RIGIDITY OF NON-COMPACT STATIC DOMAINS IN HYPERBOLIC SPACE VIA POSITIVE MASS THEOREMS

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ABSTRACT. We single out a notion of staticity which applies to any domain in hyperbolic space whose boundary is a non-compact totally umbilical hypersurface. For (time-symmetric) initial data sets modeled at infinity on any of these latter examples, we formulate and prove a positive mass theorem in the spin category under natural dominant energy conditions (both in the interior and along the boundary) whose rigidity statement retrieves, among other things, a sharper version of a recent result by Souam [Sou21] to the effect that no such hypersurface admits a compactly supported deformation keeping the original lower bound on the mean curvature. A key ingredient in our approach is the consideration of a family of elliptic boundary conditions on spinors interpolating between chirality and MIT bag boundary conditions.

1. INTRODUCTION AND STATEMENTS OF THE RIGIDITY RESULTS

A vacuum spacetime is a Lorentzian manifold $(\tilde{M}^{n+1}, \tilde{g})$ with signature $(-, +, \ldots, +)$ satisfying the vacuum Einstein field equation

\begin{equation}
\text{Ric}_{\tilde{g}} - \frac{1}{2} R_{\tilde{g}} + \tilde{\Lambda}_{\tilde{g}} = 0,
\end{equation}

where $\tilde{\Lambda} \in \mathbb{R}$ is said to be the cosmological constant. By taking trace we see that the scalar curvature $R_{\tilde{g}}$ is constant and hence (1.1) is equivalent to

\begin{equation}
\text{Ric}_{\tilde{g}} = \Lambda_{\tilde{g}}, \quad \Lambda = \frac{2}{n-1} \tilde{\Lambda}.
\end{equation}

We may assume that $\Lambda = \epsilon n$, $\epsilon = 0, \pm 1$. Here, we will be mainly interested in the case $\epsilon = -1$, so that $\tilde{\Lambda} = -n(n - 1)/2$, the negative cosmological constant case.

If $\tilde{M}$ carries a nonzero time-like Killing vector field $X$ such that its orthogonal distribution is integrable, then $\tilde{M}$ is said to be a static spacetime. In this case, if one fixes a space-like slice $M \hookrightarrow \tilde{M}$ where $X$ never vanishes, then it inherits a Riemannian metric $g$ and one can write

\begin{equation}
\tilde{g} = -V^2 dt^2 + g, \quad V = \sqrt{-\tilde{g}(X, X)}.
\end{equation}

The Einstein equation (1.2) can be written on this slice as

\begin{equation}
\begin{cases}
\nabla^2 V + AV g - V \text{Ric}_g = 0, \\
\Delta_g V + AV = 0.
\end{cases}
\end{equation}

S. Almaraz has been supported by FAPERJ-202.802/2019, L.L. de Lima has been supported by CNPq 312485/2018-2, and both authors have been supported by FUNCAP/CNPq/PRONEX 00068.01.00/15. Part of this article was written while the second author was visiting IHES (Bures-sur-Yvette) and CAMGSD/IST (Lisbon). He would like to thank these institutions for the financial support.
The discussion above motivates the following classical concept.

**Definition 1.1.** A Riemannian manifold \((M, g)\) is said to be *static* if there is a non-trivial solution \(V\) of (1.4), so called a *static potential*.

Static manifolds play a central role in the theory. For instance, they may be used as background spaces when defining mass-type invariants for initial data sets in the context of isolated gravitational systems. Roughly speaking, one imposes that the given (time-symmetric) initial data set \((M', g')\) approaches at infinity a static manifold \((M, g)\), so that the corresponding mass-type invariant somehow extracts the rate of convergence of \(g'\) to \(g\) as one goes to infinity. One should emphasize that in general the mass invariant so obtained must be thought of as a linear functional on the space of static potentials of \((M, g)\). We refer to [dL21, Section 2] for a discussion of this approach to defining conserved quantities in General Relativity.

The simplest example of a static manifold is \(\mathbb{R}^n\) with the canonical flat metric \(\delta\). A geometric invariant, called the *ADM mass*, is defined for asymptotically flat manifolds (i.e. manifolds whose geometry at infinity approaches, in a suitable sense, that of \((\mathbb{R}^n, \delta)\); in this case, \(\epsilon = 0\)) and positive mass theorems have been proved in this setting [SY79, Wit81, Bar86, SY17, Loh16]. More generally, we can consider other static background manifolds, the simplest one being the hyperbolic space \(H^n\), which is a static manifold with \(\epsilon = -1\). Its space of static potentials is identified with the \((n + 1)\)-dimensional Minkowski spacetime and positive mass theorems have been formulated and proved in this setting as well [Wan01, CH03, CD19].

Concerning these contributions recall that, from the dynamical viewpoint, static manifolds may be regarded as the stationary solutions of the theory. This naturally leads to the basic question on whether such a manifold may be deformed into another initial data set satisfying the relevant dominant energy condition and having the same asymptotic behavior at infinity. If no such deformation exists we say that the given static manifold is *rigid*. Since in General Relativity the total energy is measured by means of a certain surface integral at spatial infinity, rigidity rules out the existence of exotic initial data sets lying in the lowest energy level. From this perspective, the rigidity statements in the positive mass theorems referred to above make sure that \(\mathbb{R}^n\) and \(H^n\) are rigid in this sense.

Partially motivated by the so-called AdS/BCFT correspondence [Tak11], a notion of mass for asymptotically hyperbolic manifolds with a non-compact boundary, modeled at infinity on the half-hyperbolic space \(\mathbb{H}^n_0\), which is the bordered non-compact manifold obtained by cutting the standard hyperbolic space \(\mathbb{H}^n\) along a totally geodesic hypersurface, has been introduced in [AdL20a]. In fact, this kind of initial data set may be viewed as an example of a static manifold with boundary, a notion we isolate in Definition 1.2 below. Indeed, if our initial data set \((M, g)\) carries a (possibly non-compact) boundary, say \(\Sigma\), we argue that it is natural to add to (1.4) the boundary conditions

\[
\begin{align*}
\pi_g - \lambda \bar{g} &= 0, \\
\frac{\partial}{\partial \eta} V - \lambda V &= 0,
\end{align*}
\]

where \(\lambda \in \mathbb{R}\), \(\pi_g\) is the second fundamental form associated to the shape operator \(\nabla \eta\) of \(\Sigma\) with respect to the outward unit normal vector \(\eta\), and \(\bar{g}\) is the restriction of
It is known that $\tilde{g}$ satisfies (1.1) if and only if it is critical for the Einstein-Hilbert functional

$$\tilde{g} \mapsto \int_{\tilde{M}} (R_{\tilde{g}} - 2\tilde{\Lambda}) d\tilde{g},$$

defined on the space of Lorentzian metrics on $\tilde{M}^{n+1}$. In case $\tilde{M}$ has a nonempty boundary $\partial \tilde{M}$, it is natural to consider instead the Gibbons-Hawking-York functional

$$F : \tilde{g} \mapsto \int_{\tilde{M}} (R_{\tilde{g}} - 2\tilde{\Lambda}) d\tilde{g} + 2 \int_{\partial \tilde{M}} (H_{\tilde{g}} - \tilde{\lambda}) d\sigma_{\tilde{g}},$$

(1.6)

where $\tilde{g}$ runs over the space of all Lorentzian metrics on $\tilde{M}$ with respect to which $\partial \tilde{M}$ is time-like and $H_{\tilde{g}} = \text{tr}_{\partial \tilde{M}} \pi_{\tilde{g}}$ is the mean curvature of the embedding $\partial \tilde{M} \rightarrow \tilde{M}$. In particular, the geometry of $\partial \tilde{M}$ now should play a role. Indeed, critical metrics for $F$ are solutions of (1.1) which additionally satisfy the boundary condition

$$\pi_{\tilde{g}} - H_{\tilde{g}}|_{\partial \tilde{M}} + \tilde{\lambda}g|_{\partial \tilde{M}} = 0,$$

or equivalently,

$$\pi_{\tilde{g}} = \lambda g|_{\partial \tilde{M}}, \quad \lambda = \frac{1}{n-1} \tilde{\lambda}.$$

Now, if $X$ is a time-like Killing vector field on $\tilde{M}$, tangent to $\partial \tilde{M}$ and with integrable orthogonal distribution, we can write $\tilde{g}$ in the form (1.3) and the triple $(M, g, \Sigma)$, where $\Sigma = \partial M = \partial \tilde{M} \cap M$, inherits the equations (1.4) and (1.5). This discussion naturally leads to the following concept, which plays a central role in this work.

**Definition 1.2.** We say that the triple $(M, g, \Sigma)$, $\Sigma = \partial M$, is a static manifold with boundary with the pair $(\tilde{\Lambda}, \tilde{\lambda})$ as cosmological constants if there exists $V \neq 0$ such that (1.4) and (1.5) are satisfied. Any such $V$ is termed a static potential.

**Remark 1.3.** A related notion of staticity appears in [CS20, HH20] in connection with deformation properties of the scalar and mean curvatures.

The simplest example of such a manifold is the Euclidean half-space $\mathbb{R}^n_+$, which is obtained by cutting $\mathbb{R}^n$ along a hyperplane. In this case, $(\tilde{\Lambda}, \tilde{\lambda}) = (0, 0)$ and the corresponding positive mass theorem is proved in [Abl16]; see Remark 1.5 below. Next to it we find the hyperbolic half-space $\mathbb{H}^n_0$ mentioned earlier, where $(\tilde{\Lambda}, \tilde{\lambda}) = (-n(n-1)/2, 0)$. A positive mass theorem in this setting has been proved in the spin category [AdL20b, Theorem 5.4]. An immediate consequence of the corresponding rigidity statement [AdL20b, Theorem 1.1] is the following non-deformability result which has been rediscovered in [Sou21].

**Theorem 1.4.** A totally geodesic hypersurface in $\mathbb{H}^n_0$ can not be compactly deformed (as a hypersurface of $\mathbb{H}^n_0$) while keeping it mean convex (that is, with non-negative mean curvature everywhere).
**Remark 1.5.** The corresponding statement for a hyperplane in $\mathbb{R}^n$ follows immediately from the rigidity statement of the positive mass theorem for asymptotically flat manifolds with a non-compact boundary proved in [A bdL16]. This non-existence of compactly supported, mean convex deformations of $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ has also been recovered using an appropriate symmetrization process in [Gro19], which contains an in-depth investigation of the interplay between lower bounds for the scalar curvature (in the interior) and the mean curvature (along the boundary) in more general spaces. As the action (1.6) makes it clear, this fruitful interaction between scalar and mean curvatures already manifests itself in General Relativity.

The argument in [Sou21] leading to Theorem 1.4 is quite elementary in the sense that it relies on Aleksandrov’s Tangency Principle. In fact, this same reasoning also yields a corresponding rigidity statement for all non-compact totally umbilical hypersurfaces in $\mathbb{H}^n$ [Sou21, Theorem 2]. As we shall see in Propositions 1.6 and 5.1 below, the non-compact domains in $\mathbb{H}^n$ having such hypersurfaces as boundaries constitute examples of static manifolds with boundary with cosmological constants of the type $(-n(n-1)/2, \tilde{\lambda})$, for some $\tilde{\lambda} \in [-n(n-1), n-1]$. This suggests that, similarly to the contents of Theorem 1.4 and Remark 1.5 above, the results in [Sou21] may alternatively be retrieved as a consequence of more general rigidity statements associated to appropriate positive mass theorems for initial data sets modeled at infinity on such static domains. The purpose of this paper is precisely to confirm this expectation. Besides placing all the rigidity results mentioned above in their proper conceptual framework, our main contributions (Theorems 2.8 and 5.5) represent a substantial improvement in the sense that their applications to rigidity phenomena comprise not only purely extrinsic, compactly supported deformations of the given non-compact boundary but also more general (intrinsic and not necessarily compactly supported) deformations preserving suitable dominant energy conditions (both in the interior and along the boundary); see Theorem 1.8 and Remarks 1.10 and 1.12. Our approach relies on the adoption of a family of elliptic boundary conditions for spinors which somehow interpolates between the well-known chirality and MIT bag boundary conditions (Definition 3.3), as this guarantees, among other things, that the boundary integrals in the Witten-type mass formulas (4.43) and (5.53) have the expected shapes.

We now explain our main results in detail. Recall that the so-called hyperboloid model for hyperbolic space is given by

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{1,n} | \langle x, x \rangle_{1,n} = -1 \} \subset \mathbb{R}^{1,n},$$

where $\mathbb{R}^{1,n}$ is the Minkowski space with the flat Lorentzian metric

$$\langle x, x \rangle_{1,n} = -x_0^2 + x_1^2 + \ldots + x_n^2, \quad x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{1,n}.$$

The induced (Riemannian) metric on $\mathbb{H}^n$ is

$$b = \frac{dr^2}{1 + r^2} + r^2 g_{S^{n-1}},$$

where $g_{S^{n-1}}$ stands for the round metric on the unit sphere $S^{n-1}$, $r = |x'|_\delta$, $x' = (0, x_1, \ldots, x_n)$ and

$$|x'|_\delta = \sqrt{x_1^2 + \ldots + x_n^2}.$$
It is immediate that \((\mathbb{H}^n, b)\) is a complete static manifold with \(\overline{\kappa} = -n(n-1)/2\) whose space of static potentials is given by \(\mathcal{N}_b = [V(0), V(1), \ldots, V(n)]\), where \(V(i) = x_i|_{\partial \Sigma_0}\).

For each \(s \in \mathbb{R}\) consider \(\mathbb{H}^n_s = \{x \in \mathbb{H}^n; x_1 \leq s\}\). We will see in next proposition that \(\Sigma_s = V(1)_i(s)\), the boundary of \(\mathbb{H}^n_s\), is a totally umbilical hypersurface of \(\mathbb{H}^n\).

