Pole inflation in dRGT theory

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Abstract

Although the dRGT theory successfully explains the late-time cosmic acceleration, it cannot justify inflation. On the other hand, and in the framework of General Relativity and modified gravity, the interests and attempts to describe dark energy and inflation by using Lagrangians including poles, called pole dark energy, and pole inflation, respectively, have recently been enhanced. Subsequently, we are going to show that a pole Lagrangian may justify inflation in the framework of dRGT theory. The study is done focusing on power and exponential potentials, and the results show a plausible consistency with the Planck 2015 data. All power potential results lie within the 95% CL region of the Planck 2015 TT,TE,EE+lowP data. Furthermore, depending on the value of the order of pole, the exponential potential results also lie within the 95% CL region of the Planck 2015 TT,lowP data.

I. INTRODUCTION

Despite its enormous successes, General Relativity (GR) cannot be considered as the final theory of gravity, and needs to be modified [1]. Two of the most important motivations for modifying GR are the explanation of inflation, and the late-time cosmic acceleration [2]. Massive gravity theory is an approach to modify GR which includes a graviton of small mass [3]. In 1939, Fierz and Pauli (FP) [4] proposed massive gravity theory for the first time. By adding the fine-tuned mass term to the linearized Einstein-Hilbert action, they developed a theory that correctly leads to 5 degrees of freedom for the massive spin-2 particle [1][5]. In 1970, van Dam, Veltman [6], and Zakharov (vDVZ) [7] discovered that, in the massless limit, the FP theory does not render GR [5]. In other words, the principle of conformity is violated leading to a deviation in the gravitational lensing prediction of theory about 25% compared to that of GR [8][9]. This discontinuity is known as the vDVZ discontinuity [8].

Focusing on the nonlinear extensions of the FP theory, Vainshtein [10] found out a distance scale, called the Vainshtein radius $r_V = (\frac{M}{M_{Pl} m_g})^{\frac{1}{2}}$, where $M_{Pl}$ is the Planck mass, and $m_g$ denotes the graviton mass [11][13]. For $r < r_V$, the nonlinear effects become highly important [11][13] meaning that the outcomes differ from those predicted by FP. As the graviton mass becomes less, the $r_V$ is boosted and at the massless limit, $r_V$ tends to infinity. Consequently, the results of the FP theory are no longer valid [11][13]. So, it seems possible to rectify the vDVZ discontinuity by considering the nonlinear effects [11][13], an approach called the Vainshtein mechanism [13].

According to the Vainshtein’s idea, Boulware and Deser (BD) [14] find that all nonlinear extensions of the FP theory have an additional degree of freedom (a ghost-like scalar mode), known as the BD ghost [15][16]. The discovery of the late-time cosmic acceleration in 1998, and the unknown nature of cosmological constant led scientists to investigate modified gravity theories, such as massive gravity theory, more precisely. It is also realized that a non-renormalizable theory, such as the FP theory and even one with apparent instabilities, can be understood as an effective field theory, valid only at energies below an ultraviolet cutoff scale [17].

In 2003, Arkani-Hamed, Georgi, and Schwartz [18] could return the gauge invariance to massive gravity theory by employing the Stuckelberg trick leading to develop an effective field theory for massive gravity. The Stuckelberg trick also helps us solve vDVZ discontinuity. In the massless limit, the effects of the strong decoupling neutralize the ghost-like scalar mode, and consequently the results of GR are reproduced. Moreover, in certain nonlinear extensions of FP, the cutoff scale $\Lambda_5 = (M_{Pl} m_g^2)^{1/5}$ of FP can also be raised to $\Lambda_A = (M_{Pl} m_g^2)^{1/3}$. De Rham, Gabadadze, and Tolley [20] could propose a covariant nonlinear theory of massive gravity that correctly describes the massive spin-2 field; the birth of dRGT theory. The dRGT theory is indeed ghost-free in the decoupling limit to all orders.

