Bose–Einstein–Young condensates

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We demonstrate a possibility to create a new state of ultracold atoms which we call a Bose–Einstein–Young condensate. Atoms are supposed to be in different hyperfine state of the same isotope. The wave function of such a state, although totally symmetric with respect to simultaneous permutation of co-ordinates and spins of any pair of atoms, has more complicated structure than a simple product of totally symmetric co-ordinate and spin parts. Its properties with respect to permutations of only co-ordinates or only spins are characterized by a particular Young diagram, a symbol denoting an irreducible representation of the permutation group.

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Creation of binary mixtures of Bose–Einstein condensates (BECs) of ensembles of atoms of the same isotope in different hyperfine states [1,2] and spinor condensates in optical traps [3,4] is one of the remarkable advances in experiments with BECs. The results of these experiments are in good agreement with the existing theories of a multicomponent BEC [5,6]. Even if one deals with a spinor BEC, where an external magnetic field gives rise to magnetization and, hence, non-trivial spin properties, including condensate fragmentation [6], the translational motion of atoms at sufficiently low temperature (disregarding small quantum depletion) is essentially the ground state of a trap that can be determined by energy minimization.

However, a question arises, is such a ground state always a final state reached at the end of a process of cooling of atoms. In the present Rapid Communication, we demonstrate that the answer is negative. It is possible to find various states those should appear as final states of a sympathetic cooling process and are not coupled by interatomic interactions to the ground state determined in Refs. [5,6].

Due to the Bose-Einstein statistics, the wave function of the atoms must be invariant with respect to permutation of both the co-ordinates \( \{ r_1, \ldots, r_n \} \) and spins \( \{ \sigma_1, \ldots, \sigma_n \} \) of any pair of atoms. In general case, it can be constructed in different ways, namely,

\[
\Psi(r_1, \ldots, r_n; \sigma_1, \ldots, \sigma_n) = \frac{1}{\sqrt{s(\lambda)}} \sum_{k=1}^{s(\lambda)} \phi_k^{\{ \lambda \}}(r_1, \ldots, r_n)
\times \chi_k^{\{ \lambda \}}(\sigma_1, \ldots, \sigma_n). \tag{1}
\]

Here \( \{ \lambda \} \) denotes an irreducible representation of the symmetric group of the order \( n \) (i.e., the group of permutations of \( n \) objects), \( n \) is the number of identical atoms in the system, and \( s(\lambda) \) is the dimension of the representation. The theory of symmetric group representations and its applications in physics are described, e.g., in classical textbooks by Wigner [7] and by Lyubarskii [8]. It is worth to note that theory of symmetric group representation is widely used in theory of atomic spectra [9] as well as of rotational and vibrational spectra of symmetric polyatomic molecules containing identical nuclei. Here we remind briefly the necessary subjects from the theory of symmetric group representations.

Any irreducible representation of the symmetric group of order \( n \) is characterized by a partition \( \{ \lambda \} = \{ \lambda_1, \lambda_2, \ldots, \lambda_r \} \), i.e., representation of the number \( n \) as a sum of integer positive summands arranged in the non-increasing order:

\[
n = \lambda_1 + \lambda_2 + \ldots + \lambda_r,
\]

\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0.
\]

Graphically the partition \( \{ \lambda \} \) can be represented by a so-called Young diagram consisting of \( n \) square boxes arranged in rows, the number of boxes in \( i \)-th row is equal to \( \lambda_i \). To get a basis for this representation, one has to associate each co-ordinate or spin variable to one of the boxes. Then it is necessary to apply to a co-ordinate or, respectively, spin wave function a certain symmetrization-antisymmetrization procedure that involves particular permutations of the function arguments, according to their distribution between the boxes, and summation of the obtained terms with the sign of a term depending on a given permutation operator. This procedure is explicitly described in various textbooks, e.g., in Ref. [8]. The number of linearly independent functions obtained in such a way starting from different distribution of variables between the boxes of the Young diagram is the dimension of the representation. It can be calculated using the following formula [9]:

\[
s^{(\lambda)} = n! \prod_{i \leq i'} (l_i - l_{i'}) \prod_{i} l_i!, \tag{2}
\]

where \( l_i = \lambda_i + r - i \), \( i = 1, 2, \ldots, r \). The sum of squares of dimensions of all the irreducible representations equals to the number of operations in the group:

\[
\sum_{\{ \lambda \}} s^{(\lambda)}^2 = n!.
\tag{3}
\]
A basis of a given representation can be orthogonalized and normalized (by means of linear transformation) with respect to the usual quantum mechanical definition of the scalar product, implying integration over coordinates and summation over spin variables. Thus we get the two sets of functions $\phi_k^{(\lambda)}$ and $\chi_k^{(\lambda)}$, depending on atomic co-ordinates and spins, respectively. If we denote the operators of permutations of the co-ordinates and spins of the $s$-th and $s'$-th atoms by $P(i, i')$ and $Q(i, i')$, respectively, the transformation law can be written as follows:

\[
\begin{align*}
\hat{P}(i, i')\phi_k^{(\lambda)} & = \sum_m T_{km}(i, i')\phi_m^{(\lambda)}, \\
\hat{Q}(i, i')\chi_k^{(\lambda)} & = \sum_m T_{km}(i, i')\chi_m^{(\lambda)}.
\end{align*}
\]

