General proof of the entropy principle for self-gravitating fluid in static spacetimes

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Abstract
We show that for any perfect fluid in a static spacetime, if the Einstein constraint equation is satisfied and the temperature of the fluid obeys Tolman’s law, then the other components of Einstein’s equation are implied by the assumption that the total entropy of the fluid achieves an extremum for fixed total particle number and for all variations of metric with certain boundary conditions. Conversely, one can show that the extrema of the total entropy of the fluid are implied by Einstein’s equation. Compared to previous works on this issue, we do not require spherical symmetry for the spacetime. Our results suggest a general and solid connection between thermodynamics and general relativity.

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1 Introduction
The mathematical analogy between laws of black physics and the ordinary laws of thermodynamics leads to the discovery of black hole thermodynamics [1]-[3]. In the past four decades, black hole thermodynamics has become an important and fascinating subject in general relativity and other theories of gravity[4]-[7]. A related but different issue is to the study of the thermodynamics of ordinary matter in curved spacetime, without the presence of black hole. In contrast to the mystic origin of black hole entropy, local thermodynamic quantities of matter in curved spacetime, like entropy density $s$, energy density $\rho$, local temperature $T$, are well defined. Gravity only affects the distribution of these quantities. There are two apparently

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independent ways to determine the distribution of matter. From the thermodynamic point of view, the fluid should be configured such that its total entropy attains a maximum value. From the gravitational point of view, the distribution of matter must obey Einstein’s field equation. Since entropy plays no role in Einstein’s equation, there is no guarantee that the two ways should give rise to the same result. However, an early work by Sorkin, Wald and Zhang [8] showed that if the total entropy of a spherical radiation is an extremum and the Einstein constraint equation holds, then the Tolman-Oppenheimer-Volkoff (TOV) equation of hydrostatic equilibrium can be derived, which was originally derived from Einstein’s equation. Recently, Gao [9] extended SWZ’s proof from radiation to a general perfect fluid. This issue has been further explored in the past year [10]-[14].

The above works reveal a certain relationship between thermodynamics and gravity. But all these results only apply to spherically symmetric spacetimes. It is unclear whether the entropy principle is consistent with general relativity beyond spherical symmetry. In this paper, we propose and prove two theorems, showing that, under a few natural conditions, the extrema of total entropy is equivalent to Einstein’s equation in any static spacetime. A static spacetime admits a timelike Killing vector field which is hypersurface orthogonal. Our proof only involves general properties of spacetime geometry and thermodynamics for ordinary fluid.

It is worth noting that recently a very comprehensive discussion on the equivalence of thermodynamic equilibrium and Einstein’s equation was provided by Green, Schiffrin and Wald [15]. Although some of the results in [15] appear to be similar to ours, there are considerable differences in both assumptions and arguments. For instance, a crucial assumption in [15] is that the spacetime be asymptotically flat, while our theorems apply to any spacetime region, imposing no global conditions on the spacetime. On the other hand, the static condition in our argument is replaced by the more general stationary condition in [15].

2 Properties of perfect fluid in static spacetimes

We consider a general perfect fluid as discussed in [9]. The entropy density $s$ is taken to be a function of the energy density $\rho$ and particle number density $n$, i.e., $s = s(\rho, n)$. From the first law of thermodynamics, one can derive the integrated form of the Gibbs-Duhem relation,

$$s = \frac{1}{T(\rho + p - \mu n)},$$

where $\rho$ and $\mu$ represent the pressure and chemical potential, respectively. All the quantities are measured by static observers with four-velocity $u^a$. These observers are orthogonal to the hypersurface $\Sigma$. Therefore, the in-
duced metric on $\Sigma$ is given by

$$h_{ab} = g_{ab} + u_a u_b.$$  \hfill (2)

The stress-energy tensor $T_{ab}$ for perfect fluid takes the form

$$T_{ab} = \rho u_a u_b + p h_{ab}.$$  \hfill (3)

We shall assume that Tolman’s law holds, which states that the local temperature $T$ of the fluid satisfies

$$T \chi = T_0,$$  \hfill (4)

where $\chi$ is the redshift factor for static observers and $T_0$ is a constant. This law establishes the relationship between the fluid temperature to the metric, which was also used in [15]. Now we show that another similar relation for chemical potential $\mu$ is implied by the Tolman law.

