We show that the transformations Hill and Cox introduce, between inertial observers moving faster than light with respect to each other, are consistent with Einstein’s principle of relativity only if the space–time is two dimensional.

1. Introduction

Hill and Cox introduce the following two transformations to extend Lorentz transformations for inertial observers moving faster than light (FTL) with respect to each other, see equations (3.16) and (3.18) in Hill & Cox [1]:

\[
HC_1 : t = \frac{-T + vX/c^2}{\sqrt{v^2/c^2 - 1}}, \quad x = \frac{-X + vT}{\sqrt{v^2/c^2 - 1}}, \quad y = Y, \quad z = Z
\]

(1.1)

and

\[
HC_2 : t = \frac{T - vX/c^2}{\sqrt{v^2/c^2 - 1}}, \quad x = \frac{X - vT}{\sqrt{v^2/c^2 - 1}}, \quad y = Y, \quad z = Z
\]

(1.2)

where \(v\) is the superluminal relative speed of the two observers.

In the present paper, we show that Hill–Cox transformations give a consistent extension of Einstein’s special theory of relativity only if the dimension \(d\) of space–time is 2 (i.e. there are one space and one time dimensions).

2. Consistency of Hill–Cox transformations with the principle of relativity implies \(d = 2\)

Einstein originally formulated his principle as ‘The laws by which the states of physical systems undergo change are not affected, whether these changes of state be
referred to the one or the other of two systems of co-ordinates in uniform translatory motion.’ [2]. Clearly, this principle implies that inertial observers cannot be distinguished by physical experiments, e.g. by experiments based on sending out light signals.

We show that in the HC1-transformed worldview, the light cone is ‘flipped over’ so that its axis is the x-axis. Hence, x is a unique direction in which the speed of light is smallest (namely, c); in all other directions, either one cannot send out a light signal, or the speed of the light signal is greater than c (there are plenty of these two types of direction). From this, the FTL observer belonging to the transformed reference frame can know/‘observe’ that he is moving FTL, and x is a distinguished unique direction in his worldview. Further, there are directions in which no light signals can be mirrored back and forth (as in Einstein’s light clock) but not all directions are such, whereas in the worldview of any slower than light observer, there are no such directions. This is a quite strong violation of Einstein’s principle of relativity (e.g. because the space of each slower than light observer is isotropic, whereas this is not so for the FTL observers).

To see what the Hill–Cox transformations do with the light cones if \( d \geq 2 \), first we show that they can be written as the composition of a Lorentz transformation and a transformation exchanging the time axis and a space axis, if we use relativistic units (i.e. if the speed c of light is set to be 1),\(^1\) otherwise a scaling of time enters the picture, too.

Let \( v, c \) be as in (1.1) and (1.2). Let \( L \) be the Lorentz boost, in relativistic units, corresponding to velocity \( c/v \), i.e. \( L \) takes \((T, X, Y, Z)\) to \((t, x, y, z)\), where

\[
L: t = \frac{T - (c/v)X}{\sqrt{1 - (c/v)^2}}, \quad x = \frac{X - (c/v)T}{\sqrt{1 - (c/v)^2}}, \quad y = Y, \quad z = Z \quad (2.1)
\]

Let us note that \( c/v < 1 \) if \( v > c \). Let \( \sigma_1 \) be the transformation that interchanges the first two coordinates, i.e. \( \sigma_1 \) takes \((T, X, Y, Z)\) to \((X, T, Y, Z)\), let \( \sigma_2 \) be the transformation that takes \((T, X, Y, Z)\) to \((-X, -T, Y, Z)\), and finally let \( \rho \) be the transformation that scales the time coordinate with \( c \), i.e. \( \rho \) takes \((T, X, Y, Z)\) to \((cT, X, Y, Z)\). Then a straightforward computation shows that

\[
HC_1 = \rho^{-1} \circ \sigma_1 \circ L \circ \rho, \quad (2.2)
\]

see figure 1 and the computation below. This decomposition will make transparent what HC1 do with the light cones. The key point will be that transformations \( \sigma_1 \) and \( \sigma_2 \) radically deform the light cones in \( d \) dimensions if \( d > 2 \), they only preserve the light cones in two dimensions.

