On the Diophantine equation $F_n - F_m = 2^a$

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Abstract

In this paper, we solve Diophantine equation in the title in nonnegative integers $m$, $n$, and $a$. In order to prove our result, we use lower bounds for linear forms in logarithms and a version of the Baker-Davenport reduction method in diophantine approximation.

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1 Introduction

Fibonacci sequence ($F_n$) is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence ($L_n$), which is similar to the Fibonacci sequence, is defined by the same recursive pattern with initial conditions $L_0 = 2$, $L_1 = 1$. The terms of the Fibonacci and Lucas sequences are called Fibonacci and Lucas numbers, respectively. The Fibonacci and Lucas numbers for negative indices are defined by $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^nL_n$ for $n \geq 1$. The Fibonacci and Lucas sequences have many interesting properties and have been studied in the literature by many researchers. A brief history of Fibonacci and Lucas sequences one can consult reference [10]. Firstly, square terms and later perfect powers in the Fibonacci and Lucas sequences have attracted the attention of the researchers. The perfect power in the Fibonacci and Lucas sequences has been determined in 2006 by Bugeaud, Mignotte and Siksek in [3] (see Theorem 3 below). The Diophantine equation $L_n + L_m = 2^a$ has been tackled in [11] by Bravo and Luca. Two years later, the same authors solved Diophantine equation $F_n + F_m = 2^a$ in [12]. Besides, Luca and Patel, in [9], found that the Diophantine equation $F_n - F_m = y^p$ in integers $(n, m, y, p)$ with $p \geq 2$ has solution either $\max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$ if $n \equiv m(\text{mod } 2)$. 
But, it is still an open problem for the case \( n \neq m (\text{mod} 2) \). Motivated by the studies of Bravo and Luca, in this paper, we consider the Diophantine equation

\[ F_n - F_m = 2^a \]  

in nonnegative integers \( m, n, \) and \( a \). We follow the approach and the method presented in [12]. In section 2, we introduce necessary lemmas and theorems. Then in section 3, we prove our main theorem.

2 Auxiliary results

Lately, in many articles, to solve Diophantine equations such as the equation (1), authors have used Baker’s theory lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving of Diophantine equations, we start with recalling some basic notions from algebraic number theory.

Let \( \eta \) be an algebraic number of degree \( d \) with minimal polynomial

\[ a_0 x^d + a_1 x^{d-1} + ... + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[x], \]

where the \( a_i \)'s are relatively prime integers with \( a_0 > 0 \) and \( \eta^{(i)} \)'s are conjugates of \( \eta \). Then

\[ h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log \left( \max \left\{ |\eta^{(i)}|, 1 \right\} \right) \right) \]  

(2)

is called the logarithmic height of \( \eta \). In particularly, if \( \eta = a/b \) is a rational number with \( \gcd(a, b) = 1 \) and \( b > 1 \), then \( h(\eta) = \log \left( \max \{ |a|, b \} \right) \).

The following properties of logarithmic height are found in many works stated in references:

\[ h(\eta + \gamma) \leq h(\eta) + h(\gamma) + \log 2, \]  

(3)

\[ h(\eta \gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \]  

(4)

\[ h(\eta^s) \leq |s|h(\eta). \]  

(5)

The following theorem is deduced from Corollary 2.3 of Matveev [4], provides a large upper bound for the subscript \( n \) in the equation (1) (also see Theorem 9.4 in [3]).

