Analytic properties of the electromagnetic Green’s function

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The electromagnetic Green’s function is expressed from the inverse Helmholtz operator, where a second frequency has been introduced as a new degree of freedom. The first frequency results from the frequency decomposition of the electromagnetic field while the second frequency is associated with the dispersion of the dielectric permittivity. Then, it is shown that the electromagnetic Green’s function is analytic with respect to these two complex frequencies as soon as they have positive imaginary part. Such analytic properties are also extended to complex wavevectors. Next, Kramers-Kronig expressions for the inverse Helmholtz operator and the electromagnetic Green’s function are derived. In addition, these Kramers-Kronig expressions are shown to correspond to the classical eigengenmodes expansion of the Green’s function established in simple situations. Finally, the second frequency introduced as a new degree of freedom is exploited to characterize non-dispersive systems.

I. INTRODUCTION

The electromagnetic Green’s function is a fundamental quantity in the analysis of systems described by macroscopic Maxwell’s equations. It is defined from the inverse of the Helmholtz operator, which provides the electric field radiated by a current source. For a medium described by the electric permittivity $\varepsilon(x, z)$ depending on the space variable $x \in \mathbb{R}^3$ and the frequency $z$, the Helmholtz operator $H(z)$ is given by

$$[H(z)E](x) = z^2 \varepsilon(x, z) \mu_0 E(x) - \partial_x \times \partial_z E(x),$$

where $\partial_x \times$ is the curl operator, and $\mu_0$ the vacuum permeability. Then, the electromagnetic Green’s function can be defined from the inverse of the Helmholtz operator by

$$[H(z)^{-1} S](x) = \int_{\mathbb{R}^3} dy G(x, y; z) \cdot S(y),$$

where $S(x)$ is proportional to a current source density. In this paper, analytic properties of the electromagnetic Green’s function $G(x, y; z)$ are rigorously deduced from those of the inverse Helmholtz operator.

It is well-known that the Green’s function $G(x, y; z)$ is an analytic function in the upper half space of complex frequencies $z$. This is a direct consequence of the causality principle and passivity. Notice that the frequency dependence of the electromagnetic Green’s function has two different origins in Maxwell’s equations: the first one is the consequence of the frequency decomposition of the time derivative of the fields in Maxwell’s equations, and the second one is the frequency dispersion which results in the frequency dependence of the permittivity $\varepsilon(x, z)$. Here, it is proposed to show that the analytic properties of $G(x, y; z)$ can be established for these two frequencies independently (section V). In particular, this is exploited in section VI to provide a rigorous proof of the analyticity and causality in the non-dispersive case. Also, in the cases of homogeneous or periodic geometry, the analytic properties are extended to complex wavevectors (section VII).

The analytic properties of the electromagnetic Green’s function can be used to compute Sommerfeld integrals and time-dependent electromagnetic fields, for instance defining analytic continuation in the plane of complex frequencies. Also, the Kramers-Kronig relations are based on such properties of analyticity. For instance, new Kramers-Kronig relations have been established in reference for the reflection and transmission coefficients in non-normal incidence. In this paper, it is proposed in section VII to derive Kramers-Kronig expressions for the inverse Helmholtz operator and the electromagnetic Green’s function. Arguments are provided to interpret these expressions as generalizations to dispersive and absorptive systems of the well-known eigenmodes expansion established for simple closed cavity (without dispersion and absorption).

II. GENERALIZED HELMHOLTZ OPERATOR

A. Maxwell’s equations

We start with Maxwell’s equations in dielectric media. Let $E(x, t)$, $H(x, t)$ and $P(x, t)$ be respectively the time-dependent electric, magnetic and polarization fields. Then, equations of macroscopic electromagnetics are

$$\varepsilon_0 \partial_t E(x, t) + \partial_t P(x, t) = \partial_x \times H(x, t) - J(x, t),$$

$$\mu_0 \partial_t H(x, t) = -\partial_x \times E(x, t),$$

where $\partial_t$ is the partial derivative with respect to time, and $J(x, t)$ is the current source density. In addition, the electric field is related to the polarization through the
constitutive equation
\[ P(x, t) = \int_{-\infty}^{t} ds \, \chi(x, t-s) E(x, s), \]  \tag{4}

where \( \chi(x, t) \) is the electric susceptibility. A Fourier decomposition with respect to the time of the equations above leads to
\[ -i\omega \varepsilon(x, \omega) E(x, \omega) = \partial_x \times H(x, \omega) - J(x, \omega), \]  \tag{5}
where the dielectric permittivity is defined as
\[ \varepsilon(x, \omega) - \varepsilon_0 = \int_0^\infty dt \exp[i\omega t] \chi(x, t). \]  \tag{6}

Here, according to the causality principle, it has been used that the susceptibility \( \chi(x, t) \) vanishes for negative times, i.e. \( \chi(x, t) = 0 \) if \( t < 0 \). Also, without loss of generality, it is assumed that the susceptibility \( \chi(x, t) \) must be bounded with respect to the time. Consequently, \( t \) is always positive in the integral above, and the permittivity remains well-defined if the real frequency is replaced by the complex frequency \( z = \omega + i\eta \) with positive imaginary part \( \text{Im}(z) = \eta > 0 \). Moreover, its derivative with respect to the complex frequency is well-defined since the function \( \text{Re}(\varepsilon(z, \omega)) = \text{Re} \chi(z, t) \) has exponential decay for \( \text{Im}(z) > 0 \). It follows that the permittivity \( \varepsilon(x, z) \) is an analytic function in the half plane of complex frequencies \( z \) with positive imaginary part, which will be denominated by “upper half plane” from now on. Finally, the susceptibility can be retrieved through the inverse Fourier (Laplace) transform
\[ \chi(x, t) = \frac{1}{2\pi} \int_{\Gamma_\eta} dz \exp[-i\eta z] [\chi(x, z) - \chi_0], \]  \tag{8}
where \( \Gamma_\eta \) is an horizontal line of complex numbers \( z = \omega + i\eta \) with \( \omega \in \mathbb{R} \), at a distance \( \eta \) from the real axis.

