A COMMUTATOR METHOD FOR COMPUTATION OF HEAT INVARIANTS

IOSIF POLTEROVICH

Abstract. We introduce a new method for computing heat invariants $a_n(x)$ of a 2-dimensional Riemannian manifold based on a commutator formula derived by S. Agmon and Y. Kannai. Two explicit expressions for $a_n(x)$ are presented. The first one depends on the choice of a certain coordinate system; the second involves only invariant terms but has some restrictions on its validity, though in a “generic” case it is well-defined.

1. Introduction and main results

Let $M$ be a $d$-dimensional compact Riemannian manifold without boundary with a metric $\{g_{ij}\}$. Let $\Delta$ be the Laplace–Beltrami operator (or simply the Laplacian) on $M$. In the coordinate chart $\{u_i\}$ it is written in the form

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^d \frac{\partial(\sqrt{g} g^{ij} (\partial f / \partial u_i))}{\partial u_j},$$

(1.1)

where $g = \text{det}\{g_{ij}\}$, and $\{g^{ij}\}$ denotes the inverse of the matrix $\{g_{ij}\}$.

The Laplacian is a self-adjoint elliptic operator acting on smooth functions $f : M \to \mathbb{R}$. Let $\{\lambda_i, \varphi_i\}_{i=1}^\infty$ be the spectral decomposition of $\Delta$ (considered as an operator in $L^2(M)$) into a complete orthonormal basis of eigenfunctions $\varphi_i$ and eigenvalues $\lambda_i$, $0 \leq \lambda_1 \leq \lambda_2 \ldots$. Consider the heat operator $e^{-t\Delta}$ which is the solution of the heat equation $(\partial / \partial t + \Delta)f = 0$. For $t$ positive, it is an infinitely smoothing operator from $L^2(M)$ into itself and has a kernel function

$$K(t, x_1, x_2) = \sum e^{-\lambda_i t} \varphi_i(x_1) \otimes \varphi_i(x_2),$$

analytic in $t$ and $C^\infty$ in $x_1$ and $x_2$, such that

$$(e^{-t\Delta}f)(x_1) = \int_M K(t, x_1, x_2) f(x_2) dx_2.$$

The following asymptotic expansion holds for the trace of the heat kernel:

$$K(t, x, x) \sim \sum_{n=0}^\infty a_n(x) t^{n-\frac{d}{2}}$$

as $t \to 0^+$ ([G1]). The coefficients $a_n(x)$ are called heat invariants of the manifold $M$; they depend only on the germs of the metric. Moreover, it is known that the heat invariants are homogeneous polynomials of degree $2n$ in the derivatives of $\{g^{ij}\}$ at the point $x$ ([G2]).
Integrating $a_n(x)$ over the manifold one gets the coefficients of the expansion of the trace of the heat operator:

$$\sum_{i} e^{-t\lambda_i} \sim \sum_{n=0}^{\infty} \left( \int_{M} a_n(x, \Delta) \rho(x) dx \right) t^{n-d/2} \sim \sum_{n=0}^{\infty} a_n t^{n-d/2},$$

where $\rho(x)dx$ is the volume form on the manifold $M$.

Heat invariants are metric invariants of the given manifold $M$ and contain a lot of information about its geometry and topology (see [Be], [G2], [G3], etc.). For example, $a_0$ determines the volume of $M$, $a_1$ its scalar curvature (and hence the Euler characteristic in dimension 2). Higher coefficients are of a considerable interest to physicists since they are connected with many notions of quantum gravity (see [F]).

The structure of the heat invariants becomes more and more complicated as $n$ grows; compare the formula for $a_3$ ([Sa], 1971) with the formulas for $a_4$ ([ABC], 1989; [Av], 1990) and $a_5$ ([vdV], 1997). For $n > 5$ no explicit expression for the heat invariants is known, even though there are many algorithms to derive them — the computational complexity is so high that even with the help of modern computers it is impossible to get the result.

In the present paper we introduce a new method for computing heat invariants. We call it the commutator method since it is substantially based on a commutator formula by S. Agmon and Y. Kannai ([AK]) which deals with asymptotic expansions of the resolvent kernels of elliptic operators (let us note that the asymptotics of the relevant kernels associated with a self-adjoint realization of the elliptic operator depend only on the local behavior of its coefficients).

