General $\mathcal{N} = 2$ Supersymmetric Quantum Mechanical Model: Supervariable Approach to its Off-Shell Nilpotent Symmetries

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Abstract: Using the supersymmetric (SUSY) invariant restrictions on the (anti-)chiral supervariables, we derive the off-shell nilpotent symmetries of the general one (0 + 1)-dimensional $\mathcal{N} = 2$ SUSY quantum mechanical (QM) model which is considered on a (1, 2)-dimensional supermanifold (parametrized by a bosonic variable $t$ and a pair of Grassmannian variables $\theta$ and $\bar{\theta}$ with $\theta^2 = \bar{\theta}^2 = 0$, $\theta \bar{\theta} + \bar{\theta} \theta = 0$). We provide the geometrical meanings to the two SUSY transformations of our present theory which are valid for any arbitrary type of superpotential. We express the conserved charges and Lagrangian of the theory in terms of the supervariables (that are obtained after the application of SUSY invariant restrictions) and provide the geometrical interpretation for the nilpotency property and SUSY invariance of the Lagrangian for the general $\mathcal{N} = 2$ SUSY quantum theory. We also comment on the mathematical interpretation of the above symmetry transformations.

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1 Introduction

The local non-Abelian 1-form gauge theories are at the heart of theoretical description of three out of four fundamental interactions of nature. The Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most intuitive approaches for the covariant canonical quantization of the $p$-form ($p = 1, 2, 3, ...$) gauge theories where the local gauge symmetry of the original theory is traded with the “quantum” gauge [i.e. (anti-)BRST] symmetries. A couple of decisive features of the (anti-)BRST symmetries are their nilpotency and absolute anticommutativity. These mathematical properties are explained geometrically by the well-known superfield formalism [1-8] where the horizontality condition (HC) plays a key role. The HC is important only in the context of BRST description of the $p$-form (non-)Abelian theories where there is no coupling between the gauge and matter fields.

For the interacting gauge theories, one requires more restrictions than the celebrated HC. In a set of papers (see, e.g. [9-12]), the additional gauge invariant restrictions (GIRs) have been exploited, besides HC, to obtain the full set of off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetries for the gauge, matter and (anti-)ghost fields of a given interacting gauge theory. It has been a challenging problem to apply the appropriate form of the above superfield formalisms [1-12] to the supersymmetric (SUSY) theories where the nilpotency property exists but the absolute anticommutativity property does not. In our present paper, we address this problem in the context of $\mathcal{N} = 2$ SUSY quantum mechanical (QM) model which happens to be a one ($0 + 1$)-dimensional (1D) SUSY system. We observe that it is the generalization of the idea of GIRs [9-12] to the SUSY invariant restrictions that plays a key role in our whole theoretical discussions.

The central theme of our present investigation is to exploit the strength of SUSY invariant restrictions on the (anti-)chiral supervariables to capture the nilpotency property of the SUSY symmetry transformations for the general $\mathcal{N} = 2$ SUSY QM model and derive the full set of SUSY symmetries in an accurate manner. We also provide the geometrical basis for the SUSY symmetry invariance of the Lagrangian of the $\mathcal{N} = 2$ SUSY QM system. We lay emphasis on the fact that, to avoid the absolute anticommutativity property of the $\mathcal{N} = 2$ SUSY transformations, we are theoretically compelled to choose the (anti-)chiral supervariables defined on the (1, 1)-dimensional super-submanifolds of the full (1, 2)-dimensional supermanifold. The latter is parameterized by the superspace coordinate $Z^M = (t, \theta, \bar{\theta})$ with a pair of Grassmannian variables $\theta$ and $\bar{\theta}$ (with $\theta^2 = 0$, $\bar{\theta}^2 = 0$, $\theta \bar{\theta} + \bar{\theta} \theta = 0$) and an evolution parameter $t$.

The (anti-)BRST and $\mathcal{N} = 2$ SUSY symmetry transformations are nilpotent of order two. However, they differ drastically in their anticommutativity property. Whereas the former symmetries turn out to be absolutely anticommuting, the latter symmetries do not obey the same rule. The similarity between the above two types of symmetries is only at the level of nilpotency. The latter could be of off-shell and/or on-shell variety. Furthermore, the $\mathcal{N} = 2$ SUSY transformations are two in number as is the case with nilpotent BRST and anti-BRST symmetries for a given local gauge transformation of a (non-)Abelian $p$-form gauge theory. We discuss these issues (i.e. nilpotency and anticommutativity properties) in our Appendix A, too.

Within the framework of superfield approach to BRST formalism [1-12], the nilpotent symmetries have been identified with the translational generators ($\partial_{\theta}$, $\partial_{\bar{\theta}}$) along the Grass-
mannian directions of the (D, 2)-dimensional supermanifold on which a given D-dimensional (non-)Abelian gauge theory is generalized. In this approach, the superfields are expanded along all \((1, \theta, \bar{\theta}, \theta \bar{\theta})\) directions of the (D, 2)-dimensional supermanifold. The key features of this expansion are the fact that we obtain the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetries for a given D-dimensional gauge theory due to HC and GIRs on the superfields. The nilpotency and absolute anticommutativity owe their origin to the properties \(\partial^2_\theta = \partial^2_{\bar{\theta}} = 0\) and \(\partial_\theta \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_\theta = 0\), respectively, of the translational generators \(\partial_\theta\) and \(\partial_{\bar{\theta}}\). The (anti-)chiral supervariables have been chosen in our present theory so that the translational generators \((\partial_\theta\) or \(\partial_{\bar{\theta}})\) along the Grassmannian direction \((\bar{\theta}\) or \(\theta)) of the \((1, 1)\)-dimensional (anti-)chiral supermanifolds could be identified with one of the two \(\mathcal{N} = 2\) SUSY transformations. This choice, in fact, enables us to capture only the nilpotency property of the \(\mathcal{N} = 2\) SUSY transformations and it avoids any assertion about the anticommutativity property (see also Appendix A).

The contents of our present endeavor are organized as follows. In Sec. 2, we discuss the continuous symmetries of the Lagrangian for the general \(\mathcal{N} = 2\) SUSY QM model. Our Sec. 3 contains the discussion on the derivation of the first SUSY symmetry by imposing the appropriate SUSY invariant restrictions (SUSYIRs) on the anti-chiral supervariables. Our Sec. 4 is devoted to the derivation of the second SUSY symmetry from the SUSYIRs on the chiral supervariables. In the forthcoming Sec. 5, we deal with the proof of nilpotency of the SUSY transformations and invariance of the Lagrangian in the language of supervariables. We provide mathematical interpretation of the off-shell nilpotent \(\mathcal{N} = 2\) SUSY transformations in the language of cohomological operators of differential geometry in Sec. 6. Finally, we make some concluding remarks in our Sec. 7.

