Controlled Quantum Open Systems

Robert Alicki

Institute of Theoretical Physics and Astrophysics, University of Gdańsk, Wita Stwosza 57, PL 80-952 Gdańsk, Poland

(January 13, 2022)

The theory of controlled quantum open systems describes quantum systems interacting with quantum environments and influenced by external forces varying according to given algorithms. It is aimed, for instance, to model quantum devices which can find applications in the future technology based on quantum information processing. One of the main problems making difficult the practical implementations of quantum information theory is the fragility of quantum states under external perturbations. The aim of this note is to present the relevant results concerning ergodic properties of open quantum systems which are useful for the optimization of quantum devices and noise (errors) reduction. In particular we present mathematical characterization of the so-called "decoherence-free subspaces" for discrete and continuous-time quantum dynamical semigroups in terms of $C^*$-algebras and group representations. We analyze the non-Markovian models also, presenting the formulas for errors in the Born approximation. The obtained results are used to discuss the proposed different strategies of error reduction.

I. INTRODUCTION

There exists a widespread believe that sooner or later the technology of the future will employ typical quantum features of microscopic or mesoscopic systems like entanglement or superpositions of quantum states. The most prominent examples are the rapid development of the quantum information and computation theory [1,2] and the corresponding progress in experimental techniques. The later includes realization of quantum teleportation [3], monitoring of dissipation and decoherence for mesoscopic quantum systems [4], construction of quantum gates [5], etc.

The theory and experiment show that entanglement of separated systems and superpositions of well distinguishable states are very sensitive to perturbations by environment. The interaction with environment introduces noise (errors) which should be taken into account by designing any quantum devices and their operations. The appropriate theory has been developed in the 70-ties and 80-ties as the Quantum Theory of Open Systems (see [6-11] for the survey of this theory). The basic idea of this theory is the decomposition of the Universe into the open system $S$ and the environment (reservoir , heat bath) $R$ with their corresponding Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively. Assuming that the interaction between $S$ and $R$ is weak we can always write, within a reasonable approximation, that for the initial moment $t_0 \equiv 0$ the initial state of the total system is given by the product formula

$$\rho_{SR}(0) = \rho \otimes \omega_R$$

where $\rho$ is an arbitrarily prepared initial state of $S$ and $\omega_R$ is a stable reference state of $R$. Our interest is concentrated on the time evolution of $S$ for times $t \geq 0$ given by the reduced density matrix (we omit subscript $S$)

$$\rho_t = \text{Tr}_R(U_t \rho \otimes \omega_R U_t^*) = \Lambda_t(\rho)$$

where $\{U_t; t \in \mathbb{R}\}$ is a family of unitary operators acting on $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$ and describing the reversible (Hamiltonian) evolution of the total system. $\text{Tr}_R$ is a partial trace over $\mathcal{H}_R$. The dynamical maps $\{\Lambda_t; t \geq 0\}$, their properties and different approximation schemes for them (e.g. Markovian approximations) are the main topics of the Quantum Theory of Open Systems.

In order to deal with the problems posed by quantum information processing and quantum computing one has to extend the standard approach introducing, beside the open system $S$ and the reservoir $R$, the third element – the controller $C$. The role of $C$ is to prepare an initial state, to measure a final one and to control, as far as it is possible, the time evolution of $S$ by means of the time-dependent Hamiltonian $H_S(t)$. The presence of the time-dependent Hamiltonian makes, for example, the application of Markovian approximation often impossible and demands new mathematical techniques. Moreover, the physical structure of $C$, the resources used for state preparation and measurements must be taken into account in the estimation of efficiency of quantum devices. We shall call such systems Controlled Quantum Open Systems (CQOP).

One of the main problems in the theory of CQOP’s is to reduce errors due to the interaction of a quantum device with an environment. At least three elements of the presented scheme can be optimized: the choice of the subspace in $\mathcal{H}_S$ which supports the initial states of $S$, the trajectory $\{H_S(s); 0 \leq s \leq t\}$ which leads to a given unitary operation
on $\mathcal{H}_S$ and the structure of the coupling of $S$ to $R$. In the recent literature one can find a number of proposals of such optimization schemes [12-14]. However, some of these ideas are essentially based on the independent derivations of the earlier results, some other employ oversimplified phenomenological models which should be scrutinized from the point of view of a more fundamental theory. Therefore, it seems to be useful to collect, improve and generalize the relevant results concerning mainly the ergodic properties of quantum open systems and discuss their consequences for the optimization schemes mentioned above.

In the Section 2 we present mathematical characterization of the subalgebras of observables which evolve according to reversible dynamics for a given irreversible quantum dynamical map or quantum dynamical semigroup. Such subalgebras correspond to "decoherence-free subspaces" in the language of modern quantum information theory. The general ideas are illustrated by examples of collective dissipation models in Section 3. Section 4 is devoted to the derivation and discussion of the formula describing errors in CQOP in the non-Markovian weak coupling Born approximation.

II. ERGODIC PROPERTIES OF QUANTUM DYNAMICAL MAPS AND SEMIGROUPS

In this Section we restrict ourselves in most cases to quantum open systems with finite-dimensional Hilbert space $\mathcal{H}_S = \mathbb{C}^n$. We discuss the irreversible dynamics either in terms of the single dynamical map $\Gamma$ and its iterations $\Gamma^k$, $k = 0, 1, 2, ...$ or the dynamical semigroup $\{T_t; t \geq 0\}$ with the composition law

$$T_t T_s = T_{t+s}, \ t, s \geq 0, \ T_0 = 1. \quad (3)$$

Both $\Gamma$ and $T_t$ describe the dynamics in Heisenberg picture i.e. they act on a matrix algebra ($C^*$-algebra) $\mathcal{M}_n$ representing quantum observables of the system. In the following all results independent on the semigroup property (3) are formulated in terms of a single dynamical map $\Gamma$. The Schrödinger picture evolution is given by the dual map $\Gamma^*$ acting on $\mathcal{M}_n$ which is treated now as the smallest complex linear space containing density matrices (quantum states) of the system and satisfying

$$\text{Tr}(A \Gamma(B)) = \text{Tr}(\Gamma^*(A)B). \quad (4)$$

Dynamical maps must preserve positivity and normalization i.e.

$$\Gamma(A^*A) \geq 0, \ \Gamma^*(B^*B) \geq 0,$$

$$\Gamma(1) = 1, \ \text{Tr}\Gamma^*(B) = \text{Tr}(B) \quad (5)$$

for all $A, B \in \mathcal{M}_n$.

