Dimensional transport inequalities and Brascamp-Lieb inequalities

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Abstract

The goal of the present paper is to discuss new transport inequalities for convex measures. We retrieve some dimensional forms of Brascamp-Lieb inequalities. We also give some quantitative forms involving the Wasserstein’s distances.

1 Introduction

We shall begin by recalling Borell’s terminology [Bor1, Bor2] about convex measures. Although we will not use explicitly Borell’s results, it allows to explain the values and internal relations between the parameters appearing in our study.

Let $\alpha \in [-\infty, +\infty]$ . A Radon probability measure $\mu$ on $\mathbb{R}^n$ (or on an open convex set $\Omega \subseteq \mathbb{R}^n$) is called $\alpha$-concave, if it satisfies

$$
\mu(tA + (1-t)B) \geq \left(t \mu(A)^\alpha + (1-t) \mu(B)^\alpha\right)^{\frac{1}{\alpha}},
$$

(1)

for all $t \in (0,1)$ and for all Borel sets $A, B \subseteq \mathbb{R}^n$. When $\alpha = 0$, the right-hand side of (1) is understood as $\mu(A)^t \mu(B)^{1-t}$: $\mu$ is a log-concave measure. When $\alpha = -\infty$, the right-hand side is understood as $\min\{\mu(A), \mu(B)\}$ and when $\alpha = +\infty$ as $\max\{\mu(A), \mu(B)\}$. We remark that the inequality (1) is getting stronger when $\alpha$ increases, so the case $\alpha = -\infty$ describes the largest class whose members are called convex or hyperbolic probability measures. In [Bor1, Bor2], Borell proved that a measure $\mu$ on $\mathbb{R}^n$ absolutely continuous with respect to the Lebesgue measure is $\alpha$-concave (and verifies (1)) if and only if $\alpha \leq \frac{1}{n}$ and $\mu$ is supported on some open convex subset $\Omega \subseteq \mathbb{R}^n$ where it has a nonnegative density $p$ which satisfies, for all $t \in (0,1)$,
\[ p(tx + (1 - t)y) \geq \left( tp\left(x\right)^{\alpha_n} + (1 - t) p\left(y\right)^{\alpha_n}\right)^{\frac{1}{\alpha_n}}, \quad \forall x, y \in \Omega, \quad (2) \]

where \( \alpha_n := \frac{\alpha}{1 + \lambda_n} \in \left[-\frac{1}{n}, +\infty\right] \). Note that this amounts to the concavity of \( \alpha_n p^{\alpha_n} \). In particular, \( \mu \) is log-concave if and only if it has a log-concave density \( (\alpha = \alpha_n = 0) \).

We shall focus on the densities rather than on the measures, so let us reverse the perspective. We are given \( \kappa \in \left(\frac{-1}{n}, 0\right] \), or more precisely,

\[ \kappa \in \left[ -\frac{1}{n}, +\infty \right] = \left[ -\frac{1}{n}, 0 \right] \cup \{0\} \cup [0, +\infty[ , \quad (3) \]

and a probability density \( \rho \) on \( \mathbb{R}^n \) (by this we mean a nonnegative Borel function with \( \int \rho = 1 \)), with the property that \( \kappa \rho \) is concave on its support. Borell’s result then tells us that the (probability) measure with density \( \rho \) is \( \kappa \)-concave measure on \( \mathbb{R}^n \). This suggests two different behaviors depending on the sign of \( \kappa \) since \( \rho^{\kappa(n)} \) is convex or concave. Let us describe them.

**Case 1**

This corresponds to \( \kappa \in [0, +\infty] \) (that is, for measures, \( 0 < \kappa(n) \leq \frac{1}{n} \)). We set \( \beta := \frac{1}{\kappa} \in [0, +\infty) \) and we work with densities of the form \( \rho_\beta(x) = \frac{W(x)^\beta}{\int_\Omega W^\beta} \) where \( W : \mathbb{R}^n \to \mathbb{R}_+ \) is concave on its support. Note that the measure is supported on \( \Omega = \{ W > 0 \} \subset \mathbb{R}^n \), which is an open bounded convex set. The typical examples are the measures defined by

\[ d\tau_{\sigma,\beta}(x) = \frac{1}{C_{\sigma,\beta}} (\sigma^2 - |x|^2)^\beta_+ dx, \quad \beta > 0, \sigma > 0, \]

where \( C_{\sigma,\beta} = \int_{\mathbb{R}^n} (\sigma^2 - |x|^2)^\beta_+ dx = \sigma^{2\beta} + n \frac{\pi^{\frac{n}{2}} \Gamma(\beta + \frac{n}{2} + 1)}{\Gamma(\beta + 1)} \) is a normalizing constant.

**Case 2**

This corresponds to \( \kappa \in \left[ -\frac{1}{n}, 0 \right] \) (that is, for measures, \( \kappa(n) \leq 0 \)). We set \( \beta := \frac{-1}{\kappa} = n - \frac{1}{\kappa(n)} \geq n \) and we work with densities of the form \( \rho_\beta(x) = W(x)^{-\beta} \) where \( W : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\} \) is a convex function. Note that the support of the measure is given by the convex set \( \{ W < +\infty \} \). The typical examples are the (generalized) Cauchy probability measures defined by

\[ d\mu_{\beta}(x) = \frac{1}{C_\beta} \left( 1 + |x|^2 \right)^{-\beta} dx, \quad \beta > \frac{n}{2}, \]

where \( C_\beta = \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{-\beta} dx = \pi^{\frac{n}{2}} \frac{\Gamma(\beta - \frac{n}{2})}{\Gamma(\beta + 1)} \) is a normalizing constant.
In the sequel, we shall adopt the following unified notation. Given $\kappa$ as in (3), we consider a nonnegative function $W : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with the convention that
\[
\begin{cases}
\text{when } \kappa > 0, & W \text{ is concave on the bounded open convex set } \{ W > 0 \} \\
\text{when } \kappa < 0, & W \text{ is convex on } \mathbb{R}^n,
\end{cases}
\]
with the property that
\[
\int W^{1/\kappa} < +\infty;
\]
we then define the density $\rho_{\kappa, W}(x) = \frac{1}{\int W^{1/\kappa}} W^{1/\kappa}(x).$ (4)

Our first goal is to study generalized transport inequalities for these probability measures (which we identify with the density).

