Planar anti-Ramsey numbers of paths and cycles

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Abstract

Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. Given a positive integer \( n \) and a planar graph \( H \), let \( T_n(H) \) be the family of all plane triangulations \( T \) on \( n \) vertices such that \( T \) contains a subgraph isomorphic to \( H \). The \textit{planar anti-Ramsey number} of \( H \), denoted \( \text{ar}_p(n, H) \), is the maximum number of colors in an edge-coloring of a plane triangulation \( T \in T_n(H) \) such that \( T \) contains no rainbow copy of \( H \). Analogous to anti-Ramsey numbers and Turán numbers, planar anti-Ramsey numbers are closely related to planar Turán numbers, where the \textit{planar Turán number} of \( H \) is the maximum number of edges of a planar graph on \( n \) vertices without containing \( H \) as a subgraph. The study of \( \text{ar}_p(n, H) \) (under the name of rainbow numbers) was initiated by Horňák, Jendrol', Schiermeyer and Soták [J Graph Theory 78 (2015) 248–257]. In this paper we study planar anti-Ramsey numbers for paths and cycles. We first establish lower bounds for \( \text{ar}_p(n, P_k) \) when \( n \geq k \geq 8 \). We then improve the existing lower bound for \( \text{ar}_p(n, C_k) \) when \( k \geq 5 \) and \( n \geq k^2 - k \). Finally, using the main ideas in the above-mentioned paper, we obtain upper bounds for \( \text{ar}_p(n, C_6) \) when \( n \geq 8 \) and \( \text{ar}_p(n, C_7) \) when \( n \geq 13 \), respectively.

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1 Introduction

All graphs considered in this paper are finite and simple. Motivated by anti-Ramsey numbers introduced by Erdős, Simonovits and Sós [4] in 1975, we study the anti-Ramsey problem when host graphs are plane triangulations. A subgraph of an edge-colored graph is rainbow if all of its edges have different colors. Let $\mathcal{F}$ be a family of planar graphs. For the purpose of this paper, we call an edge-coloring that contains no rainbow copy of any graph in $\mathcal{F}$ an $\mathcal{F}$-free edge-coloring. A graph $G$ is $\mathcal{F}$-free if no subgraph of $G$ is isomorphic to any graph in $\mathcal{F}$. Let $n_\mathcal{F}$ be the smallest integer such that for any $n \geq n_\mathcal{F}$, there exists a plane triangulation on $n$ vertices that is not $\mathcal{F}$-free. Such an integer $n_\mathcal{F}$ is well-defined, because for any $F \in \mathcal{F}$, we can obtain a plane triangulation from a plane drawing of $F$ by adding new edges. When $\mathcal{F} = \{F\}$, then $n_\mathcal{F} = |F|$. For any integer $n \geq n_\mathcal{F}$, let $T_n(\mathcal{F})$ be the family of all plane triangulations $T$ on $n$ vertices such that $T$ is not $\mathcal{F}$-free. The planar anti-Ramsey number of $\mathcal{F}$, denoted $ar_p(n, \mathcal{F})$, is the maximum number of colors in an $\mathcal{F}$-free edge-coloring of a plane triangulation in $T_n(\mathcal{F})$. Clearly, $ar_p(n, \mathcal{F}) < 3n - 6$. It is worth noting that this problem becomes trivial if the host plane triangulation on $n$ vertices is $\mathcal{F}$-free, because $3n - 6$ colors can be used.

Analogous to the relation between anti-Ramsey numbers and Turán numbers proved in [4], planar anti-Ramsey numbers are closely related to planar Turán numbers [3]. The planar Turán number of $\mathcal{F}$, denoted $ex_p(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free planar graph on $n$ vertices. Given an edge-coloring $c$ of a host graph $T$ in $T_n(\mathcal{F})$, we define a representing graph $R$ of $c$ to be a spanning subgraph $R$ of $T$ obtained by taking one edge of each color under the coloring $c$ (where $R$ may contain isolated vertices). Clearly, if $c$ is an $\mathcal{F}$-free edge-coloring of $T$, then $R$ is $\mathcal{F}$-free. Thus $ar_p(n, \mathcal{F}) \leq ex_p(n, \mathcal{F})$ for any $n \geq n_\mathcal{F}$. When $\mathcal{F}$ consists of a single graph $H$, we write $ar_p(n, H)$ and $ex_p(n, H)$ instead of $ar_p(n, \{H\})$ and $ex_p(n, \{H\})$. Given a planar graph $H$, let $\mathcal{H} = \{H - e : e \in E(H)\}$. Let $G$ be an $\mathcal{H}$-free plane subgraph of a plane triangulation $T \in T_n(H)$ with $e(G) = ex_p(n, H)$. We then obtain an $H$-free edge-coloring of $T$ by coloring the edges of $G$ with distinct colors and then coloring the edges in $E(T) \setminus E(G)$ with a new color. Hence, $1 + ex_p(n, \mathcal{H}) \leq ar_p(n, H)$ for any $n \geq |H|$. We obtain the following analogous result.

**Proposition 1.1** Given a planar graph $H$ and a positive integer $n \geq |H|$, 

$$1 + ex_p(n, \mathcal{H}) \leq ar_p(n, H) \leq ex_p(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

Colorings of plane graphs that avoid rainbow faces have also been studied, see, e.g., [5, 7, 15, 16]. Various results on anti-Ramsey numbers can be found in: [1, 2, 8, 9, 10, 11, 13, 14] to
name a few. The study of planar anti-Ramsey numbers \(ar_p(n, H)\) was initiated by Horňák, Jendrol’, Schierzeyer and Soták [6] (under the name of rainbow numbers). We summarize their results in [6] as follows, where given two positive integers \(a\) and \(b\), we use \(a \mod b\) to denote the remainder when \(a\) is divided by \(b\). We use \(P_k\) and \(C_k\) to denote the path and cycle on \(k\) vertices, respectively.

**Theorem 1.2 ([6])** Let \(n, k\) be positive integers.

(a) \(ar_p(n, C_3) = \lfloor (3n - 6)/2 \rfloor\) for \(n \geq 4\).

(b) \(ar_p(n, C_4) \leq 2(n - 2)\) for \(n \geq 4\), and \(ar_p(n, C_4) \geq (9(n - 2) - 4r)/5\) for \(n \geq 42\) and \(r = (n - 2) \mod 20\).

(c) \(ar_p(n, C_5) \leq 5(n - 2)/2\) for \(n \geq 5\), and \(ar_p(n, C_5) \geq (19(n - 2) - 10r)/9\) for \(n \geq 20\) and \(r = (n - 2) \mod 18\).

