STRUCTURE FOR REGULAR INCLUSIONS. II
CARTAN ENVELOPES, PSEUDO-EXPECTATIONS AND TWISTS

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Abstract. We introduce the notion of a Cartan envelope for a regular inclusion \((\mathcal{C}, \mathcal{D})\). When a Cartan envelope exists, it is the unique, minimal Cartan pair into which \((\mathcal{C}, \mathcal{D})\) regularly embeds. We prove a Cartan envelope exists if and only if \((\mathcal{C}, \mathcal{D})\) has the unique faithful pseudo-expectation property and also give a characterization of the Cartan envelope using the ideal intersection property.

For any covering inclusion, we construct a Hausdorff twisted groupoid using appropriate linear functionals and we give a description of the Cartan envelope for \((\mathcal{C}, \mathcal{D})\) in terms of a twist whose unit space is a set of states on \(\mathcal{C}\) constructed using the unique pseudo-expectation. For a regular MASA inclusion, this twist differs from the Weyl twist; in this setting, we show that the Weyl twist is Hausdorff precisely when there exists a conditional expectation of \(\mathcal{C}\) onto \(\mathcal{D}\).

We show that a regular inclusion with the unique pseudo-expectation property is a covering inclusion and give other consequences of the unique pseudo-expectation property.

Contents

1. Introduction 1
2. General Preliminaries 3
3. Additional Preliminaries: Twists and their \(C^*\)-algebras 8
4. Conditional Expectations and Hausdorff Weyl Groupoids 15
5. Cartan Envelopes 18
6. Some Consequences of the Unique Pseudo-Expectation Property 29
7. Twists Associated to a Regular Covering Inclusion 35
References 46

1. Introduction

In their influential 1977 paper [8], Feldman and Moore showed that the collection of pairs \((\mathcal{M}, \mathcal{D})\) consisting of a Cartan MASA \(\mathcal{D}\) in the separably acting von Neumann algebra \(\mathcal{M}\) is, up to suitable notions of equivalence, equivalent to the family \((R, \sigma)\) consisting of measured equivalence relations and 2-cocycles \(\sigma\) on \(R\). The success of the Feldman-Moore program naturally led to attempts to find appropriate \(C^*\)-algebraic analogs. An early attempt was by Kumjian in 1986 [12], who introduced the notion of \(C^*\)-diagonals and proved a Feldman-Moore type result for them using suitable twists. However, Kumjian’s setting was somewhat restrictive, and excluded several classes of desirable examples. In a 2008 paper, Renault [18] extended Kumjian’s work. Renault gave a definition of a Cartan MASA \(\mathcal{D}\) in a \(C^*\)-algebra \(\mathcal{C}\) and gave a method for associating a twist \((\Sigma, G)\)

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to each such pair \((C, D)\). The philosophy is to loosely regard the passage from \((C, D)\) to \((\Sigma, G)\) as somewhat akin to “analysis” in harmonic analysis. It is of course an interesting problem to determine when the original regular inclusion can be reconstructed (“synthesized”) from \((\Sigma, G)\). In [15], Renault shows the class of Cartan inclusions is, to use Leibnitz’s immortal phrase, ‘the best of all possible worlds.’ Indeed, for any Cartan inclusion \((C, D)\), the topologies on \(\Sigma\) and \(G\) are Hausdorff, and the associated twist \((\Sigma, G)\) contains enough of the information about \((C, D)\) to completely recover \((C, D)\). More precisely, Renault shows that if \(G(0)\) is the unit space of \(G\) and \(C_r^*(\Sigma, G)\) denotes the reduced \(C^*\)-algebra of \((\Sigma, G)\), then \((C_r^*(\Sigma, G), C(G(0)))\) is a Cartan inclusion isomorphic to the original Cartan inclusion \((C, D)\). Thus, for Cartan inclusions, both analysis and synthesis are possible. With his results, Renault makes a very convincing case that his definition of Cartan MASA for \(C^*\)-algebras is the appropriate analog of the Feldman-Moore notion of a Cartan MASA in a von Neumann algebra.

While Renault’s notion of Cartan MASA appears in a wide variety of examples, there are also quite natural examples of regular MASA inclusions \((C, D)\) which are not Cartan because they lack a conditional expectation of \(C\) onto \(D\). A large class of examples of regular MASA inclusions with no conditional expectation which arise from crossed products of abelian \(C^*\)-algebras by discrete groups is constructed in [15, Section 6.1]. The lack of a conditional expectation leads to serious problems when one attempts to apply the Kumjian-Renault methods to coordinatize \((C, D)\) using a twist. Indeed, Theorem 4.4 below shows that for a regular MASA inclusion \((C, D)\), the associated Weyl groupoid \(G\) is Hausdorff if and only if there is a conditional expectation of \(C\) onto \(D\). Thus we are confronted with the problem of whether a suitable coordinatization of such pairs \((C, D)\) exists, and what that would mean. If one is willing to utilize non-Hausdorff twists, it is possible to obtain a Kumjian-Renault type characterization of a class of non-Cartan inclusions, and this was recently done in [7]. However, here we shall primarily be interested in Hausdorff twists.

One approach to analyzing a non-Cartan inclusion is to attempt to embed it into a Cartan inclusion. In [15, Theorem 5.7] we characterized when a regular inclusion \((C, D)\) regularly embeds into a \(C^*\)-diagonal, or equivalently, when it embeds into a Cartan inclusion. Applying this result produces a Cartan pair \((C_1, D_1)\) into which \((C, D)\) embeds, but \((C_1, D_1)\) is in general not closely related to the original pair \((C, D)\).

To address this issue, we introduce the notion of a Cartan envelope for a regular inclusion \((C, D)\), see Definition 5.1. This is the “smallest” Cartan pair \((C_1, D_1)\) into which the original pair \((C, D)\) can be regularly embedded. We show the Cartan envelope is unique when it exists, and that the image of \(C_1\) in \(C_1\) is dense in a suitable pointwise topology.

In [15], we introduced the notion of a pseudo-expectation for an inclusion \((C, D)\). For some purposes, pseudo-expectations can be used as a replacement for a conditional expectation. The advantage of pseudo-expectations is that they always exist, and for regular MASA inclusions, are unique [15, Theorem 3.5]. Furthermore, a regular inclusion is a Cartan inclusion if and only if it has a unique pseudo-expectation which is actually a faithful conditional expectation (see Proposition 5.5(b) below). Thus, regular inclusions with a unique and faithful pseudo-expectation are a natural class of regular inclusions containing the Cartan inclusions. We do not know a characterization of those regular inclusions \((C, D)\) for which the pseudo-expectation is unique.

The issue of existence of a Cartan envelope for \((C, D)\) is addressed in Theorem 5.2, we characterize the regular inclusions \((C, D)\) which have Cartan envelope as those which have a unique pseudo-expectation which is also faithful. We also characterize the existence of the Cartan envelope in terms of the ideal intersection property.

Suppose \((C, D)\) is a regular inclusion having Cartan envelope \((C_1, D_1)\). If \((\Sigma_1, G_1)\) is the twist associated to \((C_1, D_1)\), elements of \(\Sigma_1\) and \(G_1\) can be viewed as functions (non-linear in the case of \(G_1\)) on \(C_1\), and by restricting these functions to the image of \(C\) under the embedding of \((C, D)\) into \((C_1, D_1)\), we obtain families of functions on \(C\). These restriction mappings are both one-to-one. In
this way, $(\Sigma_1, G_1)$ may be thought of as a “weak-coordinatization” of $(\mathcal{C}, \mathcal{D})$, or as a weak form of “spectral analysis” for $(\mathcal{C}, \mathcal{D})$. Unsurprisingly, it is possible for two distinct regular inclusions to have the same Cartan envelope, so in general it is not possible to synthesize the original inclusion from a weak-coordinatization without further data. We give examples of this phenomena in Example 5.30.

For a Cartan MASA $\mathcal{D}$ in a von Neumann algebra $\mathcal{M}$, Aoi’s theorem shows that $\mathcal{D}$ is also a Cartan MASA in any intermediate von Neumann subalgebra $\mathcal{D} \subseteq N \subseteq \mathcal{M}$, see [1, Theorem 1.1] or [11, Theorem 2.5.9] for an alternate approach which does not require separability of the predual. While Aoi’s theorem is not true in full generality in the $C^*$-algebra setting, partial results are obtained in [2]. In a sense, Proposition 5.31 of the present paper complements these results: it gives a description of those regular subinclusions $(\mathcal{C}_0, \mathcal{D}_0)$ of a given Cartan pair $(\mathcal{C}, \mathcal{D})$ which are “nearly intermediate” in the sense that $\mathcal{D}_0$ is an essential subalgebra of $\mathcal{D}$.

In [15, Section 4], we introduced the notion of a compatible state for a regular inclusion $(\mathcal{C}, \mathcal{D})$. The restriction of any compatible state on $\mathcal{C}$ to $\mathcal{D}$ is a pure state on $\mathcal{D}$, and when the regular inclusion $(\mathcal{C}, \mathcal{D})$ has enough compatible states to cover $\mathcal{D}$, it is a covering inclusion. We define the notion of a compatible cover for $\mathcal{D}$ (see Definition 2.9) and Theorem 7.24 shows that associated to each compatible cover, there is a Hausdorff twist. When $(\mathcal{C}, \mathcal{D})$ has a Cartan envelope, Theorem 6.9 shows it has a minimal (necessarily compatible) cover, and by Corollary 7.31 the twist associated to the minimal cover is the twist for the Cartan envelope.

We now give an outline of the sections of the paper. Section 2 gives provides a reference for some notation and preliminary results. Section 3 is also a preliminary section, but deals with twists and reduced $C^*$-algebras associated to Hausdorff twists. Section 4 establishes our motivational result that the Weyl groupoid of a regular MASA inclusion is Hausdorff if and only if there is a conditional expectation.

Our main results are in Sections 5, 6, and 7. In Section 5 we introduce and describe the Cartan envelope. Theorem 5.2 shows uniqueness and minimality of the Cartan envelope and characterizes its existence in terms of essential inclusions and also the unique faithful pseudo-expectation property. Section 6 provides some interesting structural consequences of the unique pseudo-expectation property and we propose Conjecture 6.13 as possible characterizations for regular inclusions with the unique pseudo-expectation property. Example 6.10 provides a negative answer to [16, Question 5]. Finally, Section 7 contains a main result, Theorem 7.24 which associates a twist to each compatible cover for an inclusion $(\mathcal{C}, \mathcal{D})$. A consequence of this result is description of the twist associated to the Cartan envelope of a regular inclusion. The results of Section 7 are refinements and improvements of the results contained in Section 8 of our preprint [14].

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2. General Preliminaries

Throughout the paper, unless otherwise stated, all $C^*$-algebras will be assumed unital, and a $C^*$-subalgebra $\mathcal{A}$ of the $C^*$-algebra $\mathcal{B}$ will usually be assumed to contain the identity of $\mathcal{B}$. We will often use the notation $(\mathcal{B}, \mathcal{A})$ to indicate that $\mathcal{A}$ is a unital $C^*$-subalgebra of the $C^*$-algebra $\mathcal{B}$. For any $C^*$-algebra $\mathcal{A}$, we let $\mathcal{U}(\mathcal{A})$ be the unitary group of $\mathcal{A}$.

We recall some terminology and notation. For a Banach space $A$, we use $A^#$ for the Banach space dual; likewise when $u : A \rightarrow B$ is a bounded linear mapping between Banach spaces $A$ and $B$, $u^# : B^# \rightarrow A^#$ is the usual Banach space adjoint of $u$.

Let $A$ be a $C^*$-algebra. For a state $f$ on $A$, $L_f$ denotes its left kernel,

$$L_f := \{ x \in A : f(x^* x) = 0 \}.$$

A pair $(\mathcal{B}, \alpha)$ consisting of a $C^*$-algebra $\mathcal{B}$ and a $*$-monomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ is an extension of $\mathcal{A}$. The extension $(\mathcal{B}, \alpha)$ is a $C^*$-essential extension, or more simply an essential extension, of $\mathcal{A}$.
morphism

conditional expectation

semigroup of all partial automorphisms of \( \alpha \)

We will sometimes write \( d \) and for \( \subseteq \) \( \mathcal{B} \rightarrow \mathcal{A} \) is essential or that the extension \( (\mathcal{B}, \subseteq) \) is essential. If \( \mathcal{A} \subseteq \mathcal{B} \), some authors say \( (\mathcal{B}, \mathcal{A}) \) has the *ideal intersection property* when \( (\mathcal{B}, \mathcal{A}) \) is an essential inclusion.

The commutative setting will play a role in the sequel. When \( X \) and \( Y \) are compact Hausdorff spaces and \( h : Y \rightarrow X \) is a continuous surjection, the pair \( (Y, h) \) is called a *cover* for \( X \). The cover \( (Y, h) \) is an *essential cover* if \( Y \) is the only closed subset \( F \) of \( Y \) such that \( h(F) = X \). The following fact is left to the reader.

**Lemma 2.1.** Suppose \( X \) and \( Y \) are compact Hausdorff spaces and \( (C(Y), \alpha) \) is an extension of \( C(X) \). Then \( (C(Y), \alpha) \) is an essential extension of \( C(X) \) if and only if \( (Y, \alpha\#|_Y) \) is an essential cover for \( X \).

**Remark 2.2.** In the setting of Lemma 2.1 we will usually abuse notation and write \( \alpha\# \) instead of \( \alpha\#|_Y \).

For a topological space \( X \) and a continuous function \( f : X \rightarrow \mathbb{C} \), we shall break with convention and write

\[
\text{supp}(f) := \{ x \in X : f(x) \neq 0 \}
\]

for the *open* support of \( f \). When \( \mathcal{D} \) is an abelian \( C^* \)-algebra and \( d \in \mathcal{D} \), we will sometimes write \( \text{supp}(d) \) instead of \( \text{supp}(\hat{d}) \).

A *partial homeomorphism* of a topological space \( X \) is a homeomorphism between two open subsets of \( X \). If \( s_1 \) and \( s_2 \) are partial homeomorphisms, their product \( s_1s_2 \) has domain \( s_2^{-1}(\text{dom}(s_1) \cap \text{range}(s_2)) \) and for \( x \in X \), \( (s_1s_2)(x) = s_1(s_2(x)) \). We shall use the symbol \( \text{Inv}(X) \) for the inverse semigroup of all partial homeomorphisms of \( X \).

Dually, if \( \mathcal{D} \) is an abelian \( C^* \)-algebra, a partial automorphism is a *-isomorphism* between two closed ideals of \( \mathcal{D} \). If \( \alpha_1 \) and \( \alpha_2 \) are partial automorphisms, their product \( \alpha_1 \alpha_2 \) has domain

\[
\alpha_2^{-1}(\text{dom}(\alpha_1) \cap \text{range}(\alpha_2))
\]

and for \( d \in \text{dom}(\alpha_1\alpha_2) \), \( (\alpha_1\alpha_2)(d) = \alpha_1(\alpha_2(d)) \). We use the symbol \( \text{PAut}(\mathcal{D}) \) for the inverse semigroup of all partial automorphisms of \( \mathcal{D} \). The semigroups \( \text{PAut}(\mathcal{D}) \) and \( \text{Inv}(\mathcal{D}) \) are isomorphic via the map \( \tau \mapsto \tau^{-1}\#|_\mathcal{D} \).

An *inclusion* is a pair \( (\mathcal{E}, \mathcal{D}) \) of unital \( C^* \)-algebras (with the same unit) with \( \mathcal{D} \) abelian, and \( \mathcal{D} \subseteq \mathcal{E} \). Let

\[
\mathcal{D}^c := \{ x \in \mathcal{E} : xd = dx \ \forall \ d \in \mathcal{D} \}
\]

be the relative commutant of \( \mathcal{D} \) in \( \mathcal{E} \). When \( \mathcal{D} \) is a MASA in \( \mathcal{E} \), we refer to \( (\mathcal{E}, \mathcal{D}) \) as a *MASA inclusion*. For any inclusion, an element of the set

\[
\mathcal{N}(\mathcal{E}, \mathcal{D}) := \{ v \in \mathcal{E} : v^* \mathcal{D} v \cup v \mathcal{D} v^* \subseteq \mathcal{D} \}
\]

is called a *normalizer*. The inclusion \( (\mathcal{E}, \mathcal{D}) \) is a *regular inclusion* if \( \mathcal{N}(\mathcal{E}, \mathcal{D}) \) has dense span in \( \mathcal{E} \). Observe that when \( (\mathcal{E}, \mathcal{D}) \) is an inclusion, \( \mathcal{U}(\mathcal{D}^c) \subseteq \mathcal{N}(\mathcal{E}, \mathcal{D}) \). Clearly \( \mathcal{N}(\mathcal{D}^c, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{E}, \mathcal{D}) \), and a routine argument shows

\[
\mathcal{N}(\mathcal{D}^c, \mathcal{D}) = \{ v \in \mathcal{D}^c : v^* v = vv^* \in \mathcal{D} \}. \quad (2.3)
\]

Given two inclusions \( (\mathcal{E}_1, \mathcal{D}_1) \) and \( (\mathcal{E}_2, \mathcal{D}_2) \), a *-homomorphism* \( \alpha : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) is a *regular homomorphism* if

\[
\alpha(\mathcal{N}(\mathcal{E}_1, \mathcal{D}_1)) \subseteq \mathcal{N}(\mathcal{E}_2, \mathcal{D}_2).
\]

We will sometimes write \( \alpha : (\mathcal{E}_1, \mathcal{D}_1) \rightarrow (\mathcal{E}_2, \mathcal{D}_2) \) to indicate \( \alpha \) is a regular *-homomorphism.

The following will be used so frequently that we state it as a formal definition.

**Definition 2.4.** A regular MASA inclusion \( (\mathcal{E}, \mathcal{D}) \) is a *Cartan inclusion* if there is a faithful conditional expectation \( \mathbb{E} : \mathcal{E} \rightarrow \mathcal{D} \).
The following notions will play an important role in the sequel.

**Definition 2.5.** Let \((\mathcal{C}, \mathcal{D})\) be an inclusion and let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\).

(a) A pseudo-expectation relative to \((I(\mathcal{D}), \iota)\) is a unital completely positive map \(E : \mathcal{C} \to I(\mathcal{D})\) such that \(E|_\mathcal{D} = \iota\). When the choice of injective envelope is clear from the context, we will use the simpler term, pseudo-expectation.

The injectivity of \(I(\mathcal{D})\) ensures pseudo-expectations always exist, but in general, there are many pseudo-expectations.

(b) When there is a unique pseudo-expectation, we will say \((\mathcal{C}, \mathcal{D})\) has the *unique pseudo-expectation property*.

(c) If \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property and the pseudo-expectation is faithful, \((\mathcal{C}, \mathcal{D})\) is said to have the *faithful unique pseudo-expectation property*.

Every regular MASA inclusion has the unique pseudo-expectation property \([15, \text{Theorem 3.5}]\). When \((\mathcal{C}, \mathcal{D})\) is a Cartan inclusion, and \(E : \mathcal{C} \to \mathcal{D}\) is the conditional expectation, then \(\iota \circ E\) is the unique pseudo-expectation for \((\mathcal{C}, \mathcal{D})\), so \((\mathcal{C}, \mathcal{D})\) has the faithful unique pseudo-expectation property. Interestingly, the faithful unique pseudo-expectation property can be used to characterize Cartan inclusions as we shall see in Proposition 5.5 below.

For Cartan inclusions, we will sometimes abuse terminology and say that \(E\) is the pseudo-expectation.

Every \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) determines a partial homeomorphism \(\beta_v\) of the Gelfand space \(\hat{\mathcal{D}}\) of \(\mathcal{D}\) in the following way: \(\text{dom} \beta_v = \{\sigma \in \hat{\mathcal{D}} : \sigma(v^*v) > 0\}\), \(\text{range} \beta_v = \{\sigma \in \hat{\mathcal{D}} : \sigma(vv^*) > 0\}\), and for \(\sigma \in \text{dom} \beta_v\) and \(d \in \mathcal{D}\),

\[\beta_v(\sigma)(d) = \frac{\sigma(v^*dv)}{\sigma(v^*v)}.
\]

Then \(\beta_v^{-1} = \beta_{v^*}\). The map \(\mathcal{N}(\mathcal{C}, \mathcal{D}) \ni v \mapsto \beta_v\) is multiplicative on \(\mathcal{N}(\mathcal{C}, \mathcal{D})\) and the collection

\[\mathcal{W}(\mathcal{C}, \mathcal{D}) := \{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}\]

is an inverse semigroup of partial homeomorphisms of \(\hat{\mathcal{D}}\). The semi-group \(\mathcal{W}(\mathcal{C}, \mathcal{D})\) is sometimes called the *Weyl pseudo-group* or *Weyl semi-group* for the inclusion (see \([15])\).

Given \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\), we now describe a partial isomorphism \(\theta_v\) of \(\mathcal{D}\) which is dual to the homeomorphism \(\beta_v\) of \(\hat{\mathcal{D}}\) discussed above. The map \(\mathcal{D}vv^* \ni vv^*h \mapsto v^*hv\) extends uniquely to a \(*\)-isomorphism \(\theta_v\) of \(\mathcal{D}vv^*\) onto \(\mathcal{D}v^*v\), see \([15, \text{Lemma 2.1}]\). Note that for every \(\sigma \in \text{dom} \beta_v\),

\[\sigma \circ \theta_v = \beta_v(\sigma).
\]

For any inclusion \((\mathcal{C}, \mathcal{D})\) and \(\sigma \in \hat{\mathcal{D}}\), let

\[\text{Mod}(\mathcal{C}, \mathcal{D}) = \{\rho \in \text{States}(\mathcal{C}) : \rho|_\mathcal{D} \in \hat{\mathcal{D}}\}\]

be the set of extensions of pure states on \(\mathcal{D}\) to \(\mathcal{C}\). Equipped with the relative weak-\(*\) topology, \(\text{Mod}(\mathcal{C}, \mathcal{D})\) is a compact set. Elements of \(\text{Mod}(\mathcal{C}, \mathcal{D})\) have a property reminiscent of module homomorphisms: for \(\rho \in \text{Mod}(\mathcal{C}, \mathcal{D})\), the Cauchy-Schwartz inequality shows that for \(h, k \in \mathcal{D}\) and \(x \in \mathcal{C}\),

\[\rho(hxk) = \rho(h)\rho(x)\rho(k). \tag{2.6}\]

For any \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) the map \(\beta_v\) extends to a partial homeomorphism \(\tilde{\beta}_v\) of \(\text{Mod}(\mathcal{C}, \mathcal{D})\): if \(\rho \in \text{Mod}(\mathcal{C}, \mathcal{D})\) satisfies \(\rho(v^*v) > 0\), put \(\tilde{\beta}_v(\rho)(x) = \frac{\rho(v^*xv)}{\rho(v^*v)}\). If \(r : \text{Mod}(\mathcal{C}, \mathcal{D}) \to \hat{\mathcal{D}}\) is the restriction map, \(r(\rho) = \rho|_\mathcal{D}\), then

\[\beta_v \circ r = r \circ \tilde{\beta}_v.
\]
Definition 2.7. A subset $X \subseteq \text{Mod}(\mathcal{E}, \mathcal{D})$ covers $\hat{\mathcal{D}}$ if $r(X) = \hat{\mathcal{D}}$ and is $N(\mathcal{E}, \mathcal{D})$-invariant if for every $v \in N(\mathcal{E}, \mathcal{D})$, $\hat{\beta}_v(X \cap \text{dom} \hat{\beta}_v) \subseteq X$.

Proposition 2.8 ([15 Proposition 2.7]). Let $(\mathcal{E}, \mathcal{D})$ be a regular inclusion and suppose that $F \subseteq \text{Mod}(\mathcal{E}, \mathcal{D})$ is $N(\mathcal{E}, \mathcal{D})$-invariant. Then the set

$$\mathcal{K}_F := \{x \in \mathcal{E} : \rho(x^* x) = 0 \text{ for all } \rho \in F\}$$

is a closed, two-sided ideal in $\mathcal{E}$. Moreover, if $\{\rho|_\mathcal{D} : \rho \in F\}$ is weak-* dense in $\hat{\mathcal{D}}$, then $\mathcal{K}_F \cap \mathcal{D} = (0)$.

In [15], we introduced the compatible states, defined by

$$\mathcal{S}(\mathcal{E}, \mathcal{D}) := \{\rho \in \text{States}(\mathcal{E}) : \text{ for every } v \in N(\mathcal{E}, \mathcal{D}), |\rho(v)|^2 \in \{0, \rho(v^* v)\}\}.$$  

Then $\mathcal{S}(\mathcal{E}, \mathcal{D}) \subseteq \text{Mod}(\mathcal{E}, \mathcal{D})$ and $\mathcal{S}(\mathcal{E}, \mathcal{D})$ is a weak-* closed, $N(\mathcal{E}, \mathcal{D})$-invariant set of states. However, it is not in general the case that $\mathcal{S}(\mathcal{E}, \mathcal{D})$ covers $\hat{\mathcal{D}}$; in fact, there exist regular inclusions for which $\mathcal{S}(\mathcal{E}, \mathcal{D})$ is empty, see [15] Theorem 4.13.

Definition 2.9. A regular inclusion $(\mathcal{E}, \mathcal{D})$ is a covering inclusion if $\mathcal{S}(\mathcal{E}, \mathcal{D})$ covers $\hat{\mathcal{D}}$. A subset $F \subseteq \mathcal{S}(\mathcal{E}, \mathcal{D})$ is a compatible cover for $\mathcal{D}$ if $F$ is weak-* closed, $N(\mathcal{E}, \mathcal{D})$-invariant, and covers $\hat{\mathcal{D}}$.

Examples 2.10.

(a) Theorem 6.9 below shows that any regular inclusion with the unique pseudo-expectation property is a covering inclusion. In particular, it follows from [15] Lemma 2.10 and Proposition 4.6 that every regular inclusion $(\mathcal{E}, \mathcal{D})$ for which $\mathcal{D}^c$ is abelian is a covering inclusion.

Example 6.10 gives a construction of a regular inclusion $(\mathcal{E}, \mathcal{D})$ with the unique pseudo-expectation property such that $\mathcal{D}^c$ is non-abelian.

(b) Here is an elementary example of a covering inclusion $(\mathcal{E}, \mathcal{D})$ such that $\mathcal{D}^c$ is non-abelian and which does not have the unique pseudo-expectation property. Let $\mathcal{E} := M_2(\mathbb{C}) \oplus \mathbb{C}$ and $\mathcal{D} := C\mathcal{I}_\mathbb{C}$. Then $(\mathcal{E}, \mathcal{D})$ is a regular inclusion and $M_2(\mathbb{C}) \oplus \mathbb{C} \ni x \oplus \lambda \mapsto \lambda$ is a compatible state. Thus $(\mathcal{E}, \mathcal{D})$ is a covering inclusion, yet $\mathcal{D}^c = \mathcal{E}$, is not abelian. Any state on $\mathcal{E}$ is a pseudo-expectation.

(c) By definition, no regular inclusion $(\mathcal{E}, \mathcal{D})$ with $\mathcal{S}(\mathcal{E}, \mathcal{D}) = \emptyset$ is a covering inclusion. When $\mathcal{E}$ is simple $C^*$-algebra, then $(\mathcal{E}, \mathcal{C}\mathcal{I})$ is a regular inclusion with $\mathcal{S}(\mathcal{E}, \mathcal{C}\mathcal{I}) = \emptyset$ (see [15] Theorem 4.13)).

Associated to an ideal $J$ of a unital abelian $C^*$-algebra $\mathcal{D}$ are the two ideals,

$$J^\perp := \{d \in \mathcal{D} : dJ = 0\} \quad \text{and} \quad J^{\perp \perp} := (J^\perp)^\perp.$$  

When $J = J^{\perp \perp}$, $J$ called a regular ideal. Also for $d \in \mathcal{D}$, we will write $d^\perp$ for the ideal $\{h \in \mathcal{D} : dh = 0\}$; $d^{\perp \perp}$ is defined similarly. Let

$$\text{supp}(J) := \{\tau \in \hat{\mathcal{D}} : \tau|_J \neq 0\}.$$  

Then $J$ is a regular ideal if and only if $\text{supp}(J)$ is a regular open set. Furthermore, for any ideal $J \subseteq \mathcal{D}$,

$$\text{supp}(J^{\perp \perp}) = \left(\text{supp}(J)\right)^\circ.$$  

On the other hand, for an open set $G \subseteq \hat{\mathcal{D}}$, let

$$\text{ideal}(G) := \{d \in \mathcal{D} : \text{supp}(d) \subseteq G\}.$$  

Then $G$ is a regular open set if and only if $\text{ideal}(G)$ is a regular ideal.

Let $\text{Rideal}(\mathcal{D})$ and $\text{Ropen}(\hat{\mathcal{D}})$ be the Boolean algebras of regular ideals of $\mathcal{D}$ and regular open sets in $\hat{\mathcal{D}}$ respectively. These are isomorphic Boolean algebras under the map $\text{Rideal}(\mathcal{D}) \ni J \mapsto \text{supp}(J)$. 
Essential extensions of abelian $C^*$-algebras have isomorphic Boolean algebras of regular ideals (and hence isomorphic Boolean algebras of regular open sets). The proof of the following fact is not trivial, but due to length considerations, we leave the proof to the reader.

**Lemma 2.11.** Suppose $A$ and $B$ are abelian $C^*$-algebras, $(B, \alpha)$ is an essential extension of $A$, and $r : \hat{B} \to \hat{A}$ is the continuous surjection, $\rho \in \hat{B} \mapsto \rho \circ \alpha$. Then the maps

$$\text{Rideal}(B) \ni J \mapsto \alpha^{-1}(J) \quad \text{and} \quad \text{Ropen}(\hat{A}) \ni G \mapsto r^{-1}(G)$$

are Boolean algebra isomorphisms of Rideal($B$) onto Rideal($A$) and Ropen($\hat{A}$) onto Ropen(\hat{B}) respectively. The inverses of these maps are

$$\text{Rideal}(A) \ni K \mapsto \alpha(K)^{\perp \perp} \quad \text{and} \quad \text{Ropen}(\hat{B}) \ni H \mapsto (r(H))^\circ$$

respectively. Furthermore, for $J \in \text{Rideal}(B)$ and $G \in \text{Ropen}(\hat{A})$,

$$\text{supp}(\alpha^{-1}(J)) = (r(\text{supp}(J)))^\circ \quad \text{and} \quad \text{ideal}(r^{-1}(G)) = \alpha(\text{ideal}(G))^{\perp \perp}.$$

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and $v \in N(\mathcal{C}, \mathcal{D})$. In [15, Definition 2.13], we introduced the notion of a Frolík family of ideals for $v$. This is a set of five regular ideals $\{K_i\}_{i=0}^4$ of $\mathcal{D}$, with the property that for $i = 1, 2, 3, K_i \theta_i(K_i) = 0$, $K_4 = (vv^*)^{\perp \perp}$, and $K_0$ is the fixed point ideal for $v$. The following describes the fixed point ideal.

**Lemma 2.12** ([15, Lemma 2.15]). Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $v \in N(\mathcal{C}, \mathcal{D})$. Then

$$K_0 = \{d \in (vv^*\mathcal{D})^{\perp \perp} : vd = dv \in \mathcal{D}^c\} = \{d \in (v^*v\mathcal{D})^{\perp \perp} : vd = dv \in \mathcal{D}^c\}.$$ 

For $v \in N(\mathcal{C}, \mathcal{D})$, $\text{dom} \beta_v = \text{supp}(v^*v\mathcal{D})$. In general, $\text{supp}(K_0)$ need not be contained in $\text{dom} \beta_v$, but they are intimately related. Indeed, since $\text{dom} \beta_v \cap \text{supp}(K_0) = \text{supp}(K_0 \cap v^*v\mathcal{D})$, the next lemma shows that $\text{dom} \beta_v \cap \text{supp}(K_0) = (\text{fix } \beta_v)^\circ$, and furthermore, $\text{supp}(K_0)$ is the interior of the closure of $(\text{fix } \beta_v)^\circ$. This technical fact will play a useful role in Section 4 but it seems convenient to place it here.

