INTERSECTION OF A PARTITIONAL AND A GENERAL INFINITE MATROID

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ABSTRACT. Let $E$ be a possibly infinite set and let $M$ and $N$ be matroids defined on $E$. We say that the pair $\{M, N\}$ has the Intersection property if $M$ and $N$ share an independent set $I$ admitting a bipartition $I_M \sqcup I_N$ such that $\text{span}_M(I_M) \cup \text{span}_N(I_N) = E$. The Matroid Intersection Conjecture of Nash-Williams says that every matroid pair has the Intersection property.

The conjecture is known and easy to prove in the case when one of the matroids is uniform and it was shown by Bowler and Carmesin that the conjecture is implied by its special case where one of the matroids is a direct sum of uniform matroids, i.e., is a partitional matroid. We show that if $M$ is an arbitrary matroid and $N$ is the direct sum of finitely many uniform matroids, then $\{M, N\}$ has the Intersection property.

1. Introduction

1.1. Infinite matroids and the Matroid Intersection Conjecture. Some of the motivating examples of matroids are vector-systems with linear independence and graphs with graph theoretic cycles as circuits. Both types of structures can be infinite in which case the resulting matroid is infinite as well. An axiomatization of matroids (in the language of circuits) that allows infinite ground sets can be obtained from the axiomatization of finite matroids in a natural way: $\mathcal{C}$ is the set of the circuits of a finitary matroid if $\mathcal{C}$ is a family of finite nonempty pairwise $\subseteq$-incomparable subsets of a possible infinite set $E$ satisfying the Circuit elimination axiom. Working with this definition, Nash-Williams proposed his Matroid Intersection Conjecture [1] which has been the most important open problem in infinite matroid theory for decades. It generalizes the Matroid Intersection Theorem of Edmonds [8] to infinite matroids capturing the combinatorial structure corresponding to the largest common independent sets instead of dealing with infinite quantities (cardinality usually turns out to be an overly rough measure for problems in infinite combinatorics). Adopting a terminology of Bowler and Carmesin, for a pair $\{M, N\}$ of matroids defined on the same edge set $E$ we say that it has the Intersection property if $M$ and $N$ has a common independent set $I$ admitting a bipartition $I_M \sqcup I_N$ such that $\text{span}_M(I_M) \cup \text{span}_N(I_N) = E$.

Conjecture 1.1 (Matroid Intersection Conjecture, [1, Conjecture 1.2]). Every pair $\{M, N\}$ of matroids defined on the same (potentially infinite) ground set has the Intersection property.

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1 A matroid with only finite circuits called finitary.

2 If $C_0, C_1 \in \mathcal{C}$ are distinct and $e \in C_0 \cap C_1$, then $\exists C_2 \in \mathcal{C}$ with $C_2 \subseteq C_0 \cup C_1 - e$. 

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The concept of infinite matroids described in the first paragraph was not entirely satisfying for the experts in matroid theory. Indeed, matroids may fail to have a dual although duality is a fundamental phenomenon of the finite theory. More precisely, their formally defined duals fail to be a matroid. For example in an infinite connected graph the bonds are supposed to be the circuits of the dual of its cycle matroid because they are the minimal edge sets meeting with every base. But these bonds are usually not even finite. Rado asked in 1966 for a more general definition of infinite matroids that allows infinite circuits (see [16, Problem P531]). Among other attempts Higgs [12] introduced a class of structures he called “B-matroids” that solves Rado’s problem. Oxley gave an axiomatization of B-matroids and showed that this is the broadest class of structures satisfying some natural ‘matroid-like’ axioms (namely (i)-(iii) below) for which the usual definition of dual and minors are meaningful and results in a matroid (see [14] and [15]). Despite these discoveries of Higgs and Oxley, the systematic investigation of B-matroids started only around 2010 when Bruhn, Diestel, Kriesell, Pendavingh and Wollan found a set of cryptographic axioms for them, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms for finite matroids (see [7]). They also showed that several well-known facts of the theory of finite matroids are preserved. Their axiomatization in the language of independent sets is the following:

$$M = (E, \mathcal{I})$$ is a B-matroid (or simply matroid) if

(i) $\emptyset \in \mathcal{I}$;
(ii) $\mathcal{I}$ is downward closed;
(iii) For every $I, J \in \mathcal{I}$ where $J$ is $\subseteq$-maximal in $\mathcal{I}$ but $I$ is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(iv) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

After this success of Rado’s program the name ‘Matroid Intersection Conjecture’ gained a new interpretation by applying the definition above instead of the more restrictive infinite matroid concept that allows only finite circuits. Although several partial results have been obtained about the Matroid Intersection Conjecture (see [1], [4], [10], [2], [5], [13]), even its original version considering only finitary matroids remains wide open.

