LOG CONTRACTIONS AND EQUIDIMENSIONAL MODELS OF ELLIPTIC THREEFOLDS

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This work was initially motivated by Miranda’s work on elliptic Weierstrass threefolds. An elliptic variety is a complex (irreducible and reduced) projective variety together with a morphism $\pi : X \to S$ whose fiber over a general point is a smooth elliptic curve. For example, if the elliptic fibration has a section, $X$ can be locally expressed as a hypersurface [De]. This is called the Weierstrass form of the fibration; all such fibrations are (by definition) equidimensional. If $\dim(X) = 2$, the fibration is of course always equidimensional. This is in general not true in higher dimension (see [U] for a non trivial example).

Miranda [Mi] describes a smooth equidimensional (flat) model for any elliptic Weierstrass threefold; such models occur naturally in the study of moduli spaces.

Each birational class of equivalent elliptic fibrations over $\bar{S}$ is associated to a log variety $(\bar{S}, \Lambda_{\bar{S}})$ (1.9.2)-(1.10). Here (in §2) we use Minimal model theory to link birational maps of log surfaces (log contractions) to equidimensional fibrations of elliptic threefolds. In particular we give a necessary and sufficient condition for an elliptic threefold $X \to S$ to be birationally equivalent to an equidimensional elliptic fibration $\bar{X} \to \bar{S}$, where $\bar{X}$ has terminal singularities and $\bar{S}$ is the original basis of the fibration (Theorem 2.5). We call this an equidimensional model.

As a corollary we show that an elliptic fibration of positive Kodaira dimension has a minimal model with an equidimensional birationally equivalent elliptic fibration satisfying certain good properties (Corollary 2.7) that naturally generalize the case of elliptic fibrations of surfaces.

Miranda derives his result analyzing the local equation of the Weierstrass model. He shows, checking case by case, that after blowing up the surface a sufficient number of times it is possible to resolve the singularities of the threefold preserving the flatness. We present a summary of Miranda’s results in §3.

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In §4 we apply the results in §2 to the case of Weierstrass models and give a global explanation of Miranda’s algorithm (see 4.6). The argument provided is intrinsic, and the proof does not involve any case by case checking.

This paper originates from a section of my 1990 Ph.D. thesis at Duke University. The bulk of section 4 and of the arguments of 2.4 and 2.5 are part of my thesis. The remaining part of the research was started while traveling in the Italian Alps in the Summer of ’92 and continued at MSRI in the Spring 1993. I would like to thank D. Morrison, who also supervised the beginning of this project, M. Gross, for reading various versions of this work, and the people at MSRI for providing a serene work environment.

§1. Some facts

(1.1) Let $\pi : X \to S$ be any elliptic fibration.

$$ \Sigma_{X/S} = \{ P \in S \text{ such that } \pi \text{ is not smooth over } P \} $$

is called the ramification locus of $\pi$. I. Dolgachev [Do] has showed that if $S$ is smooth, then $\Sigma_{X/S}$ is a divisor.

We write $\Sigma$ for $\Sigma_{X/S}$ when the fibration is well understood.

Miranda considers elliptic fibration of threefolds in Weierstrass form. Then the standard equation is:

$$(1.1.1) \quad y^2 = x^3 + a(s, t)x + b(s, t),$$

where the elliptic morphism sends the point $(x, y, s, t)$ to $(s, t)$.

The corresponding equation for the ramification locus turns out to be:

$$4(a(s, t))^3 + 27(b(s, t))^2 = 0.$$ 

If $a$ and $b$ are chosen generically, this fails to have simple normal crossings.

(1.2) We can associate to each point $Q \in S - \Sigma$ the J-invariant of the smooth elliptic curve $\pi^{-1}(Q)$, where $J$ is the elliptic modular function defined on the complex upper half plane: $J : S \to \mathbb{P}^1$ [Kd].

If the elliptic threefold is written is the Weierstrass form (1.1.1), then

$$J(s, t) = \frac{4a^3}{4(a(s, t))^3 + 27(b(s, t))^2}.$$ 

In general $J$ is not a morphism:

**Example 1.2.1** If the elliptic variety is defined by

$$y^2 = x^3 + ax + b,$$

then $J = (4a^3)/(4a^3 + 27b^2)$. If $a = s$ and $b = t$ then the $J$ map is not defined at $s = t = 0$. Note that $\Sigma = (4s^3 + 27t^2)$ has a cusp at $(0, 0)$. 
Equidimensional models of elliptic threefolds

Definition 1.3. A morphism $\pi : X \to S$ is equidimensional if every fiber of $\pi$ has dimension equal to $\text{dim}(X) - \text{dim}(S)$.

If $\pi : X \to S$ is a smooth elliptic surface, then $J : S \to \mathbb{P}^1$ is a morphism and $\pi$ is equidimensional [Kd]. This is not the case in higher dimension [U] and Kodaira’s analysis of the elliptic surfaces cannot be extended directly.

(1.4) Fix a general (smooth) $P \in D_k$ a reduced component of $\Sigma$ and denote by $X_k$ the singular fiber over the point $P$. By taking $P$ “general” we restrict ourselves to an elliptic fibration over a small disc, which is the case of elliptic fibration of surfaces. Kodaira has classified all such $X_k$ [Kd; Th. 6.2, p. 1270].

Consider a small loop around $P$ and the induced monodromy transformation (Kodaira shows that $J$ is holomorphic in a neighborhood of the point $P$).

The following table [Kd; Table I, p. 1310–Table II, p. 1346] relates the value of $J$ to the eigenvalues of the monodromy matrices and the type of singular fiber $X_k$. The two eigenvalues of the monodromy matrix $A_k$ (when finite) are complex conjugates of each other: we write the one which lies in the closed upper-half plane as $e^{2\pi i a_k}$.

If the monodromy is finite $12a_k = \chi(X_k)$ is the Euler characteristic of the singular fiber. Otherwise $J(P) = \infty$, in fact the matrix has finite period if and only if $J \neq \infty$ [Kd]. Note that $J$ has pole of order $b = \chi(X_k)$ at $P$ if $X_k$ is of type $mI_b$ and of order $b = \chi(X_k) - 6$ if $X_k$ is of type $I^*_b$.

Set $a_k(mI_b) = 0$ and $a_k(I^*_b) = 1/2$. Kodaira shows that if $\pi : X \to S$ is a smooth elliptic surface, then

(1.4.1) $12\pi_*(K_{X/S}) \simeq \mathcal{O}_S(\sum 12a_kD_k) \otimes J_\infty$

where $D_k$ and $a_k = a(D_k)$ are defined as above and $J_\infty$ is the divisor of poles of the elliptic modular function $J$. 
TABLE 1: Kodaira’s Singular Fibers

| J(P)       | A_k         | χ(X_k) | e-value             | a_k = a(D_k) | type       |
|------------|-------------|--------|---------------------|---------------|------------|
| J(P) = 0   | (+1 +1 0)   | 2      | e^{2\pi i2/12}    | 1/6           | II         |
| J(P) = 0   | (0 -1 1)    | 10     | e^{2\pi i10/12}   | 5/6           | II*        |
| J(P) = 0   | (-1 -1 0)   | 8      | e^{2\pi i8/12}    | 2/3           | IV*        |
| J(P) = 0   | (0 +1 1)    | 4      | e^{2\pi i4/12}    | 1/3           | IV         |
| J(P) = 1   | (0 +1 0)    | 3      | e^{2\pi i3/12}    | 1/4           | III        |
| J(P) = 1   | (0 -1 0)    | 9      | e^{2\pi i9/12}    | 3/4           | III*       |
| regular    | (+1 0 1)    | 1      | + 1                | 0             | mI_0       |
| regular    | (-1 0 -1)   | 6      | - 1                | 1/2           | I_0^*      |
| pole of order b | (+1 b 1)      | b    |         | 0             | mI_b       |
| pole of order b | (-1 -b 1)    | b + 6 |         | 1/2           | I_b^*      |

Kawamata and Fujita have generalized (1.4.1) to higher dimensions.

Definition 1.5. A divisor $D$ has simple normal crossings if it is the union of smooth irreducible components intersecting transversely.

