PURE BRAID GROUP ACTIONS ON CATEGORY $\mathcal{O}$ MODULES

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To Corrado De Concini

Abstract. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra and $U_\hbar\mathfrak{g}$ its quantised enveloping algebra. Answering a question of P. Etingof, we prove that the quantum Weyl group operators of $U_\hbar\mathfrak{g}$ give rise to a canonical action of the pure braid group of $\mathfrak{g}$ on any category $\mathcal{O}$ (not necessarily integrable) $U_\hbar\mathfrak{g}$–module $V$. By relying on our recent results [ATL15], we show that this action describes the monodromy of the rational Casimir connection on the $\mathfrak{g}$–module $V$ corresponding to $V$. We also extend these results to yield equivalent representations of parabolic pure braid groups on parabolic category $\mathcal{O}$ for $U_\hbar\mathfrak{g}$ and $\mathfrak{g}$.

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1. Introduction

1.1. Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, $U_\hbar\mathfrak{g}$ its quantized enveloping algebra and $W$ their Weyl group. We denote by $\mathcal{O}$ the category of deformation highest weight modules of $\mathfrak{g}$, by $\mathcal{O}^{\text{int}} \subset \mathcal{O}$ the full subcategory of integrable ones, and by $\mathcal{O}^{\text{int}}_\hbar \subset \mathcal{O}_\hbar$ the corresponding categories for $U_\hbar\mathfrak{g}$. In [ATL15], we constructed an equivalence $\mathcal{O}^{\text{int}} \to \mathcal{O}^{\text{int}}_\hbar$ which intertwines the monodromy of the rational Casimir connection of $\mathfrak{g}$ and the quantum Weyl group action of the braid group $B_W$ of $\mathfrak{g}$, respectively, thus extending the equivalence obtained in [TL02, TL08, TL16] when $\mathfrak{g}$ is finite–dimensional. P. Etingof asked whether this equivalence extends to suitable categories of modules which are not necessarily integrable, while remaining equivariant under the pure braid group $P_W$ of $\mathfrak{g}$.

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The goal of the present paper is to answer this question in the affirmative. Specifically, we prove that the quantum Weyl group action of $\mathcal{P}_W$ on category $O^\text{int}_\hbar$ modules can be extended to all category $O_{\hbar}$ modules. We then show that this action is equivalent to the restriction to $\mathcal{P}_W$ of the equivariant monodromy of the Casimir connection, which is defined on any category $O$ module. Our results hold more generally for the category $O_\infty$ of modules which are locally finite under the action of the Borel subalgebra, though for simplicity we restrict to category $O$ in the Introduction.

1.2. We turn now to a more detailed description of our results. Endow $O$ with the associativity and commutativity constraints arising from the KZ equations [Dri90]. In [EK96, EK98, EK08], Etingof–Kazhdan constructed a braided tensor equivalence $F : O \rightarrow O_{\hbar}$ which is Tannakian, that is endowed with a natural isomorphism $\alpha$ fitting in the diagram

\[
\begin{array}{ccc}
O & \xrightarrow{F} & O_{\hbar} \\
\downarrow f & & \downarrow f_{\hbar} \\
\text{Vect}_{\hbar} & \xleftarrow{\alpha} & f_{\hbar}
\end{array}
\]

where $\text{Vect}_{\hbar}$ is the category of topologically free modules over $\mathbb{C}[\hbar]$, $f, f_{\hbar}$ are the forgetful functors, and $f$ is endowed with an appropriate tensor structure. The pair $(F, \alpha)$ gives rise to an isomorphism $\Psi_\alpha : \text{End}(f_{\hbar}) \rightarrow \text{End}(f)$ via the composition

\[
\text{End}(f_{\hbar}) \rightarrow \text{End}(f_{\hbar} \circ F) \rightarrow \text{End}(f)
\]

where the first isomorphism is induced by $F$, and the second is given by $\text{Ad}(\alpha)$. Note that $\alpha$ is only unique up to an automorphism $\gamma$ of $f$, and that $\Psi_{\gamma \circ \alpha} = \text{Ad}(\gamma) \circ \Psi_\alpha$.

1.3. Building on our earlier work [ATL18, ATL19a, ATL19b], we constructed in [ATL15] an automorphism $\gamma \in \text{Aut}(f)$ such that $\Psi_{\gamma \circ \alpha}$ is equivariant with respect to the action of the braid group $B_W$ on integrable category $O$ modules. Specifically, the Etingof–Kazhdan functor $F$ restricts to an equivalence $O^\text{int} \rightarrow O^\text{int}_{\hbar}$ and therefore leads to an isomorphism $\Psi_{\gamma \circ \alpha} : \text{End}(f_{\hbar}^\text{int}) \rightarrow \text{End}(f^\text{int})$ for any $\alpha' : f_{\hbar} \circ F \Rightarrow f$. Regard the quantum Weyl group action of $B_W$ on objects in $O^\text{int}_{\hbar}$ as a morphism $\lambda : B_W \rightarrow \text{End}(f_{\hbar}^\text{int})$, and the monodromy of the Casimir connection as a morphism $\mu : B_W \rightarrow \text{End}(f^\text{int})$. Then, $\gamma$ may be chosen so that the following is a commutative triangle [ATL15]

\[
\begin{array}{ccc}
\text{End}(f_{\hbar}^\text{int}) & \xrightarrow{\lambda} & B_W \\
\downarrow \Psi_{\gamma \circ \alpha} & & \downarrow \mu \\
\text{End}(f^\text{int}) & \xrightarrow{\psi} & \text{End}(f^\text{int})
\end{array}
\]

(1.1)

As a consequence, the monodromy of the Casimir connection on a module $V \in O^\text{int}$ is equivalent to the quantum Weyl group action of $B_W$ on $F(V)$.

1.4. P. Etingof asked us whether such an equivalence holds for a larger class of not necessarily integrable modules, provided $B_W$ is replaced by the pure braid group $\mathcal{P}_W$. The choice of the latter is suggested by the fact that $B_W$ does not act on all category $O$ modules for either $\mathfrak{g}$ or $U_h \mathfrak{g}$, while $\mathcal{P}_W$ does on category $O \mathfrak{g}$–modules via the monodromy of the Casimir connection.
To the best of our knowledge, no action of \( \mathcal{P}_W \) on category \( \mathcal{O}_h \) modules has been previously constructed. The main result of the present paper is to construct such an action, and then show the commutativity of the resulting diagram

\[
\begin{array}{ccc}
\text{End}(f_h) & \overset{\lambda}{\longrightarrow} & \mathcal{P}_W \\
\downarrow & & \downarrow \phi^{\gamma=\alpha} \\
& \overset{\mu}{\longrightarrow} & \text{End}(f)
\end{array}
\] (1.2)

1.5. To state our results in more detail, recall first that the abelianisation \( \mathcal{P}_W^\text{ab} = \mathcal{P}_W/\mathcal{[P}_W, \mathcal{P}_W] \) of the pure braid group is isomorphic to the free abelian group with a generator \( p_\alpha \) for each positive real root \( \alpha \) [Tit66, Dig15]. Set \( \iota = \sqrt{-1} \), and define the sign character to be the morphism

\[
\varepsilon_h: \mathcal{P}_W^\text{ab} \to \text{Aut}(\mathfrak{f}_h^\text{ab}) \\
p_\alpha \to \exp(\iota \pi h_\alpha)
\]

where \( \exp(\iota \pi h_\alpha) \) acts as multiplication by \( \exp(\iota \pi \nu(h_\alpha)) \) on the \( \nu \)-weight space of an integrable category \( \mathcal{O}_h \) module. The morphism \( \varepsilon_h \) arises as the reduction mod \( h \) of the quantum Weyl group action of \( \mathcal{P}_W^\text{ab} \) on category \( \mathcal{O}_h^\text{int} \).

As a subgroup of \( \mathcal{B}_W \), \( \mathcal{P}_W \) is generated by the elements \( S_{w,i}^2 = S_w S_i^2 S_w^{-1} \), where \( S_i \) is a generator of \( \mathcal{B}_W \), \( w \in W \) is such that \( w\alpha_i \) is a positive root, and \( S_w \in \mathcal{B}_W \) is the canonical lift of \( w \) [DG01]. Moreover, the quantum Weyl group action of \( S_{w,i} \) on a module \( V \in \mathcal{O}_h^\text{int} \) is given by

\[
\lambda(S_{w,i}^2) = \exp(\iota \pi h_{w\alpha_i}) q^{K_{w,i}} = \varepsilon_h(S_{w,i}) q^{K_{w,i}}
\]

where the second factor is the truncated quantum Casimir operator for the copy of \( U_h \mathfrak{g}_2 \subset U_h \mathfrak{g} \) corresponding to the pair \( (w, i) \) [Lus93], and \( q = \exp(h/2) \).

1.6. To extend this action to an arbitrary category \( \mathcal{O}_h \) module, we lift the sign character \( \varepsilon_h \) to a morphism

\[
\mathcal{P}_W^\text{ab} \to \text{Aut}(\mathfrak{f}_h^\text{ab}) \\
p_\alpha \to \exp(\iota \pi h_\alpha)
\]

which we denote by the same symbol. We then prove that the quantum Casimirs \( q^{K_{w,i}} \in U_h \mathfrak{g} \) give rise to a morphism \( \mathcal{X}: \mathcal{P}_W \to (U_h \mathfrak{g})^b \). It follows that

\[
\lambda: \mathcal{P}_W \to \text{Aut}(\mathfrak{f}_h^\text{ab}) \\
S_{w,i}^2 \to \exp(\iota \pi h_{w\alpha_i}) q^{K_{w,i}}
\] (1.3)

is an extension of the quantum Weyl group action of \( \mathcal{P}_W \) to all category \( \mathcal{O}_h \) modules.

1.7. The fact that \( \mathcal{X} \) is a morphism would follow at once if \( \text{End}(f_h) \) acted faithfully on \( \mathfrak{f}_h^\text{ab} \). This, however, is clearly false: if \( \varphi \) is any function on \( \mathfrak{h}^* \) which vanishes on integral weights, then \( \varphi \in \text{End}(f_h) \), but \( \varphi \) maps to zero in \( \text{End}(\mathfrak{f}_h^\text{ab}) \). To remedy this, we rely on the fact that \( U_h \mathfrak{g} \) acts faithfully on \( \mathfrak{f}_h^\text{ab} \), whose proof is due to Etingof. This implies that any \( \lambda(p) \in \text{End}(\mathfrak{f}_h^\text{ab}) \), \( p \in \mathcal{P}_W \), arises from the action of a unique element of \( U_h \mathfrak{g} \), thereby yielding the required action of \( \mathcal{P}_W \) on \( \text{End}(f_h) \).  

A similar argument works for the quantum group \( U_q \mathfrak{g} \), where \( q \) is either an indeterminate, or not a root of unity. In that case, the quantum Casimirs \( q^{K_{w,i}} \) do not lie in \( U_q \mathfrak{g} \), but in a variant \( \mathcal{D}_q \) of an algebra originally introduced by Drinfeld [Dri92, Sect. 8], which consists of formal, infinite series of the form \( \sum c_X X \), where \( X \) runs over a weight basis of \( U_q \mathfrak{g}^+ \) and \( c_X \in U_q \mathfrak{b}^- \). Etingof’s faithfulness result also applies to \( \mathcal{D}_q \), and yields an action of \( \mathcal{P}_W \) on any category \( \mathcal{O} \) module for \( U_q \mathfrak{g} \).

\footnote{Note that this bypasses having to explicitly check that the quantum Casimirs satisfy the relations of the generators \( S_{w,i} \) given in [DG01, Cor. 6].}
1.8. Let now $Y$ be the complexification of the Tits cone of $g$, $X \subset Y$ its set of regular points, and $x_0 \in X$ a basepoint. By a theorem of van der Lek [vdL83], which generalises Brieskorn’s [Bri71], the pure and full braid groups may be realised as
\[ P_W \cong \Pi_1(X; x_0) \quad \text{and} \quad B_W \cong \Pi_1(X/W; [x_0]) \]

The Casimir connection is the $Ug$–valued formal meromorphic connection on $X$ with logarithmic singularities on the root hyperplanes given by
\[ \nabla_K = d - h \sum_{\alpha \neq 0} \frac{d\alpha}{\alpha} \cdot \mathcal{K}_\alpha^+ \] (1.4)
where $\mathcal{K}_\alpha^+ = \sum_{i=1}^{m_\alpha} e^{(i)}_e e^{(i)}_\alpha$ is the normally ordered truncated Casimir operator corresponding to the positive root $\alpha$, and $h = \hbar/2\pi t$ [MTL05, TL02, Pro96, FMTV00]. The sum (1.4) over $\alpha$ is locally finite on any (not necessarily integrable) category $\mathcal{O}$ module $V$, and gives rise to a well–defined flat connection on the holomorphically trivial vector bundle $V$ on $Y$ with fibre $V$. Its monodromy therefore gives rise to a morphism
\[ \mathcal{P} : \Pi_1(X; x_0) \rightarrow \text{End}(f) \] (1.5)

1.9. The normal ordering in (1.4) breaks the equivariance of $\nabla_K$ with respect to the action of $W$ on $X$ and the subalgebra of $h$–invariants $Ug^h \subset Ug$, which contains the Casimirs $\mathcal{K}_\alpha^+$. Nevertheless, it is possible to modify the monodromy of $\nabla_K$ so that it gives rise to a representation of the braid group $B_W$ on integrable category $\mathcal{O}$ modules [ATL15, Sect. 4] (see also Section 5). This relies on the equivalence of groupoids
\[ \mathcal{E}_{x_0} : \Pi_1(X/W; [x_0]) \rightarrow W \ltimes \Pi_1(X; W x_0) \] (1.6)
where the right–hand side is the semi–direct product of $W$ with the fundamental groupoid of $X$ based at the orbit $W x_0$, and $\mathcal{E}_{x_0}$ is given by the unique lifting of loops through $x_0$, and proceeds as follows.

- Extend the monodromy of $\nabla_K$ to a morphism
\[ \mathcal{P} : \Pi_1(X; W x_0) \rightarrow \text{End}(f) \] (1.7)
- Replace the target of $\mathcal{P}$ by a subalgebra $\mathcal{T}_g \subset \text{End}(f)$ which, unlike $\text{End}(f)$, is acted upon by $W$. $\mathcal{T}_g$ is the image of the holonomy algebra of the root arrangement of $g$, and is a completion of the subalgebra of $Ug^h[h]$ generated by the Casimirs $h\mathcal{K}_\alpha^+$ and the Cartan subalgebra $h$.
- The lack of equivariance of $\nabla_K$ can then be measured by a 1–cocycle
\[ \mathcal{A} : W \rightarrow \text{Hom}(\Pi_1(X; W x_0), \mathcal{T}_g) \]
defined by $\mathcal{A}(\gamma) = \mathcal{P}(\gamma)^{-1} \cdot w^{-1} \mathcal{P}(w\gamma)$.

- We prove that $\mathcal{A}$ is abelian i.e., takes values in
\[ M = \text{Hom}(\Pi_1(X; W x_0), \exp(h)) \]
and that it is the coboundary of an essentially unique cochain $\mathcal{B} \in M$ i.e., that $\mathcal{A}_w = \mathcal{B} \cdot (w^{-1} \mathcal{B})^{-1}$ for any $w \in W$.

- As a consequence, $\mathcal{P}$ can be modified to a $W$–equivariant morphism
\[ \mathcal{P}_\mathcal{B} : \Pi_1(X; W x_0) \rightarrow \mathcal{T}_g \quad \mathcal{P}_\mathcal{B}(\gamma) = \mathcal{P}(\gamma) \cdot \mathcal{B}(\gamma) \]
- Composing $\mathcal{P}_\mathcal{B}$ with the morphism $\mathcal{E}_{x_0}$ (1.6) then yields an action of $B_W$ on any $W \ltimes \mathcal{T}_g$–module.
• It is well–known that $W$ does not act on an integrable module $V$, but that the triple exponentials

$$\tau_i = \exp(e_i) \cdot \exp(-f_i) \cdot \exp(e_i)$$  

(1.8)

are well–defined on $V$, permute its weight spaces according to the $W$–action, and give rise to a morphism $\tau : B_W \to \text{Aut}(V)$.

• Finally, lifting $E_{x_0}$ to $E_{x_0} : P_1(X/W; [x_0]) \to B_W \times P_1(X; Wx_0)$, and composing with $\tau \otimes \mathcal{P}_\mathcal{A}$ yields a morphism

$$\mathcal{P}_{\tau, \mathcal{A}} : B_W \to \text{Aut}(V)$$

$$\gamma \to \tau(\gamma) \cdot \mathcal{P}(\gamma) \cdot \mathcal{A}(\gamma)$$

which we term the equivariant monodromy of $\nabla_K$.

1.10. By [ATL15], the equivariant monodromy of $\nabla_K$ on an integrable module $V \in \mathcal{O}\text{int}$ is canonically equivalent to the quantum Weyl group action of $B_W$ on the Etingof–Kazhdan quantisation $F(V) \in \mathcal{O}_h$ i.e., the diagram (1.1) is commutative for $\mu = \mathcal{P}_{\tau, \mathcal{A}}$. This can be used to give a monodromic description of the action $\lambda$ (1.3) of $P_W$ on category $\mathcal{O}_h$ modules as follows.

The restriction of the triple exponential map $\tau$ (1.8) to $P_W$ is the sign character

$$\varepsilon : P_W \to \text{Aut}(V)$$

$$p \to \exp(\pi \mathfrak{h}a)$$

Lifting it to $\varepsilon : P_W \to \text{Aut}(V)$ as in 1.6 therefore lifts the equivariant monodromy action of $P_W$ to

$$\mathcal{P}_{\varepsilon, \mathcal{A}} : P_W \to \text{Aut}(V)$$

$$\gamma \to \varepsilon(\gamma) \cdot \mathcal{P}(\gamma) \cdot \mathcal{A}(\gamma)$$

i.e., extends the restriction of $\mathcal{P}_{\tau, \mathcal{A}}$ to $P_W$ to any category $\mathcal{O}$ module.

To relate $\mathcal{P}_{\varepsilon, \mathcal{A}}$ to $\lambda$, denote the restriction morphisms by

$$\text{Res} : \text{End}(f) \to \text{End}(f)$$

and $\text{Res}_h : \text{End}(f) \to \text{End}(f)$

The commutativity of (1.1) implies that, for any $p \in P_W$

$$\varepsilon \circ \mathcal{P}_{\varepsilon, \mathcal{A}} \circ \lambda(p) = \mathcal{P}_{\varepsilon, \mathcal{A}} \circ \lambda(p) = \mathcal{P}_{\tau, \mathcal{A}}(p)$$

and therefore that $\text{Res} \circ \Psi_{\gamma_0} \circ \lambda(p) = \mathcal{P}_{\varepsilon, \mathcal{A}} \circ \lambda(p)$, since $\varepsilon = \Psi_{\gamma_0}(\mathfrak{h})$. In turn, this implies that $\Psi_{\gamma_0} \circ \lambda(p) = \mathcal{P}(p)\mathcal{A}(p)$, so that $\Psi_{\gamma_0}$ intertwines $\lambda$ and $\mathcal{P}_{\varepsilon, \mathcal{A}}$, since $\Psi_{\gamma_0}$ maps the Drinfeld algebra $D_h \supset U_h$ to its classical analogue $D$, the latter acts faithfully on $\mathfrak{f}$, and the algebra $T_h \supset \mathcal{P}(p), \mathcal{A}(p)$ is contained in $D$.

