Blockers for Non-Crossing Spanning Trees in Complete Geometric Graphs

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Abstract
In this paper we present a complete characterization of the smallest sets that block all the simple spanning trees (SSTs) in a complete geometric graph. We also show that if a subgraph is a blocker for all SSTs of diameter at most 4, then it must block all simple spanning subgraphs, and in particular, all SSTs. For convex geometric graphs, we obtain an even stronger result: being a blocker for all SSTs of diameter at most 3 is already sufficient for blocking all simple spanning subgraphs.

1 Introduction

A geometric graph is a graph whose vertices are points in general position in the plane and whose edges are segments connecting pairs of vertices. Let $G = (V(G), E(G))$ be a complete geometric graph and let $\mathcal{F}$ be a family of subgraphs of $G$. We say that a subgraph $B$ of $G$ blocks $\mathcal{F}$ if it has at least one edge in common with each member of $\mathcal{F}$. We denote by $B(\mathcal{F})$ the collection of all smallest (i.e., having the smallest possible number of edges) subgraphs of $G$ that block $\mathcal{F}$, and call its elements blockers of $\mathcal{F}$.

Blockers for several families of subgraphs were studied in previous papers. For example, the set $B(SPM)$ of blockers for the family of all simple (i.e., non-crossing) perfect matchings in a complete convex geometric graph of even order was characterized in [5], and the family of corresponding co-blockers (i.e., $B(B(SPM))$) was characterized in [6]. The characterizations give raise to interesting structures, such as classes of caterpillars [1, 2].

In this paper we study the set $B(SST)$ of blockers for the family of simple spanning trees (SSTs) of a complete geometric graph, and give the following characterization:

Definition 1.1 A simple spanning subgraph $B$ of a complete geometric graph $G$ is a comb of $G$ if:

\footnote{Formally, the assumption is that an edge never contains a vertex in its relative interior. In the case of a complete geometric graph which we consider in this paper, this implies that the vertices are in general position (i.e., that no three vertices lie on the same line).}
1. The intersection of $B$ with the boundary of $\text{conv}(G)$ is a simple path $P$. We call this path the spine of $B$.

2. Each vertex in $V(G) \setminus P$ is connected by a unique edge to an interior vertex of $P$.

3. For each edge $e$ of $B$, the line $l(e)$ spanned by $e$ does not cross any edge of $B$.\footnote{Note that the line $l(e)$ avoids all vertices of $G$ except the endpoints of $e$, since $V(G)$ is in general position.}

Figure 1: A comb in a non-convex geometric graph on 12 vertices.

Theorem 1.2 A graph $B$ is a blocker for the family of all simple spanning trees of a complete geometric graph $G$ if and only if $B$ is either a star (i.e., the set of all edges in $G$ that emanate from a single vertex) or a comb of $G$. We note that in the convex case, this characterization can be derived by combining a result of Hernando \cite{3} that characterizes those SSTs that meet all other SSTs, with a result of Károlyi et al. \cite{4} that shows that any two-coloring of a complete geometric graph contains a monochromatic SST. Theorem 1.2 was recently used in \cite{7} to show that if $G$ is a complete geometric graph with $n$ vertices in which exactly one vertex does not lie on the boundary of $\text{conv}(G)$, and $c$ is a coloring of the edges of $G$ with $n(n-1)/2 - n + 1$ colors, then $G$ has a simple spanning tree all of whose edges have different colors.

We also present several refinements of Theorem 1.2.

For a complete geometric graph $G$ and for $k \in \mathbb{N}$, denote by $T_{\leq k}(G)$ the family of all simple spanning trees of $G$ with diameter at most $k$.

Theorem 1.3 Let $B$ be a subgraph of a complete geometric graph $G$. If $B \in \mathcal{B}(T_{\leq 4}(G))$, then $B$ is either a star or a comb of $G$.

In the case of complete convex geometric graphs we can replace diameter 4 by diameter 3, as follows:

Theorem 1.4 Let $B$ be a subgraph of a complete convex geometric graph $G$. If $B \in \mathcal{B}(T_{\leq 3}(G))$, then $B$ is a comb of $G$.\footnote{Note that the line $l(e)$ avoids all vertices of $G$ except the endpoints of $e$, since $V(G)$ is in general position.}
The two latter results improve Theorem 1.2 by showing that being a blocker for SSTs of diameter at most 4 (or even at most 3 in the convex case) is sufficient for being a blocker for all SSTs. These results are tight in the sense that $T \leq 4(G)$ cannot be replaced by $T \leq 3(G)$ in Theorem 1.3, as we show by an example in Section 5, and $T \leq 3(G)$ cannot be replaced by $T \leq 2(G)$ in Theorem 1.4, since any spanning subgraph blocks all trees in $T \leq 2(G)$ but not all SSTs of $G$.

