FREE SUBGROUPS
IN ALMOST SUBNORMAL SUBGROUPS
OF GENERAL SKEW LINEAR GROUPS

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Abstract. Let $D$ be a weakly locally finite division ring and $n$ a positive integer. The problem under study concerns the existence of noncyclic free subgroups in non-central almost subnormal subgroups of the general linear group $\text{GL}_n(D)$. Further, some applications are also investigated. In particular, all infinite finitely generated almost subnormal subgroups of $\text{GL}_n(D)$ are described.

§1. Introduction and preliminaries

Let $G$ be a group and $H$ a subgroup of $G$. Following Hartley [18], we say that $H$ is almost subnormal in $G$, and write $H$ as $G$ for short, if there is a family of subgroups

$$H = H_r \leq H_{r-1} \leq \cdots \leq H_1 = G$$

of $G$ such that for each $1 < i \leq r$, either $H_i$ is normal in $H_{i-1}$, or $H_i$ has finite index in $H_{i-1}$. Such a series of subgroups is called an almost normal series in $G$. In this paper, we study the problem on the existence of noncyclic free subgroups in almost subnormal subgroups of the general linear group $\text{GL}_n(D)$ over a division ring $D$ and the related problems.

The question on the existence of noncyclic free subgroups in linear groups over a field was studied by Tits in [26]. The main theorems of Tits assert that in the characteristic 0, every subgroup of the general linear group $\text{GL}_n(F)$ over a field $F$ either contains a noncyclic free subgroup or is soluble-by-finite, and the same conclusion is true for finitely generated subgroups in the case of prime characteristic. This famous result of Tits is now often referred as the Tits Alternative. The question as to whether the Tits Alternative would remain true for skew linear groups was posed by S. Bachshmut at the Second International Conference on the Theory of Groups (see [2, p. 736]). In [22], Lichtman proved that there exists a finitely generated group which is not soluble-by-finite and does not contain a noncyclic free subgroup, but whose group ring over any field can be embedded in a division ring of quotients. Therefore, the Tits Alternative fails even for matrices of degree one, i.e., for $D^* = \text{GL}_1(D)$, where $D$ is a noncommutative division ring. In [22], Lichtman remarked that it was not known whether the multiplicative group of a noncommutative division ring contains a noncyclic free subgroup. In [12], Gonçalves and Mandel posed a more general question: does a noncentral subnormal subgroup of the multiplicative group of a division ring contain a noncyclic free subgroup? This question was studied by several authors. Gonçalves [10] proved that the multiplicative group $D^*$ of a division ring $D$ with center $F$ contains a noncyclic free subgroup if $D$ is centrally finite.

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that is, $D$ is a finite dimensional vector space over $F$. The same result was obtained by Reichstein and Vonessen in [28] if $F$ is uncountable and there exists a noncentral element $a$ in $D$ that is algebraic over $F$. Later, Chiba [7] proved the same result but without the assumption on the existence of such an element $a$ in $D$. In [11], Gonçalves proved that any noncentral subnormal subgroup of $D^\ast$ contains a noncyclic free subgroup provided $D$ is centrally finite. Recently, B. X. Hai and N. K. Ngoc [16] proved the same result for weakly locally finite division rings. Recall that a division ring $D$ is called \textit{weakly locally finite} if every finite subset of $D$ generates a centrally finite division subring. It was proved that every locally finite division ring is weakly locally finite, and there exist infinitely many weakly locally finite division rings that are not even algebraic over their centers (see [16] and [17]), so they are not locally finite. Logically, it is natural to carry over the above results for subnormal subgroups of $GL_1(D)$ to those of $GL_n(D)$, $n \geq 2$ (see [14,24,30]). In the present paper, we investigate the question on the existence of free subgroups in almost subnormal subgroups of the group $GL_n(D)$ with $n \geq 1$ and a not necessarily commutative division ring $D$.

Note that in [10] Example 8], Hazrat and Wadsworth gave examples of division rings whose multiplicative groups contain nonnormal maximal subgroups of finite index. Hence, the existence of almost subnormal subgroups in $D^\ast := GL_1(D)$ is doubtless. Concerning the group $GL_n(D)$, $n \geq 2$, we shall prove that if $D$ is infinite then every almost subnormal subgroup of $GL_n(D)$ is normal (see Theorem 3.3 in the text). However, we shall continue to use “almost subnormal” instead of “normal” to compare the results with the corresponding ones in the case of $n = 1$.

All symbols and notation we use in this paper are standard. In particular, if $A$ is a ring or a group, then $Z(A)$ denotes the center of $A$. If $D$ is a division ring with center $F$ and $S$ is a subset of $D$, then $F(S)$ denotes the division subring of $D$ generated by the set $F \cup S$. We say that $F(S)$ is the division subring of $D$ \textit{generated by $S$ over $F$}. Finally, $D^\prime := [D^\ast, D^\ast]$ is the commutator subgroup of $D^\ast$. The following lemma, which will be used frequently in this paper, is almost evident, so we omit its proof.

\textbf{Lemma 1.1.} Let $H$ be an almost subnormal subgroup of a subgroup $G$. If $N$ is a subgroup of $G$ containing $H$, then $H$ is an almost subnormal subgroup of $N$.