In section 2 we obtain the following concise reformulation of the Agmon–Kannai formula:

**Theorem 1.2.** Let $H$ be a a self–adjoint elliptic differential operator of order $s$ on a manifold $M$ of dimension $d < s$, and $H_0$ be the operator obtained by freezing the coefficients of the principal part $H'$ of the operator $H$ at some point $x \in M$: $H_0 = H'_x$. Denote by $R_\lambda(x_1, x_2)$ the kernel of the resolvent $R_\lambda = (H - \lambda)^{-1}$, and by $F_\lambda(x_1, x_2)$ — the kernel of $F_\lambda = (H_0 - \lambda)^{-1}$. Then for $\lambda \to \infty$ there is a following asymptotic representation on the diagonal:

$$R_\lambda(x, x) \sim \rho(x)^{-1} \sum_{m=0}^{\infty} X_m F_\lambda^{m+1}(x, x),$$

and the operators $X_m$ are defined inductively by:

$$X_0 = I; \quad X_m = X_{m-1} H_0 - H X_{m-1}.$$

The commutator method may be applied to various problems of “heat kernel”–type (one may consider other operators instead of Laplacian, Laplacian on forms, etc. — see also [Kan] for the Agmon–Kannai formula in case
of matrix operators). In this paper we are focusing on the simplest non-trivial question — what are the heat invariants of a 2-dimensional Riemannian manifold?

An advantage of the commutator method is that it yields very explicit formulas for all 2-dimensional heat invariants (cf. [Xu] where heat invariants of manifolds of arbitrary dimension are considered). They are given by the following

**Theorem 1.4.** Let $(u, v)$ be local coordinates in a neighborhood of the point $x = (0, 0)$ on the Riemannian 2-manifold $M$, in which the metric is conformal:

$$ds^2 = \rho(u, v)(du^2 + dv^2).$$

Then the heat invariants $a_n(x)$ are given by

$$a_n(x) = \sum_{m=n+1}^{4n} \sum_{k=n+1}^{k-n} \sum_{s=0}^{k-n} C_{nksm} \rho(u, v)^k \Delta^k \left( u^{2k-n-2s} v^{2s} \right) \bigg|_{u=0, v=0},$$

where $C_{nksm}$ are constants given by (3.12).  

(1.5)

The proof of this theorem is presented in section 3. We also perform a test of the formula (1.5) which is described in the Appendix.

Let us note that local coordinates in which the metric is conformal always exist (see [DNF]). We also observe that the derivatives of $\rho(u, v)$ appear in (1.5) through the formula (1.1) for the coefficients of the Laplacian.

The expression (1.5) depends on a coordinate representation but it is possible to make it invariant rewriting it in special curvature coordinates which are introduced in section 4. The resulting formula is no longer valid in general but is true “generically”.

**Acknowledgments.** This paper is a part of my Ph.D. research conducted under the supervision of Professor Y. Kannai. I am very grateful to Professor Kannai for his constant help and encouragement. I would also like to thank Professor L. Polterovich and Professor M. Solomyak for many fruitful discussions, and my colleagues J. Greenstein and A. Grigoriev for helpful advice.

2. **Multiple commutators**

We recall the notion of a multiple commutator of two operators, introduced in ([AK]), which plays a fundamental role in the sequel.

**Definition 2.1.** Let $J$ be a finite vector with nonnegative integer components and let $A, B$ be linear operators on some linear space $M$. The multiple commutator $[B, A; J]$ is inductively defined for all such vectors $J$ in the following way:

1. $[B, A; 0] = B$
2. $[B, A; j] = [[B, A; j - 1], A]$
3. \([B, A; J \cup j] = [B[B, A, J], A; j]\),

where \(J \cup j\) denotes the vector obtained by adding the component \(j\) to the vector \(J\) to the right.

