Geometry of compact complex homogeneous spaces with vanishing first Chern class

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Abstract

We prove that any compact complex homogeneous space with vanishing first Chern class after an appropriate deformation of the complex structure admits a homogeneous Calabi-Yau with torsion structure, provided that it also has an invariant volume form. A description of such spaces among the homogeneous C-spaces is given as well as many examples and a classification in the 3-dimensional case. We calculate the cohomology ring of some of the examples and show that in dimension 14 there are infinitely many simply-connected spaces with the same Hodge numbers and torsional Chern classes admitting such structure. We provide also an example solving the Strominger’s equations in heterotic string theory.

1 Introduction

A complex homogeneous space is a complex manifold which admits a transitive action of a complex Lie group of biholomorphisms. In this paper we consider the geometry of compact complex homogeneous spaces with vanishing first Chern class. Except the tori such spaces are necessarily non-Kähler, so we are interested in the properties of an appropriate Hermitian connections with torsion.

On Hermitian manifolds, there is a one-parameter family of Hermitian connections canonically depending on the complex structure $J$ and the Riemannian metric $g$ \cite{13}. Among them is the Chern connection on the holomorphic tangent bundle. In this paper, we are interested in what physicists call the Kähler-with-torsion connection (a.k.a. KT connection) \cite{27}. It is the unique Hermitian connection whose torsion tensor is totally skew-symmetric when 1-forms are identified to their dual vectors with respect to the Riemannian metric. If $T$ is the torsion tensor of a KT connection, it is characterized by the identity \cite{13}

$$g(T(A, B), C) = dF(JA, JB, JC)$$

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where $F$ is the Kähler form; $F(A, B) = g(JA, B)$, and $A, B, C$ are any smooth vector fields.

As a Hermitian connection, the holonomy of a KT connection is contained in the unitary group $U(n)$. If the holonomy of the KT connection is reduced to $SU(n)$, the Hermitian structure is said to be Calabi-Yau with torsion (a.k.a. CYT).

Such geometry in physical context was considered first by A. Strominger [27] and C. Hull [22]. Since then various examples were found and this led to a conjecture [21] that any compact complex manifold with vanishing first Chern class admits a Hermitian metric and connection with totally skew-symmetric torsion and (restricted) holonomy in $SU(n)$. Counterexamples to this conjecture appear in [10]. There are also examples of CYT connections unstable under deformations. These two features of the CYT connections are in sharp contrast to well known moduli theory of Calabi-Yau (Kähler) metrics.

The main result of this paper is that after a homogeneous deformation of the complex structure any compact complex homogeneous space with vanishing first Chern class admits such a structure if it admits a volume form, invariant under some transitive Lie group of transformations. Moreover the CYT metric could be chosen to be invariant under the same group. It is interesting to compare the result with the counterexamples in [10] which are locally homogeneous. By a result of D. Guan [20] any compact complex homogeneous space with an invariant volume is a toric bundle over a product of complex parallelizable space and compact homogeneous Kähler space. The proof of the main result is in Section 6 and uses [20] and the torus bundle construction of CYT structure in [17]. We provide also a characterization of the vanishing of the first Chern class. For the complex homogeneous spaces admitting compact transitive group of transformations $G$ (C-spaces) it is expressed in terms of the Koszul form and the properties of the Tits fibration. This condition could be interpreted as the vanishing of the Koszul form on the set of complementary roots of the Lie algebra of $G$. Such characterization allows us to find many explicit examples. This is the content of Section 2. In Sections 3 and 4 we consider the existence of CYT structures on C-spaces and compact complex parallelizable manifolds. We show in Section 4 that on the compact complex parallelizable manifolds every invariant metric is CYT and in Section 3 that on C-spaces one can always deform the complex structure through a family of homogeneous complex structures to obtain a structure admitting homogeneous CYT metric. In Section 3.3 we completely characterize the left-invariant complex structures on $SU(2) \times SU(2)$ which admit a compatible homogeneous CYT metric. It is an open set and includes the Calabi-Eckmann complex structure. In section 5 we provide the classification of compact complex homogeneous spaces in dimension three and determine the existence of homogeneous CYT structure there. In particular we show that they admit CYT structure except for some complex structures on $SU(2) \times SU(2)$ which are described in Section 3. In section 7
we consider the topology of some of the examples and as a corollary we obtain that in dimension 14 there are infinitely many simply-connected nonflat CYT manifolds with same Betti and Hodge numbers and torsional Chern classes. In the last section we provide also a solution to the Strominger’s equations on a locally homogeneous nilmanifold.

2 Compact complex homogeneous spaces with vanishing first Chern class.

2.1 Characterization of the vanishing of the first Chern class

We concentrate here on the compact complex homogeneous spaces admitting transitive action of a (real) compact Lie group. Such spaces are called $C$–spaces and are investigated first by Wang [29]. In [29] is proved the following result:

**Theorem 1** Let $G$ be a compact semi-simple Lie group and $H$ be a closed connected subgroup whose semi-simple part coincides with the semisimple part of the centralizer of a toral subgroup of $G$, such that the coset space $G/H$ is even-dimensional. Then $G/H$ has a homogeneous complex structure and each $C$-space is homeomorphic to such coset.

In order to detect the spaces with vanishing first Chern class we need some basic facts about semisimple Lie algebras and their parabolic subalgebras.

Let $\mathfrak{g}^c$ is a complex semisimple Lie algebra. Fix a Cartan subalgebra (maximal toral subalgebra) $\mathfrak{t}$ in $\mathfrak{g}^c$. Then we have a system of roots $\mathfrak{R} \subseteq \mathfrak{t}^*$ defined by $\mathfrak{t}$ in $\mathfrak{g}^c$. There is also a distinguished set of simple roots $\mathfrak{P}$ in $\mathfrak{R}$ which forms a basis for $\mathfrak{t}^*$ as a (complex) vector space and defines a splitting $\mathfrak{R} = \mathfrak{R}^+ \cup \mathfrak{R}^-$ of $\mathfrak{R}$ into positive and negative roots. Let $\mathfrak{g}$ and $\mathfrak{h}$ are the (real) Lie algebras of $G$ and $H$. The complex structure on $G/H$ is defined in the following way: As a complex manifold $G/H = G^c/H^c$ where $G^c$ and $H^c$ are complex Lie groups with Lie algebras $\mathfrak{g}^c$ and $\mathfrak{h}^c$ and $G$ is a compact real form of $G^c$, while $H = H^c \cap G$. Moreover the above mentioned article of Wang provides the inclusion $\mathfrak{h}_{ss}^c \subset \mathfrak{h}^c \subset j^c$, where $j^c$ is a parabolic subalgebra, which is a centralizer of a torus and $j_{ss}^c = \mathfrak{h}_{ss}^c$. Here the subscript ”$ss$” denotes the semisimple part. In particular

$$\mathfrak{h}^c = \mathfrak{a} + \mathfrak{h}_{ss}^c$$

where $\mathfrak{a}$ is a commutative subalgebra of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}^c$. The parabolic algebra $j^c$ is $j^c = \mathfrak{t} + j_{ss}^c$ and is equal to the normalizer of $\mathfrak{h}^c$ in $\mathfrak{g}^c$. The sum here is not direct because part of $\mathfrak{t}$ is contained in $j_{ss}^c$. Let $J^c$ be a parabolic subgroup if $G^c$
with algebra \( j^c \) and \( G^c/J^c \) is the corresponding (generalized) flag manifold (called also rational homogeneous space). Then according to Wang \[29\] there is a holomorphic fibration \( G^c/H^c \to G^c/J^c \) with fiber a complex torus, which determines the complex structure on \( G/H \).

As is shown by Tits \[28\], for general complex Lie group \( G^c \) this fibration is canonically defined and unique in a sense that any other homogeneous fibration over \( G^c/H^c \) factors through \( G^c/J^c \). It is called the \textit{Tits fibration}. Moreover he shows that the fiber is complex parallelizable manifold and in the case of compact \( G \) it is a torus as above. This reduces the study of \( G/H \) to a study of the torus fibrations over the flag manifold \( G/J \), where \( J = J^c \cap G \).

Now every complex structure on the flag manifold is determined by an ordering of the system of roots of \( g^c \). The complex structure on \( G/J \) defines subset \( \Pi_0 \) in \( \Pi \) which corresponds to \( j^c \). Since this correspondence determines the second cohomology and the first Chern class of \( G/H \) we provide more details about it. In general \( j^c_{ss} \) is determined by the span of all roots in \( R \) which are positive with respect to \( \Pi_0 \). Then the complement \( \Pi - \Pi_0 = \Pi' \) provides a basis for the center \( \zeta \) of \( j^c \) and there is an identification \( \text{span}_\mathbb{Z}(\Pi') = H^2(G/J, \mathbb{Z}) \). The identification \[3\] is:

\[
\xi \to \frac{i}{2\pi} d\xi
\]

where \( \xi \) is considered as a left invariant 1-form on \( G \) which is a subgroup of \( G^c \) and \( d\xi \) is \( ad(j) \)-invariant, hence defines a 2-form on \( G/J \). This form is obviously closed and in fact defines non-zero element in \( H^2(G/J, \mathbb{Z}) \). Moreover every class in \( H^2(G/J, \mathbb{Z}) \) has unique representative of this form.

