Electrodynamics in the ’t Hooft gauge, a covariant operator approach

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Abstract

This article is devoted to describe the ’t Hooft gauge electrodynamics by means of non-perturbative methods together with Heisenberg perturbation theory insights. The point is that this specific gauge choice introduces a non-linear photon field self interaction. So, this model is analyzed in a BRST framework in analogy to the non-Abelian cases. Finally, we show that the asymptotic transverse photon modes are the same as that of the linear gauge well known case while just the non-physical longitudinal sector get renormalized by such gauge self interaction. Then the physical gauge independence is explicitly verified.

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1. Introduction

The ’t Hooft-Veltman gauge in four-dimensional quantum electrodynamics (QED₄) was proposed in the seventies to model Yang-Mills theories with complicated gauge structures [1]. The aim of the approach was to define good Lagrangians, that is, non-singular Lagrangians with well-defined propagators in theories with gauge symmetries, and to show that those definitions do not modify the physical content of the original singular Lagrangian. Since the investigation of Yang-Mills theories is a harder problem, QED₄ in this gauge can be considered as a testing ground for four-dimensional quantum chromodynamics (QCD₄). In fact, the manipulation of this gauge by means of path integral methods implies the addition of ghost fields that will be coupled to the

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electromagnetic field as well as self-interaction, in analogy to what happens in QCD4 in which the gluons are also coupled to the ghost fields of the theory.

In general, gauge-ghost couplings may turn the quantization of the system trickier since, besides the appearance of new vertices, there are more subtle problems as the possibility of Gribov copies [2], that is, that even fixing the gauge condition the path integral may take into account more than one representative element of the gauge orbits [3]. Nevertheless, the version of QED4 in the ’t Hooft-Veltman gauge is simple enough to give us control over this issue [4].

It is possible to show that the use of this gauge has no phenomenological consequences, as it is expected from its attainability and well-definiteness. ’t Hooft and Veltman established this by an explicit computation of the contributions to the photon-photon scattering at tree level which must vanish in ordinary electromagnetism. They showed that the required cancellation indeed occurs. For the general case of loops, they showed this by using diagrammatic Slavnov-Taylor identities [5,6]. However, many of works such as [7–9] explore the gauge structure of the theory by means of perturbation theory.

Although the use of linear gauges are, in general, the most useful ones in practical loop calculations, non-linear gauges has received recent interest since they are adequate to study some aspects of the infrared dynamics of non-Abelian gauge theories [10]. In this context, we can cite the Curci-Ferrari [11] gauge and the Maximal Abelian gauge (MAG) [12]. They are used to analyze the Abelian dominance hypothesis, which basically refers to the idea that diagonal gluons receive a smaller dynamically generated mass than the off-diagonal ones. In view of this Abelian dominance possibility, an investigation of \( \text{QED}4 \) in a non-linear gauge may be welcome to develop insight and a well defined consistent path. Since the ’t Hooft parameter is unphysical, there is no bound in its value, so it can be big and then just non-perturbative general methods can be employed to explicitly verify that it really renormalize just non-physical sectors of the Hilbert space. In this paper, we are concerned with the development of a completely non-perturbative study of the two-point function of the gauge field. The authors had this particular interest since in references [13,14] there is an apparent longitudinal contribution to the vacuum polarization tensor which is controlled by Ward identities. However, the fictitious \( g \) parameter (see equation (1) below) just renormalize the longitudinal non-physical sectors of the photon propagator. Since we want to compare our results with [13,14], then besides the non-perturbative approach, which is the most adequate in this case, we also extend the formalism to a perturbative Heisenberg quantization in the situation of small \( g \). In theoretical grounds, this is an important achievement since, as previously mentioned, this theory can be used as an insightful toy model for \( \text{QCD}_4 \), and there is not yet a complete understanding of a Heisenberg perturbation for quantum chromodynamics [15].

In order to proceed we will use the Kujo-Ojima-Nakanishi formalism [16,17]. It consists of an indefinite metric quantization in which an auxiliary \( B \)-field is suitably introduced to provide a second class system in the sense of the Dirac-Bergman Hamiltonian analysis [18]. The resulting theory is then free of quantum ordering ambiguities and it is described by Dirac brackets which may be turned into equal-time (anti)commutators via correspondence principle. Since we are dealing with indefinite metric, we must find a way to well-define the positive norm subspace or physical space. In the case of QED4 in linear gauges, the positive frequency part of the \( B \)-field does this job by annihilating the physical states. On the other hand, however, the use of the ’t Hooft-Veltman gauge does not allow the \( B \)-field to obey a free-field equation and, thus, the previous condition can no longer be used. Yet, we can still use the BRST charge to define physical states as expected.
We also show that the indefinite metric formalism have a tool to avoid the presence of the $B$-field, ghost and longitudinal gauge fields in the physical subspace by the so called quartet mechanism. The canonical quantization is employed and the propagator of the gauge fields is obtained. In order to do so, we had to use the quantum Cauchy problem for the propagator of ghosts [19,15] and use the BRST symmetry to relate it to the $B$ field and the longitudinal gauge field projection. Latter, we also show that in a general case, all the amplitudes are independent from the g ’t Hooft parameter [20].

This work is organized as follows. In section 2, we construct the Lagrangian of QED$_4$ in the ’t Hooft-Veltman gauge, we deduce the corresponding equations of motion and the BRST charge, and we define the physical states. In section 3, we perform the covariant quantization of the theory. We derive the canonical structure and infer the propagator of the photon field. In section 4, we present a discussion about ghost fields and the relation between the BRST charge and the ghost number operator. We also present a discussion about the quartet structure. Section 5 is devoted to show that all the amplitudes are independent from the g ’t Hooft gauge parameter. Finally, we conclude in section 6.

2. The Lagrangian and its residual symmetry

The ’t Hooft-Veltman gauge fixing condition is given by the following relation

\[ \partial_\mu A^\mu(x) + g A_\mu(x) A^\mu(x) = 0, \]  

where $g$ is an arbitrary parameter. It is important to mention that this is a good gauge condition since it is attainable, that is, by suitable gauge transformations it is always possible to write down relation (1), and since it satisfies the Dirac criteria.

Let us start by generalizing equation (1) to its $\alpha$-gauge condition using the following Lagrangian density [17]

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \left( \partial_\mu A^\mu + g A_\mu A^\mu \right) - \frac{\alpha}{2} B^2, \]  

where $\alpha$ is the gauge parameter. The equations of motion for the Lagrange multiplier or $B$-field provides an operator equation

\[ \partial_\mu A^\mu(x) + g A^\mu(x) A_\mu(x) - \alpha B(x) = 0, \]  

which depends on $\alpha$ and controls the gauge fixing condition. In particular, for $\alpha = 0$ we reproduce the original ’t Hooft-Veltman choice.