In our Appendix A, we provide convincing and cogent reasons behind our choice of the (anti-)chiral supervariables.

2 Preliminaries: General \(\mathcal{N} = 2\) SUSY QM System with Any Arbitrary Superpotential

Let us begin with the Lagrangian for the general \(\mathcal{N} = 2\) SUSY QM model as follows (see, e.g. [14])

\[
L_0 = \frac{1}{2} \dot{x}^2 + i \bar{\psi} \dot{\psi} + W' A + \frac{1}{2} A^2 + W'' \bar{\psi} \psi,
\]

where the overdot and primes (i.e. \(\dot{x} = dx/dt, \dot{\psi} = d\psi/dt, W'(x) = dW/dx, W'' = d^2W/dx^2\)) are the notations for the time derivative and space derivatives, respectively. Here \(x(t)\) is the bosonic variable and its \(\mathcal{N} = 2\) SUSY fermionic \((\psi^2 = \bar{\psi}^2 = 0, \bar{\psi} \psi + \psi \bar{\psi} = 0)\) counterparts are \(\bar{\psi}(t)\) and \(\bar{\psi}(t)\). The evolution parameter in our theory is \(t\) and classically we have the absolute anticommutativity property between the fermionic variables \(\bar{\psi}\) and \(\psi\). The superpotential \(W(x)\) is usually an even function of \(x(t)\) [i.e. \(W(-x) = W(x)\)] and is not explicitly dependent on the evolution parameter \(t\). This function is arbitrary for the case of general \(\mathcal{N} = 2\) SUSY QM model and the auxiliary variable \(A(t)\) is connected [i.e. \(A(t) = -W'(x)\)] with the space derivative on the superpotential \(W(x)\). The above
Lagrangian is actually derived from the general \( \mathcal{N} = 2 \) superspace approach (see, e.g. [15,16] for details) to SUSY quantum mechanics and its form is quite general.

Our theory being \( \mathcal{N} = 2 \) SUSY QM model, we have the following two nilpotent (\( s_1^2 = s_2^2 = 0 \)) SUSY transformations \( s_1 \) and \( s_2 \) (see, e.g. [14]):

\[
\begin{align*}
 s_1 x &= i \psi, & s_1 \psi &= 0, & s_1 \bar{\psi} &= -(\dot{x} + i A), & s_1 A &= -\dot{\psi}, \\
 s_2 x &= i \bar{\psi}, & s_2 \bar{\psi} &= 0, & s_2 \psi &= -(\dot{x} - i A), & s_2 A &= \dot{\bar{\psi}},
\end{align*}
\]

(2)

under which the Lagrangian (1) transforms to the total time derivatives, as:

\[
\begin{align*}
 s_1 L_0 &= \frac{d}{dt} \left[ -W' \psi \right], & s_2 L_0 &= \frac{d}{dt} \left[ i \bar{\psi} (\dot{x} - iA) + \bar{\psi} W' \right].
\end{align*}
\]

(3)

As a consequence, the action integral \( (S = \int dt L_0) \) remains invariant. It should be noted that \( s_1 \) and \( s_2 \) are off-shell nilpotent (\( s_2^2 s_1 = s_2 s_2^2 = 0 \)) because we do not use anywhere the following Euler-Lagrange (EL) equations of motion:

\[
\begin{align*}
 \ddot{\psi} + (W'')^2 \psi - i W''' \bar{x} \psi &= 0, & \ddot{x} = W'' A + W'' \bar{\psi} \psi, & \ddot{\psi} &= i W'' \psi,
 \ddot{\bar{\psi}} + (W'')^2 \bar{\psi} + i W''' \bar{x} \bar{\psi} &= 0, & A &= -W', & \dot{\bar{\psi}} &= -i W'' \bar{\psi},
\end{align*}
\]

(4)

(emerging from the Lagrangian (1)) in the proof of their nilpotency.

According to Noether’s theorem, the invariance of the action integral leads to the derivation of the conserved charges as listed below:

\[
Q = (i \dot{x} - A) \psi \equiv (ip - A) \psi, \quad \bar{Q} = \bar{\psi} (i \dot{x} + A) \equiv \bar{\psi} (ip + A),
\]

(5)

where \( p = \dot{x} \) is the momentum corresponding to the bosonic variable. These charges turn out to be the generators of transformations \( s_1 \) and \( s_2 \) because we have the following:

\[
\begin{align*}
 s_1 \Phi &= -i [\Phi, Q]_\pm, & s_2 \Phi &= -i [\Phi, \bar{Q}]_\pm, & \Phi &= x, \psi, \bar{\psi},
\end{align*}
\]

(6)

where the subscripts \( (\pm) \), on the square bracket, correspond to the (anti)co-mmutator for the generic variable \( \Phi \) being (fermionic)bosonic in nature. The above charges \( Q \) and \( \bar{Q} \) are conserved (\( \dot{Q} = \dot{\bar{Q}} = 0 \)) as can be directly checked by using the EL equations of motion (4).

One of the decisive features of the general \( \mathcal{N} = 2 \) SUSY QM model is the observation that the anticommutator of \( s_1 \) and \( s_2 \) should not be zero and it must generate the time translation. This can be checked to be true in our theory as we have the following:

\[
\{s_1, s_2\} \Phi = s_\omega \Phi = (-2i) \dot{\Phi}, \quad s_\omega = \{s_1, s_2\}, \quad \Phi = x, \psi, \bar{\psi}, A, W', W''.
\]

(7)

The above equation establishes that the two successive operations of SUSY transformations \( s_1 \) and \( s_2 \) leads to the time derivative on a specific variable of the theory [modulo a factor of \((-2i)\)]. Thus, we have the continuous symmetry transformation \( s_\omega \) that transforms \( L_0 \) as:

\[
\begin{align*}
 s_\omega L_0 &= (s_1 s_2 + s_2 s_1) L_0 = \frac{dL_0}{dt}.
\end{align*}
\]

(8)
According to Noether’s theorem, this continuous transformation, too, leads to the derivation of a conserved charge $Q_\omega$ as:

$$Q_\omega = \frac{p^2}{2} - \frac{1}{2} A^2 - AW' - W''\bar{\psi}\psi \equiv H,$$

(9)

where $H$ is the Hamiltonian of the theory.

