Sample path behaviors of Lévy processes conditioned to avoid zero

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Abstract

Takeda–Yano [14] determined the limit of Lévy processes conditioned to avoid zero via various random clocks in terms of Doob’s $h$-transform, where the limit processes may differ according to the choice of random clocks. The purpose of this paper is to investigate sample path behaviors of the limit processes in long time and in short time.

Keywords and phrases: one-dimensional Lévy process; conditioning; sample path property; limit theorem

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1 Introduction

For a measure $\mu$ and for a non-negative measurable or an integrable function $f$, we write $\mu[f]$ for the integral $\int f \, d\mu$ for simplicity.

Let $\{\mathbb{P}_{x}^{\text{Bes}^+} : x \geq 0\}$ (resp. $\{\mathbb{P}_{x}^{\text{Bes}^-} : x \leq 0\}$) denote the law of (resp. the negative of) the three-dimensional Bessel process, starting from $x$. For $x \in \mathbb{R}$, let $\mathbb{P}_{x}^{\text{Bes}^\pm}$ denote the law of the symmetrized three-dimensional Bessel process starting from $x$, i.e., it holds that $\mathbb{P}_{x}^{\text{Bes}} = \mathbb{P}_{x}^{\text{Bes}^+}$ for $x > 0$, $\mathbb{P}_{x}^{\text{Bes}} = \mathbb{P}_{x}^{\text{Bes}^-}$ for $x < 0$ and $\mathbb{P}_{0}^{\text{Bes}} = \frac{1}{2}\mathbb{P}_{0}^{\text{Bes}^+} + \frac{1}{2}\mathbb{P}_{0}^{\text{Bes}^-}$. Let $x \in \mathbb{R}$ and let $(X = (X_t, t \geq 0), \mathbb{F})$ be the canonical representation of a standard Brownian motion starting from $x$ and $(\mathcal{F}_t)$ denote the right-continuous filtration generated by the natural filtration. We denote by $T_0 = \inf\{t > 0 : X_t = 0\}$ the first hitting time of the origin. Then, in the case $x \neq 0$, we have the following conditioning limit theorem: for any bounded $\mathcal{F}_t$-measurable functional $F_t$, it holds that

$$\lim_{s \to \infty} \mathbb{P}_{x}^{\text{Bes}^+}[F_t | T_0 > s] = \mathbb{P}_{x}^{\text{Bes}^\pm}[F_t].$$

(1.1)

In the case $x = 0$, we have a similar conditioning limit theorem if we replace $\mathbb{P}_{0}^{\text{Bes}^\pm}$ with the law of the Brownian meander. This means that the Brownian motion conditioned to avoid zero up to time $t$ converges in law to the symmetrized three-dimensional Bessel process. The left-hand side of (1.1) can be regarded as the Brownian motion conditioned to avoid zero. We remark that the three-dimensional Bessel process is transient and never hits the origin. For $x \in \mathbb{R} \setminus \{0\}$, the process $\mathbb{P}_{x}^{\text{Bes}^\pm}$ can be written via Doob’s $h$-transform with

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respect to the non-negative harmonic function \( h(x) = |x| \) of the killed Brownian motion \( \{p^B_x: x \in \mathbb{R} \setminus \{0\}\} \) as follows:

\[
P^B_x|_{\mathcal{F}_t} = \frac{|X_t|}{|x|} p^B_x|_{\mathcal{F}_t}, \quad t > 0.
\]

(1.2)

Let \( n^B \) stand for the Brownian excursion measure. Then \( P_{0}^{\text{Bes}} \) can also be written as

\[
P_{0}^{\text{Bes}}|_{\mathcal{F}_t} = \frac{|X_t|}{n^B[|X_t|]} \cdot n^B|_{\mathcal{F}_t}, \quad t > 0.
\]

(1.3)

These results for Brownian motions were generalized to one-dimensional Lévy processes. Yano [18] constructed and investigated one-dimensional Lévy processes conditioned to avoid zero under the conditions that the process is symmetric and has no Gaussian part. He also investigated path behaviors of the process. Yano [19] extended their results to asymmetric Lévy processes. He also showed the existence of a non-negative harmonic function for asymmetric killed Lévy processes under some technical conditions. Pantí [11] and Tsukada [15] showed the existence of the harmonic function under more general conditions and Pantí [11] investigated asymmetric Lévy processes conditioned to avoid zero using \( h \)-transform with respect to its harmonic function. Recently, Takeda–Yano [14] obtained a family of harmonic functions \( h(\gamma) \) parametrized by \(-1 \leq \gamma \leq 1\) for the killed Lévy process under more general conditions. They also constructed the Lévy process conditioned to avoid zero, using the \( h(\gamma) \)-transform.

In this paper, we investigate the path behaviors of Lévy processes conditioned to avoid zero which is constructed in Takeda–Yano [14].

1.1 Lévy processes conditioned to avoid zero

We shall recall the construction of Lévy processes conditioned to avoid zero in Takeda–Yano [14]. For more details of the notation of this section, see Section 2. Let (\( X = (X_t, t \geq 0), (P_x)_{x \in \mathbb{R}} \)) denote the canonical representation of a one-dimensional Lévy process and we write \( P = P_0 \). Throughout this paper, we always assume the following condition \( (A) \):

\[(A)\] The process \((X, P)\) is recurrent and, for each \( q > 0 \), it holds that

\[
\int_0^\infty \frac{1}{q + \Psi(\lambda)} \, d\lambda < \infty,
\]

(1.4)

where \( \Psi(\lambda) \) denotes the characteristic exponent given by \( P[e^{i\lambda X_t}] = e^{-\Phi(\lambda)} \).

Let \( T_A = \inf\{t > 0: X_t \in A\} \) stand for the hitting time of a set \( A \subset \mathbb{R} \) and we simply write \( T_a := T_{\{a\}} \) for the hitting time of a point \( a \in \mathbb{R} \). The condition \( (A) \) implies that \( P(T^0_0 = 0) = 1 \) and \((X, P)\) is not compound Poisson. In addition, there exists the \( q \)-resolvent density \( r_q \) for \( q > 0 \). For \( x \in \mathbb{R} \), we define \( h_q(x) = r_q(0) - r_q(-x) \geq 0 \) and

\[
h(x) = \lim_{q \to 0^+} h_q(x), \quad x \in \mathbb{R},
\]

(1.5)
which is called the \textit{renormalized zero resolvent}; see \cite{[22]} of Lemma \cite{[22]}. The function \( h \) is subadditive; see \cite{[22]} of Lemma \cite{[22]}. We denote the second moment of \( X_1 \) by

\[
m^2 = \mathbb{P}[X_1^2] \in (0, \infty]. \tag{1.6}
\]

For \(-1 \leq \gamma \leq 1\), define

\[
h^{(\gamma)}(x) = h(x) + \frac{\gamma}{m^2} x, \quad x \in \mathbb{R}. \tag{1.7}
\]

If \( m^2 = \infty \), the functions \( h^{(\gamma)} \) coincide with \( h \) for all \(-1 \leq \gamma \leq 1\). The function \( h^{(\gamma)} \) is non-negative (see \( \text{(2.13)} \)) and subadditive. Let \( \mathbb{P}_x^0 \) denote the law under \( \mathbb{P} \) of the killed process

\[
X^0_t = \begin{cases} X_t & \text{if } t < T_0, \\ \Delta & \text{if } t \geq T_0, \end{cases} \tag{1.8}
\]

where \( \Delta \) stands for a cemetery point. Let \( n \) denote Itô’s excursion measure normalized by the equation

\[
n[1 - e^{-qT_0}] = \frac{1}{r_q(0)}, \quad q > 0; \tag{1.9}
\]

see Section \cite{[2]}. The next lemma says \( h^{(\gamma)} \) is harmonic for the killed process.

**Lemma 1.1** (\cite{[14]}). Assume the condition \((\text{A})\) is satisfied. For \(-1 \leq \gamma \leq 1\) and \( x \in \mathbb{R} \), it holds that

\[
\mathbb{P}_x^0[h^{(\gamma)}(X_t)] = h^{(\gamma)}(x) \quad \text{and} \quad n[h^{(\gamma)}(X_t)] = 1, \quad t > 0. \tag{1.10}
\]

In particular, the process \((h^{(\gamma)}(X_t), t > 0)\) is a non-negative \( \mathbb{P}_x^0 \)-martingale.

The proof of Lemma \cite{[14]} can be found in \cite{[14]} Theorem 8.1 and \cite{[11]} (iii) of Theorem 2.2.

For \(-1 \leq \gamma \leq 1\), define \( \mathcal{H}^{(\gamma)} = \{ x \in \mathbb{R}; h^{(\gamma)}(x) > 0 \} \) and \( \mathcal{H}^{(0)} = \mathcal{H}^{(\gamma)} \cup \{ 0 \} \). If \( m^2 = \infty \), we have \( \mathcal{H}^{(\gamma)} = \mathcal{H}^{(0)} \). If \( m^2 < \infty \), we have \( \mathcal{H}^{(1)} \cap \mathcal{H}^{(-1)} \subset \mathcal{H}^{(0)} \subset \mathbb{R} \) for \(-1 < \gamma < 1\) by \( \text{(2.13)} \). Adopting Doob’s \( h \)-transform approach, we construct the \( h^{(\gamma)} \)-transform by

\[
\mathbb{P}_x^{(\gamma)}|_{\mathcal{F}_t} = \begin{cases} \frac{h^{(\gamma)}(X_t)}{h^{(\gamma)}(x)} \cdot \mathbb{P}_x^0|_{\mathcal{F}_t} & \text{if } x \in \mathcal{H}^{(\gamma)}, \\ h^{(\gamma)}(X_t) \cdot n|_{\mathcal{F}_t} & \text{if } x = 0. \end{cases} \tag{1.11}
\]

Note that, if \( m^2 = \infty \), we have \( \mathbb{P}_x^{(\gamma)} = \mathbb{P}_x^0 \) for all \(-1 \leq \gamma \leq 1\). By Lemma \cite{[11]}, we see that \( \mathbb{P}_x^{(\gamma)}|_{\mathcal{F}_t} \) is consistent in \( t > 0 \) and thus \( \mathbb{P}_x^{(\gamma)} \) is well-defined and is a probability measure on \( \mathcal{F}_\infty \); for more details, see Yano \cite{[17]} Theorem 9.1. We can see \( \mathbb{P}_x^{(\gamma)}(T_{\mathbb{R} \setminus \mathcal{H}^{(\gamma)}} > t) = 1 \) for all \( t > 0 \) and consequently it holds that \( \mathbb{P}_x^{(\gamma)}(T_{\mathbb{R} \setminus \mathcal{H}^{(\gamma)}} = \infty) = 1 \). Hence the process \((X, \mathbb{P}_x^{(\gamma)})\) never hits zero. We remark that, for \( x \in \mathcal{H}^{(\gamma)} \), the measure \( \mathbb{P}_x^{(\gamma)} \) is absolutely continuous on \( \mathcal{F}_t \) with respect to \( \mathbb{P}_x \), but is singular on \( \mathcal{F}_\infty \) to \( \mathbb{P}_x \) since \( \mathbb{P}_x^{(\gamma)}(T_0 = \infty) = \mathbb{P}_x(T_0 < \infty) = 1 \).

