Packing of spanning mixed arborescences

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Funding information
National Natural Science Foundation of China, Grant/Award Numbers: 11871439, U20A2068

MATHEMATICAL SUBJECT CLASSIFICATION
05B35, 05C40, 05C70

1 | INTRODUCTION

In this paper, we consider graphs which may have multiple edges or (and) arcs but not loops. A mixed graph $F = (V; E, A)$ is a graph consisting of the set $E$ of undirected edges and the set $A$ of directed arcs. For $X \subseteq V$, let $d^X_+(X) := \{|y^X \in A: y \notin X \text{ and } x \in X\}$. Let $X_1, \ldots, X_t$ be disjoint subsets of $V$, we call $\mathcal{P} = \{X_1, \ldots, X_t\}$ a subpartition of $V$ and particularly a partition of $V$ if $V = \bigcup_{j=1}^t X_j$. Denote $e_E(\mathcal{P}) = \{|e \in E: \text{one end of } e \text{ belongs to some } X_i \text{ and the other end belongs to another } X_j \text{ with } j \neq i \text{ or } V \setminus \bigcup_{j=1}^t X_j\}$. Denote the set $\{1, \ldots, k\}$ by $[k]$. For a function $f: V \to \mathbb{N}$, define a set function $\tilde{f}: 2^V \to \mathbb{N}$ as $\tilde{f}(X) = \sum_{x \in X} f(x)$, where $X \subseteq V$.

Nash-Williams [13] and Tutte [14] independently characterized when an undirected graph has $k$ edge-disjoint spanning trees.

**Theorem 1.1** (Nash-Williams [13] and Tutte [14]). For a graph $G = (V, E)$, there exist $k$ edge-disjoint spanning trees, if and only if for any partition $\mathcal{P} = \{X_0, X_1, \ldots, X_t\}$ of $V$,

$$e_E(\mathcal{P}) \geq kt.$$
Let \( D = (V, A) \) be a digraph. A subdigraph of \( D \) is spanning if its vertex set is \( V \). A subdigraph \( F \) (it may not be spanning) of \( D \) is an \( r \)-\textit{arborescence} if its underlying graph is a tree and for any \( u \in V(F) \), there is exactly one directed path in \( F \) from \( r \) to \( u \). We say that the vertex \( r \) is the root of arborescence \( F \).

As a directed version of Theorem 1.1, Edmonds’ theorem [4] characterizes directed graphs that contain \( k \) arc-disjoint spanning arborescences with prescribed roots in terms of a cut condition.

**Theorem 1.2** (Edmonds [4]). For a digraph \( D = (V, A) \), let \( R = \{r_1, ..., r_k\} \subseteq V \) be a multiset. For \( i = 1, ..., k \), there exist arc-disjoint spanning \( r_i \)-arborescences in \( D \), if and only if for any \( \emptyset \neq X \subseteq V \),

\[
d^+_r(X) \geq |\{\eta; \eta \notin X\}|
\]

Throughout this paper, \( F = (V, E, A) \) is a mixed graph, \( R = \{r_1, ..., r_k\} \subseteq V \) is a multiset. By regarding each undirected edge as a directed arc in both directions, each concept in directed graphs can be naturally extended to mixed graphs. Especially, a subdigraph \( P \) of \( F \) is a \textit{mixed path} if its underlying graph is a path and one end of \( P \) can be reached from the other. A subdigraph \( T \) (it may not be spanning) of \( F \) is called an \( r \)-\textit{mixed arborescence} if its underlying graph is a tree and for any \( u \in V(T) \), there is exactly one mixed path in \( T \) from \( r \) to \( u \). Equivalently, a subgraph \( T \) of \( F \) is an \( r \)-mixed arborescence if there exists an orientation of the undirected edges of \( T \) such that the obtained subgraph (whose arc set is the union of original arc set and oriented arc set of \( T \)) is a \( r \)-arborescence.

The following result is due to Frank [6], it generalized Theorems 1.1 and 1.2 to mixed graphs when \( r_1 = r_2 = \cdots = r_k \).

**Theorem 1.3** (Frank [6]). Let \( F = (V, E, A) \) be a mixed graph, \( r \in V \), and \( k \) a positive integer. There exist \( k \) edge and arc-disjoint spanning \( r \)-mixed arborescences in \( F \), if and only if, for any subpartition \( \{X_1, ..., X_t\} \) of \( V - r \),

\[
e_E(P) + \sum_{j=1}^t d^+_r(X_j) \geq kt.
\]

Let \( U_i \) be the set of vertices reachable from \( r_i \) in \( F \). For \( u, v \in V \), we say \( u \sim v \) if \( \{i: u \in U_i\} = \{i: v \in U_i\} \); \( \sim \) is an equivalence relation. Denote equivalence classes for \( \sim \) by \( \Gamma_1, ..., \Gamma_t \), and we call each \( \Gamma_j \) an atom.

