NEW AND OLD TESTS OF COSMOLOGICAL MODELS AND THE EVOLUTION OF GALAXIES

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ABSTRACT

We describe classical cosmological tests, such as the log $N$–log $S$, redshift-magnitude, and angular diameter tests, and propose some new tests of the evolution of galaxies and the universe. Most analyses of these tests treat the problem in terms of a luminosity function and its evolution. The main thrust of this paper is to show that this is inadequate and can lead to incorrect conclusions when dealing with high-redshift sources. We develop a proper treatment in three parts. First, we describe these tests based on the isophotal values of quantities such as flux, size, or surface brightness. We show the shortcomings of the simple point-source approximation based solely on the luminosity function and consideration of the flux limit. We emphasize the multivariate nature of the problem and quantify the effects of other selection biases arising from the surface brightness and angular size limitations. In these considerations, the surface brightness profile and the distribution of the basic parameters describing it play a critical role in modeling the problem. In general, in the isophotal scheme the data analysis and comparison with the model predictions is complicated. Next, we show that considerable simplification is achieved if these tests are carried out in some sort of metric scheme, for example that suggested by Petrosian. This scheme, however, is limited to well-resolved sources. Finally, we describe the new tests and compare them to the traditional tests, demonstrating the observational and modeling ease that they provide. These new procedures, which can use the data to a fuller extent than the isophotal or metric-based tests, amount to simply counting the pixels or adding their intensities as a function of the surface brightness of all galaxies instead of dealing with surface brightness, size, and flux (or magnitude) of individual galaxies. We also show that a comparison of the data with the theoretical models of the distributions and evolution of galaxies has the simplicity of the metric test and utilizes the data as fully as the isophotal test.

Subject headings: cosmology: theory — galaxies: evolution — galaxies: photometry

1. INTRODUCTION

Galaxies and other extragalactic sources provide the most direct means for studying evolution in the universe, usually using classic cosmological tests such as the angular diameter–redshift magnitude relation or source counts (also known as the log $N$–log $S$) test. (For a general description of these tests, see, e.g., Weinberg 1972.) These tests, which rely primarily on the distribution of the magnitudes or fluxes of the sources, have had limited success in determining the cosmological parameters and/or the evolution of galaxies. There are two fundamental reasons for this failure. The first is the well-known difficulty of disentangling the evolution of the sources (see, e.g., Tinsley 1968 or Tinsley & Gunn 1976) from the evolution of the universe (Weinberg 1972). As a result, over the years the focus of such studies has shifted from determining the cosmological parameters to studying the evolution of galaxies in different assumed cosmological models. The second difficulty arises from the fact that galaxies are extended (i.e., resolved) sources, and there is ambiguity in defining proper magnitudes (or luminosities $L$ and fluxes $I$) and diameters. In addition, the samples of sources are not merely limited by their fluxes (or magnitudes); there are also other selection biases or data truncations resulting from surface brightness or size limitations. These aspects of the problem are usually ignored. This may be an acceptable approximation for high surface brightness sources at low redshifts, but it is woefully inadequate when dealing with data at high redshifts extending to low surface brightness sources. The main purpose of this paper is to present a proper analysis of the various observational biases that are encountered in this process. There are two ways to carry out this task. From the original data sample, one can select a subsample with fewer and simpler biases (as we do in § 3), or one can correct the model expectation fully for all known selection biases, which is the approach we take in the rest of the paper. The first method is more appropriate for tests dealing with the moments of the distributions, such as flux-redshift or size-redshift relations. The second method is preferable when dealing with the various source counts, and uses all of the valuable data.

The bias arising from the magnitude or flux limit is accounted for by various means. The most common practice is to use isophotal values, i.e., the values of these quantities up to or at some limiting apparent surface brightness $b_{lim}$. However, because of the rapid decline of the apparent surface brightness $b$ (defined as flux per unit angular area of a resolved source) with redshift $z$ (see, e.g., Tolman 1934),

$$b = \frac{B}{(1+z)^2},$$

where $B$ is the intensity or the absolute surface brightness at the source, the biases arising from the surface brightness and size limits of the observations become important at high redshifts and/or for low surface brightness sources. These effects are often ignored or are dealt with indirectly by using a limited portion of the available data.

The corrections for these effects, sometimes referred to as aperture corrections, inevitably require knowledge of the surface brightness profile,

$$B(r) = B_0 f\left(\frac{r}{r_0}\right), \quad \text{with } f(0) = 1,$$

the distributions of the central surface brightness $B_0$, the characteristic or core radius $r_0$, and other parameters $\alpha_i$.
defining the profile \( f(x) \). For example, \( x = 1 \) or \( \frac{1}{2} \) for disks or spheroids, respectively, where the profile is described by the simple relation \( \ln f(x) = -x^q \). Early examples of methods for correcting the redshift-magnitude relation for the aperture effect were described by Sandage (1972), using an iterative procedure, and by Gunn & Oak (1975), assuming a fiducial cosmological model. These authors used empirical relations for the luminosity within the radius \( r \),

\[
L(r) = (4\pi)r^2 B_0 F \left( \frac{r}{r_0} \right), \quad \text{with} \quad F(t) = \int_0^t 2x f(x) dx.
\]

It was shown by Petroian (1976, hereafter P76) that these corrections can be carried out more directly. P76 also demonstrated that it is possible to separate the evolution of the surface brightness \( B_0 \) from the evolution of universe and so avoid some of the above difficulties by dealing with the angular sizes and magnitudes up to and within a proper metric radius, \( r_p \), obtained from a specified value of the quantity

\[
\eta = \frac{F(x)}{x^2 f(x)} = 2 \frac{d \ln r}{d \ln L(r)}, \quad \text{with} \quad x = \frac{r}{r_0},
\]

that is equal to the ratio of the average surface brightness within \( r \) to the surface brightness at \( r \).

The above equations describe the source brightness profile and its basic properties in terms of two convenient parameters, the central surface brightness \( B_0 \) and core radius \( r_0 \). These parameters are not easily accessible to observation, and their relative values for different values of \( x \) are difficult to interpret. This difficulty can be overcome if we transfer the above relations to observationally more meaningful parameters. One commonly used set of parameters is the effective radius and surface brightness. The effective radius is the radius containing half the total luminosity \( L \), which means \( F(\eta_{eff}/r_0) = 0.5F(\infty) \). The ratios of the effective to central values of the surface brightness and radius are \( B_{eff}/B_0 = 0.189, \ 2.54 \times 10^{-2}, \ 3.45 \times 10^{-3} \), and \( 4.66 \times 10^{-4} \), and \( \eta_{eff}/r_0 = 1.66, \ 13.5, \ 1.82 \times 10^2, \) and \( 3.46 \times 10^3 \) for \( x = 1, \frac{1}{2}, \frac{1}{3}, \) and \( \frac{1}{4} \), respectively. Figure 1 shows the profile \( f \), the curves of growth of luminosity \( F/\infty \), and the function \( \eta \) as a function of \( (r/r_{eff})^x \) for the above values of \( x \). This shows the general and relative characteristics of these functions. To demonstrate the effects of the redshift, in Figures 2a and 2b we show the variation with redshift of the fraction of the luminosity and the area (expressed in magnitude units) within a surface brightness limit \( b \) for \( x = 1 \) and \( 1/4 \) profiles. Instead of the surface brightness \( B_0 \) and \( b \), we use the more familiar magnitudes \( \mu = -2.5 \log b + \text{const.} \) and \( \mu_{eff} = -2.5 \log B_{eff} + \text{const.} \)

It is apparent from Figure 2 that the observable area and luminosity vary rapidly with redshift, especially for low surface brightnesses, and have a different behavior for the two profiles. This will produce a variation with redshift in the relative abundances of disks and spheroids. Another way of representing these graphs is in terms of the proper metric radius \( r_p \), defined above. This approach is preferable because this definition of radius relies on the data within some measured isophote and not on the unobserved outer parts, which are needed to determine the effective radius \( r_{eff} \). This procedure will be developed further in §3.

