ON THE EXCURSION SETS OF SPHERICAL GAUSSIAN
EIGENFUNCTIONS

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Abstract. The high frequency behaviour for random eigenfunctions
of the spherical Laplacian has been recently the object of considerable
interest, also because of strong motivations arising from Physics and
Cosmology. In this paper, we are concerned with the high frequency
behaviour of excursion sets; in particular, we establish a Uniform Central
Limit Theorem for the empirical measure, i.e., the proportion of spherical
surface where spherical Gaussian eigenfunctions lie below a level \( z \). Our
proofs borrows some techniques from the literature on stationary long
memory processes; in particular, we expand the empirical measure into
Hermite polynomials, and establish a uniform weak reduction principle,
entailing that the asymptotic behaviour is asymptotically dominated by
a single term in the expansion. As a result, we establish a functional
central limit theorem; the limiting process is fully degenerate.

Keywords and Phrases: Gaussian Eigenfunctions, Excursion
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Central Limit Theorem

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1. Introduction

1.1. Background. Let \( \Delta_{S^2} \) be the usual Laplacian on the sphere \( S^2 \), i.e.,
given by

\[
\Delta_{S^2} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi
\]

in the usual spherical coordinates \((\theta, \phi)\). An orthonormal family of eigen-
functions for the Laplacian is well-known to be given by the (complex-valued) spherical harmonics \( \{Y_{lm}\}_{m=-l,...,l} \), i.e.

\[
\Delta_{S^2} Y_{lm} = -(l+1)Y_{lm}, \quad l = 1, 2, 3, ..., \int_{S^2} Y_{lm}(\theta, \phi)Y_{lm}(\theta, \phi) \sin \theta d\theta d\phi = \delta_{mm'},
\]

where \( \delta_{mm'} \) is the usual Kronecker delta. For definiteness, we recall that in
coordinates, the (complex-valued) spherical harmonics are defined by

\[
Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta) \exp(im\phi),
\]

\[
P_{lm}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l,
\]

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{P_{lm}(.)} denoting associated Legendre polynomials. The \((2l + 1)\) spherical harmonics \(\{Y_{lm}\}\) thus provide an orthonormal system for the space \(\mathcal{H}_l\) of eigenfunctions corresponding to the eigenvalue \(-l(l + 1)\), and we have the expansion

\[
L^2(S^2) = \bigoplus_{l=1}^{\infty} \mathcal{H}_l.
\]

We shall equip each of the spaces \(\{\mathcal{H}_l\}\) with a rotation-invariant Gaussian probability measure, and focus on the asymptotic behaviour of the random eigenfunctions \(\{f_t(.)\}_{t=1,2,...}\), i.e.

\[
f_t(x) = \sum_{m=-l}^{l} a_{tm} Y_{lm}(x), \quad x \in S^2,
\]

where the coefficients \(\{a_{tm}\}\) are mean zero (complex) Gaussian random variables such that \(Ea_{tm} = 0\), \(Ea_{tm}a_{m't} = (2l + 1)^{-1} \delta_{m'm} \), \(a_{lm} = (-1)^{m} a_{l,-m}\), whence

\[
Ef_t(x) \equiv 0, \quad Ef_t^2(x) = (2l + 1)E|a_{tm}|^2 = 1.
\]

Equivalently, \(f_t\) could be expanded in a real-valued basis; throughout this paper, however, we stick to the complex-valued representation, which has simpler symmetry properties. The random functions \(\{f_t(.)\}\) are easily seen to be isotropic, i.e. for all \(g \in SO(3)\) (the rotation group on \(\mathbb{R}^3\)) we have

\[
f_t(gx) \overset{d}{=} f_t(x),
\]

\(d\) denoting identity in law, in the sense of stochastic processes.

As well-known, the spherical Gaussian eigenfunctions emerge very naturally also in the analysis of random fields, since they provide the Fourier components in spectral representation expansions. The geometry of random fields has been of course extensively studied in the mathematical literature: we refer to the well-known book [1] for a complete list of references. Several statistics for Gaussian fields and their excursion sets have been investigated and many applications have also been implemented on experimental data, especially in a cosmological framework, see for instance [8] and the references therein. Geometric features of random spherical harmonics have been studied for instance by ([27], [28]) where sharp estimates for the variance of the nodal length (i.e. the boundary of excursion sets for \(z = 0\)) are provided.

In this paper, we focus on a different characterization of the excursion sets, i.e. the behaviour of their empirical measure, to be defined below.

1.2. Main results. Let us define the spherical harmonics empirical measure as follows: for all \(z \in (-\infty, \infty)\),

\[
\Phi_l(z) := \int_{S^2} \mathbb{I}(f_t(x) \leq z) dx,
\]

where \(\mathbb{I}(\cdot)\) is, as usual, the indicator function which takes value one if the condition in the argument is satisfied, zero otherwise. In words, the function \(\Phi_l(z)\) provides the (random) measure of the set where the eigenfunction lie
below the value $z$. For example, the value of $\Phi_l(z)$ at $z = 0$ is related to the so-called Defect

$$D_l := \text{meas} \left( f_l^{-1}(0, \infty) \right) - \text{meas} \left( f_l^{-1}(-\infty, 0) \right)$$

by the straightforward transformation

$$D_l = 4\pi - 2\Phi_l(0).$$

Of course, $4\pi - \Phi_l(z)$ provides the area of the excursion set $A_l(z) := \{ x : f_l(x) > z \}$. The empirical measure can be viewed as a continuous analogue of the empirical distribution function for sequences of random variables, and it is simply related to the first Minkowski functional from convex geometry, see [1]. Clearly, for all $z \in \mathbb{R}$,

$$\mathbb{E} [\Phi_l(z)] = 4\pi \Phi(z),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian. Our aim in this paper is to study the distribution of $\{ \Phi_l(z) - \Phi(z) \}$, uniformly w.r.t. $z$, asymptotically as $l \to \infty$. The following lemma deals with the variance of $\Phi_l(z)$ as $l \to \infty$.

**Lemma 1.1.** For every $z \in \mathbb{R}$,

$$\text{Var}(\Phi_l(z)) = z^2 \phi(z)^2 \cdot \frac{1}{l} + O \left( \frac{\log l}{l^2} \right),$$

where $\phi$ is the standard Gaussian probability density function.

In particular, for $z \neq 0$, Lemma 1.1 gives the asymptotic form of the variance as $l \to \infty$. In contrast, for $z = 0$ (this case corresponds to the Defect), this yields only a “$o$”-bound and one needs to work harder to obtain a precise estimate (see [29]). Lemma 1.1 follows rather easily from the Hermite expansion approach in section 3, and (9), and we omit a formal proof.

From Lemma 1.1 it is then natural to normalize $\Phi_l(z)$ and define the **spherical harmonics empirical process** by

$$G_l(z) := \sqrt{l} \left[ \int_{S^2} \mathbb{I} (f_l(x) \leq z) \, dx - \{ 4\pi \times \Phi(z) \} \right]$$

for $l = 1, 2, \ldots$, $z \in (-\infty, \infty)$. In a sense to be made rigorous later, we are then led to investigate the high-frequency ergodicity properties of Gaussian eigenfunctions (compare [17]). Indeed, as $l$ diverges, (2) amounts to the distance between an average over a trajectory $\left( \int_{S^2} \mathbb{I}(f_l(x) \leq z) \, dx \right)$ and an average over the ensemble determined by the probability law

$$\Phi(z) = \int_{\Omega} \mathbb{I}(f_l(x) \leq z) \, dP(\omega).$$

For $G_l(\cdot)$, we establish a functional central limit theorem on a suitable Skorohod space.

