Lower Bound on the Size-Ramsey Number of Tight Paths

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Abstract

The size-Ramsey number $\hat{R}(k)(H)$ of a $k$-uniform hypergraph $H$ is the minimum number of edges in a $k$-uniform hypergraph $G$ with the property that every ‘2-edge coloring’ of $G$ contains a monochromatic copy of $H$. For $k \geq 2$ and $n \in \mathbb{N}$, a $k$-uniform tight path on $n$ vertices $P_n^{(k)}$ is defined as a $k$-uniform hypergraph on $n$ vertices for which there is an ordering of its vertices such that the edges are all sets of $k$ consecutive vertices with respect to this order.

We prove a lower bound on the size-Ramsey number of $k$-uniform tight paths, which is, considered asymptotically in both the uniformity $k$ and the number of vertices $n$, $\hat{R}(k)(P_n^{(k)}) = \Omega(\log(k)n)$.

Keywords – size-Ramsey, Ramsey theory, tight path, uniform hypergraph

1 Introduction

For a $k$-graph $G = (V,E)$, i.e. a $k$-uniform hypergraph on a vertex set $V$ and an edge set $E \subseteq \binom{V}{k}$, a 2-edge coloring of $G$ is a function $c: E(G) \to \{\text{red}, \text{blue}\}$ that maps every edge to one of the given colors red or blue. In the following we refer to such a function simply as a coloring of $G$. We say that a $k$-graph $G$ has the Ramsey property $G \rightarrow H$ for some $k$-graph $H$ if every coloring of $G$ contains a monochromatic copy of $H$. The size-Ramsey number of a $k$-graph $H$ is defined as

$$\hat{R}(k)(H) = \min \{|E(G)| : G \text{ $k$-graph with } G \rightarrow H\}.$$

Size-Ramsey problems were introduced by Erdős, Faudree, Rousseau and Schelp [7] for graphs. One of the focus points of studies on the graph case is estimating the size-Ramsey number of paths. Beck [2] disproved a conjecture of Erdős [6] by showing that $\hat{R}^{(2)}(P_n) = O(n)$. Since then, estimates on this number have been gradually improved, with the current best known bounds being $(3.75 - o(1))n \leq \hat{R}^{(2)}(P_n) \leq 74n$ given by Bal, DeBiasio [1] and Dudek, Pralat [5], respectively.

Let $n, k \in \mathbb{N}$ with $k \geq 2$. A $k$-uniform tight path on $n$ vertices $P_n^{(k)}$ is a $k$-graph on $n$ vertices for which there exists an ordering of its vertices such that every edge is a $k$-element set of consecutive vertices with respect to this order, two consecutive edges have precisely
$k - 1$ vertices in common, and there are no isolated vertices. Equivalently, $\mathcal{P}_n^{(k)}$ is a $k$-graph isomorphic to the hypergraph $\left(\{1, \ldots, n\}, E\right)$ with edge set

$$E = \left\{\{i, \ldots, i + k - 1\} : i \in \{1, \ldots, n - k + 1\}\right\}.$$ 

If the uniformity is clear from the context we omit the prefix ‘$k$-uniform’ when referring to tight paths.

Research on the size-Ramsey number of hypergraphs has been substantially driven forward by Dudek, La Fleur, Mubayi and Rödl [4]. Among other results, they conjectured that the size-Ramsey number of tight paths is linear in terms of $n$. This conjecture was recently verified by Letzter, Pokrovskiy and Yepremyan [8].

**Theorem 1** ([8]). Let $k \geq 2$ be fixed. Then

$$\hat{R}^{(k)}(\mathcal{P}_n^{(k)}) = O(n).$$

Regarding a lower bound on this number, the following is a simple observation.

**Observation.** Let $n, k \in \mathbb{N}$, $k \geq 2$. Then

$$\hat{R}^{(k)}(\mathcal{P}_n^{(k)}) \geq 2n - 2k + 1.$$ 

In this paper we show an improved lower bound on the size-Ramsey number of tight paths.

