TOPOLOGICAL CLASSIFICATION OF LAGRANGIAN FIBRATIONS

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Abstract. We define topological invariants of regular Lagrangian fibrations using the integral affine structure on the base space and we show that these coincide with the classes known in the literature. We also classify all symplectic types of Lagrangian fibrations with base $\mathbb{RP}^2 \times \mathbb{R}$ and fixed monodromy representation, generalising a construction due to Bates in [2].

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1. Introduction

The geometry and topology of completely integrable Hamiltonian systems have been of interest since the 19th century. By a completely integrable Hamiltonian system, in this paper we mean a $2n$-dimensional symplectic manifold $(M, \omega)$ and functions $f_1, \ldots, f_n : M \to \mathbb{R}$ satisfying the following conditions

1. they are in involution, meaning that $\{f_i, f_j\} = 0$ for all $i, j$, where $\{., .\}$ denotes the Poisson bracket induced by the symplectic form;
2. they are functionally independent almost everywhere - i.e. $df_1 \wedge \ldots \wedge df_n \neq 0$ on a set of full measure.

We remark that we have only used the Poisson structure on $M$ to define a completely integrable Hamiltonian system and so there is an associated notion for Poisson manifolds, which we shall not consider here.

In this paper, we study the global topology of Lagrangian fibrations, first introduced by Duistermaat in [8] in the theory of completely integrable Hamiltonian systems.

I would like to thank Toby Bailey for his insightful comments on earlier versions of this note.
Definition 1.1. We say a fibration $F \xrightarrow{\iota} M \xrightarrow{\pi} B$ is a Lagrangian fibration if $(M, \omega)$ is a $2n$-dimensional symplectic manifold, $B$ is an $n$-dimensional connected manifold, $F$ is compact and $\omega|_{\pi^{-1}(b)} = 0$ for all $b \in B$.

Duistermaat in [8] showed that, locally, a Lagrangian fibration is given by a completely integrable Hamiltonian system; hence, local properties of the latter are still true for these fibrations. The Liouville-Arnold theorem completely determines the geometric structure of a completely integrable Hamiltonian system in a neighbourhood of a regular level of the map $f = (f_1, \ldots, f_n) : M \rightarrow \mathbb{R}^n$. For completeness, we state this classical theorem, referring the reader to [1] or [8] for a proof.

Theorem 1.1 (Liouville-Arnold Theorem). Let $f = (f_1, \ldots, f_n) : M \rightarrow \mathbb{R}^n$ be a completely integrable Hamiltonian system with $n$ degrees of freedom and let $x \in \mathbb{R}^n$ be a regular value in the image of $f$. Suppose that $f^{-1}(x)$ has a compact, connected component, denoted by $F_x$. Then

- $F_x$ is a Lagrangian submanifold of $M$ and is diffeomorphic to $T^n$;
- there is a neighbourhood $V$ of $F_x$ in $M$ that is symplectomorphic to an open neighbourhood $W$ of $T^n$ in $T^*T^n$, as shown in the diagram below.

Clearly, $W \cong D^n \times T^n$. Let $\varphi(p) = (a^1(p), \ldots, a^n(p), \alpha^1(p), \ldots, \alpha^n(p))$. The coordinates $a^i$ are called the actions and depend smoothly on the functions $f_i$. The coordinates $\alpha^i$ are called the angles.

In [8], Duistermaat found that there are two topological obstructions to finding a global set of action-angle coordinates in a Lagrangian fibration, namely

- the monodromy - which can be seen as the obstruction for the fibration to be a principal $T^n$-bundle (Theorem 1.1 guarantees that a Lagrangian fibration is locally a principal $T^n$-bundle);
- the Chern class - a generalisation of the Chern class for principal $T^n$-bundles, which can be seen as the obstruction for the fibration to admit a global section $s : B \rightarrow M$.

More recent work by Dazord and Delzant [7] (where they deal with the more general concept of isotropic fibrations) and by Zung in [11] and [13] has shone more light on the global geometric and topological properties of Lagrangian fibrations. In [13] Zung introduced a symplectic invariant of Lagrangian fibrations called the Lagrangian class, which can be seen as the obstruction to the existence of a symplectomorphism between two Lagrangian fibrations with the same monodromy and Chern class. He also generalised these invariants to singular Lagrangian fibrations, using a sheaf theoretic approach to the definition of these invariants (in line with the traditional approach). In this paper, we shall only be concerned with regular Lagrangian fibrations (i.e. with no singularities) and we will present an alternative approach to the definition of monodromy and Chern class using a classifying space. This method relies on the observation that Lagrangian fibrations over a base space $B$ are intimately related to integral affine structures on $B$, a relation that is illustrated by Lemma 2.1. We remark that the importance of the integral affine
structure in studying Lagrangian fibrations has been highlighted in various papers (see Bates’ paper [3] and Zung’s [13]).

There are various well-known examples of twisted Lagrangian fibrations in the literature. Cushman in [4] and Duistermaat in [8] explained how monodromy arises in a neighbourhood of a singularity of a real-world integrable system - the spherical pendulum. Many other examples of real-world systems with monodromy are now known (see [5] for some more examples) and in [12] Zung went as far as proving that in systems with two degrees of freedom monodromy can only arise around focus-focus singularities. Bates in [2] constructed examples of Lagrangian fibrations over $\mathbb{R}^3 \setminus \{0\} \cong S^2 \times \mathbb{R}$ with non-trivial Chern class (and trivial monodromy, since $\pi_1(\mathbb{R}^3 \setminus \{0\}) = 0$). In this paper, we explicitly construct examples of Lagrangian fibrations with non-trivial Chern classes and non-trivial monodromy (but trivial Lagrangian classes). We remark that, once we show that the base space of our fibrations possesses an integral affine structure, results of Dazord and Delzant (in [7]) and of Zung (in [13]) show that these fibrations can be constructed. In our construction, we use methods from [2] and [10].

The structure of this paper is the following. In section 2, we explain the relation between integral affine structures and Lagrangian fibrations, presenting a well-known lemma (Lemma 2.1), which we have not found proved in the literature and is therefore included for completeness. In section 3, we use the integral affine structure induced by the Lagrangian fibration on the fibres to construct the monodromy and Chern class of Lagrangian fibrations using classifying spaces. Examples of twisted Lagrangian fibrations are presented in section 4.

2. Integral Affine Structures and Lagrangian Fibrations

In this section we establish the relation between integral affine structures on a manifold $B$ and Lagrangian fibrations with base space $B$.

**Definition 2.1.** The semidirect product $\text{Aff}(\mathbb{R}^n) = \text{GL}(n, \mathbb{Z}) \rtimes \mathbb{R}^n$ where multiplication is defined by

\[(A, x) \cdot (B, y) = (AB, Ay + x)\]

is called the group of integral affine transformations.