Although dRGT theory does not have flat, and closed FRW solutions, the open FRW solution yields an effective cosmological constant proportional to the graviton mass ($m_g$) [21]. In the report of Ligo-Virgo Collaboration about the discovery of the first gravitational wave, the constraint $m_g < 1.2 \times 10^{-22}$eV is determined [22]. If $m_g \sim H_0 \sim 10^{-33}$eV, then the effective cosmological constant can explain the late-time cosmic acceleration [23]. Since the graviton mass cannot be of the order of the Hubble constant during the inflation, dRGT theory is not capable to produce enough number of e-foldings [24].
Recently, the interests to employ a new type of Lagrangian, the pole Lagrangian, have been attracted in order to describe the universe expansion history [25–28]. This kind of Lagrangian even supports evolving traversable wormholes satisfying energy conditions in the framework of GR [29]. Motivated by the power of pole Lagrangian as well the weakness of dRGT theory in describing inflation, we combine the dRGT theory with pole Lagrangian to build a pole inflation.

The action of the scalar field \(\sigma\) with a pole of order \(p\) and residue \(a_p\) (in the kinetic term) is written as

\[
S_\sigma = \int d^4x \sqrt{-g} \left[ -\frac{a_p}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right],
\]

where \(V(\sigma)\) denotes the potential and the pole can reside at \(\sigma = 0\) [25–28]. The case \(p = 2\), and \(V(\sigma) = 0\) corresponds to minimal \(\kappa\)-essence model [29]. The kinetic term can also be brought into canonical form

\[
S_\varphi = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right],
\]

using the transformations

\[
\varphi = \begin{cases} 
\frac{2\sqrt{|\sigma|}}{|2-p|} \frac{(2-p)}{2} \sigma & \text{for } p \neq 2 \\
\pm \sqrt{|\sigma|} \log \sigma & \text{for } p = 2
\end{cases} \tag{3}
\]

As the field does not cross zero, due to the pole, the branch \(\sigma \geq 0\) is employable [25–28], and additionally, we consider \(a_p > 0\) since \(\sqrt{|\sigma|}\) should be real.

The paper is organized as follows. In section II, we briefly review the dRGT theory. In section III, we first find open FRW solutions. Then, the number of e-foldings \(N\), scalar spectral index \(n_s\), and tensor-to-scalar ratio \(r\) necessary to compare with observational data, are obtained. Our goal in the fourth section is also to show the power of model in being in line with observations by employing the Planck 2015 data. A summary is finally presented in the last section.

II. THE DRGT THEORY

The dRGT theory is described by the physical metric \(g_{\mu\nu}\), and the non-dynamical (or fludical) metric \(f_{\mu\nu}\) so that

\[
g_{\mu\nu} = f_{\mu\nu} + H_{\mu\nu}. \tag{4}
\]

Here, \(H_{\mu\nu}\) denotes the covariantization of the metric perturbation \((h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu})\), and the fludical metric is defined as

\[
f_{\mu\nu} = f_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \tag{5}
\]

where \(f_{\mu\nu}\) is the reference metric, and \(\phi^a (a = 0, 1, 2, 3)\) are the four fields introduced to restore general covariance. For a Minkowskian reference metric, \(\phi^a\) form a 4-vector, implying that the dRGT theory carries the global Poincare symmetry [30].

The gravitational action consists of the Einstein-Hilbert term, and the mass term

\[
S_g = S_{EH} + S_{mass},
\]

\[
S_g = M_{Pl}^2 \int d^4x \sqrt{-g} \left( \frac{R}{2} + m^2 \mathcal{U}(g_{\mu\nu}, H_{\mu\nu}) \right) \tag{6}
\]

in which \(R = g^{\mu\nu} R_{\mu\nu}\) is the Ricci scalar, and \(\mathcal{U}\) is the potential without derivatives of terms originated from the interaction between \(H_{\mu\nu}\) and \(g_{\mu\nu}\) [23]. To prevent the appearance of the BD ghost, \(\mathcal{U}\) is constructed as

\[
\mathcal{U}(g_{\mu\nu}, H_{\mu\nu}) = \frac{4}{3} \sum \alpha_n \mathcal{L}_n. \tag{7}
\]

