Existence of Richardson elements for seaweed
Lie algebras of finite type

Bernt Tore Jensen and Xiuping Su

Abstract

Seaweed Lie algebras are a natural generalisation of parabolic subalgebras of reductive Lie algebras. A well-known theorem of Richardson says that the adjoint action of a parabolic group has a dense open orbit in the nilpotent radical of its Lie algebra (Richardson, Bull. Lond. Math. Soc. 6 (1974) 21–24.). Elements in the open orbit are called Richardson elements. In (Jensen, Su and Yu, Bull. Lond. Math. Soc. 42 (2009) 1–15.) together with Yu, we generalised Richardson’s Theorem and showed that Richardson elements exist for seaweed Lie algebras of type $A$. Using GAP, we checked that Richardson elements exist for all exceptional simple Lie algebras except $E_8$, where we found a counterexample.

In this paper, we complete the story on Richardson elements for seaweed Lie algebras of finite type, by showing that they exist for any seaweed Lie algebra of type $B$, $C$ and $D$. By decomposing a seaweed Lie algebra into a sum of subalgebras and analysing their stabilisers, we obtain a sufficient condition for the existence of Richardson elements. The sufficient condition is then verified using quiver representation theory. More precisely, using the categorical construction of Richardson elements in type $A$, we prove that the sufficient condition is satisfied for all seaweed Lie algebras of type $B$, $C$ and $D$, except in two special cases, where we give a direct proof.

1. Introduction

Throughout, we assume that $k$ is the field of complex numbers, $\mathfrak{g}$ is a reductive Lie algebra and $G$ is a connected reductive algebraic group with Lie algebra $\mathfrak{g}$.

Definition 1.1 [5, 12]. A Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a seaweed Lie algebra (or simply a seaweed) if there exists a pair $(\mathfrak{p}, \mathfrak{p}')$ of parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ and $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$. Two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ of $\mathfrak{g}$ such that $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$ are said to be opposite.

Seaweed Lie algebras (later also called biparabolic algebras, see, for example, [9, 10]) were defined by Dergachev and Kirillov in their study of indices of Lie algebras of type $A$ [5] and were generalised by Panyshev to arbitrary reductive Lie algebras [12]. By definition, parabolic subalgebras of $\mathfrak{g}$ are seaweed Lie algebras. Substantial work on seaweeds has been done on generalising results on parabolic subalgebras and beyond. Among others, there is work on indices by Joseph [9, 10], on affine slices for the coadjoint action by Yu and Tauvel [17] and Joseph [10, 11]. Also, Panyshev and Yakimova study meander graphs in [13, 14]. Along the line of generalising results on parabolic subalgebras, Baur and Moreau study quasi-reductive biparabolic algebras in [2].

We are interested in the adjoint action of a seaweed Lie algebra on its nilpotent radical and the density of the action, with a view from quiver representation theory.
Definition 1.2. An element $x$ in the nilpotent radical $n$ of a seaweed Lie algebra $q$ is called a Richardson element if $[q, x] = n$.

In the case where $q$ is parabolic, a well-known theorem of Richardson [15] (see also [16, Chapter 33]) says that Richardson elements exist. In this case, Brüstle, Hille, Ringel and Röhrle gave a categorical construction of Richardson elements, when $g$ is of type $A$, using representations of a double quiver with relations of a linear quiver [4]. The following natural question was raised independently by Duflo and Panyushev.

Question 1.3 [8]. Does a seaweed Lie algebra have a Richardson element?

Surprisingly, a quiver model can also be constructed from a given seaweed Lie algebra to understand the adjoint action on the nilpotent radical. In [8] together with Yu, we made use of the construction in [4] to build rigid modules for the quiver model and obtained a positive answer to Question 1.3 for all seaweeds in Lie algebras of type $A$. An example (Example 3.6) is given in Section 3 to illustrate the construction. This quiver model is the double quiver with relations of a quiver of type $A$. The path algebra of the double quiver with relations is quasi-hereditary and had been studied by Hille and Vossieck in work on the radical bimodule of a hereditary algebra [6].

In this paper, we prove the existence of Richardson elements for seaweed Lie algebras of type $B, C$ and $D$.

Theorem 1.4. Let $g$ be a Lie algebra of type $B, C$ or $D$. Then any seaweed Lie algebra in $g$ has a Richardson element.

A natural approach to answer Question 1.3 would be to adapt Richardson’s proof for the case of parabolic subalgebras. However, the fact that the corresponding parabolic group is the normaliser of the nilpotent radical plays an important role in Richardson’s proof, but fails for seaweed Lie algebras. At this point, we emphasise the advantages of techniques from quiver representation theory, which do not require this fact, as can be seen in Brüstle, Hille, Ringel and Röhrle’s construction for parabolic subalgebras [4] and our construction for seaweeds [8]. The key ingredient of the categorical approach is the interplay between Lie algebras and quiver representation theory. For instance, when $g$ is of type $A$, seaweed Lie algebras are endomorphism algebras of projective representations of quivers of type $A$. Furthermore, endomorphism algebras of representations that give us Richardson elements correspond to stabilisers of Richardson elements. In this paper, we exclusively analyse local properties of endomorphisms at a vertex of the quiver and apply these properties to prove the main theorem.

As a consequence of Theorem 1.4 and the results in [8], Richardson elements exist for all seaweed Lie algebras of finite type except for $E_8$. We remark that the existence of Richardson elements in type $E_6$ and $E_7$, and the counterexample in $E_8$ (see [8]) were verified using GAP [18, version 4.8.7].

The remainder of this paper is organised as follows. In Section 2, we recall the notion of a standard seaweed Lie algebra and prove some useful lemmas. We decompose standard seaweeds, including standard parabolic subalgebras, as sums of subalgebras and analyse how their stabilisers act. Furthermore, we show that a local property of stabilisers of Richardson elements for seaweeds of type $A$ is sufficient for the existence of Richardson elements for all seaweeds of other types, except in two special cases. This condition can be verified using the categorical construction of Richardson elements [4, 7, 8] in type $A$. So in Section 3, we recall the construction in type $A$ and explain the link between seaweed Lie algebras and representations of quivers. We prove essential results on stabilisers in Section 4. In Section 5, we prove the main results. Techniques from quiver representation theory play an important role in the proofs.
2. Richardson elements and decomposition of seaweeds

2.1. Standard seaweeds and parabolic subalgebras

We fix a Borel subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \) and a Cartan subalgebra \( \mathfrak{h} \) contained in \( \mathfrak{b} \). Denote by \( \Phi, \Phi^+, \Phi^- \) and \( \Pi \), respectively, the root system, the set of positive roots, the set of negative roots and the set of positive simple roots, determined by \( \mathfrak{h}, \mathfrak{b} \) and \( \mathfrak{g} \). For \( \alpha \in \Phi \), denote by \( g_\alpha \) the root space corresponding to \( \alpha \). Write

\[
\alpha = \sum_{\alpha_i \in \Pi} x_i \alpha_i.
\]

We say that \( \alpha \) is supported at a positive simple root \( \alpha_i \) if \( x_i \neq 0 \) and call the set of all such simple roots the support of \( \alpha \).

For \( S, T \subset \Pi \), let \( \Phi_S \) be the set of roots with support in \( S \), \( \Phi_{-S} \) be the set of roots with support in \( -S \), \( \Phi_{\pm S} = \Phi_{S} \cup \Phi_{-S} \), \( \Phi_S^\pm = \Phi_{S} \cap \Phi_{-S} \), \( \Phi_{\pm S}^\pm = \Phi_{\pm S} \cap \Phi_{\pm S} \), \( \Phi_{S,T}^\pm = \Phi_{S,T} \cap \Phi_{\pm S} \), \( \Phi_{S,T} = \Phi_{S,T}^+ \cup \Phi_{S,T}^- \), \( \Phi_{S,T}^\pm = \Phi_{S,T}^+ \cap \Phi_{S,T}^- \).

We have

\[
q_{S,T} = n_{S,T}^+ \oplus l_{S,T} \oplus n_{S,T}^-,
\]

where \( n_{S,T}^\pm = \bigoplus_{\alpha \in \Phi_{\pm S}^\pm} g_\alpha \) and \( l_{S,T} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{S,T}^\pm} g_\alpha \). Then \( l_{S,T} \) is the Levi-subalgebra of \( q_{S,T} \) and \( n_{S,T} = n_{S,T}^+ \oplus n_{S,T}^- \) is the nilpotent radical of \( q_{S,T} \).

In the sequel, we will assume that neither \( S \) or \( T \) is equal to \( \emptyset \) or \( \Pi \). Note that two Lie algebras of the same type may have different rank and we view \( \mathbb{B}_1 = \mathbb{C}_1 = \mathbb{D}_1 = \mathbb{A}_1 \).

By the type of a seaweed Lie algebra \( q \subseteq \mathfrak{g} \), we mean the type of \( \mathfrak{g} \) and thus the type of \( q \) is not well-defined without an embedding \( q \subseteq \mathfrak{g} \).

We note the following symmetry with respect to the choice of \( S \) and \( T \).

**Lemma 2.2.** The seaweed Lie algebra \( q_{S,T} \) has a Richardson element if and only if so does the seaweed \( q_{T,S} \).

