 0\) otherwise, and define \(t_{i,k} = t_k + \varepsilon_{i,k} \frac{3^m+1}{2} \pmod{3^m + 1}\), where \(i = 1,2\) and \(k = 1,2,3,4,5\). Using the way described in (11) and (13), let the codeword constructed from \((v_{i,1},v_{i,2},v_{i,3},v_{i,4},v_{i,5})\) be \(c^i, i = 1,2\). Suppose, on the contrary, that \(c_1 = c^2\). Then, we have

\[
\{t_{1,k} \mid k = 1,2,3,4,5\} = \{t_{2,k} \mid k = 1,2,3,4,5\}.
\]
From (44), we can deduce that \( t_{1,k} = t_{2,k} \) for each \( k \in \{1, 2, 3, 4, 5\} \). Otherwise assume that \( t_{1,i} = t_{2,j} \) for some \( i, j \in \{1, 2, 3, 4, 5\} \) and \( i \neq j \). Then, we conclude that \( (t_i - t_j) \) (mod \( 3^m + 1 \)) is equal to 0 or \( \frac{3^m+1}{2} \), a contradiction to Lemma 10 (i). Since \( t_{1,k} = t_{2,k} \) for each \( k \in \{1, 2, 3, 4, 5\} \), we must have \( \varepsilon_{1,k} = \varepsilon_{2,k} \), i.e., \( v_{1,k} = v_{2,k} \) for each \( k \in \{1, 2, 3, 4, 5\} \), a contradiction to the assumption that \( (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}) \) and \( (v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{2,5}) \) are different.

Moreover, note that the total number of vectors in \( (\mathbb{F}_3^*)^5 \) is \( 2^5 \). From the above discussion, it follows Claim (i).

**Proof of Claim (ii).** Let \( (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \) be another element of \( \mathcal{T}_5 \), which is different from \( (t_1, t_2, t_3, t_4, t_5) \). Let \( (v_1, v_2, v_3, v_4, v_5) \) and \( (w_1, w_2, w_3, w_4, w_5) \) are two vectors in \( (\mathbb{F}_3^*)^5 \). Define \( \eta_k = 1 \) if \( w_k = -1 \) and \( \eta_k = 0 \) otherwise, and let \( \tau'_k \equiv \tau_k + \varepsilon_k \frac{3^m+1}{2} \) (mod \( 3^m + 1 \)), \( 1 \leq k \leq 5 \). Using the method in (41 and 43) and the notation there, let \( c \) be the codeword constructed from \( (t_1, t_2, t_3, t_4, t_5) \) and \( (v_1, v_2, v_3, v_4, v_5) \), and \( c' = (c'_0, c'_1, \cdots, c'_{3m}) \) the codeword constructed from \( (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \) and \( (w_1, w_2, w_3, w_4, w_5) \). We assume \( c' = c \) and obtain a contradiction.

From \( c' = c \), we have

\[
\{ t'_k \mid k = 1, 2, 3, 4, 5 \} = \{ \tau'_k \mid k = 1, 2, 3, 4, 5 \}
\]

which implies that for each \( i \in \{1, 2, 3, 4, 5\} \), there exists a unique \( \pi(i) \in \{1, 2, 3, 4, 5\} \) such that

\[
\tau_i + \eta_i \frac{3^m+1}{2} \equiv t_{\pi(i)} + \varepsilon_{\pi(i)} \frac{3^m+1}{2} \pmod{3^m + 1}, \quad (45)
\]

where \( \pi \) is a permutation of the set \( \{1, 2, 3, 4, 5\} \). We rewrite (45) as

\[
\tau_i \equiv t_{\pi(i)} + (\varepsilon_{\pi(i)} - \eta_i) \frac{3^m+1}{2} \pmod{3^m + 1} \quad (46)
\]

for each \( i \in \{1, 2, 3, 4, 5\} \). Note that both \( (t_1, t_2, t_3, t_4, t_5) \) and \( (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \) are in \( \mathcal{T}_5 \). By (46) and Lemma 10 (ii), we have

\[
\varepsilon_{\pi(i)} - \eta_i \equiv 1 \pmod{2} \text{ for each } i \in \{1, 2, 3, 4, 5\}.
\] (47)

Furthermore, if \( c' = c \), by (45) we also have \( c'_{\pi(i)} = w_i = c_{\pi(i)} = v_{\pi(i)} \) for each \( i \in \{1, 2, 3, 4, 5\} \), which contradicts with (47). Thus, \( c' \neq c \) and Claim (ii) holds.

