GROMOV-WITTEN INVARIANTS OF TORIC CALABI-YAU THREEFOLDS

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ABSTRACT. Based on the large $N$ duality relating topological string theory on Calabi-Yau 3-folds and Chern-Simons theory on 3-manifolds, M. Aganagic, A. Klemm, M. Mariño and C. Vafa proposed the topological vertex, an algorithm on computing Gromov-Witten invariants in all genera of any non-singular toric Calabi-Yau 3-fold. In this expository article, we describe the mathematical theory of the topological vertex developed by J. Li, K. Liu, J. Zhou, and the author.

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1. Gromov-Witten invariants of Calabi-Yau 3-folds

1.1. Symplectic and algebraic Gromov-Witten invariants. We start with a general setting. Let \((X, \omega, J)\) be a compact Kähler manifold, where \(\omega\) is the Kähler form, and \(J\) is the complex structure. For our purpose, we may assume that \(X\) is a projective manifold, i.e., a compact complex submanifold of some complex projective space \(\mathbb{P}^m\), and \(\omega\) is the restriction of the Fubini-Study Kähler form on \(\mathbb{P}^m\). In this case, one may work in complex algebraic geometry.

Intuitively, symplectic Gromov-Witten invariants count parametrized holomorphic curves in \(X\). When \(X\) is projective, algebraic Gromov-Witten invariants count parametrized complex algebraic curves in \(X\), and should coincide with symplectic Gromov-Witten invariants.

1.2. Moduli space of stable maps. Gromov-Witten invariants can be viewed as intersection numbers on moduli spaces of parametrized holomorphic (complex algebraic) curves in \(X\). Let \(\mathcal{M}_{g,0}(X, \vec{d})\) be the moduli space of holomorphic maps (morphisms) \(f : C \to X\), where \(C\) is a compact Riemann surface (smooth complex algebraic curve) of genus \(g\), and \(f_*[C] = \vec{d} \in H_2(X; \mathbb{Z})\). We call \(\vec{d}\) the degree of the map. Two maps are equivalent if they differ by an automorphism of the domain \(C\).

To do intersection theory, we should compactify \(\mathcal{M}_{g,0}(X, \vec{d})\). The standard compactification in Gromov-Witten theory is \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\), the Kontsevich’s moduli space of stable maps \(f : C \to X\) of genus \(g\), degree \(\vec{d}\) [20], where the domain curve \(C\) has at most nodal singularities, and the map \(f\) is stable in the sense that the automorphism group of \(f\) is finite. When \(X\) is projective, the compactified moduli space \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\) is a proper Deligne-Mumford stack (in algebraic geometry), or a compact, Hausdorff, singular orbifold (in differential geometry). (See [6,12].) Roughly, “proper” corresponds to “compact and Hausdorff”, and “Deligne-Mumford stack” corresponds to “singular orbifold”. In our context, a singular orbifold is a space which is locally of the form \(V/\Gamma\), where \(V\) is the zero locus of polynomials defined on an open set in \(\mathbb{C}^N\), and \(\Gamma\) is a finite group acting on \(V\).

The moduli space \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\) is not a smooth manifold, so it does not have a tangent bundle. However, it has a virtual tangent bundle which is the difference \(E_0 - E_1\) of two complex vector bundles \(E_0\) and \(E_1\) over \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\). The virtual (complex) dimension of \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\) is defined to be the rank of the virtual tangent bundle:

\[
\text{vir. dim.} = \text{rank}_\mathbb{C} E_0 - \text{rank}_\mathbb{C} E_1 = \int_{\vec{d}} c_1(T_X) + (\dim_\mathbb{C} X - 3)(1 - g).
\]

The structure of a virtual tangent bundle in symplectic Gromov-Witten theory corresponds to the structure of a perfect obstruction theory in algebraic Gromov-Witten theory. The rank of the virtual tangent bundle is the virtual dimension of the perfect obstruction theory.

1.3. Gromov-Witten invariants of compact Calabi-Yau 3-folds. When \(X\) is a Calabi-Yau \(n\)-fold, in the sense that \(K_X = \Lambda^n T^* X\) is a trivial holomorphic line bundle over \(X\), we have \(c_1(T_X) = 0\). By the formula (1), the virtual dimension of \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\) is \((n - 3)(1 - g)\), which is independent of the degree \(\vec{d}\). In particular, when \(X\) is a Calabi-Yau 3-fold, the virtual dimension of \(\overline{\mathcal{M}}_{g,0}(X, \vec{d})\) is zero for any
genus $g$ and any degree $\vec{d}$. In this case, there is a virtual fundamental class
\[
[M_{g,0}(X, \vec{d})]^{vir} \in H_0(M_{g,0}(X, \vec{d}); \mathbb{Q}).
\]

The virtual fundamental class has been constructed in a much more general setting by Li-Tian [32], Behrend-Fantechi [5] in algebraic Gromov-Witten theory, and by Li-Tian [33], Fukaya-Ono [11], Ruan [39], Siebert [52] (more recently, Hofer-Wysocki-Zehnder [16, 17, 18]) in symplectic Gromov-Witten theory.

The genus $g$, degree $\vec{d}$ Gromov-Witten invariant of a Calabi-Yau 3-fold $X$ is defined by
\[
N_{g,\vec{d}}^X = \int [M_{g,0}(X, \vec{d})]^{vir}.
\]

where $\int$ stands for the pairing between $H_0(M_{g,0}(X, \vec{d}); \mathbb{Q})$ and $H^0(M_{g,0}(X, \vec{d}); \mathbb{Q})$. If $M_{g,0}(X, \vec{d})$ were a compact complex manifold of dimension zero, it would consist of finitely many points, and the right hand side of (2) would be the number of points in $M_{g,0}(X, \vec{d})$. In general, $M_{g,0}(X, \vec{d})$ can be singular and can have positive actual dimension. Then the right hand side of (2) defines the “virtual number” of points in $M_{g,0}(X, \vec{d})$. It is usually a rational number instead of an integer because $M_{g,0}(X, \vec{d})$ is an orbifold. For example, if a map has an automorphism group of order 2, we count it as one half of a map instead of one map. This is how fractional numbers arise.

To summarize, Gromov-Witten invariants are defined for any smooth projective Calabi-Yau 3-folds, or more generally, any compact Kähler Calabi-Yau 3-folds, for any genus and any degree.

