Classical and quantum solutions from string theory

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Abstract. Using the ontological interpretation of quantum mechanics in a particular sense, we obtain the classical behaviour of the scale factor and two scalar fields, derived from a string effective action for the FRW cosmological model. Besides, the Wheeler-DeWitt equation is solved exactly.

1. Introduction
There are several attempts to understand diverse aspects of time dependent models in the framework of string theory and branes [for a review see Ref. [1]]

The purpose of this paper is

• to present a mechanism to obtain classical solutions for time dependent models from the quantum regime, without having to solve directly the field equations of motion.
• Obtain the behaviour of the scale factor using the WKB semiclassical approximation evaluating the Bohmian trajectories in the ontological formulation of quantum mechanics [2], where the system follows a real trajectory given by \( \Pi_q = \frac{\partial \Phi}{\partial q} \), where the index \( q \) designates one of the degrees of freedom of the system, and \( \Phi \) is the phase of the wave function \( \Psi = W e^{i\Phi} \), where \( W \) and \( \Phi \) are real functions.

Recently a new class of objects were introduced in string theory named spacelike or S-brane [3], objects that exist only for a moment of time. The main motivation for the introduction of the S-branes was the conjectured dS/CFT correspondence [4]. The expectative is that, using the analogy with p branes, S-branes can also be found as explicit solutions of Einstein equations (coupled to scalar fields), the S-branes solutions are then time-dependent backgrounds of the theory. Therefore, the approach we shall follow for a cosmological model could also be analyzed for a S-brane solution (see [5] and references therein).

For the purpose of the talk, we consider the canonical quantization of a low-energy string effective action for a particular cosmological metric.

We begin by considering the compactification of the NS-NS sector of string effective action which contains the dilaton field, the graviton and a 2-form potential [6].

\[
S = \int d^{D+d}x \sqrt{g} g^{AB} \left[ R_{D+d} + \frac{1}{12} H_{M_1 M_2 M_3} H_{M_1' M_2' M_3'} g^{M_1 M_1'} g^{M_2 M_2'} g^{M_3 M_3'} \right].
\]
In our case we compactify on 6-torus $T^6$ with internal metric

$$h_{ab} = \text{diag} \left( e^{-2\sigma}, e^{-2\sigma}, e^{-2\sigma}, 1, 1, 1 \right).$$  \hspace{1cm} (2)

where $a, b = (4, 5, 6, 7, 8, 9)$. Choosing $H_{456} = F = \text{constant}$., and by compactifying the coupling parameter $\Phi$ becomes

$$\Phi = 2\phi - \frac{1}{2}\ln(\det h_{ab}),$$  \hspace{1cm} (3)

where $d = 6, D = 4$. In the internal space we have $h_{44} = h_{55} = h_{66} = e^{-2\sigma}$.

Choosing $F = 0$, and by straightforward calculations we arrive to an action in the string frame with two scalar fields (dilatonic and moduli field) \cite{7}.

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} e^{-2\phi - 3\sigma} \left[ R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 12g^{\mu\nu} \partial_\mu \phi \partial_\nu \sigma + 6g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right].$$  \hspace{1cm} (4)

Using the FRW metric is

$$ds^2 = -N^2(t)dt^2 + A^2(t) \left[ dr^2 + r^2d\Omega^2 \right],$$  \hspace{1cm} (5)

where $N$ is the lapse function, $A$ is the scale factor of the model, and $\kappa$ is the curvature index of the universe, we write the action (4) as

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{\bar{g}} \int dt \ e^{-2\phi - 3\sigma} \left\{ 6 \left[ -\left( \frac{\dot{A}^2 \ddot{A}}{N} \right) + \frac{A(\dot{A})^2}{N} - NkA \right] - \frac{A^3}{N} \left[ 4\dot{\phi}^2 + 12\dot{\phi}\dot{\sigma} + 6\sigma^2 \right] \right\}. \hspace{1cm} (6)$$

