TILTING MODULES FOR THE CURRENT ALGEBRA OF A SIMPLE LIE ALGEBRA

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Abstract. The category of level zero representations of current and affine Lie algebras shares many of the properties of other well–known categories which appear in Lie theory and in algebraic groups in characteristic $p$ and in this paper we explore further similarities. The role of the standard and co–standard module is played by the finite–dimensional local Weyl module and the dual of the infinite–dimensional global Weyl module respectively. We define the canonical filtration of a graded module for the current algebra. In the case when $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$ we show that the well–known necessary and sufficient homological condition for a canonical filtration to be a good (or a $\nabla$–filtration) also holds in our situation. Finally, we construct the indecomposable tilting modules in our category and show that any tilting module is isomorphic to a direct sum of indecomposables.

Introduction

The study of the representation theory of current algebras was largely motivated by its relationship to the representation theory of affine and quantum affine algebras associated to a simple Lie algebra $\mathfrak{g}$. However, it is also, now of independent interest since it yields connections with problems arising in mathematical physics, for instance the $X = M$ conjectures, see [1], [11], [18]. These connections arise from the fact that the current algebra is graded by the non–negative integers and that studying graded modules and their characters give rise to interesting combinatorics. The work of [14] for instance, also relates certain graded characters to the Poincare polynomials of quiver varieties.

The current Lie algebra is just the Lie algebra of polynomial maps from $\mathbb{C} \to \mathfrak{g}$ and can be identified with the space $\mathfrak{g} \otimes \mathbb{C}[t]$ with the obvious commutator. The Lie algebra and its universal enveloping algebra inherit a grading coming from the natural grading on $\mathbb{C}[t]$. One is interested in the category $\mathcal{I}$ of $\mathbb{Z}$–graded modules of $\mathfrak{g}[t]$ with the restriction that the graded pieces are finite–dimensional. The simple objects in the category are just the graded shifts of the irreducible modules for $\mathfrak{g}$ and so are parametrized by a set $\Lambda$ consisting of pairs $(\lambda, r)$, where $\lambda$ is a dominant integral weight and $r$ is an integer. However, the main interest of this category is that it has reducible but indecomposable objects. Many of these objects are either defined in a way similar to, or play a role which is analogous to well–known constructions in Lie theory, say in the BGG category $\mathcal{O}$ associated to a simple Lie algebra or to representations of algebraic groups in characteristic $p$. Our work has some similarity with [16] although our set up is quite different. In particular the grade zero piece of the algebra $U(\mathfrak{g}[t])$ is infinite–dimensional.

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The category \( \mathcal{I} \) contains the projective cover and the injective envelope of a simple object. Moreover, if we define a suitable partial order on \( \Lambda \), then we can define the appropriate analog of the standard and costandard objects in \( \mathcal{I} \). An interesting feature in our case is that the standard object \( \Delta(\lambda, r) \) is a finite–dimensional module called the local Weyl module which has been extensively studied (see [6], [10], [17], for instance). The co-standard object \( \nabla(\lambda, r) \) however is infinite–dimensional and is the (appropriately defined) dual of the global Weyl module. Both modules lie in a nice subcategory of \( \mathcal{I} \) which we call \( \mathcal{I}_{\text{bdd}} \). It is the full subcategory consisting of objects whose weights are in a finite union of cones (as in \( \mathcal{O} \)) and whose grades are bounded above.

The main goal of this paper is to construct another family of non–isomorphic modules indexed by \( \Lambda \) and which are in \( \mathcal{I}_{\text{bdd}} \). These modules are denoted by \( T(\lambda, r) \) and have an infinite filtration in which the successive quotients are of the form \( \Delta(\mu, s) \) for \( (\mu, s) \in \Lambda \). The filtration multiplicity of any given \( \Delta(\mu, s) \) is finite. We also show that these modules satisfying a nice homological property, namely that

\[
\operatorname{Ext}^1_\mathcal{I}(\Delta(\mu, s), T(\lambda, r)) = 0, \quad (\mu, s), (\lambda, r) \in \Lambda.
\]

In the case of algebraic groups for instance, see [9],[15] it is shown that the preceding condition is equivalent to the module having a filtration by \( \nabla(\mu, s) \) and the module \( T(\lambda, r) \) is then called tilting. A crucial tool in that situation to proving this equivalence is to show that every module can be embedded into a module admitting a \( \nabla \)–filtration.

In our case, we first have to modify slightly the definition of the \( \nabla \)–filtration, but the more serious problem is to show that any object embeds into one which has a \( \nabla \)–filtration. If we restrict our attention to \( \mathcal{I}_{\text{bdd}} \) then we are able to prove that any \( M \) embeds into an injective object of \( \mathcal{I}_{\text{bdd}} \). We show that if these injective objects admit a \( \nabla \)–filtration, then the modules \( T(\lambda, r) \) are tilting, and are all the indecomposable tilting modules. In the case when \( g \) is of type \( \mathfrak{sl}_{n+1} \) (see [3] for the \( n = 1 \) case and [2] for general \( n \)), it is shown in those papers that the injective envelopes of simple objects do have \( \nabla \)–filtrations. In fact, it is also shown in those papers that the injective envelopes of simple objects in \( \mathcal{I}_{\text{bdd}} \) (which is usually smaller) also has a \( \nabla \)–filtration. As a consequence, one see that for \( \mathfrak{sl}_{n+1} \) the modules \( T(\lambda, r) \) are indeed tilting modules. There are obviously a number of interesting questions one could ask about these modules which we will pursue elsewhere.

1. Preliminaries

1.1. Throughout this paper we denote by \( \mathbb{C} \) the field of complex numbers and \( \mathbb{Z} \) (resp. \( \mathbb{Z}_+ \)) the set of integers (resp. nonnegative integers). For any Lie algebra \( \mathfrak{a} \), we denote by \( \mathbb{U}(\mathfrak{a}) \) the universal enveloping algebra of \( \mathfrak{a} \). Let \( t \) be an indeterminate and let \( \mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t] \) be the Lie algebra with commutator given by,

\[
[a \otimes f, b \otimes g] = [a, b] \otimes fg, \quad a, b \in \mathfrak{a}, \quad f, g \in \mathbb{C}[t].
\]

We identify \( \mathfrak{a} \) with the Lie subalgebra \( \mathfrak{a} \otimes 1 \) of \( \mathfrak{a}[t] \). The Lie algebra \( \mathfrak{a}[t] \) has a natural \( \mathbb{Z} \)–grading given by the powers of \( t \) and this also induces a \( \mathbb{Z} \)–grading on \( \mathbb{U}(\mathfrak{a}[t]) \), and

\[
\mathbb{U}(\mathfrak{a}[t])[s] = 0, \quad s < 0, \quad \mathbb{U}(\mathfrak{a}[t])[0] = \mathbb{U}(\mathfrak{a}).
\]
The graded pieces are \( \mathfrak{a} \)-module under left and right multiplication by elements of \( \mathfrak{a} \) and hence also under the adjoint action of \( \mathfrak{a} \). In particular, if \( \dim \mathfrak{a} < \infty \), then \( \mathbf{U}(\mathfrak{a}[t])[r] \) is a free module for \( \mathfrak{a} \) (via left or right multiplication) of finite rank.

1.2. From now on, \( \mathfrak{g} \) denotes a finite–dimensional complex simple Lie algebra of rank \( n \) and \( \mathfrak{h} \) a fixed Cartan subalgebra of \( \mathfrak{g} \). Let \( I = \{1, \ldots, n\} \) and fix a set \( \{\alpha_i : i \in I\} \) of simple roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) and a set \( \{\omega_i : i \in I\} \) of fundamental weights. Let \( Q \) (resp. \( Q^+ \)) be the integer span (resp. the nonnegative integer span) of \( \{\alpha_i : i \in I\} \) and similarly define \( P \) (resp. \( P^+ \)) to be the \( \mathbb{Z} \) (resp. \( \mathbb{Z}_+ \)) span of \( \{\omega_i : i \in I\} \). Let \( \{x_i^+, h_i : i \in I\} \) be a set of Chevalley generators of \( \mathfrak{g} \) and let \( \mathfrak{n}^\pm \) be the Lie subalgebra of \( \mathfrak{g} \) generated by the elements \( x_i^\pm, i \in I \). We have,

\[
\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathbf{U}(\mathfrak{g}) = \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+).
\]

Given \( \lambda, \mu \in \mathfrak{h}^* \), we say that \( \lambda \leq \mu \) iff \( \lambda - \mu \in Q^+ \). Let \( W \) be the Weyl group of \( \mathfrak{g} \) and let \( w_0 \in W \) be the longest element of \( W \). Given \( \lambda \in P^+ \), let \( \operatorname{conv} W\lambda \subset \mathfrak{h}^* \) be the convex hull of the set \( W\lambda \).