If $X$ and $S$ are smooth and $\Sigma_{X/S}$ has simple normal crossings $J$ is a morphism
and $\pi_*(K_{X/S})$ is an invertible sheaf [Ka]. Note that in the example 1.2.1 $J$ is not a morphism: in fact $\Sigma$ does not have simple normal crossings (see also §5).

Assuming $\Sigma$ to have simple normal crossings is a way to control how the components of $\Sigma$ intersect. In particular this assumption is essential for the proof of Kawamata’s results, in extending the variation of Hodge structure over the ramification locus.

Kawamata [Ka] also shows that

\begin{equation}
12\pi_*(K_{X/S}) \simeq \mathcal{O}_S\left(\sum 12a_kD_k\right) \otimes J_\infty
\end{equation}

where $D_k$ are irreducible components of $\Sigma$ and $a_k = a_k(P)$ are defined in Table 1 and $P \in D_k$ is a general point.

If we write $J_\infty = \sum b_jB_j$, then $J$ has a pole of order $b_j$ on the generic point of $B_j$.

Thus $12\pi_*(K_{X/S})$ is a divisor supported on $\Sigma_{X/S}$.

Under the same assumptions, Fujita [F] has proved the following formula for the canonical bundle of a smooth elliptic threefold:

\begin{equation}
mK_X = \pi^*\left\{mK_S + m\pi_*(K_{X/S}) + m\sum \frac{n_i-1}{n_i}Y_i\right\} + mE - mg
\end{equation}

where the fiber over the general point of $Y_i$ is a multiple fiber of multiplicity $n_i$, $m$ is a multiple of the $\{n_i\}$’s; $mE$ and $mg$ are effective divisors and $\text{codim}\pi(G) \geq 2$. Furthermore $m\pi^*(\sum \frac{n_i-1}{n_i}Y_i) + mE - mg$ is an effective divisor. If the fibration has a section, then $n_i = 0, \forall i$.

For convenience, we give the following:

**Definition 1.8.** Let $\pi : X \to S$ be an elliptic fibration between smooth varieties. Assume that $\Sigma$ is a divisor with simple normal crossings. Let:

\begin{align*}
\Delta_{X/S} &= \sum a_kD_k + (1/12)J_\infty \\
\Lambda_{X/S} &= \Delta_{X/S} + \sum \frac{n_i-1}{n_i}Y_i.
\end{align*}

**Remarks.**

(1.9.1) $\Delta$ and $\Lambda$ are effective $\mathbb{Q}$-divisors (see for example, [Wi]).

(1.9.2) Since (1.8) and (1.6’) involve only the generic point of an irreducible component of the ramification locus, we can define $\Lambda_{X/S}$ for any elliptic fibration between varieties with isolated singularities. Then $\Lambda_{X/S} = \Lambda_{\tilde{S}}$ depends only on the class of birationally equivalent elliptic fibration over $\tilde{S}$.

(1.9.3) We say that $\bar{\pi} : \bar{X} \to \bar{S}$ has no multiple fiber over divisors, if $n_i = 0$, in $\Lambda_{\bar{X}/\bar{S}}$, $\forall i$.

(1.9.4) If $\psi : \bar{S} \to \bar{S}_n$ is a birational morphism, consider the induced elliptic fibration $\bar{X} \to \bar{S}_n$; then $\Lambda_{\bar{S}} = \psi_*(\Lambda_{\bar{S}_n})$. 
In the notation of 1.8 we have

\[(1.7') \quad K_X \equiv \pi^*(K_S + \Lambda_S) + E - G,\]

where = indicates equality of divisors or sheaves, while \(\equiv\) denotes numerical equivalence of \(\mathbb{Q}\)-divisors.

Note that the coefficient of the irreducible components of \(\Lambda\) in (1.8) are non negative rational numbers smaller than 1: \((S, \Lambda)\) is thus a log variety (see for example [Ko et al]).

Then each birational class of equivalent elliptic fibrations over \(\bar{S}\), as in 1.9.2, is associated to a log variety \((\bar{S}, \Lambda_{\bar{S}})\). \(\psi\) as in (1.9.4) is a (log) morphism between the log varieties \((\bar{S}, \Lambda_{\bar{S}})\) and \((\bar{S}_n, \Lambda_{\bar{S}_n})\).

If \(\phi : S \to (\bar{S}, \Lambda_{\bar{S}})\) is the blow up of a log variety, then there is not a unique choice for \(\Lambda_S\) (the birational transform of \(\Lambda_{\bar{S}}\)) so that \(\phi\) is a log morphism between \((S, \Lambda_S)\) and \((\bar{S}, \Lambda_{\bar{S}})\) (see [Ko et al; Ch 2]). When the log variety is associated to an elliptic fibration, the birational transform of \(\Lambda_{\bar{S}}\) is determined by the pullback of the elliptic fibration. We will see in section 4 that \(\Lambda_S\) is determined by \(\Lambda_{\bar{S}}\) if \(S\) is smooth and the fibration as a section. This is not the case if there are multiple fibers over divisors.

**Definition 1.10.** Let \((S, \Lambda)\) be a log surface with log terminal singularities (see for example [KMM] or [Ko. et. al]). A birational morphism \(\psi : S \to S_0\), with exceptional divisor \(\Gamma\) is called a \((K_S + \Lambda)\)-extremal contraction or a log extremal contraction if \((K_S + \Lambda) \cdot \Gamma < 0\).

If \(X\) is any smooth minimal elliptic surface, then \(K_X \equiv \pi^*(K_S + \Lambda)\) and the fibration is obviously equidimensional. Although this is not the always case in higher dimension (see 1.11 below and more generally 1.14), it is known that there exists some suitable birational model of the fibration where at least one of the two properties above is satisfied [G2] and [N2].

In §2 we link log contractions \((S, \Lambda_S) \to (\bar{S}, \Lambda_{\bar{S}})\) to the existence of equidimensional models over \(S\).

In the last part of this section we use Mori’s algorithm [Mo] to characterize those elliptic fibrations \(X \to S\) which have birationally equivalent fibration \(\bar{X} \to S\) satisfying the pullback property.

\(\bar{X}\) has at most terminal singularities, and the canonical bundle is numerically equivalent to the pullback of a divisor on the surface so that the canonical divisor is \(\bar{\pi}\)-nef. \((K_{\bar{X}}\) is said to be \(\bar{\pi}\)-nef when \(K_{\bar{X}} \cdot \Gamma \geq 0\) for all curves \(\Gamma\) contracted by \(\bar{\pi}\), see for example, [W].)
Example 1.11 (M. Gross). Take \( E \times \mathbb{P}^1 \), where \( E \) is a smooth elliptic curve and \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) the minimal ruled surface with exceptional \(-1\) curve \( \Gamma \). Performing a “suitable” logarithmic transformation on 2 different fibers of \( f \) (say \( f_1 \) and \( f_2 \)) we obtain a smooth elliptic threefold \( \pi : X \to \mathbb{P}^1 \) with \( \Sigma \) the (disjoint) curves \( f_1 \) and \( f_2 \). The type of the general fiber over each \( f_i \) is \( 3I_0 \). Furthermore \( K_X \equiv \pi^*(K_S + \Lambda) \).

Consider the elliptic fibration \( \epsilon : X \to \mathbb{P}^2 \), induced by \( \psi : \mathbb{P}^1 \to \mathbb{P}^2 \) (\( \Gamma \) is contracted by \( \psi \)). Then \( \Lambda_{\mathbb{P}^2} = \frac{2}{3}h + \frac{2}{3}h \) (\( h \) is the class of a line), \( \Lambda_{\mathbb{P}^1} = \frac{2}{3}f_1 + \frac{2}{3}f_2 \) while \( \psi^*(\Lambda_{\mathbb{P}^2}) = \frac{2}{3}f_1 + \frac{2}{3}f_2 + \frac{4}{3}\Gamma \).

\( X \) cannot have a birational equivalent model \( \tilde{\pi} : \tilde{X} \to \mathbb{P}^2 \) such that \( K_{\tilde{X}} \equiv \tilde{\pi}^*(K_{\mathbb{P}^2} + \Lambda) \). In fact \( K_X \equiv \epsilon^*(K_S + \Lambda) - \frac{1}{3}\Gamma \) (1.12). See also 4.3.2.