Instrumental for the main result is a uniform weak reduction principle, where we show that the behaviour of the empirical process is uniformly approximated by a quadratic polynomial in $\{ f_l(\cdot) \}$ times a simple deterministic function. As a consequence, the weak limit is a fully degenerate random process, which can be realized as a single random variable times
the derivative of the Gaussian density. In the sequel, we use ⇒ to denote weak convergence in the Skorohod space $D([-\infty, \infty])$, equipped with the sup-norm, and $G_\infty(z)$ to label the mean zero, degenerate Gaussian process

$$G_\infty(z) = z\phi(z)Z, \ Z \sim N(0,1).$$

**Theorem 1.2.** (The Uniform Central Limit Theorem) As $l \to \infty$,

$$G_l(z) \Rightarrow G_\infty(z).$$

Considering the limiting expression for $G_\infty(z)$, we remark the existence of a singularity for $z = 0$; indeed, here $J_q(0) = 0$, whence it is easily seen that

$$(3) \quad G_l(0) \to_p 0.$$

From (3), we immediately gather that the weak reduction argument no longer holds for the Defect (see (29)); indeed, the limiting behaviour of the sequence $\{D_l\}$ is considerably more complicated and will be considered in our forthcoming paper (29).

The material in this paper attempts to provide a characterization as complete as possible on the high energy behaviour of the sample realizations of Gaussian eigenfunctions. A further natural question is to investigate whether the weak convergence result may enjoy some form of uniformity across different frequencies $l$. For the following result, ⇒ denotes weak convergence in the Skorohod space $D \{[-\infty, \infty] \times [0,1]\}$, and $W_\infty(z;r)$ denotes the mean zero Gaussian process with covariance function

$$E[W_\infty(z_1;r_1)W_\infty(z_2;r_2)] = \{r_1 \wedge r_2\} z_1 z_2 \phi(z_1) \phi(z_2).$$

**Theorem 1.3.** (Partial Sum Processes) As $L \to \infty$

$$W_L(z;r) \Rightarrow W_\infty(z;r).$$

**Remark 1.4.** It is possible to envisage some statistical applications of Theorem 1.3 for instance in the analysis of isotropic spherical random fields (such as those related to the analysis of the Cosmic Microwave Background radiation (CMB), see [7] for details). In such circumstances, the Gaussian eigenfunctions can be identified with the Fourier components of the field at frequency $l$, and it is straightforward to exploit Theorems 1.2 and 1.3 to construct tests for Gaussianity and isotropy, of Kolmogorov-Smirnov ($S_L$) or Cramér-von Mises ($K_L$) type, by means of the statistics

$$S_L := \sup_z \sup_r |W_L(z;r)|, \ K_L := \int_0^1 \int_{-\infty}^\infty |W_L(z;r)|^2 \, dz \, dr$$

where weak convergence ensures that

$$S_L \to_d \sup_z \sup_r |W_\infty(z;r)|, \ K_L \to_d \int_0^1 \int_{-\infty}^\infty |W_\infty(z;r)|^2 \, dz \, dr,$$

whence threshold values at given significance level can be easily tabulated. We refer for instance to [13], [14] for more discussion on these issues.
1.3. Plan of the Paper. In Section 2 we discuss polynomial transforms of spherical random eigenfunctions and we establish their asymptotic behaviour under Gaussianity. Particular care is devoted to the derivation of exact asymptotic rates. In Section 3 we discuss the structure of our proofs and the significance of main results. In Section 4 and 5 we provide the proofs of Theorems (1.2) and (1.3). Some background material and technical lemmas are collected in an Appendix.

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2. Polynomial transforms and their asymptotic behaviour

Throughout this paper, we shall make extensive use of well-known results on Hermite polynomials and higher-order moments of Gaussian random variables. To fix notation, we first recall the standard definition of Hermite polynomials

\[ H_q(x) = (-1)^q \phi^{-1}(x) \left\{ \frac{d^q}{dx^q} \phi(x) \right\} , \]

the first few examples being provided by

\[ H_1(x) = x , \quad H_2(x) = x^2 - 1 , \quad H_3(x) = x^3 - 3x , \quad H_4(x) = x^4 - 6x^2 + 3 , \]
\[ H_5(x) = x^5 - 10x^3 + 15x , \quad H_6(x) = x^6 - 15x^4 + 45x^2 + 15 , \ldots \]

We recall also the basic formulae

\[ E[H_q(f_l(x))] \equiv 0 \]

and

\[ E[H_{q_1}(f_l(x))H_{q_2}(f_l(y))] = \delta_{q_1}^{q_2} q_1! \{ Ef_l(x)f_l(y) \}^{q_1} = \delta_{q_1}^{q_2} q_1! R_l(\langle x, y \rangle)^{q_1} , \]

where

\[ R_l(\langle x, y \rangle) := \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} Y_{lm}(x)Y_{lm}(y) = E \{ f_l(x)f_l(y) \} \]

are the Legendre polynomials and \( \langle x, y \rangle = \cos(d(x, y)) \) where \( d(x, y) \) is the usual spherical distance on \( S^2 \). For our arguments in the sequel, we shall need a few basic facts on the asymptotic behavior of the polynomial transforms

\[ h_{l,q} := \int_{S^2} H_q(f_l(x))dx , \text{ for } q = 1, 2, 3, \ldots \]

Note first that \( Eh_{l,q} = 0 \), for all \( l, q \); also

\[ h_{l,1} = \int_{S^2} f_l(x)dx \equiv 0 , \text{ for all } l \geq 1 ; \]
more generally, and $h_{l;q} \equiv 0$, when $l$ and $q$ are both odd, by symmetry conditions. The behaviour of quadratic transforms is discussed in [17], where non-Gaussian behaviour is also covered and uniform bounds on the speed of convergence to the limiting distribution are discussed, by means of a detailed analysis of the cumulants for spherical harmonics coefficients. We write here a different proof based on Legendre polynomials to introduce the techniques we shall use extensively, later in this paper.

**Proposition 2.1. (Proposition 2, [17])** As $l \to \infty$ we have

$$\sqrt{l}h_{l;2} \to_d N(0,1).$$

**Proof.** Note first that

$$\text{Var}(h_{l;2}) = \text{Var} \left\{ \int_{S^2} H_2(f_l(x))dx \right\} = E \left\{ \int_{S^2} H_2(f_l(x))dx \right\}^2$$

$$= \int_{S^2 \times S^2} E[H_2(f_l(x))H_2(f_l(y))] dxdy$$

$$= \int_0^1 P_l^2(t)dt = \frac{2}{2l+1}.$$

In view of results [19, 20], to investigate the asymptotic behaviour of the total variation distance $d_{TV}(\sqrt{l}h_{l;2}, Z)$ between $\sqrt{l}h_{l;2}$ and a standard Gaussian random variable $Z \sim N(0,1)$, it is enough to show that the normalized fourth-order cumulants converge to zero, i.e., that

$$\frac{\text{cum}_4(h_{l;2})}{(\text{Var}(h_{l;2}))^2} = o_{l \to \infty}(1).$$

Using the well-known Diagram Formula (see e.g. [23]), we have

$$\text{cum}_4(h_{l;2}) = 48 \int_{S^2 \times S^2 \times S^2} P_l(\langle x, y \rangle)P_l(\langle y, z \rangle)P_l(\langle z, w \rangle)P_l(\langle w, x \rangle)dxdydzdw.$$

Now recall the reproducing kernel formula

$$\int_{S^2} P_l(\langle x, y \rangle)P_l(\langle y, z \rangle)dy = \frac{4\pi}{2l+1} P_l(\langle x, y \rangle),$$

whence,

$$\text{cum}_4(h_{l;2}) = 48 \left\{ \frac{4\pi}{2l+1} \right\}^2 \int_{S^2 \times S^2} P_l^2(\langle w, x \rangle)dx dw = 48 \frac{(4\pi)^4}{(2l+1)^3},$$

We then have

$$d_{TV}(\sqrt{l}h_{l;2}, Z) = O \left( \frac{\text{cum}_4(h_{l;2})}{(\text{Var}(h_{l;2}))^2} \right) = O(l^{-1}),$$

by the result mentioned above. □

The next Proposition covers the case $q = 3$ and follows easily from results in [13, 14].
Proposition 2.2. As $l \to \infty$, we have

\[ lh_{l;3} \to_d N \left( 0, \frac{\sqrt{3}}{\pi^2} \right). \]

