Measuring Momentum-Dependent Flow Fluctuations in Heavy-Ion Collisions

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Momentum-dependent two-particle correlations in heavy-ion collisions can be fully characterized by a principal component analysis (PCA). In particular, subleading PCA modes of azimuthal Fourier harmonics are expected to reveal new information about hydrodynamic flow fluctuations. However, we find that as currently measured, it can instead be dominated by fluctuations of particle number, which can be measured independently and therefore represent redundant information that serves as an unwanted background. Here we propose a redefinition of the PCA observables which is insensitive to multiplicity fluctuations and thus free of redundancies. The proposed observables isolate novel sources of flow fluctuations from two-particle correlations and can provide fresh insight into the properties of the initial stages of the system at small length scales.

INTRODUCTION

According to the hydrodynamic picture of relativistic heavy-ion collisions, observed momentum anisotropies arise from the response of the quark-gluon plasma (QGP) to the geometry of the initial state [1,3]. In particular, local inhomogeneities of the initial transverse profile of the system are expected to produce momentum-dependent final-state anisotropies [4,8]. As the initial conditions fluctuate event by event, momentum-dependent azimuthal correlations develop among particles in their final state. The detailed momentum dependence of two-particle correlations is, so far, the only available probe of the granularity of initial-state fluctuations, to which other more standard observables are insensitive [6–8]. Therefore, a thorough study of two-particle correlations and their momentum dependence is needed in order to resolve the relevant sub-nucleonic length scales present in the initial state of heavy-ion collisions.

In Ref. [4], diagonal and off-diagonal momentum-dependent correlations were investigated with the introduction of the factorization breaking ratio $r_n(p_T^a, p_T^b)$, which measures the correlation of the anisotropic flow between two different transverse momenta [4,7,9,12]. In Ref. [13], the principal component analysis (PCA) of flow fluctuations was proposed to characterize the same correlations in a more concise and physically transparent way [13,17]. Measurements of the factorization breaking ratio and the PCA of event-by-event fluctuations were presented in Refs. [18,21].

Principal component analysis is a statistical method used to find linearly uncorrelated combinations of correlated variables through the spectral decomposition of the covariance matrix [22]. These combinations, referred to as principal components, are ordered according to their variances, which are eigenvalues of the covariance matrix. Hence, this method not only sorts out independent fluctuations but also arranges them according to importance, while requiring no model or assumptions whatsoever. In the case of anisotropic flow, the leading PCA component is related to more traditional measures of the flow harmonics, while the subleading ones are supposed to carry more detailed information on momentum-dependent flow fluctuations [13,16,18].

In this Letter, we discuss a crucial issue with the interpretation of the PCA observables as defined in [13]. This issue, related to multiplicity fluctuations, leads to unexpected redundancies, which we illustrate using experimental data from the CMS collaboration [18]. In this paper we introduce a new version of the PCA observables that factors out those redundancies and properly reflects momentum-dependent anisotropic flow fluctuations. We believe the redefined observables to be the ultimate tool for investigating two-particle correlations. By isolating new sources of correlation, they could be particularly useful for constraining properties of the QGP and its initial stages.

After hydrodynamic expansion, particles at a given transverse momentum $p_T$ and rapidity $y$ are azimuthally distributed according to a probability distribution

$$E \frac{dN}{d\phi} = \frac{1}{2\pi} N(p_T, y) \sum_{n=-\infty}^{\infty} V_n(p_T, y) e^{-in\phi}, \quad (1)$$

where $V_n(p_T, y)$ are complex flow vectors and $N(p_T, y)$ is the particle density as a function of rapidity and transverse momentum. Due to the random orientation of Fluids, it is often necessary to account for the orientation of the fluid in the calculations. This can be done using the concept of pseudo-rapidity, which is defined as

$$\eta = \frac{1}{2} \ln \left( \frac{1+\tan(y/2)}{1-\tan(y/2)} \right).$$

For simplicity we do not distinguish momentum rapidity and pseudo-rapidity in our convention.
events, \(\langle V_n(p_T, y) \rangle = 0\), where \(\langle \cdots \rangle\) represents an average over events. While particles are understood as independent samples of distribution \(\text{I}\), the limited number of particles makes it impossible to accurately measure \(V_n(p)\) in each event. One must instead study its statistical properties in an ensemble of events via multiparticle correlations. The most studied are two-particle correlations.

We are interested in fluctuations of the flow anisotropy as a function of momentum, given by the harmonics \(V_n(p_T, y)\). The main features of these fluctuations are captured by the covariance matrix

\[
V_{n\Delta}(p_1, p_2) \equiv \langle V_n^*(p_1) V_n(p_2) \rangle, \tag{2}
\]

where we denote \(p = (p_T, y)\). Note that flow fluctuations are correlated across different momenta and \(V_{n\Delta}(p_1, p_2)\) is a nondiagonal matrix. In case there is only one source of fluctuations (e.g., fluctuations of the system orientation), \(V_{n\Delta}\) has only one nonvanishing eigenvalue and factorizes as

\[
V_{n\Delta}(p_1, p_2) \overset{\text{fact.}}{=} \sqrt{\langle |V_n(p_1)|^2 \rangle} \sqrt{\langle |V_n(p_2)|^2 \rangle}. \tag{3}
\]

However, this factorization is not perfect, indicating more than one relevant eigenvalue \(\text{I}\). In general, each nonvanishing eigenvalue of a covariance matrix corresponds to a linearly uncorrelated fluctuation mode. The sub-dominant fluctuations signaled by the breaking of Eq. \(\text{I}\) imply that particles of different momenta respond differently to initial-state fluctuations.