**Lemma 2.13.** Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and $v \in N(\mathcal{C}, \mathcal{D})$. Then

$$\text{supp}(K_0 \cap \overline{v^*v\mathcal{D}}) = (\text{fix } \beta_v)^\circ \quad \text{and} \quad K_0 = (K_0 \cap \\overline{v^*v\mathcal{D}})^{\perp \perp}. \quad (2.14)$$

**Proof.** Let $J := K_0 \cap v^*v\mathcal{D}$. Suppose $\sigma \in \mathcal{D}$ and $\sigma \setminus J \neq 0$. Let $d \in J$ be such that $\sigma(d) \neq 0$. Suppose $\rho \in \mathcal{D}$ and $\rho(d) \neq 0$. Then for any $h \in \mathcal{D}$, $dh \in K_0$, so $dhv = vdh \in \mathcal{D}^c$. Thus,

$$\beta_v(\rho)(h) = \frac{\beta_v(\rho)(dh)}{\rho(d)} = \frac{\rho(dh)}{\rho(d)} = \rho(h),$$

whence $\rho = \text{fix } \beta_v$. As this holds for every such $\rho$, $\sigma \in (\text{fix } \beta_v)^\circ$. Therefore, $\text{supp}(J) \subseteq (\text{fix } \beta_v)^\circ$.

For the converse, suppose $d \in \mathcal{D}$ satisfies $\text{supp } d \subseteq (\text{fix } \beta_v)^\circ$. We first show that for every $\rho \in \mathcal{D}$,

$$\rho(v^*vd) = \rho(v^*vd). \quad (2.15)$$

Let $\rho \in \mathcal{D}$. There are three cases. First suppose $\rho(v^*v) = 0$. The Cauchy-Schwartz inequality gives,

$$|\rho(v^*vd)| \leq \rho(v^*d^*vd)^{1/2}\rho(v^*v)^{1/2} = 0,$$

so (2.15) holds when $\rho(v^*v) = 0$.

Suppose next that $\rho(v^*v) > 0$ and $\beta_v(\rho)(d) \neq 0$. Then $\beta_v(\rho) \in \text{supp}(\hat{d})$, so $\beta_v(\rho) \in \text{fix } \beta_v$. Thus, we get $\beta_v(\rho) = \beta_v(\beta_v(\rho)) = \rho$, and hence $\rho(v^*vd) = \rho(v^*v)\rho(d) = \rho(v^*vd)$.

Finally suppose that $\rho(v^*v) > 0$ and $\beta_v(\rho)(d) = 0$. Then $\rho(v^*vd) = 0$. We shall show that $\rho(d) = 0$. If not, the hypothesis on $d$ shows that $\rho \in \text{fix } \beta_v$. Hence, $0 \neq \rho(d) = \beta_v(\rho)(d) = \frac{\rho(v^*vd)}{\rho(v^*v)} = 0$, which is absurd. So $\rho(d) = 0$, and (2.15) holds in this case also. Thus we have established (2.15) in all cases.
Thus \( v^* dv = v^* vd \). So for every \( n \in \mathbb{N} \),

\[
0 = v^* dv - v^* vd = v^* (dv - vd) = vv^* (dv - vd) = (vv^*)^n (dv - vd).
\]

It follows that for every polynomial \( p \) with \( p(0) = 0 \), \( p(vv^*) (dv - vd) = 0 \). Therefore, for every \( n \in \mathbb{N} \),

\[
0 = (vv^*)^{1/n} (dv - vd) = d (vv^*)^{1/n} v - (vv^*)^{1/n} vd.
\]

Since \( \lim_{n \to \infty} (vv^*)^{1/n} v = v \), we have \( vd = dv \). Clearly if \( h \in D \), then \( \text{supp} \left( \hat{d} h \right) \subseteq (\text{fix} \beta_v)^o \), so that \( vdh = hdv = hvd \), whence \( vd \in D^c \). The first equality in (2.14) now follows.

To prove the second equality in (2.14), we will show that

\[
K_0 \cap J^\perp = (0). \tag{2.16}
\]

So suppose that \( d \in K_0 \) and \( d \in J^\perp \). As \( d \in K_0 \), \( d \in \{ v^* v \}^{\perp \perp} \), so

\[
\text{supp} \, \hat{d} \subseteq \text{supp} (\hat{v^* v}). \tag{2.17}
\]

We claim that

\[
\text{supp} \, \hat{d} \cap \text{supp} (\hat{v^* v}) \subseteq (\text{fix} \beta_v)^o. \tag{2.18}
\]

Notice that for \( h \in D \),

\[
(v^* hv - v^* vh) d = v^* hvd - v^* vhd = 0 \tag{2.19}
\]

because \( vd \in D^c \). Let \( \rho \in \text{supp} \, \hat{d} \cap \text{supp} (\hat{v^* v}) \). Since \( \rho(d) \rho(v^* v) \neq 0 \), applying \( \rho \) to (2.19) gives \( \beta_v (\rho)(h) = \rho(h) \) for every \( h \in D \). The inclusion (2.18) follows.

If \( \rho \in (\text{fix} \beta_v)^o \), we may choose \( h \in J \) such that \( \rho(h) \neq 0 \). Since \( d \in J^\perp \), \( dh = 0 \), so \( \rho(d) = 0 \). Thus \( \hat{d} \) vanishes on \( (\text{fix} \beta_v)^o \). Using (2.17) and (2.18), we conclude that

\[
\text{supp} \, \hat{d} \subseteq \text{supp} (\hat{v^* v}) \setminus \text{supp} (\hat{v^* v}),
\]

which is a set with empty interior. Thus \( d = 0 \). \( \square \)

3. Additional Preliminaries: Twists and their C*-algebras

In this section, we collect some generalities on twists and the (reduced) C*-algebras associated to them for use in Sections 4 and 7. Much of this material can be found in [18] or [19]. The description of groupoids is standard, and we include it for notational purposes. Associated to a twist are a line bundle and its conjugate, and a well-known construction associates a reduced C*-algebra to each. These C*-algebras are anti-isomorphic. This unsurprising fact is doubtless known, but because we have not found a proof of this fact in the literature, we provide a sketch of one in Proposition 3.13 below.

There is inconsistency in the literature regarding which line bundle to choose when constructing the C*-algebra associated to a twist—for example, the line bundle described in [12, Definition 4.5] is the conjugate of the line bundle used in [18, Section 4, p. 39]. However, Kumjian uses homogeneous functions to define the convolution algebra while Renault uses conjugate-homogeneous functions, so (as also noted in Proposition 3.13) the reduced C*-algebras discussed by Kumjian and Renault are the same. For the Weyl twist (described in Example 3.7), it seems more natural to use Renault’s choice of line bundle because the action of \( T \) arises from scalar multiplication on \( N(C, D) \). On the other hand, in Section 7 we construct twists from certain families of linear functionals on \( C \). For these twists, it seems natural to use Kumjian’s choice of line bundle because the action of \( T \) arises from scalar multiplication on linear functionals.

Let \( G \) be a groupoid. We use \( G^{(0)} := \{ g \in G : g = g^{-1} g \} \) for the unit space of \( G \). We often use \( r \) and \( s \) for the range and source maps: \( r(g) = gg^{-1} \) and \( s(g) = g^{-1} g \). Also if \( x \in G^{(0)} \),

\[
G^x := \{ g \in G : r(g) = x \} \quad \text{and} \quad G_x := \{ g \in G : s(g) = x \}.
\]

We will frequently write \( xG \) (resp. \( Gx \)) instead of \( G^x \) (resp \( G_x \)).
Definition 3.1. A locally compact topological groupoid $G$ is an étale groupoid if the range map (or equivalently the source map) is a local homeomorphism. A subset $S \subseteq G$ is a bisec tion if there is an open set $V \subseteq G$ with $S \subseteq V$ such that $r|_V$ and $s|_V$ are both homeomorphisms onto open subsets of $G^{(0)}$. An open bisection is sometimes called a slice (see [19, Section 3]).

Étale groupoids may be regarded as the analog of discrete groups. They have a number of pleasant properties, including:

- the unit space of an étale groupoid is open;
- for an étale groupoid $G$, if $x \in G^{(0)}$, then $G_x$ and $G^x$ are discrete sets.

A twist is the analog of a central extension of a discrete group by the circle $\mathbb{T}$. Here is the formal definition.

Definition 3.2 ([19, Definition 5.1.1]). Let $\Sigma$ and $G$ be (not necessarily Hausdorff) locally compact groupoids with $G$ étale, and let $\mathbb{T} \times G^{(0)}$ be the product groupoid. (That is, $(z_1, e_1)(z_2, e_2)$ is defined if and only if $e_1 = e_2$, in which case the product is $(z_1z_2, e_1)$, inversion is given by $(z, e)^{-1} = (z^{-1}, e)$, and the topology is the product topology.) Note that the unit space of $\mathbb{T} \times G^{(0)}$ is $\{1\} \times G^{(0)}$. A twist is a sequence

$$\mathbb{T} \times G^{(0)} \hookrightarrow \Sigma \xrightarrow{q} G$$

(3.3)

where

(a) $\iota$ and $q$ are continuous groupoid homomorphisms with $\iota$ one-to-one and $q$ onto;
(b) $q^{-1}(G^{(0)}) = \iota(\mathbb{T} \times G^{(0)})$;
(c) $\iota|_{\{1\} \times G^{(0)}}$ and $q|_{\Sigma^{(0)}}$ are homeomorphisms onto $\Sigma^{(0)}$ and $G^{(0)}$ respectively (thus we may, and do, identify $\Sigma^{(0)}$ and $G^{(0)}$ using $q$);
(d) for every $\gamma \in \Sigma$ and $z \in \mathbb{T}$, $\iota(z, r(\gamma)) \gamma = \gamma \iota(z, s(\gamma))$; and
(e) for every $g \in G$ there is an open bisection $U \subseteq G$ with $g \in U$ and a continuous function $j_U : U \to \Sigma$ such that $q \circ j_U = \text{id}_U$ and the map $\mathbb{T} \times U \ni (z \times h) \mapsto \iota(z, r(h)) j_U(h)$ is a homeomorphism of $\mathbb{T} \times U$ onto $q^{-1}(U)$.

For $\gamma \in \Sigma$, we will often denote $q(\gamma)$ by $\hat{\gamma}$; indeed, we will usually use the name $\hat{\gamma}$ for an arbitrary element of $G$.

When $G$ is Hausdorff, we shall say the twist $\mathbb{T} \times G^{(0)} \hookrightarrow \Sigma \xrightarrow{q} G$ is a Hausdorff twist. For $z \in \mathbb{T}$ and $\gamma \in \Sigma$ we will write

$$z \cdot \gamma := \iota(z, r(\gamma)) \gamma \quad \text{and} \quad \gamma \cdot z := \gamma \iota(z, s(\gamma))$$

This action of $\mathbb{T}$ on $\Sigma$ is free.

The map $q : \gamma \to G$ is necessarily a quotient map (see [20, Exercise 9K(3)]).

Definition 3.4. The twist $\mathbb{T} \times G_2^{(0)} \hookrightarrow \Sigma_2 \xrightarrow{q_2} G_2$ is an extension of the twist $\mathbb{T} \times G_1^{(0)} \hookrightarrow \Sigma_1 \xrightarrow{q_1} G_1$ if there are continuous groupoid monomorphisms $\theta : G_1 \hookrightarrow G_2$ and $\alpha : \Sigma_1 \hookrightarrow \Sigma_2$ such that $\theta(G_1)$ is closed in $G_2$ and

$$q_2 \circ \alpha = \theta \circ q_1 \quad \text{and} \quad \iota_2 \circ (\text{id}_\mathbb{T} \times \theta|_{G_1^{(0)}}) = \alpha \circ \iota_1.$$

When $\theta$ and $\alpha$ are homeomorphisms and groupoid isomorphisms, these twists are equivalent or isomorphic. Finally, if $\theta$ and $\alpha$ are inclusion maps, $\mathbb{T} \times G_1^{(0)} \hookrightarrow \Sigma_1 \xrightarrow{q_1} G_1$ is a subtwist of $\mathbb{T} \times G_2^{(0)} \hookrightarrow \Sigma_2 \xrightarrow{q_2} G_2$.

Notation 3.5. We will denote a twist in several ways: i) by explicitly writing a sequence such as $\mathbb{T} \times G_2^{(0)} \hookrightarrow \Sigma_2 \xrightarrow{q_2} G_2$; ii) writing $(\Sigma, G, \iota, q)$, or iii) when the maps $\iota$ and $q$ are understood, simply by $(\Sigma, G)$. 
Definition 3.6. Given a twist $(\Sigma, G, \iota, q)$, define \( \tau : \mathbb{T} \times G^{(0)} \to \Sigma \) by \( \tau(z, e) = \iota(\tau, e) \). The conjugate twist to \((\Sigma, G, \iota, q)\) is the twist \((\Sigma, G, \tau, q)\). We will sometimes denote it by \((\Sigma, G, \tau, q)\).

Example 3.7. An important example of a twist associated to a regular inclusion \((C, \mathcal{D})\) is the Weyl twist, which we briefly describe here. The construction is due to Kumjian and Renault (see [12] and [18]). Kumjian and Renault use different descriptions of the equivalence relation determining \(G\), but it is known that their descriptions agree for regular MASA inclusions, a fact which also follows from Lemma 4.2 below. We use Kumjian’s description because it yields a twist for every regular inclusion, whereas Renault’s description need not.

Given the regular inclusion \((C, \mathcal{D})\), set
\[
X := \{ (\sigma_2, v, \sigma_1) \in \hat{\mathcal{D}} \times N(C, \mathcal{D}) \times \hat{\mathcal{D}} : \sigma_2 \in \text{range } \beta_v, \sigma_1 \in \text{dom } \beta_v \text{ and } \sigma_2 = \beta_v(\sigma_1) \}.
\]
Define
\[
\mathcal{R}_1 := \{ ((\sigma_2, v', \sigma_1), (\sigma_2, v, \sigma_1)) \in X \times X : \exists d, d' \in \mathcal{D} \text{ with } \sigma_1(d) > 0, \sigma_1(d') > 0 \text{ and } v'd' = vd \}
\]
and
\[
\mathcal{R}_T := \{ ((\sigma_2, v', \sigma_1), (\sigma_2, v, \sigma_1)) \in X \times X : \exists d, d' \in \mathcal{D} \text{ with } \sigma_1(d) \neq 0, \sigma_1(d') \neq 0 \text{ and } v'd' = vd \}.
\]
(The notation \(\mathcal{R}_1\) and \(\mathcal{R}_T\) reflect that fact that the polar parts of \(\sigma(d)\) and \(\sigma(d')\) belong to the group \(\{1\} \text{ or } \mathbb{T}\).) Then \(\mathcal{R}_1\) and \(\mathcal{R}_T\) are equivalence relations on \(X\) with \(\mathcal{R}_1 \subseteq \mathcal{R}_T\). Denote the equivalence class of \((\sigma_2, v, \sigma_1)\) \(\in X\) relative to \(\mathcal{R}_1\) and \(\mathcal{R}_T\) by \([\sigma_2, v, \sigma_1]_1\) and \([\sigma_2, v, \sigma_1]_T\) respectively. Use
\[
\Sigma := X/\mathcal{R}_1 \quad \text{and} \quad G := X/\mathcal{R}_T
\]
to denote the sets of equivalence classes for \(\mathcal{R}_1\) and \(\mathcal{R}_T\). Then \(\Sigma\) and \(G\) become groupoids with operations defined as follows. For \(\kappa \in \{1, \mathbb{T}\}\), the pair \([\sigma_2, v, \sigma_1]\kappa, [\sigma_2', v', \sigma_1']\kappa] \in (X/\mathcal{R}_\kappa)^2\) is composable if \(\sigma_1 = \sigma'_1\) in which case, the product is \([\sigma_2, v'v', \sigma_1']_\kappa\); inverses are defined by \([\sigma_2, v, \sigma_1]^{-1}_\kappa := [\sigma_1, v', \sigma_2]_\kappa\). The unit spaces are:
\[
G^{(0)} = \{ [\sigma, d, \sigma]_T : d \in \mathcal{D}, \sigma(d) \neq 0 \} \quad \text{and} \quad \Sigma^{(0)} = \{ [\sigma, d, \sigma]_1 : d \in \mathcal{D}, \sigma(d) > 0 \}.
\]
Define \(\iota : \mathbb{T} \times G^{(0)} \to \Sigma\) by \(\iota(z, [\sigma, d, \sigma]_T) = [\sigma, zd^*d, \sigma]_1\). Note that \(\iota\) maps \(\{1\} \times G^{(0)}\) bijectively onto \(\Sigma^{(0)}\). The projection map \(q : \Sigma \to G\) given by \([\sigma, v, \sigma_1]_1 \mapsto [\sigma, v, \sigma_1]_T\) is a groupoid homomorphism with \(q|_{\Sigma^{(0)}}\) a bijection of \(\Sigma^{(0)}\) onto \(G^{(0)}\).

Suppose \([\sigma_2, v, \sigma_1]\) and \([\sigma_2', v', \sigma_1']\) satisfy \(q([\sigma_2, v, \sigma_1]) = q([\sigma_2', v', \sigma_1'])\). Then for \(i = 1, 2\), \(\sigma_i = \sigma'_i\) and there are \(d, d' \in \mathcal{D}\) with \(\sigma_i(d)\) \(\sigma_i(d')\) both non-zero such that \(vd = v'd'\). Choosing \(z, z' \in \mathbb{T}\) so that \(\sigma_1(d) > 0\) and \(z'\sigma_1(d') > 0\) we obtain \([\sigma_2, v', \sigma_1]_1 = [\sigma_2, vz'v, \sigma_1]_1 = \iota(\sigma, z, \sigma_1]_1) [\sigma_2, v, \sigma_1]_1\). In particular, if \([\sigma_2, v, \sigma_1]_T = [\sigma_2, v', \sigma_1]_T\), then there exists a unique \(\lambda \in \mathbb{T}\) such that \([\sigma_2, v', \sigma_1]_1 = [\sigma_2, \lambda v, \sigma_1]_1\). Thus, we obtain the sequence,
\[
\mathbb{T} \times G^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{q} G
\]
satisfying conditions (d) and (h) of Definition 3.2.

The next task is to describe the topologies on \(G\) and \(\Sigma\). Let \(v \in N(C, \mathcal{D})\) and set
\[
N_T(v) := \{ g \in G : \exists \sigma_1, \sigma_2 \in \hat{\mathcal{D}} \text{ such that } g = [\sigma_2, v, \sigma_1]_T \}.
\]
The collection of such sets forms a base for a topology on \(G\); also for \(g = [\sigma_2, v, \sigma_1]_T \in G\),
\[
\{ N_T(vh) : h \in \mathcal{D} \text{ and } \sigma_1(h) \neq 0 \}
\]
is a local base at \(g\). With this topology, \(G\) becomes an étale topological groupoid, but in general, it need not be Hausdorff. In fact, we show in Theorem 4.4 below that when \((C, \mathcal{D})\) is a regular MASA inclusion, \(G\) is Hausdorff if and only if there exists a conditional expectation \(E : C \to \mathcal{D}\).
To describe the topology on Σ, observe that for \( v \in N(\mathcal{C}, \mathcal{D}) \), \( T \times N_T(v) \) is homeomorphic to \( q^{-1}(N_T(v)) \) via the map \( T \times N_T(v) \ni (z, [\sigma_2, w, \sigma_1]_T) \mapsto [\sigma_2, zvw^*w, \sigma_1]_T \); use this map to identify \( T \times N_T(v) \) with \( q^{-1}(N_T(v)) \). A base for a topology on \( \Sigma \) is the collection of subsets of \( \Sigma \) of the form \( \emptyset \times N_T(v) \) where \( \emptyset \subseteq T \) is open. With these topologies, (3.8) becomes a twist, which is the Weyl twist for the regular inclusion \( (\mathcal{C}, \mathcal{D}) \).

**Line Bundles Over Twists.** Associated to a twist \( (\Sigma, G, \iota, q) \) and an integer \( k \in \{-1, 1\} \) is a line bundle \( L_k \) over \( G \), which can then be used to construct convolution algebras, which in turn produce \( C^* \)-algebras of interest. This process has been previously studied by various authors \( \text{e.g. [18, 19]} \) and a description of \( L_{-1} \) has also been described in [3]. For convenience, we give a brief description here.

Fix a twist \( (\Sigma, G, \iota, q) \) and \( k \in \{-1, 1\} \). The group \( T \) acts freely on \( C \times \Sigma \): for \( z \in T \), send \( (\lambda, \gamma) \) to \( (\lambda z^k, z \cdot \gamma) \). Let \( L_k \) be the quotient of \( C \times \Sigma \) by the equivalence relation determined by this action. (To be explicit, this equivalence relation is given by \( (\lambda_1, \gamma_1) \sim_k (\lambda_2, \gamma_2) \) if and only if there exists \( z \in T \) such that \( (\lambda_2, \gamma_2) = (\lambda_1 z^k, z \cdot \gamma_1) \).)

Let \([\lambda, \gamma]_k\) denote the equivalence class of \((\lambda, \gamma)\) and equip \( L_k \) with the quotient topology. When \( k \) is clear from the context, we shall simplify notation and write \([\lambda, \gamma]\) instead of \([\lambda, \gamma]_k\).

Notice that for \( z \in T \) and \([\lambda, \gamma] \in L_k\),

\[
[\lambda, z \cdot \gamma] = [\lambda z^k, \gamma].
\]

The map \( P : L_k \to G \) given by \( P([\lambda, \gamma]) = \tilde{\gamma} \) is a continuous surjection whose fibers are homeomorphic to \( C \). For \( \tilde{\gamma} \in G \), there is no canonical choice of \( \gamma \in P^{-1}(\tilde{\gamma}) \), however, when \( x \in G(0) \), \( P^{-1}(x) \cap \Sigma^{(0)} \) is a singleton set, and, recalling that we have previously identified \( G(0) \) and \( \Sigma^{(0)} \), we will usually identify \( P^{-1}(x) \) with \( C \) using the map \( \lambda \mapsto [\lambda, x] \). When given the following operations, \( L_k \) becomes a Fell line bundle over \( G \) ([11, Definition 2.1]):

- **product:** the product \([\lambda_1, \gamma_1][\lambda_2, \gamma_2]\) is defined when the product \( \gamma_1 \gamma_2 \) is defined in \( \gamma \), in which case, \([\lambda_1, \gamma_1][\lambda_2, \gamma_2] := [\lambda_1, \lambda_2, \gamma_1 \gamma_2]\);
- **conjugation:** \([\lambda, \gamma] := \overline{[\lambda, \gamma]}^{-1}\);
- **scalar multiplication:** for \( \mu \in \mathbb{C} \), \( \mu[\lambda, \gamma] := [\mu, r(\gamma)][\lambda, \gamma] = [\mu \lambda, \gamma]\); and
- **addition:** addition of \([\lambda_1, \gamma_1]\) and \([\lambda_2, \gamma_2]\) is defined when \( \gamma_1 = \gamma_2 \), in which case, \([\lambda_1, \gamma_1] + [\lambda_2, \gamma_2] = [\lambda_1 + z^k \lambda_2, \gamma_1]\), where \( z \) is the (necessarily unique) element of \( T \) such that \( \gamma_2 = z \cdot \gamma_1 \).

Note that since \( G \) is locally trivial, so is \( L_k \) (see [3, Section 2.2]).

Finally, there is a continuous map \( \varpi : L \to [0, \infty) \) given by

\[
\varpi([\lambda, \gamma]) := |\lambda|.
\]

When \( f : G \to L \) is a section and \( \tilde{\gamma} \in G \), we will often write \( |f(\tilde{\gamma})| \) instead of \( \varpi(f(\tilde{\gamma})) \).

**\( C^* \)-algebras associated to Hausdorff twists.** While it is possible to define \( C^* \)- algebras associated to non-Hausdorff groupoids or twists (a discussion of this process may be found in [7]), we will not need \( C^* \)-algebras arising from non-Hausdorff groupoids in the sequel. Thus, for the remainder of this section, we will assume \( G \) is Hausdorff.

Fix a twist \( (\Sigma, G, \iota, q) \) and \( k \in \{-1, 1\} \). The open support of a continuous section \( f : G \to L_k \) is

\[
\text{supp}(f) := \{ \tilde{\gamma} \in G : \varpi(f(\tilde{\gamma})) \neq 0 \},
\]

and \( f \) is compactly supported if the closure of \( \text{supp}(f) \) is a compact subset of \( G \). Let \( C_c(\Sigma, G, k) \) denote the linear space of all compactly supported continuous sections of \( L_k \). For \( f, g \in C_c(\Sigma, G, k) \), and \( \tilde{\gamma} \in G \), define

\[
(f \ast g)(\tilde{\gamma}) := \sum_{\gamma_1 \gamma_2 \in G, \gamma_1 \gamma_2 = \tilde{\gamma}} f(\gamma_1)g(\gamma_2) \quad \text{and} \quad f^*(\tilde{\gamma}) := \overline{f(\tilde{\gamma}^{-1})}.
\]
These operations make $C_c(\Sigma, G, k)$ into a $*$-algebra. In addition, given $f \in C_c(\Sigma, G, k)$ we also write $\overline{f}$ for the function $\hat{f} \mapsto \overline{\hat{f}(\hat{\gamma})}$.

As described in [3 Section 2], we shall sometimes find it convenient to view elements of $C_c(\Sigma, G, k)$ as compactly supported functions on $\Sigma$. Here is a brief outline of how this is done. If $f \in C_c(\Sigma, G, k)$, then given $\hat{f} \in G$, we may choose $\hat{\gamma} \in P^{-1}(\hat{f})$ and a scalar $\tilde{f}(\hat{\gamma})$ so that $f(\hat{\gamma}) = [\tilde{f}(\hat{\gamma}), \hat{\gamma}]$. For $z \in \mathbb{T}$, we may replace $\hat{\gamma}$ with $z \cdot \hat{\gamma}$, so we also find $f(\hat{\gamma}) = [\tilde{f}(z \cdot \hat{\gamma}), z \cdot \hat{\gamma}]$. Using (3.9), we see that $\tilde{f}$ is $k$-equivariant in the sense that for every $\hat{\gamma} \in \gamma$ and $z \in \mathbb{T}$,

$$\tilde{f}(z \cdot \hat{\gamma}) = z^k \tilde{f}(\hat{\gamma}).$$

On the other hand, if $\tilde{f}$ is a compactly supported continuous $k$-equivariant scalar-valued function on $\Sigma$, then defining $f(\hat{\gamma}) := [\tilde{f}(\hat{\gamma}), \hat{\gamma}]$ gives an element of $C_c(\Sigma, G, k)$. The map $f \mapsto \tilde{f}$ is a linear bijection between $C_c(\Sigma, G, k)$ and the space of compactly supported continuous $k$-equivariant functions on $\Sigma$.

When viewed as functions on $\Sigma$, the operations of addition and scalar multiplication are pointwise, involution becomes $f^*(\hat{\gamma}) = \overline{\tilde{f}(\hat{\gamma})}$ and the convolution multiplication is

$$(f \ast g)(\hat{\gamma}) = \sum_{\hat{\gamma}_1 \in G \atop r(\hat{\gamma}_1) = r(\hat{\gamma})} f(\hat{\gamma}_1)g(\hat{\gamma}_1^{-1}),$$

where for each $\hat{\gamma}_1$ with $r(\hat{\gamma}_1) = r(\hat{\gamma})$, only one representative $\hat{\gamma}_1$ of $\hat{\gamma}_1$ is chosen. Note that (3.11) gives a well-defined product.

For $x \in G^{(0)}$ and $f \in C_c(\Sigma, G, k)$, let $\varepsilon_{x,k}(f) := \tilde{f}(x)$. Then $\varepsilon_{x,k}$ is a positive linear functional in the sense that $\varepsilon_{x,k}(f^* \ast f) \geq 0$ for every $f \in C_c(\Sigma, G, k)$. The GNS construction produces a $*$-representation $(\pi_{x,k}, \mathcal{K}_{x,k})$ of $C_c(\Sigma, G, k)$, and $C^*_r(\Sigma, G, k)$ is defined to be the completion of $C_c(\Sigma, G, k)$ with respect to the norm

$$\|f\| := \sup_{x \in G^{(0)}} \|\pi_{x,k}(f)\|, \quad f \in C_c(\Sigma, G, k).$$

We will generally write the product in $C^*_r(\Sigma, G, k)$ using concatenation rather than using the symbol $\ast$, as in the definition of $C_c(\Sigma, G, k)$.

We now show the $C^*$-algebras $C^*_r(\Sigma, G, k)$ and $C^*_r(\Sigma, G, -k)$ are anti-isomorphic. To begin, let $c : \mathbb{C} \times \Sigma \rightarrow \mathbb{C} \times \Sigma$ be defined by

$$c(\lambda, \gamma) = (\overline{\lambda}, \gamma).$$

Note that $(\lambda_1, \gamma_1) \sim_k (\lambda_2, \gamma_2)$ if and only if $c(\lambda_1, \gamma_1) \sim_{-k} c(\lambda_2, \gamma_2)$. In particular, $c$ induces a homeomorphism, again called $c$, from $L_k$ to $L_{-k}$ satisfying $c^2 = \text{id}_{L_k}$. For $k \in \{-1, 1\}$, $i \in \{1, 2\}$, $\gamma_i \in \Sigma$, $\lambda_i \in \mathbb{C}$ and $\mu \in \mathbb{C}$, calculations yield,

- $c(\mu[\lambda_1, \gamma_1]k) = \overline{\mu}c(\lambda_1, \gamma_1)k$;
- when $\gamma_1 = \gamma_2$, $c([\lambda_1, \gamma_1]k + [\lambda_2, \gamma_1]k) = c([\lambda_1, \gamma_1]k) + c([\lambda_2, \gamma_1]k)$;
- when $\gamma_1 \gamma_2$ is defined, $c([\lambda_1, \gamma_1]k[\lambda_2, \gamma_2]k) = c([\lambda_1, \gamma_1]k)c([\lambda_2, \gamma_2]k)$; and
- $c([\lambda_1, \gamma_1]k) = \overline{c([\lambda_1, \gamma_1]k)}$.

In this sense, the bundles $L_k$ and $L_{-k}$ are conjugate.

Next define $V : C_c(\Sigma, G, k) \rightarrow C_c(\Sigma, G, -k)$ by

$$(Vf)(\hat{\gamma}) = c(f(\hat{\gamma})).$$

Clearly $V^2$ is the identity mapping on $C_c(\Sigma, G, k)$.

Define the transpose map, $\tau : C_c(\Sigma, G, k) \rightarrow C_c(\Sigma, G, -k)$ by

$$\tau(f) := V(f^*).$$
Then $\tau$ is a linear, anti-isomorphism of $C_c(\Sigma,G,k)$ onto $C_c(\Sigma,G,-k)$. For $f \in C_c(\Sigma,G,k)$ and $\dot{\gamma} \in G$, a calculation shows

$$\tau(f^*(\dot{\gamma})) = \epsilon(f(\dot{\gamma})) = \tau(f^*)(\dot{\gamma}).$$

Thus $\tau$ is also adjoint-preserving.