(d) \(ar_p(n, C_k) \geq (3n - 6) \cdot \frac{k-3}{k-2} - \frac{2k-7}{k-2}\) for \(6 \leq k \leq n\).

Finding exact values of \(ar_p(n, H)\) is far from trivial. As observed in [6], an induction argument in general cannot be applied to compute \(ar_p(n, H)\) because deleting a vertex from a plane triangulation may result in a graph that is no longer a plane triangulation.

Dowden [3] began the study of planar Turán numbers \(ex_p(n, H)\) (under the name of “extremal” planar graphs) and proved Theorem 1.3 below, where each bound is tight.

**Theorem 1.3 ([3])** Let \(n\) be a positive integer.

(a) \(ex_p(n, C_3) = 2n - 4\) for \(n \geq 3\).

(b) \(ex_p(n, C_4) \leq 15(n - 2)/7\) for \(n \geq 4\).

(c) \(ex_p(n, C_5) \leq (12n - 33)/5\) for \(n \geq 11\).

By Proposition [1.1] and Theorem 1.3(c), we see that \(ar_p(n, C_5) \leq (12n - 33)/5\) for \(n \geq 11\). This improves the upper bound for \(ar_p(n, C_5)\) in Theorem 1.2(c) when \(n \geq 11\). Notice that the upper bound in Proposition 1.1 in general is quite loose, for example, \(ex_p(n, C_3) - ar_p(n, C_3) = \lfloor n/2 \rfloor - 1\) for all \(n \geq 4\). In this paper we study planar anti-Ramsey numbers for paths and cycles. In Section 2 we establish lower bounds for \(ar_p(n, P_k)\) when \(n \geq k \geq 8\). In Section 3 we first improve the existing lower bounds for \(ar_p(n, C_k)\) when \(k \geq 5\) and \(n \geq k^2 - k\), which improves Theorem 1.2(c,d). We then use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels to obtain upper bounds for \(ar_p(n, C_6)\) when \(n \geq 8\) and \(ar_p(n, C_7)\) when \(n \geq 13\), respectively.

We need to introduce more notation. For a graph \(G\) we use \(V(G)\), \(|G|\), \(E(G)\), \(e(G)\), \(\delta(G)\) and \(\alpha(G)\) to denote the vertex set, number of vertices, edge set, number of edges,
minimum degree, and independence number of $G$, respectively. For a vertex $x \in V(G)$, we will use $N_G(x)$ to denote the set of vertices in $G$ which are adjacent to $x$. We define $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$. The subgraph of $G$ induced by $A$, denoted $G[A]$, is the graph with vertex set $A$ and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$ and $G \setminus A$ the subgraph of $G$ induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$ and $G \setminus a$, respectively. Given two graphs $G$ and $H$, the union of $G$ and $H$, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs $G$ and $H$, we may (with a slight but common abuse of notation) write $G = H$. Given a plane graph $G$ and an integer $i \geq 3$, an $i$-face in $G$ is a face of size $i$. Let $f_i(G)$ denote the number of $i$-faces in $G$ and $n_i(G)$ denote the number of vertices of degree $i$ in $G$. Given an edge-coloring $c$ of $G$, let $c(G)$ denote the number of colors used under $c$. For any positive integer $k$, let $\{k\} := \{1, 2, \ldots, k\}$.

2 Rainbow Paths

In this section, we study planar anti-Ramsey numbers for paths. We begin with a construction of a plane triangulation $T_H$ that will be needed in the proof of Theorem 2.3.

Lemma 2.1 For any integers $p \geq 1$ and $n = 3p + 2$, there exist plane triangulations $H$ on $p + 2$ vertices and $T_H$ on $n$ vertices such that $H$ and $T_H$ satisfy the following.

(a) $H \subseteq T_H$ and $H$ is hamiltonian;
(b) $V(T_H) \setminus V(H)$ is an independent set in $T_H$;
(c) The longest path in $T_H$ has $2p + 5 - \max\{0, 3 - p\}$ vertices; and
(d) The longest path in $T_H$ with both endpoints in $V(H)$ has $2p + 3$ vertices.

Proof. Let $P$ be a path with vertices $v_1, v_2, \ldots, v_p$ in order. Let $H$ be the plane triangulation obtained from $P$ by adding two adjacent vertices $x, y$ and joining each of $x$ and $y$ to all vertices on $P$ with the outer face of $H$ having vertices $x, y, v_p$ on its boundary. Then $|H| = p + 2$ and $H$ is hamiltonian. Let $T_H$ be the plane triangulation obtained from $H$ by adding a new vertex to each 3-face $F$ of $H$ and then joining it to all vertices on the boundary of $F$. For each $i \in \{1, 2, \ldots, p - 1\}$, let $u_i$ and $w_i$ be the new vertices added to the faces with vertices $v_i, v_{i+1}, x$ and $v_i, v_{i+1}, y$ on the boundary, respectively. Let $w, z$ be the new vertices added to the outer face of $H$ and the face of $H$ with vertices $x, y, v_1$ on its boundary. The construction of $T_H$ when $p = 5$ is depicted in Figure 1. Then $|T_H| = |H| + f_3(H) = 3p + 2 = n$ and $V(T_H) \setminus V(H) = \{u_1, u_2, \ldots, u_{p-1}, w_1, \ldots, w_{p-1}, v_1, z\}$. Clearly, $V(T_H) \setminus V(H)$ is a maximal independent set of $T_H$ with $|V(T_H) \setminus V(H)| = f_3(H) = 2p$ and $|V(T_H) \setminus V(H)| \geq |H| + 1 - \max\{0, 3 - p\}$. 

\[ 4 \]
Figure 1: The construction of $T_H$ when $p = 5$.

It can be easily checked that the longest path in $T_H$ has $2p + 5 - \max\{0, 3 - p\}$ vertices, and the longest path with both endpoints in $V(H)$ has $2p + 3$ vertices.

**Theorem 2.2** For any $k \in \{8, 9\}$, let $\varepsilon = k \mod 2$ and $n \geq k$ be an integer. Then $\ar_{p}(n, P_{k}) \geq (3n + 3\varepsilon - \varepsilon^{*} - 3)/2$, where $\varepsilon^{*} = (n + 1 + \varepsilon) \mod 2$.

