Bouncing Cosmological Models in a Functional form of $F(R)$ Gravity

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Abstract—We investigate some bouncing cosmological models in an isotropic and homogeneous space-time with in $F(R)$ theory of gravity. Two functional forms of $F(R)$ have are studied with a bouncing scale factor. The dynamical parameters are derived and analyzed along with cosmographic parameters. A violation of the strong energy conditions in both bouncing models is also shown. We show that both models exhibit stable behavior with respect to cosmic time.

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1. INTRODUCTION

The initial singularity is an important issue that General Relativity (GR) has encountered among other issues during the early Universe. Friedmann [1, 2] claimed that the occurrence of an initial singularity was at the beginning of the evolution of Universe. It is believed that a singularity occurred before inflation, because the inflationary scenario resolved certain key issues of early Universe [3–5]. One possible solution might be that the Universe did not attained a singularity during contraction, but expanded after experiencing a bounce. This concept is known as the big bounce. Recent discoveries [6–12] have revealed that our Universe is undergoing a late-time accelerated expansion, which is explained by dark energy (DE), time-independent vacuum energy (according to the ACDM model). The cosmological constant [13], scalar fields (including quintessence, phantom, quintom, tachyon, and others) [14–19] and holographic models [20] are possibilities for describing DE scenarios. Modified gravity theories have advantages over other models since they avoid expensive numerical computations and are consistent with current data for a late phase accelerating Universe and DE. So models with such theories are designed to modify the standard nature of GR by replacing the Ricci scalar $R$ in Einstein–Hilbert action with, e.g., $f(R)$. Several such modified theories of gravity have been developed, such as $f(R)$ gravity [21–29], $f(G)$ gravity [30], $f(T)$ gravity [31, 32] and $f(R,T)$ gravity [33–43], teleparallel gravity [44, 45], where $T$ denotes the torsion scalar, and $G$ is the Gauss–Bonnet invariant. Some other important work on modified theories of gravity [46–49] are available in the literature. The most recent $f(Q)$ gravity or symmetric teleparallel gravity [50] and $f(Q, T)$ [51] gravity have been proposed, where $Q$ and $T$, respectively, mean the non-metricity and the trace of the energy momentum tensor.

The inflationary scenario has been challenged, and the matter bounce scenario has been presented as a possible alternative to address the initial singularity issue. The Universe goes through an initial matter dominated contraction phase, then a nonsingular bounce, and finally a causal generation for fluctuation in the bouncing scenario. For this, the bouncing scenario is a typical example, and the null energy condition (NEC) has to be violated to realize a solution in a spatially flat FLRW metric in GR. The matter bounce scenario has gained a lot of attention among numerous bouncing models proposed because it creates a scale-invariant power spectrum. Additionally, the Universe passes through a matter-dominated epoch at late times in a matter bounce scenario. Alternative gravity theories like $f(R)$ gravity [52–55], $f(G)$ gravity [56, 57], $f(R, T)$ gravity [58–62], $f(Q, T)$ gravity [63], $f(T)$ gravity [64], $f(Q)$ gravity [65] and $f(R, G)$ gravity [66] have all successfully studied bouncing cosmologies. The present work is on a bouncing...
In this paper, our objective is to study some bouncing cosmological models avoiding the initial singularity issue with some of the functional forms of $F(R) = R + f(R)$, $f(R)$ being a deviation of $F(R)$ from Einstein gravity. To explain the late-time cosmic speed-up issue, the models look at geometrical degrees of freedom. The explanation of $F(R)$ gravity and the derivation of $F(R)$ field equations are presented in Section 2. In Section 3, the bouncing scale factor and Hubble parameter are introduced. Two models with a bouncing scale factor and a functional form of $F(R)$ are provided in Section 4. The cosmographic parameters are discussed in Section 5, and the energy conditions of both models in Section 6. Stability analysis is carried out in Section 7. The results and conclusions are presented in Section 8.