It is straightforward to show, from the conservation law $\nabla_a T^{ab} = 0$ and stationary conditions, that

$$\nabla_a p = - (\rho + p) A_a,$$  \hfill (5)

where $A^a$ is the four-acceleration of the observer. Since

$$u^a = \frac{\xi^a}{\chi},$$  \hfill (6)

where $\xi^a$ is the Killing vector and $\chi$ is the redshift factor, one can show that

$$A_a = \nabla_a \chi / \chi.$$  \hfill (7)

and thus

$$\nabla_a p = - (\rho + p) \nabla_a \chi / \chi.$$  \hfill (8)

On the other hand, the local first law can also be expressed in the form

$$dp = sdT + nd\mu$$  \hfill (9)

Comparing Eqs. (8) and (9), and using Eqs. (4) and (1), we find

$$\frac{\nabla_a \mu}{\mu} = - \frac{\nabla_a \chi}{\chi}$$  \hfill (10)

which leads to

$$\mu \chi = \text{const.}$$  \hfill (11)

Thus, from the Tolman’s law and local thermodynamic laws for perfect fluid, we derive the constancy of the redshifited chemical potential. This also yields

$$\frac{\mu}{T} = \text{const.}$$  \hfill (12)

This relation will be used later.
3 Two theorems

In this section, we present two theorems on the relationship between the extrema of total entropy of fluid and static solutions to Einstein’s equation.

**Theorem 1** Consider a perfect fluid in a static spacetime $(M,g_{ab})$ and $\Sigma$ is a three dimensional hypersurface denoting a moment of the static observers. Let $C$ be a region on $\Sigma$ with boundary $\bar{C}$. Assume that the temperature of the fluid obeys Tolman’s law and the Einstein constraint equation is satisfied in $C$. Then the other components of Einstein’s equation are implied by the extrema of the total fluid entropy for fixed particle number and for all variations where $h_{ab}$ and its first derivatives are fixed on $C$.

**Proof.** The total entropy $S$ is an integral of the entropy density $s$ over the region $C$ on $\Sigma$

$$S = \int_C \sqrt{h} s(\rho, n), \quad (13)$$

where $h$ is the determinant of $h_{ab}$ in any coordinates of $\Sigma$. Without loss of generality, we can fix the coordinates on $\Sigma$ for all variations. Thus, the variation of total entropy is written in the form

$$\delta S = \int_C s \delta \sqrt{h} + \sqrt{h} \delta s. \quad (14)$$

Applying the local first law of thermodynamics,

$$T ds = d\rho - \mu dn, \quad (15)$$

we find

$$\delta S = \int_C s \delta \sqrt{h} + \sqrt{h} \left( \frac{\partial s}{\partial \rho} \delta \rho + \frac{\partial s}{\partial n} \delta n \right)$$

$$= \int_C s \delta \sqrt{h} + \sqrt{h} \left( \frac{1}{T} \delta \rho - \frac{\mu}{T} \delta n \right). \quad (16)$$

The total number of particle $N$ is the integral

$$N = \int_C \sqrt{h} n, \quad (17)$$

which yields the variation

$$\delta N = \int_C \sqrt{h} \delta n + n \delta \sqrt{h}. \quad (18)$$

Therefore, the constraint $\delta N = 0$ is equivalent to

$$\int_C \sqrt{h} \delta n = - \int_C n \delta \sqrt{h}. \quad (19)$$
With this constraint as well as Eq. (12), Eq. (16) can be written as

\[
\delta S = \int_C \left( s + \frac{n\mu}{T} \right) \delta \sqrt{h} + \sqrt{h} \frac{1}{T} \delta \rho \\
= \int_C \frac{\rho + p}{T} \delta \sqrt{h} + \sqrt{h} \frac{1}{T} \delta \rho.
\]  