**Proof.** The lemma follows from the involution \( \mathfrak{g} \to \mathfrak{g} \) mapping \( g_\alpha \) onto \( g_{-\alpha} \). \( \square \)

2.2. A decomposition of seaweed Lie algebras and Richardson elements

Let \( q_{S,T} \) be a standard seaweed in a simple Lie algebra \( \mathfrak{g} \) of type \( \mathbb{B}, \mathbb{C} \) or \( \mathbb{D} \). Note that

\[
\mathfrak{g} = \left( \bigoplus_{\alpha \in \Phi} g_\alpha \right) \oplus \left( \bigoplus_{\alpha \in \Pi} [g_\alpha, g_{-\alpha}] \right),
\]
and when $\alpha + \beta \not\in \Phi \cup \{0\}$,
\[ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0. \quad (a) \]

Denote the positive simple roots of $\mathfrak{g}$ by $\alpha_1, \ldots, \alpha_n$, with the corresponding Dynkin graph numbered as follows.

- $\mathbb{B}$: \hspace{1cm} $n \rightarrow n - 1 \rightarrow n - 2 \rightarrow \cdots \rightarrow 3 \rightarrow 2 \rightarrow 1$
- $\mathbb{C}$: \hspace{1cm} $n \rightarrow n - 1 \rightarrow n - 2 \rightarrow \cdots \rightarrow 3 \leftarrow 2 \rightarrow 1$
- $\mathbb{D}$: \hspace{1cm} $n \rightarrow n - 1 \rightarrow n - 2 \rightarrow \cdots \rightarrow 4 \rightarrow 3 \rightarrow 2$

Let $C = (c_{ij})$ be the Cartan matrix of $\mathfrak{g}$. For instance, when $\mathfrak{g}$ is of type $\mathbb{B}_3$, the Cartan matrix is
\[ C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}. \]

The Cartan matrix of type $\mathbb{C}_n$ is the transpose of that of type $\mathbb{B}_n$.

Let $e_i, f_i, h_i$ be the Chevalley generators of $\mathfrak{g}$. That is, $\mathfrak{g}$ is generated by these generators subject to the relations
\begin{align*}
(1) \ [e_i, f_j] & = \delta_{ij}h_i; \\
(2) \ [h_i, e_j] & = c_{ji}e_j; \\
(3) \ [h_i, f_j] & = -c_{ji}f_j.
\end{align*}

We choose a new basis $h_1, \ldots, h_n$ of the Cartan subalgebra of $\mathfrak{g}$ as follows. First, let
\[ h_1 = \begin{cases} 
\frac{1}{2}h_1, & \text{if } \mathfrak{g} \text{ is of type } \mathbb{B}; \\
h_3, & \text{if } \mathfrak{g} \text{ is of type } \mathbb{C}; \\
\frac{1}{2}(h_3 - h_2), & \text{if } \mathfrak{g} \text{ is of type } \mathbb{D}.
\end{cases} \]

Next, for $i \geq 2$, let
\[ h_i = h_i + h_{i-1}. \]

**Lemma 2.3.** When $i = 2$ and $\mathfrak{g}$ is of type $\mathbb{D}$,
\[ [h_i, \mathfrak{g}_{\pm \alpha_j}] = \begin{cases} 
\mathfrak{g}_{\pm \alpha_j}, & \text{if } j = 1, 2, 3, \\
0, & \text{otherwise}.
\end{cases} \]

When $\mathfrak{g}$ is of type $\mathbb{D}$ and $i \neq 2$, or $\mathfrak{g}$ is of type $\mathbb{B}$ or $\mathbb{C}$,
\[ [h_i, \mathfrak{g}_{\pm \alpha_j}] = \begin{cases} 
\mathfrak{g}_{\pm \alpha_j}, & \text{if } j = i, i + 1, \\
0, & \text{otherwise}.
\end{cases} \]
Proof. Direct computation gives the following,

\[
[h_1, e_1] = \begin{cases} 
2e_1, & \text{if } \mathfrak{g} \text{ is of type } C; \\
e_1, & \text{otherwise.}
\end{cases}
\]

For \(i > 1\),

\[
[h_i, e_j] = \begin{cases} 
e_j, & \text{if } j = i; \\
-e_j, & \text{if } j = i + 1; \\
0, & \text{otherwise,}
\end{cases}
\]

except when \(i = 2\) and \(\mathfrak{g}\) is of type \(D\), where we have

\[
[h_2, e_j] = \begin{cases} 
e_j, & \text{if } j = 1 \text{ or } 2; \\
-e_j, & \text{if } j = 3; \\
0, & \text{otherwise.}
\end{cases}
\]

So the lemma follows. \(\square\)

Let \(h_i\) be the subspace spanned by \(h_i\). Let

\[
\epsilon = \min\{i \mid \alpha_i \not\in S\} \text{ and } \eta = \min\{i \mid \alpha_i \not\in T\}.
\]

By Lemma 2.2, we may assume that \(\epsilon \geq \eta \geq 1\). Let

\[
\omega = \max\{i \mid i \leq \epsilon, \alpha_i \not\in T\}.
\]

We define two subspaces of \(\mathfrak{g}\),

\[
\mathfrak{g}_1 = \bigoplus_{\alpha \in \Phi, \{\alpha_i \mid i < \epsilon\}} \mathfrak{g}_\alpha \bigoplus_{i < \epsilon} h_i,
\]

and

\[
\mathfrak{g}_2 = \bigoplus_{\alpha \in \Phi, \{\alpha_i \mid i > \omega\}} \mathfrak{g}_\alpha \bigoplus_{i \geq \omega} h_i.
\]

**Lemma 2.4.** (1) If \(\epsilon > 2\), when \(\mathfrak{g}\) is of type \(D\) or \(\epsilon > 1\) for other types, then \(\mathfrak{g}_1\) is a Lie subalgebra of the same type as \(\mathfrak{g}\).

(2) The subspace \(\mathfrak{g}_2\) is a Lie subalgebra isomorphic to \(\mathfrak{gl}_{n-\omega+1}\).

**Proof.** By the definition of \(h_i\), we have

\[
\bigoplus_{i < \epsilon} h_i = \bigoplus_{i < \epsilon} [\mathfrak{g}_\alpha_i, \mathfrak{g}_{-\alpha_i}],
\]

and

\[
\bigoplus_{i \geq \omega} h_i = \bigoplus_{i \geq \omega} [\mathfrak{g}_\alpha_i, \mathfrak{g}_{-\alpha_i}] \oplus h_\omega.
\]

So the lemma follows. \(\square\)

**Lemma 2.5.** If \(\epsilon = \omega\), then \(q_{S,T}\) has a Richardson element.

**Proof.** If \(\omega = \epsilon = 1\), then each root space \(\mathfrak{g}_\alpha\) in \(q_{S,T}\) is supported on a subgraph of type \(A\), that is, the support of \(\alpha\) contains only simple roots corresponding to vertices in the subgraph.
So \( q_{S,T} \) is isomorphic to a seaweed of type \( A \). Similarly, if \( \omega = \epsilon = 2 \) and \( g \) is of type \( D \), then \( q_{S,T} \) is also isomorphic to a seaweed of type \( A \). If \( \omega = \epsilon = n \), then \( q_{S,T} \) is a reductive Lie algebra of the same type as \( g \) with the nilpotent radical 0 and thus obviously 0 is the Richardson element. In the first two cases, by [8, Theorem 1.2] or Richardson’s Theorem [15], \( q_{S,T} \) has a Richardson element.

In all the other cases,

\[
q_{S,T} = q_2 \oplus q_1,
\]

where \( q_1 \subseteq g_1 \) is a parabolic subalgebra of the same type as \( g \), and \( q_2 \subseteq g_2 \) is a seaweed of type \( A \). Furthermore, by equation (a),

\[
[g_1, g_2] = 0 \quad \text{and so} \quad [q_1, q_2] = 0.
\]

Since both \( q_1 \) and \( q_2 \) have Richardson elements, we can conclude that \( q_{S,T} \) has a Richardson element.

\[ \square \]

Consequently, we may assume that \( \epsilon > \omega \) for the remainder of the paper. When \( g \) is of type \( D \), we also assume \( (\epsilon, \omega) \neq (2, 1) \), in which case we prove separately the existence of Richardson elements in Theorem 5.4. Let

\[
S' = \{ \alpha_i \in S \mid i < \epsilon \}, \quad T' = \{ \alpha_i \in T \mid i < \epsilon \};
\]

and

\[
S'' = \{ \alpha_i \in S \mid i > \omega \}, \quad T'' = \{ \alpha_i \in T \mid i > \omega \}.
\]

Note that by the definition of \( \epsilon \), \( S' \) contains all the simple roots \( \alpha_i \) with \( i < \epsilon \). These subsets determine two subalgebras of \( q_{S,T} \), namely the positive parabolic subalgebra \( c_{S,T} \) of \( g_1 \) determined by \( T' \) and the seaweed Lie subalgebra \( a_{S,T} \) of \( g_2 \) determined by \( S'' \) and \( T'' \).

**Example 2.6.** Let \( g \) be a Lie algebra of type \( D_6 \), \( S = \{ \alpha_5, \alpha_3, \alpha_2, \alpha_1 \} \) and \( T = \{ \alpha_6, \alpha_4, \alpha_2, \alpha_1 \} \). Then \( \epsilon = 4, \ \omega = \eta = 3 \). The subalgebras \( a_{S,T} \) and \( c_{S,T} \) can for instance be described using matrices as follows, where \( a_{S,T} \) is marked by * and †, and \( c_{S,T} \) is marked by * and †. The one-dimensional intersection is marked by †, and there is an anti-symmetry to the anti-diagonal.