**Theorem 2.** Let $n \geq 7$. Then

$$\hat{R}^{(3)}(\mathcal{P}_n^{(3)}) \geq \frac{8}{3}n - \frac{28}{3}.$$ 

**Theorem 3.** Let $k \geq 4$ and $n > \frac{k^2 + k - 2}{2}$. Then

$$\hat{R}^{(k)}(\mathcal{P}_n^{(k)}) \geq \left\lceil \log_2(k + 1) \right\rceil \cdot n - 2k^2.$$ 

Section 2 discusses some properties which are useful for the main proofs. In Section 3 the proofs of Theorem 1 and Theorem 2 are presented.

## 2 Preliminaries

Let $\mathcal{G}$ be a $k$-graph and $Z \subseteq E(\mathcal{G})$ be an edge set. Let $\cup Z = \{v \in e : e \in Z\}$ be the set of vertices that are covered by $Z$. We say that the $k$-graph $(\cup Z, Z)$ is formed by $Z$. Given a vertex set $W \subseteq V(\mathcal{G})$ the subhypergraph induced by $W$ is $\mathcal{G}[W] = (W, \{e \in E(\mathcal{G}) : e \subseteq W\})$. For $q \in \mathbb{R}$, $0 \leq q < k$, the $q$-neighborhood of $Z$ is the edge set

$$N_{>q}(Z) = \{e \in E(\mathcal{G}) : \exists e' \in Z \text{ with } |e \cap e'| > q\}.$$ 

Note that we allow $e = e'$, thus $Z \subseteq N_{>q}(Z)$ for all $0 \leq q < k$.

For each $k$-uniform tight path $\mathcal{P}$ on $n$ vertices we fix an ordering of the vertices such that each edge is a set of consecutive vertices. We say that such an enumeration $V(\mathcal{P}) = \{v_1, \ldots, v_n\}$ is according to $\mathcal{P}$. For a $k$-graph $\mathcal{G}$, we define $e(\mathcal{G}) = |E(\mathcal{G})|$, e.g. $e(\mathcal{P}_n^{(k)}) = n - k + 1$. Furthermore, let $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. For any other notation, see Diestel [3].
Proposition 4. Let \( n, k \in \mathbb{N} \) such that \( k \geq 2 \) and \( n > \frac{k^2 + k - 2}{2} \). Let \( \mathcal{P} \) be a \( k \)-uniform tight path on \( n \) vertices. Furthermore, let \( \alpha \in \mathbb{R} \) such that \( 1 \leq \alpha \leq k \) and \( W \subseteq V(\mathcal{P}) \) be a vertex set such that for every edge \( e \in E(\mathcal{P}) \) we have \( |e \cap W| \geq \alpha \). Then
\[
|W| \geq \frac{\alpha(n - k + 1)}{k}.
\]
In particular, if for each \( e \in E(\mathcal{P}) \), \( |e \cap W| > \frac{k + 1}{2} \), then for \( n > \frac{k^2 + k - 2}{2} \),
\[
|W| > \frac{n}{2}.
\]

Proof. We estimate the size of \( W \) by double-counting ordered pairs \((v, e)\) consisting of a vertex \( v \in W \) and an edge \( e \in E(\mathcal{P}) \) with \( v \in e \). Let \( \rho(v, e) \) be the number of such pairs.
Considering the edges of \( \mathcal{P} \) it is immediate that
\[
\rho(v, e) \geq \alpha \cdot e(\mathcal{P}) = \alpha(n - k + 1).
\]
Now consider the vertices in \( W \subseteq V(\mathcal{P}) \). The maximum degree of the tight path \( \mathcal{P} \) is at most \( k \), so
\[
\rho(v, e) \leq k \cdot |W|.
\]
Combining both inequations, we obtain
\[
|W| \geq \frac{\alpha(n - k + 1)}{k}.
\]
Now consider the case that for each edge \( e \in E(\mathcal{P}) \) we have \( |e \cap W| > \frac{k + 1}{2} \), then also \( |e \cap W| \geq \frac{k + 2}{2} \). Therefore we obtain for sufficiently large \( n \),
\[
|W| \geq \frac{k + 2}{2} \cdot \frac{n - k + 1}{k} > \frac{n}{2}.
\]