(20)

Using (16)

\[
\delta \sqrt{h} = \frac{1}{2} \sqrt{h} h^{ab} \delta h_{ab},
\]

(21)

we obtain

\[
\delta S = \int_C \delta L,
\]

(22)

where

\[
\delta L = \frac{1}{2} \frac{\rho + p}{T} \sqrt{h} h^{ab} \delta h_{ab} + \sqrt{h} \frac{1}{T} \delta \rho.
\]

(23)

Our purpose is to derive \(\delta L = 0\) from Einstein’s equation. First note that the extrinsic curvature of \(\Sigma\) is defined by

\[
\hat{B}_{ab} \equiv h^c_a h^d_b \nabla_d u_c,
\]

(24)

where \(\nabla_a\) is the derivative operator associated with \(g_{ab}\), satisfying \(\nabla_a g_{bc} = 0\). It is then straightforward to show

\[
\hat{B}_{ab} = \nabla_b u_a + A_a u_b,
\]

(25)

where \(A^a\) is the four-acceleration of the observer. Since

\[
u^a = \frac{\xi^a}{\chi},
\]

(26)

thus

\[
\nabla_b u_a = \frac{1}{\chi} \nabla_b \xi_a - u_a A_b,
\]

(27)

which leads to

\[
\hat{B}_{(ab)} = 0,
\]

(28)

where \(B_{(ab)}\) is the symmetrization of \(B_{ab}\). The antisymmetrization of \(B_{ab}\) also vanishes due to the fact that \(u^a\) is hypersurface orthogonal[16]. Consequently, \(B_{ab} = 0\) and

\[
\nabla_b u_a = -A_a u_b.
\]

(29)
This formula will be very helpful in the later calculation.

One can show that the curvature $R_{abc}^d$ of $\Sigma$ is related to the spacetime curvature $R_{\alpha\beta\gamma\delta}$ by

$$R_{abc}^d = h^f_i h^g_j h^k_l h^d_j R_{fgkl}. \quad (30)$$

Note that there would be $\hat{B}_{ab}$ terms on the right-hand side if the spacetime were not static\[16\].

It is not difficult to find that

$$R_{ab}^{(3)} = R_{ab} + R_{ace} u^c u^l + R_{fbu^f u^a} + u^k u^b R_{ak} + u^a u^b R_{fk} u^f u^k, \quad (31)$$

and

$$R^{(3)} = R + 2R_{ab} u^a u^b. \quad (32)$$

To calculate $\delta \rho$, we start with constraint Einstein’s equation

$$G_{\alpha\beta} u^\alpha u^\beta = 8\pi T_{\alpha\beta} u^\alpha u^\beta, \quad (33)$$

where the stress-energy tensor $T_{\alpha\beta}$ for perfect fluid has been given in Eq. \[3\]. Thus

$$\rho = \frac{1}{8\pi} G_{\alpha\beta} u^\alpha u^\beta. \quad (34)$$

Together with Eqs. \[31\] and \[32\], we obtain \[17\]

$$\rho = \frac{1}{16\pi} R^{(3)}. \quad (35)$$

This tells us that the variation of $\rho$ is actually determined by the geometry of $\Sigma$. Denote the last term in Eq. \[23\] by $\delta L_1$, which then gives

$$\delta L_1 = \frac{1}{16\pi T} \sqrt{h} \delta R^{(3)} = \sqrt{h} \frac{1}{16\pi T} \left( h^{ab} \delta R_{ab}^{(3)} + R_{ab}^{(3)} \delta h^{ab} \right). \quad (36)$$

Denote the first term on the right-hand side by $\delta L_1'$, i.e.,

$$\delta L_1' = \sqrt{h} \frac{1}{16\pi T} h^{ab} \delta R_{ab}^{(3)}. \quad (37)$$