3 Off-Shell Nilpotent Continuous SUSY Symmetry Transformations: Anti-chiral Supervariables

It is clear from (8) that the $\mathcal{N} = 2$ SUSY transformations $s_1$ and $s_2$ are not absolutely anticommuting. Thus, to derive the SUSY transformations $s_1$, we have to concentrate on the $(1, 1)$-dimensional super-submanifold that is parametrized by the superspace variables $(t, \bar{\theta})$. We have to impose SUSY invariant restrictions on the anti-chiral supervariables which are function of $(t, \bar{\theta})$ only. The first step, towards our main goal of deriving $s_1$, is to generalize all the ordinary (explicitly time-dependent) variables [cf. (1)] to their counterpart supervariables as

$$x(t) \rightarrow X(t, \theta, \bar{\theta}) \mid_{\theta = 0} \equiv X(t, \bar{\theta}) = x(t) + \bar{\theta} f_1(t),$$

$$\psi(t) \rightarrow \Psi(t, \theta, \bar{\theta}) \mid_{\theta = 0} \equiv \Psi(t, \bar{\theta}) = \psi(t) + i \bar{\theta} b_1(t),$$

$$\bar{\psi}(t) \rightarrow \bar{\Psi}(t, \theta, \bar{\theta}) \mid_{\theta = 0} \equiv \bar{\Psi}(t, \bar{\theta}) = \bar{\psi}(t) + i \bar{\theta} b_2(t),$$

$$A(t) \rightarrow \bar{A}(t, \theta, \bar{\theta}) \mid_{\theta = 0} \equiv \bar{A}(t, \bar{\theta}) = A(t) + \bar{\theta} f_2(t),$$

(10)

where the pair of secondary variables $(b_1, b_2)$ and $(f_1, f_2)$ are bosonic and fermionic in nature, respectively. We also observe that the total number of bosonic $(x, A, b_1, b_2)$ and fermionic $(\psi, \bar{\psi}, f_1, f_2)$ variables (and their corresponding degrees of freedom) do match which is one of the basic requirements of any arbitrary general SUSY theory. The expansion (10) should be contrasted with the expansions that are used in the context of BRST formalism where the superfields are expanded along all the Grassmannian directions $(1, \theta, \bar{\theta}, \theta \bar{\theta})$ of the (D, 2)-dimensional supermanifold (cf. Appendix A) for a given D-dimensional gauge theory [9-12].

It is obvious from (2) that $s_1 \psi = 0$. Hence, the fermionic variable $\psi$ is a SUSY invariant quantity under $s_1$. We demand that this quantity should remain independent of the “soul” variable $\bar{\theta}$. As a consequence, we have the SUSY invariant restriction

$$\Psi(t, \theta, \bar{\theta}) \mid_{\theta = 0} \equiv \Psi(t, \bar{\theta}) = \psi(t) \implies b_1(t) = 0.$$

(11)

Furthermore, we note that $s_1 (x \psi) = 0$ and $s_1 (\dot{x} \psi) = 0$ (primarily due to the fermionic nature of $\psi$ where $\psi^2 = 0$). Thus, we also have the other SUSY restrictions as

$$X(t, \bar{\theta}) \Psi(t, \bar{\theta}) = x(t) \psi(t), \quad \dot{X}(t, \bar{\theta}) \Psi(t, \bar{\theta}) = \dot{x}(t) \psi(t).$$

(12)

Using the result from (11), we obtain (from the above SUSY restrictions) the following:

$$f_1(t) \psi(t) = 0, \quad \dot{f}_1(t) \psi(t) = 0.$$

(13)
The non-trivial solution of the above restrictions is \( f_1(t) \propto \psi(t) \). For the algebraic convenience, however, we choose \( f_1(t) = i \psi(t) \) which will be useful later [cf. (19) below].

It is worthwhile to mention here about the analogy between the above restrictions and the gauge invariant restrictions exploited in the context of “augmented” superfield approach to BRST formalism [see, e.g. (9-12)]. In fact, in the latter approach, the gauge invariant (physical) quantities are taken to be independent of the “soul” (i.e. Grassmannian) variables because the latter are merely a mathematical artifact and they have no physical realizations. This requirement leads, in particular, to the precise derivation of the (anti-)BRST symmetries for the matter fields in an interacting gauge theory (see, e.g. [9-12] for details). This idea has been promoted in our SUSY invariant theory where we have tapped the potential and power of SUSY invariant restrictions.

A close look at the transformations (2) shows that the nilpotency of \( s_1 \) [i.e. \( s_1^2 \bar{\psi} = -s_1(\dot{x} + i A) = 0 \)] implies that we have a SUSY invariant quantity \( (\dot{x} + i A) \) under \( s_1 \). Thus, we impose the following SUSY invariant restriction

\[
\dot{X}(t, \bar{\theta}) + i \dot{\bar{A}}(t, \bar{\theta}) = \dot{x}(t) + i A(t),
\]

which leads to the relationship \( f_2 + \psi = 0 \). This implies that \( f_2 = -\dot{\psi} \). Finally, from the symmetry invariance of \( L_0 \), we observe that the following specific combination

\[
\frac{1}{2} \dot{x}^2(t) + i \bar{\psi}(t) \dot{\psi}(t) + \frac{1}{2} A^2(t) \equiv C(t),
\]

is a SUSY invariant quantity (i.e. \( s_1 C(t) = 0 \)). Thus, we have the following SUSY invariant restriction that incorporates sum of the composite (super)variables:

\[
\frac{1}{2} \dot{X}^2(t, \bar{\theta}) + i \bar{\Psi}(t, \bar{\theta}) \dot{\Psi}(t, \bar{\theta}) + \frac{1}{2} \dot{A}^2(t, \bar{\theta})
\]

\[
= \frac{1}{2} \dot{x}^2(t) + i \bar{\psi}(t) \dot{\psi}(t) + \frac{1}{2} A^2(t).
\]

The above restriction leads to the following relationship

\[
\dot{x} \dot{f}_1 - b_2 \psi + f_2 A = 0.
\]

The substitution of \( f_1 = i \psi \) and \( f_2 = -\dot{\psi} \) in the above, implies the following

\[
b_2 = i \dot{x} - A.
\]