\section{Almost subnormal subgroups with generalized group identities}

Let $G$ be a group with center $Z(G) = \{a \in G \mid ab = ba \text{ for any } b \in G\}$. An expression

$$w(x_1, x_2, \ldots, x_m) = a_1x_1^{\alpha_1}a_2x_2^{\alpha_2} \ldots a_tx_t^{\alpha_t}a_{t+1},$$

where $t, m$ are positive integers, $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, m\}$, $a_1, a_2, \ldots, a_{t+1} \in G$, and $\alpha_1, \alpha_2, \ldots, \alpha_t \in Z \setminus \{0\}$, is called a \textit{generalized group monomial} over $G$ if, whenever $i_j = i_{j+1}$ with $\alpha_j \alpha_{j+1} < 0$ ($j \in \{1, 2, \ldots, t-1\}$), we have $a_j \notin Z(G)$ (see [27]). Moreover, if $i_j \neq i_{j+1}$ whenever $\alpha_j \alpha_{j+1} < 0$, then we say that $w$ is a \textit{strict generalized group monomial} over $G$. If $G = \{1\}$, then we simply call $w$ a \textit{group monomial}. It is clear that a group monomial is a strict generalized group monomial.

Let $H$ be a subgroup of $G$. We say that $H$ satisfies the \textit{generalized group identity} $w(x_1, x_2, \ldots, x_m) = 1$ if $w(c_1, c_2, \ldots, c_m) = 1$ for any $c_1, c_2, \ldots, c_m \in H$. In particular, we say that $H$ satisfies a \textit{group identity} (respectively, a \textit{strict generalized group identity}) $w = 1$ if $w$ is a group monomial (respectively, a strict generalized group monomial) and $w(c_1, c_2, \ldots, c_m) = 1$ for any $c_1, c_2, \ldots, c_m \in H$.

In this section, we prove some properties of almost subnormal subgroups with generalized group identities to be used subsequently. We begin with the following useful lemma.
Lemma 2.1. Let $G$ be a group and assume that $H$ is a noncentral almost subnormal subgroup of $G$. If $H$ satisfies a generalized group identity over $G$, then so does $G$.

Proof. Assume that $H$ is a noncentral almost subnormal subgroup of $G$ satisfying a generalized group identity over $G$. Let $H = H_r \leq H_{r-1} \leq \cdots \leq H_1 = G$ be an almost normal series in $G$. To prove the lemma, that is, to prove that $H_1$ satisfies a generalized group identity, it suffices to prove that $H_{r-1}$ satisfies a generalized group identity over $G$.

Let $w(x_1, x_2, \ldots, x_m) = a_1 x_1^{n_1} a_2 x_2^{n_2} \cdots a_t x_t^{n_t} a_{t+1}$ be a generalized group identity of $H$ over $G$. By replacing $x_i = y_i y_{i+m}$, we get

$$u(y_1, y_2, \ldots, y_{2m}) = w(y_1 y_{1+m}, y_2 y_{2+m}, \ldots, y_{m} y_{2m}) = b_1 y_{j_1}^{\delta_1} b_2 y_{j_2}^{\delta_2} \cdots b_s y_{j_s}^{\delta_s} b_{s+1},$$

where $\delta_i \in \{-1,1\}$, which is also a generalized group identity of $H$. Hence, without loss of generality, we may assume that in $w$, the powers $n_i$ belong to $\{-1,1\}$ for any $1 \leq i \leq t$. There are two cases to examine.

Case 1. $H_r$ has finite index $k$ in $H_{r-1}$.

Then, for any $c_1, c_2, \ldots, c_m \in H_{r-1}$, we have $c_1^{k_1}, c_2^{k_1}, \ldots, c_m^{k_1} \in H_r$. By assumption, $w(c_1^{k_1}, c_2^{k_1}, \ldots, c_m^{k_1}) = 1$. But this means that $H_{r-1}$ satisfies $w(x_1^{k_1}, x_2^{k_1}, \ldots, x_m^{k_1}) = 1$.

Case 2. $H_r$ is normal in $H_{r-1}$.

Since $H = H_r$ is noncentral, there exists a noncentral element $a \in H_r$. Replacing $x_j$ by $x_j ax_j^{-1}$ for any $1 \leq j \leq m$, we get

$$w_1(x_1, x_2, \ldots, x_m) = w(x_1 ax_1^{-1}, x_2 ax_2^{-1}, \ldots, x_m ax_m^{-1}),$$

which is a generalized group monomial over $G$ by [3] Lemma 3.2. Since $H_r$ is normal in $H_{r-1}$, we have $c_i ac_i^{-1} \in H_r$ for any $c_i \in H_{r-1}$. Therefore,

$$w_1(c_1, c_2, \ldots, c_m) = w(c_1 ac_1^{-1}, c_2 ac_2^{-1}, \ldots, c_m ac_m^{-1}) = 1$$

for any $c_1, c_2, \ldots, c_m \in H_{r-1}$, which shows that $w_1 = 1$ is a generalized group identity of $H_{r-1}$. Therefore, the proof of the lemma is complete.

