Renormalization group flows between Gaussian fixed points

Diego Buccio and Roberto Percacci

SISSA, International School for Advanced Studies,
via Bonomea 265, 34136 Trieste, Italy
INFN, Sezione di Trieste,
Trieste, Italy

E-mail: dbuccio@sissa.it, percacci@sissa.it

ABSTRACT: A scalar theory can have many Gaussian (free) fixed points, corresponding to Lagrangians of the form $\phi^p \phi$. We use the non-perturbative RG to study examples of flows between such fixed points. We show that the anomalous dimension changes continuously in such a way that at the endpoints the fields have the correct dimensions of the respective free theories. These models exhibit various pathologies, but are nonetheless interesting as examples of theories that are asymptotically free both in the infrared and in the ultraviolet. Furthermore, they illustrate the fact that a diverging coupling can actually correspond to a free theory.

KEYWORDS: Renormalization Group, Scale and Conformal Symmetries

ArXiv ePrint: 2207.10596
1 Introduction

Perturbative methods are powerful tools to study the properties of quantum or statistical field theories in the neighborhood of a fixed point (FP) but they do not say much about the global properties of the theory space. For example, one would like to know which FP can be joined by an RG trajectory to another FP. Such questions can sometimes be answered, for example by the c-theorem in two dimensions or the a-theorem in four. Another possibility is to simply solve the RG equations. This is impossible in the full theory space, but it can be done within approximations. For example, in the $\mathbb{Z}_2$-invariant scalar theory in three
dimensions, one can find trajectories that join the (free) Gaussian FP in the UV to the Wilson-Fisher (WF) FP in the IR.

There are cases that would seem more trivial, but that are harder to visualize. For example, remaining in the context of $\mathbb{Z}_2$-invariant scalar theory in four dimensions, one can think of infinitely many Gaussian FP’s corresponding to the Lagrangians $\phi \square^p \phi$. We will refer to them as GFP$_n$. Most of these do not correspond to unitary theories in Minkowski space but they still make sense in Euclidean space as statistical models. Each of them can be viewed as sitting in the origin of theory space, but then all the others are nowhere to be seen. Which GFP we see is related to which GFP we take as the starting point of a perturbative expansion, and hence to the canonical dimension of the field. For example (in four dimensions) GFP$_1$ is in the origin of a theory space for a field of canonical dimension one, GFP$_2$ in the origin of theory space when the field is dimensionless, and so on. In this way it would almost seem that for each choice of field dimension we have a different theory space, and that these spaces are unrelated to each other. There is some physical basis for this point of view, because different GFP’s have different numbers of propagating degrees of freedom. One could view the theory space where a given GFP is in the origin as describing the interactions of a particular set of physical degrees of freedom. For example, whereas in Minkowski space GFP$_1$ describes a single propagating scalar degree of freedom, GFP$_2$ describes two. When GFP$_2$ is infinitesimally deformed by adding a term of the form $\phi \square \phi$, one of the two fields is massive and the other is massless. By integrating out the massive degree of freedom one remains with the free massless one. Thus there should be an RG trajectory joining GFP$_2$ in the UV to GFP$_1$ in the IR.

There is one obvious trajectory that does this: it consists of “generalized free theories” [1, 2] with Lagrangians of the form

$$\frac{1}{2} \phi \left( Z_1 \square + Z_2 \square^2 \right) \phi. \tag{1.1}$$

In the RG one has to parametrize the theory space with dimensionless coordinates, and if we choose for example the field to have canonical dimension of mass, $Z_1$ is already dimensionless, and the other direction is parametrized by $\tilde{Z}_2 = Z_2 k^2$, where $k$ is some external “renormalization group scale”, that in the present situation we can identify with the momentum $p$. In these free theories, $Z_1$ and $Z_2$ do not run, so $\tilde{Z}_2$ is negligible at low energy, but dominant at high energy. Note that choosing a different dimension for the field does not change this conclusion.$^2$ For example, if the field is dimensionless, $Z_2$ is already dimensionless and the other direction has to be parameterized by $\tilde{Z}_1 = Z_1 / k^2$. So, again, $Z_1$ is dominant at low energy and negligible at high energy. In both cases, this “classical RG” just tells us that the four derivative term dominates over the two-derivative term at high energy.

We are interested in trajectories that go through interacting regions of theory space. In this paper we will focus mainly on a shift-symmetric and $\mathbb{Z}_2$-symmetric scalar field. These

---

$^1$This type of RG flows has been considered before in [3, 4].

$^2$We show in general in appendix A that physical properties of the RG flow are independent of the choice of dimension of the field.
symmetries restrict the Euclidean free energy, or effective action, to have the form

$$\Gamma[\phi] = \int d^4x \left[ \frac{1}{2} Z_1 (\partial \phi)^2 + \frac{1}{2} Z_2 (\Box \phi)^2 + \frac{1}{4} g ((\partial \phi)^2)^2 + \ldots \right]$$

(1.2)

where the ellipses stand for terms with six or more derivatives.

In perturbation theory, one would normally consider two cases: either $Z_1 = 0$ or $Z_2 = 0$. These CFTs have been discussed recently in [5]. Beyond perturbation theory, the action (1.2) and its generalizations containing higher derivative terms, are part of a single “theory space”, where all terms can be present simultaneously. One is then interested in understanding the mutual relations between different fixed points. In particular, the question we shall investigate is whether there exist nontrivial RG trajectories joining them.

The tool we shall use is the Wetterich-Morris form of the non-perturbative RG equation for the 1-PI effective action, a.k.a. the effective average action (EAA) [6, 7]. The EAA is a functional of the fields depending on an external scale $k$ that acts as an IR cutoff. By making an ansatz for the EAA of the form (1.2), the constants $Z_1$, $Z_2$ and $g$ become $k$-dependent running couplings. Inserting the ansatz in the RG equation one can read off the beta functions and anomalous dimensions.

We shall calculate the RG flow based on two different choices of field dimension, which are in turn related to different Gaussian FP’s, and show how these flows are related by a coordinate transformation in theory space. This yields a global picture of the flow where both GFP’s are simultaneously present. In the neighborhood of a Gaussian fixed point, the anomalous dimension must be small. If, following the RG flow, we end at another FP, we can in principle calculate the anomalous dimension of the field at this endpoint. Such calculations are always based on some approximations and therefore the calculated anomalous dimension is generally not exact. Remarkably, we shall see that in the case of flows between Gaussian FPs the result is exact, in the following sense: the canonical dimension of the field at the UV FP plus the calculated anomalous dimension gives exactly the canonical dimension of the field at the IR FP.

These are the main results of the paper and occupy section 2. In section 3 we discuss the trivial FP with $n = 0$. Further discussion of the results is contained in section 4. Some appendices contain technical results concerning the choice of regulator in the definition of the RG, and the effect of non-canonical field dimensions on the RG flow.

2 Flowing from GFP$_2$ to GFP$_1$

Implicit in the definition of free particle states is the choice of a Gaussian FP. Then, it is natural to give the field the canonical dimension that pertains to that free theory. When one contemplates flows interpolating between different FP’s, the choice of dimension of the field is no longer so natural. In this section we will discuss flows joining GFP$_1$ and GFP$_2$, where the fields have canonical dimension one and zero respectively. We will therefore exhibit the flow equations in both cases. The power counting in the two cases is very different, but the calculation of the loop contributions is essentially the same.$^3$ The differences arise from the

---

$^3$This point is discussed in general in appendix A.
choice of dimensionless coordinates for theory space, that come natural in the neighborhood of each FP. For each GFP, \(i = 1, 2\) we will therefore define a chart, consisting of an open subset of theory space \(U_i\) and suitable coordinate functions. We will then discuss the transformation between the two charts and give a global picture of the RG flow.

In order to write an explicit RG equation we have to choose a form for the cutoff (or “regulator”) function \(R_k\) that suppresses the low momentum modes in the path integral. We choose:

\[
R_k^{(24)} = Z_1(k^2 - q^2) \theta(k^2 - q^2) + Z_2(k^4 - q^4) \theta(k^4 - q^4).
\]

(2.1)

The presence of the couplings \(Z_1\) and \(Z_2\) makes it an “adaptive” cutoff (in contrast to a “non-adaptive” or “pure” cutoff [8]). Normally only one of the two terms is considered, but for our purposes this choice is preferable, because it treats the two possible kinetic terms on an equal footing. Additionally, it leads to the simplest beta functions, among the choices we have tried. We discuss in the appendix different choices of the cutoff.

