We have recently presented an extension of the standard variational calculus to include the presence of deformed derivatives in the Lagrangian of a system of particles and in the Lagrangian density of field-theoretic models. Classical Euler-Lagrange equations and the Hamiltonian formalism have been re-assessed in this approach. Whenever applied to a number of physical systems, the resulting dynamical equations come out to be the correct ones found in the literature, specially with mass-dependent and with non-linear equations for classical and quantum-mechanical systems. In the present contribution, we extend the variational approach with the intervalar form of deformed derivatives to study higher-order dissipative systems, with application to concrete situations, such as an accelerated point charge - this is the problem of the Abraham-Lorentz-Dirac force - to stochastic dynamics like the Langevin, the advection-convection-reaction and Fokker-Planck equations, Korteweg–de Vries equation, Landau-Lifshits-Gilbert equation and the Caldirola-Kanai Hamiltonian. By considering these different applications, we show that the formulation investigated in this paper may be a simple and promising path for dealing with dissipative, non-linear and stochastic systems through the variational approach.
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† josewebet@gmail.com
‡ helayel@cbpf.br
I. INTRODUCTION

In this contribution, we suggest the possibility that, with the same variational formalism, it is possible to propose Lagrangians that provide the equations describing the dynamics for diverse systems, including dissipative systems, non-linear systems, and the description of stochastic processes. By adopting the intervalar form of deformed derivatives embedded into the Lagrangian, we show that our formalism yields the correct equations of motion of such systems.

Here, we apply the least action principle for dissipative systems with the use of intervalar form of deformed derivatives, without the necessity of classical Rayleigh-Lagrange equations, that are known to account for resistive forces linear in the velocities.

Although there are proposals for methods to study non-conservative and dissipative systems \[1, 11\], we claim that the approach with deformed derivatives is a simple and efficient option to obtain the equations describing the dynamics for a broad variety of linear, non-linear and stochastic systems.

The main purpose of this paper is to present the developments that go beyond the issue of Lagrangian mechanics in the classical sense, but reach the fields of dissipative, stochastic and non-linear systems. To achieve these goals, we extend the contents of our previous work of Ref. \[5\] and extend the variational calculus with deformed derivatives embedded into the Lagrangian to consider higher-order derivatives. By using the intervalar form of deformed conformable derivatives, we obtain the Euler-Lagrange (E-L) equations for each case, showing the strong concordance with the literature.

By presenting some different cases in different areas, instead of going deeper inside the solutions for each system, we suggest that with the formalism presented here we can obtain the dynamical equations that describe different physical systems. With this purpose, we apply our formulation, as exemplification, to the problem of the accelerated point charge - the Abraham-Lorentz-Dirac force, to stochastic dynamics like the Langevin, the advection-convection-reaction and Fokker-Planck(FP) equations, Korteweg-de Vries (KDV) equation, the Landau-Lifshits-Gilbert (LLG) problem and the Caldirola-Kanai (KK) Hamiltonian.

Concerning the dynamics of stochastic systems, we show that our approach does not need the Nelson-Yasue’s stochastic variational method nor Itô’s stochastic calculus and neither has the necessity of heuristics auxiliary fields, in such a way that we can use our variational approach to describe classical and quantum behaviors in broad rage of situations \[2\].
Also, it is worth to note that with our approach, the Lagrangian function does not need to be duplicated in order to describe the coupling of the dynamics with an additional process [3].

The justifications for the use of the deformed derivatives is inter-connected to the different degrees of freedom and also related to the reversibility/irreversibility process; some justifying details can be found in some previous publications in Ref. [4–7].

Our paper is outlined as follows. Section 2 addresses some mathematical aspects and justification for the use of deformed derivative. In Section 3, we focus on the extension of the variational formulations. Section 4 addresses the case of relativistic fields. In Section 5, we apply our formalism to different systems. Finally, in Section 6, we cast our general conclusions and possible paths for further investigation.

Keywords: Variational methods; Stochastic processes; Lagrangian densities, Deformed derivatives.

II. SOME MATHEMATICAL ASPECTS AND COMMENTS ON THE USE OF STRUCTURAL DERIVATIVES

Conformable Derivative

Recently, a promising new definition of local deformed derivative, called conformable fractional derivative, has been proposed by the authors in Ref. [16] that preserve classical properties and is given by

\[ T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \]  

(1)

If the function is differentiable in a classical sense, the definition above yields

\[ T_\alpha f(t) = t^{1-\alpha} \frac{df(t)}{dt}. \]  

(2)

Changing the variable \( t \to 1 + \frac{x}{t_0} \), we should write (2) as \( t_0 \left(1 + \frac{x}{t_0}\right)^{1-\alpha} \frac{df}{dx} \), that is nothing but the Hausdorff derivative up to a constant and valid for differentiable functions.

