FULLERENE-LIKE SPHERES WITH FACES OF NEGATIVE CURVATURE

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ABSTRACT. Given $R \subset \mathbb{N}$, an $(R,k)$-sphere is a $k$-regular map on the sphere whose faces have gonalities $i \in R$. The most interesting/useful are (geometric) fullerenes, i.e., $([5,6],3)$-spheres.

Call $\kappa_i = 1 + \frac{i}{k} - \frac{2}{k}$ the curvature of $i$-gonal faces. $(R,k)$-spheres admitting $\kappa_i < 0$ are much harder to study. We consider the symmetries and construction for three new instances of such spheres: $([a,b],k)$-spheres with $p_b \leq 3$ (they are listed), icosahedrites (i.e., $([3,4],5)$-spheres) and, for any $c \in \mathbb{N}$, fullerene $c$-disks, i.e., $([5,6,c],3)$-spheres with $p_c = 1$.

1. Introduction

Given $R \subset \mathbb{N}$, an $(R,k)$-sphere $S$ is a $k$-regular map on the sphere whose faces have gonalities (numbers of sides) $i \in R$. Let $v$, $e$ and $f = \sum_i p_i$ be the numbers of vertices, edges and faces of $S$, where $p_i$ is the number of $i$-gonal faces. A graph is called $k$-connected if after removing and $k - 1$ vertices, it remains connected.

Clearly, $k$-regularity implies $kv = 2e = \sum_i ip_i$ and the Euler formula $2 = v - e + f$ become Gauss-Bonnet-like one $2 = \sum_i \kappa_ip_i$, where $\kappa_i = 1 + \frac{i}{k} - \frac{2}{k}$ is called (dualizing the definition in [Hi01]) the curvature of the $i$-gonal faces.

Let $a = \min\{i \in R\}$. Then, besides the cases $k = 2$ ($a$-cycle) and exotic cases $a = 1, 2$, it holds

$$\frac{2k}{k - 2} > a > 2 < k < \frac{2a}{a - 2},$$

i.e., $(a,k)$ should belong to the five Platonic parameter pairs $(3,3)$, $(4,3)$, $(3,4)$, $(5,3)$ or $(3,5)$.

Call an $(R,k)$-sphere standard if $\min_{i \in R} \kappa_i = 0$, i.e., $b = \frac{2k}{k - 2}$, where $b$ denotes $\max\{i \in R\}$. Such spheres have $(b,k) = (3,6)$, $(4,4)$, $(6,3)$, i.e., the three Euclidean parameter pairs. Exclusion of faces of negative curvature simplifies enumeration, while the number $p_b$ of faces of curvature zero not being restricted, there is an infinity of such $(R,k)$-spheres.

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Figure 1. The decomposition of a (5, 15, 3)-sphere seen as a (5, 3)-polycycle with two, 15- and 18-gonal, holes, into elementary polycycles: two edge-split \(\{5, 3\}\)'s, \(P_{en_1}, C_2\) and \(\{5, 3\} - e\) in the middle.

An \((a, b, k)\)-sphere is an \((R, k)\)-sphere with \(R = \{a, b\}, 1 \leq a, b\). Clearly, all possible \((a, b; k)\) for the standard \((\{a, b\}, k)\)-spheres are:

\((5, 6; 3), (4, 6; 3), (3, 6; 3), (2, 6; 3), (3, 4; 4), (2, 4; 4), (2, 3; 6), (1, 3; 6)\).

Those eight families can be seen as spheric analogs of the regular plane partitions \(\{6, 3\}, \{4, 4\}, \{3, 6\}\) with \(p_a = \frac{2a}{a-2}\) \(a\)-gonal “defects” \(\kappa_a\) added to get the total curvature 2 of the sphere. \((5, 6), 3\)-spheres are (geometric) fullerences, important in Chemistry, while \((a, b, 4)\)-spheres are minimal projections of alternating links, whose components are their central circuits (those going only ahead) and crossings are the vertices.

We considered above eight families in [DeDu05, DeDuSh03, DHL02, DeSt03, DuDe11, DeDu11] and the book [DeDu08], where \(i\)-faces with \(\kappa_i < 0\) are allowed, mainly in Chapters 15-19. Here we consider three new natural instances of \((\{a, b, c\}, k)\)-spheres, each allowing faces of negative curvature. The first Section describes such \((\{a, b\}, k)\)-spheres with \(p_b \leq 3\). The second Section concerns the icosahedrites, i.e., \((\{3, 4, 5\})\)-spheres, in which 4-gonal faces have \(\kappa_4 = -\frac{1}{2}\). The third Section treats \((\{a, b, c\}, k)\)-spheres with \(p_c = 1\), in which unique \(c\)-gonal face can be of negative curvature, especially, fullerene c-disks, i.e., \((\{5, 6, c\}, 3)\)-spheres with \(p_c = 1\).

Note that all \((R, k)\)-spheres with \(1, 2 \notin R\) and \(\kappa_i > 0\) for all \(i \in R\) have \(k = 4, 5\) or 3 and, respectively, \(\kappa_i = \frac{1}{4}\), \(\frac{1}{10}\) or \(\kappa_i \in \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\}\). So, they are only Octahedron, Icosahedron and eleven \((\{3, 4, 5\}, 3)\)-spheres: eight dual deltahedra, Cube and its truncations on one or two opposite vertices (Dürer octahedron).