We introduce a filtration on the space of all vectors with non-negative integer entries. Denote by \(V_m = \{J = (j_1, \ldots, j_r) : |J| + r = m\}\), where \(|J| = j_1 + \ldots + j_r\). Let

\[ X_m = \sum_{J \in V_m} [B, A; J], \quad m \geq 1. \quad (2.2) \]

**Theorem 2.3.** The operators \(X_m\) satisfy the following recurrent relation,

\[ X_1 = B; \quad X_m = BX_{m-1} + [X_{m-1}, A], \quad (2.4) \]

and are given explicitly by

\[ X_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (A - B)^k A^{m-k}. \quad (2.5) \]

**Proof.** Let us proceed by induction. Indeed, \(X_1 = B\) since \(V_1\) consists of a single vector \((0)\). In order to pass from \(X_{m-1}\) to \(X_m\) we do the following. Let \(V_0^m\) denote the set of all vectors in \(V_m\) whose last entry is 0. Consider a mapping \(p : V_m \rightarrow V_{m-1}\) defined as follows: for \(J = (j_1, \ldots, j_r, 0) \in V_0^m\) we have \(p(J) = (j_1, \ldots, j_r)\) and for \(J = (j_1, \ldots, j_r) \in V_m \setminus V_0^m\) we have \(p(J) = (j_1, \ldots, j_r - 1)\). It is clear that the mapping \(p\) is well-defined. Moreover, it is a surjection and every element of \(V_{m-1}\) has exactly two preimages — one in \(V_0^m\) and one in its complement. Applying the definition of a multiple commutator we prove the inductive step — \(BX_{m-1}\) is the contribution of vectors from \(V_0^m\) and \([X_{m-1}, A]\) is the contribution of vectors belonging to \(V_m \setminus V_0^m\). This completes the proof of the first part of the theorem. For obtaining the closed formula, let us rewrite the recurrent relations in the following way: \(X_1 = A - (A - B) = B;\) \(X_m = X_{m-1}A - (A - B)X_{m-1}\). Proceeding by induction and recalling that \(\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}\) we get (2.5). \(\square\)

Let us present the original Agmon–Kannai formula (see [AK]):

**Theorem 2.6.** In the notations and conditions of the Theorem 1.2 the following asymptotic representation on the diagonal holds for the kernel of the resolvent \(R_\lambda\):

\[ R_\lambda(x, x) \sim \rho(x)^{-1} (F_\lambda(x, x) + \sum_{J} ([H_0 - H; J] F^{[J]_{|r+1}}(x, x))), \quad (2.7) \]

where the sum is taken over all vectors \(J\) of length \(\geq 1\) with nonnegative integer entries.
Now we may prove the Theorem 1.2.

**Proof of the Theorem 1.2.** Let us divide the above sum into sums over vectors \( J \in V_m, m \geq 1 \). Then, setting \( A = H_0, B = H_0 - H \) and applying Theorem 2.3, we get formula (1.3), which completes the proof.

3. The commutator method in dimension 2

Theorem 1.2 is not valid for the Laplacian if \( d \geq 2 \). A way to overcome this difficulty for \( d = 2 \) is to consider differences of resolvents (this was pointed out to me by Y.Kannai).

**Lemma 3.1.** Let \( R_\lambda = (\Delta - \lambda)^{-1} \) be the resolvent of the Laplacian on a 2-manifold \( M \). Consider the difference of the resolvents \( R_{\lambda, 2\lambda} = R_\lambda - R_{2\lambda} \). This operator has a continuous kernel \( R_{\lambda, 2\lambda}(x_1, x_2) \) which has the following asymptotic representation on the diagonal as \( \lambda \to \infty \)

\[
R_{\lambda, 2\lambda}(x, x) \sim \sum_{n=0}^{\infty} b_n(x)(-\lambda)^{-n},
\]

where

\[
b_n(x) = (n-1)! \cdot a_n(x), \quad n > 1.
\]

**Proof.** The difference of the resolvents \( R_{\lambda, 2\lambda} \) is a self–adjoint smoothing operator from \( L^2(M) \) into the Sobolev space \( H^4(M) \), and \( \dim M < 4 \), therefore it has a continuous kernel (see [AK]).

Let us now prove the expansion (3.2).

