On the Feynman path integrals for the Schrödinger equations with polynomially growing potentials in the spatial direction

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Abstract

The Feynman path integrals for the Schrödinger equations are defined mathematically, in particular, with polynomially growing potentials in the spatial direction. For example, we can handle potentials $A_j = 0$ and $V(t, x) = |x|^{2l} + \text{"a lower order polynomial"} \ (l = 1, 2, \ldots)$. The Feynman path integrals are defined as $L^2$-valued continuous functions with respect to the time variable. This problem has not been solved for a long time.

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1 Introduction

Let $T > 0$ be an arbitrary constant, $0 \leq t \leq T$ and $x = (x_1, \ldots, x_d) \in R^d$. Let $E(t, x) = (E_1, \ldots, E_d) \in R^d$ and $B(t, x) = (B_{jk}(t, x))_{1 \leq j < k \leq d} \in R^{d(d-1)/2}$ denote the electric strength and the magnetic strength tensor, respectively and $(V(t, x), A(t, x)) = (V, A_1, \ldots, A_d) \in R^{d+1}$ an electromagnetic potential, i.e.

$$E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x},$$

$$B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad (1 \leq j < k \leq d),$$

(1.1)

where $\partial V/\partial x = (\partial V/\partial x_1, \ldots, \partial V/\partial x_d)$. Then the Lagrangian function is given by

$$\mathcal{L}(t, x, \dot{x}) = \frac{m}{2} |\dot{x}|^2 + e\dot{x} \cdot A(t, x) - eV(t, x), \quad \dot{x} \in R^d$$

(1.2)

with mass $m > 0$ and charge $e \in R$. Then the corresponding Schrödinger equation is given by

$$i\hbar \frac{\partial u}{\partial t}(t) = H(t)u(t)$$

$$:= \left[ \frac{1}{2m} \sum_{j=1}^{d} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_j(t, x) \right)^2 + eV(t, x) \right] u(t).$$

(1.3)

Hereafter we suppose $\hbar = 1$ and $e = 1$ for simplicity.

Let $L^2 = L^2(R^d)$ denote the space of all square integrable functions on $R^d$ with inner product $(f, g) := \int f(x)g(x)^*dx$ and norm $\|f\|$, where $g(x)^*$ denotes the complex conjugate of $g(x)$. Let $S(t, s; q)$ be the classical action

$$S(t, s; q) = \int_{s}^{t} \mathcal{L}(\theta, q(\theta), \dot{q}(\theta))d\theta$$

(1.4)

for a path $q(\theta) \in R^d \ (s \leq \theta \leq t)$, where $\dot{q}(\theta) = dq(\theta)/d\theta$. Our aim in the present paper is to prove that for any $f \in L^2$ we can determine the Feynman
path integral
\[ K(t, 0)f = \int e^{iS(t, 0; q)} f(q(0)) \mathcal{D}q \] (1.5)
in \( L^2(\mathbb{R}^d) \) for the system (1.2) with a potential \((V, A)\) growing polynomially in the spatial direction \(x\). As shown in Example 3.1, a typical example of potentials that we can handle is
\[ V(t, x) = |x|^{2(M+1)} + \sum_{|\alpha| \leq 2M+1} a_\alpha(t) x^\alpha, \] (1.6)
\[ A_j(t, x) = \sum_{|\alpha| \leq M} b_{j\alpha}(t) x^\alpha \quad (j = 1, 2, \ldots, d) \] (1.7)
with an integer \(M \geq 0\) and functions \(a_\alpha(t) \in \mathbb{R}, b_{j\alpha}(t) \in \mathbb{R}\) in \( C^1([0, T])\), i.e. continuously differentiable functions, where \(|x|^2 = \sum_{j=1}^d x_j^2\) and for a multi-index \(\alpha = (\alpha_1, \ldots, \alpha_d)\) we write \(|\alpha| = \sum_{j=1}^d \alpha_j\), \(x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}\), \(\partial_{x_j} = \partial/\partial x_j\) and \(\partial^\alpha_x = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}\).

In the present paper the Feynman path integral (1.5) is defined by the time-slicing method in terms of piecewise free moving paths or piecewise straight lines. This method is familiar in physics and simple in mathematics (cf. p.32 in [8] and p.278 in [19]).

The Feynman path integral for the system (1.2) with potentials \(A = 0\) and \(V\) satisfying \(|V(t, x)| \leq C(1 + |x|^2)\) has been studied mathematically by many authors for a long time since Feynman had published his famous paper [7] in 1948. See §10 in [1] and [4] for detailed knowledge of it. In addition, see [9], [10], [12] and their references for the recent study.

On the other hand, if \(|V(t, x)| \geq C(1+|x|^2)^{1+\delta}\) holds with positive constants \(C\) and \(\delta\), it may be not simple to construct the Feynman path integral for (1.2) mathematically as stated in §10.2 of [1] and in §3.5 of [16]. In fact, there seems to be only a few papers on it, which will be referred below, as far as the author
knows. In addition, as well known, the uniqueness of solutions to (1.3) doesn’t hold in general if \( V(t, x) \) satisfies \( V(t, x) \leq -C(1 + |x|^2)^{1+\delta} \) with positive constants \( C \) and \( \delta \) (cf. pp. 157-159 in [5] or the introduction in [13]).

Nelson in [18] has constructed the Feynman path integral (1.5) for (1.2) in \( L^2 \) for \( f \in L^2 \) with \( A = 0 \) and a continuous function \( V(x) \) outside a set of capacity 0 in \( \mathbb{R}^d \), independent of \( t \in [0, T] \), by using the Trotter product formula. It is to be noted that in [18] the classical action \( S(t, 0; q) \) is replaced with a certain approximation. See (9) on p. 333 of [18].

Daubechies and Klauder in [6] have showed the following. Take \((V, A) = (V(x), A(x))\), independent of \( t \), satisfying \( |V(x)| \leq C(1 + |x|^2)^M |A(x)| \leq C(1 + |x|^2)^M \) with constants \( C \geq 0 \) and \( M \geq 0 \). Let \( H \) be the operator defined by (1.3) with a core consisting of finite linear spans generated by eigenvalues of \((-\Delta + |x|^2)/2\), where \( \Delta = \sum_{j=1}^d \partial^2_{x_j} \). Let \( F_0(x) \) be the ground state of \((-\Delta + |x|^2)/2\) and define the canonical coherent states \(|p, q > = e^{ip x} F_0(x-q)\) for all \((q, p) \in \mathbb{R}^d\), where \( p \cdot x = \sum_{j=1}^d p_j x_j \). Let \( H' \) be a maximal extension of \( H \) on \( L^2 \) and denote the deficiency indices of \( H' \) by \( n_+(H') \) and \( n_-(H') \). Daubechies and Klauder have constructed the phase space Feynman path integral in the form of weak topology of \( L^2 \), i.e. giving \((|p'', q'' >, e^{-itH'} |p', q' >)\) if \( n_+(H') = 0 \) and \((|p'', q'' >, e^{-itH'^{\dagger}} |p', q' >)\) if \( n_-(H') = 0 \) in terms of the Wiener measure pinned at \((p', q')\) at \( t = 0 \) and at \((p'', q'')\) at \( t \), where \( H'^{\dagger} \) denotes the adjoint operator of \( H' \).

Albeverio and Mazzucchi in [2] and [3] have studied the Feynman path integrals for the systems (1.2) with \( A = 0 \), \( V(x) = |\Omega x|^2/2 + \lambda C(x, x, x, x) \) (\( \lambda \in \mathbb{R} \)) and with \( A = 0 \), a positive homogeneous polynomial \( V(x) \) of \( 2M \)-order (\( M = 1, 2, \ldots \)), respectively in terms of infinite dimensional oscillatory integrals and the Wiener measure, where \( \Omega \) is a \( d \times d \) regular matrix and \( C(x, y, w, z) \) is
a completely symmetric positive fourth-order covariant tensor on $\mathbb{R}^d$. It is noted that all Feynman path integrals in [2, 3] are defined in the form of weak topology of $L^2$. See §10.2 in [1] and §3.5 in [16] for topics relating to [2, 3].

The present paper is having four points to be emphasized: (1) In our system (1.2) there exists a magnetic field $B(t, x)$. (2) Our magnetic field $B(t, x)$ and electric field $E(t, x)$ can vary on time $t$. (3) Our Feynman path integral can be defined as an $L^2$-valued function on $[0, T]$ as in [18], not in the form of weak topology of $L^2$ as in [2, 3, 6]. (4) Our method of constructing the Feynman path integral can not be applied to systems with potentials satisfying $V(t, x) \leq -C(1 + |x|^2)^{1+\delta}$ ($C > 0, \delta > 0$). In [2, 6, 18] the Feynman path integrals for such systems are constructed. Then, it is unclear which solutions to (1.3) are being expressed by their Feynman integrals, because the uniqueness of solutions to (1.3) doesn’t hold as stated in the above.

In the present paper the Feynman path integrals will be constructed not only for the one-particle systems (1.2), but also the multi-particle systems with spin. In addition, we will construct the Feynman path integrals for bosons and fermions, i.e. quantum systems consisting of many identical particles with spin.

We will prove the results in the present paper as in the proofs in [10, 11, 12, 14]. That is, we introduce the fundamental operator $C(t, s)$ in §5, and prove its stability and consistency. Combining these results and the existence theorem proved in [13] to the Schrödinger equations (1.3) in both of $L^2$ and the Schwartz space $S(\mathbb{R}^d)$ of all rapidly decreasing functions on $\mathbb{R}^d$, we can prove our main results. In particular, in the present paper we will use the delicate result below concerning the $L^2$-boundedness of pseudo-differential operators, which is stated as Theorem 13.13 on p. 322 in [22].
Theorem 1.A. Suppose \( p(x, \xi, x') \in S^0(\mathbb{R}^d) \), i.e.

\[
\sup_{x,\xi,x'} |\partial_\alpha^\beta \partial_\gamma^r p(x, \xi, x')| \leq C_{\alpha,\beta,\gamma} < \infty
\]

(1.8)

for all \( \alpha, \beta \) and \( \gamma \). Let \( P(X, hD_x, X') \) be the pseudo-differential operator defined by

\[
\int e^{ix\cdot\xi} \int e^{-ix'\cdot\xi} p(x, h\xi, x') f(x') dx' \quad d\xi = (2\pi)^{-d} d\xi
\]

for \( f \in S(\mathbb{R}^d) \). Then we have

\[
\| P(X, hD_x, X') \|_{L^2 \rightarrow L^2} = \sup_{x,\xi,x'} |p(x, \xi, x')| + O(h),
\]

(1.9)

where \( \| P \|_{L^2 \rightarrow L^2} \) denotes the operator norm from \( L^2 \) into \( L^2 \).

The plan of the present paper is as follows. In §2 our main results are stated. In §3 we will state examples to which our results can be applied. In §4 we will construct the Feynman path integrals for bosons and fermions. In §5 and §6 the stability and the consistency of \( \mathcal{C}(t, s) \) will be proved, respectively. In §7 Theorems 2.1 - 2.2 and in §8 Theorems 2.3 - 2.4 will be proved.

2 Main theorems

Let \( t \) in \([0, T]\). For an arbitrary integer \( \nu \geq 1 \) we take \( \tau_j \in [0, T] \) (\( j = 1, 2, \ldots, \nu - 1 \)) satisfying \( 0 = \tau_0 < \tau_1 < \cdots < \tau_{\nu-1} < \tau_\nu = t \), set \( \Delta := \{ \tau_j \}_{j=1}^{\nu-1} \) and write \( |\Delta| := \max\{\tau_{j+1} - \tau_j; j = 0, 1, \ldots, \nu - 1\} \). Let \( x \in \mathbb{R}^d \) be fixed. We take arbitrary points \( x^{(j)} \in \mathbb{R}^d \) (\( j = 0, 1, \ldots, \nu - 1 \)) and determine the piecewise free moving path or the piecewise straight line \( q_\Delta(\theta; x^{(0)}, \ldots, x^{(\nu-1)}, x) \in \mathbb{R}^d \) (\( 0 \leq \theta \leq t \)) by joining \( x^{(j)} \) at \( \tau_j \) (\( j = 0, 1, \ldots, \nu, x^{(\nu)} = x \)) in order. Let \( \mathcal{L}(t, x, \dot{x}) \) be the Lagrangian function defined by (1.2) and \( S(t, s; q) \) the classical action defined by (1.4). Take \( \chi \in C_0^\infty(\mathbb{R}^d) \), i.e. an infinitely differentiable
function on \( \mathbb{R}^d \) with compact support, such that \( \chi(0) = 1 \) and determine the approximation of the Feynman path integral (1.5) for \( f \in C_0^\infty(\mathbb{R}^d) \) by

\[
K_\Delta(t,0)f = \lim_{\epsilon \to 0^+} \prod_{j=0}^{\nu-1} \sqrt{\frac{m}{2\pi i(\tau_{j+1} - \tau_j)}} \int \cdots \int_{\mathbb{R}^d} e^{iS(t,0;\Delta)} f(x(0)) \prod_{j=1}^{\nu-1} \chi(\epsilon x(j)) dx(0) dx(1) \cdots dx^{(\nu-1)}.
\]

(2.1)

From now on we always suppose that \( \chi \) is a real-valued function belonging to \( C_0^\infty(\mathbb{R}^d) \) such that \( \chi(0) = 1 \). The right-hand side of (2.1) is called the oscillatory integral and written as

\[
\prod_{j=0}^{\nu-1} \sqrt{\frac{m}{2\pi i(\tau_{j+1} - \tau_j)}} \int \cdots \int_{\mathbb{R}^d} e^{iS(t,0;\Delta)} f(x(0)) \prod_{j=1}^{\nu-1} \chi(\epsilon x(j)) dx(0) dx(1) \cdots dx^{(\nu-1)}
\]

(cf. p. 45 of \[15\]).

In the present paper we often use symbols \( C, C_\alpha, C_{\alpha,\beta}, C_{\alpha} \) and \( \delta \) to write down constants, though these values are different in general.

**Assumption 2.1.** We assume that \( \partial_x^n \partial_t^k V(t,x) \) and \( \partial_x^n \partial_t^k A_j(t,x) \) \((j = 1, 2, \ldots, d)\) are continuous in \([0, T] \times \mathbb{R}^d\) for all \( \alpha \) and \( k = 0, 1 \). There exists a constant \( M_\ast \geq 0 \) such that the following hold. We have

\[
C_0 < x > 2^{(M_\ast+1)} - C_1 \leq V(t,x) \leq C_2 < x > 2^{(M_\ast+1)}
\]

(2.2)

in \([0, T] \times \mathbb{R}^d\) with constants \( C_0 > 0, C_1 \geq 0 \) and \( C_2 \geq 0 \), where \( < x >= \sqrt{1 + |x|^2} \). We also have

\[
|\partial_x^n V(t,x)| \leq C_\alpha < x > 2^{(M_\ast+1)}, |\alpha| \geq 1,
\]

(2.3)

\[
|\partial_x^n \partial_t V(t,x)| \leq C_\alpha < x > 2^{(M_\ast+1)}
\]

(2.4)

for all \( \alpha \),

\[
|A_j(t,x)| \leq C < x >^{M_\ast+1-\delta}
\]

(2.5)
with a constant $\delta > 0$ and

$$|\partial_x^\alpha \partial_t A_j(t,x)| \leq C_\alpha < x >^{M_*+1} \quad (2.6)$$

for all $\alpha$.

