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The impact of the face mask on SARS-CoV-2 disease: Mathematical modeling with a case study

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A B S T R A C T

The mathematical modeling of new emerging infectious diseases gaining high attention from the researcher’s around the world. We formulate a mathematical model with the statistical data of coronavirus observed in Saudi Arabia from March 01, 2021 to September 30, 2021, is considered. Particularly, we focus on studying recent infected data in Saudi Arabia and obtaining reasonable fitting to the model by investigating its realistic parameters. We show the model-related results and their dynamic analysis. The stability results for the disease-free and endemic cases are shown. We show when $R_0 < 1$, then the system at the disease-free case is locally asymptotically stable (LAS). When $R_0 \leq 1$, we show that the model is globally asymptotically stable (GAS) at the disease free case. We study the endemic equilibria and analyze its global dynamics whenever $R_0 > 1$. We determine that the model gives the forward bifurcation. Moreover, we do data fitting to the model and then perform numerical experiments to justify the theoretical results. The sensitive parameters are used to study the behavior of the model and provide some illustrating results for the possible elimination of infection in the Kingdom of Saudi Arabia.

Introduction

Mathematical model has various applications in many areas, such as engineering, see [1–3]. In dynamics of fluids and other physical problems, we can refer the authors to see [4–7]. Application of mathematical modeling can be seen in medical imaging, and smart phone, [8–10]. Mathematical modeling are used to understand the disease dynamics, prediction and their possible controls, see [11]. The infection that started in the city of Wuhan China, has been named CoVID-19 and later became pandemic by giving a huge number of cases and death. Besides this, it has been responsible for a big financial loss to the world countries. The researchers and other health experts start thinking by studying the data and provided time to time updates about the cases and other necessary information about disease prevention and control. The COVID-19 infection is also faced by Saudi Arabia, and a significant number of death and infected cases were recorded. Among the infected cases to date, there are 550,088 infected cases are recorded. The total number of deaths reported is 8850 which is comparatively less than in other countries. The reason is the strict actions of the government in the form of lockdown. At the end of December 31, 2020, the daily infected cases were reported to be 140. After that, a little increase in January, and then a second wave has been observed from March 2021, till the end of September 2021. It was found that the cases in the Kingdom vanishing since September 2021 and we assume not to analyze those in our data analysis [12].

The health experts and the World health organization (WHO) provided some recommendations regarding the coronavirus to protect the individuals from the infection as prevention, such as making social distances, wearing a face mask, cleaning or washing their hands, using the sanitizer, etc. Besides this, numerous research work are available in literature on the coronavirus infection using different aspects, such
as clinical, epidemiological, statistical, and mathematical investigation, to keep protect the individuals from the infection. The coronavirus infection that is becoming dangerous day by day, has put people into miserable conditions with many infections and deaths. There are several articles in the literature to study this infection as well as other diseases and especially from mathematicians who presented some useful mathematical models to study the infection and gave recommendations regarding its early infection curb [13–16]. In the recent past, a very rich literature on coronavirus has been shown in literature, see [17–20]. Moreover, some work on the COVID-19 that discuss their dynamics, see for example, a mathematical model has been formulated to analyze the initial infected cases reported in Wuhan is studied in [21]. The extended data of Wuhan about COVID-19 infection are considered by the authors and presented the dynamical results. The authors provided reasonable fit to the cases and estimated the threshold which is found close to the actual one [22]. Some important results regarding the disease vanishing were shown graphically. The isolation and quarantines that are currently the best way to handle this infection are suggested by the WHO, and has been studied mathematically in the work given in [23]. The authors considered the initially reported cases in Saudi Arabia from March to June 2020 and formulated a mathematical model by investigating their dynamical analysis in [24]. The control strategies regarding the minimization or elimination of the virus in Pakistan have been explored in [25]. Further, they provided reasonable recommendations regarding the elimination of the virus. Further, the authors suggested important information about the elimination of infection via optimal control strategies. The coronavirus using the concept of carriers is incorporated and studied in [26]. The important instructions about the disease elimination or its future control have been explained in details in the work given in [27]. The spreading of infection, and infection control through mathematical modeling study has been suggested in [28]. The coronavirus was studied through a fractal–fractional new approach in [29]. The authors in [30] studied the high infected country’s data through a mathematical model in [31]. There are many others published material on the dynamics of coronavirus disease, for example, the coronavirus with heterogeneous diffusion [32], analysis of the COVID-19 with third-wave cases [33], novel numerical procedure using an alternative legender polynomials approach [34], a stochastic model for novel coronavirus [35], the predictions of SARS-CoV-2 through an SEIR model [36], existence of solution for SARS-CoV-2 model [37], the modeling of coronavirus infection in fractional order study [38]. A mathematical model has been formulated in [39] to analyzed the coronavirus infection with suggested cases in Pakistan. A mathematical model incorporates the awareness among individuals in order to minimize the disease spread is discussed in [40]. Coronavirus infection model that incorporates the waning immunity is discussed in [41]. The two strains of the coronavirus infection model has been formulated by the authors in [42], and obtained their dynamical results. The COVID-19 model under vaccination is given in [43]. The COVID-19 model that incorporate the fractional derivative approach has been considered in [44]. The reported cases of COVID-19 in Russia has been analyzed in [45]. The fuzzy modeling approach to analyze the COVID-19 disease is given in [46]. The vaccination and their impact on the COVID-19 has been discussed in [47]. A new definition of Caputo derivative to study coronavirus disease has been studied in [48]. The coronavirus infection with control interventions is discussed in [49]. The COVID-19 disease under the ABC fractional model with fuzzy environment is discussed in [50]. The authors used the concept of exponential and Mittag-Leffler in the study of COVID-19 disease in [51]. The impact of quarantine on the disease COVID-19 controlling has been explored in [52]. The exponential law based fractional model for COVID-19 disease and their dynamics has been analyzed in [53]. The anxiety among the people during the COVID-19 period has been documented in [54]. Resonance and bifurcation analysis of a fractional system is investigated in [55]. Some other recent applications of mathematical modeling can be seen in [56–58].