We note that this is equivalent to saying that there is a short exact sequence

\[1 \rightarrow \mathbb{R}^n \rightarrow \text{Aff}(\mathbb{R}^n) \rightarrow \text{GL}(n, \mathbb{Z}) \rightarrow 1\]

**Definition 2.2.** An $n$-dimensional manifold $B$ is an integral affine manifold if it has an $\text{Aff}(\mathbb{R}^n)$-atlas - i.e. if there is an open cover $\{U_\alpha\}$ of $B$ by coordinate charts so that the transition functions lie in $\text{Aff}(\mathbb{R}^n)$.

**Example 2.1.**
1. $\mathbb{R}^n$ is an integral affine manifold;
2. any manifold $B$ admitting a local diffeomorphism $B \rightarrow \mathbb{R}^n$ (see [8]);
3. $T^n$ inherits an integral affine structure from $\mathbb{R}^n$, since the action of $\mathbb{Z}^n$ is through integral affine transformations;
(4) more generally, if $B$ has an integral affine structure and we have a free and proper action of a group $\Gamma$ acting by integral affine transformations (i.e. we have a representation $\Gamma \to \text{Aff}(B)$), then the quotient $B/\Gamma$ inherits an integral affine structure).

It is well-known that the base space $B$ of a Lagrangian fibration is an integral affine manifold (see [8], [3], [13]). In the following lemma we prove this fact and its converse, namely that an integral affine manifold $B$ is the base space of some Lagrangian fibration.

**Lemma 2.1.** A manifold $B$ is the base space of a Lagrangian fibration if and only if it is an integral affine manifold.

**Proof.** First we prove that the base space of a Lagrangian fibration naturally inherits an integral affine structure. Suppose $B$ is the base of a Lagrangian fibration. Choosing an open cover $\{U_i\}$ of $B$ by coordinate neighbourhoods is equivalent to choosing local action-angle coordinates $(a_1^i, \ldots, a_n^i, \alpha_1^i, \ldots, \alpha_n^i)$ over each $U_i$. If needed, refine this cover so that it is a good cover in the sense of Leray. We wish to determine how two sets of local action coordinates are related. Let $(a_1^i, \ldots, a_n^i)$ and $(\tilde{a}_1^i, \ldots, \tilde{a}_n^i)$ be local action coordinates over $U_i \cap U_j$. Theorem 1.1 shows that each set induces a system-preserving Hamiltonian $T^n$-action on $\pi^{-1}(U_i \cap U_j)$. Taking $(a_1^i, \ldots, a_n^i)$ as the local action coordinates, we see that the only Hamiltonian vector fields that can induce a system-preserving $S^1$-action on $\pi^{-1}(U_i \cap U_j)$ are of the form

$$Y = \sum_{i=1}^n m_i X_i$$

where $X_i$ is the Hamiltonian vector field of the coordinate function $a^i$ and $m_i \in \mathbb{Z}$. In particular, this shows that for each $j$ there exist constants $m_{ij} \in \mathbb{Z}$ such that

$$\tilde{X}_j = \sum_{i=1}^n m_{ij} X_i$$

where $\tilde{X}_j$ is the Hamiltonian vector field of the coordinate function $\tilde{a}^j$ (here we use the fact that the overlap of any two open sets is connected). Using the definition of Hamiltonian vector fields we get that

$$d\tilde{a}^j = \iota(\tilde{X}_j)\omega = \iota(\sum_{i=1}^n m_{ij} X_i)$$

$$= \sum_{i=1}^n m_{ij} \iota(X_i)\omega = \sum_{i=1}^n m_{ij} da^i$$

Hence we have that, for each $j$, the $d\tilde{a}^j$ are integral linear combinations of the $da^i$. This shows that there exist constants $c_j \in \mathbb{R}$ such that

$$\tilde{a}^j = \sum_{i=1}^n m_{ij} a^i + c_j$$

Swapping the roles of $a^i$ and $\tilde{a}^j$, we get that, for each $i$ there exist constants $\tilde{m}_{ji} \in \mathbb{Z}$ and $\tilde{c}_i \in \mathbb{R}$ such that
Combining the two equations above, we see that the matrix \((m_{ij})\) must lie in \(\text{GL}(n, \mathbb{Z})\) and so the result follows.

We now need to show that the existence of an integral affine structure on \(B\) implies the existence of a Lagrangian fibration over \(B\). Fix an affine atlas on \(B\) and let \(\{U_\alpha\}\) be an open cover of \(B\) by sets in this atlas. By shrinking the \(U_\alpha\) if needed, we may assume that each \(U_\alpha\) is contractible and \(U_\alpha \cap U_\beta\) is connected. Let \(A_{\alpha\beta} = A_{\alpha\beta} + c_{\alpha\beta}\) denote the transition function on the overlap \(U_\alpha \cap U_\beta\), where \(A_{\alpha\beta} \in \text{GL}(n, \mathbb{Z})\) and \(c_{\alpha\beta} \in \mathbb{R}^n\).

Define over each \(U_\alpha\) the trivial Lagrangian fibration
\[
\mathbb{T}^n \times U_\alpha \to U_\alpha
\]
with symplectic form defined by
\[
\omega_\alpha = \sum_i dx_i^\alpha \wedge dt_i^\alpha
\]
where \(x_i^\alpha\) are affine coordinates over \(U_\alpha\) and \(t_i^\alpha\) are the standard coordinates on \(\mathbb{T}^n\).

Define (wherever it makes sense - i.e. on overlapping coordinate neighbourhoods) maps \(\psi_{\beta\alpha} : U_\alpha \cap U_\beta \times \mathbb{T}_\alpha^n \to U_\alpha \cap U_\beta \times \mathbb{T}_\beta^n\) by
\[
(x_\alpha, t_i^\alpha) \mapsto (A_{\beta\alpha} x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^T t_i^\alpha)
\]
Note that the maps \(\{\psi_{\beta\alpha}\}\) satisfy the cocycle condition, since the \(A_{\beta\alpha}\) clearly do and the map from the affine group to its linear subgroup is a homomorphism. Hence, these maps can be used to construct a \(\mathbb{T}^n\)-bundle over \(B\). We note that the \(\omega_\alpha\) patch together to give a globally defined symplectic form \(\omega\) on the total space of the bundle. Finally, we observe that each fibre is still Lagrangian and so we have constructed a Lagrangian fibration over \(B\) as required.

We note that using the methods of the above lemma, we can also show that if \(B\) is the base space of a Lagrangian fibration, the structure group of \(T^*B\) can be reduced to (a subgroup of) \(\text{GL}(n, \mathbb{Z})\). The converse, however, is not true, as the following example shows.