**Assumption 2.2.** Let $M_*$ be the constant in Assumption 2.1. We have

$$C_* |x|^{2M_*} - C_1 \leq -\frac{1}{2m} \left( \frac{\partial E}{\partial x}(t,x) + \frac{t \partial E}{\partial x}(t,x) \right) \quad (2.7)$$

with constants $C_* > 0$ and $C_1 \geq 0$, and

$$|\partial_x^\alpha E_j(t,x)| \leq C_\alpha < x >^{2M_*}, \ |\alpha| \geq 1, \quad (2.8)$$

where $\partial E/\partial x = (\partial E_i/\partial x_j; i \downarrow j \rightarrow 1,2,\ldots,d)$ is a $d \times d$ matrix and $t \partial E/\partial x$ its transposed matrix. We have

$$|\partial_x^\alpha A_j(t,x)| \leq C_\alpha < x >^{M_*}, \ |\alpha| \geq 1. \quad (2.9)$$

In addition, we assume either

$$|\partial_x^\alpha B_{jk}(t,x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)}, \ |\alpha| \geq 1 \quad (2.10)$$

with constants $\delta_\alpha > 0$ and

$$|\partial_x^\alpha \partial_t B_{jk}(t,x)| \leq C_\alpha < x >^{M_*} \quad (2.11)$$

for all $\alpha$, or

$$|\partial_x^\alpha \partial_t B_{jk}(t,x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)}, \ |\alpha| \geq 1 \quad (2.12)$$

with constants $\delta_\alpha > 0$ if $0 \leq M_* < 1$ and

$$|\partial_x^\alpha \partial_t B_{jk}(t,x)| \leq C_\alpha < x >^{M_*-1}, \ |\alpha| \geq 1 \quad (2.13)$$

if $M_* \geq 1$.  

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Theorem 2.1. Suppose Assumptions 2.1 and 2.2. Then there exist constants $\rho^* > 0$ and $K \geq 0$ such that we obtain for $|\Delta| \leq \rho^*$: (1) $K_{\Delta}(t,0)f$ on $C_0^\infty(\mathbb{R}^d)$ defined by (2.1) is determined independently of the choice of $\chi$ and can be extended to a bounded operator on $L^2$. We have

$$\|K_{\Delta}(t,0)f\| \leq e^{Kt}\|f\|, \quad 0 \leq t \leq T$$

for all $f \in L^2$ and all $\Delta$. (2) Let $f \in L^2$. As $|\Delta| \to 0$, $K_{\Delta}(t,0)f$ converges in $L^2$ uniformly in $t \in [0,T]$. By this limit we define the Feynman path integral $K(t,0)f$. (3) $K(t,0)f$ for $f \in L^2$ is an $L^2$-valued continuous function on $[0,T]$, i.e. belongs to $\mathcal{E}_t^0([0,T];L^2)$. In addition, $K(t,0)f$ is the solution determined uniquely to (1.3) with $u(0) = f$ in the sense of distribution. (4) Let $\psi(t,x)$ be a real-valued function such that $\partial_{x_j} \partial_{x_k} \psi(t,x)$ and $\partial_t \partial_{x_j} \psi(t,x)$ ($j,k = 1,2,\ldots,d$) are continuous in $[0,T] \times \mathbb{R}^d$ and consider the gauge transformation

$V' = V - \frac{\partial \psi}{\partial t}, \quad A'_j = A_j + \frac{\partial \psi}{\partial x_j} \quad (j = 1,2,\ldots,d)$. \hfill (2.15)

We write (2.1) for this $(V',A')$ as $K'_{\Delta}(t,0)f$. Then we have the formula

$$K'_{\Delta}(t,0)f = e^{i\psi(t,\cdot)}K_{\Delta}(t,0)\left(e^{-i\psi(0,\cdot)}f\right)$$

for all $f \in L^2$, and so have the same formula between $K'(t,0)f$ and $K(t,0)f$.

Next we consider the Lagrangian function for the spin system

$$\mathcal{L}_s(t,x,\dot{x}) = \mathcal{L}(t,x,\dot{x}) - H_1(t,x),$$

where $H_1(t,x) = (h_{1jk}(t,x); j \downarrow k \to 1,2,\ldots,l)$ is a Hermitian matrix of degree $l$ and $\mathcal{L}(t,x,\dot{x})$ the Lagrangian function defined by (1.2). Then the corresponding quantized equation is given by

$$i \frac{\partial u}{\partial t}(t) = \left[H(t)I + H_1(t)\right]u(t),$$

(2.18)
where \( u(t) = t(u_1(t), \ldots, u_l(t)) \in \mathbb{C}^l, H(t) \) is the operator defined by (1.3) and \( I \) the identity matrix of degree \( l \).

For a continuous path \( q(\theta) \in \mathbb{R}^d \) (s \( \leq \theta \leq t \)) let us define an \( l \times l \) matrix \( \mathcal{F}(\theta, s; q) \) (s \( \leq \theta \leq t \)) by the solution to

\[
\frac{d}{d\theta} A(\theta) = -iH_1(\theta, q(\theta))A(\theta), \quad A(s) = I. \tag{2.19}
\]

Then, for the piecewise free moving path \( q_\Delta(\theta; x^{(0)}, x^{(1)}, \ldots, x^{(\nu-1)}, x) \) we define the probability amplitude by

\[
\exp *iS_s(t, 0; q_\Delta) = (\exp iS(t, 0; q_\Delta)) \mathcal{F}(t, 0; q_\Delta), \tag{2.20}
\]

using \( S(t, s; q) \) defined by (1.4). Let \( f = t(f_1, f_2, \ldots, f_l) \in C_0^\infty(\mathbb{R}^d)^l \). Then we define the approximation \( K_{s\Delta}(t, 0)f \) of the Feynman path integral \( K_s(t, 0)f \) for the system (2.17) by replacing \( e^{iS(t, 0; q_\Delta)} \) in (2.11) with \( e^{*iS_s(t, 0; q_\Delta)} \).

**Theorem 2.2.** Besides Assumptions 2.1 and 2.2 we assume

\[
|\partial^\alpha_\Delta h_{1jk}(t, x)| \leq C_\alpha, \quad j, k = 1, 2, \ldots, l \tag{2.21}
\]

for all \( \alpha \). Let \( \rho^* > 0 \) be the constant in Theorem 2.1. Then we get the same assertions for \( K_{s\Delta}(t, 0)f \) as for \( K_\Delta(t, 0)f \) in Theorem 2.1 with another constant \( K \geq 0 \), where \( K_s(t, 0)f = \lim_{|\Delta| \to 0} K_{s\Delta}(t, 0)f \) for \( f \in (L^2)^l \) is the solution to (2.18) with \( u(0) = f \).

**Remark 2.1.** Since we see from (2.19) that \( e^{iS(t, 0; q)} \mathcal{F}(t, 0; q) \) is the solution to

\[
\frac{d}{dt} U(t) = i\mathcal{L}_s(t, q(t), \dot{q}(t))U(t), \quad U(0) = I,
\]

we can write \( \exp *iS_s(t, 0; q_\Delta) \) formally as \( \exp i \int_0^t \mathcal{L}_s(\theta, q_\Delta(\theta), \dot{q}_\Delta(\theta))d\theta \). This is the reason why we express the right-hand side of (2.20) as \( \exp *iS_s(t, 0; q_\Delta) \).
Remark 2.2. We write
\[ q_{x,y}^{s,t}(\theta) = y + \frac{\theta - s}{t - s}(x - y), \quad s \leq \theta \leq t \quad (2.22) \]
for \( x \) and \( y \) in \( \mathbb{R}^d \) when \( s \neq t \). Then from Lemma 2.1 of [12] we have
\[ \mathcal{F}(t, 0; q_\Delta) = \mathcal{F}(t, \tau_{\nu - 1}; q_{x,x(\nu - 1)}^{\tau_{\nu - 1}}) \mathcal{F}(\tau_{\nu - 1}, \tau_{\nu - 2}; q_{x(x(\nu - 1), x(\nu - 2))}^{\tau_{\nu - 2}}) \cdots \mathcal{F}(\tau_1, 0; q_{x(1), x(0)}^{\tau_1, \tau_0}). \]

Remark 2.3. Letting \( M_* = 0 \), we assume (2.8), (2.10) and (2.21). Let \((V(t, x), A(t, x))\) be an arbitrary potential such that \( V, \partial V/\partial x_j, \partial A_j/\partial t \) and \( \partial A_j/\partial x_k \) \((j, k = 1, 2, \ldots, d)\) are continuous in \([0, T] \times \mathbb{R}^d\). Then we have proved in [11] and [12] the same assertions as in Theorems 2.1 and 2.2. Aside from this, letting \( M_* = 0 \), we assume (2.8), (2.9), (2.12), (2.21),
\[ |\partial^\alpha x V(t, x)| \leq C_\alpha < x >, \quad |\alpha| \geq 1 \quad (2.23) \]
and
\[ |\partial^\alpha_t V(t, x)| \leq C_\alpha < x >^M, \quad |\alpha| \geq 1 \quad (2.24) \]
for a constant \( M \geq 0 \). Using (1.1), from (2.8) and (2.23) we have
\[ |\partial^\alpha_x \partial_t A_j(t, x)| = |\partial^\alpha_x E_j(t, x)| + |\partial^\alpha_x \partial x_j V(t, x)| \leq C_\alpha < x > \quad (2.25) \]
for all \( \alpha \). We note (3.3) in [14] or (5.11) in the present paper. Then, under the assumptions above we can prove the same assertions as in Theorems 2.1 and 2.2 as in the proofs of the theorems stated in [11] and [12].

In the end we will consider the multi-particle system. For simplicity we will consider the 4-particle system
\[ \mathcal{L}^4(t, x, \dot{x}) = \sum_{l=1}^{4} \left\{ \frac{m_l}{2} |\dot{x}(l)|^2 + \dot{x}(l) \cdot A^0(t, x(l)) - V_l(t, x(l)) \right\} - 2 \sum_{1 \leq j < k \leq 4} V_{jk}(t, x(j) - x(k)), \quad (2.26) \]
where \(x = (x(1), x(2), x(3), x(4)) \in \mathbb{R}^d\). The corresponding Schrödinger equation is given by

\[
\begin{align*}
  i \frac{\partial u}{\partial t} (t) &= \left[ \sum_{l=1}^{4} \left\{ \frac{1}{2m_l} \left| \frac{1}{i} \frac{\partial}{\partial x(l)} - A^{(l)}(t, x(l)) \right|^2 + V_l(t, x(l)) \right\} 
    + 2 \sum_{1 \leq j < k \leq 4} V_{jk}(t, x(j) - x(k)) \right] u(t). 
\end{align*}
\]

(2.27)

**Assumption 2.3.** (1) Each \((V_l(t, x(l)), A^{(l)}(t, x(l)))\) \((l = 1, 2)\) satisfies Assumption 2.1 with \(M_* = M_{l_*} > 0\). (2) Each \(V_l(t, x(l))\) \((l = 3, 4)\) satisfies \([2.23]\) and \([2.24]\).

We define \(E^{(l)}(t, x(l))\) and \(B^{(l)}(t, x(l))\) \((l = 1, 2, 3, 4)\) by \([1.1]\) where \(A = A^{(l)}\) and \(V = V_l\).

**Assumption 2.4.** Let \(M_{l_*}\) \((l = 1, 2)\) be the constants in Assumption 2.3 and \(M_{l_*} = 0\) \((l = 3, 4)\). (1) \(A^{(l)}\) and \((E^{(l)}, B^{(l)})\) \((l = 1, 2, 3, 4)\) satisfy Assumption 2.2 with \(M_* = M_{l_*}\). (2) \(V_{jk}(t, z)\) \((z \in \mathbb{R}^d)\) satisfies

\[
|\partial_\alpha^\alpha V_{jk}(t, z)| \leq C_\alpha, \ |\alpha| \geq 2. \tag{2.28}
\]

for all \(1 \leq j < k \leq 4\).

We define the approximation \(K_{\Delta}^\sharp(t, 0) f\) of the Feynman path integral \(K^\sharp(t, 0) f\) for the 4-particle system \([2.20]\) in the same way as \([2.1]\).

**Theorem 2.3.** Suppose Assumptions 2.3 and 2.4. Then we have the same assertions for \(K_{\Delta}^\sharp(t, 0) f\) as for \(K_{\Delta}(t, 0) f\) in Theorem 2.1 with other constants \(\rho^* > 0\) and \(K \geq 0\), where the Feynman path integral \(K^\sharp(t, 0) f\) for \(f \in L^2(\mathbb{R}^d)\) is the solution to \([2.27]\) with \(u(0) = f\).
Let us consider the spin system. Taking a Hermitian matrix \( H_1(t, x) = (h_{1jk}(t, x); j \downarrow k \rightarrow 1, 2, \ldots, l_0) \) \((x \in \mathbb{R}^{4d})\) of degree \( l_0 \) and using \( L^\sharp_1(t, x, \dot{x}) \) defined by (2.26), we determine

\[
L^\sharp_1(t, x, \dot{x}) = L^\sharp(t, x, \dot{x}) - H_1(t, x). \tag{2.29}
\]

For a path \( q(\theta) \in \mathbb{R}^{4d} \) \((s \leq \theta \leq t)\) we define \( \mathcal{F}^\sharp(\theta, s; q) \) by the solution to (2.19). Then we define \( \exp *iS^\sharp_1(t, 0; q_{\Delta}) \) by (2.20) and \( K^\sharp_1(t, 0) f \) for \( f \in C^\infty_0(\mathbb{R}^{4d}) \) in the same way as we did \( K_{s\Delta}(t, 0) f \).

**Theorem 2.4.** Besides Assumptions 2.3 and 2.4 we assume (2.21). Let \( \rho^* > 0 \) be the constant in Theorem 2.3. Then we have the same assertions for \( K^\sharp_1(t, 0) f \) as for \( K_{s\Delta}(t, 0) f \) in Theorem 2.2 with another constant \( K \geq 0 \).

### 3 Exmaples

In this section we will give some examples satisfying Assumptions 2.1 and 2.2 in §2.

