A criterion for homogeneous potentials to be 3-Calabi-Yau

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Abstract

Among the homogeneous potentials $w$ of degree $N+1$ in $n$ variables, it is an open problem to find precisely which of the $w$'s are 3-Calabi-Yau, although several examples are known. In this paper, we give a necessary and sufficient condition for this to hold when the algebra $A$ defined by the potential $w$ is $N$-Koszul of global dimension 3. As an application, we study skew polynomial algebras over non-commutative quadrics and we recover two families of 3-Calabi-Yau potentials which have recently appeared in the literature.

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1 Introduction

Following Ginzburg [17], $d$-Calabi-Yau algebras $A$ are defined by some natural finiteness constraints and a duality condition involving the Hochschild cohomology $H^*(A, A \otimes A^{op})$. After examination of various examples, several authors conjectured that any 3-Calabi-Yau algebra $A$ (satisfying some more or less natural assumptions) can be defined nicely by generators and relations from a certain non-commutative polynomial depending on $A$ and called potential. In this situation, one says that $A$ is a potential algebra or that $A$ is derived from a potential (see the precise definition in Section 2 below). The best result in this direction has been obtained by Van den Bergh, who proved that any complete 3-Calabi-Yau algebra is derived from a potential [24]. A proof for graded algebras $A$ generated in degree one had been previously given by Bocklandt [8].

Actually, a potential algebra is derived in a standard way from any potential, i.e. from any non-commutative polynomial. However it is not known which are the potentials $w$ such that the algebra $A$ derived from $w$ is 3-Calabi-Yau, even in the graded case. In this paper, we restrict ourselves to homogeneous potentials $w$ in $n \geq 1$ variables of degree one. Denote by $N+1$ the degree of $w$, where $N \geq 2$, so that the graded algebra $A$ derived from $w$ is $N$-homogeneous. It is known (Proposition 5.2 in [5]) that if $A$ is AS-Gorenstein of global dimension 3 (in particular if $A$ is 3-Calabi-Yau [7]), then $A$ is $N$-Koszul with $n \geq 2$, and its Hilbert series is the following:

$$h_A(t) = (1 - nt + nt^N - t^{N+1})^{-1}. \quad (1.1)$$
The main result of this paper is the following theorem proved in Section 2. In all the paper, the basic field $k$ has characteristic zero.

**Theorem 1.1** For any $n \geq 2$ and $N \geq 2$, let $A$ be a graded algebra in $n$ generators $x_1, \ldots, x_n$, derived from a homogeneous potential $w$ of degree $N + 1$. Assume that $A$ is $N$-Koszul. Then $A$ is 3-Calabi-Yau if and only if the Hilbert series of $A$ is given by (1.1).

Denoting by $V$ and $R$ the spaces of generators and relations of such an $N$-Koszul potential algebra $A$, condition (1.1) is equivalent to saying that the three following facts hold:

(i) $A$ has global dimension 3,
(ii) the cyclic partial derivatives $\partial_{x_1}(w), \ldots, \partial_{x_n}(w)$ are $k$-linearly independent,
(iii) the cyclic sum $c(w)$ of $w$ generates the space $R_{N+1} = (R \otimes V) \cap (V \otimes R)$.

In Theorem 6.8 of [9], Bocklandt, Schedler and Wemyss have proved that if $A$ is an algebra defined by a twisted potential $w$ and $A$ is $N$-Koszul, then $A$ is twisted $d$-Calabi-Yau if and only if a certain complex defined from $w$ is exact. In the non-twisted case with $d = 3$, our Theorem 1.1 is an improvement of their result: in order to conclude that $A$ is 3-Calabi-Yau, it is sufficient to know the Hilbert series, while exactness is in general hard to prove. It would be interesting to extend Theorem 1.1 to the general setting of [9].

The examples of applications of Theorem 1.1 we have in mind are quadratic (i.e., for $N = 2$) and they have recently appeared in various contexts. The example due to Smith is constructed from the octonions [19], and more generally those due to Suárez-Alvarez are constructed from the oriented Steiner triple systems [21]. The examples due to Berger and Pichereau come from a way to embed any non-degenerate non-commutative quadric (not necessarily 3-Calabi-Yau) into a 3-Calabi-Yau potential algebra by adding a variable [6]. Actually, as shown in the cited articles, each of these examples is a skew polynomial algebra $A$ over a non-degenerate non-commutative quadric $\Gamma$, in other words $A$ is an Ore extension of $\Gamma$. Then, using the basic properties of $\Gamma$ as stated in [3] (see also [6]), the 3-Calabi-Yau property is immediate from the following consequence (proved in Section 4 below) of Theorem 1.1.

**Corollary 1.2** For any $n \geq 2$, let $\Gamma$ be a non-degenerate non-commutative quadric in $n$ variables $x_1, \ldots, x_n$ of degree 1. Let $z$ be an extra variable of degree 1. Let $A$ be an algebra defined by a non-zero cubic potential $w$ in the variables $x_1, \ldots, x_n, z$. Assume that the graded algebra $A$ is isomorphic to a skew polynomial algebra $\Gamma[z; \sigma; \delta]$ over $\Gamma$ in the variable $z$, defined by a 0-degree homogeneous automorphism $\sigma$ of $\Gamma$ and a 1-degree homogeneous $\sigma$-derivation $\delta$ of $\Gamma$. Then $A$ is Koszul and 3-Calabi-Yau.

In the situation considered in [6], where the potential $w$ is defined by $w = uz$ for $u$ the relation of the quadric $\Gamma$, Ulrich Krähmer has asked the first author whether the automorphism $\sigma$ is related to the automorphism of Van den Bergh’s duality of $\Gamma$. We answer this question by proving in Section 4 that the dualizing bimodule of $\Gamma$ is isomorphic to $\sigma^{-1}\Gamma$.

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2 3-Calabi-Yau potential algebras

Throughout the article, $k$ will denote a field of characteristic zero and $V$ a $k$-vector space of finite dimension $n \geq 1$. We fix a basis $X = \{x_1, \ldots, x_n\}$ of $V$. The tensor algebra of $V$ is denoted by $T(V)$ and $T(V) = T(V) \otimes T(V)^{op}$. The symbol $\otimes$ will always be denoted by $\otimes$. The basis of $T(V)$ consisting of the non-commutative monomials in $x_1, \ldots, x_n$ is denoted by $\langle X \rangle$. The subspace of $T(V)$ generated by the commutators is denoted by $[T(V), T(V)]$.

The elements of the vector space $Pot(V) = T(V)/[T(V), T(V)]$ also denoted by $Pot(x_1, \ldots, x_n)$, are called potentials of $V$ or potentials in the variables $x_1, \ldots, x_n$. The algebra $T(V)$ is graded by $\text{deg}(x_i) = 1$ for all $i$. Since $[T(V), T(V)]$ is homogeneous, the space $Pot(V)$ inherits the grading. The linear map $c : T(V) \rightarrow T(V)$ defined on monomials $a = a_1 \ldots a_r$ of degree $r$ when all the $a_i$’s are in $X$, as the cyclic sum

$$c(a) = \sum_{1 \leq i \leq r} a_1 \ldots a_i a_1 \ldots a_{i-1}$$

induces $\tilde{c} : Pot(V) \rightarrow T(V)$ which actually does not depend on the choice of the basis $X$. Since the characteristic of $k$ is zero, this map $\tilde{c}$ defines a linear isomorphism from $Pot(V)$ onto $\text{Im}(c)$.