The dynamics of the lung cancer has been studied in [59]. A case study to predict biomodels using genetic programming is done in [60]. Modeling the TB and COVID dynamics has been considered in [61]. The analysis and modeling of HIV/AIDS are studied by the authors in [62]. The DNA dynamics model and their analysis has been shown in [63]. Modeling skin cancer and lumpy skin disease, one can see here [64,65]. We refer the readers for more physical applications work in [66–68].

Regarding the numerical investigation of practical problems, we refer to see [58,69,70].

The new emerging and re-emerging infectious diseases that giving and giving the potential threat to human society are needed to be controlled. In the past, it is observed that particular diseases that provided many deaths cases to the human population, and also caused so many infected cases but its period was not so long. Such outbreaks include influenza, SARS, dengue, malaria, and so many other epidemics. Due to these epidemics, humans faced so my challenges and hardships but on the other hand, the researchers and other experts give attention to developing some strategies in order to help out the community by reducing the burden of infection.

Our aim is to construct a mathematical model to analyze the behavior of the coronavirus disease by considering the infected data of the second wave observed in the Kingdom of Saudi Arabia, KSA. We consider face masks use in the modeling of the problem and consider it as a useful control for the disease. Also, we consider the efficacy of the masks and their role in disease spread and control. Using face masks was considered in the model given in [71], where the authors considered the symptomatic and asymptomatic infections, while we consider in the present study, not only the asymptomatic or symptomatic cases but also the exposed individuals that participate in the disease spread. With the use of the reported data in KSA, we estimate the parameters numerical value of the system and predict its future dynamics. Using the realistic parameters value, we compute the numerical value of the threshold quantity. The model stability results will be analyzed both mathematically and graphically. The results of the model obtained through sensitive parameters will be plotted and important suggestions will be made regarding the infection’s possible elimination.

We shall divide the manuscript section-wise given as: Section “Model construction” discusses the formulation of the model using various possible interactions and the applications of facemasks and their efficacy. The mathematical results regarding the model are discussed in Section “Analysis of the model”. Mathematical analysis of the model shall be carried out in Section “Stability analysis”. In Section “Existing of bifurcation”, we study the possible existence of bifurcation in the coronavirus model. The estimations of the parameters and the sensitivity analysis are shown in Section “Parameters estimations”. Numerical simulations of the coronavirus model under facemasks are discussed in Section “Simulations results” while the summarization of the work finally is given in Section “Conclusion”.

Model construction

Face masks have been identified to be important for the disease control of the coronavirus. Facemasks have an important role in the disease control of coronavirus infection, reducing the likelihood of the disease spreading further. As a result, in this study, we will develop a mathematical model to describe the significance of face masks. To do this, we split the entire population denoted by $P(t)$, into six compartments as follows: $S(t)$ represents the healthy individuals who become infected with COVID-19 after coming into contact with exposed (having early symptoms), infected (visible symptoms), and asymptomatic (no visible symptoms) people, $E(t)$ represents exposed individuals, $I(t)$ represents symptomatic infected people, $A(t)$ represents asymptomatic infected people, and those recovering from infection are given by $R(t)$, and hence, $P(t) = S(t) + E(t) + I(t) + A(t) + R(t)$. Individuals that die as a result of the virus are given $D(t)$. The details of the flow of the
parameters are shown in Fig. 1 in the absence of the death class \(D\). The model (1) is constructed from a description of the illness dynamics and with the above assumptions:

\[
\begin{align*}
\frac{dS}{dt} &= A - (\lambda(t) + d)S, \\
\frac{dE}{dt} &= \lambda(t)S - (\delta + d)E, \\
\frac{dI}{dt} &= (1 - \phi)\delta E - (d_1 + \eta_1 + \eta_2 + d)I, \\
\frac{dA}{dt} &= \phi E - (\eta_1 + d) A, \\
\frac{dR}{dt} &= \eta_1 I + \eta_2 A - d R, \\
\frac{dD}{dt} &= d_1 I, \\
\end{align*}
\]

and the initial conditions,

\[
S(0) \geq 0, \quad E(0) \geq 0, \quad I(0) \geq 0, \quad A(0) \geq 0, \quad R(0) \geq 0.
\]