Let $\mu$ a probability measure on $\mathbb{R}^n$, we recall that a transport inequality is an inequality of the form
\[
\alpha(W_c(\mu, \cdot)) \leq H\left(\cdot \mid \|\mu\right),
\]
where $\alpha$ is an increasing function on $[0, +\infty)$ with $\alpha(0) = 0$, $W_c(\mu, \cdot)$ is the Kantorovich distance from $\mu$ and $H(\cdot \mid \|\mu\)$ a relative entropy with respect to $\mu$. Let us recall that given a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$, the Kantorovich distance $W_c(\mu, \nu)$ between two probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$ is defined by
\[
W_c(\mu, \nu) = \inf_{\pi} \int \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) \, d\pi(x, y)
\]
where the infimum is taken over all probability measures $\pi$ on $\mathbb{R}^n \times \mathbb{R}^n$ projecting on $\mu$ and $\nu$ respectively. In case where $c(x, y) = |x - y|^p$, with $p \geq 1$, we note
\[
W_c(\mu, \nu) = W_p^p(\mu, \nu).
\]

The relative entropy is defined as follows.

**Definition 1 (Entropy).** Let $\kappa$, $W$ and $\rho_{\kappa, W}$ be given as in the paragraph before (4). Given a probability density $\rho$ on $\mathbb{R}^n$ we introduce the $(\kappa, W)$-entropy
\[
H_{\kappa, W}(\rho) := \frac{1}{\kappa} \int \left( \rho^{1+\kappa} - \rho \right) + \frac{\kappa + 1}{-\kappa} \int \rho W
\]
provided the integrands are integrable (we set $H_{\kappa, W}(\rho) = +\infty$ otherwise). The relative entropy is then defined by
\[
H_{\kappa, W}(\rho || \rho_{\kappa, W}) := H_{\kappa, W}(\rho) - H_{\kappa, W}(\rho_{\kappa, W})
= \frac{1}{\kappa} \int \left( \rho^{\kappa+1} - (\kappa + 1) \rho W \right) + \int W^{1+1/\kappa}.
\]
The reader can convince himself that the functional \( \rho \to H_{\kappa, W}(\rho) \) is convex in \( \rho \) (note the role played by the sign of \( \kappa > -1 \)) and that \( H_{\kappa, W}(\rho||\rho_{\kappa, W}) \geq 0 \). Let us emphasize that the log-concave case corresponds to the case \( \kappa \to 0 \). We can approximate it from above or from below. For instance, given a convex function \( V \) with \( V \to +\infty \) at infinity, if we set, for \( \kappa < 0 \) close to zero with \( W(x) = W_\kappa(x) = (1 - \kappa V(x))_+ \) then, as \( \kappa \to 0^- \),

\[
\rho_{\kappa, W} \to \rho_V := \frac{1}{\int e^{-V}} e^{-V}
\]

and \( H_{\kappa, W}(\rho||\rho_{\kappa, W}) \to \int \log \left( \frac{\rho}{\rho_V} \right) \rho \), the classical relative entropy of \( \rho \) with respect to \( \rho_V \).

Generalized transport inequalities have been studied in [CE-G-H] in order to study quasilinear parabolic-elliptic equations, under some uniform convexity assumption. The following result can be seen as a dimensional form of the transport inequality for log-concave measures stated in [CE] that goes back to earlier work by Bobkov and Ledoux. The cost is defined by

\[
c_{\kappa, W}(x, y) = \frac{\kappa + 1}{-\kappa} \left[ W(y) - W(x) - \nabla W(x) \cdot (y - x) \right].
\]

(5)

According to the context recalled before (4), note that \( c_{\kappa, W}(x, x) = 0 \). This cost is actually mainly independent of \( \kappa \), which is there only to distinguish between convex (\( \kappa < 0 \)) and concave (\( \kappa > 0 \)) situations. The general transport inequality is as follows.

**Theorem 1.** Let \( \kappa, W \) and \( \rho_{\kappa, W} \) be given as in the paragraph before (4). Then we have the following transport inequality, for the entropy and cost defined above:

for any probability density \( \rho \) on \( \mathbb{R}^n \),

\[
W_{c_{\kappa, W}}(\rho_{\kappa, W}, \rho) \leq H_{\kappa, W}(\rho||\rho_{\kappa, W})
\]

(6)

According to the discussion above, when \( W(x) = W_\kappa(x) = (1 + \kappa V(x))_+ \) and \( \kappa \to 0^- \), the transport inequality recalled in [CE] for \( \rho_V := e^{-V} \), namely

\[
W_{c_V}(\rho_V, \rho) \leq H_V(\rho||\rho_V),
\]

is recovered, for the cost \( c_V(x, y) = V(y) - V(x) - \nabla V(y) \cdot (y - x) \) and the relative entropy \( H_V(\rho||\rho_V) = \int \log \left( \frac{\rho}{\rho_V} \right) \rho \).

The previous inequality is therefore not surprising, and it requires only a minor work to extract it from [CE-G-H].

Interestingly enough, we will show that the previous inequality allows to reproduce, by a linearization procedure, the dimensional Brascamp-Lieb inequalities.
obtained by Bobkov and Ledoux [Bo-Le1] and Nguyen [Ng] thus providing a mass transport approach to them.

Our second goal is to obtain quantitative versions of the transport inequality above. Before announcing our results, we need some notation. Let the function $F$ be defined on $\mathbb{R}_+$ by

$$F(t) := t - \log (1 + t), \quad \forall t \geq 0.$$ 

The function $F$ is an increasing, convex function on $\mathbb{R}_+$ and it behaves like $t^2$ when $t$ is small and like $t$ when $t$ is large, more precisely:

$$\frac{1}{4} \min \{t, t^2\} \leq F(t) \leq \min \{t, t^2\}, \quad \forall t \geq 0.$$ 

We introduce next a (weighted) isoperimetric type constant. Given a probability measure $\mu$, we denote by $h_W(\mu)$ the best nonnegative constant such that the following inequality

$$\int F(|\nabla f|) Wd\mu \geq \int F(h_W(\mu)|f - m_f|) d\mu \quad (7)$$

holds for every smooth enough function $f \in L^1(\mu)$. One may hope that $h_W(\rho_{\kappa,W}) > 0$. We shall briefly discuss this in the last section.

It may be convenient to change the notation and focus rather on the parameter $\beta = \pm \frac{1}{\kappa}$ according to the Case 1 and Case 2 detailed previously. With some abuse of notation, we will denote $\rho_{\beta,W}, H_{\beta,W}$ and $c_{\beta,W}$ the corresponding quantities.