Finally, the following result improves Theorem 1.2 in the opposite direction. We say that $H \subset G$ is a simple spanning subgraph (SSS) of $G$ if $H$ is non-crossing and has no isolated vertices, i.e., every vertex of $G$ is incident to an edge of $H$.

**Theorem 1.5** Let $B$ be a subgraph of a complete geometric graph $G$. If $B$ is a star or a comb of $G$, then $B$ blocks all simple spanning subgraphs of $G$.

The paper is organized as follows: In Section 2 we give precise definitions and notations used throughout the paper. In Section 3 we prove properties of blockers for $T \leq 3(G)$ common to the general case and the convex case. In Sections 4 and 5 we prove Theorems 1.4 and 1.3 respectively, and in Section 6 we complete the proof of Theorem 1.2 and prove Theorem 1.5 by showing that $B \subset B(\text{SSS})$ for any complete geometric graph $G$, where $B$ denotes the family of all combs of $G$.

## 2 Definitions and Notations

In this section we present some definitions and notations we use in the paper.

**Geometric graphs.** Throughout the paper, $G$ is a complete geometric graph on $n$ vertices. The sets of vertices and edges of $G$ are denoted by $V(G)$ and $E(G)$, respectively. The convex hull of $V(G)$ is denoted by $\text{conv}(G)$. Vertices in $V(G)$ and edges in $E(G)$ that lie on the boundary of $\text{conv}(G)$ are called boundary vertices and boundary edges of $G$, respectively. A geometric graph is simple if it does not contain a pair of crossing edges. For more information on geometric graphs, the reader is referred to [8].

**Caterpillars.** Throughout the paper, $T$ is a tree. A tree $T$ is a caterpillar if the derived graph $T'$ (i.e., the graph obtained from $T$ by removing all leaves and their incident leaf edges) is a path (or is empty). A longest path in a caterpillar $T$ is called a spine of $T$. (Note that any edge of $T$ either belongs to every spine or is a leaf edge of $T$.) If the diameter of $T$ is 3, then $T$ contains an edge $[x, y]$ such that each vertex in $V(T)$ is at distance at most 1 from either $x$ or $y$. Such an edge $[x, y]$ is called the central edge of $T$. (Note that any tree of diameter 3 is a caterpillar.)

**General notations in the plane.** Throughout the paper, $l$ denotes a line. Each line $l$ partitions the plane into open half-planes. We denote them by $l^+$ and $l^-$, and call them the sides of the line. The unique line that contains two points $a, b \in \mathbb{R}^2$ is denoted by $l(a, b)$. The complement of a set $A$ in $\mathbb{R}^2$, i.e., the set $\mathbb{R}^2 \setminus A$, is denoted by $A^c$.

## 3 Some Properties of Blockers for SSTs of Diameter at Most 3

In this section we establish several properties of blockers for SSTs of diameter at most 3. First we show that the number of edges in these blockers is $n - 1$ (where $n$ is the number of vertices in $G$), and then we show that all such blockers are caterpillars.
3.1 The Size of the Blockers

**Proposition 3.1** Let $G$ be a complete geometric graph on $n$ vertices. Then the size (i.e., number of edges) of the blockers for SSTs of diameter at most 3 in $G$ is $n - 1$.

As any star blocks all SSTs of diameter at most 3 (and actually even all spanning subgraphs), the size of the blockers is at most $n - 1$. The other inequality is a consequence of the following unpublished result of Perles [9].

**Theorem 3.2** Let $G_1$ be a geometric graph on $n$ vertices. If $|E(G_1)| \leq n - 2$, then $G_1$ includes an SST of diameter at most 3. ($\overline{G_1}$ denotes the graph complementary to $G_1$, on the same set of vertices.)

Theorem 3.2 implies that a set of at most $n - 2$ edges cannot block all the SSTs of diameter at most 3, since its complement includes such an SST. Thus, the size of blockers is at least $n - 1$, which completes the proof of Proposition 3.1. For the sake of completeness we present here the proof of Theorem 3.2.

**Proof:** Since $|E(\overline{G_1})| \leq n - 2$, the following two statements hold:

1. $\overline{G_1}$ has a vertex of degree 0 or 1.
2. $\overline{G_1}$ is not connected.

Since the number of edges in $\overline{G_1}$ is smaller than the number of vertices, at least one connected component of $\overline{G_1}$ is a tree. Denote one such component by $A$. If $A$ is a single vertex $x'$, then $G_1$ contains the star centered at $x'$, which is an SST of diameter 2 (assuming $n \geq 2$). Otherwise, $A$ has a leaf $x$. Denote the (only) neighbor of $x$ in $A$ by $y$.

Consider the ray $\vec{xy}$, and turn it around $x$ until it hits a vertex $b \notin A$ for the first time. (There must be a vertex $b \notin A$, since $\overline{G_1}$ is not connected, and thus $A \neq V(G)$. Moreover, $b$ is unique, since a ray emanating from $x$ cannot contain two other vertices of $G$.) Denote by $H$ the closed convex cone bounded by the rays $\vec{xy}$ and $\vec{xb}$.