In the Einstein frame, we perform the conformal transformation into the metric components

$$\bar{g}_{\mu\nu} = e^{-(2\phi + 3\sigma)} g_{\mu\nu},$$  \hspace{1cm} (7)

rewritten the action (6) as

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{\bar{g}} \int dt \ \left\{ \left( \frac{\ddot{A}}{N} \right)^2 - 6k\bar{N}\ddot{A} \right\} - \frac{\bar{A}^3}{N} \left[ 10\dot{\phi}^2 + 30\dot{\phi}\dot{\sigma} + 39\dot{\sigma}^2 \right].$$  \hspace{1cm} (8)

2. The Hamiltonian

The Lagrangian density of our model is

$$L = \frac{6\dot{A}^2}{N} - 6k\bar{N}A - \frac{A^3}{N} \left( 10\dot{\phi}^2 + 30\dot{\phi}\dot{\sigma} + 39\dot{\sigma}^2 \right),$$  \hspace{1cm} (9)

The canonical momenta to coordinate fields are defined in the usual way

$$\Pi_\lambda = \frac{\partial L}{\partial \dot{A}} = \frac{12A\dot{A}}{N}, \quad \Pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{A^3}{N} (20\dot{\phi} + 30\dot{\sigma}), \quad \Pi_\sigma = \frac{\partial L}{\partial \dot{\sigma}} = -\frac{A^3}{N} (30\dot{\phi} + 39\dot{\sigma}),$$  \hspace{1cm} (10)

from which we can obtain the temporal derivative of $\phi$ and $\sigma$ fields

$$\dot{\phi} = \frac{N}{A^3} \left( \frac{13}{40} \Pi_\phi - \frac{1}{4} \Pi_\sigma \right), \quad \dot{\sigma} = \frac{N}{A^3} \left( -\frac{3}{12} \Pi_\phi + \Pi_\sigma \right).$$  \hspace{1cm} (11)
Using (10, 11), we can rewrite the Lagrangian density (9) in the canonical form

\[ L = \Pi_\lambda \dot{A} + \Pi_\phi \dot{\phi} + \Pi_\sigma \dot{\sigma} - N \left\{ \frac{\Pi_A^2}{24A} + 6kA + \frac{13}{80} \frac{\Pi_\phi^2}{A^3} + \frac{\Pi_\sigma^2}{12A^3} - \frac{1}{4} \frac{\Pi_\phi \Pi_\sigma}{A^3} \right\}. \]  

(12)

We can see that the Hamiltonian is given by

\[ H = \frac{\Pi_A^2}{24A} + 6kA + \frac{13}{80} \frac{\Pi_\phi^2}{A^3} + \frac{\Pi_\sigma^2}{12A^3} - \frac{1}{4} \frac{\Pi_\phi \Pi_\sigma}{A^3}. \]  

(13)

Following the standard procedure to get the quantum version of this Hamiltonian, we promote to operators the canonical momenta \( \Pi_\lambda \) that satisfy the commutation relation \([\Pi_\lambda, \lambda] = -i\hbar\) with the representation \( \Pi_\lambda = -i \frac{\partial}{\partial \lambda} \). We then apply the operator \( \hat{H} \) to the wave function \( \Psi \), i.e.

\[ \hat{H} \Psi = 0. \]

Notice that in principle the ambiguity order in equation (13) should be taken into account [8, 9, 10].

\[ \Pi_A^2 = -A^2 \frac{\partial^2}{\partial A^2} - B \frac{\partial}{\partial A} = -\frac{\partial^2}{\partial A^2} - \frac{B}{A} \frac{\partial}{\partial A}, \]  

(14)

where \( B \) is a real parameter that measure this ambiguity. For the other operators this problem does not arise

\[ \Pi_\phi^2 = -\frac{\partial^2}{\partial \phi^2}; \quad \Pi_\sigma^2 = -\frac{\partial^2}{\partial \sigma^2}. \]  

(15)