So, more explicitly, in Case 1, which corresponds to $\kappa > 0$, we are given a $\beta \in [0, +\infty]$ and function $W : \mathbb{R}^n \to \mathbb{R}^+$ concave on its support such that $\rho_{\beta,W}(x) := \frac{W(x)^\beta}{\int W^\beta}$ is a probability density. In this Case 1 the cost is $c_{\beta,W}(x, y) = (\beta + 1) \left( W(x) - W(y) + \nabla W(x) \cdot (y - x) \right)$ and the relative entropy $H_{\beta,W}(\rho || \rho_{\beta,W}) := \int \left( \beta \rho^{1+1/\beta} - (\beta + 1) \rho W^\beta \right) + \int W^{\beta+1}$.

**Theorem 2.** Under the notation of Case 1 recalled above, introduce the costs

$$\tilde{c}(x, y) = cF(h_W(\rho_{\beta,W}) |y - x|)$$

where $c > 0$ is some fixed numerical constant, and

$$c(x, y) = c_{\beta,W}(x, y) + \tilde{c}(x, y).$$
Then, if $\rho$ and $\rho_{\beta,W}$ have the same center of mass, we have

$$H_{\beta,W}(\rho||\rho_{\beta,W}) \geq W_c(\rho_{\beta,W}, \rho).$$

(8)

**Remark 1.** Since the inequality $W_c(\rho, \rho_{\beta,W}) = W_{c_{\beta,W} + \tilde{c}}(\rho, \rho_{\beta,W}) \geq W_{c_{\beta,W}}(\rho, \rho_{\beta,W}) + W_{\tilde{c}}(\rho, \rho_{\beta,W})$ holds, the second transport inequality gives a remainder term for the first inequality:

$$H_{\beta,W}(\rho||\rho_{\beta,W}) - W_{c_{\beta,W}}(\rho, \rho_{\beta,W}) \geq W_{\tilde{c}}(\rho, \rho_{\beta,W}).$$

(9)

In **Case 2**, which corresponds to $\kappa \in \left[ -\frac{1}{n}, 0 \right[$, we are given a $\beta \geq n$ and function $W : \mathbb{R}^n \to \mathbb{R}^+$ convex such that $\rho_{\beta,W}(x) := \frac{W(x)^{-\beta}}{\int W^{-\beta}}$ is a probability density. In this **Case 2** the cost is $c_{\beta,W}(x, y) = (\beta - 1) \left( W(y) - W(x) - \nabla W(x) \cdot (y - x) \right)$ and the relative entropy $H_{\beta,W}(\rho||\rho_{\beta,W}) := \int \left( (\beta - 1) \rho W - \beta \rho^{1-1/\beta} \right) + \int W^{1-\beta}$.

**Theorem 3.** Under the notation of **Case 2** recalled above, introduce the costs

$$\tilde{c}(x, y) = \frac{c}{\beta} \left( 1 - \frac{n}{\beta} \right)^2 \mathcal{F}(h_{\rho_{\beta,W}}(y - x))$$

where $c > 0$ is some fixed numerical constant, and

$$c(x, y) = c_{\beta,W}(x, y) + \tilde{c}(x, y).$$

Then, if $\rho$ and $\rho_{\beta,W}$ have the same center of mass, we have

$$H_{\beta,W}(\rho||\rho_{\beta,W}) \geq W_c(\rho_{\beta,W}, \rho).$$

(10)

**Remark 2.** As in the **Case 1**, this gives a remainder term for the first transport inequality:

$$H_{\beta}(\rho) - W_{c_{\beta,W}}(\rho, \rho_{\beta,W}) \geq W_{\tilde{c}}(\rho, \rho_{\beta,W}).$$

(11)

The idea of the proof is to transport the densities $\rho$ onto the measure $\rho_{\beta,W}$. Cordero in [CE] uses optimal transportation to obtain a transport inequality for log-concave measures. We recall some backgrounds about mass transportation at the beginning of the following section but we refer to [V] for a detailed approach.

In a second section, we will use transport inequalities to retrieve some dimensional versions of Brascamp-Lieb inequalities. Such inequalities had already been
studied by Bobkov and Ledoux in [Bo-Le] where they use a Prékopa-Leindler type inequality. More recently, Nguyen in [Ng] retrieve these inequalities with a $L^2$–Hörmander method. Our approach is different. From transport inequalities (Theorems 1 and 2), we will use a linearization procedure to retrieve these inequalities.

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2 Proof of Theorem 1

In this part, we do not use the notation $\beta$ because there it is useless to separate the proof between Case 1 and Case 2. We can assume that $\int W^{1/\kappa} = 1$. The proof is based on optimal transportation. Let us recall briefly what it is about. Let two probability measures $\mu$ and $\nu$ on $\mathbb{R}^n$. We say a map $T : \mathbb{R}^n \to \mathbb{R}^n$ transports the measure $\mu$ onto the measure $\nu$ if:

$$\nu(B) = \mu(T^{-1}(B)), \quad \text{for all borelian sets } B \subseteq \mathbb{R}^n.$$  

This gives a transport equation: for all nonnegative Borel function $b : \mathbb{R}^n \to \mathbb{R}^+$,

$$\int_{\mathbb{R}^n} b(y) \, d\nu(y) = \int_{\mathbb{R}^n} b(T(x)) \, d\mu(x). \tag{12}$$

When $\mu$ and $\nu$ have densities with respect to Lebesgue measure (it will be always the case in this paper), say $F$ and $G$, (12) becomes:

$$\int_{\mathbb{R}^n} b(y) \, G(y) \, dy = \int_{\mathbb{R}^n} b(T(x)) \, F(x) \, dx. \tag{13}$$

The existence of a such map $T$ is resolved by the following Theorem of Brenier [Br] and refined by McCann [Mc1].

**Theorem 4.** If $\mu$ and $\nu$ are two probability measures on $\mathbb{R}^n$ and $\mu$ is absolutely continuous with respect to Lebesgue measure, then there exists a convex function $\varphi$ defined on $\mathbb{R}^n$ such that $\nabla \varphi$ transports $\mu$ onto $\nu$. Furthermore, $\nabla \varphi$ is uniquely determined $\mu$ almost-everywhere.