Now we are interested in the first Chern class of \( G/H \). It is determined by the so-called Koszul form for \( G/H \) (see \[28\],[2]). The definition of this form is

\[
\sigma_{G/H}(X) = Tr_\mathfrak{h} (ad(I X) - I ad(X)), \quad X \in \mathfrak{g}
\]

where \( I \) is the invariant complex structure on \( G/H \), extended as 0 on \( \mathfrak{h} \). Note that the form itself is defined only on \( G \). According to \[2\], Proposition 4.1,

\[
\sigma_{G/H} = 2i(\sigma_G - \sigma_H)
\]

where \( \sigma_G \) is the sum of positive roots in \( \mathfrak{g}^c \) and \( \sigma_H \) is the sum of positive roots in \( \mathfrak{g}^c \) which are also in \( \mathfrak{h}^c \). From here one has that \( \sigma_{G/J} = \sigma_{G/H} \) for the Tits fibration \( \pi : G/H \to G/J \) since the semisimple parts of \( j^c \) and \( \mathfrak{h}^c \) coincide. As is proved by Koszul, the form \( d\sigma \) descends to \( G/H \) and represents the first Chern class of the homogeneous manifold \( G/H \). The same is true for \( G/J \).
Theorem 2 The first Chern class of $G/H$ vanishes iff

$$\sigma_{G/H}|_a = 0$$

i.e. the restriction of $\sigma$ to $a$ vanishes.

Proof: The key point is that $d\sigma$ defines the zero element in $H^2(G/H, \mathbb{Z})$ iff $\sigma_{G/H}$ descends to an invariant 1-form on $G/H$ itself. But if $d\sigma_{G/H} = df$ for some 1-form $f$ on $G/H$, than one can symmetrize $f$ as in the section 3.2 below to obtain an invariant form $f'$ on $G/H$ with $df = df'$. Then the pull-back of $f'$ is an invariant form on $G$ with the same differential as $\sigma_{G/H}$, hence it coincides with $\sigma_{G/H}$. As it is proved in Tits (see [28],[3]) $\sigma_{G/H}$ is a sum with positive integer coefficients of elements of $\Pi'$ so is in $\zeta^*$, the dual of the center $\zeta$ of $j^c$. Then it descends to $G/H$ iff it vanishes on $a$. q. e. d.

With this characterization in mind we continue with examples.

2.2 Examples of C-spaces with vanishing first Chern class

We concentrate on quotients of a compact simple classical Lie groups, although the Calabi-Eckmann manifolds show that some homogeneous manifolds may be reducible as smooth manifolds and irreducible when considered as complex homogeneous ones.

The first type of examples include the following two extreme cases:

i) $a = 0$ i.e. $\mathfrak{h}$ is semisimple itself.

ii) $H = U(1)$, where $U(1)$ is appropriately embedded in odd-dimensional $G$, so that the principal bundle $G \rightarrow G/U(1)$ is a pull back of the canonical $U(1)$-bundle over the full flag manifold of $G$.

For the first case we start with an example from the $A_\ell$-series. Consider

$$M = SU(n)/SU(n_1) \times SU(n_2) \times ... \times SU(n_k), k \text{ odd}, n_i > 1$$

Here $SU(n_1) \times ... \times SU(n_k)$ is diagonally embedded as a matrix group in $SU(n)$ and $n_1 + n_2 = ... + n_k = n$. The condition $k - \text{even}$ ensures that the space is even dimensional. Then the Tits fibration is $M \rightarrow SU(n)/S(U(n_1) \times U(n_2)... \times U(n_k))$ with fiber $T^{k-1}$. The existence of a complex structure follows by Theorem 2. The vanishing of the Chern class follows by the above theorem. These examples clearly could be generalized to

$$M = SU(n)/SU(n_1) \times SU(n_2) \times ... \times SU(n_k), n_1 + n_2 + ... + n_k \leq n, n_i > 1$$

where $n - (n_1 + n_2 + ... + n_k) + k$ is odd.
For the other classical compact Lie groups we have also the following spaces:

\[
M = SO(2n)/SU(n_1) \times \ldots \times SU(n_{2k}) \times SO(2l), \quad n_1 + \ldots + n_{2k} + l = n
\]

\[
M = SO(2n)/SU(n_1) \times \ldots \times SU(n_{2k}) \times SU(n_{2k+1}), \quad n_1 + \ldots + n_{2k+1} = n
\]

\[
M = SO(2n+1)/SU(n_1) \times \ldots \times SU(n_{2k}) \times SO(2l+1), \quad n_1 + \ldots + n_{2k} + l = n
\]

\[
M = Sp(n)/SU(n_1) \times \ldots SU(n_{2k}) \times Sp(l), \quad n_1 + \ldots + n_{2k} + l = n
\]

We assume above that not all of \( n_i \) vanish.

Case \( ii \): The example is \( G/U(1) \), but with \( U(1) \) appropriately embedded. We concentrate on \( G = SU(n) \). The Tits fibration in this case is \( SU(n) \to F_{1,1\ldots,1} \) where \( F_{1,1\ldots,1} \) is the standard flag manifold \( SU(n)/S(U(1) \times \ldots \times U(1)) \). In this case we can avoid more complicated tools like the painted Dynkin diagram below because \( \sigma_F = \sigma_{SU(n)} - \sigma_{S(U(1) \times \ldots \times U(1))} = \sigma_{SU(n)} \), where \( \sigma_G \) is the sum of all positive roots of \( G \). Then since \( S(U(1) \times \ldots \times U(1)) \) is abelian, \( \sigma_{S(U(1) \times \ldots \times U(1))} = 0 \). To describe the sum of positive roots we need some notations which appear in [12]. Let \( L_i \) be the matrix with \( i \)-th diagonal element equal to 1 and all others being 0. Then the set of all roots of the complexified Lie algebra \( sl(n, \mathbb{C}) \) is \( e_{i,j} = e_i - e_j \), where \( e_i \) are the duals of \( L_i \). A set of simple roots is \( e_{i,i+1} \) which also determines the positive roots \( e_{i,j}, i < j \). Then the sum of all positive roots is:

\[
\sum_{i<j} e_{i,j} = \sum_{k=1}^{n-1} k(n-k)e_{k,k+1} = \sum_{k=1}^{n} (n-2k+1)e_k
\]

Then we have:

**Proposition 1** The space \( SU(n)/U(1) \) for \( n \) even endowed with the homogeneous complex structure from [24] has vanishing first Chern class iff \( U(1) \) is embedded as a set of diagonal matrices:

\[
A = \text{diag}(e^{2\pi \theta_1 t}, e^{2\pi \theta_2 t}, \ldots, e^{2\pi \theta_n t})
\]

with \( \theta_n = -\theta_1 - \ldots - \theta_{n-1} \) and satisfying

\[
\sum_{k=1}^{n} (n-2k+1)\theta_k = 2\sum_{k=1}^{n-1} (n-k)\theta_k = 0
\]

Next we consider the general case of factors of \( SU(n) \). In this case the Tits fibration is of the form:

\[
SU(n)/SU(n_1) \times \ldots \times SU(n_k) \times T^l \to SU(n)/SU(n_1) \times \ldots \times SU(n_k) \times T^m
\]

with \( n_1 + \ldots + n_k + m - k = n - 1 \) and \( l < m \). At this point we notice that an invariant complex structure on the flag manifold \( SU(n)/SU(n_1) \times \ldots \times SU(n_k) \times T^m \) is not
unique and depends on the so called painted Dynkin diagram (or black-white Dynkin diagram). The painted diagrams are used to describe (generalized) flag manifolds and homogeneous Einstein metrics on them (see for example [4]). For a flag manifold, painted Dynkin diagram is obtained by blackening the vertices which correspond to $\Pi'$. We refer to [3] for the details and use directly their result for two particular examples which enlighten the general case.

Example of $A_l$-type. Choose the flag manifold for the base of the Tits fibration, to be $SU(11)/SU(4) \times SU(3) \times SU(2) \times T^4 = SU(11)/S(T^2 \times U(4) \times U(3) \times U(2))$. It corresponds to a painted Dynkin diagram:

![Dynkin diagram]

The diagram also determines the complex structure on the flag manifold above. Now we need the Koszul form $\sigma$ of this flag manifold. In case of the $A_n$-series it is described in [3], Propostinion 4.1 and Proposition 5.2. From there we have that

$$\sigma = (2 + b_1)\overline{\alpha_1} + (2 + b_2)\overline{\alpha_2} + ... + (2 + b_m)\overline{\alpha_m}$$

where $\overline{\alpha_i}$ are the fundamental weights corresponding to the roots with black circles. They are defined as

$$\frac{\langle \overline{\alpha_k}, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_j^i \quad (1)$$

where $\langle , \rangle$ is the product arising from the Killing form. The numbers $b_i$ are nonnegative and in our particular case are equal to the number of white circles of the Dynkin diagram, which are connected with the black circle corresponding to the root $\alpha_i$ by a series of white circles [3].

In particular for the above diagram we have:

$$\sigma = (2 + b_1)e_{1,2} + (2 + b_2)e_{2,3} + (2 + b_3)e_{6,7} + (2 + b_4)e_{9,10} = 2e_{1,2} + 5e_{2,3} + 7e_{6,7} + 5e_{9,10}$$

Now to obtain explicit expression of the above element $\sigma$ in terms of $e_{i,i+1}$ we need a description of the fundamental weights. This could be done following [12].