2.1 Dimensionful field and the chart \(U_1\)

If in the action (1.2) we choose the term with two derivatives to define the propagator, the field \(\phi\) has canonical dimension of mass. Then \(Z_1\) is dimensionless, \(Z_2\) has dimension of inverse mass squared and \(g\) of inverse mass to power 4. the power counting is that of a non-renormalizable theory. It is natural to parametrize theory space by

\[
\tilde{g} = \frac{g k^4}{Z_1^2}, \quad \tilde{Z}_2 = \frac{Z_2 k^2}{Z_1}.
\]

(2.2)

The powers of \(k\) make the couplings dimensionless and the powers of \(Z_1\) are such that if these definitions are inserted in the action, \(Z_1\) can be set to one by rescaling the field. This makes it clear that \(Z_1\) is a redundant coupling. The anomalous dimension \(\eta_1 = -\partial_t \log Z_1\) (where \(t = \log(k/k_0)\)) and the beta functions of \(\tilde{g}\) and \(\tilde{Z}_2\) are obtained from the flows of \(g, Z_1\) and \(Z_2\), which are presented in appendix B. We find

\[
\eta_1 = \frac{8\tilde{g} (1 + 2\tilde{Z}_2)}{\tilde{g} + 128\pi^2(1 + Z_2)^2}
\]

(2.3)

and

\[
\beta_{\tilde{g}} \equiv \partial_t \tilde{g} = (4 + 2\eta_1)\tilde{g} + \frac{10 + 20\tilde{Z}_2 - \eta_1}{64\pi^2(1 + Z_2)^2} \tilde{g}^2,
\]

(2.4)

where (2.3) has to be used. The beta function of the dimensionful \(Z_2\) vanishes, which implies

\[
\beta_{\tilde{Z}_2} \equiv \partial_t \tilde{Z}_2 = (2 + \eta_1)\tilde{Z}_2.
\]

(2.5)

These beta functions have some nontrivial zeroes. We see from (2.5) that \(\beta_{\tilde{Z}_2}\) can vanish in two ways: one is by having \(\tilde{Z}_2 = 0\), the other by \(\eta_1 = -2\). Besides GFP there are two FPs of the first type, occurring at \(\tilde{g} = 128\pi^2(2\sqrt{6} - 5)\) and \(\tilde{g} = 128\pi^2(-2\sqrt{6} - 5)\) and one FP of the second type at \(\tilde{Z}_2 = -3/5, \tilde{g} = -512\pi^2/5\). The properties of these FPs are summarized in table 1. Some of these FPs had already been observed in [9–11].
Table 1. Properties of the finite FPs seen with dimensionful field. The second and third columns give the values of the couplings (2.2); the fourth the anomalous dimension and the last two the scaling exponents.

| FP     | $\tilde{Z}_{2*}$ | $\tilde{g}_*$ | $\eta_{1*}$ | $\theta_1$ | $\theta_2$ |
|--------|------------------|---------------|-------------|------------|------------|
| GFP$_1$| 0                | 0             | 0           | $-4$       | $-2$       |
| NGFP$_1$| 0                | $-127.6$     | $-0.90$     | 4.40       | $-1.10$    |
| NGFP$_2$| 0                | $-12505$     | 8.90        | 43.60      | $-10.90$   |
| NGFP$_3$| $-0.6$           | $-1011$      | $-2$        | $-13.84$   | 10.84      |

We note that the beta functions have singularities for $\tilde{g} = -128\pi^2(1 + \tilde{Z}_2)^2$ and for $\tilde{Z}_2 = -1$. The fixed points GFP$_1$, NGFP$_1$ are on one side of the singularities, while the other two are on the other side. Thus for the purpose of studying the flows that can start/end at GFP$_1$, the area with $\tilde{Z}_2 < -1$ or $\tilde{g} < -128\pi^2(1 + \tilde{Z}_2)^2$ is unphysical.

One can get a general overview picture of the flow in the chart $U_1$ by defining

$$
\tilde{Z}_2 = \tan u \quad \text{(2.6)}
$$

$$
\tilde{g} = 128\pi^2(2\sqrt{6} - 5) \tan v \quad \text{(2.7)}
$$

The rescaling factor has been chosen in such a way that NGFP$_1$ is at $u = 0$, $v = \pi/4$, while the singularity of the flow is at $u = -\pi/4$.

If we study the function $\eta_1$ in the bottom right quadrant, we find that the condition $\eta_1 = -2$ is satisfied asymptotically for $\tilde{Z}_2 \to \infty$ and

$$
\tilde{g} \sim -16\pi^2\tilde{Z}_2 \quad \text{(2.8)}
$$

This leads us to suspect the existence of another FP in the bottom right corner, outside the domain of this chart. We also note the existence of a “separatrix”: the RG trajectory that arrives at NGFP$_1$ from the irrelevant direction, corresponding to the eigenvector with components

$$
\left(\frac{5(11\sqrt{6} - 27)}{256\pi^2(505\sqrt{6} - 1237)}\epsilon, -\epsilon\right) \approx (0.0143\epsilon, -\epsilon) \quad \text{(2.9)}
$$

This trajectory can be found numerically and it has the asymptotic behavior (2.8). In fact all other trajectories in the fourth quadrant that end at GFP$_1$ have this same asymptotic behavior, as we shall show later. If we follow these trajectories in the sense of increasing $k$ or $t$, those trajectories that emerge from GFP$_1$ at a steeper angle reach this behavior sooner, while those that come out nearly horizontally only reach this regime at higher $k$. We shall discuss the meaning of these facts later.

### 2.2 Dimensionless field and the chart $U_2$

We start again from (1.2), but we assume that the propagator is given by the four derivative kinetic term, so the field is canonically dimensionless. In order not to confuse the couplings...
of this case with those of the previous section, we shall use a different notation for the effective action:

\[
F[\phi] = \int d^4x \left[ \frac{1}{2} \zeta_1 (\partial \phi)^2 + \frac{1}{2} \zeta_2 (\Box \phi)^2 + \frac{1}{4} \gamma ((\partial \phi)^2)^2 + \ldots \right]
\]  

(2.10)

It can be obtained from (1.2) by changing the dimension of the field and redefining the couplings

\[
\phi = k \phi, \quad Z_1 = k^{-2} \zeta_1, \quad Z_2 = k^{-2} \zeta_2, \quad g = k^{-4} \gamma.
\]  

(2.11)

Now the wave function renormalization constant \( \zeta_2 \) is dimensionless, while \( \zeta_1 \) has dimension of mass squared, and the coupling \( \gamma \) is dimensionless. The power counting is that of a renormalizable theory, with \( \zeta_1 \) having the meaning of a mass. The natural variables for the parametrization of theory space are

\[
\hat{\zeta}_1 = \frac{\zeta_1}{\zeta_2 k^2}, \quad \hat{\gamma} = \frac{g}{\zeta_2}.
\]  

(2.12)

Inserting these definitions in the action, we see that \( \zeta_2 \) is a redundant coupling, because it can be set to one by a field rescaling.

The anomalous dimension \( \eta_2 = -\partial_t \log Z_2 \) (where \( t = \log(k/k_0) \)) and the beta functions of \( \hat{\gamma} \) and \( \hat{\zeta}_1 \) are obtained again from the flows given in appendix B. As in the previous section, \( \partial_t \zeta_2 = 0 \), so

\[
\eta_2 = 0,
\]  

(2.13)

while the beta functions are

\[
\beta_{\hat{\zeta}_1} = -2 \hat{\zeta}_1 - \frac{8 \hat{\gamma}(2 + \hat{\zeta}_1)}{\hat{\gamma} + 128 \pi^2 (1 + \hat{\zeta}_1)^2}
\]  

(2.14)

and

\[
\beta_{\hat{\gamma}} = \frac{(2 + \hat{\zeta}_1)(\hat{\gamma} + 640 \pi^2 (1 + \hat{\zeta}_1)^2)}{32 \pi^2 (1 + \hat{\zeta}_1)^3 \left( \hat{\gamma} + 128 \pi^2 (1 + \hat{\zeta}_1)^2 \right)} \hat{\gamma}^2.
\]  

(2.15)

The fixed points of these beta functions are listed in table 2. We note that the nontrivial FP has the same scaling exponents as NGFP3 and has been labelled accordingly. We shall soon understand this identification better.

