Coincidences and secondary Nielsen numbers

Ulrich Koschorke

To Professor Andrzej Granas

Abstract. Let \( f_1, f_2: X^m \to Y^n \) be maps between smooth connected manifolds of dimensions \( m \) and \( n \). Can \( f_1, f_2 \) be deformed by homotopies until they are coincidence free (i.e., \( f_1(x) \neq f_2(x) \) for all \( x \in X \))? The main tool for addressing such a problem is traditionally the (primary) Nielsen number \( N(f_1, f_2) \). For example, when \( m < 2n - 2 \), the question above has a positive answer precisely if \( N(f_1, f_2) = 0 \). However, when \( m = 2n - 2 \), this can be dramatically wrong, e.g. in the fixed point case when \( m = n = 2 \). Also, in a very specific setting the Kervaire invariant appears as a (full) additional obstruction.

In this paper we start exploring a fairly general new approach. This leads to secondary Nielsen numbers \( \text{Sec}N(f_1, f_2) \) which allow us to answer our question, e.g., when \( m = 2n - 2, n \neq 2 \), is even and \( Y \) is simply connected.

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1. Introduction

Throughout this paper let
\[
f_1, f_2: X^m \to Y^n
\]
be (continuous) maps between smooth connected manifolds, without boundary, of dimensions \( m, n \geq 1 \), where \( X \) is compact.

We are interested in those aspects of the coincidence subspace
\[
C(f_1, f_2) := \{ x \in X \mid f_1(x) = f_2(x) \}
\]
in \( X \) which remain unchanged by homotopies of \( f_1 \) and \( f_2 \). These aspects are
reflected to a large extent by the minimum numbers
\[ \text{MC}(f_1, f_2) := \min \{ \#C(f'_1, f'_2) \mid f'_1 \sim f_1, f'_2 \sim f_2 \} \]
and, better yet,
\[ \text{MCC}(f_1, f_2) := \min \{ \#\pi_0(C(f'_1, f'_2)) \mid f'_1 \sim f_1, f'_2 \sim f_2 \} \]
of coincidence points and of coincidence components, respectively, as the maps vary within the given homotopy classes \([f_1], [f_2]\).

The principal problem in topological coincidence theory is to determine these minimum numbers, and, in particular, to decide when they vanish, i.e., when \((f_1, f_2)\) is homotopic to a coincidence free pair. In this case we say that the pair \((f_1, f_2)\) is loose.

Generalizing a well-known notion from fixed point theory, we introduced in [Ko2] a Nielsen number \(N(f_1, f_2)\) which depends only on the homotopy classes of \(f_1\) and \(f_2\) and satisfies
\[ 0 \leq N(f_1, f_2) \leq \text{MCC}(f_1, f_2) \leq \text{MC}(f_1, f_2) \leq \infty. \]
Furthermore, we proved the following result (see [Ko2, Theorem 1.10]).

**Wecken theorem in coincidence theory.** Assume \(m < 2n - 2\). Then for all maps \(f_1, f_2 : X^m \to Y^n\) we have \(\text{MCC}(f_1, f_2) = N(f_1, f_2)\). In particular, \((f_1, f_2)\) is loose if and only if \(N(f_1, f_2) = 0\).

**Central Question.** What happens when \(m \geq 2n - 2\)? Can we pin down extra looseness obstructions (besides the “primary” Nielsen number \(N(f_1, f_2)\))?

Many specific examples are known where the last claim in the Wecken theorem fails to hold as soon as the dimension assumption is not satisfied (for a detailed discussion of some of these examples see, e.g., [Ko3, 1.18–1.29] or [KR, Table 1.18]).

Already in the first critical dimension setting (when \(m = 2n - 2\)) we encounter very interesting phenomena.

**Example 1.1 (Fixed point theory).** Here \(f = f_1\) is an arbitrary self-map of \(X = Y\), \(f_2 = \text{identity map}, m = n\). In the dimension range \(m \geq 3\) our Wecken theorem implies that
\[ N(f, \text{id}) = \text{MCC}(f, \text{id}) = \text{MC}(f, \text{id}) \]
agrees with the minimum number
\[ \text{MF}(f) = \min \{ \#C(f', \text{id}) \mid f' \sim f \} \]
of fixed points. This is the classical Wecken theorem (from 1941/42) for closed smooth manifolds (cf. [B, p. 12] and [Ko2, pp. 225–227]).