In this paper we review several old procedures and propose some new ones for the study of the evolution of galaxies, and possibly of the universe, in which instead of dealing with individual galaxies we deal with the combined brightness of all galaxies. The new methods simplify the data analysis enormously and are perfectly suited to modern digitized data. In §2 we first give a brief description of the proper analysis of the classical tests for isophotal quantities, one that includes all the observational selection effects as well as the effects of the surface brightness profile, and treats the problem in terms of the multivariate distribution \( \psi(B_0, r_0, \sigma_i, z) \) instead of the commonly used luminosity function \( \Phi(L, z) \). There are several reasons for the popularity of the latter procedure. The first is that we have accumulated considerable knowledge about the local luminosity function of galaxies (see, e.g., Ëstathiou, Ellis, & Peterson 1988;}

![Figure 1](image-url)
FIG. 2.—Variation of the fraction of the luminosity (solid lines) and projected surface area (dashed lines) in some arbitrary magnitude units, within a surface brightness isophote \( \mu \) vs. redshift for several values of the effective surface brightness \( \mu_{\text{eff}} \) for (a) an exponential (disk) profile, \( \ln f = -x^{\alpha} \), and (b) a de Vaucouleurs profile, \( \ln f = -x^{1/4} \). Note that for each value of \( \mu_{\text{eff}} \) the luminosity and area shrink rapidly as we approach the maximum redshift \( z_{\text{max}} = (B_0/b)^{1/4} - 1 \), where \( B_0 \) and \( b \) are related to \( \mu_{\text{eff}} \) and \( \mu \) as described in the text. Also note that this decline is more pronounced for disks than for spheroids, indicating that the relative populations of sources with different profiles will vary strongly with redshift.

Lin et al. 1966), but little information on the distributions of the \( B_0 \) and \( r_0 \) (see, e.g., Sandage & Perlmutter 1991). Second, until recently it was believed that the distribution of surface brightnesses was fairly narrow (Fish 1964; Freeman 1970). However, in recent years, because of increasing interest in low surface brightness dwarf galaxies, some data has been accumulated on the intrinsic distributions of these parameters that indicates broad distributions (see, e.g., McGaugh 1996; de Jong 1996; Tully & Verheijen 1997; Impey & Bothun 1997, and references therein). Of course, one can use the luminosity function by substituting the total luminosity \( L = (4\pi)nr_0^2 B_0 F(\infty) \) (see eq. [3]) for either \( B_0 \) or \( r_0 \). In any case, a multivariate description is required. In § 3 we repeat the analysis of § 2 for metric quantities. In § 4 we describe the new tests and their relations to the multivariate distribution \( \psi \) and the profile \( f(r) \). Finally, in § 5 we give a brief summary.

2. CLASSICAL TESTS: ISOPHALTAL VALUES

The classical tests use the observed relations between the magnitude (or flux \( l \)), angular size (radius \( \theta \) or area \( \pi \theta^2 \)), and redshift to determine the cosmological parameters and the evolution of sources as described by the general luminosity function \( \psi \). The cosmology is introduced via the relations

\[
\psi(\theta) = \frac{(4\pi)^2 B_0 F(\infty)}{4\pi d_L^2(z, \Omega_\Lambda)} , \quad \theta = \frac{r}{d_A(z, \Omega_\Lambda)} ,
\]

(5)

where \( d_L \) and \( d_A = d_L/(1 + z)^2 \) are the luminosity and the angular diameter distances, and \( \Omega_\Lambda \) represent the cosmological parameters, such as the density parameter \( \Omega_0 \), the deceleration parameter \( q_0 \), or the cosmological constant \( \Lambda \) (see, e.g., Weinberg 1972). All the classical tests can be described in terms of the observed distribution of flux, size, and redshift; \( n(l, \theta, z) \). For the purpose of demonstrating the
effects that we want to emphasize here, let us consider the cumulative source counts as a function of redshift, which we denote by \( N(l, z) \). The cumulative and differential counts of the log \( N-\log S \) relation are obtained by integration of the above expressions over the redshift.

For simplicity, in the above relation and in what follows we ignore cosmological attenuation (if any), assume either bolometric or monochromatic fluxes, so that we can ignore the K-correction, and assume spherical symmetry with brightness profile \( f(r) \) independent of the wavelength. The complications arising from the K-correction, asphericity, etc. can be easily included in the relations that follow. We will address some aspects of these in § 5.

2.1. Point Sources

The usual practice (see, e.g., Metcalfe et al. 1995; Tyson 1988) is to compare the observed cumulative and differential distributions \( N(l, z) \) and \( n(l, z) = -\partial N(l, z)/\partial l \) (and their integrals over redshift) with that expected from models via the relations

\[
N(l, z) = \frac{dV}{dz} \int_{4\pi d_L^2 z_1}^{\infty} \phi(L, z) dL,
\]

\[
n(l, z) = \frac{dV}{dz} 4\pi d_L^2 \phi(4\pi d_L^2 l, z),
\]

where \( V(z, \Omega) \) is the comoving density up to redshift \( z \) and \( \rho(z) = \int \phi dL \) is the comoving density of all sources at \( z \). Here and in what follows we assume a complete 4\( \pi \) sr sky coverage. These expressions are what one would expect for unresolved or point sources, where only the flux limit counts in the selection process (see, however, a modification below in § 2.3).

2.2. Extended Sources

For extended sources such as galaxies, the selection process is more complex, and additional corrections are required. We now describe these selection biases.

2.2.1. Surface Brightness Limit

To be detected, a source must have an apparent central surface brightness that exceeds the detection threshold, which must be several times the standard deviation \( \sigma \) of the fluctuations in the background brightness. We denote this limit by \( b_{\text{iso}} \). If we ignore the image degradaion due to the finite size of the instrumental and atmospheric point spread function (PSF), which can be done if the core size is \( r_0 \gg \theta_s d_A \) and if the pixel size is less than or comparable to the width \( \theta_s \) of the PSF, then the surface brightness selection criterion requires that

\[
B_0 \geq B_{\text{iso}} \equiv (1 + z)^4 b_{\text{iso}}.
\]

However, for small sources or high redshifts the effect of the finite size of the PSF cannot be ignored, and the selection bias is more severe than indicated by this relation. For a PSF \( = g(\theta/\theta_s) \), the surface brightness is modified to

\[
\zeta = \frac{B_{\text{iso}}}{B_0} = \int_0^\infty \frac{f(r)}{r_0} \frac{\theta_s d_A}{r_0} d\theta.
\]

where \( f = f * g \) is the convolution of the actual profile with the PSF. As a result, the central surface brightness is reduced by \( \zeta = B_{\text{iso}}/B_0 \), where

\[
\zeta \approx \left( \frac{\theta_s d_A}{r_0} \right) \int_0^\infty f(r) g(r) \theta d\theta.
\]

For the purpose of illustration, let us consider a square PSF with a radial width of \( \theta_s \). This reduction factor then simplifies, and equation (7) is modified to read

\[
B_0 \geq B_{\text{iso}} \frac{x_s^2}{F(x_s)}, \quad \text{with } x_s = \frac{\theta_s d_A}{r_0}.
\]

Similar expressions can be derived for other forms of the PSF. For \( x_s \ll 1 \), this reduces to equation (7), but its effects become important for \( x_s \) near unity, i.e., for partially resolved and unresolved sources. Note that \( \theta_s \) is replaced by the pixel size \( \theta_{\text{pix}} \) if \( \theta_s < \theta_{\text{pix}} \).