A. Complete positivity

Positivity preserving condition (5) is generally to weak for a mathematically and physically consistent theory. The minimal condition which must be imposed on dynamical maps to allow meaningful construction of joint dynamics for noninteracting composed systems is complete positivity. It means that for any $d = 1, 2, 3, ...$, $\Gamma \otimes I_d$ is positive, where $I_d$ is an identity map acting on $d \times d$ matrices (i.e. trivial dynamical map on $d$-level quantum system). Any reduced dynamical map obtained from the formula (2) is completely positive and any completely positive dynamical map can be obtained from the reduced dynamics scheme. On the mathematical side completely positive maps on operator algebras were studied already in the 50-ties and the celebrated Stinespring representation [15] leads to a general form of completely positive dynamical map called often Kraus decomposition [16]

$$\Gamma(A) = \sum_{\alpha} W^*_\alpha A W_{\alpha} \quad (6)$$

where $W_\alpha$ are bounded operators on $\mathcal{H}$ satisfying $\sum_\alpha W^*_\alpha W_\alpha = 1$. The decomposition (6) is highly nonunique, in particular the sum over $\alpha$ can be replaced by an integral. If $\mathcal{H}$ is $n$-dimensional then complete positivity is equivalent to $n$-positivity and one can always find Kraus decomposition in terms of at most $n^2$ terms.
From now on by the quantum dynamical semigroup (in the Heisenberg picture) we mean a one-parameter strongly continuous semigroup \( T_t, t \geq 0 \) of completely positive unity preserving maps. For finite dimensional case we have always

\[
A_t = T_t(A) = e^{iL_t}(A), \quad \frac{d}{dt}A_t = L(A_t)
\] (7)

with the generator of the standard Lindblad, Gorini, Kossakowski and Sudarshan form [17]

\[
L(A) = i[H, A] + \sum_{j=1}^{p} V_j^* A V_j - \frac{1}{2} \{ \sum_{j=1}^{p} V_j^* V_j, A \}
= i[H, A] + \frac{1}{2} \sum_{j=1}^{p} (V_j^* [A, V_j] + [V_j^*, A] V_j).
\] (8)

The choice of the Hamiltonian \( H = H^* \) and the operators (matrices) \( V_j \) is again not unique and the sum over \( j \) can be replaced by an integral. In our finite dimensional case one can always find at most \( p = n^2 - 1 \) such \( V_j \)-s. To simplify the notation we put always \( \hbar \equiv 1 \) and \( k_B \equiv 1 \) to have the same units for energy, frequency and temperature.

The irreversible character of a dynamical map or semigroup is expressed in terms of the following inequalities valid for any \( A \in \mathcal{M}_n \)

\[
\Gamma(A^* A) \geq \Gamma(A^*) \Gamma(A) \quad \text{--- Kadison inequality}
\] (9)

\[
L(A^* A) \geq L(A^*) A + A^* L(A) \quad \text{--- dissipativity}.
\] (10)

Indeed, putting equality in (9) we obtain in our case the reversible dynamical map \( A \mapsto U^* A U \) with a unitary \( U \) while the equality in (10) implies the Hamiltonian derivation structure \( L(A) = i[H, A] \). Both inequalities are the consequences of 2-positivity condition obviously satisfied for any completely positive dynamical map and proved very useful in the analysis of ergodic properties.

B. \( C^* \)-algebras and group representations

In this Section we briefly review the basic mathematical facts about finite dimensional \( C^* \)-algebras and representation theory of compact groups which will be used later on [18].

The full matrix algebra is an example of \( C^* \)-algebra i.e. a complex linear space with adjoint operation (hermitian conjugation) \( A \mapsto A^* \), multiplication \( AB \) and complete with respect to the norm satisfying \( ||AA^*|| = ||A||^2 \). We are interested in finite-dimensional \( C^* \)-algebras which can always be seen as \( C^* \)-subalgebras of full matrix algebras i.e. linear subspaces of \( \mathcal{M}_n \), closed with respect to multiplication of matrices and hermitian conjugation and containing the unit matrix \( 1 \).

Any such \( C^* \)-subalgebra \( \mathcal{N} \) can be represented as the algebra of linear maps on the finite dimensional Hilbert space in the following way. The Hilbert space is decomposed as

\[
\mathbb{C}^n = \bigoplus_{j=1}^{m} \mathbb{C}^{n_j} \otimes \mathbb{C}^{d_j}, \quad \sum_{j=1}^{m} n_j d_j = n
\] (11)

where \( n_j \) is the dimension of the irreducible representation of \( \mathcal{N} \) and \( d_j \) its multiplicity. Then any element \( A \in \mathcal{N} \) can be written in the form

\[
A = \bigoplus_{j=1}^{m} A_j \otimes 1_{d_j}
\] (12)

where \( A_j \in \mathcal{M}_{n_j} \) and \( 1_{d_j} \) is a unit matrix acting on \( \mathbb{C}^{d_j} \).

In a general algebraic framework the physical symmetry or reversible dynamical map described in the Heisenberg picture is given in terms of an automorphism of the corresponding \( C^* \)-algebra, i.e. a linear map \( A \mapsto \mathcal{U}(A) \) satisfying
\(U(A^*) = U(A)^*, U(AB) = U(A)U(B), U(1) = 1\) and reversible. For an automorphism of a finite dimensional algebra \(\mathcal{N} \subset \mathcal{M}_n\) reversibility follows from the former conditions and moreover there exists a unitary matrix \(U \in \mathcal{M}_n\) such that \(U(A) = U^*AU\). The group of all automorphisms of \(\mathcal{N}\) is denoted by \(\text{Aut}(\mathcal{N})\). A subgroup \(\text{Auto}_0(\mathcal{N}) \subset \text{Aut}(\mathcal{N})\) contains all inner automorphisms given in terms of unitaries \(U \in \mathcal{N}\). For our finite dimensional case \(\text{Auto}_0(\mathcal{N})\) coincides with the path connected component of identity of \(\text{Aut}(\mathcal{N})\). In order to illustrate the fact, that even in the finite dimensional case not all automorphisms are inner, consider the abelian subalgebra of \(\mathcal{M}_2\) of diagonal matrices and the "flip" map \(\left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \mapsto \left( \begin{array}{cc} b & 0 \\ 0 & a \end{array} \right)\). This automorphism can be written as \(A \mapsto \sigma_XA\sigma_X\), where the (unitary) Pauli matrix \(\sigma_X\) is neither diagonal nor can be continuously transformed into identity.

We denote by \(A' = \{X \in \mathcal{M}_n; [X, A] = 0\}\) the commutant of the subset \(A \subset \mathcal{M}_n\). If \(A\) is self-adjoint, i.e. \(A \in A \Rightarrow A' \in A\), then the commutant \(A'\) is a unital \(C^*\)-subalgebra of \(\mathcal{M}_n\). For a subalgebra \(\mathcal{N}\) we have \((\mathcal{N}')' = \mathcal{N}\). Obviously if \(A \subset B\) then \(B' \subset A'\). Denoting by \(\text{Alg}(\mathcal{A})\) the smallest \(C^*\)-algebra containing \(\mathcal{A}\) we have \(A' = \text{Alg}(\mathcal{A})'\). For an algebra \(\mathcal{N} = \bigoplus_{j=1}^p \mathcal{M}_{n_j} \otimes 1_{d_j}\) (see (12)) its commutant \(\mathcal{N}' = \bigoplus_{j=1}^p 1_{n_j} \otimes \mathcal{M}_{d_j}\).

The theory of group representations provides natural examples of finite dimensional \(C^*\)-algebras. Take a finite-dimensional unitary representation \(\mathcal{R}: G \mapsto \mathcal{M}_n; g \mapsto R(g)\) of the compact group \(G\) (discrete or Lie group). Such representation can be always decomposed into irreducible representations \(\mathcal{R}_j\) according to (11). Any subspace \(\mathcal{C}^{n_j}\) is a carrier of an irreducible representation and appears \(d_j\)-times in the decomposition. Hence any unitary \(R(g) \in \mathcal{R}(G)\) is decomposed as \(R(g) = \bigoplus_{j=1}^{n_j} \mathcal{R}_j(g) \otimes 1_{d_j}\). The \(C^*\)-algebra generated by \(\mathcal{R}(G)\) will be denoted by \(\text{Alg}(\mathcal{R}(G))\) and consists of elements of the form \(A = \sum_\alpha \alpha g R(g)\), \(\alpha \in \mathbb{C}\) which can be written as (12).