On $g^c = sl(11, \mathbb{C})$ the matrices $e_i$ are orthonormal with respect to $\langle , \rangle$. Then one can check directly that the following elements satisfy the condition (1)

$$L_k = e_1 + ... + e_k - k/n(e_1 + ... + e_n) = \overline{e_{k,k+1}}$$

So in our case we have

$$\sigma = 2e_1 - 2/11(e_1 + ... + e_{11}) + 5(e_1 + e_2) - 10/11(e_1 + ... + e_{11}) + 7(e_1 + ... + e_6)$$
\[ -42/11(e_1 + ... + e_{11}) + 5(e_1 + ... e_9) - 45/11(e_1 + ... + e_{11}) = 10e_1 + 8e_2 + 3(e_3 + e_4 + e_5 + e_6) - 4(e_7 + ... + e_9) - 9(e_{10} + e_{11}) \]

The dimension count gives \( \dim SU(11)/S(T^2 \times U(4) \times U(3) \times U(2)) = 90 \) and there could be 2 or 4-dimensional torus as fibers for a Tits fibration with this base. We also consider the subgroup \( J = S(T^2 \times U(4) \times U(3) \times U(2)) \) embedded in \( SU(11) \) in the standard diagonal form with unitary blocks of order 1,1,4,3, and 2 respectively so that \( G/J \) is generalized flag manifold. The 4-dimensional fiber case leads to an example of the previous type because the subgroup \( H \) will be semisimple. So we consider the two dimensional fibers. At the Lie algebra level we have to add appropriate 2-dimensional space \( a \) of diagonal matrices to the Lie algebra \( j_{ss} \), the real part of \( j_{cc} \). It has to be of the form

\[ a = \text{diag}(x_1, x_2, x_3, x_3, x_3, x_4, x_4, x_5, x_5) \]

and should obey the following conditions:

\[
egin{align*}
    x_1 + x_2 + 4x_3 + 4x_4 + 2x_5 &= 0 \\
    10x_1 + 8x_2 + 12x_3 - 12x_4 - 18x_5 &= 0
\end{align*}
\]

The first equation comes from the requirement that the matrices in \( a \) are trace-free. The second follows from Theorem 2 and the form of \( \sigma \) above. Now we can fix two linearly independent integer solutions \((v_1, ..., v_5)\) and \((w_1, ..., w_5)\) of these equations. Then the Lie algebra \( \mathfrak{h} = j_{ss} + a \) should be:

\[
\mathfrak{h} = \text{diag}(v_1t + w_s, v_2t + w_2s, (v_3t + w_3s)A, (v_4t + w_4s)B, (v_5t + w_5s)C)
\]

where \( A, B, C \) are trace-free skew-adjoint matrices of order 4,3 and 2 respectively. Then at the end we obtain that \( SU(11)/H \) is a complex homogeneous manifold with vanishing first Chern class, if \( H \) is of the form:

\[
H = \text{diag}(e^{2i\pi(v_1t+w_1s)}, e^{2i\pi(v_2t+w_2s)}, e^{2i\pi(v_3t+w_3s)}A, e^{2i\pi(v_4t+w_4s)}B, e^{2i\pi(v_5t+w_5s)}C)
\]

where \( A, B, C \) are unitary matrices with determinants one and order 4,3 and 2 respectively. Moreover any such manifold with a stationary subgroup \( H \) containing strictly \( Id_2 \times SU(4) \times SU(3) \times SU(2) \) is of this form.

**Example of C_{l-type}.** As a last example we consider the following quotient of the symplectic group - \( \frac{Sp(7)}{Sp(2) \times Sp(2) \times T} \). Let \( sp(2n, \mathbb{C}) \) be the complex Lie algebra of symplectic complex \( 2n \times 2n \) matrices. Its diagonal has a basis of the form \( L_i = E_{i,i} - E_{n+i,n+i} \), where \( E_{i,j} \) is the matrix with \((i,j)\)-th element 1 and all others 0.
Then a set of simple roots is given by the duals to the matrices \(L_i - L_{i+1}\) and \(2L_n\). Denote by \(e_i\) the dual to \(L_i\). So the simple roots are given by \(\alpha_i = e_i - e_{i+1}, \alpha_n = 2e_n\).

To calculate the Koszul form we can not use the routine from the previous example but we can adapt the algorithm from [6]. Below is the Dynkin diagram for \(sp(14, \mathbb{C})\) together with the coefficients for each simple root, with which it enters in the sum of all positive roots:

```
14 26 36 44 50 54 28
```

The similar diagram corresponding to \(sl(2, \mathbb{C}) \oplus sp(4, \mathbb{C})\) is

```
1 4 3
```

Now we take the difference of the two diagrams: Complete the second diagram with black dots corresponding to the complementary roots and take the difference of the corresponding coefficients. The result is given by (notice that our painting is opposite to the one in [6]):

```
14 25 36 44 50 50 25
```

This means that the Koszul form of the flag manifold \(\frac{Sp(7)}{SU(2) \times Sp(2) \times T^2}\) is given by

\[
\sigma = 14\alpha_1 + 25\alpha_2 + 36\alpha_3 + 44\alpha_4 + 50(\alpha_5 + \alpha_6) + 24\alpha_7 = 14e_1 + 11(e_2 + e_3) + 8e_4 + 6e_5
\]

Now we consider the compact real forms of the Lie algebras above. The compact form of \(sp(2n, \mathbb{C})\) is denoted simply by \(sp(n)\) and is the the set \(n \times n\) quaternionic matrices \(A\) with \(A + \overline{A} = 0\). Then the \(sp(2)\) which corresponds to the last two dots in the painted diagram above is represented by a \(2 \times 2\) submatrix in the lower right corner. In the same way \(su(2)\) is represented by one diagonal entry which is an imaginary quaternion. The 2-dimensional abelian subalgebra which should be added to the sum \(su(2) + sp(2)\) is the algebra of diagonal matrices with entries \(i(\lambda_1 t + \mu_1 s, \lambda_2 t + \mu_2 s, \lambda_3 t + \mu_3 s, \lambda_4 t + \mu_4 s, 0, 0)\), for which \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) and \(\mu = (\mu_1, \mu_2, \mu_3, \mu_4)\) are independent integral solutions of \(14x + 22y + 8z + 6v = 0\). Then by Theorem 2 we have that \(M = \frac{Sp(7)}{H}\) is a complex homogeneous space of vanishing first Chern class if \(SU(2) \times Sp(2) \times T^2 = H\) is embedded as \(H = diag(e^{2\pi i(\lambda_1 t + \mu_1 s)}, q, e^{2\pi i(\lambda_2 t + \mu_2 s)}, e^{2\pi i(\lambda_3 t + \mu_3 s)}, e^{2\pi i(\lambda_4 t + \mu_4 s)}, A)\), where \(A\) runs over \(Sp(2)\) and \(q\) over \(SU(2) = Sp(1)\).
Remark 1 If the integer solutions in the above two examples are chosen with greatest common divisor one, than the factor-spaces are simply-connected. The topology of the examples from Proposition 7 is discussed in Section 7.

3 CYT metrics on C-spaces

3.1 Canonical connections and CYT metrics

We provide here the properties of the canonical connections on Hermitian manifolds which we will need. Let \((M, J, g)\) be Hermitian manifold and \(F(X, Y) = g(JX, Y)\) is the Kähler form. Let \(d^c\) be the operator \((-1)^n J dJ\) on n-forms and \(D\) be the Levi-Civita connection of the metric \(g\). Then, a family of canonical connections is given by

\[
g(\nabla_t A B, C) = g(D_A B, C) + \frac{t - u}{4} (d^c F)(A, B, C) + \frac{t + u}{4} (d^c F)(A, JB, JC), \tag{2}
\]

where \(A, B, C\) are any smooth vector fields and the real number \(t\) is a free parameter. The connections \(\nabla^t\) are called canonical connections. It is known that the connection \(\nabla^1\) is the Chern connection on the holomorphic tangent bundle. The connection \(\nabla^{-1}\) is called KT connection by physicists and Bismut connection by some mathematicians. Its mathematical features and background are articulated in [13]. It is apparent in the above expression that when the Hermitian structure is a Kähler structure, this one-parameter family of canonical connections collapses to a single connection: the Levi-Civita connection.

From formula (2.7.6) in [13] we have

\[
\nabla^t s - \nabla^u s = t \frac{u}{2} \delta F \otimes s
\]

for any section \(s\) of the canonical bundle \(K^{-1}\) and \(\delta\) is the co-differential.

Let \(R^t\) be the curvature of \(\nabla^t\) and \(\rho^t(X, Y) = \Sigma g(R^t(X, Y)E_i, JE_i)\) be the corresponding trace. Then \(i\rho^t\) is the curvature of \(K^{-1}\) and from the above relation we obtain:

\[
\rho^t - \rho^u = t \frac{u}{2} d\delta F
\]

Denote by \(\rho\) and \(\rho^B\) the Ricci forms of the Chern and KT connections respectively.

In [14], formula (53) gives representation in local coordinates:

\[
\rho = -1/2dd^c \log(\det(g_{\alpha\beta})) \tag{3}
\]

in particular \(\rho\) is always (1,1)-form. In general \(d\delta F\) is not (1,1)-form.
Definition 1 A Hermitian metric is called CYT (or Calabi-Yau with torsion) if $\rho^B = 0$. Then $(g, J)$ is called a CYT structure and $\nabla^B$ a CYT connection.