**Proof.** Let $k \in \{8, 9\}, n, \varepsilon, \varepsilon^{*}$ be given as in the statement. Let $t$ be a positive integer satisfying $2t - 3 - \varepsilon + \varepsilon^{*} = n$. Then $t \geq k - 3$ because $n \geq k$. Let $H$ be a plane drawing of $K_{2} + (K_{t-3-\varepsilon} \cup K_{\varepsilon+1})$. Clearly, $H$ has 3-faces and 4-faces only. Notice that $|H| = t$, $f_{3}(H) = 2 + 2\varepsilon$, $f_{4}(H) = t - 3 - \varepsilon$, $e(H) = 2t - 3 + \varepsilon$, and $H$ is $P_{k-2}$-free but not $P_{k-3}$-free. Let $\mathcal{F}$ be a set which consists of all 4-faces of $H$ and $\varepsilon^{*}$ of the 3-faces of $H$. Let $T^{*}$ be the plane triangulation obtained from $H$ by adding a new vertex to each face $F \in \mathcal{F}$ and then joining it to all vertices on $F$. Then $|T^{*}| = |H| + |\mathcal{F}| = |H| + f_{4}(H) + \varepsilon^{*} = 2t - 3 - \varepsilon + \varepsilon^{*} = n$. Clearly, $T^{*} \in \mathcal{T}_{n}(P_{k})$. Now let $c$ be an edge-coloring of $T^{*}$ defined as follows: edges in $E(H)$ are colored with distinct colors under $c$ (that is, $T^{*}$ contains a rainbow copy of $H$ under $c$), and for each $F \in \mathcal{F}$, all the new edges added inside $F$ are colored the same, but for distinct faces $F, F' \in \mathcal{F}$, new edges inside $F$ are colored differently than the new edges inside $F'$. It can be easily checked that $T^{*}$ has no rainbow $P_{k}$ but contains a rainbow copy of $P_{k-1}$ under $c$. Then $c(T^{*}) = e(H) + f_{4}(H) + \varepsilon^{*} = 3t - 6 + \varepsilon^{*} = (3n + 3\varepsilon - \varepsilon^{*} - 3)/2$, since $n = 2t - 3 - \varepsilon + \varepsilon^{*}$. Hence, $\ar_{p}(n, P_{k}) \geq c(T^{*}) \geq (3n + 3\varepsilon - \varepsilon^{*} - 3)/2$, as desired. This completes the proof of Theorem 2.2.

We next prove a lower bound for $\ar_{p}(n, P_{k})$ when $k \geq 10$. 


Theorem 2.3 Let $k$ and $n$ be two integers such that $n \geq k \geq 10$. Let $\varepsilon = k \mod 2$. Then

$$ar_p(n, P_k) \geq \begin{cases} 
    n + 2k - 12 & \text{if } k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5, \\
    (3n + 9 \lfloor k/2 \rfloor + 3\varepsilon - 43)/2 & \text{if } 3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15, \\
    2n + k - 14 & \text{if } n > 5\lfloor k/2 \rfloor + \varepsilon - 15.
\end{cases}$$

Proof. Let $k, n, \varepsilon$ be given as in the statement. Assume first that $k \leq n < 3\lfloor k/2 \rfloor + \varepsilon - 5$. Then $k \geq 12$. Let $p = k - 5$ and let $P$ and $H$ be defined in the proof of Lemma 2.1. By Lemma 2.1, $|H| = k - 3$, $f_3(H) = 2k - 10$, $e(H) = 3k - 15$ and $H$ is hamiltonian. Since $n < 3\lfloor k/2 \rfloor + \varepsilon - 5$, we see that $n - k + 3 < f_3(H)$. Let $F$ be a set which consists of $n - k + 3$ many 3-faces of $H$. Let $T^*$ be the plane triangulation obtained from $H$ by adding a new vertex to each face $F \in F$ and then joining it to all vertices on the boundary of $F$. Clearly, $T^* \in T_n(P_k)$. Now let $c$ be an edge-coloring of $T^*$ defined as follows: edges in $E(H)$ are colored with distinct colors under $c$ (that is, $T^*$ contains a rainbow copy of $H$ under $c$), and for each $F \in F$, all the new edges added inside $F$ are colored the same, but for distinct faces $F, F' \in F$, new edges inside $F$ are colored differently than the new edges inside $F'$. It can be easily checked that $T^*$ has no rainbow $P_k$ but contains a rainbow copy of $P_{k-1}$ under $c$. Then $c(T^*) = c(H) + |F| = 3k - 15 + n - k + 3 = n + 2k - 12$. Hence, $ar_p(n, P_k) \geq c(T^*) \geq n + 2k - 12$.

Next assume that $3\lfloor k/2 \rfloor + \varepsilon - 5 \leq n \leq 5\lfloor k/2 \rfloor + \varepsilon - 15$. Let $\varepsilon^* = (n + \lfloor k/2 \rfloor) \mod 2$. By the choice of $\varepsilon^*$, let $t$ be a positive integer satisfying $2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon = n$. Since $n \geq 3\lfloor k/2 \rfloor + \varepsilon - 5$, it follows that $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$. Let $p = \lfloor k/2 \rfloor - 4$ and let $P, H, T_H, x, y, w, v_{\lfloor k/2 \rfloor - 4}$ be defined in the proof of Lemma 2.1. By Lemma 2.1, $|H| = \lfloor k/2 \rfloor - 2$, $f_3(H) = 2|H| - 4 = 2\lfloor k/2 \rfloor - 8$ and $|T_H| = |H| + f_3(H) = 3\lfloor k/2 \rfloor - 10 \geq k - 5 - \varepsilon$. Let $F^*$ be the outer face of $T_H$ and $F_0$ be the 3-face of $T_H$ with vertices $x, w, v_{\lfloor k/2 \rfloor - 4}$ on its boundary. Let $T$ be the plane graph on $t$ vertices obtained from $T_H$ by adding $t - 3\lfloor k/2 \rfloor + 10 \geq 2 + \varepsilon$ new vertices to the face $F^*$ and then joining each of the new vertices to both $x$ and $y$ (and further adding exactly one edge among the new vertices added inside $F^*$ when $\varepsilon = 1$). Then all 4-faces of $T$ are inside the face $F^*$ of $T_H$, $e(T) = e(T_H) + 2(t - 3\lfloor k/2 \rfloor + 10) + \varepsilon = 2t + 3\lfloor k/2 \rfloor - 16 + \varepsilon$ and $f_4(T) = t - 3\lfloor k/2 \rfloor + 10 - \varepsilon$. Let $F$ be a set which consists of all 4-faces of $T$ (and $F_0$ when $\varepsilon^* = 1$). Finally, let $T^*$ be the plane triangulation obtained from $T$ by adding a new vertex to each face $F \in F$ and then joining it to all vertices on the boundary of $F$. Then $|T^*| = |T| + f_4(T) + \varepsilon^* = 2t - 3\lfloor k/2 \rfloor + 10 + \varepsilon^* - \varepsilon = n$. By Lemma 2.1, the longest $(x, y)$-path in $T_H$ has $k - 5 - \varepsilon$ vertices. Clearly, the longest $(x, y)$-path in $T^*$ with all its internal vertices inside the face $F^*$ contains all the new vertices added to $F^*$. 