From Theorems 1, 2 in [3], it follows that for any division ring $D$ with infinite center $F$, if $\text{GL}_n(D)$ satisfies a generalized group identity, then $n = 1$ and $D = F$. Recently, it was proved that this result remains true if one replaces $\text{GL}_n(D)$ by any of its subnormal subgroup [5 Theorem 1.1]. Lemma 2.1 gives us the possibility to get the following strong result.

Theorem 2.2. Let $D$ be a division ring with infinite center $F$ and assume that $G$ is an almost subnormal subgroup of $\text{GL}_n(D)$. If $G$ satisfies a generalized group identity over $\text{GL}_n(D)$, then $G$ is central.

Proof. If $G$ is noncentral, then by Lemma 2.1 $\text{GL}_n(D)$ satisfies a generalized group identity. In view of [3], $n = 1$ and $D$ is commutative, so $G$ is central, a contradiction.

Corollary 2.3. Let $D$ be a division ring with infinite center $F$ and assume that $G$ is an almost subnormal subgroup of $\text{GL}_n(D)$. If $G$ is Abelian, then $G$ is central.

Proof. If $G$ is Abelian, then $G$ satisfies the group identity $xyx^{-1}y^{-1} = 1$. So, by Theorem 2.2 $G$ is central.
§3. Almost subnormal subgroups of $GL_n(D)$ are normal

Recall that a field $K$ is said to be locally finite if every subfield generated by finitely many elements of $K$ is finite. Hence, $K$ is locally finite if and only if its prime subfield $P$ is a finite field and $K$ is algebraic over $P$. The following theorem was proved in [30].

**Theorem A.** Let $D$ be a division ring that is not a locally finite field and let $n > 1$ be an integer. If $N$ is a noncentral normal subgroup of $GL_n(D)$, then $N$ contains a noncyclic free subgroup.

Our aim in this section is to prove that if $D$ is an infinite division ring and $n \geq 2$, then every almost subnormal subgroup of $GL_n(D)$ is normal in $GL_n(D)$. Hence, by Theorem A, the problem on the existence of noncyclic free subgroups in almost subnormal subgroups of general skew linear groups reduces to that in skew linear groups of degree 1. To prove this fact, we need the following results.

**Theorem 3.1.** Let $D$ be a division ring, $n$ a natural number, and $N$ a noncentral subgroup of $GL_n(D)$. Suppose that either $n \geq 3$ or $n = 2$ but $D$ contains at least four elements. If $xNx^{-1} \subseteq N$ for any $x \in SL_n(D)$, then $N$ contains $SL_n(D)$. In particular, a noncentral subgroup of $GL_n(D)$ is normal in $GL_n(D)$ if and only if it contains $SL_n(D)$.

**Proof.** The proof follows from Theorem 4.7 and Theorem 4.9 in [11]. □

**Lemma 3.2.** Let $D$ be a division ring, and let $n > 1$. Then the special linear group $SL_n(D)$ satisfies a group identity if and only if $D$ is finite.

**Proof.** If $D$ is finite, then $SL_n(D)$ is finite, so $SL_n(D)$ satisfies a group identity. Assume that $D$ is infinite. Let $K$ be a maximal subfield of $D$. If $K$ is finite, then $\dim_F D < \infty$ by [21] (15.8)], which implies that $D$ is finite, a contradiction. Therefore, $K$ is infinite. Suppose that $w(x_1, x_2, \ldots, x_m) = 1$ is a group identity of $SL_n(D)$. Then, $GL_n(D)$ satisfies the group identity

$$w(y_1, y_2, \ldots, y_{2m}) = w(y_1y_2y_1^{-1}y_2^{-1}, y_3y_4y_3^{-1}y_4^{-1}, \ldots, y_{m-1}y_my_{m-1}^{-1}y_m^{-1}) = 1.$$ 

In view of Theorem 2.2 $GL_n(D)$ is central, a contradiction. □

Let $G$ be a group and $H$ a subgroup of $G$. Denote by $Core_G(H)$ the core of $H$ in $G$, that is,

$$Core_G(H) = \bigcap_{x \in G} xHx^{-1}.$$ 

It is well known that $Core_G(H)$ is the largest normal subgroup of $G$ contained in $H$. Moreover, if $H$ is of finite index in $G$, then so is $Core_G(H)$.

The following theorem is the main result of this section.

**Theorem 3.3.** Let $D$ be an infinite division ring, and let $n \geq 2$. Assume that $N$ is a noncentral subgroup of $G := GL_n(D)$. The following conditions are equivalent:

1. $N$ is almost subnormal in $G$;
2. $N$ is subnormal in $G$;
3. $N$ is normal in $G$;
4. $N$ contains $SL_n(D)$.

**Proof.** The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are trivial. Now we show that (1) implies (4). Assume that $N$ is a noncentral almost subnormal subgroup of $G$ and $N = N_r \leq N_{r-1} \leq \cdots \leq N_1 \leq N_0 = G$ is an almost normal series of $N$ in $G$. We shall prove that $N_i$ is normal in $G$ for all $1 \leq i \leq r$ by induction on $i$. Assume that $N_1$ has finite index in $G$. Then, $Core_G(N_1)$ is
a normal subgroup, say of finite index $m$ in $G$ and it is contained in $N_1$. If $\text{Core}_G(N_1)$ is central, then $x^m y^m x^{-m} y^{-m} = 1$ for any $x, y \in G$. In view of Lemma 3.2, $D$ is finite, a contradiction. Thus, $\text{Core}_G(N_1)$ is a noncentral normal subgroup of $G$. By Theorem 3.3 $\text{SL}_n(D) \subseteq \text{Core}_G(N_1) \subseteq N_1$.

