MOTIVIC IRRATIONALITY PROOFS

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Abstract. We exhibit geometric conditions on a family of toric hypersurfaces under which the value of a canonical normal function at a point of maximal unipotent monodromy is irrational.

1. Introduction

The limit of a generalized normal function at a point where the underlying variation of Hodge structure degenerates, as recently studied in [7K], turns out to have an unexpected arithmetic application. R. Apéry’s famous proof (see [vdP]) of irrationality of \( \zeta(3) := \sum_{k \geq 1} k^{-3} \) relies on the existence of rapidly divergent sequences \( a_m \in \mathbb{Z}, b_m \in \mathbb{Q} \) (the latter having denominators of bounded growth) with \( 2a_m \zeta(3) + b_m \) converging rapidly to zero. Beukers, Peters and Stienstra [Be, BP, Pe, PS] geometrically repackaged much of the proof, noting for instance that the generating function \( \sum_{m \geq 0} a_m \lambda^m =: A(\lambda) \) records periods of a holomorphic 2-form on a family of K3 surfaces \( \{X^\lambda\}_{\lambda \in \mathbb{P}^1} \), hence must satisfy a Picard-Fuchs differential equation \( D_{PF} A(\lambda) = 0 \).

Behind the remaining details of the irrationality proof lurks a family of cycles in (algebraic) K3 of the K3. The associated higher normal function \( \tilde{V}(\lambda) \) has special value \( \tilde{V}(0) = -2\zeta(3) \), and satisfies the inhomogeneous equation \( D_{PF} \tilde{V}(\lambda) = Y(\lambda) \), where \( Y \) denotes the Yukawa coupling. Setting \( \sum_{m \geq 1} b_m \lambda^m := A(\lambda) \tilde{V}(0) - \tilde{V}(\lambda) \), one deduces from this recurrence relations on the \( \{b_m\} \) which (as presented here) give “half” of the required bounded denominator growth. The other “half” comes from the Fermi family of K3s studied, but not related to the Apéry proof, in [PS]. Finally, the behavior of the cycles at singular members of the family \( \{X^\lambda\} \) shows that \( \tilde{V}(\lambda) \) has no monodromy about the conifold singular fiber closest to \( \lambda = 0 \), implying the rapid convergence of \( 2a_m \zeta(3) + b_m \to 0 \).

In this paper, we reveal a general criterion for the irrationality of special values of certain higher normal functions. Given a Laurent polynomial \( \phi(x_1, \ldots, x_n) \) with reflexive Newton polytope \( \Delta \), the equation \( \phi(x) = \lambda \) defines a family of Calabi-Yau hypersurfaces \( \{X^\lambda\} \) in

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the toric variety $\mathbb{P}_\Delta$. Associated to this family is a pure irreducible variation $\mathcal{V}_\phi$ of weight $(n - 1)$ (over a Zariski open $\mathcal{U} \subset \mathbb{P}^1$), together with a canonical section $\{\tilde{\omega}^\lambda\}$ of the Hodge line bundle $\mathcal{F}^{n-1} \mathcal{V}_\phi$. We call $\phi$ tempered if the coordinate symbol $\{x_1, \ldots, x_n\}$ lifts to a family of motivic cohomology classes $\Xi^\lambda$ on the family, producing an extension

\[(1.1) \quad 0 \rightarrow \mathcal{V}_\phi(n) \rightarrow \mathcal{E}_\phi \rightarrow \mathbb{Q}(0)_{\mathcal{U}} \rightarrow 0\]

of admissible variations of mixed Hodge structure over $\mathcal{U}$. (This temperedness typically holds, for example, for LG-models constructed from Minkowski polynomials \[\text{[KS]}\].) Applying a variant of this hypothesis allows us to construct a canonical truncated higher normal function $\tilde{V}(\lambda)$ on $\mathbb{P}^1 \setminus \phi(\mathbb{R}^n)$ by pairing the regulator class of $\Xi^\lambda$ (i.e., the extension class of (1.1)) with $\tilde{\omega}^\lambda$ (see Theorem 4.2).

To arrange $\tilde{V}(0) \notin \mathbb{Q}$, we must impose several additional conditions on $\phi$, roughly as follows (see Theorem 3.1):

- the local system of periods of $\tilde{\omega}^\lambda$ must be of rank $n$, admit an isomorphism to its pullback by $\lambda \mapsto C/\lambda$, and have two mild singularities apart from 0 and $\infty$, one of which is very far from 0;
- $\phi(-x)$ has positive integer coefficients, and the Picard-Fuchs operator associated to $\tilde{\omega}^\lambda$ (suitably normalized) is integral; and
- a finite $(r : 1)$ pullback of the family $X^\lambda$ can be presented as a family of toric hypersurfaces in $\mathbb{P}_\diamond$, where $\diamond$ is a “facile” polytope (Definition 2.2).

The role of the last condition is to produce a basis of periods whose power-series coefficients have the right denominator bounds (see Corollary 2.5). This basis is closely tied to mirror symmetry \([\text{HLY}]\) and the Frobenius method \([\text{IKSY}]\); for $n \geq 4$, Theorem 2.3 uncovers a surprising arithmetic implication of the Hyperplane Conjecture \([\text{HLY}],[\text{LZ}]\). We also remark that, assuming only temperedness, the higher normal function $\tilde{V}(\lambda)$ can always be written as one of the chain-integral solutions of \([\text{HLYZ}]\), while $\tilde{V}(0)$ may be interpreted as an Apéry constant as studied in \([\text{Ga}],[\text{Go}],[\text{GGI}],[\text{GI}],[\text{GZ}]\).

In the last section, we exhibit Laurent polynomials which satisfy all these conditions for $n = 1, 2,$ and 3, recovering irrationality of $\log(1 + b^{-1})$ ($b \in \mathbb{N}$), $\zeta(2)$, and $\zeta(3)$. We also propose relaxations of some of the conditions, together with specific families of polynomials, for making contact with results on linear forms in more than one odd zeta value – for instance \([\text{Va}],[\text{Z1}]\), and especially \([\text{Br}]\), whose basic cellular integrals on $\mathcal{M}_{0,n+3}$ are the power series coefficients of a $\tilde{V}(\lambda)$ as above.
While the results on $\zeta(2)$ and $\zeta(3)$ complete, in a way, the story begun in [BP], the reader familiar with those works will notice (perhaps deplore?) the complete lack of reference to modular forms in what follows. The omission is strategic, as a weight-$(n-1)$ VHS $\mathcal{V}_\phi$ with maximal unipotent and conifold monodromies cannot have a modular parametrization for $n \geq 4$. This is, of course, precisely where we hope to stimulate the search for examples with Theorem 3.1, starting with the increasingly sophisticated databases of polytopes, local systems, Calabi-Yau differential operators and their geometric realizations [AESZ, Fan01, Fan02, DM].

The same reader may be puzzled by our reference to “splitting up the bound” on denominators of $b_m$, as in the Apéry story one simply shows that $2(L_m)^3b_m \in \mathbb{Z}$, where $L_m := \text{lcm}\{1, 2, \ldots, m\}$. What we are able to show under general hypotheses, using techniques from Hodge theory and mirror symmetry, is instead that (for some fixed $\varepsilon \in \mathbb{N}$) both $\varepsilon(L_m)^n b_m$ and $\varepsilon(m!)^n b_m$ are integers (with $r = 2$ for Apéry), which together are enough for an irrationality proof. With that said, the results of this article are intended to be a narrow proof of concept for a Hodge-theoretic approach to irrationality proofs, rather than to be optimal as respects either methodology or hypotheses.

We freely (though infrequently) use the notation and terminology of regulator currents and toric varieties throughout, as reviewed in §§1-2 of [DK].

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2. Facile polytopes

Let $\Delta, \Delta^\circ$ be a dual pair of reflexive polytopes in $\mathbb{R}^n$, admitting regular projective triangulations $\mathcal{T}, \mathcal{T}^\circ$. Take $\Sigma, \Sigma^\circ$ to be the fans on these triangulations, and $\mathbb{P}_\Delta, \mathbb{P}_{\Delta^\circ}$ (respectively) the toric Fano $n$-folds determined by the fans. Write

$$\mathcal{A} = \mathcal{A}_\Delta := \Delta \cap \mathbb{Z}^n = \{v^{(0)} = 0, v^{(1)}, \ldots, v^{(N)}\}$$

for the integer points, and

$$\mathbb{L} := \{\ell = (\ell_0 = \sum_{i=1}^N \ell_i, \ell_1, \ldots, \ell_N) \in \mathbb{Z}^{N+1} \mid \sum_{i=1}^N \ell_i v^{(i)} = 0\}$$
for the lattice of integral relations on them. The irreducible components \( \{D_i\}_{i=1}^N \) of \( \mathbb{P}_{\Delta^0} \cap \mathbb{G}_m^n = \mathbb{D}_{\Delta^0} \) (with \( \deg D_i(x_k) = i_k^{(i)} \)) generate \( H^2(\mathbb{P}_{\Delta^0}, \mathbb{Z}) \), and \( K_{\mathbb{P}_{\Delta^0}} = -\sum_{i=1}^N D_i =: D_0 \).

Assume that the Mori cone \( \mathcal{M} = \mathcal{M}_{\Delta^0} \subset H_2(\mathbb{P}_{\Delta^0}, \mathbb{R}) \) (of classes of effective curves) is regular simplicial, so that \( \mathcal{M} := \mathcal{M} \cap H_2(\mathbb{P}_{\Delta^0}, \mathbb{Z}) = \mathbb{Z}_{\geq 0}(C_1, \ldots, C_{M=N-n}) \). Write \( C_j \mapsto \ell^{(i)} = C_j \cdot D \) for the images under

\[
H_2(\mathbb{P}_{\Delta^0}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L} \xrightarrow{\sim} \text{Hom}(H^2(\mathbb{P}_{\Delta^0}, \mathbb{Z}), \mathbb{Z}),
\]

so that the dual (nef) cone \( \mathcal{K} \subset H^2(\mathbb{P}_{\Delta^0}, \mathbb{R}) \) has \( \mathcal{K} := \mathcal{K} \cap H^2(\mathbb{P}_{\Delta^0}, \mathbb{Z}) = \mathbb{Z}_{\geq 0}(J_1, \ldots, J_M) \) with \( J_k \cdot C_j = \delta_{kj} \) and \( D_i = \sum_{k=1}^M \ell_i^{(k)} J_k \). (To compute the \( \ell^{(k)} \), one may use primitive collections as in \([LZ]\).)

Let \( f_\alpha(x) := \sum_{i=0}^N a_i x^i \), and \( X^\alpha_\mathcal{K} := \{ f_\alpha(x) = 0 \} \subset \mathbb{G}_m^n \) with (CY \((n-1)\)-fold) Zariski closure \( X_\mathcal{K} \subset \mathbb{P}_\Delta \); take \( X^0 \subset \mathbb{P}_{\Delta^0} \) to be any smooth anticanonical hypersurface. We are interested in the \( \mathbb{A}\)-periods

\[
\pi_{\gamma}(\alpha) = \int_{\gamma} \omega_\alpha (\gamma \in H_{n-1}(X_\mathcal{K}, \mathbb{A}), \mathbb{A} = \mathbb{Z}, \mathbb{Q}, \text{ or } \mathbb{C}) \text{ of }
\]

\[
\omega_\alpha = \text{Res}_{X_\mathcal{K}} \Omega_\alpha = \frac{a_0}{(2\pi i)^n} \text{Res}_{X_\mathcal{K}} \left( \frac{dx_1/x_1 \wedge \cdots \wedge dx_n/x_\alpha}{f_\alpha(x)} \right) \in \Omega^{n-1}(X_\mathcal{K}),
\]

in the large complex-structure limit (LCSL) — i.e., where the \( t_k := \alpha_\mathcal{K}^{(k)} \) are sufficiently small. These are known to solve the GKZ system

\[
\tau_{\mathbb{A}}^{\Delta_{\mathbb{GKZ}}} : \quad \left\{ \left( \sum_{i=0}^N \Delta_{\alpha_\mathcal{K}}^{(i)} \delta_{\alpha_\mathcal{K}} \right)_{j=1, \ldots, n}, \left( \sum_{i=0}^N \delta_{\alpha_\mathcal{K}} \right)_{j=1, \ldots, n} \right\} \in \Omega^{n-1}(X_\mathcal{K}),
\]

whose remaining solutions are the other integrals of \( \frac{1}{2\pi i} \Omega_\alpha \) over relative cycles in \( H_{\mathcal{K}}(\mathbb{P}_{\Delta} \cap X_\mathcal{K}; \mathbb{C}) \) \([HLYZ]\).

