THE $q$-BANNAI-ITO ALGEBRA AND MULTIVARIATE
$(-q)$-RACAH AND BANNAI-ITO POLYNOMIALS

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Abstract. The Gasper and Rahman multivariate $(-q)$-Racah polynomials appear as connection coefficients between bases diagonalizing different abelian subalgebras of the recently defined higher rank $q$-Bannai-Ito algebra $A^q_n$. Lifting the action of the algebra to the connection coefficients, we find a realization of $A^q_n$ by means of difference operators. This provides an algebraic interpretation for the bispectrality of the multivariate $(-q)$-Racah polynomials, as was established in Iliev, Trans. Amer. Math. Soc., 363(3) (2011), 1577–1598.

Furthermore, we extend the Bannai-Ito orthogonal polynomials to multiple variables and use these to express the connection coefficients for the $q = 1$ higher rank Bannai-Ito algebra $A_n$, thereby proving a conjecture from De Bie et al., Adv. Math. 303 (2016), 390–414. We derive the orthogonality relation of these multivariate Bannai-Ito polynomials and provide a discrete realization for $A_n$.

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1. Introduction

1.1. The Bannai-Ito algebra and its generalizations. The Bannai-Ito algebra is the associative algebra over \( \mathbb{C} \) with three generators \( \Gamma_{12}, \Gamma_{23}, \Gamma_{13} \) and relations

\[
\{ \Gamma_{12}, \Gamma_{23} \} = \Gamma_{13} + \alpha_{13}, \quad \{ \Gamma_{12}, \Gamma_{13} \} = \Gamma_{23} + \alpha_{23}, \quad \{ \Gamma_{13}, \Gamma_{23} \} = \Gamma_{12} + \alpha_{12},
\]

where \( \alpha_{ij} \) are structure constants. It was introduced in [47] as the algebraic structure underlying the Bannai-Ito orthogonal polynomials, which sit atop the so-called \((-1\text{-Askey scheme})\). It appeared also in [11] as the symmetry algebra of the three-dimensional Dirac-Dunkl operator with \( \mathbb{Z}_3^2 \) reflection group. A central extension of this algebra can be embedded in the threefold tensor product of the Lie superalgebra \( \mathfrak{osp}(1\vert 2) \) [21]. This observation allowed to generalize the Bannai-Ito algebra in several ways.

One possibility is to \( q \)-deform the underlying Lie superalgebra to the quantum superalgebra \( \mathfrak{osp}_q(1\vert 2) \). This led to the construction of a \( q \)-deformed Bannai-Ito algebra \( \mathcal{A}_3^q \) in [22], where its connection with the univariate \((-q\text{-Racah polynomials})\) was uncovered. The algebra \( \mathcal{A}_3^q \) turned out to be isomorphic to the universal Askey-Wilson algebra [44], which is a central extension of the Askey-Wilson or Zhedanov algebra [51], under a transformation \( q \rightarrow -q^2 \).

A second generalization was proposed in [10], where the connection with Dirac-Dunkl operators was used to extend the Bannai-Ito algebra to higher rank. The resulting algebra \( \mathcal{A}_n \) was observed to be the symmetry algebra of a superintegrable system with reflections on the \( n \)-sphere [8] and of the \( \mathbb{Z}_n^2 \) Dirac-Dunkl equation. Generalizations to other reflection groups were obtained in [20] and [12]. Other types of higher rank extensions, notably in connection with more general Lie algebras and quantum groups, are still actively investigated. Recently a classical Askey-Wilson algebra associated to the Lie algebra \( \mathfrak{sl}_n \) was constructed in [4], see also [41].

The two mentioned techniques were combined in our recent work [7]. We introduced the higher rank \( q \)-Bannai-Ito algebra \( \mathcal{A}_n^q \), which is both a higher rank extension of \( \mathcal{A}_3^q \) and a \( q \)-deformation of \( \mathcal{A}_n \), and which will be at the center of our attention throughout this paper. To each subset \( A \subseteq \{1, \ldots, n\} \) we associated an
algebra generator $\Gamma^q_A$, yielding the algebraic relations
\begin{equation}
\{\Gamma^q_A, \Gamma^q_B\}_q = \Gamma^q_{(A\cup B)\setminus(A\cap B)} + (q^{1/2} + q^{-1/2}) \left( \Gamma^q_{A\cap B} \Gamma^q_{A\cup B} + \Gamma^q_{A\setminus(A\cap B)} \Gamma^q_{B\setminus(A\cap B)} \right),
\end{equation}
where $\{A, B\}_q = q^{1/2} AB + q^{-1/2} BA$ is the $q$-anticommutator and $A, B$ are sets of integers between 1 and $n$ subject to certain conditions. As in the rank one case, this algebra can equally be considered a higher rank universal Askey-Wilson algebra. A different extension method was studied in [36] for the rank two case. The algebra $\mathcal{A}^q_n$ is also closely related to superintegrable quantum systems. In [7] we established a model of $q$-difference operators and reflections which has the higher rank $q$-Bannai-Ito algebra as its symmetry algebra. We will repeat the definition of this so-called $\mathbb{Z}^n_q$ $q$-Dirac-Dunkl model, as well as our extension algorithm for multifold tensor products in Sections 2.1 and 2.2.

We will consider a class of abelian subalgebras for $\mathcal{A}^q_n$ and to each such subalgebra we will attach a set of joint eigenvectors in a positive discrete series representation of $\mathfrak{osp}_q(1|2)^{\otimes n}$. We give an explicit construction for these vectors in Definition 5 and motivate this definition throughout Section 2.3. They are to be compared with the canonical bases for finite-dimensional $U_q(\mathfrak{sl}_2)$-modules obtained in [16] and [28]. The overlap coefficients between two such vectors, each corresponding to a different abelian subalgebra, will be expressed in Theorem 1 in terms of a special class of $q$-orthogonal polynomials, namely the Gasper and Rahman multivariate $(-q)$-Racah polynomials [18]. As such, these polynomials arise as $3nj$-symbols for $\mathfrak{osp}_q(1|2)$, in the sense of [29] and [38].

1.2. Orthogonal polynomials in multiple variables. The $q$-Racah polynomials are amongst the most general of all $q$-hypergeometric orthogonal polynomials. They can be obtained upon reparametrization and truncation from the Askey-Wilson polynomials, which are at the top of the $q$-Askey scheme and hence have all other $q$-orthogonal polynomials as limiting cases. Two different methods have been proposed to extend the Askey-Wilson and $q$-Racah polynomials to multiple variables.

One possibility stems from the theory of symmetric functions and was exploited by Koornwinder [32] based on previous work by Macdonald [34]. Restricting the orthogonality measure to a discrete support, one obtains a class of multivariable $q$-Racah polynomials [42]. These Macdonald-Koornwinder polynomials and their connection to affine Hecke algebras have been intensively studied, see for example [40].

A second type of extension, by means of coupled products of univariate Askey-Wilson polynomials, was introduced by Gasper and Rahman [17], based on a construction by Tratnik [45] for multivariable ($q = 1$) Racah polynomials. Suitable truncation led to a class of multivariate $q$-Racah polynomials [18], which will be of interest in this paper.

The univariate Askey-Wilson and $q$-Racah polynomials are well-known to be bispectral in the sense of Duistermaat and Grünbaum [15]. Writing them as $p_n(x)$, these polynomials can be defined both through a recursion relation in $x$ as well as one in $n$. Iliev extended these bispectrality properties to the multivariate case in [25]. He constructed two commutative algebras of difference and $q$-difference operators diagonalized by these polynomials and thereby discovered a duality relation between the variables $x_i$ and the degrees $n_i$. 
So far, this bispectrality has received little algebraic foundation. In this paper we will connect the multivariate \((-q)\)-Racah polynomials to the higher rank \(q\)-Bannai-Ito algebra \(A_q^n\) and thereby cast Ilievs difference operators in a larger algebraic framework. More precisely, we will show in Theorem 2 how these operators give rise to a realization of \(A_q^n\), which is hence generated by Ilievs aforementioned commutative algebras.

1.3. **Algebraic underpinning for the Askey-Wilson polynomials.** In previous literature, profound connections were established between quantum groups and algebras on the one hand and Askey-Wilson polynomials on the other. We refer to [35], [31] and [37] for an overview in the univariate case. More recently, similar relations were unveiled for their multivariate counterparts.

Genest, Iliev and Vinet studied in [19] the coupling coefficients for multifold tensor products of \(su_q(1, 1)\). These were identified as multivariate \(q\)-Racah polynomials through their appearance as connection coefficients between bases of multivariate \(q\)-Hahn and \(q\)-Jacobi polynomials. A related problem was studied for the \(q\)-Onsager algebra [3], which is known to have the Askey-Wilson algebra as a homomorphic image [44]. In [6], an iteration of coproducts of the quantum Kac-Moody algebra \(U_q(\widehat{sl}_2)\) allowed to identify two sets of \(N\) mutually commuting \(q\)-difference operators, which together generate several copies of the \(q\)-Onsager algebra. The polynomial eigenbases of the two sets of \(q\)-difference operators were considered and their overlap coefficients were expressed as entangled products of univariate \(q\)-Racah polynomials. However, these did not coincide with the Gasper and Rahman multivariable \(q\)-Racah polynomials. Altering the difference operators nevertheless allowed to obtain the Gasper and Rahman polynomials as generators for a family of infinite-dimensional modules of the rank one \(q\)-Onsager algebra in [5].

In [24] an iteration of coproducts of twisted primitive elements was used to relate the quantum group \(U_q(su(1, 1))\) to the multivariate Askey-Wilson and Al-Salam-Chihara polynomials. The obtained difference equations coincide with the relations in [25].

A similar recoupling for the higher rank Racah algebra was considered in [9]. Its overlap coefficients were obtained in [13] in terms of Tratniks multivariate \(q = 1\) Racah polynomials and the higher rank Racah algebra was realized by means of the difference operators for Racah polynomials defined by Geronimo and Iliev [23].

Also in the classical setting, bispectral orthogonal polynomials have been given profound algebraic underpinnings. In [26] Iliev studies Krawtchouk polynomials in \(n\) variables and relates their spectral properties to representations of the Lie algebra \(sl_{n+1}(\mathbb{C})\). Iliev and Xu established the bispectrality of a class of multivariate Hahn polynomials in [27]. The corresponding difference operators arise as symmetries of the discrete generic quantum superintegrable system and were shown to generate a realization of the Kohno-Drinfeld Lie algebra.

All of these references, except for the latter two, focus solely on \((q)\)-shifts in the variables of the polynomials, thus neglecting the duality with the polynomial degrees. However, we will allow for discrete shifts in the degrees as well, thereby exploiting the full bispectrality. Another novelty in our approach is that we obtain explicit algebraic relations connecting both types of difference operators, thereby recovering the \(A_q^n\) identities (2). Moreover, none of the mentioned works goes beyond the iteration of coproducts as a technique for extension to higher rank. We
will instead use the novel extension algorithm that we introduced in [7], to provide a complete description of the considered algebras. Referring to the notation in (2), we will index our algebra generators by subsets of \( \{1, \ldots, n\} \). In previous work only sets of consecutive integers could be considered, whereas our method makes it possible to obtain explicit expressions for any possible set.

We emphasize that all results obtained in this paper are easily transferred to the higher rank Askey-Wilson algebra under the transformation \( q \rightarrow -q^2 \), by the isomorphism with \( \mathcal{A}_n^q \) established in [7]. The considered overlap coefficients then become multivariate \((q^2)\)-Racah polynomials and generate infinite-dimensional modules for the higher rank Askey-Wilson algebra.

1.4. Specialization for \( q \rightarrow 1 \). In the limit \( q \rightarrow 1 \), the univariate \((-q)\)-Racah polynomials reduce to the Bannai-Ito polynomials, a class of orthogonal polynomials first considered in [2]. They satisfy both a three-term recurrence and three-term difference relation, thus giving rise to an associated Leonard pair [43]. They appeared in [21] as connection coefficients for the original \( q = 1 \) Bannai-Ito algebra (1). In [10] a construction by means of Dunkl operators was proposed for the higher rank Bannai-Ito algebra, of which \( \mathcal{A}_n^q \) is a \( q \)-deformation. Its connection coefficients were conjecturally stated to be multivariate extensions of Bannai-Ito polynomials, which had however not been defined at the time. In Section 5 we will affirm and prove this statement: we propose a definition for multivariate Bannai-Ito polynomials following the methods of Tratnik and Gasper and Rahman. We will obtain explicit expressions for the overlap coefficients in terms of these polynomials in Theorem 3. Subsequently we derive their orthogonality and bispectrality properties. We also show that our definitions coincide with the Bannai-Ito polynomials recently introduced in [33] for the bivariate case, up to a modification of two parameters.

1.5. Outline of the contents. The paper is organized as follows. In Section 2, we recall the necessary prerequisites on the higher rank \( q \)-Bannai-Ito algebra and the discrete series representation of \( \text{osp}_q(1|2) \). We repeat the definition of the \( \mathbb{Z}_2 \) \( q \)-Dirac-Dunkl model in Section 2.2. We obtain a decomposition of a coupled \( \text{osp}_q(1|2) \)-module in Section 2.3 as a motivation for our solution to the spectral problem in Definition 5. The connection coefficients for \( \mathcal{A}_n^q \) will be derived in Section 3. We first consider the univariate case before going to multiple variables. In Section 4 we translate some of the results from [25] to \((-q)\)-Racah polynomials and use these to construct a discrete realization for \( \mathcal{A}_n^q \). Finally, we introduce the multivariate Bannai-Ito polynomials in Section 5. We motivate their definition and obtain their orthogonality and bispectrality properties. The results from previous sections are subjected to a limit \( q \rightarrow 1 \), leading to the connection coefficients and their interpretation as generators for infinite-dimensional modules of the higher rank \( q = 1 \) Bannai-Ito algebra. We end with some conclusions and an outlook.

2. The higher rank \( q \)-Bannai-Ito algebra

We will first recall some prerequisites about the \(q\)-Bannai-Ito algebra and \( \text{osp}_q(1|2) \).

2.1. The quantum algebra \( \text{osp}_q(1|2) \) and the extension process. Let \( q \) be a complex number which is not a root of unity. For any complex number or any
operator $A$ we will denote by $[A]_q$ the $q$-number

$$[A]_q = \frac{q^A - q^{-A}}{q - q^{-1}}.$$  

We will write $[i; j]$ for the set of natural numbers $\{i, i+1, \ldots, j\}$. The $q$-anticommutator of two operators $A$ and $B$ is defined as

$$\{A, B\}_q = q^{1/2}AB + q^{-1/2}BA.$$  

The quantum superalgebra $\mathfrak{osp}_q(1|2)$ is the $\mathbb{Z}_2$-graded unital associative algebra with generators $A_+, A_-, K, K^{-1}$ and the grade involution $P$, satisfying the commutation relations

$$(3) \quad KA_+K^{-1} = q^{1/2}A_+, \quad KA_-K^{-1} = q^{-1/2}A_-, \quad \{A_+, A_-\} = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}},$$

$$\{P, A_{\pm}\} = 0, \quad [P, K] = 0, \quad [P, K^{-1}] = 0, \quad KK^{-1} = K^{-1}K = 1, \quad P^2 = 1.$$  

The Casimir operator has the expression

$$(4) \quad \Gamma^q = \left(-A_+A_- + \frac{q^{-1/2}K^2 - q^{1/2}K^{-2}}{q - q^{-1}}\right)P.$$  

It is easily checked that $\Gamma^q$ commutes with all algebra elements.

One can endow $\mathfrak{osp}_q(1|2)$ with a coproduct $\Delta : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)^{\otimes 2}$ acting on the generators as $[22]$

$$(5) \quad \Delta(A_{\pm}) = A_{\pm} \otimes KP + K^{-1} \otimes A_{\pm}, \quad \Delta(K) = K \otimes K, \quad \Delta(P) = P \otimes P.$$  

We will consider the multifold tensor product algebra $\mathfrak{osp}_q(1|2)^{\otimes n}$ with its standard product law as in $[22]$

$$(6) \quad (a_1 \otimes a_2 \otimes \cdots \otimes a_n)(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = a_1 b_1 \otimes a_2 b_2 \otimes \cdots \otimes a_n b_n.$$  

The rank $n - 2$ $q$-Bannai-Ito algebra was constructed inside $\mathfrak{osp}_q(1|2)^{\otimes n}$ in $[7]$ by means of an extension process, allowing to lift the Casimir operator $\Gamma^q$ to $n$-fold tensor products. We introduce the mapping $\tau : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)^{\otimes 2}$ through its action on the elements $A_-, K, A_+, K^2P$ and $\Gamma^q$

$$\tau(A_+K) = K^2P \otimes A_-, \quad \tau(A_-K) = (K^{-2}P \otimes A_+K) + q^{-1/2}(q - q^{-1})(A_+^2P \otimes A_-K)$$

$$+ q^{-1/2}(q^{1/2} - q^{-1/2})(A_+K^{-1}P \otimes K^2P) + q^{-1/2}(q - q^{-1})(A_+K^{-1}P \otimes \Gamma^q),$$

$$\tau(K^2P) = 1 \otimes K^2P - (q - q^{-1})(A_+K \otimes A_-K),$$

$$\tau(\Gamma^q) = 1 \otimes \Gamma^q$$

and we require it to act as an algebra morphism on the subalgebra spanned by these elements.

To each set $A \subseteq [1; n]$ one can now associate an element $\Gamma_A^q \in \mathfrak{osp}_q(1|2)^{\otimes n}$, constructed as

$$(8) \quad \Gamma_A^q = 1^{\otimes (\min(A) - 1)} \otimes \prod_{k = \min(A) + 1}^{\max(A)} \tau_{k-1,k}^q (\Gamma^q) \otimes 1^{\otimes (n - \max(A))},$$
and where $id$ is the identity morphism. For sets $A$ this process reduces to the well-known iteration of coproducts

\[
\Delta^{(d)} = \left(1^\otimes(d-2) \otimes \Delta\right) \Delta^{(d-1)}, \quad \Delta^{(1)} = id.
\]

**Definition 1.** The subalgebra of $\mathfrak{osp}_q(1|2)^\otimes n$ spanned by the elements $\Gamma^q_A$, $A \subseteq [1; n]$, is said to be the $q$-Bannai-Ito algebra of rank $n-2$. We will denote it by $\mathcal{A}_q^n$.

This terminology goes back to [22], where the rank 1 $q$-Bannai-Ito algebra was introduced, and to [10], which established a construction for the $q = 1$ rank $n-2$ Bannai-Ito algebra using Dunkl operators. The algebra relations satisfied by the elements $\Gamma^q_A$ can be summarized as follows.

**Proposition 1.** Let $A, B \subseteq [1; n]$ be such that

\[
\max(A \setminus (A \cap B)) < \min(A \cap B) \text{ and } \max(A \cap B) < \min(B \setminus (A \cap B)).
\]

Let $C = (A \cup B) \setminus (A \cap B)$. Then the elements $\Gamma^q_A$, $\Gamma^q_B$ and $\Gamma^q_C$ satisfy the relations

\[
\{\Gamma^q_A, \Gamma^q_B\}_q = \Gamma^q_C + \left(q^{1/2} + q^{-1/2}\right) \left(\Gamma^q_{A \cap B} \Gamma^q_{A \cup B} + \Gamma^q_A \Gamma^q_{A \setminus (A \cap B)}\right),
\]

\[
\{\Gamma^q_B, \Gamma^q_C\}_q = \Gamma^q_A + \left(q^{1/2} + q^{-1/2}\right) \left(\Gamma^q_{B \cap C} \Gamma^q_{B \cup C} + \Gamma^q_B \Gamma^q_{B \setminus (B \cap C)}\right),
\]

\[
\{\Gamma^q_C, \Gamma^q_A\}_q = \Gamma^q_B + \left(q^{1/2} + q^{-1/2}\right) \left(\Gamma^q_{C \cap A} \Gamma^q_{C \cup A} + \Gamma^q_C \Gamma^q_{C \setminus (C \cap A)}\right).
\]

The proof was given for sets of consecutive integers in [7], which is the only case we will rely on in this paper. For a proof of the general case we refer to [14].

The iteration of coproducts (10) can equally be applied to the $\mathfrak{osp}_q(1|2)$-generators, one thus obtains from (5)

\[
(A_\pm)_{[1;n]} = \Delta^{(n)}(A_\pm) = \sum_{i=1}^n (K^{-1})^\otimes(i-1) \otimes A_\pm \otimes (KP)^\otimes(n-i),
\]

\[
(K^\pm)_{[1;n]} = \Delta^{(n)}(K^\pm) = (K^\pm)^\otimes n, \quad P_{[1;n]} = \Delta^{(n)}(P) = P^\otimes n
\]

and hence by (4) we have

\[
\Gamma^q_{[1;n]} = \left(- (A_+)_{[1;n]}(A_-)_{[1;n]} + \frac{q^{-1/2}}{q - q^{-1}} K^2_{[1;n]} - \frac{q^{1/2}}{q - q^{-1}} (K^{-1})^2_{[1;n]}\right) P_{[1;n]}.
\]

Some other useful properties of the algebra generators, proven in [7], are the following.
Corollary 1. The set of operators $\Gamma^q_{[i;j]}$, with $1 \leq i \leq j \leq n$, is a generating set for $A^q_n$.

Proposition 2. For $A, B \subseteq \{1;n\}$ sets of consecutive integers such that $A \subseteq B$ or $A \cap B = \emptyset$, one has

$$[\Gamma^q_A, \Gamma^q_B] = 0.$$ 

This allows us to state the following definition.

Definition 2. For $i \in \{1;n - 1\}, j \in \{1;n - i\}$, we denote by $\mathcal{Y}_{i,j}$ the subalgebra of $A^q_n$ with generators

$$\mathcal{Y}_{i,j} = \{\Gamma^q_{[1;2]}, \ldots, \Gamma^q_{[1;i]}, \Gamma^q_{[i+1;i+2]}, \ldots, \Gamma^q_{[i+1;i+j]}, \Gamma^q_{[i+1;i+j+1]}, \ldots, \Gamma^q_{[1;n]}\}.$$ 

It is abelian by Proposition 2.

Remark 1. In case $i = 1$, the first sequence $\Gamma^q_{[1;2]}, \ldots, \Gamma^q_{[1;i]}$ of generators is of course empty. Hence our notation $\mathcal{Y}_{1,j}$ will refer to the algebra

$$\mathcal{Y}_{1,j} = \{\Gamma^q_{[2;3]}, \ldots, \Gamma^q_{[2;j+1]}, \Gamma^q_{[1;j+1]}, \ldots, \Gamma^q_{[1;n]}\}.$$ 

For $j = 1$, the middle sequence $\Gamma^q_{[i+1;i+2]}, \ldots, \Gamma^q_{[i+1;i+j]}$ will be empty, so by $\mathcal{Y}_{i,1}$ we mean

$$\mathcal{Y}_{i,1} = \{\Gamma^q_{[1;2]}, \ldots, \Gamma^q_{[1;i]}, \Gamma^q_{[1;i+1]}, \ldots, \Gamma^q_{[1;n]}\} = \{\Gamma^q_{[1;k]} : k \in \{2;n\}\},$$

which yields the same algebra for every $i$, i.e.

$$\mathcal{Y}_{i,1} = \mathcal{Y}_{1,1}, \quad \forall i \in \{2;n - 1\}.$$ 

Remark 2. It is clear that no generators can be added to the algebras $\mathcal{Y}_{i,j}$ without losing the property of commutativity. The number of non-central generators of these maximal abelian subalgebras, $n - 2$ in this case, stands as the rank of the algebra $A^q_n$.

The generators of these subalgebras mutually commute and they will also turn out to be simultaneously diagonalizable in Section 2.3. Our main purpose in Section 3 will be the computation of the overlap coefficients between the eigenbases of two such subalgebras $\mathcal{Y}_{i,j}$.