In system (1), \(A\) defines the birth rate of the vulnerable population and the natural mortality rate of the of each compartment is given by \(d\). The contact rates \(\beta_1, \beta_2,\) and \(\beta_3\) respectively show the contact among healthy and exposed, healthy and infected (symptomatic), and healthy and asymptomatic infected. The incubation period of exposed individuals is shown by \(\delta\), where a portion of individuals after completing their incubation period and identified as infected with clinical symptoms, goes to compartment \(I\) (symptomatic) with the rate \((1 - \phi)\delta\), while the rest, who are not showing disease clinical symptoms accumulated at the rate \(\phi \delta\) in \(A\). The recovery of the infected people class \(I\) and \(A\) are given by \(\eta_1\) and \(\eta_2\) respectively. The symptomatic individuals who die from the infection are given through the rate \(d_1\). The parameter \(x\) \((0 < x \leq 1)\) defines the individuals at the infected class \(I\) and \(A\) using face masks while \(\sigma_r, \sigma_m \in (0, 1]\) is the efficacy of the face masks. One can determine the death cases reported by the following equation,

\[
\frac{dD}{dt} = d_1 I.
\]

**Analysis of the model**

This section shall consider the boundedness and model positivity as well as the computation of the threshold quality.

The dynamics of the total population, \(P(t) = S(t) + E(t) + I(t) + A(t) + R(t)\) is

\[
\frac{dP}{dt} = A - d P.
\]

Eq. (3) leads to the following,

\[
P(t) = \frac{A}{d} + \left(\frac{A}{d} - \frac{A}{d}\right)e^{-dt}.
\]

Eq. (4) tends to \(A/d\) when \(t \to \infty\). Thus, for any \(t \geq 0\), the variables in the proposed system are nonnegative. Therefore, the corresponding all solutions in the model (1) shall continue positive for every \(t \geq 0\). From this it is concluded that model (1) is well established and epidemiologically relevant. Thus, the dynamical aspects can be studied in defined below feasible region:

\[
\Gamma = \left\{ X \in \mathbb{R}_+^5 : S + E + I + A + R \leq \frac{A}{d} \right\},
\]

where \(X = (S, E, I, A, R)\).

Further, we present result for the model regarding its positivity and boundedness. Note that the class \(D\) is omitted due to the fact that the individuals are dead.

**Positivity and boundedness**

**Theorem 1.** Let the initial data of the coronavirus model (1) before the disease start, that is \(t = 0\), is given by \(S(0) > 0, E(0) > 0, I(0) > 0, A(0) > 0,\) and \(R(0) > 0\), then each solutions of the model is non-negative for all \(t > 0\).

**Proof.** Using the first equation in (1),

\[
S' = A - \lambda(t)S - dS \geq -\lambda(t) + d)S.
\]

Taking integration of Eq. (6) leads to,

\[
S(t) \geq S_0 \exp\left(-\int_0^t (\lambda(t) + d)dt\right) > 0.
\]

In Eq. (7), \(S_0\) is the population at \(t = 0\), and it is positive obviously, because its represents human, so \(S(t) > 0\). The approach used in (6) can also be applied easily to get the positive solution of the remaining equations in (1), i.e., \(E(t) \geq 0, I(t) \geq 0, A(t) \geq 0,\) and \(R(t) \geq 0\).

Furthermore, the solution positivity is confirmed by Theorem 1 and is clearly bounded, and it can be verified by evaluating Eq. (4) for \(t \to \infty\).
Disease-Free Equilibrium (DFE) and $R_0$

The DFE of the model (1) and the computation of $R_0$ shall be explored in this subsection. The DFE of the system (1), is denoted by $Q_0$, and is shown below:

$$Q_0 = \left( S^0, 0, 0, 0, 0 \right).$$

The DFE $Q_0$ will be used in the computation of $R_0$. We follow [72] to obtain the matrices that are essential in the computation of $R_0$. The essential matrices are,

$$U = \begin{bmatrix}
(1 - \sigma \omega) \beta_1 & (1 - \sigma \omega) \beta_2 & (1 - \sigma \omega) \beta_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},$$

$$B = \begin{bmatrix}
\delta(1 - \phi) & d + d_1 + \eta_1 & 0 & 0 & 0 \\
-\delta \phi & 0 & 0 & 0 & d + \eta_2.
\end{bmatrix}$$

We shall use the spectral radius of $\rho(UB^{-1})$, which provide us the required threshold quantity $R_0$, where $\rho$ denotes spectral radius,

$$R_0 = \frac{\beta_1(1 - \sigma \omega) + \beta_2(1 - \sigma \omega)(1 - \phi) + \beta_3(1 - \sigma \omega)\phi}{\delta + d}.$$

From the $R_0$ given by (8), it can be denoted that $R_E$ is the sub-basic reproduction number correspond to the contact among healthy and exposed individuals, $R_I$ for the symptomatic while $R_A$ for the asymptomatic people. The basic reproduction number is defined as “an average number of coronavirus infected cases introduced into a purely vulnerable community that produce secondary infection”. According to the findings in [72], a few infected cases will not be enough to cause a COVID-19 outbreak whenever $R_0 < 1$(the epidemic will not spread when the threshold is below unity). Also, it is important to mention that $R_0 < 1$ for an epidemiological model is sufficient only but not necessary (for example, Kermack–McKendrick epidemic models with vital/demographic dynamics) due to the fact the disease always dies out with the passage of time (regardless the value of $R_0$). If $R_0 > 1$ then the infection increases over time and reaches its peak and hence decreases with the passage of time due to prevention and control mechanisms.

Stability analysis

In the present section, we shall present results regarding the stability analysis of the system (1).

Theorem 2. At DFE $Q_0$, the coronavirus model given in (1) is LAS if $R_0 < 1$.