2. FIELD EQUATIONS OF $F(R)$ GRAVITY

The action for $F(R)$ gravity can be defined as

$$ S = \int \sqrt{-g} \frac{F(R)}{2\kappa^2} d^4x, \quad (1) $$

$$ \kappa^2 = 8\pi G/c^4, \; G \text{ is Newton's gravitational constant}, $$

$$ g \text{ the determinant of the metric tensor} g_{ij}. $$

Varying the action (1) with respect to $g_{ij}$, the $F(R)$ gravity field equations are obtained as

$$ F_R R_{ij} - \frac{1}{2} F g_{ij} - \nabla_i \nabla_j F_R + g_{ij} \Box F_R = 0. \quad (2) $$

Here $F_R = dF/dR$, $\nabla_i$ is a covariant derivative, $\Box \equiv g^{ij} \nabla_i \nabla_j$ is the d'Alembert operator. The natural system of units $8\pi G = \hbar = c = 1$ is used, where $\hbar$ and $c$ denote the reduced Planck constant and the velocity of light in vacuum, respectively.

We consider flat FLRW space-time with

$$ ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (3) $$

For this metric, the temporal and spatial components of Eq. (2) become

$$ 0 = -\frac{F}{2} + 3(H^2 + \dot{H}) F_R $$

$$ - 18(4H^2 \dot{H} + H \ddot{H}) F_{RR}, \quad (4) $$

$$ 0 = \frac{F}{2} - 3(H^2 + \dot{H}) F_R + 6(8H^2 \dot{H} + 4H^2 \ddot{H} $$

$$ + 6H^2 \dddot{H} + \dot{H} F_{RR} + 36(4H \dot{H} + \dot{H})^2 F_{RRR}, \quad (5) $$

where $F_{RR} = d^2 F/dR^2$, $F_{RRR} = d^3 F/dR^3$, and $H = \dot{a}/a$ is the Hubble parameter.

Comparing the above equations with the standard Friedmann equations, it is clear that $F(R)$ gravity model is used to describe the phenomenology of the current nonsingular bounce.
contributes to the energy–momentum tensor, with its effective energy density \( \rho_{\text{eff}} \) and pressure \( p_{\text{eff}} \) given by

\[
\rho_{\text{eff}} = -\frac{f}{2} + 3 \left( H^2 + \dot{H} \right) f_R - 18 \left( 4H^2 \dot{H} + H \ddot{H} \right) f_{RR},
\]

\[
p_{\text{eff}} = \frac{f}{2} - 3 \left( H^2 + \dot{H} \right) f_R + 6 \left( 8H^2 \dot{H} + 4\ddot{H}^2 + 6H\dddot{H} + 3H\dot{H} \dddot{H} \right) f_{RR}. \tag{6}
\]

Equations (6) and (7) can be expressed in terms of the Hubble parameter and the derivatives of functional forms of \( F(R) \) with respect to \( R \), where the Ricci scalar is \( R = 6 \left( \frac{2}{a^2} + \frac{\dot{a}^2}{a^3} \right) \). So, we need a Hubble function to obtain the energy density and pressure of matter to further study the dynamics of the Universe. Also, to study the issue of late time acceleration, the equation of state (EoS) parameter behavior to be analyzed, which is

\[
\omega_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \left[ 12(2H^2 \dddot{H} + 4\dddot{H}^2 + 3H\dot{H} + \dddot{H}) f_{RR} + 72(4H\dot{H} + \dddot{H})^2 f_{RR} \right] \left[ f - 6(H^2 + \dot{H}) f_R + 36(4H^2 \dot{H} + H \dddot{H}) f_{RR} \right]^{-1}, \tag{7}
\]

where \( f(R) \) represents the departure of \( F(R) \) gravity from Einstein gravity, \( F(R) = R + f(R) \). As expected, the effective energy–momentum tensor depends on the nature of \( F(R) \). So, in the subsequent sections, we will study the bouncing scenario and late-time cosmic acceleration of the Universe by considering a bouncing scale factor and some of the functional forms of \( F(R) \).

3. THE SCALE FACTOR

Inflationary cosmology is one of two existing theories of the early Universe, the other being bounce cosmology, in which the theoretical contradictions of the Big Bang description of our Universe are addressed. The most recent observational data imposed strict limits on inflationary models, confirming the validity of some while ruling out others. Here we intend to study the bouncing scenario in \( F(R) \) gravity.

