ON THE VISCOSITY SOLUTIONS TO A CLASS OF NONLINEAR DEGENERATE PARABOLIC DIFFERENTIAL EQUATIONS

TILAK BHATTACHARYA AND LEONARDO MARAZZI

Abstract. In this work, we show existence and uniqueness of positive solutions of $H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0$ in $\Omega \times (0, T)$ and $u = h$ on its parabolic boundary. The operator $H$ satisfies certain homogeneity conditions, $\Gamma > 0$ and depends on the degree of homogeneity of $H$, $f > 0$, increasing and meets a concavity condition. We also consider the case $f \equiv 1$ and prove existence of solutions without sign restrictions.

1. Introduction and statements of the main results

In this work, we address the issue of existence and uniqueness of viscosity solutions to a class of nonlinear degenerate parabolic differential equations that are doubly nonlinear. Our main goal is to present a unified approach to studying as diverse a group of equations as possible and could be viewed as a natural outgrowth of the previous works in [1, 2]. As a result, the current work includes as special instances many of the results proven in these works.

We now describe the class of equations of interest to us. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and $T > 0$. Let $\partial\Omega$ denote its boundary and $\overline{\Omega}$ its closure. Call $\Omega_T = \Omega \times (0, T)$ and $P_T$ its parabolic boundary.

We address existence results and comparison principles for viscosity solutions to

$$
H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0,
$$

in $\Omega_T$,

$$
u(x, 0) = i(x), \forall x \in \Omega \quad \text{and} \quad u(x, t) = j(x, t), \forall (x, t) \in \partial\Omega \times [0, T),
$$

where $\Gamma > 0$ is a constant, $\chi(t)$, $i(x)$, $j(x, t)$ and $f$ are continuous and $f > 0$. Our work also includes the case $f \equiv 1$. The conditions on $H$ and $f$ are described later in this section.

In [1], $H$ is the infinity-Laplacian and $f(u) = 3u^2$, and in [2, 9], $H$ is the $p$-Laplacian and $f(u) = (p - 1)u^{p-2}$. These are contained in this work and, in addition, are included some fully nonlinear operators such as the Pucci operators. Equations such as (1.1) are of great interest and have been studied in great detail in the weak solution setting, see the discussions in the works cited in [1] and [6]. In this context, a study of large time asymptotic behaviour of viscosity solutions to the equations in [1, 2] appears in [3].

We now state precisely the conditions placed on $H$ and also state the main results of this work. Let $o$ denote the origin in $\mathbb{R}^n$. On occasions, we write a point $x \in \mathbb{R}^n$ as $(x_1, x_2, \ldots, x_n)$. Call $S^n$ the set of all real $n \times n$ symmetric matrices. Let $I$ be the $n \times n$ identity matrix.

Keywords: degenerate, parabolic, viscosity solutions.

AMS Math Subject Classification 2010: 35K65, 35K55.
By \text{(1.2)}, the functions \( O \) the identity matrix and \( O \) the \( n \times n \) matrix with all entries being zero. We reserve \( e \) to stand for a unit vector in \( \mathbb{R}^n \).

Throughout the work we require that \( H \in C(\mathbb{R}^n \times S^n, \mathbb{R}) \) and \( H(p, O) = 0, \forall p \in \mathbb{R}^n \). We require that \( H \) satisfy the following conditions.

\textbf{Condition A (Monotonicity):} The operator \( H(p, X) \) is continuous at \( p = 0 \) for any \( X \in S^n \) and \( H(p, O) = 0 \), for any \( p \in \mathbb{R}^n \). In addition, for any \( X, Y \in S^n \) with \( X \leq Y \),

\begin{equation}
H(p, X) \leq H(p, Y), \; \forall p \in \mathbb{R}^n.
\end{equation}

Since \( H(p, O) = 0, \forall p \) and any \( X \geq 0 \). \( \square \)

\textbf{Condition B (Homogeneity):} We assume that there are constants \( k_1, k_2 \), a positive real number, and \( k_2 \), a positive odd integer, such that for any \( (p, X) \in \mathbb{R}^n \times S^n \),

\begin{equation}
H(\theta p, X) = |\theta|^{k_1} H(p, X), \; \forall \theta \in \mathbb{R}, \quad \text{and} \quad H(p, \theta X) = \theta^{k_2} H(p, X), \; \forall \theta > 0.
\end{equation}

Define

\begin{equation}
k = k_1 + k_2 \quad \text{and} \quad \gamma = k_1 + 2k_2.
\end{equation}

While our work allows \( k_2 \geq 1 \) (consistent with Condition A), we consider, mainly, the case \( k_2 = 1 \) implying \( k = k_1 + 1 \) and \( \gamma = k_1 + 2 \). \( \square \)

Before stating the third condition, we introduce the following quantities. Observe that \((e \otimes e)_{ij} = e_i e_j\) and \( e \otimes e \) is a non-negative definite matrix. For every \(-\infty < \lambda < \infty\), we set

\begin{equation}
\begin{align*}
m_{\min}(\lambda) &= \min_{|e| = 1} H(e, I - \lambda e \otimes e), \\
\mu_{\min}(\lambda) &= \min_{|e| = 1} H(e, \lambda e \otimes e - I)
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
m_{\max}(\lambda) &= \max_{|e| = 1} H(e, I - \lambda e \otimes e), \\
\mu_{\max}(\lambda) &= \max_{|e| = 1} H(e, \lambda e \otimes e - I).
\end{align*}
\end{equation}

By \text{(1.2)}, the functions \( m_{\min}(\lambda) \) and \( m_{\max}(\lambda) \) are non-increasing in \( \lambda \) while \( \mu_{\min}(\lambda) \) and \( \mu_{\max}(\lambda) \) are non-decreasing in \( \lambda \).

If \( \lambda \leq 1 \) then \( I - \lambda e \otimes e \) is a non-negative definite matrix and, by Condition A, \( m_{\max}(\lambda) \geq m_{\min}(\lambda) \geq 0 \). Also, if \( H \) is odd in \( X \) then \( m_{\max}(\lambda) = -\mu_{\min}(\lambda) \) and \( m_{\min}(\lambda) = -\mu_{\max}(\lambda) \).

However, in this work we do not require that \( H \) be odd in \( X \).

We set

\begin{equation}
m(\lambda) = \min \{ m_{\min}(\lambda), \mu_{\min}(\lambda) \} \quad \text{and} \quad \mu(\lambda) = \max \{ m_{\max}(\lambda), \mu_{\max}(\lambda) \}.
\end{equation}

Both \( \mu(\lambda) \) and \( m(\lambda) \) are non-increasing and \( \mu(\lambda) \geq m(\lambda) \geq 0 \), if \( \lambda \leq 1 \). However, if \( \lambda > 1 \) then \( I - \lambda e \otimes e \) is neither non-negative definite nor non-positive definite and it is not clear what signs do \( m(\lambda) \) and \( \mu(\lambda) \) have. To address this, we impose a coercivity condition. In Section 3 we have listed several equations that satisfy the condition including Trudinger’s equation and equations involving the Pucci operators and the infinity-Laplacian.

\textbf{Condition C (Coercivity):} We take \( H \) to be coercive in the following sense. We impose that there are \( \lambda_0 \) and \( \lambda_1 \) such that \(-\infty < 0 < \lambda_1 \leq 1 \leq \lambda_0 < \infty \) and

\begin{equation}
\begin{align*}
\text{(i)} \; m(\lambda) > 0, \; \forall \lambda \leq \lambda_1, \quad \text{and} \quad (\text{ii}) \; \mu(\lambda) < 0, \; \forall \lambda \geq \lambda_0.
\end{align*}
\end{equation}

\( \square \)
Note that this requires $H(e, I - \lambda e \otimes e)$, as a function of $\lambda$, to change sign in $(-\infty, \infty)$. As noted above, the value $\lambda = 1$ arises from the observation that $I - \lambda e \otimes e$ changes behaviour at $\lambda = 1$. As it is seen later the quantities $m(\lambda)$ and $\mu(\lambda)$ play a significant role in this work in obtaining bounds and estimates for the auxiliary functions that are used in the construction of sub-solutions and super-solutions, see Remark 2.2. Also, see below.

In the rest of the work, we distinguish between the following two cases that arise in (1.7)(ii).

Case (i): there is a $\overline{\lambda}$ such that $1 < \overline{\lambda} < 2$ such that $\mu(\overline{\lambda}) < 0,$

\begin{equation}
(1.8) \quad \text{Case (ii): there is a } \overline{\lambda} \geq 2 \text{ such that } \mu(\overline{\lambda}) < 0, \forall \lambda > \overline{\lambda}.
\end{equation}

The quantity $\overline{\lambda}$ in Case (ii) is assumed to be minimal in the sense that $\mu(\overline{\lambda}) \geq 0$, if $\lambda < \overline{\lambda}$. The value of $\overline{\lambda}$ influences greatly the construction of the sub-solutions and the super-solutions in Sections 5, 6 and 7. In particular, see (5.7), (6.4) and (7.4). Also see (8.1) in the Appendix, where a version of the weak maximum principle is derived for the class of equations under consideration.

Next, we make an observation regarding an operator $\hat{H}$ closely related to $H$. Define $\hat{H}(p, X) = -H(p, -X), \forall (p, X) \in \mathbb{R}^n \times S^n$.

**Remark 1.1.** It is clear that $\hat{H}$ satisfies Conditions A and B, see (1.2) and (1.3). Next, using definitions analogous to (1.5) and calling $\hat{m}_{\text{min}}$, $\hat{m}_{\text{max}}$, $\hat{\mu}_{\text{min}}$ and $\hat{\mu}_{\text{max}}$ the corresponding quantities for $\hat{H}$, we find that

\[
\hat{m}_{\text{min}}(\lambda) = -\mu_{\text{max}}(\lambda), \quad \hat{m}_{\text{max}}(\lambda) = -\mu_{\text{min}}(\lambda), \quad \hat{\mu}_{\text{min}}(\lambda) = -m_{\text{max}}(\lambda)
\]

and $\hat{\mu}_{\text{max}}(\lambda) = -m_{\text{min}}(\lambda), \forall \lambda \in \mathbb{R}$.

It is clear that $\hat{m}(\lambda) = m(\lambda)$ and $\hat{\mu}(\lambda) = \mu(\lambda)$. Thus, $\hat{H}$ satisfies Condition C or (1.7). \(\square\)

From hereon, we define

\[
h(x, t) = \begin{cases} 
i(x), & \forall x \in \Omega, \text{ at } t = 0, \\
j(x, t), & \forall (x, t) \in \partial \Omega \times [0, T]. \end{cases}
\]

We assume that $i(x)$ and $j(x, t)$ are continuous and $h \in C(\mathcal{H}_T)$, i.e, $\lim_{x \to y} i(x) = j(y, 0) = \lim_{(z, t) \to (y, 0^+)} j(z, t)$, for any $y \in \partial \Omega$ and where $(z, t) \in \partial \Omega \times (0, T)$.

We now state the main results of the work. Recall (1.3), (1.4)，$k = k_1 + k_2$ and $\gamma = k_1 + 2k_2$.

**Theorem 1.2.** Let $H$ satisfy Conditions A, B and C and $0 < T < \infty$. Suppose that $\chi : [0, T] \to \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$, $f > 0$ are continuous. Assume further that Case(i) of (1.8) holds and $\Omega \subset \mathbb{R}^n, n \geq 2$, is any bounded domain.

I. Let $k > 1$ and $h > 0$. Suppose that $f$ is an increasing $C^1$ function and $f^{1/(k-1)}$ is concave. Then the problem

\[
H(Du, D^2u) + \chi(t)|Du|^k - f(u)u_t = 0, \text{ in } \Omega_T \text{ and } u = h \text{ in } \mathcal{P}_T,
\]
admits a unique positive solution \( u \in C(\Omega_T \cup P_T) \).

II. Let \( k \geq 1 \). If \( 0 < \Gamma < \gamma \), then, for any continuous function \( h \), the following equation

\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \text{ in } \Omega_T \text{ and } u = h \text{ in } P_T,
\]

admits a unique solution \( u \in C(\Omega_T \cup P_T) \).

**Theorem 1.3.** Let \( H \) satisfy Conditions A, B and C and \( 0 < T < \infty \). Assume that \( \chi : [0, T] \to \mathbb{R} \) and \( f : [0, \infty) \to \mathbb{R}, f > 0 \) are continuous. Assume further that Case(ii) of (1.8) holds and \( \Omega \subset \mathbb{R}^n, n \geq 2 \), is a bounded domain that satisfies a uniform exterior ball condition.

I. Let \( k > 1 \) and \( h > 0 \). Suppose that \( f \) is an increasing \( C^1 \) function and \( f^{1/(k-1)} \) is concave. Then the problem

\[
H(Du, D^2u) + \chi(t)|Du|^k - f(u)u_t = 0, \text{ in } \Omega_T \text{ and } u = h \text{ in } P_T,
\]

admits a unique positive solution \( u \in C(\Omega_T \cup P_T) \).

II. Let \( k \geq 1 \). If \( 0 < \Gamma < \gamma \) then, for any continuous \( h \), the following problem

\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \text{ in } \Omega_T \text{ and } u = h \text{ in } P_T,
\]

admits a unique solution \( u \in C(\Omega_T \cup P_T) \).

In Theorems 1.2 and 1.3 the part I’s address the doubly nonlinear case. The part II’s require that \( \Gamma < \gamma \) in the case \( f \equiv 1 \). This restriction can be relaxed to include \( \Gamma = \gamma \) for some equations that can be converted by a transformation to a doubly nonlinear case to which Part I applies.

To illustrate the point, we take an example like Trudinger’s equation, i.e, take in Theorems 1.2 and 1.3 \( k_1 = p - 2, p \geq 2, k_2 = 1, k = p - 1, \gamma = p \) and \( f(u) = u^{p-2} \),

\[
\text{div} \left( |Du|^{p-2}Du \right) + \chi(t)|Du|^{p-1} - (p - 1)u^{p-2}u_t = 0, \text{ } u > 0.
\]

The Part I’s of the theorems imply existence. If we make a change of variables \( v = \log u \) (see Lemma 2.3) we get

\[
\text{div} \left( |Dv|^{p-2}Dv \right) + \chi(t)|Dv|^{p-1} - (p - 1)v^{p-2}v_t = 0, \text{ where } v \text{ can have any sign. Although the Part II’s do not apply here we do get existence and uniqueness.}
\]

We prove both parts I and II by taking \( h > 0 \). In part II, since adding constants to a solution yields a solution we get the claim for any \( h \). The concavity of \( f^{1/(k-1)} \) is required for a comparison principle to hold, see Section 4, and it is not clear to us if a version of the comparison principle holds if the condition fails to hold. The proof of existence employs the Perron method and a substantial part of the work is devoted to the construction of appropriate sub-solutions and super-solutions. These are so done that they are close to the boundary data \( h \) in \( P_T \) in a local sense. Section 5 contains the details for the initial data
while Sections 6 and 7 have details for the side condition. We also remark that some of our results hold for more general operators $H$. However, to keep our presentation clear, we have taken $H$ to be as described above and made remarks and comments along the way where needed.