Thus $T^*$ contains $P_k$ as a subgraph and so $T^* \in \mathcal{T}_n(P_k)$. Now let $c$ be an edge-coloring of $T^*$ defined as follows: edges in $E(T)$ are colored with distinct colors under $c$ (that is, $T^*$ contains a rainbow copy of $T$ under $c$), and for each $F \in \mathcal{F}$, all the new edges added inside $F$ are colored the same, but for distinct $F, F' \in \mathcal{F}$, new edges inside $F$ are colored differently than the new edges inside $F'$. We see that $T^*$ has no rainbow $P_k$ but contains a rainbow $P_{k-1}$ under $c$. Since $n = 2t + \varepsilon^* + 10 - 3\lfloor k/2 \rfloor - \varepsilon$, we see that
\[ c(T^*) = e(T) + f_4(T) + \varepsilon^* = (2t + 3 \lfloor k/2 \rfloor - 16 + \varepsilon) + (t - 3 \lfloor k/2 \rfloor + 10 - \varepsilon) + \varepsilon^* \]
\[ = (3n + 9 \lfloor k/2 \rfloor + 3\varepsilon - 42 - \varepsilon^*)/2 \]
\[ \geq (3n + 9 \lfloor k/2 \rfloor + 3\varepsilon - 43)/2, \]
Hence, $ar_\rho(n, P_k) \geq c(T^*) \geq (3n + 9 \lfloor k/2 \rfloor + 3\varepsilon - 43)/2$, as desired.

Finally assume that $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$. Let $n - k + 7 = 3m + r$, where $m$ is a positive integer and $r \in \{0, 1, 2\}$. Since $k \geq 10$ and $n \geq 5\lfloor k/2 \rfloor + \varepsilon - 14$, we have $m \geq 3$ or $m = r = 2$. Let $t := k + 2m - 7 + \lfloor r/2 \rfloor$. Then $t \geq k - 2$ because $m \geq 3$ or $m = r = 2$, and $t + \lfloor (t-k+7)/2 \rfloor = n - \varepsilon'$, where $\varepsilon' = 1$ when $r = 1$ and $\varepsilon' = 0$ when $r \in \{0, 2\}$. Let $p = k - 9$ and let $P, H, x, y, v_1, \ldots, v_{k-9}$ be defined in the proof of Lemma 2.1. Then $|H| = k - 7$ and the longest path between $x$ an $y$ in $H$ has $k - 7$ vertices. Let $T'$ be the plane triangulation on $t$ vertices obtained from $H$ by: adding $t - k + 7 \geq 5$ new vertices to the outer face of $H$, then adding a matching of size $\lfloor (t-k+7)/2 \rfloor \geq 2$ among the new vertices, and finally joining each of the new vertices to both $x$ and $y$. We see that $T'$ is a connected $P_{k-2}$-free plane graph with only 3-faces and 4-faces. It can be easily checked that $f_4(T') = \lfloor (t-k+7)/2 \rfloor$ and $e(T') = 2t + k - 13 + \lfloor (t-k+7)/2 \rfloor$. Let $F_0$ be the 3-face of $T'$ with vertices $x, y, v_{k-9}$ when $k = 10$ and $x, v_{k-10}, v_{k-9}$ when $k \geq 11$ on its boundary. Let $\mathcal{F}$ be a set which consists of all 4-faces of $T'$ (and $F_0$ when $\varepsilon' = 1$). Let $T^*$ be the plane triangulation obtained from $T'$ by adding a new vertex to each $F \in \mathcal{F}$ and then joining it to all vertices on the boundary of $F$. Then $|T^*| = |T'| + |\mathcal{F}| = |T'| + f_4(T') + \varepsilon' = t + \lfloor (t-k+7)/2 \rfloor + \varepsilon' = n$. Clearly, $T^*$ contains $P_k$ as a subgraph and so $T^* \in \mathcal{T}_n(P_k)$. Now let $c$ be an edge-coloring of $T^*$ defined as follows: edges in $E(T')$ are colored with distinct colors under $c$ (that is, $T^*$ contains a rainbow copy of $T'$ under $c$), and for each $F \in \mathcal{F}$, all the new edges added inside $F$ are colored the same, but for distinct faces $F, F' \in \mathcal{F}$, new edges inside $F$ are colored differently than the new edges inside $F'$. We see that $T^*$ has no rainbow $P_k$ under $c$ but contains a
rainbow copy of $P_{k-1}$. Then
\[
c(T^*) = c(T') + f_4(T') + \varepsilon' = \left(2t + k - 13 + \left\lfloor \frac{t - k + 7}{2} \right\rfloor \right) + \left( \left\lfloor \frac{t - k + 7}{2} \right\rfloor + \varepsilon' \right) \\
= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{t - k + 7}{2} \right\rfloor - \left\lfloor \frac{t - k + 7}{2} \right\rfloor \\
= 2n + k - 13 - \varepsilon' + \left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{r}{2} \right\rfloor \\
\geq 2n + k - 14,
\]
since $n = t + \left\lfloor (t - k + 7)/2 \right\rfloor + \varepsilon'$ and $t = k + 2m - 7 + \lfloor r/2 \rfloor$. Hence, $ar_p(n, P_k) \geq c(T^*) \geq 2n + k - 14$, as desired. This completes the proof of Theorem 2.3.\]

Remark. In the proofs of Theorem 2.2 and Theorem 2.3, $T^* \in T_n(P_k)$ has no rainbow $P_k$ but does contain a rainbow copy of $P_{k-1}$ under the coloring $c$ we found.

3 Rainbow Cycles

In this section, we study planar anti-Ramsey numbers for cycles.

3.1 Improving the existing lower bound for $ar_p(n, C_k)$

We first prove a lower bound for $ar_p(n, C_5)$, which improves Theorem 1.2(c).

Theorem 3.1 Let $n \geq 119$ be an integer and let $r = (n + 7) \mod 18$. Then $ar_p(n, C_5) \geq (39n - 123 - 21r)/18$.