The next theorem shows that for \( x \in \mathcal{H}^{(\gamma)} \), the measure \( \mathbb{P}_x^{(\gamma)} \) can be obtained as the limit measure of the Lévy process conditioned to avoid zero via a random clock, i.e., a certain parametrized family of random times, going to infinity. Let \( e \) be an independent exponential time with mean 1 and we write \( e_q = e/q \).
Theorem 1.2 ([14]). Assume the condition \([A]\) is satisfied. Let \(t > 0\) and \(F_t\) be a bounded \(\mathcal{F}_t\)-measurable functional. Then the following assertions hold:

\[
\begin{align*}
(i) & \lim_{q \to 0^+} \mathbb{P}_x[F_t | T_0 > e_q] = \mathbb{P}_x^{(0)}[F_t], \; \text{for} \; x \in \mathcal{H}^{(0)}; \\
(ii) & \lim_{a \to \pm\infty} \mathbb{P}_x[F_t | T_0 > T_a] = \mathbb{P}_x^{(\pm 1)}[F_t], \; \text{for} \; x \in \mathcal{H}^{(\pm 1)}; \\
(iii) & \lim_{a \to \infty, b \to \infty, \frac{b-a}{a+b} \to \gamma} \mathbb{P}_x[F_t | T_0 > T_{\{a,-b\}}] = \mathbb{P}_x^{(\gamma)}[F_t], \; \text{for} \; -1 \leq \gamma \leq 1 \; \text{and} \; x \in \mathcal{H}^{(\gamma)}.
\end{align*}
\]

The proof of Theorem 1.2 can be found in Corollary 8.2 of [14]. We remark, however, that there are computational errors in the claim (iii) of Corollary 8.2 of [14]; see [7, Section 6]. Claim (i) of Theorem 1.2 is also proved in Pantí [11, Theorem 2.7]. If \(m^2 < \infty\), then the limit measure differs according to the random clock.

We remark that the limit \(\lim_{s \to \infty} \mathbb{P}_x[F_t | T_0 > s]\) via constant clock is determined in the symmetric stable case (see Yano–Yano–Yor [22]) but the limit is an open problem in the general Lévy case.

The left-hand side of each of (i)–(iii) of Theorem 1.2 can be regarded as Lévy processes conditioned to avoid zero although the resulting processes may differ according to the choice of the clocks. We remark that the resulting processes are characterized via Doob’s \(h\)-transform. For related studies, see Chaumont [3] and Chaumont–Doney [4] for Lévy processes conditioned to stay positive. Yano–Yano [20] for diffusions conditioned to avoid zero.

Let \(D\) denote the space of càdlàg paths \(\omega: [0, \infty) \to \mathbb{R} \cup \{\Delta\}\). We denote by \(\theta\) the shift operator and by \(k\) the killing operator, i.e., we define, for \(\omega \in D\) and \(t \geq 0\), \(\theta_t \omega(s) = \omega(s + t), \; s \geq 0\), and define

\[
k_t \omega(s) = \begin{cases} 
\omega(s) & \text{if } s < t, \\
\Delta & \text{if } s \geq t.
\end{cases}
\]

(1.12)

For \(s > 0\), we denote by \(g_s = \sup\{u \leq s: X_u = 0\}\) the last hitting time of the origin up to time \(s\). Then we have, for \(\tau > 0\),

\[
k_{\tau-g_s} \circ \theta_{g_s} \omega(s) = \begin{cases} 
\omega(g_s + s) & \text{if } 0 \leq s < \tau - g_s, \\
\Delta & \text{if } s \geq \tau - g_s.
\end{cases}
\]

(1.13)

The next theorem shows that for \(x = 0\), the measure \(\mathbb{P}_x^{(\gamma)} = \mathbb{P}_0^{(\gamma)}\) can be obtained as the limit, via a random clock, of a measure similar to the Lévy meander.

Theorem 1.3. Assume the condition \([A]\) is satisfied. Let \(t > 0\) and \(F_t\) be a bounded \(\mathcal{F}_t\)-measurable functional. Then the following assertions hold:

\[
\begin{align*}
(i) & \lim_{q \to 0^+} \mathbb{P}_0[F_t \circ k_{e_q-g_{e_q}} \circ \theta_{g_{e_q}}] = \mathbb{P}_0^{(0)}[F_t]; \\
(ii) & \lim_{a \to \infty} \mathbb{P}_0[F_t \circ k_{T_a-g_{T_a}} \circ \theta_{g_{T_a}}] = \mathbb{P}_0^{(\pm 1)}[F_t];
\end{align*}
\]
\( \lim_{a \to \infty, b \to \infty} \mathbb{P}_0[F_t \circ k_{T_{(a,-b)}} \circ \theta_{g_{T_{(a,-b)}}}] = \mathbb{P}^{(\gamma)}_0[F_t], \) for \(-1 \leq \gamma \leq 1.\)

The proof of Theorem 1.3 will be given in Section 6. Claim (ii) of Theorem 1.3 is also proved in Pantí [11, Theorem 2.8].

1.2 Main results

Recall that we always assume the condition (A).

1.2.1 Long-time behaviors of the process \((X, \mathbb{P}^{(\gamma)}_x)\)

The proofs of the following Theorems 1.4, 1.5 and 1.6 will be given in Section 3.

**Theorem 1.4.** Let \(-1 \leq \gamma \leq 1\) and \(x \in \mathcal{H}_0^{(\gamma)}\). Then it holds that

\[
\mathbb{P}^{(\gamma)}_x\left(\lim_{t \to \infty} |X_t| = \infty \right) = 1. \tag{1.14}
\]

Consequently, the process \((X, \mathbb{P}^{(\gamma)}_x)\) is transient.

We discuss the result when \(m^2 < \infty\). In this case, recall that, by (2.13), we have \(\mathcal{H}_0^{(\gamma)} = \mathbb{R}\) for \(-1 < \gamma < 1.\)

**Theorem 1.5.** Assume \(m^2 < \infty.\) Then, for \(x \in \mathbb{R}\) and \(-1 < \gamma < 1,\) the measure \(\mathbb{P}^{(\gamma)}_x\) is equivalent to \(\mathbb{P}^{(0)}_x.\) Moreover, for \(x \in \mathcal{H}_0^{(\pm1)}\), the measure \(\mathbb{P}^{(\pm1)}_x\) is absolutely continuous with respect to \(\mathbb{P}^{(0)}_x.\)

We discuss long-time behaviors of the process \((X, \mathbb{P}^{(\gamma)}_x)\) in the case \(m^2 < \infty.\) Define

\[
\Omega^+ = \left\{ \lim_{t \to \infty} X_t = \infty \right\},
\]

\[
\Omega^- = \left\{ \lim_{t \to \infty} X_t = -\infty \right\},
\]

\[
\Omega^{+,-} = \left\{ \limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty \right\}. \tag{1.17}
\]

Then the sets \(\Omega^+\), \(\Omega^-\) and \(\Omega^{+,-}\) are mutually disjoint and \(\{\lim_{t \to \infty} |X_t| = \infty\} \subset \Omega^+_\infty \cup \Omega^- \cup \Omega^{+,-}.\) Hence by Theorem 1.4, it holds that

\[
\mathbb{P}^{(\gamma)}_x(\Omega^+_\infty \cup \Omega^- \cup \Omega^{+,-}) = 1. \tag{1.18}
\]

If \(m^2 < \infty,\) the process \((X, \mathbb{P}^{(\gamma)}_x)\) drifts to either \(+\infty\) or \(-\infty\) with a certain probability.

**Theorem 1.6.** Assume \(m^2 < \infty.\) Then, for \(-1 \leq \gamma \leq 1,\) it holds that

\[
\mathbb{P}^{(\gamma)}_x(\Omega^+_\infty) = \begin{cases} 
\frac{(1 + \gamma)}{2} \frac{h^{(1)}(x)}{h^{(\gamma)}(x)} & \text{if } x \in \mathcal{H}^{(\gamma)}, \\
\frac{1 + \gamma}{2} & \text{if } x = 0.
\end{cases} \tag{1.19}
\]
Consequently, for \( x \in H_0^{(1)} \cap H_0^{(-1)} \), it holds that

\[
P_x^\gamma(\Omega_\infty^+ \cup \Omega_\infty^-) = P_x^{(1)}(\Omega_\infty^+) = P_x^{(-1)}(\Omega_\infty^-) = 1,
\]

which implies \( P_x^{(1)} \) and \( P_x^{(-1)} \) are mutually singular on \( \mathcal{F}_\infty \).

Note that, if \( m^2 = \infty \), the process \( X \) can be oscillating under \( P_x^{(\gamma)} = P_x^{(0)} \), see, e.g., Theorem 1.12.