For any $x \in X$, the cyclic derivative $\partial_x : Pot(V) \rightarrow T(V)$ is the linear map defined on any $p \in \langle X \rangle$ by

$$\partial_x(p) = \sum_{p=uxv} vu,$$

where $u$ and $v$ are in $\langle X \rangle$. The “ordinary” partial derivative $\frac{\partial}{\partial x} : T(V) \rightarrow T(V) \otimes T(V)$ (see [23]) is the linear map defined on any monomial $p$ by

$$\frac{\partial p}{\partial x} = \sum_{p=uxv} u \otimes v,$$

which will be written as

$$\frac{\partial}{\partial x} = \sum_{1,2} \left( \frac{\partial p}{\partial x} \right)_1 \otimes \left( \frac{\partial p}{\partial x} \right)_2. \tag{2.1}$$

Considering $T(V)$ as the natural $T(V)$-bimodule, hence as a right $T(V)^{op}$-module, we verify that

$$1 \cdot \frac{\partial}{\partial x} = \partial_x(p).$$

Finally, for $x$ and $y$ in $X$, the second partial derivative $\frac{\partial^2}{\partial x \partial y} : Pot(V) \rightarrow T(V) \otimes T(V)$ is defined by

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial}{\partial x} \circ \partial_y.$$

For any $w \in Pot(V)$, we recall two basic formulas concerning the previous derivatives which can be easily proved. The first one is the non-commutative Euler relation (generalizing the well-known Euler relation for homogeneous commutative polynomials):

$$\sum_{1 \leq i \leq n} \partial_{x_i}(w)x_i = \sum_{1 \leq i \leq n} x_i \partial_{x_i}(w) = c(w), \tag{2.2}$$
where the constant term of \( w \) is assumed to be zero. The second one is the symmetry of the non-commutative Hessian \([23]\):

\[
\tau \left( \frac{\partial^2 w}{\partial x \partial y} \right) = \frac{\partial^2 w}{\partial y \partial x}
\]

where \( \tau : T(V) \otimes T(V) \to T(V) \otimes T(V) \) is the flip \( a \otimes b \mapsto b \otimes a \).

**Definition 2.1** For any \( w \in \text{Pot}(V) \), let \( I(\partial_x(w); x \in X) \) denote the two-sided ideal generated by all the cyclic partial derivatives of \( w \). We say that the associative \( k \)-algebra

\[
A = A(w) = T(V)/I(\partial_x(w); x \in X)
\]

is derived from the potential \( w \), or that it is the potential algebra defined from \( w \).

For the rest of this section, let us fix an integer \( N \geq 2 \) and a non-zero homogeneous potential \( w \) of degree \( N + 1 \). The space of homogeneous potentials of degree \( N + 1 \) is the following

\[
\text{Pot}(V)_{N+1} = \frac{V^{\otimes (N+1)}}{\sum_{i+j=N+1} [V^{\otimes i} \otimes V^{\otimes j}]} = \frac{V^{\otimes (N+1)}}{\sum_{i+j=N+1} V^{\otimes i} \otimes [V, V] \otimes V^{\otimes j}},
\]

and the class \( w \in \text{Pot}(V)_{N+1} \) will be often defined by a representative denoted again by \( w \). The \( k \)-algebra \( A \) derived from \( w \) is \( \mathbb{N} \)-graded and the relations \( \partial_x(w) \), \( x \in X \), are homogeneous of degree \( N \). So the graded algebra \( A \) is \( N \)-homogeneous [4]. Let us denote by \( R \) the space of relations of \( A \), i.e. the subspace of \( V^{\otimes N} \) generated by the relations \( \partial_x(w) \), \( x \in X \). The subspace \( R_{N+1} = (R \otimes V) \cap (V \otimes R) \) of \( V^{\otimes (N+1)} \) appears in the Koszul complex of \( A \) [2] and moreover, the non-commutative Euler relation (2.2) shows that

\[
c(w) \in R_{N+1}.
\]

Remark that \( c(w) \neq 0 \) since \( w \neq 0 \). The element \( c(w) \) will play the role of the volume form in the non-commutative setting. For convenience, we will sometimes omit the unadorned symbols \( \otimes \), e.g. we will write \( R_{N+1} = RV \cap VR, A \otimes A = AA, \ldots \)

Recall that the bimodule Koszul complex of \( A \) starts as follows [5]:

\[
ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AA \to 0,
\]

where \( d_1 \) and \( d_2 \) are \( A \)-linear and their restrictions to \( V \) (resp. to \( R \)) are defined by

\[
d_1(v) = v \otimes 1 - 1 \otimes v \in AA,
\]

\[
d_2(v_1 \ldots v_N) = \sum_{1 \leq i \leq N} (v_1 \ldots v_{i-1}) \otimes v_i \otimes (v_{i+1} \ldots v_N) \in AVA,
\]

for any \( v, v_1, \ldots v_N \) in \( V \). Moreover, via the multiplication \( \mu : AA \to A \), the sequence [23] can be extended to a minimal projective resolution of \( A \) in the category \( A \)-grMod-\( A \) of graded \( A \)-bimodules.

The graded vector space \( \ker d_2 \) lives in degrees \( \geq N + 1 \) and the linear map \( \varphi : R_{N+1} \to ARA \) defined by

\[
\varphi \left( \sum_{1 \leq i \leq n} x_i \otimes u_i \right) = \sum_{1 \leq i \leq n} v_i \otimes x_i = \sum_{1 \leq i \leq n} x_i \otimes u_i \otimes 1 - 1 \otimes v_i \otimes x_i
\]
where $u_i$ and $v_i$ are in $R$, is an isomorphism from $R_{N+1}$ to $(\ker d_2)_{N+1}$. Choose a graded subspace $E$ of $\ker d_2$ such that

$$\ker d_2 = E \oplus \sum_{i+j \geq 1} A_i(\ker d_2)_{N+1}A_j.$$ 

Note that $E$ lives in degrees $\geq N + 1$ and that $E_{N+1} = (\ker d_2)_{N+1}$. Then the complex

$$AEA \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AA \to 0,$$ \hspace{1cm} (2.6)

where $d_3$ is the $A$-$A$-linear extension of the inclusion of $E$ into $ARA$, can be extended via the multiplication $\mu$ of $A$ to a minimal projective resolution of $A$ in $\text{-grMod-}A$. Actually, using the isomorphism $\varphi$, we will assume that $E_{N+1} = R_{N+1}$ and that $d_3$ coincides with $\varphi$ on $R_{N+1}$. So $E_{N+1}$ contains the non-zero element $c(w)$.

**Lemma 2.2** Keep the above notation and assumptions. The global dimension of $A$ is equal to 3 if and only if $d_3$ is injective. In this case, $A$ is $N$-Koszul if and only if $E = R_{N+1}$.

**Proof.** The global dimension of $A$ is the length of a minimal projective resolution, hence the first equivalence follows. The second one is clear from the definition of $N$-Koszul algebra [2].

The next result is an immediate consequence of Proposition 5.2 in [5]. For the convenience of the reader, we give here a self-contained proof.

**Proposition 2.3** Let $k$ be a field of characteristic zero. Let $V \neq 0$ be an $n$-dimensional vector space. Let $w$ be a non-zero homogeneous potential of $V$ of degree $N + 1$ with $N \geq 2$. Let $A = A(w)$ be the potential algebra defined by $w$. Denote by $R$ the space of relations of $A$. Assume that $A$ is AS-Gorenstein of global dimension 3. Then

(i) $\dim R = n$ (with $n \geq 2$), so that $(\partial_{x_i} (w))_{1 \leq i \leq n}$ is a basis of $R$,

(ii) $E = R_{N+1}$ which is one-dimensional generated by $c(w)$,

(iii) $A$ is $N$-Koszul,

(iv) the Hilbert series of the graded algebra $A$ is given by $h_A(t) = (1 - nt + nt^N - t^{N+1})^{-1}$.