**Theorem 1** Assume that \( \gamma_1, \gamma_2, \ldots, \gamma_t \) are positive real algebraic numbers in a real algebraic number field \( \mathbb{K} \) of degree \( D \), \( b_1, b_2, \ldots, b_t \) are rational integers, and

\[ \Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \]
is not zero. Then
\[ |\Lambda| > \exp \left( -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 A_2 \ldots A_t \right), \]
where
\[ B \geq \max \{|b_1|, \ldots, |b_t|\}, \]
and \( A_i \geq \max \{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \) for all \( i = 1, \ldots, t. \)

The following lemma, was proved by Dujella and Pethö [6], is a variation of a lemma of Baker and Davenport [5]. And this lemma will be used to reduce the upper bound for the subscript \( n \) in the equation (1). In the following lemma, the function \(|\cdot||\) denotes the distance from \( x \) to the nearest integer, that is, \(||x|| = \min \{|x-n| : n \in \mathbb{Z} \}\) for a real number \( x. \)

**Lemma 2** Let \( M \) be a positive integer, let \( p/q \) be a convergent of the continued fraction of the irrational number \( \gamma \) such that \( q > 6M, \) and let \( A, B, \mu \) be some real numbers with \( A > 0 \) and \( B > 1. \) Let \( \epsilon := ||\mu q|| - M||\gamma q||. \) If \( \epsilon > 0, \) then there exists no solution to the inequality
\[ 0 < |u \gamma - v + \mu| < AB^{-w}, \]
in positive integers \( u, v, \) and \( w \) with
\[ u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}. \]

It is well known that
\[ F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n, \quad \quad (6) \]
where \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2}, \) which are the roots of the characteristic equations \( x^2 - x - 1 = 0. \) The relations between Fibonacci and Lucas number, and \( \alpha \) are given by
\[ F_{n+1} + F_{n-1} = L_n \quad \quad (7) \]
and
\[ \alpha^{n-2} \leq F_n \leq \alpha^{n-1}. \quad \quad \quad (8) \]
for \( n \geq 1. \) The inequality (8) can be proved by induction. It can be seen that \( 1 < \alpha < 2 \) and \( -1 < \beta < 0. \)

The following theorem and lemma are given in [3] and [9], respectively.

**Theorem 3** The only perfect powers in the Fibonacci sequence are \( F_0 = 0, F_1 = F_2 = 1, F_6 = 8, \) and \( F_{12} = 144. \) The only perfect powers in the Lucas sequence are \( L_1 = 1 \) and \( L_3 = 4. \)

**Lemma 4** Assume that \( n \equiv m (\mod 2). \) Then
\[ F_n - F_m = \begin{cases} \frac{F_{(n-m)/2} L_{(n+m)/2}}{L_{(n-m)/2}} & \text{if } n \equiv m (\mod 4), \\ F_{(n+m)/2} L_{(n-m)/2} & \text{if } n \equiv m + 2 (\mod 4). \end{cases} \]
3 Main theorem

Theorem 5 The only solutions of the Diophantine equation (1) in nonnegative integers \(m < n\), and \(a\), are given by

\[(n, m, a) \in \{(1, 0, 0), (2, 0, 0), (3, 0, 1), (6, 0, 3), (3, 1, 0), (4, 1, 1), (5, 1, 2), (3, 2, 0)\}\]
and

\[(n, m, a) \in \{(4, 3, 0), (4, 2, 1), (5, 2, 2), (9, 3, 5), (5, 4, 1), (7, 5, 3), (8, 5, 4), (8, 7, 3)\}\].

Proof. Assume that the equation (1) holds. With the help of Mathematica program, we obtain the solutions in Theorem 5 for \(1 \leq m < n \leq 200\). This takes a little time. From now on, assume that \(n > 200\) and \(n - m \geq 3\). Now, let us show that \(a < n\). Using (3), we get the inequality

\[2^a = F_n - F_m < F_n < \alpha^n < 2^n,\]
that is, \(a < n\).