By expanding the beta functions in \( \hat{\zeta}_1 \) and \( \hat{\gamma} \), and demanding that they point towards the origin, we find that the only direction by which one can tend to GFP2 is \((\epsilon, -16\pi^2 \epsilon)\). Also in this case in the fourth quadrant there is a separatrix. It distinguishes curves for which \( \hat{\zeta}_1 \) tends to infinity in the limit \( k \to 0 \) from those for which \( \hat{\zeta}_1 \) reaches a maximum and then turns down again. For large \( \hat{\zeta}_1 \) the separatrix has the asymptotic form

\[
\hat{\gamma} = -128 \pi^2 (2\sqrt{6} - 5) \zeta_2^2.
\]  

(2.16)

This limit corresponds to GFP1.

\textsuperscript{4}There may seem to be a contradiction between this statement, \( \partial_t Z_2 = 0 \) and the transformation (2.11). The point is that in the derivation of the beta functions from the functional RG flows \( \partial_t \Gamma[\phi] \) and \( \partial_t F[\phi] \) one always interprets the field as being independent of \( k \). This is discussed in greater generality in appendix A.
Table 2. Properties of the finite FPs seen with dimensionless field. The second and third columns give the values of the couplings (2.12); the fourth the anomalous dimension and the last two the scaling exponents.

2.3 Global properties of the flows

We shall now show that the two flows described in the previous subsections are merely coordinate transformation of each other, outside the two Gaussian FP’s, and derive various physical properties of the system.

2.3.1 The coordinate transformation

The chart $U_1$ contains GFP$_1$ but not GFP$_2$, and vice-versa. In order to understand the flows from one Gaussian FP to the other, we must understand that in each chart “the other” FP is a limiting set. To this end, we need the coordinate transformation. From the relations (2.2), (2.11) and (2.12) the two sets of coordinates for theory space are related by

$$
\hat{\zeta}_1 = \frac{1}{\tilde{Z}_2}, \quad \hat{\gamma} = \frac{\tilde{g}}{\tilde{Z}_2^2} \quad \text{or conversely} \quad \tilde{g} = \hat{\gamma} \frac{\tilde{Z}_2}{\hat{\zeta}_1^2}. \quad (2.17)
$$

From here we see that in the chart $U_2$, taking the limit $\hat{\zeta}_1 \to \infty$ for any fixed and finite $\hat{\gamma}$, gives $\tilde{Z}_2 = 0$ and $\tilde{g} = 0$. Therefore, all these limit points correspond to GFP$_1$. Conversely in the chart $U_1$, if we take the limit $\tilde{Z}_2$ and $\tilde{g} \to \infty$ with relation (2.8), we find $\hat{\zeta}_1 = 0$, $\hat{\gamma} = 0$, which corresponds to GFP$_2$. While mathematically clear, these statements may sound a bit puzzling: from the point of view of the chart $U_2$, how can it be that the theory becomes free in the IR even as the coupling $\hat{\gamma}$ remains constant? Even more dramatically, from the point of view of the chart $U_1$, how can it be that the theory becomes free in the UV even as the coupling $\tilde{g}$ diverges? We shall gain a better understanding of these statements by studying the properties of the RG trajectories. After the picture in dimensionless variables has been clarified, we will discuss the picture in dimensionful variables.

2.3.2 The mass threshold

In the chart $U_2$, $Z_1$ is a mass squared and therefore there is an obvious mass threshold located where $\hat{\zeta}_1 = 1$, with the UV located to its left and the IR to its right. By the same token, in the chart $U_1$ the mass threshold is at $\tilde{Z}_2 = 1$, with the IR to its left and the UV to its right. It is somehow natural to use the chart $U_2$ for energy scales above the threshold and the chart $U_1$ for energy scales below it, even though the validity of both charts extends far below this point.
Figure 1. The flow in the chart $U_1$ (left) and in the chart $U_2$ (right). The black dot (on the left) and the dahed black line (on the right) mark GFP$_1$, the red dot marks GFP$_2$, the blue square marks NGFP$_1$. The separatrix is the green flow line. The continuous red lines are singularities of the flow.

### 2.3.3 Perturbative vs strongly interacting trajectories

We have shown in section 2.2 that all trajectories emerge from GFP$_2$ with $\hat{\gamma} = -16\pi^2\hat{\zeta}_1$. Applying the transformation (2.17) this implies that in the chart $U_1$ all the trajectories have the asymptotic behavior (2.8), as mentioned in section 2.1.

Next note that the lines with $\tilde{g} = 0$ (in the chart $U_1$) and $\hat{\gamma} = 0$ (in the chart $U_2$) correspond to the classical trajectory (1.1), joining GFP$_2$ in the UV to GFP$_1$ in the IR, and consisting entirely of free theories. At the other extreme, the separatrix is in some sense the “strongest interacting” trajectory. In the chart $U_1$ it consists of two segments: first the line going from GFP$_2$ in the UV to NGFP$_1$ in the IR, and then the trajectory with $\tilde{Z}_2 = 0$, joining NGFP$_1$ in the UV to GFP$_1$ in the IR.$^5$ We will be interested in the infinitely many trajectories that are contained between these two extremes, see figure 1.

There are trajectories that remain entirely in the perturbative domain, i.e. are close to the classical trajectory. This is not obvious when one works in a fixed chart, because both $\tilde{g}$ and $\hat{\gamma}$ do not go to zero at both ends of the trajectory. Flowing out of GFP$_2$ in the chart $U_2$ they are the ones for which $\hat{\gamma}$ is small at least down to the mass threshold $\hat{\zeta}_1 = 1$. Eventually, if one proceeds further towards the IR, $\hat{\gamma}$ remains constant. However, around $\hat{\zeta}_1 = 1$, one can change chart: at that point $\tilde{g} = \hat{\gamma}$ is small and following the flow towards the IR in the chart $U_1$, the coupling $\tilde{g}$ tends to zero. The qualitative behavior of the trajectories is the same also when the coupling midflows is strong.

---

$^5$Strictly speaking this should be seen as two separate trajectories, since each one takes infinite RG time, but it can be seen as the limit of trajectories joining directly GFP$_2$ to GFP$_1$. 
2.3.4 The dimension of the field is automatically adjusted

We can now see the automatic change of dimensionality of the field along the flow. In the chart $U_1$, that is more appropriate to describe the low energy physics, the field has dimension of mass and $Z_1$ is dimensionless. Near GFP$_1$ the anomalous dimension is small and negative. However, if we follow any RG trajectory towards the UV, as discussed in section 2.1, the anomalous dimension grows and eventually tends to $-2$. Recalling that the canonically normalized field

$$\tilde{\phi} = \sqrt{Z_1}\phi$$

has scaling dimension $(d - 2 + \eta_1)/2$, this means that the field is effectively dimensionless in the UV limit. This is indeed the natural choice for the field at GFP$_2$ in the chart $U_2$. We observe that this automatic adjustment of the dimension is a consequence of the form (2.5) of the beta function of $\tilde{Z}_2$. The fact that in the UV limit $\eta_1 \approx -2$ also means that the wave function renormalization constant scales like $Z_1 \sim k^2$ in that limit. Thus in going from GFP$_1$ to GFP$_2$, $Z_1$ gets multiplied by an infinite factor.

One can arrive at the same conclusions in the chart $U_2$. Here the RG equation to be observed is not the one for the anomalous dimension $\eta_2$, which is identically zero, but the running of the mass $\hat{\zeta}_1$. In fact eq. (2.14) can also be written in the form

$$\beta_{\hat{\zeta}_1} = -(2 + \eta_1)\hat{\zeta}_1,$$

where

$$\eta_1 = \frac{8\hat{\gamma}(2 + \hat{\zeta}_1)}{\hat{\gamma} + 128\pi^2(1 + \hat{\zeta}_1)^2 Z_1},$$

and one recovers $\eta_1 = -2$ in the fixed point GFP$_2$.