2.2. The $\mathbb{Z}^q_2$ q-Dirac-Dunkl model. An explicit realization of the higher rank $q$-Bannai-Ito algebra was proposed in [7]. We recall here its basic features as a motivation for the forthcoming Definition 5. Let $x_1, \ldots, x_n$ be arbitrary real variables and $\gamma_1, \ldots, \gamma_n > \frac{1}{2}$ be real parameters. Let $r_i$ be the reflection with respect to the hyperplane $x_i = 0$, i.e. $r_if(x_1, \ldots, x_n) = f(x_1, \ldots, -x_i, \ldots, x_n)$, and let $T_{q,i}$ be the $q$-shift operator $T_{q,i}f(x_1, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$. Introduce the $\mathbb{Z}^q_2$ q-Dunkl operator as

$$D^q_i = \frac{q^\gamma_i - \frac{1}{2}}{q - q^{-1}} (T_{q,i} - r_i) - \frac{q^{-\gamma_i - \frac{1}{2}}}{q - q^{-1}} \left( T_{q,i}^{-1} - r_i \right).$$

A realization of $\mathfrak{osp}_q(1|2)$, acting on functions of the variable $x_i$, can be obtained by taking [22]

$$A_+ \rightarrow x_i, \quad A_- \rightarrow D^q_i, \quad K \rightarrow q^{\gamma_i/2}T_{q,i}^{1/2}, \quad P \rightarrow r_i.$$
Explicit expressions were obtained in [7] for the operators $\Gamma^q_{[i+1; i+1]}$ inside this realization. The model was completed by the definition of the operators

$$D^q_{[n]} = \sum_{i=1}^{n} D^q_{i} R^q_{[n],i}, \quad X^q_{[n]} = \sum_{i=1}^{n} x^i R^q_{[n],i}, \quad T^q_{[n]} = \prod_{i=1}^{n} T^q_{i,1},$$

where

$$R^q_{[n],i} = \left( \prod_{j=1}^{i-1} q^{-(\gamma_j + 1)} (T_{q,j})^{-1/2} \right) \left( \prod_{j=i+1}^{n} q^{\gamma_j} (T_{q,j})^{1/2} r_{j} \right),$$

and the CK-extension

$$\text{CK}^x_{[n]} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\sigma_{m}(\gamma_1) \sigma_{m-1}(\gamma_2) \cdots \sigma_{1}(\gamma_n)} x_{[1]}^{m} \left( D^q_{[n]-1} \right)^{m} (T_{q,n-1})^{m} \prod_{j=1}^{n} \left( X^q_{[j]} \right)^{\gamma_j},$$

where by $\sigma_{m}(\gamma)$ we denote the number

$$\sigma_{m}(\gamma) = \left[ \frac{m + \frac{1}{2}}{2} q \right] - (-1)^{m} \left[ \frac{m + \frac{1}{2}}{2} q \right].$$

A construction was given for the polynomial null-solutions of $D^q_{[n]}$ that diagonalize the subalgebra $\mathcal{Y}_{1,1}$. For any vector $j \in \mathbb{N}^{n-1}$ we will denote by $[j_{m}]_{n}$ the sum $j_{1} + \cdots + j_{m}$ and for $A \subseteq [1,n]$ we will write $\gamma_A = \sum_{i \in A} \gamma_{i}$. Then the functions

$$\psi_{j}(x) = \psi_{(j_{1}, \ldots, j_{n-1})}(x_{1}, \ldots, x_{n})$$

are homogeneous polynomials in $x_{1}, \ldots, x_{n}$ of degree $[j_{n-1}]_{1}$ and were shown to satisfy the eigenvalue equations

$$\Gamma^q_{[1,\ell]} \psi_{j}(x) = (-1)^{\left| j_{\ell-1} \right| + \gamma_{[1,\ell]} - \frac{1}{2}} q \psi_{j}(x),$$

for all $\ell \in \{2, \ldots, n\}$ and

$$D^q_{[n]} \psi_{j}(x) = 0,$$

and moreover turned out to be orthogonal with respect to the $q$-Dunkl-Fischer inner product

$$\langle \psi(x), \varphi(x) \rangle = \lim_{x \to 0} \frac{\langle D^q_{1}, \ldots, D^q_{n} \rangle \varphi(x_{1}, \ldots, x_{n}) \rangle}{x_{1} \cdots x_{n} \cdots x_{n} \cdots x_{1}}.$$}

Referring to the notation of the forthcoming Definition 5, they will coincide with the abstract vectors $[j_{n-1}; \ldots, j_{1}; j_{n-1}]$, up to normalization. Similarly, the basis vectors $[j_{n-1}; \ldots, j_{1}; j_{n-1}]$ diagonalizing the subalgebra

$$\mathcal{Y}_{1,n-1} = \langle \Gamma^q_{[2,3]} \rangle, \ldots, \Gamma^q_{[2,n]} \rangle, \Gamma^q_{[1,n]} \rangle$$

are realized inside the $q$-Dirac-Dunkl model by the functions

$$\psi_{j}(x_{1}, \ldots, x_{n}) = \pi (\psi_{j}(x_{1}, \ldots, x_{n})), \quad \text{where } \pi \text{ is the permutation } 1 \leftrightarrow 2, 2 \leftrightarrow 3, \ldots, n-1 \leftrightarrow n, n \leftrightarrow 1, \text{ applied simultaneously to the index } i \text{ of the variables } x_{i}, \text{ the } q\text{-Dunkl operators } D^q_{i}, \text{ the reflections } r_{i}, \text{ the } q\text{-shift operators } T^q_{i,i} \text{ and the parameters } \gamma_{i}. \text{ The overlap coefficients are realized by the functions } \psi_{j}(x)$$.  

$$\langle \varphi_{j}(x), \psi_{k}(x) \rangle$$
with respect to the inner product (18) will hence follow as special cases from the results in Section 3.

2.3. Unitary irreducible modules and the spectral problem. Let us recall some notation from [22]. Let \( \gamma > \frac{1}{2} \) be a real number and \( m \in \mathbb{N} \). Denote by \( |m\rangle_\gamma \) the orthonormal vectors satisfying

\[
\gamma \langle m|m'\rangle_\gamma = \delta_{m,m'},
\]

endowed with the following \( \mathfrak{osp}(1|2) \)-action

\[
A_+ |m\rangle_\gamma = \sqrt{\sigma_m^{(\gamma)}} |m+1\rangle_\gamma, \quad A_- |m\rangle_\gamma = \sqrt{\sigma_m^{(\gamma)}} |m-1\rangle_\gamma,
K|m\rangle_\gamma = q^{\frac{1}{2}(m+\gamma)} |m\rangle_\gamma, \quad P|m\rangle_\gamma = (-1)^m |m\rangle_\gamma,
\]

with \( \sigma_m^{(\gamma)} \) as in (17). As follows from (4), the Casimir element \( \Gamma^q \) acts on these vectors as a multiple of the identity:

\[
\Gamma^q|m\rangle_\gamma = \left[ \gamma - \frac{1}{2} \right]_q |m\rangle_\gamma.
\]

Let \( W^{(\gamma)} \) be the infinite-dimensional vector space spanned by the vectors \( |m\rangle_\gamma \), \( m \in \mathbb{N} \). It follows immediately from (19) that

\[
A_\dagger = A_+, \quad K_\dagger = K, \quad P_\dagger = P,
\]

hence also

\[
\Gamma^q_\dagger = \Gamma^q.
\]

Then by the preceding observations \( W^{(\gamma)} \) is a unitary \( \mathfrak{osp}(1|2) \)-module, and moreover it is irreducible if one requires \( |q| \neq 1 \), since then one has \( \sigma_m^{(\gamma)} \neq 0 \) for all \( m \in \mathbb{N} \setminus \{0\} \).

Remark 3. Note that the explicit \( \mathfrak{osp}(1|2) \)-realization (16) agrees with (19) under the identification

\[
|m\rangle_\gamma \rightarrow \frac{\sigma_m}{\sqrt{\sigma_1^{(\gamma)} \sigma_2^{(\gamma)} \cdots \sigma_m^{(\gamma)}}}.
\]

Like before we will let \( \gamma_1, \ldots, \gamma_n > \frac{1}{2} \) be a set of real numbers. In what follows, we will consider the action of \( \mathcal{A}_q^\gamma \) on the coupled module \( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \). Our main purpose in this section will be the derivation of several explicit bases for this \( n \)-fold tensor product module. To that end, we state the following definition, as a special case of the forthcoming general Definition 5.

Definition 3. Let \( j^{(1,1)} \in \mathbb{N}^{n-1} \) and \( N \in \mathbb{N}, N \geq j^{(1,1)}_{n-1} \). The vectors \( |j^{(1,1)}; N\rangle = |j^{(1,1)}_{n-1}, \ldots, j^{(1,1)}_1, j^{(1,1)}_1 + m\rangle_\gamma \in W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \) are recursively defined as

\[
|j^{(1,1)}_{n-1}, \ldots, j^{(1,1)}_1; j^{(1,1)}_1 + m\rangle_\gamma = \left( A_+ \right)^m \left( \sum_{k=0}^{j^{(1,1)}_n} c_{k,j^{(1,1)}} |j^{(1,1)}_{n-1}, \ldots, j^{(1,1)}_2, j^{(1,1)}_1 + k\rangle_\gamma \otimes |j^{(1,1)}_{n-1} - k\rangle_\gamma \right).
\]
with coefficients
\[ c_{k,j^{(1,1)}} = (-1)^{k(j^{(1,1)}_n+1)} q^{\frac{k}{2}} \frac{\prod_{\ell=0}^{k-1} \sqrt{\sigma_{j^{(1,1)}_n}}}{\sigma_{j^{(1,1)}_n}} \]
and initial conditions
\[ |j^{(1,1)}_{n-1}; j^{(1,1)}_n + m⟩ = \left( \prod_{\ell=1}^{m} \sqrt{\sigma_{j^{(1,1)}_{n}}^{(\gamma_{1})}} \right) |m⟩_{\gamma_{1}} \quad \text{for } n = 1. \]

The vectors \(|j^{(1,1)}; N⟩\) can be characterized through the following eigenvalue equations.

**Lemma 1.** The vectors \(|j^{(1,1)}; N⟩\) satisfy the equations

\[ \Gamma_{[1;m+1]}^q |j^{(1,1)}; N⟩ = (-1)^{j^{(1,1)}_m} \left[ |j^{(1,1)}_m⟩ + \gamma_{[1;m+1]} - \frac{1}{2} \right] q |j^{(1,1)}; N⟩, \]
for any \(m \in [1; n-1]\) and

\[ K_{[1;n]} |j^{(1,1)}; N⟩ = q^{\frac{1}{2}} (j^{(1,1)}_{n-2} + N + \gamma_{[1;n]}) |j^{(1,1)}; N⟩, \]
and

\[ P_{[1;n]} |j^{(1,1)}; N⟩ = (-1)^{j^{(1,1)}_{n-2} + N} |j^{(1,1)}; N⟩, \]
for any \(N \geq j^{(1,1)}_{n-1}\). For \(N = j^{(1,1)}_{n-1}\) we have the supplementary equation

\[ (A_-)_{[1;n]} |j^{(1,1)}; j^{(1,1)}_{n-1}⟩ = 0. \]

**Proof.** We will first prove the claims for \(N = j^{(1,1)}_{n-1}\), by induction on \(n\). Observe that for \(n = 2\) one has

\[ |j^{(1,1)}_{2}; j^{(1,1)}_{1}⟩ = \sum_{k=0}^{j^{(1,1)}_{2}} c_{k,j^{(1,1)}} (A_+)^{j^{(1,1)}_{2}} (A_-)^{j^{(1,1)}_{1}} |j^{(1,1)}_{2}; j^{(1,1)}_{1}⟩, \]

hence the actions of \(K_{[1;2]} = K \otimes K\), \(P_{[1;2]} = P \otimes P\) and \((A_-)_{[1;2]} = \Delta(A_-)\) on \(|j^{(1,1)}_{1}; j^{(1,1)}_{1}⟩\) are easily checked to coincide with (23), (24) and (25) by (5) and (19).

The equation for \(\Gamma^q_{[1;2]}\) then follows immediately from (14).

Suppose now the claim holds for \(n - 1\). Observe that

\[ |j^{(1,1)}_{n-1}; j^{(1,1)}_{n}⟩ = \sum_{k=0}^{j^{(1,1)}_{n-1}} c_{k,j^{(1,1)}} (A_+)^{j^{(1,1)}_{n-1}} (A_-)^{j^{(1,1)}_{n-2}} \cdots (A_+)^{j^{(1,1)}_{1}} (A_-)^{j^{(1,1)}_{n-2} - k} |j^{(1,1)}_{n-1}; j^{(1,1)}_{n}⟩, \]

It is immediate from (3) that

\[ A_- A_+^k = A_+^{k-1} \left( \frac{q^{-1/2}(q^k - (-1)^k)}{q - q^{-1}} K^2 - \frac{q^{1/2}(q^{-k} - (-1)^k)}{q - q^{-1}} K^{-2} \right) + (-1)^k A_+^k A_- , \]

\[ KA_+^k = q^{1/2} A_+^k K, \]

\[ PA_+^k = (-1)^k A_+^k P, \]

and of course the same identities hold after applying \(\Delta^{(n-1)}\) to both sides. The induction hypothesis together with (26) leads straightforwardly to (23), (24) and
(25). The relations (22) for \( m < n - 1 \) follow from the induction hypothesis, the case \( m = n - 1 \) arises from (14), (23), (24) and (25).

As a last step, we need to consider the case \( N > j_{n-1}^{(1,1)} \). The relations (22) follow directly from the assertion for \( N = j_{n-1}^{(1,1)} \), combined with the fact that any \( \Gamma_{[1;m+1]}^q \) commutes with \( (A_+)[1;n] \), as one sees by applying \( \Delta^{(m+1)} \otimes 1^{\otimes(n-m-1)} \) to the identity

\[
\left[ \Gamma^q \otimes 1^{\otimes(n-m-1)}, (A_+)[1;n-m] \right] = 0.
\]

The equations (23) and (24) follow similarly from the case \( N = j_{n-1}^{(1,1)} \) together with the relations (27).

\[
\square
\]

Remark 4. Note also that for any vector \( |v\rangle \) in \( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \), in particular for the \([j^{(1,1)}; N] \), one has

\[
\Gamma_{[1]}^q |v\rangle = \left[ \gamma_i - \frac{1}{2} \right]_q |v\rangle.
\]

It will turn out useful to make the following observation.

Lemma 2. Each of the operators \( \Gamma_{[k;\ell]}^q \), \( 1 \leq k \leq \ell \leq n \), as well as the operator \( K_{[1;n]} \), are self-adjoint.

Proof. Using the definition of the coproduct (5) and the hermitian conjugate (20), one can easily show by induction that

\[
\Delta^{(d)}(A_\pm)^\dagger = \Delta^{(d)}(A_\pm^\dagger), \quad \Delta^{(d)}(K)^\dagger = \Delta^{(d)}(K^\dagger), \quad \Delta^{(d)}(P)^\dagger = \Delta^{(d)}(P^\dagger),
\]

for any \( d \in \mathbb{N} \), referring to the notation (10). By the multiplicativity of the coproduct and (4) one hence also finds

\[
\Delta^{(d)}(\Gamma^q)^\dagger = \Delta^{(d)}(\Gamma^q),
\]

concluding the proof. \( \square \)

Let us introduce the notation

\[
U_{N,m} = \langle n_1 \rangle_{\gamma_1} \otimes \langle n_2 \rangle_{\gamma_2} \otimes \cdots \otimes \langle n_m \rangle_{\gamma_m} : n_1 + n_2 + \cdots + n_m = N,
\]

ei. \( U_{N,m} \) is the eigenspace of \( K^{\otimes m} \) with eigenvalue \( q^{\frac{1}{2}(N + \gamma_{[1;m]})} \).

We will also need the following subspaces of \( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \).

Definition 4. We denote by \( V_{N;1} = V_{N,i_1,\ldots,i_{n-2}} \) the joint eigenspace of the operators

\[
\Gamma_{[1;m+1]}^q \quad \text{with eigenvalue} \quad (-1)^{|m|} \left[ |m| + \gamma_{[1;m+1]} - \frac{1}{2} \right]_q, \quad m \in \{1, \ldots, n-2\}
\]

and

\[
K_{[1;n]} \quad \text{with eigenvalue} \quad q^{\frac{1}{2}(N + |i_{n-2}| + \gamma_{[1;n]})}.
\]

Note that for \( n = 2 \) only the last eigenvalue equation remains, hence in that case \( V_{N,1} \) reduces to \( U_{N,2} \). A basis for this space is obtained in the following lemma.

Lemma 3. A basis for the space \( U_{N,2} \) is given by

\[
\{ [j^{(1,1)}; N] : j^{(1,1)} \in \{0, \ldots, N\} \}.
\]
Proof. Each of the vectors $|j_1^{(1,1)}; N \rangle$, $j_1^{(1,1)} \in \{0, \ldots, N\}$, lies in $U_{N,2}$ by Lemma 1, and moreover they are linearly independent, since they correspond to different eigenvalues for $\Gamma_{[1;2]}^q$. The claim now follows from the fact that $\dim (U_{N,2}) = N + 1$, as follows immediately from (28). □

This simple observation will lead us to a decomposition for the coupled $\mathfrak{osp}(1|2)$-module. The proof will be omitted, since it essentially follows by induction from the results in [39, Theorems 2.1 and 5.1].

**Proposition 3.** The coupled module $W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)}$ can be decomposed in irreducible components as

$$W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \cong \bigoplus_{N=0}^{\infty} m_N W^{(\gamma_{[1,n]} + N)},$$

where the multiplicity is given by

$$m_N = \binom{N + n - 2}{N}.$$

Starting from the spaces $V_{N,i}$ we may also obtain a different decomposition and even an explicit basis for the coupled module. This is the subject of the following proposition.

**Proposition 4.** An orthogonal basis for the coupled module $W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)}$ is given by

$$\{ |j_{N-1}^{(1,1)}, \ldots, j_1^{(1,1)}; N \rangle : j_1^{(1,1)}, \ldots, j_{N-1}^{(1,1)} \in \mathbb{N}, N \in \mathbb{N}, N \geq j_1^{(1,1)} \}.$$

More precisely, one has the decomposition

$$W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} = \bigoplus_{i \in \mathbb{N}^{n-2}} \bigoplus_{N \in \mathbb{N}} V_{N,i}$$

and

$$B_{\gamma_1^{(1,1)}, N} = \{ |j_{N-1}^{(1,1)}, \ldots, j_1^{(1,1)}; N \rangle : j_1^{(1,1)} \in \{0, \ldots, N\} \},$$

forms an orthogonal basis for $V_{N,j_1^{(1,1)}, \ldots, j_{N-1}^{(1,1)}}$.

Proof. By induction on $n$. The case $n = 2$ follows immediately from Lemma 3 and the fact that one has

$$W^{(\gamma_1)} \otimes W^{(\gamma_2)} = \bigoplus_{N=0}^{\infty} U_{N,2},$$

as is immediate from (28).

Now suppose the claims have been proven for $n - 1$. Referring to the notation (28) we have

$$W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} = \bigoplus_{M=0}^{\infty} \left( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_{n-1})} \right) \cap U_{M,n}$$

$$= \bigoplus_{M=0}^{\infty} \bigoplus_{k=0}^{M} \left( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_{n-1})} \right) \cap U_{k,n-1} \otimes \langle |M - k\rangle_{\gamma_{n-1}} \rangle.$$

By the induction hypothesis, we have

$$W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_{n-1})} = \bigoplus_{i \in \mathbb{N}^{n-3}} \bigoplus_{N \in \mathbb{N}} V_{N,i}.$$
and moreover
\[ V_{N,i} \cap U_{k,n-1} = \begin{cases} V_{N,i} & \text{if } k = N + |i_{n-3}|, \\ \{0\} & \text{otherwise.} \end{cases} \]

Hence we find
\[
W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} = \bigoplus_{M=0}^{\infty} \bigoplus_{k=0}^{M-|i_{n-3}|} V_{k,|i_{n-3}|,i} \otimes \langle |M-k| \gamma_n \rangle
\]
\[
= \bigoplus_{M=0}^{\infty} \bigoplus_{k=0}^{M-|i_{n-3}|} \bigoplus_{i_{n-3} \leq M} V_{k,|i_{n-3}|,i} \otimes \langle |M-k| \gamma_n \rangle
\]
\[
= \bigoplus_{M=0}^{\infty} \bigoplus_{i_{n-3} \leq M} V_{k,i} \otimes \langle |M-k-|i_{n-3}| \gamma_n \rangle,
\]
where we have switched the order of summation in the second line and renamed \( k \) in the third. By the induction hypothesis, a basis for \( V \) is given by
\[ \mathcal{B}_{k,i} = \{ |i_{n-2},\ldots,i_1;k\rangle : i_{n-2} \in \{0,\ldots,k\} \}. \]

Hence we have
\[
W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} = \bigoplus_{M=0}^{\infty} \bigoplus_{i_{n-3} \leq M} \langle |i_{n-2},\ldots,i_1;k\rangle \otimes |M-k-i_{n-3}| \gamma_n \rangle
\]
\[
= \bigoplus_{M=0}^{\infty} \bigoplus_{i_{n-3} \leq M} \bigoplus_{i_{n-2} \leq M} \langle |i_{n-2},\ldots,i_1;k\rangle \otimes |M-k-i_{n-3}| \gamma_n \rangle
\]
\[
= \bigoplus_{M=0}^{\infty} \bigoplus_{i_{n-2} \leq M} \langle |i_{n-2},\ldots,i_1;k\rangle \otimes |M-k-i_{n-3}| \gamma_n : k \in \{i_{n-2},\ldots,M-i_{n-3}\} \rangle,
\]
where we have once more switched the order of summation in the third line and concatenated the second and third summation in the fourth line.

Let us now introduce the set
\[ \mathcal{C}_{i,N} = \{ |i_{n-2},\ldots,i_1;i_{n-2}+k\rangle \otimes |N-k| \gamma_n : k \in \{0,\ldots,N\} \}. \]

Observe from Lemma 1 that each of its elements belongs to \( V_{N,i} \). Hence we may write for the \( C \)-linear span of \( \mathcal{C}_{i,N} \):
\[
\langle |i_{n-2},\ldots,i_1;i_{n-2}+k\rangle \otimes |N-k| \gamma_n : k \in \{0,\ldots,N\} \rangle \subseteq V_{N,i_{n-1},\ldots,i_{n-2}}.
\]

In combination with (32), this leads to
\[
W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \subseteq \bigoplus_{M=0}^{\infty} \bigoplus_{i_{n-2} \leq M} V_{M-|i_{n-2}|,i_{n-1},\ldots,i_{n-2}}.
\]
which after switching summation order and renaming summation indices again, becomes

$$W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \subseteq \bigoplus_{N=0}^{\infty} \bigoplus_{i_1, \ldots, i_{n-2}} V_{N,i_1, \ldots, i_{n-2}}.$$  

But obviously the opposite inclusion also holds, since the spaces $V_{N,i}$ intersect trivially. Hence we get equality in (34) and so (31) is proven.

Looking back at the fourth line in (32), we see that the inclusion in (33) must also be an equality. This implies

$$\dim(V_{N,i_1, \ldots, i_{n-2}}) \leq N + 1.$$  

On the other hand, the set $B_{i,N}$ is contained in $V_{N,i}$ by Lemma 1, and its elements are linearly independent, since they correspond to different eigenvalues for $\Gamma^q_{[1;n]}$. As a consequence, the inequality in (35) must be an equality as well, and both $B_{i,N}$ and $C_{i,N}$ must be bases for $V_{N,i}$. This concludes the induction.