The standard calculation yields (see e.g., \[16\])

$$\delta L_1' = \sqrt{h} \frac{1}{16\pi T} D^a v_a, \quad (38)$$

where

$$v_a = D^b h_{ab} - h^{bc} D_a \delta h_{bc}, \quad (39)$$
and $D_a$ is the derivative operator on $\Sigma$ associated with $h_{ab}$. To get $\delta h_{ab}$ as a common factor, we perform integration by parts and find

$$
\delta L_1' = \sqrt{\hbar} \frac{1}{16\pi} D^a (v_a / T) - \sqrt{\hbar} \frac{1}{16\pi} v_a D^a (1 / T). \tag{40}
$$

According to the assumption of Theorem 1, the metric and its first derivatives are fixed on $\bar{C}$. So we may get rid of the boundary term and obtain

$$
\delta L_1' = -\frac{1}{16\pi} \sqrt{\hbar} v_a D^a (1 / T) = -\frac{1}{16\pi} \sqrt{\hbar} D^b (\delta h_{ab}) D^a (T^{-1}) + \frac{1}{16\pi} \sqrt{\hbar} h^{bc} D_a (\delta h_{bc}) D^a (T^{-1}) . \tag{41}
$$

Using integration by parts again and dropping the boundary terms, we have

$$
\delta L_1' = \frac{1}{16\pi} \sqrt{\hbar} D^b D^a (T^{-1}) \delta h_{ab} - \frac{1}{16\pi} \sqrt{\hbar} h^{ab} D_c D^c \delta h_{ab} . \tag{42}
$$

Now $\delta L_1'$ is linear in $\delta h_{ab}$, as desired.

Without loss of generality, we take $T_0 = 1$ in Eq. (4) and Eq. (42) becomes

$$
\delta L_1' = \frac{1}{16\pi} \sqrt{\hbar} D^b D^a (T^{-1}) \delta h_{ab} - \frac{1}{16\pi} \sqrt{\hbar} h^{ab} D_c D^c \delta h_{ab} . \tag{43}
$$

Note that

$$
D_a \chi = \chi A_a \tag{44}
$$

Thus,

$$
D_b D_a \chi = A_a D_b \chi + \chi D_b A_a = \chi A_a A_b + \chi D_b A_a . \tag{45}
$$

and

$$
h_{ab} D_c D^c \chi = h_{ab} (\chi A_c A_c + \chi D_c A_c) \tag{46}
$$

So

$$
\delta L_1' = \frac{1}{16\pi T} \sqrt{\hbar} M_1^{ab} \delta h_{ab} , \tag{47}
$$

