Entropy of automorphisms of $\text{II}_1$-factors arising from the dynamical systems theory

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Abstract

Let a countable amenable group $G$ acts freely and ergodically on a Lebesgue space $(X, \mu)$, preserving the measure $\mu$. If $T \in \text{Aut} (X, \mu)$ is an automorphism of the equivalence relation defined by $G$ then $T$ can be extended to an automorphism $\alpha_T$ of the $\text{II}_1$-factor $M = L^\infty(X, \mu) \rtimes G$. We prove that if $T$ commutes with the action of $G$ then $H(\alpha_T) = h(T)$, where $H(\alpha_T)$ is the Connes-Størmer entropy of $\alpha_T$, and $h(T)$ is the Kolmogorov–Sinai entropy of $T$. We prove also that for given $s$ and $t$, $0 \leq s \leq t \leq \infty$, there exists a $T$ such that $h(T) = s$ and $H(\alpha_T) = t$.

Introduction

Entropy is an important notion in classical statistical mechanics and information theory. Initially the conception of entropy for automorphism in the ergodic theory was introduced by Kolmogorov and Sinai in 1958. This invariant proved to be extremely useful in the classical dynamical systems theory and topological dynamics. The extension of this notion onto quantum dynamical systems was done by Connes, Narnhofer, Størmer and Thirring \[\text{CS, CNT}\]. At the present time there are several other promising approaches to entropy of $C^*$-dynamical systems \[\text{S, AF, V}\].

An important trend in dynamical entropy is its computation for various models. A lot of interesting results was obtained in this field in the recent years. We note several of them. Størmer, Voiculescu \[\text{SV}\], and the second author \[\text{N}\] computed the entropy of Bogoliubov automorphisms of CAR and CCR algebras (see also \[\text{BG, GN2}\]). Pimsner, Popa \[\text{PP}\], Choda \[\text{Ch1}\] computed the entropy of shifts of Temperley-Lieb algebras, Choda \[\text{Ch2}\], Hiai \[\text{H}\] and Størmer \[\text{St}\] computed the entropy of canonical shifts. The first author, Størmer \[\text{GS1, GS2}\], Price \[\text{Pr}\] computed entropy for a wide class of binary shifts.

In this paper we consider automorphisms of $\text{II}_1$ factors arising from the dynamical systems theory. Let a countable group $G$ acts freely and ergodically on a Lebesgue space $(X, \mu)$ and preserves $\mu$. Then one can construct the crossed product $M = L^\infty(X, \mu) \rtimes G$, which, as well known, is a $\text{II}_1$-factor. If $T \in \text{Aut} (X, \mu)$ defines an automorphism of the ergodic equivalence relation induced by $G$ then $T$ can be extended to an automorphism $\alpha_T$ of $M$ \[\text{FM}\]. It is a natural problem to compute the dynamical entropy $H(\alpha_T)$ in the sense of \[\text{CS}\] and to compare it with the Kolmogorov-Sinai entropy $h(T)$ of $T$. It should be noted that this last problem is a part of a more general problem. Namely, let $M$ be a $\text{II}_1$-factor, $\alpha \in \text{Aut} M$, $A$ its $\alpha$-invariant Cartan subalgebra, $\alpha(A) = A$, then it is nature to investigate when $H(\alpha)$ is equal to $H(\alpha|_A)$. 
These problems are studied in our paper. In Section 1 we prove that if $T$ commutes with the action of $G$ then $H(\alpha_T) = h(T)$. More generally, we prove that this result is valid for crossed products of arbitrary algebras for entropies of Voiculescu and of Connes-Narnhofer-Thirring. In Section 2 we consider two examples to illustrate this result. These examples give non-isomorphic ergodic automorphisms of the hyperfinite ergodic equivalence relation with the same entropy. In Section 3 we construct several examples showing that the entropies $h(T)$ and $H(\alpha_T)$ can be distinct. These systems are non-commutative analogs of dynamical systems of algebraic origin (see [A, Y, LSW, S]). In particular, some of our examples are automorphisms of non-commutative tori. In Section 4 we construct flows $T_t$ such that $H(\alpha_{T_t}) > h(T_t)$. In particular, we show that the values $h(T)$ and $H(\alpha_T)$ can be arbitrary.

1 Computation of entropy of automorphisms of crossed products

Let $(X, \mu)$ be a Lebesgue space, $G$ a countable amenable group of automorphisms $S_g, g \in G$, of $(X, \mu)$ preserving $\mu$, and $T$ an automorphism of $(X, \mu)$, $\mu \circ T = \mu$, such that

$$TS_g = S_g T, \quad g \in G.$$ 

**Theorem 1.1** Let $(X, \mu)$, $G$ and $T$ be as above. Suppose $G$ acts freely and ergodically on $(X, \mu)$. Then $M = L^\infty(X, \mu) \rtimes S G$ is the hyperfinite II$_1$-factor with the trace-state $\tau$ induced by $\mu$. The automorphism $T$ can be canonically extended to an automorphism $\alpha_T$ of $M$, and

$$H(\alpha_T) = h(T),$$

where $H(\alpha_T)$ is the Connes-Størmer entropy of $\alpha_T$, and $h(T)$ is the Kolmogorov-Sinai entropy of $T$.

We will prove the following more general result.