For $x \in G^{(0)}$, let $\eta_{x,k} : C_c(\Sigma,G,k) \to \mathcal{H}_{x,k}$ be defined by $\eta_{x,k}(f) = f + N_{x,k}$, where $N_{x,k}$ is the left kernel of $\varepsilon_{x,k}$. Calculations show that for $f,g \in C_c(\Sigma,G,k)$, $\varepsilon_{x,k}(g^* f) = \varepsilon_{x,-k}((Vf)^* (Vg))$, that is,

$$\langle \eta_{x,k}(f), \eta_{x,k}(g) \rangle_{\mathcal{H}_{x,k}} = \langle \eta_{-k}(Vg), \eta_{-k}(Vf) \rangle_{\mathcal{H}_{x,-k}}.$$

Thus $V$ induces a surjective conjugate-linear isometry $W_x : \mathcal{H}_{x,k} \to \mathcal{H}_{x,-k}$ given by $\eta_{x,k}(f) \mapsto \eta_{x,-k}(Vf)$. Therefore, $\mathcal{H}_{x,-k}$ is the conjugate Hilbert space of $\mathcal{H}_{x,k}$.

Now we observe that the transpose map is isometric. For $x \in G^{(0)}$ and $f,g \in C_c(\Sigma,G,k)$, a calculation yields

$$W_x \pi_{x,k}(f^*) \eta_{x,k}(g) = \pi_{x,-k}(\tau(f)) W_x \eta_{x,k}(g).$$

Therefore, for any $f \in C_c(\Sigma,G,k)$,

$$W_x \pi_{x,k}(f^*) W_x^{-1} = \pi_{x,-k}(\tau(f)). \quad (3.12)$$

Write $\|\cdot\|_k$ for the norm in $C^*_r(\Sigma,G,k)$. For $f \in C_c(\Sigma,G,k)$, $(3.12)$ implies

$$\|f\|_k = \|f^*\|_k = \|\tau(f)\|_{-k}.$$

These considerations yield the first part of the following observation relating the $C^*$-algebras of a twist and its conjugate.

**Proposition 3.13.** Let $(\Sigma,G,\iota,q)$ be a (Hausdorff) twist with conjugate twist $(\overline{\Sigma},G,\overline{\iota},q)$ and let $k \in \{-1,1\}$. The following statements hold.

(a) The transpose map $\tau : C_c(\Sigma,G,k) \to C_c(\Sigma,G,-k)$ extends to a $\ast$-preserving anti-isomorphism of $C^*_r(\Sigma,G,k)$ onto $C^*_r(\Sigma,G,-k)$.

(b) Let $L_k(\Sigma)$ and $L_k(\overline{\Sigma})$ denote the Fell bundles over $G$ associated to $(\Sigma,G,\iota,q)$ and $(\overline{\Sigma},G,\overline{\iota},q)$ respectively. Then $L_k(\Sigma)$ and $L_{-k}(\overline{\Sigma})$ are the same; thus $C^*_r(\Sigma,G,k) = C^*_r(\overline{\Sigma},G,-k)$.

**Proof.** We have already outlined the proof of (a) above.

(b) For $\lambda \in \mathbb{C}$, $z \in T$, $\gamma \in \Sigma$ and $e = r(\gamma)$,

$$(\overline{\lambda}, \iota(z,e)\gamma) = (\overline{\lambda}, \overline{\tau(z,e)}\gamma), \quad \text{so} \quad \{(z\lambda, \iota(z,e)\gamma) : z \in T\} = \{(\overline{\lambda}, \overline{\tau(z,e)}\gamma) : z \in T\}.$$

In other words, the $L_k(\Sigma)$ equivalence class of $(\lambda, \gamma)$ coincides with the $L_{-k}(\overline{\Sigma})$ equivalence class of $(\lambda, \gamma)$. As the identity map $\text{id} : L_k(\Sigma) \to L_{-k}(\overline{\Sigma})$ preserves the Fell bundle operations,

$$L_k(\Sigma) = L_{-k}(\overline{\Sigma}).$$

Thus $C_c(\Sigma,G,k) = C_c(\overline{\Sigma},G,-k)$, so that $C^*_r(\Sigma,G,k) = C^*_r(\overline{\Sigma},G,-k)$. $\Box$

**Remark 3.14.** While anti-isomorphic, the $C^*$-algebras $C^*_r(\Sigma,G,1)$ and $C^*_r(\Sigma,G,-1)$ need not be isomorphic.

**Notation 3.15.** In the sequel, when we write $C^*_r(\Sigma,G)$, the reader is to assume that $k \in \{-1,1\}$ has been fixed, and that the validity of the result or discussion does not depend upon the choice of $k$. However, when it is necessary to specify $k$, we will always write $C^*_r(\Sigma,G,k)$.

As observed in the remarks following [18, Proposition 4.1] (and with more detail in [3, Proposition 2.21]), elements of $C^*_r(\Sigma,G,k)$ may be regarded as $k$-equivariant continuous functions on $\Sigma$, and the formulas defining the product and involution on $C_c(\Sigma,G,k)$ remain valid for elements of $C^*_r(\Sigma,G,k)$. Also, as in [18, Proposition 4.1] and [3, Proposition 2.21], for $\gamma \in \Sigma$ and $f \in C^*_r(\Sigma,G,k)$,

$$|f(\gamma)| \leq \|f\|.$$
Thus, point evaluations are continuous linear functionals on $C^*_r(\Sigma, G, k)$.

**Definition 3.16.** We shall call the smallest topology on $C^*_r(\Sigma, G)$ such that for every $\gamma \in \Sigma$, the point evaluation functional, $C^*_r(\Sigma, G) \ni f \mapsto f(\gamma)$ is continuous, the **$\Sigma$-pointwise topology** on $C^*_r(\Sigma, G)$. Clearly this topology is Hausdorff.

We will say that an element $f \in C^*_r(\Sigma, G)$ is **supported in the slice $S$** if $\text{supp}(f) \subseteq S$. Notice that $C_0(G^{(0)})$ may be identified with

$$\{ f \in C^*_r(\Sigma, G) : \text{supp}(f) \subseteq G^{(0)} \}.$$  

We will often tacitly make this identification.

In order to remain within the unital context, we now assume that the unit space of $G$ is compact. In this case $C^*_r(\Sigma, G)$ is unital, and $C(G^{(0)}) \subseteq C^*_r(\Sigma, G)$, so that $(C^*_r(\Sigma, G), C(G^{(0)}))$ is an inclusion.

We next observe it is a regular inclusion. If $f \in C^*_r(\Sigma, G)$ is supported in a slice $U$, then a computation (see [18, Proposition 4.8]) shows that $f \in N(C^*_r(\Sigma, G), C(G^{(0)}))$, and, because the collection of slices forms a basis for the topology of $G$ ([6, Proposition 3.5]), it follows (as in [18, Corollary 4.9]) that $(C^*_r(\Sigma, G), C(G^{(0)}))$ is a regular inclusion.

For $f \in C_r(\Sigma, G)$, define

$$E(f)(\gamma) := \begin{cases} 0 & \text{if } \gamma \notin G^{(0)}; \\ f(\gamma) & \text{if } \gamma \in G^{(0)}. \end{cases}$$

As in [18, Proposition 4.3] or [17, Proposition II.4.8], $E$ extends to a faithful conditional expectation of $C^*_r(\Sigma, G)$ onto $C(G^{(0)})$.

**Proposition 3.17.** Let $(\Sigma, G, \iota, \varphi)$ be a Hausdorff twist and assume $G^{(0)}$ is compact. Then there is a faithful conditional expectation $E : C^*_r(\Sigma, G) \to C(G^{(0)})$, and the inclusion $(C^*_r(\Sigma, G), C(G^{(0)}))$ is regular. Thus, when $C(G^{(0)})$ is a MASA in $C^*_r(\Sigma, G)$, $(C^*_r(\Sigma, G), C(G^{(0)}))$ is a Cartan inclusion.

**Proof.** We have already observed that the inclusion is regular and there is a faithful conditional expectation. When $C(G^{(0)})$ is a MASA in $C^*_r(\Sigma, G)$, $(C^*_r(\Sigma, G), C(G^{(0)}))$ is a Cartan inclusion by definition.  

We conclude this section with a pair of technical results which we apply in Section 7. The first of these will be used in the proof of Theorem [7.24]. As we are regarding $C_r(\Sigma, G) \subseteq C^*_r(\Sigma, G)$, we will write products via concatenation.

**Lemma 3.18.** Let $(\Sigma, G)$ be a twist. For $i = 1, 2$ suppose $U_i$ are open bisections and $f_i \in C_r(\Sigma, G)$ satisfy $\text{supp}(f_i) \subseteq U_i$. Then for $\gamma \in G$,

$$(E(f_1f_2)f_2)(\gamma) = \begin{cases} 0 & \gamma \notin \text{supp}(f_1) \cap \text{supp}(f_2), \\ (f_1f_2)(\gamma) & \gamma \in \text{supp}(f_1) \cap \text{supp}(f_2). \end{cases}$$

**Proof.** A computation shows that when $h \in C_r(\Sigma, G)$ has support in $G^{(0)}$, then

$$(f_1h)(\gamma) = f_1(\gamma)h(s(\gamma)) \quad \text{and} \quad (hf_1)(\gamma) = h(r(\gamma))f_1(\gamma).$$

Thus, for $\gamma \in G$,

$$(E(f_1f_2)f_2)(\gamma) = E(f_1f_2)(r(\gamma))f_2(\gamma) \quad \text{and} \quad (f_1f_2f_2)(\gamma) = f_1(\gamma)(f_2f_2)(s(\gamma)). \quad (3.19)$$

Now

$$E(f_1f_2)(r(\gamma)) = (f_1f_2)(r(\gamma)) = \sum_{\gamma_1 \gamma_2 = r(\gamma)} f_1(\gamma_1)f_2(\gamma_2) \quad \text{and} \quad (f_1f_2)(\gamma) = f_1(\gamma)f_2(\gamma).$$

$$= \sum_{\gamma_1 \gamma_2 = r(\gamma)} f_1(\gamma_1)f_2(\gamma_2) = f_1(\gamma)f_2(\gamma);$$

14
the last equality holding because \( f_i \) are supported in bisections, so that \((r(\bar{\gamma})G) \cap \text{supp}(f_i) \subseteq \{\bar{\gamma}\}\). Also, \((f_2^*f_2)(s(\bar{\gamma})) = f_2(\bar{\gamma})f_2(\bar{\gamma}) \) and thus,

\[
E(f_1f_2^*)(r(\bar{\gamma})) f_2(\bar{\gamma}) = f_1(\bar{\gamma})f_2(\bar{\gamma})f_2(\bar{\gamma}) = f_1(\bar{\gamma}) (f_2^*f_2)(s(\bar{\gamma})).
\]

Combining this equality with \((3.19)\) gives the result. \(\square\)

The following lemma will be used when proving Proposition \([7,32]\).

**Lemma 3.20.** Suppose \((\Sigma, G)\) is a Hausdorff twist and \(H \subseteq G\) is a closed subgroupoid satisfying the following factorization property: if \(\bar{\gamma} \in H\) factors as \(\bar{\gamma} = \bar{\gamma}_1 \bar{\gamma}_2\) where \(\bar{\gamma}_1, \bar{\gamma}_2 \in G\), then \(\bar{\gamma}_1\) and \(\bar{\gamma}_2\) both belong to \(H\). Set \(\Sigma_H := q^{-1}(H)\).

Then \((\Sigma_H, H)\) is a subtwist of \((\Sigma, G)\) and the restriction map \(P : C_c(\Sigma, G) \rightarrow C_c(\Sigma_H, H)\) given by \(f \mapsto f|_H\) extends to a *-epimorphism of \(C^*_\gamma(\Sigma, G)\) onto \(C^*_\gamma(\Sigma_H, H)\).

**Proof.** We shall only sketch the proof. That \((\Sigma_H, H)\) is a subtwist follows from the definitions of twist and subtwist.

The linearity of \(P\) is clear and the factorization property for \(H\) implies \(P\) is a *-homomorphism. If \(f \in C_c(\Sigma_H, H)\) is supported in a slice \(U \subseteq H\), then we may find an open set \(V \subseteq G\) such that \(U = V \cap H\), so extending \(f\) by 0 to \(V\) produces an element \(f_G \in C_c(\Sigma, G)\) such that \(Pf_G = f\). As \(H\) is Hausdorff, the linear span of functions supported in slices of \(H\) is all of \(C_c(\Sigma_H, H)\); it follows that \(P\) is onto.

Let \(Y = H^{(0)}\), so \(Y \subseteq G^{(0)}\). For \(y \in Y\), we may consider the evaluation mappings \(\varepsilon_{y,H}\) (respectively \(\varepsilon_{y,G}\)) given by \(f \mapsto f(y)\), where \(f\) is a continuous section of the line bundle for \(H\) (respectively for \(G\)). Let \((\pi_{y,H}, \mathcal{H}_{y,H})\) be the GNS representation of \(C_c(\Sigma_H, H)\) arising from \(\varepsilon_{y,H}\) and let \(\eta_{y,G} : C_c(\Sigma, G) \rightarrow \mathcal{H}_{y,G}\) be the map \(f \mapsto f + N_{\varepsilon_{y,H}}\); use similar notation for \(\varepsilon_{y,G}\). A computation (again using the factorization property) shows that for \(f \in C_c(\Sigma, G)\), the map \(\eta_{y,G}(f) \rightarrow \eta_{y,H}(Pf)\) is a well-defined isometry which extends to a unitary operator \(U_y : \mathcal{H}_{y,G} \rightarrow \mathcal{H}_{y,H}\). Further, for \(f \in C_c(\Sigma, G)\),

\[
U_y\pi_{y,G}(f) = \pi_{y,H}(Pf)U_y. \tag{3.21}
\]

Therefore,

\[
\|Pf\|_{C^*_\gamma(\Sigma_H, H)} = \sup_{y \in Y} \|\pi_{y,H}(Pf)\| = \sup_{y \in Y} \|\pi_{y,G}(f)\| \leq \sup_{x \in G^{(0)}} \|\pi_x(f)\| = \|f\|_{C^*_\gamma(\Sigma, G)}.
\]

The lemma follows. \(\square\)

4. **Conditional Expectations and Hausdorff Weyl Groupoids**

The purpose of this section is to establish Theorem \([4,4]\) which shows that the groupoid of germs \(G\) for the Weyl semigroup associated to the regular MASA inclusion \((\mathcal{C}, \mathcal{D})\) is Hausdorff if and only if there exists a (necessarily unique) conditional expectation \(E : \mathcal{C} \rightarrow \mathcal{D}\). The fact that \(G\) is Hausdorff in the presence of a conditional expectation was shown by Renault in \([18]\). We include a sketch of an alternate argument establishing this fact in the proof of Theorem \([4,4]\). To our knowledge, the converse is new and is an interesting application of the pseudo-expectation on a regular MASA inclusion.

We begin with two lemmas, the first of which will be used in the proof of the second.

**Lemma 4.1.** Suppose \((\mathcal{C}, \mathcal{D})\) is an inclusion and there exists a conditional expectation \(E : \mathcal{C} \rightarrow \mathcal{D}\). If \(v \in N(\mathcal{C}, \mathcal{D})\), then \(v^*E(v) \in \mathcal{D}^c\).
Proof. Let $J = \overline{E(v)D}$. If $\tau \in \hat{D}$ and $\tau|_J \neq 0$, we have $0 < |\tau(E(v))|^2 = \tau(E(v)^*E(v)) \leq \tau(v^*v)$, so $\tau \in \text{dom}(\beta_v)$. For any $h \in J$ and $d \in D$,

$$v^*E(vh)d = v^*E(\theta_{v'}(hd)v) = v^*\theta_{v'}(hd)E(v) = hdv^*E(v) = dv^*E(vh),$$

so $v^*E(vh) = v^*E(v)h \in D$. Taking $h$ to be from an approximate unit for $J$ we obtain $v^*E(v) \in D^c$. \hfill \Box

The following lemma yields useful characterizations of the equivalence relation used in the definition of Weyl groupoid for a skeletal MASA inclusion. However, we state the lemma for general inclusions to make clear which hypothesis are needed for the equivalences.

**Lemma 4.2.** Let $(\mathcal{C}, D)$ be an inclusion and suppose $M$ is a skeleton for $(\mathcal{C}, D)$. For $i = 1, 2$, let $v_i \in M$ and suppose $\sigma \in \text{dom}(\beta_{v_1}) \cap \text{dom}(\beta_{v_2})$. Consider the following statements.

(a) There exist $h, k \in D$ with $\sigma(h)\sigma(k) \neq 0$ such that $v_1h = v_2k$.

(b) $\beta_{v_1}$ and $\beta_{v_2}$ have the same germ at $\sigma$.

Then (a) $\Rightarrow$ (b). If $M$ is a MASA skeleton for $(\mathcal{C}, D)$, then (b) $\Rightarrow$ (a).

Furthermore, if $M$ is a MASA skeleton for $(\mathcal{C}, D)$ and there exists a conditional expectation $E : \mathcal{C} \to D$, then (a) and (b) are equivalent to:

(c) $\sigma(E(v_2^*v_1)) \neq 0$.

**Proof.** Suppose (a) holds. Consider the open neighborhood of $\sigma$,

$$H := \{\rho \in \hat{D} : |\rho(v_1^*v_1v_2^*v_2)h)| > |\sigma(v_1^*v_1v_2^*v_2)h)/2\}.$$  

Choose $\rho \in H$. By hypothesis, $\sigma(v_1^*v_1v_2^*v_2)h) \neq 0$, so $\rho \in \text{dom}(\beta_{v_1}h) \cap \text{dom}(\beta_{v_2}k)$. Thus for $d \in D$,

$$\beta_{v_1}(\rho)(d) = \beta_{v_1}h(\rho)(d) = \beta_{v_2}k(\rho)(d) = \beta_{v_2}(\rho)(d),$$

so $\beta_{v_1}$ and $\beta_{v_2}$ have the same germ at $\sigma$.

Suppose $M$ is a MASA skeleton for $(\mathcal{C}, D)$ and (b) holds. Since $M$ is a skeleton, $D \subseteq \text{span}M$, so letting $M_1 := \text{span}M \cap N(\mathcal{C}, D)$, we see that $M_1$ is a skeleton containing $D$. Let $H \subseteq \text{dom}(\beta_{v_1}) \cap \text{dom}(\beta_{v_2})$ be open in $\hat{D}$ with $\sigma \in H$ and $\beta_{v_1}h = \beta_{v_2}k$. Recall that an intertwiner for $(\mathcal{C}, D)$ is an element $w \in \mathcal{C}$ such that $wD = Dw$ (no closures). The proof of [5, Proposition 3.4] shows that we may choose $h_1, k_1 \in D$ such that $\sigma(h_1)\sigma(k_1) \neq 0$ and both $v_1h_1$ and $v_2k_1$ are intertwiners. Since $\beta_{v_1}, \beta_{v_2}, v_1h_1$ and $v_2k_1$, have the same germ at $\sigma$, we may assume without loss of generality that $v_1$ and $v_2$ are intertwiners.

Let $w = v_2^*v_1$. Then $\beta_w|H = \text{id}_H$, so $\sigma = (\text{fix} \beta_w)^\circ$. Therefore there exists $d \in D$ such that $\text{supp} \hat{d} \subseteq H$ and $\sigma(d) \neq 0$. Lemmas 2.12 and 2.13 give $wd = dw \in D^c \cap M_1 = D$. Put $k := wd$. Since $v_1$ is an intertwiner, there exists $a \in D$ such that $(v_2v_1^*)v_1 = v_1a$. Set $h = ad$. Then

$$v_2k = v_2v_1^*v_1d = v_1ad = v_1h.$$  

Next, $|\sigma(k)|^2 = \sigma(d^*w^*wd) = |\sigma(d)|^2\sigma(w^*w) \neq 0$. Since

$$|\sigma(h)|^2\sigma(v_1^*v_1) = \sigma(h^*v_1^*v_1h) = \sigma(v_1^*v_2^*v_2) \neq 0,$$

we obtain $\sigma(h) \neq 0$, so (a) holds.

For the remainder of the proof, suppose $M$ is a MASA skeleton for $(\mathcal{C}, D)$ and there exists a conditional expectation $E : \mathcal{C} \to D$. If (a) holds, then

$$\sigma(k^*)\sigma(h)\sigma(E(v_2^*v_1)) = \sigma(E(k^*v_2^*v_1h)) = \sigma(E(h^*v_1^*v_1h)) \neq 0.$$  

Thus (c) holds.

Finally, suppose (c) holds and again put $w = v_2^*v_1$. By Lemma 4.11 $w^*E(w) \in D$, so $\beta_1^{-1}\beta_{v_1} = \beta_w$ and $\text{id}$ have the same germ at $\sigma$. Therefore, $\beta_{v_1}$ and $\beta_{v_2}$ have the same germ at $\sigma$. \hfill \Box
Let \((\mathcal{C}, \mathcal{D})\) be a regular MASA inclusion, let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\) and let \(E: \mathcal{C} \to I(\mathcal{D})\) be the pseudo-expectation. Recall that a state on \(\mathcal{C}\) is a strongly compatible state if it belongs to the set
\[
\mathcal{G}_s(\mathcal{C}, \mathcal{D}) := \{\rho \circ E : \rho \in I(\mathcal{D})\},
\]
and that \(\mathcal{G}_s(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D})\) ([15] Proposition 4.6) or Theorem 4.3 below. Let \(r: \mathcal{G}_s(\mathcal{C}, \mathcal{D}) \to \mathcal{D}\) be the restriction map, \(r(\tau) = \tau\mid_\mathcal{D}\). Since \(\iota = E\mid_\mathcal{D}\), \(r\) is a continuous surjection.

**Theorem 4.4.** For a regular MASA inclusion \((\mathcal{C}, \mathcal{D})\) the following statements are equivalent.

(a) The restriction map \(r\) is one-to-one.

(b) There is a conditional expectation of \(\mathcal{C}\) onto \(\mathcal{D}\).

(c) The Weyl groupoid \(G\) associated to \((\mathcal{C}, \mathcal{D})\) is Hausdorff.

**Proof.** Suppose \(r\) is one-to-one. Then \(r\) is a homeomorphism with inverse \(r^{-1}(\sigma) = \rho \circ E\) where \(\rho \in I(\mathcal{D})\) is any choice so that \(\sigma = \rho \circ E\mid_\mathcal{D}\), equivalently, \(\rho \circ \iota = \sigma\). For \(x \in \mathcal{C}\), the function \(\hat{\Delta}(x)\) is continuous and hence determines an element \(\Delta(x) \in \mathcal{D}\) whose Gelfand transform is \(\Delta(x)(\sigma) := r^{-1}(\sigma)(x)\). Then \(\Delta\) is completely positive, unital, and for every \(d \in \mathcal{D}\) and \(\sigma \in \hat{\mathcal{D}}\),
\[
\sigma(\Delta(d)) = \rho(E(d)) = \rho(\iota(d)) = \sigma(d),
\]
so \(\Delta\mid_\mathcal{D} = \text{Id}_\mathcal{D}\). Thus, \(\Delta\) is a conditional expectation and (b) holds.

As noted earlier, the implication (b) \(\Rightarrow\) (c) was established by Renault, see [18, Proposition 5.7]. Here is a sketch of an argument somewhat different from Renault’s. Let \(X_1 := \{(v, \sigma_1) : (v, \sigma_1) \in X\}\) for each \((v, \sigma) \in X_1\), consider the function on \(\mathcal{C}\) given by
\[
||[v, \sigma]||(x) = \frac{|\sigma(E(v^*x))|}{\sigma(v^*v)^{1/2}} \quad (x \in \mathcal{C}),
\]
and let \(\mathcal{G} := \{||[v, \sigma]|| : (v, \sigma) \in X_1\}\). Put the topology of pointwise convergence on \(\mathcal{G}\). Then \(\mathcal{G}\) is a Hausdorff space.

It follows from Lemma 4.1 that the map \(U\) given by \([\sigma_2, v, \sigma_1]_T \mapsto ||[v, \sigma_1]||\) is a well-defined map of the Weyl groupoid \(G\) into \(\mathcal{G}\). We shall show \(U\) is continuous and one-to-one. Indeed if \([\rho_\lambda, v_\lambda, \sigma_\lambda]_T\) is a net in \(G\) converging to \([\rho, v, \sigma]_T\), continuity of the source map gives \(\sigma_\lambda \to \sigma\). Let \(d \in \mathcal{D}\) with \(\sigma(d) \neq 0\). Then \(N(\rho vd)\) is a basic neighborhood of \([\rho, v, \sigma]_T\), so for large \(\lambda\), \([\rho_\lambda, v_\lambda, \sigma_\lambda]\) is in \(N(\rho vd)\). Thus for large enough \(\lambda\), \(v_\lambda\) and \(\sigma_\lambda\) have the same germ at \(\sigma_\lambda\). Therefore there exist \(h_\lambda, k_\lambda \in \mathcal{D}\) such that \(\sigma_\lambda(k_\lambda)\sigma_\lambda(h_\lambda) \neq 0\) and
\[
v_\lambda k_\lambda = vh_\lambda.
\]
Then for \(x \in \mathcal{C}\) computations yield,
\[
|[v_\lambda, \sigma_\lambda]| = |[v_\lambda k_\lambda, \sigma_\lambda]| = |[vh_\lambda, \sigma_\lambda]| = ||[v, \sigma_1]| \to |[v, \sigma]|.
\]
Thus \(U\) is continuous.

Suppose now that \(g = [\sigma_2, v, \sigma_1]_T\) and \(g' = [\sigma_2', v', \sigma_1']_T\) are elements of \(G\) with \(U(g) = U(g')\), that is, \([v, \sigma_1] = [v', \sigma_1]\). Then for any \(h \in \mathcal{D}\), taking \(x = vh\) in (4.5) above shows
\[
\sigma_1(E(v^*vh)) = \sigma_1(E(v^*v))\sigma_1(h) = 0 \iff \sigma_1'(E(v'^*v)\sigma_1(h)) = \sigma_1'(E(v'^*v))\sigma_1'(h) = 0.
\]
It follows that for any \(h \in \mathcal{D}\), \(\sigma_1(h) = 0\) if and only if \(\sigma_1'(h) = 0\), so \(\sigma_1 = \sigma_1'\). Taking \(h = I\) gives \(\sigma_1(E(v^*v)) \neq 0\). Thus \(\beta_\nu\) and \(\beta_{\nu'}\) have the same germ at \(\sigma_1\) by Lemma 4.1. Therefore \(g = g'\), so \(U\) is one-to-one.

Since \(U\) is a one-to-one and continuous mapping of \(G\) into the Hausdorff space \(\mathcal{G}\), it follows \(G\) is also Hausdorff.

We prove the contrapositive of (c) \(\Rightarrow\) (a). Suppose \(r\) is not one-to-one. We may then find \(\rho_1, \rho_2 \in I(\mathcal{D})\) such that \(\rho_1 \circ E \neq \rho_2 \circ E\) yet \(\rho_1 \circ \iota = \rho_2 \circ \iota\). Put \(\sigma := \rho_1 \circ \iota\).
Since $\rho_1 \circ E \neq \rho_2 \circ E$, regularity yields the existence of $v \in N(\mathcal{C}, \mathcal{D})$ such that $\rho_1(E(v)) \neq \rho_2(E(v))$. Let $K_0 \subseteq \mathcal{D}$ be the fixed point ideal for $v$.

The Cauchy-Schwartz inequality gives

$$|\rho_1(E(v))|^2 \leq \rho_1(v^*v) = \sigma(v^*v).$$

Since $\rho_1(E(v))$ cannot both vanish, $\sigma(v^*v) \neq 0$.

We claim that $\sigma|_{K_0} = 0$. For $d \in K_0$, $vd = dv \in \mathcal{D}$, so $\rho_1(E(v))\sigma(d) = \rho_1(E(vd)) = \sigma(vd) = \rho_2(E(vd)) = \rho_2(E(v))\sigma(d)$. Since $\rho_1(E(v)) \neq \rho_2(E(v))$, $\sigma(d) = 0$, as desired.

Next we show that $\sigma|_{K_0^\perp} = 0$. Choose $d \in K_0^\perp$ and let $R_0 \in I(\mathcal{D})$ be the support projection of $K_0$ in $I(\mathcal{D})$ (see [13] Lemma 1.9)). Then $\iota(d)R_0 = 0$, and by [13] Theorem 3.5], $E(v) = E(v)R_0$. Therefore,

$$\rho_1(E(v))\sigma(d) = \rho_1(E(v))\rho_1(\iota(d)) = \rho_1(E(v)R_0 \iota(d)) = 0.$$

Similarly, $\rho_2(E(v))\sigma(d) = 0$, so that $\sigma(d) = 0$.

By construction, $K_0$ is a regular ideal in $\mathcal{D}$. Thus,

$$\text{supp}(K_0)^\perp = \text{supp}(K_0^\perp) \in \text{ROPEN}(\mathcal{D}).$$

Also, since $K_0 \vee K_0^\perp = \mathcal{D}$,

$$\text{supp}(K_0) \cup \text{supp}(K_0^\perp)$$

is a dense open set in $\mathcal{D}$. Since $\sigma$ annihilates $K_0$ and $K_0^\perp$, we have

$$\sigma \in \overline{\text{supp}(K_0) \cap \text{supp}(K_0^\perp)}.$$

Let $J := v^*v\mathcal{D} \cap K_0$. By Lemma 2.13, $J^{\perp \perp} = K_0$. Therefore $\text{supp}(J) = \text{supp}(K_0)$. But $\text{supp}(J) = (\text{fix } \beta_v)^\circ$, so $\sigma \in (\text{fix } \beta_v)^\circ$. Note $\sigma \in \text{fix } \beta_v$ because $\text{fix } \beta_v$ is relatively closed in $\text{dom } \beta_v$.

Consider the ideal $\mathcal{L} := K_0^\perp \cap v^*v\mathcal{D}$. Fix $\tau \in \text{supp}(\mathcal{L})$. We claim that whenever $H \subseteq \text{supp}(\mathcal{L})$ is an open neighborhood of $\tau$, then there exists $\tau_1 \in H$ such that $\beta_v(\tau_1) \neq \tau_1$. Indeed, if otherwise, then there exists an open neighborhood $H \subseteq \text{supp}(\mathcal{L})$ of $\tau$ such that $\beta_v(\tau_1) = \tau_1$ for every $\tau_1 \in H$. But then $\tau \in (\text{fix } \beta_v)^\circ = \text{supp}(J) \subseteq \text{supp}(K_0)$. But $\text{supp}(K_0)$ and $\text{supp}(K_0^\perp)$ are disjoint, and $\text{supp}(\mathcal{L}) \subseteq \text{supp}(K_0^\perp)$. This contradiction establishes the claim.

