A Survey of
Conformally Invariant Measures
on $H^m(\Delta)$

Doug Pickrell
Department of Mathematics
University of Arizona
Tucson, Arizona 85721
pickrell@math.arizona.edu

Abstract. The universal covering of $PSU(1,1)$ acts naturally on the space of holomorphic differentials of order $m$ on the unit disk. The main purpose of this paper is to survey, as broadly as I am able, some basic sources and examples of invariant measures for this action. A problem for the future is to determine, or at least to organize in some useful way, all of the invariant measures.

§0. Introduction

The group $G = PSU(1,1) = \{g = \pm \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\}$ acts on the open unit disk $\Delta \subset \mathbb{C}$ by linear fractional transformations,

$$g : z \rightarrow \frac{az + b}{bz + \bar{a}}. \quad (0.1)$$

This identifies $G$ with the group of all conformal automorphisms of $\Delta$, or equivalently with the group of all orientation-preserving isometries of $\Delta$, equipped with the Poincare metric, $ds = \frac{|dz|}{1-|z|^2}$.

Let $\tilde{G}$ denote the universal covering of $G$. For our purposes a useful model is

$$\tilde{G} = \{\tilde{g} = \left( \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, A \right) \in SU(1,1) \times \mathbb{C} : e^A = a \}, \quad (0.2)$$
where if $\tilde{g}_3 = \tilde{g}_1 \tilde{g}_2$, then

$$A_3 = A_1 + A_2 + \log(1 + \frac{b_1 \bar{b}_2}{a_1 a_2}),$$

(0.3)

and the covering short exact sequence is

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 0,$$

(0.4)

where $n \in \mathbb{Z}$ maps to $\left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right)^n i \pi n$, and $\tilde{g}$ above maps to $g = \pm \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \in G$.

For each real number $m$, the action of $G$ on $\Delta$ lifts to an action of $\tilde{G}$ on $\kappa^m$, the ‘$m$th power’ of the canonical bundle on $\Delta$. This induces a natural action of $\tilde{G}$ on the space of holomorphic sections of $\kappa^m$, the holomorphic differentials of degree $m$. We will denote such a differential, and the space of all such differentials, by $f(z)(dz)^m$ and $H^m = H^m(\Delta)$, respectively. By slight abuse of notation, given $\tilde{g} \in \tilde{G}$ as in (0.2), we will write the action of $\tilde{g}$ on $H^m$ as

$$\tilde{g} : f(z)(dz)^m \rightarrow f(\frac{\bar{a} z - b}{-b z + a})(-\bar{b} z + a)^{-2m}(dz)^m.$$

(0.5)

The precise interpretation of $(-\bar{b} z + a)^{-2m}$ as a holomorphic function of $z \in \Delta$ is given by

$$(a - \bar{b} z)^{-2m} = e^{-2mA(1 - \frac{\bar{b}}{a} z)^{-2m}},$$

(0.6)

this is a well-defined holomorphic function, because $|a|^2 - |b|^2 = 1$ implies $|\frac{\bar{b}}{a} z| \leq 1$, and hence $|\frac{\bar{b}}{a} z| < 1$ for $z \in \Delta$. Thus, for nonintegral $m$, the action (0.5) does depend upon $\tilde{g} \in \tilde{G}$, not simply $g$.

The set of all $\tilde{G}$-invariant probability measures on $H^m(\Delta)$ is a convex set, and the extreme points are the ergodic measures. The naïve futuristic problem which we pose is the following: determine all, or at least organize in some useful way the interesting, ergodic $\tilde{G}$-invariant probability measures on $H^m(\Delta)$. Our modest goal in this paper is simply to survey some basic sources and examples of such measures. There are at least two cases which are of special interest: $m = 1$, because of its relation to the theory of classical Toeplitz and Hankel operators (§3 and §4), and $m = 2$, because of its relation to universal Teichmüller space and conformal maps (§5). The case $m = 0$ is closely related to the case $m = 1$, because of the equivariant sequence

$$0 \rightarrow \mathbb{C} \rightarrow H^0(\Delta) \xrightarrow{\partial} H^1(\Delta) \rightarrow 0.$$

(0.7)
Invariant measures on $H^m(\Delta)$ are also related to invariant measures on the space $C(\Delta)$ of (possibly infinite) configurations on $\Delta$. This correspondence arises via the natural map

$$Z : H^m \setminus \{0\} \to C(\Delta) : \theta \to Z(\theta),$$

where $Z(\theta)$ denotes the zero set. This seems to be a complete mystery at present.

The space $H^m$ is infinite dimensional, and $G$ is three dimensional. As a consequence the support of an invariant measure can range from an orbit of the group (which will resemble the sphere bundle of a finite area Riemann surface), to something infinite dimensional. Measures of the former type, transitive measures, possibly generate all invariant measures, in the sense that by applying operations, such as convolution, all such measures are limits of these combinations. We are primarily interested in these limits, and insightful interpretations which we might be able to attach to their supports, or to their (heuristic) ‘densities’, or to their Fourier transforms.

The action of $\tilde{G}$ on $H^m$, for $m > 0$, is well-known to be unitary and irreducible on a dense subspace; we will denote this Hilbert subspace by $H^m \cap L^2$ (although this notation is potentially misleading when $0 < m < 1/2$; see §1B). In this unitary context, at least as a starting point, it is perhaps useful to think of an invariant measure with ‘Lebesgue type support’ (a notion which I do not know how to formalize) as heuristically having the form

$$\frac{1}{Z} \rho(\theta) dV(\theta),$$

(0.8)

where $\rho$ (the ‘density’) is an invariant function of $\theta \in H^m$, and $dV$ denotes a fictitious Riemannian volume element corresponding to the invariant unitary structure.

The basic examples are the Gaussians, where

$$\rho(\theta) = \exp(-\frac{1}{2t} |\theta|_{H^m \cap L^2}^2).$$

(0.9)

These measures can be characterized in many ways, and they are invariant and ergodic with respect to the full unitary group of the Hilbert space $H^m \cap L^2$, and exhaust all such possibilities. A venerable result of Irving Segal, which in some ways remains mysterious, asserts that these Gaussian measures are also $\tilde{G}$-ergodic ([Se]). This is a fairly simple consequence of the obvious fact that the finite dimensional unitary representations of $\tilde{G}$ are trivial. But it seems hard to understand what ergodicity implies, in terms of the dynamics of $\tilde{G}$ acting on a random $m$-differential.
One of the main points of this paper is to show that there are many other natural invariant functions which lead to invariant measures, via the heuristic expression (0.8). The simplest non-Gaussian example is the following. If $\theta \in H^1$, let

$$x(z) = \int \theta = x_1z + x_2z^2 + .. \in H^0/\mathbb{C},$$

as in the correspondence (0.7). The Hankel operator corresponding to $x$, or $\theta$, is represented by the infinite Hankel matrix

$$B(x) = \begin{pmatrix}
\cdot & \cdot \\
\cdot & x_n \\
x_2 & \cdot \\
x_1 & x_2 & \cdot & x_n
\end{pmatrix}.$$  

This operator is Hilbert-Schmidt precisely when $\theta \in H^1 \cap L^2$, and the action of $G$ on $x$, or $\theta$, intertwines with a natural action of $G$ on $B(x)$ by unitary conjugation (this is explained in §3). Consequently the determinant

$$D(\theta) = \det(1 + B(x)B(x)^*)$$

is a well-defined invariant function of $\theta \in H^1 \cap L^2$. For each $-1 < l$, the heuristic expression

$$\frac{1}{Z} \frac{1}{\det(1 + B(x)B(x)^*)^{3+l}} \frac{dm(\theta),}{dm(\theta)},$$

has a rigorous interpretation as a limit of the finite dimensional probability measures

$$\frac{1}{Z_n} \frac{1}{\det(1 + B_n B_n^*)^{1+p_n+l}} \frac{dm(x_1,..,x_n),}{dm(x_1,..,x_n)},$$

where $B_n = B(\sum^n x_j z^{j-1})$, and by a miracle, $Z_n$ and $p_n$ can be calculated exactly. This limit defines an invariant probability measure on $H^1$.

This construction fits into a general pattern, which can be crudely formulated in the following way. Suppose that a group acts in an ‘isometric way’ on a dense subspace $X_\infty$ (with respect to a Riemannian or symplectic structure) of an infinite dimensional space $X$. Suppose that $X_\infty$ is well-approximated by finite dimensional spaces

$$X_1 \subset X_2 \subset .. \subset X_n \subset .. \subset X_\infty \subset X.$$
Suppose that $D$ is a positive invariant function on $X_\infty$ (such as (0.12)) such that for each $n$

$$\int_{X_n} \frac{1}{D^p} dV_{X_n} < \infty$$

(0.16)

for sufficiently large exponents $p$, and let $p_n$ denote the critical exponent, the smallest exponent such that (0.16) holds for $p_n < p$. Then with appropriate hypotheses one might expect that for $L > 0$

$$\lim_{n \to \infty} \frac{1}{Z_n} \frac{1}{D^{l_n}} dV_{X_n}$$

(0.17)

exists and defines an invariant measure on $X$. Although many examples are known, unfortunately a general existence result is missing.

In the Hankel example above, $p_n = 2 - \frac{1}{n}$, which leads to the heuristic expression (0.13). In turn the expression (0.13) suggests a potential interpretation (which I am unable to formalize) for the Fourier transform of the measure: up to a multiple, it is a fundamental solution for the pseudodifferential operator (involving infinitely many variables) $det(1 + B(\frac{\partial}{\partial y})B(\frac{\partial}{\partial y})^*)^{3+l}$, where $y$ is dual to $x$.

The plan of the paper is the following.

In §1 we present elementary examples of ergodic invariant probability measures. If $m$ is rational and nonnegative, then there exist transitive measures. In addition to further discussing Gaussian measures, we also present an elementary example illustrating the general pattern outlined above.

In §2 we discuss basic operations. Because $H^m$ is a linear space, the convolution of two invariant measures is another invariant measure, and one can always scale a measure. The multiplication map

$$H^m \otimes H^n \to H^{m+n} : f(z)(dz)^m \otimes g(z)(dz)^n \to fg(dz)^{m+n}$$

(0.18)

is also $\tilde{G}$-equivariant, and we can use pushforward to define the ‘product’ of two invariant measures. For example if $\nu_1^{(m)}$ denotes the standard Gaussian corresponding to $H^m \cap L^2$, as in (0.8) and (0.9), then the product $\nu_1^{(m)} \otimes \nu_1^{(n)}$ is an invariant measure on $H^{m+n}$, and its Fourier transform is given by

$$(\nu_1^{(m)} \otimes \nu_1^{(n)})(F) = \frac{1}{\det(1 + B(m, n; F)B(m, n; F)^*)},$$

(0.19)

where $B(m, n; F)$, for $F \in H^{m+n} \cap L^2$, is a kind of generalized Hankel operator which is represented by the matrix (0.11) when $m = n = 1/2$. For both convolution and ‘multiplication’, the product of two ergodic measures is another ergodic
measure. Also the product operation intertwines via the zero map \( Z \) with a natural convolution operation for measures on configuration space.

In §3 we discuss the measures in (0.13), and analogues involving the operators \( B(m, n; F) \), and more general linear equivariant operator-valued functions. At this point I do not know whether these measures are ergodic, whether they could possibly be generated from simpler measures by the operations described above, whether they remember the parameters \((m, n)\), and so on.

In §4 I briefly discuss how (0.13) likely fits into a larger context. Given a simply connected compact symmetric space \( U/K \), it is known that there exists a \( LU \)-invariant probability measure on a distributional completion of the loop space \( L(U/K) \) (see [Pi2]). This measure is believed to be reparameterization invariant. By expressing a loop in terms of its Riemann-Hilbert factorization, and by making use of a global version of the map (0.7),

\[
H^0(\Delta, U^C) \rightarrow H^1(\Delta, u^C) : g \rightarrow g^{-1}\partial g,
\]

(0.20)

this should give rise to a conformally invariant measure on \( H^1(\Delta, u^C) \), where \( u^C \) denotes the Lie algebra of the complexification of \( U \). Hence the distributions of the matrix coefficients should provide a large source of \( PSU(1, 1) \)-invariant measures on \( H^1(\Delta) \). In the special case in which \( U/K \) is a group, which we consider in §4, there is an expression for the \( H^1(\Delta, u^C) \)-distribution which displays its conformal invariance. The measures in (0.13) appear to be related to the matrix coefficient corresponding to the highest root, when \( U/K \) is \( SU(2) \). But this merely looks plausible, and in particular, I do not have any insight into how to generate formulas for other matrix coefficients.

In an analogous way invariant measures associated to \( Diff(S^1) \), which are conjectured to exist, should give rise to \( PSU(1, 1) \)-invariant measures on \( H^2(\Delta) \). This is related to an infinitesimal map

\[
H^{-1}(\Delta) \xrightarrow{dS} H^2(\Delta)
\]

(0.21)

and a corresponding global map

\[
H^0(\Delta, C) \xrightarrow{S} H^2(\Delta)
\]

(0.22)

where \( S \) is the Schwarzian derivative, which are analogous to (0.7) and (0.20), respectively. This speculation is described briefly in §5. I do not know if there is any useful connection with the Schramm-Loewner evolution process.
In the last section, §6, to compensate somewhat for the scattered character of the results in this paper, I have attempted to summarize some of the main points, and to pose some questions which point in promising directions.

Preliminaries

The space $H^m$ is a Frechet space with respect to the topology of uniform convergence on compact sets, hence has a natural standard Borel structure. With respect to this topology, the action (0.5) is continuous.

Throughout the remainder of this paper, the phrase ‘$\nu$ is an invariant measure’ will be understood as an abbreviation for ‘$\nu$ is a $\tilde{G}$-invariant probability measure on $H^m$, relative to its natural standard Borel structure, for some $m$’, unless explicitly stated otherwise.