We conclude that the SUSY restrictions (11), (12), (14) and (16) lead to the following expansions of the anti-chiral supervariables [cf. (10)] as:

\[
X^{(1)}(t, \theta, \bar{\theta}) \big|_{\theta = 0} = X^{(1)}(t, \bar{\theta}),
\]

\[
X^{(1)}(t, \bar{\theta}) = x(t) + \bar{\theta}(i \psi) \equiv x(t) + \bar{\theta}(s_1 x),
\]

\[
\Psi^{(1)}(t, \theta, \bar{\theta}) \big|_{\theta = 0} = \Psi^{(1)}(t, \bar{\theta}),
\]

\[
\Psi^{(1)}(t, \bar{\theta}) = \psi(t) + \bar{\theta}(0) \equiv \psi(t) + \bar{\theta}(s_1 \psi),
\]

\[
\bar{\Psi}^{(1)}(t, \theta, \bar{\theta}) \big|_{\theta = 0} = \bar{\Psi}^{(1)}(t, \bar{\theta}),
\]

\[
\bar{\Psi}^{(1)}(t, \bar{\theta}) = \bar{\psi}(t) + \bar{\theta}(-\dot{x} - i A) \equiv \bar{\psi}(t) + \bar{\theta}(s_1 \bar{\psi}),
\]

\[
\bar{A}^{(1)}(t, \theta, \bar{\theta}) \big|_{\theta = 0} = \bar{A}^{(1)}(t, \bar{\theta}),
\]

\[
\bar{A}^{(1)}(t, \bar{\theta}) = A(t) + \bar{\theta}(-\dot{\psi}) \equiv A(t) + \bar{\theta}(s_1 A).
\]
Here the superscript (1) denotes the expansions of supervariables obtained after the application of SUSY invariant restrictions. Thus, we have derived the SUSY transformations $s_1$ [cf. (2)] in a very clear fashion using the SUSY invariant restrictions on the anti-chiral supervariables.

From the expansion (19), it is clear that we have the following relationship between the Grassmannian derivative $\partial_{\bar{\theta}}$ and the SUSY transformations $s_1$, namely;

$$\frac{\partial}{\partial \bar{\theta}} \Omega^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} \equiv \frac{\partial}{\partial \bar{\theta}} \Omega^{(1)}(t, \bar{\theta}) = s_1 \Omega(t), \quad (20)$$

where $\Omega^{(1)}(t, \bar{\theta})$ is the generic supervariable obtained after the application of SUSY restriction and $\Omega(t)$ is the generic variable in the one (0 + 1)-dimensional ordinary space. Geometrically, it is clear that the SUSY transformations ($s_1$) for a generic one (0 + 1)-dimensional variable $\Omega(t)$ is equivalent to the translation of its corresponding supervariable $\Omega^{(1)}(t, \bar{\theta})$ along the $\bar{\theta}$-direction of super-submanifold where the anti-chiral supervariables are defined. In view of the definition of the generator (i.e. $s_1 \Phi = -i [\Phi, Q]_\pm$), it is obvious that the translational generator $\partial_{\bar{\theta}}$, along the $\bar{\theta}$-direction of the (1, 1)-dimensional anti-chiral super-submanifold, is also connected with the super charge $Q$. Finally, we have the mapping $\partial_{\bar{\theta}} \leftrightarrow s_1 \leftrightarrow Q$ where the nilpotency property of the operators $(s_1, Q, \partial_{\bar{\theta}})$ is intertwined in a beautiful fashion as they are inter-dependent on one-another.

We wrap up this section with the remark that the generic supervariable $\Omega^{(1)}(t, \bar{\theta})$ is actually the anti-chiral limit of the most general supervariable $\Omega^{(1)}(t, \theta, \bar{\theta})$ [i.e. $\Omega^{(1)}(t, \theta, \bar{\theta}) |_{\theta=0} \equiv \Omega^{(1)}(t, \bar{\theta})$ defined on the general (1, 2)-dimensional supermanifold]. This is precisely the reason that we have continued with the partial nature of the Grassmannian derivative $\partial_{\bar{\theta}}$ and have not taken the total derivative (i.e. $d/d\bar{\theta}$) w.r.t. $\bar{\theta}$ in the relationship (20).

### 4 Off-Shell Nilpotent Continuous SUSY Symmetry Transformations: Chiral Supervariables

To derive the SUSY transformations $s_2$, we focus on the chiral super-submanifold which is parametrized by the superspace variables $(t, \theta)$. The basic and auxiliary variables (depending explicitly on $t$) of the Lagrangian (1) are, first of all, generalized onto the (1, 1)-dimensional chiral super-submanifold as:

\[
\begin{align*}
  x(t) & \longrightarrow X(t, \theta, \bar{\theta}) |_{\theta=0} \equiv X(t, \theta) = x(t) + \theta \bar{f}_1(t), \\
  \psi(t) & \longrightarrow \Psi(t, \theta, \bar{\theta}) |_{\theta=0} \equiv \Psi(t, \theta) = \psi(t) + i \theta \bar{b}_1(t), \\
  \bar{\psi}(t) & \longrightarrow \bar{\Psi}(t, \theta, \bar{\theta}) |_{\theta=0} \equiv \bar{\Psi}(t, \theta) = \bar{\psi}(t) + i \theta \bar{b}_2(t), \\
  A(t) & \longrightarrow \bar{A}(t, \theta, \bar{\theta}) |_{\theta=0} \equiv \bar{A}(t, \theta) = A(t) + \theta \bar{f}_2(t),
\end{align*}
\]

(21)

where the secondary variables ($\bar{f}_1$, $\bar{f}_2$) are fermionic and their counterparts ($\bar{b}_1$, $\bar{b}_2$) are bosonic in nature. On the r.h.s. of (21), we observe that the fermionic ($\bar{f}_1$, $\bar{f}_2$, $\psi$, $\bar{\psi}$) degrees of freedom match with their counterpart bosonic ($\bar{b}_1$, $\bar{b}_2$, $x$, $A$) degrees of freedom.

The above secondary variables ($\bar{b}_1$, $\bar{b}_2$, $\bar{f}_1$, $\bar{f}_2$) can be determined in terms of the basic variables if we impose the proper SUSY invariant restrictions on the chiral supervariables.
For instance, we observe that \( s_2 \bar{\psi} = 0 \) [cf. (2)]. Thus, we impose the SUSY restriction
\[
\bar{\Psi}(t, \theta, \bar{\theta}) \mid_{\bar{\theta} = 0} \equiv \bar{\Psi}(t, \theta) = \bar{\psi}(t) \quad \implies \quad \bar{b}_2(t) = 0. \tag{22}
\]
We also note that \( s_2 (x \bar{\psi}) = 0, \, s_2 (\dot{x} \bar{\psi}) = 0 \) [cf. (2)] because of the fermionic nature of \( \bar{\psi} \). As a result, we have the following two SUSY restrictions on the composite chiral supervariables:
\[
X(t, \theta) \bar{\Psi}(t, \theta) = x(t) \bar{\psi}(t), \quad \dot{X}(t, \theta) \bar{\Psi}(t, \theta) = \dot{x}(t) \bar{\psi}(t).
\tag{23}
\]
With the help from (22), we find that
\[
\bar{f}_1(t) \bar{\psi}(t) = 0, \quad \dot{\bar{f}}_1(t) \bar{\psi}(t) = 0. \tag{24}
\]
The non-trivial solution of the above restrictions is \( \bar{f}_1 = i \bar{\psi} \). We have taken \( i \) factor for the algebraic convenience which will become clear later [cf. (28) below] in our further discussions.