An \( r_i \)-mixed arborescence \( T_i \) is said to be \textit{maximal} if \( V(T_i) = U_i \) (i.e., it spans all the vertices that are reachable from \( r_i \) in \( F \)). A \textit{packing} of maximal mixed arborescences w.r.t. \( R = \{r_1, ..., r_k\} \) is a collection \( \{T_1, ..., T_k\} \) of mutually edge and arc-disjoint mixed arborescences such that \( T_i \) has root \( r_i \) and \( V(T_i) = U_i \).

The following remarkable extension of Edmonds’ theorem (by Kamiyama, Katoh, and Takizawa [10]) enables us to find a packing of maximal arborescences \( \{T_1, ..., T_k\} \) w.r.t. \( R \) in a digraph (i.e., \( E = \emptyset \)).

For nonempty \( X, Z \subseteq V \), let \( Z \to X \) denote that \( X \) and \( Z \) are disjoint and \( X \) is \textit{reachable} from \( Z \), that is, there is a mixed path from \( Z \) to \( X \). We shall write \( v \) for \( \{v\} \) for simplicity. Let \( P(X) := X \cup \{v \in V \setminus X: v \to X\} \).
Theorem 1.4 (Kamiyama et al. [10]). Let \( D = (V, A) \) be a digraph, and \( R = \{r_1, \ldots, r_k\} \subseteq V \) be a multiset. Then there exists a packing of maximal arborescences w.r.t. \( R \) in \( D \) if and only if for any \( \emptyset \neq X \subseteq V \),

\[
d^*_A(X) \geq |\{r_i \in P(X) \setminus X\}|.
\]

Recently, Matsuoka and Tanigawa [12] extended Theorem 1.4 to mixed graphs. A bi-set \( X = \{X_0, X_1\} \) is a pair of sets satisfying \( X_i \subseteq X_0 \subseteq V \). Denote \( d^*_A(X) := |\{uv \in A : u \in V \setminus X_0, v \in X_i\}| \).

Theorem 1.5 (Matsuoka and Tanigawa [12]). Let \( F = (V; E, A) \) be a mixed graph, and \( R = \{r_1, \ldots, r_k\} \subseteq V \) be a multiset. There exists a packing of maximal mixed arborescences w.r.t. \( R \) in \( F \) if and only if

\[
e_E(P) + \sum_{q=1}^{t} d^*_A(X^q) \geq \sum_{q=1}^{t} \left| \left\{ r_i : X^q_i \subseteq U_i \setminus \{r_i\}, \left( X^q_0 \setminus X^q_i \right) \cap U_i = \emptyset \right\} \right|
\]

holds for every family of bi-sets \( \{X^1, \ldots, X^t\} \) such that \( P = \{X^1_1, \ldots, X^t_1\} \) is a subpartition of some atom \( \Gamma_j \) and that \( (X^q_0 \setminus X^q_j) \cap \Gamma_j = \emptyset \) for \( q = 1, \ldots, t \).

For packing of maximal arborescences, further extensions have been made, such as its matroidal version [11], and matroidal mixed version [9, 12]. Some other developments for packing of arborescences in the recent years include matroid-based packing [3], and its hypergraphic version [5], under cardinality constraints [8]. Readers can refer to [5] for more details.

In this paper, we are interested in the following extension of Edmonds’ theorem, which is due to Cai [2] and Frank [6] (see also Theorem 10.1.11 in Frank [7]). This extension characterized a digraph \( D \) which contains \( k \) arc-disjoint spanning arborescences \( F_1, \ldots, F_k \), such that for each \( v \in V(D) \), the cardinality of \( \{i \in [k] : v \text{ is the root of } F_i\} \) lies in some prescribed interval.

Theorem 1.6 (Cai [2] and Frank [6]). Let \( D = (V, A) \) be a digraph, \( f, g : V \to \mathbb{N} \) be functions such that \( f \leq g \). Then there exist \( k \) arc-disjoint spanning arborescences \( F_1, \ldots, F_k \) in \( D \) for which \( F_i \) is rooted at some \( n \in V \) for \( 1 \leq i \leq k \) such that \( f(v) \leq |\{i \in [k] : n = v\}| \leq g(v) \) for \( v \in V \), if and only if,

(i) \( \widetilde{f}(V) \leq k \);
(ii) for any subpartition \( \{X_1, \ldots, X_t\} \) of \( V \),

\[
\sum_{j=1}^{t} d^*_A(X_j) \geq k(t - 1) + \widetilde{f}(V \setminus \bigcup_{j=1}^{t} X_j);
\]

(iii) for any \( \emptyset \neq X \subseteq V \),

\[
d^*_A(X) \geq k - \bar{g}(X).
\]