For \( t_0 \neq 0 \), an interval form of conformable derivative, the left conformable derivative is

\[ T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon (t - a)^{1-\alpha}) - f(t)}{\varepsilon}, \]  

(3)

The use of deformed-operators was there also justified based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and the same limit scale of energy for the interactions [6, 7]. Since we are dealing with open systems, as commented in Ref. [4], the particles are
indeed dressed particles or quasi-particles that exchange energy with other particles and the environment. Depending on the energy scale an interaction may change the geometry of space–time, disturbing it at the level of its topology. A system composed by particles and the surrounding environment may be considered nonconservative due to the possible energy exchange. This energy exchange may be the responsible for the resulting non-integer dimension of space–time, giving rise then to a coarse-grained medium. This is quite reasonable since, even standard field theory, comes across a granularity in the limit of Planck scale. So, some effective limit may also exist in such a way that it should be necessary to consider a coarse-grained space–time for the description of the dynamics for the system, in this scale. Also, another perspective that may be proposed is the previous existence of a nonstandard geometry, e.g., near a cosmological black hole or even in the space nearby a pair creation, that imposes a coarse-grained view to the dynamics of the open system. Here, we argue that deformed-derivatives allows us to describe and emulate this kind of dynamics without explicit many-body, dissipation or geometrical terms in the dynamical governing equations. In some way, the formalism proposed here may yield an effective theory, with some statistical average without imposing any specific nonstandard statistics. So, deformed derivatives may be the tools that could describe, in a softer way, connections between coarse-grained medium and dissipation at a certain energy scale.

Also, we indicate that one relevant applicability of our formalism, that concerns position-dependent systems [27] (see also Ref. [17] and references therein), seems to be more adequate to describe the dynamics of many real complex systems, where there could exist long-range interactions, long-time memories, anisotropy, certain symmetry breakdown, non-linear media, etc.

An important point to emphasize is that the paradigm we adopt is different from the standard approach in the generalized statistical mechanics context, where the modification of entropy definition leads to the modification of algebra and consequently the derivative concept. We adopt that the mapping to a continuous fractal space leads naturally to the necessity of modifications in the derivatives, that we will call deformed or metric derivatives [18]. The modifications of derivatives brings to a change in the algebra involved, which in turn may conduct to a generalized statistical mechanics with some adequate definition of entropy.
III. VARIATIONAL APPROACH WITH EMBEDDED DERIVATIVES

Our problem here is to search for minimizers of a variational problem with what we now refer to as structural (or deformed/metric) derivatives embedded into the Lagrangian function $L$. After the mapping into the fractal continuum, $L$ will be a $C^2$-function with respect to all its arguments.

Remarks: (i) We consider a fractional variational problem which involves local structural-derivatives called as Hausdorff that is in some sense equivalent to the conformable derivative. Here we extend the previously treated problem in Ref. [5], generalizing for Lagrangian which will depend also on higher-order structural-derivatives.

(iii) We assume that $0 < \alpha < 1$.

(iv) Here, we adopt the Option 3 in Ref. [5]: Usual integral, $\delta$-usual, structural-derivatives embedded, similar to Ref. [21], but here with local deformed or metric derivatives.

We know that deformed-kernels used here can be replaced with other kernels, resulting in a general variational calculus, as in Ref. [22].

Now, we are ready to set up the process:

In this case, we consider the fractional action $J[y] = \int_a^b L(x, y, D^1_x y, aD^\alpha_x y, aD^\alpha_x (D^1_x y)) dx$ and the usual $\delta$-variational processes. Note the interval $[a, b]$ in action functional and the interval form of the conformable derivative [Ref.].

To derive the extended version of the Euler–Lagrange equation let us introduce the following $\alpha$—deformed functional action

We shall find the condition such that $J[y]$ has a local minimum.

To do so, we consider the new fractional functional depending on the parameter $\varepsilon$.

Consider for the variable $y(x)$:

$y(x) = y^*(x) + \varepsilon \eta(x)$; $y^*(x)$ is the objective function, and $\eta(a) = \eta(b) = 0$, $\varepsilon$ is a the parameter.