2. The list of \((\{a, b\}, k)\)-spheres with \(p_b \leq 3\)

For \(a, k \geq 3\), a \((a, k)\)-polycycle is a plane graph whose faces, besides some disjoint pairwisely, including exterior one and called holes, are \(a\)-gons, and
Figure 2. \((a,k)\)-polycycles from Chapters 4,7 of [DeDu08] used in proofs of Theorems 2 and 3.
whose vertices have degree between 2 and \( k \) with vertices not on the boundary of holes being \( k \)-valent. Let as see any \( (a,b) \), \( k \)-sphere, after merging its adjacent \( b \)-gons in larger faces, as a \((a,k)\)-polycycle. For a given \((a,b), k\)-sphere, we can first remove the edges contained in two \( b \)-gons. If a vertex has a clockwise list of incident faces of the form \( b^{x_1}a^{y_1} \ldots b^{x_N}a^{y_N} \) with \( N \geq 2 \) and \( x_i, y_i \geq 1 \), then we split it into \( N \) different vertices. The remaining faces are \( a \)-gonal and are organized into one or more \((a,k)\)-polycycles with the pair \((a,k)\) being one of \((3,3), (3,4), (4,3), (3,5)\) and \((5,3)\).

The \((3,3)\)-, \((3,4)\)- and \((4,3)\)-polycycles are easily classified (see pages 45, 46 of [DeDu08]) and this gives a method for solving the problem of this section. For the remaining two cases, we have to introduce another method. An \((a,k)\)-polycycle is called elementary if it cannot be cut along an edge into two \((a,k)\)-polycycles. An \((a,k)\)-polycycle admits a unique decomposition into elementary \((a,k)\)-polycycles and the list of elementary \((5,3)\)-, \((3,5)\)-polycycles is given in pages 75, 76 of [DeDu08].

From the set of elementary polycycles, appearing in an \((a,b), k\)-sphere, we can form a decomposition graph with the vertices being the occurring polycycles and two polycycles adjacent if they share an edge, a vertex or are connected by an edge contained in two \( b \)-gons. The connecting vertices and edges are called active. See in Figure 1 an example of such a decomposition.

See the list of polycycles used in this paper on Figure 2. Here \( S_q, Tr_i \) denote horizontal paths of \( i \), respectively, squares and triangles, while series \( Pen_i, Sun_i \) are defined similarly. Note that \( Tr_2 = \{3,3\} - e \) and \( Tr_4 = \{3,4\} - P_4 \). We define vertex-split \( \{3,5\} \) as unique \((3,5)\)-polycycle obtained from \( a_3 \) by adjoining a \( Tr_1 \), and face-split \( \{3,5\} \) as unique \((3,5)\)-polycycle obtained from \( c_1 \) by adding two \( Tr_1 \) on two opposite edges.

Note that the 5-gons of unique minimal fullerene \( c \)-disk with \( e = 4, 7, 8 \) and \( c \geq 13 \) given in Figure 18 are organized, respectively, into edge-split \( \{5,3\}, C_3 + Pen_7, Pen_7 + Pen_7 \) and \( B_2 + Pen_{c-12} + B_2 \).

**Theorem 1.** There is no \((a,b), k\)-sphere with \( p_b = 1 \).

**Proof.** The decomposition graph of such spheres is a tree. If this tree is reduced to a vertex, then the occurring polycycle has an exterior face being a \( b \)-gon and an examination of the possibilities gives \( a = b \). Otherwise, we have at least one polycycle with a unique active vertex or edge. But an inspection of the list of elementary polycycles gives that no such one satisfies the condition. \( \Box \)

Note that \((6,8), 3)\)-maps with \( p_b = 1 \) exist on an oriented surface of genus 3.

Clearly, all \((a,b), k\)-spheres with \( a, k > 2 \) and \( p_b = 0 \) are five Platonic ones denoted by \([a,k]\): Tetrahedron, Cube \((Prism_4)\), Octahedron \((APrism_3)\), Dodecahedron \((snub Prism_5)\) and Icosahedron \((snub APrism_3)\).
There exists unique 3-connected trivial \((a, b, k)\) -sphere with \(p_b = 2\) for \((4, b), 3\)-, \((3, b), 4\)-, \((5, b), 3\)-, \((3, b), 5\)-: \(Prism_b\), \(APrism_b\), \(D_{bd}\), snub \(Prism_b\), \(D_{bd}\), snub \(APrism_b\), \(D_{bd}\), i.e., respectively, two \(b\)-gons separated by \(b\)-ring of \(4\)-gons, \(2b\)-ring of \(3\)-gons, two \(b\)-rings of \(5\)-gons, two \(3b\)-rings of \(3\)-gons.

Clearly, for any \(t \geq 2\), there are 2-connected \(b\)-vertex \((\{2, b = 2t\}, 3\)- and \((\{2, b\}, 2t\)-sphere with \(p_b = 2\): a circle with \(t\) disjoint \(2\)-gons put on it and a \(b\)-gon with every edge repeated \(t\) times.

**Theorem 2.** Let \(b > a > 2\), \(k > 2\). For any non-trivial \((a, b, k)\)-sphere with \(p_b = 2\), the number \(t = \frac{b}{a}\) is an integer. The list of such spheres consists of following 10 spheres for each \(t \geq 2\):

(i) For \((a, k) = (3, 3), (4, 3), (5, 3), (3, 4), (3, 5)\), the \((a, ta, k)\)-sphere \(D_{th}\) obtained by putting on a circle \(t\) polycycles \(a, k - e\). Those polycycles are connected by an edge to their neighbors and so, only 2-connected.

(ii) For \((a, k) = (3, 4), (5, 3), (3, 5)\), the \((a, ta, k)\)-sphere \(D_{th}\) obtained by partitioning of a circle into \(t\) polycycles: respectively, vertex-split \(\{3, 4\}\), edge-split \(\{3, 5\}\), edge-split \(\{3, 3\}\).

(iii) The \((\{3, 3t\}, 5\)-spheres \(C_{th}\), \(D_t\), obtained by partitioning of a circle into \(t\) polycycles: respectively, vertex-split \(\{3, 5\}\) \((Tr_1 + a_3)\) and face-split \(\{3, 5\}\) \((Tr_1 + c_1 + Tr_1)\).