1.11. The above can also be used to give a description of the (non–equivariant) monodromy $\mathcal{P} : P_W \to \text{Aut}(V)$ of the Casimir connection $\nabla_K$ (1.5) in terms of quantum Weyl group operators as follows.

We prove that the restriction to $P_W$ of the cochain $\mathcal{A}$ is the map $P_W \to \exp(h\mathfrak{h})$ given by $\mathcal{A}(p) = \exp(ht_a/2)$, where $t_a \in \mathfrak{h}$ corresponds to $a$ via the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ induced by the chosen inner product on $\mathfrak{g}$. Define the morphism

$$\lambda_{\varepsilon, \mathcal{A}} : P_W \to \text{Aut}(V)$$

$$p \to \varepsilon(p)^{-1} \cdot \lambda(p) \cdot \mathcal{A}(p)^{-1}$$

We refer to $\lambda_{\varepsilon, \mathcal{A}}$ as the normally ordered quantum Weyl group action of $P_W$ on category $\mathcal{O}_h$ modules. The terminology is motivated by the fact that, while $\lambda(S_0^2) = \exp(\pi \mathfrak{h}a)\cdot q^{K_{\mathfrak{a}}}$ by (1.3), $\lambda_{\varepsilon, \mathcal{A}}(S_0^2) = q^{2K_{\mathfrak{a}}}$, where the latter is a normally ordered version of the quantum Casimir. The commutativity of (1.1) then implies that $\lambda_{\varepsilon, \mathcal{A}}$ computes the monodromy of $\nabla_K$, that is that $\Psi_{\gamma_0} \circ \lambda_{\varepsilon, \mathcal{A}} = \mathcal{P}$. 
1.12. The above results can be generalised to the parabolic setting as follows. Let $J$ be a subset of nodes of the Dynkin diagram of $\mathfrak{g}$, $\mathfrak{g}_J \subseteq \mathfrak{g}$ the corresponding Lie subalgebra, $W_J \subseteq W$ its Weyl group, and $\mathcal{P}B_J \subseteq B_W$ the parabolic pure braid group given by the preimage of $W_J$.

We construct a quantum Weyl group action of $\mathcal{P}B_J$ on any category $\mathcal{O}_\hbar$ module whose restriction to $U_\hbar \mathfrak{g}_J$ is integrable. This action is such that

- its restriction to the braid group $B_{W_J}$ is the quantum Weyl group action of $B_{W_J}$ on integrable $U_\hbar \mathfrak{g}_J$–modules

- its restriction to the pure braid group $\mathcal{P}W$ coincides with the quantum Weyl group action (1.3) on category $\mathcal{O}_\hbar$ modules

We also define a normally ordered version of this quantum Weyl group action, in analogy with 1.11.

We then construct a monodromy action of $\mathcal{P}B_J$ on any category $\mathcal{O}$ module whose restriction to $\mathfrak{g}_J$ is integrable. We do so by relying on the fact that $\mathcal{P}B_J$ is isomorphic to $\Pi_1(X/W_J; \mathbb{x}_0)$, and correcting the equivariance of the Casimir connection, as outlined in 1.9, but only with respect to $W_J$. The resulting $W_J$–equivariant monodromy action is such that

- its restriction to $B_{W_J}$ is the equivariant monodromy action of $B_{W_J}$ on integrable category $\mathcal{O}_{\mathfrak{g}_J}$–modules

- its restriction to $\mathcal{P}W$ coincides with the monodromy action (1.7) on category $\mathcal{O}$ modules (up to a simple correction on $\mathcal{P}W_J$).

Finally, we show that the above quantum Weyl group and monodromic actions of $\mathcal{P}B_J$ are equivalent by relying on the fact that $\mathcal{P}B_J$ is generated by $B_{W_J}$ and $\mathcal{P}W$, and using the equivalences (1.1) for $B_{W_J}$ and (1.2) for $\mathcal{P}W$.

1.13. **Outline of the paper.** In Section 2, we review the definition of quantum Weyl group operators. In Section 3, we introduce the Drinfeld algebra and prove that it acts faithfully on $\mathcal{O}_\hbar$. In Section 4, we construct the quantum Weyl group action of $\mathcal{P}W$ on category $\mathcal{O}$. Section 5 reviews the definition of the Casimir connection, and the equivariant extension of its monodromy to a representation of the braid group $B_W$. Section 6 reviews the definition of a braided Coxeter category, and Section 7 the main result of [ATL15]. In Section 8, we prove the stated equivalence. We also point out that it continues to hold if $F$ is replaced by the Etingof–Kazhdan equivalence $F^\Phi$ corresponding to an arbitrary Lie associator $\Phi$ rather than the one arising from the KZ equations. Finally, in Section 9, we generalise these results to parabolic pure braid groups.

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2. Kac–Moody algebras and quantum groups

2.1. Symmetrisable Kac–Moody algebras [Kac90]. Let $I$ be a finite set and $A = (a_{ij})_{i,j \in I}$ a generalised Cartan matrix, i.e., $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leq 0}$, $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $A$, i.e.,

- $\mathfrak{h}$ is a finite–dimensional complex vector space
- $\Pi = \{\alpha_i\}_{i \in I}$ is a linearly independent subset of $\mathfrak{h}^*$
- $\Pi^\vee = \{h_i\}_{i \in I}$ is a linearly independent subset of $\mathfrak{h}$
- $\alpha_i(h_j) = a_{ji}$ for any $i, j \in I$

The Kac–Moody algebra corresponding to $A$ and the realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ is the Lie algebra $\mathfrak{g}$ generated by $\mathfrak{h}$ and elements $\{e_i, f_i\}_{i \in I}$, with relations $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$, $[e_i, f_j] = \delta_{ij}h_i$ and, for any $i \neq j$,

$$\text{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \text{ad}(f_i)^{1-a_{ij}}(f_j)$$

Let $n^+ \subset \mathfrak{g}$ be the Lie subalgebras generated by $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$, respectively.

Assume that $A$ is symmetrisable, and fix an invertible diagonal matrix $D = \text{diag}(d_i)_{i \in I}$ with coprime entries $d_i \in \mathbb{Z}_{>0}$ such that $DA$ is symmetric. Then, there is a symmetric, non–degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$ such that $\langle h_i, - \rangle = d_i^{-1}\alpha_i$ (see, e.g., [ATL19b, Prop. 11.4]). The corresponding identification $\nu : \mathfrak{h} \to \mathfrak{h}^*$ intertwines the actions of $W$, satisfies $\nu(h_i) = d_i^{-1}\alpha_i$ and therefore restricts to an isomorphism $\mathfrak{h}' \cong Q \otimes_{\mathbb{Z}} \mathbb{C}$, where $\mathfrak{h}'$ is the span of $\{h_i\}_{i \in I}$ and $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subseteq \mathfrak{h}^*$ is the root lattice. Note that $\langle h_i, h_i \rangle = 2d_i^{-1}$, while the induced form on $\mathfrak{h}^*$ satisfies $\langle \alpha_i, \alpha_i \rangle = 2d_i \in 2\mathbb{Z}_{>0}$.

By [Kac90, Thm. 2.2], $\langle \cdot, \cdot \rangle$ uniquely extends to a non–degenerate, invariant symmetric bilinear form on $\mathfrak{g}$, which satisfies $\langle e_i, f_j \rangle = \delta_{ij}d_i^{-1}$ and $[x, y] = \langle x, y \rangle \cdot t_\alpha$ for any $x, y \in \mathfrak{g}_\alpha$, $\alpha \in \mathfrak{g}^*$, where $t_\alpha = \nu^{-1}(\alpha)$.

2.2. Category $\mathcal{O}_{\infty}$ representations. If $V$ is an $\mathfrak{h}$–module and $\lambda \in \mathfrak{h}^*$, we denote the corresponding weight space of $V$ by

$$V[\lambda] = \{v \in V \mid h v = \lambda(h)v, h \in \mathfrak{h}\}$$

and set $P(V) = \{\lambda \in \mathfrak{h}^* \mid V[\lambda] \neq 0\}$. A $\mathfrak{g}$–module $V$ is

(C1) a weight module if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

(C2) integrable if it is a weight module, and the elements $\{e_i, f_i\}_{i \in I}$ act locally nilpotently.

This implies that $\lambda(h_i) \in \mathbb{Z}$ for any $\lambda \in P(V)$ and $i \in I$, and that $V$ is completely reducible as a (possibly infinite) direct sum of simple finite–dimensional modules over $\mathfrak{sl}_2 \cong \langle e_i, h_i, f_i \rangle \subset \mathfrak{g}$.

(C3) in category $\mathcal{O}_{\infty, 0}$ if the action of $\mathfrak{h}^+$ is locally finite, i.e., any $v \in V$ is contained in a finite–dimensional $\mathfrak{h}^+$–submodule of $V$.

This implies in particular that $V$ is the direct sum of its generalised weight spaces and that, for any $v \in V$, $(Un^+)\beta v = 0$ for all but finitely many $\beta \in Q^+$.

Footnote:

1Note that, unlike [Kac90], we do not require $\mathfrak{h}$ to have minimal dimension $2|I| - \text{rank}(A)$. 


(C4) in category $\mathcal{O}_q^g$ if it is a weight module with finite-dimensional weight spaces, such that

$$P(V) \subseteq D(\lambda_1) \cup \cdots \cup D(\lambda_m)$$

(2.1)

for some $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$, where $D(\lambda) = \{ \mu \in \mathfrak{h}^* | \mu < \lambda \}$ and $\mu \leq \lambda$ iff $\lambda - \mu \in \mathbb{Q}_{\geq 0} = \bigoplus_{i \in \mathbb{N}} \mathbb{N}\alpha_i$

The categories $\mathcal{O}_q^g \subset \mathcal{O}_q^\infty$ are symmetric tensor categories. Denoting by $\mathcal{O}_g^\text{int} \subset \mathcal{O}_g^\infty$ and $\mathcal{O}_\infty^\text{int} \subset \mathcal{O}_\infty^\infty$, the full tensor subcategories of integrable representations, we have the following inclusions

$$\mathcal{O}_q^g \subset \mathcal{O}_q^\infty, \quad \mathcal{O}_g^\text{int} \subset \mathcal{O}_\infty^\text{int}, \quad \mathcal{O}_\infty^\text{int} \subset \mathcal{O}_\infty^\infty$$

2.3. **Deformation category $\mathcal{O}_\infty$ representations.** Similar notions can be defined for $\mathfrak{g}$–modules in the category $\text{Vect}_h$ of topologically free $\mathbb{C}[\hbar]$–modules. Namely, a $\mathfrak{g}$–module $V \in \text{Vect}_h$ is called

(D1) a **weight module** if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda],^1$ where $\bigoplus$ is the direct sum in $\text{Vect}_h$, i.e., the completion of the algebraic direct sum in the $\mathbb{C}$–adic topology.

(D2) **integrable** if it is a weight module and, for any $i \in \mathbb{I}$ and $v \in V$, $\lim_{n \to \infty} e_i^n v = 0 = \lim_{n \to \infty} f_i^n v$.

(D3) in category $\mathcal{O}_q^h$ if the action of $\mathfrak{b}^+$ on $V/h^n V$ is locally finite for any $n \geq 0$.

(D4) in category $\mathcal{O}_q^\infty$ if it is a weight representation with finite–rank weight spaces, and such that $P(V)$ satisfies (2.1).

It is easy to see that $V$ is a weight (resp. integrable) module in $\text{Vect}_h$ if and only if $V/h^n V$ is a weight (resp. integrable) module in $\text{Vect}$ for any $n \geq 0$. We denote by $\mathcal{O}_q^\text{int} \subset \mathcal{O}_q^h$ and $\mathcal{O}_\infty^\text{int} \subset \mathcal{O}_\infty^\infty$ the full tensor subcategories of integrable representations.

2.4. **Braid group action.** Let $W$ be the Weyl group of $\mathfrak{g}$, and $\{s_i\}_{i \in \mathbb{I}}$ its set of simple reflections. The braid group $B_W$ is the group generated by the elements $\{S_i\}_{i \in \mathbb{I}}$, with relations

$$S_i \cdot S_j \cdot S_i \cdots = S_j \cdot S_i \cdot S_j \cdots$$

(2.2)

for any $i \neq j$, where $m_{ij}$ is the order of $s_i s_j$ in $W$. If $V$ is an integrable $\mathfrak{g}$–module in $\text{Vect}$ or $\text{Vect}_h$, the operators

$$\tilde{s}_i = \exp(e_i) \cdot \exp(-f_i) \cdot \exp(e_i) \in \text{GL}(V)$$

(2.3)

are well-defined, and satisfy the braid relations (2.2) [Tit66]. The corresponding action of $B_W$ on $V$ factors through the **Tits extension** $\tilde{W}$, an extension of $W$ by the sign group $\mathbb{Z}_2$.

---

1Note that the eigenvalues of the action of $\mathfrak{h}$ on $V$ are required to lie in $\mathfrak{h}^* \subset \mathfrak{h}^*[\hbar]$. 
2.5. **The quantum group** $U_{h\mathfrak{g}}$ [Dri87, Jim85]. Let $h$ be a formal variable, set $q = \exp(h/2)$ and $q_i = q^{i_1}$, $i \in \mathbf{I}$. The Drinfeld–Jimbo quantum group of $\mathfrak{g}$ is the algebra $U_{h\mathfrak{g}}$ over $\mathbb{C}[h]$ topologically generated by $\mathfrak{h}$ and the elements $\{E_i, F_i\}_{i \in \mathbf{I}}$, subject to the relations $[h, h'] = 0$,

$$[h, E_i] = \alpha_i(h)E_i \quad [h, F_i] = -\alpha_i(h)F_i \quad [E_i, F_i] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$$

for any $h, h' \in \mathfrak{h}$, $i, j \in \mathbf{I}$, and the $q$–Serre relations

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \binom{1-a_{ij}}{m} X_i^{1-a_{ij}-m} X_j X_i^m = 0$$

for $X = E, F$, $i \neq j \in \mathbf{I}$, where $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and, for any $k \leq n$,

$$[n]_i! = [n]_i \cdot [n-1]_i \cdots [1]_i \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i! \cdot [n-k]_i}$$

Define weight, integrable, category $\mathcal{O}_\infty$ and $\mathcal{O}$ modules for $U_{h\mathfrak{g}}$ in $\text{Vect}_h$ analogously to Section 2.3, and denote by

$$\mathcal{O}^{\text{int}}_{\infty, U_{h\mathfrak{g}}} \subset \mathcal{O}_{\infty, U_{h\mathfrak{g}}} \quad \text{and} \quad \mathcal{O}^{\text{int}}_{U_{h\mathfrak{g}}} \subset \mathcal{O}_{U_{h\mathfrak{g}}}$$

the subcategories of integrable modules.\(^1\)

2.6. **Quantum Weyl group operators** [KR90, Lus90, Lus93, Sa94, So90]. For any $\mathcal{V} \in \mathcal{O}^{\text{int}}_{\infty, U_{h\mathfrak{g}}}$, define the endomorphisms $\{S_i\}_{i \in \mathbf{I}}$ of $\mathcal{V}$ as follows.\(^2\) For any $v_\mu \in \mathcal{V}[\mu]$, set

$$S_i v_\mu = \sum_{a, b, c \in \mathbb{Z}_{\geq 0}} (-1)^b q_i^{b-ae} E_i^{(a)} F_i^{(b)} E_i^{(c)} v_\mu$$

where $X_i^{(a)} = X_i^a / [a]_i!$.

Then, $S_i(\mathcal{V}[\mu]) \subseteq \mathcal{V}[S_i(\mu)]$ and the $S_i$ give rise to an action of the braid group $B_W$ on $\mathcal{V}$, which deforms the action by triple exponentials described in 2.4 [Lus93, Sec. 39.4].

2.7. **Action of $B_W$ on $U_{h\mathfrak{g}}$** ([Lus88], [Lus93, Chap. 37–39]). Consider the algebra automorphisms $\{T_i\}_{i \in \mathbf{I}}$ of $U_{h\mathfrak{g}}$ defined by

$$T_i(h) = s_i(h) \quad T_i(E_i) = -F_i q_i^{h_i} \quad T_i(F_i) = -q_i^{-h_i} E_i$$

where $h \in \mathfrak{h}$ and, for any $i \neq j \in \mathbf{I}$,

$$T_i(X_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{\sigma(X)^r} X_i^{-a_{ij}-r} X_j X_i^r$$

where $X = E, F$ and $\sigma(E) = -1 = -\sigma(F)$.

The automorphisms $\{T_i\}_{i \in \mathbf{I}}$ define an action of the braid group $B_W$ on $U_{h\mathfrak{g}}$ which we denote by $b(X)$, $b \in B_W$ and $X \in U_{h\mathfrak{g}}$. Moreover, for any $X \in U_{h\mathfrak{g}}$, $\mathcal{V} \in \mathcal{O}^{\text{int}}_{\infty, U_{h\mathfrak{g}}}$, and $v \in \mathcal{V}$, one has $S_i(X \cdot v) = T_i(X) \cdot S_i(v)$.

\(^1\)Note in particular that a representation $\mathcal{V}$ of $U_{h\mathfrak{g}}$ is in category $\mathcal{O}_{\infty}$ if the action of $U_{h\mathfrak{g}} b^+$ on $\mathcal{V}$ is locally finite for any $n \geq 0$.

\(^2\)The operator $S_i$ is the operator $T_i^\mu$ defined in [Lus93, Sec. 5.2].
3. Faithfulness of category $O$ integrable modules

Integrable $U_h\mathfrak{g}$-modules are well-known to be faithful, i.e., the only element of $U_h\mathfrak{g}$ acting trivially on every integrable module is zero [Lus93, Prop. 3.5.4]. To the best of our knowledge, the analogous result for the more restrictive class of integrable modules in category $O$ does not appear in the literature. We present here a proof due to P. Etingof, which establishes faithfulness for a larger algebra containing $U_h\mathfrak{g}$.

3.1. The Drinfeld algebra $D_h$. For any $\beta \in \mathbb{Q}_+$, let $B_\beta = \{X_{\beta,p}\}$ be a basis of $U_h n_\beta^+$ and set $B = \bigsqcup_{\beta \in \mathbb{Q}_+} B_\beta$. Set

$$D_h = \left\{ \sum_{x \in B} c_x X : c_x \in U_h b^- \right\} = \prod_{\beta \in \mathbb{Q}_+} U_h b^- \otimes U_h n_\beta^+ \supset U_h \mathfrak{g}$$

$D_h$ has an algebra structure which extends that of $U_h \mathfrak{g}$. Moreover, the action of $U_h \mathfrak{g}$ on any module $V \in O_{U_h \mathfrak{g}}$ extends to one of $D_h$ since, for any $v \in V$, $U_h n_\beta^- v = 0$ for all but finitely many $\beta \in \mathbb{Q}_+$.

**Theorem** (Etingof). Category $O$ integrable $U_h \mathfrak{g}$-modules are faithful for $D_h$.

The proof is carried out in Sections 3.2–3.4.