Finally, the set (30) is precisely

$$\bigcup_{\mathbf{j}^{(1,1)} \in \mathbb{N}^{n-2}, N \in \mathbb{N}} B_{\mathbf{j}^{(1,1)}, N},$$

which by (31) is indeed a basis for the coupled module $W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)}$. These basis vectors are orthogonal, as one can see by computing

$$\langle \mathbf{j}^{(1,1)}; N | \Gamma^q_{[1;m+1]} | \mathbf{j}'^{(1,1)}; N' \rangle$$

for $m \in [1; n-1]$ and

$$\langle \mathbf{j}^{(1,1)}; N | K_{[1;n]} | \mathbf{j}'^{(1,1)}; N' \rangle$$

using Lemmas 1 and 2. This concludes the proof. $\square$

Now that we are guaranteed that the vectors $|\mathbf{j}^{(1,1)}; N\rangle$ form a basis for the coupled module, we note that we could have also defined them through the eigenvalue equations (22) for all $m \in [1; n-1]$ and (23), as these uniquely determine the quantum numbers $j^{(1,1)}_k$ and $N$. In complete analogy, we may define a new class of vectors in $W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)}$ through a set of eigenvalue equations. Recall the notation $\mathcal{Y}_{i,j}$ for the subalgebra (15), whose generators mutually commute by Proposition 2.

**Definition 5.** We denote by $|\mathbf{j}^{(i,j)}; N\rangle = |j^{(i,j)}_{n-1}, j^{(i,j)}_{n-2}, \ldots, j^{(i,j)}_1; N\rangle$, with all $j^{(i,j)}_k \in \mathbb{N}$ and $N \in \mathbb{N}$, $N \geq j^{(i,j)}_n$, the orthonormal vectors in $W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)}$ diagonalizing the subalgebra $\mathcal{Y}_{i,j}$ through the eigenvalue equations

$$\Gamma^q_{[1;k+1]} |j^{(i,j)}_{n-1}, \ldots, j^{(i,j)}_1; N\rangle = (-1)^{\left[\frac{j^{(i,j)}_n}{q} \right]} \left[ |j^{(i,j)}_k| + \gamma_{[1;k+1]} - \frac{1}{2} \right] |j^{(i,j)}_{n-1}, \ldots, j^{(i,j)}_1; N\rangle$$

for all $k \in \{1, \ldots, i-1\} \cup \{i + j - 1, \ldots, n - 1\}$,

$$\Gamma^q_{[i+1; i+k+1]} |j^{(i,j)}_{n-1}, \ldots, j^{(i,j)}_1; N\rangle = (-1)^{\left[\frac{j^{(i,j)}_{n-1}}{q} \right]} \left[ |j^{(i,j)}_{i+k-1}| - |j^{(i,j)}_{i-1}| \right] |j^{(i,j)}_{n-1}, \ldots, j^{(i,j)}_{i+k+1}; N\rangle,$$

with $\gamma_{[1;k+1]} = \sum_{i=1}^{k+1} \gamma_i$. These equations also uniquely determine the quantum numbers $j^{(i,j)}_k$ and $N$. When $j^{(i,j)}_k$ is odd, then $|\mathbf{j}^{(i,j)}; N\rangle$ is an eigenvector of the subalgebra $\mathcal{Y}_{i,j}$. This defines the vectors $|\mathbf{j}^{(i,j)}; N\rangle$.
for all \( k \in \{1, \ldots, j - 1\} \) and
\[
K_{[1;n]}(j^{(i,j)}_n; \ldots; j^{(i,j)}_1; N) = \frac{1}{\delta_{j^{(i,j)}_n; \ldots; j^{(i,j)}_1; N}}.
\]

These vectors are again orthogonal by Lemma 2 and we may choose the normalization such that
\[
(j^{(i,j)}_n; \ldots; j^{(i,j)}_1; N) \cdot (j^{(i,j)}_n; \ldots; j^{(i,j)}_1; N') = \delta_{j^{(i,j)}_n; \ldots; j^{(i,j)}_1; N} \delta_{N,N'}.
\]

Remark 5. It follows immediately from Remark 1 that
\[
j^{(i,1)}; N = j^{(1,1)}; N,
\]
for every \( i \in [2; n - 1] \).

Remark 6. This definition agrees with Definition 3 for \( i = j = 1 \), up to rescaling by a constant imposed by the normalization (39). By a reasoning similar to Proposition 4, we may prove that the vectors \( j^{(i,j)}; N \), \( j^{(i,j)} \in \mathbb{N}^{n - 1}, N \in \mathbb{N}, N \geq j^{(i,j)}_{n-1} \), form an orthonormal basis for \( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \).

As a conclusion, we have found several orthonormal bases for the coupled module \( W^{(\gamma_1)} \otimes \cdots \otimes W^{(\gamma_n)} \). In Section 3 we will obtain explicit expressions for the Clebsch-Gordan coefficients between two such orthonormal bases.

3. Connection coefficients

In this section we will compute an explicit expression for the overlap coefficients
\[
(j^{(i,j)}; j_n^{(i,j)} + m; j^{(i,1)}_n; \ldots; j^{(i,1)}_1; j^{(i,1)}_n + m)
\]
related to the abelian subalgebras \( \mathcal{Y}_i,j \) and \( \mathcal{Y}_i,1 \). These coefficients can be shown to be independent of \( m \) [48], hence from now on we let \( m \) be a fixed natural number and write \( j^{(i,j)} \) instead of \( j^{(i,j)}; j_n^{(i,j)} + m \). We will first consider several intermediate bases in Section 3.1 before tackling the general case in Section 3.2.

3.1. Univariate \((-q\)-)Racah polynomials. The \( q\)-Racah polynomials depend on four parameters \( a, b, c, N \in \mathbb{R} \) and are defined as \[1\]
\[
r_n(x; a, b, c, N) = (aq, bcq, c^{-N}; q)_n \left( \frac{q^n}{c} \right) \frac{\phi_3}{aq, bc, c^{-N}} q^{-n}, \quad (a_1, \ldots, a_s; q)_n = \prod_{k=1}^s (a_k; q)_n
\]
and the basic hypergeometric series [30]
\[
4\phi_3 \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3} \bigg| q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, a_4)_n}{(b_1, b_2, b_3)_n} (q, q)_n z^n.
\]

However, the polynomials we will need have \(-q\) as a deformation parameter instead of \( q \), hence we will refer to the polynomials (40) with all \( q \) changed to \(-q\) as \((-q\)-)Racah polynomials. In this section we will show that the connection coefficients
\[
(j^{(i,k+1)}; j^{(i,k)}; \ldots; j^{(i,1)}; j_n^{(i,1)}; \ldots; j^{(i,1)}_1; j^{(i,1)}_n + m)
\]
are proportional to these \((-q\)-)Racah polynomials. A useful way to write these is the following.
Lemma 4. The connection coefficients between the eigenbases of the intermediate subalgebras $\mathcal{Y}_{i,k}$ and $\mathcal{Y}_{i,k+1}$ can be expressed as

$$\langle \mathbf{j}^{(i, k+1)} | \mathbf{j}^{(i, k)} \rangle = \delta_{j_{i+k-1}^{(i, k)}} j_{i+k}^{(i, k+1)} + j_{i+k}^{(i, k+1)} j_{i+k-1}^{(i, k+1)} \prod_{\ell=1}^{i+k-2} \delta_{j_{\ell}^{(i, k)}} j_{\ell}^{(i, k+1)} \prod_{\ell=i+k+1}^{n-1} \delta_{j_{\ell}^{(i, k)}} j_{\ell}^{(i, k+1)},$$

where

$$\omega_{j_{i,k+1}} G_{j_{i,k+1}}^{(i,k)} (\mathbf{j}^{(i,k+1)})$$

is a normalization factor, chosen such that

$$G_{0} (\mathbf{j}^{(i,k+1)}) = 1.$$

Proof. Writing $\lambda_{[1;m+1]}^{(i,\ell)} (\mathbf{j}^{(i,\ell)})$ for the eigenvalue

$$(44) \quad (\lambda_{[1;m+1]}^{(i,\ell)} (\mathbf{j}^{(i,\ell)}) - [j_{m}^{(i,\ell)}] + \gamma_{[1;m+1]} - \frac{1}{2} \gamma_{q}^{(i,\ell)}),$$

we find from Definition 5 and Lemma 2

$$\langle \mathbf{j}^{(i,k+1)} | \Gamma_{A}^{q} | \mathbf{j}^{(i,k)} \rangle = \lambda_{A}^{(i,k)} (\mathbf{j}^{(i,k)}) \langle \mathbf{j}^{(i,k+1)} | \mathbf{j}^{(i,k)} \rangle = \lambda_{A}^{(i,k+1)} (\mathbf{j}^{(i,k+1)}) \langle \mathbf{j}^{(i,k+1)} | \mathbf{j}^{(i,k)} \rangle$$

where $A$ is any of the sets $[1; 2], \ldots, [i; i], [i + 1; i + 2], \ldots, [i + 1; i + k + 1], \ldots, [1; n]$.

Hence we find that the connection coefficient will vanish unless

$$j_{i+k}^{(i,k)} = j_{i+k+1}^{(i,k)},$$

for any $\ell \in [1; i + k - 2] \cup [i + k + 1; n - 1]$ and

$$j_{i+k-1}^{(i,k)} + j_{i+k}^{(i,k)} = j_{i+k-1}^{(i,k+1)} + j_{i+k}^{(i,k+1)}.$$

This results in the formulation

$$\langle \mathbf{j}^{(i,k+1)} | \mathbf{j}^{(i,k)} \rangle = g(\mathbf{j}^{(i,k+1)}, j_{i+k}^{(i,k)} \delta_{j_{i+k}^{(i,k)}} j_{i+k+1}^{(i,k+1)} \prod_{\ell=1}^{i+k-2} \delta_{j_{\ell}^{(i,k)}} j_{\ell}^{(i,k+1)} \prod_{\ell=i+k+1}^{n-1} \delta_{j_{\ell}^{(i,k)}} j_{\ell}^{(i,k+1)},$$

with

$$g(\mathbf{j}^{(i,k+1)}, j_{i+k}^{(i,k)}, j_{i+k-1}^{(i,k)}) = \langle \mathbf{j}^{(i,k+1)} | j_{i+k}^{(i,k+1)} \rangle = \langle \mathbf{j}^{(i,k+1)} | j_{i+k}^{(i,k)} \rangle = \langle \mathbf{j}^{(i,k+1)} | j_{i+k}^{(i,k)} \rangle = \langle \mathbf{j}^{(i,k+1)} | j_{i+k}^{(i,k)} \rangle.$$
The only two operators that act differently on \(|j^{(i,k)}\rangle\) and \(|j^{(i,k+1)}\rangle\) are \(\Gamma^q_{[i+1;i+k+1]}\), which diagonalizes \(|j^{(i,k+1)}\rangle\) but not \(|j^{(i,k)}\rangle\), and \(\Gamma^q_{[i;i+k]}\), where we have the opposite. In the next proposition, we will show that \(\Gamma^q_{[i+1;i+k+1]}\) will act on \(|j^{(i,k)}\rangle\) in a tridiagonal fashion. Let us first introduce some notation. Let

\[
\begin{align*}
a^{(i,k)} &= (-q)|j^{(i,k)}⟩_{i,k+1} - |j^{(i,k)}⟩_{i,k-1} q^{γ_{[i+1;i+k+1]} - \frac{1}{2}} \\
b^{(i,k)} &= (-q)|j^{(i,k)}⟩_{i,k+1} + |j^{(i,k)}⟩_{i,k-1} q^{-γ_{[i+1;i+k+1]} + \frac{1}{2}} \\
c^{(i,k)} &= (-q)|j^{(i,k)}⟩_{i,k+1} + |j^{(i,k)}⟩_{i,k-1} q^{2γ_{[i+1;i+k+1]} - \frac{1}{2}} \\
d^{(i,k)} &= (-q)|j^{(i,k)}⟩_{i,k+1} - |j^{(i,k)}⟩_{i,k-1} q^{γ_{[i+1;i+k+1]} - γ_{i+1;i+k+1} + \frac{1}{2}},
\end{align*}
\]

and

\[
\begin{align*}
A^{(i,k)}_s &= -\frac{1 + (-q)^s a^{(i,k)} b^{(i,k)} (1 - (-q)^s a^{(i,k)} c^{(i,k)})(1 - (-q)^s a^{(i,k)} d^{(i,k)})}{a^{(i,k)}(1 - (-q)^{2s-1} a^{(i,k)} b^{(i,k)} c^{(i,k)} d^{(i,k)})(1 - (-q)^{2s} a^{(i,k)} b^{(i,k)} c^{(i,k)} d^{(i,k)})} \\
C^{(i,k)}_s &= \frac{a^{(i,k)} (1 - (-q)^s b^{(i,k)} c^{(i,k)})(1 - (-q)^s b^{(i,k)} d^{(i,k)})}{(1 - (-q)^{2s-1} a^{(i,k)} b^{(i,k)} c^{(i,k)} d^{(i,k)})(1 - (-q)^{2s} a^{(i,k)} b^{(i,k)} c^{(i,k)} d^{(i,k)})} \\
&\times (1 + (-q)^s b^{(i,k)} d^{(i,k)}).
\end{align*}
\]

Define also

\[
\begin{align*}
V^{(i,k)}_j &= a^{(i,k)} - (a^{(i,k)})^{-1} - A^{(i,k)}_{j_{i+k-1}} - C^{(i,k)}_{j_{i+k-1}}, \\
U^{(i,k)}_j &= \sqrt{A^{(i,k)}_{j_{i+k-1}} - C^{(i,k)}_{j_{i+k-1}}}.
\end{align*}
\]

Then one can show the following proposition.

**Proposition 5.** The operator \(\Gamma^q_{[i+1;i+k+1]}\) acts tridiagonally on the basis functions \(|j^{(i,k)}\rangle\):

\[
\Gamma^q_{[i+1;i+k+1]}|j^{(i,k)}\rangle = V^{(i,k)}_j|j^{(i,k)}\rangle + U^{(i,k)}_j|j^{(i,k)}\rangle - h_{i+k-1} + U^{(i,k)}_{j_{i+k-1}}|j^{(i,k)}\rangle + h_{i+k-1}.
\]

where \(h_{i+k-1} = (0, \ldots, 0, 1, -1, 0, \ldots, 0)\).

**Proof.** The proof follows the same steps as Theorem 2 in [7]. The eigenvalue equations (36), (37) and the algebra relations (12) allow to obtain expressions for the coefficients \(V^{(i,k)}_j\) and \(U^{(i,k)}_j\), whose factorizations (50) can be checked using any computer algebra package.

Assuming that the conditions (46) and (47) are fulfilled and using Proposition 5 and the notation (45), we obtain a three-term recurrence relation for the functions
\( G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) \), similar to [22, formula (3.33)], namely

\[
\begin{align*}
\langle j^{(i,k+1)} | \rho_{j_{i+1};j_{i+k+1}}^{q} | j^{(i,k)} \rangle \\
= \lambda_{j_{i+1};j_{i+k+1}}^{(i,k+1)} \omega_{j_{i,k+1}}^{(i,k+1)} G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) \\
= \omega_{j_{i,k+1}}^{(i,k+1)} V_{j_{i,k+1}}^{(i,k)} G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) \\
+ \omega_{j_{i,k+1}}^{(i,k+1)} \left( U_{j_{i,k}}^{(i,k)} G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) + U_{j_{i,k}}^{(i,k)} + h_{j_{i+k-1}} G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) \right).
\end{align*}
\]

We will also renormalize as in [22]: define

\[
\begin{align*}
\tilde{G}_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) &= -q^{i(k)} \left( j^{(i,k)} \right) \left( j_{i+k-1} \right) \prod_{\ell=1}^{j_{i+k-1}} \sqrt{A_{\ell-1} \ell \tilde{C}_{\ell}} \right) G_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right).
\end{align*}
\]

These functions turn out to be proportional to \((-q)\)-Racah polynomials.

**Proposition 6.** The functions \( \tilde{G}_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) \) can be expressed as

\[
\begin{align*}
\tilde{G}_{j_{i+k-1}}^{(i,k)} \left( j^{(i,k+1)} \right) &= F_{i,k} \rho_{j_{i+k-1}}^{(i,k)} \left( j_{i+k-1} ; \gamma'(i,k), \delta'(i,k), \tilde{\nu}(i,k) \right) - q,
\end{align*}
\]

where \( r_n(x; a, b, c, N) \) is the \((-q)\)-Racah polynomial (40), where \( \tilde{\nu}(i,k) = |j_{i+k} - j_{i+k-2}| \) and

\[
\begin{align*}
\alpha'(i,k) &= q^{-1} a(i,k) b(i,k), \\
\beta'(i,k) &= q^{-1} c(i,k) d(i,k), \\
\gamma'(i,k) &= q^{-1} a(i,k) d(i,k), \\
\delta'(i,k) &= -a(i,k) d(i,k),
\end{align*}
\]

and where the proportionality coefficient is given by

\[
\begin{align*}
F_{i,k} &= \frac{\left( \beta(i,k) (-q)^{-\tilde{\nu}(i,k)} \right) \left( -\gamma(i,k) q \right)}{\left( \beta(i,k) (-q)^{-j_{i+k-1}} \right) \left( -\gamma(i,k) q \right)} \left( -\gamma(i,k) q, -\beta(i,k) \delta(i,k) q, (-q)^{-\tilde{\nu}(i,k)} \right) - q \right)_{j_{i+k-1}}.
\end{align*}
\]

**Proof.** Let us introduce some more notation. Define

\[
\begin{align*}
\mu'(j_{i+k-1}) &= (-q)^{-j_{i+k-1}} + \gamma'(i,k) \delta'(i,k) (-q)^{j_{i+k-1}+1},
\end{align*}
\]

with \( \gamma'(i,k), \delta'(i,k) \) as in (54) and let

\[
\begin{align*}
\tilde{A}_s^{(i,k)} &= -a(i,k) A_s^{(i,k)}, \\
\tilde{C}_s^{(i,k)} &= -a(i,k) C_s^{(i,k)},
\end{align*}
\]

with \( A_s^{(i,k)}, C_s^{(i,k)} \) as in (49). Observe that

\[
\begin{align*}
\tilde{A}_s^{(i,k)} &= \frac{\left( 1 - (-q)^{s+1} \alpha'(i,k) \right) \left( 1 - (-q)^{s+1} \alpha'(i,k) \right) \left( 1 - (-q)^{s+1} \beta'(i,k) \right) \left( 1 - (-q)^{s+1} \beta'(i,k) \right) \left( 1 - (-q)^{2s+1} \beta'(i,k) \right) \left( 1 - (-q)^{2s+1} \beta'(i,k) \right)}{(1 - (-q)^{2s+1} \alpha'(i,k) \beta'(i,k)) \left( 1 - (-q)^{2s+1} \alpha'(i,k) \beta'(i,k) \right)} \times \left( 1 - (-q)^{s+1} \gamma'(i,k) \right),
\end{align*}
\]

\[
\begin{align*}
\tilde{C}_s^{(i,k)} &= \frac{(-q) (1 - (-q)^s) \left( 1 - (-q)^s \beta'(i,k) \right) \left( 1 - (-q)^s \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right)}{(1 - (-q)^s \alpha'(i,k) \beta'(i,k)) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right) \times \left( 1 - (-q)^s \alpha'(i,k) \beta'(i,k) \right)},
\end{align*}
\]
Using these notations and (52), the recursion relation (51) reduces to

\[
\mu \left( j_{i+k}^{(i,k+1)} \right) \mathcal{G}_{j_{i+k}^{(i,k+1)}} = \mathcal{G}_{j_{i+k}^{(i,k+1)}} \left( \mathcal{G}_{j_{i+k}^{(i,k)}} + \left( 1 - q^{(i,k)} \delta^{(i,k)} - A_{j_{i+k}^{(i,k)}} - C_{j_{i+k}^{(i,k)}} \right) \mathcal{G}_{j_{i+k}^{(i,k+1)}} + A_{j_{i+k}^{(i,k)}} \mathcal{G}_{j_{i+k}^{(i,k+1)}} \right),
\]

This is the well-known recursion relation for \((-q)\)-Racah polynomials \([30]\). Taking into account the initial condition (43), we can write its solution as

\[
\mathcal{G}_{j_{i+k}^{(i,k)}} = \frac{(-\alpha^{(i,k)} q, -\beta^{(i,k)} \delta^{(i,k)} q, -\gamma^{(i,k)} q; -q)_{j_{i+k}^{(i,k)}}}{(\alpha^{(i,k)} \beta^{(i,k)} (-q)^{j_{i+k}^{(i,k)}+1}; -q)_{j_{i+k}^{(i,k)}} \times f_{j_{i+k}^{(i,k)}} \left( \mu \left( j_{i+k}^{(i,k+1)} \right); \alpha^{(i,k)}, \beta^{(i,k)}, \gamma^{(i,k)}, \delta^{(i,k)} \right) + q),
\]

where

\[
f_{n}(\mu(x); \alpha, \beta, \gamma, \delta; -q) = 4f_{n} \left( (-q)^{-n}, \alpha \beta (-q)^{n+1}, (-q)^{-x}, \gamma \delta (-q)^{x+1} \mid -q, -q \right),
\]

and

\[
\mu(x) = (-q)^{-x} + \gamma \delta (-q)^{x+1}.
\]

Upon permuting the arguments of the basic hypergeometric function, it is seen that these functions relate to the \((-q)\)-Racah polynomials as

\[
r_{n}(x; a, b, c, N; -q) = (-aq, -bcq, (-q)^{-N}; -q)_{n} \left( \frac{(-q)^{N}}{c} \right)^{\frac{x}{q}} \times f_{x} \left( \mu(n); (-q)^{-N-1}, c, a, b \mid -q \right).
\]

Observing that \(\alpha^{(i,k)} = (-q)^{-\tilde{\nu}^{(i,k)}-1},\) we can combine (55) and (56) to find the expression (53).

\[\square\]

3.2. **Multivariate \((-q)\)-Racah polynomials.** Gasper and Rahman introduced their multivariable \(q\)-Racah polynomials in \([18]\) as \(q\)-analogs of Tratnik’s \(q = 1\) Racah polynomials \([45]\). They were defined as entangled products of univariate \(q\)-Racah polynomials, depending on \(s\) discrete indices \(n_{i}\), \(s\) continuous variables \(x_{i}\), and parameters \(a_{1}, \ldots, a_{s+1}, b, N \in \mathbb{N}:\)

\[
R_{(n_{1}, \ldots, n_{s})}(x_{1}, \ldots, x_{s}; a_{1}, \ldots, a_{s+1}, b, N|q)
\]

\[
= \prod_{k=1}^{s} r_{n_{k}} \left( x_{k} - N_{k-1}; -\frac{b A_{k}}{a_{1}} q^{2 N_{k-1}}, \frac{a_{k+1}}{q}, A_{k} q^{x_{k+1}+N_{k-1}}, x_{k+1} - N_{k-1} \mid q \right),
\]

where

\[
N_{k} = \sum_{i=1}^{k} n_{i}, \quad A_{k} = \prod_{i=1}^{k} a_{i}, \quad x_{s+1} = N.
\]

They are homogeneous polynomials of total degree \(N_{s}\) in the variables

\[
q^{-x_{k}} + A_{k} q^{x_{k}}, \quad k \in \{1, \ldots, s\}.
\]

They turn out to be orthogonal on the simplex

\[
\{(x_{1}, \ldots, x_{s}) \in \mathbb{N}^{s} : 0 \leq x_{1} \leq x_{2} \leq \cdots \leq x_{s} \leq N\},
\]
in the sense that
\begin{equation}
\sum_{x_s=0}^{N_s} \sum_{x_{s-1}=0}^{x_2} \cdots \sum_{x_1=0}^{x_2} \rho(x)R_n(x; a, b, N|q) R_{n'}(x; a, b, N|q) = h_n \delta_{n,n'},
\end{equation}

for a certain weight function \(\rho(x)\) and normalization coefficient \(h_n\). We refer to [18, (2.14) and (2.16)] for explicit expressions.

As before, the polynomials that we will make use of are multivariate \((-q)\)-Racah polynomials, i.e. the polynomials (57) where we change all \(q\) to \(-q\).