Assume that $j > 1$ and $N_j$ contains $\text{SL}_n(D)$. We must prove that $N_{j+1}$ also contains $\text{SL}_n(D)$. Indeed, assume that $N_{j+1}$ has finite index in $N_j$. Then the subgroup $\text{Core}_{N_j}(N_{j+1})$ is normal in $N_j$ and is of finite index, say $k$, and it is contained in $N_{j+1}$. Assume that $\text{Core}_{N_j}(N_{j+1})$ is central. Then, $x^k y^k x^{-k} y^{-k} = 1$ for any $x, y \in N_j$. In particular, $\text{SL}_n(D)$ satisfies the group identity $x^k y^k x^{-k} y^{-k} = 1$. In view of Lemma 3.2 $D$ is finite, a contradiction. Hence, $\text{Core}_{N_j}(N_{j+1})$ is a noncentral normal subgroup of $N_j$. Therefore, $x \text{Core}_{N_j}(N_{j+1}) x^{-1} \subseteq \text{Core}_{N_j}(N_{j+1})$ for any $x \in \text{SL}_n(D) \subseteq N_j$. In view of Theorem 3.1 $\text{Core}_{N_j}(N_{j+1})$ contains $\text{SL}_n(D)$ and so does $N_{j+1}$.

The implication (1) $\Rightarrow$ (4) is proved, and so the proof of the theorem is complete. $\square$

**Remark 3.4.** Theorem 3.3 fails if $D$ is a finite field. Indeed, let $D = F_q$ be a finite field with $q$ elements and consider the projective special linear group $\text{PSL}(n, q)$, which is different from the groups $\text{PSL}(2, 2)$ and $\text{PSL}(2, 3)$. Then it is well-known that $\text{PSL}(n, q)$ is a simple group. Assume that $k$ is a prime divisor of $|\text{PSL}(n, q)|$ and $H$ is the inverse image of a subgroup of order $k$ in $\text{PSL}(n, q)$ via the natural homomorphism

$$\text{SL}(n, q) \rightarrow \text{PSL}(n, q).$$

Then, $H$ is a noncentral proper subgroup of $\text{SL}(n, q)$. By Theorem 3.1 $H$ is not normal in $\text{GL}(n, q)$. Hence, $H$ is an almost subnormal subgroup of $\text{GL}(n, q)$ which is not normal in $\text{GL}(n, q)$.

§4. NONCYCLIC FREE SUBGROUPS IN NONCOMMUTATIVE DIVISION RINGS

As we have mentioned in the Introduction, almost subnormal subgroups that are not subnormal exist in the multiplicative group of a division ring. Our aim in this section is to show that if a noncommutative division ring $D$ is weakly locally finite, then every noncentral almost subnormal subgroup of $D^*$ contains a noncyclic free subgroup. Recall that a division ring $D$ is weakly locally finite if every finite subset in $D$ generates a centrally finite division subring in $D$. Some basic properties and the existence of noncyclic free subgroups in weakly locally finite division rings can be found in [8] and [10]. The following lemma is useful for our study.

**Lemma 4.1.** Let $D$ be a noncommutative weakly locally finite division ring with center $F$ and assume that $G$ is an almost subnormal subgroup of $D^*$. If $G$ satisfies a generalized group identity, then $G$ is central. In particular, If $G$ is Abelian, then $G$ is central.

**Proof.** Assume that $G$ is noncentral and $G$ satisfies some generalized group identity $w(x_1, x_2, \ldots, x_m) = 1$. By Lemma 2.1 $D^*$ satisfies some generalized group identity $w'(x_1, \ldots, x_m) = a_1 x_1^{t_1} a_2 x_2^{t_2} \cdots a_m x_m^{t_m} a_{m+1} = 1$. Let $x, y \in D$ such that $xy \neq yx$. Consider the division subring $D_1$ of $D$ generated by $x, y$, and all $a_i$. Then $D_1$ is noncommutative and centrally finite. Since $a_i \in D_1^* \subseteq D^*$, $w' = 1$ is a generalized group identity of $D_1^*$. In view of Theorem 2.2 $D_1$ is commutative, a contradiction. Hence, $G$ is central. $\square$

**Theorem 4.2.** Let $D$ be a weakly locally finite division ring with center $F$. Then every noncentral almost subnormal subgroup of $D^*$ contains a noncyclic free subgroup.