On the other hand, rearranging the equation (1) as \(\frac{\alpha^n}{\sqrt{5}} - 2^a = -F_m - \frac{\beta^n}{\sqrt{5}}\)
and taking absolute values, we obtain

\[\left|\frac{\alpha^n}{\sqrt{5}} - 2^a\right| = \left|F_m + \frac{\beta^n}{\sqrt{5}}\right| \leq F_m + \left|\frac{\beta^n}{\sqrt{5}}\right| < \alpha^m + \frac{1}{2}\]
by (8). If we divide both sides of the above inequality by \(\frac{\alpha^n}{\sqrt{5}}\), we get

\[\left|1 - 2^a \alpha^{-n} \sqrt{5}\right| < \frac{4}{\alpha^n - m},\]  \hspace{1cm} (9)
where we have used the facts that \(\alpha^{-m} < 1\) and \(n > m\). Now, let us apply Theorem 1 with \(\gamma_1 := 2\), \(\gamma_2 := \alpha\), \(\gamma_3 := \sqrt{5}\) and \(b_1 := a\), \(b_2 := -n\), \(b_3 := 1\). Note that the numbers \(\gamma_1\), \(\gamma_2\), and \(\gamma_3\) are positive real numbers and elements of the field \(K = \mathbb{Q}(\sqrt{5})\), so \(D = 2\). It can be shown that the number \(\Lambda_1 := 2^n \alpha^{-n} \sqrt{5} - 1\) is nonzero. For, if \(\Lambda_1 = 0\), then we get

\[2^a = \frac{\alpha^n}{\sqrt{5}} = F_n + \frac{\beta^n}{\sqrt{5}} > F_n - 1 > F_n - F_m = 2^a,\]
which is impossible. Moreover, since \(h(\gamma_1) = \log 2 = 0.6931...\), \(h(\gamma_2) = \frac{\log \alpha}{2} = 0.4812...\), and \(h(\gamma_3) = \log \sqrt{5} = 0.8047...\) by (2), we can take \(A_1 := 1.4\), \(A_2 := 0.5\), and \(A_3 := 1.7\). Also, since \(a < n\), it follows that \(B := \max \{|a|, |n|\} = n\). Thus, taking into account the inequality (9) and using Theorem 1 we obtain

\[\frac{4}{\alpha^n - m} > |\Lambda_1| > \exp (-1.4 \cdot 30^6 \cdot 3^{1.5} \cdot 2^2(1 + \log 2)(1 + \log n)(1.4)(0.5)(1.7))\]
and so

\[(n - m) \log \alpha - \log 4 < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n) (1.4) (0.5) (1.7)\]

From the last inequality, a quick computation using Mathematica yields to

\[(n - m) \log \alpha < 2.4 \cdot 10^2 \log n. \quad (10)\]

Now, we try to apply Theorem 1 a second time. Rearranging the equation (1) as

\[\alpha \sqrt{5} = \frac{\beta^n}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}}\]

and taking absolute values in here, we obtain

\[\left| \frac{\alpha n(1 - \alpha^{m-n})}{\sqrt{5}} - 2^a \right| = \left| \frac{\beta^n}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}} \right| \leq \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} < \frac{1}{3},\]

where we used the fact that \(|\beta|^n + |\beta|^m < 2/3\) for \(n > 200\). Dividing both sides of the above inequality by \(\frac{\alpha n(1 - \alpha^{m-n})}{\sqrt{5}}\), we get

\[\left| 1 - 2^a \alpha^{n} \sqrt{5}(1 - \alpha^{m-n})^{-1} \right| < \frac{\sqrt{5} \alpha^{-n}(1 - \alpha^{m-n})^{-1}}{3}. \quad (11)\]

Since

\[\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < \frac{2}{3},\]

it is seen that

\[1 - \alpha^{m-n} > 1 - \frac{2}{3} = \frac{1}{3},\]

and therefore

\[\frac{1}{1 - \alpha^{m-n}} < 3.\]

Then from (11), it follows that

\[\left| 1 - 2^a \alpha^{n} \sqrt{5}(1 - \alpha^{m-n})^{-1} \right| < \frac{3}{\alpha^n}. \quad (12)\]

