Flexible Lagrangians

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Abstract

We introduce and discuss notions of regularity and flexibility for Lagrangian manifolds with Legendrian boundary in Weinstein domains. There is a surprising abundance of flexible Lagrangians. In turn, this leads to new constructions of Legendrians submanifolds and Weinstein manifolds. For instance, many closed $n$-manifolds of dimension $n > 2$ can be realized as exact Lagrangian submanifolds of $T^*S^n$ with possibly exotic Weinstein symplectic structures. These Weinstein structures on $T^*S^n$, infinitely many of which are distinct, are formed by a single handle attachment to the standard $2n$-ball along the Legendrian boundaries of flexible Lagrangians. We also formulate a number of open problems.

1 Liouville and Weinstein cobordisms

The main goal of the paper is a discussion of two new notions of regularity and flexibility for exact Lagrangian cobordisms with Legendrian boundaries in Weinstein cobordisms, see Sections 2 and 3. In particular, we prove an existence $h$-principle for flexible Lagrangian cobordisms (Theorem 4.2), explore applications to Lagrangian and Legendrian embeddings and exotic Weinstein structures, and formulate throughout the paper numerous open problems.

A \textit{Liouville cobordism} between contact manifolds is a $2n$-dimensional cobordism $(W, \partial_-, W, \partial_+ W)$ equipped with a pair $(\omega, X)$ of a symplectic form and an expanding...
(Liouville) vector field for $\omega$, i.e. $L_X\omega = \omega$, which is outward pointing along $\partial_- W$ and inward pointing along $\partial_+ W$, such that the contact structure induced by the Liouville form $\lambda := \iota(X)\omega$ on $\partial_\pm W$ coincides with $\xi_\pm$. If in addition we are given a Morse function $\phi : W \to \mathbb{R}$ that is defining for $W$ and Lyapunov for $X$, i.e. it attains its minimum on $\partial_- W$, its maximum on $\partial_+ W$ and has no critical points on $\partial W$, and satisfies the inequality $d\phi(X) \geq c||X||^2$ for some $c > 0$, then the triple $(\omega, X, \phi)$ is called a Weinstein cobordism structure on $W$ between contact manifolds $(\partial W_-, \xi_-)$ and $(\partial W_+, \xi_+)$, see [14, 36, 15, 10]. A cobordism with $\partial_- W = \emptyset$ will be referred as a Weinstein domain.

Stable manifolds of zeroes of $X$ for a Weinstein cobordism structure $(W, \omega, X, \phi)$ are necessarily $\omega$-isotropic, see [15], and in particular the indices of all critical points of $\phi$ are always $\leq n$. A cobordism is called subcritical if there are are no critical points of index $n$.

Every Liouville or Weinstein cobordism can be canonically completed by adding cylindrical ends $(-\infty, 0] \times \partial_- W$ and $[0, \infty) \times \partial_+ W$ with $X = \partial_\delta$ and $\phi$ equal to $s$ up to an additive constant. Here we denote by $s$ the coordinate corresponding to the first factor. An important feature of completed Liouville cobordisms, see [15, 10], is that if $(\omega_t, X_t), t \in [0, 1]$, is a homotopy of completed Liouville cobordisms then there exists an isotopy $\phi_t : W \to W$ such that $\phi_t^* \omega_t = \omega_0$, $t \in [0, 1]$, which preserves the Liouville field at infinity. In other words, the symplectic structure of a completed Liouville cobordism remains unchanged up to isotopy when one deforms the Liouville structure.

Usually we will not distinguish in the notation between a Weinstein cobordism and its completion. Moreover, we will allow contact manifolds to have boundaries and will require in this case the cobordisms to be trivial over boundaries of the contact manifolds. Alternatively, a Weinstein cobordism between manifolds $\partial_\pm W$ with boundary can be viewed as a sutured manifold with corner along the suture, see Fig. 1.1 (taken from [18]). More precisely, we assume that the boundary $\partial W$ is presented as a union of two manifolds $\partial W_-$ and $\partial_+ W$ with common boundary $\partial^2 W = \partial_+ W \cap \partial_- W$, along which it has a corner. Of course, in this case the function $\phi$ cannot be chosen constant on $\partial_- W$ and $\partial_+ W$.

Given a $2n$-dimensional Weinstein cobordism $\mathcal{W} := (W, \partial_- W, \partial_+ W; \omega, X, \phi)$ we consider in it exact Lagrangian cobordisms with Legendrian boundary $(L, \partial_- L, \partial_+ L) \subset (W, \partial_- W, \partial_+ W)$. We will additionally require $L$ to be tangent to $X$ near $\partial_\pm L$. This condition can always be achieved by a $C^0$-small isotopy of $L$ fixed on $\partial_\pm L$. The boundary components $\partial_\pm L$ will always assumed to be closed and contained in $\text{Int} \partial_\pm W$. Sometimes we will talk about a parameterized Lagrangian cobordism, i.e.
Fig. 1.1: A Liouville cobordism \( W \) with corners.

a diffeomorphism of a smooth cobordism \((L, \partial_- L, \partial_+ L)\) onto an exact Lagrangian cobordism in \( W \) between Legendrian manifolds in \( \partial_\pm W \).

A Lagrangian cobordism \( L \) can be canonically completed to a submanifold with cylindrical ends in the completion of \( W \). An isotopy between two exact Lagrangian cobordisms with Legendrian boundaries will be always understood in this class, i.e. as a Hamiltonian isotopy of the completions which at infinity is required to preserve the Liouville vector field \( X \). We note that any exact Lagrangian isotopy with Legendrian boundary lifts to a Hamiltonian isotopy of completions.

A Morse decomposition for the Lyapunov function \( \phi \) yields an equivalent definition of a Weinstein cobordism as a Weinstein handlebody, formed by attaching handles with symplectically isotropic core discs along contactly isotropic sphere in the regular contact level sets of \( \phi \). A Weinstein cobordism of dimension \( 2n \geq 6 \) is called flexible if it can be presented as a Weinstein handlebody so that all critical (i.e. of index \( n \)) handles are attached along loose Legendrian links, see [10, 26] for precise definitions and discussion. A flexible Weinstein structure is a choice of such a presentation.

2 Regular Lagrangians

Let \((W, \omega, X, \phi)\) be a Weinstein cobordism, and

\[
(L, \partial_- L, \partial_+ L) \subset (W, \partial_- W, \partial_+ W, \omega, X, \phi)
\]

a Lagrangian cobordism with Legendrian boundary.

**Definition 2.1.** We say \( L \subset W \) is regular if \((W, \omega, X, \phi)\) can be deformed to a Weinstein structure \((W, \omega', X', \phi')\) through Weinstein structures for which \( L \) remains Lagrangian, such that the new Liouville vector field \( X' \) is tangent to \( L \). This is
equivalent to the condition \( \alpha'|_L = 0 \), where \( \alpha' := \iota(X')\omega' \) is the corresponding Liouville form.

We call such \((W, \omega', X', \phi')\) a tangent Weinstein structure to the regular Lagrangian \(L\). It follows that all critical points of \(\phi'|_L\) are global critical points of \(\phi'\), and the local models near such \(p\) can be described by a "coupled handle attachment" picture. Indeed, let \(k\) be the index of a critical point \(x\) of \(\phi'\) and \(l\) the index of \(x\) as a critical point of \(\phi'|_L\). We have \(l \leq k \leq n\). A Weinstein handle of index \(k\leq n\) is isomorphic to the subset \(H_k := \{|p|, |q|, |P|, |Q| \leq 1\} \subset T^*\mathbb{R}^k \times T^*\mathbb{R}^{n-k}\), where we denoted by \((p, q)\) and \((P, Q)\) the canonical coordinates in \(T^*\mathbb{R}^k\) and \(T^*\mathbb{R}^{n-k}\). The handle \(H_k\) contains a Lagrangian sub-handle \(L_l\) of index \(l\) which is the intersection with \(H_k\) of the total space of the conormal bundle to \(\mathbb{R}^l \subset \mathbb{R}^k \subset T^*\mathbb{R}^k = (T^*\mathbb{R}^k) \times 0 \subset T^*\mathbb{R}^k \subset T^*\mathbb{R}^k \times T^*\mathbb{R}^{n-k}\).