Given a Riemann surface $\Sigma$, $\kappa$, or for emphasis $\kappa|_\Sigma$, will denote the canonical holomorphic line bundle. A local holomorphic section of $\kappa$ is simply a holomorphic one form. Thus $H^1(\Delta) = H^0(\kappa|_\Delta)$, and in general

$$H^m = H^m(\Delta) = H^0((\kappa|_\Delta)^m).$$

The line bundle $\kappa|_\Delta$ has a natural, i.e. $\tilde{G}$-invariant, Hermitian norm, where

$$|(dz)^m| = (1 - |z|^2)^m.$$  

This can be used to define various $\tilde{G}$-invariant Banach subspaces, such as $H^m \cap L^p$. The global Poincare metric on $\Delta$ is given by

$$d(z, w) = \arctanh\left(\frac{z - w}{1 - \bar{z}w}\right).$$

Configuration Space.

Let $C(\Delta)$ denote the configuration space of $\Delta$. As a topological space, $C(\Delta)$ divides up into connected components

$$C(\Delta) = \bigcup_{0 \leq n \leq \infty} C(\Delta)^{(n)},$$

where $C(\Delta)^{(n)}$ denotes the space of configurations with $n$ points. If $n < \infty$, then there is a universal covering

$$Perm(n) \times \{(q_i) \in \Delta^n : q_i \neq q_j, i \neq j\} \to C(\Delta)^{(n)}.$$ 


Suppose that \( n = \infty \). By definition \( \mathcal{C}(\Delta)(\infty) \) consists of countable subsets which have finite intersection with bounded sets. The topology is the weak topology generated by functions of the following form: given a compactly supported continuous function \( f : \Delta \to \mathbb{R}, \)

\[
\mathcal{C}(\Delta)(\infty) : \mathbb{R} : \gamma \to \sum_{q \in \gamma} f(q) \tag{0.28}
\]
is continuous. It is known that this topology is standard (defined by a complete separable metric), and the fundamental group is isomorphic to \( \text{Perm}(\mathbb{N}) \). However this space is not locally simply connected, so that it does not have a universal covering.

For any \( m \), there is a continuous equivariant map

\[
Z : H^m(\Delta) \setminus \{0\} \to \mathcal{C}(\Delta) : \theta \to Z(\theta), \tag{0.29}
\]
where \( Z(\theta) \) is the zero set of \( \theta \). For an invariant measure \( \nu \) on \( H^m \), \( Z_* \nu \) will be an invariant measure on the configuration space, and if \( \nu \) is ergodic, then \( Z_* \nu \) will be ergodic. Thus an ergodic invariant measure has an invariant, \( n \), where \( Z_* \nu \) is support on \( \mathcal{C}(\Delta)^{(n)} \).

The simplest and most important measures on the infinite configuration space are the Poisson measures, \( p_\lambda \), where \( \lambda > 0 \). Given a finite disjoint collection of compact sets \( K_i \), and integers \( n_i \geq 0 \),

\[
p_\lambda \{ \gamma \in \Gamma_\Delta : \forall i, \text{card} (\gamma \cap K_i) = n_i \} = \prod_i \frac{(\lambda m(K_i))^{n_i}}{n_i!} e^{-\lambda m(K_i)} \tag{0.30}
\]
where \( m \) denotes the invariant measure (see [KMM]).

§1. Elementary Examples of Ergodic Measures.

A. Transitive Measures.

Suppose that \( \nu \) is a transitive measure. This means that there is a differential \( \theta = f(z)(dz)^m \in H^m(\Delta) \) such that the orbit \( \tilde{G} \ast \theta \) has finite volume, with respect to the essentially unique \( \tilde{G} \)-invariant volume element on this space, and \( \nu \) is the normalized volume element on this orbit.

There are two possibilities for the stability subgroup of \( \tilde{G} \) at \( \theta \). The first possibility is that it is the entire group, in which case \( m = 0 \) and \( \theta = f \) is constant, or \( m > 0 \) and \( \theta = 0 \). In these cases \( \nu = \delta_\theta \).
The second possibility is that it is a lattice \( L \), i.e. a discrete subgroup of \( \tilde{G} \) with a quotient which has finite volume. The structure of such a lattice is known: the projection of \( L \) to \( G \) is a lattice \( L \), and \( L \cap \mathbb{Z} = N \mathbb{Z} \), for some positive integer \( N \), where \( \mathbb{Z} \) is identified with the center of \( \tilde{G} \) as in (0.4) (see Corollary 4.7.3, page 278 of [T]). Let \( G^{(N)} \) denote the unique \( N \)-fold covering of \( G \) (for example \( G^{(2)} = SU(1, 1) \), \( G^{(4)} \) is the 2-fold covering of \( SU(1, 1) \), the metaplectic group, and so on). On the one hand \( L \) is the full inverse image of a lattice \( L^{(N)} \) in \( G^{(N)} \):

\[
0 \to N \mathbb{Z} \to \tilde{G} \to G^{(N)} \to 0 \tag{1.1}
\]

and on the other hand \( L^{(N)} \) is isomorphic to \( L \) with respect to the projection \( G^{(N)} \to G \), so that it represents a splitting of the induced extension:

\[
0 \to \mathbb{Z}_N \to G^{(N)} \to G \to 0 \tag{1.2}
\]

(1.3) Examples: (a) If \( m \) is an integer, so that \( G \) acts on \( H^m \), then \( L \) is simply the full inverse image of the lattice \( L \subset G \). As an example, \( L \) could be the fundamental group of a compact Riemann surface \( X \), where \( X = L \backslash \Delta \). In this context \( \theta \) could be the pullback of a holomorphic one form, or quadratic differential, etc, from \( X \) to \( \Delta \).

(b) Using a conformal isomorphism of \( \Delta \) with the upper half plane, we can identify \( G \) with \( PSL(2, \mathbb{R}) \). Let \( L = \Gamma(N) \), the principal congruence subgroup of level \( N \) (see page 54 of [FK]). As a group \( \Gamma(N) \) is free. It follows that the extension of \( L \) induced by

\[
0 \to \mathbb{Z}_N \to G^{(N)} \to G \to 0 \tag{1.4}
\]

can be split, and there is an essentially unique way to do the splitting. This determines \( L^{(N)} \) as in (1.2), and \( L \) is the inverse image of \( L^{(N)} \) in \( \tilde{G} \). In this context

\[
\theta = \eta(\tau)(d\tau)^{1/2}, \text{ where } \eta(\tau) = q^{1/24} \prod_{1}^{\infty} (1 - q^n) \tag{1.5}
\]

and \( q = exp(2\pi i \tau) \), \( 0 < Im(\tau) \), is a nonvanishing element of \( H^{1/2} \) which is invariant with respect to \( [\Gamma(1), \Gamma(1)] \), the commutator subgroup, which has index 6. By taking roots of \( \eta \), it follows that there exist transitive measures for arbitrary rational \( m \), where the stability subgroup is commensurate with the \( \Gamma(N) \).
Returning to the question of stability subgroups, the other a priori possibilities are that the quotient is a circle or a disk. But there are not invariant (finite) measures in these cases.

We can summarize this discussion in the following way.

**1.6 Proposition.** Suppose \( m > 0 \) and \( \nu \) is a transitive \( \tilde{G} \)-invariant measure on \( H^m(\Delta) \). Then either \( \nu = \delta_0 \), or \( m = \frac{M}{N} \) is rational (in reduced form), and \( \nu \) is the normalized invariant probability measure on the orbit \( G^N \theta \), where \( \theta \) is the pullback to \( \Delta \) of a holomorphic \( m \)-differential on the finite volume Riemann surface \( L \setminus \Delta \), where \( L \subset G \) is the image of \( L^N \): \[
\begin{align*}
G^N \times \kappa^{M/N} & \rightarrow \kappa^{M/N} \\
\downarrow & \quad \downarrow ;
\end{align*}
\]

here \( \kappa^{1/N} \) denotes a fixed \( N \)th root of the canonical bundle \( \kappa \) over \( L \setminus \Delta \).

Given an invariant measure, it is natural to ask for minimal information which is needed to identify the measure. At least for transitive measures this should be answered by the following

**1.7 Conjecture.** The evaluation map \( \text{eval}_0 : H^m(\Delta) \rightarrow \mathbb{C} : f(z)(dz)^m \rightarrow f(0) \)

induces an injective map from the set of transitive invariant probability measures to the set of Rot-invariant probability measures on \( \mathbb{C} \).

In the case \( m = 0 \), this asserts that a holomorphic function on \( \Delta \), which is the pullback of a function on a finite volume surface, is determined by the (finite) distribution of its values, up to an automorphism of \( \Delta \). This is analogous to the assertion that a function on a finite set is determined by a listing of its values and their multiplicities, up to a permutation of the function’s domain.

B. Gaussian Measures.

For each \( m \geq 0 \), the action \( \tilde{G} \times H^m(\Delta) \) contains an irreducible invariant unitary action. If \( m > \frac{1}{2} \), then the essentially unique invariant Hilbert space norm is given by (using the invariant norm (0.24), and the invariant hyperbolic volume element)
\[
|f(z)(dz)^m|_{H^m \cap L^2}^2 = \frac{2m-1}{2\pi} \int_{\Delta} |f(z)|^2(1-|z|^2)^{2m-2}dx \land dy =
\]
\[
\frac{2m-1}{2\pi} \sum_{n\geq 0} |f_n|^2 2\pi \int_0^1 r^{2n}(1-r^2)^{m-2}rdr = \sum_{n\geq 0} |f_n|^2 \frac{\Gamma(n+1)\Gamma(2m)}{\Gamma(2m+n)}
\]

\[
= \sum_{n\geq 0} |f_n|^2 B(n+1,2m) = |f_0|^2 + \frac{1}{2m} |f_1|^2 + \frac{2!}{(2m)(2m+1)} |f_2|^2 + .. \tag{1.8}
\]

where \( f = \sum f_n z^n \). The seemingly unnatural factor \( \frac{2m-1}{2\pi} \) is inserted because the last sum shows that this norm can be analytically continued to \( m > 0 \), since the Beta function coefficients satisfy \( B(n+1,2m) > 0 \). In the critical case \( m = \frac{1}{2} \), the norm can be understood as the \( L^2 \) norm of the \( S^1 \)-boundary values of \( f \) with respect to \( |dz| = d\theta \), which is manifestly \( \tilde{G} \)-invariant. For \( 0 < m < \frac{1}{2} \), there does not appear to be an integral representation for the norm, but we will nonetheless continue to denote the norm by \( |\cdot|_{H^m \cap L^2} \). This norm continues to be \( \tilde{G} \)-invariant (it would obviously be desirable, if it is possible, to find a geometric realization which clearly displays the invariance).

The limit \( m \downarrow 0 \) is exceptional. The action of \( \tilde{G} \) on \( H^0 \) is reducible: \( \mathbb{C} \) (the constants) is an invariant subspace, with quotient determined by \( \partial : H^0(\Delta) \rightarrow H^1(\Delta) \). The norm as defined by (1.8) is not well-defined when \( m = 0 \). However, in the definition of the norm, we can multiply by \( m \). In this case we obtain a Hilbert space substructure for the quotient \( H^0/\mathbb{C} \) (the original vacuum, the constant 1, is now a ‘ghost’), and an isometry

\[
(H^0/\mathbb{C}) \cap L^2 \xrightarrow{\partial} H^1 \cap L^2 : x \rightarrow \theta = \partial x. \tag{1.9}
\]

We will make extensive use of this natural isomorphism of \( H^0/\mathbb{C} \) and \( H^1 \).

To each of these unitary representations there is a corresponding one parameter convolution semigroup of invariant Gaussian measures, written heuristically as

\[
d\nu_T^{(m)} = \frac{1}{Z_T} \exp(-\frac{1}{2T} |\theta|^2_{H^m \cap L^2}) dV(\theta). \tag{1.10}
\]

These measures are ergodic, by a general result of Irving Segal ([Se]).

In our context the Gaussian measure (1.10) is realized as an infinite product measure

\[
\prod_{n\geq 0} \frac{B(n+1,2m)}{2\pi T} e^{-\frac{1}{2T} B(n+1,2m)|f_n|^2} dm(f_n). \tag{1.11}
\]

In other words we are given a sequence of independent standard Gaussian complex random variables \( Z_n \), and we are considering the independent random variables

\[
f_n = a_n Z_n, \quad \text{where } a_n = (\frac{1}{T} B(n+1,2m))^{-1/2} = (\frac{(2m)_n T}{n!})^{1/2}, \tag{1.12}
\]
and a random Taylor series

\[ f(z) = \sum a_n Z_n z^n. \]  

(1.13)

If \( T = 1 \), note that when \( m = 1/2 \), then \( f = \sum Z_n z^n \). When \( m = 1 \), then \( \theta = f dz \), where \( f = \sum n^{1/2} Z_n z^n \), and

\[ x = \int_0^z \theta = \sum n^{-1/2} Z_n z^n. \]  

(1.14)

The \( L^2 \) norm of this on the circle diverges logarithmically, so this almost, but does not quite, have boundary values.

The series (1.13) is considered in great detail in [K], and here we merely note some of the main points.

(1.15) **Proposition.** Let \( \nu = \nu_T^{(m)} \).

(a) The radius of convergence of \( f \) is

\( \limsup a_n^{1/n} = 1 \) a.e. \( f [\nu] \),

i.e. \( \nu \) is supported on \( H^m(\Delta) \).

(b) For a.e. \( z \in S^1 \), (1.13) diverges, and \( f \) assumes all values in \( \mathbb{C} \), for a.e. \( f [\nu] \).

Suppose that \( m = 1 \), and consider \( x(z) \) in (1.14).

(c) Fix an angle \( \alpha \). For all \( a \in \mathbb{C} \),

\[ \liminf_{r \uparrow 1} |x(re^{i\alpha}) - a| = 0 \quad \text{a.e.} \quad x [\nu], \]

and in particular the \( x \) image of each ray is dense in \( \mathbb{C} \), a.e. \( x [\nu] \).