**Example 2.2.** \(S^3\) is parallelisable and so the structure group of \(T^*S^3\) is trivial. Suppose \(S^3\) was the base space of a Lagrangian fibration. Since it is 2-connected (see [13]), the fibration need be topologically trivial - i.e. \(S^3 \times T^3 \to T^3\). Call \(\omega\) the symplectic form on this space. By the Künneth formula, the inclusion of the fibre in the total space \(\iota : T^3 \to S^3 \times T^3\) induces an isomorphism
\[
\iota^* : H^2(S^3 \times T^3; \mathbb{R}) \to H^2(T^3; \mathbb{R})
\]
Since \(S^3 \times T^3\) is closed, the cohomology class of \(\omega\) is non-zero. However, the fibre of the fibration is Lagrangian and so \(\iota^* [\omega] = 0\) and we have a contradiction.

This example generalises to the following result.

**Lemma 2.2.** No sphere \(S^n\) for \(n \geq 2\) is the base space of a Lagrangian fibration.
Proof. We only need to deal with the $n = 2$ case. Since $S^2$ is simply-connected, if it is the base space of a Lagrangian fibration then the fibration has trivial monodromy. It follows from [8] that $S^2$ is parallelisable, but this is a contradiction. □

Remark 2.1. Lemma 2.1 implies that in order to determine whether a manifold $B$ is the base space of a Lagrangian fibration, it is sufficient to determine whether it is an integral affine manifold.

Now we turn to the question of determining the structure group of a Lagrangian fibration. Choose two sets of local action-angle coordinates $(x_\alpha, t_\alpha)$, $(x_\beta, t_\beta)$ on overlapping trivialising open sets $U_\alpha$, $U_\beta$. It is well-known (see [13] and the proof of Lemma 2.1) that on $U_\alpha \cap U_\beta$ these coordinates are related by the following

\[(x_\beta, t_\beta) = (A_{\beta\alpha} x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^T t_\alpha + g_{\beta\alpha})\]

where $A_{\beta\alpha} \in \text{GL}(n, \mathbb{Z})$, $c_{\beta\alpha} \in \mathbb{R}^n$ and $g_{\beta\alpha} = (g_{\beta\alpha}^1, \ldots, g_{\beta\alpha}^n) : U_\alpha \to \mathbb{T}^n$ is a locally defined function such that

\[\sum_{i=1}^n g_{\beta\alpha}^i dx_\alpha^i\]

is a closed form.

Definition 2.3. The group of affine toral automorphisms, denoted by $\text{Aff}(\mathbb{T}^n)$, is the semidirect product $\text{GL}(n, \mathbb{Z}) \rtimes \mathbb{T}^n$ with product defined by

\[(A, x) \cdot (B, y) = (AB, Ay + x)\]

We have a commutative diagram of short exact sequences

\[
\begin{array}{cccccc}
1 & \to & \mathbb{R}^n & \to & \text{Aff}(\mathbb{R}^n) & \to & \text{GL}(n, \mathbb{Z}) & \to & 1 \\
& & q & & & & & \\
1 & \to & \mathbb{T}^n & \to & \text{Aff}(\mathbb{T}^n) & \to & \text{GL}(n, \mathbb{Z}) & \to & 1 \\
\end{array}
\]

relating the integral affine group $\text{Aff}(\mathbb{R}^n)$ with the group of toral affine automorphisms $\text{Aff}(\mathbb{T}^n)$, where the map $q : \mathbb{R}^n \to \mathbb{T}^n$ is the standard covering map. Equation (4) shows that the structure group of a Lagrangian fibration can be reduced to $\text{Aff}(\mathbb{T}^n)$. Hence, if we are given a Lagrangian fibration $F \xleftarrow{i} M \xrightarrow{\pi} B$, by passing to the associated principal $\text{Aff}(\mathbb{T}^n)$-bundle, we can study its isomorphism type by looking at the homotopy type of the classifying map

\[(5) \quad f : B \to \text{BAff}(\mathbb{T}^n)\]

where $\text{BAff}(\mathbb{T}^n)$ denotes the classifying space of the group $\text{Aff}(\mathbb{T}^n)$.

3. Classifying space and obstructions to global action-angle coordinates

As we have seen, the structure group of a Lagrangian fibration is given by $\text{Aff}(\mathbb{T}^n) = \text{GL}(n, \mathbb{Z}) \rtimes \mathbb{T}^n$. Let $\text{Aff}(\mathbb{T}^n) \xrightarrow{i} M \xrightarrow{\pi} B$ denote the principal $\text{Aff}(\mathbb{T}^n)$-bundle associated to a given Lagrangian bundle $F \xleftarrow{i} M \xrightarrow{\pi} B$ (for
The isomorphism type of this bundle is determined by the homotopy class of a classifying map
\[ f : B \rightarrow \text{BAff}(T^n) \]

In this section we use this map to define the monodromy and Chern class of a Lagrangian fibration.

We begin by identifying two important closed subgroups of \( \text{Aff}(T^n) \), \( H_1 = I \times T^n \) and \( H_2 = \text{GL}(n, \mathbb{Z}) \times 0 \), called the toral and linear subgroups respectively. For \( i = 1, 2 \), we have fibrations
\[
\text{Aff}(T^n)/H_i \rightarrow \text{BAff}(T^n)
\]
arising as follows. Let \( \text{Aff}(T^n) \rightarrow \text{EAff}(T^n) \rightarrow \text{BAff}(T^n) \) be the universal bundle for \( \text{Aff}(T^n) \). Each \( H_i \) acts freely on \( \text{EAff}(T^n) \) and so we can take this space as a model for \( E_{H_i} \).

We note that \( \text{Aff}(T^n)/(I \times T^n) \cong \text{GL}(n, \mathbb{Z}) \) (as groups, since \( I \times T^n \leq \text{Aff}(T^n) \)) and \( \text{Aff}(T^n)/(\text{GL}(n, \mathbb{Z}) \times 0) \cong \mathbb{T}^n \) (as topological spaces).