When passing through the critical level \(a = \phi'(x)\) of the critical point \(x\) the Weinstein handle \(H_k\) is attached to \(\{\phi' \leq a - \varepsilon\} \subset W\) along \(\partial_+ H_k := H_k \cap \{|q| = 1\}\), and the Lagrangian handle \(L_l\) is attached to \(\{\phi|_L \leq a - \varepsilon\} \subset L\) along \(\partial_- L := \partial_- W \cap L_l\). It turns out that given a regular \(L\), any Weinstein cobordism structure tangent to \(L\) can be further adjusted. Let us call a tangent to \(L\) Weinstein cobordism structure \((W, \omega, X, \phi)\) special if there exists a regular value \(c \in \mathbb{R}\) of the function \(\phi\) such that

- all critical points of \(\phi\) in the sublevel set \(\{\phi \leq c\}\) lie on \(L\) and the indices of these critical points for \(\phi\) and \(\phi|_L\) coincide;

- there are no critical points of \(\phi\) on \(L \cap \{\phi \geq c\}\).

In other words, \((W, \omega, X, \phi)\) has the following handlebody presentation. First, one attaches handles corresponding to critical points of \(\phi|_L\) and then the remaining handles, so that their attaching spheres do not intersect \(L\).

For a special tangent to \(L\) Weinstein cobordism \((W, \omega, X, \phi)\) we set \(W_L := \{\phi \geq c\}\) and view \(W_L\) as a Weinstein subcobordism of \((W, \omega, X, \phi)\) with the induced Weinstein structure. We call \(W_L\) the complementary Weinstein cobordism to \(L\) and note that \(\phi\) determines a presentation \(W := T^*L \cup W_L\), where \(T^*L\) is endowed with its canonical Weinstein structure. The following lemma asserts that up to homotopy of Weinstein structures for which \(L\) remains Lagrangian, the existence of such a presentation is equivalent to regularity:
Lemma 2.2. Let \((L, \partial_- L, \partial_+ L) \subset (W, \partial_- W, \partial_+ W)\) be an exact Lagrangian subcobordism in a Weinstein cobordism \((W, \omega_0, X_0, \phi_0)\) tangent to \(L\). Then there is a homotopy \((W, \omega_t, X_t, \phi_t)\) of tangent to \(L\) Weinstein structures such that \((W, \omega_1, X_1, \phi_1)\) is special.

Proof. Suppose first that for each critical point \(p\) of the function \(\phi|_L\) its index on \(L\) coincides with its index as a critical point of \(\phi\) on the whole \(W\).\(^1\) Then for any critical point \(p \in L\) of \(\phi\) its stable manifold is contained in \(L\), and hence for any critical point \(q \notin L\) there are no \(X\)-trajectories converging to \(q\) at the negative direction, and to \(p\) at the positive one. Hence, using Lemma 9.45 from [10] we can deform \(\phi\) without changing \(X\) and \(\omega\) (and hence keeping Weinstein structure tangent to \(L\)) so that the critical values corresponding to critical points from \(L\) are all smaller than the critical values corresponding to critical points of \(\phi\) which are not in \(L\). Then an intermediate regular value \(c\) has the required properties, and thus the Weinstein structure is special.

Suppose now that the index \(l\) of a critical point \(p\) of the function \(\phi|_L\) is less than the index \(k\) of \(p\) for \(\phi\) on the whole \(W\). Let \(D^k\) be the stable disc of \(p\) on \(W\) and \(D^l = D^k \cap L\) the stable disc of \(p\) for the function \(\phi|_L\). Note that that there exists a function \(\tilde{\phi} : D^k \to \mathbb{R}\) which coincides with \(\phi\) on \(D^l \cup \partial D^k\), has a critical point of index \(l\) at \(p\) and two additional critical points \(p', p''\) on \(D^k \setminus D^l\) of indices \(l + 1\) and \(k\), respectively. Hence the attaching of one handle of index \(k\) corresponding to the point \(p\) can be replaced by attaching of three handles of indices \(l, l + 1\), and \(k\) corresponding to the points \(p, p', p''\). Moreover, only the first handle intersects the Lagrangian \(L\). Hence, the claim follows from the already considered case. \(\Box\)

The following proposition characterizes regular Lagrangian discs.

Proposition 2.3. Let \((C, \partial C) \subset (W, \partial_\pm W)\) be a Lagrangian disc with Legendrian boundary. It is regular if and only if there is a Weinstein handlebody representation of the cobordism \(W\) for which \(L\) coincides with the co-core Lagrangian disc of one of the index \(n\) handles.

Proof. If \(C\) is a co-core disc for a Weinstein structure, then this structure is tangent to it, and hence \(C\) is regular. Conversely, if \(C\) is regular for a Weinstein structure \(W\), then by Lemma 2.2 this structure (after Weinstein homotopy of tangent to \(C\) Weinstein structures) admits a Weinstein handlebody consisting of the ball \(B^{2n}\) with

\(^1\) In this case the modification of the Lagrangian after passing through the corresponding critical value coincides with the ambient Legendrian surgery defined by G. Dimitroglou Rizell in [30].
Lagrangian equatorial disc $C \subset B$ and other Weinsteins handles glued to $\partial B \setminus \partial C$). One can deform the Weinstein structure on $B^{2n}$ by creating two additional critical points of index $n$ and $(n - 1)$ such that $C$ serves as the co-core disc of the corresponding index $n$ handle. It remains to note that using Proposition 10.10 from [10] one can re-order critical points of a Lyapunov function so that the handle corresponding to the critical point $C$ is the last one to attach.

The regularity property for Lagrangian submanifolds also has (at least conjecturally) a Lefschetz fibration characterization. E. Giroux and J. Pardon have suggested to us that the following can probably be proven along the lines of [19], adapting results of [7]: for any regular Lagrangian submanifold $L \subset W$ with Legendrian boundary there exists a Lefschetz fibration over $C$ which projects $L$ to a ray in $\mathbb{R} \subset \mathbb{C}$. Of course, the converse statement is true: a Lagrangian with such a Lefschetz presentation is regular.

There are many other natural examples of regular Lagrangians, including the zero section and cotangent fibers of a cotangent bundle, and more generally smooth loci of Lagrangian skeleta or ascending Lagrangian co-cores of the flow of $X$. A necessary condition for the regularity of a closed $L$, or more generally of a cobordism $L$ with $\partial_+ L = \emptyset$, is given by the following:

**Lemma 2.4.** Let $L$ be a regular Lagrangian cobordism with $\partial_+ L = \emptyset$. Then the inclusion $H_n(L, \partial_- L) \to H_n(W, \partial_+ W)$ is injective. Here the homology is taken with integer coefficients if $L$ is orientable and with $\mathbb{Z}/2$-coefficients otherwise. Moreover, in the orientable case the image of the (relative) fundamental class of $(L, \partial_- L)$ in $H_n(W, \partial_+ W)$ is indivisible.

**Proof.** Assuming the Weinstein structure is special, we observe that a generic fiber $F$ of the cotangent bundle $T^*L$ has boundary $\partial F$ which does not intersect the attaching spheres of any of the additional Weinstein handles, and hence $F$ represents a homology class $[F] \in H_n(W, \partial_+ W)$. But $[F] \cdot [L] = \pm 1$, and hence the class $[L] \in H_n(W, \partial_- W)$ is indivisible.

The results of [17] and [27] show that this injectivity condition does not necessarily hold when $\partial_- L$ is loose, or when $\partial_- L = \emptyset$ but $(\partial_- W, \xi_-)$ is overtwisted, and these therefore provide examples of non-regular Lagrangian cobordisms.

However, we do not know any counterexample to the positive answer to the following problem:
Problem 2.5. Suppose $\partial_- W = \emptyset$ (or more generally when $(\partial_- W, \xi_\omega)$ is tight and $\partial_- L$ is not loose). Is every Lagrangian cobordism $L \subset W$ regular? In particular, does the conclusion of Lemma 2.4 hold for such $L$? For instance, is the image of the fundamental class of a closed Lagrangian manifold in a Weinstein manifold necessarily indivisible (and in particular non-zero)?

3 Flexible Lagrangians

Definition 3.1. We say a Lagrangian cobordism $L \subset W$ is flexible if it is regular with a special tangent Weinstein structure $(W, \omega, X, \phi)$ for which the complementary Weinstein cobordism $(W_L, \omega|_{W_L}, X|_{W_L}, \phi|_{W_L})$ is flexible.

In the case when $\partial_+ L \neq \emptyset$ one can equivalently characterize flexibility in terms of tangent but not necessarily special Weinstein structures.

Lemma 3.2. A Lagrangian cobordism $L \subset W$ with $\partial_+ L \neq \emptyset$ is flexible if and only it is regular with a tangent to $L$ Weinstein cobordism structure which admits a partition into elementary cobordisms such that links of attaching spheres of index $n$ handles are loose in the complement of $L$.