Proof. The Jacobian matrix at $Q_0$ is constructed as follows:

$$J(Q_0) = \begin{bmatrix}
-d & -(1 - \sigma \omega) \beta_1 & -(1 - \sigma \omega) \beta_2 & -(1 - \sigma \omega) \beta_3 & 0 \\
0 & -d - \delta + (1 - \sigma \omega) \beta_1 & (1 - \sigma \omega) \beta_2 & (1 - \sigma \omega) \beta_3 & 0 \\
0 & \delta(1 - \phi) & -d - d_1 + \eta_1 & 0 & 0 \\
0 & \delta \phi & 0 & -d - \eta_2 & 0 \\
0 & 0 & \eta_1 & \eta_2 & -d
\end{bmatrix}.$$

For $J(Q_0)$, we have the characteristics equation,

$$(\lambda + d)^2(\delta^2 + \lambda \beta_1 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0,$$

where

$$a_1 = 2d + d_1 + \eta_1 + \eta_2 + (d + \delta)(1 - R_E),$$

$$a_2 = (d + \eta_2)(d + d_1 + \eta_1) + (d + \delta)(d + \eta_2)(1 - R_A - R_E) + (d + \delta)(d + d_1 + \eta_1)(1 - R_E - R_I),$$

$$a_3 = (d + \delta)(d + d_1 + \eta_1)(d + \eta_2)(1 - R_0).$$

It is easy to show that $a_1a_2 - a_3 > 0$ where $a_1 > 0$ and $a_2$ and $a_3$ can be positive when $R_0 < 1$. Hence, the cubic equation will give roots with a negative real part by following the Routh–Hurwitz criteria. The condition $a_1a_2 - a_3 > 0$ can be easily satisfied using the Software. So, the given system (1) at $Q_0$ is LAS when $R_0 < 1$. $\square$

In the next result, we shall investigate the global asymptotical stability (GAS) of the system (1) at $Q_0$.

Theorem 3. The coronavirus infection system (1) at $Q_0$ is GAS whenever $R_0 \leq 1$.

Proof. We can have the Lyapunov function in the form given by,

$$\mathcal{L} = C_1 E + C_2 I + C_3 A,$$

where $C_m > 0$, for $m = 1, 2, 3$, and later, we will fixed its values. We differentiate the Lyapunov function $\mathcal{L}$ with time, and further utilizing the model’s equations (1), we get,

$$\dot{\mathcal{L}} = C_1 \dot{E} + C_2 \dot{I} + C_3 \dot{A}.$$

Using the assumption $S \leq P$ and choosing the constants, $C_1 = (d + d_1 + \eta_1)$, $C_2 = \beta_1(1 - \sigma \omega) \frac{S}{(d + \eta_2)}$, and $C_3 = \frac{\beta_1(1 - \sigma \omega)}{(d + \eta_2)}$, then we have

$$\dot{\mathcal{L}} = (d + \delta)(d + d_1 + \eta_1)(R_0 - 1)E.$$

Here, $\dot{\mathcal{L}} \leq 0$ for $R_0 \leq 1$, and $\dot{\mathcal{L}} = 0$ iff $E = 0$. We get $X \rightarrow (S^0, 0, 0, 0, 0)$ when using $E = 0$ for $t \rightarrow \infty$ in system (1). So, the DFE $Q_0$ is the largest invariant set. Hence, it follows from the LaSalle’s Invariance Principle [73], that the COVID-19 disease system given in (1) is GAS $I^*$ whenever $R_0 \leq 1$. $\square$

Endemic equilibria

The purpose of this subsection is to determine the endemic equilibria associated to the system (1). The calculation of the endemic equilibria is important for disease model in order to determine the possible existence of the equilibria, it may have for the model, single or multi equilibria. We can then find out under what conditions the EE exists and unique. Also, it also gives us information about the possible existence of bifurcation phenomenon in the model, whether it is forward or backward bifurcation. Representing the endemic equilibrium of the system (1) by $Q^*$, and calculate in the following:

$$Q^* = \left( S^*, E^*, I^*, A^*, R^* \right),$$

where

$$S^* = \frac{A}{d + \delta},$$

$$E^* = \frac{\beta_1(1 - \sigma \omega)S}{(d + \delta)(d + \eta_1)},$$

$$I^* = \frac{\beta_1(1 - \sigma \omega)E}{(d + \delta)(d + \eta_1)},$$

$$A^* = \frac{\beta_1(1 - \sigma \omega)A}{(d + \delta)(d + \eta_1)},$$

$$R^* = \frac{\beta_1(1 - \sigma \omega)R}{(d + \delta)(d + \eta_1)}.$$
Putting the results given in Eq. (10) into the following equation, we have
\[
\lambda = \frac{\beta_1(1 - \pi \sigma)E}{P} + \frac{\beta_2(1 - \pi \sigma)I}{P} + \frac{\beta_3(1 - \pi \sigma)A}{P},
\]
and after some simplifications, we get
\[
k_1 x^* + k_2 = 0,
\]
(11)

It should be noted that \( k_1 > 0 \) and \( k_2 > 0 \) whenever \( R_0 < 1 \). In this case \( x^* = -k_2/k_1 \). To get the positive endemic equilibria, we need to have \( R_0 > 1 \). So, the existence of the positive equilibrium depends on \( R_0 > 1 \). Next, we study the GAS of the model (1) in the below theorem.