More generally the following hold:

**Lemma 1.12.** Let \( \pi_i : X_i \to S, i = 1, 2 \) be birationally equivalent elliptic fibrations, \( X_i \) with terminal singularities, \( S \) with log terminal singularities. Assume that

\[
K_{X_i} \equiv \pi_i^*(K_S + \Lambda) + D_i,
\]

for some \( \mathbb{Q} \)-divisor \( D_i \). Then \( D_1 \) is effective if and only if \( D_2 \) is effective.

**Proof.** Let \( f_i : X \to X_i \) be a common resolution with exceptional divisors \( F_k^i \). Then

\[
K_X \equiv f_1^*(K_{X_1}) + \sum e_k F_k^1 \equiv \pi^*(K_S + \Lambda) + f_1^*(D_1) + \sum e_k F_k^1,
\]

where \( \sum e_k F_k^1 \) is an effective divisor. Then \( f_1^*(D_1) + \sum e_k F_k^1 \equiv f_2^*(D_2) + \sum e_k F_k^2 \) and \( |m(f_1^*(D_1)) + \sum e_k F_k^1| = |m(f_2^*(D_1))| \forall m \gg 0 \). The statement follows. \( \square \)

**Theorem 1.13.** Let \( \pi : X \to S \) be an elliptic fibration, \( X \) with terminal and \((S, \Lambda)\) with log terminal singularities. Assume that \( K_X \equiv \pi^*(K_S + \Lambda) + E - G \), as in (1.7), with \( E - G \) effective. Then

\[
(1.13.1) \quad K_X \equiv \pi^*(K_S + \Lambda) + E.
\]

(1.13.2) Furthermore there exists a birationally equivalent fibration \( \tilde{\pi} : \tilde{X} \to S \) such that \( \tilde{X} \) has only \( \mathbb{Q} \)-factorial terminal singularities, \( K_{\tilde{X}} \) is \( \tilde{\pi} \)-nef and

\[
rK_{\tilde{X}} = \tilde{\pi}^*r(K_S + \Lambda).
\]

**Proof.** (1.13.1) Let \( E - G = D \) be an effective divisor (1.7). Write \( D = L + M_+ - M_- \), with \( L, M_+, M_- \) effective divisors, such that \( L \) is the part with no non-flat component, while \( M_+ \) and \( M_- \) map via \( \pi \) to a subset of \( S \) of codimension \( \geq 2 \).

Without loss of generality we can assume also that \( M_+ \) and \( M_- \) have no common component. \( L + M_+ \sim D + M_- \). For any curve \( C \) in \( S \) such that \( \pi^{-1}(C) \) is a smooth surface, \( L_{\pi^{-1}(C)} = \sum s \sum_k l_{k,s} \), where \( l_{k,s} \) are the exceptional curves on the surface \( \pi^{-1}(C) \) (with multiplicity) and \( s \in C \cap \Sigma \).
For a fixed point \( s \), \( \pi^{-1}(s) = \sum_k l_k + e \), where \( e \) is the non-exceptional part of the fiber and the intersection matrix of the \( \{l_k\} \)'s is negative definite (Zariski's Lemma [Z]). Hence:

\[
(L + M_+) \cdot \sum_k l_k = L \cdot \sum_k l_k = (\sum_k l_k)^2 < 0.
\]

Therefore there exists a curve \( l_{k_0} \in L \) which is in the base locus of \( L + M_+ \); also since the set of such \( s \) is dense in \( \pi(L) \), there exists a component of \( L \), say \( L_1 \), in the fixed component of \( L + M_+ \). Write

\[
|L + M_+| = L_1 + |(L - L_1) + M_+|;
\]

then the intersection matrix of the curves \( (L_1)_{|\pi^{-1}(s)} \) is negative definite. Proceeding by induction on the number of components, we conclude that \( L \) is in the fixed component of \( |L + M_+| \).

Hence \( |L + M_+| = L + |M_+| \). But the linear system \( |M_+| \) consists only of the fixed component \( M_+ \) (\( M_+ \) maps to points via \( \pi \)), therefore each component of \( M_- \) has to be a component of \( L \) or of \( M_+ \), contradicting the assumption.

(1.13.2) The above argument shows also that if \( E \neq \emptyset \), then \( K_X \) is not \( \pi \)-nef. Using the relative version of Mori’s theory we know that \( X \) is birationally equivalent to a projective threefold \( \bar{X} : \bar{X} \rightarrow S \) such that \( \bar{X} \) has only \( \mathbb{Q} \)-factorial terminal singularities, and \( K_{\bar{X}} \) is \( \bar{\pi} \)-nef [Mo; 0.3.10]. Without loss of generality we can assume that \( \mu : X \rightarrow \bar{X} \) is a morphism. Then \( K_{\bar{X}} \equiv \mu^*(K_{\bar{X}}) + \sum e_iE_i, \; e_i > 0 \forall i \).

Comparing with 1.13.1 we have

\[
\mu^*(K_{\bar{X}} - \bar{\pi}^*(K_S + \Lambda)) = E - \sum e_iE_i.
\]

We need to prove that \( E \) is exceptional for \( \mu \). This follows from [G2; proof of Theorem 1.1 (Case 2), pages 294-295]. \( \square \)

**Corollary 1.14.** \( E - G \) (as in 1.7) is an effective \( \mathbb{Q} \)-divisor, if and only if there exists a birationally equivalent fibration over \( \bar{\pi} : \bar{X} \rightarrow S \) such that \( \bar{X} \) has only \( \mathbb{Q} \)-factorial terminal singularities and \( K_{\bar{X}} \equiv \bar{\pi}^*(K_S + \Lambda) \).

If \( \pi : X \rightarrow S \) is an elliptic fibration between smooth varieties, with no multiple fiber over divisors and \( \Lambda \) with simple normal crossings, then there exists a birationally equivalent fibration \( \bar{X} \rightarrow S \) such that \( K_{\bar{X}} \equiv \bar{\pi}^*(K_S + \Lambda) \). In fact \( E - G \) is effective.

(1.15) Following Miranda we refer to the the double points of the ramification locus with simple normal crossing as the “collision points”.

The divisors of singular fibers on \( X \) which map to irreducible components of the ramification divisor intersecting at a point \( p \), are said to “collide at \( p \)”. In general the fiber over the intersection point of the ramification divisor is not one of Kodaira type. (Take a birational model of example 1.2.1, with a simple normal crossing ramification divisor.)
The varieties in consideration are assumed to be complex projective, normal and \( \mathbb{Q} \)-factorial (see [Wi]).

§2. Blowing down: minimal model program and flat models.

The goal of this paragraph is proving Theorem 2.3 and Corollary 2.7. We state the theorem in general form, and then we will apply it to the case studied by Miranda in section 4.

Lemma 2.1, [N1, 0.4], [YPG; Hr, III, ex 10.9]. Let \( \bar{\pi} : \bar{X}_0 \to \bar{S} \) be an elliptic fibration such that \( \bar{X}_0 \) has terminal singularities.

If \( K_{\bar{X}} \equiv \bar{\pi}^*(D) \), for some \( \mathbb{Q} \)-divisor \( D \) on \( \bar{S} \), then \( D \equiv K_{\bar{S}} + \Lambda_{\bar{X}_0/\bar{S}} \) and \( (\bar{S}, \Lambda_{\bar{X}_0/\bar{S}}) \) has log terminal singularities (as in 1.9.2).

If \( \bar{S} \) is smooth and \( \bar{\pi} \) is equidimensional, then \( \bar{\pi} \) is also flat.

Theorem 2.2. Let \( \bar{\pi} : \bar{X} \to S \) be an elliptic fibration, where \( \bar{X} \) is a threefold with terminal singularities such that \( K_{\bar{X}} \equiv \bar{\pi}^*(K_S + \Lambda) \).

Let \( \psi : S \to S_0 \) be a \( K_S + \Lambda \)-extremal contraction of an irreducible curve \( \Gamma \).

(2.2.1) Then there exists a collection of extremal rays \( \{ R_j \} \) on \( \bar{X} \), such that \( \bar{\pi}^*(R_j) \equiv \Gamma \), for all \( j \).