Proof. The proof repeats the steps of the proof of Lemma 2.2. If for each critical point $p$ of the function $\phi|_L$ its index on $L$ coincides with its index as a critical point of $\phi$ on $W$, then we modify the Weinstein structure into one which is special and tangent to $L$ without changing $\omega$ and $X$. For the resulting Weinstein structure the cobordism $W_L$ is automatically flexible.

Suppose that the index $l$ of a critical point point $p$ of the function $\phi|_L$ is strictly less than the index $k$ of $p$ for $\phi$ on $W$. Letting $D^k$ be the stable disc of $p$ on $W$ and $D^l = D \cap L$ the stable disc of $p$ for the function $\phi|_L$, we modify, as in the proof of Lemma 2.2 the Weinstein structure by changing the index of $p$ for $\phi$ to $l$ at the expense of creating two new critical points of index $l + 1$ and $k$ respectively on the stable $D^k$.

Finally we observe that if $k = n$, then the index $n$ handle corresponding to the point $p''$ is attached along a loose Legendrian by assumption. If $l = n - 1$, the second index $n$ handle corresponding to the point $p'$ is in canceling position with the $n - 1$ index handle corresponding to $p$ and hence is also attached along a loose knot. This implies the flexibility of $L$.

The opposite implication is straightforward. 


The next proposition gives two fundamental examples of flexible Lagrangian submanifolds. Recall that a product $\mathcal{W}_1 \times \mathcal{W}_2 = (W_1 \times W_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$ of completed Weinstein cobordisms $\mathcal{W}_1 = (W_1, \omega_1, X_1, \phi_1)$ and $\mathcal{W}_2 = (W_2, \omega_2, X_2, \phi_2)$ is again a (completed) Weinstein cobordism (of manifolds with boundary). For a Weinstein cobordism structure $\mathcal{W} = (W, \omega, X, \phi)$ we denote $\overline{\mathcal{W}} := (W, -\omega, X, \phi)$ and observe that the the structure $\mathcal{W} \times \overline{\mathcal{W}}$ is tangent to the Lagrangian diagonal $D \subset W \times W$, and hence $D$ is regular for $\mathcal{W} \times \overline{\mathcal{W}}$.

**Proposition 3.3.**  (i) Let $(W, \partial_- W, \partial_+ W)$ be a flexible Weinstein cobordism and let $\overline{W}$ denote the result of attaching an $n$-handle to $W$ along a loose Legendrian knot $\Lambda \subset \partial_+ W$. Then the co-core disc $C$ of the attached handle is flexible.

(ii) Let $\mathcal{W} = (W, \omega, X, \phi)$ be a flexible Weinstein cobordism structure. Then the diagonal $D$ is flexible for $\mathcal{W} \times \overline{\mathcal{W}}$.

**Proof.** (i) This is immediate from Lemma 3.2.

(ii) To show that $D$ is flexible, we must show that all index 2n handles determined by $\phi \oplus \phi$ are attached along Legendrians which are loose in the complement of $D$. The proof of this fact is essentially identical to the proof that the product of a flexible Weinstein manifold with any other Weinstein manifold is always flexible (and moreover, in the case of $\mathcal{W} \times \overline{\mathcal{W}}$, one can ensure by construction that the loose charts stay away from $D$). This folkloric statement was known for a while to several specialists and recently was proven by Murphy-Siegel [28, Proposition 3.7].

**Problem 3.4.** Is the converse to each of the statements in Proposition 3.3 true? In other words, is it true that

(i) If a Lagrangian co-core of an $n$-handle is flexible, then its attaching Legendrian sphere is loose? Moreover, can a flexible Weinstein cobordism remain flexible after attaching an $n$-handle along a non-loose Legendrian knot?

(ii) If the diagonal in the product $\mathcal{W} \times \overline{\mathcal{W}}$ is flexible, then $\mathcal{W}$ is flexible.

**Problem 3.5.** (around the nearby Lagrangian conjecture)

(i) Are all regular closed Lagrangians in $T^*M$ Hamiltonian isotopic?

(ii) Let $L \subset W$ be a flexible closed Lagrangian in a Weinstein domain $W$. Are all (regular) closed Lagrangian submanifolds in $W$ Hamiltonian isotopic to $L$?

To tie this to (i), we note that the 0-section in a cotangent bundle $T^*M$ is tautologically flexible.
4 Existence and classification of flexible Lagrangians

By a formal parameterized Lagrangian cobordism in \((W, \partial_-, \partial_+, \partial_+)\) we mean a pair \((f, \Phi_t)\), where \(f : (L, \partial_- L, \partial_+ L) \to (W, \partial_+ W, \partial_- W)\) is a smooth embedding of an \(n\)-dimensional cobordism \((L, \partial_- L, \partial_+ L)\), and \(\Phi_t : TL \to TW, t \in [0, 1]\), is a homotopy of injective homomorphisms such that

(i) \(\Phi_0 = df\);

(ii) \(\Phi_1\) is a Lagrangian homomorphism, i.e. \(\Phi_1(T_x L) \subset T_x W\) is a Lagrangian subspace for all \(x \in L\);

(iii) \(\Phi_1|_{TL|_{\partial \pm L}} \subset \text{Span}(X, \xi)\);

(iv) \(\Phi_t(T(\partial L)) \subset T(\partial W)\); and

(v) \(\Phi_t|_{TL|_{\partial L}}\) is transverse to \(\partial W\) for all \(t \in [0, 1]\).

A genuine parameterized Lagrangian cobordism \(f : (L, \partial_- L, \partial_+ L) \to (W, \partial_+ W, \partial_- W)\) can be viewed as formal by setting \(\Phi_t \equiv df, t \in [0, 1]\). Two formal parameterized Lagrangian cobordisms are formally Lagrangian isotopic if they are isotopic through formal parameterized Lagrangian cobordisms. Note that a formal parameterized Lagrangian cobordism has well defined formal Legendrian classes of its positive and negative boundaries. If \((L, \partial_- L, \partial_+ L) \subset (W, \partial_- W, \partial_+ W)\) is a subcobordism and \(f\) is the inclusion map \((L, \partial_- L, \partial_+ L) \hookrightarrow (W, \partial_- W, \partial_+ W)\) then we will drop the word “parameterized” from the term “formal parameterized Lagrangian cobordism”.

Remark 4.1. One can define a weaker notion of a formal parameterized Lagrangian cobordism by dropping the transversality condition (v) from the definition. We note, however, that every homotopy class of weak formal Lagrangians contains a unique homotopy class of strong ones. Indeed, the obstructions to deform a weak to a strong one lie in the homotopy groups \(\pi_k(S^{2n-1})\), where \(k \leq n\). But \(n < 2n - 1\) for \(n > 1\), and hence obstructions vanish.

Theorem 4.2. (i) (Existence) In a flexible Weinstein cobordism \((W, \omega)\), any formal Lagrangian cobordism with non-empty positive boundary of each of its components is formally Lagrangian isotopic to a flexible genuine Lagrangian cobordism.

(ii) (Uniqueness) Let

\[f_0 : (L_0, \partial_- L_0, \partial_+ L_0) \subset (W_0, \partial_- W_0, \partial_+ W_0)\]

and

\[f_1 : (L_1, \partial_- L_1, \partial_+ L_1) \subset (W_1, \partial_- W_1, \partial_+ W_1)\]

be two flexible Lagrangian cobordisms. Then given diffeomorphisms $h : W_0 \to W_1$ and $g : L_0 \to L_1$ such that $h \circ f_0 = f_1 \circ g$ and $f^* \omega_1$ is homotopic to $\omega_0$ via a homotopy of non-degenerate not necessarily closed 2-forms vanishing on $f_0(L_0)$, there exists an isotopy of $h$ to a symplectomorphism $\tilde{h} : W_0 \to W_1$ such that $h \circ f_0 = f_1 \circ g$.

**Remark 4.3.**

(i) Note that the condition $\partial_+ L \neq \emptyset$ in (i) is essential. For example, a flexible Weinstein manifold has no compact exact Lagrangians, even though it may have many formal compact Lagrangians.

(ii) If the formal embedding in Theorem 4.2(i) is genuine Lagrangian near $\partial_- L$ then the formal homotopy can be constructed fixed near $\partial_- L$.

(iii) We do not know whether the uniqueness part of Theorem 4.2 holds up to a Hamiltonian isotopy, i.e. whether formally Lagrangian isotopic flexible Lagrangians are Hamiltonian isotopic.