In Section 3.1 we found expressions for the overlap coefficients between the eigenbases of the algebras \(\mathcal{Y}_{i,k}\) and \(\mathcal{Y}_{i,k+1}\). Gluing these intermediate steps together, we can compute the overlaps
\[
\langle j^{(i,j)}_i | j^{(i,1)}_i \rangle = \langle j^{(i,j)}_{n-1} | \cdots | j^{(i,j)}_{j} | j^{(i,1)}_{j} | \cdots | j^{(i,1)}_{1} \rangle
\]
corresponding to the subalgebras \(\mathcal{Y}_{i,1}\) and \(\mathcal{Y}_{i,j}\).

**Lemma 5.** The connection coefficients between the eigenbases of the subalgebras \(\mathcal{Y}_{i,1}\) and \(\mathcal{Y}_{i,j}\) have the expression
\[
\langle j^{(i,j)}_i | j^{(i,1)}_i \rangle = \delta_{[i+1]}_{[i+1]} \prod_{\ell=1}^{i-1} \delta_{[j]^\ell_{[j]}} \prod_{\ell=i+j}^{n-1} \delta_{[j+1]_{[j]}} \prod_{\ell=i+1}^{i+j-2} \omega_{[i,j,\ell]} G_{[j^{(i)}_{\ell}] [j^{(i,1)}_{\ell+1}] [j^{(i,1)}_{\ell+2}] \cdots [j^{(i,1)}_{n-1}] [j^{(i,j)}_i] [j^{(i,1)}_i] \cdots [j^{(i,1)}_1]}
\]

with
\[
\hat{j}^{(i,j,\ell)} = \langle j^{(i,j)}_1 | \cdots | j^{(i,j)}_{\ell} | j^{(i,1)}_{\ell+1} \rangle - \langle j^{(i,j)}_1 | \cdots | j^{(i,j)}_{\ell} | j^{(i,1)}_{\ell+2} \cdots | j^{(i,1)}_{n-1} \rangle.
\]

**Proof.** By induction on \(j\). The case \(j = 2\) follows from Lemma 4 with \(k = 1\). Assuming the statement holds for \(j\), one proceeds by writing
\[
\langle j^{(i,j+1)}_i | j^{(i,1)}_i \rangle = \sum_{j^{(i,j)}} \langle j^{(i,j+1)}_i | j^{(i,j)}_i \rangle \langle j^{(i,j)}_i | j^{(i,1)}_i \rangle
\]
and using the induction hypothesis and Lemma 4. The products of Kronecker deltas reveal that each term in the sum above vanishes unless
\begin{equation}
\begin{aligned}
J^{(i,j)}_\ell &= J^{(i,j+1)}_\ell, & \ell &\in [1; i + j - 2] \\
J^{(i,j)}_{i+j-1} &= J^{(i,1)}_{i+j-1} - J^{(i,j+1)}_{i+j-2} \\
J^{(i,j)}_\ell &= J^{(i,1)}_\ell, & \ell &\in [i + j; n - 1].
\end{aligned}
\end{equation}

This concludes the induction. \(\square\)

Substituting the expressions for \(G_{[j^{(i)}_{\ell}] [j^{(i,1)}_{\ell+1}] \cdots [j^{(i,1)}_{n-1}] [j^{(i,j)}_i] [j^{(i,1)}_i] \cdots [j^{(i,1)}_1]}\) as obtained in Proposition 6, we find an explicit expression for the overlap coefficients. In order to assure that these coefficients are real-valued, we will from now on impose the supplementary requirement that \(q\) be a real number between 0 and 1.
Theorem 1. The overlap coefficients between the eigenbases of $\mathcal{Y}_{i,1}$ and $\mathcal{Y}_{i,j}$ can be expressed as
\begin{equation}
\langle j^{(i,j)} | j^{(i,1)} \rangle = \delta_{j^{(i,j)}_{i+j-1},j^{(i,1)}_{i+j-1}} \prod_{\ell=1}^{i-1} \delta_{j^{(i,j)}_{\ell},j^{(i,1)}_{\ell}} \prod_{\ell=i+1}^{n-1} \delta_{j^{(i,j)}_{\ell},j^{(i,1)}_{\ell}} \sqrt{\frac{\rho^{(i,j)}(j^{(i,1)})}{h^{(i,j)}}} R_n\left( x; (-q)^{2|j^{(i,j)}_{i-1}| - 1} q^{2\gamma_{i+1}}, q^{2\gamma_{i+2}}, \ldots, q^{2\gamma_{i+j}}, -q^{2\gamma_{i+1}-1}, |j^{(i,1)}_{1}| - |j^{(i,1)}_{i-1}| - q \right),
\end{equation}
where $n$ and $x$ stand for
\begin{equation}
n = (n_1, \ldots, n_j), \quad n_k = j^{(i,j)}_{i+k-1}, \quad x = (x_1, \ldots, x_{j-1}), \quad x_k = |j^{(i,1)}_{i+k-1}| - |j^{(i,1)}_{i-1}|.
\end{equation}
The expressions for the functions $\rho^{(i,j)}(j^{(i,1)})$ and $h^{(i,j)}$ can be found in Appendix A.

Proof. Combining Lemma 5, Proposition 6 and the expression (52), we find
\begin{equation}
\langle j^{(i,j)} | j^{(i,1)} \rangle = \delta_{j^{(i,j)}_{i+j-1},j^{(i,1)}_{i+j-1}} \prod_{\ell=1}^{i-1} \delta_{j^{(i,j)}_{\ell},j^{(i,1)}_{\ell}} \prod_{\ell=i+1}^{n-1} \delta_{j^{(i,j)}_{\ell},j^{(i,1)}_{\ell}} C_{j^{(i,1)},j^{(i,j)}} \prod_{k=i}^{i+j-2} r^{(i,j)}_{j^{(i,j)}} \left( |j^{(i,1)}_{k}| - |j^{(i,j)}_{k-1}|; \gamma_k, \beta_k, |j^{(i,1)}_{k-1}| - |j^{(i,j)}_{k-1}| - q \right),
\end{equation}
for some proportionality factor $C_{j^{(i,1)},j^{(i,j)}}$ depending on $j^{(i,1)}$ and $j^{(i,j)}$, and with
\begin{align*}
\tilde{\beta}_k &= q^{-1} c^{(i,j)}_k d^{(i,j)}_k, \\
\tilde{\gamma}_k &= -q^{-1} a^{(i,j)}_k d^{(i,j)}_k, \\
\tilde{\delta}_k &= -q^{-1} a^{(i,j)}_k d^{(i,j)}_k,
\end{align*}
where
\begin{align*}
\tilde{a}^{(i,j)}_k &= (-q)^{\tilde{b}^{(i,j)}_k - |j^{(i,j)}_{k-1}|} q^{\gamma_{i+k-1} - \frac{1}{2}} \\
\tilde{b}^{(i,j)}_k &= (-q)^{-|j^{(i,j)}_{k-1}| + \tilde{b}^{(i,j)}_k} q^{-\gamma_{i+k-1} + \frac{1}{2}} \\
\tilde{c}^{(i,j)}_k &= (-q)^{\tilde{b}^{(i,j)}_k + |j^{(i,j)}_{k-1}|} q^{2\gamma_{i+k-1} + \gamma_{i+k-1} - \frac{1}{2}} \\
\tilde{d}^{(i,j)}_k &= (-q)^{\tilde{b}^{(i,j)}_k - |j^{(i,j)}_{k-1}|} q^{2\gamma_{i+k-1} + \gamma_{i+k-1} - \frac{1}{2}},
\end{align*}
are obtained from (48) upon consecutively fixing the intermediary indices $j^{(i,k)}_k$ by (60) with $k$ instead of $j$. Comparison with (57), with all $q$ changed to $-q$, tells us that the product of univariate polynomials above is in fact a multivariate $(-q)$-Racah polynomial with parameters
\begin{equation}
s = j - 1, \quad a_1 = (-q)^{2|j^{(i,1)}_{i-1}| - 1} q^{2\gamma_{i+2}}, \quad a_k = q^{2\gamma_{i+k}}, k \in \{2, \ldots, j\},
\end{equation}
\begin{equation}
b = -q^{2\gamma_{i+1}} - 1, \quad N = |j^{(i,1)}_{i+j-1}| - |j^{(i,1)}_{i-1}|
\end{equation}
and with $n_k$ and $x_k$ as in (62).

The factor $C_{j^{(i,1)},j^{(i,j)}}$ can be obtained from the orthogonality relation for the multivariate $(-q)$-Racah polynomials. Since the vectors $|j^{(i,k)}_k\rangle$ are only determined
up to a phase factor, we may assume the $C_{j_{i+1, j_{i+j}}}$ to be real and positive. As we have chosen the functions $|j^{(i,j)}\rangle$ to be mutually orthonormal, we have

$$\langle j^{(i,j)} | j^{(i,j)} \rangle = \delta_{j_{i+1, j_{i+j}}},$$

whereas on the other hand we have

$$\langle j^{(i,j)} | j^{(i,j)} \rangle = \sum_{\tilde{j}^{(i,1)}} \langle \tilde{j}^{(i,j)} | j^{(i,1)} \rangle \langle j^{(i,1)} | \tilde{j}^{(i,j)} \rangle$$

$$= \delta_{\tilde{j}_{i+1, j_{i+j}}-1} \prod_{\ell=1}^{i-1} \delta_{\tilde{j}_{\ell} - j_{\ell}}^\prime \prod_{\ell=i+1}^{n-1} \delta_{\tilde{j}_{\ell} - j_{\ell}}$$

$$\sum_{\tilde{j}^{(i,1)}} C_{\tilde{j}^{(i,1)}, j^{(i,1)}} C_{j^{(i,1)}, \tilde{j}^{(i,1)}} R_n (x; a, b, N - q) R_{n'} (x; a, b, N - q),$$

with $a = (a_1, \ldots, a_j)$, $b$ and $N$ as in (63), $x$ as in (62) and

$$n = (j^{(i,j)}_1, \ldots, j^{(i,j)}_{i+j-2}), \quad n' = (j^{(i,j)}_1, \ldots, j^{(i,j)}_{i+j-2}).$$

The sum is over all vectors $\tilde{j}^{(i,1)}$ with

$$\tilde{j}_{\ell} = j_{\ell} - 1, \quad \ell \in \{i; i - 1\} \cup \{i + j; n - 1\}, \quad \tilde{j}_{i+1, j_{i+j}} = \tilde{j}_{i+1, j_{i+j}} - 1,$$

hence we are in fact summing over all $j^{(i,1)}_i, j^{(i,1)}_{i+1}, \ldots, j^{(i,1)}_{i+j-2} \in \mathbb{N}^{i-1}$ satisfying

$$0 \leq j^{(i,1)}_i \leq j^{(i,1)}_{i+1} \leq j^{(i,1)}_{i+2} \leq \cdots \leq |j^{(i,1)}_{i+j-2} - j^{(i,1)}_{i-1}| \leq N,$$

precisely as required in the orthogonality relation (59) of the multivariate $(-q)$-Racah polynomials. We conclude that the factor $C_{j^{(i,1)}, j^{(i,1)}} C_{j^{(i,1)}, j^{(i,1)}}$ coincides with

$$\frac{\rho(x)}{h_n}$$

with the substitutions (62) and (63), which lead to the expressions in Appendix A. This factor is positive by our requirement that $0 < q < 1$, hence $C_{j^{(i,1)}, j^{(i,1)}}$ can be identified with its positive square root. This concludes the proof. \qed

Remark 7. From the orthonormality of the vectors $|j^{(i,1)}\rangle$ one can deduce a second identity for the $(-q)$-Racah polynomials, known as the dual orthogonality relation. Indeed, equating

$$\delta_{j^{(i,1)}, j^{(i,1)}} = \langle j^{(i,1)} | j^{(i,1)} \rangle = \sum_{\tilde{j}^{(i,j)}} \langle j^{(i,1)} | \tilde{j}^{(i,j)} \rangle \langle \tilde{j}^{(i,j)} | j^{(i,1)} \rangle,$$

we obtain an expression of the form

$$\sum_{n \in \mathbb{N}^{i-1}} \frac{\rho(x)}{h_n} R_n (x; a, b, N - q) R_{n'} (x'; a, b, N - q) = \delta_{x, x'},$$

as was obtained in [19, formula (49b)], using duality properties of the multivariate $q$-Racah polynomials. This duality between the degrees $n_k$ and the variables $x_k$ will also be of use for our discrete realization of the $q$-Bannai-Ito algebra in the next section.
4. Realization with difference operators

The multivariate $q$-Racah polynomials (57), like their univariate counterparts, can be obtained upon reparametrization and truncation from the multivariate Askey-Wilson polynomials. For convenience of the reader we repeat here the definition given by Gasper and Rahman in [17]. Let $\mathbf{y} = (y_1, \ldots, y_s) \in \mathbb{R}^s$, $\mathbf{n} = (n_1, \ldots, n_s) \in \mathbb{N}^s$ and $\mathbf{\alpha} = (\alpha_0, \ldots, \alpha_{s+2}) \in \mathbb{R}^{s+3}$, then the multivariate Askey-Wilson polynomials are given by

$$P_s(\mathbf{n}; \mathbf{y}, \mathbf{\alpha} | q) = \prod_{k=1}^s p_{n_k} \left( y_k; \alpha_k q^{N_{k-1}}, \frac{\alpha_k}{\alpha_0}, \frac{\alpha_{k+1}}{\alpha_k}, \frac{\alpha_{k+1}}{\alpha_{k+2}} q^{-1} | q \right),$$

where $z_i$ is such that $y_i = \frac{1}{q} (z_i + z_i^{-1})$ for $i \in [1; s]$, and $z_{s+1} = \alpha_{s+2}$, and where the univariate Askey-Wilson polynomials are defined as [1]

$$p_n(y; a, b, c, d | q) = \frac{(ab, ac, ad; q)_n}{a^n} \phi_3 \left( \begin{array}{c} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{array} ; q, q \right),$$

with again $y = \frac{1}{q} (z + z^{-1})$.

In [25] Iliev obtained a set of $s$ mutually commuting and algebraically independent difference operators acting on the variables $z_i$, which are all diagonalized by the polynomials (64). A duality transformation between the degrees $\mathbf{n}$ and the variables $\mathbf{y}$ allows to define a second set of $s$ difference operators diagonalizing the same polynomials, again mutually commuting but this time acting by discrete shifts in the degrees $n_i$. Both these sets of difference operators serve as generating sets for a commutative algebra. In this section, we will provide a larger algebraic framework for these algebras.

4.1. Bispectrality of the multivariate $(-q)$-Racah polynomials. Throughout this section we will take $s$ to be the total number of variables, so we will consider $(-q)$-Racah polynomials in $s$ real variables $x_i$, indexed by a tuple $\mathbf{n} = (n_1, \ldots, n_s)$ of $s$ natural numbers, and difference operators in several of the variables $x_i$ and the degrees $n_i$.

The precise reparametrization that one needs to go from the multivariate $(-q)$-Askey-Wilson polynomials $P_s(\mathbf{n}; \mathbf{y}, \mathbf{\alpha} | q)$ in (64) to the multivariate $(-q)$-Racah polynomials $R_s(\mathbf{x}; a, b, N | -q)$ of (57) is the following:

$$\begin{align*}
\alpha_0 &= \sqrt{\frac{a_1}{ab}}, \\
\alpha_i &= \sqrt{\frac{a_i}{a_{i+1}}}, \quad i \in \{1, \ldots, s+1\}, \\
\alpha_{s+2} &= (-q)^N \prod_{k=1}^{s+1} \sqrt{a_k}, \\
\beta_0 &= \sqrt{\frac{a_1}{qab}}, \\
\beta_i &= (-q)^{x_i} \sqrt{\frac{a_i}{a_{i+1}}}, \quad i \in \{1, \ldots, s\}, \\
\beta_{s+1} &= (-q)^N \prod_{k=1}^{s+1} \sqrt{a_k}.
\end{align*}$$

As observed in [25, Remark 2.3], the $q$-difference operators for the Askey-Wilson polynomials translate to similar operators diagonalizing the multivariate $(-q)$-Racah polynomials under the above reparametrization. For convenience of the reader, we repeat here the statements from [25, Propositions 4.2, 4.5 and Theorem 5.5], translated to the setting of $(-q)$-Racah polynomials by means of (65). Note that a $(-q)$-shift $z_i \rightarrow -qz_i$ corresponds to a discrete shift $x_j \rightarrow x_j + 1$, as is immediate from (65). This suggests the notation $T_{+,x_j}$ for the operator

$$T_{+,x_j} f(x_1, \ldots, x_s) = f(x_1, \ldots, x_i + 1, \ldots, x_s).$$
Let us first study the shift operators in the variables \(x_i\).

**Proposition 7 ([25]).** Let us define the operator

\[
\mathcal{L}^x_j(x_1, \ldots, j; a_1, \ldots, a_{j+1}, b, x_{j+1} | - q) = \sum_{\nu \in \{-1, 0, 1\}} C_\nu(x_1, \ldots, x_{j+1}) T^\nu_{+, x_1} \cdots T^\nu_{+, x_j} - \left(1 + \frac{b A_{j+1}}{a_1} - \frac{(-q)^{-x_{j+1}}}{1+q} \right) \left(1 + A_{j+1}(-q)^{2x_{j+1}} \right),
\]

referring to the notation (58), with the convention that \(x_{s+1} = N\) and where the \(C_\nu(x_1, \ldots, x_{j+1})\) are functions in the variables \((-q)^{x_1}, \ldots, (-q)^{x_{j+1}}\), explicit expressions of which can be found in Appendix B. Then the operators

\[
\mathcal{L}^x_j(x_1, \ldots, x_j; a_1, \ldots, a_{j+1}, b, x_{j+1} | - q), \quad j \in \{1, \ldots, s\},
\]

form a set of \(s\) mutually commuting operators, with the multivariate \((-q)\)-Racah polynomials

\[
R_n(x; a, b, N | - q) = R_{(a_1, \ldots, a_s)}((x_1, \ldots, x_s); a_1, \ldots, a_{s+1}, b, N | - q)
\]

as common eigenfunctions:

\[
\mathcal{L}^x_j(x_1, \ldots, x_j; a_1, \ldots, a_{j+1}, b, x_{j+1} | - q) R_n(x; a, b, N | - q) = \mu_j R_n(x; a, b, N | - q),
\]

with eigenvalue

\[
\mu_j = - \left(1 - (-q)^{-N_j} \right) \left(1 + \frac{b A_{j+1}}{a_1} \right) (-q)^{N_j}.
\]

In analogy to [25], we will denote by \(\mathcal{D}_{a, b, N}\) the algebra

\[
\mathcal{D}_{a, b, N}^{a, b, N} = \mathbb{C} \left((-q)^{x_1}, \ldots, (-q)^{x_s}, \sqrt{a_1}, \ldots, \sqrt{a_{s+1}}, \sqrt{b}, (-q)^N \right) [T^\pm_{+, x_1}, \ldots, T^\pm_{+, x_s}]
\]

of difference operators with coefficients rational in the variables \((-q)^{x_i}\) and the parameters \(\sqrt{a_i}, \sqrt{b}\) and \((-q)^N\). The analog of Ilijev's algebra \(A_z\) in our \((-q)\)-Racah setting is the algebra

\[
A_{x, s}(x_1, \ldots, x_s) = \left\{ \mathcal{L}^x_j(x_1, \ldots, x_j; a_1, \ldots, a_{j+1}, b, x_{j+1} | - q) : j \in \{1, \ldots, s\} \right\},
\]

considered as subalgebra of \(\mathcal{D}_{a, b, N}^{a, b, N}\). By Proposition 7, \(A_{x, s}(x_1, \ldots, x_s)\) is an abelian algebra of difference operators in the variables \(x_i\). We will relate this algebra to the higher rank \(q\)-Bannai-Ito algebra in Section 4.2.

Like before, let us denote by \(T^+_{+, n_i}\) the forward shift operator in the discrete variables \(n_i\), i.e.

\[
T^+_{+, n_i} f(n_1, \ldots, n_s) = f(n_1, \ldots, n_i + 1, \ldots, n_s).
\]

Changing all \(x_i\) to \(n_i\) in (66), one obtains the algebra

\[
\mathcal{D}_{a, b, N}^{a, b, N} = \mathbb{C} \left((-q)^{n_1}, \ldots, (-q)^{n_s}, \sqrt{a_1}, \ldots, \sqrt{a_{s+1}}, \sqrt{b}, (-q)^N \right) [T^\pm_{+, n_1}, \ldots, T^\pm_{+, n_s}].
\]
Let us now introduce a transformation $b$ acting as
\[
\begin{align*}
    b((-q)^{x_j}) &= \frac{bA_{x-j+2}(-q)^{-N}}{a_1A_{x+1}}, \\
    b(a_1) &= \frac{a_1}{b}A_{x+1}(-q)^{2N}, \\
    b(T_{x,x_1}) &= T_{x,n_x},
\end{align*}
\]
and extended by linearity and multiplicativity to an isomorphism from $D_{\alpha,b,N}$ to $D_{\alpha,b,N}$. Define also
\[
g_n = g_{n,a,b,N} = \frac{(A_{s+1}(-q)^N)_{x_s}}{(A_{s+1}(-q)^N)_{x_s}^N} \left( \prod_{j=1}^N \frac{(a_j; q)_{y_j}}{(a_j; q)_{y_j}^N} \right),
\]
where we have again used the $(q)$-Pochhammer symbol (41). The analogs of Iliev's renormalized Askey-Wilson polynomials, see [25, (5.4)], are the renormalized $(q)$-Racah polynomials
\[
\text{(69) } \tilde{R}_n(x; a, b, N | - q) = g_{n,a,b,N} R_n(x; a, b, N | - q).
\]
Iliev's second set of difference operators is described in the following proposition.