C. Decoherence-free subalgebras

For an open system with a time-independent Hamiltonian interacting with a stable environment being at the equilibrium state (heat bath) we expect that for an arbitrary initial state the system returns to its thermal equilibrium state determined by the temperature of environment. In mathematical terms it means that \(\Gamma^\infty\) (or \(\Gamma_t^\infty\)) possesses a unique stationary state \(\rho_{eq}\) and \(\lim_{t \to \infty} \Gamma(t)(A) = \text{Tr}(\rho_{eq}A)1\) (\(\lim_{t \to \infty} T_t(A) = \text{Tr}(\rho_{eq}A)1\)). We say that the dynamics \(\Gamma\) (\(T_t\)) is ergodic in this case. The relaxation process can be very complex and involving many time scales of very different magnitude, some of them being much longer than the time scale relevant for the operation of our quantum device. Therefore, it is often very useful to study simplified approximative nonergodic dynamics which take into account only the fastest relaxation processes. For such a dynamics we are interested in observables and states which evolve according to the reversible (unitary, Hamiltonian) evolution i.e. in a "decoherence-free" way. We shall prove that such observables are hermitian elements of a certain unitial \(C^*\)-subalgebra of \(\mathcal{M}_n\). In practice, the manifest form of this subalgebra is difficult to find and quite often we must be satisfied with a detailed knowledge of its certain nontrivial subalgebras. It turns out that the symmetries of the system can be very useful in this context.

The mathematical techniques used here are based on the earlier results on the ergodic theory of quantum dynamical semigroups [19,20]. Some partial results of this type can be found in the recent literature also [21].

A minimal condition which should be satisfied by a decoherence-free observable \(A\) with respect to the dynamical map \(\Gamma\) is the lack of dissipation expressed by

\[
\Gamma(A^2) = (\Gamma(A))^2, \quad A = A^*.
\]  

We formulate this condition in terms of a dissipation function \(D_\Gamma(.,.)\) from \(\mathcal{M}_n \times \mathcal{M}_n\) to \(\mathcal{M}_n\) defined by

\[
D_\Gamma(A, B) = \Gamma(A^*B) - \Gamma(A^*)\Gamma(B)
\]  

Denote by \(\mathcal{N}_R(\Gamma)\) the following subset of hermitian matrices (observables) in \(\mathcal{M}_n\) and by \(\mathcal{N}(\Gamma)\) its complexification

\[
\mathcal{N}_R(\Gamma) = \{A = A^* \in \mathcal{M}_n; D_\Gamma(A, A) = 0\},
\]

\[
\mathcal{N}(\Gamma) = \{A + iB; A, B \in \mathcal{N}_R(\Gamma)\}.
\]

We shall call \(\mathcal{N}(\Gamma)\) the decoherence free subalgebra of \(\Gamma\) and this name is justified by the following theorem

**Theorem 1**

1) \(\mathcal{N}(\Gamma)\) is a unital \(C^*\)-subalgebra of \(\mathcal{M}_n\)

2) \(\Gamma\) restricted to \(\mathcal{N}(\Gamma)\) is a unit preserving homomorphism, i.e. for any \(A, B \in \mathcal{N}(\Gamma)\)

\[
\Gamma(AB) = \Gamma(A)\Gamma(B), \quad \Gamma(1) = 1
\]
Moreover if for the discrete case \( U \) holds automatically for continuous time semigroup, i.e.,

\[
\Gamma(A) = U_T^* A U_T , \quad \text{for all } A \in \mathcal{N}(\Gamma). \tag{17}
\]

**Proof** We show first that for any \( A \in \mathcal{N}_R(\Gamma) \) and arbitrary \( B \in \mathcal{M}_n \)

\[
\Gamma(AB) = \Gamma(A)\Gamma(B) , \quad \Gamma(BA) = \Gamma(B)\Gamma(A) . \tag{18}
\]

The relations (18) follow from the Kadison inequality (9) applied to \( A + \lambda B \) with an arbitrary \( \lambda \in \mathbb{C} \). Namely,

\[
0 \leq D_\Gamma(A + \lambda B, A + \lambda B) = \Gamma(A^2) + |\lambda|^2 \Gamma(B^*B) + \lambda \Gamma(AB) + \bar{\lambda} \Gamma(B^*A)
\]

\[
-\Gamma(A)^2 - |\lambda|^2 \Gamma(B^*)\Gamma(B) - \lambda \Gamma(A)\Gamma(B) - \bar{\lambda} \Gamma(B^*)\Gamma(A)
\]

\[
= |\lambda|^2 (\Gamma(B^*B) - \Gamma(B^*)\Gamma(B)) + \lambda (\Gamma(AB) - \Gamma(A)\Gamma(B)) + \bar{\lambda} (\Gamma(B^*A) - \Gamma(B^*)\Gamma(A)) . \tag{19}
\]

Dividing both sides of the inequality (19) by \(|\lambda|\) and taking \(|\lambda| \to 0\) we have for any \( z \in \mathbb{C}, |z| = 1 \)

\[
z (\Gamma(AB) - \Gamma(A)\Gamma(B)) + \bar{z} (\Gamma(B^*A) - \Gamma(B^*)\Gamma(A)) \geq 0 \tag{20}
\]

what implies (18).

It is now a simple exercise to prove using (18) that if \( A, B \in \mathcal{N}(\Gamma) \), then \( A^*, B^* \), \( A + B, AB \in \mathcal{N}(\Gamma) \). Hence \( \mathcal{N}(\Gamma) \) is a unital \( C^* \)-subalgebra and from the general results on finite dimensional \( C^* \)-algebras reviewed in Section 2.2 it follows that (17) holds also.

Theorem 1 gives us the general mathematical structure of observables evolving in a reversible way under a single step dynamics. The physical meaning of the algebra \( \mathcal{N}(\Gamma) \) is illustrated by the most elementary physical process consisting of the following stages:

1) preparation of the initial state \( \rho \),
2) irreversible evolution governed by the dynamical map \( \Gamma \),
3) measurement of a given observable with a spectral resolution

\[
A = \sum_k a_k P_k .
\]

The probability of obtaining the value \( a_k \) is equal to \( p_k = \text{Tr}(\rho P_k) \) and if \( A \in \mathcal{N}(\Gamma) \) then \( p_k = \text{Tr}(\rho U_T^* P_k U_T) \).

In the language of quantum information theory it means that if we choose decoherence-free *decoding observables* then the noisy quantum channel is equivalent to a certain noiseless one.

Analogical structures can be defined for the discrete (continuous) time dynamical semigroups \( \{\Gamma^k, k = 0, 1, 2, ...\} \) \( \{\{T_t, t \geq 0\}\} \) to obtain the global decoherence-free subalgebras which contain observables evolving in a reversible way for all times

\[
\mathcal{N}(\Gamma^{(i)}) = \bigcap_{k=1}^\infty \mathcal{N}(\Gamma^k) , \quad \mathcal{N}(T^{(i)}) = \bigcap_{t \geq 0} \mathcal{N}(T_t) \tag{21}
\]

respectively.