Note that the same matrix $T_{km}(i, i')$ appears in Eqs.(4, 5). Since the scalar product of quantum mechanical wave functions is invariant under the permutation operations, and the chosen basis is orthonormal, this matrix is unitary, $\sum_k T_{km}(i, i')T_{km'}^*(i, i') = \delta_{mm'}$. Obviously, the functions $\chi_k^{(\lambda)}$ complex conjugate to $\chi^{(\lambda)*}$ also form a basis of an irreducible representation (a conjugate representation), and the transformation matrices are $T_{km}(i, i')$. Hence, the wave function $\Psi$ given by Eq.(1) remains unchanged after application of operators $\hat{P}(i, i')\hat{Q}(i, i')$ permuting both the co-ordinates and spins of the pair of atoms. Also it is proved in the group theory that there is no way to construct the wave function having the Bose–Einstein symmetry property besides that given by Eq.(1).

However, the explicit form for the bosonic wave function in a case of an arbitrary $\lambda$ is extremely lengthy. As an example, we write the formula for three particles, $\{\lambda\} = \{2, 1\}$, $a$, $b$ being the translational motion states and 1, 2 being the spin states:

\[
\begin{align*}
\Psi & = \frac{1}{\sqrt{3}}[\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)]\psi_a(r_3) \\
& \times \{\chi_1(\sigma_1)|\chi_1(\sigma_2)\chi_2(\sigma_3) + \chi_1(\sigma_3)\chi_2(\sigma_2)| -2\chi_2(\sigma_1)|\chi_2(\sigma_2)\chi_1(\sigma_3)|
\end{align*}
\]

Consider an atomic ensemble composed of $N$ atoms in the state $|1\rangle$ and $N$ atoms in the state $|2\rangle$. The states $|1\rangle$ and $|2\rangle$ can be identified with the states $|F = 1, m_F = -1\rangle$ and $|F = 2, m_F = 1\rangle$, respectively, of the ground state of the same bosonic isotope, e.g., $^{23}\text{Na}$ or $^{87}\text{Rb}$. These states have equal magnetic momenta and, hence, experience the same trapping potential. For the sake of simplicity, we assume $N$ to be small enough to allow us to use a single-particle classification of translational motion states in the trap. Energy levels are essentially the levels of a single atom in the trap, the interatomic interaction provides only small correction to them. This assumption is not necessary for validity of the subsequent treatment, however, it allows us to present our arguments in the most compact and brief manner.

The states of translational motion will be numbered by $j$, $j = 0, 1, 2, \ldots$, in the non-decreasing order in energy, so that 0 means the ground state of the trap. The tilde symbol is used to distinguish these states from the spin states $|1\rangle$ and $|2\rangle$. We use also notation $n = 2N$ for the total number of atoms in the trap.

In the standard theory of a multicomponent BEC, even when dealing with the elementary excitations of BEC, the wave function is always assumed to correspond to the simplest possible Young diagram characterizing both the co-ordinate and spin parts. This diagram has only one row consisting of $n$ boxes, where $n$ is the total number of atoms in the system; the dimension of the corresponding (totally symmetric) irreducible representation of the permutation group is 1, so that usually one writes the wave function as a product of co-ordinate and spin parts,

\[
\Psi(r_1, \ldots, r_n; \sigma_1, \ldots, \sigma_n) = \phi(n)(r_1, \ldots, r_n) \\
\times \chi(n)(\sigma_1, \ldots, \sigma_n),
\]

and the functions in the right-hand-side of this equation are invariant with respect to permutations of only co-ordinates or only spins. Now we show a way leading to creation of complicated states corresponding to the general form of Eq.(1).

Initially, before the cooling process begins, atoms are thermally distributed. Mean occupation number for each translational state is much less than unity, so that we can assume that all the trap states occupied by the atoms are different.