**Proposition 8** ([25]). Let us define the operator
\[
\begin{align*}
    \mathcal{L}_n^\alpha(n_1, \ldots, n_s; a_1, \ldots, a_{s+1}, b, N | - q) &= b(\mathcal{L}_x^\alpha(x_1, \ldots, x_j; a_1, \ldots, a_{s+1}, b, x_{j+1} | - q)) \\
    &= \sum_{\nu \in \{-1, 0, 1\}^j} D_{\nu}(n_1, \ldots, n_s) T_{x,n_x}^{\nu_j} T_{x,n_x-j+1}^{\nu_{j+1}} \ldots T_{x,n_x-j+2}^{\nu_{2j}} \ldots T_{x,n_s}^{\nu_{2j}},
\end{align*}
\]
where the $D_{\nu}(n_1, \ldots, n_s) = b(C_{\nu}(x_1, \ldots, x_{j+1}))$ are functions in the variables $(-q)^{n_1}, \ldots, (-q)^{n_s}$, explicit expressions of which can be found in Appendix B. Then the operators
\[
\mathcal{L}_n^\alpha(n_1, \ldots, n_s; a_1, \ldots, a_{s+1}, b, N | - q), \quad j \in \{1, \ldots, s\},
\]
are mutually commuting with the renormalized $(q)$-Racah polynomials as common eigenfunctions:
\[
\mathcal{L}_n^\alpha(n_1, \ldots, n_s; a_1, \ldots, a_{s+1}, b, N | - q) \tilde{R}_n(x; a, b, N | - q) = \kappa_j \tilde{R}_n(x; a, b, N | - q),
\]
or equivalently
\[
(g_n^{-1} \mathcal{L}_n^\alpha(n_1, \ldots, n_s; a_1, \ldots, a_{s+1}, b, N | - q) g_n) \tilde{R}_n(x; a, b, N | - q) = \kappa_j \tilde{R}_n(x; a, b, N | - q),
\]
with eigenvalue
\[
\kappa_j = \frac{1}{2} \frac{A_{s+1}(-q)^N + A_{s+1}(-q)^N}{A_{s+1}(-q)^N} + \frac{(-q)^{-2s+1}}{A_{s+1}(-q)^N} + \frac{(-q)^{2s+1}}{A_{s+1}(-q)^N}.
\]

The role of Iliev's second algebra $A_{n_i}$ is now played by
\[
\mathcal{A}_{n_i, s}(n_1, \ldots, n_s) = \langle \mathcal{L}_n^\alpha(n_1, \ldots, n_s; a_1, \ldots, a_{s+1}, b, N | - q) : j \in \{1, \ldots, s\} \rangle,
\]
considered as subalgebra of $D_{n_i, s}^{a,b,N}$. By Proposition 8, $A_{n_i, s}(n_1, \ldots, n_s)$ is an abelian algebra of difference operators in the variables $n_i$. 
4.2. Discrete realization of the higher rank $q$-Bannai-Ito algebra. For ease of notation, let us from now on write $|s\rangle = |s_{n-1}, s_{n-2}, \ldots, s_1\rangle$ for the vectors
\[(71) \quad |s\rangle = |j^{(1,1)}\rangle = |j^{(1,1)}\rangle\]
diagonalizing the subalgebra
\[
\mathcal{Y}_{1,1} = \mathcal{Y}_{1,1} = \langle \Gamma^q_{[1;2]}, \Gamma^q_{[1;3]}, \ldots, \Gamma^q_{[1;n]} \rangle,
\]
with $i \in [1; n-1]$ arbitrary, where we have used the equalities from Remarks 1 and 5. Similarly, we write $|j\rangle = |j_{n-1}, j_{n-2}, \ldots, j_1\rangle$ for the vectors
\[(72) \quad |j\rangle = |j^{(1,n-1)}\rangle\]
that diagonalize
\[
\mathcal{Y}_{1,n-1} = \langle \Gamma^q_{[2;3]}, \Gamma^q_{[2;4]}, \ldots, \Gamma^q_{[2;n]}, \Gamma^q_{[1;n]} \rangle.
\]
We will now use the results from Section 4.1, for a total number of variables $s = n - 2$ and with certain substitutions for the parameters $a_i$ and $b$ and renamings for the variables $x_i$ and the degrees $n_i$. These will lead us to a realization of the higher rank $q$-Bannai-Ito algebra $A^q$. We will call this realization discrete, as it represents the algebra generators $\Gamma^q_2$ as shift operators in $2n - 2$ discrete variables $j_1, \ldots, j_{n-1}, s_1, \ldots, s_{n-1} \in \mathbb{N}$, i.e. the quantum numbers labelling the vectors $|j\rangle$ and $|s\rangle$. We will first introduce two function spaces, one of which will be our representation space.

**Definition 6.** We denote by $\mathcal{V}_q$ the infinite-dimensional vector space over $\mathbb{C}$ spanned by all overlap coefficients
\[
\langle j|s \rangle, \quad j \in \mathbb{N}^{n-1},
\]
considered as functions of $s_1, \ldots, s_{n-1} \in \mathbb{N}$. We will write $\tilde{\mathcal{V}}_q$ for the infinite-dimensional vector space spanned by the renormalized multivariate $(-q)$-Racah polynomials
\[
\tilde{R}_{(j_1, \ldots, j_{n-2})}((s_1, s_1 + s_2, \ldots, |s_{n-2}|); a, b, |s_{n-1}|; |q\rangle, \quad j \in \mathbb{N}^{n-1},
\]
again considered as functions of $s \in \mathbb{N}^{n-1}$, where $j_{n-1}$ is fixed by the constraint $|j_{n-1}| = |s_{n-1}|$ and with
\[(73) \quad a = (a_1, \ldots, a_{n-1}), \quad a_1 = -q^{2(\gamma_1 + \gamma_2) - 1}, \quad a_k = q^{2\gamma_{k+1}}, k \in [2; n-1], \quad b = -q^{2\gamma_2 - 1}.
\]

It will suffice to give an explicit realization for each of the elements $\Gamma^q_{[i+1;i+j]}$, $i \in [0; n-1], j \in [1; n-i]$, agreeing with the algebra relations in Propositions 1 and 2, since these generate the full algebra $A^q_n$ by Corollary 1. We will use the approach from [13] to find appropriate realizations. In order to pass from the abstract algebraic perspective to the setting of operators acting on functions of discrete variables, we will lift the action of the algebra to the connection coefficients. More precisely, in a first step we will look for suitable operators $\Gamma^q_A$ which act on the connection coefficients as follows:
\[(74) \quad \Gamma^q_A (|j|s \rangle = \langle j|\Gamma^q_A |s\rangle.
\]
Afterwards, we will slightly modify the considered $\Gamma^q_A$ to obtain more elegant expressions. This will be our strategy of proof in the following theorem.
Theorem 2. Let $\gamma_i > \frac{1}{2}$, $i \in [1; n]$, be a set of real parameters. Let us define

\begin{equation}
\Gamma^q_{[1;n+1]} = - \frac{(-q)^{-\frac{1}{2}m+1}q^{-\gamma_{[1;n]}-\gamma_{[m+2;n]}+\frac{1}{2}}}{q - q^{-1}} q^m m^{-1} + (-1)^{m-1} \left[ s_{n-1} + \gamma_{[1;n]} + \gamma_{[m+2;n]} - \frac{1}{2} \right] q,
\end{equation}

for $m \in [0; n - 1]$ and

\begin{equation}
\Gamma^q_{[i+1;i+j]} = \frac{(-q)^{-\gamma_{[i+1;i+j]}+\frac{1}{2}}}{q - q^{-1}} L_{i,j-1}^q + \left[ \gamma_{[i+1;i+j]} - \frac{1}{2} \right] q,
\end{equation}

for $i \in [1; n - 1]$, $j \in [1; n - i]$, where we have written $\mathcal{L}_{m-1,j-1}$ for

\begin{equation}
\mathcal{L}_{m-1,j-1} \left( s_1, \ldots, s_{i+j-2} - \left| s_{i-1} \right| \left| -q^{2\gamma_{[1;i+1]}-1}, q^{2\gamma_{i+1}} - q^{2\gamma_{i+2}} - \cdots - q^{2\gamma_{i+j-1}} - \left| s_{i+j-1} \right| - \left| s_{i-1} \right| \right) \right).
\end{equation}

with the convention that $L_{i,0} = L_{0} = 0$ and that

\begin{equation}
\mathcal{L}_{m-1} = (q - q^{-1}) q^{s_{m-1} + \gamma_{[1;n]} + 2\gamma_{[2;n]} - \frac{1}{2}} \left[ \left| s_{n-1} \right| + \gamma_{1} + 2\gamma_{[2;n]} - \frac{1}{2} \right] q^{-1} - (-1)^{s_{m-1}} \left[ \gamma_{1} - \frac{1}{2} \right] q.
\end{equation}

The algebra generated by the operators (75) and (76) forms a discrete realization of the $q$-Bannai-Ito algebra $A^q_n$ of rank $n - 2$ on the module $V_q$.

Proof. Recall from Theorem 1, with $i = 1$ and $j = n - 1$, that

\begin{equation}
\langle j | s \rangle = \langle j_{n-1}, \ldots, j_1 | s_{n-1}, \ldots, s_1 \rangle
\end{equation}

\begin{equation}
= \sqrt{\frac{\rho_{[1,n-1]}(s)}{\rho_{[1,n-1]}(s)}} R_{j_1, \ldots, j_{n-2}} (s_1, s_1 + s_2, \ldots, | s_{n-2} |; a, b, | s_{n-1} | - q),
\end{equation}

where the parameters $a$ and $b$ are given by (73). Here we assume that $| j_{n-1} | = | s_{n-1} |$, otherwise $\langle j | s \rangle = 0$.

Our first objective will be to find suitable operators $\tilde{\Gamma}_A^q$ subject to (74). If the proposed operators satisfy this requirement, then they will automatically comply with the algebra relations and thus form a proper realization of our algebra on the space $V_q$. Indeed, referring to the notation from Definition 5, Proposition 4 asserts that for any set $B$, $\Gamma^q_B | s \rangle$ can be written as a linear combination of vectors $| s' ; s'_{n-1} + m \rangle$ such that $s' \in \mathbb{N}^{n-1}$ and $m' \in \mathbb{N}$. Moreover $\Gamma^q_B$ commutes with both $K_{[1;n]}$ and $\Gamma^q_A$, such that the linear combination will only contain terms with $m' = m$ and $| s'_{n-1} | = | s_{n-1} |$. As a consequence, if we assume that (74) holds for any set $A$, then we also find

\begin{equation}
\tilde{\Gamma}_A^q \Gamma^q_B (j | s \rangle) = \langle j | \Gamma^q_A \Gamma^q_B | s \rangle,
\end{equation}

for all sets $A$ and $B$, such that the $\tilde{\Gamma}_A^q$ will satisfy the algebra relations from Propositions 1 and 2. It will suffice to fix an operator realization for each of the generators $\Gamma^q_A$ with $A$ a set of consecutive numbers. By Corollary 1, this uniquely determines the corresponding expressions for each of the remaining generators.
In case $A$ is a singleton, (74) is met by Remark 4. Hence we may restrict ourselves to sets of consecutive numbers with at least two elements. We will subdivide those sets into three different classes.

**Case 1:** $A = [1; m + 1]$ with $m \in [1; n - 1]$

The eigenvalue equations (36) for $|s\rangle$ assert that

$$\langle j| \Gamma^q_{[1; m+1]} |s\rangle = (-1)^{|s_m|} \left[ |s_m| + \gamma_{[1; m+1]} - \frac{1}{2} \right]_q \langle j|s \rangle.$$  

(78)

On the other hand, one can observe from Proposition 8 and (77) that

$$\left( h_j^{-\frac{\rho}{2}} g_j^{-1} \mathcal{C}_{n-m-1}^{n} g_j h_j^{\frac{\rho}{2}} \right) \langle j|s \rangle
\begin{align*}
&= \sqrt{\frac{\rho(1,n-1)(s)}{h_j}} \left( g_j^{-1} \mathcal{C}_{n-m-1}^{n} g_j \right) R_{j_1,\ldots,j_{n-2}} \left( (s_1, s_1 + s_2, \ldots, |s_{n-2}|); a, b, |s_{n-1}| - q \right) \\
&= \kappa_{n-m-1} \langle j|s \rangle,
\end{align*}

where the eigenvalue is given by

$$\kappa_{n-m-1} = - (q - q^{-1})(-q)^{|s_{n-1}|} q^{\gamma_{[1; n]} - \gamma_{[m+2; n]} - \frac{\rho}{2}}$$

$$\times \left[ (-1)^{|s_{n-1}|+1} \left[ |s_{n-1}| + \gamma_{[1; n]} + \gamma_{[m+2; n]} - \frac{1}{2} \right]_q + (-1)^{|s_{n-1}|} \left[ |s_{n-1}| + \gamma_{[1; m+1]} - \frac{1}{2} \right]_q \right].$$

This suggests to define

$$\Gamma^q_{[1; m+1]} = \frac{- (q - q^{-1})(-q)^{|s_{n-1}|} q^{\gamma_{[1; n]} - \gamma_{[m+2; n]} + \frac{\rho}{2}}}{q - q^{-1}} \left( h_j^{-\frac{\rho}{2}} g_j^{-1} \mathcal{C}_{n-m-1}^{n} g_j h_j^{\frac{\rho}{2}} \right)$$

(79)

$$+ (-1)^{|s_{n-1}|} \left[ |s_{n-1}| + \gamma_{[1; n]} + \gamma_{[m+2; n]} - \frac{1}{2} \right]_q \langle j|s \rangle$$

$$= \left( g_j h_j^{\frac{\rho}{2}} \right)^{-1} \Gamma^q_{[1; m+1]} \left( g_j h_j^{\frac{\rho}{2}} \right),$$

referring to the notation (75), as this leads to

$$\Gamma^q_{[1; m+1]} \langle j|s \rangle = (-1)^{|s_{m}|} \left[ |s_{m}| + \gamma_{[1; m+1]} - \frac{1}{2} \right]_q \langle j|s \rangle,$$

in agreement with (78). We will correct for the factors $g_j$ and $h_j$ in the final step.

**Case 2:** $A = [2; m + 1]$ with $m \in [2; n - 1]$

The eigenvalue equations (37) for $|s\rangle$ assert that

$$\langle j| \Gamma^q_{[2; m+1]} |s\rangle = (-1)^{|s_{m-1}|} \left[ |s_{m-1}| + \gamma_{[2; m+1]} - \frac{1}{2} \right]_q \langle j|s \rangle.$$  

(80)

On the other hand, Proposition 7 and (77) imply that

$$\left( \frac{\rho(1,n-1)(s)}{h_j} \right)^{\frac{1}{2}} \mathcal{L}_{1,m-1}^x \left( \frac{\rho(1,n-1)(s)}{h_j} \right)^{-\frac{1}{2}} \langle j|s \rangle
\begin{align*}
&= \sqrt{\frac{\rho(1,n-1)(s)}{h_j}} \mathcal{L}_{1,m-1}^x R_{j_1,\ldots,j_{n-2}} \left( (s_1, s_1 + s_2, \ldots, |s_{n-2}|); a, b, |s_{n-1}| - q \right) \\
&= \mu_{m-1} \langle j|s \rangle,
\end{align*}
This suggests the definition
\[ (81) \]
\[ \Gamma_{[2;m+1]}^q = \frac{q^{-\gamma[2;m+1]} + \frac{i}{q}}{q-q^{-1}} \left( (\rho^{(1,n-1)}(s))^\frac{1}{2} \mathcal{L}_{1,m-1}^x (\rho^{(1,n-1)}(s))^{-\frac{1}{2}} \right) + \left[ \gamma[2;m+1] - \frac{1}{2} \right]_q, \]
referring to the notation (76), as this leads to
\[ \Gamma_{[2;m+1]}^q(\mathbf{j}|s) = (1)_{[m-1]} \left[ |m-1| + \gamma[2;m+1] - \frac{1}{2} \right]_q \langle \mathbf{j}|s \rangle \]
agreeing with (80). We will correct for the factor \( \rho^{(1,n-1)}(s) \) in the final step.

Case 3: \( A = \{i + 1; j + 1\} \) with \( i \in [2; n - 2], j \in [2; n - i] \)
The operator \( \Gamma_{[i+1; i+j]}^q \) is not diagonalized by \( \mathbf{j} \) or \( |s \rangle \), but writing
\[ (82) \]
\[ |s\rangle = \sum_{\mathbf{j} \in \{i,j\}} \langle \mathbf{j}^{(i,j)}| \langle \mathbf{j}^{(i,j)}|s\rangle, \]
where the sum is over all vectors \( \mathbf{j}^{(i,j)} \in \mathbb{R}^{n-1} \), and using the eigenvalue equations (37) for \( \langle \mathbf{j}^{(i,j)}| \), we find
\[ (83) \]
\[ \langle \mathbf{j}| \Gamma_{[i+1; i+j]}^q |s\rangle = \sum_{\mathbf{j} \in \{i,j\}} (-1)^{|\mathbf{j}^{(i,j)}| - |\mathbf{j}^{(i,j)}|} \left[ |\mathbf{j}^{(i,j)}| + \gamma[i+1;i+j] - \frac{1}{2} \right]_q \langle \mathbf{j}^{(i,j)}| \langle \mathbf{j}^{(i,j)}|s\rangle. \]

Recall from Theorem 1 that
\[ \langle \mathbf{j}^{(i,j)}|s\rangle = \sqrt{\frac{\rho^{(i,j)}(s)}{h^{(i,j)}}} R_{[j^{(i,j)}, \ldots, j^{(i,j)}]} (s, \ldots, |s_i+2 - |s_i - 1|; a, b, |s_i+j+1| - |s_i - 1| - q), \]
with
\[ a = (a_1, \ldots, a_j), \quad a_1 = (-q)^{2|s_i+1| - 1} q^{2\gamma[i+1]}, \quad a_k = q^{2\gamma_i+1}, k \in [2; j], \quad b = -q^{2\gamma_i+1}, \]
in the assumption that
\[ j^{(i,j)}_\ell = s_\ell, \quad \ell \in [1; i - 1] \cup [i + j; n - 1], \quad |j^{(i,j)}_{i+j-1}| = |s_{i+j-1}|, \]
otherwise \( \langle \mathbf{j}^{(i,j)}|s\rangle = 0 \). As a consequence of Proposition 7 we thus have
\[ \left( (\rho^{(i,j)}(s))^\frac{1}{2} \mathcal{L}_{x-i+j-1}^x (\rho^{(i,j)}(s))^{-\frac{1}{2}} \right) \langle \mathbf{j}^{(i,j)}|s\rangle \]
\[ = \sqrt{\frac{\rho^{(i,j)}(s)}{h^{(i,j)}}} \mathcal{L}_{x-i+j-1}^x R_{[j^{(i,j)}, \ldots, j^{(i,j)}]} (s, \ldots, |s_i+2 - |s_i - 1|; a, b, |s_i+j+1| - |s_i - 1| - q) \]
\[ = \mu_{i,j-1} \langle \mathbf{j}^{(i,j)}|s\rangle, \]
where the eigenvalue is given by
\[ \mu_{i,j} = -(q - q^{-1})q^{\gamma[i+1;i+j]} - \frac{1}{2} \]
\[ \times \left( (-1)^{|j(i,j) - j(i,j)|} \left[ |j(i,j) - j(i,j)| + \gamma[i+1;i+j] - \frac{1}{2} \right] q - \left[ \gamma[i+1;i+j] - \frac{1}{2} \right] q \right). \]

This suggests the definition
\[ \Gamma_{q[i+1;i+j]} = -\frac{q - q^{-1}}{q - q^{-1}} \left( \left( \rho(i,j)(s) \right)^{\frac{x}{2}} \mathcal{L}_{i,j-1}^{x} \left( \rho(i,j)(s) \right)^{-\frac{x}{2}} \right) + \left[ \gamma[i+1;i+j] - \frac{1}{2} \right] q, \]

referring to the notation (76), as this leads to
\[ \Gamma_{q[i+1;i+j]}(j^{(i,j)}|s) = (-1)^{|j^{(i,j)} - j^{(i,j)}|} \left[ |j^{(i,j)} - j^{(i,j)}| + \gamma[i+1;i+j] - \frac{1}{2} \right] q \langle j^{(i,j)}|s \rangle. \]

Since \( \Gamma_{q[i+1;i+j]} \) only acts on the variables \( s_i, \ldots, s_{i+j-2} \), we may use (82) to write
\[ \Gamma_{q[i+1;i+j]}(j|s) = \sum_{j^{(i,j)}} (jj^{(i,j)}) \Gamma_{q[i+1;i+j]}(j^{(i,j)}|s) \]
\[ = \sum_{j^{(i,j)}} (-1)^{|j^{(i,j)} - j^{(i,j)}|} \left[ |j^{(i,j)} - j^{(i,j)}| + \gamma[i+1;i+j] - \frac{1}{2} \right] q \langle jj^{(i,j)}|j^{(i,j)}|s \rangle, \]
in agreement with (83).

Each of the obtained operators \( \Gamma_{q} \) satisfies the requirement (74), hence they form a representation of \( \mathcal{A}_n^q \) on the space \( V_q \) of overlap coefficients \( (jj|s) \). As a final step, we will transform these operators into the anticipated \( \Gamma_{q}^x \) in (75) and (76), thereby changing the representation space to \( V_q^x \), but without altering the algebra relations.

First observe that each of the factors
\[ \frac{\rho^{(1,n-1)}(s)}{\rho^{(i,j)}(s)}, \]
explicit expressions of which are given in Appendix C, is independent of the variables \( |s_i, \ldots, s_{i+j-2}| \). As a consequence, they commute with the difference operators \( \mathcal{L}_{i,j-1}^{x} \) acting on precisely these variables, such that one can write
\[ \Gamma_{q[i+1;i+j]} = \left( \frac{\rho^{(1,n-1)}(s)}{\rho^{(i,j)}(s)} \right)^{\frac{x}{2}} \Gamma_{q[i+1;i+j]} \left( \frac{\rho^{(1,n-1)}(s)}{\rho^{(i,j)}(s)} \right)^{-\frac{x}{2}} \]
\[ = -\frac{q - q^{-1}}{q - q^{-1}} \left( \rho^{(1,n-1)}(s) \right)^{\frac{x}{2}} \mathcal{L}_{i,j-1}^{x} \left( \rho^{(1,n-1)}(s) \right)^{-\frac{x}{2}} + \left[ \gamma[i+1;i+j] - \frac{1}{2} \right] q \]
\[ = \left( \rho^{(1,n-1)}(s) \right)^{\frac{x}{2}} \Gamma_{q[i+1;i+j]} \left( \rho^{(1,n-1)}(s) \right)^{-\frac{x}{2}}. \]
Finally, observe that \( \tilde{\Gamma}_{q_{[1:m+1]}} \) commutes with functions of \( s \), and is hence invariant under conjugation with such functions. The analogous statement holds for \( \tilde{\Gamma}_{q_{[i+1;i+j]}} \), \( i \geq 1 \), and functions of \( j \). Hence it follows from (79), (81) and (84) that conjugation with the function

\[
g_j h_j^{\frac{1}{2}} (\mu^{(1,n-1)}(s))^\frac{1}{2}
\]

acts on each of the generators as

\[
\tilde{\Gamma}_{A} \rightarrow \Gamma_{A}^q.
\]

Such a conjugation leaves the algebra relations invariant, but requires to take as a representation space the space of all functions

\[
g_j h_j^{\frac{1}{2}} (\mu^{(1,n-1)}(s))^\frac{1}{2} \langle j | s \rangle = \tilde{R}_{(j_1,\ldots,j_{n-2})} ((s_1, s_1 + s_2, \ldots, |s_{n-2}|); a, b, |s_{n-1}| - q),
\]

with the parametrization (73), i.e. the space \( \tilde{V}_q \). This concludes the proof. \( \square \)

Let us now explain the significance of the previous theorem in relation to Ilievs work. In [25] two commutative algebras of difference operators were defined, denoted here by \( A_{x,s}(x_1,\ldots,x_s) \), see (67), and \( A_{n,s}(n_1,\ldots,n_s) \), see (70). These algebras are diagonalized by the renormalized multivariate \((-q)-Racah polynomials\) (69). Restricting their action to the space \( \tilde{V}_q \), the algebraic relations between these two algebras are encoded in Theorem 2. In fact they are observed to be embedded in a larger algebra, namely the rank \( n - 2 \) \(-q\)-Bannai-Ito algebra \( A_q^n \). In the realization of Theorem 2, one observes that

\[
\langle \Gamma_{q_{[1,m+1]}}^q : m \in [0; n - 1] \rangle = A_{n,n-2}(j_1, j_2, \ldots, j_{n-2})
\]

and

\[
\langle \Gamma_{q_{[i+1;i+j]}}^q : j \in [1; n - i] \rangle = A_{x,s}(s_1, s_1 + s_{i+1}, \ldots, |s_{n-2}| - |s_{i-1}|),
\]

for all \( i \in [1; n - 2] \). Combining this observation with Corollary 1, we conclude that several such difference operator algebras are enough to generate the rank \( n - 2 \) \(-q\)-Bannai-Ito algebra, namely

\[
A_{n,n-2}(j_1, \ldots, j_{n-2}), A_{x,s}(s_1, \ldots, |s_{n-2}|), A_{x,s}(s_2, \ldots, |s_{n-2}|-s_1), \ldots, A_{x,1}(s_{n-2})
\]

together generate the whole \( A_q^n \).