### 1.2.2 Short-time behaviors of the process \((X, P_x^{(\gamma)})\)

The proofs of the following Theorems 1.7, 1.8, 1.9, and 1.10 will be given in Section 4.

We first deal with differential property at 0 of \( h \), which is used for the discussion of short-time behaviors. Since \( h \) and \( h^{(\gamma)} \) are subadditive, it holds that

\[
h'(0\pm) := \lim_{x \to 0\pm} \frac{h(x)}{x} = \pm \sup_{x > 0} \frac{h(\pm x)}{x},
\]

\[
h^{(\gamma)'}(0\pm) := \lim_{x \to 0\pm} \frac{h^{(\gamma)}(x)}{x} = \pm \sup_{x > 0} \frac{h^{(\gamma)}(\pm x)}{x} = h'(0\pm) \pm \frac{\gamma}{m^2}.
\]

By (v) of Lemma 2.2, we have

\[
|h'(0\pm)| = \pm h'(0\pm) \in \left[ \frac{1}{m^2}, \infty \right],
\]

\[
|h^{(\gamma)'}(0\pm)| = \pm h^{(\gamma)'}(0\pm) \in \left[ \frac{1 \pm \gamma}{m^2}, \infty \right].
\]

**Theorem 1.7.** It holds that

\[
h'(0+) + |h'(0-)| = \lim_{x \to 0+} \frac{h(x) + h(-x)}{x} = \frac{2}{\sigma^2} \in (0, \infty].
\]

Consequently, for \(-1 \leq \gamma \leq 1\), it holds that

\[
h^{(\gamma)'}(0+) + |h^{(\gamma)'}(0-)| = \lim_{x \to 0+} \frac{h^{(\gamma)}(x) + h^{(\gamma)}(-x)}{x} = \frac{2}{\sigma^2} \in (0, \infty].
\]

We remark that Winkel [16, Lemma 1] already showed that

\[
\lim_{x \to 0+} \frac{h_q(x) + h_q(-x)}{x} = \frac{2}{\sigma^2}, \quad q > 0.
\]

By Theorem 1.7, we see that \( \sigma^2 > 0 \) implies \( |h'(0\pm)| \leq 2/\sigma^2 < \infty \) and that \( \sigma^2 = 0 \) implies \( h'(0+) = \infty \) or \( -h'(0-) = \infty \).

Define

\[
\Omega_0^+ = \{ \exists t_0 > 0 \text{ such that } 0 < \forall t < t_0, X_t > 0 \},
\]

\[
\Omega_0^- = \{ \exists t_0 > 0 \text{ such that } 0 < \forall t < t_0, X_t < 0 \},
\]

\[
\Omega_0^{+\!-} = \{ \exists \{t_n\} \text{ with } t_n \to 0+ \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \}.
\]

Then \( \Omega_0^+, \Omega_0^- \) and \( \Omega_0^{+\!-} \) are mutually disjoint and we have \( \Omega_0^+ \cup \Omega_0^- \cup \Omega_0^{+\!-} = \mathcal{D} \).
Theorem 1.10. Assume $m^2 < \infty$ and $\sigma^2 > 0$. Then, for $-1 \leq \gamma \leq 1$, it holds that $\mathbb{P}_0^{(\gamma)}(\Omega^+_0) = 0$,
\[
\mathbb{P}_0^{(\gamma)}(\Omega^+_0) = \frac{\sigma^2}{2} h^{(\gamma)}(0+) \quad \text{and} \quad \mathbb{P}_0^{(\gamma)}(\Omega^-_0) = \frac{\sigma^2}{2} |h^{(\gamma)}(0-)|.
\] (1.31)

Theorem 1.9. Assume $m^2 < \infty$. If $h'(0+) = \infty$ and $|h'(0-)| < \infty$, then $\mathbb{P}_0^{(\gamma)}(\Omega^+_0) = 1$ for $-1 \leq \gamma \leq 1$. If $h'(0+) < \infty$ and $|h'(0-)| = \infty$, then $\mathbb{P}_0^{(\gamma)}(\Omega^-_0) = 1$ for $-1 \leq \gamma \leq 1$.

In the case $|h'(0\pm)| = \infty$, we do not obtain general properties. We obtain the oscillating short-time behavior under some technical assumptions.

Theorem 1.10. Let $-1 \leq \gamma \leq 1$. Assume the following four assertions hold:

(i) $\liminf_{x \to \infty} h^{(\gamma)}(x) > 0$ and $\liminf_{x \to -\infty} h^{(\gamma)}(x) > 0$;

(ii) $\lim_{x \to 0} \frac{h(x)}{x} = \infty$, i.e., $h'(0+) = -h'(0-) = \infty$;

(iii) $\lim_{x \to 0} \frac{h_q(x + y) - h_q(y)}{h(x)} = 1_{\{y=0\}}$ for all $q > 0$;

(iv) $0 < \liminf_{x \to 0} \frac{h(-x)}{h(x)} < \limsup_{x \to 0} \frac{h(-x)}{h(x)} < \infty$.

Then it holds that $\mathbb{P}_0^{(\gamma)}(\Omega^\pm_0) = 1$.

Note that, If $m^2 < \infty$, (iv) of Lemma 2.2 implies that the condition (ii) of Theorem 1.10 always holds for $-1 < \gamma < 1$.

1.3 Examples

Before proceeding with the proofs of the results, we introduce some examples.

1.3.1 Brownian motions

Assume $(X, \mathbb{P})$ is a standard Brownian motion. Then $\sigma^2 = m^2 = 1$ and $h(x) = |x|$. By Theorems 1.6 and 1.8 it holds that, for $-1 \leq \gamma \leq 1$,
\[
\mathbb{P}_0^{(\gamma)}(\Omega^+_0) = \frac{1 + \gamma}{2}, \quad \mathbb{P}_0^{(\gamma)}(\Omega^-_0) = \frac{1 - \gamma}{2}.
\] (1.32) (1.33)

Since the Brownian motion has no jumps, the process $(X, \mathbb{P}^{(\gamma)}_x)$ also has no jumps. Thus the avoiding zero process $(X, \mathbb{P}^{(\gamma)}_x)$ does not change the sign. In fact, we have
\[
\mathbb{P}^{(\gamma)}_0 = \frac{1 + \gamma}{2} \mathbb{P}^{\text{Bes}+}_0 + \frac{1 - \gamma}{2} \mathbb{P}^{\text{Bes}-}_0.
\] (1.34)

Moreover, by Theorem 1.8 it holds that, for $-1 \leq \gamma \leq 1$,
\[
\mathbb{P}^{(\gamma)}_x(\Omega^+_\infty) = 1_{\{x > 0\}}, \quad \mathbb{P}^{(\gamma)}_x(\Omega^-_\infty) = 1_{\{x < 0\}}, \quad x \neq 0.
\] (1.35)

In fact, we have $\mathbb{P}^{(\gamma)}_x = \mathbb{P}^{\text{Bes}+}_x$ for $x > 0$ and $\mathbb{P}^{(\gamma)}_x = \mathbb{P}^{\text{Bes}-}_x$ for $x < 0$. 

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1.3.2 Stable processes

Assume \((X, \mathbb{P})\) is strictly stable of index \(1 < \alpha < 2\). Then \(m^2 = \infty\) and the Lévy measure \(\nu\) can be written as

\[
\nu(dx) = \begin{cases} 
  c_+ x^{-\alpha} \, dx & \text{for } x \in (0, \infty), \\
  c_- |x|^{-\alpha} \, dx & \text{for } x \in (-\infty, 0),
\end{cases}
\]  

(1.36)

where \(c_+\) and \(c_-\) are non-negative constants such that \(c_+ + c_- > 0\). The characteristic exponent \(\Psi\) can be expressed as

\[
\Psi(\lambda) = c|\lambda|^{\alpha} \left( 1 - i\beta \text{sgn}(\lambda) \tan \frac{\alpha \pi}{2} \right),
\]  

(1.37)

where \(c = -(c_+ + c_-) \Gamma(-\alpha) \cos(\pi \alpha/2)\) and \(\beta = (c_+ - c_-)/(c_+ + c_-)\); see, e.g., [10, Section 2]. We write \(c' = -c\beta \tan(\alpha \pi/2)\). Then as a special case of (2.8), it holds that

\[
h_q(x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - e^{i\lambda x}}{q + (c + ic')|\lambda|^{\alpha}} \right) d\lambda.
\]  

(1.38)

The dominated convergence theorem implies that

\[
h(x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - e^{i\lambda x}}{(c + ic')|\lambda|^{\alpha}} \right) d\lambda.
\]  

(1.39)

Thus we have

\[
h(x) - h_q(x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - e^{i\lambda x}}{(c + ic')|\lambda|^{\alpha}} \frac{q}{q + (c + ic')|\lambda|^{\alpha}} \right) d\lambda.
\]  

(1.40)

Hence it is obvious that

\[
h'_q(x) = h'(x) + v(x), \quad x \in \mathbb{R} \setminus \{0\},
\]  

(1.41)

where \(v(x)\) is a bounded continuous function. Yano [19] calculated (1.39) and obtained

\[
h(x) = -\frac{\alpha \Gamma(-\alpha) \sin(\pi \alpha/2) (1 - \beta \text{sgn}(x))}{c(1 + \beta^2 \tan^2(\pi \alpha/2))} |x|^{\alpha-1};
\]  

(1.42)

see also Pantí [11] Example 5.1.