- In the case of a nonsingular bounce, the scenario assumes a contracting phase where the scale factor decreases with time, i.e., \( \dot{a} < 0 \) and the Hubble parameter is negative in the contacting phase, \( H = \dot{a}/a < 0 \).
- Near the bouncing epoch, the Hubble parameter increases, \( H > 0 \) which corresponds to a ghost (phantom) behavior of the model.
- The scale factor increases with time in a positive acceleration, so that the Hubble parameter becomes positive after bounce, \( \dot{a} > 0 \).
- Also, to provide a bouncing model, the EoS evolves in such a phantom region and changes twice, once before and again after the bounce.

Our bouncing model with an assumed scale factor obeys the above bouncing conditions and the corresponding dynamical behavior. We consider a bouncing scale factor \( a(t) = (\alpha/\beta + t^2)^{1/(2\beta)} \), where \( \alpha \) and \( \beta \) are positive constants, and subsequently \( H = t/(\alpha + \beta t^2) \).

Figure 1 presents the behavior of the scale factor and Hubble parameter at representative values of the parameter \( \alpha = 1.2, 1.3, 1.4 \). It is observed that the bounce occurs at \( t = 0 \), and the parameter \( \alpha \) controls the slope of the curve. A higher value of \( \alpha \) yields a larger slope. The bounce appears to be symmetric, the scale factor decreases from a higher value at
The horizon shrinks and becomes the horizon only at that time. The modes escape time are formed at cosmic times near the bouncing primordial perturbation modes relevant to the present bouncing era as in the prior cases. As a result, the regime of the contracting era rather than near the atractive cosmic times, corresponding to a low curvature the perturbation modes are created at very large neg-

indicating a decelerating Universe. In such scenarios,规模因数，胡柏尔半径在晚期会发

于零，这表明宇宙在早期是加速的。在这种情况下，我们考虑两种不同的功能形式的 F(R) 和对称

性弹跳的尺度因素，根据上述给出的弹跳模型来获得两种不同的弹跳情况。

4. MODELS

We need to have a functional form of F(R) here, thus we will use two well-known functional forms as Model I and Model II,

4.1. Model I

To construct the cosmological model, Eqs. (6) and

(7) are required to be solved, so that the dynamical parameters can be obtained. To do so, a functional

form of F(R) will be considered.

We consider the form of F(R) \[72, 75\]

\[ F(R) = R + \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right], \tag{9} \]

where \( R_0 \) is a constant characteristic curvature, \( \lambda \) and \( n \) are also constants. We chose the value of the exponent \( n = 1 \) in order to match with the Starobinsky model. Using \( F(R) \) from (9), Eqs. (6)–(8) are reduced to

the Hubble radius approaches zero asymptotically because all perturbation modes are within the horizon at that time, the perturbations occur near the bounce.

The Planck restrictions \[84–86\] become compatible with \( F(R) \) gravity. Odintsov et al. \[87\] discovered that \( F(R) \) gravity leads to a feasible bounce only when perturbations are generated near the bounce, directly from observational indices using a bottom-up approach. To check whether the scale factor considered in the present work is in conformity with generation of perturbation modes and whether it is viable for considering it in the framework of \( F(R) \) gravity, we show the cosmic Hubble radius as a function of time for different choices of the parameter \( \alpha \) and a specific choice of \( \beta = 0.9 \) (Fig. 2). With these choices, we obtain that the cosmic Hubble radius monotonically decreases symmetrically around the bouncing epoch and tends to zero asymptotically in both positive and negative time domains. This finding is in agreement with that of Odintsov et al. in \[83\], where they mentioned that the cosmic Hubble radius for the choice of the scale factor \( a_F(t_F) = (a_0 t_F^2 + 1)^n \) drops monotonically on both sides of the bounce for \( n > 1/2 \). Thus we may infer that our present bouncing model may be compatible with the Planck constraints.

In the following sections, we will consider two different functional forms of \( F(R) \) and the symmetric bouncing scale factor given above to obtain two different bouncing scenarios.

early times (\( t < 0 \)), bounces at \( t = 0 \), and increases at \( t > 0 \). The Hubble parameter increases from a higher negative value, crosses the bouncing point at \( t = 0 \) and increases further over the evolution. The behavior of the parameters supports the occurrence of bouncing trajectory, avoiding an initial singularity.