In this formula, \(\alpha_n\) are the dimensionless parameters, and

\[
\mathcal{L}_0 = 1, \quad \mathcal{L}_1 = [K], \quad \mathcal{L}_2 = \frac{1}{2} \left( |K|^2 - |\mathcal{K}|^2 \right), \quad \mathcal{L}_3 = \frac{1}{6} \left( |K|^3 - 3 |K| |\mathcal{K}|^2 + 2 |\mathcal{K}|^3 \right), \\
\mathcal{L}_4 = \frac{1}{24} \left( |K|^4 - 6 |K|^2 |\mathcal{K}|^2 + 3 |\mathcal{K}|^2 |\mathcal{K}|^2 + 8 |K| |\mathcal{K}|^3 - 6 |\mathcal{K}|^4 \right) \tag{8}
\]

where \(K_{\mu\nu}\) is defined as

\[
K_{\mu\nu} = \delta_{\mu}^\lambda - \sqrt{g^{|\lambda|} f_{\lambda\mu},} \tag{9}
\]

and the square brackets denote trace operation i.e.,

\[
[K] = K_{\mu\mu}, \quad [K^2] = K_{\mu\nu} K_{\nu\mu}, \\
[K^3] = K_{\mu\nu} K_{\alpha\beta} K_{\mu\alpha} K_{\nu\beta}, \quad [K^4] = K_{\mu\nu} K_{\alpha\beta} K_{\gamma\delta} K_{\rho\sigma}. \tag{10}
\]

\(g_{\mu\nu}\) raises, and lowers the indices, and moreover, \(\mathcal{L}_0\) corresponds to a cosmological constant, \(\mathcal{L}_1\) and \(\mathcal{L}_2\) to a tadpole, and the mass term, respectively. \(\mathcal{L}_3, 4\) also include higher-order interaction terms [10]. Setting \(\alpha_0 = \alpha_1 = 0\) Minkowskian spacetime becomes a vacuum solution [10], and in addition, if \(\alpha_2 = 1\), then the FP theory, at a linearized level, is recovered [31]. \(\alpha_3\) and \(\alpha_4\) are the free parameters of the theory. Accordingly, the gravitational action (6) is finally rewritten as

\[
S_g = M_{Pl}^2 \int d^4x \sqrt{-g} \left( \frac{R}{2} + m^2 \left( \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 \right) \right). \tag{11}
\]
III. BACKGROUND COSMOLOGY

The total action consists of three parts including i) the gravitational action (11), ii) the pole Lagrangian (1), and iii) the matter action $S_m$ so that

$$S_{tot} = S_g + S_\sigma + S_m,$$

where $S_m$ corresponds to a perfect fluid, described by the energy-momentum tensor

$$T^\mu_\nu = diag(-\rho_m, p_m, p_m, p_m),$$

in which $\rho_m$, and $p_m$ denote the energy density, and the pressure of matter, respectively.

Now, assuming the Universe is homogeneous and isotropic, the physical metric $g_{\mu\nu}$ is chosen as the Friedmann- Robertson-Walker (FRW) metric

$$g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2(dt)^2 + a(t)^2 \Omega_{ij} dx^i dx^j.$$  \hspace{1cm} (14)

Here, $N$ is the cosmological time, $a$ is the scale factor, and $\Omega_{ij}$ denotes the metric of unit 3-sphere

$$\Omega_{ij} = \delta_{ij} + \frac{k}{1-kr^2} x^i x^j,$$

where $k$ is the spatial curvature, and $r^2 = x^2 + y^2 + z^2$. In this paper, the Minkowskian form of the reference metric is chosen as

$$\tilde{f}_{\mu\nu} = \eta_{\mu\nu} = diag(-1,1,1,1).$$

As demonstrated in Ref [16], the flat FRW solution of the dRGT theory is equal to $a = \text{const}$, inconsistent with a dynamic Cosmos. Furthermore, the dRGT theory lacks closed FRW solutions, since the fiducial Minkowski metric cannot be foliated by closed slices [21]. Therefore, we only focus on open FRW of negative $k$ very close to zero, in accordance with the Planck collaboration prescription [22]. In order to carry the symmetries of an open FRW metric by the fiducial metric, the Stückelberg fields should be chosen as

$$\phi^0 = f(t) \sqrt{1-kr^2}; \quad \phi^i = \sqrt{-kf(t)} x^i;$$

leading to

$$f_{\mu\nu} dx^\mu dx^\nu = -(\dot{f}(t))^2(dt)^2 - kf(t)^2 \Omega_{ij} dx^i dx^j,$$

where $\dot{f} = \frac{df(t)}{dt}$. Without loss of generality, we also assume $a > 0$, $N > 0$, $f \geq 0$, $\dot{f} \geq 0$.

