ASYMPTOTICS FOR PILLAI’S PROBLEM WITH POLYNOMIALS

SEBASTIAN HEINTZE

Abstract. Let \( a_1(x)p_1(x)^n + \cdots + a_k(x)p_k(x)^n \) as well as \( b_1(x)q_1(x)^m + \cdots + b_l(x)q_l(x)^m \) be two polynomial power sums where the complex polynomials \( p_i(x) \) and \( q_j(x) \) are all non-constant. Then in the present paper we will give an asymptotic for the number of pairs \((n, m)\) such that the degree of the sum of these two power sums is between 0 and \( d \) when \( d \) goes to infinity.

1. Introduction

About hundred years ago Pillai considered in [5] exponential Diophantine equations of the shape

\[ a^n - b^m = f \]

for given positive integers \( a, b, f \) to be solved in integers \( n, m \geq 2 \). He proved for fixed integers \( a > 1 \) and \( b > 1 \) the asymptotic

\[ \# \{(n, m) \in \mathbb{N}^2 : 0 < a^n - b^m \leq x \} \sim \frac{(\log x)^2}{2 \log a \log b} \]

as \( x \to \infty \).

In [4] Kreso and Tichy considered the analogous situation for polynomials. Namely, they prove that for fixed non-constant coprime complex polynomials \( p(x) \) and \( q(x) \) we have the asymptotic

\[ \# \{(n, m) \in \mathbb{N}^2 : 0 \leq \deg (p(x)^n - q(x)^m) \leq d \} \sim \frac{d^2}{\deg p \deg q} \]

as \( d \to \infty \). It remains an open question in [4] to generalize the asymptotic result to more general polynomial power sums. The purpose of the present paper is to provide such a generalization of the asymptotic result to polynomial power sums having dominant roots, see Theorem 1 below.

2. Notation and results

Let us denote by \( F \) a function field in one variable over \( \mathbb{C} \) and by \( g \) the genus of \( F \). We will work with valuations and give here for the readers convenience a short wrap-up of this notion that can e.g. also be found in [2]: For \( c \in \mathbb{C} \) and \( f(x) \in \mathbb{C}(x) \), where \( \mathbb{C}(x) \) is the rational function field over \( \mathbb{C} \), we denote by \( \nu_c(f) \) the unique integer such that \( f(x) = (x - c)^{\nu_c(f)} p(x)/q(x) \) with \( p(x), q(x) \in \mathbb{C}[x] \) such that \( p(c)q(c) \neq 0 \). Further we write \( \nu_\infty(f) = \deg q - \deg p \) if \( f(x) = p(x)/q(x) \). These functions \( \nu : \mathbb{C}(x) \to \mathbb{Z} \) are up to equivalence all valuations in \( \mathbb{C}(x) \). If

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\( \nu_c(f) > 0 \), then \( c \) is called a zero of \( f \), and if \( \nu_c(f) < 0 \), then \( c \) is called a pole of \( f \), where \( c \in \mathbb{C} \cup \{ \infty \} \). For a finite extension \( F \) of \( \mathbb{C}(x) \) each valuation in \( \mathbb{C}(x) \) can be extended to no more than \([ F : \mathbb{C}(x) ] \) valuations in \( F \). This again gives up to equivalence all valuations in \( F \). Both, in \( \mathbb{C}(x) \) as well as in \( F \) the sum-formula

\[
\sum_\nu \nu(f) = 0
\]

holds, where the sum is taken over all valuations (up to equivalence) in the considered function field. For a finite set \( S \) of valuations on \( F \), we denote by \( \mathcal{O}_S^* \) the set

\[ \mathcal{O}_S^* = \{ f \in F^* : \nu(f) = 0 \text{ for all } \nu \notin S \}. \]

We will also use the Landau symbol \( \mathcal{O} \), i.e. “Big-O”, in the usual way and the symbol \( \sim \) to denote asymptotic equality. Our result is now the following theorem:

**Theorem 1.** Let \( p_1, \ldots, p_k, q_1, \ldots, q_l \) be non-constant complex polynomials and \( a_1, \ldots, a_k, b_1, \ldots, b_l \) be non-zero complex polynomials. Furthermore, assume that \( \deg p_1 > \max_{i=2, \ldots, k} \deg p_i \) and \( \deg q_1 > \max_{j=2, \ldots, l} \deg q_j \). Using the notation

\[
D(n, m) := \deg \left( \sum_{i=1}^k a_i(x) p_i(x)^n + \sum_{j=1}^l b_j(x) q_j(x)^m \right)
\]

we have

\[
A_d := \# \{ (n, m) \in \mathbb{N}^2 : 0 \leq D(n, m) \leq d \} \sim \frac{d^2}{\deg p_1 \cdot \deg q_1}
\]

as \( d \to \infty \).