So, applying the deformed derivatives and the integer one, we obtain:

$$aD^\alpha_x y(x) = aD^\alpha_x y^*(x) + \varepsilon aD^\alpha_x \eta(x).$$

$$D^1_x y(x) = D^1_x y^*(x) + \varepsilon D^1_x \eta(x).$$

$$aD^\alpha_x (D^1_x y(x)) = aD^\alpha_x (D^1_x y^*(x)) + \varepsilon aD^\alpha_x (D^1_x \eta(x)), \text{ with } aD^\alpha_x \eta = (x - a)^{1-\alpha} \frac{d\eta}{dx}, D^1_x = \frac{dx}{dt}.$$

Using the chain rule and the well known $\delta$—variational processes relative to the $\varepsilon$ parameter, we can
write
\[ \delta \varepsilon L = \frac{\partial L}{\partial y} \eta + \frac{\partial L}{\partial (D^1 x y)} D^1_x \eta + \frac{\partial L}{\partial (a D^a x y)} D^a_x \eta + \frac{\partial L}{\partial (a D^a_x (D^1 x y))} a D^a_x ((D^1 x \eta(x))), \]

Since integration by part holds with deformed integral, similarly to the integer case and, using that the usual transversality condition for an extreme value, one obtains that \( \delta \varepsilon J = 0 \) implies that

The resulting \( E - L \) equations are:

\[ \frac{\partial L}{\partial y} - D^1_x \left( \frac{\partial L}{\partial (D^1 x y)} \right) - D^1_x [(x - a)^{1-\alpha} \frac{\partial L}{\partial (a D^a x y)}] + D^2_x [(x - a)^{1-\alpha} \frac{\partial L}{\partial (a D^a_x (D^1 x y))}] = 0, \] (4)

IV. RELATIVISTIC, INDEPENDENT FIELDS

Now, we can proceed to pursue equivalent approaches to field theory, based on independent relativistic fields.

Here, \( \phi = \tilde{\phi} + \epsilon^\mu_1 \delta \phi, \)

\[ \partial_\mu \phi = \partial_\mu \tilde{\phi} + \epsilon^\mu_1 \partial_\mu \delta \phi, \]

Here, \( \delta \phi, \delta \psi \) are arbitrary, \( \tilde{\phi}, \tilde{\psi} \) are the objective fields and \( \mu = 0, 1, 2, 3, \) following the index spatial-temporal derivative, \( \partial_\mu. \)

With deformed standard derivative, we usually consider the fractional action

\[ S = \int dt \int d^3x L(\phi, \partial_\mu \phi, \partial_\mu^{\alpha \lambda} \phi, x^\mu), \]

with the usual \( \delta - \) process

\[ \delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\partial L}{\partial \partial_\mu^{\alpha \lambda} \phi} \delta \partial_\mu^{\alpha \lambda} \phi. \]

\( \partial_\mu \rightarrow \partial_\mu^{\alpha \lambda}, \lambda = 0, 1, 2, 3; L(\phi, \partial_\mu \phi, \partial_\mu^{\alpha \lambda} \phi, x^\mu). \)

For the option 3 approach on our recent article Ref. [5], we have:

\[ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) - \partial_\mu [(x^\mu)^{1-\alpha} \frac{\partial L}{\partial (\partial_\mu^{\alpha \lambda} \phi)}] = 0. \] (5)
In the case of higher-order derivative, \( L(\phi, \partial_\mu \phi, \partial_{\mu\lambda} \phi, \partial^{\alpha\lambda}_{\mu\lambda} \phi, x^\mu) \), it can be shown that the E-L equation result as

\[
\frac{\partial L}{\partial \phi} - \partial_\mu [\frac{\partial L}{\partial (\partial_\mu \phi)}] - \partial_\mu [(x^\mu)^{1-\alpha\lambda} \frac{\partial L}{\partial (\partial^{\alpha\lambda}_{\mu\lambda} \phi)}] + \partial_\mu^2 [(x^\mu)^{1-\alpha\lambda} \frac{\partial L}{\partial (\partial^{\alpha\lambda}_{\mu\lambda} (\partial_\mu \phi))}] = 0.
\]

Also, the derivatives can be considered in the intervalar form, \( a \partial_\mu \phi \), for relativistic fields.

V. APPLICATIONS:

In this Section, we apply our formalism to diverse physical problems.

A. Dissipative Forces

For details on obstruction to standard variational principles, the reader is referred to Ref. \[28\].

Here, starting off with a deformed Lagrangian and to gain some insight, we apply the formalism to a simple case. We will show in the sequel that dissipative systems can be treated with our formalism.