In this paper, we generalize Theorems 1.1 and 1.6 to mixed graphs as follows.
Theorem 1.7. Let $F = (V; E, A)$ be a mixed graph, $f, g: V \to \mathbb{N}$ be functions such that $f \leq g$. Then there exist $k$ edge and arc-disjoint spanning mixed arborescences $F_1, ..., F_k$ in $F$ for which $F_i$ is rooted at some $r_i \in V$ for $1 \leq i \leq k$ and $f(v) \leq |\{i \in [k]: r_i = v\}| \leq g(v)$ for $v \in V$, if and only if,

(i) $\tilde{f}(V) \leq k$;

(ii) for any subpartition $\mathcal{P} = \{X_1, ..., X_t\}$ of $V$,

$$e_E(\mathcal{P}) + \sum_{j=1}^{t} d_A(X_j) \geq k(t - 1) + \tilde{f}(V \setminus \bigcup_{j=1}^{t} X_j);$$

(iii) for any subpartition $\mathcal{P} = \{X_1, ..., X_t\}$ of $V$,

$$e_E(\mathcal{P}) + \sum_{j=1}^{t} d_A(X_j) \geq kt - \tilde{g}(\bigcup_{j=1}^{t} X_j).$$

For the proof of our result, besides what have been used by Nash-Williams [13], Tutte [14], Cai [2], and Frank [6], we adopt a technique named as properly intersecting elimination operation, which was first introduced by Bérczi and Frank [1] (to the best of our knowledge), studied and used again by the present authors [8]. Indeed, we shall use some similar approaches to [8] in our proofs of Theorem 1.7.

2 | PROOF OF THEOREM 1.7

We shall use some definitions and propositions that have been presented in [8]. For reader’s convenience, we have made this section self-contained.

Let $\Omega$ be a finite set. Two subsets $X, Y \subseteq \Omega$ are intersecting if $X \cap Y \neq \emptyset$ and properly intersecting if $X \cap Y, X \setminus Y,$ and $Y \setminus X \neq \emptyset$. A function $p: 2^{\Omega} \to \mathbb{Z}$ is supermodular (intersecting supermodular), where $2^{\Omega}$ denotes the power set of $\Omega$, if the inequality

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y)$$

holds for all subsets (intersecting subsets, respectively) of $\Omega$. A function $b$ is submodular if $-b$ is supermodular.

Let $\mathcal{F}$ be a multiset, which consists of some subsets of $\Omega$ (these subsets do not have to be different). Let $\bigcup \mathcal{F}$ be the union of elements in $\mathcal{F}$ (then $\bigcup \mathcal{F} \subseteq \Omega$). Let $x \in \Omega$ and $\mathcal{F}(x)$ denote the number of elements in $\mathcal{F}$ containing $x$. If there exist no properly intersecting pairs in $\mathcal{F}$, then $\mathcal{F}$ is laminar.

Suppose $\mathcal{F}$ is not laminar, then there exists a properly intersecting pair $X$ and $Y$ in $\mathcal{F}$. Define multiset $\mathcal{F}' := \mathcal{F} - \{X, Y\} + \{X \cup Y, X \cap Y\}$ (i.e., we obtain $\mathcal{F}'$ from $\mathcal{F}$ by replacing $X$ and $Y$ with $X \cup Y$ and $X \cap Y$). To correspond what have already been used in [8], we say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ through a properly intersecting elimination operation of type 1, for simplicity, we name this operation as PIEO$^1$ (on a properly intersecting pair $X$ and $Y$ in $\mathcal{F}$).
Let $Z_1$ and $Z_2$ be multisets. Denote by $Z_1 \uplus Z_2$ the \textit{multiset union} of $Z_1$ and $Z_2$, that is, for any $z$, the number of $z$ in $Z_1 \uplus Z_2$ is the total number of $z$ in $Z_1$ and $Z_2$.

Let $D(\Omega)$ be the set that consists of all families of disjoint subsets of $\Omega$. From now on, we suppose $F_1, F_2 \in D(\Omega)$. We adopt PIEO$^1$s in $G_0 = F_1 \uplus F_2$, step by step, and obtain families $G_0, \ldots, G_{i-1}, G_i, \ldots$ of subsets of $\Omega$.

Let $G'_i$ be the family of maximal elements in $G_i$ with respect to inclusion. Note that it could happen that more than one element in $G_i$, say $X$ and $Y$ in $G_i$ with $X = Y$, are the same subset of $\Omega$ and maximal in $G_i$; in such cases, we add exactly one of the same maximal elements to $G'_i$. In such a way, all elements of $G'_i$ are distinct from each other; this is used in Proposition 2.3 to show that $F_3 \in D(\Omega)$ (where $F_3 := G'_i$).