As $\varphi$ is convex on its domain, it is differentiable $\mu$ almost-everywhere. If we assume $\varphi$ of class $C^2$, the change of variables $y = \nabla \varphi(x)$ in (13) shows that $\varphi$ satisfies the Monge-Ampère equation, for $\mu$ almost-every $x \in \mathbb{R}^n$:

$$F(x) = G(\nabla \varphi(x)) \det D^2 \varphi(x). \tag{14}$$

Here $D^2 \varphi(x)$ stands for the Hessian matrix of $\varphi$ at the point $x$. Cafarelli’s Theorems [Ca1] and [Ca2] asserts the validity of (14) in classical sense when the
functions $F$ and $G$ are Hölder-continuous and strictly positive on their respective supports. Generally speaking, the matrix $D^2 \varphi(x)$ can be defined with the Taylor expansion of $\varphi$ ($\mu$ almost-everywhere)

$$\varphi(x+h) = \varphi(x) + \nabla \varphi(x) \cdot h + \frac{1}{2} D^2 \varphi(x)(h) \cdot h + o(|h|^2).$$

In our case we are given a probability density $\rho$ on $\mathbb{R}^n$, which we can assume to be, by approximation, continuous and strictly positive. Let $T = \nabla \varphi$ the Brenier map between $\rho_{\kappa,W}$ and $\rho$. Because $\rho_{\kappa,W}$ has a convex support, and is continuous on its support, we know that $\varphi \in W^{2,1}_{\text{loc}}$. Then the following integration by parts formula

$$\int f \Delta \varphi = -\int \nabla \varphi \cdot \nabla f$$

is valid for any smooth enough function $f : \Omega \to \mathbb{R}$. We begin by writing Monge-Ampère equation:

$$\rho_{\kappa,V}(x) = \rho(T(x)) \det D^2 \varphi.$$  \hfill (15)

It follows that for

$$\rho(T(x))^\kappa = \rho_{\kappa,W}(x)^\kappa \left( \det D^2 \varphi \right)^{-\kappa} = W(x) \left( \det D^2 \varphi \right)^{-\kappa}$$  \hfill (16)

Recall that for $\kappa \in [-\frac{1}{n}, +\infty]$, the functional

$$M \mapsto \frac{1}{\kappa} \det^{-\kappa}(M)$$

is concave on the set of nonnegative symmetric $n \times n$ matrices. If we consider the tangent at the identity matrix $I$ we find that

$$\frac{1}{\kappa} \det^{-\kappa}(M) \geq \frac{1}{\kappa} - \text{tr}(M - I).$$

Actually, for future use, let us introduce

$$\mathcal{G}_\kappa(M) := \frac{1}{\kappa} \det^{-\kappa}(M) - \frac{1}{\kappa} + \text{tr}(M - I) \geq 0.$$  \hfill (17)

So if we introduce the displacement function function $\theta(x) = \varphi(x) - |x|^2/2$ so that $T(x) = \nabla \varphi(x) = x + \nabla \theta(x)$ we have

$$\frac{1}{\kappa} \rho(T(x))^\kappa \geq W(x) \left( \frac{1}{\kappa} - \Delta \theta(x) \right) + W(x) \mathcal{G}_\kappa(D^2 \theta(x))$$

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Integrating with respect to $\rho\kappa W = W^{1/\kappa}$ and performing an integration by parts (note that $W^{1+\frac{1}{\kappa}} \to 0$ at infinity) we find

$$\frac{1}{\kappa} \int \rho^{1+\kappa} \geq \frac{1}{\kappa} \int \rho^{1+\kappa}_{\kappa,W} - \int W^{1+\frac{1}{\kappa}} \Delta \theta + \int W G_\kappa (D^2 \theta(x)) \rho_{\kappa,W}$$

$$= \frac{1}{\kappa} \int \rho^{1+\kappa}_{\kappa,W} + \frac{1 + \kappa}{\kappa} \int W^{\frac{1}{\kappa}} \nabla W \cdot \nabla \theta + \int W G_\kappa (D^2 \theta(x)) \rho_{\kappa,W}$$

By definition of mass transport we have

$$\frac{1 + \kappa}{\kappa} \int W(y) \rho(y) \, dy = \frac{1 + \kappa}{\kappa} \int W(T(x)) W^{\frac{1}{\kappa}}(x) \, dx$$

so adding the left-hand expression to the left and the right-hand expression to the right we find (adding also the required cosmetic constant) that

$$H_{\kappa,W}(\rho) = H_{\kappa,W}(\rho_{\kappa,W}) + \int c_{\kappa,W}(x,T(x)) \rho_{\kappa,W} + \int W G_\kappa (D^2 \theta(x)) \rho_{\kappa,W},$$

or equivalently

$$H_{\kappa,W}(\rho||\rho_{\kappa,W}) \geq \int c_{\kappa,W}(x,T(x)) \rho_{\kappa,W} + \int W G_\kappa (D^2 \theta(x)) \rho_{\kappa,W}. \quad (18)$$

In particular, since $G_\kappa \geq 0$, we find, by the definition of the transportation cost, the inequality stated in Theorem 1.

### 3 Remainder terms (Theorems 2 and 3)

The main step is to obtain a quantitative form of the inequality (17) and the approach is not the same whether we are in Case 1 or in Case 2. The rest of the proof is exactly the same.

#### 3.1 Case 1

We start from (18) and try to exploit the last term in order to get an improved inequality. The following Lemma gives a quantitative form of the inequality (17).

**Lemma 1.** Under the notation of Case 1, for any symmetric $n \times n$ matrix $M$, we have

$$G_\kappa (M) \geq c \sum_{i=1}^{n} \min \left\{ \mu_i^2, |\mu_i| \right\}, \quad (19)$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of $M - I$ and for some numerical constant $c > 0$.  

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Proof. The main point is the following inequality valid for $t \geq -1$,

$$\log (1 + t) \leq t - c \min \left\{ t^2, |t| \right\},$$

where $c > 0$ is a numerical constant (for instance $c = \frac{3}{10}$ works). Then, applying it with $\mu_i$ and after summing, this gives

$$\sum_{i=1}^{n} \log (1 + \mu_i) \leq \sum_{i=1}^{n} \mu_i - c \sum_{i=1}^{n} \min \left\{ |\mu_i|, \mu_i^2 \right\},$$

and

$$\prod_{i=1}^{n} (1 + \mu_i)^{-1/\beta} \geq \exp \left( - \frac{1}{\beta} \sum_{i=1}^{n} \mu_i + c \frac{1}{\beta} \sum_{i=1}^{n} \min \left\{ |\mu_i|, \mu_i^2 \right\} \right)$$

$$\geq 1 - \frac{1}{\beta} \sum_{i=1}^{n} \mu_i + c \frac{1}{\beta} \sum_{i=1}^{n} \min \left\{ |\mu_i|, \mu_i^2 \right\}.$$  

Since $\prod_{i=1}^{n} (1 + \mu_i)^{-1/\beta} = \det^{-1/\beta} (M)$ and $\sum_{i=1}^{n} \mu_i = \text{tr} (M - I)$ dividing by $\frac{1}{\beta} > 0$ ends the proof.