A. Equations of motion

Inserting the physical metric (14), and the fluidic metric (18) in Eq. (12), the total action is obtained as

$$S_{tot} = \int d^4x \left[ 3M_p^2 \left[ -\frac{\dot{a}^2}{N} + kNa + m_n^2 (NF(a,f) - \dot{f}G(a,f)) \right] + \frac{a^3}{2N} (\frac{a}{\sigma})^2 - NV(\sigma) \right] + S_m,$$

in which

$$F(a,f) = a(a-\sqrt{-k}) (2a-\sqrt{-k}) + \frac{\alpha_3}{3}(a-\sqrt{-k})^2 (4a-\sqrt{-k}) + \frac{\alpha_4}{3}(a-\sqrt{-k})^3,$$

and

$$G(a,f) = a^2(a-\sqrt{-k}) + \alpha_3 a (a-\sqrt{-k})^2 + \frac{\alpha_4}{3}(a-\sqrt{-k})^3.$$\hspace{1cm} (21)

Now, variation of the total action (19) with respect to the field variables $f$, $N$, $a$, and $\sigma$ renders

$$\delta f |_{N=1} : (\dot{a} - \sqrt{-k}) \frac{\partial G(a,f)}{\partial a} = 0,$$

$$\delta N |_{N=1} : 3M_p^2 (H^2 + \frac{k}{a^2}) = \rho_MG + \rho_\sigma + \rho_m,$$

$$\delta a |_{N=1} : -M_p^2 (2\dot{H} + 3H^2 + \frac{k}{a^2}) = \rho_MG + p_\sigma + p_m,$$

$$\delta \sigma |_{N=1} : -\frac{1}{2\sigma^a + 1} \frac{a \rho M^2}{a^2} (\frac{\dot{\sigma}}{\sigma^a} + 3\dot{H}\sigma + V'(\sigma)) = 0,$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, and $V'(\sigma) = \frac{dV(\sigma)}{d\sigma}$. In Eqs. (22)-(25), the effective energy density, and pressure of dRGT theory are given by

$$\rho_MG = -3M_p^2 m_n^2 \frac{F(a,f)}{a^3},$$

and

$$p_MG = 3M_p^2 m_n^2 \frac{F(a,f)}{a^4} + (\frac{f-\beta}{\sigma}) \frac{\partial G(a,f)}{\partial a},$$

respectively. Also, we have

$$\rho_\sigma = \frac{1}{2} \frac{\alpha_3}{\sigma^a} \frac{\dot{\sigma}}{\sigma^a} + V(\sigma),$$

$$p_\sigma = \frac{1}{2} \frac{\alpha_4}{\sigma^a} \frac{\dot{\sigma}}{\sigma^a} - V(\sigma),$$
for the energy density and pressure of \( \sigma \). If the pole action is considered as Eq. (2), then the equation of motion of \( \varphi \) yields

\[
\delta \phi_{\text{tot}}|_{N=1} : \ddot{\varphi} + 3H \dot{\varphi} + \frac{dV(\varphi)}{d\varphi} = 0. 
\]

(30)

In this case, the energy density and pressure of \( \varphi \) are also given by

\[
\rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \quad (31)
\]

and

\[
p_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi), \quad (32)
\]

respectively.

Now, we examine the equations of motion (22)-(25). Eq. (22) has three solutions. For one of these solutions, one obtains \( \ddot{a} = -\sqrt{\alpha} k \) which leads to \( a = \sqrt{-k t + \text{const}} \) that signal us to an open FRW universe as the physical metric \( g_{\mu
u} \). Similar to flat FRW solution, this solution is unacceptable. The other two solutions are obtained from \( \frac{\partial G(\alpha, \beta)}{\partial \alpha} = 0 \) which leads to

\[
\beta_{\pm} = \frac{1 + 2 \alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_4^2 - \alpha_4}}{\alpha_3 + \alpha_4}, 
\]

(33)

and

\[
a = \frac{\sqrt{-f}}{\beta_{\pm}}. 
\]

(34)

Now, assuming \( a \) is positive, one finds that, according to Eq. (34), \( \beta_{\pm} \) should also be positive. For the negative scale factor, Eq. (34) becomes \( a = -\frac{\sqrt{-f}}{\beta_{\pm}} \). So, \( \beta_{\pm} \) is always positive. If \( f(t) \propto a(t) \), then Eqs. (26)-(29) give us

\[
\rho_{\text{MG}} = -\rho_{\text{MG}} = M^2_P \Lambda_{\pm} = M^2_P a^2(\beta_{\pm} - 1)[(3 - \beta_{\pm}) + \alpha_3(1 - \beta_{\pm})],
\]