Proof. Note first that

\[
h_{l;3} = \int_{S^2} f_l^3(x) dx = \sum_{m_1 m_2 m_3} a_{l m_1} a_{l m_2} a_{l m_3} \int_{S^2} Y_{l m_1} Y_{l m_2} Y_{l m_3} dx
\]

\[
= \sum_{m_1 m_2 m_3} a_{l m_1} a_{l m_2} a_{l m_3} \left( \begin{array}{ccc} l & l & l \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right) \sqrt{\frac{(2l+1)^3}{4\pi}}
\]

\[
= I_{III} \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right) \frac{1}{\sqrt{4\pi}}.
\]

where

\[ I_{III} = \sum_{m_1 m_2 m_3} a_{l m_1} a_{l m_2} a_{l m_3} \sqrt{\frac{(2l+1)^3}{4\pi}} \left( \begin{array}{ccc} l & l & l \\ m_1 & m_2 & m_3 \end{array} \right), \]

is the normalized bispectrum (see [9, 13, 14] for definitions and further discussions); the symbol on the right-hand side denotes the so-called Wigner coefficients, for which we refer to [24] and the Appendix below. It was shown in [14] that

\[ EI_{III} = 0, \quad EI_{III}^2 = 6 \quad \text{and} \quad I_{III} \to_d N(0, 6) \quad \text{as} \quad l \to \infty. \]

Hence, in view of Lemma A.1,

\[
\lim_{l \to \infty} l^2 Eh_{l;3}^2 = \frac{3}{2\pi} \lim_{l \to \infty} l^2 \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{3}{2\pi} \times \frac{2}{\pi\sqrt{3}} = \frac{\sqrt{3}}{\pi^2},
\]

as claimed. \hfill \Box

For the higher order terms, we have the following:

Lemma 2.3. For all $q \geq 3$, as $l \to \infty$ we have

\[ \int_{S^2 \times S^2} P_l(\langle x, y \rangle)^q dx dy = O \left( \frac{\log l}{l^2} \right). \]

The proof of Lemma 2.3 is given in Appendix B for completeness. In fact, it is possible to obtain the asymptotics of the LHS of (5):

\[
\int_{S^2 \times S^2} P_l(\langle x, y \rangle)^3 dx dy \sim c_3 \cdot \frac{\log l}{l^2},
\]

and for $q = 3$ or $q \geq 5$ we have

\[
\int_{S^2 \times S^2} P_l(\langle x, y \rangle)^q dx dy \sim c_q \cdot \frac{1}{l^2},
\]

for some constants $c_q$ (in particular, we get rid of the logarithm on the RHS of (5)). We use the precise estimates in our forthcoming paper [29] in order to compute the variance of $\{D_l\}$. 


3. ON THE PROOFS OF THE MAIN RESULTS

Our aim in this Section is to discuss heuristically the expansion of the empirical measure and process (1)-(2) into Hermite polynomials. Recall first that \( \{ f_1(x) \} \) is defined pointwise as a Gaussian variable, whence the function \( \mathbb{I}(f_1(x) \leq z) \) belongs for each \( x, z \) to the \( L^2 \) space of square integrable functions of Gaussian variables. In particular, we have that

\[
\mathbb{I}(f_1(x) \leq z) = \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} H_q(f_1(x)),
\]

where the right hand side converges in the \( L^2 \) sense (i.e.

\[
\lim_{Q \to \infty} E \left\{ \sum_{q=Q}^{\infty} \frac{J_q(z)}{q!} H_q(f_1(x)) \right\}^2 = 0,
\]

uniformly w.r.t. \( z, x \). It is possible to provide analytic expressions for the coefficients \( \{ J_q(.) \} \), indeed for \( q \geq 1 \)

\[
J_q(z) = \int_R \mathbb{I}(u \leq z) H_q(u) \phi(u) du = \int_{-\infty}^{z} (-1)^q \left( \frac{d^q}{du^q} \phi(u) \right) \phi^{-1}(u) \phi(u) du = (-1)^q \int_{-\infty}^{z} \left( \frac{d^q}{du^q} \phi(u) \right) du = (-1)^q \Phi(q)(z) = (-1)^q \phi^{(q-1)}(z),
\]

where \( \phi(z) := d\Phi(z)/dz \) is the standard Gaussian density. Hence for instance \( J_0(z) = \Phi(z), J_1(z) = -\phi(z), J_2(z) = -z\phi(z), J_3(z) = (1 - z^2)\phi(z) \) and in general

\[
J_q(z) = -H_q(z)\phi(z).
\]

For \( z = 0 \) it follows that \( J_1(0) = -(2\pi)^{-1/2}, \)

\[
J_q(0) = \begin{cases} 
(-1)^q (q-2)!! & q = 2k+1, \ k = 1, 2, \ldots \\
0 & q = 2k, \ k = 1, 2, \ldots 
\end{cases}
\]

We have, formally

\[
\int_{S^2} \mathbb{I}(f_1(x) \leq z) \, dx = \int_{S^2} \sum_{q=0}^{\infty} \frac{J_q(z)}{q!} H_q(f_1(x)) \, dx = 4\pi \times \Phi(z) + \int_{S^2} \sum_{q=2}^{\infty} \frac{J_q(z)}{q!} H_q(f_1(x)) \, dx,
\]

whence

\[
G_l(z) = \sqrt{l} \int_{S^2} \sum_{q=2}^{\infty} \frac{J_q(z)}{q!} H_q(f_1(x)) \, dx,
\]
and heuristically
\[ G_l(z) = -\frac{z\phi(z)}{2} \sqrt{l} \int_{S^2} H_2(f_l(x)) dx \]
\[ + \frac{(1 - z^2)\phi(z)}{6} \sqrt{l} \int_{S^2} H_3(f_l(x)) dx + ... \]
\[ = -\frac{z\phi(z)}{2} \sqrt{l} h_{l;2} + \frac{(1 - z^2)\phi(z)}{6} \sqrt{l} h_{l;3} - \frac{(3z - z^3)\phi(z)}{24} \sqrt{l} h_{l;4} + ..., \]
where \( h_{l;q} \) are as earlier in (4). The meaning of the exchange between the integral and the series is discussed below. From the results in the previous section, we know that the variance of \( \sqrt{l} h_{l;2} \) is given by
\[ \text{Var}\left\{ \sqrt{l} h_{l;2} \right\} = \frac{2l}{2l + 1}, \]
and the variances of the other terms are bounded by
\[ \text{Var}\left\{ \sqrt{l} h_{l;3} \right\} = O\left( \frac{1}{l} \right), \quad \text{Var}\left\{ \sqrt{l} h_{l;q} \right\} = O\left( \frac{\log l}{l} \right), \quad q \geq 4. \]
In fact, one can compute the precise asymptotic shape of the latter (see the discussion after Lemma 2.3).

Our idea in the proofs to follow is hence to show that the term \( \sqrt{l} h_{l;2} \) is asymptotically dominant in the expansion of \( G_l \), all the other summands being uniformly of smaller order. Note that, as remarked in the Introduction, the first term in the expansion (5), \( (q = 2) \) vanishes for \( z = 0 \); as a consequence, in the Defect statistic \( D_l \) the quadratic term \( 2^{-1} z\phi(z) \sqrt{l} h_{l;2} \) disappears. From the arguments below it is easy to realize that \( D_l \) has hence a lower order asymptotic variance than the empirical process; no single term is asymptotically dominant and the determination of the limiting behaviour becomes much more challenging. The asymptotic behaviour of the sequence \( \{D_l\} \) is hence delayed to the forthcoming paper [29].