(iv) The formal Lagrangian isotopy in Theorem 4.2(i) need not be $C^0$-small, because the proof relies on the flexibility of the ambient structure. The situation here is similar to that of Theorem 7.19 of [10]: any formal Legendrian embedding in an overtwisted contact manifold is formally Legendrian isotopic to a genuine Legendrian embedding, but the isotopy need not be $C^0$-small.

**Proof of Theorem 4.2.** To prove the existence part, let $(j, \Phi_t) : (L, \partial_- L, \partial_+ L) \hookrightarrow (W, \partial_- W, \partial_+ W)$ be a formal Lagrangian submanifold and $(U, \eta)$ denote a tubular neighborhood of $L$ in $T^* L$ with its canonical symplectic structure $\eta = d(pdq)$. There exist an extension of $j$ to an embedding $\hat{j} : U \to W$ and a homotopy $\hat{\Phi}_t : TU \to TW$, $t \in [0, 1]$, of fiberwise isomorphisms extending $\Phi_t$, such that $\hat{\Phi}_0 = d\hat{j}$ and $\hat{\Phi}_1$ is a symplectic bundle isomorphism $(TU, \eta) \to (TW, \omega)$. The homotopy $(\Phi_t)^* \eta := ((\Phi_t^{-1})^* \eta)$ of non-degenerate but not necessarily closed 2-forms on $\hat{j}(U)$ extends to a homotopy $\omega_t$, $t \in [0, 1]$, of non-degenerate 2-forms on $W$ such that $\omega_1 = \omega$ and $\hat{j}^* \omega_0 = \eta$. In particular, $\omega_0$ is a genuine symplectic structure on a neighborhood of $L$ and $L$ is Lagrangian. We denote by $W_L := (W_L, (\omega_0)|_{W_L})$ the complement of $\hat{j}(U)$ with its induced formal symplectic structure. We wish to now show that $W_L$ admits a Weinstein structure in the same almost symplectic class as $(\omega_0)|_{W_L}$ relative boundary.

The condition $\partial_+ L \neq \emptyset$ for each component of $L$ implies that $\pi_j(W_L, \partial_+ W_L) = 0$ for $j \leq n - 1$. Indeed, the pair $(W, \partial_+ W)$ is $(n - 1)$-connected because the cobordism

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*Footnote*: Proposition 6.6 below provides a partial result in this direction.
\( W \) is Weinstein. Hence, any relative spheroid \( \psi : (D^j, \partial D^j) \to (W_L, \partial_+ W_L) \) extends for \( j \leq n - 1 \) to a spheroid \( \Psi : (D^j_{+1}, \partial_- D^j_{+1}) \to (W, \partial_+ W) \), where we denoted

\[
D^j_{+1} := \left\{ \sum_{k=1}^{j+1} x_k^2 \leq 1, \ x_{j+1} \geq 0 \right\} \subset \mathbb{R}^{j+1}, \ \partial_- D^j_{+1} = D^j_{+1} \cap \{ x_{j+1} = 0 \}
\]

and identified the the upper-half sphere \( \left\{ \sum_{k=1}^{j+1} x_k^2 = 1, \ x_{j+1} = 0 \right\} = \partial D^j_{+1} \setminus \text{Int} (\partial_- D^j_{+1}) \) with the disc \( D^j \). Assuming without loss of generality that \( \Psi \) is transverse to \( L \) we conclude that if \( j < n - 1 \) then \( \Psi(D^j_{+1}) \subset W_L \), and hence the homotopy class \([\psi] \in \pi_j(W_L, \partial_+ W_L)\) is trivial, and if \( j = n - 1 \) the image \( \Psi(D^j_{+1}) \) intersects \( L \) transversely in finitely many points. Hence, \( \pi_{n-1}(W_L, \partial_+ W_L) \) is generated by small \((n - 1)\)-spheres \( S \) linked with \( L \) and transported to the base point in \( \partial_+ W_L \) by some paths in \( W_L \). Moreover, the condition \( \partial_+ L \neq \emptyset \) for each connected component of \( L \) allows us to choose \( S \subset \partial_+ W_L \) and the condition \( \pi_1(W_L, \partial_+ W_L) = 0 \) provides a homotopy of the connecting path to \( \partial_+ W \), i.e. the generating spheroid is trivial in \( \pi_{n-1}(W_L, \partial_+ W_L) \). Hence, the pair \( (W_L, \partial_+ W_L) \) is \((n - 1)\)-connected. Then the classical Whitehead-Smale’s handle exchange argument, see \([32]\), allows us to construct a defining function on the cobordism \( W_L \) without critical points of index \( \geq n \).

Using Theorem 13.1 from \([10]\), which holds for cobordisms with corners, we construct a flexible Weinstein cobordism structure on \( W_L \) which agrees with the standard symplectic structure on \((\text{the boundary of}) \) the given neighborhood \( \mathcal{O} p L := \hat{j}(U) \) of the Lagrangian \( L \). Together with the canonical subcritical Weinstein structure on \( \mathcal{O} p L \), it yields a flexible Weinstein structure \( \eta \) on \( W \) which is in the same almost symplectic homotopy class as the original symplectic structure on \( W \). Using Theorem 14.3 and Proposition 11.8 from \([10]\), we can construct a diffeotopy \( h_t : W \to W \) connecting the identity \( h_0 = \text{Id} \) with a symplectomorphism \( h_1 : (W, \eta) \to (W, \omega) \). Then the parameterized Lagrangian cobordism \( h_1 \circ f : (L, \partial_- L, \partial_+ L) \to (W, \partial_- W, \partial_+ W) \) is in the prescribed formal class.

To prove the second part denote \( \hat{L}_0 := f_0(L_0) \) and \( \hat{L}_1 := f_1(L_1) \). Let us observe that the Lagrangian neighborhood theorem allows us to assume that \( h \) is symplectic on \( \mathcal{O} p \hat{L}_0 \). Let \( \mathfrak{W}_0 \), and \( \mathfrak{W}_1 \) denote the given Weinstein structures on \( W_0 \) and \( W_1 \). By assumption the Weinstein structures \( h_0 \mathfrak{W}_0 \) and \( h_1 \mathfrak{W}_1 \) restricted to \( W_{L_0} \) are in the same relative to \( \partial_- W \) almost symplectic class, and hence according to Theorem 14.3 and Proposition 11.8 from \([10]\) \( h \) is isotopic to a symplectomorphism \( \tilde{h} : W \to W \) via an isotopy fixed on \( \mathcal{O} p L_0 \), and in particular, we have \( \tilde{h} \circ f_0 = g \circ f_1 \).
Remark 4.4. An interesting aspect of Theorem 4.2 is that when $\partial W = \emptyset$, the positive Legendrian boundaries of a flexible Lagrangians $L$ necessarily cannot be loose in the sense of [26] (as they are filled by exact Lagrangians), and indeed must have non-trivial holomorphic curve invariants. For instance, the wrapped Floer homology $WFH_\ast(L,L; W)$ of $L$ must be 0 by i.e., Lemma 6.3, or more directly, one can note that $W$ is flexible, hence $SH_\ast(W) = 0$ and $WFH_\ast(L,L; W)$ is a unital module over $SH_\ast(M)$. It follows [11] that there is an isomorphism between the (linearized) Legendrian contact homology $WFH^{+}_\ast(\partial L)$ and the relative homology $H_\ast(L, \partial L)$. See Problem 4.13 for further discussion.

Corollary 4.5. Let $L$ be an $n$-manifold with non-empty boundary, equipped with a fixed trivialization $\eta$ of its complexified tangent bundle $TL \otimes \mathbb{C}$. Then there exists a flexible Lagrangian embedding with Legendrian boundary $(L, \partial L) \rightarrow (B^{2n}, \partial B^{2n})$ where $B^{2n}$ is the standard symplectic $2n$-ball, realizing the trivialization $\eta$. In particular, any 3-manifold with boundary can be realized as a flexible Lagrangian submanifold of $B^6$ with Legendrian boundary in $\partial B^6$.

Proof. With respect to a reference trivialization $f : TL \otimes \mathbb{C} \cong L \times \mathbb{C}^n$, the trivialization $\eta$ is equivalent to the data of a map $\phi : L \rightarrow U(n)$ such that $\phi(\partial L) \subset U(n-1) \subset U(n)$. Given such data, Gromov’s h-principle for Lagrangian immersions produces a Lagrangian immersion $f : (L, \partial L) \rightarrow (B^{2n}, \partial B^{2n})$ transverse to $\partial B^{2n}$. Moreover, using Whitney’s cancellation technique and the fact that $\partial L \neq \emptyset$ we can regularly (but not symplectically) homotope $f$ to an embedding $f'$. The resulting embedding $f'$ inherits a formal Lagrangian structure from the immersion $f$. We then complete the proof using Theorem 4.2.