The Helmholtz equation is directly deduced from the set of equations \( \mathbf{9} \), where the \( \omega \)-dependence of the fields has been omitted:
\[ [H_\omega E](x) = \omega^2 \varepsilon(x, \omega) \mu_0 E(x) - \partial_x \times \partial_x \times E(x) = -i\omega \mu_0 J(x). \]  \tag{9}

Since the permittivity is well-defined in the upper half plane \( \text{Im}(z) > 0 \), the definition of the Helmholtz operator \( H_\omega(\omega) \) can be also extended to all complex frequency \( z \) with \( \text{Im}(z) > 0 \). It can be shown rigorously that the inverse of \( H_\omega(z) \) exists and is analytic in this domain \( \text{Im}z > 0 \) using the the auxiliary field formalism. Indeed, adding a new “auxiliary” field \( \mathbf{A}(t) \) to the electromagnetic field to form the total vector field \( F(t) = [\mathbf{E}(t), H(t), \mathbf{A}(t)] \), the set of Maxwell’s equations [without the source \( \mathbf{J}(x, t) \)] can be written as the unitary time-evolution equation \( \partial_t F(t) = -iK F(t) \), where \( K \) is a time-independent selfadjoint operator. The inverse \( [z - K]^{-1} \) is then well-defined for all complex number \( z \) with \( \text{Im}(z) > 0 \), and is moreover an analytic function of \( z \). Next, the inverse of the Helmholtz operator is retrieved by projecting the total field \( F(z) \) on the electric fields \( \mathbf{E}(z) \), and using the Feshbach projection formula. Since the projector on electric fields is \( z \)-independent, the inverse of the Helmholtz operator has the same analytic properties as the inverse \( [z - K]^{-1} \).

In this paper, it is proposed to use arguments based on the properties of the permittivity \( \varepsilon(x, z) \), and then to transpose them directly to the Helmholtz operator and the electromagnetic Green’s function.

B. The permittivity

Properties of the permittivity are derived in this subsection. The key point is the generalized expression of Kramers-Kronig relations for the permittivity:
\[ \varepsilon(x, z) = \varepsilon_0 - \int_{\mathbb{R}} dv \frac{\sigma(x, v)}{z^2 - v^2}, \]  \tag{10}
where
\[ \sigma(x, v) = \text{Im} \frac{\varepsilon(x, v) - \varepsilon_0}{\pi} \geq 0. \]  \tag{11}

One can checked that, when the limit \( \text{Im}(z) = \eta \downarrow 0 \) is considered in \( \mathbf{10} \), a Dirac function appears in the integral \( \mathbf{10} \) and the relation \( \mathbf{11} \) is retrieved. Notice that it has been assumed that only passive media are considered. Under this condition, the electromagnetic energy must decrease with time, and thus the permittivity must have positive imaginary part \( \eta \); hence the function \( \sigma(x, v) \) is positive. At the microscopic scale, this function corresponds to the oscillator strength \( \mathbf{14} \) which must be positive. Notice that this passivity requirement \( \mathbf{11} \) can be extended to all complex frequency in the upper half plane. Indeed, using that \( \sigma(x, v) = \sigma(x, -v) \), the expression \( \mathbf{11} \) can be written as
\[ z[\varepsilon(x, z) - \varepsilon_0] = -\int_{\mathbb{R}} dv \frac{\sigma(x, v)}{z - v}, \]  \tag{12}
which implies
\[ \text{Im}\{z[\varepsilon(x, z) - \varepsilon_0]\} = \text{Im}(z) \int_{\mathbb{R}} dv \frac{\sigma(x, v)}{|z - v|^2} \geq 0. \]  \tag{13}
Thus, from now on, it is assumed that the second derivative of $\chi(x, t)$ vanishes for $t < 0$ and, for $t > 0$:

$$
(\partial_t^2 \chi)(x, t) = \int_{\Gamma} dv \sigma(x, \nu) \cos[\nu t].
$$

(15)

This implies in particular that the integral of the function $\sigma(x, \nu)$ is finite since

$$
\int_{\Gamma} dv \sigma(x, \nu) = [\partial_t \chi](x, 0^+) < \infty.
$$

(16)

Also, it can be checked that $[\partial_t \chi](x, t)$ is continuous of $t$ in the general case (except at $t = 0$), and that $[\partial_t^2 \chi](x, t)$ is bounded by $[\partial_t \chi](x, 0)$.

Finally, an important estimate of the permittivity is provided at the limit of high frequencies. The function $[\partial_t \chi](x, t)$ being bounded and continuous (except at $t = 0$), the second derivative $[\partial_t^2 \chi](x, t)$ can be defined for all $t \neq 0$. Hence the following relation holds

$$
z^2[\varepsilon(x, z) - \varepsilon_0] = -[\partial_t \chi](x, 0^+) - \int_0^\infty dt \text{exp} [izt] [\partial_t^2 \chi](x, t).
$$

(17)

Since the derivative $[\partial_t \chi](x, t)$ corresponds to the microscopic currents, it is related to the impulsion of the charges: it is then reasonable to assume that its variations are bounded because of the inertia (charges have a mass), otherwise an infinite power is required. Notice that this argument does not apply to the initial time $t = 0$, where the causality requirement switches on (or switches off) instantaneously a force on the charges. Thus, from now on, it is assumed that the second derivative $[\partial_t^2 \chi](x, t)$ is bounded for $t > 0$. Using that the integral in the relation (17) is the Fourier transform of an integrable function, this relation implies for fixed $\eta$ in $z = \omega + i\eta$ the important estimate

$$
z^2[\varepsilon(x, z) - \varepsilon_0] \rightarrow -[\partial_t \chi](x, 0^+)\text{ as } \omega \rightarrow \infty.
$$

(18)

C. Helmholtz operator and extension of its definition

In order to define rigorously the inverse of the Helmholtz operator for complex frequencies $z$, the set of equation (17) is written as

$$
[M_0(z) + V(x, z)] F(x) = S(x),
$$

(19)

where $F(x)$ contains the electromagnetic field and $S(x)$ the current source:

$$
F(x) = \begin{bmatrix} E(x) \\ H(x) \end{bmatrix}, \quad S(x) = \begin{bmatrix} -iz_0^{-1}J(x) \\ 0 \end{bmatrix}.
$$

(20)

The operator $M_0(z)$ corresponds to Maxwell’s equations in vacuum and $V(x, z)$ contains the response of the material:

$$
M_0(z) = \begin{bmatrix} z & iz_0^{-1} \partial_x \times \\ -i\mu_0^{-1} \partial_x \cdot \\ 0 & 0 \end{bmatrix},
$$