**Lemma 3.1.** Let \( f \in C^2([0, \infty)) \) and set

\[
V(x) = f(|x|^2), \ x \in \mathbb{R}^d. \tag{3.1}
\]

Let \( x \neq 0 \) be an arbitrary point in \( \mathbb{R}^d \). Then there exists an orthogonal matrix \( \mathfrak{R} \) such that

\[
\frac{\partial^2 V}{\partial x^2}(x) := \begin{pmatrix}
\frac{\partial^2 V}{\partial x_i \partial x_j}; i \downarrow j \rightarrow 1, 2, \ldots, d
\end{pmatrix} = 2 \mathfrak{R} f'(|x|^2) \mathfrak{R} + 4 \mathfrak{R} \begin{pmatrix}
|x|^2 f''(|x|^2) & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix} \mathfrak{R}. \tag{3.2}
\]
Proof. Let $\mathbf{R} = (R_{ij}; i \downarrow j \rightarrow 1, 2, \ldots, d)$ be an orthogonal matrix. Then we have

$$V(\mathbf{R}x) = f(|\mathbf{R}x|^2) = f(|x|^2) = V(x), \quad (3.3)$$

which shows

$$\frac{\partial V}{\partial x_j}(x) = \frac{\partial}{\partial x_j} V(\mathbf{R}x) = \sum_{k=1}^{d} \frac{\partial V}{\partial x_k}(\mathbf{R}x) R_{kj}$$

and so

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x) = \sum_{k,l=1}^{d} R_{li} \frac{\partial^2 V}{\partial x_l \partial x_k}(\mathbf{R}x) R_{kj}.$$ 

Hence

$$\frac{\partial^2 V}{\partial x^2}(x) = \mathbf{R} \frac{\partial^2 V}{\partial x^2}(\mathbf{R}x) \mathbf{R}. \quad (3.4)$$

On the other hand, from (3.1) we see

$$\frac{\partial V}{\partial x_j}(x) = 2x_j f'(|x|^2)$$

and so

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(x) = 2\delta_{ij} f'(|x|^2) + 4x_i x_j f''(|x|^2).$$

Consequently, letting $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$, we have

$$\frac{\partial^2 V}{\partial x_i \partial x_j}(|x|\mathbf{e}_1) = 2\delta_{ij} f'(|x|^2) + 4\delta_{1i} \delta_{1j} |x|^2 f''(|x|^2). \quad (3.5)$$

Let $x \neq 0$ be an arbitrary point in $\mathbb{R}^d$. Then we can take an orthogonal matrix $\mathbf{R}$ such that $\mathbf{R}x = |x|\mathbf{e}_1$. Then from (3.3) we have

$$\frac{\partial^2 V}{\partial x^2}(x) = \mathbf{R} \frac{\partial^2 V}{\partial x^2}(|x|\mathbf{e}_1) \mathbf{R}$$

and hence have (3.2) from (3.5). \qed

From Lemma 3.1 we can easily get the following.
Corollary 3.2. Let \( f \in C^2([0, \infty)) \) such that
\[
f''(\theta) \geq 0 \quad (0 \leq \theta < \infty).
\] (3.6)
We define \( V(x) \) by (3.1). Then we have
\[
\frac{\partial^2 V}{\partial x^2}(x) \geq 2f'(|x|^2)I.
\] (3.7)

Proposition 3.3. Let \( f \in C^2([0, \infty)) \) satisfying (3.6) and \( f'(0) \geq 0 \). Let \( A \) be a regular real matrix. We denote the smallest eigenvalue of \( tA^tA \) by \( \beta > 0 \).

We set
\[
V(x) = f(|A|x^2).
\] (3.8)
Then we have
\[
\frac{\partial^2 V}{\partial x^2}(x) \geq 2\beta f'(|A|x^2)I \geq 2\beta f'(|x|^2)I.
\] (3.9)

Proof. Setting \( W(x) = f(|x|^2) \), we have \( V(x) = W(Ax) \), which shows
\[
\frac{\partial^2 V}{\partial x^2}(x) = tA^t \frac{\partial^2 W}{\partial x^2}(Ax)A
\]
as in the proof of (3.4). Hence
\[
\left( \frac{\partial^2 V}{\partial x^2}(x)u, u \right) = \left( \frac{\partial^2 W}{\partial x^2}(Ax)A, Au \right)
\]
for \( u \in \mathbb{R}^d \), where \((\cdot, \cdot)\) is the inner product in \( \mathbb{R}^d \). Since we have \( \partial^2 W(x)/\partial x^2 \geq 2f'(|x|^2)I \) from (3.7), we have
\[
\left( \frac{\partial^2 V}{\partial x^2}(x)u, u \right) \geq 2 \left( f'(|A|x^2)A, Au \right),
\]
which leads to
\[
\left( \frac{\partial^2 V}{\partial x^2}(x)u, u \right) \geq 2f'(|A|x^2) \left( tA^tA, u \right) \geq 2\beta f'(|A|x^2)|u|^2
\]
because of \( f'(\theta) \geq 0 \) \((0 \leq \theta < \infty)\). Hence we obtain the first inequality of (3.9). The second inequality follows from the fact that \( f'(\theta) \) is an increasing function and \( |A|x^2 = (tA^tA, x) \geq \beta|x|^2 \). \( \square \)
Example 3.1. Let \((V, A)\) be the potential defined by (1.6) and (1.7) with an integer \(M \geq 0\), real-valued \(a_\alpha(t)\) and \(b_j(t)\) in \(C^1([0, T])\). I will prove that this \((V, A)\) satisfies Assumptions 2.1 and 2.2 with the integer \(M^* = M\).

Noting (1.1), we can easily see that we have only to prove (2.7). Letting \(f(\theta) = \theta^{M+1}\) in Corollary 3.2, we have

\[
\frac{\partial^2}{\partial x^2} |x|^{2(M+1)} \geq 2(M + 1)|x|^{2M + 1}.
\] (3.10)

Hence we can prove (2.7), because we have

\[-\frac{\partial E}{\partial x} = \frac{\partial^2 A}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2} \geq |x|^{2(M+1)} + O(< x >^{2M - 1})\]

from (1.1), (1.6) and (1.7).

Example 3.2. Let \(\mathfrak{A}(t)\) be a regular real matrix whose components are continuously differentiable on \([0, T]\). We set

\[V(t, x) = |\mathfrak{A}(t)x|^{2(M+1)} + V_1(t, x)\] (3.11)

with an integer \(M \geq 0\), where we assume

\[|\partial^\alpha_x V_1(t, x)| \leq C_\alpha < x >^{2(M+1) - |\alpha| - \delta_\alpha}\] (3.12)

for all \(\alpha\) with constants \(\delta_\alpha > 0\) and

\[|\partial^\alpha_x \partial_t V_1(t, x)| \leq C_\alpha < x >^{2(M+1)}\] (3.13)

for all \(\alpha\). In addition, we assume

\[|\partial^\alpha_x A_j(t, x)| \leq C_\alpha < x >^{M+1 - |\alpha| - \delta_\alpha} \quad (j = 1, 2, \ldots, d)\] (3.14)

for all \(\alpha\) and

\[|\partial^\alpha_x \partial_t A_j(t, x)| \leq C_\alpha < x >^{M+1 - |\alpha|} \quad (j = 1, 2, \ldots, d)\] (3.15)
for all $\alpha$. Then this potential $(V, A)$ satisfies Assumptions 2.1 and 2.2 with $M_* = M$.

In fact we have only to prove (2.7) as in the arguments in Example 3.1. Letting $f(\theta) = \theta^{M+1}$ in Proposition 3.3, we have

$$\frac{\partial^2}{\partial x^2} |A(t)x|^{2(M+1)} \geq 2(M + 1)\beta |A(t)x|^{2M} \geq 2(M + 1)\beta^{M+1}|x|^{2M}$$

(3.16)

with $\beta > 0$. Hence we can prove (2.7) as in the proof of Example 3.1.

**Example 3.3.** Let $A(t)$ be the matrix in Example 3.2. We set

$$V(t, x) = (1 + |A(t)x|^2)^{M+1} + V_1(t, x)$$

(3.17)

with a constant $M \geq 0$, where $V_1(t, x)$ is assumed to satisfy (3.12) and (3.13). Suppose that $A(t, x)$ satisfies (3.14) and (3.15). In addition, when $M$ in (3.17) is in $(0, 1)$, we assume (3.14) with $M = 0$. Then this potential $(V, A)$ satisfies Assumptions 2.1 and 2.2. In fact we have only to prove (2.7). Letting $f(\theta) = (1 + \theta)^{M+1}$ in Proposition 3.3, we have

$$\frac{\partial^2}{\partial x^2} \left(1 + |A(t)x|^2\right)^{M+1} \geq 2(M + 1)\beta \left(1 + |A(t)x|^2\right)^M$$

$$\geq 2(M + 1)\beta \left(1 + \beta |x|^2\right)^M \geq 2(M + 1)\beta^{M+1}|x|^{2M}.$$ 

(3.18)

Hence we can prove (2.7) as in the proof of Example 3.2.
The Feynman path integrals for bosons and fermions

In this section we consider the quantum spin system consisting of $N$ particles. We write $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N}$. The Lagrangian function is given by

$$L^\#(t, x, \dot{x}) = \sum_{i=1}^{N} \left\{ \frac{m_i}{2} |\dot{x}_i|^2 + e_i \dot{x}_i \cdot A_i(t, x_i) - e_i V_i(t, x_i) + I_1 \otimes \cdots \otimes I_{i-1} \otimes \hat{s}_i \otimes I_{i+1} \otimes \cdots \otimes I_N \right\} - \sum_{j,k=1, j \neq k}^{N} e_j e_k V_{jk}(t, x_j - x_k)$$

in terms of the tensor product, where $A_i \in \mathbb{R}^3, B_i \in \mathbb{R}^3, V_i \in \mathbb{R}, V_{jk} \in \mathbb{R}$, $\hat{s}_i = (\hat{s}_1, \hat{s}_2, \hat{s}_3)$ are spin matrices with three components and $I_j$ the identity matrix for the $j$-th particle. In particular we suppose that all particles are identical. Hence we suppose $m_i = m, e_i = e, A_i = A, B_i = B, V_i = V, V_{jk} = W$ and $\hat{s}_i = \hat{s}$. Let $L$ be the magnitude of spin of particles. We note that the $N$-fold tensor product $L^2(\mathbb{R}^3)^{2L+1} \otimes \cdots \otimes L^2(\mathbb{R}^3)^{2L+1}$ is isomorphic to $L^2(\mathbb{R}^{3N})^l$ with $l = (2L + 1)^N$ (cf. Theorem II.10 on p. 52 in [20]), which we write as $\mathcal{H}$.

The Schrödinger equation for the Lagrangian (4.1) is given by

$$i \frac{\partial u}{\partial t}(t) = \left[ \sum_{j=1}^{N} \left\{ \frac{1}{2m} \frac{1}{i} \frac{\partial}{\partial x_j} - eA(t, x_j) \right\}^2 + eV_j(t, x_j) - I_1 \otimes \cdots \otimes I_{j-1} \otimes \hat{s} \otimes I_{j+1} \otimes \cdots \otimes I_N \right] u(t).$$

We note that if $W = 0$ and $u(t, x) = u_1(t, x_1) \otimes \cdots \otimes u_N(t, x_N)$, (4.2) is written as

$$0 = \sum_{j=1}^{N} u_1(t) \otimes \cdots \otimes u_{j-1}(t) \otimes \left[ i \frac{\partial}{\partial t} - H_j(t) \right] u_j(t) \otimes \cdots \otimes u_N(t),$$

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where $H_j(t) = |i^{-1} \partial_{x_j} - eA(t, x_j)|^2/(2m) + eV(t, x_j) - eB(t, x_j) \cdot \hat{s}/m$.

Let $S^\#_i(t, s; q)$ be the classical action for $L^\#_i(t, x, \dot{x})$ defined by (4.1). We define the approximation $K^\#_i s\Delta(t, 0)f$ of the Feynman path integral $K^\#_i(t, 0)f$ for (4.1) in the same way as we did $K^\#_s\Delta(t, 0)f$ before Theorem 2.4, where $f = \{ f(x_1, s_1, x_2, s_2, \ldots, x_N, s_N); s_j = -L, -L + 1, \ldots, L (j = 1, 2, \ldots, N) \} \in C_0^\infty(\mathbb{R}^{3N})$. That is, we define $\mathcal{F}^\#_i(\theta, s; q)$ for a path $q(\theta) \in \mathbb{R}^3 (s \leq \theta \leq t)$ by the solution

$$\frac{d}{d\theta} A(\theta) = -iH_1(\theta, q(\theta)), A(s) = I,$$

where $H_1(t, x) = -\sum_{j=1}^{N} I_1 \otimes \cdots \otimes I_{j-1} \otimes eB(t, x_j) \cdot \hat{s}/m \otimes I_{j+1} \otimes \cdots \otimes I_N$. Next we define the probability amplitude by (2.20) and eventually define $K^\#_i s\Delta(t, 0)f$ by (2.1).

We define $\mathcal{F}(\theta, s; q) (s \leq \theta \leq t)$ for a continuous path $q(\theta) \in \mathbb{R}^3 (s \leq \theta \leq t)$ by the solution

$$\frac{d}{d\theta} A'(\theta) = i\frac{e}{m} (B(\theta, q(\theta)) \cdot \hat{s}) A'(\theta), A'(s) = I. \quad (4.3)$$

Then we can easily have

$$\frac{d}{d\theta} \mathcal{F}(\theta, s; q_1) \otimes \cdots \otimes \mathcal{F}(\theta, s; q_N) = \sum_{j=1}^{N} \mathcal{F}(\theta, s; q_1) \otimes \cdots \otimes \mathcal{F}(\theta, s; q_{j-1})$$

$$\otimes \frac{d}{d\theta} \mathcal{F}(\theta, s; q_j) \otimes \mathcal{F}(\theta, s; q_{j+1}) \otimes \cdots \mathcal{F}(\theta, s; q_N) = \sum_{j=1}^{N} \left( I_1 \otimes \cdots \otimes i\frac{e}{m} B(\theta, q_j(\theta)) \cdot \hat{s} \right)$$

$$\otimes I_{j+1} \otimes \cdots \otimes I_N) \mathcal{F}(\theta, s; q_1) \otimes \cdots \otimes \mathcal{F}(\theta, s; q_N)$$

$$= -iH_1(\theta, q(\theta)) \mathcal{F}(\theta, s; q_1) \otimes \cdots \otimes \mathcal{F}(\theta, s; q_N).$$

Hence we have

$$\mathcal{F}^\#_i(\theta, s; q) = \mathcal{F}(\theta, s; q_1) \otimes \cdots \otimes \mathcal{F}(\theta, s; q_N) \quad (4.4)$$

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because of the uniqueness of solutions to the ordinary differential equation, where \( B(t, x) \) are assumed to be continuous in \([0, T] \times \mathbb{R}^3\).