**Proof of Claim (iii).** Let \( c = (c_0, c_1, \cdots, c_q) \) be a codeword of weight five in \( Z(q) \) and assume that \( c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}, c_{i_5} \in \mathbb{F}_3^* \), \( 0 \leq i_1 < i_2 < i_3 < i_4 < i_5 < 3^m + 1 \) and \( c_j = 0 \) for
According to the definition of $Z(q)$, we must have
\[ c_1 \delta^{i_1} + c_2 \delta^{i_2} + c_3 \delta^{i_3} + c_4 \delta^{i_4} + c_5 \delta^{i_5} = 0. \tag{48} \]

Let \( i'_k = i_k + \frac{3^m + 1}{2} \pmod{3^m + 1} \) if \( c_{i_k} = -1 \) and \( i'_k = i_k \) otherwise, \( k = 1, 2, 3, 4, 5 \). Then, (48) can be rewritten as
\[ \delta^{i'_1} + \delta^{i'_2} + \delta^{i'_3} + \delta^{i'_4} + \delta^{i'_5} = 0. \tag{49} \]

In the sequel we will show that \( i'_k \), \( 1 \leq k \leq 5 \), are pairwise distinct, and then by sorting them in increasing order, we get an element of \( T_5 \). From this element of \( T_5 \) and \((c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}, c_{i_5})\), one can see that the codeword \( \mathbf{c} \) given here can be constructed via the method described in (41) and (43).

Now we turn to proving that \( i'_k \), \( 1 \leq k \leq 5 \), in (49) are pairwise distinct. We assume without loss of generality that at most two of \( c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4} \) and \( c_{i_5} \) in (48) take the value \(-1\). Otherwise, replace \((c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}, c_{i_5})\) by \((-c_{i_1}, -c_{i_2}, -c_{i_3}, -c_{i_4}, -c_{i_5})\) in (48). Thus, we only need to consider the following two cases.

Case 1: Only one of \( c_{i_1}, c_{i_2}, c_{i_4}, c_{i_5} \) and \( c_{i_3} \) takes the value \(-1\). Without loss of generality, we assume that \( c_{i_1} = -1 \). Then, (49) becomes
\[ \delta^{i'_1} + \delta^{i'_2} + \delta^{i'_3} + \delta^{i'_4} + \delta^{i'_5} = 0, \tag{50} \]
where \( i'_1 = i_1 + \frac{3^m + 1}{2} \pmod{3^m + 1} \), and we only need to show that none of \( i_2, i_3, i_4 \) and \( i_5 \) are equal to \( i'_1 \). Without loss of generality, suppose that \( i_2 = i'_1 \). Then, \( \delta^{i'_1} + \delta^{i'_2} = \delta^{i_1} \) and (50) can be rewritten as
\[ \delta^{i_1} + \delta^{i_3} + \delta^{i_4} + \delta^{i_5} = 0. \tag{51} \]

According to Lemma 5 holds if and only if
\[ 0 \leq i_1 < i_3 < \frac{3^m + 1}{2}, \quad i_4 = \frac{3^m + 1}{2} + i_1 \quad \text{and} \quad i_5 = \frac{3^m + 1}{2} + i_3. \]

Thus, we have \( i_2 = i_4 \), a contradiction to the assumption \( 0 \leq i_1 < i_2 < i_3 < i_4 < i_5 < 3^m + 1 \). Therefore, in this case we have proved that \( i'_k \), \( 1 \leq k \leq 5 \), in (49) are pairwise distinct.

Case 2: Two of \( c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4} \) and \( c_{i_5} \) take the value \(-1\). Without loss of generality, we assume that \( c_{i_1} = c_{i_2} = -1 \). Then, (49) can be rewritten as
\[ \delta^{i'_1} + \delta^{i'_2} + \delta^{i'_3} + \delta^{i'_4} + \delta^{i'_5} = 0, \tag{52} \]
where $i'_k = i_k + \frac{3^m+1}{2}$ (mod $3^m + 1$), $k = 1, 2$. Note that $i'_1 \neq i'_2$ and $i_3 < i_4 < i_5$. Thus, the number of distinct values in the multi-set $\{i'_1, i'_2, i_3, i_4, i_5\}$ is at least 3. To prove $i'_1, i'_2, i_3, i_4$ and $i_5$ in (52) are pairwise distinct, we need to exclude the following two cases.

(a) The number of distinct values in the multi-set $\{i'_1, i'_2, i_3, i_4, i_5\}$ is 4. Without loss of generality, suppose that $i'_1 = i_3$. Then, (52) becomes

$$\delta^{i_1} + \delta^{i_2} + \delta^{i_4} + \delta^{i_5} = 0.$$  (53)

Then, by a discussion similar to Case 1, (53) also leads to a contradiction. Thus, case (a) cannot occur.

(b) The number of distinct values in the set $\{i'_1, i'_2, i_3, i_4, i_5\}$ is 3. Without loss of generality, we assume $i'_1 = i_3$ and $i'_2 = i_4$. Then, (52) becomes

$$\delta^{i_1} + \delta^{i_2} + \delta^{i_5} = 0,$$

which cannot hold due to Lemma 4 and thus case (b) cannot occur either.

Combining the discussion in Case 1 and 2, we conclude that $i'_k$, $1 \leq k \leq 5$, in (49) are pairwise distinct. □

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