To explore further consequences of the above constructions we first recall a theorem of Michèle Audin. Given a connected closed n-dimensional manifold $L$ and an immersion $f : L \rightarrow \mathbb{R}^{2n}$ with transverse double points we denote by $d(f)$ the algebraic count of double points. This is an integer if $L$ is orientable and $n$ is even and an element of $\mathbb{Z}/2$ otherwise. $d(f)$ is an invariant of the regular homotopy class of $f$ which vanishes if and only if the class contains an embedding. For a closed connected manifold $L$ of dimension $n = 2k + 1$ we denote by $\chi^1_2 (L)$ Kervaire’s semi-characteristic

$$\chi^1_2 (L) := \sum_{i=0}^{k} \text{rank}H_i (L) \pmod{2}.$$ 

A relationship between $\chi^1_2 (L)$ and $d(L)$ is given (in nice cases) by the following result:
Theorem 4.6. [M. Audin, [6]] Let $L$ be a closed manifold of odd dimension $\neq 1, 3$. Then for any Lagrangian immersion $f : L \to \mathbb{R}^{2n}$ with transverse double points, we have $d(f) = \chi_{\frac{1}{2}}(L)$ at least in the following cases:

(i) $L$ is stably parallelizable;

(ii) $n = 4k + 1$ and $L$ is orientable;

(iii) $n = 8k + 3, k \neq 2q$ and $L$ is spin.

Let us call an $n$-dimensional, $n > 2$, connected closed manifold $L$ with trivial complexified tangent bundle $TL \otimes \mathbb{C}$ admissible if at least one of the following conditions holds:

- $n = 3$;
- $n$ is even, $L$ is orientable and $\chi(L) = 2$;
- $n$ is even, $L$ is not orientable and $\chi(L)$ is even;
- $n$ is odd, $L$ satisfies one of the conditions (i)–(iii) of Audin’s theorem and $\chi_{\frac{1}{2}}(L) = 1$.

Theorem 4.7. Let $L$ be a closed admissible $n$-dimensional manifold. Then there exists a Weinstein structure $W(L) = (\omega_L, X_L, \phi_L)$ on $T^*S^n$ in the same formal homotopy class as the standard one, which contains $L$ as a flexible Lagrangian submanifold in the homology class of the 0-section (with $\mathbb{Z}/2$-coefficients in the non-orientable case). Moreover, infinitely many of the $W(L)$ are distinct as Weinstein manifolds.

Remark 4.8. Conversely, any closed regular Lagrangian in $T^*S^n$ with a possibly exotic, but formally standard Weinstein structure must have a trivial complexified tangent bundle and realizes the generator homology class (with $\mathbb{Z}/2$-coefficients if $L$ is not orientable), see Lemma 2.4. Furthermore, if $n$ is even then $\chi(L) = 2$ if $L$ is orientable and $\chi(L)$ is even otherwise. Indeed, we have $\chi(L) = -[L] \cdot [L] = -[S^n] \cdot [S^n] = \chi(S^n) = 2$, and if $L$ is not orientable this holds mod 2. If $n$ is odd then one can deduce from Audin’s theorem that for all admissible $L$, i.e. in all cases listed in that theorem the condition $\chi_{\frac{1}{2}}(L) = 1$ is also necessary.

The proof of Theorem 4.7 roughly will proceed as follows: we remove a disc from $L$ to obtain a manifold $\hat{L}$ with spherical boundary. Corollary 4.5 produces a flexible Lagrangian embedding $\hat{L} \hookrightarrow \mathbb{C}^n$ with parametrized Legendrian boundary $S^{n-1} \cong$
\( \partial \hat{L} \hookrightarrow S^{2n-1} \); if this Legendrian lies in the same formal Legendrian isotopy class as the standard unknot, then the result \( W(L) \) of attaching a handle to \( \mathbb{C}^n \) along \( \partial \hat{L} \) (which contains \( L \) as a regular Lagrangian) will be formally homotopic to \( T^*S^n \). It will thus be necessary to understand the Legendrian isotopy class of the aforementioned Legendrian embedding.

Recall that the formal Legendrian isotopy class of a parameterized Legendrian sphere \( g : S^{n-1} \to S^{2n-1} \) in the standard contact \( S^{2n-1} \) is determined by two invariants (see [26, 10, 33, 20]): the rotation class \( r(g) \in \pi_{n-1}(U(n)) \) and the generalized Thurston-Bennequin invariant \( \text{tb}(g) \). If \( n \) is even then \( \text{tb}(g) \) can be defined as the linking number between \( g \) and its push-off by the Reeb flow. If \( n \) is odd the rotation class identically vanishes, while the above definition of \( \text{tb}(g) \) always yields \( \pm \frac{\chi(S^{n-1})}{2} = \pm 1 \), where the sign depends only on dimension. When \( n = 3 \) there is indeed only 1 formal Legendrian isotopy class of spheres. However, for all odd \( n > 3 \) there are exactly two classes, see [26, 10]. They are distinguished by a modified Thurston-Bennequin invariant, which we will continue to denote by \( \text{tb} \), and which can be defined as follows, see [10].

The vanishing of \( r(g) \) allows us to connect \( g \) with the Legendrian unknot \( g_0 \) by a regular Legendrian homotopy. Viewing the homotopy as an immersed cylinder in \( S^{2n-1} \times [0, 1] \), and assuming that the immersion has transverse double points, we set \( \text{tb}(g) := k + 1 \pmod{2} \), where \( k \) is the number of double points. It turns out that this residue is independent of the choice of a regular homotopy.

In order to prove Theorem 4.7 we will need the following two Lemmas:

**Lemma 4.9.** Let \( \hat{L} \) be a non-orientable manifold of dimension \( n = 2k > 2 \) bounded by a sphere, and \( h : S^{n-1} \to \partial \hat{L} \) a parameterization of its boundary. Suppose that the complexified tangent bundle \( TL \otimes \mathbb{C} \) is trivial. Then for any \( k \equiv \chi(L) \pmod{2} \) there exists a Lagrangian embedding \( f : (\hat{L}, \partial \hat{L}) \to (B^{2n}, \partial B^{2n}) \) with Legendrian boundary such that \( \text{tb}(f \circ h) = k \) and \( r(f \circ h) = r(f \circ h) \).

**Proof.** Let \( \tilde{f} : (\hat{L}, \partial \hat{L}) \to (B^{2n}, \partial B^{2n}) \) be a Lagrangian embedding with Legendrian boundary provided by Corollary 4.5. Using the stabilization procedure, see [10], one can modify for any integer \( m \) the Legendrian knot \( \tilde{f}|_{\partial \hat{L}} \) by a Legendrian regular homotopy to a Legendrian embedding \( f : \partial \hat{L} \to \partial B^{2n} \) with \( \text{tb}(f) = \text{tb}(\tilde{f}|_{\partial \hat{L}}) + m \). Note that \( r(\tilde{f}) = r(f) \). Let \( F : \partial \hat{L} \times [0, 1] \to \partial B^{2n} \times [0, 1] \) be a Lagrangian immersion, corresponding to this regular homotopy which connects \( F|_{\partial \hat{L} \times 0} = \tilde{f}|_{\partial \hat{L}} \) and \( F|_{\partial \hat{L} \times 1} = f \). The algebraic number of double points of \( F \) is equal to \( m \). Gluing the Lagrangian cylinder \( F \) with the embedding \( \tilde{f} \) we get a Lagrangian immersion \( \tilde{f} \).
of $\hat{L}$ whose boundary Legendrian sphere has its Thurston-Bennequin invariant equal to $\text{tb}(\hat{f}|_{\partial\hat{L}}) + m$ and has the same rotation class as $f$.

Suppose now that $k$ has the same parity as $\chi(\hat{L})$. Since $\text{tb}(\hat{f}|_{\partial\hat{L}})$ also has the same parity as $\chi(\hat{L})$, we have $k = \text{tb}(\hat{f}|_{\partial\hat{L}}) + 2l$ for some integer $l$. Apply the above construction to $m := 2l$. Then, using non-orientability of $\hat{L}$ one can cancel all $2l$ double points in pairs by a smooth (not necessarily Lagrangian) isotopy, thus obtaining a formal Lagrangian embedding with the required Legendrian boundary invariants. Applying again Corollary 4.5 we construct a genuine Lagrangian embedding $f: \hat{L} \to B^{2n}$ with the prescribed invariants of the boundary.