**Remark.** A variant $Q_h$ of the algebra $D_h$ was introduced by Drinfeld in [Dri92, Sect. 8] as follows. For any $\beta \in \mathbb{Q}_+$, let $I_\beta \subset U_h \mathfrak{g}$ be the left ideal generated by $U_h \mathfrak{n}_{\beta'}$ for any $\beta' > \beta$, or equivalently by $\{U_h n_{\beta'}^+\}_{\beta' > \beta}$, and set $Q_h = \text{lim}_\beta U_h \mathfrak{g}/I_\beta$. Since $U_h \mathfrak{g}/I_\beta \cong \bigoplus_{\beta' > \beta} U_h b^- \otimes U_h n_{\beta'}^+$, $Q_h$ embeds into $D_h$ as the subalgebra consisting of series $\sum_{\beta \in \mathbb{Q}_+} X_{\beta}$, $X_\beta \in U_h b^- \otimes U_h n_\beta^+$, where for any $\beta \in \mathbb{Q}_+$, $X_{\beta'} = 0$ for all but finitely many $\beta' \neq \beta$. The algebra $Q_h$ is less natural than $D_h$, however. For instance, if $\emptyset \subsetneq J \subsetneq I$ is a proper non-empty subset, $\mathfrak{g}_J \subset \mathfrak{g}$ the corresponding subalgebra, and $Q_{J,h}$ (resp. $D_{J,h}$) the analogue of $Q_h$ (resp. $D_h$) for $\mathfrak{g}_J$, then $D_{J,h} \subset D_h$ while $Q_{J,h}$ does not map to $Q_h$.

3.2. Verma modules. For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of highest weight $\lambda$ and $v_\lambda \in M(\lambda)$ its cyclic vector. For any $\beta \in \mathbb{Q}_+$, let $M(\lambda)_\beta \subset M(\lambda)$ be the weight space of weight $\lambda - \beta$. Note that there is a natural identification $M(\lambda)_\beta \simeq (U_h n^-)_\beta$. Recall that the contragredient Verma module $M^\vee(\lambda)$ is the pullback through the Chevalley involution of the restricted dual $M^*(\lambda) = \bigoplus_{\beta \in \mathbb{Q}_+} M(\lambda)^*_\beta$, where $M(\lambda)^*_\beta$ denotes the dual in Vect$_h$. The contragredient Verma module is equipped with a morphism $M(\lambda) \to M^\vee(\lambda)$, $v_\lambda \mapsto v_\lambda^*$. The Shapovalov form on $M(\lambda)$ is defined by

$$\langle \cdot, \cdot \rangle_\lambda : M(\lambda) \otimes M(\lambda) \to M(\lambda) \otimes M^\vee(\lambda) \to \mathbb{C}[h]$$

By construction, it satisfies $\langle v_\lambda, v_\lambda \rangle_\lambda = 1$, $\langle M(\lambda)_\beta, (M(\lambda)_{\beta'})_\lambda \rangle = 0$ if $\beta \neq \beta'$, and $\langle xv, w \rangle_\lambda = -\langle x, \omega(x)w \rangle_\lambda$ for any $x \in \mathfrak{g}$, $v, w \in M(\lambda)$. It is well-known that $\langle \cdot, \cdot \rangle_\lambda$ is symmetric and non-degenerate only for generic $\lambda \in \mathfrak{h}^*$.

For generic $\lambda \in \mathfrak{h}^*$, let $B'_{\lambda,\beta} = \{X_{\beta,p}^*\}$ be the dual basis of $U_h n_\beta$ with respect to the Shapovalov form. In particular, one has $\langle X_{\beta,p}^* v_\lambda, \omega(S(X_{\beta,j})) v_\lambda \rangle = \delta_{ij}$. Thus, modulo elements of weights lower than $\lambda$, $X_{\beta,j} X_{\beta,i}^* v_\lambda = \delta_{ij} v_\lambda$.

**Proposition.** Verma modules are faithful for $D_h$. 

Therefore, it is possible to choose \( \lambda \) module, so that
\[
\lambda \text{ check that this holds for } O \text{ integrable}
\]
i.e.
\[
\text{Proof.}
\]
Assume that \( x \in D_\hbar \) acts trivially on every \( M(\lambda) \), and write
\[
x = \sum g x_{\beta,i} x_{\beta,i}^0 X_{\beta,i}
\]
where \( x_{\beta,i}^0, x_{\beta,i}^- \in U_\hbar \hbar \) and \( x_{\beta,i}^- \in U_\hbar \hbar n^- \). Note that, for any \( \lambda \in \hbar^* \), the action of \( x \) on the cyclic vector \( v_\lambda \in M(\lambda) \) gives
\[
0 = x \cdot v_\lambda = \lambda(\varphi_0) x_0 \cdot v_\lambda
\]
Therefore, \( x_0^0 = 0 = x_0^- \). We shall prove that, for any \( X_{\beta,i} \in B \), it holds \( x_{\beta,i}^0 = 0 = x_{\beta,i}^- \). Proceeding by induction, we assume that \( x_{\gamma,j} = 0 = x_{\gamma,j}^0 \) for any \( X_{\gamma,j} \in B \) such that \( h \gamma < n \). Fix \( \beta \in \mathbb{Q}_+ \) with \( h \beta = n \). Then, for generic \( \lambda \in \hbar^* \), we have \( X_{\beta,i}^* v_\lambda \in M(\lambda)_\beta \) and, since \( X_{\beta,j} X_{\beta,i}^* v_\lambda = \delta_{ij} v_\lambda \),
\[
0 = x \cdot X_{\beta,i}^* v_\lambda = \sum_j x_{\beta,j} x_{\beta,j}^0 X_{\beta,j} X_{\beta,i}^* v_\lambda = \lambda(x_{\beta,i}^0) x_{\beta,i}^- v_\lambda
\]
Therefore, \( x_{\beta,i}^0 = 0 = x_{\beta,i}^- \). \( \square \)

3.3. Regularity of the matrix coefficients on \( M(\lambda) \). For any \( \lambda \in \hbar^* \), let \( M^*(\lambda) \) be the (restricted) dual Verma module and \((\cdot, \cdot)_{M(\lambda)} : M(\lambda) \otimes M^*(\lambda) \to \mathbb{C}[\hbar] \) the natural pairing.

Proposition. For any \( \lambda \in \hbar^* \), \( v \in M(\lambda) \), and \( f \in M(\lambda)^* \), the matrix coefficient \((xv, f)_{M(\lambda)} \) lies in \( \mathbb{C}[\lambda][\hbar] \).

Proof. Note that, for any \( x^\pm \in U_\hbar \hbar n^\pm \), the coefficient \((x^- v, x^+ f) \in \mathbb{C}[\hbar] \) is independent of \( \lambda \). We can write \( x = \sum_i x_i^+ x_i^- \), for some \( x_i^+ \in U_\hbar \hbar n^+, \ x_i^- \in U_\hbar \hbar \), and \( x_i^- \in (U_\hbar \hbar n^-)_\beta \), with \( \beta_i \in \mathbb{Q}_+ \). Then, we have
\[
(xv, f)_{M(\lambda)} = \sum_i (x_i^+ x_i^- v, S(x_i^+) f)_{M(\lambda)} = \sum_i (\lambda - \beta_i)(x_i^+ v, S(x_i^+) f)_{M(\lambda)}.
\]
The result follows. \( \square \)

3.4. Proof of Theorem 3.1. Assume that \( x \in D_\hbar \) acts trivially on every category \( \mathcal{O} \) integrable \( U_\hbar \mathfrak{g} \) module. We shall prove that \( x \) acts trivially on any Verma module, so that \( x = 0 \) by Proposition 3.2.

Clearly, \( x \) acts trivially on \( M(\lambda) \) if and only if, for any \( v \in M(\lambda) \) and \( f \in M(\lambda)^* \), the matrix coefficient \((xv, f)_{M(\lambda)} \) vanishes. By Proposition 3.3, it is enough to check that this holds for \( \lambda \) in a Zariski open subset of \( \hbar^* \). To this end, note that, if \( v \in M(\lambda)_\beta \), then \( xv = x(\beta)v \), where \( x(\beta) \in U_\mathfrak{g} \) is the truncation of \( x \) at \( \beta \). Therefore, it is possible to choose \( \lambda \in P_+ \) large enough such that
\[
(xv, f)_{M(\lambda)} = (xv, f)_{L(\lambda)} = 0
\]
i.e., \((xv, f)_{M(\lambda)} \) is equal to the matrix coefficient of \( x \) on the unique irreducible quotient \( L(\lambda) \) of \( M(\lambda) \). By assumption on \( x \), the latter is zero, since \( L(\lambda) \) is integrable for \( \lambda \in P_+ \). The result follows.
4. Quantum Weyl group actions of pure braid groups

4.1. Completions. Let $A$ be an algebra, $C \subset \text{Rep}(A)$ a full subcategory, and $\text{End}(f_C)$ the algebra of endomorphisms of the forgetful functor $f_C : C \to \text{Vect}$. By definition, an element of $\text{End}(f_C)$ is a collection

$$\varphi = \{ \varphi_V \}_{V \in C} \in \prod_{V \in C} \text{End}(V)$$

which is natural, i.e., such that $f \circ \varphi_V = \varphi_W \circ f$ for any $f : V \to W$ in $C$. The action of $A$ on any $V \in C$ yields a morphism of algebras $A \to \text{End}(f_C)$, and factors through the action of $\text{End}(f_C)$ on $V$. We shall refer to $\text{End}(f_C)$ as the completion of $A$ with respect to the category $C$.

4.2. Braid groups and completions. The braid group actions considered in Section 2 can be concisely described in terms of completions. For instance, let $\text{End}(f^w_C)$ be the algebra of endomorphisms of the forgetful functor $f^w_C : \mathcal{O}_{\infty, U_h \mathfrak{g}} \to \text{Vect}_h$. The quantum Weyl group operators $S_i$ defined by (2.4) are elements of $\text{Aut}(f^w_C)$, and yield a group homomorphism $\lambda : B_W \to \text{Aut}(f^w_C)$.

4.3. Sign character of the pure braid group. Let $Z$ be the free abelian group with a generator $p_\alpha$ for each positive real root $\alpha$, endowed with the $W$–action given by $w p_\alpha = p_{|w\alpha|}$, where $|w\alpha| = \pm w\alpha$ according to whether $w\alpha$ is positive or negative.

Let $P_W \subset B_W$ be the pure braid group. Its abelianisation $P^0_W = P_W/[P_W, P_W]$ is acted upon by $B_W/P_W \simeq W$. By [Tit66, Thm. 2.5] and [Dig15] the assignment $p_\alpha \to S_\alpha^2$ uniquely extends to a $W$–equivariant isomorphism $Z \to P^0_W$.

Define the sign character of $P_W$ to be the morphism

$$\varepsilon_h : P^0_W \to \text{Aut}(f^w_C) \quad \varepsilon_h(p_\alpha) = \exp(i \pi h_\alpha) \quad (4.1)$$

where $\exp(i \pi h_\alpha)$ is the operator acting on a weight space of (integral) weight $\lambda$ as multiplication by $\exp(i \pi \lambda(h_\alpha))$.

4.4. Canonical lift of the sign character. Let $f_h : \mathcal{O}_{\infty, U_h \mathfrak{g}} \to \text{Vect}_h$ be the forgetful functor, and consider the morphism $\text{Aut}(f_h) \to \text{Aut}(f^w_C)$ corresponding to the inclusion $\mathcal{O}_{\infty, U_h \mathfrak{g}} \subset \mathcal{O}_{\infty, U_h \mathfrak{g}}$. The sign character $\varepsilon_h$ has a canonical lift

$$P^0_W \to \text{Aut}(f_h) \quad p_\alpha \to \exp(i \pi h_\alpha)$$

which is well–defined since for any $V \in \mathcal{O}_{\infty, U_h \mathfrak{g}}$ and $n \geq 0$, $V/\hbar^n V$ is a locally finite $\mathfrak{h}$–module. We denote this lift by the same symbol.

4.5. Pure braid group action on category $\mathcal{O}_{\infty}$. The following is one of the main results of this paper.

**Theorem.** Let $\lambda : B_W \to \text{End}(f^w_C)$ be the quantum Weyl group action of the braid group $B_W$. Then, the following holds.

1. For any $p \in P_W$,

$$\lambda(p) = \varepsilon_h(p) \cdot \mathcal{K}(p)$$

where $\varepsilon_h(p)$ is the sign character (4.1), and $\mathcal{K}(p)$ is a unique element of $U_h \mathfrak{g}$ which is invertible and of weight zero.

2. The assignment $p \to \mathcal{K}(p)$ is a homomorphism $P_W \to (U_h \mathfrak{g})^h$ which is $B_W$–equivariant.
(3) The quantum Weyl group action of the pure braid group $\mathcal{P}_W$ on integrable modules extends to an action

$$\lambda : \mathcal{P}_W \rightarrow \text{Aut}(f_h) \quad \text{given by} \quad \lambda(p) = \varepsilon_h(p) \cdot \mathcal{X}(p) \quad (4.2)$$

(4) The map $\lambda$ intertwines the inner action of $\mathcal{P}_W$ on $U_h\mathfrak{g}$ i.e., for any element $Y \in U_h\mathfrak{g}$ and $p \in \mathcal{P}_W$

$$\lambda(p) Y \lambda(p)^{-1} = p(Y)$$

in $\text{End}(f_h)$.

**Proof.** (2),(3) and (4) follow from (1).

(1) It suffices to prove the existence of $\mathcal{X}(p)$ for a set of generators of $\mathcal{P}_W$. The uniqueness of $\mathcal{X}(p)$ for any $p \in \mathcal{P}_W$ then follows from Theorem 3.1. By [DG01, Cor. 6] (see also [Dig15, Prop. 2.5]), $\mathcal{P}_W$ is generated by the elements $S_w S_w^2 S_w^{-1}$, where $w \in W$.

Consider first the case $w = 1$. By [Lus93, Sec. 5.2], the square of the operator $S_i$ is related to the quantum Casimir operator of $U_h \mathfrak{sl}_2^i = \langle E_i, F_i, h_i \rangle \subset U_h \mathfrak{g}$ as follows. Let $f_{w,i} : \text{Out}_{\mathfrak{sl}_2^i} \rightarrow \text{Vect}$ be the forgetful functor. An element of $\text{End}(f_{w,i}^m)$ is determined by its action on each of the indecomposable representations $\{V_r^i\}_{r \geq 0}$, where $V_r^i$ is of rank $r + 1$. The Casimir operator $C_i$ of $U_h \mathfrak{sl}_2^i$ acts on $V_r^i$ as multiplication by $d_i r(r + 2)/2$. Set $K_i = C_i - d_i h_i^2/2$, so that $K_i$ acts on the subspace of $V_r^i$ of weight $m a_i$ as multiplication by $d_i (r(r + 2) - m^2)/2$. Then,

$$S_i^2 = \exp(i \pi h_i) \cdot q^{K_i} \quad (4.3)$$

By [Dri89, Sec. 5],

$$q^{K_i} = \sum_{m \geq 0} F_i^m \phi_m E_i^m$$

for some explicit $\phi_m \in U_h[h]$. It follows that $q^{K_i}$ lies in $U_h \mathfrak{g}$, and therefore so does $q^{K_i}$. Thus, setting $\mathcal{X}(S_i^2) = q^{K_i} \in \mathcal{D}_h$, we get

$$\lambda(S_i^2) = S_i^2 = \exp(i \pi h_i) \cdot q^{K_i} = \varepsilon_h(S_i^2) \cdot \mathcal{X}(S_i^2)$$

Note next that if $w \in W$ satisfies $w a_i > 0$, then $T_w = \text{Ad}(S_w)$ satisfies $T_w(E_i) \in U_h \mathfrak{b}^+_w a_i$, and $T_w(F_i) \in U_h \mathfrak{b}^-_w a_i$. [Lus93, Sec. 37.1]. It follows that $q^{K_{w,i}} = T_w(q^{K_i})$ is a weight zero element in $\mathcal{D}_h$, and if we set $\mathcal{X}(S_w S_w^2 S_w^{-1}) = q^{K_{w,i}}$, then

$$\lambda(S_w S_w^2 S_w^{-1}) = S_w S_w^2 S_w^{-1} = \exp(i \pi h_{w,i}) \cdot q^{K_{w,i}} = \varepsilon_h(p) \cdot \mathcal{X}(p)$$

□

**Remarks.**

(1) The proof of Theorem 4.5 shows that the action $\lambda$ on category $\mathcal{O}_\infty$ modules for $U_h \mathfrak{g}$ is explicitly given on the generators of $\mathcal{P}_W$ by

$$\lambda(S_w S_w^2 S_w^{-1}) = \exp(i \pi h_{w,i}) \cdot q^{K_{w,i}}$$

(2) Since $\mathcal{X}$ maps to $U_h \mathfrak{g}$, it defines a (signless quantum Weyl group) action of $\mathcal{P}_W$ on any $U_h \mathfrak{g}$–module.
4.6. The normally ordered quantum Weyl group action. We shall be interested in the following modification of the action (4.2). Let
\[ \mathcal{B} : \mathcal{P}_W \to \exp(\hbar h) \subset U_h g \] be given by \[ \mathcal{B}(p) = q^{i_n} = \exp(\hbar t_n/2) \]
(cf. Section 1.11). Define the morphism
\[ \lambda_{\varepsilon,\mathcal{B}} : \mathcal{P}_W \to U_h g \] by \[ \lambda_{\varepsilon,\mathcal{B}}(p) = \mathcal{X}(p) \cdot \mathcal{B}(p)^{-1} \]
so that \( \lambda(p) = \varepsilon(p) \cdot \lambda_{\varepsilon,\mathcal{B}}(p) \cdot \mathcal{B}(p) \) for any \( p \in \mathcal{P}_W \).

We refer to \( \lambda_{\varepsilon,\mathcal{B}} \) as the normally ordered quantum Weyl group action of \( \mathcal{P}_W \).

The terminology is justified by the fact that, for any \( i \in I \), \( \lambda_{\varepsilon,\mathcal{B}}(S_i^2) \) acts as the normally ordered quantum Casimir operator, in contrast with (4.3). Namely, one has
\[ \lambda_{\varepsilon,\mathcal{B}}(S_i^2) = \mathcal{X}(S_i^2) \cdot \mathcal{B}(p)_{\alpha_i}^{-1} = q^{2\mathcal{K}_i^+} \]
where \( \mathcal{K}_i^+ = (\kappa_i - t_n)/2 \). This modified action will be relevant in Theorem 8.2.

Note also that for any element \( Y \in U_h g \) of weight \( \gamma \in \mathbb{Q} \) and \( p \in \mathcal{P}_W \), one has
\[ \text{Ad}(\lambda_{\varepsilon,\mathcal{B}}(p))(Y) = p(Y) \cdot (\varepsilon(h)(p), \gamma)^{-1} \cdot (\mathcal{B}(p), \gamma)^{-1} \]
in \( \text{End}(f_h) \).

4.7. Pure braid group actions for \( U_q g \). Let \( \mathbb{K} \) be a field of characteristic zero, \( q \in \mathbb{K}^\times \) an element of infinite order, \( e.g., q \in \mathbb{C}^\times \) not a root of unity or \( q \in \mathbb{Q}(q) \), and \( U_q g \) the corresponding quantum group over \( \mathbb{K} \).

The definition of (integrable) category \( \mathcal{O}_\infty U_q g \)-modules is similar to the formal case (see \( e.g., [Lus93, Ch. 3] \)). The analogues of Theorem 4.5 and Section 4.6 hold for \( U_q g \) and defines actions of \( \mathcal{P}_W \) on category \( \mathcal{O}_\infty \) modules.