**Proposition 2.1.** For $x \in \Omega$ and $i \in \mathbb{N}$, $G_i(x) = G_{i+1}(x)$.

\textbf{Proof.} Suppose we obtain $G_{i+1}$ from $G_i$ by replacing $X$ and $Y$ with $X \cup Y$ and $X \cap Y$. Then, for $x \in \Omega$, $\{X, Y\}(x) = \{X \cup Y, X \cap Y\}(x)$. Since $G_i - \{X, Y\} = G_{i+1} - \{X \cup Y, X \cap Y\}$, we have

$$G_i(x) = (G_i - \{X, Y\})(x) + \{X, Y\}(x)$$

$$= (G_{i+1} - \{X \cup Y, X \cap Y\})(x) + \{X \cup Y, X \cap Y\}(x)$$

$$= G_{i+1}(x).$$

\hfill \Box

**Proposition 2.2.** If $X, Y \in G_i$ are properly intersecting, then $X, Y \in G'_i$.

\textbf{Proof.} Suppose there exists $Z \in G_i$ such that $X \subseteq Z$. Then, for $x \in X \cap Y$, $G_i(x) \geq \{X, Y, Z\}(x) \geq 3$. However, since $F_1, F_2 \in D(\Omega)$, $F_1(x), F_2(x) \leq 1$. By Proposition 2.1, $G_i(x) = G_0(x) = F_1(x) + F_2(x) \leq 2$. This is a contradiction. So $X$ is maximal in $G_i$, and the same for $Y$.

Note once we adopt PIEO$^1$ on a properly intersecting pair in $G_i$, using Proposition 2.2, the number of maximal elements in $G_{i+1}$ is less than that in $G_i$. Thus the process of PIEO$^1$s will terminate. Suppose the obtained families of subsets of $\Omega$ are $G_0, \ldots, G_n$. Then $G_n$ is laminar. Define $F_3 := G_n$ and $F_4 := G_n \setminus F_3$. Note that, since $G_n$ is laminar, we have $F_3 \in D(\Omega)$ and $\cup F_3 = \cup G_n$.

**Proposition 2.3.** The following hold true:

(i) $F_3, F_4 \in D(\Omega)$.

(ii) $\cup F_4 = (\cup F_1) \cap (\cup F_2)$.

\textbf{Proof.} Let $u \in \cup F_4$. Since $\cup F_4 \subseteq \cup G_n = \cup F_3$, we know that $F_3(u), F_4(u) \geq 1$. By Proposition 2.1 and $F_1, F_2 \in D(\Omega)$, we have

$$2 \leq F_3(u) + F_4(u) = G_n(u) = G_0(u) = F_1(u) + F_2(u) \leq 2.$$ 

Therefore, $F_4(u) = 1$ and $F_1(u) = F_2(u) = 1$. This proves that $F_4 \in D(\Omega)$; and $u \in (\cup F_1) \cap (\cup F_2)$, hence $\cup F_4 \subseteq (\cup F_1) \cap (\cup F_2)$.
Suppose $v \in (\cup F_1) \cap (\cup F_2)$, that is, $G_0(v) = 2$. By Proposition 2.1, $G_n(v) = G_0(v) = 2$. Since $F_3(v) = 1$ and $F_4 = G_n \setminus F_3$, we have $v \in \cup F_4$. This shows $(\cup F_1) \cap (\cup F_2) \subseteq \cup F_4$, therefore $\cup F_4 = (\cup F_1) \cap (\cup F_2)$.

Now we are ready for the proof of Theorem 1.7.

($\Rightarrow$) Necessity: Suppose there exist $k$ edge and arc-disjoint spanning mixed arborescences $F_1$, ..., $F_k$ in $F$ for which $F_i$ is rooted at some $\eta_i \in V$ for $1 \leq i \leq k$ and $f(v) \leq |\{i \in [k]: \eta_i = v\}| \leq g(v)$ for $v \in V$. Then, for $1 \leq i \leq k$, orient edges in $E(F_i)$ making $F_i$ a spanning arborescence $F'_i$ rooted at $\eta_i$; orient edges in $E \setminus \cup_{i=1}^k E(F_i)$ arbitrarily. We obtain an oriented arc set $A'$ of $E$, such that $F'_1$, ..., $F'_k$ are $k$ arc-disjoint spanning arborescences in $D = (V, A \cup A')$, $F'_i$ is rooted at $\eta_i$ for $1 \leq i \leq k$, and $f(v) \leq |\{i \in [k]: \eta_i = v\}| \leq g(v)$ for $v \in V$.

Obviously, $\tilde{f}(V) \leq k$. By Theorem 1.6, (1) and (2) hold in $D$. Let $P = \{X_0, X_1, ..., X_t\}$ be a partition of $V$. Since $A'$ is an oriented arc set of $E$, $e_E(P) \geq \sum_{j=1}^t d_{A'}(X_j)$. Hence, by (1),

$$e_E(P) + \sum_{j=1}^t d_{A'}(X_j) \geq \sum_{j=1}^t d_{A'}(X_j) + \sum_{j=1}^t d_{A}(X_j) = \sum_{j=1}^t d_{A \cup A'}(X_j) \geq k(t - 1) + \tilde{f}(X_0),$$

this is (3). By (2),

$$e_E(P) + \sum_{j=1}^t d_{A}(X_j) \geq \sum_{j=1}^t d_{A \cup A'}(X_j) \geq \sum_{j=1}^t (k - \tilde{g}(X_j)) = kt - \tilde{g}(\cup_{j=1}^t X_j),$$

this is (4).