As a direct consequence of this definition, the Theorem 1 and the general results of Section 2.2 we obtain

**Colloary** \( \mathcal{N}(\Gamma^{(i)}) \) and \( \mathcal{N}(T^{(i)}) \) are globally invariant under the action of \( \Gamma^k \) and \( T_t \) respectively. The both semigroups restricted to decoherence-free algebras produce groups of unitary automorphisms \( \{U_T^k, k \in \mathbb{Z}\} \) and \( \{U_t; t \in \mathbb{R}\} \). Moreover if for the discrete case \( U_T \) can be continuously transformed into identity then \( U_T \in \mathcal{N}(\Gamma^{(i)}) \). Analogical result holds automatically for continuous time semigroup, i.e., \( U_t \in \mathcal{N}(T^{(i)}) \).
For the future applications it is useful to define a set of fixed points of \( \Gamma \) as \( \mathcal{F}(\Gamma) = \{ A \in \mathcal{M}_n; \Gamma(A) = A \} \). In general, \( \mathcal{F}(\Gamma) \) is a self-adjoint linear subspace of \( \mathcal{M}_n \) but under additional assumptions it becomes \( C^* \)-algebra. For the dynamical semigroup \( \mathcal{F}(T_\tau) = Ker(L) = \{ A \in \mathcal{M}_n; L(A) = 0 \} \).

The algebras \( \mathcal{N}(\Gamma), \mathcal{N}(\Gamma^{(i)}) \) and \( \mathcal{N}(T_\tau) \) are the largest ones with the decoherence-free property but their definition is not constructive and often difficult to apply. Therefore, it is sometimes convenient to consider smaller subalgebras which are easier to construct explicitly. Theorems 2,3 provide us with the appropriate examples.

**Theorem 2**
Assume that \( \Gamma = \mathcal{U} \Gamma_D \) where \( \mathcal{U}(X) = U^* X U \) is a unitary automorphism and
\[
\Gamma_D(X) = \sum_{\alpha=1}^r W_\alpha^* X W_\alpha .
\]
Then the commutants defined below are \( C^* \)-subalgebras and
\[
\mathcal{W}_1 = \{ W_\alpha, W_\alpha^*; \alpha = 1,2,\ldots ,r \}' \subset \mathcal{W}_2 = \{ W_\alpha W_\beta^*; \alpha, \beta = 1,2,\ldots ,r \}' \subset \mathcal{N}(\Gamma) .
\]
If additionally \( \mathcal{U} \) commutes with \( \Gamma_D \) then
\[
\mathcal{W}_1 \subset \mathcal{W}_3 = \{ W_\alpha W_\beta, W_\alpha^* W_\beta^*; W_\alpha W_\beta^*, W_\alpha^* W_\beta; \alpha, \beta = 1,2,\ldots ,r \}' \subset \mathcal{N}(\Gamma^{(i)}) .
\]
For continuous time dynamical semigroups we have:

**Theorem 3**
Assume that the semigroup generator (8) decomposed as \( L = L_H + L_D , L_H = i[H,\cdot] \) satisfies
\[
L_H L_D = L_D L_H .
\]
Then the commutant defined below is a \( C^* \)-subalgebra and
\[
\{ V_j, V_j^*; j = 1,2,\ldots ,p \}' \subset \mathcal{N}(T_\tau) .
\]

**Proof of Theorems 2 and 3** From the definition of the commutant \( \mathcal{W}_1 \subset \mathcal{W}_2 \subset \mathcal{N}(\Gamma_D) = \mathcal{N}(\Gamma) \).

Due to the condition \( \mathcal{U} \Gamma_D = \Gamma_D \mathcal{U}, \mathcal{N}(\Gamma^{(i)}) = \mathcal{N}(\Gamma) \). Finally, if \( X = X^* \) commutes with all \( W_\alpha W_\beta, W_\alpha^* W_\beta^* \), \( W_\alpha^* W_\beta^* , \alpha, \beta = 1,2,\ldots ,r \) then \( \Gamma^k(X^2) = (\Gamma^k(X))^2 \) for all \( k = 0,1,2,\ldots \)

Similarly, for continuous time semigroups the commutation of the Hamiltonian part \( L_H \) with \( L_D \) implies that the elements of the commutant \( \{ V_j, V_j^*; j = 1,2,\ldots ,p \}' \), which lie in a kernel of \( L_D \), are decoherence-free.

**Remark** The decomposition of the dynamical map into reversible and dissipative parts in Theorems 2,3 is not unique and can be optimized to obtain the largest commutants.

To apply Theorems 2, 3 it is often convenient to use the results of the theory of group representations. Assume that the matrices \( W_\alpha \) belong to \( Alg(\mathcal{R}(G)) \subset \mathcal{M}_n \) of a certain unitary representation of a finite or compact Lie group \( G \). In this case the Hilbert space is decomposed into carriers of irreducible representations of \( G \) as in (11) where \( n_j \) are dimensions of irreducible representations and \( d_j \) their multiplicities. The commutant
\[
\{ \mathcal{R}(G) \}' = \bigoplus_{j=1}^m 1_{n_j} \otimes \mathcal{M}_{d_j} \subset \mathcal{W}_1
\]
in this case.

Similarly, for quantum dynamical semigroup satisfying the assumptions of the Theorem 3 the matrices \( V_j, V_j^* ; j = 1,2,\ldots ,p \) generate a certain Lie algebra in \( \mathcal{M}_n \) and hence the representation \( \mathcal{R}(G) \) of the corresponding group \( G \). In this case the commutant (25) coincides with \( \{ V_j, V_j^* ; j = 1,2,\ldots ,p \}' \).

Another application of the group theory is possible when the finite or compact Lie group \( G \) is a symmetry group for the dynamical map \( \Gamma \) i.e.
\[
\mathcal{R}_g \Gamma \mathcal{R}_g^{-1} = \Gamma , \quad \mathcal{R}_g(X) = R(g)^* X R(g)
\]
for any unitary \( R(g) \in \mathcal{R}(G), g \in G \). We call the condition (26) **global invariance** of the dynamical map \( \Gamma \) with respect to the group \( G \). Using the definition of \( \mathcal{N}(\Gamma^{(i)}) \) and then applying (26) we can prove that from \( A \in \mathcal{N}(\Gamma^{(i)}) \) it follows that \( \mathcal{R}_g(A) \in \mathcal{N}(\Gamma^{(i)}) \) too. Therefore, the invariance holds
\[
\mathcal{R}_g(\mathcal{N}(\Gamma^{(i)})) = \mathcal{N}(\Gamma^{(i)}) .
\]
If $G$ is a connected compact Lie group then any $R(g)$ can be continuously connected to identity and hence by general theorems from Section 2.2 $R_g$ is an inner automorphism of $\mathcal{N}(\Gamma^{(1)})$, i.e. $R(g) \in \mathcal{N}(\Gamma^{(1)})$. Hence the $C^*$-algebra generated by $R(G)$ is included in in the decoherence-free algebra $\mathcal{N}(\Gamma^{(1)})$. The situation for discrete symmetry may be quite different due to "incidental degeneracies" of $\Gamma$ which admit larger group algebras. All that is true for the dynamical semigroup $\{T_t; t \geq 0\}$ also and will be illustrated by examples in Section 3.

The stronger invariance conditions with respect to the symmetry group called local invariance can be formulated for the dynamical map $\Gamma = UT_D = \Gamma_D U$ as in the Theorem 2. It reads for all $\alpha$ and $g \in G$

$$R(g)^* W_\alpha R(g) = W_\alpha$$

(28)

Therefore the algebra generated by the group representation $Alg(R(G)) \subset \mathcal{N}(\Gamma^{(1)})$. Similarly, for quantum dynamical semigroup as in the Theorem 3 the condition

$$R(g)^* V_\alpha R(g) = V_\alpha$$

(29)

implies $Alg(R(G)) \subset \mathcal{N}(T_{(1)}) = Ker(L_D)$.