2.3.5 The picture in dimensionful variables

Even though the charts $U_1$ and $U_2$ are defined only for the dimensionless coordinates on theory space, when we consider the flow in the dimensionful parameters appearing in the Lagrangian, there is still a vestige of these coordinates in the choice of the dimension of the field.

Because terms with fewer derivatives dominate at low energy, it is natural to describe the IR physics in terms of the dimensionful field $\phi$ with two-derivative kinetic term. In the limit $k \to 0$ both $Z_1$ and $g$ become constants, see equations (B.2), (B.4), and recall that $Z_2$ is also a constant. Therefore the effective action is (1.2) with arbitrary fixed coefficients. This does not look like a free theory, but if we identify the scale $k$ with a characteristic external momentum $p$, by mere momentum counting the first term is the dominant one in the IR limit. The interaction is of order $gp^4$ and goes to zero much faster, and the same happens for the higher-derivative kinetic term.$^6$ It is noteworthy that in order to identify the $k \to 0$ limit as a free theory it is necessary to identify $k$ as a physical momentum scale.

---

$^6$The identification $k = p$ is unambiguous for the two point function, but may require further qualifications for more complicated physical processes.
On the other hand, using the asymptotic behavior (2.8), and solving the flow equation for $\tilde{Z}_2$, we find that the behavior for large $k$ is $\tilde{Z}_2 \sim \frac{11}{4} \log k$ and therefore\footnote{This gives the anomalous dimension $-2 + 1/\log k$.}

$$Z_1 = \frac{4Z_2k^2}{11 \log k},$$

(2.21)

where $Z_2$ is fixed and arbitrary. Then using (2.8) one obtains

$$g = -\frac{64\pi^2 Z_2^2}{11 \log k}.$$  

(2.22)

Thus for $k \to \infty$, $g$ goes to zero and we remain with a free theory. Identifying again $k$ with the momentum in the two-point function, the four-derivative kinetic term has an overall momentum-dependence $p^4$ whereas the two-derivative one goes like $p^4/\log p$. Thus in the UV limit the four-derivative kinetic term is the dominant one, but only logarithmically.

For large momentum it is natural to redefine the field as in (2.11), thus absorbing in the field the power in the running of $Z_1$ at high energy. Then we find that $\tilde{\zeta}_1 = 4/(11 \log k)$ and

$$\zeta_1 = \frac{4\zeta_2 k^2}{11 \log k},$$

(2.23)

where $\zeta_2$ is an arbitrary dimensionless constant that can be set to one. Thus the “mass squared” $\zeta_1$ has the expected power behavior, with a logarithmic correction. For the coupling $\gamma$ we get

$$\gamma = -\frac{64\pi^2 \zeta_2^2}{11 \log k},$$

(2.24)

which is the expected behavior of a renormalizable coupling and demonstrates asymptotic freedom at high energy.

If we now look at the IR limit using the field $\varphi$ we find that $\zeta_1$, $\zeta_2$ and $\gamma$ become all constants and we recover the previous statement that the (free) two-derivative term is the dominant one. Once again, the understanding of this limit as a free theory hinges on identifying the RG scale $k$ with a characteristic external momentum $p$.

### 2.3.6 The mass of the ghost

In this section we think of the theory in Minkowski space, where the classical action differs from (1.1) by an overall sign.\footnote{We use signature $- + + +$.} It describes two propagating particles: a normal massless scalar and a massive scalar ghost with mass $m_{gh}^2 = Z_1/Z_2$ (Notice that this statement is independent of the dimension of the field). If $g = 0$, $Z_1$ and $Z_2$ do not run and the statement holds \textit{verbatim} also at the quantum level. If we now switch on $g$, the value of the physical (pole) mass will receive quantum corrections. The two-point function of the theory is defined as the limit for $k \to 0$ of the two point function of the EAA. In general, the dependence of the $n$-point functions on the external momenta and on the parameter $k$ are not interchangeable, but in the case of the two-point function, given that the external momentum $p$ has the effect of an IR cutoff in the integrations over the loop momenta, the
$k$-dependence is a good proxy for the $p$-dependence. We can therefore reliably calculate the pole mass from the running of the parameters $Z_1$ and $Z_2$ with the cutoff scale $k$.

In the presence of a running (renormalized) mass $m_R^2(k)$, the pole mass can be defined by

$$m_{\text{pole}}^2 = m_R^2(k = m_{\text{pole}})$$

and corresponds to the threshold discussed in the previous section. There is an old argument that if $m_R$ grows sufficiently fast, there may be no pole at all.\footnote{See e.g.\cite{12}.} This would remove the unwanted ghost state.

Working in the chart $U_1$, the location of the pole is defined by

$$\hat{Z}_2 = 1.$$ \hfill (2.26)

Since all the trajectories run from $\hat{Z}_2 = \infty$ to $\hat{Z}_2 = 0$, they inevitably hit the pole and the argument mentioned above cannot apply. However, the pole may be shifted to arbitrarily high scale.

To see this we start by setting $Z_1 = 1$ in the IR limit. Then on the “classical” RG trajectory (1.1), $Z_1 = 1$ everywhere and the pole mass is at

$$k_P^2 = \frac{Z_1}{Z_2} = \frac{1}{Z_2}.$$ \hfill (2.27)

Let us now switch on the interaction. The anomalous dimension $\eta_1$ is negative and therefore $Z_1$ becomes larger than one. Thus the pole is shifted to a larger value, compared to the “classical” trajectory. This effect becomes stronger as one considers trajectories that are further away from the classical one. In the limit, consider a trajectory that is close to the separatrix. Already for small $k$, $\hat{g}$ becomes quickly very negative, until one gets close to NGFP$_1$. There the running of $\hat{g}$ almost stops, but $Z_1$ grows like

$$Z_1(k) \sim k^{0.90}.$$ \hfill (2.28)

This behavior can last for many orders of magnitude of $k$. By the time the RG trajectory finally leaves the vicinity of NGFP$_1$ and reaches $\hat{Z}_2 = 1$, $Z_1$ can be arbitrarily large. Thus the mass of the ghost grows continuously from $1/\sqrt{Z_2}$ to infinity as one moves from the classical RG trajectory to the separatrix.

### 2.3.7 Redundant couplings and the essential RG

One says that a coupling in the Lagrangian is “redundant” or “inessential” \textit{at a specified FP}, if it can be removed from the Lagrangian by means of a local field redefinition \cite{13, 14}. There has been recently an interesting discussion of the “essential RG”, which is a way of simplifying the RG flow by eliminating all redundant couplings \cite{15}.

The prime example of a redundant coupling is the wave function renormalization: the parameter $Z_1$ is redundant at GFP$_1$ and $\zeta_2$ is redundant at GFP$_2$, because they can be fixed to 1 by rescaling the field. We have already taken this into account by putting suitable
powers of $Z_1$ or $\zeta_2$ in the definition of the coordinates in theory space. It is shown in appendix A that doing so is necessary if we demand that the beta functions be independent of the dimension of the field.

However, also the parameter $Z_2$ is redundant at GFP$_1$. Indeed, if $\tilde{Z}_2$ is infinitesimal, it can be removed by an infinitesimal field redefinition of the form

$$\delta \phi = \frac{Z_2}{2Z_1} \Box \phi.$$  \hfill (2.29)

One could therefore eliminate also $Z_2$ and get the essential flow equation for the single (in our approximation) coupling $\hat{g}$: in figure 1, left panel, it would be a flow in the vertical direction only, and would lead to different properties of NGFP$_1$. Similar considerations can be used to prove, in a much more general setting than a mere scalar theory, that if the kinetic term is the standard one containing two derivatives, then in perturbation theory one will never generate higher-derivative kinetic terms [16].

On the other hand when one considers GFP$_2$, $\zeta_1$ is not redundant there because it cannot be removed by a local field redefinition. We conclude that by studying only the essential RG at GFP$_1$ we would not realize the possibility of flowing to GFP$_2$ in the UV, which would imply an increase in the number of propagating degrees of freedom, but we would still have the possibility of flowing to the non-Gaussian FP [15].

