Universal stability of Banach spaces for $\varepsilon$-isometries

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Abstract

Let $X$, $Y$ be two real Banach spaces and $\varepsilon > 0$. A standard $\varepsilon$-isometry $f : X \to Y$ is said to be $(\alpha, \gamma)$-stable (with respect to $T : L(f) \equiv \text{span} f(X) \to X$ for some $\alpha, \gamma > 0$) if $T$ is a linear operator with $\|T\| \leq \alpha$ so that $Tf - Id$ is uniformly bounded by $\gamma \varepsilon$ on $X$. The pair $(X,Y)$ is said to be stable if every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable for some $\alpha, \gamma > 0$. $X(Y)$ is said to be universally left (right)-stable, if $(X,Y)$ is always stable for every $Y(X)$. In this paper, we show that universal right-stability spaces are just Hilbert spaces; every injective space is universally left-stable; a Banach space $X$ isomorphic to a subspace of $\ell_\infty$ is universally left-stable if and only if it is isomorphic to $\ell_\infty$; and that a separable space $X$ satisfies the condition that $(X,Y)$ is left-stable for every separable $Y$ if and only if it is isomorphic to $c_0$.

1 Introduction

The study of properties of isometries and its generalizations between Banach spaces has continued for 80 years since Mazur and Ulam 1932’s celebrated results [18]: Every surjective isometry between two Banach spaces $X$ and $Y$ is necessarily affine. While a simple example $f : \mathbb{R} \to \ell_\infty^2$ defined by $f(t) = (t, \sin t)$ shows that it is not always possible if the mapping is not surjective. In 1968, Figiel [10] showed the following remarkable result: For every standard isometry $f : X \to Y$ there is a linear operator $T : L(f) \to X$ with $\|T\| = 1$ so that $Tf = Id$ on $X$, where $L(f)$ is the closure of $\text{span} f(X)$ in $Y$ (see, also, [3] and [8]). In 2003, Godefroy and Kalton [12] studied the relationship between isometries and linear isometries and resolved a long-standing problem: Whether existence of an isometry $f : X \to Y$ implies the existence of a linear isometry $U : X \to Y$?

Definition 1.1. Let $X, Y$ be two Banach spaces, $\varepsilon \geq 0$, and let $f : X \to Y$ be a mapping.

(1) $f$ is said to be an $\varepsilon$-isometry if

\begin{equation}
\|f(x) - f(y)\| - \|x - y\| \leq \varepsilon \quad \text{for all } x, y \in X.
\end{equation}

In particular, a 0-isometry $f$ is simply called an isometry.

(2) We say an $\varepsilon$-isometry $f$ is standard if $f(0) = 0$.

(3) A standard $\varepsilon$-isometry is $(\alpha, \gamma)$-stable if there exist $\alpha, \gamma > 0$ and a bounded linear operator $T : L(f) \to X$ with $\|T\| \leq \alpha$ such that

\begin{equation}
\|Tf(x) - x\| \leq \gamma \varepsilon, \quad \text{for all } x \in X.
\end{equation}
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In this case, we also simply say $f$ is stable, if no confusion arises.

(4) A pair $(X, Y)$ of Banach spaces $X$ and $Y$ is said to be stable if every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable for some $\alpha, \gamma > 0$.

(5) A pair $(X, Y)$ of Banach spaces $X$ and $Y$ is called $(\alpha, \gamma)$-stable for some $\alpha, \gamma > 0$ if every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable.

In 1945, Hyers and Ulam proposed the following question \[15\] (see, also \[19\]): Whether for every pair of Banach spaces $(X, Y)$ there is $\gamma > 0$ such that for every standard surjective $\varepsilon$-isometry $f : X \to Y$ there exists a surjective linear isometry $U : X \to Y$ so that

$$\|f(x) - Ux\| \leq \gamma \varepsilon, \quad \text{for all } x \in X. \quad (1.3)$$

After many years efforts of a number of mathematicians (see, for instance, \[11\], \[13\], \[15\], and \[19\]), Omladič and Šemrl \[19\] finally achieved the sharp estimate $\gamma = 2$ in (1.3).