**Remark 8.** Another discrete realization could be obtained upon replacing (75) by

\[
\Gamma_{q_{[1:m+1]}}^q = (-1)^{|s_m|} \left( |s_m| + \gamma_{[1:m+1]} - \frac{1}{2} \right)_q,
\]

in analogy with [13], which by (78) immediately agrees with (74). This reduces the total number of variables from \( 2n - 2 \) to \( n - 1 \), but has the disadvantage that neither Ilievs algebras \( A_{n,s}(n_1,\ldots,n_s) \) nor the bispectrality of the considered polynomials will play a role.
5. The limit $q \to 1$

The results obtained so far establish a strong connection between the higher rank $q$-Bannai-Ito algebra $\mathcal{A}_n^q$ on the one hand and the multivariate $(-q)$-Racah polynomials on the other. In the limit $q \to 1$, $\mathcal{A}_n^q$ reduces to the rank $n-2$ Bannai-Ito algebra, which was introduced in [10] and has been in its own right the subject of intensive study, see for example [8]. In this section we will subject our results to this limiting process and show how this establishes an algebraic framework for a novel class of multivariate $(-1)$-orthogonal polynomials.

5.1. Multivariate Bannai-Ito polynomials. The Bannai-Ito polynomials were introduced in [2] in the context of algebraic combinatorics. More specifically, Bannai and Ito obtained them as limits $q \to -1$ of the $q$-Racah polynomials, in their classification of orthogonal polynomials satisfying the Leonard duality property. Following the conventions of most recent literature on the subject, for example [47], we will denote them as $B_n(x) = B_n(x; \rho_1, \rho_2, r_1, r_2, N)$, where $\rho_1, \rho_2, r_1, r_2$ are real parameters and $N \in \mathbb{N}$ is a truncation parameter. They are invariant under the transformations $\rho_1 \leftrightarrow \rho_2$ and $r_1 \leftrightarrow r_2$ and can be defined through the three-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_nB_{n-1}(x)$$

with initial conditions $B_{-1}(x) = 0, B_0(x) = 1$ and recurrence coefficients

$$A_n = \frac{(n + 2\rho_1 - 2r_1 + n_p(2\rho_2 - 2r_2) + 1)(n + 2\rho_1 - 2r_2 + n_p(2r_2 + 2\rho_2) + 1)}{4(n + \rho_1 + \rho_2 - r_1 - r_2 + 1)},$$

$$C_n = \frac{(n + n_p(2\rho_2 - 2r_2) - 2r_1 - 2r_2 + n_p(2r_2 + 2\rho_2))}{4(n + \rho_1 + \rho_2 - r_1 - r_2)},$$

where we have written

$$n = 2n_c + n_p, \quad n_c, n_p \in \mathbb{N}, n_p \in \{0, 1\}$$

as in [33], in order to combine the expressions for even and odd $n$. For explicit expressions in terms of hypergeometric series $\mathbf{4}_F\mathbf{3}$ we refer the reader to [47].

The Bannai-Ito polynomials were observed to satisfy a discrete orthogonality relation of the form

$$\sum_{k=0}^{N} w_k B_n(x_k; \rho_1, \rho_2, r_1, r_2, N)B_m(x_k; \rho_1, \rho_2, r_1, r_2, N) = h_n \delta_{n,m},$$

given that the positivity condition

$$A_{n-1}C_n > 0, \quad \forall n \in \{1, \ldots, N\}$$

is satisfied, as well as one of the following truncation conditions. For $N$ even we must have

$$i) \quad r_j - \rho_\ell = \frac{N + 1}{2},$$

for some $j, \ell \in \{1, 2\}$, whereas for $N$ odd one of the following requirements must be met:

$$i) \quad \rho_1 + \rho_2 = -\frac{N+1}{2}, \quad iii) \quad r_1 + r_2 = \frac{N+1}{2}, \quad iv) \quad \rho_1 + \rho_2 - r_1 - r_2 = -\frac{N+1}{2}.$$
These conditions are typically referred to as type i) to iv). The cases of interest to us will be type i) with \( j = \ell = 1 \) and type ii), so

\[
\begin{cases}
  r_1 - \rho_1 = \frac{N + 1}{2}, & \text{for } N \text{ even}, \\
  \rho_1 + \rho_2 = -\frac{N + 1}{2}, & \text{for } N \text{ odd}.
\end{cases}
\]

If (87) and (90) are fulfilled, then the Bannai-Ito polynomials satisfy the relation (86), with explicit expressions [21]

\[ x_k = (-1)^k \left( \frac{k}{2} + \rho_1 + \frac{1}{4} \right) - \frac{1}{4} \]

for the grid points,

\[ w_k = (-1)^k \frac{(\rho_1 + \rho_2 + 1)_{k_e} (\rho_1 - r_1 + \frac{1}{2})_{k_e+k_p} (\rho_1 - r_2 + \frac{1}{2})_{k_e+k_p} (2\rho_1 + 1)_{k_e}}{k_e! (\rho_1 + r_2 + \frac{1}{2})_{k_e+k_p} (\rho_1 + r_1 + \frac{1}{2})_{k_e+k_p} (\rho_1 - \rho_2 + 1)_{k_e}} \]

for the weight function and

\[
    h_n = \frac{N_e!n_p!}{(N_e - n_e - n_p(1-N_e))! (\rho_2 - r_1 - r_2 + n_e + 1)_{N_e+N_p-n_e}} \\
    \times \frac{(2\rho_1 + 1)_{N_e+N_p} (\rho_1 - r_2 - r_1 + r_2 + \rho_2 - r_2 + \frac{1}{2})_{n_e+n_p}}{(\rho_1 + r_2 + \frac{1}{2})_{N_e-n_e+N_p(1-n_p)} (\rho_1 + r_2 + r_1 + r_2 + \rho_1 + \rho_2 + 1)_{n_e}}
\]

for the normalization coefficient. Here we have used the notation

\[ (a)_n = \prod_{\ell=0}^{n-1} (a + \ell) \]

for the Pochhammer symbol and as before we have written

\[ N = 2N_e + N_p, \quad k = 2k_e + k_p, \quad n = 2n_e + n_p. \]

with \( N_e, k_e, n_e \in \mathbb{N} \) and \( N_p, k_p, n_p \in \{0, 1\} \). In what follows we will also use the hypergeometric series

\[ _4F_3 \left( \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{array} \right| z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, a_3, a_4)_n}{(b_1, b_2, b_3)_n} \frac{z^n}{n!}, \]

with

\[ (a_1, \ldots, a_s)_n = \prod_{k=1}^{s} (a_k)_n. \]

In order to establish the connection with the previously obtained results, we want to provide an extension of these polynomials to multiple variables. In the spirit of Tratnik [45] and Gasper and Rahman [17, 18], these multivariate Bannai-Ito polynomials should be entangled products of their univariate counterparts, which moreover should correspond to a limit \( q \to -1 \) of the multivariate \( q \)-Racah polynomials (57). In order to see how the parameters should be related and to motivate our upcoming Definition 7, we compute in the next lemma the limit \( q \to 1 \) of the
basic hypergeometric functions that arise in the definition (57) of the multivariate 
\((-q)\)-Racah polynomials. The proof uses the same techniques as [33, Section 2.1].

**Lemma 6.** Let \(n_1, \ldots, n_k\) and \(x_k, x_{k+1}\) be integers and let \(\alpha_1, \ldots, \alpha_{k+1}\) and \(\beta\) be real parameters. In the limit \(q \to 1\), the basic hypergeometric function

\[
4\phi_3 \left( \begin{array}{c}
\frac{(-q)^{-n_k} - (-q)^{n_k} q^{2N_{k-1} + \alpha_1 + \cdots + \alpha_k}}{q^{2N_{k-1} + \alpha_1 + \cdots + \alpha_k + 1}} \end{array} \right) = (-q, -q),
\]

becomes

\[
\ell \prod_{t=0}^{k} \left( \sum_{n=0}^{\infty} \left( \begin{array}{c}
\frac{(1 - (-q)^{-n_k + \ell}) (1 - (-q)^{n_k + \ell + 1} q^{2N_{k-1} + 2n_k + \alpha_1 + \cdots + \alpha_k + 1})}{(1 - (-q)^{n_k + \ell + 1} q^{2N_{k-1} + 2n_k + \alpha_1 + \cdots + \alpha_k + 1})} \end{array} \right) (-q)^n \right).
\]

The factors in this product have a remarkable symmetry: they have all been arranged in the form

\[
\frac{(1 - (-q)^d + \ell) (1 - (-q)^d + \ell + 1) q^d}{(1 - (-q)^{d + \ell + 1}) (1 - (-q)^{d + \ell + 1} q^d)}.
\]
with \(d, e \in \mathbb{Z}\) and \(\delta, \epsilon \in \mathbb{R}\). Irrespective of the parities of \(d\) and \(e\), we will always find \(\frac{8}{3}\) in the limit \(q \to 1\), which by L'Hôpital's rule reduces to

\[
\frac{2d + 2e + (1 - (-1)^{d+\ell}) (\delta + 1)}{2e + 2\ell + (1 - (-1)^{e+\ell}) (\epsilon + 1)}.
\]

Applying this to (94), we find that the function (91) in the limit \(q \to 1\) becomes

\[
\sum_{n=0}^{\infty} \left(-1\right)^n n^{-1} f(\ell),
\]

with

\[
f(\ell) = \frac{2\ell - 2n_k + (1 - (-1)^{-n_k+\ell}) (2N_k + \alpha_{[2,k+1]} + \beta)}{2\ell + 2 + (1 - (-1)^{\ell+1}) (2N_{k-1} + \alpha_{[2,k]} + \beta)}
\times \frac{2\ell + 2N_{k-1} - 2x_k + (1 - (-1)^{N_{k-1} - x_k +\ell}) (2x_k + \alpha_{[1,k]})}{2\ell + 2N_{k-1} - 2x_k + 1 + (1 - (-1)^{N_{k-1} - x_k +\ell}) (2x_{k+1} + \alpha_{[1,k+1]})}
\]

In order to recognize a hypergeometric series \(\text{F}_3\) in this expression, we proceed as follows:

1. We split the sum over \(n\) in a sum over \(n\) even and one over \(n\) odd, thereby eliminating the \((-1)^n\). Then we rename \(n \to 2n\) in the first sum, and \(n \to 2n + 1\) in the second, such that we obtain

\[
\sum_{n=0}^{\infty} \frac{2\ell - 2n_k + (1 - (-1)^{-n_k+\ell}) (2N_k + \alpha_{[2,k+1]} + \beta)}{2\ell + 2 + (1 - (-1)^{\ell+1}) (2N_{k-1} + \alpha_{[2,k]} + \beta)} \times \frac{2\ell + 2N_{k-1} - 2x_k + (1 - (-1)^{N_{k-1} - x_k +\ell}) (2x_k + \alpha_{[1,k]})}{2\ell + 2N_{k-1} - 2x_k + 1 + (1 - (-1)^{N_{k-1} - x_k +\ell}) (2x_{k+1} + \alpha_{[1,k+1]})}
\]

2. We split the first product over \(\ell\) above into a product for \(\ell\) even, where we rename \(\ell \to 2\ell\), and one for \(\ell\) odd, where we rename \(\ell \to 2\ell + 1\). We do the same thing for the second product, but we first separate the factor corresponding to \(\ell = 0\), which will be contained in \(t_{n,x}\). The limit now becomes

\[
\sum_{n=0}^{\infty} n^{-1} f(2\ell) f(2\ell + 1) - f(0) \sum_{n=0}^{\infty} n^{-1} f(2\ell + 2) f(2\ell + 1)
\]

3. In all obtained fractions we divide numerator and denominator by 4, such we are left with only numerators and denominators of the form \(\prod_{\ell=0}^{n-1} (\ell+c)\), for \(c \in \mathbb{R}\) independent of \(\ell\), which we can write as a Pochhammer symbol \((c)_n\).

Combining all these steps, the anticipated result follows.

We are now ready to state our definition of the multivariate Bannai-Ito polynomials.

**Definition 7.** The Bannai-Ito polynomials in \(s\) real variables \(x_k\) are defined as

\[
B_{(n_1,\ldots,n_s)}((x_1,\ldots,x_s);\alpha_1,\ldots,\alpha_{s+1},\beta,N) = \prod_{k=1}^{s} B_{n_k} \left( \frac{(-1)^{N_{k-1} + x_k}}{2} \left( x_k + \frac{\alpha_{[1,k]}}{2} \right) - \frac{1}{4} \rho_1^{(k)} \rho_2^{(k)} r_1^{(k)} r_2^{(k)} M^{(k)} \right)
\]
where the parameters are given by

\begin{align}
\rho_1^{(k)} &= \frac{N_{k-1}}{2} + \frac{\alpha_{[1,k]} - 1}{4}, \\
\rho_2^{(k)} &= \frac{(-1)^N_{k-1+x_{k+1}}}{2} \left( x_{k+1} + \frac{\alpha_{[1,k+1]}}{2} \right) + \frac{\alpha_{k+1} - 1}{4}, \\
r_1^{(k)} &= \frac{(-1)^N_{k-1+x_{k+1}}}{2} \left( x_{k+1} + \frac{\alpha_{[1,k+1]}}{2} \right) - \left( \frac{\alpha_{k+1} - 1}{4} \right), \\
r_2^{(k)} &= -\frac{N_{k-1}}{2} + \frac{\alpha_1 - \alpha_{[2,k]} - 1}{4} - \frac{\beta}{2}, \\
M^{(k)} &= x_{k+1} - N_{k-1},
\end{align}

with \(x_{s+1} = N \in \mathbb{N}\), where \(n_1, \ldots, n_s\) are natural numbers and \(\alpha_1, \ldots, \alpha_{s+1}\) and \(\beta\) are real parameters subject to the conditions

\begin{equation}
\alpha_1 - 1 > \beta > 0, \quad \alpha_i > 1, i \in \{2, \ldots, s + 1\}.
\end{equation}

Remark 9. Note that the conditions (87) and (90) for orthogonality are satisfied by each of the univariate Bannai-Ito polynomials in (95), if we require each \(M^{(k)}\) to be a natural number. Indeed, the truncation follows immediately from (96). As shown in [21, (1.16) and (1.24)], the positivity condition (87) is fulfilled if

\[
\rho_1^{(k)} - r_2^{(k)} > 0, \quad \rho_1^{(k)} + r_2^{(k)} > 0, \quad \rho_2^{(k)} - r_1^{(k)} > 0.
\]

With the parametrization (96), this is tantamount to (97).

Remark 10. In the special case \(s = 2\), these polynomials coincide with the bivariate Bannai-Ito polynomials recently defined by Lemay and Vinet, up to a change in expression for \(\rho_2^{(2)}\) and \(r_2^{(2)}\). The parametrization in [33] can be obtained through the formulas

\[
p_1 = \frac{-\alpha_1 + 2\beta + 3}{4}, \quad p_i = \frac{\alpha_i}{4}, i \in \{2, 3\}, \quad c = \frac{\beta}{2}, \quad z_k = \frac{(-1)^x_k}{2} \left( x_k + \frac{\alpha_{[1,k]}}{2} \right).
\]

The difference in expression of \(\rho_2^{(2)}\) and \(r_2^{(2)}\) can be explained by the truncation: for odd \(N\) the truncation condition of type \(iii\) was used in [33], whereas we use type \(ii\).

Remark 11. We acknowledge that the arguments of the univariate Bannai-Ito polynomials in (95) will take on complex values in case the \(x_k\) are not integers. Slightly different polynomials would be obtained if one instead considers

\[
z_k = \frac{(-1)^x_k}{2} \left( x_k + \frac{\alpha_{[1,k]}}{2} \right)
\]

to be the actual variables. This would lead to real-valued polynomials, but has the disadvantage that the truncation parameters \(M^{(k)}\) cannot be expressed directly in terms of the \(z_k\) and hence should be added in a slightly artificial fashion. We hence prefer to work with the polynomials as constructed in Definition 7.

Our definition originates from the following lemma, where we compute the limit \(q \to 1\) of a renormalized multivariate \((-q)\)-Racah polynomial with a certain parametrization.
Lemma 7. Let \( \widehat{R}_n(x; a, b, N| - q) \) be a renormalized \((-q\)-Racah polynomial (69)) in \( s \) integer variables, where the parameters are such that

\[
a_1 = -q^{\alpha_1}, \quad a_k = q^{\alpha_k}, k \in [2; s + 1], \quad b = -q^\beta,
\]

for certain \( \alpha_1, \beta \in \mathbb{R} \) subject to (97). Then in the limit \( q \to 1 \) these polynomials become proportional to the multivariate Bannai-Ito polynomials:

\[
\lim_{q \to 1} \left( \widehat{R}_n(x; a, b, N| - q) \right) = k_n, x, \alpha, \alpha, \beta, N B_n (x; \alpha_1, \ldots, \alpha_{s+1}, \beta, N),
\]

with \( n = (n_1, \ldots, n_s) \in \mathbb{N}^s \) and \( x = (x_1, \ldots, x_s) \in \mathbb{Z}^s \), and where the proportionality coefficient is given by

\[
k_n, x, \alpha, \beta, N = \frac{(-1)^{N_s} N_s \sum_{k=2}^{n_s} n_k N_k^{-1} \prod_{k=1}^s \eta_{n,k} \zeta_{n,x,k} \xi_{n,x,k}}{\prod_{\ell=0}^{n_k-1} (-2\ell + 2N_k + \alpha_1 + 2\alpha_{[2,s+1]} + \beta + 1) + (-1)^{N+k}(\alpha_1 - \beta - 1)}
\]

with

\[
\eta_{n,k} = \prod_{\ell=0}^{n_k-1} \frac{2\ell + 2N_k + \alpha_{[2,k]} + \beta + 1}{2\ell + \alpha_{k+1}}
\]

\[
\zeta_{n,x,k} = \frac{4^{n_k} (\rho_1^{(k)} + \rho_2^{(k)} - r_1^{(k)} - r_2^{(k)} + 1) n_k}{\prod_{\ell=0}^{n_k-1} \left( \ell + 2\rho_1^{(k)} - 2r_1^{(k)} + 1 + (1 - (-1)^\ell) (\rho_2^{(k)} - r_2^{(k)}) \right)}
\]

\[
\times \prod_{\ell=0}^{n_k-1} \left( \ell + 2\rho_1^{(k)} - 2r_1^{(k)} + 1 + (1 - (-1)^\ell) (\rho_2^{(k)} + r_2^{(k)}) \right)^{-1}
\]

\[
\xi_{n,x,k} = \prod_{\ell=0}^{n_k-1} \left( (-2\ell + 2N_k - \alpha_{[1,k+1]}) + (-1)^{x_k+1+N_k-1+\ell}(2x_k+1 + \alpha_{[1,k+1]}) \right)
\]

and with \( \rho_i^{(k)} \) and \( r_i^{(k)} \) as in (96).

Proof. Comparing the definitions (40), (57) and (69) and the parametrization (98), we find that \( \widehat{R}_n(x; a, b, N| - q) \) is proportional to

\[
\prod_{k=1}^s \frac{(-q)^{n_k} q^{N_k-1+\alpha_{[2,k]}+\beta}}{q^{2N_k-1+\alpha_{[2,k]}+\beta+1}(-q)^{N_k-x_k} q^{\alpha_{1,k}}} \bigg| - q, -q \bigg),
\]

where we recognize the basic hypergeometric function considered in Lemma 6. In the limit \( q \to 1 \), the polynomials \( \widehat{R}_n(x; a, b, N| - q) \) thus become proportional to products of hypergeometric functions of the form (92) with (93). Each such function determines a polynomial \( u_i(\theta_j; s, s^*, r_1, r_2, r_3, N) \) as originally defined by Bannai and Ito in [2, p. 272–273], under the reparametrizations

\[
i = n_k, \quad j = x_k - N_k-1
\]

\[
s = -2N_k-1 - \alpha_{[1,k+1]} + 1, \quad s^* = -2N_k-1 - \alpha_{[2,k+1]} - \beta + 1
\]

\[
r_1 = 2N_k-1 + \alpha_{[2,k]} + \beta, \quad N = x_k + 1 - N_k-1
\]

\[
r_2 = \begin{cases} -x_k+1 + N_k-1 & \text{if } N \text{ is even} \\ N_k-1 + x_k+1 + \alpha_{[1,k+1]} - 1 & \text{if } N \text{ is odd} \end{cases}
\]

\[
r_3 = \begin{cases} -N_k-1 - x_k+1 - \alpha_{[1,k+1]} + 1 & \text{if } N \text{ is even} \\ x_k+1 - N_k-1 + 1 & \text{if } N \text{ is odd} \end{cases}
\]
and where

\[ \theta_j = \begin{cases} -\frac{j}{4} + \frac{j}{2} & \text{if } j \text{ is even} \\ -\frac{j}{4} - \frac{j}{2}(j + 1 - s) & \text{if } j \text{ is odd.} \end{cases} \]

This can be checked upon comparing (92) and (93) with the hypergeometric expressions in [2]. Following [49], it is clear that the polynomials \( u_s(\theta_j) \) are rescaled Bannai-Ito polynomials

\[ \zeta_n; k \cdot B_{n_k} \left( \frac{(-1)^{N_{k-1}+x_k}}{2} \left( x_k + \frac{\alpha_{[1:k]}}{2} \right) - \frac{1}{4} \cdot \rho_1^{(k)} \cdot \rho_2^{(k)} \cdot r_1^{(k)} \cdot r_2^{(k)} \cdot M^{(k)} \right), \]

with \( \zeta_n; k \) as in (100). The other factors in \( k_n; x, \alpha, \beta, N \) follow by taking limits of the proportionality coefficients in (40) and (69). This proves our claim. \( \square \)

**Remark 12.** Note that the same choice of parametrization (98) was made in [33, (2.11)], for the bivariate case.

Like their univariate counterparts, the multivariate Bannai-Ito polynomials satisfy a discrete orthogonality relation, which we derive in the next proposition.

**Proposition 9.** The multivariate Bannai-Ito polynomials satisfy the relation

\[ \sum_{\ell_s=0}^N \sum_{\ell_{s-1}=0}^{\ell_s} \cdots \sum_{\ell_1=0}^{\ell_2} \Omega(\ell, n; \alpha, \beta, N) B_{n}(\ell; \alpha, \beta, N) B_{m}(\ell; \alpha, \beta, N) = H(n; \alpha, \beta, N) \delta_{n,m} \]

where the orthogonality grid is given by the simplex

\[ \{ \ell = (\ell_1, \ldots, \ell_s) \in \mathbb{N}^s : 0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_s \leq N \} \]

and we have the expressions

\[ \Omega(\ell, n; \alpha, \beta, N) = \frac{\prod_{i=1}^s w^{(i)}_{\ell_i-N_{i-1}}(\ell_1, \ell_i+1, N_{i-1}, \alpha_1, \ldots, \alpha_i+1, \beta)}{\prod_{i=1}^{s-1} h^{(i)}_{n_i}(\ell_i+1, N_{i-1}, n_i, \alpha_1, \ldots, \alpha_i+1, \beta)} \]

for the weight function and

\[ H(n; \alpha, \beta, N) = h^{(s)}_{n_s}(N, N_{s-1}, n_s, \alpha_1, \ldots, \alpha_{s+1}, \beta), \]

for the normalization coefficient, where we use the convention that \( \ell_{s+1} = N \). Expressions for the functions \( w^{(i)}_{\ell_i-N_{i-1}} \) and \( h^{(i)}_{n_i} \) can be found in Appendix D.

**Proof.** We prove this by induction on \( s \). For \( s = 1 \) the relation (101) is precisely the univariate orthogonality relation (86), which is applicable by Remark 9 and where the parameters are given by (96) with \( k = 1 \).