where

$$
M_1^{ab} = A^a A^b + D^b A^a - h^{ab} (A^c A_c + D_c A^c) . \tag{48}
$$
We calculate
\[ D_c A^c = h^c_j h^c_c \nabla_c A^f \]
\[ = \nabla_c A^c + u^c u_f \nabla_c A^f \]
\[ = \nabla_c A^c + u^c u_f \nabla_c (u^b \nabla_b u^f) \]
\[ = \nabla_c A^c + u^c \nabla_c (u_f u^b \nabla_b u^f) - u^c (\nabla_c u_f) (u^b \nabla_b u^f) \]
\[ = \nabla_c A^c - A^c A_c, \tag{49} \]
and Eq. (48) can be rewritten as
\[ M_{ab}^{1} = A^a A^b + D^b A^a - h^{ab} \nabla_c A^c. \tag{50} \]
Substituting these results into Eq. (23) yields
\[
\delta L = \frac{1}{2} \rho + \rho T \sqrt{\frac{h}{16\pi}} \delta h_{ab} + \sqrt{\frac{h}{16\pi}} \left( -R^{(3)ab} \delta h_{ab} + M_{ab}^{1} \delta h_{ab} \right)
\]
\[ = \sqrt{\frac{h}{T}} \left( \frac{\rho + p}{2} h_{ab}^{ab} - \frac{1}{16\pi} R^{(3)ab} + \frac{1}{16\pi} M_{ab}^{1} \right) \delta h_{ab}. \tag{51} \]
This shows explicitly that \( \delta S \) is determined by the variation of \( h_{ab} \). Since \( \delta S = 0 \) by the assumption of Theorem 1, we have
\[ \frac{\rho + p}{2} h_{ab}^{ab} - \frac{1}{16\pi} R^{(3)ab} + \frac{1}{16\pi} (A^a A^b + D^b A^a - h^{ab} \nabla_c A^c) = 0 \tag{52} \]
So
\[ 8\pi p h_{ab}^{ab} = R^{(3)ab} - A^a A^b - D^b A^a + h^{ab} \nabla_c A^c - 8\pi p h_{ab}^{ab} \tag{53} \]
Substitution of Eqs. (31), (32) and (35) yields
\[ 8\pi p h_{ab}^{ab} = h^{ac} h^{bd} R_{cd} + h^{ac} h^{bd} R_{ced} u^c u_l - A^a A^b - D^b A^a \]
\[ + h^{ab} \nabla_c A^c - \frac{1}{2} Rh^{ab} - R_{ced} u^c u^d h_{ab} \]
\[ = h^{ac} h^{bd} R_{cd} - \frac{1}{2} Rh^{ab} - P_{1}^{ab} - P_{2}^{ab}, \tag{54} \]
where
\[ P_{1}^{ab} = h^{ab} R_{ced} u^c u^d - h^{ab} (\nabla_c A^c) \tag{55} \]
\[ P_{2}^{ab} = -h^{ac} h^{bd} R_{ced} u^c u^d + A^a A^b + D^b A^a. \tag{56} \]
Now we show that \( P_{1}^{ab} \) and \( P_{2}^{ab} \) vanish respectively. We first calculate
\[ \nabla_c A^c = \nabla_c (u^b \nabla_b u^c) \]
\[ = (\nabla_c u^b) \nabla_b u^c + u^b \nabla_c \nabla_b u^c. \tag{57} \]
Note that
\[ \nabla_c \nabla_b u^d - \nabla_b \nabla_c u^d = -R_{cde} u^e. \tag{58} \]
Hence,
\[ \nabla_c A^e = (\nabla_c u^b)\nabla_b u^e + u^b \nabla_b \nabla_c u^e + R_{be} u^b u^e. \quad (59) \]

Then Eq. (55) can be written in the form
\[ P_{ab}^{1} = h^{ab} \left[ - (\nabla_c u^d) \nabla_d u^e - u^d \nabla_d \nabla_c u^e \right] . \quad (60) \]

Since \( u^a = \xi^a / \chi \) where \( \xi^a \) is the Killing vector field, we have
\[ \nabla_c u^e = 0. \quad (61) \]

So
\[ P_{ab}^{1} = h^{ab} \left[ - (\nabla_c u^b) \nabla_b u^e \right] . \quad (62) \]

By Eq. (29), we find immediately
\[ P_{ab}^{1} = 0. \quad (63) \]

To deal with the first term on the right-hand side of Eq. (56), we start from
\[ R_{cedl} u^e u^l = u^e \nabla_c \nabla_e u_d - u^e \nabla_e \nabla_c u_d \]
\[ = \nabla_c (u^e \nabla_e u_d) - (\nabla_c u^e) (\nabla_e u_d) - u^e \nabla_c \nabla_e u_d \]
\[ = \nabla_c A_d - u_c A^e u_e A_d + u^e \nabla_e (u_c A_d) \]
\[ = \nabla_c A_d + u_c u^e \nabla_e A_d + A_d A_e, \quad (64) \]

where we have used Eq. (29) repeatedly. Hence, \( P_{ab}^{2} \) in Eq. (56) can be written in the form
\[ P_{ab}^{2} = - h^{ac} h^{bd} \nabla_c A_d - A^a A^b + A^a A^b + D^b A^a \]
\[ = - D^a A^b + D^b A^a. \quad (65) \]

Eq. (44) implies that \( D^a A^b \) is symmetric in \( a, b \) and thus
\[ P_{ab}^{2} = 0. \quad (66) \]

Therefore, Eq. (54) just gives the projection of Einstein’s equation into \( \Sigma \)
\[ h^{ac} h^{bd} R_{cd} - \frac{1}{2} R h^{ab} = 8\pi p h^{ab} \quad (67) \]

This completes the proof of Theorem 1.