1.2. Uniform matroids. Let $E$ be a (possibly infinite) set of size at least $n \in \mathbb{N}$. Then the subsets of $E$ of size at most $n$ are the independent sets of a matroid $U_{E,n}$ which is called the $n$-uniform matroid on $E$. Talking about $\kappa$-uniform matroids for an infinite cardinal $\kappa$ is not meaningful in general. Indeed, if $|E| > \kappa$, then the set of subsets of $E$ of size at most $\kappa$ fails to be the set of the independent sets of a matroid because there is no maximal among these sets which is a violation of axiom (iv) for $I = \emptyset$ and $X = E$. The “right” concept of (potentially infinite) uniform matroids was found by Bowler and Geschke:

**Definition 1.2** ([6, Definition 2]). A matroid $(E, \mathcal{I})$ is uniform if for every $I \in \mathcal{I}$, $e \in I$ and $f \in E \setminus I$: $I - e + f \in \mathcal{I}$.

The matroids $\{U_{E,n} : n \in \mathbb{N}\}$ and their duals are uniform according to Definition 1.2. It is not too hard to show (see Proposition 2.5) that if a uniform matroid is finitary then it is
either free or $n$-uniform for some $n\in\mathbb{N}$. A natural question if there are uniform matroids that are neither finitary nor cofinitary\(^3\). The positive answer (under some set theoretic assumptions) was shown by Bowler and Geschke in [6]. They developed an efficient method to build uniform matroids while having a large degree of freedom during the construction. Their technique turned out to be a powerful tool to prove relative consistency results in infinite matroid theory. Higgs showed in [11] that under the Generalized Continuum Hypotheses for every fixed matroid $M$ the bases of $M$ have the same cardinality and asked if it is provable already in set theory ZFC. The negative answer was shown by Bowler and Geschke which means that the question about the equicardinality of the bases is undecidable in ZFC. Another example for an application of uniform matroids is corresponding to the following problem. Suppose that $M$ and $N$ are matroids on a common ground set and $M$ has a base contained in a base of $N$ and vice versa. Do $M$ and $N$ necessarily share a base? The positive answer is known for the special case where each of the matroids is either finitary or cofinitary [9, Corollary 1.4]. The unprovability of the general case [9, Theorem 5.1] was demonstrated via the method of Bowler and Geschke.

A question of undoubtedly great importance in the field is if the existence of uniform matroids that are neither finitary nor cofinitary is provable in ZFC alone. The existence of such a matroid on a $\kappa$-sized ground set is known to be equivalent with the positive answer for the following set theoretic question:

**Question 1.3.** Let $\mathcal{P}(\kappa)/\text{fin}$ be the power set of $\kappa$ factorized by the equivalence relation $\sim$ where $X \sim Y$ iff the symmetric different $X \triangle Y$ is finite. For $[X], [Y] \in \mathcal{P}(\kappa)/\text{fin}$ we let $[X] \subseteq^* [Y]$ iff $X \setminus Y$ is finite. Is there a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)/\text{fin}$ satisfying the following conditions?

1. $\mathcal{A}$ is an antichain, i.e., it consists of $\subseteq^*$-incomparable elements;
2. $\mathcal{A}$ is line-dense in $\mathcal{P}(\kappa)/\text{fin}$, i.e., for every $[X], [Y] \in \mathcal{P}(\kappa)/\text{fin}$ with $[X] \subseteq^* [Y]$, there is an $[A] \in \mathcal{A}$ such that at least one of the following holds
   - $[A] \subseteq^* [X]$;
   - $[X] \subseteq^* [A] \subseteq^* [Y]$;
   - $[Y] \subseteq^* [A]$;
3. $\mathcal{A}$ is non-trivial, i.e., $\mathcal{A}$ is neither $\{[\varnothing]\}$ nor $\{[\kappa]\}$.