(21)

and that

$$
V(x, z) = \begin{bmatrix} z\{\varepsilon(x, z)/\varepsilon_0 - 1\} \\ 0 \\ 0 \end{bmatrix}.
$$

Let $(\cdot, \cdot)$ be the standard inner product in the Hilbert space of square integrable electromagnetic fields:

$$
\langle F_1, F_2 \rangle = \int_{\mathbb{R}^3} dx \varepsilon_0 E_1(x) \cdot E_2(x) + \mu_0 H_1(x) \cdot H_2(x).
$$

(22)

If for all field $F$ there exists a constant $\alpha > 0$ such that

$$
|\langle F, [M_0(z) + V(x, z)] F \rangle| \geq \alpha \langle F, F \rangle,
$$

(23)

then the operator $[M_0(z) + V(x, z)]$ is invertible, and its inverse is bounded by $\alpha^{-1}$. Using that the curl is self-adjoint, the following relationship is obtained

$$
\text{Im} \langle F, M_0(z) F \rangle = \text{Im}(z) \int_{\mathbb{R}^3} dx |\varepsilon_0| E(x)|^2 + \mu_0 |H(x)|^2
$$

$$
= \text{Im}(z) \langle F, F \rangle.
$$

(24)

The term with $V(x, z)$ containing the relative permittivity is estimated using the extended passivity requirement (13):

$$
\text{Im} \langle F, V(x, z) F \rangle = \int_{\mathbb{R}^3} dx |E(x)|^2 \text{Im}(z|\varepsilon(x, z) - \varepsilon_0|) \geq 0.
$$

(25)

The combination of the two equations leads to

$$
|\langle F, [M_0(z) + V(x, z)] F \rangle| \geq \text{Im}(z) \langle F, F \rangle,
$$

(26)

which implies that the inverse $[M_0(z) + V(x, z)]^{-1}$ is well-defined for $\text{Im}(z) > 0$ and bounded by $\alpha^{-1} = 1/\text{Im}(z)$. In addition, this inverse is an analytic function of the complex frequency $z$ for $\text{Im}(z) > 0$ as well as the permittivity function $\varepsilon(x, z)$ in $V(x, z)$, and $M_0(z)$. The analyticity property can be also shown using the first resolvent formula (12). Denoting $M(z) = M_0(z) + V(x, z)$, the difference of the inverses at $z$ and $z_0$ is

$$
M(z)^{-1} - M(z_0)^{-1} = M(z)^{-1}(z_0 - z)A M(z_0)^{-1},
$$

(27)

where, using expression (12) for $V(x, z)$,

$$
A = \begin{bmatrix} 1 + m & 0 \\ 0 & 1 \end{bmatrix}, \quad m = \varepsilon_0^{-1} \int_{\mathbb{R}} dv \frac{\sigma(x, \nu)}{(z - \nu)(z_0 - \nu)}.
$$

(28)
The operator $A$ is bounded thanks to the condition (16), and $M(z_0)^{-1}$ is bounded by $1/\text{Im}(z_0)$. The identity (27) implies
\[
M(z)^{-1} = M(z_0)^{-1}[1 - (z_0 - z)AM(z_0)^{-1}]^{-1} = M(z_0)^{-1}\left\{1 + \sum_p (z_0 - z)^p[AM(z_0)^{-1}]^p\right\}.
\]

The last series converges in norm provided $|z - z_0|$ is smaller than the inverse of the norm of $[AM(z_0)^{-1}]$. This power series expansion shows that the inverse $[M_0(z) + V(x, z)]^{-1}$ is an analytic function if $\text{Im}(z_0) > 0$.

In the last step, the Helmholtz operator is retrieved using the projector on electric fields $P$, defined by $PF(x) = E(x)$. Then, equation (19) yields
\[
E(x) = PF(x) = P[M_0(z) + V(x, z)]^{-1}S(x) = -i\varepsilon_0^{-1}P[M_0(z) + V(x, z)]^{-1}PS(x),
\]
where, according to (20), $S(x) = PS(x)$ has the single "electric component" $-i\varepsilon_0^{-1}J(x)$. According to (9), the inverse Helmholtz operator is given by $E(x) = -i\varepsilon_0\mu_0 H_e(z)^{-1}J(x)$, and the comparison with the equation above provides
\[
H_e(z)^{-1} = \frac{1}{\varepsilon_0\mu_0}P[M_0(z) + V(x, z)]^{-1}P,
\]
This expression shows that all the properties of the inverse $[M_0(z) + V(x, z)]^{-1}$ are directly transposable to the inverse Helmholtz operator. In particular the inverse Helmholtz operator is bounded by
\[
\|H_e(z)^{-1}\| \leq \frac{1}{|\varepsilon_0\mu_0\text{Im}(z)|}.
\]

It is stressed that the bound $\alpha = \text{Im}(z)$ in (20) is governed by the imaginary part of $z$ in $M_0(z)$ only, and thus is independent of the complex number $z$ in $V(x, z)$. Hence the possibility to consider the two complex frequencies in $M_0(z)$ and $V(x, z)$ independently. As a result, it is obtained that the inverse
\[
[M_0(z) + V(x, \xi)]^{-1} \leq [\text{Im}(z)]^{-1},
\]
exists and is analytic with respect to both complex frequencies $z$ and $\xi$ as soon as $\text{Im}(z) > 0$ and $\text{Im}(\xi) > 0$. In particular power series expansions like (29) can be stated for both variables $z$ and $\xi$ independently. This property can be transposed to the inverse of a generalized version of the Helmholtz operator. The expression of this operator, denoted by $H(z, \xi)$, can be obtained by replacing $V(x, z)$ by $V(x, \xi)$ and then by eliminating the magnetic field $H(x)$ in equation (19):
\[
H(z, \xi) = z^2\varepsilon_0\mu_0 + \varepsilon_0\mu_0[\varepsilon(x, \xi) - \varepsilon_0] - \partial_x \times \partial_\xi x
\]
A relation similar to (31) shows that the inverse of $H(z, \xi)$ exists and is analytic of both complex variables $z$ and $\xi$ with $\text{Im}(z) > 0$ and $\text{Im}(\xi) > 0$. It is bounded by
\[
\|H(z, \xi)^{-1}\| \leq \frac{1}{|\varepsilon_0\mu_0\text{Im}(z)|}.
\]
This extended definition of Helmholtz operator is used in section V to analyze non dispersive systems.