4. PROOF OF THEOREM 1.2

Proof. The key step in the proof is the following uniform weak reduction principle: there exist a constant \( C \) such that for all \( 0 < \varepsilon \leq 1 \), as \( l \to \infty \),
\[ \text{Pr}\left\{ \sup_z \left| G_l(z) - J_2(z) \sqrt{l} h_{l;2} \right| > \varepsilon \right\} \leq \frac{C \log^6 l}{\varepsilon^4 l}, \]
which is the statement of Lemma 4.2. We deduce from this that
\[ \sup_z \left| G_l(z) - J_2(z) \sqrt{l} h_{l;2} \right| = o_p(1), \]
and the result follows immediately from
\[ \sqrt{l} h_{l;2} \to_d N(0, 1) \]
(see [15, 17]).

It then remains to prove Lemma 1.2. Our approach will follow closely the ideas by [5], which were developed in a rather different context (the
empirical process for long range dependent sequences on \( \mathbb{R} \). We will need the following notation:

\[
S_l(z) := G_l(z) - J_2(z) \sqrt{l} h_{l,2},
\]

(10)

and

\[
S_l(z_1, z_2) := S_l(z_2) - S_l(z_1), \quad z_1 \leq z_2,
\]

\[
J_q(z_2, z_1) := J_q(z_2) - J_q(z_1), \quad z_1 \leq z_2.
\]

Lemma 4.1. \textit{There exist a constant }\( C > 0 \) \textit{such that}

\[
E \left[ S_l(z_1, z_2) \right]^2 \leq \frac{C \log l}{l} \left[ \Phi(z_2) - \Phi(z_1) \right].
\]

(11)

\[\text{Proof.}\] For notational simplicity and without loss of generality we take \( z_1 = -\infty \), i.e. \( S_l(z_1, z_2) = S_l(z_2) = S_l(z) \); the argument in the general case is identical. We have trivially

\[
E \left\{ \int_{S^2} \left[ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right] \, dx \right\}^2
\]

(12)

\[
= E \left\{ \int_{S^2 \times S^2} \left[ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right] \times \left[ \mathbb{I}(f_l(y) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(y)) \right] \, dxdy \right\}
\]

\[
= \left\{ \int_{S^2 \times S^2} E \left[ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right] \right. \times \left. \left[ \mathbb{I}(f_l(y) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(y)) \right] \, dxdy \right\}
\]

(13)

\[
= E \left\{ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right\} \cdot \left\{ \mathbb{I}(f_l(y) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(y)) \right\}
\]

\[
= E \left[ \sum_{q_1 = 3}^{\infty} \frac{J_{q_1}(z)}{q_1!} H_{q_1}(f_l(x)) \right] \cdot \left[ \sum_{q_2 = 3}^{\infty} \frac{J_{q_2}(z)}{q_2!} H_{q_2}(f_l(y)) \right] = \sum_{q = 3}^{\infty} \frac{J_q^2(z)}{q!} P_q^1((x, y)).
\]

This result is standard, as \( \sum_{q_1 = 3}^{\infty} \frac{J_{q_1}(z)}{q_1!} H_{q_1}(f_l(x)) \) belongs to the Hilbert space of Gaussian subordinated random variables \( G(f_l) \), with inner product \( \langle X, Y \rangle := E_X Y \). Plugging (13) into (12) and using Lemma 2.3 we finally obtain

\[
E \left\{ \int_{S^2} \left[ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right] \, dx \right\}^2
\]

(14)

\[
= \int_{S^2 \times S^2} E \left[ \mathbb{I}(f_l(x) \leq z) - \Phi(z) + \frac{z \phi(z)}{2} H_2(f_l(x)) \right] \times \left\{ \sup_{q \geq 3} \int_{S^2 \times S^2} P_q^1((x, y)) \, dxdy \right\}
\]

\[
= O \left( \frac{\log l}{l} \right).
\]
since
\[
\sum_{q=3}^{\infty} \frac{J_q^2(z)}{q!} = \text{Var} \left( \mathbb{I}(f_1(x) \leq z) \right) = \Phi(z)(1 - \Phi(z)) < \infty,
\]
is the variance of a Bernoulli random variable. \qed

**Lemma 4.2.** *(Chaining argument)* For all \(0 < \varepsilon \leq 1\)
\[
\Pr \left\{ \sup_z |S_t(z)| > \varepsilon \right\} \leq \frac{C}{t} \left\{ \frac{\log^6 t}{\varepsilon^3} \right\},
\]
and for \(\varepsilon \geq 1\)
\[
\Pr \left\{ \sup_z |S_t(z)| > \varepsilon \right\} \leq \frac{C}{t} \left\{ \varepsilon^{-4/3} + \frac{\log^6 t}{\varepsilon^{4/3}} \right\}.
\]

**Proof.** For \(14\), we follow again closely \[5\]. We define
\[
\Lambda(z) = \Phi(z) + \frac{1}{2} \int_{-\infty}^{\xi} |x^2 - 1| \phi(x) dx
\]
\[
= \Phi(z) + \frac{1}{2\sqrt{2\pi}} \left\{ -z \exp(-\frac{z^2}{2}) \mathbb{I}(z \leq -1) \right\}
\]
\[
+ \frac{1}{2\sqrt{2\pi}} \left\{ \left[ z \exp(-\frac{z^2}{2}) + \exp(-\frac{1}{2}) \right] \mathbb{I}(-1 < z \leq 1) + \left[ -z \exp(-\frac{z^2}{2}) + 4 \exp(-\frac{1}{2}) \right] \mathbb{I}(z > 1) \right\}.
\]
Clearly \(\Lambda(z)\) is continuous, \(\lim_{z \to -\infty} \Lambda(z) = 0, \lim_{z \to \infty} \Lambda(z) = 1.483943... =: \Lambda(\infty)\). Also, for \(\Phi(z_1, z_2) = \Phi(z_2) - \Phi(z_1) = J_0(z_1, z_2)\) we have
\[
\Phi(z_1, z_2) + |J_2(z_1, z_2)| \leq \Lambda(z_2) - \Lambda(z_1) =: \Lambda(z_2, z_1),\text{ for all } z_2 \geq z_1,
\]
by \(6\). Now define refining partitions
\(\infty = z_0(k) \leq z_1(k) \leq ... \leq z_{2^k}(k) = \infty\),
where
\[
z_i(k) = \inf \left\{ z : \Lambda(z) \geq \frac{\Lambda(\infty)i}{2^k} \right\}, \text{ } i = 0, ..., 2^k - 1,
\]
implying that
\[
\Lambda(z_i(k)) - \Lambda(z_{i-1}(k)) \leq \frac{1.5}{2^k}.
\]
Define \(i_k(z)\) by
\[
z_{i_k}^{(z)}(k) \leq z \leq z_{i_k}^{(z)+1}(k),
\]
and note that the sequence \(\{z_{i_k}^{(z)}(k)\}\) is increasing, while \(\{z_{i_k}^{(z)+1}(k)\}\) is decreasing; then write, for some integer \(K = K(l)\) to be determined later
\[
S_l(z) = S_l(z_{i_0}(z)(0), z_{i_1}(z)(1)) + S_l(z_{i_1}(z)(1), z_{i_2}(z)(2)) + ... + S_l(z_{i_K}(z)(K), z).
\]
The idea is to bound uniformly each term of (17) by means of (11) and (16). For the last term of (17) we have, by a simple algebraic manipulation

\[ |S_l(z_{iK}(z))| \leq \sqrt{T} \left\{ \int_{S^2} \{ \mathbb{I}(z_{iK}(K) \leq f_i(x) \leq z_{iK+1}(K)) \} dx + 4 \pi \Phi(z_{iK}(K), z_{iK+1}(K)) \right\} \\
+ \sqrt{T} \mathbb{I}(z_{iK}(K), z_{iK+1}(K)) |h_{iL}| \\
\leq \sqrt{T} |S_l(z_{iK}(z), z_{iK+1}(K))| + 2 \times 4 \pi \sqrt{T} \mathbb{I}(z_{iK}(K), z_{iK+1}(K)) \\
+ \sqrt{T} |S_l(z_{iK}(K), z_{iK+1}(K))| + \sqrt{T} \frac{8 \pi \times 1.5}{2K} \\
+ \sqrt{T} \frac{1.5}{2K} |h_{L}|.