In this case, the quantum Casimirs \( q^\mathcal{K}_i \) do not lie in \( U_q g \), but in the Drinfeld algebra \( D_q \) of \( U_q g \), and the morphism \( \mathcal{X} \) takes values in \( D_q \). Note that the latter acts on any category \( \mathcal{O}_\infty \) module \( V \) since, for any \( v \in V \), \( (U_q n^+)_{\beta} v = 0 \) for all but finitely many \( \beta \in Q^+ \).

5. The Casimir connection

5.1. Fundamental group of root system arrangements. Let \( A \) be a symmetrisable generalised Cartan matrix, \( (\mathfrak{h}_R, \Pi, \Pi^+) \) a realisation of \( A \) over \( \mathbb{R} \), and \( (\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_R, \Pi, \Pi^+) \) its complexification. Let \( \Pi^+ \subset \mathfrak{h} \) be the annihilator of \( \Pi \), set \( \mathfrak{h}^c = \mathfrak{h}/\Pi^+ \), and note that \( \mathfrak{h}^c \) is independent of the realisation of \( A \). Let
\[ \mathcal{C} = \{ h \in \mathfrak{h}^c_R \ | \ \forall i \in I, \alpha_i(h) > 0 \} \]
be the fundamental Weyl chamber in \( \mathfrak{h}^c_R \), and \( \Upsilon_R = \bigcup_{w \in W} w(\mathcal{C}) \) the Tits cone. \( \Upsilon_R \) is a convex cone, and the Weyl group \( W \) acts properly discontinuously on its interior \( \Upsilon_R \) and complexification \( \mathcal{Y} = \Upsilon_R + i\mathfrak{h}_R^c \subseteq \mathfrak{h}^c \) [Loo80, Vin71]. The regular points of this action are given by
\[ \mathcal{X} = \mathcal{Y} \setminus \bigcup_{\alpha \in \Delta^+} \text{Ker}(\alpha) \]
The action of \( W \) on \( \mathcal{X} \) is proper and free, and the space \( \mathcal{X}/W \) is a complex manifold. The following result is due to van der Lek [vdL83], and generalises Brieskorn’s Theorem [Bri71] to the case of an arbitrary Weyl group.

Theorem. The fundamental groups of \( \mathcal{X}/W \) and \( \mathcal{X} \) are isomorphic to \( B_W \) and \( \mathcal{P}_W \) respectively.
The generators \{S_i\}_{i \in I} of \mathcal{B}_W may be described as follows. Let \( p : X \to X/W \) be the canonical projection, fix a point \( x_0 \in C \) and use \([x_0] = p(x_0)\) as a base point in \( X/W \). For any \( i \in I \), choose an open disk \( D_i \) in \( x_0 + C h_i \), centered in \( x_0 - \frac{a_i(x_0)}{2} h_i \), and such that \( \overline{D_i} \) does not intersect any root hyperplane other than \( \text{Ker}(\alpha_i) \). Let \( \gamma_i : [0, 1] \to x_0 + C h_i \) be the path from \( x_0 \) to \( s_i(x_0) \) in \( X \) determined by \( \gamma_i|_{[0,1/3]} \cup [2/3,1] \) is affine and lies in \( x_0 + \mathbb{R} h_i \setminus D_i \), the points \( \gamma_i(1/3), \gamma_i(2/3) \) are in \( \partial \overline{D_i} \), and \( \gamma_i|_{1/3,2/3} \) is a semicircular arc in \( \partial \overline{D_i} \), positively oriented with respect to the natural orientation of \( x_0 + C h_i \). Then, \( S_i = p \circ \gamma_i \).

5.2. The Casimir connection. For any positive root \( \alpha \in \Delta_+ \), let \( \{e^{(i)}_{\pm \alpha}\}_{i = 1}^{m_{\pm \alpha}} \) be bases of \( g_{\pm \alpha} \) which are dual with respect to \( \langle \cdot, \cdot \rangle \), and

\[ K^+_\alpha = \sum_{i=1}^{m_{\pm \alpha}} e^{(i)}_{-\alpha} e^{(i)}_{\alpha} \]

the corresponding truncated and normally ordered Casimir operator. Let \( \mathcal{V} \) be a \( g \)-module in category \( \mathcal{O}^{h}_{\infty,g} \) and \( \mathcal{V} = X \times \mathcal{V} \) the holomorphically trivial vector bundle over \( X \) with fibre \( V \). Finally, set \( h = \frac{\hbar}{2\pi i} \).

**Definition.** The Casimir connection of \( g \) is the connection on \( \mathcal{V} \) given by

\[ \nabla_K = d - h \sum_{\alpha \in \Delta_+} \frac{d\alpha}{\alpha} \cdot K^+_\alpha \]

Note that the sum converges in the \( h \)-adic topology since, for any \( v \in \mathcal{V} \) and \( n \geq 0 \), \( K^+_\alpha v \in h^n\mathcal{V} \) for all but finitely many \( \alpha \in \Delta_+ \).

The Casimir connection for a semisimple Lie algebra was discovered by De Concini around ’95 (unpublished, though the connection is referenced in [Pro96]) and, independently, Millson–Toledano Laredo [TL02, MTL05] and Felder–Markov–Tarasov–Varchenko [FMTV00]. In [FMTV00], the case of an arbitrary symmetrisable Kac–Moody algebra is considered.

The connection \( \nabla_K \) is flat (see [FMTV00] and [ATL15, Thm. 3.4]) and therefore yields a monodromy representation

\[ \mathcal{P} : \mathcal{P}_W = \Pi_1(X; x_0) \to \text{GL}(\mathcal{V}) \]

Moreover, since the coefficients of \( \nabla_K \) have weight zero, the action of \( \mathcal{P}_W \) preserves the generalised weight spaces of \( \mathcal{V} \).

This is more conveniently expressed in terms of completions. Let \( f : \mathcal{O}^{h}_{\infty,g} \to \text{Vect}_h \) be the forgetful functor. Then, the monodromy of \( \nabla_K \) yields an action

\[ \mathcal{P} : \mathcal{P}_W = \Pi_1(X; x_0) \to \text{Aut}(f) \]

5.3. The orbifold fundamental groupoid of \( X \). Let \( \Pi_1(X; Wx_0) \) be the fundamental groupoid of \( X \) based at the \( W \)-orbit of \( x_0 \). Then, \( \Pi_1(X/W; [x_0]) \) is equivalent to the orbifold fundamental groupoid \( W \ltimes \Pi_1(X; Wx_0) \), which is defined as follows.

- Its set of objects is \( Wx_0 \).
- A morphism between \( x, y \in Wx_0 \) is a pair \( (w, \gamma) \), where \( w \in W \) and \( \gamma \) is a path in \( X \) from \( x \) to \( w^{-1}y \).
- The composition of \( (w, \gamma) : x \to y \) and \( (w', \gamma') : y \to z \) is given by \( (w', \gamma') \circ (w, \gamma) = (w'w, w^{-1}(\gamma') \circ \gamma) : x \to z \).
The projection functor
\[ P : W \cong \Pi_1(X; Wx_0) \to \Pi_1(X/W; [x_0]) \] 
(5.1)
given by \( P(wx_0) = [x_0] \) and \( P(w, \gamma) = [\gamma] \) is fully faithful since, for any given \( x, y \in Wx_0 \), a loop \( [\gamma] \in \Pi_1(X/W; [x_0]) \) lifts uniquely to a path \( \gamma : x \to w^{-1}y \), for a unique \( w \in W \). Any \( x \in Wx_0 \) therefore determines a right inverse \( E_x \) of \( P \) given by \( E_x([x_0]) = x \) and \( E_x([\gamma]) = (w, \gamma) \), where \( \gamma \) is the lift of \([\gamma]\) through \( x \), and \( w \) is such that \( \gamma(1) = w^{-1}x \).

5.4. Obstruction to \( W \)-equivariance [ATL15, Sec. 4]. Extend the monodromy of \( \nabla_K \) to \( \Pi_1(X; Wx_0) \), and lift it to a map \( \mathcal{P} : \Pi_1(X; Wx_0) \to T_g \), where \( T_g \) is the holonomy algebra of the root arrangement of \( g \). The lack of \( W \)-equivariance of \( \nabla_K \) can then be described by the 1–cocycle
\[ \mathcal{A} : W \to \text{Hom}(\Pi_1(X; Wx_0), T_g) \]
defined by \( \mathcal{A}_w(\gamma) = \mathcal{P}(\gamma)^{-1} \cdot w^{-1} \mathcal{P}(w\gamma) \).

The following summarises the main properties of \( \mathcal{A} \).

**Theorem.**

1. \( \mathcal{A} \) is abelian, that is takes values in \( M = \text{Hom}(\Pi_1(X; Wx_0), \exp(hh)) \).
2. \( \mathcal{A} \) is a coboundary, that is \( \mathcal{A}_w = dB_w = B \cdot (w^{-1}B)^{-1} \) for some \( B \in M \), and any \( w \in W \).
3. The cochain \( B \) can be normalised so that \( B(\gamma_i) = \exp(ha_1, t_{\alpha_i}) \) for any given choice of \( \{a_i\}_{i \in I} \subset \mathbb{C} \), and is then unique.

**Remark.** (1) follows from the fact that \( w^{-1} \mathcal{P}(w\gamma) \) is the parallel transport of \( w^*\nabla_K = \nabla_K - ha_w \) where \( a_w = \sum_{\alpha \in \Delta_+; w\alpha \in \Delta_+} \frac{d\alpha}{\alpha} \cdot t_{\alpha} \).

Since \( \nabla_K \) and the \( h \)-valued 1–form \( a_w \) commute, \( \mathcal{A}_w \) is the parallel transport of \( d - ha_w \), and in particular takes values in \( M \).

5.5. Equivariant monodromy [ATL15, Sec. 4]. For any \( b \in B_W \), let \( \tau(b) \in \text{Aut}(f^m) \) be its action by the triple exponentials (2.3), and \( \tilde{b} \in \Pi_1(X; Wx_0) \) the unique lift of \( b \) through \( x_0 \). The following is a direct consequence of Theorem 5.4

**Theorem.** There is a unique morphism \( \mathcal{B} : \Pi_1(X; Wx_0) \to \exp(hh) \) such that

1. The assignment
   \[ \mathcal{P}_{\tau, \mathcal{B}} : B_W \to \text{Aut}(f^m) \quad \mathcal{P}_{\tau, \mathcal{B}}(b) = \tau(b) \cdot \mathcal{P}(\tilde{b}) \cdot \mathcal{B}(\tilde{b}) \]
is a group homomorphism.
2. For any \( i \in I \), \( \mathcal{B}(\gamma_i) = \exp(ht_{\alpha_i}/4) \).
Remarks.

- The normalisation of $\mathcal{B}(\gamma_i)$ is chosen so that, if $\mathfrak{g} = \mathfrak{sl}_2$ with simple root $\alpha_i$,
  $$\mathcal{P}_{\tau,\mathcal{B}}(S_i) = \tilde{s_i} \cdot \exp(hK_{\alpha_i}^+ / 2) \cdot \exp(ht_{\alpha_i} / 4) = \tilde{s_i} \cdot \exp(hK_{\alpha_i} / 4) \quad (5.2)$$
  where $K_{\alpha_i} = e_i f_i + f_i e_i$ is the truncated Casimir of $\mathfrak{sl}_2$.

- We shall refer to $\mathcal{P}_{\tau,\mathcal{B}}$ as the monodromy action of $\mathcal{B}_W$. This is justified by the fact that, when $\mathfrak{g}$ is of finite or affine type, $\mathcal{B}$ is the monodromy of the connection $\nabla - hA$, where $A$ is a resummation of the formal abelian 1–form
  $$\hat{A} = \frac{1}{2} \sum_{\alpha \in \Delta_+} \frac{d\alpha}{\alpha} \cdot m_\alpha t_\alpha$$
  (cf. [ATL15, Prop. 4.9 and Appendix A]). Thus, in these cases, $\mathcal{P}_{\tau,\mathcal{B}}$ is the monodromy of the pushdown of the connection $\nabla_K - hA$ to the quotient $X/W$.

5.6. Monodromy action of the pure braid group on category $\mathcal{O}_\infty$. Let
  $$\epsilon : \mathcal{P}_W^h \to \text{Aut}(f^m) \quad \epsilon(p_\alpha) = \exp(i\pi h_\alpha) \quad (5.3)$$
  be the sign character (cf. 4.3), $f : \mathcal{O}_{\infty,0}^h \to \text{Vect}_h$ the forgetful functor, and lift $\epsilon$ to a morphism $\mathcal{P}_W^h \to \text{Aut}(f)$ as in 4.4.

Proposition. The following holds.

1. For any $\alpha \in \Delta_+$, $\tau(p_\alpha) = \epsilon(p_\alpha)$ and $\mathcal{B}(p_\alpha) = \exp(ht_{\alpha}/2)$.

2. The restriction of $\mathcal{P}_{\tau,\mathcal{B}}$ to $\mathcal{P}_W$ lifts to an action
  $$\mathcal{P}_{\tau,\mathcal{B}} : \mathcal{P}_W \to \text{Aut}(f)$$
  given by
  $$\mathcal{P}_{\tau,\mathcal{B}}(p) = \epsilon(p) \cdot \mathcal{P}(p) \cdot \mathcal{B}(p)$$

Proof. (1) For any $i \in I$, $\tau(S_i^2) = \tilde{s}_i^2 = \exp(i\pi h_i)$ so that, for any $w \in W$ such that $w\alpha_i > 0$, $\tau(S_w S_i^2 S_w^{-1}) = \exp(i\pi h_{w\alpha_i})$. Thus, $\tau(p) = \epsilon(p)$ for any $p \in \mathcal{P}_W$.

For the second identity, it is enough to verify the relation on the loops $p_{w\alpha_i} = w(p_{\alpha_i}) \in \Pi_1(X; wx_0)$, where $p_{\alpha_i} = s_i(\gamma_i) \circ \gamma_i$, for $i \in I$, and $w \in W$ is such that $w\alpha_i > 0$ (cf. Section 5.1). For $w = \text{id}$, one has
  $$\mathcal{B}(p_{\alpha_i}) = \mathcal{B}(s_i(\gamma_i)) \mathcal{B}(\gamma_i) = s_i(\mathcal{A}_{\alpha_i}(\gamma_i)^{-1} \mathcal{B}(\gamma_i)) \mathcal{B}(\gamma_i) = s_i(\mathcal{A}_{\alpha_i}(\gamma_i))^{-1}$$
  where the second equality follows from $\mathcal{A} = d\mathcal{B}$, and the third one from $\mathcal{B}(\gamma_i) \in \exp(\text{Ch}t_{\alpha_i})$. By Remark 5.4, $\mathcal{A}_\alpha$ is the parallel transport of the abelian connection
  $$d - h \sum_{\alpha \in \Delta_+: \alpha_i \in \Delta_+} \frac{d\alpha}{\alpha} \cdot t_\alpha \quad (5.4)$$
  For $v = s_i$, this is $d - h d \log \alpha_i \cdot t_{\alpha_i}$, so that $\mathcal{A}_{\alpha_i}(\gamma_i) = \exp(ht_{\alpha_i}/2)$.

For $w \neq \text{id}$, one has
  $$\mathcal{B}(w(p_{\alpha_i})) = w(\mathcal{A}_{\alpha_i}(p_{\alpha_i})^{-1} \mathcal{B}(p_{\alpha_i})) = w(\mathcal{A}_{w}(p_{\alpha_i}))^{-1} \exp(ht_{w\alpha_i}/2)$$
  Note that $\alpha_i/\alpha$ has a non–zero residue on the hyperplane $\alpha_i = 0$ only if $\alpha = \pm \alpha_i$. It follows from (5.4) for $v = w$, and $w\alpha_i \in \Delta_+$ that $\mathcal{A}_w(p_{\alpha_i}) = 1$, whence the result.

(2) follows from (1) and Theorem 5.5. \qed
6. Braided Coxeter categories

We review below the notion of braided Coxeter category introduced in [ATL19a]. Informally speaking, such an object is a collection of braided monoidal categories labelled by the subdiagrams of a given diagram $\mathbb{D}$ – in the relevant examples the Coxeter graph of $\mathfrak{g}$. These are equipped with relative fiber functors corresponding to the inclusions of subdiagrams and an additional combinatorial datum – a maximal nested set – which labels points at infinity in the De Concini–Procesi model of the Cartan subalgebra of $\mathfrak{g}$ [DCP95]. The functors corresponding to the inclusion $\emptyset \subset \mathbb{D}$ additionally carry distinguished automorphisms – the local monodromies – which give rise to an action of the generalised braid group $B_W$.

For $U_{\hbar}\mathfrak{g}$, such a structure arises on $\mathcal{O}_{\hbar,\infty}^{\text{int}} \mathfrak{g}$ from the $R$–matrix and quantum Weyl group operators. For the category $\mathcal{O}_{\hbar,\infty}^{\text{int}} \mathfrak{g}$, it arises from the dynamical coupling of the KZ and Casimir connections of $\mathfrak{g}$ [TL16]. This is analogous to the fact that the monodromy of the KZ equations gives rise to a braided tensor category structure on $\mathcal{O}_{\hbar,\infty}^{\text{int}} \mathfrak{g}$ [Dri89], and the fact that the canonical fundamental solutions of the Casimir equations constructed by Cherednik and De Concini–Procesi [Che89, DCP95] give rise to a Coxeter structure on $\mathcal{O}_{\hbar,\infty}^{\text{int}} \mathfrak{g}$ [TL08].

6.1. Nested sets [ATL15, Sec. 5]. A diagram is an undirected graph $\mathbb{D}$ with no multiple edges or loops. A subdiagram $B \subseteq \mathbb{D}$ is a full subgraph that is, a graph consisting of a (possibly empty) subset of vertices of $\mathbb{D}$, together with all edges of $\mathbb{D}$ joining any two elements of it.

Two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are orthogonal if they have no vertices in common, and no two vertices $i_1 \in B_1, i_2 \in B_2$ are joined by an edge in $\mathbb{D}$. Two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are compatible if either one contains the other or they are orthogonal.

A nested set on $\mathbb{D}$ is a collection $H$ of pairwise compatible, connected subdiagrams of $\mathbb{D}$ which contains the empty subdiagram and the connected components of $\mathbb{D}$. We denote by $\text{Mns}(\mathbb{D})$ the collections of maximal nested sets on $\mathbb{D}$.

More generally, if $B' \subseteq B \subseteq \mathbb{D}$ are two subdiagrams, a nested set on $B$ relative to $B'$ is a collection of pairwise compatible subdiagrams of $B$ which contains the connected components of $B$ and $B'$, and in which every element is compatible with, but not properly contained in any of the connected components of $B'$. We denote by $\text{Mns}(B, B')$ the collections of maximal nested sets on $B$ relative to $B'$.

Remark. It is well–known that when $\mathbb{D}$ is a diagram of type $A_{n-1}$

maximal nested sets on $\mathbb{D}$ are in bijection with complete bracketings on the non–associative monomial $x_1 x_2 \cdots x_n$. Specifically, for any $1 \leq i \leq j \leq n$, the connected subdiagram $[i, j] \subseteq \mathbb{D}$ corresponds to the brackets $x_1 \cdots (x_i \cdots x_{j+1}) \cdots x_n$, and two subdiagrams $B_1, B_2 \subseteq \mathbb{D}$ are compatible if and only if the corresponding brackets are consistent. Similarly, maximal nested sets on $\mathbb{D}$ relative to a subdiagram $B \subseteq \mathbb{D}$ are in bijection with partially complete bracketings, i.e., complete except for the monomials $(x_i \cdots x_{j+1})$, where $[i, j]$ is a connected component of $B$.