1.3. For any \( \mathfrak{g} \)-module \( M \) and \( \mu \in \mathfrak{h}^* \), set

\[
M_\mu = \{m \in M : hm = \mu(h)m, \; h \in \mathfrak{h}\}.
\]

We say \( M \) is a weight module for \( \mathfrak{g} \) if

\[
M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu,
\]

and we set \( \operatorname{wt}(M) = \{\mu \in \mathfrak{h}^* : M_\mu \neq 0\} \). Any finite–dimensional \( \mathfrak{g} \)-module is a weight module. It is well-known that the set of isomorphism classes of irreducible finite-dimensional \( \mathfrak{g} \)-modules is in bijective correspondence with \( P^+ \). For \( \lambda \in P^+ \) we denote by \( V(\lambda) \) the representative of the corresponding isomorphism class which is generated by a vector \( v_\lambda \) with defining relations

\[
n^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1}v_\lambda = 0, \quad h \in \mathfrak{h}, \; i \in I.
\]

and recall that \( \operatorname{wt} V(\lambda) \subset \operatorname{conv} W\lambda \). The module \( V(0) \) is the trivial module for \( \mathfrak{g} \) and we shall write it as \( C \). The character of \( M \) is the element of the integral group ring \( \mathbb{Z}[P] \) defined by,

\[
\operatorname{ch}_gM = \sum_{\mu \in P} \dim \mathbb{C} M_\mu e(\mu),
\]

where \( e(\mu) \in \mathbb{Z}[P] \) is the generator of the group ring corresponding to \( \mu \). The set \( \{\operatorname{ch}_g V(\mu) : \mu \in P^+\} \) is a linearly independent subset of \( \mathbb{Z}[P] \).

We say that \( M \) is a locally finite-dimensional \( \mathfrak{g} \)-module if it is a direct sum of finite-dimensional \( \mathfrak{g} \)-modules, in which case \( M \) is necessarily a weight module. Using Weyl’s theorem one knows that a locally finite-dimensional \( \mathfrak{g} \)-module \( M \) is isomorphic to a direct sum of modules of the form \( V(\lambda), \lambda \in P^+ \) and hence \( \operatorname{wt} M \subset P \). Set

\[
M^{n^+} = \{m \in M : n^+ m = 0, \} \quad M_\lambda^{n^+} = M^{n^+} \cap M_\lambda \cong \operatorname{Hom}_g(V(\lambda), M).
\]
1.4. Let $\mathcal{I}$ be the category whose objects are graded $\mathfrak{g}[t]$-modules $V$ with finite-dimensional graded components and where the morphisms are maps of graded $\mathfrak{g}[t]$-modules. Thus an object $V$ of $\mathcal{I}$, is a $\mathbb{Z}$–graded vector space $V = \oplus_{s \in \mathbb{Z}} V[s]$ which admits a left action of $\mathfrak{g}[t]$ satisfying

$$(\mathfrak{g} \otimes t^r)V[s] \subset V[s + r], \quad s, r \in \mathbb{Z}.$$ 

A morphism between two objects $V$, $W$ of $\mathcal{I}$ is a degree zero map of graded $\mathfrak{g}[t]$–modules. Clearly $\mathcal{I}$ is closed under taking submodules, quotients and finite direct sums. For any $r \in \mathbb{Z}$ we let $\tau_r$ be the grade shifting operator.

If $V \in \text{Ob} \mathcal{I}$ and $\mu \in P^+$, then

$$V[\mu] = \bigoplus_{r \in \mathbb{Z}} V[r]^n, \quad V[r]^{-n} = V[r] \cap V[\mu].$$

The graded character of $V \in \text{Ob} \mathcal{I}$ is an element of the power series ring $\mathbb{Z}[P][[u, u^{-1}]]$, given by

$$\text{ch}_{gr} V := \sum_{r \in \mathbb{Z}} \text{ch}_g(V[r]) u^r,$$

where we observe that for all $r \in \mathbb{Z}$ the subspace $V[r]$ is a $\mathfrak{g}$–module. Given $V \in \text{Ob} \mathcal{I}$, the restricted dual is

$$V^* = \bigoplus_{r \in \mathbb{Z}} V[r]^*, \quad V^*[r] = V[-r]^*.$$ 

Then $V^* \in \text{Ob} \mathcal{I}$ with the usual action:

$$(xt^s)v^*(w) = -v^*(xt^sw),$$

and $(V^*)^* \cong V$ as objects of $\mathcal{I}$. Note that if $V \in \text{Ob} \mathcal{I}$, then

$$\text{ch}_{gr} V^* := \sum_{r \in \mathbb{Z}} \text{ch}_g(V[r]^*) u^{-r}.$$ 

2. The main result

2.1. Let $\mathcal{I}_{\text{bdd}}$ be the full subcategory of $\mathcal{I}$ consisting of objects $M$ satisfying the following two conditions:

(i) there exists $\mu_1, \ldots, \mu_s \in P^+$ (depending on $M$) such that

$$\text{wt} M \subset \bigcup_{\ell=1}^s \text{conv} W\mu_\ell,$$

(ii) there exists $r \in \mathbb{Z}$ (depending on $M$) such that $M[\ell] = 0$ if $\ell \geq r$.

Notice that $\mathcal{I}_{\text{bdd}}$ is not closed under taking duals. We now define three natural families of objects of $\mathcal{I}_{\text{bdd}}$ which are all indexed by $P^+ \times \mathbb{Z}$.
2.2. Let \( ev_0 : g[t] \to g \) be the homomorphism of Lie algebras which maps \( x \otimes f \mapsto f(0)x \). The kernel of this map is a graded ideal in \( g[t] \) and hence any \( g \)-module \( V \) can be regarded as a graded \( g[t] \)-module by pulling back through \( ev_0 \) and \( ev_0 V \in \text{Ob} \, \mathcal{I} \) if \( \dim V < \infty \). The pull back of \( V(\lambda) \) is denoted \( V(\lambda, 0) \) and we set \( \tau_r V(\lambda, 0) = V(\lambda, r) \) and we let \( v_{\lambda,r} \in V(\lambda, r) \) be the element corresponding to \( v_\lambda \). For any module \( M \) denote by \( \text{soc} \, M \) the maximal semisimple submodule of \( M \). The next proposition gives an explanation for restricting our study to \( \mathcal{I}_{\text{bdd}} \).

**Proposition.** (i) Any irreducible object in \( \mathcal{I} \) (or \( \mathcal{I}_{\text{bdd}} \)) is isomorphic to \( V(\mu, r) \) for a unique element \( (\mu, r) \in P^+ \times \mathbb{Z} \). Moreover

\[
V(\mu, r)^* \cong V(-w_0 \mu, -r).
\]

(ii) Let \( M \in \text{Ob} \, \mathcal{I}_{\text{bdd}} \) be non-trivial. Then \( \text{soc} \, M \neq 0 \) and we have

\[
\text{soc} \, M \cong \bigoplus_{(\lambda, r) \in P^+ \times \mathbb{Z}} V(\lambda, r)^{m(\lambda, r)}, \quad m(\lambda, r) = \dim \text{Hom}_\mathcal{I}(V(\lambda, r), M).
\]

**Proof.** Part (i) is straightforward and a proof can be found in \([5, \text{Proposition 1.3}]\). For (ii), choose \( s \in \mathbb{Z} \) such that \( M[s] \neq 0 \) and \( M[\ell] = 0 \) for all \( \ell > s \). Since \( M[s] \) is a finite-dimensional \( g \)-module, there exists \( \mu \in P^+ \) such that \( \text{Hom}_g(V(\mu), M[s]) \neq 0 \). Since

\[
(g \otimes t \mathbb{C}[t])M[s] = 0,
\]

it follows that \( \text{Hom}_g(V(\mu, s), M) \neq 0 \) proving that \( \text{soc} \, M \neq 0 \). The rest of (ii) is now immediate. \( \square \)

2.3. The next family we need are the local Weyl modules which were originally defined in \([8]\). For the purposes of this paper, we shall denote them as \( \Delta(\lambda, r) \), \( (\lambda, r) \in P^+ \times \mathbb{Z} \). Thus, \( \Delta(\lambda, r) \) is generated as a \( g[t] \)-module by an element \( w_{\lambda,r} \) with relations:

\[
\begin{align*}
\text{n}^+[t] w_{\lambda,r} &= 0, \\
(x_i^-)^{\lambda(h_i)+1} w_{\lambda,r} &= 0, \\
(h \otimes t^s) w_{\lambda,r} &= \delta_{s,0} \lambda(h) w_{\lambda,r},
\end{align*}
\]

where \( i \in I \), \( h \in h \) and \( s \in \mathbb{Z}_+ \). The following proposition summarizes the properties of \( \Delta(\lambda, r) \) which are necessary for this paper.

**Proposition.** Let \( (\lambda, r) \in P^+ \times \mathbb{Z} \).

(i) The module \( \Delta(\lambda, r) \) is indecomposable and finite-dimensional and hence an object of \( \mathcal{I}_{\text{bdd}} \).

(ii) \( \dim \Delta(\lambda, r) = \dim \Delta(\lambda, r)[r] = 1 \),

(iii) \( \text{wt} \, \Delta(\lambda, r) \subset \text{conv} \, W \lambda \),

(iv) The module \( V(\lambda, r) \) is the unique irreducible quotient of \( \Delta(\lambda, r) \),

(v) \( \{ \text{ch}_{gr} \Delta(\lambda, r) : (\lambda, r) \in P^+ \times \mathbb{Z} \} \) is a linearly independent subset of \( \mathbb{Z}[P][u, u^{-1}] \).