3 The flow from GFP$_1$ to GFP$_0$

We now show that the preceding results are closely related to a more familiar example. Here we consider the case of a theory with an action of the form

$$S = \int d^4x \left[ \frac{1}{2} Z_1 (\partial \phi)^2 + \frac{1}{2} Z_0 \phi^2 + \frac{1}{4!} \lambda \phi^4 + \ldots \right]$$  \hfill (3.1)

There are two potential terms that do not have shift symmetry. The mass squared has been called $Z_0$ since it is a member of the family of quadratic Lagrangians.

As in the case of the shift symmetric theories, there are two natural choices of coordinates. In the standard approach the field is assigned dimension of mass, in which case the wave function renormalization $Z_1$ is redundant and the coordinates on theory space are

$$\tilde{Z}_0 = \frac{Z_0}{k^2 Z_1}, \quad \tilde{\lambda} = \frac{\lambda}{Z_1^2}.$$  \hfill (3.2)

This is the same chart $U_1$ considered above, extended to a shift-non-invariant interaction, and it has GFP$_1$ in the origin. Using the cutoff $R_k(z) = Z_1(k^2 - z)\theta(k^2 - z)$, the beta functions have the familiar form

$$\beta_{Z_0} = -2 \tilde{Z}_0 - \frac{\tilde{\lambda}}{32\pi^2 (1 + \tilde{Z}_0)^2},$$  \hfill (3.3)

$$\beta_{\tilde{\lambda}} = \frac{3\tilde{\lambda}^2}{16\pi^2 (1 + \tilde{Z}_0)^3}.$$  \hfill (3.4)
Figure 2. The flow in the chart $U_1$ (left) and in the chart $U_0$ (right). Flow lines in the lower right quadrant go from GFP$_1$ (black) to GFP$_0$ (green). The vertical red lines are singularities of the flow.

The beta function of $Z_1$ is zero, and so is the anomalous dimension $\eta_1 = -\partial_t \log Z_1$. There are no FPs in this chart except for GFP$_1$. Using the rescaling $u = \tan(\tilde{Z}_0)$ and $v = \tan(\tilde{\lambda})$, the flow lines have the form shown in figure 2. We recognize that $\lambda$ is asymptotically free for $\lambda < 0$, as was noticed by Symanzik [17]. The fact that the flow lines asymptote horizontally is due to the decoupling effect of the denominators: for sufficiently small $k$, the running of $\lambda$ stops whereas $\tilde{Z}_0$ continues to run due to the classical term.

Another chart $U_0$ is centered on the fixed point GFP$_0$ that is a free conformal field theory where the field $\phi$ has canonical dimension of mass squared. This is sometimes called a trivial fixed point, because in Minkowski signature it has no propagating degrees of freedom, whereas in the Euclidean theory the correlation length at the fixed point is zero. In this case the “squared mass” $Z_0$ is actually dimensionless and redundant, $Z_1$ is an irrelevant coupling of dimension $-2$ while $\lambda$ has dimension $-4$. The coordinates on theory space are

$$\tilde{Z}_1 = \frac{Z_1 k^2}{Z_0}, \quad \tilde{\lambda} = \frac{\lambda k^4}{Z_0^2}.$$  \hfill (3.5)

The running of $Z_0$ is described by the anomalous dimension

$$\eta_0 = -\partial_t \log Z_0 = -\frac{\tilde{\lambda} \tilde{Z}_1}{32\pi^2(1 + \tilde{Z}_1)^2}$$  \hfill (3.6)

whereas the beta functions are

$$\partial_t \tilde{Z}_1 = (2 + \eta_0) \tilde{Z}_1,$$  \hfill (3.7)

$$\partial_t \tilde{\lambda} = (4 + 2\eta_0) \tilde{\lambda} + \frac{3\tilde{\lambda}^2 \tilde{Z}_1}{16\pi^2(1 + \tilde{Z}_1)^3}.$$  \hfill (3.8)
Again, there are no nontrivial finite fixed points. The beta function of $\bar{Z}_1$ can vanish either because $\bar{Z}_1 = 0$, or because $\eta_0 = -2$, which is satisfied asymptotically for $\bar{\lambda} \sim -64\pi^2 \bar{Z}_1$ and $\bar{Z}_1 \to \infty$. These asymptotes correspond to GFP$_1$.

We note that, aside from the absence of other FP’s, the picture of the flow is very similar to the one of the shift-symmetric theory. In the chart $U_1$, the origin GFP$_1$ is the source of all flow lines with $\bar{\lambda} < 0$ and GFP$_0$ corresponds to all points with $\bar{\lambda} < 0$ finite and $\bar{Z}_0 \to \infty$, so all the RG flow lines that are visible in the fourth quadrant joint GFP$_1$ to GFP$_0$. The same lines are visible in the chart $U_0$, where GFP$_1$ is in the bottom right corner and GFP$_0$ in the center.

The coordinate transformation between the charts $U_0$ and $U_1$ is

$$\bar{Z}_1 = \frac{1}{Z_0}, \quad \bar{\lambda} = \frac{\bar{\lambda}}{Z_0^2}$$

or conversely $\bar{\lambda} = \frac{\bar{\lambda}}{Z_1^2}$, (3.9)

and the beta functions transform as vectors under this transformation.

We observe that also in this case the kinetic term of the UV fixed point (GFP$_1$), which gives rise to a propagating degree of freedom, is a redundant operator from the point of view of the IR fixed point (GFP$_0$), where nothing propagates. In fact, every local perturbation of GFP$_0$ is redundant.

4 Discussion

We now review our main findings and then comment on possible extensions.

The general theory space contains all possible kinetic terms and none of them plays an a priori preferred role. It is only when one considers the perturbative expansion around a Gaussian fixed point that the corresponding kinetic term acquires a special meaning. One then has a clear choice for the canonical dimensionality of the field.$^{10}$ Otherwise, the choice of the dimension of the field is essentially arbitrary: physical conclusions are independent of this choice, as we discuss in appendix A. However, the picture of the flow that follows from different choices can be quite different. For a fixed $n \geq 0$ the choice of kinetic term $Z_n \phi \mathcal{C}^n \phi$ dictates that the field has canonical dimension $(d - n)/2$ and this fixes the dimension of all the couplings $g_i$ in the Lagrangian. When suitably rescaled by powers of $n$ and $Z_n$, these couplings define coordinates on an open subdomain $U_n$ of theory space. Thus theory space is a manifold covered by infinitely many charts. In the origin of the chart $U_n$ there sits GFP$_n$, while all the other Gaussian FPs are outside this chart, but in its closure.

We have discussed mainly the RG trajectories joining GFP$_2$ to GFP$_1$. They describe the unfamiliar situation of interacting theories that are free both in the UV and in the IR.$^{11}$ Starting in the perturbative regime near GFP$_1$ at low energy, the coupling $\hat{g}$ grows without bound as one goes towards the UV. Superficially one may conclude that the theory does not have a good UV limit. However, one has to take into account the infinite amount of running of the wave function renormalization constants: while $\hat{g}$ increases, $\hat{Z}_2$ also increases at a similar rate, in such a way that the combination $\hat{\gamma}$ goes to zero, as seen from (2.17).

$^{10}$In the case of “generalized free theories” (1.1), this is not the case.

$^{11}$For examples of gauge couplings in semisimple gauge theories that have this behavior see [18].
At the same time, the anomalous dimension also becomes large and tends to $-2$, which is a sign that the scaling dimension of the field becomes exactly zero, as appropriate to GFP$_2$. Thus, at some point, it becomes natural to move to the chart $U_2$ where one sees again a perturbative theory, this time governed by the four-derivative kinetic term.