(d) Again fix the angle \( \alpha \). Then

\[ x(re^{i\alpha}) = O(\sqrt{\rho(r)ln(\frac{\rho(\sqrt{r})}{1-r})} \quad \text{as} \quad r \uparrow 1, \]

a.e. \( x [\nu] \), where \( \rho(r) = -ln(1 - r^2) \).

For (a), see section 2, for (b), see Theorem 1 of chapter 13, for (c), see Theorem 3, page 184, and for (d), see (22) on page 187 of [K].

The conditions (c) and (d) give us some feeling for what a random \( x(\theta) \) is like for the \( H^1 \)-Gaussian. It should be possible to refine these conditions to give criteria for a given \( x \) to have a dense \( G \)-orbit (see (6.3) below).

The central limit theorem implies the following
(1.16) Proposition. Suppose that $\nu$ is an invariant measure with the property that $\phi \ast \nu$ has finite variance for each $\phi \in (H^m \cap L^2)^*$. Then

(a) The variance of $\phi \ast \nu$ is independent of $\phi$, provided $|\phi|_{H^m \cap L^2} = 1$; denote this common value by $T$; and

(b) the scaled $n$-fold convolution

$$\nu \ast \nu \ast \ldots \ast \nu(\sqrt{n} \cdot) \to d\nu^H T_m \quad \text{as} \quad n \to \infty$$

in a weak sense.

Proof of (1.16). Let $\nu_n$ denote the scaled $n$-fold convolution: $\nu_n(E) = \nu^{n*}(\sqrt{n}E)$. The invariance of $\nu$ implies that the mean of $\nu$ is zero. Given $\phi \in (H^m \cap L^2)^*$, let $\sigma^2 = \sigma^2(\phi, \nu)$, the variance of the $\phi$ distribution of $\nu$. This is finite, by assumption.

The central limit theorem implies that

$$\phi \ast \nu_n \to \frac{1}{\sigma \sqrt{\pi}} e^{-\frac{|\phi|^2}{2\sigma^2}} dm(\phi) \quad \text{(1.17)}$$

Given $\phi$ and $\psi$, this implies that

$$\int \phi \overline{\psi} \nu_n \to \langle \phi, \psi \rangle \quad \text{as} \quad n \to \infty, \quad \text{(1.18)}$$

where $\langle \cdot, \cdot \rangle$ defines an invariant Hermitian inner product. By irreducibility, this inner product is necessarily a multiple of $| \cdot |_{H^m \cap L^2}$. This determines $T$ and proves (a).

Given $\phi$ and $\psi$ of unit length with $\phi \perp \psi$, by considering the distribution for $s\phi + t\psi$, we see that in the limit $n \to \infty$, $\phi$ and $\psi$ are independent. This implies that the pair $\phi, \psi$ has a Gaussian distribution. This implies (b). $\Box$

Note that if $\nu$ is a transitive measure, then $\nu$, and its scalings, have finite variance, and hence the Gaussians are in the closure of the convolution algebra generated by $\nu$ and its scalings.

For later use we recall the $n$-point functions for the standard Gaussians. Suppose that we fix a point $z_0 \in \Delta$. The continuous linear functional

$$eval_{z_0} : H^m(\Delta) \to \mathbb{C} : f(z)(dz)^m \to f(z_0), \quad \text{(1.18)}$$

restricted to $L^2$ differentials, is represented by the differential $(1 - \overline{z_0}z)^{-2m}(dz)^m$; that is,

$$f(z_0) = \langle f(dz)^m, (1 - \overline{z_0}z)^{-2m}(dz)^m \rangle_{H^m \cap L^2}. \quad \text{(1.19)}$$
Given \( n \)-points \( z_i \in \Delta \), the covariance matrix \( C \) has entries given by

\[
C_{ij} = \langle \text{eval}_{z_i}, \text{eval}_{z_j} \rangle = (1 - \bar{z}_i z_j)^{-2m}.
\] (1.20)

The \( \nu_1^{(m)} \)-distribution for the \( n \)-point function

\[
H^m(\Delta) \xrightarrow{\text{eval}(z_1, \ldots, z_n)} \mathbb{C}^n : f(z)(dz)^m \to (f(z_i))_{i=1, \ldots, n}
\] (1.21)

is given by the Gaussian measure

\[
2^{-n} \det C C^{-1} \exp\left(-\frac{1}{4} (C^{-1})_{jk} \bar{z}_j \bar{z}_k \right) \prod_{1}^{n} dm(z_j).
\] (1.22)

Suppose that \( m = 1 \). In this case it is more to the point to consider the distribution

\[
H^1 \to \mathbb{C}^n : \theta = f(z)dz \to (x(z_i) - x(0))_{1 \leq i \leq n},
\] (1.23)

where \( x \in H^0/\mathbb{C} \) is related to \( \theta \) as in (1.14). Now

\[
x(z_0) - x(0) = \sum x_n z_0^n = \sum nx_n(\frac{\bar{z}_0^n}{n})^* = \langle \sum x_n z^n, \sum(\frac{\bar{z}_0^n}{n})z^n \rangle_{H^n/\mathbb{C} \cap L^2}
\]

\[
= \langle x, \ln\left(\frac{1}{1 - \bar{z}_0 z}\right) \rangle
\] (1.24)

Therefore

\[
C_{ij} = \langle \ln(\frac{1}{1 - \bar{z}_i z}), \ln(\frac{1}{1 - \bar{z}_j z}) \rangle = \ln\left(\frac{1}{1 - \bar{z}_i z_j}\right).
\] (1.25)

C. Another Example.

The following elementary example is included to illustrate what is involved in constructing and analyzing an invariant measure, and the general scheme outlined in the introduction.

For simplicity suppose \( m = 1 \). Write \( \theta \in H^1 \) as

\[
\theta = (\theta_1 + \theta_2 z + \ldots)dz,
\] (1.26)

and define the projection \( P_n \theta = (\theta_1 + \ldots + \theta_n z^{n-1})dz \). Define a probability measure on \( P_n H^1 \) by

\[
d\mu_{l}^{(n)} = \frac{1}{Z} (1 + |P_n \theta|_{H^1 \cap L^2})^{-n-1-l} dm(P_n \theta) =
\]
\[
\frac{\Gamma(n + l + 1)}{\pi^n \Gamma(l + 1)} (1 + |\theta_1|^2 + \frac{1}{2} |\theta_2|^2 + \ldots + \frac{1}{n} |\theta_n|^2)^{-n-1-l} dm(P_n \theta).
\] (1.27)

This measure is finite precisely when \(-1 < l\), for each \(n\). In the context of the general scheme discussed in the introduction,

\[
D = 1 + |\theta|^2_{H^1 \cap L^2} = 1 + tr(B(x)B(x)^*),
\] (1.28)

and the critical exponent is \(p_n = n\).

These measures are coherent in the sense that

\[
(P_n)_* \mu_l^{(n)} = \mu_l^{(n-1)}.
\] (1.29)

The formal completion of \(H^1\) is the infinite product space which we write formally as

\[
H^1_{\text{formal}} = \{ \theta = (\sum_1^\infty \theta_j z^{j-1})dz : (\theta_j) \in \prod_1^\infty \mathbb{C} \}.
\] (1.30)

The coherence property, together with Kolmogorov’s theorem (page 228 of [B]), implies that there is a unique probability measure \(\mu_l\) on the formal completion such that

\[
(P_n)_* \mu_l = \mu_l^{(n)}
\] (1.31)

for all \(n\). There are a number of slightly different ways of showing that \(\mu_l\) is supported on \(H^1\): (1) it is a convex combination of Gaussians ((1.34) below), (2) one can compute its Fourier transform in spherical coordinates and observe that it is continuous on the dual of \(H^1\) (see (1.35)), and (3) in the next section we will see that \(\mu_l\) is a ‘quotient of Gaussians’.

Along the lines of (0.8), the measure \(\mu_l\) has a heuristic expression

\[
d\mu_l = \frac{1}{Z} \frac{1}{(1 + |\theta|^2_{H^1 \cap L^2})^{1+p+l}} dm(\theta),
\] (1.32)

where \(p = \infty\) in this case. Although of limited use, this expression does suggest that \(\mu_l\) depends only upon the unitary structure of \(H^1\), and hence that \(\mu_l\) should be a convex combination of Gaussians. In fact, for each \(n\)

\[
\int_0^\infty \left\{ \left(\frac{\beta}{\pi}\right)^n e^{-\beta |P_n \theta|^2} dm(P_n \theta) \right\} \frac{1}{\Gamma(1 + l)} \beta^l e^{-\beta} d\beta
\] (1.33)
and hence
\[ d\mu_l = \int_{\beta=0}^{\infty} \left( d\nu_{\beta-1}^{(1)} \right) \frac{1}{\Gamma(1+l)} \beta^l e^{-\beta} d\beta. \] (1.34)

These expressions are very interesting in the way that normalization constants and exponents interact.

Note that this implies that \( \mu_l \) is not ergodic, a fact which is not so easy to understand in a naive way.

In general, given an invariant measure \( \nu \) on \( H^m \), for \( m > 0 \), we define its Fourier transform using the unitary substructure,
\[ \hat{\nu}(F) = \int e^{-i\text{Re}(F,f)}_{H^m \cap L^2} d\nu(f) \] (1.35)
for \( F \in H^m_{\text{alg}} \) (this is well-defined for any finite measure on \( H^m_{\text{formal}} \), because \( H^m_{\text{alg}} \) is dual to the formal completion).

It is interesting to note that the Fourier transform of \( \mu_l \) can be written in two ways
\[ \int e^{-i\text{Re}(\theta,\theta')}_{H^1} d\mu_l(\theta) = \int_{0}^{\infty} \frac{1}{(1+r^2)^2} J_0(r|\theta'|_{H^1 \cap L^2}) r dr 
= \int_{0}^{\infty} e^{-|\theta'|^2/T} \frac{1}{\Gamma(1+l)} T^l e^{-T} dT. \] (1.36)

The first expression is obtained by integrating with respect to polar coordinates; the second expression follows from (1.34). If we think of this Fourier transform as a function of \( n \) variables, then it is a multiple of the fundamental solution of \( (1 + \Delta)^{1+n+l} \).

Note that both expressions clearly show that the Fourier transform is defined and continuous on \( (H^1)^* \). Hence the support properties of \( \mu_l \) and the Gaussians are roughly the same.

§2. Operations and Further Examples.

There are a number of basic operations which we can perform to obtain new invariant measures from known invariant measures.

Because \( \tilde{G} \) acts linearly on \( H^m \), we can scale a given measure, we can form the convolution of two invariant measures, and we can form convex combinations.

The multiplication map
\[ M : H^m \times H^n \to H^{m+n} : (\theta, \theta') \to \theta \theta' = f(z)g(z)(dz)^{m+n} \] (2.1)
is also $\tilde{G}$-equivariant, where $\theta = f(z)(dz)^m$ and $\theta' = g(z)(dz)^m$. Hence if $\nu$ and $\mu$ are invariant measures on $H^m$ and $H^n$, respectively, then $M_*(\nu \times \mu)$ is an invariant measure on $H^{m+n}$. We will refer to this measure as the multiplicative image of $\nu$ and $\mu$, and we will write $\nu \otimes \mu = M_*(\nu \times \mu)$.

We can also consider the quotient map

$$Q : (\theta, \theta') \mapsto \theta / \theta' = \frac{f(z)}{g(z)}(dz)^{m-n},$$

which is defined, provided $\theta'$ is nonvanishing. If $\mu$ is supported on nonvanishing differentials, then $Q_*(\nu \times \mu)$ will be a well-defined invariant measure on $H^{m-n}$.

It is definitely the case that there exist transitive $\mu$ on $H^n$ which are supported on nonvanishing differentials, assuming that $n$ is rational; this was one of the main points of example (b) of (1.3). However for measures having infinite dimensional support, one expects that the sample properties will be similar to the Gaussian case. For the Gaussians, provided $m > 0$, part (b) of (1.15) implies that with probability one, a typical $f$ vanishes at some points.

For this reason, in thinking about $Q$, it is useful to consider the formal completions of the spaces $H^m$. The formal completion of $H^m$, denote $H^m_{\text{formal}}$, is the space consisting of differentials $\theta = f(dz)^m$, where $f$ is simple a formal power series (see (1.30)). We view $Q$ as a map

$$Q : H^m_{\text{formal}} \times \{ \theta' \in H^n_{\text{formal}} : g_0 \neq 0 \} \rightarrow H^{m-n}_{\text{formal}}$$

where $f/g$ in (2.2) is interpreted as the formal power series

$$\frac{f}{g} = \frac{f_0}{g_0} + \frac{f_1g_0 - f_0g_1}{g_0^2}z + ..$$

(2.5) Proposition. If $\nu$ and $\mu$ are ergodic measures, then, whenever defined, the convolution $\nu * \mu$, the multiplicative image $\nu \otimes \mu$, and the quotient image $Q_*(\nu \times \mu)$ are ergodic invariant measures.

This follows from the fact that $\tilde{G}$ does not have any finite dimensional unitary representations, see [Se].

Now recall the natural map $H^m \rightarrow C(\Delta)$. There is a natural operation of convolution for measures on $C(\Delta)$, corresponding to the natural union operation

$$C(\Delta) \times C(\Delta) \rightarrow C(\Delta) : \gamma_1, \gamma_2 \mapsto \gamma_1 \cup \gamma_2.$$

It is well-known, for example, that for the Poisson measures

$$p_{\lambda_1} * p_{\lambda_2} = p_{\lambda_1} * p_{\lambda_2}.$$
(2.6) Proposition. If $\lambda > 0$, then $Z_\ast(\lambda_*\nu) = Z_\ast(\nu)$. Also $Z_\ast$ intertwines $\otimes$ and convolution.

In this paper we will not make any use of this proposition, simply because we do not know how to compute $Z$ for any nontransitive examples.

Some Calculations.