The study of non-surjective $\varepsilon$-isometries has also brought to mathematicians’ attention (see, for instance, \[2\], \[4\], \[5\], \[7\], \[19\], \[20\], \[21\] and \[23\]). Qian \[20\] first proposed the following problem in 1995.

**Problem 1.2.** Whether for every pair $(X, Y)$ of Banach spaces $X$ and $Y$ there exists $\gamma > 0$ such that every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable for some $\alpha > 0$.

Then he showed that the answer is affirmative if both $X$ and $Y$ are $L_p$ spaces. Šemrl and Väisälä \[21\] further presented a sharp estimate of (1.2) with $\gamma = 2$ if both $X$ and $Y$ are $L_p$ spaces for $1 < p < \infty$. However, Qian \[20\] presented a counterexample showing that if a separable Banach space $Y$ contains a uncomplemented closed subspace $X$ then for every $\varepsilon > 0$ there is a standard $\varepsilon$-isometry $f : X \to Y$ which is not stable. Cheng, Dong and Zhang \[4\] showed the following weak stability version.

**Theorem 1.3** (Cheng-Dong-Zhang). Let $X$ and $Y$ be Banach spaces, and let $f : X \to Y$ be a standard $\varepsilon$-isometry for some $\varepsilon \geq 0$. Then for every $x^* \in X^*$, there exists $\phi \in Y^*$ with $\|\phi\| = \|x^*\| \equiv r$ such that

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r, \quad \text{for all } x \in X. \quad (1.4)$$

For study of the stability of $\varepsilon$-isometries of Banach spaces, the following two questions are very natural.

**Problem 1.4.** Is there a characterization for the class of Banach spaces $\mathcal{X}$ satisfying given any $X \in \mathcal{X}$ and Banach space $Y$, the pair $(X, Y)$ is $(\alpha, \gamma)$-, resp.) stable?
Every space $X$ in this class is said to be a universal $((\alpha, \gamma), \text{resp.})$ left-stability space.

**Problem 1.5.** *Can we characterize the class of Banach spaces $\mathcal{Y}$, such that given any $Y \in \mathcal{Y}$ and Banach space $X$, the pair $(X, Y)$ is $((\alpha, \gamma), \text{resp.})$ stable?*

Every space $Y$ in this class is called a universal $((\alpha, \gamma), \text{resp.})$ right-stability space.

In this paper, we study universal stability and universal right-stability of Banach spaces. As a result, with the aim of Qian’s counterexample and Theorem 1.3 incorporating of Lindenstrauss-Tzafriri’s characterization of Hilbert spaces [17], we show that universal stability spaces are spaces of finite dimensions; and up to an isomorphism, a universal right-stability space is just a Hilbert space. By using Theorem 1.3 we then prove that every injective space is universally left-stable; and a Banach space $X$ which is isomorphic to a subspace of $\ell_\infty$ is universally left-stable if and only if it is isomorphic to $\ell_\infty$. Finally, applying Zippin’s theorem [25] we verify that a separable space $X$ satisfies that $(X, Y)$ is stable for every separable $Y$ if and only if it is isomorphic to $c_0$.

All symbols and notations in this paper are standard. We use $X$ to denote a real Banach space and $X^*$ its dual. $B_X$ and $S_X$ denote the closed unit ball and the unit sphere of $X$, respectively. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$, $\overline{A}$ stands for the closure of $A$, and $\text{card}(A)$, the cardinality of $A$.

## 2 Universal (right-) stability spaces for $\varepsilon$-isometries

In this section, we search for some properties of the class of universal left (right)-stability spaces for $\varepsilon$-isometries.

Recall that a Banach space $X$ ($Y$) is universally left (right)-stable if it satisfies that for every Banach space $Y$ ($X$) and for every standard $\varepsilon$-isometry $f : X \to Y$, there exist $\alpha, \gamma > 0$ and a bounded operator $T : L(f) \to X$ with $\|T\| \leq \alpha$ so that

\[ \|Tf(x) - x\| \leq \gamma \varepsilon, \text{ for all } x \in X. \]

(2.1)

A universal stability space is a Banach space which is both universally left and right stable. As a result, we show inequality (2.1) holds for every Banach
space $X$ if and only if $Y$ is, up to linear isomorphism, a Hilbert space; and universal stability spaces are just finite dimensional spaces.