\( \square \)

We denote by \( [\Delta(\lambda, r) : V(\mu, s)] \) the multiplicity of \( V(\mu, s) \) in a Jordan–Holder series of \( \Delta(\lambda, r) \).
2.4. We now define the modules $\nabla(\lambda, r)$. These modules are usually defined to be the dual of the modules $\Delta(\lambda, r)$, but in our situation the resulting modules would be too small. The correct definition is to take $\nabla(\lambda, r)$ to be the dual of the global Weyl modules $W(\lambda, r)$. Here $W(\lambda, r)$ is generated as a $g[t]$–module by an element $w_{\lambda,r}$ with relations:

\[
\begin{align*}
 n^+[t]w_{\lambda,r} &= 0, \\
 (x_i^-)^{\lambda(h_i)+1}w_{\lambda,r} &= 0, \\
 hw_{\lambda,r} &= \lambda(h)w_{\lambda,r},
\end{align*}
\]

where $i \in I$ and $h \in h$. Clearly the module $\Delta(\lambda, r)$ is a quotient of $W(\lambda, r)$ and moreover $V(\lambda, r)$ is the unique irreducible quotient of $W(\lambda, r)$. It is known (see [4] or [8]) that $W(0, r) \cong \mathbb{C}$ and that if $\lambda \neq 0$, the modules $W(\lambda, r)$ are infinite-dimensional and satisfy $wt W(\lambda, r) \subset \text{conv } W\lambda$.

It follows that if we set $\nabla(\lambda, r) = W(-w_0\lambda, -r)^*$, then $\nabla(\lambda, r) \in \text{Ob}_{\mathcal{I}_{\text{bdd}}}$ and $\text{soc } \nabla(\lambda, r) \cong V(\lambda, r)$. The following proposition summarizes the main results on $\nabla(\lambda, r)$ that are needed for this paper.

**Proposition.** Let $(\lambda, r) \in P^+ \times \mathbb{Z}$.

(i) The module $\nabla(\lambda, r)$ is an indecomposable object of $\mathcal{I}_{\text{bdd}}$.
(ii) $\dim \nabla(\lambda, r)[r]_\lambda = 1$, and $\dim \nabla(\lambda, r)[s]_\lambda \neq 0 \iff s \leq r$,
(iii) $wt \nabla(\lambda, r) \subset \text{conv } W\lambda$,
(iv) Any submodule of $\nabla(\lambda, r)$ contains $\nabla(\lambda, r)[r]_\lambda$ and the socle of $\nabla(\lambda, r)$ is $V(\lambda, r)$.
(v) $\{ chgr \nabla(\lambda, r) : (\lambda, r) \in P^+ \times \mathbb{Z} \}$ is a linearly independent subset of $\mathbb{Z}[P][u, u^{-1}]$. 

\[\square\]

2.5. **Definition.** We say that $M \in \text{Ob } \mathcal{I}$ admits a $\Delta$ (resp. $\nabla$)–filtration if there exists an increasing family of submodules

\[
0 \subset M_1 \subset M_2 \subset \cdots, \quad M = \bigcup_k M_k,
\]

such that

\[
M_k/M_{k-1} \cong \bigoplus_{(\lambda, r) \in P^+ \times \mathbb{Z}} \Delta(\lambda, r)^{m_k(\lambda, r)}, \quad \text{resp., } M_k/M_{k-1} \cong \bigoplus_{(\lambda, r) \in P^+ \times \mathbb{Z}} \nabla(\lambda, r)^{m_k(\lambda, r)},
\]

for some choice of $m_k(\lambda, r) \in \mathbb{Z}_+$. We say that $M$ is tilting if $M$ has both a $\Delta$ and a $\nabla$–filtration.

Since $\dim M[r]_\lambda < \infty$ for all $(\lambda, r) \in P^+ \times \mathbb{Z}$, we see that if $M$ has a $\Delta$–filtration (resp. $\nabla$–filtration) $M_k \subset M_{k+1}$, then $m_k(\lambda, r) = 0$ for all but finitely many $k$. Since

\[
chgr M = \sum_{k \geq 0} chgr M_k/M_{k-1} = \sum_{(\lambda, r) \in \mathbb{Z}} \left( \sum_{k \geq 0} m_k(\lambda, r) \right) chgr \Delta(\lambda, r),
\]
(where we understand that \( M_{-1} = 0 \)) it follows from Proposition 2.3 that the filtration multiplicity

\[
[M : \Delta(\lambda, r)] = \sum_{k \geq 0} m_k(\lambda, r),
\]

is well-defined and independent of the choice of the filtration. An analogous statement holds for modules admitting a \( \nabla \)-filtration.

2.6. The main goal of this paper is to understand tilting modules in \( I_{\text{bdd}} \). In the case of algebraic groups (see [9], [15]) a crucial necessary result is to give a cohomological characterization of modules admitting a \( \Delta \)-filtration. The analogous result in our situation is to prove the following statement:

An object \( M \) of \( I_{\text{bdd}} \) admits a \( \nabla \)-filtration iff \( \text{Ext}^1_I(\Delta(\lambda, r), M) = 0 \) for all \((\lambda, r) \in P^+ \times \mathbb{Z}\).

It is not hard to see that the forward implication is true. The converse statement however requires one to prove that any object of \( I_{\text{bdd}} \) be embedded in a module which admits a \( \nabla \)-filtration. At this point we can only prove the result for \( \mathfrak{s}l_{n+1} \) and we explain the reason for these limitations in the next section. Summarizing, the first main result that we shall prove in this paper is:

**Proposition.** Let \( M \in \text{Ob} I_{\text{bdd}} \).

(i) If \( M \) admits a \( \nabla \)-filtration, then for all \((\lambda, r) \in P^+ \times \mathbb{Z}\), we have

\[\text{Ext}^1_I(\Delta(\lambda, r), M) = 0.\]

(ii) Let \( \mathfrak{g} \) be of type \( A_n \), and assume that \( M \in I_{\text{bdd}} \) satisfies \( \text{Ext}^1_I(\Delta(\lambda, r), M) = 0 \) for all \((\lambda, r) \in P^+ \times \mathbb{Z}\). Then \( M \) admits a \( \nabla \)-filtration.

2.7. The second main result that we shall prove in this paper is the following:

**Theorem.**

(i) Given \((\lambda, r) \in P^+ \times \mathbb{Z}\), there exists an indecomposable module \( T(\lambda, r) \in \text{Ob} I_{\text{bdd}} \) which admits a \( \Delta \)-filtration and satisfies

\[
\text{Ext}^1_I(\Delta(\mu, s), T(\lambda, r)) = 0, \quad (\mu, s) \in P^+ \times \mathbb{Z},
\]

\[
T(\lambda, r)[r]_\lambda = 1, \quad \text{wt} T(\lambda, r) \subset \text{conv} W \lambda,
\]

and \( T(\lambda, r) \cong T(\mu, s) \) iff \((\lambda, r) = (\mu, s)\).

(ii) If \( \mathfrak{g} \) is of type \( \mathfrak{s}l_{n+1} \), then \( T(\lambda, r) \) is tilting. Moreover any indecomposable tilting module in \( I_{\text{bdd}} \) is isomorphic to \( T(\lambda, r) \) for some \((\lambda, r) \in P^+ \times \mathbb{Z}\). Finally any tilting module in \( I_{\text{bdd}} \) is isomorphic to a direct sum of indecomposable tilting modules.

3. The canonical filtration and proof of Proposition 2.6

In this section we show that one can define in a canonical way a filtration on any object of \( I_{\text{bdd}} \) such that the successive quotients embed into a direct sum of modules \( \nabla(\mu, s), (\mu, s) \in P^+ \times \mathbb{Z} \). To do this we need to understand the projective and injective objects of \( I \) although these are not objects of \( I_{\text{bdd}} \). Using the canonical filtration we get an upper bound for the character of any object of \( I_{\text{bdd}} \). We then use this bound along with the BGG–reciprocity result proved in [7] and [3] to establish Proposition 2.6.
3.1. The category \( I \) contains the projective cover and the injective envelope of a simple object. For \( (\lambda, r) \in P^+ \times \mathbb{Z} \), set

\[
P(\lambda, r) = U(\mathfrak{g}[l]) \otimes_{U(\mathfrak{g})} V(\lambda, r), \quad I(\lambda, r) = P(-w_0 \lambda, -r)^*.
\]

Note that

\[
P(\lambda, r)[r] \cong \mathfrak{g} V(\lambda) \cong \mathfrak{g} I(\lambda, r)[r],
\]

\[
P(\lambda, r)[s] = 0 = I(\lambda, -r)[-s], \quad s < r.
\]

Clearly \( P(\lambda, r) \) is generated by the element \( p_{\lambda, r} = 1 \otimes v_{\lambda} \) with defining relations:

\[
n^{+}p_{\lambda, r} = 0, \quad hp_{\lambda, r} = \lambda(\mathfrak{h})p_{\lambda, r}, \quad (x_i^-)^{\lambda(h_i)+1}p_{\lambda, r} = 0.
\]

The following was proved in [5, Proposition 2.1].