(35)

where \( \Lambda_{\pm} \) denotes the effective cosmological constant of the theory. Although \( \alpha_3 \) and \( \alpha_4 \) are free parameters, Eq. (33) renders some constraints. Moreover, if \( \alpha_3 = -\alpha_4(\pm(1 + \alpha_3) > 0) \), then the cosmological constant \( \Lambda_{\pm} \) becomes infinite. On the other hand, \( \alpha_4 = \frac{1}{4} (1 + \alpha_3) > 0 \) leads to \( \Lambda_{\pm} = 0 \). Finally, the values of free parameters that lead to negative radicand in Eq. (33) are not acceptable (see Ref. [21]).

B. Inflation

In order to provide a general outlook on the possibility of inflation in the dRGT theory, we derive the corresponding formulation considering the general potential \( V(\sigma) \). Moreover, a slow-roll regime is taken into consideration for which the scale factor has an exponential behavior \( (a \sim e^{Ht} (H \sim \text{const})) \), and grows rapidly. Therefore, after a few number of e-foldings, the term \( \frac{k}{a^2} \) becomes ignorable. Hence, Eq. (23) can be rewritten as

\[
3M^2_P H^2 = \frac{1}{2M^2_P} \frac{\dot{\sigma}^2}{\sigma} + V(\sigma) + M^2_P \Lambda_{\pm}.
\]

(36)

Moreover, combining Eq. (23) with Eq. (24), one reaches at

\[
\dot{H} = -\frac{1}{2M^2_P} \frac{\dot{\sigma}^2}{\sigma}.
\]

(37)

The Hubble slow-roll parameters are also defined as

\[
\varepsilon_H = -\frac{\dot{H}}{H^2}, \quad (38)
\]

and

\[
\eta_H = -\frac{\ddot{\varphi}}{H \dot{\varphi}}. \quad (39)
\]

In this regard, as the slow-roll conditions, we have \( \varepsilon_H < 1 \), and \( |\eta_H| < 1 \), while the first condition leads to

\[
\frac{\dot{\varphi}}{\varphi} < \frac{a}{|\sigma|^2} \lesssim V(\sigma) + M^2_P \Lambda_{\pm},
\]

(40)

meaning that Eq. (36) can be recast as

\[
H^2 \approx \frac{V(\sigma) + M^2_P \Lambda_{\pm}}{3M^2_P}. \quad (41)
\]

Additionally, for the relation between the first time-derivatives of \( \sigma \) and \( \varphi \), we have

\[
\dot{\varphi} = \begin{cases} 
\frac{-\sqrt{|\sigma|} a p}{|p|} \sigma^{-p} & \text{for } p \geq 2 \\
\frac{\sqrt{|\sigma|} a p}{|p|} \sigma^{-p} & \text{for } p \leq 2
\end{cases}
\]

(42)

While on the other hand, the second time-derivatives of \( \sigma \) and \( \varphi \) address us to

\[
\ddot{\varphi} = \begin{cases} 
\frac{-\sqrt{|\sigma|} a p}{|p|} (\sigma - \frac{p}{2} \sigma^{-1} a^2) & \text{for } p \geq 2 \\
\frac{\sqrt{|\sigma|} a p}{|p|} (\sigma - \frac{p}{2} \sigma^{-1} a^2) & \text{for } p \leq 2
\end{cases}
\]

(43)

Now, inserting \( \dot{\varphi} \) and \( \ddot{\varphi} \) in Eq. (39), \( \eta_H \) is obtained as

\[
\eta_H = -\frac{\ddot{\varphi}}{H \dot{\varphi}}.
\]

(44)

Using \( |\eta_H| < 1 \), Eq. (25) is approximately equal to
Combining Eqs. (37), (38), and (41) with each other, we get

\[ \varepsilon_H = \frac{1}{2M_{Pl}^2 \sigma_P H} \approx \frac{M_{Pl}^2 \sigma_P}{2} \left( \frac{V'(\sigma)}{V(\sigma) + M_{Pl}^2 \Lambda_{\pm}} \right)^2. \]  

and thus

\[ 3H \dot{\sigma} + 3H \sigma \approx -\frac{\sigma_P}{a_P} V'(\sigma). \]  

where \( V''(\sigma) = \frac{d^2V(\sigma)}{d\sigma^2} \). Finally, it is straightforward to achieve

\[ \eta_H \approx M_{Pl}^2 \frac{\sigma_P}{a_P} \left( \frac{V''(\sigma) + \frac{\sigma}{2} \sigma^{-1} V'(\sigma)}{V(\sigma) + M_{Pl}^2 \Lambda_{\pm}} \right) + \frac{H}{H^2}. \]  