Now suppose the claim has been proven for \( s - 1 \). Let us write

\[ \ell = (\ell_1, \ldots, \ell_s), \quad \bar{\ell} = (\ell_1, \ldots, \ell_{s-1}) \]
\[ n = (n_1, \ldots, n_s), \quad \bar{n} = (n_1, \ldots, n_{s-1}) \]
\[ \alpha = (\alpha_1, \ldots, \alpha_{s+1}), \quad \bar{\alpha} = (\alpha_1, \ldots, \alpha_s), \]

Observe that

\[ B_{n}(\ell; \alpha, \beta, N) = B_{\bar{n}}(\bar{\ell}; \bar{\alpha}, \beta, \ell_s) B_{n_s} \left( \frac{(-1)^{N_{s-1}+\ell_s}}{2} \left( \ell_s + \frac{\alpha_{[1:s]}}{2} \right) - \frac{1}{4} \cdot \rho_1^{(s)} \cdot \rho_2^{(s)} \cdot r_1^{(s)} \cdot r_2^{(s)} \cdot N - N_{s-1} \right), \]

and

\[ B_{\bar{n}}(\bar{\ell}; \bar{\alpha}, \beta, \ell_s) B_{n_s} \left( \frac{(-1)^{N_{s-1}+\ell_s}}{2} \left( \ell_s + \frac{\alpha_{[1:s]}}{2} \right) - \frac{1}{4} \cdot \rho_1^{(s)} \cdot \rho_2^{(s)} \cdot r_1^{(s)} \cdot r_2^{(s)} \cdot N - N_{s-1} \right), \]
with $\rho_i^{(s)}$, $r_i^{(s)}$ as in (96). It is immediate that

$$\Omega(\ell, n; \alpha, \beta, N) = \Omega(\overline{\ell}, \overline{n}; \overline{\alpha}, \overline{\beta}, \overline{\ell}) \frac{u_{\ell_s-N_s-1}^{(s)}}{h_{\ell_s-N_s-1}^{(s-1)}} ,$$

where the factor $\frac{u_{\ell_s-N_s-1}^{(s)}}{h_{\ell_s-N_s-1}^{(s-1)}}$ is independent of $\ell_1, \ldots, \ell_{s-1}$. Hence we find that the left-hand side of (101) equals

$$\sum_{\ell_s=0}^{h^{(s-1)}_{\ell_s-N_s-1}} \frac{\sum_{k=0}^{\ell_s-N_s-1} u_k^{(s)} B_{n_s}(x_k; \rho_1^{(s)}, \rho_2^{(s)}, r_1^{(s)}, r_2^{(s)}, N - N_s-1)}{h^{(s-1)}_{\ell_s-N_s-1}}$$

where we have written $M_{s-1}$ for $\sum_{i=1}^{s-1} m_i$. By the induction hypothesis, the sum in the third line yields

$$\delta_{n_1, m_1} \ldots \delta_{n_{s-1}, m_{s-1}} H(\overline{n}; \overline{\alpha}, \overline{\beta}, \overline{\ell})$$

and by definition, we have $H(\overline{n}; \overline{\alpha}, \overline{\beta}, \overline{\ell}) = h^{(s-1)}_{\ell_s-N_s-1}(\ell_s, N_{s-2}, n_{s-1}, \overline{\alpha}, \overline{\beta})$. Hence the left hand side of (101) reduces to

$$\delta_{n_1, m_1} \ldots \delta_{n_{s-1}, m_{s-1}} h^{N-N_s-1} \sum_{k=0}^{\ell_s-N_s-1} u_k^{(s)} B_{n_s}(x_k; \rho_1^{(s)}, \rho_2^{(s)}, r_1^{(s)}, r_2^{(s)}, N - N_s-1)$$

$$\times B_{m_s}(x_k; \rho_1^{(s)}, \rho_2^{(s)}, r_1^{(s)}, r_2^{(s)}, N - N_s-1) ,$$

with

$$x_k = (-1)^{k} \left( k + \rho_1^{(s)} + \frac{1}{4} \right) - \frac{1}{4}$$

and where we have used (96) and the fact that $u_{\ell_s-N_s-1}^{(s)} = 0$ for $\ell_s \in \{0, \ldots, N_{s-1} - 1\}$, as explicitly stated in Appendix D. The assertion now follows immediately from the orthogonality relation (86) in the univariate case. \hfill \Box

5.2. The higher rank $q = 1$ Bannai-Ito algebra and connection coefficients. The rank $n - 2$ Bannai-Ito algebra $\mathcal{A}_n$ was introduced in [10] as the abstract associative algebra with generators $\Gamma_A$, indexed by all possible subsets $A \subseteq [1; n]$, subject to the relations

$$\{ \Gamma_A, \Gamma_B \} = \Gamma_{(A \cup B) \setminus (A \cap B)} + 2 \Gamma_{A \cup B} \Gamma_{A \cap B} + 2 \Gamma_{A \setminus (A \cap B)} \Gamma_{B \setminus (A \cap B)} ,$$

for all sets $A, B \subseteq [1; n]$. It was originally constructed as the symmetry algebra of the so-called Dirac-Dunkl operator

$$\mathcal{D} = \sum_{i=1}^{n} e_i T_i, \quad T_i = \partial_{x_i} + (\gamma_i - \frac{1}{2}) \frac{1 - r_i}{x_i},$$

where the $e_i$ are Clifford elements, satisfying $\{ e_i, e_j \} = -2 \delta_{ij}$, the $\gamma_i > \frac{1}{2}$ are real parameters and the $r_i$ are the reflections defined in Section 2.2. The scalar version
of this model, with all $e_i$ replaced by products of reflections $\prod_{j=i+1}^n r_j$, coincides with the $\mathbb{Z}_q^N$ $q$-Dirac-Dunkl model introduced in Section 2.2 in the limit $q \to 1$.

Under the specialization $q \to 1$, the quantum superalgebra $\mathfrak{osp}_q(1|2)$ reduces to the Lie superalgebra $\mathfrak{osp}(1|2)$. This is manifest from the alternative presentation for $\mathfrak{osp}_q(1|2)$, with generators $A_0, A_\pm, P$ and relations
\[
[A_0, A_\pm] = \pm A_\pm, \quad \{A_+, A_-\} = [2A_0]_{q^{1/2}}, \quad [P, A_0] = 0, \quad \{P, A_\pm\} = 0, \quad P^2 = 1,
\]
which clearly reduce to the defining relations for $\mathfrak{osp}(1|2)$ in the limit $q \to 1$. This presentation relates to (3) by taking $K = q^{A_0/2}$. The coproduct action on $A_0$ compatible with (5) is hence
\[
\Delta(A_0) = A_0 \otimes 1 + 1 \otimes A_0.
\]

Comparison of the relations (102) with the defining relations (12) of the algebra $A_n^q$ explains why the latter is considered a $q$-deformation of $A_n$. As a consequence, the limit $q \to 1$ of the extension process (8)–(9) should yield a different construction method for $A_n$ by means of embedding in $\mathfrak{osp}(1|2)^{\otimes n}$. Note that the expression (7) for the extension morphism $\tau$ simplifies remarkably under this limit. As a consequence, the requirement (11) on the sets $A$ and $B$ in Proposition 1 can be omitted in case $q = 1$.

Taking $q \to 1$ in (19) we obtain unitary irreducible modules for the Lie superalgebra $\mathfrak{osp}(1|2)$, which we will denote by $\tilde{W}(\gamma)$. The solution to the spectral problem proposed in Definition 5 survives the limit $q \to 1$ and the expressions for the eigenvalues follow immediately from the fact that
\[
\lim_{q \to 1} (|\alpha\rangle_q) = |\alpha\rangle,
\]
for any $\alpha \in \mathbb{R}$. Hence we are led to propose the following analog of Definition 5 for $q = 1$. For ease of notation, we will consider only the $q \to 1$ limits of the vectors (71) and (72), although one could similarly define $|\tilde{j}^{(i,j)}; N\rangle$ for any $i, j$.

**Definition 8.** We denote by $|\tilde{s}; N\rangle = |\tilde{s}_{n-1}, \tilde{s}_{n-2}, \ldots, \tilde{s}_1; N\rangle$, with all $\tilde{s}_k \in \mathbb{N}$ and $N \in \mathbb{N}$, $N \geq \tilde{s}_n-1$, the vectors $\in \tilde{W}(\gamma_1) \otimes \cdots \otimes \tilde{W}(\gamma_n)$ diagonalizing the abelian subalgebra
\[
\langle \Gamma_{1:2}, \Gamma_{1:3}, \ldots, \Gamma_{1:n}\rangle
\]
of $A_n$ through the eigenvalue equations
\[
\Gamma_{1:k+1}|\tilde{s}_{n-1}, \ldots, \tilde{s}_1; N\rangle = (-1)^{|\tilde{s}_k|} \left( |\tilde{s}_k| + \gamma_{[1:k+1]} - \frac{1}{2} \right) |\tilde{s}_{n-1}, \ldots, \tilde{s}_1; N\rangle
\]
for all $k \in [1; n-1]$, and satisfying
\[
(A_0)|_{[1:n]} |\tilde{s}_{n-1}, \ldots, \tilde{s}_1; N\rangle = (|\tilde{s}_{n-2}| + N + \gamma_{[1:n]} ) |\tilde{s}_{n-1}, \ldots, \tilde{s}_1; N\rangle,
\]
where
\[
(A_0)|_{[1:n]} = \Delta^{(n)}(A_0) = \sum_{i=0}^{n-1} 1^i \otimes A_0 \otimes 1^{n-i-1},
\]
with $A_0$ as in (103). Similarly, the vectors $|\tilde{j}; N\rangle = |\tilde{j}_{n-1}, \tilde{j}_{n-2}, \ldots, \tilde{j}_1; N\rangle$ diagonalize the abelian subalgebra
\[
\langle \Gamma_{2:3}, \Gamma_{2:4}, \ldots, \Gamma_{2:n}, \Gamma_{1:n}\rangle
\]
of $A_n$ through the relations

$$\Gamma_{[2;k+1]}[\tilde{j}_{n-1}, \ldots, \tilde{j}_1; N] = (-1)^{\tilde{j}_{k-1}} \left( \tilde{j}_{k-1} + \gamma_{[2;k+1]} - \frac{1}{2} \right) \tilde{j}_{n-1}, \ldots, \tilde{j}_1; N),$$

for all $k \in [2; n-1]$ and

$$\Gamma_{[1;n]}[\tilde{j}_{n-1}, \ldots, \tilde{j}_1; N] = (-1)^{\tilde{j}_{n-1}} \left( \tilde{j}_{n-1} + \gamma_{[1;n]} - \frac{1}{2} \right) \tilde{j}_{n-1}, \ldots, \tilde{j}_1; N).$$

The overlap coefficients have a nice expression in terms of orthogonal polynomials. The proportionality coefficient is given by

$$\text{coefficients have a nice expression in terms of orthogonal polynomials.}$$

We will now focus on the overlap coefficients $\langle \tilde{s}; N \mid \tilde{j}_{n-1} + m \mid \tilde{s}; s_{n-1} + m \rangle$, which by the previously established connections are in fact the Racah coefficients of the algebra $\mathfrak{osp}(1|2)$. As before, these coefficients can be shown to be independent of $m$, hence we let $m$ be a fixed natural number and write $\langle \tilde{s}; N \mid \tilde{j} \rangle$ instead of $\langle \tilde{s}; N \mid \tilde{j}_{n-1} + m \rangle$ and similarly $\langle \tilde{s} \rangle$ instead of $\langle \tilde{s}; s_{n-1} + m \rangle$. Like in the $q$-deformed case, the overlap coefficients have a nice expression in terms of orthogonal polynomials.

**Theorem 3.** The overlap coefficients $\langle \tilde{j} \rangle$ are proportional to multivariate Bannai-Ito polynomials

$$\langle \tilde{j}_{\gamma_{1}}, \ldots, \tilde{j}_{\gamma_{N}} | \tilde{s}_{\gamma_{1}}, \ldots, \tilde{s}_{\gamma_{N}} \rangle = \delta_{\gamma_{1}, \gamma_{N}} \delta_{\gamma_{N}, \gamma_{N}}, \quad \langle \tilde{j} \rangle = \delta_{\gamma_{1}, \gamma_{N}},$$

The proportionality coefficient is given by

$$C_{\tilde{s}, \tilde{j}} = \sqrt{\frac{\Omega((\tilde{s}_{\gamma_{1}}, \ldots, \tilde{s}_{\gamma_{N}}); (\tilde{j}_{\gamma_{1}}, \ldots, \tilde{j}_{\gamma_{N}}); 2\gamma_{[1;2]} - 1, 2\gamma_{3}, \ldots, 2\gamma_{N}; 2\gamma_{2} - 1, |\tilde{s}_{\gamma_{N}}|)}{H((\tilde{j}_{\gamma_{1}}, \ldots, \tilde{j}_{\gamma_{N}}); 2\gamma_{[1;2]} - 1, 2\gamma_{3}, \ldots, 2\gamma_{N}; 2\gamma_{2} - 1, |\tilde{s}_{\gamma_{N}}|)}},$$

where the functions $\Omega$ and $H$ have been defined in Proposition 9.

**Proof.** By definition, these are the limits $q \to 1$ of the overlap coefficients (77). The choice of parameters (73) coincides with (98), and (97) is satisfied since we require all $\gamma_i > \frac{1}{2}$. Hence we find from Lemma 7

$$\langle \tilde{j} \rangle \propto B_{(\tilde{j}_{\gamma_{1}}, \ldots, \tilde{j}_{\gamma_{N}})}((\tilde{s}_{\gamma_{1}}, \ldots, \tilde{s}_{\gamma_{N}}); 2\gamma_{[1;2]} - 1, 2\gamma_{3}, \ldots, 2\gamma_{N}; 2\gamma_{2} - 1, |\tilde{s}_{\gamma_{N}}|).$$

The proportionality coefficient $C_{\tilde{s}, \tilde{j}}$ has been derived from the orthonormality of the vectors $\langle \tilde{j} \rangle$ in combination with Proposition 9, following the same lines as the proof of Theorem 1.

Note that this agrees with the results obtained in [21] for the univariate case, i.e. the case $n = 3$, modulo a transformation $\gamma_1 \leftrightarrow \gamma_3$ of the representation parameters.
Remark 14. If we denote by \( \tilde{j}^{(i,j)}; N \) the vector introduced in Definition 5 for \( q = 1 \), then as before we would find that \( \tilde{j}^{(i,j)}; \tilde{j}_{n-1} + m; \tilde{s}_{n-1} + m \) is proportional to a multivariate Bannai-Ito polynomial

\[
B_{\tilde{j}^{(i,j)}, \ldots, \tilde{j}_{i+j-2}}^{(i,j)} (\tilde{s}_i, \ldots, |\tilde{s}_{i+j-2}| - |\tilde{s}_{i-1}|; \alpha, 2\gamma_{i+1} - 1, |\tilde{s}_{i+j-1}| - |\tilde{s}_{i-1}|),
\]

with

\[
\alpha = (\alpha_1, \ldots, \alpha_j), \quad \alpha_1 = 2|\tilde{s}_{i-1}| + 2\gamma_{1;i+1} - 1, \quad \alpha_k = 2\gamma_{i+k}, \ k \in [2; j].
\]

5.3. Discrete realization. In this final section we will investigate the bispectrality of the multivariate Bannai-Ito polynomials from Definition 7. Like before, we will first renormalize our polynomials as follows:

\[
\tilde{B}_n(x; \alpha, \beta, N) = k_{n,x,\alpha,\beta,N} B_n(x; \alpha, \beta, N),
\]

with \( k_{n,x,\alpha,\beta,N} \) as in (99). By Lemma 7, these are the renormalized multivariate \((-q\rangle\)-Racah polynomials (69) with parametrization (98) in the limit \( q \to 1 \). The following is the \( q = 1 \) analog of Proposition 7, i.e. the \( q = -1 \) analog of [25, Propositions 4.2 and 4.5].

**Proposition 10.** Let \( x_1, \ldots, x_s \) be integer variables. Let us define the operator

\[
\tilde{E}_j^x (x_1, \ldots, x_j; \alpha_1, \ldots, \alpha_{j+1}, \beta, x_{j+1})
\]

\[
= \sum_{\nu \in \{-1,0,1\}^j} \tilde{C}_\nu(x_1, \ldots, x_{j+1}) T_{+;x_1}^{\nu_1} \cdots T_{+;x_j}^{\nu_j}
\]

\[
+ \left( \frac{\alpha_{2;j+1} + \beta}{2} + (-1)^{x_{j+1}}(\alpha_1 - \beta - 1) \left( x_{j+1} + \frac{\alpha_{1;j+1}}{2} \right) \right),
\]

with the convention that \( x_{s+1} = N \) and where the \( \tilde{C}_\nu(x_1, \ldots, x_{j+1}) \) are functions in the variables \( x_1, \ldots, x_{j+1}, \) explicit expressions of which can be found in Appendix E. Then the operators

\[
\tilde{E}_j^x (x_1, \ldots, x_j; \alpha_1, \ldots, \alpha_{j+1}, \beta, x_{j+1}), \quad j \in \{1, \ldots, s\},
\]

are mutually commuting operators, with the renormalized Bannai-Ito polynomials in \( s \) variables as common eigenfunctions:

\[
\tilde{E}_j^x (x_1, \ldots, x_j; \alpha_1, \ldots, \alpha_{j+1}, \beta, x_{j+1}) \tilde{B}_n(x; \alpha, \beta, N) = \tilde{\mu}_j \tilde{B}_n(x; \alpha, \beta, N),
\]

with eigenvalue

\[
\tilde{\mu}_j = \frac{\alpha_{2;j+1} + \beta}{2} - (-1)^{N_j} \left( N_j + \frac{\alpha_{1;j+1}}{2} \right).
\]

**Proof.** This is immediate from Proposition 7, Lemma 7 and the fact that with the parametrization (98) one finds

\[
\tilde{E}_j^x (x_1, \ldots, x_j; \alpha_1, \ldots, \alpha_{j+1}, \beta, x_{j+1}) = \lim_{q \to 1} \left( \frac{\tilde{E}_j^x(x_1, \ldots, x_j; a_1, \ldots, a_{j+1}, b, x_{j+1} | - q)}{q - q^{-1}} \right),
\]

\[
\tilde{B}_n(x; \alpha, \beta, N) = \lim_{q \to 1} \left( \tilde{R}_n(x; a, b, N | - q) \right),
\]

\[
\tilde{\mu}_j = \lim_{q \to 1} \left( \frac{\mu_j}{q - q^{-1}} \right).
\]

Note that the restriction that all \( x_i \) be integer is necessary for the limits to converge. \( \square \)
Observe that the equation (107) establishes in fact a $3^j$-term difference relation for the multivariate Bannai-Ito polynomials. Similarly we can define a second commutative algebra of shift operators in the discrete variables $n_i$ that diagonalize the polynomials $\tilde{B}_n(x; \alpha, \beta, N)$, as a $q = 1$ analog of Proposition 8. This will lead to a $3^j$-term recurrence relation for the $s$-variate Bannai-Ito polynomials, for any $j \in \{1, \ldots, s\}$.

**Proposition 11.** Let $x_1, \ldots, x_s, n_1, \ldots, n_s$ be a set of integer variables. Let us define the operator

$$\tilde{S}^n_j(n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_{s+1}, \beta, N)$$

where the $\tilde{D}_\nu(n_1, \ldots, n_s)$ are functions in the variables $n_1, \ldots, n_s$, explicit expressions of which can be found in Appendix E. Then the operators

$$\tilde{S}^n_j(n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_{s+1}, \beta, N), \quad j \in \{1, \ldots, s\},$$

are mutually commuting operators, with the renormalized Bannai-Ito polynomials in $s$ variables as common eigenfunctions:

$$\tilde{S}^n_j(n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_{s+1}, \beta, N) \tilde{B}_n(x; \alpha, \beta, N) = \tilde{\kappa}_j \tilde{B}_n(x; \alpha, \beta, N),$$

with eigenvalue

$$\tilde{\kappa}_j = \left(\frac{\alpha [1; s+1] + \alpha [s-j+2; s+1]}{2} + N - (-1)^{N+x_{s-j+1}} (x_{s-j+1} + \frac{\alpha [1; s+1]}{2})\right).$$

**Proof.** This is again immediate from Proposition 8, Lemma 7 and the fact that with the parametrization (98) one finds

$$\tilde{S}^n_j(n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_{s+1}, \beta, N) = \lim_{q \to 1} \left(\frac{\tilde{S}_j^n(n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_{s+1}, \beta, N|q)}{q - q^{-1}}\right),$$

$$\tilde{B}_n(x; \alpha, \beta, N) = \lim_{q \to 1} \left(\frac{\tilde{R}_n(x; \alpha, \beta, N)}{q - q^{-1}}\right),$$

$$\tilde{\kappa}_j = \lim_{q \to 1} \left(\frac{\kappa_j}{q - q^{-1}}\right).$$

We will conclude with a discrete realization of the rank $n-2$ Bannai-Ito algebra. Our representation space will be defined as follows.

**Definition 9.** We will denote by $\tilde{V}$ the infinite-dimensional vector space spanned by the renormalized multivariate Bannai-Ito polynomials

$$\tilde{B}_{\tilde{j}_1, \ldots, \tilde{j}_{n-2}}((\tilde{s}_1, \tilde{s}_1 + 1, \tilde{s}_2, \ldots, |\tilde{s}_{n-2}|); \alpha, \beta, |\tilde{s}_{n-1}|), \quad \tilde{j} \in \mathbb{N}^{n-1},$$

considered as functions of $\tilde{s} \in \mathbb{N}^{n-1}$, where $\tilde{j}_{n-1}$ is fixed by the constraint $|\tilde{j}_{n-1}| = |\tilde{s}_{n-1}|$ and with

$$\alpha = (\alpha_1, \ldots, \alpha_{n-1}), \quad \alpha_1 = 2\gamma_1 + 2\gamma_2 - 1, \quad \alpha_k = 2\gamma_{k+1}, k \in [2; n-1], \quad \beta = 2\gamma_2 - 1.$$
By Lemma 7, this is the $q \to 1$ analog of the space $\tilde{V}_q$ from Definition 6. Finally, taking limits in Theorem 2, we obtain a realization of the algebra $A_n$ on the space $\hat{V}$ by means of difference operators.

**Theorem 4.** Let $\gamma_i > \frac{1}{2}$, $i \in [1; n]$, be a set of real parameters. Let us define

$$
\Gamma_{[1;m+1]} = (-1)^i \tilde{s}_{n-1} \left( -\tilde{\alpha}_{n-m-1} + \left( |\tilde{s}_{n-1}| + \gamma_{[1;n]} + \gamma_{[m+2;n]} - \frac{1}{2} \right) \right)
$$

for $m \in [0; n-1]$ and

$$
\Gamma_{[i+1;i+j]} = -\bar{L}_{i,j-1}^x + \left( \gamma_{[i+1;i+j]} - \frac{1}{2} \right)
$$

for $i \in [1; n-1], j \in [1; n-i]$, with the abbreviations $\tilde{\alpha}_{n-m-1}$ for

$$
\tilde{\alpha}_{n-m-1} \left( \tilde{j}, \ldots, \tilde{j}_n, 2\gamma_{[1;2]} - 1, 2\gamma_3, \ldots, 2\gamma_n, 2\gamma_2 - 1, \tilde{j}_n, \tilde{j}_1 \right)
$$

and $\bar{L}_{i,j-1}$ for

$$
\bar{L}_{i,j-1}^x \left( \tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_1-j-1 \right) - |\tilde{s}_{i-1}| \left( 2|\tilde{s}_{i-1}| + 2\gamma_{[1;i+1]} - 1, 2\gamma_{[i+2]}, \ldots, 2\gamma_{[i+j-1]}, 2\gamma_{[i+1]} - 1, |\tilde{s}_{i+j-1}| - |\tilde{s}_{i-1}| \right),
$$

with the convention that $\bar{L}_{i,0}^x = \tilde{\alpha}_{n-i}^i = 0$ and that

$$
\tilde{\alpha}_{n-i}^i = \left( |\tilde{s}_{i-1}| + \gamma_{[1;2]} - \frac{1}{2} \right) - (-1)^i |\tilde{s}_{i-1}| \left( \gamma_{1} - \frac{1}{2} \right).
$$

Then the algebra generated by the operators (109) and (110) forms a discrete realization of the rank $n-2$ Bannai-Ito algebra $A_n$ on the module $\hat{V}$.