We would like to be able to calculate the Fourier transform of the measure 

$$\nu_1^{(m_1)} \otimes \cdots \otimes \nu_1^{(m_n)}$$

(2.7)

We are able to do this in a completely satisfactory way only for pairs. We will need the fact that there is a essentially unique $\tilde{G}$-equivariant embedding

$$H^{m+n} \cap L^2 \to (H^m \cap L^2) \otimes (H^n \cap L^2) = L_2(H^m(\Delta^*) \cap L^2, H^n \cap L^2),$$

and this embedding is completely determined by requiring that it map $(dz)^{m+n}$ to $(dz)^m \otimes (dz)^n$.

The is a consequence of the following standard result.

(2.9) Lemma. As a $\tilde{G}$-representation

$$(H^m \cap L^2) \otimes (H^n \cap L^2) = (H^{m+n} \cap L^2) \oplus (H^{m+n+1} \cap L^2) \oplus \cdots$$

(2.10)

Furthermore multiplication, $M$ in (2.1), is the orthogonal projection

$$0 \to \text{Ker} \to (H^m \cap L^2) \otimes (H^n \cap L^2) \xrightarrow{M} (H^{m+n} \cap L^2) \to 0.$$

(2.11)

Proof of (2.9). The action of (the covering in $\tilde{G}$) of rotations extends to a holomorphic contraction representation on $H^m \cap L^2$. The corresponding character of $H^m$ is $q^m/(1-q)$, where $q = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$, and this character is a complete invariant of this (lowest weight) representation. Now simply observe that the characters of the two sides of (2.10) are equal:

$$\frac{q^m}{1-q} \cdot \frac{q^n}{1-q} = \frac{q^{m+n} + q^{m+n+1} + \cdots}{1-q}. $$

(2.12)

The second claim follows from irreducibility of $H^{m+n}$. □

Given $F \in H^{m+n} \cap L^2$, we will denote the Hilbert- Schmidt operator corresponding to $F$ via (2.8) by $B(m, n; F)$. We will write out the matrix for this operator in (2.17) below. This calculation will make it clear that $B(m, n; F)$ is a kind of generalized Hankel operator (note that we are also using $B$ for the Beta function, but the number of arguments and context should always make it clear which is intended).
(2.13) Proposition.
\[
(v_S^{(m)} \otimes v_T^{(n)})(F) = \frac{1}{\det(1 + STB(m, n; F)B(m, n; F *))}
\]
for \( F \in H^{m+n} \cap L^2 \).

Proof of (2.13). Suppose \( f = f(z)(dz)^m \in H^m \), \( g = g(z)(dz)^n \in H^n \), and \( F = F(z)(dz)^{m+n} \in H^{m+n}_{alg} \). Then
\[
\langle F, fg \rangle_{H^{m+n}} = \sum_l \sum_{k \leq l} F_l \bar{f}_{l-k} \bar{g}_k B(l + 1, 2(m + n))
\]
\[
= \sum_k \left( \sum_j F_{k+j} \bar{f}_j \frac{B(k + j + 1, 2m + 2n)}{B(k + 1, 2n)} \right) \bar{g}_k B(k + 1, 2n). \tag{2.14}
\]
This last sum is the \( H^n \) inner product of \( f \) and \( h \), where \( h_k \) is the term in braces in (2.14) (which depends upon both \( F \) and \( f \)). The equivariant properties of the linear mapping, depending upon \( F \), which sends \( \bar{f} \) to \( h \) are the same as those for the operator \( B(m, n; F) \). The uniqueness of the embedding (2.8) implies that
\[
\langle F, fg \rangle_{H^{m+n}} = \langle B(m, n; F)\bar{f}, g \rangle_{H^n}. \tag{2.15}
\]

(2.16) Remark. The calculation (2.14) implies that the matrix of \( B(m, n; F) \), relative to the orthogonal (but not necessarily orthonormal) bases \( z^k(dz)^m \) and \( z^l(dz)^n \) for \((H^m \cap L^2)^* \) and \( H^n \cap L^2 \), respectively, has entries given by
\[
B(m, n; F)_{j,k} = F_{k+j} \frac{B(k + j + 1, 2m + 2n)}{B(k + 1, 2n)} \tag{2.17}
\]
If we scale these bases to obtain orthonormal bases, then the matrix entries are
\[
\frac{B(k + j + 1, 2m + 2n)}{\sqrt{B(j + 1, 2m)B(k + 1, 2n)}}. \tag{2.18}
\]
In the special case that \( m = n = \frac{1}{2} \), \( B(\cdot, 2m) = B(\cdot, 2n) = 1 \), \( B(k + j + 1, 2m + 2n) = \frac{1}{k+j+1} \), and hence \( B(\frac{1}{2}, \frac{1}{2}, F) = B(x) \), where \( x \) is the antiderivative of \( F(z)dz \).

Using (2.15), we now see, by first integrating with respect to \( g \), that
\[
(v_S^{(m)} \otimes v_T^{(n)})(F) = \int e^{-\frac{1}{2}(A\bar{f}, f)} d\nu_S^{(m)}(f), \tag{2.19}
\]
where \( A = B(m, n; F)B(m, n; F)^* \). There is a standard formula for the Gaussian integral of the exponential of a quadratic functional, as in (2.19), and this implies (2.13). \( \square \)
(2.20) Corollary. (a) \( \det(1 + B(m, n; F))B(m, n; F)^{-j} \) is a positive definite function of \( F \in H^{m+n} \cap L^2 \), for \( j = 1, 2, \ldots \).

(b) The Fourier transform of \( \nu_S^{(m)} \otimes \nu_T^{(n)} \) is continuous on \( H^{m+n} \cap L^2 \).

It is definitely interesting to ask whether (a) is true for nonintegral \( j > 0 \) (and what kind of measure would correspond to this Fourier transform). It is not so hard to see that (b) holds for any of the measures (2.7). But whether there is a tractable formula for the Fourier transform of (2.7), in general, is unclear.

Now we consider the quotient operation on formal completions.

Here is a particular application of this construction which shows that in general an invariant probability on the formal completion of \( H^m \) is not determined by its one-point function (which we conjectured is the case for transitive measures, in (1.7)).

(2.21) Proposition. Fix \( T > 0 \), and \( m \geq 1 \). The image measure

\[ \nu_m = Q_*(\nu^H_m \times \nu^H_{m-1}) \]

is a \( G \)-invariant probability measure on \( H^1(\Delta) \) with the property that

\[ (eval_0)_* \nu_m = \frac{1}{Z} \frac{dm(q_0)}{1 + |q_0|^2}. \]

Proof of (2.21). For all \( m \), \( (eval_0)_* \nu^H_m = \frac{1}{Z} e^{-\frac{1}{2T} |q_0|^2} dm(q_0) \). The proposition now follows from the commutativity of the diagram

\[
\begin{array}{ccc}
H^m \times H^{m-1} & \xrightarrow{Q} & H^1 \\
\downarrow eval_0 & & \downarrow eval_0 \\
\mathbb{C} \times \mathbb{C} & \rightarrow & \mathbb{C} : (z, w) \rightarrow \frac{z}{w}
\end{array}
\]  

(2.22)

together with the fact that a ratio of independent Gaussians (with the same temperature) is the measure above.

\[ \square \]

(2.23) Example. The simplest example is when \( m = 1 \), where we interpret \( H^0 \) to be \( \mathbb{C} \) with the usual Gaussian. In this case we obtain the measure on \( H^1 \) discussed in §1C. This explains in another way why that measure is supported on \( H^1 \).

Although a little tedious, it is a straightforward matter to compute the \( n \)-point function for \( \nu_m \) in general (using the well-known fact that the \( n \)-point functions for a Gaussian are determined by the 2-point functions and note that when we
say \( n \)-point function, we actually have \( n \) complex values, so that we have \( 2n \) real values). It is given by

\[
\frac{1}{2^n \det C(m)C(m-1)} \sum_{i,j \in \text{Perm}(n)} \prod_{l=1}^{n} [(1+C(m-1)\text{diag}(q_k)C(m)^{-1}\text{diag}(\bar{q}_k))^{-1}C(m-1)]_{ij}\]

where \( C(m) \) denotes the covariance for the \( n \)-point function corresponding to the points \( q_k \) for the Gaussian \( \nu^H_m \). From this expression, say by looking at the case \( n = 2 \), one can see that the \( \nu_m \) are distinct measures, as \( m \) varies. Thus the one-point function certainly does not determine an invariant probability (on the formal completion).

Using this construction it is possible to construct invariant probabilities on \( H^m \) with many different one-point functions. But given a candidate for a one-point function, i.e. a \( \text{Rot} \)-invariant probability on \( \mathbb{C} \), I have no idea how to decide whether it comes from an invariant probability on \( H^m \).

§3. Linear Operator-Valued Functions, and Associated Measures

In the first part of this section, after some further discussion of the sense in which the map from \( \theta \in H^1 \cap L^2 \) to the Hankel matrix (0.11) is \( PSU(1,1) \)-equivariant, we will discuss the limit (0.13). In the second part of the section, using the operators \( B(m,n;F) \) of §2 (and more general, generalized Hankel operators), we indicate one type of generalization (which also involve more complexity).

Hankel Operators and \( m = 1 \).

The circle has a unique nontrivial (or Moebius) spin structure: the \( (C^r) \) sections are of the form \( f(\theta)(d\theta)^{1/2} \), where \( f \) is a \textit{real-valued} \( (C^r) \) function satisfying \( f(\theta + 2\pi) = -f(\theta) \) (Note that such an odd real-valued function must vanish, and this explains why the bundle is trivial). The complexification of this real line is the restriction to \( S^1 \) of \( \kappa^{1/2} \), the unique holomorphic square root of the canonical line bundle \( \kappa \) for \( \hat{\mathbb{C}} \). This complexified bundle is trivial: a global section is

\[
(dz)^{1/2} = \sqrt{i} e^{i\theta/2}(d\theta)^{1/2},
\]

where we fix a choice of square root for \( i \). The group \( SU(1,1) \) (the double cover of \( G! \)) acts on this bundle, as in (0.5). We will denote the space of sections of this
complexification by $\Omega^{1/2}(S^1)$ (or $\Omega^0(\kappa^{1/2}|_{S^1})$). There is a natural Hermitian inner product

$$\Omega^{1/2}(S^1) \otimes \Omega^{1/2}(S^1)^{\text{conj}} \rightarrow \mathbb{C} : \theta \otimes \bar{\eta} \rightarrow \int_{S^1} \theta \bar{\eta}$$

(3.2)

The point is that if $\theta = f(z)(dz)^{1/2}$ and $\eta = g(z)(dz)^{1/2}$ are odd spinors on $S^1$, then $\theta \bar{\eta} = f\bar{g}|dz|$ is naturally a one density on $S^1$, which can be integrated. We will write $\Omega^{1/2}(S^1) \cap L^2$ when we are thinking of this space as a Hilbert space.

There is a Hilbert space isomorphism

$$L^2(S^1, \mathbb{C}) \rightarrow \Omega^{1/2}(S^1) \cap L^2 : f(z) \rightarrow f(z)(dz)^{1/2}.$$  

(3.3)

In this identification the Hardy polarization of $L^2(S^1, \mathbb{C})$ used in [PS] is identified with the $SU(1,1)$-equivariant polarization

$$\Omega^{1/2}(S^1) \cap L^2 = H^{1/2} \cap L^2 \bigoplus H^{1/2}(\Delta^*) \cap L^2.$$  

(3.4)

The identification (3.3) will be fixed from this point onward.

As in [PS], given $x \in L^\infty(S^1, \mathbb{C})$, we obtain a multiplication operator $M_x$ on $\Omega^{1/2}(S^1)$. With respect to the Hardy polarization, we write

$$M_x = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

(3.5)

where $A$ is referred to as the Toeplitz operator associated to $g$, and $B = B(x)$ (or $C$) is the Hankel operator associated to $x$. Relative to the basis $z^j(dz)^{1/2}$, the matrix of $M_x$ is constant along diagonals, and in particular the matrix of $B$ is given by (0.11).

Note that if $x \in H^0/\mathbb{C}$, then the $L^2$ norm of $x$ is the same as the Hilbert-Schmidt norm of $B(x)$. These considerations imply the following

**(3.6) Proposition.** The map $x \rightarrow B(x)$ is equivariant with respect the natural unitary action of $PSU(1,1)$ on $x \in H^0/\mathbb{C} \cap L^2$ and the action by unitary conjugation of $SU(1,1)$ on

$$B(x) \in L_2(H^{1/2}(\Delta^*) \cap L^2, H^{1/2} \cap L^2) = H^{1/2} \cap L^2 \otimes H^{1/2} \cap L^2.$$
Now fix \( N \). Define \( B_N(x) \) to be the finite Hankel matrix

\[
B_N(x) = \begin{pmatrix}
  x_N & 0 & \ldots \\
  . & x_N & 0 & \ldots \\
  . & . & \ddots & \ddots \\
  x_2 & x_3 & \ldots \\
  x_1 & x_2 & x_3 & \ldots & x_N
\end{pmatrix}
\]  \tag{3.7}

It is clear that for sufficiently large \( p \),

\[
\frac{1}{\det(1 + B_N B_N^*)^p} \in L^1(dm(x_1, \ldots, x_N)).
\]  \tag{3.8}

Let \( p_N \) denote the infimum of these \( p \), and

\[
Z(p, N) = \int \frac{1}{\det(1 + B_N B_N^*)^p} dm(x_1, \ldots, x_N) < \infty, \quad p_N < p.
\]  \tag{3.9}

It is clear that \( p_1 \leq p_2 \leq \ldots \).