3 Proofs of the main results

Proof of Theorem 3. Let \( \mathcal{G} \) be a \( k \)-uniform hypergraph with \( \mathcal{G} \to \mathcal{P}_n^{(k)} \), i.e. such that every 2-coloring contains a monochromatic \( k \)-uniform tight path on \( n \) vertices. We show that there are at least \( \lceil \log_2(k + 1) \rceil \cdot n - 2k^2 \) many edges in \( \mathcal{G} \) by iteratively constructing many edge-disjoint tight paths of length \( n \). Let \( \lambda = \lfloor \log_2(k + 1) \rfloor - 1 \), this number indicates how many iteration steps are executed. Additionally, we define the function \( q: \{0, \ldots, \lambda\} \to \mathbb{R} \),
\[
q(i) = \left(1 - \frac{1}{2^i}\right)(k + 1),
\]
which will be the parameter of the \( q \)-neighborhoods considered in each iteration step. Clearly, \( q \) is an increasing function and \( q(i) \geq 0 \) for \( i \in \{0, \ldots, \lambda\} \). For \( i \leq \lambda \) (or equivalently \( i < \log_2(k + 1) \)) it can be seen that \( q(i) < k \), which implies that the \( q(i) \)-neighborhood is well-defined for all \( i \in \{0, \ldots, \lambda\} \).

As an initial step of the iteration, the Ramsey property \( \mathcal{G} \to \mathcal{P}_n^{(k)} \) provides that there is some tight path on \( n \) vertices in \( \mathcal{G} \), which we denote by \( \mathcal{P}_0 \).

From now on we proceed iteratively, so let \( i = 1, \ldots, \lambda \) and suppose that the iteration has been performed for all smaller values of \( i \). In each step of the iteration we construct the following:
• Edge sets $Z^1_i, Z^2_i \subseteq E(P_{t-1})$ such that $\cup Z^1_i \cap \cup Z^2_i = \emptyset$ and each of the sets forms a tight path in $G$ on precisely $\left\lceil \frac{n}{2} \right\rceil$ vertices.

• A tight path $P_t$ on $n$ vertices with $E(P_t) \cap N_{>q(i)}(Z_b^i) = \emptyset$ for all $a \in [2], b \in [i]$. First we construct $Z^1_i$ and $Z^2_i$ by dividing the tight path $P_{t-1}$ into two parts of equal length and considering the edge sets of the two created shorter tight paths. For this purpose, consider an ordering of the vertices $V(P_{t-1}) = \{v_1, \ldots, v_n\}$ according to $P_{t-1}$. Let

$$V^1_i = \{v_1, \ldots, v_{\left\lfloor \frac{n}{2} \right\rfloor}\} \quad \text{and} \quad V^2_i = \{v_{\left\lceil \frac{n}{2} \right\rceil+1}, \ldots, v_n\}.$$  

Then $|V^1_i| = \left\lfloor \frac{n}{2} \right\rfloor = |V^2_i|$. Now let $Z^1_i = E(P_{t-1}|V^1_i)$ and $Z^2_i = E(P_{t-1}|V^2_i)$. Clearly, these two sets form vertex-disjoint tight paths on $\left\lceil \frac{n}{2} \right\rceil$ vertices in $G$. The size of $Z^1_i$ and $Z^2_i$ is

$$|Z^1_i| = |Z^2_i| = e(P^{(k)}_{\left\lceil \frac{n}{2} \right\rceil}) = \left\lceil \frac{n}{2} \right\rceil - k + 1 \geq \frac{n - 2k + 1}{2}.$$  

In the next step we show a key property of the edge sets $Z^a_i$ for $a \in [2], b \in [i]$.

**Claim.** Let $a_1, a_2 \in [2], b_1, b_2 \in [i]$ such that $(a_1, b_1) \neq (a_2, b_2)$. Then for any two edges $e_1 \in N_{>q(i)}(Z_{b_1}^{a_1})$ and $e_2 \in N_{>q(i)}(Z_{b_2}^{a_2})$ we have

$$|e_1 \cap e_2| < k - 1.$$  

**Proof of the claim.** Assume that there are edges $e_1 \in N_{>q(i)}(Z_{b_1}^{a_1}), e_2 \in N_{>q(i)}(Z_{b_2}^{a_2})$ with $|e_1 \cap e_2| \geq k - 1$. By definition, there is an edge $z_1 \in Z_{b_1}^{a_1}$ such that $|e_1 \cap z_1| > q(i)$ and an edge $z_2 \in Z_{b_2}^{a_2}$ with $|e_2 \cap z_2| > q(i)$.