A very similar picture was found for the flow from GFP$_1$ to GFP$_0$, which corresponds to an asymptotically free massive scalar theory à la Symanzik, that flows towards a trivial FP in the IR. It is natural to conjecture the existence of flows between GFP$_k$ and GFP$_\ell$ with $n > \ell$. However, theories with large $n$ have negative canonical field dimension and infinitely many relevant couplings, a problematic situation. In spacetime dimension $d = 4$ this already occurs for $n = 3$, and this is the reason why this case has not been discussed here. One may think of restricting the number of relevant operators by imposing higher order symmetries of the form

$$\delta^{(n)} \phi = c^0 + c_1^\mu x^\mu + \ldots + c_{n-1}^{\mu_1 \ldots \mu_{n-1}} x^{\mu_1} \ldots x^{\mu_{n-1}},$$

(4.1)

which is the symmetry of the kinetic term $\phi \Box^n \phi$, for $n > 0$. More generally, this will be a symmetry for Lagrangians where every field appears under at least $n$ derivatives. By choosing the cutoff appropriately one can obtain flows that respect the symmetry and therefore remain in the symmetric subspaces of theory space, which would acquire a complicated stratified structure.

All theories with $n > 1$ have ghosts at perturbative level, but we have shown that its mass depends on the trajectory and there are trajectories where it is arbitrarily high. The limiting case is the trajectory connecting GFP$_1$ in the IR to NGFP$_1$ in the UV, which is free of ghosts. Another pathology is that the coupling must be negative, leading to negative interaction energy. This was already well known in the case of Symanzik’s asymptotically free scalar theory, and it is generally agreed that in spite of the coupling going asymptotically to zero, this is an unphysical feature [19]. Thus these theories are probably not very useful, even as statistical models, but we think that they still offer some interesting lessons in quantum field theory.

Finally, let us discuss the limitations of our analysis, starting in the chart $U_1$. We have found that when we run the RG towards the UV, all the trajectories above the separatrix in the fourth quadrant tend to GFP$_2$. However, going beyond the truncation (1.2), infinitely many other irrelevant terms will be turned on. For generic initial conditions near GFP$_1$, these couplings will go to infinity in the UV, signalling that these are just effective field theories. It is only for a very special subset of trajectories that UV completeness can be achieved. This is best seen by running the RG in the other direction, starting from GFP$_2$. Also in this case all other couplings compatible with the symmetries will be generated when one looks beyond linear order. As an example, working in the chart $U_2$, one can consider the coupling $\gamma_2$ that multiplies the six-derivative operator $(\Box \phi)^2 (\partial \phi)^2$. The beta function of the dimensionless $\hat{\gamma}_2 = \gamma_2 k^2 / \zeta_2^2$ is

$$\partial_t \hat{\gamma}_2 = 2 \hat{\gamma}_2 + \frac{1024 \hat{\gamma}_2^2 (2 + \hat{\zeta}_1)}{3(1 + \hat{\zeta}_1)(\hat{\gamma} + 128\pi^2(1 + \hat{\zeta}_1)^2)} + O(\hat{\gamma}_2^2).$$

(4.2)
As soon as $\hat{\gamma}$ is turned on, this beta function becomes nonzero and $\hat{\gamma}_2$ starts to grow. However, assuming that $\hat{\gamma}_2$ does not change too much the behavior of the other two couplings, in the IR $\hat{\gamma}_1$ goes to infinity and suppresses the loop term, while the classical term remains. Thus $\hat{\gamma}_2$ is expected to go again to zero in the IR. This is confirmed by numerically solving the equations. Similarly, all the other local couplings will be generated, but they are irrelevant for GFP$_2$ and even more so for GFP$_1$. Thus, one expects that they all go to zero as one flows towards the IR.

In the recent paper [11], the shift-invariant scalar theory has been studied in a truncation involving potentially infinitely many terms, all powers of $(\partial \phi)^2$. This gives more insight in the flow along the axis $\tilde{Z}_2 = 0$, in particular on the properties of NGFP$_1$. However, we observe that the term $(\Box \phi)^2$ will be generated by quantum fluctuations: first, one loop effects of the coupling $\tilde{g}$ will generate the coupling $\gamma_2$ as indicated above (this happens independently of the form of the kinetic term and of the cutoff) and then one loop effects involving $\gamma_2$ will give a nonzero beta function for $Z_2$. Since all the additional terms $((\partial \phi)^2)^n$, $n > 2$, are irrelevant at GFP$_2$, our conclusions will not be modified by the inclusion of such terms in the truncation, except for changes in the properties of the trajectories at strong coupling, and in particular near the fixed point NGFP$_1$.

Acknowledgments

We thank G.P. Vacca, T. Morris, D. Litim and M. Reuter for useful discussions at various stages of this work.

A The dimension of the field is immaterial

Assume that the effective action is a quasi-local functional of the field of the form:

$$\Gamma_k[\phi] = \sum_i g_i O_i(\phi) \quad (A.1)$$

For the sake of power counting, the operators $O_i$ have the general form

$$O_i(\phi) = \int d^d x \, \partial^{m_i} \phi^{n_i}, \quad (A.2)$$

where the integrand stands for any scalar constructed with $m_i$ derivatives and $n_i$ fields. Assuming that the field has dimension $[\phi] = d_\phi$, $O_i$ has dimension $[O_i] = -d + m_i + n_i d_\phi \equiv -d_i$ and the coupling $g_i$ has dimension $d_i$.

Now let us change variable from $\phi$ to a new field $\phi'$ of dimension $d'_\phi$:

$$\phi' = \phi k^{\Delta d_\phi}, \quad (A.3)$$

with $\Delta d_\phi = d'_\phi - d_\phi$. The effective action of the new field is related to that of the old field by

$$\Gamma'_k[\phi'] = \Gamma_k[\phi]. \quad (A.4)$$

\[12\] If one gives up $Z_2$ symmetry, $Z_2$ is generated by quantum fluctuations involving the interaction $\partial_\nu \phi \partial_\nu \phi \phi^a \partial^a \phi$ [20].
This means that while the two functionals have numerically the same values when the fields are related as in (A.3), \( \Gamma'_k \) is a different functional of its argument from \( \Gamma_k \). In particular, writing
\[
\Gamma'_k[\phi'] = \sum_i g'_i \mathcal{O}_i(\phi'),
\]
we find that
\[
g'_i = g_i k^{-n_i \Delta d_\phi}.
\]

At this point it is important to understand that once the functional \( \Gamma'_k \) has been defined by (A.4), we are free to attribute all the \( k \)-dependence to the couplings \( g'_i \) and to think of the field \( \phi' \) as being \( k \)-independent. If we do so, we can use (A.6) as a change of coordinates, but not to calculate the \( k \)-dependence of \( g'_i \). Indeed, when we extract the beta functions from the generating functionals \( \partial_t \Gamma_k \) and \( \partial_t \Gamma'_k \), where \( t = \log k \), in both cases we will keep the field fixed. In this way we arrive at the following relation between the beta functions:
\[
\partial_t g'_i = \partial_t g_i k^{-n_i \Delta d_\phi}.
\]
The apparent contradiction with (A.6) is due to our keeping \( \phi' \) fixed. Thus the missing contribution is compensated by the fact that at the same time we also ignore the \( k \)-term in (A.3). Equation (A.7) expresses the fact that the calculation of the loop corrections is the same, for any dimension of the field, up to an overall factor of \( k \) that accounts for the different dimensions.