(3.10) Examples. (1) For \( N = 1 \)

\[
Z(p, 1) = \int_{\mathbb{C}} \frac{1}{(1 + |x_1|^2)^p} dm(x_1) = \frac{\pi}{p - 1}.
\]  \tag{3.11}

Hence, following the notation in the introduction, \( p_1 = 1 \). The corresponding normalized measure, as in (0.14), is

\[
\frac{l + 1}{\pi} \frac{1}{(1 + |x|^2)^{1+1+l}} dm(x_1)
\]  \tag{3.12}

The Fourier transform \( u(y_1) \) is the multiple of the fundamental solution for

\[
(1 + \Delta_{y_1})^{1+1+l}
\]  \tag{3.13}

normalized so that \( u(0) = 1 \), i.e.

\[
(1 + \Delta)^{1+1+l} u(y_1) = \frac{l + 1}{\pi} \delta_0(y_1).
\]  \tag{3.14}

(2) For \( N = 2 \)

\[
\int \frac{1}{(1 + |x_1|^2 + 2|x_2|^2 + |x_2|^4)^p} dm(x_1, x_2) =
\]  \tag{3.15}
\[ \int \frac{1}{(|x_1|^2 + (1 + |x_2|^2)^2)^p} dm(x_1, x_2) = \quad (3.16) \]

\[ \int \frac{dm(x'_1)}{(1 + |x'_1|^2)^p} \cdot \int \frac{dm(x_2)}{(1 + |x_2|^2)^{2p-2}} \quad (3.17) \]

where we factored out the \((1 + |x_2|^2)^2\) term, and made a change of variable in \(x_1\). Thus

\[ Z(p, 2) = \frac{\pi^2}{(p - 1)(2p - 3)}. \quad (3.18) \]

and \(p_2 = 3/2\). Thus the normalized measure, as in (0.14), is

\[ 2(l + 1)(\frac{3}{2} + l) \cdot \frac{1}{\pi^2} \cdot \frac{1}{(|x_1|^2 + (1 + |x_2|^2)^2)^{1+\frac{3}{2}+l}} dm(x_1, x_2) \quad (3.19) \]

The Fourier transform \(u(y_1, y_2)\) is the multiple of the fundamental solution for

\[ (\Delta_{y_1} + (1 + \Delta_{y_2}) \cdot (1 + \frac{3}{2} + l)^{1+\frac{3}{2}+l} \quad (3.20) \]

normalized so that \(u(0) = 1\), i.e.

\[ (\Delta_{y_1} + (1 + \Delta_{y_2}) \cdot (1 + \frac{3}{2} + l)^{1+\frac{3}{2}+l} u(y_1, y_2) = \frac{2(l + 1)(\frac{3}{2} + \frac{3}{2})}{\pi^2} \delta_0(y_1, y_2). \quad (3.21) \]

It seems unlikely that there is a simple factorization which allows one to directly evaluate the integrals when \(2 < N\). However a miracle occurs. There is a very simple torus action

\[ \mathbb{T}^2 \times \{x = \sum_1^N x_j z^j\} \rightarrow \{(\sum_1^N x_j z^j) : ((\lambda, \mu), x) \rightarrow \sum_1^N \mu x_j \lambda^j z^j. \quad (3.22) \]

It turns out that this is a Hamiltonian action, relative to a nonobvious symplectic form, with origins in the theory of loop groups, and \(logdet(1 + BB^\ast)\) is a component of the momentum map. Consequently one can apply the Duistermaat-Heckman theorem to evaluate the integrals. We will simply state the result; the details will appear in [Pi3].

\textbf{(3.23) Proposition. In general}

\[ \int \frac{1}{\det(1 + B_N B_N^\ast)^p} dm(x_1, \ldots, x_N) = \frac{\pi^N}{N!(p - 1)(p - (2 - \frac{1}{2}) \ldots (p - (2 - \frac{1}{N}))}. \]

\textit{Hence, in the notation of (0.14),} \(p_N = 2 - \frac{1}{N}\).
(3.24) Remarks. (1) In addition to having an explicit formula, this says that we have an incredibly simple recursion relation
\[ Z(p, N + 1) = \frac{\pi}{(N + 1)(p - (2 - \frac{1}{N}))} Z(p, N). \] (3.25)

There must be some direct proof of this.

(2) Note that ignoring the \( \pi^N \) factor, (3.23) equals
\[ \prod_{k=1}^{N} (kp - (2k - 1))^{-1} = \prod_{k} (1 + (p - 2)k)^{-1} = \frac{z^N}{N!} \prod_{k=1}^{N} \frac{1}{1 + \frac{z}{k}} \]
\[ = \frac{z^N \Gamma(z + 1)}{\Gamma(z + N + 1)} \] (3.26)
where \( z = \frac{1}{p-2} \). This is related to the Γ function,
\[ \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left( 1 + \frac{1}{1 + \frac{z}{k}} \right) e^{z/k}, \] (3.27)

(3.28) Theorem. The sequence of probability measures
\[ \frac{1}{\pi^N} \prod_{k=1}^{N} (1 + (l + 1)k)^{-1} \frac{1}{\det(1 + B_N(x)B_N(x)^*)^{1+p_N+l}} dm(x_1, \ldots, x_N) \]

converge to an invariant measure on \( H^1 = H^0 / \mathbb{C} \), for each \(-1 < l \).

(3.29) Remarks. (a) This would be a beautifully simple result if these measures indexed by \( N \) were coherent. But unfortunately this is not the case (although when one calculates (using Maple, say), one finds that they are close to being coherent, even for small \( N \)).

(a) It is clearly important to compute the exact \( P_N \) distributions, or their Fourier transforms. Another possibility is to consider the values of \( \theta \) at points of a configuration in \( \Delta \), and try to compute the distribution for \((\theta(z_1), \ldots, \theta(z_n))\).

Corresponding to this configuration we consider
\[ \theta = \sum \frac{\theta(z_j)}{(1 - \bar{z}_j z)^2} \] (3.30)

This is the function that represents evaluation at the points. The corresponding
\[ x = \sum \frac{\theta(z_j)}{\bar{z}_j (1 - \bar{z}_j z)} + \text{constant} \] (3.31)
\[ = \sum \theta(z_j) \frac{z}{1 - \bar{z}_j z} \quad (3.32) \]

Unfortunately it is not clear whether this yields any advantage in trying to understand the corresponding integrals involving \( \text{det}(1 + BB^*) \) for this form of \( x \).

Generalized Hankel Operators.

Fix \( m, n > 0 \). Define

\[ B_N(m, n; F) = B(m, n; (\sum_{j=0}^{N-1} F_j z^j)(dz)^{m+n}) \quad (3.33) \]

We will abbreviate this to \( B_N \).

It is clear that for sufficiently large \( p \),

\[ \frac{1}{\text{det}(1 + B_N B_N^*)^p} \in L^1(dm(F_0, \ldots, F_{N-1})). \quad (3.34) \]

Let \( p_N \) denote the infimum of these \( p \), and

\[ \mathcal{Z}(m, n; p) = \int \frac{1}{\text{det}(1 + B_N B_N^*)^p} dm(F_0, \ldots, F_{N-1}) < \infty, \quad p_N < p. \quad (3.35) \]

With these additional parameters floating around, it seems less likely that a localization argument will apply to calculate \( \mathcal{Z} \) exactly. However there is some remote possibility that there is some relation to Selberg integrals.

\textbf{(3.23) Question.} Is there some obvious way to generalize our formula for \( \mathcal{Z} \) so that when \( m = n = \frac{1}{2} \)

\[ \mathcal{Z}(m, n, N; p) = \frac{\pi^N}{N!(p-1)(p-(2-\frac{1}{N})) \ldots (p-(2-\frac{1}{N}))}. \]

Regardless of whether there is an exact formula in this generality or not, it should always be the case that

\[ p_c = p_c(m, n) = \sup_N p_N < \infty. \]
(3.28) Conjecture. For $p_c < p$, the sequence of probability measures

$$\frac{1}{Z(m, n, N; p)} \det(1 + B_N(m, n; F)B_N(m, n; F)^*)^p dm(P_N F)$$

converge to an invariant measure on $H^{m+n}$.

It is straightforward to generalize these conjectures in various ways. For example using the decomposition

$$H^{m_1} \cap L^2 \otimes \ldots \otimes H^{m_r} \cap L^2 \otimes H^{n_1} \cap L^2 \otimes \ldots \otimes H^{n_s} \cap L^2 = H^{\sum(m_j+n_j)} \cap L^2 \oplus \ldots,$$ (3.29)

generalizing (3.9), there is a Hilbert-Schmidt operator

$$B(\vec{m}, \vec{n}; F) \in L_2((H^{m_1} \cap L^2 \otimes \ldots \otimes H^{m_r} \cap L^2)^*, H^{n_1} \cap L^2 \otimes \ldots \otimes H^{n_s} \cap L^2)$$ (3.30)

for $F \in H^{\sum(m_j+n_j)} \cap L^2$. All of these generalizations involve equivariant operator-valued maps which are linear.

Other Generalizations.

Recall from Lemma (2.9) that

$$H^{1/2} \cap L^2 \otimes H^{1/2} \cap L^2 = H^1 \cap L^2 \oplus H^2 \cap L^2 \oplus \ldots$$

From this perspective the map

$$H^m \cap L^2 \to H^{1/2} \cap L^2 \otimes H^{1/2} \cap L^2$$

is a natural generalization of the classical Hankel map $x \to B(x)$, which corresponds to $m = 1$. To appreciate the special nature of these maps, we go back to the beginning.

The infinitesimal action of $sl(2, \mathbb{C})$ on $H^m$ is given by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to -\frac{\partial}{\partial z}$$

$$A^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to z^2 \frac{\partial}{\partial z} + 2mz$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to 2z \frac{\partial}{\partial z} + 2m$$ (A.1)
The spectrum of the Hamiltonian is
\[ 2m, 2m + 2, 2m + 4, \ldots \] (A.2)

From this one can read off that the action is essentially irreducible, except when \( m \) is a nonpositive half-integer. For \( s = 0, \frac{1}{2}, 1, \ldots \), and \( m = -s \), there is a short exact sequence
\[ 0 \to \mathbb{C}^{2s+1} = \text{span}\{z^0(dz)^m, \ldots, z^{2s}(dz)^m\} \to H^m \to H^{1+s} \to 0. \] (A.7)

We are interested in the ‘bosonic cases’ \( s = 0, 1, \ldots \). The case \( s = 0 \) is classical Hankel
\[ H^0 \xrightarrow{\theta} H^1 : x \to \theta \] (A.3)

As we have emphasized, the Hankel operator is thought of most naturally in terms of \( x \) (this is also the linearization of the mapping we will consider in §4).

Suppose \( s = 1 \). In this case () is realized as the map
\[ 0 \to sl(2, \mathbb{C}) \to H^{-1} \xrightarrow{dS} H^2 \to 0 \] (A.5)

where
\[ dS : v = (v_2 z^3 + \ldots) \frac{\partial}{\partial z} \rightarrow Q = v'''(dz)^2 = (Q_2 + Q_3 z + \ldots)(dz)^2, \]
\[ Q_2 = 3 \cdot 2v_2, Q_3 = 4 \cdot 3v_3, \ldots \]
(This is the linearization of the Schwarzian
\[ SL(2, \mathbb{C}) \to \exp(H^{-1}) \xrightarrow{S} H^2 \] (A.6)

where we think of \( \exp(H^{-1}) \) as formal holomorphic automorphisms of the disk, and which we pursue in §5).

In this case, instead of considering the multiplicative action of \( x \in H^0 \) on \( \Omega^{1/2} \), we consider the natural infinitesimal action of the vector field \( v \). Because
\[ \mathcal{L}_v(z^n(dz)^{1/2}) = (v(z)nz^{n-1} + \frac{1}{2}v'(z)z^n)(dz)^{1/2} \]
\[ = (z^n \sum_{j=0}^{\infty} \frac{1}{2}(2n + j + 1)v_j z^j)(dz)^{1/2} \]
This leads to

\[ B(v) = \frac{1}{2} \begin{pmatrix}
4v_5 & 2v_5 & \ldots \\
3v_4 & 2v_5 & 0 \\
v_2 & v_4 & 0 \\
0 & -v_4 & -2v_5 \\
0 & -v_2 & -2v_3 & -3v_4 & -4v_5 & \ldots
\end{pmatrix} \]

At this point I do not have any feeling for the behavior of the critical exponents.

In the next case we should seek a map making the following diagram commute:

\[ H^{-2} \quad \vdash \quad H^3 \quad \rightarrow \quad H^{1/2} \otimes H^{1/2} \]

and so on.

§4. Loop Groups and Conformal Invariance.

As discussed in the introduction, the theory of loop groups yields further examples of conformally invariant measures in \( H^1 \). In place of (0.11) and (0.12), we will consider a more complicated equivariant operator-valued function

\[ \theta_+ \rightarrow W(x_+) \]

and the associated invariant function

\[ \det(1 + W(x_+)W(x_+)^*) \] (4.1)

where \( \theta_+ \in H^1(\Delta, \mathfrak{k}^\mathbb{C}) \) and \( x_+ = \int \theta_+ \). This leads to a family of invariant measures on \( H^1(\Delta, \mathfrak{k}^\mathbb{C}) \) of the form

\[ \frac{1}{Z} \frac{1}{\det(1 + WW^*)^{\frac{1}{2}(2g+l)}} dm(\theta_+), \] (4.2)

where \( l \geq 0 \), and the other symbols will be explained below.

Let \( K \) denote a connected compact Lie group, and let \( K^\mathbb{C} \) denote the complexification. We are mainly interested in the case when the Lie algebra \( \mathfrak{k} \) is simple, e.g. \( K = SU(n) \). But we will want to compare with the abelian case, \( K = U(1) \), so we will not assume simplicity at the outset.