Summarizing, global invariance (26) for connected compact Lie group implies $Alg(R(G)) \subset \mathcal{N}(\Gamma^{(1)})$ while local invariance (28) or (29) implies for any compact group $G$ that $Alg(R(G)) \subset \mathcal{N}(\Gamma^{(1)})$ or $Alg(R(G)) \subset \mathcal{N}(T_{(1)})$ respectively.

The concrete examples of semigroups and their decoherence-free observables obtained using the methods proposed above will be discussed in Section 3.

**D. Limited relaxation**

We expect that the open system governed by a non-ergodic dynamics shows a behaviour which can be called limited relaxation. It means that in the Heisenberg picture any initial observable becomes for long times decoherence-free and the set of decoherence-free observables is nontrivial. We are able to prove this property under additional assumptions which are typically fulfilled for open quantum systems weakly coupled to heat baths.

We begin with the dynamical map $\Gamma$ satisfying the following conditions:

1) There exist a faithful (i.e. given by a strictly positive density matrix) stationary state $\sigma$, $\Gamma^*(\sigma) = \sigma$.

2) Detailed balance condition with respect to $\sigma$ holds, i.e. $\Gamma = UT_D = \Gamma_D U$ where $U(X) = U^* X U$ is a unitary automorphism, $U(\sigma) = \sigma$ and the dynamical map $\Gamma_D$ is a hermitian operator on the Hilbert space $(\mathcal{M}_n; <\cdot, \cdot>_\sigma)$ equipped with the scalar product

$$< A, B >_\sigma = \text{Tr}(\sigma A^* B) \ , \ A, B \in \mathcal{M}_n$$

(30)

and the norm $\|A\|_\sigma = < A, A >^{1/2}_\sigma$. This Hilbert space is often called Liouville space. More explicitly $U$ is unitary, $\Gamma_D$ is hermitian and $\Gamma_D U = \Gamma_D U$ is a normal contraction on the Liouville space $(\mathcal{M}_n; <\cdot, \cdot>_\sigma)$.

The existence of a stationary state, generally non unique, is guaranteed by the finite-dimensionality of the system’s Hilbert space but its faithfulness is a nontrivial assumption.

**Lemma 1**

If the dynamical map $\Gamma$ possesses a faithful stationary state $\sigma$ then the fixed point set $\mathcal{F}(\Gamma)$ is a $C^*$-subalgebra of $\mathcal{M}_n$ and $\mathcal{F}(\Gamma) \subset \mathcal{N}(\Gamma^{(1)})$.

**Proof** Take $A = A^* \in \mathcal{F}(\Gamma)$ then, $A^2 = (\Gamma(A))^2$ and hence

$$0 = \text{Tr}(\sigma[A^2 - \Gamma(A) \Gamma(A)]) = \text{Tr}(\sigma(\Gamma(A^2) - \Gamma(A) \Gamma(A))) \ .$$

(31)

As $\Gamma(A^2) - \Gamma(A) \Gamma(A) \geq 0$ and $\sigma$ is faithful then (31) is equivalent to $\Gamma(A^2) = (\Gamma(A))^2$. It follows that $A \in \mathcal{N}(\Gamma)$. If $A = A^* \in \mathcal{N}(\Gamma)$ then we can reverse the reasoning to prove that $A \in \mathcal{F}(\Gamma)$. For a non-hermitian $A$ we can use the decomposition $A = A_1 + i A_2$ into hermitian ones.

We shall denote by $P_T$ the orthogonal projector acting on the Liouville space $(\mathcal{M}_n; <\cdot, \cdot>_\sigma)$ and projecting on the subspace $\mathcal{N}(\Gamma^{(1)})$.

Limited relaxation process is characterized by the following theorem.

**Theorem 4**

Consider a dynamical map $\Gamma$ satisfying the detailed balance condition of above. Then $\mathcal{N}(\Gamma^{(1)}) = \mathcal{F}(\Gamma_D)$ and

$$\lim_{k \to \infty} \|\Gamma^k(A) - U^k(P_T A)\| = 0 \ , \ \text{for all} \ A \in \mathcal{M}_n \ .$$

(32)
Proof From $\Gamma^k = U^k \Gamma^k_D$, it follows that $\mathcal{N}(\Gamma^{(i)}) = \mathcal{N}(\Gamma^{(i)}_D)$. For any $A \in \mathcal{N}(\Gamma^{(i)})$ we have $\|\Gamma^k_D(A)\|_\sigma = \|A\|_\sigma$. Therefore, because $\Gamma_D$ is a hermitian contraction on the Liouville space, $\mathcal{N}(\Gamma^{(i)}_D) = \mathcal{F}(\Gamma_D)$.

For the dynamical semigroup with the generator $L = i[H, \cdot] + L_D$ the detailed balance condition means that the Hamiltonian part $L_H$ commutes with $L_D$. It is also a Hermitian operator on the Liouville space $(\mathcal{M}_n; < \cdot, \cdot > _\sigma)$ [7]. Introducing a kernel $\text{Ker}(L_D) = \{ A \in \mathcal{M}_n; L_D(A) = 0 \}$, the orthogonal projection on $\text{Ker}(L_D)$ denoted by $P_{L_D}$ and applying the same idea of the proof we obtain the following analog of Theorem 4.

**Theorem 5**

Consider the dynamical semigroup $T_t$ with the generator satisfying detailed balance condition of above. Then $\mathcal{N}(T_{\omega}) = \text{Ker}(L_D)$ and

$$\lim_{t \to \infty} \|T_t(A) - e^{itH}P_{L_D}(A)e^{-itH}\| = 0, \text{ for all } A \in \mathcal{M}_n.$$

**Example**

The particular case of detailed balance generator with the stationary Gibbs state

$$\sigma = Z^{-1}e^{-H/T}$$

leads to a very specific structure of $L_D$ [7]

$$L_D(A) = \sum_{j: \omega_j \geq 0} \left( V_j^*A V_j - \frac{1}{2}\{V_j^*V_j, A\} + e^{-\omega_j/T}(V_jAV_j^* - \frac{1}{2}\{V_jV_j^*, A\}) \right)$$

with $[H, V_j] = \omega_j V_j$. Simple calculation involving the dissipativity condition (10) and hermicity of $L_D$ on the Liouville space shows that in this case

$$0 \leq \text{Tr}(\sigma[L(A^*A) - L(A^*A) + A^*L(A)]) = -< A, L_D(A) > _\sigma$$

$$= \sum_{j: \omega_j \geq 0} \left( < [V_j, A], [V_j, A] > _\sigma + e^{-\omega_j/T} < [V_j^*, A], [V_j^*, A] > _\sigma \right).$$

Therefore, the commutant $\{V_j, V_j^*; \omega_j \geq 0\}' = \text{Ker}(L_D) = \mathcal{N}(T_{\omega}).$

**III. EXAMPLES**

In this Section we give examples of physical systems governed by nonergodic quantum dynamical semigroups and illustrate the different methods of searching for decoherence-free observables introduced in Section 2.

**A. N-particle systems with permutation invariance**

Consider a system which consists of $N$ identical particles each of them is described by a finite dimensional Hilbert space $\mathcal{H}_j \equiv \mathbb{C}^d, j = 1, 2, \ldots, N$. There are two possibilities of constructing a quantum dynamical semigroup generator invariant with respect to the permutation group $S_N$. The first one corresponds to particles interacting with identical "private" reservoirs and is given by the Master equation in the Schrödinger picture for the $N$-particle density matrix $\rho_t$

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \sum_{m=1}^{N} L_m^*(\rho_t)$$

where $H$ is a $N$-particle permutation invariant Hamiltonian and $L_m^*$ are identical copies of a single-particle dynamical semigroup generator.

