ON TZITZEICA CURVES IN EUCLIDEAN 3-SPACE $\mathbb{E}^3$

Bengü Bayram, Emrah Tunç, Kadri Arslan and Günay Öztürk

Abstract. In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. We characterize such curves according to their curvatures. We show that there is no Tz-curve with constant curvatures (W-curves). We consider Salkowski (TC-curve) and anti-Salkowski curves.

Keywords: Tz-curves, W-curves, TC-curves

1. Introduction

Gheorgha Tzitzeica, a Romanian mathematician (1872-1939), introduced a class of curves, nowadays called Tzitzeica curves, and a class of surfaces of the Euclidean 3-space called Tzitzeica surfaces. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ for which the ratio of its torsion $\kappa_2$ and the square of the distance $d_{osc}$ from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e.,

$$\frac{\kappa_2}{d_{osc}^2} = a$$

where $d_{osc} = \langle N_2, x \rangle$ and $a \neq 0$ is a real constant, $N_2$ is the binormal vector of $x$.

In [3] the authors gave the connections between the Tzitzeica curve and the Tzitzeica surface in a Minkowski 3-space and the original ones from the Euclidean 3-space. In [7] the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in a Euclidean space. In [12], the elliptic cylindrical curves verifying Tzitzeica condition were adapted to the Minkowski 3-space. In [2], the authors gave the necessary and sufficient condition for a space curve to become a Tzitzeica curve. The new classes of symmetry reductions for the Tzitzeica curve equation were determined. In [1], the authors were interested in the curves of Tzitzeica type and they investigated the conditions for non-null general helices, pseudo-spherical curves and pseudo-spherical general helices to become of Tzitzeica type in a Minkowski space $\mathbb{E}_1^3$. 

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A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface $M$ given with the parametrization $X(u,v)$ for which the ratio of its Gaussian curvature $K$ and the distance $d_{\tan}$ from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e.,

$$\frac{K}{d_{\tan}^4} = a_1$$

for a constant $a_1$. The orthogonal distance from the origin to the tangent plane is defined by

$$d_{\tan} = \langle X, \vec{U} \rangle$$

where $X$ is the position vector of the surface and $\vec{U}$ is a unit normal vector of the surface.

The asymptotic lines of a Tzitzeica surface with a negative Gaussian curvature are Tzitzeica curves [7]. In [18], the authors gave the necessary and sufficient condition for the Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. In addition, a new Tzitzeica hypersurface was obtained in parametric, implicit and explicit forms in [8].

In this study, we consider Tzitzeica curves (Tz-curves) in a Euclidean 3-space $\mathbb{E}^3$. Furthermore, we investigate a Tzitzeica curve in a Euclidean 3-space $\mathbb{E}^3$ whose position vector $x(s) = x'(s)$ satisfies the parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s),$$

for some differentiable functions, $m_i(s)$, $0 \leq i \leq 2$, where $\{T, N_1, N_2\}$ is the Frenet frame of $x$. We characterize such curves according to their curvatures. We show that there is no Tzitzeica curve in $\mathbb{E}^3$ with constant curvatures (W-curves). We give the relations between the curvatures of the Tz-Salkowski curve (TC-curve) and the Tz-anti-Salkowski curve.

2. Basic Notations

Let $x: I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in a Euclidean 3-space $\mathbb{E}^3$. Let us denote $T(s) = x'(s)$ and call $T(s)$ a unit tangent vector of $x$ at $s$. We denote the curvature of $x$ by $\kappa_1(s) = ||x''(s)||$. If $\kappa_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve $x$ at $s$ is given by $x''(s) = \kappa_1(s)N_1(s)$. The unit vector $N_2(s) = T(s) \times N_1(s)$ is called the unit binormal vector of $x$ at $s$. Then we have the Serret-Frenet formulae:

$$T'(s) = \kappa_1(s)N_1(s),$$

$$N_1'(s) = -\kappa_1(s)T(s) + \kappa_2(s)N_2(s),$$

$$N_2'(s) = -\kappa_2(s)N_1(s),$$

where $\kappa_2(s)$ is the torsion of the curve $x$ at $s$ (see, [10]).
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If the Frenet curvature $\kappa_1(s)$ and torsion $\kappa_2(s)$ of $x$ are constant functions then $x$ is called a screw line or a helix [9]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations then F. Klein and S. Lie called them W-curves [14]. It is known that a curve $x$ in $E^3$ is called a general helix if the ratio $\kappa_2(s)/\kappa_1(s)$ is a nonzero constant [16]. Salkowski (resp. anti-Salkowski) curves in a Euclidean space $E^3$ are generally known as the family of curves with a constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization [15, 17] (for T.C-curve see also [13]).