2.2.2. Size Limit

Another selection criterion for extended sources such as galaxies is that their size must exceed some limit. One way to quantify this is to have the isophotal angular radius (namely, the radius at which the surface brightness has dropped to the specified isophotal value \( b_{\text{iso}} \)) be larger than some specified size \( \theta \). If \( \theta \gg \theta_s \), then the isophotal angular radius is given by

\[
\theta_{\text{iso}} = \frac{r_0}{d_A} f^{-1} \left( \frac{B_{\text{iso}}}{B_0} \right),
\]

where the function \( f^{-1} \) is the inverse of the profile function \( f \). Then the selection condition \( \theta_{\text{iso}} \geq \theta \) is satisfied if

\[
B_0 \geq \frac{B_{\text{iso}}}{f(4\pi d_A/r_0)}. \quad \text{(12)}
\]

However, as \( \theta \) decreases toward \( \theta_s \), one should use the modified profile \( \hat{f} \) of equation (8) in place of \( f \). In any case, it is clear that for \( \theta > \theta_s \), this inequality will provide a more restrictive limit than the surface brightness limit, because it requires that more than one pixel exceed the surface brightness limit \( b_{\text{iso}} \). This can be demonstrated mathematically by setting \( \theta = \theta_s \) in the last equation and comparing it with the limit in equation (10). The ratio of the two limits is equal to \( \eta(x_s) \), which according to equation (4) is greater than 1, except in the unlikely event of the surface brightness increasing with \( r \). In the opposite case, when \( \theta < \theta_s \), one is dealing with unresolved or pointlike sources, in which case a size limit does not make sense.

For nonspherical sources, we can follow a similar procedure by dealing with the isophotal angular area \( \omega \) (which for spherical sources is equal to \( \pi \theta^2 \)) as the area of the sources with apparent surface brightness \( b > b_{\text{iso}} \). However, the relation of this area to the surface brightness profile will be more complicated. For example, for elliptical sources with a constant ellipticity, we can express the profile \( f \) as a function of the area \( a/A_0 \), with \( A_0 = \pi r_1 r_2 \), where \( r_1 \) and \( r_2 \) are the core radii along the major and minor axes, respectively. For randomly oriented elliptical sources, this will amount to replacing the quantity \( 4\pi d_A/r_0 \) in equation (12) by \( (\omega d_A/A_0)^{1/2} \). The distribution function \( \psi \) will now be a function of \( a/A_0 \) and the ellipticity, or the ratio \( r_1/r_2 \).

2.2.3. Flux Limit

Finally, the sample of sources is subject to a flux limit. In this section we consider the flux \( I_{\text{iso}} \) within the isophotal angular radius \( r_{\text{iso}} \) or up to the surface brightness limit \( b_{\text{iso}} \). The flux limit then implies that \( I_{\text{iso}} = I(\theta_{\text{iso}}, d_A)/4\pi d_L^2 \) is greater than some specified flux \( I \). Using equations (3) and
(5), we can write this limit as

\[ B_0 F(x_{\text{iso}}) \geq \frac{d_L^2 l}{\pi r_0^2}, \quad \text{with} \quad x_{\text{iso}} = \frac{\theta_{\text{iso}} d_A}{r_0} = f^{-1} \left( \frac{B_{\text{iso}}}{B_0} \right). \]  

(13)

Note that the left-hand side of this inequality is independent of \( r_0 \). However, if \( \theta_{\text{iso}} \theta_s \), then one should replace the profiles \( f \) and its integral, the luminosity growth curve \( F \), by the corresponding values \( \tilde{f} \) and \( \tilde{F} \) modified by the PSF. In this case, the above relation becomes more complex with the involvement of the additional variable \( \theta_s d_A/r_0 \) (see § 2.4).

### 2.2.4. Combined Limits

Thus, for given values of \( \theta, l, b_{\text{iso}}, \) and \( \theta_s \), the above three inequalities determine the region of the \( B_0 r_0 \) plane that is accessible at these particular conditions. This region varies with redshift, becoming smaller at higher redshifts. The redshift dependences are hidden in \( B_{\text{iso}} \) and \( d_A \). Figures 3a and 3b show the three boundary conditions obtained by the equality sign in equations (10), (12), and (13) for the exponential (\( \alpha = 1 \)) and de Vaucouleurs (\( \alpha = \frac{1}{4} \)) profiles, respectively. We plot \( B_0/B_{\text{iso}} \) versus \( r_0/(\theta_s d_A) \), which is valid at all redshifts. The heavy solid line in Figure 3 shows the truncation due to the surface brightness limit. The lighter solid lines show the effects of the size limit for several values of \( \theta/\theta_s \). The dashed lines show the truncation due to the flux limit for different values of the ratio \( l/(\pi \theta_s^2 b_{\text{iso}}) \). Sources lying in the region above all three lines satisfy all the selection criteria. It is clear that as long as \( \theta > \theta_s \), or \( l > \pi \theta_s^2 b_{\text{iso}} \), which obviously will be the case for resolved sources, the surface brightness limit due to the PSF described by equation (10) is never important. However, both size and flux limits could be important depending on the relative values of the observational limits. For larger values of the

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**Fig. 3.** Central surface brightness \( B_0 \) in units of \( B_{\text{iso}} = b_{\text{iso}}(1 + z)^4 \) vs. the size \( r_0 \), in units of \( \theta_s d_A \), depicting the surface brightness limit (heavy solid line), the angular size limit for \( \theta/\theta_s = 1, 2, 5, \) and 10 (thin solid lines), and the flux limit for \( l/(\pi \theta_s^2 b_{\text{iso}}) = 1, 2, 5, 10, 100, \) and 1000 (dashed lines) for (a) an exponential (disk) profile, \( \ln f = -x \), and (b) a de Vaucouleurs profile, \( \ln f = -x^{1/4} \). Note that the solid and dashed lines cross each other where the limiting factor changes from size to flux. The region to the right of the appropriate solid line before these intersections and to the right of the appropriate dashed line after the intersections are accessible to observation. Note also that because \( B_{\text{iso}} \) and \( d_A \) increase with redshifts, the observable region shrinks systematically, moving toward the upper right-hand corner as redshift increases.
ratio \( l/(\pi \theta^2 b_{\text{iso}}) \), the flux limit provides the major constraint. In the opposite case, the size becomes more important, and as evident from the above figures, for \( l = \pi \theta^2 b_{\text{iso}} \) only the size limit is relevant.

2.2.5. Extended Source Counts

We can now relate the observable \( N(>l, >\theta, z) \) to the distribution function \( \psi(B_0, r_0, z) \) by integration of the latter over the accessible region as determined by the observational limits. For the general case, this gives

\[
N(>l, >\theta, z) = \frac{dV}{dz} \left[ \int_{B_{\text{iso}}} B_0 \int_{r_0}^{\infty} d\theta \psi(B_0, r_0, z) \right.
+ \left. \int_{B_{\text{iso}}} B_0 \int_{r_0}^{\infty} d\theta \psi(B_0, r_0, z) \right].
\]

Here \( r_{0,1} \) and \( r_{0,2} \) are obtained by solving equations (13) and (12) for \( r_0 \) in terms of \( B_0 \) and other observables, and

\[
B_{0,cr} = \frac{B_{\text{iso}}}{f(\eta^{-1}(1/\pi \theta^2 b_{\text{iso}}))}
\]

is the intersection point of the two boundary conditions described by equations (12) and (13), or the intersection of a solid and a dashed line in Figures 3a and 3b; \( \eta^{-1} \) is the inverse function of the function \( \eta \) defined in equation (4). Note that as stated above for \( l = \pi \theta^2 b_{\text{iso}} \) this critical value of the surface brightness becomes equal to \( B_{\text{iso}} \), the first double integral in the left-hand side of equation (14) vanishes, and we are left with the size-limited part of this expression only.