Set \( z = -\lambda \), and let \( \Re z > 0 \). We have (formally):

\[
\int_0^\infty e^{-t(\Delta + z)} dt = \frac{1}{z + \Delta}
\]

Therefore

\[
\frac{d}{dz} \left( \frac{1}{z + \Delta} \right) = -\int_0^\infty t e^{-t(\Delta + z)} dt \sim -\int_0^\infty t \sum_{n=0}^\infty t^{-1+n} a_n e^{-zt} dt.
\]

Let us integrate both parts from \( z \) to \( 2z \):

\[
\frac{1}{z + \Delta} - \frac{1}{2z + \Delta} \sim \int_0^2 \sum_{n=0}^\infty a_n t^n e^{-zt} dt.
\]

For \( n = 0 \) we get:

\[
\int_z^{2z} a_0 \int_0^\infty e^{-zt} dt dz = \int_z^{2z} a_0 \frac{dz}{z} = a_0 \log 2 = a_0'
\]
For \( n \geq 1 \) we get:

\[
\int_0^\infty t^n e^{-z t} \, dt \, dz = \Gamma(n + 1) a_n \int_0^\infty \frac{1}{z^n} (\frac{1}{z^n} - \frac{1}{(2z)^n}) \, dz = (n - 1)! a_n \frac{1}{z^n}.
\]

This completes the proof of the lemma. \( \square \)

**Lemma 3.4.** Let \( R_\lambda(x_1, x_2) \) be the kernel of the resolvent \( R_\lambda = (\Delta - \lambda)^{-1} \)

Then for \( \lambda \to \infty \) the difference of the resolvents has the following asymptotic representation on the diagonal:

\[
R_{\lambda, 2\lambda}(x, x) \sim \rho(x)^{-1} \sum_{m=0}^{\infty} X_m \left( F^{m+1}_\lambda(x, x) - F^{m+1}_{2\lambda}(x, x) \right), \quad (3.5)
\]

where

\[
X_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \Delta^k \Delta_0^{m-k}. \quad (3.6)
\]

**Proof.** Setting \( \Delta = H \) and applying Theorem 1.2 and Lemma 3.1 completes the proof. Note that we can define \( X_0 = I \) without loss of consistence of the recurrent formula (2.4.). \( \square \)

Now we are ready to prove an explicit formula for the heat invariants in two dimensions.

**Proof of the Theorem 1.4.** Let \( (u, v) \) be conformal local coordinates in the neighborhood of the point \( x = (0, 0) \) (we can assume that \( x \) is the origin without loss of generality) on the 2-manifold \( M \). As it was mentioned in the introduction this is always possible. The metric in such coordinates has the form

\[
ds^2 = \rho(u, v) (du^2 + dv^2), \quad (3.7)
\]

and hence the Laplacian is given by

\[
\Delta = -\rho(u, v)^{-1} (\partial^2 / \partial u^2 + \partial^2 / \partial v^2). \quad (3.8)
\]

Our aim is to find the coefficient \( a_n(x) \). In order to do this we have to collect all terms in the sum (3.5) containing \((-\lambda)^{-n}\). We use the following well-known formula (see [AK]):

\[
\frac{\partial^{\alpha_1 + \alpha_2}}{\partial u^{\alpha_1} \partial v^{\alpha_2}} F^{m+1}_\lambda(x, x) = (-\lambda)^{|\alpha|} \frac{1}{2^m} (-1)^{\frac{|\alpha|}{2}} \int_{\mathbb{R}^2} \frac{\xi_1^{\alpha_1} \xi_2^{\alpha_2}}{(\Delta'_x(\xi_1, \xi_2) + 1)^{m+1}} \, d\xi_1 d\xi_2, \quad (3.9)
\]

where \( \Delta'_x(\xi_1, \xi_2) \) denotes the symbol of the operator \( \Delta'_x \), and \((-\lambda)^{|\alpha|} \frac{1}{2^m} \) is the analytic branch of the power which is positive on the negative axis.