In this way (13) can be written as

\[ \frac{1}{24A} \left( -\frac{\partial^2}{\partial A^2} - \frac{B}{A} \frac{\partial}{\partial A} \right) \Psi + 6kA \Psi - \frac{13}{80} \frac{\Pi_\phi^2}{A^3} \Psi - \frac{1}{12A^3} \frac{\Pi_\sigma^2}{A^3} \Psi + \frac{1}{4A^3} \frac{\partial^2}{\partial \phi \partial \sigma} \Psi = 0, \]  

(16)

and multiplying (16) by \( 24A^3 \), we obtain the equation

\[ -A^2 \frac{\partial^2}{\partial A^2} \Psi - BA \frac{\partial}{\partial A} \Psi + 144kA^4 \Psi - \frac{39}{10} \frac{\partial^2}{\partial \phi^2} \Psi - 2 \frac{\partial^2}{\partial \sigma^2} \Psi + 6 \frac{\partial^2}{\partial \phi \partial \sigma} \Psi = 0. \]  

(17)

3. Quantum solution

Employing the separation of variables method to solve (17), \( \Psi(A, \phi, \sigma) = A(A)C(\phi, \sigma) \) and rearranging, we get

\[ \frac{1}{A} \left( -A^2 \frac{\partial^2}{\partial A^2} - BA \frac{\partial}{\partial A} + 144kA^4 A \right) + \frac{1}{C} \left( -\frac{39}{10} \frac{\partial^2}{\partial \phi^2} - 2 \frac{\partial^2}{\partial \sigma^2} + 6 \frac{\partial^2}{\partial \phi \partial \sigma} \right) = 0. \]  

(18)

This equation is equivalent to the set of partial differential equations

\[ A^2 \frac{\partial^2}{\partial A^2} + BA \frac{\partial}{\partial A} - \left( 144kA^4 \pm \nu^2 \right) A = 0, \]

\[ \frac{39}{10} \frac{\partial^2}{\partial \phi^2} + 2 \frac{\partial^2}{\partial \sigma^2} - 6 \frac{\partial^2}{\partial \phi \partial \sigma} = \pm \nu^2 C, \]  

(19)

where \( \nu \) is a separation constant.

Equation (19) can be transformed to a Bessel differential equation for the function \( \Phi \), performing the transformations \( z = 6\sqrt{-kA^2}, A = \frac{1}{A^2} \Phi(z) \), whose solution for \( k \neq 0 \) becomes

\[ A = \frac{1}{A^2} Z_{\alpha} \left( 6\sqrt{-kA^2} \right), \]  

(20)
where $Z_\alpha$ is a generic Bessel function, with order $\alpha = \frac{1}{4} \sqrt{(1 - B)^2 - 4(\mp \nu^2)}$. For $k = 1$, we have the modified Bessel functions $I_\alpha$ and $K_\alpha$. For $k = -1$, if $\alpha$ is not integer, the solutions become the ordinary Bessel function $J_{\pm \alpha}$, in other case we have a combination of the Bessel functions $J_\alpha$ and $Y_\alpha$ [11].

At this point, we need some conditions on the wave function, in such a way that the classical solutions are not forbidden for the scale factor $A$. Then, we need a wave function having a decreasing behaviour with respect to the scale factor $A$, it can be shown that this is the case when the parameter $B \geq 1$ and the order of the generic Bessel function $\alpha > 0$; besides we choose that the generic Bessel function will be $(K_\alpha$ or $J_\alpha)$, the modified Bessel function or ordinary Bessel function, according to case. With these conditions on the parameters, we postulate that the classical behaviour will be obtained, at least in the semiclassical approximation.

For the particular case of $k = 0$, the solution for the scale factor has the behaviour

$$A = C_1 A^{\frac{1}{2} - \frac{B + \mu}{2}} + C_2 A^{\frac{1}{2} - \frac{B - \mu}{2}}, \quad (21)$$

where $\mu$ is given by

$$\mu = \sqrt{(1 - B)^2 - 4(\mp \nu^2)} \neq 0, \quad (22)$$

with $B > 2 + \nu$ in order to get a decreasing behaviour.