6.2. Braided Coxeter categories [ATL15, Sec. 9]. A labelling $m$ of a diagram $\mathbb{D}$ is the assignment of an element $m_{ij} \in \{2, 3, \ldots, \infty\}$ to any pair $i, j$ of distinct vertices of $\mathbb{D}$ such that $m_{ij} = m_{ji}$ and $m_{ij} = 2$ if $i$ and $j$ are orthogonal.
Let \((D, m)\) be a labelled diagram. A braided Coxeter category \(\mathcal{C}\) of type \((D, m)\) consists of the following data

- **Diagrammatic categories.** For any subdiagram \(B \subseteq D\), a braided monoidal category \(\mathcal{C}_B\).
- **Restriction functors.** For any pair of subdiagrams \(B' \subseteq B\) and relative maximal nested set \(\mathcal{F} \in \text{Mns}(B, B')\), a tensor functor \(F_{\mathcal{F}} : \mathcal{C}_B \to \mathcal{C}_{B'}\).
- **Generalised associators.** For any pair of subdiagrams \(B' \subseteq B\) and relative maximal nested sets \(\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')\), an isomorphism of tensor functors \(\Upsilon_{\mathcal{G}, \mathcal{F}} : F_{\mathcal{G}} \Rightarrow F_{\mathcal{F}}\).
- **Vertical joins.** For any chain of inclusions \(B'' \subseteq B' \subseteq B, \mathcal{F} \in \text{Mns}(B, B'), \) and \(\mathcal{F'} \in \text{Mns}(B', B'')\), an isomorphism of tensor functors \(a_{\mathcal{F}, \mathcal{F}'} : F_{\mathcal{F}} \circ F_{\mathcal{F}'} \Rightarrow F_{\mathcal{F}''}\).
- **Transitivity.** For any vertex \(i \in D\) with corresponding restriction functor \(F_{(i)} : \mathcal{C}_i \to \mathcal{C}_\emptyset\), a distinguished automorphism \(S_i \in \text{Aut}(F_{(i)})\).

These data are assumed to satisfy the following properties.

- **Normalisation.** If \(\mathcal{F} = \{B\}\) is the unique element in \(\text{Mns}(B, B)\), then \(F_{\mathcal{F}} = \text{id}_{\mathcal{C}_B}\) with the trivial tensor structure.
- **Transitivity.** For any \(B' \subseteq B\) and \(\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{Mns}(B, B')\), \(\Upsilon_{\mathcal{H}, \mathcal{F}} = \Upsilon_{\mathcal{H}, \mathcal{G}} \circ \Upsilon_{\mathcal{G}, \mathcal{F}}\) as isomorphisms \(F_{\mathcal{F}} \Rightarrow F_{\mathcal{H}}\). In particular, \(\Upsilon_{\mathcal{F}, \mathcal{F}} = \text{id}_{\mathcal{C}_B}\) and \(\Upsilon_{\mathcal{G}, \mathcal{F}} = \Upsilon_{\mathcal{F}, \mathcal{F}}^{-1}\).
- **Associateativity.** For any \(B'' \subseteq B' \subseteq B\), \(\mathcal{F} \in \text{Mns}(B, B')\), \(\mathcal{F'} \in \text{Mns}(B', B'')\), and \(\mathcal{F''} \in \text{Mns}(B'', B''')\),
\[
a_{\mathcal{F'}, \mathcal{F}''} \cdot a_{\mathcal{F}, \mathcal{F'}} = a_{\mathcal{F}, \mathcal{F}''} \cdot a_{\mathcal{F'}, \mathcal{F}''}
\]
as isomorphisms \(F_{\mathcal{F}''} \circ F_{\mathcal{F'}} \circ F_{\mathcal{F}} \Rightarrow F_{\mathcal{F}'' \cup \mathcal{F} ' \cup \mathcal{F}}\).
- **Vertical factorisation.** For any \(B'' \subseteq B' \subseteq B\), \(\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')\) and \(\mathcal{F'}, \mathcal{G'} \in \text{Mns}(B', B'')\),
\[
\Upsilon_{(\mathcal{G} \cup \mathcal{G'}) \cup (\mathcal{F} \cup \mathcal{F'})} \circ a_{\mathcal{F'}, \mathcal{F}'} = a_{\mathcal{G'}, \mathcal{G}} \circ \left( \begin{array}{c} \Upsilon_{\mathcal{G}, \mathcal{F}} \\ \Upsilon_{\mathcal{G'}, \mathcal{F'}} \end{array} \right)
\]
as isomorphisms \(F_{\mathcal{F}''} \circ F_{\mathcal{F'}} \Rightarrow F_{\mathcal{G} \cup \mathcal{G'} \cup (\mathcal{F} \cup \mathcal{F}')}\).
- **Generalised braid relations.** For any \(B \subseteq D, i \neq j \in B\) and maximal nested sets \(\mathcal{K}_i, \mathcal{K}_j\) on \(B\) such that \(\{i\} \in \mathcal{K}_i\), \(\{j\} \in \mathcal{K}_j\), the following holds in \(\text{Aut} F_{\mathcal{K}_i}\)
\[
\text{Ad}(\Upsilon_{ij}) (S_i^m) \cdot S_i^m \cdot \text{Ad}(\Upsilon_{ij}) (S_j^m) \cdot S_j^m \cdots = S_i^m \cdot \text{Ad}(\Upsilon_{ij}) (S_j^m) \cdot S_j^m \cdots
\]
where \(\Upsilon_{ij} = \Upsilon_{\mathcal{K}_i \cap \mathcal{K}_j}\) and \(S_i^m = \text{Ad} a_{\mathcal{K}_i}^{\mathcal{K}_j} (S_i) \in \text{Aut} F_{\mathcal{K}_i}\).

\footnote{\text{Note that} \(F_{\mathcal{F}}\) is not assumed to be braided.}
\footnote{\text{Note that} \(S_i\) is not assumed to be a tensor automorphism of \(F_{(i)}\).}
\footnote{\(\mathcal{K}_i\) and \(\mathcal{K}_i^*\) denote the truncations of \(\mathcal{K}_i\) at \(\{i\}\).}
• **Coproduct identity.** For any \( i \in D \), the following holds in \( \text{Aut} \left( F_{(i)} \otimes F_{(i)} \right) \)

\[
J_i^{-1} \circ F_{(i)}(c_i) \circ \Delta(S_i) \circ J_i = c_{\emptyset} \circ (S_i \otimes S_i)
\]

where \( J_i \) is the tensor structure on \( F_{(i)} \) and \( c_i, c_{\emptyset} \) are the opposite braidings in \( C_i \) and \( C_{\emptyset} \), respectively.\(^1\)

6.3. **Representations of braid groups.** Let \( B_{\mathbb{D}}^{\mathbb{D}} \) be the braid group with generators \( S_i, \ i \in \mathbb{D} \), and relations (2.2) for the labelling \( m \). Let \( B_{\mathbb{D}}^{\mathbb{D}} \leq B_{\mathbb{D}}^{\mathbb{D}} \) be the subgroup generated by \( S_i \) with \( i \in B \). Finally, let \( B_n \) be the braid group associated to the symmetric group \( \mathfrak{S}_n \), with generators \( T_1, \ldots, T_{n-1} \), and \( \mathfrak{b}_n \) the set of complete bracketings on the non-commutative monomial \( x_1 x_2 \cdots x_n \).

Let \( \mathcal{C} = (C_B, F_F, Y_F G, a_F, S_i) \) be a braided Coxeter category. Then, there is a family of representations

\[
\lambda_{F,b}^{\mathcal{C}} : B_{\mathbb{D}}^{\mathbb{D}} \times B_n \to \text{Aut}(F_{\mathbb{D}}^{\mathbb{D}})
\]

labelled by \( B \subseteq \mathbb{D}, F \in \text{Mns}(B) \), and \( b \in \mathfrak{b}_n \), which is uniquely determined by the conditions

- \( \lambda_{F,b}^{\mathcal{C}}(S_i) = \text{Ad}(a_F)(S_i)_{1 \cdots n} \) if \( \{i\} \in F \) and \( \lambda_{F,b}^{\mathcal{C}} = \text{Ad}(Y_F G)_{1 \cdots n} \circ \lambda_{F,b}^{\mathcal{C}} \).
- \( \lambda_{F,b}^{\mathcal{C}}(T_i) = R_{B,i,i+1}^{\mathbb{D}} \) if \( b = x_1 \cdots (x_i x_{i+1}) \cdots x_n \) and \( \lambda_{F,b}^{\mathcal{C}} = \text{Ad}(\Phi_{B,B,b}) \circ \lambda_{F,b}^{\mathcal{C}} \), where \( \Phi_B \) and \( R_{B}^{\mathbb{D}} \) are the associativity and commutativity constraints of \( C_B \).

6.4. **Equivalence of braided Coxeter categories.** Let \( \mathcal{C}, \mathcal{C}' \) be two braided Coxeter categories of type \((\mathbb{D}, m)\). An equivalence \( H : \mathcal{C} \to \mathcal{C}' \) is the data of

- For any \( B \subseteq \mathbb{D} \), a braided tensor equivalence \( H_B : C_B \to C_B' \)
- For any \( B' \subseteq B \) and \( F \in \text{Mns}(B, B') \), an isomorphism \( \gamma_F \) of tensor functors

\[
\begin{array}{ccc}
C_B & \xrightarrow{H_B} & C_B' \\
F_F & \overset{\gamma_F}{\swarrow} & F_F' \\
C_{B'} & \xleftarrow{H_{B'}} & C_{B'}'
\end{array}
\]

(6.2)

These are required to preserve the generalised associators, vertical joins, and local monodromies.

- For any \( B' \subseteq B \subseteq \mathbb{D} \) and \( F, G \in \text{Mns}(B, B') \),

\[
Y_G F \circ \gamma_F = \gamma_G \circ Y_F G
\]

as isomorphisms \( F_F' \circ H_B \Rightarrow H_{B'} \circ F_G \).
- For any \( B'' \subseteq B' \subseteq B \subseteq \mathbb{D}, F \in \text{Mns}(B, B') \), and \( F' \in \text{Mns}(B', B'') \),

\[
\gamma_{F,F'} \circ (a_{F'}^{F_F'})' = a_{F''}^{F_F}, \circ \begin{pmatrix} \gamma_F \\ \gamma_{F'} \end{pmatrix}\]

as isomorphisms \( F_F' \circ (F_F')' \circ H_B \Rightarrow H_{B'} \circ F_{F''} \).

\(^1\)Given a braided monoidal category with braiding \( \beta \), we set \( \beta_{X,Y}^{\mathbb{D}} := \beta_{Y,X}^{-1} \).
Similarly, we denote by $\gamma_{\partial_i} = \gamma_{\partial_i} \circ S_i'$ as isomorphisms $F_i' \circ H_i \mapsto H_\partial \circ F_i$.

Let $H : \mathcal{C} \to \mathcal{C}'$ be an equivalence of braided Coxeter categories. Then, the representations of the braid groups $\lambda_{\mathcal{C}, B}$ and $\lambda_{\mathcal{C}', B}$ are equivalent through the natural isomorphism $\gamma : F'_B \circ H_B \mapsto F_B$.

6.5. The braided Coxeter category $\mathcal{O}^{\text{int}}_{U_h \mathfrak{g}, \mathbf{R}, \mathbf{S}}$.

Let now $\mathfrak{A}$ be a symmetrisable generalised Cartan matrix, $(\mathfrak{h}, \Pi, \Pi')$ a realisation of $\mathfrak{A}$, $\mathfrak{g}$ the corresponding Kac–Moody algebra and $\mathfrak{D}$ its Dynkin diagram with the standard labelling (2.2), thus $\mathcal{B}_\mathfrak{D}^\text{int} = \mathcal{B}_W$. To simplify the exposition, we assume that $\mathfrak{A}$ is of finite or affine type, and $\mathfrak{h}$ is its minimal realisation.

For any proper subdiagram $B \subset \mathfrak{D}$, we denote by $\mathfrak{g}_B \subset \mathfrak{g}$ the subalgebra generated by $\{e_i, f_i, h_i\}_{i \in B}$, and set $\mathfrak{g}_\mathfrak{D} = \mathfrak{g}$. 1 Similarly, we denote by $U_h \mathfrak{g}_B \subset U_h \mathfrak{g}$ the subalgebra topologically generated by $\{E_i, F_i, h_i\}_{i \in B}$, and set $U_h \mathfrak{g}_\mathfrak{D} = U_h \mathfrak{g}$.

Then, the braided Coxeter category $\mathcal{O}^{\text{int}}_{U_h \mathfrak{g}, \mathbf{R}, \mathbf{S}}$ is given by the following data.

- The diagrammatic category corresponding to $B \subset \mathfrak{D}$ is the monoidal category $\mathcal{O}^{\text{int}}_{U_h \mathfrak{g}_B, \mathbf{R}}$ with braiding induced by the universal $R$–matrix $\mathbf{R}_B$ of $U_h \mathfrak{g}_B$.

- For any $B' \subset B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$, $F_\mathcal{F}$ is the restriction functor $\mathsf{Res}_B \mathbf{R}_B : \mathcal{O}^{\text{int}}_{U_h \mathfrak{g}_B, \mathbf{R}} \to \mathcal{O}^{\text{int}}_{U_h \mathfrak{g}_{B'} , \mathbf{R}}$ with the trivial tensor structure.

- The generalised associators and vertical joins are trivial.

- The local monodromy corresponding to $i \in \mathfrak{D}$ is the quantum Weyl group operator $S_i \in \text{Aut}(f^\mathfrak{h}_{B,i})$.

Remarks.

(1) The braided Coxeter structure on $\mathcal{O}^{\text{int}}_{U_h \mathfrak{g}, \mathbf{R}, \mathbf{S}}$ is particularly simple in that the restriction functors, the generalised associators, and the vertical join do not depend upon the choice of a maximal nested set $\mathcal{F} \in \mathsf{Mns}(B, B')$, but only on the subdiagrams $B' \subset B$.

(2) The category $\mathcal{O}^{\text{int}}_{U_h \mathfrak{g}, \mathbf{R}, \mathbf{S}}$ gives rise to a single representation of the braid group $B_W$ (independent of $\mathcal{F}$) which is the quantum Weyl group action $\rho : B_W \to \text{Aut}(f^\mathfrak{h}_B)$ from Section 4.2.

(3) Strictly speaking, for the coproduct identity (6.1) to hold, it is necessary to consider a Cartan correction of the quantum Weyl group operator $S_i$ (cf. [ATL15, Sec. 17.3]). For simplicity, we shall gloss over this technical detail and refer the reader to [ATL15].

6.6. The braided Coxeter category $\mathcal{O}^{\mathfrak{h}, \text{int}}_{\mathfrak{g}, \text{int}}$.

In [ATL15, Sec. 16], we defined a braided Coxeter category $\mathcal{O}^{\mathfrak{h}, \text{int}}_{\mathfrak{g}, \text{int}}$ which underlies the equivariant monodromy of the Casimir connection, together with that of the KZ equations for all the subalgebras $\mathfrak{g}_B \subset \mathfrak{g}$. In outline, $\mathcal{O}^{\mathfrak{h}, \text{int}}_{\mathfrak{g}, \text{int}}$ is described as follows.

1Since $\mathfrak{A}$ is assumed to be of finite or affine type, $\mathfrak{g}_B = \mathfrak{g}_B'$ is the Kac–Moody algebra corresponding to the Cartan submatrix $\mathfrak{A}_B$. For a general $\mathfrak{A}$, the definition of $\mathfrak{g}_B$ and $U_h \mathfrak{g}_B$ requires a realisation which is diagrammatic in the sense of [ATL15, Sect. 2.4].
• The diagrammatic category corresponding to \( B \subseteq \mathbb{D} \) is the braided monoidal category \( \mathcal{O}^\text{int}_{\infty, \mathfrak{g}_B} \), with associativity and commutativity constraints given by the KZ associator \( \Phi^\nabla_B \) and \( R \)-matrix \( R^\nabla_B = \exp(h\Omega_B/2) \), where \( \Omega_B \in \mathfrak{g}_B \otimes \mathfrak{g}_B \) is the Casimir tensor of \( \mathfrak{g}_B \), cf. [Dri90].

• For any \( B' \subseteq B \) and \( \mathcal{F} \in \text{Mns}(B, B') \), \( F_\mathcal{F} \) is the standard restriction functor \( f_{B'B}: \mathcal{O}^\text{int}_{\infty, \mathfrak{g}_B} \to \mathcal{O}^\text{int}_{\infty, \mathfrak{g}_{B'}} \), with tensor structure given by the relative twists \( J^\nabla_\mathcal{F} \) constructed in [TL16], see also [ATL15, Sec. 13].

• For any \( B' \subseteq B \) and \( \mathcal{F}, \mathcal{G} \in \text{Mns}(B, B') \), the natural isomorphism of tensor functors \( F_\mathcal{G} \Rightarrow F_\mathcal{F} \) is given by the De Concini–Procesi (relative) associator \( \Upsilon^\nabla_{\mathcal{F}\mathcal{G}} \) constructed in [DCP95], see also [ATL15, Sec. 8].

• The vertical joins are trivial.

• The local monodromy corresponding to any \( i \in \mathbb{D} \) is the operator (cf. (5.2))

\[
S^\nabla_i = \tilde{s}_i \cdot \exp(hK_{\alpha_i}/4)
\]  

(6.3)

**Remark.** Contrary to the local monodromies \( S^\nabla_i \), the data \( (\Phi^\nabla_B, R^\nabla_B, J^\nabla_\mathcal{F}, \Upsilon^\nabla_{\mathcal{F}\mathcal{G}}) \) acts on category \( \mathcal{O}_{\infty} \) modules. By replacing the diagrammatic categories \( \mathcal{O}^\text{int}_{\infty, \mathfrak{g}_B} \), with \( \mathcal{O}^\text{h, int}_{\infty, \mathfrak{g}_B} \) and excluding the \( S^\nabla_i \), one obtains a braided pre–Coxeter category \( \mathcal{O}^\text{h, int}_{\infty, \mathfrak{g}_B} \) [ATL15, Sec. 15].

In 6.7–6.9, we briefly outline the construction of the relative De Concini–Procesi associators \( \Upsilon^\nabla_{\mathcal{F}\mathcal{G}} \) and the relative twists \( J^\nabla_\mathcal{F} \).

6.7. **Monodromy data of the Casimir connection.** Following Cherednik [Che89, Che91] and De Concini–Procesi [DCP95] (see also [ATL15, Sec. 8]), for any \( \mathcal{F} \in \text{Mns}(\mathbb{D}) \), there is a canonical universal solution \( G_\mathcal{F} \) of \( \nabla_K \) valued in \( \text{Aut}(f) \). It is uniquely determined by its prescribed asymptotics on a point at infinity \( p_\mathcal{F} \) corresponding to a choice of blow–up coordinates on \( X \) associated to \( \mathcal{F} \).