![Possible constellation of the edges in iteration step $i = 1$ where $k = 6$](image)

We estimate the size of $z_1 \cap z_2$ in order to find a contradiction to our assumption. Since $|e_1 \cap e_2| \geq k - 1$, we have $|e_1 \\backslash e_2| \leq 1$ and so $|e_2 \\backslash z_1| > q(i) - 1$. Applying this, we obtain:

$$|z_1 \cap z_2| \geq |e_2 \cap z_1 \cap z_2| \geq |e_2| - |e_2 \\backslash z_1| - |e_2 \\backslash z_2| = |e_2| + |e_2 \cap z_1| + |e_2 \cap z_2|$$

$$> -k + q(i) - 1 + q(i) = \left(1 - \frac{1}{2^{q(i)}}\right)(k+1) = q(i) - 1.$$  

If $b_1 = b_2$, we have $\cup Z_{b_1}^{a_1} \cap \cup Z_{b_2}^{a_2} = \emptyset$ by construction. But then $q(i-1) < |z_1 \cap z_2| = 0$, which is a contradiction.
We suppose that \( b_1 \neq b_2 \), then without loss of generality \( b_1 > b_2 \) (and by this \( b_1 - 1 \geq 1 \)). By construction we know \( z_1 \in Z_{b_1}^a \subseteq E(\mathcal{P}_{b_1-1}) \). In the iteration step \( b_1 - 1 \) the tight path \( \mathcal{P}_{b_1-1} \) was chosen to be edge-disjoint from \( \bigcup_{a \in [2], b < b_1} N_{b+b-1}(Z_{b_1}^a) \). This yields that \( z_1 \notin N_{b+b-1}(Z_{b_1}^a) \) and so

\[
|z_1 \cap z_2| \leq q(b_1 - 1) \leq q(i - 1),
\]

where the last inequality holds because \( q \) is an increasing function, and we again reach a contradiction. This concludes the proof of the claim.

Now we find the next tight path \( \mathcal{P}_i \) in \( \mathcal{G} \) by considering the following coloring of \( \mathcal{G} \). For all \( a \in [2] \) and \( b \in [i] \), assign the color red to each edge in \( N_{\{b\}}(Z_b^a) \). The remaining edges are colored blue. We will prove that there is a monochromatic blue path \( \mathcal{P} \) on \( n \) vertices in \( \mathcal{G} \).

Clearly, each edge in \( E(\mathcal{R}) \) is in some neighborhood \( N_{\{b\}}(Z_b^a) \), \( a \in [2], b \in [i] \). Now the above claim provides that any two edges which are consecutive in \( \mathcal{R} \), so intersect in precisely \( k - 1 \) vertices, belong to the same neighborhood \( N_{\{b\}}(Z_b^a) \) for some \( a \in [2], b \in [i] \). By repeating this argument, we obtain that \( E(\mathcal{R}) \subseteq N_{\{i\}}(Z_b^a) \) for some \( a \in [2], b \in [i] \). This implies that for all \( e \in E(\mathcal{R}) \),

\[
|e \cap \mathcal{Z}_b^a| > q(i) \geq q(1) = \frac{k+1}{2}.
\]

Then applying Proposition 4 for the tight path \( \mathcal{R} \) and the vertex set \( \mathcal{U} \mathcal{Z}_b^a \) yields \( |\mathcal{U} \mathcal{Z}_b^a| > \frac{k}{2} \). But by construction \( Z_b^a \) forms a \( k \)-graph on precisely \( \left\lfloor \frac{n}{2} \right\rfloor \) vertices, a contradiction. This concludes the proof of the claim.

Consequently, there is no red tight path on \( n \) vertices in the coloring, so the Ramsey property \( \mathcal{G} \rightarrow \mathcal{P}_{n(k)} \) implies the existence of a monochromatic blue \( \mathcal{P}_{n(k)} \), which we denote \( \mathcal{P}_i \). Observe that for all \( e \in E(\mathcal{P}_i) \) and for all \( a \in [2], b \in [i] \), we have \( e \notin N_{\{i\}}(Z_b^a) \), since all edges in these neighborhoods are colored red.