(2.2.2) Let \( \mu : \bar{X} \to \bar{X}_0 \) be the contraction map induced by all the extremal rays mapping to \( \psi^*(\Gamma) \), where \( \bar{X}_0 \) is a threefold with terminal singularities. If the support of a divisor \( E \) dominates \( \Gamma \), \( E \) is exceptional for \( \mu \).

(2.2.3) The induced fibration \( \bar{\pi}_0 : \bar{X}_0 \to S_0 \) is 1-dimensional over \( \psi(\Gamma) \) and \( K_{\bar{X}_0} \equiv \bar{\pi}_0^*(K_{S_0} + \Lambda_{\bar{X}_0/S_0}) \).

Proof.

(2.2.1) Let \( \varepsilon = \psi \cdot \bar{\pi} \); any curve \( \Gamma_\alpha \) in \( \bar{\pi}^{-1}(\Gamma) \) is mapped by \( \varepsilon \) to a point; furthermore if \( \Gamma_\alpha \) is not a fiber of \( \bar{\pi} \),

\[
K_{\bar{X}} \cdot \Gamma_\alpha = \bar{\pi}^*(K_S + \Lambda) \cdot \Gamma_\alpha < 0.
\]

Thus \( K_{\bar{X}} \) is not \( \varepsilon \)-nef and there exist a threefold \( \tilde{X}_0 \) and a morphism \( \pi_0 \) as in the diagram

\[
\begin{array}{ccc}
\tilde{X}_0 & \xleftarrow{\mu} & \bar{X} \\
\pi_0 \downarrow & & \bar{\pi} \downarrow \\
S_0 & \xleftarrow{\psi} & S
\end{array}
\]

such that \( K_{\tilde{X}_0} \) is \( \bar{\pi}_0 \)-nef [Mo].

\( \Gamma_\alpha \) belongs to the negative part of the cone \( \overline{NE}(\bar{X}) \) and we can write \( \Gamma_\alpha = \sum_i q_i + \sum_k R_k^\bar{X} \), where \( R_k^\bar{X} \) are extremal rays and \( q_i \) other non extremal generators of the cone. Now,

\[
\Gamma = \bar{\pi}_*(\Gamma) = \sum_j \bar{\pi}_*(q_j) + \sum_k R_k^\bar{X}.
\]
since $\Gamma$ is extremal the above equality implies that $\Gamma$ and $\bar{\pi}_* R_i^X$ belong to the same extremal ray on $S$, for all $i$. In particular $\Gamma \sim c \bar{\pi}_* R_i^X$, for some constant $c$ and thus $\Gamma = \pi_* R_i^X$.

(2.2.2) Let $E$ be a divisor which dominates $\Gamma$; without loss of generality we can assume $E$ irreducible. We will assume that $E$ is not contracted by $\mu$ and derive a contradiction.

By hypothesis there exists an infinite collection of curves $\Gamma_\alpha$ on $E$ such that $K_{X_0} \cdot \Gamma_\alpha < 0$. Note that $\mu$ is a composition of divisorial contraction and flips [Mo] and denote by $\mu_i : \bar{X}_{(i-1)} \to \bar{X}_i$ the first divisorial contraction, $E_i$ the exceptional divisor and $\bar{\pi}_i$ the induced fibration. There exists an infinite subcollection $\{\Gamma_\beta\}$ such that $K_{X_{(i-1)}} \cdot \Gamma_\beta < 0$ [G2, Lemma 0.6] and it is thus possible to choose the $\Gamma_\beta$ so that no one of them is contained in any of the $E_i$ (we have assumed that $E$ is not contracted by $\mu$). Therefore:

$$0 > K_{X_{(i-1)}} \cdot \Gamma_\beta - E_i \cdot \Gamma_\beta = K_{\bar{X}_i} \mu_{i*} \cdot (\Gamma_\beta), \forall \Gamma_\beta.$$ 

Iterating this process we get:

$$0 > K_{X_0} \cdot \mu_*(\Gamma_\gamma), \text{ for some } \gamma.$$

As above, write $\mu_*(\Gamma_\gamma) = \sum_i g_i + \sum_k R_k^X$, where $R_k^X$ are extremal rays. Then:

$$0 = \sum_j (\bar{\pi}_0)_*(g_j) + \sum_k (\bar{\pi}_0)_* R_k^X.$$

Hence $(\bar{\pi}_0)_* R_k = 0$, and $R_k$ belongs to the fiber over $\psi(\Gamma)$ contradicting the definition of $\mu$. Note that the extremal rays mapping to $\psi(\Gamma)$ are all the extremal rays mapping to points, because $K_{\bar{X}} \equiv \bar{\pi}_* (K_S + \Lambda)$.

(2.2.3) By 2.2.2, $\mu$ contracts all divisors dominating $\Gamma$. The idea of the proof is showing that all the divisors in $\bar{\pi}^{-1}(\Gamma)$ intersecting this first set of exceptional divisors are contracted as well, and so on.

Set:

$$\mathcal{E}_X = \{\bar{E} \subset \bar{X}, \text{ irreducible divisors such that } \bar{E} \in \bar{\pi}^{-1}(\Gamma)\}$$

$$\mathcal{S}_E = \{\bar{E}^k \in \mathcal{E}_X \text{ such that there exits a chain } (\bar{E}^k, \ldots, \bar{E}^1 = \bar{E}) \text{ of divisors} \}$$

$$\in \mathcal{E}_X \text{ such that } \bar{E}^j \cap \bar{E}^{j-1} \neq \emptyset \text{ and } \bar{E}^k \text{ dominates } \Gamma\}.$$ 

For any divisor $\bar{E} \in \mathcal{E}_X$, $\mathcal{S}_E \neq \emptyset$, because the fibers are connected. In fact, the connected component of $\mathcal{E}_X$ containing $\bar{E}$ would consist only of divisors mapping to points via $\bar{\pi}$. For any divisor $\bar{E} \in \mathcal{E}_X$ define the $\text{alt}$ of $\bar{E}$ as:

$$\text{alt}(\bar{E}) = \min\{M \text{ such that } (\bar{E}^M, \ldots, \bar{E}^0 = \bar{E}) \text{ is a chain as above }\}.$$ 

Note that such $M$ is bounded.
By (2.2.2) $\mu$ contracts all divisors dominating $\Gamma$, i.e. all divisors of alt zero. Assume by induction that $\mu$ contracts all the divisors of alt $(M - 1)$ but not the ones of alt $M$, and derive a contradiction.

Let $\bar{E}$ be a divisor such that alt$(E) = M$ and $(\bar{E}^M, \ldots, \bar{E}^0 = \bar{E})$ the corresponding minimal chain. Then

$$K_{\bar{X}} = [\mu^*(K_{\bar{X}^0}) + \sum b_i\bar{E}^i],$$

where $b_i > 0$; note that each $\bar{E}^i \in \mathcal{S}_{\bar{E}}$ is in the summand and that by assumption $\bar{E}^i \neq \bar{E}$, $\forall i$.

Note that there exists a family of curves $\{C_\gamma\}$ in $\bar{E}$ such that $\bar{E}^i \cdot C_\gamma \geq 0$, for any $\bar{E}^i$ in the summation and $\bar{E}^j \cdot C_\gamma > 0$ for some $\bar{E}^j$ in the chain. Note also that $K_{\bar{X}} \cdot C_\gamma = 0$, since alt$(\bar{E}) > 0$. By [G2, Lemma 1.6] (stated below) this implies that $0 > \mu^*(K_{\bar{X}^0}) \cdot C_\beta = K_{\bar{X}^0} \cdot \mu_*(C_\beta)$ for any curve $C_\beta$ in some subfamily $\{C_\beta\}$ of $\{C_\gamma\}$, contradicting $(K_{\bar{X}^0}) \pi^0$- nef.

It follows also that $K_{\bar{X}^0} \equiv \pi^0_*(K_{S^0} + \Lambda_{X^0/S^0})$. $\Box$

Lemma [G2, 1.6]. Let $\mu : X \to X^+$ be a “flip” map between threefolds. If $\{C_\Gamma\}$ is an infinite collection of curves such that $K_X \cdot C_\Gamma < 0$, for all $\Gamma$, then there exists an infinite subcollection $\{C_\beta\}$ of $\{C_\Gamma\}$ such that for every $C_\beta$, $K_X + \mu_*(C_\beta) < 0$.