Consider the Lagrangian of a particle with mass \( m \), submitted to a position-dependent potential \( U \):

\[
L = \frac{1}{2} m (\dot{x})^2 - U(x) - \frac{1}{2} \gamma (a D^{1/2}_t x)^2,
\]

where the \( \gamma \) is some parameter whose the physical meaning will appear forward.

The corresponding deformed \( E-L \) equation is

\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial D^{1/2}_t x} - \frac{d}{dt} [(t-a)^{1/2} \frac{\partial L}{\partial a D^{1/2}_t x}] = 0,
\]

and leads to the movement equation

\[
m \frac{d^2 x}{dt^2} + \frac{dU(x)}{dx} + \gamma \frac{dx}{dt} [(t-a) \frac{dx}{dt}] = 0,
\]

that can be rewritten as

\[
m \frac{d^2 x}{dt^2} + \frac{dU(x)}{dx} + \gamma \frac{dx}{dt} + (t-a) \frac{d^2 x}{dt^2} = 0.
\]

We now follow the limiting procedure indicated in Ref. \[11\] with FC and in Ref. \[31\], with conformable derivatives. In the limit \( a \to b \implies (t-a) \to 0 \), the resulting equation is
Same result in Ref. [31], but without any redefinition for the Lagrangian.

The equation stated above is a standard equation with the presence of friction, showing the appearance of dissipation.

So, the results are evidently consistent with the classical Newtonian mechanics with dissipation.

B. Langevin Equation

We shall show here that some the results of Ref. [29] for under-damped scaled Brownian motion can also be obtained by the deformed variational procedure.

To this aim, let us now consider the Lagrangian:

\[
L = \frac{1}{2} m \dot{x}^2 - U(x) - \frac{1}{2} \gamma(aD^{1/2}_t x^2) - \sqrt{2D(t)} \gamma(t) x(t) \zeta(t),
\]

where \( \zeta(t) \) is a Gaussian noise [29]. Here we consider the time dependent diffusion coefficient \( D(t) \) and the time dependent damping coefficient \( \gamma(t) \).

The E-L equation results as

\[
m \frac{d^2 x}{dt^2} + \frac{dU(x)}{dx} + \left\{ \frac{d^2 \gamma(t)}{dt^2} (t-a) \frac{dx}{dt} + \gamma(t) \left[ \frac{dx}{dt} + (t-a) \frac{d^2 x}{dt^2} \right] \right\} = \sqrt{2D(t)} \gamma(t) \zeta(t).
\]

In the limit \( a \to b \), we obtain the Langevin equation

\[
m \frac{d^2 x}{dt^2} + \frac{dU(x)}{dx} + \gamma(t) \frac{dx}{dt} = \sqrt{2D(t)} \gamma(t) \zeta(t).
\]  

(7)

Now, following again Ref. [29], if we consider the case of relevant parameters and potential as

\[
\gamma(t) = \gamma_0 (1 + \frac{t}{\tau})^{\alpha-1}
\]
\[
D(t) = D_0 (1 + \frac{t}{\tau})^{\alpha-1}
\]
\[
U(x) = 0,
\]

The Langevin equation now reads
\[ m \frac{d^2x}{dt^2} + \gamma_0 (1 + \frac{t}{\tau})^{\alpha - 1} \frac{dx}{dt} = \sqrt{2D_0 \gamma_0 (1 + \frac{t}{\tau})^2(\alpha-1)} \zeta(t). \]

That is the result in the Ref. \[29\] for underdamped scaled Brownian motion.

**C. Abraham-Lorentz Lagrangian**

We now proceed with application related to a Abraham-Lorentz force.

To pursue this objective, that is, to obtain the back-reaction equation, let us now consider the Lagrangian

\[ \mathcal{L} = \frac{1}{2} m (\dot{x})^2 - U(x) + \frac{e^2}{6c^3} \left( \left[ a D_1^{1/2} \left( \frac{dx}{dt} \right) \right]^2 \right). \]

Using now eq.(4), we obtain the E-L equation as

\[ m \frac{d^2x}{dt^2} - \frac{dU(x)}{dt} + \frac{2e^2}{6c^3} \left( \frac{d^3x}{dt^3} + \frac{d^3x}{dt^3} + (t-a) \frac{d^4x}{dt^4} \right) = 0, \]

that can be rewritten as

\[ m \frac{d^2x}{dt^2} - \frac{dU(x)}{dt} + \frac{2e^2}{6c^3} \left( \frac{d^3x}{dt^3} + \frac{d^3x}{dt^3} + (t-a) \frac{d^4x}{dt^4} \right) = 0. \]

Taking the limit \( a \to b \Rightarrow (t-a) \to 0 \), the resultant equation is the one with the Abraham Lorentz term for the radiation reaction,

\[ m \frac{d^2x}{dt^2} - \frac{dU(x)}{dt} + \frac{2e^2}{3c^3} \left( \frac{d^3x}{dt^3} \right) = 0. \]

**D. Adapted Galley Method**

Here, we make an attempt to apply a simplified form of the method due to Galley, in Ref. \[1\].