Let $T$ be the subgraph of $G$ whose edges are

$$\{[x, z] : z \in V(G) \cap H^c\} \cup \{[b, w] : w \in (V(G) \setminus \{b\}) \cap H\},$$

as illustrated in Figure 2. It is clear by the construction that $T$ is an SST of diameter 3 (with central edge $[x, b]$), or of diameter 2 (if $V(G) \subset H$). We claim that $T \subset G_1$. Indeed, the edges $[x, z]$, where $z \in H^c \cap V(G)$ are all in $G_1$, as the only edge in $\overline{G_1}$ that contains $x$ is $[x, y]$. The edges $[b, w]$, where $w \in H \cap (V(G) \setminus \{b\})$ are also in $G_1$, since the vertices $\{w : w \in (V(G) \setminus \{b\}) \cap H\}$ belong to $A$, whereas $b$ belongs to another connected component of $\overline{G_1}$. Therefore, $T \subset G_1$, which completes the proof. □

We note that the proof of Theorem 3.2 implies a stronger statement:
Figure 3: An illustration of the proof of Observation 3.4. The edges of $B$ are depicted by heavy lines, and the edges of $T$ by dotted lines.

**Proposition 3.3** Let $G$ be a complete geometric graph on $n$ vertices. Then the blockers for SSTs of diameter at most 3 in $G$ are spanning trees.

**Proof:** Let $B$ be a blocker for SSTs of diameter at most 3 in $G$. By Proposition 3.1, $|E(B)| = n - 1$. It is clear that $B$ is a spanning subgraph of $G$ without isolated vertices, since otherwise it avoids a star which is an SST of diameter 2. If $B$ is not a tree, then the two statements at the beginning of the proof of Theorem 3.2 clearly hold (i.e., $B$ has a vertex of degree 0 or 1, and is not connected). Thus, by the proof of Theorem 3.2, $\overline{B}$ contains an SST of diameter at most 3, a contradiction. □

### 3.2 The Blockers are Caterpillars

In order to further characterize the blockers, we use two observations.

**Observation 3.4** Let $B$ be a subgraph of $G$. Assume there exist two vertices $a, b \in V(G)$ and a line $l$, such that:

1. $[a, b] \notin E(B)$, and
2. $a$ and all neighbors of $b$ in $B$ lie on one (open) side $l^+$ of $l$, while $b$ and all neighbors of $a$ in $B$ lie on the other (open) side $l^-$ of $l$.

Then $B \notin \mathcal{B}(T_{\leq 3})$.

**Proof:** If Conditions (1) and (2) hold, then $B$ avoids the following SST, as illustrated in Figure 3: $T = (V(G), E(T))$, where

$$E(T) = \{[a, b] \cup \{[a, x] : x \in V(G) \cap (l^+ \cup l), x \neq a\} \cup \{[b, y] : y \in V(G) \cap l^-, y \neq b\}.$$

It is clear that $\text{diam}(T) \leq 3$ (as the distance (in $T$) of all vertices from the edge $[a, b]$ is at most 1), that $T$ is crossing-free, and that $T$ avoids $B$. The assertion follows. □

**Remark 3.5** It is clear that the observation holds also if $a$ and $b$ have neighbors on $l$, as long as they do not have a common neighbor on $l$. 5
Figure 4: An illustration to the proof of Observation 3.7

**Corollary 3.6** Assume $B \in \mathcal{B}(T_{\leq 3})$, and let $a, b$ be two leaves of $B$. Let the corresponding leaf edges be $[a, c]$ and $[b, d]$. If $a, b, c, d$ are mutually distinct, then the points $a, b, d, c$ (in this order) are the vertices of a convex quadrilateral.

**Proof:** Recall that $a, b, c, d$ are vertices of $G$ and therefore, are in general position. If they are not in convex position, then the segments $[a, d]$ and $[b, c]$ are disjoint. The same holds if they form a convex quadrilateral in a different order, say $a, c, b, d$ or $a, b, c, d$. If $[a, d] \cap [b, c] = \emptyset$, then these two segments can be separated by a line $l$. This means that the conditions of Observation 3.4 hold for $a, b,$ and $l$, and thus $B \notin \mathcal{B}(T_{\leq 3})$, a contradiction. □

**Observation 3.7** Suppose $B \in \mathcal{B}(T_{\leq 3})$. Let $a$ be a leaf of $B$ with leaf edge $[a, b]$. Let $c \in V(G) \cap l(a, b)^+$ be the vertex for which the angle $\angle abc$ is maximal (among all vertices in $l(a, b)^+$). Then $[b, c] \in B$.