Thus, taking \(\gamma_1 := 2, \gamma_2 := \alpha, \gamma_3 := \sqrt{5}(1 - \alpha^{m-n})^{-1}\) and \(b_1 := a, b_2 := -n, b_3 := 1\), we can apply Theorem 1. As one can see that, the numbers \(\gamma_1, \gamma_2, \) and \(\gamma_3\) are positive real numbers and elements of the field \(\mathbb{K} = \mathbb{Q}(\sqrt{5})\), so \(D = 2\). Since

\[\frac{\alpha^n}{\sqrt{5}} - \frac{\alpha^m}{\sqrt{5}} = F_n + \frac{\beta^n}{\sqrt{5}} - F_m - \frac{\beta^m}{\sqrt{5}} \neq 2^a\]

for \(n > m\), the number \(\Lambda_2 := 2^a \alpha^{n} \sqrt{5}(1 - \alpha^{m-n})^{-1} - 1\) is nonzero. Similarly, since \(h(\gamma_1) = \log 2 = 0.6931...\), and \(h(\gamma_2) = \frac{\log \alpha}{2} = 0.4812...\) by (2), we can take \(A_1 := 1.4\) and \(A_2 := 0.5\). Besides, using (3), (4), and (5), we get that

\[h(\gamma_3) \leq \log 2 \sqrt{5} + (n - m) \frac{\log \alpha}{2},\]

and so we can take \(A_3 := \log 20 + (n - m) \log \alpha.\)
Also, since $a < n$, it follows that $B := \max \{|a|, |n|, 1\} = n$. Thus, taking into account the inequality (12) and using Theorem 1, we obtain

$$\frac{3}{\alpha^n} > |\Lambda_2| > \exp(-C)(1 + \log 2)(1 + \log n) (1.4) (0.5) (\log 20 + (n - m) \log \alpha)$$

or

$$n \log \alpha - \log 3 < C(1 + \log 2)(1 + \log n) (1.4) (0.5) (\log 20 + (n - m) \log \alpha),$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$. Inserting the inequality (11) into the last inequality, we get

$$n \log \alpha - \log 3 < C(1 + \log 2)(1 + \log n) (1.4) (0.5) (\log 20 + 2.4 \cdot 10^{12} \log n)$$

and so $n < 2.91 \cdot 10^{28}$.

Now, let us try to reduce the upper bound on $n$ applying Lemma 2 two times. Let

$$z_1 := a \log 2 - n \log \alpha + \log \sqrt{5}.$$ 

Then

$$|1 - e^{z_1}| < \frac{4}{\alpha^{n-m}}$$

by (11). The inequality

$$\frac{\alpha^n}{\sqrt{5}} = F_n + \frac{\beta^n}{\sqrt{5}} > F_n - 1 > F_n - F_m = 2^n$$

implies that $z_1 < 0$. In that case, since $\frac{4}{\alpha^{n-m}} < 0.95$ for $n - m \geq 3$, it follows that $e^{\vert z_1 \vert} < 20$. Hence, we get

$$0 < |z_1| < e^{\vert z_1 \vert} - 1 \leq e^{\vert z_1 \vert} |1 - e^{\vert z_1 \vert}| < \frac{80}{\alpha^{n-m}},$$

or

$$0 < |a \log 2 - n \log \alpha + \log \sqrt{5}| < \frac{80}{\alpha^{n-m}}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < |a \left( \frac{\log 2}{\log \alpha} \right) - n + \frac{\log \sqrt{5}}{\log \alpha}| < 50 \cdot \alpha^{-(n-m)}. \quad (14)$$

Now, we can apply Lemma 2. Put

$$\gamma := \log \frac{2}{\log \alpha} / \in \mathbb{Q}, \quad \mu := \log \frac{\sqrt{5}}{\log \alpha}, \quad A := 50, \quad B := \alpha, \quad \text{and} \quad w := n - m.$$

Taking $M := 2.91 \cdot 10^{28}$, we found that $q_{64}$, the denominator of the 64th convergent of $\gamma$ exceeds $6M$. Furthermore,

$$\epsilon = ||\mu q_{64}|| - M||\gamma q_{64}|| \geq 0.184.$$
Thus, we can say that the inequality (14) has no solutions for

\[ n - m \geq \frac{\log (Aq_{44}/\epsilon)}{\log B} \]

A computer search with Mathematica yields to \( n - m \geq 146.408 \). So \( n - m \leq 146 \).