**Proposition.** For \( (\lambda, r) \in P^+ \times \mathbb{Z} \), the object \( P(\lambda, r) \) is the projective cover in \( I \) of \( V(\lambda, r) \). Analogously, the object \( I(\lambda, r) \) is the injective envelope of \( V(\lambda, r) \) in \( I \). □

Notice that \( P(\lambda, r)_{\mu} \neq 0 \) for infinitely many \( \mu \geq \lambda \) and hence \( P(\lambda, r) \) (and also \( I(\lambda, r) \)) is not an object of \( I_{\text{bdd}} \). However, we shall introduce quotients (resp. submodules) of these objects which do lie in \( I_{\text{bdd}} \).

3.2. The object \( W(\lambda, r) \) defined in Section 2.4 is the the maximal quotient (in \( I \)) of \( P(\lambda, r) \) such that

\[
\text{wt } W(\lambda, r) \subset \lambda - Q^+,
\]

or equivalently the maximal quotient whose weights are contained in \( \text{conv } W_{\lambda} \). Similarly, \( \nabla(\lambda, r) \) is the maximal submodule of \( I(\lambda, r) \) whose weights are in the \( \text{conv } W_{\lambda} \). The following is now trivially proved.

**Lemma.** For \( \lambda, \mu \in P^+ \) with \( \lambda \not\leq \mu \), we have

\[
\text{Ext}_I^1(W(\lambda, r), W(\mu, s)) = 0 = \text{Ext}_I^1(\nabla(\mu, r), \nabla(\lambda, s)), \quad \text{for all } r, s \in \mathbb{Z}.
\]

□

3.3. At this stage it is worth making the following remark. Define a partial order \( \preceq \) on \( P^+ \times \mathbb{Z} \) by: \( (\lambda, r) \preceq (\mu, s) \) if either \( \lambda < \mu \) or \( \lambda = \mu \) and \( r \leq s \). Then it is not hard to see that, \( \Delta(\lambda, r) \) is the maximal quotient of \( P(\lambda, r) \) such that

\[
\Delta(\lambda, r)[s]_{\mu} \neq 0 \implies (\mu, s) \preceq (\lambda, r).
\]

On the other hand, \( \nabla(\lambda, r) \) is the maximal submodule of \( I(\lambda, r) \) satisfying,

\[
\nabla(\lambda, r)[s]_{\mu} \neq 0 \implies (\mu, s) \preceq (\lambda, r),
\]

and hence our choices are consistent with the ones usually made in the literature.
3.4. Given $\Gamma \subset P^+$, let $\mathcal{I}(\Gamma)$ be the full subcategory of $\mathcal{I}$ consisting of objects $M$ such that

$$ \text{wt } M \subset \bigcup_{\lambda \in \Gamma} \text{conv } W\lambda. $$

The category $\mathcal{I}_{\text{bdd}}(\Gamma)$ is defined similarly. Given $M \in \mathcal{I}$, let $M_\Gamma$ be the maximal submodule of $M$ that lies in $\mathcal{I}(\Gamma)$. We shall say that a subset $\Gamma$ of $P^+$ is closed with respect to $\leq$ if $\lambda \in \Gamma$ and $\mu \leq \lambda$ implies $\mu \in \Gamma$.

**Proposition.** Let $\Gamma \subset P^+$ be closed with respect to $\leq$. Then $I(\lambda, r)_\Gamma$ is an injective object of $\mathcal{I}_{\text{bdd}}(\Gamma)$ for $(\lambda, r) \in \Gamma \times \mathbb{Z}$. Moreover, if $M \in \text{Ob} \mathcal{I}_{\text{bdd}}(\Gamma)$ for some finite closed subset $\Gamma \subset P^+$, there exists an injective morphism

$$ M \hookrightarrow \bigoplus_{(\lambda, r) \in \Gamma \times \mathbb{Z}} I(\lambda, r)_\Gamma^{\oplus m(\lambda, r)}, \quad m(\lambda, r) = \dim \text{Hom}_\mathcal{I}(V(\lambda, r), M). $$

**Proof.** The first statement is immediate from the fact that $I(\lambda, r)$ is injective in $\mathcal{I}$ and the observation that if $\pi : M \to N$ is a morphism of objects in $\mathcal{I}$ and $M \in \text{Ob} \mathcal{I}_{\text{bdd}}(\Gamma)$, then $\pi(M) \in \text{Ob} \mathcal{I}_{\text{bdd}}(\Gamma)$. For the second statement let $(\lambda, r) \in \Gamma \times \mathbb{Z}$ and $M \in \text{Ob} \mathcal{I}_{\text{bdd}}(\Gamma)$. Corresponding to any non–zero morphism $\varphi : V(\lambda, r) \to M$, we have a morphism $\tilde{\varphi} : M \to I(\lambda, r)_\Gamma$ whose image is clearly in $I(\lambda, r)_\Gamma$. If $m(\lambda, r) > 0$ it follows that by fixing a basis for $\text{Hom}_\mathcal{I}(V(\lambda, r), M)$ we have a morphism $\varphi_{\lambda, r} : M \to (I(\lambda, r)_\Gamma)^{\oplus m(\lambda, r)}$.

Notice that

$$ \varphi_{\lambda, r} M[s] \neq 0 \implies s \leq r. $$

Since $M[\ell] = 0$ for all $\ell >> 0$, it follows that we have a well-defined map

$$ \Phi : M \to \bigoplus_{(\lambda, r) \in \Gamma \times \mathbb{Z}} (I(\lambda, r)_\Gamma)^{m(\lambda, r)}, \quad m \to \{\varphi_{\lambda, r}(m)\}_{(\lambda, r) \in \Gamma \times \mathbb{Z}}. $$

It remains to prove that $\Phi$ is injective. If $\ker \Phi \neq 0$ then we have $\text{soc} \ker \Phi \neq 0$ by Proposition 2.2. On the other hand, $\text{soc} \Phi \subset \text{soc } M$ and the restriction of $\Phi$ to $\text{soc } M$ is injective by design. The proof is complete. $\square$

3.5. From now on we fix an enumeration $\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots$ of $P^+$ satisfying:

$$ \lambda_r - \lambda_s \in Q^+ \implies r \geq s. $$

Given $M \in \mathcal{I}_{\text{bdd}}$, define $k(M) \in \mathbb{Z}_+$ to be minimal such that

$$ \text{wt } M \subset \bigcup_{s=0}^{k(M)} \text{conv } W\lambda_s. $$

For $0 \leq s \leq k(M)$, let $M_s$ be the maximal submodule of $M$ whose weights lie in the union of the sets $\{\text{conv } W\lambda_r : r \leq s\}$. Clearly

$$ M_s \subset M_{s+1}, \quad M = \bigcup_{s=0}^{k(M)} M_s, \quad \text{wt } M_{s+1}/M_s \subset \text{conv } W\lambda_{s+1}. $$
We call the filtration $M_0 \subset M_1 \subset \cdots \subset M_k(M) = M$ the canonical filtration of $M$. It follows from Proposition 3.4 that $M_{s+1}/M_s$ embeds into a direct sum of modules of the form $\nabla(\lambda_{s+1}, r)$, $r \in \mathbb{Z}$, and in fact we get
\[
\text{ch}_{\text{gr}} M = \sum_{s \geq 0} \text{ch}_{\text{gr}} M_s/M_{s-1} \leq \sum_{s \geq 0} \sum_{r \in \mathbb{Z}} \text{dim} \text{Hom}_I(V(\lambda_s, r), M_s/M_{s-1}) \text{ch}(\lambda_s, r). \tag{3.1}
\]
We claim that this is equivalent to,
\[
\text{ch}_{\text{gr}} M \leq \sum_{s \geq 0} \sum_{r \in \mathbb{Z}} \text{dim} \text{Hom}_I(\Delta(\lambda_s, r), M) \text{ch}(\lambda_s, r). \tag{3.2}
\]
For the claim, observe that any non-zero map $\varphi : \Delta(\lambda_s, r) \to M$ has its image in $M_s$. Moreover $\varphi$ maps the unique maximal submodule of $\Delta(\lambda_s, r)$ to $M_{s-1}$ and hence induces a non-zero map from $V(\lambda_s, r) \to M_s/M_{s-1}$, which proves that
\[
\text{dim} \text{Hom}_I(\Delta(\lambda_s, r), M) \leq \text{dim} \text{Hom}_I(V(\lambda_s, r), M_s/M_{s-1}).
\]
For the reverse inequality, suppose that we have a non-zero map $\psi : V(\lambda_s, r) \to M_s/M_{s-1}$ and choose $m \in M_s[r]_{\lambda_s}$ such that $\psi(v_{\lambda_s, r}) = \bar{m}$ where $\bar{m}$ is the image of $m$ in $M_s/M_{s-1}$. Since $\text{wt} M_s \subset \text{conv} W_{\lambda_s}$ it follows that
\[
n^+[t]m = 0.
\]
On the other hand since $(\mathfrak{h} \otimes t \mathbb{C}[t])\bar{m} = 0$, we must have that
\[
(\mathfrak{h} \otimes t \mathbb{C}[t])m \in (M_{s-1})_{\lambda_s} = 0.
\]
Hence there exists a non-zero map from $\Delta(\lambda_s, r) \to M_s$ which proves the claim. Finally, note that equality holds in (3.2) iff the canonical filtration is a $\nabla$–filtration.