1.3. **Our main result.** The special case of Conjecture 1.1 where $N$ is assumed to be uniform is well-known and easy to prove while the special case where $N$ is the direct sum of uniform matroids (known as partitional matroids) is already equivalent with the conjecture itself (see [4, Corollary 3.9 (b)]). Our main result is deciding the problem between these two extremal cases:

**Theorem 1.4.** Let $M$ and $N$ be matroids on $E$ such that $N = \bigoplus_{i<n} U_i$ where $n \in \mathbb{N}$ and $U_i$ is a uniform matroid on $E_i$ with $E_i \cap E_j = \varnothing$ for $0 \leq i < j < n$. Then $\{M, N\}$ has the Intersection property.

Having the unprovability results mentioned in the previous subsection in mind, we phrase the following question in a careful way:

\(^3\) A matroid is cofinitary if its dual is finitary.
Question 1.5. Is it provable in ZFC that a pair \( \{M, N\} \) of partitional matroids on a common countable ground set always has the Intersection property?

2. Notation and Preliminaries

We apply the standard set theoretic convention that natural numbers are identified with the set of smaller natural numbers, i.e., \( n = \{0, \ldots, n - 1\} \). For ease of presentation, when there is no chance of misunderstanding, we write simply \( X - y + z \) instead of \( X \setminus \{y\} \cup \{z\} \). The defined terms and symbols are highlighted with italic and bold respectively for convenience.

2.1. General matroidal terms. A pair \( M = (E, I) \) is a matroid if \( I \subseteq \mathcal{P}(E) \) satisfies the axioms (i)-(iv). A matroid is trivial if \( E = \emptyset \) and free if \( I = \mathcal{P}(E) \). The sets in \( I \) are called independent while the elements of \( \mathcal{P}(E) \setminus I \) are the dependent sets. The maximal independent sets are the bases.

Fact 2.1 ([7, Lemma 3.7]). If matroid \( M \) has a finite base, then every base of \( M \) has the same size.\(^4\)

Every dependent set contains a minimal dependent set, these are called circuits. For an \( X \subseteq E \), \( M \lvert X := (X, I \cap \mathcal{P}(X)) \) is a matroid referred as the restriction of \( M \) to \( X \). We write \( M - X \) for \( M \lvert (E \setminus X) \) and call it the minor obtained by the deletion of \( X \).

Let \( B_X \) be a maximal independent subset of \( X \). The contraction \( M/X \) of \( X \) in \( M \) is the matroid on \( E \setminus X \) where \( I \subseteq E \setminus X \) is independent iff \( I \cup B_X \) is independent in \( M \). One can show that the definition does not depend on the choice of \( B_X \). Contraction and deletion commute, i.e., for disjoint \( X, Y \subseteq E \), we have \( (M/X) - Y = (M - Y)/X \). Matroids of this form are the minors of \( M \). The set \( \text{span}_M(X) \) of edges spanned by \( X \) in \( M \) consists of the edges in \( X \) and of those \( e \in E \setminus X \) for which \( \{e\} \) is dependent in \( M/X \). The operator \( \text{span}_M \) is idempotent and hence (since extensivity and monotonicity are straightforward from the definition) is a closure operator. An \( S \subseteq E \) is spanning in \( M \) if \( \text{span}_M(S) = E \). Bases are exactly the independent spanning sets.

Let \( \Theta \) be some index set. For \( i \in \Theta \), let \( M_i \) be a matroid on \( E_i \) such that \( E_i \cap E_j = \emptyset \) for \( i \neq j \). Then the direct sum \( \bigoplus_{i \in \Theta} M_i \) is the matroid defined on \( E := \bigcup_{i \in \Theta} E_i \) where \( I \subseteq E \) is independent in \( \bigoplus_{i \in \Theta} M_i \) iff \( I \cap E_i \) is independent in \( M_i \) for every \( i \in \Theta \). For a detailed introduction to the theory of infinite matroids we refer to [3].

2.2. Basic facts about uniform matroids. We will make use of the following characterisation of uniform matroids (recall Definition 1.2):

**Proposition 2.2.** A matroid \( U \) on \( E \) is uniform if and only if for every \( F \subseteq E \), \( F \) is either independent or spanning in \( U \).