To solve the equation (19) for the fields $\phi, \sigma$, we propose the following ansatz

$$C = Ge^{m_1 \phi}e^{m_2 \sigma}, \quad (23)$$

where $G$ is a constant, and $m_1, m_2$ are two complex parameters, obtaining

$$C = e^{m_1 (\phi + \frac{\mu}{2} \sigma)} \left( A_1 e^{\frac{1}{2} \sqrt{\frac{2}{5} m_1^2 + 2 \nu^2} \sigma} + A_2 e^{\frac{1}{2} \sqrt{\frac{2}{5} m_1^2 + 2 \nu^2} \sigma} \right). \quad (24)$$

4. Classical solutions a la WKB

In the WKB approximation, we propose a solution as $\Psi = \exp(iS(A, \phi, \sigma))$, where the function $S = S_1(A) + S_2(\phi) + S_3(\sigma)$ is known as the superpotential function, being a real function, and fulfilling the usual following conditions

$$\left( \frac{\partial S}{\partial A} \right)^2 \gg \left| \frac{\partial^2 S}{\partial A^2} \right|, \quad \left( \frac{\partial S}{\partial \phi} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \phi^2} \right|, \quad \left( \frac{\partial S}{\partial \sigma} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \sigma^2} \right|, \quad (25)$$

in this way, the Einstein-Hamilton-Jacobi equation (EHJ) is obtained

$$A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 + \frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 + 2 \left( \frac{dS_3}{d\sigma} \right)^2 - 6 \frac{dS_2}{d\phi} \frac{dS_3}{d\sigma} = 0. \quad (26)$$

Employing the separation of variables method, we have

$$A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 = -\frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 - 2 \left( \frac{dS_3}{d\sigma} \right)^2 + 6 \frac{dS_2}{d\phi} \frac{dS_3}{d\sigma} = \alpha, \quad (27)$$

where $\alpha$ is a separation constant

$$A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 = \alpha, \quad (28)$$

$$-\frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 - 2 \left( \frac{dS_3}{d\sigma} \right)^2 + 6 \frac{dS_2}{d\phi} \frac{dS_3}{d\sigma} = \alpha. \quad (29)$$
Taking into account that the canonical momentum $\Pi_A$ was defined as $\Pi_A = \frac{12}{N} A \frac{dA}{d\tau}$, the equations (10) and (28) give us the following relation

$$\frac{dA}{Ndt} = \frac{\sqrt{\alpha - 144kA^4}}{12A^2}. \quad (30)$$

Defining $d\tau = Ndt$ as a physical time, we can rewrite the solution for the scale factor $A$ with respect to the coordinate $\tau$

$$\tau = \int d\tau = \int \frac{12A^2 dA}{\sqrt{\alpha - 144kA^4}}. \quad (31)$$

The relation between $A$ and $\tau$, (31) can be expressed in terms of elliptic integrals [11], in this way by rewriting the integral, we have

$$\tau = \frac{1}{\sqrt{k}} \int^A_0 \frac{z^2 dz}{\sqrt{\alpha \frac{144k}{A^2} - z^2 \sqrt{\alpha \frac{144k}{A^2} + z^2}}} = \frac{1}{\sqrt{k}} \left( \sqrt{2} \sqrt{\alpha \frac{144k}{A^2}} E(\gamma, r) - \frac{\alpha \frac{144k}{A^2}}{\sqrt{2} \sqrt{\alpha \frac{144k}{A^2}}} F(\gamma, r) - A \left[ \sqrt{\alpha \frac{144k}{A^2} - A^2} \right] \right), \quad (32)$$

where $F$ and $E$ are the elliptic integrals of first and second class, respectively, and the parameters

$$\gamma = \arccos \left( \frac{A}{\sqrt{\alpha \frac{144k}{A^2}}} \right) \frac{\sqrt{2} \sqrt{\alpha \frac{144k}{A^2}}}{\sqrt{\alpha \frac{144k}{A^2} + A^2}}, \quad r = \frac{1}{\sqrt{2}}. \quad (33)$$