**GAS of EE**

We demonstrate the model’s GAS (1) in by excluding the compartment \( R \), as it has no involvement in the rest of the model equations. Before showing the results, the endemic equilibria related to syste (1) at a steady-state, can be obtained:

\[
\begin{bmatrix}
\delta & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & \delta
\end{bmatrix}
\begin{bmatrix}
S \\
I \\
A
\end{bmatrix}
= 0.
\]

(12)

The expression given in (12) will be used later in the proof of Theorem 4 below at the endemic state \( Q^* \).

**Theorem 4.** The COVID-19 model (1) is GAS if \( R_0 > 1 \) and further the condition holds
\[
\left( 4 - \frac{S^*}{S} - I - \frac{A^*}{A} - \frac{\delta}{\lambda} \left( \frac{S^*}{S} E - 1 \right) - \frac{E}{E} \left( I + \frac{A^*}{A} - 1 \right) \right) \leq 0.
\]

**Proof.** Assume the Lyapunov given by,
\[
L(t) = \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + \left( E - E^* - E^* \ln \frac{E}{E^*} \right) + \frac{\lambda^* S^*}{\delta E^*} \left( \frac{S}{S^*} - 1 \right)
\]
\[
\times (1 - \frac{\delta}{\lambda} \frac{E}{E^*} + \frac{\lambda^* S^*}{\delta E^*} \left( A - A^* - A^* \ln \frac{A}{A^*} \right)).
\]

(13)

The time differentiating of Eq. (13) gives
\[
L'(t) = \left( 1 - \frac{S^*}{S} \right) S' + \left( 1 - \frac{E^*}{E} \right) E' + \frac{\lambda^* S^*}{\delta E^*} \left( 1 - \frac{I^*}{I} \right) I'
\]
\[
+ \frac{\lambda^* S^*}{\delta E^*} \left( 1 - \frac{A^*}{A} \right) A'.
\]

(14)

\[\text{The terms on the right in (14) can be achieved as:}\]
\[
\left( 1 - \frac{S^*}{S} \right) S' = \left( 1 - \frac{S^*}{S} \right) \left[ A - \lambda S - d S \right],
\]
\[
= \left( 1 - \frac{S^*}{S} \right) \left[ \lambda^* S^* + d S^* - \lambda S - d S \right],
\]
\[
= d S^* \left( 2 - \frac{S^*}{S} - \frac{S}{S^*} \right) + \lambda^* S^* \left( 1 - \frac{S^*}{S} - \frac{S}{S^*} - \frac{\lambda}{\lambda^*} \right),
\]
\[
\leq \lambda^* S^* \left( 1 - \frac{S}{S^*} - \frac{\lambda}{\lambda^*} \right).
\]

(15)

\[\text{Substituting Eqs. (15)–(18) into Eq. (14), and after simplifications, we get}\]
\[
L' = \lambda^* S^* \left( 4 - \frac{S^*}{S} - I - \frac{A^*}{A} - \frac{\delta}{\lambda} \left( \frac{S^*}{S} E - 1 \right) - \frac{E}{E} \left( I + \frac{A^*}{A} - 1 \right) \right) \leq 0.
\]

(19)

Here, \( L' \leq 0 \) if
\[
\left( 4 - \frac{S^*}{S} - I - \frac{A^*}{A} - \frac{\delta}{\lambda} \left( \frac{S^*}{S} E - 1 \right) - \frac{E}{E} \left( I + \frac{A^*}{A} - 1 \right) \right) \leq 0.
\]

(20)

So, \( Q^* \) is to be the largest invariant subset for which \( L' \leq 0 \). Utilizing the result in \([73]\), it can be concluded that \( Q^* \) is GAS if \( R_0 > 1 \) and further the condition in (20) holds.

**Existing of bifurcation**

This section will study the possibilities of the existence of the backward or forward bifurcation in model (1). As the equilibria shows that the model will not give backward bifurcation as from Eq. (11) one can see that it is linear and hence we may have forward bifurcation. We can check the result by using the center manifold theory (CMT) \([74]\). In order to use this method, we shall write the variables in the system (1) as:
\[
S = u_1, \quad E = u_2, \quad I = u_3, \quad A = u_4 \quad \text{and} \quad R = u_5.
\]

Suppose \( U = (u_1, u_2, u_3, u_4, u_5)^T \), then we shall write the model in the form
\[
dU/dt = K(U),
\]
where \( K(U) = (k_1, k_2, k_3, k_4, k_5)^T \), given by
\[
\frac{dU}{dt} = \lambda(t)u_1 - d_1u_1 := k_1,
\]
\[
\frac{dU}{dt} = \lambda(t)u_1 - (d + d_1)u_2 := k_2,
\]
\[
\frac{dU}{dt} = (1 - \lambda(t))u_2 - (d_1 + \eta_1 + d)u_3 := k_3,
\]
\[
\frac{dU}{dt} = \delta u_2 - (\eta_2 + d_1)u_4 := k_4,
\]
\[
\frac{dU}{dt} = \eta_1 u_1 + \eta_2 u_4 - d_5u_5 := k_5,
\]
where
\[
\lambda(t) = \frac{\beta_1(1 - \pi \sigma)u_1}{P} + \frac{\beta_2(1 - \pi \sigma)u_2}{P} + \frac{\beta_3(1 - \pi \sigma)u_4}{P},
\]
and \( P = u_1 + u_2 + u_3 + u_4 + u_5 \). Let us consider that \( \beta_2 = \beta_2^* \) is a bifurcation parameter, then using \( R_0 = 1 \), we shall get
\[
\beta_2^* = \frac{d + \eta_1 + d_1}{(d + \delta) \left( d + n_2 \right) - \left( 1 - \lambda(t) \right) \left( \eta_2 \delta + \beta_1 \left( d + n_2 \right) \right)}.
\]