In dimension \(m = 2\), J. Nielsen had already shown in the 1920s that \(\text{MF}(f) = N(f, \text{id})\) holds whenever \(X\) is a closed connected surface with Euler characteristic \(\chi(X) \geq 0\). For a long time this restriction was believed to be merely technical.
However, in 1985 B. Jiang proved that each surface $X$ having a strictly negative Euler characteristic allows a self-map $f$ such that $MF(f) \neq N(f, \text{id})$ (cf. [J]). Actually, later X. Zhang, M. Kelly and B. Jiang showed much more: whenever $\chi(X) < 0$ the difference $MF(f) - N(f, \text{id})$ becomes arbitrarily large for suitable self-maps $f$ of the surface $X$ (cf. [B, p. 16]).

B. Jiang used an approach via braid groups. Could the phenomena described above also be captured by secondary obstructions?

**Example 1.2.** Consider the case $X = S^{2n-2}$, $Y = \mathbb{R}P(n)$. The following result was established in [Ko3, 1.27] and [KR, 1.13].

**Theorem 1.3.** Assume $n$ is even, $n \neq 2, 4, 8$. Let $\tilde{f} : S^{2n-2} \to S^n$ be a lifting of a map $f : S^{2n-2} \to \mathbb{R}P(n)$.

Then the pair $(f, f)$ is loose if and only if both $N(f, f)$ and the Kervaire invariant $K(\tilde{f})$ vanish.

Originally, M. Kervaire introduced his ($\mathbb{Z}_2$-valued) invariant in order to exhibit a triangulable closed manifold which does not admit any differentiable structure (cf. [Ke]). Subsequently, M. Kervaire and J. Milnor used it in their classification of exotic spheres (cf. [KM]). And now the Kervaire invariant makes a somewhat surprising appearance as a full secondary looseness obstruction in a very specific self-coincidence setting.

In this paper we start exploring the following general approach to constructing secondary looseness obstructions when $m \geq 2n - 2$. Given a pair of maps $f_1, f_2 : X^m \to Y^n$ such that $N(f_1, f_2) = 0$, let us try to mimic the proof of the Wecken theorem in [Ko2] and measure somehow the obstacles which we encounter in the process.

We concentrate on the first critical dimension setting $m = 2n - 2$. Here the first two steps of the proof in [Ko2] (embedding a nullbordism and describing its normal bundle via a suitable desuspension) present no difficulties. But the third step leads to a pair of maps $w_1, w_2 : W^n \to Y^n$ which is coincidence free on the boundary of a given compact $n$-manifold $W$ and must be made coincidence free on all of $W$. This “secondary coincidence problem” leads to our definition of the secondary Nielsen number $\text{Sec}N(f_1, f_2)$, a non-negative integer which depends only on the homotopy classes of $f_1$ and $f_2$. Clearly, if the pair $(f_1, f_2)$ is loose to begin with, then

$$N(f_1, f_2) = 0 = \text{Sec}N(f_1, f_2).$$

In turn we have the following result.

**Theorem 1.4.** Assume that $Y^n$ is simply connected, with $n$ even, $n \neq 2$. Given arbitrary maps $f_1, f_2 : X^{2n-2} \to Y^n$ we have: the pair $(f_1, f_2)$ is loose if and only if both the (primary) Nielsen number $N(f_1, f_2)$ and the secondary Nielsen number $\text{Sec}N(f_1, f_2)$ vanish.

**Example 1.5.** Let $f : S^{2n-2} \to S^n$ be a map between spheres of the indicated dimensions. Then $N(f, f)$ is known to be a full looseness obstruction (this was shown, e.g., in [Ko3, 1.19]). Hence $\text{Sec}N(f, f) = 0$. 
Example 1.6. Let $Y = \mathbb{K}P(n')$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, $n' \geq 2$, be complex or quaternionic projective space of (real) dimension $n = dn'$, where
\[ d = \dim_{\mathbb{R}}(\mathbb{K}) \in \{2, 4\}. \]
Let $\partial_Y : \pi_m(Y) \to \pi_{m-1}(S^{n-1})$ denote the boundary homomorphism in the exact homotopy sequence of the tangent sphere bundle $ST(Y)$ over $Y$. Then, given a map $f : S^{2n-2} \to Y$, it is well known (cf. [Ko3, 1.19]) that the pair $(f, f)$ is loose precisely if $\partial_Y([f]) = 0$; in contrast, the Nielsen number $N(f, f)$ vanishes precisely if the suspended value $E(\partial_Y([f])) \in \pi_{2n-2}(S^n)$ is trivial.

