AN $L^2$ HARTOGS-TYPE EXTENSION THEOREM FOR UNBOUNDED DOMAINS

BO-YONG CHEN

ABSTRACT. In this note, we prove an $L^2$ Hartogs-type extension theorem for unbounded domains.

1. INTRODUCTION

In the landmark paper [9], Hartogs proved the following celebrated result.

Theorem 1.1 (Hartogs, 1906). Let $\Omega$ be a domain in $\mathbb{C}^n$ with $n \geq 2$ and $E$ a compact subset in $\Omega$. If $\Omega \setminus E$ is connected, then every holomorphic function on $\Omega \setminus E$ can be extended holomorphically to $\Omega$.

There are at least three approaches for the Hartogs extension theorem. The first one, which is the original proof of Hartogs, was completed only recently by Merker-Porton [12]; the second one is based on the Bochner-Martinelli formula (cf. [3], [11]); the third one, which is the most popular, is by using the $\bar{\partial}$-method (cf. [8]). Generalizations to complex manifolds and complex spaces are also available (see e.g., [7], [13, 14, 15]). We refer to the paper of Range [16] for a very interesting historical recollection on this topic.

The Hartogs extension phenomenon for the case when $E$ is an unbounded closed subset seems to be more involved. A classic example in this direction is the following tube theorem obtained by Bochner [2].

Theorem 1.2 (Bochner, 1938). Every holomorphic function defined on the tube $D \times i\mathbb{R}^n$, where $D$ is an open set in $\mathbb{R}^n$, can be extended holomorphically to $(\text{convex hull of } D) \times i\mathbb{R}^n$.

There are some generalizations to certain "tube-like" domains (cf. [4, 5]). In particular, [5] indicates the complexity of the Hartogs extension phenomenon for unbounded cases.

Actually, Bochner had proved an $L^2$ version of Theorem 1.2 in an earlier paper [1], which seems to be less known.

Theorem 1.3 (Bochner, 1937). Every $L^2$ holomorphic function defined on $D \times i\mathbb{R}^n$ can be extended to an $L^2$ holomorphic function on $(\text{convex hull of } D) \times i\mathbb{R}^n$.

Motivated by this theorem, we shall prove the following $L^2$ Hartogs extension theorem.

Theorem 1.4. Let $\Omega$ be a domain in $\mathbb{C}^n$ and $E$ a closed set in $\mathbb{C}^n$ such that

1. there exists $r > 0$ such that $E_r := \{z \in \mathbb{C}^n : d(z, E) \leq r\} \subset \Omega$;
2. there exist an affine-linear subspace $H \subset \mathbb{R}^{2n} = \mathbb{C}^n$ of real codimension $\geq 3$ and a number $R > 0$ such that
   \[ E \subset H_R := \{z \in \mathbb{C}^n : d(z, H) < R\}; \]
(3) $\Omega \backslash E$ is connected. Then every $L^2$ holomorphic function defined on $\Omega \backslash E$ can be extended to an $L^2$ holomorphic function on $\Omega$.

Theorem 1.1 follows directly from Theorem 1.4. To see this, first take a domain $\Omega' : E \subset \Omega' \subset \subset \Omega$ and $H = \{0\}$, then apply Theorem 1.4 to the pair $(\Omega', E)$. It is also easy to see that Theorem 1.4 contains some special cases of Theorem 1.3, e.g., $D = D' \backslash E$ where $D'$ is a convex domain in $\mathbb{R}^n$ and $E$ is a compact subset in $D'$.

The proof of Theorem 1.4 relies heavily on the classic Hardy inequality, which also reveals that the basic reason for the Hartogs extension phenomenon is nothing but the non-parabolicity of $\mathbb{C}^n = \mathbb{R}^{2n}$ when $n \geq 2$.