We point out that the work in [5] also addresses issues that overlap with our work. In [5], besides homogeneity, $H$ satisfies $\forall (x, p, Y) \in \Omega \times \mathbb{R}^n \times S^n$,

$$a|p|^{k_1} \text{Trace}(X) \leq H(x, p, Y + X) - H(x, p, Y) \leq b|p|^{k_1} \text{Trace}(X), \ \forall X \in S^n, \ X \geq 0,$$

where $0 < a \leq b < \infty$ and $k_1 > -1$. Thus, $k = k_1 + 1$ and $\gamma = k_1 + 2$. The author considers equations of the type

$$H(x, Du, D^2u) + \langle \chi(t), Du \rangle |Du|^{k_1} = g(x, t), \ \text{in } \Omega_T \text{ and } u = h \text{ in } P_T,$$

where $H$ and $\chi$ satisfy additional conditions in $x$ and in $t$. The work contains a comparison principle and regularity results under further conditions on $g$ and $h$. The author also shows existence of solutions of the above in domains with exterior cone condition. Clearly, singular cases are also included. We direct the reader to the work for a more detailed discussion.

We now compare and contrast [5] with the current work. The condition in (1.9) implies that

$$(i) \ a(t-s) \leq H(x, e, I - se \otimes e) - H(x, e, I - te \otimes e) \leq b(t-s), \ t \geq s, \ \text{and}$$

$$(ii) \ a \leq \frac{H(x, e, I - e \otimes e)}{n-1} \leq b.$$

Our conditions require that $H(p, X + Y) \geq H(p, X)$, for $Y \geq 0$, and coercivity as stated in condition C. Thus, $H(e, I - se \otimes e)$ is continuous and non-increasing in $s$ (see condition A) and (1.3) and (1.7) hold. The conditions in (1.3) and (1.7) are also satisfied by the operators in [5]. However, we do not require that $H$ be Lipschitz continuous, see (1.11)(i). Also, unlike (1.11)(ii), we allow the possibility that $H(e, I - e \otimes e) = 0$, as in the case of the infinity-Laplacian which is a very degenerate operator. In addition, the class of operators $H$ includes some fully nonlinear operators such as the Pucci operators (as does [5]). See Section 3 for examples. Equally importantly, our work addresses the doubly nonlinear case where $f(u) \neq 1$. The second term involving the gradient, in the doubly nonlinear case, has the same power as in (1.10). However, we allow a greater range of powers if $f \equiv 1$, see Theorems 1.2 and 1.3. Equations of the kind discussed following the statements of Theorems 1.2 and 1.3 involving two terms in $|Du|$ with differing powers are also included here.

On the other hand, our work takes $g = 0$ (see (1.10)) and while Theorem 1.2 applies to any general domain, Theorem 1.3 is proven for domains with exterior ball condition. We do not address any regularity results and the operator $H$ does not depend on $x$ although the results here would hold (modifying the definitions appropriately) if it depended on $t$.

We describe the layout of the paper. Section 2 contains additional notations, definitions and some auxiliary results. Sections 3 lists examples of $H$ covered by the work. We prove
T. BHATTACHARYA AND L. MARAZZI

various versions of the comparison principle in Section 4. Sections 5, 6 and 7 provide details of the constructions of the sub-solutions and super-solutions and lead to the proofs of Theorems 1.2 and 1.3. These lead to the existence of a unique solution by using Perron’s method. In the Appendix, we have included a version of the weak maximum principle for (1.1).

We thank the referees for reading the work and for their many suggestions that have helped improve the work.

2. Notations, definitions and preliminary results

Throughout this work, \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), is a bounded domain and \( \partial \Omega \) its boundary. For \( 0 < T < \infty \), we define the cylinder

\[
\Omega_T = \Omega \times (0, T) = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in \Omega, \ 0 < t < T \}.
\]

The parabolic boundary of \( \Omega_T \), denoted by \( P_T \), is the set

\[
P_T = (\Omega \times \{0\}) \cup (\partial \Omega \times [0, T)).
\]

Let \( B_r(x) \subset \mathbb{R}^n \) be the ball of radius \( r \), centered at \( x \). For \( r > 0 \) and \( \tau > 0 \), we define the following open cylinder

\[
D_{r,\tau}(x, t) = B_r(x) \times (t - \tau, t + \tau).
\]

Our goal in this work is to show existence of positive solutions of (1.1), that is,

\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0, \text{ in } \Omega_T, \text{ and } u = h, \text{ in } P_T,
\]

where \( \chi : [0, T] \to \mathbb{R} \) is continuous, \( f \) is \( C^1 \) and \( f > 0 \), and \( \Gamma \geq 0 \). Also,

\[
h(x, t) = \begin{cases} 
i(x), & \forall x \in \Omega, \text{ at } t = 0, \\ j(x, t), & \forall (x, t) \in \partial \Omega \times (0, T). \end{cases}
\]

We assume that \( i(x) \) and \( j(x, t) \) are continuous and \( h \in C(P_T) \), i.e., \( \lim_{x \to y} i(x) = j(y, 0) = \lim_{(z, t) \to (y, 0^+)} j(z, t) \), where \( y \in \partial \Omega \) and \( (z, t) \in \partial \Omega \times (0, T) \).

For a set \( A \subset \mathbb{R}^{n+1} \), the function class \( \text{usc}(A) \) is the set of all functions that are upper semi-continuous on \( A \). Similarly, \( \text{lsc}(A) \) is the set of all functions that are lower semi-continuous on \( A \).

We discuss the notion of a viscosity sub-solution and a super-solution of the parabolic equation

\[
H(Dw, D^2w) + \chi(t)|Dw|^\Gamma - f(w)w_t = 0, \text{ in } \Omega_T.
\]

For these definitions, we assume that \( H \) satisfies Condition A, see (1.2), and \( f \) is a continuous function of one variable and \( f > 0 \).

Through out this work, by a test function \( \psi \) we mean a function that is \( C^2 \) in \( x \) and \( C^1 \) in \( t \).
We say that \( u \in \text{usc}(\Omega_T) \) is a sub-solution of (2.5) in \( \Omega_T \) if, for any test function \( \psi \), \( u - \psi \) has a maximum at a point \((y, s) \in \Omega_T \), we have
\[
H(D\psi(y, s), D^2\psi(y, s)) + \chi(s)|D\psi(y, s)|^\Gamma - f(u(y, s))(\psi_t)(y, s) \geq 0.
\]
(2.6) In this case, we write \( H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t \geq 0 \). A function \( v \in \text{lsc}(\Omega_T) \) is a super-solution of (2.5) in \( \Omega_T \) if, for any test function \( \psi \), \( v - \psi \) has a minimum at a point \((y, s) \in \Omega_T \), we have
\[
H(D\psi(y, s), D^2\psi(y, s)) + \chi(s)|D\psi(y, s)|^\Gamma - f(v(y, s))(\psi_t)(y, s) \leq 0.
\]
(2.7) In this case, we write \( H(Dv, D^2v) + \chi(t)|Dv|^\Gamma - f(v)v_t \leq 0 \). If \( u \) is a sub-solution and a super-solution of (2.5) then \( u \in C(\Omega_T) \) and is a solution of (2.5) in \( \Omega_T \).

Next, \( u \) is a sub-solution of (2.4) if \( u \in \text{usc}(\Omega_T \cup P_T) \), \( u \) is a sub-solution of (2.5) and \( u \leq h \) in \( P_T \). Similarly, \( u \) is a super-solution of (2.4) if \( u \in \text{lsc}(\Omega_T \cup P_T) \), \( u \) is a super-solution of (2.5) and \( u \geq h \) in \( P_T \). We say \( u \) is a solution of (2.4) if \( u \in C(\Omega_T \cup P_T) \), \( u \) is a solution of (2.5) and \( u = h \).

In this work, we construct sub-solutions and super-solutions that are \( C^2 \) functions of \( x \) and \( t \). With (2.4) in mind, we state an expression for the operator \( H \) and this will be applied quite frequently in this work. Let \( \sigma(t) > 0 \) and \( v(x) \) be a \( C^2 \) function. Using (1.3) and (1.4),
\[
H(D\sigma v, D^2\sigma v) = \sigma^k H(Dv, D^2v).
\]
(2.8) Let \( v(x) = v(r) \) where \( r = |x - z| \), for some \( z \in \mathbb{R}^n \). Set \( e = (e_1, e_2, \cdots, e_n) \) where \( e_i = (x - z)_i/r, \forall i = 1, 2, \cdots, n \). Then for \( x \neq z \),
\[
H(Dv, D^2v + dDv \otimes Dv) = H \left( v'(r)e, \frac{v'}{r} I + \left( v'' + d \left( \frac{v'}{r} \right)^2 - \frac{v'}{r} \right) e \otimes e \right),
\]
where \( I \) is the \( n \times n \) identity matrix and \( d = 0 \) or \( 1 \). We now take \( d = 0 \) and use Condition B. If \( v' \geq 0 \) then (2.9) shows that
\[
H(Dv, D^2v) = \frac{(v')^k}{r^{k_2}} H \left( e, I + \left( \frac{rv''}{v'} - 1 \right) e \otimes e \right).
\]
(2.10) If \( v' \leq 0 \) then (2.9) leads to
\[
H(Dv, D^2v) = \frac{|v'|^k}{r^{k_2}} H \left( e, - \left( I + \left( \frac{rv''}{v'} - 1 \right) e \otimes e \right) \right).
\]
(2.11) We apply (2.10) and (2.11) to the function \( v(r) = a + br^\beta \) where \( a + br^\beta > 0 \). We note
\[
\frac{rv''}{v'} - 1 = \beta - 2.
\]
(2.12) Using (2.10), (2.12) and recalling that \( k = k_1 + k_2 \) and \( \gamma = k_1 + 2k_2 \) (see (1.4)), we get
\[
H(Dv, D^2v) = r^{-k_2} \left( b\beta r^{\beta - 1} \right)^k H(e, I + (\beta - 2)e \otimes e)
\]
\[
(2.13) = \left( b\beta \right)^k r^{\beta k - \gamma} H(e, I - (2 - \beta)e \otimes e), \quad \text{if } b\beta > 0.
\]
Similarly, using (2.11) and (2.12), we get
\[ H(Dv, D^2v + Dv \otimes Dv) = r^{-k_2} \left( |b\beta| r^{\beta-1} \right)^k H(e, -I - (\beta - 2)e \otimes e) \]
\[ = (|b\beta|^k r^{\beta-k-\gamma} H(e, (2 - \beta)e \otimes e - I), \quad \text{if } b\beta < 0. \] 

(2.14)

**Remark 2.1.** In this work, we take \( d = 0 \) and we make use of (2.13) and (2.14) in Sections 5, 6 and 7.

The expressions in (2.9)-(2.14) hold if \( H \) depends on \( t, u, Du \) and \( D^2u \). However, to keep our exposition clearer, we will take \( H \) to depend on \( Du \) and \( D^2u \) and make comments about more general situations as and when the need arises. □

**Remark 2.2.** Recall (1.6), (2.13) and (2.14). Let \( v = a + b r^\beta \) then the following hold.

(i) \[ \frac{(b\beta)^k m(2 - \beta)}{r^{\gamma-\beta k}} \leq H(Dv, D^2v) \leq \frac{(b\beta)^k \mu(2 - \beta)}{r^{\gamma-\beta k}}, \quad \text{if } b\beta > 0. \]

(ii) \[ -\frac{(|b\beta|^k \mu(2 - \beta)}{r^{\gamma-\beta k}} \leq H(Dv, D^2v) \leq -\frac{(|b\beta|^k m(2 - \beta)}{r^{\gamma-\beta k}}, \quad \text{if } b\beta < 0. \]

We make use of the above estimates in Sections 5, 6 and 7. □

We now discuss a change of variables formula needed for a version of the comparison principle for equations of the kind
\[ H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0, \]
where \( \chi : [0, T] \rightarrow \mathbb{R} \) is continuous. Recall from (1.3) and (1.4) that \( k = k_1 + k_2 \) and \( \gamma = k_1 + 2k_2 \). In this work, we take (a) \( \Gamma = k \) for a non-constant \( f \) and \( k > 1 \), and (b) any \( 0 < \Gamma < \gamma \) for \( f \equiv 1 \) and \( k \geq 1 \).

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^1 \) function and \( f > 0 \). For \( k > 1 \), define \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) to be a \( C^2 \) solution of
\[ \frac{d\phi}{d\tau} = \left( (f \circ \phi)(\tau) \right)^{1/(k-1)} \]
(2.15)

Thus, \( \phi \) is increasing. For proving the comparison principle in Section 4, we will assume further that
\[ f^{1/(k-1)} \text{ is concave, i.e. } \left\{ f^{1/(k-1)} \right\}'(\tau) \text{ is non-increasing in } \tau. \]

(2.16)

Combining (2.15) and (2.16) the above reads
\[ \frac{d \log \phi'(\tau)}{d\tau} = \frac{\phi''(\tau)}{\phi'(\tau)} = \left[ \frac{f'(\phi(\tau))}{(k-1)} \right] \left\{ (f \circ \phi)(\tau) \right\}^{(2-k)/(k-1)} \text{ is non-increasing in } \tau. \]

The facts that \( f \) is positive and \( f^{1/(k-1)} \) is concave impose restrictions on the domain of \( f \). From hereon, for all the main results we take
\[ f \text{ is defined on } [c, \infty), c \geq 0, f > 0 \text{ and } f \text{ is increasing.} \]
We now prove the following change of variables lemma. We do this for a somewhat more general case and do not require that (2.16) hold.

Lemma 2.3. Let $H$ satisfy Conditions A and B, see (1.2) and (1.3), $f : [0, \infty) \to \mathbb{R}^+$ be a $C^1$ function and $g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\chi : [0, T] \to \mathbb{R}$ be continuous.

Let $k = k_1 + k_2$ (see (1.4)) and $\phi : \mathbb{R} \to \mathbb{R}$ is a positive $C^2$ increasing function. Set $\tilde{f}(v) = \{(f \circ \phi)(v)\}^{k/(k-1)}$ and $\tilde{g}(x, t, v) = g(x, t, \phi(v))$.

Case (i): Suppose that $k > 1$ and $\phi$ is as in (2.13). We assume that $f$ is non-constant.

(a) If $u \in \text{usc}(\Omega_T)$, $u > 0$, solves $H(Du, D^2u) + \chi(t)|Du|^k + g(x, t, u) \geq f(u)u_t$ in $\Omega_T$ and $v = \phi^{-1}(u)$ then $v \in \text{usc}(\Omega_T)$ and

$$H \left( Dv, D^2v + \frac{\phi''(v)}{\phi'(v)} Dv \otimes Dv \right) + \chi(t)|Dv|^k + \frac{\tilde{g}(x, t, v)}{\tilde{f}(v)} \geq v_t, \text{ in } \Omega_T.$$  

The converse also holds.

(b) If $u \in \text{lsc}(\Omega_T)$, $u > 0$, solves $H(Du, D^2u) + \chi(t)|Du|^k + g(x, t, u) \leq f(u)u_t$ in $\Omega_T$ and $v = \phi^{-1}(u)$ then $v \in \text{lsc}(\Omega_T)$ and

$$H \left( Dv, D^2v + \frac{\phi''(v)}{\phi'(v)} Dv \otimes Dv \right) + \chi(t)|Dv|^k + \frac{\tilde{g}(x, t, v)}{\tilde{f}(v)} \leq v_t, \text{ in } \Omega_T,$$

and conversely.