By formula (3.9) we get \(|\alpha| = 2(|J| + r - n) = 2m - n\). Since the integral (3.9) vanishes unless both \( \alpha_1 \) and \( \alpha_2 \) are even we may put \( \alpha_1 = 2m - 2n - 2p \),
\( \alpha_2 = 2p, \) where \( 0 \leq p \leq 2(m - n). \) In polar coordinates this integral transforms into the product of two integrals:

\[
\int_{\mathbb{R}^2} \frac{(\Delta_x'(\xi_1, \xi_2) + 1)^{m+1}}{d\xi_1 d\xi_2} = \frac{1}{2} \int_0^{2\pi} \cos^{2m-2n-2p} \varphi \sin^{2p} \varphi \, d\varphi \int_0^\infty \frac{\rho^{m-n} dt}{(t/\rho_0 + 1)^{m+1}} = \rho_0^{m-n+1} B(m - n - p + 1/2, p + 1/2) B(m - n + 1, n),
\]

where \( \rho_0 = \rho(0,0), \) and \( B \)-functions stand for the values of the first and the second integral in the product respectively (cf. [GR], formulas 3.621(5) and 3.194(3); note that formula 4.368(3) which could be used directly for computation of the integral in (3.9) is inaccurate). Expressing the \( B \)-functions in terms of \( \Gamma \)-functions we find that (3.9) may be represented as \((-\lambda)^{-n} I_{mnp},\) where:

\[
I_{mnp} = (-1)^{m-n} \frac{1}{4\pi^2 \rho_0^{m-n+1}} \frac{\Gamma(m - n - p + 1/2) \Gamma(p + 1/2) \Gamma(n)}{\Gamma(m + 1)}.
\]

Since \( |\alpha| \) should be positive, we have \( m > n. \) On the other hand, as it was proved in ([AK]) the order of every operator \([B, A; J]\) is not greater than \( |J| + 2r \) and hence \( |\alpha| \leq |J| + 2r \) which implies \( |J| \leq 2n. \) Another statement from ([AK]) gives the same estimate on the length of the vector: \( r \leq 2n. \)

Therefore if we are interested in the coefficient \( a_n \) it is sufficient to consider \( X_m \) with \( m \leq 4n. \)

Together with (3.3), (3.5) and (3.6) this implies:

\[
a_n(x) = \frac{1}{(n-1)!} \rho(u, v)^{-1} \cdot \\
\cdot \sum_{m=n+1}^{4n} \sum_{k=0}^{m-n} (-1)^k \binom{m}{k} \Delta^k \Delta_0^{m-k} \left( \sum_{p=0}^{m-n} I_{mnp} \frac{u^{2m-2n-2p} v^{2p}}{(2m-2n-2p)! (2p)!} \right) |_{u=0, v=0, t=0}^{t=0} \tag{3.10}
\]

Computing explicitly \( \Delta_0^{m-k} \) using (3.8) we may apply the obtained differential operator to the last sum in (3.10). Afterwards we change for convenience one of the summation indices and omit the vanishing terms. Finally we obtain:

\[
a_n(x) = \sum_{m=n+1}^{4n} \sum_{k=n+1}^{m} \sum_{s=0}^{k-n} C_{nksm} \rho(u, v)^{k-n} \Delta^k \left( u^{2k-2n-2s} v^{2s} \right) |_{u=0, v=0, t=0}, \tag{3.11}
\]

where the constants \( C_{nksm} \) are given by

\[
C_{nksm} = \frac{(-1)^n}{4\pi^2} \sum_{l=0}^{m-k} \frac{\Gamma(k + l - n - s + 1/2) \Gamma(s + m - k - l + 1/2)}{k! l! (m - k - l)! (2k - 2n - 2s)! (2s)!}. \tag{3.12}
\]
This completes the proof of the Theorem 1.4.

4. CURVATURE COORDINATES AND AN INVARIANT EXPRESSION FOR $a_n(x)$ IN A “GENERIC” CASE

The commutator method essentially involves non-invariant coordinate representation. Nevertheless, using a special substitution it is possible to get the result in an invariant form, though in such a way that it is valid for in some sense “generic” germs of curvature functions on 2-manifolds.

**Definition 4.1.** Let $x$ be a point on the manifold $M$, and $K(x) = K_0$ be the Gaussian curvature at this point. Let $(z, w)$ be coordinates near the point $x$ and suppose that in a neighborhood of $x$ the map

$$(z, w) \rightarrow (z_K, w_K) = (K - K_0, \Delta K - \Delta K_0),$$

where $\Delta K_0 = (\Delta K)(x)$, has a nonvanishing Jacobian. The coordinates $(z_K, w_K)$ are called the curvature coordinates in the neighborhood of the point $x$.