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\]

Let \( l = g_2 \cap g_1 \). Then

\[
l = \bigoplus_{\alpha \in \Phi \setminus \{\alpha_i \mid i > \omega \}} g_\alpha \oplus \bigoplus_{i > \omega} h_i.
\]

Let \( n_a \) and \( n_c \) be the nilpotent radicals of \( a_{S,T} \) and \( c_{S,T} \), respectively.
LEMMA 2.7. Assume \((\epsilon, \omega) \neq (2, 1)\) if \(g\) is of type \(D\). Then

1. \(q_{S,T} = a_{S,T} + c_{S,T}\);
2. \(a_{S,T} \cap c_{S,T} = I\) is a block in the Levi subalgebra of \(q_{S,T}\);
3. \(n_{S,T} = n_a \oplus n_c\).

Proof. Let \(\alpha\) be a positive root such that \(g_{\alpha} \subseteq q_{S,T}\). Then \(\alpha\) is not supported at \(\alpha_i\), since \(\alpha_i \notin S\). By the assumption on the root system, \(\alpha\) must be supported on simple roots \(\alpha_i\) with all \(i < \epsilon\) or all \(i > \epsilon\). So \(g_{\alpha} \subseteq a_{S,T}\) or \(g_{\alpha} \subseteq c_{S,T}\). Similarly, a negative root \(\beta\) with \(g_{\beta} \subseteq q_{S,T}\) is supported on simple roots \(\alpha_i\) with all \(i < \omega\) or all \(i > \omega\), and so \(g_{\beta} \subseteq a_{S,T}\) or \(g_{\beta} \subseteq c_{S,T}\). By construction, \([g_{\alpha_1}, g_{-\alpha_1}] \subseteq a_{S,T} + c_{S,T}\) for all simple roots \(\alpha_i\), and so \(q_{S,T} = a_{S,T} + c_{S,T}\).

(2) follows from the construction;
(3) follows from (1) and (2).

By [8, Theorem 1.2], Richardson elements exist in \(a_{S,T}\). Let \(r_2 \in a_{S,T}\) be a Richardson element and denote by

\[
\text{stab}_{a_{S,T}}(r_2) = \{x \in a_{S,T} | [x, r_2] = 0\},
\]

the stabiliser of \(r_2\) in \(a_{S,T}\). For a subalgebra \(u \subseteq g\) given as a direct sum of root spaces and subspaces \(h_i\), let \(x_{|u}\) be the canonical projection of \(x \in g\) onto \(u\). Let

\[
c_{r_2} = \{x \in c_{S,T} | x_{|u} = y_{|u} \text{ for some } y \in \text{stab}_{a_{S,T}}(r_2)\}.
\]

LEMMA 2.8. Assume \((\epsilon, \omega) \neq (2, 1)\) if \(g\) is of type \(D\) and let \(r_1 \in n_c\). If \([c_{r_2}, r_1] = n_c\), then \(r_1 + r_2\) is a Richardson element of the seaweed \(q_{S,T}\).

Proof. Assume \([c_{r_2}, r_1] = n_c\). Take any \((x_a, x_i) \in n_a \oplus n_c\). There exists \(y_a \in a_{S,T}\) such that

\[
[y_a, r_2] = x_a.
\]

Write \(y_a = y_a' + (y_a)_{|u}\). Note that \(r_1|_{e_a} \neq 0\) can only occur for positive roots \(\alpha\) with support contained in \(\{\alpha_{e-1}, \ldots, \alpha_1\}\), and \(y_a' \neq 0\) can only occur for positive roots \(\beta\) with support contained in \(\{\alpha_n, \ldots, \alpha_{e+1}\}\), or negative roots with support contained in \(\{\alpha_n, \ldots, \alpha_{\omega+1}\}\) and containing at least one \(\alpha_j\) for some \(j \geq \epsilon\). So by equation (a) in Subsection 2.2 and the fact from Lemma 2.3 that

\[
[h_i, g_{\alpha_j}] = 0 \text{ for } i \geq \epsilon \text{ and } j < \epsilon,
\]

we have

\[
[y_a', r_1] = 0
\]

and so

\[
[y_a, r_1] = [(y_a)_{|u}, r_1] \in n_c.
\]

Let \(y_\epsilon \in c_{r_2}\) be such that

\[
[y_\epsilon, r_1] = x_\epsilon - [y_a, r_1].
\]

Let \(z \in \text{stab}_{a_{S,T}}(r_2)\) with \(z_{|u} = (y_\epsilon)_{|u}\) and \(z' = z - z_{|u}\). Then similar to equation (b),

\[
[z', r_1] = 0
\]

and

\[
[y_\epsilon - (y_\epsilon)_{|u}, r_2] = 0.
\]
Therefore,
\[
[y_a + z' + y_c, r_1 + r_2] \\
= [y_a, r_1 + r_2] + [z', r_1 + r_2] + [y_c, r_1 + r_2] \\
= x_a + [y_a, r_1] + [z', r_2] + x_c - [y_a, r_1] \\
= x_a + x_c + [z, r_2] \\
= x_a + x_c.
\]
This completes the proof of the lemma. \(\square\)

2.3. A decomposition of parabolic subalgebras and Richardson elements

The main goal in this subsection is to present a key sufficient condition for the existence of Richardson elements in general, except in the two special cases, namely when \(\epsilon = \omega\) as in Lemma 2.5 and when \((\epsilon, \omega) = (2, 1)\) in type \(D\). We transfer the problem of the existence of Richardson elements to a local relationship (see Lemma 2.12) between a parabolic subalgebra constructed from a given seaweed Lie algebra and the seaweed Lie algebra itself. We first give a decomposition of parabolic subalgebras, discuss properties of subalgebras in the decomposition and then state and prove the sufficient condition at the end of the section.

Let \(S, T, \epsilon, \omega, g_1, g_2\) and \(l\) be defined as in Subsection 2.2 with \(\omega < \epsilon\). In the remainder of this section, for type \(D\), besides the assumption that \((\epsilon, \omega) \neq (2, 1)\), we also assume that

1. \((\epsilon, \omega) \neq (2, 1),\)

Assumption (ii) is purely a technical issue, to avoid a complication in the description of the decomposition discussed in this subsection and it does not compromise the completeness of the existence of Richardson elements for \(q_{S,T}\) with \((\epsilon, \omega) \neq (2, 1)\), due to the symmetry between \(\alpha_1\) and \(\alpha_2\) when \(g\) is of type \(D\).

Let \(g'\) be a Lie algebra of the same type as \(g\), with rank at least \(\epsilon\) and root system denoted by \(\Phi'\). We may assume that both \(g\) and \(g'\) are subalgebras of a Lie algebra of the same type as \(g\) such that \(g \subseteq g'\) or \(g' \subseteq g\). Here all inclusions are induced by inclusions of Dynkin diagrams. This, in particular, implies that \(g\) and \(g'\) have the simple roots, \(\alpha_i\) for \(1 \leq i \leq \epsilon\), in common.

Let \(p_U^+\subseteq g'\) be the standard parabolic subalgebra determined by \(U\) with \(\alpha_\epsilon \notin U\) and\n\[
\{\alpha_i| i < \epsilon\} \cap U = \{\alpha_i| i < \epsilon\} \cap T.
\]
We choose a basis \(\{h'_i\}_i\) for the Cartan subalgebra \(h'\) of \(g'\) in the same manner as we did for the basis \(\{h_i\}_i\) of the Cartan subalgebra of \(g\). Let \(g'_1 = g_1\) and let \(g'_2 \subseteq g'\) be defined similarly to \(g_2 \subseteq g\), that is,
\[
\left( \bigoplus_{\alpha \in \Phi' \setminus U'_{(\alpha_i | i > \omega)}} g'_\alpha \right) \oplus \bigoplus_{i > \omega} h'_i,
\]
which is of type \(A\). Furthermore, let \(U''_{(\alpha_i | i > \omega)} = \{\alpha_i \in U | i > \omega\}\), \(U''_{(\alpha_i | i < \epsilon)} = \{\alpha_i \in U | i < \epsilon\}\). These two sets determine the following standard parabolic subalgebras of \(g_2'\) and \(g_1'\),
\[
a_U = \left( \bigoplus_{\alpha \in \Phi'_{U''_{(\alpha_i | i > \omega)}}} g'_\alpha \right) \oplus \bigoplus_{i > \omega} h'_i \subseteq g_2'.
\]
and

$$c_U = \bigoplus_{\alpha \in \Phi_{\gamma_0} \cup \Phi_{\gamma_1} \cap i < \epsilon} g'_{\alpha} \oplus \bigoplus_{i < \epsilon} h'_{i} \subseteq g'_1.$$ 

Note that $c_U = c_{S,T}$.

Let $d_U \subseteq p_U^+$ be the direct sum of all root spaces $g_{\alpha}$ with $\alpha$ a positive root such that $g_{\alpha}$ is neither contained in $a_U$ nor in $c_U$. Let $n'_a$ be the nilpotent radical of $a_U$. Recall that $n'_c$ is the nilpotent radical of $c_U = c_{S,T}$.