\[
(\rho^{-1} \circ \sigma_1 \circ L \circ \rho)(T, X, Y, Z) = (\rho^{-1} \circ \sigma_1 \circ L)(cT, X, Y, Z)
\]

\[
= (\rho^{-1} \circ \sigma_1) \left( \frac{cT - (c/v)X}{\sqrt{1 - (c/v)^2}}, \frac{X - (c/v)cT}{\sqrt{1 - (c/v)^2}}, Y, Z \right)
\]

\[
= \rho^{-1} \left( \frac{X - (c/v)cT}{\sqrt{1 - (c/v)^2}}, \frac{cT - (c/v)X}{\sqrt{1 - (c/v)^2}}, Y, Z \right)
\]

\[
= \frac{X/c - (c/v)cT}{\sqrt{1 - (c/v)^2}}, \frac{cT - (c/v)X}{\sqrt{1 - (c/v)^2}}, Y, Z
\]

\[
= \left( \frac{(c/v)(vX/c^2 - T)}{(c/v)(v/c^2 - 1)}, \frac{(c/v)(vT - X)}{(c/v)(v/c^2 - 1)}, Y, Z \right)
\]

\[
= \frac{T + vX/c^2 - 1}{\sqrt{v^2/c^2 - 1}}, \frac{-X + vT}{\sqrt{v^2/c^2 - 1}}, Y, Z
\]

\[
= HC_1(T, X, Y, Z).
\]

The computation for HC2 is similar.

\(^1\)The physical meaning of this choice is to fit the units of measuring time and distance together, i.e. measuring time in years and distance in light years, or measuring time in Planck time and distance in Planck length.
Now let us see what the $HC_i$-transformed light cones are like. First, we give a geometric visual proof for our original claim, and then we supplement this proof with computations valid for any $d > 2$. The idea of our proof is depicted in figure 1 and is based on the ‘step-by-step’ understanding of what Hill–Cox transformations do with the light cones. Let us concentrate on the light cone emanating from the origin. In the original, non-transformed worldview, this light cone is a regular cone with width $c$ (because the speed of light is $c$ in each direction). Now, $\rho$ scales the time axis such that this cone becomes a right-angle one, i.e. one with width 1. Transformation $L$, being a simple Lorentz boost acts non-trivially only in plane $TX$ and takes this light cone to itself (basic property of Lorentz transformations). Then $\sigma_i$ flips this cone over by exchanging coordinates $x$ and $t$. Finally, $\rho^{-1}$ compresses the flipped-over cone in the $t$ direction, so that the ‘height’ of this flipped-over cone is $c$, whereas its ‘width’ remains 1. Thus, after this final compression, the steepest line of this flipped-over cone goes in the $x$ direction and the corresponding speed is $c$; all spatial projections of the lines of this cone enclose at most a $45^\circ$ angle with the $x$-axis. Hence, there are no light signals in any direction enclosing an angle greater than $45^\circ$ with the $x$-axis, and the speed of light is greater than $c$ for every direction enclosing an angle (strictly) between $0^\circ$ and $45^\circ$ with the $x$-axis. The speed of light is infinite in the directions enclosing an exactly $45^\circ$ angle with the $x$-axis. By using some right-angle triangles, one can compute the exact dependence of the speed $c(\alpha)$ of light going in a direction that encloses an angle $\alpha$ with the $x$-axis:

$$c(\alpha) = c \cdot \sqrt{\frac{1 + \tan(\alpha)^2}{1 - \tan(\alpha)^2}}, \quad \text{where } 0 \leq \alpha < 45^\circ. \quad (2.3)$$

Let us now give the calculations belonging to the above chain of thought. We use $d = 4$, but the computations we give are completely analogous for any $d > 2$.