For any \( \mathcal{F}, \mathcal{G} \in \text{Mns}(\mathbb{D}) \), the De Concini–Procesi associator \( \Upsilon^\nabla_{\mathcal{F}\mathcal{G}} \) is the element of \( \text{Aut}(f) \) defined by

\[
G_\mathcal{G}(x) = G_\mathcal{F}(x) \cdot \Upsilon^\nabla_{\mathcal{F}\mathcal{G}}
\]

where \( x \) lies in the fundamental Weyl chamber. The datum of the De Concini–Procesi associators yields a combinatorial description of the equivariant monodromy of \( \nabla_K \) as follows (cf. [ATL15, Thm. 9.3]). Let \( S^\nabla_i \) be given by (6.3). Then, there is a family of representations

\[
\mu_\mathcal{F}: \mathcal{B}_W \to \text{Aut}(f^\text{int})
\]

labelled by \( \mathcal{F} \in \text{Mns}(\mathbb{D}) \), which is uniquely determined by the conditions

• \( \mu_\mathcal{F}(S_i) = S^\nabla_i \) if \( \{i\} \in \mathcal{F} \)

• \( \mu_\mathcal{G} = \text{Ad}(\Upsilon_{\mathcal{G}x}) \circ \mu_\mathcal{F} \)

The representation \( \mu_\mathcal{F} \) is the equivariant monodromy of \( \nabla_K \) computed with respect to the fundamental solution \( G_\mathcal{F} \).

6.8. **Generalised associators.** For any \( B \subseteq \mathbb{D} \), one similarly obtains the associators \( \Upsilon^\nabla_{\mathcal{F}\mathcal{G}} \in \text{Aut}(f_B) \) with \( \mathcal{F}, \mathcal{G} \in \text{Mns}(B) \) which, together with the local monodromies \( \{S^\nabla_i\}_{i \in B} \), describe the equivariant monodromy of the Casimir connection.
of $g_B$. These associators are related to those for $g$ as follows. Let $H \in \mathsf{Mns}(\mathbb{D}, B)$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, \emptyset)$. Then, [DCP95, Thm. 3.6] implies that

$$\Upsilon^\nabla_{\mathcal{H} \cup \mathcal{G} \cup \mathcal{F}} = \iota_{\mathcal{D} B} (\Upsilon^\nabla_{\mathcal{G} \cup \mathcal{F}})$$

(6.4)

where $\iota_{DB} : \text{End}(f_B) \to \text{End}(f_D)$ is induced by the equality $f_D = f_{DB} \circ f_B$.

The relative associators corresponding to an inclusion $B' \subseteq B$ are constructed as follows. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Mns}(B, B')$, choose $H \in \mathsf{Mns}(B', \emptyset)$, and set

$$\Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}} = \Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}} \cdot \Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}} = \iota_{BB'} (\Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}} \cdot \Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}})$$

where the second equality follows from (6.4) and the definition of $\Upsilon^\nabla_{\mathcal{G} \cup H \cup \mathcal{F}}$.

### 6.9. Monodromy data of the joint KZ-Casimir system

The tensor structures $\{ J^\nabla_{\mathcal{F}} \}_{\mathcal{F} \in \mathsf{Mns}(\mathbb{D})}$ on the forgetful functor $f = f_D$ are obtained from the dynamical KZ equations in $n = 2$ points

$$d - \left( \frac{h}{z} + \mu^{(1)} \right) dz$$

where $z = z_1 - z_2$, $\mu \in \mathfrak{h}$ and $\mu^{(1)} = \mu \otimes 1$ as follows.

These admit a canonical solution $G_0$ which is asymptotic to $z^h \Omega$ near $z = 0$. If $\mu$ is regular and real, they also admit two canonical solutions $G_{\pm}$ which are asymptotic to $z^{h_0} \cdot \exp(z\mu^{(1)})$ as $z \to \infty$ with $\text{Im} z \geq 0$, where $\Omega_0$ is the projection of $\Omega$ onto $\mathfrak{h} \otimes \mathfrak{g}$ [TL16, Sect. 6]. Define the differential twist $J_{\pm}(\mu)$ by

$$J_{\pm}(\mu) = G_0^{-1}(z) \cdot G_{\pm}(z)$$

where $\text{Im} z \geq 0$.

Then, $J_{\pm}(\mu)$ kills the KZ associator for $g$. As a function of $\mu \in \mathcal{C}$, where $\mathcal{C}$ is the fundamental Weyl chamber, $J_{\pm}(\mu)$ is real analytic and varies according to the Casimir equations [TL16, Sect. 7]

$$d_\mathfrak{h} J_{\pm} = \frac{h}{2} \sum_{\alpha \in \Delta_+} \frac{d\alpha}{\alpha} \left( \Delta(K^\pm_\alpha) J_{\pm} - J_{\pm} (K^\pm_\alpha \otimes 1 + 1 \otimes K^\pm_\alpha) \right)$$

It follows that, for any maximal nested set $\mathcal{F} \in \mathsf{Mns}(\mathbb{D})$, the twist

$$J^\nabla_{\mathcal{F}} = \Delta(G_{\mathcal{F}}(\mu))^{-1} \cdot J_{\pm}(\mu) \cdot G_{\mathcal{F}}(\mu)^{\otimes 2}$$

where $G_{\mathcal{F}}(\mu)$ is the fundamental solution of the Casimir connection corresponding to $\mathcal{F}$ (see 6.7), is independent of $\mu \in \mathcal{C}$, and a tensor structure on $f_D$.

The relative twists $J^\nabla_{\mathcal{F}}$ corresponding to any $B' \subseteq B$ and $\mathcal{F} \in \mathsf{Mns}(B, B')$ are obtained by relying on vertical factorisation as follows. Fix $H \in \mathsf{Mns}(B', \emptyset)$, let $F_{\mathcal{F} \cup \mathcal{H}}$ and $F_{\mathcal{H}}$ be the tensor structures on $f_B, f_{B'}$ corresponding to $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{H}$ respectively. Then, define $J^\nabla_{\mathcal{F}}$ by

$$f_{B'}(J^\nabla_{\mathcal{F}}) = J^\nabla_{\mathcal{F} \cup \mathcal{H}} \cdot (J^\nabla_{\mathcal{H}})^{-1}$$
More precisely, the right-hand side is a collection of natural isomorphisms
\[ f_B' (f_{B'B}(U) \otimes f_{B'B}(V)) \rightarrow f_B (f_{B'B}(U \otimes V)) = f_B' (f_{B'B}(U \otimes V)) \]
defined for any \( U, V \in \mathcal{O}_{\infty, U \mathfrak{g}}^h \). One can prove that it satisfies the centraliser property, i.e., commutes with the action of \( \mathfrak{g}_{B'} \) [TL16, Sect. 8]. Since \( f_B' \) is faithful, it follows that it is of the form \( f_B' (J_{\mathcal{Y}}) \) for a unique \( J_{\mathcal{Y}} \). Moreover, the latter is independent of the choice of \( \mathcal{Y} \).

7. The equivariant monodromy theorem

We review in this section the main result of [ATL15], which extends that of [TL08, TL16] to the case of an arbitrary symmetrisable Kac–Moody algebra, and yields an equivalence of braided Coxeter categories \( \mathcal{O}_{\infty, \mathfrak{g}}^h \rightarrow \mathcal{O}_{\infty, U \mathfrak{g}}^h \). Its proof relies on the Etingof–Kazhdan equivalence, which is briefly reviewed in 7.1–7.2.

7.1. The Etingof–Kazhdan equivalence. In [EK08, Thm. 4.2], Etingof and Kazhdan construct an equivalence of categories \( F : \mathcal{O}_{\infty, \mathfrak{g}}^h \rightarrow \mathcal{O}_{\infty, U \mathfrak{g}}^h \), together with an isomorphism \( \alpha \) of functors \( \mathcal{O}_{\infty, \mathfrak{g}}^h \rightarrow \mathcal{O}_{\infty, U \mathfrak{g}}^h \). It is the identity on \( \mathfrak{h} \)–modules and preserves integrability [ATL15, Lemma 22.9]. It therefore gives rise to a diagram of functors in which every face commutes

\[
\begin{array}{c}
\mathcal{O}_{\infty, \mathfrak{g}}^h \xrightarrow{F} \mathcal{O}_{\infty, U \mathfrak{g}}^h \\
\downarrow \alpha \quad \downarrow f_h \\
\text{Vect}_h \end{array}
\]

where \( f \) and \( f_h \) are the forgetful functors.\(^1\) The equivalence \( F \) is the identity on \( \mathfrak{h} \)–modules and preserves integrability [ATL15, Lemma 22.9]. It therefore gives rise to a diagram of functors in which every face commutes

\[
\begin{array}{c}
\mathcal{O}_{\infty, \mathfrak{g}}^{h, \text{int}} \xrightarrow{F^{\text{int}}} \mathcal{O}_{\infty, U \mathfrak{g}}^{h, \text{int}} \\
\downarrow \alpha^{\text{int}} \quad \downarrow f_{h, \text{int}} \\
\text{Mod}_\mathfrak{h} \xrightarrow{f_h} \text{Mod}_\mathfrak{h} \\
\downarrow f_{h, \text{int}} \quad \downarrow f_h \\
\text{Vect}_h \end{array}
\]

where the vertical arrows are restriction functors, and the natural isomorphisms are either trivial or induced from \( \alpha \).\(^2\)

\(^1\)More precisely, in [EK08] Etingof–Kazhdan construct an equivalence \( F \) between the larger categories of Drinfeld–Yetter modules over the negative Borel subalgebra \( \mathfrak{b}^- \), and admissible Drinfeld–Yetter modules over \( U \mathfrak{b}^- \) (see also [ATL18, 6.13]). It easily follows that \( F \) restricts to an equivalence \( \mathcal{O}_{\infty, \mathfrak{g}}^h \rightarrow \mathcal{O}_{\infty, U \mathfrak{g}}^h \), since it is the identity on Drinfeld–Yetter \( \mathfrak{h} \)–modules, see [ATL15, Lemma 22.11]. By the same argument, it also restricts to an equivalence \( \mathcal{O}_\mathfrak{g}^h \rightarrow \mathcal{O}_{U \mathfrak{g}}^h \).

\(^2\)The categories \( \mathcal{W}_\mathfrak{h}, \mathcal{O}_{\infty, \mathfrak{g}}^h \) and \( \mathcal{O}_{\infty, U \mathfrak{g}}^h \) naturally fit within the diagram (7.1), but are omitted for simplicity.
7.2. The Etingof–Kazhdan isomorphism. In terms of completions, the Etingof–Kazhdan equivalence $\left( F, \alpha \right)$ gives rise to an isomorphism $\Psi : \text{End}(f_h) \to \text{End}(f)$ via the composition

$$\text{End}(f_h) \longrightarrow \text{End}(f_h \circ F) \to \text{End}(f)$$

where the first isomorphism is induced by $F$, and the second is given by $\text{Ad}(\alpha)$. By (7.1), $\Psi$ restricts to an isomorphism $\Psi^{\text{int}} : \text{End}(f^\text{int}_h) \to \text{End}(f^\text{int})$ such that

$$\begin{array}{c}
\text{End}(f^\text{int}_h) \\
\downarrow \\
\text{End}(f^\text{int})
\end{array} \xrightarrow{\Psi^{\text{int}}} \begin{array}{c}
\text{End}(f) \\
\downarrow \\
\text{End}(f)
\end{array}$$

where the vertical arrows are restriction to category $O_\infty$ and integrable modules.

7.3. The classical Drinfeld algebra. Let $D$ be the analogue of the Drinfeld algebra $D_h$ for $U\mathfrak{g}[h]$ (cf. Section 3.1). Namely, for any $\beta \in \mathbb{Q}_+$, let $B_\beta = \{ X_{\beta, p} \}$ be a basis of $U\mathfrak{n}^+_{\beta}$ and $B = \bigsqcup_{\beta \in \mathbb{Q}_+} B_\beta$. Set

$$D_0 = \left\{ \sum_{X \in B} c_X X : c_X \in U\mathfrak{b}^- \right\} = \prod_{\beta \in \mathbb{Q}_+} U\mathfrak{b}^- \otimes U\mathfrak{n}^+_{\beta} \supset U\mathfrak{g}$$

and $D = D_0[h]$. The algebra structure of $U\mathfrak{g}[h]$ extends to one on $D$ and yields a chain of morphisms $U\mathfrak{g}[h] \subset D \to \text{End}(f)$. Proceeding as in Section 3 one shows that $D$ embeds into $\text{End}(f)$ and $\text{End}(f^\text{int}).$

7.4. The monodromy theorem. In [ATL15, Thm. 22.1] we prove the following.

Theorem.

(1) There is a canonical equivalence of braided pre–Coxeter categories (cf. Remark 6.6)

$$H_0 = (H_B, \gamma_F) : \mathcal{O}^h_{\mathfrak{g}, V} \to \mathcal{O}_{U\mathfrak{g}, R, S}$$

such that

- for any $B \subseteq \mathbb{D}$, the equivalence $H_B$ is the Etingof–Kazhdan functor

$$F_{\mathfrak{g}, B} : \mathcal{O}^h_{\infty, \mathfrak{g}, \mathfrak{b}} \to \mathcal{O}_{\infty, U\mathfrak{g}, \mathfrak{b}}$$

- for any $B' \subseteq B$ and $F \in \text{Mns}(B, B')$, the natural isomorphism $\gamma_F$ is induced by the action of an invertible weight zero element $g_F$ in the Drinfeld
algebra of $\mathfrak{g}_B$, i.e., there is a commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{O}^h_{\infty, \mathfrak{g}B} & \xrightarrow{\gamma_F} & \mathcal{O}^h_{\infty, \mathfrak{g}B'} \\
\mathfrak{f}_B & \xrightarrow{\mathfrak{g}_F} & \mathfrak{f}_{B'} \\
\text{Vect}_B & \xleftarrow{\text{End}(f^\text{int})} & \text{End}(f^\text{int}) \\
\end{array}
\]

(7.3)

where the unmarked back face is the identity and the two unmarked lateral faces are the isomorphisms $\alpha$ for $\mathfrak{g}_B$ and $\mathfrak{g}_{B'}$.

(2) $H_\mathfrak{g}$ restricts to an equivalence of braided Coxeter categories

\[
H_\mathfrak{g}^\text{int} = (H_B^\text{int}, \gamma_F) : \mathcal{O}^h_{\infty, \mathfrak{g}B} \rightarrow \mathcal{O}^\text{int}_{U_h \mathfrak{g}, \mathfrak{R}, \mathfrak{S}}
\]

where $H_B^\text{int} = F_{\mathfrak{g}_B}^\text{int}$.

(3) For any $F \in \text{Mns}(\mathfrak{D})$, the isomorphism

\[
\Psi_F^\text{int} = \text{Ad}(g_F) \circ \Psi^\text{int} : \text{End}(f^\text{int}) \rightarrow \text{End}(f^\text{int})
\]

intertwines the quantum Weyl group and the monodromy actions of $B_W$, i.e.,

\[
\begin{array}{ccc}
\lambda & \circ & \mu_F \\
\text{End}(f^\text{int}) & \xrightarrow{\Psi_F^\text{int}} & \text{End}(f^\text{int})
\end{array}
\]

(7.4)

where $\mu_F = \mathcal{R}_F^\text{int}_{B_W}$ denotes the monodromy action of $B_W$ around the point at infinity in the De Concini–Procesi compactification of $X$ corresponding to $F$.

Since the diagrammatic equivalences are fixed, the proof amounts to constructing suitable isomorphisms (6.2). The construction is in two steps. First, we prove that $\mathcal{O}^\text{int}_{U_h \mathfrak{g}, \mathfrak{R}, \mathfrak{S}}$ is equivalent to a braided Coxeter category $\mathcal{O}^h_{\mathfrak{g}_B, \mathfrak{R}, \mathfrak{S}}$ with diagrammatic categories $\mathcal{O}^h_{\infty, \mathfrak{g}B}, B \subseteq \mathfrak{D}$, and standard restriction functors with non–trivial tensor structures. The equivalence is given by the diagrammatic Etingof–Kazhdan functors, equipped with natural isomorphisms $\tilde{\gamma}_F$ whose construction is carried out in [ATL18, ATL19a]. Then, relying on the rigidity result from [ATL19b], we prove that $\mathcal{O}^h_{\mathfrak{g}_B, \mathfrak{R}, \mathfrak{S}}$ is equivalent to $\mathcal{O}^\text{int}_{\mathfrak{g}_B, \mathfrak{V}}$ with diagrammatic equivalences given by the identity functors. Finally, we observe that, by [ATL19a, Thm. 10.7], the resulting isomorphisms $\gamma_F$ satisfy (7.3) for weight zero elements $\mathfrak{g}_F$ in the Drinfeld algebra.
8. The monodromy theorem in category $\mathcal{O}_\infty$

In this section, we show the equivalence of the actions of $\mathcal{P}_W$ constructed in Sections 4 and 5. The proof relies on the equivariant monodromy Theorem 7.4, the explicit description of the actions of $\mathcal{P}_W$ from Sections 4 and 5, and the following auxiliary result.

8.1. Isomorphism between Drinfeld algebras. We show below that the isomorphism $\Psi: \text{End}(f_h) \to \text{End}(f)$ (7.2) restricts to an isomorphism $\Psi^D: \mathcal{D}_h \to \mathcal{D}$. Our proof closely follows Etingof and Kazhdan’s argument [EK08, Rem. p. 535] for the analogous algebra $\mathcal{Q}_h = \lim_\beta U_h g/I_\beta$ (cf. Remark 3.1), and completes their affirmative answer to a question raised by Drinfeld [Dri92, Question 8.2].

For any $\beta = \sum_i k_i \alpha_i \in \mathbb{Q}_+$, define the height of $\beta$ by $\text{ht} \beta = \sum_i k_i$. For any $n \geq 0$, let $J_n \subseteq U_h g$ be the left ideal generated by $(U_h n^+)_{\beta}$ with $\text{ht}(\beta) > n$. Set $U_{\beta}^{(n)} = U_h g/J_n$, and denote by $\phi_{mn}^h: U_{\beta}^{(n)} \to U_{\beta}^{(m)}$ ($m \leq n$) the natural morphisms. Their classical analogues $U^{(n)}$ and $\phi_{mn}: U^{(n)} \to U^{(m)}$ ($m \leq n$) are defined similarly for $U g[[h]]$.

Theorem.

1. There is a canonical isomorphism of $U_h g$-modules $\mathcal{D}_h \simeq \lim_n U_{\beta}^{(n)}$.
2. There is a canonical isomorphism of $U g[[h]]$-modules $\mathcal{D} \simeq \lim_n U^{(n)}$.
3. $\Psi$ restricts to an isomorphism of algebras $\Psi^D: \mathcal{D}_h \to \mathcal{D}$.

Proof. (1)–(2) The action of $\mathcal{D}_h$ on the cyclic vector yields surjective morphisms $\phi_n: \mathcal{D}_h \to U_{\beta}^{(n)}$ of $U_h g$-modules such that $\phi_{mn}^h \circ \phi_n = \phi_m$. The corresponding morphism $\phi: \mathcal{D}_h \to \lim_n U_{\beta}^{(n)}$ is easily seen to be an isomorphism.

(3) The algebra structure of $\mathcal{D}_h$ is encoded by the morphisms between the modules $U_{\beta}^{(n)}$. Namely, we have a natural isomorphism

$$\mathcal{D}_h^{op} \simeq \text{End}_{U_h g} \left( \lim_n U_{\beta}^{(n)} \right) \simeq \lim_m \text{colim}_n \text{Hom}_{U_h g}(U_{\beta}^{(n)}, U_{\beta}^{(m)})$$

(see also [App13, Appendix A.1]). A similar results holds for $\mathcal{D}$.