Proof. Let $r, n$ be given as in the statement. Let $t \geq 6$ be a positive integer satisfying $18t + 11 + r = n$. This is possible because $n \geq 119$ and $r = (n + 7) \mod 18$. Let $H$ be a connected $C_5$-free plane graph with $15t + 9$ vertices and $(12|H| - 33)/5$ edges such that $H$ has only 3-faces and 6-faces, and no two 6-faces share an edge in common. The existence of such a graph $H$ is due to Dowden (see Theorem 4 in [3]). Notice that $f_6(H) = 3t + 2$ and $f_3(H) = 18t + 6$. Let $F$ be a set which consists of all 6-faces and $r$ of the 3-faces of $H$. Then $|F| = f_6(H) + r$. Let $T^*$ be the plane triangulation obtained from $H$ by adding a new vertex to each face $F \in F$ and then joining it to all vertices on the boundary of $F$. Then $|T^*| = |H| + |F| = |H| + f_6(H) + r = (15t + 9) + (3t + 2) + r = 18t + 11 + r = n$ and so $T^* \in T_n(C_5)$. Finally let $c$ be an edge-coloring of $T^*$ defined as follows: edges in $E(H)$ are colored with distinct colors under $c$ (that is, $T^*$ contains a rainbow copy of $H$ under $c$), and for each $F \in F$, all the new edges added inside $F$ are colored the same, but for distinct
$F, F' \in \mathcal{F}$, new edges inside $F$ are colored differently than the new edges inside $F'$. We see that $T^*$ has no rainbow $C_5$ under $c$ because $H$ is $C_5$-free and no rainbow $C_5$ in $T^*$ can contain any new edges added to $H$. Then

$$c(T^*) = e(H) + f_6(H) + r = (36t + 15) + (3t + 2) + r = (39n - 123 - 21r)/18,$$

since $n = 18t + 11 + r$. Therefore, $ar_p(n, C_5) \geq c(T^*) \geq (39n - 123 - 21r)/18$, as desired. This completes the proof of Theorem 3.1.

**Remark.** By Proposition 1.1 and Theorem 1.3(c), $ar_p(n, C_5) \leq ex_p(n, C_5) \leq (12n - 33)/5$ for all $n \geq 11$. It then follows from Theorem 3.1 that $(39n - 123 - 21r)/18 \leq ar_p(n, C_5) \leq ex_p(n, C_5) \leq (12n - 33)/5$ for all $n \geq 119$, where $r = (n + 7) \mod 18$.

Theorem 3.2 below provides a new lower bound for $ar_p(n, C_k)$ when $k \geq 5$, which improves Theorem 1.2(d).

**Theorem 3.2** For integers $k \geq 5$, $n \geq k^2 - k$, and $r = (n - 2) \mod (k^2 - k - 2)$,

$$ar_p(n, C_k) \geq \left(\frac{k - 3}{k - 2} + \frac{2}{3(k + 1)(k - 2)}\right)(3n - 6) - \frac{2k^2 - 5k - 5}{k^2 - k - 2}r.$$

**Proof.** Let $n, k, r$ be given as in the statement. Let $t \geq 3$ be an integer satisfying $(k^2 - k - 2)(t - 2) + 2 + r = n$. This is possible because $r = (n - 2) \mod (k^2 - k - 2)$ and $n \geq k^2 - k$. Let $T$ be a plane triangulation on $t$ vertices. Then $f_3(T) = 2t - 4$. Let $k := 3m + q$, where $q \in \{0, 1, 2\}$ and $m \geq 1$ is an integer. Let $T'$ be obtained from $T$ as follows. For each face $F$ in $T$: first subdivide each of the $q + 1$ of the edges of $F$ $m$ times; next, subdivide each of the remaining $2 - q$ edges of $F$ $m - 1$ times; and finally, replace each edge from the subdivision of $T$ by any plane triangulation on $k - 1$ vertices. Examples of constructions of $T'$ when $k \in \{5, 6, 7\}$ are depicted in Figure 2 and Figure 3.

![Figure 2: Subdividing one 3-face of T when k \in \{5, 6, 7\}.](image)

It is worth noting that different edges of the subdivision of $T$ may be replaced by different plane triangulations on $k - 1$ vertices. Such a subdivision of $T$ is possible when $q \in \{0, 1, 2\}$.
because when \( q = 2 \), every edge of \( T \) is subdivided \( m \) times; and when \( q \in \{0, 1\} \), the dual of \( T \) has a perfect matching, say \( M \). Let \( M^* \) be the dual edges of \( M \) in \( T \). Then every face \( F \) in \( T \) contains exactly one edge in \( M^* \) and \( |M^*| = t - 2 \). When \( q = 0 \), each edge in \( M^* \) is divided \( m \) times, and when \( q = 1 \), each edge in \( M^* \) is divided \( m - 1 \) times. Thus \((q + 1)(t - 2)\) many edges of \( T \) are each subdivided \( m \) times and \((2 - q)(t - 2)\) many edges of \( T \) are each subdivided \( m - 1 \) times. One can check that

\[
|T'| = t + (q + 1)(t - 2)[(m + 1)(k - 3) + m] + (2 - q)(t - 2)[((m - 1) + 1)(k - 3) + (m - 1)]
\]

\[
= t + (t - 2)[(q + 1)(mk - 2m + k - 3) + (2 - q)(mk - 2m - 1)]
\]

\[
= t + (t - 2)[(q + 1 + 2 - q)(mk - 2m) + (q + 1)(k - 3) - (2 - q)]
\]

\[
= t + (k^2 - k - 5)(t - 2)
\]

and

\[
e(T') = (q + 1)(t - 2)(m + 1)[(3(k - 1) - 6] + (2 - q)(t - 2)m[3(k - 1) - 6]
\]

\[
= (t - 2)(3k - 9)[(q + 1 + 2 - q)m + q + 1]
\]

\[
= (t - 2)(3k - 9)(3m + q + 1)
\]

\[
= (t - 2)(3k - 9)(k + 1) = 3(k^2 - 2k - 3)(t - 2).
\]