Case (ii): Let $k = 1$. If $f \equiv 1$ then the claims in (a) and (b) hold if $\phi(\tau)$ is any increasing positive $C^2$ function (define $\tilde{f} \equiv 1$). In particular, if $\phi(\tau) = e^\tau$ and $u \in \text{usc}(\Omega_T)$ then $H(Du, D^2u) + \chi(t)|Du|^k + g(x, t, u) - u_t \geq (\leq)0$ if and only if

$$H(Dv, D^2v + Dv \otimes Dv) + \chi(t)|Dv|^k + \frac{\tilde{g}(x, t, v)}{\phi'(v)} - v_t \geq (\leq)0.$$

Proof. We prove Case (i) part (a) and start with the converse. Let $\phi$ be as in (2.15) and $v \in \text{usc}(\Omega_T)$ solve

$$H \left( Dv, D^2v + \frac{\phi''(v)}{\phi'(v)} Dv \otimes Dv \right) + \chi(t)|Dv|^k + \frac{\tilde{g}(x, t, v)}{\tilde{f}(v)} - v_t \geq 0.$$  

Take $u = \phi(v)$ and let $\psi$ to be a test function such that $u - \psi$ has a maximum at a point $(y, s) \in \Omega_T$. Clearly,

$$v(x, t) \leq \phi^{-1}( u(y, s) + \psi(x, t) - \psi(y, s) ), \forall (x, t) \in \Omega_T.$$  

Calling $\zeta(x, t) = \phi^{-1}( u(y, s) + \psi(x, t) - \psi(y, s) )$, we get $(v - \zeta)(x, t) \leq (v - \zeta)(y, s) = 0$. Thus, $v - \zeta$ has a maximum at $(y, s)$ and hence,

$$H \left( D\zeta(y, s), \left( D^2\zeta + \frac{\phi''(v)}{\phi'(v)} D\zeta \otimes D\zeta \right)(y, s) \right) + \chi(s)|D\zeta(y, s)|^k + \frac{\tilde{g}(y, s, \psi(y, s))}{\tilde{f}(v(y, s))} - \zeta_t(y, s) \geq 0.$$  

We note
\[ D\zeta(y, s) = \frac{D\psi(y, s)}{\partial'(\zeta(y, s))}, \quad \zeta_t(y, s) = \frac{\psi_t(y, s)}{\partial'(\zeta(y, s))} \quad \text{and} \]
\[ D^2\zeta(y, s) = \frac{D^2\psi(y, s)}{\partial''(\zeta(y, s))} - \left[ \frac{\partial''(\zeta(y, s))}{\partial'(\zeta(y, s))} \right] D\zeta(y, s) \otimes D\zeta(y, s). \]

Recalling that \( \zeta(y, s) = v(y, s) \) and using the above, we get
\[ (2.18) \quad \frac{D^2\psi(y, s)}{\partial'(v(y, s))} = D^2\zeta(y, s) + \frac{\partial''(v(y, s))}{\partial'(v(y, s))} D\zeta(y, s) \otimes D\zeta(y, s). \]

Using (2.15), we get
\[ 0 \leq H \left( \frac{D\psi(y, s)}{\partial'(v(y, s))}, \frac{D^2\psi(y, s)}{\partial'(v(y, s))} \right) + \chi(s) \left( \frac{|D\psi(y, s)|}{\partial'(v(y, s))} \right)^k + \frac{g(y, s, u(y, s))}{\{f(u(y, s))\}^{k/(k-1)}} - \frac{\psi_t(y, s)}{\partial'(v(y, s))}, \]
\[ = H \left( D\psi(y, s), D^2\psi(y, s) \right) \frac{\partial'(v(y, s))}{\{\partial'(v(y, s))\}^k} + \chi(s) \left( \frac{|D\psi(y, s)|}{\partial'(v(y, s))} \right)^k + \frac{g(y, s, u(y, s))}{\{f(u(y, s))\}^{k/(k-1)}} \frac{\psi_t(y, s)}{\partial'(v(y, s))}. \]

Using (2.15), we get \( H(Du, D^2u) + \chi(t)|Du|^k + g(x, t, u) - f(u)u_t \geq 0. \)

Suppose that \( u \in \text{usc}(\Omega_T) \) solves \( H(Du, D^2u) + \chi(t)|Du|^k + g(x, t, u) - f(u)u_t \geq 0. \) Define \( v = \phi^{-1}(u). \)

Let \( \psi \) be a test function such that \( v - \psi \) has a maximum at \((y, s), \) i.e., \( v(x, t) \leq v(y, s) + \psi(x, t) - \psi(y, s), \forall (x, t) \in \Omega_T. \) Thus,
\[ u(x, t) \leq \phi(v(y, s) + \psi(x, t) - \psi(y, s)), \quad \forall (x, t) \in \Omega_T. \]

Let \( \eta(x, t) = \phi(v(y, s) + \psi(x, t) - \psi(y, s)) \) implying that \( \eta(y, s) = u(y, s), (u - \eta)(x, t) \leq (u - \eta)(y, s) = 0 \) and
\[ H(D\eta(u, s), D^2\eta(y, s)) + \chi(s)|D\eta(y, s)|^k + g(y, s, u(y, s)) - f(u(y, s))\eta_t(y, s) \geq 0. \]

Calculating,
\[ 0 \leq H(D\eta(y, s), D^2\eta(y, s)) + \chi(s)|D\eta(y, s)|^k + g(y, s, u(y, s)) - f(u(y, s))\eta_t(y, s) \]
\[ = |\phi'(v(y, s))|^k H \left( D\psi(y, s), D^2\psi(y, s) + \frac{\phi''(v(y, s))}{\phi'(v(y, s))} D\psi(y, s) \otimes D\psi(y, s) \right) \]
\[ + \chi(s)|\phi'(v(y, s))|^k |D\psi(y, s)|^k + g(y, s, (\phi \circ v)(y, s)) - (f \circ \phi \circ v)(y, s)\phi'(v(y, s)) \psi_t. \]

Simplifying, we see that the claim holds. The claims in Case (i) (b) and Case (ii) follow analogously.

\[ \square \]

**Remark 2.4.** (i) Lemma 2.3 does not address the case \( f \equiv 1 \) and \( k > 1 \) since the comparison principle for \( H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \) where \( \Gamma \geq 0, \) follows from a general result. See Section 4.

(ii) We now address the example that was referred to in the discussion following Theorem 1.3 see Section 1. Let \( Tr(X) \) be the trace of a matrix \( X. \) Set
\[ H(p, X) = |p|^{q-2}Tr(X) + (q - 2)|p|^{q-4}p_i p_j X_{ij}, \quad q \geq 2. \]
Clearly, $H(Du, D^2u) = \text{div}(|Du|^{q-2}Du)$. If $X = Y + p \otimes p$ then
\[
H(p, Y + p \otimes p) = |p|^{q-2}Tr(Y) + (q-2)|p|^{q-4}p_ip_jY_{ij} + (q-1)|p|^q = H(p, Y) + (q-1)|p|^q.
\]
Suppose that $u > 0$ solves $\text{div}(|Du|^{q-2}Du) + \chi(t)|Du|^{q-1} - (q-1)u^{q-2}u_t = 0$. It follows from (2.15), $\phi(s) = e^s$. If $v = \log u$ then Lemma 2.3 and the above observations imply that
\[
\text{div}(|Dv|^{q-2}Dv) + (q-1)|Dv|^q + \chi(t)|Dv|^{q-1} - (q-1)v_t = 0.
\]
Thus, showing the existence of $u$ is equivalent to showing the existence of $v$. See Section 3. □

**Remark 2.5.** It is clear from Lemma 2.3 that analogous results hold if $H$ satisfies Condition A and $B$ and depends on $x$, $t$, $u$, $Du$ and $D^2u$. □

Finally, we state a lemma that will be used Sections 5, 6 and 7. Note that the result holds if $H$ depends on $t$, $u$, $Du$, $D^2u$ and $H(t, u, Du, O) = 0$.

**Lemma 2.6.** Let $O \subset \Omega_T$ be a sub-domain. Suppose that $\ell : IR \to IR$, $\chi : [0, T] \to IR$ and $f : IR \to IR$ are continuous. Assume that $H$ satisfies Condition A (see (1.2)) and $\Gamma \geq 0$. Suppose that $u \in \text{usc}(lsc)(\Omega_T \cup P_T)$ satisfies
\[
H(Du, D^2u + \ell(u)Du \otimes Du) + \chi(t)|Du|^\Gamma - f(u)u_t \geq (\leq)0, \text{ in } O.
\]
Assume that for some $c \in IR$, $u \geq (\leq)c$ in $O$, $u = c$ on $\partial O \cap \Omega_T$, and $u = c$ in $\Omega_T \setminus O$. Then $u$ satisfies
\[
H(Du, D^2u + \ell(u)Du \otimes Du) + \chi(t)|Du|^\Gamma - f(u)u_t \geq (\leq)0, \text{ in } \Omega_T.
\]

**Proof.** We prove the statement when $u$ is a sub-solution. We check at points on $\partial O \cap \Omega_T$.

Let $(y, \tau) \in \partial O \cap \Omega_T$, with $\tau > 0$. Suppose that $\psi$ is a test function such that $u - \psi$ has a maximum at $(y, \tau)$. Since $u \geq c$ and $u(y, \tau) = c$, we have
\[
0 \leq u(x, t) - u(y, \tau) \leq \langle D\psi(y, \tau), x - y \rangle + \psi_t(y, \tau)(t - \tau) + \frac{\langle D^2\psi(y, \tau)(x - y), x - y \rangle}{2} + o(|x - y| + |t - \tau|),
\]
as $(x, t) \to (y, \tau)$. Clearly, $D\psi(y, \tau) = 0$, $\psi_t(y, \tau) = 0$ and $D^2\psi(y, \tau) \geq 0$. Thus, using Condition A,
\[
H \left( D\psi(y, \tau), D^2\psi(y, \tau) + \ell(u(y, \tau))D\psi(y, \tau) \otimes D\psi(y, \tau) \right) + \chi(\tau)|D\psi(y, \tau)|^\Gamma - f(\psi(y, \tau))(\psi_t)(y, \tau) = H(0, D^2\psi(y, \tau)) \geq 0.
\]
The conclusion holds. The proof when $u$ is a super-solution is analogous. □
In this section, we list examples of operators $H$ that satisfy Conditions A, B and C and to which our results apply. Let $\lambda \in \mathbb{R}$ and $e \in \mathbb{R}^n$ be such that $|e| = 1$. Set $r = |x|, \forall x \in \mathbb{R}^n$. Recall the definitions of $k_1, k_2, k, m(\lambda)$ and $\mu(\lambda)$ from (1.3), (1.4), (1.6) and (1.7).

**Example 1: The $p$-Laplacian and the pseudo $p$-Laplacian.** Recall that the $p$-Laplacian $\Delta_p$, for $p \geq 2$, is $D_p u = |Du|^{p-2} \Delta u + (p-2)|Du|^{p-4} \Delta_\infty u$, where $\Delta_\infty u = \sum_{i,j=1}^n D_i u D_j u D_{ij} u$ is the infinity-Laplacian. We consider a somewhat more general version. Define

$$H(Du, D^2 u) = |Du|^q \Delta u + a|Du|^{q-2} \Delta_\infty u,$$

where $q \geq 0$ and $a > -1$. Then $H(e, I - \lambda e \otimes e) = n + a - \lambda(1+a)$. Clearly, Conditions A, B and C are met.

Next we discuss a version of the pseudo $p$-Laplacian, denoted by $\Delta^s_{p,q}$, where

$$H(Du, D^2 u) = \Delta^s_{p,q} u = |Du|^q \sum_{i=1}^n |D_i u|^p D_{ii} u, \text{ where } p, q \geq 0.$$

Thus, $H(e, I - \lambda e \otimes e) = \sum_{i=1}^n |e_i|^p - \lambda \sum_{i=1}^n |e_i|^{p+2}$ and $H > 0$, if $\lambda \leq 0$.

Let $\lambda > 0$. Note that $H(e, I - \lambda e \otimes e) \geq (1 - \lambda) \sum_{i=1}^n |e_i|^p$, since $|e_i| \leq 1$. By Hölder’s inequality, if $r \geq 0$ then

$$(3.1) \quad \min \left(1, n^{(2-r)/2}\right) \leq \sum_{i=1}^n |e_i|^r \leq \left(\sum_{i=1}^n |e_i|^{r+2}\right)^{r/(r+2)} n^{2(r+2)/r}.$$

Apply (3.1) with $r = p$ to get a lower bound for $H$, that is,

$$H(e, I - \lambda e \otimes e) = \sum_{i=1}^n |e_i|^p - \lambda \sum_{i=1}^n |e_i|^{p+2} \geq (1 - \lambda) \sum_{i=1}^n |e_i|^p \geq \begin{cases} (1 - \lambda)n^{-(|2-p|)/2}, & 0 \leq \lambda \leq 1, \\ (1 - \lambda)n, & \lambda \geq 1. \end{cases}$$

Set $E = E(e) = (\sum_{i=1}^n |e_i|^{p+2})^{p/(p+2)}$. Use (3.1) first with $r = p$ and then with $r = p + 2$ to get an upper bound for $H$, that is,

$$H(e, I - \lambda e \otimes e) = \sum_{i=1}^n |e_i|^p - \lambda \sum_{i=1}^n |e_i|^{p+2} \leq \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{p/(p+2)} n^{2(p+2)/(p+2)} - \lambda \sum_{i=1}^n |e_i|^{p+2}
$$

$$\leq E \left[n^{2(p+2)} - \lambda \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{2(p+2)}\right] \leq E \left[n^{2(p+2)} - \frac{\lambda}{n^{p/(p+2)}}\right] = E \left(n - \frac{\lambda}{n^{p/(p+2)}}\right)$$

$$= \left(\sum_{i=1}^n |e_i|^{p+2}\right)^{p/(p+2)} (n - \lambda) \leq I(\lambda) (n - \lambda),$$

where $I(\lambda) = 1$, if $\lambda \leq n$, and $I(\lambda) = n^{-p/2}$, if $\lambda \geq n$. Observe that if $e_i = 1$, for some $i$, then $H(e, I - \lambda e \otimes e) = 1 - \lambda$. Also, if $e_i = n^{-1/2}$, for $i = 1, 2, 3, \cdots, n$, and then $H(e, I - \lambda e \otimes e) = n^{-p/2}(n - \lambda)$. Conditions A, B and C hold.
Example 2: The ∞-Laplacean and a related operator. Setting $H(Du, D^2u) := \Delta_\infty u = \sum_{i,j=1}^n D_i u D_j u D_{ij} u$, we get $H(e, I - \lambda e \otimes e) = 1 - \lambda$.

Next, we consider $q \geq 0$ and define $H(Du, D^2u) := \sum_{i,j=1}^n |D_i u|^q |D_j u|^q D_i u D_j u D_{ij} u$. Then

$$H(e, I - \lambda e \otimes e) = \sum_{i=1}^n |e_i|^{2q+2} - \lambda \left( \sum_{i=1}^n |e_i|^{q+2} \right)^2.$$  

We use (3.1) for estimating $H(e, I - \lambda e \otimes e)$. If $\lambda \leq 0$ then $H > 0$. Taking $\lambda \geq 0$ and observing that $(\sum_{i=1}^n |e_i|^{q+2})^2 \leq \sum_{i=1}^n |e_i|^{2q+2} \leq 1$, we get

$$H(e, I - \lambda e \otimes e) \geq (1 - \lambda) \sum_{i=1}^n |e_i|^{2q+2} \geq \begin{cases} (1 - \lambda)n^{-q}, & 0 \leq \lambda \leq 1, \\ 1 - \lambda, & \lambda \geq 1. \end{cases}$$

Noting that $\sum_{i=1}^n |e_i|^{2q+2} \leq \sum_{i=1}^n |e_i|^{q+2}$ and using (3.1), we get

$$H(e, I - \lambda e \otimes e) \leq \sum_{i=1}^n |e_i|^{q+2} \left( 1 - \lambda \sum_{i=1}^n |e_i|^{q+2} \right) \leq I(\lambda) \left( 1 - \frac{\lambda}{n^{q/2}} \right),$$

where $I(\lambda) = 1$, if $\lambda \leq n^{q/2}$ and $I(\lambda) = n^{-q/2}$, if $\lambda \geq n^{q/2}$. Conditions A, B and C are satisfied. See also [3].