**Proof.** Suppose that \( U \) is uniform and let \( F \subseteq E \) be arbitrary. We take be a maximal independent subset \( B \) of \( F \). We may assume that \( B \subsetneq F \) since otherwise \( F = B \) is independent and we are done. Suppose for a contradiction that \( B \) is not a base of \( U \). Then

\(^4\)The equicardinality of bases can be proved for some larger classes of matroids but as already mentioned is independent of set theory ZFC considering the class of all matroids.
there is some \( e \in E \setminus B \) such that \( B + e \) is independent. By the choice of \( B \) we know that \( e \in E \setminus F \). Let \( f \in F \setminus B \) be arbitrary. By uniformity (Definition 1.2), \( B + f \) must be independent, since we can obtain it from \( B + e \) by deleting \( e \) and adding \( f \). But then \( B + f \) is an independent subset of \( F \) contradicting the maximality of \( B \). Thus \( B \) is a base and therefore \( F \) is spanning.

Conversely, assume that every \( F \subseteq E \) is either independent or spanning in the matroid \( U = (E, \mathcal{I}) \). Suppose for a contradiction that there are \( I \in \mathcal{I} \), \( e \in I \) and \( f \in E \setminus I \) such that \( I - e + f \notin \mathcal{I} \). Then by assumption \( I - e + f \) is spanning, thus contains some base \( B \). We must have \( f \in B \) since otherwise \( B \subseteq I - e \subsetneq I \) contradicts the fact that \( I \) is independent. We also know that \( B \subseteq I - e + f \) because \( I - e + f \) is dependent, therefore we can fix some \( h \in (I - e + f) \setminus B \). But then \( \{ f \} \) is a base of \( U/(B - f) \) and \( \{ e, h \} \in \mathcal{I}_{U/(B - f)} \) because \( B - f + e + h \subseteq I \in \mathcal{I} \) which contradicts Fact 2.1 after extending \( \{ e, h \} \) to a base of \( U/(B - f) \).

\[
\text{Corollary 2.3. The class of uniform matroids is closed under duality and under taking minors.}
\]

\[
\text{Corollary 2.4. If } M \text{ and } U \text{ are matroids on } E \text{ and } U \text{ is uniform, then there is either an } M\text{-independent base of } U \text{ or an } U\text{-independent base of } M.
\]

\[
\text{Proof. Let } B \text{ be an arbitrary base of } M. \text{ If } B \text{ is independent in } U, \text{ then we are done. Otherwise } B \text{ must be spanning in } U, \text{ thus } B \text{ contains a base of } U \text{ which is then independent in } M. \hspace{1cm} \Box
\]

\[
\text{Proposition 2.5. If the uniform matroid } U = (E, \mathcal{I}) \text{ is finitary, then } U \text{ is either free or of the form } U_{E, n} \text{ for suitable } n \in \mathbb{N}.
\]

\[
\text{Proof. If every finite set is independent in } U \text{ then it has no circuits (because every circuit of } U \text{ is finite by assumption), thus } U \text{ is free. If } F \subseteq E \text{ is a finite dependent set, then } F \text{ is spanning by Proposition 2.2, thus there is a finite base } B \subseteq F. \text{ But then Fact 2.1 ensures that every base has size } |B| =: n. \text{ If an } H \subseteq E \text{ of size } n \text{ were not a base, then by applying Proposition 2.2 there would be a base } B' \text{ such that either } H \subsetneq B' \text{ or } B' \subsetneq H \text{ holds in both of which cases } |B'| \neq n, \text{ a contradiction. Thus every } n\text{-element subset is a base and therefore } U = U_{E, n}. \hspace{1cm} \Box
\]

\[
\text{Proposition 2.6. Assume that } U \text{ is a uniform matroid on } E, I \text{ is independent in } U \text{ and } I \text{ spans } e \in E \setminus I \text{ in } U. \text{ Then } I \text{ must be a base of } U.
\]

\[
\text{Proof. By assumption } I + e \text{ is dependent and therefore also spanning by Proposition 2.2. But then } I \text{ is spanning because it spans the spanning set } I + e. \text{ Since } I \text{ is independent as well, it is a base.} \hspace{1cm} \Box
\]

3. Proof of the main result

3.1. Preparatory lemmas.

\[
\text{Observation 3.1. If } M \text{ and } N \text{ are arbitrary matroids on } E \text{ and there is an } M\text{-independent base } B_N \text{ of } N \text{ (} N\text{-independent base } B_M \text{ of } M), \text{ then } B_N (B_M) \text{ witnesses the Intersection property of } \{ M, N \} \text{ via the trivial bipartition } \emptyset \sqcup B_N \text{ (}B_M \sqcup \emptyset).\]