**Proof.** If the two \(b\)-gonal faces are separated by an elementary polycycle, then we are in the case of the snub \(Prism_b\) or snub \(APrism_b\). Otherwise, the decomposition graph should contain one cycle separating two \(b\)-gons. Any nontrivial path connected to this cycle would have a vertex of degree 1 which we have seen to be impossible. So, the decomposition graph is reduced to this cycle. Examination of the list of polycycles with exactly two connecting edges/vertices and consideration of all possibilities gives the above list.

Among above spheres, only those coming from edge-split \(\{3, 3\}\) edge-split \(\{3, 5\}\) and face-split \(\{3, 5\}\) are 3-connected. Those coming from vertex-split \(\{3, 4\}\) and vertex-split \(\{3, 5\}\) are 3-edge connected, but only 2-(vertex)-connected.

Let us address now the case \(p_b = 3\). Note that \(3 \times K_2\) with \(t\) disjoint \(2\)-gons put on each edge is a \((\{2, b = 2 + 4t\}, 3\)-sphere with \(p_b = 3\).

**Theorem 3.** Let \(b \geq 2, b \neq a\) and let \((a, k) = (3, 3), (4, 3), (3, 4), (5, 3)\). Then \((a, b, k)\)-spheres with \(p_b = 3\) exists if and only if \(b \equiv 2, a, 2a - 2\) \((\mod 2a)\) and \(b \equiv 4, 6\) \((\mod 10)\) for \(a = 5\).

Such spheres are unique if \(b \equiv a\) \((\mod 2a)\) and their symmetry is \(D_{sh}\). Let \(t = \lfloor \frac{b}{a} \rfloor\). There are seven such spheres with \(t = 0\) and \(3 + 4 + 5 + 9\) of them for any \(t \geq 1\); see corresponding Figures 4 5 6 7 and detailed description below.
Figure 3. All non-trivial \((a, ta, k)\)-spheres with \(p_{2a} = t = 2\); see Theorem 2.

(i) The three \((3, b, 3)\)-sphere with \(p_b = 3\) and \(b = 2 + 6t, 4 + 6t, 3 + 6t\) come by putting \(t\) polycycles \(\{3, 3\} - e\) on 3 edges of, respectively, \(3 \times K_2, \text{Prism}_3\) (three 4-4 edges) and Tetrahedron \(\{3, 3\}\). Only for \(b = 2, 4\), the graph is 3-connected. The symmetry is \(C_{3v}\), if \(b = 3 + 6t\).

(ii) All four but one \((4, b, 3)\)-spheres with \(p_b = 3\) and \(b = 2 + 8t, 6 + 8t, 4 + 8t\) come by putting \(t\) polycycles \(\{4, 3\} - e\) on 3 edges of, respectively, \(3 \times K_2, 14\)-vertex \((4, 6, 3)\)-sphere (three 6-6 edges) and Cube \(\{4, 3\}\) (3 incident edges). The remaining sphere is \(S_{q_{8t-1}}\) with its two end edges connected on each of two sides by a chain of \(t\) polycycles \(\{4, 3\} - e\). This graph has symmetry \(C_{2v}\), while other graph coming from Cube has symmetry \(C_{3v}\). Only for \(b = 6\), the graph is 3-connected.

(iii) All five but two \((3, b, 4)\)-spheres with \(p_b = 3\) and \(b = 2 + 6t, 4 + 6t, 3 + 6t\) come when replacing by \(t\) vertex-split \(\{3, 4\}\)’s, 3 vertices of, respectively, edge-doubled triangle, 9-vertex \((3, 4, 4)\)-sphere (3 vertices common to two 4-gons) and Octahedron \(\{3, 4\}\) (3 vertices of a triangle). First of remaining spheres consists of \(Tr_{6t+5}\) connected to its other end on one side by a chain of \(t\) vertex-split \(\{3, 4\}\)’s and on the other side by a chain of \(t\) polycycles \(\{3, 4\} - e\). Second remaining sphere consists of a vertex and \(\{3, 4\} - P_3\).
connected by two chains of \(t\) polycycles \(\{3, 4\} - e\) and one chain of \(t\) vertex-split \(\{3, 4\}\)’s. Those graphs have symmetry \(C_3\), while other graph coming from Octahedron has symmetry \(C_{3v}\). Only for \(b = 2\) and \(b = 4\), the graphs are 3-connected.

There are nine \((5, b), 3\)-spheres with \(p_b = 3\) for each \(t = \lfloor \frac{b}{10} \rfloor \geq 1\).

(a) Three spheres with \(b = 2 + 10t, b = 8 + 10t, b = 5 + 10t\) come by putting \(t\) polycycles \(\{5, 3\} - e\) on 3 edges of \(3 \times K_2\), 22-vertex \((5, 8), 3\)-sphere (three 8-8 edges) and Dodecahedron \(\{5, 3\}\) (3 incident edges). Symmetry is \(C_{3v}\) in last case and \(D_{3h}\), otherwise. Only for \(b = 8\), the graph is 3-connected.

(b) Three 3-connected spheres with \(b = 4 + 10t, 6 + 10t, 5 + 10t\) come by putting 3 chains of \(t\) edge-split \(\{5, 3\}\)’s between two polycycles, \(\{\text{Pen}_3, \text{Pen}_3\}\), \(\{C_3, C_3\}\) and \(\{\text{Pen}_3, C_3\}\). Symmetry is \(C_{3v}\) in last case and \(D_{3h}\), otherwise.

(c) Three 2-connected \((5, 5 + 10t), 3\)-spheres of symmetry \(C_3\) come from, respectively:

\[c_1\) a chain \(B_2 + t\) times edge-split \(\{5, 3\}\) connected on both ends by \(t\) times \(\{5, 3\} - e\);

\[c_2\) \(C_2 + t(\{5, 3\} - e) + \text{Pen}_1\) connected on both ends by \(t\) times edge-split \(\{5, 3\}\);

\[c_3\) \(\text{Pen}_{10t+9}\) connected on one side by \(t\) times edge-split \(\{5, 3\}\) and on the other one by \(t\) times \(\{5, 3\} - e\).