It is clear, therefore, that the relation for the counts of extended sources is considerably more complicated than the commonly used relation given by equation (6). In order to see these differences more clearly, we can rewrite the above expressions in terms of the luminosity \( L \). For example, if we replace \( r_0 \) by \( L = (4\pi r_0^2 \lambda B_0 F(\infty)) \), we can rewrite all the boundary conditions in terms of \( B_0 \) and \( L \) instead of \( B_0 \) and \( r_0 \). The three limits in equations (10), (12), and (13) now give, respectively, the conditions

\[
B_0 \geq \frac{B_{\text{iso}}(F^{-1}(\lambda))^2}{\lambda}, \quad \text{with} \quad \lambda = \frac{4\pi d_s^2(\pi \theta^2 b_{\text{iso}})F(\infty)}{L},
\]

\[
L \geq L_{\text{min},\theta} = \frac{4\pi d_s^2(\pi \theta^2 b_{\text{iso}})F(\infty)}{x_{\text{iso}}^2 f(x_{\text{iso}})},
\]

\[
L \geq L_{\text{min}}, \equiv \frac{4\pi d_s^2 L F(\infty)}{F(x_{\text{iso}})}.
\]

Here \( F^{-1} \) is the inverse function of the luminosity curve of growth \( F \), and \( x_{\text{iso}} \) is a function of \( B_0/B_{\text{iso}} \) eq. [13]. Figures 4a and 4b show the truncations produced by the above selection criteria in the \( B_0/L \) plane. If we define the distribution \( \psi(B_0, L, z) = \psi(B_0, L, z)dr_0/dL \), equation (14) then becomes

\[
N(>l, >\theta, z) = \frac{dV}{dz} \left[ \int_{B_{\text{iso}}} B_0 \int_{L_{\text{min}}}^{\infty} dL \psi(B_0, L, z) \right.
+ \left. \int_{B_{\text{iso}}} B_0 \int_{L_{\text{min}}}^{\infty} dL \psi(B_0, L, z) \right].
\]

So far, we have expressed our results in terms of the central surface brightness \( B_0 \) and the core radius \( r_0 \). As mentioned in § 1, these parameters are not convenient for comparing the results with observations. This task can be carried out more readily if we express the above relations in terms of observationally more meaningful parameters such as the effective radius and surface brightness, as defined in § 1. In Figure 4c we show the size and flux limits in the surface brightness–luminosity plane for both profiles, where instead of the surface brightnesses \( b_{\text{iso}} \) and \( B_0 \) we use the more familiar magnitudes, \( \mu_{\text{iso}} = -2.5 \log b_{\text{iso}} + \text{const.} \) and \( \mu_{\text{eff},z} = -2.5 \log B_{\text{eff},z} + 10 \log (1 + z) + \text{const.} \) This figure demonstrates that the region of the \( B_0/L \) plane accessible to observations is different for the two profiles and shrinks with increasing redshift.

2.2.6. Comparison with the Point-Source Approximation

There are several ways in which this correct description differs from the approximate expression given by equation (6). Some of these have been discussed by Yoshii (1993). The first difference is the existence of the second set of integrals in equations (14) and (19), which we discussed above. Even at high values of the flux limit, \( l > \pi \theta^2 b_{\text{iso}} \), when this additional term is negligible there are two other important differences. The first difference is the presence of the ratio \( F(\infty)/F(x_{\text{iso}}) \) in the lower limit of the luminosity \( L_{\text{min}}, \), which is absent from the lower limit in equation (6). The second effect arises from the breadth of the distribution of \( B_0 \). For narrower distributions this effect is smaller, and it disappears for a delta function distribution of \( B_0 \).

These differences can be seen in Figures 4a and 4b as follows. For a given value of \( l \) or the ratio \( l/(\pi \theta^2 b_{\text{iso}}) \), the point-source approximation given by equation (6) truncates the \( B_0/L \) plane by a vertical line at the asymptote of the dashed line appropriate for this ratio. It then counts all the sources to the right of this line \( (L \geq 4\pi d_s^2 l) \), irrespective of their surface brightness \( B_0 \), and uses \( \psi(L, z) = [\psi(B_0, L, z)] \). Equation (19), on the other hand, indicates that this is an overestimation of the counts and that we should count only sources that lie to the right of both the solid and the dashed lines appropriate to the ratios \( \theta/\theta_0 \) and \( l/(\pi \theta^2 b_{\text{iso}}) \), respectively. It is therefore clear that ignoring the surface brightness and size limitations can cause significant errors, the extent of which can be quantified only if we know the distribution function \( \psi \). We have a good knowledge of the dependence of \( \psi \) on \( L \) at low \( z \) and high \( B_0 \), but we have only scanty information on its form at high \( z \) and low values of \( B_0 \). The primary aim of the cosmological tests under discussion here is to determine the variation with redshift of the general distribution function \( \psi \). A detailed investigation of these aspects is beyond the scope of this paper. Here we make some simple comparisons between the extended and point-source results that do not require a knowledge of the distribution of \( B_0 \).

Luminosity limits.—The ratio of the limiting luminosities \( L_{\text{min}}, \) and/or \( L_{\text{min},\theta} \) to the limit \( 4\pi d_s^2 l \) of the point-source approximation, which depends only on the surface profile, are shown in Figures 5a and 5b for \( \tau = 1 \) and \( \frac{1}{2} \), respectively, and for several values of the surface brightness (actually the ratio \( B_0/B_{\text{iso}} \) and the ratio \( \mu = l/(\pi \theta^2 b_{\text{iso}}) \)). We use the effective rather the central surface brightness, and express the above ratios in magnitudes. As evident for high values of the ratio \( \mu \), i.e., for higher flux limits, there is an increasing bias against detection of extended sources at higher redshifts. At lower values of this ratio, there is additional bias against detection of galaxies at low redshifts because of the size limit.
In (c), we plot the combined size and flux limits for both profiles in terms of the effective values of the surface brightness in magnitudes; \( \mu_{\text{iso}} = -2.5 \log h_{\text{iso}} + \text{const.} \) and \( \mu_{\text{eff, x}} = -2.5 \log B_{\text{eff}} + 10 \log (1 + z) + \text{const.} \) For clarity, only curves for \( \theta = \theta_0 \) and \( l = n \theta_0^2 h_{\text{iso}} \) are shown.
Fig. 5.—Ratio of the minimum detectable luminosity for extended sources in the isophotal scheme (eqs. [17] and [18]) to that of point sources vs. redshift, for indicated values of the effective surface brightnesses (more precisely, the ratio $B_0/b_{iso}$ and for four values of the ratio $\theta = l/(l^2 b_{iso}) = 1, 3.16, 10, \text{ and } \infty$ (long-dashed, short-dashed, dotted, and solid lines, respectively) for (a) an exponential (disk) profile, $\ln f = -x$, and (b) a de Vaucouleurs profile, $\ln f = -x^{1/4}$.