**Lemma 3.1.** The homotopy groups of \( \text{BAff}(T^n) \) are given by
\[
\pi_i(\text{BAff}(T^n)) = \begin{cases} 
\text{GL}(n, \mathbb{Z}) & \text{if } i = 1 \\
\mathbb{Z}^n & \text{if } i = 2 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** Consider the fibration arising from the action of the toral subgroup
\[
\text{GL}(n, \mathbb{Z}) \rightarrow \text{B}(I \times T^n) \rightarrow \text{BAff}(T^n)
\]
Note that \( \text{B}(I \times T^n) \) is a \( K(\mathbb{Z}^n, 2) \) as it is the classifying space of an \( n \)-torus. Using the long exact sequence in homotopy (see [6]), we obtain the result. \( \square \)

We now deal with the other fibration, arising from the action of the linear subgroup \( \text{GL}(n, \mathbb{Z}) \times 0 \)
\[
\mathbb{T}^n \rightarrow \text{B}(\text{GL}(n, \mathbb{Z}) \times 0) \rightarrow \text{BAff}(T^n)
\]
The long exact sequence in homotopy for this fibration ends as follows
\[
\pi_1(\mathbb{T}^n) \rightarrow \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \rightarrow \pi_1(\text{BAff}(T^n)) \rightarrow 0
\]
since \( \pi_0(\mathbb{T}^n) = 0 \). Note that \( \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \) is a \( K(\text{GL}(n, \mathbb{Z}), 1) \) since \( \text{GL}(n, \mathbb{Z}) \) is discrete group. In particular, we have that \( \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \cong \pi_1(\text{BAff}(T^n)) \). However, we also have that \( \pi_* \) is a surjection and so
\[
\pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \cong \pi_1(\text{BAff}(T^n)) \cong \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0))/\ker \pi_*
\]
where the first isomorphism follows from Lemma [3.1]. Equation (9) shows that \( \pi_* \) is actually an isomorphism and we will use this fact when defining the Chern class of a Lagrangian fibration.

The key idea that we use to define these invariants is to define them as obstructions to the existence of a lift of the classifying map \( f \) to maps \( \tilde{f} : B \rightarrow \text{B}(I \times T^n) \)
and \( \hat{f} : B \to B(GL(n, \mathbb{Z}) \times 0) \) into the total spaces of the fibrations constructed above. We will use the following theorem, quoted here without proof (see [9]).

**Theorem 3.1.** Let \( H \leq G \) be a closed subgroup of a topological group \( G \). The structure group of a principal \( G \)-bundle \( G \rightarrow P \rightarrow B \) can be reduced to \( H \) if and only if there exists a lift \( \bar{F} : B \to BH \) of the classifying map \( F : B \to BG \), as illustrated below.

\[
\begin{array}{ccc}
B & \xrightarrow{\bar{F}} & BH \\
\downarrow{F} & & \downarrow{\pi_B} \\
B & \xrightarrow{\pi} & BG
\end{array}
\]

where \( G/H \rightarrow BH \rightarrow BG \) is a fibration obtained as above.

3.1. **Monodromy.** We first recall the classical definition (see [8], [3], [5] and [13]) of monodromy and then give an entirely topological definition. Associated to a Lagrangian fibration \( F \rightarrow M \rightarrow B \), there is a period lattice bundle \( H_1(F, \mathbb{Z}) \rightarrow P \rightarrow B \) where the generators of \( H_1(F, \mathbb{Z}) \) define a local Hamiltonian \( T^n \)-action on \( M \). An explicit construction of this period lattice bundle is explained in [5]. We want to remark that if we fix an good (in the sense of Leray) open cover \( \{U_\alpha\} \) of \( B \) by trivialising neighbourhoods for \( F \rightarrow M \rightarrow B \), then we can locally trivialise the period lattice bundle over this cover, with transition functions over \( U_\alpha \cap U_\beta \) given by

\[
(x_\beta, z_\beta) = (A_{\beta\alpha}x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^Tz_\alpha)
\]

where \( z \in H_1(F, \mathbb{Z}) \). In particular, it is important to notice that the trivialisations and transition functions for the period lattice bundle are completely determined by those for the associated Lagrangian fibration.

The classical definition of monodromy is precisely as an obstruction for these local Hamiltonian \( T^n \)-actions to patch together to yield a global \( T^n \)-action. If the monodromy vanishes, then this \( T^n \)-action is actually free (this is most evident in the proof of the Liouville-Arnold theorem (Theorem 1.1) or in [5]) and so the Lagrangian fibration is a principal \( T^n \)-bundle. Conversely, if there exists a globally defined free Hamiltonian \( T^n \)-action on the total space \( M \) of a Lagrangian fibration, then the fibration is a principal bundle and the period lattice bundle is trivial. It follows therefore from the classical definition of monodromy (see [5]) that the monodromy of the Lagrangian fibration is trivial. In particular, this shows that the monodromy can be seen as the obstruction to reducing the structure group of the fibration to the toral subgroup \( I \times T^n \).

We have the fibration

\[
GL(n, \mathbb{Z}) \rightarrow B(I \times T^n) \rightarrow B\text{Aff}(T^n)
\]

where the fibre is discrete and the total space is simply connected. A lift \( \hat{f} : B \to B(I \times T^n) \) of the classifying map
exists if and only if
\[(10) \quad f_* : \pi_1(B) \to \pi_1(\mathrm{BAff}(\mathbb{T}^n)) \cong \mathrm{GL}(n, \mathbb{Z})\]
is trivial.

It is tempting to define \( f_* \) as the monodromy associated to the Lagrangian fibration, but this would be a naive definition since we would like monodromy to be an invariant of the isomorphism type of the fibration. Note that if two Lagrangian fibrations \( F \xrightarrow{\iota} M \xrightarrow{\pi} B \) and \( F' \xrightarrow{\iota'} M' \xrightarrow{\pi'} B \) over \( B \) are isomorphic, then clearly the associated period lattice bundles \( H_1(F, \mathbb{Z}) \xrightarrow{\Phi} \mathbb{P} \xrightarrow{\pi} B \) and \( H_1(F', \mathbb{Z}) \xrightarrow{\Phi'} \mathbb{P}' \xrightarrow{\pi'} B \) are isomorphic. It is well-known (see [4]) that the structure group of these discrete bundles is \( \mathrm{GL}(n, \mathbb{Z}) \) and, hence, their isomorphism type is determined by the homotopy class of a map
\[
\phi : B \to \mathrm{BGL}(n, \mathbb{Z})
\]
Since \( \mathrm{BGL}(n, \mathbb{Z}) \) is a \( K(\mathrm{GL}(n, \mathbb{Z}), 1) \), then we have that such homotopy classes are parametrised by
\[
[B, \mathrm{BGL}(n, \mathbb{Z})] = \mathrm{Hom}(\pi_1(B), \mathrm{GL}(n, \mathbb{Z}))/\sim
\]
Hence if we let \( f, f' \) be the classifying maps associated to the Lagrangian fibrations \( F \xrightarrow{\iota} M \xrightarrow{\pi} B \) and \( F' \xrightarrow{\iota'} M' \xrightarrow{\pi'} B \) respectively, we have that \( f_*(\pi_1(B)) \) and \( f'_*(\pi_1(B)) \) are conjugate subgroups of \( \mathrm{GL}(n, \mathbb{Z}) \).