For later use we recall also the toric bundle construction of CYT structures described in [17]. Suppose that $X \to B$ is a $T^{2k}$-bundle over complex manifold $B$ with characteristic classes $\omega_1, \ldots, \omega_{2k}$. If $\eta_1, \ldots, \eta_{2k}$ are connection 1-forms with $d\eta_i = \omega_i$, then there is an almost complex structure on $X$ defined in the following way. On horizontal cotangent subspaces it is the horizontal lift of the structure on the base $B$. The vertical subspaces are spanned by $\text{Ker}(\eta_i)$ and the definition is just $J\eta_{2i-1} = \eta_{2i}$. The result is an almost complex structure $J$ on $X$ which is integrable if $\omega_{2i-1} + i\omega_{2i}$ is a 2-form of type $(1,1)+(2,0)$. Moreover any toric invariant complex structure on $X$ for which the bundle projection is holomorphic arises in this way for some choice of the connection.

Suppose now that $g_X$ is a Hermitian metric on the base manifold $X$ with Kähler form $F_X$. Consider a Hermitian metric on $M$ defined by

$$g_M = \pi^* g_X + \sum_{\ell=1}^{2k} \theta_{\ell}^2.$$ 

Since $J\theta_{2j-1} = \theta_{2j}$, the Kähler form for this Hermitian metric is

$$F_M = \pi^* F_X + \sum_{j=1}^{k} \theta_{2j-1} \wedge \theta_{2j}.$$ 

By [17] the Ricci forms of the canonical connections of $X$ and $M$ are related by:

$$\rho^X_M = \pi^* \rho^X_X + \frac{t-1}{2} \sum_{\ell=1}^{2k} d((\Lambda \omega_{\ell})\theta_{\ell}).$$

where $\Lambda \omega_{\ell} = g(F_X, \omega_{\ell})$. In particular when $g(F, \omega_{\ell}) = \text{constant}$ we have

$$\rho^B_M = \pi^*(\rho^B_X - \sum_{\ell=1}^{2k} g(F_X, \omega_{\ell})\omega_{\ell}).$$

(4)

For more details see [17, 15].

3.2 Existence of CYT metrics on C-spaces

Many examples of CYT spaces are provided by the compact homogeneous C-spaces of Wang. Assume that $M = G^\circ/L$ is a compact complex homogeneous space defined
by a transitive action of a complex Lie group $G^c$ with finite fundamental group. Then by an observation in [29], Remark (2.3), $M$ is analytically isomorphic to $G/(G \cap L)$, where $G$ is a maximal connected compact semisimple subgroup of $G^c$. From here we have:

**Theorem 3** On any compact simply connected complex non-Kähler homogeneous space with vanishing first Chern class there is a 1-parameter family of homogeneous complex structures which contains the given structure and a structure that admits a compatible CYT metric.

**Proof:** First we notice that $M$ is a coset space $M = G/H$ of a compact semisimple Lie group $G$. Then consider the splitting of the Lie algebra $g = h + m$ of $G$, where $m$ is the orthogonal compliment of $h$ in $g$ with respect to the metric $B(X,Y)$ given by the negative of the Killing form on $G$. Now $B$ is bi-invariant and in particular $\text{Ad}G$-invariant and when restricted to $h$ and $m$ is non-degenerate. Then by theorem 3.4, Chapter 10 of [24], we have that $M$ is a naturally reductive homogeneous space. Such spaces have a canonical homogeneous connection which is metric and its torsion at the identity coset is given by $T(X,Y)_0 = -[X,Y]|_m$. Then for naturally reductive spaces $B(X,[Y,Z]|_m)$ is totally skew-symmetric 3-form. Suppose for the moment that the complex structure is Hermitian with respect to the metric induced by $B$. Now since any invariant tensor field is parallel with respect to the canonical connection, it is the Bismut connection for the naturally reductive complex homogeneous space $M$. We will show that it is CYT. We recall first that the canonical bundle $K$ for the complex structure is topologically trivial because its first Chern class vanishes. Then we can find a $(n,0)$-form which is nowhere zero and by symmetrization, we obtain a global $G$-invariant $(n,0)$-form. More precisely if $\omega$ is non-vanishing $(n,0)$-form we define:

$$\mu(\omega)_x(A_1, ..., A_n) = \int_G \omega_{g(x)}(L_g^*(A_1), ..., L_g^*(A_n))dm_g$$

where $dm_g$ is bi-invariant volume form on $G$. Then $\mu(\omega)$ is invariant. Since the invariant tensors are parallel, it is parallel with respect to the canonical connection. Then the only point is to prove that it is everywhere non-zero. However $\mu(\omega) \wedge \mu(\omega) = f dm_g$ for some function $f$. Since $\mu(\omega)$, $\mu(\omega)$ and $dm_g$ are $G$-invariant, $f = \text{constant}$. So $\mu(\omega)$ is either nowhere zero or vanishing identically. However in the definition of $\omega$ we have the freedom to multiply it by arbitrary nonvanishing real function. Assume that $\mu(\omega) = 0$ for any choice of $\omega$. Fix tangent $(1,0)$-vectors $(A_1, ..., A_n)$ at the identity coset in $G/H$. Consider the function $h(g) = L_g^*\omega(A_1, ..., A_n)$. Then our assumption implies that $\int_G f(g)h(g)dm_g = 0$ for every smooth function $f(g)$ such that $f(gh_1) = f(gh_2)$ for every $h_1, h_2 \in H$ and $f(g) \neq 0$. By taking limit, we can actually extend our assumption to every integrable function $f$. Since $h(e) \neq 0$, we can assume
that its real part is $\Re h(e) > 0$ in some neighborhood of $H$ in $G$. Then by taking cut-off real function $f$ we can restrict the above integral on a smaller neighborhood of $H$ and see that its real part is strictly positive, which is a contradiction. Then the proof follows from the next Lemma

$q. e. d.

Lemma 1 Every homogeneous complex structure on a C-space $G/H$ is either compatible with $B$ or belongs to a 1-parameter family of homogeneous complex structures which contains a structure compatible with $B$

Proof: Since any homogeneous complex structure admits a Tits fibration, it is determined by an invariant complex structure on the base generalized flag manifold and a complex structure on the torus fiber. The main observation is that any invariant complex structure on the base is compatible with the induced $B$. Fixing a $B$-orthonormal invariant 1-forms which provide a basis for $a^*$ will determine a connection 1-forms of a ”canonical” connection for the Tits fibration. Then the given complex structure on the fiber is determined by a constant matrix $J$ of square minus identity in such basis. The structure is $B$-compatible if and only if this matrix is orthogonal. However if it is not, we can find a path of such matrices which connects $J$ to an orthogonal one, because the set $O(2n)/U(n)$ is a deformation retract of $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. Such a path will define the required 1-parameter family on $G/H$. $q. e. d.$

Remark 2 The arguments in the theorem above extend to any quotient of a compact Lie group $G$ and closed subgroup $H$, not just semisimple one. This will be used later in Section 6.

3.3 Deformations of the bi-invariant metric on SU(3) and CYT structures on $S^3 \times S^3$

Suppose that we have a complex semisimple Lie algebra $\mathfrak{g}^c$ and a root decomposition with root basis $(E_\alpha, E_{-\alpha}, H_k)$, where $H_k$ is a basis of a fixed Cartan subalgebra $\mathfrak{k}$. Suppose its compact form has associated Lie group $G$ which admits complex structure $I$ with $I(E_\alpha) = iE_\alpha, I(E_{-\alpha}) = -iE_{-\alpha}$. Then consider a left invariant Hermitian metric on $G$ such that it is also right-invariant for the maximal tori. In such case it is of the form $g(X, Y) = B(\Lambda(X), Y)$ where $B(., .)$ is the Hermitian extension of the bi-invariant metric on $G$ and $\Lambda$ is a Hermitian positive operator defined by $\Lambda(E_\alpha) = \lambda_\alpha E_\alpha$ for $\alpha > 0$ and any appropriate matrix on $H_k$. Consider the case when it is identity for $H_k$. Then we calculate $\delta F$ as follows:

$$\delta F \circ I = 1/2 F \bar{\partial} dF$$
Because \( dF(X, Y, Z) = F([X, Y], Z) + F([Y, Z], X) + F([Z, X], Y) \) for invariant metrics, the sum \( \sum_i dF(E_i, IE_i, Z) \) has contributions only from \( dF(E_\alpha, E_{-\alpha}, H) = -\lambda_\alpha \alpha(IH) \), for \( H \in \mathfrak{k} \). Then

\[
\delta F \circ I = -1/2 \sum _\alpha \lambda_\alpha \alpha \circ I
\]

where the sum is over all positive roots. By Koszul [23] the Ricci form of the Chern connection for \( I \) is the exterior derivative of the half-sum of all positive roots for any invariant metric. So the Ricci form \( \rho^B \) is given by

\[
\rho^B = 1/2 \sum _\alpha (1 - \lambda_\alpha) d\alpha
\]

for the sum running over all positive roots again.

In the case of \( SU(3) \), the roots are \( (\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta) \). Then the sum is \( 2\alpha + 2\beta \) and we have:

\[
\rho^B = 0 \quad i f f \quad \lambda_\alpha + \lambda_{\alpha+\beta} = \lambda_\beta + \lambda_{\alpha+\beta} = 2
\]

Then taking \( \lambda_\alpha \) as a parameter, we have that:

**Proposition 2** There is a 1-parameter family parametrized by \((0,2)\) of CYT metrics on \( SU(3) \) with respect to a fixed complex structure.

One can compare the deformation above with the fact that any small hypercomplex deformation of the HKT structure on \( SU(3) \) is HKT [18]. Since every HKT structure is CYT, this implies that for many non-homogeneous complex structures on \( SU(3) \) there are CYT metrics.