In the discussion of RG flows and fixed points we must use the dimensionless variables
\[
\tilde{g}_i = g_i k^{-d_i}, \quad \tilde{g}'_i = g'_i k^{-d'_i}.
\]
However, using (A.7), one finds that the beta functions of these dimensionless variables are different, namely
\[
\partial_t \tilde{g}'_i = \partial_t g'_i k^{-d'_i} - d'_i \tilde{g}'_i = \partial_t \tilde{g}_i + n_i \Delta d_\phi \tilde{g}_i.
\]
This does not happen if we properly take into account the normalization of the field. Among the couplings \( g_i \) there is the wave function renormalization constant \( Z \) or \( Z' \), that has dimension \( d_Z = d - 2 - 2d_\phi \) or \( d'_Z = d - 2 - 2d'_\phi \) respectively. Let us therefore define
\[
\tilde{g}_i = g_i Z^{-n_i/2} k^{-d_i+n_i d_Z/2}, \quad \tilde{g}'_i = g'_i Z'^{-n_i/2} k^{-d'_i+n_i d'_Z/2}.
\]
Note that
\[
-d'_i + n_i d'_Z/2 = -d + m_i + n_i \frac{d - 2}{2} = -d + n_i d_Z/2,
\]
and then, if we use (A.6), \( \tilde{g}'_i = \tilde{g}_i \). Equation (A.7) implies that \( \partial_t Z' = \partial_t Z k^{-2 \Delta d_\phi} \) and therefore
\[
\partial_t \log Z' = \partial_t \log Z.
\]
So, using the preceding formulae,
\[
\partial_t \tilde{g}'_i = \partial_t g'_i Z'^{-n_i/2} k^{-d'_i+n_i d'_Z/2} + \left( -d + m_i + n_i \frac{d - 2}{2} - \partial_t \log Z' \right) \tilde{g}'_i
\]
\[
= \partial_t g_i Z^{-n_i/2} k^{-d_i+n_i d_Z/2} + \left( -d + m_i + n_i \frac{d - 2}{2} - \partial_t \log Z \right) \tilde{g}_i = \partial_t \tilde{g}_i.
\]
Thus the flows of the dimensionless and canonically normalized couplings is the same, independently of the dimension of the field. This underlines the importance of including the redundant wave function renormalization constants in the definition of the coordinates on theory space.

B Beta functions of the dimensionful couplings

The beta functions are extracted from the functional RG equation [6, 7]

\[ k \frac{d \Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right)^{-1} k \frac{d R_k}{dk} \]  

(B.1)

by inserting the Ansatz (1.2) for $\Gamma_k$, evaluating the functional trace Tr (which for a scalar in flat space is a momentum integral) and extracting from it the coefficients of the three operators $(\partial \phi)^2$, $(\Box \phi)^2$ and $((\partial \phi)^2)^2$.

As mentioned in the preceding appendix, the beta functions of the original, generally dimensionful, parameters in the Lagrangian are related as in (A.6). This is because the calculation of the loop contributions is the same, up to a redefinition of the dimensions. We can see this explicitly in the case of the shift-symmetric theory. Using a dimension-one field and action (1.2), the beta functions are

\[ \partial t Z_1 = \frac{(8 - \eta_1)Z_1 + 16k^2Z_2}{128\pi^2(Z_1 + k^2Z_2)^2} gk^4, \]  

(B.2)

\[ \partial t Z_2 = 0, \]  

(B.3)

\[ \partial t g = \frac{(10 - \eta_1)Z_1 + 20k^2Z_2}{64\pi^2(Z_1 + k^2Z_2)^3} g^2k^4. \]  

(B.4)

With dimensionless field and action (2.10), the beta functions are the same, with the replacements $Z_i \rightarrow \zeta_i$, $g \rightarrow \gamma$.

C Other cutoffs

We considered also cutoffs different from $R_k^{(24)}$ in (2.1): either the two-derivative cutoff:

\[ R_k^{(2)} = Z_1(k^2 - q^2)\theta(k^2 - q^2), \]  

(C.1)

which is the standard choice, or the four-derivatives cutoff:

\[ R_k^{(4)} = Z_2(k^4 - q^4)(k^4 - q^4), \]  

(C.2)

which is sometimes used when the kinetic term has four derivatives.
C.1 Dimensionful fields and two-derivative cutoff

With the field dimension set to 1 and the cutoff $R^{(2)}_k$, we find

\[
\eta_1 = \frac{-6\tilde{Z}_2 + 6\sqrt{\tilde{Z}_2(1 + \tilde{Z}_2)} \arctan \left( \sqrt{\tilde{Z}_2} \right)}{(1 + \tilde{Z}_2) \left( 64\pi^2 \tilde{Z}_2^2 + 3\tilde{g} \sqrt{\tilde{Z}_2} \arctan \left( \sqrt{\tilde{Z}_2} \right) - 3\tilde{g} \log(1 + \tilde{Z}_2) \right)} \tilde{g}, \tag{C.3}
\]

while the beta function is

\[
\beta_{\tilde{g}} = (4 + 2\eta_1)\tilde{g} + \left[ \frac{15\eta_1}{128\pi^2 \tilde{Z}_2^5} \arctan \sqrt{\tilde{Z}_2} \right] \tilde{g}^2, \tag{C.4}
\]

where (C.3) has to be used. These are real only if $\tilde{Z}_2 \geq 0$, hence this set of flow equation has a restricted domain. Concerning the beta function of $Z_2$, we have $\partial_t Z_2 = 0$, hence the equation is independent of the cutoff and we have again

\[
\beta_{Z_2} \equiv \partial_t \tilde{Z}_2 = (2 + \eta_1)\tilde{Z}_2. \tag{C.5}
\]

In the limit $\tilde{Z}_2 \to 0$, we recover the same nontrivial NGFP$_1$ and NGFP$_2$ of the cutoff $R^{(24)}_k$, while NGFP$_3$ is in the half-plane where the RG equations are complex. In the right half-plane the condition $\eta_1 = -2$ is satisfied on the curve

\[
\tilde{g} = -\frac{64}{3} \pi \tilde{Z}_2^{3/2}. \tag{C.6}
\]

The separatrix is not a straight line anymore, but we still have $\hat{g} = 0$ after the coordinate transformation from $U_1$ to $U_2$, hence a region with a flow between GFP$_1$ and GFP$_2$ exists also with this cutoff.

C.2 Dimensionful field and four-derivative cutoff

If we use the cutoff (C.2), the anomalous dimension is

\[
\eta_1 = \frac{3\tilde{g} \tilde{Z}_2 \left( 1 + 2\tilde{Z}_2 \right)}{8\pi^2 (1 + \tilde{Z}_2)} - \frac{3\tilde{g} \tilde{Z}_2^2}{4\pi^2} \log \left( \frac{1 + \tilde{Z}_2}{\tilde{Z}_2} \right), \tag{C.7}
\]

and the beta function is

\[
\beta_{\tilde{g}} = 4\tilde{g} + \frac{16 + 21\tilde{Z}_2(3 + 2\tilde{Z}_2)}{8\pi^2 (1 + Z_2)^2} \tilde{g}^2 \tilde{Z}_2 - \frac{42}{8\pi^2} \tilde{g}^2 \tilde{Z}_2^2 \log \left( \frac{1 + \tilde{Z}_2}{Z_2} \right), \tag{C.8}
\]

where (C.7) has been used. The beta function of $Z_2$ is still given by (C.5). The log $\left( \frac{1 + \tilde{Z}_2}{Z_2} \right)$ is ill-defined or complex in the region $-1 < \tilde{Z}_2 \leq 0$, therefore the nontrivial FPs can not be found and the only finite FP of these beta functions is GFP$_1$.

The beta functions are influenced by the logs on $\tilde{Z}_2 = 0$, but, up to a small neighborhood of this axis, the general behaviour of the RG flow in the fourth quadrant is very similar to the one described in section 2.1. The condition $\eta_1 = -2$ is satisfied for

\[
\tilde{g} \approx -16\pi^2 \tilde{Z}_2, \tag{C.9}
\]

giving exactly the same asymptotic behaviour as $R^{(24)}_k$. 