**Example 2.9.** (1) Let $g$ be a Lie algebra of type $D_6$, $S = \{\alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$ and $T = \{\alpha_6, \alpha_5, \alpha_4, \alpha_2\}$. Then $\epsilon = 5$, $\omega = 3$, $\eta = 1$. The subalgebras $a_{S,T}$ and $c_{S,T}$ are as below, marked by $\ast, \dagger$ and $\hat{\dagger}, \ast$, respectively, where the intersection is marked by $\hat{\dagger}$, and

$$q_{S,T} = \left(\begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \hat{\dagger} & \hat{\dagger} & \ast & \ast & \ast & \ast \\
\ast & \ast & \hat{\dagger} & \hat{\dagger} & \ast & \ast & \ast & \ast \\
\ast & \ast & \hat{\dagger} & \hat{\dagger} & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}\right).$$

(2) Let $g' = g$ and $U = \{\alpha_6, \alpha_4, \alpha_2\}$, which satisfies the conditions $\alpha_6 \notin U$ and $U \cap \{\alpha_i \mid i < \epsilon\} = T \cap \{\alpha_i \mid i < \epsilon\} = \{\alpha_4, \alpha_2\}$. The subalgebras $a_U$ marked by $\ast$ and $\dagger$, $c_U$ by $\hat{\dagger}$ and $\ast$, and $d_U$ by $-$ are as below.

$$p_U = \left(\begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}\right).$$

(3) $a_{S,T} \cap c_{S,T} = a_U \cap c_U$, marked by $\hat{\dagger}$, and $c_{S,T} = c_U$. Note that in both (1) and (2) there is an anti-symmetry to the anti-diagonal.

**Lemma 2.10.** The following are true.

(1) $[p_U^+, d_U] \subseteq d_U$.
(2) $[p_U^+, n'_a] \subseteq n'_a + d_U$.
(3) $p_U^+ = (a_U + c_U) \oplus d_U$. 
(4) $\alpha_U \cap \epsilon_U = I$.

(5) $\alpha_U = n_\alpha + n_\epsilon + \delta_U$.

Proof. By the construction, $\delta_U$ is the direct sum of the root spaces $g_\alpha$ with $\alpha$ positive and supported at both simple roots $\alpha_\epsilon$ and $\alpha_\omega$. For any $g_\alpha \subseteq p_U^+$ with $-\alpha$ a negative root, the root $-\alpha$ is not supported at $\alpha_\epsilon$ and $\alpha_\omega$. So (1) follows. Similarly, (2) holds.

(3) and (4) follow from the construction. (5) follows from (3) and (4).

By Richardson’s theorem, there exists

$$r = r_1 + r_2' + r_d$$

with $(r_1, r_2', r_d) \in n_\epsilon \oplus n'_\alpha \oplus \delta_U$ such that $[p_U^+, r] = n_U$. By Lemma 2.10 (1) and (2), we may assume that $r_1$ is the Richardson element for $\epsilon_U$. Again by Lemma 2.10, we can identify

$$(\alpha_U + \epsilon_U) = p_U^+ / \delta_U$$

and $n'_\alpha + n_\epsilon = n_U / \delta_U$.

So we have a well-defined action $p_U^+$ on $n'_\alpha \oplus n_\epsilon$ and

$$n'_\alpha \oplus n_\epsilon = [p_U^+, r_1 + r_2'] = [\alpha_U + \epsilon_U, r_1 + r_2']$$

Let $c_{r_2} = \{x \in \epsilon_U | x|_1 = y_1 | 1 \text{ for some } y \in \text{stab}_{\epsilon_U}(r_2') \}$.

**Lemma 2.11.** We have $[c_{r_2}, r_1] = n_\epsilon$.

Proof. Let $x \in n_\epsilon$. There exists $y \in \epsilon_U + \epsilon_U$ such that $[y, r_1 + r_2'] = x$. We write

$$y = y'' + y_1 + y'$$

where

$$y'' + y_1 \in \alpha_U \text{ and } y_1 + y' \in \epsilon_U.$$ Then similar to equation (b) in the proof of Lemma 2.8,

$$[y'', r_1] = 0, \quad [y', r_2'] = 0$$

and so

$$[y, r_2'] = [y'' + y_1, r_2'] \in n'_\alpha \text{ and } [y, r_1] = [y_1 + y', r_1] \in n_\epsilon.$$

Since $n'_\alpha \cap n_\epsilon = 0$ and $[y, r_2' + r_1] = x \in n_\epsilon$, we have

$$[y, r_2'] = [y'' + y_1, r_2'] = 0 \text{ and } [y, r_1] = [y_1 + y', r_1] = x.$$ It follows that $y_1 + y' \in \epsilon_{r_2}$ and so $[c_{r_2}, r_1] = n_\epsilon$. □

Recall that $r_2$ is a Richardson element for $\epsilon_{S,T}$. We have the following key observation, which gives a sufficient condition for the existence of Richardson elements.

**Lemma 2.12.** If $\text{stab}_{\epsilon_{U}}(r_2')|_1 = \text{stab}_{\epsilon_{S,T}}(r_2)|_1$, then $q_{S,T}$ has a Richardson element.

Proof. Assume $\text{stab}_{\epsilon_{U}}(r_2')|_1 = \text{stab}_{\epsilon_{S,T}}(r_2)|_1$. Then $c_{r_2} = c_{r_2}$ and so $[c_{r_2}, r_1] = n_\epsilon$ by Lemma 2.11. Then $q_{S,T}$ has a Richardson element, by Lemma 2.8. □

Verifying the condition in Lemma 2.12 is a key step in the proof of the existence of Richardson elements in the seaweed $q_{S,T}$. We will make use of the categorical construction of Richardson elements [8], using quiver representation theory. That is, we will analyse the properties of local endomorphisms (that is, restrictions of endomorphisms to a vertex) of rigid modules constructed in [8]. So we recall the construction in the next section.
3. Rigid $D$-modules and Richardson elements in type $A$

In this section, we first recall a quasi-hereditary algebra $D$, which is the path algebras of a double quiver with relations of a quiver $Q$ of type $A$, and the construction of rigid good $D$-modules [8]. We then explain in examples how to construct Richardson elements for the corresponding seaweed Lie algebras from rigid modules. We remark that the quiver $Q$ can be constructed from a given seaweed and the relations defining the algebra $D$ can be read off from the seaweed as well [8].

3.1. The path algebra $D$ of a double quiver with relations

Let $Q$ be a quiver of type $A_m$ with vertices $Q_0 = \{1, \ldots, m\}$ and arrows $Q_1 = \{\alpha_i \mid i \to i + 1 \text{ or } i \leftarrow i + 1 \text{ for } i = 1, \ldots, m - 1\}$.

Let $A = kQ$, the path algebra of $Q$. We denote the projective indecomposable $A$-module associated to vertex $i$ by $P_i$. Let

$$P(d) = \bigoplus_{i=1}^{m} P_i^{d_i}$$

for any $d \in \mathbb{Z}_{\geq 0}^m$. Note that $\text{End}_A P(d)$ is a seaweed in a Lie algebra of type $A$, and a Richardson element in $\text{radEnd}_A P(d)$ can be constructed from a certain representation $X(d)$ (to be made precise in Subsections 3.2 and 3.3) of a double quiver of $Q$ with relations [7, 8]. We recall the double quiver with relations from [3, 6] and some related basic definitions.

Let $\tilde{Q}$ be the double quiver of $Q$, that is, $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = Q_1 \cup Q_1^*$ with

$$Q_1^* = \{\alpha^*: i \to j \mid \alpha: j \to i \in Q_1\}.$$

Let $\mathcal{I}$ be the ideal of $k\tilde{Q}$ generated by

$$\alpha^*\alpha - \sum_{\beta \in Q_1, t(\beta) = s(\alpha)} \beta^*\beta$$

for any arrow $\alpha \in Q_1$, where $s(\alpha)$ is the starting vertex of $\alpha$ and $t(\beta)$ is the terminating vertex of $\beta$, and

$$\alpha^*\beta$$

for pairs of arrows $\alpha \neq \beta$ in $Q_1$ terminating at the same vertex. Let

$$D = k\tilde{Q}/\mathcal{I}.$$ 

Any $D$-module is an $A$-module via the inclusion $A \subseteq D$ and any $A$-module is a $D$-module via the surjection $D \to A$ mapping all arrows in $Q_1^*$ to zero. We use the notation $A_X$ to indicate the $A$-module structure of a $D$-module $X$ and note that

$$\text{Hom}_A(M, N) = \text{Hom}_D(M, N)$$

for two $A$-modules $M$ and $N$.

The algebra $D$ is quasi-hereditary with Verma modules $\Delta(i) = P_i$ for all $i$ (see [4]). The modules filtered by the Verma modules are called good modules. So for any good $D$-module $M$, we have

$$A_M \cong P(d)$$

as $A$-modules for some $d \in \mathbb{Z}_{\geq 0}^m$, where $d = (d_i)$ with $d_i$ the multiplicity of $P_i$ in $A_M$. We call $d$ the $\Delta$-dimension vector of $M$ and denote it by $\dim_{\Delta} M$. Let

$$\text{supp}_{\Delta}(M) = \{i \mid (\dim_{\Delta} M)_i \neq 0\}$$
be the $\Delta$-support of $M$. This definition is similar to the support of a module, which is defined using the usual dimension vector.

We identify modules with the corresponding quiver representations. So a $D$-module $M$ is a collection of vector spaces $M_i, i \in Q_0$ and linear maps $M_\beta, \beta \in Q_1 \cup Q_1^*$, satisfying the relations $I$, and a homomorphism $f: M \to N$ of $D$-modules is a collection of linear maps $(f_i)_{i \in Q_0}$ commuting with the module structure on $M$ and $N$.

Note that a $D$-module $M$ is rigid if it has no self-extensions, that is,

$$\text{Ext}_D^1(M, M) = 0.$$ 

In the remainder of this section, we briefly recall the construction of rigid $D$-modules and their corresponding Richardson elements [8].