The best method for investigating a specific potential is the calculation of slow-role parameters including \( \varepsilon_V \) and \( \eta_V \) combined with \( \varepsilon_V \approx \varepsilon_H \), and \( \eta_V \approx \eta_H + \epsilon_H \) to reach

\[ \varepsilon_V = \frac{M_{Pl}^2 \sigma_P}{2} \left( \frac{V'(\sigma)}{V(\sigma) + M_{Pl}^2 \Lambda_{\pm}} \right)^2, \]  

and

\[ \eta_V = M_{Pl}^2 \frac{\sigma_P}{a_P} \left( \frac{V''(\sigma) + \frac{\sigma}{2} \sigma^{-1} V'(\sigma)}{V(\sigma) + M_{Pl}^2 \Lambda_{\pm}} \right). \]  

In addition, the number of e-foldings \( N \) is also obtained as

\[ N = \int_{\Phi_f}^{\Phi_i} \frac{d\sigma}{M_{Pl} \sqrt{2\epsilon_V}} = \frac{a_P}{M_{Pl}} \int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p} \left| \frac{V(\sigma) + M_{Pl}^2 \Lambda_{\pm}}{V'(\sigma)} \right|, \]  

and the scalar spectral index \( n_s \), and tensor-to-scalar ratio \( r \) are finally calculated as

\[ n_s = 1 - 6\varepsilon_V + 2\eta_V, \]  

and

\[ r = 16\epsilon_V, \]  

respectively. Here, \( r \) is a significant observational quantity, and it is used to distinguish different inflationary models.

\section{IV. NUMERICAL DYNAMICS}

\subsection{A. The Power Potential}

First, let us consider a power potential as \( V(\sigma) = V_0 \sigma^n \), where \( V_0 \) is a positive constant with dimension (mass)^{3-n}, and \( n \) is a dimensionless constant. Inserting \( V_0 \sigma^n \) in Eq. (49), and using Eq. (50), \( \varepsilon_V \) and \( \eta_V \) are obtained as

\[ \varepsilon_V = \frac{M_{Pl}^2 \sigma_P}{2} \frac{n}{a_P} \frac{n-1}{\sigma^n + c}, \]  

and

\[ \eta_V = \frac{M_{Pl}^2 \sigma_P}{2} \left( \frac{n(2n + p - 2)\sigma^{n-2}}{\sigma^n + c} \right), \]  

respectively, where \( c \) is defined as \( c = \frac{M_{Pl}^2 \Lambda_{\pm}}{V_0} \). Since \( M_{Pl}^2 \Lambda_{\pm} \) has dimension (mass)^4, the dimension of \( c \) is equal to (mass)^n. As inflation ends when \( \varepsilon_V = \eta_V = 1 \), \( a_P \) and \( \sigma_f \) are achieved as

\[ \sigma_f = \left( 2n+p+2, \frac{1}{n-p+2} \right), \]  

and

\[ a_P = \frac{M_{Pl}^2}{2} \left( \frac{c}{-n-p+2} \right)^{\frac{p-2}{n}} \left( 2n + p - 2 \right)^2 \frac{2n+p-2}{n}, \]  

respectively. The pole Lagrangian, according to Eqs. (54) and (55), makes the condition of \( \varepsilon_V = \eta_V = 1 \) is satisfied at the end of inflation for all \( n \), whereas for the scalar Lagrangian, equivalent to fix \( a_P = 1 \) and \( p = 0 \), this condition is valid only for \( n = 2 \).