Let us prove now Theorem 2.

Proof. We go back to (18) and we use the previous Lemma to minimize $\int Wg_\kappa \left( D^2 \theta (x) \right) \rho_{\beta, W}$:

$$\int Wg_\kappa \left( D^2 \theta (x) \right) \rho_{\beta, W} \geq c \int \text{tr} \left( F (D^2 \theta (x)) \right) W \rho_{\beta, W}$$

Now, we follow the approach of Cordero-Erausquin in [CE].

**Lemma 2.** [CE] For any $n \times n$ symmetric matrix $M$ with eigenvalues larger than $-1$, we have:

$$\text{tr} (F (M)) \geq \frac{1}{8} \int_{S^{n-1}} F \left( \sqrt{n} |M u| \right) d\sigma (u).$$

This gives

$$\int Wg_\kappa \left( D^2 \theta (x) \right) \rho_{\beta, W} \geq c \int_{S^{n-1}} \left( \int F \left( \sqrt{n} |D^2 \theta (x) u| \right) W \rho_{\beta, W} \right) d\sigma (u) \quad (20)$$

Since $D^2 \theta (x) u = \nabla (\nabla \theta (x) \cdot u)$, and using (7) in (20), we find

$$\int Wg_\kappa \left( D^2 \theta (x) \right) \rho_{\beta, W} \geq c \int_{S^{n-1}} \int F \left( h_W (\rho_{\beta, W}) \sqrt{n} |\nabla \theta \cdot u| \right) \rho_{\beta, W} d\sigma (u) \quad (21)$$
Note that since $\rho$ and $\rho_{\beta, W}$ have the same center of mass, we have
\[
\int \nabla \theta(x) \cdot u \, \rho_{\beta, W} = 0.
\]

Before going on, let us use the following Fact.

**Fact 1.** There exists $c_n > 0$, such that for all $x \in \mathbb{R}^n$, we have
\[
\int_{S^{n-1}} |x \cdot u| \, d\sigma(u) = c_n |x|.
\]
Moreover, one can prove that there exists two positive numerical constants, say $c$ and $C$, such that $c \leq c_n \sqrt{n} \leq C$.

**Proof.** It is easy to see that $N(x) := \int_{S^{n-1}} |x \cdot u| \, d\sigma(u)$ is a norm invariant with rotations, then it is a multiple of the Euclidean norm. It is classical, see [Bor1, Bor2], that $c_n \simeq \sqrt{\int_{S^{n-1}} |x \cdot u|^2 \, d\sigma(u)}$ (i.e. up to numerical constants). Then, one can prove, using concentration of measures, that
\[
\sqrt{\int_{S^{n-1}} |x \cdot u|^2 \, d\sigma(u)} \simeq \frac{1}{\sqrt{n}}.
\]

Using Fubini’s theorem, Jensen’s inequality ($\mathcal{F}$ is convex) and Fact 1 in (21), we find
\[
\int W G_{\kappa} \left( D^2 \theta(x) \right) \rho_{\beta, W} \geq c \int \mathcal{F} \left( h_W (\rho_{\beta, W}) \sqrt{n} \int_{S^{n-1}} |\nabla \theta(x) \cdot u| \, d\sigma(u) \right) \rho_{\beta, W} \\
\geq C \int \mathcal{F} \left( h_W (\rho_{\kappa, W}) |\nabla \theta(x)| \right) \rho_{\beta, W} \\
= C \int \tilde{e} (x, T(x)) \rho_{\beta, W}
\]
Replacing this inequality in (18) finishes the proof of Theorem 2.

**3.2 Case 2**

As we say at the beginning of this part, the proof of Theorem 3 is very similar as the one for Theorem 2, the only difference is proof of the quantitative form of (17). That is the goal of the following Lemma.
Lemma 3. Under the notation of Case 2, for any nonnegative, symmetric \( n \times n \) matrix \( M \) and for all \( \beta \geq n \), we have:

\[
G_\kappa (M) \geq \frac{3}{64 \beta} \left( 1 - \frac{n}{\beta} \right)^2 \mathcal{F}(\| M - I \|_{\text{HS}}).
\]  \hfill (22)

**Proof.** We introduce the probability measure \( \mu \) defined on \( \mathbb{R} \) by

\[
d\mu = \frac{1}{\beta} \delta_1 + \cdots + \frac{1}{\beta} \delta_n + \left( 1 - \frac{n}{\beta} \right) \delta_{n+1},
\]

the function \( \phi \) defined on \([-1, +\infty)\) by \( \phi(x) = \log(1 + x) \) and the function \( f \) defined on \( \mathbb{R} \) by:

\[
f(x) = \begin{cases} 
\mu_i & \text{if } x = i \text{ with } i \in \{1, \ldots, n\}, \\
0 & \text{else.}
\end{cases}
\]

Let us note that \( \phi (\int_{\mathbb{R}} f d\mu) = \log \left( 1 + \frac{1}{\beta} \sum_{i=1}^{n} \mu_i \right) \) and \( \int_{\mathbb{R}} \phi(f) d\mu = \log \left( \prod_{i=1}^{n} (1 + \mu_i)^{\frac{1}{\beta}} \right) \).

We start with this inequality: for all \( s, t \in \mathbb{R}_+ \),

\[
\log(s) \leq \log(t) + \frac{s-t}{t} - \frac{(s-t)^2}{2 \max\{s,t\}^2}.
\]  \hfill (23)

In (23), taking \( s = 1 + f \) and \( t = 1 + m = 1 + \int_{\mathbb{R}} f d\mu \), then integrating with respect to the measure \( \mu \), it gives:

\[
\int_{\mathbb{R}} \phi(f) d\mu \leq \phi(m) - \frac{1}{2} \int_{\mathbb{R}} \frac{(f-m)^2}{\max\{1+m,1+f\}^2} d\mu.
\]

Then, we have:

\[
\phi(m) - \int_{\mathbb{R}} \phi(f) d\mu \geq \frac{1}{2} \int_{\mathbb{R}} \frac{(f-m)^2}{\max\{1+m,1+f\}^2} d\mu \\
\geq \frac{1}{4} \frac{\mu_{\text{max}}^2}{(1+\mu_{\text{max}}^2)} \int_{\mathbb{R}} (f-m)^2 d\mu.
\]