At the DFE, we can get the Jacobian system at \( \beta_2 = \beta_2^* \), and shall have
\[
J^* = \begin{pmatrix}
-\delta & -(\lambda - \delta) & 0 & 0 & 0 \\
0 & -\delta & (\lambda - \delta) & 1 & 0 \\
0 & 0 & -\delta & 1 & 0 \\
0 & 0 & 0 & -d & 0 \\
0 & 0 & 0 & 0 & -d
\end{pmatrix}.
\]
where
\[
J_1^T = \left( (\pi \sigma - 1) (d + d_1 + \eta_1) (d + \delta) (d + \eta_2) - (1 - \pi \sigma) (\delta \phi \beta_1 + \beta_1 (d + \eta_1)) \right) \frac{d(1 - \phi)}{(d + \eta_2)},
\]
\[
J_2^T = \left( d + d_1 + \eta_1 \right) (d + \delta) (d + \eta_2) - (1 - \pi \sigma) (\delta \phi \beta_1 + \beta_1 (d + \eta_1)) \frac{d(1 - \phi)}{(d + \eta_2)}.
\]

The characteristics equation of \(J^T\) gives one simple eigenvalue while the rest four eigenvalues contain negative real parts. So, the requirements of the (CMT) is fulfilled and it can be employed to model (1) to analyze their dynamics near \(\beta_2 = \beta_2^*\). We shall compute the right eigenvectors of \(J^T\), so denoted by \(V = (w_1, w_2, w_3, w_4, w_5)^T\), given by
\[
w_1 = -\frac{w_2(d + \delta)}{d}, \quad w_2 = -\frac{\delta w_2 (\phi - 1)}{d + \eta_1 + d_1},
\]
\[
w_3 = \frac{w_3 (\phi - 1)}{d + \eta_2}, \quad w_4 = -\frac{w_4 (\phi - 1)}{d + \eta_1 + \Phi} - \frac{\delta w_2 (\phi - 1)}{d + \eta_2}, \quad w_5 = w_2 > 0.
\]

We shall compute the left eigenvectors which is denoted by \(V^* = (v_1, v_2, v_3, v_4, v_5)^T\) associated to the matrix \(J^T\), given by
\[
v_1 = v_2 = 0, \quad v_3 = \frac{v_2 \left( (d + \delta) (d + \eta_2) - (1 - \pi \sigma) (\beta_2 \delta \phi + \phi_1 (d + \eta_1)) \right) \delta(1 - \phi) (d + \eta_2)}{\eta_2}, \quad v_4 = \frac{\beta_1 v_2 (\pi \sigma - 1)}{d + \eta_2}, \quad v_5 = v_2 > 0.
\]

Using the formula mentioned in [74], we can calculate \(a \) and \(b \), that shows the bifurcation coefficients, and is shown below,
\[
a = \frac{2 v_2 w_2 (\pi \sigma - 1) \left( \beta_1 J_2^T (d + \eta_2) (d + \eta_1 + d_1) + \delta J_2^T \right)}{A (d + \eta_2) (d + \eta_1 + d_1)^2}
\]
where
\[
J_2^T = \beta_1 \phi (d + \eta_1 + d_1) \left( d^2 + \eta_2 (d + \delta) + d (\delta + d_1) + \delta d_1 \phi \right)
\]
\[
- \beta_2 d (\phi - 1) \left( d^2 + \eta_2 (-\delta \phi + d + \delta + d_1) + \eta_1 (\delta \phi + d + n_2) \right)
\]
\[
+ \delta d + d d_1 + \delta d_1 \phi \right),
\]
\[
J_1^T = d^2 + \eta_2 (d + \delta) + d (\delta + \Phi) + \delta d_1 \phi \right.
\]
and
\[
b = \frac{\delta v_2 w_2 (1 - \phi)(1 - \pi \sigma)}{d + \eta_1 + d_1} > 0.
\]

The numerical values of the parameters listed in Table 1, it is showing obviously that \(a < 0 \). Now according to the result in Theorem 4.1 in [74] there exits forward bifurcation and the result is given graphically in Fig. 2, where parameters values are taken from Table 1.