Now assume that $n \neq 4, 8$ and that $N(f, f) = 0$. Then
\[ \partial_Y([f]) \in \text{Ker} E \cong \mathbb{Z}_2 = \{0, 1\} \]
agrees with the secondary Nielsen number $\text{Sec}N(f, f)$. It can take a nontrivial value here if and only if
\[ \{0\} \neq \partial_Y(\pi_{2n-2}(\mathbb{K}P(n'))) \cap \text{Ker} E: \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n) \]
(i.e., the Wecken condition for $(2n - 2, \mathbb{K}P(n'))$, cf. [Ko3, Definition 1.18], fails to hold), or, equivalently, if and only if
\[ 0 = j_K^*([\tau_{n-1}, \tau_{n-1}]) \in \pi_{2n-3}(V_{n'+1,2}(\mathbb{K})); \]
here
\[ j_K : S^{n-1} \subset V_{n'+1,2}(\mathbb{K}), \quad j_K(v) := ((0, \ldots, 0, 1), (v, 0)), \quad v \in S^{n-1}, \]
denotes the fiber inclusion into the Stiefel manifold of orthonormal 2-frames in $\mathbb{K}^{n'+1}$.

2. Primary Nielsen numbers

In this section we recall basic geometric facts about coincidences and the resulting definition of the Nielsen number $N(f_1, f_2)$ of a given pair of maps $f_1, f_2 : X^m \to Y^n$, $m, n \geq 1$. (For details see [Ko2].)

After suitable approximations we may assume that $(f_1, f_2)$ is a generic pair; i.e., $(f_1, f_2) : X \to Y \times Y$ is smooth and transverse to the diagonal $\Delta := \{(y_1, y_2) \in Y \times Y \mid y_1 = y_2\}$. Then the coincidence locus
\[ C := C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta) \]
is a closed smooth submanifold of $X$. It is naturally equipped with two important geometric “coincidence data.” On the one hand, we have a map
\[ \tilde{g} : C \to E(f_1, f_2) := \{(x, \theta) \in X \times Y^I \mid \theta(0) = f_1(x), \theta(1) = f_2(x)\} \quad (2.1) \]
such that
\[ g := \text{pr} \circ \tilde{g} = \text{inclusion} : C \subset X, \]
where $\text{pr}$ denotes the projection from the “path space” $E(f_1, f_2)$ to $X$; $\tilde{g}$ is defined by
\[ \tilde{g}(x) = (x, \text{constant path at } f_1(x) = f_2(x)), \quad x \in C. \]
On the other hand, the normal bundle \( \nu(C, X) \) of \( C \) in \( X \) is described by the vector bundle isomorphism
\[
\nu(C, X) \xrightarrow{T(f_1, f_2)} ((f_1, f_2)|C)^* (\nu(\Delta, Y \times Y)) \cong f_1^*(TY)|C
\]
(2.2')
over \( C \); in turn this yields the (stable) isomorphism
\[
\bar{g}: TC \oplus f_1^*(TY)|C \cong TX|C.
\]
The resulting normal bordism class
\[
\bar{\omega}(f_1, f_2) := [(C, \bar{g}, g)] \in \Omega_{m-n}(E(f_1, f_2); \bar{\varphi})
\]
is our basic primary looseness obstruction (cf. [Ko2]).

The decomposition of the path space \( E(f_1, f_2) \) into its path-components \( A \) yields the decomposition of the coincidence manifold \( C \) into the closed manifolds
\[
C_A := \bar{g}^{-1}(A), \quad A \in \pi_0(E(f_1, f_2))
\]
(named Nielsen classes of the pair \( (f_1, f_2) \)). The path-component
\[
A \in \pi_0(E(f_1, f_2))
\]
is called either inessential or essential according as the bordism class
\[
\bar{\omega}_A(f_1, f_2) := [C_A, \bar{g}|C_A, g] \in \Omega_{m-n}(E(f_1, f_2); \bar{\varphi})
\]
(of the coincidence data restricted to \( C_A \)) vanishes or not. By definition, the Nielsen number \( N(f_1, f_2) \) is the number of essential path-components \( A \) of \( E(f_1, f_2) \). It is finite and it depends only on the homotopy classes of \( f_1 \) and \( f_2 \).

**Remark.** The Nielsen number \( N(f_1, f_2) \) is denoted by \( \tilde{N}(f_1, f_2) \) in [Ko3, Ko4]. An important role is also played by the refined (nonstabilized) version \( N^#(f_1, f_2) \) of our Nielsen number (cf., e.g., [Ko3, Ko4]). Furthermore, in [Ko4] a whole intermediate hierarchy
\[
(MC \geq MCC \geq) N^# \equiv N_0 \geq N_1 \geq N_2 \geq \cdots \geq N_r \geq \cdots \geq N_\infty \equiv \tilde{N} \geq 0
\]
of (primary) Nielsen numbers is discussed. However, they all coincide in the special dimension setting \( m = 2n - 2 \) which will interest us in the remainder of this paper.

3. **Secondary Nielsen numbers**

Now we concentrate on the case \( m = 2n - 2, n \geq 2 \).