2. A $L^2$-estimate for the $\bar{\partial}$-equation in $\mathbb{C}^n$

Let $\nabla$ and $\Delta$ denote the standard gradient and real Laplacian. We shall prove the following

**Theorem 2.1** (compare [6]). Suppose that there exists a measurable function $\omega \geq 0$ on $\mathbb{C}^n$ such that

$$\int_{\mathbb{C}^n} \phi^2 \omega \leq \int_{\mathbb{C}^n} |\nabla \phi|^2 \tag{2.1}$$

holds for any real-valued smooth function $\phi$ with compact support in $\mathbb{C}^n$. Then for any $\bar{\partial}$-closed $(0, q)$-form $v$ on $\mathbb{C}^n$ with $\int_{\mathbb{C}^n} |v|^2 < \infty$ and $\int_{\mathbb{C}^n} |v|^2 / \omega < \infty$, there exists a $(0, q - 1)$-form $u$ on $\mathbb{C}^n$ such that $\bar{\partial} u = v$ and

$$\int_{\mathbb{C}^n} |u|^2 \leq 4 \int_{\mathbb{C}^n} |v|^2 / \omega.$$

**Remark.** (2.1) is usually called a Hardy-type inequality in literature (the special case when $\omega = (n-1)^2 / |z|^2$ is the standard Hardy inequality), which is of particular importance in real analysis and partial differential equations.

**Proof.** Let $\phi = \phi_1 + i\phi_2$, where $\phi_1, \phi_2$ are real-valued smooth functions with compact supports in $\mathbb{C}^n$. By (2.1) we have

$$\int_{\mathbb{C}^n} |\phi|^2 \omega = \int_{\mathbb{C}^n} \phi_1^2 \omega + \int_{\mathbb{C}^n} \phi_2^2 \omega \leq \int_{\mathbb{C}^n} (|\nabla \phi_1|^2 + |\nabla \phi_2|^2) = -\int_{\mathbb{C}^n} (\phi_1 \Delta \phi_1 + \phi_2 \Delta \phi_2) \quad (\text{Stokes formula})$$

$$= -\int_{\mathbb{C}^n} \phi \Delta \overline{\phi}, \tag{2.2}$$

where the last equality follows from

$$\int_{\mathbb{C}^n} \phi_1 \Delta \phi_2 = -\int_{\mathbb{C}^n} \nabla \phi_1 \cdot \nabla \phi_2 = \int_{\mathbb{C}^n} \phi_2 \Delta \phi_1.$$
Let \( D_{(0,q)}(\mathbb{C}^n) \) denote the set of smooth \((0,q)\)-forms with compact supports in \( \mathbb{C}^n \) and \( L^2_{(0,q)}(\mathbb{C}^n) \) the completion of \( D_{(0,q)}(\mathbb{C}^n) \) with respect to the standard \( L^2 \) norm. Let \( \bar{\partial} \) be the formal adjoint of \( \partial \) and \( \Box := \bar{\partial} \partial + \bar{\partial} \partial \bar{\partial} \) the complex Laplacian. In what follows we shall use standard terminologies in Hörmander’s classic book [10]. For any \( u = \sum_{I,J} u_{IJ} dz_I \wedge d\bar{z}_J \in D_{(0,q)}(\mathbb{C}^n) \), we infer from (2.2) that

\[
\int_{\mathbb{C}^n} |u|^2 \omega \leq - \sum_{I,J} \int_{\mathbb{C}^n} u_{IJ} \Delta u_{IJ}
\]

(2.3)

Now we can apply the standard duality argument. Consider the mapping (2.4)

\[
\int_{\mathbb{C}^n} |u|^2 \omega \leq 4 \liminf_{j \to \infty} \int_{\mathbb{C}^n} |u_j|^2 \omega
\]

\[
= 4 \left( \|\bar{\partial} u_j\|^2 + \|\bar{\partial}^* u_j\|^2 \right)
\]

(2.4)

Note that \( \bar{\partial} : L^2_{(0,q)}(\mathbb{C}^n) \to L^2_{(0,q+1)}(\mathbb{C}^n) \) gives a densely defined and closed operator. Let \( \bar{\partial}^* \) be the Hilbert space adjoint of \( \bar{\partial} \). It is well-known that \( D_{(0,q)}(\mathbb{C}^n) \) lies dense in \( \text{Dom} \bar{\partial} \cap \text{Dom} \bar{\partial}^* \) under the following graph norm:

\[
u \mapsto \|u\| + \|\bar{\partial} u\| + \|\bar{\partial}^* u\|.
\]