Now, combining Eqs. (56), (57) with Eq. (3), one finds

\[ \varphi_f = \begin{cases} \frac{M_{Pl}}{2-p} \sqrt{2(2n+p-2)} & \text{for } p \neq 2 \\ \pm \frac{M_{Pl}}{\sqrt{2}} (2n) \log (-2c) \frac{1}{\pi} & \text{for } p = 2 \end{cases}, \]  

Since the logarithm of negative numbers is not defined, \( p = 2 \) is not acceptable here. Moreover, considering the conditions \( \sigma_f \geq 0 \) (we take the branch \( \sigma \geq 0 \)), and \( a_P > 0 \), the allowed range for \( p \) is obtained as

\[ \begin{cases} -2n+2 \leq p < -n+2 & \text{for } n > 0 \\ -n+2 \leq p < -2n+2 & \text{for } n < 0 \end{cases} \]  

Note that \( p = -n+2 \) is not acceptable, because it leads to \( a_P = 0 \). Inserting \( V(\sigma) = V_0 \sigma^n \) in Eq. (51), the number of e-foldings \( N \) are
where

\[
\int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p+1} = \begin{cases} 
\frac{1}{-p+2} (\sigma_i^{-p+2} - \sigma_f^{-p+2}) & \text{for } p \neq 2 \\
\ln \left( \frac{\sigma_i}{\sigma_f} \right) & \text{for } p = 2 
\end{cases}
\]

and

\[
\int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p-n+1} = \begin{cases} 
\frac{1}{p-n+2} (\sigma_i^{-p-n+2} - \sigma_f^{-p-n+2}) & \text{for } p + n \neq 2 \\
\ln \left( \frac{\sigma_i}{\sigma_f} \right) & \text{for } p + n = 2 
\end{cases}
\]

Now, equipped with Eqs. (52), (53), (54), and (55), the scalar spectral index \( n_s \), and tensor-to-scalar ratio \( r \) are obtained as

\[
n_s = 1 + nM_{Pl}^2 \frac{\sigma_p}{\sigma_0} \frac{(-n + p - 2)\sigma^2n - 2 + (2n + p - 2)\sigma n - 2}{(\sigma^2 + \sigma)^2},
\]

and

\[
r = 8M_{Pl}^2 \frac{\sigma_p}{\sigma_0} \frac{n\sigma n - 1}{(\sigma^2 + \sigma)}.
\]

Finally, the slow-roll parameters are written as the functions of \( \varphi \) in the form of

\[
\varepsilon_V = \frac{M_{Pl}^2}{2} n_s^2 \times 
\left( \frac{p - 2}{2p} \right)^{\frac{2(p + 2n - 2)}{2p - p}} \left( \frac{M_{Pl}^2}{2p} \right)^{\frac{2n}{2p}} \left( -n - p + 2 \right) \left( 2n + p - 2 \right) \left( \frac{2n - p - 2}{2p} \right)^{\frac{2n - p - 2}{2p}}.
\]

and

\[
\eta_V = \frac{M_{Pl}^2}{2} \times 
\left( \frac{p - 2}{2p} \right)^{\frac{2n}{2p}} \left( \frac{M_{Pl}^2}{2p} \right)^{\frac{2n}{2p}} \left( -n - p + 2 \right) \left( 2n + p - 2 \right) \left( \frac{2n + p - 2}{2p} \right)^{\frac{2n + p - 2}{2p}}.
\]

In Figs. (1) and (2), \( r(n_s) \) and \( N(n_s) \) are shown for \( V(\sigma) = V_0 \sigma^n \) (\( n = 2 \)) and several values of \( p \), respectively. \(-2 \leq p < 0\) is the range of \( p \) for which desired results are achievable. The results depicted in Figs. (1) and (2) are repeatable for other values of \( n \), depending on the values of \( p \). In this regard, two examples are given in table. (I).

**FIG. 1.** Plot of the tensor-to-scalar ratio \( r \) versus the scalar spectral index \( n_s \) for the power potential \( V(\sigma) = V_0 \sigma^2 \) (\( n = 2 \)). Gray, red, and blue regions are related to the Planck 2013, Planck 2015 TT+lowP, and Planck 2015 TT,TE, EE+lowP data, respectively [32]. In this figure, the 68% CL regions are distinguished from the 95% CL regions by highlighting the corresponding regions. The results lie within the 95% CL region of the Planck 2015 TT,TE,EE+lowP data.

**FIG. 2.** The number of e-foldings \( N \) versus the scalar spectral index \( n_s \) for the power potential \( V(\sigma) = V_0 \sigma^2 \) (\( n = 2 \)).

| Power \( n \) | Allowed range of \( p \) |
|-------------|---------------------|
| -1         | \(-6 \leq p < -2\) |
| -4         | \(-3 < p \leq 4\) |
| 4          | \(-6 \leq p < -2\) |

**TABLE I.** Different values of \( n \) and their corresponding intervals of \( p \) generating the same results as those of Figs. (1) and (2).