1.4. Gromov-Witten invariants of noncompact Calabi-Yau 3-folds. The construction of the virtual fundamental class requires two properties of the moduli space $\overline{M}_{g,0}(X, \vec{d})$: compactness (properness) and the structure of a virtual tangent bundle (perfect obstruction theory). When $X$ is not compact, the moduli space $\overline{M}_{g,0}(X, \vec{d})$ is usually noncompact, but still equipped with a virtual tangent bundle (perfect obstruction theory). Therefore, if $X$ is noncompact but $\overline{M}_{g,0}(X, \vec{d})$ is compact for a particular genus $g$ and degree $\vec{d}$, then the Gromov-Witten invariant $N_{g,\vec{d}}^X$ is defined for the particular genus $g$ and degree $\vec{d}$.

Example 1. Let $X$ be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then $X$ is a noncompact Calabi-Yau 3-fold. We have $H_2(X; \mathbb{Z}) \cong H_2(\mathbb{P}^1; \mathbb{Z}) = \mathbb{Z}[\mathbb{P}^1]$, where $[\mathbb{P}^1]$ is the class of the zero section. Any nonconstant holomorphic map from a compact Riemann surface to $X$ factors through the embedding $i_0 : \mathbb{P}^1 \to X$ by the zero section. Therefore, when $d \neq 0$, the following two moduli spaces are identical as topological spaces (or Deligne-Mumford stacks):
\[
\overline{M}_{g,0}(X, d[\mathbb{P}^1]) = \overline{M}_{g,0}(\mathbb{P}^1, d[\mathbb{P}^1]).
\]

(They both are empty when $d < 0$.) However, they have different virtual tangent bundles (perfect obstruction theories). The virtual dimension of $\overline{M}_{g,0}(X, d[\mathbb{P}^1])$ is 0 while that of $\overline{M}_{g,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ is $2(d + g - 1)$. For any $d > 0$, the right hand side of (3) is compact because $\mathbb{P}^1$ is compact. So $N_{g,d[\mathbb{P}^1]}^X$ is defined for any genus $g$ and any $d \neq 0$, and is 0 when $d < 0$. 


Example 2. Let $X$ be the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$. Then $X$ is a noncompact Calabi-Yau 3-fold. We have $H_2(X; \mathbb{Z}) \cong H_2(\mathbb{P}^2; \mathbb{Z}) = \mathbb{Z}\ell$, where $\ell$ is the class of a projective line $\mathbb{P}^1$ in the zero section $\mathbb{P}^2$. Any nonconstant holomorphic map from a compact Riemann surface to $X$ must factor through the embedding $i_0 : \mathbb{P}^2 \to X$ by the zero section. Therefore, when $d \neq 0$, the following two moduli spaces are identical as topological spaces (or Deligne-Mumford stacks):

\begin{equation}
\overline{M}_{g,0}(X, d\ell) = \overline{M}_{g,0}(\mathbb{P}^2, d\ell).
\end{equation}

(They both are empty when $d < 0$.) However, they have different virtual tangent bundles (perfect obstruction theories). The virtual dimension of $\overline{M}_{g,0}(X, d\ell)$ is 0 while that of $\overline{M}_{g,0}(\mathbb{P}^2, d\ell)$ is $3d + g - 1$. For any $d > 0$, the right hand side of (4) is compact because $\mathbb{P}^2$ is compact. So $N^X_{g, d\ell}$ is defined for any genus $g$ and any $d \neq 0$, and is 0 when $d < 0$.

Example 1 and Example 2 are examples of toric Calabi-Yau 3-folds. A Calabi-Yau 3-fold $X$ is toric if it contains $(\mathbb{C}^*)^3$ as an open dense subset, and the action of $(\mathbb{C}^*)^3$ on itself extends to $X$. In Example 1 removing $\infty$ from $\mathbb{P}^1$, we obtain a rank 2 holomorphic vector bundle over $\mathbb{P}^1 - \{\infty\} = \mathbb{C}$, which must be the trivial bundle, so the total space is $\mathbb{C} \times \mathbb{C}^2 = \mathbb{C}^3$. In Example 2 removing the line at infinity, we obtain a holomorphic line bundle over $\mathbb{P}^2 - \mathbb{P}^1 \cong \mathbb{C}^2$, which must be the trivial line bundle, so the total space is $\mathbb{C}^2 \times \mathbb{C} = \mathbb{C}^3$. In both examples, the inclusions $(\mathbb{C}^*)^3 \subset \mathbb{C}^3 \subset X$ are open and dense, and the $(\mathbb{C}^*)^3$-action on itself extends to $X$.

2. Traditional Algorithm in the Toric Case

The traditional algorithm (the algorithm before the “topological vertex”) of computing Gromov-Witten invariants of toric Calabi-Yau 3-folds consists of two steps:

T1. Localization reduces Gromov-Witten invariants of toric Calabi-Yau 3-folds to Hodge integrals, which are intersection numbers on moduli spaces of curves.

T2. Hodge integrals can be computed recursively.

We will explain these two steps in this section.

2.1. Localization. The $(\mathbb{C}^*)^3$-action on $X$ induces a $(\mathbb{C}^*)^3$-action on the moduli space $\overline{M}_{g,0}(X, d\vec{d})$ by moving the image of a stable map. Let $T$ be a subtorus of $(\mathbb{C}^*)^3$ such that

$$\overline{M}_{g,0}(X, d\vec{d})^T = \overline{M}_{g,0}(X, d\vec{d})(\mathbb{C}^*)^T$$

where the left (resp. right) hand side is the set of $T$ (resp. $(\mathbb{C}^*)^3$) fixed points in $\overline{M}_{g,0}(X, d\vec{d})$. We have

\begin{equation}
N^X_{g, d} = \int_{[\overline{M}_{g,0}(X, d\vec{d})]^T} \frac{1}{e_T(N_{F}^{vir})} \frac{1}{e_T(N_{F}^{vir})} \frac{1}{e_T(N_{F}^{vir})}
\end{equation}

where the first equality is the definition (2), and the second equality follows from the virtual localization formula proved by Graber-Pandharipande [15]. The sum is over connected components $F$ of the fixed points set $[\overline{M}_{g,0}(X, d\vec{d})]^T = \overline{M}_{g,0}(X, d\vec{d})(\mathbb{C}^*)^T$. $[F]^{vir}$ is the virtual fundamental class of $F$ and $e_T(N_{F}^{vir})$ is the $T$-equivariant Euler class of the virtual normal bundle $N_{F}^{vir}$ of $F$ in $\overline{M}_{g,0}(X, d\vec{d})$. More explicitly, the virtual tangent bundle of $\overline{M}_{g,0}(X, d\vec{d})$ is $T$-equivariant: it is of the form $E_0 - E_1$, where $E_0$ and $E_1$ are $T$-equivariant complex vector bundles on $\overline{M}_{g,0}(X, d\vec{d})$. For
The \( \lambda \) is the virtual normal bundle of \( F \), respectively, so that \( E^i \) is the fixed and moving parts of \( E_i \). The \( T^\text{vir} \) of \( F \) in \( \overline{\mathcal{M}}_{g,0}(X,d) \).