While considering bouncing scenarios in \( F(R) \) gravity, it is useful to consider the generation era of perturbation modes. Usually, in many bouncing models considered with a choice of some scale factors, the Hubble parameter vanishes at bounce, which leads to the divergence of the comoving Hubble radius defined by \( r_h = 1/aH \). The asymptotic behavior of the comoving Hubble radius, on the other hand, shows the accelerating or decelerating nature of the Universe. For some specific choices of the scale factors, the Hubble radius drops monotonically on both sides of the bounce before asymptotically shrinking to zero. Such a behavior indicates an accelerating Universe at late times. As a result, in such instances, the Hubble horizon shrinks to zero at large values of cosmic time, and only the Hubble horizon has an infinite size near the bouncing point. However, with some other choices of the bouncing scale factors, the Hubble radius diverges at late times, indicating a decelerating Universe. In such scenarios, the perturbation modes are created at very large negative cosmic times, corresponding to a low curvature regime of the contracting era rather than near the bouncing era as in the prior cases. As a result, the primordial perturbation modes relevant to the present time are formed at cosmic times near the bouncing point, because all primordial modes are contained in the horizon only at that time. The modes escape the horizon when the horizon shrinks and become relevant for the present-day observations \[83\].
as we move away from the bounce epoch. The size of phantom-like behavior in the bounce epoch, and a occurs mostly in the quintessence area, exhibiting a and once after it. It is evident that the current model also showing a symmetric behavior and crosses the creases in the expanding phase of the evolution. It is phantom-like behavior in the bounce epoch and in-

\[
\rho_{\text{eff}} = \frac{\lambda R_0 \left[ 72H R_0^2 (R_0^2 - 3R^2)(4H \dot{H} + \ddot{H}) - 12R_0^2 R(\dot{H} + H^2)(R_0^2 + R^2) + R^2(R_0^2 + R^2)^2 \right]}{2(R_0^2 + R^2)^3},
\]

\[
p_{\text{eff}} = \frac{4\lambda R_0^3 R(\dot{H} + 3H^2)(R_0^2 + R^2)^2 - 24\lambda R_0^3 (R_0^2 - 3R^2)(R_0^2 + R^2)(4\dot{H}(\dot{H} + 2H^2) + 6H\ddot{H} + \dot{H}) + 1728\lambda R_0^3 R(R_0 - R)(R_0 + R)(4H\dot{H} + \ddot{H})^2 - \lambda R_0 R^2(R_0^2 + R^2)^3}{2(R_0^2 + R^2)^4},
\]

\[
\omega_{\text{eff}} = \frac{4R_0^2(\dot{H} + 3H^2)(R_0^2 + R^2)^2 + 1728R_0^2 R(R_0 - R)(R_0 + R)(4H\dot{H} + \ddot{H})^2 - R^2(R_0^2 + R^2)^3}{(R_0^2 + R^2)(72H R_0^2 (R_0^2 - 3R^2)(4H\dot{H} + \ddot{H}) - 12R_0^2 R(\dot{H} + H^2)(R_0^2 + R^2) + R^2(R_0^2 + R^2)^2)} - \frac{24R_0^2(R_0^2 - 3R^2)(R_0^2 + R^2)(4\dot{H}(\dot{H} + 2H^2) + 6H\ddot{H} + \dot{H})}{(R_0^2 + R^2)(72H R_0^2 (R_0^2 - 3R^2)(4H\dot{H} + \ddot{H}) - 12R_0^2 R(\dot{H} + H^2)(R_0^2 + R^2) + R^2(R_0^2 + R^2)^2)}.\]

For Model I, the effective pressure remains negative near the bouncing point \( t = 0 \). The energy density remains positive throughout the evolution. For higher values of \( \alpha \), the bounce at \( t = 0 \) becomes more prominent. The energy density increasing initially, shows a kind of ditch, reduces near and at the bounce, and then subsequently decreases. The EoS parameter decreases in the contracting phase and crosses the phantom-divide line. The EoS parameter shows a phantom-like behavior in the bounce epoch and increases in the expanding phase of the evolution. It is also showing a symmetric behavior and crosses the phantom divide two times, once before the bounce and once after it. It is evident that the current model occurs mostly in the quintessence area, exhibiting a phantom-like behavior in the bounce epoch, and a \( \Lambda \)CDM behavior at both positive and negative times as we move away from the bounce epoch. The size of the well occurred here depends on the value of \( \alpha \), the larger \( \alpha \), the deeper is the well. When the value of \( \alpha \) is small, only near the bounce the well is visible, else it remains mostly in the quintessence phase.