Redshift Distributions.—These differences can also be seen when we compare the redshift distributions expected for point and extended sources. For the purpose of illustration, let us assume that $L$ and $B_0$ are uncorrelated (clearly not a good assumption), so that we can separate the distribution function as $\psi = \rho(z) h(B_0) \phi(L/L^*)/L^*$, where $\rho(z)$ and $L^*(z)$ describe the density and luminosity evolution of the sources, respectively, and $h(B_0)$ gives the distributions of the central surface brightness. If we define the cumulative functions $H(B_0) = \int_0^\infty h(B_0) dB_0$ and $\Phi(x) = \int_0^\infty \phi(x') dx'$, with $H(0) = \Phi(0) = 1$, then equations (6) and (19) become, respectively

$$N(>l, >\theta, z) = \frac{dV}{dz} \rho(z) \Phi \left( \frac{4\pi d^2 l}{L^*} \right) \Phi \left( \frac{L_{\min,l}}{L^*} \right) dB_0 \left[ h(B_0) \Phi \left( \frac{L_{\min,l}}{L^*} \right) dB_0 \right].$$

In the absence of an exact knowledge of the distribution $h(B_0)$, we compare these expressions for different assumed values of the central or effective surface brightness, which amounts to a delta function approximation of $h(B_0)$. In this case, equation (21) simplifies to

$$N(>l, >\theta, z) = \frac{dV}{dz} \rho(z) \left\{ \Phi \left( \frac{L_{\min,l}}{L^*} \right), \quad \text{if } B_{0,cr} > B_0, \right. \left. \Phi \left( \frac{L_{\min,0}}{L^*} \right), \quad \text{if } B_{0,cr} < B_0. \right.$$
Assuming a Schechter luminosity function, \( \phi(x) \propto x^p e^{-x} \), we evaluate the redshift distributions for some representative values of the surface brightness \( B_0 \) (or \( \mu_{\text{det}} \)) and for several combinations of the limits \( l \) (or magnitude \( m \)), \( \theta \), and \( b_{\text{iso}} \) (or \( \mu_{\text{iso}} \)). The results are shown in Figures 6a–6d for \( p = -1 \) and for a cosmological model with \( \Omega = 1 \) and \( \Lambda = 0 \). We also assume the absence of any density or luminosity evolution (\( p \) and \( L^* \) constants). It should be noted that the relative shapes of the point-source and various extended source distributions are independent of the cosmological model or the density evolution \( \rho(z) \). It is evident that equations (6) and (20) give quite incorrect redshift dependences, overestimating the number of sources by a large factor at high redshifts, especially for low values of the surface brightness, because of the surface brightness limit, and at high values of surface brightness, because of the size limit. Clearly, ignoring these effects could lead to incorrect results. For example, if these expressions were used to derive the extent of the luminosity evolution, \( L^*(z) \), they would underestimate this evolution by factors equal to \( F(x_{\text{iso}})/F(x_{\text{iso}}) \) and \( x_{\text{iso}} f(x_{\text{iso}})/F(x_{\text{iso}}) \), depending on whether \( l/(\pi \theta^2 b_{\text{iso}}) \geq 1 \) or is equal to 1, respectively. The situation is more complicated when the effects of the dispersion of the surface brightness or its correlation with the luminosity (or core size \( r_0 \)) are included. In such cases, there would be errors in the determination of the density evolution as well.

**Source Counts.**—Integrating the redshift distributions over \( z \) gives the cumulative counts. Differentiation of this number gives the differential counts. For example, for point sources \( n(l) = \int_0^\infty (dV/dz) \phi(4\pi d^2 l, z)(4\pi d^2)dz \). Figures 7a and

![Figure 6](image-url)

**Fig. 6.**—Redshift distribution \( N(l, z) \) from eq. (20) of the point sources (heavy solid line) and extended sources (eq. [22]) for seven different values of the effective surface brightness. All curves are normalized by the peak value of the point-source distribution. In each case, the dotted lines show low surface brightnesses (\( n = 1, 1, \) and 3) falling below the turning point of the solid lines in Fig. 4c, where the surface brightness limit is most important; the dashed lines show high surface brightnesses (\( n = 7, 9, \) and 11) above this turning point, where the size limit is important. The thin solid line (with \( n = 5 \)) generally lies near the turning point. We assume a Schechter luminosity function with the exponent \( p = -1 \), a \( \Omega = 1 \), \( \Lambda = 0 \) cosmological model, and no evolution. The limiting magnitude is \( m = 2.5 \log l + \text{const.} \) and \( m_{\text{iso}} = 2.5 \log (\pi \theta^2 b_{\text{iso}}) + \text{const.} \). The surface brightness is in units of mag arcsec\(^{-2} \). (a) For an exponential (disk) profile, in \( f = -x \), with limits such that only the size limit (Fig. 4c, solid lines) is important. (b) Same as (a), but for a de Vaucouleurs profile, in \( f = -e^{x} \). (c) Same as (b), but for a higher flux limit. (d) Same as (a), but for higher surface brightness and flux limits.
show the differential magnitude counts of extended sources with various values of the surface brightness as well as of point sources. We use the same model parameters as above. Again, it is evident that neglecting the selection effects discussed above can cause considerable error in the determination of the evolution of the general luminosity function or the cosmological parameters.

### 2.3. Point Sources Revisited

It should be noted that some of these effects are present even for unresolved or point sources. However, in this case the correct equation is only slightly different from equations (6) and (20). When \( \theta_d \gg \theta_{\text{eff}} \), the sources are unresolved and all have essentially the same profile, \( g \), and size, \( \theta_0 \), as the PSF: \( \bar{B}(\theta) = B_0 \zeta(\theta_0, \theta_d/\theta_0) g(\theta/\theta_0) \). The total luminosity can be written as \( L = 4\pi d_A^2(\pi \theta_s^2) B_0 \zeta \). If we limit ourselves to the isophotal fluxes \( > l \) and sizes \( > \theta \), then the three selection criteria (eqs. [10], [12], and [13]) become almost identical in form: \( B_0 \geq B_{\text{iso}}/[K_1 \zeta(\theta, d/\theta_0)] \), where \( \zeta \) is defined in equation (9) and \( K_1 \approx 1 \). \( g(\theta/\theta_0) \) and \((4\pi)(\pi \theta_s^2 b_{\text{iso}}) G(\theta_{\text{iso}}/\theta_s)/l \), respectively, for the three limits. In

The above relations, we have defined the cumulative PSF as \( G(x) = \int_0^x 2g(x) \, dx \), \( G(\infty) = 1 \), and \( \theta_{\text{iso}} \) is obtained from \( g(\theta_{\text{iso}}/\theta_0) = B_{\text{iso}}(B_0, \zeta) = L_{\text{iso}}/L \). Clearly, the size limit does not make sense for unresolved sources, and the flux limit is the most restrictive limit. It can be shown then that the correct expression for the source counts is

\[
N(>l, \theta) = \frac{dV}{dz} \int_{b_{\text{iso}}}^{\infty} dB_0 \int_{L_{\text{min}}}^{\infty} dL \bar{\psi}(B_0, L, z),
\]

where the lower limit of the luminosity is obtained from the solution of

\[
LG\left[ g^{-1}\left(\frac{L_{\text{iso}}}{L}\right) \right] = 4\pi d_A^2 l, \quad \text{with} \ L_{\text{iso}} = 4\pi d_A^2(\pi \theta_s^2 b_{\text{iso}}).
\]

Here \( g^{-1} \) is the inverse function of the PSF. In addition to the surface brightness cutoff and the integration over \( B_0 \), this expression also differs from the simple equation (6) by the presence of the term involving the cumulative PSF \( G \) in the integration limit. This difference becomes important.
only for flux limits very near the isophotal values, \( l = \pi \theta_o^2 b_{iso} \). The surface brightness limit can be important for unresolved galaxies because of their low intrinsic surface brightness or effective brightness temperatures. But for other point sources, such as quasars, whose surface brightness temperature is equal to that of a hot accretion disk, this effect is negligible (becoming important only at extremely high redshifts), and the point-source approximation of equation (6) is very accurate.