### Parameters estimations

The curve fitting to the real data has been widely used in literature to get the realistic values [75]. Parameters estimations for an epidemic model play a vital role in disease epidemiology. Due to these, one can use the real data and obtain the realistic parameters and upon those realistic values, the dynamics of the model and its predictions can be studied effectively. In this regard, a nonlinear least square fitting method is used widely by the researchers to fit their proposed data to their epidemic models. We shall use the approach of nonlinear least-square curve fitting technique to the system (1), to have the realistic parameters. The infected cases are considered for the period from March 01, 2021, to September 30, 2021 in Saudi Arabia. In these proposed months March to September, a high number of cases are observed. We consider the cases in the data fitting in days. The model (1) has 12 parameters, among these, we take some of the parameter’s values from the literature, such as \(d_1, \eta_1, \) and \(\eta_2\) from [76] and \(x\) and \(\sigma\) from [77], see for more details Table 1. The value of \(d\) is \(d = 1/74.87 \approx 365\) is taken from [24], where \(1/74.87\) is the average life expectancy in Saudi Arabia. The birth rate \(A\) is calculated from \(A = d \times P(0)\) and has been shown in Table 1. For the rest of the five parameters, we performed the numerical experiments and their numerical values are obtained from the data matched to the model. The model versus the data and its fitted/estimated value of the parameters are listed in Table 1, while the experimental result is shown graphically in Fig. 3. In subgraph (a), the cumulative cases are well fitted to the model while the long-time behavior of the model versus data is depicted in (b). We use the parameter values listed in Table 1 and estimate the numerical value of \(R_0\) as approximately \(R_0 \approx 1.0974\). In the data fitting and in onward numerical results, the values of the initial conditions for the control system to determine the parameters that have great impact for the model variables are chosen as: The population of the Saudi Arabia in 2021 was \(P(0) = 35587827\), the number of infected people on March 1, 2021, was \(I(0) = 317\). The values for the exposed \(E(0) = 8000\) and for asymptomatic infected was \(A(0) = 1200\), while for \(R(0) = 0\) and we can obtain the healthy people \(S(0) = 35578310\).

### Sensitivity analysis

Here, we shall present the sensitivity analysis of parameters in \(R_0\). Sensitivity analysis is considered useful in epidemic models and also in control system to determine the parameters that have great impact on \(R_0\). In control system, one can use the sensitive parameters to be constants and time dependent to make control strategies for elimination of infection in society. The sensitivity indices determine the important parameters in the coronavirus model and its impact on the infection dynamics. This analysis show, how certain parameters that can enhance \(R_0\), and identify the most sensitive parameters that greatly affect \(R_0\). We shall consider the method in literature [78] to obtain the sensitivity analysis of \(R_0\).

#### Definition 1

The sensitivity index (SI) of a variable \(x\) with parameter \(\sigma\) is given by
\[
\prod_{\sigma} := \frac{\partial x}{\partial \sigma} \times \frac{\sigma}{x}. \tag{22}
\]

Using the formula (22), we shall obtain the analytical expression of \(R_0\),
\[
\prod_{\sigma} = \frac{\partial R_0}{\partial \sigma} \times \frac{\sigma}{R_0} \tag{23}
\]
for each parameters in \(R_0\) and their numerical values shown in Table 1. We shall have the SI of \(R_0\) with \(\beta_1\), given by
\[
\prod_{\beta_1} = \frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0} = 0.5593. \tag{24}
\]
Using this way, we can get the sensitivity indices of the remaining parameters and are listed in Table 2.
Fig. 2. The diagram for the Forward bifurcation in the COVID-19 model.

Fig. 3. The plot shows (a) cases versus model, (b) prediction of the model versus cases.

Fig. 4. The graph represents the parameters sensitivity given in $R_0$, using PRCC.

### Table 2

| Notation | SI  |
|----------|-----|
| $d$      | -0.0001 |
| $\delta$ | -0.5592 |
| $\phi$  | 0.0306 |
| $d_1$    | -0.0288 |
| $\eta_1$ | -0.3768 |
| $\eta_2$ | -0.0349 |
| $\beta_1$ | 0.5593 |
| $\beta_2$ | 0.4057 |
| $\beta_3$ | 0.0349 |
| $\pi$    | -0.0526 |
| $\sigma$ | -0.0526 |
The parameters $\beta_1$, $\beta_2$, $\beta_3$, $\phi$ and $\delta$ which have positive indices can increase the basic reproductive number while $\delta$, $\eta_1$, $\eta_2$, $\pi$, $\sigma$, $d_1$ and $d$ shall decrease the basic reproduction number. The increase and decrease in the values of the parameters shown in Table 2 can cause increase and decrease in the basic reproduction number $R_0$. The parameters $\beta_1$, $\delta$, $\beta_2$, $\beta_3$, $\phi$, $\pi$, $\sigma$ are considered to be the most sensitive parameters that can cause increase or decrease in $R_0$. The sensitivity of the parameters involve in the above compactions are shown graphically using the Partial rank correlation coefficient (PRCC) method is given in Fig. 4.

Simulations results

We simulate the model (1) using the parameter values mentioned in Table 1. We consider the time unit in days. The model is solved numerically using Runge–Kutta order four scheme and the desired results are depicted in Figs. 5–13. Fig. 5 describes the impact of the parameters of the contact on the exposed populations. It can be observed from Fig. 5 that decreasing the contacts rates $\beta_i$ for $i = 1, 2, 3$, the number of exposed cases is decreasing. It should be required that individuals that are in an exposed period should restrict themselves at home quarantine or isolate from other healthy people.

Figs. 6 and 7 depict respectively the behavior of asymptomatic and symptomatic individuals with the variation in contact rates $\beta_1$, $\beta_2$ and $\beta_3$. Using the facemasks, home quarantine, repeated use of the hands with sanitizer shall reduce the disease spread in the human community as well as the future cases. Decreasing the contact of infected and asymptomatic infected with those who are vulnerable shall be minimized in order to reduce the burden of the infection on human community.