3.2. Construction of rigid $D$-modules: the linear case [4]

Let $Q$ be a linear quiver with $m$ the unique sink vertex. Then $I$ is generated by commutative relations at $2, \ldots, m - 1$, and a zero relation at $1$. In this case, the indecomposable projective $D$-module $R_m$, at vertex $m$, is injective. A submodule $X$ of $R_m$ is uniquely determined by its $A$-structure $AX \cong \oplus_{i=1}^m P_i^d$ with $d_i \in \{0, 1\}$. Thus, there is a natural bijection between subsets $I \subseteq Q_0$ and submodules of $R_m$. More precisely, under this bijection a subset $I$ corresponds to the unique submodule $X(I) \subseteq R_m$ with $\Delta$-support $I$. For any vector $d \in \mathbb{Z}_{\geq 0}^m$, define

$$X(d) = \sum_{i=1}^t X(I_i),$$

with $\dim \Delta X(d) = d$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$. Then $X(d)$ is a rigid $D$-module. We give an example to illustrate the construction. See [4] for more details.

**Example 3.1.** Let $m = 3$ and $d = (2, 1, 2)$. The algebra $D$ is given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,$$

with the ideal $I$ generated by $\alpha_1^* \alpha_1$ and $\alpha_1 \alpha_2^* - \alpha_2^* \alpha_2$. The projective-injective $D$-module $R_3$ has the following seven nonzero submodules with the first one $R_3$,

$$X(d) = X(\{1, 2, 3\}) \oplus X(\{1, 3\}).$$
3.3. Construction of rigid $D$-modules: the general case

Now suppose that $Q$ has an arbitrary orientation. Recall that a vertex is admissible if it is a source or a sink. Let

$$i_1 < i_2 < \cdots < i_{t-1} < i_t$$

be the complete list of interior admissible vertices in $Q$ and let $i_0 = 1$ and $i_{t+1} = m$. Each interval $[i_s, \ldots, i_{s+1}]$ has a unique sink and a unique source. Similar to the linear case, each subset $I \subseteq \{i_s, \ldots, i_{s+1}\}$ determines a unique (up to isomorphism) indecomposable rigid good $D$-module, which has $\Delta$-support $I$.

Two indecomposable rigid good $D$-modules $M$ and $N$ with

$$\text{supp}_\Delta(M) \cap \text{supp}_\Delta(N) = \{i_j\},$$

$\text{supp}_\Delta(M) \subseteq \{i|i \leq i_j\}$ and $\text{supp}_\Delta(N) \subseteq \{i|i \geq i_j\}$, can be glued by identifying $P_{ij}$ to obtain a new indecomposable rigid good $D$-module.

**Definition 3.2** [7]. Let $u$ be a vertex with $i_v < u \leq i_{v+1}$ for some $v \in \{0, 1, \ldots, t\}$. Suppose that two indecomposable rigid $D$-modules $M$ and $N$, glued from modules $X(I_s)$ and modules $X(J_s)$, respectively, are supported (but not necessarily $\Delta$-supported) at $u$. We define $M \leq u N$ if for any $s$ with both $I_s$ and $J_s$ non-empty, $I_s \subseteq J_s$ when $s - v$ is even and $I_s \supseteq J_s$ when $s - v$ is odd.

**Remark 3.3.** The order $\leq_u$ depends on the base interval, that is, the $v$th interval $[i_v, i_{v+1}]$. When $u = i_{v+1}$ is admissible, $u$ is contained in both the $v$th and the $(v+1)$th intervals. We can also define $\leq_u$ based on the $(v+1)$th interval and obtain an order that is opposite to the one defined in Definition 3.2. This explains for instance in Example 3.4, why the summands $M^1$ versus $M^2$ and $N^1$ versus $N^2$ are ordered the way they are.

Now using the order $\leq_u$, we can construct rigid good $D$-modules as follows. Let $M$ and $N$ be two good rigid $D$-modules satisfying

$$(\dim_\Delta M)_i = 0 \text{ for } i > i_j, (\dim_\Delta N)_i = 0 \text{ for } i < i_j, \text{ and } (\dim_\Delta N)_{i_j} = (\dim_\Delta M)_{i_j}.$$ 

With respect to $\leq_{i_j}$, we glue the $i$th biggest summand of $M$ to the $i$th biggest summand of $N$ and then take the direct sum of all the glued modules. In this way, we obtain a rigid good $D$-module $X(d)$ for any $\Delta$-dimension vector $d$. We illustrate the construction by an example. See [7, 8] for more details.

**Example 3.4.** Let $Q$ be the quiver

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5.$$ 

Let $d = (2, 1, 2, 1, 2)$. Let $d^1 = (2, 1, 2, 0, 0)$ and $d^2 = (0, 0, 2, 1, 2)$. Then $X(d^1) = M^1 \oplus M^2$ and $X(d^2) = N^1 \oplus N^2$ with $M^1 = X(\{1, 3\}), M^2 = X(\{1, 2, 3\}), N^1 = X(\{3, 4, 5\})$ and $N^2 = X(\{3, 5\})$ as follows.
We have $M^1 \leq M^2$ and $N^1 \leq N^2$. So $X(d)$ is the direct sum of the gluings of $M^1$, $M^2$ with $N^1$ and $N^2$, respectively, that is,

![Diagram]

By the construction of rigid good modules, we have the following lemma.

**Lemma 3.5 [7, 8].** The indecomposable summands of a rigid good $D$-module $X(d)$ that are supported at a vertex $u$ are totally ordered by $\leq_u$.

### 3.4. Construction of Richardson elements

Observe that a standard parabolic subalgebra in $\mathfrak{gl}_n$ can be naturally identified with the endomorphism algebra of a projective representation of a linear quiver. For instance, $p^\downarrow_U \leq \mathfrak{gl}_5$ with $U = \Pi \setminus \{\alpha_2\}$ can viewed as $\text{End}(P^2_1 \oplus P^2_2)$ for the projective representation $P^2_1 \oplus P^3_2$ of the quiver

$$1 \longrightarrow 2,$$

where the number of vertices is the number of Levi-blocks of $p^\downarrow_U$ and the multiplicities 2 and 3 of $P_1$ and $P_2$ are the sizes of the Levi-blocks. This identification is due to the following,

$$\text{Hom}(P_1, P_1) = \text{Hom}(P_2, P_2) = \text{Hom}(P_2, P_1) = k \text{ and } \text{Hom}(P_1, P_2) = 0.$$  

Similarly, a seaweed Lie algebra $q_{S,T}$ can be viewed as an endomorphism algebra of a projective module of a quiver of type $A$. In both cases, the nilpotent radical can then be identified with the Jacobson radical of the endomorphism algebra.

**Example 3.6.** Consider $q_{S,T} \leq \mathfrak{gl}_8 = \mathfrak{gl}(V)$ with $S = \{\alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ and $T = \Pi \setminus \{\alpha_5, \alpha_6\}$.

$$q_{S,T} = \begin{pmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{pmatrix}.$$  

Let $V = \oplus_i V_i$ with spaces $V_i$ determined by the Levi-blocks and of dimension 2, 1, 2, 1, 2, respectively. We have the following embeddings:

$$V_1 \oplus V_2 \oplus V_3 \leftarrow V_2 \oplus V_3 \leftarrow V_3 \oplus V_4 \leftarrow V_3 \oplus V_4 \oplus V_5.$$  

This is the projective representation $P = P^2_1 \oplus P_2 \oplus P^2_3 \oplus P_4 \oplus P^2_5$ of the quiver $Q$

$$1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5.$$  

Denote by $W_i$ the subspace at step $i$ (starting from the left) in the embedding sequence. Restricting an element in the nilpotent radical $n_{S,T}$ to the subspaces $W_i$ produces a map of the
neighbouring spaces (in the direction opposite to the inclusions) and the relations of generic maps are exactly the defining relations of $D$ associated to $Q$. Let $Q_{S,T}$ be the Lie group associated to $q_{S,T}$, $c_i = \dim W_i$ and $c = (c_i)$. Let $\text{Rep}(D,P)$ be the variety of good $D$-modules with dimension vector $c$,

$$\{ M \in \Pi_{i \to j \in Q_{1}} \text{Hom}(k^{c_i},k^{c_j}) \mid M \text{ satisfies the relations } \mathcal{I} \text{ and } M|_Q = P \},$$

where $\mathcal{I}$ is as defined in Subsection 3.1. We have

1. $n_{S,T} = \text{radEnd}P$ and $Q_{S,T} = \text{Aut}P$, the automorphism group of $P$;
2. the three adjoint actions $Q_{S,T}$ on $n_{S,T}$, $\text{Aut}P$ on $\text{radEnd}P$ and $\text{Aut}P$ on $\text{Rep}(D,P)$ are essentially the same. In particular, open orbits correspond to open orbits;
3. by Voigt's Lemma [19], a rigid $D$-module in $\text{Rep}(D,P)$ yields an open $\text{Aut}P$-orbit in $\text{Rep}(D,P)$. This then implies the existence of Richardson elements in $n_{S,T}$.

We now describe how to construct a Richardson element $r(d)$ for $q_{S,T} \subseteq \mathfrak{gl}_n$ from the rigid module $X(d)$. Note that $X(d)$ is constructed based on data contained in $q_{S,T}$. Let

$$X(d) = \bigoplus_i X^i$$

be a decomposition of $X(d)$ into indecomposable summands and let $n = \sum_i d_i$. For each summand $X^i$ that is $\Delta$-supported at $j$, choose an integer $x_{ij}$, where

$$\sum_{l < j} d_l < x_{ij} \leq \sum_{l \leq j} d_l$$

such that $x_{ij} \neq x_{lj}$ for two different summands $X^i$ and $X^l$. If $X^i$ is $\Delta$-supported at both $s$ and $t$ with $s < t$, but not at $s+1, \ldots, t-1$, then the matrix $r(d)$ has a 1 at either $(x_{is}, x_{it})$ or $(x_{it}, x_{is})$, depending on which root space belongs to $q_{S,T} = \text{End}_A(P(d))$. All other entries in $r(d)$ are equal to 0. Note that

$$\text{End}_D(X(d)) \cong \text{stab}_{q_{S,T}}(r(d)).$$

The matrix $r(d)$ is also the adjacency matrix of an oriented graph with components corresponding to indecomposable summands of $X(d)$. See the example below for an illustration and [1, 4, 8] for more detail.