6. Conclusions and outlook

In this paper we have obtained the connection coefficients between the eigenbases of several abelian subalgebras of the higher rank $q$-Bannai-Ito algebra $A_n^q$. We have given an explicit expression of these coefficients in terms of multivariable $(-q)$-Racah polynomials and shown how these generate a family of infinite-dimensional modules for $A_n^q$, thereby providing an algebraic underpinning for [25]. The limit $q \to 1$ led to similar results for the higher rank $q = 1$ Bannai-Ito algebra, and suggested a natural extension of the Bannai-Ito polynomials to multiple variables.

In Definition 7 the univariate Bannai-Ito polynomials were chosen to satisfy specific truncation conditions. Other choices of truncation (88)–(89), as well as different parametrizations in (98), should lead to other types of multivariate Bannai-Ito polynomials, such as the polynomials obtained in [33] for the bivariate case. It might be of interest to determine the influence of these choices on the orthogonality and bispectrality of the multivariate Bannai-Ito polynomials. Moreover, limiting processes such as established in [49] should lead to $q = -1$ analogs of several other multivariate $q$-orthogonal polynomials, such as $q$-Hahn, $q$-Jacobi and many others, thereby providing multivariate extensions for the whole $q = -1$ Askey scheme.

In Theorem 3 we have obtained a realization of the higher rank $q = 1$ Bannai-Ito algebra by operators acting on multivariate polynomials through discrete shifts in the variables. It should be possible to obtain a different realization, with both difference operators and reflections, as a multivariate extension of [47], thereby complementing the bispectrality of the multivariate Bannai-Ito polynomials. The same principles should work for the $q$-deformed case.
Throughout this paper, we have restricted our attention to abelian subalgebras of the type (15). An interesting but undoubtedly very technical problem would be the generalization of Theorem 1 to other types of abelian subalgebras. In the terminology of [9], this would correspond to asking whether the recoupling graph is connected.

In [19, Section 4] a method is suggested to construct superintegrable quantum Calogero-Gaudin models with \( \mathfrak{osp}_q(1|2) \)-symmetry from the \( q \)-Bannai-Ito generators. This would establish a \( q \)-deformation of the superintegrable systems with reflections in [8].

Finally, the results presented here strengthen the suggestion made in [22] of a deeper connection between quantum algebras and quantum superalgebras under the transformation \( q \rightarrow -q \), based on [46] and [50].

We hope to report on many of these issues in the near future.

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Appendix A: Weight function and normalization coefficient for the multivariate \((-q)\)-Racah polynomials in Theorem 1

The functions \(\rho^{(i,j)}(j^{(i,1)})\) and \(h^{(i,j)}\) in Theorem 1 have the expression

\[
\rho^{(i,j)}(j^{(i,1)}) = \frac{(-q, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+j}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \prod_{k=1}^{j-1} \left[ \frac{(-q, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+k}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \right]
\]

\[
= \frac{(-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+j}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \prod_{k=1}^{j-1} \left[ \frac{(-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+k}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \right]
\]

and

\[
h^{(i,j)} = \frac{(-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+j}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+j-1]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \prod_{k=1}^{j-1} \left[ \frac{(-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}}{(q 2\gamma_{i+k}, (-q)^2 j^{(i,1)}| q 2\gamma_{[i,i+k]}; -q)_j^{(i,1)} - j^{(i,1)}; -q)_j^{(i,1)}} \right]
\]
APPENDIX B: EXPLICIT EXPRESSIONS FOR THE FUNCTIONS $C_{\nu}(x)$ IN PROPOSITION 7 AND $D_{\nu}(n)$ IN PROPOSITION 8

The following expressions for the functions $C_{\nu}(x_1, \ldots, x_{j+1})$ can be obtained from [25, (4.2)–(4.4)], upon using the reparametrization (65). For $\nu \in \{-1, 0, 1\}^j$, let

$$\nu^\pm = (\nu_1^\pm, \ldots, \nu_j^\pm) \in \{0, 1\}^j, \quad \nu_j^+ = \max(\nu_j, 0), \quad \nu_j^- = -\min(\nu_j, 0).$$

Let also $|\nu^\pm| = \sum_{k=1}^j \nu_k^\pm$. Then we have

$$C_{\nu}(x_1, \ldots, x_{j+1}) = (q(q - 1))^j |\nu^+| - |\nu^-| \prod_{\ell=0}^{j-1} B_{\ell}^{(\nu_\ell, \nu_{\ell+1})}(x),$$

with the convention that $\nu_0 = \nu_{j+1} = 0$ and where

$$B_{0}^{(0,0)}(x) = 1 + b + \frac{1}{q - 1} \left(1 - \frac{q b}{a_1}\right) (a_1(-q)x_1 + (-q)^{-x_1}),$$

$$B_{0}^{(0,1)}(x) = (1 - a_1(-q)x_1)(1 - b(-q)^{x_{\ell+1}}),$$

$$B_{0}^{(0,-1)}(x) = (1 - (-q)^{-x_1}) \left(1 - \frac{b}{a_1}(-q)^{-x_{\ell+1}}\right),$$

and where for $\ell \in \{1, \ldots, j\}$ we have

$$B_{\ell}^{(0,0)}(x) = \frac{1}{q - 1} \left(\frac{(-q)^{-x_\ell-x_{\ell+1}}}{A_\ell}\right) \left(1 + (-q)^{-2x_\ell A_\ell} - 1 - \frac{a_{\ell+1}}{q}\right),$$

$$B_{\ell}^{(0,1)}(x) = (1 - A_{\ell+1})(-q)^{x_\ell+x_{\ell+1}} \left(1 - a_{\ell+1}(-q)^{x_{\ell+1}-x_\ell}\right),$$

$$B_{\ell}^{(1,0)}(x) = (1 - A_{\ell+1})(-q)^{x_\ell+x_{\ell+1}} \left(1 - (-q)^{x_\ell-x_{\ell+1}}\right),$$

$$B_{\ell}^{(1,1)}(x) = (1 - A_{\ell+1})(-q)^{x_\ell+x_{\ell+1}} \left(1 - (-q)^{x_{\ell+1}+x_{\ell+1}}\right),$$

$$B_{\ell}^{(-1,0)}(x) = (1 - a_{\ell+1})(-q)^{-x_\ell+x_{\ell+1}} \left(1 - \frac{(-q)^{-x_\ell-x_{\ell+1}}}{A_\ell}\right),$$

$$B_{\ell}^{(-1,1)}(x) = (1 - a_{\ell+1})(-q)^{-x_\ell+x_{\ell+1}} \left(1 - a_{\ell+1}(-q)^{-x_{\ell+1}-x_\ell}\right),$$

$$B_{\ell}^{(-1,-1)}(x) = (1 - (-q)^{-x_\ell-x_{\ell+1}}) \left(1 - \frac{(-q)^{-x_\ell-x_{\ell+1}}}{A_\ell}\right),$$

$$B_{\ell}^{(-1,0)}(x) = (1 - (-q)^{-x_\ell-x_{\ell+1}}) \left(1 - \frac{(-q)^{-x_\ell-x_{\ell+1}}}{A_\ell}\right),$$

and

$$b_{\ell}^{(0)}(x) = (1 - A_{\ell+1})(-q)^{2x_{\ell+1}} \left(1 - \frac{(-q)^{-2x_{\ell+1}}}{A_\ell}\right),$$

$$b_{\ell}^{(1)}(x) = (1 - A_{\ell+1})(-q)^{2x_{\ell+1}} \left(1 - A_{\ell}(-q)^{2x_{\ell+1}}\right),$$

$$b_{\ell}^{(-1)}(x) = \left(1 - \frac{(-q)^{-2x_{\ell+1}}}{A_\ell}\right) \left(1 - \frac{(-q)^{-2x_{\ell+1}}}{A_\ell}\right),$$

Similarly, the functions $D_{\nu}(n_1, \ldots, n_s)$ are given by

$$D_{\nu}(n_1, \ldots, n_s) = (q(q - 1))^j |\nu^+| - |\nu^-| \prod_{\ell=0}^{j-1} E_{\ell}^{(\nu_\ell, \nu_{\ell+1})}(n) \prod_{\ell=1}^{j} e_\ell^{\nu}(n),$$
with the convention that $\nu_0 = \nu_{j+1} = 0$ and where

$$E_{0}^{(0,0)}(n) = 1 + A_{s+1}(-q)^{2n} + \frac{1}{q - 1} \left( 1 - \frac{q b}{a_1} \right) \left( \frac{a_1}{b}(-q)^{N_{s}-N_{e}} + A_{s+1}(-q)^{N_{s}+N_{e}} \right),$$

$$E_{0}^{(0,1)}(n) = \left( 1 - A_{s+1}(-q)^{N_{s}+N_{e}} \right) \left( 1 - \frac{b}{a_1}A_{s+1}(-q)^{N_{s}+1} \right),$$

$$E_{0}^{(0,-1)}(n) = \left( 1 - \frac{a_1}{b}(-q)^{N_{s}-N_{e}} \right) \left( 1 - (-q)^{N_{s}+1} \right),$$

and where for $\ell \in \{1, \ldots, j\}$ we have

$$E_{\ell}^{(0,0)}(n) = 1 - \frac{a_{s+2}}{q} + \frac{1}{q - 1} \left( (-q)^{-N_{s}+\ell+1} + \frac{bA_{s+2}}{a_1}(-q)^{N_{s}-\ell+1} \right) \times \left( \frac{a_1}{bA_{s-\ell+1}}(-q)^{-N_{s}+\ell} \right),$$

$$E_{\ell}^{(0,1)}(n) = \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+\ell+n_{s+1}} \right) \left( 1 - (-q)^{-n_{s+1}} \right),$$

$$E_{\ell}^{(1,0)}(n) = \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+\ell+n_{s+1}} \right) \left( 1 - a_{s-\ell+2}(-q)^{n_{s+1}} \right),$$

$$E_{\ell}^{(1,1)}(n) = \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+\ell+n_{s+1}} \right) \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+\ell+n_{s+1}+1} \right),$$

$$E_{\ell}^{(-1,0)}(n) = \left( 1 - (-q)^{-n_{s+1}} \right) \left( 1 - \frac{a_1(-q)^{-2N_{s}+\ell-n_{s+1}}}{bA_{s-\ell+1}} \right),$$

$$E_{\ell}^{(-1,1)}(n) = \left( 1 - (-q)^{-n_{s+1}} \right) \left( 1 - (-q)^{-n_{s+1}+1} \right),$$

$$E_{\ell}^{(0,-1)}(n) = \left( 1 - \frac{a_{s-\ell+2}}{b}(-q)^{n_{s+1}} \right) \left( 1 - a_1(-q)^{-2N_{s}+\ell-n_{s+1}} \right),$$

$$E_{\ell}^{(-1,-1)}(n) = \left( 1 - a_1(-q)^{-2N_{s}+\ell-n_{s+1}} \right) \left( 1 - \frac{a_1(-q)^{-2N_{s}+\ell-n_{s+1}+1}}{bA_{s-\ell+1}} \right),$$

and

$$e_{\ell}^{0}(n) = \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+\ell+1} \right) \left( 1 - a_1(-q)^{-2N_{s}+\ell+1} \right),$$

$$e_{\ell}^{1}(n) = \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+1} \right) \left( 1 - \frac{b}{a_1}A_{s-\ell+2}(-q)^{2N_{s}+1} \right),$$

$$e_{\ell}^{-1}(n) = \left( 1 - a_1(-q)^{-2N_{s}+\ell+1} \right) \left( 1 - \frac{a_1(-q)^{-2N_{s}+\ell+1}}{bA_{s-\ell+2}} \right).$$
APPENDIX C: GAUGE COEFFICIENTS IN THE PROOF OF THEOREM 2

Let us denote by \( A_{i,j} \) the set \([1; i - 2] \cup [i + j - 1; n - 2]\). Starting from the definition (111), a long but straightforward calculation shows

\[
\frac{\rho(1,n-1)(s)}{\rho(1,j)(s)} = \frac{(-q, q^{2}[1; n-1]; -q)_{[s_{n-1}]} (q^{2}[j+1], (-q)^{2}[s_{n-1}]^{-1} q^{2[1;j+1]}; -q)_{[s_{n-1}]}}{(-q, (-q)^{2}[s_{n-1}] q^{2[1;j+1]}; -q)_{[s_{n-1}]}} \times \prod_{k \in A_{i,j}} \frac{(-q, q^{2!}; -q)_{sk_{k+1}} (q^{2[1;k+2]}; -q)_{sk_{k+1}}^{-1}}{(-q, -q)_{sk_{k+1}} (q^{2[1;k+1]}; -q)_{sk_{k+1}}^{-1}} \times \prod_{k=0}^{n-2} \frac{(1 + q^{2[1;k+1]}; -q)_{[s_{n-1}]-1}}{1 + q^{2[1;k+1]} - 1} \times \prod_{k=j}^{n-2} \frac{(q^{2[1;k+2]}; -q)_{sk_{k+1}}^{-1}}{(q^{2[1;k+1]}; -q)_{sk_{k+1}}^{-1}} \frac{1 + q^{2[1;k+1]}; -q)_{[s_{n-1}]}}{1 + q^{2[1;k+1]} - 1},
\]

for \( i \in \{2, \ldots, n-1\} \), whereas for \( i = 1 \) one finds

\[
\frac{\rho(1,n-1)(s)}{\rho(1,1)(s)} = \frac{(-q, q^{2[1; n-1]}; -q)_{[s_{n-1}]} (q^{2[1+1]}, -q^{2[1;j+1]}; -q)_{[s_{n-1}]}}{(-q, q^{2[1]; -q)_{[s_{n-1}]} (q^{2[1,j+1]}; -q)_{[s_{n-1}]}} \times \prod_{k=j}^{n-2} \frac{(q^{2[1;k+2]; -q)_{sk_{k+1}}^{-1}}}{(q^{2[1;k+1]; -q)_{sk_{k+1}}^{-1}} \frac{1 + q^{2[1;k+1]}}{1 + q^{2[1;k+1]} - 1}.
\]
APPENDIX D: Weight function and normalization coefficient for the multivariate Bannai-Ito polynomials in Proposition 9

The functions $w_{\ell_i-N_{i-1}}^{(i)}$ in Proposition 9 vanish if $\ell_i < N_{i-1}$, whereas if $\ell_i \geq N_{i-1}$, they have the expression

$$w_{\ell_i-N_{i-1}}^{(i)} (\ell_i, \ell_{i+1}, N_{i-1}, \alpha_1, \ldots, \alpha_{i+1}, \beta)$$

$$= (-1)^{\ell_i-N_{i-1}} \frac{(N_{i-1} - \frac{\alpha_1}{2} + \frac{1}{2} + \frac{(-1)^{\ell_i+1+N_{i-1}}}{2} (\ell_{i+1} + \frac{\alpha_i+1}{2}))}{(\ell_i-N_{i-1})! (\ell_{i-1})! (\ell_{i+1})!}$$

$$\times \frac{(N_{i-1} - \frac{\alpha_1}{2} + \frac{\alpha_i+1}{4} + \frac{(-1)^{\ell_i+1+N_{i-1}}}{2} (\ell_{i+1} + \frac{\alpha_i+1}{2}))}{(\ell_i-N_{i-1})! (\ell_{i-1})! (\ell_{i+1})!}$$

The functions $h_{ni}^{(i)}$ are given by

$$h_{ni}^{(i)} (\ell_{i+1}, N_{i-1}, n_i, \alpha_1, \ldots, \alpha_{i+1}, \beta)$$

$$= \frac{(N_{i-1} - \frac{\alpha_1}{2} + \frac{\alpha_i+1}{4} + \frac{(-1)^{\ell_i+1+N_{i-1}}}{2} (\ell_{i+1} + \frac{\alpha_i+1}{2}))}{(\ell_i-N_{i-1})! (\ell_{i-1})! (\ell_{i+1})!}$$

$$\times \left[ \frac{(N_{i-1} - \frac{\alpha_1}{2} + \frac{\alpha_i+1}{4} + \frac{(-1)^{\ell_i+1+N_{i-1}}}{2} (\ell_{i+1} + \frac{\alpha_i+1}{2}))}{(\ell_i-N_{i-1})! (\ell_{i-1})! (\ell_{i+1})!} \right]$$

$$\times \frac{(n_i)_{\ell_{i+1}-N_i-1}! (n_i)_{\ell_{i-1}-N_i-1}! (n_i)_{\ell_{i+1}-N_i-1}!}{(\ell_{i+1})! (\ell_{i-1})! (\ell_{i+1})!}$$
APPENDIX E: Explicit expressions for the functions $\tilde{C}_{\nu}(x)$ in Proposition 10 and $\tilde{D}_{\nu}(n)$ in Proposition 11

For $\nu \in \{-1,0,1\}^j$ we have

$$\tilde{C}_{\nu}(x_1,\ldots,x_{j+1}) = \frac{\prod_{j=0}^{j} \tilde{B}_{\ell}^{(\nu_{j},\nu_{j+1})}(x)}{\prod_{j=1}^{j} \tilde{b}_{\ell}^{\nu}(x)},$$

with the convention that $\nu_0 = \nu_{j+1} = 0$ and where

$$\tilde{B}_{0}^{(0,0)}(x) = \frac{1}{2} (-\beta + (1) x_1 (2x_1 + \alpha_1)(\beta - \alpha_1 + 1)),
\tilde{B}_{0}^{(0,1)}(x) = -\frac{1}{2} ((2x_1 + \alpha_1 + \beta + 1) + (1) x_1 (\beta - \alpha_1 + 1)),
\tilde{B}_{0}^{(0,-1)}(x) = \frac{1}{2} ((2x_1 + \alpha_1 - \beta - 1) + (1) x_1 (\beta - \alpha_1 + 1)),$$

and where for $\ell \in \{1,\ldots,j\}$ we have

$$\tilde{B}_{\ell}^{(0,0)}(x) = (1 - \alpha_{\ell+1}) - (-1)^{x_\ell+1}(2x_\ell + \alpha_{[1;\ell]})(2x_{\ell+1} + \alpha_{[1;\ell+1]}),
\tilde{B}_{\ell}^{(0,1)}(x) = -(2x_{\ell+1} + \alpha_{[1;\ell+1]} + \alpha_{\ell+1}) + (-1)^{x_\ell+1}(2x_\ell + \alpha_{[1;\ell]}),
\tilde{B}_{\ell}^{(1,0)}(x) = -(2x_\ell + \alpha_{[1;\ell+1]} + (1) x_\ell+1(2x_{\ell+1} + \alpha_{[1;\ell+1]}),
\tilde{B}_{\ell}^{(1,1)}(x) = -(-2x_\ell + 2x_{\ell+1} + 2\alpha_{[1;\ell+1]} + 1) + (-1)^{x_\ell+1}(2x_{\ell+1} + \alpha_{[1;\ell+1]}),
\tilde{B}_{\ell}^{(-1,0)}(x) = (2x_\ell + \alpha_{[1;\ell]} - \alpha_{\ell+1}) - (-1)^{x_\ell+1}(2x_{\ell+1} + \alpha_{[1;\ell+1]}),
\tilde{B}_{\ell}^{(-1,1)}(x) = (2x_\ell - 2x_{\ell+1} - 2\alpha_{\ell+1} - 1) + (-1)^{x_\ell+1},
\tilde{B}_{\ell}^{(0,-1)}(x) = (2x_{\ell+1} + \alpha_{[1;\ell]} - (-1)^{x_\ell+1}(2x_\ell + \alpha_{[1;\ell]}),
\tilde{B}_{\ell}^{(1,-1)}(x) = -(2x_\ell - 2x_{\ell+1} + 1) + (-1)^{x_\ell+1},
\tilde{B}_{\ell}^{(-1,-1)}(x) = (2x_\ell + 2x_{\ell+1} + 2\alpha_{[1;\ell]} - 1) - (-1)^{x_\ell+1},$$

and

$$\tilde{b}_{\ell}^{0}(x) = -(2x_\ell + \alpha_{[1;\ell]} + 1)(2x_\ell + \alpha_{[1;\ell]} - 1),
\tilde{b}_{\ell}^{1}(x) = -2(2x_\ell + \alpha_{[1;\ell]} + 1),
\tilde{b}_{\ell}^{-1}(x) = 2(2x_\ell + \alpha_{[1;\ell]} - 1).$$

Similarly, the functions $\tilde{D}_{\nu}(n_1,\ldots,n_s)$, with $\nu \in \{-1,0,1\}^j$, are given by

$$\tilde{D}_{\nu}(n_1,\ldots,n_s) = \frac{\prod_{j=0}^{j} \tilde{E}^{(\nu_{j},\nu_{j+1})}_{\ell}(n)}{\prod_{j=1}^{j} \tilde{e}_{\ell}^{\nu}(n)},$$

with the convention that $\nu_0 = \nu_{j+1} = 0$ and where

$$\tilde{E}^{(0,0)}_{0}(n) = -\frac{1}{2} ((2N + \alpha_{[1;s+1]} + (-1)^{N+N_s}(2N_\alpha + \alpha_{[2;s+1]} + \beta)(\alpha_1 - \beta - 1)),
\tilde{E}^{(0,1)}_{0}(n) = -\frac{1}{2} ((2N + 2N_\alpha + 2\alpha_{[2;s+1]} + \alpha_1 + \beta + 1) + (-1)^{N-N_s}(\beta - \alpha_1 + 1)),
\tilde{E}^{(0,-1)}_{0}(n) = -\frac{1}{2} ((2N - 2N_\alpha + \alpha_1 - \beta + 1) - (-1)^{N-N_s}(\beta - \alpha_1 + 1)),$$
and where for $\ell \in \{1, \ldots, j\}$ we have
\[
\tilde{E}_\ell^{(0,0)}(n) = (1 - \alpha_{s-\ell+2}) - (-1)^{n_s - \ell + 1}(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta)(2N_{s-\ell} + \alpha_{[2:s-\ell+1]} + \beta),
\[
\tilde{E}_\ell^{(0,1)}(n) = - (2N_{s-\ell} + \alpha_{[2:s-\ell+2]} + \beta) + (-1)^{n_s - \ell + 1}(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta),
\]
\[
\tilde{E}_\ell^{(1,0)}(n) = - (2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \alpha_{s-\ell+2} + \beta) + (-1)^{n_s - \ell + 1}(2N_{s-\ell} + \alpha_{[2:s-\ell+1]} + \beta),
\]
\[
\tilde{E}_\ell^{(1,1)}(n) = - (4N_{s-\ell} + 2n_{s-\ell+1} + 2\alpha_{[2:s-\ell+2]} + 2\beta + 1) - (-1)^{n_s - \ell + 1},
\]
\[
\tilde{E}_\ell^{(-1,0)}(n) = (2N_{s-\ell+1} + \alpha_{[2:s-\ell+1]} + \beta) - (-1)^{n_s - \ell + 1}(2N_{s-\ell} + \alpha_{[2:s-\ell+1]} + \beta),
\]
\[
\tilde{E}_\ell^{(-1,1)}(n) = 2n_{s-\ell+1} - 1 + (-1)^{n_s - \ell + 1},
\]
\[
\tilde{E}_\ell^{(0,-1)}(n) = (2N_{s-\ell} + \alpha_{[2:s-\ell+1]} - \alpha_{s-\ell+2} + \beta) - (-1)^{n_s - \ell + 1}(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta),
\]
\[
\tilde{E}_\ell^{(1,-1)}(n) = -(2n_{s-\ell+1} + 2\alpha_{s-\ell+2} + 1) + (-1)^{n_s - \ell + 1},
\]
\[
\tilde{E}_\ell^{(-1,-1)}(n) = (4N_{s-\ell} + 2n_{s-\ell+1} + 2\alpha_{[2:s-\ell+1]} + 2\beta - 1) - (-1)^{n_s - \ell + 1},
\]
and
\[
\tilde{e}_\ell^0(n) = -(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta + 1)(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta - 1),
\]
\[
\tilde{e}_\ell^1(n) = -2(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta + 1),
\]
\[
\tilde{e}_\ell^{-1}(n) = 2(2N_{s-\ell+1} + \alpha_{[2:s-\ell+2]} + \beta - 1).
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