Thus for any \( u \in \text{Dom} \bar{\partial} \cap \text{Dom} \bar{\partial}^* \) there exists a sequence \( \{u_j\} \subset D_{(0,q)}(\mathbb{C}^n) \) such that \( u_j \to u \) under the graph norm. Replace \( \{u_j\} \) by a subsequence, we may assume furthermore that \( u_j \to u \) a.e. in \( \mathbb{C}^n \). It follows from Fatou’s lemma that

\[
\int_{\mathbb{C}^n} |u|^2 \omega \leq \liminf_{j \to \infty} \int_{\mathbb{C}^n} |u_j|^2 \omega
\]

\[
\leq 4 \liminf_{j \to \infty} \left( \|\bar{\partial} u_j\|^2 + \|\bar{\partial}^* u_j\|^2 \right)
\]

(2.4)

Now we can apply the standard duality argument. Consider the mapping

\[
T : \bar{\partial}^* w \mapsto (w, v), \quad w \in \text{Dom} \bar{\partial}^* \cap \text{Ker} \bar{\partial}.
\]

The Cauchy-Schwarz inequality gives

\[
|\langle w, v \rangle|^2 \leq \int_{\mathbb{C}^n} |w|^2 \omega \cdot \int_{\mathbb{C}^n} |v|^2 / \omega
\]

(2.5)

in view of (2.4). Thus \( T \) is a well-defined continuous linear functional on \( \text{Dom} \bar{\partial}^* \cap \text{Ker} \bar{\partial} \) with

\[
\|T\|^2 \leq 4 \int_{\mathbb{C}^n} |v|^2 / \omega.
\]

(2.6)

Since \( v \in \text{Ker} \bar{\partial} \), we have \( \langle w, v \rangle = 0 \) for all \( w \perp \text{Ker} \bar{\partial} \), so that \( T \) extends to a continuous linear functional on the range of \( \bar{\partial}^* \) which still satisfies (2.6). The Hahn-Banach theorem combined with the Riesz representation theorem gives a unique \( u \in L^2_{(0,q-1)}(\mathbb{C}^n) \) such that \( \|u\| \leq 2 \|v/\sqrt{\omega}\| \) and

\[
\langle w, v \rangle = \langle \bar{\partial}^* w, u \rangle, \quad w \in \text{Dom} \bar{\partial}^*;
\]

i.e., \( \bar{\partial} u = v \) holds in the sense of distributions. \( \Box \)
Let us recall the following classic result.

**Lemma 2.2** (Hardy inequality). Let $H \subset \mathbb{R}^N$ be an affine-linear subspace with codimension $m \geq 3$. Define $d_H = d(\cdot , H)$. Then we have

$$\frac{(m-2)^2}{4} \int_{\mathbb{R}^N} \phi^2 / d_H^2 \leq \int_{\mathbb{R}^N} |\nabla \phi|^2$$

for any smooth real-valued function with compact support in $\mathbb{R}^N$.

**Proof.** For the sake of completeness, we still provide a proof here. We may assume $H = \{ x' = 0 \}$ where $x' = (x_1, \cdots , x_m)$. Then we have $d_H(x) = |x'|$ and the function $\psi(x) = \psi(x') = -|x'|^{2-m}$ is subharmonic on $\mathbb{R}^m$, hence is subharmonic on $\mathbb{R}^N$ (in particular, $\mathbb{R}^N$ is non-parabolic). Thus

$$0 \leq \int_{\mathbb{R}^N} \frac{\phi^2}{\psi^2} \cdot \frac{\Delta \psi}{-\psi} = - \int_{\mathbb{R}^N} \nabla \psi \cdot \nabla \left( \frac{\phi^2}{\psi} \right) = 2 \int_{\mathbb{R}^N} \frac{\phi}{\psi} \cdot \nabla \psi \cdot \nabla \phi - \int_{\mathbb{R}^N} \frac{\phi^2}{\psi^2} \cdot |\nabla \psi|^2,$$