**CYT structures on** \( SU(2) \times SU(2) \). The space \( SU(2) \times SU(2) \) carries a Calabi-Eckmann complex structure and has a basis of global left-invariant 1-forms \( \alpha_i, e^+_i, e^-_i \) where \( i = 1, 2 \) correspond to the two factors. It satisfies the Maurer-Cartan equations \( d\alpha_i = e^-_i \wedge e^+_i, de^+_i = \alpha_i \wedge e^+_i, de^-_i = -\alpha_i \wedge e^-_i \). The forms \( \alpha_1, \alpha_2 \) define the canonical connection for the Tits fibration which in this case is \( SU(2) \times SU(2) \to \mathbb{CP}^1 \times \mathbb{CP}^1 \). The forms \( e^+_i \) are pull-backs from forms on the base. Every left-invariant complex structure on \( SU(2) \times SU(2) \) is also right-invariant with respect to the torus fibers of the Tits fibration. Then, up to sign on the factors of the base \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), it is determined by the following:

\[
J(\alpha_1) = a\alpha_1 + b\alpha_2, J(\alpha_2) = c\alpha_1 + d\alpha_2, J(e^+_i) = e^-_i, J(e^-_i) = -e^+_i
\]

(5)

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -Id \), so \( d = -a, b \neq 0, c = -\frac{a^2 + 1}{b} \). From the previous sections we know that the Ricci form of the Chern connection for any invariant Hermitian
metric is given by $d\sigma_{SU(2)\times SU(2)} = d(\alpha_1 + \alpha_2)$. If the Kähler form of the metric is $F$ then from $0 = \rho^B = \rho^{Ch} - d\delta F$ we have $\rho^{Ch} = d\delta F$ for the equation of a CYT metric. Since $\rho^{Ch} = d\sigma_{SU(2)\times SU(2)}$ and both $\sigma_{SU(2)\times SU(2)}$ and $\delta F$ are invariant 1-forms then $\sigma_{SU(2)\times SU(2)} = \delta F + \beta$ for an invariant and closed 1-form $\beta$. However $\beta$ is not exact and since $SU(2) \times SU(2)$ is simply-connected, $\beta = 0$. This considerations are obviously valid for any compact semisimple Lie group $G$, so we obtain:

**Lemma 2** The CYT equation for any fixed left-invariant complex structure on a compact semisimple Lie group $G$ and any compatible left-invariant metric $g$ is

$$\sigma_G = \delta F$$

where $F$ is the Kähler form of $g$ and $\delta$ is the co-differential.

To solve this equation on $SU(2) \times SU(2)$ we denote by $g$ the metric induced on the cotangent bundle and notice that (6) is equivalent to $g(\alpha_1 + \alpha_2, \xi_i) = g(\delta F, \xi_i) = g(F, d\xi_i)$ for any basis $\xi_i$ of left-invariant 1-forms. The last equation follows by integration and the fact that all functions in the equalities are constants. We choose the basis $\xi_i$ to be $\alpha_i, e_i^\pm$. Using the Maurer-Cartan equations, we have $g(\alpha_1 + \alpha_2, \alpha_i) = g(F, e_i^- \wedge e_i^+) = g(F, J(e_i^+) \wedge e_i^+) = g(e_i^+, e_i^+)^2 = G_i > 0$. Now from $g(\alpha_i, J(\alpha_i)) = 0$ we obtain after a short calculation that $g(\alpha_1, \alpha_2) = -\frac{a}{b}g(\alpha_1, \alpha_1) = \frac{a}{c}g(\alpha_2, \alpha_2)$. Then from the previous equation we have

$$
\begin{align*}
g(\alpha_1 + \alpha_2, \alpha_1) &= \frac{b-a}{b}g(\alpha_1, \alpha_1) = G_1 \\
g(\alpha_1 + \alpha_2, \alpha_2) &= \frac{a+c}{c}g(\alpha_2, \alpha_2) = G_2
\end{align*}
$$

From the fact that $G_i$ and $g(\alpha_i, \alpha_i)$ are positive and $b, c \neq 0$ we obtain that $b(b-a) > 0$, $\frac{a^2+1-ab}{a^2+1} > 0$. This is necessary condition for the existence of homogeneous CYT structure on $SU(2) \times SU(2)$. Moreover one can check that it is also sufficient because we can determine the metric to be the form $g = g_\alpha + \sum G_i|e_i^+|^2 + |e_i^-|^2$ where $g_\alpha$ is Hermitian metric on Span$(\alpha_1, \alpha_2)$. This follows if we fix $g(\alpha_1, \alpha_1) = 1$ and use the above equations to determine $G_i$ and $g_\alpha$. The result can be formulated as

**Proposition 3** The invariant complex structure $J$ on $SU(2) \times SU(2)$ determined by (3) admits a homogeneous CYT structure if and only if $b(b-a) > 0$, $a^2+1-ab > 0$.

We notice that for the space $S^1 \times SU(2)$ any invariant complex structure carries an invariant CYT metric and the same is true for $T^3 \times SU(2)$. This will be used in Section 5. It is a natural question to ask whether the complex structures on $SU(2) \times SU(2)$ which do not admit an invariant CYT metric admit any such metric. This is still an open question and the symmetrization above can not be applied.
4 Compact complex parallelizable manifolds

Another type of compact complex homogeneous manifolds with vanishing first Chern class are the complex parallelizable manifolds - i.e. the manifolds with holomorphic parallelization of its holomorphic tangent bundle.

If the canonical bundle is holomorphically trivial, then from formula (3) follows that

$$\rho^B = dd^c \log |\Omega|$$

for a nonvanishing holomorphic section $\Omega$ and

$$\rho^B = dd^c \log |\Omega| - d\delta F$$  \hspace{1cm} (7)

It is well known that complex paralelizable manifolds are of the form $G^c/\Gamma$, where $G^c$ is a complex Lie group and $\Gamma$ is a cocompact lattice. For the Hermitian geometry of such manifolds there is the following result which is easily deduced from the main theorem in [1]:

**Theorem 4** For a compact complex parallelisable manifold any left invariant metric is a balanced metric, i.e $\delta F = 0$.

A sketch of the proof is as follows: The Levi-Civita connection satisfies $g((D_X J)Y, Z) = g(ad_{JZ}X, Y)$ from the standard formulas. Then $\delta F(X) = tr(ad_{JX})$, which does not depend on $g$. But a Lie group $G$ admits a compact quotient only if it is unimodular and for such groups $tr(ad_X) = 0$ for any left-invariant $X$.

Now using it we can prove the following:

**Theorem 5** For a compact complex parallelisable manifold any left invariant metric is a CYT metric. Moreover all Ricci forms of the canonical Hermitian connections vanish.

**Proof:** Let $M = G^c/\Gamma$ and let $g^c$ is the complex Lie algebra of $G^c$. Then there exist a holomorphic left invariant vector fields $X_1, X_2, ... X_n$ on $G$ which form a basis for $g^{(1,0)}$ and moreover $[X_i, X_j] = c_{ij}^k X_k, [X_i, \overline{X}_j] = 0$. If we define the (1,0)-forms $\alpha_i$ to be dual of $X_i$ i.e. $\alpha_i(X_j) = \delta_{ij}$, $\alpha_i(\overline{X}_j) = 0$, then because of the last identity, $\alpha_i$ constitute a holomorphic 1-forms. Now consider any left invariant metric $g$ on $G$ (and hence on $M$). The form $\Omega = \alpha_1 \wedge \alpha_2 \wedge ... \alpha_n$ is a holomorphic (n,0)-form with constant norm. From (7) and the previous theorem we conclude that all Ricci forms for the canonical connections vanish. q.e.d.

**Examples:** There is a classification of all complex Lie algebras in (complex) dimension three [26]. These are the following:

i) The abelian algebra $\mathbb{C}^3$

ii) The complex Heisenberg algebra which is two-step nilpotent.
iii) The direct sum $s_2(\mathbb{C}) \oplus \mathbb{C}$ of the 2-dimensional solvable algebra $s_2(\mathbb{C})$ defined by $[X_1, X_2] = X_2$ for the basis $X_1, X_2$ and a center $\mathbb{C}$.

iv) Two irreducible solvable Lie algebras defined as follows:

$$s_3(\mathbb{C}) = \text{span}\{X_1, X_2, X_3 | [X_1, X_2] = X_2, [X_1, X_3] = X_2 + X_3\}$$

and

$$s_{3,\lambda}(\mathbb{C}) = \text{span}\{X_1, X_2, X_3 | [X_1, X_2] = X_2, [X_1, X_3] = \lambda X_3\}$$

v) $sl(2, \mathbb{C})$.

It is known that the Lie groups of the complex Heisenberg algebra, the solvable Lie algebra $s_3_{-1}$ and $sl(2, \mathbb{C})$ admit cocompact lattices, the others do not since they are not unimodular. In particular one have a classification of the 3-dimensional compact complex parallelizable manifolds.

Remark 3 In [25] are considered the deformations of some of the compact complex parallelisable manifolds above. In particular it is shown that there exists a deformation of a parallelizable manifolds which provides non-parallelizable ones. These are again complex manifolds of vanishing first Chern class. Whether they admit a CYT metric is an open question.