Let us compute \( \int_{\mathbb{R}} (f-m)^2 d\mu \).
\[
\int_{\mathbb{R}} (f - m)^2 \, d\mu = \int_{\mathbb{R}} f^2 \, d\mu - \left( \int_{\mathbb{R}} f \, d\mu \right)^2
\]

\[
= \frac{1}{\beta} \sum_{i=1}^{n} \mu_i^2 - \frac{1}{\beta^2} \left( \sum_{i=1}^{n} \mu_i \right)^2.
\]

\[
\geq \frac{1}{\beta} \left( 1 - \frac{n}{\beta} \right) \sum_{i=1}^{n} \mu_i^2,
\]

Cauchy-Schwarz inequality

\[
= \frac{1}{\beta} \left( 1 - \frac{n}{\beta} \right) \|M - I\|_{HS}^2.
\]

So, we have:

\[
\phi(m) - \int_{\mathbb{R}} \phi(f) \, d\mu \geq \frac{1}{4\beta} \left( 1 - \frac{n}{\beta} \right) \|M - I\|_{HS}^2 =: z.
\]

Taking the exponential, this yields \( e^{\int_{\mathbb{R}} \phi(f) \, d\mu} \leq e^{-z} e^{\phi(m)} \) then \( e^{\phi(m)} - e^{\int_{\mathbb{R}} \phi(f) \, d\mu} \geq (1 - e^{-z}) e^{\phi(m)} \). It is easy to see that \( z \in \left[ 0, \frac{1}{2} \right] \), so the inequality \( 1 - e^{-z} \geq \frac{3}{4}z \) holds. Finally, we have established the following inequality:

\[
1 + \text{tr} (M - I) - \det(M)^{\frac{1}{2}} \geq \frac{3}{16\beta} \left( 1 - \frac{n}{\beta} \right) \|M - I\|_{HS}^2 \left( 1 + \frac{1}{\beta} \text{tr} (M - I) \right).
\]

To conclude, we discuss whether \( \mu_{\text{max}} \) is bigger than 1 or not.

We assume \( \mu_{\text{max}} \leq 1 \). In this case, we have:

\[
1 + \text{tr} (M - I) - \det(M)^{\frac{1}{2}} \geq \frac{3}{32\beta} \left( 1 - \frac{n}{\beta} \right)^2 \|M - I\|_{HS}^2.
\]

We assume \( \mu_{\text{max}} \geq 1 \). First, we work on \( 1 + \frac{1}{\beta} \text{tr} (M - I) \). This yields the following lines:
\[
\beta \left( 1 + \frac{1}{\beta} \text{tr} (M - I) \right) = \beta + \sum_{i=1}^{n} \mu_i \\
\geq (\beta - (n - 1)) + \mu_{\max} \\
\geq \mu_{\max} \\
\geq \frac{1}{n} \sum_{i=1}^{n} |\mu_i| \\
\geq \frac{1}{n} \sqrt{\sum_{i=1}^{n} \mu_i^2} \\
= \frac{1}{n} \|M - I\|_{\text{HS}}.
\]

Consequently,

\[
1 + \text{tr} (M - I) - \det (M)^{\frac{1}{\beta}} \geq \frac{3}{16 n \beta^2} \left( 1 - \frac{n}{\beta} \right) \|M - I\|_{\text{HS}} \frac{\|M - I\|_{\text{HS}}^2}{1 + \mu_{\max}^2}.
\]

We finally conclude thanks to

\[
\frac{\|M - I\|_{\text{HS}}^2}{1 + \mu_{\max}^2} \geq \frac{n \mu_{\max}^2}{1 + \mu_{\max}^2} \geq \frac{1}{2} n
\]

and

\[
1 + \text{tr} (M - I) - \det (M)^{\frac{1}{\beta}} \geq \frac{3}{32 \beta^2} \left( 1 - \frac{n}{\beta} \right) \|M - I\|_{\text{HS}}.
\]

In the two cases, we have at the same time the inequality:

\[
1 + \text{tr} (M - I) - \det (M)^{\frac{1}{\beta}} \geq \frac{3}{32 \beta^2} \left( 1 - \frac{n}{\beta} \right)^2 \min \left\{ \|M - I\|_{\text{HS}}, \|M - I\|_{\text{HS}}^2 \right\}
\]

\[
\geq \frac{3}{32 \beta^2} \left( 1 - \frac{n}{\beta} \right)^2 \mathcal{F} (\|M - I\|_{\text{HS}}).
\]

Multiplying by \( \beta > 0 \), this concludes the proof of the Lemma.

\[\square\]

Let us end the proof of Theorem 3.
Proof. Let us plug (22) in (18), we obtain,

$$\int W G_\kappa(D^2\theta(x))\rho_{\beta,W} \geq \frac{3}{64\beta} \left(1 - \frac{n}{\beta}\right)^2 \int \mathcal{F}\left(\|M - I\|_{\text{HS}}\right) W \rho_{\beta,W}. $$

The rest of the proof is the same as the one for Theorem 2. \qed

4 Linearization and dimensional Brascamp-Lieb inequalities

4.1 Dimensional Brascamp-Lieb inequalities

The goal of this part is to recover dimensional Brascamp-Lieb inequalities. For that, we linearize our transport inequality we established in Theorem 1. Let us cite the result we need for that.

Lemma 4. \[CE]\n
Let $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ a function such that $c(y, y) = 0$ and $c(x, y) \geq \delta_0 |x - y|^2$ for all $x, y \in \mathbb{R}^n$ and for some $\delta_0 > 0$. We assume that for every $y \in \mathbb{R}^n$, there exists a nonnegative, symmetric matrix $n \times n$, say $H_y$, such that

$$c(y, y + h) = h \to 0 \frac{1}{2} H_y h \cdot h + |h|^2 o(1).$$

Then, if $\mu$ is a probability measure on $\mathbb{R}^n$ and $g$ is $C^1$ compactly supported with $\int_{\mathbb{R}^n} g \, d\mu = 0$, we have

$$\liminf_{\epsilon \to 0} \epsilon^2 W_c(\mu, (1 + \epsilon g) \mu) \geq \frac{1}{2} \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} g \, f \, d\mu)^2 H_y^{-1} |\nabla f| \cdot |\nabla f| \, d\mu,$$

for any function $C^1$ compactly supported $f$.

Using this Lemma to linearize inequality \[R\] gives

Theorem 5. With the notation of Theorem 1 and assuming $\int W^{1/\kappa} = 1$, we have the following inequality

$$-\kappa \int \left(D^2 W\right)^{-1} \nabla f \cdot \nabla f \rho_{\kappa,W} \geq \int g^2 W \rho_{\kappa,W}, \quad (24)$$

with $\int g \rho_{\kappa,W} = 0$ and $f = g W$. 