By iterating the described procedure for \( i = 1, \ldots, \lambda \), we obtain edge sets \( Z_b^a \) for \( a \in [2], b \in [\lambda] \) which are pairwise disjoint and additionally a tight path \( \mathcal{P}_\lambda \) on \( n \) vertices such that each edge in \( E(\mathcal{P}_\lambda) \) is not contained in any set \( Z_b^a \). This allows for the following estimate on the number of edges in \( \mathcal{G} \)

\[
e(\mathcal{G}) \geq \sum_{b \in [\lambda]} (|Z_b^1| + |Z_b^2|) + e(\mathcal{P}_\lambda) \geq \lambda(n - 2k - 1) + (n - k + 1)
\]

\[
\geq \left\lceil \log_2(k + 1) \right\rceil \cdot n - (k - 1)(2k + 2) \geq \left\lceil \log_2(k + 1) \right\rceil \cdot n - 2k^2,
\]

where in the last line we used \( \left\lceil \log_2(k + 1) \right\rceil \leq k \).

We point out that the above proof also applies to 3-uniform tight paths, but does not yield an improvement of the trivial bound. In order to obtain a refined bound in this case, we instead use a non-iterative adaption of the above proof.

**Proof of Theorem 2.** Let \( \mathcal{G} \) be an arbitrary 3-uniform hypergraph which has the Ramsey property \( \mathcal{G} \rightarrow \mathcal{P}_{n(3)} \). As before, we show that \( \mathcal{G} \) is a 3-graph on at least \( \frac{5}{3}n - \frac{2n}{3} \) many edges.
Using the Ramsey property $G \to P_n^{(3)}$, there exists some tight path on $n$ vertices in $G$. In particular, we find a shorter tight path $P_0$ on only $\lceil \frac{2}{3}n - \frac{2}{3} \rceil$ many vertices. Observe that $e(P_0) = \lceil \frac{2}{3}n - \frac{2}{3} \rceil - 2 \geq \frac{2}{3}n - \frac{14}{3}$.

In order to find a tight path $P_1$ which is edge-disjoint from $P_0$, we consider the following coloring. Color all edges in the $1$-neighborhood $N_{>1}(E(P_0))$ in red and the remaining edges in blue. Assume for a contradiction that in this coloring there is a monochromatic red tight path on $n$ vertices, say $R$. Then Proposition 4 applied to the tight path $R$ and the vertex set $V(P_0)$ provides a contradiction. Since $G \to P_n^{(3)}$, there is a monochromatic blue tight path on $n$ vertices in $G$. This implies that there is also a blue tight path on $n - 1$ vertices, i.e. on $n - 3$ edges. We fix such a tight path $P_1$ with $e(P_1) = n - 3$. Note that $N_{>1}(E(P_0))$ and $E(P_1)$ are disjoint edge sets.

In the following, in order to find a third edge-disjoint tight path, we consider another coloring of $G$. From now on, let each edge in $E(P_0) \cup E(P_1)$ be colored red and all other edges blue. Assume for a contradiction that there is a red tight path $R$ on $n$ vertices in this coloring. Then neither $E(R) \subseteq E(P_0)$ nor $E(R) \subseteq E(P_1)$, because the two edge sets have size strictly less than $e(P_n^{(3)})$. Therefore, $R$ consists of edges of both $E(P_0)$ and $E(P_1)$. Both of these edge sets are disjoint, so there exist two edges $e_1 \in E(P_0) \cap E(R), e_2 \in E(P_1) \cap E(R)$ which are consecutive in $R$, i.e. $|e_1 \cap e_2| = 2$. But that is a contradiction to the fact that $N_{>1}(E(P_0))$ and $E(P_1)$ are disjoint. Consequently, there is no red $P_n^{(3)}$ in this coloring. By the same argument as before, there is a blue tight path $P_2$ on $n$ vertices in $G$.

Then the three edge sets $E(P_0), E(P_1), E(P_2)$ are pairwise disjoint. Thus,

$$e(G) \geq e(P_0) + e(P_1) + e(P_2) \geq \frac{5}{3}n - \frac{28}{3}.$$

\[\square\]

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