If \( \overline{\mathcal{M}}_{g,0}(X,d) \) were a compact complex manifold, and each \( F \) were a compact complex submanifold, then \( [F]^{\text{vir}} \) would be the usual fundamental class \( [F] \) of \( F \), and the second equality in (5) would be the classical Atiyah-Bott localization formula [1].

In our case, \( F \) is (up to finite cover) a product of moduli spaces of stable curves (see Section 2.2 below), which are smooth orbifolds (smooth Deligne-Mumford stacks), so it has a fundamental class \( [F] \in H_*(F;\mathbb{Q}) \). We have \( [F] = [F]^{\text{vir}} \), and

\[
\int_{[F]^{\text{vir}}} \frac{1}{e_T(E^m)^n} = \int_{[F]} e_T(E^m).
\]

The integral on the right hand side of (6) can be expressed in terms of Hodge integrals. The definition of Hodge integrals will be reviewed in the next subsection.

2.2. Hodge integrals. Let \( \overline{\mathcal{M}}_{g,n} \) be the Deligne-Mumford compactification of the moduli space of complex algebraic curves of genus \( g \) with \( n \) marked points. A point in \( \overline{\mathcal{M}}_{g,n} \) is represented by \( (C,x_1,\ldots,x_n) \), where \( C \) is a connected complex algebraic curve of arithmetic genus \( g \) with at most nodal singularities, \( x_1,\ldots,x_n \) are distinct smooth points on \( C \), and \( (C,x_1,\ldots,x_n) \) is stable in the sense that its automorphism group is finite. When \( C \) is smooth, it can be viewed as a connected compact Riemann surface of genus \( g \). The moduli space \( \overline{\mathcal{M}}_{g,n} \) is a proper, smooth Deligne-Mumford stack (in algebraic geometry), or a compact, complex, smooth orbifold (in differential geometry), of complex dimension \( 3g-3+n \). It is empty when \( 3g-3+n < 0 \).

The Hodge bundle \( E \) is a rank \( g \) vector bundle over \( \overline{\mathcal{M}}_{g,n} \) whose fiber over the moduli point \( [(C,x_1,\ldots,x_n)] \) is \( H^0(C,\omega_C) \), where \( \omega_C \) is the dualizing sheaf over \( C \). When \( C \) is smooth, \( C \) can be viewed as a compact Riemann surface and \( H^0(C,\omega_C) \) is the space of holomorphic 1-forms on \( C \). The \( \lambda \) classes are the Chern classes of the Hodge bundle:

\[
\lambda_j = c_j(E) \in H^{2j}(\overline{\mathcal{M}}_{g,n};\mathbb{Q}), \quad j = 1,\ldots,g.
\]

The cotangent line \( T^*_x C \) of \( C \) at the \( i \)-th marked point \( x_i \) gives rise to a line bundle \( L_i \) over \( \overline{\mathcal{M}}_{g,n} \). The \( \psi \) classes are the first Chern classes of these line bundles:

\[
\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q}), \quad i = 1,\ldots,n.
\]

The \( \lambda \) and \( \psi \) classes lie in \( H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q}) \) instead of \( H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Z}) \) because \( E \) and \( L_i \) are orbibundles on the orbifold \( \overline{\mathcal{M}}_{g,n} \).

Hodge integrals are intersection numbers of \( \lambda \)-classes and \( \psi \)-classes on \( \overline{\mathcal{M}}_{g,n} \):

\[
\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_1^{b_1} \cdots \lambda_g^{b_g} \in \mathbb{Q}.
\]

The \( \psi \) integrals (also known as descendant integrals)

\[
\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}
\]
can be computed recursively by the Witten’s conjecture \[53\]. Witten’s conjecture was first proved by Kontsevich \[25\], and there are now several alternative proofs \[42, 40, 24, 23, 7\]. Using Mumford’s Grothendieck-Riemann-Roch calculations in \[41\], Faber showed in \[9\] that general Hodge integrals \(7\) can be uniquely reconstructed from descendant integrals \(8\).

3. Physical Theory of the Topological Vertex

Based on the large \(N\) duality \[13\] between the topological string theory on Calabi-Yau 3-folds and the Chern-Simons theory on 3-manifolds, Aganagic, Klemm, Mariño, and Vafa proposed the topological vertex \[2\], an algorithm of computing Gromov-Witten invariants in all genera of any smooth toric Calabi-Yau 3-folds. Their algorithm can be summarized in the following three steps.

O1. Topological vertex. There exist certain open Gromov-Witten invariants that count holomorphic maps from bordered Riemann surfaces to \(\mathbb{C}^3\) with boundaries mapped to three Lagrangian submanifolds \(L_1, L_2, L_3\) (see Figure 1). Such invariants depend on the following discrete data:

(i) the topological type of the domain, classified by the genus and the number of boundary circles;

(ii) the topological type of the map, described by a triple of partitions \(\vec{\mu} = (\mu^1, \mu^2, \mu^3)\) where \(\mu^i = (\mu^i_1, \mu^i_2, \ldots)\) are degrees (“winding numbers”) of boundary circles in \(L_i \cong S^1 \times \mathbb{C}\);

(iii) the “framing” \(n_i \in \mathbb{Z}\) of the Lagrangian submanifolds \(L_i\) \((i = 1, 2, 3)\). The topological vertex

\[ C_{\vec{\mu}}(\vec{n}; \lambda) \]

is a generating function of such invariants where one fixes the winding numbers \(\vec{\mu} = (\mu^1, \mu^2, \mu^3)\) and the framings \(\vec{n} = (n_1, n_2, n_3)\) and sums over the genus of the domain. It can be viewed as local open Gromov-Witten invariants of \(D_1 \cup D_2 \cup D_3\) embedded in \((\mathbb{C}^3, L_1 \cup L_2 \cup L_3)\) as in Figure 1.