Proof. It is not possible to have an elementary polycycle separating three \(b\)-gonal faces. Hence, the decomposition graph has three faces and is either formed of two vertices of degree 3 connected by chains of vertices of degree 2, or a vertex of degree 4 connected by two chains of vertices of degree 2 on each side. An examination of the possibilities along the same lines gives the above result. \(\square\)

Note that all \((\{a, b\}, k)\)-spheres with \(p_b = 3\) and symmetry \(\neq C_3\), are \(bR_j\) (i.e., each \(b\)-gon has exactly \(j\) edges of adjacency with \(b\)-gons); see face-regularity in Section 3. \(j = 2 \left\lfloor \frac{b}{2a} \right\rfloor\) for \((a, k) = (3, 3), (4, 3)\) and \(j = 0\) for \((a, k) = (3, 4), (3, 5)\). For \((a, k) = (5, 3)\), we have \(j = 0\) or \(2 \left\lfloor \frac{b}{2a} \right\rfloor\).

In case \((a, k) = (3, 5)\), we have 17 infinite series of spheres but no proof that the list is complete. See below the list of \((\{3, b\}, 5)\)-spheres obtained and in Figure 8, 9 their pictures for small \(b\). All but \((a)\) have \(b = 3 + 6t\). All but \((a)\) and \((d_2)\) are only 2-connected. By \(R_e, V_{sp}, E_{sp}\), and \(F_{sp}\) we denote a chain of \(t\) polycycles \(A\) with \(A\) being \(\{3, 5\} - e\), vertex-split \(\{3, 5\}\), edge-split \(\{3, 5\}\) and face-split \(\{3, 5\}\), respectively.

(a) Three spheres with \(b = 2 + 6t, 4 + 6t, 3 + 6t\) obtained by putting three \(E_{sp}\) between two polycycles, \(\{Tr_1 + 3Tr_1, Tr_1 + 3Tr_1\}\), \(\{c_4 +...\)
Note also that all 15 orbits of \((3, b, 3)\)-spheres with \(p_b = 3\) for \(2 \leq b \leq 10\). Symmetry is \(C_3\) in last case and \(D_3\), otherwise.

\((b_1)\) \(C_1\): \(c_1 + Tr_1 + F_{sp}\) with ends connected by \(R_e\) and \(V_{sp}\).

\((b_2)\) \(C_s\): \(Sun_{6s+5}\) with ends connected by \(R_e\) and \(E_{sp}\).

\((b_3)\) \(C_1\): \(b_3 + Tr_1 + E_{sp}\) with ends connected by \(R_e\) and \(F_{sp}\).

\((b_4)\) \(C_s\): \(b_3 + 2Tr_1 + E_{sp}\) with ends connected by \(V_{sp}\) and \(E_{sp}\).

\((b_5)\) \(C_1\): \(Sun_{4s+6r} + 3Tr_1\) with ends connected by \(V_{sp}\) and \(F_{sp}\).

\((c_1)\) \(C_s\): a vertex and \(\{3, 5\} - v\) + \(2Tr_1\) connected by \(R_e\) and two \(V_{sp}\)’s.

\((c_2)\) \(C_s\): a vertex and \(a_4\) connected by two \(R_e\)’s and one \(V_{sp}\).

\((c_3)\) \(C_s\): a vertex and \(c_2 + 2Tr_1\) connected by two \(V_{sp}\)’s and one \(F_{sp}\).

\((d_1)\) \(C_s\): \(Sun_1\) and \(Sun_3 + Tr_1\) connected by \(V_{sp}\) and two \(E_{sp}\)’s.

\((d_2)\) \(C_s\): \(Sun_2\) and \(Sun_1 + 2Tr_1\) and connected by \(E_{sp}\) and two \(F_{sp}\)’s.

\((e_1)\) \(C_{3v}\): \(Tr_1\) and \(a_4\) connected by three \(V_{sp}\)’s.

\((e_2)\) \(C_s\): \(Tr_1\) and \(c_2\) connected by \(R_e\) and two \(F_{sp}\)’s.

\((e_3)\) \(C_s\): \(2Tr_1\) and \(c_3 + 3Tr_1\) connected by \(V_{sp}\) and two \(F_{sp}\)’s.

\((e_4)\) \(C_1\): \(3Tr_1\) and \(b_4 + Tr_1\) connected by \(V_{sp}\), \(E_{sp}\) and \(F_{sp}\).

Note that all 15 orbits of \((\{3, 3 + 6t\}, 5)\)-spheres constructed were built from the 15 orbits of triangles of Icosahedron. Note also that \(\binom{20}{3} = 5 \times 120 + 8 \times 60 + 40 + 20\), since there are 5, 8, 1, 1 cases with symmetry \(C_1, C_s, C_3, C_{3v}\), respectively. There may be other spheres in this case and for other values of \(b\).
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Figure 6. All \( (\{3, b\}, 4) \)-spheres with \( p_b = 3 \) for \( 2 \leq b \leq 14 \)

3. Icosahedrites

We call icosahedrite the \((\{3, 4\}, 5)\)-spheres. They are in a sense the simplest 5-valent plane graphs. Clearly, for them it holds \( p_3 = 20 + 2p_4 \) and \( v = 12 + 2p_4 \). Note that all \((\{a, 3\}, 5)\)-spheres with \( a < 3 \) are: \((\{1, 3\}, 5)\)-sphere with \((p_1, p_3; v) = (2, 6; 4)\) and four \((\{2, 3\}, 5)\)-spheres with \((p_2, p_3; v) = (4, 4; 4)\) (two), \((3, 8; 6)\), \((2, 12; 8)\). Remaining \((\{1, 2, 3\}, 5)\)-spheres should have \((p_1, p_2, p_3; v) = (1, 3, 1; 2), (2, 1, 2; 2), (1, 2, 5; 4)\) or \((1, 1, 9; 6)\); only 2nd and 3rd exist.