III. THE ELECTROMAGNETIC GREEN’S FUNCTION

The electromagnetic Green’s function can be introduced from the inverse Helmholtz operator as shown by equation (2). While the left side of the equation, $H_e(z)^{-1}S$, is well-defined, there is no argument which ensures the existence of the Green’s function in the right side in Electromagnetism. Indeed, the existence of the Green’s function is usually the consequence of the compact or Hilbert-Schmidt nature of the corresponding operator. This compact nature could have been obtained by considering the difference of the original inverse $[M_0(z) + V(x, z)]^{-1}$ with the free inverse $M_0(z)^{-1}$ [see reference (32)]. However, in the case of Electromagnetism, this technique is not suitable because of the presence of the “static” modes which generate a “Dirac” singularity in the Green’s function (see reference (34) for investigations on the singularity). In practice, for square integrable functions $\phi$ and $\psi$, it is always possible to define coefficients like
\[
\langle \phi, H_e(z)^{-1}\psi \rangle = \int_{\mathbb{R}^3} dx \overline{\phi(x)}[H_e(z)^{-1}\psi](x)
\]
corresponding to
\[
\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy \overline{\phi(x)G_e(x, y; z)\psi(y)}.
\]

[From now, the brackets $\langle \cdot, \cdot \rangle$ denotes the standard inner product (33) for the solely electric fields.] The functions $\phi$ and $\psi$ can be the elements of an orthonormal basis $\{\phi_p(x)\}$ of the square integrable functions. Then, the coefficients $\langle \phi_p, H_e(z)^{-1}\phi_q \rangle$ can be used to define “formally”
\[
G_e(x, y; z) = \sum_{n,m} \langle \phi_p, H_e(z)^{-1}\phi_q \rangle \phi_p(x) \otimes \overline{\phi_q(y)},
\]
where the symbol $\otimes$ means that the tensor product is considered. Note that this definition is only “formal” because there is no argument ensuring that the convergence of the sum in the case $H_e(z)^{-1}$ is not Hilbert-Schmidt. The functions $\phi$ and $\psi$ in (35) and (37) can be also chosen to approach the identity. Let be function $\phi_a$ be defined by $\phi_a(x) = (3/4\pi)a^{-3}$ if $|x| \leq a$ and $\phi_a(x) = 0$ if $|x| \geq a$ in order to approach the Dirac function $\delta(x)$ when $a \downarrow 0$.

Then, for $a$ small enough, $\phi_{a_0}(x) = \phi_a(x - x_0) \approx \delta(x - x_0)$ and $\phi_{a_0}(y) = \phi_a(y - y_0) \approx \delta(y - y_0)$, and the coefficient
\[ \langle \phi_{x_0}, H_e(z)^{-1}\phi_{y_0} \rangle \] can approach the Green’s function:
\[ \langle \phi_{x_0}, H_e(z)^{-1}\phi_{y_0} \rangle = \int d\mathbf{x} \int \mathbb{R}^3 dy \phi_o(x-x_0)G_e(x,y;z)\phi_o(y-y_0) \]
\[ \approx G_e(x_0,y_0;z). \] (39)

Here, it is stressed that nothing authorizes to take the limit \( a \downarrow 0 \) since the function \( \phi_o \) is not square integrable at this limit. Thus the coefficient \( \langle \phi_{x_0}, H_e(z)^{-1}\phi_{y_0} \rangle \) just allows to address an approximation of the Green’s function \( G_e(x_0,y_0;z) \).

According to these arguments above, it is assumed that the electromagnetic Green’s function can be defined, and that its properties can be established from the coefficients \( \langle \phi, H_e(z)^{-1}\psi \rangle \).

First, it is clear that all the analytic properties of \( H_e(z)^{-1} \) and \( H(z,\xi)^{-1} \) are directly transposable to coefficients \( \langle \phi, H_e(z)^{-1}\psi \rangle \) and \( \langle \phi, H(z,\xi)^{-1}\psi \rangle \). Indeed, it is enough to expand the inverse in \( \langle \phi, H_e(z)^{-1}\psi \rangle \) in a power series like (32) and then to check that the resulting power series with the coefficients converges. Another important property of the Green’s function is the behavior for large frequency \( z \). First, it is shown in the appendix A that it decreases like \( 1/(z^2\varepsilon_0\mu_0) \) or, equivalently,
\[ \lim_{|z| \to \infty} z^2\varepsilon_0\mu_0 \langle \phi, H_e(z)^{-1}\psi \rangle = \langle \phi, \psi \rangle. \] (40)

It has to be noticed that this asymptotic behavior is for the modulus \(|z|\) of the complex frequency which tends to infinity. An important asymptotic regime is the limit \( \omega \to \infty \) in the complex frequency \( z \), i.e. for fixed imaginary part \( \text{Im}(z) \). In this case, one can show that
\[ z^2[H_e(z)^{-1} - H_0(z)^{-1}] \] (41)
is bounded when \( \omega \to \infty \). This can be established writing the difference
\[ H_e(z)^{-1} - H_0(z)^{-1} = -H_0(z)^{-1}z^2\mu_0[\varepsilon(x,z)-\varepsilon_0]H_e(z)^{-1}, \] (42)
and then using the bound (32) and the estimate (18). Hence it is found that
\[ z^2[H_e(z)^{-1} - H_0(z)^{-1}] \approx \omega \to \infty zH_0(z)^{-1}[\partial_x\chi(x,0^+)]zH_e(z)^{-1} \] (43)
\[ \leq \frac{|\partial_x\chi(x,0^+)|}{|\varepsilon_0\mu_0 \text{Im}(z)|^2}. \]

These properties and asymptotic behaviors will be used in the next section to derive a version of Kramers-Kronig relations for the electromagnetic Green’s function.

IV. KRAMERS-KRONIG RELATIONS FOR THE ELECTROMAGNETIC GREEN’S FUNCTION

The Kramers-Kronig relations can be applied to all function derived from a causal signal. It is generally used to analyze the dielectric permittivity, the permeability or the optical indices. A new version of Kramers-Kronig relations, given by equation (10), has been proposed recently. This version shows that the general expression of the permittivity is a continuous superposition of elementary resonances given by the elastically bound electron model. Thus it extends the classical Drude-Lorentz expression of the permittivity, and also its quantum mechanical justification based on the electric dipole approximation. In particular, the continuous superposition of resonances in (10) describes a regime with absorption, while the quantum mechanics model is reduced to a discrete superposition of resonances and thus to the description of systems without absorption.