**Example 3.7.** The rigid modules $X(d)$ in Example 3.1 and 3.4 correspond, respectively, to the parabolic subalgebra of $\mathfrak{gl}_5$ with Richardson element $r_1$ and the seaweed Lie algebra in $\mathfrak{gl}_8$, which is exactly the seaweed given in Example 3.6, with Richardson element $r_2$ as follows

$$r_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
4. Stabilisers of Richardson elements in type \( A \)

Consider the quiver \( Q \) of type \( A_m \) of arbitrary orientation and the algebras \( A \) and \( D \) as in Section 3. For \( D \)-modules \( M \) and \( N \), let

\[
\text{Hom}_D(N,M) = \{ f_i \mid f \in \text{Hom}_D(N,M) \}
\]

be the space of homomorphisms from \( N \) to \( M \) restricted to vertex \( i \). Let \( \text{End}_D(M)_i \) and \( \text{Aut}_D(M)_i \) be defined similarly. We study the structure of the endomorphism algebra of a good rigid \( D \)-module and its restriction to a vertex.

Let

\[
X(d) = \bigoplus_i (X^i)^{n_i}
\]

be a good rigid \( D \)-module with \( A X(d) = P(d) \). We order the summands such that \( X^i <_{m} X^{i+1} \) for all indecomposable summands \( X^i \).

4.1. Restriction to \( m \), when \( m \) is a source

Let \( V = X(d)_m \), the vector space of \( X(d) \) at vertex \( m \). Then

\[
\text{End}_D(X(d))_m \subseteq \mathfrak{gl}(V).
\]

**Lemma 4.1.** The subalgebra \( \text{End}_D(X(d))_m \subseteq \mathfrak{gl}(V) \) is parabolic.

**Proof.** By the construction of \( X(d) \), \( \text{Hom}_D(X^i,X^j)_m \) is one dimensional for \( i \leq j \) and zero otherwise. Furthermore, we may choose basis elements \( r_{ji} \in \text{Hom}_D(X^i,X^j)_m \) for all \( i \) and \( j \), such that \( r_{ji}r_{kl} = r_{lj} \) if \( i = k \) and zero otherwise. The lemma follows. \( \square \)

4.2. Restriction to \( m \), when \( m \) is a sink

Recall that \( \alpha_{m-1} : m-1 \to m \) is the arrow that ends at \( m \). For \( D \)-modules \( M \) and \( N \), let

\[
\text{Hom}_D(M,N)_m^0 = \{ f_m : M_m/\alpha_{m-1}(M_{m-1}) \to N_m/\alpha_{m-1}(N_{m-1}) \mid f \in \text{Hom}_D(M,N) \}.
\]

Let

\[
V = X(d)_m/\alpha_{m-1}(X(d)_{m-1}).
\]

Then \( \text{End}_D(X(d))_m^0 \subseteq \mathfrak{gl}(V) \).

**Lemma 4.2.** The subalgebra \( \text{End}_D(X(d))_m^0 \subseteq \mathfrak{gl}(V) \) is parabolic.

**Proof.** The proof is similar to the proof of Lemma 4.1. \( \square \)

We remark that \( \mathfrak{gl}(V) \) is in fact isomorphic to \( \text{End}_A(P_{m}^{d_m}) \).

4.3. Stabilisers of indecomposable rigid modules

There is an obvious embedding of endomorphism rings

\[
\prod_i \text{End}_D(X^i)^{n_i} \subseteq \text{End}_D(X(d)).
\]

As before, let \( 1 = i_0 < i_1 < \cdots < i_{t+1} = m \) be the admissible vertices of \( Q \).

Let \( v \) be a sink in \( Q \) and let \( \alpha \) be an arrow ending at \( v \). Then \( \alpha \alpha^* \) induces a nilpotent endomorphism \( x \) of \( X^i \). We have such an endomorphism \( x_s : X^i \to X^i \) for each interval \([i_{s-1}, i_s] \) with \( s = 1, \ldots, t+1 \), where \( x_s \) can be zero, depending on the intersection of \( \text{supp}_D(X^i) \) and
Note that $x_s$ is zero on vertices that are not in the interval $[i_{s-1}, i_s]$. Let $m_s \geq 1$ be the smallest integer such that $x_s^{m_s} = 0$. In fact,

$$m_s = \max\{1, \supp(X^i) \cap [i_{s-1}, i_s]\}.$$ 

**Lemma 4.3.** The map $y_s \mapsto x_s$ induces an isomorphism

$$k[y_1, \ldots, y_{t+1}] / (y_s y_l = 0 \text{ for } s \neq l, y_s^{m_s} = 0) \cong \text{End}_D(X^i).$$

**Proof.** Let $x_s$ and $x_l$ be two arbitrary endomorphisms, induced by $\alpha^*$ and $\beta^*$, respectively, where $\alpha$ and $\beta$ are two different arrows. Clearly, $x_s x_l = 0$ if $\alpha$ and $\beta$ end at different sinks.

Otherwise, $x_s x_l = 0$ follows from the relation $\alpha^* \beta = 0$. By definition, $x_s^{m_s} = 0$. So the map is well defined.

By the construction of $X^i$, $\text{End}_D(X^i)$ is generated by the $x_s$ and thus the map is surjective. The intersection of the images of $x_s$ and $x_l$ is zero if $s \neq l$, and so the injectivity follows. □

By the embedding,

$$\text{End}_D(X(d)) \subseteq \text{End}_A(P(d)) \subseteq \mathfrak{gl}_n$$

and the construction of Richardson elements discussed in Subsection 3.4, each element of $\prod_i \text{End}_D(X^i)^{n_i}$ can be explicitly described in terms of matrices. They can also be described in terms of the oriented graphs constructed from $X^i$.

**Example 4.4.** We use the two Richardson elements from Example 3.7. In both cases, there are two indecomposable summands $X^1$ and $X^2$ in $X(d)$, and we use black entries to describe $\text{End}_D(X^1)$ and blue entries to describe $\text{End}_D(X^2)$ as follows,

$$\text{End}_D(X^1) \oplus \text{End}_D(X^2): \begin{pmatrix} a & 0 & b & c & 0 \\ 0 & d & 0 & e & 0 \\ a & b & 0 & 0 & d \\ a & 0 & 0 & e & 0 \\ 0 & d & 0 & 0 & e \end{pmatrix}$$

and

$$\text{End}_D(X^1) \oplus \text{End}_D(X^2): \begin{pmatrix} a & 0 & 0 & e & 0 \\ 0 & e & 0 & 0 & h \\ b & 0 & 0 & a & 0 \\ 0 & d & 0 & 0 & e \\ g & f & 0 & e & 0 \\ a & c & 0 & 0 & e \\ a & 0 & 0 & e & 0 \end{pmatrix}.$$

To illustrate, for instance in the first matrix, the $a$-entries are $a \cdot 1_{X^1}$, the $b$-entries are $b \cdot x^1_1$ with $x^1_1 : X^1 \to X^1$, the $c$-entries are $c \cdot (x^1_1)^2$ and the $e$-entry is $e \cdot x^1_2$ with $x^1_2 : X^2 \to X^2$.

The non-zero off-diagonal entries in the matrices correspond to non-trivial paths in the oriented graphs as follows.

$$P_1 \xleftarrow{c} b \xrightarrow{c} P_2 \xrightarrow{b} P_3$$

$$P_1 \xrightarrow{d} b \xleftarrow{c} P_3 \xrightarrow{c} P_4 \xleftarrow{c} P_3$$

$$P_1 \xleftarrow{e} P_3$$

$$P_1 \xrightarrow{f} b \xrightarrow{f} P_2 \xrightarrow{h} P_3 \xrightarrow{h} P_5.$$
5. The main result

Theorem 5.1. Let \( g \) be a simple Lie algebra of type \( \mathbb{B}, \mathbb{C} \) or \( \mathbb{D} \). Then any seaweed in \( g \) has a Richardson element.

We continue to use the notation from Section 2. The proof of the theorem is split into two cases. The first case (see Theorem 5.2) deals with type \( \mathbb{B}, \mathbb{C} \) and \( \mathbb{D} \), under the assumptions \((i)\) and \((ii)\) in Subsection 2.3 for type \( \mathbb{D} \). We will use quiver representations and results from Section 4 to verify that the sufficient condition in Lemma 2.12 holds, and so Richardson elements exist. The second case deals with the special situation, where \( g \) is of type \( \mathbb{D} \) and \((\epsilon, \omega) = (2, 1)\). Unlike the first case, there can be root spaces that are not contained in \( \mathfrak{a}_{S,T} + \mathfrak{c}_{S,T} \) (cf Lemma 2.7). That is, \( q_{S,T} \) is not necessarily equal to \( \mathfrak{a}_{S,T} + \mathfrak{c}_{S,T} \).

Theorem 5.2. Let \( q_{S,T} \subseteq g \) be a seaweed, where \( g \) is of type \( \mathbb{B} \) or \( \mathbb{C} \), or of type \( \mathbb{D} \) with \((\epsilon, \omega) \neq (2, 1)\). Then \( q_{S,T} \) has a Richardson element.

