Boolean Dimension and Local Dimension

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Abstract

Dimension is a standard and well-studied measure of complexity of posets. Recent research has provided many new upper bounds on the dimension for various structurally restricted classes of posets. Bounded dimension gives a succinct representation of the poset, admitting constant response time for queries of the form “is $x < y$?”. This application motivates looking for stronger notions of dimension, possibly leading to succinct representations for more general classes of posets. We focus on two: boolean dimension, introduced in the 1980s and revisited in recent research, and local dimension, a very new one. We determine precisely which values of dimension/boolean dimension/local dimension imply that the two other parameters are bounded. This is an extended abstract; see arXiv:1705.09167 for a full version.

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1 Introduction

Dimension

The dimension of a poset $P = (X, \leq)$ is the minimum number of linear extensions of $\leq$ on $X$ whose intersection is $\leq$. More precisely, a realizer of a poset $P = (X, \leq)$ is a set $\{\leq_1, \ldots, \leq_d\}$ of linear extensions of $\leq$ on $X$ such that

$$x \leq y \iff (x \leq_1 y) \land \cdots \land (x \leq_d y),$$

for any $x, y \in X$,

and the dimension is the minimum size of a realizer. The concept of dimension was introduced by Dushnik and Miller [5] and has been widely studied since. There are posets with arbitrarily large dimension: the standard example $S_k = (\{a_1, \ldots, a_k, b_1, \ldots, b_k\}, \leq)$, where $a_1, \ldots, a_k$ are minimal elements, $b_1, \ldots, b_k$ are maximal elements, and $a_i < b_j$ if and only if $i \neq j$, has dimension $k$ when $k \geq 2$ [5]. On the other hand, the dimension of a poset is at most the width [9], and it is at most $\frac{n}{2}$ when $n \geq 4$, where $n$ is the number of elements [9].

The cover graph of a poset $P = (X, \leq)$ is the graph on $X$ with edge set $\{xy: x < y$ and there is no $z$ with $x < z < y\}$. A poset is planar if its cover graph has a non-crossing upward drawing in the plane, which means that every cover graph edge $xy$ with $x < y$ is drawn as a curve that goes monotonically up from $x$ to $y$. Planar posets that contain a least element and a greatest element are well known to have dimension at most 2 [1]. Trotter and Moore [17] proved that planar posets that contain a least element have dimension at most 3 (and so do posets whose cover graphs are forests) and asked whether all planar posets have bounded dimension. The answer is no—Kelly [13] constructed planar posets with arbitrarily large dimension (pictured). Another property of Kelly’s posets is that their cover graphs have path-width and tree-width 3. Recent research brought a plethora of new bounds on dimension for structurally restricted posets. In particular, dimension is bounded for posets with

- height 2 and planar cover graphs [6],
- bounded height and planar cover graphs [16],
- bounded height and cover graphs of bounded tree-width [10],
- bounded height and cover graphs excluding a topological minor [18],
- bounded height and cover graphs of bounded expansion [12],
- cover graphs of path-width 2 [2] or, more generally, tree-width 2 [11].
Boolean dimension

The boolean dimension of a poset \( P = (X, \leq) \) is the minimum number of linear orders on \( X \) a boolean combination of which gives \( \leq \). More precisely, a boolean realizer of \( P \) is a set \( \{\leq_1, \ldots, \leq_d\} \) of linear orders on \( X \) for which there is a \( d \)-ary boolean formula \( \phi \) such that

\[
x \leq y \iff \phi((x \leq_1 y), \ldots, (x \leq_d y)) \quad \text{for any } x, y \in X, \tag{1}
\]

and the boolean dimension is the minimum size of a boolean realizer. The boolean dimension is at most the dimension, because a realizer is a boolean realizer for the formula \( \phi(\alpha_1, \ldots, \alpha_d) = \alpha_1 \land \cdots \land \alpha_d \). Beware that the relation \( \leq \) defined by (1) from arbitrary linear orders \( \leq_1, \ldots, \leq_d \) on \( X \) and formula \( \phi \) is not necessarily a partial order.