Using the latter together with (17), we may bound the latter as

\[ \Pr \left\{ \sup_z |S_l(z)| > \epsilon \right\} \leq \Pr \left\{ |S_l(z_{i0}(z), z_{i1}(1))| + \ldots + |S_l(z_{iK}(z), z_{iK+1}(K))| > \frac{\epsilon}{2} \right\} \\
+ \Pr \left( \sqrt{T} \frac{8 \pi \times 1.5}{2K} + \sqrt{T} \frac{1.5}{2K} |h_{L}| > \frac{\epsilon}{2} \right).

Since \( \sum_{k=0}^\infty \frac{\epsilon/(k+3)^2}{\epsilon/2} \leq \epsilon/2 \), we may bound

\[ \Pr \left\{ \sup_z |S_l(z)| > \epsilon \right\} \leq \Pr \left\{ \sup_z |S_l(z_{i0}(z), z_{i1}(1))| > \frac{\epsilon}{9} \right\} \\
+ \Pr \left\{ \sup_z |S_l(z_{i1}(1), z_{i2}(2))| > \frac{\epsilon}{16} \right\} + \ldots \\
+ \Pr \left\{ \sup_z |S_l(z_{iK}(z), z_{iK+1}(K))| > \frac{\epsilon}{(K+3)^2} \right\} \\
+ \Pr \left( \sqrt{T} \frac{1.5}{2K} |h_{L}| > \frac{\epsilon}{2} - \frac{12 \pi}{2K} \sqrt{T} \right).

Here, using the idea of refining partitions, we effectively reduced the supremum over a continuous set (\( \mathbb{R} \)) to the supremum over the possible nearest neighbors of \( z \) in these partitions. In view of (14), the Chebyshev inequality yields

\[ \Pr \left\{ \sup_z |S_l(z_{iK}(z), z_{iK+1}(z)(k+1))| > \frac{\epsilon}{(k+3)^2} \right\} \]

\[ \leq \sum_{i=0}^{2K-1} \Pr \left\{ |S_l(z_{iK}(z), z_{iK+1}(z)(k+1))| > \frac{\epsilon}{(k+3)^2} \right\} \]

\[ \leq \frac{C \log l (k+3)^4}{\epsilon^2} \sum_{i=0}^{2K-1} \left[ \Phi(z_{iK+1}(z)(k+1)) - \Phi(z_{iK}(z)(k+1)) \right] \leq \frac{C \log l (k+3)^4}{\epsilon^2} \ldots 

We choose

\[ K = \left\lfloor \log_2 \left( \frac{48 \pi \sqrt{l}}{\epsilon} \right) \right\rfloor + 1, \]

so that

\[ 2^K \approx \frac{48 \pi \sqrt{l}}{\epsilon}, \quad \sqrt{l} \frac{1.5}{2K} \approx \frac{\epsilon}{32 \pi}, \quad \frac{\epsilon}{2}, \quad \frac{12 \pi}{2K} \sqrt{l} \approx \frac{\epsilon}{4}. \]
Hence we obtain
\[(22)\]
\[
\Pr \left\{ \sqrt{\frac{1.5}{2K}} |h_{l;2}| > \frac{\varepsilon}{2} - \frac{12\pi}{2K} \sqrt{l} \right\} \leq \Pr \left\{ \sqrt{\frac{1.5}{2K}} |h_{l;2}| > \frac{\varepsilon}{4} \right\} \leq \Pr \{ |h_{l;2}| > 8\pi \} \\
\leq \frac{1}{(8\pi)^2} l^{-1} E \left( \frac{l^{1/2} h_{l;2}}{2} \right)^2 = \frac{1}{(8\pi)^2} l^{-1} \text{Var} \left\{ \frac{l^{1/2} \sum_m |a_{lm}|^2}{2} \right\} \leq \frac{1}{(8\pi)^2} l^{-1}.
\]

To conclude the proof of (14), we plug the bounds (19) and (22) into (18); we obtain
\[
\Pr \left\{ \sup_z |S_l(z)| > \varepsilon \right\} \leq \frac{C \log l}{l} \sum_{k=0}^{K} \frac{(k + 3)^4}{\varepsilon^2} + \frac{l^{-1}}{(8\pi)^2} \leq \frac{C' \log l}{l} \frac{(K + 3)^5}{5\varepsilon^2} + \frac{l^{-1}}{(8\pi)^2} \leq \frac{C''}{l} \left\{ \frac{\log l}{\varepsilon^3} + 1 \right\}, \text{ some } C, C', C'' > 0.
\]

For the proof of (15), we repeat exactly the same argument as before, replacing $K$ in (21) by
\[
K' = \left\{ \log_2 \left( \frac{48\pi}{\varepsilon^{1/3}} \sqrt{l} \right) \right\}, \quad 2K' = \frac{48\pi}{\varepsilon^{1/3}} \sqrt{l},
\]
\[
\frac{\varepsilon}{2} - \frac{12\pi}{2K'} \sqrt{l} \geq \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4}, \text{ because } \varepsilon \geq 1.
\]

It follows easily that
\[
\Pr \left\{ \sqrt{\frac{1.5}{2K}} \left| \int_{S_2} \{ f_l^2(x) - 1 \} dx \right| > \frac{\varepsilon}{2} - \frac{12\pi}{2K} \sqrt{l} \right\} \\
\leq \Pr \left\{ \frac{\varepsilon^{1/3}}{32\pi} \left| \int_{S_2} \{ f_l^2(x) - 1 \} dx \right| > \frac{\varepsilon}{4} \right\} \\
\leq \Pr \left\{ \left| \int_{S_2} \{ f_l^2(x) - 1 \} dx \right| > 8\pi \varepsilon^{2/3} \right\} \leq \frac{\varepsilon^{-4/3}}{(8\pi)^2} l^{-1},
\]
and
\[
\Pr \left\{ \sup_z |S_l(z)| > \varepsilon \right\} \leq \frac{C \log l}{l} \frac{(K' + 3)^5}{5\varepsilon^2} + C' \frac{\varepsilon^{-4/3}}{(8\pi)^2} l^{-1} \\
\leq \frac{C''}{l} \left\{ \frac{\log l}{\varepsilon^{4/3}} + \varepsilon^{-4/3} \right\}, \text{ some } C, C', C'' > 0.
\]

\[\square\]

Remark 4.3. The proof above is uses the integral
\[
\frac{1}{2} \int_{-\infty}^{\varepsilon} |x^2 - 1| \phi(x) dx = \frac{1}{2} \int_{-\infty}^{\varepsilon} (x^2 - 1) \phi(x) dx (z \leq -1) \\
+ \left\{ \frac{1}{2} \int_{-\infty}^{-1} (x^2 - 1) \phi(x) dx + \frac{1}{2} \int_{-1}^{\varepsilon} (1 - x^2) \phi(x) dx \right\} I(1 < z \leq 1)
\]
\[
\begin{align*}
&+ \left\{ \frac{1}{2} \int_{-\infty}^{1} (x^2 - 1) \phi(x) dx + \frac{1}{2} \int_{-1}^{1} (1 - x^2) \phi(x) dx + \frac{1}{2} \int_{1}^{z} (x^2 - 1) \phi(x) dx \right\} \mathbb{I}(z > 1) \\
&= \frac{1}{2 \sqrt{2 \pi}} \left\{ -z \exp\left(-\frac{z^2}{2}\right) \mathbb{I}(z \leq -1) + \left[ z \exp\left(-\frac{z^2}{2}\right) + \exp\left(-\frac{1}{2}\right) \right] \mathbb{I}(-1 < z \leq 1) \right\} \\
&\quad + \frac{1}{2 \sqrt{2 \pi}} \left[ -z \exp\left(-\frac{z^2}{2}\right) + 4 \exp\left(-\frac{1}{2}\right) \right] \mathbb{I}(z > 1),
\end{align*}
\]

which follows directly from
\[
\begin{align*}
&\int_{-\infty}^{z} (x^2 - 1) \phi(x) dx \mathbb{I}(z \leq -1) = [x \phi(x)]_{-\infty}^{z} = -z \exp\left(-\frac{z^2}{2}\right) \mathbb{I}(z \leq -1), \\
&\int_{-1}^{z} (1 - x^2) \phi(x) dx \mathbb{I}(z \leq 1) = \frac{1}{\sqrt{2 \pi}} \left[ z \exp\left(-\frac{z^2}{2}\right) + \exp\left(-\frac{1}{2}\right) \right] \mathbb{I}(z \leq 1),
\end{align*}
\]

and
\[
\begin{align*}
&\int_{1}^{z} (x^2 - 1) \phi(x) dx \mathbb{I}(z > 1) = \frac{1}{\sqrt{2 \pi}} \left[ -z \exp\left(-\frac{z^2}{2}\right) + \exp\left(-\frac{1}{2}\right) \right] \mathbb{I}(z > 1).
\end{align*}
\]