A formula for the solutions to \( \tau_{\mathbb{A}}^{\Delta_{\mathbb{GKZ}}} \) in the LCSL was given by \([HLY]\): writing \( \tau_{\mathbb{L}} := \frac{\log(t_k)}{2\pi i} \), \( \mathcal{O} := \mathbb{C}[\ell^k] \), they are precisely the functions \( \psi(B_\Delta) \in \mathcal{O} \) where \( \psi \in H^* (\mathbb{P}_{\Delta^0}, \mathbb{C})^\vee \) and

\[
\mathcal{B}_\Delta := \sum_{\ell^k \in \mathbb{M}} B_{\ell^k}(D) \ell_\mathcal{K}^{\ell^k} = \sum_{\ell^k \in \mathbb{M}} B_{\ell^k}(J) \ell_\mathcal{K}^{\ell^k} \in H^* (\mathbb{P}_{\Delta^0}, \mathcal{O})
\]

with \( B_{\alpha_\mathcal{K}}(J) := B \sum_{n \in \mathbb{Z}_0} \left( \sum_{i=1}^M \ell_i^{(k)} J_k \right) \) and

\[
B_{\ell^k}(D) := \prod_{i=1;}^{N_{\ell_i < 0}} D_i (D_i - 1) \cdots (D_i + \ell_i + 1) \prod_{i=1;}^{N_{\ell_i > 0}} (D_i + 1) \cdots (D_i + \ell_i) \times (D_0 - 1) \cdots (D_0 + \ell_0).
\]

According to the Hyperplane Conjecture \([HLY], [LZ]\), \( \psi(B_\Delta) \) is a \( \mathbb{C} \)-period (in the above sense) precisely when \( \psi \) belongs to

\[
\text{im } \{ t^*_\mathbb{L} : H^* (X^0, \mathbb{C})^\vee \rightarrow H^* (\mathbb{P}_{\Delta^0}, \mathbb{C})^\vee \}.\]
More precisely, for each $\kappa := (\kappa_1, \ldots, \kappa_M) \in \mathbb{Z}_{\geq 0}^M$ with $|\kappa| := \sum \kappa_j \leq n$, we compute

$$B_\kappa(t, \tau) := \frac{1}{(2\pi i)^{\kappa}} \left( \partial_{\ell_1}^{\kappa_1} \cdots \partial_{\ell_M}^{\kappa_M} B_\Delta \right)_{|\ell| = 0} = \sum_{\kappa' + \kappa'' = \kappa} (2\pi i)^{-|\kappa''|} \tau^{\kappa'} \sum_{\mu \in \mathbb{Z}_{\geq 0}^M} b_\mu^{\kappa''} t^\mu$$

where $b_\mu^{\kappa''} = \left( \partial_{\ell_1}^{\kappa_1''} \cdots \partial_{\ell_M}^{\kappa_M''} B_\Delta \right) (0)$. Given bases $\{\tilde{\psi}^r_i\} \subset H_{2r}(\mathbb{P}_{\Delta^c}, \mathbb{Z})$ resp. $\{\hat{\psi}^r_i\} \subset \text{im}(i^{2r}_*) \mathbb{Z}$, we obtain $\mathbb{C}$-bases for the solutions to $\tau^A_{\Delta \text{GKZ}}$ resp. for the $\mathbb{C}$-periods (assuming the Hyperplane Conjecture) which are $\mathbb{Z}$-linear combinations of the $\{B_\kappa\}$. That is, writing $B_\kappa^0(t) = \sum b_\mu^{\kappa''} t^\mu =: \sum a_n t^n =: A(t)$, we have

$$(\wedge) \psi^r_i(B_\Delta) = A(t) P^r_i(\tau) + \sum_{|\kappa'| < r} (2\pi i)^{|\kappa'| - r} \tau^{\kappa'} \sum_{\mu} \left( \sum_{|\kappa''| = r - |\kappa'|} c_{\kappa'\kappa''} b_\mu^{\kappa''} \right) t^\mu$$

where $(\wedge) P^r_i$ are $\mathbb{Z}$-homogeneous polynomials of degree $r$ and $c_{\kappa'\kappa''} \in \mathbb{Z}$.

**Remark 2.1.** (i) The full assertion for the $\mathbb{C}$-periods holds without the Hyperplane Conjecture for $r \leq \min \left\{ \frac{n-1}{2}, 1 \right\}$. Writing $A' \subset A$ for the points (if any) interior to facets, the periods cannot depend on the $\{\log(a_i)\}_{i \in A'}$ because taking $a_i \to 0$ does not make $X_{\tilde{a}}$ singular. Moreover, $X^0$ avoids the corresponding (exceptional) $\{D_i\}$; so there are $M - |A'|$ independent $\{\tilde{\psi}^1_i\}$ (with leading terms $A(t)$ times the $M - |A'|$ independent linear combinations of the $\{\tau_j\}$ with no such $\log(a_i)$’s). But these must all be periods since (by mirror symmetry) there are $h_{tr}^{1,1}(X_{\tilde{a}}) = h_{alg}^{1,1}(X^0) = M - |A'|$ independent periods.

(ii) Moreover, applying $r \{N_i = \log(T_i)\}$ to a $\mathbb{Z}$-period with “$\log^r(t)$” leading term must yield a $\mathbb{Z}$-multiple of $A(t)$. So (a fixed integer multiple of) this period must be a $\mathbb{Z}$-linear combination of the $\{\psi^r_i(B_\Delta)\}$ plus a $\mathbb{C}$-linear combination of the $\{\psi^r_i(B_\Delta)\}_{r < r}$. If all of the $\mathbb{C}$-linear combinations that can appear are themselves $\mathbb{C}$-periods, they can be subtracted off. So one in fact has a basis of $\mathbb{C}$-periods of the form $\{2.2\}$ for $r \leq \min \left\{ \frac{n+1}{2}, 2 \right\}$.

Call $\ell(n) := \sum n_k \ell(k)$ effective (resp. quasi-effective) if all $\ell_i(n)$ $(i > 0)$ are $\geq 0$ (resp. at most one $< 0$). Clearly the

$$a_n = \begin{cases} \frac{(-\ell(\ell(n)))!}{\prod_{i > 0} \ell_i(n)!}, & \ell(n) \text{ effective} \\ 0, & \text{otherwise}, \end{cases}$$

being multinomial coefficients, are all integers. Now using [2.1]:
• If $\ell(n)$ is effective, then the $\left(\partial_{D_0}^{\ell(n)} \cdots \partial_{D_N}^{\ell(n)} B_{\ell(n)}(\mathfrak{a})\right)/(a_\mathfrak{a})$ are $\mathbb{Z}$-linear combinations of products

$$\prod_{i=0}^{N} \prod_{P \in \mathbb{Z}_{\ge 0}(\ell(n))} \ell_i(n) / \prod_{j=1}^{\ell_i(n)} j^{-p_j}.$$ 

Since $b_{\mathfrak{a}}^{\ell''}/a_\mathfrak{a}$ is a $\mathbb{Z}$-linear combination of these with $|\ell| = |\ell''|$, $b_{\mathfrak{a}}^{\ell''}$ can be written as $P/Q$ ($P, Q \in \mathbb{Z}$) with $Q \mid L_{|\ell_0(\mathfrak{a})|}$, where $L_n := \text{lcm}\{1, \ldots, s\}$. 

• If $\ell(n)$ is quasi-effective, with (say) $\ell_1(n) < 0$, then $\left(\partial_{D_1} B_{\ell(n)}(\mathfrak{a})\right) = \frac{(-\ell_0(n))! (-\ell_1(n))!}{\Pi_{i>1} \ell_i(n)!} = \frac{(\Sigma_i \ell_i(n))!}{(\Sigma_i \ell_1(n))! (\ell_i(n) - 1)!}$ is the quotient of a multinomial coefficient by an integer of the form $A!/(B - 1)! (A - B)!$, which always divides $L_A$. Repeated differentiation as above now shows that $b_{\mathfrak{a}}^{\ell} = P/Q$ with $Q \mid L_{|\ell_0(\mathfrak{a})|}$, where $|\ell| = \sum_{E \ell_i > 0} \ell_i (\geq -\ell_0)$.

**Definition 2.2.** A reflexive polytope $\Delta \subset \mathbb{R}^n$ is *facile* if:

• $\Delta$, $\Delta^o$ admit regular projective triangulations;
• the Mori cone $M_{\Delta^o}$ is regular simplicial, with generators $\ell^{(k)}$;
• $\ell(n) := \sum n_k \ell^{(k)}(n)$ is quasi-effective for each $n \in \mathbb{Z}^M_{\ge 0}$; and
• $n \leq 3$, or the Hyperplane Conjecture holds for $\Delta$.

Now let $W_*$ denote the monodromy weight filtration for the large complex structure limit, with

$$h_i := \text{rk} \left(\text{Gr}_{2i}^W H^{n-1}(X_2)\right) = \text{rk} \left(\text{Gr}_{2i}^W H_{n-1}(X_2)\right) = \text{rk} \left(H^{i,n-i-1}(X_2)\right)$$

(a very general), which is 1 for $i = 0$ or $n - 1$.

**Theorem 2.3.** If $\Delta$ is facile, there exists a basis of $H_{n-1}(X_2, \mathbb{C})$, of the form $\gamma_{r,\mu}^* \in W_2(r-n+1) H_{n-1}(X_2, \mathbb{C})$ ($r = 0, \ldots, n - 1; \mu = 1, \ldots, \ell_r$), with periods

$$(2.3) \quad \pi_{r,\mu} := \int_{\gamma_{r,\mu}^*} \omega_2 = \sum_{|z|=r} c_{r,\mu}^z \sum_{a_\mathfrak{a}} a_\mathfrak{a} t_\mathfrak{a} + \sum_{|z|<r} \frac{1}{(2\pi i)^{|z|+2}} \sum_{n \in \mathbb{Z}_{\ge 0}^M} \beta_{r,\mu}^* \mathfrak{a}_n \mathfrak{a}_n^*,$$

where each $a_\mathfrak{a} \in \mathbb{Z}$, each $c_{r,\mu}^z \in \mathbb{Z}$, $\beta_{r,\mu}^* = P/Q$ ($P, Q \in \mathbb{Z}$) with $Q \mid L_{|\ell_0(\mathfrak{a})|}$, and (for each $r, \mu$) some $c_{r,\mu}^z \neq 0$. Moreover, this basis becomes $Q$-rational in the associated graded $\oplus G_{2i}^W$.

For the irrationality proofs, we need to apply this to certain 1-parameter families.
Definition 2.4. A facile CY pencil is a family of anticanonical hypersurfaces $X_\xi = \{ \phi(\xi) = 0 \} \subset \mathbb{P}_\Delta$ parametrized by $\xi \in \mathbb{P}^1$, where:

- $\Delta \subset \mathbb{R}^n$ is a facile polytope;
- $\phi(\xi, z) := \sum_{m \in A_\Delta} \xi^m P_m(\xi) z^m$, with $a_0 = 0$ and $a_m > 0$ for $m \neq 0$, $P_m(\xi) \in \mathbb{Z}[\xi]$ with $\gcd_{m \in A}\{P_m(\xi)\} = 1$, and (if $n > 1$) $\prod_{i=0}^N P_{\ell(i)}(0)^{\ell(k)} = 1 \ (\forall k), \ P_0(0) = 1$;
- the VHS on $H^{n-1}(X_\xi)$ is pure, with a factor $H_{tr}^{n-1}(X_\xi) =: \mathcal{W}_\phi$ with Hodge numbers $h^{n-1,0} = h^{n-2,1} = \ldots = h^{0,n-1} = 1$; and
- $\sum_{i=0}^N a_{\ell(i)}(n) \geq |\ell(n)| \ + \ \forall n \in \mathbb{Z}_{\geq 0}$.

In particular, note that $\mathcal{W}_\phi$ has maximal unipotent monodromy at $\xi = 0$. Since the resulting $t_k = \xi \sum a_{\ell_i} g_k(\xi)$ with $g_k(\xi) \in 1 + \xi \mathbb{Z}[\xi]$, and $2\pi i t_k = (\sum a_{\ell_i} \xi^{\ell_i}) \log \xi + \sum m > 0 \frac{h_{km}}{m} \xi^m \ (h_{km} \in \mathbb{Z})$, while also $P_0(0) = 1 \in 1 + \xi \mathbb{Z}[\xi]$, substituting into (2.3) and normalizing yields at once the

Corollary 2.5. Near $\xi = 0$, any facile CY pencil admits a multivalued basis $\{ \gamma_j \}_{j=0}^{n-1}$ of $\mathcal{W}_\phi^{\gamma_0} \cong H^{n-1}_{tr}(X_\xi, \mathbb{C})$ ($\mathbb{Q}$-rational in $\text{Gr}^W_*$), holomorphic functions $f^{(j)}(\xi) = \sum_{m \geq 0} \xi^m f_m^{(j)},$ and an integer $\varepsilon \in \mathbb{N}$, such that the periods

$$\Pi_\ell := \int_{\gamma_\ell} t_k := \int_{\gamma_\ell} \frac{1}{(2\pi i)^n} \text{Res}_{X_\xi} \left( \frac{dx_1 \wedge \ldots \wedge dx_n}{\phi(\xi, z)} \right)$$

take the form

$$\Pi_\ell(\xi) = \frac{1}{(2\pi i)^\ell} \sum_{j=0}^\ell \frac{1}{j!} \log^j(\xi) f^{(\ell-j)}(\xi),$$

with each $\varepsilon L^{(j)} \in \mathbb{Z}$, and $f^{(0)} = 1$.