2.4. Combined Counts

In principle, we can combine the counts of the resolved and unresolved sources by replacing \( B(r) \) with the modified profile \( \tilde{B}(r) \) from equation (8). We can then repeat the procedure carried out for the extended sources, replacing the profiles \( f \) and \( F \) with \( \tilde{f} \) and \( \tilde{F} \) and changing the limits correspondingly, except that now the profiles are functions of the additional parameter \( \theta_o d_\alpha r_0 \). However, the size limit is now unnecessary, because we can include all sources. Of course, one must make sure that the sample of galaxies, for example, is not contaminated by other unresolved sources (e.g., stars). We therefore have the simpler expression

\[
N(l, z) = \frac{dV}{dz} \int_{B_{iso}}^{\infty} dB_0 \int_{L_{min, l}}^{\infty} dL \tilde{\psi}(B_0, L, z),
\]

(25)

where \( L_{min, l} \) is given by equation (18), with \( F(x_{iso}) \to \tilde{F}(x_{iso}) \); note that \( \tilde{F}(x) = F(x) \). The parameter \( x_{iso} \) is now obtained from \( \tilde{f}(x_{iso}, \theta_o d_\alpha r_0) = B_{iso}/B_0 \), with \( \pi r_o^2 = L/[4\pi B_0 F(x)] \).

2.5. Other Tests

Differentiation of \( N(l, \theta, z) \) gives the differential distribution \( n(l, \theta, z) \), from which we can calculate various moments and compare them to observations. For example, the flux-redshift relation can be obtained from

\[
\langle l(z) \rangle = \int_0^\infty \frac{[LF(x_{iso})/[4\pi d_\alpha^2 F(x)]]n(l, \theta, z)dl}{N(l, \theta, z)}. \quad (26)
\]
In a similar fashion, one can derive $\langle \theta \rangle$-z or $\langle \theta \rangle$-$\langle l \rangle$ relations.

3. CLASSICAL TESTS: METRIC VALUES

Some of the complications evident in the above analysis can be avoided if instead of the isophotal sizes and fluxes we deal with some metric values of these quantities. For example, if we define a proper metric size $r_p$, corresponding to a constant value of the function $\eta$, say $\eta_0$, as defined in P76 and equation (4), then the expressions for the surface brightness, size, and flux limits (or truncations) become considerably simplified. For a limiting surface brightness $b_{iso}$ a limiting angular radius $\theta$, and a limiting flux $l$, these truncations are described by

$$B_0 \geq \frac{B_{iso}}{f(\xi)}, \quad r_0 \geq \frac{d_A \theta}{\xi}, \quad L \geq \frac{4 \pi d_A^2 l F(\infty)}{F(\xi)} ,$$

(27)

where $\eta(\xi) = \eta_0$ and $\xi = r_p/r_0$.

Note that in contrast to the complicated truncations we found for the isophotal case (eqs. [10], [12], and [13]), the current truncations are much simpler; they depend on only one observational limit and the redshift. The above limits are good for $\theta \gg \theta_0$, the size of the PSF. This will always be true in this case, because of the need to have a well-defined surface brightness distribution.

If the truncation limits are chosen so that $l = \pi \theta^2 b_{iso}$, then the last limit in equation (27) due to the flux limitation falls below the other two and can be ignored. In this case, the data truncation in the $B_0$-$r_0$ plane is parallel to the axis, making the calculation of the observables straightforward and free of the complex limits of integration. Since the flux limit $l$ does not enter into the determination of the observed distribution of the sources, such a sample will not provide appropriate data for tests based on source counts as a function of the flux $l$. But such a sample can be used to obtain the distributions of the angular size, average surface brightness, or redshift. For example, the redshift is simply

$$N(\zeta, z) = \frac{dV}{dz} \int_{B_{iso}/f(\xi)}^\infty dB_0 \int_{0}^{d_A \theta/\xi} dr_0 \psi(B_0, r_0, z) .$$

(28)

Because of this simplification, such samples are well suited to tests based on the moments of the observed distributions. For example, the angular size–redshift relation is given simply as

$$\langle \pi \theta^2(z) \rangle = \frac{\zeta}{N(\zeta, z) d_A^2} \frac{dV}{dz} \int_{B_{iso}/f(\xi)}^\infty dB_0 B_0 \int_{0}^{d_A \theta/\xi} dr_0 \pi r_0^2 \psi(B_0, r_0, z) .$$

(29)

Similarly, for the flux-redshift relation we have

$$\langle l(z) \rangle = \frac{F(\xi)}{N(\zeta, z) d_A^2} \frac{dV}{dz} \int_{B_{iso}/f(\xi)}^\infty dB_0 B_0 \int_{0}^{d_A \theta/\xi} dr_0 \pi r_0^2 \psi(B_0, r_0, z) ,$$

(30)

or, in the terms of the luminosity $L$

$$\langle L(z) \rangle = \frac{F(\xi)/F(\infty)}{N(\zeta, z) d_A^2} \frac{dV}{dz} \int_{B_{iso}/f(\xi)}^\infty dB_0 \int_{L_{min}}^{\infty} dLL \psi(B_0, L, z) ,$$

(31)

with a similar expression for the size-z relation. The integration limit is given by

$$L_{min, z} = 4 \pi d_A^2 (\pi \theta^2 b_{iso}) \left[ \frac{F(\infty)}{\zeta^2} \right] \left( \frac{B_0}{B_{iso}} \right) .$$

(32)

These expressions are considerably simpler than the corresponding equations for the isophotal analysis.

Similar expressions can be derived for other definitions of the metric quantities. For example, as mentioned in § 1 in connection with Figures 1 and 2, instead of $r_p$ one can use the effective radius $r_{eff}$, within which resides a certain fraction (usually half) of the total light. This would amount to a new definition of the constant $\xi$ as $F(\xi) = F(\infty)/2$. This may be a convenient procedure for nearby galaxies, but not at high redshifts, because it relies on a knowledge of the total flux, the determination of which lies at the heart of the difficulty associated with these tests. The procedure proposed in P76 relies only on data within a specified radius, not using data from the outer, invisible parts; $\theta_p$ is obtained by setting the ratio of the average to the limiting surface brightnesses at a fixed value.

4. NEW TESTS

The discussions in the above two sections demonstrate that an accurate analysis of the extragalactic data for the purpose of cosmological tests is complicated and must include all of the above-mentioned considerations. In particular, it is imperative to keep in mind the multivariate nature of the problem and to account for the surface brightness limitation (eq. [7]) common to all of the above expressions. The dispersions in the distributions of $B_0$, $r_0$, or $L$, and the correlations between these, can have substantial effects on the final results. These effects are more pronounced when dealing with the isophotal quantities than with the metric. However, the metric procedure must be limited to well-resolved sources, while the isophotal, in principle, could be extended to unresolved sources if a good knowledge of the PSF is at hand.

This task, however complicated, can be carried out given a knowledge of the distribution function and the brightness profile. With sufficient care in the data analysis and in the modeling, one can determine either the cosmological evolution of the sources (i.e., the redshift variation of the distribution $\psi$) or the cosmological parameters. Such analyses, which may be simple or appropriate for data limited to low redshifts, are not the simplest method for determining cosmological or Galactic evolution. For example, the traditional method for identifying sources with some apparent flux may not be necessary or may be the most straightforward way of carrying out this task. The complexities described in the previous sections are the result of the multiple selection criteria needed for counting individual galaxies.