– 19 –
C.3 Dimensionless field and two-derivative cutoff

The dimensionless field with the cutoff $R_k^{(2)}$ gives

$$\beta_{\hat{\zeta}_1} = -2\hat{\zeta}_1 + \frac{6\hat{\gamma}\hat{\zeta}_1 \left( \sqrt{\hat{\zeta}_1} - (1 + \hat{\zeta}_1) \arccot \sqrt{\hat{\zeta}_1} \right)}{(1 + \hat{\zeta}_1)(64\pi^2 \sqrt{\hat{\zeta}_1} - 3\hat{\gamma}\sqrt{\hat{\zeta}_1} \log \left( \frac{1 + \hat{\zeta}_1}{\hat{\zeta}_1} \right) + 3\hat{\gamma} \arccot \sqrt{\hat{\zeta}_1})}$$

(C.10)

The beta function of $\hat{\gamma}$ is

$$\beta_{\hat{\gamma}} = \frac{5\hat{\gamma}^2 \sqrt{\hat{\zeta}_1} \left[ 128\pi^2 + \frac{\hat{\gamma}}{6\sqrt{\hat{\zeta}_1} (4 + 3\hat{\zeta}_1) \arccot \sqrt{\hat{\zeta}_1} + 9\sqrt{\hat{\zeta}_1} (1 + \hat{\zeta}_1)^2 \arccot^2 \sqrt{\hat{\zeta}_1} - 6\log \left( \frac{1 + \hat{\zeta}_1}{\hat{\zeta}_1} \right) \right]}{64\pi^2 (1 + \hat{\zeta}_1)^2 \left( 64\pi^2 \sqrt{\hat{\zeta}_1} - 3\hat{\gamma}\sqrt{\hat{\zeta}_1} \log \left( \frac{1 + \hat{\zeta}_1}{\hat{\zeta}_1} \right) + 3\hat{\gamma} \arccot \sqrt{\hat{\zeta}_1} \right)}$$

(C.11)

while the anomalous dimension, which does not receive any contributions from one loop diagrams, is still zero, as in section 2.2. Also with $|\varphi| = 0$, the cutoff $R_k^{(2)}$ introduces some $\sqrt{\hat{\zeta}_1}$ that reduce the domain of reality of the beta functions to the half plane $\hat{\zeta}_1 > 0$. Moreover now there are also some logs which may give problems in the interval $-1 < \hat{\zeta}_2 < 0$. The qualitative behaviour in the bottom right quadrant is the same of the regulator $R_k^{(24)}$, with a separatrix leading to $FP_1$ at $\hat{\zeta}_1 \rightarrow \infty$ and delimiting the attractive basin of GFP₁. The main difference is its trajectory near to GFP₂, which is not linear, as could be expected from the different asymptotic behaviour observed in $U_1$ for large $\hat{Z}_2$.

C.4 Dimensionless field and four-derivative cutoff

Using $R_k^{(4)}$, we have again $\eta_2 = 0$. Then,

$$\beta_{\hat{\zeta}_1} = -2\hat{\zeta}_1 + \frac{3\hat{\gamma}}{8\pi^2 \hat{\zeta}_1^3} \left[ -\frac{\hat{\zeta}_1 (2 + \hat{\zeta}_1)}{1 + \hat{\zeta}_1} + 2 \log(1 + \hat{\zeta}_1) \right]$$

(C.12)

$$\beta_{\hat{\gamma}} = \frac{5\hat{\gamma}^2}{8\pi^2 \hat{\zeta}_1^4} \left[ \frac{\hat{\zeta}_1 (6 + 9\hat{\zeta}_1 + 2\hat{\zeta}_1^2)}{(1 + \hat{\zeta}_1)^2} - 6 \log(1 + \hat{\zeta}_1) \right]$$

(C.13)

The terms $\log(1 + \hat{\zeta}_1)$ becomes complex for $\hat{\zeta}_1 < -1$, so these beta functions are real only in the region $\hat{\zeta}_1 > -1$. The RG flow is similar to $R_k^{(24)}$ near GFP₂, but the behaviour for large $\hat{\zeta}_1$ is different: one can still observe a region where there is a flow from GFP₂ in the UV to GFP₁ at infinity in the IR, however there is no clear separatrix, because with this regulator there is no NGFP₁ in the chart $U_1$.

C.5 Vanishing regulator

We may put a prefactor $a$ in front of the regulator (2.1):

$$R_k^{(24)} = aZ_1(k^2 - q^2)\theta(k^2 - q^2) + aZ_2(k^4 - q^4)\theta(k^4 - q^4) \cdot$$

The beta functions of dimensionful couplings in the action depend on this parameter, but the qualitative features of the flow are the same for a large range of values of $a$. For our
discussion the main point to observe is that the value of $\tilde{g}$ at NGFP$_1$ decreases when $a$ increases, and increases when $a$ decreases.

The limit of vanishing regulator $a \to 0$ is interesting because it is related to dimensional regularization [21, 22]. In this limit NGFP$_1$ goes to $-\infty$, the point in the middle of the bottom side in figure 1, and the separatrix also disappears at infinity. The trajectories we have been discussing now fill up the bottom right quadrant and the conclusions regarding the mass of the ghost remain valid.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP$^3$ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

[1] A.L. Licht, A generalized asymptotic condition, *Annals Phys.* **34** (1965) 161.

[2] R. Jost, *The General Theory of Quantized Fields*, AMS, Providence, RI, U.S.A. (1965) [DOI].

[3] O.J. Rosten, Relationships Between Exact RGs and some Comments on Asymptotic Safety, *arXiv:1106.2544* [INSPIRE].

[4] D. Benedetti, R. Gurau, S. Harribey and D. Lettera, The F-theorem in the melonic limit, *JHEP* **02** (2022) 147 [arXiv:2111.11792] [INSPIRE].

[5] M. Safari, A. Stergiou, G.P. Vacca and O. Zanusso, Scale and conformal invariance in higher derivative shift symmetric theories, *JHEP* **02** (2022) 034 [arXiv:2112.01084] [INSPIRE].

[6] C. Wetterich, Exact evolution equation for the effective potential, *Phys. Lett. B* **301** (1993) 90 [arXiv:1710.05815] [INSPIRE].

[7] T.R. Morris, The Exact renormalization group and approximate solutions, *Int. J. Mod. Phys. A* **9** (1994) 2411 [hep-ph/9308265] [INSPIRE].

[8] G. Narain and R. Percacci, On the scheme dependence of gravitational $\beta$-functions, *Acta Phys. Polon. B* **40** (2009) 3439 [arXiv:0910.5390] [INSPIRE].

[9] G.P. de Brito, A. Eichhorn and R.R.L.d. Santos, The weak-gravity bound and the need for spin in asymptotically safe matter-gravity models, *JHEP* **11** (2021) 110 [arXiv:2107.03839] [INSPIRE].

[10] C. Laporte, A.D. Pereira, F. Saueressig and J. Wang, Scalar-tensor theories within Asymptotic Safety, *JHEP* **12** (2021) 001 [arXiv:2109.09566] [INSPIRE].

[11] C. Laporte, N. Locht, A.D. Pereira and F. Saueressig, Evidence for a novel shift-symmetric universality class from the functional renormalization group, *arXiv:2207.06749* [INSPIRE].

[12] R. Floreanini and R. Percacci, The Renormalization group flow of the Dilaton potential, *Phys. Rev. D* **52** (1995) 896 [hep-th/9412181] [INSPIRE].

[13] F.J. Wegner Some invariance properties of the renormalization group, *J. Phys. C* **7** (1974) 2098.

[14] S. Weinberg, *Ultraviolet Divergences In Quantum Theories Of Gravitation*, in *General Relativity*, S.W. Hawking and W. Israel eds., Cambridge University Press (1980), pp. 790–831 [INSPIRE].
[15] A. Baldazzi, R.B.A. Zinati and K. Falls, *Essential renormalisation group*, arXiv:2105.11482 [inSPIRE].

[16] D. Anselmi, *Absence of higher derivatives in the renormalization of propagators in quantum field theories with infinitely many couplings*, Class. Quant. Grav. 20 (2003) 2355 [hep-th/0212013] [inSPIRE].

[17] K. Symanzik, *A field theory with computable large-momenta behavior*, Lett. Nuovo Cim. 6S2 (1973) 77 [inSPIRE].

[18] A.D. Bond and D.F. Litim, *More asymptotic safety guaranteed*, Phys. Rev. D 97 (2018) 085008 [arXiv:1707.04217] [inSPIRE].

[19] G. ’t Hooft, *The birth of asymptotic freedom*, Nucl. Phys. B 254 (1985) 11 [inSPIRE].

[20] C.F. Steinwachs, *Non-perturbative renormalization of shift-symmetric scalar field theories*, unpublished.

[21] A. Baldazzi, R. Percacci and L. Zambelli, *Functional renormalization and the $\overline{\text{MS}}$ scheme*, Phys. Rev. D 103 (2021) 076012 [arXiv:2009.03255] [inSPIRE].

[22] A. Baldazzi, R. Percacci and L. Zambelli, *Limit of vanishing regulator in the functional renormalization group*, Phys. Rev. D 104 (2021) 076026 [arXiv:2105.05778] [inSPIRE].