**Proof.** Since the global dimension of $A$ is equal to 3, the map $d_3$ in (2.6) is injective, so

$$0 \to AEA \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AA \to 0$$

is a minimal projective resolution of $A$ via $\mu$. Thus the AS-Gorenstein symmetry shows that $\dim R = \dim V$ and $\dim E = 1$, hence we get (i) (where $n = 1$ is easily ruled out) and (ii). Next we use the Lemma. The Hilbert series is immediately obtained from the exact complex

$$0 \to AEA \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AA \xrightarrow{\mu} A \to 0.$$ 

Throughout the rest of this section, $A$ is derived from a non-zero homogeneous potential $w$ of degree $N + 1$ ($N \geq 2$) in $n \geq 1$ variables, and we use the previous notation. Before proving Theorem 1.1 of the Introduction, we obtain some general results without assuming yet that $h_A(t) = (1 - nt + nt^N - t^{N+1})^{-1}$. In particular, we just have $\dim R \leq n$. We want to study the self-duality of the following complex denoted by $C_w$:

$$0 \to A \ker c(w) \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} A \ker A \to 0,$$ \hspace{1cm} (2.7)
where $k$ in $AA = AkA$ is the subspace of $AA$ generated by $1 \otimes 1$, and $kc(w)$ in $Ak(w)A$ is the subspace of $AR_{N+1}A$ generated by $c(w)$. Actually, (2.7) is a subcomplex of

$$0 \to E_\ast A \xrightarrow{d_1} A_\ast A \xrightarrow{d_2} A_\ast V A \xrightarrow{d_3} AkA \to 0.$$ We already know that it is exact at $AVA$ and that its homology at $AkA$ is isomorphic to $A$ (using the multiplication $\mu : AkA \to A$). Let $r_i = \partial_{x_i}(w)$ for $1 \leq i \leq n$. According to (2.2), we have

$$d_3(c(w)) = \sum_{1 \leq i \leq n} x_i \otimes r_i \otimes 1 - 1 \otimes r_i \otimes x_i.$$ (2.8)

Next we recall some general facts about duality of bimodules: let $A_1$ and $A_2$ be associative $k$-algebras and let $M$ be an $A_1$-$A_2$-bimodule. Denote by $A_1 \overset{1}{\otimes} A_2$ (resp. $A_1 \overset{i}{\otimes} A_2$) the $A_1$-$A_2$ (resp. $A_2$-$A_1$) bimodule $A_1 \otimes A_2$ for the outer (resp. inner) action. Set

$$M^\ast = \text{Hom}_{A_1 \rightarrow A_2}(M, A_1 \overset{o}{\otimes} A_2),$$

so that $M^\ast$ is an $A_2$-$A_1$-bimodule whose action comes from $A_1 \overset{i}{\otimes} A_2$. For any finite-dimensional $k$-vector space $E$ whose dual will be denoted by $E^\ast$, we have a natural isomorphism

$$\theta : (A_1 EA_2)^\ast \rightarrow A_2 E^\ast A_1$$

of $A_2$-$A_1$-bimodules that we are going to describe. Firstly, we have naturally

$$\text{Hom}_{A_1 \rightarrow A_2}(A_1 EA_2, A_1 A_2) \cong \text{Hom}_k(E, A_1 A_2).$$

Secondly, if $\gamma \in \text{Hom}_k(E, A_1 A_2)$, we define $\tilde{\gamma} \in E^\ast A_1 A_2$ by

$$\tilde{\gamma} = \sum_{i \in I} v_i^* \otimes \gamma(v_i),$$

where $(v_i)_{i \in I}$ is a basis of $E$ and $(v_i^*)_{i \in I}$ is its dual basis. Then

$$\text{Hom}_k(E, A_1 A_2) \cong E^\ast A_1 A_2$$

by the $k$-linear isomorphism $\gamma \mapsto \tilde{\gamma}$ whose inverse isomorphism is given by

$$\phi a_1 a_2 \mapsto (v \mapsto \phi(v)a_1 a_2), \quad \phi \in E^\ast, \quad a_1 \in A_1, \quad a_2 \in A_2.$$

Finally $E^\ast A_1 A_2 \cong A_2 E^\ast A_1$ is natural with respect to the obvious $A_2$-$A_1$-bimodule structures. Composing all these maps, we get the isomorphism $\theta$.

Throughout the sequel, we take $A_1 = A_2 = A$. For any chain complex $(C, d)$ of $A$-bimodules, the dual complex

$$C^\ast = \text{Hom}_{A \rightarrow A}(C, A \overset{o}{\otimes} A)$$

(as usual, Hom is graded) is a chain complex of $A$-bimodules whose differential

$$d^\ast_{\ast-n} : C^\ast_{n} = \text{Hom}_{A \rightarrow A}(C_{-n}, AA) \rightarrow C^\ast_{n-1}$$

is defined for any $n \in \mathbb{Z}$ by

$$d^\ast_{\ast-n}(f) = -(-1)^n f \circ d_{-1-n},$$ (2.9)
where \(d_{1-n}: C_{1-n} \to C_{-n} \) and \(f: C_{-n} \to AA\). Note that the sign \((-1)^n\) in this definition comes from the Koszul rule (see for example [11], formula (1), p. 81).

Let us compute the differential of the dual complex \(C^\vee_w\).

\[
0 \to Ak^*A \xrightarrow{d_1} AV^*A \xrightarrow{d_2} AR^*A \xrightarrow{d_3} Akc(w)^*A \to 0,
\]

(2.10)

where \(d_1^*\) denotes the image of \(d_1\) via the isomorphism \(\theta\). Firstly, we have \(d_1^* = \theta \circ d_1^* \circ \theta^{-1}\). From the dual basis \(1^* \in k^*\) of \(1 \in k\), we get that \(\theta^{-1}(1^*)\) coincides with the identity map \(1_{AA}\) of \(AA\), thus \(\gamma = d_1^*(\theta^{-1}(1^*))\) is defined in \(V\) by \(v \mapsto -1_{AA} \circ d_1(v) = 1 \otimes v - v \otimes 1\) (note that \(n = 0\) in (2.9)). We get \(\tilde{\gamma} = \sum_{1 \leq i \leq n} x_i^* \otimes (1 \otimes x_i - x_i \otimes 1)\), where \((x_i^*)_{1 \leq i \leq n}\) is the dual basis of the basis \((x_i)_{1 \leq i \leq n}\) of \(V\). We conclude that

\[
d_1^*(1^*) = \sum_{1 \leq i \leq n} x_i \otimes x_i^* \otimes (1 - 1 \otimes x_i^* \otimes x_i).
\]

(2.11)

Secondly, we have \(d_2^* = \theta \circ d_2^* \circ \theta^{-1}\). Recall that \(r_1 = \partial_x, \ldots, r_n = \partial_{x_n}\) generate \(R\). Fix a part \(J\) of \(\{1, \ldots, n\}\) such that \((r_j)_{j \in J}\) is a basis of \(R\), and denote by \((c_i^*)_{j \in J}\) its dual basis. We get that \(\theta^{-1}(x_i^*)\) coincides with the map \(AVA \to AA\), \(axb \mapsto \delta_{ij}ab\), thus \(\gamma_i = d_2^*(\theta^{-1}(x_i^*))\) is defined on \(R\) by \(r_j \mapsto \theta^{-1}(x_i^*) \circ d_2(r_j)\) since \(n = -1\) in (2.9). Using the symbolic writing (2.11), we have

\[
d_2(r_j) = \sum_{1 \leq i \leq n} \sum_{j \in J} \left( \frac{\partial r_j}{\partial x_s} \right)_{1} \otimes x_s \otimes \left( \frac{\partial r_j}{\partial x_s} \right)_{2}.
\]

(2.12)

Therefore \(\gamma_i(r_j) = \sum_{j \in J} \left( \frac{\partial r_j}{\partial x_s} \right)_{1} \otimes \left( \frac{\partial r_j}{\partial x_s} \right)_{2}\), which together with the equality \(\tilde{\gamma}_i = \sum_{j \in J} r_j^* \otimes \gamma_i(r_j)\) implies that, for any \(1 \leq i \leq n\),

\[
d_2^*(x_i^*) = \sum_{j \in J} \sum_{1 \leq i \leq n} \left( \frac{\partial r_j}{\partial x_s} \right)_{1} \otimes r_j^* \otimes \left( \frac{\partial r_j}{\partial x_s} \right)_{2}.
\]