Fix a representation of \( K \) in \( U(N) \). As in §3, there is a Hilbert space isomorphism

\[ L^2(S^1, \mathbb{C}^N) \rightarrow \Omega^0(\kappa^{1/2} \otimes \mathbb{C}^N) : f(z) \rightarrow f(z)(dz)^{1/2}, \] (4.3)
and the Hardy polarization of $L^2(S^1, \mathbb{C}^N)$ is identified with the $SU(1,1)$-equivariant polarization

$$
\Omega^{1/2} \cap L^2 \otimes \mathbb{C}^N = (H^{1/2} \cap L^2 \otimes \mathbb{C}^N) \oplus (H^{1/2}(\Delta^*) \cap L^2 \otimes \mathbb{C}^N).
$$

(4.4)

As in Chapter 6 of [PS], given $g \in LK^\mathbb{C}$, we obtain a multiplication operator $M_g$ on $\Omega^{1/2} \cap L^2 \otimes \mathbb{C}^N$. Relative to the Hardy polarization (4.4),

$$
M_g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
$$

(4.5)

where $A$ is referred to as the Toeplitz operator associated to $g$, and $B$ is the Hankel operator. If we introduce the standard ordered orthonormal basis

$$
\ldots, z^{2}(dz)^{1/2}, z^{1}(dz)^{1/2}, z^{0}(dz)^{1/2}, z^{-1}(dz)^{1/2}, z^{-2}(dz)^{1/2}, \ldots
$$

then $M_g$, and its Toeplitz and Hankel operators, are represented by infinite block matrices,

$$
M_g = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & g_0 & g_1 & g_2 & \cdot & \cdot \\
\cdot & g_0 & g_1 & g_2 & g_3 & \cdot \\
\cdot & g_0 & g_1 & \cdot & \cdot & \cdot \\
\cdot & g_0 & g_1 & g_2 & \cdot & \cdot \\
\cdot & g_0 & g_1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}
$$

(4.5')

which are constant along diagonals, where $g = \sum g_n z^n$ is the Fourier series of $g$ as a $\mathcal{L}(\mathbb{C}^N)$-valued function. We also introduce the graph operators $Z = CA^{-1}$ and $W = A^{-1}B$; these arise when we consider the matrix LDU factorization

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - ZB \end{pmatrix} \begin{pmatrix} 1 & W \\ 0 & 1 \end{pmatrix}.
$$

(4.6)

Because the Hardy polarization identifies with a $SU(1,1)$-invariant polarization, the operators $A$, $B$, $C$, $D$, $Z$ and $W$, are all equivariant operator-valued functions, in an appropriate sense; for example, the map to the Toeplitz operator

$$
L K^\mathbb{C} \to \mathcal{L}(H^{1/2} \cap L^2 \otimes \mathbb{C}^N) : g \to A(g)
$$

(4.7)

intertwines the natural automorphic action of $PSU(1,1)$ and the action of $SU(1,1)$ by conjugation. We will mainly focus on $W$ below.
It is a basic fact that \( A = A(g) \) is invertible if and only if \( g \in LK^C \) has a Riemann-Hilbert factorization

\[
g = g_- \cdot g_0 \cdot g_+, \tag{4.8}
\]

where \( g_- \in H^0(D^*, \infty; K^C, 1) \), \( g_0 \in K^C \), and \( g_+ \in H^0(D, 0; K^C, 1) \), and in this case

\[
A((g_0 g_+)^{-1}) = 1_{N \times N} \quad \text{or} \quad g_0 g_+ = (A^{-1}(\epsilon_1), \ldots, A^{-1}(\epsilon_N))^{-1} \tag{4.9}
\]

where \( 1_{N \times N} \) denotes the identity \( N \times N \) matrix, we are applying the operator \( A \) column by column in the first equation, and \( \epsilon_i \) denotes the standard basis for \( \mathbb{C}^N \) in the second equation. Consequently there is a \( PSU(1, 1) \times K^C \)-equivariant map

\[
\{ g \in LK^C : \det |A(g)|^2 \neq 0 \} \rightarrow H^1(D, \mathfrak{k}^C) : g \rightarrow \theta_+ = g_+^{-1} \partial g_+.
\]

In the following Lemma, which summarizes well-known facts, \( c(\cdot, \cdot) \) denotes the Kac-Moody cocycle defined in Proposition (6.6.4), page 88 of [PS].

**Lemma.** For \( g \in LK \), with Riemann-Hilbert factorization as in (4.8),

\[
det (A(g)^* A(g)) = \det (1 + WW^*)^{-1}, \quad W = A^{-1} B
\]

\[
= \det (A(g) A(g)^*) = \det (1 + Z^* Z)^{-1}, \quad Z = C A^{-1}
\]

\[
= \det \{ A(g_+) A(g_+^* A(g_+) A(g_+ g_+^*)^{-1}\} = c(g_+, g_+^*).
\]

and

\[
Z(g) = Z(g_-), \quad W(g) = W(g_+).
\]

These determinants are finite precisely when \( g \) belongs to the Sobolev space \( W^{1/2} \).

**Proof of (4.11).** Note that

\[
M_g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} = \begin{pmatrix} A(g_-) & 0 \\ C(g_-) & D(g_-) \end{pmatrix} \begin{pmatrix} A(g_0 g_+) & B(g_0 g_+) \\ 0 & D(g_0 g_+) \end{pmatrix}\]

\]

(4.12)

This implies that

\[
Z(g) = C(g) A(g)^{-1} = C(g_-) A(g_0 g_+) \{ A(g_-) A(g_0 g_+)^{-1}\} = Z(g_-) \tag{4.13}
\]

and similarly \( W(g) = W(g_+) \). Therefore

\[
1 + Z^* Z = 1 + (C(g_-) A(g_-)^{-1})*C(g_-) A(g_-)^{-1}
\]
\[ (A(g_\ast)A(g_\ast))^{-1}A(g_\ast g_\ast) = A(g_\ast)^{-1}A(g_\ast)A(g_\ast)^{-1}A(g_\ast g_\ast). \quad (4.14) \]

with a similar expression involving \( g_+ \).

By inspecting the matrix expression \((4.5')\), it is apparent that \( B \) and \( C \) are Hilbert-Schmidt if and only if \( g \) belongs to \( W^{1/2} \). Since \( g \) is automatically bounded (because it is \( K \)-valued), this leads to the Sobolev condition. \( \square \)

Before discussing determinants and measures, we will first consider the operator

\[ W(g_+) = A(g_+)^{-1}B(g_+) = A(g_+^{-1})B(g_+). \quad (4.15) \]

Ultimately we are interested in understanding this as an equivariant holomorphic operator-valued function of \( \theta_+ \) in \((4.10)\), but we will first compute it in terms of \( g_+ \). We can picture \( W \) as a block matrix, \( W = (W_{i,-j}) \), where each block \( W_{i,-j} \in L(\mathbb{C}^N) \), and the index \( i \geq 0 \) denotes the row (starting from the bottom) and \( j \geq 1 \) denotes the column (starting from the left). If

\[ g_+ = 1 + g_1 z + g_2 z^2 + .. \quad (4.16) \]

where \( g_j \in L(\mathbb{C}^N) \), then

\[ g_+^{-1} = 1 + (-g_1)z + (-g_2 + g_1^2)z^2 + (-g_3 + g_1 g_2 + g_2 g_1 - g_1^3)z^3 + .. \quad (4.17) \]

and in general

\[ (g_+^{-1})_n = \sum (-1)^l g_{i_1} .. g_{i_l} \quad (4.18) \]

where the sum is over all positive multi-indices \( I = (i_1, .., i_l) \) of order \( n \), i.e. \( i_m > 0 \) and \( i_1 + .. + i_l = n \). By \((4.15)\),

\[
W_{i,-j} = 1 \cdot g_{i+j} + (g_+^{-1})_1 g_{i+j-1} + .. + (g_+^{-1})_i g_j \\
= \sum_{n=j}^{i+j} \sum (-1)^l g_{i_1} .. g_{i_l} g_n \quad (4.19)
\]

where given \( n \), the sum is over all positive multi-indices of order \( i + j - n \). This can also be written as

\[ W_{i,-j} = \sum (-1)^{l+1} g_{i_1} .. g_{i_l} \quad (4.20) \]
where the sum is now over all positive multi-indices of order $i + j$ satisfying $i_l \geq j$.

Thus, in terms of the representation dependent expression (4.16) for $g_+$, $W$ has the form

$$
\begin{pmatrix}
\cdots & \cdots & g_3 - g_1 g_2 - g_2 g_1 + g_1^3 & g_4 - g_1 g_3 - g_2 g_2 + g_1^2 g_2 \\
g_3 - g_1 g_2 & g_2 - g_1 g_1 & g_3 - g_1 g_2 & g_4 - g_1 g_3 \\
g_1 & g_2 & g_3 & \cdots
\end{pmatrix}
$$

(4.21)

Now write

$$
\theta_+ = (\theta_1 + \theta_2 z + ..)dz \in H^1(D, g),
$$

(4.22)

where $\theta_i \in g$. Since $g_+$ is the solution of the integral equation

$$
g_+(z) = 1 + \int_0^z g_+(w)\theta(w), \quad g_+(0) = 1,
$$

(4.23)

it can be expressed in terms of iterated integrals:

$$
g_+(z) = 1 + \int \theta + \int \{\int \theta\} + \int \{\int \theta\} \int \cdots
$$

$$
= 1 + g^{(1)}(\theta) + g^{(2)}(\theta) + ..
$$

(4.24)

where

$$
g^{(n)}(\theta) = \int g^{(n-1)}(\theta)\theta = \sum_{I} \frac{1}{i_1 \cdots i_n} \theta_{i_1 \cdots i_n} z^{|I|}
$$

(4.25)

and the sum is over all positive multi-indices $I = (i_1, .., i_n)$.

Given a positive multi-index $I = (i_1, i_2, .., i_l)$, define

$$
c(I) = \frac{1}{i_1 i_1 + i_2 \cdots i_1 + \cdots + i_l}.
$$

(4.26)

Observe that there is a bijective correspondence between positive multi-indices $I$ of order $n$ and subsets of $S \subset \{1, .., n-1\}$: A multi-index $I$ induces a strictly increasing sequence

$$
\lambda_1 = i_1 \prec \lambda_2 = i_1 + i_2 \prec .. \prec \lambda_l = i_1 + .. + i_l = n
$$

(4.27)

which is uniquely determined by the complement

$$
S = \{1, .., n\} \setminus \{\lambda_1, .., \lambda_l\} \subset \{1, .., n-1\}.
$$

(4.28)
In terms of $S$, the $i_j$ are of the form

$$i_j = 1 + |S_j|,$$  \hspace{1cm} (4.29)

where $S_j$ is the $j$th connected component of $S$, and two integers are connected if they are adjacent. We can then write

$$c(I) = \frac{\prod S \lambda}{n!}$$  \hspace{1cm} (4.30)

We have

$$g_n = \sum c(I)\theta_I = \sum c(I)\theta_{i_1}..\theta_{i_l}$$  \hspace{1cm} (4.31)

where the sum is over all positive multi-indices of order $n$. Plugging this into (4.20) implies the following formula.

**Proposition.** As a function of $\theta_+ \in H^1(D, g)$, for $i \geq 0$ and $j \geq 1$,

$$W_{i,-j} = \sum C(I)\theta_{i_1}..\theta_{i_l},$$

where $I$ ranges over all positive multi-indices of order $i + j$, and

$$C(I) = \sum (-1)^{l+1}c(I_1)..c(I_l)$$

$$= \sum (-1)^{l+1} \frac{\prod S_1 \lambda_1}{n!}..\frac{\prod S_l \lambda_l}{n!}$$

where the sum is over all ways of representing $I$ as a tuple $(I_1, .., I_l)$ with $|I_i| \geq j$.

Thus $W(\theta_+)$ has the form

$$
\begin{pmatrix}
\frac{1}{3!}(2\theta_3 - \theta_1\theta_2 - 2\theta_2\theta_1 + \theta_1^3) \\
\frac{1}{2}(\theta_2 - \theta_1^2) \\
\theta_1
\end{pmatrix}
\begin{pmatrix}
\ddots & \ddots & \ddots \\
\frac{1}{3!}(2\theta_3 - \theta_1\theta_2 - 2\theta_2\theta_1 + \theta_1^3) \\
\frac{1}{2}(\theta_2 + \theta_1^2) \\
\frac{1}{3!}(2\theta_3 + 2\theta_1\theta_2 + \theta_2\theta_1 + \theta_1^3)
\end{pmatrix}
$$

Now that we have a formula, the natural question is whether we can understand the terms of the Taylor series, in terms of the representation theory of $PSU(1,1)$. The Taylor series is

$$W = W^{(1)} + W^{(2)} + W^{(3)} + ..$$

where $W^{(n)}$ is a homogeneous symmetric function of the $\theta_i$. 
For example \( W^{(1)} \) is the natural inclusion
\[
H^1(D; \mathfrak{g}) \to H^{1/2} \otimes H^{1/2} \otimes \mathcal{L}(C^N) = (H^1 + H^2 + \ldots) \otimes \mathcal{L}(C^N).
\]
The second derivative \( W^{(2)} \) can be viewed as a linear function
\[
S^2(H^1(D; \mathfrak{g})) \to (H^1 + \ldots) \otimes \mathcal{L}(C^N)
\]
and this time the function is not determined by equivariance. The calculations indicate that there should something very interesting to be said about these functions. But what is it?

We now consider the determinants in (4.9). These reduce to familiar invariant functions in special cases. In the abelian case, \( K = U(1) \),
\[
A(g_+)A(g_+^*)A(g_+g_+^*)^{-1} =
\]
\[
= A(g_+)A(g_+^*)A(g_+^*g_+)^{-1} =
\]
\[
= A(g_+)A(g_+^*)A(g_+)^{-1}A(g_+^*)^{-1} =
\]
\[
= e^{A(x_+)}e^{A(x_+^*)}e^{-A(x_+)}e^{-A(x_+^*)}.
\]
\[ (4.33) \]
This last expression is a commutator, and consequently the determinant can be calculated using the Helton-Howe formula ([HH]):
\[
c(g_+, g_+^*) = det([e^{A(x_+)}, e^{A(x_+^*)}]) =
\]
\[
e^{tr[A(x_+), A(x_+^*)]} = e^{\sum_{n=1}^{\infty} n|x_n|^2} = e^{\|\theta_+\|_{H^1(\Delta)}^2}.
\]
\[ (4.34) \]

Now suppose that \( K = SU(2) \). If \( g \in LK \) has a diagonal form, \( g = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \), then the abelian case implies
\[
det|A(g)|^2 = exp(\|\theta_+\|^2_{H^1 \cap L^2}),
\]
\[ (4.35) \]
where \( \theta_+ \) is now a matrix of the special form \( \theta_+ = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \alpha \in H^1(\Delta) \).