4.2. Model II

We consider another form of the function \( F(R) \) [73]

\[
F(R) = R + R_0 \lambda \left[ e^{-R/R_0} - 1 \right],
\]

where \( R_0 \) and \( \lambda \) are constants. The same scale factor has been considered here as in Model I. From Eqs. (6) and (7), the effective energy density, effective pressure and EoS parameter for the exponential \( F(R) \) form (13) are obtained as

\[
\rho_{\text{eff}} = \frac{\lambda e^{-R/R_0}(-6(24H^2 + R_0)\dot{H} - R_0(6H^2 + R_0) - 36H\ddot{H}) + R_0^2}{2R_0},
\]

\[
p_{\text{eff}} = \frac{\lambda e^{-R/R_0} \left[ R_0(R_0(6H^2 - R_0 e^{R/R_0} + R_0) + 12\dot{H}) + 2\dot{H}(R_0(48H^2 + R_0) - 288H\ddot{H}) \right]}{2R_0^2} + \frac{\lambda e^{-R/R_0} \left[ 48\dot{H}^2(R_0 - 24H^2) + 36H\ddot{H}(R_0 - 36\ddot{H}) \right]}{2R_0^2},
\]

\[
\omega_{\text{eff}} = -1 - \frac{4 \left[ \ddot{H}(R_0(12H^2 + R_0) + 144H\ddot{H}) + 12\dot{H}^2(24H^2 - R_0) - 3(3HR_0\ddot{H} + \dot{H}R_0 - 6\ddot{H}^2) \right]}{R_0^2 e^{R/R_0} - R_0(6(24H^2 + R_0)\dot{H} + R_0(6H^2 + R_0) + 36H\ddot{H})}.\]
In Fig. 4 we observe that the effective pressure remains entirely negative throughout the evolution of the Universe. The energy density remains entirely positive, and a well appears at the bounce point for a lower $\alpha$. The EoS parameter mostly remains in the phantom phase near the bounce epoch. It crosses the phantom divide twice, as required in a bounce model, before and after the bounce. As in Model I, the EoS parameter gets a deeper well for higher values of $\alpha$. It has a phantom-like behavior at bounce and shows a quintessence behavior as it moves away from the bounce epoch.

5. COSMOGRAPHIC PARAMETERS

In cosmology research, two families of models are being studied, the DE models and modified gravity models. They are fundamentally different in the sense that it is possible to distinguish between these models that are having same cosmic expansion history. In a usual manner, the growth rate of cosmological density perturbations are calculated, and even if the models have an identical expansion history, the models are distinguished depending on different gravity theories.

One approach in discriminating DE and modified gravity models is with the use of the growth factor of matter density perturbation [88]. Another approach is concerned with the statefinder pair $(j, s)$ [89].

It is known that the expansion rate of the Universe can be expressed using the scale factor $a$, and the deceleration parameter $q$ corresponds to the second derivative of $a$. The jerk parameter $j$ and the snap parameter $s$ correspond to the third and fourth derivatives of $a$, while the fifth derivative is associated with the lerk parameter $(l)$. These quantities can be well defined in the Taylor series expansion around the scale factor,

$$a(t) = a(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a}{dt^n} \bigg|_{t=t_0} (t-t_0)^n,$$

where $t_0$ is the present cosmic time. The coefficients of the expansion give these parameters, called the cosmographic parameters. We can derive these geometrical parameters from the scale factor as