Proof. Note that we only need to consider the situation, where \( \epsilon > \omega \) and neither \( S \) nor \( T \) is equal to \( \Pi \) or \( \emptyset \). We also assume \((ii)\) in Subsection 2.3.

Let \( P = P(d) \) be a projective \( A \)-module such that

\[
\text{End}_A(P(d)) \cong \mathfrak{a}_{S,T},
\]

where \( P(d) \) is a projective representation of a quiver \( Q \) of type \( A_m \) and the labelling of the vertices and arrows of \( Q \) is as in Subsection 3.1. Let \( r \in \mathfrak{n}_A \) be a Richardson element constructed from the good rigid \( D \)-module \( X(d) \) as in Subsection 3.4. Fix an embedding

\[
\text{End}_A(P(d)) \subseteq g
\]

such that

\[
\text{End}_A(P(d)) = \mathfrak{a}_{S,T} \quad \text{and} \quad \text{End}_A(P(d))_m = \mathfrak{l},
\]

where \( \mathfrak{l} = \mathfrak{a}_{S,T} \cap \mathfrak{c}_{S,T} \) is as in Subsection 2.2. By the condition \( \epsilon > \omega \) on \( \mathfrak{a}_{S,T} \), the vertex \( m \) is a source. So

\[
\text{stab}_{\mathfrak{a}_{S,T}}(r)_\mathfrak{l} = \text{End}_D(X(d))_m
\]

is a parabolic in \( \mathfrak{l} \), by Lemma 4.1.

We order the summands \( X(d) \) from big to small with respect to the order \( \leq_m \), so that \( \text{End}_D(X(d))_m \) is standard upper triangular. Suppose that the sizes of the blocks in the Levi-subalgebra of \( \text{End}_D(X(d))_m \) are \( c_1, \ldots, c_l \) and let

\[
\hat{c} = \left( c_l, c_{l-1} + c_l, \ldots, \sum_{j \geq i} c_j, \ldots, \sum_{j \geq 1} c_j \right).
\]

Let \( B \) be the path algebra of the linearly oriented quiver \( A_l \) with the unique sink \( l \) and let

\[
P(\hat{c}) = P^{\hat{c}}_1 \oplus \cdots \oplus P^{\hat{c}}_l,
\]

a projective representation of this linear quiver. Denote by \( F \) the algebra of the associated double quiver with relations, defined in the same way as the algebra \( D \) in Subsection 3.1, and let \( X(\hat{c}) \) be the rigid good \( F \)-module.

Note that

\[
\text{End}_B(P(\hat{c})) = \text{End}_F(P(\hat{c}))
\]

and by abuse of notation we let

\[
\text{End}_B(P(\hat{c}))_l^0 = \text{End}_F(P(\hat{c}))_l^0,
\]
which is in fact isomorphic to $\text{End}_B(P^\circ i)$. Choose $g'$ and $U$ (see Subsection 2.3), and an embedding

$$\text{End}_B(P(\hat{c})) \subseteq g'$$

such that

$$\text{End}_B(P(\hat{c})) = a_U \text{ and } \text{End}_B(P(\hat{c}))^0 = I.$$

Let $r' \in a_U$ be a Richardson element corresponding to $X(\hat{c})$, where the summands of $X(\hat{c})$ are ordered from small to big with respect to the order $\leq l$, so that

$$\text{End}_F(X(\hat{c}))^0 = \text{stab}_{a_U}(r'|l)$$

is standard upper triangular. Both $\text{End}_F(X(\hat{c})) \subseteq l$ and $\text{End}_D(X(d))_m \subseteq l$ are standard upper triangular with Levi blocks of equal sizes, and so

$$\text{stab}_{a,S,T}(r)|l = \text{stab}_{a_U}(r'|l).$$

Then $q_{S,T}$ has a Richardson element by Lemma 2.12.

The following example illustrates the construction of $X(\hat{c})$ in the proof above.

**Example 5.3.** Let $Q$ be the quiver

$$1 \to 2 \to 3 \to 4$$

where $m = 4$ is a source. Let $d = (3, 1, 3, 4)$. Then the rigid module $X(d)$ is

where the summands are ordered from big to small with respect to $\leq 4$. In the base interval $[3, 4]$, the last summand has the smallest $\Delta$-support $\{4\}$ and so is the smallest one, the other summands have the same $\Delta$-support $\{3, 4\}$ and so they are compared at next interval $[1, 3]$, in which case representations with smaller supports are actually bigger with respect to $\leq 4$. We have

(1) $\text{End}_D(X(d))_4 = \left(\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{array}\right)$, a standard upper triangular parabolic in $\mathfrak{gl}_4$;

(2) the number of blocks is $l = 3$ with the sizes 2, 1, 1 and $\hat{c} = (1, 2, 4)$;

(3) the representation $P(\hat{c}) = P_1 \oplus P_2 \oplus P_3$ is the projective representation of the quiver $1 \to 2 \to 3$;

(4) the rigid $F$-module $X(\hat{c})$ is as follows
The summands are ordered from small to big with respect to $\leq_3$. The space $\text{End}(X(\hat{c}))_{\geq 3}$ is the space of induced homomorphisms between the top 3 in the summands and indeed

$$\text{End}_F(X(\hat{c}))_{\geq 3} = \text{End}_D(X(d))_4.$$ 

For the proof of the following theorem, we denote by $E_{ij}$ the elementary matrix with 1 at $(i,j)$-entry and 0 elsewhere.

**Theorem 5.4.** If $\mathfrak{g}$ has type $D$ and $(\epsilon, \omega) = (2,1)$, then the seaweed $q_{S,T} \subseteq \mathfrak{g}$ has a Richardson element.

**Proof.** Let $W = S \setminus \{1\}$ and

$$q_{S,T}^1 = \bigoplus_{\alpha \text{ supported at } \alpha_1, \alpha \in \Phi^+_S} g_{\alpha}.$$ 

Then

$$q_{S,T} = q_{W,T} \oplus q_{S,T}^1 \text{ and } n_{S,T} = n_{W,T} \oplus q_{S,T}^1.$$ 

Note that $q_{W,T}$ is a seaweed of type $A$ and so it has a Richardson element $r \in n_{W,T}$.

Since $\alpha_1 \not\in T$, we have $[q_{S,T}, q_{S,T}^1] \subseteq q_{S,T}^1$. Then $r + r'$ with $r' \in q_{S,T}^1$ is a Richardson element in $q_{S,T}$ if $[\text{stab}_{q_{W,T}}(r), r'] = q_{S,T}^1$. Indeed, let $x + x' \in n_{W,T} \oplus q_{S,T}^1$ and let $y \in q_{W,T}$ and $y' \in \text{stab}_{q_{W,T}}(r)$ such that

$$[y, r] = x \text{ and } [y', r'] = x' - [y, r'].$$

Then

$$[y + y', r + r'] = x + x'$$

and so $r + r'$ is a Richardson element. Hence, to prove the theorem, it suffices to show

$$[\text{stab}_{q_{W,T}}(r), r'] = q_{S,T}^1.$$ 

We choose the representation of $\mathfrak{g}$ given by $2n \times 2n$-matrices anti-symmetric to the anti-diagonal,

$$g_{\alpha_1} = k \cdot (E_{n-1,n+1} - E_{n,n+2}) \text{ and } g_{\alpha_2} = k \cdot (E_{n-1,n} - E_{n+1,n+3}).$$

Then $q_{S,T}^1$ has a basis $E_{l,n+1} - E_{n,n-l+1}$ for $l = n - a + 2, \ldots, n - 1$, where

$$a = \begin{cases} 
\min\{p | p > \epsilon, \alpha_p \not\in S\}, & \text{if such a } p \text{ exists}, \\
 n + 1, & \text{otherwise}.
\end{cases}$$

Fix an embedding $\mathfrak{gl}_n \subseteq \mathfrak{g}$ induced by

$$E_{ij} \mapsto E_{ij} - E_{2n-j+1,2n-i+1}.$$ 

We may assume the Richardson element $r$ is $r(d)$, constructed from a rigid good module $X(d) = \oplus_i X^i$. Recall that the underlying quiver $Q$ is of type $A_m$, where $m$ is the number of Levi-blocks in $q_{W,T}$.

Choose an embedding

$$\text{End}_A(P(d)) \subseteq \mathfrak{g}$$

such that

$$\text{End}_A(P(d)) = q_{W,T} \text{ and } \text{End}_D(X(d)) = \text{stab}_{q_{W,T}}(r).$$
Then

\[ \text{End}_D(X^i) \subseteq \text{stab}_{q_{W,T}}(r), \]

with

\[ 1_{X^i} = \sum_{j \in \text{Supp}_\Delta(X^i)} (E_{x_{ij},x_{ij}} - E_{2n-x_{ij}+1,2n-x_{ij}+1}), \]

where the \( x_{ij} \) are constructed from the summand \( X^i \) as in Subsection 3.4. As \((\epsilon, \omega) = (2, 1)\), the vertex \( m \) is a source and the \( m \)th Levi-block in \( q_{W,T} \) is of rank 1, that is, \( d_m = 1 \). Furthermore, the \( m \)th Levi-block is spanned by \( E_{n,n} - E_{n+1,n+1} \).