($\Leftarrow$) Sufficiency: We prove the sufficiency by induction on $|E|$. For the base step, suppose $E = \emptyset$; and then apply Theorem 1.6.

For the induction step, suppose $E \neq \emptyset$. We shall prove that we can orient an edge $e \in E$ to $\vec{e}$, such that after we do $A := A + \vec{e}$, $E := E - e$, $F' := (V; A, E)$, assumptions (3) and (4) still hold for the new mixed graph $F'$. Then by the induction hypothesis, there exist $k$ edge and arc-disjoint mixed arborescences $F_1$, ..., $F_k$ in $F'$ for which $F_i$ is rooted at some $\eta_i \in V$ for $1 \leq i \leq k$ such that $f(v) \leq |\{i \in [k]: \eta_i = v\}| \leq g(v)$ for $v \in V$. If $\vec{e} \not\in \cup_{i=1}^k F_i$, then $F$ includes $F_1$, ..., $F_k$ as demanded; if $\vec{e} \in E(F_i)$ for some $i_0 \in [k]$, then $F$ includes $F_1$, ..., $F_{i_0} - \vec{e}$, $\vec{e}$, ..., $F_k$ as demanded.

The critical point that determines an orientation of $e$ lies on the subpartitions of $V$ that make assumptions (3) or (4) tight in $F$. These critical subpartitions are defined next; $E^1$ is aimed at (3), $E^2$ is aimed at (4). This explains why the subpartitions in $E^1$ and $E^2$ will play some central roles next. Define

$$E^1 := \left\{ F \in D(V): e_E(F) + \sum_{X \in F} d_{A}(X) = k(t - 1) + \tilde{f}(V \setminus F) \right\},$$

$$E^2 := \left\{ F \in D(V): e_E(F) + \sum_{X \in F} d_{A}(X) = kt - \tilde{g}(\cup F) \right\}.$$

Suppose $F_1, F_2 \in E^1 \cup E^2$, denote

$$E(F_1, F_2) := \{ e \in E: \text{ one end of } e \text{ is in } \cup F_1 \setminus \cup F_2 \text{ and the other is in } \cup F_2 \setminus \cup F_1 \}.$$
Process of PIEO's. Let $G_0 = F_1 \cup F_2$. We adopt PIEO's in $G_0 = F_1 \cup F_2$, step by step, and obtain families $G_0, \ldots, G_{i-1}, G_i, \ldots, G_n$ of subsets of $V$. Recall that $G'_1$ is the family of maximal elements in $G_i$, $F_3 := G'_n$ and $F_4 := G_n \setminus F_3$.

Claim 2.4.

(i) $|F_1| + |F_2| = |F_3| + |F_4|$.  
(ii) $\cup F_3 = (\cup F_1) \cup (\cup F_2)$.

Proof. For $i \in [n]$, suppose we replace a properly intersecting pair $X$ and $Y$ in $G_{i-1}$ with $X \cup Y$ and $X \cap Y$, and obtain $G_i$. Clearly, $|G_{i-1}| = |G_i|$. It follows that $|F_1| + |F_2| = |G_0| = |G_n| = |F_3| + |F_4|$. By Proposition 2.2, $X, Y \in G_{i-1}$, and thus $X \cup Y \in G'_i$. So $G'_i$ consists of $X \cup Y$ and all the subsets in $G_{i-1}$ not contained in $X \cup Y$; this proves $\cup G_{i-1} = \cup G'_i$. Hence $G_0 = G'_n$. It follows that $(\cup F_1) \cup (\cup F_2) = G'_0 = G'_n = \cup F_3$. □

Claim 2.5. For $F_1, F_2 \in E^1 \cup E^2$, we have

$$e_E(F_1) + \sum_{X \in F_1} d_A(X) + e_E(F_2) + \sum_{X \in F_2} d_A(X) \geq e_E(F_3) + \sum_{X \in F_3} d_A(X) + e_E(F_4) + \sum_{X \in F_4} d_A(X) + |E(F_1, F_2)|.$$  

Proof. Define an orientation $A''$ of $E$ as follows:

- if $e = uv \in E$ is such that $u \notin \cup F_1$ and $v \in \cup F_1$, orient $e$ from $u$ to $v$ in $A''$;
- else if $e = uv \in E$ is such that $u \notin \cup F_2$ and $v \in \cup F_2$, orient $e$ from $u$ to $v$ in $A''$;
- else, orient the rest of $E$ arbitrarily in $A''$.