Let \( \hat{P}_{ij} (i, j = 1, 2, \ldots, N) \) be the operator exchanging the \( i \)-th particle and the \( j \)-th one. That is, we define

\[
\hat{P}_{ij} (f_1(x_1) \otimes \cdots \otimes f_i(x_i) \otimes \cdots \otimes f_j(x_j) \otimes \cdots \otimes f_N(x_N))
= f_1(x_1) \otimes \cdots \otimes f_j(x_i) \otimes \cdots \otimes f_i(x_j) \otimes \cdots \otimes f_N(x_N) \quad (4.5)
\]

for \( f_j(x_j) \in L^2(\mathbb{R}^3)^{2L+1} \) \( (j = 1, 2, \ldots, N) \) and extend \( \hat{P}_{ij} \) for \( f = \sum_{n=1}^{\infty} f_1^{(n)}(x_1) \otimes \cdots \otimes f_N^{(n)}(x_N) \in \mathcal{F} \).

The following theorem shows that the Feynman path integrals \( K_{\text{is}}^\sharp (t, 0) f \) are expressing bosons and fermions.

**Theorem 4.1.** Assume that \( (V(t, x), A(t, x)) (x \in \mathbb{R}^3), B(t, x) \) and \( W(t, x) \) satisfy Assumptions 2.1-2.2, (2.21) and (2.28), respectively. Then we have: (1) The same assertions for \( K_{\text{is}}^\sharp (t, 0) f \) as for \( K_{\Delta} (t, 0) f \) in Theorem 2.1 hold, where \( K_{\text{is}}^\sharp (t, 0) f \) \( (f \in \mathcal{F}) \) is the solution to (4.2) with \( u(0) = f \). (2) If \( f \in \mathcal{F} \) is symmetric, i.e. \( \hat{P}_{ij} f = f \) for all \( i \) and \( j \), so is \( K_{\text{is}}^\sharp (t, 0) f \). (3) If \( f \in \mathcal{F} \) is antisymmetric, i.e. \( \hat{P}_{ij} f = -f \) for all \( i \neq j \), so is \( K_{\text{is}}^\sharp (t, 0) f \).

**Proof.** The first assertion (1) follows from Theorem 2.4. Let’s prove the second assertion. For simplicity suppose \( N = 2 \). Let \( c(x_1, x_2) \in \mathbb{C} \) be a bounded measurable function. Then we can prove

\[
\hat{P}_{12} c(x_1, x_2) f_1(x_1) \otimes f_2(x_2) = c(x_2, x_1) f_2(x_1) \otimes f_1(x_2) \quad (4.6)
\]

from the definition of \( \hat{P}_{12} \), approximating \( c(x_1, x_2) \) by \( \sum_{j=1}^{n} c_1^{(j)}(x_1) c_2^{(j)}(x_2) \).

Let \( 0 \leq t - s \leq \rho^* \). Setting \( q_j(\theta) := q_{x_j, y_j}^s(\theta) (s \leq \theta \leq t, j = 1, 2) \), we
We have proved in \((1)\) of Theorem 2.4 that
\[
C_{is}^t(t, s)(f_1 \otimes f_2)
\]
\[
= \int \int e^{iS_i^t(t, s; q_2, q_1)} F_i^t(t, s; q_1, q_2)(f_1(q_1(s)) \otimes f_2(q_2(s))) dy_1 dy_2 \quad (4.7)
\]
for \(f_j(x_j) \in C_0^\infty(\mathbb{R}^3)^{2L+1} \quad (j = 1, 2)\), which belongs to \(L^2(\mathbb{R}^6)^l\) from \((1)\) of Theorem 2.4. From \((4.4)\) we can write
\[
C_{is}^t(t, s)(f_1 \otimes f_2) = \int \int e^{iS_i^t(t, s; q_1, q_2)} F(t, s; q_1) f_1(q_1(s))
\]
\[
\otimes F(t, s; q_2) f_2(q_2(s)) dy_1 dy_2.
\]
Making the same arguments as in the proof of \((4.6)\), by the exchange of \((x_1, y_1)\) and \((x_2, y_2)\) in the above equation we can prove
\[
\hat{P}_{12} \left( \chi(\epsilon x_1) \chi(\epsilon x_2) C_{is}^t(t, s)(f_1 \otimes f_2) \right) = \chi(\epsilon x_2) \chi(\epsilon x_1) \int \int e^{iS_i^t(t, s; q_2, q_1)}
\]
\[
\times F(t, s; q_1) f_2(q_1(s)) \otimes F(t, s; q_2) f_1(q_2(s)) dy_1 dy_2.
\]
Letting \(\epsilon \to 0\), we have
\[
\hat{P}_{12} C_{is}^t(t, s)(f_1 \otimes f_2) = \int \int e^{iS_i^t(t, s; q_2, q_1)} F(t, s; q_1) f_2(q_1(s))
\]
\[
\otimes F(t, s; q_2) f_1(q_2(s)) dy_1 dy_2. \quad (4.8)
\]
Using \(S_i^t(t, s; q_1, q_2) = S_i^t(t, s; q_2, q_1)\), we have
\[
\hat{P}_{12} C_{is}^t(t, s)(f_1 \otimes f_2) = \int \int e^{iS_i^t(t, s; q_1, q_2)} F(t, s; q_1) \otimes F(t, s; q_2)
\]
\[
\cdot (f_2(q_1(s)) \otimes f_1(q_2(s))) dy_1 dy_2 = \int \int e^{iS_i^t(t, s; q_1, q_2)} F(t, s; q_1)
\]
\[
\otimes F(t, s; q_2) \hat{P}_{12} (f_1 \otimes f_2) dy_1 dy_2 = C_{is}^t(t, s) \hat{P}_{12} (f_1 \otimes f_2). \quad (4.9)
\]
We have proved in \((1)\) of Theorem 2.4 that \(C_{is}^t(t, s)\) is a bounded operator on \(\mathcal{F}\). Hence from \((4.9)\) we see
\[
\hat{P}_{12} C_{is}^t(t, s) f = C_{is}^t(t, s) \hat{P}_{12} f \quad (4.10)
\]
for \( f \in \mathcal{H} \).

Noting Remark 2.2, from (2.1) and (2.20) we can write

\[
K^\varepsilon_{is\Delta}(t, 0) f = \lim_{\varepsilon \to 0^+} C^\varepsilon_{is\varepsilon}(t, \tau_{\nu - 1}) \chi(\varepsilon) \cdots \chi(\varepsilon) C^\varepsilon_{is\varepsilon}(\tau_1, 0) f
\]

for \( f \in C_0^\infty(\mathbb{R}^6)^l \). Since we have

\[
C^\varepsilon_{is\varepsilon}(t, \tau_{\nu - 1}) \chi(\varepsilon) C^\varepsilon_{is\varepsilon}(\tau_{\nu - 1}, \tau_{\nu - 2}) \chi(\varepsilon) \cdots \chi(\varepsilon) C^\varepsilon_{is\varepsilon}(\tau_1, 0) f - C^\varepsilon_{is\varepsilon}(t, \tau_{\nu - 1})
\]

\[
\cdot C^\varepsilon_{is\varepsilon}(\tau_{\nu - 1}, \tau_{\nu - 2}) \cdots C^\varepsilon_{is\varepsilon}(\tau_1, 0) f = \sum_{j=1}^{\nu - 1} C^\varepsilon_{is\varepsilon}(t, \tau_{\nu - 1}) \chi(\varepsilon) C^\varepsilon_{is\varepsilon}(\tau_{\nu - 1}, \tau_{\nu - 2}) \chi(\varepsilon) \cdots
\]

\[
\cdot \chi(\varepsilon) C^\varepsilon_{is\varepsilon}(\tau_{j + 1}, \tau_j) \{ \chi(\varepsilon) - 1 \} C^\varepsilon_{is\varepsilon}(\tau_j, \tau_{j - 1}) C^\varepsilon_{is\varepsilon}(\tau_{j - 1}, \tau_{j - 2}) \cdots C^\varepsilon_{is\varepsilon}(\tau_1, 0) f,
\]

by (1) of Theorem 2.4 we obtain

\[
K^\varepsilon_{is\Delta}(t, 0) f = C^\varepsilon_{is\varepsilon}(t, \tau_{\nu - 1}) C^\varepsilon_{is\varepsilon}(\tau_{\nu - 1}, \tau_{\nu - 2}) \cdots C^\varepsilon_{is\varepsilon}(\tau_1, 0) f
\]

(4.11)

for \( f \in C_0^\infty(\mathbb{R}^6)^l \) and so for \( f \in \mathcal{H} \). Therefore, by (4.10) we have

\[
\hat{P}_{12} K^\varepsilon_{is\Delta}(t, 0) f = K^\varepsilon_{is\Delta}(t, 0) \hat{P}_{12} f.
\]

(4.12)

Since \( \hat{P}_{12} f = f \) holds from the assumption, we obtain

\[
\hat{P}_{12} K^\varepsilon_{is\Delta}(t, s) f = K^\varepsilon_{is\Delta}(t, s) f.
\]

This shows that \( K^\varepsilon_{is\Delta}(t, 0) f \) is symmetric, which completes the proof of the second assertion. In the same way the third assertion is proved from (4.12).

Remark 4.1. We have supposed Assumptions 2.1 and 2.2 for \((V(t, x), A(t, x))\) in Theorem 4.1. In place of these assumptions we suppose the assumptions stated in Remark 2.3 for \((V, A)\). Then we can prove the same assertions as in Theorem 4.1 as in the proof of Theorem 4.1.
5 Stability of $C(t, s)$

Let $\mathcal{L}(t, x, \dot{x})$ and $S(t, s; q)$ be the Lagrangian function and the classical action defined by (1.2) and (1.4), respectively. Let $q_{t,s}^{x,y}$ be the path defined by (2.22) and write

$$\gamma_{t,s}^{x,y}(\theta) = (\theta, q_{t,s}^{x,y}(\theta)) \in \mathbb{R}^{d+1}. \tag{5.1}$$

Then we have

$$S(t, s; q_{t,s}^{x,y}) = \frac{m|x - y|^2}{2(t - s)} + \int_{\gamma_{t,s}^{x,y}} (A \cdot dx - V dt)$$

$$= \frac{m|x - y|^2}{2(t - s)} + (x - y) \cdot \int_0^1 A(s + \theta \rho, y + \theta(x - y))d\theta$$

$$- \int_s^t V(\theta, y + \frac{\theta - s}{t - s}(x - y))d\theta$$

$$= \frac{m|x - y|^2}{2(t - s)} + (x - y) \cdot \int_0^1 A(t - \theta \rho, x - \theta(x - y))d\theta$$

$$- \rho \int_0^1 V(t - \theta \rho, x - \theta(x - y))d\theta, \rho = t - s. \tag{5.2}$$

Let $M \geq 0$ and suppose that $p(x, w) \in C^\infty(\mathbb{R}^{2d})$ satisfies

$$|\partial^\alpha_w \partial^\beta_x p(x, w)| \leq C_{\alpha,\beta} < x; w >^M, (x, w) \in \mathbb{R}^{2d} \tag{5.3}$$

for all $\alpha$ and $\beta$, where $< x; w > = \sqrt{1 + |x|^2 + |w|^2}$. We write the semi-norms of $S = S(\mathbb{R}^d)$ as $|f|_l = \sum_{|\alpha + \beta| \leq l} \sup\{|x^\alpha \partial^\beta_x f(x)|; x \in \mathbb{R}^d\} \ (l = 0, 1, 2, \ldots)$. For $f \in S$ we define

$$P(t, s)f = \begin{cases} \sqrt{m/(2\pi i)}^d \int (\exp iS(t, s; q_{t,s}^{x,y})) \\ \times p(x, (x - y)/\sqrt{\rho})f(y)dy, \quad s < t, \\ \sqrt{m/(2\pi i)}^d Os - \int (\exp im|w|^2/2) \\ \times p(x, w)dwf(x), \quad s = t \end{cases} \tag{5.4}$$
Then the formal adjoint operator $P(t, s)^\dagger$ of $P(t, s)$ on $S$ is given by

$$P(t, s)^\dagger f = \begin{cases} \sqrt{im/(2\pi\rho)} \int (\exp -iS(t, s; q^{t,s}_{y,x})) f(y)dy, & s < t, \\ \sqrt{im/(2\pi)} Os - \int (\exp -im|w|^2/2) p(x, w)^* dw f(x), & s = t. \end{cases}$$ (5.5)

We have the following from Lemma 2.1 of [11].

**Lemma 5.1.** We define $P(t, s)f$ by (5.4) for $f \in S$. Assume (2.3) and (2.9). Then, $\partial_{\alpha}^x (P(t, s)f)$ are continuous in $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}^d$ for all $\alpha$.

Taking 1 as $p(x, w)$ in (5.4), for $f \in S$ we define

$$C(t, s)f = \begin{cases} \sqrt{m/(2\pi i\rho)} \int (\exp iS(t, s; q_{x,y}^{t,s})) f(y)dy, & s < t, \\ f, & s = t. \end{cases}$$ (5.6)

Using Lemma 5.1, we can write $K_\Delta(t, 0)f$ defined by (2.1) as

$$K_\Delta(t, 0)f = \lim_{\epsilon \to 0^+} C(t, \tau_{\nu-1})\chi(\epsilon)C(\tau_{\nu-2})\chi(\epsilon)\cdots C(\tau_1, 0)f$$ (5.7)

for $f \in S$ under the assumptions of Lemma 5.1.