**Proof.** Assume that $G$ is a noncentral almost subnormal subgroup of $D^*$. By Lemma 4.1 $G$ is non-Abelian. Hence, there exist $a, b \in G$ such that $ab \neq ba$. Denote by $D_1$ the division subring of $D$ generated by $a, b$. Then, $D_1$ is centrally finite. By Lemma 1.1
$N = G \cap D_1^*$ is an almost subnormal subgroup of $D_1^*$. We claim that $N$ contains a noncyclic free subgroup. Indeed, if this is not the case, then, by [20, Theorem 2.21], $N$ satisfies a group identity $w(x_1, x_2, \ldots, x_m) = 1$. Observe that the center $F_1$ of $D_1$ is infinite, so by Theorem 2.2, $N$ is central. In particular, $ab = ba$, which is a contradiction. Thus, the claim is proved, and this implies that $G$ contains a noncyclic free subgroup.

Now, combining Theorem 1.2 and Theorem A, we summarize the results on the existence of noncyclic free subgroups in almost subnormal subgroups of the general linear group over a weakly locally finite division ring.

**Theorem 4.3.** Let $D$ be a weakly locally finite division ring, $n$ a natural number and $N$ a noncentral almost subnormal subgroup of $GL_n(D)$. Then, $N$ contains a noncyclic free subgroup if one of the following conditions satisfies:

1. $D$ is noncommutative;
2. $n = 1$;
3. $n \geq 2$ and $D$ is not a locally finite field.

The following proposition extends [12, Corollary 3.4].

**Proposition 4.4.** Let $D$ be a division ring with center $F$ and $G$ an almost subnormal subgroup of $D^*$. If $G \setminus F$ contains a torsion element, then $G$ contains a noncyclic free subgroup.

**Proof.** Assume that $a \in G \setminus F$ is such that $a^n = 1$ for some positive integer $n$. By [4 Proposition 2.1], there exists a centrally finite division ring $D_1$ such that $a \notin F_1 = Z(D_1)$. Using the same argument as in the proof of Theorem 1.2, we see that $M = G \cap D_1^*$ is an almost subnormal subgroup of $D_1^*$. Since $a \in M$, the subgroup $M$ is noncentral. By Theorem 1.2, $M$ contains a noncyclic free subgroup, so does $G$.

**Theorem 4.5.** Let $D$ be a weakly locally finite division ring with center $F$, and $G$ an almost subnormal subgroup of $D^*$. If $G$ is soluble-by-periodic, then $G$ is central.

**Proof.** Assume that $G$ is noncentral. Let $H$ be a soluble normal subgroup of $G$ such that $G/H$ is periodic. Then, $H$ is almost subnormal in $D^*$. Since $H$ is soluble, $H$ is central by Lemma 4.1. Hence, for any $x \in G$, there exists a natural number $n_x$ such that $x^{n_x} \in H \subseteq F$. But this is impossible since $G$ contains a noncyclic free subgroup by Theorem 1.2.

Note that previously in [23, Proposition 1], Lichtman proved the same result as in Theorem 4.5 for normal subgroups in centrally finite division rings. The class of weakly locally finite division rings we consider in Theorem 4.5 is very large. Indeed, in [16] it was indicated that this class strictly contains the class of locally finite division rings. Recently, in [17], we have constructed infinitely many examples of weakly locally finite division rings that are not even algebraic over the center.

Now, let $D$ be a centrally finite division ring. The following theorem gives useful characterization of a subgroup of $D^*$ that contains no noncyclic free subgroups.

**Theorem 4.6.** Let $D$ be a centrally finite division ring and $G$ a subgroup of $D^*$. The following conditions are equivalent:

1. $G$ contains no noncyclic free subgroups;
2. $G$ is soluble-by-finite;
3. $G$ is Abelian-by-finite;
4. $G$ satisfies a group identity;
5. $G$ contains a soluble subgroup of finite index;
6. $G$ contains an Abelian subgroup of finite index;
(7) $G$ satisfies a strict generalized group identity.

Proof. For (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) see [20, Theorem 2.21]. The implications (2) $\Rightarrow$ (5), (3) $\Rightarrow$ (6), (6) $\Rightarrow$ (5), and (4) $\Rightarrow$ (7) are trivial.

To prove (5) $\Rightarrow$ (4), assume that $G$ contains a soluble subgroup $H$ of finite index $[G : H] = m$. Since $H$ is soluble, $H$ satisfies a group identity $w(x_1, x_2, \ldots, x_n) = 1$. Then, $w(c_i^m, c_i^m, \ldots, c_i^m) = 1$ for any $c_i \in G$, so a group identity $w(x_1^m, x_2^m, \ldots, x_n^m) = 1$ holds for $G$.

Finally, the equivalence (5) $\iff$ (7) follows from [27, Theorem 1]. The proof of a theorem is now complete.

In [23], Lichtman showed that for a normal subgroup $G$ of $D^*$, if there exists a non-Abelian nilpotent-by-finite subgroup in $G$, then $G$ contains a noncyclic free subgroup. Recently, Gonçalves and Passman gave another proof and an explicit construction of noncyclic free subgroups (see [13]). In the following theorem, we consider the case where $D$ is algebraic over its center and generalize this result to almost subnormal subgroups.

**Theorem 4.7.** Let $D$ be a division ring algebraic over its center $F$ and assume that $G$ is an almost subnormal subgroup of $D^*$. If $G$ contains a non-Abelian nilpotent-by-finite subgroup, then $G$ contains a noncyclic free subgroup.