\[ \begin{array}{c}
\text{z}_2\text{-axis} \\
\text{D}_1 \quad \text{D}_2 \quad \text{D}_3
\end{array} \]

\[ \begin{array}{c}
\text{L}_1 \cong S^1 \times \mathbb{C} \\
\text{L}_2 \\
\text{L}_3
\end{array} \]

\[ \begin{array}{c}
\text{z}_1\text{-axis} \\
\text{z}_3\text{-axis}
\end{array} \]

Figure 1. \(D_i\) is a holomorphic disk in \(\mathbb{C}^3\) with boundary in the Lagrangian submanifold \(L_i\): \(D_1 = \{(z_1, 0, 0) \mid z_1 \in \mathbb{C}, |z_1| \leq 1\}\), and \(L_1 = \{(\sqrt{1 + |u|^2} e^{i\theta}, u, e^{-i\theta} \bar{u}) \mid e^{i\theta} \in S^1, u \in \mathbb{C}\}\). \(D_2, D_3\) and \(L_2, L_3\) can be obtained from \(D_1\) and \(L_1\) by cyclic permutation of the three coordinates \(z_1, z_2, z_3\).
Gluing algorithm. Any toric Calabi-Yau 3-fold $X$ can be constructed by gluing $\mathbb{C}^3$ charts. The Gromov-Witten invariants of $X$ can be expressed in terms of local open Gromov-Witten invariants $C_{\vec{\mu}}(\lambda; n)$ of $\mathbb{C}^3$ by explicit gluing algorithm.

Closed formula. By the large $N$ duality, the topological vertex is given by

$$C_{\vec{\mu}}(\lambda; n) = q^{\frac{1}{2} \sum_{i=1}^3 \kappa_{\mu_i} n_i} W_{\vec{\mu}}(q), \quad q = e^{\sqrt{-1} \lambda},$$

where $\kappa_{\mu} = \sum \mu_i (\mu_i - 2i + 1)$ for a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$, and $W_{\vec{\mu}}(q)$ is a combinatorial expression related to the Chern-Simons link invariants. (Precise definition of $W_{\vec{\mu}}(q)$ will be given in Section 4.9.) The left hand side of (9) is an infinite series while the right hand side of (9) is a finite sum.

The above algorithm is significantly more efficient than the traditional algorithm described in Section 2. For example, to compute the degree 2 invariants of the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$ using the traditional algorithm, one computes one genus at a time; as the genus increases, the computations soon become too complicated to do by hand, so one needs the aid of a computer. On the other hand, using the algorithm of the topological vertex, one can compute the generating function of degree 2 invariants in all genera by hand.

Assuming O1 and the validity of open string virtual localization, Diaconescu and Florea related $C_{\vec{\mu}}(\lambda; n)$ (at certain fractional $n_i$) to Hodge integrals, and derived the gluing algorithms in O2 by localization [8].

When the toric Calabi-Yau threefold is the total space of the canonical line bundle $K_S$ of a toric surfaces $S$ (e.g. $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$), only $\tilde{C}_{\vec{\mu}}^{1, 2}$ are required to evaluate their Gromov-Witten invariants. The algorithm in this case was described by Aganagic-Mariño-Vafa [3]; an explicit formula was given by Iqbal [19] and derived by Zhou by localization, assuming a formula of Hodge integrals [54, 55].

4. MATHEMATICAL THEORY OF THE TOPOLOGICAL VERTEX

J. Li, K. Liu, J. Zhou and the author developed a mathematical theory of the topological vertex [30] based on relative Gromov-Witten theory. The relative Gromov-Witten theory has been developed in symplectic geometry by Li-Ruan [27] and Ionel-Parker [20, 21]. In our context, we need to use the algebraic version developed by J. Li [28, 29]. Our algorithm can be summarized as follows.

R1. We defined formal relative Gromov-Witten invariants for relative formal toric Calabi-Yau (FTCY) 3-folds. These invariants are refinements and generalizations of Gromov-Witten invariants of smooth toric Calabi-Yau 3-folds.

R2. Formal relative Gromov-Witten invariants satisfy the degeneration formula. In particular, they can be expressed in terms of $\tilde{C}_{\vec{\mu}}(\lambda; n)$, formal relative Gromov-Witten invariants of an indecomposable relative FTCY 3-fold. The degeneration formula agrees with the gluing formula in O2, with $C_{\vec{\mu}}(\lambda; n)$ replaced by $\tilde{C}_{\vec{\mu}}(\lambda; n)$.

R3. $\tilde{C}_{\vec{\mu}}(\lambda; n) = q^{\frac{1}{2} \sum_{i=1}^3 \kappa_{\mu_i} n_i} \tilde{W}_{\vec{\mu}}(q)$, where $\tilde{W}_{\vec{\mu}}(q)$ is a combinatorial expression in terms of representations of symmetric groups. (The precise definition of $\tilde{W}_{\vec{\mu}}(q)$ will be given in Section 4.7.)

We will describe this algorithm in detail in the remainder of this section.
4.1. Locally planar trivalent graph. Let $X$ be a smooth toric Calabi-Yau 3-fold $X$. For $k = 0,1,2,3$, define $X^k$, the $k$-skeleton of $X$, to be the union of $k$-dimensional $(\mathbb{C}^*)^3$ orbit closures. Then

$$X^0 \subset X^1 \subset X^2 \subset X^3 = X$$

where $X^0 = X(\mathbb{C})^3$, the set of $(\mathbb{C}^*)^3$ fixed points in $X$.

When $X^0$ is nonempty, we may choose a distinguished rank 2 subtorus $T$ of $(\mathbb{C}^*)^3$ as follows. Pick any $(\mathbb{C}^*)^3$ fixed point $p \in X^0$. Then $(\mathbb{C}^*)^3$ acts on $T_pX$, and its representation on $\Lambda^3T_pX \cong \mathbb{C}$ gives a nontrivial irreducible character $\alpha : (\mathbb{C}^*)^3 \rightarrow \mathbb{C}^*$. Define $T = \text{Ker}(\alpha) \cong (\mathbb{C}^*)^2$. Note that the definition is independent of the choice of the fixed point $p$ because $\Lambda^3TX = K_X^{-1}$ is a trivial line bundle over $X$.