Since $\sigma \in \text{supp}(K_0^\perp)$ and $\text{supp}(\mathcal{L}) = \text{supp}(K_0^\perp) \cap \text{dom } \beta_v$, every neighborhood of $\sigma$ contains an element of $\text{supp}(\mathcal{L})$. Thus, the preceding discussion shows that every open neighborhood of $\sigma$ has non-empty intersection with both $(\text{fix } \beta_v)^\circ$ and the set $\{\tau \in \text{dom } \beta_v : \beta_v(\tau) \neq \tau\}$. Therefore

$$[\sigma, I, \sigma]_T \neq [\sigma, v, \sigma]_T.$$

Suppose now that $V_1$ and $V_2$ are open neighborhoods of $[\sigma, I, \sigma]_T$ and $[\sigma, v, \sigma]_T$ respectively. We may choose $d_1, d_2 \in v^*v\mathcal{D}$ such that $\sigma(d_i) = 1$ so that $N_T(d_1) \subseteq V_1$ and $N_T(vd_2) \subseteq V_2$; recall $N_T(d_1)$ and $N_T(vd_2)$ are basic open neighborhoods of $[\sigma, I, \sigma]_T$ and $[\sigma, v, \sigma]_T$ respectively. We shall show that $N_T(d_1) \cap N_T(vd_2) \neq \emptyset$. Let $d = d_1d_2$. Since $\sigma \in \text{supp}(J)$ and $\sigma(d) = 1$, we may find $\tau \in \text{supp}(J)$ such that $|\tau(d)| > 1/2$. Then $\tau((vd)^*(vd)) \neq 0$, and $\tau \in (\text{fix } \beta_v)^\circ$. Hence $[\tau, vd, \tau]_T = [\tau, d, \tau]_T \in N_T(d_1) \cap N_T(vd_2) \subseteq V_1 \cap V_2$. Therefore, $G$ is not Hausdorff.

\[\square\]

5. Cartan Envelopes

It follows from [15] Theorem 5.7 that a regular inclusion $(\mathcal{C}, \mathcal{D})$ regularly embeds into a Cartan pair $(\mathcal{C}_1, \mathcal{D}_1)$ precisely when the ideal $\text{Rad}(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : \rho(x^*x) = 0 \forall \rho \in \mathcal{G}(\mathcal{C}, \mathcal{D})\}$ vanishes. In general, the construction given in the proof of [15] Theorem 5.7] produces a Cartan pair $(\mathcal{C}_1, \mathcal{D}_1)$ having little connection with the original pair $(\mathcal{C}, \mathcal{D})$. An example of this behavior is the inclusion $(C[0,1], \mathcal{C}I)$, where the Cartan pair into which $(C[0,1], \mathcal{C}I)$ embeds is $(C[0,1], C[0,1])$. However
in some cases, the image of $\mathcal{C}$ under the embedding generates $\mathcal{C}_1$ as a $\mathcal{D}_1$-bimodule and $(\mathcal{C}_1, \mathcal{D}_1)$ is minimal in a sense made precise below. When this occurs, we call such a minimal pair $(\mathcal{C}_1, \mathcal{D}_1)$ a Cartan envelope for $(\mathcal{C}, \mathcal{D})$ (in analogy with the $C^*$-envelope of an operator system).

The purpose of this section is to establish a main result, Theorem 5.2, which characterizes the existence and uniqueness of the Cartan envelope for a regular inclusion in terms of the ideal intersection property and also in terms of the unique faithful pseudo-expectation property. We begin with definitions.

**Definition 5.1.** Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion.

(a) An extension of $(\mathcal{C}, \mathcal{D})$ is a triple $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ consisting of the regular inclusion $(\mathcal{C}_1, \mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : (\mathcal{C}, \mathcal{D}) \rightarrow (\mathcal{C}_1, \mathcal{D}_1)$. In addition, if $(\mathcal{C}_1, \mathcal{D}_1)$ is a Cartan pair, we say $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ is a Cartan extension of $(\mathcal{C}, \mathcal{D})$.

(b) A package for $(\mathcal{C}, \mathcal{D})$ is an extension $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ such that there exists a faithful conditional expectation $E_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ and the image of $(\mathcal{C}, \mathcal{D})$ under $\alpha$ generates $(\mathcal{C}_1, \mathcal{D}_1)$ in the sense that

$$\mathcal{C}_1 = C^*(\alpha(\mathcal{C}) \cup E_1(\alpha(\mathcal{C})))$$

and

$$\mathcal{D}_1 = C^*(E_1(\mathcal{C})).$$

(c) The package $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ is a Cartan package when $(\mathcal{C}_1, \mathcal{D}_1)$ is a Cartan pair (the additional restriction is that $\mathcal{D}_1$ is a MASA in $\mathcal{C}_1$).

(d) An envelope for $(\mathcal{C}, \mathcal{D})$ is a package $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ for $(\mathcal{C}, \mathcal{D})$ such that $(\mathcal{D}_1, \alpha|_{\mathcal{D}})$ is an essential extension of $\mathcal{D}$. An envelope $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ is a Cartan envelope when $(\mathcal{C}_1, \mathcal{D}_1)$ is a Cartan pair.

Two extensions $(\mathcal{C}_i, \mathcal{D}_i, \alpha_i) \ (i = 1, 2)$ of $(\mathcal{C}, \mathcal{D})$ are equivalent if there is a regular $*$-isomorphism $\psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that $\psi \circ \alpha_1 = \alpha_2$.

Not every regular inclusion $(\mathcal{C}, \mathcal{D})$ has a Cartan extension. Indeed, if $\mathcal{D}^c$ is not abelian, [15, Theorem 5.4] shows that $(\mathcal{C}, \mathcal{D})$ cannot have a Cartan extension. However, as noted above, [15, Theorem 5.7] characterizes when $(\mathcal{C}, \mathcal{D})$ has a Cartan extension.

We can now state the main result of this section.

**Theorem 5.2.** Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and let $\mathcal{D}^c$ be the relative commutant of $\mathcal{D}$ in $\mathcal{C}$. The following statements are equivalent:

(a) $(\mathcal{C}, \mathcal{D})$ has a Cartan envelope;

(b) $(\mathcal{C}, \mathcal{D})$ has the faithful unique pseudo-expectation property;

(c) $(\mathcal{D}^c, \mathcal{D})$ and $(\mathcal{C}, \mathcal{D}^c)$ are essential inclusions and $\mathcal{D}^c$ is abelian.

When $(\mathcal{C}, \mathcal{D})$ satisfies any of conditions (a)–(c), the following statements hold.

Uniqueness: If for $j = 1, 2$, $(\mathcal{C}_j, \mathcal{D}_j, \alpha_j)$ are Cartan envelopes for $(\mathcal{C}, \mathcal{D})$, there exists a unique regular $*$-isomorphism $\psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that $\psi \circ \alpha_1 = \alpha_2$.

Minimality: If $(\mathcal{C}_1, \mathcal{D}_1, \alpha)$ is a Cartan package for $(\mathcal{C}, \mathcal{D})$, there is an ideal $\mathfrak{J} \subseteq \mathcal{C}_1$ such that $\mathfrak{J} \cap \alpha(\mathcal{C}) = \{0\}$ and, letting $q : \mathcal{C}_1 \rightarrow \mathcal{C}_1/\mathfrak{J}$ denote the quotient map, $(\mathcal{C}_1/\mathfrak{J}, \mathcal{D}_1/(\mathfrak{J} \cap \mathcal{D}_1), q \circ \alpha)$ is a Cartan envelope for $(\mathcal{C}, \mathcal{D})$.

**Remark 5.3.** It is possible to construct a regular MASA inclusion whose pseudo-expectation is not faithful, so the condition in part (c) that $(\mathcal{C}, \mathcal{D}^c)$ is essential is needed.

We shall give a groupoid description of the Cartan envelope for a regular inclusion with the faithful unique pseudo-expectation property in Section 7 (see Theorem 7.24).

The proof of Theorem 5.2 will be accomplished in several steps. We begin with a lemma on essential inclusions for abelian $C^*$-algebras. It is possible to give a proof of the lemma from the definitions, but we prefer to use properties of pseudo-expectations.
Lemma 5.4. For $i = 1, 2, 3$, let $D_i$ be abelian $C^*$-algebras with $D_1 \subseteq D_2 \subseteq D_3$. Then $(D_3, D_1)$ is an essential inclusion if and only if both $(D_3, D_2)$ and $(D_2, D_1)$ are essential inclusions.

Proof. Suppose $(D_3, D_1)$ is an essential inclusion. That $(D_3, D_2)$ is an essential inclusion follows readily from the definition of essential inclusion. By [16, Corollary 3.22], $(D_3, D_1)$ has the faithful unique pseudo-expectation property, so [16, Proposition 2.6], shows $(D_2, D_1)$ also has the faithful unique pseudo-expectation property. Then $(D_2, D_1)$ is an essential inclusion by [16, Corollary 3.22].

The converse is left to the reader. \hfill $\square$

The following gives the equivalence of parts (b) and (c) of Theorem 5.2.

Proposition 5.5. Suppose $(C, D)$ is a regular inclusion and $(I(D), \iota)$ is an injective envelope for $D$. The following statements hold.

(a) $(C, D)$ has the faithful unique pseudo-expectation property if and only if the the following conditions hold:
   (i) the relative commutant of $D$ in $C$ is abelian; and
   (ii) both $(D^c, D)$ and $(C, D^c)$ are essential inclusions.

(b) $(C, D)$ is a Cartan inclusion if and only if there is a faithful conditional expectation $E : C \to D$ and $\iota \circ E$ is the only pseudo-expectation for $(C, D)$.

Remark 5.6. An inclusion with a unique conditional expectation need not have the unique pseudo-expectation property; an example of this behavior is given by Zarikian in [21].

Proof. (a) Suppose $(C, D)$ has a unique pseudo expectation $E : C \to I(D)$ which is faithful. By [16, Corollary 3.14], $D^c$ is abelian, and [16, Proposition 2.6] shows that $E|_{D^c}$ is the unique pseudo-expectation for $(D^c, D)$. Then [16, Corollary 3.22] shows that $(D^c, D)$ is essential and $E|_{D^c}$ is a $*$-monomorphism which is the unique pseudo-expectation for $(D^c, D)$. As $(I(D), \iota(D))$ is an essential inclusion and $\iota(D) \subseteq E(D^c) \subseteq I(D)$, Lemma 5.4 shows the inclusion $(I(D), E(D^c))$ is also essential. It follows from the “Moreover” portion of [16, Theorem 2.16] that $(I(D), E(D^c))$ is an injective envelope for $D^c$.

By [15, Lemma 2.10], the identity mapping on $C$ is a regular $*$-monomorphism of $(C, D)$ into $(C, D^c)$, whence $(C, D^c)$ is a regular MASA inclusion. Note that $E$ is a pseudo-expectation for $(C, D^c)$ (relative to $(I(D), E|_{D^c})$). By [15, Theorem 3.5 and Theorem 3.15], $L(C, D^c)$ is the left kernel of $E$ and is the unique ideal of $C$ maximal with respect to having trivial intersection with $D^c$. Since $E$ is faithful, $L(C, D^c) = (0)$, and it follows that $(C, D^c)$ is an essential inclusion.

For the converse, suppose $(C, D)$ is a regular inclusion satisfying conditions (i) and (ii). By [15, Corollary 3.7], $(C, D)$ has a unique pseudo-expectation $E : C \to I(D)$. As $(D^c, D)$ is an essential inclusion, [16, Corollary 3.22] shows $E|_{D^c}$ is multiplicative and is the unique pseudo-expectation for $(D^c, D)$. It follows that $(I(D), E|_{D^c})$ is an injective envelope for $D^c$, whence $E$ is also the pseudo-expectation for $(C, D^c)$. As $(C, D^c)$ is essential and $L(C, D^c) \cap D^c = (0)$, we obtain $L(C, D^c) = (0)$. Thus, $E$ is faithful.

(b) If $(C, D)$ is a Cartan inclusion, then by definition, it is a regular MASA inclusion with a faithful conditional expectation $E : C \to D$. By [15, Theorem 3.5], $\iota \circ E$ is the unique pseudo-expectation for $(C, D)$.

Suppose now that $(C, D)$ has a faithful conditional expectation $E : C \to D$ and $\iota \circ E$ is the only pseudo-expectation. As $\iota \circ E$ is faithful, part (a) shows that $D^c$ is abelian and $(D^c, D)$ is an essential extension. An application of [16, Corollary 3.22] shows that there exists a unique and faithful pseudo-expectation $u : D^c \to I(D)$ which is multiplicative. Then $\iota \circ E|_{D^c} = u$, so $E|_{D^c}$ is a homomorphism of $D^c$ onto $D$. As $E$ is faithful, $E|_{D^c}$ is an isomorphism, whence $D^c = D$. So $(C, D)$ is a regular MASA inclusion with a faithful conditional expectation, that is, $(C, D)$ is a Cartan inclusion.
Next we prove the implication (a)⇒(b) of Theorem 5.2.

**Proposition 5.7.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion and \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a Cartan envelope for \((\mathcal{C}, \mathcal{D})\). Then \((\mathcal{C}, \mathcal{D})\) has the faithful unique pseudo-expectation property.

**Proof.** By [15, Proposition 5.3(ii)], the relative commutant, \(\mathcal{D}^c\), of \(\mathcal{D}\) in \(\mathcal{C}\) is abelian and \(\alpha(\mathcal{D}^c) \subseteq \mathcal{D}_1\). Since \((\mathcal{D}_1, \alpha|_{\mathcal{D}})\) is an essential extension of \(\mathcal{D}\), Lemma 5.4 implies \((\alpha(\mathcal{D}^c), \alpha|_{\mathcal{D}})\) is also an essential extension.

Let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\) and use \(E_1\) to denote the (faithful) conditional expectation of \(\mathcal{C}_1\) onto \(\mathcal{D}_1\). As \(\alpha\) is one-to-one, an application of [15, Corollary 3.2] shows that \((\mathcal{C}, \mathcal{D})\) has a unique pseudo-expectation \(E\). Injectivity shows there is a \(*\)-homomorphism \(u : \mathcal{D}_1 \to I(\mathcal{D})\) such that \(\iota = u \circ \alpha|_{\mathcal{D}}\) (see diagram 5.10). Since \((\mathcal{D}_1, \alpha|_{\mathcal{D}})\) is an essential extension of \(\mathcal{D}\), \(u\) is faithful. As \(u \circ E_1 \circ \alpha : \mathcal{C} \to I(\mathcal{D})\) satisfies \(\iota = u \circ E_1 \circ \alpha|_{\mathcal{D}}\), it is a pseudo-expectation. Then \(E = u \circ E_1 \circ \alpha\) is a composition of faithful completely positive maps. Thus \(E\) is faithful.

The converse of Proposition 5.7 will require more effort. To begin, we observe that the faithful unique pseudo-expectation property implies the existence of Cartan extensions.

**Lemma 5.8.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the faithful unique pseudo-expectation property. Then \((\mathcal{C}, \mathcal{D})\) has a Cartan extension.

**Proof.** Proposition 5.5 implies \((\mathcal{C}, \mathcal{D}^c)\) is a regular MASA inclusion and \((\mathcal{D}^c, \mathcal{D})\) is an essential inclusion. Let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\) and let \(E : \mathcal{C} \to I(\mathcal{D})\) be the pseudo-expectation. As \((\mathcal{D}^c, \mathcal{D})\) is essential, \(E|_{\mathcal{D}^c}\) is a faithful \(*\)-monomorphism of \(\mathcal{D}^c\) into \(I(\mathcal{D})\) (see [16, Corollary 3.22]). In particular, \((I(\mathcal{D}), E|_{\mathcal{D}^c})\) is an injective envelope for \(\mathcal{D}^c\). It follows that \(E\) is also the unique pseudo-expectation for \((\mathcal{C}, \mathcal{D}^c)\) (relative to \((I(\mathcal{D}), E|_{\mathcal{D}^c})\)). Since \(E\) is faithful, the ideal \(\mathcal{L}(\mathcal{C}, \mathcal{D}^c)\) is trivial. As \(\text{Rad}(\mathcal{C}, \mathcal{D}^c) \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D}^c)\), [15, Theorem 5.7] shows \((\mathcal{C}, \mathcal{D}^c)\) regularly embeds into a \(C^*\)-diagonal. By [16, Lemma 2.10], the identity map on \(\mathcal{C}\) is regular when viewed as a map of \((\mathcal{C}, \mathcal{D})\) into \((\mathcal{C}, \mathcal{D}^c)\). As the composition of regular maps is again regular, we conclude that \((\mathcal{C}, \mathcal{D})\) regularly embeds into a \(C^*\)-diagonal. But every \(C^*\)-diagonal is a Cartan inclusion, so we are done.

**Lemma 5.9.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the unique pseudo-expectation property, let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\), and let \(E : \mathcal{C} \to I(\mathcal{D})\) be the pseudo-expectation. Suppose \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a package. Then there exists a unique \(*\)-homomorphism \(u : \mathcal{D}_1 \to I(\mathcal{D})\) such that \(u \circ \alpha|_{\mathcal{D}} = \iota\).

**Proof.** Let \(E_1 : \mathcal{C}_1 \to \mathcal{D}_1\) be a faithful conditional expectation such that \(\mathcal{D}_1 = C^*(E_1(\alpha(\mathcal{C})))\) and \(\mathcal{C}_1\) is generated by \(\alpha(\mathcal{C})\) and \(\mathcal{D}_1\). Injectivity gives the existence of a \(*\)-homomorphism \(u : \mathcal{D}_1 \to I(\mathcal{D})\) such that \(u \circ \alpha|_{\mathcal{D}} = \iota\). We thus have the following diagram (the vertical upward-pointing arrows are the inclusion maps).

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\alpha} & \mathcal{C}_1 \\
\downarrow E & & \downarrow u \\
\subseteq & & \subseteq E_1 \\
\mathcal{D} & \xleftarrow{\iota} & \mathcal{D}_1 \\
\downarrow \alpha|_{\mathcal{D}} & & \downarrow \\
\end{array}
\]

Observe that \(u \circ E_1 \circ \alpha\) is a pseudo-expectation for \((\mathcal{C}, \mathcal{D})\) because \(u \circ \alpha|_{\mathcal{D}} = \iota\). The uniqueness hypothesis on \(E\) then yields,

\[
E = u \circ E_1 \circ \alpha.
\]
Proposition 5.12. Suppose \((\mathcal{E}, \mathcal{D})\) is a regular inclusion with the unique pseudo-expectation property and suppose \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is a package. The following are equivalent.

(a) \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is an envelope.
(b) \((\mathcal{D}_1, \alpha|_{\mathcal{D}})\) is an essential extension of \(\mathcal{D}\).
(c) \((\mathcal{E}, \mathcal{D})\) has the faithful unique pseudo-expectation property.
(d) \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is a Cartan envelope.

Proof. Let \((I(\mathcal{D}), \iota)\) be an injective envelope for \((\mathcal{E}, \mathcal{D})\), let \(E : \mathcal{E} \to \mathcal{D}\) be the pseudo-expectation and let \(u : \mathcal{D}_1 \to I(\mathcal{D})\) be the *-homomorphism obtained from Lemma 5.9). Also, let \(E_1 : \mathcal{E}_1 \to \mathcal{D}_1\) be a faithful conditional expectation. Diagram (5.10) and equation (5.11) show \(E\) is faithful if and only if \(u\) is faithful.

The equivalence of (a) and (b) is the definition of envelope.

Assume (b) holds. Then \(u\) is faithful, so \(E\) is faithful, which is (c).

Next suppose \(E\) is faithful. Then \(u\) is faithful, so Lemma 5.4 implies that \((I(\mathcal{D}), u)\) is an essential extension of \(\mathcal{D}_1\). Therefore, \((I(\mathcal{D}), u)\) is an injective envelope for \(\mathcal{D}_1\). Let \(\Delta : \mathcal{E}_1 \to I(\mathcal{D})\) be a pseudo-expectation for \((\mathcal{E}_1, \mathcal{D}_1)\) relative to \((I(\mathcal{D}), u)\). Then

\[
\Delta|_{\mathcal{D}_1} = u = (u \circ E_1)|_{\mathcal{D}_1}.
\]

Note that \((\Delta \circ \alpha)|_{\mathcal{D}} = (u \circ \alpha)|_{\mathcal{D}} = \iota\), so \(\Delta \circ \alpha\) is a pseudo-expectation for \((\mathcal{E}, \mathcal{D})\). Since \((\mathcal{E}, \mathcal{D})\) has the unique pseudo-expectation property,

\[
\Delta \circ \alpha = E = u \circ E_1 \circ \alpha.
\]

By Choi’s Lemma (see [13, Corollary 3.19]), both \(\Delta\) and \(u \circ E_1\) are \(\mathcal{D}_1\)-bimodule maps. As \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is a package, we conclude

\[
\Delta = u \circ E_1.
\]

Lemma 5.5(b) shows \((\mathcal{E}_1, \mathcal{D}_1)\) is a Cartan envelope. Thus \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is a Cartan envelope.

As every Cartan envelope is an envelope, the proof is complete. \(\square\)

Next we construct a package from a Cartan extension.

Lemma 5.13. Suppose \((\mathcal{E}, \mathcal{D})\) is a regular inclusion. If \((\mathcal{E}, \mathcal{D})\) has a Cartan extension, then it has a package. More specifically, if \((\mathcal{E}_1, \mathcal{D}_1, \alpha)\) is a Cartan extension for \((\mathcal{E}, \mathcal{D})\) with conditional expectation \(E_1 : \mathcal{E}_1 \to \mathcal{D}_1\), put

\[
\mathcal{E}_\alpha := C^*(\alpha(\mathcal{E})) \cup E_1(\alpha(\mathcal{E})) \quad \text{and} \quad \mathcal{D}_\alpha := C^*(E_1(\alpha(\mathcal{E}))).
\]

Then \((\mathcal{E}_\alpha, \mathcal{D}_\alpha, \alpha)\) is a package for \((\mathcal{E}, \mathcal{D})\).

Proof. Thinking of \(\mathcal{D}_1\) as a subalgebra of \(I(\mathcal{D}_1)\), we may view \(E_1\) as the pseudo-expectation for \((\mathcal{E}_1, \mathcal{D}_1)\). Then [15 Proposition 3.14] implies that for any \(w \in N(\mathcal{E}_1, \mathcal{D}_1)\) and \(z \in \mathcal{E}_1\),

\[
w^*E_1(z)w = E_1(w^*z)w.
\]

Fix \(v \in N(\mathcal{E}, \mathcal{D})\). We claim that for \(n \in \mathbb{N}\) and any \(x_1, \ldots, x_n \in \mathcal{E}\),

\[
\alpha(v) \left( \prod_{j=1}^{n} E_1(\alpha(x_j)) \right) \alpha(v^*) \in \mathcal{D}_\alpha. \quad (5.14)
\]

When \(n = 1\), as \(\alpha(v) \in N(\mathcal{E}_1, \mathcal{D}_1)\), we find

\[
\alpha(v)E_1(\alpha(x_1))\alpha(v)^* = E_1(\alpha(vx_1v^*)) \in \mathcal{D}_\alpha.
\]
Now suppose \([5.14]\) holds for some \(n\), let \(\{x_j\}_{j=1}^{n+1} \subseteq \mathcal{C}\) and put

\[
y = \prod_{j=1}^{n} E_1(\alpha(x_j)).
\]

For \(d \in \mathcal{D}\),

\[
\alpha(v)\alpha(v^* d v) y E_1(\alpha(x_{n+1})) \alpha(v)^* = \alpha(v) y \alpha(v)^* \alpha(d) \alpha(v) E_1(\alpha(x_{n+1})) \alpha(v)^* \in \mathcal{D}_\alpha.
\]

Noting that \(\overline{v^* D v} = \overline{v^* v D}\), continuity shows that for any \(h \in \overline{v^* v D}\),

\[
\alpha(v) \alpha(h) y E_1(\alpha(x_{n+1})) \alpha(v)^* \in \mathcal{D}_\alpha.
\]

Replacing \(h\) with \((\alpha(v)^*)^{1/n}\) and then taking the limit as \(n \to \infty\) yields

\[
\alpha(v) y E_1(\alpha(x_{n+1})) \alpha(v)^* \in \mathcal{D}_\alpha. \quad (5.15)
\]

Induction completes the proof of \([5.14]\). Since \(\mathcal{D}_\alpha\) is generated by \(E_1(\alpha(\mathcal{C}))\), \([5.14]\) yields \(\alpha(v) \in N(\mathcal{C}_\alpha, \mathcal{D}_\alpha)\). It follows that \(\alpha(N(\mathcal{C}, \mathcal{D}) \subseteq N(\mathcal{C}_\alpha, \mathcal{D}_\alpha)\).

Since \(\alpha(N(\mathcal{C}, \mathcal{D})) \cup \mathcal{D}_\alpha\) generates \(\mathcal{C}_\alpha\), we conclude that \((\mathcal{C}_\alpha, \mathcal{D}_\alpha)\) is a regular inclusion and \(\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_\alpha, \mathcal{D}_\alpha)\) is a regular homomorphism. As \(E_1|_{\mathcal{C}_\alpha}\) is a faithful conditional expectation, \((\mathcal{C}_\alpha, \mathcal{D}_\alpha, \alpha)\) is a package for \((\mathcal{C}, \mathcal{D})\).

\[\square\]

**Remark 5.16.** It would simply our arguments if we could show that \((\mathcal{C}_\alpha, \mathcal{D}_\alpha)\) is a MASA inclusion, for it would then follow that \((\mathcal{C}_\alpha, \mathcal{D}_\alpha, \alpha)\) is a Cartan package. We have been unable to do this is because we do not know whether \(I(\mathcal{D})\) and \(I(\mathcal{D}_\alpha)\) agree, so we do not know that \(u \circ E_\alpha\) is the unique faithful pseudo-expectation for \((\mathcal{C}_\alpha, \mathcal{D}_\alpha)\). Thus we cannot apply Proposition \(5.5\).

**Proposition 5.17.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the faithful unique pseudo-expectation property. Then \((\mathcal{C}, \mathcal{D})\) has a Cartan envelope.

**Proof.** Lemma \([5.8]\) shows the existence of a Cartan extension \((\mathcal{C}_2, \mathcal{D}_2, \alpha)\) for \((\mathcal{C}, \mathcal{D})\). Let \(\mathcal{E}_2 : \mathcal{C}_2 \to \mathcal{D}_2\) be the conditional expectation. Put

\[
\mathcal{D}_1 := C^*(\mathcal{E}_2(\alpha(\mathcal{C}))), \quad \mathcal{C}_1 := C^*(\alpha(\mathcal{C}), \mathcal{D}_1), \quad \text{and} \quad \mathcal{E}_1 := \mathcal{E}_2|_{\mathcal{C}_1}.
\]

Then \(\mathcal{E}_1 : \mathcal{C}_1 \to \mathcal{D}_1\) is a faithful conditional expectation and Lemma \([5.13]\) shows \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a package for \((\mathcal{C}, \mathcal{D})\). Furthermore, the regularity of \(\alpha\) and the definition of \(\mathcal{E}_1\) show that for any \(x \in \mathcal{C}_1\) and \(v \in N(\mathcal{C}, \mathcal{D})\),

\[
\alpha(v) \mathcal{E}_1(x) \alpha(v)^* = \mathcal{E}_1(\alpha(v) x \alpha(v)^*). \quad (5.18)
\]

Let \(u : \mathcal{D}_1 \to I(\mathcal{D})\) be the (unique) \(*\)-homomorphism with \(\iota = u \circ \alpha|_\mathcal{D}\) obtained from Lemma \([5.9]\).

Set

\[
\mathcal{J} := \{x \in \mathcal{C}_1 : \mathcal{E}_1(x^* x) \in \ker u\}.
\]

We shall show that the following statements hold.

(a) \(\mathcal{J}\) is a closed, two-sided ideal of \(\mathcal{C}_1\) such that \(\alpha(\mathcal{C}) \cap \mathcal{J} = (0)\) and \(\mathcal{D}_1 \cap \mathcal{J} = \ker u\).

(b) If \(\tilde{\alpha} : \mathcal{C} \to \mathcal{C}_1/\mathcal{J}\) is the map \(x \mapsto \alpha(x) + \mathcal{J}\), then \((\mathcal{C}_1/\mathcal{J}, \mathcal{D}_1/\ker u, \tilde{\alpha})\) is a Cartan envelope for \((\mathcal{C}, \mathcal{D})\).

To this end, first note that \(\mathcal{J}\) is a closed left ideal of \(\mathcal{C}_1\): indeed, for \(x \in \mathcal{J}\) and \(y \in \mathcal{C}_1\), \(\mathcal{E}_1(x^* y^* y x) \leq \|y\|^2 \mathcal{E}_1(x^* x) = 0\). To show that \(\mathcal{J}\) is a right ideal, we will require the following invariance property of \(\ker u\): for every \(v \in N(\mathcal{C}, \mathcal{D})\),

\[
\alpha(v)^* (\ker u) \alpha(v) \subseteq \ker u. \quad (5.19)
\]

For this, we first establish some properties of \(\ker u\).

Consider the inclusion \((\mathcal{D}_1, \alpha(\mathcal{D}))\). The uniqueness of \(u\) allows us to apply \([16, \text{Corollary 3.21}]\) to conclude that \(\ker u\) is the unique ideal of \(\mathcal{D}_1\) maximal with respect to having trivial intersection with \(\alpha(\mathcal{D})\).
Let \( S := \{ \sigma_1 \in \hat{D}_1 : \sigma_1 \text{ annihilates } \ker u \} \). Then \( S \) is closed and
\[
\ker u = \{ h \in D_1 : \hat{h} \text{ vanishes on } S \}. \tag{5.20}
\]

Using \([16, \text{Corollary 3.21}]\) again, \( S \) is the unique minimal closed subset of \( \hat{D}_1 \) such that \( \alpha^*(S) = \hat{D} \).

Since \( \hat{v}^* = \alpha^* \circ u^* \), it follows that \( \alpha^*(u^*(I(\hat{D}))) = \hat{D} \), so \( S \subseteq u^*(I(\hat{D})) \). On the other hand, if \( \sigma_1 = \rho \circ u \) for some \( \rho \in I(\hat{D}) \), then \( \sigma_1 \) annihilates \( \ker u \), whence \( S \supseteq u^*(I(\hat{D})) \). Thus,
\[
S = u^*(I(\hat{D})). \tag{5.21}
\]

Next, fix \( v \in N(C, D) \) and suppose \( \sigma_1 \in S \) satisfies \( \sigma_1(\alpha(v^*v)) \neq 0 \). We will show
\[
\beta_{\alpha(v)}(\sigma_1) \in S. \tag{5.22}
\]

Recall from \([15, \text{Proposition 1.11}]\) that the partial automorphism \( \theta_v \) of \( D \) (see \([15, \text{Lemma 2.1}]\)) extends uniquely to a partial automorphism \( \theta_v \) of \( I(\hat{D}) \) such that \( \theta_v \circ t = t \circ \theta_v \). By \((5.21)\), there exists \( \rho \in I(\hat{D}) \) such that
\[
\sigma_1 = \rho \circ u.
\]

Since \( \theta_v(vv^*) = v^*v \),
\[
(\rho \circ \hat{\theta})(\iota(vv^*)) = \rho(\iota(v^*v)) = \rho(u(\alpha(v^*v))) = \sigma_1(\alpha(v^*v)) \neq 0.
\]
Thus, setting \( \rho' := \rho \circ \theta_v \) we find \( \rho' \in I(\hat{D}) \). Let \( \sigma'_1 = \rho' \circ u \in \hat{D}_1 \). By construction, \( \sigma'_1 \in S \). We shall show that
\[
\sigma'_1 = \beta_{\alpha(v)}(\sigma_1). \tag{5.23}
\]

For \( x \in C \),
\[
\beta_{\alpha(v)}(\sigma_1)(E_1(\alpha(x))) = \frac{\sigma_1(\alpha(v^*xv))}{\sigma_1(\alpha(v^*v))} = \frac{\rho(\iota(xv))}{\rho(\iota(v^*v))},
\]
and applying \([15, \text{Proposition 3.14}]\) to the numerator,
\[
= \frac{\rho(\hat{\theta})(E(vv^*)x)}{\rho'(\iota(vv^*)x)} = \frac{\rho'(\iota(vv^*)x)}{\rho'(\iota(vv^*))} = \rho'(E(x)),
\]
As \( \beta_{\alpha(v)}(\sigma_1) \) and \( \sigma'_1 \) belong to \( \hat{D}_1 \) and \( D_1 \) is generated by \( E_1(\alpha(C)) \), \((5.23)\) holds. This establishes \((5.22)\).