3.6. The following result was proved in [3] when $\mathfrak{g}$ is of type $\mathfrak{sl}_2$ and in [2] when $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. More precisely the dual of the following result was proved in these papers, i.e. it was shown that the projective objects had a canonical decreasing filtration with successive quotients being the global Weyl modules $W(\mu, s)$. It is conjectured in [3] that the result is true in general.

**Theorem.** Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. Let $\Gamma$ be a finite subset of $P^+$. For all $(\lambda, r) \in \Gamma \times \mathbb{Z}$ the canonical filtration of $I(\lambda, r)_\Gamma$ is a $\nabla$–filtration. Moreover for all $(\mu, s) \in P^+ \times \mathbb{Z}$, we have
\[
[I(\lambda, r)_\Gamma : \nabla(\mu, s)] = [\Delta(\mu, s) : V(\lambda, r)] = \text{dim} \text{Hom}_I(\Delta(\mu, s), I(\lambda, r)_\Gamma). \tag{3.3}
\]

3.7. We note the following consequence Proposition 3.4 and Theorem 3.6.

**Proposition.** Assume that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$ and let $M \in \text{Ob} \mathcal{I}_{\text{bdd}}$. Then $M$ embeds into an object $I(M)$ of $\mathcal{I}_{\text{bdd}}$ which admits a $\nabla$–filtration.
3.8. To prove (ii) of Proposition 2.6, suppose that $M \in \text{Ob} \mathcal{I}_{\text{bdd}}$ satisfies
\[ \text{Ext}^1_\mathcal{I}(\Delta(\lambda, r), M) = 0, \quad (\lambda, r) \in P^+ \times \mathbb{Z}. \]
Assume also that we have an embedding
\[ 0 \to M \to I(M) \to Q \to 0, \]
where $I(M) \in \text{Ob} \mathcal{I}_{\text{bdd}}$ has a $\nabla$–filtration, in which case $Q \in \text{Ob} \mathcal{I}_{\text{bdd}}$. (In particular Proposition 3.7 shows that we can do this when $g$ is of type $\mathfrak{sl}_{n+1}$.) Applying $\text{Hom}_\mathcal{I}(\Delta(\lambda, r), \cdot)$ to the short exact sequence shows that
\[ \dim \text{Hom}_\mathcal{I}(\Delta(\lambda, r), I(M)) = \dim \text{Hom}_\mathcal{I}(\Delta(\lambda, r), M) + \dim \text{Hom}_\mathcal{I}(\Delta(\lambda, r), Q). \]
Since
\[ \text{ch}_{\text{gr}} I(M) = \text{ch}_{\text{gr}} M + \text{ch}_{\text{gr}} Q, \]
part (ii) now follows by using equation (3.2).

3.9. We need one more standard result (whose proof we include for convenience) to prove part (i) of Proposition 2.6.

Lemma. Suppose that $M \in \mathcal{I}_{\text{bdd}}$ has a $\nabla$–filtration. Then $M$ admits a $\nabla$–filtration
\[ 0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M, \quad M_s/M_{s-1} \cong \bigoplus_{r \in \mathbb{Z}} \nabla(\lambda, r)^{\nabla[M, \nabla(\lambda, r)]}. \]
In particular there exists $(\mu, s) \in P^+ \times \mathbb{Z}$ with $\mu \in P^+$ maximal such that $M_\mu \neq 0$ and a surjective map $M \to \nabla(\mu, s)$ such that the kernel of this map also admits a $\nabla$–filtration.

Proof. Let $N_k \subset N_{k+1}$ be a $\nabla$–filtration of $M$ and assume that $\lambda \in P^+$ is minimal such that $[M : \nabla(\lambda, r)] \neq 0$. Using Lemma 3.2 and an induction on $k$, we see that
\[ \text{Ext}^1(\nabla(\lambda, r), N_k) = 0, \quad k \geq 1, \quad r \geq \mathbb{Z}. \]
This implies that for each $k$, we have $\tilde{N}_k \subset N_k$ such that
\[ \tilde{N}_k \cap N_{k-1} = 0, \quad \tilde{N}_k \cong \bigoplus_{r} \nabla(\lambda, r)^{\nabla[M, \nabla(\lambda, r)]}, \quad \frac{N_k}{N_{k-1} \oplus \tilde{N}_k} \cong \bigoplus_{(\mu, s) : \mu \neq \lambda} \nabla(\mu, s)^{\nabla[m_k(\mu, s)]}. \]
Define a filtration $M_k \subset M_{k+1}, k \geq 1$ of $M$ by,
\[ M_k = N_{k-1} \bigoplus_{s \geq k} \tilde{N}_s, \]
where we recall that $N_0 = 0$. Then
\[ \frac{M_k}{M_{k-1}} \cong \frac{N_k}{N_{k-1} \oplus \tilde{N}_k} \cong \bigoplus_{(\mu, s) : \mu \neq \lambda} \nabla(\mu, s)^{\nabla[m_k(\mu, s)]}. \]
An iteration of this argument completes the proof. \qed
3.10. The following Lemma establishes Proposition 2.6(i).

Lemma. We have $\text{Ext}^1_\mathcal{I}(\Delta(\lambda, r), \nabla(\mu, s)) = 0$ for all $(\lambda, r), (\mu, s) \in P^+ \times \mathbb{Z}$. In particular if $N \in \text{Ob} \mathcal{I}_{\text{bdd}}$ has a $\nabla$–filtration then $\text{Ext}^1_\mathcal{I}(\Delta(\lambda, r), N) = 0$.

Proof. The proof is standard. Thus, suppose that we have a short exact sequence

$$0 \to \nabla(\mu, s) \xrightarrow{\iota} M \xrightarrow{\tau} \Delta(\lambda, r) \to 0.$$ 

Then $M_\lambda \neq 0$ and if $\mu \not\geq \lambda$ we have

$$(n^+[t])M_\lambda = 0 = (\mathfrak{h} \otimes tC[t])M_\lambda.$$ 

It follows from the defining relations of $\Delta(\lambda, r)$ that if $m \in M[r]_\lambda$ is such that $\tau(m) = w_{\lambda,r}$, then $U(\mathfrak{g}[t])m$ is a quotient of $\Delta(\lambda, r)$ via the map $w_{\lambda,r} \to m$ and hence the sequence splits. If $\mu \geq \lambda$, then by taking duals we have a short exact sequence

$$0 \to \Delta(\lambda, r)^* \xrightarrow{\tau^*} M^* \xrightarrow{\iota^*} W(-w_{0\mu}, -s) \to 0.$$ 

Since $-w_{0\mu} \geq -w_{0\lambda}$ we have $n^+[t]M^*_{-w_{0\mu}} = 0$ and using the defining relations of $W(-w_{0\mu}, -s)$ we see that $\iota^*$ splits.

Suppose that $N \in \text{Ob} \mathcal{I}_{\text{bdd}}$ admits a $\nabla$–filtration and let $p \in \mathbb{Z}$ be such that $N[s] = 0$ if $s > p$. It follows from Lemma 3.9 that there exists $k \in \mathbb{Z}_+$ and a filtration $0 \subset N_0 \subset N_1 \subset \cdots \subset N_k = N$ such that

$$N_s/N_{s-1} \cong \bigoplus_{\ell \leq p} \nabla(\lambda_s, \ell)^{m(\lambda_s, \ell)},$$ 

for some $m(\lambda_s, \ell) \in \mathbb{Z}_+$. Since

$$\text{Ext}^1_\mathcal{I}(\Delta(\lambda, r), N_s/N_{s-1}) \hookrightarrow \prod_{s \leq p}(\text{Ext}^1_\mathcal{I}(\Delta(\lambda, r), \nabla(\lambda_s, \ell)^{\otimes m(\lambda_s, \ell)}))$$

it follows that $\text{Ext}^1(\Delta(\lambda, r), N_s/N_{s-1}) = 0$. An obvious induction on $s$ proves the Lemma. □

3.11.

Proposition. Suppose that $\mathfrak{g}$ is of type $\mathfrak{sl}_{n+1}$. An object $M$ of $\mathcal{I}_{\text{bdd}}$ has a $\nabla$–filtration iff the canonical filtration of $M$ is a $\nabla$–filtration. □

Proof. Suppose that $M$ has a $\nabla$–filtration. Then we have proved in Section 3.8 that

$$\text{ch}_{\text{gr}} M = \sum_{(\lambda, r) \in P^+ \times \mathbb{Z}} \dim \text{Hom}_\mathcal{I}(\Delta(\lambda, r), M)\text{ch}_{\text{gr}} \nabla(\lambda, r).$$

Hence equality must hold in (3.2) which was written for the canonical filtration. This proves that the canonical filtration is a $\nabla$–filtration. The converse is obvious. □
4. Modules with $\Delta$--filtrations

In our situation the fact that the dual of a $\Delta$--module is not a $\nabla$--module means that we have to also study properties of modules admitting a $\Delta$--filtration. We also need some results on the vanishing of $\text{Ext}^1_I(\Delta(\lambda, r), \Delta(\mu, s))$ which will be used to construct the tilting modules in the next section.