**Proof:** If $[b, c] \notin B$, then $B$ avoids the following SST:

$$T = \{[a, x] : x \in V(G) \setminus \{a, b\}\} \cup \{[b, c]\}.$$

It is clear that $T$ is a spanning tree of diameter at most 3. $T$ avoids $B$ since the only edge in $B$ that emanates from $a$ is $[a, b]$, and since $[b, c] \notin B$. Finally, $T$ is simple, since if two edges in $T$ cross, then these must be edges of the form $[a, d]$ and $[b, c]$ for some $d \in V(G)$, and in such case, $\angle abd > \angle abc$ (see Figure 4), contradicting the choice of $c$. Thus, $B \notin \mathcal{B}(T_{\leq 3})$, a contradiction. □

Clearly, the same holds for the vertex $c \in V(G) \cap l(a, b)^-$ for which the angle $\angle abc$ is maximal among all vertices in $l(a, b)^-$.

If the leaf edge $[a, b]$ lies on the boundary of $\text{conv}(G)$, then the line $l(a, b)$ supports $V(G)$, and thus only one of the sides of $l(a, b)$ (w.l.o.g. $l(a, b)^+$) contains vertices of $G$. The vertex $c \in V(G) \cap l(a, b)^+$ for which the angle $\angle abc$ is maximal is the vertex that follows $b$ on the boundary of $\text{conv}(G)$, and thus $[b, c]$ is a boundary edge.

If $[a, b]$ is not a boundary edge, then there exist two vertices $c \in V(G) \cap l(a, b)^+$ and $c' \in V(G) \cap l(a, b)^-$ such that the angles $\angle abc, \angle abc'$ are maximal (each with respect to its side of $l(a, b)$), and $[b, c], [b, c'] \in B$. This observation is used in the proof of the theorem below.

Now we are ready to present the main result of this section.
Theorem 3.8 Let $G$ be a complete geometric graph. Then any blocker for SSTs of diameter at most 3 in $G$ is either a star or a caterpillar with a spine whose terminal edges lie on the boundary of $\text{conv}(G)$.\footnote{Note that usually the term “terminal edges of the spine” of a caterpillar is not defined uniquely. Here and in the sequel we mean that there exists a spine whose terminal edges are boundary edges, and in all proofs where we consider the spine of $B$, we refer to a particular spine whose terminal edges are boundary edges.}

Proof: Suppose $B \in B(\mathcal{T}_{\leq 3})$. By Proposition 3.3, $B$ is a tree. If $B$ is a star, we are done. Otherwise, the derived graph $B'$ is a tree with more than one vertex, and thus it has at least two leaves. Let $a, b$ be distinct leaves of $B'$. By Corollary 3.6 (with $a, b$ playing the role of $c, d$), all the leaf edges of $B$ that emanate from $a$ and $b$ lie on the same side of the line $l(a, b)$, w.l.o.g., $l(a, b)^+$ (see Figure 5). We claim that the extremal leaf edges emanating from $a$ and $b$, denoted in the figure by $[a, f]$ and $[b, g]$, are boundary edges.

Assume on the contrary that $[a, f]$ is not a boundary edge. As described above, it follows from Observation 3.7 that if $c \in l(a, f)^+$ with $\angle fac$ maximal, and $e \in l(a, f)^-$ with $\angle fae$ maximal, then $[a, c], [a, e] \in B$. (Here $l(a, f)^+$ is the side of $l(a, f)$ that contains $b$.) We have $c \notin l(a, b)^+$, as otherwise $\angle fac < \angle fab$, contradicting the choice of $c$. Thus, $[a, c]$ is not a leaf edge (since all the leaf edges that emanate from $a$ lie in $l(a, b)^+$). On the other hand, since $a$ is a leaf of $B'$, all the edges in $B$ that emanate from $a$ except one are leaf edges, and thus, $[a, e]$ is a leaf edge. Therefore, $e \in l(a, b)^+$ (and not as shown in the figure), which contradicts the assumption that $[a, f]$ is the extremal (i.e., the leftmost) leaf edge emanating from $a$.

So far we have shown that any leaf of $B'$ is contained in a leaf edge of $B$ that is a boundary edge. In order to complete the proof, it suffices to show that $B'$ has only two leaves, which will imply that $B'$ is a path, and hence $B$ is a caterpillar. As the terminal edges of the spine of a caterpillar $B$ emanate from the two leaves of $B'$, it will follow that these edges can be chosen to be boundary edges of $G$, completing the proof of the theorem.

Assume on the contrary that $B'$ has at least three leaves, say $a, b, c$. By the previous steps of the proof, $B$ has leaf edges $[a, d], [b, e], [c, f]$, which all lie on the boundary of $\text{conv}(G)$. By Corollary 3.6 these edges must satisfy the following three conditions:

- $d$ and $e$ lie on the same side of $l(a, b)$.
- $d$ and $f$ lie on the same side of $l(a, c)$.
- $e$ and $f$ lie on the same side of $l(b, c)$.