Example 3: Pucci operators. Let $a_i$, $i = 1, 2, \cdots, n$, denote the eigenvalues of the matrix $D^2u$.

For $0 < \theta \leq \hat{\theta}$ and $q \geq 0$ define

$$M_{\theta, \hat{\theta}}^{+, q}(u) = |Du|^q \left( \hat{\theta} \sum_{a_i \geq 0} a_i + \theta \sum_{a_i \leq 0} a_i \right) \quad \text{and} \quad M_{\theta, \hat{\theta}}^{-, q}(u) = |Du|^q \left( \theta \sum_{a_i \geq 0} a_i + \hat{\theta} \sum_{a_i \leq 0} a_i \right).$$

For any $e$ with $|e| = 1$, the eigenvalues of $I - \lambda e \otimes e$ are 1, with multiplicity $n - 1$, and $1 - \lambda$. Set $H^\pm(Du, D^2u) = M_{\theta, \hat{\theta}}^{+, q}(u)$ and observe that $H^+(e, \pm(I - \lambda e \otimes e)) = -H^-(e, \mp(I - \lambda e \otimes e))$. Clearly,

$$H^+(e, I - \lambda e \otimes e) = \begin{cases} \hat{\theta}(n - \lambda), & \lambda \leq 1, \\ \hat{\theta}(n - 1) + \theta(1 - \lambda), & \lambda \geq 1 \end{cases}$$

and

$$H^-(e, I - \lambda e \otimes e) = \begin{cases} \theta(n - \lambda), & \lambda \leq 1, \\ \theta(n - 1) + \hat{\theta}(1 - \lambda), & \lambda \geq 1. \end{cases}$$

Thus, $H^\pm$ satisfy Conditions A, B and C. The maximal and minimal Pucci operators are also included here, see [7].

4. Comparison principles

In this section we prove a version of the comparison principle that applies to the class of parabolic equations addressed in the work. If $k > 1$ and $f$ is an increasing function and $f^{1/(k-1)}$ is concave (the equation is doubly nonlinear) then the comparison principle is proven under the condition that sub-solutions and super-solutions are positive. However, if
If $f \equiv 1$ and $k \geq 1$ then a comparison principle holds without any restrictions on the sign of the sub-solutions and super-solutions.

We now state a comparison principle which is a slight variant of the version in \cite{[4]} and the statement is influenced by the change of variables Lemma 2.3. We consider a more general operator than $H$. Let $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ be continuous and satisfy

$$\text{(i) } F(t, r, p, X) \leq F(t', r, p, Y), \forall (t, r, p) \in \Omega_T \times \mathbb{R}^n,$$

with $X, Y \in S^n$ and $\forall X \leq Y$,

$$\text{(ii) } \forall (t, p, X) \in \mathbb{R}^+ \times \mathbb{R} \times S^n, F(t, r_1, p, X) \leq F(t, r_2, p, Y), \text{ if } r_1 \geq r_2.$$  

In Lemma 4.1 the only condition imposed on $F$ is \eqref{4.1}.

**Lemma 4.1.** (Comparison principle) Let $F$ be as in \eqref{4.1}, $G : \mathbb{R} \to \mathbb{R}$ be a bounded non-increasing continuous function and $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. Let $u \in \text{usc}(\Omega_T \cup P_T)$ and $v \in \text{lsc}(\Omega_T \cup P_T)$ satisfy in $\Omega_T$,

$$F(t, u, Du, D^2u + g(u)Du \otimes Du) - \kappa(t)u_\xi \geq 0$$

and

$$F(t, v, Dv, D^2v + g(v)Dv \otimes Dv) - \kappa(t)v_\xi \leq 0.$$ 

If $\sup_{P_T} v < \infty$ and $u \leq v$ on $P_T$ then $u \leq v$ in $\Omega_T$.

**Proof.** We note that $X + g(u)p \otimes p \leq Y + g(v)p \otimes p$, for any $p \in \mathbb{R}^n$, $X \leq Y$ and $u \geq v$. The claim follows from Theorem 3.3 on page 18 of \cite{[4]}. \hfill $\Box$

**Remark 4.2.** (a) Let $F$, $\kappa$, $u$ and $v$ be as in Lemma 4.1. Let $k = \sup_{P_T}(u - v)$ and $v_k = v + k$. Since $v_k \geq v$, by \eqref{4.1}(ii),

$$F(t, v_k, Dv_k, D^2v_k + g(v_k)Dv_k \otimes Dv_k) - \kappa(t)v_k \leq 0,$$ 

in $\Omega_T$ and $u \leq v_k$, in $P_T$.

By Lemma 4.1 $u - v \leq \sup_{P_T}(u - v)^+.$

(b) Suppose that $F = F(t, p, X)$ where $p \in \mathbb{R}^n$ and $X \in S^n$. Take $d \geq 0$, a constant. Let $u \in \text{usc}(\Omega_T)$ and $v \in \text{lsc}(\Omega_T)$ solve

$$F(t, Du, D^2u + dDu \otimes Du) - u_t \geq 0,$$

and

$$F(t, Dv, D^2v + dDv \otimes Dv) - v_t \leq 0.$$ 

Then $u - v \leq \sup_{P_T}(u - v)$. To see this, set $k = \sup_{P_T}(u - v)$ and take $v_k = v + k$. Lemma 4.1 shows that $u \leq v_k$ in $\Omega_T$ and the claim holds. \hfill $\square$

As an application of the above result we get a comparison principle for parabolic equations of the type (see \eqref{1.1})

$$H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0,$$ 

in $\Omega_T$.

Recall \eqref{2.15}, \eqref{2.16} and Lemma 2.3.

**Theorem 4.3.** (Comparison principle) Let $H$ satisfy Conditions A and B, see \eqref{1.2}, \eqref{1.3} and \eqref{1.4}. Suppose $f : [0, \infty) \to [0, \infty)$, is a $C^1$ function and $\Gamma \geq 0$.

Case (i): $k > 1$, $f$ is a non-constant increasing function and $f^{1/(k-1)}(\theta)$ is concave in $\theta$.

Let $u \in \text{usc}(\Omega_T \cup P_T)$ and $v \in \text{lsc}(\Omega_T \cup P_T)$ satisfy

$$H(Du, D^2u) + \chi(t)|Du|^k - f(u)u_t \geq 0,$$

and

$$H(Dv, D^2v) + \chi(t)|Dv|^k - f(v)v_t \leq 0,$$ 

in $\Omega_T$. 

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing $C^2$ function such that $\phi'(\tau) = f(\phi(\tau))^{1/(k-1)}$, see (2.13). If $u > 0$, $v > 0$, $\sup_{P_T} v < \infty$ and $u \leq v$ on $P_T$ then $\phi^{-1}(u) \leq \phi^{-1}(v)$ and $u \leq v$ in $\Omega_T$. In general,

$$u \leq \phi \left( \phi^{-1}(v) + \sup_{P_T} \{ \phi^{-1}(u) - \phi^{-1}(v) \}^+ \right).$$

Case (ii): $k \geq 1$ and any $1 \geq 0$. Let $u \in \text{usc}(\Omega_T \cup P_T)$ and $v \in \text{lsc}(\Omega_T \cup P_T)$ satisfy

$$H(Du, D^2u) + \chi(t)|Du|^k - u_t \geq 0, \text{ and } H(Dv, D^2v) + \chi(t)|Dv|^k - v_t \leq 0, \text{ in } \Omega_T.$$

If $u \leq v$, in $P_T$ and $\sup_{P_T} v < \infty$ then $u \leq v$ in $\Omega_T$. More generally, $u - v \leq \sup_{P_T} (u - v)$. The result holds regardless of the signs of $u$ and $v$.

Proof. The claims follow from the change of variables Lemma [2.3], the comparison principle in Lemma [4.4] Remark [4.2] and that $\phi''(s)/\phi'(s)$ is decreasing in $s$. □

Remark 4.4. (a) Theorem 4.3(i) holds for operators $H$ that depend on $t$, $u$, $Du$, $D^2u$, $H$ is decreasing in $u$ and satisfy Conditions A and B. See Remark 4.2(a).

(b) Theorem 4.3(ii) holds for the more general operator $F$ as in Lemma 4.2. □

Lemma 4.5. (Maximum principle) Let $F$ satisfy (4.1) and $F(t, r, p, O) = 0$, for any $t \geq 0$, any $r \in \mathbb{R}$ and any $p \in \mathbb{R}^n$.

(a) Suppose that $u \in \text{usc} \text{lsc}(\Omega_T)$ solves $F(t, u, Du, D^2u) - u_t \geq (\leq)0$, in $\Omega_T$. Then $u \leq \sup_{P_T} u \geq \inf_{P_T} u$.

(b) Let $k > 1$. Suppose that, in addition, $F$ satisfies Condition B. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a $C^1$ increasing function and $f^{1/(k-1)}$ be concave. Assume that $u \in \text{usc} \text{lsc}(\Omega_T)$, $u > 0$, solves

$$F(t, u, Du, D^2u) + \chi(t)|Du|^k - f(u)u_t \geq (\leq)0, \text{ in } \Omega_T,$$

where $\chi$ is a continuous function. Then $u \leq \sup_{P_T} u \geq \inf_{P_T} u$.

Proof. Since $F(t, r, p, O) = 0$, for any $(t, r, p)$, $t \geq 0$, the function $\phi = \sup_{P_T} u$ is a solution. Similarly, $\eta = \inf_{P_T} u$ is also a solution. Using Remark 2.3, Theorem 4.3 and Remark 4.4 the claims hold. □

Remark 4.6. Let $F$ satisfy (4.1), Condition B and $F(t, r, p, O) = 0$. Suppose that $u \in \text{usc}(\Omega_T \cup P_T)$ solves

$$(*) \quad F(t, u, Du, D^2u) + \chi(t)|Du|^k - u^{k-1}u_t \geq (\leq)0, \text{ in } \Omega_T.$$

If $u > 0$ and $\phi = \log u$ then by Lemma 2.3

$$F(t, e^{\phi}, D\phi, D^2\phi + D\phi \otimes D\phi) + \chi|D\phi|^k - \phi_t \geq (\leq)0,$$

in $\Omega_T$. Remark 4.2(a) and (1.1)(ii) show that if $u > 0$ is a sub-solution of $(*)$ and $v > 0$ is super-solution of $(*)$ then

$$\frac{u}{v} \leq \max \left( \sup_{P_T} \frac{u}{v}, 1 \right).$$
If \( F = F(t, p, X) \) then Remark 4.2(b) shows that \( u/v \leq \sup_{P_T} (u/v) \).

A similar quotient type comparison principle was derived for the doubly nonlinear parabolic equations studied in [1,2]. □

**Remark 4.7.** Let \( H \) be as in Theorem 4.3, \( k \geq 1 \), and \( f(u) = u^m \), \( m \geq 0 \). The condition \( f^{1/(k-1)}(\theta) \), \( k > 1 \), is concave in \( \theta \) implies that \( 0 \leq m \leq k-1 \) and Theorem 4.3 holds. For \( k = 1 \), we require that \( m = 0 \). For \( m < 0 \) or \( m > k-1 \), it is not clear to us if a comparison principle holds. □

5. **Initial data \( t = 0 \). Constructions for Theorems 1.2 and 1.3**

In Sections 5, 6 and 7, we address the existence of positive solutions to (1.1), i.e,

\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0, \quad \text{in } \Omega_T \quad \text{and} \quad u(x, t) = h(x, t), \quad \forall (x, t) \in P_T,
\]

where \( h \) is as in (2.4), \( \Gamma > 0 \) and \( f : [c, \infty) \to [0, \infty), \quad c \geq 0 \), is \( C^1 \). Let us recall that

\[
h(x, t) = \begin{cases} 
i(x), & \forall x \in \Omega, \quad \text{at } t = 0, \\
j(x, t), & \forall (x, t) \in \partial \Omega \times [0, T),
\end{cases}
\]

where \( i(x) \) and \( j(x, t) \) are positive and continuous and, for any \( y \in \partial \Omega \), \( \lim_{x \to y} i(x) = \lim_{(z, t) \to (y, 0)} j(z, t) = j(y, 0) \), where \( x \in \Omega \) and \( z \in \partial \Omega \).

We assume in Sections 5, 6 and 7 that

(i) \( k > 1 \) and \( f : [0, \infty) \to [0, \infty) \) is an increasing \( C^1 \) function and \( f^{1/(k-1)}(\theta) \) is concave in \( \theta \), and \( \Gamma = k \), or

(ii) \( k \geq 1 \), \( f(\theta) = 1 \), \( \forall \theta \in \mathbb{R} \), and \( 0 < \Gamma < \gamma \).

This will ensure that the problem in (5.1) has a comparison principle. We also assume through out that

\[
\text{(5.3)} \quad \text{either (i) } \inf_{0 \leq \theta < \infty} f(\theta) > 0, \quad \text{or (ii) } f \geq 0 \text{ and } f(\theta) = 0 \text{ iff } \theta = 0.
\]

Our proof of the existence of a positive continuous solution to the problem (5.1) involves constructing positive sub-solutions and super-solutions for the problem that are arbitrarily close, in a local sense, to the data specified on the parabolic boundary \( P_T \). Existence then follows by using Perron’s method [4], see also [1]. Uniqueness is implied by Theorem 4.3.

The ideas used are an adaptation of the works in [1,2].

We have divided our work into three sections. In this section we take up the construction for the initial data at \( t = 0 \). Our work is valid for any bounded domain \( \Omega \).

Set \( \vartheta = \inf_{P_T} h \) and \( M = \sup_{P_T} h \). Assume that

\[
0 < \vartheta \leq M < \infty, \quad \text{and} \quad 0 < \omega = \inf_{[\vartheta/2, 2M]} f(\theta) \leq \sup_{[\vartheta/2, 2M]} f(\theta) = \nu < \infty.
\]
If \( \vartheta = M \) then \( M \) is the solution. Through out the rest of the work, the quantity \( \varepsilon > 0 \) is small and so chosen that

\[
0 < \frac{\vartheta}{2} < \vartheta - 2\varepsilon \leq M - 2\varepsilon \leq 2M.
\]

Also, set

\[
(5.6) \quad B_0 = \sup_{[0,T]} |\chi(t)|.
\]

Our constructions will ensure that the sub-solutions and the super-solutions \( \eta \) of (5.1) are bounded below by \( \vartheta/2 \) and bounded above by \( 2M \).