(2.13)

Thirdly, we have \(d_3^* = \theta \circ d_3^* \circ \theta^{-1}\). We get that \(\theta^{-1}(x_i^*)\) (for \(i \in J\)) coincides with the map \(ARA \to AA\), \(ar_jb \mapsto \delta_{ij}ab\) with \(j \in J\), thus \(\gamma_i = d_3^*(\theta^{-1}(x_i^*))\) is defined on \(kc(w)\) by \(c(w) \mapsto -\theta^{-1}(x_i^*) \circ d_3(c(w))\) since \(n = -2\) in (2.9). Using (2.8), we obtain for any \(i \in J\),

\[
\gamma_i(c(w)) = 1 \otimes x_i - x_i \otimes 1 + \sum_{j \notin J} r_j^*(r_j)(1 \otimes x_j - x_j \otimes 1).
\]

Thus, for any \(i \in J\), we conclude that

\[
d_3^*(r_j^*) = x_i \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_i + \sum_{j \notin J} r_j^*(r_j)(x_j \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_j).
\]

(2.14)

In particular, if the \(A-A\)-linear map \(\mu_w: A(kc(w)^*)A \to A\) is defined by \(\mu_w(c(w)^*) = 1\), then we form the augmented complex

\[
C^\vee_w \xrightarrow{\mu_w} A \to 0.
\]

Moreover, the \(A-A\)-linear map \(f_0: AkA \to A(kc(w)^*)A\) defined by \(f_0(1) = c(w)^*\) is such that \(\mu_w \circ f_0 = \mu\).
Consider the diagram

\[
0 \to A(kc(w))A \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AkA \to 0 \\
\begin{array}{cccc}
\quad & f_3 & \quad & f_2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to Ak^* A & \xrightarrow{d_1^*} & AV^* A & \xrightarrow{d_2^*} & AR^* A & \xrightarrow{d_3^*} & A(kc(w))^* A & \to 0
\end{array}
\] (2.15)

where the \( A \)-linear maps \( f_1, f_2 \) and \( f_3 \) are given by

\[
f_1(x_i) = \begin{cases} 
  r_i^* & \text{if } i \in J \\
  0 & \text{if } i \notin J 
\end{cases}, \quad f_2(r_i) = x_i^* \text{ for any } i \in J, \quad f_3(c(w)) = 1^*.
\] (2.16)

Recall that \( J \) is a part of \( \{1, \ldots, n\} \) such that \( (r_j)_{j \in J} \) is a basis of \( R \) and \( (r_j^*)_{j \in J} \) is its dual basis. Clearly, \( f_0 \) and \( f_3 \) are bijective, \( f_1 \) is surjective and \( f_2 \) is injective. The following proposition shows that the diagram (2.15) is commutative if and only if \( \dim R = n \).

**Proposition 2.4**

(i) The central square in (2.15), i.e. the square limited by \( f_1 \) and \( f_2 \), is commutative.

(ii) Each of both remaining squares is commutative if and only if \( \dim R = n \).

(iii) If \( C^w \) is exact at \( AV^* A \), i.e. if \( H^1(A, A \otimes A) = 0 \), then \( \dim R = n \).

**Proof.**

(i) For any \( i \in J \), we have

\[
f_1 \circ d_2(r_i) = f_1(\sum_{1 \leq j \leq n} \sum_{1,2} (\frac{\partial r_i}{\partial x_j})_1 \otimes x_j \otimes (\frac{\partial r_i}{\partial x_j})_2) = \sum_{j \in J} \sum_{1,2} (\frac{\partial r_i}{\partial x_j})_1 \otimes r_j^* \otimes (\frac{\partial r_i}{\partial x_j})_2.
\]

\[
d_2^* \circ f_2(r_i) = d_2^*(x_i^*) = \sum_{j \in J} \sum_{1,2} (\frac{\partial r_i}{\partial x_i})_2 \otimes r_j^* \otimes (\frac{\partial r_j}{\partial x_i})_1.
\]

So, in these both sums, the coefficients of \( r_j^* \) are respectively

\[
\frac{\partial}{\partial x_j} \circ \frac{\partial}{\partial x_i}(w) = \frac{\partial^2 w}{\partial x_j \partial x_i},
\]

\[
\tau \left( \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_j}(w) \right) = \tau \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right).
\]

Thus they are equal by the symmetry (2.3) of the non-commutative Hessian.

(ii) On one hand, from \( c(w) = \sum_{1 \leq i \leq n} x_i \otimes r_i \otimes 1 - 1 \otimes r_i \otimes x_i \), we get

\[
f_2 \circ d_3(c(w)) = \sum_{i \in J} x_i \otimes x_i^* \otimes 1 - 1 \otimes x_i^* \otimes x_i + \sum_{i \notin J} r_i^*(r_i)(x_i \otimes x_i^* \otimes 1 - 1 \otimes x_i^* \otimes x_i).
\]

From (2.11), we have

\[
d_1^* \circ f_3(c(w)) = d_1^*(1^*) = \sum_{i \in J} x_i \otimes x_i^* \otimes 1 - 1 \otimes x_i^* \otimes x_i + \sum_{i \notin J} x_i \otimes x_i^* \otimes 1 - 1 \otimes x_i^* \otimes x_i.
\]

If \( \dim R = n \), then \( f_2 \circ d_3 = d_1^* \circ f_3 \). The converse comes from the linear independence of \( x_1^*, \ldots, x_n \).

On the other hand, we have for any \( 1 \leq i \leq n \),

\[
f_0 \circ d_1(x_i) = x_i \otimes (c(w))^* \otimes 1 - 1 \otimes c(w)^* \otimes x_i.
\]
If \( i \notin J \), then \( d^*_3 \circ f_1(x_1) = 0 \). If \( i \in J \), we have
\[
d^*_3 \circ f_1(x_i) = d^*_3(r^*_i) = x_i \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_i + \sum_{j \notin J} r^*_j(x_j \otimes c(w)^* \otimes 1 - 1 \otimes c(w)^* \otimes x_j).
\]

If \( \text{dim } R = n \), then \( f_0 \circ d_1 = d_3^* \circ f_1 \). The converse comes from \( c(w)^* \neq 0 \).

(iii) Assume that \( C_w' \) is exact at \( AV^*A \) and that there exists \( i \notin J \). Set \( r_i = \sum_{j \in J} \lambda_{ij} r_j \) where \( \lambda_{ij} \in k \), and apply \( \frac{\partial}{\partial x_s} \) to this equality for \( 1 \leq s \leq n \). We obtain
\[
\sum_{1,2} \left( \frac{\partial r_i}{\partial x_s} \right)_1 \otimes \left( \frac{\partial r_j}{\partial x_s} \right)_2 = \sum_{j \in J} \sum_{1,2} \lambda_{ij} \left( \frac{\partial r_j}{\partial x_s} \right)_1 \otimes \left( \frac{\partial r_j}{\partial x_s} \right)_2.
\]
(2.17)

However [8.13] shows that \( d^*_2(x_i^* - \sum_{j \in J} \lambda_{ij} x_j^*) \) is equal to
\[
\sum_{1 \leq s \leq n} \left( \sum_{1,2} \left( \frac{\partial r_s}{\partial x_j} \right)_2 \otimes r_s^* \otimes \left( \frac{\partial r_s}{\partial x_j} \right)_1 \right) - \sum_{j \in J} \sum_{1,2} \lambda_{ij} \left( \frac{\partial r_s}{\partial x_j} \right)_2 \otimes r_s^* \otimes \left( \frac{\partial r_s}{\partial x_j} \right)_1.
\]

For each \( s \), the terms with \( r_s^* \) vanish by applying the Hessian symmetry [2.3] to (2.17).