If $\psi$ is a $(K_S + \Lambda)$-extremal contraction of $\Gamma$, then in particular $K_S + \Lambda \neq \psi^*(K_{S^0} + \Lambda_0)$ and $K_X \neq \epsilon^*(K_{S^0} + \Lambda_0)$.

The following proposition is in some sense the converse of the previous one.

**Proposition 2.3.** Let $X$ and $\bar{X}$ be threefolds with terminal singularities and

$$\begin{array}{ccc}
X & \bar{X} \\
\pi \searrow & \nearrow \epsilon \\
S^0
\end{array}$$

two birationally equivalent elliptic fibrations. Assume that $K_{\bar{X}}$ and $K_X$ are numerically equivalent to the pullback of a $\mathbb{Q}$-divisor on $S^0$.

Then $\epsilon$ is equidimensional if and only if $\pi$ is equidimensional (flat if $S^0$ is smooth).

**Proof.** Proposition 2.3 is a particular case of the following Proposition 2.4.

In fact $K_{\bar{X}} \equiv \epsilon^*(K_{S^0} + \Lambda_{S^0})$ and $K_X \equiv \pi^*(K_{S^0} + \Lambda_{S^0})$, by 2.1 [N1, 0.6] and (1.9). $\Box$

**Proposition 2.4.** Let $\bar{X}$ be a variety with terminal singularities and

$$\begin{array}{ccc}
X & \bar{X} \\
\pi \searrow & \nearrow \epsilon \\
S^0
\end{array}$$

two birationally equivalent elliptic fibrations. Assume that $K_{\bar{X}}$ and $K_X$ are numerically equivalent to the pullback of a $\mathbb{Q}$-divisor on $S^0$.

Then $\epsilon$ is equidimensional if and only if $\pi$ is equidimensional (flat if $S^0$ is smooth).
two birationally equivalent fibrations. Assume that \( K_{\tilde{X}} \equiv \epsilon^*(D), \) \( K_X \equiv \pi^*(D), \)
for some \( \mathbb{Q} \)-divisors \( D \) on \( S_0 \) and that the fiber of \( \epsilon \) over a point \( Q \) on \( S_0 \) does not contain any divisor. Then the fiber of \( \pi \) over a point \( Q \) does not contain any divisor.

**Proof.** Let \( \tilde{X} \) be a common resolution of \( X \) and \( X \) and let \( \{F_i\} \) be the exceptional irreducible divisors of the morphism \( f : \tilde{X} \to \bar{X} \) and \( \{G_i\} \) of \( g : \tilde{X} \to X \).

Then

\[
K_{\tilde{X}} \equiv f^*(K_X) + \sum d_i F_i \equiv f^* \cdot \epsilon^*(D) + \sum d_i F_i \\
\equiv g^*(K_X) + \sum c_i G_i \equiv f^* \cdot \epsilon^*(D) + \sum c_i G_i
\]

If \( T \) is an irreducible divisor \( X \) mapping to the point \( Q \), then \( T = F_i \), for some index \( i \). Since \( X \) has terminal singularities, then \( F_i \) has a positive coefficient in the formula of the canonical bundle and has to be exceptional also for \( g \). Contradiction. \( \square \)

**Remark.** The above proposition 2.4 has no assumption on the dimensions and the nature of the fibration.

**Theorem 2.5.** Let \( \tilde{\pi} : \tilde{X} \to S \) an elliptic fibration of threefolds with terminal singularities such that \( K_{\tilde{X}} \equiv \tilde{\pi}^*(K_S + \Lambda_{X/S}) \).

Let \( \psi : S \to S_0 \) be a birational morphism with irreducible exceptional divisor \( \Gamma \) and write:

\[
K_S + \Lambda_{X/S} = \psi^*(K_{S_0} + \Lambda_{\tilde{X}/S_0}) + \delta \Gamma.
\]

(2.5.1) If \( \delta \geq 0 \) there exists a birationally equivalent elliptic fibration \( \tilde{\pi}_0 : \tilde{X}_0 \to S_0 \), such that \( \tilde{X}_0 \) is a threefold with terminal singularities and

\[
K_{\tilde{X}_0} \equiv \tilde{\pi}_0^*(K_{S_0} + \Lambda_{\tilde{X}_0/S_0}).
\]

Moreover, \( \delta > 0 \) if and only if the fiber over \( \psi(\Gamma) \) of any such \( \tilde{\pi}_0 \) is 1-dimensional.

(2.5.2) If \( \delta < 0 \), no birational equivalent fibration \( \tilde{\pi}_0 : \tilde{X}_0 \to \bar{S}_0 \) (\( \tilde{X}_0 \) with terminal singularities) can satisfy \( K_{\tilde{X}_0} \equiv \tilde{\pi}_0^*(K_{S_0} + \Lambda_{\tilde{X}_0/S_0}) \). Furthermore \( \psi \cdot \tilde{\pi} \) cannot have an equidimensional model.

**Remarks.**

(2.6.1) In the above theorem \( \delta > 0 \) if and only if \( \psi \) is a \( (K_S + \Lambda) \)-extremal contraction (1.13).

(2.6.2) We do not make assumptions on the equidimensionality of \( \tilde{\pi} \) over \( \Gamma \).

(2.6.2) 2.5.1 always holds if \( S_0 \) is smooth, \( \Sigma_0 \) has normal crossing and the fibration has a section (1.13), (4.3). 2.5.2 can actually occur, see 4.3.2 and Example 1.11.

**Proof.** Set \( \epsilon = \psi \cdot \tilde{\pi} : \tilde{X} \to \bar{S}_0 \).

(2.5.1): If \( \delta = 0 \), then

\[
K_{\tilde{X}} \equiv \tilde{\pi}^*(K_S + \Lambda_{X/S}) \equiv \tilde{\pi}^*(\psi^*(K_{S_0} + \Lambda_{\tilde{X}_0/S_0})) \equiv \epsilon^*(K_{S_0} + \Lambda_{\tilde{X}_0/S_0})
\]

and the fiber of \( \psi \cdot \tilde{\pi} \) over \( \psi(\Gamma) \) contains a divisor. The statement follows from 2.3. Proposition 2.3 proves the other part of the statement.
(2.5.2): Set $\beta = -b < 0$. Then $K_{\bar{X}} \equiv \epsilon^*(K_{\bar{S}_0} + \Lambda_{\bar{X}_0/S_0}) - b\pi^*(\Gamma)$. $K_{\bar{X}}$ is $\pi$-nef by hypothesis and $\epsilon$-nef. All birationally equivalent fibration over $S_0$ have a 2-dimensional component over $\psi^*(\Gamma)$, [Hm, 3.4].

Assume that $\bar{\pi}_0 : \bar{X}_0 \to S_0$ is a birationally equivalent fibration such that $\bar{X}_0$ has terminal singularities and $K_{\bar{X}_0} \equiv \pi_0^*(K_{\bar{S}_0} + \Lambda_{\bar{X}_0/S_0})$. Let $X$ be a common resolution, as in the diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \bar{X} \\
\downarrow{\epsilon} & & \downarrow{g} \\
S_0 & \xrightarrow{\pi_0} & \bar{X}_0
\end{array}
\]

Then $f^*(K_{\bar{X}}) = g^*(K_{\bar{X}_0})$ [Ko, 4.4]. That is, by commutativity of the diagram

\[
f^* \cdot \epsilon^*(K_{\bar{S}_0} + \Lambda_{\bar{X}_0/S_0}) - bg^*\pi_0^*(\Gamma) \equiv f^* \cdot \epsilon^*(K_{\bar{S}_0} + \Lambda_{\bar{X}_0/S_0}),
\]

contradicting $b \neq 0$. \hfill \Box

Nakayama shows the existence of an equidimensional model but with a different base surface [N2].