We start with the initial data \( h(x,0) \). We select points \( y \in \Omega \) at \( t = 0 \). There are two cases to consider: (a) \( y \in \Omega \), and (b) \( y \in \partial \Omega \). We assume that \( h(y,0) > \vartheta \). If \( h(y,0) = \vartheta \), we take the sub-solution to be \( \vartheta \). Similarly, if \( h(y,0) = M \), we take the super-solution to be \( M \).

We recall the following calculation. Let \( g^\pm(x) = a \pm br^2 \), \( a, b \geq 0 \), where \( r = |x - z| \) for some \( z \in \mathbb{R}^n \). By (1.3), (1.4) and Remark 2.2,

\[
(5.7) \quad (i) \quad (2b) r k_1 m(0) \leq H(Dg^+, D^2g^+) \leq (2b) r k_1 \mu(0), \quad \text{and}
\]

\[
(ii) \quad -(2b) r k_1 \mu(0) \leq H(Dg^-, D^2g^-) \leq -(2b) r k_1 m(0).
\]

Recall the definitions of \( m(\lambda) \) and \( \mu(\lambda) \), see (1.6) and (1.7). Thus, \( \mu(0) \geq m(0) > 0 \).

**Part I's of Theorems 1.2 and 1.3** Case (5.2)(i): \( k > 1 \), \( f \) increasing \( C^1 \) function, \( f^{1/(k-1)} \) is concave and \( \Gamma = k \).

Case (a): Let \( y \in \Omega \) and \( \varepsilon > 0 \), small, so that (5.5) holds. By continuity, there is a \( 0 < \delta_0 \leq \text{dist}(y, \partial \Omega) \) such that

\[
h(y,0) - \varepsilon \leq h(x,0) \leq h(y,0) + \varepsilon, \quad \forall x \in B_{\delta_0}(y).
\]

Recall by the comment right after (5.7) that \( \mu(0) > 0 \). Set \( r = |x - y| \).

**Sub-solution:** Note that \( k_1 > 0 \), see (1.3) and (1.4). Define

\[
(5.8) \quad \tau = \frac{1}{\ell} \log \left( \frac{h(y,0) - 2\varepsilon}{\vartheta - 2\varepsilon} \right), \quad b = \frac{1 - e^{-\ell \tau}}{\delta^2} \quad \text{and} \quad \ell = \frac{3(8b) k_1 \mu(0) M^{2k-1}}{\omega \eta^k}
\]

where \( 0 < \delta \leq \delta_0 \). Note that \( \ell = E \delta^{-k_1-2k_2} \), where \( E \) is independent of \( \tau \) and \( \delta \). First we choose \( \delta > 0 \), small, and calculate \( \ell, b \) and \( \tau \). In particular, choose \( \delta \) small so that \( \tau < T \).

Using (5.8), let \( R \) be the region

\[
(5.9) \quad R = \{(x,t) : e^{\ell(t-t)}(1 - br^2) \geq 1, \quad 0 \leq t \leq \tau \}.
\]

The base of \( R \) is a spatial sphere of radius \( \delta \) at \( t = 0 \), tapers as \( t \) increases and has an apex at \( (y,\tau) \). We construct a bump like function at \( (y,0) \) which decreases in \( t \).
Next, define

\[(5.10) \quad \eta(x,t) = \begin{cases} \varphi^{(t-t_0)}(1 - br^2), & \forall (x,t) \in R, \\ \varphi - 2\varepsilon, & \forall (x,t) \in (\Omega_T \cup P_T) \setminus R. \end{cases} \]

By (5.8), (5.9) and (5.10), \(0 \leq \frac{b\varepsilon}{2} \leq 1 - e^{\varepsilon(\tau-t)}, 0 \leq t \leq \tau,\)

(i) \(\eta(y,0) = \sup \eta = h(y,0) - 2\varepsilon,\)

(ii) \(\eta = \varphi - 2\varepsilon,\) in \(\partial R \cap \Omega_T,\)

(iii) \(\eta \leq h,\) in \(P_T,\)

(5.11) and (5.12) \(\varphi \leq \frac{\eta}{2} \leq \eta \leq M.\)

Recalling (5.4), (5.6), (5.7)(ii), (5.8), (5.10), setting \(A_1 = (\varphi - 2\varepsilon)e^{\varepsilon(\tau-t)}\) and estimating \(B_0\delta^k \leq 2\mu(0)\delta^{k_1}\) (take \(\delta\) small), we calculate in \(0 \leq \tau \leq \delta\) and \(0 \leq t \leq \tau,\)

\[
H(D\eta, D^2\eta) + \chi(t)|D\eta|^k - f(\eta) \eta_t \geq A_1 \ell f(\eta)(1 - br^2) - A_1 \ell B_0(2br)^k - A_1 \ell (2b)^k \tau^{k_1} \mu(0)
\]

\[
\geq A_1 \ell \left[\frac{\ell f(\eta)(1 - b\delta^2)}{\ell(\varphi - 2\varepsilon)e^{\varepsilon(\tau-t)}} - (2b)^k B_0 \delta^k + \delta^{k_1} \mu(0)\right]
\]

\[
\geq A_1 \ell \left[\frac{\ell f(\eta)e^{-\ell\tau}}{(\varphi - 2\varepsilon)e^{\varepsilon(\tau-t)}} - (2b)^k B_0 \delta^k + \delta^{k_1} \mu(0)\right]
\]

\[
\geq A_1 \ell \left[\frac{\ell \omega}{(\varphi - 2\varepsilon)e^{\varepsilon(\tau-t)}} - 3(2b)^k \delta^{k_1} \mu(0)\right]
\]

\[(5.12) \quad 0 \geq A_1 \ell \left[\frac{\ell \omega}{M^{k-1}(4M/\varphi)^k} - 3(2b)^k \delta^{k_1} \mu(0)\right] = 0,
\]

where we have used (5.5) and (5.8) (i.e, \(1 - b\delta^2 = e^{-\ell\tau} \geq \varphi/(4M))\). Thus, \(\eta\) is a sub-solution in \(R\) and Lemma 2.6 shows that \(\eta\) is a sub-solution in \(\Omega_T\).

**Super-solution:** The work is similar to what we did for the sub-solution. Define

\[(5.13) \quad \tau = \frac{1}{\ell} \log \left(\frac{M + 2\varepsilon}{h(y,0) + 2\varepsilon}\right), \quad b = \frac{e^{\ell\tau} - 1}{\delta^2} \quad \text{and} \quad \ell = \frac{3(4b)^k \delta^{k_1} M^{k-1} \mu(0)}{\omega},
\]

where \(0 < \delta \leq \delta_0.\) Again, \(\ell = O(\delta^{k_1-2k_2})\) implying that \(\tau \to 0\) if \(\delta \to 0.\)

Let

\[R = \{(x,t) \in (1 + br^2)e^{\varepsilon(\tau-t)} \leq 1, \ 0 \leq t \leq \tau\}.
\]

Then, \(b\varepsilon^2 \leq e^{\varepsilon(\tau-t)} - 1\) and, at \(t = 0, R\) is a ball of radius \(\delta.\) As \(t\) increases \(R\) tapers to \((y, \tau).\)

Define

\[(5.14) \quad \phi(x,t) = \begin{cases} (M + 2\varepsilon)e^{\varepsilon(\tau-t)}(1 + br^2), & \forall (x,t) \in R, \\ M + 2\varepsilon, & \forall (x,t) \in (\Omega_T \cup P_T) \setminus R. \end{cases} \]

It is clear that

(i) \(\phi(y,0) = \inf \phi = h(y,0) + 2\varepsilon,\)

(ii) \(\phi = M + 2\varepsilon,\) in \(\partial R \cap \Omega_T,\)

(iii) \(\phi \geq h,\) in \(P_T,\)

(5.15) and (iv) \(\varphi \leq \phi \leq 2M.\)
Set $A_2 = (M + 2\varepsilon)e^{\ell(t-\tau)}$. We calculate in $R$ using (5.4), (5.6), (5.7)(i), (5.13), (5.15) and the comment after (5.8), and see that, for small $\delta$,

$$H(D\phi, D^2\phi) + \chi(t)|D\phi|^k - f(\phi)\phi_t \leq A_2^k(2b)^k\mu(0) + A_2^k B_0(2br)^k - \ell\omega A_2(1 + br^2)$$

$$\leq A_2^k \left[(2b)^k \delta k\mu(0) + B_0 \delta k^3 - \frac{\ell\omega}{1 + (M + 2\varepsilon)e^{\ell(t-\tau)}}\right]$$

$$\leq A_2^k \left[(2b)^k \delta k\mu(0) - \frac{\ell\omega}{(2M)^k}\right] \leq 0.$$  

(5.16)

Thus, $\phi$ is a super-solution in $R \cap \Omega_T$. Recalling (5.14) and using Lemma 2.6, $\phi$ is a super-solution in $\Omega_T$.

**Case (b)** Let $y \in \partial\Omega$: By continuity, there are $\delta > 0$ and $s > 0$ such that

$$h(y,0) - \varepsilon \leq h(x,t) \leq h(y,0) + \varepsilon, \quad \forall (x,t) \in P_T \cap (B_\delta(y) \times [0,s]).$$

We utilize the quantities in (5.8) and (5.13) in our constructions. For both the sub-solution and the super-solution, we take the $\ell$’s large enough so that $\tau \leq s$ and the apex $(y,\tau) \in B_\delta(y) \times [0,s]$.

Next, we define the sub-solution $\eta$ as in (5.10) and the super-solution $\phi$ as in (5.14). The rest of the work is similar to part (a). □

**The Part II’s of Theorems 1.2 and 1.3.** Case (5.2)(ii): $k \geq 1$, $f(\theta) = 1$, $\forall \theta \in \mathbb{R}$, and any $0 < \Gamma < \gamma$.

We consider

$$H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \quad \text{in} \ \Omega_T \text{ and } u = h \text{ in } P_T.$$  

(5.17)

For both the sub-solution and the super-solution we proceed as in Part I.

Let $\eta$ be as in (5.10), and $\phi$ be as in (5.14). We discuss the changes needed in (5.8) and (5.13). Note that unlike Part I, $k_1 = 0$ may occur, i.e., $k = 1$. We show that the calculations in the corresponding regions $R$ continue to apply by modifying the quantity $\ell$. The proof for the rest of $\Omega_T$ is as in Part I.

We address the sub-solution $\eta$. Let $(y,0)$ be as in Part I. Setting $A_3 = (\vartheta - 2\varepsilon)e^{\ell(\tau-t)}$ in $R$ (see (5.9) and (5.12)),

$$H(D\eta, D^2\eta) + \chi(t)|D\eta|^\Gamma - \eta_t \geq A_3 \ell(1 - br^2) - A_3^k B_0(2br)^\Gamma - A_3^k(2b)^k r^{k_1} \mu(0)$$

$$= A_3^k \left(\ell(1 - br^2) - A_3^k B_0(2br)^\Gamma - A_3^k(2b)^k r^{k_1} \mu(0)\right)$$

$$\geq A_3^k \left(\frac{\ell(1 - \vartheta^2)}{A_3^{k_1}} - A_3^k B_0(2\vartheta)^\Gamma - A_3^k(2b)^k r^{k_1} \mu(0)\right).$$

Since $\vartheta/2 \leq A_3 \leq 2M$, using the appropriate estimates for $A_3$ (depending on whether $\Gamma > k$ or $\Gamma < k$) and choosing $\ell$ large, it follows that $\eta$ is a sub-solution in $R$ and hence in $\Omega_T$. 

We now discuss the super-solution \( \phi \). Setting \( A_4 = (M + 2\varepsilon)e^{\ell(t-\tau)} \) and calculating in \( R \) (see (5.10)),
\[
H(D\phi, D^2\phi) + \chi(t)|D\phi|^k - \phi_t = A_4^k(2b)^k r^{k_1} \mu(0) + A_4^\ell B_0(2br)^\Gamma - \ell A_4(1 + br^2)
\leq A_4^k \left( (2b)^k \delta^{k_1} \mu(0) + A_4^{\Gamma-k} B_0(2b\delta)^\Gamma - \frac{\ell}{A_4^{k-1}} \right).
\]
Since \( \vartheta / 2 \leq A_4 \leq 2M \), arguing as done above, one can choose \( \ell \) large enough so that \( \phi \) is a super-solution in \( R \) and thus in \( \Omega_T \).

6. SIDE BOUNDARY: CASE (1.8)(i). CONSTRUCTION FOR THEOREM 1.2.

We construct positive sub-solutions and super-solutions for the side boundary \( \partial \Omega \times (0,T) \) when Case (i) in (1.8) holds. Our results hold for any bounded \( \Omega \).

As in Section 5, we assume that \( f : [0, \infty) \to [0, \infty) \) and (5.3) holds. We present the work for Parts I and II of the theorem below.

We recall (5.1) for easy reference:
\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0, \quad \text{in } \Omega_T, \quad \text{and } u = h \text{ in } P_T.
\]

Combining the constructions in this section with the set of sub-solutions and super-solutions in Section 5 and applying the Perron method one obtains the existence of positive solutions of (6.1) when (1.8)(i) holds. Recall the notations and the conditions stated in (5.4) and (5.5).

We recall (1.8)(i): there is a
\[
1 < \bar{\lambda} < 2 \quad \text{such that } \mu(\bar{\lambda}) < 0,
\]
where \( \mu(\lambda) = \max\{m_{\max}(\lambda), -\mu_{\min}(\lambda)\} \), see (1.5) and (1.6).

Fix \( \varepsilon > 0 \), small, and \( (y,s) \in P_T \) where \( s > 0 \). By continuity, there is a \( \delta_0 > 0 \) and \( \tau_0 > 0 \), depending on \( y \) and \( s \), such that
\[
h(y,s) - \varepsilon \leq h(x,t) \leq h(y,s) + \varepsilon, \quad \forall (x,t) \in \overline{D}_{\delta_0,2\tau_0}(y,s) \cap P_T.
\]

Recall from (1.3) and (1.4) that \( k = k_1 + k_2 \) and \( \gamma = k_1 + 2k_2 \). Set \( r = |x-y| \) and \( v^\pm(r) = a \pm br^\beta \), where \( b > 0 \) and \( \beta > 0 \). From Remark 2.2
\[
(i) \quad \frac{(b\beta)^k}{r^{\gamma-\beta k}} m(2-\beta) \leq H(Dv^+, D^2v^+) \leq \frac{(b\beta)^k}{r^{\gamma-\beta k}} \mu(2-\beta),
\]
\[
(ii) \quad -\frac{(b\beta)^k}{r^{\gamma-\beta k}} \mu(2-\beta) \leq H(Dv^-, D^2v^-) \leq -\frac{(b\beta)^k}{r^{\gamma-\beta k}} m(2-\beta).
\]

Also, the assumption in (6.2) shows that if \( 2-\beta = \bar{\lambda} \) then \( \beta = 2 - \bar{\lambda} \) and
\[
\mu(2-\beta) = \mu(\bar{\lambda}) < 0, \quad 0 < \beta < 1 \quad \text{and} \quad \gamma - \beta k > 0.
\]

Recall that \( 0 < \theta \leq h \leq M < \infty \), \( \omega = \inf_{[\theta/2,2M]} f(\theta) \) and \( \nu = \sup_{[\theta/2,2M]} f(\theta) \).