4.1. Consider the projection map $\text{pr}: U(g[t]) \to U(h[t]) \to 0$ corresponding to the vector space decomposition,

$$U(g[t]) = U(h[t]) \bigoplus (n^-[t]U(g[t]) + U(g[t])n^+[t]).$$

For $i \in I$, define elements $P_{i,s} \in U(h[t])$ recursively, by

$$P_{i,0} = 1, \quad P_{i,s} = -\frac{1}{s} \sum_{r=1}^{s} (h_i \otimes t^r)P_{i,s-r}.$$

The following was proved in [12] (see [8]) for the current formulation:

**Lemma.** For $i \in I$ and $s \geq 1$, we have,

$$\text{pr}(x_i^+ \otimes t^s(x_i^-)^s) = (-1)^s(s!)^2 P_{i,s}.$$

4.2. **Proposition.**

(i) Let $(\lambda, r) \in P^+ \times \mathbb{Z}$ and assume that $N \in \text{Ob} \mathcal{I}$ satisfies,

$$N[s]_\lambda = 0 \text{ if } r \leq s \leq r + 1 + \sum_{i=1}^{n} \lambda(h_i). \quad (4.1)$$

Then,

$$\text{Ext}^1_I(\Delta(\lambda, r), N) = 0.$$

(ii) If $\lambda, \mu \in P^+$ and $\mu \npreceq \lambda$, we have

$$\text{Ext}^1_I(\Delta(\lambda, r), \Delta(\mu, \ell)) = 0, \text{ for all } r, \ell \in \mathbb{Z},$$

and

$$\text{Ext}^1_I(\Delta(\lambda, r), \Delta(\lambda, r)) = 0, \text{ for all } r \in \mathbb{Z},$$

(iii) Given $\lambda, \mu \in P^+$ there exists $d(\lambda, \mu) \in \mathbb{Z}_+$ such that

$$\text{Ext}^1_I(\Delta(\lambda, r), \Delta(\mu, s)) \neq 0 \implies |r - s| \leq d_{\lambda, \mu}.$$

**Proof.** Consider a short exact sequence,

$$0 \to N \xrightarrow{\lambda} M \xrightarrow{\lambda} \Delta(\lambda, r) \to 0.$$

Choose $m \in M[r]_\lambda^{n^+}$ such that $\tau(m) = \omega_\lambda r$. Then $\tau((h_i \otimes t^s)m) = 0$ for all $s > 0$ or equivalently $(h \otimes t^s)m \in N$. Using equation (4.1) we get

$$(h_i \otimes t^s)m = 0, \quad 0 < s \leq 1 + \sum_{i=1}^{n} \lambda(h_i).$$
Taking $s = 1$ gives
\[ 2(x^+_{\alpha_i} \otimes t)m = [h_i \otimes t, x_{\alpha_i}] = 0, \]
and repeating we find that for all all $i \in I$ and $k \in \mathbb{Z}_+$ we have $(x^+_{\alpha_i} \otimes t^k)m = 0$. Applying Lemma 4.1 we have
\[ (x^+_{\alpha_i} \otimes t^s(x^-_{\alpha_i})^s m = P_{i,s}m = 0, \quad s > \lambda(h_i). \]
Since $P_{i,s}$ is a polynomial in $h_i \otimes t^k$, $1 \leq k \leq s$, it follows by an obvious induction that $(h_i \otimes t^s)m = 0$ for all $i$ and $s$. Hence we have proved that $m$ satisfies the defining relations of $\Delta(\lambda, r)$ which means that $\tau$ splits.

The proof of (ii) is similar and easier and we omit the details. Part (iii) is immediate from part (i) and the fact that $\Delta(\mu, s)$ is finite–dimensional. \hfill \Box

**Corollary.** If $M \in \mathcal{I}$ has a $\Delta$–filtration then $\text{Ext}^1_I(M, \nabla(\lambda, r)) = 0$ for all $(\lambda, r) \in P^+ \times \mathbb{Z}$.

**Proof.** If $M$ has a finite $\Delta$–filtration then a obvious induction on the length of the filtration gives the result. The proof of the infinite case is a simple exercise and we omit the details. \hfill \Box

**4.3. Lemma.** Let $(\lambda, r), (\mu, s) \in P^+ \times \mathbb{Z}$. We have
\[ \text{Hom}_I(\Delta(\lambda, r), \nabla(\mu, s)) \cong \begin{cases} \mathbb{C}, & (\lambda, r) = (\mu, s), \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** Suppose that $\varphi : \Delta(\lambda, r) \to \nabla(\mu, s)$ is non–zero. Then $\varphi(w_{\lambda, r}) \neq 0$ and hence we have $\lambda \leq \mu$ and $r \leq s$. Moreover since any submodule of $\nabla(\mu, s)$ has non–zero socle it follows that $\nabla(\mu, s)[s]_{\mu}$ must be in the image of $\varphi$ which shows that $\mu \leq \lambda$ and $s \geq r$. \hfill \Box

**4.4.** We end the section with a final result needed to construct $T(\lambda, r)$. It can be deduced from the fact (proved in [5]) that the space of extensions between irreducible objects of $\mathcal{I}$ is finite–dimensional, but we include a proof for convenience.

**Proposition.** For $(\lambda, r), (\mu, s) \in P^+ \times \mathbb{Z}$, we have $\dim \text{Ext}_I^1(\Delta(\lambda, r), \Delta(\mu, s)) < \infty$.

**Proof.** Let $\pi : P(\lambda, r) \to \Delta(\lambda, r) \to 0$ be the canonical projection which maps $p_{\lambda, r}$ to $w_{\lambda, r}$. Apply $\text{Hom}(\pi, \Delta(\mu, s))$ to the short exact sequence
\[ 0 \to \ker \pi \to P(\lambda, r) \to \Delta(\lambda, r) \to 0. \]
Since $P(\lambda, r)$ is a projective object of $\mathcal{I}$, the result follows if we prove that
\[ \dim \text{Hom}_I(\ker \pi, \Delta(\mu, s)) < \infty. \quad (4.2) \]
Choose $\ell \in \mathbb{Z}$ such that $\Delta(\mu, s)[p] = 0$ for all $p > \ell$, in which case we have an injective map
\[ \text{Hom}_I(\ker \pi, \Delta(\mu, s)) \to \dim \text{Hom}_I \left( \frac{\ker \pi}{\oplus_{p > \ell} \ker \pi[p]}, \Delta(\mu, s) \right). \]
Since
\[
\dim \left( \bigoplus_{p > \ell} \ker \pi[p] \right) = \sum_{p=r}^{\ell} \dim \ker \pi[p] < \infty,
\]
equation 4.2 is proved.

5. The modules \( T(\lambda, r) \)

In this section we construct a family of indecomposable modules \( T(\lambda_k, r) \), \( k \geq 0, r \in \mathbb{Z} \), satisfying:
\[
\dim T(\lambda_k, r)[r]_\lambda = 1, \quad T(\lambda_k, r)_{\mu} \neq 0 \implies \mu \leq \lambda_k.
\]
The construction is similar to the one given in [15] but there are several difficulties to be overcome in our situation.

5.1. We begin by noting the following elementary result.

**Lemma.** Suppose that \( M, N \in \text{Ob} \mathcal{I} \) are such that \( 0 < \dim \text{Ext}^1_\mathcal{I}(M, N) < \infty \) and \( \text{Ext}^1_\mathcal{I}(M, M) = 0 \). Then, there exists \( U \in \text{Ob} \mathcal{I} \), \( d \in \mathbb{Z}_+ \) and a non-split short exact sequence
\[
0 \to N \to U \to M^\oplus d \to 0
\]
so that \( \text{Ext}^1_\mathcal{I}(M, U) = 0 \). \( \square \)

5.2. Set \( r_k = 0 \) and for \( 0 < s \leq k \), choose \( r_s \in \mathbb{Z} \) with \( 0 < r_{k-1} < r_2 < \cdots < r_1 \) such that the following two conditions are satisfied:
\[
\Delta(\lambda_s, r_s)[\ell] \neq 0 \implies \Delta(\lambda_p, r_p)[\ell] = 0, \quad 0 \leq p < s \leq k.
\]
This choice can be made since \( \Delta(\lambda, r) \) is finite-dimensional and so has only finitely many graded pieces. In particular, what we are doing here is to choose the integers \( r_s \) so that the following holds. Set
\[
V = \bigoplus_{s=0}^{k} \Delta(\lambda_s, r_s).
\]
Then \( V[p] \neq 0 \) implies that there exists an unique \( 0 \leq s \leq k \) such that \( \Delta(\lambda_s, r_s)[p] \neq 0 \). Moreover, if \( p < p' \) and \( V[p'] \) is also non-zero, and \( s' \) is such that \( \Delta(\lambda_{s'}, r_{s'})[p'] \neq 0 \) then we must have \( s' < s \). Proposition 4.2(iii) gives and
\[
\text{Ext}^1_\mathcal{I}(\Delta(\lambda_s, r), \Delta(\lambda_p, \ell)) = 0 \quad \text{for all } s < p, \quad r_s < r, \quad \ell \leq r_p.
\]
There are obviously infinitely many choices for the \( r_s \), and from now on we fix the choice by requiring \( r_s \) to be minimal so that (5.1) is satisfies.