Also
\[
\frac{1}{2} \int_{-\infty}^{\infty} |x^2 - 1| \phi(x) dx = \frac{4e^{-1/2}}{2\sqrt{2 \pi}} \simeq 0.48394.
\]

5. Proof of Theorem 1.3

In view of (13), we can write
\[
W_L(z; r) = \frac{J_2(z)}{\sqrt{L}} \sum_{l=1}^{[Lr]} \left\{ \sqrt{l} \int_{S^2} \{ f_i^2(x) - 1 \} \ dx \right\} + \frac{1}{\sqrt{L}} \sum_{l=1}^{[Lr]} S_l(z)
\]

\[= W_{AL}(z; r) + W_{BL}(z; r).
\]

We shall prove that, as \( L \to \infty, \)
\[
W_{AL}(z; r) \Rightarrow W_{\infty}(z; r), \quad \sup_{z} \sup_{r} W_{BL}(z; r) = o_p(1).
\]

5.1. Step 1: the proof that \( W_{AL}(z; r) \Rightarrow W_{\infty}(z; r), \) as \( L \to \infty. \)

Proof. To prove convergence of the finite-dimensional distributions for \( W_{AL}(z; r), \)

it is enough to note that, for \( r_1 \leq r_2 \leq r_3 \leq r_4 \)

\[
EW_{AL}(z; r) = \frac{J_2(z)}{\sqrt{L}} \sum_{l=1}^{[Lr]} \sqrt{l} \int_{S^2} E \{ f_i^2(x) - 1 \} \ dx = 0,
\]

\[
EW_{AL}(z_1; r_1) W_{AL}(z_2; r_2) = \frac{J_2(z_1) J_2(z_2)}{L} \sum_{l=1}^{[Lr_1]} l E h_{1,2}^2 \to J_2(z_1) J_2(z_2) r_1,
\]

and

\[
cum \{ W_{AL}(z_1; r_1), W_{AL}(z_2; r_2), W_{AL}(z_3; r_3), W_{AL}(z_4; r_4), \} = \frac{J_2(z_1) J_2(z_2) J_3(z_1) J_4(z_2)}{L^2} \sum_{l=1}^{[Lr_1]} l^2 cum_4(h_{1,2}) \to 0, \text{ as } L \to \infty.
\]

The multivariate extension is trivial. To establish tightness, in view of the results from [3] (see also [18]), it is enough to prove that, for all \( r_1 \leq r \leq \)}
For (24), we have

\[
\mu_r \leq C \{ \mu([r_1, r_2] \times [z_1, z_2]) \}^2.
\]

Indeed, for (23), it is enough to exploit independence over \( l \) to show that

\[
E \left\{ \left[ \frac{J_2(z_2) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right]^2 \left[ \frac{J_2(z_2) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right]^2 \right\} 
\]

\[
\leq \left\{ \frac{J_2(z_2) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right\}^2 \left\{ \frac{J_2(z_2) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right\}^2 E \left\{ \sqrt{lh_{l,2}} \right\}^2 
\]

\[
\leq (r - r_1)(r_2 - r) \{ J_2(z_2) - J_2(z_1) \}^2 \leq (r_2 - r_1)^2 \{ A_2(z_2) - A_1(z_1) \}^2.
\]

For (24), we have

\[
E \left\{ \left[ \frac{J_2(z) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right]^2 \left[ \frac{J_2(z) - J_2(z_1)}{\sqrt{L}} \sum_{l=[Lr_1]}^{[Lr_2]} \sqrt{lh_{l,2}} \right]^2 \right\} 
\]

\[
= \frac{\{ J_2(z) - J_2(z_1) \}^2 \{ J_2(z) - J_2(z_1) \}^2}{L^2} \sum_{l=[Lr_1]}^{[Lr_2]} E \left\{ \sqrt{lh_{l,2}} \right\}^4 
\]

\[
+ 6 \frac{\{ J_2(z) - J_2(z_1) \}^2 \{ J_2(z) - J_2(z_1) \}^2}{L^2} \sum_{l_1 < l_2 = [Lr_1]}^{[Lr_2]} E \left\{ \sqrt{lh_{l_1,2}} \right\}^2 E \left\{ \sqrt{lh_{l_2,2}} \right\}^2.
\]
Now \( E \left\{ \sqrt{\hat{h}_{1:2}} \right\}^2, E \left\{ \sqrt{\hat{h}_{1:2}} \right\}^4 \leq C \) uniformly w.r.t. \( l \); note also that \( L^{-1} \leq (r_2 - r_1) \), for all \( r_1, r_2 \) in \( \{1/L, 2/L, \ldots, 1\} \), whence

\[
\frac{(J_2(z) - J_2(z_1))^2}{L^2} \frac{(J_2(z) - J_2(z_1))^2}{L^2} \sum_{l=\lfloor l \rfloor}^{\lfloor Lr \rfloor} E \left\{ \sqrt{l} h_{1:2} \right\}^4 \\
\leq C(r_2 - r_1)^2 \{ Λ_2(z_2) - Λ_1(z_1) \}^2,
\]

\[
\frac{(J_2(z) - J_2(z_1))^2}{L^2} \frac{(J_2(z) - J_2(z_1))^2}{L^2} \sum_{l_1 < l_2 = \lfloor Ll \rfloor} E \left\{ \sqrt{l} h_{1:2} \right\}^2 E \left\{ \sqrt{l} h_{1:2} \right\}^2 \\
\leq (r_2 - r_1)^2 \{ Λ_2(z_2) - Λ_1(z_1) \}^2,
\]

as needed.

5.2. Step 2: the proof that \( \sup_z \sup_r W_{BL}(z; r) = o_p(1) \), as \( L \to \infty \).

Proof. For fixed \( z, r \) it is enough to show that

\[
\sup_{0 \leq r \leq 1} \sup_{-\infty < z < \infty} |W_{BL}(z; r)| = o_p(1), \text{ as } L \to \infty.
\]

To establish this result, we note that

\[
E \left\{ \sup_{0 \leq r \leq 1} \sup_{-\infty < z < \infty} |W_{BL}(z; r)| \right\} \leq \frac{1}{\sqrt{L}} E \left\{ \sup_{0 \leq r \leq 1} \sum_{l=1}^{\lfloor Lr \rfloor} \sup_{-\infty < z < \infty} |S_l(z)| \right\} \\
\leq \frac{1}{\sqrt{L}} \sum_{l=1}^{L} E \left\{ \sup_{-\infty < z < \infty} |S_l(z)| \right\} \\
\leq \frac{C}{\sqrt{L}} \sum_{l=1}^{L} \frac{\log^{6} l}{l} = C \frac{\log^{7} L}{\sqrt{L}} \to 0, \text{ as } L \to \infty,
\]

because

\[
E \left\{ \sup_{-\infty < z < \infty} |S_l(z)| \right\} \leq 1 + \frac{C \log l}{l} \int_{1}^{\infty} \left\{ \varepsilon^{-4/3} + \frac{\log^{5} l}{\varepsilon^{3}} \right\} d\varepsilon \\
\leq C \frac{\log^{6} l}{l}.
\]

\[
W_{BL}(z; r) = O_p \left( \left[ EW_{BL}^2(z; r) \right]^{1/2} \right) = O_p \left( \frac{1}{L} \sum_{l=1}^{\lfloor Lr \rfloor} ES_l^2(z) \right) = O_p \left( \frac{1}{L} \sum_{l=1}^{\lfloor Lr \rfloor} \frac{\log l}{l} \right) = o_p(1).
\]

□
Appendix A. Background on Wigner Coefficients

Throughout this paper, we made a heavy use of Wigner’s 3j coefficients. In this appendix, we review briefly some of their features and provide some results on their asymptotic properties. We refer to [25, 24] and [4] for a much more detailed discussion, in particular concerning the relationships with the quantum theory of angular momentum and group representation properties of SO(3).