To determine all the secondary variables, we note further that \( s_2 [\dot{x} - iA(t)] = 0 \). This invariance emerges from the nilpotency of \( s_2 \) because we observe that \( s_2^2 \bar{\psi} = s_2 (- [\dot{x} - iA]) = 0 \) in equation (2). This shows that \( [\dot{x} - iA] \) is a SUSY invariant quantity. Thus, we have the following SUSY invariant restriction on the chiral supervariables:
\[
\dot{X}(t, \theta) - i \dot{A}(t, \theta) = \dot{x}(t) - iA(t). \tag{25}
\]
The above condition yields \( \bar{f}_2 = \dot{\bar{\psi}} \). A part of the modified form of Lagrangian (1) also remains invariant under \( s_2 \). In fact, we note that the following sum of the composite variables are invariant under \( s_2 \), namely;
\[
s_2 \left[ \frac{1}{2} \dot{x}^2(t) - i \dot{\bar{\psi}}(t) \psi(t) + \frac{1}{2} A^2(t) \right] = 0.
\tag{26}
\]
Thus, we have the following SUSY invariant restriction on the specific combination of composite chiral supervariables:
\[
\frac{1}{2} \dot{X}^2(t, \theta) - i \dot{\bar{\Psi}}(t, \theta) \Psi(t, \theta) + \frac{1}{2} \dot{A}^2(t, \theta)
= \frac{1}{2} \dot{x}^2(t) - i \dot{\bar{\psi}}(t) \psi(t) + \frac{1}{2} A^2(t),
\tag{27}
\]
which leads to the determination of \( \bar{b}_1 = i\dot{x} + A \).

Plugging in the value \( \bar{f}_1 = i\bar{\psi}, \, \bar{b}_2 = 0, \, \bar{f}_2 = \dot{\bar{\psi}} \) and \( \bar{b}_1 = i\dot{x} + A \), we obtain the following expansions
\[
X^{(2)}(t, \theta, \bar{\theta}) \mid_{\bar{\theta} = 0} = X^{(2)}(t, \theta),
X^{(2)}(t, \theta) = x(t) + \theta (i \bar{\psi}) \equiv x(t) + \theta (s_2 x),
\Psi^{(2)}(t, \theta, \bar{\theta}) \mid_{\bar{\theta} = 0} = \Psi^{(2)}(t, \theta),
\Psi^{(2)}(t, \theta) = \bar{\psi}(t) + \theta (-\dot{x} + iA) \equiv \bar{\psi}(t) + \theta (s_2 \bar{\psi}),
\bar{\Psi}^{(2)}(t, \theta, \bar{\theta}) \mid_{\bar{\theta} = 0} = \bar{\Psi}^{(2)}(t, \theta),
\bar{\Psi}^{(2)}(t, \theta) = \bar{\psi}(t) + \theta (0) \equiv \bar{\psi}(t) + \theta (s_2 \bar{\psi}),
\bar{A}^{(2)}(t, \theta, \bar{\theta}) \mid_{\bar{\theta} = 0} = \bar{A}^{(2)}(t, \theta),
\bar{A}^{(2)}(t, \theta) = A(t) + \theta (\dot{\bar{\psi}}) \equiv A(t) + \theta (s_2 A). \tag{28}
\]
Furthermore, we have found that the following relationship is true, namely;

\[ \frac{\partial}{\partial \theta} \Omega^{(2)}(t, \theta, \bar{\theta}) \bigg|_{\bar{\theta}=0} \equiv \frac{\partial}{\partial \theta} \Omega^{(2)}(t, \theta) = s_2 \Omega(t). \]  

(29)

The above relation demonstrates that the translation of the generic chiral supervariable \( \Omega^{(2)}(t, \theta) \equiv X^{(2)}(t, \theta), \Psi^{(2)}(t, \theta), \bar{\Psi}^{(2)}(t, \theta), A^{(2)}(t, \theta) \) along the Grassmannian direction \( \theta \) of the chiral (1, 1)-dimensional super-submanifold generates the SUSY transformations \( s_2 \) on the 1D ordinary generic variable \( \Omega(t) \) [cf. (1)]. However, as we know from (6), \( \bar{Q} \) is also the generator for \( s_2 \) because \( s_2 \Omega = -i[\Omega, \bar{Q}]_\pm \). Thus, we conclude that the following mapping

\[ \frac{\partial}{\partial \theta} \longleftrightarrow s_2 \longleftrightarrow \bar{Q}, \]  

(30)

exists amongst the translation generator (\( \partial_\theta \)), symmetry transformation (\( s_2 \)) and conserved charge \( \bar{Q} \). The nilpotency of \( s_2 \) (i.e. \( s_2^2 = 0 \)) is also encoded in the nilpotency of SUSY charge \( \bar{Q} \) which, in turn, is deeply related to the nilpotency (\( \partial_\theta^2 = 0 \)) of the Grassmannian derivative (\( \partial_\theta \)). Thus, the nilpotency of (\( s_2, \bar{Q}, \partial_\theta \)) are inter-related. Within the framework of supervariable approach, the nilpotency of \( s_2 \) and \( \bar{Q} \) is encoded in the two successive translations along \( \theta \)-direction [cf. (29), (30)] because \( \partial_\theta^2 = 0 \).

5 Symmetry Invariance and Off-Shell Nilpotency: Supervariable Approach

In this section, we capture the symmetry invariance of the Lagrangian under SUSY transformations \( s_1 \) and \( s_2 \) and the off-shell nilpotency of the charges \( Q \) and \( \bar{Q} \) in the language of supervariables obtained after the application of SUSYIRs. Using the expansion (19), it can be seen that the Lagrangian (1) can be generalized [onto (1, 1)-dimensional chiral super-submanifold] in terms of the anti-chiral supervariables as:

\[ L_0 \Rightarrow \tilde{L}_0^{(ac)} = \frac{1}{2} X^{(1)}(t, \bar{\theta}) X^{(1)}(t, \bar{\theta}) + i \bar{\Psi}^{(1)}(t, \bar{\theta}) \dot{\Psi}^{(1)}(t, \bar{\theta}) \]

\[ + \frac{1}{2} \bar{A}^{(1)}(t, \bar{\theta}) \bar{A}^{(1)}(t, \bar{\theta}) + \bar{W}''(X^{(1)}) \bar{A}^{(1)}(t, \bar{\theta}) \]

\[ + \bar{W}'''(X^{(1)}) \bar{\Psi}^{(1)}(t, \bar{\theta}) \Psi^{(1)}(t, \bar{\theta}), \]  