The second example corresponds to the collective coupling of $N$-particle system to a single "common" reservoir

$$\frac{d}{dt}\rho_t = -i[H, \rho_t] + \frac{1}{2} \sum_{\alpha} ([V_\alpha, \rho_t V_\alpha] + [V_\alpha \rho_t, V_\alpha])$$

(37)
where $V_{a} = \sum_{m=1}^{N} v_{a}^{(m)}$ with identical copies of a single particle operator $v_{a}$.

The Hilbert space of the system possesses the structure of $N$-fold tensor product $\bigotimes_{N} \mathbb{C}^{d}$ with the basis $\{ e_{j_{1}} \otimes e_{j_{2}} \otimes \cdots \otimes e_{j_{N}} \}$. Any permutation $\pi \in S_{N}$ is represented by the operator $R(\pi)$ defined in terms of the basis as

$$R(\pi)e_{j_{1},j_{2},\ldots,j_{N}} = e_{\pi(j_{1}),\pi(j_{2}),\ldots,\pi(j_{N})}.$$  \hspace{1cm} (38)

It is well-known that this representation is highly reducible and therefore $Alg(\mathcal{R}(S_{N}))$ is nontrivial. The explicite construction of the decomposition (11)(12) is given in terms of Young tables and described in the textbooks on group theory [22].

The semigroup generated by the master equation(36) is globally invariant with respect to $S_{N}$ but because $S_{N}$ is a discrete group according to the results of Section 2.3 we can have no nontrivial decoherence-free observables i.e. $\mathcal{N}(T_{(i)}) = \mathbb{C}$. Indeed, if a single-particle semigroup generated by $L_{m}^{*}$ is ergodic and the Hamiltonian $H = 0$, then the whole $N$-particle dynamics governed by (36) is ergodic too.

For the case of the dynamics governed by the master equation (37) the situation is different. The corresponding semigroup is locally invariant with respect to $S_{N}$ and therefore according to the results of Section 2.3 $Alg(\mathcal{R}(S_{N})) \subset \mathcal{N}(T_{(i)})$.

**B. Superradiance model with $SU(2)$ symmetry**

The superradiance [23] and subradiance phenomena [24] are the first examples of collective dissipative phenomena in quantum systems which leads to a nonergodic behaviour (on a certain time scale) due to (approximative) permutation symmetry of the interaction of atoms with the electromagnetic field. In the simplest case of 2-level atoms in open space and for the temperature $T = 0$, we obtain the master equation of the form (37) with

$$H = \frac{1}{2} \omega \sum_{m=1}^{N} \sigma_{z}^{(m)} , \quad v_{a} = \frac{1}{2} \sqrt{\gamma}(\sigma_{x} + i\sigma_{y})$$ \hspace{1cm} (39)

where $\omega$ is an atomic frequency, $\gamma$ is a radiation damping constant and $\sigma_{j}^{(m)}, j = x, y, z,$ are Pauli matrices for the $m$-th atom.

In order to find decoherence-free observables we can use either the local symmetry of the master equation with respect to the permutation group $S_{N}$ or the fact that the collective operators $\sum_{m=1}^{N} \sigma_{j}^{(m)}$ which appear in the semigroup generator define a reducible $N$-fold product representation of the group $SU(2)$ on $\bigotimes_{N} \mathbb{C}^{2}$. Then we obtain

$$Alg(\mathcal{R}(S_{N})) \subset Alg(\mathcal{R}(SU(2))) \subset \mathcal{N}(T_{(i)})$$ \hspace{1cm} (40)

where $\mathcal{R}(S_{N})$ is a standard representation of $S_{N}$ defined by (38) and $\mathcal{R}(SU(2))$ is the mentioned above product representation of $SU(2)$. For our case of $T = 0$ the explicite structure of $\mathcal{N}(T_{(i)})$ is unknown while for $T > 0$ we can use detailed balance condition and the Theorem 5 to characterize $\mathcal{N}(T_{(i)})$ [20].

**IV. NON-MARKOVIAN CONTROLLED OPEN SYSTEMS**

The analysis of the previous Section has a rather mathematical and phenomenological character and its most developed part is applicable to dynamical semigroups (discrete or continuous) and hence involve Markovian assumption or at least stationary external conditions. For a generic controlled system we expect that the external conditions like, for instance, external electromagnetic fields used to control the quantum device vary in time. Therefore, we need other approximation schemes which are based on the assumption of the weak influence of the environment on the system and express the dynamics in terms of the fundamental ingredients (controlled Hamiltonian, correlation functions of the bath)[25].

**A. Errors in CQOP**

We would like to describe a quantum device operating during the time interval $[0,t]$. We assume that in the initial moment the prepared state can be given by a product formula (1) with a pure state $\rho = |\psi><\psi|$ while in the final
moment the state of our open system is described by the reduced density matrix (2). The actual dynamics is compared
with the ideal one given in terms of the controlled unitary evolution $U_S(t)$ which solves the Schrödinger equation
governed by the controlled Hamiltonian $H_S(s); 0 \leq s \leq t$. The error $\epsilon$ due to the interaction with environment can
be defined as

$$\epsilon = 1 - \langle U_S(t)\psi, \Lambda_s(|\psi\rangle < \psi|) U_S(t)\psi \rangle.$$  (41)

The choice of the Hamiltonian $H_S(t)$ is a tricky problem. Typically, one takes a sum of the time-independent Hamiltonian
$H_S^0$ of the bare system $S$ and a time-dependent contributions $H_S^1(s)$ describing the influence of controlled
external fields on our system. However the interaction with environment induces generally time-dependent Hamiltonian corrections.
For example, vacuum fluctuations of the electromagnetic field produce Lamb shift of the atomic energy levels and van der Waals type interactions between neutral atoms [24]. Similar phenomena appear in solid state physics due to interactions with phonons and other quasi-particles. In the following we adopt the optimistic point of view assuming that we can include these corrections in the process of designing quantum algorithms. In practice one should expect the presence of the Hamiltonian errors due to an approximative knowledge of the true physical Hamiltonian $H_S(s)$.

B. Reduced dynamics in Born approximation

We consider the system which consists of the quantum device $S$ driven by the time-dependent Hamiltonian $H_S(t)$
and the bath with its Hamiltonian $H_R$. Then the total Hamiltonian reads

$$H(t) = H_S(t) + H_R + H_{int}$$  (42)

with the interaction Hamiltonian of the form

$$H_{int} = \sum_\alpha S_\alpha \otimes R_\alpha$$  (43)

where $S_\alpha, R_\alpha$ are self-adjoint operators.