By the construction of \( T' \), we see that \( T' \) is \( C_k \)-free (but contains \( C_{k+1} \) as a subgraph), \( T' \) has \( f_3(T) \) many \( i \)-faces with \( i > 3 \) and at least

\[
(q + 1)(t - 2)(m + 1)[(2(k - 1) - 5] + (2 - q)(t - 2)m[(2(k - 1) - 5]
\]

\[
= (t - 2)(2k - 7)[(q + 1 + 2 - q)m + q + 1]
\]

\[
= (t - 2)(2k^2 - 5k - 7)
\]

\[
\ge k^2 - k - 2
\]
many 3-faces because \( t \geq 3 \) and \( k \geq 5 \). Let \( \mathcal{F} \) be a set which consists of all \( i \)-faces of \( T' \) with \( i > 4 \) and \( r \) of the 3-faces of \( T' \). Let \( T^* \) be the plane triangulation obtained from \( T' \) by adding a new vertex to each face \( F \in \mathcal{F} \) and then joining it to all vertices on the boundary of \( F \). Then \( |T^*| = |T'| + f_3(T) + r = [t+(k^2-k-5)(t-2)]+(2t-4)+r = (k^2-k-2)(t-2)+2+r = n \) and so \( T^* \in \mathcal{T}_n(C_k) \). Now let \( c \) be an edge-coloring of \( T^* \) defined as follows: edges in \( E(T') \) are colored with distinct colors under \( c \) (that is, \( T^* \) contains a rainbow copy of \( T' \) under \( c \)), and for each \( F \in \mathcal{F} \), all the new edges added inside \( F \) are colored the same, but for distinct \( F, F' \in \mathcal{F} \), new edges inside \( F \) are colored differently than the new edges inside \( F' \). We see that \( T^* \) has no rainbow \( C_k \) (but contains a rainbow copy of \( C_{k+1} \)) under \( c \) because \( T' \) is \( C_k \)-free (but contains \( C_{k+1} \) as a subgraph) and no rainbow \( C_k \) in \( T^* \) can contain any new edges added to \( T' \). Hence,

\[
c(T^*) = e(T') + f_3(T) + r = 3(k^2 - 2k - 3)(t - 2) + 2(t - 2) + r
\]

\[
= (3k^2 - 6k - 7)(t - 2) + r
\]

\[
= (3k^2 - 6k - 7) \frac{n - r - 2}{k^2 - k - 2} + r
\]

\[
= \left( \frac{k - 3}{k - 2} + \frac{2}{3(k + 1)(k - 2)} \right)(3n - 6) - \frac{2k^2 - 5k - 5}{k^2 - k - 2} r,
\]

since \( n = (k^2-k-2)(t-2)+2+r \). Therefore, \( ar_p(n, C_k) \geq c(T^*) \geq \left( \frac{k - 3}{k - 2} + \frac{2}{3(k + 1)(k - 2)} \right)(3n - 6) - \frac{2k^2 - 5k - 5}{k^2 - k - 2} r \). This completes the proof of Theorem 3.2.

### 3.2 New upper bounds for \( ar_p(n, C_k) \) when \( k \in \{6, 7\} \)

Finally, we use the main ideas in [6] to establish upper bounds for \( ar_p(n, C_k) \) when \( k \in \{6, 7\} \). We need to introduce more notation. Let \( C_q \) be a cycle with vertices \( v_1, v_2, \ldots, v_q \) in order, where \( q \geq 3 \). Let \( W_q \) be a wheel obtained from \( C_q \) by adding a new vertex \( v \), the central vertex of \( W_q \), and joining it to all vertices of \( C_q \). Vertices \( v_1, v_2, \ldots, v_q \) are called rim vertices of \( W_q \). A cycle \( C \subseteq W_q \) is a central \( k \)-cycle if it contains the central vertex of \( W_q \) and \( |C| = k \). For any plane triangulation \( T \) with at least four vertices and any \( v \in V(T) \), the subgraph of \( T \) induced by \( N_T[v] \) contains the wheel \( W_{d_T(v)} \) with central vertex \( v \) as a subgraph. Let \( c(v) \) be the set of all colors such that each is used to color the edges of \( W_{d_T(v)} \) under any edge-coloring \( c \) of \( T \). Lemma 3.3 below will be used in our proof.

**Lemma 3.3 ([6])** Let \( T \) be a plane triangulation and let \( c : E(T) \to [m] \) be a surjection, where \( m \) is a positive integer. Then

\[
\sum_{v \in V(T)} |c(v)| \geq 4m.
\]
To establish an upper bound for $ar_p(n, C_k)$ when $k \in \{6, 7\}$, we use the main ideas in [6] by studying lower and upper bounds for the planar anti-Ramsey numbers when host graphs are wheels. For integers $k \geq 4$ and $q \geq k - 1$, we define $ar_p(W_q, C_k)$ to be the maximum number of colors in an edge-coloring of $W_q$ that has no rainbow copy of $C_k$.

**Theorem 3.4** For integers $k \geq 5$ and $q \geq k - 1$, $\lfloor \frac{2k-7}{k-3}q \rfloor \leq ar_p(W_q, C_k) \leq \lfloor \frac{2k-5}{k-2}q \rfloor$.

**Proof.** Let $W_q$ be a wheel with rim vertices $v_1, v_2, \ldots, v_q$ and central vertex $v$. To obtain the desired lower bound, let $c : E(W_q) \to [(2k-7)q/(k-3)]$ be an edge-coloring of $W_q$ defined as follows: for each $i \in [q]$, let $r := i \mod (k-3)$ and $c(vv_i) := i$,

$$c(v_i v_{i+1}) = \begin{cases} (k-4) \cdot \frac{i-r}{k-3} + q + r - 1, & \text{if } r \in \{3, 4, \ldots, k-4\}, \\ (k-4) \cdot \frac{i-2}{k-3} + q + 1, & \text{if } r = 2, \\ (k-4) \cdot \frac{i}{k-3} + q, & \text{if } r = 0, \end{cases}$$

and

$$c(v_i v_{i+1}) = \begin{cases} (k-4) \cdot \frac{i+1}{k-3} + q + 1, & \text{if } i \neq q \text{ and } r = 1, \\ (k-4) \cdot \frac{i+1}{k-3} + q, & \text{if } i = q \text{ and } r = 1, \end{cases}$$

where all arithmetic on the index $i+1$ here and henceforth is done modulo $q$. It can be easily checked that $c$ is a surjection and $W_q$ has no rainbow $C_k$ (but contains a rainbow copy of $C_{k-1}$) under the coloring $c$. Hence, $ar_p(W_q, C_k) \geq \lfloor (2k-7)q/(k-3) \rfloor$.