In fact, the family \(\{\Sigma_s\}_{s \in \mathbb{R}}\) constitutes the equidistant hypersurfaces of \(\Sigma_0\), which is totally geodesic. Notice that the outward pointing unit vector field along \(\Sigma_s\) is \(\eta_s = V_n V(1)_i / V_n \nabla V(1)_i\).

**Proposition 1.6.** Along \(\Sigma_s\) we have \(|\nabla b V(1)_i| = \sqrt{1 + s^2}\). Also, the second fundamental form \(\Pi_s\) of \(\Sigma_s\) is given by

\[
\Pi_s = \lambda_s \gamma_s, \quad \lambda_s = \frac{V(1)_i}{|\nabla b V(1)_i|} = \frac{s}{\sqrt{1 + s^2}}.
\]

where \(\gamma_s = b|_{\Sigma_s}\) is the induced metric. Moreover, for each \(i \neq 1\),

\[
\frac{\partial V(1)_i}{\partial n} = \lambda_s V(i).
\]

In particular, the triple \((\mathbb{H}^n_s, \Sigma_s, b)\) is a static manifold with boundary satisfying

\[
(\widetilde{\Lambda}, \widetilde{\lambda}) = \left(-\frac{n(n-1)}{2}, (n-1)\lambda_s\right),
\]

and the corresponding space of static potentials is

\[
\mathcal{N}_{b,s} = [V(0), V(2), \ldots, V(n)].
\]

**Proof.** A direct computation shows that

\[
\nabla b V(1)_i = \sum_j \langle \partial_{x_j}, \partial_{x_j}\rangle_{1,n} \partial_{x_j} + x_i x_i,
\]

where \(\{\partial_{x_i}\}_{i=0}^n\) is the standard orthonormal basis. It follows that

\[
(\nabla b V(1)_i, \nabla b V(1)_j) = \langle \partial_{x_i}, \partial_{x_j}\rangle_{1,n} x_i x_j,
\]

so that \(|\nabla b V(1)_i| = \sqrt{1 + s^2}\) along \(\Sigma_s\). Thus, if \(X, Y\) are vector fields tangent to \(\Sigma_s\),

\[
\Pi_s(X, Y) = -\langle (\nabla b)_{XY}, \eta_s \rangle = -(1 + s^2)^{-1/2} \langle (\nabla b)_{XY}, \nabla b V(1)_i \rangle = (1 + s^2)^{-1/2} \langle Y, (\nabla b)_{XY} \nabla b V(1)_i \rangle = (1 + s^2)^{-1/2} (\nabla b V^2(1)_i)(X, Y).
\]

We now remark that the staticity equation (1.4) implies

\[
\nabla b V = \nabla b V, \quad V \in \mathcal{N}_b,
\]

so we get

\[
\Pi_s(X, Y) = s(1 + s^2)^{-1/2} \gamma_s(X, Y),
\]

as desired. Finally, note that (1.7) is equivalent to the identity

\[
\langle \nabla b V(1)_i, \nabla b V(1)_i \rangle = V(i)_i V(1)_i, \quad i \neq 1,
\]

which is immediate from (1.8). \qed
We now define the concept of an asymptotically hyperbolic manifold with non-compact boundary having \((\mathbb{H}^n, b, \Sigma_s)\) as an asymptotic model. Recall that one can parameterize \(\mathbb{H}^n\) by polar coordinates \((r, \varphi)\), where \(r = |x|\) as before and \(\varphi = (\varphi_1, ..., \varphi_{n-1}) \in S^{n-1}\). Let \(\{\partial_{\varphi_1}, ..., \partial_{\varphi_{n-1}}\}\) be an orthonormal frame for \(g_{S^{n-1}}\). Then \(\{f_i\}_{i=1}^n\), with \(f_a = r^{-1} \partial_{\varphi_a}\), \(a = 1, ..., n-1\), and \(f_n = \sqrt{r^2 + 1}\partial_r\), is an orthonormal frame for \(b\). Given \(s\) as above, for all \(r_0 > 0\) large enough let us set \(\mathbb{H}^n_{s, r_0} = \{x \in \mathbb{H}^n: r(x) \geq r_0\}\).

**Definition 1.7.** We say that \((M^n, g, \Sigma)\) is an \(s\)-asymptotically hyperbolic (briefly, \(s\)-AH) manifold if there exist a region \(M_{\text{ext}} \subset M\) and a diffeomorphism (a chart at infinity) \(F: \mathbb{H}^n_{s, r_0} \rightarrow M_{\text{ext}}\), for some \(r_0 > 0\), such that the induced metric \(F^* g\) on \(\mathbb{H}^n_{s, r_0}\) satisfies the asymptotic conditions

\[
|F^* g - b|_b + \sum_{i=1}^n |f_i (F^* g)|_b + \sum_{i,j=1}^n |f_i f_j (F^* g)|_b = O(r^{-\sigma}), \quad \sigma > n/2.
\]

We further assume that \(r(R_g + n(n-1)) \in L^1(M)\) and \(r(H_g - (n-1)\lambda) \in L^1(\Sigma)\), where the radial function \(r\) has been smoothly extended to the whole of \(M\).

In the next section we define a notion of mass for this kind of asymptotically hyperbolic manifold with a non-compact boundary. In case the underlying manifold is spin, we will be able to establish the corresponding positive mass theorem under suitable dominant energy conditions. This is the content of our main result, Theorem 2.8 below. As a consequence of the corresponding rigidity statement, the following result is easily obtained.

**Theorem 1.8.** Let \((M^n, g, \Sigma)\) be an \(s\)-AH spin manifold with \(R_g \geq -n(n-1)\) and \(H_g \geq (n-1)\lambda\). Assume further that (1.9) holds with \(\sigma > n\). Then \((M^n, g, \Sigma) = (\mathbb{H}^n, b, \Sigma_s)\) isometrically.

**Proof.** Since any \(V \in \mathcal{N}_{b,s}\) satisfies \(V = O(r)\) as \(r \to +\infty\), it is immediate that the decay assumption (1.9) with \(\sigma > n\) implies that the mass vector \(P_s(F)\) in (2.16) vanishes. The result then follows from Theorem 2.8 and Remark 2.7. \(\square\)

Clearly, this result means that the static manifold with boundary \((\mathbb{H}^n, b, \Sigma_s)\) is rigid in the sense discussed previously.

**Corollary 1.9.** [Sou21, Theorem 2] The embedding \(\Sigma_s \hookrightarrow \mathbb{H}^n\) can not be compactly deformed (as a hypersurface of \(\mathbb{H}^n\)) while keeping its mean curvature at least \((n-1)\lambda\) everywhere.

**Remark 1.10.** Theorem 1.8 actually implies a sharper version of Corollary 1.9 in the sense that the deformation does not have to be compactly supported. This follows from the fact that, in the specific case of deformations of the embedding \(\Sigma_s \hookrightarrow \mathbb{H}^n\), the assumption \(\sigma > n\) may be rephrased in terms of suitable decay rates for the fundamental forms of the deformation viewed as a graph over \(\Sigma_s\) in a neighborhood of infinity.

**Remark 1.11.** It is well-known that the condition \(R_g \geq -n(n-1)\) may be interpreted as a dominant energy condition in the interior of \(M\). Similarly, it turns out that the condition \(H_g \geq (n-1)\lambda\) may also be viewed as a dominant energy condition along \(\Sigma\) in the spirit of [AdLM21], which treats the case \(s = 0\).
Remark 1.12. As stated above, Theorem 1.8 and Corollary 1.9 do not contemplate the case of a non-compact (static) domain in $\mathbb{H}^n$ whose boundary is a horosphere, a situation that, at least for compactly supported deformations, has also been considered in [Sou21]. It turns out that this case may be approached by means of a somewhat involved variation of the methods leading to Theorem 1.8. More precisely, it is possible to formulate and prove a positive mass theorem for asymptotically hyperbolic spin manifolds modeled at infinity on such static domains. As expected, its rigidity statement retrieves a sharper version (in the sense of Remark 1.10 above) of the corresponding result in [Sou21 Theorem 2]; see Section 5 for details.

This paper is organized as follows. In Section 2, we introduce the concept of mass for $s$-AH manifolds and establish its geometric invariance. The corresponding positive mass theorem, including its rigidity statement, is proved in Section 3 under the spin assumption. This uses in a crucial way the $\vartheta$-boundary conditions on spinors, whose properties are discussed in Section 3. Finally, in Section 5 we indicate how our main theorem may be extended to the horospherical case.

2. THE GEOMETRIC INVARIANCE OF THE MASS FUNCTIONAL

Here we define a mass-type invariant for an $s$-AH manifold $(M, g, \Sigma)$ as in Definition 1.7, and study its invariance properties under the group of isometries of the model space. The argument here is quite similar to that appearing in [AdL20b, Section 3], so we omit some details.

Since ultimately the mass will depend only on the asymptotic geometry of the manifold, we may appeal to the identification provided by the chart $F$ to work in $\mathbb{H}^n_{s, r_0}$. We set, for $r_0 < r_1 < r_2$,

$$A_{r_1, r_2} = \{ x \in \mathbb{H}^n_{s, r_0}; r_1 \leq |x'|_b \leq r_2 \}, \quad \Sigma_{r_1, r_2} = \{ x \in \mathbb{H}^n_{s, r_0}; \Sigma_s; r_1 \leq |x'|_b \leq r_2 \},$$

and $S^{n-1}_{r, \vartheta} = \{ x \in \mathbb{H}^n_{s, r_0}; |x'|_b = r \}$, so that

$$\partial A_{r_1, r_2} = S^{n-1}_{r_1, \vartheta} \cup \Sigma_{r_1, r_2} \cup S^{n-1}_{r_2, \vartheta}.$$ 

In other words, $S^{n-1}_{r, \vartheta}$ is the portion of the geodesic sphere centered at the “origin” $(1, 0, \ldots, 0)$ in $\mathbb{R}^{1, n}$ and with radius $\sinh^{-1} r$ lying inside $\mathbb{H}^n_{s, r_0}$. We represent by $\mu$ the outward unit normal vector field to $S^{n-1}_{r_1, \vartheta}$ or $S^{n-1}_{r_2, \vartheta}$ with respect to the metric $b$. Also, we set $S^{n-2}_{r, \vartheta} = \partial S^{n-1}_{r, \vartheta} \rightarrow \Sigma_s$, oriented by its outward unit conormal field $\vartheta$, again with respect to $b$. Finally, we set $e = g - b$ and define the 1-form

$$\Psi(V, e) = V (\text{div}_b e - \text{tr}_b e) - \nabla_b V \uparrow e + \text{tr}_b e \, dV.$$ 

Theorem 2.1. If $(M, g, \Sigma)$ is an $s$-AH manifold then the quantity

$$m_{s, F}(V) = \lim_{r \to +\infty} \left[ \int_{S^{n-1}_{r, \vartheta}} (U(V, e), \mu) dS^{n-1}_{r, \vartheta} - \int_{S^{n-2}_{r}} V e(\eta_s, \vartheta) dS^{n-2}_{r} \right], \quad V \in \mathcal{N}_{b, s},$$

exists and is finite.
Proof. As in the proof of [AdL20b Theorem 3.1], the argument is based on the expansions of both the scalar curvature and the mean curvature around the background metric $b$. To begin with, a well-known computation gives, for any $V \in \mathcal{N}_{b,s}$,
\[
V(R_g + n(n-1)) = \text{div}_b U(V,e) + Q_b(e,V),
\]
where $Q_b(e,V)$ is linear in $V$ and at least quadratic in $e$. We now perform the integration of this identity over the half-annular region $A_{r',r} \subset \mathbb{H}^{n}_{s,r_0}$ and explore the imposed boundary conditions on the underlying static domain, namely,

\[
(2.12) \quad \frac{\partial V}{\partial \eta_s} = \lambda_s V, \quad \Pi_s = \lambda_s \gamma_s;
\]

compare with Proposition 1.6. Using the well-known first variation formula for the mean curvature of the boundary we obtain

\[
\mathcal{F}_{r_1,r_2}(g) := \int_{A_{r_1,r_2}} V(R_g + n(n-1)) dM_b + 2 \int_{\Sigma_{r_1,r_2}} V(H_g - (n-1)\lambda_s) d\Sigma_{\gamma_s}
\approx \int_{S_{r_1}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_2}^{n-1} + \int_{S_{r_1}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_1}^{n-1}
- \int_{\Sigma_{r_1,r_2}} (\nabla_b V \cdot e)(\eta_s) d\Sigma_{\gamma_s} + \int_{\Sigma_{r_1,r_2}} \text{tr}_b e V(\eta_s) d\Sigma_{\gamma_s}
- \int_{\Sigma_{r_1,r_2}} V \text{div}_{\gamma_s} X d\Sigma_{\gamma_s} - \int_{\Sigma_{r_1,r_2}} V \langle \Pi_b, e \rangle \gamma d\Sigma_{\gamma_s}
\]

where $X$ is the vector field dual to the 1-form $(\eta_s, \cdot)|_{\Sigma^*}$ the symbol $\approx$ means that we are discarding certain integrals over $\Sigma_{r_1,r_2}$ or $A_{r_1,r_2}$ which vanish as $r_1 \to +\infty$ due to the assumption $\sigma > n/2$. Also, we are omitting the restriction symbol on $e|_{\Sigma}$. Now observe that (2.12) leads to
\[
\text{tr}_b e V(\eta_s) = V(\Pi_b, e) \gamma_s,
\]
so that, after a little manipulation, we end up with

\[
\mathcal{F}_{r_1,r_2}(g) \approx \int_{S_{r_1}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_2}^{n-1} + \int_{S_{r_1}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_1}^{n-1}
- \int_{\Sigma_{r_1,r_2}} \text{div}_{\gamma_s} (V X) d\Sigma_{\gamma_s}
= \int_{S_{r_1}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_2}^{n-1} + \int_{S_{r_2}^{n-2}} V e(\eta_s, \vartheta) dS_{r_2}^{n-2}
- \left( \int_{S_{r_2}^{n-1}} \langle U(V,e), \mu \rangle dS_{r_1}^{n-1} - \int_{S_{r_1}^{n-2}} V e(\eta_s, \vartheta) dS_{r_1}^{n-2} \right).
\]

Bearing in mind that $V = O(r)$, the integrability assumptions on $r(R_g + n(n-1))$ and $r(H_g - (n-1)\lambda_s)$ in Definition 1.7 clearly imply that $\mathcal{F}_{r_1,r_2}(g) \to 0$ as $r_1 \to +\infty$, which completes the proof.