We start from the analytic expression (valid for \( m_1 + m_2 + m_3 = 0 \), see [24], expression (8.2.1.5))

\[
\begin{pmatrix}
\ell_1 & \ell_2 & \ell_3 \\
m_1 & m_2 & m_3
\end{pmatrix}
= (-1)^{l_1+m_1} \sqrt{2l_3 + 1} \frac{\left[ (l_1 + l_2 - l_3)! (l_1 - l_2 + l_3)! (l_1 - l_2 + l_3)! \right]}{(l_1 + l_2 + l_3 + 1)!} \left[ \frac{(l_3 + m_3)! (l_3 - m_3)!}{(l_1 + m_1)! (l_1 - m_1)! (l_2 + m_2)! (l_2 - m_2)!} \right]^{1/2} \\
\times \sum_z \frac{(-1)^z (l_2 + l_3 + m_1 - z)! (l_1 - m_1 + z)!}{2! (l_2 + l_3 - l_1 - z)! (l_3 + m_3 - z)! (l_1 - l_2 - m_3 + z)!},
\]

where the summation runs over all \( z \)'s such that the factorials are non-negative. This expression becomes much neater for \( m_1 = m_2 = m_3 = 0 \), where we have

\[
\begin{pmatrix}
\ell_1 & \ell_2 & \ell_3 \\
0 & 0 & 0
\end{pmatrix}
= (-1)^{l_1/2 - l_3/2} \frac{1}{[l_1 + l_2 - l_3]/2! [l_1 - l_2 + l_3]/2! [l_1 + l_2 + l_3]/2!} \left[ \frac{(l_1 + l_2 - l_3)! (l_1 - l_2 + l_3)! (-l_1 + l_2 + l_3)!}{(l_1 + l_2 + l_3 + 1)!} \right]^{1/2}.
\]

Some of the properties to follow become neater when expressed in terms of the so-called Clebsch-Gordan coefficients, which are defined by the identities (see [24], Chapter 8)

\[
\begin{align*}
\begin{pmatrix}
\ell_1 & \ell_2 & \ell_3 \\
m_1 & m_2 & -m_3
\end{pmatrix}
&= (-1)^{l_3 + m_3} \frac{1}{\sqrt{2l_3 + 1}} C^{l_3 m_3}_{l_1 m_1 l_2 m_2} \\
\begin{pmatrix}
\ell_1 & \ell_2 & \ell_3 \\
m_1 & m_2 & -m_3
\end{pmatrix}
&= (-1)^{l_1 - l_2 + m_3} \sqrt{2l_3 + 1} \begin{pmatrix}
\ell_1 & \ell_2 & \ell_3 \\
m_1 & m_2 & -m_3
\end{pmatrix}
\end{align*}
\]

We have the following orthonormality conditions:

\[
\begin{align*}
\sum_{m_1, m_2} C^{l m}_{l_1 m_1 l_2 m_2} C^{l' m'}_{l_1 m_1 l_2 m_2} &= \delta^l_{l'} \delta^m_{m'} \\
\sum_{l, m} C^{l m}_{l_1 m_1 l_2 m_2} C^{l m}_{l_1 m_1 l_2 m_2} &= \delta^{m_1}_{m_1} \delta^{m_2}_{m_2}.
\end{align*}
\]

Now recall the general formula ([21], eqs. (5.6.2.12-13))

\[
\int_{S^2} Y_{l_1 m_1}(x) \cdots Y_{l_n m_n}(x) dx
\]
\[
\sqrt{\frac{4\pi}{2l + 1} \sum_{L_1 \ldots L_{n-3} M_1 \ldots M_{n-3}} \left[ C_{L_1 M_1} C_{L_2 M_2} \cdots C_{L_{n-3} M_{n-3}} C_{M_{n-3} M_{n-1} M_{n-1}} \right]}
\times \sqrt{\prod_{l_i = 1}^{l_{n-3}} (2l_i + 1) \frac{(4\pi)^{n-1}}{(4\pi)^{n-1}}} \left\{ C_{L_1 0} C_{L_2 0} \cdots C_{L_{n-3} 0} C_{M_{n-3} 0} \right\}.
\]

Hence we have
\[
\int_0^1 P_l^3(t) dt = \sqrt{\frac{(4\pi)^{n-2}}{(2l + 1)^n}} \int_{S^2} Y_{l_0}^3(x) dx = \sum_{L_1 \ldots L_{n-3}} \left\{ C_{L_1 0} C_{L_2 0} \cdots C_{L_{n-3} 0} \right\}^2,
\]
in the notation of [15, 16]. Special cases are
\[
\int_0^1 P_l^3(t) dt = \frac{1}{2} \int_0^\pi P_l^3(\cos \vartheta) d\cos \vartheta
\]
\[
= \sqrt{\frac{4\pi}{(2l + 1)^3}} \int_0^\pi \int_0^\pi Y_{l_0}^3(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi
\]
\[
= \frac{1}{2l + 1} \left\{ C_{l_0 0} \right\}^2 \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2,
\]
\[
\int_0^1 P_l^4(t) dt = \frac{1}{2l + 1} \sum_{L=0}^{2l} \left\{ C_{l_0 0 0 0} \right\}^2 = \frac{1}{2l + 1} \sum_{L=0}^{2l} \left\{ C_{l_0 0 0 0} \right\}^2
\]
\[
= \frac{1}{2l + 1} \sum_{L=0}^{2l} (2L + 1)(2l + 1) \left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^4
\]
\[
(30) = \sum_{L=0}^{2l} (2L + 1) \left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^4,
\]

compare [24], equation (8.9.4.20). The following result is certainly known, but we failed to locate a reference and we provide a proof for completeness.

**Lemma A.1.** 1) As \( l \to \infty \)
\[
\lim_{l \to \infty} l^2 \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{2}{\pi \sqrt{3}} \approx 0.367.
\]
2) For all \( l = 1, 2, \ldots \) and even \( L = 2, \ldots, 2l - 2 \) we have
\[
\left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^2 = \gamma_{lL} \times \frac{2}{\pi} \times \frac{1}{L(2l - L)^{1/2}(2l + L)^{1/2}},
\]
where \( 0.596 = 1.09^{-6} \leq \gamma_{lL} \leq 1.09^5 = 1.539. \)
3) For \( L = 0, 2l \) we have
\[
\left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{1}{2l + 1}, \left( \begin{array}{ccc} l & l & 2l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{\sqrt{2}}{\sqrt{\pi(4l + 1)^{1/2}}} \{1 + O(l^{-1})\}. \]
Proof. 1) Note that (see [25] and [24], equation (8.5.2.32))

\[
\left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right) = \frac{(-1)^{l/2}(3l/2)!}{(l/2)!^3} \left[ \frac{[l!]^3}{(3l+1)!} \right]^{1/2}.
\]