In combination with Theorems 1 und 3 in [2], which state that under suitable assumptions for a fixed non-zero polynomial \( f(x) \) there are only finitely many representations of \( f(x) \) of the form

(1) \[ f(x) = \sum_{i=1}^k a_i(x) p_i(x)^n + \sum_{j=1}^l b_j(x) q_j(x)^m \]

and that there are only finitely many such \( f(x) \) with more than one representation of the shape (1), respectively, Theorem 1 above gives also an asymptotic for the number of polynomials \( f(x) \) with \( 0 \leq \deg f \leq d \) having a representation of the form (1).

### 3. Preliminaries

The proof of our theorem given in the next section will use height functions in function fields. Hence, let us define the height of an element \( f \in F^* \) by

\[
\mathcal{H}(f) := -\sum_\nu \min(0, \nu(f)) = \sum_\nu \max(0, \nu(f))
\]

where the sum is taken over all valuations (up to equivalence) on the function field \( F/\mathbb{C} \). Additionally we define \( \mathcal{H}(0) = \infty \). This height function satisfies some basic properties, listed in the lemma below which is proven in [3]:

**Lemma 2.** Denote as above by \( \mathcal{H} \) the height on \( F/\mathbb{C} \). Then for \( f, g \in F^* \) the following properties hold:
a) \( \mathcal{H}(f) \geq 0 \) and \( \mathcal{H}(f) = \mathcal{H}(1/f) \),
b) \( \mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(f + g) \leq \mathcal{H}(f) + \mathcal{H}(g) \),
c) \( \mathcal{H}(f) - \mathcal{H}(g) \leq \mathcal{H}(fg) \leq \mathcal{H}(f) + \mathcal{H}(g) \),
d) \( \mathcal{H}(f^n) = |n| \cdot \mathcal{H}(f) \),
e) \( \mathcal{H}(f) = 0 \iff f \in \mathbb{C}^* \),
f) \( \mathcal{H}(A(f)) = \deg A \cdot \mathcal{H}(f) \) for any \( A \in \mathbb{C}[T] \setminus \{0\} \).

Moreover, the following theorem due to Brownawell and Masser is an important ingredient for the proof section. It is an immediate consequence of Theorem B in [4]:

**Theorem 3** (Brownawell-Masser). Let \( F/\mathbb{C} \) be a function field in one variable of genus \( g \). Moreover, for a finite set \( S \) of valuations, let \( u_1, \ldots, u_k \) be \( S \)-units and
\[
1 + u_1 + \cdots + u_k = 0,
\]
where no proper subsum of the left hand side vanishes. Then we have
\[
\max_{i=1, \ldots, k} \mathcal{H}(u_i) \leq \binom{k}{2} (|S| + \max(0, 2g - 2)).
\]

4. **Proof**

We are now ready to prove our theorem about the asymptotic number of solutions to the Pillai-type equation.

**Proof of Theorem 4.** By the dominant root condition there exist positive integers \( N \) and \( M \) such that for \( n \geq N \) we have \( \deg(a_i(x)p_i(x)^n) > \deg(a_i(x)p_i(x)^n) \) for \( i = 2, \ldots, k \) and for \( m \geq M \) we have \( \deg(b_j(x)q_j(x)^m) > \deg(b_j(x)q_j(x)^m) \) for \( j = 2, \ldots, l \).

Since we aim for proving an asymptotic result for \( d \to \infty \), we may assume that \( d \) is large enough such that the following four inequalities are valid:

- \( \forall n < N : \deg \left( \sum_{i=1}^k a_i(x)p_i(x)^n \right) < d \);
- \( \forall m < M : \deg \left( \sum_{j=1}^l b_j(x)q_j(x)^m \right) < d \);
- \( \frac{d-\deg a_1}{\deg p_1} > N + 2 \);
- \( \frac{d-\deg b_1}{\deg q_1} > M + 2 \).

We start by proving a lower bound for \( A_d \). For \( (n, m) \in \mathbb{N}^2 \) with \( N \leq n \leq \frac{d-\deg a_1}{\deg p_1} \) and \( M \leq m \leq \frac{d-\deg b_1}{\deg q_1} \) we clearly have \( D(n, m) \leq d \). Moreover, for each such \( n \) there is at most one \( m \) with \( D(n, m) < 0 \). Thus we get
\[
A_d \geq \left( \frac{d - \deg a_1}{\deg p_1} - 1 - N \right) \left( \frac{d - \deg b_1}{\deg q_1} - 1 - M \right) - \left( \frac{d - \deg a_1}{\deg p_1} - (N - 1) \right)
\]
\[
= \frac{d^2}{\deg p_1 \cdot \deg q_1} + O(d).
\]