**Lemma 5.2.** We assume that $\partial_{x}^\alpha V(t, x), \partial_{x}^\alpha A_j(t, x)$ and $\partial_{x}^\alpha \partial_t A_j(t, x)$ are continuous in $[0, T] \times \mathbb{R}^d$ for $|\alpha| \leq 1$. Let $p(x, w)$ be a function satisfying (5.3). Then for any $0 < \epsilon \leq 1$ and $0 \leq s < t \leq T$ we have

$$P(t, s)^\dagger \chi(\epsilon)^2 P(t, s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2$$

$$\times \left(\exp i(x - y) \cdot \frac{m\Phi}{t - s}\right) p\left(z, \frac{z - x}{\sqrt{t - s}}\right)^* p\left(z, \frac{z - y}{\sqrt{t - s}}\right) dz,$$ (5.8)
\[
\Phi = \Phi(t, s; x, y, z) = (\Phi_1, \ldots, \Phi_d),
\]
\[
\Phi_j = z_j - \frac{x_j + y_j}{2} + \frac{t - s}{m} \int_0^1 A_j(s, x + \theta(y - x))d\theta
\]
\[
- \frac{(t - s)^2}{m} \int_0^1 \int_0^1 \sigma_1 E_j(\tau(\sigma), \zeta(\sigma))d\sigma_1d\sigma_2
\]
\[
- \frac{t - s}{m} \sum_{k=1}^d (z_k - x_k) \int_0^1 \int_0^1 \sigma_1 B_{jk}(\tau(\sigma), \zeta(\sigma))d\sigma_1d\sigma_2
\]
\[
\text{or}
\]
\[
\Phi_j = z_j - \frac{x_j + y_j}{2} + \frac{t - s}{m} \int_0^1 A_j(s, x + \theta(y - x))d\theta
\]
\[
- \frac{(t - s)^2}{m} \int_0^1 \int_0^1 \sigma_1 E_j(\tau(\sigma), \zeta(\sigma))d\sigma_1d\sigma_2 - \frac{(t - s)^2}{m} \int_0^1 d\theta \sum_{k=1}^d (z_k - x_k)
\]
\[
\times \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \frac{\partial B_{jk}}{\partial t}(s + \theta(1 - \sigma_1)\rho, \zeta(\sigma))d\sigma_1d\sigma_2,
\]
where
\[
(\tau(\sigma), \zeta(\sigma)) = (t - \sigma_1(t - s), z + \sigma_1(x - z) + \sigma_1\sigma_2(y - x)) \in \mathbb{R}^{d+1}.
\]

Proof. We have proved (5.8), (5.9) and (5.10) in Proposition 3.3 of [10] and Lemma 2.2 of [11]. So we will prove (5.8), (5.9) and (5.11), though these have been proved in Lemma 3.1 of [14] in essentials. Let \( \Delta \) be the 2-dimensional plane with oriented boundary consisting of \(-\{(s, y + \theta(x - y)); 0 \leq \theta \leq 1\}, \{(s + \theta\rho, y + \theta(z - y)); 0 \leq \theta \leq 1\}\) and \(-\{(s + \theta\rho, x + \theta(z - x)); 0 \leq \theta \leq 1\}\). Then we have
\[
\lim_{t \to s+0} \iint_{\Delta} d(A \cdot x - Vdt) = 0.
\]
Hence from the proof of Lemma 3.2 in [10] we have
\[
\sum_{j=1}^d (x_j - y_j) \sum_{k=1}^d (z_k - x_k) \int_0^1 \int_0^1 \sigma_1 B_{jk}(s, \zeta(\sigma))d\sigma_1d\sigma_2 = 0
\]
for all \( x, y \) and \( z \) in \( \mathbb{R}^d \). Multiplying (5.13) by \((t - s)/m\) and adding this to \((x - y) \cdot \Phi\) where \( \Phi \) is defined by (5.10), we have (5.8), (5.9) and (5.11). \( \square \)

**Lemma 5.3.** Assume (2.7) and let \( C_1 \geq 0 \) be the constant in (2.7). Then, there exists a constant \( C'_* > 0 \) such that for all \( X = (x, y, z) \in \mathbb{R}^3d \) we have

\[
- \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \left\{ \frac{1}{2m} \left( \frac{\partial E}{\partial x} (t, x) + i \frac{\partial E}{\partial x} (t, x) \right) (\tau(\sigma), \zeta(\sigma)) - C_1 \right\} d\sigma_1 d\sigma_2 \\
\geq C'_* |X|^{2M_*}.
\]  

(5.14)

**Proof.** For a while we write

\[
Q(t, x) = - \frac{1}{2m} \left( \frac{\partial E}{\partial x} (t, x) + i \frac{\partial E}{\partial x} (t, x) \right) + C_1
\]

and \( X' = (z, x - z, y - x) = |X'| (\omega'_1, \omega'_2, \omega'_3) \in \mathbb{R}^3d \). From (2.7) we have

\[
|X'|^{−2M_*} \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1)Q(\tau(\sigma), z + \sigma_1(x - z) + \sigma_1 \sigma_2(y - x)) d\sigma_1 d\sigma_2 \\
\geq C_* \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1)|\omega'_1 + \sigma_1 \omega'_2 + \sigma_1 \sigma_2 \omega'_3|^{2M_*} d\sigma_1 d\sigma_2
\]

(5.15)

for \( X' \neq 0 \). If the right-hand side of (5.15) is equal to zero for a point \((\omega'_1, \omega'_2, \omega'_3)\) such that \(|\omega'_1|^2 + |\omega'_2|^2 + |\omega'_3|^2 = 1\), we have

\[
\omega'_1 + \sigma_1 \omega'_2 + \sigma_1 \sigma_2 \omega'_3 = 0
\]

for all \( 0 < \sigma_1 < 1 \) and \( 0 \leq \sigma_2 \leq 1 \), which means \( \omega'_1 = \omega'_2 = \omega'_3 = 0 \). This is contradiction. Hence there exists a constant \( C'_* > 0 \) such that

\[
\int_0^1 \int_0^1 \sigma_1(1 - \sigma_1)Q(\tau(\sigma), z + \sigma_1(x - z) + \sigma_1 \sigma_2(y - x)) d\sigma_1 d\sigma_2 \geq C'_* |X'|^{2M_*},
\]

which shows (5.14) with another constant \( C'_* > 0 \) because of \( |X'| \geq C |X| \) with a constant \( C > 0 \). \( \square \)
For a while we write the constant $C_1$ in (2.11) and (3.14) as $a$. Let us write

$$E'_0(t, s; x, y, z) = -\int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \times \left\{ \frac{1}{2m} \left( \frac{\partial E}{\partial x}(\tau(\sigma), \zeta(\sigma)) + \frac{i}{t} \frac{\partial E}{\partial x}(\tau(\sigma), \zeta(\sigma)) \right) - a \right\} d\sigma_1 d\sigma_2,$$

(5.16)

$$E'_1(t, s; x, y, z) = -\int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \times \frac{1}{2m} \left( \frac{\partial E}{\partial x}(\tau(\sigma), \zeta(\sigma)) - \frac{i}{t} \frac{\partial E}{\partial x}(\tau(\sigma), \zeta(\sigma)) \right) d\sigma_1 d\sigma_2.$$  

(5.17)

We can easily have the following.

**Lemma 5.4.** (1) Let us define $\Phi(t, s; x, y, z)$ by (5.10). Then we have

$$\frac{\partial \Phi}{\partial z}(t, s; x, y, z) = I + \rho^2 E'_0(t, s; x, y, z) - \frac{\rho^2 a}{6} + \rho^2 E'_1(t, s; x, y, z) + B'(t, s; x, y, z),$$

(5.18)

$$B'(t, s; x, y, z) = -\rho \frac{\partial}{\partial z} \frac{1}{m} \sum_{k=1}^d (z_k - x_k) \int_0^1 \int_0^1 \sigma_1 B_{jk}(\tau(\sigma), \zeta(\sigma)) d\sigma_1 d\sigma_2.$$  

(5.19)

(2) Let us define $\Phi(t, s; x, y, z)$ by (5.11). Then we have (5.18) where $B'(t, s; x, y, z)$ is given by

$$B'(t, s; x, y, z) = -\rho \frac{\partial}{\partial z} \frac{1}{m} \int_0^1 d\theta \sum_{k=1}^d (z_k - x_k) \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \frac{\partial B_{jk}}{\partial t}(s + \theta(1 - \sigma_1)\rho, \zeta(\sigma)) d\sigma_1 d\sigma_2.$$  

(5.20)

**Lemma 5.5.** (1) Let us write the fifth term on the right-hand side of (5.10) as $\rho \tilde{B}(t, s; x, y, z)$. Assume (2.10). Then we have

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \tilde{B}(t, s; x, y, z)| \leq C_{\alpha \beta \gamma}, \ |\alpha + \beta + \gamma| \geq 1.$$  

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Let us write the fifth term on the right-hand side of (5.11) as $\tilde{B}(t, s; x, y, z)$. Assume (2.12). Then we have

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_z \tilde{B}(t, s; x, y, z)| \leq \rho C_{\alpha \beta \gamma}, \quad |\alpha + \beta + \gamma| \geq 1.$$  

Proof. Both of (1) and (2) are proved by Lemma 3.5 in [10]. □

The following lemma is crucial in the present paper.

**Lemma 5.6.** We assume (2.7) and (2.8). (1) Let us define $\Phi$ by (5.11). Assume (2.12) if $0 \leq M_* < 1$ and (2.13) if $M_* \geq 1$. Then, there exist constant $\rho^*>0, \delta >0$ and $C \geq 0$ such that for $0 \leq t - s \leq \rho^*$ and $X = (x, y, z) \in \mathbb{R}^{3d}$ we have

$$\det \frac{\partial \Phi}{\partial z}(t, s, x, y, z) \geq \delta (1 + \rho^2 |X|^{2M_*})^d, \quad \rho = t - s, \quad (5.21)$$

$$\left| \frac{\partial \Phi}{\partial z}(t, s, x, y, z)^{-1} \right| \leq C (1 + \rho^2 |X|^{2M_*})^{-1} \quad (5.22)$$

and the mapping : $\mathbb{R}^d \ni z \rightarrow \xi = \Phi(t, s; x, y, z) \in \mathbb{R}^d$ is homeomorphic, where $|\Omega|$ for a matrix $\Omega = (\Omega_{ij}; i \downarrow j \rightarrow 1, 2, \ldots, d)$ denotes the matrix norm $(\sum_{i, j=1}^d |\Omega_{ij}|^2)^{1/2}$. We write its inverse mapping as $\mathbb{R}^d \ni \xi \rightarrow z = z(t, s; x, \xi, z) \in \mathbb{R}^d$. (2) Let us define $\Phi$ by (5.10). Assume (2.10) and (2.11). Then we have the same assertions as in (1).

Proof. We will first prove (1). Lemma 5.3 and (5.16) show

$$I + \rho^2 E'_0(t, s; x, y, z) \geq 1 + C'_* \rho^2 |X|^{2M_*} \quad (5.23)$$

for all $X = (x, y, z) \in \mathbb{R}^{3d}$. Hence we have

$$\left| (I + \rho^2 E'_0(t, s; x, y, z))^{-1} \right| \leq \frac{C}{1 + \rho^2 |X|^{2M_*}} \quad (5.24)$$

together with (2.8). We note Faraday’s law

$$\frac{\partial E}{\partial x}(t, x) - \frac{t \partial E}{\partial x}(t, x) = - \left( \frac{\partial B_{ji}}{\partial t} ; i \downarrow j \rightarrow 1, 2, \ldots, d \right), \quad (5.25)$$
which follows from (1.1). Hence, using the assumption (2.12) if $0 \leq M_* < 1$ and (2.13) if $1 \leq M_*$, from (5.17) we have

$$|\partial_\alpha \partial_\beta \partial_\gamma E'_1(t, s; x, y, z)| \leq C_{\alpha\beta\gamma} < X^{M_*}$$  \hspace{1cm} (5.26)

for all $\alpha, \beta$ and $\gamma$. Here we used that if (2.12) holds, $\partial_\alpha \partial_t B(t, x)$ are bounded on $\mathbb{R}^d$ for all $\alpha$. This follows from Lemma 3.5 in [10]. From (5.20) we also get

$$|\partial_\alpha \partial_\beta \partial_\gamma B'(t, s; x, y, z)| \leq C_{\alpha\beta\gamma} \rho^2 < X^{M_*}$$  \hspace{1cm} (5.27)

for all $\alpha, \beta$ and $\gamma$ together with (2) of Lemma 5.5.

Noting (5.23), we can rewrite (5.18) as

$$\frac{\partial \Phi}{\partial z}(t, s; x, y, z) = (I + \rho^2 E'_0) \left\{ I - (I + \rho^2 E'_0)^{-1} \frac{\rho^2 a}{6} + \rho \left( I + \rho^2 E'_0 \right)^{-1} \right. \left. \times \rho E'_1(t, s; x, y, z) + \rho \left( I + \rho^2 E'_0 \right)^{-1} \rho^{-1} B'(t, s; x, y, z) \right\}.$$  \hspace{1cm} (5.28)

Noting $\theta/(1 + \theta^2) \leq 1$ for all $\theta \geq 0$, from (2.8), (5.24) and (5.26) we have

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_z (I + \rho^2 E'_0)^{-1} \rho E'_1(t, s; x, y, z)| \leq C_{\alpha\beta\gamma} < \infty$$  \hspace{1cm} (5.29)

for all $\alpha, \beta$ and $\gamma$. In the same way from (5.21) we also have

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_z (I + \rho^2 E'_0)^{-1} \rho^{-1} B'(t, s; x, y, z)| \leq C_{\alpha\beta\gamma} < \infty$$  \hspace{1cm} (5.30)

for all $\alpha, \beta$ and $\gamma$. Therefore, from (5.23) and (5.28) we have

$$\det \frac{\partial \Phi}{\partial z} \geq (I + C' \rho^2 |X|^{2M_*})^d (1 - C\rho)$$

with a constant $C \geq 0$. Thereby we can see together with (5.21) and (5.28)-(5.30) that there exists a constant $\rho^* > 0$ satisfying (5.21) and (5.22). Hence we can complete the proof of the assertion (1) by using Theorem 1.22 on p. 16 in [21].

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We will prove (2). As in the proof of (1) we can prove (5.23) - (5.26), and so prove (5.29). Now, $B'$ is given by (5.19). Then from (1) of Lemma 5.5 we have

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_z B'(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma} \rho \quad (5.31)$$

for all $\alpha, \beta$ and $\gamma$. Consequently we can prove (5.30). Hence we can complete the proof of (2) as in the proof of (1).

The constant $\rho^* > 0$ defined in Lemma 5.6 is fixed from now on through sections 5, 6 and 7.

**Proposition 5.7.** We assume (2.7) - (2.9). (1) Let us define $\Phi$ by (5.11). Assume (2.12) if $0 \leq M_* < 1$ and (2.13) if $M_* \geq 1$. Let $0 \leq t - s \leq \rho^*$ and $z(t, s; x, \xi, y)$ the function defined in Lemma 5.6. Then we have

$$|\partial^{\alpha}_\xi \partial^{\beta}_x \partial^{\gamma}_y z_j(t, s; x, \xi, y)| \leq C_{\alpha, \beta, \gamma, |\alpha + \beta + \gamma|} \geq 1 \quad (5.32)$$

for $(x, \xi, y) \in \mathbb{R}^{3d}$. (2) Let us define $\Phi$ by (5.10). Assume (2.10) and (2.11). Then we have the same assertions as in (1).