Part I: \( k > 1, \quad f > 0 \) is an increasing \( C^1 \) function, \( f^{1/(k-1)} \) concave and \( \Gamma = k \).
**Sub-solutions:** Our idea is to construct a sub-solution \( \eta \) that will be defined in a region \( R \) that lies in \( \bar{D}_{b_0,2\tau_0}(y,s) \) and extended to the rest of \( \Omega_T \) as a sub-solution. Moreover, \( \vartheta/2 \leq \eta \leq M \). Choose

\[
\ell_T = \log \left( \frac{h(y,s) - 2\varepsilon}{\vartheta - 2\varepsilon} \right), \quad \beta = 2 - \lambda, \quad B_0 = \sup_{[0,T]} |\chi(t)|, \quad 0 < \delta^{k_2} \leq \min \left\{ 1, \delta_0^{k_2}, \frac{|\mu(\lambda)|}{2B_0} \right\},
\]

(6.6) \( bL = 1 - e^{-\ell_T} \), and \( b \geq \left( \frac{2\ell\nu}{|\mu(\lambda)||\beta^k(\vartheta/2)^{k-1}|} \right)^{1/k} \).

Choose \( \ell \) large so that \( 0 < \vartheta \leq \tau_0 \). Next, choose \( b \) large so that the lower bound holds and \( \delta \) satisfies the conditions.

Set \( r = |x - y| \). By (6.6), \( 1 - br^{\beta} > 0 \), in \( [0, \delta] \). Define, in \([0, \delta] \times [s - \tau, s + \tau] \),

(6.7) \( \text{the region } R \text{ to be the set: } \exp((\ell_T - \ell)|s - t|)(1 - br^{\beta}) \geq 1, \quad |s - t| \leq \tau, \)

In \( R \), \( br^{\beta} \leq 1 - e^{\ell(|t-s|-\tau)} \) and thus, \( \bar{R} \) lies in the cylinder \( \bar{B}_\delta(y) \times [s - \tau, s + \tau] \). The set \( R \) at the level \( t = s \) is the spatial ball \( B_\delta(y) \) (see (6.6)) and tapers to the points \((y, s \pm \tau)\) as \( |s - t| \to \tau \).

In \( \bar{\Omega}_T \), define the bump function

(6.8) \( \eta(x,y) = \eta(r,t) = \begin{cases} (\vartheta - 2\varepsilon)\exp((\ell_T - \ell)|s - t|)(1 - br^{\beta}), & \forall (x,t) \in R, \\ \vartheta - 2\varepsilon, & \forall (x,t) \in \bar{\Omega}_T \setminus R. \end{cases} \)

From (6.3), (6.5), (6.6), (6.7) and (6.8) we see that

(i) \( \max_{\Omega_T} \eta = \eta(y,s) = h(y,s) - 2\varepsilon \), (ii) \( \eta \geq \vartheta - 2\varepsilon \), in \( \bar{\Omega}_T \), (iii) \( \eta = \vartheta - 2\varepsilon \), in \( \partial R \cap \Omega_T \),

(iv) \( \vartheta - 2\varepsilon \leq \eta \leq h(y,s) - 2\varepsilon \leq h \), in \( R \cap \Gamma_T \), and (v) \( \eta \leq h \), in \( \Gamma_T \).

If we show that \( \eta \) is a sub-solution in \( R \cap \Omega_T \) then by Lemma 2.6 \( \eta \) is a sub-solution in \( \Omega_T \). This together with the above listed observations in (i)-(iv) would imply that \( \eta \) is a sub-solution of (6.1).

Let \((x,t) \in R \cap \Omega_T \). We discuss separately the two cases: (a) \( t \neq s \), and (b) \( t = s \). Recall that in \( 0 < r < \delta, \eta \) is (i) \( C^\infty \) in \( x \), and (ii) in \( t \), for \( t \neq s \).

**Case (a) \( t \neq s \):** Call \( A_5 = (\vartheta - 2\varepsilon)e^{\ell_T - \ell|s-t|} \) and write \( \eta = A_5(1 - br^{\beta}) \). Using (6.4), (6.5), (6.6), (6.8) and \( \gamma - k\beta - k(1 - \beta) = \gamma - k = k_2 \), we get

\[
H(D\eta, D^2\eta) + \chi(t)\left| D\eta \right|^k - f(\eta)^k \eta_k \geq -A_5^k (b\beta)^k \mu(2 - \beta) \frac{r^{\gamma - \beta k}}{r^{\gamma - \beta k}} - A_5^k b_0 (b\beta)^{k-1} (1 - \beta) \frac{f(\eta)}{A_5^{k-1}} - \ell A_5 f(\eta)(1 - br^{\beta})
\]

\[
\geq A_5^k \left( \frac{(b\beta)^k |\mu(\lambda)|}{r^{\gamma - \beta k}} - \frac{b_0 (b\beta)^k}{r^{k(1 - \beta)}} - \frac{\ell f(\eta)}{A_5^{k-1}} \right) = A_5^k \left[ \frac{(b\beta)^k |\mu(\lambda)| \left| - B_0 r^{k_2} \right|}{A_5^{k-1}} \right]
\]

(6.9) \( \geq A_5^k \left( \frac{(b\beta)^k |\mu(\lambda)|}{2\delta^{k_2}} - \frac{\ell \nu}{(\vartheta/2)^{k-1}} \right) \geq 0, \)

where we have used that \( \vartheta/2 \leq A_5 \leq M, \gamma - \beta k > 0 \) and \( \delta^{\gamma - k\beta} > 1 \) (since \( \delta^{k_2} \eta \leq 1 \)). Hence, \( \eta \) is a sub-solution.
Clearly, \( \phi \) listed above, would then imply that \( \eta - \psi \) has a maximum at \( (z, s) \). Then for \( (x, t) \to (z, s) \),

\[
(6.10) \quad \eta(x, t) \leq \eta(z, s) + \psi_t(z, s)(t-s) + (D\psi(z, s), x-z) + \frac{(D^2\psi(z, s)(x-z), x-z)}{2} + o(|t-s| + |x-z|^2).
\]

Since \( r > 0 \), \( \eta \) is \( C^\infty \) in \( x \) Using \( t = s \) in \( (6.10) \) we get \( D\psi(z, s) = D\eta(z, s) \) and \( D^2\psi(z, s) \geq D^2\eta(z, s) \). Using \( (6.8) \), taking \( x = z \) in \( (6.10) \) and \( r = |z - y| \).

\[
\psi_t(z, s)(t-s) + o(|t-s|) \geq (\vartheta - 2\varepsilon)(1 - br^\beta) [\exp(\ell\tau - \ell|t - s|) - \exp(\ell\tau)], \quad \text{as} \ t \to s.
\]

Hence,

\[
|\psi_t(z, s)| \leq \ell(\vartheta - 2\varepsilon)e^{\ell\tau}(1 - br^\beta).
\]

Using the observations made above and arguing as in Case (a) (see \( (6.9) \)), we get

\[
H(D\psi, D^2\psi)(z, s) + \chi(s)|D\psi(z, s)|^k - f(\eta(z, s))\psi_t(z, s) \geq H(D\eta, D^2\eta)(z, s) + \chi(s)|D\eta(z, s)|^k - f(\eta(z, s))\ell(\vartheta - 2\varepsilon)e^{\ell\tau}(1 - br^\beta) \geq 0.
\]

Thus \( \eta \) is a sub-solution in \( R \cap \Omega_T \).

**Super-solutions:** In this part, we construct a super-solution \( \eta \) of \( (6.1) \). Our work is quite similar to the work for the sub-solution. Choose

\[
\ell\tau = \log \left( \frac{M + 2\varepsilon}{h(y, s) + 2\varepsilon} \right), \quad \beta = 2 - \bar{\lambda}, \quad B_0 = \sup_{[0,T]} |\chi(t)|, \quad 0 < \delta^k \leq \min \left\{ 1, \delta_0, \frac{|\mu(\bar{\lambda})|}{2B_0} \right\},
\]

\[
(6.11) \quad b\delta^\beta = e^{e\tau} - 1, \quad \text{and} \quad b \geq \left( \frac{8M\ell\nu}{(\vartheta \beta)^k |\mu(\lambda)|} \right)^{1/k}.
\]

We choose \( \ell > 0 \) and \( b \) so that \( 0 < \tau \leq \tau_0 \) and \( \delta \) small.

The region \( R \) is defined as follows.

\( R \) is the set: \( (1 + br^\beta)\exp(\ell|t - s| - \ell\tau) \leq 1, \quad |s - t| \leq \tau. \)

Clearly, \( br^\beta \leq e^{(\ell|t - s| - \ell\tau)} - 1 \), and thus, \( \bar{R} \subset \overline{B_\delta(y)} \times [s - \tau, s + \tau] \).

Define the **indent function** in \( \Omega_T \) as follows:

\[
(6.12) \quad \phi(x, t) = \begin{cases} (M + 2\varepsilon)(1 + br^\beta)\exp(\ell|t - s| - \ell\tau), & \forall(x, t) \in R, \\ M + 2\varepsilon, & \forall(x, t) \in \overline{\Omega_T} \setminus R. \end{cases}
\]

Using \( (6.11) \) and \( (6.12) \),

\( (i) \quad \min_{\Omega_T} \phi = \phi(y, s) = h(y, s) + 2\varepsilon, \quad (ii) \quad \phi \leq M + 2\varepsilon, \quad \text{in} \ \overline{\Omega_T}, \quad (iii) \quad \phi = M + 2\varepsilon, \quad \text{in} \ \partial R \cap \Omega_T, \quad (iv) \quad h \leq h(y, s) + 2\varepsilon \leq \phi \leq M + 2\varepsilon, \quad \text{in} \ R \cap P_T, \quad \text{and} \quad (iv) \quad \phi \geq h, \quad \text{in} \ P_T. \)

We show that \( \phi \) is a super-solution in \( R \cap \Omega_T \). Lemma 2.5 and the observations (i)-(v), listed above, would then imply that \( \phi \) is a super-solution of \( (6.1) \). We consider the two cases:

(a) \( t \neq s \), and (b) \( t = s \).
We take $x$ and $\phi$ as in (6.14), set $r > A_6$, then as (6.12) to see that $$
abla |\psi_t(z,s) - \phi_t(z,s)| \leq (M + 2\varepsilon)(1 + br^\beta)[\exp(\ell |t-s| - \ell r) - \exp(-\ell r)]$$ as $t \to s$.

Thus, $$|\psi_t(z,s)| \leq \ell (M + 2\varepsilon)(1 + br^\beta)e^{-\ell r}.$$

Since $r > 0$, $\phi$ is $C^2$ in $x$. Hence, (6.14) shows that $D\psi(z,s) = D\phi(z,s)$ and $D^2\psi(z,s) \leq D^2\phi(z,s)$. Using (6.11) and arguing as in (a),

$$H(D\psi, D^2\psi)(z,s) + \chi(s)|D\psi(z,s)|k - f(\phi)\psi_t(z,s) \leq H(D\phi, D^2\phi)(z,s) + \chi(s)|D\phi(z,s)|k + \ell f((\phi(z,s))(M + 2\varepsilon)(1 + br^\beta)e^{-\ell |s-t|/\ell r} \leq 0.$$

Thus, $\phi$ is a super-solution in the interior of $\Omega_T$.

**Part II:** $k \geq 1$, $f(\theta) = 1$, $\forall \theta \in \mathbb{R}$, and any $0 < \Gamma < \gamma$.

As done in Section 5, we provide an outline of the constructions. The value of $b$ in the functions $\eta$ and $\phi$ (see (6.8) and (6.12)) will undergo a slight change. The differential equation reads

$$H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \text{ in } \Omega_T \text{ and } u = h \text{ in } P_T.$$

Set $a = \gamma - k\beta - \Gamma(1 - \beta)$, where $\beta$ is as in (6.5). Then

$$a - \beta(\Gamma - k) = \gamma - k\beta - \Gamma(1 - \beta) - \beta(\Gamma - k) = \gamma - \Gamma.$$

We show Case (a) ($t \neq s$) for both $\eta$ and $\phi$ in Part I. Case (b) ($t = s$) is quite similar to what was done in Part I.
We start with \( \eta \) and use \( (6.6), (6.9) \) and \( (6.15) \), to get

\[
H(D\eta, D^2\eta) + \chi(t)|D\eta|^\Gamma - \eta_t \geq -\frac{A_5^2(b\beta)^k \mu(2 - \beta)}{r^{\gamma - \beta k}} - B_0(A_5 b\beta r^{\beta - 1})^\Gamma - \ell A_5(1 - br^\beta)
\]

\[
\geq A_5^6 \left( \frac{(b\beta)^k |\mu(\bar{\lambda})|}{r^{\gamma - \beta k}} - \frac{\ell}{A_5^{k-1}} \right) - B_0(A_5 b\beta r^{\beta - 1})^\Gamma \geq A_5^6 \left( \frac{(b\beta)^k |\mu(\bar{\lambda})|}{r^{\gamma - \beta k}} - \frac{\ell}{A_5^{k-1}} \right) \geq 0,
\]

where (in the second term \( B_0(A_5 b\beta r^{\beta - 1})^\Gamma \)) we have used that \( b \leq \delta - \beta \leq r - \beta, r^{a - \beta (\gamma - k)} = r^{\gamma - \Gamma}, \delta \) is small and \( b \) is large enough. This verifies that \( \eta \) is a sub-solution.

Next, we use \( (6.11), (6.12), (6.15), (6.13) \), \( e^{\ell r} \leq 2M/\vartheta \) and see that

\[
H(D\phi, D^2\phi) + \chi(t)|D\phi|^\Gamma - \phi_t \leq \frac{A_6^k (b\beta)^k \mu(2 - \beta)}{r^{\gamma - \beta k}} + B_0(A_6 b\beta r^{(\beta - 1)\Gamma}) + A_6(1 + br^\beta)
\]

\[
\leq A_6^6 \left( \frac{\ell(1 + b\delta^{\beta})}{A_6^{k-1}} + B_0(A_6 b\beta r^{(\beta - 1)\Gamma}) - \frac{(b\beta)^k |\mu(\bar{\lambda})|}{r^{\gamma - \beta k}} \right)
\]

\[
\leq A_6^6 \left[ \frac{\ell e^{\ell r}}{A_6^{k-1}} + \frac{(b\beta)^k}{r^{\gamma - \beta k}} \left( B_0(A_6 b\beta r^{(\beta - 1)\Gamma}) - |\mu(\bar{\lambda})| \right) \right]
\]

\[
\leq A_6^6 \left[ \frac{\ell e^{\ell r}}{A_6^{k-1}} + \frac{(b\beta)^k}{r^{\gamma - \beta k}} \left( \frac{2MA_6\beta}{\vartheta} r^{(\gamma - k)\Gamma} - |\mu(\bar{\lambda})| \right) \right] \leq A_6^6 \left( \frac{4M\ell}{\vartheta^{k-1}} - \frac{(b\beta)^k |\mu(\bar{\lambda})|}{2\delta^{\gamma - \beta k}} \right) \leq 0,
\]

by using in the third line \( b = (e^{\ell r} - 1)\delta - \beta \leq (2M/\vartheta)\delta - \beta \leq (2M/\vartheta)r - \beta, r^{a - \beta (\gamma - k)} = r^{\gamma - \Gamma} \), taking \( \delta \) small enough and then \( b \) large enough.

**Remark 6.1.** The discussion above shows the existence of positive solutions of

\[
H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \quad \text{in } \Omega_T \text{ and } u = h \text{ in } P_T,
\]

where \( h > 0 \). For a general \( h \), define \( \hat{h} = h + 2\vartheta \). Then \( \hat{h} > 0 \) and the above has a positive solution \( \hat{u} \). Thus, \( u = \hat{u} - 2\vartheta \) solves the required differential equation. \( \square \).