Boolean dimension was first considered by Gambosi, Nešetřil, and Talamo [7] and by Nešetřil and Pudlák [15]. Boolean dimension \( d \) and dimension \( d \) are equivalent for \( d \in \{1, 2, 3\} \) (this is essentially proved in [7], although the actual statement is more restricted), while the standard examples \( S_k \) with \( k \geq 4 \) have boolean dimension \( 4 \) [7]. The boolean dimension of an \( n \)-element poset can be as large as \( \Theta(\log n) \) [15]. Nešetřil and Pudlák [15] asked whether boolean dimension is bounded for planar posets. It was proved already in [7] that posets with height 2 and planar cover graphs have bounded boolean dimension. This and the recent progress on dimension of structurally restricted posets have motivated revisiting boolean dimension in current research.

Local dimension

A partial linear extension of a partial order \( \leq \) on \( X \) is a linear extension of the restriction of \( \leq \) to some subset of \( X \). A local realizer of \( P \) of width \( d \) is a set \( \{\leq_1, \ldots, \leq_d\} \) of partial linear extensions of \( \leq \) such that every element of \( X \) occurs in at most \( d \) of \( \leq_1, \ldots, \leq_d \) and

\[
x \leq y \iff \text{there is no } i \in \{1, \ldots, t\} \text{ with } x >_i y, \quad \text{for any } x, y \in X. \tag{2}
\]

The local dimension of \( P \) is the minimum width of a local realizer of \( P \). Thus, instead of the size of a local realizer, we bound the number of times any element of \( X \) occurs in it. A set of linear extensions of \( \leq \) is a local realizer if and only if it is a realizer. In particular, the local dimension is at most the dimension. For arbitrary partial linear extensions \( \leq_1, \ldots, \leq_t \) of \( \leq \) on subsets of \( X \), the relation \( \leq \) defined by (2) is not necessarily a partial order—it may fail to be antisymmetric or transitive. It is antisymmetric, for example, if one
of \(\leq_1, \ldots, \leq_t\) is a linear extension of \(\leq\) on \(X\). The concept of local dimension was proposed by Ueckerdt (Order & Geometry Workshop, Gultowy, 2016) and originates from concepts studied in [3,14].

**Results**

Extending the results on boolean dimension from [7], for each \(d\), we determine whether posets with dimension/boolean dimension/local dimension \(d\) have the other two parameters bounded or unbounded. Here is the full picture:

A. **Boolean dimension \(d\) and dimension \(d\) are equivalent for \(d \in \{1, 2, 3\}\)** [7].

B. **Local dimension \(d\) and dimension \(d\) are equivalent for \(d \in \{1, 2\}\).**

C. **Standard examples have boolean dimension 4** [7] **and local dimension 3.**

D. **There are posets with boolean dimension 4 and unbounded local dimension.**

E. **Posets with local dimension 3 have bounded boolean dimension.**

F. **There are posets with local dimension 4 and unbounded boolean dimension.**

**2 Proofs**

B. **Local dimension \(d\) and dimension \(d\) are equivalent for \(d \in \{1, 2\}\).**

If a poset \(P = (X, \leq)\) has local dimension 1, then its local realizer of width 1 must consist of a single full linear order on \(X\). Now, let \(P = (X, \leq)\) be a poset with local dimension 2, and consider a local realizer of \(P\) of width 2. If \(x, y \in X\) are incomparable in \(\leq\), then both occurrences of \(x\) and \(y\) are in the same two partial linear extensions, where \(x < y\) in one and \(x > y\) in the other. Therefore, the subposet of \(P\) induced on every connected component \(C\) of the incomparability graph of \(P\) is witnessed by two partial linear extensions, which restricted to \(C\) form a realizer of that subposet. These realizers stacked according to the order \(\leq\) form a realizer of \(P\) of size 2.