(31)

where the superscript \((ac)\) denotes the expression for the Lagrangian in terms of the anti-chiral supervariables. It can be checked explicitly that:

\[ \bar{W}''(X^{(1)}) = W''(x) + \bar{\theta} \left[ i W'''(x) \psi(t) \right], \]

\[ \bar{W}'''(X^{(1)}) = W'''(x) + \bar{\theta} \left[ i W'''(x) \psi(t) \right], \]  

(32)

where we have used the Taylor expansion with \( X^{(1)}(t, \bar{\theta}) = x(t) + i \bar{\theta} \psi(t) \equiv x(t) + s_1 x(t) \) around \( x(t) \). In view of the mapping \( s_1 \leftrightarrow \partial_\theta \) [cf. (20)], we note that the invariance of the
Lagrangian (1) under \( s_1 \) can be expressed in the following fashion as the Grassmannian derivative on \( \tilde{L}^{(ac)}_0 \):

\[
\frac{\partial}{\partial \theta} \tilde{L}^{(ac)}_0 = \frac{d}{dt} \left[ -W' \psi \right] \quad \Leftrightarrow \quad s_1 L_0 = \frac{d}{dt} \left[ -W' \psi \right].
\]

Geometrically, the invariance \( s_1 L_0 = d/dt (-W' \psi) \) can be explained in the following manner in the language of the supervariables obtained after the application of SUSY restrictions [cf. (20)]. The translation of the super Lagrangian (31) along the direction of \( \bar{\theta} \) is such that the result is a total derivative. In other words, the super Lagrangian (31) is a combination of composite supervariables, obtained after the application of SUSY restrictions, such that its translation along \( \bar{\theta} \)-direction of the (1, 1)-dimensional super-submanifold produces a result which is nothing but the total time derivative.

In exactly similar fashion, the starting Lagrangian (1) can also be expressed in terms of the chiral supervariables, obtained after SUSY restrictions [cf. (28)], as

\[
L_0 \quad \Rightarrow \quad \tilde{L}^{(c)}_0 = \frac{1}{2} \dot{X}^{(2)}(t, \theta) X^{(2)}(t, \theta) + i \tilde{\Psi}^{(2)}(t, \theta) \tilde{\Psi}^{(2)}(t, \theta) + \frac{1}{2} \dot{A}^{(2)}(t, \theta) \dot{A}^{(2)}(t, \theta) + \tilde{W}'(X^{(2)}) \tilde{A}^{(2)}(t, \theta) + \tilde{W}''(X^{(2)}) \tilde{\Psi}^{(2)}(t, \theta),
\]

where \( \tilde{W}'(X^{(2)}) \) and \( \tilde{W}''(X^{(2)}) \) have the same expansions as quoted in (32) with the replacements: \( \bar{\theta} \rightarrow \theta \) and \( \psi \rightarrow \tilde{\psi} \). The invariance of the original Lagrangian (1) under \( s_2 \) can be captured in the following fashion:

\[
\frac{\partial}{\partial \theta} \tilde{L}^{(c)}_0 = \frac{d}{dt} \left[ i \tilde{\psi} (\dot{x} - iA - iW') \right] \quad \Leftrightarrow \quad s_2 L_0 = \frac{d}{dt} \left[ i \tilde{\psi} (\dot{x} - iA - iW') \right].
\]

Geometrically, the SUSY invariance of Lagrangian (1) is equivalent to the translation of the composite supervariables (present in \( \tilde{L}^{(c)}_0 \) [cf. (34)]) such that the outcome of the translation is a total derivative. Finally, we observe that the action integral can be expressed as:

\[
S = \int dt \; L_0 \leftrightarrow S = \int dt \; \tilde{L}^{(ac)}_0 \leftrightarrow S = \int dt \; \tilde{L}^{(c)}_0,
\]

which is self-evident from (31) and (34) because we observe that \( \tilde{L}^{(ac)}_0 = L_0 + \bar{\theta} \frac{d}{dt} [-W' \psi] \) and \( \tilde{L}^{(c)}_0 = L_0 + \theta \frac{d}{dt} [i \tilde{\psi} (\dot{x} - iA - iW')] \). Thus, the inter-relationships, given in (36), are correct because the total derivative terms vanish due to Gauss’s divergence theorem.

We can express the supercharge \( Q \) in terms of the supervariables, obtained after the application of SUSY restrictions, in two different ways as:

\[
Q = \frac{\partial}{\partial \theta} \left[ -i \tilde{\Psi}^{(1)}(t, \bar{\theta}) \tilde{\Psi}^{(1)}(t, \bar{\theta}) \right] = \int d\bar{\theta} \left[ -i \tilde{\Psi}^{(1)}(t, \bar{\theta}) \tilde{\Psi}^{(1)}(t, \bar{\theta}) \right],
\]

\[
Q = \frac{\partial}{\partial \theta} \left[ (\dot{x}(t) + iA(t)) X^{(1)}(t, \bar{\theta}) \right] = \int d\bar{\theta} \left[ (\dot{x}(t) + iA(t)) X^{(1)}(t, \bar{\theta}) \right].
\]

\[
(37)
\]
In view of the mappings (20) and (29), the above charges can be also expressed as follows:

\[ Q = s_1 \left[ -i \bar{\psi} \psi \right], \quad Q = s_1 \left[ (\dot{x} + iA)x \right], \tag{38} \]

which prove the nilpotency of the charge \( Q \) in the language of the nilpotency of transformations (2) as well as in terms of the nilpotency \( (\partial^2_\theta = 0) \) of the translational generator \( (\partial_\theta) \). This can be seen by \( s_1 Q = -i \{ Q, Q \} = 0 \) and \( \partial_\theta Q = 0 \) (by exploiting the relationships (38) and (37), respectively).

In exactly similar fashion, we can express the supercharge \( \bar{Q} \) in terms of the supervariables (28), obtained after the application of SUSY invariant restrictions, in two different ways as illustrated below:

\[
\bar{Q} = \frac{\partial}{\partial \theta} \left[ i\bar{\Psi}(2)(t, \theta) \Psi(2)(t, \theta) \right] \\
\equiv \int d\theta \left[ i\bar{\Psi}(2)(t, \theta) \Psi(2)(t, \theta) \right],
\]

\[
\bar{Q} = \frac{\partial}{\partial \theta} \left[ (\dot{x}(t) - iA(t))X(2)(t, \theta) \right] \\
\equiv \int d\theta \left[ (\dot{x}(t) - iA(t))X(2)(t, \theta) \right]. \tag{39}
\]

The above relationships can be re-expressed in terms of the ordinary 1D variables and transformations \( s_2 \) of (2) as follows

\[ \bar{Q} = s_2 \left[ i\bar{\psi} \psi \right], \quad \bar{Q} = s_2 \left[ (\dot{x} - iA)x \right], \tag{40} \]

which establish the nilpotency of \( \bar{Q} \) in the ordinary space due to \( s_2 \bar{Q} = -i \{ \bar{Q}, \bar{Q} \} = 0 \).