Hence, recalling Stirling’s formula

\[ l! = \sqrt{2\pi}(l)^{l+1/2}\exp(-l) + O(l^{-1}), \]

we obtain

\[
\lim_{l \to \infty} l^2 \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{\sqrt{2\pi} \lim_{l \to \infty} l^2 (3l/2)^{3l+1} e^{-3l}}{\sqrt{2\pi} \lim_{l \to \infty} (l/2)^{3l+3} e^{-3l}} \left[ \frac{l^{3l+3/2} e^{-3l}}{(3l+1)^{3l+3/2} e^{-3l-1}} \right] = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \lim_{l \to \infty} \frac{3^{3l+1}}{(3l+1)^{3l+3/2}} = \frac{6e}{3^{3/2}} \lim_{l \to \infty} \frac{3^l}{(3l+1)^3} = \frac{6e}{3^{3/2} \pi} \lim_{l \to \infty} \left[ \frac{1}{(1 + l^{-1})^3} \right] = \frac{2}{\pi \sqrt{3}} \approx 0.367.
\]

2) From [24], equation (8.5.2.32) we have

\[
\left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^2 = \left\{ \frac{(l + L/2)!}{(l/2)!^2(l - L/2)!} \right\}^2 \frac{(L!)^2((2l - L)!)}{(2l + L + 1)!}.
\]

We use repeatedly Stirling’s formula, for \( n = 1, 2, ... \)

\[ 1 < \exp(\frac{1}{12n+1}) \leq \frac{n!}{n^n} e^{-n\sqrt{2\pi n}} \leq \exp(\frac{1}{12n}) \leq 1.09, \]

also we write \( a_n \simeq b_n \) for sequences such that \( a_n/b_n, b_n/a_n = O(1) \). Hence we have

\[
\left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^2 = \left\{ \frac{(l + L/2)!}{(l/2)!^2(l - L/2)!} \right\}^2 \frac{(L!)^2(2l - L)!}{(2l + L + 1)!} = \frac{\gamma_{lL}}{2\pi} \frac{e^{-2l - L} e^{-2l + L} e^{-2l + L - 1}}{(L/2)^{2l+2} (l - L/2)^{2l-2l+4} (2l + L + 1)^{2l + 1/2}} \\
= \frac{\gamma_{lL}}{2\pi} \frac{(l + L/2)^{2l+2l+1} (2l - L)^{2l - L + 1/2}}{(L/2)^{2l+2} (l - L/2)^{2l-2l+4} (2l + L + 1)^{2l + 1/2}}.
\]

where

\[ 0.596 = 1.09^{-6} < \exp(-\frac{6}{13}) \leq \gamma_{lL} \leq \exp(\frac{5}{12}) < 1.09^{5} = 1.539. \]
Now
\[
\frac{(l + L/2)^{2l+2L+1}}{(L/2)^{2L+2}(l - L/2)^{2l-2L+1}} \frac{(L)^2L+1}{(2l + L + 1)^{2L+L+3/2}}
\]
\[
= \frac{2^{2l+2L-L+1/2}}{2^{2l+L+3/2}} \frac{(l + L/2)^{2l+L+1}}{L^{2L+2}(l - L/2)^{2l-L+1/2}} \frac{L^{2L+1}}{(l + L/2 + 1/2)^{2L+L+3/2}}
\]
\[
= \frac{2}{L(l - L/2)^{1/2}(l + L/2)^{1/2}} \frac{1}{(l/2)^{2l+L+3/2}} = \frac{2e^{-1}}{L(l - L/2)^{1/2}(l + L/2)^{1/2}},
\]
whence the proof of the first is completed.

3) The first part is equation (8.5.1.1) from [24]. For the second part, it is sufficient to recall from [24] equation (8.5.2.33) to deduce that
\[
\left( \begin{array}{ccc} l & l & 2l \\ 0 & 0 & 0 \end{array} \right)^2 = \frac{1}{4l + 1} \left\{ \frac{(2l)!}{l!!} \left[ \frac{(2l)!}{(2l)!} \right]^{1/2} \right\}^2
\]
\[
= \frac{1}{4l + 1} \left\{ \frac{(2l)^{2l+1/2}}{\sqrt{2\pi l^{1/2} l^{1/2}}} \left[ \frac{\sqrt{2\pi} (2l)^{2l+1/2} (2l)^{2l+1/2}}{(4l)^{4l+1/2}} \right]^{1/2} \right\}^2
\]
\[
= \frac{\sqrt{2}}{\sqrt{\pi (4l + 1) \sqrt{l}}},
\]
as claimed. □

Remark A.2. Lemma [A.1] implies immediately
\[
\left( \begin{array}{ccc} l & l & L \\ 0 & 0 & 0 \end{array} \right)^4 \leq \frac{1}{L^2 (4l^2 - L^2)} \quad \text{for } L = 2, 4, \ldots, 2l - 2
\]
\[
= O(l^{-1}) \quad \text{for } L = 2l.
\]
Also, for \( L = l \) we obtain
\[
\frac{1}{L(l - L/2)^{1/2}(l + L/2)^{1/2}} = \frac{1}{l(1/2)^{1/2} (3l/2)^{1/2}} = \frac{2}{\sqrt{3l^2}}
\]
leading to the special case
\[
\lim_{l \to \infty} \left[ \frac{2}{\pi \sqrt{3l^2}} \right]^{-1} \left( \begin{array}{ccc} l & l & l \\ 0 & 0 & 0 \end{array} \right)^2 = 1,
\]
see also [24], equation (8.9.4.20) for different asymptotic approximations.

Appendix B. Proof of Lemma 2.3

Proof. First, we note that, since for every \( t \in [-1, 1] \), \( |P_t(t)| \leq 1 \), it is sufficient to prove the statement of Lemma 2.3 for \( q = 4 \). In this case, we have
\[
E \left[ h_t^{4} \right] = 4! \int_0^1 P_t^4(t)dt = 12 \int_0^\pi P_t^4(\cos \vartheta) \cos \vartheta d\vartheta
\]
\[
= \frac{12}{4\pi} \int_0^{2\pi} \int_0^\pi P_t^4(\cos \vartheta) \cos \vartheta d\vartheta d\varphi = \frac{12}{4\pi} \left\{ \left[ \frac{4\pi}{2l + 1} \right]^{4} \right\} \int_{S^2} Y_{10}^4(x) dx.
\]
Now from (30) and Lemma A.1
\[
\int_0^{2\pi} \int_0^\pi P^4_l(\cos \vartheta) d\cos \vartheta d\varphi = 4\pi \int_0^1 P^4_0(t) dt = \sum_{L=0}^{2l} (2L + 1) \begin{pmatrix} l & 0 & 0 & L \end{pmatrix}^4
\leq \frac{1}{(2l + 1)^4} + \sum_{L=2}^{2L-2} (2L + 1) \frac{1}{L^2 \{4l^2 - L^2\}} + O(l^{-3}).
\]
For \(L \leq l\)
\[
\sum_{L=2}^{l} (2L + 1) \frac{1}{L^2 \{4l^2 - L^2\}} \leq \frac{C}{l^2} \sum_{L=2}^{l} \frac{1}{L} \leq \frac{C \log l}{l^2},
\]
while for \(l \leq L < 2l\)
\[
\sum_{L=l}^{2l-2} (2L + 1) \frac{1}{L^2 \{4l^2 - L^2\}} \leq \frac{C}{2l^2} \sum_{L=l}^{2l-2} \frac{1}{\{2l - L\}} \leq \frac{C \log l}{l^2}.
\]
Note also that
\[
\sum_{L=2}^{2l-2} (2L + 1) \frac{1}{L^2 \{4l^2 - L^2\}} \geq \frac{1}{4l^2} \sum_{L=2}^{2l-2} (2L + 1) \frac{1}{L^2} \geq C' \frac{\log l}{l^2},
\]
so this order cannot be improved. Also, using the second part of Lemma A.1 we have easily
\[
(4l + 1) \begin{pmatrix} l & 0 & 2l \end{pmatrix}^4 = \frac{2}{\pi (4l + 1)} \{1 + O(l^{-1})\} = O(l^{-2}).
\]

\[\square\]

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