**Proof.** Let $0 \leq t - s \leq \rho^*$. We will first prove (1). Let $w = x, \xi$ or $y$. It follows from $\xi = \Phi(t, s; x, y, z(t, s; x, \xi, y))$ that we have

$$\frac{\partial \xi}{\partial w_j} = \frac{\partial \Phi}{\partial z}(t, s; x, y, z) \frac{\partial z}{\partial w_j} + \frac{\partial \Phi}{\partial w_j}(t, s; x, y, z) \quad (5.33)$$

and so from (1) of Lemma 5.6

$$\frac{\partial z}{\partial w_j}(t, s; x, \xi, y) = \left(\frac{\partial \Phi}{\partial z}\right)^{-1} \left(\frac{\partial \xi}{\partial w_j} - \frac{\partial \Phi}{\partial w_j}\right).$$

Using (2.8)-(2.9) and (2.12)-(2.13), from (5.11) and (2) of Lemma 5.5 we get

$$\left|\frac{\partial \Phi}{\partial w_j}(t, s; x, y, z)\right| \leq C(1 + \rho |X|^{M_*} + \rho^2 |X|^{2M_*} + \rho^2 |X|^{M_*})$$

$$\leq C'(1 + \rho |X|^{M_*} + \rho^2 |X|^{2M_*}) \quad (5.34)$$
with non-negative constants $C$ and $C'$. Hence, using (5.22), we can prove
\[
\left| \frac{\partial z}{\partial w_j}(t, s; x, \xi, y) \right| \leq C'' < \infty
\]
with a constant $C'' \geq 0$. Next from (5.33) we have
\[
0 = \frac{\partial \Phi}{\partial z} \frac{\partial^2 z}{\partial w_k \partial w_j} + \left( \frac{\partial^2 \Phi}{\partial w_k \partial z} + \frac{\partial^2 \Phi}{\partial z^2} \frac{\partial z}{\partial w_j} \right) \frac{\partial z}{\partial w_j} + \frac{\partial^2 \Phi}{\partial z \partial w_j} \frac{\partial z}{\partial w_k}.
\]
Hence, using (5.32) with $|\alpha + \beta + \gamma| = 1$, we can prove (5.32) with $|\alpha + \beta + \gamma| = 2$ as in the proof of the case of $|\alpha + \beta + \gamma| = 1$. In the same way we can complete the proof of (5.32) by induction.

We consider the assertion (2). $\Phi$ is given by (5.10). Then we see from (1) of Lemma 5.5 that the corresponding inequalities to (5.34) are given by
\[
\left| \frac{\partial \Phi}{\partial w_j}(t, s; x, y, z) \right| \leq C(1 + \rho |X|^M + \rho^2 |X|^{2M} + \rho)
\]
Hence we can prove (5.32) as in the proof of (1).

**Theorem 5.8.** Suppose (2.3) and Assumption 2.2. Let $C(t, s)$ be the operator on $S(\mathbb{R}^d)$ defined by (5.6) and $0 \leq t - s \leq \rho^*$. Then $C(t, s)$ can be extended to a bounded operator on $L^2(\mathbb{R}^d)$ and satisfies
\[
\|C(t, s)f\| \leq e^{K(t-s)}\|f\|
\] (5.35)
for all $f \in L^2$ with a constant $K \geq 0$.

**Proof.** Since we can prove from (2.3) and (2.8) as in the proof of (2.25) that $\partial_x^* \partial_t A_j(t, x)$ are continuous in $[0, T] \times \mathbb{R}^d$, Lemma 5.2 holds. We will first prove the case that (2.12) and (2.13) are assumed. Let us define $\Phi$ by (5.11). Then
from Lemma 5.2 we have

$$C(t, s)^\dagger \chi(\epsilon)^2 C(t, s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \times \int \chi(\epsilon z)^2 \left(\exp i(x - y) \cdot \frac{m\Phi}{t-s}\right) dz$$

and so, changing variables from $z$ to $\xi = \Phi(t, s; x, y, z)$ by Lemma 5.6,

$$C(t, s)^\dagger \chi(\epsilon)^2 C(t, s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2 \left(\exp i(x - y) \cdot \frac{m\xi}{t-s}\right) dz \times \det \frac{\partial z}{\partial \xi}(t, s; x, \xi, y)f(y)dy d\eta, \ \xi = \frac{t-s}{m} \eta. \quad (5.36)$$

From (2.8), (5.24) and (5.28) - (5.30) we have

$$0 < \det \frac{\partial z}{\partial \xi}(t, s; x, \xi, y) = \det(I + \rho^2 E_0^{-1}) + (t-s)p_1(t, s; x, \xi, y)$$

$$\equiv p_0(t, s; x, \xi, y) + (t-s)p_1(t, s; x, \xi, y) \quad (5.37)$$

with $p_j(t, s; x, \xi, y) \in S^0(\mathbb{R}^3)$ ($j = 0, 1$). In particular, from (5.23) we have

$$0 \leq p_0(t, s; x, \xi, y) \leq 1. \quad (5.38)$$

Noting (5.32), from (5.36) and (5.37) we can prove

$$\lim_{\epsilon \to 0^+} C(t, s)^\dagger \chi(\epsilon)^2 C(t, s)f = \int \int e^{i(x - y) \cdot \eta}p_0(t, s; x, \frac{t-s}{m} \eta, y)f(y)dy d\eta$$

$$+ (t-s) \int \int e^{i(x - y) \cdot \eta}p_1(t, s; x, \frac{t-s}{m} \eta, y)f(y)dy d\eta$$

in $S$ for $f \in S$. Therefore, applying Theorem 1.A to the above, we have

$$\|C(t, s)f\|^2 \leq \liminf_{\epsilon \to 0^+} (C(t, s)^\dagger \chi(\epsilon)^2 C(t, s)f, f) \leq (1 + K(t-s))\|f\|^2$$

$$+ K(t-s)\|f\|^2 = (1 + 2K(t-s))\|f\|^2 \leq e^{2K(t-s)}\|f\|^2$$

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with a constant $K \geq 0$, which shows (5.35).

Next we consider the case that (2.10) and (2.11) are assumed. Let us define $\Phi$ by (5.10). Then we can prove Theorem 5.8 as in the proof of the first case.

\textbf{Corollary 5.9.} Suppose the assumptions of Theorem 5.8. Let $K_{\Delta}(t,0)f$ for $f \in C_0^\infty(\mathbb{R}^d)$ be the approximation defined by (2.11) of the Feynman path integral. Let $|\Delta| \leq \rho^*$. Then $K_{\Delta}(t,0)f$ can be extended to a bounded operator on $L^2(\mathbb{R}^d)$, are written as

$$K_{\Delta}(t,0)f = C(t, \tau_{\nu-1})C(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C(\tau_1,0)f$$

(5.39)

for $f \in L^2$ and satisfies (2.14) with the constant $K$ in Theorem 5.8.

\textbf{Proof.} As in the proof of (4.11) from Theorem 5.8 we can prove (5.39), which shows (2.14) by (5.35). \hfill \square

\textbf{Theorem 5.10.} Suppose (2.3) and Assumption 2.2. Let $q(x,w)$ be a function satisfying (5.3) with $M = 0$. We set $p(t,s;x,w) = q(x, \sqrt{t-s}w)$ and define $P(t,s)$ by (5.4). Then we have

$$\|P(t,s)f\| \leq C\|f\|, \quad 0 \leq t - s \leq \rho^*$$

(5.40)

for $f \in L^2$ with a constant $C \geq 0$.

\textbf{Proof.} Let $f \in S$. As in the proof of (5.36) we have

$$P(t,s)^{\dagger}\chi(\epsilon)^2 P(t,s)f = \left(\frac{m}{2\pi(t-s)}\right)^d \int f(y)dy \int \chi(\epsilon z)^2 \left(\exp i(x-y) \cdot \frac{m\xi}{t-s}\right)$$

$$ \times q(z,z-x)^* q(z,z-y) \det \frac{\partial}{\partial \xi}(t,s;x,\xi,y) d\xi.$$

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Noting (5.32), we have
\[
\lim_{\epsilon \to 0^+} P(t, s)^\dagger \chi(\epsilon)^2 P(t, s)f = \iint e^{i(x-y)\cdot \eta} q(z, z-x)^* q(z, z-y) \\
\times \det \frac{\partial z}{\partial \xi}(t, s; x, \frac{t-s}{m} \eta, y) f(y) dy d\eta
\]  
in $S$ with $z = z(t, s; x, (t-s)\eta/m, y)$ and hence we can prove (5.40) by Theorem 1.A as in the proof of (5.35).

6 Consistency of $C(t, s)$

Lemma 6.1. Suppose the assumptions of Proposition 5.7. Let $0 \leq t - s \leq \rho^*$ and $z(t, s; x, \xi, y)$ the function defined in Lemma 5.6. Then we have
\[
|z(t, s; x, \rho \eta/m + \sqrt{\rho} \zeta/m, x + \sqrt{\rho} y) - x| \leq C \sqrt{\rho} (1 + \sqrt{\rho}) |x|^{2M_*+1} \\
+ |y|^{2M_*+1} + \sqrt{\rho} |\eta|^{2M_*+1} + |\zeta|^{2M_*+1}.
\]  
(6.1)

Proof. We first consider the case that $\Phi$ is given by (5.11), which we write as
\[
\Phi(t, s; x, y, z) = z - \frac{x + y}{2} + \rho A(s; x, y) + \rho^2 \tilde{E}(t, s; x, y, z) \\
+ \rho \tilde{B}(t, s; x, y, z).
\]  
(6.2)

Then, using (2) of Lemma 5.5, from the assumptions (2.8) - (2.9) and (2.12) - (2.13) we have
\[
|\partial_x^\alpha \partial_y^\beta \tilde{A}(s; x, y)| \leq C_{\alpha, \beta} (1 + |x| + |y|)^{M_*}, \ |\alpha + \beta| \geq 1, 
\]  
(6.3)
\[
|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \tilde{E}_j(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma} (1 + |x| + |y| + |z|)^{2M_*}, \ |\alpha + \beta + \gamma| \geq 1, 
\]  
(6.4)
\[
|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \tilde{B}_j(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma} \rho (1 + |x| + |y| + |z|)^{M_*}, \ |\alpha + \beta + \gamma| \geq 1. 
\]  
(6.5)

From (5.32) we have
\[
|z(t, s; x, \xi, y)| \leq C (1 + |x| + |\xi| + |y|),
\]  
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which shows

\[ |z(t, s; x, \rho\eta/m + \sqrt{\rho}\zeta/m, x + \sqrt{\rho}y)| \]
\[ \leq C(1 + |x| + \sqrt{\rho}|y| + \rho|\eta| + \sqrt{\rho}|\zeta|). \quad (6.6) \]

We take \( z = z(t, s; x, \rho\eta/m + \sqrt{\rho}\zeta/m, x + \sqrt{\rho}y) \) in (6.2). Then we have

\[ \frac{\rho\eta}{m} + \frac{\sqrt{\rho}\zeta}{m} = z - \frac{2x + \sqrt{\rho}y}{2} + \rho\tilde{A}(s; x, x + \sqrt{\rho}y) \]
\[ + \rho^2 E(t, s; x, x + \sqrt{\rho}y, z) + \rho^2 \tilde{B}(t, s; x, x + \sqrt{\rho}y, z). \]

Hence we get

\[ \frac{z - x}{\sqrt{\rho}} = \frac{1}{2} y + \frac{\sqrt{\rho}\eta}{m} + \frac{\zeta}{m} - \sqrt{\rho}\tilde{A}(s; x, x + \sqrt{\rho}y) \]
\[ - \rho^{3/2} E(t, s; x, x + \sqrt{\rho}y, z) - \sqrt{\rho}\tilde{B}(t, s; x, x + \sqrt{\rho}y, z). \quad (6.7) \]

Applying (6.3) - (6.6) to (6.7), we have (6.1).

We consider the case that \( \Phi \) is given by (5.10). We write \( \Phi \) as (6.2). Then from (1) of Lemma 5.5 we have

\[ |\partial_\alpha^\eta \partial_\beta^y \partial_\gamma^z \tilde{B}_j(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma}, \quad |\alpha + \beta + \gamma| \geq 1 \quad (6.8) \]

correspondingly to (6.5). Hence we can also prove (6.1). \( \Box \)

From (5.32) we have the following.

**Lemma 6.2.** Suppose the assumptions of Proposition 5.7. Let \( 0 \leq t - s \leq \rho^* \). Then we have

\[ |\partial_\alpha^\eta \partial_\beta^y \partial_\gamma^z (z(t, s; x, \rho\eta/m + \sqrt{\rho}\zeta/m, x + \sqrt{\rho}y))| \leq C_{\alpha, \beta, \gamma} \sqrt{\rho}^{2|\alpha + \beta + \gamma|}, \]
\[ |\alpha + \beta + \gamma| \geq 1. \quad (6.9) \]
Lemma 6.3. Suppose the assumptions of Proposition 5.7. Let $0 \leq t - s \leq \rho^*$. Take a $p(x,w)$ satisfying (5.3) and set

$$q_c(t,s;x,\eta) = Os - \int \int e^{-iy\zeta} p \left( z, \frac{z - x}{\sqrt{\rho}} \right)^* \chi(\epsilon z)^2 p \left( z, \frac{z - x - \sqrt{\rho}y}{\sqrt{\rho}} \right) \times \det \frac{\partial z}{\partial \xi} (t, s; x, \rho\eta/m + \sqrt{\rho}\zeta/m, x + \sqrt{\rho}y) dy d\zeta$$

(6.10)

for $0 < \epsilon \leq 1$, where $z = z(t, s; x, \rho\eta/m + \sqrt{\rho}\zeta/m, x + \sqrt{\rho}y)$. Then we have

$$|\partial_\eta^\alpha q_c(t, s; x, \eta)| \leq C_\alpha (1 + |x| + |\eta|)^{2M(2M_* + 1)}$$

(6.11)

for all $\alpha$ with constants $C_\alpha$ independent of $\epsilon$.

Proof. We write $< D_y >^2 = 1 - \sum_{j=1}^{d} \partial_{y_j}^2$. Let $l_j \geq 0$ ($j = 0, 1$) be integers. Using (6.1), (6.6) and (6.9), from (6.10) we have

$$|q_c(t, s; x, \eta)| \leq \int \int |y|^{-2l_0} < D_\zeta >^{2l_0} < \zeta >^{2l_1} < D_y >^{2l_1} \left\{ p \left( z, \frac{z - x}{\sqrt{\rho}} \right)^* \chi(\epsilon z)^2 \times \left( 1 + |z| + \left| \frac{z - x}{\sqrt{\rho}} \right| + |y| \right)^{2M} \right\} dy d\zeta \leq C_1 \int \int |y|^{-2l_0} < \zeta >^{-2l_1} \times \left( 1 + |x|^{2M_* + 1} + |y|^{2M_* + 1} + |\eta|^{2M_* + 1} + |\zeta|^{2M_* + 1} \right)^{2M} dy d\zeta \leq C_2 \int \int |y|^{-2l_0} < \zeta >^{-2l_1} \times < \zeta >^{-2l_1} < y >^{2M(M_* + 1)} < \zeta >^{2M(M_* + 1)} \left( 1 + |x|^{2M_* + 1} + |\eta|^{2M_* + 1} \right)^{2M} dy d\zeta.$$

Hence, taking $l_0$ and $l_1$ so that $2l_j - 2M(2M_* + 1) > d$, we get

$$|q_c(t, s; x, \eta)| \leq C_4 (1 + |x| + |\eta|)^{2M(2M_* + 1)}.$$

In the same way we can prove (6.11), using (6.9). □

Proposition 6.4. Suppose (2.3) and Assumption 2.2. Let $p(x,w)$ be a function satisfying (5.3) and define $P(t,s)$ by (5.4). Let $0 \leq t - s \leq \rho^*$. Then
there exists an integer $l \geq 0$ such that we have

\[ \| P(t, s)f \| \leq C|f|_l \quad (6.12) \]

for $f \in \mathcal{S}(\mathbb{R}^d)$.