The following lemma follows from Qian’s counterexample\[20\].

**Lemma 2.1.** Let $X$ be a closed subspace of a Banach space $Y$. If $\text{card}(X) = \text{card}(Y)$, then for every $\varepsilon > 0$ there is a standard $\varepsilon$-isometry $f : X \rightarrow Y$ such that

1. $L(f) \equiv \text{span}_f(X) = Y$;
2. $X$ is complemented whenever $f$ is stable.

**Theorem 2.2.** Let $Y$ be a Banach space. Then the following statements are equivalent.

i) $Y$ is universally right-stable;
ii) $Y$ is isomorphic to a Hilbert space;
iii) $Y$ is universally $(\alpha, 4)$-right-stable for some $\alpha > 0$.

**Proof.** i) $\implies$ ii). By definition of universal right-stability, every closed subspace of $Y$ is again universally right-stable. Fix any closed separable subspace $Z$ of $Y$. By Lemma 2.1, universal right-stability of $Z$ entails that every closed subspace of $Z$ is complemented in $Z$. According to Lindenstrauss-Tzafriri’s theorem \[17\]: ”a Banach space satisfying that every closed subspace is complemented is isomorphic to a Hilbert space”, $Z$ is isomorphic to a (separable) Hilbert space. Hence, $Y$ itself is isomorphic to a Hilbert space.

ii) $\implies$ iii). Suppose that $Y$ is isomorphic to a Hilbert space $H$. Let $\alpha = \text{dist}(Y, H)$, the Banach-Mazur distance between $Y$ and $H$. Then every closed subspace of $Y$ is $\alpha$-complemented in $Y$. Given $\varepsilon > 0$ and any standard $\varepsilon$-isometry $f : X \rightarrow Y$, according to Theorem 4.8 of \[4\], inequality (2.1) holds for some $T : L(f) \rightarrow X$ with $\|T\| \leq \alpha$ and with $\gamma = 4$, i.e., $Y$ is universally $(\alpha, 4)$-right stable.

iii) $\implies$ i). It is trivial. \qed

**Theorem 2.3.** A normed space $X$ is universally-stable if and only if it is finite dimensional.

**Proof.** Sufficiency. Since every finite dimensional normed space is isomorphic to an Euclidean space, Theorem 2.2 entails that it is universally right-stable. While Theorem 3.4 of \[4\] says that $n$ dimensional spaces are universally left-stable with the parameter $\gamma = 4n$.

Necessity. Suppose, to the contrary, that $X$ is infinite dimensional. Since $X$ is also universally right-stable, according to Theorem 2.2 we have just
proven, it is isomorphic to a Hilbert space. Since every closed subspace of a universally right-stable space is again universally right-stable, we can assume that $X$ is separable. Thus, $X$ is isometric to a subspace of $\ell_\infty$. Since $\ell_\infty$ is prime \cite{17} (i.e. every complemented infinite dimensional subspace is isomorphic to it), $X$ is uncomplemented in $\ell_\infty$. Note $\text{card}(X) = \text{card}\ell_\infty$. By Lemma \ref{23} there is an unstable standard $\varepsilon$-isometry $f : X \to \ell_\infty$ for every $\varepsilon > 0$, which is a contradiction to universal stability of $X$.

\section{Universal left-stability spaces}

In this section, we consider properties of universal left-stability spaces. We shall show that (1) an injective Banach space is universally left-stable; (2) a Banach space isomorphic to a subspace of $\ell_\infty$ is universally left-stable if and only if it is isomorphic to $\ell_\infty$ and (3) for a separable Banach space $X$, $(X,Y)$ is stable for every separable Banach space $Y$ if and only if $X$ is a separably injective Banach space.