Before proceeding with the quantization of this theory, we must identify which is the residual gauge invariance of the theory. Since the non-gauge sector of the Lagrangian is invariant under the local gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$, where $\Lambda(x)$ is an a priori c-number field, it is enough to vary the equation of motion (3) with respect to the previous transformation in order to find a condition that defines the residual symmetry. Explicitly, we obtain

\[ \delta \left[ \partial_\mu A^\mu(x) + g A^\mu(x) A_\mu(x) - \alpha B(x) \right] = 0 \rightarrow \left( \Box + 2 g A_\mu(x) \partial^\mu \right) \Lambda(x) = 0. \]  

These residual gauge transformations are defined by a class of $\Lambda(x)$ “functions” satisfying equation (4). However, we must be careful. Unlike in linear gauges, the residual symmetry (4) is defined by an operator equation, thus, the set of $\Lambda(x)$ must be operators as well instead of ordinary functions. Therefore, equation (4) plays the role of a constraint and should be incorporated
in the Lagrangian density (2) by means of a Lagrange multiplier operator. In other words, we end up with the following modified Lagrangian density

\[ \mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + B \left( \partial_\mu A^\mu + g A^\mu A_\mu \right) - \frac{\alpha}{2} B^2 - i \tilde{c} \left( \Box + 2g A^\mu \partial_\mu \right) c. \]  

(5)

With regard to \( \tilde{c}(x) \), it is the aforementioned Lagrange multiplier field whose equation of motion furnishes the residual symmetry (4) and has a Grassmann character. Moreover, in order to reproduce equation (4) we must have \( c(x) = \varepsilon \Lambda(x) \) where \( \varepsilon \) is a Grassmann number and \( \Lambda(x) \) has acquired an operator field status. This Grassmannian nature is necessary if one wants a conserved residual-symmetry generating charge. It also has an important relation with unitarity where non-physical fields can be arranged in a non-observable structure called quartets. We will call \( \tilde{c}(x) \) and \( c(x) \) as ghost fields. Needless to say, the imaginary unit has been added to recover the Hermitian characteristic of the Lagrangian.

After these observations, it is straightforward to show that the system presents the following operational global symmetry transformations, known as BRST transformations,

\[ Q_B A_\mu(x) = \partial_\mu c(x), \quad Q_B c(x) = 0, \quad Q_B \tilde{c}(x) = i B(x), \quad Q_B B(x) = 0, \]  

(6)

where \( Q_B \) denotes the BRST generating charge. From (6), we note that the BRST charge is nilpotent, that is, \( Q_B^2 = 0 \). This nilpotency property allows us to write the Lagrangian (5) in an explicit BRST invariant way

\[ \mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + Q_B \left[ -i \tilde{c} \left( \partial_\mu A^\mu + g A^\mu A_\mu \right) + i \frac{\alpha}{2} \tilde{c} B \right] + A_\mu J^\mu, \]  

(7)

where we have added to the system a fermionic matter current denoted by \( J_\mu \).

From (7) it follows the equation of motion

\[ \partial_\mu F^{\mu\nu}(x) + Q_B \tilde{J}^\nu(x) + J^\nu(x) = 0, \]  

(8)

wherein

\[ Q_B \tilde{J}^\nu(x) = -\partial^\nu B(x) + 2g B(x) A^\nu(x) - 2i g \tilde{c}(x) \partial^\nu c(x). \]  

(9)

With \( \tilde{J}^\nu(x) = i \partial^\nu \tilde{c}(x) - 2i g \tilde{c}(x) A^\nu(x) \).

Since the first term on the left-hand side of (8) and the matter current are transverse operators, we conclude that \( Q_B J_\mu(x) \) must also be transverse.

It is important to comment that with covariant linear gauges, the \( B \)-field satisfies a free-field equation, thus, the physical Hilbert space, \( \mathcal{H}_{\text{phys}} \), is constructed by imposing the following subsidiary condition

\[ B^+(x) |\text{phys} \rangle = 0, \quad \forall |\text{phys} \rangle \in \mathcal{H}_{\text{phys}}. \]  

(10)

This expression is Poincaré invariant since it is defined in terms of the positive frequency part of the \( B \)-field. However, in the present case, the use of the ’t Hooft-Veltman gauge does not permit \( B \) to obey a free equation as we can see by acting with the differential operator \( \partial_\mu \) on equation (8). The generalization of (10) is given by

\[ Q_B |\text{phys} \rangle = 0, \quad \forall |\text{phys} \rangle \in \mathcal{H}_{\text{phys}} \equiv \frac{\mathcal{V}}{\mathcal{V}_0}. \]  

(11)

This definition is also Poincaré invariant. It means that physical states must be invariant by residual gauge transformations. This is a consistent definition due to its nilpotency. In fact, the quotient
space \( V/V_0 \) above tells us that the physical states are defined by \( Q_B |\text{phys}\rangle = 0 \) and also by the requirement that it should not be of the form \( Q_B |\Psi\rangle \) for any state \( \Psi \) since the nilpotency of \( Q_B \) would make it unobservable.

3. Covariant quantization

In order to quantize the theory we compute the corresponding canonical momenta variables and write down the equal-time commutators by using the correspondence principle. We start from Lagrangian (5) to obtain that

\[
\pi^i(x) = \frac{\partial L}{\partial (\partial_0 A_i(x))} = \partial_0 A^i(x) - \partial^i A^0(x),
\]

(12)

\[
\pi^0(x) = \frac{\partial L}{\partial (\partial_0 A_0(x))} = B(x),
\]

(13)

are the momenta for the gauge field, that

\[
\pi_{\bar{c}}(x) = \frac{\partial L}{\partial (\partial_0 \bar{c}(x))} = i\partial_0 \bar{c}(x),
\]

(14)

\[
\pi_c(x) = \frac{\partial L}{\partial (\partial_0 c(x))} = i\partial_0 c(x) - i2g\bar{c}(x)A^0(x),
\]

(15)

are those for the ghost fields, and that

\[
\pi_B(x) = \frac{\partial L}{\partial (\partial_0 B(x))} = 0,
\]

(16)

is the one for the auxiliary \( B \)-field.

The expressions above prove that also in the 't Hooft gauge \( QED_4 \) case, the careful introduction of the \( B \) field structure furnishes already a well defined system without any first class ambiguities [18].