**B. The Exponential Potential**

In this subsection, we study the inflationary era generated by the exponential potential \( V(\sigma) = V_0 e^{-\gamma \sigma} \) where \( \gamma \) is a constant with dimension \((\text{mass})^{-1}\). Inserting \( V_0 e^{-\gamma \sigma} \) in Eq. (49), and Eq. (50), the slow roll parameters are obtained as
At the point of $\varepsilon_V = \eta_V = 1$, inflation is ended, a fact helps us find out

$$
\varepsilon_V = \frac{M^2_{pl}}{2} \left(\frac{\gamma}{a_p} + \frac{1}{1 + ce^{\gamma \sigma}}\right)^2, \hspace{1cm} (68)
$$

and

$$
\eta_V = \frac{M^2_{pl}}{2} \frac{\gamma (\gamma - \frac{p}{2} \sigma^{-1})}{1 + ce^{\gamma \sigma}}. \hspace{1cm} (69)
$$

At the point of $\varepsilon_V = \eta_V = 1$, our investigations indicate that the second state does not lead to proper $N$, and $r$. Hence, $p > 0$, and $\gamma > 0$ is the only state studied here. Inserting $V_0 e^{-\gamma \sigma}$ in Eq. (51), the number of e-foldings ($N$) are obtained as

$$
N = \frac{a_p}{|\gamma| M^2_{pl}} \int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p}(1 + ce^{\gamma \sigma}), \hspace{1cm} (72)
$$

in which

$$
\int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p} = \begin{cases} 
\frac{1}{-p+1} \left(\sigma_i^{-p+1} - \sigma_f^{-p+1}\right) & \text{for } p \neq 1 \\
\ln\left(\frac{\sigma_i}{\sigma_f}\right) & \text{for } p = 1
\end{cases} \hspace{1cm} (73)
$$

and

$$
\int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-p} e^{\gamma \sigma} = \frac{1}{\gamma} \left(\sigma_i^{-p+\gamma} - \sigma_f^{-p+\gamma}\right) + \frac{p}{\gamma} \int_{\sigma_f}^{\sigma_i} d\sigma \sigma^{-(p+1)} e^{\gamma \sigma}. \hspace{1cm} (74)
$$

Now, combining Eqs. (52), (53), and (67) with Eq. (68), the scalar spectral index $n_s$, and tensor-to-scalar ratio $r$ are calculated as

$$
n_s = 1 + M^2_{pl} \frac{a_p}{\sigma_f} \left[\frac{-\gamma^2 - p \gamma \sigma - 1 + c (2 \gamma^2 - p \gamma^{-1}) e^{\gamma \sigma}}{(1 + ce^{\gamma \sigma})^2}\right], \hspace{1cm} (75)
$$

and

$$
r = 8 M^2_{pl} \frac{a_p}{\sigma_f} \left(\frac{\gamma}{1 + ce^{\gamma \sigma}}\right)^2. \hspace{1cm} (76)
$$

FIG. 3. $r$ versus $n_s$ for the exponential potential $V(\sigma) = V_0 e^{-\gamma \sigma}$ when $c = 1$ and $\gamma = -1$. Gray, red, and blue regions are related to the Planck 2013, Planck 2015 TT+lowP, and Planck 2015 TT, TE, EE+lowP data, respectively [32]. In this figure, the 68% CL regions are highlighted compared to the 95% CL regions. The results lie within the 95% CL region of the Planck 2015 TT,lowP data, depending on the values of $p$.

FIG. 4. $N$ versus $n_s$ for the exponential potential $V(\sigma) = V_0 e^{-\gamma \sigma}$ when $c = 1$ and $\gamma = -1$. As it is obvious from the result, the curves get away from data.

V. SUMMARY AND DISCUSSIONS

dRGT theory cannot provide a unique source for both dark energy and inflation. Motivated by this shortcoming of dRGT theory, and also, the successes of pole Lagrangian in describing the Universe [25–29], we addressed some pole inflation scenarios in the framework of dRGT theory. Throughout our analysis, we only focused on power and exponential potentials, both of which can satisfactorily produce $50 - 70$ number of e-foldings, required to solve the flatness problem. Also, comparing the results with the Planck 2015 data, it has been obtained that the established model provide acceptable outcomes, although this results do not lie within the 68% CL of the Planck 2015 data.
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