Substituting this upper bound for \( n - m \) into (13), we obtain \( n < 7.56 \cdot 10^{15} \).

Now, let us apply again Lemma 2 to reduce a little bit the upper bound on \( n \). Let

\[ z_2 := a \log 2 - n \log \alpha + \log \left( \sqrt{\frac{5}{2}}(1 - \alpha^{m-n})^{-1} \right) \]

In this case,

\[ |1 - e^{z_2}| < \frac{3}{\alpha^n} \]

by (11). It is seen that \( \frac{3}{\alpha^n} < \frac{1}{2} \). If \( z_2 > 0 \), then \( 0 < z_2 < e^{z_2} - 1 < \frac{3}{\alpha^n} \). If \( z_2 < 0 \), then \( |1 - e^{z_2}| = 1 - e^{z_2} < \frac{3}{\alpha^n} < \frac{1}{2} \). From this, we get \( e^{|z_2|} < 2 \) and therefore

\[ 0 < |z_2| < e^{|z_2|} - 1 \leq e^{|z_2|} \left| 1 - e^{z_2} \right| < \frac{6}{\alpha^n}. \]

In any case, the inequality

\[ 0 < |z_2| < \frac{6}{\alpha^n} \]

is true. That is,

\[ 0 < \left| a \log 2 - n \log \alpha + \log \left( \sqrt{\frac{5}{2}}(1 - \alpha^{m-n})^{-1} \right) \right| < \frac{6}{\alpha^n}. \]

Dividing both sides of the above inequality by \( \log \alpha \), we get

\[ 0 < \left| a \frac{\log 2}{\log \alpha} - n + \frac{\log \left( \sqrt{\frac{5}{2}}(1 - \alpha^{m-n})^{-1} \right)}{\log \alpha} \right| < 13 \cdot \alpha^{-n}. \] (15)

Putting \( \gamma := \frac{\log 2}{\log \alpha} \) and taking \( M := 7.6 \cdot 10^{15} \), we found that \( q_{44} \), the denominator of the 44th convergent of \( \gamma \) exceeds \( 6M \). Also, taking

\[ \mu := \frac{\log \left( \sqrt{\frac{5}{2}}(1 - \alpha^{m-n})^{-1} \right)}{\log \alpha} \]

with \( n - m \in [3, 146] \) except for \( n - m \neq 4, 12 \), a quick computation using Mathematica gives us the inequality

\[ \epsilon = ||\mu q_{44}|| - M||\gamma q_{44}|| \geq 0.49939. \]

Let \( A := 13 \), \( B := \alpha \), and \( w := n \) in Lemma 2. Thus, with the help of Mathematica, we can say that the inequality (15) has no solution for \( n \geq 98.1915 \) with \( n - m \neq 4, 12 \). In that case \( n \leq 98 \). This contradicts our assumption that
Thus, we have to consider the cases $n - m = 1, 2, 4,$ and $12$ to complete the proof. If $n - m = 1$, then we have the equation $2^a = F_{m+1} - F_m = F_{m-1}$, which implies that $(n, m, a) = (3, 2, 0), (4, 3, 0), (8, 7, 3)$. If $n - m = 2$, then we have the equation $2^a = F_{m+2} - F_m = F_{m+1}$, which implies that $(n, m, a) = (3, 1, 0)$. If $n - m = 4$, then we have the equation $2^a = F_{m+4} - F_m = F_{m+3} + F_{m+1} = L_{m+2}$ by (7). By Theorem 3 this is only possible for $m = 1$, which implies that $n = 5$ and $a = 2$. Now, assume that $n - m = 12$. Then, we have the equation $F_{m+12} - F_m = 2^a$. Since $m + 12 \equiv m \pmod{4}$, it follows that $2^a = F_{m+12} - F_m = F_6 L_{m+6}$ by Lemma 4. This implies that $L_{m+6} = 2^{a-3}$, which is impossible by Theorem 3 since $m > 0$.

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