Next we prove that $ar_p(W_q, C_k) \leq (2k-5)q/(k-2)$. Let $c : E(W_q) \to [m]$ be any surjection such that $W_q$ contains no rainbow $C_k$ under the coloring $c$. It suffices to show that $m \leq (2k-5)q/(k-2)$. For any integer $\ell$, let $A_\ell$ be the set of colors used $\ell$ times under the coloring $c$. For integers $\alpha \in [m]$ and $j \geq 1$, let: $\eta_j(\alpha)$ be the number of central $k$-cycles in $W_q$ containing $j$ edges colored $\alpha$ under $c$, $\eta(\alpha) := \sum_{j=2}^k \eta_j(\alpha)$, $\beta(\alpha) := |\{i \in [q] : c(vv_i) = \alpha\}|$ and $\beta'(\alpha) := |\{i \in [q] : c(v_i v_{i+1}) = \alpha\}|$. For any integer $\ell$, it is easy to check that $\beta(\alpha) + \beta'(\alpha) = \ell$ for any $\alpha \in A_\ell$. Notice that for any integer $i \in [q]$, $vv_i$ belongs to exactly two central $k$-cycles and $v_i v_{i+1}$ belongs to exactly $k-2$ central $k$-cycles in $W_q$. For any $\alpha \in A_\ell$, we see that

$$2\eta(\alpha) \leq 2\eta(\alpha) + \eta_1(\alpha) \leq \sum_{j \geq 1} j\eta_j(\alpha) = 2\beta(\alpha) + (k-2)\beta'(\alpha) \leq (k-2)\ell,$$

which implies that $\eta(\alpha) \leq (k-2)\ell/2$. Since each of the $q$ central $k$-cycles of $W_q$ contains a color $\alpha$ with $\eta(\alpha) \geq 1$, we have

$$q \leq \sum_{\ell \geq 2} \sum_{\alpha \in A_\ell} \eta(\alpha) \leq \sum_{\ell \geq 2} (k-2)\ell|A_\ell|/2,$$

12
which implies $2q/(k-2) \leq \sum_{\ell \geq 2} \ell |A_\ell|$. This, together with $2q = e(W_q) = \sum_{\ell \geq 1} \ell |A_\ell|$, implies that $|A_1| \leq (2k - 6)q/(k - 2)$. Then

$$m = |A_1| + \sum_{\ell \geq 2} |A_\ell| \leq |A_1| + \sum_{\ell \geq 2} |A_\ell|/2 = |A_1|/2 + \sum_{\ell \geq 1} \ell |A_\ell|/2 \leq (2k - 5)q/(k - 2),$$

as desired. \[\blacksquare\]

Corollary 3.5 below follows from the fact that $\left\lceil \frac{2k-7}{k-3} q \right\rceil = 2q - \left\lfloor \frac{q}{k-3} \right\rfloor$, $\left\lfloor \frac{2k-5}{k-2} q \right\rfloor = 2q - \left\lfloor \frac{q}{k-2} \right\rfloor$ and $ar_p(W_q, C_k) = 2q - \left\lfloor \frac{q}{k-2} \right\rfloor$ if $\left\lfloor \frac{q}{k-2} \right\rfloor = \left\lfloor \frac{q}{k-3} \right\rfloor$. One can see that $\left\lfloor \frac{q}{k-3} \right\rfloor = \left\lfloor \frac{q}{k-2} \right\rfloor$ when $q \in \{t(k - 2), \ldots, t(k - 2) + k - 4 - t\}$ for any integer $t \in [k - 4]$.

Corollary 3.5 Let $k \geq 5$ and $q \geq k - 1$ be integers. If $q \in \{t(k - 2), \ldots, t(k - 2) + k - 4 - t\}$ for some integer $t \in [k - 4]$, then $ar_p(W_q, C_k) = 2q - \left\lfloor \frac{q}{k-3} \right\rfloor$.

We are ready to determine the exact value for $ar_p(W_q, C_6)$ when $q \geq 5$.

Theorem 3.6 For integer $q \geq 5$, $ar_p(W_q, C_6) = \lfloor 5q/3 \rfloor$.

Proof. By Theorem 3.3, $ar_p(W_q, C_6) \geq \lfloor 5q/3 \rfloor$. To prove that $ar_p(W_q, C_6) \leq \lfloor 5q/3 \rfloor$, it suffices to show that for any surjection $c : E(W_q) \to [m]$ such that $W_q$ contains no rainbow $C_6$ under the coloring $c$, we must have $m \leq \lfloor 5q/3 \rfloor$. We do that next.

Let $A_\ell$ be the set of colors used $\ell$ times under the coloring $c$. For $\alpha \in [m]$, let $\eta_j(\alpha)$ be the number of central 6-cycles in $W_q$ containing $j$ edges colored $\alpha$ under $c$, $\eta(\alpha) := \sum_{j=2}^6 \eta_j(\alpha)$, $\beta(\alpha) := |\{i \in [q] : c(v_iv_{i+1}) = \alpha\}|$ and $\beta'(\alpha) := |\{i \in [q] : c(v_{i+1}v_{i+2}) = \alpha\}|$. Then $\beta(\alpha) + \beta'(\alpha) = \ell$ for all $\alpha \in A_\ell$. Notice that for any integer $i \in [q]$, $v_iv_i$ belongs to exactly two central 6-cycles and $v_iv_{i+1}$ belongs to exactly four central 6-cycles. For any $\alpha \in A_\ell$, we see that

$$2\eta(\alpha) \leq 2\eta(\alpha) + \eta_1(\alpha) \leq \sum_{j \geq 1} j \eta_j(\alpha) = 2\beta(\alpha) + 4\beta'(\alpha) \leq 4\ell.$$

This implies that $\eta(\alpha) \leq 2\ell$. Notice that for any $\alpha \in A_2$, two edges of $W_q$ colored by $\alpha$ can prevent at most three central 6-cycles from being rainbow under the coloring $c$, and so $\eta(\alpha) = \eta_2(\alpha) \leq 3$. Since each of the $q$ central 6-cycles of $W_q$ contains a color, say $\alpha \in [m]$, with $\eta(\alpha) \geq 1$, it follows that

$$q \leq \sum_{\ell \geq 2} \sum_{\alpha \in A_\ell} \eta(\alpha) \leq 3|A_2| + \sum_{\ell \geq 3} 2\ell |A_\ell|.$$
Thus $q/2 \leq 3|A_2|/2 + \sum_{\ell \geq 3} \ell |A_\ell|$. This, together with $2q = e(W_q) = \sum_{\ell \geq 1} \ell |A_\ell|$, implies that $2|A_1| + |A_2| \leq 3q$. Then

$$m = |A_1| + |A_2| + \sum_{\ell \geq 3} |A_\ell| \leq |A_1| + |A_2| + \sum_{\ell \geq 3} \ell |A_\ell|/3$$

$$= (2|A_1| + |A_2|)/3 + \sum_{\ell \geq 1} \ell |A_\ell|/3 = (2|A_1| + |A_2|)/3 + 2q/3 \leq 5q/3,$$

as desired.