The non-vanishing Poisson brackets

\[
\{ A_\mu(x), \pi^\nu(y) \}_B = \delta^\nu_\mu \delta^3(x - y)
\]

(17)

\[
\{ B(x), \pi_B(y) \}_B = \delta^3(x - y)
\]

(18)

\[
\{ c(x), \pi_c(y) \}_B = \delta^3(x - y)
\]

(19)

\[
\{ \bar{c}(x), \pi_{\bar{c}}(y) \}_B = \delta^3(x - y),
\]

(20)

give the fundamental (anti)commutation relations that will serve as starting points to derive initial conditions in our analysis below. They are

\[
\left[ A_i(x), \partial_0 A^j(y) \right] = i\delta^j_i \delta^3(x - y)
\]

(21)

\[
\left[ A_0(x), B(y) \right] = i\delta^3(x - y)
\]

(22)

\[
\left[ \bar{c}(x), \partial_0 c(y) \right] = \delta^3(x - y)
\]

(23)

\[
\left[ c(x), \partial_0 \bar{c}(y) \right] = \delta^3(x - y)
\]

(24)

\[
\left[ B(x), B(y) \right] = 0,
\]

(25)

where the last commutator has been added for completeness and the subscript 0 means equal-
time, that is, \( x_0 = y_0 \).

From the gauge condition (3), we obtain

\[
\left[ A_\mu(x), \partial_0 A_0(y) \right]_0 = i \alpha \delta_\mu^0 \delta^3(x - y).
\]  
(26)

Using the \( v = 0 \) component of the equation of motion (8) to obtain the time derivative of the
\( B \)-field in terms of the other fields, we get

\[
\left[ A_\mu(x), \partial_0 B(y) \right]_0 = -i \partial_\nu^\nu \delta^3(x - y) + 2i g A_0(x) \delta^3(x - y)
\]  
(27)

\[
\left[ B(x), \partial_0 B(y) \right]_0 = -2i g B(x) \delta^3(x - y).
\]  
(28)

In order to find the general structure for the vacuum expectation value of the photon field we
first manipulate the equation of motion to write it in the following form

\[
\Box A_\mu(x) + g \partial_\mu \left( A_\beta(x) A^\beta(x) \right) = \alpha \partial_\mu B(x) - Q_B J_\mu(x).
\]  
(29)

Thus, we have

\[
\Box^\nu \Box^\nu \left[ A_\mu(x), A_\nu(y) \right]_0 = \langle 0 \left[ -g \partial_\mu \left( A_\beta(x) A^\beta(x) \right) + \alpha \partial_\mu B(x) - Q_B J_\mu(x) - J_\mu(x), -g \partial_\nu \left( A_\gamma(y) A^\gamma(y) \right) + \alpha \partial_\nu B(y) - Q_B J_\nu(y) \right] \rangle_0.
\]  
(30)

This seemingly complicated expression can be simplified if one considers that \( B \) has a vanishing
norm and is a \( Q_B \)-boundary term, i.e., \( Q_B B(x) = 0 \), that the matter current \( J_\mu(x) \) is physical
in the sense of \( Q_B J_\mu(x) \text{phys} = 0 \), and that both currents \( J_\mu(x) \) and \( Q_B J_\mu(x) \) commute with \( \partial_\nu A^\nu(x) \).\(^1\) In fact, we have

\[
\Box^\nu \Box^\nu \left[ A_\mu(x), A_\nu(y) \right]_0 \\
= g^2 \partial_\mu^\nu \partial_\nu^\nu \langle 0 \left[ A_\beta(x) A^\beta(x), A_\gamma(y) A^\gamma(y) \right]_0 \rangle_0 + \langle 0 \left[ J_\mu(x), J_\nu(y) \right]_0 \rangle_0 \nonumber \\
- \alpha g \partial_\mu^\nu \partial_\nu^\nu \langle 0 \left[ A_\beta(x) A^\beta(x), B(y) \right]_0 \rangle_0 - \alpha g \partial_\nu^\nu \partial_\mu^\nu \langle 0 \left[ B(x), A_\beta(y) A^\beta(y) \right]_0 \rangle_0.
\]  
(31)

Using, again, the gauge condition (3) on the second line of this expression in order to relate it
with \( \partial_\mu A^\mu(x) \), we obtain

\[
\Box^\nu \Box^\nu \left[ A_\mu(x), A_\nu(y) \right]_0 \\
= g^2 \partial_\mu^\nu \partial_\nu^\nu \langle 0 \left[ A_\beta(x) A^\beta(x), A_\gamma(y) A^\gamma(y) \right]_0 \rangle_0 + \langle 0 \left[ J_\mu(x), J_\nu(y) \right]_0 \rangle_0 \\
+ \alpha \partial_\mu^\nu \partial_\nu^\nu \langle 0 \left[ \partial_\beta^\beta A^\beta(x), B(y) \right]_0 \rangle_0 + \alpha \partial_\nu^\nu \partial_\mu^\nu \langle 0 \left[ B(x), \partial_\beta^\beta A^\beta(y) \right]_0 \rangle_0.
\]  
(32)

\(^1\) Explicitly, we have \([\partial^\mu A_\mu(x), J^\nu(y)] = [\partial^\mu A_\mu(x), Q_B J^\nu(y)] = 0 \). This result is obtained by commuting the longitudinal part of \( A_\mu(x) \) with \( J^\nu(y) \) and \( Q_B J^\nu(y) \), respectively. The transversality in the \( v \) index implies that \([Q_B J^\nu(x), A^\beta(y)] = C_\nu \partial_\beta \Delta(x - y, 0) \) and \([J_\nu(x), A^\beta(y)] = D_\nu \partial_\beta \Delta(x - y, 0) \) where \( \Delta(x - y, 0) \) is defined in
(39). \( C \) and \( D \) are indefinite constants.
From this we can immediately conclude that the non-linear self-interaction part contributes to the renormalization of the longitudinal non-physical sector. Therefore, we expect that the physical sector is independent of the fictitious parameter $g$.

To find the general expression for the two-point function it is mandatory to compute the commutator between the photon field $A_\mu$ and the $B$-field, as we can see in (32). Since the use of the ’t Hooft-Veltman gauge forbids the harmonic character of the $B$-field, the desired commutator cannot be easily found by means of the initial data. An alternative approach is to use BRST symmetry. In fact,

$$0 = \langle 0 | Q_B [A_\mu(x) , \bar{c}(y)] | 0 \rangle = \partial_\mu^x \langle 0 | \{c(x) , \bar{c}(y) \} | 0 \rangle + i \langle 0 | [A_\mu(x) , B(y)] | 0 \rangle. \quad (33)$$

This expression shows us that in order to find the vacuum expectation value of the commutator between $A_\mu$ and $B$ we must first compute the two-point function for ghosts which is defined by the following Cauchy problem

$$\Box^x \langle 0 | \{c(x) , \bar{c}(y) \} | 0 \rangle = -2g \langle 0 | A^\mu(x) \partial_\mu^x c(x) , \bar{c}(y) \} | 0 \rangle \quad (34)$$

$$\langle 0 | \{c(x) , \bar{c}(y) \} | 0 \rangle = 0 \quad (35)$$

$$\partial_0^x \langle 0 | \{c(x) , \bar{c}(y) \} | 0 \rangle = -\delta^3(x - y). \quad (36)$$