Consider the set \( \mathcal{S} = \{ (\lambda_s, r) : 0 \leq s \leq k, \ r \leq r_s \} \) and let \( \eta : \mathcal{S} \to \mathbb{Z}_+ \) be the enumeration given by
\[
\eta(\lambda_s, r_s - \ell) = k - s + (k + 1)\ell, \quad s \geq 0.
\]
Given \( s \in \mathbb{Z}_+ \), let \( (\mu_s, p_s) \in \mathcal{S} \), be the unique element such that
\[
\eta(\mu_s, p_s) = s, \quad (\mu_0, p_0) = (\lambda_k, 0).
\]
Since $\tau_r(\Delta(\lambda, s) = \Delta(\lambda, r + s)$, we find by using equation (5.1) that
\[
\Delta(\mu_s, p_s)[p_t]_{\mu_t} \neq 0 \implies s \leq \ell.
\] (5.3)
Further, Proposition 4.2(ii) and (5.2), gives
\[
\text{Ext}_T^1(\Delta(\lambda, r), \Delta(\mu_s, p_s)) = 0, \quad s \geq 0, \quad (\lambda, r) \notin S.
\] (5.4)

5.3. Proposition 4.4 implies that if $M \in \text{Ob} \mathcal{I}$ admits a finite $\Delta$–filtration, then
\[
\dim \text{Ext}_T^1(\Delta(\lambda, r), M) < \infty \quad \text{for all} \quad (\lambda, r) \in P^+ \times \mathbb{Z}.
\]
We now use Proposition 4.2 and Lemma 5.1 to define finite–dimensional objects $M_s, s \geq 0$, of $\mathcal{I}$, recursively as follows.

Set $M_0 = \Delta(\lambda_0, 0)$. If $\text{Ext}_T^1(\Delta(\mu_1, p_1), M_0) = 0$, take $M_0 = M_1$. Otherwise, let $U \in \text{Ob} \mathcal{I}_{\text{bdd}}$ be chosen as in Lemma 5.1 (with $M = \Delta(\mu_1, p_1)$ and $N = M_0$) and let $M_1$ be the indecomposable summand of $U$ which contains $U[0]_{\lambda_k} \cong (M_0)_{\lambda_k}$ and note that
\[
\text{Ext}_T^1(\Delta(\mu_1, p_1), M_1) = 0.
\]
Since $U$ is finite–dimensional and $\Delta(\mu_1, p_1)$ is indecomposable, it follows that there exists $d_1 \in \mathbb{Z}_+$ and a non–split short exact sequence such that
\[
0 \rightarrow M_0 \rightarrow M_1 \rightarrow \Delta(\mu_1, p_1)^{\oplus d_1} \rightarrow 0.
\]
Clearly, $M_1$ is generated as a $g[\ell]$–module by the spaces $M_1[0]_{\lambda_k}$ and $M_1[p_1]_{\mu_1}$, and
\[
\text{Ext}_T^1(\Delta(\mu_j, p_j), M_1) = 0, \quad j = 0, 1, \quad \text{Ext}_T^1(\Delta(\lambda, r), M_1) = 0, \quad (\lambda, r) \notin S.
\] (5.5)
Repeating this procedure, we can construct a family $M_s, s \geq 0$, of indecomposable finite–dimensional modules and injective morphisms $\iota_s : M_s \rightarrow M_{s+1}$ of objects of $\mathcal{I}$. Each $M_s$ admits a finite $\Delta$–filtration, and satisfies
\[
\dim M_s[0]_{\lambda_k} = 1, \quad \text{wt} \ M_s \subset \text{conv} \ W\lambda_k,
\]
and,
\[
\text{Ext}_T^1(\Delta(\mu_\ell, p_\ell), M_s) = 0, \quad 0 \leq \ell \leq s, \quad \text{Ext}_T^1(\Delta(\lambda, r), M_s) = 0, \quad (\lambda, r) \notin S.
\]
Choose $\ell_0$ maximal such that $M_s[\ell_0] \neq 0$ for $0 \leq s \leq k$, in fact with our choices $\ell_0$ is maximal such that $\Delta(\mu_k, \tau_k)[\ell_0] \neq 0$. It follows that
\[
\Delta(\mu_s, p_s)[\ell] \neq 0 \implies \ell \leq \ell_0, \quad ss \geq 0,
\]
and so we have
\[
M_s[p] = 0, \quad \text{for all} \quad s \geq 0, \quad p > \ell_0.
\] (5.6)
Set $\iota_{r,s} = \iota_{s-1} \cdots \iota_r : M_r \rightarrow M_s, \quad r < s, \quad \iota_{r,r} = \text{id}$. Then,
\[
M_s[p_\ell]_{\mu_\ell} = \iota_{\ell,s}(M_s[p_\ell]_{\mu_\ell}), \quad s \geq \ell,
\] (5.7)
and $M_s$ is generated as a $g[\ell]$–module by the spaces $\{M_s[p_\ell]_{\mu_\ell} : \ell \leq s\}$. Let $T(\lambda_k, 0)$ be the direct limit of $\{M_{s, \iota_{r,s}} : r, s \in \mathbb{Z}_{+}, r \leq s\}$.

It is straightforward to see that the preceding discussion establishes the following.

Lemma. For $k \geq 0$, $T(\lambda_k, 0)$ is an object of $\mathcal{I}_{\text{bdd}}$ and we set $T(\lambda_k, r) = \tau_r T(\lambda_k, 0)$. We have
\[
\text{wt} T(\lambda_k, r) \subset \text{conv} \ W\lambda_k \quad \text{and} \quad \dim T(\lambda_k, r)[r]_{\lambda_k} = 1, \quad \text{and} \quad T(\lambda_k, r) \cong T(\lambda_p, s) \quad \text{iff} \quad k = p \quad \text{and} \quad r = s.
\]
5.4. Since the maps $\iota_{r,s}$ are injective morphisms it follows that the canonical morphism $M_s \to T(\lambda_k,0)$ is injective and we have an isomorphism of $M_s$ with a submodule $\tilde{M}_s$ of $T(\lambda_k,0)$. Moreover, we have inclusions $\tilde{M}_s \subset \tilde{M}_{s+1}$ and

$$T(\lambda_k,0) = \bigcup_{s \geq 0} \tilde{M}_s, \quad \tilde{M}_s / \tilde{M}_{s-1} \cong M_s / M_{s-1}, \quad s \geq 0,$$

proving that $T(\lambda_k,0)$ has a $\Delta$-filtration. From now on, by abuse of notation, we write $M_s$ for $\tilde{M}_s$. Then, (5.7) gives,

$$T(\lambda_k,0)[p_{\ell}]_{\mu,s} = M_{\ell}[p_{\ell}]_{\mu,s}. \quad (5.8)$$

To prove that $T(\lambda_k,0)$ is indecomposable, suppose that

$$T(\lambda_k,r) = U_1 \oplus U_2.$$

Since $\dim T(\lambda_k,0)[0]_{\lambda_k} = 1$, we may assume without loss of generality that $T(\lambda_k,0)[0]_{\lambda_k} \subset U_1$ and hence $M_0 \subset U_1$. Assume that we have proved by induction that $M_{s-1} \subset U_1$. Since $M_s$ is generated as a $\mathfrak{g}[t]$-module by the spaces $\{M_s[p_{\ell}]_{\mu,s} : \ell \leq s\}$, it suffices to prove that $M_s[p_{s}]_{\mu,s} \subset U_1$. By (5.8), we have $U_2[p_s]_{\mu,s} \subset M_s$ and hence

$$M_s = (M_{s-1} + U(\mathfrak{g}[t])U_1[p_s]_{\mu,s}) \bigoplus U(\mathfrak{g}[t])U_2[p_s]_{\mu,s}.$$

Since $M_s$ is indecomposable by construction, it follows that $U_2[p_s]_{\mu,s} = 0$ and $M_s \subset U_1$ which completes the inductive step.

5.5.

**Proposition.** For $k \geq 0$, and for all $(\mu,s) \in P^+ \times \mathbf{Z}$, we have

$$\Ext^1_T(\Delta(\mu,s), T(\lambda_k,r)) = 0, \quad (5.9)$$

**Proof.** Consider a short exact sequence

$$0 \to T(\lambda_k,0) \to U \to \Delta(\mu,s) \to 0. \quad (5.10)$$

If $\mu \not\leq \lambda_k$, an argument identical to the one given in the proof of Lemma 3.10 proves that the short exact sequence in (5.10) must split. If $\mu \leq \lambda_k$ so that $p \leq k$, choose $r >> 0$ so that

$$(T(\lambda_k,0)/M_r)[\ell] = 0, \quad s \leq \ell \leq s + 1 + \sum_{i=1}^n \mu(h_i), \quad (5.11)$$

$$\Ext^1_T(\Delta(\mu,s), M_r) = 0. \quad (5.12)$$

We can choose such an $r$ for the following reasons. Since $T(\lambda_k,0)$ has finite-dimensional graded pieces there exists $p$ such that $M_p[\ell] = M_r[\ell]$ for all $s \leq \ell \leq s + 1 + \sum_{i=1}^n \lambda_p(h_i)$ and all $r \geq p$. If $(\mu,s) \not\in \mathcal{S}$ then (5.12) is automatically satisfied. If $(\mu,s) \in \mathcal{S}$, say $\eta((\mu,s)) = \tilde{s}$, then if $r > \tilde{s}$ we see that (5.12) holds because of the way $M_r$ was constructed.