Assume \((X, \mathbb{P})\) is spectrally positive (resp. negative), i.e., \(\beta = 1\) (resp. \(\beta = -1\)). Then by (1.42), we have \(H_0^0 = (-\infty, 0]\) (resp. \(H_0^0 = [0, \infty)\)). On the other hand, assume \((X, \mathbb{P})\) is not spectrally one-sided, i.e., \(-1 < \beta < 1\). Then the functions \(h\) and \(h_q\) satisfy the assumptions of Theorem 1.10. Hence we obtain the following theorem:

**Theorem 1.11.** Assume \((X, \mathbb{P})\) is a strictly stable process of index \(1 < \alpha < 2\). If \((X, \mathbb{P})\) is spectrally positive (resp. negative), it holds that

\[
\mathbb{P}_0^0(\Omega_0^0) = 1 \quad (\text{resp. } \mathbb{P}_0^0(\Omega_0^+)) = 1).
\]  

(1.43)

If \((X, \mathbb{P})\) is not spectrally one-sided, it holds that

\[
\mathbb{P}_0^0(\Omega_0^-) = 1.
\]  

(1.44)
Furthermore, we obtain the following long-time behavior:

**Theorem 1.12.** Assume \((X, \mathbb{P})\) is a strictly stable process of index \(1 < \alpha < 2\). If \((X, \mathbb{P})\) is spectrally positive (resp. negative), it holds that

\[
\mathbb{P}^{(0)}_0(\Omega_{\infty}^-) = 1 \quad (\text{resp. } \mathbb{P}^{(0)}_0(\Omega_{\infty}^+) = 1).
\]  

(1.45)

If \((X, \mathbb{P})\) is not spectrally one-sided, it holds that

\[
\mathbb{P}^{(0)}_0(\Omega_{\infty}^{+,-}) = 1.
\]  

(1.46)

To prove (1.46), we use the same discussion as the proof of [18, Corollary 1.4].

1.3.3 Recurrent spectrally negative processes

Let \((X, \mathbb{P})\) be a spectrally negative Lévy process, i.e., \(\nu(0, \infty) = 0\), satisfying the assumption \((A)\). Then, [11, Example 5.2] says that

\[
h(x) = W(x) - \frac{x}{m^2},
\]

(1.47)

where \(W(x)\) stands for the scale function of \(X\). Since \(W(x) = 0\) for \(x \leq 0\), we have \(h(x) = |x|/m^2 \in [0, \infty)\) for \(x \leq 0\). If \(m^2 = \infty\), we have \(\mathcal{H}_0^{(0)} = [0, \infty)\) and hence it holds that

\[
\mathbb{P}^{(0)}_x(\Omega_{\infty}^+) = \mathbb{P}^{(0)}_0(\Omega_{\infty}^+) = 1, \quad x \in [0, \infty).
\]  

(1.48)

1.3.4 Symmetric processes

We consider the case \((X, \mathbb{P})\) is symmetric and satisfies the condition \((A)\). Then \(h(x) = h(-x)\). If \(\sigma^2 > 0\), we have \(h'(0+) = -h'(0-) = 1/\sigma^2\) and hence, by Theorem 1.8

\[
\mathbb{P}^{(\gamma)}_0(\Omega_{\infty}^+) = \mathbb{P}^{(\gamma)}_0(\Omega_{\infty}^-) = \frac{1}{2}, \quad -1 \leq \gamma \leq 1.
\]  

(1.49)

On the other hand, we assume \(\sigma^2 = 0\) and \(\lambda \mapsto \text{Re}\, \Psi(\lambda)\) is eventually non-decreasing. Then, Theorem 1.7 implies that the function \(h\) satisfies the assumption \((ii)\) of Theorem 1.10. Moreover, Lemma 4.4 and (i) of Lemma 6.2 of Yano [18] states that \(h\) and \(h_q\) satisfy the assumption \((iii)\) of Theorem 1.10. The function \(h\) obviously satisfies the assumption \((iv)\) of Theorem 1.10. Hence, if \((i)\) of Theorem 1.10 also holds, it holds that

\[
\mathbb{P}^{(\gamma)}_0(\Omega_{\infty}^{+,-}) = 1, \quad -1 \leq \gamma \leq 1.
\]  

(1.50)

1.4 Outline of the paper

The paper is organized as follows. In Section 2, we prepare general properties of Lévy processes and some preliminary facts of the renormalized zero resolvent \(h\). In Sections 3 and 4, we prove the main results for long-time behaviors and short-time behaviors, respectively. In Section 5 as an appendix, we investigate the resolvent density under \(\mathbb{P}^{(\gamma)}_x\). In Section 6 as another appendix, we will give the proof of Theorem 1.3.
2 Preliminaries

2.1 General properties of Lévy processes

Let \((X = (X_t, t \geq 0), \mathbb{P}_x)\) denote the canonical representation of a one-dimensional Lévy process starting from \(x \in \mathbb{R}\) on the càdlàg path space \(D\) and we write \(\mathbb{P} = \mathbb{P}_0\). For \(t \geq 0\), we denote by \(\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)\) the natural filtration of \(X\) and we write \(\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s^X\) and \(\mathcal{F}_\infty = \sigma(\cup_{t>0} \mathcal{F}_t)\). It is well-known that we have

\[
P[e^{i\lambda X_t}] = e^{-t\Phi(\lambda)}, \quad \text{for } t \geq 0 \text{ and } \lambda \in \mathbb{R},
\]  

where \(\Phi(\lambda)\) denotes the characteristic exponent of \(X\) given by the Lévy–Khintchine formula

\[
\Phi(\lambda) = iv\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x 1_{|x|<1}\right)\nu(dx)
\]  

for some constants \(v \in \mathbb{R}\) and \(\sigma^2 \geq 0\) and a characteristic measure \(\nu\) on \(\mathbb{R}\) which satisfies \(\nu(\{0\}) = 0\) and

\[
\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty.
\]

The measure \(\nu\) is called a Lévy measure. See, e.g., [1, 10].

We consider the following four conditions:

**(A1)** The process \((X, \mathbb{P})\) is not a compound Poisson process;

**(A2)** \(0\) is regular for itself, i.e., \(\mathbb{P}(T_0 = 0) = 1\);

**(A3)** \(\int_{\mathbb{R}} \text{Re}\left(\frac{1}{q + \Phi(\lambda)}\right) d\lambda < \infty\) for all \(q > 0\);

**(A4)** We have either \(\sigma^2 > 0\) or \(\int_{(-1,1)} |x| \nu(dx) = \infty\).

Then the following lemma is well-known:

**Lemma 2.1.** The following three assertions hold:

(i) The conditions **(A1)** and **(A2)** hold if and only if the conditions **(A3)** and **(A4)** hold;

(ii) Under the condition **(A3)**, the condition **(A2)** holds if and only if the condition **(A4)** holds;

(iii) The condition **(A3)** holds if and only if \((X, \mathbb{P})\) has the bounded \(q\)-resolvent density \(r_q\), which satisfies

\[
\int_{\mathbb{R}} f(x) r_q(x) dx = \mathbb{P}\left[\int_0^\infty e^{-qt} f(X_t) dt\right], \quad q > 0,
\]  

for all non-negative measurable functions \(f\). Moreover, under the condition **(A3)**, the condition **(A2)** holds if and only if \(x \mapsto r_q(x)\) is continuous.
For the proofs of (i) and (ii) of Lemma 2.1 see Kesten [9] and Bretagnolle [2]. For the proof of (iii) of Lemma 2.1 see Theorems II.16 and II.19 of Bertoin [1].

Throughout this paper, we always assume the condition \((A)\). This implies that \((X, P)\) has the bounded continuous resolvent density which is given by

\[
q(x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\lambda x}}{q + \Psi(\lambda)} \right) d\lambda \tag{2.5}
\]

for all \(q > 0\) and \(x \in \mathbb{R}\); see, e.g., Winkel [16, Lemma 2] and Tsukada [15, Colorally 15.1]. Combining this and Lemma 2.1 we see the condition \((A)\) implies \((A1)-(A4)\). Tsukada [15, Lemma 15.5] also proved that the condition \((A)\) implies

\[
\int_0^\infty \left| \frac{1 \wedge \lambda^2}{\Psi(\lambda)} \right| d\lambda < \infty; \tag{2.6}
\]

see also [14, Lemma 2.4].

Under the condition \((A)\), we denote by \(L = (L_t, t \geq 0)\) local time at 0 normalized by the equation

\[
P_x \left[ \int_0^\infty \exp(-qt) dL_t \right] = r_q(-x), \quad x \in \mathbb{R}; \tag{2.7}
\]

see, e.g., [1, Section V]. Let \(n\) denote the characteristic measure of excursions away from 0, called Itô’s excursion measure (see, e.g., [1, Section IV.4]). Then the equation (1.9) holds.

### 2.2 The renormalized zero resolvent

We define

\[
h_q(x) = r_q(0) - r_q(-x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)} \right) d\lambda. \tag{2.8}
\]

Since we have

\[
P_x[e^{-qT_0}] = \frac{r_q(-x)}{r_q(0)} \geq 0 \tag{2.9}
\]

(see, e.g., Bertoin [1, Colorally II.18]), the function \(h_q\) is non-negative. In addition, \(h_q\) is subadditive, i.e., \(h_q(x + y) \leq h_q(x) + h_q(y)\) for \(x, y \in \mathbb{R}\); see, e.g., the proof of Lemma 3.3 in [11] and the proofs of (ii) and (iii) of Theorem 1.1 in [14].

We denote the second moment of \(X_1\) by

\[
m^2 = \mathbb{E}[X_1^2] = \sigma^2 + \int_\mathbb{R} x^2 \nu(dx) \in (0, \infty]. \tag{2.10}
\]

**Lemma 2.2** (The renormalized zero resolvent). Assume the condition \((A)\) is satisfied. Then the following assertions hold:

1. for \(x \in \mathbb{R}\), the limit \(h(x) := \lim_{q \to 0^+} h_q(x)\) exists and is finite, which is called the renormalized zero resolvent;
(ii) $h$ is non-negative, continuous and subadditive ($h(x + y) \leq h(x) + h(y)$ for $x, y \in \mathbb{R}$) and $h(0) = 0$;

(iii) $h(x) + h(-x) = \frac{2}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda$, for $x \in \mathbb{R}$;

(iv) if $m^2 < \infty$, it holds that $h(x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - e^{\lambda x}}{\Psi(\lambda)} \right) d\lambda$, for $x \in \mathbb{R}$;

(v) $\lim_{x \to \pm\infty} h(x) = \frac{1}{m^2} \in [0, \infty)$;

(vi) $\lim_{y \to \pm\infty} \{h(x + y) - h(y)\} = \pm \frac{x}{m^2} \in \mathbb{R}$.