In the superspace, we observe that \( \partial_\theta \bar{Q} = 0 \) due to the nilpotency \( (\partial^2_\theta = 0) \) of translational generator \( \partial_\theta \) along the Grassmannian direction \( \theta \) of the \((1, 1)\)-dimensional chiral supermanifold. Hence, we have proven the nilpotency property in a clear fashion.

### 6 Off-Shell Nilpotent \( \mathcal{N} = 2 \) SUSY Transformations: Towards Cohomological Interpretation

For the sake of completeness, we shall discuss here the mathematical implications of the off-shell nilpotent \((s_1^2 = 0, \ s_2^2 = 0)\) \( \mathcal{N} = 2 \) SUSY transformations \( s_1 \) and \( s_2 \) in the language of de Rham cohomological operators of differential geometry which have been discussed thoroughly in [14]. We note that the Lagrangian (1) remains invariant under the following unique discrete symmetry transformations [14]

\[
\begin{align*}
x & \to -x, & t & \to -t, & \psi & \to +\bar{\psi}, & \bar{\psi} & \to -\psi, \\
A & \to -A, & W' & \to -W', & W'' & \to +W''.
\end{align*} \tag{41}
\]

where there is an explicit presence of the time-reversal as well as reflection (i.e. parity) symmetries. Furthermore, the above transformations are physically interesting because
the superpotential $W(x)$ is even under parity [i.e. $W(-x) = W(x)$] which is required for the existence of square-integrable eigenfunctions. The above discrete transformations are unique because we observe the validity of the following [14]

$$s_2 \Phi_1 = -\ast s_1 \ast \Phi_1, \quad s_1 \Phi_1 = \ast s_2 \ast \Phi_1, \quad s_2 \Phi_2 = +\ast s_1 \ast \Phi_2,$$

$$s_1 \Phi_2 = -\ast s_2 \ast \Phi_2, \quad \Phi_1 = x, A, W', W'', \quad \Phi_2 = \psi, \bar{\psi}. \quad (42)$$

The above relationships are the analogue of the relationship $\delta = \pm \ast d \ast$ of differential geometry where $d = dt \partial_t (d^2 = 0)$ is the exterior derivative and $\delta$ (with $\delta^2 = 0$) is the co-exterior derivative. The ($\pm$) signs in (42) are dictated by two successive operations of (41) on the specific variable, namely;

$$\ast (\ast \Phi_1) = + \Phi_1, \quad \ast (\ast \Phi_2) = - \Phi_2, \quad (43)$$

where ($\ast$) corresponds to the discrete symmetry transformations (41) and the generic variables $\Phi_1$ and $\Phi_2$ have been explained in (42).

Now we concentrate on the physical meaning of $d$ and $\delta$ in the language of the symmetry transformations. It is straightforward to check that the continuous symmetry transformations ($s_1, s_2, s_\omega$) of Sec. 2 satisfy the following algebra in their operator form, namely;

$$s_1^2 = 0, \quad s_2^2 = 0, \quad \{s_1, s_2\} = s_\omega = (s_1 + s_2)^2,$$

$$[s_\omega, s_1] = 0, \quad [s_\omega, s_2] = 0, \quad \{s_1, s_2\} \neq 0, \quad (44)$$

where the operator $s_\omega$ is defined modulo a factor of $(-2i)$. Furthermore, we note that $s_\omega$ is like the Casimir operator because it commutes with $s_1$ and $s_2$. A close look at (44) exemplifies that this algebra is reminiscent of the algebra satisfied by the de Rham cohomological operators ($d, \delta, \Delta$) of differential geometry [17-21]. The latter algebra is [17-21]:

$$d^2 = 0, \quad \delta^2 = 0, \quad \{d, \delta\} = \Delta = (d + \delta)^2,$$

$$[\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \{d, \delta\} = 0. \quad (45)$$

Thus, we conclude that there exists one-to-one correspondence between the algebraic structures of (44) and (45). As a result, we have provided the physical meaning to the abstract mathematical properties associated with the cohomological operators ($d, \delta, \Delta$) of differential geometry in the language of continuous symmetry transformations.

To summarize, we add that the algebra (44) can be shown to be emulated by the conserved charges $(Q, \bar{Q}, Q_\omega)$ (cf. Sec. 2) if we modify a bit the transformations (2) by an overall constant factor [14]. The other properties of ($d, \delta, \Delta$) can be captured by the above charges where the eigenvalues and eigenfunctions (defined in the quantum Hilbert space) play important roles. Thus, ultimately, we observe that the general $\mathcal{N} = 2$ SUSY quantum mechanical model (with any arbitrary superpotential) provides the physical realizations of the cohomological operators. We wish to add that we have not focused here on the formal mathematical quantities (see, e.g. [22-25]) like the spin complex structure, $\mathbb{Z}_2$-grading, Witten’s parity operator, etc., in the discussion of our $\mathcal{N} = 2$ theory and its symmetries. We shall discuss about these formal aspects of the cohomological features in our future publication.
7 Conclusions

The main result of our present investigation is the derivation of the full set of off-shell nilpotent SUSY symmetries \( s_1 \) and \( s_2 \) [cf. (2)] for the general \( \mathcal{N} = 2 \) SUSY QM model (with any arbitrary superpotential \( W(x) \)) using the supervariable approach. We have defined the supervariables [corresponding to the 1D ordinary variables of Lagrangian (1)] on the \((1, 1)\)-dimensional (anti-)chiral super-submanifolds of the general \((1, 2)\)-dimensional supermanifold. It is the strength of the SUSYIRs on the (anti-)chiral supervariables that we have been able to derive the above SUSY transformations \( s_1 \) and \( s_2 \) accurately. Primarily, we have demanded that the SUSY invariant 1D quantities must remain independent of the “soul” coordinates \( \theta \) and \( \bar{\theta} \) when the former are generalized onto the appropriate super-submanifolds. This requirement is physically cogent and logically appealing. It is pertinent to point out that, in the old literature (see, e.g. [19]), the space coordinate \( x(t) \) has been christened as the “body” coordinate and the Grassmannian variables \( (\theta \text{ and } \bar{\theta}) \) have been named as “soul” coordinates. The former could be realized physically but the latter are totally mathematical and abstract in nature. Thus, a SUSY invariant quantity must remain independent of the latter coordinates as they are only mathematical artifacts.