Note that in this scenario, $\gamma_j$ generates $\text{Gr}^{\varepsilon}_{2(j-n+1)}$, the monodromy logarithm $N$ sends $\gamma_{n-1} \mapsto \gamma_{n-2} \mapsto \ldots \mapsto \gamma_1 \mapsto \gamma_0$, and $\gamma_0$ generates the local invariant cycles in $H^{n-1}_{tr}(X_\xi, \mathbb{Z})$.

3. Very special values

Let $\phi \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \ (n \geq 1)$ be an integral Laurent polynomial, with reflexive Newton polytope $\Delta = \Delta_\phi$, and $\mathbb{P}_\Delta$ be the (possibly singular) toric variety associated to a maximal projective triangulation of $\Delta^*$ \text{[Ba]}. We begin by defining several notions we shall require for the general irrationality statement.

\footnote{this is automatic if $\ell(n)$ is effective.}
Denote by $\mathcal{X}_\phi \to \mathbb{P}^1$ the Zariski closure in $\mathbb{P}_\Delta \times \mathbb{P}^1_t$ of $\mathcal{X}_\phi^\lambda := \{1-t\phi(x) = 0\} \subset \mathbb{G}_m^\alpha \times (\mathbb{P}^1_\xi \setminus \{0\})$, and write $t = \lambda^{-1}$, $X_{\phi,t} := X_{\phi}^\lambda := \rho^{-1}(t)$. (Note that $\rho$ is given by $1/\phi$ resp. $\phi$ when working in $t$ resp. $\lambda$.) We shall call $\phi$ involutive if there exists a birational map $\mathcal{I} : \mathcal{X}_\phi \to \mathcal{X}_\phi$ over $t \mapsto \pm t^{-1}$, defined over $\mathbb{Q}$. Further, $\phi$ is said to admit a facile $r$-cover ($r \in \mathbb{N}$) if there is a facile CY pencil $X = \{\phi(\xi, y) = 0\} \subset \mathbb{P}_\Delta \times \mathbb{P}^1_\xi$ and a dominant (genrically 1) rational map $\mathcal{J} : X \to \mathcal{X}_\phi$ over $\xi \mapsto \xi^r = t$. We say that $\phi$ is of conifold type if\footnote{When $\phi$ is not $\Delta$-regular, we also assume that no non-generic singularities of $X_\phi^\lambda$ along the base locus $X_\phi^\lambda \cap \Delta_\Delta$ occur for values $\lambda \in D_\phi$ (hence don’t affect the local system $\mathcal{V}_\phi$), see below.} the nonzero critical values of $\phi : (\mathbb{C}^\times)^n \to \mathbb{C}$ underlie (isolated) ordinary double points in the fibers.

While we will not assume that $1-t\phi$ is $\Delta$-regular for general $t$, or even that the generic fiber of $\rho$ is smooth, we shall impose the conditions that the variation $\mathcal{H}_\phi$ underlain by $R^{n-1}\rho_* \mathbb{Q}$ is pure (of weight $(n-1)$)\footnote{For $n = 1$, the generic $X_{\phi,t}$ is a pair of points, and $H^0[\text{resp. } H_0]$ means everywhere the augmentation cokernel [resp. kernel], i.e., reduced (co)homology. Involutivity and reflexivity imply that $x\phi(x)$ is a quadratic polynomial with two distinct roots.} and that the minimal sub-VHS $\mathcal{V}_\phi$ containing the class of

$$\omega_{\phi,t} := \omega_\phi^\lambda := \frac{1}{(2\pi i)^n} \text{Res}_{\mathcal{X}_\phi^\lambda} \left( \frac{dx_1/x_1 \wedge \cdots \wedge dx_n/x_n}{1-t\phi(x)} \right) \in \Omega^{n-1}(X_\phi^\lambda)$$

is of rank $n$, with Hodge numbers $\{h_{i,n-i-1}\}_{0 \leq i \leq n-1}$ all 1. In this case we call $\phi$ principal (since $\mathcal{V}_\phi$ is a “principal VHS”\footnote{With no information on exponents, one would have $P_\phi(t) \delta_t^n + \sum_{k=0}^{n-1} f_{\phi,k}(t) \delta_t^k \in \mathbb{C}[t, \delta_t]$. The existence of holomorphic solutions with orders 0 thru $n-2$ about each root $t$ of $P_\phi$ forces $(t-t)^{n-k-1} \mid F_k$. Note that if $n$ is even, the exponent $\frac{n-2}{2}$ is repeated.} $[\text{Ro}]$). Denote the singularity (discriminant) locus of $\mathcal{V}_\phi$ by $D_\phi$; since $X_{\phi,0} = \Delta_\Delta$, $\mathcal{V}_\phi$ has maximal unipotent monodromy at $t = 0$, and so $0 \in D_\phi$. Write $D_\phi^\times := D_\phi \cap \mathbb{G}_m$, $P_\phi(t) := \prod_{t_0 \in D_\phi^\times} \left( \frac{t}{t_0} - 1 \right)$, $\delta_\phi := |D_\phi^\times| = \deg(P_\phi)$, and $\tau_\phi = \min\{|t| \mid t \in D_\phi^\times\}$. For $\phi$ of conifold type, the monodromy of $\mathcal{V}_\phi$ about each point of $D_\phi^\times$ is given by a single Picard-Lefschetz transformation. Since the local exponents are then given by $\{0, 1, \ldots, n-2\} \cup \{\frac{n-2}{2}\}$, an easy calculation\footnote{With no information on exponents, one would have $P_\phi(t) \delta_t^n + \sum_{k=0}^{n-1} f_{\phi,k}(t) \delta_t^k \in \mathbb{C}[t, \delta_t]$. The existence of holomorphic solutions with orders 0 thru $n-2$ about each root $t$ of $P_\phi$ forces $(t-t)^{n-k-1} \mid F_k$. Note that if $n$ is even, the exponent $\frac{n-2}{2}$ is repeated.} shows that the Picard-Fuchs operator for $\omega_\phi$ takes the form

$$D_\phi(t) := P_\phi(t) \delta_t^n + \sum_{k=0}^{n-1} f_{\phi,k}(t) \delta_t^k \in \mathbb{C}[t, \delta_t].$$
For $n > 2$ we remark that $\rho$ is not semistable over 0 (and possibly at other points of $D_\phi$) without blowing up $\mathbb{P}_\Delta$ along the singularities of the base locus, but this won’t be an issue for us.

The isomorphism $X^\phi_\phi \to \mathbb{G}_m^n$ provides $n$ coordinates $x_i \in \mathcal{O}(X^\phi_\phi) \cong H^1_{\mathcal{M}}(X^\phi_\phi, \mathbb{Q}(1))$ whose cup product produces the coordinate symbol $\{\exists\} = \{x_1, \ldots, x_n\} \in H^1_{\mathcal{M}}(X^\phi_\phi, \mathbb{Q}(n))$. We shall call $\phi$ tempered if this lifts to a motivic cohomology class $\Xi \in H^1_{\mathcal{M}}(X^\phi_\phi, \mathbb{Q}(n))$, where $X^\phi_\phi := X_\phi \setminus X^\infty_\phi$. Writing $T_{x_i}$ [resp. $T_{\{\exists\}}$] for the analytic chain of $x_i^{-1}(\mathbb{R}_-)$ [resp. $\cap_{i=1}^n x_i^{-1}(\mathbb{R}_-)$] and $U_\phi := \mathbb{P}^1 \setminus \rho(T_{\{\exists\}})$, we term $\phi$ strongly tempered if the higher normal function (HNF) defined by

$$\nu_{\phi,t} := \text{A}J_{X^\phi_\phi,t}(\Xi|_{X^\phi_\phi}) \in H^{n-1}(X^\phi_\phi, \mathbb{C}/\mathbb{Q}(n))$$

has a single-valued holomorphic family of lifts

$$\tilde{\nu}_{\phi,t} \in H^{n-1}(X^\phi_\phi, \mathbb{C})$$

over $U_\phi$. We shall only really care about the $V_\phi$-component of $\nu_\phi$ below, so one might as well regard it as an element of $\text{ANF}(V_\phi(n))$. Note that any lift (such as $\tilde{\nu}_\phi$) of this component must have nontrivial monodromy on $\mathbb{P}^1 \setminus D_\phi$ (as opposed to just $V_\phi$): otherwise its topological invariant $[\nu_\phi] \in \text{Hom}(\mathbb{Q}(0), H^n(\mathbb{P}^1 \setminus D_\phi, \mathbb{Q}(n)))$ would vanish. Since the latter is computed by $\Omega_{\Xi}$, which restricts to the Haar form on $X^\phi_\phi \cong \mathbb{G}_m^n$, this is absurd.

Now either $\omega_{\phi,t}$ or $\tilde{\omega}_{\phi,t} := t\omega_{\phi,t}$ is a holomorphic section of the canonically extended Hodge bundle $\mathcal{F}_\phi := F^{n-1}V_{\phi,e}$. As residue forms, they are really most naturally regarded as elements of $H_{n-1}(X^\phi_\phi, \mathbb{C})$ for any $t \in \mathbb{P}^1$. If $t \notin D_\phi$, then purity of $\mathcal{H}_\phi$ makes $H^{n-1} \to H_{n-1}$ an isomorphism, allowing us to treat them as cohomology classes; but over all of $\mathbb{P}^1$ or $U_\phi$, they only make sense as sections of $H_{n-1}$. Conveniently enough, this pairs with $H^{n-1}(X^\phi_\phi, \mathbb{C})$, allowing us to define

$$\tilde{V}_\phi(\lambda) := \langle \tilde{\nu}^\lambda_\phi, \tilde{\omega}^\lambda_\phi \rangle \in \mathcal{O}(U_\phi).$$

This extends to a multivalued-holomorphic function

$$V_\phi(\lambda) := \langle \nu^\lambda_\phi, \omega^\lambda_\phi \rangle \mod \mathcal{F}_\phi \mathbb{Q}(n)$$

\footnote{Note that if we assume this only over $U_\phi \setminus U_\phi \cap D_\phi$, the lift extends to $U_\phi$ anyway: the single-valuedness of $\tilde{\nu}$ on a punctured disk about $t_0$ means $\nu$ has no singularity at the center, so that $\tilde{\nu}$ uniquely extends to the whole disk. The value $\tilde{\nu}(t_0)$ lies in $\ker(T - I)$ in the canonical extension $H^\text{sm}_\phi$, which contains the image of $H^{n-1}(X^\phi_\phi, \mathbb{C})$; see [FKK §5].}

\footnote{Here and below, we use the polarization $Q$ to make this identification (up to twist).}
on $\mathbb{P}^1 \setminus D_\phi$, defined up to $\mathbb{Q}(n)$-periods of $\tilde{\omega}_\phi^\lambda$; we shall refer to both $V$ and $\tilde{V}$ as the truncated higher normal function (THNF) associated to $\Xi$ and $\tilde{\omega}_\phi$. Note that $\tilde{V}_\phi$ cannot extend to an entire function, since $\tilde{u}_\phi$ has nontrivial monodromy on $\mathbb{P}^1 \setminus D_\phi$ and $\tilde{\omega}_\phi$ has $n$ independent periods (courtesy of the maximal unipotent monodromy).

We can now state the main result of this section:

**Theorem 3.1.** Let $\phi(x_1, \ldots, x_n)$ be an integral Laurent polynomial, such that $\phi(-\bar{x})$ has all positive coefficients, which is reflexive, involutive, principal, strongly tempered, of conifold type, and admits a facile $r$-cover. Assume that $\delta_\phi = 2$, $D_\phi \in \mathbb{Z}[t, \delta_i]$, and $\tau_\phi < e^{-n}$. Then $\tilde{V}_\phi(0) \notin \mathbb{Q}^\times$.