Now we need an upper bound for \( A_d \). Assuming \( D(n, m) \leq d \), we distinguish four cases. In the first case let \( n \leq \frac{d-\deg a_1}{\deg p_1} \) and \( m \leq \frac{d-\deg b_1}{\deg q_1} \). Then there are at most
\[
\left( \frac{d - \deg a_1}{\deg p_1} \right) \left( \frac{d - \deg b_1}{\deg q_1} \right)
\]
such pairs.
In the second case we assume \( n \leq \frac{d - \deg a_1}{\deg p_1} \) and \( m > \frac{d - \deg b_1}{\deg q_1} \). Then \( b_1(x)q_1(x)^m \) is the term with largest degree and we get the contradiction \( D(n, m) = \deg b_1 + m \deg q_1 > d \). Analogously, the third case \( n > \frac{d - \deg a_1}{\deg p_1} \) and \( m \leq \frac{d - \deg b_1}{\deg q_1} \) ends up in a contradiction.

Lastly, we consider the case \( n > \frac{d - \deg a_1}{\deg p_1} \) and \( m > \frac{d - \deg b_1}{\deg q_1} \). Here \( D(n, m) \leq d \) can only be possible if \( \deg a_1 + n \deg p_1 = \deg b_1 + m \deg q_1 \). Hence \( n \) and \( m \) uniquely determine each other. Writing

\[
\sum_{i=1}^{k} a_i(x)p_i(x)^n + \sum_{j=1}^{l} b_j(x)q_j(x)^m = f(x)
\]

with \( 0 \leq \deg f \leq d \) we aim for applying Theorem 3. Choosing a finite set \( S \) of discrete valuations such that all \( a_i, p_i, b_j, q_j \) as well as \( f \) are \( S \)-units is possible with

\[
|S| \leq 1 + d + \sum_{i=1}^{k} \deg a_i + \sum_{i=1}^{k} \deg p_i + \sum_{j=1}^{l} \deg b_j + \sum_{j=1}^{l} \deg q_j.
\]

Let us define

\[
C_{BM} := \left( \frac{k + l}{2} \right) \left( 1 + d + \sum_{i=1}^{k} \deg a_i + \sum_{i=1}^{k} \deg p_i + \sum_{j=1}^{l} \deg b_j + \sum_{j=1}^{l} \deg q_j \right)
\]

and rewrite equation (3) as

\[
1 - \sum_{i=1}^{k} \frac{a_i}{f} p_i^n - \sum_{j=1}^{l} \frac{b_j}{f} q_j^m = 0.
\]

Now we take a closer look at a minimal vanishing subsum containing the summand 1. There must be at least one further summand in this subsum. Assume first that this summand has the form \( \frac{b_{j_0}}{f} q_{j_0}^m \) for some \( j_0 \). Then, by Theorem 3 we get

\[
\mathcal{H} \left( \frac{b_{j_0}}{f} q_{j_0}^m \right) \leq C_{BM}
\]

and by some standard calculations using properties of the height function from Lemma 2 (cf. e.g. the calculations in [2]) the bound

\[
m \leq \frac{C_{BM} + d + \deg b_{j_0}}{\deg q_{j_0}} \leq \frac{2C_{BM}}{\min_{j=1,\ldots,l} \deg q_j}.
\]

If the summand in the subsum has the form \( \frac{a_{i_0}}{f} p_{i_0}^n \) for some \( i_0 \), we analogously get

\[
n \leq \frac{2C_{BM}}{\min_{i=1,\ldots,k} \deg p_i}.
\]

Thus in both subcases we have the bound

\[
\min(n, m) \leq \frac{2C_{BM}}{\min(\min_{i=1,\ldots,k} \deg p_i, \min_{j=1,\ldots,l} \deg q_j)} \leq 2C_{BM}.
\]

Recalling that \( n \) and \( m \) determine each other uniquely, yields that there are no more than \( 4C_{BM} \) pairs \((n, m)\) with \( D(n, m) \leq d \) in this case.
Putting the things together from all the four analyzed cases we get the final upper bound

\[ A_d \leq \left( \frac{d - \deg a_1}{\deg p_1} \right) \left( \frac{d - \deg b_1}{\deg q_1} \right) + 4C_{BM} \]

\[ = \frac{d^2}{\deg p_1 \cdot \deg q_1} + O(d). \]

From the lower bound (2) and the upper bound (4) the statement of the theorem follows immediately. □

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Sebastian Heintze
Graz University of Technology
Institute of Analysis and Number Theory
Steyrergasse 30/II
A-8010 Graz, Austria

Email address: heintze@math.tugraz.at