The proof of Lemma 2.2 can be found in Theorems 1.1 and 1.2 and Lemma 3.3 of [14].

We define, for $-1 \leq \gamma \leq 1$,

$$h^{(\gamma)}(x) = h(x) + \frac{\gamma}{m^2} x, \quad x \in \mathbb{R}. \quad (2.11)$$

By Lemma 2.2, the function $h^{(\gamma)}$ is subadditive, $h^{(\gamma)}(0) = 0$ and

$$\lim_{x \to \pm\infty} \frac{h^{(\gamma)}(x)}{|x|} = \frac{1 \pm \gamma}{m^2}. \quad (2.12)$$

By subadditivity of $h^{(\gamma)}$ and by (2.12), we also have

$$h^{(\gamma)}(\pm x) \geq \frac{1 \pm \gamma}{m^2} x \geq 0, \quad \text{for all } x \geq 0. \quad (2.13)$$

For $-1 \leq \gamma \leq 1$, we define $\mathcal{H}^{(\gamma)} = \{x \in \mathbb{R}; h^{(\gamma)}(x) > 0\}$ and $\mathcal{H}_0^{(\gamma)} = \mathcal{H}^{(\gamma)} \cup \{0\}$. By recurrence of $X$, continuity and subadditivity of $h$ and (1.10), $\mathcal{H}_0^{(\gamma)}$ is either $\mathbb{R}$, $[0, \infty)$ or $(-\infty, 0]$. In addition, if $m^2 < \infty$ and $-1 < \gamma < 1$, then (2.13) implies that $\mathcal{H}_0^{(\gamma)} = \mathbb{R}$.

Then we can define the $h^{(\gamma)}$-transformed process given by (1.11).

3 The long-time behaviors

We prepare some important $P_x^{(\gamma)}$-martingale which is used for investigating the path behaviors of the process. Recall that we always assume the condition (A).

**Lemma 3.1.** Let $-1 \leq \gamma \leq 1$ and $x \in \mathcal{H}_0^{(\gamma)}$. Then $(\frac{1}{h^{(\gamma)}(X_t)}, t > 0)$ is a non-negative $P_x^{(\gamma)}$-supermartingale. Moreover, for $\gamma_1, \gamma_2 \in [-1, 1]$, the process $(\frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}, t > 0)$ is a non-negative $P_x^{(\gamma_1)}$-martingale, and its mean is $\frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)}$ if $x \in \mathcal{H}^{(\gamma_1)}$ and is 1 if $x = 0$.

**Proof.** Recall that we have $P_x^{(\gamma)}(T_{\mathbb{R}\setminus\mathcal{H}^{(\gamma)}} = \infty) = 1$ for $x \in \mathcal{H}_0^{(\gamma)}$; see just after (1.11). This implies $h^{(\gamma)}(X_t) \neq 0$, $P_x^{(\gamma)}$-a.s. We first assume $x \in \mathcal{H}^{(\gamma)}$. Let $0 < s < t$ and let $F_s$ be a
non-negative bounded \( \mathcal{F}_x \)-measurable functional. Then we have

\[
\mathbb{P}^{(\gamma)}_x \left[ \frac{1}{h^{(\gamma)}(X_t)} F_s \right] = \frac{1}{h^{(\gamma)}(x)} \mathbb{P}_x[F_s; T_0 > t] \leq \frac{1}{h^{(\gamma)}(x)} \mathbb{P}_x[F_s; T_0 > s] = \mathbb{P}^{(\gamma)}_x \left[ \frac{1}{h^{(\gamma)}(X_s)} F_s \right], \tag{3.1}
\]

which implies that \( \left( \frac{1}{h^{(\gamma)}(X_t)}; t > 0 \right) \) is a non-negative \( \mathbb{P}^{(\gamma)}_x \)-supermartingale. By Lemma 1.1 we have, for \( \gamma_1, \gamma_2 \in [-1, 1] \) and \( x \in \mathcal{H}^{(\gamma)} \),

\[
\mathbb{P}^{(\gamma_1)}_x \left[ \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} F_s \right] = \frac{1}{h^{(\gamma_1)}(x)} \mathbb{P}_x \left[ h^{(\gamma_2)}(X_t) F_s; T_0 > t \right] \tag{3.2}
\]

\[
= \frac{1}{h^{(\gamma_1)}(x)} \mathbb{P}_x \left[ h^{(\gamma_2)}(X_t) F_s; T_0 > s \right] \tag{3.3}
\]

\[
= \mathbb{P}^{(\gamma_1)}_x \left[ \frac{h^{(\gamma_2)}(X_s)}{h^{(\gamma_1)}(X_s)} F_s \right], \tag{3.4}
\]

which implies that \( \left( \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}; t > 0 \right) \) is a non-negative \( \mathbb{P}^{(\gamma_1)}_x \)-martingale with mean \( \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} \).

We next assume \( x = 0 \). Then we have

\[
\mathbb{P}^{(\gamma)}_0 \left[ \frac{1}{h^{(\gamma)}(X_t)} F_s \right] = n[F_s; T_0 > t] \leq n[F_s; T_0 > s] = \mathbb{P}^{(\gamma)}_0 \left[ \frac{1}{h^{(\gamma)}(X_s)} F_s \right], \tag{3.5}
\]

which implies that \( \left( \frac{1}{h^{(\gamma)}(X_t)}; t > 0 \right) \) is a non-negative \( \mathbb{P}^{(\gamma)}_0 \)-supermartingale. By Lemma 1.1 we have

\[
\mathbb{P}^{(\gamma_1)}_0 \left[ \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} F_s \right] = n \left[ h^{(\gamma_2)}(X_t) F_s; T_0 > t \right] = n \left[ h^{(\gamma_2)}(X_s) F_s; T_0 > s \right] = \mathbb{P}^{(\gamma_1)}_0 \left[ \frac{h^{(\gamma_2)}(X_s)}{h^{(\gamma_1)}(X_s)} F_s \right], \tag{3.6}
\]

which implies that \( \left( \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}; t > 0 \right) \) is a non-negative \( \mathbb{P}^{(\gamma_1)}_0 \)-martingale with mean 1. \( \square \)

\textbf{Proof of Theorem 1.4} We first prove Theorem 1.4 in the case \( x \in \mathcal{H}^{(\gamma)} \). Let \( F_s \) be a non-negative bounded \( \mathcal{F}_x \)-measurable functional. Then, since \( \left( \frac{1}{h^{(\gamma)}(X_t)}; t \geq 0 \right) \) is a non-negative \( \mathbb{P}^{(\gamma)}_x \)-martingale by Lemma 3.1, we see \( \lim_{t \to \infty} \frac{1}{h^{(\gamma)}(X_t)} \) exists and its limit is non-negative \( \mathbb{P}^{(\gamma)}_x \)-a.s. By Fatou’s lemma, we obtain

\[
\mathbb{P}^{(\gamma)}_x \left[ \lim_{t \to \infty} \frac{1}{h^{(\gamma)}(X_t)} \right] \leq \liminf_{t \to \infty} \mathbb{P}^{(\gamma)}_x \left[ \frac{1}{h^{(\gamma)}(X_t)} \right] = \frac{1}{h^{(\gamma)}(x)} \liminf_{t \to \infty} \mathbb{P}_x(T_0 > t) = 0, \tag{3.7}
\]

here the last equality follows from the fact that \( (X, \mathbb{P}_x) \) is recurrent. Hence it holds that \( \lim_{t \to \infty} \frac{1}{h^{(\gamma)}(X_t)} = 0, \mathbb{P}^{(\gamma)}_x \)-a.s. This implies \( \lim_{t \to \infty} |X_t| = \infty, \mathbb{P}^{(\gamma)}_x \)-a.s. The proof in the case \( x = 0 \) is similar. Hence we omit it. \( \square \)

\textbf{Proof of Theorem 1.5} We first consider the case \( x \in \mathcal{H}^{(\gamma)} \). Let \( F_t \) be a non-negative bounded \( \mathcal{F}_t \)-measurable functional. Then we have

\[
\mathbb{P}^{(\gamma)}_x[F_t] = \mathbb{P}_x \left[ \frac{h^{(\gamma)}(X_t)}{h^{(\gamma)}(x)} F_t; T_0 > t \right] = \frac{h(x)}{h^{(\gamma)}(x)} \mathbb{P}^{(\gamma)}_0 \left[ \frac{h^{(\gamma)}(X_t)}{h^{(\gamma)}(x)} F_t \right]. \tag{3.8}
\]
Since \( h(x) \geq |x|/m^2 \) and \( h^{(\gamma)}(x) = h(x) + \gamma x/m^2 \), it holds that
\[
1 - |\gamma| \leq \frac{h^{(\gamma)}(X_t)}{h(X_t)} \leq 1 + |\gamma|.
\] (3.9)

Thus, we have
\[
(1 - |\gamma|) \frac{h(x)}{h^{(\gamma)}(x)} \mathbb{P}^x_0[F_t] \leq \mathbb{P}^{\gamma}[F_t] \leq (1 + |\gamma|) \frac{h(x)}{h^{(\gamma)}(x)} \mathbb{P}^x_0[F_t].
\] (3.10)

By the extension theorem, it holds that
\[
(1 - |\gamma|) \frac{h(x)}{h^{(\gamma)}(x)} \mathbb{P}^0_0 \leq \mathbb{P}^{\gamma} \leq (1 + |\gamma|) \frac{h(x)}{h^{(\gamma)}(x)} \mathbb{P}^0_0, \quad \text{on } \mathcal{F}_\infty.
\] (3.11)

By the similar discussion, we also have
\[
(1 - |\gamma|) \mathbb{P}^0_0 \leq \mathbb{P}^{\gamma} \leq (1 + |\gamma|) \mathbb{P}^0_0, \quad \text{on } \mathcal{F}_\infty.
\] (3.12)