We are now prepared to establish \((5.19)\). Choose \( h \in \ker u \). When \( \sigma_1 \in S \) satisfies \( \sigma_1(\alpha(v^*v)) \neq 0 \),
\[
\sigma_1(\alpha(v^*h\alpha(v)) = \beta_{\alpha(v)}(\sigma_1)(h) \sigma_1(\alpha(v^*v)) = \sigma'(h) \sigma_1(\alpha(v^*v)) = 0.
\]

On the other hand, when \( \sigma_1(\alpha(v^*v)) = 0 \), \( \sigma_1(\alpha(v^*h\alpha(v))) = 0 \) because
\[
|\sigma_1(\alpha(v^*h\alpha(v)))|^2 = \sigma_1(\alpha(v^*h\alpha(v^*v)h\alpha(v))) \leq \|\alpha(v^*h\alpha(v^*v))\|^2 \sigma_1(\alpha(v^*v)) = 0.
\]

We conclude that whenever \( \sigma_1 \in S \), \( \sigma_1(\alpha(v^*h\alpha(v))) = 0 \). By \((5.20)\), \( \alpha(v^*h\alpha(v)) \in \ker u \), so \((5.19)\) holds.

Next we show \( \mathcal{J} \) is a right ideal. If \( x \in \mathcal{J} \), \( v \in N(C, D) \), and \( h \in D_1 \), then \( x\alpha(v)h \in \mathcal{J} \) because
\[
E_1(h^*\alpha(v^*x^*\alpha(v)h) = h^*\alpha(v^*)E_1(x^*x)\alpha(v)h \in \ker u.
\]
As \( \alpha(N(C, D)) \subseteq N(C_1, D_1) \), \( \{ \alpha(v)h : v \in N(C, D), h \in D_1 \} \) is a *-semigroup whose span is dense in \( C_1 \). It follows that \( \mathcal{J} \) is a right ideal. So \( \mathcal{J} \) is a closed, two-sided ideal in \( C_1 \).
If \( y \in \alpha(\mathcal{C}) \cap \mathfrak{J} \), then there is some \( x \in \mathcal{C} \) so that \( y = \alpha(x) \). Then \( 0 = u(\mathbb{E}_1(\alpha(x^*x))) = E(x^*x) \), so \( x = 0 \) because the pseudo-expectation for \((\mathcal{C}, \mathcal{D})\) is faithful. Thus \( \alpha(\mathcal{C}) \cap \mathfrak{J} = \{0\} \). The fact that \( \mathfrak{J} \) is an essential extension of \( \mathcal{D}_1 \) is clear. This completes the proof of assertion (a).

We turn now to assertion (b). Let
\[
\tilde{\mathcal{C}}_1 := \mathcal{C}_1/\mathfrak{J}, \quad \tilde{\mathcal{D}}_1 := \mathcal{D}_1/\ker u, \quad \text{and} \quad \tilde{u} : \tilde{\mathcal{D}}_1 \to I(\tilde{\mathcal{D}})
\]
be the map \( \tilde{\mathcal{D}}_1 \ni h + \ker u \mapsto u(h) \). Then if \( \mathfrak{A} \) is the \( C^* \)-subalgebra of \( I(\tilde{\mathcal{D}}) \) generated by \( E(\mathcal{C}) \), \( \tilde{u} \) is a \( * \)-isomorphism of \( \tilde{\mathcal{D}}_1 \) onto \( \mathfrak{A} \). Since \( \iota = \tilde{u} \circ \tilde{\alpha}|_{\tilde{\mathcal{D}}_1} \), \( (\tilde{\mathcal{D}}_1, \tilde{\alpha}|_{\tilde{\mathcal{D}}_1}) \) is an essential extension of \( \mathcal{D} \) and \((I(\tilde{\mathcal{D}}), \tilde{u})\) is an injective envelope for \( \tilde{\mathcal{D}}_1 \).

Clearly, \( \tilde{\alpha} \) is a regular \( * \)-monomorphism of \((\tilde{\mathcal{C}}, \tilde{\mathcal{D}})\) into \((\tilde{\mathcal{C}}_1, \tilde{\mathcal{D}}_1)\). If \( x \in \mathfrak{J} \), then the operator inequality \( \mathbb{E}_1(x^*x) \leq \mathbb{E}_1(x^*x) \), shows that \( \mathbb{E}_1(\mathfrak{J}) \subseteq \mathfrak{J} \). Thus the map \( \tilde{\mathbb{E}}_1 : \tilde{\mathcal{C}}_1 \to \tilde{\mathcal{D}}_1 \) given by \( x + \mathfrak{J} \mapsto \mathbb{E}_1(x^*x) + \ker u \) is well-defined and is a conditional expectation. If \( \mathbb{E}_1(x^*x + \mathfrak{J}) = 0 \), then \( \mathbb{E}_1(x^*x) \in \ker u \), so \( x \in \mathfrak{J} \). It follows that \( \tilde{\mathbb{E}}_1 \) is faithful. Note also that \( \tilde{u} \circ \tilde{\mathbb{E}}_1 \) is then a faithful pseudo-expectation for \((\tilde{\mathcal{C}}, \tilde{\mathcal{D}}_1)\).

Suppose \( \Delta : \tilde{\mathcal{C}}_1 \to I(\tilde{\mathcal{D}}) \) satisfies \( \Delta|_{\tilde{\mathcal{D}}_1} = \tilde{u} \). Then \( \Delta \circ \tilde{\alpha} = \iota \), so \( \Delta \circ \tilde{\alpha} \) is a pseudo-expectation for \((\mathcal{C}, \mathcal{D})\). As \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property,
\[
\Delta \circ \tilde{\alpha} = E = \tilde{u} \circ \tilde{\mathbb{E}}_1 \circ \tilde{\alpha}.
\]
Since \( \Delta|_{\tilde{\mathcal{D}}_1} = (\tilde{u} \circ \tilde{\mathbb{E}}_1)|_{\tilde{\mathcal{D}}_1} \), and \( \tilde{\mathcal{C}}_1 \) is generated by \( \tilde{\mathcal{D}}_1 \cup \tilde{\alpha}(\mathcal{N}(\mathcal{C}, \mathcal{D})) \), \( \Delta = \tilde{u} \circ \tilde{\mathbb{E}}_1 \). Thus \( \tilde{u} \circ \tilde{\mathbb{E}}_1 \) is the unique pseudo-expectation for \((\tilde{\mathcal{C}}_1, \tilde{\mathcal{D}}_1)\). By Proposition 5.23(b), \((\tilde{\mathcal{C}}_1, \tilde{\mathcal{D}}_1)\) is a Cartan inclusion.

Finally, as \( \tilde{\mathcal{C}}_1 \) is regular, it is generated by \( \tilde{\alpha}(\mathcal{C}) \cup \tilde{\mathcal{D}}_1 \), and by construction, \( \tilde{\mathcal{D}}_1 \) is generated by \( \tilde{\mathbb{E}}_1(\tilde{\alpha}(\mathcal{C})) \). Thus \((\tilde{\mathcal{C}}_1, \tilde{\mathcal{D}}_1, \tilde{\alpha})\) is a Cartan package for \((\mathcal{C}, \mathcal{D})\). As \( \tilde{u} \) is a faithful \( * \)-homomorphism of \( \tilde{\mathcal{D}}_1 \) into \( I(\tilde{\mathcal{D}}) \), it follows that \((\tilde{\mathcal{D}}_1, \tilde{\alpha})\) is an essential extension for \( \mathcal{D} \), so the proof of assertion (b), and hence the proposition, is complete.

Next we show uniqueness (up to equivalence) of Cartan envelopes.

**Proposition 5.24.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion, and assume that for \( i = 1, 2 \), \((\mathcal{C}_i, \mathcal{D}_i, \alpha_i)\) are Cartan envelopes for \((\mathcal{C}, \mathcal{D})\). Then there is a unique \( * \)-isomorphism \( \psi : \mathcal{C}_1 \to \mathcal{C}_2 \) such that \( \psi \circ \alpha_1 = \alpha_2 \). Furthermore, if \( \mathbb{E}_i : \mathcal{C}_i \to \mathcal{D}_i \) are the conditional expectations, then for every \( x \in \mathcal{C} \) and \( v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) \),
\[
\psi(\alpha_1(v)\mathbb{E}_1(\alpha_1(x))) = \alpha_2(v)\mathbb{E}_2(\alpha_2(x)).
\]

**Proof.** Let \((I(\mathcal{D}), \iota)\) be an injective envelope for \( \mathcal{D} \). Lemma 5.7 gives a unique (and faithful) pseudo-expectation \( E : \mathcal{C} \to I(\mathcal{D}) \) for \((\mathcal{C}, \mathcal{D})\). For \( i = 1, 2 \), let \( \mathbb{E}_i : \mathcal{C}_i \to \mathcal{D}_i \) be the (necessarily unique) faithful conditional expectation.

By Lemma 5.9 there exist unique \( * \)-homomorphisms \( u_i : \mathcal{D}_i \to I(\mathcal{D}) \) such that \( u_i \circ \alpha_i|^{|\mathcal{D}} = \iota \). By the definition of Cartan envelope, \((\mathcal{D}_1, \alpha_1|^{|\mathcal{D}})\) are essential extensions of \( \mathcal{D}_2 \), so \( u_i \) are actually \( * \)-monomorphisms.

Examining a variant of the diagram (5.10) and using the fact that \( E \) is the unique pseudo-expectation, observe that for every \( x \in \mathcal{C} \),
\[
u_i(\mathbb{E}_i(\alpha_i(x))) = E(x).
\]
Since \( \mathcal{D}_i \) is generated by \( \{\mathbb{E}_i(\alpha_i(x)) : x \in \mathcal{C}_i\} \), we conclude that the range of \( u_i \) is the \( C^* \)-subalgebra \( \mathfrak{A} \subseteq I(\mathcal{D}) \) generated by \( E(\mathcal{C}) \). Thus \( \psi := u_2^{-1} \circ u_1 \) is a \( * \)-isomorphism of \( \mathcal{D}_1 \) onto \( \mathcal{D}_2 \) such that for every \( x \in \mathcal{C} \),
\[
\psi(\mathbb{E}_1(\alpha_1(x))) = \mathbb{E}_2(\alpha_2(x)).
\]
Let \( M_i := \{\alpha_i(v)h : v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), h \in \alpha_i(v^*v)\mathcal{D}_i\} \).
It follows from [15, Lemma 2.1] that $M_i$ is a $*$-semigroup and, as $C_i$ is generated by $\alpha_i(C) \cup \mathcal{E}_i(\alpha_i(C))$, $M_i$ is a MASA skeleton for $(C_i, D_i)$ (see [15, Definitions 1.7 and 3.1]). Let $n \in \mathbb{N}$ and for $1 \leq k \leq n$, suppose $v_k \in N(C, D)$ and $h_k \in \alpha(v_k^*v_k) \mathcal{D}_1$. Then

$$0 = \sum_{k=1}^{n} \alpha_1(v_k)h_k \iff \mathbb{E}_1 \left( \sum_{k, \ell=1}^{n} h_k^* \alpha_1(v_k^*v_\ell)h_\ell \right) = 0$$

$$\iff \psi \left( \sum_{k, \ell=1}^{n} h_k^* \mathbb{E}_1(\alpha_1(v_k^*v_\ell))h_\ell \right) = 0$$

$$\iff \sum_{k, \ell=1}^{n} \psi(h_k^*)\mathbb{E}_2(\alpha_2(v_k^*v_\ell))\psi(h_\ell) = 0$$

$$\iff \mathbb{E}_2 \left( \sum_{k, \ell=1}^{n} \psi(h_k^*)\alpha_2(v_k^*v_\ell)\psi(h_\ell) \right) = 0$$

$$\iff \sum_{k=1}^{n} \alpha_2(v_k)\psi(h_k) = 0.$$

Thus, we may extend $\psi$ uniquely to a linear mapping, again called $\psi$, of span $M_1$ onto span $M_2$ determined by

$$\sum_{k=1}^{n} \alpha_1(v_k)h_k \mapsto \sum_{k=1}^{n} \alpha_2(v_k)\psi(h_k).$$

Since $M_i$ are $*$-semigroups, it follows that span $M_i$ are $*$-algebras and $\psi$ is a $*$-isomorphism of span $M_1$ onto span $M_2$.

Now define two $C^*$-norms on span $M_1$:

$$\nu_1(x) = \|x\|_{C_1} \quad \text{and} \quad \nu_2(x) = \|\psi(x)\|_{C_2}.$$ 

Since $(C_1, D_1)$ is a Cartan pair, $\mathcal{L}(C_1, D_1) = (0)$, so [15, Theorem 7.4] implies that $\nu_1$ is the minimal $C^*$-norm on span $M_1$. Thus, for every $x \in \text{span } M_1$, $\|x\|_{C_1} \leq \|\psi(x)\|_{C_2}$. A symmetric argument yields $\|\psi(x)\|_{C_2} \leq \|x\|_{C_1}$, so $\psi$ is an isometric $*$-isomorphism of span $M_1$ onto span $M_2$. Therefore, $\psi$ extends to a $*$-isomorphism (once again called $\psi$) of $C_1$ onto $C_2$. As $\psi(D_1) = D_2$, $\psi$ is regular, and by construction, $\psi \circ \alpha_1 = \alpha_2$. Thus $(C_1, D_1, \alpha_1)$ is equivalent to $(C_2, D_2, \alpha_2)$ via a $*$-isomorphism satisfying (5.25).

Turning to the uniqueness statement, suppose $\psi' : (C_1, D_1) \to (C_2, D_2)$ is a regular $*$-isomorphism such that $\psi' \circ \alpha_1 = \alpha_2$. The regularity of $\psi'$ yields $\psi'(D_1) \subseteq D_2$. We now show equality. If $y \in C_2$ commutes with $\psi'(D_1)$, then $\psi'^{-1}(y)$ commutes with $D_1$, whence $\psi'^{-1}(y) \in D_1$. Thus, $y \in \psi'(D_1)$, so $\psi'(D_1)$ is a MASA in $C_2$. This gives $\psi'(D_1) = D_2$.

Note that $\psi'^{-1} \circ \mathbb{E}_2 \circ \psi'$ is a faithful conditional expectation of $C_1$ onto $D_1$. By uniqueness of $\mathbb{E}_1$, we obtain $\mathbb{E}_2 \circ \psi' = \psi' \circ \mathbb{E}_1$. As this relation also holds for $\psi$,

$$\psi \circ \mathbb{E}_1 \circ \alpha_1 = \mathbb{E}_2 \circ \psi \circ \alpha_1 = \mathbb{E}_2 \circ \alpha_2 = \mathbb{E}_2 \circ \psi' \circ \alpha_1 = \psi' \circ \mathbb{E}_1 \circ \alpha_1.$$

Therefore, $\psi|_{D_1} = \psi'|_{D_1}$, because $D_1$ is generated by $\mathbb{E}_1(\alpha_1(C))$. Since $C_1$ is the closed $D_1$-bimodule generated by $\alpha_1(C)$, we obtain $\psi' = \psi$, completing the proof.

We have now established most of Theorem 5.2 and we now finish its proof.

**Proof of Theorem 5.2** Proposition 5.5 gives (b)$\iff$(c) and (a)$\iff$(b) is Proposition 5.7 combined with Proposition 5.17. Uniqueness of the Cartan envelope is Proposition 5.24.
Finally, suppose \((\mathcal{C}, \mathcal{D})\) has the faithful unique pseudo-expectation property and \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a Cartan package for \((\mathcal{C}, \mathcal{D})\). The proof of Proposition 5.17 establishes the existence of an ideal \(\mathfrak{I}\) in \(\mathcal{C}_1\) with the requisite properties. \(\square\)

Suppose \((\mathcal{C}, \mathcal{D})\) has the unique faithful unique pseudo-expectation property. Then \((\mathcal{C}, \mathcal{D}^c)\) is a virtual Cartan inclusion. Our next goal is Proposition 5.29 below, which shows that \((\mathcal{C}, \mathcal{D})\) and \((\mathcal{C}, \mathcal{D}^c)\) have the same Cartan envelope.

**Lemma 5.26.** Suppose \((\mathcal{C}, \mathcal{D})\) is an inclusion with the faithful unique pseudo-expectation property, let \(\mathcal{D}^c\) be the relative commutant of \(\mathcal{D}\) in \(\mathcal{C}\), and let \(\mathcal{D}_0 \subseteq \mathcal{D}\) be a \(\mathcal{C}^*\)-subalgebra such that \((\mathcal{D}, \mathcal{D}_0)\) is an essential inclusion. If \(\mathcal{C}_0\) is a \(\mathcal{C}^*\)-subalgebra of \(\mathcal{C}\) containing \(\mathcal{D}_0\), then the following statements hold.

(a) The inclusions \((\mathcal{C}, \mathcal{D}_0)\) and \((\mathcal{C}_0, \mathcal{D}_0)\) both have the faithful unique pseudo-expectation property.

(b) The inclusion mapping is a regular homomorphism of \((\mathcal{C}_0, \mathcal{D}_0)\) into \((\mathcal{C}, \mathcal{D}^c)\); in other words, \(\mathcal{N}(\mathcal{C}_0, \mathcal{D}_0) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}^c)\).

**Proof.** We begin with some preliminary observations. Let \((I(\mathcal{D}_0), \iota_0)\) be an injective envelope for \(\mathcal{D}_0\). By [16, Corollary 3.22], there exists a unique pseudo-expectation \(\iota : \mathcal{D} \to I(\mathcal{D}_0)\) for \((\mathcal{D}, \mathcal{D}_0)\); furthermore, \(\iota\) is multiplicative and faithful. We claim that \((I(\mathcal{D}_0), \iota)\) is an injective envelope for \(\mathcal{D}\). As \(\iota|_{\mathcal{D}_0} = \iota_0\),

\[
I(\mathcal{D}_0) \supseteq \iota(\mathcal{D}) \supseteq \iota_0(\mathcal{D}_0).
\]

Lemma 5.4 implies \((I(\mathcal{D}_0), \iota)\) is an essential extension of \(\mathcal{D}\). Thus as \(I(\mathcal{D}_0)\) is injective, \((I(\mathcal{D}_0), \iota)\) is an injective envelope for \(\mathcal{D}\). Let \(E : \mathcal{C} \to I(\mathcal{D}_0)\) be the pseudo-expectation for \((\mathcal{C}, \mathcal{D})\) with respect to \((I(\mathcal{D}_0), \iota)\).

a) Let \(\Delta : \mathcal{C} \to I(\mathcal{D}_0)\) be a pseudo-expectation for \((\mathcal{C}, \mathcal{D}_0)\) relative to \((I(\mathcal{D}_0), \iota_0)\). Then \(\Delta|_{\mathcal{D}}\) is a pseudo-expectation for \((\mathcal{D}, \mathcal{D}_0)\), so \(\Delta|_{\mathcal{D}} = \iota\). Therefore, \(\Delta\) is a pseudo-expectation for \((\mathcal{C}, \mathcal{D})\) relative to \((I(\mathcal{D}_0), \iota)\). Since \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property, \(\Delta = E\). As \(E\) is faithful, \((\mathcal{C}, \mathcal{D}_0)\) has the faithful unique pseudo-expectation property. That \((\mathcal{C}_0, \mathcal{D}_0)\) also has the faithful unique pseudo-expectation property follows from [16, Proposition 2.6].

b) Let \(\mathcal{D}_0^c\) be the relative commutant of \(\mathcal{D}_0\) in \(\mathcal{C}\). By [16, Corollary 3.14], \(\mathcal{D}^c\) and \(\mathcal{D}_0^c\) are abelian. Therefore, \((\mathcal{C}, \mathcal{D}^c)\) is a MASA inclusion. Since \(\mathcal{D}^c \subseteq \mathcal{D}_0^c\), we obtain

\[
\mathcal{D}^c = \mathcal{D}_0^c.
\]  (5.27)

Now suppose that \(v \in \mathcal{N}(\mathcal{C}_0, \mathcal{D}_0)\) and let \(h \in \mathcal{D}^c\). Let \(\theta^*\) be the \(*\)-isomorphism of \(v^*v\mathcal{D}_0^c\) onto \(v^*v\mathcal{D}_0^c\) determined by \(v^*vdv^* \mapsto v^*vdv^*\). Then for every \(k \in v^*v\mathcal{D}_0\), \(vk = \theta^*(k)v\). Let \(d \in \mathcal{D}_0\). For every \(u \in v^*v\mathcal{D}_0^c\),

\[
v^*hvud = v^*h\theta^*(du)v = v^*\theta^*(du)hv = dv^*hv.
\]  (5.28)

Taking \((u_\lambda)\) to be an approximate unit for \(v^*v\mathcal{D}_0^c\), we have \(v^* = \lim v^*u_\lambda\) and \(v^* = \lim u_\lambda v^*\). Then (5.28) gives \(v^*hv \in \mathcal{D}_0^c\), so \(v^*hv \in \mathcal{D}^c\) by (5.27). A similar argument shows that for every \(h \in \mathcal{D}^c\), \(v^*hv \in \mathcal{D}^c\). Thus, \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D}^c)\). \(\square\)

We now give the relationships between Cartan envelopes and Cartan extensions for \((\mathcal{C}, \mathcal{D})\) and \((\mathcal{C}, \mathcal{D}^c)\).

**Proposition 5.29.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the unique faithful pseudo-expectation property. The following statements hold.

(a) \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a Cartan extension for \((\mathcal{C}, \mathcal{D})\) if and only if \((\mathcal{C}_1, \mathcal{D}_1, \alpha)\) is a Cartan extension for \((\mathcal{C}, \mathcal{D}^c)\).
(b) \((\mathcal{C}, \mathcal{D}, \alpha)\) is a Cartan envelope for \((\mathcal{E}, \mathcal{D})\) if and only if \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is a Cartan envelope for \((\mathcal{E}, \mathcal{D}^c)\).

**Proof.** If \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is a Cartan extension for \((\mathcal{E}, \mathcal{D}^c)\), it clearly is a Cartan extension for \((\mathcal{E}, \mathcal{D})\). On the other hand, if \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is a Cartan extension for \((\mathcal{E}, \mathcal{D})\), \cite{15} Proposition 5.3(b)] shows that it is also a Cartan extension for \((\mathcal{E}, \mathcal{D}^c)\).

Turning to part (b), suppose \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is a Cartan envelope for \((\mathcal{E}, \mathcal{D})\). Let \(\mathcal{E}_1\) be the conditional expectation for \((\mathcal{C}, \mathcal{D}_1)\). We have already observed that \(\alpha(\mathcal{D}^c) \subseteq \mathcal{D}_1\) \cite{15} Proposition 5.3(b)). By definition, \(\mathcal{D}_1\) is generated by \(E_1(\alpha(\mathcal{E}))\) and \(\mathcal{E}_1\) is generated by \(\alpha(\mathcal{E}) \cup \mathcal{D}_1\). As \((\mathcal{D}_1, \alpha(\mathcal{D}))\) is an essential extension, so is \((\mathcal{D}_1, \alpha(\mathcal{D}^c))\). Applying Lemma 5.20(b) to \((\alpha(\mathcal{E}), \alpha(\mathcal{D}^c))\) yields \(\alpha(N(\mathcal{E}, \mathcal{D}^c)) \subseteq N(\mathcal{E}_1, \mathcal{D}_1)\). Thus, \(\alpha : (\mathcal{E}, \mathcal{D}^c) \rightarrow (\mathcal{E}_1, \mathcal{D}_1)\) is a regular \(\ast\)-monomorphism. We conclude that \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is Cartan envelope for \((\mathcal{E}, \mathcal{D}^c)\).

For the converse, suppose \((\mathcal{C}, \mathcal{D}_1, \alpha)\) is a Cartan envelope for \((\mathcal{E}, \mathcal{D}^c)\). Then \(\alpha(\mathcal{D}) \subseteq \alpha(\mathcal{D}^c) \subseteq \mathcal{D}_1\). As \((\mathcal{D}_1, \alpha(\mathcal{D}^c))\) and \((\mathcal{D}^c, \mathcal{D})\) are both essential inclusions, Lemma 5.4 shows \((\mathcal{D}_1, \alpha|_{\mathcal{D}})\) is an essential extension of \(\mathcal{D}\). As \(\mathcal{D}_1\) is generated by \(E_1(\alpha(\mathcal{E}))\), the proposition follows. \(\square\)

Theorem 5.2 characterizes which inclusions have a Cartan envelope. Of course, the process of passing from an inclusion to its Cartan envelope loses information: for example, if \(\mathcal{D}\) is an abelian \(C^*\)-algebra, and \(\mathcal{D}_0\) is any essential \(C^*\)-subalgebra of \(\mathcal{D}\), then \((\mathcal{D}, \mathcal{D}_0)\) is the Cartan envelope for \((\mathcal{D}, \mathcal{D}_0)\). Here is a more interesting example of very different inclusions with the same Cartan envelope.

**Example 5.30.** Let \(A\) and \(B\) be self-adjoint unitaries in \(M_2(\mathbb{C})\) such that

\[ AB = -BA; \]

we have in mind the concrete examples, \(A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\). The spectra of \(A\) and \(B\) are both \([{-1, 1}]\), and the set \(\{A, B\}\) generates \(M_2(\mathbb{C})\). For \(k = 0, 1\), let \(P_k\) be the projection onto the \((-1)^k\)-eigenspace for \(A\), likewise let \(Q_k\) be the projection onto the \((-1)^k\)-eigenspace for \(B\). Write \(C^*(A)\) and \(C^*(B)\) for the \(C^\ast\)-subalgebras of \(M_2(\mathbb{C})\) generated by \(A\) and \(B\). The inclusions \((M_2(\mathbb{C}), C^*(A))\) and \((M_2(\mathbb{C}), C^*(B))\) are \(C^\ast\)-diagonals, and we let \(E_A\) and \(E_B\) denote the conditional expectations of \(M_2(\mathbb{C})\) onto \(C^*(A)\) and \(C^*(B)\) respectively.

Let

\[ \mathcal{X} := [-2, -1] \cup [1, 2] \quad \text{and} \quad \mathcal{E} := C(\mathcal{X}) \otimes M_2(\mathbb{C}); \]

regard \(\mathcal{E}\) as continuous \(M_2(\mathbb{C})\)-valued functions on \(\mathcal{X}\). Set

\[ \mathcal{D} := \{ f \in \mathcal{E} : f(t) \in C^*(A) \text{ if } t < 0; \ f(t) \in C^*(B) \text{ if } t > 0 \}. \]

Then \((\mathcal{E}, \mathcal{D})\) is a Cartan inclusion.

We now define two inclusions whose Cartan envelope is \((\mathcal{E}, \mathcal{D})\). Let

\[ \mathcal{D}_0 := \{ f \in \mathcal{D} : f(-1) = f(1) \} \quad \text{and} \quad \mathcal{E}_0 := \{ f \in \mathcal{E} : f(-1) = f(1) \}. \]

Then \((\mathcal{D}, \mathcal{D}_0)\) is an essential inclusion and a computation shows \((\mathcal{E}_0, \mathcal{D}_0)\) is a MASA inclusion. Also,

\[ \{ f \otimes A : f \in C(\mathcal{X}) \} \cup \{ f \otimes B : f \in C(\mathcal{X}) \} \subseteq N(\mathcal{E}_0, \mathcal{D}_0). \]

It now follows readily that both \((\mathcal{E}_0, \mathcal{D}_0)\) and \((\mathcal{E}, \mathcal{D}_0)\) are regular inclusions with the unique pseudo-expectation property, but in both cases, the pseudo-expectation is not a conditional expectation.

Note that unlike \((\mathcal{E}_0, \mathcal{D}_0)\), \((\mathcal{E}, \mathcal{D}_0)\) is not a MASA inclusion. Nevertheless, the Cartan envelopes of \((\mathcal{E}_0, \mathcal{D}_0)\) and \((\mathcal{E}, \mathcal{D}_0)\) are both \((\mathcal{E}, \mathcal{D})\).
Thus, the question of when two inclusions have isomorphic Cartan envelopes arises. The following result gives a method for constructing regular inclusions with the faithful unique pseudo-expectation property as sub-inclusions of a given Cartan inclusion. This result also provides a partial answer to the question.

**Proposition 5.31.** Suppose the following:

- \((C, D)\) is a Cartan inclusion with conditional expectation \(E : C \to D\);
- \(D_0 \subseteq D\) is a \(C^*\)-subalgebra such that \((D, D_0)\) is an essential inclusion; and
- \(M \subseteq N(C, D_0)\) is a \(*\)-monoid such that \(D_0 \subseteq \text{span} M\).

If \(C_0 := \text{span} M\), then \((C_0, D_0)\) is a regular inclusion with the faithful unique pseudo-expectation property.

Furthermore, if

\[ D_1 = C^*(E(C_0)), \quad C_1 = C^*(C_0 \cup D_1), \]

and \(\alpha : C_0 \to C_1\) is the inclusion map, then:

- (a) \((C_1, D_1, \alpha)\) is the Cartan envelope for \((C_0, D_0)\); and
- (b) \((C_0, C_0 \cap D)\) is a virtual Cartan inclusion.

**Proof.** By construction, \((C_0, D_0)\) is a regular inclusion and Lemma 5.26(a) shows it has the faithful unique pseudo-expectation property.

We next show \((C_1, D_1)\) is a Cartan inclusion. Part (b) of Lemma 5.26 shows \(N(C, D_0) \subseteq N(C, D)\), so since \(C_1\) is generated by \(D_1 \cup N(C, D_0)\), \((C_1, D_1)\) is a regular inclusion. As \((D, D_0)\) is an essential inclusion, so is \((D, D_1)\). Therefore, Lemma 5.26(a) shows \((C_1, D_1)\) has the faithful unique pseudo-expectation property. As \(E|_{C_1}\) is a faithful conditional expectation of \(C_1\) onto \(D_1\), Proposition 5.31(b) shows \((C_1, D_1)\) is a Cartan inclusion.

Lemma 5.26(b) shows \(N(C_0, D_0) \subseteq N(C, D_1)\), so the inclusion map \(\alpha : C_0 \to C_1\) is a regular homomorphism. Thus, \((C_1, D_1, \alpha)\) is the Cartan envelope for \((C_0, D_0)\). This completes the proof of (a).

We turn to (b). Since \((D, D_0)\) is an essential inclusion, both \((C_0 \cap D, D_0)\) and \((D, C_0 \cap D)\) are essential inclusions. Lemma 5.26 shows \(N(C_0, D_0) \subseteq N(C, D)\), whence \(N(C_0, D_0) \subseteq N(C_0, C_0 \cap D)\). It follows that \((C_0, C_0 \cap D)\) is a regular inclusion. Also, Lemma 5.26 shows \(\alpha\) is a regular homomorphism of \((C_0, C_0 \cap D)\) into \((C, D)\). Let \(A\) be the relative commutant of \(C_0 \cap D\) in \(C_0\). By [15], Proposition 5.3(b)], \(A \subseteq D\). Thus,

\[ C_0 \cap D \subseteq A \subseteq C_0 \cap D, \]

so \((C_0, C_0 \cap D)\) is a regular MASA inclusion. Finally, as the faithful unique pseudo-expectation property is hereditary from above and \((C_0, D_0)\) has that property, so does \((C_0, C_0 \cap D)\). Thus \((C_0, C_0 \cap D)\) is a virtual Cartan inclusion.

\[ \square \]

**6. Some Consequences of the Unique Pseudo-Expectation Property**

This section has two main purposes. One goal is to show that regular inclusions with the unique pseudo-expectation property are covering inclusions. This gives a class of regular inclusions for which the results of Section 7 can be used for descriptions of packages via twists. The second goal is to record some consequences of the unique pseudo-expectation property for regular inclusion with the hope that they may be useful in obtaining a characterization of the unique pseudo-expectation property.