$X^1$ is a configuration of rational curves. For example, let $X$ be the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Then $X^1$ is the union of a projective line $C_0 \cong \mathbb{P}^1$ and four complex lines $E_i \cong \mathbb{C}$, $i = 1,2,3,4$, as shown on the left hand side of Figure 2. $C_0 \cong \mathbb{P}^1$ is the zero section, $p_0,p_1 \in C_0$ are the two $T$-fixed points on $X$, and $E_1,E_2$ (resp. $E_3,E_4$) are the two $T$-invariant lines in the fiber of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $p_0$ (resp. $p_1$).

We now associate to $X$ a locally planar trivalent graph $\Gamma_X$. As an abstract graph, $\Gamma_X$ is determined by the configuration $X^1$: each $T$-fixed point corresponds to a vertex in $\Gamma_X$; each $T$-invariant $\mathbb{P}^1$ connecting two fixed points corresponds to a compact edge (line segment) connecting two vertices; each $T$-invariant $\mathbb{C}$ containing a fixed point corresponds to a noncompact edge (ray) emanating from a vertex. Therefore all the vertices in $\Gamma_X$ are trivalent. For example, the graph for $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is shown on the right hand side of Figure 2. The (local) embedding of $\Gamma_X$ into $\mathbb{Z}^2$ is determined by the $T$-action on $X^1$: the slope of an edge is determined by the $T$-action on the corresponding irreducible component of $X^1$. On the right hand side of Figure 2, the vectors $(1,0),(0,1),(-1,1)$ (resp. $(-1,0),(0,-1),(1,1)$) at the vertex $v_0$ (resp. $v_1$) correspond to weights $t_1,t_2,t_1^{-1}t_2^{-1}$ (resp. $t_1^{-1},t_2^{-1},t_1t_2$) of the $T$-actions on $T_{p_0}C_0,T_{p_0}E_1,T_{p_0}E_2$ (resp. $T_{p_1}C_0,T_{p_1}E_3,T_{p_1}E_4$). The sum of the three vectors at $v_0$ (resp. $v_1$) is zero because $T$ acts trivially on $\Lambda^3T_{p_0}X$ (resp. $\Lambda^3T_{p_1}X$).

![Figure 2. $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.](image)

More generally, given any toric Calabi-Yau 3-fold $X$ and any $C_0 \cong \mathbb{P}^1$ which is an irreducible component of $X^1$, the degree of the normal bundle $N_{C_0/X}$ must be $-2$, so $N_{C_0/X} \cong \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n-2)$ for some $n \in \mathbb{Z}$. We have a $T$-equivariant open embedding $N_{C_0/X} \hookrightarrow X$. The weights at one fixed point are $w_1,w_2,w_3 \in \mathbb{Z}^2$, where $w_1,w_2$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ and $w_3 = -w_1 - w_2$. The weights at the other
fixed point is determined by $w_1, w_2, w_3$ and the degree $n$ (see Figure 3). Putting together the graphs of all $T$-invariant $\mathbb{P}^1$ in $X$, we obtain a locally planar trivalent graph $\Gamma_X$. Some examples are shown in Figure 4.

$\begin{align*}
w_3 &= -w_1 - w_2 \\
w_2 &= n w_1 \\
w_2 &= (n + 2) w_1
\end{align*}$

**Figure 3.** $N_{C_0/X} = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n - 2)$

**Figure 4.** Locally planar trivalent graphs

Conversely, from the graph in Figure 3 we can read off the degrees of the two summands of $N_{C_0/X}$ together with the $T$-action on $N_{C_0/X}$. Therefore, from $\Gamma_X$ we may recover a $T$-equivariant formal neighborhood $\hat{X}^1$ of $X^1$ in $X$ (in algebraic geometry), or a $T_\mathbb{R}$-equivariant tubular neighborhood of $X^1$ in $X$ (in differential geometry), where $T_\mathbb{R} \cong U(1)^2$ is the maximal compact subgroup of $T \cong (\mathbb{C}^*)^2$. The $T$-equivariant formal neighborhood $\hat{X}^1$ contains all the information needed to compute all Gromov-Witten invariants $N_{g,d}^X$ of $X$ (using the traditional algorithm described in Section 2), because a point in $\overline{M}_{g,0}(X, \vec{d})^T$ is represented by a stable map with image in $X^1$, and $e_T(N_{g,d}^X)$ is determined by the $T$-equivariant formal scheme $\hat{X}^1$. We summarize this paragraph in Figure 5.

**Figure 5.** Gromov-Witten invariants from a locally trivalent graph
4.2. **Formal Toric Calabi-Yau (FTCY) graphs.** In order to develop the mathematical theory of the topological vertex, we need to generalize local planar trivalent graphs associated to toric Calabi-Yau 3-folds. The generalization will be FTCY (formal toric Calabi-Yau) graphs. The reverse procedure $\Gamma_X \to X^1$ will be generalized to construction of relative FTCY 3-folds. We define relative FTCY invariants by localization. These three ingredients are summarized in Figure 6, which is generalization of equivariant parameters, so they are topological invariants instead of equivariant invariants. It is crucial to pick the subtorus $T$ because $F^{\Gamma}_{g,\vec{d},\vec{\mu}}$ would depend on equivariant parameters if we used the rank 3 torus $(\mathbb{C}^*)^3$.

**Figure 6.** formal relative Gromov-Witten invariants from a FTCY (formal toric Calabi-Yau) graph

As an example, we start with the graph of $\mathcal{O}_{p^1}(-1) \oplus \mathcal{O}_{p^1}(-1)$ on the left hand side of Figure 5. We first compactify the rays by adding a univalent vertex at the end of each ray. We obtain a graph with 2 trivalent vertices and 4 univalent vertices, as on the left hand side of of Figure 6. This corresponds to adding a point of infinity to each $E_i \cong \mathbb{C}$ in Figure 5 so that it becomes $C_i \cong \mathbb{P}^1$ in Figure 6. We obtain a configuration of five $\mathbb{P}^1$'s, as on the right hand side of Figure 6. This configuration sits in a 3-dimensional formal scheme $\hat{Y}$, and the 4 points added are the intersection points with the 4 connected components $D_1, D_2, D_3, D_4$ of the relative divisor $\hat{D} \subset \hat{Y}$. By the relative Calabi-Yau condition $K_{\hat{Y}} + \hat{D} = 0$, the degrees of $N_{C_i/\hat{Y}}$ are $-1$ for $i = 1, 2, 3, 4$ and $-2$ for $i = 0$. We introduce, at each univalent vertex, a framing vector which determines $n_i$ together with the $T$-action on $N_{C_i/\hat{Y}}$ for $i = 1, 2, 3, 4$ (see Figure 6).