### 7. Side Boundary: The case (1.8) (ii). Construction for Theorem 1.3

In this section we assume that (1.8) (ii) holds, that is,

\[
\text{(7.1)} \quad \text{there is a smallest } \bar{\lambda} \geq 2 \text{ such that } \mu(\lambda) < 0, \forall \lambda > \bar{\lambda}.
\]

Also, recall (1.5), (1.6) and (1.7). In addition, we impose that \( \Omega \) satisfy a uniform outer ball condition. More precisely: there is a \( \rho_0 > 0 \) such that, for each \( y \in \partial \Omega \), if \( 0 < \rho \leq \rho_0 \) then there is a \( z \in \mathbb{R}^n \setminus \Omega \) such that the ball \( B_\rho(z) \subset \mathbb{R}^n \setminus \Omega \) and \( y \in \partial B_\rho(z) \cap \partial \Omega \).

Our goal is to construct sub-solutions \( \eta \) and super-solutions \( \phi \) of

\[
\text{(7.2)} \quad H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = 0, \quad \text{in } \Omega_T \text{ and } u = h, \text{ in } P_T.
\]
Let \((y, s) \in P_T\) where \(s > 0\). There is a \(\delta_0 > 0\) and \(\tau_0 > 0\), small, depending on \(y\) and \(s\), such that

\[
(7.3) \quad h(y, s) - \varepsilon \leq h(x, t) \leq h(y, s) + \varepsilon, \quad \forall (x, t) \in \overline{D}_{\delta_0, 2\tau_0}(y, s) \cap P_T.
\]

Recall that \(\vartheta = \inf_{P_T} h\), \(M = \sup_{P_T} h\), and assume that \(0 < \vartheta \leq M < \infty\). Fix \(\varepsilon > 0\), small, such that \(\vartheta - 2\varepsilon > 0\).

As done in Section 6, we recall Remark 2.2: let \(b > 0\), \(\beta > 0\) and \(v^\pm(r) = a \pm br^{-\beta}\). Then

\[
(i) \quad \frac{(b\beta)^k}{r^{\beta k + \gamma}} \mu(\beta + 2) \leq H(Dv^+, D^2v^+) \leq -\frac{(b\beta)^k}{r^{\beta k + \gamma}} m(\beta + 2),
\]

\[
(7.4) \quad (ii) \quad \frac{(b\beta)^k}{r^{\beta k + \gamma}} m(\beta + 2) \leq H(Dv^-, D^2v^-) \leq \frac{(b\beta)^k}{r^{\beta k + \gamma}} \mu(\beta + 2).
\]

Recall (7.1) and set \(\lambda = \beta + 2 > \bar{\lambda}\). Then \(\beta > \bar{\lambda} - 2\) and

\[
(7.5) \quad \mu(\beta + 2) = \mu(\lambda) < 0, \quad \beta > 0 \text{ and } \beta k + \gamma > 0.
\]

Recall that \(\omega = \inf_{[\vartheta/2, 2M]} f(\theta)\) and \(\nu = \sup_{[\vartheta/2, 2M]} f(\theta)\).

**Part I: \(k > 1\), \(f\) is a \(C^1\) increasing function, \(f^{1/(k-1)}\) is concave and \(\Gamma = k\).**

**Sub-solutions:** By our hypothesis, let \(z \in \mathbb{R}^n \setminus \Omega\) and \(0 < \rho\) be such that \(B_\rho(z) \subset \mathbb{R}^n \setminus \Omega\) and \(y \in \partial B_\rho(z) \cap \partial \Omega\). Set \(r = |x - z|\); the region \(R\) will be in the cylindrical shell \((\overline{B}_2\rho(z) \setminus B_\rho(z)) \times [s - \tau, s + \tau]\), where \(\rho\) and \(\tau\) will be determined below. We require that this shell be in \(D_{\delta_0, 2\tau_0}(y, s)\) and this is achieved if \(4\rho \leq \delta_0\).

Set \(B_0 = \sup_{[0, T]} |\chi(t)|\) and choose

\[
(7.6) \quad \ell \tau = \log \left(\frac{h(y, s) - 2\varepsilon}{\vartheta - 2\varepsilon}\right), \quad \beta > \bar{\lambda} - 2 \quad \text{and} \quad 0 < \rho \leq \min \left\{ \frac{\delta_0}{4}, \frac{1}{2} \left( \frac{\mu(\lambda)}{2B_0} \right)^{1/(\gamma - k)} \right\},
\]

where \(\lambda = \beta + 2\). We choose \(\ell\), large, so that \(0 < \tau \leq \tau_0\). A value of \(\rho\) will be chosen later.

We define the region \(R\) as follows: for \(\rho \leq r \leq 2\rho\) and \(|s - t| \leq \tau\), let

\[
(7.7) \quad R \text{ is the region: } \exp(\ell \tau - \ell |s - t|) \left[1 - \left(\frac{1 - e^{-\ell \tau}}{1 - 2^{-\beta}}\right) \left(1 - \frac{\rho^\beta}{\tau^\beta}\right)\right] \geq 1.
\]

At \(t = s\) the region \(R\) is the spatial annulus \(\rho \leq r \leq 2\rho\), it tapers as \(|t - s| \to \tau\) and at \(|s - t| = \tau\) we get \(r = \rho\). Also, \(\overline{R} \subset (\overline{B}_{2\rho}(z) \setminus B_\rho(z)) \times [s - \tau, s + \tau]\).

Define the following *bump function* in \(\Omega_T:

\[
(7.8) \quad \eta(x, t) = \eta(r, t) = \begin{cases} (\vartheta - 2\varepsilon) \exp(\ell \tau - \ell |s - t|) \left[1 - \left(\frac{1 - e^{-\ell \tau}}{1 - 2^{-\beta}}\right) \left(1 - \frac{\rho^\beta}{\tau^\beta}\right)\right], & \text{in } R, \\ \vartheta - 2\varepsilon, & \text{in } \overline{\Omega_T} \setminus R. \end{cases}
\]

Note that \(\vartheta/2 \leq \eta \leq M\). Using (7.3), (7.6), (7.7) and (7.8), we get

\[
(i) \quad \eta(y, s) = \sup \eta = h(y, s) - 2\varepsilon, \quad (ii) \quad \eta \geq \vartheta - 2\varepsilon, \quad \text{in } \overline{\Omega_T}, \quad (iii) \quad \eta \leq h, \quad \text{in } P_T, \quad \text{and} \quad (iv) \quad \vartheta - 2\varepsilon \leq \eta \leq h(y, s) - 2\varepsilon \leq h, \quad \text{in } R \cap P_T.
\]


Clearly, if \( \eta \) is a sub-solution in \( \Omega_T \), the observations (7.3) (i)-(iv), listed above, would then imply that \( \eta \) is a sub-solution of (7.2). We first show that \( \eta \) is a sub-solution in \( R \cap \Omega_T \). We consider: (a) \( t \neq s \), and (b) \( t = s \). Lemma 2.6 then shows \( \eta \) is a sub-solution in \( \Omega_T \).

(a) \( t \neq s \): Set \( \hat{A}_0 = (\vartheta - 2 \varepsilon)e^{-\ell(t-s-t)} \) and \( \hat{C}_0 = (1 - e^{-\ell}) \frac{1}{2 - \beta}^{-1} \). Note \( \eta \) is \( C^\infty \) (in \( x \)) in \( R \cap \Omega_T \) and \( \eta \leq A_0 \). Using (7.4) (i), (7.5), (7.8) and bounding the spatial part of \( \eta \) from above by 1, we get in \( \rho \leq r \leq 2 \rho \), \( 0 < |s-t| \leq \tau \),

\[
 H(D\eta, D^2\eta) + \chi(t)|D\eta|^k - f(\eta)|\eta| \geq \frac{\hat{A}_0 \hat{C}_0^k (\beta \rho^\beta |\mu(2 + \beta)|)}{r^{k+\gamma}} - B_0(\hat{A}_0 \hat{C}_0\beta \rho^\beta)^k - \nu \ell \hat{A}_0 \\
 \geq (\hat{A}_0 \hat{C}_0)^k \left( \frac{(\beta \rho^\beta |\mu(\lambda)|)}{r^{k+\gamma}} - B_0(\beta \rho^\beta)^k - \nu \ell \hat{A}_0 \right) \\
 = (\hat{A}_0 \hat{C}_0)^k \left( \frac{(\beta \rho^\beta |\mu(\lambda)|)}{2(\rho)^{k+\gamma}} - \nu \ell \hat{A}_0 \right) \geq (\hat{A}_0 \hat{C}_0)^k \left( \frac{\beta |\mu(\lambda)|}{2^{k+\gamma+1} \rho} - \frac{2^{k-1} \nu \ell \hat{A}_0}{\hat{C}_0^{k+\gamma-1}} \right) \geq 0,
\]

where \( \hat{A}_0 \geq \vartheta/2 \) and \( \rho \) is chosen small enough. Thus, \( \eta \) is sub-solution in \( R \cap \Omega_T \).

Part (b) and the rest of the proof is similar to that in Part I of Section 6.

Super-solutions: We now construct a super-solution \( \phi > 0 \) to (7.2). The ideas are similar to those in Part I and we make use of (7.4) (i). The ball \( B_\rho(z) \) is the outer ball at \( y \in \partial \Omega \), see the discussion for sub-solutions.

Take \( \lambda > \hat{\lambda} \). Set

\[
 (7.11) \quad \beta = \lambda - 2, \quad \ell \tau = \log \left( \frac{M + 2\varepsilon}{h(y, s) + 2\varepsilon} \right), \quad 0 < \rho \leq \delta_0/4.
\]

Select \( \ell \), large, so that \( 0 < \tau \leq \tau_0 \). A more precise (and smaller) value of \( \rho \) is chosen later.

Define \( r = |x - z| \). Let \( R \) be the region in \( \rho \leq r \leq 2 \rho \), \( |s-t| \leq \tau \), defined as follows.

\[
 (7.12) \quad R \text{ is the region: } \exp(\ell |s-t| - \ell \tau) \left[ 1 + \left( \frac{e^{\ell \tau} - 1}{1 - 2^{-\beta}} \right) \left( 1 - \left( \frac{\rho}{r} \right)^\beta \right) \right] \leq 1.
\]

Note that if \( t = s \) then the spatial annulus \( \rho \leq r \leq 2 \rho \) is in \( R \). The region tapers as \( |s-t| \to \tau \) and at \( |s-t| = \tau \) we have \( r = \rho \).

In \( \overline{\Omega}_T \), define the indent function

\[
 (7.13) \quad \phi(x, t) = \begin{cases} 
 (M + 2\varepsilon) \exp(\ell |s-t| - \ell \tau) \left[ 1 + \left( \frac{e^{\ell \tau} - 1}{1 - 2^{-\beta}} \right) \left( 1 - \left( \frac{\rho}{r} \right)^\beta \right) \right], & \forall (x, t) \in R \\
 M + 2\varepsilon, & \forall (x, t) \in \overline{\Omega}_T \setminus R.
\end{cases}
\]

Using (7.3), (7.11) and (7.12) we see that

(i) \( \phi(y, s) = \inf \phi = h(y, s) + 2\varepsilon \), (ii) \( \phi \leq M + 2\varepsilon \), in \( \overline{\Omega}_T \), (iii) \( \phi \geq h \), in \( P_T \),

(iv) \( h \leq h(y, s) + 2\varepsilon \leq \phi \leq 2M \), in \( R \cap P_T \).
Note that $\theta/2 \leq \phi \leq 2M$. We now show that $\phi$ is a super-solution in $R \cap \Omega_T$. We consider:

(a) $t \neq s$, and (b) $t = s$. Lemma 2.7 will then show that $\eta$ is a super-solution in $\Omega_T$.

(a) $t \neq s$: Set $A_1 = (M + 2\varepsilon)e^{(s-t)|-\nu^f}$, $\tilde{C}_1 = (e^{\nu^f} - 1)(1 - 2^{-\beta})^{-1}$ and $B_0 = \sup_{[0,T]} |\chi(t)|^2$. Using (7.3)(ii), (7.11), (7.13) and bounding the spatial part of $0$ where we have used $\tilde{\Gamma}$, we get

$$H(D\phi, D^2\phi) + \chi(t)|D\phi|^\gamma - f(\phi) \phi t \leq \frac{(\tilde{A}_1 \tilde{C}_1)^k (\beta \rho^3)^k}{\nu^k(1+\gamma)} \mu(\beta + 2) + \frac{(\tilde{A}_1 \tilde{C}_1)^k B_0 (\beta \rho^3)^k}{\nu^k(1+\gamma)} + 2\nu \ell \tilde{A}_1 \leq \left(\tilde{A}_1 \tilde{C}_1\right)^k \left(\frac{2\nu \ell \epsilon^f}{\tilde{A}_1 \tilde{C}_1} + \frac{B_0 (\beta \rho^3)^k}{\nu^k(1+\gamma)} \mu(\lambda)\right)$$

(7.14)

$$\leq \left(\tilde{A}_1 \tilde{C}_1\right)^k \left(\frac{2\nu \ell \epsilon^f}{\tilde{C}_1^k(\theta/2)^k-1} - \frac{B_0 (\beta \rho^3)^k}{2(2\rho)^{k(1+\gamma)}} \mu(\lambda)\right) = \left(\tilde{A}_1 \tilde{C}_1\right)^k \left(\frac{2\nu \ell \epsilon^f}{\tilde{C}_1^k(\theta/2)^k-1} - \frac{B_0 |\mu(\lambda)|}{2(2\rho)^{k(1+\gamma)+1}}\right) \leq 0$$

where we have used $\tilde{A}_1 \geq \theta/2$ and $\rho$ is small. We see that $\phi$ is a super-solution.

The proof of Part (b) and the rest of the proof is similar to that in Part I.

