SOME NEW IDENTITIES OF GENERALIZED FIBONACCI AND GENERALIZED PELL NUMBERS VIA A NEW TYPE OF NUMBERS

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Abstract

This paper is concerned with developing some new identities of generalized Fibonacci numbers and generalized Pell numbers. A new class of generalized numbers is introduced for this purpose. The two well-known identities of Sury and Marques which are recently developed are deduced as special cases. Moreover, some other interesting identities involving the celebrated Fibonacci, Lucas, Pell and Pell-Lucas numbers are also deduced.

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1 Introduction

Many number sequences can be generated by recurrence relations of order two, among these numbers are the celebrated Fibonacci, Lucas, Pell and Pell-Lucas numbers. It is well-known that these sequences of numbers have important parts in mathematics. They are of fundamental importance in the fields of combinatorics and number theory, see for example [3,4]. Recently, the studies concerned with various generalizations of Fibonacci and Lucas numbers have attracted a number of authors, see for example [1,2,7,9,10].

The \(n\)-th Fibonacci and Lucas numbers can be generated respectively, with the aid of the two recurrence relations:

\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \ F_1 = 1,
\]

and

\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \ L_1 = 1.
\]

A few terms of Fibonacci sequence are

\[0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,\]

while a few terms of Lucas sequence are

\[2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots.\]

Note that the Lucas numbers \(L_n\) are linked with the Fibonacci numbers \(F_n\) by the relation

\[L_n = F_{n-1} + F_{n+1}, \ n \geq 1.\]
The generation of the two sequences of Fibonacci and Lucas numbers can be unified via the sequence of generalized Fibonacci numbers \((G_n^{a,b})_{n \geq 0}\). These numbers can be generated with the aid of the following recurrence relation:

\[
G_{n+2}^{a,b} = G_{n+1}^{a,b} + G_n^{a,b}, \quad G_0^{a,b} = b - a, G_1^{a,b} = a.
\]  

(1)

We note that the Fibonacci and Lucas sequences are special classes of \((G_n^{a,b})_{n \geq 0}\). Explicitly, we have

\[
F_n = G_{n,1}^{1,1}, \quad L_n = G_{n,1}^{1,3}.
\]

An important identity of \(G_n^{a,b}\) is

\[
G_{n+2}^{a,b} = a F_n + b F_{n+1}.
\]

For properties of Fibonacci, Lucas and generalized Fibonacci numbers, one can be referred to the beautiful book of Koshy [4].

There are other two important sequences of numbers, namely Pell numbers \((P_n)_{n \geq 0}\), and Pell-Lucas numbers \((Q_n)_{n \geq 0}\). These two sequences may be generated by the following two recurrence relations:

\[
P_{n+2} = 2 P_{n+1} + P_n, \quad P_0 = 0, P_1 = 1,
\]

and

\[
Q_{n+2} = 2 Q_{n+1} + Q_n, \quad Q_0 = 2, Q_1 = 2.
\]

A few terms of Pell sequence are

\[0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots,\]

while a few terms of Pell-Lucas sequence are

\[2, 2, 6, 14, 34, 82, 198, 478, 1154, \ldots.\]

Note that the numbers \(Q_n\) are linked with the numbers \(P_n\) by the relation:

\[Q_n = P_{n-1} + P_{n+1}, n \geq 1.\]

Now, we introduce the sequence of generalized Pell numbers, which we denote \((P_n^{a,b})\). This sequence may be generated by the recurrence relation

\[
P_{n+2}^{a,b} = 2 P_{n+1}^{a,b} + P_n^{a,b}, \quad P_0^{a,b} = b - 2a, P_1^{a,b} = a.
\]  

(2)

It is clear that the generalized Pell sequence \((P_n^{a,b})_{n \geq 0}\) generalizes the two sequences of Pell and Pell Lucas numbers. In fact, we have

\[P_n = P_n^{1,2}, \quad Q_n = P_n^{2,6}.
\]

In this letter, we construct a more general sequence of numbers aiming to unify the construction of the two sequences of the generalized Fibonacci numbers \((G_n^{a,b})_{n \geq 0}\) in (1) and the generalized Pell numbers \((P_n^{a,b})_{n \geq 0}\) in (2). For this purpose, we consider the generalized sequence of numbers \((U_n^{a,b,r})\) generated by the recurrence relation:

\[
U_{n+2}^{a,b,r} = r U_{n+1}^{a,b,r} + U_n^{a,b,r}, \quad U_0^{a,b,r} = b - r a, U_1^{a,b,r} = a.
\]  

(3)

It is clear from (3) that the two sequences \((G_n^{a,b})\) and \((P_n^{a,b})\) are particular sequences of the more general sequence \((U_n^{a,b,r})\). In fact, we have

\[C_n^{a,b} = U_n^{a,b,1}, \quad P_n^{a,b} = U_n^{a,b,2}.
\]

Thus, the main advantage of introducing the sequence \((U_n^{a,b,r})\) is that the four sequences of Fibonacci \((F_n)\), Lucas \((L_n)\), Pell \((P_n)\), Pell-Lucas \((Q_n)\) numbers can be deduced as particular cases of the generalized sequence of numbers \((U_n^{a,b,r})\).
2 New identities of some generalized numbers

This section is devoted to presenting some new identities of the three generalized sequences of numbers \((G_n^{a,b}, P_n^{a,b})\) and \((U_n^{a,b,r})\) which introduced in Section 1. We show that one of the presented identities generalize the two identities of Sury and Marques which are recently developed. In addition, some other new identities involving the celebrated Fibonacci, Lucas, Pell and Pell-Lucas numbers are also deduced.