We must think of $m_{s,F}$ as a linear functional on the space $\mathcal{N}_{b,s}$ of static potentials satisfying the given boundary conditions. We must be aware, however, that the decomposition $g = b + e$ used above depends on the choice of a chart at infinity (namely, the diffeomorphism $F$), so we need to check that $m_{s,F}$ behaves as
expected when we pass from one such chart to another. For this we need a couple of results whose proofs follow from well-known principles \[ \text{[CH03, AdL20b].} \]

**Lemma 2.2.** If \( V \in \mathcal{N}_{b,s} \) and \( X \) is a vector field then
\[
(2.13) \quad \mathcal{U}(V, \mathcal{L}X b) = \text{div}_b V(V, X, b),
\]
where the 2-form is explicitly given by
\[
(2.14) \quad \mathcal{V}_{ik} = V(X_{i;k} - X_{k;i}) + 2(X_k V_i - X_i V_k).
\]
Here, the semicolon denotes covariant differentiation with respect to \( b \).

**Lemma 2.3.** If \( F: \mathbb{H}^n \rightarrow \mathbb{H}^n \) is a diffeomorphism such that \( F^* b = b + O(r^{-\sigma}) \) then there exists an isometry \( A \) of \( \mathbb{H}^n \) such that
\[
F = A + O(r^{-\sigma}).
\]

**Remark 2.4.** In the hyperboloid model \( \mathbb{H}^n \rightarrow \mathbb{R}^{1,n} \), the isometry group of hyperbolic \( n \)-space gets identified to \( O^+(1,n) \), the subgroup of linear isometries of \( (\mathbb{R}^{1,n}, \langle , \rangle_{1,n}) \) preserving time orientation. It is clear that any \( A \in O^+(1,n) \) preserving \( \Sigma_{s} \), and hence defining an isometry of \( \mathbb{H}^n_s \), also preserves \( \Sigma_0 \), which is an isometric copy of hyperbolic \( (n-1) \)-space. Thus, the group of isometries of \( \mathbb{H}^n_s \), which appears in Lemma 2.2 above, may be identified to \( O^+(1,n-1) \). We thus obtain a natural representation \( \rho^s \) of \( O^+(1,n-1) \) on \( \mathcal{N}_{b,s} \) by setting \( \rho^s_A(V) = V \circ A^{-1} \), which is easily shown to be irreducible.

Suppose now that we have two diffeomorphisms, say \( F_1, F_2: \mathbb{H}^n_{s,r_0} \rightarrow M_{\text{ext}} \), defining charts at infinity as above and consider \( F = F_1^{-1} \circ F_2: \mathbb{H}^n_{s,r_0} \rightarrow \mathbb{H}^n_{s,r_0} \). It is clear that \( F^* b = b + O(r^{-\sigma}) \), \( \sigma > n/2 \), so by Lemma 2.3 \( F = A + O(r^{-\sigma}) \) for some isometry \( A \). The next result establishes the geometric invariance of the mass-type invariant appearing in Theorem 2.1

**Theorem 2.5.** Under the conditions above, there holds
\[
m_{s,F_1}(V) = m_{s,F_2}(\rho^s_A(V)), \quad V \in \mathcal{N}_{b,s}.
\]

**Proof.** As in the proof of \[ \text{[AdL20b, Theorem 3.4]}, \ we may assume that \( A \) is the identity, so that \( F = \exp \circ \zeta \), where \( \zeta \) is a vector field on \( \mathbb{H}^n_{s,r_0} \) which vanishes at infinity and is tangent to \( \Sigma_s \) everywhere. Now set
\[
e_1 = g_1 - b, \quad g_1 = F_1^* g, \quad e_2 = F_2^* g - b = F^* g_1 - b,
\]
so that
\[
e := e_2 - e_1 = \mathcal{L}_\zeta b + R_1,
\]
where \( \mathcal{L} \) is Lie derivative and \( R_1 \) is a certain remainder. It follows that
\[
\mathcal{U}(V, e) = \mathcal{U}(V, \mathcal{L}_\zeta b) + R_2,
\]
where \( R_2 \) is a remainder that vanishes, as \( r \rightarrow +\infty \), after integration over \( S^{n-1}_{r^*} \).

Hence,
\[
\lim_{r \rightarrow +\infty} \int_{S^{n-1}_{r^*}} \langle \mathcal{U}(V, e), \mu \rangle dS^{n-1}_{r^*} = \lim_{r \rightarrow +\infty} \int_{S^{n-1}_{r^*}} \langle \mathcal{U}(V, \mathcal{L}_\zeta b), \mu \rangle dS^{n-1}_{r^*}
\]
\[
= - \lim_{r \rightarrow +\infty} \int_{\Sigma(r)} \langle \mathcal{U}(V, \mathcal{L}_\zeta b), \eta \rangle d\Sigma_s,
\]
where we used (2.13) in the last step to transfer the integral to $\Sigma_{r}$, the compact domain of $\Sigma_{r}$ enclosed by $S_{r}^{n-2}$. We fix an adapted orthonormal frame $\{e_{i}\}_{i=1}^{n}$ so that $e_{1} = -\eta_{s}$ is the inward unit normal to $\Sigma_{s}$, so that $(\Pi_{s})_{\alpha\beta} = \Gamma_{\alpha\beta}^{1} = -\Gamma_{1\beta}^{\alpha}$, $2 \leq \alpha, \beta \leq n$. By (2.13) with $X = \zeta$, 

$$
(\cup (V, L_{\zeta} b), \eta_{s}) = b^{i}V_{i;k} \eta_{s} = -V_{1;k} = -\nabla_{1;\alpha} = \text{div}_{\gamma}(\eta_{s} + V)
$$

and hence,

$$
\int_{\Sigma_{r}} (\cup (V, L_{\zeta} b), \eta_{s}) d\Sigma_{r} = \int_{\Sigma_{r}} \nabla(\eta_{s}, \vartheta) d\Sigma_{r} = -\int_{S_{r}^{n-2}} V_{1;\alpha} \vartheta^{\alpha} dS_{r}^{n-2}.
$$

We now observe that (2.12) may be used to check that $V_{1} = -\lambda_{s} V$ and

$$
\zeta_{1;\alpha} = \zeta_{1,\alpha} + \Gamma_{1\alpha}^{1} \zeta_{1} = \Pi_{s \alpha \beta} \zeta_{\beta} = \lambda_{s} \zeta_{\alpha},
$$

where we used that $\zeta_{1} = 0$ (because $\zeta$ is tangent to $\Sigma$). Similarly, $\zeta_{1;\alpha} = \zeta_{1,\alpha} - \lambda_{s} \zeta_{\alpha}$.

Thus,

$$
\nabla_{1;\alpha} = V(\zeta_{1;\alpha} - \zeta_{1,\alpha}) + 2(\zeta_{\alpha} V_{1} - \zeta_{1} V_{\alpha}) = -V \zeta_{1;\alpha},
$$

so that

$$
\lim_{r \to +\infty} \int_{S_{r}^{n-2}} (\cup (V, e), \mu) dS_{r}^{n-1} = -\lim_{r \to +\infty} \int_{S_{r}^{n-2}} V \zeta_{1;\alpha} \vartheta^{\alpha} dS_{r}^{n-2}.
$$

The argument is completed by noticing that

$$
\left( -\int_{S_{r}^{n-2}} V e_{2}(\eta_{s}, \vartheta) dS_{r}^{n-2} \right) - \left( -\int_{S_{r}^{n-2}} V e_{1}(\eta_{s}, \vartheta) dS_{r}^{n-2} \right) = -\int_{S_{r}^{n-2}} V e(\eta_{s}, \vartheta) dS_{r}^{n-2},
$$

and this equals

$$
-\int_{S_{r}^{n-2}} V (L_{\zeta} b)(\eta_{s}, \vartheta) dS_{r}^{n-2} - \int_{S_{r}^{n-2}} V R_{1}(\eta_{s}, \vartheta) dS_{r}^{n-2},
$$

where the last integral vanishes at infinity. Finally, the remaining integral may be evaluated as

$$
-\int_{S_{r}^{n-2}} V (L_{\zeta} b)(\eta_{s}, \vartheta) dS_{r}^{n-2} = \int_{S_{r}^{n-2}} V (\zeta_{1;\alpha} + \zeta_{1,\alpha}) \vartheta^{\alpha} dS_{r}^{n-2} = \int_{S_{r}^{n-2}} V \zeta_{1;\alpha} \vartheta^{\alpha} dS_{r}^{n-2},
$$

which cancels out the right-hand side of (2.13) as $r \to +\infty$.

We now explore the consequences of Theorem 2.5. For this we introduce a “Lorentzian” inner product $(\cdot, \cdot)_{s}$ on $N_{b,s}$ by declaring that $(V_{(0)}, V_{(2)}, \cdots, V_{(n)})$ is an orthonormal basis with $(V_{(0)}, V_{(0)})_{s} = 1$ and $(V_{(j)}, V_{(j)})_{s} = -1$, $j \geq 2$. Thus, we agree that $V \in N_{b,s}$ is future-directed if $(V, V_{(0)})_{s} > 0$. The key point now is that the isometry group $O^{1}(1, n-1)$ of the background static space $(H_{b}^{n}, b, \Sigma_{s})$ acts isometrically on $(N_{b,s}, (\cdot, \cdot)_{s})$ by means of the representation $\rho^{s}$ considered in Remark 2.4. Since $\rho^{s}$ is irreducible, all vectors in $N_{b,s}$ should equally contribute to defining a single vector-valued mass invariant. Precisely, if for any chart at infinity $F$ as above we set

$$
P_{s}(F)_{a} = m_{s,F}(V_{(a)}), \quad a = 0, 2, \cdots, n,
$$

then Theorem 2.5 guarantees that the causal properties of $P_{s}(F)$ (e.g., whether it is space-like, isotropic or time-like, its past/future-directed nature in the two latter
cases, its Lorentzian length with respect to $\langle \cdot, \cdot \rangle$ etc.) are chart independent indeed. This suggests the following conjecture.

**Conjecture 2.6.** Let $(M, g, \Sigma)$ be an $s$-AH manifold as in Theorem 2.1 above with $R_g \geq -n(n-1)$ and $H_g \geq (n-1)\lambda_s$. Then for any chart at infinity $F$ the vector $P_s(F)$ is time-like and future-directed unless it vanishes, in which case $(M, g, \Sigma)$ is isometric to $(\mathbb{H}^n_s, b, \Sigma_s)$.

**Remark 2.7.** Whenever this conjecture holds true we may define the numerical invariant

$$m_s := \sqrt{\langle P_s(F), P_s(F) \rangle} = \sqrt{\sum_{a=2}^{n} P_a(F)^2 - \sum_{a=2}^{n} P_a(F)^2}.$$

which happens to be independent of the chosen chart. This may be regarded as the total mass of the isolated system whose (time-symmetric) initial data set is $(M, g, \Sigma)$. Notice that $m_s \geq 0$ with the equality holding if and only if $(M, g, \Sigma)$ is isometric to $(\mathbb{H}^n_s, b, \Sigma_s)$.

As remarked in the introduction, our main result here confirms the conjecture (and hence the physical interpretation for $m_s$ above) in case $M$ is spin.

**Theorem 2.8.** Conjecture 2.6 holds true in case $M$ is spin.

As already observed, Theorem 1.8 (and hence Corollary 1.9) is an immediate consequence of Theorem 2.8.

3. Spinors on manifolds with boundary

In this section we review the results in the theory of spinors on manifolds carrying a (possibly non-compact) boundary $\Sigma$ which are needed in the rest of the paper, including the appropriate integral version of the celebrated Lichnerowicz formula. Also, we introduce a family of boundary conditions for spinors which interpolates between chirality and MIT bag boundary conditions (Definition 3.3 below) and plays a central role in the proof of our main results.

3.1. The integral Lichnerowicz formula on spin manifolds with boundary. We assume that the given manifold $(M, g)$ is spin and fix once and for all a spin structure on $TM$. We denote by $\mathbb{S}M$ the associated (Hermitian) spinor bundle and by $\nabla$ both the Levi-Civita connection of $TM$ and its compatible lift to $\mathbb{S}M$. Also, each tangent vector $X$ induces a linear map $c(X) : \mathbb{S}M \to \mathbb{S}M$, the (left) Clifford multiplication by $X$. For our purposes, it suffices to know that these structures satisfy a few compatibility conditions:

1. $c(X)c(Y) + c(Y)c(X) = -2\langle X, Y \rangle g$;
2. $\langle c(X)\Psi, c(X)\Phi \rangle = |X|^2 \langle \Psi, \Phi \rangle$;
3. $\nabla_X (c(Y)\Psi) = c(\nabla_X Y)\Psi + c(Y)(\nabla_X \Psi)$.