5 Compact complex homogeneous manifolds of dimension 3 with vanishing first Chern class

We use in this section the result due to Tits about the classification of the homogeneous compact complex 3-manifolds. It is described as follows (Theorem 6.3 in Tits [28]) : Any compact complex homogeneous space of dimension 3 is one of the following:

i) a compact complex parallelizable manifold classified in Section 4.

ii) a torus fibration of appropriate fiber dimension over $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$, or $\mathbb{CP}^1$.

iii) a generalized flag manifold

iv) reducible compact complex manifold

From this list i) ii) and possibly iv) are with vanishing first Chern class. This follows form Proposition 2 for i) and is standard for ii). Moreover in iv) we have products of a two-dimensional torus and a homogeneous compact complex surface. One can check (again in Tits [28]) that the homogeneous compact complex surfaces with vanishing first Chern class are the following:

i) torus

ii) Hopf surface fibered over $\mathbb{CP}^1$ with a homogeneous complex structure.

Combining the results of this and the previous sections we obtain:
Theorem 6 Any compact complex homogeneous manifold of complex dimension 3 with vanishing first Chern class is biholomorphic to one of the above. Except for the $SU(2) \times SU(2)$ it admits a homogeneous CYT structure. The set of invariant complex structures on $SU(2) \times SU(2)$ which admit a homogeneous CYT structure is open and is described in Proposition 3.

The proof is case by case with only the torus bundles in case ii) above are not evident. But all of them are finitely covered by $SU(2) \times SU(2)$ or the product of a 2-tori with a Hopf surface. This cases were considered in Section 3.3.

Remark 4 Some real homogeneous spaces which admit left invariant complex structure are not complex homogeneous in this context. For example the product of 5 dimensional real Heisenberg group with $S^1$ admits a compact quotient $M$ which is real homogeneous and will be used in Section 8. There is also a left invariant complex structure which descends to $M$. However in our terminology $M$ is only complex locally homogeneous.

6 Complex homogeneous manifolds with invariant volumes

In [20] D.Guan proved the following:

Theorem 7 Every compact complex homogeneous space with an invariant volume form is a principal homogeneous complex torus bundle over the product of a projective rational homogeneous space and a parallelizable manifold.

In [20] there is more information about such bundles. It is shown that the bundle $\pi : M \to G/K \times D$, where $G$ is a compact Lie group such that $G/K$ is a rational homogeneous space and $D$ is a complex parallelizable space, arises as a factor of the product of two principal complex torus bundles. One is $\pi_1 : G/H \to G/K$, which is the Tits fibration for $G/H$ and the other is $\pi_2 : D_1 \to D$ where $D_1$ is again compact complex parallelizable and the fiber is a complex torus, which is in the center of $D_1$. The action for the factor bundle is the anti-diagonal one.

The main goal of this section is to prove the following:

Theorem 8 Every compact complex homogeneous space with invariant volume and a vanishing first Chern class is a principal torus bundle over a product of a standard homogeneous CR manifold and a complex parallelizable manifold. After a homogeneous complex deformation it admits a homogeneous CYT structure.
The standard homogeneous CR space here is obtained from the principle circle bundle over a generalized flag manifold with characteristic class equal to the first Chern class of the flag space. Such fibre bundle is called Boothby-Wang fibration. By homogeneous CYT structure we mean a Hermitian metric invariant under the action of some transitive Lie group of biholomorphisms and a homogeneous complex deformation is a 1-parameter family of homogeneous complex structures. Before we start the proof of the theorem we need the following two lemmas:

**Lemma 3** For any \((a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k\) with \(\gcd(a_1, \ldots, a_k) = 1\), there are \((k - 1)\) elements of \(\mathbb{Z}^k\) which together with \((a_1, a_2, \ldots, a_k)\) determine a matrix in \(SL(k, \mathbb{Z})\).

We sketch the proof of the Lemma for sake of completeness. We use the Bezout Lemma, that for any \((a_1, a_2)\) there are \((x, y)\) with \(\gcd(x, y) = 1\), such that \(a_1x - a_2y = \gcd(a_1, a_2)\). Start with the standard basis \((e_1, \ldots, e_k)\) of \(\mathbb{Z}\) and change it first to \(e_1' = (\frac{a_1}{\gcd(a_1, a_2)} e_1, \frac{a_2}{\gcd(a_1, a_2)} e_2, e_2', e_3 = x e_1 + y e_2, e_3' = e_3, \ldots, e_k' = e_k)\). Then similarly we change the second basis to a new one using the solution of the equation \(\gcd(a_1, a_2)x_2 - a_3y_2 = \gcd(a_1, a_2, a_3)\) with \(e_1'' = \frac{\gcd(a_1, a_2)}{\gcd(a_1, a_2, a_3)} e_1' + \frac{a_3}{\gcd(a_1, a_2, a_3)} e_3' e_2'' = e_2'\) and \(e_3'' = x_2 e_1' + y_2 e_3'\).

We see that at each step the matrix of the basis is in \(SL(k, \mathbb{Z})\) and we can continue until the \(k\)th step will produce the necessary basis.

**Lemma 4** Let \(M \to B\) be a principal complex torus bundle with real characteristic classes \(\{\omega_1, \omega_2, \ldots, \omega_{2k}\}\) over a complex base \(B\), such that \(M\) carries a torus invariant complex structure. If \(A = (a_{ij})\) is a matrix in \(SL(2k, \mathbb{Z})\), then the classes \(\omega_i' = \sum_j a_{ij} \omega_j\) determine equivalent torus bundle equipped with equivalent complex structure on \(M\). In particular one can choose the classes in such a way that the first \(m\) are \(\mathbb{Z}\)-independent and the rest \(2k - m\) are zero.

**Proof:** Let the torus action on \(M\) which gives a bundle with characteristic classes \((\omega_1, \ldots, \omega_{2k})\) is given by \((e^{i\theta_1}, \ldots, e^{i\theta_{2k}}) : M \to M\). There are connection forms \(\eta_1, \ldots, \eta_{2k}\), such that the curvatures are forms in \((\omega_1, \ldots, \omega_{2k})\) and the complex structure on \(M\) is given by the standard construction (see [17]). Then the torus action determined by \((e^{i\theta'}, \ldots, e^{i\theta_{2k}})\), where \(\theta'_i = \sum_j a_{ij}\theta_j\) is free. Topologically the two actions determine the same fibers and the same base. Now consider the forms \(\eta'_1, \ldots, \eta'_{2k}\) given by \(\eta'_i = \sum_j a_{ij}\eta_j\) which provide a connection in the second bundle. Then the characteristic classes are \((\omega'_1, \ldots, \omega'_{2k})\) and the horizontal spaces are the same. It also induces an automorphism of the complex structure on the fiber, since the matrix \((a_{ij})\) is in \(SL(2k, \mathbb{Z})\) and corresponds to an automorphism of the complex torus (the fiber) induced by an automorphism of its underlying lattice. So the two complex structures on \(M\) are equivalent. For the second claim we apply the previous Lemma. q.e.d.

**Proof of Theorem** From D. Guan’s result any compact complex homogeneous space with invariant volume admits a complex torus fiber bundle structure as \(M \to\)
$G/K \times D$ with characteristic classes $(\omega_1 + \alpha_1, \omega_2 + \alpha_2, \ldots \omega_{2k} + \alpha_{2k})$, where $(\omega_1, \ldots, \omega_{2k})$ are the characteristic classes of $G/H \to G/K$ which are $(1,1)$ and $(\alpha_1, \ldots, \alpha_{2k})$ are the characteristic classes of $D_1 \to D$. Notice that by the properties of the averaging map $\mu$ in Section 4 there is a unique $G$-invariant representative in each class $\omega_k$. The second fibration is a complex torus fibration and its (complex) characteristic classes are of type $(2,0)$ with respect to the complex structure on the base $D$. In particular $\alpha_i$ are of type $(2,0)+(0,2)$, i.e they have representatives of this type. From now on we denote the invariant representatives with the same letters. Now the first Chern class of $D$ is a complex torus fibration and its (complex) characteristic classes are of type $(2,0)$ with respect to the complex structure on the base $D$. In particular $\alpha_i$ are of type $(2,0)+(0,2)$, i.e they have representatives of this type. From now on we denote the invariant representatives with the same letters. Now the first Chern class of $M$ vanishes iff $c_1(G/K \times D)$ is represented as a $\mathbb{Z}$-combination of the characteristic classes of $M \to G/K \times D$. Since $c_1(D) = 0$, because of the type, there are integer constants $(d_1, d_2, \ldots, d_{2k})$ such that $d_1\alpha_1 + \ldots + d_{2k}\alpha_{2k} = 0$ and $d_1\omega_1 + \ldots + c_{2k}\omega_{2k} = \rho$, where $\rho = c_1(G/K)$ is the class of the Koszul form of $G/K$. Moreover since $\alpha_1, \ldots, \alpha_{2k}$ are real and imaginary parts of the $(2,0)$-forms $\alpha_{2i+1} + i\alpha_{2i+2}$, then $\alpha_{2i+2}(JX,Y) = \alpha_{2i+1}(X,Y)$ for any tangent vectors $X, Y$. From here we have $-d_2\alpha_1 + d_1\alpha_2 - d_1\alpha_4 + \ldots + d_{2k-1}\alpha_{2k} = 0$. Now we change the basis for the characteristic classes of the fibration $M \to G/K \times D$ by choosing $m\omega'_i = d_1(\omega_1 + \alpha_1) + \ldots + d_{2k}(\omega_{2k} + \alpha_{2k}) = \rho$, where $m$ is $\text{gcd}(d_1, \ldots, d_{2k})$. Then we complete $\omega'_i$ to a basis of characteristic classes $(\omega'_2 + \alpha'_2, \ldots, \omega'_k + \alpha'_k)$ which determine the same bundle $M$ such that $\omega'_i$ are on $G/K$ and $\alpha'_i$ are on $D$. Since $\omega'_1$ is the first Chern class of $G/K$, it defines a bundle $G/H_1 \to G/K$ which is a standard homogeneous CR space and the bundle structure is the Boothby-Wang fibration. To obtain a CYT metric we use the formula (4) for the relation of the Ricci forms in generic torus bundle $X \to B$ over Hermitian manifold $B$:

$$
\rho_X^B = \pi^*(\rho_B^B - \sum g(F, \omega_i)\omega_i)
$$

when $g(F, \omega_i) = \text{constant}$ and $F$ is the fundamental form on $B$. Since $g(F, \alpha_i) = 0$ we need to find $g$ on $G/K \times D$, such that $\rho_{G/K \times D}^B = \sum g(F, \omega_i)(\omega_i + \alpha_i)$. Because $\alpha_i$ is of type $(2,0)+(0,2)$, we have to find $g$ or a basis of characteristic classes, such that $g(F, \omega_i) = 0$ for $i > 1$. Now consider the bundle $G/H \to G/K$ from the Guan’s construction with characteristic classes $(\omega_1, \ldots, \omega_{2k})$. Then $G/H$ is itself a complex manifold of vanishing first Chern class and the same bundle is determined also by some classes $(\omega'_1, \ldots, \omega'_{2k})$, with $\omega'_1 = (1/m)c_1(G/K)$ by Lemma [4]. Then by Theorem [3] and Remark [2] the complex structure on $G/H$ can be deformed to one that admits a homogeneous CYT metric $h$. The deformation preserves the Tits fibration and doesn’t change the complex structure on $G/K$. In particular $\rho_{G/K}^B = \sum g(F, \omega'_i)\omega'_i$ for the fundamental form $F$ of the metric $g$ induced on $G/K$ from $h$. However we can choose the classes $\omega'_i$ to be such that part of them are zero and the rest are linearly independent and represented by invariant forms. Since $\rho^B$ is proportional to $\omega'_1$, all $g(F, \omega'_i) = 0$ for $i \geq 2$ and $\rho^B = g(F, \omega'_1)\omega'_1$. Then the metric we are looking for on $G/K \times D$ is a product of $g$ and any invariant Hermitian metric on $D$. The above
mentioned toric bundle construction produces a CYT metric on $M$.

The last step of the proof is to show that the metric could be chosen so that it is homogeneous. We noticed that $\omega_i$ are chosen $G$-invariant. For the second factor $D$, there is a Lie group $H$ acting transitively, such that the isotropy subgroup is a cocompact lattice. Then by the averaging which is used in [10] one can find $H$-invariant representatives for $\alpha_i$. This will produce a metric which is invariant under the action of some real transitive Lie group obtained as an abelian extension of $G \times H$ which will also leave the volume form invariant.

q. e. d.

7 Topological properties

In this section we discuss the integral cohomology of the space $SU(4)/U(1)$ from Proposition 1 although our considerations work for $SU(2n)/U(1)$ in general. The calculations follow [3] and are based on the classical methods developed by Borel and Serre, see [7]. Let $U(1)$ be the subgroup defined by $(e^{2k\pi it}, e^{2l\pi it}, e^{2m\pi it}, e^{-2(k+l+m)\pi it})$ in $SU(4)$. Assume that $gcd(k, l, m) = 1$ so that the action is free. The first Chern class vanishes if $m = -3k - 2l$, or $U(1)$ acts with weights $(k, l, -3k - 2l, 2k + l)$ by Proposition 1. The embedding induces a projection of the corresponding classifying spaces $\rho : BU(1) \to BSU(4)$. The integral cohomology $H^*(BU(1))$ are identified with the ring of polynomials $\mathbb{Z}[s]$ of one variable of degree 2. Also $H^*(BSU(4))$ is a subring of $H^*(BU(1)^4)$ generated by the elementary symmetric polynomials in 4 variables with the restriction that the first one vanishes. More precisely, if $(t_1, t_2, t_3, t_4)$ are the independent variables corresponding to the generators of $H^*(BU(1)^4)$, then on $H^*(BSU(4))$, $t_4 = -t_1 - t_2 - t_3$ and the generators are

\[
\begin{align*}
  u_4 &= -t_1^2 - t_2^2 - t_3^2 + t_1t_2 + t_2t_3 + t_1t_3 \\
  u_6 &= -2t_1t_2t_3 + t_1^2t_2 + t_1t_2^2 + t_1t_3^2 + t_1t_2t_3 + t_2t_3^2 + t_3^2t_1 + t_2^2t_3 + t_2t_3^2 \\
  u_8 &= -t_1t_2t_3(t_1 + t_2 + t_3)
\end{align*}
\]

As a result the induced map $\rho^*$ on the cohomology has the following characterization:

**Lemma 5** The map $\rho^* : H^*(BSU(4)) \to H^*(BU(1))$ is determined by $\rho^*(u_4) = N_4s^2, \rho^*(u_6) = M_6s^3, \rho^*(u_8) = K_8s^4$ where $M_4 = -k^2 - l^2 - m^2 + kl + lm + kl, N_6 = -2klm + kl(k + l) + lm(l + m) + km(k + m), K_8 = -klm(k + l + m)$

Let $EU(1) \to BU(1)$ be the universal $U(1)$ bundle and $ESU(4) \to BSU(4)$ the universal $SU(4)$-bundle. Consider the following diagram:
Then right column is the principal bundle defining our homogeneous space $X = SU(4)/U(1)$. The low horizontal arrow is a projection with fibre the acyclic space $EU(1) \cong ESU(4)$ so determines an isomorphism on cohomology. Then the space $M = EU(1) \times SU(4)/U(1)$ is included in the diagram:

$$
ESU(4) \times SU(4) \rightarrow SU(4) \\
\downarrow M \rightarrow \downarrow X
$$

The fibres of the horizontal arrows could be identified with $SU(4)$. According to Borel, the spectral sequence $E$ of $M \rightarrow BU(1)$ has second term $E_2 = H^*(BU(1)) \otimes H^*(SU(4))$. The ring $H^*(SU(4))$ is $\Lambda(u_3, u_5, u_7)$ such that $(u_4, u_6, u_8)$ are images under the transgression of $(u_3, u_5, u_7)$ in the universal bundle $ESU(4) \rightarrow BSU(4)$.

**Lemma 6** The nonzero differentials of the spectral sequence of the bundle $M \rightarrow BU(1)$ are given by

$$d_j(1 \otimes u_i) = 0, j \leq i$$

$$d_{j+1}(1 \otimes u_j) = \rho^*(u_{j+1}) \otimes 1$$

*Proof:* Consider the diagram:

$$
M = ESU(4) \times SU(4)/U(1) \rightarrow ESU(4) \\
\downarrow BU(1) \rightarrow BSU(4)
$$

The projection of the lower row is $\rho$ and it is covered by a map which induces the identity on the cohomology of the fibers. The nonzero differentials of the universal bundle $ESU(4) \rightarrow BSU(4)$ are given by:

$$d_j(1 \otimes u_i) = 0, j \leq i$$

$$d_{j+1}(1 \otimes u_j) = u_{j+1} \otimes 1$$

Then the Lemma follows by naturality. q. e. d.

From the previous two lemmas one can use $E$ to compute the cohomology of $M$ and hence of $X$. First notice that $E_2 = E_3 = E_4$. Since $d_4(1 \otimes u_3) = M_4s^2 \otimes 1$ by Lemma 6 it follows that $Imd_4 = \langle M_4s^2 \rangle \otimes 1$ and $Kerd_4 = H^*(BU(1)) \otimes \langle 1, u_5, u_7 \rangle$.
where the last notation means the ideal generated by the decomposable elements with first factor in $H^*(BU(1)) = \mathbb{Z}[s]$ and second factor one of the $1, u_5$ or $u_7$. Hence $E_5 = \text{Kerd}_4/\text{Imd}_4 = \mathbb{Z}[s]/(M_4 s^2) \otimes < 1, u_5, u_7 >$. Since again $E_5 = E_6$, the next step to consider is $d_6(1 \otimes u_5) = N_6 s^3 \otimes 1$. Now assume that $gcd(M_4, N_6) = L$. Then $1 \otimes (M_4/L)u_5 \in \text{Kerd}_6$ and $d_6$ acts on $E_6$. So $\text{Kerd}_6 = \mathbb{Z}[s]/(M_4 s^2) \otimes < 1, (M_4/L)u_5, u_7 >$ and $\text{Imd}_6 = < N_6 s^3 \otimes 1 >$. By the assumption, $< M_4 s^2, N_6 s^3 > = < M_4 s^2, Ls^3 >$, since there are $a$ and $b$ such that $aM_4 + bN_6 = L$. Then we have $\text{Kerd}_6/\text{Imd}_6 = \mathbb{Z}[s]/< M_4 s^2, Ls^3 > \otimes < 1, (M_4/L)u_5, u_7 >= E_7 = E_8$. We can also see that $d_8(1 \otimes u_7) = K_8 s^4 \otimes 1 = 0$ in $E_8$ if $gcd(L, K_8) = gcd(M_4, N_6, K_8) = 1$. Then all other differentials vanish and $E_{\infty} = E_7$. Now if we denote by $w = s \otimes 1, v_5 = 1 \otimes (M_4/L)u_5, v_7 = 1 \otimes u_7$, then we have:

**Theorem 9** Let $k, l$ are relatively prime such that $M_4 = 13k^2 + 7l^2 + 16kl, N_6 = 6k^3 + 2l^3 + 26kl^2 + 18kl^2, K_8 = 6k^3l + 2kl^3 + 7k^2l^2$ are also relatively prime and $X = SU(4)/U(1)$ is given by the action of $U(1)$ with weights $(k, l, -3k - 2l, 2k + l)$. The cohomology ring $H^*(M, \mathbb{Z})$ is generated by $w \in H^2, v_5 \in H^3, v_7 \in H^7$ with the relations $(M_4/L)w^2 = w^3 = w^5v_5 = w^2v_7 = 0$ where $L = gcd(M_4, N_6)$. In particular the nonzero cohomology are $H^2 = \mathbb{Z}, H^4 = \mathbb{Z}_{[M_4/L]}, H^7 = \mathbb{Z}_2, H^8 = H^9 = H^{10} = H^{12} = H^{14} = \mathbb{Z}$

The choice of the weights is such that $c_1(X) = 0$. From here and the condition $c_1 = 0$, we see that all rational Chern numbers and hence all rational Pontriagin numbers of $M$ vanish. A simple considerations regarding the relations of the characteristic classes in the Tits fibration show that only $c_2$ and $p_1$ could be nonzero but torsional. When we vary $k$ and $l$ we obtain:

**Corollary 1** There are infinitely many simply-connected non-flat CYT manifolds of dimension 14 with same Hodge and Betti numbers and torsional Chern classes.

**Proof:** The curvature of the space $SU(4)/U(1)$ is the curvature of the canonical connection which is given in [24]. It vanishes on $SU(4)$ but one can check directly that is nonzero on the quotient. The statement for the Betti numbers follow from Theorem 9. The Hodge numbers for such quotients are calculated in [19], Proposition 5.1 and one can check directly that they are equal. The only part which remains to be proven is that there are infinitely many choices of numbers $k$ and $l$ as in the Theorem 9 for which $|M_4/L|$ is unbounded. First take $l = 1$ and notice that $S = gcd(M_4, K_8) = gcd(13k^2 + 16k + 7, 6k^3 + 7k^2 + 2k)$ divides $5k + 16$. There are infinitely many $k$ for which this number is prime, which easily gives that $S = 1$. Then $gcd(M_4, N_6, K_8) = 1$ and the conditions of Theorem 9 are met. Similarly one can check that $L$ divides a linear polynomial of $k$, so $M_4/L$ will grow infinitely. q. e. d.
Remark 5 The Dolbeaut cohomology of the compact complex parallelizable manifolds are subject of intensive investigation. In general $H^{(p,q)}(G/\Gamma) = \Lambda^q(g^+) \otimes H^{(0,\cdot)}(G/\Gamma)$, where $g^+$ is the $+\sqrt{-1}$-eigenspace of the Lie algebra of $G$. The second factor depends both on $G$ and $\Gamma$. In case $G = SL(2, \mathbb{C})$ there are different $\Gamma$ which lead to quotients with different cohomology rings and different types of Kuranishi spaces of deformations [20].

8 Relation to the Strominger’s equations in heterotic string theory

In 1986 A. Strominger [27] analyzed heterotic superstring background with spacetime supersymmetry. His model is based on Hermitian manifolds which are CYT spaces with holomorphically trivial canonical bundle. In terms of Hermitian geometry it consists of conformally balanced complex 3-manifold with holomorphic $(3,0)$-form of constant norm and an anomaly cancelation condition. The manifold is endowed with an auxiliary semistable bundle with Hermitian-Einstein connection $A$ with curvature $F_A$ and the anomaly cancelation condition is:

$$dH = 2i\partial\bar{\partial}F = d\iota F = \frac{\alpha'}{4}[\text{tr}(R \wedge R) - \text{tr}(F_A \wedge F_A)]$$

for the Kähler form $F$. Here $R$ is the curvature of some metric connection. In fact any metric connection with curvature $R$ solving the anomaly cancelation condition with $\alpha' > 0$ leads to a physically meaningful solution as noted in [5]. The first solutions on non-Kähler manifolds of this system were constructed only recently by J. Fu and S.T. Yau [11].

We start here with solutions of the Strominger’s system in which the anomaly cancellation equation have trivial instanton $F_A = 0$. Later we provide also a nontrivial instanton solutions. Let $\{e^1, Je^1, e^2, Je^2, e^3, Je^3\}$ be an ordered unitary co-basis for a complex structure $J$ and Hermitian metric $g$ on a vector space. Consider the Lie algebra whose sole non-trivial structure equation is as follow.

$$d(Je^3) = e^1 \wedge Je^1 - e^2 \wedge Je^2.$$

With the given ordered basis, we could consider this algebra as $(0, 0, 0, 0, 12 - 34)$. If we change the order of the basis to $\{e^1, Je^1, Je^2, e^2, e^3, Je^3\}$, we obtain the “standard basis” in classification of six-dimensional nilpotent algebras. The structure equation is $(0, 0, 0, 0, 12 + 34)$. As a real Lie algebra, it is the direct sum of a real five-dimensional Heisenberg algebra and a one-dimensional trivial algebra.
Since $d(e^3 + iJe^3) \in \Lambda^{(1,1)}$ and the exterior differential of $e^1 + iJe^1$ and $e^2 + iJe^2$ are equal to zero, the complex structure is integrable. As $d\Lambda^{(1,0)} \subset \Lambda^{(1,1)}$, the complex structure is abelian.

A direct calculation gives,

$$ F^2 = -2 \sum_{1 \leq j < k \leq 3} ( (e^j + iJe^j) \wedge (e^j - iJe^j) \wedge (e^k + iJe^k) \wedge (e^k - iJe^k) ) $$

$$ = 8 \sum_{1 \leq j < k \leq 3} (e^j \wedge Je^j \wedge e^k \wedge Je^k). $$

Therefore,

$$ dF^2 = 8 \sum_{1 \leq j < k \leq 3} d(e^j \wedge Je^j \wedge e^k \wedge Je^k) $$

$$ = -8 (e^1 \wedge Je^1 \wedge e^3 \wedge dJe^3 + e^2 \wedge Je^2 \wedge e^3 \wedge dJe^3) $$

$$ = -8 \left( -e^1 \wedge Je^1 \wedge e^3 \wedge (e^2 \wedge Je^2) + e^2 \wedge Je^2 \wedge e^3 \wedge (e^1 \wedge Je^1) \right) $$

$$ = 0 $$

so the metric is balanced. Moreover the form $(e^1 + iJe^1) \wedge (e^2 + iJe^2) \wedge (e^3 + iJe^3)$ is easily seen to be closed and holomorphic.

Next, note that $d(Je^3)$ is a type (1,1)-form, we have $Jd(Je^3) = dJe^3$. Therefore,

$$ dd^c F = dJdJF = dJdF $$

$$ = 2dJd(e^3 \wedge Je^3) = 2dJ(-e^3 \wedge dJe^3) $$

$$ = -2d(Je^3 \wedge JdJe^3) $$

$$ = -2d(Je^3 \wedge dJe^3) $$

$$ = -2d(Je^3) \wedge (dJe^3). $$

Now we take a connection form $\omega$ to be

$$ \omega = \begin{pmatrix}
0 & aJe^3 & 0 & 0 & 0 & 0 \\
-aJe^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. $$

with $a = constant$. Then we obtain $tr(R \wedge R) = -a^2 (dJe^3)^2 = +a^2 dd^c F = 2a^2 i\partial\overline{\partial} F$. So this provides a solution to the Strominger’s system.
Now we can also construct an instanton bundle which still satisfies the anomaly cancellation condition. We take any Hermitian Yang-Mills connection on a holomorphic vector bundle on the base 4-tori. To ensure its existence we may assume that the base is an abelian surface. Then we can take its pull-back to $M$. The curvature satisfies $\text{tr}(F \wedge F) = b e^1 \wedge J e^1 \wedge e^2 \wedge J e^2$ for some function $b$. If we take the bundle to be homogeneous, $b = \text{constant}$. So $d d^c F = (\frac{1}{a^2} - b) e^1 \wedge J e^1 \wedge e^2 \wedge J e^2$. By choosing $a$ sufficiently small, we have a solution for the anomaly cancellation condition with positive $\alpha'$.

Remark 6 The solution of [11] uses the Chern connection and produces a nonconstant dilaton. The connection is chosen because the curvature term has appropriate type $(2,2)$. In physics a preferred choice of connection is a metric non-Hermitian connection with skew-symmetric torsion equal to the negative of the torsion of the Bismut connection. Unfortunately the example above doesn’t provide a solution for this connection since for its curvature $R$ one obtains $\text{tr}(R \wedge R) = 0$.

The compact factors of $H_5 \times \mathbb{R}$ are toric bundles over 4-tori. For such there is a vanishing theorem in [3] whenever the contribution of $\text{tr}(R \wedge R)$ is zero. After the results of this section were reported the paper [9] appeared, where similar solutions of the Strominger’s equations were given on many 6-dimensional nilmanifolds, using different invariant connection and instanton. Their solutions, like ours have constant dilaton and instanton with vanishing Euler number. From physics point of view more realistic models require this Euler number to be six. Such non-Kähler solutions are still unknown. It is worth to notice that the 3-dimensional compact complex parallelizable manifolds admit solutions of the Strominger’s equations, but with $\alpha' < 0$. The role of such solutions is unclear and we don’t provide any details here.

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