Finally, we obtain new upper bounds for $ar_p(n, C_6)$ when $n \geq 8$ and $ar_p(n, C_7)$ when $n \geq 13$, respectively.

**Theorem 3.7** $ar_p(n, C_6) \leq 17(n - 2)/6$ for all $n \geq 8$, and $ar_p(n, C_7) \leq (59n - 113)/20$ for all $n \geq 13$.

**Proof.** We first prove that $ar_p(n, C_6) \leq 17(n - 2)/6$ for all integers $n \geq 8$. Let $n \geq 8$ be given and let $T$ be any plane triangulation on $n$ vertices such that $T$ contains $C_6$ as a subgraph. Let $c : E(T) \to [m]$ be any surjection such that $T$ contains no rainbow $C_6$ under the coloring $c$. It suffices to show that $m \leq 17(n - 2)/6$. Since $e(T) = 3n - 6$ and $n \geq 8$, $T$ must have at least two vertices each with degree at least five. Thus, $n_4(T) \leq n - 2 - n_3(T)$ and $n_3(T) \geq 0$. For any $v \in V(T)$, we see that $|c(v)| \leq e(W_{d_T(v)}; C_6) = 2d_T(v)$. But for any $v \in V(T)$ with $d_T(v) \geq 5$, by Theorem 3.6, $|c(v)| \leq ar_p(W_{d_T(v)}; C_6) = 5d_T(v)/3$. By Lemma 3.3

$$4m \leq \sum_{v \in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + \sum_{v \in V(T), d_T(v) \geq 5} 5d_T(v)/3$$

$$\leq n_3(T) + 4n_4(T)/3 + 5/3 \cdot \sum_{v \in V(T)} d_T(v)$$

$$\leq 4(n - 2)/3 - n_3(T)/3 + 5/3 \cdot 2(3n - 6) \leq 34(n - 2)/3,$$

which implies that $m \leq 17(n - 2)/6$, as desired.

It remains to prove that $ar_p(n, C_7) \leq (59n - 113)/20$ for all $n \geq 13$. The proof is similar to the proof of $ar_p(n, C_6) \leq 17(n - 2)/6$. We include a proof here for completeness. Let $n \geq 13$ be given and let $T$ be any plane triangulation on $n$ vertices such that $T$ contains $C_7$ as a subgraph. Let $c : E(T) \to [m]$ be any surjection such that $T$ contains no rainbow $C_7$ under the coloring $c$. It suffices to show that $m \leq (59n - 113)/20$. Since $e(T) = 3n - 6$ and $n \geq 13$, $T$ must have at least one vertex of degree six. Thus, $n_5(T) \leq n - 1 - n_3(T) - n_4(T)$ and $n_i(T) \geq 0$ $(i = 3, 4)$. For any $v \in V(T)$, we see that $|c(v)| \leq e(W_{d_T(v)}; C_7) = 2d_T(v)$. But
for any $v \in V(T)$ with $d_T(v) \geq 6$, by Theorem 3.4, $|c(v)| \leq ar_p(W_{d_T(v)}, C_7) \leq \lfloor 9d_T(v)/5 \rfloor$.

By Lemma 3.3,

$$4m \leq \sum_{v \in V(T)} |c(v)| \leq 6n_3(T) + 8n_4(T) + 10n_5(T) + \sum_{v \in V(T), d_T(v) \geq 6} \lfloor 9d_T(v)/5 \rfloor \leq 3n_3(T)/5 + 4n_4(T)/5 + n_5(T) + 9/5 \cdot \sum_{v \in V(T)} d_T(v) \leq n - 1 - 2n_3(T)/5 - n_4(T)/5 + 9/5 \cdot 2(3n - 6) \leq 59(n - 2)/5 + 1,$$

which implies that $m \leq (59n - 113)/20$, as desired.

This completes the proof of Theorem 3.7.

**Remark.** A better upper bound for $ar_p(n, C_6)$ can be obtained using a result in [12] that $ex_p(n, C_6) \leq 18(n - 2)/7$ when $n \geq 6$. By Proposition 1.1 and Theorem 3.2, we see that $65(n - 2)/28 - 37r/28 \leq ar_p(n, C_6) \leq ex_p(n, C_6) \leq 72(n - 2)/28$ for all $n \geq 30$, where $r = (n - 2) \mod 28$.

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**References**

[1] N. Alon, On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems, J Graph Theory 7 (1983) 91–94.

[2] M. Axenovich, T. Jiang and A. Kündgen, Bipartite anti-Ramsey numbers of cycles, J Graph Theory 47 (2004) 9–28.

[3] C. Dowden, Extremal $C_4$-free/$C_5$-free planar graphs, J Graph Theory 83 (2016) 213–230.

[4] P. Erdős, M. Simonovits and V.T. Sós, Anti-Ramsey theorems, Colloq Math Soc Janos Bolyai 10 (1975) 633–643.

[5] Z. Dvořák, D. Král’ and R. Škrekovski, Non-rainbow coloring 3-, 4- and 5-connected plane graphs, J Graph Theory 63 (2010) 129–145.

[6] M. Horňák, S. Jendrol’, I. Schiermeyer and R. Soták, Rainbow numbers for cycles in plane triangulations, J Graph Theory 63 (2010) 129–145.

[7] S. Jendrol’, J. Miškuf, R. Soták and E. Škrabul’áková, Rainbow faces in edge-colored plane graphs, J Graph Theory 62 (2009) 84–99.

[8] S. Jendrol’, I. Schiermeyer and J. Tu, Rainbow numbers for matchings in plane triangulations, Discrete Math. 331 (2014) 158–164.
[9] T. Jiang, Anti-Ramsey numbers of subdivided graphs, J Combin. Theory Ser. B 85 (2002) 361–366.

[10] T. Jiang and O. Pikhurko, Anti-Ramsey numbers of doubly edge-critical graphs, J Graph Theory 61 (2009) 210–218.

[11] Z. Jin and X. Li, Anti-Ramsey numbers for graphs with independent cycles, Electron. J Combin. 16 (2009) #R85.

[12] Y. Lan, S. O, Y. Shi and Z-X. Song, Planar Turán numbers for paths and cycles, to be submitted.

[13] J. J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem, Combinatorica 22 (2002) 445–449.

[14] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, Discrete Math. 286 (2004) 157–162.

[15] R. Ramamurthi and D. B. West, Maximum face-constrained coloring of plane graphs, Discrete Math. 274 (2004) 233–240.

[16] A. A. Zykov, Hypergraphs, Uspekhi Mat Nauk 29 (1974) 89–154.