For a space curve $x: I \subset \mathbb{R} \rightarrow E^3$, the planes at each point of $x(s)$ spanned by $\{T, N_1\}$, $\{T, N_2\}$ and $\{N_1, N_2\}$ are known as the osculating plane, the rectifying plane and normal plane, respectively. If the position vector $x$ lies on its rectifying plane, then $x(s)$ is called rectifying curve [5]. Similarly, the curve for which the position vector $x$ always lies in its osculating plane is called osculating curve. Finally, $x$ is called normal curve if its position vector $x$ lies in its normal plane.

Rectifying curves characterized by the simple equation

\[ x(s) = \lambda(s)T(s) + \mu(s)N_2(s), \]

where $\lambda(s)$ and $\mu(s)$ are smooth functions and $T(s)$ and $N_2(s)$ are tangent and binormal vector fields of $x$, respectively [5, 6].

For a regular curve $x(s)$, the position vector $x$ can be decomposed into its tangential and normal components at each point:

\[ x = x^T + x^N. \]

A curve in $E^3$ is called $N$-constant if the normal component $x^N$ of its position vector $x$ is of constant length [4, 11]. It is known that a curve in $E^3$ is congruent to an $N$-constant curve if and only if the ratio $\kappa_2/\kappa_1$ is a non-constant linear function of an arc-length function $s$, i.e., $\kappa_2/\kappa_1(s) = c_1 s + c_2$ for some constants $c_1$ and $c_2$ with $c_1 \neq 0$ [4]. Further, an $N$-constant curve $x$ is called first kind if $\|x^N\| = 0$, otherwise second kind [11].

3. Tzitzeica Curves in $E^3$

In the present section we characterize Tzitzeica curves in $E^3$ in terms of their curvatures.

**Definition 3.1.** Let $x: I \subset \mathbb{R} \rightarrow E^3$ be a unit speed curve with curvatures $\kappa_1(s) > 0$ and $\kappa_2(s) \neq 0$. If the torsion of $x$ satisfies the condition

\[ \kappa_2(s) = a d_{osc}^2, \]

for some real constant $a$ then $x$ is called Tzitzeica curve (Tz-curve), where

\[ d_{osc} = \langle N_2, x \rangle \]

is the orthogonal distance from the origin to the osculating plane of $x$. 

We have the following result.

**Proposition 3.1.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \). If \( x \) is a Tz-curve, then the equation

\[
\kappa_2' \langle x, N_2 \rangle + 2\kappa_2^2 \langle x, N_1 \rangle = 0
\]

holds.

**Proof.** Let \( x \) be a unit speed curve in \( \mathbb{E}^3 \), then by the use of the equations (3.1) and (3.2) we get

\[
\frac{\kappa_2(s)}{\langle N_2, x \rangle^2} = a \neq 0.
\]

Further, differentiating the equation (3.4), we obtain the result. \( \Box \)

**Definition 3.2.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve with curvatures \( \kappa_1(s) > 0 \) and \( \kappa_2(s) \neq 0 \). Then \( x \) is a spherical curve if and only if

\[
\left(\frac{\kappa_2(s)}{\kappa_1(s)}\right)' = \left(\frac{\kappa_1(s)}{\kappa_2(s)}\right) \left(\frac{\kappa_2(s)}{\kappa_1(s)}\right)'
\]

holds [9].

**Theorem 3.1.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed spherical curve in \( \mathbb{E}^3 \). If \( x \) is a Tz-curve then the equation

\[
\kappa_2'(s) \frac{2\kappa_2^2(s)}{\kappa_1(s)} = \frac{\kappa_1(s)}{\kappa_1'(s)}
\]

holds between the curvatures of \( x \).