**Theorem 1.2** Let $M$ be an approximately finite-dimensional $W^*$-algebra, $\sigma$ its normal state, $T$ a $\sigma$-preserving automorphism. Suppose a discrete amenable group $G$ acts on $M$ by automorphisms $S_g$ that commute with $T$ and preserve $\sigma$. The automorphism $T$ defines an automorphism $\alpha_T$ of $M \rtimes_S G$, and the state $\sigma$ is extended to the dual state which we continue to denote by $\sigma$. Then

(i) $hcpa_\sigma(\alpha_T) = hcpa_\sigma(T)$, where $hcpa_\sigma$ is the completely positive approximation entropy of Voiculescu;

(ii) $h_\sigma(\alpha_T) = h_\sigma(T)$, where $h_\sigma$ is the dynamical entropy of Connes-Narnhofer-Thirring.

Since CNT-entropy coincides with KS-entropy in the classical case, and with CS-entropy for tracial $\sigma$ and approximately finite-dimensional $M$, Theorem 1.1 follows from Theorem 1.2.

To prove Theorem 1.2 we will generalize a construction of Voiculescu.

**Lemma 1.3** Let $B$ be a $C^*$-algebra, $x_1, \ldots, x_n \in B$. Then the mapping $\Psi: \text{Mat}_n(\mathbb{C}) \otimes B \to B$,

$$\Psi(e_{ij} \otimes b) = x_i b x_j^*,$$

is completely positive.
Proof. Consider the element \( V \in \text{Mat}_n(B) = \text{Mat}_n(\mathbb{C}) \otimes B, \)

\[
V = \begin{pmatrix} x_1 & \ldots & x_n \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}.
\]

Consider also the projection \( p = e_{11} \otimes 1 \in \text{Mat}_n(\mathbb{C}) \otimes B. \) Then \( \Psi \) is the mapping \( \text{Mat}_n(B) \to p\text{Mat}_n(B)p = B, \) \( x \mapsto VXV^*. \)

Let \( \lambda \) be the canonical representation of \( G \) in \( M \times G, \) so that \( (\text{Ad}\, \lambda)(a) = S_\sigma(a) \) for \( a \in M. \)

Lemma 1.4 For any finite subset \( F \) of \( G, \) there exist normal unital completely positive mappings \( I_F: B(l^2(F)) \otimes M \to M \times G \) and \( J_F: M \times G \to B(l^2(F)) \otimes M \) such that

\[
I_F(e_{g,h} \otimes a) = \frac{1}{|F|} \lambda(g)a \lambda(h)^* = \frac{1}{|F|} \lambda(gh^{-1})S_h(a),
\]

\[
J_F(\lambda(g)a) = \sum_{h \in F \cap g^{-1}F} e_{gh,h} \otimes S_h^{-1}(a),
\]

\[
(I_F \circ J_F)(\lambda(g)a) = \frac{|F \cap g^{-1}F|}{|F|} \lambda(g)a,
\]

\[
\sigma \circ I_F = \text{tr}_F \otimes \sigma, \quad \alpha_T \circ I_F = I_F \circ (\text{id} \otimes T),
\]

\[
(\text{tr}_F \otimes \sigma) \circ J_F = \sigma, \quad (\text{id} \otimes T) \circ J_F = J_F \circ \alpha_T,
\]

where \( \text{tr}_F \) is the unique tracial state on \( B(l^2(F)). \)

Proof. The complete positivity of \( I_F \) follows from Lemma 1.3. Consider \( J_F. \) Suppose that \( M \subset B(H), \) and consider the regular representation of \( M \times G \) on \( l^2(G) \otimes H: \)

\[
\lambda(g)(\delta_h \otimes \xi) = \delta_{gh} \otimes \xi, \quad a(\delta_h \otimes \xi) = \delta_h \otimes S_h^{-1}(a)\xi \quad (a \in M).
\]

Let \( P_F \) be the projection onto \( l^2(F) \otimes H. \) Then a direct computation shows that the mapping \( J_F(x) = P_F P_F, \) \( x \in M \times G, \) has the form written above. All others assertions follow immediately.

Proof of Theorem 1.2.

(i) Since there exists a \( \tau \)-preserving conditional expectation \( M \times G \to M, \) we have \( hcpa_\sigma(\alpha_T) \geq hcpa_\sigma(T). \) To prove the opposite inequality we have to show that \( hcpa_\sigma(\alpha_T, \omega) \leq hcpa_\sigma(T) \) for any finite subset \( \omega \) of \( M \times G. \) Fix \( \varepsilon > 0. \) We can find a finite subset \( F \) of \( G \) such that \( \|(I_F \circ J_F)(x) - x\|_\sigma < \varepsilon \) for any \( x \in \omega. \) Let \( (\psi, \phi, B) \in CPA(B(l^2(F)) \otimes M, \text{tr}_F \otimes \sigma). \) Then \( (I_F \circ \psi, \phi \circ J_F, B) \in CPA(M \times G, \sigma). \) Suppose

\[
\|(\psi \circ \phi)(J_F(x)) - J_F(x)\|_{\text{tr}_F \otimes \sigma} < \delta
\]

for some \( x \in \alpha_T^k(\omega) \) and \( k \in \mathbb{N}. \) Then

\[
\|(I_F \circ \psi \circ \phi \circ J_F)(x) - x\|_\sigma \leq \|(\psi \circ \phi)(J_F(x)) - J_F(x)\|_{\sigma \circ I_F} + \|(I_F \circ J_F)(x) - x\|_\sigma < \delta + \varepsilon,
\]