In the above proof, we used the Einstein constraint equation (35) to derive Eq. (51). Then by applying \( \delta S = 0 \), we obtained the spatial components of Einstein’s equation. It is not difficult to check that the proof is reversible, i.e., From the projected Einstein’s equation (67), one can show \( \delta L = 0 \) in Eq. (51), which makes the total entropy be an extremum. Thus, we arrive at the following theorem:
Theorem 2. Consider a perfect fluid in a static spacetime \((M, g_{ab})\) and \(\Sigma\) is a three dimensional hypersurface denoting a moment of the static observers. Let \(C\) be a region on \(\Sigma\) with a boundary \(\bar{C}\) and \(h_{ab}\) be the induced metric on \(\Sigma\). Assume that the temperature of the fluid obeys Tolman’s law and Einstein’s equation is satisfied in \(C\). Then the fluid is distributed such that its total entropy in \(C\) is an extremum for fixed total particle number and for all variations where \(h_{ab}\) and its first derivatives are fixed on \(\bar{C}\).

Note that the Einstein constraint equation usually refers to

\[
G_{ab}u^b = 8\pi T_{ab}u^b,
\]

while we only used its time component, i.e., Eq. (35), throughout the paper. The remaining part of Eq. (68) reads

\[
G_{cb}u^b h^c_a = 8\pi T_{cb}u^b h^c_a.
\]

By Eq. (3), the right-hand side simply vanishes for perfect fluid. With the help of Eqs. (58) and (29), one can show that the left-hand side of Eq. (69) also vanishes. Thus, Eq. (69) is automatically satisfied in static spacetimes.

4 Conclusions

We have rigorously proven the equivalence of the extrema of the entropy and Einstein’s equation under a few natural and necessary conditions. The significant improvement from previous works is that no spherical symmetry or any other symmetry is needed on the spacelike hypersurface. Our work suggests a clear connection between Einstein’s equation and the thermodynamics of perfect fluid in static spacetimes.

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References

[1] J.D. Bekenstein, Phys. Rev. D, 7, 2333 (1973).
[2] J.M. Bardeen, B. Carter, and S.W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[3] S.W. Hawking, Commun. Math. Phys. 43, 199(1975).
[4] V. Iyer and R. M. Wald, Phys. Rev. 50, 846 (1994).
[5] R. M. Wald, Living Rev. Relativ. 4, 2001-6 (2001).
[6] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995).
[7] Y. Gong, A. Wang, Phys. Rev. Lett. 99, 211301 (2007).
[8] R. D. Sorkin, R. M. Wald and Z. J. Zhang, Gen. Rel. Grav. 13, 1127 (1981).
[9] S. Gao, Phys. Rev. D 84, 104023 (2011); 85, 027503 (2012).
[10] Z. Roupas, Class. Quantum Grav. 30, 115018 (2013).
[11] L. M. Cao, J. Xu, Z. Zeng, Phys. Rev. D. 87, 064005 (2013).
[12] Z. Roupas, arXiv:1305.4851
[13] N. Savvidou, C. Anastopoulos, Class. Quantum Grav. 31, 055003 (2014).
[14] R. Yang, Entropy 15(1), 156 (2013).
[15] S. R. Green, J. S. Schiffrin and R. M. Wald, Class. Quantum Grav. 31, 035023 (2014).
[16] R. M. Wald, General Relativity (The University of Chicago Press, Chicago, 1984).
[17] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (W. H. Freeman and Company, San Francisco, 1973).