The system of thermal atoms is not in a pure state, but rather is described by a statistical mixture of states characterized by all the possible Young diagrams. The statistical weight of each Young diagram in this mixture is directly proportional to the number of different basis sets of functions those can be associated with the diagram. The basis sets are different in the sense that in any subspace of the total Hilbert space spanned by each basis there are Hilbert vectors not belonging to any of the remaining subspaces spanned by other basis sets. This is rather a subtle point. When we discussed earlier a question of dimensionality of an irreducible representation of the permutation group, we dealt with permutation of only function arguments within a Young diagram and assumed implicitly that each single-particle state $j$ was associated with a fixed box of the diagram. However, to determine a statistical weight of any Young diagram, we have to allow also permutation of state indices within the diagram. As it was assumed before, the probability that a given state of translational motion is occupied by more than 1 atom is negligible before the cooling process.
starts. And the number of non-trivial ways to assign \( n \) different state indices to a Young diagram consisting of \( n \) boxes equals exactly to the number of non-trivial distribution of \( n \) co-ordinate variables between the boxes of the same diagram, i.e., to \( s^{(\lambda)} \). So that for a given \( \{\lambda\} \) there are \( s^{(\lambda)} \) ways to construct different \( s^{(\lambda)} \)-dimensional basis sets of co-ordinate functions. All this sets together comprise, according to Eq.(3), \( n! \) functions [cf. Eq.(3)].

However, not all of the Young diagrams appear in our problem. The restriction on possible Young diagrams arises from that fact that only two spin states are present in this Bose gas mixture. Only diagrams consisting of two rows, at maximum, can appear, and \( \{\lambda\} = \{2N - K, K\}, K = 0, 1, 2, ..., N \). If one tries to construct spin wave function of \( N \) atoms in the state \( \{1\} \) and \( N \) atoms in the state \( \{2\} \) characterized by a Young diagrams with 3 or more rows, the result will be identically zero [7,9]. Hence, only Young diagrams with not more than two rows will apply also to functions of co-ordinates in our problem.

There is only one non-trivial way to assign a spin state label (1 or 2) to each of the boxes of the diagram, so that spin effects do not increase the statistical weights of diagrams.

Thus, the statistical weight of the Young diagram \( \{2N - K, K\} \) in the initial thermal mixed state of the atomic ensemble is, according to Eq.(2) (see also Ref. [7]), equals to \( s^{(2N-K,k)} = C_{2N}^K - C_{2N}^{K-1} \), where \( C_n^k = n!/[k!(n-k)!] \) is the binomial coefficient, and \( K = 0, 1, 2, ..., N \). The maximum of this probability distribution locates very close to \( K = N \). Calculations show that the average length of the second (shorter) row of the Young diagram will be approximately (for \( N \gg 1 \))

\[
\langle K \rangle \approx N - \sqrt{\pi N/2}. \quad (8)
\]

In other words, most probably, an observation will result in finding a state characterized by a Young diagram with two almost equal rows, which is very far from the Young diagram with only one row, and the probability to get the latter diagram tends to zero when \( N \) increases.

Then consider a cooling process. As it was said before, we assume a sympathetic cooling of our two-component atomic ensemble due to interaction with the reservoir of atoms of a different isotope kept at a sufficiently low temperature.

An interaction operator acting only to atomic co-ordinates or only to atomic spins cannot couple a total wave function of \( n \) identical bosons characterized by the Young diagram \( \{\lambda\} \) to a wave function characterized by \( \{\lambda'\} \neq \{\lambda\} \). Even an interatomic potential depending on an atomic spin orientation cannot cause such a transition during an elastic collision. Only if both the spin and translational motion states of an atom change, a transition between states characterized by different Young diagrams becomes possible. But such a process is, in fact, a spin-flip collision leading to inelastic losses but not facilitating the cooling process. The cooling can be accomplished successfully if the inelastic processes are inefficient on the cooling time scale.

Thus we can see that during the sympathetic cooling process the type of the wave function symmetry with respect to permutations of co-ordinates only is conserved. This means that at the final stage of the cooling process an observer also will find mostly states characterized by Young diagrams with two rows of almost equal length. But the translational motion wave functions corresponding to such diagrams cannot be constructed using only ground state of the trap as a one-particle state appearing \( n \) times. If one tries to construct a co-ordinate wave function by assigning to every box of a two-row diagram the same state, one will end up with identical zero.

Therefore, the lowest energy state corresponding to the Young diagram \( \{2N - K, K\} \) corresponds to filling of \( 2N - K \) boxes in the first row of the diagram with the ground one-particle state of the trap (\( |0\rangle \)) and \( K \) boxes in the second row of the diagram with the first excited state (\( |1\rangle \)). Thus the total energy is \( 2NE_0 + K(E_1 - E_0) \), where \( E_j \) is the energy of the \( j \)-th one-particle state. Usual BEC (corresponding to the Young diagram \( \{2N\} \)) would have the energy \( 2NE_0 \).