A Banach space $X$ is said to be $\lambda$-injective (or, simply, injective) if it has the following extension property: Every bounded linear operator $T$ from a closed subspace of a Banach space into $X$ can be extended to be a bounded operator on the whole space with its norm at most $\lambda \|T\|$ (see, for instance, \cite{II}). Goondner \cite{14} introduced a family of Banach spaces coinciding with the family of injective spaces: for any $\lambda \geq 1$, a Banach space $X$ is a $P_\lambda$-space if, whenever $X$ is isometrically embedded in another Banach space, there is a projection onto the image of $X$ with norm not larger than $\lambda$. The following result is due to Day \cite{6} (see, also, Wolfe \cite{24}, Fabian et al. \cite{9}, p. 242).

\begin{proposition}
A Banach space $X$ is $\lambda$-injective if and only if it is a $P_\lambda$-space.
\end{proposition}

\begin{remark}
For any set $\Gamma$, that $\ell_\infty(\Gamma)$ is 1-injective follows from the Hahn-Banach theorem.
\end{remark}

\begin{theorem}
Every $\lambda$-injective space is universally $(\lambda, 4\lambda)$-left-stable.
\end{theorem}

\begin{proof}
Let $X$ be a $\lambda$-injective Banach space. We can assume that $X$ is a closed complemented subspace of $\ell_\infty(\Gamma)$; otherwise, we can identify $X$ for its canonical embedding $J(X)$ as a subspace of $\ell_\infty(\Gamma)$, where $\Gamma$ denotes the closed ball $B_{X^*}$ of $X^*$. Hence, it is $\lambda$-complemented in $\ell_\infty(\Gamma)$. Let $P : \ell_\infty(\Gamma) \to X$ be a projection such that $\|P\| \leq \lambda$. Given any $\beta \in \Gamma$, let $\delta_\beta \in \ell_\infty(\Gamma)^*$ be defined for $x = (x(\gamma))_{\gamma \in \Gamma} \in \ell_\infty(\Gamma)$ by $\delta_\beta(x) = x(\beta)$. Assume
that $f : X \to Y$ is a standard $\varepsilon$-isometry. For every $x^* \in X^*$, by Theorem
[1,3] there is $\phi \in Y^*$ with $\|\phi\| = \|x^*\|$ such that
\begin{equation}
\|\langle \phi, f(x) \rangle - \langle x^*, x \rangle\| \leq 4\varepsilon\|x^*\|, \text{ for all } x \in X.
\end{equation}
In particular, letting $x^* = \delta_\gamma$ in (3.1) for every fixed $\gamma \in \Gamma$, we obtain a
linear functional $\phi_\gamma \in Y^*$ satisfying (3.1) with $\|\phi_\gamma\| = \|\delta_\gamma\|_X \leq 1$. Therefore,
$(\phi_\gamma(y))_{\gamma \in \Gamma} \in \ell_\infty(\Gamma)$ for every $y \in Y$.

Let $T(y) = P(\phi_\gamma(y))_{\gamma \in \Gamma}$, for all $y \in Y$, and note $P|_X = I_X$, the identity
from $X$ to itself. Then $\|T\| \leq \|P\| \leq \lambda$ and for all $x \in X$,
\begin{align*}
\|Tf(x) - x\| &= \|P(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - (\delta_\gamma(x))_{\gamma \in \Gamma}\| \\
&= \|P(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - P((\delta_\gamma(x))_{\gamma \in \Gamma})\| \\
&\leq \|P\| \cdot \|(\phi_\gamma(f(x)))_{\gamma \in \Gamma} - (\delta_\gamma(x))_{\gamma \in \Gamma}\|_\infty \leq 4\lambda\varepsilon.
\end{align*}

\[\Box\]

**Theorem 3.4.** Let $X$ be a Banach space $X$ isomorphic to an infinite di-
menional subspace of $\ell_\infty$. Then the following statements are equivalent.

i) $X$ is universally left-stable;

ii) $X$ is isomorphic to $\ell_\infty$.

iii) $X$ is universally $(\lambda,4\lambda)$-left stable, where $\lambda = \text{dist}(X,\ell_\infty)$.