for some \( x \in \alpha_T^k(\omega). \)
where we have used the facts that \( \sigma \circ I_F = \text{tr}_F \otimes \sigma \) and that \( \alpha_T \) commutes with \( I_F \circ J_F \). Since \( J_F \circ \alpha_T = (\text{id} \otimes T) \circ J_F \), we infer that

\[
  rcp_{\sigma}(\omega \cup \alpha_T(\omega) \cup \ldots \cup \alpha_T^{n-1}(\omega); \delta + \epsilon) \leq rcp_{\text{tr}_F \otimes \sigma}(J_F(\omega) \cup \ldots \cup (\text{id} \otimes T)^n(\omega); \delta),
\]

so that (for \( \delta < \epsilon \))

\[
  hcp_{\sigma}(\alpha_T, \omega; 2\epsilon) \leq hcp_{\sigma}(\alpha_T, \omega; \epsilon + \delta) \leq hcp_{\text{tr}_F \otimes \sigma}(\text{id} \otimes T, J_F(\omega); \delta) \leq hcp_{\text{tr}_F \otimes \sigma}(\text{id} \otimes T) = hcp_{\sigma}(T),
\]

where the last equality follows from the subadditivity of the entropy [V]. Since \( \epsilon > 0 \) was arbitrary, the proof of the inequality \( hcp_{\sigma}(\alpha_T, \omega) \leq hcp_{\sigma}(T) \) is complete.

(ii) We always have \( h_{\sigma}(\alpha_T) \geq h_{\sigma}(T) \). To prove the opposite inequality consider a channel \( \gamma: B \to M \otimes G \), i.e., a unital completely positive mapping of a finite-dimensional \( C^* \)-algebra \( B \). We have to prove that \( h_{\sigma}(\alpha_T; \gamma) \leq h_{\sigma}(T) \). Fix \( \epsilon > 0 \). We can choose \( F \) such that

\[
  ||(I_F \circ J_F \circ \gamma - \gamma)(x)||_\sigma \leq \epsilon ||x|| \quad \text{for any } x \in B.
\]

By [CNT, Theorem IV.3],

\[
  \frac{1}{n} H_{\sigma}(\gamma, \alpha_T \circ \gamma, \ldots, \alpha_T^{n-1} \circ \gamma) \leq \delta + \frac{1}{n} H_{\sigma}(I_F \circ J_F \circ \gamma, \alpha_T \circ I_F \circ J_F \circ \gamma, \ldots, \alpha_T^{n-1} \circ I_F \circ J_F \circ \gamma),
\]

where \( \delta = \delta(\epsilon, \text{rank } B) \to 0 \) as \( \epsilon \to 0 \). Since \( \sigma \circ I_F = \text{tr}_F \otimes \sigma \), it is easy to see from the definition of mutual entropy \( H_{\sigma} \) [CNT] that

\[
  H_{\sigma}(I_F \circ J_F \circ \gamma, I_F \circ J_F \circ \alpha_T \circ \gamma, \ldots, I_F \circ J_F \circ \alpha_T^{n-1} \circ \gamma) \leq H_{\text{tr}_F \otimes \sigma}(J_F \circ \gamma, J_F \circ \alpha_T \circ \gamma, \ldots, J_F \circ \alpha_T^{n-1} \circ \gamma),
\]

Since \( I_F \circ J_F \) commutes with \( \alpha_T \), and \( J_F \circ \alpha_T = (\text{id} \otimes T) \circ J_F \), we infer from (1.1) and (1.2) that

\[
  h_{\sigma}(\alpha_T; \gamma) \leq \delta + h_{\text{tr}_F \otimes \sigma}(\text{id} \otimes T; J_F \circ \gamma) \leq \delta + h_{\text{tr}_F \otimes \sigma}(\text{id} \otimes T).
\]

Since we could choose \( F \) such that \( \delta \) was arbitrary small, we see that it suffices to prove that \( h_{\text{tr}_F \otimes \sigma}(\text{id} \otimes T) = h_{\sigma}(T) \). For abelian \( M \) this is proved by standard arguments, using [CNT, Corollary VIII.8]. To handle the general case we need the following lemma.

**Lemma 1.5** For any finite-dimensional \( C^* \)-algebra \( B \), any state \( \phi \) of \( B \), and any positive linear functional \( \psi \) on \( \text{Mat}_n(\mathbb{C}) \otimes B \), we have

\[
  S(\text{tr}_n \otimes \phi, \psi) \leq S(\phi, \psi|_B) + 2\psi(1) \log n.
\]

**Proof.** By [OP, Theorem 1.13],

\[
  S(\text{tr}_n \otimes \phi, \psi) = S(\phi, \psi|_B) + S(\psi \circ E, \psi),
\]

where \( E = \text{tr}_n \otimes \text{id}: \text{Mat}_n(\mathbb{C}) \otimes B \to B \) is the \( (\text{tr}_n \otimes \phi) \)-preserving conditional expectation (note that we adopt the notations of [CNT], so we denote by \( S(\omega_1, \omega_2) \) the quantity which is denoted by \( S(\omega_2, \omega_1) \) in [OP]). By the Pimsner-Popa inequality [PP, Theorem 2.2], we have

\[
  E(x) \geq \frac{1}{n^2} x \quad \text{for any } x \in \text{Mat}_n(\mathbb{C}) \otimes B, \ x \geq 0.
\]
In particular, $\psi \circ E \geq \frac{1}{n^2} \psi$, whence $S(\psi \circ E, \psi) \leq 2\psi(1) \log n$.