Geometrically, we have shown that the translation of the supervariables, obtained after the application of SUSY invariant restrictions, along the Grassmannian directions \( \bar{\theta} \) and \( \theta \) produces the SUSY transformations \( s_1 \) and \( s_2 \) (cf. Sec. 3 and 4). The nilpotency of \( s_1 \) and \( s_2 \) is deeply connected with two successive translations along the Grassmannian directions \( \bar{\theta} \) and \( \theta \) which are generated by the nilpotent \( (\partial_{\bar{\theta}}^2 = \partial_{\theta}^2 = 0) \) translational generators \( \partial_{\bar{\theta}} \) and \( \partial_{\theta} \) on the (anti-)chiral \((1, 1)\)-dimensional super-submanifolds. The symmetry invariance of the Lagrangian, under \( s_1 \) and \( s_2 \), is connected with the translation of some combination of composite supervariables (obtained after SUSY invariant restrictions) along \( \bar{\theta} \) and \( \theta \)-directions such that the outcome of these translations is a total time derivative in the ordinary one \((0 + 1)\)-dimensional (1D) space.

One of the decisive features of our supervariable approach is the intelligent choice of \((1, 1)\)-dimensional (anti-)chiral super-submanifolds of the general \((1, 2)\)-dimensional supermanifold on which the (anti-)chiral supervariables are defined. The latter are subjected to the SUSY invariant restrictions which lead to the derivation of off-shell nilpotent \( \mathcal{N} = 2 \) SUSY transformations \( s_1 \) and \( s_2 \). The off-shell nilpotency property is encoded in the nilpotency \( (\partial_{\bar{\theta}}^2 = \partial_{\theta}^2 = 0) \) of the translational generators \( \partial_{\bar{\theta}} \) or \( \partial_{\theta} \) along the \( \theta \) or \( \bar{\theta} \)-direction of the (anti-)chiral \((1, 1)\)-dimensional super-submanifolds (which accommodate the existence of the above (anti-)chiral supervariables). This choice has been responsible in demonstrating that \( s_1 \) and \( s_2 \) do not respect the anticommutativity property (i.e. \( \{ s_1, s_2 \} \neq 0 \)).

We hope to generalize our analysis for the description of the extended SUSY quantum mechanical models (with \( \mathcal{N} = 4, 6, 8, \ldots \)). Furthermore, we can implement the procedure of dimensional reduction to obtain the one \((0 + 1)\)-dimensional \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) Yang-Mills models thereby establishing a connection with the SUSY gauge theories. An excellent set of works [22-25] exist in this regard which we wish to apply in our future endeavors. We are also trying to apply SUSY version of HC to obtain the proper (anti-)BRST symmetries for the SUSY gauge theories [26]. In this context, we are sure that the idea of SUSY invariant restrictions would play very important role as they would provide additional useful restrictions. Thus, the methodology and ideas used in our present text would be
useful in deriving SUSY as well as (anti-)BRST symmetries for the SUSY gauge theories. Currently, we are devoting time on it and our results would be reported elsewhere [27].

Before we close this section, we would like to briefly mention here the cohomological implications of the $\mathcal{N} = 2$ SUSY quantum mechanical algebra. The presence of this algebra provides a $Z_2$-grading of the quantum Hilbert space of states and it also generates transformations between even-odd parity states (w.r.t. the Witten parity operator). This is why, the spacetime manifold turns out to be a globally graded manifold (but not a supermanifold). Thus, even though the cohomological operators $(d, \delta, \Delta)$ are identified with the $\mathcal{N} = 2$ SUSY quantum mechanical charges ($Q, \bar{Q}, H \equiv Q_\omega$), the Hodge decomposition theorem can not be defined in the quantum Hilbert space of $Z_2$-graded quantum states. However, for the even or odd parity states of the total quantum Hilbert space, the (co-)cohomology w.r.t. the $\mathcal{N} = 2$ supercharges can be defined in the corresponding subspace.

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A On the Choice of (Anti-)chiral Supervariables

Here we explain the reasons behind our intelligent choice of the (anti-)chiral supervariables in the context of derivation of the SUSY nilpotent transformations for the general $\mathcal{N} = 2$ SUSY quantum mechanical model of our present investigation. As pointed out (and emphasized in the main body of our text), the (anti-)BRST symmetry transformations for a gauge theory are nilpotent and absolutely anticommuting. Thus, corresponding to a given bosonic field $\phi(x)$ of this D-dimensional gauge theory, one has to generalize it onto a (D, 2)-dimensional supermanifold with the following general expansion [4-7].

$$\hat{\Phi}(x, \theta, \bar{\theta}) = \phi(x) + \theta \hat{R}(x) + \bar{\theta} \hat{R}(x) + \theta \bar{\theta} S(x), \quad (A.1)$$

where $\hat{\Phi}(x, \theta, \bar{\theta})$ is the superfield defined on the (D, 2)-dimensional supermanifold and $(\hat{R}(x), \bar{\hat{R}}(x))$ are the fermionic secondary fields and $S(x)$ is a bosonic secondary field. As it turns out, the translational generators $(\partial_{\theta}, \partial_{\bar{\theta}})$ are found to correspond to the (anti-)BRST symmetry transformations $s_{(a)b}$ which are nilpotent of order two due to $(\partial_{\theta}^2 = \partial_{\bar{\theta}}^2 = 0)$ and they are absolutely anticommuting because it is straightforward to check that:

$$\partial_{\bar{\theta}} \partial_{\theta} \left( \hat{\Phi}(x, \theta, \bar{\theta}) \right) = i S(x) \quad \Leftrightarrow \quad s_b s_{ab} \phi(x), \quad (A.2)$$

$$\partial_{\theta} \partial_{\bar{\theta}} \left( \hat{\Phi}(x, \theta, \bar{\theta}) \right) = -i S(x) \quad \Leftrightarrow \quad s_{ab} s_b \phi(x). \quad (A.3)$$

Thus, we observe that $(s_b s_{ab} + s_{ab} s_b) \phi(x) = 0$ due to the above relations (A.2) and (A.3) which are also implied by $(\partial_{\theta} \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_{\theta}) = 0$. In our present investigation (connected with the SUSY QM theory), we are compelled to avoid relations of the type (A.2) and (A.3) so that our nilpotent SUSY symmetries could not become absolutely anticommuting in nature.
We wrap up this Appendix A with the remarks that our SUSY nilpotent symmetries are geometrically identified with the translational generators \((\partial_\theta, \partial_{\bar{\theta}})\) along the Grassmannian directions of the (anti-)chiral super-submanifolds which encapsulate only the nilpotency of the symmetry transformations. The above choice also makes it clear that one can derive both the \(\mathcal{N} = 2\) SUSY symmetries independently. As far as the computation of the anticommutativity property is concerned, one has to compute it and check its nature separately after derivation of the \(\mathcal{N} = 2\) SUSY symmetries by our proposed method of supervariable approach.

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