### 3.1. Proof of Theorem 3.1

**Step 1: The power series.** By involutivity, the four points of $D_\phi$ have $\lambda$-values $0, \lambda_0, \pm \lambda_0^{-1}, \infty$, with $|\lambda_0| = \tau_\phi < 1 < |\lambda_0|^{-1}$. Moreover, the $\mathbb{Z}$-local system $V_\phi$ underlying $V_\phi$ has maximal unipotent monodromy at $\lambda = 0$, with (rank 1) invariant subsystem on the disk $D_{\tau_\phi}$ generated by a family of $(n - 1)$-cycles $\varphi_0^\lambda$. Indeed, we may assume that $\varphi_0^0 = \mathcal{I}_e^* \varphi_0^I(\lambda)$ where Tube($\varphi_0$) = $\mathcal{T}_n := \{|x_1| = \cdots = |x_n| = 1\} \subset \mathbb{P}_\Delta \setminus X_\phi^I(\lambda)$. As the Hodge bundle $\mathcal{F}_e$ has degree 1 [GGK], and $\omega_\phi \in \Gamma(\mathbb{P}^1, \mathcal{F}_e)$ has a zero at $\infty$, we have $(\mathcal{I}^*\omega)/\omega = M\lambda^{-1}$ for some $M \in \mathbb{Q}^\times$. But then $(\mathcal{I}^*\omega)^2 = \mathcal{I}^* M^2 \lambda^2 \omega = \pm M^2 \omega$, and since $(\mathcal{I}^*)^2$ acts on $\mathcal{V}_\phi$ (and $\mathcal{V}_\phi^C$ is irreducible), we must have $M^2 \in \mathbb{Z}^\times = \{\pm 1\}$. This forces $M = \pm 1$ hence $\mathcal{I}^*\omega = \pm \tilde{\omega}$. The holomorphic period at $\lambda = 0$ is therefore

$$A_\phi(\lambda) := \int_{\varphi_0^0} \tilde{\omega}_\phi^\lambda = \int_{\varphi_0^I(\lambda)} \mathcal{I}^* \tilde{\omega}_\phi^\lambda = \pm \int_{\varphi_0^I(\lambda)} \omega_\phi, \pm \lambda = \pm \int_{\varphi_0^I(\lambda)} \omega_\phi, \pm \lambda \omega = \pm \int_{\varphi_0^I(\lambda)} \omega_\phi, \pm \lambda \omega = \pm \int_{\varphi_0^I(\lambda)} \omega_\phi, \pm \lambda \omega = \pm \int_{\varphi_0^I(\lambda)} \omega_\phi, \pm \lambda \omega,$$

and of course we may change the signs of $\varphi_0$ and $\lambda$ if needed so that

(3.1) $$A(\lambda) = \sum_{k \geq 0} |\varphi^k|_0 \lambda^k := \sum_{k \geq 0} a_k \lambda^k$$

(where $[\cdot]_0$ denotes “constant term”), with $a_0 = 1$. By the fundamental result of [DvdK], we have $r_A := \left(\limsup_{k \to \infty} |a_k|^{1/k}\right)^{-1} \in \mathbb{C}^\times$; and since $A$ is a period of $(V_\phi, \tilde{\omega}_\phi)$, $r_A$ must be the modulus of an element of $D_{\phi}^\times$ (i.e., $r_\phi$ or $r_\phi^{-1}$). Since the $[\varphi^k]_0$ are all (positive) integers, $r_A = r_\phi$. 


On the other hand, positivity of the coefficients of \( \phi(-x) \) forces \( \phi(T_x) \subset [1, \infty] \). Now the global minimum of \( \phi(T_x) \) necessarily occurs at a critical point of \( \phi \); more precisely, it may be regarded as the terminus of a Lefschetz thimble on the generator \( \partial T_x \) of \( V_\phi^{0, \text{lim}} / N V_\phi^{0, \text{lim}} \cong \text{Gr}_0^W H_{n-1}(X^0_\phi) \), and thus is \( \pm \) the \( \lambda \)-coordinate of a point of \( D_\phi^* \). Clearly this must be the larger one, so that \( U_\phi \) contains the disk \( D_{\phi^*} \) about \( \lambda = 0 \). Writing

\[
\tilde{V}_\phi(\lambda) =: \sum_{k \geq 0} v_k \lambda^k,
\]

it follows that \( r_V := \left( \limsup_{k \to \infty} |v_k|^{1/k} \right)^{-1} = r_\phi^{-1} \). Moreover, since \( A(\lambda) \) is (up to scale) the only period of \( \tilde{w}_\phi \) invariant about \( \lambda = 0 \), and it is not invariant about \( \lambda = \lambda_0 \), we conclude that \( \tilde{V}_\phi(\lambda) \) [resp. \( \tilde{V}_\phi(\lambda) \)] is the unique analytic continuation of \( V_\phi(\lambda) \) [resp. holomorphic lift of \( v_\phi \)] which is well-defined on \( U_\phi \).

In particular, \( \tilde{V}_\phi(0) = v_0 \in \mathbb{C} \) is well-defined. Assuming henceforth that \( v_0 \neq 0 \), we may consider

\[
B_\phi(\lambda) := -\tilde{V}_\phi(\lambda) + \tilde{V}_\phi(0) A_\phi(0) =: \sum_{k \geq 1} b_k \lambda^k.
\]

Obviously this has radius of convergence \( r_B = r_A < 1 \), while \( v_k = v_0 a_k - b_k \to 0 \) as \( k \to 0 \). Restricting if necessary to a subsequence \( a_{k_j} \to \infty \), we therefore have

\[
\lim_{j \to \infty} b_{k_j} a_{k_j} = v_0.
\]

**Step 2: The inhomogeneous equation.** Since the holomorphic period of \( \tilde{w} \) about \( \lambda = 0 \) is obtained by substituting \( \lambda \) for \( t \) in the holomorphic period of \( \omega \) about \( t = 0 \), the same goes for the (inhomogeneous) Picard-Fuchs equations: that is,

\[
D_\phi(\lambda) \tilde{\rho}_\phi^C = 0.
\]

Consequently, in \( \mathcal{V}_\phi \) we have

\[
P_\phi(\lambda) \nabla^n_{\delta_\phi} [\tilde{w}_\phi] = - \sum_{k=0}^{n-1} f_{\phi,k}(\lambda) \nabla^k_{\delta_\phi} [\tilde{w}_\phi];
\]

and regarding \( \tilde{w}_\phi \) as a section of \( \mathcal{V}_{\phi,e} \), maximal unipotent monodromy at \( \lambda = 0 \) and \( \text{Res}_0 \nabla = \frac{1}{2\pi i} N \) imply the independence of the \( \{ \nabla^k_{\delta_\phi} [\tilde{w}_\phi] \}_{k=0}^{n-1} \) in the fiber \( \mathcal{V}_{\phi,e} \). As \( N^n = 0, \nabla^n_{\delta_\phi} [\tilde{w}_\phi] \) vanishes in \( \mathcal{V}_{\phi,e}^0 \), and so all

\footnote{Recall that \( \tilde{V}_\phi \) cannot be entire.}
basis punctured disk

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(3.3) \[ Y(\lambda) := Q \left( \tilde{\omega}_\phi^\lambda, \nabla_{\tilde{\omega}_\phi^\lambda}^{-1} \tilde{\omega}_\phi^\lambda \right) \in \mathbb{C}(\lambda)^\times. \]

Let \( w \) be a local coordinate at a point \( p \in D_\phi^\circ \), about which monodromy is described by \( C \mapsto C - \langle C, \sigma \rangle \sigma \) (since \( \phi \) is of confoid type). Writing \( \tilde{\omega}_\phi^\lambda \) with respect to a local basis \( \{ \sigma_0, \ldots, \sigma_{n-1} \} \) of \( \mathcal{V}_\phi \) with \( \sigma_1, \ldots, \sigma_{n-1} \) invariant about \( p \) and \( \sigma = \sigma_0 \) [resp. \( \sigma_1 \)] for \( n \) odd [resp. even] (with \( \sigma_0 \mapsto -\sigma_0 \) resp. \( \sigma_0 \mapsto \sigma_0 - \sigma_1 \)), the coefficients \( u_i(w) \) of \( \{ \sigma_1, \ldots, \sigma_n \} \) are holomorphic and we have (near \( p \)) \( u_0(w) \sim w^{-n/2} \) [resp. \( u_0(w) \sim u_1(w) \log(w) \sim w^{-n/2} \log(w) \)]. From this we compute \( Y \sim w^{n/2-n(n-1)} \times w^{-n/2} = w^{-1} \), so that \( Y \) has (at worst) simple poles at \( D_\phi^\circ \). Moreover, the pairing (3.3) makes sense in \( \mathcal{V}_{\phi,e}^0 \), so that \( Y(0) \neq \infty \); while \( \mathcal{T}^* \tilde{\omega}_\phi^\lambda = \pm \lambda \tilde{\omega}_\phi^\lambda \implies Y(\mathcal{T}(\lambda)) = \pm \lambda^2 Y(\lambda) \implies \text{ord}_\infty(Y) \geq 2 \). Since \( Y \) is rational with only 2 simple poles, it can have only the double zero at \( \infty \), and thus

\[ Y(\lambda) = \frac{Y(0)}{P_\phi(\lambda)}. \]

To evaluate \( Y(0) \), and also anticipating Step 3, we extend \( \varphi_0^\lambda \) to a basis \( \{ \varphi_j^\lambda \}_{j=0}^{n-1} \) of \( (\mathcal{V}_\phi^\lambda)^* \cong H_{n-1}(X_\phi^\lambda, \mathbb{Q}) \) on an angular sector of the punctured disk \( D_\phi^\circ \) satisfying \( \varphi_{n-j} = N^j \varphi_{n-1} \) hence \( \varphi_j \in W_{2(j-n+1)} \) (\( N = \log(T) \) the monodromy logarithm at \( 0 \), and \( W_* = W(N)[n-1] \)). From \( NQ = -QN \) and \( N^n = 0 \), the \( Q(\varphi_j, \varphi_k) \) are zero for \( j + k < n - 1 \), and the \( (-1)^j Q(\varphi_j, \varphi_{n-j-1}) \) equal a common constant \( Q_0^{-1} \in \mathbb{Q}^\times \). We may then modify \( \varphi_j \mapsto \hat{\varphi}_j = \varphi_j + \sum_{i<j} \alpha_{ij} \varphi_i \in W_{2(j-n+1)} \) (\( \alpha_{ij} \in \mathbb{Q} \)) so that \( Q(\hat{\varphi}_j, \hat{\varphi}_{n-j-1}) = (-1)^j Q_0^{-1} \delta_{ij} \), with \( N\hat{\varphi}_j = \hat{\varphi}_{j+1} + \sum \eta_{ij} \hat{\varphi}_i \) (\( \eta_{ij} \in \mathbb{Q} \)) and dual basis \( \{ \hat{\varphi}_j \} \) of \( \mathcal{V}_\phi^Q \). (Note that \( \hat{\varphi}_j \in W_{2(n-j-1)} \mathcal{V}_\phi \), \( -N\hat{\varphi}_j = \hat{\varphi}_{j+1} + \sum \eta_{ij} \hat{\varphi}_i \), while \( Q(\hat{\varphi}_{n-1}, \hat{\varphi}_j) = (-1)^j Q_0 \delta_{ij} \). Writing locally

\[ [\tilde{\omega}_\phi^\lambda] = \sum_{j=0}^{n-1} \left( \int \hat{\varphi}_j \tilde{\omega}_\phi^\lambda \right) \hat{\varphi}_j =: \sum_{j=0}^{n-1} \hat{\pi}_j(\lambda) \hat{\varphi}_j \]

One may also show that \( g_{\phi,n-1}(\lambda) = \pi P_\phi'(\lambda) \), but we won’t need this.
in \( V_\phi \), and \( \pi_j(\lambda) := \int_{\varphi_j} \tilde{\omega}_\phi^{\lambda} \), we have
\[
\pi_0(\lambda) = \tilde{\pi}_0(\lambda) = A_\phi(\lambda) = A_\phi^{(0)}(\lambda)
\]
and
\[
\tilde{\pi}_j = \pi_j(\lambda) + \sum_{i<j} \alpha_{ij} \pi_i(\lambda);
\]
while in accordance with the monodromy properties of \( \varphi_j \),
\[
\pi_j(\lambda) = \sum_{k=0}^{j} \frac{\log^k(\lambda)}{(2\pi i)^k k!} A_\phi^{(j-k)}(\lambda)
\]
for some functions \( A_\phi^{(\ell)}(\lambda) \) holomorphic on \( D_\phi \). Clearly, the limits
\[
\lim_{\lambda \to 0} \log^j(\lambda) (\delta^{n-1}_\lambda \tilde{\pi}_j)(\lambda)
\]
are zero for \( j < n - 1 \), while we have
\[
\lim_{\lambda \to 0} (2\pi i)^{n-1} (\delta^{n-1}_\lambda \tilde{\pi}_{n-1})(\lambda) = 1 = A_\phi^{(0)}(0)
\]
and so \( Y(0) = (2\pi i)^{1-n} \times Q(\tilde{\varphi}_0, \tilde{\varphi}_{n-1}) = \pm Q_0 \).