**Lemma 4.10.** Suppose that $L$ is admissible, and $\hat{L}$ is obtained from $L$ by removing an $n$-ball, $\hat{L} := L \setminus \text{Int} D^n$. Suppose that the boundary $\partial\hat{L}$ is parameterized by a diffeomorphism $h : S^{n-1} \to \partial\hat{L}$. If $L$ is orientable then for any Lagrangian embedding with Legendrian boundary $f : (\hat{L}, \partial\hat{L}) \to (B^{2n}, S^{2n-1})$, the Legendrian embedding $f \circ h : S^{n-1} \to S^{2n-1}$ is in the formal Legendrian isotopy class of the Legendrian unknot $g_0 : S^{n-1} \to S^{2n-1}$. If $L$ is non-orientable then there exists a Lagrangian embedding $f : (\hat{L}, \partial\hat{L}) \to (B^{2n}, S^{2n-1})$ with Legendrian boundary such that $f \circ h$ is in the formal Legendrian isotopy class of the Legendrian unknot $g_0$.

**Proof.** Now, suppose that $g = f \circ h$ for a Lagrangian embedding $f : \hat{L} \to B^{2n}$ and a diffeomorphism $h : S^{n-1} \to \partial\hat{L}$. Then $r(g) = 0$. We already noted that this is always the case when $n$ is odd. If $n$ is even and $L$ is orientable then the Hurewicz homomorphism $\pi_{n-1}(U(n)) \to H_{n-1}(U(n))$ is injective (see e.g. Theorem 20.9.6 in [21]). Hence $r(g) = 0$ if the bounding Lagrangian is orientable. But the same argument applies in the non-orientable case to $2r(g) \in \pi_{n-1}(U(n)) = \mathbb{Z}$.

If $n$ is even and $L$ is orientable then $\text{tb}(g) = \pm \chi(\hat{L}) = \text{tb}(g_0)$. This is proven in [20] but here is another argument. Consider a vector field $v$ tangent to $\hat{L}$ such that $v|_{\partial\hat{L}}$ agrees the Liouville vector field $X$. Then the push-off of $f(\hat{L})$ along $w := Jdf(v)$ intersects $f(\hat{L})$ at $|\chi(\hat{L})|$ points. But $w|_{\delta\hat{L}}$ is the Reeb vector field, so the linking number entering the definition of $\text{tb}(g)$ is equal to $\chi(\hat{L})$ up to sign. If $\hat{L}$ is not orientable then the above argument implies only that $\text{tb}(g) \equiv \chi(\hat{L}) \pmod{2}$. But in that case Lemma 4.9 allows us to modify the embedding $f$ to ensure that $\text{tb}(g) = \chi(\hat{L})$. Suppose now that $n$ is odd and $n > 3$. Then we can use Audin’s Theorem 4.6 to deduce that $\text{tb}(g)$ coincides with the Kervaire semi-characteristic $\chi_1(\hat{L})$ for all admissible $L$. Indeed, let $G : S^{n-1} \times [0, 1] \to B^{2n}(R) \setminus B^{2n}(1)$ be a Lagrangian immersion realizing a regular
Legendrian homotopy connecting $G|_{S^{n-1} \times 0} = g : S^{n-1} \rightarrow \partial B^{2n}(1)$ with the Legendrian unknot $g_0 = G|_{S^{n-1} \times 1} : S^{n-1} \rightarrow \partial B^{2n}(R)$. Such immersion exists for sufficiently large $R$, see [16]. It has $tb(g) + 1$ (mod 2) intersection points. In turn, the unknot $g_0$ bounds an immersed Lagrangian disc $g_1 : (D^n, \partial D^n) \rightarrow (\mathbb{C}^n \setminus \text{Int} B^{2n}(R), \partial B^{2n}(R))$ with 1 intersection point. Gluing together the Lagrangian embedding $f$ with Lagrangian immersions $G$ and $g_1$ we get a Lagrangian immersion of $L = \hat{L} \cup D^n$ to $\mathbb{C}^n$ with $tb(g)$ (mod 2) intersection points, and hence the claim follows from Audin’s Theorem 4.6.

Proof of Theorem 4.7. Using Corollary 4.5 we realize $(\hat{L} := L \setminus \text{Int} D^n, \partial \hat{L})$ as a flexible Lagrangian submanifold with Legendrian boundary in the standard symplectic ball $(B^{2n}, \partial B^{2n})$. According to Lemma 4.10 the gluing diffeomorphism $h$ viewed as a Legendrian embedding $\partial D^n \rightarrow \partial B^{2n}$ is formally Legendrian isotopic to the standard Legendrian unknot. Hence, by attaching to the ball $B^{2n}$ a Weinstein handle of index $n$ along $\partial L$ using $h$ we get a Weinstein domain $W(L)$ diffeomorphic to the disk cotangent bundle $UT^* S^n$ with its symplectic structure in the standard formal symplectic homotopy class. The Weinstein domain $W(L)$ contains $L$ as a closed flexible Lagrangian submanifold in the homology class of the 0-section.

To prove the second part of the theorem, which concerns with infinitely many symplectomorphic types of the resulting Weinstein domains $W(L)$, we first observe that the Viterbo transfer map on symplectic homology $SH(W(L)) \rightarrow SH(UT^*(L))$ is an isomorphism preserving symplectic cohomology’s TQFT operations and BV algebra structure, see e.g. [29] (the part about compatibility of Viterbo transfer map with the BV operator is folkloric). In turn, $SH(UT^*(L))$ as a BV algebra is isomorphic to $H(\Lambda(L); \mathbb{Z})$, the homology of the free loop space $\Lambda(L)$, with the Chas-Sullivan string topology BV algebra structure at least whenever $L$ is Spin, see [31, 34, 4, 5, 11]. Since $c_1(T^*S^n) = 0$ and $H^1(T^*S^n; \mathbb{Z}) = 0$, there is a canonical grading on $SH(W(L))$; if $H^1(L; \mathbb{Z}) = 0$, then $SH(W(L)) \cong H(\Lambda(L); \mathbb{Z})$ is grading-preserving. In particular, assuming $L, L'$ are spin with $H^1(L; \mathbb{Z}) = H^1(L'; \mathbb{Z}) = 0$, the Weinstein domains $W(L)$ and $W(L')$ are not symplectomorphic whenever the BV algebras $H_*(\Lambda(L); \mathbb{Z})$ and $H_*(\Lambda(L'); \mathbb{Z})$ are non-isomorphic. Thus, the above construction provides a rich source of exotic symplectic structures on $T^* S^n$; it is at least as rich as the collection of string topology BV algebra structure on various $n$-manifolds.

Hence, to get infinitely many non-symplectomorphic structures it suffices to find for any $n \geq 3$ infinitely many closed stably parallelizable manifolds $L$ with $H^1(L; \mathbb{Z}) = 0$, $\chi(L) = 2$ for $n$ even, $\chi_L(\hat{L}) = 1$ for $n$ odd and $\neq 3$, and different $H_0(\Lambda(L); \mathbb{Z})$.

Since the rank of $H_0(\Lambda(L); \mathbb{Z})$ equals the number of conjugacy classes of $\pi_1(L)$, it suffices to find $L$ with fundamental groups $\pi_1(L)$ with different numbers of conjugacy
classes; if we furthermore assume that $\pi_1(L)$ is finite, then $H^1(L) = 0$ automatically since $H^1(L)$ is always torsion-free. For $n = 3$ we can take the collection of Lens spaces $L(k, 1)$. To get examples for $n \geq 4$, consider the CW complex $X$ which has one 0, 1, and 2-cell such that the attaching map for the 2-cell wraps $k$ times around the 1-cell so that $\pi_1(X) \cong \mathbb{Z}/k\mathbb{Z}$ and $\chi(X) = 1$. We can then embed $X$ into $\mathbb{R}^{n+1}$, $n \geq 4$, and take a regular neighborhood $W$ of $X \subset \mathbb{R}^{n+1}$. Then the closed $n$-dimensional manifold $\partial W$ satisfies all the required conditions. Indeed, it is stably parallelizable, $\chi(\partial W) = 2\chi(W) = 2$ for $n$ even, $\chi_2(\partial W) = 1$ for $n$ odd, and $\pi_1(\partial W) \cong \pi_1(W) \cong \pi_1(X) = \mathbb{Z}/k\mathbb{Z}$.