But if we somehow orient the boundary of $\text{conv}(G)$ (say, counterclockwise), then at least two of the directed edges $\overrightarrow{ad}, \overrightarrow{be}$ and $\overrightarrow{cf}$ point the same way (both forward or both backward), and thus fail the condition above. This completes the proof of the theorem. □
Figure 6: Three leaf edges on the boundary of conv(G).

Figure 7: The terminal edges of the spine of a blocker for SSTs of diameter at most 3 in a complete convex geometric graph divide the boundary into two arcs.

4 The Convex Case – Proof of Theorem 1.4

In this section we assume in addition that $G$ is convex, i.e., that the vertices of $G$ are in convex position in the plane. By Theorem 3.8, a blocker for SSTs of diameter at most 3 in $G$ is a caterpillar, and the terminal edges of its spine lie on the boundary of conv($G$). We wish to show that the whole spine of $B$ lies on the boundary of conv($G$). This is clear when $B$ is a star. Assume, therefore, that $B$ is not a star. Denote the terminal edges by $[a, a']$ and $[b', b]$, where $a$ and $b$ are the leaves, and $a' \neq b'$, as shown in Figure 7 (Note that by Corollary 3.6, $a$ and $b$ lie on the same side of the line $l(a', b')$.) Let $\alpha$ and $\beta$ denote the two closed arcs with endpoints $a, b$ on the boundary of conv($G$), as shown in the figure.

Claim 4.1 If $c$ is a leaf of $B$, then $c \in \beta$.

Proof: Assume, w.l.o.g., that $c \neq a$ and $c \neq b$. Denote the leaf edge of $B$ that emanates from $c$ by $[c, c']$. By Corollary 3.6, the following two conditions hold:

- Either $a' = c'$ or $a$ and $c$ lie on the same side of $l(a', c')$.
- Either $b' = c'$ or $b$ and $c$ lie on the same side of $l(b', c')$.

If $c \in \alpha$ (as in the figure), then the two conditions clearly contradict each other. Hence, $c \in \beta$, as claimed. □
Now we are ready to prove the main result of this section.

**Theorem 4.2** Any blocker for SSTs of diameter at most 3 in a complete convex geometric graph $G$ is a comb of $G$.

**Proof:** Following the notations of Figure 7, denote the vertices of $G$ on the arc $\alpha$ by $a = a_0, a_1, a_2, a_3, \ldots, a_k = b'$, as shown in Figure 8. Since any vertex of a caterpillar that does not belong to its spine is a leaf, it follows from Claim 4.1 that all the vertices $a_0, a_1, a_2, \ldots, a_k$ belong to the spine of $B$. Let the spine of $B$ be $\langle d_0, d_1, \ldots, d_l, d_{l+1} \rangle$, where $d_0 = a_0 = a$, $d_1 = a_1 = a'$, $d_l = a_k = b'$, and $d_{l+1} = a_{k+1} = b$. If the spine of $B$ is not $\langle a_0, a_1, \ldots, a_k, a_{k+1} \rangle$, then there is a first index $\nu$, $1 \leq \nu \leq l - 1$, such that either $d_{\nu+1} \notin \alpha$, or $d_{\nu+1} \in \alpha$, but $d_{\nu+1}$ precedes $d_\nu$ on $\alpha$. Assume $d_\nu = a_i$, $0 < i < k$. Let $l$ be a line that crosses the segments $[a_{i-1}, a_i]$ and $[a_k, b]$ (see Figure 8). On one side of $l$ (call it $l^-$), we have $a_i$ and the only neighbor of $b$ in $B$, i.e., $a_k$. On the other side $l^+$ of $l$ we have $b$, all the vertices $a_0, a_1, \ldots, a_{i-1}$, and the whole path $\beta$. The only neighbors of $a_i(= d_\nu)$ in $B$ are its predecessor $d_{\nu-1}$ (which lies in $\{a_0, a_1, \ldots, a_{i-1}\}$), its successor $d_{\nu+1}$ (which lies either in $\{a_0, a_1, \ldots, a_{i-1}\}$ or in $\beta$), and possibly some leaves (which all lie in $\beta$). Thus, all neighbors of $a_i$ in $B$ lie in $l^+$. Hence, the conditions of Observation 3.4 hold for $d_\nu = a_i, b$ and the line $l$, and thus, by the observation, $B$ avoids an SST of diameter at most 3, a contradiction.

Therefore, the spine of $B$ is the boundary path $\langle a_0, a_1, \ldots, a_k, b \rangle$. Finally, $B$ is simple, since the only edges in $B$ that can cross are the leaf edges, and these edges do not cross, again by Corollary 3.6. This completes the proof. □

### 5 The General Case – Proof of Theorem 1.3

In this section we consider again general geometric graphs. It turns out that in this case, blocking SSTs of diameter at most 3 is not sufficient even for blocking SSTs of diameter 4, as we demonstrate by an example at the end of this section. Thus, we strengthen the assumption, and assume now that $B$ is a blocker for all SSTs of diameter at most 4. This allows us to use the following observation.
Lemma 5.1 Let $B$ be a blocker for SSTs of diameter at most 4 in $G$. Let $b$ be a boundary vertex of $G$, and let $[a, b]$ and $[b, c]$ be the two boundary edges of $G$ that contain $b$. If at least one of these edges is not in $B$, then $b$ is a leaf of $B$.