Now the fiberwise restrictions \( \Xi^\lambda := \Xi|_{X_\phi^\lambda} (\lambda \neq \infty) \) have trivial
class in \( \text{Hom}_{\text{HHS}} \left( \mathbb{Q}(0), H^n(X_\phi^\lambda, \mathbb{Q}(n)) \right) \), so that \( T_\Xi \) is a coboundary
over any sufficiently small \( B \subset \mathbb{A}^1_\lambda \). Since the regulator current \( R_\Xi \) has
\( dR_\Xi = \frac{dx_1}{x_1} \wedge \frac{dx_n}{x_n} - (2\pi i)^n \delta_{T_\Xi} \) on \( X_\phi^{\lambda} \), writing \( T_\Xi|_{\rho^{-1}(B)} \equiv \partial \Gamma_B \) (mod
\( \partial \rho^{-1}(B) \)) yields a current \( \tilde{R}_B = R_\Xi|_{\rho^{-1}(B)} + (2\pi i)^n \delta_{\Gamma_B} \) with closed
fiberwise pullbacks. This yields
\[
\nabla_{\delta_\lambda} \nu^\lambda_\phi = \nabla_{\delta_\lambda} [\tilde{R}_B] = \left[ \left( \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right) |_{X_\phi^\lambda} \right] = -(2\pi i)^{n-1} [\omega_\phi^\lambda].
\]
Using this (and \( Q(\omega, \nabla_{\delta_\lambda} \omega) = 0 \) for \( k < n - 1 \)), one computes that
\[
\delta^k_{\lambda} V(\lambda) = \delta^k_{\lambda} Q(\nu, \tilde{\omega}) = Q(\nu, \delta^k_{\lambda} \tilde{\omega})
\]
for \( k < n \) and \( -(2\pi i)^{n-1} Q(\omega, \nabla_{\delta_\lambda} \omega) = Q(\nu, \nabla_{\delta_\lambda} \omega) \) for \( k = n \). Using \( (3.3) \), we have at once
\[
D_\phi(\lambda) V_\phi(\lambda) = \pm (2\pi i)^{n-1} \lambda Y(\lambda) P_\phi(\lambda) = \pm Q_0 \lambda.
\]
Notice that while \( V_\phi(\lambda) \) is multivalued, the ambiguities are killed by
\( D_\phi(\lambda) \). Moreover, \( \tilde{V}_\phi(\lambda) \) satisfies the same equation, so that
\( D_\phi(\lambda) A_\phi(\lambda) = 0 \implies \)
\[
D_\phi(\lambda) B_\phi(\lambda) = \mp Q_0 \lambda.
\]
Step 3: Arithmetic of coefficients. By \( (3.1) \) and the integrality of \( \phi \), we have \( a_m \in \mathbb{Z} \) (\( \forall m \)). Expressing \( b_m \) as \( \frac{p_m}{q_m} \) with \( p_m \in \mathbb{Z} \) and \( q_m \in \mathbb{N} \)
relatively prime, we claim that for some fixed \( \varepsilon_B \in \mathbb{N} \),
\[
q_m \mid \varepsilon_B (m!)^n \quad (\forall m).
\]
Indeed, if we write $P_\phi(\lambda) = c'_n \lambda^2 + c'_n \lambda + 1$ and $g_{\phi,i}(\lambda) = c'_j \lambda + c''_j$ (all $c'_n, c''_n \in \mathbb{Z}; c'_n = \pm 1$), substituting $\sum_{m \geq 1} b_m \lambda^m = B_\phi(\lambda)$ in (3.6) yields $b_1 = \pm Q_0 \in \mathbb{Q}$ and the recurrence

$$-m^n b_m = \left( \sum_{j=0}^n c'_j (m-2)^j \right) b_{m-2} + \left( \sum_{j=0}^n c''_j (m-1)^j \right) b_{m-1}.$$  

Taking $\varepsilon_B = q_1$, this establishes (3.7).

We claim that, in addition (modifying $\varepsilon_B$ if necessary),

$$q_m \mid \varepsilon_B L_{\ell,m}^n \quad (\forall m).$$

To show this, we make use of the facile $r$-cover $J$, which induces an isomorphism of VHS $J^* \mathcal{V}_\phi \cong \mathcal{W}_\phi$ hence of their extended Hodge bundles $\mathcal{O}_{\mathbb{P}^1}(r) \cong J^* \mathcal{F}_{\phi, \ell}^{n-1} \cong \mathcal{F}_{\phi, \ell}^{n-1}$ over $\mathbb{P}^1$, of which $J^* \omega_{\phi, \ell}$ and

$$'\omega_\ell = (2\pi i)^{1-n} \text{Res}_{X_\ell} \left( \prod_{x=2}^{n} \frac{dy_1/y_1 \wedge \cdots \wedge dy_n/y_n}{\phi(x_n)} \right)$$

are sections. Since $\text{gcd}_{m \in \mathbb{A}} \{ P_m(\xi) \} = 1$, $'\omega_\ell$ has no zeroes over $\mathbb{A}_{\ell}$, so that both sections share the divisor $r[\infty]$. Since $(J^\pm_\ell)_*$ exchanges the generators $\mathcal{I}_*(\mathcal{V}_\phi)$ and $\gamma_0$ of the integral invariant cycles about $\xi = 0$ and $\lambda = 0$, and $\lim_{t \to 0} \int_{\mathcal{I}_*}(\phi, t) = 1 = \lim_{\gamma \to 0} \int_{\gamma} '\omega_\ell$, we find that $'\omega_\ell = (J^* \omega_{\phi, \ell})$. Via $J^* \omega_\phi^{r, \ell} = \omega_{\phi, \ell}$, Corollary 2.5, and (3.4), we therefore have $(0 \leq \ell \leq n - 1)$

$$\hat{\pi}_\ell(\xi^r) = \int \tilde{\omega}_\phi^{r, \ell} = \int_{J^* \mathcal{I}_*} \omega_{\phi, \ell} = \int '\omega_\ell$$

for some $\hat{\gamma}_\ell = \sum_{k \leq \ell} (2\pi i)^{k-\ell} \beta_k \gamma_k \in W_{2(\ell-n+1)} \mathcal{W}_\phi' (\beta_k \ell \in \mathbb{C}; \beta_k \ell = r^\ell)$. That is,

$$\hat{\pi}(\xi^r) = \frac{1}{(2\pi i)^\ell} \sum_{m \geq 0, 0 \leq j \leq k \leq \ell} \frac{1}{j!} j_m(k-j) \beta_k \xi^m \log^j(\xi).$$

Returning to the $\lambda$-disk for a moment, we have $[\hat{\omega}_0^0] \equiv Q_0 \hat{\phi}_{n-1}$ in $H_{n-1}(X_\phi^0)$ so that $\hat{\nu}_0^0 = Q_0^{-1} v_0 \hat{\phi}_{n-1}$ in $H^{n-1}(X_\phi^0)$. This fixes the constant of integration, so that (3.5) gives

$$\hat{\nu}_\phi^\lambda - Q_0^{-1} v_0 \hat{\phi}_{n-1} = \int_0^\lambda \left( \nabla_\delta \nu_\phi^0 \right) \frac{d\lambda}{\lambda}$$

$$= -(2\pi i)^{n-1} \int_0^\lambda \left[ \hat{\omega}_0^0 \right] \frac{d\lambda}{\lambda}$$

$$= (2\pi i)^{n-1} \sum_{\ell=0}^{n-1} \int_0^\lambda \hat{\pi}_\ell(\lambda) d\lambda \hat{\phi}_{\ell}$$

(3.10)
in $V_\phi$. Since $Q(Q_0^{-1}v_0\hat{\omega}_{n-1},\hat{\omega}_\phi^n) = v_0A_\phi(\lambda)$ and $Q(\tilde{\nu}_\phi^n,\hat{\omega}_\phi^n) = \tilde{V}_\phi(\lambda)$, pairing (3.10) with $[\hat{\omega}_\phi^n] = \sum_{\ell=0}^{n-1}\hat{\nu}_\phi^\ell(\nu_\ell)\hat{\omega}_\phi^\ell$ yields the key formula

$$-B_\phi(\lambda) = (-2\pi i)^n q_0 \sum_{\ell=0}^{n-1} (-1)^\ell \hat{\nu}_{n-\ell-1}(\lambda) \int_0^1 \hat{\nu}_\ell(\lambda) d\lambda.$$  

Substituting $\xi^r = \lambda$, this becomes

$$\pm \sum_{\mu \geq 1} b_\mu \xi^{r\mu} = (2\pi i)^n q_0 \sum_{\ell=0}^{n-1} (-1)^\ell \hat{\nu}_{n-\ell-1}(\xi^r) \int_0^1 \hat{\nu}_\ell(\xi^r) r \xi^{r-1} d\xi,$$

and using $\int_0^1 \log^a(x) x^b dx = x^b \sum_{c=0}^a (\log x)^{a-c} (x)$ together with (3.9),

$$= q_0^r \sum_{m \geq 0} \xi^m \sum_{(*)_m} \frac{(-1)^\ell}{(m'+r)j'+1} \sum_{(k'^r)_{m'}} \beta_{k'^r,n-\ell-1} \beta_{k',\ell} \log^{j'+j'-i'}(\xi),$$

where the $\sum_{(*)_m}$ (finite for each $m$) is over $m' + m'' = m - r$, $0 \leq \ell \leq n - 1$, $0 \leq j' \leq k'' \leq n - \ell - 1$, and $0 \leq i' \leq j' \leq k' \leq \ell$. Since this plainly has to be a power series in $\xi^r$, the log $^a$ $\xi$ terms must cancel out (forcing $j'' = 0 = j' - i'$), leaving us with

$$(3.11) \quad = q_0^r \sum_{m \geq 0} \xi^m \sum_{(*)_m} \frac{(-1)^\ell}{(m'+r)j'+1} \sum_{(k'^r)_{m'}} \beta_{k'^r,n-\ell-1} \beta_{k',\ell}.$$  

Let $\{\sigma_1, \ldots, \sigma_d\}$ ($\sigma_1 = 1$) be a basis of the $\mathbb{Q}$-vector space $\mathcal{B}$ generated by all the products $\{\beta_{ij}\beta_{kl}\}$, and write $\beta_{ij}\beta_{kl} = \sum_{s=1}^d q_{ij}^{(a)} \sigma_s$. Since (3.11) $= \pm \sum_{\mu} b_\mu \xi^{r\mu}$ with $b_\mu \in \mathbb{Q}$ (and $f^{(a)}_b \in \mathbb{Q}$), the resulting “$\sigma_s$-components” of (3.11) vanish for $s > 1$, while

$$(3.12) \quad \pm b_\mu = q_0^r \sum_{(*)_{r\mu}} \frac{(-1)^\ell}{(m'+r)j'+1} \sum_{(k'^r)_{m'}} \beta_{k'^r,n-\ell-1} \beta_{k',\ell}.$$  

Choose $\epsilon \in \mathbb{N}$ sufficiently large that all the $\{\epsilon q_0^{ij\ell}\}$ (a finite set) are integers. Now $m' + r \leq r\mu \implies \frac{L_{m'+r}^{j'+1}}{(m'+r)^{j'+1}} \in \mathbb{Z}$, while Corollary 2.5

$$\implies \sum_{r=0}^{\ell} \epsilon^{r(k'^r)} L_{m'}^{j'+r} \in \mathbb{Z}.$$  

Since in each term of RHS (3.12) we have $k'' + k' - j' - 1 = k'' + k' + 1 \leq (n - \ell - 1) + \ell + 1 = n$, multiplying the original $\epsilon_B$ by $\epsilon^2$ gives $\epsilon_B b_\mu L_{m'}^n \in \mathbb{Z}$, as desired.