The formal relative Gromov-Witten invariants $F^{\Gamma}_{g,\vec{d},\vec{\mu}}$ count morphisms $u : C \to \hat{Y}$, where $C$ is a curve of arithmetic genus $g$, $u_*[C] = \vec{d} = d[C_0] + \sum_{i=1}^4 d_i[C_i]$, and the ramification patterns of $u$ along $\hat{D}$ are $\vec{\mu} = (\mu^1, \ldots, \mu^4)$. In our example, $d_i = |\mu^i|$, where $|\mu^i| = \mu_1^i + \mu_2^i + \cdots$ is the size of the partition $\mu^i$. A priori the formal relative Gromov-Witten invariants $F^{\Gamma}_{g,\vec{d},\vec{\mu}}$ depend on equivariant parameters. It was proved in [30] that $F^{\Gamma}_{g,\vec{d},\vec{\mu}}$ are rational numbers independent of equivariant parameters, so they are topological invariants instead of equivariant invariants.
For our purpose, we would like to introduce normal crossing singularities of the form
\[(x, y) \in \mathbb{C}^2 \mid xy = 0\] × \mathbb{C}^2.

For example, we degenerate the smooth relative FTCY 3-fold \(\hat{Y}\) in Figure 6, so that \(C_0\) degenerates to \(C'_0\) and \(C''_0\) intersecting at a node \(p\), as on the right of Figure 7. This degeneration corresponds to inserting a bivalent vertex \(v\), as shown on the left of Figure 7. The normal bundle \(N_{C_0/\hat{Y}} \cong \mathcal{O}_{\mathbb{P}^1}(n_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-n_0 - 2)\) degenerates into two degree \(-1\), rank 2 bundles over \(C'_0\) and \(C''_0\):

\[N_{C'_0/\hat{Y}'} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a - 1), \quad N_{C''_0/\hat{Y}''} \cong \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(-b - 1), \quad a + b = n_0,
\]

where \(\hat{Y}'\) and \(\hat{Y}''\) are the two irreducible components of the singular relative FTCY 3-fold. To specify the splitting \(11\) \(T\)-equivariantly, we introduce a framing vector \(f\) at the bivalent vertex \(v\).

\[\text{Figure 7.}\]

4.3. Degeneration formula. Given a FTCY graph \(\Gamma\) with a bivalent vertex (see Figure 7), so that the relative FTCY 3-fold \(\hat{Y}_{\Gamma_{\text{rel}}}^\text{rel}\) has a normal crossing singularity, we may either deform or resolve this singularity. Figure 8 shows the corresponding operation on the graph \(\Gamma\): \(\Gamma_0\) is the smoothing of \(\Gamma\), and \(\Gamma_1 \cup \Gamma_2\) is the resolution of \(\Gamma\). The relative FTCY 3-folds

\[\hat{Y}_{\Gamma_0}^\text{rel}, \quad \hat{Y}_{\Gamma_1 \cup \Gamma_2}^\text{rel} = \hat{Y}_{\Gamma_1}^\text{rel} \cup \hat{Y}_{\Gamma_2}^\text{rel} \text{ (disjoint union)}\]

are smooth. The degeneration formula relates the relative formal Gromov-Witten invariants of the smoothing \(\hat{Y}_{\Gamma_0}^\text{rel}\) to those of the resolution \(\hat{Y}_{\Gamma_1}^\text{rel} \cup \hat{Y}_{\Gamma_2}^\text{rel}\).

Geometrically, given a relative stable map \(\hat{Y}_{\Gamma_0}^\text{rel}\) with

\[\vec{d} = d[C_0] + d_1[C_1] + \cdots + d_4[C_4] , \quad \vec{\mu} = (\mu^1, \mu^2, \mu^3, \mu^4),\]

we degenerate and resolve to obtain a pair of maps to \(\hat{Y}_{\Gamma_1}^\text{rel}\) and to \(\hat{Y}_{\Gamma_2}^\text{rel}\) with degrees

\[d[C'_0] + d_1[C_1] + d_2[C_2], \quad d[C''_0] + d_3[C_1] + d_4[C_4],\]

and ramification patterns

\[(\mu^1, \mu^2, \nu), \quad (\nu, \mu^3, \mu^4).\]

(See Figure 8.) In this example, \(d_i = |\mu^i|\) for \(i = 1, 2, 3, 4\), so we may suppress \(d_1, d_2, d_3, d_4\).
Conversely, given a pair of maps to $\hat{Y}_{\Gamma_1}^{\text{rel}}$ and $\hat{Y}_{\Gamma_2}^{\text{rel}}$, with ramifications matching in the middle, we may glue and smooth to obtain a map to $\hat{Y}_{\Gamma_0}^{\text{rel}}$. The degeneration formula says exactly this:

$$F_{\Gamma_0}^{\chi,d,\mu} = \sum_{\chi_1, \chi_2, \nu \vdash d} z_{\nu} F_{\chi_1,\mu_1,\mu_2,\nu}^{\Gamma_1} F_{\chi_2,\nu,\mu_3,\mu_4}^{\Gamma_2}$$

where $z_{\nu}$ is some combinatorial factor. Here we consider $F_{\Gamma,\ldots}$ which counts maps from possibly disconnected curves $C$ with $\chi = 2\chi(\mathcal{O}_C)$ instead of $F_{g,\ldots}$ which counts maps from connected curves of genus $g$, because the degeneration formula of $F_{\Gamma,\ldots}$ is neater than that of $F_{g,\ldots}$.

4.4. Topological vertex. By degeneration and resolution, it remains to compute the formal invariants of the graph on the left hand side of Figure 9. The graph depends on three integers $n_1, n_2, n_3$ and two weights $w_1, w_2$; note that $w_3 = -w_1 - w_2$ and the framings $f_i$'s are determined by $n_i$'s and $w_i$'s.