Hence \( x_i^* - \sum_{j \in J} \lambda_{ij} x_j^* \) is a 1-cocycle of \( C_w' \). Since the \( d_i \)'s are homogeneous of degree 0, the same holds for the \( d^*_i \)'s, where \( k^*, V^*, R^*, kc(w)^* \) are respectively concentrated in degrees \( 0, -1, -N \) and \( -N - 1 \). In particular, the 1-coboundaries live in degrees \( \geq 0 \). Thus the exactness of \( C_w' \) at \( AV^*A \) implies that \( x_i^* - \sum_{j \in J} \lambda_{ij} x_j^* = 0 \), which is a contradiction. \( \blacksquare \)

It is possible to have \( \text{dim } R < n \), for example when \( w = x_1^{N+1} \) and \( n \geq 2 \), since in this case \( \partial_s w = (N+1)x_1^N \) and \( \partial_s w = 0 \) for \( i > 1 \). Remark that if \( \text{dim } R < n \), then the complex \( C_w \) is not isomorphic to its dual \( C_w' \) for a dimensional reason, therefore our complex \( C_w \) is different from the complex considered by Bocklandt, Schedler and Wemyss (Lemma 6.4 in [9]). According to the previous proposition, the assumption \( \text{dim } R = n \) is natural as far as the self-duality of \( C_w \) is concerned.

**Corollary 2.5** Let \( k \) be a field of characteristic zero. Let \( V \) be an \( n \)-dimensional space with \( n \geq 1 \). Let \( \bar{w} \) be a non-zero homogeneous potential of \( V \) of degree \( N + 1 \) with \( N \geq 2 \). Let \( \bar{A} = A(\bar{w}) \) be the potential algebra defined by \( w \), so that the space of generators of \( A \) is \( V \). Assume that the space of relations \( R \) of \( A \) is \( n \)-dimensional. Then \( f_0, f_1, f_2 \) and \( f_3 \) form an isomorphism of complexes of graded \( A \)-bimodules \( f : C_w \rightarrow C_w' \), which is homogeneous of degree \( -N - 1 \). Moreover, we have \( H^2(\bar{A}, \bar{A} \otimes A) = 0 \). If the global dimension of \( A \) is equal to 3, then \( H^0(\bar{A}, \bar{A} \otimes A) = 0 \) and \( H^3(\bar{A}, \bar{A} \otimes A) \) surjects onto \( A \).

**Proof.** The first statement is immediate from Proposition 2.4. Let us denote by \( d_3 \) : \( AEA \rightarrow ARA \) the first arrow in (2.6), while we keep \( d_3^* : Akc(w)A \rightarrow ARA \) for the first arrow in (2.7). So \( d_3 = d_3^* \circ i \) where \( i \) denotes an obvious inclusion, and we have \( d_3^* = i^* \circ d_3 \), hence \( \ker d_3^* \) is contained in \( \ker d_3 \). But the dual complex of (2.6) shows that \( \text{im } d_3^* \) is contained in \( \ker d_3^* \). Carrying on the exactness of \( C_w \) at \( AVA \) by \( f \), we finally get that \( \text{im } d_3^* = \ker d_3 \), i.e. \( H^2(A, A \otimes A) = 0 \). The last statement is clear. \( \blacksquare \)

**Theorem 2.6** Let \( k \) be a field of characteristic zero. Let \( V \) be an \( n \)-dimensional space with \( n \geq 1 \). Let \( \bar{w} \) be a non-zero homogeneous potential of \( V \) of degree \( N + 1 \) with \( N \geq 2 \). Let \( \bar{A} = A(\bar{w}) \) be the potential algebra defined by \( w \), so that the space of generators of \( A \) is \( V \). If the space of relations \( R \) of \( A \) is \( n \)-dimensional, the following are equivalent.

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(i) $A$ is 3-Calabi-Yau.
(ii) $A$ is AS-Gorenstein of global dimension 3.
(iii) $A$ is $N$-Koszul of global dimension 3 and $\dim R_{N+1} = 1$.
(iv) The complex $C_w$ (see (2.7)) is exact in positive degrees.

Proof. (i)⇒(ii) comes from Proposition 4.3 in [7]. (ii)⇒(iii) follows from Proposition 2.3.
(iii)⇒(iv) is obvious. If $C_w$ is exact in positive degrees, then it coincides with the Koszul resolution of $A$ which is self-dual by $f$ (Corollary 2.5), hence $H^i(A, A \otimes A) = 0$ whenever $i \neq 3$ and $H^3(A, A \otimes A)$ is isomorphic to $A$ as $A$-bimodule. Thus $A$ is 3-Calabi-Yau. This argument of self-duality was used by Bocklandt [8] (see also [7]).

We are now ready to prove Theorem 1.1 of the Introduction.

**Theorem 2.7** Let $k$ be a field of characteristic zero. Let $V$ be an $n$-dimensional space with $n \geq 1$. Let $w$ be a non-zero homogeneous potential of $V$ of degree $N+1$ with $N \geq 2$. Let $A = A(w)$ be the potential algebra defined by $w$. Assume that $A$ is $N$-Koszul. Then $A$ is 3-Calabi-Yau if and only the Hilbert series of the graded algebra $A$ is given by

$$h_A(t) = (1 - nt + nt^N - t^{N+1})^{-1}.$$

**Proof.** Suppose that $h_A(t) = (1 - nt + nt^N - t^{N+1})^{-1}$. Recall that the complex (2.6):

$$\begin{align*}
AEA & \xrightarrow{d_3} ARA & \xrightarrow{d_2} AVA & \xrightarrow{d_1} AA & \rightarrow 0,
\end{align*}$$

(2.18)

is the beginning of a minimal projective resolution of $A$ in $A\text{-grMod-A}$. Since $A$ is $N$-Koszul, the assumption on $h_A(t)$ implies that $\dim R = n$, $\dim E = 1$, and the global dimension of $A$ is equal to 3. In particular, one has $E = R_{N+1} = kc(w)$. Thus $A$ is 3-Calabi-Yau by Theorem 2.6.

The converse is immediate from Proposition 2.3 since any 3-Calabi-Yau algebra is AS-Gorenstein of global dimension 3.

We will examine in Section 4 the recent examples which have motivated us to state this theorem. Now we want to show how this theorem allows us to recover some important examples of 3-Calabi-Yau homogeneous potentials.

**Example 2.8**

Let $S_n$ be the symmetric group of $\{1, \ldots, n\}$ and let $\text{sgn}$ be the sign of a permutation. Suppose that $n \geq 3$ is odd. Then the potential

$$c(w) = \text{Ant}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) x_{\sigma(1)}, \ldots, x_{\sigma(n)}$$

is 3-Calabi-Yau. In fact, it is known [2] that $A(w) = T(V)/I(\Lambda^{n-1}V)$ is $(n-1)$-Koszul of global dimension 3, $\dim R = n$ and $\dim R_n = 1$. Note that $w = \text{Ant}(x_1, \ldots, x_{n-1})x_n$. The algebra $A(w)$ is called an antisymmetrizer algebra or an $(n-1)$-symmetric algebra in $n$ variables. It coincides with the polynomial algebra when $n = 3$. If $n \geq 4$ is even, then $A(w)$ is not derived from a potential [7], but $A(w)$ can be derived from a super-potential defined by a super-cyclic sum $c$ given by the formula (1.2) in [9].
Example 2.9
Assume that \( n \geq 2 \) and that \( V \) is endowed with a non-degenerate symmetric bilinear form \((\ ,\ )\). Setting \( g_{ij} = (x_i, x_j) \), the inverse matrix of \((g_{ij})_{i,j \leq n}\) is denoted by \((g^{ij})_{i,j \leq n}\). Krieg and Van den Bergh have shown in [15] that the space \( \text{Pot}(V)^{O(V)}_4 \) of the 4-degree potentials invariant by the orthogonal group \( O(V) \) is 2-dimensional, generated by the following

\[
\begin{align*}
    w_1 &= \sum_{1 \leq i,j,p,q \leq n} g^{ip} g^{jq} [x_i, x_j][x_p, x_q], \\
    w_2 &= (\sum_{1 \leq i,j \leq n} g^{ij} x_i x_j)^2,
\end{align*}
\]

where \([a, b]\) denotes the commutator of \( a \) and \( b \) in \( T(V) \). For any \( \lambda \in k \) with \( \lambda \neq \frac{n-1}{n+1} \), the potential \( w = w_1 + \lambda w_2 \) is 3-Calabi-Yau. In fact, Krieg and Van den Bergh have proved that, for any \( \lambda \in k \), \( A(w) \) is 3-Koszul and \( \dim R = n \). For \( \lambda \neq \frac{n-1}{n+1} \), Connes and Dubois-Violette have proved that \( \dim R_4 = 1 \) and \( R_5 = 0 \), which implies that the global dimension of \( A(w) \) is equal to 3. The algebras \( A(w) \) are called deformed Yang-Mills algebras (one omits “deformed” if \( \lambda = 0 \)) and were introduced by Connes and Dubois-Violette [14, 15].