**Corollary 2.7.** Let $X$ be any elliptic threefold. Then then there exists an equidimensional birational equivalent elliptic fibration $\bar{\pi} : \bar{X} \to \bar{S}_n$ such that $\bar{X}_n$ has terminal singularities, $K_{\bar{X}_n} \equiv \pi_n^*(K_{\bar{S}_n} + \Lambda_{\bar{S}_n})$, and $(\bar{S}_n, \Lambda_{\bar{S}_n})$ has log terminal singularities. Then, either

(2.7.1) the Kodaira dimension of $X$ is non negative and $\bar{X}_n$ is minimal. Or

(2.7.2) there exists a morphism $\phi : \bar{S}_n \to C$ such that $\dim(\bar{S}_n) < \dim(C)$ and the general fiber of $\phi \cdot \bar{\pi}$ is Fano. If $\bar{S}_n$ is smooth, then $\bar{\pi}$ is flat.

**Proof.** Nakayama [N2] proves that there always exists an equidimensional birational equivalent fibration $\bar{X} \to \bar{S}$ satisfying the first part of the statement. We need to show that (2.7.1) and (2.7.2) hold for some particular model.

If $K_{\bar{X}}$ is nef (that is, $K_{\bar{S}} + \Lambda_{\bar{X}/\bar{S}}$ is nef), then $\bar{X}$ is minimal and we are done. Otherwise we apply Mori’s algorithm to $(\bar{S}, (K_{\bar{S}} + \Lambda_{\bar{S}}))$. Each birational contraction $\bar{S} \to \bar{S}_1$ is thus log extremal and by 2.3.2 there exists an elliptic fibration $\bar{X}_1 \to \bar{S}_1$ satisfying the same part of the theorem. We can repeat the process until we obtain a minimal elliptic threefold $\bar{X}_n \to S_n$ or there exists a morphism $\phi : \bar{S}_n \to C$ such that $\dim(\bar{S}_n) < \dim(C)$. In the second case $X$ is uniruled [Mo]. The remaining statements follow from Lemma 2.1. \hfill \Box

Corollary 2.7 is a stronger version of Theorem 1.1 [G2], but the proof presented there is different. Nakayama’s argument [N2] implies 2.7.1, but not 2.7.2.
Proposition 2.8. Let $\tilde{\pi}: \tilde{X} \to S$ be an elliptic fibration where $\tilde{X}$ is a threefolds with terminal singularities and $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_S + \Lambda_{\tilde{X}/S}) + D$, with $D$ non effective.

Let $\psi: S \to S_0$ be a birational morphism with irreducible exceptional divisor $\Gamma$ and write:
\[ K_S + \Lambda_{\tilde{X}/S} = \psi^*(K_{S_0} + \Lambda_{\tilde{X}/S_0}) + \delta \Gamma. \]

If $\delta \leq 0$, no birational equivalent fibration $\tilde{\pi}_0: \tilde{X}_0 \to S_0$ ($\tilde{X}_0$ with terminal singularities) can satisfy $K_{\tilde{X}_0} \equiv \tilde{\pi}_0^*(K_{S_0} + \Lambda_{\tilde{X}_0/S_0})$. In particular $\tilde{\pi}$ cannot have an equidimensional birationally equivalent fibration over $S_0$.

If $\delta > 0$ such a $\tilde{\pi}_0: \tilde{X}_0 \to S_0$ exists only if $\delta \tilde{\pi}^*(\Gamma) - G$ is effective.

Proof. It follows from 1.14. Note that if an elliptic fibration is equidimensional then $E - G$, as in 1.13, is effective. \[ \square \]

§3. Blowing up: Miranda’s work.

In this section we review properties of Weierstrass models and Miranda’s construction: given an elliptic fibration with section, it is possible to choose a Weierstrass model $W \to T$, where $W$ has rational singularities and $T$ is smooth. The fibration $W \to T$ is thus flat, because it is equidimensional [Hr, III, ex 10.9].

The starting point of Miranda’s construction is blowing up $T$ to get a birationally equivalent fibration $\tilde{W} \to S_0$ with ramification locus with simple normal crossings, and then resolve the singularities of the threefold $\tilde{W}$. Note that the pullback of a Weiestrass model is a Weiestrass model.

Over a general point of each component of $\Sigma_{\tilde{W}/S_0}$ the singularities are uniform, and so can be resolved as surface singularities, maintaining the flatness of the fibration. Over the collision points the task is not as easy, but Miranda shows, checking case by case, that after blowing up the surface a sufficient number of times it is possible to resolve the singularities of the new threefold preserving the flatness.

Let $\tilde{\pi}: \tilde{X} \to S_0$ be the smooth threefold thus obtained (the smooth Miranda model). Then
\[ (3.1) \quad K_{\tilde{X}} \equiv \tilde{\pi}_0^*(K_{S_0} + \Lambda_0), \]

in fact $G = E = \emptyset$ in formula 1.7, because the $\tilde{\pi}$ is flat and the singularities over the general points of $\Sigma_{\tilde{W}/S_0}$ are resolved as surface singularities.

We say that $\tilde{X}_0$ is a terminal Miranda model if $\tilde{X}_0 \to S_0$ is a flat birationally equivalent elliptic fibration, satisfying 3.1 with terminal singularities along the collision. (Smooth points are terminal singularities, see for example [W].)

Miranda writes explicitly a local analytic equation for each collision. Using the explicit equations, he shows that the singularities of the threefold at the collisions denoted by $\bullet$ in the table below can be resolved preserving the flatness of the fibration. The next step in his algorithm is blowing up the double points of the
ramification divisor and describe the type of the singular fiber on (the generic point of) the exceptional locus.

The possible collisions and the types of the singular fiber on the exceptional locus are listed in the following table. Thus by further blowing up the surface all the collisions are reduced to the one denoted by • in the table below.

|   | II | IV | I₀* | IV* | II* | III | I₀* | III* | I₀*  |
|---|----|----|-----|-----|-----|-----|-----|------|------|
| II* | I₀ | II | IV | I₀* | IV* | II* | III | I₀*  | IV*  |
| IV* | •II* | I₀ | II | IV | I₀* | IV* | II* | III | I₀* |
| I₀* | •IV* | •II* | I₀ | II* | III | I₀* | IV* | II* | III |
| I₀ | IV | •I₀* | •IV* | III | I₀* | IV* | II* | III |
| I₀* | •IV* | III | I₀* | IV* | II* | III |
| III* | I₀ | III | I₀* | IV* | II* | III |
| I₀ | I₀* | •III* | I₀ | II* | III |
| III | •I₀* | II* | III |
| I₀* | •I₀* | II* |
| I₀ | •I₀* | II* |
| I₀ | •I₀* | II* |

It is easy to verify that also the collisions denoted by ◦, have a Miranda model. Note that the only terminal (and not smooth) singularities at the collisions occur at II-II and IV-IV (see Remark after Corollary 4.6); flatness follows from equidimensionality also in this case, (see also 2.1).

§4. Variations on an algorithm of Miranda.

We now apply Theorem 2.5 to the case of elliptic fibration $X_0 \to S_0$ with section over a smooth surface and ramification divisor with simple normal crossing. If $\pi_0$ has a section, then $n_i = 0$ in (1.7) and $\mathcal{O}_{S_0}(\Lambda_{S_0}) = \mathcal{O}_{S_0}(\Delta_{S_0})$ is a line bundle.

**Hypothesis 4.0.** All throughout §4, $\pi_0 : X_0 \to S_0$ is an elliptic fibration between smooth varieties, $\dim(X) = 3$ and $\Sigma_0$ is a divisor of simple normal crossings. $Q$ is a double point of $\Sigma_0 = \Sigma_{X/S\bar{0}}$, and $\psi : S \to S_0$ be the blow up of $S_0$ at $Q$, with exceptional divisor $\Gamma$.