Therefore we obtain the desired result. \( \square \)

**Proof of Theorem 1.6.** We first consider the case \( x \in \mathcal{H}(\gamma) \). Let \( \gamma_1, \gamma_2 \in [-1, 1] \) be different constants. Since \( \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}, t > 0 \) is a non-negative \( \mathbb{P}^{(\gamma_1)}_x \)-martingale by Lemma 3.1, the limit \( \lim_{t \to \infty} \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} \) exists and is finite \( \mathbb{P}^{(\gamma_1)}_0 \)-a.s. By (v) of Lemma 2.2, we see
\[
\lim_{x \to \pm \infty} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} = \frac{1 \pm \gamma_2}{1 \pm \gamma_1} \in [0, \infty].
\] (3.13)

Hence the limits \( \lim_{x \to \infty} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} \) and \( \lim_{x \to -\infty} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} \) are different. Combining this and (1.18), we have \( \mathbb{P}^{(\gamma_1)}_x(\Omega_+^+ \cup \Omega_-^-) = 1 \). Since \( \lim_{t \to \infty} \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} \) is finite \( \mathbb{P}^{(\gamma_1)}_x \)-a.s., and \( \lim_{x \to \pm \infty} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} = \infty \) for \( \gamma_2 \neq \mp 1 \), we have \( \mathbb{P}^{(\gamma_1)}_x(\Omega_+^+) = \mathbb{P}^{(\gamma_1)}_x(\Omega_-^-) = 1 \). Suppose \(-1 < \gamma_1 < 1 \). Then, since \( \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} \leq \frac{1 + |\gamma_2|}{1 - |\gamma_1|}, \) we may apply the dominated convergence theorem to obtain
\[
\mathbb{P}^{(\gamma_1)}_x \left[ \lim_{t \to \infty} \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} \right] = \lim_{t \to \infty} \mathbb{P}^{(\gamma_1)}_x \left[ \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)} \right] = \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)}. \tag{3.14}
\]

By (3.13) and (3.14), we have
\[
\frac{1 + \gamma_2}{1 + \gamma_1} \mathbb{P}^{(\gamma_1)}_x(\Omega_+^+) + \frac{1 - \gamma_2}{1 - \gamma_1} \mathbb{P}^{(\gamma_1)}_x(\Omega_-^-) = \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)}. \tag{3.15}
\]

Since \( \mathbb{P}^{(\gamma_1)}_x(\Omega_+^+) + \mathbb{P}^{(\gamma_1)}_x(\Omega_-^-) = 1, \) (3.15) implies that
\[
\mathbb{P}^{(\gamma_1)}_x(\Omega_+^+) = \frac{1 + \gamma_1}{2} \frac{h^{(1)}(x)}{h^{(\gamma_1)}(x)} \quad \text{and} \quad \mathbb{P}^{(\gamma_1)}_x(\Omega_-^-) = \frac{1 - \gamma_1}{2} \frac{h^{(-1)}(x)}{h^{(\gamma_1)}(x)}. \tag{3.16}
\]

The proof in the case \( x = 0 \) is similar. So we omit it. \( \square \)
4 The short-time behaviors

First, we offer the proof of Theorem 1.7.

Proof of Theorem 1.7. We write $h^S(x) = h(x) + h(-x)$. Since we have

$$\text{Re} \Psi(\lambda) = \frac{1}{2} \sigma^2 \lambda^2 + \int_{\mathbb{R}} (1 - \cos \lambda x) \nu(dx) \geq 0, \quad (4.1)$$

it holds that $\text{Re}(\frac{1}{\Psi(\lambda)}) \geq 0$. In addition, it holds that $\lim_{x \to 0^+} x^2 \Psi(\lambda/x) = \sigma^2 \lambda^2 / 2$. By (iii) of Lemma 2.2, we have

$$\frac{h^S(x)}{x} = \frac{2}{\pi x} \int_0^\infty \text{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda = \frac{2}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - \cos \xi}{x^2 \Psi(\xi/x)} \right) d\xi. \quad (4.2)$$

We first assume $\sigma^2 = 0$. By Fatou’s lemma, we obtain

$$\liminf_{x \to 0^+} \frac{h^S(x)}{x} = \liminf_{x \to 0^+} \frac{2}{\pi} \int_0^\infty \text{Re} \left( \frac{1 - \cos \xi}{x^2 \Psi(\xi/x)} \right) d\xi \geq \frac{2}{\pi} \int_0^\infty \liminf_{x \to 0^+} \text{Re} \left( \frac{1 - \cos \xi}{x^2 \Psi(\xi/x)} \right) d\xi = \infty, \quad (4.4)$$

which implies (1.25).

We next assume $\sigma^2 > 0$. Since $|x^2 \Psi(\xi/x)| \geq |\text{Re}(x^2 \Psi(\xi/x))| \geq \sigma^2 \xi^2 / 2$, we have

$$\left| \text{Re} \left( \frac{1 - \cos \xi}{x^2 \Psi(\xi/x)} \right) \right| \leq \frac{1 - \cos \xi}{x^2 \Psi(\xi/x)} \leq \frac{2(1 \wedge \xi^2)}{\sigma^2 \xi^2}, \quad (4.6)$$

which is integrable in $\xi > 0$. Hence we may apply the dominated convergence theorem to obtain

$$\lim_{x \to 0^+} \frac{h^S(x)}{x} = \frac{4}{\pi \sigma^2} \int_0^\infty \frac{1 - \cos \xi}{\xi^2} d\xi = \frac{2}{\sigma^2}, \quad (4.7)$$

which implies (1.25).

Proof of Theorem 1.8. Let $\gamma_1, \gamma_2 \in [-1, 1]$ be different constants. Then, since $(\frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}, t > 0)$ is a non-negative $\mathbb{P}_0^{(\gamma_1)}$-martingale by Lemma 3.1, the limit $\lim_{t \to 0^+} \frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}$ exists $\mathbb{P}_0^{(\gamma_1)}$-a.s. for all $t > 0$. We have

$$\lim_{x \to 0^\pm} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)} = \frac{|h^{(\gamma_2)}(0^\pm)|}{|h^{(\gamma_1)}(0^\pm)|} = \frac{|h'(0^\pm)| \pm \gamma_2/m^2}{|h'(0^\pm)| \pm \gamma_1/m^2} \in [0, \infty]. \quad (4.8)$$

Consequently, the limits $\lim_{x \to 0^+} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)}$ and $\lim_{x \to 0^-} \frac{h^{(\gamma_2)}(x)}{h^{(\gamma_1)}(x)}$ are different, which yields that $\mathbb{P}_0^{(\gamma)}(\Omega_0^{+, -}) = 0$. Since the $\mathbb{P}_0^{(\gamma_1)}$-martingale $(\frac{h^{(\gamma_2)}(X_t)}{h^{(\gamma_1)}(X_t)}, t > 0)$ has mean 1, it holds that

$$\frac{h^{(\gamma_2)}(0^+)}{h^{(\gamma_1)}(0^+)} \mathbb{P}_0^{(\gamma_1)}(\Omega_0^+) + \frac{h^{(\gamma_2)}(0^-)}{h^{(\gamma_1)}(0^-)} \mathbb{P}_0^{(\gamma_1)}(\Omega_0^-) = 1. \quad (4.9)$$
Since \( \mathbb{P}_0^{(\gamma_1)}(\Omega_0^+) + \mathbb{P}_0^{(\gamma_1)}(\Omega_0^-) = 1 \) and by Theorem 1.7, we obtain
\[
\mathbb{P}_0^{(\gamma_1)}(\Omega_0^+) = \frac{\sigma^2}{2} h_{\gamma_1}'(0+) \quad \text{and} \quad \mathbb{P}_0^{(\gamma_1)}(\Omega_0^-) = \frac{\sigma^2}{2} |h_{\gamma_1}'(0^-)|. \tag{4.10}
\]
Hence the proof is complete. \( \square \)

**Proof of Theorem 1.9.** Assume \( h'(0+) = \infty \) and \( |h'(0-)| < \infty \). Let \( \gamma_1, \gamma_2 \in [-1, 1] \) be different constants. By the same discussion as the proof of Theorem 1.8, we obtain (4.9).

By the assumption, we also have \( \frac{h_{\gamma_2}'(0+)}{h_{\gamma_1}'(0+)} = 1 \) for all \(-1 < \gamma_2 < 1\). Thus we have
\[
\mathbb{P}_0^{(\gamma_1)}(\Omega_0^+) = 1 \quad \text{and} \quad \mathbb{P}_0^{(\gamma_1)}(\Omega_0^-) = 0. \tag{4.11}
\]
The proof in the case \( h'(0+) < \infty \) and \( |h'(0-)| = \infty \) is similar. So we omit it. \( \square \)

**Proof of Theorem 1.10.** Ikeda–Watanabe [8, Theorem 3.3] proved that
\[
\mathbb{P}_x(\Omega_1^{+, -}|T_0 < \infty) = 1, \quad x \in \mathbb{R} \setminus \{0\}, \tag{4.12}
\]
where
\[
\Omega_1^{+, -} := \{ \exists \{t_n\} \text{ with } t_n \to T_0^- \text{ such that } \forall n, X_{t_n}X_{t_{n+1}} < 0 \}. \tag{4.13}
\]
This implies that
\[
n((\Omega_1^{+, -})^c \cap \{T_0 < \infty\}) = 0. \tag{4.14}
\]
By time reversal property of excursion paths (see [5, Lemma 5.2]), it holds that
\[
n((\Omega_0^{+, -})^c \cap \{T_0 < \infty\}) = 0. \tag{4.15}
\]
Since \((X, \mathbb{P})\) is recurrent, it holds that \( n(\{T_0 < \infty\}) = 0 \). Thus we have \( n((\Omega_0^{+, -})^c) = 0 \), which implies \( \mathbb{P}_0^{(\gamma)}(\Omega_0^{+, -}) = 1 \). \( \square \)

5 **Appendix A: Resolvent density under \( \mathbb{P}_x^{(\gamma)} \)**

We calculate the resolvent density under \( \mathbb{P}_x^{(\gamma)} \) and show some Feller property. Recall that we always assume the assumption (A).