Since $M$ is an AFD-algebra, to compute the entropy of $\text{id} \otimes T$ it suffices to consider subalgebras of the form $B(l^2(F)) \otimes B$, where $B \subset M$. From Lemma \cite{L} and the definitions \cite{CNT}, we immediately get

$$h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T; B(l^2(F)) \otimes B) \leq h_\sigma(T; B) + 2 \log |F|.$$ 

Hence $h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T) \leq h_\sigma(T) + 2 \log |F|$. Applying this inequality to $T^m$, we obtain

$$h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T)^m \leq h_\sigma(T^m) + 2 \log |F| \quad \forall m \in \mathbb{N}.$$ 

But since $M$ is an AFD-algebra, we have $h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T)^m = m \cdot h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T)$ and $h_\sigma(T^m) = m \cdot h_\sigma(T)$. So dividing the above inequality by $m$, and letting $m \to \infty$, we obtain $h_{\text{tr}_F\otimes \sigma}(\text{id} \otimes T) \leq h_\sigma(T)$, and the proof of Theorem is complete.

Remarks.

(i) For any AFD-algebra $N$ and any normal state $\omega$ of $N$, we have $h_{\omega \otimes \sigma}(\text{id} \otimes T) = h_\sigma(T)$. Indeed, we may suppose that $N$ is finite-dimensional and $\omega$ is faithful (because if $p$ is the support of $\omega$, then $h_{\omega \otimes \sigma}(\text{id} \otimes T) = h_{\omega \otimes \sigma}((\text{id} \otimes T)|_p N_{p \otimes M})$). Now the only thing we need is a generalization of the Pimsner-Popa inequality. Let $p_1, \ldots, p_m$ be the atoms of a maximal abelian subalgebra of the centralizer of the state $\omega$. Then

$$(\omega \otimes \text{id})(x) \geq \left( \sum_{i=1}^{m} \frac{1}{\omega(p_i)} \right)^{-1} x \quad \text{for any } x \in N \otimes M, \ x \geq 0,$$

by \cite{L} Theorem 4.1 and Proposition 5.4.

(ii) By Corollary 3.8 in \cite{V}, $\text{hcpa}_\mu(T) = h(T)$ for ergodic $T$. For non-ergodic $T$, the entropies can be distinct. Indeed, let $X_1$ be a $T$-invariant measurable subset of $X$, $\lambda = \mu(X_1), 0 < \lambda < 1$. Set $\mu_1 = \lambda^{-1} \mu|_{X_1}, T_1 = T|_{X_1}, X_2 = X \setminus X_1, \mu_2 = (1 - \lambda)^{-1} \mu|_{X_2}, T_2 = T|_{X_2}$. It is easy to see that $h(T) = \lambda h(T_1) + (1 - \lambda) h(T_2)$. On the other hand, it can be proved that

$$\text{hcpa}_\mu(T) = \max\{\text{hcpa}_\mu(T_1), \text{hcpa}_\mu(T_2)\}.$$ 

So if $h(T_1), h(T_2) < \infty$, $h(T_1) \neq h(T_2)$, then $h(T) < \text{hcpa}_\mu(T)$.

To obtain an invariant which coincides with KS-entropy in the classical case, one can modify Voiculescu’s definition replacing rank by $\exp S(\sigma \circ \psi)$ in \cite{V}, Definition 3.1]. Theorem \cite{I} remains true for this modified entropy.

2 Examples

We present two examples to illustrate Theorem \cite{I}. These examples give non-isomorphic ergodic automorphisms of amenable equivalence relations with the same KS-entropy.

Let us first describe a general construction.

**Proposition 2.1** Let $S_0, S_1, S_2$ be ergodic automorphisms of $(X, \mu)$ such that $S_0$ commutes with $S_1$ and $S_2$, and $S_1$ is conjugate with neither $S_2$, nor $S_2^{-1}$ by an automorphism commuting with $S_0$. Set $M_i = L^\infty(X, \mu) \rtimes S_i$, $i = 1, 2$, and let $\alpha_i$ be the automorphism of $M_i$ induced by $S_0$. Then there is no isomorphism $\phi$ of $M_1$ onto $M_2$ such that $\phi \circ \alpha_1 = \alpha_2 \circ \phi$ and $\phi(L^\infty(X, \mu)) = L^\infty(X, \mu)$.
Proof. Suppose such a $\phi$ exists. Let $U_i \in M_i$ be the unitary corresponding to $\alpha_i$, $i = 1, 2$, $A = L^\infty(X, \mu) \subset M_1$. Set $U = \phi^{-1}(U_2)$. Since $U$ is a unitary operator from $M_1$ such that $(\text{Ad } U)(A) = A$, it is easy to check that $U$ has the form

$$U = \sum_{i \in \mathbb{Z}} a_i U_1^i E_i, \quad a_i \in \mathbb{T},$$

where $\{E_i\}$ is a family of projections from $A$, $E_i E_j = 0$, for $i \neq j$, $\sum_i E_i = \sum_i U_1^i E_i U_1^{-i} = I$. Since $\alpha_1(U) = U$, we have $\alpha_1(E_i) = E_i$, $i \in \mathbb{Z}$. But $S_0$ is ergodic, therefore $E_i = I$ or $E_i = 0$. Hence $U = a_i U_1^i$ for some $i \in \mathbb{Z}$ and $a_i \in \mathbb{T}$. Since $\phi$ is an isomorphism, we have either $i = -1$, or $i = 1$. We see that $\phi |_{L^\infty(X, \mu)}$ is an automorphism that commutes with $S_0$ and conjugates $S_2$ with either $S_1^{-1}$, or $S_1$.