Following Ref. \[5\], considering the Lagrangian and the action functional as

\[ \mathcal{L} = L(x, y, D_{x,y}^1, D_{x,y}^\alpha D_{x,z}^\alpha D_{x,z}^\alpha (D_1^1 y), D_{x,z}^1 D_{x,z}^\alpha), \]

\[ J[L] = \int_a^b L(x, y, D_{x,y}^1, D_{x,y}^\alpha D_{x,z}^\alpha D_{x,z}^\alpha (D_1^1 y), D_{x,z}^1 D_{x,z}^\alpha) dx. \]
Here the variable \( z \) can be considered as the duplicated variable in the context of Galley formalism. So, there are two E-L equations

\[
\frac{\partial L}{\partial y} - D^1_x \left( \frac{\partial L}{\partial (D^1_x y)} \right) - D^1_x [(x - a)^{1-\alpha} \frac{\partial L}{\partial (a D^1_x y)}] + D^2_x [(x - a)^{1-\alpha} \frac{\partial L}{\partial (a D^2_x (D^1_x y))}] = 0, \tag{8}
\]

\[
\frac{\partial L}{\partial z} - D^1_x \left( \frac{\partial L}{\partial (D^1_x z)} \right) - D^1_x [(x - a)^{1-\gamma} \frac{\partial L}{\partial (a D^1_x z)}] = 0. \tag{9}
\]

Considering now the Lagrangian as

\[
L = L_x - L_z + L_{xz} = \frac{1}{2} m \ddot{x} - U(x) - \frac{1}{2} m \ddot{z} + U(z) + \frac{2e^2}{3c^3} [a D^\alpha_t \frac{dx}{dt}] [a D^\alpha_t z],
\]

we can write, using the standard notation for first order derivative \( D^1_x = \frac{dx}{dt} \), the E-L Equations as

\[
m \ddot{x} - \frac{dU(x)}{dx} + \frac{2e^2}{3c^3} \frac{d}{dt} [(t - a)^{1-\alpha} a D^\alpha_t z] = 0,
\]

\[-m \ddot{z} + \frac{dU(z)}{dx} - \frac{2e^2}{3c^3} [(t - a)^{1-\alpha} a D^\alpha_t \frac{dx}{dt}] = 0.
\]

or

\[
m \ddot{x} - \frac{dU(x)}{dx} + \frac{2e^2}{3c^3} \frac{d}{dt} \left\{ (1 - 2\alpha)(2 - 2\alpha)(t - a)^{-2\alpha} \frac{d^2 z}{dt^2} + (2 - 2\alpha)(t - a)^{-2\alpha} \frac{d^2 z}{dt^2} + (1 - \alpha)(t - a)^{-\alpha} \frac{d^2 z}{dt^2} + (t - a)^{-\alpha} \frac{d^2 z}{dt^2} \right\} = 0
\]

\[-m \ddot{z} + \frac{dU(z)}{dx} + \frac{2e^2}{3c^3} [(2 - 2\alpha) (t - a)^{1-2\alpha} \frac{d^2 x}{dt^2} + (t - a)^{1-2\alpha} \frac{d^2 x}{dt^2} = 0
\]

Considering now the limit \( \alpha \to 1 \) and for physical meaning, collapse the variables \( x \) and \( z \) into one, that is \( x = z \).

So, we obtain a unique equation

\[
m \ddot{x} - \frac{dU(x)}{dx} + \frac{2e^2}{3c^3} \frac{d^3 x}{dt^3} = 0.
\]

This may indicate the equivalence between Galley’s method for integer derivatives and the simple one with structural derivatives, confirming that the variational calculus with structural derivatives (here, the conformable derivative is in its “intervalar” form) treats the hidden degrees of freedom in a simple way.
E. Reaction-Convection-Diffusion Equation

It is known \([28]\) that many physical systems may be modeled by the convection-diffusion equation. When the laws of thermodynamics predict a different behavior for macroscopic systems, compared to the behavior of individual molecules, the concept of energy dissipation may appear. This is the case of systems such as fluid particles, e.g., in the fields of confined and free-surface flows \([30]\). We here claim that the correct treatment of dissipative forces is the deformed Lagrangian methods such as those with the use of intervalar form of conformable derivatives.