In the following we investigate the dynamical map $\Gamma^*$ in the Schrödinger picture which describes the evolution of
$S$ from the initial moment $t_{in} = -\tau$ to to the final one $t_{fin} = \tau$ in Born approximation. The exact dynamical map can be expressed in terms of $H_{int}$, the unitary propagator for the total systems $U(t, s)$, the free propagator for the total system $U_0(t, s)$ and the propagator $U_S(t, s)$ defined by

$$U(t, s) = Te^{-i\int_s^t H(u) du}, U_0(t, s) = Te^{-i\int_s^t (H_S(u) + H_R) du},$$

$$U_S(t, s) = Te^{-i\int_s^t H_S(u) du}$$  (44)

where $T$ is the time ordering operator. Introducing the ”superoperator” notation: $\hat{U} \rho = U \rho U^*$ and $\hat{H}_{int} \rho = [H_{int}, \rho]$, choosing the initial state similarly to (1)

$$\rho_{SR}(-\tau) = \rho \otimes \omega_R, \quad [H_R, \omega_R] = 0, \quad \text{Tr}(\omega_R R_\alpha) = 0$$  (45)

and applying the second order integral identity for $\hat{U}(t, s)$ we obtain the exact equation

$$\Gamma^*(\rho) = \hat{U}_S(\tau, -\tau) \rho$$

$$-\text{Tr}\left\{\int_{-\tau}^{\tau} ds \int_s^t du \hat{U}_0(\tau, s) \hat{H}_{int} \hat{U}_0(s, u) \hat{H}_{int} \hat{U}(u, -\tau) \rho \otimes \omega_R \right\}.$$  (46)

The Born approximation consists in replacing $\hat{U}(u, -\tau) \rho \otimes \omega_R$ by $\hat{U}_S(u, -\tau) \rho \otimes \omega_R$ in eq.(46). Our main assumption is that for a given initial state $\rho$ and a given trajectory $\{H_S(t); -\tau \leq t \leq \tau\}$ of the controlling Hamiltonian the influence of the reservoir on the state of our system $S$ given by the second term on the RHS of eq.(46) is small, say of the order $\epsilon \ll 1$. Then up to the higher order terms $\sim \epsilon^2$ we can use the proposed approximation. One should notice that we
do not necessarily impose weak coupling or/and short time $\tau$ regime but we can search for the optimal choice of $\rho$ and $H_S(t)$ to fulfill the Born approximation beyond this regime.

The final formula for $\Gamma^*(\rho)$ in Born approximation reads

$$\Gamma^*(\rho) = \hat{U}_S \left( \rho + \Phi^*(\rho) - \frac{1}{2}\{K,\rho\} \right)$$  \hspace{1cm} (47)$$

where $\hat{U}_S \equiv \hat{U}_S(\tau, -\tau)$. The superoperator $\Phi^*$ called error map is a completely positive map given by

$$\Phi^*(\rho) = \int_{-\tau}^{\tau} ds \int_{-\tau}^{\tau} du \text{Tr}(\omega R_\alpha R_\beta(s - u)) S_\beta(s, -\tau) \rho S_\alpha(u, -\tau)$$  \hspace{1cm} (48)$$

where $S_\alpha(s, u) = \hat{U}_S(u, s) S_\alpha$. The operator $K \geq 0$ is given by $K = \Phi(1)$. Strictly speaking one has also a contribution of the form $-i[h,\rho]$ which contains Hamiltonian corrections due to the interaction with $R$ like for example Lamb shift, collective Lamb shift, etc. but we put $h \equiv 0$. This can be justified by starting with a bare Hamiltonian $H^0_S(t)$ containing appropriate counterterms and performing renormalization procedure. It is, however, equivalent to treating $H_S(t)$ as a full physical Hamiltonian and putting $h \equiv 0$.

The equation (47) resembles in its structure the short time approximation to quantum dynamical semigroup and indeed, if the conditions of Markovian approximation are fulfilled we can derive from (47) different types of the Markovian master equations [7,9,11]. The approximative map (47) is trace preserving but complete positivity is satisfied up to the higher order terms in $\epsilon$.

Finally, we pass to the frequency (energy) domain introducing spectral density of the reservoir $R_{\alpha\beta}(\omega)$ by

$$\text{Tr}(\omega R_\alpha R_\beta(t)) = \int_{-\infty}^{\infty} R_{\alpha\beta}(\omega) e^{-i\omega t} d\omega$$  \hspace{1cm} (49)$$

and defining

$$Y_\alpha(\omega) = \int_{-\tau}^{\tau} S_\alpha(s, -\tau) e^{-i\omega s} ds.$$  \hspace{1cm} (50)$$

We obtain the following formula for the error map

$$\Phi^*(\rho) = \sum_{\alpha,\beta} \int_{-\infty}^{\infty} d\omega R_{\alpha\beta}(\omega) Y_\beta(\omega) \rho Y_\alpha^*(\omega)$$  \hspace{1cm} (51)$$

which is a kind of quantum fluctuation-dissipation theorem in the linear response regime [9,10].

C. Error formula

The general definition of the error due to the interaction with the environment (41) can be applied to our approximative evolution

$$\epsilon = 1 - \langle U_S \psi, \Gamma^*(|\psi\rangle <\psi|) U_S \psi \rangle.$$  \hspace{1cm} (52)$$

Putting the expression (47) we obtain the error formula in Born approximation

$$\epsilon = \langle \psi, K \psi \rangle - \langle \psi, \Phi^*(|\psi\rangle <\psi|) \psi \rangle$$  \hspace{1cm} (53)$$

which can be the starting point for the analysis of efficiency of quantum devices. Writing it in terms of (49-51) we obtain

$$\epsilon = \sum_{\alpha,\beta} \int_{-\infty}^{\infty} d\omega R_{\alpha\beta}(\omega) \left( \langle \psi, Y_\alpha^*(\omega) Y_\beta(\omega) \psi \rangle \right.$$  

$$\left. - \langle \psi, Y_\alpha^*(\omega) \psi \rangle <\psi, Y_\beta(\omega) \psi \rangle \right).$$  \hspace{1cm} (54)$$
The interpretation of (54) becomes simple and elegant if we introduce the following notation

\[ \langle \psi, Y_\alpha^*(\omega)Y_\beta(\omega)\psi \rangle = -\langle \psi, Y_\alpha^*(\omega)\psi \rangle > \langle \psi, Y_\beta(\omega)\psi \rangle > = 2\tau S_{\alpha\beta}(\omega). \]  

(55)

Now we can write the error formula in a compact form as an overlap of two "correlators" \( R(\omega) = [R_{\alpha\beta}(\omega)] \) and \( S(\omega) = [S_{\alpha\beta}(\omega)] \) multiplied by the duration of the operation time of the quantum device \( 2\tau \)

\[ \epsilon = 2\tau \int_{-\infty}^{\infty} d\omega \text{Tr}(R(\omega)S(\omega)) \]  

(56)

Both correlators encode the information about the dynamics and the initial state of the baths and the quantum device, respectively. Moreover, one can see that their mathematical structure is essentially the same. To show this we consider a sequence of random (quantum or classical) time-dependent real variables \( f_\alpha(t) \) and denote by \( < \cdot >_F \) the appropriate average. Define a correlator \( F(\omega) = [F_{\alpha\beta}(\omega)] \) by the following formula

\[ \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} ds \left( < f_\alpha(t+s)f_\beta(s) >_F - < f_\alpha(t+s) >_F < f_\beta(s) >_F \right) \]

\[ = \int_{-\infty}^{\infty} F_{\alpha\beta}(\omega) e^{-i\omega t} d\omega. \]  

(57)

Then putting \( f_\alpha \equiv R_\alpha, < \cdot >_F = \text{Tr}(\omega R) \) we obtain \( F(\omega) = R(\omega) \). Similarly, \( F(\omega) = S(\omega) \) for \( f_\alpha \equiv S_\alpha, < \cdot >_F = < \psi, \psi > \) if only \( \tau \) is long enough such that the limit \( \tau \to \infty \) in (57) can be used.