Let \( p_t(dx) \) denote the transition law of \( X_t \) under \( \mathbb{P} \) and let \( p_t^0(x, dy) \) denote the transition law of \( X_t \) under \( \mathbb{P}_0 \). By the Markov property, we have, for \( x, y \in \mathbb{R} \setminus \{0\}, \)
\[
p_t^0(x, dy) = \mathbb{P}_x(dy - x) - \int_{[0,t]} \mathbb{P}_x(T_0 \in ds) p_{t-s}(dy). \tag{5.1}
\]
For \( t, q > 0 \) and \( x, y \in \mathbb{R} \setminus \{0\} \), we denote the \( q \)-resolvent for killed process by
\[
r_q^0(x, y) = \int_0^\infty e^{-qt} p_t^0(x, dy) \, dt / dy \tag{5.2}
\]
\[
= r_q(y - x) - \frac{r_q(-x) r_q(y)}{r_q(0)}, \tag{5.3}
\]
\[
= h_q(x) + h_q(-y) - h_q(x - y) - \frac{h_q(x) h_q(-y)}{r_q(0)}. \tag{5.4}
\]
Note that the second identity follows from (5.1) and (2.9). This implies that the killed process \( (X, \mathbb{P}_x^0) \) has the continuous \( q \)-resolvent density.

Let \(-1 \leq \gamma \leq 1\). For \( x \in \mathcal{H}_0^{(\gamma)} \) and \( y \in \mathcal{H}^{(\gamma)} \), we denote the transition law of \( X_t \) under \( \mathbb{P}_x^{(\gamma)} \) by

\[
p_t^{(\gamma)}(x, dy) = \begin{cases} 
\frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} p_t^{0}(x, dy) & \text{if } x \in \mathcal{H}^{(\gamma)}, \\
\frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} n(X_t \in dy) & \text{if } x = 0.
\end{cases}
\]  

(5.5)

Then the \( q \)-resolvent density \( r_q^{(\gamma)}(x, y) \) of \( (X, \mathbb{P}_x^{(\gamma)}) \) can be expressed as follows: if \( x \in \mathcal{H}^{(\gamma)} \),

\[
r_q^{(\gamma)}(x, y) = \int_0^\infty e^{-q t} p_t^{(\gamma)}(x, dy) \, dt/\, dy
\]

(5.6)

\[
= \frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} \left( h_q(x) + h_q(y) - h_q(x - y) - \frac{h_q(x)h_q(-y)}{r_q(0)} \right),
\]

(5.7)

and

\[
r_q^{(\gamma)}(0, y) = \int_0^\infty e^{-q t} p_t^{(\gamma)}(0, dy) \, dt/\, dy = \frac{h^{(\gamma)}(y)r_q(y)}{r_q(0)} = h^{(\gamma)}(y) \left( 1 - \frac{h_q(-y)}{r_q(0)} \right),
\]

(5.8)

where the second identity follows from the formula: for any non-negative measurable function \( f \), it holds that

\[
\int_0^\infty e^{-q t} n[f(X_t)] \, dt = \int_\mathbb{R} f(x) \hat{\mathbb{P}}_x \left[ e^{-q T_0} \right] \, dx,
\]

(5.9)

where \( \mathbb{P}_x \) and \( \hat{\mathbb{P}}_x \) are in weak duality, i.e., the probability measure \( \hat{\mathbb{P}}_x \) denotes the law of \( (-X_t, t \geq 0) \) under \( \mathbb{P}_x \). For more details, see Chen–Fukushima–Ying [5] and Fitzsimmons–Getoor [6]. See also Yano–Yano–Yor [21, Theorem 3.3].

Summarizing the above computations, we obtain the following results:

**Proposition 5.1.** Let \( q > 0 \) and \(-1 \leq \gamma \leq 1\). Let \( r_q^{(\gamma)}(x, y) \) denote the \( q \)-resolvent density of \( (X, \mathbb{P}_x^{(\gamma)}) \). Then, for \( y \in \mathcal{H}^{(\gamma)} \), it holds that

\[
r_q^{(\gamma)}(x, y) = \begin{cases} 
\frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} \left( h_q(x) + h_q(y) - h_q(x - y) - \frac{h_q(x)h_q(-y)}{r_q(0)} \right) & \text{if } x \in \mathcal{H}^{(\gamma)}, \\
\frac{h^{(\gamma)}(y)}{r_q(0)} & \text{if } x = 0.
\end{cases}
\]

(5.10)

Letting \( q \to 0^+ \) in (5.10), we obtain the zero resolvent

\[
r_0^{(\gamma)}(x, y) := \lim_{q \to 0^+} r_q^{(\gamma)}(x, y).
\]

(5.11)

Note that it holds that \( \lim_{q \to 0^+} \frac{1}{r_q(0)} = 0 \) if \( (X, \mathbb{P}) \) is recurrent; see, e.g., [11, Theorem I.17] and [13, Theorem 37.5]. (Recall that we always assume \( (X, \mathbb{P}) \) is recurrent.)

**Corollary 5.2.** For \( y \in \mathcal{H}^{(\gamma)} \), it holds that

\[
r_0^{(\gamma)}(x, y) = \begin{cases} 
\frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} (h(x) + h(y) - h(x - y)) & \text{if } x \in \mathcal{H}^{(\gamma)}, \\
\frac{h^{(\gamma)}(y)}{r_q(0)} & \text{if } x = 0.
\end{cases}
\]

(5.12)
Set
\[ T_t^{(\gamma)} f(x) = \mathbb{P}^{(\gamma)}_x [f(X_t)], \quad t \geq 0, \quad f \in \mathcal{B}_{+,b}(\mathcal{H}_0^{(\gamma)}), \] (5.13)
where \( \mathcal{B}_{+,b}(\mathcal{H}_0^{(\gamma)}) \) denotes the set of non-negative bounded measurable functions. Then the family \( \{T_t^{(\gamma)}, t \geq 0\} \) forms a transition semigroup. We define the resolvent operator of the semigroup \( T_t^{(\gamma)} \) as
\[ R_q^{(\gamma)} f(x) = \int_0^\infty e^{-qt} T_t^{(\gamma)} f(x) \, dt, \quad q > 0, \quad f \in \mathcal{B}_{+,b}(\mathcal{H}_0^{(\gamma)}). \] (5.14)
For Theorem 5.3 we are inspired by Yano [13, Theorem 1.5].

**Theorem 5.3.** Assume the conditions (1)–(3) of Theorem 1.4 hold. (Note that (1) of Theorem 1.4 and subadditivity of \( h^{(\gamma)} \) imply that \( \mathcal{H}_0^{(\gamma)} = \mathbb{R} \)). Then the semigroup \( (T_t^{(\gamma)})_{t \geq 0} \) enjoys Feller property, i.e.,

1. \( T_t^{(\gamma)} C_0(\mathbb{R}) \subset C_0(\mathbb{R}) \);
2. \( \|T_t^{(\gamma)} f - f\| \to 0 \) as \( t \to 0^+ \) for all \( f \in C_0(\mathbb{R}) \),

where \( C_0(\mathbb{R}) \) stands for the class of continuous functions vanishing at infinity.

**Proof.** Note that the condition (2) implies that
\[ \lim_{x \to 0} \frac{h^{(\gamma)}(x)}{|x|} = \infty \quad \text{and} \quad \lim_{x \to 0} \frac{h^{(\gamma)}(x)}{h(x)} = 1, \] (5.15)
for \(-1 \leq \gamma \leq 1\). To show Feller property, it is sufficient to show that

1. \( T_t^{(\gamma)} f(x) \to f(x) \) as \( t \to 0^+ \) for all \( x \in \mathbb{R}, \ f \in C_0(\mathbb{R}) \),
2. \( R_q^{(\gamma)} C_0(\mathbb{R}) \subset C_0(\mathbb{R}) \).

For more details see [12, Proposition III.2.4]. Since \( T_t^{(\gamma)} f(x) = \mathbb{P}^{(\gamma)}_x [f(X_t)] \) and since \( \mathbb{P}^{(\gamma)}_x \) is a probability measure on the càdlàg space, (F3) is obvious. We proceed the proof of (F4). Let \( C_c(\mathbb{R}) \) stand for the set of continuous functions with compact support on \( \mathbb{R} \). Since \( \|q R_q^{(\gamma)} f\| \leq \|f\| \) and since the closure of \( C_c(\mathbb{R}) \) is \( C_0(\mathbb{R}) \), it is sufficient to show that \( R_q^{(\gamma)} C_c(\mathbb{R}) \subset C_0(\mathbb{R}) \). Recall that, for \( x \in \mathbb{R} \),
\[ R_q^{(\gamma)} f(x) = \int_{\mathcal{H}^{(\gamma)}} f(y) r_q^{(\gamma)}(x, y) \, dy, \quad f \in C_c(\mathbb{R}). \] (5.16)
Since \( f \) has compact support and is continuous, and \( r_q^{(\gamma)} \) is continuous in \( (x, y) \in \mathcal{H}^{(\gamma)} \times \mathcal{H}^{(\gamma)} \), the function \( R_q^{(\gamma)} f(x) \) is continuous in \( x \in \mathcal{H}^{(\gamma)} \).

Let the set \( A \subset \mathbb{R} \) stand for the support of \( f \). Since \( r_q^{(\gamma)}(x, y) = \frac{h^{(\gamma)}(y)}{h^{(\gamma)}(x)} q_0(x, y) \), it holds that
\[ R_q^{(\gamma)} f(x) = \frac{1}{h^{(\gamma)}(x)} \int_{\mathcal{H}^{(\gamma)}} f(y) h^{(\gamma)}(y) r_q^{(\gamma)}(x, y) \, dy \] (5.17)
\[ \leq \sup_{y \in A} h^{(\gamma)}(y) \frac{1}{q h^{(\gamma)}(x)} \int_{\mathcal{H}^{(\gamma)}} f(y) r_q^{(\gamma)}(x, y) \, dy \] (5.18)
\[ \leq \sup_{y \in A} h^{(\gamma)}(y) \frac{\|f\|}{q h^{(\gamma)}(x)} \mathbb{P}^{(\gamma)}_x [\mathbb{e}^{-q T_A}] \] (5.19)
\[ \to 0 \quad \text{as} \quad x \to \pm \infty. \] (5.20)
Here we used the assumption (i). Hence $R_q^{\gamma}f(x)$ vanishes at infinity.