From now on, let $M$ and $N$ be matroids on $E$ such that $N = \bigoplus_{i<n} U_i$ where $n \in \mathbb{N}$ and $U_i$ is a uniform matroid on $E_i$ with $E_i \cap E_j = \emptyset$ for $0 \leq i < j < n$. The union of an $\subseteq$-increasing sequence of common independent sets of $M$ and $N$ is not a common independent set in general because both $M$ and $N$ may have infinite circuits. Therefore Zorn’s lemma cannot be used to extend common independent sets to maximal ones as in the case of finitary matroids. Even so, such an extension is still always possible:

**Lemma 3.2.** Every common independent set $I$ of $M$ and $N$ can be extended to a maximal common independent set.

*Proof.* Let $i < n$ be arbitrary. By applying Corollary 2.4 with $M/I \upharpoonright (E_i \setminus I)$ and $U_i/(I \cap E_i)$ we obtain a common independent set $B$ of these matroids which is in addition a base in at least one of them. It follows directly from the construction that $I \cup B \in I_M \cap I_N$, furthermore, $I \cup B$ either $M$-spans or $N$-spans $E_i$ depending on whose base was $B$. Iterating this with all indices yields a maximal common independent set. \(\square\)

In the previous proof we obtained a seemingly stronger property than maximality. Let us point out that it is actually equivalent:

**Lemma 3.3.** An $I \in I_M \cap I_N$ is maximal in $I_M \cap I_N$ if and only if every $E_i$ is spanned by $I$ in at least one of the matroids.

*Proof.* The “if” direction is straightforward. To show the “only if” part let $I$ be a maximal common independent set and let $i < n$. On the one hand, every $e \in E_i \setminus I$ is spanned by $I$ in at least one of the matroids because $I + e \notin I_M \cap I_N$. On the other hand, either $I \cap E_i$ is a base of $U_i$ in which case $E_i \subseteq \text{span}_N(I)$, or $I \cap E_i$ is not a base of $U_i$ but then by Proposition 2.6 $I$ does not $N$-span any edges from $E_i \setminus I$ and hence we must have $E_i \subseteq \text{span}_M(I)$. \(\square\)

### 3.2. Proof of the main theorem.

Let us repeat here our main theorem for convenience:

**Theorem 1.4.** Let $M$ and $N$ be matroids on $E$ such that $N = \bigoplus_{i<n} U_i$ where $n \in \mathbb{N}$ and $U_i$ is a uniform matroid on $E_i$ with $E_i \cap E_j = \emptyset$ for $0 \leq i < j < n$. Then $\{M, N\}$ has the Intersection property.

We apply induction on $n$. For $n = 0$, both $M$ and $N$ are trivial and $\emptyset$ witnesses the Intersection property of $\{M, N\}$ via $\emptyset \cup \emptyset$.

Suppose now that $n \geq 1$. Assume first that there is a nonempty $W \subseteq E$ which is the union of some of the sets $E_i$ such that $M \upharpoonright W$ admits an $N$-independent base $B$. Then $N/W$ can be written as the direct sum of strictly less than $n$ uniform matroids. We apply the induction hypothesis with $M/W$ and $N/W$ to obtain a witness $I_{M/W} \cup I_{N/W}$ showing that $\{M/W, N/W\}$ has the Intersection property. We define $I_M := I_{M/W} \cup B$ and $I_N := I_{N/W}$. Then

$$I_M \cup I_N \in I_M \cap I_N$$

because

$$I_{M/W} \cup I_{N/W} \in I_{M/W} \cap I_{N/W}$$

and

$$B \in I_{M/W} \cap I_{N/W}.$$
Since $I_{M/W} \sqcup I_{N/W}$ shows the Intersection property of $\{M/W, N/W\}$, set $I_{M/W}$ spans
\[ E \setminus (W \cup \text{span}_{N/W}(I_{N/W})) \]
in $M/W$.