Example 2.10
Fix \( k = \mathbb{C} \) and three generators \( x, y, z \). Set

\[
S = \{ (\alpha : \beta : \gamma) \in \mathbb{P}^2(\mathbb{C}) ; \alpha^3 = \beta^3 = 27 \gamma^3 \} \cup \{ (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1) \}.
\]

For any \( (\alpha : \beta : \gamma) \in \mathbb{P}^2(\mathbb{C}) \setminus S \), the potential

\[
w = \alpha xyz + \beta yxz + \gamma(x^3 + y^3 + z^3)
\]

is 3-Calabi-Yau. In fact, the algebras \( A(w) \) are exactly the generic quadratic AS-regular algebras of global dimension 3 and of type \( A \) (also called Sklyanin algebras in three generators), and one deduces from [1] that \( A(w) \) is Koszul with \( h_A(t) = (1 - t)^{-3} \).

Example 2.11
Fix \( k = \mathbb{C} \) and two generators \( x, y \). Set

\[
S = \{ (\alpha : \beta : \gamma) \in \mathbb{P}^2(\mathbb{C}) ; \alpha^2 = 4 \beta^2 = 16 \gamma^2 \} \cup \{ (0 : 1 : 0), (0 : 0 : 1) \}.
\]

For any \( (\alpha : \beta : \gamma) \in \mathbb{P}^2(\mathbb{C}) \setminus S \), the potential

\[
w = \alpha x^2 y^2 + \beta (xy)^2 + \gamma(x^4 + y^4)
\]

is 3-Calabi-Yau. In fact, the algebras \( A(w) \) are exactly the generic cubic AS-regular algebras of global dimension 3 and of type \( A \), and one deduces from [1] that \( A(w) \) is 3-Koszul with \( h_A(t) = (1 - 2t + 2t^3 - t^4)^{-1} \).

Example 2.12
Consider \( V \) of dimension 1, \( V = kx \) and \( w = x^{N+1} \). Then, \( \dim R = \dim R_{N+1} = 1 \) and \( A(w) \) is \( N \)-Koszul, but the global dimension of \( A(w) \) is infinite. So \( w \) is not 3-Calabi-Yau. Here is a non-trivial way to get other examples of \( w \) which are not 3-Calabi-Yau. According to Theorem 2.6 it suffices to assume that \( \dim R = n \), \( A(w) \) is \( N \)-Koszul of global dimension 3 and \( \dim R_{N+1} > 1 \). Question: find such a potential.
3 Van den Bergh’s duality

From now on $k$ will be a field of characteristic zero. Let $V$ be a vector space of dimension $n \geq 2$, $w$ is a non-zero homogeneous potential of $V$ of degree $N + 1$ with $N \geq 2$. Assume that the algebra $A = A(w)$ is 3-Calabi-Yau. Then, since $A$ is $N$-Koszul and AS-Gorenstein, it satisfies Van den Bergh’s duality (Theorem 6.3 in [5]), and the dualizing bimodule is $A$ itself. This means that for any $A$-bimodule $M$, there are linear isomorphisms between Hochschild homology and cohomology: $H_*(A, M) \cong H^{3-*}(A, M)$. We are going to construct an explicit isomorphism of complexes giving the above duality, from the self-duality $f : C_w \rightarrow C_w'$ of the previous section. We do not use Van den Bergh’s duality theorem [22].

Replace the assumption that $A$ is 3-Calabi-Yau by the weaker assumption $\dim R = n$ with $n \geq 1$, so that the self-duality $f$ still holds according to Corollary [2.4]. From the chain complex isomorphism $f$, we define for any $A$-bimodule $M$ an isomorphism of complexes of $k$-vector spaces

$$M \otimes_{A^e} f : M \otimes_{A^e} C_w \rightarrow M \otimes_{A^e} C_w', \quad (3.1)$$

and since $f$ has an inverse morphism $g$, then $M \otimes_{A^e} f$ has an inverse morphism which is $M \otimes_{A^e} g$. The flip

$$\tau : M \otimes_{A^e} C_w' \rightarrow C_w' \otimes_{A^e} M$$

is an isomorphism of complexes. Since $C_w$ is a projective left $A^e$-module of finite type, one has canonical isomorphisms of $k$-vector spaces ([11], Ch. 2, Prop. 2, p.75):

$$C_w' \otimes_{A^e} M = \text{Hom}_{A^e}(C_w', A^e) \otimes_{A^e} M \cong \text{Hom}_{A^e}(C_w, A^e \otimes_{A^e} M) \cong \text{Hom}_{A^e}(C_w, M), \quad (3.2)$$

so that the homology of the complex $M \otimes_{A^e} C_w'$ is the Hochschild cohomology $H^*(A, M)$. Thus, if $A$ is 3-Calabi-Yau, the isomorphism (3.1) in homology provides the expected linear isomorphisms $H_*(A, M) \cong H^{3-*}(A, M)$. Remark that if $A$ is not 3-Calabi-Yau, the complex $M \otimes_{A^e} C_w$ (resp. $M \otimes_{A^e} C_w'$) only computes $H_0(A, M)$ and $H_1(A, M)$ (resp. $H^0(A, M)$ and $H^1(A, M)$).

Let us express explicitly the isomorphism (3.1). We keep the weaker assumption $\dim R = n$ with $n \geq 1$. As previously, we omit the unadorned symbols $\otimes$ when they separate spaces. Moreover, $M \otimes kc(w)$ is denoted by $MC(w)$. Firstly $M \otimes_{A^e} C_w$ is written down

$$0 \rightarrow MC(w) \xrightarrow{d_1} MR \xrightarrow{d_2} MV \xrightarrow{d_1} M \rightarrow 0, \quad (3.3)$$

where $M \otimes_{A^e} d$ is denoted by $\tilde{d}$. For any $m \in M$ and $1 \leq i \leq n$, one has

$$\tilde{d}_1(m \otimes x_i) = mx_i - x_im = [m, x_i]. \quad (3.4)$$

From (2.12), we get

$$\tilde{d}_2(m \otimes r_i) = \sum_{1 \leq j \leq n} \sum_{1,2} \left( \frac{\partial r_i}{\partial x_j} \right)_1 \left( \frac{\partial r_j}{\partial x_i} \right)_2 m \left( \frac{\partial r_i}{\partial x_j} \right)_1 \otimes x_j. \quad (3.5)$$

Recalling that the entries of the Hessian matrix are given by

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = \sum_{1,2} \left( \frac{\partial r_j}{\partial x_i} \right)_1 \otimes \left( \frac{\partial r_j}{\partial x_i} \right)_2 = \sum_{1,2} \left( \frac{\partial r_i}{\partial x_j} \right)_1 \otimes \left( \frac{\partial r_i}{\partial x_j} \right)_1,$$

we see that

$$\tilde{d}_2(m \otimes r_i) = \sum_{1 \leq j \leq n} \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \cdot m \right) \otimes x_j. \quad (3.5)$$
where \( k \) directed triple is denoted by \((i, j, k)\). Let \( u \) the matrix \((u_{ij})_{i,j=1}^{n} \) be a non-zero cubic homogeneous potential in the variables \( x_{1}, \ldots, x_{n} \). We have immediately

\[
\tilde{f}_{0}(m) = m \otimes c(w)^{*}, \quad \tilde{f}_{1}(m \otimes x_{i}) = m \otimes x_{i}^{*}, \quad \tilde{f}_{2}(m \otimes r_{i}) = m \otimes r_{i}^{*}, \quad \tilde{f}_{3}(m \otimes c(w)) = m.
\]