Let \( f_1, f_2: X^{2n-2} \to Y^n \) be a generic pair. Assume that the Nielsen coincidence class \( C_A = \bar{g}^{-1}(A) \) corresponding to some path-component \( A \) of \( E(f_1, f_2) \) is inessential. Then we can choose a connected, \((n-1)\)-dimensional nullbordism \( B \) of \( C_A \), together with maps
\[
\tilde{G}: B \to E(f_1, f_2) \quad \text{and} \quad G := \text{pr} \circ \tilde{G}: B \to X
\]
(3.1)
extending \( \bar{g} \) and \( g \), respectively (compare (2.1)) on the one hand, as well as
a (stable) vector bundle isomorphism
\[ \overline{G} : TB \oplus G^*(f_1^*(TY)) \cong G^*(TX) \oplus \mathbb{R} \] extending \( \overline{g} \) (cf. (2.2)) on the other hand.

Due to our dimension assumption the \( n \)-plane bundle \( G^*(f_1^*(TY)) \) allows a nowhere zero section which spans a trivial line bundle \( \mathbb{R} \) over \( B \). In view of (3.2) and according to Smale–Hirsch theory we can deform \( G \) until it is an immersion. Generically its self-intersection set consists of finitely many isolated points. We may “push” them all along suitable arcs across the boundary \( \partial B = C \). So in the end \( G \) is a smooth embedding which extends the inclusion \( C \subset X \).

Next let \( \nu(G) \) be the normal bundle of \( G \). Compose the obvious isomorphism
\[ TB \oplus \nu(G) \oplus \mathbb{R} \cong TX|B \oplus \mathbb{R} \]
with the isomorphism \( \overline{G} \) in (3.2). Again in view of our dimension assumption we can desuspend to get the isomorphism
\[ \nu(G) \oplus \mathbb{R} \cong f_1^*(TY)|B \] (3.2')
which extends (2.2') when we put \( \nu(C_A, B) = \mathbb{R}|C_A \).

As in the proof of the Wecken theorem in [Ko2, pp. 223–224], let \( i_2 : B \to [0, 1] \) be a smooth function which is essentially defined by the second projection on a collar \( C_A \times [0, \frac{1}{2}] \) of \( C_A \) in \( B \), and takes the constant value \( \frac{1}{2} \) outside of this collar. Consider the embedding
\[ i := (G, i_2) : B \hookrightarrow X \times [0, 1] \]
which extends the embedding \( g : \partial B = C_A \hookrightarrow X = X \times \{0\} \) (compare [Ko1, Figure 3.8]). As in [Ko2, pp. 223–224], the isomorphism in (3.2') allows us to extend \((f_1, f_2)\), defined on \( X = X \times \{0\} \), to a map
\[ (F_1, F_2) : X \times \{0\} \cup T \to Y \times Y, \]
where \( T \) is a suitable (compact) tubular neighborhood of \( i(B) \) in \( X \times [0, 1] \); the coincidence locus of the pair \((F_1, F_2)\) is just \( i(B) \).

Now let \( W \subset X \times I \) denote that part of the “shadow”
\[ R_1 = \{(x, t) \in G(B) \times I \mid 0 \leq t \leq i_2(x)\} \]
(compare [Ko1, pp. 38–39]) which lies outside of the interior \( \hat{T} \) of \( T \). The map \( \overline{G} \) (cf. (3.1)) determines an extension of the coincidence free pair
\[ (F_1|\partial W, F_2|\partial W) \]
to the whole \( n \)-dimensional manifold \( W \). But now coincidences (which are generically 0 dimensional) may occur in the interior of \( W \). They lead to
\[ A \in \pi_0(E(f_1, f_2)) \]
being called either 2-essential (i.e., essential of second order) or not. If \( C_A \) can be made empty by suitable homotopies of \( f_1 \) and \( f_2 \), then \( A \) is certainly 2-inessential.
Now assume that the (primary) Nielsen number \( N(f_1, f_2) \) vanishes. Then each path-component \( A \) of \( E(f_1, f_2) \) is inessential in the classical (primary) sense, and hence either 2-essential or not. We define the secondary Nielsen number \( \text{Sec} N(f_1, f_2) \) to be the number of 2-essential path-components \( A \in \pi_0(E(f_1, f_2)) \). Clearly \( \text{Sec} N(f_1, f_2) \) is a nonnegative integer smaller than or equal to the (not necessarily finite) number of path-components of \( E(f_1, f_2) \) (which is also known as the Reidemeister number of \( (f_1, f_2) \); see [Ko2, 2.1] for a way to compute this Reidemeister number).

If \( \pi_1(Y) = 0 \), then not only \( E(f_1, f_2) \) but also \( E(F_1, F_2) \) (in the construction discussed above) is path-connected. This allows us to prove Theorem 1.4. Details and generalizations will be given in a future paper.

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