The simplest icosahedrite is Icosahedron, which is a \((\{3\}, 5)\)-sphere of symmetry \(I_h\). One way to obtain icosahedrite is from an octahedrite, i.e. a \((\{3, 4\}, 4)\)-sphere \(G\). To every vertex of \(G\) we associate a square, to every edge (coherently) a pair of adjacent triangles and faces are preserved. Only the rotational symmetries of \(G\) are preserved in the final icosahedrite \(C(G)\). If one applies \(C\) to Octahedron, then one gets the smallest (24 vertices) icosahedrite of symmetry \(O\) (see Figure 14). Applying it to the infinite regular plane tiling \(\{4, 4\}\) by squares, one gets the Archimedean snub square tiling \((3.3.3.4.3.4)\). Note that there is only one other infinite Archimedean icosahedrite, i.e., a vertex-transitive 5-valent tiling of the plane by regular 3- and 4-gons only: elongated triangular tiling \((3.3.3.4.4)\).

For a given icosahedrite, a weak zigzag \(WZ\) is a circuit of edges such that one alternate between the left and right way but never extreme left or right. The usual zigzag is a circuit such that one alternate between the extreme left and extreme right way.
A zigzag or weak zigzag is called *edge-simple*, respectively *vertex-simple*, if any edge, respectively vertex, of it occurs only once. A vertex-simple zigzag is also edge-simple. If $WZ$ is vertex-simple of length $l$, then one can construct another icosahedrite with $l$ more vertices.

Clearly, the weak zigzags (as well as the usual zigzags) doubly cover the edge set. For example, 30 edges of Icosahedron are doubly covered by 10 weak vertex-simple zigzags of length 6, as well as by 6 usual vertex-simple zigzags of length 10. Clearly, a 3-gon surrounded by 9 3-gons or a 4-gon surrounded by 12 3-gons corresponds to weak zigzags of length six or eight, respectively. In fact, no other vertex-simple weak zigzags exist.

In Table 1 we list the number of $v$-vertex icosahedrites for $v \leq 32$. From this list it appears likely that any icosahedrite is 3-connected.

**Theorem 4.** A $([3, 4], 5)$-sphere exist if $v$ is even, $v \geq 12$ and $v \neq 14$. Their number grows at least exponentially with $v$. 

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**Figure 7.** All $([5, b], 3)$-spheres with $p_b = 3$ for $2 \leq b \leq 15$
Figure 8. All known \((3, b, 5)\)-spheres with \(p_b = 3\) and \(2 \leq b \leq 10\) (part 1)
Figure 9. All known \(([3, b], 5)\)-spheres with \(p_b = 3\) and \(2 \leq b \leq 10\) (part 2)

Table 1. The number of \(v\)-vertex icosahedrites

| \(v\) | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| \(\text{Nr}\) | 1 | 0 | 1 | 1 | 5 | 12 | 63 | 246 | 1395 | 7668 | 45460 |
Proof. If there is an weak zigzag of length 6 in an icosahedrite $G$, then we can insert a corona (6-ring of three 4-gons alternated by three pairs of adjacent 3-gons) instead of it and get an icosahedrite with 6 more vertices. Since there are such icosahedrites for $n = 18, 20, 22$ (see Figure 10, 11), we can generate the required graphs. There is always two options when inserting the corona and so the number grows exponentially as required. □

In the case of fullerenes or other standard $\{a, b\}, k$-spheres, the number of $a$-gons is fixed and the structure is made up of patches of $b$-gons. The parametrization of graphs, generalizing one in [Thur98], including the case of one complex parameter - Goldberg-Coxeter construction ([DuDe04]) - are thus built. A consequence of this is the polynomial growth of the number of such spheres. This does not happen for the case of icosahedrites and their parametrization, if any, looks elusive.

As a consequence of this increased freedom, we can more easily build a new icosahedrite from a given one by an expansion operation, while in the case of standard $\{a, b\}, k$-spheres, one is essentially restricted to the Goldberg-Coxeter construction. The operation $A$, respectively $A'$, replaces each vertex of an icosahedrite $G$ by 6, respectively 21, vertices and gives icosahedrites $A(G), A'(G)$ with the same symmetries as $G$ (see Figure 12). Moreover, for any $m \geq 2$ we can define an operation that replaces every face $F$ by a patch and add $m - 1$ rings of squares and pairs of triangles. The operation $B_2$ on a 4-gon is shown in Figure 12. The resulting map $B_m(G)$ has only the rotational symmetries of $G$ and associate to every vertex of $G$ $1 + 5m(m - 1)$ vertices in $B_m(G)$.

**Theorem 5.** Any symmetry group of icosahedrites if one of following 38:

$C_1, C_i, C_s, S_4, S_6, S_8, S_{10}, C_2, C_{2h}, C_{2v}, C_3, C_{3h}, C_{3v}, C_4, C_{4h}, C_{4v}, C_5,$
$C_{5h}, C_{5v}, D_2, D_{2h}, D_{2d}, D_3, D_{3h}, D_{3d}, D_4, D_{4h}, D_{4d}, D_5, D_{5h}, D_{5d}, O, O_h, T,$
$T_d, T_h, I, I_h.$
Proof. By the face sizes and vertex sizes, the list of possibilities is the one indicated. We used the enumeration up to 32 vertices to find many groups and their minimal representative. If a sphere has a 3-fold axis, then it necessarily passes through two 3-gons. Those two 3-gons can be replaced by 4-gons and ipso facto we get examples of a sphere with 4-fold axis. From this we got the icosahedrites with groups $C_4, C_{4h}, C_{4v}, D_4, D_{4h}, D_{4d}$ and $S_8$. All but 40-vertex one $C_4, C_{4v}, D_{4h}$ are minimal.