Having the formula (56) one can try to design different strategies reducing errors. Although very roughly the error seems to be proportional to the operation time \( 2\tau \) the situation is much more complicated. For example, if the leading error source is described by the correlator satisfying for small \( \omega \) \( R(\omega) \sim \omega^\kappa, \kappa > 1 \) \n
(58)

the optimal strategy is to slow down the time evolution of \( S \) in order to move the correlator \( S(\omega) \) into the low frequency domain. On the contrary, if \( R(\omega) \) is flat "fast gates" are optimal. Generally, one can try to shift the support of \( S(\omega) \) into the frequency domain where \( R(\omega) \) is small. However, because both correlators are positively defined matrices we cannot expect dramatic reduction of errors by some "cancelation effects". Moreover, different type of reservoirs and interactions produce different shapes of \( R(\omega) \), e.g. scattering with gas particles gives flat shapes while the linear coupling to quantum bosonic field produces the relations of type (58) with some cut-offs for large \( \omega \) (see [25] and the authors contribution to [11]).

Another different strategy is to select an optimal subspace of initial states \( \psi \) which are "decoherence-free" with respect to the leading error source. The following simple criterion is a generalization of the idea of quantum correcting codes [12].

Assume that the matrices \( R(\omega) \) is strictly positive for \( \omega \in \Omega \). Than any \( \psi \) satisfying the eigenvector condition for all \( \alpha \) and \( \omega \in \Omega \)

\[ Y_\alpha(\omega)\psi = \lambda_\alpha(\omega)\psi, \lambda_\alpha(\omega) \in \mathbb{C} \]  

(59)

evolves without errors.

Another partial results and examples of applications can be found in [25].

V. CONCLUSIONS

The author believes that the presented results, although often formulated in a rather abstract mathematical form, can serve as a guide in the discussion concerning the optimal strategies in controlling quantum devices. These strategies can be roughly divided into passive and active ones. At present, it seems that the passive ones can be realized in three apparently different ways:

I) using the symmetries of the coupling to the baths and systems’ Hamiltonian to obtain large enough decoherence-free subalgebras as presented in Section 2.3,

II) slowing down the action of quantum algorithm (slow gates) for the environments which satisfy the low frequency scaling (58),

III) using the symmetries of the coupling to the baths and systems’ Hamiltonian to obtain large enough decoherence-free subalgebras as presented in Section 2.3.
III) Speeding up the quantum algorithm (fast gates) for the environments with flat $R(\omega) \simeq R$.

The best known models of the type I) are given by the "superradiance" systems governed by the equations (37). In principle, they can be realized in different settings using collective coupling to bosonic fields (photons, phonons, etc.). Such reservoirs satisfy also the condition (58) what suggests that both I) and II) demand the similar technology. Indeed, the collective coupling of qubits to the bosonic field, which is a necessary condition for the validity of (37) is possible if the wave-length of the relevant bosonic modes is long in comparison with the diameter of the device [24]. It means that only the low frequencies can appear in the correlator $S(\omega)$ (55),(56). This is equivalent to the slow gates strategy II). Moreover, in both cases I) and II) the time scale of gates must grow with the size of the device (proportional to the number of qubits) in order to keep the total error constant.

The main obstacle for the strategy I) is the approximative character of the symmetry which becomes less accurate with the growing size of the device. Moreover, the Hamiltonian corrections due to the renormalization effects discussed in Section 4.2 do not possess this symmetry except of the cases with the very special geometry [24].

The strategies II) and III) are obviously contradictory and therefore for real systems, for which both types of reservoirs are at work, the optimization is needed.

The most elaborated active strategy is the idea of quantum error corrections for quantum computers [12]. The existing models of error corrections possess, however, several drawbacks. First of all, the noise is assumed to be independent of the algorithm what is equivalent to the assumption of the strategy III) ($R(\omega) \simeq R$). Moreover, the entropy produced in the computer must be very precisely taken away by "fresh" qubits prepared in pure states. Finally, initial errors for all qubits used in this scheme are not taken into account seriously.

Another active approach, called dynamical decoupling [13], involves the existence of the frequency domain for which the correlator $R(\omega) \simeq 0$. Then applying a suitable fast modulation of the system’s dynamics we can, in principle, move the support of the correlator $S(\omega)$ into this domain. Although for the specific environments the correlator $R(\omega)$ may have some "valleys", the overall tendency for $R(\omega)$ is to grow with $\omega$. This is related to the fact that the number of accessible quantum states of the environment typically grows with increasing energy scale. In any case, dynamical decoupling strategy demands fast gates technology as for III).

In authors’ opinion the optimized combination of the passive strategies I,II and III may be the most realistic attempt to deal with quantum noise and associated errors. It would be interesting to apply the formulas of the type (56) to estimate the minimal error per a single gate within the different proposed physical implementations of quantum computers (e.g. trapped ions, NMR, quantum dots, etc).

ACKNOWLEDGMENTS

A part of this paper (Sections 4,5) is based on the collaboration with Michał, Paweł and Ryszard Horodecki.

[1] M.A. Nielsen and I.L. Chuang: Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, 2000).
[2] G. Alber et.al.: Quantum Information (Springer, Berlin 2001)
[3] D. Bouwmaester et.al.: Nature 390, 575 (1997)
[4] M. Brune et.al.: Phys.Rev.Lett. 77, 4887 (1996) M. Arndt et.al.: Nature 401, 680 (1999)
[5] R. Blatt: Nature 404, 231 (2000)
[6] E.B. Davies, Quantum Theory of Open Systems, (Academic Press, London 1976)
[7] R. Alicki and K. Lendi: Quantum Dynamical Semigroups and Applications, (Springer, Berlin, 1987)
[8] R. Alicki and M. Fannes: Quantum Dynamical Systems, (Oxford University Press, Oxford, 2001)
[9] H-P. Breuer and F. Petruccione: Theory of Open Quantum Systems, (Oxford University Press, Oxford, 2002)
[10] G. Sewell: Quantum Mechanics and its Emergent Macrophysics, (Princeton University Press, Princeton, 2002)
[11] P. Garbaczewski and R. Olkiewicz (Eds.): Dynamics of Dissipation, LNP 597, (Springer, Berlin, 2002)
[12] E. Knill, R. Laflamme and W.H. Zurek: Science 279, 342 (1998)
[13] L. Viola and S. Lloyd: Phys.Rev. A 58, 2733 (1998)
[14] P. Zanardi and M. Rassetti: Phys.Rev.Lett. 79, 3306 (1997)
[15] W.F. Stinespring: Proc. Am. Math. Soc. 6, 211 (1955)
[16] K. Kraus: Annals of Physics 64, 311 (1971)
[17] V. Gorini, A. Kossakowski and E.C.G Sudarshan: J. Math. Phys.**17**, 821 (1976) G. Lindblad: Commun. Math. Phys.**48**, 119 (1976)
[18] K.R. Davidson: *C*-Algebras by Example, (AMS, Providence, 1996)
[19] H. Spohn: Lett.Math.Phys.**2**, 33 (1977) A. Frigerio: Lett.Math.Phys.**2**, 79 (1977)
[20] R. Alicki: Physica A **150**, 455 (1988)
[21] P. Zanardi: Phys.Rev. A **63**, 012301-1 (2001) S. De Filippo: Phys.Rev. A **62**, 052307-1 (2000) D.A. Lidar: *Contribution to this volume*
[22] M. Hamermesh: *Group Theory and Its Applications to Physical Problems*, (Addison-Wesley, London, 1964)
[23] R.H. Dicke: Phys.Rev.**93**, 99 (1954)
[24] M. Gross and S. Haroche: Phys.Rep.**93**, 301 (1982)
[25] R.Alicki, M. Horodecki, P. Horodecki and R. Horodecki: Phys.Rev. A **65**, 062101 (2002)