The invariants depend on the genus $g$ of the domain and three partitions $\mu^1, \mu^2, \mu^3$ corresponding to the ramification patterns along $D^1, D^2, D^3$. Note that in this case the degrees $\vec{d}$ is determined by ramification patterns $\vec{\mu}$.
Figure 9. The topological vertex

Define

\[ F^\bullet_{\chi, \vec{\mu}}(n) := F^\bullet_{\chi, \vec{\mu}}(n_1, w_2, n) \]

where \( \vec{\mu} = (\mu^1, \mu^2, \mu^3) \), \( n = (n_1, n_2, n_3) \). A priori it depends on both the topological data \( n_i \) and the equivariant data \( w_i, w_2 \). In [30], the authors proved that it is indeed topological: it is a rational number depending on \( n_i \) but not on \( w_i \).

\[ F^\bullet_{\chi, \vec{\mu}}(n) \] can be viewed as local relative Gromov-Witten invariants of a configuration of three \( \mathbb{P}^1 \)'s embedded in a relative Calabi-Yau 3-fold such that the formal neighborhood is the relative FTCY 3-fold defined by the graph of a topological vertex. We expect the following two counting problems to be equivalent:

(i) Relative topological vertex. Counting multiple covers of a configuration of three spheres \( C_1 \cup C_2 \cup C_3 \) (Figure 9) embedded in a Calabi-Yau 3-fold \( Y \) relative to three divisors \( D_1, D_2, D_3 \) with ramification patterns \( \mu^1, \mu^2, \mu^3 \).

(ii) Open topological vertex. Counting multiple covers of a configuration of three discs \( D_1 \cup D_2 \cup D_3 \) (Figure 1) embedded in a Calabi-Yau 3-fold \( Y \) relative to three Lagrangian submanifolds \( L_1, L_2, L_3 \) with winding numbers \( \mu^1, \mu^2, \mu^3 \).

It is interesting to compare the above (i) and (ii) with the classical Hurwitz problem:

(i)' Counting ramified covers of the sphere by compact Riemann surfaces, with prescribed ramification pattern \( \mu \) over \( \infty \).

(ii)' Counting ramified covers of the disk by bordered Riemann surfaces, with prescribed winding numbers \( \mu \).

The above two counting problems (i)' and (ii)' are equivalent; they both give rise to Hurwitz numbers.

We now introduce some notation. Given a partition \( \mu = (\mu_1 \geq \cdots \geq \mu_n > 0) \), let \( \ell(\mu) = n \) be the length of the partition. Given a triple of partitions \( \vec{\mu} = (\mu^1, \mu^2, \mu^3) \), let \( \ell(\vec{\mu}) = \ell(\mu^1) + \ell(\mu^2) + \ell(\mu^3) \), and define a generating function

\[ F^\bullet_{\vec{\mu}}(\lambda; n) = \sum_{\chi} \lambda^{-\chi + \ell(\vec{\mu})} F^\bullet_{\chi, \vec{\mu}}(n). \]
4.5. Localization. For later convenience, we consider a slightly modified generating function
\[ \tilde{F}_{\mu}^{\bullet}(\lambda; n) = (-1)^{\sum_{i=1}^{3} (n_i - 1) |\mu^i|} \sqrt{-1} F_{\mu}^{\bullet}(\lambda; n). \]

This is a generating function of relative Gromov-Witten invariants in the **winding basis.** By localization calculations, these invariants can be expressed in terms of **three-partition Hodge integrals** and **double Hurwitz numbers,** which we define now.

The **three-partition Hodge integrals** are defined to be
\[ \chi_{\gamma}(\sigma) = \int_{\mathcal{M}_{g,\bar{\mu}}} \prod_{i=1}^{3} \Lambda_{\gamma_i}^{\nu_i}(u_i) \left( w_i - \mu_i^j \psi_{i,j} - 1 \right)^{\ell_j} \]
where
\[ \ell_1 = 0, \quad \ell_2 = \ell(\mu^1), \quad \ell_3 = \ell(\mu^1) + \ell(\mu^2) \]
\[ \Lambda_{\gamma_i}(u_i) = u_i^g - \lambda_{i1} u_i^{g-1} + \cdots + (-1)^g \lambda_g \]

Define a generating function
\[ G_{\bar{\mu}}^{\bullet}(\lambda; w_1, w_2, w_3) = \sum \lambda^{2g - 2 + \ell(\bar{\mu})} G_{\bar{\mu}}^{\bullet}(\lambda; w_1, w_2, w_3). \]

Let \( G_{\bar{\mu}}^{\bullet}(\lambda; w_1, w_2, w_3) \) be the disconnected version of \( G_{\bar{\mu}}^{\bullet}(\lambda; w_1, w_2, w_3). \)

Let \( H_{\chi,\nu,\mu}^{\bullet} \) be disconnected double Hurwitz numbers which count possibly disconnected covers of \( \mathbb{P}^1 \) with ramification patterns \( \nu \) and \( \mu \) over 0 and \( \infty. \) We define a generating function
\[ \Phi_{\nu,\mu}^{\bullet}(\lambda) = \sum_{\chi} \chi_{\gamma}(\sigma) \left( \frac{\lambda^{\chi + \ell(\nu) + \ell(\mu)}}{(-\chi + \ell(\nu) + \ell(\mu)!} \right). \]

Localization calculations yield the following expression:
\[ \tilde{F}_{\mu}^{\bullet}(\lambda; n) = \sum_{|\nu^i| = |\mu^i|} G_{\nu^i}^{\bullet}(\lambda; w_1, w_2, w_3) \prod_{i=1}^{3} z_{\nu^i} \Phi_{\nu^i,\mu^i}^{\bullet} \left( \sqrt{-1} \lambda (n_i - \frac{w_i + 1}{w_i}) \right). \]

4.6. Framing dependence. Let \( d = |\nu| = |\mu|. \) Recall that for each partition \( \mu \) of \( d, \) there is an associated irreducible character \( \chi_{\mu} \) of the symmetric group \( S_d \) and an associated conjugacy class \( C_{\mu} \) of \( S_d. \) By Burnside formula,
\[ \Phi_{\nu,\mu}^{\bullet}(\lambda) = \sum_{\sigma \vdash d} \sigma_{\lambda/2} \chi_{\sigma}(C_{\nu}) \chi_{\sigma}(C_{\mu}) \frac{z_{\nu}}{z_{\mu}}. \]

Equations (12) and (13) imply the following.