Several of the results of this section extend results of [15] from the setting of regular MASA inclusions (or skeletal MASA inclusions) considered there to simply assuming the unique pseudo-expectation property. While some of these are routine adaptations of proofs in [15], others are more complicated.
When \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property and \(E : \mathcal{C} \to I(\mathcal{D})\) is the pseudo-expectation, recall from (6.3) that
\[
\mathfrak{G}_s(\mathcal{C}, \mathcal{D}) = \{ \rho \circ E : \rho \in \hat{I}(\mathcal{D}) \}.
\]
We already have observed that if \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the \textit{faithful} unique pseudo-expectation property, then \(\mathfrak{G}_s(\mathcal{C}, \mathcal{D})\) is a compatible cover for \(\hat{\mathcal{D}}\). We shall see that this holds when the faithfulness hypothesis on the unique pseudo-expectation is removed: Theorem 6.9 shows that in the presence of the unique pseudo-expectation property, \((\mathcal{C}, \mathcal{D})\) is a covering inclusion and that \(\mathfrak{G}_s(\mathcal{C}, \mathcal{D})\) is a compatible cover for \(\hat{\mathcal{D}}\) which is contained in all compatible covers.

The proof of the following statements are straightforward adaptations of the proofs of corresponding results found in [15].

**Theorem 6.1.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion with the unique pseudo-expectation property, let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\) and let \(E\) be the pseudo-expectation. The following statements hold.

(a) The map \(E^\# : \hat{I}(\mathcal{D}) \to \text{Mod}(\mathcal{C}, \mathcal{D})\) is the unique continuous map of \(\hat{I}(\mathcal{D})\) into \(\text{Mod}(\mathcal{C}, \mathcal{D})\) such that for every \(\rho \in \hat{I}(\mathcal{D})\), \(E^\#(\rho)_{|\mathcal{D}} = \rho \circ \iota\).

(b) Suppose \(F\) is a closed subset of \(\text{Mod}(\mathcal{C}, \mathcal{D})\) such that \(\hat{\mathcal{D}} = \{ \rho_{|\mathcal{D}} : \rho \in F \}\). Then \(\mathfrak{G}_s(\mathcal{C}, \mathcal{D}) \subseteq F\).

**Proof.** Part (a) follows as in the proof of [15] Theorem 3.9, and part (b) follows as in the proof of the first part of [15] Theorem 3.12. \(\square\)

When \((\mathcal{C}, \mathcal{D})\) is a regular MASA inclusion, it has the unique pseudo-expectation property and the left kernel of the pseudo-expectation is a two-sided ideal of \(\mathcal{C}\) maximal with respect to having trivial intersection with \(\mathcal{D}\) ([15] Theorem 3.15]). Our next goal is to show that this result holds when the hypothesis that \(\mathcal{D}\) is a MASA in \(\mathcal{C}\) is removed, that is, we extend this result to any inclusion with the unique pseudo-expectation property.

To do this, we require the following result, which is the version of [15] Proposition 3.14 with the skeletal MASA hypothesis removed. For \(v \in N(\mathcal{C}, \mathcal{D})\), we have already defined the partial automorphism, \(\theta_v : vv^*\mathcal{D} \to v^*v\mathcal{D}\). In the following, \(\tilde{\theta}_v\) is the unique extension of \(\theta_v\) to a partial automorphism of the regular ideals in \(I(\mathcal{D})\) generated by \(vv^*\) and \(v^*v\) respectively, see [15] Definition 2.13.

**Proposition 6.2.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion with the unique pseudo-expectation property, let \((I(\mathcal{D}), \iota)\) be an injective envelope for \(\mathcal{D}\) and let \(E : \mathcal{C} \to I(\mathcal{D})\) be the pseudo-expectation. Then for every \(v \in N(\mathcal{C}, \mathcal{D})\) and \(x \in \mathcal{C}\),
\[
E(v^*vx) = \tilde{\theta}_v(E(vv^*x)).
\]

**Proof.** We begin by establishing (6.3) for \(x \in \mathcal{D}^c\). First suppose \(x^* = x \in \mathcal{D}^c\) and put
\[
S := \{ d \in \mathcal{D}_{s.a.} : d \geq x \}.
\]
We claim that
\[
\tilde{\theta}_v(E(vv^*x)) = \inf_{I(\mathcal{D})_{s.a.}} \iota(v^*Sv) \geq E(v^*vx). \tag{6.4}
\]
Recall that for any \(f \in vv^*\mathcal{D}\), \(\iota(\theta_v(f)) = \tilde{\theta}_v(\iota(f))\). So
\[
d \in S \Rightarrow \iota(d) \geq E(x) \Rightarrow \iota(dvv^*) \geq E(xvv^*)
\]
\[
\Rightarrow \iota(\theta_v(dvv^*)) = \tilde{\theta}_v(\iota(dvv^*)) \geq \tilde{\theta}_v(E(xvv^*))
\]
\[
\Rightarrow \iota(v^*d) \geq \tilde{\theta}_v(E(xvv^*))
\]
\[
\Rightarrow \inf_{I(\mathcal{D})_{s.a.}} v^*Sv \geq \tilde{\theta}_v(E(xvv^*)). \tag{6.4}
\]
Next, let $Q \in I(\mathcal{D})$ be the support projection for $v^*v \mathcal{D}$, that is, $Q = \sup_{I(\mathcal{D})_{s.a.}} \iota(u_{\lambda})$ where $(u_{\lambda})$ is an approximate unit for $v^*v \mathcal{D}$. Suppose $y \in I(\mathcal{D})_{s.a.}$ satisfies
$$y \leq \iota(v^*dv)$$
for every $d \in S$. Then $Qy \leq Q\iota(v^*dv) = \iota(v^*dv)$ for $d \in S$. Thus, for every $d \in S$,
$$\tilde{\theta}_v^{-1}(Qy) \leq \tilde{\theta}_v^{-1}(\iota(v^*dv)) = \tilde{\theta}_v^{-1}(\tilde{\theta}_v(\iota(vv^*d))) = \iota(vv^*d).$$
Therefore,
$$\tilde{\theta}_v^{-1}(Qy) \leq \inf_{I(\mathcal{D})_{s.a.}} (vv^*S) \inf_{I(\mathcal{D})_{s.a.}} \iota(vv^*) \inf_{I(\mathcal{D})_{s.a.}} S \iota(vv^*)E(x) = E(vv^*)E(x),$$
with equality (1) following from [10, Lemma 1.9] and equality (2) by [16, Theorem 3.16]. Thus,
$$Qy \leq \tilde{\theta}_v(E(xvv^*)).$$
Observe that $Q^\perp y \leq 0$ because $Q\iota(v^*v) = \iota(v^*v)$ and $Q^\perp y$ is a lower bound for $Q^\perp \iota(v^*Sv) = \{0\}$. Therefore, $y \leq Qy$, so $y \leq \tilde{\theta}_v(E(xvv^*)).$ Thus, $\inf_{I(\mathcal{D})_{s.a.}} \iota(v^*Sv) \leq \tilde{\theta}_v(E(xvv^*))$, which completes the proof of the equality in (6.4).

If $d \in S$, then $v^*dv \geq v^*xv$, so $\iota(v^*dv) \geq E(v^*xv)$. Thus $E(v^*xv)$ is a lower bound for $\iota(v^*Sv)$, which gives the inequality in (6.4).

Replacing $x$ with $-x$ in (6.4) yields $\tilde{\theta}_v(E(vv^*x)) \leq E(v^*xv)$, whence (6.3) holds for any $x \in (\mathcal{D}^c)_{s.a.}$. It follows from linearity that (6.3) holds for all $x \in \mathcal{D}^c$.

The remainder of the proof is a modification of the proof of [15 Proposition 3.14]. Since $(\mathcal{E}, \mathcal{D})$ is a regular inclusion, it suffices to show (6.3) holds for any $w \in N(\mathcal{E}, \mathcal{D})$.

By Lemma [15 Lemma 3.3], it suffices to show that the ideal $H := \{d \in \mathcal{D} : (E(v^*wv) - \tilde{\theta}_v(E(vv^*w))\iota(d) = 0\}$ is an essential ideal of $\mathcal{D}$.

So let $w \in N(\mathcal{E}, \mathcal{D})$. Let $\{K_i\}_{i=0}^4$ be a left Frolik family of ideals for $w$, let $J = vv^*D$ and let $P$ be the support projection in $I(\mathcal{D})$ for $J$. Let
$$A := \theta_v(J^\perp) \cup \bigcup_{i=0}^4 \theta_v(J \cap K_i).$$
Then $A^\perp = \{0\}$, so $A$ generates an essential ideal of $\mathcal{D}$. To show $H$ is an essential ideal, we show $A \subseteq H$.

The following facts follow as in the corresponding facts in the proof of [15 Theorem 3.14]:
- $\theta_v(J^\perp) \subseteq H$; and
- for $1 \leq i \leq 4$, $\theta_v(J \cap K_i) \subseteq H$.

We now show that $\theta_v(J \cap K_0) \subseteq H$. Let $d \in J \cap K_0$. Lemma 2.12 gives $wd = dw \in (\mathcal{D}^c)$. Thus,
$$E(v^*wv)\iota(\theta_v(d)) = E(v^*wdv) = \tilde{\theta}_v(E(vv^*wd)) = \tilde{\theta}_v(E(vv^*w))\iota(\theta_v(d)).$$
Therefore, $\theta_v(J \cap K_0) \subseteq H$ as well.

We conclude that $A \subseteq H$, which completes the proof.

With the previous proposition in hand, we have the following result which extends [15 Theorem 3.15].

**Theorem 6.5.** Let $(\mathcal{E}, \mathcal{D})$ be a regular inclusion with the unique pseudo-expectation property. Let $E$ be the pseudo-expectation and set $L(\mathcal{E}, \mathcal{D}) := \{x \in \mathcal{E} : E(x^*x) = 0\}$. Then $L(\mathcal{E}, \mathcal{D})$ is an ideal of $\mathcal{E}$ such that $L(\mathcal{E}, \mathcal{D}) \cap \mathcal{D} = (0)$.

Moreover, if $\mathcal{K} \subseteq \mathcal{E}$ is an ideal such that $\mathcal{K} \cap \mathcal{D} = (0)$, then $\mathcal{K} \subseteq L(\mathcal{E}, \mathcal{D})$. 

31
Proof. The proof is the same as the proof of [15, Theorem 3.15], except that one uses Proposition 6.2 instead of [15, Proposition 3.14].

Remark 6.6. Without the regularity assumption on (C, D), Theorem 6.5 is false, see [16, Remark 3.11].

By [16, Proposition 3.6], the quotient of an inclusion with the unique pseudo-expectation property by a D-disjoint ideal also has the unique pseudo-expectation property. The maximality of \( \mathcal{L}(C, D) \) allows us to conclude that the quotient by \( \mathcal{L}(C, D) \) has the faithful unique pseudo-expectation property.

Corollary 6.7. Suppose (C, D) is a regular inclusion with the unique pseudo-expectation property. Let \( \mathcal{C}_q := C/\mathcal{L}(C, D) \), let \( q : C \to \mathcal{C}_q \) be the quotient map, and \( D_q = q(D) \). Then \( q|_D \) is an isomorphism of \( D \) onto \( D_q \) and \( (\mathcal{C}_q, D_q) \) has the faithful unique pseudo-expectation property.

Proof. Since \( \mathcal{L}(C, D) \cap D = \{0\} \), \( q|_D \) is one-to-one, so \( q|_D \) is an isomorphism of \( D \) onto \( D_q \), and as noted above, \( (\mathcal{C}_q, D_q) \) has the unique pseudo-expectation property. Consider the ideal \( J := q^{-1}(\mathcal{L}(C_q, D_q)) \). If \( d \in D \cap J \), then \( q(d) \in \mathcal{L}(C_q, D_q) \cap D_q = \{0\} \), hence \( d = 0 \). Therefore \( J \subseteq \mathcal{L}(C, D) \), whence \( \mathcal{L}(C_q, D_q) = \{0\} \). It follows that \( E_q \) is faithful, as desired.

Now we turn to showing that inclusions with the unique pseudo-expectation property are covering inclusions. We begin with a special case.

Lemma 6.8. Suppose (C, D) is an inclusion with the unique pseudo-expectation property. If \( D \) is contained in the center of \( C \), then the unique pseudo-expectation is multiplicative on \( C \) and the ideal \( \mathcal{L}(C, D) \) contains the commutator ideal of \( C \).

Proof. Put \( J := \mathcal{L}(C, D) \), let \( q : C \to C/J \) be the quotient map, \( \mathcal{C}_q := C/J \), and \( D_q := q(D) \). Corollary 6.7 shows that \( D_q \simeq D \) and \( (\mathcal{C}_q, D_q) \) has the faithful unique pseudo-expectation property. Let \( E_q \) be the unique pseudo-expectation for \( (\mathcal{C}_q, D_q) \).

Note that \( D_q \) is contained in the center of \( \mathcal{C}_q \), so the relative commutant of \( D_q \) in \( \mathcal{C}_q \) is all of \( \mathcal{C}_q \). By [16, Corollary 3.14], \( \mathcal{C}_q \) is abelian, so \( \mathcal{L}(C, D) \) contains the commutator ideal of \( C \). By [16, Corollary 3.21], \( E_q \) is multiplicative. But \( E_q \circ q \) is multiplicative and is the pseudo-expectation for \( (C, D) \).

We now are equipped to show that \( \mathcal{S}_s(C, D) \) is the minimal compatible cover for a regular inclusion with the unique pseudo-expectation property.

Theorem 6.9. Suppose (C, D) is a regular inclusion with the unique pseudo-expectation property. Then (C, D) is a covering inclusion and \( \mathcal{S}_s(C, D) \) is a compatible cover for \( \hat{D} \). Furthermore, if \( F \) is any closed subset of \( \text{Mod}(C, D) \) which covers \( \hat{D} \), then \( \mathcal{S}_s(C, D) \subseteq F \).

Proof. We claim that whenever \( v \in \text{N}(C, D) \) and \( \rho_0 \in \widehat{I(D)} \) satisfies \( \rho_0(E(v)) \neq 0 \), then \( \rho_0 \circ \iota \in (\text{fix} \beta_v)^c \). Denote by \( r \) the “restriction” map, \( \widehat{I(D)} \ni \rho \mapsto \rho \circ \iota \in \hat{D} \) and let

\[
X := \{ \rho' \in \widehat{I(D)} : |\rho'(E(v))| > |\rho_0(E(v))|/2 \}.
\]

As \( \widehat{I(D)} \) is Stonean, \( X \) is clopen, so \( \overline{X} \in \text{ROPEN}(\widehat{I(D)}) \). Let \( G := (r(\overline{X}))^c \). Lemma 2.11 shows that \( r^{-1}(G) = X \). In particular, \( \rho_0 \circ \iota \in G \).

If \( \sigma \in G \), then \( \sigma = \rho' \circ \iota \) for some \( \rho' \in \overline{X} \), and, by definition of \( X \), \( |\rho'(E(v))| \neq 0 \). By [15, Lemma 2.5], \( (\rho' \circ E)|_D \in \text{fix} \beta_v \). But \( (\rho' \circ E)|_D = \rho' \circ \iota \). Thus,

\[
\rho_0 \circ \iota \in G \subseteq (\text{fix} \beta_v)^c.
\]

Therefore, the claim holds.
Let $\tau \in \mathcal{G}_s(\mathcal{C}, \mathcal{D})$ and suppose $\tau(v) \neq 0$ for some $v \in N(\mathcal{C}, \mathcal{D})$. Write $\tau = \rho \circ E$ for some $\rho \in \overline{I(\mathcal{D})}$. Set $\sigma = \tau|_{\mathcal{D}}$; note that $\sigma \in \mathcal{D}$ and $\sigma(v^*v) > 0$. By the claim, $\sigma \in (\text{fix } \beta_v)^\circ$. By Lemmas 2.13 and 2.12 there exists $d \in \mathcal{D}$ such that $\sigma(d) = 1$ and supp $d \subseteq (\text{fix } \beta_v)^\circ$. Then $d^*v = vd \in \mathcal{D}^c$. By Lemma 6.8, $E$ is multiplicative on $\mathcal{D}^c$, so

$$|	au(v)|^2 = |\tau(vd)|^2 = \rho(E(d^*v))\rho(vd) = \rho(E(d^*vd)) = \tau(v^*v).$$

It follows that $\tau$ is a compatible state.

Let us show the invariance of $\mathcal{G}_s(\mathcal{C}, \mathcal{D})$. Choose $\tau \in \mathcal{G}_s(\mathcal{C}, \mathcal{D})$ and write $\tau = \rho \circ E$ for some $\rho \in \overline{I(\mathcal{D})}$. Suppose $v \in N(\mathcal{C}, \mathcal{D})$ is such that $\rho(v^*v) \neq 0$ and let $P$ and $Q$ be the support projections in $I(\mathcal{D})$ for $vv^*\mathcal{D}$ and $v^*v\mathcal{D}$ respectively. Then $\tilde{\theta}_v$ is a partial automorphism with domain $PI(\mathcal{D})$ and range $QI(\mathcal{D})$. Define $\tau' \in I(\mathcal{D})$ by

$$\tau'(h) = \rho(\tilde{\theta}_v(Ph)), \quad h \in I(\mathcal{D}).$$

For $x \in \mathcal{C}$, Proposition 5.2 gives,

$$\rho(E(v^*xv)) = \rho(\tilde{\theta}_v(E(vv^*x))) = \rho(\tilde{\theta}_v(\iota(vv^*)PE(x))) = \rho(\iota(v^*v))\rho(\tilde{\theta}_v(PE(x))) = \rho(\iota(v^*v))(\tau' \circ E)(x).$$

Thus $\mathcal{G}_s(\mathcal{C}, \mathcal{D})$ is invariant.

If $\sigma \in \mathcal{D}$, choose any $\rho \in \overline{I(\mathcal{D})}$ such that $\rho \circ \iota = \sigma$. Then $\sigma = (\rho \circ E)|_{\mathcal{D}}$, so $\mathcal{G}_s(\mathcal{C}, \mathcal{D})$ covers $\mathcal{D}$. Thus, $\mathcal{G}_s(\mathcal{C}, \mathcal{D})$ is a compatible cover for $\mathcal{D}$ and $(\mathcal{C}, \mathcal{D})$ is a covering inclusion.

Finally, if $F \subseteq \text{Mod}(\mathcal{C}, \mathcal{D})$ is closed and covers $\mathcal{D}$, then $\mathcal{G}_s(\mathcal{C}, \mathcal{D}) \subseteq F$ by Theorem 6.11(b).

\[\square\]

**Example 6.10.** As noted previously, when $(\mathcal{C}, \mathcal{D})$ has the faithful unique pseudo-expectation property, $\mathcal{D}^c$ is abelian, but when $(\mathcal{C}, \mathcal{D})$ merely has the unique pseudo-expectation property, $\mathcal{D}^c$ need not be abelian. We now outline a method for constructing examples of this behavior. This also provides a negative answer to [16] Question 5.

Suppose $(\mathcal{A}, \mathcal{B})$ is an inclusion with the unique pseudo-expectation property and suppose $\mathcal{A}$ is abelian. Let $J := L(\mathcal{A}, \mathcal{B})$ and assume $J \neq 0$. (See [16] Corollary 3.21 for a characterization of such inclusions.) Define

$$\mathcal{C} := \left\{ \left[ \begin{array}{cc} b & j_1 \\ j_2 & b + j_{22} \end{array} \right] : b \in \mathcal{B}, j_{mn} \in J \right\} \quad \text{and} \quad \mathcal{D} := \left\{ \left[ \begin{array}{cc} b & 0 \\ 0 & b \end{array} \right] : b \in \mathcal{B} \right\} \simeq \mathcal{B}.$$

Then $\mathcal{D}$ is contained in the center of $\mathcal{C}$, so $(\mathcal{C}, \mathcal{D})$ is a regular inclusion and $\mathcal{D}^c = \mathcal{C} \supseteq M_2(J)$. Thus $\mathcal{D}^c$ is not abelian. Let $E$ be the pseudo-expectation for $(\mathcal{A}, \mathcal{B})$ and suppose $\Delta$ is a pseudo-expectation for $(\mathcal{C}, \mathcal{D})$. Then $a \in \mathcal{A} \mapsto \Delta(a \oplus a)$ is a pseudo-expectation for $(\mathcal{A}, \mathcal{B})$. So for $j \in J = \ker E$, the fact that $(\mathcal{A}, \mathcal{B})$ has the unique pseudo-expectation property gives $\Delta(j \oplus j) = 0$. Also, for $j_1, j_2 \in J$, applying $\Delta$ to each operator in the inequality,

$$-|j_1| + |j_2| \oplus -|j_1| + |j_2| \leq j_1 \oplus j_2 \leq (|j_1| + |j_2|) \oplus (|j_1| + |j_2|)$$

gives $\Delta(j_1 \oplus j_2) = 0$. Note that $\Delta \left( \left[ \begin{array}{c} 0 \\ j_1 \end{array} \right] \right) = 0$ because

$$\Delta \left( \left[ \begin{array}{c} 0 \\ j_1 \end{array} \right] \right)^* \Delta \left( \left[ \begin{array}{c} 0 \\ j_1 \end{array} \right] \right) \leq \Delta \left( \left[ \begin{array}{cc} 0 & j_1 \\ j_2 & 0 \end{array} \right] \right)^* \Delta \left( \left[ \begin{array}{cc} 0 & j_1 \\ j_2 & 0 \end{array} \right] \right) = \Delta(|j_2|^2 \oplus |j_1|^2) = 0.$$

Thus $M_2(J) \subseteq \ker \Delta$. If $x \in \mathcal{C}$, write $x = (b \oplus b) + y$ where $b \in \mathcal{B}$ and $y \in M_2(J)$. Then $\Delta(x) = E(b)$, and it follows that $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property.
Remark 6.14. Given an inclusion \((\mathcal{C}, \mathcal{D})\) and an injective envelope \((I(\mathcal{D}), \iota)\) for \(\mathcal{D}\), we use \(\text{PsExp}(\mathcal{C}, \mathcal{D})\) for the collection of all pseudo-expectations for \((\mathcal{C}, \mathcal{D})\) relative to \((I(\mathcal{D}), \iota)\).

A consequence of the following result is that \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property if and only if \((\mathcal{D}^e, \mathcal{D})\) does.

**Proposition 6.11.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion. The map \(\Phi : \text{PsExp}(\mathcal{C}, \mathcal{D}) \to \text{PsExp}(\mathcal{D}^e, \mathcal{D})\) given by \(\Delta \mapsto \Delta|_{\mathcal{D}^e}\) is a bijection.

**Proof.** Clearly when \(\Delta \in \text{PsExp}(\mathcal{C}, \mathcal{D})\), \(\Delta|_{\mathcal{D}^e} \in \text{PsExp}(\mathcal{D}^e, \mathcal{D})\). The fact that \(\Phi\) is onto follows from injectivity. To show \(\Psi\) is one-to-one, suppose \(\Delta_1, \Delta_2 \in \text{PsExp}(\mathcal{C}, \mathcal{D})\) and \(\Delta_1|_{\mathcal{D}^e} = \Delta_2|_{\mathcal{D}^e}\). We now argue as in the proof of [15, Proposition 3.4]. Here is an outline. Let \(v \in N(\mathcal{C}, \mathcal{D})\) and let

\[
J := \{d \in \mathcal{D} : (\Delta_1(v) - \Delta_2(v))\iota(d) = 0\}.
\]

To obtain \(\Delta_1 = \Delta_2\), it suffices to show \(J\) is an essential ideal of \(\mathcal{D}\). Let \(\{K_i\}_{i=0}^4\) be a left Frölik family of ideals for \(v\). For \(d \in K_0\), \(vd = dv \in \mathcal{D}^e\), so

\[
\Delta_1(v)\iota(d) = \Delta_1(vd) = \Delta_2(vd) = \Delta_2(v)\iota(d).
\]

Thus, \(K_0 \subseteq J\). Establishing \(K_i \subseteq J\) for \(1 \leq i \leq 4\) is done exactly as in the proof of [15, Proposition 3.4]. Thus \(\cup_{i=0}^4 K_i \subseteq J\). As \(\cup_{i=0}^4 K_i\) generates an essential ideal of \(\mathcal{D}\), \(J\) is essential. By [15, Lemma 3.3], \(\Delta_1(v) = \Delta_2(v)\). As span \(N(\mathcal{C}, \mathcal{D})\) is dense in \(\mathcal{C}\), \(\Delta_1 = \Delta_2\). Thus \(\Phi\) is one-to-one. \(\square\)

The following result extends [15, Theorem 3.5] to the setting of regular inclusions \((\mathcal{C}, \mathcal{D})\) for which \(\mathcal{D}^e\) is abelian.

**Corollary 6.12.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion such that \(\mathcal{D}^e\) is abelian. Let \(X := \hat{\mathcal{D}}, Y := \hat{\mathcal{D}}^e\), and let \(r : Y \to X\) be the restriction mapping. The following statements are equivalent.

(a) \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property.
(b) There exists a unique minimal closed set \(F \subseteq Y\) such that \(r(F) = X\).
(c) There exists a unique maximal \(\mathcal{D}\)-disjoint ideal of \(\mathcal{D}^e\).

**Proof.** Combine Proposition 6.11 with [16, Corollary 3.21]. \(\square\)

We conclude with the following conjecture regarding characterizations of the unique pseudo-expectation property.

**Conjecture 6.13.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion and let \(\mathcal{M}\) be the set of all multiplicative linear functionals on \(\mathcal{D}^e\). The following statements are equivalent.

(a) \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property.
(b) \((\mathcal{C}, \mathcal{D})\) is a covering inclusion and there exists a \(\mathcal{D}\)-disjoint ideal \(J \subseteq \mathcal{C}\) with the following property: if \(I \subseteq \mathcal{C}\) is a \(\mathcal{D}\)-disjoint ideal of \(\mathcal{C}\), then \(I \subseteq J\).
(c) \((\mathcal{C}, \mathcal{D})\) is a covering inclusion and there exists a compatible cover \(F\) for \(\hat{\mathcal{D}}\) with the following property: if \(C \subseteq \text{Mod}(\mathcal{C}, \mathcal{D})\) is closed and covers \(\hat{\mathcal{D}}\), then \(F \subseteq C\).
(d) \((\mathcal{D}^e, \mathcal{D})\) has the unique pseudo-expectation property.
(e) Every multiplicative linear functional on \(\mathcal{D}\) extends to an element of \(\mathcal{M}\) and there exists a \(\mathcal{D}\)-disjoint ideal \(J \subseteq \mathcal{D}^e\) with the following property: if \(I \subseteq \mathcal{D}^e\) is a \(\mathcal{D}\)-disjoint ideal of \(\mathcal{D}^e\), then \(I \subseteq J\).
(f) Every multiplicative linear functional on \(\mathcal{D}\) extends to an element of \(\mathcal{M}\) and there exists a closed subset \(F \subseteq \mathcal{M}\) with the following properties: \(F\) covers \(\hat{\mathcal{D}}\) and if \(C \subseteq \text{Mod}(\mathcal{D}^e, \mathcal{D})\) is closed and covers \(\hat{\mathcal{D}}\), then \(F \subseteq C\).

**Remark 6.14.** Here are some comments regarding this conjecture.

- (a)\(\Rightarrow\)(b) and (a)\(\Rightarrow\)(c) by Theorems 6.5 and 6.9.
Suppose (c) holds. Then \( \mathcal{X}_F := \{ x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in F \} \) is an ideal of \( \mathcal{C} \) such that \( \mathcal{X}_F \cap \mathcal{D} = (0) \). We expect this ideal to have the property in (b). Furthermore, \((F,r)\) is an essential cover for \( \hat{\mathcal{D}} \), and if \((\mathcal{C}_q, \mathcal{D}_q) := (\mathcal{C}/\mathcal{X}_F, \mathcal{D}/(\mathcal{X}_F \cap \mathcal{D}))\), it seems reasonable to expect \((\mathcal{C}_q, \mathcal{D}_q)\) has the faithful unique pseudo-expectation property (see Proposition 7.32 below). Because we have been unable to establish that every pseudo-expectation annihilates \( \mathcal{X}_F \), it is not clear this implies \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property.

- (a)\(\iff\)(d) by Proposition 6.11
- The sets \( \mathcal{G}(\mathcal{D}^c, \mathcal{D}) \) and \( \mathcal{M} \) coincide by [15] Theorem 4.13. In particular \((\mathcal{D}^c, \mathcal{D})\) is a covering inclusion if and only if \( \mathcal{M} \) covers \( \hat{\mathcal{D}} \).

7. Twists Associated to a Regular Covering Inclusion

Let \((\mathcal{C}, \mathcal{D})\) be an inclusion for which there exists a compatible cover \( F \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D}) \) for \( \hat{\mathcal{D}} \). The main result of this section, Theorem 7.24, shows that given this data, there exists a twist \((\Sigma, G)\) and a regular \(*\)-homomorphism \( \theta \) of \((\mathcal{C}, \mathcal{D})\) into the the inclusion \((C^*_r(\Sigma, G), C(G^{(0)}))\) such that \( F \) is identified with the unit space of \( G \) and \( \theta(\mathcal{C}) \) is dense in \( C^*_r(\Sigma) \) with respect to a pointwise topology. The kernel of \( \theta \) is the ideal \( \mathcal{X}_F = \{ x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in F \} \), and when this ideal vanishes, \((C^*_r(\Sigma, G), C(G^{(0)}), \theta)\) is a package for \((\mathcal{C}, \mathcal{D})\). The fact that elements of \( F \) are compatible states is the key ingredient in our construction of our groupoids and \( \theta \).

Also, when \((\mathcal{C}, \mathcal{D})\) has the faithful unique pseudo-expectation property, and \( F \) is taken to be the family of strongly compatible states, this construction produces the Cartan envelope for \((\mathcal{C}, \mathcal{D})\). Theorem 7.24 is a refinement of the embedding results of Section 5 of [15].

Our construction is inspired by the constructions by Kumjian and Renault, but the construction here differs from theirs in significant ways. Most significantly, in the Renault and Kumjian contexts, a conditional expectation \( E : \mathcal{C} \to \mathcal{D} \) is present, and since \( \{ \rho \circ E : \rho \in \mathcal{D} \} \) is homeomorphic to \( \hat{\mathcal{D}} \), Renault and Kumjian use \( \mathcal{D} \) as the unit space for the twists they construct. In our context, we need not have a conditional expectation, so instead of using \( \mathcal{D} \) as the unit space, we use the set \( F \) instead.

In order that the coordinates determined by the twist \((\Sigma_F, G_F)\) reflect as many of the properties of the original covering inclusion \((\mathcal{C}, \mathcal{D})\) as possible, it is desirable that the choice of \( F \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D}) \) be made as small as possible. If \( F_1 \subseteq F_2 \) are two \( \mathcal{N}(\mathcal{C}, \mathcal{D})\)-invariant, closed and \( \mathcal{D}\)-covering subsets of \( \mathcal{G}(\mathcal{C}, \mathcal{D}) \), \( \mathcal{J}_{F_2} \subseteq \mathcal{J}_{F_1} \). However, it is not clear whether the triviality of the ideal \( \mathcal{J}_{F_2} \) implies \( \mathcal{J}_{F_1} \) is also trivial. Thus, it may be that \( \mathcal{C} \) regularly embeds into \( C^*_r(\Sigma_{F_2}, G_{F_2}) \) but does not regularly embed into \( C^*_r(\Sigma_{F_1}, G_{F_1}) \). This leads to the following question.