We shall define

$$\mathcal{D}(x - y; A) \equiv \langle 0 | \{c(x) , \bar{c}(y) \} | 0 \rangle , \quad (37)$$

for convenience. It only depends on the difference of the coordinates due to Poincaré invariance [22]. Hence, the formal solution of (34) is the following integro-differential equation [19]

$$\mathcal{D}(x - y; A) = -2g \int d^4u \, \varepsilon(x, y, u) \Delta(x - u) \langle 0 | A^\beta(u) \partial_\beta^u c(u) , \bar{c}(y) \} | 0 \rangle - \int d^3u \left[ \Delta(x - u) \partial_0^u \mathcal{D}(u - y; A) - \partial_0^u \Delta(x - u) \mathcal{D}(u - y; A) \right]_{u^0 = y^0}$$

$$= -2g \int d^4u \, \varepsilon(x, y, u) \Delta(x - u) \langle 0 | A^\beta(u) \partial_\beta^u c(u) , \bar{c}(y) \} | 0 \rangle - \Delta(x - y) \quad (38)$$

where $\Delta(x - y; s)$ is defined by the following Cauchy data

$$\Box \Delta(x - y; s) = -s \Delta(x - y; s), \quad \Delta(x - y; s)|_0 = 0, \quad \partial_0^x \Delta(x - y; s)|_0 = -\delta^3(x - y), \quad (39)$$

and we have defined $\varepsilon(x, y, u)$ in terms of the Heaviside function as

$$\varepsilon(x, y, u) = \Theta(x_0 - u_0) - \Theta(y_0 - u_0). \quad (40)$$

Summing up, we have from (33) that

$$\langle 0 | [A_\mu(x) , B(y)] | 0 \rangle = i \partial_\mu^x \mathcal{D}(x - y; A), \quad (41)$$

with $\mathcal{D}$ defined in (38).
Having these results in mind, it follows from (32) that
\[
\square^x \square^y \langle 0 \left| A_\mu(x), A_\nu(y) \right| 0 \rangle = g^2 \partial_\mu^x \partial_\nu^y \langle 0 \left| A_\mu(x), A_\nu(y) \right| 0 \rangle + i \alpha \partial_\mu^x \partial_\nu^y \square^y \mathcal{D}(y - x; A) - i \alpha \partial_\mu^x \partial_\nu^y \square^y \mathcal{D}(x - y; A),
\]
(42)
is the differential equation we must solve in order to compute the vacuum expectation value of the gauge field commutator. In view of this, we shall define the following spectral representation [17]
\[
\langle 0 \left| J_\mu(x), J_\nu(y) \right| 0 \rangle \equiv -i \int ds \rho(s) \left( s \eta^{\mu\nu} + \partial_\mu \partial_\nu \right) \Delta(x - y; s)
\]
(43)
and use (38) to obtain that the general solution of equation (42) is given by
\[
\langle 0 \left| A_\mu(x), A_\nu(y) \right| 0 \rangle = a \eta^{\mu\nu} \Delta(x - y) + b \partial_\mu \partial_\nu E(x - y) + c \partial_\mu \partial_\nu \Delta(x - y) + d \eta^{\mu\nu} E(x - y) - i \int ds \rho(s) \left( s \eta^{\mu\nu} + \partial_\mu \partial_\nu \right) \Delta(x - y; s)
\]
(44)
where \( E(x - y) \) is defined as
\[
(\square + s) E(x - y; s) = \Delta(x - y; s), \quad E(x - y; s)|_0 = 0, \quad (\partial_0^{\alpha})^3 E(x - y; s)|_0 = -\delta^3(x - y).
\]
(45)
The unknown coefficients \( a, b \) and \( c \) are determined using the initial conditions as follows. Take the spatial components \( \mu = k, \nu = l \) of (44) and act to it with \( \partial_0^x \), set \( x_0 = y_0 \), and finally use (21) to obtain that
\[
a = -i \left( 1 - \int ds s \rho(s) \right) \equiv -i Z
\]
(46)
\[
c = i \int ds \left( s \rho(s) \right).
\]
(47)
Moreover, let us act with \( \partial_0^y \) to (44), set \( x_0 = y_0 \), and use the \( \mu = 0 \) component of (26) to find that
\[
a + b = -i \alpha
\]
(48)
or, more precisely,
\[
b = i (Z - \alpha).
\]
(49)
Thus, the vacuum expectation value of the gauge field commutator reads as
\[
\langle 0 \lvert [A_\mu(x), A_\nu(y)] \rvert 0 \rangle
= a \left( \eta_{\mu\nu} \Delta(x - y) - \partial_\mu \partial_\nu E(x - y) \right) - i \alpha \partial_\mu \partial_\nu E(x - y) + d \eta_{\mu\nu} E(x - y)
+ c \partial_\mu \partial_\nu \Delta(x - y) - i \int ds \rho(s)(s \eta_{\mu\nu} + \partial_\mu \partial_\nu) \Delta(x - y; s) \\
- g^2 \partial_\mu^x \partial_\nu^x \int d^4\omega d^4u \epsilon(y, x; u) \epsilon(x, u; \omega) \Delta(x - \omega) \Delta(y - u)
\times \langle 0 \lvert [A_\mu(\omega) A^\mu(\omega), A_\rho(u) A^\rho(u)] \rvert 0 \rangle
- i \alpha \partial_\mu^x \partial_\nu^x \int d^4u \epsilon(x, y, u) \Delta(y - u) \mathcal{D}(x - u; A)
- i \alpha \partial_\mu^x \partial_\nu^x \lim_{g^2 \to 0} \int d^4u \epsilon(x, y, u) \left( \Delta(y - u) \mathcal{D}(x - u; A) + \Delta(x - u) \mathcal{D}(y - u; A) \right).
\] (50)

This is a good point to stop developing and to introduce two consistency checks to test that our solution (50) is correct. Let us first consider the absence of matter current, namely, \(\rho(s) \to 0\), and the limit \(g^2 \to 0\). We should expect to end up in the theory of a free electromagnetic field with Lorenz gauge condition. In other words, we expect that (50) becomes
\[
\langle 0 \lvert [A_\mu(x), A_\nu(y)] \rvert 0 \rangle_{\text{Free}} = -i \eta_{\mu\nu} \Delta(x - y) + i (1 + \alpha) \partial_\nu E(x - y)
\] (51)