**Proof.** i) $\implies$ ii). Since dim $X = \infty$ and since it is isomorphic to a subspace
of $\ell_\infty$, we have
\begin{equation}
\text{card}(X) \geq \aleph_0 = \aleph_0^\mathbb{N} = \text{card}(\mathbb{R}^\mathbb{N}) = \text{card}(\ell_\infty) \geq \text{card}(X).
\end{equation}
Assume that $X$ is universally left-stable. We can put an equivalent norm
$\|\cdot\|$ on $\ell_\infty$ such that $X$ is isometric to a closed subspace of $(\ell_\infty,||\cdot||)$.\n
Indeed, Let $T : X \to \ell_\infty$ be a linear embedding and let $|\cdot|$ on $Z \equiv T(X)$
be defined by $|z| = \|x\|$ for all $z = Tx \in Z$. Then, we choose a sufficiently
large $\lambda > 0$ and define $||\cdot||$ on $\ell_\infty$ by $||u|| = \inf\{|v| + \lambda\|u - v\| : v \in Z\}$.
Clearly, the norm $||\cdot||$ has the property we desired. Applying Lemma
[2,1] we observe that $X$ is complemented in $(\ell_\infty,||\cdot||)$, hence, in $\ell_\infty$. By
Lindenstrauss' theorem [16], $X$ is isomorphic to $\ell_\infty$.

ii) $\implies$ iii). Suppose that $X$ is isomorphic to $\ell_\infty$. Since $\ell_\infty$ is $1$-injective
( Remark [3,2]), $X$ is necessarily $\lambda$-injective ($\lambda = \text{dist}(X,\ell_\infty)$). By Theorem
3.3, $X$ is universally $(\lambda,4\lambda)$-left stable.

iii) $\implies$ i). It is trivial. \[\Box\]
A separable Banach space $X$ is said to be separably injective if it has the following extension property: Every bounded linear operator from a closed subspace of a separable Banach space into $X$ can be extended to be a bounded operator on the whole space. In 1941, Sobczyk [22] showed that $c_0$ is separably injective, and Zippin ([25], 1977) further proved that $c_0$ is, up to isomorphism, the only separable separably injective space.

With the aim of Zippin’s theorem, we can prove the following theorem, which says that $c_0$ is (up to isomorphism) the only space satisfying inequality (2.1) for every separable $Y$.

**Theorem 3.5.** Let $X$ be a separable Banach space. Then the following statements are equivalent.

i) $(X,Y)$ is stable for every separable Banach space $Y$;

ii) $X$ is isomorphic to $c_0$;

iii) $(X,Y)$ is $(2\alpha,8\alpha)$-stable for every separable Banach space $Y$, where $\alpha = \|T\|\|T^{-1}\|$ for any isomorphism $T : X \to c_0$.

**Proof.** i) $\implies$ ii). Suppose that $X$ is not isomorphic to $c_0$. Then by Zippin’s theorem, $X$ is not separably injective. Therefore, there exists a separable Banach space $Y$, which contains $X$ as an uncomplemented subspace. Clearly, $\text{card}(X) = \text{card}(Y)$. By Lemma 2.1 again, for every $\varepsilon > 0$, there is a standard $\varepsilon$-isometry $f : X \to Y$ which is not stable.

ii) $\implies$ iii). Let $X$ be a Banach space isomorphic to $c_0$ and $T : X \to c_0$ be an isomorphism. Assume that $(e_n)_{n=1}^{\infty}$ is the canonical basis of $c_0$ with the standard biorthogonal functionals $(e^*_n)_{n=1}^{\infty} \subset \ell_1$. Let $(x_n) \subset X$ satisfy $Tx_n = e_n$ for all $n \in \mathbb{N}$, and let $T^* : \ell_1 \to X^*$ be the conjugate operator of $T$. Then

$$Tx = \sum (T^*e^*_n)(x)e_n \text{ and } x = \sum (T^*e^*_n)(x)T^{-1}e_n, \text{ for all } x \in X.$$  

Let $\alpha = \|T\|\|T^{-1}\|$, $x^*_n = T^*e^*_n \in \|T\|B_{X^*}$ for all $n \in \mathbb{N}$, and note $x_n = T^{-1}e_n \in X$. By Theorem 1.3 there exists $\phi_n \in \|T\|B_{Y^*}$ with $\|\phi_n\| = \|x_n^*\|$ such that

$$|\langle \phi_n, f(x) \rangle - \langle x^*_n, x \rangle| \leq 4\varepsilon \|T\|, \text{ for all } x \in X. \tag{3.3}$$