The equation of the light cone (whose apex is the origin) in the $(T, X, Y, Z)$ coordinate system is

$$(cT)^2 = X^2 + Y^2 + Z^2. \quad (2.4)$$

We get the $HC_i$-image of this by successively applying $\rho, L, \sigma_i$; and $\rho^{-1}$ to this equation, and we get in both cases of $i = 1, 2$

$$x^2 = (ct)^2 + y^2 + z^2. \quad (2.5)$$

Figure 1. Decomposition of Hill–Cox transformations. (Online version in colour.)
Equation (2.5) corresponds to the flipped-over light cone depicted in the bottom right corner of figure 1. Let $\ell$ be any line going through the origin and orthogonal to the time axis (figure 2). Then, there are $A, B, C$ such that $\ell$’s equation is

\[(0, As, Bs, Cs), \quad s \in \mathbb{R}.\]  

(2.6)

Thus the point $P$ of the light cone in direction $\ell$ and with time coordinate 1 is

\[(1, As, Bs, Cs) \quad \text{with} \quad (As)^2 = c^2 + (Bs)^2 + (Cs)^2,

from where we get

\[s = \frac{c}{\sqrt{A^2 - B^2 - C^2}},\]  

(2.8)

thus the speed of light in direction $\ell$ is

\[c(\ell) = \frac{c \cdot \sqrt{A^2 + B^2 + C^2}}{\sqrt{A^2 - B^2 - C^2}}.\]  

(2.9)

Let $\alpha$ denote the angle between $\ell$ and the $x$-axis, then $\tan(\alpha) = \sqrt{B^2 + C^2}/A$. Substituting this to (2.9) we get (2.3).

The fact that the Hill–Cox transformations do not work in the case $d > 2$ is not surprising. It can be shown in a strictly axiomatic framework, with using only a few assumptions of special relativity theory that inertial observers cannot move faster than the speed of light, if $d > 2$, see, e.g. theorem 2.1 in Andréka et al. [3]. By observers we mean reference frames as, e.g. the standard relativity book d’Inverno [4] does. So the difference between particles and observers is that particles do not need to have worldviews (frames of reference), hence dealing with particles does not require dealing with worldview transformations.

For $d = 2$, transformations HC$_1$ and HC$_2$ are perfectly consistent with Einstein’s special relativity. In this case, exchanging time and space is the usual way for constructing models satisfying the axioms of special relativity in which there are FTL observers. This construction is investigated in §2.4 in Andréka et al. [5].
3. Do we need faster than light observers in a theory of faster than light particles?

The existence of particles moving with the speed of light (photons) does not imply the existence of observers moving with the speed of light. In the same way, the existence of FTL particles does not imply (logically) the existence of FTL observers. This fact suggests that in order to elaborate a theory of superluminal particles, we do not necessarily have to introduce superluminal observers.

Indeed, even though observers cannot move FTL if $d > 2$, the superluminal motion of particles is consistent with both the kinematics and the dynamics of special relativity, see Andréka et al. [5], Székely [6] and Madarász & Székely [7].

In Hill & Cox [1], it is shown that the relativistic mass ($m$) depends on the speed ($v$) of a superluminal particle and an observer-independent quantity $p_\infty$ as follows:

$$m = \frac{p_\infty}{c\sqrt{v^2/c^2 - 1}}. \quad (3.1)$$

The dynamical results of Hill and Cox can also be proved to hold in a strictly axiomatic framework without using FTL observers using very few, simple assumptions. For example, their formula (3.1) can be derived because a natural, consistent axiom system of special relativistic particle dynamics containing Einstein’s principle of relativity implies that

$$m_k(b)\sqrt{|1 - v_k^2(b)|} = m_h(b)\sqrt{|1 - v_h^2(b)|}, \quad (3.2)$$

where $m_k(b)$ and $m_h(b)$ are the relativistic masses and $v_k(b)$ and $v_h(b)$ are the speeds of a (possibly FTL) particle $b$ with respect to (ordinary slower than light) inertial observers $k$ and $h$ (figure 3). This is done in Madarász & Székely [7], relying on Andréka et al. [8]. We get formula (3.1) in relativistic units, by introducing an observer-independent quantity for FTL particle $b$ as

$$p_\infty(b) := m_k(b)\sqrt{v_k^2(b) - 1}. \quad (3.3)$$

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