**Proof.** Let \( x \) be a unit speed spherical curve in \( \mathbb{E}^3 \). Then we have

\[
\|x\| = r
\]

where \( r \) is the radius of the sphere. Differentiating the equation (3.7) with respect to \( s \), we get

\[
\langle x, T \rangle = 0.
\]

Further, differentiating the equation (3.8), we have

\[
\langle x, N_1 \rangle = -\frac{1}{\kappa_1}.
\]

By differentiating the equation (3.9), we obtain

\[
\langle x, N_2 \rangle = \frac{\kappa_1'}{\kappa_1^2\kappa_2}.
\]

Finally, substituting (3.9) and (3.10) into (3.3), we get the result. \( \Box \)
Corollary 3.1. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed spherical Tz-curve in \( \mathbb{E}^3 \). Then the torsion of \( x \) satisfies the equation

\[
(3.11) \quad \kappa_2 = \sqrt{\frac{\kappa_1'' \kappa_1 - 2 (\kappa_1')^2}{3 \kappa_1^3}}.
\]

Proof. Substituting (3.6) into (3.5), we get the result. \( \square \)

Corollary 3.2. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski spherical Tz-curve in \( \mathbb{E}^3 \). Then the curvature of \( x \) is given by

\[
(3.12) \quad \kappa_1 = \frac{\sqrt{3 \kappa_2}}{c_1 \sin (\sqrt{3 \kappa_2} s) - c_2 \cos (\sqrt{3 \kappa_2} s)}
\]

where \( c_1, c_2 \) are integral constants and \( \kappa_2 \) is the constant torsion of \( x \).

Proof. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski spherical Tz-curve in \( \mathbb{E}^3 \). Then from (3.11), we obtain the differential equation

\[
(3.13) \quad \kappa_1'' \kappa_1 - 2 (\kappa_1')^2 - 3 \kappa_1^3 \kappa_2^2 = 0
\]

which has the solution (3.12). \( \square \)

Lemma 3.1. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \) whose position vector satisfies the parametric equation

\[
(3.14) \quad x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s)
\]

for some differentiable functions, \( m_i(s), 0 \leq i \leq 2 \). If \( x \) is a Tz-curve then we get

\[
(3.15) \quad \begin{align*}
    m_0' - \kappa_1 m_1 &= 1, \\
    m_1' + \kappa_1 m_0 - \kappa_2 m_2 &= 0, \\
    m_2' + \kappa_2 m_1 &= 0, \\
    \kappa_2^2 m_2 + 2 \kappa_1^2 m_1 &= 0.
\end{align*}
\]

Proof. Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed curve in \( \mathbb{E}^3 \). Then, by taking the derivative of (3.14) with respect to the parameter \( s \) and using the Frenet formulae, we obtain

\[
(3.16) \quad x'(s) = (m_0'(s) - \kappa_1(s)m_1(s))T(s) + (m_1'(s) + \kappa_1(s)m_0(s) - \kappa_2(s)m_2(s))N_1(s) + (m_2'(s) + \kappa_2(s)m_1(s))N_2(s).
\]

Further, using the equations (3.3) and (3.16), we get (3.15). \( \square \)
**Theorem 3.2.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \) (with the curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then \( x \) is congruent to a rectifying curve with the parametrization

\[
x(s) = (s + c_1) T(s) + c_2 N_2(s)
\]

where \( c_1 \) and \( c_2 \) are integral constants.

**Proof.** Let \( x \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \). Then, the torsion \( \kappa_2 \) of \( x \) is constant. From the equation (3.15), we get

\[
\begin{align*}
    m_0 &= s + c_1 \\
    m_1 &= 0 \\
    m_2 &= c_2
\end{align*}
\]

where \( c_1 \) and \( c_2 \) are integral constants. Finally, substituting (3.18) into (3.14), we get the result. \( \square \)

**Corollary 3.3.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed anti-Salkowski Tz-curve in \( \mathbb{E}^3 \) (with curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then \( x \) is congruent to \( N \)-constant curve of second kind.

**Corollary 3.4.** Let \( x : I \subset \mathbb{R} \to \mathbb{E}^3 \) be a unit speed Salkowski Tz-curve in \( \mathbb{E}^3 \) (with the curvatures \( \kappa_1 > 0 \) and \( \kappa_2 \neq 0 \)) given with the parametrization (3.14). Then we have

\[
m''_1 + (\kappa_1^2 + 3\kappa_2^2) m_1 + \kappa_1 = 0
\]

where the curvature \( \kappa_1 \) of \( x \) is a real constant.