Then $e_E(F_1) = \sum_{X \in F_1} d_A''(X)$, and $e_E(F_2) = \sum_{X \in F_2} d_A''(X) + |E(F_1, F_2)|$. Hence,

$$e_E(F_1) + \sum_{X \in F_1} d_A(X) + e_E(F_2) + \sum_{X \in F_2} d_A(X) = \sum_{X \in F_1} d_A''(X) + \sum_{X \in F_1} d_A(X) + \sum_{X \in F_2} d_A''(X) + \sum_{X \in F_2} d_A(X) + |E(F_1, F_2)| = \sum_{X \in F_0} d_A''(X) + |E(F_1, F_2)| \quad (5)$$

By Claim 2.4(ii), $\cup F_3 = (\cup F_1) \cup (\cup F_2)$. For every $e = uv \in E$ such that $u \notin (\cup F_1) \cup (\cup F_2)$ and $v \in (\cup F_1) \cup (\cup F_2)$, by the definition of $A''$, $e$ is oriented from $u$ to $v$. Therefore $e_E(F_3) = \sum_{X \in F_1} d_A''(X)$.

By Proposition 2.3, $\cup F_4 = (\cup F_1) \cup (\cup F_2)$. For every $e = uv \in E$ such that $u \notin (\cup F_1) \cup (\cup F_2)$ and $v \in (\cup F_1) \cup (\cup F_2)$, by the definition of $A''$, $e$ is oriented from $u$ to $v$. Therefore $e_E(F_4) = \sum_{X \in F_4} d_A''(X)$. Hence,
\[ e_E(\mathcal{F}_3) + \sum_{X \in \mathcal{F}_3} d_A(X) + e_E(\mathcal{F}_4) + \sum_{X \in \mathcal{F}_4} d_A(X) \]
\[ = \sum_{X \in \mathcal{F}_3} d_A^-(X) + \sum_{X \in \mathcal{F}_3} d_A^+(X) + \sum_{X \in \mathcal{F}_4} d_A^+(X) \]
\[ = \sum_{X \in \mathcal{F}_3} d_{AUA}^-(X) + \sum_{X \in \mathcal{F}_3} d_{AUA}^+(X) \]
\[ = \sum_{X \in \mathcal{G}_i} d_{AUA}^-(X) \quad \text{(since } \mathcal{G}_n = \mathcal{F}_3 \cup \mathcal{F}_4). \tag{6} \]

In the process of PIEO, for \( i \in [n] \), suppose we obtain \( \mathcal{G}_i \) by replacing a properly intersecting pair \( X \) and \( Y \) in \( \mathcal{G}_{i-1} \) with \( X \cup Y \) and \( X \cap Y \). Then \( \mathcal{G}_{i-1} \setminus \{X, Y\} = \mathcal{G}_i \setminus \{X \cup Y, X \cap Y\} \). Since \( d_{AUA}^- \) is submodular on \( 2^\mathcal{V} \), \( d_{AUA}^- (X) + d_{AUA}^- (Y) \geq d_{AUA}^- (X \cup Y) + d_{AUA}^- (X \cap Y) \). Therefore \( \sum_{X \in \mathcal{G}_{i-1}} d_{AUA}^- (X) \geq \sum_{X \in \mathcal{G}_i} d_{AUA}^- (X) \). It follows that
\[ \sum_{X \in \mathcal{G}_0} d_{AUA}^- (X) \geq \sum_{X \in \mathcal{G}_1} d_{AUA}^- (X) \geq \cdots \geq \sum_{X \in \mathcal{G}_n} d_{AUA}^- (X). \tag{7} \]

Hence, we have
\[ e_E(\mathcal{F}_3) + \sum_{X \in \mathcal{F}_3} d_A(X) + e_E(\mathcal{F}_4) + \sum_{X \in \mathcal{F}_4} d_A(X) \]
\[ = \sum_{X \in \mathcal{G}_0} d_{AUA}^-(X) + |E(\mathcal{F}_1, \mathcal{F}_2)| \quad \text{(by (5))} \]
\[ \geq \sum_{X \in \mathcal{G}_0} d_{AUA}^-(X) + |E(\mathcal{F}_1, \mathcal{F}_2)| \quad \text{(by (7))} \]
\[ = e_E(\mathcal{F}_3) + \sum_{X \in \mathcal{F}_3} d_A(X) + e_E(\mathcal{F}_4) + \sum_{X \in \mathcal{F}_4} d_A(X) + |E(\mathcal{F}_1, \mathcal{F}_2)| \quad \text{(by (6))}. \]

The following lemma will be used in the final step to explain why we can orient edges \( e \in E \) to take care of all these critical subpartitions in \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \).

**Lemma 2.6.** For \( \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}_1 \cup \mathcal{E}_2 \), we have \( E(\mathcal{F}_1, \mathcal{F}_2) = \emptyset \).