**Definition 3.1.** Let \( f : B \to \mathrm{BAff}(\mathbb{T}^n) \) be the classifying map associated to a Lagrangian fibration and let \([f_*]\) denote the equivalence class of subgroups of \( \mathrm{GL}(n, \mathbb{Z}) \) that are conjugate to \( f_*(\pi_1(B)) \). Then we call \([f_*]\) the **free monodromy** of the Lagrangian fibration.

From now on, whenever we mention the monodromy of a Lagrangian fibration we will mean it in the sense of **Definition 3.1**, although we abuse notation and still denote it as a map \( f_* : \pi_1(B) \to \pi_1(\mathrm{BAff}(\mathbb{T}^n)) \).

### 3.2. Chern Class

Fix a Lagrangian fibration \( F \xrightarrow{\iota} M \xrightarrow{\pi} B \) with monodromy \( f_* : \pi_1(B) \to \pi_1(\mathrm{BAff}(\mathbb{T}^n)) \). Using the methods of Lemma [2.1] we can construct the **trivial \( f_* \)-fibration** as follows. Let \( \{U_\alpha\} \) be an open cover of \( B \) by trivialising open sets for \( F \xrightarrow{\iota} M \xrightarrow{\pi} B \). We know from equation (11) that the transition functions on \( U_\alpha \cap U_\beta \) are of the form
\[
(x_\beta, t_\beta) = (A_{\beta\alpha}x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^T t_\alpha + g_{\beta\alpha})
\]
Define a new fibration with same trivialisations as the original one and transition functions given by
\[
(x_\beta, t_\beta) = (A_{\beta\alpha}x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^T t_\alpha)
\]
This new fibration is still Lagrangian since the local symplectic structures patch together and the fibres are Lagrangian. Note that its structure group lies entirely in the linear subgroup $\text{GL}(n, \mathbb{Z}) \times 0$ of $\text{Aff}(\mathbb{T}^n)$. This new Lagrangian fibration has the same monodromy as the original fibration, since they have isomorphic period lattice bundles, as can be seen by the construction of a period lattice bundle in [5].

We can use the trivialising cover $\{ U_\alpha \}$ of $B$ to define local trivialisations for $T^*B$, given by

$$ (x_\beta, y_\beta) = (A_{\beta\alpha} x_\alpha + c_{\beta\alpha}, (A_{\beta\alpha}^{-1})^T y_\alpha) $$

where $y \in T^*_B$. We have also seen that the transition functions for the period lattice bundle $\mathcal{P} \to B$ are precisely the same and so we deduce that the trivial $f_*$-fibration is isomorphic to the Lagrangian bundle $T^*B/\mathcal{P} \to B$ (since they have the same transition functions and trivialisations over a common trivialising cover). This explains why we call this fibration the trivial $f_*$-fibration, as it has trivial Chern class in the classical definition (see [8]). Indeed, Duistermaat defined the Chern class of a Lagrangian fibration to be the obstruction to the existence of a bundle isomorphism

$$ M \xrightarrow{\theta} T^*B/\mathcal{P} \xrightarrow{\pi} B $$

We note that the structure group of $T^*B/\mathcal{P} \to B$ can be reduced to the linear subgroup $\text{GL}(n, \mathbb{Z}) \times 0$ of $\text{GL}(n, \mathbb{Z})$. Hence the structure group of a Lagrangian fibration that is isomorphic to the trivial $f_*$-fibration can be reduced to $\text{GL}(n, \mathbb{Z}) \times 0$. Conversely, suppose that the structure group of a Lagrangian fibration $F \xrightarrow{\iota} M \xrightarrow{\pi} B$ reduces to $\text{GL}(n, \mathbb{Z}) \times 0$. Then Theorem [3] shows that we have a lift $g : B \to B(\text{GL}(n, \mathbb{Z}) \times 0)$ as shown below

$$ B(\text{GL}(n, \mathbb{Z}) \times 0) \xrightarrow{\pi} B $$

We wish to show that this Lagrangian fibration is isomorphic to the trivial $f_*$-fibration. To this end, let $f_0 : B \to \text{Baff}(\mathbb{T}^n)$ denote the (homotopy type of the) classifying map associated to the trivial $f_*$-fibration. Equation [9] shows that the map $\pi_* : \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \to \pi_1(\text{Baff}(\mathbb{T}^n))$ is an isomorphism. Since the structure group of the trivial $f_*$-fibration can be reduced to $\text{GL}(n, \mathbb{Z}) \times 0$, then $f_0$ has a lift $G : B \to \text{B}(\text{GL}(n, \mathbb{Z}) \times 0)$. Since $\pi_*$ is an isomorphism, it follows that $g_* = G_*$ (up to conjugation).

We wish to show that this Lagrangian fibration is isomorphic to the trivial $f_*$-fibration. To this end, let $f_0 : B \to \text{Baff}(\mathbb{T}^n)$ denote the (homotopy type of the) classifying map associated to the trivial $f_*$-fibration. Equation [9] shows that the map $\pi_* : \pi_1(\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)) \to \pi_1(\text{Baff}(\mathbb{T}^n))$ is an isomorphism. Since the structure group of the trivial $f_*$-fibration can be reduced to $\text{GL}(n, \mathbb{Z}) \times 0$, then $f_0$ has a lift $G : B \to \text{B}(\text{GL}(n, \mathbb{Z}) \times 0)$. Since $\pi_*$ is an isomorphism, it follows that $g_* = G_*$ (up to conjugation). The isomorphism type of maps from $B$ to $\text{B}(\text{GL}(n, \mathbb{Z}) \times 0)$ is classified precisely by $\text{Hom}(\pi_1(B), \text{B}(\text{GL}(n, \mathbb{Z})))$ up to conjugation and so $g$ is homotopic to $G$. It is therefore straightforward to see that $f$ is homotopic to $f_0$ and so the original Lagrangian fibration is isomorphic to the trivial $f_*$-fibration.

From here on, until the end of the section, we assume that $B$ is a CW-complex. The obstruction to finding a lift $g : B \to \text{B}(\text{GL}(n, \mathbb{Z}) \times 0)$ of the classifying map $f$
of a Lagrangian fibration is an element of a twisted cohomology group (see [6])
(11) \[ c \in H^2(B, \mathbb{Z}_n^\rho) \]
where the map \( \rho : \pi_1(B) \to \mathbb{Z}^n \) is the composite

\[ \pi_1(B) \xrightarrow{f_*} \pi_1(\text{BAff}(\mathbb{T}^n)) \xrightarrow{\chi} \text{Aut}(\pi_1(\mathbb{T}^n)) \]

Here \( \chi \) denotes the action of the fundamental group of \( \text{BAff}(\mathbb{T}^n) \) on the fundamental

\( \text{group of the fibre } \mathbb{T}^n \) (see [6]). We will abuse notation and write \( f_* \) also to mean \( \chi \circ f_* \).