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As we said in Introduction, we can now retrieve dimensional Brascamp-Lieb inequalities. For example, in Case 1, (24) becomes
\[ \int \left(-D^2W\right)^{-1} \nabla f \cdot \nabla f \rho_{\beta,W} \geq \beta \int g^2 W \rho_{\beta,W}, \]
whereas in Case 2
\[ \int \left(D^2W\right)^{-1} \nabla f \cdot \nabla f \rho_{\beta,W} \geq \beta \int g^2 W \rho_{\beta,W}. \]

Proof. Let us remark first that, when \( h \to 0 \),
\[ c(y, y + h) = \frac{1}{2} \left( \frac{\kappa + 1}{-\kappa} D^2W(y) \right) (h) \cdot h + |h|^2 o(1). \] (25)
Let us compute, for \( g \) verifying \( \int g \rho_{\kappa,W} = 0 \)
\[ \liminf_{\epsilon \to 0} \frac{1}{\epsilon^2} H_{\kappa,W} \left((1 + \epsilon g) \rho_{\kappa,W} \| \rho_{\kappa,W}\right). \]
Thanks to Theorem 1, it we will have a maximization of
\[ \liminf_{\epsilon \to 0} \frac{1}{\epsilon^2} W_{c_{\kappa,W}} \left(\rho_{\kappa,W}, (1 + \epsilon g) \rho_{\kappa,W}\right). \]
Using the definition of the entropy, we have
\[ H_{\kappa,W} \left((1 + \epsilon g) \rho_{\kappa,W} \| \rho_{\kappa,W}\right) = \frac{\kappa + 1}{2} \epsilon^2 \int g^2 \rho_{1+\kappa} + o(\epsilon^2). \] (26)
Putting together (25) and (26) thanks to the above Lemma gives the following inequality, for all function \( f C^1 \) compactly supported
\[ \frac{\kappa + 1}{2} \int g^2 W \rho_{\kappa,W} \geq \frac{1}{2} \frac{(\int f g f \rho_{\kappa,W})^2}{\int \left(\frac{\kappa + 1}{-\kappa} D^2W\right)^{-1} \nabla f \cdot \nabla f \rho_{\kappa,W}}. \]
Taking \( f = gW \) concludes the proof of the Theorem.

4.2 Quantitative forms

In this section, we are interested by giving some quantitative forms of the inequalities stated in (8) and (10). The main argument is, once again, Lemma 4: we use it with the costs we introduced in Theorems 2 and 3. We separate our result whether we are in Case 1 or in Case 2.
Theorem 6. Under the notation of Case 1, we have the following inequality

$$\int \left(-D^2W + \frac{c}{\beta + 1} h_W(\rho_{\beta,W}) I\right)^{-1} \nabla f \cdot \nabla f \rho_{\beta,W} \geq \beta \int g^2 W \rho_{\beta,W},$$

for some numerical constant $c > 0$ and with $\int g \rho_{\beta,W} = 0$, $\int xg(x) \rho_{\beta,W}(x) = 0$ and $f = g W$.

And

Theorem 7. Under the notation of Case 2, we have the following inequality

$$\int \left(D^2W + \frac{c}{\beta (\beta - 1)} \left(1 - \frac{n}{\beta}\right)^2 h_W(\rho_{\beta,W}) I\right)^{-1} \nabla f \cdot \nabla f \rho_{\beta,W} \geq \beta \int g^2 W \rho_{\beta,W},$$

for some numerical constant $c > 0$ and with $\int g \rho_{\beta,W} = 0$, $\int xg(x) \rho_{\beta,W}(x) = 0$ and $f = g W$.

The proofs are very similar as the one for Theorem 5. Anyway, let us proof Theorem 6.

Proof. We keep the notation of Theorem 2. As $\int g \rho_{\beta,W} = 0$ and $\int xg(x) \rho_{\beta,W}(x) = 0$, the measures $\rho_{\beta,W}$ and $(1 + \epsilon g) \rho_{\beta,W}$ are both probability measures with the same center of mass. Thanks to Theorem 2, it is enough to give an estimation of the relative entropy instead of $W_c(\rho_{\beta,W}, (1 + \epsilon g) \rho_{\beta,W})$. Proof of Theorem 5 gives for the relative entropy:

$$H_{\kappa,W}((1 + \epsilon g) \rho_{\kappa,W} \| \rho_{\kappa,W}) = \frac{\kappa + 1}{2\kappa} \epsilon^2 \int g^2 \rho_{1+\kappa}^\kappa W + o(\epsilon^2).$$

Thanks to the definition of $F$, one have

$$\lim_{h \to 0} c(y, y + h) = \frac{1}{2} \left(- (\beta + 1) D^2W(y) + ch_W(\rho_{\beta,W}) I\right),$$

for some numerical constant $c > 0$. Using Lemma 4 with $f = g W$ permits to conclude the proof.
5 Further remarks on weighted Poincaré inequalities

5.1 Generality on weighted Poincaré inequalities

In (7), we introduced $h_W(\mu)$ as the best nonnegative constant such that the inequality
\[
\int F(|\nabla f|) Wd\mu \geq \int F(h_W(\mu) |f - mf|) d\mu
\]
holds for every smooth enough $f \in L^1(\mu)$. Nevertheless, we are convinced that the following definition for weighted Poincaré inequality (note that the weight has not the same place):
\[
\int F \left( \frac{1}{h_W(\mu)} W |\nabla f| \right) d\mu \geq \int F \left( |f - mf| \right) d\mu, \quad (27)
\]
is more natural. The next Proposition goes in this way.

Proposition 1. Let $\mu$ a probability measure with a support $\Omega \subseteq \mathbb{R}^n$ and let $\omega : \Omega \to \mathbb{R}_+$ a function. If we assume that there exists $h(\mu) > 0$ such that
\[
\int_{\Omega} \left| \frac{1}{h(\mu)} \nabla f \right| \omega d\mu \geq \int_{\Omega} |f - mf| d\mu \quad (28)
\]
for every smooth enough function $f \in L^1(\mu)$ then the following inequality
\[
\int_{\Omega} F \left( \frac{1}{h(\mu)} \omega |\nabla f| \right) d\mu \geq \int_{\Omega} F \left( |f - mf| \right) d\mu
\]
holds.

The proof is identical to the one of Bobkov-Houdré [B-H] (see [CE]). In the next section, we give an example where inequality (7) is fulfilled.

5.2 Example of weighted Poincaré inequality

Let us recall the result we will use.