Now, suppose that a 2-fold axis of rotation passes by two edges, which are both contained in two triangles. Then we insert a vertex on those two edges and replace the 2-fold symmetry by a 5-fold symmetry. By iterating over all known isocahedrites, and all such 2-fold axis, we get the symmetries $C_5, C_{5h}, C_{5v}, S_{10}$ and $D_{5h}, D_{5d}, D_5$; last three are minimal. For the cases of $O_h, T, T_d$ and $T_h$, we obtained examples by hand drawing. The 132 vertices icosahedrite of symmetry $I$ in Figure 14 is obtained from Isocahedron by operation $B_2$. The minimal known (actually, examples are minimal whenever $v \leq 32$) are given in Figure 13, 14.

Aggregating groups $C_1=\{C_1, C_5, C_i\}, C_m=\{C_m, C_{mv}, C_{mh}, S_{2m}\}, D_m=\{D_m, D_{mh}, D_{md}\}, T=\{T, T_d, T_h\}, O=\{O, O_h\}, I=\{I, I_h\}$, all 38 symmetries of $\{3, 4\}, 5$-spheres are: $C_1, C_m, D_m$ for $2 \leq m \leq 5$ and $T, O, I$. 5-, 4- and 3-fold symmetry exists if and only if, respectively, $p_4 \equiv 0 \pmod{5}$ (i.e., $v = 2p_4 + 12 \equiv 2 \pmod{10}$), $p_4 \equiv 2 \pmod{4}$ (i.e., $v = 2p_4 + 12 \equiv 0 \pmod{8}$) and $p_4 \equiv 0 \pmod{3}$ (i.e., $v = 2p_4 + 12 \equiv 0 \pmod{6}$).
Figure 13. Examples of icosahedrites for all possible symmetry groups (part 1); all 21 with at most 32 vertices are minimal ones.
Any group appear an infinite number of times since, for example, one gets an infinity by applying construction $A$ iteratively.

From the above result it appears that the only limitations for the group are coming from the rotation axis. It seems possible that this also occurs for all $((a, b), k)$-spheres with $b$-gons being of negative curvature.

A map is said to be face-regular, or, specifically, $pR_i$, if every face of size $p$ is adjacent to exactly $i$ faces of the same size $p$. 

**Figure 14.** Examples of icosahedrites for all possible symmetry groups (part 2); all 5 with at most 32 vertices are minimal ones
Theorem 6. (i) The only icosahedrite which is 3R$_i$ is Icosahedron which is 3R$_3$
(ii) For $i = 0, 1$ and 2 there is an infinity of icosahedrites that are 4R$_i$.

Proof. Let $N_{ij}$ denote the number of $i$-j edges, i.e., those which are common to an $i$-gon and an $j$-gon; so, $e = N_{33} + N_{34} + N_{44}$. But $N_{34} + N_{44} \leq 4p_4$ with equality if and only if our icosahedrite is 4R$_0$. So,

$$2e = 3p_3 + 4p_4 = p_3 + 2(20 + 2p_4) + 4p_4 \leq 2N_{33} + 8p_4,$$

i.e., $2N_{33} \geq p_3 + 40$. It excludes the cases 3R$_0$ and 3R$_1$, since then $2N_{33} = 0$ and $p_3$, respectively.

If our icosahedrite is 3R$_2$, then $N_{33}+N_{34} = p_3+p_3$, implying $N_{44} = p_4-10$. Any 4R$_0$ icosahedrite with 32 vertices (i.e., with $p_4 = 10$) has $\frac{N_{33}}{p_3} = 2$, i.e., it is 3R$_2$ in average. Now, 3R$_2$ means that the 3-gons are organized in rings separated by 4-gons. Such rings can be either five 3-gons with common vertex, or 12 3-gons around a 4-gon, or APrism$_m$, $m > 4$. In each case, such ring should be completed to an icosahedrite by extruding edge to keep 5-regularity. In order to be isolated from other 3-gons, the ring should have twice longer ring of 4-gons around it. The faces touching the ring of 3-gons in a vertex only, could not be 3-gonal since they have two 4-gonal neighbors. So, the isolating ring of 4-gons consists of 4-gons adjacent (to the ring of 3-gons) alternated by 4-gons touching it only in a vertex. Easy to see that such process cannot be closed.

The number of icosahedrites 4R$_0$ is infinite; such series can be obtained by the operation $A$ (see Figure 12) from, say, unique 16-vertex icosahedrite. An infinity of 4R$_1$ icosahedrites can be obtained in the following way. We take the 4R$_1$ icosahedrite of Figure 15 and the ring bounded by two over-lined circuits can be transformed into any number of concentrated rings. An infinity of icosahedrites 4R$_2$ can be obtained from the 22-vertices icosahedrite of symmetry $D_{5h}$ in Figure 14. It suffices to add layers of five 4-gons alternated by layers of ten 3-gons (as in Archimedean elongated triangular plane tiling (3.3.3.4.4)).

Among all 54, 851 icosahedrites with $v \leq 32$, there are no 4R$_3$ and only four specimens 4R$_2$. Those with 4R$_3$ should have $2N_{33} = 3p_3 - p_4$ and $v$ divisible by 4, but we doubt they exist. Note that part (i) of Theorem 6 can be easily generalized on ([3, b], 5)-sphere with any $b \geq 4$.

All icosahedrites with only edge-simple usual zigzags and $v \leq 32$ are three of those seven (see Figure 16) having only edge-simple weak zigzags: 12, $I_h$ with (10$^3$), 24, $D_{3d}$ with (6, 10$^6$, 18$^3$) and 28, $D_{2h}$ with (10$^6$, 20$^4$). We expect that any icosahedrite with only edge-simple usual or weak zigzags, if such ones exist for $v \geq 34$, has $v$ divisible by four. Also, for $v \leq 32$, the
Figure 15. The icosahedrite used in proof of Theorem 6 as the first one in an infinite series with $4R_1$

maximal number of zigzags and weak zigzags are realized only by edge-simple ones whenever they exist.