**Question 7.1.** Suppose \((\mathcal{C}, \mathcal{D})\) is a regular inclusion and \( F \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D}) \) is closed, invariant and covers \( \mathcal{D} \). Let \( r : F \to \hat{\mathcal{D}} \) be the restriction map. If \((\mathcal{G}(\mathcal{C}, \mathcal{D}), r)\) is an essential cover for \( \hat{\mathcal{D}} \), must \( \text{Rad}(\mathcal{C}, \mathcal{D}) = \mathcal{J}_F \)? If \( \text{Rad}(\mathcal{C}, \mathcal{D}) = (0) \), is \( \mathcal{J}_F = (0) \) also?

**Example 7.2.** This example illustrates the need for the hypothesis that \((\mathcal{G}(\mathcal{C}, \mathcal{D}), r)\) is an essential cover for \( \hat{\mathcal{D}} \) in Question 7.1. Take \( \mathcal{C} = C[0, 1] \) and \( \mathcal{D} = CI \). Then \( \mathcal{N}(\mathcal{C}, \mathcal{D}) = \{ \lambda U : \lambda \in \mathcal{C}, U \in \mathcal{U}(\mathcal{C}) \} \). By [15] Theorem 4.13, \( \mathcal{G}(\mathcal{C}, \mathcal{D}) \) is the set of all multiplicative linear functionals on \( \mathcal{C} \). Thus \( \text{Rad}(\mathcal{C}, \mathcal{D}) = (0) \) However, every non-empty closed subset \( F \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D}) \) is invariant and covers the singleton set \( \mathcal{D} \). In particular, it is possible for \( \mathcal{J}_F \) to be a maximal ideal.

However, when \( \mathcal{C} \) is abelian and \((\mathcal{C}, \mathcal{D})\) is an essential inclusion, Question 7.1 has an affirmative answer. As before, \( \mathcal{G}(\mathcal{C}, \mathcal{D}) = \mathcal{C} \), and a closed set \( F \subseteq \mathcal{G}(\mathcal{C}, \mathcal{D}) \) is a compatible cover for \( \hat{\mathcal{D}} \) when \( \mathcal{J}_F \cap \mathcal{D} = (0) \). As \((\mathcal{C}, \mathcal{D})\) is essential, \( \mathcal{J}_F = (0) = \text{Rad}(\mathcal{C}, \mathcal{D}) \).

Recall (see [5]) that an eigenfunctional is a non-zero element \( \phi \in \mathcal{C}^# \) which is an eigenvector for both the left and right actions of \( \mathcal{D} \) on \( \mathcal{C}^# \); when this occurs, there exist unique elements \( \rho, \sigma \in \hat{\mathcal{D}} \).
so that whenever \(d_1, d_2 \in \mathcal{D}\) and \(x \in \mathcal{C}\), we have
\[
\phi(d_1 xd_2) = \rho(d_1) \phi(x) \sigma(d_2). \tag{7.3}
\]

As a simple example, consider the inclusion, \((M_n(\mathbb{C}), D_n)\), where \(D_n\) is the set of \(n \times n\) diagonal matrices. The eigenfunctionals for this inclusion are non-zero scalar multiples of functionals of the form \(\phi_{ij}: A \mapsto \langle Ae_j, e_i \rangle\); here \(\{e_i\}\) is the usual orthonormal basis for \(\mathbb{C}^n\). For \(\phi_{ij}\) and \(d \in D_n\), \(\rho(d) = \langle de_i, e_i \rangle\) and \(\sigma(d) = \langle de_j, e_j \rangle\).

For a general regular inclusion and eigenfunctional \(\phi\), the elements \(\rho, \sigma \in \mathcal{D}\) appearing in (7.3) are to be regarded as range and source maps for \(\phi\), and we write
\[
s(\phi) := \sigma \quad \text{and} \quad r(\phi) := \rho.
\]

It will be clear from the context whether \(r\) refers to the range of an eigenfunctional or a restriction mapping (as used earlier).

**Remark 7.4.** Notice that when \(\phi\) is an eigenfunctional and \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) is such that \(\phi(v) \neq 0\), then for every \(d \in \mathcal{D}\),
\[
\frac{s(\phi)(v^* dv)}{s(\phi)(v^* v)} = r(\phi)(d). \tag{7.5}
\]

Indeed, \(\phi(v) s(\phi)(v^* dv) = \phi(vv^* dv) = \phi(dvv^* v) = r(\phi)(d) \phi(v) s(\phi)(v^* v)\).

**Definition 7.6.** A *compatible eigenfunctional* is an eigenfunctional \(\phi\) such that for every \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\),
\[
|\phi(v)|^2 \in \{0, s(\phi)(v^* v)\} \quad \text{(equivalently, } |\phi(v)|^2 \in \{0, r(\phi)(v v^*)\}). \tag{7.7}
\]

**Notation 7.8.** We use the following notation.

(a) \(E(\mathcal{C}, \mathcal{D})\) (respectively \(E_\mathcal{C}(\mathcal{C}, \mathcal{D})\)) will denote the set consisting of the zero functional together with the set of all eigenfunctionals (resp. the set of all compatible eigenfunctionals together with the zero functional).

(b) Let \(E^1(\mathcal{C}, \mathcal{D})\) be the set of all eigenfunctionals with unit norm. Likewise \(E^1_\mathcal{C}(\mathcal{C}, \mathcal{D})\) will denote the compatible eigenfunctionals of unit norm.

Equip \(E(\mathcal{C}, \mathcal{D}), E^1(\mathcal{C}, \mathcal{D}), E_\mathcal{C}(\mathcal{C}, \mathcal{D})\) and \(E^1_\mathcal{C}(\mathcal{C}, \mathcal{D})\) with the relative \(\sigma(\mathcal{C}^\# , \mathcal{C})\) topology.

(c) For \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) and \(f \in \text{Mod}(\mathcal{C}, \mathcal{D})\) such that \(f(v^* v) > 0\), let \([v, f] \in \mathcal{C}^\#\) be defined by
\[
[v, f](x) := \frac{f(v^* x)}{f(v^* v)^{1/2}} = \left( x + L_f \cdot \frac{v + L_f}{\|v + L_f\|_{\beta_v}} \right)_{\beta_v}.
\]

(This notation is borrowed from Kumjian [12]. There, Kumjian works in the context of \(C^\ast\)-diagonals and uses states on \(\mathcal{C}\) of the form \(\sigma \circ E\) with \(\sigma \in \mathcal{D}\).)

A calculation shows that if \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) and \(f \in \text{Mod}(\mathcal{C}, \mathcal{D})\) satisfy \(f(v^* v) \neq 0\), then
\[
\phi := [v, f] \quad \text{belongs to } E^1(\mathcal{C}, \mathcal{D}), \quad s(\phi) = f|_\mathcal{D} \quad \text{and} \quad r(\phi) = \beta_v(s(\phi)).
\]

Our goal is to show that in fact, all elements of \(E^1(\mathcal{C}, \mathcal{D})\) arise in this way, and furthermore, all elements of \(E^1_\mathcal{C}(\mathcal{C}, \mathcal{D})\) have the form \([v, f]\), where \(f \in \mathcal{S}(\mathcal{C}, \mathcal{D})\). We first show that associated with each \(\phi \in E^1(\mathcal{C}, \mathcal{D})\) is a pair \(f, g \in \text{Mod}(\mathcal{C}, \mathcal{D})\) which extend \(r(\phi)\) and \(s(\phi)\) and describe some properties of these preferred extensions. Note that regularity of the inclusion \((\mathcal{C}, \mathcal{D})\) ensures the existence of \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\) such that \(\phi(v) > 0\).

**Theorem 7.9.** Let \((\mathcal{C}, \mathcal{D})\) be a regular inclusion and let \(\phi \in E^1(\mathcal{C}, \mathcal{D})\). The following statements hold.
(a) There are unique elements \( s(\phi), t(\phi) \in \text{Mod}(C, D) \) such that whenever \( v \in N(C, D) \) satisfies \( \phi(v) \neq 0 \) and \( x \in C \),

\[
\phi(vx) = \phi(v)s(\phi)(x), \quad \phi(xv) = t(\phi)(x)\phi(v).
\]

The functionals \( s(\phi) \) and \( t(\phi) \) satisfy,

(i) \( s(\phi)=s(\phi)|_D \) and \( t(\phi)=t(\phi)|_D; \)

(ii) if \( v \in N(C, D) \) and \( \phi(v) \neq 0 \), then for every \( x \in C \),

\[
s(\phi)(vxv) = s(\phi)(v^*v)t(\phi)(x) \quad \text{and} \quad t(\phi)(vxv^*) = t(vv^*)s(\phi)(x).
\]

(b) If \( v \in N(C, D) \) satisfies \( \phi(v) > 0 \), then \( \phi = [v, s(\phi)] \).

(c) If \( \phi \) is represented in two ways, \( \phi = [v, f] = [w, g] \) \((f, g \in \text{Mod}(C, D), v, w \in N(C, D))\), then

\[
f = g = s(\phi) \quad \text{and} \quad s(\phi)(v^*w) > 0.
\]

(d) If \( s(\phi) \in S(C, D) \), then \( \phi \in E_C^1(C, D) \).

In addition, when \( \phi \in E_C^1(C, D) \) the following hold.

(e) Both \( s(\phi) \) and \( t(\phi) \) belong to \( S(C, D) \).

(f) Suppose \( v \in N(C, D) \), satisfies \( \phi(v) > 0 \). If \( w \in N(C, D) \) and \( s(\phi)(v^*w) > 0 \), then \( \phi = [w, s(\phi)] \).

Proof. Begin by fixing \( v \in N(C, D) \) such that \( \phi(v) > 0 \). Define linear functionals on \( C \) by

\[
s(\phi)(x) := \frac{\phi(vx)}{\phi(v)} \quad \text{and} \quad t(\phi)(x) := \frac{\phi(xv)}{\phi(v)}.
\]

As \( \phi \in E_C^1(C, D) \), \( t(\phi)|_D = r(\phi) \) and \( s(\phi)|_D = s(\phi) \). We next claim that \( \|s(\phi)\| = \|t(\phi)\| = 1 \). For any \( d \in D \) with \( s(\phi)(d) = 1 \), replacing \( v \) by \( vd \) in the definition of \( t(\phi) \) does not change \( t(\phi) \). Thus, if \( x \in C \) and \( \|x\| \leq 1 \), we have \( |t(\phi)(x)| \leq \left\{ \frac{\|vd\|}{\|\phi(v)\|} : d \in D, s(\phi)(d) = 1 \right\} \) = 1 (because \( d \) may be chosen so that \( \|vd\| = \|d^*v^*vd\|^{1/2} \) is as close to \( s(\phi)(v^*v) \) as desired). This shows \( \|t(\phi)\| = 1 \). Likewise \( \|s(\phi)\| = 1 \). As \( t(\phi)(1) = s(\phi)(1) = 1 \), both \( t(\phi) \) and \( s(\phi) \) are states on \( C \) and hence belong to \( \text{Mod}(C, D) \). This gives the existence portion of (a) and also item (i) of part (a).

We continue with the choice of \( v \) made above. Note that \( s(\phi)(v^*v) > 0 \) because

\[
0 \neq \phi(v) = \lim \phi(v(v^*v)^{1/n}) = \lim \phi(v)(s(\phi)(v^*v))^{1/n}.
\]

Thus \( \psi := [v, s(\phi)] \) is defined. A calculation shows that for any \( x \in C \),

\[
\psi(x) = \frac{r(\phi)(v^*v)^{1/2}}{\phi(v)} \phi(x).
\]

Thus \( \psi \) is a positive scalar multiple of \( \phi \) and as \( \|\psi\| = \|\phi\| = 1 \), we obtain \( \phi = \psi \). This establishes part (b).

To verify item (ii) of part (a), taking \( x = v \) in the representation, \( \phi = [v, s(\phi)] \), we find \( r(\phi)(v^*v)^{1/2} = \phi(v) \). Similarly, \( s(\phi)(v^*v)^{1/2} = \phi(v) \). A calculation now shows that for \( x \in C \),

\[
s(\phi)(v^*xv) = s(\phi)(v^*v)t(\phi)(x).
\]

A similar argument yields \( t(\phi)(vxv^*) = t(\phi)(v^*v)s(\phi)(x) \) for each \( x \in C \).

The uniqueness portion of part (a) will follow from part (c), so we turn to part (c) now. Suppose that \( \phi = [v, f] = [w, g] \). Computations using Notation \( CE \) show that for every \( x \in C \),

\[
f(x) = \frac{\phi(vx)}{\phi(v)} \quad \text{and} \quad g(x) = \frac{\phi(wx)}{\phi(w)}.
\]

Since \( \frac{g(w^*v)}{g(w^*w)^{1/2}} = \phi(v) = f(v^*v)^{1/2} \), we obtain

\[
g(w^*v) = f(v^*v)^{1/2}g(w^*w)^{1/2} > 0.
\]

37
Likewise, \( f(v^*w) > 0 \). Also,
\[
    f(x) = \frac{\phi(vx)}{\phi(v)} = \frac{[w, g(vx)]}{[v, f(v)]} = \frac{g(w^*vx)}{f(v^*v)} = \frac{g(w^*v)g(x)}{g(w^*v)} = g(x),
\]
where the fourth equality follows from Proposition 4.4. Part (c) now follows.

For (d), suppose \( s(\phi) \in \mathcal{S}(\mathcal{C}, \mathcal{D}) \). If \( w \in N(\mathcal{C}, \mathcal{D}) \) and \( \phi(w) \neq 0 \), then
\[
    |\phi(w)|^2 = \frac{|s(\phi)(v^*w)|^2}{s(\phi)(v^*v)} = \frac{s(\phi)(v^*w^vw)}{s(\phi)(v^*v)} = \beta_v(s(\phi))(ww^*) = r(\phi)(ww^*),
\]
so \( \phi \) belongs to \( \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) \) by Proposition 4.4.

Next we establish part (e). Suppose \( \phi \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) \). If \( w \in N(\mathcal{C}, \mathcal{D}) \) and \( \tau(\phi)(w) \neq 0 \), we have (using (7.13))
\[
    |\tau(\phi)(w)|^2 = \frac{|\phi(wv)|^2}{|\phi(v)|^2} = \frac{s(\phi)(v^*w^vwv)}{s(\phi)(v^*v)} = r(\phi)(w^*w) = f(w^*w),
\]
and it follows that \( \tau(\phi) \in \mathcal{S}(\mathcal{C}, \mathcal{D}) \). Likewise, \( s(\phi) \in \mathcal{S}(\mathcal{C}, \mathcal{D}) \).

Turning to (f), suppose \( v, w \in N(\mathcal{C}, \mathcal{D}) \) satisfy \( s(\phi)(v) > 0 \) and \( s(\phi)(v^*w) > 0 \). Since \( s(\phi) \in \mathcal{S}(\mathcal{C}, \mathcal{D}) \), Proposition 4.4 shows that \( s(\phi)(v^*w)^2 = s(\phi)(w^*w)s(\phi)(v^*v) \). Thus in the GNS Hilbert space \( \mathcal{H}_{s(\phi)} \), we have \( (v + Ls(\phi), w + Ls(\phi)) = \langle v + Ls(\phi), w + Ls(\phi) \rangle \). By the Cauchy-Schwartz inequality, there exists a positive real number \( t \) so that \( v + Ls(\phi) = tw + Ls(\phi) \). But then for any \( x \in \mathcal{C} \),
\[
    [v, s(\phi)](x) = \frac{\langle x + Ls(\phi), v + Ls(\phi) \rangle}{\|v + Ls(\phi)\|} = \frac{\langle x + Ls(\phi), tw + Ls(\phi) \rangle}{\|tw + Ls(\phi)\|} = [w, s(\phi)](x).\]

The following is immediate.

**Corollary 7.11.** For a regular inclusion \( (\mathcal{C}, \mathcal{D}) \),
\[
    \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) = \{[v, f] : v \in N(\mathcal{C}, \mathcal{D}), f \in Mod(\mathcal{C}, \mathcal{D}) \text{ and } f(v^*v) \neq 0\} \quad \text{and} \quad (7.12)
\]
\[
    \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) = \{[v, f] : v \in N(\mathcal{C}, \mathcal{D}), f \in \mathcal{S}(\mathcal{C}, \mathcal{D}) \text{ and } f(v^*v) \neq 0\}. \quad (7.13)
\]

**Standing Assumption 7.14.** Unless stated otherwise, for the remainder of this section, \( (\mathcal{C}, \mathcal{D}) \) will be a regular covering inclusion and \( F \subseteq \mathcal{S}(\mathcal{C}, \mathcal{D}) \) will be a compatible cover for \( \mathcal{D} \).

By item(ii) of Theorem 7.9(a), the \( N(\mathcal{C}, \mathcal{D}) \)-invariance of \( F \) shows that for \( \phi \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) \), we have \( s(\phi) \in F \) if and only if \( r(\phi) \in F \).

**Definition 7.15.** Let \( \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) := \{\phi \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D}) : s(\phi) \in F\} \). We shall call \( \phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \) an \( F \)-compatible eigenfunctional. By Theorem 7.9
\[
    \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) = \{[v, f] : f \in F \text{ and } f(v^*v) \neq 0\}.
\]

The proof of the following fact is essentially the same as that of [5, Proposition 2.3] (the continuity of the range and source maps follows from their definition).

**Proposition 7.16.** The set \( \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \cup \{0\} \) is a weak-* compact subset of \( \mathcal{C}^\# \), and the maps \( s, r: \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \to \mathcal{S}(\mathcal{C}, \mathcal{D}) \) are weak-*–weak-* continuous.

We now show that \( \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \) forms a topological groupoid. The topology has already been defined, so we need to define the source and range maps, composition and inverses. The hypothesis that \( F \subseteq \mathcal{S}(\mathcal{C}, \mathcal{D}) \) in the following definition ensures that the product on \( \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \) is well-defined.
Definition 7.17. Given $\phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$, let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ be such that $\phi(v) > 0$. We make the following definitions.

(a) We say that $s(\phi)$ and $r(\phi)$ are the source and range of $\phi$ respectively.

(b) Define the inverse, $\phi^{-1}$ by the formula,
$$
\phi^{-1}(x) := \phi(x^*).
$$

If $\phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is such that $\phi(v) > 0$, (so that $\phi = [v, s(\phi)]$), then a calculation shows that $\phi^{-1} = [v^*, r(\phi)]$. The fact that $F$ is $\mathcal{N}(\mathcal{C}, \mathcal{D})$-invariant ensures that $\phi^{-1} \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$. Thus, our definition of $\phi^{-1}$ is consistent with the definition of inverse in the definition of the twist of a $C^*$-diagonal arising in [12] and the twist of a Cartan MASA from [15].

(c) For $i = 1, 2$, let $\phi_i \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$. We say that the pair $(\phi_1, \phi_2)$ is a composable pair if $s(\phi_1) = r(\phi_2)$. As is customary, we write $\mathcal{E}_F^1(\mathcal{C}, \mathcal{D})^{(2)}$ for the set of composable pairs. To define the composition, choose $v_i \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ with $\phi_i(v_i) > 0$, so that $\phi_i = [v_i, s(\phi_i)]$. By item (ii) of Theorem 7.9, we have
$$
\phi_1 \phi_2 := [v_1 v_2, s(\phi_2)].
$$

We show now that this product is well defined. Suppose that $(\phi_1, \phi_2) \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})^{(2)}$, $f = s(\phi_2)$, $r(\phi_2) = g = s(\phi_1)$, and that for $i = 1, 2$, $v_i, w_i \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ are such that $\phi_1 = [v_1, g] = [w_1, g]$ and $[v_2, f] = [w_2, f]$. Then using parts (c) and (f) of Theorem 7.9, we have $g(w_1^* v_1) > 0$ and $f(v_2^* w_2) > 0$, so, as $f \in \mathcal{G}(\mathcal{C}, \mathcal{D})$, there exists a positive scalar $t$ such that $v_2 + L_f = t w_2 + L_f$. Hence,
$$
f((w_1 w_2)^*(v_1 v_2)) = \langle \pi_f(v_1)(v_2 + L_f), \pi_f(w_1)(w_2 + L_f) \rangle
$$
$$
= t \langle \pi_f(v_1)(w_2 + L_f), \pi_f(w_1)(w_2 + L_f) \rangle
$$
$$
= t f(w_1^* v_1) w_2
$$
$$
= t f(w_1^* v_1) s(\phi_1)(w_1^* v_1)
$$
$$
= t f(w_1^* v_1) g(w_1^* v_1) > 0.
$$

By Theorem 7.9, $[v_1 v_2, f] = [w_1 w_2, f]$, so that the product is well defined.

(d) For $\phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$, denote the map $\mathcal{C} \ni x \mapsto |\phi(x)|$ by $|\phi|$. Observe that for $\phi, \psi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})$, $|\phi| = |\psi|$ if and only if there exists $z \in \mathbb{T}$ such that $\phi = z \psi$; clearly $z$ is unique. Let
$$
\mathcal{R}_F(\mathcal{C}, \mathcal{D}) := \{|\phi| : \phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})\}
$$
and define $q : \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \to \mathcal{R}_F(\mathcal{C}, \mathcal{D})$ by
$$
q(\phi) := |\phi|.
$$

We now define inverse and product maps in $\mathcal{R}_F(\mathcal{C}, \mathcal{D})$, as well as source and range maps.

Since a state on $\mathcal{C}$ is determined by its values on the positive elements of $\mathcal{C}$, we may identify $f \in F$ with $|f| \in \mathcal{R}_F(\mathcal{C}, \mathcal{D})$. Define $s(\phi) = s(\phi)$ and $r(\phi) = r(\phi)$. Next we define inversion in $\mathcal{R}_F(\mathcal{C}, \mathcal{D})$ by $|\phi|^{-1} = |\phi|^{-1}$, and composable pairs by $\mathcal{R}_F(\mathcal{C}, \mathcal{D})^{(2)} := \{|(\phi, |\psi|) : (\phi, \psi) \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})^{(2)}\}$, and the product by $\mathcal{R}_F(\mathcal{C}, \mathcal{D})^{(2)} \ni (|\phi|, |\psi|) \mapsto |\phi \psi|$. Topologize $\mathcal{R}_F(\mathcal{C}, \mathcal{D})$ with the topology of point-wise convergence: $|\phi_\lambda| \to |\phi|$ if and only if $|\phi_\lambda|(x) \to |\phi|(x)$ for every $x \in \mathcal{C}$. This topology is the quotient topology arising from $q$. We call $\mathcal{R}_F(\mathcal{C}, \mathcal{D})$ the spectral groupoid over $F$ for $(\mathcal{C}, \mathcal{D})$.  

39
(e) We have already identified $\mathcal{R}_F(\mathbb{C}, \mathcal{D})^{(0)}$ with $F$. Define $\nu: T \times F \to \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ by

$$\nu(z, f) = zf.$$  

Then the action of $T$ on $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ is $(z \cdot \phi)(x) = \phi(zx)$. Notice that if $\phi$ is written as $\phi = [v, f]$, where $v \in N(\mathbb{C}, \mathcal{D})$ and $f \in F$, then $z \cdot \phi = [zv, f]$.

We now show that $(\mathcal{E}_F^1(\mathbb{C}, \mathcal{D}), \mathcal{R}_F(\mathbb{C}, \mathcal{D}), \nu, q)$ is a twist.

**Theorem 7.18.** Let $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ and $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ be as above. Then $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ and $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ are locally compact Hausdorff topological groupoids and $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ is an étale groupoid. Their unit spaces are $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})^{(0)} = \mathcal{R}_F(\mathbb{C}, \mathcal{D})^{(0)} = F$. Moreover,

$$T \times F \overset{\nu}{\to} \mathcal{E}_F^1(\mathbb{C}, \mathcal{D}) \overset{q}{\to} \mathcal{R}_F(\mathbb{C}, \mathcal{D})$$

is a locally trivial Hausdorff topological twist.

**Proof.** That inversion on $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ is continuous follows readily from the definition of inverse map and the weak-$*$ topology. Suppose $(\phi_\lambda)_{\lambda \in \Lambda}$ and $(\psi_\lambda)_{\lambda \in \Lambda}$ are nets in $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ converging to $\phi, \psi \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ respectively, and such that $(\phi_\lambda, \psi_\lambda) \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})^{(2)}$ for all $\lambda$. Since the source and range maps are continuous, we find that $s(\phi) = \lim_\lambda s(\phi_\lambda) = \lim_\lambda t(\psi_\lambda) = t(\psi)$, so $(\phi, \psi) \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})^{(2)}$. Let $v, w \in N(\mathbb{C}, \mathcal{D})$ be such that $\phi(v) > 0$ and $\psi(w) > 0$. There exists $\lambda_0$, so that $\lambda \geq \lambda_0$ implies $\phi_\lambda(v)$ and $\psi_\lambda(w)$ are non-zero. For each $\lambda \geq \lambda_0$, there exists scalars $\xi_\lambda, \eta_\lambda \in T$ such that $\phi_\lambda(v) = \xi_\lambda[v, s(\phi_\lambda)]$ and $\psi_\lambda = \eta_\lambda[v, s(\psi_\lambda)]$. Since

$$\lim_\lambda \phi_\lambda(v) = \phi(v) = \lim_\lambda [v, s(\phi_\lambda)](v) \quad \text{and} \quad \lim_\lambda \psi_\lambda(v) = \psi(v) = \lim_\lambda [v, s(\psi_\lambda)](v),$$

we conclude that $\lim_\lambda \eta_\lambda = 1 = \lim_\lambda \xi_\lambda$. So for any $x \in \mathbb{C},$

$$(\phi \psi)(x) = \frac{s(\psi)((vw)^* x)}{(s(\psi)((vw)^* (vw)))^{1/2}} = \lim_\lambda \frac{s(\psi_\lambda)((vw)^* x)}{(s(\psi_\lambda)((vw)^* (vw)))^{1/2}} = \lim_\lambda \frac{[v, s(\phi_\lambda)][w, s(\psi_\lambda)](x)}{\lambda}$$

giving continuity of multiplication. Notice that for $\phi \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$, $s(\phi) = \phi^{-1} \phi$ and $t(\phi) = \phi \phi^{-1}$, and $F \subseteq \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$. Thus, $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ is a locally compact Hausdorff topological groupoid with unit space $F$.

The definitions show that $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ is a groupoid. By construction, the map $q$ is continuous and is a surjective groupoid homomorphism. The topology on $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ is clearly Hausdorff. If $\phi \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$, and $v \in N(\mathbb{C}, \mathcal{D})$ is such that $\phi(v) \neq 0$, then $W := \{ \alpha \in \mathcal{R}_F(\mathbb{C}, \mathcal{D}) : \alpha(v) > |\phi(v)|/2 \}$ has compact closure so $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ is locally compact. Also, if $\alpha_1, \alpha_2 \in W$ and $t(\alpha_1) = t(\alpha_2) = f$, then writing $\alpha_i = |\psi_i|$ for $\psi_i \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$, we see that $\psi_i(v) \neq 0$, so there exist $z_1, z_2 \in T$ so that for $i = 1, 2$ and every $x \in \psi_i$, $\psi_i(x) = z_i f(xy^*) f(v)^{-1}$. Hence $\alpha_1 = \alpha_2$ showing that the range map is locally injective. We already know that the range map is continuous, so by local compactness, the range map is a local homeomorphism.

Note that convergent nets in $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ can be lifted to convergent nets in $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$. Indeed, if $q(\phi_\lambda) \to q(\phi)$ for some net $(\phi_\lambda)$ and $\phi \in \mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$, choose $v \in N(\mathbb{C}, \mathcal{D})$ so that $\phi(v) > 0$. Then for large enough $\lambda$, $\phi_\lambda(v) \neq 0$. Then $q([v, s(\phi_\lambda)]) = q(\phi_\lambda)$ and $[v, s(\phi_\lambda)] \to \phi$. The fact that the groupoid operations on $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ are continuous now follows easily from the continuity of the groupoid operations on $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$. Thus $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$ is a locally compact Hausdorff étale groupoid.

Finally, $q(\phi_1) = q(\phi_2)$ if and only if there exist $z \in T$ so that $\phi_1 = z \phi_2$. Moreover, for each $v \in N(\mathbb{C}, \mathcal{D})$, let $F_v := \{ \rho \in F : \rho(v^* v) > 0 \}$ and set $O_v := \{ [v, \rho] : \rho \in F_v \}$. Then the map $f \mapsto [v, f]$, where $f \in F_v$ is a continuous section for $q|_{O_v}$, so $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ is locally trivial. Also, the action of $\mathbb{T}$ on $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ given above makes $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ into a $\mathbb{T}$-groupoid. So $\mathcal{E}_F^1(\mathbb{C}, \mathcal{D})$ is a twist over $\mathcal{R}_F(\mathbb{C}, \mathcal{D})$.  

40
Remark 7.19. Let \((\mathcal{C}, \mathcal{D})\) be a Cartan pair. Consider two twists associated with \((\mathcal{C}, \mathcal{D})\): the Weyl twist, and the twist obtained from Theorem 7.18 applied with the family of strongly compatible states. These twists can be seen to be isomorphic as follows. Let \(E : \mathcal{C} \to \mathcal{D}\) be the conditional expectation, let \((\Sigma_W, G_W, \iota_W, q_W)\) be the Weyl twist associated to \((\mathcal{C}, \mathcal{D})\), and let \(F = \{\sigma \circ E : \sigma \in \hat{D}\}\) be the family of strongly compatible states on \(\mathcal{C}\). Using direct arguments (or the results in Section 4.2), one can show that the maps \(\Sigma_W \ni [\sigma_1, v, \sigma_2]_1 \mapsto [v, \sigma_2] \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})\) and \(G_W \ni [\sigma_1, v, \sigma_2]_1 \mapsto [v, \sigma_2] \in \mathcal{R}_F(\mathcal{C}, \mathcal{D})\) are isomorphisms of topological groupoids. Further, notice that for \(z \in \mathbb{T}, [\sigma_1, zv, \sigma_2]_1 \mapsto z[v, \sigma_2]\). Thus the twist \((\mathcal{E}_F^1(\mathcal{C}, \mathcal{D}), \mathcal{R}_F(\mathcal{C}, \mathcal{D}), \iota, q)\) is isomorphic to the conjugate Weyl twist, \((\Sigma_W, G_W, \overline{\iota}_W, q_W)\). In particular, by Renault’s theorem and Proposition 3.13,

\[(\mathcal{C}, \mathcal{D}) \simeq \mathcal{C}_r^*(\Sigma_W, G_W, -1) \simeq \mathcal{C}_r^*(\mathcal{E}_F^1(\mathcal{C}, \mathcal{D}), \mathcal{R}_F(\mathcal{C}, \mathcal{D}), 1). \tag{7.20}\]

**Notation 7.21.** For the remainder of this section, we use the following notation.

(a) Write

\[\Sigma = \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})\] and \(G = \mathcal{R}_F(\mathcal{C}, \mathcal{D})\), so that \(G^{(0)} = F\).

As in Section 3, for \(\phi \in \Sigma\), we will sometimes write \(\hat{\phi}\) instead of \(|\phi|\).

(b) For \(a \in \mathcal{C}\), define \(g(a) : \mathcal{E}_F^1(\mathcal{C}, \mathcal{D}) \to \mathbb{C}\) to be the ‘Gelfand’ map: for \(\phi \in \mathcal{E}_F^1(\mathcal{C}, \mathcal{D})\),

\[g(a)(\phi) = \phi(a).\]

Then \(g(a)\) is a continuous 1-equivariant function on \(\mathcal{E}_F^1(\mathcal{C}, \mathcal{D})\).