Here, $X, Y \in \Gamma(TM)$ are tangent vector fields and $\Psi, \Phi \in \Gamma(\mathbb{S}M)$ are spinors. When emphasizing the dependence of $c$ on $g$ is needed, we append a sub/superscript and write $c = c^g$ for instance (and similarly for other geometric invariants associated to the given spin structure).
We define the corresponding Dirac operator acting on spinors by
\[ D = c \circ \nabla, \]
where we view the connection as a linear map
\[ \nabla : \Gamma(SM) \to \Gamma(T^*M \otimes SM) = \Gamma(TM \otimes SM) \]
and the identification \( T^*M = TM \) comes from the metric. For our purposes, it is convenient to slightly modify this classical construction. Thus, we define the Killing connections on \( SM \) by
\[ \nabla_X = \nabla_X \pm \frac{i}{2} c(X), \quad X \in \Gamma(TM), \]
so that the corresponding Killing-Dirac operators are defined in the usual way, namely,
\[ D^\pm := c \circ \nabla^\pm = D \mp \frac{n i}{2}. \]
As a consequence, we obtain the integral version of the fundamental Lichnerowicz formula:
\[ \int_{\Omega} \left( |D^\pm \Psi|^2 - |D^\pm \Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2 \right) dM = \text{Re} \int_{\partial \Omega} \langle W^\pm(\nu) \Psi, \Psi \rangle d\Sigma. \]
Here, \( \Psi \in \Gamma(SM) \), \( \Omega \subset M \) is a compact domain with a nonempty boundary \( \partial \Omega \), which we assume endowed with its inward pointing unit normal \( \nu \), and
\[ W^\pm(\nu) = - (\nabla^\pm_{\nu} + c(\nu) D^\pm). \]
A key step in our argument is to rewrite the right-hand side of (3.18) along the portion of \( \partial \Omega \) lying on \( \Sigma = \partial M \) in terms of the corresponding extrinsic geometry. To proceed, note that \( SM|_{\Sigma} \) becomes a Dirac bundle if endowed with the Clifford multiplication
\[ c^T(X) \Psi = c(X) c(\nu) \Psi, \]
and the connection
\[ \nabla^T_X \Psi = \nabla_X \Psi + \frac{1}{2} c^T(\nabla_X \nu) \Psi, \]
so the corresponding Dirac operator \( D^T : \Gamma(SM|_{\Sigma}) \to \Gamma(SM|_{\Sigma}) \) is
\[ D^T = c^T \circ \nabla^T. \]
It follows that
\[ D^T - \frac{H_g}{2} = - (\nabla_{\nu} + c(\nu) D), \]
which combined with (3.18) yields the following important result for our arguments.

**Proposition 3.1.** Under the conditions above,
\[ \int_{\Omega} \left( |\nabla^\pm \Psi|^2 - |D^\pm \Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2 \right) d\Omega = \int_{\partial \Omega \cap \Sigma} \langle (D^{T,\pm} \Psi, \Psi) - \frac{H_g}{2} |\Psi|^2 \rangle d\Sigma + \text{Re} \int_{\partial \Omega \cap \text{int} M} \langle W^\pm(\nu) \Psi, \Psi \rangle d\partial \Omega, \]
where
\[ D^{T,\pm} = D^T \pm \frac{(n-1)i}{2} c(\nu) : \Gamma(\Sigma \Sigma) \to \Gamma(\Sigma \Sigma). \]
3.2. \textbf{\textit{\theta}-boundary conditions.} To further simplify the integral over $\partial \Omega \cap \Sigma$ in (3.21) we must impose suitable boundary conditions on $\Psi$.

\textbf{Definition 3.2.} A \textit{chirality operator} on a spin manifold $(M,g)$ is a (pointwise) self-adjoint involution $Q : \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ which is parallel and anti-commutes with Clifford multiplication by tangent vectors.

To simplify the exposition, we henceforth assume that $n$ is even, as in this case it is well-known that Clifford multiplication by the complex volume element $\omega = i^{n/2}e(e_1)\cdots(e_n)$ provides a natural chirality operator, namely, $Q = \omega$. Nevertheless, see Remark 3.8 below for the indication on how a chirality operator may be constructed in odd dimensions as well, so that the argument below may be carried out in full generality.

We now fix $\kappa \in (0,1]$ and set $\tau = \pm \sqrt{1-\kappa^2} \in (-1,1)$, so that $e^{i\theta} = \kappa + \tau i$, where $\kappa = \cos \theta$ and $\tau = \sin \theta$, $\theta \in (-\pi/2, \pi/2)$.

\textbf{Definition 3.3.} The $\theta$-boundary operator $Q_{\theta,g} : \Gamma(\mathbb{S}M|\Sigma) \to \Gamma(\mathbb{S}M|\Sigma)$ associated to $Q = \omega$ as above is
\begin{equation}
Q_{\theta,g} = e^{i\theta Q} Q \xi(\nu).
\end{equation}

\textbf{Remark 3.4.} The involutiveness of $Q$ ($Q^2 = I$) implies that $\sin(\theta Q) = (\sin \theta)Q$ and $\cos(\theta Q) = (\cos \theta)I$, so that
\begin{equation}
Q_{\theta,g} = \kappa Q \xi(\nu) + \tau \text{ic}(\nu).
\end{equation}

Hence, as $\theta$ varies between $\theta = 0$ and $\theta = \pm \pi/2$, $Q_{\theta,g}$ interpolates between the chirality boundary operator $Q \xi(\nu)$ used in [ABdL16] and the MIT bag boundary operators $\pm \text{ic}(\nu)$ used in [ABdL16].

\textbf{Remark 3.5.} The complex phase $e^{i\theta} = \kappa + \tau i$ acquires a geometric meaning if we impose that
\begin{equation}
\lambda_s = \sin \theta,
\end{equation}
where $s \in \mathbb{R}$ is the parameter appearing in Proposition 1.6. This matching of parameters, which we always assume to hold throughout this work, means that $\Sigma_s$ has constant mean curvature equal to $(n-1)\tau$ and is intrinsically isometric to $\mathbb{H}^{n-1}(-\kappa^2)$, the hyperbolic $(n-1)$-space with curvature $-\kappa^2$. From this perspective, the fundamental trigonometric identity
\begin{equation}
\kappa^2 + \tau^2 = 1
\end{equation}
is just Gauss equation in disguise.

\textbf{Remark 3.6.} The extrinsic data $(\mathbb{S}M|\Sigma; \xi, \nabla^\gamma)$ associated to a hypersurface embedding $\Sigma \to M$ can be identified to objects constructed out of the \textit{intrinsic} data $(\Sigma, \xi, \nabla^\gamma)$, where $\gamma = g|_{\Sigma}$ is the induced metric along $\Sigma$; see [Mor11] Section 2.2 or [HM15] Section 2.2]. Since we are assuming that $n$ is even, the fact that $Q = \omega$ as above is an involution may be used to split the spinor bundle of $M$ as an orthogonal direct sum of its $\pm 1$-eigenbundles. In this way we obtain the \textit{chiral decomposition}
\[
\mathbb{S}M = \mathbb{S}M^+ \oplus \mathbb{S}M^-,
\]
so that
\[
Q \Psi^\pm = \pm \Psi^\pm, \quad \Psi = \Psi^+ + \Psi^- , \quad \Psi^\pm \in \Gamma(\mathbb{S}M^\pm).
\]
Upon restriction to $\Sigma$ this yields the first identification, namely,

$$SM|_{\Sigma} = SM^+|_{\Sigma} \oplus SM^-|_{\Sigma} = S\Sigma \oplus S\Sigma,$$

with

$$\Psi \in \Gamma(SM|_{\Sigma}) \mapsto \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi_i \in \Gamma(S\Sigma),$$

satisfying

$$Q \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -\Psi_1 \\ -\Psi_2 \end{pmatrix}.$$  

Moreover, under (3.28),

$$c^\top = \begin{pmatrix} c^\gamma & 0 \\ 0 & -c^\gamma \end{pmatrix}, \quad \nabla^\top = \begin{pmatrix} \nabla^\gamma & 0 \\ 0 & -\nabla^\gamma \end{pmatrix},$$

so that

$$D^\top = \begin{pmatrix} D^\gamma & 0 \\ 0 & -D^\gamma \end{pmatrix},$$

where $D^\gamma = c^\gamma \circ \nabla^\gamma$ is the intrinsic Dirac operator associated to $\gamma$. Finally, we may agree that

$$\epsilon(\nu) = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$  

The identifications in Remark 3.6 provide a rather explicit description of the $\theta$-boundary operator which will be useful later on.

**Proposition 3.7.** With the notation in Remark 3.6 the action of $Q_{\theta,g}$ on a (restricted) spinor is

$$Q_{\theta,g} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} -ie^{i\theta} \Psi_2 \\ ie^{-i\theta} \Psi_1 \end{pmatrix} = \begin{pmatrix} (\tau - \kappa i) \Psi_2 \\ (\tau + \kappa i) \Psi_1 \end{pmatrix}.$$  

**Proof.** Use (3.24), (3.29) and (3.30). \qed

**Remark 3.8.** In the odd dimensional case, we may always define a chirality operator in the direct sum bundle $SM \oplus SM$ by simply switching the factors. Even though we do not carry out the details, it turns out that this simple trick allows us to straightforwardly extend our main results here to this case; see [AdL20b] for the pertinent details when $\theta = 0$.

For later reference, we now isolate a few algebraic facts concerning this formalism. In what follows, we denote by $[A, B]$ (respectively, $\{A, B\}$) the commutator (respectively, the anti-commutator) of the operators $A$ and $B$.

**Proposition 3.9.** The following properties hold:

1. $Q_{\theta,g}$ is a self-adjoint involution;
2. $\{Q, Q_{\theta,g}\} = 0$;
3. $\{D^\top, Q_{\theta,g}\} = 0$;
4. $\{\epsilon(\nu), Q_{\theta,g}\} = -2\tau I;$
Proposition 3.11. If then holds everywhere along \( \Sigma \).

We say that \( \{ \rangle \) gives the third item. Finally, the fourth item is proved by first checking that \( \langle \), so that \( \] follows. The second item is an immediate consequence of the definitions. To proceed, note that \([D^\top, e^{i\theta Q}] = 0\) as well. Also, \( \{D^\top, e^{i\theta Q}\} = 0\), so that \( D^\top Q_{\theta,g} = D^\top e^{i\theta Q} e^{i\theta Q} = -e^{i\theta Q} Q_{\theta,g} D^\top \), which gives the third item. Finally, the fourth item is proved by first checking that \( \{e^{i\theta Q}, e^{i\theta Q}\} = 0 \) and \( \{e^{i\theta Q}, e^{\pm i\theta Q}\} = -2i \) and then using this in (3.24).

Proof. This already follows (for \( n \) even) from the explicit description of these objects in Remark 3.6 and Proposition 3.7 but we provide here an alternate abstract reasoning that works whenever a chirality operator is available. First, self-adjointness of \( Q_{\theta,g} \) follows from it being a real linear combination of \( Q \) and \( ic(\nu) \), which are easily checked to meet this property. Since \( \{Q, c(\nu)\} = 0 \) we have \( c(\nu) e^{i\theta Q} = e^{-i\theta Q} c(\nu) \) so that

\[
Q_{\theta,\nu}^2 = e^{i\theta Q} Q c(\nu) e^{i\theta Q} c(\nu) = Q c(\nu) e^{i\theta Q} Q c(\nu) = Q c(\nu) c(\nu) = -Q^2 c(\nu)^2,
\]

from which the involutiveness of \( Q_{\theta,g} \) follows. The second item is an immediate consequence of the definitions. To proceed, note that \([D^\top, e^{i\theta Q}] = 0\) so that \([D^\top, e^{i\theta Q}] = 0\) as well. Also, \( \{D^\top, e^{i\theta Q}\} = 0\), so that

\[
D^\top Q_{\theta,g} = D^\top e^{i\theta Q} e^{i\theta Q} = -e^{i\theta Q} Q_{\theta,g} D^\top,
\]

which gives the third item. Finally, the fourth item is proved by first checking that \( \{e^{i\theta Q}, e^{i\theta Q}\} = 0 \) and \( \{e^{i\theta Q}, e^{\pm i\theta Q}\} = -2i \) and then using this in (3.24). \( \square \)

Since \( Q_{\theta,g} \) is a self-adjoint involution, we may consider the projections

\[
P^{(\pm)}_\theta = \frac{1}{2} \left( I_{\Sigma |\Sigma} \pm Q_{\theta,g} \right) : \Gamma(SM|\Sigma) \to \Gamma(V_\theta^{(\pm)})
\]

onto the \( \pm 1 \)-eigenbundles \( V_\theta^{(\pm)} \) of \( Q_{\theta,g} \). Thus, \( \Psi \in \Gamma(V_\theta^{(\pm)}) \) if and only if \( Q_{\theta,g} \Psi = \pm \Psi \). For any \( \Psi \in \Gamma(SM|\Sigma) \) we set \( \Psi^{(\pm)}_\theta = P^{(\pm)}_\theta \Psi \in \Gamma(V_\theta^{(\pm)}) \), so that

\[
\Psi = \Psi^{(+)}_\theta + \Psi^{(-)}_\theta,
\]

an orthogonal decomposition. From Proposition 3.7 (3), we have \( D^\top P^{(\pm)}_\theta = P^{(\pm)}_\theta D^\top \), which gives

\[
(D^\top \Psi, \Psi) = (D^\top \Psi^{(+)}_\theta, \Psi^{(+)}_\theta) + (D^\top \Psi^{(-)}_\theta, \Psi^{(-)}_\theta).
\]

Definition 3.10. We say that \( \Psi \in \Gamma(SM) \) satisfies a \( \theta \)-boundary condition if any of the identities

\[
Q_{\theta,g} \Psi = \pm \Psi
\]

holds everywhere along \( \Sigma = \partial M \) (equivalently, \( \Psi^{(\pm)}_\theta = 0 \)).