In this section, it is proposed to express the new version of Kramers-Kronig relations for the electromagnetic Green’s function or, equivalently, for the inverse operator \( H_e(z)^{-1} \). The objective is to transpose all the properties of the permittivity, and to make it possible to use all the knowledge on permittivity \( \varepsilon(x,z) \) for the electromagnetic Green’s function.

The inverse operators \( H_e(z)^{-1} \) and \( H_0(z)^{-1} \) are expected to have properties similar to those of permittivities \( \varepsilon(x,z) \) and \( \varepsilon_0 \). Thus the following operator is considered:
\[ R(z) = H_e(z)^{-1} - H_0(z)^{-1}, \] (44)
First, it is noticed that, as well as \( H_e(z)^{-1} \) and \( H_0(z)^{-1} \), the adjoint operator of \( R(z)^{-1} \) is
\[ [R(z)^{-1}]^\dagger = R(-\overline{\varepsilon})^{-1}, \] (45)
which is related to \( \overline{\varepsilon(z)} = \varepsilon(-\overline{\varepsilon}) \). Next, let the operator \( X(t) \) be defined by
\[ X(t) = \int_{\Gamma_{\eta}} dz \exp[-izt]R(z), \] (46)
where \( \Gamma_{\eta} \) is the horizontal line parallel to the real axis at a distance \( \eta \) of complex numbers \( z = \omega + in \) with \( \eta > 0 \). It is stressed that this integral is well defined since, thanks to (43), \( R(z) \) is bounded and decreases like \( 1/\omega^2 \). Also, this decrease in \( 1/\omega^2 \) implies that \( X(t) \) is the Fourier transform of an integrable function, and thus \( X(t) \) is continuous of \( t \). The integral expression of \( X(t) \) is independent of \( \eta \) thanks to the analytic nature of the operator under the integral. The operator \( X(t) \) is selfadjoint since, for \( z = \omega + in \),
\[ X(t)^\dagger = \int_{\mathbb{R}} d\omega \exp[i\omega t]R(-\overline{\varepsilon}) = \int_{\mathbb{R}} d\omega \exp[-i\omega t]R(z), \] (47)
where (45) has been used and the change \( \omega \to -\omega \) has been performed to obtain the last expression. In addition, it can be checked that \( X(t) \) vanishes for negative times. Indeed, in that case, the integral (46) can be computed by closing the line \( \Gamma_{\eta} \) by a semi circle with infinite radius in the upper half plane. Since all the functions are
analytic, it is found that \( X(t) = 0 \) if \( t < 0 \). Hence, the quantity \( X(t) \) is similar to the real susceptibility \( \chi(x, t) \): it is selfadjoint and is associated with causality principle.

Next, it is always possible to write for \( \text{Im} z = \eta > 0 \)

\[
R(z) = \frac{1}{2\pi} \int_{0}^{\infty} dt \exp[izt]X(t) .
\]

Using integration by parts, this expression above becomes

\[
R(z) = -\frac{1}{2\pi} \int_{0}^{\infty} dt \frac{\exp[izt]}{iz} \partial_{t}X(t)
\]

(49)

since the values at the bounds vanish \( [X(t) \) is continuous and vanishes at \( t = 0] \). Let \( \xi = \nu + i\zeta \) be a complex number with positive imaginary part such that \( \text{Im}(\xi) = \eta > \xi > 0 \). Then, equation (49) implies

\[
i\xi R(\xi) = -\frac{1}{2\pi} \int_{0}^{\infty} dt \exp[i\xi t] \partial_{t}X(t) .
\]

(50)

and the selfadjoint part is

\[
i\xi R(\xi) - \overline{\xi} R(\xi)^{\dagger} = -\frac{1}{2\pi} \int_{0}^{\infty} dt \exp[-\xi t] 2\cos[\nu t] \partial_{t}X(t) = -\frac{1}{2\pi} \int_{0}^{\infty} dt \exp[i\nu t] \exp[-\xi t] \{ \partial_{t}X(t) + \partial_{t}X(-t) \} .
\]

(51)

The inverse Fourier transform is used: for \( t > 0 \),

\[
\frac{1}{2\pi} \partial_{t}X(t) = \int_{\mathbb{R}} d\nu \exp[-i\xi t] \frac{\xi R(\xi) - \overline{\xi} R(\xi)^{\dagger}}{2i\pi} .
\]

(52)

Injecting this expression in (49) yields

\[
R(z) = -\int_{0}^{\infty} dt \frac{\exp[izt]}{iz} \int_{\mathbb{R}} d\nu \exp[-i\xi t] \frac{\xi R(\xi) - \overline{\xi} R(\xi)^{\dagger}}{2i\pi} .
\]

(53)

Next, by analogy with the function \( \sigma(x, \nu) \) defined by (11), the following quantity is considered:

\[
D(\nu) = \frac{1}{2i\pi} \left[ \xi R(\xi) - \overline{\xi} R(\overline{\xi}) \right] ,
\]

(54)

which is selfadjoint (and, contrary to \( \sigma(x, \nu) \), it is not positive since it is the difference of the one in the media minus the one in vacuum). The integral over the time in (53) is performed:

\[
R(z) = -\int_{\mathbb{R}} d\nu \frac{1}{iz \frac{1}{z - \xi}} D(\nu) = \frac{1}{2i\pi} \int_{\mathbb{R}} d\nu \frac{D(\xi)}{z - \xi} .
\]

(55)

The integral over \( \nu \) in (55) and above is independent of the imaginary part \( \zeta \) of \( \xi \). Hence the integral in (55) and above can be considered in the limit \( \zeta \downarrow 0 \). Let \( D(\nu) \) be defined by

\[
D(\nu) = \lim_{\zeta \downarrow 0} D(\nu + i\zeta) ,
\]

(56)

it is left invariant when \( \nu \) is changed in \( -\nu \) [as well as in \( D(\xi) \)]. It implies the following expressions

\[
R(z) = -\int_{\mathbb{R}} d\nu \frac{D(\nu)}{z^{2} - \nu^{2}} \quad \text{Im} z > 0 ,
\]

\[
zR(z) = -\int_{\mathbb{R}} d\nu \frac{D(\xi)}{z - \xi} \quad \text{Im} z > \text{Im} \xi \geq 0 .
\]