At the opposite extreme is the nilpotent case:
\[
\theta_+ = \partial x_+ = \begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}, \quad \theta \in H^1(\Delta),
\]
\[ (4.36) \]
where
\[
g_+ = \begin{pmatrix} 1 & \int_0^x \theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\]
\[ (4.37) \]
(4.38) Lemma. Relative to the defining representation,

\[ W(g_+) = B(g_+) = \begin{pmatrix}
0 & x_n & . & . & . & . & . & . & . \\
0 & 0 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
0 & x_2 & . & . & . & . & . & . & . \\
0 & 0 & . & . & . & . & . & . & . \\
0 & x_1 & 0 & x_2 & . & . & . & . & . \\
0 & 0 & 0 & 0 & .. & 0 & 0 & 0 & . \\
\end{pmatrix} \]

which, modulo vanishing columns and rows, is the same

as \( B(x) \) in (0.11).

Proof. One can either read this off from (4.32), or argue directly as follows. Suppose that \( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^- \).

\[
A(g_+)^{-1}B(g_+) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P_+ g_+^{-1} P_+ g_+ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =
\]

\[
= P_+ g_+^{-1} \begin{pmatrix} f_1 + P_+ (x f_2) \\ f_2 \end{pmatrix} = P_+ \begin{pmatrix} f_1 + P_+ (x f_2) - x f_2 \\ f_2 \end{pmatrix} =
\]

\[
\begin{pmatrix} P_+ (x f_2) \\ 0 \end{pmatrix} = B(g_+) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \quad (4.39)
\]

The matrix for \( B \) is linear as a function of \( g_+ \), and it vanishes on the identity. Since \( g_+ = 1 + xe^*_2 \otimes e_1 \), this implies that \( B(g_+) \) is \( B(x) \), modulo vanishing columns and rows. □

These calculations show that in the abelian case the measure (4.2) is a Gaussian (the number \( \check{\gamma} = 0 \) in the abelian case, and we must require \( l > 0 \)), and if we consider nilpotent \( \theta_+ \), when \( g = sl_2 \), then (4.2) is to be interpreted as in §3.

We now suppose that \( k \) is a simple Lie algebra. Let \( \check{\gamma} \) denote the dual Coxeter number of \( k \); if \( K = SU(n) \), then \( \check{\gamma} = n \). The number \( m \) in (4.2) depends upon \( N \): for \( X, Y \in k^C \),

\[
tr_{CN}(XY) = \frac{1}{m} \kappa(X, Y), \quad (4.40)
\]

where \( \kappa \) denotes the Killing form for \( k^C \) (if \( C^N \) is the adjoint representation, then \( m = \check{\gamma} \), and if \( K = SU(n) \), and \( N = n \), then \( m = 1 \)).
(4.41) Theorem. Fix $l \geq 0$. (a) Fix $M$. Given $\theta_+ = P_M \theta_+ = \sum^M \theta_i z^{-i} dz^{-1}$, let $g \in LK$ satisfy $g = g_-$ modulo $H^0(D, G)$. The measure
\[
\frac{1}{\det(1 + W(\theta_+) W(\theta_+)^*)} dm(P_M \theta_+) \tag{4.42}
\]
is finite.

(b) Let $Z_M$ denote the total mass. The limit of the probability measures
\[
\frac{1}{Z_M} \frac{1}{\det(1 + WW^*)} dm(P_M \theta_+) \tag{4.43}
\]
exists (in a weak sense), and defines a $PSU(1, 1)$-invariant measure on $H^1(\Delta, \mathfrak{k}^C)$. Consequently, for each matrix coefficient of $\theta_+$, we obtain an invariant measure on $H^1$.

This will appear in [Pi4].

Because these formulas involve limits, they seem to be of limited utility, beyond heuristically explaining why the measures are $PSU(1, 1)$-invariant. For example it seems plausible that the measure in (0.14) is the invariant measure that corresponds to the highest root, but this is uncertain. These measures are so natural, from a group point of view, that one has to believe there is some direct way of computing them.

§5. Universal Teichmuller Space.

In this section we will consider another invariant function, this time associated with diffeomorphisms of the circle. The function will have the same form as in (4.1), but it is far more complicated.

The first object of this section is to recall the Bers embedding of universal Teichmuller space into $H^2(\Delta)$. This embedding is equivariant with respect to the natural actions of $PSU(1, 1)$ on these spaces. The hypothetical invariant measures which we will discuss should be supported on the image of this embedding.

We will use the following notations. The invariant norm (0.24) induces a natural $PSU(1, 1)$-invariant $L^\infty$ norm on quadratic differentials:
\[
|Q(z)(dz)^2|_{L^\infty} = \sup_{\Delta} \{1 - |z|^2\}^2 |Q(z)|. \tag{5.1}
\]
To gain a feeling for this norm, note that a quadratic differential which extends meromorphically to a neighborhood of $\bar{\Delta}$, and which is bounded in this $PSU$-invariant sense, can have at most double poles around the boundary, $S^1$. We will denote this Banach space by $H^2 \cap L^\infty$. 
We will denote the space of normalized univalent holomorphic functions on \( \Delta \) by \( S \): \( u \in S \) means that \( u \) is a \( 1 - 1 \) holomorphic function on \( \Delta \) having a Taylor series expansion of the form

\[
    u(z) = z(1 + \sum_{1}^{\infty} u_n z^n).
\]

The space of orientation-preserving homeomorphisms of \( S^1 \) will be denoted by \( \mathcal{D} \), and we will use subscripts to denote further restrictions on the degree of smoothness. For our purposes the most important example is \( \mathcal{D}_{qs} \), the group of quasi-symmetric homeomorphisms. The condition that \( \sigma \) is quasi-symmetric can be expressed in several equivalent ways: (1) \( \sigma \) is the restriction to \( S^1 \) of a quasi-conformal automorphism of \( \hat{\mathbb{C}} \) (or \( \Delta \)) mapping \( S^1 \) into itself; (2) \( \sigma \) satisfies the Beurling-Ahlfors criterion (see chapter 16 of [GL], especially (16.1)); and (3) \( \sigma \) stabilizes \( W^{1/2}(S^1) \) (see [NS], especially Theorem 3.1).

The Bers embedding depends upon the following theorem of Bers (see [L], page 100).

**Theorem.** If \( \sigma \in \mathcal{D}_{qs} \), then there is a unique factorization (where the multiplication is composition of maps)

\[
    \sigma = l \circ \text{diag} \circ u,
\]

where \( \text{diag} \) is multiplication by a constant \( \lambda \in \mathbb{C}_{\leq 1}^* \),

\[
    u = z(1 + \sum_{1}^{\infty} u_n z^n), \quad \frac{1}{l^{-1}(\frac{1}{w})} = w(1 + \sum_{1}^{\infty} l_n w^n),
\]

\( u \) is univalent in \( \Delta \) and admits a quasi-conformal extension to \( \hat{\mathbb{C}} \), \( l^{-1} \) is univalent in \( \Delta^* \) and admits a quasi-conformal extension to \( \hat{\mathbb{C}} \), and the compatibility condition

\[
    \lambda u(S^1) = l^{-1}(S^1)
\]

is satisfied.

The complement of \( l^{-1}(\Delta^*) \), \( \lambda u(\Delta) \), has unit transfinite diameter, \( \rho(\lambda u(\Delta)) = 1 \) (see §16.2 of [Hi] for the original definition of \( \rho \), Fekete’s theorem 16.2.2, page 270 for the basic characterization of \( \rho \), and see the second paragraph of page 347 for this specific fact), hence \( \frac{1}{|\lambda|} = \rho(u(\Delta)) \). An immediate consequence of this is
that \( u \) (or \( l \)) determine \( \sigma \) up to a phase factor. In particular we can think of \( u \) as a parameter for the homogeneous space \( \text{Rot}\backslash \mathcal{D}_{qs} \).

This leads to the following diagram:

\[
\begin{array}{cccc}
\text{Rot}\backslash \mathcal{D}_{qs} & \leftrightarrow & \mathcal{S}_{qc} & \leftrightarrow & \mathcal{S} & \leftrightarrow & \mathcal{S}_{\text{formal}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{PSU}\backslash \mathcal{D}_{qs} & \leftrightarrow & \mathcal{T} & \leftrightarrow & \mathcal{U} & \leftrightarrow & \mathcal{H}^2(\Delta) \leftrightarrow \mathcal{H}^2(\Delta)_{\text{formal}}
\end{array}
\]

(5.7)

Here \( \mathcal{S} \) denotes the space of all univalent holomorphic functions \( u \) on \( \Delta \) of the form (1.11), \( \mathcal{S}_{qc} \) denotes the subspace of all such \( u \) which have quasi-conformal extensions to \( \hat{\mathbb{C}} \), \( \mathcal{S}_{\text{formal}} \) consists of formal power series \( u \) as in (5.5), \( \mathcal{S} \) is the Schwarzian derivative, and \( \mathcal{T} \), the universal Teichmüller space, and \( \mathcal{U} \) are the \( \mathcal{S} \)-images of \( \mathcal{S}_{qc} \) and \( \mathcal{S} \), respectively.

The action of \( \text{PSU} \subset \mathcal{D}_{qs} \) from the right on \( \mathcal{S}_{qc} \) extends to an action on \( \mathcal{S} \) which is given as follows. Suppose that \( u \in \mathcal{S} \) and \( g \in \text{PSU} \). There exists a unique \( G \in \text{PSL}(2, \mathbb{C}) \) such that (i) \( G(0) = u(g(0)) \), (ii) \( G'(0) = (u \circ g)'(0) \) and (iii) \( G(\infty) = \infty \). The right action by \( g \) is given by

\[
g : u \rightarrow G^{-1} \circ u \circ g.
\]

(5.8)

The cocycle condition

\[
S(f \circ g) = S(f)|_{g(z)}g'(z)^2 + S(g),
\]

(5.9)

and the vanishing of \( S \) on linear fractional maps implies that \( S(G^{-1} \circ u \circ g) = S(u \circ g) \).

Hence the down arrows above are \( \text{PSU} \)-equivariant (excluding the rightmost arrow, because the action of \( \text{PSU} \) does not extend to the formal completions). Thus both \( \mathcal{T} \) and \( \mathcal{U} \) are invariant for the natural (right) pullback action of \( \text{PSU} \) on \( \mathcal{H}^2(\Delta) \).

We have implicitly observed here that \( \mathcal{S} \) extends to an algebraic map of the formal completions (see below for some speculation about the algebraic properties of this mapping).

It is known that

\[
\{ q \in \mathcal{H}^2(\Delta) \cap L^\infty : |q|_{L^\infty} < 2 \} \subset \mathcal{T} \subset \mathcal{U} \subset \{ q \in \mathcal{H}^2(\Delta) \cap L^\infty : |q|_{L^\infty} < 6 \}
\]

(5.10)

Also \( \mathcal{T} \) is a bounded open subset of the Banach space \( \mathcal{H}^2 \cap L^\infty \), so in this sense it is a homogeneous bounded domain. A lot of work has gone into understanding the nature of the boundary of \( \mathcal{T} \), and its subspaces \( \mathcal{T}_\Sigma \), as \( \Sigma \) varies over all hyperbolic type Riemann surfaces. The state of the art in understanding these spaces is described in [GL].
To understand the Bers embedding, or the Schwarzian derivative, algebraically, we introduce the following mappings:

\[
S \xrightarrow{c} H^0(\Delta) : u \to \log(u'), \quad (5.11)
\]

and

\[
S \xrightarrow{N} H^1(\Delta) : u \to \log(u')'. \quad (5.12)
\]

These mappings satisfy the cocycle conditions

\[
c(f \circ g) \mid_z = c(f) \mid_{g(z)} + c(g) \mid_z \quad (5.13)
\]

\[
N(f \circ g) = N(f) \mid_{g(z)} g'(z) + N(g) \mid_z \quad (5.14)
\]

and they have the property that they are equivariant with respect to the actions of \(PSU(1, 1)\) on \(S\) and the natural actions on the \(H^*(\Delta)\) spaces.