**Proof.** If $t = s$, the inequality (6.12) follows from (5.4). Let $0 < t - s \leq \rho^*$. Let us define $q_\epsilon(t, s; x, \eta)$ by (6.10) for $p(x, w)$. Then, using Lemma 5.2 and (5.32), we can prove

\[ P(t, s)\dagger \chi(\epsilon)^2 P(t, s)f = Q_\epsilon(t, s; X, D_x)f \quad (6.13) \]

for $f \in \mathcal{S}$, which has been proved in (4.12) of [11]. Hence we see

\[ \| \chi(\epsilon)P(t, s)f \|^2 = (P(t, s)\dagger \chi(\epsilon)^2 P(t, s)f, f) = (Q_\epsilon(t, s)f, f) \]

\[ = \int f(x)^*dx \int e^{i(x-y)\cdot \eta} q_\epsilon(t, s; x, \eta)f(y)d\eta \]

\[ = \int f(x)^*dx \int e^{i(x-y)\cdot \eta} <\eta >^{-2l_0} < D_y >^{2l_0} q_\epsilon(t, s; x, \eta)f(y)d\eta. \]

Consequently, using (6.11), we have

\[ \| P(t, s)f \|^2 \leq C_1 \int |f(x)|dx \int <\eta >^{-2l_0} (1 + |x| + |\eta|)^{2M(2M_\ast + 1)} \]

\[ \times \int < D_y >^{2l_0} f(y)|d\eta \leq C_1 \int x >^{2M(2M_\ast + 1)} |f(x)|dx \]

\[ \times \int <\eta >^{-2l_0+2M(2M_\ast + 1)}d\eta \int < D_y >^{2l_0} f(y)|dy \]

\[ \leq C_2 \int <\eta >^{-2l_0+2M(2M_\ast + 1)}d\eta |f|_l^2 \quad (6.14) \]

with an integer $l$. Taking $l_0$ so that $2l_0 - 2M(2M_\ast + 1) > d$, we obtain (6.12). \qed

**Theorem 6.5.** Suppose (2.3) and Assumption 2.2. Let $p(x, w)$ be a function satisfying (5.3) and define $P(t, s)$ by (5.4). Let $0 \leq t - s \leq \rho^*$. Then, for any $\alpha$ there exists an integer $l(\alpha) \geq 0$ such that we have

\[ \| x^\alpha (P(t, s)f) \| \leq C|f|_{l(\alpha)}, \quad \| \partial_x^\alpha (P(t, s)f) \| \leq C|f|_{l(\alpha)} \quad (6.15) \]
for $f \in S$.

Proof. Setting $p'(x, s) = x^\alpha p(x, w)$, we have $P'(t, s) f = x^\alpha P(t, s) f$ from (5.4). Hence from Proposition 6.4 we can prove
\[
\|x^\alpha (P(t, s) f)\| = \|P'(t, s) f\| \leq C |f|_{l(\alpha)}
\]
for $f \in S$ with an $l(\alpha) \geq 0$, which shows the first inequality of (6.15).

Next we can write $P(t, s) f$ as
\[
P(t, s) f = \sqrt{\frac{m}{2\pi i}} \operatorname{Os} - \int e^{i\phi(t; s; x, w)} p(x, w) f(x - \sqrt{\rho} w) dw,
\]
\[
\phi(t; s; x, w) = \frac{m}{2} |w|^2 + \sqrt{\rho} w \cdot \int_0^1 A(t - \theta \rho, x - \theta \sqrt{\rho} w) d\theta - \rho \int_0^1 V(t - \theta \rho, x - \theta \sqrt{\rho} w) d\theta, \rho = t - s
\]
as in §2 of [11]. Then we have
\[
\partial_x^\alpha (P(t, s) f) = \sum_{\beta \leq \alpha} P_\beta(t, s) (\partial_x^{\alpha-\beta} f),
\]  
(6.17)
where $\beta \leq \alpha$ indicates $\beta_j \leq \alpha_j$ for all $j = 1, 2, \ldots, d$. Using the assumptions (2.3) and (2.9), from (6.16) we have
\[
|\partial_x^{\alpha'} \partial_x^\beta p(t; s; x, w)| \leq C_{\alpha', \beta'} (1 + |x| + |w|)^{M+2(M_*+1)|\beta|}
\]  
(6.18)
for all $\alpha'$ and $\beta'$. Hence, applying Proposition 6.4 to $P_\beta(t, s)$, from (6.17) we obtain the second inequality of (6.15).

Proposition 6.6. We assume (2.3), (2.8) and (2.9). Let $H(t)$ and $C(t, s)$ be the operators defined by (1.3) and (5.6), respectively. Then there exists a continuous function $r(t, s; x, w)$ in $0 \leq s \leq t \leq T$ and $(x, w) \in \mathbb{R}^{2d}$ satisfying (5.3) for an $M \geq 0$ such that
\[
\left\{ \frac{\partial}{\partial t} - H(t) \right\} C(t, s) f = \sqrt{t - s} R(t, s) f
\]  
(6.19)
for $f \in S(\mathbb{R}^d)$.  

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Proof. From (2.3) and (2.8) we have
\[ |\partial_x^2 \partial_t A(t,x)| \leq C_\alpha < x >^{2(M_*+1)} \]
for all \( \alpha \) as in the proof of (2.25). Consequently we get Proposition 6.6 from Proposition 3.5 in [12] or Proposition 2.3 in [10]. \( \square \)

7 Proofs of Theorems 2.1 and 2.2

We suppose Assumption 2.1 and let \( M_* \geq 0 \) be the constant in Assumption 2.1. Let us introduce the weighted Sobolev spaces
\[ B^a(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d); \| f \|_a := \| f \| + \sum_{|\alpha| \leq 2a} \| \partial_x^\alpha f \| \]
\[ + \| < \cdot >^{2a(M_*+1)} f \| < \infty \} \quad (a = 1, 2, \ldots). \] (7.1)

We denote the dual space of \( B^a \) by \( B^{-a} \) and the \( L^2 \) space by \( B^0 \).

We have proved the following in Theorem 2.1 of [13].

**Theorem 7.A.** Suppose Assumption 2.1 and (2.9). Then for any \( f \in B^a \ (a = 0, \pm 1, \pm 2, \ldots) \) there exists the unique solution \( u(t) = U(t,0)f \in E^0_t([0,T];B^a) \cap E^1_t([0,T];B^{a-1}) \) with \( u(0) = f \) to the equation (1.3). This solution \( u(t) \) satisfies
\[ \| u(t) \|_a \leq C_a \| f \|_a, \quad 0 \leq t \leq T \] (7.2)
and in particular
\[ \| u(t) \| = \| f \|, \quad 0 \leq t \leq T. \] (7.3)

**Corollary 7.B.** Suppose Assumption 2.1 and (2.9). Then for any integer \( l \geq 0 \) there exists an integer \( l' \geq 0 \) such that
\[ |U(t,0)f|_l \leq C_l |f|_{l'}, \quad 0 \leq t \leq T \] (7.4)
for all \( f \in \mathcal{S} \).

**Proof.** The Sobolev lemma indicates

\[
\sup_{x \in \mathbb{R}^d} |f(x)| \leq C \sum_{|\alpha| \leq [d/2]+1} \|\partial_x^\alpha f\|
\]

where \([\cdot]\) denotes the Gauss symbol (cf. (2.24) on p. 78 in [17]). Hence, for any integer \( l \geq 0 \) there exist integers \( l_1 \geq 0 \) and \( l_2 \geq 0 \) such that

\[
|f|_l \leq C\|f\|_{l_1}, \quad \|f\|_l \leq C'\|f\|_{l_2}
\]

(7.5)

for \( f \in \mathcal{S} \). Therefore from (7.2) we have

\[
|U(t, 0)f|_l \leq C\|U(t, 0)f\|_{l_1} \leq CC\|f\|_{l_1} \leq C'\|f\|_{l_2}
\]

(7.6)

with an integer \( l' \).

**Lemma 7.1.** Suppose (2.3) - (2.4) and Assumption 2.2. Let \( H(t) \) and \( \mathcal{C}(t, s) \) be the operators defined by (1.3) and (5.6), respectively. Then there exists an integer \( l \geq 0 \) such that

\[
\left\| \frac{\mathcal{C}(t, s)f - f}{t - s} - H(t)f \right\| \leq C\sqrt{\rho}|f|_{l_1}, \quad 0 < t - s \leq \rho^*
\]

(7.7)

for all \( f \in \mathcal{S} \).

**Proof.** Using (6.19), we can write

\[
i\{C(t, s)f - f\} = i\{C(s + \rho, s)f - f\} = i\rho \int_0^1 \frac{\partial C}{\partial t}(s + \theta \rho, s)f d\theta
\]

\[
= \rho \int_0^1 \{H(s + \theta \rho)C(s + \theta \rho, s)f + \sqrt{\rho}R(s + \theta \rho, s)f\} d\theta
\]

(7.7)

and so

\[
i\frac{\mathcal{C}(t, s)f - f}{\rho} - H(t)f = \sqrt{\rho} \int_0^1 R(s + \theta \rho, s)f d\theta + \int_0^1 H(s + \theta \rho)
\]

\[
\cdot \{C(s + \theta \rho, s)f - f\} d\theta + \int_0^1 \{H(s + \theta \rho) - H(t)\} f d\theta.
\]

(7.8)
From (2.3) and (2.9) we can see that for $a = 0, 1, 2, \ldots$ there exist integers $l(a) \geq 0$ satisfying
\[ \|H(t)f\|_a \leq C_a\|f\|_{l(a)}. \]
Consequently we see that the $L^2$ norm of the second term on the right-hand side of (7.8) is bounded by
\[ C\int_0^1 \|C(s + \theta\rho, s)f - f\|_{l(0)} d\theta. \]
Applying (7.7) to this term, and applying (2.4), (2.9) and (6.20) to the third term on the right-hand side of (7.8), we have
\[ \left\| \frac{i}{\rho} C(t, s)f - f - H(t)f \right\| \leq \sqrt{\rho} \int_0^1 \|R(s + \theta\rho, s)f\|d\theta + C_1\int_0^1 d\theta \int_0^1 \rho \times \left\{ \|C(s + \theta'\rho, s)f\|_{l(0)} + \sqrt{\rho}\|R(s + \theta'\rho, s)f\|_{l(0)} \right\}d\theta' + C_2\rho\|f\|_{l(0)} \quad (7.9) \]
with an integer $l'(0) \geq 0$. Hence, applying Theorem 6.5 to $C(t, s)f$ and $R(t, s)f$, and using (7.5), we can prove (7.6).

**Lemma 7.2.** Suppose Assumptions 2.1 and 2.2. Then there exists an integer $l \geq 0$ such that we have
\[ \|C(t, s)f - U(t, s)f\| \leq C\sqrt{\rho^3}|f|_l, \quad 0 < t - s \leq \rho^* \quad (7.10) \]
for $f \in S$.

**Proof.** Correspondingly to (7.7) - (7.9) we have
\[ i\left\{ U(t, s)f - f \right\} = \rho \int_0^1 H(s + \theta\rho)U(s + \theta\rho, s)f d\theta, \]
\[ i\left\{ \frac{U(t, s)f - f}{\rho} - H(t)f \right\} = \int_0^1 H(s + \theta\rho)\left\{ U(s + \theta\rho, s)f - f \right\} d\theta \]
\[ + \int_0^1 \left\{ H(s + \theta\rho) - H(t) \right\} f d\theta, \]
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\[
\left\| \frac{U(t, s)f - f}{\rho} - H(t)f \right\| \leq C_1 \int_0^1 d\theta \int_0^1 \rho \|U(s + \theta' \rho, s)f\|_r(0)d\theta' + C_2 \rho \|f\|_r(0).
\]

Hence from Theorem 7.4 and (7.5) we can see

\[
\left\| \frac{U(t, s)f - f}{\rho} - H(t)f \right\| \leq C\rho |f|_{l'}
\]

for all \(f \in S\) with an integer \(l' \geq 0\). Writing

\[
C(t, s)f - U(t, s)f = \rho \left\{ \frac{C(t, s)f - f}{\rho} - H(t)f \right\}
\]

\[
-C(t, s)f - U(t, s)f = \rho \left\{ \frac{U(t, s)f - f}{\rho} - H(t)f \right\},
\]

we can prove (7.10) from (7.6) and (7.11). \(\square\)

Now we will prove Theorem 2.1. Hereafter we assume \(|\Delta| \leq \rho^*\). We have proved (2.14) in Corollary 5.9. First we assume \(f \in S\). From (5.39) we can write

\[
K_\Delta(t, 0)f - U(t, 0)f = C(t, \tau_{\nu-1})C(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C(\tau_1, 0)f
\]

\[
- U(t, \tau_{\nu-1})U(\tau_{\nu-1}, \tau_{\nu-2}) \cdots U(\tau_1, 0)f = \sum_{j=1}^{\nu} C(t, \tau_{\nu-1})C(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C(\tau_{j+1}, \tau_j)\{C(\tau_j, \tau_{j-1}) - U(\tau_j, \tau_{j-1})\}U(\tau_{j-1}, 0)f.
\]

Using (5.35), we have

\[
\|K_\Delta(t, 0)f - U(t, 0)f\| \leq e^{Kt} \sum_{j=1}^{\nu} \|\{C(\tau_j, \tau_{j-1}) - U(\tau_j, \tau_{j-1})\}U(\tau_{j-1}, 0)f\|
\]

which leads to

\[
\|K_\Delta(t, 0)f - U(t, 0)f\| \leq Ce^{Kt} \sum_{j=1}^{\nu} (\tau_j - \tau_{j-1})^{3/2} |U(\tau_{j-1}, 0)f|_l
\]

\[
\leq C' \sqrt{\|\Delta\|_e} e^{K'T}|f|_{l'}
\]

(7.14)
from (7.4) and (7.10). Hence we see that as $|\Delta| \to 0$, $K_\Delta(t,0)f$ for $f \in S$ converges to $U(t,0)f$ in $L^2$ uniformly in $t \in [0,T]$.

Let $f \in L^2$ be arbitrary. For any $\epsilon > 0$ we take a $g \in S$ such that $\|g - f\| < \epsilon$. Then from (2.14) and (7.3) we see

$$\|K_\Delta(t,0)f - U(t,0)f\| \leq \|K_\Delta(t,0)(f-g)\| + \|K_\Delta(t,0)g - U(t,0)g\| + (e^{KT} + 1)\|f-g\|,$$  

which shows

$$\lim_{|\Delta| \to 0} \sup_{0 \leq t \leq T} \|K_\Delta(t,0)f - U(t,0)f\| \leq (e^{KT} + 1)\epsilon$$

because of $g \in S$. This indicates

$$\lim_{|\Delta| \to 0} \sup_{0 \leq t \leq T} \|K_\Delta(t,0)f - U(t,0)f\| = 0.$$

In the end, to complete the proof of Theorem 2.1 we have only to prove (2.16). From (2.15) and (5.2) we have

$$S'(t,s; q_{x,y}^{t,s}) = \frac{m|x-y|^2}{2(t-s)} + \int_{\gamma_{x,y}^{t,s}} (A' \cdot dx - V'dt)$$

$$= S(t,s; q_{x,y}^{t,s}) + \psi(t,x) - \psi(s,y),$$

which shows (2.10) from (5.6) and (5.7).