C. **Standard examples have boolean dimension 4 and local dimension 3.**

The standard example \(S_k\) has boolean dimension 4 (when \(k \geq 4\)), witnessed by the formula \(\phi(\alpha) = \alpha_1 \land \alpha_2 \land (\alpha_3 \lor \alpha_4)\) and the following four linear orders:

\[
\begin{align*}
  a_1 < \cdots < a_k < b_1 < \cdots < b_k, \\
  a_k < \cdots < a_1 < b_k < \cdots < b_1,
\end{align*}
\]

\[
\begin{align*}
  b_1 < a_1 < \cdots < b_k < a_k, \\
  b_k < a_k < \cdots < b_1 < a_1.
\end{align*}
\]

It has local dimension 3 (when \(k \geq 3\)), witnessed by the two linear extensions above on the left and \(k\) partial linear extensions each of the form \(b_i < a_i\).
D  There are posets with boolean dimension 4 and unbounded local dimension.

Another well-known construction of posets with arbitrarily large dimension [5] involves incidence posets of complete graphs: \( P_n = (V \cup E, \leq) \), where \( V = \{v_1, \ldots, v_n\} \) are the minimal elements, \( E = \{v_1 v_2, v_1 v_3, \ldots, v_{n-1} v_n\} \) are the maximal elements, and the only comparable pairs are \( v_i < v_j \) and \( v_j < v_i \) for \( i \neq j \). The boolean dimension of \( P_n \) is at most 4, witnessed by the formula \( \phi(\alpha) = (\alpha_1 \land \alpha_2) \lor (\alpha_3 \land \alpha_4) \) and the following four linear orders:

\[
A_1 < \cdots < A_n, \text{ where each } A_i \text{ has form } v_i < v_i v_{i+1} < \cdots < v_i v_n;
\]
\[
B_n < \cdots < B_1, \text{ where each } B_i \text{ has form } v_i < v_i v_n < \cdots < v_i v_{i+1};
\]
\[
C_1 < \cdots < C_n, \text{ where each } C_i \text{ has form } v_i < v_1 v_i < \cdots < v_{i-1} v_i;
\]
\[
D_n < \cdots < D_1, \text{ where each } D_i \text{ has form } v_i < v_{i-1} v_i < \cdots < v_1 v_i.
\]

The local dimension of \( P_n \) is unbounded as \( n \to \infty \). For suppose \( P_n \) has a local realizer of width \( d \). Enumerate the occurrences of each element of \( V \cup E \) in the local realizer from 1 to (at most) \( d \). Each triple \( v_i v_j v_k \) \((i < j < k)\) can be assigned a color \((p, q)\) so that \( v_i v_k < v_j \) in a partial linear extension containing the \( p \)th occurrence of \( v_j \) and the \( q \)th occurrence of \( v_i v_k \). By Ramsey’s theorem, if \( n \) is large enough, then there is a quadruple \( v_i v_j v_k v_\ell \) \((i < j < k < \ell)\) with all four triples of the same color \((p, q)\). Therefore, the \( p \)th occurrences of \( v_j \) and \( v_k \) and the \( q \)th occurrences of \( v_i v_\ell \), \( v_i v_k \), and \( v_\ell v_k \) are all in the same partial linear extension, which contains a cycle \( v_j < v_\ell v_k < v_k < v_i v_k < v_j \), a contradiction.

E  Posets with local dimension 3 have bounded boolean dimension.

Let \( P = (X, \leq) \) be a poset with a local realizer of width 3 consisting of partial linear extensions that we call gadgets. We construct a boolean realizer \( \{\leq^*, \leq_1, \leq'_1, \ldots, \leq_d, \leq'_d\} \) for a formula of the form \( \alpha^* \land (\alpha_1 \lor \alpha'_1) \land \cdots \land (\alpha_d \lor \alpha'_d) \). The order \( \leq^* \) is an arbitrary linear extension of \( \leq \) on \( X \). Each pair \( \leq_1, \leq'_1 \) is defined by \( X_1 <_1 \cdots <_1 X_t \) and \( X_\ell <'_1 \cdots <'_1 X_1 \), where \( \{X_1, \ldots, X_t\} \) is some partition of \( X \) into blocks such that every block \( X_j \) is completely ordered by some gadget and that order is inherited by \( \leq_1 \) and \( \leq'_1 \). It remains to construct a bounded number of partitions of \( X \) into blocks so that for any \( x, y \in X \), if \( x <^* y \) and \( x > y \) in some gadget, then \( x > y \) in some block.