We now describe two new and much simpler tests that combine the good aspects and avoid the complexities of the two methods described above, and are much better suited to the analysis of modern digitized CCD data. The essence of these tests is to reduce the selection criteria to one, namely the surface brightness, and to deal with the distribution of $B_0$. In practice, this amounts to simply counting the number of pixels at a given surface brightness (or adding up their intensities), independent of which galaxy they belong to. This way, one can avoid the complexities arising from the
need to define the sizes and fluxes (isophotal or metric) for every galaxy in the field.

4.1. Sky Covered by Galaxies

The first of these tests, which is related to the angular diameter test, involves computation of the fraction of the sky that is covered by all galaxies above (cumulative) or within (differential) a given range of surface brightness, \( b \) to \( b + db \). Observational determination of this fraction is simple; it is accomplished by counting the number of pixels with a given intensity value. The expressions relating this quantity to the cosmological models and to the properties of the galaxies are decidedly simpler than for the isophotal treatment, and somewhat simpler than for the metric. Let us first consider well-resolved sources, namely, those with angular radii \( \geq \theta_g \), which we take to be \( \gg \theta_s \). In this case, the only data truncation arises from the size limit, which is same as described by equations (11) and (12), with the isophotal quantities replaced by those for an arbitrary value of \( b \):

\[
B_z = (1 + z)^4 b, \quad x_b = \frac{\theta_g d_A}{r_0} = f^{-1}(B_z/B_0),
\]

with \( B_0 \geq B_z/\theta d_A/d ). Following the same steps as in the previous sections, the sky fraction covered by all galaxies down to a given apparent surface brightness \( b \) is obtained by adding the contribution \( \pi \theta_b^2 = x_b r_0/d_A \) of each galaxy. The result is

\[
\mathcal{F}_{\text{sky}}(b, z) = \frac{d\mathcal{X}}{dz} \int_{B_z}^{\infty} dB_0 \psi(b_0, r_0, z),
\]

(34)

where the line element

\[
\frac{d\mathcal{X}}{dz} = \frac{dV}{dz} \frac{1}{4\pi d_A^2}.
\]

Alternatively, in terms of the luminosity distribution,

\[
\mathcal{F}_{\text{sky}}(b, z) = \frac{d\mathcal{X}}{dz} \int_{B_z}^{\infty} dB_0 \frac{x_b^2}{B_0 F(\infty)} \int_{L_{\text{min}}}^{\infty} dLL \psi(b_0, L, z),
\]

(36)

where

\[
L_{\text{min}} = \frac{4\pi d_A^2(\pi \theta_b^2 b)}{x_b^2 f(x_b)}. \quad (37)
\]

Note that these equations have the simplicity of the tests based on metric values; equation (34), except for the term \( x_b^2 \), is identical to \( \langle \pi \theta_b^2(z) \rangle \tilde{N}(\zeta, z) \), shown in equation (29). More importantly, the data analysis is enormously simpler because it does not require determination of the surface brightness profile and the metric or isophotal values for each galaxy.

As in the classical tests, this test can also be carried out for unresolved sources. But this is not too different from counting sources, because all unresolved sources have essentially the same area, \( \pi \theta_b^2 \). Following the procedure given in § 2.3, it can be shown that the sole truncation due to the surface brightness limit, \( B_0 \zeta \geq B_z \), is equivalent to

\[
L = 4\pi d_A^2(\pi \theta_b^2 b_0) \geq L_0 \equiv 4\pi d_A^2(\pi \theta_b^2 b),
\]

and the actual angular radius of each source is given by \( \theta_g/\theta_s = g^{-1}(B_z/B_0) \zeta = g^{-1}(L_0/L) \). Thus, the fraction of the sky covered by unresolved sources becomes

\[
\mathcal{F}_{\text{sky}}(b, z) = \frac{d\mathcal{X}}{dz} \int_{B_z}^{\infty} dB_0 \int_{L_0}^{\infty} dL \times \left[ g^{-1}(\frac{L_0}{L}) \right]^2 \psi(B_0, L, z). \quad (39)
\]

Similarly, as in § 2.4, if we have a good knowledge of the form of the PSF, we can combine resolved and unresolved sources by expressing the above relations in terms of the modified profile \( \tilde{f} \) as the convolution of the profile \( f \) and the PSF \( g \) (eq. [8]). Then we do not need to specify a size limit, and the only truncation comes from the value of the apparent surface brightness \( b \). However, in this case we have the added complication of the dependence of the characteristics on the ratio \( \theta_d d_A/d_0 \).

The differential distribution can be obtained from

\[
\mathcal{F}_{\text{sky}}(b, z) = \frac{d\mathcal{X}}{dz} \int_{B_z}^{\infty} dB_0 \frac{F(x_b)}{F(\infty)} \int_{L_{\text{min}}}^{\infty} dLL \tilde{\psi}(b_0, L, z).
\]

(40)

where \( L_{\text{min}} \) is given in equation (37), with \( f \rightarrow \tilde{f}, F \rightarrow \tilde{F} \), and \( x_b = \theta_d d_A/d_0 \) is a function of both \( B_0 \) and \( L \) (or \( d_0 \)) and is obtained from the inversion of the relation \( f(x_b, \theta_d d_A/d_0) = B_0/B_0 \zeta \), with \( \pi r_0^2 = L_0/B_0 F(\infty) \).

The differential distribution can be obtained from

\[
i_{\text{sky}}(b, z) = -\frac{d}{db} \mathcal{F}_{\text{sky}}(b, z)/db. The integration of either distribution over \( z \) gives the differential or cumulative distribution, respectively, of all galaxies irrespective of their redshift.

4.2. Total Sky Brightness

The second test, which is related to the flux-redshift test, deals with the contribution of all galaxies to the sky brightness within a range of (or above) a given surface brightness \( b \). This amounts to adding all of the intensity values of the appropriate pixels. This sum is then to be compared with the expression for the total intensity (flux per steradian) as the sum of the contribution \( \langle \pi \theta_b^2 b_0 F(x_b)/(4\pi d_A^2) \rangle \) of all resolved galaxies with \( \theta_b \geq \theta_g \):

\[
i_{\text{sky}}(b, z) = \frac{1}{(1 + z)^2} \int_{B_z}^{\infty} dB_0 \frac{F(x_b)}{F(\infty)} \int_{L_{\text{min}}}^{\infty} dLL \tilde{\psi}(b_0, L, z). \quad (41)
\]

Note again the similarity of this expression to equation (31). As above, if we redefine the profile as the convolved \( \tilde{f} \) and include all resolved and unresolved galaxies in the analysis, we obtain the relation

\[
i_{\text{sky}}(b, z) = \frac{1}{(1 + z)^2} \int_{B_z}^{\infty} dB_0 \frac{F(x_b)}{F(\infty)} \int_{L_{\text{min}}}^{\infty} dLL \tilde{\psi}(b_0, L, z). \quad (42)
\]

The differential distribution is obtained as \( i_{\text{sky}}(b, z) = -\frac{d}{db} i_{\text{sky}}(b, z)/db \), and the integrals of these over \( z \) give the cumulative and differential total sky brightness as a function of \( b \).
4.3. Average Sky Brightness

The ratio of the quantities described in the two above tests gives the average surface brightness at or down to some surface brightness \( b \). This quantity, as expected, is independent of the cosmological model parameters and depends only on the surface brightness profile and redshift, and consequently, as pointed out in P76, can be used to determine the evolution of the surface brightness. In general, this relation is more complicated than envisioned in P76, where the discussion was aimed at the brightest cluster galaxies. For a larger and more varied sample of sources, this relation is more complex and not as obvious. If, for the purpose of illustration, we assume that \( B_0 \) and \( r_0 \) or \( L \) are not correlated, then from equation (40) and its counterpart for \( \delta \) this ratio becomes

\[
\langle b(z, b) \rangle = b \left[ \int \frac{y}{x} f(x) h(B_0) dB_0 \right],
\]

where \( h(B_0) \) describes the distribution of the central surface brightness. Note that for a delta function or a relatively narrow distribution, the above expression simplifies to \( \langle b/z \rangle = \eta(x_0) \), as is the case for individual galaxies. This demonstrates that the surface brightness profile, or the function \( \eta \) based on it, plays a central role in cosmological studies of extended sources.