**Definition 3.2.** \( c \) is the Chern class associated to the (isomorphism type of a)
Lagrangian fibration determined by a classifying map \( f : B \to \text{BAff}(\mathbb{T}^n) \).

### 4. Examples of Twisted Lagrangian Fibrations

In this section we explicitly construct examples of regular Lagrangian fibrations
with non-trivial monodromy and non-trivial Chern class, following ideas of Bates
(see [2]) and Sakamoto and Fukuhara (see [10]). First we show how to construct
a Lagrangian fibration with base space \( \mathbb{R}P^2 \times \mathbb{R} \) and calculate its monodromy \( f_* \),
which is non-trivial since \( \mathbb{R}P^2 \times \mathbb{R} \) is not parallelisable (see [8]). Then, using a
result of Dazord and Delzant (see [7]) adapted by Zung in [13], we deduce that all
cohomology classes in \( H^2(\mathbb{R}P^2 \times \mathbb{R}; \mathbb{Z}_2) \) can be realised as the Chern class of some
Lagrangian fibration with base \( \mathbb{R}P^2 \times \mathbb{R} \). Finally, we construct these explicitly, first
just topologically following ideas of [10] and then also taking care of the symplectic
form, generalising a construction carried out in [2].

For notational ease, set \( X = \mathbb{R}P^2 \times \mathbb{R} \) and let \( \tilde{X} = \mathbb{R}^3 - \{0\} \) denote its universal
cover. \( \tilde{X} \) is the base space of a trivial Lagrangian fibration

\[ \mathbb{T}^3 \longleftarrow \tilde{X} \times \mathbb{T}^3 \xrightarrow{\tilde{z}} \tilde{X} \]

We remark that we are using the standard integral affine structure on \( \tilde{X} \). There is
a free and proper \( \mathbb{Z}_2 \)-action on \( \tilde{X} \), given by

\[ a \cdot \mathbf{x} = -\mathbf{x} \]

where \( a \in \mathbb{Z}_2 \) is the non-trivial element. We can lift this action to the total space
of (12) by setting

\[ a \cdot (\mathbf{x}, t) = (-\mathbf{x}, -t) \]

This defines a free and proper \( \mathbb{Z}_2 \)-action on \( \tilde{X} \times \mathbb{T}^3 \), which is by bundle automor-
phisms (the bundle is trivial) and symplectomorphisms (since the symplectic form
\( \omega \) is just \( \sum_{i=1}^{3} dx^i \wedge dt^i \)). Hence, if we quotient the bundle (12) by this action, we
obtain a Lagrangian fibration

\[ \mathbb{T}^3 \longleftarrow M_0 \longrightarrow X \]

Since \( X \) is not parallelisable (let alone orientable), this fibration need have non-
trivial monodromy.
Claim 4.1. The monodromy of this fibration is given by the map
\[ f_* : \pi_1(\mathbb{RP}^2 \times \mathbb{R}) \to \text{GL}(3, \mathbb{Z}) \]
\[ a \mapsto -I \]

Proof. We work on the double covering
\[ (16) \quad \mathbb{T}^3 \xrightarrow{\pi} \tilde{X} \xrightarrow{\tilde{q}} \tilde{X} \]
Let \( c : [0, 1] \to \tilde{X} \) be a lift of the generator \( a \) of the fundamental group of \( \mathbb{RP}^2 \times \mathbb{R} \) - take for instance the path \( c(t) = (\cos(\pi t), \sin(\pi t), 0) \). It is well-known (see [3]) that an integral affine manifold \( N \) admits a flat, torsion-free connection \( \nabla \), whose holonomy equals the monodromy of the Lagrangian fibration over \( N \) constructed in Lemma 2.3. Let \( \nabla \) denote the connection on \( X \) arising from the Lagrangian fibration we have just constructed and let \( \nabla \) denote the pullback of this connection by the covering map \( q : \tilde{X} \to X \). By construction, \( \nabla \) is the standard connection on \( X \) - here we recall that we started with the standard integral affine structure on \( \tilde{X} \) to construct the Lagrangian fibration with base \( X \). We wish to calculate the holonomy of \( \nabla \). Consider any non-zero \( v \in T^*_+(1_0, 0) \tilde{X} \) and parallel transport it along \( c \). Recall that the cotangent bundle \( T^* \tilde{X} \) is trivial, so that we can canonically identify any two of the fibres via the standard connection \( \nabla \). Since parallel transport is trivial, \( v \) maps to \( v \in T^*_+(-1_0, 0) \tilde{X} \). However, under the lift of the \( \mathbb{Z}_2 \)-action on \( \tilde{X} \), we have the following identification
\[ ((1, 0, 0), v) \sim ((-1, 0, 0), -v) \]
Hence, projecting down to \( X \), we get that the holonomy of the connection \( \nabla \) is given by
\[ a \mapsto -I \]
as required. \( \square \)

In section 3.2, we showed that the Chern class of Lagrangian fibrations with base \( X \) and monodromy given by \( f_* \) are classified by elements of the twisted cohomology group \( H^2(\mathbb{RP}^2 \times \mathbb{R}; \mathbb{Z}_f^3) \). A natural question to ask is which twisted cohomology classes in \( H^2(\mathbb{RP}^2 \times \mathbb{R}; \mathbb{Z}_f^3) \) can be realised as the Chern class of a Lagrangian fibration over \( X \) with the given monodromy. In the general case, we have no concrete answer, although in this case we can use a result of Dazord and Delzant (see [2]) and Zung (see [13]) to deduce that all twisted cohomology classes can be realised.

Theorem 4.1 (Dazord-Delzant, Zung). Let \( Y \) be an \( n \)-dimensional manifold that is the base space of a Lagrangian fibration with monodromy given by \( f_* \). If \( H^2(Y; \mathbb{R}) = 0 \), then all cohomology classes in \( H^2(Y; \mathbb{Z}_f^n) \) can be realised as the Chern class of some Lagrangian fibration with base \( Y \) and monodromy as before. Furthermore, if \( H^2(Y; \mathbb{R}) = 0 \), then the Lagrangian class vanishes for any Chern class.