**Step 4: Irrationality of $v_0$.** Let $\Lambda_m := \gcd(m!, L_{m})$, so that by Step 3 we have $\epsilon_B \Lambda_m b_m = B_m \in \mathbb{Z} \ (\forall m)$. Writing $P_m := \{p \text{ prime} \mid p \leq m\}$, set

$$\pi(m) = |P_m| \text{ and } \chi(m) := \sum_{p \in P_m} \log(p).$$
Evidently
\[ e^{\chi(m)} \leq \Lambda_m \leq \prod_{p \leq m \text{ prime}} p^{|\log_p(r_m)|} \leq (rm)^{\pi(m)}, \]
hence
\[ (3.13) \quad e^{\chi(m)/m} \leq \Lambda^{1/m}_m \leq (rm)^{\pi(m)/m}. \]

By the Prime Number Theorem and its proof, \( \frac{\pi(m)}{m} \sim \frac{1}{\log(m)} \) and \( \lim_{m \to \infty} \frac{\chi(m)}{m} = 1 \). So the outer terms of (3.13) limit to \( e \), and so does \( \Lambda^{1/m}_m \).

Finally, suppose \( v_0 = \frac{P}{Q} \) (\( P \in \mathbb{Z}, Q \in \mathbb{Z} \setminus \{0\} \)). Then
\[
\limsup_{m \to \infty} |\varepsilon_B \Lambda^a_m a_m P - B_m Q|^\frac{1}{m}
\]
\[ \quad = \left( \lim_{m \to \infty} \Lambda^{1/m}_m \right)^n \cdot \lim_{m \to \infty} |\varepsilon_B Q|^\frac{1}{m} \cdot \limsup_{m \to \infty} |a_m v_0 - b_m|^\frac{1}{m}
\]
\[ \quad = e^n \cdot 1 \cdot \varepsilon_B \quad \text{(by Step 1)}
\]
\[ \quad < 1 \quad \text{(by assumption),}
\]
and for some sufficiently large \( m \) we therefore have
\[ 0 < |\varepsilon_B \Lambda^a_m a_m P - B_m Q| < 1, \]
a contradiction. Q.E.D.

3.2. Casting a wider net. By [7K, Thm. 5.2], we can think of \( \tilde{\nu}_0^\phi \) as computing the extension of \( \mathbb{Q}(0) \) by \( W_{-2n} \text{H}^{n-1}(X_0^\phi, \mathbb{Q}(n)) \cong \mathbb{Q}(n) \) associated to \( \Xi|_{X_0^\phi} \in H^1_{\lambda_0}(X_0^\phi, \mathbb{Q}(0)) \). Since \( (I^*\omega)|_{X_0^\phi} \) generates the top graded piece \( \text{Gr}^W H_{n-1}(X_0^\phi, \mathbb{Q}) \), pairing with it sends the generator of \( \mathbb{Q}(n) \) to \( (2\pi i)^n \). So \( \tilde{V}_0^\phi(0) \) realizes \( AJ \left( \Xi|_{X_0^\phi} \right) \in \mathbb{C}/(2\pi i)^n \mathbb{Q} \), which one interprets (arguing as in [op. cit.], at least for \( n \leq 3 \)) as a Borel regulator value \( r_{\text{Bor}} \) for \( K_{2n-1} \) of the number field \( k \) required to resolve \( X_0^\phi \). Since this is just the field of definition of \( I \), the numbers \( \tilde{V}_0^\phi(0) \) appearing in Theorem 3.1 are limited (at best) to \( \zeta(n) \).

The first step in a generalization of this result would be to expand the notion of involutivity:

- drop the requirement \( k = \mathbb{Q} \),
- replace \( t \mapsto \pm t^{-1} \) by \( t \mapsto \frac{1}{C} (C \in \mathbb{Z}) \), and
- allow \( I \) to be a correspondence inducing an isomorphism between the \( \mathbb{Q} \)-VHS \( V_\phi \) and its pullback.

Unfortunately, we have to pay for this expansion with a stronger bound:
Proposition 3.2. Let $\phi$ be as in Theorem 3.1, but with the weaker involutivity just described. Assume in addition that $r_{\phi} < \frac{e^{-n}}{|C|}$. Then $\tilde{V}_\phi(0) \notin \mathbb{Q}$.

The proof is a straightforward, but tedious, generalization of the above. One nice formal consequence is that $I^* \omega = c \lambda \omega$ for $c \in \mathbb{C}^\times$ with $c^2 \mathbb{C}^\times = \pm 1$, so that $\tilde{V}_\phi(0) = \langle \tilde{\nu}_0, \tilde{\omega}_0 \rangle = c^{-1} \langle \tilde{\nu}_0, I^* \omega_{\phi,0} \rangle = c^{-1} r_{Bor}$. For example, if $n = 2$ and $I$ is defined over $\mathbb{Q}(i)$, then one expects $r_{Bor} \sim iG$, where $G = L(\chi_{-4}, 2)$ is Catalan’s constant. But then, one also expects $c \sim i$, so that $\tilde{V}_\phi(0) \sim G$ — a not-insignificant “calibration”, as irrationality of $iG$ and of $G$ are rather different things. Naturally, we don’t have a proof of the latter, but we will briefly discuss some higher normal functions with $G$ as a special value below.

Remark 3.3. The function of $I$ and $J$ in the above proof is to match periods of a pullback of $\mathcal{V}_\phi$ with those of $\mathcal{V}_\phi$ resp. the facile family. If one has such a matching by other means, there is obviously no need for the maps of varieties.

Another natural way to relax the hypotheses is to permit the Newton polytope of $\phi$ to be non-reflexive, as long as $0$ remains its unique interior integral point. This ensures that, while $X_{\phi}$ may not be Calabi-Yau, $h^{n-1,0}(X_{\phi})$ remains 1. More significantly, one could abandon the “principality” constraint that Hodge numbers of $\mathcal{V}_\phi$ all equal 1, in order to make contact with results (such as [22]) involving linear forms in more than one zeta value; two likely sources of interesting examples will be discussed in §5.4.

4. Low dimension

In order to implement Theorem 3.1 in any specific cases, we must be able to check strong temperedness and compute $\tilde{V}_\phi(0)$. To this end, we examine the boundary structure more closely. Let $\phi$ and $\mathbb{P}_\Delta$ be as in the first paragraph of §3, with associated family $X_\phi \to \mathbb{P}^1$ of anticanonical hypersurfaces.

Consider the toric variety $\hat{\mathbb{P}}_\Delta$ associated to $\Delta$ without the triangulation, with canonical blow-down morphisms $\mathbb{P}_\Delta \to \hat{\mathbb{P}}_\Delta$, $\mathbb{D}_\Delta \to \hat{\mathbb{D}}_\Delta = \hat{\mathbb{P}}_\Delta \setminus \mathbb{G}^n_m$. Let $\sigma \subset \Delta$ be a codimension-$j$ face with $((j-1)$-dimensional) dual $\sigma^o \subset \Delta^o$, and $\mathbb{G}^{n-j}_m \cong \hat{\mathbb{D}}^\times_\sigma \subset \hat{\mathbb{D}}_\sigma \subset \hat{\mathbb{D}}_\Delta$ the corresponding (open resp. closed) codimension-$j$ stratum. The $((n-j+1)$-dimensional) strata $\mathbb{D}^a_\sigma \subset b^{-1}(\hat{\mathbb{D}}_\sigma)$ correspond to the $((j-i-1)$-dimensional) faces $\sigma^o_\alpha \subset \sigma^o$ in the triangulation of $\Delta^o$. A basis of
the flag $\text{ann}(\sigma^\circ) \subseteq \text{ann}(\sigma^\circ_\alpha) \subseteq \mathbb{Z}^n$ produces $n - j + 1$ toric coordinates 
\[ x_1^\circ, \ldots, x_n^\circ_{n-j}; y_1, \ldots, y_j \] on $\mathbb{D}^\alpha_{\sigma}$, with $b$ induced by forgetting the $\{y_i\}$. 
Writing $Y_\phi := X_\phi \cap \mathbb{D}_\Delta \xrightarrow{b} \hat{Y}_\phi$, each 
\[ Y_\alpha^\sigma := Y_\phi \cap \mathbb{D}^\alpha_\sigma = \bigcup_{i=1}^{\mu_\sigma} Y_\alpha^\sigma,i \xrightarrow{b} \bigcup_{i=1}^{\mu_\sigma} \hat{Y}_\sigma^i = \hat{Y}_\phi \cap \mathbb{D}_{\sigma} 
\]
is cut out of $\mathbb{D}^\alpha_\sigma$ resp. $\hat{D}_\sigma$ by a “face polynomial” $\phi_\sigma(x_1^\circ, \ldots, x_n^\circ_{n-j})$ given (up to a shift) by rewriting the terms \( \{cx^m | m \in \sigma \} \) of $\phi$ in terms of $\mathbb{Z}^\sigma$.

A precondition for temperedness of $\phi$ (as defined above) is that the iterated residues of the coordinate symbol \( \{x \} |_{(X_\phi^\lambda)\times} \) along strata $Y_\alpha^\sigma$ of $X_\phi^\lambda \setminus (X_\phi^\lambda)^\times$ all vanish, which is equivalent to the vanishing of the $\{x^\sigma\}$ on $\hat{Y}_\sigma^\times$:

**Definition 4.1.** $\phi$ is weakly tempered\(^{10}\) if the symbols 
\begin{equation}
\{x_1^\sigma, \ldots, x_n^\sigma_{n-j}\}|_{\hat{Y}_\sigma^\times} = 0 \in K_{n-j}^M(\bar{Q}(\hat{Y}_\sigma^i)) \otimes \bar{Q}
\end{equation}
for all $1 \leq j \leq n - 1$, $\sigma \in \Delta(j)$, and $i = 1, \ldots, \mu_\sigma$.

This is a condition on the face polynomials $\phi_\sigma = \prod \phi_{\sigma,i}$. For example, (4.1) holds:
\begin{itemize}
  \item for $j = n - 1$ if all edge polynomials are cyclotomic; and
  \item for $j = n - 2$ if (for all 2-faces) the $\phi_{\sigma,i}$ are Steinberg (i.e. $\phi_{\sigma,i}(x, y) = 0$ makes $\{x, y\} = 0$ in $K_2$).
\end{itemize}

We will say that $\phi_\sigma$ is $\mathbb{Q}$-Steinberg if its factors are Steinberg and defined over $\mathbb{Q}$.

For $n \geq 4$, it may not even be the case that weak temperedness implies the existence of pointwise lifts $\Xi_\phi^\lambda \in \text{CH}^n(X_\phi^\lambda, n) \otimes \bar{Q}$ of $\{x\}|_{X_\phi^\lambda\times}$ to a smoothing. This is always true for $n \leq 3$; for $n = 4, 5$ it holds if (for instance) all boundary strata are rational and defined over a totally real number field. (See [DK, §3] for other refinements.) For particular families one can certainly check temperedness in any dimension. However, in the remainder of this section, we prefer to restrict to the case $n \leq 3$.

Let $K_\phi := \rho(T_{\{x\}}) \subset \mathbb{P}^1$ and $U_\phi := \mathbb{P}^1 \setminus K_\phi$ (as in §3). We are interested in conditions under which not only is $\phi$ strongly tempered,

\[^9\]The $\{\hat{Y}_\sigma, Y_\sigma^\alpha\}$ need not be irreducible, as $\lambda - \phi$ is not assumed $\Delta$-regular for generic $\lambda$.

\[^{10}\]Note that this is the definition of temperedness in [DK, §3]. Our use of “tempered” here correlates to the property of “$\{x\}$ completes to a family of motivic cohomology classes” in [op. cit.].
but where the (unique) single-valued lift $\bar{\nu}_\phi^\lambda$ over $U_\phi$ is given by fiberwise restrictions of

$$R(\underline{x}) = \sum_{j=1}^n (-1)^{(n-1)(j-1)} (2\pi i)^{j-1} \log(x_j) \frac{dx_{j+1}}{x_{j+1}} \wedge \cdots \wedge \frac{dx_n}{x_n} \delta_{T_{s_1} \cap \cdots \cap T_{s_{j-1}}}$$

up to push-forwards of currents from $Y_\phi$ and coboundary currents. The biggest nuisance turns out to be the correction term $(2\pi i)^n \delta_\Gamma$ added to $R_{\Sigma}$ to produce a closed current; part of this correction is (the current of integration over) a chain on $X_\phi^\lambda$ bounding on an unknown $(n-2)$-cycle on $Y_\phi$. In order that this not contribute to $\bar{V}_\phi$, we have to assume that $H_{n-2}$ of $Y_\phi$ (or part of it) vanishes.

To make these conditions relatively weak, we define some “bad” subsets of the boundary. Let $\mathcal{I}_\phi \subset Y_\phi$ be the generic $\Delta$-irregularity locus of $\lambda - \phi$ — that is, the closure of the union of all singularities and nonreduced components of all $Y_{\alpha}^\phi$ (computable by taking partials of $\phi$). Let $\mathcal{A} \subset \mathcal{I}$ be the locus of generic singularities of $X_\phi^\lambda$. Denoting by $\mathcal{I}_\Delta$ the intersection with $D_\Delta$ of the closure of the locus $\cup_{i=1}^n \{ x_i = 1 \}$ in $\mathbb{P}_\Delta$, we write $\mathcal{J}_\phi$ for the union of all components of $Y_\phi$ not contained in $\mathcal{I}_\Delta$, and (for $n = 3$) not of “Steinberg type” $\{ x_{i_1} + x_{i_2} = 1, x_{i_3} = 0 \text{ or } \infty \}$.