Proof. Let $N$ be a non-Abelian nilpotent-by-finite subgroup of $G$. Then there exists a nilpotent normal subgroup $A$ of $N$ such that $[N : A] = m$.

**Case 1.** $A$ is non-Abelian:

Since $A$ is nilpotent, there exist $x, y \in A$ such that

\[1 \neq z = y^{-1}x^{-1}yx, \quad zx = xz, \quad zy = yz.\]

Let $D_1$ be a division subring of $D$ generated by $x, y$. By [23, Lemma 1], $D_1$ is centrally finite. Using the same argument as in the proof of Theorem 4.2, we can conclude that $M = G \cap D_1^*$ is a non-Abelian almost subnormal subgroup in $D_1^*$. By Theorem 4.2, $M$ contains a noncyclic free subgroup.

**Case 2.** $A$ is Abelian:

Let $D_1$ be a division subring of $D$ generated by $F$ and $N$, and let $F_1 = Z(D_1)$. Since $[N : A] = m$, $D_1$ is finite dimensional over the subfield $F(A)$. By [6], $D_1$ is centrally finite. Using the same argument as in the proof of Theorem 4.2, we conclude that $M = G \cap D_1^*$ is a non-Abelian almost subnormal subgroup in $D_1^*$. By Theorem 4.2, $M$ contains a noncyclic free subgroup.

\[\S 5. \text{Finitely generated almost subnormal subgroups of } \text{GL}_n(D)\]

In this section, we investigate finitely generated subgroups of $\text{GL}_n(D)$ with some additional conditions. Recall that if $D = F$ is a field then the Tits Alternative asserts that every finitely generated subgroup of $\text{GL}_n(F)$ either contains a noncyclic free subgroup or is soluble-by-finite. Now, assume that $G$ is a finitely generated subgroup of $\text{GL}_n(D)$, where $D$ is a noncommutative division ring. It was shown in [25] that if $D$ is centrally finite and $G$ is a subnormal subgroup in $\text{GL}_n(D)$, then $G$ is central. In the case where $n = 1$, it was proved in [13, Theorem 2.5] that if $D$ is of type 2, then there are no finitely generated subgroups of $D^*$ containing the center $F^*$. Recall that a division ring $D$ is said to be of type 2 if the division subring $F(x, y)$ of $D$ generated over its center $F$ by any two elements $x, y \in D$ is a finite-dimensional vector space over $F$. The aim of this section is to carry over these results to almost subnormal subgroups of $\text{GL}_n(D)$, where $D$ is a weakly locally finite division ring. Recall that Theorem 3.3 implies that any almost
subnormal subgroup of \( GL_n(D) \) is normal if \( n \geq 2 \), but we shall continue to use “almost subnormal” instead of “normal” to compare the results with the corresponding ones in the case \( n = 1 \).

The following result, which is an easy consequence of Theorem 2, gives the characterization of finite almost subnormal subgroups in division rings.

**Lemma 5.1.** Let \( D \) be a division ring with center \( F \) and assume that \( G \) is an almost subnormal subgroup of \( D^* \). If \( G \) is finite, then \( G \) is central.

**Proof.** Let \( D_1 = F(G) \) be the division subring of \( D \) generated by \( G \) over \( F \). By Lemma 1.1, \( G \) is almost subnormal in \( D_1 \). If \( F \) is finite, then \( D_1 \) is a field. In particular, \( G \) is Abelian, so by Theorem 2, \( G \) is central. If \( F \) is infinite, then so is the center of \( D_1 \). Hence, in view of Theorem 2, \( G \) is central. \( \square \)

**Remark 5.2.** Let \( H = \langle a_1, a_2, \ldots, a_m \rangle \) be a finitely generated subgroup of \( GL_n(D) \), where \( D \) is a division ring. Denote by \( S \) the set of all entries of all matrices \( a_i, a_i^{-1} \), and by \( R \) the subring of \( D \) generated by \( S \). Then, \( H \) is contained in \( GL_n(R) \). In particular, \( H \) is contained in \( GL_n(D_1) \), where \( D_1 \) is the division subring of \( D \) generated by \( S \). This fact will be used frequently in this section.

Let \( G \) be a subgroup of \( GL_n(F) \), where \( F \) is a field. Suppose that \( a \in GL_n(F) \). It is easy to see that \( a + xI_n \) is noninvertible for finitely many elements \( x \in F \) if \( a + xI_n \) is noninvertible, then the determinant \( |a + xI_n| \) of \( a + xI_n \), a polynomial of degree \( n \) in \( x \), is 0. By the Vandermonde argument [29, Propositions 2.3.26 and 2.3.27], there are finitely many elements \( x \in F \) such that \( |a + xI_n| = 0 \).

Now assume that \( H \) is an almost subnormal subgroup of \( G \), and

\[
H = H_r \leq H_{r-1} \leq \cdots \leq H_1 = G
\]

is an almost normal series of subgroups of \( G \). For any \( a, b \in H \) and \( x \in F \) such that \( b + xI_n \) is invertible, put \( c_1(a, b, x) := (b + xI_n)a(b + xI_n)^{-1} \), and for \( 1 < i \leq r \), we define \( c_i \) inductively as follows: if \( H_i \) is normal in \( H_{i-1} \), then \( c_i(a, b, x) := c_{i-1}bc_{i-1}^{-1} \), otherwise \( c_i(a, b, x) := \ell_i \), where \( r_i \) is the index of \( H_i \) in \( H_{i-1} \).