Proof: Clearly, the degree of $b$ in $B$ is at least 1, as otherwise, $B$ avoids the star centered at $b$. Assume, on the contrary, that the degree of $b$ in $B$ is at least 2, and that (w.l.o.g.) $[a, b] \notin B$. Let $G_1$ be the graph obtained from $G$ by omitting the vertex $b$ and all edges that contain it, and let $B_1 = G_1 \cap B$. By the assumption, $B_1$ is a graph on $n - 1$ vertices (where $n = |V(G)|$) that has at most $n - 3$ edges. Therefore, by Theorem 3.2, $B_1$ avoids an SST $T_1$ of diameter at most 3 in $G_1$. Since $[a, b]$ is a boundary edge of $G$, it does not cross any edge of $T_1$, and thus $T = T_1 \cup \{[a, b]\}$ is an SST of $G$ of diameter at most 4 that avoids $B$, a contradiction. $\square$

Now we are ready to prove the main result of this section.

Theorem 5.2 Let $G$ be a complete geometric graph, and let $B$ be a blocker for SSTs of diameter at most 4 in $G$. Then $B$ is either a star or a comb of $G$.

Proof: Assume $B$ is not a star. By Theorem 3.8, $B$ is a caterpillar, and it has a spine $\{b_0, b_1, \ldots, b_{k+1}\}$ ($k \geq 2$) whose extreme edges $[b_0, b_1]$ and $[b_k, b_{k+1}]$ lie on the boundary of conv($G$). We would like to show that all the edges $[b_i, b_{i+1}]$ are boundary edges of $G$. Assume on the contrary that this is not true, and let $i$, $0 < i < k$, be the smallest index such that $[b_i, b_{i+1}]$ is not a boundary edge. By the assumption, $[b_{i-1}, b_i]$ is a boundary edge of $G$. Denote the other boundary edge that contains $b_i$ by $[b_i, c]$. We claim that $[b_i, c] \notin B$. Indeed, as the spine edge $[b_i, b_{i+1}]$ is not a boundary edge, if we had $[b_i, c] \in B$ then this edge would be a leaf edge of $B$. But this is impossible, since by the proof of Theorem 3.8, $B$ cannot contain three boundary leaf edges. Therefore, $b_i$ satisfies the condition of Lemma 5.1 and thus, by that lemma, $b_i$ is a leaf of $B$, a contradiction.

So far we proved that the spine of $B$ lies on the boundary of conv($G$). Consequently, if two edges of $B$ cross, then both must be leaf edges of $B$. However, a leaf edge of $G$ cannot cross another leaf edge, by Corollary 3.6. Thus, $B$ is simple. Finally, if an edge of $G$ crosses the line spanned by another edge, then there are two possibilities:

1. Both edges are leaf edges. In this case, the convex hull of the union of these two leaf edges is a triangle and not a quadrilateral, contrary to Corollary 3.6.

2. One of the edges is a leaf edge $[b_i, d]$, and the line $l(b_i, d)$ crosses the boundary edge $[b_j, b_{j+1}]$, for some $i, j$. Assume, w.l.o.g., that $k > j > i$ (and thus, in particular, $i < k - 1$). Consider the edges $[b_i, d]$ and $[b_k, b_{k+1}]$. Both are leaf edges of $B$, and they lie on different sides of the line $l(b_i, b_k)$. This contradicts Corollary 3.6, since $B \in B(T_{\leq 4}) \subset B(T_{\leq 3})$.

This completes the proof of the theorem. $\square$

5.1 Blocking SSTs of Diameter at most 3 is Insufficient

The example presented in Figure 9 shows that blocking SSTs of diameter at most 3 is not sufficient even for blocking SSTs of diameter 4. In the example, it is clear that $T$ (whose edges are represented by dotted lines) is an SST of diameter 4, and that the path $B$ (whose edges are represented by full lines) avoids it. It is also clear that $B$ blocks any SST of diameter 2, since such SSTs are stars, and $B$ is a spanning subgraph of $G$. In order to prove that $B$ meets all SSTs of diameter 3, we show that no edge in $E(G)$ can be the central edge of an SST of diameter 3 that avoids $B$. We do this using the following observation.
Figure 9: An example of a subgraph $B$ of a complete geometric graph $G$ that blocks all SSTs of diameter at most 3, but avoids an SST $T$ of diameter 4. The full lines represent edges of $B$, and the dotted lines represent edges of $T$.