**Observation 3.4.** $N - W = N/W$ since
\[ N = (N \upharpoonright W) \oplus (N - W). \]

By $I_N = I_{N/W}$ and by Observation 3.4 we obtain
\[ \text{span}_{N/W}(I_{N/W}) = \text{span}_{N/W}(I_N) = \text{span}_{N-W}(I_N) = \text{span}_N(I_N). \]

Thus $I_{M/W}$ spans actually
\[ E \setminus (W \cup \text{span}_N(I_N)) \]
in $M/W$. But then, since $B$ is a base of $M \upharpoonright W$, $I_M = I_{M/W} \cup B$ spans $E \setminus \text{span}_N(I_N)$ in $M$. Therefore $I_M \sqcup I_N$ witnesses the Intersection property of $\{M, N\}$.

Suppose now that there is no such a $W$, i.e., the following holds:

**Condition 3.5.** There is no nonempty $W \subseteq E$ which is the union of some of the sets $E_i$ such that $M \upharpoonright W$ admits an $N$-independent base $B$.

In this case we can finish the proof by applying the following theorem with $J = \emptyset$:

**Theorem 3.6.** Let $M$ and $N$ be matroids on $E$ such that $N = \bigoplus_{i<n} U_i$ where $n \in \mathbb{N}$ and $U_i$ is a uniform matroid on $E_i$ with $E_i \cap E_j = \emptyset$ for $0 \leq i < j < n$. Assume that Condition 3.5 holds. Then for every $J \in I_M \cap I_N$ there exists an $M$-independent base $B$ of $N$ with $J \subseteq \text{span}_M(B)$.

Indeed, a $B$ provided by Theorem 3.6 for $J = \emptyset$ witnesses the Intersection property of $\{M, N\}$ by Observation 3.1.

### 3.3. Proof of Theorem 3.6

We use induction on $n$. For $n = 0$, the matroids are trivial, we must have $J = \emptyset$ and $\emptyset$ is a desired $M$-independent base of $N$. Suppose that $n \geq 1$. For $I \in \mathcal{I}_M \cap \mathcal{I}_N$, let
\[ \Theta(I) := \{i < n : E_i \subseteq \text{span}_M(I)\}. \]

We take an $I \in \mathcal{I}_M \cap \mathcal{I}_N$ with $\text{span}_M(I) \supseteq J$ that maximizes $\Theta(I)$ in the sense that $\Theta(K) = \Theta(I)$ holds for every $K \in \mathcal{I}_M \cap \mathcal{I}_N$ with $\text{span}_M(K) \supseteq I$. Note that $\Theta(I) \subseteq n$ since otherwise $W := E$ would violate condition 3.5. By symmetry we can assume without loss of generality that $\Theta(I) = \{0, \ldots, k - 1\}$ for some $k < n$. We consider $E' := \bigcup_{i<k} E_i$ and $J' := I \cap E'$ together with matroids $M' := M \upharpoonright E'$ and $N' := N \upharpoonright E'$. Then $N' = \bigoplus_{i<k} U_i$ and Condition 3.5 is satisfied by $M'$ and $N'$ because a violating $W$ would be also a violation with respect to $M$ and $N$. By induction we get an $M'$-independent base $B'$ of $N'$ with
$J' \subseteq \text{span}_{M'}(B')$. Clearly

$$(I \setminus E') \cup B' \in \mathcal{I}_N$$

because

$$B' \in \mathcal{I}_{N|E'}, (I \setminus E') \in \mathcal{I}_{N|(E \setminus E')}$$

and

$$N = (N \upharpoonright E') \oplus (N - E').$$

We also know that $(I \setminus E') \cup B'$ spans $I$ in $M$ because $I \cap E' = J'$ is spanned by $B'$ in $M$. We extend $B'$ to a maximal $M$-independent subset of $(I \setminus E') \cup B'$ and then further extend the resulting set via Lemma 3.2 to a maximal element $B$ of $\mathcal{I}_M \cap \mathcal{I}_N$. Then $\text{span}_M(B) \supseteq I$ by construction which implies that $\text{span}_M(B) \supseteq J$ and $\Theta(B) = \Theta(I)$. On the one hand, it follows from $\Theta(B) = \Theta(I)$ by Lemma 3.3 via the maximality of $B$ that $E_i \subseteq \text{span}_N(B)$ for every $k \leq i < n$. On the other hand, $B' \subseteq B$ is a base of $N' = \bigoplus_{i<k} U_i$. By combining these, it follows that $E_i \subseteq \text{span}_N(B)$ for every $i < n$. Thus $B$ is an $M$-independent base of $N$ with $\text{span}_M(B) \supseteq J$ and therefore the proof of Theorem 3.6 is complete.

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