Snub $APrism_b$, Snub Cube and Snub Dodecahedron with $(v, b) = (4b, b)$, (24, 4) and (60, 5), respectively, are $bR_0$ and only known $b$-gon-transitive $([3, b], 5)$-spheres. They are also, besides Icosahedron, only known $([3, b], 5)$-spheres with at most two orbits of 3-gons. The Archimedean $([3, b], 5)$-plane tilings (3.3.4.3.4), (3.3.3.4.4), (3.3.3.3.6) have $b = 4$, 4, 6, respectively. They are transitive on $b$-gons and vertices. They are also, respectively, $(4R_1, 3R_1)$, $(4R_2, 3R_2)$, $6R_0$ and have 1, 1, 2 orbits of 3-gons.
4. On \((a, b, c, k)\)-spheres with \(p_c = 1\)

Clearly, an \((a, b, c, k)\)-sphere with \(p_c = 1\) has

\[ v = \frac{2}{k-2}(p_a - 1 + p_b) = \frac{2}{2k - a(k - 2)}(a + c + p_b(b - a)) \]

vertices and (setting \(b' = \frac{2k}{k-2}\)) \(p_a = b'/b + p_b b'^{-1} a\)-gons. So, \(p_a = b'+c\) if \(b = b'\), i.e., \((a, b, k)\)-sphere is standard.

We are especially interested in fullerene \(c\)-disks, i.e., \((5, 6, c, 3)\)-spheres with \(p_c = 1\). It exists for any \(c \geq 1\) and has \(p_5 = c + 6, v = 2(p_6 + c + 5)\). Clearly, there is an infinity of fullerene \(c\)-disks for any \(c \geq 1\).

The only way to get fullerene 1-disks, is to get a \((5, 6, s, r)\), \(3\)-sphere with \(p_r = p_s = 1\), \((r, s) = (3, 4), (3, 3), (2, 3), (2, 4)\) and add 4 vertices: 2-gon on the \(r - s\) edge and then erect an edge with 1-gon from the middle of 2 - \(r\) edge. In fact, only \((3, 4)\) is possible and minimal such graph has 36 vertices, proving minimality of 40-vertex 1-disk.

One can check that all \((a, b, c, 3)\)-spheres with \(p_c = 1, 2 \leq a, b, c \leq 6\) and \(c \neq a, b\) are four series with \((a, b, c) = (4, 6, 2), (5, 6, 2), (5, 6, 3), (5, 6, 4)\) and following two 10-vertex spheres of symmetry \(C_{3v}\) having \(p_a = p_b = 3, p_c = 1\): \((4, 5, 3)\)- (Cube truncated on one vertex) and \((3, 5, 6)\)- (Tetrahedron truncated on three vertices). There are also series of \((1, 4, 2, 4)\)- and \((3, 4, 2, 4)\)-spheres with \(p_2 = 1\).

Theorem 1 implies that \((a, b, c, k)\)-sphere with \(p_c = 1\) and \(p_b = 0\) has \(c = a\), i.e., it is the \(k\)-regular map \(\{a, k\}\) on the sphere. We conjecture that a \((a, b, c, k)\)-sphere with \(p_c = p_b = 1 < a, c\) has \(c = b\), i.e., it is a \((\{a, b\}, k)\)-sphere with \(p_b = 2\). Note that the 2-vertex \((1, 4, 2, 4)\)-sphere has \(p_2 = p_4 = 1 = a < c < b\).

Call \((a, b, c, k)\)-thimble any \((a, b, c, k)\)-sphere with \(p_c = 1\) such that the \(c\)-gon is adjacent only to \(a\)-gons. Fullerene \(c\)-thimble is the case \((a, b, c; k) = (5, 6, c; 3)\) of above. It exists if and only if \(c \geq 5\).

Moreover, we conjecture that for odd or even \(c \geq 5\), the following \((5c - 5)\)- or \((5c - 6)\)-vertex \(c\)-thimble is a minimal one; it holds for \(5 \leq c \leq 10\) since this construction generalizes cases 7, 9 - 1 and 8, 10 - 3 in Figure 18 as well as cases \(c = 5\) and 6, of minimal \(c\)-disks. Take the \(c\)-ring of 5-gons, then put inside a concentric \(c\)-ring of 5- or 6-gons: 3-path of 5-gons and, on opposite side, 3-path or 3-ring, for even or odd \(c\), of 5-gons. Remaining \(c - 6\) or \(c - 5\) 5- or 6-gons of interior \(c\)-ring are 6-gons. Finally, fill inside of the interior \(c\)-ring by the \(\frac{5}{2}c\)- or \(\frac{3}{2}c\)-path of 6-gons. Another generalization of the minimal 6-disk is, for \(c = 6t\), \(\frac{c(c+18)}{6}\)-vertex \(c\)-thimble of symmetry (for \(c > 6\)) \(C_6\) or \(C_{6v}\). In fact, take \(6t\)-ring of 5-gons, then, inside of it, 6\(t\)-ring, where six equispaced 5-gons are alternated by \((t - 1)\)-tuples of 6-gons. Finally, put inside, for \(i = t - 1, t - 2, \ldots, 1\), consecutively 6\(i\)-rings of 6-gons.
Any such $c$-thimble can be elongated by adding an outside ring 5-gons along the $c$-gon and transforming inside ring of 5-gons along it into a ring of 6-gons. Let a $([5, 6, c], 3)$-sphere with $p_c = 1$ have a simple zigzag (left-right circuit without self-intersections). A railroad is a circuit of 6-gons, each of which is adjacent to its neighbors on opposite edges. Let us elongate above sphere by a railroad along zigzag and then let us cut elongated sphere in the middle of this ring. We will get two $c$-thimbles.