Consider the Lagrangian density Ref. \([28]\). As in the indicated reference, \(U(t, x)\) is some field concentration or field density that is space and time dependent. The tensor \(K\) represents the diffusivity, \(f\) is the source term and \(\gamma \in \mathbb{R}^d\) is some flow velocity.

\[
L = L(t, x, U(t, x), \nabla U(t, x), x_a D_t^a U, \gamma \cdot \nabla \alpha U) = f(t, x)U(t, x) - \frac{1}{2} \beta U(t, x)^2 + \frac{1}{2}(a D_t^{1/2}U)^2 + \frac{1}{2}(\gamma \times x_a \nabla_x^{1/2} U \cdot x_a \nabla_x^{1/2} U) - \frac{1}{2}(K \cdot \nabla U(t, x))(\nabla U(t, x)),
\]

where \(x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\).

\[
\nabla U(t, x) = \begin{pmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \\ \frac{\partial U}{\partial x_3} \end{pmatrix}, \quad x_a \nabla_x^\alpha U = \begin{pmatrix} x_a \alpha D_{x_1}^\alpha U \\ x_a \alpha D_{x_2}^\alpha U \\ x_a \alpha D_{x_3}^\alpha U \end{pmatrix} = \begin{pmatrix} (x_1 - x_a_1)^{1-a} \frac{\partial U}{\partial x_1} \\ (x_2 - x_a_2)^{1-a} \frac{\partial U}{\partial x_2} \\ (x_3 - x_a_3)^{1-a} \frac{\partial U}{\partial x_3} \end{pmatrix}.
\]

\[
\frac{\partial L}{\partial U} = f(t, x) - \beta U
\]

\[
(t - a)^{1/2} \frac{\partial L}{\partial (a D_t^{1/2}U)} = (t - a)^{1/2} a D_t^{1/2}U = (t - a) \frac{\partial L}{\partial (t - a) \dot{U}} = -[\dot{U} + (t - a) \dot{U}] = -[\dot{U} + (t - a) \dot{U}].
\]

In the limit \(a \to b\), the term is \(-\dot{U}\).

Also,

\[
-\nabla[(x - x_a)^{1/2} \frac{\partial L}{\partial x_a \nabla_x^{1/2} U}] = -\nabla[(x - x_a)^{1/2} \gamma \cdot (x_a \nabla_x^{1/2} U)] = -[\gamma \cdot (\nabla U(t, x)) + \gamma (x - x_a) \Delta^2 U].
\]

Taking the limit \(a \to b\), we obtain the term

\[-\gamma \cdot (\nabla U(t, x)).\]

The last parcel in E-L equation is
\[-\nabla.]\left[\frac{\partial L}{\partial (\nabla U(t,x))}\right] = \nabla . (K.\nabla U).

The EL equation gives the reaction-convection-diffusion Equation [28], that have not been obtained from a variational principle with standard derivatives.

\[\frac{\partial U}{\partial t} + \gamma . (\nabla U(t,x)) - \nabla . (K.\nabla U) + \beta U = f(x,t).\]

**F. Linear Fokker-Planck Equation**

Consider now the Lagrangian density

\[L = \frac{1}{2} \left( aD^{1/2}P \right)^2 - \frac{1}{2}(D.\nabla P).\nabla P - \frac{1}{2}(\nabla f(x)).P^2 + \frac{1}{2}[f(x) \times (x_n \nabla x^{1/2}P)] \times (x_n \nabla x^{1/2}P).\]

The resulting EL equation gives

\[\frac{\partial P}{\partial t} = - \nabla . [f(x) \times P(x,t)] + D.\Delta P.\]

In one dimension, it gives the linear Fokker-Planck equation

\[\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} [f(x).P(x,t)] + D \frac{\partial^2 P}{\partial x^2}.\]

**G. Non-linear Fokker-Planck Equation**

Consider now the Lagrangian \(L\), with \(P\) as an event probability in the statistical physics context of Fokker-Planck equations,

\[L = \frac{1}{2} \left( aD^{1/2}P \right)^2 - \frac{1}{2}[\nabla . (D.\nabla P).\nabla P]^\mu - \frac{1}{2}(\nabla f(x)).P^2 + \frac{1}{2}[f(x) \times (x_n \nabla x^{1/2}P)] \times (x_n \nabla x^{1/2}P).\]

The resultant equation, in one dimensional form, is

\[\frac{\partial P}{\partial t} = - \frac{\partial}{\partial x} [f(x).P(x,t)] + D \frac{\partial}{\partial x} \left[ P^{\mu - 1} \frac{\partial P}{\partial x} \right] + \frac{1}{2}(\mu - 1)D \left( \frac{\partial P}{\partial x} \right)^2 P^{\mu - 2}.\]

This is similar to the proposed equation in Ref. [26], but here with the additional last term.
In the case of $\mu - 1 = \varepsilon \approx 1$, the lest term drops and the resultant equation is identical of Ref. [26], but with the condition of smooth non-linearity indicated.