From (46) and (47) we have that \(a \to -i\) and \(c \to 0\), respectively. Thus, from (50) we obtain
\[
\langle 0 \lvert [A_\mu(x), A_\nu(y)] \rvert 0 \rangle
\to -i \left( \eta_{\mu\nu} \Delta(x - y) - \partial_\mu \partial_\nu E(x - y) \right) - i \alpha \partial_\nu E(x - y) + d \eta_{\mu\nu} E(x - y)
- i \alpha \partial_\mu^x \partial_\nu^x \lim_{g^2 \to 0} \int d^4u \epsilon(x, y, u) \left( \Delta(y - u) \mathcal{D}(x - u; A) + \Delta(x - u) \mathcal{D}(y - u; A) \right).
\] (52)

For the last term we use the fact that
\[
E(x - y) = - \int d^4u \epsilon(x, y, u) \Delta(x - u) \Delta(y - u),
\] (53)
and that in the \(g^2 \to 0\) limit the function \(\mathcal{D}(x - y; A)\) goes to \(-\Delta(x - y)\) to obtain that
\[
- i \alpha \partial_\mu^x \partial_\nu^x \lim_{g^2 \to 0} \int d^4u \epsilon(x, y, u) \left( \Delta(y - u) \mathcal{D}(x - u; A) + \Delta(x - u) \mathcal{D}(y - u; A) \right)
= 2i \alpha \partial_\mu \partial_\nu E(x - y).
\] (54)

Plugging this result back in equation (52) we realize that it becomes the expected solution (51) if \(d = 0\), a constant that was undetermined until this point. We are going to see that this condition can be obtained also by making our solution compatible with the gauge condition.

Our second consistency check is to consider again the limit \(g^2 \to 0\) but in the presence of the fermionic current. This time we should expect the theory of an interacting electromagnetic field with Lorenz gauge condition, namely,
\begin{align}
0 \left[ A_\mu(x), A_\nu(y) \right] 0 \rangle_{\text{int}} \\
&= -i \left( Z \eta_{\mu\nu} - \int ds \rho(s) \partial_\mu \partial_\nu \right) \Delta(x - y) + i (Z + \alpha) \partial_\mu \partial_\nu E(x - y) \\
&\quad - i \int ds \rho(s) (s \eta_{\mu\nu} + \partial_\mu \partial_\nu) \Delta(x - y; s). \tag{55}
\end{align}

Similarly, from (50) we obtain
\begin{align}
0 \left[ A_\mu(x), A_\nu(y) \right] 0 \rangle &\longrightarrow -i \left( \eta_{\mu\nu} \Delta(x - y) - \partial_\mu \partial_\nu E(x - y) \right) - i \alpha \partial_\mu \partial_\nu E(x - y) \\
&\quad + i \int ds \rho(s) \partial_\mu \partial_\nu \Delta(x - y) - i \int ds \rho(s) (s \eta_{\mu\nu} + \partial_\mu \partial_\nu) \Delta(x - y; s) \\
&\quad - i \alpha \partial_\nu^\chi \partial_\nu \lim_{g^2 \to 0} \int d^4 u \epsilon(x, y, u) \left( \Delta(y - u) D(x - u; A) + \Delta(x - u) D(y - u; A) \right). \tag{56}
\end{align}

Using again (54) we end up with the expected solution (55). These results show that (50) is the non-perturbative two-point function of an interacting electromagnetic field in the 't Hooft-Veltman gauge.

It is also noteworthy to mention that these results are compatible with the gauge condition. In fact, on the one hand we have\(^2\)
\begin{align}
0 \left[ \partial_\nu^\mu A_\mu(x), A_\nu(y) \right] 0 \rangle \\
&= d \partial_\nu E(x - y) + g^2 \partial_\nu^x \int d^4 \omega \epsilon(x; \omega) \Delta(x - \omega) 0 \left[ A_\mu(\omega) A_\mu^\chi(\omega), A_\beta(y) A_\beta^\chi(y) \right] 0 \rangle \\
&\quad - i \alpha \partial_\nu \int d^4 u \epsilon(x, y, u) \Delta(y - u) \square^x D(x - u; A) - i \alpha \partial_\nu D(y - x), \tag{57}
\end{align}

but on the other hand, by using the gauge condition, we obtain
\begin{align}
0 \left[ \partial_\nu^\mu A_\mu(x), A_\nu(y) \right] 0 \rangle &= 0 \left[ - g A_\mu(x) A_\mu(x) + \alpha B(x), A_\nu(y) \right] 0 \rangle \\
&= - g 0 \left[ A_\mu(x) A_\mu(x), A_\nu(y) \right] 0 \rangle - i \alpha \partial_\nu D(y - x; A). \tag{58}
\end{align}

For the first term we use the equations of motion in the following way
\begin{align}
- g 0 \left[ A_\mu(x) A_\mu(x), A_\nu(y) \right] 0 \rangle \\
&= - g 0 \left[ A_\mu(x) A_\mu(x), - g \frac{\partial_\nu^x (A_\beta(y) A_\beta^\chi(y))}{\square^y} - \frac{Q_B J_\nu(y)}{\square^y} - \frac{J_\nu(y)}{\square^y} + \alpha \frac{\partial_\nu B(y)}{\square^y} \right] 0 \rangle \\
&= g^2 \partial_\nu^x \int d^4 \omega \epsilon(x, y; \omega) \Delta(x - \omega) 0 \left[ A_\mu(\omega) A_\mu(\omega), A_\beta(y) A_\beta(y) \right] 0 \rangle \\
&\quad - \alpha g \partial_\nu^x \int d^4 u \epsilon(x, y, u) \Delta(y - u) 0 \left[ A_\mu(\omega) A_\mu(\omega), B(y) \right] 0 \rangle
\end{align}

\(^2\) Use had been made of \(- i \alpha \square^x \int d^4 u \epsilon(x, y, u) \Delta(y - u) D(x - u) = + i \alpha \Delta(x - y) - i \alpha \int d^4 \omega \epsilon(x, y, u) \square^x D(x - u) \Delta(y - u) \) and \(- i \alpha \square^x \int d^4 u \epsilon(x, y, u) \Delta(x - u) D(y - u) = i \alpha D(y - x) \).
\[-i\alpha \partial^x_i \int d^4 u \, \epsilon(x, y, u) \Delta(y - u) \Box^x \mathcal{D}(x - u; A).\]  

(59)

The \( J_\mu(x) \) and \( Q_B \tilde{J}_\mu(x) \) did not contribute to the previous expressions\(^3\) due to the same reasons as in (31).