**Theorem 1.** For every nonnegative integer \(m\) and for all \(c \in \mathbb{R} - \{0\}\), the following identity is valid

\[
c^m + 1 U_{m+1}^{a,b,r} = b - r a + \sum_{i=0}^{m} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\}. \quad (4)
\]

**Proof.** We will proceed by induction. It is clear that each of the two sides of (4) is equal to:

\[
(a c) \text{ in case of } m = 0. \text{ Now, assume that (4) is valid, hence to complete the proof, we have to show the validity of the following identity:}
\]

\[
c^m + 2 U_{m+2}^{a,b,r} = b - r a + \sum_{i=0}^{m+1} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\}. \quad (5)
\]

Now, we write the right hand side of (5) in the form

\[
b - r a + \sum_{i=0}^{m+1} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\} =
\]

\[
b - r a + \sum_{i=0}^{m} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\} + c^{m+1} \left[ (r - 1) U_{m+1}^{a,b,r} + (c - 1) U_{m+2}^{a,b,r} + U_m^{a,b,r} \right]. \quad (6)
\]

If we make use of the valid relation (4), then the latter equation can be turned into

\[
b - r a + \sum_{i=0}^{m} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\} =
\]

\[
c^{m+1} U_{m+1}^{a,b,r} + c^{m+1} \left[ (r - 1) U_{m+1}^{a,b,r} + (c - 1) U_{m+2}^{a,b,r} + U_m^{a,b,r} \right]. \quad (7)
\]

Based on the recurrence relation (3), it is easy to see that (7) can be written alternatively as

\[
b - r a + \sum_{i=0}^{m+1} c^i \left\{ (r - 1) U_i^{a,b,r} + (c - 1) U_{i+1}^{a,b,r} + U_{i-1}^{a,b,r} \right\} = c^{m+2} U_{m+2}^{a,b,r}. \quad (8)
\]

This completes the proof of Theorem 1. \qed

The following identity of the generalized Fibonacci numbers \(G_n^{a,b}\) is a direct consequence of identity (4).

**Theorem 2.** For every nonnegative integer \(m\) and for all \(c \in \mathbb{R} - \{0\}\), the following identity holds for generalized Fibonacci numbers

\[
c^{m+1} G_{m+1}^{a,b} = b - a + \sum_{i=0}^{m} c^i \left\{ (c - 1) G_{i+1}^{a,b} + G_{i-1}^{a,b} \right\}. \quad (9)
\]
Proof. Identity (9) follows immediately from identity (4) by setting $r = 1$. \hfill \square

Several important identities involving Fibonacci and Lucas numbers can be deduced as special cases of (9). These identities are stated in the following corollaries.

**Corollary 1.** If we set $a = b = 1$ in (9), then the following identity is obtained:

$$c^{m+1} F_{m+1} = \sum_{i=0}^{m} c^i \left\{ (c - 1) F_{i+1} + F_{i-1} \right\},$$

and in particular, we have the following two important identities:

$$2^{m+1} F_{m+1} = \sum_{i=0}^{m} 2^i L_i,$$

and

$$3^{m+1} F_{m+1} = \sum_{i=0}^{m} 3^i L_i + \sum_{i=0}^{m+1} 3^{i-1} F_i.$$

**Remark 1.** The identity in (11) is in agreement with that obtained in [5,8], while the identity (12) coincides with that obtained in [9].

**Corollary 2.** If we set $a = 1, b = 3$ in (9), then the following identity is obtained:

$$c^{m+1} L_{m+1} = 2 + \sum_{i=0}^{m} c^i \left\{ (c - 1) L_{i+1} + L_{i-1} \right\}.$$

**Corollary 3.** If we set $c = 1$ in (9), then the following identity holds for generalized Fibonacci numbers:

$$G_{m+1}^{a,b} = b - a + \sum_{i=0}^{m} G_{i-1}^{a,b}.$$

Now, we state an identity involving the generalized Pell numbers $P_n^{a,b}$.

**Theorem 3.** For every nonnegative integer $m$ and for all $c \in \mathbb{R} - \{0\}$, the following identity holds for generalized Pell numbers

$$c^{m+1} P_{m+1}^{a,b} = b - 2a + \sum_{i=0}^{m} c^i \left\{ P_i^{a,b} + (c - 1) P_{i+1}^{a,b} + P_{i-1}^{a,b} \right\}.$$  \hfill (15)

**Proof.** Identity (15) follows immediately from identity (4) by setting $r = 2$. \hfill \square

The following identities follow from identity (15) for particular choices of its parameters.

**Corollary 4.** If we set $a = 1, b = 2$ in (15), then the following identity is obtained:

$$c^{m+1} P_{m+1} = \sum_{i=0}^{m} c^i \left\{ P_i + (c - 2) P_{i+1} + Q_i \right\},$$

and in particular, we have

$$2^{m+1} P_{m+1} = \sum_{i=0}^{m} 2^i \left\{ P_i + Q_i \right\}.$$
Corollary 5. If we set $a = 2, b = 6$ in (15), then the following identity is obtained:

$$c^{m+1}Q_{m+1} = 2 + \sum_{i=0}^{m} c^i \left\{ Q_i + (c - 1) Q_{i+1} + Q_{i-1} \right\}. \quad (18)$$

Corollary 6. If we set $c = 1$ in (15), then the following identity holds for generalized Pell numbers

$$P^{a,b}_{m+1} = b - 2a + \sum_{i=0}^{m} \left\{ P^{a,b}_{i} + P^{a,b}_{i-1} \right\}. \quad (19)$$

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