Remark 4.11. There are by now many constructions of exotic Weinstein structures on $T^*S^n$, for instance [23, 25, 24, 2]. A notable feature of the examples $W(L)$ given above is that they are each constructed from a single handle attachment on standard $B^{2n}$. Hence they have minimal complexity, meaning the defining function for the resulting Weinstein structure can be chosen with exactly two critical points (in particular, the Weinstein geometry of each $W(L)$ is entirely determined by the Legendrian isotopy class $\partial \hat{L} \subset S^{2n-1}$; recall $\hat{L} = L \setminus D^n$). In addition, each example $W(L)$ contains an exact Lagrangian $L$; and hence has non-vanishing symplectic homology with any coefficients (at least whenever $L$ is Spin).

It is conceivable that the examples $W(L)$ are as diverse as diffeomorphism types of manifolds $L$:

**Problem 4.12.** *Does symplectic topology of $W(L)$ remember the diffeomorphism type of $L$, or even of $L$?*

As Remark 4.11 recalls, the symplectic topology of $W(L)$ is entirely determined by the differing Legendrian topology of embeddings $\partial \hat{L} \hookrightarrow \partial B^{2n}$; in particular the examples in Theorem 4.7 produce infinitely many non-isotopic Legendrians $\partial \hat{L} \subset S^{2n-1}$ in the same formal isotopy class. Hence, one can recast Problem 4.12 in terms of more general questions about the richness of the Legendrian topology of boundaries $\partial L$ of flexible Lagrangians $L$. For instance,

**Problem 4.13.** *Does the Legendrian boundary $\partial L$ of a flexible Lagrangian $L$ remember the topology of the filling? For instance, when $\partial L = S^2$ there exists a unique formal class of Legendrian 2-spheres in contact $S^5$. Does the genuine Legendrian isotopy class of $\partial L$ remember the topology of $L$?*

Work in progress of T. Ekholm and Y. Lekili, [12], see also [22], implies that the Legendrian boundary of a flexible Lagrangian knows a lot about the topology of its
filling. For instance, if $L$ is simply connected the Legendrian isotopy class of $\partial L$ remembers the rational homotopy type of $L$.

Suppose we are given two $n$-dimensional closed manifolds $M, N$ such that $M$ admits a formal Lagrangian embedding $N \to TM$ which intersects a cotangent fiber at 1 point. Let $\hat{M}$ and $\hat{N}$ be manifolds with boundary obtained by removing small $n$-discs from $M$ and $N$. Then the cotangent bundle $W := T^*\hat{M}$ is a subcritical Weinstein manifold, and hence Theorem 4.2 provides a flexible Lagrangian embedding $f : (\hat{N}, \partial\hat{N}) \to (W, \partial W)$ with Legendrian boundary. We also have the canonical inclusion $(\hat{M}, \partial\hat{M}) \hookrightarrow (W, \partial W)$. This leads to an alternative:

*either Legendrian spheres $f(\partial\hat{N})$ and $j(\partial\hat{M})$ are not Legendrian isotopic, or the nearby Lagrangian conjecture fails.*

In particular, an intriguing case is when $M, N$ are two homeomorphic 4-manifolds distinguished by gauge-theoretic invariants.

5 Murphy-Siegel example

![Diagram](Fig. 5.1: Caving out a neighborhood of a Legendrian submanifold)

We note that when $(D, \partial D) \subset (W, \partial W)$ is a regular Lagrangian disc, then besides the complementary cobordism $W_D$ we can also consider a Weinstein domain $W^D$, the result of Weinstein anti-surgery. In other words, $W$ is obtained from $W^D$ by a
Weinstein surgery with the co-core disc $D$. We denote by $\Gamma$ the Legendrian sphere in $W^D$ along which the handle with co-core $D$ is attached.

**Lemma 5.1.** If $W^D$ is a flexible cobordism, then $W^D$ is a flexible domain. If $W^D$ is a flexible domain and $W$ is obtained from $W^D$ by attaching a handle along a loose Legendrian sphere $\partial D \subset \partial W^D$, then $W_D$ is flexible.

Lemma 5.1 follows from the more general Lemma 5.2 below. (Note alternately that the second part of Lemma 5.1 is equivalent to Proposition 3.3(i)).

Suppose $W$ is a flexible Weinstein domain and $\Lambda \subset \partial W$ is a Legendrian submanifold. Then one can canonically construct a Liouville cobordism $\Lambda W$ by caving out a neighborhood of $\Lambda$, see Fig. 5.1. Thus $\partial_-(\Lambda W)$ is the canonical Darboux neighborhood $J^1(\Lambda)$ of $\Lambda$ in $\partial W$. Note that in the situation of Lemma 5.1, the Weinstein cobordism $W_D$ coincides with the cobordism $\Gamma W^D$.

**Lemma 5.2.** $\Lambda W$ is always a Weinstein cobordism. If $\Lambda W$ is flexible then $W$ is flexible. If $W$ is flexible and $\Lambda$ is loose then $\Lambda W$ is flexible.

**Proof.** Take the cotangent bundle $T^*(\Lambda \times [-1, 1])$ with its canonical subcritical Weinstein domain structure $U$ of a cotangent bundle of a manifold with boundary. The boundary $\partial U$ contains $\Lambda \times 1$ as its Legendrian submanifold. Consider a tubular neighborhood $\Sigma$ of $\Lambda \times 1$ in $\partial U$. It can be identified with a neighborhood of $\Lambda$ in $J^1(\Lambda)$. Consider a trivial Weinstein cobordism $V$ over $\Sigma$, or rather its sutured
version, see Fig. 5.1. Note that the positive boundary \( \partial_+ V \) is a copy of \( \Sigma \) and it coincides with the negative boundary \( \partial_- (\Lambda W) \). The Weinstein handlebody presentation of \( \Lambda W \) builds \( \Lambda W \) by a sequence of handle attachments to \( \partial_+ V \). Making the same handle attachments to \( \Sigma \subset \partial U \) we build instead the original Weinstein domain \( W \). If \( \Lambda W \) is flexible then the result of the gluing is flexible as well, because we added a flexible cobordism to a subcritical domain.

Let us now assume that \( W \) is flexible and \( \Lambda \) is loose. Then we can get \( \Lambda W \) by attaching a handle with the cylindrical Lagrangian core \( \Lambda \times [-1, 1] \) to the cobordism \( V \sqcup W \) by gluing \( \Lambda \times (-1) \) to the 0 section \( \Lambda \subset \Sigma \times 1 \) and \( \Lambda \times (+1) \) to \( \Lambda \subset \partial W \), see Fig. 5.2. By choosing a handlebody decomposition of \( \Lambda \times [-1, 1] \) with exactly one \( n \)-handle, we can decompose the attachment of the handle with a cylindrical core to a sequence of attachments of handles corresponding to the handles of the decomposition of \( \Lambda \times [-1, 1] \). One can arrange that the stable manifolds of the subcritical handles are in the complement of the intersection of \( \Lambda \subset W \) with its loose chart. It then follows that the only index \( n \) handle is attached along a loose knot, and hence the resulting cobordism is flexible. For general \( \Lambda \), this construction shows that \( \Lambda W \) is always a Weinstein cobordism.

**Problem 5.3.** Suppose \( \Lambda W \) (and hence \( W \)) is flexible. Is it true that then \( \Lambda \) is a loose Legendrian?

Note that an affirmative answer to this problem would also give an affirmative answer to Problem 3.4(i).

Emmy Murphy and Kyler Siegel produced an example, see [28], of a non-flexible Weinstein domain \( W \) which becomes flexible after attaching a Weinstein \( n \)-handle \( H \) (they call domains with this property subflexible). Denote \( \hat{W} := W \sqcup H \). Let \((C, \partial C) \subset (\hat{W}, \partial \hat{W})\) be the Lagrangian co-core of the handle \( H \).

**Proposition 5.4.** \((C, \partial C) \subset (\hat{W}, \partial \hat{W})\) is non-flexible.

**Proof.** We can identify \( W \) with \( \hat{W}^C \). But by Lemma 5.1 the non-flexibility of \( \hat{W}^C \) implies the non-flexibility of \( \hat{W}_C \). \( \square \)

Thus, in a flexible domain there could be non-flexible regular Lagrangian discs.

Moreover, Murphy-Siegel’s results imply that even in the standard symplectic ball there exists a non-flexible but regular union of Lagrangian discs.

Let \((\hat{C}, \partial \hat{C}) \subset (\hat{W}, \partial \hat{W})\) be the flexible realization of the same formal Lagrangian class using Theorem 4.2. It follows that \((C, \partial C)\) and \((\hat{C}, \partial \hat{C})\) are not Hamiltonian isotopic or even symplectomorphic.
Problem 5.5. Are the spheres $\partial C, \partial \tilde{C} \subset \partial \tilde{W}$ Legendrian isotopic?