$$p_{\text{eff}}$$

$\omega_{\text{eff}}$
The behavior of the cosmographic parameters is presented in Fig. 5. All these parameters are symmetric around the bounce point and experience a singularity at the bounce subsequent to the representative values of $\alpha$. The positive value of the deceleration parameter indicates a decelerated Universe, whereas its negative value indicates an accelerated Universe. The deceleration parameter at early and late times approaches $-0.6$, thereby confirming the accelerated behavior of the models and aligned with the present observed value of $q$ being close to $-0.6$ at 13.8 Gyrs, matching with the current observational data at present age of the Universe [90]. It entirely remains in the negative region, initially decreases and after a singularity at the bounce subsequently increases, and settles at $-0.6$. The jerk parameter exhibits a negative behavior throughout and evolves from a large value, rapidly decreases to experience a singularity and again drastically increases. At the same time, the snap parameter exhibits a singularity in the negative profile at bounce; when we move away from the bounce, it crosses the zero value and merges to zero again in between, reaching its maximum value. The jerk parameter shows the opposite behavior from the snap parameter.

6. ENERGY CONDITIONS

In GR, Einstein’s field equations address the causal metric and geodesic structure of space-time, so the energy-momentum tensor has to satisfy some conditions. For a space-time $(-, +, +, +)$, we can take a timelike vector $u^i$ to be normalized as $u_i u^i = -1$ and a future-directed null $k^i$ as $k^i k_i = 0$. We can define the energy conditions as contractions of timelike or null vector fields with the Einstein tensor and the energy-momentum tensor from the matter side of Einstein’s field equations [91–93]. We can obtain four energy conditions:

- For a future directed null vector $k^i$, $T_{ij} u^i u^j \geq 0$: Null Energy Condition (NEC). So, $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- Matter flows along timelike or null line, contracted with the energy-momentum tensor, the quantity $-T_{ij} u^i u^j$ becomes a future-directed timelike or null vector field: Dominant Energy Condition (DEC). So, $\rho_{\text{eff}} - p_{\text{eff}} \geq 0$.
- $(T_{ij} - \frac{1}{3} T g_{ij}) u^i u^j \geq 0$ says gravity must be attractive: Strong Energy Condition (SEC). So, $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$.

Extended theories of gravity are straightforward extensions of Einstein’s GR, and such is $F(R)$ gravity. Any such extended theory should be confronted with the energy conditions.

6.1. Model I

The energy conditions of the bouncing $F(R)$ model can be obtained by using (10) and (11).

Graphically, the energy conditions are presented in Fig. 6 with varying $\alpha$. At bounce, both the NEC and SEC are violated, and DEC is satisfied. The symmetric behavior around the bounce has been obtained in all energy conditions. The NEC shows a transition behavior, it mostly remains in the positive domain, both in negative and positive time zones, but near the bounce epoch it remains in the negative domain. This violation of NEC realizes the bouncing model [82].

The SEC is violated at bounce and near the bounce epoch of the evolution. A violation of the SEC is another requirement for the extended theory of gravity, hence we claim that the model under discussion also favor the late-time cosmic acceleration. As expected, the DEC is satisfied entirely, it increases in the negative time zone and decreases after bounce. In addition, the energy conditions are in accordance with the EoS parameter in the sense that NEC violates at $\omega_{\text{eff}} \leq -1$, SEC at $\omega_{\text{eff}} \leq -1/3$ and DEC is satisfied if $\omega_{\text{eff}} \leq 1$. This enables us to further claim the validity of the model in the context of recent cosmic dynamics.
6.2. Model II

The energy conditions for Model II can be obtained from (14) and (15).

Figure 7 depicts the behavior of energy conditions for Model II. The model violates the NEC in the bounce epoch, and while moving away it fails to violate it in both negative and positive time zones, a similar kind of behavior was found by Nojiri et al., [82]. The SEC is violated entirely, it initially decreases and increases after the bounce. In contrast, the DEC fails...
to be violated throughout the evolution. This result for the energy conditions confirms the bouncing nature of the model and its validity in extended gravity. It has been observed that the violation of energy conditions depend on the parametric $\alpha$. For a smaller value of $\alpha$, the well is deeper at bounce, while at larger values of $\alpha$, the bounce is almost flat.