4 Skew polynomial algebras over n.-c. quadrics

Let \( u = \sum_{1 \leq i,j \leq n} u_{ij} x_{i} x_{j} \) denote a quadratic polynomial in the non-commutative one-degree variables \( x_{1}, \ldots, x_{n} \) with \( n \geq 2 \). We assume that \( u \) is non-degenerate, meaning that the matrix \((u_{ij})_{1 \leq i,j \leq n} \) is invertible. Let \( \Gamma \) be the non-commutative quadric defined by \( u \), i.e. \( \Gamma \) is defined by

\[
\Gamma = k(x_{1}, \ldots, x_{n})/I(u)
\]

where \( k(x_{1}, \ldots, x_{n}) \) denotes the free associative algebra in \( x_{1}, \ldots, x_{n} \) and \( I(u) \) the two-sided ideal generated by \( u \). Then the graded algebra \( \Gamma \) is Koszul, AS-Gorenstein of global dimension 2, and \( \Gamma \) is 2-Calabi-Yau if and only if \( u \) is skew-symmetric [3]. Moreover, \( \Gamma \) is left (right) noetherian if and only if \( n = 2 \), and \( \Gamma \) is always a domain [25]. Its Hilbert series is given by \( h_{\Gamma}(t) = (1 - nt + t^{2})^{-1} \). Following [6], let \( z \) be an extra generator of degree 1, and let \( w \) be a non-zero cubic homogeneous potential in the variables \( x_{1}, \ldots, x_{n}, z \). In the next proposition, \( w \) is not necessarily equal to \( uz \) as in [6]. Actually, we want to include in the same proposition the examples coming from [19, 21]. In [21], Suárez Alvarez treats the case of algebras coming from Steiner triple systems, generalizing the example studied in [19]. Let us explain briefly the form of the potentials considered in [21]. Fix an oriented Steiner triple system \( (E; S) \) of order \( n + 1 \), where \( E = \{1, \ldots, n + 1\} \) and \( S \) is a set of directed triples in \( E \) such that any pair in \( E \) is included in a unique (non-directed) triple belonging to \( S \). A directed triple is denoted by \((i, j, k)\), so that \((j, k, i)\) and \((k, i, j)\) are also directed (see [21]).
for details on the choice of the orientation of the triples). Then Suárez Alvarez associates to \((E; S)\) the following potential

\[ w = \sum_{(i,j,k) \in S} (x_i x_j - x_j x_i) x_k, \]

where \(x_{n+1} = z\).

Let us go back to the generic case of a non-zero cubic homogeneous potential \(w\) in the variables \(x_1, \ldots, x_n, z\). Let \(A\) be the quadratic graded algebra derived from the potential \(w\).

**Proposition 4.1** We keep the above notation and assumptions. Assume that the graded algebra \(A\) is isomorphic to a skew polynomial algebra \(\Gamma[z; \sigma; \delta]\), where \(\sigma\) is a 0-degree homogeneous automorphism of \(\Gamma\) and \(\delta\) is a 1-degree homogeneous \(\sigma\)-derivation of \(\Gamma\). Then \(A\) is Koszul and 3-Calabi-Yau, and it is a domain. Moreover, \(A\) is left (right) noetherian if and only if \(n = 2\).

**Proof.** For a sketch on skew polynomial algebras, the reader is referred to [12], pp. 8-9. From \(A \cong \Gamma[z; \sigma; \delta]\), we see that \(h_A(t) = h_\Gamma(t)/1 - t\), hence

\[ h_A(t) = (1 - (n + 1)t + (n + 1)t^2 - t^3)^{-1}. \]

Moreover, since \(\Gamma\) is Koszul, then \(A \cong \Gamma[z; \sigma; \delta]\) is Koszul (see Theorem 10.2 in [13]). Thus \(A\) is 3-Calabi-Yau from Theorem 2.7. Since \(\Gamma\) is a domain, \(A \cong \Gamma[z; \sigma; \delta]\) is a domain. If \(\Gamma\) is left (right) noetherian, the same holds for \(A \cong \Gamma[z; \sigma; \delta]\). If \(A\) is left (right) noetherian, then the Gelfand-Kirillov dimension of \(A\) is finite [20], hence the poles of \(h_A\) are complex numbers of module 1, implying that \(n = 2\). □

In the algebras considered in [6, 19, 21], the assumptions of Proposition 4.1 are satisfied (see the cited articles). Thus we recover that these algebras are Koszul and 3-Calabi-Yau. Note that \(\delta = 0\) in [6], while \(\sigma = \text{Id}_{\Gamma}\) and \(\delta \neq 0\) in [19, 21].

For the rest of this section, we focus on the situation considered in [6], that is, we assume that

\[ w = uz = \sum_{1 \leq i,j \leq n} u_{ij} z. \]

Our aim is to show that the automorphism \(\sigma\) in this case is related to Van den Bergh’s duality of \(\Gamma\). Denote by \(V\) (resp. \(V_\Gamma\)) the space of generators of \(A\) (resp. \(\Gamma\)), and by \(R\) (resp. \(R_\Gamma\)) the corresponding space of relations. A basis of \(V_\Gamma\) consists of \(x_1, \ldots, x_n\), and by \(V = V_\Gamma \oplus kz\). A basis of \(R_\Gamma\) consists of \(\partial z w = u\), and it suffices to add \(r_1 = \partial x_1 w, \ldots, r_n = \partial x_n w\) to get a basis of \(R\). The graded algebra \(\Gamma\) is isomorphic to the subalgebra of \(A\) generated by \(x_1, \ldots, x_n\) and it is also isomorphic to \(A/I(z)\). Recall from [6] that the automorphism \(\sigma\) of \(A\) is defined for \(1 \leq i \leq n\) by

\[ zx_i = \sigma(x_i) z. \]

For \(1 \leq i \leq n\), one has

\[ r_i = \sum_{1 \leq j \leq n} (u_{ij} x_j z + u_{ji} z x_j). \]

Then it is easy to compute the entries of the Hessian matrix

\[ H = \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq n+1} = \left( \frac{\partial r_j}{\partial x_i} \right)_{1 \leq i,j \leq n+1}. \]
for our potential \( u \), where \( x_{n+1} = z \) and \( r_{n+1} = u \). For \( 1 \leq i, j \leq n \), we find that

\[
\frac{\partial r_j}{\partial x_i} = u_{ji} 1 \otimes z + u_{ij} z \otimes 1, \\
\frac{\partial r_j}{\partial z} = \sum_{1 \leq i \leq n} (u_{ij} 1 \otimes x_i + u_{ji} x_i \otimes 1), \\
\frac{\partial u}{\partial x_i} = \sum_{1 \leq j \leq n} (u_{ij} 1 \otimes x_j + u_{ji} x_j \otimes 1), \\
\frac{\partial u}{\partial z} = 0.
\]

The bimodule Koszul resolution \( C_w \) of \( A \) is given by (2.7), that is

\[
0 \to Ak(c(w))A \xrightarrow{d_3} ARA \xrightarrow{d_2} AVA \xrightarrow{d_1} AkA \to 0.
\]