Since \( \mathbb{RP}^2 \times \mathbb{R} \) satisfies the assumptions of Theorem 4.1, we deduce that all elements of \( H^2(\mathbb{RP}^2 \times \mathbb{R}; \mathbb{Z}_f^n) \) arise as the Chern class of some Lagrangian fibration with base \( \mathbb{RP}^2 \times \mathbb{R} \) and that the symplectic structure on the total space of these fibrations preserving the Lagrangian structure is unique up to symplectomorphism. We now compute \( H^2(\mathbb{RP}^2 \times \mathbb{R}; \mathbb{Z}_f^3) \) and show that
\[ H^2(\mathbb{RP}^2 \times \mathbb{R}; \mathbb{Z}_f^3) \cong \mathbb{Z}^3 \]
using standard techniques in algebraic topology (see [6]). Since $\mathbb{R}P^2 \times \mathbb{R}$ is homotopy equivalent to $\mathbb{R}P^2$, it will suffice to show that $H^2(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}^3$. Recall that $\mathbb{R}P^2$ has a standard cell decomposition

$$\mathbb{R}P^2 = e^0 \cup e^1 \cup e^2$$

inducing a cell decomposition on its universal cover

$$S^2 = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup e^2_+ \cup e^2_-$$

Set $\pi_1(\mathbb{R}P^2) = \pi$. Denote by $C_i(S^2)$ the $\mathbb{Z}[\pi]$-module of $i$-cells in $S^2$. For $i = 0, 1, 2$ $C_i(S^2)$ is a 1-dimensional $\mathbb{Z}[\pi]$-module, generated by $e^+_i$. We denote the action of $\pi$ on $C_i(S^2)$ by multiplication by $t$, following the notation in [6]. The twisted cellular chain complex for $\mathbb{R}P^2$ is given by

$$0 \longrightarrow C_2(S^2) \xrightarrow{\partial_2} C_1(S^2) \xrightarrow{\partial_1} C_0(S^2) \longrightarrow 0$$

where the differentials are defined as follows

$$\partial_2(e^+_2) = (1 + t)e^+_1$$
$$\partial_1(e^+_1) = (1 - t)e^+_0$$

The associated twisted cochain complex is given by

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}[\pi]}(C_0(S^2), \mathbb{Z}^3) \xrightarrow{\delta_1} \text{Hom}_{\mathbb{Z}[\pi]}(C_1(S^2), \mathbb{Z}^3) \xrightarrow{\delta_2} \text{Hom}_{\mathbb{Z}[\pi]}(C_2(S^2), \mathbb{Z}^3) \longrightarrow 0$$

where $\delta_1, \delta_2$ are adjoint maps to $\partial_1, \partial_2$ respectively and we think of $\mathbb{Z}^3$ as a $\mathbb{Z}[\pi]$-module via the representation $f_* : \pi \rightarrow \text{GL}(3, \mathbb{Z})$ (extended naturally to the whole ring).

We are interested in

$$H^2(\mathbb{R}P^2; \mathbb{Z}_2) = \frac{\text{Hom}_{\mathbb{Z}[\pi]}(C_2(S^2), \mathbb{Z}^3)}{\text{im}\delta_2}$$

Note that $\text{Hom}_{\mathbb{Z}[\pi]}(C_2(S^2), \mathbb{Z}^3) \cong \mathbb{Z}^3$, since any $\mathbb{Z}[\pi]$-module homomorphism $\chi : C_2(S^2) \rightarrow \mathbb{Z}^3$ is completely determined by its value on the generator $e^+_2$ of $C_2(S^2)$. If $\tau \in \text{Hom}_{\mathbb{Z}[\pi]}(C_1(S^2), \mathbb{Z}^3)$, then

$$(\delta_2\tau)(e^+_2) = \tau(\partial_2 e^+_2) = \tau((1 + t)e^+_1)$$
$$= \tau(e^+_1) + f_*(t)\tau(e^+_0) = \tau(e^+_1) - \tau(e^+_0) = 0$$

and so $\text{im}\delta_2 = \{0\}$. Hence, $H^2(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}^3$ as claimed.

This is the space of Chern classes of Lagrangian fibrations with base $\mathbb{R}P^2 \times \mathbb{R}$ and monodromy given by $f_*$. We can now construct these fibrations explicitly, first topologically (using ideas from [10]), and then showing how to construct the symplectic form. Fix an element $(m, n, p) \in \mathbb{Z}^3$. Think of $\mathbb{R}P^2 \times \mathbb{R}$ as $D^2/\sim \times \mathbb{R}$, where $D^2$ denotes the closed disk in two dimensions and the equivalence relation is defined by $x \sim -x$ if $|x| = 1$. Let $B(c)$ be a closed disk of radius $c > 0$ about $0 \in D^2$ and set $B = B(c) \times \mathbb{R}$. Recall that we have a Lagrangian bundle

$$T^3 \longrightarrow M_0 \longrightarrow X$$
whose Chern class is clearly $(0, 0, 0) \in \mathbb{Z}$. Construct a new bundle $M \to \mathbb{R}P^2 \times \mathbb{R}$ by defining the total space by

$$M = (M_0 - \pi^{-1}(\text{int}B)) \cup (T^3 \times B)$$

where $T^3 \times \partial B$ is attached to $\pi^{-1}(\partial B)$ via the map

$$h((\epsilon \cos(\theta), \epsilon \sin(\theta), s), t) = ((\epsilon \cos(\theta), \epsilon \sin(\theta), s), \frac{\theta}{2\pi}(m, n, p) + t)$$

(the fibre is thought of as $T^3 = \mathbb{R}^3/\mathbb{Z}^3$). This defines a $T^3$-bundle with base space $\mathbb{R}P^2 \times \mathbb{R}$, the monodromy is unchanged and its Chern class (seen in this case as the obstruction to the existence of a global section) is $(m, n, p) \in \mathbb{Z}$. Theorem 4.1 ensures that there exists a unique (up to symplectomorphism) symplectic structure on $M$ that makes it Lagrangian. Since the choice of $(m, n, p)$ was arbitrary, we have constructed all the symplectic types of Lagrangian fibrations with base space $\mathbb{R}P^2 \times \mathbb{R}$ and monodromy given by $f$. 