**Theorem 7.** All simple zigzags of any elongated $c$-thimble are parallel and its railroads are parallel $c$-rings of 6-gons, forming a cylinder.

In fact, suppose that there is a railroad in an $c$-thimble not belonging to the cylinder of parallel $c$-rings of 6-gons along the boundary $c$-ring of 5-gons.

Let us cut it in the middle. The thimble is separated into an smaller $c$-thimble with $p_5 = c$ and a $([5, 6, c], 3)$-sphere with $p_5 = 6$. But this sphere can not contain at least $c + 1$ of $c + 6$ original 5-gons: $c$ boundary 5-gons and, at other side of the cylinder, at least one 5-gon. Only a railroad belonging to the cylinder can go around this 5-gon. □

Given a $c$-disk, call its *type*, the sequence of gonalities of its consecutive neighbors. So, any $c$-thimble has type $(5, \ldots, 5)$.

There are bijections between $v$-vertex 1-disks, $(v - 2)$-vertex 2-disks of type $(5, 6)$ and $(v - 4)$-vertex $([5, 6, 3, 4], 3)$-spheres with unique and adjacent 3 and 4-gon. Using it, we found the minimal 1-disk, given in Figure 18. There is a bijection between $v$-vertex 3-disks of type $(5, 6, 6)$ and $(v - 2)$-vertex fullerenes with $(5, 5, 5)$-vertex (collapse the 3-gon). There is a bijection between $v$-vertex 3-disks of type $(5, 6, 6)$ and $(v - 2)$-vertex fullerenes with $(5, 5, 6)$-vertex (delete the edge adjacent to 5-gon). So, the possible types (and minimal number of vertices for examples) are: $(6) (v = 40)$ for 1-disks; $(6, 6) (v = 26)$, $(6, 5) (v = 40 - 2)$ for 2-disks; $(6, 6, 6) (v = 22)$, $(6, 5, 5) (v = 40 - 6)$, $(6, 6, 5) (v = 26$, see Figure 17) for 3-disks.
One can see unique \((5, 6, 3)\)-spheres with \(v = 20, 24\) as minimal fullerene \(c\)-disks with \(c = 5, 6\). Clearly, the minimal fullerene 3- and 4-disks are 22-vertex spheres obtained from Dodecahedron by truncation on a vertex or an edge. As well, \(C_1\)-minimal 3-disk and \(C_2\)-minimal 4-disk are truncations of 28-vertex \((D_2)\) or 24-vertex \((5, 6, 3)\)-sphere on, respectively, a vertex or an edge.

**Conjecture 1.** An \(v\)-vertex fullerene \(c\)-disk with \(c \geq 1\), except the cases \((c, v) = (1, 42), (3, 24)\) and \((5, 22)\), exists if and only if \(v\) is even and \(v \geq 2(p_6(c) + c + 5)\). Here \(p_6(c)\) denotes the minimal possible number of 6-gons in a fullerene \(c\)-disk.

We have \(p(1) = 14, p(2) = 6, p(3) = 3, p(4) = 2, p(5) = 0, p(6) = 1, p(7) = 3, p(8) = 4, p(9) = 6, p(10) = 7, p(11) = 8, p(12) = 5\), and \(p(c) = 6\) for \(c \geq 13\). For \(2 \leq c \leq 20\), the conjecture was checked by computation and all minimal fullerene \(c\)-disks \((2, 3, 10\) for \(c = 9, 10, 11\) and unique, otherwise) are listed; see Figure 18 for \(1 \leq c \leq 14\), \(c \neq 5, 6\). All \(c\)-disks there having \(c \geq 3\), except three among 11-disks, are 3-connected.

The \(c\)-pentatube is \(2(c + 11)\)-vertex fullerene \(c\)-disk of symmetry \(C_2\), \(C_s\) for even, odd \(c\), respectively. Its 5-gons are organized in two \((5, 3)\)-polycycles \(B_2\) separated by \(Pen_{c-12}\) and its six 6-gons are organized into two 3-rings each shielding a \(B_2\) from \(Pen_{c-12}\). The \(c\)-pentatube is unique minimal for \(13 \leq c \leq 20\); we expect that it remains so for any \(c \geq 13\).

**Theorem 8.** The possible symmetry groups of a \((5, 6, c)\)-sphere with \(p_c = 1\) and \(c \neq 5, 6\) are \(C_n, C_{nv}\) with \(n \in \{1, 2, 3, 5, 6\}\) and \(n\) dividing \(c\).

In fact, any symmetry of a \((5, 6, c)\)-sphere should stabilize unique c-gon. So, the possible groups are only \(C_n\) and \(C_{nv}\) with \(n (1 \leq n \leq c)\) dividing \(c\). Moreover, \(n \in \{1, 2, 3, 5, 6\}\) since, on the axis has to pass by a vertex, edge or face. Remind that \(C_s = C_{1v}\). □

Cases (xi) and (xvi) in [DDFo09] show that the possible symmetry groups (minimal \(v\)) of a fullerene \(c\)-disk with \(c = 3\) and \(c = 4\) are \(C_1\) (30), \(C_s\) (26), \(C_3\) (34), \(C_{3v}\) (22) and \(C_1\) (28), \(C_s\) (24), \(C_2\) (26), \(C_{2v}\) (22). Minimal examples are given there. For \(c = 1, 2, 7, 8, 9\) see minimal examples on Figure 19. For \(c = 2\), those examples are all of type \((6, 6)\) and coming from one-edge truncation of \((5, 6, 3)\)-spheres with, respectively, 28 \((D_2)\), 26, 24 and 20 vertices.

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Figure 18. Minimal fullerene $c$-disks with $1 \leq c \leq 14$, $c \neq 5, 6$
FULLERENE-LIKE SPHERES WITH FACES OF NEGATIVE CURVATURE

Figure 19. Minimal fullerene $c$-disks for each possible group with $c = 1, 2, 7, 8, \text{ and } 9$

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