Let $X$ be the resolution of the pullback of $X_0$ by $\psi$, as in the diagram:

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\pi_0} & X \\
\downarrow \pi & & \downarrow \pi \\
S_0 & \xleftarrow{\psi} & S
\end{array}
$$

As in §1, we set $\Lambda = \Lambda_{X/S}$ and $\Lambda_0 = \Lambda_{X_0/S_0}$.
It is clear that $\Lambda_0 = \psi_*(\Lambda)$ (1.9.4); in this section we compare $\psi^*(\Lambda_0)$ with $\Lambda$. Write

$$\psi^*(\Lambda_0) - \Lambda = \alpha \Gamma, \text{ for some } \alpha \in \mathbb{Q}. $$

**Notation 4.1.** Following 1.8, we write

$$\Lambda_0 = \sum a_k D_k + \frac{1}{12} J_\infty(S_0) + \sum \frac{n_i - 1}{n_i} Y_i, $$

and

$$\Lambda = \sum a_k \hat{D}_k + a(\Gamma) \Gamma + \frac{1}{12} J_\infty(S) + \sum \frac{n_i - 1}{n_i} \hat{Y}_i + \frac{n(\Gamma) - 1}{n(\Gamma)} \Gamma, $$

where $\hat{D}_i$ ($\hat{Y}_i$) denotes the strict transforms of the divisor $D_i$ ($Y_i$), $n(\Gamma)$ the multiplicity of the general fiber along $\Gamma$ and $a(\Gamma)$ is the coefficient associated to $\Gamma$ (as in Table 1, §1). We can write

$$\psi^*(\Lambda_0) = \sum a_k \hat{D}_k + \frac{1}{12} J_\infty(S) + \beta \Gamma, $$

for some $0 < \beta \in \mathbb{Q}$; in fact $J_\infty(S) = \psi^* J_\infty(S_0)$, see §1.

Comparing the two formulas, we have

$$\beta - a(\Gamma) - \frac{n(\Gamma) - 1}{n(\Gamma)} = \alpha. $$

**Lemma 4.2.** Suppose that $\pi_0$ has a section. Let $a_\ell$, $a_m$ be the coefficients determined by the monodromy around each of the two branches of the ramification divisor containing $Q$. Then $\beta = (a_\ell + a_m)$, $\alpha = \lceil \beta \rceil$ and thus $a(\Gamma) = \{ \beta \} = \{ a_\ell + a_m \}$, where $\{ x \}$ denotes the fractional part of $x$ and $\lceil x \rceil = x - \{ x \}$.

In particular $\alpha = 0$ or 1 and $\alpha \Gamma$ is an effective divisor.

**Note:** $\psi^*(\Lambda_0) - \Lambda = \alpha \Gamma$ is in general only a $\mathbb{Q}$-divisor. For example this is the case when the fibers over the generic points of the components of the ramification divisor containing $Q$ are multiple, or more generally when the fibration does not admit a section (see 4.3).

The proof of the Lemma shows that the monodromy around the 2 branches of the ramification divisor containing $Q$ determine the type of monodromy around the exceptional divisor $\Gamma$. If there exists a section in a neighborhood of $Q$ the monodromy matrices and the type of singular fiber determine each other uniquely (as we see in the proof of the Lemma below). This is not true in general, see Proposition 4.3.

**Proof.**

(4.2.0) Choose local coordinates such that $Q = (0,0)$ and the two branches of the ramification divisor at $Q$ are defined by $\Sigma_0^\ell : \{ x_0 = 0 \}$ and $\Sigma_0^m : \{ y_0 = 0 \}$. Denote by $\gamma_{x_0}$, $\gamma_{y_0}$, the monodromy matrices over the generic point of $\Sigma_0^\ell$ and $\Sigma_0^m$. 
The blow up of $S_0$ at $Q$, in the coordinate chart $x_1 = x_o$, $y_1 = y_o/x_o$, induces the change in monodromy $\gamma_{x_1} = \gamma_{x_o} \cdot \gamma_{y_1} = \gamma_{y_o}$. Therefore the monodromy matrix around the generic point of $\Gamma$, the exceptional divisor of the blow up, is the composition of the two monodromy matrices $\gamma_{x_o}$ and $\gamma_{y_o}$.

Case (4.2.1): The monodromy matrix around the general point of $\Sigma_0^\ell$ (resp. $\Sigma_0^m$) has finite order ([Kd], 7.3 pg. 1281).

The only monodromy matrices in $SL(2, \mathbb{Z})$ of finite order (2, 4, 3, 6) are the ones listed in [Kd]. (It is enough to show it for $SL(2, \mathbb{Z})/\pm\text{Id}$, see [L, §4 Thm 1].) Thus, the type of fiber, the monodromy matrix and its eigenvalue (in the closed upper half plane) determine each other uniquely. Furthermore, for each fixed $J$ value, any two monodromy matrices commute. The eigenvalue of the matrix around the general point of $\Sigma_0^\ell$ (resp. $\Sigma_0^m$) is $e^{2\pi ia_\ell}$ (resp. $e^{2\pi ia_m}$) (see Table 1, §1). Therefore the eigenvalue of the matrix around the general point of $\Gamma$ is $e^{2\pi i(a_\ell+a_m)}$, and the coefficient of $\Gamma$ in 2.2.3. is $\{a_\ell+a_m\}$.

In the notation of 4.2, we have $a(\Gamma) = \{a_\ell+a_m\}$, $\beta = (a_\ell+a_m)$ and $\alpha = \land a_\ell+a_m \land$. Note that, since $a_\ell$ and $a_m$ are rational numbers both strictly less then 1, then $\land a_\ell+a_m \land = 1$ or 0 and $\psi^*(\Lambda_0) - \Lambda$ is an effective divisor.

Case (4.2.2): $J$ has a pole at $Q$.

Let $B_j$ be the irreducible divisor supported on $\Sigma_0^j$. Recall that $J_\infty(S) = \psi^*J_\infty(S_0)$, see §1. By 4.2.0, the monodromy actions around $\Sigma_0^\ell$ and $\Sigma_0^m$ determine the monodromy action around $\Gamma$ and thus $a(\Gamma)$, by Table 1, in Section 1.

Case (i): The fibers over the general point of $\Sigma_0^j$, $j = \ell, m$ is of type $I_{b_j}$. In this case $a(B_m) = a(B_\ell) = 0$ and thus $\beta = a(\Gamma) = 0$. Then $\psi^*(\Delta_0) = \Delta$.

Case (ii) The fiber over the general point of $\Sigma_0^\ell$ is of type while the fiber over the general point of $\Sigma_0^m$ is of type $I_{b_m}^*$. Then $a(B_m) = 1/2$, $a(B_\ell) = 0$ and $\beta = a(\Gamma) = 1/2$.

Case (iii) The fibers over the general point of $\Sigma_0^j$, $j = \ell, m$ is of type $I_{b_j}^*$, $j = \ell, m$. In this case $a(B_m) = a(B_\ell) = 1/2$ and thus $\beta = 1$, while the combined monodromy actions give $a(\Gamma) = 0$. Consequently $\alpha = 1$. □

More generally, the following hold,

**Proposition 4.3.** Notation as in 4.0. Then $0 \leq \alpha < 2$, with $\alpha \in \mathbb{Q}$; actually, $0 \leq \alpha \leq 1$ except in case 4.3.2, below.

**Proof.** Lemma 4.2. relates the behavior of the monodromy transformations. This determines the type of the singular fiber over $\Gamma$ if there is a section around $Q$. In that case $\alpha = 0, 1$.

Case (4.3.1): The general fiber over the two branches of the ramification divisor at $Q$ is not multiple, but the general fiber over $\Gamma$ is multiple.
Such examples can occur [DG], but the collision has to be of type $III-III^*$, $IV-IV^*$, $I_0^*-I_0$, because $a(\Gamma) = 0$, by Lemma 4.2 and Table 1, §1. In this case $\beta = 1$ and $0 \leq \alpha = \frac{1}{n(\Gamma)} < 1$.

Case (4.3.2): The general fibers over the two branches of the ramification divisor are multiple, of multiplicity $n_1$ and $n_2$.

Then $a_\ell = a_m = a(\Gamma) = 0$ and $\beta = 2 - \frac{1}{n_1} - \frac{1}{n_2} > 1$. Thus

$$0 \leq \alpha = 2 - \frac{1}{n_1} - \frac{1}{n_2} - 1 + \frac{1}{n(\Gamma)} < 2.$$ 

The cases with $\alpha > 1$ actually occur (see example 1.11); for a description of the relations between $n_1$, $n_2$ and $n(\Gamma)$, see [Gm].