We have to prove $R_q^{\gamma}f(x)$ is continuous at $x = 0$. By Proposition 5.1 and assumptions (ii) and (iii), we have $r_q(x, y) \to r_q(0, y)$ as $x \to 0$. Moreover, since $h_q$ is subadditive, it holds that, for $x, y \in H^{(\gamma)}$, 
\[ r_q^{(\gamma)}(x, y) \leq h^{(\gamma)}(y) \frac{h_q(x) + h_q(-x)}{h^{(\gamma)}(x)}. \] (5.21)

The conditions (iii) and (iv) implies that the right hand side of (5.21) is bounded near $x = 0$ and $y \in A$. Thus we may apply the dominated convergence theorem to deduce that $R_q^{\gamma}f(x)$ is also continuous at $x = 0$. Hence $(T_t^{\gamma})$ has Feller property.

6 Appendix B: Proof of Theorem 1.3

Recall that we always assume the assumption (A).

Lemma 6.1. For any $t > 0$, it holds that $n[h(-X_t)] < \infty$.

Recall that $n[h(X_t)] = 1$ for all $t > 0$; see Lemma 1.1.

Proof of Lemma 6.1. We write $\hat{h}(x) = h(-x)$. By the formula (5.9) and by (2.9), we have
\[ \int_0^\infty e^{-qt}n[\hat{h}(X_t)] \, dt = \int_\mathbb{R} \hat{h}(x) \hat{P}_x[e^{-qT_0}] \, dx \] (6.1)
\[ = \int_\mathbb{R} \hat{h}(x) r_q(x) \frac{r_q(0)}{r_q(0)} \, dx. \] (6.2)

By (11, (3.20)), the equation (6.2) is finite. (Note that the assumptions in (11) is stronger, but this remains true since its proof is valid if Lemma 2.2 holds.) Hence, for almost any $t > 0$, it holds that $n[\hat{h}(X_t)] < \infty$. Thus for any $t > 0$, there exists $0 < s < t$ such that $n[\hat{h}(X_s)] < \infty$. By the Markov property of the excursion measure $n$, we have
\[ n[\hat{h}(X_t)] = n[\mathbb{P}_X[\hat{h}(X_{t-s}); T_0 > t-s]] \] (6.3)
\[ \leq n[\mathbb{P}_X[\hat{h}(X_{t-s})]] \] (6.4)
\[ = n[\bar{\mathbb{P}}_0[\hat{h}(X_s + \hat{X}_{t-s})]], \] (6.5)
where the symbol $\sim$ means independence. Since $\hat{h}$ is subadditive, we obtain
\[ n[\hat{h}(X_t)] \leq n[\hat{h}(X_s)] + \mathbb{P}_0[\hat{h}(X_{t-s})]. \] (6.6)

By Tsukada [15, Proof of Theorem 15.2] (see also Takeda–Yano [14, Lemma 4.3]), we have $\mathbb{P}_0[\hat{h}(X_{t-s})] < \infty$. Consequently, it holds that $n[\hat{h}(X_t)] < \infty$ for all $t > 0$.

For the proof of Theorem 1.3 we introduce the following lemma, whose proof is in [14, Lemmas 3.4 and 6.2].
Lemma 6.2 ([14]). For \( a, b \in \mathbb{R} \setminus \{0\} \) and \( a \neq b \), it holds that

\[
h^B(a) := \mathbb{P}_0[L_{T_a}] = h(a) + h(-a),
\]
(6.7)

\[
h^B(a) \mathbb{P}_x(T_a < T_0) = h(x) + h(-a) - h(x - a),
\]
(6.8)

\[
\mathbb{P}_0[L_{T_{(a,-b)}}] \mathbb{P}_x(T_{(a,-b)} < T_0)
\]
\[
= h(x) + \frac{1}{h^B(a + b)} \left( (h(-a) - h(x - a))h(a + b) + (h(b) - h(x + b))h(-a - b) \right)
\]
\[
- \left( (h(a) - h(0)) \left( (h(-a) - h(x - a)) + (h(b) + h(x + b)) \right) \right). \quad (6.10)
\]

Proof of Theorem 1.3. For \( s > 0 \), we define \( d_s = \inf \{ u > s : X_u = 0 \} \). We also define \( G = \{ g_s : g_s \neq d_s, s > 0 \} \).

\( \square \) For any \( q > 0 \), we have

\[
\mathbb{P}_0[F_t \circ k_{e_x - g_{e_q}} \circ \theta_{g_{e_q}}] = \mathbb{P}_0\left[ \int_0^{\infty} q e^{-qu} F_t \circ k_{u - g_{e_q}} \circ \theta_{g_{e_q}} \, du \right]
\]
(6.11)

\[
= \mathbb{P}_0\left[ \sum_{L \in G} e^{-qL} \int_0^{d_s} q e^{-q(u-s)} F_t \circ k_{u - s} \circ \theta_{s} \, du \right].
\]
(6.12)

Using the compensation formula in excursion theory (see e.g., Bertoin [1, Corollary IV.11]), we obtain

\[
\mathbb{P}_0[F_t \circ k_{e_x - g_{e_q}} \circ \theta_{g_{e_q}}] = \mathbb{P}_0\left[ \int_0^{\infty} e^{-qs} \, dL_s \right] n \left[ \int_0^{T_0} q e^{-qu} F_t 1_{\{u > t\}} \, du \right].
\]
(6.13)

By (2.7) and the Markov property of the excursion measure \( n \), it holds that

\[
\mathbb{P}_0[F_t \circ k_{e_x - g_{e_q}} \circ \theta_{g_{e_q}}] = r_q(0)n[F_t; t < e_q < T_0]
\]
(6.14)

\[
= r_q(0)e^{-qt}n[F_t\mathbb{P}_X[T_0 > e_q]]. \quad (6.15)
\]

It follows from (2.9) that

\[
\mathbb{P}_{X_t}[T_0 > e_q] = 1 - \mathbb{P}_{X_t}[e^{-QT_0}] = 1 - \frac{r_q(-X_t)}{r_q(0)} = \frac{h_q(X_t)}{r_q(0)}. \quad (6.16)
\]

Hence we have

\[
\mathbb{P}_0[F_t \circ k_{e_x - g_{e_q}} \circ \theta_{g_{e_q}}] = e^{-qt}n[F_t h_q(X_t)]. \quad (6.17)
\]

By (2.8), (4.1) and (iii) of Lemma 2.2, it holds that

\[
h_q(X_t) \leq h_q(X_t) + h_q(-X_t) = \frac{2}{\pi} \int_0^{\infty} \text{Re} \left( \frac{1 - \cos \lambda x}{\lambda + \Psi(\lambda)} \right) d\lambda
\]
\[
\leq \frac{2}{\pi} \int_0^{\infty} \text{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda = h(X_t) + h(-X_t).
\]
(6.18)

(6.19)

By Lemmas 1.1 and 6.1, the function \( h(X_t) + h(-X_t) \) is integrable with respect to the measure \( n \). Thus we may apply the dominated convergence theorem to deduce

\[
\lim_{q \to 0^+} \mathbb{P}_0[F_t \circ k_{e_x - g_{e_q}} \circ \theta_{g_{e_q}}] = \lim_{q \to 0^+} e^{-qt}n[F_t h_q(X_t)] = n[F_t h(X_t)] = \mathbb{P}_0^{(0)}[F_t]. \quad (6.20)
\]

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For $a \in \mathbb{R} \setminus \{0\}$, we have

$$
P_0[F_t \circ k_{T_a-g_{T_a}} \circ \theta_{g_{T_a}}] = \mathbb{P}_0 \left[ \sum_{s \in G} 1_{\{s < T_a < d_a\}} F_t \circ k_{T_a-s} \circ \theta_s \right].$$

(6.21)

Using the compensation formula in excursion theory, the Markov property of the excursion measure $n$ and Lemma 6.2, it holds that

$$
P_0[F_t \circ k_{T_a-g_{T_a}} \circ \theta_{g_{T_a}}] = \mathbb{P}_0 \left[ \int_{T_a}^{T_0} dL_s \right] n[F_t; t < T_a < T_0]
= \mathbb{P}_0[L_{T_a}] n[F_t \mathbb{P}_{X_t}(T_a < T_0); t < T_a]
= n[F_t(h(-a) + h(X_t) - h(X_t - a)); t < T_a].$$

(6.22)

Since $h$ is subadditive, we have

$$
h(-a) + h(X_t) - h(X_t - a) \leq h(X_t) + h(-X_t),$$

(6.25)

which is integrable with respect to the measure $n$. Thus we may apply the dominated convergence theorem to deduce

$$
\lim_{a \to \pm \infty} \mathbb{P}_0[F_t \circ k_{T_a-g_{T_a}} \circ \theta_{g_{T_a}}] = n[F_t h^{(\pm)}(X_t)] = \mathbb{P}_0^{(\pm)}[F_t],
$$

(6.26)

here we used (vi) of Lemma 2.2.

(iii) By the same discussion as the proof of (ii), it holds that

$$
P_0[F_t \circ k_{T(a,-b)-g_{T(a,-b)}} \circ \theta_{g_{T(a,-b)}}] = \mathbb{P}_0[L_{T(a,-b)}] n[F_t \mathbb{P}_{X_t}(T_{(a,-b)} < T_0); t < T_{(a,-b)}].$$

(6.27)

By Lemma 6.2 and by the dominated convergence theorem, we obtain the desired result.

(We omit the details.)

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Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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