We have

\[
u = z(1 + \sum_{1}^{\infty} u_n z^n) \xrightarrow{S} Q = \sum_{0}^{\infty} Q_{n+2} z^n (dz)^2 \quad (5.15)
\]

\[
Q = \ln(u'') - \frac{1}{2} \ln(u')^2 = N(u)' - \frac{1}{2} N(u)^2. \quad (5.16)
\]

Here are some sample calculations:

\[
c(u) = \ln(u') = p_1(u)z + p_2(u)z^2 + ..,
\]

\[
p_1 = 2u_1, p_2 = 3u_2 - 2u_1^2, p_3 = 4u_3 - 6u_1 u_2 + \frac{8}{3} u_1^3
\]

\[
p_4 = 5u_4 - \frac{1}{2} (2 \cdot 2u_1 4u_3 + 3u_2 3u_2) + \frac{1}{3} (3 \cdot 2u_1 2u_1 3u_2) - \frac{1}{4} (16u_1^4), ..
\]

\[
N(u) = (lnu')' = p_1 + 2p_2 z + 3p_3 z^2 + 4p_4 z^3 + ..
\]

\[
Q = S(u) = N(u)' - \frac{1}{2} N(u)^2
\]

\[
= (2p_2 + 3 \cdot 2p_3 z + 4 \cdot 3p_4 z^2 + ..)
\]

\[
- \frac{1}{2} (p_1^2 + 2p_1 2p_2 z + [2p_1 3p_3 + +2p_2 2p_2] z^2 + [p_1 4p_4 + 2p_2 3p_3 + 3p_3 2p_2 + 4p_4 p_1] z^3 + ..)
\]

\[
Q_2 = 2p_2 - \frac{1}{2} p_1^2 = 6(u_2 - u_1^2),
\]
\[ Q_3 = 6p_3 - 2p_1p_2 = 6(4u_3 - 6u_1u_2 + \frac{8}{3}u_1^3) - 2(2u_1)(2u_2 - 2u_1^2) \]
\[ = 24u_3 - 14u_1u_2 + 20u_1^3. \]

\[ Q_4 = 12p_4 - 3p_1p_3 - 2p_2^2 \]
\[ = 12(5u_4 - 8u_1u_3 + \frac{9}{2}u_2^2 + 12u_1^2u_2 - 4u_1^4) - 3(2u_1)(4u_3 - 6u_1u_2 + \frac{8}{3}u_1^3) - 2(3u_2 - 2u_1^2)^2 \]

If we set \( u_1 = 0 \), then it seems fairly certain that there should be some tractable combinatorial description of the coefficients of \( u \) in terms of \( Q \):
\[ u_2 = \frac{1}{6}Q_2, \quad u_3 = \frac{1}{24}Q_3 \]
\[ u_4 = \frac{1}{5!}(2Q_4 - 2^23Q_2^2), \ldots \]

Our goal now is to develop this section in analogy with the loop group case. As we noted in §1B, the representation
\[ SU(1,1) \times H^{1/2}(\Delta) \to H^{1/2}(\Delta) \] (5.18)
is special. If we consider the odd spin structure on \( S^1 \), then we obtain an action
\[ D^{(2)} \times \Omega^{1/2}_{odd} \to \Omega^{1/2}_{odd} \] (5.19)
In concrete terms, if we introduce the double cover
\[ p : (S^1)^{(2)} \to S^1 : \zeta \to z = \zeta^2, \] (5.20)
then
\[ D^{(2)} = \{ \sigma \in Diff((S^1)^{(2)}) : \sigma(-\zeta) = \sigma(\zeta) \} \] (5.21)
\[ \Omega^{1/2}_{odd} = \{ f(\zeta)d\zeta : f(-\zeta) = -f(\zeta) \} \] (5.22)
and \( D^{(2)} \) acts in the natural way.
There is a \( SU(1,1) \)-invariant polarization
\[ \Omega^{1/2}_{odd} = H^{1/2}(\Delta^*) \oplus H^{1/2}(\Delta) \] (5.23)
of Hilbert spaces. Given $\sigma \in \mathcal{D}^{(2)}$, we write

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in analogy with the loop group case (hence we refer to $A$ as the Toeplitz operator corresponding to $\sigma$, etc.).

The analogue of (4.11) holds, where the Riemann-Hilbert factorization is replaced by the Bers factorization. Whereas it seems not so enlightening to examine a matrix representation for $(5.24)$, it is quite interesting to stare that the matrix representation for $u^{-1}$, relative to the standard basis

$$\ldots, z^2(dz)^{1/2}, \ldots, z^{-2}(dz)^{1/2}, \ldots$$

(5.25)

It is a unipotent upper triangular matrix which the $(i,j)$-entry is homogeneous of degree $j - i$, where $u_l$ has degree $l$. It is very easy to generate this matrix, but not so easy to display it:

$$
\begin{pmatrix}
3u_1 & \frac{5}{2}u_2 + \frac{1}{2}u_1^2 & \frac{5}{2}u_4 - 2u_1u_3 - \frac{9}{8}u_2^2 + \frac{9}{4}u_1u_2 - \frac{5}{8}u_1^4 \\
1 & 2u_1 & 2u_3 - \frac{3}{2}u_1u_2 + \frac{1}{2}u_1^3 \\
- & - & \frac{3}{2}u_4 - 3u_1u_3 - \frac{13}{8}u_1^4 + \frac{19}{4}u_1^2u_2 - \frac{13}{8}u_2^2
\end{pmatrix}
$$

The key point is that there is a precise analogue of the linear coordinate for $g_+$, $\theta_+ = g_+^{-1}\partial g_+$ (see (4.10)). In terms of the Bers embedding theorem, the analogue is the map

$$\sigma \rightarrow Q = S(u)(dz)^2 \in \mathcal{T} \subset H^2 \cap L^\infty$$

(5.26)

Just as we studied the equivariant mapping

$$\theta_+ \rightarrow W(g_+) = A(g_+^{-1})B(g_+)$$

we would like to be able to say something about the combinatorics of the mapping

$$Q \rightarrow W(u) = A(u^{-1})B(u)$$
Here we will only say that it has the homogeneous form

\[
\begin{pmatrix}
\frac{1}{12}Q_4 + cQ_\frac{3}{2} & \ldots \\
-\frac{1}{12}Q_3 & -\frac{1}{36}Q_4 + c'Q_\frac{3}{2} & 0 & \ldots \\
\frac{1}{3}Q_2 & 0 & \frac{1}{36}Q_4 + c''Q_\frac{3}{2} & \ldots \\
0 & -\frac{1}{3}Q_2 & \frac{1}{12}Q_3 & -\frac{1}{12}Q_4 + dQ_\frac{3}{2} \\
\end{pmatrix}
\]

The major new complications are that (1) whereas we have at least a crude grasp of the correspondence between \(\theta_+\) and the analytic properties of \(W(g_+)^\ast\), this is quite mysterious in the case of \(Q\) and \(W(u)\), and (2) whereas it was reasonable to use ‘infinite dimensional Lebesgue measure’ as a heuristic background measure for \(\theta_+\), we now need a background measure for \(Q\) which is presumably supported on \(\mathcal{T}\).

We want to consider the critical exponents for the expression

\[
\frac{1}{\det(1 + WW^\ast)^p}dV(Q)
\]

over a sequence of finite dimensional space \(\mathcal{T}_N\) which tend to \(\mathcal{T}\) in the limit. One suspects that there must be many different interesting possibilities; the hard question is whether any of them lead to tractable calculations. At this point I will just mention one obvious possibility.

By (5.7) we can identify \(\mathcal{T}\) with domains in \(\mathbb{C}\) with Jordan curve boundaries, where two regions are identified if one can be obtained from another by a linear fractional transformation of \(\hat{\mathbb{C}}\). We let \(\mathcal{T}_N\) denote the subset corresponding to \(N\)-gons.

Given an \(N\)-gon, let \(z_1, z_2, \ldots, z_N\) denote the vertices listed in the counterclockwise direction. We can write the interior angle of the polygon at \(z_j\) in the form \(\alpha_j\pi\), \(0 < \alpha_j < 2\). We also let \(\beta_j = 1 - \alpha_j\).

In this case we can say exactly what the \(u\) and \(l\) are (we do not quite know \(\lambda\), using the Schwarz-Christoffel transform theory, namely we have

\[
du = \prod_{j=1}^n (1 - \frac{z}{z_j})^{-\beta_j} = 1 + \sum \frac{\beta_j}{z_j}z + \ldots \implies u = z(1 + u_1z + \ldots) \quad (5.29)
\]

\[
dl^{-1} = \prod_{j=1}^n (1 - \frac{z_j'}{z})^{\beta_j} = 1 + \sum \frac{\beta_j z_j'}{z} + \ldots \implies l^{-1} = z + b_0 + b_1z^{-1} + \ldots \quad (5.30)
\]
This enables one to do calculations. But the main thing that is lacking is a feeling for the background geometry, which should approximate the invariant (Weil-Petersson) geometry on universal Teichmüller space.

§6. Comments and Questions

In §1 we established the existence of transitive measures, hence of measures having finite dimensional support, for $H^m$, when $m > 0$ and rational.

(6.1). Do there exist measures having finite dimensional support when $m$ is irrational? (We know there do not exist transitive measures in this case, so one suspects the answer is ‘no’).

(6.2). Suppose that $m > 0$ is rational. Is the convolution algebra generated by transitive measures dense in the space of all invariant (not necessarily positive) measures on $H^m(\Delta)$? If yes, we could further ask if there is some canonical way to decompose an invariant measure of finite dimensional support. If no, we could allow multiplication of such measures and ask the same questions.

The results in [K], summarized in (1.15), go a long way toward describing the support of the $G$-ergodic Gaussian measure corresponding to the Hilbert space $H^1 \cap L^2$.

(6.3a). Can this be refined to say in an explicit way how the dynamics of $G$ is chaotic?

(6.3b). Is there something interesting to be said about $\{\Re e(x) = 0\}$. This is a curve that starts at zero and goes out to $\infty$ in a random. This is the opposite of chordal SLE, where one starts at infinity and goes towards the origin. We are also looking at the whole curve at once; there is not a time parameter. We could ask this question for any of the invariant distributions on functions.

(6.). Are there examples of invariant distributions for which one can compute the distribution of zeroes? Does the distribution of zeroes determine the distribution on $H^m$, or at the other extreme, possibly there are conditions under which it is independent of the $H^m$-distribution.

The following table is a rather feeble attempt to indicate some of what we do and do not know in an explicit way, regarding invariant measures for $H^1 = H^0 / \mathbb{C}$ ($y$ is a variable dual to $x \in H^0 / \mathbb{C}$):
(6.4). In the second to last line, we have indicated that the Fourier transform is heuristically a ‘fundamental solution’ for a constant coefficient operator (involving infinitely many variables). Can one make sense of this?

(6.5). If we could fill in the question mark on the fourth line, the table should continue in some interesting way.

The Lebesgue type measures which we have come across in this paper have the property that their Fourier transforms are continuous on \((H^m \cap L^2)^*\). It appears that the Fourier transform for an invariant measure of finite dimensional support is not continuous on \(H^m \cap L^2\).

(6.6) Does this continuity property capture what should be meant by having ‘Lebesgue type support’?

In this paper we have encountered a lot of invariant measures on \(H^1\) which have \(x_1\)-density of the form \(Z^{-1}(1 + |x_1|^2)^{-p}\). The measures in §4, involving loops into \(SU(2)\), or more generally a compact Lie group, have matrix coefficients which have this property.

(6.7) If one considers loops into \(S^2\), one expects to encounter a measure having a matrix coefficient with \(x_1\)-density

\[
\frac{1}{Z} \frac{1}{(1 + |x_1|^2)^{3/2}} F\left(1 + \frac{|x_1|^2}{1 + |x_1|^2}\right)
\]

where \(F(\rho)\) is the function

\[
F(\rho) = \int_0^\infty \frac{\rho}{(\rho + (x_1^2)^{3/2})} dx
\]

How does this conjectural measure fit into our scheme?

(6.8) Referring to §4, is \(\frac{1}{Z} \frac{1}{(1 + WW^*)}\) a positive definite function on \(H^1 \cap L^2 \otimes \mathbb{C}\)? If so, what is the corresponding measure?
(6.9) Consider the finite Hankel matrix $B_n(x)$. Can one find a reasonable formula for the distribution of singular values, with respect to the background Lebesgue measure (hence with respect to more general invariant measures)? If so, it might be interesting to develop a theory of random Hankel matrices in analogy with the theory of random Hermitian matrices.

(6.10) We have considered a number of invariant functions, $\det(1+B(x)B(x)^*)$, $\det(1+W(\theta_+)W(\theta_+)^*)$, $\det(1+W(q_+)W(q_+)^*)$, and so on. It would be interesting to understand the Taylor series of these functions in a $G$-equivariant way.

References

[B] P. Billingsleys, Convergence of Probability Measures, John Wiley and Sons (1968).

[Bo] V.I. Bogachev, Gaussian Measures, A.M.S. Math. Surveys and Monographs, Vol. 62 (1998).

[FK] H. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group, Graduate Studies in Mathematics Vol. 37, AMS (2001).

[HH] W. Helton and R. Howe, Traces of commutators of integral operators, Acta Math. 135, No. 3-4 (1975) 271-305.

[He] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press (1984).

[Hi] E. Hille, Analytic Function Theory, Vol. II (1973).

[GK], F. Gardiner and N. Lakic, Quasiconformal Teichmuller Theory, A.M.S. Math. Surveys and Monographs, Vol. 76 (2000).

[K] J.P. Kahane, Some Random Series of Functions, Cambridge Tracts in Advanced Mathematics (1985).

[Kra] I. Kra, Automorphic Forms and Kleinian Groups, Benjamin (1972).

[L] O. Lehto, Univalent Functions and Teichmuller Spaces, Springer Verlag (1986).

[MKM] K. Matthes, J. Kerstan and J. Mecke, Infinitely Divisible Point Processes, John Wiley (1978).

[NS] S. Nag and D. Sullivan, Teichmuller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, Osaka J. Math. 32 (1995) 1-34.

[Pe] V. Peller, Hankel Operators and Their Applications, Springer-Verlag (2003).

[Pi1] D. Pickrell, Invariant measures for unitary groups associated to Kac-Moody Lie algebras, Memoirs of the A.M.S., No. 693 (2000).
[Pi2] ——, An invariant measure for the loop space of a simply connected compact symmetric space, J. Funct. Anal. 234 (2006) 321-363.

[Pi3] ——, Loop spaces, diagonal distributions, and the Duistermaat-Heckman theorem, in progress.

[Pi4] ——, A coordinate expression for the invariant measure of a loop group, in progress.

[PS] A. Pressley and G. Segal, Loop Groups, Oxford University Press (1986).

[Se] I.E. Segal, Ergodic subgroups of the orthogonal group on a real Hilbert space, Annals of Mathematics, Vol. 66, no. 2 (1957) 297-303.

[St] D. Stroock, Gaussian measures in traditional and not so traditional settings, Bull. A.M.S., Vol 33, No. 2 (1996) 135-156.

[T] W. Thurston, Three-dimensional Geometry and Topology, edited by S. Levy, Princeton Mathematical Series 35, Princeton University Press (1997).

[Z] R. Zimmer, Ergodic Theory and Semisimple Groups, Birkhauser (1984).