(57)

These equations define Kramers-Kronig expressions for the inverse Helmholtz operator. The first line of the equation above implies for the coefficients

\[
\langle \phi, H_{\nu}(z)^{-1}\psi \rangle = \langle \phi, H_{0}(z)^{-1}\psi \rangle - \int_{\mathbb{R}} d\nu \int \frac{\langle \phi, D(\nu)\psi \rangle}{z^{2} - \nu^{2}} ,
\]

(58)

or, for the Green’s function

\[
G_{\nu}(x, y; z) = G_{0}(x, y; z) - \int_{\mathbb{R}} d\nu \frac{\rho(x, y; \nu)}{z^{2} - \nu^{2}} ,
\]

(59)

where

\[
\rho(x, y; \nu) = \frac{\nu \text{Im}[G_{\nu}(x, y; z) - G_{0}(x, y; z)]}{\pi}
\]

(60)

is related to the imaginary part of the relative Green’s function [modulo the free Green’s function \( G_{0}(x, y; z) \)]. It is stressed that the quantity \( \rho(x, y; \nu) \) is closely related to the local density of states which is just proportional to the trace of \( \rho(x, x; \nu) \) [modulo the density of states in vacuum]. Hence the Kramers-Kronig expression (59) may be a first step for a generalization of the usual eigenmodes expansion of the Green’s function\( ^{6} \). Indeed, consider the case of a closed cavity filled with a non dispersive and non absorptive dielectric media. Then, the Helmholtz equation without source term can be written

\[
\mathbf{L}E = z^{2} E , \quad H_{\nu}(z) = z^{2} - \mathbf{L}
\]

(61)

where \( \mathbf{L} \) is a selfadjoint operator independent of the complex frequency \( z \). In the considered closed cavity, the operator \( \mathbf{L} \) has a discrete set of eigenfunctions \( |\phi_{n}\rangle \) and (real) eigenvalues \( \omega_{n}^{2} \)

\[
\mathbf{L}|\phi_{n}\rangle = \omega_{n}^{2}|\phi_{n}\rangle ,
\]

(62)

which can be used to develop the inverse Helmholtz operator and the Green’s function. Let the operator \( \mathbf{L} \) act on the eigenfunctions (here, for the sake of simplicity, the free inverse operator is not substracted):

\[
D(\xi)|\phi_{n}\rangle = \frac{1}{2\pi} \left[ \frac{\xi}{\xi^{2} - \omega_{n}^{2}} - \frac{\overline{\xi}}{\overline{\xi}^{2} - \omega_{n}^{2}} \right] |\phi_{n}\rangle = \rho_{n}(\xi)|\phi_{n}\rangle ,
\]

(63)

which, for \( \xi = \nu + i\zeta \), leads to

\[
\rho_{n}(\xi) = \frac{1}{2\pi} \left[ \frac{\xi}{(\nu - \omega_{n})^{2} + \zeta^{2}} + \frac{\zeta}{(\nu + \omega_{n})^{2} + \zeta^{2}} \right] .
\]

(64)
When the imaginary part of $\xi$ tends to zero, the eigenvalue $\rho_n(\xi)$ becomes
\[
\lim_{\xi \to 0} \rho_n(\xi) = \rho_n(\nu) = \frac{1}{2} \left[ \delta(\nu - \omega_n) + \delta(\nu + \omega_n) \right],
\]
and the operator $D(\nu)$ can be written
\[
D(\nu) = \sum_n \rho_n(\nu) |\phi_n\rangle \langle \phi_n|.
\]
The function under the integral in (59) is then
\[
\rho(x, y; \nu) = \sum_n \rho_n(\nu) \phi_n(x) \otimes \phi_n(y),
\]
where the symbol $\otimes$ means that the tensor product is considered. Finally, This expression is replaced into the equation (59) without the free part $G_0(x, y; z)$:
\[
G_v(x, y; z) = -\sum_n \frac{1}{z^2 - \omega_n^2} \phi_n(x) \otimes \phi_n(y),
\]
Hence the classical expression of the Green’s function is retrieved. This confirms that the Kramers-Kronig expression (59) may be considered as an extension of the classical expansion in the general case of frequency dispersive and absorptive systems.

V. ANALYTIC PROPERTIES IN NON DISPERSIVE SYSTEMS

In non dispersive systems, the dielectric permittivity is independent of the frequency. Here, it is assumed that the dielectric constant is given by the expression (10) of the permittivity where the frequency has been fixed to the real value $\omega$:
\[
\varepsilon(x, \nu_0) = \varepsilon_0 - \int_{\mathbb{R}} d\nu \frac{\sigma(x, \nu)}{\omega_0^2 - \nu^2}.
\]
The Helmholtz operator for the corresponding non dispersive system is
\[
[H_{\mu}(z)E](x) = z^2 \varepsilon(x, \omega_0) \mu_0 E(x) - \partial_x \times \partial_x \times E(x).
\]
In this paper, it is proposed to exploit the new degree of freedom offered by the second complex frequency $\zeta$ in $H(\zeta, \xi)$ to analyze the analytic properties of the non dispersive Helmholtz operator $H_d(z)$. The identification of $H_d(z)$ with the expression (33) yields
\[
z[\varepsilon(x, \omega_0) - \varepsilon_0] = \xi[\varepsilon(x, \xi) - \varepsilon_0].
\]
The permittivities are replaced by their Kramers-Kronig expressions (69) and (12), which provides the following condition:
\[
z \int_{\mathbb{R}} d\nu \frac{\sigma(x, \nu)}{\omega_0^2 - \nu^2} = \int_{\mathbb{R}} d\nu \frac{\sigma(x, \nu)}{\xi - \nu}.
\]
A sufficient condition to ensure this equation is
\[
z(\xi - \nu) = \omega_0^2 - \nu^2 \iff \xi = \nu + \frac{\omega_0^2 - \nu^2}{z}.
\]
Finally, the imaginary part of $\xi$ can be related to the one of $z$:
\[
\text{Im}(\xi) = \text{Im}(z) \frac{\nu^2 - \omega_0^2}{|z|^2}.
\]
In order to preserve the analytic properties of the inverse Helmholtz operator, it is necessary to have $\text{Im}(\xi) > 0$. Since $\text{Im}(z) > 0$, it is found that the difference $\nu^2 - \omega_0^2$ has to be positive. This condition is realized if the function $\sigma(x, \nu)$ vanishes for frequency $\nu$ smaller than a frequency $\nu_0 > \omega_0$. The resulting dielectric constant is given by
\[
\varepsilon(x, \omega_0) = \varepsilon_0 + \int_{|\nu| > \omega_0} db \frac{\sigma(x, \nu)}{\nu^2 - \omega_0^2}.
\]
The function under the integral is strictly positive and purely real. Thus the dielectric constant $\varepsilon(x, \omega_0)$ is real, positive, and takes values greater than $\varepsilon_0$: it describes a transparent dielectric.