Observe first that for any \( x \in g_\alpha \subseteq q_{S,T}^1 \),

\[ [E_{n,n} - E_{n+1,n+1}, x] = x \]

and when \( i < n \),

\[ [E_{i,i} - E_{2n-i+1,2n-i+1}, x] = \begin{cases} x, & \text{if } \alpha = \alpha_1 + \cdots + \alpha_{n-i+1}; \\ 0, & \text{otherwise}. \end{cases} \]

Note also that there is a unique summand of \( X(d) \), say \( X_{i_0} \), such that \( A X_{i_0} \) contains \( P_m \) as a summand and \( X_{i_0} \neq P_m \), that is, \( A X_{i_0} \) has at least two summands and so

\[ 1_{X_{i_0}} - (E_{n,n} - E_{n+1,n+1}) \neq 0. \]

We will decompose \( q_{S,T}^1 \) as direct sum of subspaces \( V_i \), determined by the summands of \( X(d) \), and then construct a ‘Richardson element’ \( r_i \) in each subspace, in the sense that

\[ [\text{stab}_{q_{W,T}}(r), r_i] = V_i. \]

The orientation of the arrows at vertex \( m - 1 \) determines the construction of \( V_i \). There are two cases to be considered.

Case 1. The vertex \( m - 1 \) is non-admissible and so

\[ \dim q_{S,T}^1 = d_{m-1}. \]

Note that each summand \( P_{m-1} \) is contained in a different indecomposable summand of \( X(d) \) and we may assume

\[ x_{i(m-1)} = \left( \sum_{s \leq m-1} d_s \right) - i + 1. \]

For \( 1 \leq i \leq d_{m-1} \), let \( V_i \) be the unique root space \( g_\alpha \) such that

(i) when \( i \neq i_0 \), \([1_{X^i}, g_\alpha] \neq 0\);
(ii) when \( i = i_0 \), \([1_{X^i} - (E_{n,n} - E_{n+1,n+1}), g_\alpha] \neq 0\). In this case, \( x_{im} = n = \sum_s d_s \).

Then

\[ q_{S,T}^1 = \bigoplus_i V_i. \]

Let \( \delta_{ij} \) be Kronecker numbers. For any \( r_j \in V_j \), we have

\[ [1_{X^i}, r_j] = \begin{cases} \delta_{ij} r_j, & \text{if } i \neq i_0; \\ 2 r_{i_0}, & \text{if } i = j = i_0; \\ r_j, & \text{if } i = i_0 \text{ and } j \neq i_0. \end{cases} \]
Now choose a non-zero element \( r_i \in V_i \) and let
\[
    r' = \sum_i r_i.
\]

**Case 2.** The vertex \( m - 1 \) is admissible. Then it is a sink and
\[
    \dim q^1_{S,T} > d_{m-1}.
\]
Let
\[
    V_i = \begin{cases}
        q^1_{S,T} \cap \bigoplus \{ [X_i, g_\alpha] \neq 0 : \alpha \}, & \text{if } i \neq i_0; \\
        q^1_{S,T} \cap \bigoplus \{ [X_i - E_{n,n}, g_\alpha] = 0 : \alpha \}, & \text{if } i = i_0.
    \end{cases}
\]
Note that \( V_i \) can be 0 and we have
\[
    q^1_{S,T} = \bigoplus_i V_i.
\]
When \( V_i \neq 0 \), we let \( \beta_i \) be the smallest root such that
(i) when \( i \neq i_0 \), \( [X_i, g_\beta_i] \neq 0 \);
(ii) when \( i = i_0 \), \( [X_i - E_{n,n}, g_\beta_i] \neq 0 \). As in Case (1) (ii), \( x_{i,m} = n \).

Then for any non-zero \( r_i \in g_\beta_i \),
\[
    V_i = \sum_j [k \cdot x^j, r_i],
\]
where \( x \) is the endomorphism of \( X^i \) induced by \( \alpha \alpha^* \) with \( \alpha \) the arrow from vertex \( m - 2 \) to vertex \( m - 1 \) (see Lemma 4.3). By convention, \( x^0 \) denotes the identity endomorphism \( 1_{X^i} \).

Choose a non-zero element \( r_i \in g_\beta_i \) and let
\[
    r' = \sum_i r_i.
\]
In both cases,
\[
    [\text{stab}_{q_{W,T}}(r), r'] = \bigoplus_i V_i = q^1_{S,T}.
\]
Therefore \( q_{S,T} \) has a Richardson element.

**Remark 5.5.** (a) The Richardson elements and their stabilisers can be explicitly constructed using results from [8], the proofs of Theorem 5.2 and Theorem 5.4. The work of Baur [1] on parabolic subalgebras is also needed in the case of Theorem 5.2.

(b) The method of Theorem 5.4 could be generalised to Lie algebras of exceptional types and therefore provide an explanation to why Richardson elements do not exist for some seaweed Lie algebras of type \( E_8 \).

We end this paper with an example of constructing Richardson elements, using the method discussed in Theorem 5.4.

**Example 5.6.** Let \( g = sl_{10} \), a Lie algebra of type \( D_5 \).

1. Consider the seaweed Lie algebras \( q_{S,T} \) and \( q_{K,L} \) with \( T = \{ \alpha_2, \alpha_4, \alpha_5 \}, S = \{ \alpha_1, \alpha_3, \alpha_4 \}, \)
\( L = \{ \alpha_2, \alpha_3, \alpha_4 \} \) and \( K = \{ \alpha_1, \alpha_3, \alpha_5 \} \). These two seaweeds are like those discussed in the proof of Theorem 5.4 and have the following shapes, where the first one is \( q_{S,T} \) as in Case
(2) and the second one is \( q_{K,L} \) as in Case (1). The matrices are anti-symmetric to the anti-diagonal.

\[
\begin{pmatrix}
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{pmatrix},
\begin{pmatrix}
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast
\end{pmatrix}
\]

(2) The seaweeds \( q_{S\{1\},T} \) and \( q_{K\{1\},L} \) are of type \( A \). They are isomorphic to \( \text{End}_A(P(c)) \) and \( \text{End}_B(P(d)) \) of the following quivers, respectively, where \( c = (1, 2, 1, 1) \) and \( d = (1, 1, 2, 1) \),

\[
A: 1 \rightarrow 2 \rightarrow 3 \rightarrow 4,
B: 1 \rightarrow 2 \rightarrow 3 \rightarrow 4.
\]

In both cases, the vertex \( m \) in the proof of Theorem 5.4 is 4. The vertex 3 (= \( m - 1 \)) is a sink for \( q_{S\{1\},T} \) and is non-admissible for \( q_{K\{1\},L} \). We have

\[
q_{S,T}^1 = V_1 \oplus V_2 \quad \text{and} \quad q_{K,L}^1 = W_1 \oplus W_2
\]

with

\[
V_1 = g_{\alpha_1 + \alpha_3 + \alpha_4}, \quad V_2 = g_{\alpha_1} \oplus g_{\alpha_1 + \alpha_3}, \quad W_1 = g_{\alpha_1 + \alpha_3} \quad \text{and} \quad W_2 = g_{\alpha_1}.
\]

(3) The Richardson elements of \( q_{S\{1\},T} \) and \( q_{K\{1\},L} \) are as below. Entries of the same colour come from the same indecomposable summand. Note that for \( X(d) \), one of the indecomposable summands is a Verma module, so both the fourth column and row are zero.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

(4) The stabilisers of the two Richardson elements are as follows.

\[
\begin{pmatrix}
a & b & a & 0 & g & 0 \\
b & a & 0 & g & 0 & 0 \\
f & 0 & c & d & 0 & 0 \\
c & 0 & e & c & 0 & 0 \\
e & c & 0 & 0 & 0 & 0 \\
-\ast & -\ast & -\ast & -\ast & -\ast & -\ast \end{pmatrix},
\begin{pmatrix}
a & b & a & 0 & 0 & 0 \\
a & b & a & 0 & 0 & 0 \\
c & a & 0 & g & 0 & 0 \\
f & 0 & e & 0 & 0 & 0 \\
d & c & g & a & 0 & 0 \\
d & c & g & a & 0 & 0 \end{pmatrix}
\]

There are two indecomposable direct summands in each of \( X(c) \) and \( X(d) \). The different colours indicate homomorphisms between different pairs of summands.
(5) Denote the two Richardson element in (3) by \( r_{S \setminus \{1\},T} \) and \( r_{K \setminus \{1\},L} \). The action of \( \text{stab}_{q_{S \setminus \{1\},T}}(r_{S \setminus \{1\},T}) \) on \( q_{S,T}^1 \) is equivalent to the natural action of
\[
\begin{pmatrix}
a + c & 0 & g \\
0 & 2c & d \\
0 & 0 & 2c
\end{pmatrix}
\]
on \( k^3 \), although in the proof of Theorem 5.4, we only use the action of
\[
\begin{pmatrix}
a + c & 0 & 0 \\
0 & 2c & d \\
0 & 0 & 2c
\end{pmatrix}
\]
with
\[
r' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in q_{S,T}^1.
\]
The action of \( \text{stab}_{q_{K \setminus \{1\},L}}(r_{K \setminus \{1\},L}) \) on \( q_{K,L}^1 \) is equivalent to the natural action of \((a + e \ 0 \ 0 \ 2e \ 0))\) on \( k^2 \), and \( r' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in q_{K,L}^1 \). So we have the following Richardson elements for the two seaweeds \( q_{S,T} \) and \( q_{K,L} \), with the red entries coming from the contributions of \( r' \) in \( q_{S,T}^1 \) and \( q_{K,L}^1 \), respectively.
\[
\begin{pmatrix}
0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 \\
1 & 0 & -1 & 0 & -1 \\
0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

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Bernt Tore Jensen
Department of Mathematical Sciences
Norwegian University of Science and Technology
2802 Gjøvik
Norway
bernt.jensen@ntnu.no

Xiuping Su
Department of Mathematical Sciences
University of Bath
Bath BA2 7JY
United Kingdom
xs214@bath.ac.uk

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