**Theorem 4.2.** Let $\phi \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a reflexive Laurent polynomial for $n = 1$, 2, or 3.

(a) Assume that edge polynomials [resp. 2-face polynomials] of $\phi$ are cyclotomic [resp. $Q$-Steinberg], and that $\mathcal{I} \subset \mathcal{I}_\Delta$ and $\mathcal{I} \cap \mathcal{J} \subset \mathcal{A}$; if $n = 3$, assume $\mathcal{A}$ consists only of $A_1$ singularities and lies in $\mathcal{I}_\Delta \cap \text{sing}(D_\Delta)$. Then $\phi$ is tempered.

(b) If $n = 2$, assume in addition that $\mathcal{J}$ is one point. If $n = 3$, assume in addition that $H_1(\mathcal{J} \setminus \mathcal{J} \cap \mathcal{A}) = \{0\}$. Then $\phi$ is strongly tempered, with single-valued THNF

$$\bar{V}_\phi(\lambda) = \int_{X_\phi^\lambda} R(\underline{x}) \wedge \bar{\omega}_\phi^\lambda$$

over $U_\phi$.

**Sketch.** (a) is a restatement of part of [DK, Thm. 3.8]. The strong-temperedness part of (b) recapitulates [DK, Prop. 4.12]. We now show that (4.2) is consistent with the lift constructed in the proof of [loc. cit.]. Assuming first that $\mathcal{A} = \emptyset$, we can extend $\{ \underline{x} \}$ to a closed precycle on $Z^n(X_{\phi}^- \setminus \mathcal{J} \times \mathbb{A}^1, n)$ by taking Zariski closure and (for $n = 3$) adding precycles of the form $(t, 1-t, 1 - \frac{x_{i_1} t}{t})_{t \in \mathbb{P}^1}$ over the “Steinberg type” locus $\mathcal{J}$. As in [loc. cit.], one then has $\gamma \in Z^n(X_{\phi}^- \setminus \mathcal{J} \times \mathbb{A}^1, n+1)$

\[ \mathcal{A} \] is empty for $n = 1$ or 2.
and (closed) \( \Xi \in \mathbb{Z}^n(\mathcal{X}_\phi, n) \) with \( \Xi|_{\mathcal{X}_\phi \setminus \mathcal{J} \times \mathbb{A}^1} = \{ \mathcal{L} \} + i_\gamma^* Z + \partial \gamma \implies R_{\Xi}|_{\mathcal{X}_\phi \setminus \mathcal{J} \times \mathbb{A}^1} = R(\mathcal{L}) + \frac{1}{2\pi i} dR_\gamma + i_\gamma^* R_Z - (2\pi i) n \delta_{T_\gamma} \) and \( T_\Xi \equiv \partial T_\gamma + \tau \mod \rho^{-1}(\mathcal{K}) \), with \( \tau \) supported on \( \mathcal{J} \times (\mathbb{P}^1, \mathcal{K}) \). Arguing as in [loc. cit.], our hypotheses give that \( \tau \equiv 0 \) on \( \mathcal{J} \times (\mathbb{P}^1, \mathcal{K}) \). Writing this as \( \tau \equiv \partial C_\mathcal{J} \), the “lift” of \( R_\Xi \) given there is \( R'_\Xi = R_\Xi + (2\pi i)^n \delta_{T_\gamma + C_\mathcal{J}} \), so that

\[
R'_\Xi|_{\mathcal{X}_\phi \setminus \mathcal{J} \times \mathbb{A}^1} = R(\mathcal{L}) + \frac{1}{2\pi i} dR_\gamma.
\]

Writing \( X^\lambda_{\phi, \epsilon} \) for \( X^\lambda_\phi \) minus a small tubular neighborhood of \( \mathcal{J} \), this leads to

\[
\bar{V}_\phi(\lambda) = \int_{X^\lambda_{\phi, \epsilon}} R'_\Xi \wedge \bar{\omega}^\lambda_\phi = \lim_{\epsilon \to 0} \left\{ \int_{X^\lambda_{\phi, \epsilon}} R(\mathcal{L}) \wedge \bar{\omega} + \frac{1}{2\pi i} \int_{\partial X^\lambda_{\phi, \epsilon}} R_\gamma \wedge \bar{\omega} \right\}
\]

hence \((4.2)\), provided \( \lim_{\epsilon \to 0} \int_{\partial X^\lambda_{\phi, \epsilon}} R_\gamma \wedge \bar{\omega} = 0 \). Viewing \( \gamma \) as a family of curves in \( \Box^{n+1} \) over \( \mathcal{X}_\phi \), the \((n - 2)\)-current \( R_\gamma \) is the push-forward of \( R(\mathcal{L}) \) \((\mathcal{L} = \text{coordinates on } \Box^{n+1})\). If \( n = 2, \) and \( n \) is a local holomorphic coordinate about \( \mathcal{J} \), one checks that \( R_\gamma \) is locally \( \mathcal{O}(\log^2 u) \); clearly \( \int u |_{[u]} \log^2(u)du \to 0 \). For \( n = 3 \) (with \( |(u)[u]| = \mathcal{J} \) locally), \( R_\gamma \) is the current of integration over a 3-chain times a locally-\( \mathcal{O}(\log^2(u)) \) function, with the same result. Finally, in the event that \( \mathcal{J} \) is nonempty, and \( \mathcal{X}_\phi^{-} \) the blowup along \( \mathcal{J} \times \mathbb{P}^1 \) (with exceptional divisor \( \mathcal{E} \)), one replaces all complexes (of higher Chow cycles and currents) on \( \mathcal{X}_\phi^{-} \) by cone (double-)complexes for the morphism \( \mathcal{E} \to \mathcal{J} \times \mathcal{X}_\phi^{-} \). The assumption that \( H_1(\mathcal{J} \setminus \mathcal{J} \cap \mathcal{J}) = \{ 0 \} \) allows \( C_\mathcal{J} \) to be drawn so as to avoid \( \mathcal{E} \).

This result is likely far from optimal, but suffices for the applications in the next section.

**Corollary 4.3.** For \( \phi \) as in Theorem 4.2(b), if \( \phi(-x) \) has all positive coefficients, then

\[
\bar{V}_\phi(0) = \int_\psi R(\mathcal{L})|_{X^\lambda_\phi}
\]

for any \((n - 1)\)-cycle \( \psi \subset X^\lambda_\phi \setminus \mathcal{J} \) representing \( Q_0 \phi_{n-1} \in H_{n-1}(X^\lambda_\phi) \).

**Sketch.** The additional hypothesis on \( \phi \) ensures that \( \lambda = 0 \) belongs to \( U_\phi \). Now \( Z^0 = \text{Res}_{X^0}(T^* \Xi) = T^* \text{Res}_{X^0}(\Xi) = T^* Z_0 \) has \( \Omega_{Z^0} = T^* \Omega_{Z_0} = (2\pi i)^{n-1} T^* \omega_0 = (2\pi i)^{n-1} \bar{\omega}^0 \), while \( T_{Z^0} \) is an \((n - 1)\)-cycle with \( dR_{Z^0} = \Omega_{Z^0} - (2\pi i)^{n-1} \delta_{T_{Z^0}} \implies [T_{Z^0}] = [\bar{\omega}^0] = Q_0 \phi_{n-1} \) in homology. So for \( \psi \)

\(^{12}Q_0 \phi_{n-1} \) is the class with intersection number \((\pm)1\) against \( \phi_0 \); one should think of a membrane “stretched once around” \( X^\lambda_\phi \).
as above, there exists an \((n-2)\)-current \(\mathcal{R}\) on \(X^0\) with \(d\mathcal{R} = \omega^0 - \delta_\phi\), with closed hence (by hypothesis on \(\mathcal{J}\)) exact restriction to \(\mathcal{J}\). We may therefore assume that \(\mathcal{R}\) (is of intersection type with respect to \(\mathcal{J}\) and) pulls back to 0 on \(\mathcal{J}\), so that \(\lim_{\epsilon \to 0} \int_{\partial X^\epsilon_0} R_{\{\xi\}} \wedge \mathcal{R} = 0\). It now follows that \(\tilde{V}_\phi(0) = \lim_{\epsilon \to 0} \int X^\epsilon_0 R_{\{\xi\}} \wedge \tilde{\omega}_\phi^0 = \lim_{\epsilon \to 0} \int X^\epsilon_0 R_{\{\xi\}} \wedge \delta_\psi\) as claimed. □

5. Examples and near-examples

Here we record some Laurent polynomials that satisfy the conditions of Theorems 3.1 and 4.2, as well as a few which stray close enough to warrant attention.

5.1. \(n = 1\). Let \(b \in \mathbb{Z}_+, a = 2b + 1\), and set

\[ \phi(x) := -x + a + \frac{1 - a^2}{4x}. \]

\(X^\lambda_\phi\) is a pair of points \(\{p^\lambda_+, p^\lambda_-\}\) which are distinct unless \(\lambda\) is a root of \(P_a(\lambda) := \lambda^2 - 2a\lambda + 1\), and \(X^\lambda_\phi\) has involution

\[ (x, \lambda) \mapsto (\lambda^{-1}x + \frac{1}{2}(1 - \lambda^{-1})(a + 1), \lambda^{-1}). \]

We have \(p^\lambda_\pm = \frac{1}{2}(a - \lambda \pm P_a(\lambda)^{1/2})\), and the 0-form

\[ \tilde{\omega}^\lambda_\phi = \lambda^{-1}\omega^\lambda_\phi = \frac{\pm 1}{2\sqrt{P_a(\lambda)}} \text{ on } p^\lambda_\pm \]

has period

\[ A(\lambda) = \int_{p^\lambda_+ - p^\lambda_-} \tilde{\omega}^\lambda_\phi = \frac{1}{\sqrt{P_a(\lambda)}}. \]

The regulator 0-current \(R_\lambda = \log(p^\lambda_+ )\) on \(p^\lambda_\pm\), and so the HNF is

\[ \tilde{V}_\phi(\lambda) = \frac{1}{\sqrt{P_a(\lambda)}} \log(p^\lambda_+ / p^\lambda_-). \]

Since \(r_\phi = a - \sqrt{a^2 - 1} < e^{-1}\) for all \(b \geq 1\), we conclude that

\[ \tilde{V}_\phi(0) = \log \left( \frac{b + 1}{b} \right) \notin \mathbb{Q}. \]
5.2. $n = 2$. Let

$$\phi(x_1, x_2) := x_1^{-1} x_2^{-1} (1 - x_1)(1 - x_2)(1 - x_1 - x_2);$$

the picture indicates the (reflexive) Newton polytope $\Delta$ and $X^\lambda_\phi \subset \mathbb{P}_\Delta$. The green [resp. red] dots represent $Y_\phi \setminus \mathcal{I}$ [resp. $\mathcal{I}$]; edge polynomials are $x^a - 1$ or $(x^a - 1)^2$, so $\phi$ is strongly tempered.