Remark. It follows from Proposition 2.1 that $S_0$ defines non-isomorphic automorphisms of the ergodic equivalence relations induced by $S_1$ and $S_2$ on $X$ correspondingly, despite of $H(\alpha_1) = H(\alpha_2) = h(S_0)$.

Example 2.2 Let $X = [0, 1]$ be the unit interval, $\mu$ the Lebesgue measure on $X$, $t_0$, $t_1$ and $t_2$ irrational numbers from $[0, 1]$ such that $t_2 \neq t_1$, $1 - t_1$. Consider the shifts $S_i x = x + t_i \pmod{1}$, $x \in [0, 1]$. Any automorphism of $X$ commuting with $S_0$ commutes with $S_1$ and $S_2$. Since $S_1 \neq S_2^{-1}$, Proposition 2.1 is applicable. Note that $h(S_0) = 0$.

Example 2.3 Let $(X, \mu)$ be a Lebesgue space, $T_t$ a Bernoulli flow on $(X, \mu)$ with $h(T_1) = \log 2$. Choose $t_i \in \mathbb{R}$, $t_i \neq 0$ ($i = 0, 1, 2$), $t_1 \neq \pm t_2$, and set $S_i = T_{t_i}$. Then $h(S_1) \neq h(S_2)$, and we can apply Proposition 2.1.

3 Entropy of automorphisms and their restrictions to a Cartan subalgebra

Let $M$ be a II$_1$-factor, $A$ its Cartan subalgebra, $\alpha \in \text{Aut } M$ such that $\alpha(A) = A$. We consider cases when $H(\alpha) > H(\alpha |_A)$.

Suppose a discrete abelian group $G$ acts freely and ergodically by automorphisms $S_g$ on $(X, \mu)$, $\beta$ an automorphism of $G$, and $S$ an automorphism of $(X, \mu)$ such that $TS_g = S_{\beta(g)}T$. Then $T$ induces an automorphism $\alpha_T$ of $M = L^\infty(X, \mu) \rtimes_{S} G$. Explicitly,

$$\alpha_T(f)(x) = f(T^{-1} x) \text{ for } f \in L^\infty(X, \mu), \quad \alpha_T(\lambda(g)) = \lambda(\beta(g)).$$

The algebra $A = L^\infty(X, \mu)$ is a Cartan subalgebra of $M$. On the other hand, the operators $\lambda(g)$ generate a maximal abelian subalgebra $B \cong L^\infty(\hat{G})$ of $M$, and $\alpha_T |_B = \hat{\beta}$, the dual automorphism of $\hat{G}$. We have

$$H(\alpha_T) \geq \max\{h(T), h(\hat{\beta})\},$$

so if $h(\hat{\beta}) > h(T)$, then $H(\alpha_T) > H(\alpha_T |_A)$.

To construct such examples we consider systems of algebraic origin.
Let $G_1$ and $G_2$ be discrete abelian groups, and $T_1$ an automorphism of $G_1$. Suppose there exists an embedding $l: G_2 \hookrightarrow \hat{G}_1$ such that $l(G_2)$ is a dense $\hat{T}_1$-invariant subgroup. Set $T_2 = T_1|_{G_2}$. The group $G_2$ acts by translations on $\hat{G}_1$ ($g_2 \cdot \chi_1 = \chi_1 + l(g_2)$), and we fall into the situation described above (with $X = \hat{G}_1$, $G = G_2$, $T = \hat{T}_1$ and $\beta = T_2$).

The roles of $G_1$ and $G_2$ above are almost symmetric. Indeed, to be given an embedding $G_2 \hookrightarrow \hat{G}_1$ with dense range is just the same as to be given a non-degenerate pairing $\langle \cdot, \cdot \rangle: G_1 \times G_2 \rightarrow \mathbb{T}$, then the equality $T_2 = T_1|_{G_2}$ means that this pairing is $T_1 \times T_2$-invariant. The pairing gives rise to an embedding $r: G_1 \hookrightarrow \hat{G}_2$. Then $G_1$ acts on $\hat{G}_2$ by translations $g_1 \cdot \chi_2 = \chi_2 - r(g_1)$, and $L^\infty(\hat{G}_1) \rtimes G_2 \cong G_1 \rtimes L^\infty(\hat{G}_2)$. In fact, both algebras are canonically isomorphic to the twisted group $W^*$-algebra $W^*(G_1 \times G_2, \omega)$, where $\omega$ is the bicharacter defined by

$$\omega((g_1', g_2'), (g_1'', g_2'')) = \langle g_1', g_2' \rangle.$$ 

Then $\alpha_T$ is nothing else than the automorphism induced by the $\omega$-preserving automorphism $T_1 \times T_2$.