Figs. 8–10 presents the population of exposed, asymptomatic, and symptomatic individuals. Decreasing the value of $\delta$ and $\phi$, the number of cases ultimately decreases. Individuals that use face masks correctly and the efficacy of the masks have been shown graphically in Figs. 11–13. In the subgraphs (a) in Figs. 11–13, one can observe that by increasing the use of face masks correctly the models provide the best results for the decrease in the cases only for 18%. It is a useful result suggested when the face masks are used till the pandemic is over, then the new cases will be less as compared to no-masks use. The sub-figures (b) in Figs. 11–13 show the dynamics of exposed, asymptomatic, and symptomatic individuals with the efficacy of face masks. With the increase in the efficacy of face masks, the number of infected populations is decreasing effectively.

Conclusion

A mathematical study with the use of the face masks together with different contact rates and their impact on the coronavirus infection
dynamics in KSA has been investigated. We formulated a novel mathematical model by considering the different contacts with the use of the face masks as well as their efficacy. We considered the infected data of the coronavirus infection cases reported in Saudi Arabia. The reported data have been used to get data fitting results and the numerical values of the parameters of the model. Then, we presented the local asymptotical stability of the model and provided the results when the threshold $R_0 < 1$. We proved for disease-free that the system is locally asymptotically stable whenever the threshold quantity is less than one.

We obtained the global asymptotical stability of the model. We proved at the disease-free equilibrium, when $R_0 \leq 1$, the system will be globally asymptotically stable. The endemic equilibrium is obtained and have been analyzed. We showed at endemic state that the system is globally asymptotically stable when $R_0 > 1$. The global asymptotical stability ensures the impossibilities of the backward bifurcation in our model. Because, if there is a backward bifurcation for the model, then it is difficult to eliminate the infections, otherwise some control, vaccinations and other preventive mechanism shall be required.

The model is then used to find out the numerical values of the parameters. In this regard, we considered the real cases of coronavirus in the Kingdom of Saudi Arabia from March 01, 2021, to September 30, 2021. We performed least-squares curve fitting technique to get the needed parameters from data fitting. Among these parameter values, some were taken from literature, see Table 1. We found that the basic reproduction number for the infected cases in Saudi Arabia is $R_0 = 1.0974$. Furthermore, we used the model with the obtained fitted and estimated numerical values of the parameters and some useful graphical results have been presented. The model exhibits forward bifurcation which has been analyzed in detail.

The impact of the prevention parameters such as $\beta_i$ for $i = 1, 2, 3$ on the dynamics of exposed, asymptomatic, and symptomatic populations have been drawn. With the decrease in the values of the contact rates, we observed a decrease in the infected population. We studied the effects of using the face masks and their efficacy graphically. The use of face masks can decrease the infected population as well as decrease the number of new cases in the Kingdom of Saudi Arabia in the future. Our graphical results suggest that only 18% of people that use the face masks probably decrease well the future infected cases. It does not mean that only 18% are advised to use the facemasks but it is the recommendation of this work. If we increase the use of the facemasks and enhance the efficacy, we shall have more better results. Increasing the efficacy of the face masks can decrease the number of infected cases in the future which is the recommendation of the WHO. Face masks are one of the important prevention controls that is suggested by the World Health Organization (WHO) where their use can possibly reduce the transmission of COVID-19 among other healthy individuals.
Fig. 7. The graphical result of the symptomatic infected population with variations in the contact rates $\beta_1$, $\beta_2$, and $\beta_3$. Subgraphs (a-c) show the population behavior of the symptomatic people respectively, when varying $\beta_1$, $\beta_2$, and $\beta_3$.

Fig. 8. The graphical result of the exposed population on the variations of the parameters $\delta$ and $\phi$. Subfigures (a) and (b) shows respectively, the dynamics of exposed population with the variation in $\delta$ and $\phi$. 
Fig. 9. The graphical result of the asymptomatic population with the variations on \( \delta \) and \( \phi \). Subfigures (a) and (b) shows respectively, the dynamics of asymptomatic population with the variation in \( \delta \) and \( \phi \).

Fig. 10. The parameters \( \delta \) and \( \phi \), and their impact on the infected population. Subfigures (a) and (b) shows respectively, the dynamics of symptomatic population with the variation in \( \delta \) and \( \phi \).

Fig. 11. The impact of parameters \( x \) and \( \eta \) on exposed population. Subfigures (a) and (b) shows respectively, the dynamics of exposed population with the variation in \( x \) and \( \eta \).
Fig. 12. The impact of parameters $\kappa$ and $\sigma$ on asymptomatic population. Subfigures (a) and (b) show respectively, the dynamics of exposed population with the variation in $\kappa$ and $\sigma$.

Fig. 13. Graphical solution of the infected people for the parameters $\kappa$ and $\sigma$. Subfigures (a) and (b) show respectively, the dynamics of infected population with the variation in $\kappa$ and $\sigma$.

CRediT authorship contribution statement

Mahmoud H. DarAssi: Investigation, Data curation, Writing – original draft. Irfan Ahmad: Investigation, Data curation, Writing – review & editing. Mansoor Alsulami: Investigation, Data curation, Writing – review & editing. Muhammad Altaf Khan: Conceptualization, Investigation, Data curation, Writing – original draft, Reviewing and editing, Supervision. Elsayed M. Tag-eldin: Investigation, Data curation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no conflict of interest regarding the publications of this work.

Data availability

Data will be made available on request.

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