**Lemma 5.3** Suppose $[x, y] \in E(G) \setminus E(B)$. If there exist $z, w \in V(G)$, such that:

1. The points $x, y, z, w$ are distinct and in convex position,
2. $[x, w], [y, z] \in E(B)$, and
3. The segments $[x, y]$ and $[z, w]$ do not cross,

then $[x, y]$ cannot be the central edge in an SST of diameter 3 that avoids $B$.

**Proof:** Assume, on the contrary, that $[x, y]$ is the central edge of such an SST $T$. Then $z$ and $w$ must be at distance 1 in $T$ from $[x, y]$. As $[x, w], [y, z] \in E(B)$ and $T$ avoids $B$, this can happen only if $[x, z], [y, w] \in E(T)$. However, since $x, y, z, w$ are in convex position and the pairs of edges $\{[x, y], [z, w]\}$ and $\{[x, w], [y, z]\}$ do not cross, the pair $\{[x, z], [y, w]\}$ must cross, contradicting the assumption that $T$ is simple. □

It can be seen, by checking all pairs $(i, j)$ with $1 \leq i < j \leq 7$, that in the example, no edge $[a_i, a_j]$ can be the central edge of an SST $T$ of diameter 3 that avoids $B$, since for any edge $[a_i, a_j]$, at least one of the following holds:

1. $[a_i, a_j] \in E(B)$. This happens when $j = i + 1$.
2. There exists a $k$ such that $[a_i, a_k], [a_j, a_k] \in E(B)$ (and thus, $a_k$ cannot be at distance 1 in $T$ from $[a_i, a_j]$). This happens when $j = i + 2$.
3. The vertices $x = a_i, y = a_j, z = a_{j+1}$, and $w = a_{i-1}$ satisfy the conditions of Lemma 5.3. This happens when $1 < i$ and $i + 3 \leq j < 7$. Note that in this case we never obtain $\{3, 4, 5\} \subset \{i - 1, i, j, j + 1\}$.
4. The vertices $x = a_i, y = a_j, z = a_{j-1}$, and $w = a_{i+1}$ satisfy the conditions of Lemma 5.3. This happens when $1 \leq i < i + 3 \leq j \leq 7$, and $i = 1$ or $j = 7$ (or both).

Therefore, $B$ blocks all SSTs of diameter 3, as asserted.

We note that the example can be enlarged arbitrarily: the edges $[a_1, a_2]$ and $[a_6, a_7]$ can be replaced by longer convex polygonal arcs.
6 The Converse Direction

In this section we prove Theorem 1.5, which is an improved variant of the converse direction of Theorem 1.2.

**Theorem 6.1** Let $G$ be a complete geometric graph, and let $B \subset G$ be a comb in $G$. Then $B$ meets every simple spanning subgraph of $G$.

**Proof:** Assume, on the contrary, that $H$ is a simple spanning subgraph of $G$ such that $E(H) \cap E(B) = \emptyset$. Denote the spine of $B$ by $(a_{-1}, a_0, a_1, \ldots, a_k, a_{k+1})$. First we show that there is no loss of generality in assuming that $H$ does not contain edges of the form $[a_i, a_j]$, where $-1 \leq i, j \leq k + 1$ (except, possibly, $[a_{-1}, a_{k+1}]$).

Assume that $H$ contains such edges, and let $[a_{i_0}, a_{j_0}] \in H$, where $i_0 < j_0$, be such an edge that minimizes the difference $j - i$. Consider the subgraph of $G$ defined by:

$$G_1 = G \cap \text{conv}([a_{i_0}, a_{i_0+1}, \ldots, a_{j_0}]).$$

Note that $H$ does not contain edges that connect vertices in $V(G_1) \setminus [a_{i_0}, a_{j_0}]$ with vertices in $V(G) \setminus V(G_1)$, as such edges would cross the edge $[a_{i_0}, a_{j_0}]$, contradicting the simplicity of $H$. Thus, the restriction of $H$ to $\text{conv}([a_{i_0}, a_{i_0+1}, \ldots, a_{j_0}])$, i.e., $H_1 = H \cap G_1$, is a simple spanning subgraph of $G_1$. Similarly, it is clear that $B_1 = B \cap G_1$ is a simple spanning caterpillar of $G_1$ such that the line spanned by an edge $e$ of $B_1$ never crosses another edge of $B_1$. Finally, $H_1$ does not contain edges of the form $[a_i, a_j]$ for $i_0 \leq i, j \leq j_0$ (except for $[a_{i_0}, a_{j_0}]$) by the minimality of the edge $[a_{i_0}, a_{j_0}]$. Thus, by reduction to $G_1, B_1, H_1$, we can assume w.l.o.g. that $H$ does not contain edges of the form $[a_i, a_j]$ (except, possibly, for $[a_{-1}, a_{k+1}]$).