Next we consider the Lagrangian function defined by (2.17). Let $F(\theta,s; q_{x,y}^{t,s})$ ($s \leq \theta \leq t$) be the $l \times l$ matrix defined as the solution to (2.19). The following has been proved in Lemma 3.1 of [12].

**Lemma 7.3.** We assume

$$|\partial_\theta^\alpha h_{1jk}(t,x)| \leq C_\alpha, \ |\alpha| \geq 1$$  

(7.16)
in $[0,T] \times \mathbb{R}^d$ for all $j, k = 1, 2, \ldots, l$. Then we have
\[
|\partial_x^\alpha \partial_y^\beta F(t', s; q_{x,y}^{t,s})| \leq C_{\alpha\beta} < \infty \tag{7.17}
\]
in $0 \leq s \leq t' \leq t \leq T$ for all $\alpha$ and $\beta$.

Using $S(t, s; q_{x,y}^{t,s})$, we define
\[
C(t, s)f = \sqrt{m/(2\pi i\rho)} \int \left( \exp iS(t, s; q_{x,y}^{t,s}) \right) F(t, s; q_{x,y}^{t,s})f(y)dy \tag{7.18}
\]
if $0 \leq s < t \leq T$ and $C(t, t)f = f$ for $f = t(f_1, \ldots, f_l) \in \mathcal{S}(\mathbb{R}^d)^l$. Then we have the following correspondingly to Theorem 5.8.

**Proposition 7.4.** We assume $\text{(2.3)}$, Assumption 2.2 and $\text{(2.21)}$. Then $C(t, s)$ on $\mathcal{S}^l$ can be extended to a bounded operator on $(L^2)^l$ and satisfies
\[
\|C(t, s)f\| \leq e^{K'(t-s)}\|f\|, \ 0 \leq t - s \leq \rho^* \tag{7.19}
\]
for $f = t(f_1, \ldots, f_l) \in (L^2)^l$ with a constant $K' \geq 0$, where $\|f\|^2 = \sum_{j=1}^l \|f_j\|^2$.

**Proof.** From (2.19) we have
\[
F(t', s; q_{x,y}^{t,s}) - I = -i \int_0^{t'} H_1(\theta, q_{x,y}^{t,s}(\theta)) F(\theta, s; q_{x,y}^{t,s}) d\theta. \tag{7.20}
\]
Then from the assumption $\text{(2.21)}$ and Lemma 7.3 we have
\[
|\partial_x^\alpha \partial_y^\beta \left\{ F(t, s; q_{x,y}^{t,s}) - I \right\} | \leq C_{\alpha\beta}(t - s) \tag{7.21}
\]
for all $\alpha$ and $\beta$. Using $C(t, s)$ defined by (5.6), we can write
\[
C(t, s)f = C(t, s)f + \sqrt{m/2\pi i\rho} \int \left( \exp iS(t, s; q_{x,y}^{t,s}) \right) \times \left\{ F(t, s; q_{x,y}^{t,s}) - I \right\} f(y)dy \equiv C(t, s)f + C'(t, s)f. \tag{7.22}
\]
Then, noting (7.21), from Theorems 5.8 and 5.10 we obtain
\[
\|C(t, s)f\| \leq e^{K'(t-s)}\|f\| + C_0(t - s)\|f\| \leq e^{K'(t-s)}\|f\|
\]
with constants $C_0 \geq 0$ and $K' \geq 0$, which shows (7.19).
We have the following correspondingly to Proposition 6.6.

**Lemma 7.5.** We consider the equation (2.18). Assume (2.3), (2.8), (2.9) and (7.16). Then there exist $r_{jk}(t, s; x, w)$ ($j, k = 1, 2, \ldots, l$) satisfying (5.3) with an $M \geq 0$ such that

\[
\left( i \frac{\partial}{\partial t} - H(t)I - H_1(t) \right) C_s(t, s)f
= \sqrt{t - s} \left( R_{sjk}(t, s); j \downarrow k \rightarrow 1, 2, \ldots, l \right) f
\]  

(7.23)

for $f \in S(\mathbb{R}^d)^l$.

**Proof.** From (2.3) and (2.8) we had (6.20). Hence we can prove (7.23) from Proposition 3.5 of [12].

Now we will prove Theorem 2.2. Let $|\Delta| \leq \rho^*$. Using Proposition 7.4, from (2.20), Remark 2.2 and (7.18) we can write

\[
K_{s\Delta}(t, 0)f = C_s(t, \tau_{\nu-1})C_s(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C_s(\tau_1, 0)f
\]  

(7.24)

for $f \in (L^2)^l$ and get the estimates (2.14) by (7.19). Next consider the equation (2.18). Then since we are assuming (2.21), we get the same assertions as in Theorem 7.A and so get (7.4). Hence we can complete the proof of Theorem 2.2 as in the proof of Theorem 2.1, using Theorem 6.5, Proposition 7.4 and Lemma 7.5.

**8 Proofs of Theorems 2.3 and 2.4**

In this section we always suppose Assumptions 2.3 and 2.4. Let $L^\sharp(t, x, \dot{x})$ be the Lagrangian function defined by (2.26). We write

\[
W(t, x) := 2 \sum_{1 \leq j < k \leq 4} V_{jk}(t, x(j) - x(k)).
\]  

(8.1)

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Let’s define $q_{t,s}^{l,x,y}$ by (2.22) and write $\gamma^{l,x,y}_{t,s}$ as (5.1). Then the classical action for $q_{t,s}^{l,x,y}$ is written as

$$S(t, s; q_{t,s}^{l,x,y}) = \sum_{l=1}^{4} \left\{ \frac{m_l |x(l) - y(l)|^2}{2(t-s)} + \int_{\gamma^{l,x,y}_{t,s}} A^{(l)}(t, x(l)) \cdot dx(l) \right\} - \int_{\gamma^{l,x,y}_{t,s}} V_l(t, x(l)) dt$$

(8.2)
correspondingly to (5.2).

Let $p(x, w)$ be a function satisfying (5.3) and define $P(t, s)$ by (5.4). Then we have the same assertions as in Lemma 5.1. We define $\tau(\sigma)$ and $\zeta^{(l)}(\sigma)$ by (5.12) for $x = x(l)$, $y = y(l)$ and $z = z(l)$. Hereafter for simplicity we suppose $m = m_l$ ($l = 1, 2, 3, 4$).

**Lemma 8.1.** Let $p(x, w)$ be a function satisfying (5.3). Then for any $0 < \epsilon \leq 1$ and $0 \leq s < t \leq T$ we have (5.8), where

$$\Phi(t, s; x, y, z) = (\Phi^{(l)}_1, \Phi^{(l)}_2, \Phi^{(l)}_3, \Phi^{(l)}_4) \in \mathbb{R}^{4d},$$

(8.3)

$$\Phi^{(l)}_j = z(l)_j - \frac{x(l)_j + y(l)_j}{2} + \frac{t-s}{m} \int_{0}^{1} A^{(l)}_j(s, x(l) + \theta(y(l) - x(l))) d\theta$$

$$- \frac{(t-s)^2}{m} \sum_{k=1}^{d} \phi^{(l)}_{jk} \int_{0}^{1} \int_{0}^{1} \sigma_1 B^{(l)}_{jk}(\tau(\sigma), \zeta^{(l)}(\sigma)) d\sigma_1 d\sigma_2$$

$$- \frac{t-s}{m} \sum_{k=1}^{d} (z(l)_k - x(l)_k) \int_{0}^{1} \int_{0}^{1} \sigma_1 B^{(l)}_{jk}(\tau(\sigma), \zeta^{(l)}(\sigma)) d\sigma_1 d\sigma_2$$

$$+ \frac{(t-s)^2}{m} \int_{0}^{1} \int_{0}^{1} \sigma_1 \frac{\partial W}{\partial x(l)}_j(\tau(\sigma), \zeta^{(l)}(\sigma)) d\sigma_1 d\sigma_2.$$
or

\[
\Phi_j^{(l)} = z(l)_j - \frac{x(l)_j + y(l)_j}{2} + \frac{t-s}{m} \int_0^1 A_j^{(l)}(s, x(l) + \theta(y(l) - x(l))) d\theta \\
- \frac{(t-s)^2}{m} \int_0^1 \int_0^1 \sigma_1 E_j^{(l)}(\tau(\sigma), \zeta(l)(\sigma)) d\sigma_1 d\sigma_2 \\
- \frac{(t-s)^2}{m} \int_0^1 d\theta \sum_{k=1}^d (z(l)_k - x(l)_k) \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \\
\times \frac{\partial B_{jk}^{(l)}}{\partial t}(s + \theta(1 - \sigma_1) \rho, \zeta(l)(\sigma)) d\sigma_1 d\sigma_2 \\
+ \frac{(t-s)^2}{m} \int_0^1 \int_0^1 \sigma_1 \frac{\partial W}{\partial x(l)_j}(\tau(\sigma), \zeta(l)(\sigma)) d\sigma_1 d\sigma_2.
\]

(8.5)

Proof. We note

\[
d\left(\sum_{l=1}^4 \sum_{j=1}^d A_j^{(l)}(t, x(l)) dx(l)_j - \sum_{l=1}^4 V_l(t, x(l)) dt - W(t, x) dt\right) \\
= \sum_{l,j} \left( \frac{\partial}{\partial t} A_j^{(l)} + \frac{\partial}{\partial x(l)_j} V_l + \frac{\partial W}{\partial x(l)_j} \right) dt \wedge dx(l)_j \\
+ \sum_{l,j,k=1}^d \frac{\partial}{\partial x(l)_k} A_j^{(l)} dx(l)_k \wedge dx(l)_j = - \sum_{l,j} E_j^{(l)} dt \wedge dx(l)_j \\
+ \sum_{l=1}^d \sum_{1 \leq j < k \leq d} B_{jk}^{(l)} dx(l)_j \wedge dx(l)_k + \sum_{l,j} \frac{\partial W}{\partial x(l)_j} dt \wedge dx(l)_j.
\]

(8.6)

Then we can prove Lemma 8.1 as in the proof of Lemma 5.2.

We have the following from Lemma 5.3.

**Lemma 8.2.** We have \(E = E^{(l)}(t, x(l)) \ (l = 1, 2, 3, 4), C_1 = C_1(l) \geq 0, x = x(l), \zeta(\sigma) = \zeta^{(l)}(\sigma), X = X(l) = (x(l), y(l), z(l)) \in \mathbb{R}^{3d} \) and \(M_* = M_{Is} \).

We set \(a = \max\{C_1(l); l = 1, 2, 3, 4\} \geq 0\). Let us write (5.16) and (5.17) for \(E = E^{(l)}(t, x(l)) \ (l = 1, 2, 3, 4)\) as \(E_0^{(l)}(t, s; x(l), y(l), z(l))\) and \(E_1^{(l)}(t, s; x(l), y(l))\).
, \(z(l)\), respectively. In the same way we write (5.19) and (5.20) for \(B_{jk} = B_{jk}^{(l)}(t, x(l))\) as \(B^{(l)}(t, s; x(l), y(l), z(l))\). We define

\[
E_{0s}(t, s; x, y, z) = \begin{pmatrix}
E_0^{(1)} & 0 & 0 & 0 \\
0 & E_0^{(2)} & 0 & 0 \\
0 & 0 & E_0^{(3)} & 0 \\
0 & 0 & 0 & E_0^{(4)}
\end{pmatrix}.
\] (8.7)

In the same way we define \(E_{1s}(t, s; x, y, z)\) and \(B_{s}(t, s; x, y, z)\). Then from (8.8) and (8.9) we have

\[
\frac{\partial \Phi}{\partial z}(t, s; x, y, z) = I + \rho^2 E_{0s}(t, s; x, y, z) - \frac{\rho^2 a}{6} + \rho^2 E_{1s}(t, s; x, y, z) + B_{s}(t, s; x, y, z) + \frac{\rho^2}{m} \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \frac{\partial^2 W}{\partial x^2}(\tau(\sigma), \zeta(\sigma)) d\sigma_1 d\sigma_2,
\] (8.8)

which is correspondent to (5.18).

**Lemma 8.3.** There exist constants \(\rho^* > 0\) and \(\delta > 0\) such that for \(0 \leq t - s \leq \rho^*\) and \((x, y, z) \in \mathbb{R}^{12d}\) we have (5.21) and

\[
\left| \frac{\partial \Phi}{\partial z}(t, s; x, y, z)^{-1} \right| \leq C (1 + \rho^2)^{-1},
\]

where \(1 + \rho^2 |X|^{2M_*}\) in (5.21) is replaced with \(\prod_{l=1}^4 (1 + \rho^2 |X(l)|^{2M_*})\).

**Proof.** Noting the assumption (2.28), we can easily prove Lemma 8.3 from Lemma 8.2 and (8.8) as in the proof of Lemma 5.6. \(\square\)

We see from Lemma 8.3 that the mapping \(\mathbb{R}^{4d} \ni z \rightarrow \xi = \Phi(t, s; x, y, z) \in \mathbb{R}^{4d}\) is homeomorphic if \(0 \leq t - s \leq \rho^*\). We write its inverse mapping as \(\mathbb{R}^{4d} \ni \xi \rightarrow z = z(t, s; x, \xi, y) \in \mathbb{R}^{4d}\). Then, noting the assumption (2.28), we can prove (5.32), (5.35) and (5.40) as in the proofs of Proposition 5.7,
Theorems 5.8 and 5.10. In the same way we can prove (6.15) and (6.19) as in the proofs of Theorem 6.5 and Proposition 6.6.

We introduce the weighted Sobolev spaces

\[ B^a_\omega(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d); \| f \|^a := \| f \| + \sum_{|\alpha| \leq 2a} \| \partial^\alpha_x f \| + \| \omega_a(\cdot)f \| < \infty \right\} \tag{8.9} \]

for \( a = 1, 2, \ldots \), where \( \omega_a(x) = \sum_{l=1}^d < x(l) > 2^{a(M_l+1)} \). We denote the dual space of \( B^a_\omega \) by \( B^{t-a}_\omega \) and the \( L^2 \) space by \( B^0 \). Then from Theorem 2.4 in [13] we get the same assertions as in Theorem 7.A. Joining the results above, we can prove Theorems 2.3 and 2.4 as in the proofs of Theorems 2.1 and 2.2.

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