Thus we see that the evaporative cooling process of a random mixture of the same isotope in the hyperfine states \( \{1\} \) and \( \{2\} \) with almost 100 % probability results in creation of a new state of degenerate atomic Bose gas, where almost half of atoms are in the first excited state of the trap, and no further cooling can put them into the ground state. We propose to call such a macroscopic quantum object a Bose–Einstein–Young condensate (BEYC).

We would like to stress once again that BEYC can be converted to an ordinary BEC only by processes where both the spin and translation motion of atoms is changed. But spin-flip collisions being the most obvious example of such a process lead to losses and heating of the atomic sample, due to hyperfine energy release. Spin-flip collisions, hence, simply destroy BEYC as well as BEC rather than facilitate transitions between them.

The situation with BEYC and ordinary BEC is similar to that of orth- and para-hydrogen molecules. Transition between these molecular states also requires change in the type of a Young diagram (\( \{2\} \) or \( \{1, 1\} \)) and, hence, requires a rather long time even for room temperature and atmospheric pressure.

The number of atoms in the first excited trap in the BEYC state far exceeds both the quantum and thermal (at experimentally reachable temperatures) condensate depletion. The first excited state is, in fact, the dipole oscillation state. If the number of atoms in each of the two hyperfine states is well defined, the phase of these oscillations is indeterminate. However, it must become well-defined if the BEYC undergoes a continuous measurement of atomic positions, in close analogy with formation of an interference pattern produced by two counterpropagating condensates each of them was initially in the Fock state [12].

Now we have to say few words on experimental per-
pects for a BEYC. First of all, we must answer, why BEYC has not been obtained up to now. In Ref. [1], the evaporative cooling technique was used. However, $^{87}$Rb was present in that experiment in the states $|F = 1, m_F = -1\rangle$ and $|F = 2, m_F = 2\rangle$. Their magnetic momenta differ by a factor 2. Hence, they experience different trapping potentials, and, due to the gravitation, the centers of the potential wells are displaced one from another. The displacement is of such a magnitude that, on the cooling stage, the thermal clouds of atoms in different spin states overlap significantly, but the two BEC do not. In fact, condensation of rubidium atoms takes place in different states for different $m_F$’s. In such a situation, it is hard to distinguish between the states characterized by different Young schemes, since all of them have the same energy, $N_{-1}E_{0,-1}+N_{2}E_{0,2}$, where $N_{m_F}$ is the number of atoms with the magnetic quantum number $m_F$ and $E_{0,m_F}$ is the energy of the ground state of these atoms in the corresponding trapping potential influenced by the gravitation. In Ref. [2] an ordinary (single-component) BEC was prepared first. Then a fraction of the atomic ensemble was transferred to another hyperfine state by a two-photon microwave-RF pulse. Although the magnetic momenta of the two states were the same, creation of a BEYC was prevented by that circumstance that the transferred kinetic momentum (corresponding to the hyperfine splitting plus Zeeman shift) was negligible compared to the inverse trap size. Thus the translational state of atoms was not changed, and, hence, the wave function retained its symmetry properties with respect to permutation of only co-ordinates or only spins (i.e., remained a product of totally symmetric co-ordinate and spin parts). In the experiment of Ref. [3], the atoms were prepared initially in the ordinary (single-component) BEC state in a optical trap, and then the atomic spins were allowed to evolve due to interatomic interactions. The BEYC state was not reached there since it had a significantly higher energy than that stored in the BEC. The most obscure case is the case of all-optical formation of a BEC [4]. In this experiment, an evaporative cooling technique was used, and the number of atoms in BEC was more than by order of magnitude smaller that the number of atoms just before the beginning of the cooling process. This case is beyond the scope of the present Rapid Communication and requires straightforward numerical (Monte Carlo) modeling. Only performing such a modeling, it is possible to understand how Young diagrams, in average, change. A scenario, when more than 90 % of atoms are lost during the evaporative cooling process, complicated Young diagrams do not “survive”, and the Young diagram consisting of only one row of boxes prevails on the final stage, looks quite reasonable.

Thus, to observe BEYC one needs to repeat the experiment like that of Ref. [1] (where the atoms in the two different hyperfine states were prepared independently), but with the states $|F = 1, m_F = -1\rangle$ and $|F = 2, m_F = 2\rangle$ those have equal magnetic momenta. The combined magnetic trapping and gravitational potential will be the same, no spatial separation of condensed atoms in different hyperfine state will occur, and difference of BEYC from ordinary BEC will become apparent. An alternative approach would imply transfer of atoms into another hyperfine state not by a microwave-RF pulse as in Ref. [2] but by a Raman transition in two crossed laser beams. A non-zero angle of crossing will provide a kinetic momentum transfer needed for a translational motion change.

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