**Lemma 4.2.** The operator $\Delta_0 = \Delta'_x$ may be written in curvature coordinates as

$$\Delta_0 = -(E \partial^2 / \partial z_K^2 + 2F \partial^2 / \partial z_K \partial w_K + G \partial^2 / \partial w_K^2),$$

where

$$2E = 2g^{11} = -\Delta(z_K^2) = 2K \Delta K - \Delta(K^2);$$

$$2F = 2g^{12} = -\Delta(z_K w_K) = K \Delta^2 K + (\Delta K)^2 - \Delta(K \Delta K);$$

$$2G = 2g^{22} = -\Delta(w_K^2) = 2\Delta K \Delta^2 K - \Delta(\Delta K)^2,$$

and all the values are taken at the point $x = (0, 0)$.

**Lemma 4.3.** In the new coordinates

$u = \sqrt{EG - F^2} z_K \quad v = Ew_K - F z_K$,

the Riemannian metric $\{g_{ij}\}$ takes the following form at the point $x = (0, 0)$:

$$ds^2 = \frac{1}{E(EG - F^2)} (du^2 + dv^2), \quad (4.4)$$

where $E$, $F$, $G$ are as in Lemma 4.2.

Lemmas 4.2. and 4.3. are proved by easy computations.

The proof of Theorem 1.4. uses the conformal form of the metric in fact only at the point $x = (0, 0)$, therefore we may apply this theorem to (4.4), and expressing the result in curvature coordinates we obtain:
Theorem 4.5. Suppose that curvature coordinates exist in some neighborhood of the point \( x = (0, 0) \in M \). Then the heat invariants \( a_n(x) \) are given by the following formula

\[
a_n(x) = \sum_{m=n+1}^{4n} \sum_{k=n+1}^{m} \sum_{s=0}^{k-n} \sum_{p=0}^{2s} \frac{(-1)^p}{p} \binom{2s}{p} C_{nksm} \frac{E^{n-k+p} F^{2s-p}}{(GE - F^2)^p} \cdot \Delta^k \left( (K - K_0)^{2k-2n-p}(\Delta K - \Delta K_0)^p \right) |_{x=(0,0)},
\]

where \( C_{nksm} \) are the same as in formula (3.12).

Note that all terms in this expression are invariant — indeed, it is constructed from the powers of the Laplacian and the Gaussian curvature taken at the given point.

It is not always possible to introduce curvature coordinates — for example, on a standard round sphere the Gaussian curvature is constant and therefore such coordinates are degenerate. Nevertheless, they do exist “generically”:

Theorem 4.7. In the space \( J^r \) of \( r \)-jets \((r \geq 3)\) of germs of analytic functions \( K(u, v) \) at the point \( x = (0, 0) \), the set of \( r \)-jets for which the “curvature coordinates” are degenerate forms an algebraic set of codimension at least 1.

Proof. We demand the curvature function \( K(x), x \in M \) be analytic near the point \( x \in M \) in some local coordinates (and hence in any). One may show that every germ of an analytic function maybe taken as a germ of some curvature function. Indeed, if we write the metric locally in conformal form (3.7) the Gaussian curvature is given by

\[
K(u, v) = \frac{1}{2} \Delta(\log(\rho(u, v))).
\]

Clearly, for every analytic function \( K(u, v) \) there exists a function \( \rho(u, v) \) satisfying the above PDE with initial conditions \( K(0, 0) = K_0 \) (by Cauchy–Kovalevskaya theorem — see, for example, [ES]).

Consider the space \( J^r \) of \( r \)-jets \((r \geq 3)\) of germs of analytic functions \( K(u, v) \) at the point \( x = (0, 0) \). Non-degeneracy of the “curvature coordinates” is equivalent to the nonvanishing condition for the determinant of the Jacobi matrix \( \text{Jac}(K, \Delta K) \) at the point \( x \). The equation \( \text{Jac}(K, \Delta K) = 0 \) is an algebraic equation in the space \( J^r \), therefore (since the equality is obviously not identical) the set of its solutions is an algebraic set of codimension at least 1 (see [Br]) in the space \( J^r \).