It has been shown that the analytic properties of the inverse Helmholtz operator can be preserved in non dispersive systems if the permittivity is the one of transparent dielectric. This analyticity property implies that such non dispersive systems are physically acceptable. Indeed, it can be checked that the causality principle is preserved for the solution $E(x, t)$ of Maxwell’s equation. Let $J(x, t)$ be a current source switched on at $t = 0$, hence $J(x, t) = 0$ for $t < 0$. Then, after the frequency decomposition, the current source
\[
J(x, t) = \int_0^\infty dt \exp[izt] J(x, t)
\]
is an analytic function in the upper half plane of the complex frequencies $z$. The time dependent electric field is given by
\[
E(x, t) = \frac{1}{2\pi} \int_{\Gamma} dz \exp[-izt]H_e(z)^{-1}(iz\mu_0)J(x, z),
\]
where $\Gamma$ is an horizontal line, parallel to the real axis, in the upper half plane. For negative times $t$, the integral can be computed by closing the line $\Gamma$ by a semi circle with infinite radius in the upper half plane. Since all the functions are analytic, it is found that the electric field vanishes for negative times: $E(x, t) = 0$ if $t < 0$. Hence, the causality principle is preserved.

In addition, it can be checked that the light velocity $v$ is always smaller than the one vacuum $c$ since the dielectric constant takes values larger than $\varepsilon_0$:
\[
v = \frac{1}{\sqrt{\varepsilon(x, \omega_0)\mu_0}} \leq c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}.
\]
Notice that the expression (75) obtained for the permittivity, and derived from equation (71), is just a sufficient condition which might be too strong. At this stage, this condition means that a physically acceptable description of a metallic behavior ($\omega_0^2 > \nu^2$) or absorptive materials must include frequency dispersion.

VI. ANALYTIC PROPERTIES WITH RESPECT TO THE WAVEVECTOR

In this section, it is assumed that a wavevector $k$ can be defined, which requires for the geometry of the system to be invariant under a group of translations. The starting point is the expression of Maxwell’s equations (21) introduced in section II C. In the case the group of translations is discrete (periodic structure), the curl operator becomes $\partial_x + i k \times$ after a Floquet-Bloch decomposition and, in case the group of translation is continuous (homogeneous structure), the curl operator becomes $i k \times$ after a Fourier transform. The resulting free operator introduced in (21) becomes $M_0(k, z)$ while the potential $V(x, z)$ is left invariant. The wavevector $k$ can have one, two or three components according to the geometry of the system which can be invariant under translations in one, two or three dimensions respectively. Let $k'$ and $k''$ be the real and imaginary part of the wavevector: $k = k' + ik''$. Then the “imaginary” part of the free operator $M_0(k, z)$ can be computed:

$$M_0(k, z) - M_0(k, z) \| \frac{i}{2} \left[ \begin{array}{c} \text{Im}(z) - \varepsilon_0^{-1} k' \times \\ \mu_0^{-1} k'' \times \end{array} \right] \right].$$

The eigenvalues of this matrix are

$$\lambda_0 = \text{Im}(z), \quad \lambda_{\pm} = \text{Im}(z) \pm c k'' ,$$

where $c$ is the light velocity in vacuum (78), and $(k'')^2 = k'' \cdot k''$. Consequently, this imaginary part of the free operator is strictly positive as soon as

$$\text{Im}(z) > c |k''| > 0 .$$

Since the imaginary part of the potential $V(x, z)$ is also positive (23), the inverse $[M_0(k, z) + V(x, z)]$ is well-defined and analytic in the domain $\text{Im}(z) - c |k''| > 0$ of complex frequencies $z$ and wavevectors $k$. And it is straightforward to extend this property to the inverse $[M_0(k, z) + V(\xi, z)]$ in the domain defined by i) $\text{Im}(z) - c |k''| > 0$ and ii) $\text{Im}(\xi) > 0$.

For instance, similar property has been used in [2] to derive new Kramers-Kronig relations for the reflection and transmission coefficients (via the Green’s function) in the case of multilayered stacks illuminated with incident angle $\theta \neq 0$. Indeed, in this situation, the square of the wavevector appears to be $k = k = (z^2/c^2) \sin^2 \theta$, which always meets the requirement (51).

VII. CONCLUSION

Analytic properties of the inverse Helmholtz operator and the Electromagnetic Green’s function have been established. These properties are strongly related to the causality principle and the passivity requirement in frequency dispersive and absorptive media. Notably, the consequences for the permittivity make it possible to extend all the analytic properties of the free inverse Helmholtz operator (and the free electromagnetic Green’s function) to the general inverse Helmholtz operator (and the Green’s function). Hence the general inverse Helmholtz operator has been shown to be analytic in the domain $\text{Im}(z) - c |k''| > 0$ of complex frequencies $z$ and complex wavevectors $k = k' + ik''$ (sections III and VI). Moreover, it has been shown that an additional degree of freedom (the second frequency $\xi$) can be introduced in the inverse Helmholtz operator which remains analytic as soon as $\text{Im}(\xi) > 0$ (section IV). This additional frequency has been then exploited to retrieve that causal systems with non dispersive permittivity must have purely real dielectric constant taking values above the vacuum permittivity $\varepsilon_0$ (section V). Finally, asymptotic estimates and Kramers-Kronig expressions have been established for the inverse Helmholtz operator and the electromagnetic Green’s function (sections III and IV). Such Kramers-Kronig expressions can be considered as an extension of the well-known eigenmodes expansion of the Green’s function in the case of a closed cavity filled with non dispersive and non absorptive media.