Enumerate the occurrences of each \( x \in X \) in the gadgets as \( x^1, x^2, x^3 \) according to a fixed order of the gadgets. For each \( p \in \{1, 2, 3\} \), form a partition of \( X \) by restricting every gadget to the \( x^p \)s. These three partitions witness all comparabilities of the form \( x^p > y^p \) within gadgets. Now, let \( G \) be a graph on \( X \) where \( xy \) (with \( x <^* y \)) is an edge if and only if \( x^p > y^q \) \((p \neq q)\) in
some gadget. The fact that every element of $X$ has only 3 occurrences in the gadgets implies that $G$ has bounded chromatic number (we omit the details). Let $c$ be a coloring of $G$ with a bounded number of colors. For $1 \leq p < q \leq 3$ and any distinct colors $a, b$, form a partition of $X$ by restricting every gadget to the $x^p$'s with $c(x) = a$ and the $y^q$'s with $c(y) = b$ (adding singletons if necessary to obtain a full partition of $X$). These partitions witness all comparabilities of the form $x^p > y^q$ (where $x <^* y$ and $p \neq q$) within gadgets. The overall number of partitions thus obtained is bounded.

F There are posets with local dimension 4 and unbounded boolean dimension.

When $(V, E)$ is an acyclic digraph and $v \in V$, let $E^+(v) = \{uv \in E : u \in V\}$ and $E^-(v) = \{vw \in E : w \in V\}$ ($uv$ denotes a directed edge from $u$ to $v$). For $k \geq 1$, we construct an acyclic digraph $G = (V, E)$ with $\chi(G) > k$, a poset $P = (E, \leq)$, and its local realizer $\{\leq_A, \leq_B\} \cup \{\leq_v : v \in V\}$ of width 4, where

- $\leq_A$ and $\leq_B$ are (full) linear extensions of $\leq$ on $E$ such that $E^+(v) <_A E^-(v)$ and $E^+(v) <_B E^-(v)$ for every $v \in V$,
- $\leq_v$ is a partial linear extension of the form $E^-(v) <_v E^+(v)$, for each $v \in V$.

This is achieved by using Tutte’s construction of a triangle-free graph $G$ with $\chi(G) > k$ [4]. We omit the details, noting that the main challenge is to ensure transitivity of the relation $\leq$ defined by (2) (it could not be achieved when $G$ was, for instance, a transitive tournament). We show that when $k = 2^{2^d}$, the resulting poset $P$ has boolean dimension greater than $d$. For suppose $\{d_1, \ldots, d_d\}$ is a boolean realizer of $P$ for a formula $\phi$. Let $H = (A, B)$, where $A = \{uvw : uv, vw \in E\}$ and $B = \{uvw : uwx, vxw \in A\}$. It follows that $\chi(H) \geq \log_2 \log_2 \chi(H) > 2^d$ [8, Theorem 9]. For $uvw \in A$, let $\alpha(uvw) = ((uv <_1 vw), \ldots, (uv <_d vw)) \in \{0, 1\}^d$; since $uv >_v vw$, we have $uv \not\leq vw$ and thus $\phi(\alpha(uvw)) = 0$. Let $uvw\in B$. We have $uw <_A vw <_A wx$ and $uw <_B vw <_B wx$, which implies $uw < wx$, because no partial linear extension from $\{\leq_v : v \in V\}$ contains both $uw$ and $wx$. If $\alpha(uvw) = \alpha(vwx) = \alpha$, then transitivity of $d_1, \ldots, d_d$ implies $((uv <_1 wx), \ldots, (uv <_d wx)) = \alpha$. This, $\phi(\alpha) = 0$, and $uw < wx$ result in a contradiction. Therefore, $\alpha : A \rightarrow \{0, 1\}^d$ is a $2^d$-coloring of $H$. This contradicts the fact that $\chi(H) > 2^d$.

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