**Lemma 5.3.** Let \( c_i(a, b, x) \) be as above. Then, \( c_i = (b + xI_n)w_i(a, b)(b + xI_n)^{-1} \), where \( w_i(a, b) \) is a reduced word in \( a, b, a^{-1}, b^{-1} \) which begins and ends by \( a \) or \( a^{-1} \).

**Proof.** We prove the lemma by induction on \( 1 \leq i \leq r \). If \( i = 1 \), then \( w_1(a, b) = (b + xI_n)^{-1}c_1(b + xI_n) = a \). Assume that \( c_i = (b + xI_n)w_i(a, b)(b + xI_n)^{-1} \), where a reduced word \( w_i(a, b) \) begins and ends by \( a \) or \( a^{-1} \) for any \( i \leq 1 \). We must show that \( c_{i+1} \) has a same property, that is, \( c_{i+1}(a, b, x) = (b + xI_n)w_{i+1}(a, b)(b + xI_n)^{-1} \) with a reduced word \( w_{i+1}(a, b) \) ending by \( a \) or \( a^{-1} \). Indeed, there are two cases to examine.

**Case 1.** If \( H_{i+1} \) is normal in \( H_i \), then

\[
c_{i+1}(a, b, x) = c_{i}c_{i-1}^{-1}
\]

\[
= ((b + xI_n)w_i(a, b)(b + xI_n)^{-1})b((b + xI_n)w_i(a, b)(b + xI_n)^{-1})^{-1}
\]

\[
= (b + xI_n)w_{i+1}(a, b)(b + xI_n)^{-1}
\]

\[
= (b + xI_n)w_{i+1}(a, b)(b + xI_n)^{-1}, \text{ where } w_{i+1} = w_{i}(a, b)bw_{i}(a, b)^{-1}.
\]
Case 2. If $H_{i+1}$ has finite index $r_{i+1}$ in $H_i$ and $\ell_i+1 = r_{i+1}$!, then
\[
\begin{align*}
c_{i+1}(a, b, x) &= c_i(a, b, x)^{\ell_i+1} \\
&= ((b + xI_n)w(a, b)(b + xI_n)^{-1})^{\ell_i+1} \\
&= (b + xI_n)w_i(a, b)^{\ell_i+1}(b + xI_n)^{-1} \\
&= (b + xI_n)w_{i+1}(a, b)(b + xI_n)^{-1}, \ \text{where } w_{i+1} = w_i(a, b)^{\ell_i+1}.
\end{align*}
\]

The proof of the lemma is complete. □

Notice that in the proof of main results in [25], the authors considered two cases $n = 1$ and $n > 1$ separately with two difference arguments. By modifying the proof of the case where $n = 1$ in [25], we will prove our main result in the general case as described below.

**Theorem 5.4.** Let $D$ be a weakly locally finite division ring, and assume that $G$ is an infinite almost subnormal subgroup of $\text{GL}_n(D)$. If $G$ is finitely generated, then $G$ is central.

**Proof.** Suppose that, to the contrary, $G$ is noncentral finitely generated subgroup of $\text{GL}_n(D)$. Since $D$ is weakly locally finite, by Remark 5.2 without loss of generality, we may assume that $D$ is centrally finite with $[D : F] = t < \infty$, where $F$ is the center of $D$.

We first claim that $G$ contains a noncyclic free subgroup by showing that $D$ and $G$ satisfy the requirements of Theorem 4.3. Indeed, if $\text{char}(F) = p > 0$ and $D = F$ is a field algebraic over its prime subfield, then every element of $\text{GL}_n(F)$ is torsion. By Schur’s Theorem [21, Theorem (9.9), p. 154], $G$ is finite, which contradicts the assumption. Therefore, the claim is proved.

Clearly the group $\text{GL}_n(D)$ may be viewed as a subgroup of $\text{GL}_m(F)$, so $G$ is also a subgroup of $\text{GL}_m(F)$. By Remark 5.2, $G$ is a subgroup of $\text{GL}_m(P(S))$, where $P$ is the prime subfield of $F$ and $S$ is a finite subset of $F$. Since $G$ is infinite, so is $P(S)$. We consider two cases.