Proposition 3.11. If \( \Psi \) as in Proposition 3.7 satisfies the \( \theta \)-boundary condition (3.33) then

\[
\text{Re} \int_{\partial \Omega \cap \text{int} M} \langle W^\pm (\nu) \Psi, \Psi \rangle d\partial \Omega = \int_{\Omega} \left( |\nabla^\pm \Psi|^2 - |D^\pm \Psi|^2 + \frac{R_g + n(n - 1)}{4} |\Psi|^2 \right) d\Omega
\]

\[
+ \int_{\partial \Omega \cap \Sigma} \frac{1}{2} (H_g - (n - 1) \tau) |\Psi|^2 d\Sigma.
\]
Proof. From Proposition 3.9 (4), we find that

\[ i(c(\nu)\Psi, \Psi) = \pm i(c(\nu)Q_{\theta,g}\Psi, \Psi) \]

\[ = \pm i(Q_{\theta,g} c(\nu)\Psi, \Psi) \pm 2\tau|\Psi|^2 \]

\[ = \pm i(c(\nu)\Psi, Q_{\theta,g}\Psi) \pm 2\tau|\Psi|^2 \]

\[ = \pm i(c(\nu)\Psi, \pm \Psi) \pm 2\tau|\Psi|^2 \]

\[ = \pm i(c(\nu)\Psi, \Psi) \pm 2\tau|\Psi|^2 , \]

which gives

(3.35)

\[ i(c(\nu)\Psi, \Psi) = \pm \tau|\Psi|^2 . \]

On the other hand, from (3.32) we have

\[ \langle D^{\top} \Psi, \Psi \rangle = 0 . \]

Thus, from (3.22),

\[ \langle D^{\top}, \pm \Psi, \Psi \rangle = \frac{n-1}{2} (\pm \tau|\Psi|^2 ) , \]

so that

(3.36)

\[ \langle D^{\top}, \Psi \rangle = \frac{(n-1)\tau}{2}|\Psi|^2 . \]

Together with (3.21), this proves (3.34). \( \Box \)

Remark 3.12. The sign cancellation leading to (3.36) shows that the integral over \( \partial \Omega \cap \Sigma \) in (3.34) only has the expected shape if the sign of the Killing connection defined in (3.17) matches the sign of the \( \theta \)-boundary condition in (3.33). In the following we always assume that this sign convention pairing objects constructed out of the Killing connection \( \nabla^\pm \) to the \( \theta \)-boundary condition involving the projection \( P_{\theta}^{(\pm)} \) holds true both in statements and computations; see Remark 3.14 for an illustrative example.

We now observe that, by Proposition 3.9 (2), the eigenbundles \( V_{\theta}^{(\pm)} \) are interchanged by \( Q \), so that

\[ \text{rank} V_{\theta}^{(\pm)} = \frac{1}{2} \text{rank} \Sigma M|\Sigma . \]

It then follows from the analysis in [Gil03, Subsection 1.4.5] that the projections \( P_{\theta}^{(\pm)} \) define elliptic boundary conditions for the Dirac operator \( D^s \) considered above. This is the key input in establishing the following existence result.

Proposition 3.13. Let \( (M, g, \Sigma) \) be \( s \)-AH as in Definition 1.7 with \( s \) satisfying (3.25) and assume further that \( R_g \geq -n(n-1) \) and \( H_g \geq (n-1)\lambda_s \). Then for any \( \Phi \in \Gamma(\Sigma M) \) such that \( D^s \Phi \in L^2(\Sigma M) \) there exists a unique \( \Xi \in L^2_1(\Sigma M) \) solving the boundary value problem

(3.37)

\[ \begin{cases} 
D^s \Xi = -D^s \Phi & \text{in } M, \\
Q_{\theta,g} \Xi = \pm \Xi & \text{on } \Sigma. 
\end{cases} \]

Proof. As already remarked, (3.25) means that \( (M, g, \Sigma) \) is modeled at infinity on \( (\mathbb{H}^n_s, b, \Sigma_s) \). With this information at hand, the proof is a simple adaptation of the argument leading to [AdL20b Proposition 4.7], which treats the case \( s = 0 \). \( \Box \)
Remark 3.14. The statement of Proposition 3.13 displays an example of the sign matching convention alluded to in Remark 3.12. For instance, the plus sign in $D^+ = \psi \circ \nabla^+$ should match the plus sign in the right-hand side of the bottom line of (3.37) and similarly for $D^-$ and the minus sign. By convention, the remaining possibilities do not occur in (3.37). This illustrates the kind of sign convention we henceforth adopt.

3.3. Imaginary Killing spinors. We start by recalling a well-known definition which in a sense justifies the consideration of the Killing connection $\nabla^\pm$ in (3.17).

Definition 3.15. We say that $\Phi \in \Gamma(\mathcal{S}M)$ is an imaginary Killing spinor if it is parallel with respect to $\nabla^\pm$, that is,

$$\nabla_X \Phi \pm \frac{1}{2}i(X)\Phi = 0, \quad X \in \Gamma(TM).$$

The space of all such spinors is denoted by $\mathcal{K}^{\theta,\pm}(\mathcal{S}M)$.

Remark 3.16. If $\{e_i\}_{i=1}^n$ is a local orthonormal basis of tangent vectors, it is known that for any $\Phi \in \mathcal{K}^{\theta,\pm}(\mathcal{S}M)$ the quantity

$$q_\Phi := |\Phi|^2 + \sum_{i=1}^n (\epsilon(e_i)\Phi, \Phi)^2$$

is a non-negative constant [Bau89, Lemma 5]. If $q_\Phi = 0$ we say that $\Phi$ is of type I.

Remark 3.17. The hyperbolic $n$-space $\mathbb{H}^n$ can be described in terms of the so-called Poincaré ball model, which is given by the unit $n$-disk

$$\mathbb{B}^n = \{ y \in \mathbb{R}^n; |y|_\theta < 1 \}$$

endowed with the conformal metric

$$\tilde{b} = \Omega(y)^{-2}\delta, \quad \Omega(y) = \frac{1 - |y|_\theta^2}{2}.$$

It is easy to check that under the corresponding isometry $I : \mathbb{H}^n(-1) \to \mathbb{B}^n(1)$, the equidistant hypersurface $\Sigma_s$ is mapped onto $\Sigma_s = V^{-1}_s(\psi)$, where here $V_s(1) = \Omega(y)^{-1}y_1$. More importantly for our purposes, this conformal relation between $(\mathbb{B}^n, \tilde{b})$ and $(\mathbb{B}^n, \delta)$ allows us to canonically identify the corresponding spin bundles $\mathbb{S}B^n_\delta$ and $\mathbb{S}B^n_\tilde{b}$, so that $\phi \in \Gamma(\mathbb{S}B^n_\delta)$ corresponds to a certain $\tilde{\phi} \in \Gamma(\mathbb{S}B^n_\tilde{b})$. Under this identification, if $u \in \Gamma(\mathbb{S}B^n_\delta)$ is a $\nabla^\delta$-parallel spinor then the prescription

$$\Phi_{u,\pm}(y) := \Omega(y)^{-1/2}(I \pm i\epsilon(y))u \in \Gamma(\mathbb{S}B^n_\tilde{b})$$

exhausts the space $\mathcal{K}^{\theta,\pm}(\mathbb{S}B^n_\delta)$ [Bau89, Theorem 1]. As usual, we assume that $\Phi_{u,\pm}$ is of type I, which here means that

$$|u|_\delta^2 + \sum_{i=1}^n (\epsilon^i(\partial_{y_i})u, u)^2 = 0;$$

see Remark 3.16.

In general, if $(M, g, \Sigma)$ is s-AH with $s$ as in (3.25), we may consider the space

$$\mathcal{K}^{\theta,\pm,\beta}(\mathcal{S}M) = \{ \Phi \in \mathcal{K}^{\theta,\pm}(\mathcal{S}M); Q_{g,\beta}\Phi = \pm \Phi \}$$

of all imaginary Killing spinors satisfying the corresponding $\beta$-boundary condition along $\Sigma_s$. Notice that again we use here the sign matching convention in
Remark 3.12 so that only two possibilities for the various signs involved actually occur, namely, $K^{g,+}(-)(\mathcal{M})$ and $K^{g,-}(-)(\mathcal{M})$. Our next goal is to check that our model space $(M, g, \Sigma) = (\mathbb{H}^n_\omega, b, \Sigma_s)$ carries the maximal number of linearly independent such spinors.

**Proposition 3.18.** We have

$$\dim \mathcal{K}^{b,\pm,(\pm)\omega}(\mathbb{H}^n_\omega) = 2^{k-1}, \quad n = 2k.$$  

In particular, $\mathcal{K}^{b,\pm,(\pm)\omega}(\mathbb{H}^n_\omega) \neq \{0\}$.

**Proof.** We use the notation of Remark 3.17 and represent by $(\mathbb{B}^n, \tilde{b}, \tilde{\Sigma}_s)$ the realization of $(\mathbb{H}^n, b, \Sigma_s)$ in the Poincaré ball model. Let $\mathbb{S}^{n}_{s,\delta}$ be the spinor bundle of $\mathbb{B}^n$ with respect to the flat metric $\delta$ and $\mathbb{S}^{n\pm}_{s,\delta}$ its chiral factors. The identification in (3.27) with $M = \mathbb{S}^{n\pm}_{s,\delta}$ and $\Sigma = \tilde{\Sigma}_s$, yields a natural bundle isomorphism $\mathbb{S}^{n\pm}_{s,\delta} \cong \mathbb{S}^{n\pm}_{s,\delta}$. Since these bundles can be trivialized by $\nabla^\delta$-parallel spinors, this gives

$$\ker \nabla^\delta \cap \Gamma(\mathbb{S}^{n\pm}_{s,\delta}) \equiv \ker \nabla^\delta \cap \Gamma(\mathbb{S}^{n-\pm}_{s,\delta}),$$

so that a further appeal to parallel transport provides the natural identification (3.40)

$$\ker \nabla^\delta \cap \Gamma(\mathbb{S}^{n\pm}_{s,\delta}) \equiv \ker \nabla^\delta \cap \Gamma(\mathbb{S}^{n\pm}_{s,\delta}).$$

On the other hand, we may define an involution $Q_{\theta,\delta} : \ker \nabla^\delta \to \ker \nabla^\delta$ by first applying $Q_{\theta,\delta}$ to the restriction of $u = (u^+, u^-) \in \ker \nabla^\delta$ to $\tilde{\Sigma}_s$ and then extending the resulting spinor back to $\mathbb{B}^n$ by parallel transport. From (3.31) we see that

$$Q_{\theta,\delta} = \begin{pmatrix} u^+ & -i e^{i\theta} u^- \\ i e^{-i\theta} u^+ & u^- \end{pmatrix}.$$  

It is immediate to check that this involution satisfies the following properties:

- $[Q_{\theta,\delta}, e^\delta(y)] = 0$ for any $y \in \mathbb{B}^n$;
- when restricted to $\tilde{\Sigma}_s$, $Q_{\theta,\delta}$ corresponds to $Q_{\theta,\delta}$ under the identification in Remark 3.17 in the sense that $Q_{\theta,\delta}(\tilde{\eta}) = Q_{\theta,\delta}(\eta)$.

Thus, if $u \in \ker \nabla^\delta$ satisfies $Q_{\theta,\delta}u = \pm u$, which is allowed by (3.40), then $\Phi_{u,\pm}$ given by (3.38) lies in $\mathcal{K}^{b,\pm,(\pm)\omega}(\mathbb{H}^n_\omega)$ due to the properties above. Since the homomorphism $u \mapsto \Phi_{u,\pm}$ is obviously injective, this completes the proof. \hfill \Box

### 4. The proof of Theorem 2.8

We make use of Witten’s spinorial approach as adapted to the asymptotically hyperbolic spin case [MO89, AD98, Wan01, CH03, AdL20b]. The first step is the following result, which links imaginary Killing spinors to static potentials in the model space. Recall that we are always assuming that the relation (3.25) holds true. Also, we set

$$\mathcal{C}^1_{b,\omega} = \{ V \in \mathcal{N}_{b,\omega} ; \langle (V, V) \rangle_s = 0, \langle (V, V_{(0)}) \rangle_s > 0 \}$$

to be the future-pointing isotropic cone.

**Proposition 4.1.** For any $\Phi \in \mathcal{K}^{b,\pm,(\pm)\omega}(\mathbb{H}^n_\omega)$ we have that $V_\Phi := |\Phi|^2 \in \mathcal{N}_{b,\omega}$. If $\Phi \neq 0$ is of type I then $V_\Phi \in \mathcal{C}^1_{b,\omega}$ and moreover any $V \in \mathcal{C}^1_{b,\omega}$ can be written as $V = V_\Phi$ for some such $\Phi$. 
Proof. We only check the first assertion, since the rest follows as in the proof of [AdL20b Proposition 5.1]. Thus, if we start with $\Phi = \Phi_{u,z} \in K^{b,+,\pm}(\mathbb{H}^n)$, where $\nabla^\delta u = 0$ as in (3.38), a computation shows that

\begin{equation}
V_\Phi(y) = |\Phi_{u,z}|^2 = |u_0|^2 V_{(0)}(y) + i \sum_{j=1}^n \langle \epsilon^\delta(\partial_{y_j})u, u \rangle \delta V_{(j)}(y),
\end{equation}

with $u = u^+ + u^-$, $u^- = \pm i e^{-i\theta} u^+$. Here,

\begin{equation}
V_{(0)}(y) = \frac{1 + |y_j|^2}{1 - |y_j|^2}, \quad V_{(j)}(y) = \frac{2y_j}{1 - |y_j|^2},
\end{equation}

are the expressions of the static potentials in the Poincaré ball model. It follows that

\begin{align*}
\langle \epsilon^\delta(\partial_{y_1})u, u \rangle \delta &= \langle \epsilon^\delta(\partial_{y_1})u^+, u^+ \rangle \delta + \langle \epsilon^\delta(\partial_{y_1})u^-, u^- \rangle \delta \\
&= 2 \langle \epsilon^\delta(\partial_{y_1})u^+, u^+ \rangle \delta \\
&= \langle \epsilon^\delta(\partial_{y_1})v, v \rangle \delta,
\end{align*}

where $v = u^+ \pm i u^-$, so that $\Phi_{v,z} \in K^{b,+,\pm}(\mathbb{H}^n)$. Noticing that the unit normal to $\Sigma_0$ is everywhere aligned to $\partial_{y_1}$, we may use (3.35) with $\theta = 0$ to see that, along $\Sigma_0$,

\begin{equation*}
\langle \epsilon^\delta(\partial_{y_1})\Phi_{v,z}, \Phi_{v,z} \rangle \delta = 0.
\end{equation*}