Appendix

As a further check of the formula (3.11) we programmed it (together with its slight modification for diagonalized metrics) using the Mathematica software ([Wo]).
Here is the result of the work of the program for $a_1(x)$:

$$a_1(x) = \frac{1}{24\pi} \left( \frac{\rho_u^2 + \rho_v^2 - \rho_u\rho - \rho_v\rho}{\rho^3} \right)$$

On the other hand, using the formula (4.8) we get that

$$K = \frac{\rho_u^2 + \rho_v^2 - \rho_u\rho - \rho_v\rho}{2\rho^3},$$

and hence

$$a_1(x) = \frac{K}{12\pi} = \frac{\tau}{24\pi},$$

where $\tau = 2K$ is the scalar curvature at the point $x$. This coincides with the well-known formula for $a_1(x)$ (cf. [G3]).

Let us emphasize that our aim was just to test the formula (3.11); for computational purposes *Mathematica* is not an efficient tool for our method — it does not manage to calculate even $a_2(x)$ for a general metric in reasonable time (though in particular cases of a “simple” conformal metric with $\rho(u,v) = 1/(a_0 + a_1u + a_2v)$, or a spherical metric $ds^2 = R^2(d\theta^2 + \sin^2 \theta d\phi^2)$ — here we used a modification of (3.11) for diagonalized metrics, — we found $a_2(x)$, and the result also agreed with the already known (see [G1],[G3])).

The main difficulty is to compute high powers of the Laplacian, which is a differential operator with non-constant coefficients, and therefore the number of terms in the expression for $\Delta^k$ grows exponentially in $k$. However, we think that programming the commutator method in a language like C or in some lower level language one may obtain completely explicit expressions for higher heat invariants and, hopefully, even for some unknown ones.

References

[ABC] P.Amsterdamski, A.Berkin, and D.O’Connor, $b_8$ “Hamidew” coefficient for a scalar field, Class. Quant. Grav. vol.6, (1989), 1981-1991.

[AK] S.Agmon and Y.Kannai, On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators, Israel J. Math. 5 (1967), 1-30.

[Av] I.V.Avramidi, The covariant technique for the calculation of the heat kernel asymptotic expansion, Phys. Let. B 238 (1990), 92-97.

[Be] M.Berger, Geometry of the spectrum, Proc. Symp. Pure Math. 27 (1975), 129-152.

[Br] Th. Bröcker, Differential germs and catastrophes, London Math. Soc. Lect. Note Series, Cambridge Univ. Press, (1975).

[DNF] B.A.Dubrovin, S.P.Novikov and A.T.Fomenko, Modern geometry: methods and applications, Nauka, (1986) (in Russian).

[ES] Yu.V. Egorov, M.A. Shubin, Partial Differential Equations I, Encycl. Math. Sci. vol. 30, Springer–Verlag, (1992).
[F] S.A. Fulling ed., Heat Kernel Techniques and Quantum Gravity, Discourses in Math. and its Appl., No. 4, Texas A&M Univ., (1995).

[G1] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Diff. Geom. 10 (1975), 601-618.

[G2] P. Gilkey, The index theorem and the heat equation, Math. Lect. Series, Publish or Perish, (1974).

[G3] P. Gilkey, Heat equation asymptotics, Proc. Symp. Pure Math. 54 (1993), 317-326

[G4] P. Gilkey, Invariance theory, the heat equation and the Atiyah–Singer index theorem, Math. Lect. Series, Publish or Perish, (1984)

[GR] I.S. Gradshtein, I.M. Ryzhik, Table of integrals, series and products, Academic Press, (1980).

[Kan] Y. Kannai, On the asymptotic behavior of resolvent kernels, spectral functions and eigenvalues of semi–elliptic systems, Ann. Sc. Norm. Sup. Pisa, Classe di Scienze XXIII, (1969), 563-634.

[Sa] T. Sakai, On the eigenvalues of the Laplacian and curvature of Riemannian manifold, Tôhoku Math. J. 23 (1971), 585-603.

[vdV] A.E.M. van de Ven, Index–free heat kernel coefficients, hep-th/9708152 (1997), 1–38.

[Wo] S. Wolfram, Mathematica: a system for doing mathematics by computer, Addison–Wesley, (1991).

[Xu] C. Xu, Heat kernels and geometric invariants I, Bull. Sc. Math. 117 (1993), 287-312.

Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail address: iossif@wisdom.weizmann.ac.il