**Proposition 3** (framing dependence in the winding basis).
\[ \tilde{F}_{\mu}^{\bullet}(\lambda; n) = \sum_{|\nu^i| = |\mu^i|} \tilde{F}_{\nu^i}^{\bullet}(\lambda; 0) \prod_{i=1}^{3} z_{\nu^i} \Phi_{\nu^i,\mu^i}^{\bullet} \left( \sqrt{-1} \lambda n_i \right). \]
We introduce a generating function
\[
\tilde{C}_\mu(\lambda; \mathbf{n}) = \sum_{|\mu'|=|\mu|} \tilde{F}_\mu(\lambda; \mathbf{n}) \prod_{i=1}^{3} \chi_{\mu'_i}(C_{\nu'_i}).
\]
These are relative Gromov-Witten invariants in the representation basis. The framing dependence in the representation basis is simple:

**Proposition 4** (framing dependence in the representation basis).
\[
\tilde{C}_\mu(\lambda; \mathbf{n}) = q^{\sum \kappa_{\mu_i} n_i/2} \tilde{C}_\mu(\lambda; 0), \quad q = e^{-\lambda}.
\]

Note that O3 in Section 3 implies
\[
C_\mu(\lambda; \mathbf{n}) = q^{\sum \kappa_{\mu_i} n_i/2} C_\mu(\lambda; 0).
\]

Up to now, we have defined invariants \(\tilde{C}_\mu\) which have the same gluing formula and framing dependence as \(C_\mu\) do.

**4.7. Combinatorial expression.** We have

**Lemma 5.**
\[
G_{g,\bar{\mu}}(\lambda; 1, 1, -2) = (-1)^{|\mu|} (-\ell(\mu^i)) \bar{z}_{\mu_1} \cdot z_{\mu_2} G_{g,\emptyset,\mu_1,\mu_2,\mu_3}(\lambda; 1, 1, -2)
\]
\[
+ \delta_{g0} \sum_{m \geq 1} \delta_{\mu_1'(m)} \delta_{\mu_2'(2m)} (-1)^{m-1} \frac{1}{m}
\]

A formula of the two-partition Hodge integrals \(G_{g,0,\mu_1,\mu_2}\) in terms of \(W_{\mu_1,\mu_2}\) (the Chern-Simons invariants of the Hopf link) was derived in [35]. \(W_{\mu_1,\mu_2}\) can be expressed in terms of the skew functions \(s_{\mu'}/\lambda\) (see [44]):
\[
W_{\mu_1,\mu_2}(q) = q^{(\kappa_{\mu_1} + \kappa_{\mu_2})/2} \sum_{\lambda} s_{\mu'_1}/\lambda(q^{-\frac{1}{2}}, q^{-\frac{3}{2}}, \ldots) s_{\mu'_2}/\lambda(q^{-\frac{1}{2}}, q^{-\frac{3}{2}}, \ldots).
\]

This allows one to evaluate the relative Gromov-Witten invariants \(\tilde{C}_\mu\) of the topological vertex in terms of \(W_{\mu_1,\mu_2}(q)\). More precisely, let \(W_\mu(q) = W_{\mu,0}(q)\), and let \(c^\mu_{\eta,\rho}\) be the Littlewood-Richardson coefficients. Then

R3. \(\tilde{C}_\mu(\lambda; 0) = \tilde{W}_\mu(q)\), where
\[
\tilde{W}_{\mu_1,\mu_2,\mu_3}(q) = q^{-(\kappa_{\mu_1} - 2\kappa_{\mu_2} - \frac{1}{2}\kappa_{\mu_3})/2} \sum_{\eta,\rho} c_{\eta^1,\rho}^+ c_{\eta^2}^+ c_{\eta^3}^+ c_{\rho}^3
\]
\[
q^{-(\kappa_{\mu_1} + \kappa_{\mu_2} - \frac{1}{2}\kappa_{\mu_3})/2} W_{\mu_1,\mu_2,\mu_3}(q) \frac{1}{z_{\mu}} \chi_{\eta^1}(\mu) \chi_{\eta^2}(2\mu) \chi_{\eta^3}(2\mu)
\]

**4.8. Applications.** We now have an explicit formula of all formal relative Gromov-Witten invariants of relative FT CY 3-folds (and in particular, Gromov-Witten invariants of all toric Calabi-Yau 3-folds) in terms of \(\tilde{W}_\mu\). This formula has computational and theoretical applications:

A1. It is significantly more efficient than the traditional algorithm described in Section 2. Given a toric Calabi-Yau 3-fold, the generating function of its Gromov-Witten invariants in all genera in a fixed degree is an infinite series. By this formula, this infinite series is equal to a finite sum in terms of symmetric functions.
A2. This formula can be used to prove structural theorems of Gromov-Witten invariants. For example, P. Peng used this formula to prove the Gopakumar-Vafa conjecture [14] for toric Calabi-Yau 3-folds [17].

A3. One can use this formula to verify enumerative predictions by other string dualities, for example the geometric engineering (see [31]).

4.9. Comparison. The physical theory of the topological vertex predicts the following formula for \( C_{\vec{\mu}}(q) \):

\[
C_{\vec{\mu}}(\lambda; 0) = W_{\vec{\mu}}(q),
\]

where

\[
W_{\mu_1, \mu_2, \mu_3}(q) = q^{\kappa_{\mu_2} + \kappa_{\mu_3}}/2 \sum c_{\mu_1}^{\mu_2} c_{\mu_2}^{\mu_3} \frac{W_{(\mu_2, \rho_3)}(q)W_{(\mu_3, \rho_1)}(q)}{W_{\mu_2}(q)}.
\]

The equivalence of the physical theory and the mathematical theory of the topological vertex boils down to the following identity of classical symmetric functions:

\[
(15) \quad \hat{W}_{\vec{\mu}}(q) = W_{\vec{\mu}}(q).
\]

Equation (15) follows from the results in [39] (see Section 5 below).

5. GW/DT Correspondence and the Topological Vertex

Maulik, Nekrasov, Okounkov, and Pandharipande conjectured a correspondence between the GW (Gromov-Witten) and DT (Donaldson-Thomas) theories for any non-singular projective 3-fold [37, 38]. This correspondence can also be formulated for certain noncompact 3-folds in the presence of a torus action; the correspondence for toric Calabi-Yau 3-folds is equivalent to the algorithm of the topological vertex [37, 44]. For non-Calabi-Yau toric 3-folds the building block is the equivariant vertex (see [37, 45, 46]) which depends on equivariant parameters. The GW/DT correspondence for all toric 3-folds has been proved in a recent work by Maulik-Oblomkov-Okounkov-Pandharipande [39].

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