Therefore, comparing with equation (57) we obtain again that \( d = 0 \)

### 4. Perturbative analysis in powers of \( g^2 \)

Note that our well-established solution (50) is only formal since it depends on the photon field commutator. In order to have information about the change in the structure of the two-point function due solely to this part, we set \( \alpha = 0 \) since in this case equation (3) reduces to the original \( \text{'t} \) Hooft-Veltman choice,\(^4\) and we will make use of the Heisenberg perturbation theory \([15]\). We also consider the system without interaction with the fermions. We must call attention to the fact that \( g \) is an unphysical parameter and it may have any value and then non-perturbative methods may be used. But since we want to compare our results to the diagramatic approach \([13]\), we consider the limit of small \( g \) in this section. Thus, from our formal solution we expect that for \( \alpha = 0 \) we must have

\[
\langle 0 \rangle A_\mu(x), A_\nu(y) \langle 0 \rangle
= -i \left( \eta_{\mu\nu} \Delta(x - y) - \partial_\mu \partial_\nu \Delta(x - y) \right)
- g^2 \partial_\mu^x \partial_\nu^x \int d^4 u d^4 v \, \epsilon(y, v, u) \epsilon(x, u; \omega) \Delta(x - \omega) \Delta(y - \omega) \Delta(x - u) \Delta(y - u)
\times \langle 0 \rangle A_\mu(\omega) A_\nu(\omega) A_\beta(u) A_\gamma(u) \langle 0 \rangle
\]

(60)
as the solution of the following differential equation

\[
\Box^x \Box^y \langle 0 \rangle A_\mu(x), A_\nu(y) \langle 0 \rangle = - g^2 \partial_\mu^x \partial_\nu^x \langle 0 \rangle A_\beta(\omega) A_\gamma(u) A_\beta(u) A_\gamma(u) \langle 0 \rangle,
\]

(61)

which will serve us as a guiding block in the perturbative expansion of (61).

Let us consider \( g^2 \) as an infinitesimal parameter and expand the gauge field commutator as follows [23]

\[
\left[ A_\mu(x), A_\nu(y) \right] = \sum_{n=0}^{\infty} \left( g^2 \right)^n \mathcal{O}^{(n)}_{\mu\nu}(x, y),
\]

(62)

where \( \mathcal{O}^{(n)}_{\mu\nu}(x, y) \) are operators\(^5\) whose vacuum expectation values must be computed by solving a chain of coupled differential equations arising from (61). Hence, the initial conditions (21) and (26) with \( \alpha = 0 \) translate to

\[
\langle 0 \rangle \partial_0^x \mathcal{O}^{(n)}_{k}(x, y) \langle 0 \rangle = i \delta_{k}^3 \delta^3(x - y)
\]

(63)

\(^3\) It can be understood by using the gauge condition to relate \( gA_\mu A^\mu \) with \( \partial_\mu A^\mu \) and the \( B \) field.

\(^4\) Moreover, we chose to work with \( \alpha = 0 \) for maximum simplicity since the term proportional to \( \alpha \) in (50) may also have corrections in powers of \( g \).

\(^5\) We cannot specify a given coordinate dependence for this operator, although it can be shown that the first term of the approximation depends on its difference. However, their vacuum expectation values must depend on the mentioned difference due to Poincare invariance.
and

\[ \langle 0 | \hat{\partial}_0^\gamma \mathcal{O}^{(n)}_{\nu}(x, y) | 0 \rangle_0 = 0 \quad \text{for any } n, \quad (64) \]

respectively. Using identities for the self-interaction commutator we have that

\[ \left[ \mathcal{A}_\beta(x) \mathcal{A}_\beta(x), \mathcal{A}_\gamma(y) \mathcal{A}_\gamma(y) \right] \\
= \sum_{n=0}^{\infty} \left( g^2 \right)^n \left[ \mathcal{O}^{(n)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{O}^{(n)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) \\
+ \mathcal{A}_\beta(x) \mathcal{O}^{(n)}_{\beta\gamma}(x, y) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{A}_\beta(x) \mathcal{O}^{(n)}_{\beta\gamma}(x, y) \right]. \quad (65) \]

Thus, plugging these results back in (61) we obtain

\[ \sum_{n=0}^{\infty} \left( g^2 \right)^n \hat{\square}^{\mu} \hat{\square}^{\nu} \langle 0 | \mathcal{O}^{(n)}_{\mu\nu}(x, y) | 0 \rangle_0 = - \sum_{k=0}^{\infty} \left( g^2 \right)^{k+1} \partial_{\mu}^{x} \partial_{\nu}^{x} \langle 0 | \mathcal{O}^{(k)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) \mathcal{A}_\gamma(y) \\
+ \mathcal{A}_\gamma(y) \mathcal{O}^{(k)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) + \mathcal{A}_\beta(x) \mathcal{O}^{(k)}_{\beta\gamma}(x, y) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{A}_\beta(x) \mathcal{O}^{(k)}_{\beta\gamma}(x, y) \rangle | 0 \rangle_0, \quad (66) \]

from which we obtain the following differential equations

\[ \hat{\square}^{\mu} \hat{\square}^{\nu} \langle 0 | \mathcal{O}^{(0)}_{\mu\nu}(x, y) | 0 \rangle_0 = 0 \]

\[ g^2 \hat{\square}^{\mu} \hat{\square}^{\nu} \langle 0 | \mathcal{O}^{(1)}_{\mu\nu}(x, y) | 0 \rangle_0 \\
= - g^2 \partial_{\mu}^{x} \partial_{\nu}^{x} \langle 0 | \mathcal{O}^{(0)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{O}^{(0)}_{\beta\gamma}(x, y) \mathcal{A}_\beta(x) \\
+ \mathcal{A}_\beta(x) \mathcal{O}^{(0)}_{\beta\gamma}(x, y) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{A}_\beta(x) \mathcal{O}^{(0)}_{\beta\gamma}(x, y) \rangle | 0 \rangle_0. \quad (68) \]

for the zeroth order and \( n = 1 \) contributions, respectively.

The former differential equation is easily solved by taking the \( g^2 \rightarrow 0 \) limit in the non-perturbative solution (60). In fact, we have

\[ \mathcal{O}^{(0)}_{\mu\nu}(x, y) = - i \left[ \eta_{\mu\nu} \Delta(x - y) - \partial_{\mu} \partial_{\nu} E(x - y) \right] \quad (69) \]

which satisfies the requirements (63) and (64). Since this is the solution of a free theory we conclude that \( \mathcal{O}^{(0)}_{\mu\nu}(x, y) \) is actually a c-number (which depends on the coordinate difference) which we will denote it as \( D^{(0)}_{\mu\nu}(x - y) \) as it is usually found in the literature. Hence, the latter differential equation is reduced to

\[ g^2 \hat{\square}^{\mu} \hat{\square}^{\nu} \langle 0 | \mathcal{O}^{(1)}_{\mu\nu}(x, y) | 0 \rangle_0 = - 2 g^2 \partial_{\mu}^{x} \partial_{\nu}^{x} \left( D^{(0)}_{\beta\gamma}(x, y) \langle 0 | \mathcal{A}_\beta(x) \mathcal{A}_\gamma(y) + \mathcal{A}_\gamma(y) \mathcal{A}_\beta(x) \rangle | 0 \rangle \right). \quad (70) \]