Case (4.3.3): The general fiber over only one branch (say $\Sigma_{0}^n$) is multiple, of multiplicity $n_1$.

(i) Assume $a_\ell > 0$. Then $n(\Gamma) = 0$, $a(\Gamma) = a(\ell)$ and

$$0 \leq \alpha = a_\ell - a(\Gamma) + 1 - \frac{1}{n_1} < 1.$$ 

(ii) Assume $a_\ell = 0$. Then $a(\Gamma) = a(\ell) = 0$ and $n(\Gamma) = n_1$. Thus $\alpha = 0$. □

Note that $\Gamma$ is log extremal if and only if $(K_S + \Lambda) \cdot \Gamma < 0$, that is

$$\psi^*(K_{S_0} + \Lambda_0) \cdot \Gamma + (1 - \alpha)\Gamma \cdot \Gamma = (1 - \alpha)\Gamma \cdot \Gamma < 0.$$ 

Then $1 - \alpha > 0$, that is $\alpha < 1$.

**Corollary 4.4.** With the notation of 4.0 and 4.2, assume that $\pi_0$ has a section. The $\psi$ (the blow up of the collision point $Q$) is $(K_S + \Lambda)$-extremal if and only if $\alpha = 0$. In particular

(4.4.1) If $J$ has a pole at $Q$, then $\psi$ is a log extremal contraction unless the type of the fiber over the generic point of each component of the ramification divisor containing $Q$ is of type $I^*_{b_j}$, for some $b_j$.

(4.4.1) If $J(Q) \neq \infty$, then $\psi$ is a log extremal contraction if and only if $\sqcup a_\ell + a_m = \alpha = 0$.

**Proof.** $\psi$ is log extremal if and only if $0 \leq \alpha < 1$, that is $\alpha = 0$, by Lemma 4.2 □

**Corollary 4.5.** If $\pi_0$ has a section, $\alpha = 0$ if and only if $X_0$ is birationally equivalent to an elliptic threefold $\tilde{\pi}_0 : \tilde{X}_0 \rightarrow S_0$, with 1-dimensional fiber over $\psi(\Gamma)$, such that $\tilde{X}_0$ is a threefold with terminal singularities and $K_{\tilde{X}_0} \equiv \tilde{\pi}^*(K_{S_0} + \Lambda \tilde{X}_0/S_0)$.

**Proof.** Since $X \rightarrow S$ does not have multiple fiber in codimension 1, there exists a birationally equivalent fibration $\tilde{\pi} : \tilde{X} \rightarrow S$ such that $K_{\tilde{X}} \equiv \tilde{\pi}^*(K_S + \Lambda)$ (Theorem 1.13). By the above proposition 4.3, $\psi$ is a log extremal contraction if and only if $\alpha = 0$. Then the statement follows from Theorem 2.5. □

The following corollary shows which of the collisions have a flat (terminal) resolution and which do not (the “bad collisions”).
Corollary 4.6. Let $\pi_0 : X_0 \to S_0$ be a resolution of a Weierstrass model $W \to S_0$ (as in §3). Then:

(4.6.1) The collision above and on the diagonal in the following tables, together with the collisions of type $I_{b_1}^* - I_{b_m}^*$, cannot have a Miranda model over $S_0$. (These are the “bad” collisions.)

(4.6.2) The other collisions have a flat (over $Q$) terminal model $\bar{\pi}_0 : \bar{X}_0 \to S_0$ such that $K_{\bar{X}_0} \equiv \bar{\pi}^*(K_{S_0} + \Lambda_{X_0/S_0})$.

(4.6.3) Blowing up a finite number of times it is possible to replace the “bad” collisions, with the other ones. The number of blowups needed depends only on $\alpha = a_\ell + a_m$.

| $a_\ell$(type) | $1/4(III)$ | $1/2(I_0^*)$ | $3/4(III^*)$ |
|-----------------|------------|--------------|--------------|
| $a_m$(type)     | $\beta$(type) |              |              |
| 3/4(III*)       | $1(I_0)$   | $5/4(III)$   | $3/2(I_0^*)$ |
| 1/2(I_0^*)      | $3/4(III^*)$ | $1(I_0)$     | $5/4(III)$   |
| 1/4(III)        | $1/2(I_0^*)$ | $3/4(III^*)$ | $1(I_0)$     |
| 1/6(II)         | $1/3(IV)$  | $1/2(I_0^*)$ | $2/3(IV^*)$  | $5/6(II^*)$ |
| 5/6(II*)        | $1(I_0)$   | $7/6(II)$   | $4/3(IV)$   | $3/2(I_0^*)$ | $5/3(IV^*)$ |
| 2/3(IV*)        | $5/6(II^*)$ | $1(I_0)$   | $7/6(II)$   | $4/3(IV)$   | $3/2(I_0^*)$ |
| 1/2(I_0^*)      | $2/3(IV^*)$ | $5/6(II^*)$ | $1(I_0)$   | $7/6(II)$   | $4/3(IV)$ |
| 1/3(IV)         | $1/2(I_0^*)$ | $2/3(IV^*)$ | $5/6(II^*)$ | $1(I_0)$   | $7/6$ |
| 1/6(II)         | $1/3(IV)$  | $1/2(I_0^*)$ | $2/3(IV^*)$ | $5/6(II^*)$ | $1(I_0)$ |

Proof. Recall that $\beta = (a_\ell + a_m)$, $\alpha = \cup \beta \cup$ and $X_0 \to S_0$ has 1-dimensional fibers outside the collision points (see §3).

Let $Q$ be a collision point and $\psi$ the blow up at $Q$. The induced fibration $\pi : X \to S$ has also 1-dimensional fibers outside the collision points and $\Gamma$. 
(4.6.1) In case (4.6.1) \( \psi \) is not a log extremal contraction (4.2-4.4). If a Miranda model \( \tilde{\pi} : \tilde{X} \to S_0 \) exists, then \( \tilde{\pi} \) would be flat and

\[ K_{\tilde{X}} \equiv \tilde{\pi}^*(K_{S_0} + \Lambda_0), \]

contradicting 2.5.

(4.6.2) follows from Corollary 4.5.

(4.6.3) If \( \alpha = \downarrow \beta \downarrow = 1 \), then \( a(\Gamma) < \max\{a_\ell, a_m\} \). □

Mori’s contraction algorithm outlined in 2.2 gives an explicit way to construct a Miranda (terminal) model \( \bar{\pi} : \bar{X}_0 \to S_0 \) for all the “good” collisions.

All the Miranda (terminal) models are isomorphic outside the point \( Q \) and related by flop transformations on the collision fiber over \( Q \) (they are all minimal models relative to the respective fibrations) [Ko].

In particular the flop transformations preserve the type of analytic singularities [Ko].

The Weierstrass model around the collisions \( II-II \) is a terminal (but not smooth) Miranda model; the resolution of the equisingular locus of the collision of type \( IV-IV \) is also a terminal (but not smooth) Miranda model. All Miranda models (over the same base) of these collisions have to be terminal and not smooth [Ko].

All the other “good” collisions have smooth Miranda models.

§5. One more remark

The equation of example 1.2.1, \( y^2 = x^3 + sx + t \) defines a smooth threefold \( \pi_0 : X_0 \to S_0 \) in Weierstrass form.

Kawamata’s and Fujita’s starting points (described in §1) use the hypothesis of the ramification divisor with simple normal crossing. This is not the case for the above threefold, in fact there exists a cusp at \( s = t = 0 \).

One can blow up \( S_0 \) three times to resolve the cusp, pull back the fibration, and at each step resolve the singularities of the threefold. Let \( \psi_i : S_i \to S_{i-1}, i = 1, 2, 3 \) be the blow ups.

We then obtain smooth threefolds \( \pi_i : X_i \to S_i, i = 1, 2, 3 \). It is easy to see that

\[ \psi^*(\Lambda_{i-1}) = \Lambda_i \quad \forall i. \]

Each blow up \( \psi_i : S_i \to S_{i-1} \) is then in a \( K_{S_i} + \Lambda_i \)-extremal contraction (1.10). Theorem 2.5 applies and the Mori’s algorithm used in 2.2 gives us back the original threefold \( X_0 \). See [G1] for the explicit computations.

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