Consider the short exact sequence

$$0 \to M_r \to T(\lambda_k,0) \to T(\lambda_k,0)/M_r \to 0.$$

Applying $\text{Hom}_T(\Delta(\mu,s), -)$ to the short exact sequence we get from Proposition 3.21) that

$$\Ext^1_T(\Delta(\mu,s), T(\lambda,k)/M_r) = 0.$$

Using (5.12) we see that equation (5.9) is proved. \qed
Corollary. If \( g \) is of type \( \mathfrak{s}_{\nu+1} \), the objects \( T(\lambda_k, r) \) are tilting.

5.6. Assume from now on that Proposition 2.6 is true in which case \( T(\lambda, r) \) is tilting. The following result which is proved in the section completes the proof of Theorem 2.7.

Proposition. Assume that \( T(\lambda, r) \) is a tilting module for all \( (\lambda, r) \in P^+ \times \mathbb{Z} \). Then any tilting module in \( \mathcal{I}_{\text{bdd}} \) is isomorphic to direct sum of modules \( T(\lambda, r), (\lambda, r) \in P^+ \times \mathbb{Z} \).

5.7. Let \( T \in \mathcal{I}_{\text{bdd}} \) be a fixed tilting module. Using Proposition 2.6 and Corollary 4.2, we have

\[
\text{Ext}^1_T(T, \nabla(\mu, r)) = \text{Ext}^1_T(\Delta(\lambda, r), T)) = 0, \quad (\lambda, r) \in P^+ \times \mathbb{Z}. \tag{5.13}
\]

Lemma. Suppose that \( T_1 \) any summand of \( T \). Then \( T_1 \) admits a \( \nabla \)-filtration and

\[
\text{Ext}^1_T(T_1, \nabla(\lambda, r)) = 0,
\]

for all \( (\lambda, r) \in \mathbb{Z} \).

Proof. Since \( \text{Ext}^1 \) commutes with finite direct sums, we get

\[
\text{Ext}^1_T(T_1, \nabla(\lambda, r)) = 0, \quad \text{Ext}^1_T(\Delta(\lambda, r), T)) = 0, \quad (\lambda, r) \in P^+ \times \mathbb{Z}.
\]

Under the assumption that Proposition 2.6 is true, the second equality implies that \( T_1 \) has a \( \nabla \)-filtration and the proof of the Lemma is complete.

5.8. The preceding lemma illustrates one of the difficulties we face in our situation. Namely, we cannot conclude that \( M_1 \) has a \( \Delta \)-filtration from the vanishing \( \text{Ext} \)-condition by using Proposition 2.6. However, we can prove,

Proposition. Suppose that \( M \in \mathcal{I}_{\text{bdd}} \) has a \( \nabla \)-filtration and satisfies

\[
\text{Ext}^1_T(M, \nabla(\lambda, r)) = 0, \quad \text{for all } (\lambda, r) \in P^+ \times \mathbb{Z}.
\]

There exists \( (\mu, s) \in P^+ \times \mathbb{Z} \) such that \( T(\mu, s) \) is a summand of \( M \).

Proof. Since \( M \) has a \( \nabla \)-filtration we can choose \( (\mu, s) \in P^+ \times \mathbb{Z} \) so that we have a non–zero surjective maps \( \varphi : M \rightarrow \nabla(\mu, s) \rightarrow 0 \) and we can also choose \( \pi : T(\mu, s) \rightarrow \nabla(\mu, s) \rightarrow 0 \). We may assume also that \( \ker \varphi \) and \( \ker \pi \) also have \( \nabla \)-filtrations. Let \( v_{\mu, s} \) be a non–zero element of \( \nabla(\mu, s)[s]_\mu \) and choose \( m \in M[s]_\mu \) and \( u \in T(\mu, s)[s]_\mu \) so that

\[
\varphi(m) = v_{\mu, s} = \pi(u).
\]

Consider the short exact sequences

\[
0 \rightarrow \ker \varphi \rightarrow M \rightarrow \nabla(\mu, s) \rightarrow 0,
\]

and

\[
0 \rightarrow \ker \pi \rightarrow T(\mu, s) \rightarrow \nabla(\mu, s) \rightarrow 0.
\]

Apply \( \text{Hom}_T(T(\mu, s), -) \) to the first sequence and \( \text{Hom}_T(M, -) \) to the second sequence. Since \( \ker \varphi \) and \( \ker \pi \) admit a \( \nabla \)-filtration, equation (5.13) gives \( \text{Ext}^1_T(T(\mu, s), \ker \varphi) = 0 \). By hypothesis, we also have \( \text{Ext}^1_T(M, \ker \pi) = 0 \) and so we have surjective maps

\[
\text{Hom}_T(T(\mu, s), M) \rightarrow \text{Hom}_T(T(\mu, s), \nabla(\mu, s)) \rightarrow 0, \text{Hom}_T(M, T(\mu, s)) \rightarrow \text{Hom}_T(M, \nabla(\mu, s)) \rightarrow 0.
\]
Choose $\tilde{\phi} \in \text{Hom}_T(M, T(\mu, s))$ and $\tilde{\pi} \in \text{Hom}_T(T(\mu, s), M)$ such that
\[
\pi.\tilde{\phi} = \phi, \quad \phi.\tilde{\pi} = \pi.
\]
This gives that
\[
\pi.\tilde{\phi}.\tilde{\pi} = \pi.
\]
Setting $\psi = \tilde{\phi}.\tilde{\pi}$, we see that $\psi(u) = u$ and hence $\psi$ is a non-nilpotent endomorphism of $T(\mu, s)$. Moreover, for any $s$, it follows from (5.8) that
\[
0 \neq \psi(M_s) \subset M_s.
\]
Since $M_s$ is indecomposable and finite-dimensional we can use Fitting’s Lemma to conclude that $\psi_s : M_s \to M_s$ is an isomorphism. It follows that $\psi$ is an isomorphism of $T(\mu, s)$ and hence that $\tilde{\pi}\psi^{-1}$ is a splitting of $\tilde{\phi} : M \to T(\mu, s)$. \hfill $\Box$

**Corollary.** Any indecomposable tilting module is isomorphic to $T(\lambda, r)$ for some $(\lambda, r) \in P^+ \times \mathbb{Z}$. Further if $T$ is tilting and $(\lambda, r) \in P^+ \times \mathbb{Z}$ is such that $T \to \nabla(\lambda, r)$ then $T(\lambda, r)$ is isomorphic to a direct summand of $M$. \hfill $\Box$

**Proof.** Since $T$ is tilting it satisfies (5.13) and the corollary follows. \hfill $\Box$

**5.9.** Suppose now that $T \in \mathcal{I}_{\text{bdd}}$ is a tilting module and let $\lambda_k \in P^+$ be maximal such that $[T : \nabla(\lambda_k, r)] \neq 0$ for some $r \in \mathbb{Z}$. Fix also a decreasing sequence $r_1 \geq r_2 \geq \cdots$ such that
\[
[T : \nabla(\lambda_k, s)] \neq 0 \implies s = r_j \text{ for some } j \geq 1.
\]
Then we have a surjective map $T \to \nabla(\lambda_k, r_1)$ and so
\[
T = t_1 T(\lambda_k, r_1) \oplus T_1.
\]
By Lemma 5.7, we see that $T_1$ has a $\nabla$–filtration and that $T_1$ maps onto $\nabla(\lambda_k, r_2)$ and hence $T(\lambda_k, r_2)$ is isomorphic to a summand of $T_1$. Continuing, we find that for $j \geq 1$, there exists a summand $T_j$ of $T$ with
\[
T = T_j \bigoplus_{s=1}^j t_s T(\lambda_k, r_s).
\]
Let $\pi_j : T \to t_j(T(\lambda_k, r_j))$ be the canonical projection. Since $T$ has finite-dimensional graded pieces and $r_j \leq r_{j-1}$ are decreasing and the modules $T(\lambda_k, r_j)$ are all graded shifts, it follows that for any $m \in T$ we have $\pi_j(m) = 0$ for all but finitely many $j$. Hence we have a surjective map
\[
\pi : T \to \bigoplus_{j \geq 1} t_j T(\lambda_k, r_j) \to 0, \quad \text{and } \ker \pi = \bigcap_{j \geq 1} T_j.
\]
In particular, it follows that
\[
T = \bigoplus_{j \geq 1} t_j T(\lambda_k, r_j) \oplus \ker \pi.
\]
Repeat the argument with $\ker \pi$. Since $(\ker \pi)_{\lambda_k} = 0$, the argument stops eventually and Proposition 5.6 is proved.
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