Theorem 8. [Bo-Le3]

Let $\kappa \in (-\infty, 0]$ and let $\mu$ a $\kappa$-concave measure defined on $\mathbb{R}^n$ (i.e. with the notation introduced in Introduction, we are in Case 2). Let $m = \exp \left( \int_{\mathbb{R}^n} \log(|x|) d\mu(x) \right)$ (note that $m$ is finite). Then, for any non-empty Borel sets $A$ and $B$ in $\mathbb{R}^n$ located at distance $\epsilon = \text{dist}(A, B) > 0$.
\[
\mu (A) \mu (B) \leq \frac{C_\kappa}{2e} \int_{\mathbb{R}^n \setminus (A \cup B)} (m - \kappa |x|) \, d\mu (x)
\]

(29)

with \( C_\kappa \) depending continuously in \( \kappa \) in the indicated range.

Thanks to (29), let us deduce a weighted Poincaré inequality. Let us take \( K \) a non-empty compact set with smooth enough boundary and if set \( A = K \setminus S \) and \( B = (\mathbb{R}^n \setminus K) \setminus S \) where \( S \) is the closure of the \( \frac{1}{2} \)-neighborhood of \( \partial K \) in (29) and letting \( \epsilon \to 0 \), we have:

\[
\mu (K) (1 - \mu (K)) \leq \frac{C_\kappa}{2} \mu_\omega^+ (K),
\]

(30)

where \( \omega (x) = m - \kappa |x| \) and \( m = \exp (\int_{\mathbb{R}^n} \log (|x|) \, d\mu (x)) \). If we use the coarea formula in (30), we finally have

\[
\int_{\mathbb{R}^n} |f (x) - m_f| \, d\mu (x) \leq \frac{C_\kappa}{2} \int_{\mathbb{R}^n} |\nabla f (x)| \omega (x) \, d\mu (x).
\]

(31)

Now, we give an example of density \( \rho_\beta \) such that the measure \( \rho_\beta (x) \, dx \) \( \kappa \) \( (n) \)-concave which satisfy (7). For this, let for \( \beta > n \),

\[
\rho_\beta (x) = \frac{1}{Z_\beta} \left( 1 + |x|^2 \right)^{-\beta} = \frac{1}{Z_\beta} W (x)^{-\beta}
\]

where \( Z_\beta = \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{-\beta} \, dx \). Recalling the notation we used in the introduction, we have \( \beta = -\frac{1}{\kappa} = \frac{\kappa (n) - 1}{\kappa (n)} \) then \( \kappa (n) = \frac{1}{n-\beta} \). Then, the measure \( \rho_\beta (x) \, dx \) is \( \kappa \)-concave with \( \kappa = \frac{1}{n-\beta} \). If \( m = \exp (f \log (|x|) \rho_\beta) \), using (31), we can write, for all function \( f : \mathbb{R}^n \to \mathbb{R} \) smooth enough (noting \( C_\beta \) instead of \( C_\kappa \))

\[
\int |f (x) - m_f| \rho_\beta \leq \frac{C_\beta}{2} \int |\nabla f (x)| \left( m + \frac{1}{\beta - n} |x| \right) \rho_\beta
\]

\[
\leq \frac{C_\beta}{2} \max \left\{ m, \frac{1}{\beta - n} \right\} \int |\nabla f (x)| (1 + |x|) \rho_\beta.
\]

Proposition 1 gives

\[
\int \mathcal{F} \left( |f (x) - m_f| \rho_\beta \right) \leq \int \mathcal{F} \left( \frac{C_\beta}{2} \max \left\{ m, \frac{1}{\beta - n} \right\} |\nabla f| (1 + |x|) \rho_\beta \right).
\]

Remarking that \( \mathcal{F} (ab) \leq \max \{a, a^2\} \mathcal{F} (b) \) and \( (1 + |x|)^2 \leq 3 \left( 1 + |x|^2 \right) \), this gives

\[
\frac{1}{3} \int \mathcal{F} \left( |f (x) - m_f| \rho_\beta \right) \leq \int \mathcal{F} \left( \frac{C_\beta}{2} \max \left\{ m, \frac{1}{\beta - n} \right\} |\nabla f| \right) \left( 1 + |x|^2 \right) \rho_\beta.
\]
Since \( f_{ort} \geq 0, F(t/12) \leq \frac{1}{3} F(t) \), we have

\[
\int F\left(\frac{1}{12} |f(x) - m_f|\right) \rho_{\beta, W} \leq \int F\left(\frac{C_{\beta}}{2} \max \left\{ m, \frac{1}{\beta - n} \right\} |\nabla f|\right) W \rho_{\beta},
\]

or equivalently

\[
\int F\left(\frac{1}{6C_{\beta} \max \{ m, \frac{1}{\beta - n} \}} |f - m_f|\right) \rho_{\beta} \leq \int F(|\nabla f|) W \rho_{\beta}. \tag{32}
\]

Thus (32) provides an example where (7) is satisfied with the constant \( h_W (\rho_{\beta, W}) = \frac{1}{6C_{\beta} \max \{ m, \frac{1}{\beta - n} \}} > 0 \). To conclude properly, one can give an estimation of \( m \). If we note, for \( q \geq 0 \)

\[ m_q = \left( \int_{\mathbb{R}^n} |x|^q \rho_{\beta} (x) \, dx \right)^{1/q}, \]

then we have the following lines

\[
m \leq m_1 \leq m_2 \leq \cdots \leq m_n.
\]

It remains to give an estimation of

\[
\frac{\int_{0}^{+\infty} r^n \, dr}{\int_{0}^{+\infty} r^n \log(1 + r^2) \, dr}.
\]

If we note

\[
I_n (\beta) = \int_{0}^{+\infty} \frac{r^n}{(1 + r^2)^\beta} \, dr = \int_{0}^{+\infty} r^n e^{-\beta \log(1 + r^2)} \, dr,
\]

one can have, thanks to Laplace’s method (see [D]),

\[
I_n (\beta) \overset{\beta \to +\infty}{\sim} \frac{1}{2} \Gamma \left( \frac{n + 1}{2} \right) \beta^{-\frac{n+1}{2}}.
\]
and

\[ \int_0^\infty \frac{r^n}{(1+r^2)^\beta} dr \sim \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \sqrt{\beta}. \]

Since \( \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \xrightarrow{n \to \infty} \sqrt{\frac{n}{\pi}} \), we have \( m \leq C\sqrt{n/\beta} \) for some numerical constant \( C \) and for all \( \beta \geq n \), this finally gives

\[ h_W (\rho_\beta, W) \geq \frac{1}{6C_\beta \max \left\{ C\sqrt{n\beta}, \frac{1}{\beta-n} \right\}}. \]

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