The singular fibers of $X_\phi$ are over $0$, $\infty$, and $t_\pm = \frac{-11 \pm 5\sqrt{5}}{2}$, with $\tau_\phi = t_+ < e^{-2}$ (and $X_{\phi, t_\pm}$ of type $I_1$), while

$$D_\phi(t) = (t^2 + 11t - 1)\delta_t^2 + t(2t + 11)\delta_t + t(t + 3)$$

is integral\(^{14}\) involutivity is ensured by the (order 4) automorphism

$$\mathcal{I} : (x_1, x_2, t) \mapsto \left( \frac{x_1}{x_2 - 1}, \frac{1 - x_2}{1 - x_1 - x_2}, \frac{1}{t} \right),$$

while the facile 2-cover given by

$$(1 - \xi^2) + \xi y_1 + \xi^2 y_2 - \xi y_1 y_2 = 0$$

maps down by

$$\mathcal{J} : (y_1, y_2, \xi) \mapsto \left( \frac{y_1}{\xi}, \frac{y_1 - \xi}{y_1 y_2 - \xi}, \xi^2 \right).$$

To compute the special value, note that $X^0 = \{x_1 = 1\} \cup \{x_2 = 1\} \cup \{x_1 + x_2 = 1\}$, with $\psi$ going “once around” the figure. Furthermore, $R_{\{x\}} = \log(x_1) \frac{dx_1}{x_1^2} - 2\pi i \log(x_2) \delta_{x_1}$ vanishes on the first two components. Parametrizing the remaining component of $-\psi$ by $[0, 1] \ni s \mapsto (1 - s, s)$,

$$\tilde{V}_\phi(0) = - \int_0^1 \log(1 - s) \frac{ds}{s} = \text{Li}_2(1) = \frac{\pi^2}{6} \notin \mathbb{Q}.$$

In view of the results of Zagier’s search for recurrences of Apéry type [Za], it seems likely that this $\phi$ is the unique example for $n = 2$ that satisfies the conditions of Proposition \([3.2]\). One can match tempered Laurent polynomials (hence higher normal functions) to the sporadic examples of [op. cit.], but outside case “D” (just treated), both the

\(^{14}\)The coefficients of the holomorphic solution are the “baby Apéry” sequence $a_m = \sum_{k=0}^m \binom{m}{k}^2 \binom{m+k}{k} = 1, 3, 19, 147, \ldots.$
bound on \( r_\phi \) and involutivity fail. For instance, case “E” of [op. cit.] is \( \phi(\varepsilon) = (x_1 + x_1^{-1})(x_2 + x_2^{-1}) + 4 \); this has \( \tilde{V}_\phi(0) = 2G \) (Catalan), but \( r_\phi C = \frac{1}{2} \cdot 32 = 4 \) (too big), and the Kodaira fiber types at \( \lambda_0 \) and \( C/\lambda_0 \) (or 0 and \( \infty \)) don’t match, so that \( V_\phi \not\cong I^*V_\phi \). This non-involutivity is not a problem for the approach via modular forms, which gives a different means for obtaining period expansions about any cusp; we are trading off this advantage for (at least in principle) the ability to treat non-modular families in higher dimension.

Remark 5.1. One other “near-example” related to Catalan’s constant arises from work of Zudilin [Z3], who found an Apéry-like recurrence with rational solutions \( a_m, b_m \) whose ratios \( b_m/a_m \) converge rapidly to \( G \). With some work, one can write the generating series \( \sum_{m \geq 0} (a_m G - b_m) \lambda^m \) as a normal function associated to a higher cycle on a family of open\(^{15} \) genus-9 curve, which are branched 4:1 covers of the “baby Apéry” family of elliptic curves above! By construction, this has \( V(0) = G \).

5.3. \( n = 3 \). The Newton polytope \( \Delta \) of

\[
\phi(x_1, x_2, x_3) = x_1^{-1}x_2^{-1}x_3^{-1}(x_1 - 1)(x_2 - 1)(x_3 - 1)(1 - x_1 - x_2 + x_1x_2 - x_1x_2x_3)
\]

and its dual are

\[\Delta\]

\[\Delta^\circ\]

showing reflexivity; maximal triangulation adds the green edge. The edge and facet polynomials are products of \( (x_1^\sigma - 1), (x_2^\sigma - 1)^2 \), and

\(^{15}\)The corresponding Laurent polynomial is neither reflexive nor tempered; 2 points are removed from each fiber.
$(1 - x_1^a \pm x_2^b)$, and $\mathcal{A}$ (red), $\mathcal{J}$ (blue), and $\mathcal{I}$ (green) are as depicted:

\[ \bullet = A_1 \text{ singularities of generic fiber} \]

In particular, $\mathcal{J} \setminus \mathcal{J} \cap \mathcal{A}$ is two copies of $\mathbb{A}^1$ attached at a point, and we conclude that $\phi$ is strongly tempered.

Singular fibers are at $0$, $\infty$, $t_\pm = (\sqrt{2} \pm 1)^4$, and $1$; the last of these does not contribute to monodromy of $\mathbb{V}_\phi$, and so $\delta_\phi = 2$, while $r_\phi = t_- < e^{-3}$. To see that the generic Picard rank is 19, one can use a torically-induced elliptic fibration (cf. [Ke, §2]). The Picard-Fuchs operator is

\[ D_\phi(t) = (t^2 - 34t + 1)\delta_t^3 + 3t(t - 17)\delta_t^2 + 3t(t - 9)\delta_t + t(t - 5), \]

and the $\{a_m\}$ the famous Apéry sequence $a_m = \sum_{k=0}^{m} \binom{m}{k}^2 \binom{m+k}{k}^2 = 1, 5, 73, 1445, \ldots$.

Changing coordinates by $X_i = \frac{x_i}{x_i - 1}$ brings $1 - t\phi(x) = 0$ into the form studied by Beukers and Peters [BP]. By the results of Peters and Stienstra [PS], $X_{\phi,t}$ thus has a (facile) 2-cover by the Fermi family

\[ X_\xi := \left\{ \xi \sum_{i=1}^{3} (y_i + y_i^{-1}) + 1 + \xi^2 = 0 \right\}. \]

It also has an involution, by

\[ \mathcal{I} : (x_1, x_2, x_3, t) \mapsto \left( \frac{x_3}{x_3 - 1}, \frac{(x_1 - 1)(x_2 - 1)}{1 - x_1 - x_2 + x_1 x_2 - x_1 x_2 x_3}, \frac{x_1}{x_1 - 1}, \frac{1}{t} \right). \]

The 2-current

\[ R_{\pm} = \log(x_1) \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} + (2\pi i) \log(x_2) \frac{dx_3}{x_3} \delta_{T_{x_1}} + (2\pi i)^2 \log(x_3) \delta_{T_{x_1} \cap T_{x_2}} \]

vanishes on the components $\{x_i = 1\}$ of $X^0_{\phi}$. The piece of $\psi$ on the remaining component $x_3 = \frac{(1-x_1)(1-x_2)}{x_1 x_2}$ is parametrized by $0 \leq r \leq s \leq \infty$. 

\[ \mathcal{A} = \left\{ \begin{array}{ll}
\end{array} \right\} \]
1 \mapsto (1-r, s, \frac{s(1-s)}{s(1-r)}), and so

\tilde{V}_\phi(0) = \int_{0 \leq r \leq s \leq 1} \log(1-r)d\log(s) \land d\log(\frac{r}{1-r})

= - \int_0^1 \log(1-r) \log(r) d\log(\frac{r}{1-r})

= -2 \int_0^1 \log(1-r) \log(r) d\log(r)

= 2 \sum_{k \geq 1} \frac{1}{k} \int_0^1 r^{k-1} \log(r) dr = -2\zeta(3) \notin \mathbb{Q}.

There are at least three “near-examples” for \( n = 3 \), identified in [dS] (and closely related to [Go]), which satisfy all the criteria in Proposition 3.2 that we have checked, except for the bound: writing \( \phi_I \) for (5.1), these are

\phi_{II} = (1 - x_1 - x_2 - x_3)(1 - x_1^{-1})(1 - x_2^{-1})(1 - x_3^{-1})

\phi_{III} = (x_1 + x_2 + x_3)(-1 + x_1^{-1} + x_2^{-1} + x_3^{-1} - x_1^{-1} x_2^{-1} - x_1^{-1} x_3^{-1} - x_2^{-1} x_3^{-1})

\phi_{IV} = (1 - x_1 - x_2 - x_3)(1 - x_1^{-1} - x_2^{-1} - x_3^{-1}).

They are reflexive, tempered, and involutive, with \( C_1 = 1, C_{II} = 16, C_{III} = -27, \) and \( C_{IV} = 64 \); while \( \tilde{V}(0) \sim \zeta(3) \) except for \( \phi_{III} \), where

\tilde{V}(0) \sim L(\chi_{-3}, 3) \sim \frac{\pi^3}{\sqrt{3}}. \) For \( r_\phi|C|e^3 \) we obtain \( \approx 0.59, 13.78, 27.97, \) resp. 80.34, which satisfies the required bound \(< 1\) only in the first case.

5.4. Higher dimension? Here we propose two sources for examples with \( n \geq 4 \), if one is prepared to weaken the hypotheses as in the last paragraph of §3.2. In both cases, the Laurent polynomials considered, while not in general reflexive, all have Newton polytope \( \Delta \subset [-1, 1]^n \) having the origin as unique integer interior point. Details, proofs, and further developments will appear elsewhere.

Define the VZ polynomials \( \{\phi_n\} \) inductively by \( \psi_1 = 1, \)

\( \psi_n(x_1, \ldots, x_n) := x_1 \cdots x_n + (1 - x_n)\psi_{n-1}(x_1, \ldots, x_{n-1}), \)

\( \phi_n(\underline{x}) := (1 - x_1^{-1}) \cdots (1 - x_n^{-1})\psi_n(\underline{x}); \)

they are obtained (by substituting \( X := \frac{\underline{x}}{x_1^{-1}} \)) from denominators of integrals first considered by Vasilyev [Va] and Zudilin [Z2] in their works on linear forms in zeta values. For \( n = 2 \) and 3, this recovers (up to inversion and permutation of coordinates) the Apéry polynomials above. For \( n = 5 \), we expect that \( \phi_5 \) is strongly tempered, and conjecture that
Hodge numbers of $V_\phi$ are $(1, 1, 2, 1, 1)$. This would result in two invariant periods $A(\lambda) = 1 + \sum_{m \geq 1} a_m \lambda^m$, $B(\lambda) = \lambda + \sum_{m \geq 2} b_m \lambda^m$ about the maximal unipotent monodromy point, as $N$ has two primitive classes. Writing $V(\lambda)$ for the HNF, one then expects $r_\phi < e^{-5}$, and

$$C(\lambda) := -V(\lambda) + A(\lambda)V(0) + B(\lambda)(-V(0)A'(0) + V''(0))$$

$$= \sum_{m \geq 2} c_m \lambda^m$$

to satisfy an inhomogeneous equation as above. Combining this with Vasilyev’s results, one would conclude that

$$V(\lambda) = \sum_{m \geq 0} (2a_m \zeta(5) + b_m \zeta(3) - c_m),$$

where $a_m, L_m^2 b_m, L_m^5 c_m \in \mathbb{Z}$, with the innocuous consequence that “at least one of $\zeta(3)$ and $\zeta(5)$ is irrational.”

Recent work of F. Brown [Br] provides another expected source of interesting Laurent polynomials. Given a permutation $\pi \in S_{n+3}$, write formally

$$\theta_\pi(z_1, \ldots, z_{n+3}) := \prod_{i \in \mathbb{Z}/(n+3)\mathbb{Z}} (z_{\pi(i)} - z_{\pi(i+1)}),$$

$$x_\pi^j := -CR \left( z_{\pi(1)}, z_{\pi(n+2-j)}, z_{\pi(n+3-j)}, z_{\pi(n+4-j)} \right),$$

where $j = 1, \ldots, n$, and

$$\Omega_\pi := \frac{dx_\pi^1}{x_\pi^1} \wedge \cdots \wedge \frac{dx_\pi^n}{x_\pi^n};$$

if $\pi = \text{Id}$ then we drop the sub- and superscripts. Now let $\sigma \in S_{n+3}$ be a convergent permutation in the sense of [op. cit.]; namely, we assume that for any $i \in \mathbb{Z}/(n+3)\mathbb{Z}$ and $2 \leq k \leq n + 1$, $\{\sigma(i), \ldots, \sigma(i+k)\}$ is not a consecutive sequence of integers mod $(n+3)$. It turns out that $\theta_\sigma(\underline{z})/\theta(\underline{z})$ can be written as a Laurent polynomial $\phi_\sigma(x_1, \ldots, x_n)$ (with Newton polytope $\Delta_\sigma$), and the basic cellular integrals on $M_{0,n}$ of [op. cit.] become the integrals

$$I^{(k)}_\sigma := \int_{T(\underline{z})} \phi_\sigma(\underline{x})^{-k} \Omega_\sigma$$

on $\mathbb{P}_{\Delta_\sigma}$. Defining $X_\lambda^\sigma \subset \mathbb{P}_{\Delta_\sigma}$ by $\lambda = \phi_\sigma(\underline{x})$, the generating series

$$V_\sigma(\lambda) := (2\pi i)^{1-n} \sum_{k \geq 0} I^{(k)}_\sigma$$

$$= (2\pi i)^{1-n} \int_{T(\underline{z})} \frac{dx_1/x_1 \wedge \cdots \wedge dx_n/x_n}{\lambda - \phi_\sigma(\underline{x})}$$
may be rewritten (using integration by parts) in the form

\[ \int_{X^\lambda} R_{\{z\}} \big|_{X^\lambda} \wedge \tilde{\omega}^\lambda, \]

which is a truncated HNF under a strong temperedness hypothesis. Finally, involutivity may be arranged via the additional hypothesis that \( \sigma^{-1} = \pi_1 \circ \sigma \circ \pi_2 \), with \( \pi_1, \pi_2 \) belonging to the dihedral group \( D_{n+3} \).

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