Since $e_n^* \to 0$ in the $w^*$-topology of $\ell_1 = c_0^*$, $x_n^* = T^*e^*_n \to 0$ in the $w^*$-topology of $X^*$. Let

$$K = \{\psi \in \|T\|B(Y^*) : |\langle \psi, f(x) \rangle| \leq 4\varepsilon \|T\|, \text{ for all } x \in X\}. \tag{3.4}$$
Then $K$ is a nonempty $w^*$-closed compact subset of $Y^*$. Since $Y$ is separable, $(\|T\|_{B_{Y^*}}, w^*)$ is metrizable. Let $\rho$ be a metric such that $(\|T\|_{B_{Y^*}}, \rho)$ is isomorphic to $(\|T\|_{B_{Y^*}}, w^*)$. Since $(\|T\|_{B_{Y^*}}, \rho)$ is a compact metric space and since $K$ is a compact subset of it, $(\phi_n) \subset K$ has at least one $\rho$-sequentially cluster point. Since $(x_n^*)$ is a $w^*$-null sequence in $X^*$, inequality (3.3) entails that any $\rho$-cluster point $\phi$ of $(\phi_n)$ is in $K$ and with $\|\phi\| \leq \|T^*\| = \|T\|$. This further implies that $\text{dist}_\rho(\phi_n, K) \to 0$. Consequently, there is a sequence $(\psi_n) \subset K$ such that $\text{dist}_\rho(\phi_n, \psi_n) \to 0$, or equivalently, $\phi_n - \psi_n \to 0$ in the $w^*$-topology of $Y^*$. Hence, for every $y \in Y$,

\[(3.5) \quad Uy \equiv \sum_{n=1}^{\infty} (\phi_n - \psi_n, y)e_n \in c_0\]

and with

\[(3.6) \quad \|Uy\| \leq (\sup_{n \in \mathbb{N}} \|\phi_n - \psi_n\|)\|y\| \leq 2\|T\|\|y\|,\]

that is, $\|U\| \leq 2\|T\|$.

Finally, let

\[(3.7) \quad S(y) = T^{-1}(Uy) = \sum_{n=1}^{\infty} (\phi_n - \psi_n, y)x_n \text{ for all } y \in Y.\]

Then

\[\|S\| = \|T^{-1}U\| \leq 2\|T\| \cdot \|T^{-1}\| = 2\alpha\]

and
\[\|Sf(x) - x\| = \|\sum_{n=1}^{\infty} \langle \phi_n - \psi_n, f(x) \rangle x_n - \sum_{n=1}^{\infty} \langle x_n^*, x \rangle x_n\|\]

\[= \lim_{n \to \infty} \|\sum_{i=1}^{n} \langle \phi_i - \psi_i, f(x) \rangle x_i - \sum_{i=1}^{n} \langle x_i^*, x \rangle x_i\|\]

\[= \lim_{n \to \infty} \|\sum_{i=1}^{n} (\langle \phi_i, f(x) \rangle - \langle x_i^*, x \rangle) x_i - \sum_{i=1}^{n} \langle \psi_i, f(x) \rangle x_i\|\]

\[\leq \limsup_{n \to \infty} \|\sum_{i=1}^{n} (\langle \phi_i, f(x) \rangle - \langle x_i^*, x \rangle) x_i\| + \limsup_{n \to \infty} \|\sum_{i=1}^{n} \langle \psi_i, f(x) \rangle x_i\|\]

\[= \limsup_{n \to \infty} \|T^{-1} \sum_{i=1}^{n} (\langle \phi_i, f(x) \rangle - \langle x_i^*, x \rangle) e_i\| + \limsup_{n \to \infty} \|T^{-1} \sum_{i=1}^{n} \langle \psi_i, f(x) \rangle e_i\|\]

\[\leq \|T^{-1}\| \cdot \limsup_{n \to \infty} (\|\sum_{i=1}^{n} (\langle \phi_i, f(x) \rangle - \langle x_i^*, x \rangle) e_i\| + \|\sum_{i=1}^{n} \langle \psi_i, f(x) \rangle e_i\|)\]

\[\leq \|T^{-1}\| (\sup_{n} |\langle \phi_i, f(x) \rangle - \langle x_i^*, x \rangle| + \sup_{n} |\langle \psi_i, f(x) \rangle|)\]

\[\leq 8\varepsilon \|T\| \cdot \|T^{-1}\| = 8\varepsilon \alpha.\]

Thus, our proof is complete. \(\Box\)

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