Another type of non-linear Fokker-Planck equation, can result from the Lagrangian

$$L = \frac{1}{2}(aD_t^{1/2}P)(aD_t^{1/2}P^\mu) - \frac{1}{2}[(D,\nabla P),\nabla P]P^{\mu - 1} - \frac{\nabla f(x),P^{\mu + 1}}{\mu + 1} + \frac{1}{2}[f(x) \times (x_a\nabla_x^{1/2}P)] \times (x_a\nabla_x^{1/2}P)^\mu.$$

The resulting dynamical equation, in one dimension, from EL equation is

$$\frac{\partial P^\mu}{\partial t} = -\frac{\partial}{\partial x} [f(x),(P(x,t))^\mu] + D \frac{\partial}{\partial x} [P^{\mu - 1}\frac{\partial P}{\partial x}] + \frac{1}{2}(\nu - 1)D(\frac{\partial P}{\partial x})^2 P^{\nu - 2}.$$  

The equation above is similar to Ref. [25]. If we put $\nu = 1$, the equation above coincide with that of Ref. [25], for this value of $\nu$.

### H. KDV Equation - Lagrangian formulation without auxiliary potential $\psi$

Consider now a Lagrangian described by three terms as follows:

$$L = L_1 + L_2 + L_3,$$

with

$$L_1 = \frac{1}{4}((x_aD_x^{1/2}(\frac{\partial \phi}{\partial x})))^2, \quad (10)$$

$$L_2 = -\frac{1}{2}(aD_t^{1/2}\phi), \quad (11)$$

$$L_3 = 3\phi(x_aD_x^{1/2}\phi)^2. \quad (12)$$

The resulting E-L is nothing but the the KDV equation:

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} - 6\phi \frac{\partial \phi}{\partial x} = 0. \quad (13)$$

Note that no auxiliary potential was introduced.

Now, by considering in the sequence some deformations of the previous Lagrangian, we will show that some Deformed KDV equation emerges. Consider the Lagrangian terms
\[ L_1 = \frac{1}{4}(x_a D_x^{1/2}(\frac{\partial \phi}{\partial x}))^2, \]  
\[ L_2 = -\frac{1}{2}(a D_t^{1/2}\phi)(a D_t^{1/2}\phi^\mu), \]  
\[ L_3 = 3\phi''(x_a D_x^{1/2}\phi)^2. \]  

The resulting deformed E-L is the deformed-KDV equation:
\[ \frac{\partial \phi^\mu}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} - 6\phi^\nu \frac{\partial \phi}{\partial x} = 0. \]  

I. LLG Equation - Lagrangian Formulation with Conformable Derivatives

Following steps in Ref. [32], we will show that the Landau-Lifshits-Gilbert equation could also be obtained by our approach.

To this intend, consider the Lagrangian:
\[ L = 2A_\nu \dot{m}_\nu - \frac{1}{2}\kappa c (a D_t^{1/2} m_\beta)^2 - H_{eff_\beta} m_\beta; \]
\[ H_{eff_\beta} = H_\beta - \sigma \cdot \dot{m}_\beta. \]

E-L LLG(component \( \beta \)):  
\[ [\vec{m} \wedge (\nabla_m \wedge \vec{A}(\vec{m}))]_\beta - \vec{H}_{eff_\beta} + \kappa c (\vec{m})_\beta = 0. \]

With [32] \( (\nabla_m \wedge \vec{A}(\vec{m})) = g\vec{m} \), and \( m^2 = 1 \), the equation above can be rewritten as the LLG equation. That is,
\[ \frac{\partial \vec{m}}{\partial t} = \frac{1}{g}[\vec{m} \wedge \vec{H}_{eff}] - \frac{\kappa c}{g}[\vec{m} \wedge \vec{m}]. \]  