Let $R = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$, $f \in \mathbb{Z}[t]$, $f \neq 1$, a polynomial whose irreducible factors are not cyclotomic, equivalently, $f$ has no roots of modulus 1. Fix $n \in \{2, 3, \ldots, \infty\}$. Set $G_1 = R/(f^\sim)$ and $G_2 = \bigoplus_{k=1}^n R/(f^k)$, where $f^\sim(t) = f(t^{-1})$. Let $T_i$ be the automorphism of $G_i$ of multiplication by $t$. Let $\chi$ be a character of $G_2$. Then the mapping $R \ni f_i \mapsto f_i(T_2)\chi \in \hat{G}_2$ defines an equivariant homomorphism $G_1 \rightarrow \hat{G}_2$. In other words, if $\chi = (\chi_1, \ldots, \chi_n) \in G_2 \subset \mathbb{C}^n$, then the pairing is given by

$$\langle f_i, (g_1, \ldots, g_n) \rangle = \prod_{k=1}^n \chi_k(f_i^\sim \cdot g_k),$$

where $(f_i^\sim \cdot g_k)(t) = f_i(t^{-1})g_k(t)$. This pairing is non-degenerate iff the orbit of $\chi$ under the action of $T_2$ generates a dense subgroup of $\hat{G}_2$. Since $T_2$ is aperiodic, the dual automorphism is ergodic. Hence the orbit is dense for almost every choice of $\chi$.

Now let us estimate entropy. First, by Yuzvinskii’s formula [12, LM], $h(T_1) = m(f)$, $h(T_2) = n \cdot m(f)$, where $m(f)$ is the logarithmic Mahler measure of $f$,

$$m(f) = \int_0^1 \log |f(e^{2\pi is})| ds = \log |a_m| + \sum_{j:|\lambda_j|>1} \log |\lambda_j|,$$

where $a_m$ is the leading coefficient of $f$, and $\{\lambda_j\}_j$ are the roots of $f$. Now suppose that the coefficients of the leading and the lowest terms of $f$ are equal to 1. Then $G_1 \times G_2$ is a free abelian group of rank $(n + 1) \deg f$, and by a result of Voiculescu [12] we have $H(\alpha_T) \leq h(T_1 \times T_2) = (n + 1)m(f)$.

Note also that since the automorphism $T_1 \times T_2$ is aperiodic, the automorphism $\alpha_T$ is mixing. Let us summarize what we have proved.

**Theorem 3.1** For given $n \in \{2, 3, \ldots, \infty\}$ and a polynomial $f \in \mathbb{Z}[t]$, $f \neq 1$, whose coefficients of the leading and the lowest terms are equal to 1 and which has no roots of modulus 1, there exists a mixing automorphism $\alpha$ of the hyperfinite $III_1$-factor and an $\alpha$-invariant Cartan subalgebra $A$ such that

$$H(\alpha|_A) = m(f), \quad n \cdot m(f) \leq H(\alpha) \leq (n + 1)m(f).$$
The possibility of constructing on this way systems with arbitrary values $H(\alpha|_A) < H(\alpha)$ is closely related to the question, whether 0 is a cluster point of the set $\{ m(f) | f \in \mathbb{Z}[t] \}$ (note that it suffices to consider irreducible polynomials whose leading coefficients and constant terms are equal to 1). This question is known as Lehmer’s problem, and there is an evidence that the answer is negative (see [LSW] for a discussion).

In estimating the entropy above we used the result of Voiculescu stating that the entropy of an automorphism of a non-commutative torus is not greater than the entropy of its abelian counterpart. It is clear that this result should be true for a wider class of systems. Consider the most simple case where the polynomial $f$ is a constant.

**Example 3.2** Let $f = 2$ and $n = 2$. Then $G_1 = R/(2) \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, $G_2 = G_1 \oplus G_1$, $T_1$ is the shift to the right, $T_2 = T_1 \oplus T_1$. Let $G_1(0) = \mathbb{Z}/2\mathbb{Z} \subset G_1$ and $G_2(0) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \subset G_2$ be the subgroups sitting at the 0th place. Set

$$G_i^{(n)} = G_i(0) \oplus T_i G_i(0) \oplus \ldots \oplus T_i^n G_i(0).$$

Then $H(\alpha_T) \leq \text{hcap}_\tau(\alpha_T) \leq \lim_{n \to \infty} \frac{1}{n} \log \text{rank} C^*(G_1^{(n)} \times G_2^{(n)}, \omega) \leq 3 \log 2$, so (for $A = L^\infty(\hat{G}_1)$)

$$H(\alpha_T|_A) = \log 2 \quad \text{and} \quad 2 \log 2 \leq H(\alpha_T) \leq 3 \log 2.$$

The actual value of $H(\alpha_T)$ is probably depends on the choice of the character $\chi \in \hat{G}_2$. We want to show that $H(\alpha_T) = 2 \log 2$ for some special choice of $\chi$. For this it suffices to require the pairing $\langle \cdot, \cdot \rangle|_{G_1^{(n)} \times G_2^{(n)}}$ be non-degenerate in the first variable for any $n \geq 0$ (so that $C^*(G_2^{(n)})$ is a maximal abelian subalgebra of $C^*(G_1^{(n)} \times G_2^{(n)}, \omega)$, and the rank of the latter algebra is equal to $4(n+1)$). The embedding $G_1 \hookrightarrow G_2$ is given by

$$g_1 \mapsto \prod_{n \in \mathbb{Z}, g_1(n) \neq 0} \hat{T}_2^n \chi, \quad g_1 = (g_1(n))_n \in \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