**Part II:** $k \geq 1$, $f(\theta) = 1$, $\forall \theta \in R$, and any $0 < \Gamma < \gamma$.

Our discussion is similar to Part II in Section 6. We will verify that $\eta$ and $\phi$ as in (7.8) and (7.13), with slight modifications, continue to be sub-solutions and super-solutions. The differential equation reads

$$H(Du, D^2u) + \chi(t)|Du|^\Gamma - u_t = 0, \text{ in } \Omega_T, \text{ and } u = h \text{ in } P_T.$$}

We compute with $\eta$, see (7.8) and (7.10). The definitions of $\tilde{A}_0$, $B_0$, and $\tilde{C}_0$ continue to be the same.

$$H(D\eta, D^2\eta) + \chi(t)|D\eta|^\Gamma - \eta_t \geq \frac{(\tilde{A}_0 \tilde{C}_0)^k (\beta \rho^3)^k |\mu(\beta + 2)|}{\nu^k(1+\gamma)} - \frac{B_0 (\tilde{A}_0 \tilde{C}_0)^k (\beta \rho^3)^k}{\nu^k(1+\gamma)}$$

$$\geq \left(\tilde{A}_0 \tilde{C}_0\right)^k \left(\frac{(\beta \rho^3)^k |\mu(\lambda)|}{\nu^k(1+\gamma)} - \frac{B_0 (\tilde{A}_0 \tilde{C}_0)^k (\beta \rho^3)^k}{\nu^k(1+\gamma)}\right) \geq \left(\tilde{A}_0 \tilde{C}_0\right)^k \left(\frac{(\beta \rho^3)^k |\mu(\lambda)|}{\nu^k(1+\gamma)} - \frac{\ell}{\tilde{A}_0^{k-1} \tilde{C}_0}\right)$$

$$\geq \left(\tilde{A}_0 \tilde{C}_0\right)^k \left(\frac{(\beta \rho^3)^k |\mu(\lambda)|}{\nu^k(1+\gamma)} - \frac{\ell}{\tilde{A}_0^{k-1} \tilde{C}_0}\right) \geq \left(\tilde{A}_0 \tilde{C}_0\right)^k \left(\frac{\beta^k |\mu(\lambda)|}{\nu^k(1+\gamma)} - \frac{2k \ell}{\tilde{C}_0^{k-1} \tilde{C}_0}\right) \geq 0,$$

where we have used $\Gamma < \gamma$, $\rho \leq r \leq 2\rho$ and $\rho$ is chosen small. The rest is as in Part I.
Next, we calculate using $\phi$, see (7.13) and (7.14). The definitions of $\hat{A}_1$, $B_0$ and $\hat{C}_1$ continue to be the same. In what follows $\rho \leq r \leq 2\rho$ and $\rho$ is small.

\[
H(D\phi, D^2\phi) + \chi(t)|D\phi|^\Gamma - \phi_t \leq \frac{(\hat{A}_1\hat{C}_1)^k (\beta\rho^3) \mu}{r^{\beta k + \gamma}}(\beta + 2) + \frac{B_0(\hat{A}_1\hat{C}_1\beta\rho^3)^\Gamma}{r^{\Gamma(1 + \beta)}} + 2\ell\hat{A}_1\ell^\Gamma \\
\leq (\hat{A}_1\hat{C}_1)^k \left( \frac{2\ell^\Gamma}{\hat{A}_1\hat{C}_1} + \frac{B_0(\hat{A}_1\hat{C}_1)^{-k}(\beta\rho^3)^\Gamma}{r^{\Gamma(1 + \beta)}} - \frac{(\beta\rho^3)^k}{r^{\beta k + \gamma} |\mu(\lambda)|} \right) \\
= (\hat{A}_1\hat{C}_1)^k \left[ \frac{2\ell^\Gamma}{\hat{A}_1\hat{C}_1} + \frac{(\beta\rho^3)^k}{r^{\beta k + \gamma}} + \left( B_0(\hat{A}_1\hat{C}_1\beta)^{-k}u\gamma - \Gamma \left( \frac{2}{r} \right)^{\beta(\Gamma-k)} - |\mu(\lambda)| \right) \right] \\
\leq (\hat{A}_1\hat{C}_1)^k \left( \frac{2\ell^\Gamma}{\hat{A}_1\hat{C}_1} - \frac{(\beta\rho^3)^k |\mu(\lambda)|}{2} \right) \leq (\hat{A}_1\hat{C}_1)^k \left( \frac{2\ell^\Gamma}{\hat{C}_1^k(\vartheta/2)^{k-1}} - \frac{B_0(\beta\rho^3)^k |\mu(\lambda)|}{2(2\rho)^{\beta k + \gamma}} \right) \\
= (\hat{A}_1\hat{C}_1)^k \left( \frac{2\ell^\Gamma}{\hat{C}_1^k(\vartheta/2)^{k-1}} - \frac{B_0|\mu(\lambda)|}{2\beta k + \gamma + 1} \right) \leq 0.
\]

The rest of the proof is as in Part I. Now apply Remark 6.1 to get the general statement.

8. Appendix

We discuss a maximum principle that applies to the case where $f$ is a positive continuous function. No sign conditions are imposed on the sub-solutions and super-solutions.

Recall Conditions A and B, (1.2)-(1.6). From (1.5), we have

\[
m_{\min}(\lambda) = \min_{|\epsilon|=1} H(e, I - \lambda\epsilon \otimes e), \quad \mu_{\max} = \max_{|\epsilon|=1} H(e, \lambda\epsilon \otimes e - I)
\]

and $m(\lambda) = \min(m_{\min}(\lambda), -\mu_{\max}(\lambda))$. In place of Condition C, we assume that

\[(8.1) \quad m(0) > 0 \text{ and } \lim_{\lambda \to -\infty} m(\lambda) = \infty.\]

Recall the notation, $\hat{H}(p, X) = -H(p, -X), \forall (p, X) \in \mathbb{R}^n \times S^n$, see Remark 1.9

**Lemma 8.1. (Weak Maximum Principle)** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain and $T > 0$. Suppose that $H$ satisfies Conditions A, B and (8.7). Suppose that $\chi : [0, T] \to \mathbb{R}$ and $f : \mathbb{R} \to [0, \infty)$ and $f \not\equiv 0$, are continuous functions.

Let $\Gamma > 0$ and $\phi \in \text{usc}(\text{usc}(\text{usc}))^{(\Omega_T \cup P_T)}$ solve

\[
H(D\phi, D^2\phi) + \chi(t)|D\phi|^\Gamma - f(\phi)\phi_t \geq (\leq) 0, \text{ in } \Omega_T.
\]

(a) If $\Gamma \geq k$ then $\sup_{\Omega_T} \phi \leq \sup_{P_T} \phi = \sup_{\Omega_T \cup P_T} \phi (\inf_{\Omega_T} \phi \geq \inf_{P_T} \phi = \inf_{\Omega_T \cup P_T} \phi)$.

(b) If $0 < \Gamma < k$ and $\inf f > 0$ then the conclusion in (a) holds.

(c) If $\chi \equiv 0$ then the conclusion in (a) holds even if $\inf f = 0$.

**Proof.** Let $0 < \tilde{\tau} < \tau < T$, $\Omega_{\tilde{\tau}, \tau} = \Omega \times [\tilde{\tau}, \tau]$ and $P$ the parabolic boundary of $\Omega_{\tilde{\tau}, \tau}$. Our goal is to prove the weak maximum principle in $\Omega_{\tilde{\tau}, \tau}$ for any $0 < \tilde{\tau} < \tau < T$ and then extend it to $\Omega_T$. Note that $u$ is bounded from above in $\overline{\Omega_{\tilde{\tau}, \tau}}$ since $u \in \text{usc}(\Omega_T \cup P_T)$. 
Choose $z \in \mathbb{R}^n \setminus \Omega$ and $R > 0$ such that $\Omega \subset B_R(z) \setminus B_{R/2}(z)$. Call $r = |x - z|$; clearly, $R/2 \leq r \leq R, \forall x \in \Omega$.

Set
\[
\vartheta = \sup_{\Omega_{\tau, \tau}} \phi, \quad \ell = \sup_P \phi, \quad \delta = \vartheta - \ell, \quad c = \sup_{\Omega} \phi(x, \tau), \quad \eta = \max(\delta, \ c - \ell)
\]
(8.2) and $\nu = \max(c, \vartheta, \ell)$.

We recall from Remark 2.2 (ii) and (6.4)(ii) that if $v = a - br^\beta$, where $b > 0$ and $\beta > 0$, then
\[
- \frac{(b\beta)^k}{r^{\gamma - \beta k}} \mu(2 - \beta) \leq H(Dv, D^2v) \leq - \frac{(b\beta)^k}{r^{\gamma - \beta k}} m(2 - \beta).
\]

We argue by contradiction and assume that $\delta > 0$. Since $\Omega_{\tau, \tau}$ is an open set there is a point $(\xi, \theta) \in \Omega_{\tau, \tau}$ such that $\phi(\xi, \theta) > \ell + 3\delta/4$ and $0 < \hat{\tau} < \theta < \tau$. Define
\[
g(t) = 0, \quad \forall t \in [\hat{\tau}, \theta] \quad \text{and} \quad g(t) = (t - \theta)^4/(\tau - \theta)^4, \quad \forall t \in [\theta, \tau].
\]

Select $0 < \varepsilon \leq \min(0.5, \delta/4)$. For $\beta > 0$, set
\[
\psi(x, t) = \psi(r, t) = \ell + \frac{\varepsilon}{4} + \eta g(t) - \frac{\varepsilon \beta}{32 \beta} r^\beta, \quad \forall(x, t) \in \overline{\Omega}_{\tau, \tau}.
\]

Thus, $\psi(x, t) \geq \ell + \varepsilon/8, \forall(x, t) \in \overline{\Omega}_{\tau, \tau}$, and $\psi(x, \tau) \geq \ell + \eta + \varepsilon/8 \geq c + \varepsilon/8, \forall x \in \overline{\Omega}$. Moreover,
\[
\phi(\xi, \theta) - \psi(\xi, \theta) \geq \ell + \frac{3\delta}{4} - \ell - \frac{\varepsilon}{4} = \frac{3\delta}{4} - \frac{\varepsilon}{4} \geq \frac{\delta}{4} > 0.
\]

Since $\phi - \psi \leq 0$ on $\partial \overline{\Omega}_{\tau, \tau}$ and $(\phi - \psi)(\xi, \theta) > 0$, the function $\phi - \psi$ has a positive maximum at some point $(y, s) \in \Omega_{\tau, \tau}$.

Set $B_0 = \sup_{[0, \tau]} |\chi(t)|$, call $\rho = |y - z|$ and use (8.3) to get
\[
H(D\psi(y, s), D^2\psi(y, s)) + \chi(s)D\psi(y, s)^\Gamma \leq - \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^k \frac{m(2 - \beta)}{r^{\gamma - \beta k}} + B_0 \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^\Gamma \rho^{(\beta - 1)\Gamma}
\]
\[
= \rho^{\beta k - \gamma - 1} \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^k \left[ B_0 \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^\Gamma \rho^{\gamma - \Gamma + \beta(\Gamma - k) - m(2 - \beta)} \right]
\]
(8.5)
\[
= \rho^{\beta k - \gamma - 1} \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^k \left[ B_0 \left( \frac{\varepsilon \beta}{32 \beta^\gamma} \right)^\Gamma \rho^{\gamma - \Gamma - m(2 - \beta)} \right]
\]

Call $I$ the right hand side of the third line in (8.5) and note that $1/2 \leq \rho/R \leq 1$. We now show part (a) of the lemma. Note that $\psi_1(y, s) \geq 0$.

(i) If $\Gamma > k$ then taking $\beta = 2$ (see (8.1)) and $\varepsilon$ small enough we can make $I < 0$. We conclude from (8.5) that $I < 0 \leq f(\phi(y, s))\psi_1(y, s)$ implying that the lemma holds for $0 < \hat{\tau} < \tau < T$.

(ii) If $\Gamma = k$ then $\gamma - k = k_2$ (see (1.3) and (1.4)). Taking $\beta$ large and using (8.1) we can make $I < 0$. We conclude from (8.5) that $I < 0 \leq f(\phi(y, s))\psi_1(y, s)$ implying that the lemma holds for $0 < \hat{\tau} < \tau < T$.

Taking $B_0 = 0$ and arguing as above we get part (c) of the lemma.
To see part (b), set $\omega = \inf f$ and modify $g(t) = (t/\tau)^\alpha$, where $\alpha$ is large so that $\eta g(\theta) \leq \varepsilon/8$. Since $\varepsilon \leq \delta/4$, this ensures that in (8.1)

$$
\phi(\xi, \theta) - \psi(\xi, \theta) \geq \ell + \frac{3\delta}{4} - (\ell + \frac{\varepsilon}{4} + \eta g(\theta)) \geq \frac{3\delta}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{8} \geq \frac{\delta}{4} > 0.
$$

Using (8.5) estimate $I$ (disregard the second term in the parenthesis) as

$$
I \leq A \left( \frac{\varepsilon \beta}{32 R^\beta} \right)^\Gamma \rho^{(\beta-1)\Gamma} = A \left( \frac{\varepsilon \beta R^\beta}{32 R^\beta} \right)^\Gamma \leq 2^\Gamma A \left( \frac{\beta}{32} \right)^\Gamma \varepsilon R^\Gamma.
$$

Next, $\psi_t(y,s) = \alpha \eta s^{\alpha-1}/s^{\alpha} \geq \alpha \eta \tau^{\alpha-1}/\tau^{\alpha}$ implying that

$$
I - f(\hat{\phi}(y,s))\psi_t(y,s) \leq 2^\Gamma A \left( \frac{\beta}{32} \right)^\Gamma \varepsilon R^\Gamma - \alpha \omega (\tau^{\alpha-1}/\tau^{\alpha}) < 0,
$$

if $R$ is chosen large enough. Using (8.5), we get a contradiction and $\phi \leq \ell$ in $\Omega_{\tau, \tau}$.

If $\sup_{\Omega_{\tau}} \phi > \sup_{\Omega_{\tau}} \phi$ then there is a point $(y,s) \in \Omega_{\tau}$ (with $0 < s < T$) such that $\phi(y,s) > \sup_{\Omega_{\tau}} \phi$. Select $0 < \bar{s} < s < \bar{\tau}$ and call $P$ the parabolic boundary of $\Omega_{\bar{s}, \bar{\tau}}$. Then, $\sup_{P_{\tau}} \phi < \phi(y,s) \leq \sup_{\Omega_{\bar{s}, \bar{\tau}}} \phi \leq \sup P \phi \leq \sup_{P_{\tau}} \phi$. This is a contradiction and the lemma holds.

To show the weak minimum principle, take $v = -\phi$ and conclude that $H(-Dv, -D^2v) \leq f(-v)/(-v_t)$. If $\hat{f}(v) = f(-v)$ then (1.3) shows that $\hat{H}(Dv, D^2v) \geq \hat{f}(v)v_t$. As noted in Remark [9], $\hat{H}$ satisfies Conditions A, B and (8.1) and the minimum principle follows. \(\square\)

**Remark 8.2.** Suppose that $u$ solves $H(Du, D^2u) + \chi(t)|Du|^\Gamma - f(u)u_t = g(x,t)$, where $L = \sup_{\Omega_{\tau}} |g| < \infty$ and $\omega = \inf_{\Omega_{\tau}} f > 0$. Using $u + \ell t, \ell \geq L/\omega$ large, one gets $\inf_{P_{\tau}} (u + \ell t) - \ell t \leq u \leq \sup_{P_{\tau}} (u - \ell t) + \ell t$.

**References**

[1] T. Bhattacharya and L. Marazzi, *On the viscosity solutions to a degenerate parabolic differential equation*, Annali di Matematica Pura ed Applicata, vol 194, no 5, 2014, DOI:10.1007/s10231-014-0427-1

[2] T. Bhattacharya and L. Marazzi, *On the viscosity solutions to Trudinger’s equation*. Nonlinear Differential Equations and Applications (NoDEA), vol 22, no 5, 2015, DOI:10.1007/s00030-015-0315-4

[3] T. Bhattacharya and L. Marazzi, *Asymptotics of viscosity solutions to some doubly nonlinear parabolic equations*, vol 16. issue 4. J. Evol. Eqns. Dec 2016. DOI:10.1007/s00028-015-0319-x

[4] M. G. Crandall, H. Ishii and P. L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27(1992) 1-67.

[5] F. Demengel, *Existence results for parabolic problems related to fully non linear operators degenerate or singular*, Pot. Anal. 35(2011), no 1, 1-38.

[6] DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer (1993)

[7] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998 2nd Edition, Classics in Mathematics.

[8] B. Kawohl and P. Juutinen, *On the evolution equation governed by the Infinity-Laplacian*, Math. Ann. 335 (2006) no 4, 819-851.

[9] N. S. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math., 21(1968), 205-226