The module $U^{(n)}$ (resp. $U_{\beta}^{(n)}$) does not lie in $\mathcal{O}_\infty g$ (resp. $\mathcal{O}_\infty U_h g$) since it is free over $U g[[h]]$. However, the fact that $U n^+_h v = 0$ (resp. $U_h n^+_h v = 0$) for all but finitely many $\beta \in \mathbb{Q}_+$ for any weight vector $v \in U^{(n)}$ (resp. $v \in U_{\beta}^{(n)}$) implies that $U^{(n)}$ is an equicontinuous $g$-module and therefore a Drinfeld–Yetter module over $b^+$, and that $U_{\beta}^{(n)}$ is an admissible Drinfeld–Yetter module over $U_h b^+$. One can therefore apply the equivalence $F$ to $U^{(n)}$, and finds that $F(U^{(n)}) = U_{\beta}^{(n)}$ and $F(\phi_{mn}) = \phi_{mn}^h$ ([EK08, Thms. 4.1–4.2]). This yields a collection of natural isomorphisms

$$\text{Hom}_{U g[[h]]}(U^{(n)}, U^{(m)}) \simeq \text{Hom}_{U_h g}(U_{\beta}^{(n)}, U_{\beta}^{(m)})$$

and the desired isomorphism $\Psi^D: \mathcal{D}_h \to \mathcal{D}$. □

---

1. The argument in [EK08] is not complete since the modules $U g/I_\beta$ are not equicontinuous for an arbitrary Kac–Moody algebra $g$, so that the Etingof–Kazhdan equivalence $F$ cannot be applied to them. In particular, the existence of an isomorphism between $\mathcal{Q}_h$ and its classical counterpart raised in [Dri92, Question 8.2] is not settled by [EK08]. Theorem 8.1 yields such an isomorphism for the algebra $\mathcal{D}_h$. 
8.2. The monodromy theorem.

**Theorem.** The monodromy of the normally ordered Casimir connection on a $\mathfrak{g}$–module $V \in \mathcal{O}_{\infty, \mathfrak{g}}^\hbar$ is canonically equivalent to the normally ordered quantum Weyl group action of the pure braid group $\mathcal{P}_W$ on the Etingof–Kazhdan quantisation $F(V) \in \mathcal{O}_{\infty, U\mathfrak{g}}^\hbar$.

**Proof.** Let $\mathcal{F} \in \text{Mns}(\mathcal{D})$. By Theorem 7.4 (3), there is a weight zero element $g_\mathcal{F} \in \mathcal{D}^\times \subset \text{Aut}(\mathcal{f})$ such that $\Psi^\text{int} \circ \Psi^\text{int}$ intertwines the quantum Weyl group and the monodromy actions of $\mathcal{B}_W$, cf. (7.4). We claim that this yields a commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_W & \xrightarrow{\lambda_x} & \mathcal{P}_W \\
\downarrow{\Psi^\text{int}} & & \downarrow{\Psi^\text{int}} \\
\mathcal{D}_h & \xrightarrow{\lambda_x} & \mathcal{D}_h \\
\downarrow{\Psi\mathcal{F}} & & \downarrow{\Psi\mathcal{F}} \\
\mathcal{F} & \xrightarrow{\mathcal{F}} & \mathcal{F} \\
\end{array}
\]

where $\Psi\mathcal{F} = \text{Ad}(g_\mathcal{F}) \circ \Psi\mathcal{D}$, $\mathcal{F}$ denotes the normally ordered monodromy action of $\mathcal{P}_W$ around the point at infinity corresponding to $\mathcal{F}$, and every face is commutative. Then, the result follows from the commutativity of the back face.

We first prove the commutativity of the top face. Since $g_\mathcal{F} \in \mathcal{D}$ is weight zero and $\Psi^\text{int} : \mathcal{O}_{\infty, \mathfrak{g}}^\hbar \to \mathcal{O}_{\infty, U\mathfrak{g}}^\hbar$ is the identity at the level of $\mathfrak{h}$–modules in $\text{Vect}_\mathfrak{h}$, $\Psi^\text{int} = \text{Ad}(g_\mathcal{F}) \circ \Psi^\text{int}$ intertwines the characters of $\mathcal{P}_W$ given by $\epsilon(p_\mathcal{F}) = \exp(i\pi h_\mathcal{F})$, and $\mathcal{B}(p_\mathcal{F}) = \exp(\hbar t_\mathcal{F}/2)$. Therefore, by Theorem 4.5 (1) and Proposition 5.6 (3), we can remove $\epsilon$ and $\mathcal{B}$, and obtain the result.

The commutativity of the lateral faces follows from Sections 4 and 5. Namely, by Theorem 4.5 (2) and Section 4.6, the normally ordered quantum Weyl group action of the pure braid group $\mathcal{P}_W \subset \mathcal{B}_W$ factors through the Drinfeld algebra $\mathcal{D}_h \subset \text{End}(\mathcal{f}_h)$. Moreover, by definition, $\mathcal{F}$ is the normally ordered monodromy action of $\mathcal{P}_W$, which readily factors through the classical Drinfeld algebra $\mathcal{D} \subset \text{End}(\mathcal{f})$.

The commutativity of the bottom and front faces follows from Section 8.1. Namely, by Theorem 3.1 (and its analogue for $U\mathfrak{g}[\hbar]$), the restriction to integrable category $\mathcal{O}_{\infty}$ modules yields the embeddings $\mathcal{D}_h \hookrightarrow \text{End}(\mathcal{f}_h^\text{int})$ and $\mathcal{D} \hookrightarrow \text{End}(\mathcal{f}^\text{int})$. Since $g_\mathcal{F} \in \mathcal{D}$, it follows from Theorem 8.1 that $\Psi^\text{int}_\mathcal{F}$ also restricts to an isomorphism $\Psi^\text{int}_\mathcal{F} = \text{Ad}(g_\mathcal{F}) \circ \Psi\mathcal{D} : \mathcal{D}_h \to \mathcal{D}$.

Finally, since $\mathcal{D}$ embeds in $\text{End}(\mathcal{f}^\text{int})$, the commutativity of the top, lateral, bottom, and front faces yields that of the diagram

\[
\begin{array}{ccc}
\mathcal{P}_W & \xrightarrow{\mathcal{F}} & \mathcal{P}_W \\
\downarrow{\Psi\mathcal{F}} & & \downarrow{\Psi\mathcal{F}} \\
\mathcal{D}_h & \xrightarrow{\mathcal{F}} & \mathcal{D}_h \\
\end{array}
\]

and the result follows. \qed
8.3. The equivariant braid monodromy theorem. The following is a direct consequence of Theorem 8.2.

**Theorem.** Let \( V \) be a \( g \)-module in category \( \mathcal{O}_\infty^h \), \( F(V) \) its Etingof–Kazhdan quantisation,

\[
\mathcal{P}_{r,B} : \mathcal{P}_W \to GL(V) \quad \text{and} \quad \lambda : \mathcal{P}_W \to GL(F(V))
\]

the equivariant monodromy of the Casimir connection given by Proposition 5.6, and quantum Weyl group action given by Theorem 4.5.

Then, \( \mathcal{P}_{r,B} \) and \( \lambda \) are canonically equivalent. Specifically, for any \( F \in \text{Mns}(\mathbb{D}) \) the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{P}_W & \xrightarrow{\mathcal{P}_{r,B}} & \mathcal{P}_W^F \\
\Phi & \xrightarrow{\circ} & \Phi \\
\text{End}(f_h) & \xleftarrow{\lambda} & \text{End}(f)
\end{array}
\]

8.4. Extension to other Lie associators. Although Theorem 8.2 and Corollary 8.3 are formulated in terms of the tensor equivalence \( F : \mathcal{O}^h_\infty \to \mathcal{O}^h_{\infty,U_h g} \) corresponding to the KZ associator, they hold true for the Etingof–Kazhdan equivalence corresponding to an arbitrary Lie associator \( \Phi \).

Indeed, by [ATL18, ATL19a] the braided Coxeter category \( \mathcal{O}^\text{int}_{U_h g,R,S} \) underlying the \( R \)-matrix and quantum Weyl group of \( U_h g \) (see 6.5) is equivalent to a braided Coxeter category \( \mathcal{O}^\text{h,int}_{\text{g,B}} \) with diagrammatic categories \( \{ \mathcal{O}^\text{h,int}_{\text{g,B}} \}_{B \subseteq \mathbb{D}} \), and standard restriction functors, with the corresponding horizontal equivalences \( \mathcal{O}^\text{h,int}_{\text{g,B}} \to \mathcal{O}^\infty_{\infty,U_h g} \) given by the Etingof–Kazhdan tensor equivalence \( \Phi^\text{F} \) corresponding to \( \Phi \).

By the rigidity result of [ATL19b], \( \mathcal{O}^\text{h,int}_{\text{g,B}} \) is equivalent to \( \mathcal{O}^\text{h,int}_{R,S} \), with diagrammatic equivalences given by the identity functors endowed with a non-trivial tensor structure.

Composing yields an equivalence \( \mathcal{O}^\text{h,int}_{R,S} \to \mathcal{O}^\text{int}_{U_h g,R,S} \) whose diagrammatic equivalences are the Etingof–Kazhdan functors corresponding to \( \Phi \), which then yields Theorem 8.2 and Corollary 8.3 for \( \mathcal{F}^\Phi \).

9. Parabolic pure braid group actions

In this section, we extend the results of Sections 4 and 8 to parabolic pure braid groups.

9.1. The group \( PB_J \). For any subset \( J \subseteq I \), let \( PB_J \subseteq B_W \) be the preimage of \( W_J = \langle s_j \rangle_{j \in J} \) under the projection \( B_W \to W \). Thus, \( PB_{\emptyset} = \mathcal{P}_W \) and \( PB_1 = B_W \). The **parabolic pure braid group** \( PB_J \) is generated by the braid group \( B_{W_J} \) and the pure braid group \( \mathcal{P}_W \). Moreover, as an abstract group,

\[
PB_J \simeq (\mathcal{P}_W \times B_{W_J})/\overline{PB}_{W_J} \quad \text{where} \quad \overline{PB}_{W_J} = \{ (p,p^{-1}) \mid p \in PB_{W_J} \} \subseteq \mathcal{P}_W \times B_{W_J}
\]

9.2. Quantum Weyl group action of \( PB_J \). Let \( U_h g_J \subseteq U_h g \) be the Hopf subalgebra generated by \( \{ E_j, F_j, h_j \}_{j \in J} \), and \( \mathcal{O}^\text{int}_{\infty,U_h g_J} \subseteq \mathcal{O}^\infty_{\infty,U_h g} \) the full subcategory of modules whose restriction to \( U_h g_J \) is integrable. We have the inclusions

\[
\mathcal{O}^\text{int}_{\infty,U_h g_J} \subseteq \mathcal{O}^\text{int}_{\infty,U_h g} \subseteq \mathcal{O}^\infty_{\infty,U_h g}
\]

together with the equalities \( \mathcal{O}^\text{h,int}_{U_h g_J} = \mathcal{O}^\infty_{\infty,U_h g} \) and \( \mathcal{O}^\text{h,int}_{U_h g} = \mathcal{O}^\infty_{\infty,U_h g} \).
Let \( f^{\text{int}}_J : \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \to \text{Vect}_h \) be the forgetful functor. We define below and in 9.3 two actions
\[
\lambda, \lambda_{\varepsilon, \mathcal{B}}[\mathfrak{g}]: \mathcal{PB}_J \to \text{Aut}(f^{\text{int}}_J)
\]
such that

- for \( J = \emptyset \), they recover the quantum Weyl group action \( \lambda : \mathcal{P}_W \to \text{Aut}(f^\text{int}_\mathfrak{g}) \) from Theorem 4.5 (3) and the normally ordered quantum Weyl group action \( \lambda_{\varepsilon, \mathcal{B}}[\mathfrak{g}] : \mathcal{P}_W \to U_h \mathfrak{g} \) from Section 4.6, respectively.

- for \( J = I \), both give the quantum Weyl group action \( \lambda : \mathcal{B}_W \to \text{Aut}(f^\text{int}_\mathfrak{g}) \).

Let \( f^\text{int}_{J, \mathfrak{h}} : \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}, J} \to \text{Vect}_h \) be the forgetful functor and \( \lambda_J : \mathcal{B}_{W_J} \to \text{Aut}(f^\text{int}_{J, \mathfrak{h}}) \) the quantum Weyl group action of \( \mathcal{B}_{W_J} \). Let \( \lambda^{\text{int}} : \mathcal{B}_{W_J} \to \text{Aut}(f^{\text{int}}_{J, \mathfrak{h}}) \) be its lift through the restriction functor \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}, J} \to \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \).

**Theorem.** The following holds.

1. The quantum Weyl group action of \( \mathcal{PB}_J \) on integrable modules in category \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \) has a unique extension to an action \( \lambda : \mathcal{PB}_J \to \text{Aut}(f^{\text{int}}_J) \) such that \( \lambda_{\mathcal{B}_{W_J}} = \lambda^{\text{int}} \) and \( \lambda_{\mathcal{P}_W} \) is the restriction of the action (4.2) to \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}, J} \subset \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \).

2. The map \( \lambda \) intertwines the inner action of \( \mathcal{PB}_J \) on \( U_h \mathfrak{g} \), i.e., for any element \( Y \in U_h \mathfrak{g} \) and \( b \in \mathcal{PB}_J \)
\[
\lambda(b) Y \lambda(b)^{-1} = b(Y)
\]
in \( \text{End}(f^{\text{int}}_J) \).

**Proof.** The uniqueness of \( \lambda \) follows from the fact that \( \mathcal{PB}_J \) is generated by \( \mathcal{P}_W \) and \( \mathcal{B}_{W_J} \). To prove the existence of \( \lambda \), it is enough to observe that on the one hand there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{B}_W & \xrightarrow{\lambda} & \text{Aut}(f^\text{int}_\mathfrak{g}) \\
\uparrow & & \uparrow \\
\mathcal{B}_{W_J} & \xrightarrow{\lambda^{\text{int}}} & \text{Aut}(f^{\text{int}}_{J, \mathfrak{h}}) \\
\end{array}
\]
where the right vertical arrow is induced by the inclusion \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}, J} \subset \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \). On the other, by Theorem 4.5, the quantum Weyl group action of \( \mathcal{P}_W \) on integrable modules extends canonically to \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \) and therefore to \( \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}, J} \subset \mathcal{O}^{\text{int}}_{\infty, U_h \mathfrak{g}} \), i.e., there is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{B}_W & \xrightarrow{\lambda} & \text{Aut}(f^\text{int}_\mathfrak{g}) \\
\uparrow & & \uparrow \\
\mathcal{P}_W & \xrightarrow{\lambda} & \text{Aut}(f^{\text{int}}_{J, \mathfrak{h}}) \\
\end{array}
\]

The actions of \( \mathcal{B}_{W_J} \) and \( \mathcal{P}_W \) on \( f^\text{int}_\mathfrak{g} \) give rise to an action of \( \mathcal{P}_W \rtimes \mathcal{B}_{W_J} \), since, for any \( p \in \mathcal{P}_W \) and \( b \in \mathcal{B}_{W_J} \), one has
\[
\lambda^{\text{int}}(b) \cdot \lambda(p) = \lambda^{\text{int}}(b) \cdot \varepsilon_h(p) \cdot \mathcal{X}(p) \\
= b(\varepsilon_h(p)) \cdot b(\mathcal{X}(p)) \cdot \lambda^{\text{int}}(b) \\
= \varepsilon_h(bp^{-1}) \cdot \mathcal{X}(bp^{-1}) \cdot \lambda^{\text{int}}(b) \\
= \lambda(bp^{-1}) \cdot \lambda^{\text{int}}(b)
\]
where the third equality follows the $B_W$-equivalence of $\mathcal{X}$ (Theorem 4.5 (2)). Moreover, they coincide on $\mathcal{P}_{W_J} = \mathcal{P}_W \cap B_{W_J}$, and therefore give rise to the desired action $\lambda : \mathcal{P}B_J \to \text{Aut}(f^J_{h^{-1}})$.

\[ \square \]

9.3. **Normally ordered quantum Weyl group action of $\mathcal{P}B_J$.** Let $\Delta_J \subseteq \Delta$ be the root subsystem generated by $\{\alpha_j\}_{j \in J}$, and let

$$\varepsilon^J_h : \mathcal{P}B_J \to \text{Aut}(f^J_{h^{-1}}) \quad \text{and} \quad h^J : \mathcal{P}B_J \to \exp(h \hbar) \quad (9.1)$$

be the morphisms uniquely defined by the following conditions.

- For any $b \in B_{W_J}$, $\varepsilon^J_h(b) = 1 = h^J(b)$.
- For any $\alpha \in \Delta^+_J$, $\varepsilon^J_h(p_\alpha) = 1 = h^J(p_\alpha)$.
- For any $\alpha \in \Delta^- \setminus \Delta^+_J$, $\varepsilon^J_h(p_\alpha) = \exp(i \pi h_\alpha)$ and $h^J(p_\alpha) = \exp(h \hbar \alpha / 2)$.

Note that $\varepsilon^J_h$ and $h^J$ are both $B_{W_J}$-equivariant. They therefore give rise to a morphism

$$\lambda_{\varepsilon^J_h, h^J} : \mathcal{P}B_J \to \text{Aut}(f^J_{h^{-1}}) \quad \text{by} \quad \lambda(b) = \varepsilon^J_h(b) \cdot \lambda_{\varepsilon^J_h, h^J}(b) \cdot h^J(b)$$

for any $b \in \mathcal{P}B_J$, which we shall refer to as the normally ordered quantum Weyl group action of $\mathcal{P}B_J$ on $\mathcal{O}^{\text{int}}_{h \hbar, B}$.

If $J = \emptyset$, $\varepsilon^J_h, h^J$ is the action of $\mathcal{P}_W$ constructed in 4.6 while, if $J = I$, $\lambda_{\varepsilon^J_h, h^J}$ is the quantum Weyl group action of $\mathcal{B}_W$ on $\mathcal{O}^{\text{int}}_{\infty, \hbar}$.

9.4. **Tits extension and $\mathcal{P}B_J$.** Let now $\mathfrak{g}_J \subseteq \mathfrak{g}$ be the subalgebra generated by $\{e_j, f_j\}_{j \in J} \subseteq \mathfrak{h}^{\text{int}}$, $\mathcal{O}^{\text{int}}_{\mathfrak{h}, \mathfrak{g}} \subseteq \mathcal{O}^h_{\infty, \mathfrak{g}}$ the full subcategory of modules whose restriction to $\mathfrak{g}_J$ is integrable, and $f^J_{\text{int}} : \mathcal{O}^{\text{int}}_{\mathfrak{h}, \mathfrak{g}} \to \text{Vect}_h$ the forgetful functor.

Let $\varepsilon^J : \mathcal{P}B_J \to \text{Aut}(f^J_{\text{int}})$ be the sign character defined as in (9.1), and define $\varepsilon^J : \mathcal{P}_W \to \text{Aut}(f^J_{\text{int}})$ by the relation

$$\varepsilon(p) = \varepsilon^J(p) \cdot \varepsilon^J(p)$$

for any $p \in \mathcal{P}_W$, where $\varepsilon$ is given by (5.3). Thus, $\varepsilon^J(p_\alpha) = \exp(i \pi h_\alpha)$ if $\alpha \in \Delta^+_J$, and $\varepsilon(p_\alpha) = 1$ if $\alpha \notin \Delta^+_J$.

**Lemma.** Let $\mathcal{V}$ be a module in $\mathcal{O}^{\text{int}}_{\mathfrak{h}, \mathfrak{g}}$. Then, there is an action $\tau^J$ of $\mathcal{P}B_J$ on $\mathcal{V}$ uniquely determined by the following conditions.