So we must choose $\chi$ in a way such that the character $\prod_{k=1}^m \hat{T}_2^{n_k} \chi$ is non-trivial on $G_2^{(n)}$ for any $0 \leq n_1 < \ldots < n_m \leq n$. Identify $\hat{G}_2$ with $\prod_{n \in \mathbb{Z}} (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. Then $\hat{T}_2$ is the shift to the right, and we may take any $\chi = (\chi_n)_n$ such that

(i) $\chi_n = 0$ for $n < 0$, $\chi_0 \neq 0$;

(ii) the group generated by $\hat{T}_2^n \chi$ is dense in $\hat{G}_2$.

Finally, we will show that it is possible to construct systems with positive entropy, which have zero entropy on a Cartan subalgebra.

**Example 3.3** Let $p$ be a prime number, $p \neq 2$, $\hat{G}_1 = \mathbb{Z}_p$ (the group of $p$-adic integers), $G_2 = \bigcup_{n \in \mathbb{N}} 2^{-n}\mathbb{Z} \subset \hat{G}_1$, $T_1$ and $T_2$ the automorphisms of multiplication by 2. The group $G_1$ is the inductive limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$, and $T_1$ acts on them as the automorphism of division by 2. Hence

$$H(\alpha_T|_A) = \lim_{n \to \infty} H(\alpha_T|_{C^*(\mathbb{Z}/p^n\mathbb{Z})}) = 0.$$

Since $G_2 = R/(t - 2)$, we have $h(T_2) = \log 2$, so $H(\alpha_T) \geq \log 2$. We state that

$$H(\alpha_T) = \text{hcap}_\tau(\alpha_T) = \log 2.$$
The automorphism $T_1^{p^{n-1}(p-1)}$ is identical on $\mathbb{Z}/p^n\mathbb{Z}$. Since
\[ W^*(\mathbb{Z}/p^n\mathbb{Z} \times G_2, \omega) = \mathbb{Z}/p^n\mathbb{Z} \ltimes L^\infty(\hat{G}_2), \]
by Theorem 1.2 we infer
\[ hcpa_\tau(\alpha_T^{p^{n-1}(p-1)}|_{W^*(\mathbb{Z}/p^n\mathbb{Z} \times G_2, \omega)}) = h(\hat{T}_2^{p^{n-1}(p-1)}), \]
whence $hcpa_\tau(\alpha_T|_{W^*(\mathbb{Z}/p^n\mathbb{Z} \times G_2, \omega)}) = \log 2$, and
\[ hcpa_\tau(\alpha_T) = \lim_{n \to \infty} hcpa_\tau(\alpha_T|_{W^*(\mathbb{Z}/p^n\mathbb{Z} \times G_2, \omega)}) = \log 2. \]

4 Flows on II$_1$-factors with invariant Cartan subalgebras

Using examples of previous sections and the construction of associated flow we will construct systems with arbitrary values of $H(\alpha|_A)$ and $H(\alpha)$ ($0 \leq H(\alpha|_A) \leq H(\alpha) \leq \infty$).

Suppose a discrete amenable group $G$ acts freely and ergodically by measure-preserving transformations $S_t$ on $(X, \mu)$, $T$ an automorphism of $(X, \mu)$ and $\beta$ an automorphism of $G$ such that $TS_g = S_{\beta(g)}T$. Consider the flow $F_t$ associated with $T$. So $Y = \mathbb{R}/\mathbb{Z} \times X$, $d\nu = dt \times d\mu$,
\[ F_t(\hat{\tau}, x) = (\hat{\tau} + t, T^{[r+\hat{\tau}]}x) \quad \text{for} \quad r \in [0, 1), x \in X, \]
where $t \mapsto \hat{\tau}$ is the factorization mapping $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. The semidirect product group $G_0 = G \times \beta\mathbb{Z}$ acts on $(X, \mu)$, This action is ergodic. It is also free, if
\[ \text{there exist no } g \in G \text{ and no } n \in \mathbb{N} \text{ such that } S_g = T^n \text{ on a set of positive measure}. \quad (4.1) \]
Let $\Gamma$ be a countable dense subgroup of $\mathbb{R}/\mathbb{Z}$, it acts by translations on $\mathbb{R}/\mathbb{Z}$. Set $\mathcal{G} = \Gamma \times G_0$. The group $\mathcal{G}$ is amenable. It acts freely and ergodically on $(Y, \nu)$. The corresponding equivalence relation is invariant under the flow, so we obtain a flow $\alpha_t$ on $L^\infty(Y, \nu) \rtimes \mathcal{G}$. Compute its entropy. Let $\alpha_T$ be the automorphism of $L^\infty(X, \mu) \rtimes G$ defined by $T$. We state that
\[ H(\alpha_t) = |t|H(\alpha_T), \quad hcpa_\tau(\alpha_t) = |t|hcpa_\tau(\alpha_T), \quad \text{and} \quad H(\alpha_t|_{L^\infty(Y, \nu)}) = |t|h(T). \quad (4.2) \]
Since $h(F_t) = |t|h(F_1) = |t|h(id \times T)$, the last equality in (4.2) is evident. To prove the first two note that
\[ H(\alpha_t) = |t|H(\alpha_1) \quad \text{and} \quad hcpa_\tau(\alpha_t) = |t|hcpa_\tau(\alpha_1) \]
(see [OP1, Proposition 10.16] for the first equality, the second is proved analogously). We have
\[ L^\infty(Y, \nu) \rtimes \mathcal{G} = (L^\infty(\mathbb{R}/\mathbb{Z} \times \Gamma) \otimes (L^\infty(X, \mu) \rtimes G_0), \quad \alpha_1 = id \otimes \check{\alpha}_T, \]
where $\check{\alpha}_T$ is the automorphism of $L^\infty(X, \mu) \rtimes G_0$ defined by $T$. Since completely positive approximation entropy is subadditive and monotone [V], we have $hcpa_\tau(id \otimes \check{\alpha}_T) = hcpa_\tau(\check{\alpha}_T)$. We have also $H(id \otimes \check{\alpha}_T) = H(\check{\alpha}_T)$ by virtue of Theorem 1.2. Since
\[ L^\infty(X, \mu) \rtimes G_0 = (L^\infty(X, \mu) \rtimes S G) \rtimes_{\alpha_T} \mathbb{Z}, \]
we obtain $hcpa_\tau(\check{\alpha}_T) = hcpa_\tau(\alpha_T)$ and $H(\check{\alpha}_T) = H(\alpha_T)$ by virtue of Theorem 1.2. So $hcpa_\tau(\alpha_1) = hcpa_\tau(\alpha_T)$ and $H(\alpha_1) = H(\alpha_T)$, and the proof of the equalities (4.2) is complete.
**Theorem 4.1** For any \( s \) and \( t \), \( 0 \leq s < t \leq \infty \), there exist an automorphism \( \alpha \) of the hyperfinite \( \text{II}_1 \)-factor and an \( \alpha \)-invariant Cartan subalgebra \( A \) such that
\[
H(\alpha|_A) = s \quad \text{and} \quad H(\alpha) = t.
\]