Let us apply the natural projections \( A \to \Gamma \), \( V \to V_\Gamma \) and \( R \to R_\Gamma \) to this complex, in order to obtain the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & A(kc(w))A & \xrightarrow{d_3} & ARA & \xrightarrow{d_2} \ AVA & \xrightarrow{d_1} \ AkA & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & 0 & \to & \Gamma R_\Gamma \Gamma & \xrightarrow{d'_2} \ \Gamma V_\Gamma \Gamma & \xrightarrow{d'_1} \ \Gamma k\Gamma & \to & 0
\end{array}
\]

in which the second row is defined as follows. Since \( c(w) \) vanishes modulo \( z \), \( d_3 \) factors out to \( 0 \to \Gamma R_\Gamma \Gamma \). Clearly, \( d_1 \) factors out to a \( \Gamma - \Gamma \)-linear map \( d'_1 : \Gamma V_\Gamma \Gamma \to \Gamma \Gamma \) defined by

\[
d'_1(x_i) = x_i \otimes 1 - 1 \otimes x_i, \quad 1 \leq i \leq n.
\]

From (2.12) and the entries of \( H \) given above, one obtains for \( 1 \leq j \leq n \), that

\[
d'_2(r_j) = \sum_{1 \leq i \leq n} u_{ij}(z \otimes x_i \otimes 1 + 1 \otimes z \otimes x_i) + \sum_{1 \leq i \leq n} u_{ji}(1 \otimes x_i \otimes z + x_i \otimes z \otimes 1).
\]

Thus \( d'_2 \) factors out to a \( \Gamma - \Gamma \)-linear map \( d'_2 : \Gamma R_\Gamma \Gamma \to \Gamma V_\Gamma \Gamma \). Using (4.3) and (4.4), \( d'_2 \) is defined by

\[
d'_2(u) = \sum_{1 \leq i,j \leq n} u_{ij} (1 \otimes x_i \otimes x_j + x_i \otimes x_j \otimes 1).
\]

Consequently, the so-obtained quotient complex

\[
0 \to \Gamma R_\Gamma \Gamma \xrightarrow{d'_2} \Gamma V_\Gamma \Gamma \xrightarrow{d'_1} \Gamma k\Gamma \to 0
\]

coincides with the bimodule Koszul resolution of \( \Gamma \).

Let us proceed similarly with the dual complex \( C'_w \) given by (2.10):

\[
0 \to Ak^*A \xrightarrow{d'_3} AV^*A \xrightarrow{d'_2} AR^*A \xrightarrow{d'_1} Ak(c(w))^*A \to 0.
\]

The natural inclusion \( V_\Gamma \to V \) (resp. \( R_\Gamma \to R \)) provides the projection \( V^* \to V^*_\Gamma \) (resp. \( R^* \to R^*_\Gamma \)). The image of the dual basis \( (x_1^*, \ldots, x_n^*, z^*) \) (resp. \( (r_1^*, \ldots, r_n^*, u^*) \)) by this projection consists of 0 and the basis \( (x_1^*, \ldots, x_n^*) \) of \( V^*_\Gamma \) (resp. the basis \( u^* \) of \( R^*_\Gamma \)). Clearly, \( d'_3 \) factors out to \( \Gamma R_\Gamma^* \Gamma \to 0 \) and \( d'_1 \) factors out to \( d'_1^* : \Gamma k^* \Gamma \to \Gamma V^*_\Gamma \Gamma \) defined by

\[
d'_1^*(1^*) = \sum_{1 \leq i \leq n} x_i \otimes x_i^* \otimes 1 - 1 \otimes x_i^* \otimes x_i.
\]
From (2.13) and the entries of \( H \), one deduces that
\[
d_2^\Gamma(z^\ast) = \sum_{1 \leq i,j \leq n} u_{ij}(1 \otimes r_j^* \otimes x_i + x_j \otimes r_i^* \otimes 1).
\]
Since the RHS does not contain the element \( u^* \), \( d_2^\Gamma \) factors out to \( d_2^\Gamma : \Gamma V_i^* \Gamma \rightarrow \Gamma \Gamma_i^* \Gamma \). Using (2.13) and (4.3), \( d_2^\Gamma \) is defined for \( 1 \leq i \leq n \) by
\[
d_2^\Gamma(x_i^*) = \sum_{1 \leq j \leq n} (u_{ij}x_j \otimes u^* \otimes 1 + u_{ji}1 \otimes u^* \otimes x_j).
\] (4.8)

Then, it is easy to show that the complex
\[
0 \rightarrow \Gamma k^* \Gamma \xrightarrow{d_1^\Gamma} \Gamma V_i^* \Gamma \xrightarrow{d_2^\Gamma} \Gamma \Gamma_i^* \Gamma \rightarrow 0
\] (4.9)
is isomorphic to the dual complex of the complex of bimodules (4.7). In fact, following along the same lines of Section 2, it suffices to apply the isomorphism \( \theta \) to this dual complex and to verify that we obtain the complex (4.9). The verification is left to the reader.

Since \( \Gamma \) is Koszul and AS-Gorenstein, \( \Gamma \) satisfies Van den Bergh’s duality (Proposition 2.11 in [6]). More precisely, there is an automorphism \( \nu \) of the graded algebra \( \Gamma \) such that, for any \( \Gamma \)-bimodule \( M \), we have linear isomorphisms
\[
H^*(\Gamma, M) \cong H_{\ast-\bullet}(\Gamma, \nu M).
\]
As usual, the bimodule \( \nu M \) coincides with \( M \) as right module but the left action of \( a \in \Gamma \) upon \( m \in \nu M \) is given by \( \nu(a)m \). In [22], \( \nu \) is expressed in terms of the Nakayama automorphism of the dual Koszul algebra \( \Gamma^\ast \) (note that the same results hold for AS-Gorenstein \( N \)-Koszul algebras [5]). In our situation, the following description of \( \nu \) does not need the use of \( \Gamma^\ast \).

**Proposition 4.2** We have \( \nu = \sigma^{-1} \). In particular, \( \sigma = \text{Id}_\Gamma \) if and only if \( u \) is skew-symmetric.

**Proof.** Since \( \Gamma \) satisfies Van den Bergh’s duality, the homology of the complex (4.9) at \( \Gamma k^* \Gamma \) and at \( \Gamma V_i^* \Gamma \) vanishes, and it is isomorphic to \( \nu \Gamma \) at \( \Gamma \Gamma_i^* \Gamma \). Define the \( \Gamma \)-linear map \( \mu_u : \Gamma \Gamma_i^* \Gamma \rightarrow \Gamma_{\sigma} \) by
\[
\mu_u(a \otimes u^* \otimes b) = a \sigma(b)
\]
for any \( a \) and \( b \) in \( \Gamma \). Choose \( 1 \in \Gamma_{\sigma} \) of degree \(-2\), so that \( \mu_u \) is homogeneous of degree \( 0 \). Let us check that \( \mu_u \circ d_2^\Gamma = 0 \). Fix \( i \in \{1,\ldots,n\} \) and set \( X_i = \mu_u \circ d_2^\Gamma(1 \otimes x_i^* \otimes 1) \). From (4.8), we get that
\[
X_i = \sum_{1 \leq j \leq n} (u_{ij}x_j + u_{ji} \sigma(x_j)).
\]
Therefore \( X_i z = \sum_{1 \leq j \leq n} (u_{ij}x_j z + u_{ji}z x_j) = r_i \), hence \( X_i z = 0 \) in \( A \). But \( z \) is not a zero-divisor in \( A \), thus \( X_i = 0 \) as desired.

Next, examining the surjective homogeneous natural map
\[
\nu \Gamma \cong \frac{\Gamma \Gamma_i^* \Gamma}{\ker \mu_u} \rightarrow \frac{\Gamma \Gamma_i^* \Gamma}{\ker \mu_u} \cong \Gamma_{\sigma}
\]
degree by degree, we see that it is an isomorphism and that \( \nu \Gamma \cong \Gamma_{\sigma} \). Thus \( \nu = \sigma^{-1} \). In particular, \( \sigma = \text{Id}_\Gamma \) if and only if \( \Gamma \) is Calabi-Yau, i.e. if and only if \( u \) is skew-symmetric, recovering 2) of Proposition 2.11 in [6].
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