**Proof.** Suppose to the contrary that \( E(\mathcal{F}_1, \mathcal{F}_2) \neq \emptyset \). Then, by Claim 2.5, we have
\[ e_E(\mathcal{F}_3) + \sum_{X \in \mathcal{F}_3} d_A(X) + e_E(\mathcal{F}_4) + \sum_{X \in \mathcal{F}_4} d_A(X) \]
\[ > e_E(\mathcal{F}_3) + \sum_{X \in \mathcal{F}_3} d_A(X) + e_E(\mathcal{F}_4) + \sum_{X \in \mathcal{F}_4} d_A(X). \tag{8} \]

**Case 1:** Assume \( \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}_1 \).

By Claim 2.4(ii), \( V \setminus (\mathcal{F}_3 \cup (V \setminus \mathcal{F}_1)) \cap (V \setminus (\mathcal{F}_4 \cup (V \setminus \mathcal{F}_2))) \). By Proposition 2.3, \( V \setminus (\mathcal{F}_3 \cup (V \setminus \mathcal{F}_1)) \cup (V \setminus (\mathcal{F}_4 \cup (V \setminus \mathcal{F}_2))) \). Thus,
\[ \tilde{\mathcal{F}}(V \setminus (\mathcal{F}_3 \cup (V \setminus \mathcal{F}_1))) + \tilde{\mathcal{F}}(V \setminus (\mathcal{F}_4 \cup (V \setminus \mathcal{F}_2))) = \tilde{\mathcal{F}}(V \setminus (\mathcal{F}_1 \cup \mathcal{F}_3)) + \tilde{\mathcal{F}}(V \setminus (\mathcal{F}_2 \cup \mathcal{F}_4)). \tag{9} \]
Hence,

\[ e_E(\mathcal{F}_3) + \sum_{x \in \mathcal{F}_1} d(x) + e_E(\mathcal{F}_4) + \sum_{x \in \mathcal{F}_2} d(x) \geq k(|\mathcal{F}_3| - 1) + \tilde{f}(V \setminus \mathcal{F}_3) + k(|\mathcal{F}_4| - 1) + \tilde{f}(V \setminus \mathcal{F}_4) \] (by (3))

\[ = k(|\mathcal{F}_3| - 1) + \tilde{f}(V \setminus \mathcal{F}_3) + k(|\mathcal{F}_4| - 1) \]

\[ + \tilde{f}(V \setminus \mathcal{F}_4) \] (by (9) and Claim (2.4)(i))

\[ = e_E(\mathcal{F}_3) + \sum_{x \in \mathcal{F}_1} d(x) + e_E(\mathcal{F}_4) + \sum_{x \in \mathcal{F}_2} d(x) \quad \text{ (since } \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}^1), \]

but this is a contradiction to (8).

**Case 2**: Assume \( \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{E}^2 \).

The proof will use the function \( g \) and the assumption (4), also the definition of \( \mathcal{E}^2 \).

The process is similar to Case 1, details are skipped here.

**Case 3**: Assume \( \mathcal{F}_1 \in \mathcal{E}^1 \) and \( \mathcal{F}_2 \in \mathcal{E}^2 \).

By Claim 2.4(ii), \( V \setminus \mathcal{F}_3 \subseteq V \setminus \mathcal{F}_1 \), and \( (V \setminus \mathcal{F}_3) \setminus (V \setminus \mathcal{F}_1) = (V \setminus \mathcal{F}_3) \setminus \mathcal{F}_1 \). By Proposition 2.3, \( \mathcal{F}_4 \subseteq \mathcal{F}_2 \), and \( \mathcal{F}_4 \subseteq \mathcal{F}_2 \). By the assumption \( f \leq g \), using \( V \setminus \mathcal{F}_4 \subseteq V \setminus \mathcal{F}_1 \) and \( \mathcal{F}_4 \subseteq \mathcal{F}_2 \), we have

\[ \tilde{f}(V \setminus \mathcal{F}_3) - \tilde{f}(V \setminus \mathcal{F}_1) = \tilde{f}((V \setminus \mathcal{F}_4) \setminus (V \setminus \mathcal{F}_1)) \leq \tilde{g}((V \setminus \mathcal{F}_4) \setminus (V \setminus \mathcal{F}_1)) = \tilde{g}(\mathcal{F}_4) - \tilde{g}(\mathcal{F}_2). \] (10)

Hence,

\[ e_E(\mathcal{F}_3) + \sum_{x \in \mathcal{F}_1} d(x) + e_E(\mathcal{F}_4) + \sum_{x \in \mathcal{F}_2} d(x) \geq k(|\mathcal{F}_3| - 1) + \tilde{f}(V \setminus \mathcal{F}_3) + k(|\mathcal{F}_4| - 1) + \tilde{g}(\mathcal{F}_2) \] (by (3) and (4))

\[ \geq k(|\mathcal{F}_3| - 1) + \tilde{f}(V \setminus \mathcal{F}_3) + k(|\mathcal{F}_4| - 1) \]

\[ + \tilde{g}(\mathcal{F}_2) \] (by (10) and Claim 2.4(i))

\[ = e_E(\mathcal{F}_3) + \sum_{x \in \mathcal{F}_1} d(x) + e_E(\mathcal{F}_4) + \sum_{x \in \mathcal{F}_2} d(x) \quad \text{ (since } \mathcal{F}_1 \in \mathcal{E}^1, \mathcal{F}_2 \in \mathcal{E}^2), \]

but this is a contradiction to (8). This proves the lemma.