Using (3.38) and taking the real part we get, for $y \in \Sigma_0$,

\begin{align*}
\langle \epsilon^\delta(\partial_{y_1})v, v \rangle \delta &= -\langle \epsilon^\delta(\partial_{y_1})c^\delta(y)v, c^\delta(y)v \rangle \delta \\
&= \langle \epsilon^\delta(y)c^\delta(\partial_{y_1})v, c^\delta(y)v \rangle \delta \\
&= |y_j|^2 \langle \epsilon^\delta(\partial_{y_1})v, v \rangle \delta,
\end{align*}

where we used that $\langle \partial_{y_1}, v \rangle \delta = 0$ in the second step. This shows that $\langle \epsilon^\delta(\partial_{y_1})v, v \rangle \delta = 0$, so $V_\Phi$ is a linear combination of $V_{(0)}, V_{(2)}, \ldots, V_{(n)}$, as desired. \hfill \Box

We now take an imaginary Killing spinor $\Phi \in K^{b,+,\pm}(\mathbb{H}^n)$ so that $V_\Phi \in \mathcal{C}^1_{b,s}$ as in Proposition 4.1. Using the given chart at infinity $F$ we may transplant this spinor to the whole of $M$ in the usual way so that the corresponding $\theta$-boundary condition is satisfied along $\Sigma$. It is easy to check that this transplanted spinor, say $\Phi_\ast$, satisfies $D^\ast \Phi_\ast \in L^2(\mathcal{S}M)$, so we may apply Proposition 3.13 to obtain $\Xi \in L^2(\mathcal{S}M)$ such that $D^\ast \Xi = -D^\ast \Phi_\ast$ and $Q_\ast \Xi = \pm \Xi$ along $\Sigma$. Thus, $\Psi_\Phi := \Phi_\ast + \Xi$ is Killing harmonic ($D^\ast \Psi_\Phi = 0$), satisfies $Q_\ast \Psi_\Phi = \pm \Psi_\Phi$ along $\Sigma$ and asymptotes $\Phi$ at infinity in the sense that $\Psi_\Phi - \Phi \in L^2(\mathcal{S}M)$. We may now state our first main result, which provides a Witten-type formula for the mass functional restricted to the cone $\mathcal{C}^1_{b,s}$, compare with [AdL20b] Theorem 5.2 which considers the case $s = 0$.

**Theorem 4.2.** *With the notation above,*

\begin{equation}
\frac{1}{4} m_{s,F}(V_\Phi) = \int_M \left( |\nabla^m \Psi_\Phi|^2 + \frac{R_g + n(n-1)}{4} |\Psi_\Phi|^2 \right) dM + \frac{1}{2} \int_{\Sigma} (H_g - (n-1)\lambda_s) |\Psi_\Phi|^2 d\Sigma,
\end{equation}

*for any $\Phi \in K^{b,+,\pm}(\mathbb{H}^n)$ so that $V_\Phi \in \mathcal{C}^1_{b,s}$.***
We now explain how this mass formula implies Theorem 2.8. The dominant energy conditions \( R_{ab} \geq -n(n-1) \) and \( H_y \geq (n-1)\lambda_\nu \) combined with Proposition 4.1 clearly imply that

\[ \theta \in \text{Proposition 4.1}. \] (4.44)

\[ \text{for any } V \in \mathcal{C}_{b,\nu}. \] Hence, \( P_s(F) \) is time-like and future-directed, unless there exists \( V \neq 0 \) such that the equality holds in (4.44), in which case there exists, by Proposition 4.1, a nonzero imaginary Killing spinor on \( M \), say \( \Psi^\theta \), satisfying the corresponding \( \theta \)-boundary condition along \( \Sigma \).

**Proposition 4.3.** The existence of \( \Psi^\theta \) implies that \( g \) is Einstein (with \( \text{Ric}_g = -(n-1)g \)) and \( \Sigma \) is totally umbilical (with \( H_y = (n-1)\lambda_\nu \)).

**Proof.** That \( g \) is Einstein is a classical fact [BFGK91, Theorem 8]. In order to check the assertion regarding \( \Sigma \), take \( X \in \Gamma(T\Sigma) \), so that \( \{\epsilon(X), \epsilon(\nu)\} = 0 \), which implies that \( Q_\theta \epsilon(X) = -\epsilon(X)Q^*_\theta \). In the rest of this argument we set \( Q_\theta = Q_{\theta,\nu} \) for simplicity.

It is convenient here to factor out \( \epsilon(\nu) \) from \( Q_\theta \) so we consider

\[ Q^*_\theta = -Q_\theta \epsilon(\nu) = \kappa Q + \tau I, \]

which satisfies \( \nabla Q^*_\theta = 0 \) and \( Q^*_\theta \epsilon(Y) = -\epsilon(Y)Q^*_\theta \) for any \( Y \in \Gamma(TM) \). Applying \( \nabla_X \) to the \( \theta \)-boundary condition \( \pm \Psi^\theta = Q_\theta \Psi^\theta \), we find that

\[ 0 = \nabla_X \left( Q^*_\theta \epsilon(\nu) \Psi^\theta \right) + \frac{i}{2} \epsilon(X)Q^*_\theta \epsilon(\nu) \Psi^\theta \]

\[ = Q^*_\theta \nabla_X (\epsilon(\nu) \Psi^\theta) + \frac{i}{2} Q^*_\theta \epsilon(X) \epsilon(\nu) \Psi^\theta \]

\[ = Q^*_\theta \epsilon(\nabla_X \nu) \Psi^\theta + Q^*_\theta \epsilon(\nu) \nabla_X \Psi^\theta + \frac{i}{2} Q^*_\theta \epsilon(\nu) \epsilon(X) \Psi^\theta \]

\[ = Q^*_\theta \epsilon(\nabla_X \nu) \Psi^\theta + \frac{1}{2} Q^*_\theta \epsilon(\nu) \epsilon(X) \Psi^\theta + \frac{i}{2} Q^*_\theta \epsilon(\nu) \epsilon(X) \Psi^\theta \]

\[ = Q^*_\theta \epsilon(\nabla_X \nu) \Psi^\theta + \tau \epsilon(\nu) \epsilon(X) \Psi^\theta. \]

We now restore the \( \theta \)-boundary operator by applying \( \epsilon(\nu) \) to this identity. We get

\[ 0 = -Q_{-\theta} \epsilon(\nabla_X \nu) \Psi^\theta \mp \tau \epsilon(X) \Psi^\theta \]

\[ = -Q_{-\theta} \epsilon(\nabla_X \nu) \Psi^\theta \mp \tau \epsilon(X) Q_\theta \Psi^\theta \]

\[ = -Q_{-\theta} \epsilon(\nabla_X \nu + \tau X) \Psi^\theta, \]

which implies \( \epsilon(\nabla_X \nu + \tau X) \Psi^\theta = 0 \) since \( Q_{-\theta} \) is invertible. On the other hand, it is easy to check that

\[ (\nabla_X \nu + \tau X) |\Psi^\theta|^2 = \mp i(\epsilon(Y) \Psi^\theta, \Psi^\theta), \]

so we obtain

\[ (\nabla_X \nu + \tau X) |\Psi^\theta|^2 = \mp i(\epsilon(\nabla_X \nu + \tau X) \Psi^\theta, \Psi^\theta) = 0 \]

and since \( |\Psi^\theta|^2 \) is known to vary exponentially along geodesics, we conclude that \( -\nabla_X \nu = \tau X \), as desired. \( \square \)
This proposition implies that the embedding $\Sigma \to M$ has the same second fundamental form as the model embedding $\Sigma_s \to \mathbb{H}^n_s$. The next result takes care of the corresponding first fundamental forms. Recall that $\Psi^\theta$ satisfies $\nabla^\Sigma \Psi^\theta = 0$ and $Q_{\theta^*} \Psi^\theta = \pm \Psi^\theta$.

**Proposition 4.4.** Restricted to $\Sigma$, $\Psi^\theta$ satisfies
\[
\nabla_X^\Sigma \Psi^\theta = \frac{\kappa_1}{2} c^\gamma(X) \Psi^\theta = 0, \quad X \in \Gamma(T\Sigma).
\]

**Proof.** By (3.20),
\[
\nabla_X^\Sigma \Psi^\theta = \mp \frac{i}{2} c(X) \Psi^\theta - \frac{\tau}{2} c^\gamma(X)
\]
\[
= \pm \frac{i}{2} c^\gamma(X) c(\nu) \Psi^\theta - \frac{\tau}{2} c^\gamma(X) \Psi^\theta.
\]
If we decompose $\Psi^\theta = (\Psi^\theta_1, \Psi^\theta_2) \in \Gamma(\Sigma M|_\Sigma)$ as in (3.28) and use the identifications in Remark 3.6 we get
\[
\nabla_X^\gamma \begin{pmatrix} \Psi^\theta_1 \\ \Psi^\theta_2 \end{pmatrix} = \pm \frac{1}{2} \begin{pmatrix} c^\gamma(X) \Psi^\theta_2 \\ -c^\gamma(X) \Psi^\theta_1 \end{pmatrix} - \frac{\tau}{2} \begin{pmatrix} c^\gamma(X) \Psi^\theta_1 \\ -c^\gamma(X) \Psi^\theta_2 \end{pmatrix}
\]
The result follows if we note that by (3.31) the corresponding $\theta$-boundary condition says that $\Psi^\theta_2 = \pm (1 + \kappa_1) \Psi^\theta_1$.

Now, from Proposition 4.3 Gauss equation and (3.26) we find that the scalar curvature of $\gamma$ is $R_\gamma = -(n - 1)(n - 2)\kappa^2$. Also, it follows from (1.9) that $\gamma$ has the appropriate decay at infinity to the hyperbolic metric in $\mathbb{H}^{n-1}(-\kappa^2)$, the hyperbolic $(n-1)$-space with sectional curvature $-\kappa^2$. Hence, $(\Sigma, \gamma)$ is asymptotically hyperbolic in the sense of [CH03] (that is, as a boundaryless $(n-1)$-manifold having $\mathbb{H}^{n-1}(-\kappa^2)$ as its model at infinity) and therefore has a well defined mass vector. Since Proposition 4.4 and [BFGK91] Theorem 8 imply that $\gamma$ is Einstein with $\text{Ric}_\gamma = -(n - 2)\kappa^2 \gamma$, the Ashtekar-Hansen-type formula in [Her16, Theorem 3.3] may be used to check that this mass vector vanishes. By the rigidity part of the positive mass theorem in [CH03], $(\Sigma, \gamma)$ is isometric to $\mathbb{H}^{n-1}(-\kappa^2)$. Thus, we have seen that the embedding $\Sigma \to M$ has the same first and second fundamental forms as the model embedding $\Sigma_s \to \mathbb{H}^n_s$. This allows us to glue $(M, g, \Sigma)$ to $(\mathbb{H}^n_s, b, \Sigma_s)$ along the common boundary to obtain a (smooth) boundaryless $n$-manifold which is asymptotically hyperbolic (with $(\mathbb{H}^n_s, b)$ as its model at infinity) and Einstein (it actually carries a nonzero imaginary Killing spinor, an appropriate extension of $\Psi^\theta$). Again appealing to [Her16, CH03], we find that this glued manifold is isometric to $(\mathbb{H}^n, b)$ and hence $(M, g, \Sigma)$ is isometric to $(\mathbb{H}^n, b, \Sigma_s)$, which completes the proof of Theorem 2.8.

**Remark 4.5.** Although the claimed regularity of the glued metric, say $\tilde{g}$, in the argument above may be verified by means of the standard elliptic machinery as in [DK81], for the specific gluing above a quite elementary approach is available, as we now describe. In principle, $\tilde{g}$ is only $C^{1,1}$ along the common boundary $\Sigma$ where the gluing takes place. We note however that as we approach a given point $p \in \Sigma$ from each side then $\tilde{g}$ is actually smooth all the way up to $\Sigma$. Thus, we can choose a small neighborhood $V \subset \Sigma$ of $p$ and neighborhoods $U^\pm$ in each side of $\Sigma$ such that $U^+ \cap U^- = V$. We set $\tilde{g}^\pm = \tilde{g}|_{U^\pm}$ which, being smooth, allows
us to consider Fermi coordinates \( x^\pm_i \) on \( U^\pm \) and “centered” at \( V \), with the convention that \( x^\pm_n \) corresponds to the normal geodesic direction. In the following computations, we assume that, when decorating a geometric invariant, the symbol \( \pm \) refers to \( \tilde{g}^\pm \). Also, we use commas to denote partial differentiation with respect to the Fermi coordinates above. By letting greek indices vary from 1 to \( n-1 \), we are left with the task of initially checking that \( \tilde{g}_{\alpha\beta,nn} = \tilde{g}_{\beta\alpha,nn} \) at \( p \), as these are the only second order derivatives of \( \tilde{g}^\pm \) for which this possibly fails to hold (here we use that \( \tilde{g}_{in}^\pm = \delta_{in} \) in these coordinates). We start by recalling that

\[
(4.46) \quad \text{Ric}_{\tilde{g}^\pm_{ij}} \approx -\frac{1}{2} \tilde{g}^{kl}_{\alpha} \tilde{g}_{ikl,j} + I_{ij}^\pm,
\]

where

\[
(4.47) \quad I_{ij}^\pm = \frac{1}{2} \left( \tilde{g}^k_{\alpha} \tilde{\Gamma}^k_{i,j,\alpha} + \tilde{g}^k_{\beta} \tilde{\Gamma}^k_{i,j,\beta} \right), \quad \tilde{\Gamma}^k_{i,j} = g_z^{\alpha} \tilde{\Gamma}^k_{\alpha rs},
\]

with \( \tilde{\Gamma}^k_{\alpha rs} \) being the Christoffel symbols and \( \approx \) meaning here that we are discarding terms of at most first order in \( \tilde{g}^\pm \), as they agree to each other as we approach \( p \). By inverting \( (4.46) \) we obtain

\[
\tilde{g}_{\alpha\beta,nn}^\pm \approx -2\tilde{g}_{\alpha\beta}^\pm \text{Ric}_{\tilde{g}^\pm_{ij}} + 2\tilde{g}_{nn}^\pm I_{ij}^\pm \approx 2\tilde{g}_{nn}^\pm I_{ij}^\pm,
\]

where in the last step we used that \( \tilde{g}^\pm_{ij} \) is Einstein with \( \text{Ric}_{\tilde{g}^\pm_{ij}} = -(n-1)\tilde{g}^\pm_{ij} \). By \( (4.47) \),

\[
\tilde{g}_{\alpha\beta,nn}^\pm \approx \tilde{g}_{nn}^\pm \left( \tilde{g}^\pm_{\phi\alpha} \tilde{\Gamma}^\phi_{\alpha,\pm,\beta} + \tilde{g}^\pm_{\phi\beta} \tilde{\Gamma}^\phi_{\beta,\pm,\alpha} \right) + \tilde{g}_{nn}^\pm \left( \tilde{g}^\pm_{\phi\alpha} \tilde{\Gamma}^\phi_{\alpha,n,\beta} + \tilde{g}^\pm_{\phi\beta} \tilde{\Gamma}^\phi_{\beta,n,\alpha} \right) + \tilde{g}_{nn}^\pm \left( \tilde{g}^\pm_{\phi\alpha} \tilde{\Gamma}^\phi_{\alpha,\pm,\beta} + \tilde{g}^\pm_{\phi\beta} \tilde{\Gamma}^\phi_{\beta,\pm,\alpha} \right).
\]