**Case 1.** $\text{char}(P) > 0$.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If all elements of $S$ are algebraic over $P$, then $P(S)$ is finite, a contradiction. Let $i_0$ be the largest index such that $\alpha := \alpha_{i_0}$ is not algebraic over $P$. If $K = P(\alpha_1, \alpha_2, \ldots, \alpha_{i_0-1})$ and $L = K(\alpha)$, then $[P(S) : L] = s < \infty$. Therefore, $G$ may be viewed as a subgroup of $\text{GL}_{nts}(L)$. Recall that $\alpha$ is not algebraic over $K$, so $L$ can be regarded as the field of fractions of $K[\alpha]$. Again by Remark 5.2 we can find a finite subset
\[T = \left\{ \frac{u_1(\alpha)}{v_1(\alpha)}, \frac{u_2(\alpha)}{v_2(\alpha)}, \ldots, \frac{u_m(\alpha)}{v_m(\alpha)} \right\}\]
of $L$ such that $\text{GL}_{nts}(K[\alpha][T])$ contains $G$. Note that $v_1(\alpha), u_1(\alpha), \ldots, v_m(\alpha), u_m(\alpha)$ are elements of $K[\alpha]$ such that $v_i(\alpha)$ and $u_i(\alpha)$ are coprime for any $i$. Let $a, b \in G$ be two elements such that $(a, b)$ is a non-Abelian free subgroup of $G$ and let $G = G_r \subseteq G_{r-1} \subseteq \cdots \subseteq G_1 = \text{GL}_n(D)$ be an almost normal series of $G$ in $\text{GL}_{nts}(K[\alpha][T])$. Observe that $b + xI_{nts}$ is noninvertible for finitely many elements $x \in K[\alpha][T]$. Now, take an $x \in K[\alpha][T]$ such that $b + xI_{nts}$ is invertible.

By Lemma 5.3, $c_r(a, b, x) = (b + xI_{nts})w_r(a, b)(b + xI_{nts})^{-1} \in G_r = G$. We claim that all the entries of $c_r(a, b, x)$ do not depend on $x$. Indeed, assume that there exists an entry with indices $(i, j)$ of $c_r(a, b, x)$ depending on $x$. Without loss of generality, we may assume that $(i, j) = (1, 1)$. Notice that the determinant of $b + xI_{nts}$ is a polynomial $f(x)$ in $x$ of degree $q = nts$, so the entry in question has the form
\[
g(x) = \frac{b_qx^q + b_{q-1}x^{q-1} + \cdots + b_0}{x^q + c_{q-1}x^{q-1} + \cdots + c_0} \in K[\alpha][T].
\]
Assume that \( b_q = \frac{u_{m+1}(a)}{v_{m+1}(a)} \). Then,

\[
\frac{g(x)}{f(x)} - b_q \in K[\alpha]\left[T \cup \left\{ \frac{u_{m+1}(a)}{v_{m+1}(a)} \right\} \right],
\]

and clearly, the degree of the numerator of \( \frac{g(x)}{f(x)} - b_q \) is less than \( q \). Hence, without loss of generality, we may assume that \( b_q = 0 \), that is,

\[
g(x) = \frac{b_{q-1}x^{q-1} + \cdots + b_0}{x^q + c_{q-1}x^{q-1} + \cdots + c_0} \in K[\alpha][T].
\]

Observe that \( c_0 \) is the determinant of \( b \), which is an invertible element in \( \text{GL}_{nts}(K[\alpha][T]) \), so \( c_0 \neq 0 \). Put

\[
g_1(x) = \frac{c_0^{-1}g(x)}{c_0^{-1}f(x)},
\]

that is, \( g_1(x) = c_0^{-1}b_{q-1}x^{q-1} + \cdots + c_0^{-1}b_0 \) and \( f_1(x) = c_0^{-1}x^q + c_0^{-1}c_{q-1}x^{q-1} + \cdots + 1 \). Let \( w_1(\alpha), \ldots, w_l(\alpha) \) be all prime factors of \( v_1(\alpha), u_1(\alpha), \ldots, v_m(\alpha), u_m(\alpha) \) and put

\[
x(\alpha) = (w_1(\alpha)w_2(\alpha) \ldots w_l(\alpha))^p.
\]

Then, since the degree of \( g_1(x) \) is less than that of \( f_1(x) \) when \( p \) is sufficiently large, the degree of the denominator \( f_1(x(\alpha)) \) in \( \alpha \) is greater than that of the numerator which implies that there exists \( i \) such that \( f_1(x(\alpha)) \) is a multiple of \( w_i(\alpha) \). Hence, 1 is also a multiple of \( w_i(\alpha) \), which is a contradiction. Thus, the claim is proved. Therefore, \( c_r(a, b, x) \) does not depend on \( x \). Now, one has \( c_r(a, b, 0) = c_r(a, b, y) \) for some \( y \in K[\alpha] \setminus \{0\} \) such that \( b + yI_q \) is invertible. Hence, \( bw_r(a, b)b^{-1} = (b + yI_q)w_r(a, b)(b + yI_q) \), equivalently, \( bw_r(a, b)b^{-1} = w_r(a, b) \), which contradicts the fact that \( a, b \) are generators of a noncyclic free group.

**Case 2.** \( \text{char}(F) = 0 \).

If \( P(S) \) is not algebraic over \( \mathbb{Q} \), then by the same procedure as in the first part of **Case 1**, we conclude that the field \( P(S) \) contains a subfield \( L_1 = K_1(\beta) \) such that \( [P(S) : L_1] = s_1 < \infty \), where \( K_1 \) is a subfield of \( P(S) \) and \( \beta \) is not algebraic over \( K_1 \).

Now, again by the same procedure as in **Case 1** with replacing \( K(\alpha) \) by \( K_1(\beta) \), one arrives at a contradiction.

Therefore, the proof of the theorem is complete. \( \square \)

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