Now, we define a function $f : \{0, 1, \ldots, k\} \to \{0, 1, \ldots, k\}$ by the following procedure, performed for each $0 \leq i \leq k$.

1. Consider the vertex $a_i$. Since $H$ is a spanning subgraph, there exist edges in $E(H)$ that emanate from $a_i$. Pick one such edge $[a_i, y]$. Note that $y \neq a_l$ for $-1 \leq l \leq k + 1$, since by assumption, $H$ does not contain edges of the form $[a_i, a_l]$.

2. Since $y \neq a_l$ for all $-1 \leq l \leq k + 1$, $y$ is a leaf of $B$. Hence, the only edge of $B$ that emanates from $y$ connects it to an interior vertex of the spine of $B$, i.e., is of the form $[y, a_j]$ for some $0 \leq j \leq k$.

3. Define $f(i) = j$, where $j$ is determined by the two previous steps.
Note that we indeed have $0 \leq f(i) \leq k$ for all $i$, since $a_{i-1}$ and $a_{k+1}$ are leaves of $B$, and thus are connected only to $a_0$ and $a_k$, respectively. Also note that $f(i) \neq i$ for all $i$, since otherwise, $B$ and $H$ would share the edge $[a_i, y]$, for some $y$.

Consequently, we have $f(0) > 0$ and $f(k) < k$, and thus, there exists an $i$, $0 \leq i \leq k$, such that $j = f(i) > i$ and $h = f(i+1) < i+1$. Denote the vertices that were used in the generation of $f(i)$ and of $f(i+1)$ by $y$ and $x$, respectively, as illustrated in Figure 10. We claim that the edges $[a_i, y]$ and $[a_{i+1}, x]$ cross, which contradicts the assumption that $H$ is simple.

In order to prove this claim, consider the following polygon:

$$P = \langle x, a_h, a_{h+1}, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}, a_j, y, x \rangle.$$  

(Note that the highest possible value of $h$ is $i$. If $h = i$, then there are no edges between $a_h$ and $a_i$. Similarly for $a_{i+1}$ and $a_j$.) We claim that the path $P_0 = \langle x, a_h, \ldots, a_i, a_{i+1}, \ldots, a_j, y \rangle$ lies on the boundary of $\text{conv}(P)$. Indeed, all the edges of $P_0$ except for $[x, a_h]$ and $[a_j, y]$ lie on the boundary of $\text{conv}(G)$, so they clearly support $P$. The edge $[a_h, x]$ also supports $P$, since the line $l(a_h, x)$ meets the path $\langle a_h, \ldots, a_i, a_{i+1}, \ldots, a_j, y \rangle$ only at its endpoint $a_h$. For similar reasons, $[a_j, y]$ must also support $P$. It follows that the path $P_0$ is part of the boundary of $\text{conv}(P)$.

Finally, $a_{i+1}$ lies on the boundary of $\text{conv}(P)$ strictly between $a_i$ and $y$, and $a_i$ lies on the boundary of $\text{conv}(P)$ strictly between $a_{i+1}$ and $x$. This implies that the two edges $[a_i, y]$ and $[a_{i+1}, x]$ must cross. This contradicts the assumption that $H$ is simple, and thus completes the proof of the theorem. □

### References

1. F. Harary and A.J. Schwenk, Trees with Hamiltonian Square, *Mathematika* 18 (1971), pp. 138-140.
2. F. Harary and A.J. Schwenk, The Number of Caterpillars, *Disc. Math.* 6 (1973), pp. 359–365.
3. M. Carmen Hernando, Complejidad de Estructuras Geométricas y Combinatorias, Ph.D. Thesis, Universitat Politècnica de Catalunya, 1999 (in Spanish). Available online at: http://www.tdx.cat/TDX-0402108-120036/
4. G. Károlyi, J. Pach, and G. Tóth, Ramsey-type Results for Geometric Graphs I, *Discrete and Computational Geometry*, 18 (1997), pp. 247–255.
5. C. Keller and M. A. Perles, On the Smallest Sets Blocking Simple Perfect Matchings in a Convex Geometric Graph, *Israel J. of Math.*, in press. Available on-line at: http://arxiv.org/abs/0911.3350.
6. C. Keller and M. A. Perles, Characterization of Co-Blockers for Simple Perfect Matchings in a Convex Geometric Graph, submitted. Available online at: http://arxiv.org/abs/1011.5883.
7. J. J. Montellano-Ballesteros and E. Rivera-Campo, On the Heterochromatic Number of Hypergraphs Associated to Geometric Graphs and to Matroids, submitted. Available online at: http://arxiv.org/abs/1011.4888.
[8] J. Pach, Geometric Graph Theory, Chapter 10 in: Handbook of Discrete and Computational Geometry, Second Edition, J. E. Goodman and J. O’Rourke, eds., CRC Press, Boca Raton, 2004, pp. 219–238.

[9] M. A. Perles, unpublished, 1987.