For the right-hand side we introduce the following expansion\(^6\)

\[^6\text{The use of the notation } + \text{ is to emphasize that its vacuum expectation value gives the positive frequency part of the associated distributions.}\]
\begin{equation}
A_\beta(x)A_\gamma(y) = \sum_{n=0}^{\infty} \left(g^2\right)^n \mathcal{O}_{\beta\gamma}^{+(n)}(x, y),
\end{equation}

to obtain that the first order contribution is given by
\begin{equation}
\Box^x \Box^y \langle 0| \mathcal{O}^{(1)}_{\mu\nu}(x, y)|0\rangle = -2 \partial_\mu \partial_\nu \left( D^{(0)\beta\gamma}(x, y) \left( \mathcal{O}_{\beta\gamma}^{+(0)}(x, y) + \mathcal{O}_{\beta\gamma}^{+(0)}(y, x) \right) \right).
\end{equation}

Since we have concluded that the zeroth contribution is a c-number we finally have\footnote{The distribution $D^{+(0)}_{\beta\gamma}(x-y)$ denotes the positive frequency part of $D^{(0)\beta\gamma}(x-y)$ which is defined as $i D^{(0)\beta\gamma}(x-y) = D^{+(0)}_{\beta\gamma}(x-y) + D^{+(0)}_{\beta\gamma}(y-x)$.}
\begin{equation}
\Box^x \Box^y \langle 0| \mathcal{O}^{(1)}_{\mu\nu}(x, y)|0\rangle = -2 \partial_\mu \partial_\nu \left( D^{(0)\beta\gamma}(x, y) \left( D^{+(0)\beta\gamma}(x, y) + D^{+(0)\beta\gamma}(y, x) \right) \right),
\end{equation}

from which it follows that
\begin{equation}
\langle 0| \mathcal{O}^{(1)}_{\mu\nu}(x, y)|0\rangle = -2 \partial_\mu \partial_\nu \left( D^{(0)\beta\gamma}(x, y) \left( D^{+(0)\beta\gamma}(x - y) + D^{+(0)\beta\gamma}(y - x) \right) \right).
\end{equation}

where we have defined
\begin{equation}
M^{(0)}(x - y) = D^{(0)\beta\gamma}(x - y) \left( D^{+(0)\beta\gamma}(x - y) + D^{+(0)\beta\gamma}(y - x) \right).
\end{equation}

Now we set $\tilde{a} = \tilde{b} = \tilde{d} = 0$ since it can be verified that the loop term do not contribute to the initial conditions. We also set $\tilde{c} = 0$ in order to be compatible with the ’t Hooft gauge condition. Therefore, we have that
\begin{equation}
\langle 0| \mathcal{O}^{(1)}_{\mu\nu}(x, y)|0\rangle = -2 \partial_\mu \partial_\nu \int d^4 \omega d^4 u \epsilon(y, x; u) \epsilon(x, u; \omega) \Delta(x - \omega) \Delta(y - u) M^{(0)}(\omega - u),
\end{equation}

Then, the propagator evaluated until order $g^2$ has the form
\begin{equation}
\langle 0| A_\mu(x), A_\nu(y) \rangle |0\rangle = -i \left( \eta_{\mu\nu} \Delta(x - y) - \partial_\mu \partial_\nu E(x - y) \right)
- 2g^2 \partial_\mu \partial_\nu \int d^4 \omega d^4 u \epsilon(y, x; u) \epsilon(x, u; \omega) \Delta(x - \omega) \Delta(y - u) M^{(0)}(\omega - u)
\end{equation}

Again, now in the Heisenberg perturbative framework, we conclude that the ’t Hooft parameter affects just non-physical longitudinal contributions in accordance with the diagramatic approach [13,14].
5. A digression about the ghost fields

This section has the purpose of exposing some important theoretical content about the auxiliary Grassmann fields usually called ghosts. Firstly, the need of an odd parity for those fields is inferred demanding that the BRST symmetry current must be conserved. It is given below

\[ J^B_\mu (x) = - F_{\mu\nu}(x) \partial^\nu c(x) + B(x) \partial_\mu c(x) \] (78)

Its divergence is evaluated as

\[ \partial^\mu J^B_\mu (x) = - \partial^\mu F_{\mu\nu}(x) \partial^\nu c(x) - F_{\mu\nu}(x) \partial^\mu \partial^\nu c(x) + \partial^\mu B(x) \partial_\mu c(x) + B(x) \Box c(x) \] (79)

Using the operator equations of motion we have:

\[ \partial^\mu J^B_\mu (x) = B(x) \left( \Box + 2 g A_\nu(x) \partial^\nu \right) c(x) + i 2 g \bar{c}(x) \partial_\nu c(x) \partial^\nu c(x) = 0 \] (80)

The first term vanishes in all Hilbert space due to the ghost field equation. The second one do not contribute if \( c(x) \) has a Grassmann character. That is the reason to consider this kind of parity.

Regarding the ghost field and its Lagrangian one may find a global ghost number symmetry expressed by the invariance under the transformations

\[ c(x) \rightarrow c(x) e^{i \theta} \quad ; \quad \bar{c}(x) \rightarrow \bar{c}(x) e^{i \theta} \] (81)

The \( \theta \) parameter that appears above must be real in order to preserve the field’s Hermitian nature. Its associate current reads

\[ J^c_\mu(x) = i \left( \partial_\mu \bar{c}(x) - 2 g \bar{c}(x) A_\mu(x) \right) \theta c(x) - i \partial_\mu c(x) \theta \bar{c}(x) = J^B_\mu(x) \] (82)

We can show that the charge \( Q_c = \int d^3 x J^c_0 \) indeed generate this symmetry

\[ \left\{ i Q_c, c(x) \right\} = c(x) \quad ; \quad \left\{ i Q_c, \bar{c}(x) \right\} = - \bar{c}(x) \] (83)

It is possible to show that there is a relation between \( Q_c \) and the BRST charge

\[ Q_B J^c_0(x) = - \partial_0 B(x) c(x) + B(x) \partial_0 c(x) + 2 g B(x) A_0(x) c(x) \] (84)

Indeed, the BRST charge can be rewritten as

\[ J^B_0(x) = B(x) \partial_0 c(x) - \partial_0 B(x) c(x) + 2 g B(x) A_0(x) c(x) \] (85)

So we have the relation

\[ Q_B Q_c = Q_B \] (86)

It shows that the application of the BSRT charge raises the ghost number eigenvalue in one unit.