With minor modification, the above expressions will be valid for elliptical sources with constant ellipticity. For sources with major and minor core radii \( r_1 \) and \( r_2 \), respectively, the area within any isophotal limit is proportional to \( \pi r_1 r_2 \), so that if we define \( r = (r_1 r_2)^{1/2} \), the above expressions would apply; but we now have the additional integration over the possible dispersion of the ellipticities, or the ratio \( r_1/r_2 \).

5. SUMMARY

In this paper we address the analysis of the distribution of redshift-size-flux (or magnitude) data for extragalactic sources in particular galaxies, which analysis is often used to test the evolution of sources and/or the universe. The usual practice is to describe the characteristics and evolution of galaxies in terms of a simple luminosity function \( \phi(L, z) \), as if the galaxies are point sources and the data is simply flux or magnitude limited. We emphasize that in reality these tests are more complicated. A proper analysis must involve variation of the surface brightness profile and the multidimensional distribution function of the parameters that describe this profile, such as the core radius, central surface brightness, luminosity, etc.: \( \psi(B_0, r_0, L, x, \ldots, z) \). Neglecting these complexities and the truncations of the data arising from the surface brightness and angular size limits can lead to grossly misleading results.

There are different ways to account for these effects. The most efficient use of the observations lies in a comparison of the full data set with model predictions that include the effects of all biases encountered in the observational selection processes. Use of the isophotal values down to the lowest possible isophote and size is a good example of this approach. We have described the correct analysis of such data in terms of the multivariate luminosity function \( \psi \). We derive the relevant expressions, in terms of the surface brightness profile of spherical sources, to be compared with observations of the isophotal values of the fluxes, sizes, and surface brightnesses. In general, because more than just the usual flux (or magnitude) limit enters in the analysis, these expressions are relatively complex. Truncation of the data through other selection effects, such as angular extent and surface brightness thresholds, occurs and can play the dominant role in defining the content of a sample of sources. The surface brightness profile of the galaxies plays a pivotal role in these calculations.

A second approach would be to select a subset of the data that yields to a more straightforward comparison with models. For example, if we select the more limited sample of large and well-resolved sources, we can use fluxes and sizes related to a metric (instead of isophotal) radius, such as the radius defined in P76 for a constant value of the \( \eta \) function. We derive the relevant expressions for this case and show that they are much simpler than the isophotal ones, and resemble more closely the simple expressions for point sources. This method, therefore, would be more appropriate for evaluating the moments of the observed distributions in tests such as the redshift-flux or angular size–redshift tests. The isophotal method is a more appropriate approach for tests based on source counts, and in principle can be used for samples of sources that include resolved as well as unresolved sources. With a knowledge of the PSF, one can use a modified surface brightness profile as a convolution of the actual profile and the PSF.

The above tests, aside from the complications in the modeling, suffer the additional shortcoming of requiring elaborate procedures for analysis of the data. To overcome some of the difficulties in both of these areas, we propose a new method for the analysis and modeling of extended extragalactic sources for the purpose of determining either their evolution or the cosmological parameters. This method is very well suited to modern digitized data, and amounts to counting the number of pixels or summing their intensity values. It is capable of using all the data, as in the isophotal case, but the expressions relating the data to models are considerably simpler, similar to the metric ones. Their simplicity stems from the fact that they deal with the surface brightness limit alone and do not include selection based on flux or size. As already shown in P76, tests based on surface brightness tend to be more robust and simpler.

To reiterate, these new methods clearly have the following advantages:

1. The data analysis is considerably simpler.
2. The expressions relating the observations to the distributions of the basic properties of galaxies are simpler; compare, e.g., equations (14) and (40).
3. The dependence on the cosmological parameters is also considerably more straightforward. Instead of a dependence on the volume \( V(z) \) and the luminosity and angular diameter distances, \( d_s \) and \( d_d \), the new tests depend primarily on the redshift and the much simpler line element \( dx/dz \). For example, for models with zero cosmological constant, this is equal to \((c/H_0)(1+z)(\Omega z + 1)^{-1/2}\), where \( \Omega \) is the density parameter and \( H_0 \) is the Hubble constant.
4. Because we are dealing with surface brightness, to first order the resulting expressions are independent of the weak gravitational lensing effects arising from the inhomogeneities of the intervening matter distribution (clumpiness due to galaxies and clusters).

As mentioned above, modern digitized CCD data are ideally suited to the task proposed here. However, several conditions are required for the proper application of all the
tests proposed here. A good knowledge of the background sky brightness from all other sources except the extragalactic sources under consideration is needed, because we wish to go to as low a surface brightness as possible. Application of the new methods to integrated (over redshift) values of \( \mathcal{F}_{\text{sky}} \) and \( \mathcal{F}_{\text{sky}} \) to the whole data, say, in a CCD frame, will require accurate flat-fielding. However, for redshift-dependent analysis, the requirements are similar to those of galaxy-based analysis, because redshifts are known for galaxies as a whole and not for individual pixels. For this kind of study, some identification of the pixels with galaxies is necessary so that we can assign redshifts to pixels. Second, when combining resolved and unresolved galaxies it is important that contamination from other unresolved sources such as stars is kept to an acceptable value. Finally, in all these tests one must deal with several profiles: disks, spheroids, and possibly a continuum of superpositions of both. In future works we hope to apply these new methods to data from the Hubble Deep Field (see Williams et al. 1996) as well as to data from ground-based observations.

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REFERENCES

de Jong, R. S. 1996, A&A, 313, 45
Efstanthiou, G., Ellis, R. S., & Peterson, B. A. 1988, MNRAS, 232, 431
Fish, R. A. 1964, ApJ, 139, 284
Freeman, K. C. 1970, ApJ, 160, 811
Gunn, J. E., & Oke, J. B. 1975, ApJ, 195, 255
Impey, C., & Bothun, G. 1997, A&A Rev., 35,
Lin, H., et al. 1996, ApJ, 464, 60
McGaugh, S. S. 1996, MNRAS, 280, 337
Metcalfe, N., Shanks, T., Fong, R., & Roche, N. 1995, MNRAS, 273, 257
Petrosian, V. 1976, ApJ, 209, L1 (P76)
Sandage, A. R. 1972, ApJ, 173, 485
Sandage, A., & Perelmutter, J.-M. 1991, ApJ, 370, 455
Tinsley, B. M. 1968, ApJ, 151, 547
Tinsley, B. M., & Gunn, J. E. 1976, ApJ, 203, 52
Tolman, R. C. 1934, Relativity, Thermodynamics, and Cosmology (Oxford: Clarendon Press)
Tully, R. B., & Verheijen, M. A. W. 1997, ApJ, 484, 145
Tyson, T. 1988, ApJ, 96, 1
Weinberg, S. 1972, Gravitation and Cosmology (New York: Wiley)
Williams, R. E., et al. 1996, AJ, 112, 1335
Yoshii, Y. 1993, ApJ, 403, 552