It is stressed that all this results can be extended to magneto-electric materials. Indeed, in that case, it is enough to add the permeability $\mu(x, z)$ in the expression of the matrix $V(x, z)$

$$V(x, z) = \left[ \begin{array}{cc} z[z(c(x, z) - \varepsilon_0) & 0 \\ 0 & z[\mu(x, z) - \mu_0] \end{array} \right],$$

and then to apply the arguments proposed in this paper. Also, the analytic properties with respect to the wavevector can be used to extend the Kramers-Kronig expressions (like in section IV) to spatial dispersion.

The results established in this paper may be used to calculate time-dependent electromagnetic field$\ddag$ and to establish rigorous eigenmodes expansion in dispersive and absorptive systems.

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where $k^2 = k \times k$. The inverse of the free Helmholtz operator is then

$$H_0(z) = z^2 \varepsilon_0 \mu_0 - k^2 [1 - kk/k^2],$$

(A3)

where $kk$ is the tensor product of $k$ with $k$, and $k^2 = k \cdot k$. The inverse of the free Helmholtz operator is then

$$H_0(z)^{-1} = \frac{1}{z^2 \varepsilon_0 \mu_0 - k^2} \left[ kk/k^2 \right]$$

or, equivalently,

$$z^2 \varepsilon_0 \mu_0 \ H_0(z)^{-1} = 1 + \frac{k^2}{z^2 \varepsilon_0 \mu_0 - k^2} [1 - kk/k^2].$$

(A5)

Let $\phi$ and $\psi$ be square integrable fields, then the following coefficients are considered:

$$\langle \phi, H_0(z)^{-1} \psi \rangle = \int_{\mathbb{R}^3} dk \hat{\phi}(k) \tilde{H}_0(z)^{-1} \hat{\psi}(k).$$

(A6)

The expression (A6) of the free resolvent leads to

$$z^2 \varepsilon_0 \mu_0 \langle \phi, H_0(z)^{-1} \psi \rangle = \langle \phi, \psi \rangle + I(z),$$

(A7)

with

$$I(z) = \int_{\mathbb{R}^3} dk \hat{\phi}(k) \frac{k^2}{z^2 \varepsilon_0 \mu_0 - k^2} [1 - kk/k^2] \hat{\psi}(k).$$

(A8)

In order to obtain the property (A10) for the free resolvent, it is enough to show that the number $I(z)$ tends to zero for large $|z|$. It is used that $|1 - kk/k^2|$ is bounded by unity, and then the integral in

$$|I(z)| \leq \int_{\mathbb{R}^3} dk |\hat{\phi}(k)| |\hat{\psi}(k)| \frac{k^2}{z^2 \varepsilon_0 \mu_0 - k^2},$$

(A9)

is splited in two parts: $I_+(z)$ for $|k| \geq K$ and $I_-(z)$ for $|k| \leq K$. Writing the complex number as $z = |z| \exp[i \vartheta]$, it is obtained that $|z^2 \varepsilon_0 \mu_0 - k^2| \geq k^2 \sin^2 \vartheta$. The first part of the integral becomes

$$|I_+(z)| \leq \int_{|k| \geq K} dk |\hat{\phi}(k)| |\hat{\psi}(k)| \frac{1}{\sin^2 \vartheta},$$

(A10)

and, since the function $\hat{\phi}(k)\hat{\psi}(k)$ is integrable [i.e. in $L^1(\mathbb{R}^3)$], it can be made arbitrary small for large enough $K$. Next, the number $|z|$ is chosen large enough to have $|z|^2 \varepsilon_0 \mu_0 > K^2$, and the second part is bounded by

$$|I_-(z)| \leq \frac{K^2}{|z|^2 \varepsilon_0 \mu_0 - K^2} \int_{|k| \leq K} dk |\hat{\phi}(k)| |\hat{\psi}(k)|.$$  

(A11)

This second part of the integral can be made arbitrary small when $|z| \to \infty$. Thus it is concluded that

$$\lim_{|z| \to \infty} z^2 \varepsilon_0 \mu_0 \langle \phi, H_0(z)^{-1} \psi \rangle = \langle \phi, \psi \rangle.$$  

(A12)

In order to extend this asymptotic behavior to $H_\varepsilon(z)^{-1}$, it is proposed in the second step to show that the coefficient

$$C(z) = z^2 \varepsilon_0 \mu_0 \ [\phi, [H_\varepsilon(z)^{-1} - H_0(z)^{-1}] \psi]$$

(A13)

vanishes at the limit of infinite $|z|$. The second resolvent identity is used to express the difference

$$H_\varepsilon(z)^{-1} - H_0(z)^{-1} = -H_0(z)^{-1} z^2 \mu_0 [\varepsilon(x, z) - \varepsilon_0] H_\varepsilon(z)^{-1}.$$  

(A14)

Let the function $\Psi$ be given by

$$\Psi(z) = -z^2 \mu_0 [\varepsilon(x, z) - \varepsilon_0] H_\varepsilon(z)^{-1} \psi.$$  

(A15)

It is well-defined for all $z$ since $z H_\varepsilon(z)^{-1}$ is bounded by $[\varepsilon_0 \mu_0 \text{Im}(z)]^{-1}$ and, from (12), $z [\varepsilon(x, z) - \varepsilon_0]$ is bounded.
by $[\partial_t \chi(x, 0^+)] / \text{Im}(z)$. Also, the norm of $\Psi(z)$ can be made arbitrarily small for large $|z|$ thanks to the estimate for $z^2 [\varepsilon(x, z) - \varepsilon_0]$: 

$$
\|\Psi(z)\| \approx \frac{[\partial_t \chi](x, 0^+) \|\mu_0 \mathcal{H}_e(z)^{-1} \psi\|}{|z| \varepsilon_0 \text{Im}(z)} \|\psi\|. 
$$

(A16)

Finally, the preceding equations (A12–A16) imply 

$$
|C(z)| = z^2 \varepsilon_0 \mu_0 \left| \langle \phi, \mathcal{H}_0(z)^{-1} \Psi(z) \rangle \right| 
\approx \frac{|\langle \phi, \Psi(z) \rangle|}{|z| \varepsilon_0 \text{Im}(z)} \leq \frac{|[\partial_t \chi](x, 0^+)\|\phi\| \|\psi\|}{|z| \varepsilon_0 \text{Im}(z)} 
\rightarrow |z| \varepsilon_0 \text{Im}(z) \rightarrow 0.
$$

(A17)