5.1. Quartet structure

An interesting property shared by \( Q_c \) eigenstates with a given \( N \) eigenvalue is the fact that it obeys orthogonality relations of the kind [17]

\[ \langle \phi_N | \psi_M \rangle \sim \delta_{M-N} \] (87)
This result can be inferred by the Hermitian nature of $Q_c$. Having this result in mind we can introduce the concept of quartet configuration. It is nothing more than a way to dispose the non-physical fields in an non-observable structure. Since the physical Hilbert space is defined as

$$\mathcal{H}_{phys.} = \frac{\mathcal{V}}{\mathcal{V}_0}$$

We must confine the auxiliary fields outside $\mathcal{H}_{phys.}$, in other words, in the negative or null norm sectors.

So, if non-physical fields such as $B(x)$ and the longitudinal $A_{\mu}(x)$ part are in the quartet configuration they cannot be detected.

In order to define such a structure, suppose four states defined in the following manner

$$|\pi_k^N\rangle, \quad |\delta_k^{N-1}\rangle = Q_B|\pi_k^N\rangle, \quad |\pi_k^{N-1}\rangle, \quad |\delta_k^{N-1}\rangle = Q_B|\pi_k^{N-1}\rangle$$

Where $N$ denotes a given eigenvalue of the ghost number symmetry charge.

Evidently, two of those states are non-physical while the other two has vanishing norm. From the previously introduced orthogonality relations the first state is not orthogonal to the last one while the second is not orthogonal just to the third. We can write those states as asymptotic creation operator acting on the vacuum state

$$|\pi_k^N\rangle = \chi_k^\dagger |0\rangle, \quad |\delta_k^{N-1}\rangle = -i\gamma_k^\dagger |0\rangle, \quad |\pi_k^{N-1}\rangle, \quad |\delta_k^{N-1}\rangle = -\beta_k^\dagger |0\rangle$$

Regarding the symmetry transformations, according to the quartet definition, considering $N = 0$, we have

$$[Q_B, \chi_k] = \gamma_k, \quad \{Q_B, \gamma_k\} = i\beta_k, \quad [Q_B, \beta_k] = \{Q_B, \gamma_k\} = 0$$

Those previous relations are the same as considering $A_{\mu}(x) = \delta_{\mu}\chi(x)$, $B(x) = \beta(x)$, $c(x) = \gamma(x)$ and $\bar{c}(x) = \bar{\gamma}(x)$. It allows us to conclude that the non-physical auxiliary fields in ’t Hooft gauge electrodynamics are harmless since they are confined in a quartet non-observable structure.

6. Parameter independence

In this final section we intend to use the BRST symmetry to show that all physical amplitudes are independent of the $g$ parameter that appears in the expression defining the ’t Hooft gauge.

In order to do so we define the generator functional of connected Green function [17,21]:

$$\exp iW(J) \equiv \langle 0|\mathcal{T}\exp iS(J)|0\rangle$$

The vacuum above is still the Heisenberg one, $\mathcal{T}$ denotes the time ordering operator and $S(J)$ is given by:

$$S(J) = \int d^4x \Big( \Sigma^\mu(x)A_{\mu}(x) + \beta(x)\bar{c}(x) + \bar{\beta}c(x) + \sigma(x)B(x) \Big)$$

Where $\Sigma^\mu(x)$, $\beta(x)$, $\bar{\beta}$ and $\sigma(x)$ are c-number external sources with the appropriate Grassmann parity.

The c-number fields are defined collectively as:

$$\phi_I(x) = \frac{\delta W(J)}{\delta J_I(x)} = \langle 0|\mathcal{T}\hat{\phi}_I(x)\exp iS(J)|0\rangle \exp (-iW(J))$$

Where we emphasize the operator nature of the fields by using the hat.
Since the $\mathcal{T}\exp iS(J)$ is a gauge invariant object, we use this fact explicitly to derive the generator of the Ward identities:

$$0 = \langle 0 \left[ Q_B, \mathcal{T}\exp iS(J) \right] \rangle 0 = \langle 0 \mathcal{T} \int d^4x \left( \Sigma^\mu \delta^\mu \beta + i\beta B \right) \exp iS(J) \rangle 0 \tag{95}$$

It is possible to show that the generator of connected Green functions can be written in terms of a path integral [21]

$$e^{iW(J)} \equiv Z = \int DA_\mu(x)DB(x)Dc(x)\exp i(S_i + iS(J)) \tag{96}$$

Where $S_i = \int d^4x L(x)$ with $L(x)$ defined in equation (5).

If we impose that this expression is invariant with relation to the $g$ parameter the following condition arises

$$\langle 0 \mathcal{T} \int d^4x \left( A^\mu(x)A_\mu(x)B(x) - 2i\bar{c}(x)A_\mu(x)\partial^\mu c(x) \right) \rangle 0 = 0 \tag{97}$$

The relation above is the $B$ field version of [20].

Now we show that this expression indeed follows from BRST invariance. Then, we vary the generator of connected Green functions with relation to $\frac{\delta^3}{\delta i\Sigma^\mu(x)\Sigma^\nu(x)\delta\beta(x)}$ and taking the limit of vanishing sources in the end of the process

$$\langle 0 \mathcal{T} \int d^4x \left( A^\mu(x)A_\mu(x)B(x) - 2i\bar{c}(x)A_\mu(x)\partial^\mu c(x) \right) \rangle 0 = 0 \tag{98}$$

The same expression as the one required for amplitudes to be independent from the parameter $g$ is recovered. So, the BRST symmetry indeed furnishes a sufficient condition to it.

7. Conclusion

Throughout this article we have shown that the non-linear ’t Hooft gauge do not have any influence on the physical output of the theory, as it should be. In order to do so we have employed the Nakanishi $B$-field formalism, in its non-perturbative and perturbative versions [23]. Then, we concluded that just longitudinal sectors of the two point function are dependent of the $g$ parameter, a result in agreement with [13,14].

To generalize the result to an alpha gauge it was required to obtain the vacuum projection between the $B$ and photon field. It was achieved by relating this projection with the ghost fields two point function by means of BRST symmetry.

The paper had a constructive approach in which the ghost fields and the BRST and ghost number charges were introduced in a didactical way. We also show that the Maxwell equations are recovered between physical states and that the auxiliar fields are harmless since they are confined in the quartet non-observable configuration. The proof of $g$ parameter independence was obtained in its $B$-field version and also in a Heisenberg point of view which was latter related to the conventional path integral formalism. We also employed Heisenberg perturbation theory to show that the ’t Hooft gauge self interaction just generates longitudinal contributions which do not have any physical influence as one can infer by the action of the BRST symmetry and the definition of the positive Hilbert space sector.

For future perspectives we intend to extend the $B$ field formalism to also contemplate Ostrogadskian systems [24,25] in order to study, for example, the Podolsky electrodynamics [26–29].
We also want to use this non perturbative approach to clearly point out some quantum field theory phenomena, such as the elimination of the negative norm Ostrogradsky ghost in the free and interacting cases. Latter, we intend to use a well defined perturbation theory to give an explicit numerical representation to our efforts.

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