(c) Because of (7.20), we will write

\[C_c(\Sigma, G)\ (\text{resp. } C_c^*(\Sigma, G)) \quad \text{instead of } \quad C_c(\Sigma, G, 1)\ (\text{resp. } C_c^*(\Sigma, G, 1))\]

unless there is danger of confusion. Also, instead of writing \(L_1\) for the 1-equivariant line bundle associated to \((\Sigma, G)\), we will write \(L\). Furthermore, when referring to elements of \(L\), we drop the subscript and write \([\lambda, \phi]\) instead of \([\lambda, \phi]_1\) for the equivalence class in \(L\) associated to \((\lambda, \phi) \in \mathbb{C} \times \Sigma\).

We aim to show that \(g\) determines a regular homomorphism of \(\mathcal{C}\) into \(C_c^*(\Sigma, G)\). For this, it would be convenient if whenever \(v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), g(v) \in C_c(\Sigma, G, 1)\). However, this need not be the case. (For an example, let \(\mathcal{H} = \ell^2(\mathbb{Z})\) with standard orthonormal basis \(\{e_n\}_{n \in \mathbb{Z}}\), let \(\mathcal{K}\) be the compact operators on \(\mathcal{H}\), put \(\mathcal{C} := C^*(\mathcal{K} \cup \{1\})\) and \(\mathcal{D} = C^*(\{1\} \cup \{e_n e_n^* : n \in \mathbb{Z}\})\). Then \((\mathcal{C}, \mathcal{D})\) is a regular MASA inclusion and \(v := \sum_{n \in \mathbb{N}} \frac{e_n e_n^*}{n}\) is a normalizer whose support is clopen and not compact.) To circumvent this issue, we use the following technical tool.

**Lemma 7.22.** Let \(w \in \mathcal{N}(\mathcal{C}, \mathcal{D})\). The following statements hold.

(a) \(\text{supp}(g(w)) = \{[w, f] : f(w^*w) > 0\}\) and hence \(\text{supp}(g(w))\) is an open bisection of \(G\).

(b) If \(d \in \mathcal{D}\) satisfies

\[\{\sigma \in \mathcal{D} : \sigma(d) \neq 0\} \subseteq \{\sigma \in \hat{D} : \sigma(w^*w) \neq 0\}, \tag{7.23}\]

then \(\text{supp}(g(wd))\) is compact, that is, \(g(wd) \in C_c(\Sigma, G, 1)\).

(c) If \(\varepsilon > 0\), there exists \(d \in \mathcal{D}\) satisfying (7.23) such that \(\|w - wd\| < \varepsilon\).
Proof. (a) We have defined $\supp g(w) = \{ \phi \in G : |\phi(w)| \neq 0 \}$, so $\supp g(w)$ is open. By definition, for any $f \in F$ with $f(w^*w) \neq 0$, $|[w,f]| \in \supp g(w)$. Conversely, for $\phi = [v,f] \in \mathcal{E}_F^+(\mathcal{C},\mathcal{D})$,
\[
|\phi(w)| \neq 0 \Rightarrow f(v^*w) \neq 0 \\
\Rightarrow |[w,f]| = |[v,f]| \quad (\text{Theorem 7.29}(f)) \\
\Rightarrow f(w^*w) \neq 0.
\]
Thus, $\supp(g(w)) = \{|[w,f]| : f(w^*w) > 0\}$.

We now show $\tau_{\supp g(w)}$ and $s_{\supp g(w)}$ are one-to-one. Let $|\phi_1|, |\phi_2|$ belong to $\supp(g(w))$, let $f_i := s(|\phi_i|)$ and $g_i := \tau(|\phi_i|)$. As just observed, we may write $|\phi_i| = |[w,f_i]|$. If $f_1 = f_2$, then $|\phi_1| = |[w,f]| = |\phi_2|$, so $s_{\supp g(w)}$ is one-to-one. On the other hand, if $g_1 = g_2$, then $|\phi_1|^* = |\phi_2|^* = |[w^*,g_1]| = |[w^*,g_2]| = |\phi_2|^*$. Therefore, $|\phi_1| = |\phi_2|$ so $\tau_{\supp g(w)}$ is one-to-one. Thus $\supp(g(w))$ is an open bisection.

(b) Let $B := \{ f \in F : f(w^*w) \neq 0 \}$. Then $s_{\supp g(w)} : \supp g(w) \rightarrow B$ is a homeomorphism.

Suppose $d \in \mathcal{D}$ satisfies \(\ref{2.23}\). Let $A := \{ f \in F : f(d) \neq 0 \}$ and observe that $s(\supp(g(wd))) = A$. For $g \in \mathcal{A}$, we may find a net $(g_\lambda)$ in $A$ such that $g_\lambda \rightarrow g$. Then $(g_\lambda)_|\mathcal{D} \rightarrow g|\mathcal{D}$, so $g|\mathcal{D} \in \{ \sigma \in \mathcal{D} : \sigma(w^*w) \neq 0 \}$. Thus, $g(w^*w) \neq 0$, whence $\overline{A} \subseteq B$. Since $F$ is compact, so is $\mathcal{A}$. But $s^{-1}$ is a homeomorphism of $B$ onto $\supp(g(w))$ and therefore $s^{-1}(\overline{A}) = \supp(g(wd))$ is compact, as desired.

(c) If $w = 0$, this is obvious, so assume $w \neq 0$. Without loss of generality, we may assume $\|w\| = 1$. Let $S \subseteq [0,1]$ be the spectrum of $w^*w$. Suppose first that 0 is not an isolated point of $S \cup \{0\}$. The sets $X_\epsilon := \{ \sigma \in \mathcal{D} : \sigma(w^*w) \geq \epsilon^2 \}$ and $Y_\epsilon := \{ \sigma \in \mathcal{D} : \sigma(w^*w) \leq \epsilon^2/4 \}$ are closed, disjoint, and non-empty, so there exists an element $d \in \mathcal{D}$ with $0 \leq d \leq I$ such that $\sigma(d) = 1$ for $\sigma \in X_\epsilon$ and $\sigma(d) = 0$ for $\sigma \in Y_\epsilon$. Then for any $\sigma \in \mathcal{D}$,
\[
\sigma((I-d)w^*w(I-d)) = \sigma(w^*w)\sigma(I-d)^2 < \epsilon^2,
\]
so the result holds in this case.

If 0 is an isolated point of $S \cup \{0\}$, then there is a projection $d \in \mathcal{D}$ such that $\hat{d}$ is the characteristic function of $S \setminus \{0\}$. Then $v = vd$ and $d$ satisfies \(\ref{2.23}\).

\[ \square \]

Before stating the main result of this section, recall that Proposition \ref{2.23} shows that $\mathcal{X}_F = \{ x \in \mathcal{C} : f(x^*x) = 0 \text{ for all } f \in F \}$ is an ideal of $\mathcal{C}$ whose intersection with $\mathcal{D}$ is trivial.

**Theorem 7.24.** Let $(\mathcal{C},\mathcal{D})$ be a regular covering inclusion, $F$ a compatible cover for $\mathcal{D}$, and let $G := \mathcal{R}_F(\mathcal{C},\mathcal{D})$ and $\Sigma := \mathcal{E}_F^+(\mathcal{C},\mathcal{D})$. The map $g$ extends uniquely to a regular $*$-homomorphism $\theta_F : (\mathcal{C},\mathcal{D}) \rightarrow (C_r^*(\Sigma,G),C(G^{(0)}))$ with $\ker \theta = \mathcal{X}_F$. Furthermore, the following statements hold.

(a) The $C^*$-algebra generated by $\theta_F(\mathcal{C}) \cup C(G^{(0)})$ is $C_r^*(\Sigma,G)$.

(b) $\theta_F(\mathcal{C})$ is dense in $C_r^*(\Sigma,G)$ in the $\Sigma$-pointwise topology.

**Proof.** Throughout the proof, $\mathbb{E}$ will denote the (necessarily faithful) conditional expectation of $C_r^*(\Sigma,G)$ onto $C(G^{(0)})$. Also, let $N_0(\mathcal{C},\mathcal{D}) := \{ v \in \mathcal{N}(\mathcal{C},\mathcal{D}) : g(v) \in C_c(\Sigma,G) \}$ and $\mathcal{C}_0 := \text{span}_\mathbb{F} N_0(\mathcal{C},\mathcal{D})$. Finally, during the proof, we will write $\theta$ instead of $\theta_F$.

Once again, recall $F = G^{(0)}$. We regard $C_c(\Sigma,G)$ as a dense subalgebra of $C_r^*(\Sigma,G)$. A computation shows that any element of $C_r(\Sigma,G)$ supported in an open bisection of $G$ belongs to $N(C_r^*(\Sigma,G),C(G^{(0)}))$. Clearly $g : \mathcal{C}_0 \rightarrow C_c(\Sigma,G)$. A computation shows that for $d \in \mathcal{D}$, $g(d)$ is supported on $G^{(0)}$. So $g(\mathcal{D}) \subseteq C(G^{(0)})$.

Next we show that $g$ is a $*$-homomorphism of $\mathcal{C}_0$ into $C_c(\Sigma,G)$. To do this, it suffices to show that for $w,w_1,w_2 \in N_0(\mathcal{C},\mathcal{D})$,
\[
g(w_1w_2) = g(w_1)g(w_2) \quad (\text{7.25})
\]
Indeed, these equalities imply \( N_0(\mathfrak{C}, \mathcal{D}) \) is a \( * \)-semigroup, and as \( g \) is linear, it will follow that \( g \) is a \( * \)-homomorphism. We now establish (7.25) and (7.26).

For \( i = 1, 2 \), view \( g(w_i) \) as a continuous section of the line bundle \( L \), that is, \( g(w_i)(\hat{\phi}) = [\phi(w_i), \phi_i]_L \), where we have (temporarily) added a subscript to aid in distinguishing elements of \( L \) from elements \( \phi = [v, s(\phi)] \in \Sigma \).

Suppose \( \hat{\phi} \in \text{supp}(g(w_1)g(w_2)) \). Lemma (7.22) shows that for \( i = 1, 2 \), \( g(w_i) \) are supported on open bisections. An examination of the definition of multiplication in \( C_*(\Sigma, G) \) (see (3.10)) shows that there is exactly one composable pair \((\hat{\phi}_1, \hat{\phi}_2) \in G^2\) which contributes to the sum defining the product; that is, for \( i = 1, 2 \), there are unique \( \hat{\phi}_i \in \text{supp}(g(w_i)) \) with

\[
s(\hat{\phi}_2) = s(\hat{\phi}), \quad s(\hat{\phi}_1) = r(\hat{\phi}_2) \quad \text{and} \quad \hat{\phi} = \hat{\phi}_1 \hat{\phi}_2.
\]

As \( \hat{\phi} \in \text{supp}(g(w_1)g(w_2)) \), \( g(w_i)(\hat{\phi}_i) \neq 0 \), so regardless of the choice of \( \phi_i \in q^{-1}(\hat{\phi}_i) \), we have \( \phi_i(w_i) \neq 0 \). By multiplying by appropriate elements of \( T \), we may choose \( \phi_i \in q^{-1}(\hat{\phi}_i) \) so that \( \phi_i(w_i) > 0 \). With these choices of \( \phi_i \), we now take \( \phi := \phi_1 \phi_2 \in q^{-1}(\hat{\phi}) \). In particular, by Theorem (7.9) we may represent

\[
\phi_i = [w_i, s(\phi_i)], \quad \text{so that} \quad \phi = [w_1 w_2, s(\phi)].
\]

Then

\[
(g(w_1)g(w_2))(\hat{\phi}) = g(w_1)(\hat{\phi}_1)g(w_2)(\hat{\phi}_2) = [\phi_1(w_1), \phi_1]_L [\phi_2(w_2), \phi_2]_L = [\phi_2(w_1)\phi_2(w_2), \phi]_L. \quad (7.27)
\]

As \( s(\hat{\phi}_2) = s(\hat{\phi}) \),

\[
\phi_1(w_1)\phi_2(w_2) = \sqrt{s(\hat{\phi}_1)(w_1^* w_1) s(\hat{\phi}_2)(w_2^* w_2)}
\]

\[
= \sqrt{r(\hat{\phi}_2)(w_1^* w_1) s(\hat{\phi}_2)(w_2^* w_2)} = \left( \frac{s(\hat{\phi}_2)(w_2^* w_1^* w_1 w_2)}{s(\hat{\phi}_2)(w_2^* w_2)} \right)^{1/2} \left( s(\hat{\phi}_2)(w_2^* w_2) \right)^{1/2}
\]

\[
= \sqrt{s(\hat{\phi}_2)(w_2^* w_1^* w_1 w_2)} = [w_1 w_2, s(\phi)](w_1 w_2) = \phi(w_1 w_2).
\]

Therefore, when \( \hat{\phi} \in \text{supp}(g(w_1)g(w_2)) \),

\[
(g(w_1)g(w_2))(\hat{\phi}) = [\phi_1(w_1), \phi_2(w_2), \phi]_L = [\phi(w_1 w_2), \phi]_L = g(w_1 w_2)(\hat{\phi}), \quad (7.28)
\]

where the first equality uses (7.27). It follows that \( \text{supp}(g(w_1)g(w_2)) \subseteq \text{supp}(g(w_1 w_2)) \).

Now suppose \( \hat{\phi} \in \text{supp}(g(w_1 w_2)) \). Choose \( \phi \in q^{-1}(\hat{\phi}) \) so that \( \phi(w_1 w_2) > 0 \) and write \( \phi = [w_1 w_2, s(\phi)] \), so that

\[
0 < \phi(w_1 w_2) = \sqrt{s(\phi)(w_2^* w_1^* w_1 w_2)}. \quad (7.29)
\]

The Cauchy-Schwartz inequality gives \( s(\phi)(w_2^* w_2) \neq 0 \), so \( \phi_2 := [w_2, s(\phi)] \in \Sigma \). Furthermore, (7.29) shows \( r(\hat{\phi}_2)(w_1^* w_1) \neq 0 \), so \( \phi_1 := [w_1, r(\hat{\phi}_2)] \in \Sigma \). This yields the factorization \( \hat{\phi} = \hat{\phi}_1 \hat{\phi}_2 \). Also, for \( i = 1, 2 \), \( \phi_i(w_i) \neq 0 \), so \( \phi_i \in \text{supp}(g(w_i)) \). But then

\[
0 \neq g(w_1)(\hat{\phi}_1) g(w_2)(\hat{\phi}_2) = (g(w_1)g(w_2))(\hat{\phi}),
\]

so \( \hat{\phi} \in \text{supp}(g(w_1)g(w_2)) \). We have now shown that \( \text{supp}(g(w_1)g(w_2)) = \text{supp}(g(w_1 w_2)) \). Then (7.28) gives (7.25), as desired.

For any \( w \in N_0(\mathfrak{C}, \mathcal{D}) \) and \( \phi = [v, s(\phi)] \in \Sigma \), \( \phi^{-1} = [v^*, r(\phi)] \), so

\[
\phi^{-1}(w) = \frac{r(\phi)(vw)}{r(\phi)(vw^*)}^{1/2} = \frac{s(\phi)(vw)}{s(\phi)(v^* v)^{1/2}} = [v, s(\phi)](w^*) = \phi(w^*).
\]
Therefore,
\[(\mathfrak{g}(w^*)(\hat{\phi})) = \mathfrak{g}(w)(\hat{\phi}^{-1}) = (\hat{\phi}^{-1}(w), \phi^{-1})_L = (\hat{\phi}^{-1}(w), \phi)_L = \mathfrak{g}(w^*)(\hat{\phi}).\]

Thus (7.26) holds, and, as noted earlier, we conclude \(\mathfrak{g}\) is a \(*\)-homomorphism.

We now turn to showing that \(\mathfrak{g}\) is contractive. The point is that the norms on \(\mathcal{C}/\mathcal{K}_F\) and \(C_r^*(\Sigma, G)\) both arise from the left regular representation on appropriate spaces. Here are the details.

Let \(f \in F\). Then \(f\) can be regarded as either a state on \(\mathcal{C}\) or as determining a state on \(C_r^*(\Sigma, G)\) via evaluation at \(f\). We write \(f_c\) when viewing \(f\) as a state on \(\mathcal{C}\), and \(f_\Sigma\) when viewing \(f\) as a state on \(C_r^*(\Sigma, G)\).

Let \((\pi_{\Sigma, f}, \mathcal{H}_{\Sigma, f})\) be the GNS representation of \(\mathcal{C}\) arising from \(f_c\), and let \((\pi_{\Sigma, f}, \mathcal{H}_{\Sigma, f})\) be the GNS representation of \(C_r^*(\Sigma, G)\) determined by \(f_\Sigma\).

Now fix \(f \in G(0)\). For \(a_1, a_2 \in \mathcal{C}_0\),
\[
\langle a_1 + L_f, a_2 + L_f \rangle_{\mathcal{H}_c} = f_c(a_2^* a_1) = \mathfrak{g}(a^* a_1)(f) = (\mathfrak{g}(a)^* \mathfrak{g}(a_1))(f) = f_\Sigma(\mathfrak{g}(a)^* \mathfrak{g}(a_1)) = \langle \mathfrak{g}(a_1) + N_f, \mathfrak{g}(a_2) + N_f \rangle_{\mathcal{H}_\Sigma}.
\]

It follows that the map \(a + L_f \mapsto \mathfrak{g}(a) + N_f\) extends to an isometry \(W_f : \mathcal{H}_c \to \mathcal{H}_{\Sigma, f}\).

Next, notice that for \(a_1, a_2 \in \mathcal{C}_0\),
\[
\pi_{\Sigma, f}(\mathfrak{g}(a_1)) W_f(a_2 + L_f) = \mathfrak{g}(a_1 a_2) + N_f = W_f \pi_{\Sigma, f}(a_1)(a_2 + L_f).
\]

Thus for every \(a \in \mathcal{C}_0\),
\[
\pi_{\Sigma, f}(a) = W_f^* \pi_{\Sigma, f}(\mathfrak{g}(a)) W_f, \text{ so } \|a\|_c \geq \|\pi_{\Sigma, f}(a)\| \geq \|\pi_{\Sigma, f}(\mathfrak{g}(a))\|.
\]

We conclude that for \(a \in \mathcal{C}_0\),
\[
\|a\|_c \geq \sup_{f \in F} \|\pi_{\Sigma, f}(\mathfrak{g}(a))\| = \|\mathfrak{g}(a)\|_{C_r^*(\Sigma, G)}.
\]

Lemma (7.22) implies \(\mathcal{C}_0\) is norm-dense in \(\mathcal{C}\), so \(\mathfrak{g}\) extends by continuity to a \(*\)-homomorphism \(\theta : \mathcal{C} \to C_r^*(\Sigma, G)\). For \(v \in N(\mathcal{C}, \mathcal{D})\), Lemma (7.22) shows that \(v \in N_0(\mathcal{C}, \mathcal{D})\), so \(\theta(v) \in \mathfrak{g}(N_0(\mathcal{C}, \mathcal{D})) \subseteq N(C_r^*(\Sigma, G), C(G(0))).\) Thus, \(\theta\) is a regular \(*\)-homomorphism.

Let us show \(\mathcal{K}_F = \ker \theta\). Then for \(f \in F\), \(f_\Sigma \circ \mathfrak{E} = f_\Sigma\). Furthermore, \(f_c\) and \(f_\Sigma \circ \theta\) agree on the dense set \(\mathcal{C}_0\), so \(f_c = f_\Sigma \circ \theta = f_\Sigma \circ \mathfrak{E} \circ \theta\). For \(x \in \mathcal{C}\),
\[
x \in \mathcal{K}_F \iff \mathfrak{E}(\theta(x^*x)) = 0 \iff x \in \ker \theta.
\]

We now establish statement (a), that is, the \(C^*\)-algebra generated by \(\theta(\mathcal{C}) \cup C(G(0))\) is \(C_r^*(\Sigma, G)\). Let
\[
\mathcal{M} := \{h \mathfrak{g}(v) k : v \in N_0(\mathcal{C}, \mathcal{D}) \text{ and } h, k \in C(G(0))\}.
\]
Then \(\mathcal{M}\) contains \(C(G(0))\) and is a \(*\)-semigroup of normalizers in \(C_r^*(\Sigma, G)\). We will show span \(\mathcal{M}\) is dense in \(C_r^*(\Sigma, G)\). We require the following fact.

**Fact.** Suppose \(U \subseteq G\) is an open bisection and \(u \in C_c(\Sigma, G)\) satisfies \(\text{supp}(u) \subseteq U\). If \(\hat{\phi} \in \text{supp}(u)\), then there exists \(k \in C(G(0))\) such that \(uk \in \mathcal{M}\) and
\[
\hat{\phi} \in \text{supp}(uk) \subseteq \text{supp}(uk) \subseteq \text{supp}(u).
\]

**Proof of the Fact.** Note that if \(w \in C_c(\Sigma, G)\) satisfies \(\text{supp} w \subseteq U\), then \(\hat{w} \in \text{supp}(w) \iff s(\hat{w}) \in \text{supp}(w^*w)\). We will repeatedly use this.

Choose \(\phi \in \mathfrak{g}^{-1}(\hat{\phi})\) and write \(\phi = [v, s(\phi)]\) for some \(v \in N(\mathcal{C}, \mathcal{D})\). As \(\phi(v) > 0\), by Lemma (7.22) we may assume without loss of generality that \(v \in N_0(\mathcal{C}, \mathcal{D})\). Then \(\hat{\phi} \in \text{supp}(\mathfrak{g}(v)) \cap \text{supp}(u)\).
Therefore, there exists \( h \in C(G(0)) \) so that \( h(\phi) = 1 \) and \( \text{supp}(\overline{h}) \subseteq (\text{supp}(g(v^*v)) \cap \text{supp}(u^*u)) \). Thus,
\[
\phi \in \text{supp}(g(v^*)h) \subseteq \text{supp}(u).
\]
By Lemma 3.18, \( E(uh^*(g(v^*)) g(v^*)h) = uh^*g(v^*v)h \). Take \( k = h^*g(v^*v)h \). Then \( k \in C(G(0)) \) and \( uk \in M \). Since \( s(\phi) \in \text{supp}(g(v^*v)) \cap \text{supp}(h) \subseteq \text{supp}(g(v^*v)) \cap \text{supp}(u^*u) \), we have \( \phi \in \text{supp}(uk) \). This establishes the fact.

Now fix \( u \in C_c(\Sigma, G) \) with the property that its closed support is contained in an open bisection \( U \subseteq G \). Let
\[
J := \{ k \in C(G(0)) : uk \in \text{span}M \}.
\]
As \( M \) is a \( * \)-semigroup, \( J \) is a closed ideal of \( C(G(0)) \). The fact shows that if \( \phi \in \text{supp}(u) \), then there exists \( k \in J \) such that \( \phi \in \text{supp}(uk) \). Thus \( s(\phi) \) does not annihilate \( J \) because \( (uk)(\phi) = u(\phi) k(g(\phi)) \). Therefore, \( C_0(\text{supp}(u^*u)) \subseteq J \). Let \( (h_\lambda)_{\lambda \in \Lambda} \) be an approximate unit for \( C_0(\text{supp}(u^*u)) \). Then
\[
\| u - uh_\lambda \|^2 = \| u^*u - 2u^*uh_\lambda + h_\lambda^2u^*u \| \to 0.
\]
Thus, \( u \in \text{span}M \).

It follows from [6, Proposition 3.10] that \( C_c(\Sigma, G) \subseteq \text{span}M \), and therefore \( \text{span}M \) is dense in \( C^*_r(\Sigma, G) \). This establishes part (a).

Finally, we turn to establishing part (b), the \( \Sigma \)-pointwise density of \( \theta(\mathcal{C}) \) in \( C^*_r(\Sigma, G) \). Let \( \mathcal{X} \subseteq C^*_r(\Sigma, G)^\# \) be the linear span of the evaluation functionals \( \xi \mapsto \xi(\phi) \) where \( \xi \in C^*_r(\Sigma, G) \) and \( \phi \in \Sigma \). Suppose \( \mu \in \mathcal{X} \) annihilates \( \theta(\mathcal{C}) \). Then there exists \( n \in \mathbb{N} \), scalars \( \lambda_1, \ldots, \lambda_n \), elements \( v_1, \ldots, v_n \in \mathcal{N}(\mathcal{C}, \mathcal{D}) \) and \( f_1, \ldots, f_n \in F \) such that for any \( \xi \in C^*_r(\Sigma, G) \),
\[
\mu(\xi) = \sum_{k=1}^n \lambda_k \xi([v_k, f_k]).
\]
Without loss of generality, we may assume that \( [v_i, f_i] \neq [v_j, f_j] \) if \( i \neq j \). Since \( \mu \) annihilates \( \theta(\mathcal{C}) \), for every \( a \in \mathcal{C}_0 \),
\[
0 = \mu(g(a)) = \sum_{k=1}^n \lambda_k [v_k, f_k](a).
\]
Fix \( 1 \leq j \leq n \), and let \( d, e \in \mathcal{D} \) be such that \( \beta_{v_j}(f_j)(d) = f_j(e) = 1 \). For \( i \neq j \), since \( [v_i, f_i] \neq [v_j, f_j] \), either \( f_j \neq f_i \) or \( \beta_{v_i}(f_i) \neq \beta_{v_j}(f_j) \). Hence we assume that \( d \) and \( e \) have been chosen so that if \( i \neq j \), then \([v_i, f_i] (d) = 0 \). Then
\[
\mu(g(v_j)) = \lambda_j f_j (v_j^*v_j)^{1/2} = 0.
\]
As \( f_j(v_j^*v_j) \neq 0 \), we obtain \( \lambda_j = 0 \). It follows that \( \mu = 0 \). Since the dual of \( C^*_r(\Sigma, G) \) equipped with the \( \Sigma \)-pointwise topology is \( \mathcal{X} \), we conclude that \( \theta(\mathcal{C}) \) is dense in the \( \Sigma \)-pointwise topology on \( C^*_r(\Sigma, G) \). This completes the proof.

Recalling that \( \mathcal{K}_F \cap \mathcal{D} = (0) \), we will abuse notation and view \( \mathcal{D} \) as a subalgebra of \( \mathcal{C} \) or \( \mathcal{C}/\mathcal{K}_F \) depending on context. The following is immediate.

**Corollary 7.30.** \((C^*_r(\Sigma, G), C(G(0)), \theta_F) \) is a package for \((\mathcal{C}/\mathcal{K}_F, \mathcal{D})\).

Recall that when \((\mathcal{C}, \mathcal{D}) \) has the unique pseudo-expectation property, then \( \mathcal{S}_s(\mathcal{C}, \mathcal{D}) := E^\#(\mathcal{I}(\mathcal{D})) \), where \( E \) is the pseudo-expectation.

When \((\mathcal{C}, \mathcal{D}) \) has a Cartan envelope, the following gives a description of the Cartan envelope as the \( C^* \)-algebra of a twist.

45
Corollary 7.31. Suppose \((\mathcal{C}, \mathcal{D})\) has the unique pseudo-expectation property, and let \(F := \mathcal{S}_s(\mathcal{C}, \mathcal{D})\). Then \((C^*(\Sigma_F, G_F), C(\Sigma_F^0), \theta_F)\) is the Cartan envelope for \((\mathcal{C}/\mathcal{K}_F, \mathcal{D})\).

Proof. Theorem 6.39 shows \(F = G^0\) is a compatible cover for \(\hat{\mathcal{D}}\) and Theorem 6.11(b) implies that \((F, r)\) is an essential cover for \(\hat{\mathcal{D}}\). Therefore, the map \(\alpha : \mathcal{D} \rightarrow C(F)\) given by \(d \mapsto d \circ r\) yields an essential extension \((C(F), \alpha)\) for \(\mathcal{D}\). As \(\theta_F|_F = \alpha\), Theorem 7.32 shows that \((C^*(\Sigma_F, G_F), C(F), \theta)\) is a Cartan envelope for \(\mathcal{C}/\mathcal{K}_F, \mathcal{D}\).

Different compatible covers yield different twists, and hence different reduced \(C^*\)-algebras. We now describe the relationship between these objects when given a (set-theoretic) inclusion of compatible covers.

Proposition 7.32. Let \((\mathcal{C}, \mathcal{D})\) be a covering inclusion and for \(i = 1, 2\), suppose \(F_i \subseteq \mathcal{S}_s(\mathcal{C}, \mathcal{D})\) are compatible covers for \(\hat{\mathcal{D}}\). Put \(\Sigma_i = \mathcal{E}_{F_i}^1(\mathcal{C}, \mathcal{D}), G_i = \mathfrak{R}_{F_i}(\mathcal{C}, \mathcal{D})\) and let \(\theta_i : \mathcal{C} \rightarrow C^*_r(\Sigma_i, G_i)\) be the homomorphism described in Theorem 7.23. If \(F_1 \subseteq F_2\), then \(\Sigma_1 \subseteq \Sigma_2\) and there exists a \(*\)-epimorphism \(q : C^*_r(\Sigma_2, G_2) \rightarrow C^*_r(\Sigma_1, G_1)\) such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\theta_2} & C^*_r(\Sigma_2, G_2) \\
\downarrow{\theta_1} & & \downarrow{q} \\
\mathcal{C}/\mathcal{K}_F & \xrightarrow{\tilde{\theta}_2} & C^*_r(\Sigma_1, G_1)
\end{array}
\]

Proof. Let \(F_1\) and \(F_2\) be compatible covers for \(\hat{\mathcal{D}}\) with \(F_1 \subseteq F_2\). Then

\[\{[v, \rho_1] : \rho_1 \in F_1, v \in N(\mathcal{C}, \mathcal{D}), \rho_1(v^*v) \neq 0\} \subseteq \{[v, \rho_2] : \rho_2 \in F_2, v \in N(\mathcal{C}, \mathcal{D}), \rho_2(v^*v) \neq 0\}.\]

The continuity of the source map and the fact that \(F_1\) is closed implies that \(\mathcal{E}_{F_1}^1(\mathcal{C}, \mathcal{D})\) is a closed subgroupoid of \(\mathcal{E}_{F_2}^1(\mathcal{C}, \mathcal{D})\). Similarly, \(\mathfrak{R}_{F_1}(\mathcal{C}, \mathcal{D})\) is a closed subgroupoid of \(\mathfrak{R}_{F_2}(\mathcal{C}, \mathcal{D})\). In other words, \(G_1\) is a closed subgroupoid of \(G_2\) and \(\Sigma_1\) is a closed subgroupoid of \(\Sigma_2\).

Suppose \([v, \rho_1] \in G_1\) and that for some \([w, \rho_2], [w', \rho_2'] \in G_2\), \([v, \rho_1]\) factors as

\[|[v, \rho_1]| = |[w, \rho_2]| [w', \rho_2'].\]

Then \(\rho_2' = \rho_2\) and \(\tilde{\theta}_w(\rho_2') = \rho_2\). As \(\rho_1 \in F_1\) and \(w' \in N(\mathcal{C}, \mathcal{D})\), the invariance of \(F_1\) gives \(\rho_2' \in F_1\). Thus \([w, \rho_1]\) and \([w', \rho_2]\) belong to \(G_1\). An application of Lemma 3.20 shows that the restriction mapping extends to a \(*\)-epimorphism \(q : C_c(\Sigma_2, G_2) \rightarrow C_c(\Sigma_1, G_1)\) such that \(q \circ \tilde{\theta}_2 = \theta_1\) follows from the definition of \(\theta_i\).

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