**Proof.** Consider a system from Example 3.3. Then the condition (4.1) is satisfied, so the construction above leads to a flow \( \alpha_t \) and an \( \alpha_t \)-invariant Cartan subalgebra \( A_1 \) such that
\[
H(\alpha_t|_{A_1}) = 0 \quad \text{and} \quad H(\alpha_t) = hcpa_t(\alpha_t) = |t| \log 2.
\]
As in Example 2.3, consider a Bernoulli flow \( S_t \) on \( (X, \mu) \) with \( h(S_1) = \log 2 \). Then for the corresponding flow \( \beta_t \) on \( L^\infty(X, \mu) \times_{S_1} \mathbb{Z} \) we have (with \( A_2 = L^\infty(X, \mu) \))
\[
H(\beta_t|_{A_2}) = H(\beta_t) = hcpa_t(\beta_t) = |t| \log 2.
\]
Since Connes-Størmer’ entropy is superadditive [SV] and Voiculescu’s entropies are subadditive, we conclude that
\[
H((\alpha_t \otimes \beta_s)|_{A_1 \otimes A_2}) = |s| \log 2, \quad H(\alpha_t \otimes \beta_s) = H(\alpha_t) + H(\beta_s) = (|t| + |s|) \log 2.
\]
Finally, consider an infinite tensor product of systems from Example 3.3. Thus we obtain an automorphism \( \gamma \) and an \( \alpha \)-invariant Cartan subalgebra \( A_3 \) such that
\[
H(\gamma|_{A_3}) = 0 \quad \text{and} \quad H(\gamma) = \infty.
\]
Then \( H(\beta_s \otimes \gamma)|_{A_2 \otimes A_3} = |s| \log 2, \ H(\beta_s \otimes \gamma) = \infty. \]

**5 Final remarks**

5.1. Let \( p_1 \) and \( p_2 \) be prime numbers, \( p_i \geq 3, \ i = 1, 2 \). Construct automorphisms \( \alpha_1 \) and \( \alpha_2 \) as in Example 3.3.

**Proposition 5.1** If \( p_1 \neq p_2 \), then \( \alpha_1 \) and \( \alpha_2 \) are not conjugate as automorphisms of the hyperfinite \( \text{II}_1 \)-factor, though \( H(\alpha_1) = H(\alpha_2) = \log 2. \)

**Proof.** Indeed, the automorphisms define unitary operators \( U_i \) on \( L^2(M, \tau) \). As we can see, the point part \( S_i \) of the spectrum of \( U_i \) is non-trivial. If \( p_1 \neq p_2 \), then \( S_1 \neq S_2 \), so \( \alpha_1 \) and \( \alpha_2 \) are not conjugate.

5.2. The automorphisms of Theorem 4.1 and Example 3.2 are ergodic. On the other hand, the automorphisms of Example 3.3 are not ergodic, even on the Cartan subalgebra. Moreover, any ergodic automorphism of compact abelian group has positive entropy (it is even Bernoullian), so with the methods of Section 3 we can not construct ergodic automorphisms with positive entropy and zero entropy restriction to a Cartan subalgebra (however, for actions of \( \mathbb{Z}^d, \ d \geq 2 \), we are able to construct such examples).

The construction of Section 4 leads to non-ergodic automorphisms also, even if we start with an ergodic automorphism (such as in Example 3.2).

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