Since

\[
\tilde{\Gamma}^\phi_{\alpha\beta,nn} = \frac{1}{2} g_z^{\phi} \left( 2g_z^{\phi,n,n} - g_z^{\phi,n,i} \right)
\]

vanishes identically, we see that the first term in the right-hand side above vanishes identically as well. On the other hand, by [DK81] Lemma 1.2] we may assume that \( \{x_n\} \) is harmonic with respect to \( \tilde{g}^\pm |_V \), so that the second term vanishes in the limit as we approach \( p \). We conclude that \( \tilde{g}_{\alpha\beta,nn}^\pm = \tilde{g}_{\alpha\beta,nn}^\pm \) at \( p \), that is, \( \tilde{g}_{\alpha\beta} \) is \( C^2 \) (in fact, smooth), as desired.

5. The horospherical case

Here we briefly discuss how the methods above can be modified to establish a positive mass theorem whose rigidity statement implies the horospherical case in [Son21] Theorem 2]; see Theorem 5.5 below and compare with Corollary 1.9 and Remark 1.12. Roughly speaking, this corresponds to taking the limits \( \theta \to \pm \pi/2 \) in the results from the previous sections, but new phenomena emerge upon analysis of this formal limit, so a separate proof is required; see Remark 5.7 for more on this point. Needless to say, at least from a conceptual viewpoint the approach below is quite similar to the one leading to our previous main result (Theorem 2.8) so our presentation will be brief at some points.
By means of the hyperboloid model $\mathbb{H}^n \to \mathbb{R}^{1,n}$ described in Section 1, we consider the horoball

$$\mathbb{H}^n_{b,\chi} = \{ x \in \mathbb{H}^n; V_h(x) \leq \chi \}, \quad \chi > 0,$$

where $V_h = V(0) - V(1)$. We denote by $\Sigma_{b,\chi}$ its boundary, so that as $\chi$ varies we obtain the so-called horospherical foliation of $\mathbb{H}^n$. This terminology is justified by the following result.

**Proposition 5.1.** The triple $(\mathbb{H}^n_{b,\chi}, b, \Sigma_{b,\chi})$ is a static domain whose boundary $\Sigma_{b,\chi}$ is a horosphere. In this case, $(\Lambda, \lambda) = (-n(n-1)/2, n-1)$ and the corresponding space of static potentials $N_{b,\chi}^\infty$ has dimension $n$ and is generated by $V_0, V_2, \ldots, V_n$.

**Proof.** Along $\Sigma_{b,\chi}$ we have $|\nabla b V_h| = \chi$ so its outward pointing unit normal is $\eta_h = \chi^{-1} \nabla b V_h$. As in the proof of Proposition 1.6 we compute that

$$\nabla^b \eta_h = X, \quad X \in \Gamma(T\Sigma_{b,\chi}),$$

which proves that $\Sigma_{b,\chi}$ is a horosphere indeed (with constant mean curvature equal to $n-1$). In particular, $N_{b,\chi}^\infty$ is formed by those static potentials of $(\mathbb{H}^n, b)$ which, when restricted to $\Sigma_{b,\chi}$, satisfy $\partial V / \partial \eta = V$. A calculation using (1.8) shows that, restricted to $\Sigma_{b,\chi}$,

$$\frac{\partial V_0}{\partial \eta} = x_0 - \chi^{-1}, \quad \frac{\partial V_1}{\partial \eta} = x_1 - \chi^{-1}, \quad \frac{\partial V_j}{\partial \eta} = x_j, \quad j \geq 2,$$

which completes the proof. \qed

In analogy with the discussion preceding Conjecture 2.4, we now seek to determine the structure of the natural representation of the relevant subgroup of isometries of $(\mathbb{H}^n, b)$ on the space of static potentials $N_{b,\chi}^\infty$. We start with the parabolic subgroup $P$ of isometries intertwining the horospherical leaves $\Sigma_{b,\chi}$ above. In the hyperboloid model $\mathbb{H}^n \to \mathbb{R}^{1,n}$, it may be accessed as follows [Tay86, Section 10.2]. As in Remark 2.4, any element of the isometry group $O^1(1,n)$ of $(\mathbb{H}^n, b)$, viewed as (the restriction of) a time-orientation preserving linear isometry of $(\mathbb{R}^{1,n}, \langle \cdot, \cdot \rangle_{1,n})$, defines a conformal diffeomorphism of the sphere at infinity $S^\infty_{\infty}$ of $\mathbb{H}^n$, and conversely. Here, we view $S^\infty_{\infty}$ as the set of isotropic lines in the future-directed cone $C^+: = \{ x \in \mathbb{R}^{1,n}; \langle x, x \rangle_{1,n} = 0, x_0 > 0 \}$. Under this identification, the horospherical foliation $\Sigma_{b,\chi}$ picks the distinguished element $o = \{ x \in \mathbb{R}^{1,n}; x_0 = x_1 \} \cap C^+ \in S^\infty_{\infty}$. In this picture, $P$ emerges as the stabilizer of $o$ under this conformal action of $O^1(1,n)$ on $S^\infty_{\infty}$, so that $S^\infty_{\infty} = O^1(1,n)/P$, the homogeneous flat model of Conformal Geometry. Explicitly, we have the Langlands decomposition

$$P = O(n-1) \mathbb{R}N,$$

where $O(n-1)$ acts by rotations on $[\partial_{x_2}, \ldots, \partial_{x_n}]$, $\mathbb{R}$ comes from boosts

$$b_\varrho = \begin{pmatrix} \cosh \varrho & \sinh \varrho \\ \sinh \varrho & \cosh \varrho \end{pmatrix}, \quad \varrho \in \mathbb{R},$$

where $O(n-1)$ acts by rotations on $[\partial_{x_2}, \ldots, \partial_{x_n}]$, $\mathbb{R}$ comes from boosts.
acting on the Lorentzian plane $[\partial z_0, \partial z_1]$ and the last factor $N$ comprises the isomorphic image of $\mathbb{R}^{n-1}$ under exponentiation:

$$U \in \mathbb{R}^{n-1} \mapsto \begin{pmatrix} 1 + \frac{|U|^2}{2} & -\frac{|U|^2}{2} & U \\ \frac{|U|^2}{2} & 1 - \frac{|U|^2}{2} & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \in N.$$ 

Here, the superscript means transpose of a row vector. Since $N = \mathbb{R}^{n-1} \subset \mathcal{P}$ is normal and intersects $O(n-1)\mathbb{R}$ only at the identity element, we actually have the semi-direct product representation

$$\mathcal{P} = (0(n-1)\mathbb{R}) \ltimes \mathbb{R}^{n-1}.$$ 

As can be easily checked, each $b_o$ induces an (ambient) isometry between $(\mathbb{H}_n^{+}, b)$ and $(\mathbb{H}_n^{+}, b^e \chi)$ corresponding to a multiplicative shift of $e^{-\epsilon} \chi$ on the parameter $\chi$. We conclude that $\mathcal{P}^h := \mathcal{P}/\mathbb{R} = O(n-1) \ltimes \mathbb{R}^{n-1}$, the group of Euclidean isometries in dimension $n-1$, is the full isometry group of each $(\mathbb{H}_n^{+}, b)$. In particular, since the $\mathcal{P}^h$-action on $\Sigma_{n, h} = \partial \mathbb{H}_n^{+}$ is transitive, this confirms that each horosphere is intrinsically flat. As in Remark 2.4 we obtain a natural representation $\rho^b$ of $\mathcal{P}^h$ on $\mathcal{N}^\chi_{b, h}$ by setting $\rho^b_A(V) = V \circ A^{-1}$.

**Proposition 5.2.** The space of static potentials $\mathcal{N}^\chi_{b, h}$ splits into two irreducible representations under $\rho^b$, namely,

$$\mathcal{N}^\chi_{b, h} = [V_h] \oplus [V_{(2)}, \cdots, V_{(n)}],$$

with $\rho^h|_{[V_h]}$ being trivial (that is, $\rho^h_A(V_h) = V_h$ for any $A$).

**Proof.** From the discussion above it is clear that the only elements of $\mathcal{N}^\chi_{b, h}$ left fixed by $\rho^b$ are precisely those which are constant along each horospherical leaf. Hence, the proof essentially amounts to checking that the only static potentials of $(\mathbb{H}_n^{+}, b)$ meeting this property are proportional to $V_h$. This claim may be readily verified if we pass to the Poincaré half-space model $(\mathbb{R}^n_+, \overline{b})$ of hyperbolic $n$-space, where $\mathbb{R}^n_+ = \{ z = (z_1, z_2, \cdots, z_n) \in \mathbb{R}^n; z_1 > 0 \}$ and $\overline{b} = z_1^{-1} \delta$. We assume here that the distinguished point $o \in \mathbb{S}^{n-1}_\infty$ is mapped onto the point at infinity $\{ z_1 = +\infty \}$, so that the horospheres above correspond to the hyperplanes in the family $\{ z_1 = c \}_{c > 0}$ and the associated parabolic group $\mathcal{P}$ acts on the hyperplane at infinity $\mathbb{R}^n_{\infty-1} = \{ z \in \mathbb{R}^n; z_1 = 0 \}$ by similitudes. As expected, when acting on $(\mathbb{R}^n_+, \overline{b})$ by isometries, $\mathcal{P}^h$ preserves each horospherical leaf so that the only coordinate function left fixed is $z_1$. On the other hand, it is known that the isometry relating $\overline{b}$ and $\overline{b}$ satisfies

$$\overline{z_1} = \frac{1 - |y|_G^2}{|y_1|_G^2}, \quad \partial y_1 = (1, 0, \cdots, 0), \quad z_j = -\frac{2y_j}{|y_1|_G^2}, \quad j \geq 2.$$ 

We may combine this with (4.42) to check that

$$V_h(z) = z_1^{-1}, \quad V_{(j)}(z) = z_j z_1^{-1}, \quad j \geq 2,$$

which finishes the proof. \(\square\)

Henceforth we drop the symbol $\chi$ from the notation, so the corresponding static manifold will be denoted by $(\mathbb{H}_n^{+}, b, \Sigma_b)$, etc. Alternatively, we may take $\chi = 1$ for simplicity. Proceeding in analogy with Definition 3.7 we may consider an
where \( \nu \) may define a mass invariant for Definition 1. Also, as in Theorem 2.1 and with a self-explanatory notation, we may define a mass invariant for \((M, g, \Sigma)\) by setting

\[
m_{h,F}(V_h) = \lim_{r \to +\infty} \left[ \int_{S_{r-1}} (U(V_h, r), \mu) dS_{r-1} - \int_{S_{r-2}} V_h(e(\eta, \partial)) dS_{r-2} \right],
\]

where as usual \( F \) is a chart at infinity. Of course, here we rely on the decomposition (5.49) to restrict the mass functional to \([V_h]\) and disregard its action on the complement \([V_{i(2)}, \ldots, V_{i(n)}]\) of \([V_h]\) in \(N_{h,b} \). However, we argue in Remark 5.8 below that the (vector-valued) invariant which arises by restricting the mass functional to this complement qualifies as a "center of mass" of the underlying manifold. Also, we point out that the model \( \mathbb{H}^n \) is compactified by adding a single point at infinity, so the picture here is quite similar to the definition of the mass invariant in [ABdL16]. The important point now is that an analogue of Lemma 2.3 guarantees that two charts at infinity differ by an element of \( \mathcal{P}^h \) (up to a term that vanishes as \( r \to +\infty \)). Proceeding as in the proof of Theorem 2.5 and taking Proposition 5.2 into account, we thus see that the right-hand side in (5.52) actually does not depend on which chart is used and the conclusion is that the mass

\[
m_h := m_{h,F}(V_h)
\]

is a numerical invariant.

In order to obtain a Witten-type formula for \( m_h \) in the spin category under suitable dominant energy conditions, we must impose an appropriate boundary condition on spinors along \( \Sigma \). In view of Remark 3.4 it is natural to implement this by considering the MIT bag boundary operator \( Q_h = \text{ic}(\nu) : \Gamma(\mathbb{S}M|_{\Sigma}) \to \Gamma(\mathbb{S}M|_{\Sigma}) \), where \( \nu \) is the inward unit normal along \( \Sigma \). Notice that, differently from the \( \theta \)-boundary operator in (3.24) which involves the chirality operator \( \zeta = \omega \) and therefore only makes sense for \( n \) even, \( Q_h \) is well defined in any dimension. In any case, spinors are required to satisfy \( Q_h \Psi = \pm \Psi \) along \( \Sigma \). Now we must check that there exist plenty of such spinors in the model space.