If the answer was positive, then by attaching an $n$-handle we would get a Weinstein manifold containing two regular Lagrangian spheres in the same formal class, one with a flexible and the other with a non-flexible complement, thus providing a negative answer to the last part of Problem 3.5.

Remark 5.6. One can ask if there is a relative version of the Murphy–Siegel phenomenon [28], i.e., are there non-flexible regular Lagrangians which become flexible after attaching a handle to the Weinstein domain? One example of this kind is immediate from [28]. Namely, let $W$ be a subflexible (but not flexible) Weinstein domain, which becomes flexible after attaching a handle $H$. As work of Murphy-Siegel shows, one can attach to $W$ an index $n$ handle $\tilde{H}$ along a loose knot in such a way that the new domain $\tilde{W} := W \cup \tilde{H}$ remains non-flexible. Note that the co-core Lagrangian disc $C \subset \tilde{H}$ is regular but not flexible in $\tilde{W}$. However, it is flexible in $\tilde{W} \cup H$ (by the same direct argument that shows that $W \cup H$ is flexible). This construction is not a completely satisfactory answer to the above problem; note that while the co-core disc $C$ is not flexible in $\tilde{W}$, it is semi-flexible there, see Section 6 below, and thus already has vanishing Floer-theoretic invariants.

6 Semi-flexible Lagrangians

Suppose $L \subset W$ is a flexible Lagrangian cobordism in a Weinstein cobordism $W$, and a $W'$ any other Weinstein cobordism. Suppose that $\tilde{W}$ obtained from $W$ and $W'$ by gluing some connected components of $\partial_- W$ not intersecting $\partial_- L$ with the corresponding components of $\partial_+ W'$. We then say that $L \subset \tilde{W}$ is a semi-flexible Lagrangian cobordism.

A prototypical example of a semi-flexible Lagrangian is the co-core of an index $n$ Weinstein handle attached along a loose Legendrian sphere $S \subset \partial_+ W$ to a Weinstein cobordism $W$.

Problem 6.1. Are there semi-flexible but not flexible Lagrangians in a flexible Weinstein cobordism?

In particular, is it possible to make a non-flexible Weinstein cobordism flexible by attaching handle along a loose knot?

Proposition 6.2. If $L$ is semi-flexible Lagrangian with $\partial L \neq \emptyset$ in a Weinstein domain $W$, then there are no closed Lagrangians $L'$ with non-zero homological intersection index $L \cap L'$.
Lemma 6.3. Let $L$ be a semi-flexible Lagrangian in a Weinstein domain $W$. Suppose that $\partial L \neq \emptyset$. Then the wrapped Floer homology $WFH(L, L; W)$ vanishes.

Proof. For the canonical subcritical neighborhood $U \supset L$ we have $WFH(L, L; U) = 0$ because it is a (unital) module over symplectic homology $SH(U) = 0$. By assumption $W$ is obtained by attaching subcritical or flexible handles to the disjoint union $U \sqcup W'$ along spheres in the complement of $\partial L$. But $WFH(L, L; U \sqcup W') = WFH(L, L; U) = 0$, and attaching of subcritical or flexible handles does not change $WFH(L, L)$. For index $n$ handles this last fact is proven in [8]. In the subcritical case this is a part of symplectic folklore (see [9, 37, 13] for the closest statements in the literature). Hence, $WFH(L, L; W) = 0$. 

Proof of Proposition 6.2. For any pair $L, L'$ with $L'$ closed, the Euler characteristic of the Lagrangian Floer homology $FH(L, L')$ coincides with the homological intersection index $L \cap L'$. On the other hand, $FH(L, L') = WFH(L, L')$ is a (unital) module over $WFH(L, L)$ which vanishes by Lemma 6.3, hence $FH(L, L')$ must vanish as well.

Corollary 6.4. Semi-flexible Lagrangians do not satisfy the existence $h$-principle. For instance, for any closed $L$ the class of a cotangent fiber in $T^*L$ cannot be realized by a semi-flexible Lagrangian disc with Legendrian boundary.

It seems similarly unlikely that semi-flexible Lagrangians satisfy any non-trivial form of the uniqueness statement. However, we formulate this question as an open problem.

Problem 6.5. Consider two semi-flexible Lagrangians $L_0, L_1 \subset (W, \omega)$. Suppose that there exists a diffeomorphism $W \to W$ such that $f(L_0) = L_1$, and $f^*\omega$ and $\omega$ are homotopic as non-degenerate not necessarily closed 2-forms. Is $f$ isotopic to a symplectomorphism $g : W \to W$ with $g(L_0) = L_1$?

We finish this section with Proposition 6.6 which gives a Hamiltonian isotopy classification of semi-flexible Lagrangian discs smoothly isotopic to a disc in $\partial_+ W$. Consider $\mathbb{R}^{2n}$ with the standard Liouville form $\lambda = \frac{1}{2} \left( \sum_{i=1}^{n} x_i dy_i - y_i dx_i \right)$. In the unit sphere $S := \{ \sum_{i=1}^{n} x_i^2 + y_i^2 = 1 \}$ consider a Legendrian equator $E := S \cap \{ y = (y_1, \ldots, y_n) = 0 \}$ and the Lagrangian disc $\tilde{E}$ bounded by $S$ in the unit ball $B := \{ \sum_{i=1}^{n} x_i^2 + y_i^2 \leq 1 \}$. 


Denote by $C$ the pre-Lagrangian disc $S \cap \{y_1 \geq 0, y_2 = \cdots = y_n = 0\}$ which bounds $E$ in $S$. Note that

$$\lambda|_C = \frac{1}{2}(x_1 dy_1 - y_1 dx_1)|_C = -y_1^2 d\left(\frac{x_1}{y_1}\right)|_C = \left(-1 + \sum_1^n x_j^2\right) d\left(\frac{x_1}{\sqrt{1 - \sum_1^n x_i^2}}\right).$$

The disc $C$ is foliated by Legendrian discs $x_1 = cy_1$, $c \in \mathbb{R}$. The Lagrangian lift of $C$ to $\mathbb{R}^{2n} \setminus 0$ (thought of as the symplectization of $(S, \{\lambda|_S = 0\})$) is the plane $\{y_1 = 1, y_2 = \cdots = y_n = 0\}$. We say that a Legendrian sphere in a contact manifold $(Y, \xi = \{\beta = 0\})$ is a Legendrian unknot if it bounds in $Y$ a disc $D$ such that $\beta|_D$ is isomorphic to a form proportional to $\lambda|_C$. This definition is equivalent to the usual definition of the unknot as a Legendrian sphere which is Legendrian isotopic to a sphere in a Darboux chart with the saucer-like front projection. Any unknot bounds in the symplectization of $Y$ (and hence in any Liouville filling of $Y$) a Lagrangian disc $\tilde{D}$ which is the lift of the pre-Lagrangian disc $D$. We call any Lagrangian disc Hamiltonian isotopic to $\tilde{D}$ small. We recall that by Hamiltonian isotopy we mean the Hamiltonian isotopy of completed Lagrangians, which is equivalent to a Lagrangian homotopy of discs with Legendrian boundary.

**Proposition 6.6.** Any semi-flexible Lagrangian disc $(D, \partial D) \subset (W, \partial_+ W)$ with Legendrian boundary which is smoothly isotopic to a disc in $\partial_+ W$ is small.

**Proof.** By assumption, $W$ can be built by attaching a flexible cobordism $V$ to the disjoint union of $T^*D \sqcup W'$, where $W'$ is a Weinstein domain. Note that $D$ is small in $T^*D$ and thus the Legendrian sphere $\partial D$ bounds a pre-Lagrangian disc $E \subset \partial T^*D$. The attaching spheres of all subcritical handles forming the flexible cobordism $V$ are generically disjoint from $E$ by dimension reasons. On the other hand, the attaching spheres of index $n$ handle can be disjoined from $E$ by a smooth isotopy in view of the condition that $D$ can be pushed by a smooth isotopy to the boundary $\partial_+ W$. But then the flexibility condition, meaning here the looseness of the Legendrian link formed by the attaching spheres of all index $n$ handles, allows us to disjoin the attaching spheres of all index $n$ handles from $E$ via a Legendrian isotopy. Hence, $\partial D$ bounds the required pre-Lagrangian disc $E$ in $\partial_+ W$, and thus $D$ is small. \qed
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