7. STABILITY ANALYSIS

The model parameters $(R_0, \lambda)$ are considered as $(2, 0.01)$ and $(2.5, 0.01)$ for Starobinsky and exponential gravity models, respectively, and the scale factor parameters $\beta = 0.9$ and $\alpha$ having three values $1.2, 1.3,$ and $1.4$. We are now analyzing the behavior of $F_R$ vs. cosmic time to assess the stability of the model. Figure 8 shows these variations with a bouncing behavior at $t = 0$. Moreover, the values of the model parameters and the scale factor parameter are chosen so that the effective energy density lies in the favorable profile. On the other hand, the effective pressure is negative throughout the evolution of the universe. Furthermore, with the chosen bouncing scale factor, the parameter $\beta = 0.9$ makes $n > 1/2 \ln (\alpha q_0^2 + 1)^{\alpha}$, with $\alpha = 1.2, 1.3,$ and $1.4$, for which the Hubble radius diverges at bounce and falls monotonically on both sides of the bounce before asymptotically shrinking to zero, indicating an accelerating late-time Universe [83]. Also, such a behavior of the scale factor is required for compatibility of the above $F(R)$ theory with Planck constraints and generates the required perturbation modes near the bounce. For the cosmographic parameters, it has been noticed that the deceleration, jerk, snap and lerk parameter have a singularity at the bounce. The deceleration parameter is negative throughout the evolution, confirming the accelerating nature of the Universe. The EoS parameter curve twice crosses the phantom divide line in both models, as should be with a bounce. Further, the accelerated expansion of the models is validated by the EoS and deceleration parameters. We wish to mention here that the presence of a finite nonzero parameter $R_0$ removes a singularity of the EoS parameter in the bouncing epoch.

The EoS parameter evolution is determined by that of the scale factor. NEC and SEC violations in both models are shown. These violations are inevitable in the context of modified theories of gravity and a bouncing scale factor. To note, the phantom phase might develop in the model with a positive Hubble parameter slope due to NEC violation. Moreover, stability of the model has been detected from the behavior of $F_R$ vs. cosmic time, both models show a stable behavior throughout the evolution. In conclusion, we comment that these two models may give some more insight in resolving the issue of initial singularity in the early Universe.

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Fig. 8. Variations of $F_R = dF/dR$ vs. $t$ for Starobinsky model (left) and Exponential model (right) for different values of $\alpha$ with $\beta = 0.9$. 

8. RESULTS AND CONCLUSION

Two cosmological models, the Starobinsky ($n = 1$) [72] and exponential gravity [73] models have been presented in $F(R)$ theory of gravity. $F(R)$ gravity is derived from an action where the usual Ricci scalar is replaced by a minimally coupled function in $R$. Two well-recognized forms of $F(R)$ have been considered with a bouncing scale factor. Both Model I and Model II show a bouncing behavior at $t = 0$. Moreover, the values of the model parameters and the scale factor parameter are chosen so that the effective energy density lies in the favorable profile. On the other hand, the effective pressure is negative throughout the evolution of the universe. Furthermore, with the chosen bouncing scale factor, the parameter $\beta = 0.9$ makes $n > 1/2 \ln (\alpha q_0^2 + 1)^{\alpha}$, with $\alpha = 1.2, 1.3,$ and $1.4$, for which the Hubble radius diverges at bounce and falls monotonically on both sides of the bounce before asymptotically shrinking to zero, indicating an accelerating late-time Universe [83]. Also, such a behavior of the scale factor is required for compatibility of the above $F(R)$ theory with Planck constraints and generates the required perturbation modes near the bounce. For the cosmographic parameters, it has been noticed that the deceleration, jerk, snap and lerk parameter have a singularity at the bounce. The deceleration parameter is negative throughout the evolution, confirming the accelerating nature of the Universe. The EoS parameter curve twice crosses the phantom divide line in both models, as should be with a bounce. Further, the accelerated expansion of the models is validated by the EoS and deceleration parameters. We wish to mention here that the presence of a finite nonzero parameter $R_0$ removes a singularity of the EoS parameter in the bouncing epoch.

The EoS parameter evolution is determined by that of the scale factor. NEC and SEC violations in both models are shown. These violations are inevitable in the context of modified theories of gravity and a bouncing scale factor. To note, the phantom phase might develop in the model with a positive Hubble parameter slope due to NEC violation. Moreover, stability of the model has been detected from the behavior of $F_R$ vs. cosmic time, both models show a stable behavior throughout the evolution. In conclusion, we comment that these two models may give some more insight in resolving the issue of initial singularity in the early Universe.
CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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