For the fields $\phi$ and $\sigma$, we use the equation (29), where we recognize a quadratic equation in the associated momenta. This equation can diagonalized, defining the parameters $x = \frac{dS_2}{d\phi}$, and $y = \frac{dS_3}{d\sigma}$, where $x$ and $y$ will be taken as constants. The solution show us that $x$ is proportional to $y$. Solving the quadratic equation (29) we get

$$y = \frac{3x \pm \sqrt{9x^2 - 2\alpha}}{2}. \quad (34)$$

Now, we have two integration constants ($x, \alpha$), which could be determined with appropriate initial boundary conditions.

Using $\frac{d\sigma}{d\tau}$ and $\frac{d\phi}{d\tau}$ as function of $x = \Pi_\phi, y = \Pi_\sigma$ (see equations ((10) and (11))) and integrating, we obtain:

$$\sigma - \sigma_0 \quad = \quad \int d\sigma = \left( -\frac{3}{12} x + \frac{y}{6} \right) \int \frac{d\tau}{A^3(\tau)}, \quad (35)$$

$$\phi - \phi_0 \quad = \quad \int d\phi = \left( \frac{13}{40} x - \frac{y}{4} \right) \int \frac{d\tau}{A^3(\tau)}. \quad (36)$$

For $k = 0$, a flat universe, equation (31) is very simple, its solution is

$$\tau = \frac{4A^3}{\sqrt{\alpha}}. \quad (37)$$
So, the scale factor $A$ has the following behaviour

$$A = \left( \frac{\sqrt{\alpha}}{4} \right)^{1/3} \tau^{\frac{1}{3}}. \quad (38)$$

Introducing this result into (35, 36) we have

$$\sigma - \sigma_0 = \left( -\frac{3}{12} x + \frac{\gamma}{6} \right) \left( \frac{4}{\sqrt{\alpha}} \right)^{1/3} \int \tau^{-1} d\tau = \left( -\frac{3}{12} x + \frac{\gamma}{6} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \ln \tau, \quad (39)$$

$$\phi - \phi_0 = \left( \frac{13}{40} x - \frac{\gamma}{4} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \ln \tau. \quad (40)$$

In this manner, we can obtain the behaviour of the fields $\phi$ and $\sigma$ as functions of the scale factor $A$, as follows

$$\sigma - \sigma_0 = \left( -\frac{3}{12} x + \frac{\gamma}{6} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \left( 3 \ln A + \ln \frac{4}{\sqrt{\alpha}} \right), \quad (41)$$

$$\phi - \phi_0 = \left( \frac{13}{40} x - \frac{\gamma}{4} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \left( 3 \ln A + \ln \frac{4}{\sqrt{\alpha}} \right). \quad (42)$$

This procedure can be also used for the cases, $k = \pm 1$.

5. Final Remarks

It is well known that the Wheeler-DeWitt cosmological equation is not an evolution equation and therefore the associated quantum states do not evolve in time (stationary Schrödinger equation like). A possible way to connect some parameters of the ‘quantum’ WDW solutions with classical Einstein ones is by phenomenological restrictions imposed on the superpotential functions or final conditions over the wave function as we did. By these means we get a decreasing function in the gravitational part of the wave function of the Universe. Using this method, we find the classical behaviour for the scale factor, scalar fields $\phi$ and $\sigma$ in terms of $A$. So even not knowing a time WDW quantum equation, our physical assumptions allow us to connect the quantum behaviour with the classical one. Under dynamical compactification condition, the moduli scalar $\sigma$ will be an increasing function, in such away that the radius for the extra dimension vanish, see Eq. (2). On the other hand, our procedure applied to a time dependent object in the context of a low-energy string effective action (7) provides a quantization interesting in its own right and that is expected to provide information on the quantum objects depending on time in string theory. Their classical behaviour was also presented.

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