1. The restriction of $\tau^J$ to $\mathcal{B}_{W_J}$ is given by the action $\tau^J$ of the triple exponentials (2.3) indexed by $J$.

2. The restriction of $\tau^J$ to $\mathcal{P}_W$ is given by the sign character $\varepsilon^J$.

**Proof.** The result follows at once from Proposition 5.6 (1).

**Remark.** Equivalently, $\tau^J$ is given by a projection of $\mathcal{P}B_J$ onto the Tits extension $\tilde{W}_J$. Note also that, for $J = \emptyset$, $\tau_J$ is trivial, while, for $J = I$, $\tau_J = \tau$.

9.5. **Monodromy action of $\mathcal{P}B_J$ on category $\mathcal{O}^{\text{int}}_{h}$.** We construct below an action

$$\mathcal{P}^J : \mathcal{P}B_J \to \text{Aut}(f^J_{\text{int}})$$

by making the monodromy of the Casimir connection $\nabla_K$ of $\mathfrak{g}$ equivariant, as described in 1.9 and 5.4–5.6, but only with respect to the parabolic subgroup $W_J$. For $J = \emptyset$, $\mathcal{P}^J : \mathcal{P}_W \to \text{Aut}(f) \circ \nabla_K$ (cf. Section 5.2) while,
for \( J = 1 \), \( \mathcal{P}_{\tau_1;J,\mathcal{B}_J} \) is the equivariant monodromy action \( \mathcal{P}_{\tau,J} : \mathcal{B}_W \to \text{Aut}(f^{\text{int}}) \) of Theorem 5.5.

Let \( \mathcal{P} : \Pi_1(X;Wx_0) \to \mathcal{T}_0 \) be the monodromy of \( \nabla_K \), where \( \mathcal{T}_0 \) is the image of the holonomy algebra (cf. 1.9), and consider its restriction to \( \Pi_1(X;W_Jx_0) \). The lack of equivariance of \( \mathcal{P} \) under \( W \) is controlled by the 1-cocycle

\[
\mathcal{A}[J] = i_J^* \mathcal{A}|_{W_J} : W_J \to \text{Hom}(\Pi_1(X;W_Jx_0), \exp(hh))
\]

where \( i_J : \Pi_1(X;W_Jx_0) \to \Pi_1(X;Wx_0) \) is the inclusion.

The obstruction \( \mathcal{A}[J] \) is related to the one for the Casimir connection of \( g_J \) as follows. Consider the quotient map

\[
p_J : \mathfrak{e}/\mathfrak{h} \to \mathfrak{e}/\bigcap_{\alpha \in \Delta_J} \ker(\alpha) \cong \mathfrak{b}_J
\]

\( p_J \) is equivariant under \( W_J \) and, by [Kac90, Prop. 3.12], restricts to a map \( X \to X_J \) of Tits cones. It therefore induces a morphism of groupoids \( p_J : \Pi_1(X;W_Jx_0) \to \Pi_1(X_J;W_J[x_0];J) \), where \( [x_0]_J = p_J(x_0) \), which we denote by the same symbol.

**Lemma.** Let

\[
\mathcal{A}_w : W_J \to \text{Hom}(\Pi_1(X_J;W_J[x_0];J), \exp(hh_J))
\]

be the 1-cocycle measuring the lack of equivariance of the Casimir connection of \( g_J \) with respect to \( W_J \). Then, \( \mathcal{A}[J] = p_J^* \mathcal{A}_w \).

**Proof.** Let \( w \in W_J \). By Remark 5.4, \( \mathcal{A}_w \) is the monodromy of the connection \( d - \hbar a_w \), where

\[
a_w = \nabla_K - w^*\nabla_K = \hbar \sum_{\alpha \in \Delta^+: \omega \alpha < 0} \frac{d\alpha}{\alpha} t_\alpha = \hbar \sum_{\alpha \in \Delta^+: \omega \alpha < 0} \left( \frac{d\alpha}{\alpha} \right) t_\alpha = p_J^*(\nabla_K - w^*\nabla_K;J)
\]

\( \square \)

By Theorem 5.5 for \( g_J \), \( \mathcal{A}_J = d\mathcal{B}_J \), where \( \mathcal{B}_J \in \text{Hom}(\Pi_1(X_J;W_J[x_0];J), \exp(hh_J)) \). Set \( \mathcal{B}[J] = p_J^* \mathcal{B}_J \). Then,

\[
\mathcal{A}[J] = p_J^* \mathcal{A}_J = p_J^* d\mathcal{B}_J = dp_J^* \mathcal{B}_J = d\mathcal{B}[J]
\]

It follows that \( \mathcal{B}[J] \) gives rise to a \( W_J \)-equivariant morphism

\[
\mathcal{P}_{\mathcal{B}[J]} : \Pi_1(X;W_Jx_0) \to \mathcal{T}_0 \quad \mathcal{P}_{\mathcal{B}[J]}(\gamma) = \mathcal{P}(\gamma) \cdot \mathcal{B}[J](\gamma)
\]

Consider next the equivalence of groupoids

\[
P_J : W_J \ltimes \Pi_1(X;W_Jx_0) \to \Pi_1(X/W_J;[x_0]) \cong \mathcal{P}_{\mathcal{B}_{\tau}}
\]

generalising (5.1). Composing with \( P_J^{-1} \) yields a morphism \( \mathcal{P}_{\mathcal{B}_J} : W_J \ltimes \mathcal{T}_0 \) and its lift \( \mathcal{P}_{\mathcal{B}_J} \to \mathcal{P}_{\mathcal{B}_J} \ltimes \mathcal{T}_0 \). Combining this with the action \( \tau[J] \) of \( \mathcal{P}_{\mathcal{B}_J} \) on \( f^{\text{int}} \) defined in Lemma 9.4, yields the following generalisation of Theorem 5.5.

**Theorem.** There is a morphism \( \mathcal{P}_{\tau[J],\mathcal{B}[J]} : \mathcal{P}_{\mathcal{B}_J} \to \text{Aut}(f^{\text{int}}) \) given by

\[
\mathcal{P}_{\tau[J],\mathcal{B}[J]}(b) = \tau[J](b) \cdot \mathcal{P}(b) \cdot \mathcal{B}[J](\bar{b})
\]

where \( \bar{b} \in \Pi_1(X;W_Jx_0) \) is the unique lift of \( b \) through \( x_0 \).
Remark. Note that, for any \( j \in J \), \( \mathcal{B}^J(\gamma_j) = \exp( \hbar t \alpha_j / 4) \), since \( p_J \) maps \( \gamma_j \) to the corresponding generator of \( \gamma_J \), \( \in \Pi_1(\mathcal{X}_J; W_J[x_0]) \) and \( \mathcal{B}^J(\gamma_j) = \exp( \hbar t \alpha_j / 4) \) by construction.

9.6. The monodromy theorem for \( \mathcal{P}B_J \).

Theorem. The \( W_J \)-equivariant monodromy of the Casimir connection on a \( \mathfrak{g} \)-module \( V \in \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \) is canonically equivalent to the normally ordered quantum Weyl group action of the parabolic braid group \( \mathcal{P}B_J \) on the Etingof–Kazhdan quantisation \( F(V) \in \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \).

Proof. The result follows from the combination of Theorem 7.4 for \( \mathcal{B}_W \) and Theorems 8.2–8.3 for \( \mathcal{P}_W \).

Specifically, let \( B \subseteq D \) be the subdiagram corresponding to \( J \), \( \mathcal{F} \) a maximal nested set containing \( B \subseteq D \) corresponding to \( J \), and \( \mathcal{F}_J \) the induced maximal nested set on \( B \). Let

\[
\begin{align*}
\mathfrak{f}_J : \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} & \to \text{Vect}_h, \\
\mathfrak{f}_{J,J} : \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} & \to \text{Vect}_h
\end{align*}
\]

be the forgetful functors. By Theorem 7.4 (1) and (7.3), the isomorphism \( \Psi_\mathcal{F} \) restricts to \( \Psi_{\mathcal{F}_J} \), i.e., there is a commutative diagram

\[
\begin{CD}
\text{End}(\mathfrak{f}_{J,J}) @> \Psi_{\mathcal{F}_J} >> \text{End}(\mathfrak{f}_{J,J}^{\text{int}}) \\
@VV \circ V @VV \circ V \\
\text{End}(\mathfrak{f}_J) @> \Psi_\mathcal{F} >> \text{End}(\mathfrak{f})
\end{CD}
\]

where the vertical arrows are induced by the restriction functors \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \) and \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \), respectively.

Further, since the Etingof–Kazhdan equivalence preserves the categories of \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \) modules, \( \Psi_\mathcal{F} \) restricts to an isomorphism \( \Psi_{\mathcal{F}_{\text{int}}} : \text{End}(\mathfrak{f}_{J,J}^{\text{int}}) \to \text{End}(\mathfrak{f}_{J,J}^{\text{int}}) \) such that

1. There is a commutative diagram

\[
\begin{CD}
\text{End}(\mathfrak{f}_{J,J}^{\text{int}}) @> \Psi_{\mathcal{F}_{\text{int}}} >> \text{End}(\mathfrak{f}_{J,J}^{\text{int}}) \\
@VV \circ V @VV \circ V \\
\text{End}(\mathfrak{f}_{J,J}^{\text{int}}) @> \Psi_{\mathcal{F}_{\text{int}}} >> \text{End}(\mathfrak{f}_{J,J}^{\text{int}})
\end{CD}
\]

where the vertical arrows are induced by the restriction functors \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \) and \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \), respectively.

2. There is a commutative diagram

\[
\begin{CD}
\text{End}(\mathfrak{f}_J) @> \Psi_\mathcal{F} >> \text{End}(\mathfrak{f}) \\
@VV \circ V @VV \circ V \\
\text{End}(\mathfrak{f}_{J,J}^{\text{int}}) @> \Psi_{\mathcal{F}_{\text{int}}} >> \text{End}(\mathfrak{f}_{J,J}^{\text{int}})
\end{CD}
\]

where the vertical arrows are induced by the inclusions \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \) and \( \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \to \mathcal{O}_{\mathfrak{g}_{\infty}^{\hbar,J}} \), respectively.
We claim that $\Psi^\text{int}_J$ intertwines the actions of $B_W$ and $P_W$, and therefore that of $PB_J$. To this end, consider the diagram

$$
\begin{array}{c}
\text{End}(f_{J,n}^\text{int}) \rightarrow \text{End}(f_{J}^\text{int}) \\
\lambda_J \downarrow \quad \Psi^\text{int}_J \quad \uparrow \tau_{J,J}\cdot \text{End}(f_{J}^\text{int}) \\
\text{PB}_J \rightarrow \text{PB}_J \\
\text{End}(f_{J}^\text{int}) \rightarrow \text{End}(f_{J}^\text{int}) \\
\end{array}
$$

The front face commutes by (1); the top face by Theorem 7.4 (3) for $g_J$; the left lateral face by Theorem 9.2 (1). For the right lateral face, recall that, for any $b \in PB_J$,

$$\mathcal{P}_X(b) = \tau_J(b) \cdot \mathcal{P}_J(b) \cdot \mathcal{B}_J(b)$$

Let $b \in B_W$. By Lemma 9.4 (1), we have that $\tau_J(b) = \tau_J(b)$. Then, by Remark 9.5, $\mathcal{B}_J(b) = \mathcal{B}_J(b)$, where $b \in \Pi_1(X_J, W_J[x_0,J])$ denotes the unique lift of $b$ through $[x_0]_J$. Finally, $\mathcal{F}_J(b) = \mathcal{F}_J(b)$ since the monodromy in the De Concini–Procesi compactification is recursive in nature [DCP95, Thm. 3.6]. Thus, $\Psi^\text{int}_J$ intertwines the actions of $B_W$ through $PB_J$.

Similarly, consider the diagram

$$
\begin{array}{c}
\text{End}(f_h) \rightarrow \text{End}(f) \\
\lambda_{\varepsilon,J} \downarrow \quad \Psi_J \quad \uparrow \tau_{J,J} \cdot \text{End}(f) \\
\text{PB}_J \rightarrow \text{PB}_J \\
\text{End}(f_{h}^\text{int}) \rightarrow \text{End}(f_{h}^\text{int}) \\
\end{array}
$$

Let $p \in P_W$ and recall the identities

$$\varepsilon(p) = \varepsilon_J(p) \cdot \varepsilon_J(p) \quad \text{and} \quad \mathcal{B}(p) = \mathcal{B}_J(p) \cdot \mathcal{B}_J(p)$$

from 9.4 and Remark 9.5. The commutativity of the top face then follows from Theorem 8.2 by correcting simultaneously $\lambda_{\varepsilon,J}$ and $\mathcal{F}_J$ by $\varepsilon_J$ and $\mathcal{B}_J$. The left lateral face commutes by Theorem 9.2 (1). The right lateral face commutes by Lemma 9.4 (2). Thus, $\Psi^\text{int}_J$ intertwines the actions of $P_W$ through $PB_J$. \hfill \square

The (non normally ordered) quantum Weyl group action of $PB_J$ admits a similar monodromic interpretation, in analogy with Theorem 8.3. Namely, define $\mathcal{P}_{\tau,J,J} : PB_J \rightarrow \text{Aut}(f_{h}^\text{int})$ by

$$\mathcal{P}_{\tau,J,J}(b) = \varepsilon_J(b) \cdot \mathcal{P}_{\tau(J,J),J}(b) \cdot \mathcal{B}_J(b)$$  \hspace{1cm} (9.2)

for any $b \in PB_J$. Then, the following holds.
Corollary. Let $V$ be a $g$–module in category $\mathcal{O}^{h,\text{-int}}$, $F(V)$ its Etingof–Kazhdan quantisation,

\[ \mathcal{P}_{r,B} : \mathcal{P}B \rightarrow GL(V) \quad \text{and} \quad \lambda : \mathcal{P}B \rightarrow GL(F(V)) \]

the corrected $W_\lambda$–equivariant monodromy of the Casimir connection (9.2), and the quantum Weyl group action given by Theorem 9.2 respectively. Then, $\mathcal{P}_{r,B}$ and $\lambda$ are canonically equivalent.

References

[App13] A. Appel, Monodromy theorems in the affine setting, Ph.D. Thesis, Northeastern University (2013).

[ATL15] A. Appel and V. Toledano Laredo, Monodromy of the Casimir connection of a symmetrisable Kac–Moody algebras, arXiv:1512.03041 (2015), 104 pp.

[ATL18] A. Appel and V. Toledano Laredo, Coxeter categories and quantum groups, Selecta Math. (N.S.) 24 (2018), 3529–3617.

[ATL19a] A. Appel and V. Toledano Laredo, The corrected $W_\lambda$–equivariant monodromy of the Casimir connection (9.2), and the quantum Weyl group action given by Theorem 9.2 respectively. Then, $\mathcal{P}_{r,B}$ and $\lambda$ are canonically equivalent.

[Cam71] E. Brieskorn, Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe, Invent. Math. 12 (1971), 57–61.

[Che89] I. Cherednik, Generalized braid groups and local $r$–matrix systems, Dokl. Akad. Nauk SSSR 307 (1989), 49–53.

[Che91] I. Cherednik, Monodromy representations for generalized Knizhnik–Zamolodchikov equations and Hecke algebras, Publ. Res. Inst. Math. Sci. 27 (1991), 711–726.

[DCP95] C. De Concini and C. Procesi, Hyperplane arrangements and holonomy equations, Selecta Math. (N.S.) 1 (1995), 495–535.

[DG01] F. Digne and Y. Gomi, Presentation of pure braid groups, J. Knot Theory Ramifications 10 (2001), 609–623.

[Dig15] F. Digne, Présentation des groupes de tresses purs et de certaines de leurs extensions, arXiv:1511.08731 (2015), 16 pp.

[Dri87] V. Drinfeld, Quantum groups, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.

[Dri89] V. Drinfeld, Almost cocommutative Hopf algebras, Algebra i Analiz 1 (1989), 30–46.

[Dri90] V. Drinfeld, On quasitriangular quasi–Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Algebra i Analiz 2 (1990), 149–181.

[Dri92] V. Drinfeld, On some unsolved problems in quantum group theory, Quantum groups (Leningrad, 1990), Lecture Notes in Math., vol. 1510, Springer, Berlin, 1992, 1–8.

[EK96] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. I, Selecta Math. (N.S.) 2 (1996), 1–41.

[EK98] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. II, Selecta Math. (N.S.) 4 (1998), 213–231.

[EK08] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras. VI, Quantization of generalized Kac–Moody algebras, Transform. Groups 13 (2008), 527–539.

[FMTV00] G. Felder, Y. Markov, V. Tarasov, and A. Varchenko, Differential equations compatible with $KZ$ equations, Math. Phys. Anal. Geom. 3 (2000), 139–177.

[Jim85] M. Jimbo, A $q$–difference analogue of $U(g)$ and the Yang–Baxter equation, Lett. Math. Phys. 10 (1985), 63–69.

[Kac90] V. Kac, Infinite–dimensional Lie algebras, third ed., Cambridge University Press, Cambridge, 1990.

[KR90] A. Kirillov and N. Reshetikhin, $q$–Weyl group and a multiplicative formula for universal $R$–matrices, Comm. Math. Phys. 134 (1990), 421–431.

[Loo80] E. Looijenga, Invariant theory for generalized root systems, Invent. Math. 61 (1980), 1–32.
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[127x690] 36 A. APPEL AND V. TOLEDANO LAREDO

[G. Lusztig, Quantum deformations of certain simple modules over enveloping algebras, Adv. in Math. 70 (1988), 237–249,]

[Lus90] G. Lusztig, Canonical bases arising from quantized enveloping algebras, II, Common trends in mathematics and quantum field theories (Kyoto, 1990), Progr. Theoret. Phys. Suppl., 102 (1990), 175–201.

[Lus93] G. Lusztig, Introduction to quantum groups, Progress in Mathematics, 110. Birkhäuser, 1993.

[MTL05] J. Millson and V. Toledano Laredo, Casimir operators and monodromy representations of generalised braid groups, Transform. Groups 10 (2005), 217–254.

[Pro96] C. Procesi, Complementi di sottospazi e singolarità coniche, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9 7 (1996), 113–123.

[Sa94] Y. Saito, PBW basis of quantized universal enveloping algebras, Publ. Res. Inst. Math. Sci. 30 (1994), 209–232.

[So90] Ya. S. Soibelman, Algebra of functions on a compact quantum group and its representations, Algebra i Analiz 2 (1990), 190–212.

[Tit66] J. Tits, Normalisateurs de tores. I. Groupes de Coxeter étendus, J. Algebra 4 (1966), 96–116.

[TL02] V. Toledano Laredo, A Kohno–Drinfeld theorem for quantum Weyl groups, Duke Math. J. 112 (2002), 421–451.

[TL08] V. Toledano Laredo, Quasi–Coxeter algebras, Dynkin diagram cohomology, and quantum Weyl groups, Int. Math. Res. Pap. IMRP (2008), Art. ID rpu009, 167 pp.

[TL16] V. Toledano Laredo, Quasi–Coxeter quasitriangular quasibialgebras and the Casimir connection, arXiv:1601.04076 (2016), 55 pp.

[vdL83] H. van der Lek, The homotopy type of complex hyperplane complements, Ph.D. Thesis, University of Nijmegen (1983).

[Vin71] E. Vinberg, Discrete linear groups that are generated by reflections, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1072–1112.

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