so that

$$\int_{\mathbb{R}^N} \frac{\phi^2}{\psi^2} \cdot |\nabla \psi|^2 \leq -2 \int_{\mathbb{R}^N} \frac{\phi}{\psi} \cdot \nabla \psi \cdot \nabla \phi \leq \frac{1}{2} \int_{\mathbb{R}^N} \frac{\phi^2}{\psi^2} \cdot |\nabla \psi|^2 + 2 \int_{\mathbb{R}^N} |\nabla \phi|^2,$$

that is

$$\int_{\mathbb{R}^N} \frac{\phi^2}{\psi^2} \cdot |\nabla \psi|^2 \leq 4 \int_{\mathbb{R}^N} |\nabla \phi|^2.$$

On the other hand, a straightforward calculation gives $|\nabla \psi|^2 / \psi^2 = (m-2)^2 |x'|^{-2}$, hence we are done. \( \square \)

**Theorem 2.1** combined with **Lemma 2.2** immediately yields

**Corollary 2.3.** Let $H \subset \mathbb{R}^{2n} = \mathbb{C}^n$ be an affine-linear subspace with codimension $m \geq 3$. Then for any $\partial -$closed $(0, q)$–form on $\mathbb{C}^n$ with $\int_{\mathbb{C}^n} |v|^2 < \infty$ and $\int_{\mathbb{C}^n} |v|^2 d_H^2 < \infty$, there exists a $(0, q-1)$–form $u$ on $\mathbb{C}^n$ such that $\bar{\partial} u = v$ and

$$\int_{\mathbb{C}^n} |u|^2 \leq \frac{16}{(m-2)^2} \int_{\mathbb{C}^n} |v|^2 d_H^2.$$

3. **Proof of Theorem 1.4**

Set $d_E(z) := d(z, E)$ and $d_H(z) := d(z, H)$. Choose a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi|_{(-\infty, 1/2]} = 0$ and $\chi|_{[1, \infty)} = 1$. Given $f \in A^2(\Omega \setminus E) := L^2 \cap \mathcal{O}(\Omega \setminus E)$, define $v := \bar{\partial} \{ \chi(d_E / r) f \}$ on $\Omega \setminus E$ and $v = 0$ on $E \cup (\mathbb{C}^n \setminus \Omega)$. Clearly, $v$ is a smooth $\partial -$closed $(0, 1)$–form on $\mathbb{C}^n$. Moreover, since $|\nabla d_E| \leq 1$ a.e., we have

$$\int_{\mathbb{C}^n} |v|^2 \leq \frac{\sup |\chi'|^2}{r^2} \cdot \int_{r/2 \leq d_E \leq r} |f|^2 \leq \frac{\sup |\chi'|^2}{r^2} \cdot \int_{\Omega \setminus E} |f|^2 < \infty,$$
and since $E \subset H_R$, it follows that
\[ \int_{\mathbb{C}^n} |v|^2 d^2_H \leq \frac{\sup |\chi'|^2}{r^2} \cdot \int_{r/2 \leq d \leq r} |f|^2 d^2_H \]
\[ \leq \frac{(R + r)^2}{r^2} \cdot \sup |\chi'|^2 \cdot \int_{\Omega \setminus E} |f|^2 < \infty. \]

Thanks to Corollary 2.3 we obtain a solution of $\bar{\partial} u = v$ which satisfies
\[ \int_{\mathbb{C}^n} |u|^2 \lesssim \int_{\Omega \setminus E} |f|^2. \]