**Proof.** Let \( x \) be a unit speed Salkowski Tz-curve in \( \mathbb{E}^3 \). Hence, the curvature \( \kappa_1 \) of \( x \) is constant, from the equation (3.15), we get the result. \( \square \)

**Corollary 3.5.** There is no Tz-curve with a constant curvature and a constant torsion. (i.e. Tz-W-curve)

**Proof.** Let \( x \) be a unit speed Tz-curve in \( \mathbb{E}^3 \) with a constant curvature and a constant torsion. (i.e. Tz-W-curve). Then, using (3.15), we obtain

\[
\frac{\kappa_1(s)}{\kappa_2(s)} = \frac{c_2}{s + c_1}
\]

which is a contradiction. \( \square \)
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REFERENCES

1. M.E. Aydın, M. Ergüt: *Non-null curves of Tzitzeica type in Minkowski 3-space*. Romanian J. of Math. and Comp. Science 4(1) (2014), 81-90.

2. N. Bila: *Symmetry reductions for the Tzitzeica curve equation*. Math. and Comp. Sci. Workin Papers 16 (2012).

3. A. Bobe, W. G. Boskoff and M. G. Ciuca: *Tzitzeica type centro-affine invariants in Minkowski space*. An. St. Univ. Ovidius Constanta 20(2) (2012), 27-34.

4. B. Y. Chen: *Geometry of warped products as Riemannian submanifolds and related problems*. Soochow J. Math. 28 (2002), 125-156.

5. B. Y. Chen: *Convolution of Riemannian manifolds and its applications*. Bull. Aust. Math. Soc. 66 (2002), 177-191.

6. B.Y. Chen: *When does the position vector of a space curve always lies in its rectifying plane?*. Amer. Math. Monthly 110 (2003), 147-152.

7. M. Craşmareanu: *Cylindrical Tzitzeica curves implies forced harmonic oscillators*. Balkan J. of Geom. and Its App. 7(1) (2002), 37-42.

8. O. Constantinescu, M. Craşmareanu,: *A new Tzitzeica hypersurface and cubic Finslerian metrics of Berwald type*. Balkan J. of Geom. and Its App. 16(2) (2011), 27-34.

9. A. Gray: *Modern differential geometry of curves and surface*, CRS Press, Inc. 1993.

10. H. Gluck: *Higher curvatures of curves in Euclidean space*. Amer. Math. Monthly 73 (1966), 699-704.

11. S. Gürpinar, K. Arslan, G. Öztürk: *A Characterization of Constant-ratio Curves in Euclidean 3-space $\mathbb{E}^3$*. Acta Universitatis Apulensis 44 (2015), 39-51.

12. M. K. Karacan, B. Bükçü: *On the elliptic cylindrical Tzitzeica curves in Minkowski 3-space*. Sci. Manga 5 (2009), 44-48.

13. B. Kılıç, K. Arslan and G. Öztürk: *Tangentially cubic curves in Euclidean spaces*. Differential Geometry-Dynamical Systems 10 (2008), 186-196.

14. F. Klein, S. Lie: *Über diejenigen ebene Kurven welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen*. Math. Ann. 4 (1871), 50-84.

15. J. Monterde: *Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion*. Computer Aided Geometric Design. 26 (2009) 271–278.

16. G. Öztürk, K. Arslan and H. Haci salihoglu: *A characterization of ccr-curves in $\mathbb{R}^n$*. Proc. Estonian Acad. Sciences 57 (2008), 217-224.

17. E. Salkowski: *Zur transformation von raum-krümmungen*. Mathematische Annalen. 66(4) (1909) 517–557.

18. G. E. Vilcu: *A geometric perspective on the generalized Cobb-Douglas production function*. Appl. Math. Lett. 24 (2011), 777-783.
Bengü Bayram, Emrah Tunç
Department of Mathematics
Balıkesir University
Balıkesir, TURKEY
benguk@balikesir.edu.tr, emrahtunc172@gmail.com

Kadri Arslan
Uludağ University
Department of Mathematics
Bursa, TURKEY
arslan@uludag.edu.tr

Gülay Öztürk
Izmir Democracy University
Department of Mathematics
Izmir, TURKEY
gunay.ozturk@idu.edu.tr