To finish the proof, we pick an edge \( e_0 \in E \), orient \( e_0 \) to \( \overrightarrow{e_0} \) as following: If there exists an \( \mathcal{F}_0 \in \mathcal{E}^1 \cup \mathcal{E}^2 \) such that one end of \( e_0 \), say \( v \in \mathcal{V}_0 \) and the other end \( u \notin \mathcal{V}_0 \), then we orient \( e_0 \) from \( u \) to \( v \) (i.e., \( \overrightarrow{e_0} = \overrightarrow{uv} \)); otherwise, orient \( e_0 \) to \( \overrightarrow{e_0} \) arbitrarily. Then we define \( A := A + \overrightarrow{e_0} \),
$E := E - e_0$, $F' := (V; A, E)$. It suffices for us to prove that for any $F \in \mathcal{D}(V)$, assumptions (3) and (4) still hold for this new mixed graph $F'$.

Note that the subpartitions in $\mathcal{E}^1$ and $\mathcal{E}^2$ are the ones that make assumptions (3) and (4) tight in the mixed graph $F$. If $F \notin \mathcal{E}^1 \cup \mathcal{E}^2$, since $e_F(F) + \sum_{X \in F} d_A(X)$ is decreased by at most 1, (3) and (4) still hold in $F'$. Otherwise, we have $F \in \mathcal{E}^1 \cup \mathcal{E}^2$. Then we prove (next) that $e_F(F) + \sum_{X \in F} d_A(X)$ is the same in $F$ and $F'$. Thus (3) and (4) still hold in $F'$.

Suppose $\vec{e}_0 = \vec{u}v$ is in $F'$. If both $u, v \in X$ for some $X \in F$, or both $u, v \notin \cup F$, then $e_F(F)$ and $\sum_{X \in F} d_A(X)$ are the same in $F$ and $F'$.

If, for some $X, Y \in F$ and $X \neq Y$, $v \in X$ and $u \in Y$, then $e_F(F)$ is decreased by 1 and $\sum_{X \in F} d_A(X)$ is increased by 1 in $F'$. Therefore $e_F(F) + \sum_{X \in F} d_A(X)$ is the same in $F$ and $F'$.

The left cases are either (A) $v \in \cup F$ and $u \notin \cup F$; or (B) $u \in \cup F$ and $v \notin \cup F$. But Case (B) cannot happen. Indeed, assume on the contrary that $u \notin \cup F$ and $v \notin \cup F$. Since $\vec{e}_0$ is oriented from $u$ to $v$, there exists an $F_0 \in \mathcal{E}^1 \cup \mathcal{E}^2$ such that $v \in \cup F_0$ and $u \notin \cup F_0$. We conclude that $e_0 \in E(F_0, F)$. By Lemma 2.6, $E(F_0, F) = \emptyset$. This is a contradiction. So the only left case is (A) $v \in \cup F$ and $u \notin \cup F$. Then $e_F(F)$ is decreased by 1 and $\sum_{X \in F} d_A(X)$ is increased by 1 in $F'$. Therefore $e_F(F) + \sum_{X \in F} d_A(X)$ is the same in $F$ and $F'$. This finishes the proof of Theorem 1.7.

Finally, combining Theorem 1.7 with some recent developments in this field, we think the following questions are interesting for further studies.

**Question 2.7.** Is there an extension of both Theorem 1.4 and Theorem 1.6? Specifically, let $D = (V, A)$ be a digraph, $f, g: V \to N$ be functions such that $f \leq g$. How to characterize the digraph $D$ which contains $k$ arc-disjoint arborescences $F_1, ..., F_k$ such that

(i) each $F_i$ is maximal;
(ii) for any $v \in V$, $f(v) \leq ||i \in [k]: v$ is the root of $F_i|| \leq g(v)$?

**Question 2.8.** If there is a solution to Question 2.7, then does it have a mixed version, which should also be a generalization of Theorem 1.5 and Theorem 1.7?

**ACKNOWLEDGMENT**

We thank an anonymous referee for careful reading of the paper and the detailed and helpful suggestions that helped improve the expression and clarity of the paper (Grant numbers NSFC 11871439 and U20A2068).

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**How to cite this article:** H. Gao and D. Yang, *Packing of spanning mixed arborescences*, J Graph Theory. 2021;98:367–377. [https://doi.org/10.1002/jgt.22702](https://doi.org/10.1002/jgt.22702)