Since $\text{supp } v \subset E_r$, we conclude that $u \in O(\mathbb{C}^n \setminus E_r)$. Also, since $H_{R+r} \not\subset \mathbb{C}^n$ is convex, so there exists a real hyperplane $H$ in $\mathbb{C}^n$ such that $d(H, H_{R+r}) > 1$. Since $n \geq 2$, so $H$ contains at least one complex line $l$. Without loss of generality, we assume $l = \{z' = 0\}$ where $z' = (z_1, \cdots, z_{n-1})$. Thus the cylinder $C := l \times \mathbb{B}^{n-1} \subset \mathbb{C}^n \setminus H_{R+r} \subset \mathbb{C}^n \setminus E_r$, where $\mathbb{B}^{n-1}$ is the unit ball in $\mathbb{C}^{n-1}$. Now $u \in A^2(C)$, so $u(z', \cdot) \in A^2(\mathbb{C})$ for every $z' \in \mathbb{B}^{n-1}$, and it has to vanish in view of the ($L^2$) Liouville theorem. By the theorem of unique continuation, $u = 0$ in an unbounded component of $\mathbb{C}^n \setminus E_r$, which naturally intersects with $\Omega \setminus E$. Finally, the function $F := \chi(d_E/r)f - u$ is holomorphic on $\Omega$ and satisfies $F = f$ on a nonempty open subset in $\Omega \setminus E$. Since $\Omega \setminus E$ is connected, it follows that $F = f$ on $\Omega \setminus E$. Clearly,
\[ \int_{\Omega} |F|^2 \leq 2 \int_{\Omega} |\chi(d_E/r)f|^2 + 2 \int_{\Omega} |u|^2 \leq \int_{\Omega \setminus E} |f|^2 + 2 \int_{\mathbb{C}^n} |u|^2 < \infty. \]

REFERENCES

[1] S. Bochner, Bounded analytic functions in several variables and multiple Laplace integrals, Amer. Math. J. 59 (1937), 732–738.
[2] S. Bochner, A theorem on analytic continuation of functions in several variables, Ann. Math. 39 (1938), 14–19.
[3] S. Bochner, Analytic and meromorphic continuation by means of Green’s formula, Ann. Math. 44 (1943), 652–673.
[4] A. Boggess, R. Dwilewicz and Z. Slodkowski, Hartogs extension for generalized tubes in $\mathbb{C}^n$, J. Math. Anal. Appl. 402 (2013), 574–578.
[5] A. Boggess, R. Dwilewicz and Z. Slodkowski, Hartogs-type extension for tube-like domains in $\mathbb{C}^2$, Math. Ann. 363 (2015), 35–60.
[6] B. Y. Chen, Hardy-Sobolev type inequalities and their applications, arXiv:1712.02044.
[7] M. Coltoiu and J. Ruppenthal, On Hartogs’ extension theorem on $(n - 1)$–complete complex spaces, J. Reine Angew. Math. 637 (2009), 41–47.
[8] L. Ehrenpreis, A new proof and an extension of Hartogs’ theorem, Bull. Amer. Math. Soc. 67 (1961), 507–509.
[9] F. Hartogs, Zur Theorie der analytischen Functionen mehrerer unabhängiger Veränderlicher insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, Math. Ann. 62 (1906), 1–88.
[10] L. Hörmander, An Introduction to Complex Analysis in Several Variables, Elsevier, 1990.
[11] E. Martinelli, Sopra una dimostrazione di R. Fueter per un teorema di Hartogs, Comm. Math. Helv. 15 (1942/43), 340–349.
[12] J. Merker and E. Porten, A Morse-theoretical proof of the Hartogs extension theorem, J. Geom. Anal. 17 (2007), 513–546.
[13] J. Merker and E. Porten, *The Hartogs extension theorem on $(n-1)-$complete complex spaces*, J. Reine Angew. Math. **637** (2009), 23–39.

[14] T. Ohsawa, *Hartogs type extension theorems on some domains in Kähler manifolds*, Ann. Polon. Math. **106** (2012), 243–254.

[15] N. Ovrelid and S. Vassiliadou, *Hartogs extension theorems on Stein spaces*, J. Geom. Anal. **20** (2010), 817–836.

[16] R. M. Range, *Extension phenomena in multidimensional complex analysis: correction of the historical record*, Math. Intelligencer **24** (2002), 4–12.

Department of Mathematical Sciences, Fudan University, Shanghai, 200433, China

Email address: boychen@fudan.edu.cn