With \( \gamma = -\frac{1}{g}, \alpha = -\frac{\kappa c}{g} = \kappa c \gamma \), the equation can be converted to usual LLG form
\[ \frac{\partial \vec{m}}{\partial t} = -\frac{\gamma}{(1 + \alpha^2)}[\vec{m} \wedge \vec{H}_{eff}] - \frac{\alpha \gamma}{(1 + \alpha^2)}[\vec{m} \wedge (\vec{m} \wedge \vec{H}_{eff})]. \]  

J. Caldirola-Kanai Hamiltonian by \( \lambda - \text{exp} \) Metric Derivative

New \( \lambda \)-Exponential Metric (or Conformable) Derivative
Let us define a new metric derivative, that will be related, as will be shown, to Caldirola-Kanai Hamiltonian

\[ D^\lambda_t(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{-\lambda t}) - f(t)}{\epsilon}, \]

With a change in variable \( \epsilon e^{-\lambda t} = \epsilon', \) we can write the \( \lambda - \text{exp} \) metric derivative as

\[ D^\lambda_t(f)(t) = e^{-\lambda t} \frac{df(t)}{dt}. \]  

(20)

We can see that the above form of deformed derivative, for differentiable functions, has the same mathematical structure of the generic deformed metric derivative, so we can use the variational approach indicated in our recent work [5], to obtain interesting result related to dissipative systems, particularly interesting is the Caldirola-Kanai Hamiltonian.

Consider the \( \lambda - \text{exp} \) metric derivative given by eq. (20). We can use the variational approach proposed and write for the E-L equation as

\[ \frac{\partial L}{\partial q} - \frac{d}{dt}(e^{-\frac{\lambda t}{2}} \frac{\partial L}{\partial D^\lambda_t q}) = 0, \]

With a Hamiltonian \[ H = H(p^\lambda, q, t) \equiv p^\lambda(D^\lambda_t q) - L, \] where \( p^\lambda = \frac{\partial L}{\partial (D^\lambda_t q)} \). Here \( q \) is some generalized coordinate and have not to be confused with the entropic parameter [5].

Now, consider a “quasi-particle”, with a Lagrangian given by \( L = \frac{1}{2} m(D^\lambda_t q)^2 + V(t), \) where \( V(t) \) is a time dependent “potential-like” term given in by similarity with an harmonic oscillator with an elastic time-dependent constant. So, the time-dependent potential can be written as \( V(t) = \frac{1}{2} k(t)q^2 = \frac{1}{2} me^{\lambda t}\omega_0 q^2. \)

With the above expressions, we can write for the Hamiltonian \( H = p^\lambda(D^\lambda_t q) - L = \frac{1}{2} m(D^\lambda_t q)^2 + \frac{1}{2} me^{\lambda t}\omega_0 q^2. \)

Now, the \( \lambda - \text{exp} \) metric derivative of the generalized coordinate is

\[ D^\lambda_t q = e^{-\frac{\lambda t}{2}} \frac{dq}{dt} = e^{-\frac{\lambda t}{2}} \frac{p}{m}, \]

where \( p \) is the generalized moment. With this, the Hamiltonian can be written as

\[ H = e^{-\lambda t} \frac{p^2}{2m} + \frac{1}{2} me^{\lambda t}\omega_0 q^2, \]

(21)
that is nothing but the Caldirola-Kanai Hamiltonian, with important implications in the study of dissipative systems.

This is a strong indication that the metric involved is the responsible for an apparent mass-time dependent term and that the approach in [5] is an adequate variational approach to deal with dissipative systems.

VI. CONCLUSIONS AND OUTLOOK

In conclusion, we have extended the approach in Ref. [5] and employed the variational calculus to obtain E-L equations with deformed derivatives. We believe that with this formalism can set up a systematic way to obtain nonstandard equations in several areas of science, without an excessive heuristics. This can avoid to introduce ad hoc fields and unnecessary suppositions about strange dynamics.

We have shown that our approach is suitable to describe, within the Lagrangian formalism, stochastic systems, Langevin, Reaction- Convection-Diffusion equation, Fokker-Planck equations, Abraham-Lorentz radiation reaction force, KDV equation and so on. By building up Lagrangians with deformed derivatives for several dynamical systems, we have shown that the mathematical tool can be applied to the study and development of several areas in science.

For the future works, we shall apply the approach to quantum systems, by following, e.g., the canonical method, where the time derivative of physical quantities can be obtained via a modified Poisson bracket formalism, and then perform the quantization. We will also apply a Hamilton formalism for dissipative systems using conformable derivatives.

The inclusion of higher-order derivatives could lead to an interesting study of Podolsky-like systems and shall be the matter of a future publication.

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