**Proposition 5.3.** If \( n = 2k \) or \( n = 2k + 1 \) then, with self-explanatory notation,

\[
\dim \mathcal{K}^{h,+,(+)h}(\mathbb{H}^n_h) = 2^{k-1}
\]

In particular, \( \mathcal{K}^{h,+,(+)h}(\mathbb{H}^n_h) \neq \{0\} \).

**Proof.** Just repeat the proof of Proposition 3.18 with \( \theta = \pi/2 \), which, as already remarked, morally corresponds to the case considered here. \( \square \)

We now confirm that the well-known link between imaginary Killing spinors and static potentials works fine here as well; compare with Proposition 4.1.

**Proposition 5.4.** For any \( \Phi \in \mathcal{K}^{h,+,(+)h}(\mathbb{H}^n_h) \) we have that \( V_\Phi := |\Phi|^2 \in [V_h] \). In fact, if \( \Phi = \Phi_{u,s} \) for some \( \nabla^\delta \)-parallel spinor \( u \) as in (3.38) then \( V_\Phi = |u|^2 [V_h] \).

**Proof.** We take \( \Phi = \Phi_{u,s} \) and note that the derivation of (3.35) only uses the corresponding boundary condition. Thus, if we use it with \( \tau = 1 \) we get \( i(\epsilon^\tau(\nu) \Phi, \Phi) = \pm |\Phi|^2 \) along \( \Sigma_{h,b} \), where here \( \nu = -\nu_b \). Bearing in mind that \( \nu^+ \Phi = 0 \) and comparing with (4.35) we find that \( V_\Phi = -\nu V_\Phi \). By bringing back the parameter \( \chi > 0 \) and
letting it vary, we see that this holds true everywhere along \( \mathbb{H}^n \) if we think of \( \nu \) as being the (globally defined) unit normal along the leaves of the horospherical foliation. To investigate the constraints imposed on \( V_\Phi \) by this equation, we pass to the half-space model appearing in the proof of Proposition 5.2. We may invert \((5.50)\) to obtain
\[
y_1 = \frac{|z|_\delta^2 - 1}{|z + \partial z_1|_\delta}, \quad y_j = \frac{2z_j}{|z + \partial z_1|_\delta}, \quad j \geq 2,
\]
and then compute that
\[
V(1)_z = \frac{2y_1}{1 - |y_2|^2} = \frac{2(|z|_\delta^2 - 1)|z + \partial z_1|_\delta}{|z + \partial z_1|_\delta^2 - ((|z|_\delta^2 - 1)^2 - 4(|z|_\delta^2 - z_1^2))} = \frac{|z|_\delta^2 - 1}{2z_1}.
\]
Combining this with \((5.51)\) we see that, in this half-space model, \((4.41)\) may be rewritten as
\[
V_\Phi(z) = |u|_\delta^2 z_1^{-1} + \frac{1}{2} (i \epsilon (\partial_{y_1})u, u)_{\delta} + |u|_\delta^2) (|z|_\delta^2 - 1) z_1^{-1} - i \sum_{j \geq 2} (\epsilon (\partial_{y_j})u, u)_{\delta} z_j z_1^{-1}.
\]
Since \( \nu = z_1 \partial_{z_1} \) we get
\[
(\nu V_\Phi)(z) = -|u|_\delta^2 z_1^{-1} + \frac{1}{2} (i \epsilon (\partial_{y_1})u, u)_{\delta} + |u|_\delta^2) (2 - (|z|_\delta^2 - 1) z_1^{-1}) - i \sum_{j \geq 2} (\epsilon (\partial_{y_j})u, u)_{\delta} z_j z_1^{-1},
\]
which gives
\[
i \epsilon (\partial_{y_1})u, u)_{\delta} + |u|_\delta^2 = (\nu V_\Phi + V_\Phi)(z) = 0.
\]
Thus, if we set \((\epsilon (\partial_{y_j})u, u) = a_j, a_j \in \mathbb{R}, \) it follows from this and \((5.39)\) that
\[
\sum_{j \geq 2} a_j^2 = \sum_{j \geq 2} (\epsilon (\partial_{y_j})u, u)^2 = |u|_\delta^2 + (\epsilon (\partial_u)z_1 u, u)^2 = 0.
\]
Hence, \( V_\Phi(z) = |u|_\delta^2 z_1^{-1} \) which completes the proof in view of \((5.51).\) \( \square \)

We may now outline the argument leading to a positive mass inequality for \( m_h, \) with a rigidity statement included. As before, we start with some \( \Phi \in \mathcal{K}^{h,+,+}(\mathbb{H}^n) \). We may assume that \( \Phi = \Phi_{u,\pm} \) where \( \nabla^\pm u = 0 \) and \( |u|_\delta = 1 \), so that \( V_\Phi = V_h \) by Proposition 5.4. Under the appropriate dominant energy conditions as in Theorem 5.5 below, the usual analytical machinery may be employed to obtain another spinor \( \Psi_u \in \Gamma(SM) \) which is Killing harmonic, satisfies the corresponding MIT bag boundary condition along \( \Sigma \) and asymptotes \( \Phi_{u,\pm} \) at infinity. A standard computation provides the corresponding Witten-type formula:
\[
\frac{1}{4} m_h = \int_M \left( |\nabla^+ \Phi|^2 + \frac{R_g + n(n-1)}{4} |\Phi|^2 \right) dM + \int_\Sigma \left( H_g - (n-1) \right) |\Phi|^2 d\Sigma.
\]
This is the key ingredient in establishing the following positive mass theorem.

**Theorem 5.5.** Let \((M, g, \Sigma)\) be an asymptotically hyperbolic spin manifold (modeled at infinity on \((\mathbb{H}^n, b, \Sigma_h)\)) with \( R_g \geq -n(n-1) \) and \( H_g \geq (n-1) \). Then \( m_h \geq 0 \) and the equality holds if and only if \((M, g, \Sigma)\) is isometric to \((\mathbb{H}^n, b, \Sigma_h).\)
Proof. That $m_h \geq 0$ already follows from (5.53). As for the rigidity statement, if $m_h = 0$ then again 
by (5.53), this time combined with Propositions 5.3 and 5.4, 
$(M, g, \Sigma)$ carries as many imaginary Killing spinors satisfying a MIT bag boundary condition as $(\mathbb{H}^n_h, b, \Sigma_h)$ does, which implies that $g$ is locally hyperbolic, $(\Sigma, \gamma) \rightarrow (M, g)$, $\gamma = g|_{\Sigma}$ is totally umbilical (with $H_g = (n - 1)$) and $\gamma$ is flat. It follows 
that $(\Sigma, \gamma) = (\mathbb{R}^{n-1}, \delta)$ isometrically, so we may glue together $(M, g, \Sigma)$ and the 
complement of the corresponding horoball model along the common boundary to conclude, again appealing to \cite{Her16, CH03}, that $(M, g, \Sigma) = (\mathbb{H}^n_h, b, \Sigma_h)$ isometrically. 

Clearly, the rigidity statement here covers the horospherical case in \cite{Sou21} Theorem 2]; see also the companion result in Remark 5.6 below. We emphasize, however, that it actually provides a much sharper non-deformability result for the embedding $\Sigma_h \rightarrow \mathbb{H}^n$; compare with Remark 1.10.

Remark 5.6. Essentially the same argument as above yields a mass formula similar to (5.53) 
for asymptotically hyperbolic manifolds modeled at infinity on the 
complement of a horoball $\mathbb{H}^n_h$ in $\mathbb{H}^n$, the only difference being that $H_g - (n - 1)$ gets 
replaced by $H_g + (n - 1)$ in the boundary integral. This corresponds to the case 
$\theta = -\pi/2$ ($\tau = -1$) in the notation of the previous sections, which means that we 
must use $-i\kappa(\nu)$ as the associated boundary operator. Perhaps the most noticeable 
modification in the argument occurs while repeating the proof of Proposition 5.3 
with $V_h = \nu' V_\delta$, where $\nu' = -\nu = -z_i \partial_{z_i}$ is the appropriate unit normal along the 
horospherical foliation, so we end up with the same conclusion.

Remark 5.7. The definition (5.52) suggests that we should think of $m_h$ as a multiple 
of $V_h$. On the other hand, we may equip $\mathcal{N}_h$ with a degenerate symmetric bi-linear 
form, say $(\cdot, \cdot)_h$, by declaring that $V_h$ is null (that is, $(V_h, V)_h = 0$ for any $V \in \mathcal{N}_h$) 
and that $(V(j), V(j'))_h = -\delta_{jj'}$, $j, j' \geq 2$, so that $\mathcal{P}^h$ acts isometrically on $(\mathcal{N}_h, (\cdot, \cdot)_h)$ 
via $\rho^h$. Intuitively, we may view $(\mathcal{N}_h, (\cdot, \cdot)_h)$ as obtained from $(\mathcal{N}_h, (\cdot, \cdot)_h)$ by collapsing 
its isotropic cone into a single null line (namely, $[V_h]$), so that no time-like 
direction survives in the process. In terms of the isometry groups of the underlying 
static spaces, this amounts to passing from $O(1, n-1)$ to $O(n-1) \times \mathbb{R}^{n-1}$, with 
this latter group acting on $\{(V_2, \cdots, V_{n-1}), (\cdot, \cdot)_h\} = (\mathbb{R}^{n-1}, -\delta)$ by isometries. Using 
this setup, we may interpret Theorem 5.5 as saying that, under the given dominant 
energy conditions, $m_h$ is null-positive (that is, a positive multiple of $V_h$) unless it 
vanishes, in which case the underlying manifold is isometric to the model. In this 
way, Theorem 5.5 may be regarded as the natural rewording of Theorem 2.8 as 
$\theta \rightarrow \pi/2$.

Remark 5.8. The splitting (5.49) of $\mathcal{N}_h$ under the natural $\mathcal{P}^h$-action suggests defining 
a complementary (vector-valued) Hamiltonian charge for any asymptotically 
hyperbolic manifold modeled at infinity on $\mathbb{H}^n_h$ (or on its complement) by simply 
restricting the mass functional to $[V_2, \cdots, V_{n-1}]$. More precisely, in the presence of a 
chart at infinity $F$, a vector $c_{h,F} \in \left[V_2, \cdots, V_{n-1}\right]$ is defined by 

$$(C_{h,F}, V_{(a)})_h = \lim_{r \rightarrow +\infty} \left[ \int_{S^{n-1}_{r+1}} \langle \Omega(V_{(a)}, e), \mu \rangle dS^{n-1}_{r+1} - \int_{S^{n-2}_{r+1}} V_{(a)}(\eta, \vartheta) dS^{n-2}_{r+1} \right];$$

compare with (5.52). As expected, $c_{h,F}$ transforms as a vector in the representation $\rho^h$ as we pass from one chart at infinity to another. It would be interesting to
investigate the basic properties of this invariant and how it relates to the asymptotic geometry of the underlying manifold. In this regard, we note that [Cha21], Theorem 2) provides an Ashtekar-Hansen-type formula for this invariant (and for our mass \( m_h \) as well), very much in the spirit of results found in [dLGM19, Cha22] for the case \( s = 0 \). Specifically, in \( \mathbb{R}^{1,n} \) consider the vector fields

\[
X = C + (C, x)_{1,n}x - Y_{01}
\]

and

\[
X_a = C_a + (C, x)_{1,n}x - Y_{01} - Y_{0a}, \quad 2 \leq a \leq n,
\]

where

\[
C = \partial_{x_0} + \partial_{x_1}, \quad C_a = \partial_a,
\]

and

\[
Y_{01} = x_0 \partial_{x_1} + x_1 \partial_{x_0}, \quad Y_{0a} = x_0 \partial_{x_a} + x_a \partial_{x_0}.
\]

Upon restriction to \( \mathbb{H}^n \), \( X \) and \( X_a \) are conformal fields which remain tangent to \( \Sigma_h = \{ x_0 = x_1 \} \). With this notation, and under the obvious identifications induced by \( F \), Chai’s formulae read as

\[
m_h = c_n \lim_{r \to +\infty} \left[ \int_{S_{r,n}^{n+1}} \tilde{E}_g(X, \mu) dS_{r,n}^{n-1} - \int_{S_{r,n-2}} \tilde{\Pi}_g(X, \vartheta) dS_{r,n-2} \right],
\]

and

\[
C_{h,F}^n = c'_n \lim_{r \to +\infty} \left[ \int_{S_{r,n}^{n+1}} \tilde{E}_g(X_a, \mu) dS_{n}^{n-1} - \int_{S_{r,n-2}} \tilde{\Pi}_g(X_a, \vartheta) dS_{r,n-2} \right],
\]

where

\[
\tilde{E}_g = \text{Ric}_g - \frac{R_g}{2} g - \frac{(n-1)(n-2)}{2} g
\]

is the modified Einstein tensor of \( g \) and

\[
\tilde{\Pi}_g = \Pi_g - H_{g}^\gamma + (n-2) \gamma
\]

is the modified Newton tensor of the embedding \((\Sigma, \gamma) \to (M, g)\). Also, \( S_{r,n}^{n-2} = \partial S_{r,n}^{n-1} \) and, in the half-space model above, we take \( S_{r,n}^{n-1} \) to be the Euclidean hemisphere of radius \( r > 1 \) centered at the “origin” \((1, 0, 0, \ldots, 0)\) and lying above the horosphere \( \Sigma_h = \{ z_1 = 1 \} \), so that \( S_{r,n}^{n-1} \) is a piece of an equidistant hypersurface with mean curvature \( H_{r} = (n-1)/r \). This shows that \( C_h \) qualifies as a kind of “center of mass” for the given asymptotically hyperbolic manifold. In any case, at least in dimension \( n = 3 \), \( C_h \) should play a role in solving the relative isoperimetric problem for large values of the enclosed volume in such manifolds; compare with [AdL20a, Theorem 2.28].

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