These bundles can also be constructed in a different fashion, which elucidates how the symplectic structure comes about. This method generalises ideas from Bates’ paper [2]. The universal cover $\tilde{X}$ of $\mathbb{R}P^2 \times \mathbb{R}$ is diffeomorphic to $S^2 \times \mathbb{R}$. Consider the trivial Lagrangian fibration

$$\begin{array}{c}
T^3 \xrightarrow{\pi} \tilde{X} \times T^3 \xrightarrow{\pi} \tilde{X}
\end{array}$$

with the standard symplectic form on the total space. Fix some small $\epsilon > 0$. Let $B_\pm(\epsilon) \subset S^2$ be closed discs of radius $\epsilon$ centred at $(1, 0, 0)$ and $(-1, 0, 0)$ respectively (here we identify the open upper and lower hemispheres of $S^2$ with $\mathbb{R}^2$). Set $B_\pm = B_\pm(\epsilon) \times \mathbb{R} = \bigcup_{r > 0} B_\pm(\epsilon r) \times \{r\}$. We define a new bundle with base space $\tilde{X}$ by constructing the following total space

$$\tilde{M} = ((\tilde{X} \times T^3) - (\pi^{-1}(\text{int}B_+) \cup \pi^{-1}(\text{int}B_-))) \cup (T^3 \times B_+) \cup (T^3 \times B_-)$$

where $T^3 \times \partial B_+$ is attached to $\pi^{-1}(\partial B_+)$ via the attaching map

$$h_+(x, t) = (x, t^1 + \arg(x^1 + ix^2), t^2 + \frac{1}{2} \log\left(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}\right), t^3)$$

and $T^3 \times \partial B_-$ is attached to $\pi^{-1}(\partial B_-)$ via the attaching map

$$h_-(x, t) = (x, t^1 - \arg(x^1 + ix^2) + \pi, t^2 - \frac{1}{2} \log\left(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}\right), t^3)$$

Note that here we think of $T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ and so there is no ambiguity when defining the argument function $\arg$. This defines a $T^3$-bundle with base space $\tilde{X}$. Moreover, the local (natural) symplectic forms patch together (since $\partial_t(\arg(x^1 + ix^2)) = \partial_t\left(\frac{1}{2} \log\left(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}\right)\right)$) to yield a global symplectic form $\omega$ with respect to which the fibres are Lagrangian. Hence this new bundle is a Lagrangian fibration. The standard antipodal action on $\tilde{X} \times T^3$ sends $(\tilde{X} \times T^3) - (\pi^{-1}(\text{int}B_+) \cup \pi^{-1}(\text{int}B_-))$ to itself and swaps $(T^3 \times B_+)$ and $(T^3 \times B_-)$. Note further that the action is compatible with the attaching maps, since
\[ a \cdot h_+(x, t) = a \cdot (x, t^1 + \arg(x^1 + ix^2), t^2 + \frac{1}{2} \arg(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}), t^3) \]

\[ = (-x, -t^1 - \arg(x^1 + ix^2), -t^2 - \frac{1}{2} \log(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}), -t^3) \]

and

\[ h_-(a \cdot (x, t)) = h_-(-x, -t) \]

\[ = (-x, -t^1 - \arg(-x^1 - ix^2) + \pi, -t^2 - \frac{1}{2} \log(\frac{(-x^1)^2 + (-x^2)^2}{1 - (x^2)^2}), -t^3) \]

\[ = (-x, -t^1 - \arg(x^1 + ix^2), -t^2 - \frac{1}{2} \log(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}), -t^3) \]

since \( \arg(-x^1 - ix^2) = \arg(x^1 + ix^2) + \pi \) (mod \( 2\pi \)). Furthermore, the same proof shows that it is a \( \mathbb{Z}_2 \)-action on the total space of the new Lagrangian fibration. In particular, the above also shows that this \( \mathbb{Z}_2 \)-action is by bundle automorphisms (as it commutes with the attaching maps) and by symplectomorphisms. Taking quotients, we obtain a Lagrangian fibration

\[ \mathbb{T}^3 \xrightarrow{\cdot} M(1, 0, 0) \xrightarrow{\cdot} \mathbb{R}P^2 \times \mathbb{R} \]

Its monodromy is again given by \( f_+ \) and, by construction, its Chern class is precisely \((1, 0, 0)\).

We now show how to generalise this construction to produce a Lagrangian fibration with base \( \mathbb{R}P^2 \times \mathbb{R} \) and arbitrary Chern class. Let \( G \in \text{GL}(3, \mathbb{Z}) \) and define disjoint cones \( B^G_\pm \) so that \( B_\pm = GB^G_\pm \). Set \( \phi_+(x) = (\arg(x^1 + ix^2), \frac{1}{2} \log(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}), 0) \) and \( \phi_-(x) = (-\arg(x^1 + ix^2) + \pi, -\frac{1}{2} \log(\frac{(x^1)^2 + (x^2)^2}{1 + (x^2)^2}), 0) \). We define a new bundle with base space \( \tilde{X} \) whose base space is given by

\[ \tilde{M}^G = ((\tilde{X} \times \mathbb{T}^3) - (\tilde{\pi}^{-1}(\text{int}B^G_+) \cup \tilde{\pi}^{-1}(\text{int}B^G_-))) \cup (\mathbb{T}^3 \times B^G_+) \cup (\mathbb{T}^3 \times B^G_-) \]

where \( \mathbb{T}^3 \times \partial B^G_+ \) is attached to \( \tilde{\pi}^{-1}(\partial B^G_+) \) via the attaching map

\[ h^G_+(x, t) = (x, t + (G^{-1})^T \phi_+(Gx)) \]

and \( \mathbb{T}^3 \times \partial B^G_- \) is attached to \( \tilde{\pi}^{-1}(\partial B^G_-) \) via the attaching map

\[ h^G_-(x, t) = (x, t + (G^{-1})^T \phi_-(Gx)) \]

Since \( G \) is linear, we can still define a \( \mathbb{Z}_2 \)-action as we did in the previous example and all the claims made above follow through as well (again, by linearity of \( G \)). Set \( (m, n, p) = (G^{-1})^T(1, 0, 0) \). In particular, by passing to the quotient, we obtain a Lagrangian fibration

\[ \mathbb{T}^3 \xrightarrow{\cdot} M(m, n, p) \xrightarrow{\cdot} \mathbb{R}P^2 \times \mathbb{R} \]

whose monodromy is still given by \( f_+ \) and whose Chern class is precisely given by \((m, n, p)\). Since \( G \in \text{GL}(3, \mathbb{Z}) \) was arbitrary and the orbit of \((1, 0, 0)\) under the natural action of \( \text{GL}(3, \mathbb{Z}) \) on \( \mathbb{Z}^3 \) is \( \mathbb{Z}^3 \), we can construct in this fashion all the topological (and symplectic) types of Lagrangian fibrations with base space \( \mathbb{R}P^2 \times \mathbb{R} \).
and monodromy given by $f_\ast$. It is important to notice that we exploit in a crucial fashion the natural linear action of \(GL(3, \mathbb{Z})\) on $\mathbb{T}^3$ in this example.

5. Concluding Remarks

While the construction carried in section 4 might be a simple exercise in differential topology, we believe it has nonetheless highlighted the importance of the integral affine structure on the base space of a Lagrangian fibration. This is also why the whole paper has been centred on understanding topological invariants of Lagrangian fibrations primarily using this rigid structure on the base. Furthermore, this remark might be of some use in understanding special Lagrangian fibrations, which play a fundamental role in mirror symmetry.

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