In Eq. \(\text{I}\), the distribution of particles also depends on the density of particles \(N(p)\). Fluctuations of \(N(p)\) can be studied similarly with the covariance matrix \(\text{I}\)

\[
N_{\Delta}(p_1, p_2) \equiv \langle \Delta N(p_1) \Delta N(p_2) \rangle, \tag{4}
\]

where \(\Delta(\cdots) \equiv (\cdots) - \langle (\cdots) \rangle\). This covariance matrix contains nontrivial information about fluctuations of mean transverse momentum and their correlation with global multiplicity fluctuations \(\text{I}\).

**PRINCIPAL COMPONENT ANALYSIS**

The principal component analysis of event-by-event fluctuations was introduced in Ref. \(\text{I}\). The idea is to isolate the linearly independent fluctuation modes contributing to \(V_{n\Delta}\). According to the spectral theorem,

\[
V_{n\Delta}(p_a, p_b) = \sum_{\alpha=1}^{\infty} \lambda_n^{(\alpha)}(p_a) \lambda_n^{(\alpha)}(p_b)
= \sum_{\alpha=1}^{\infty} V_n^{(\alpha)}(p_a) V_n^{(\alpha)}(p_b), \tag{5}
\]

where \(\lambda_n^{(\alpha)}(p)\) and \(\psi_n^{(\alpha)}(p)\) are the eigenvalues and normalized eigenvectors of \(V_{n\Delta}\). Since \(\text{Im}[V_n^*(p_1) V_n(p_2)]\) is not invariant under parity transformations, \(V_{n\Delta}\) is taken to be real \(\text{I}\). Moreover, a covariance matrix must have positive eigenvalues. Labeling the eigenvalues in descending order, \(\lambda_n^{(\alpha)} \geq \lambda_n^{(\alpha+1)}\), an approximation can be found by truncating the sum in Eq. \(\text{I}\) at \(\alpha = k\). The principal components, or modes, of the flow fluctuations are defined as

\[
V_n^{(\alpha)}(p) \equiv \sqrt{\lambda_n^{(\alpha)}} \psi_n^{(\alpha)}(p). \tag{6}
\]

The physical interpretation of the PCA can be clarified by projecting \(V_n(p)\) onto the basis defined by \(\{V_n^{(\alpha)}(p)\}\):

\[
V_n(p) \approx \sum_{\alpha=1}^{k} \xi_n^{(\alpha)} V_n^{(\alpha)}(p), \tag{7}
\]

where \(\xi_n^{(\alpha)} = 0\). From Eqs. \(\text{I}\) and \(\text{I}\) one finds that \(\langle \xi_n^{(\alpha)} \rangle \delta_{\alpha\beta} = \delta_{\alpha\beta}.\) Indeed, the PCA isolates linearly uncorrelated fluctuation modes, with both their magnitudes and momentum dependence characterized by \(V_n^{(\alpha)}(p)\). Together with Eq. \(\text{I}\), it allows for an event-by-event description of \(V_n(p)\), providing unique insight into the momentum dependence of flow fluctuations.

There is an important subtlety regarding the measurement \(V_{n\Delta}\) and the PCA. In principle, one could estimate the flow covariance matrix from

\[
V_{n\Delta}(p_a, p_b) = \left\{ \frac{N_{\text{pairs}}(p_a, p_b)}{N_{\text{pairs}}(p_a, p_b)} \right\}, \tag{8}
\]

where we sum over the \(N_{\text{pairs}}\) pairs of particles \(a \neq b\) that can be formed between two bins, centered around \(p_a\) for particle \(a\) and \(p_b\) for particle \(b\). In the hydrodynamic picture, particles are independently emitted from the fluid and Eq. \(\text{I}\) is recovered \(\text{I}\). However, the principal component analysis of Refs. \(\text{I}\) is considered instead the covariance matrix

\[
V_{n\Delta}^N(p_a, p_b) \equiv \frac{1}{(2\pi \Delta p_T \Delta y)^2} \left\langle \sum_{a \neq b} e^{-i \phi_a - \phi_b} \right\rangle \tag{9}
= \langle N(p_a) V_n^*(p_a) N(p_b) V_n(p_b) \rangle.
\]

Here, \(N(p_a) = N_a\) is the multiplicity in bin \(a\) normalized by \(2\pi \Delta p_T \Delta y^a\), where \(\Delta p_T^2\) and \(\Delta y^a\) are the bin widths in transverse momentum and pseudo-rapidity, respectively.\(^{\text{I}}\) The extra factors of particle number are only compensated at the end of the analysis, by dividing the

\(^2\) Detector imperfections might lead to slightly non-vanishing \(\langle V_n \rangle\), in which case one can manually subtract \(e^{-i \phi_a} e^{i \phi_b}\), from \(e^{-i \phi_a - \phi_b}\).\(^{\text{I}}\)

\(^3\) The normalization of \(N\) with \((2\pi \Delta p_T \Delta y)^{-1}\) is chosen for compatibility with \(\text{I}\), but is not relevant for \(V_{n\Delta}^R\) and \(V_{n\Delta}^R\) below.
resulting modes by $\langle N(p) \rangle$. Thus, one obtains a quantity to be compared to the usual “per particle” flow:

$$V_n^{N(\alpha)}(p) \equiv \sqrt{\lambda_n^{N(\alpha)} \psi_n^{N(\alpha)}(p) / \langle N(p) \rangle}, \quad (10)$$

where $\lambda_n^{N(\alpha)}$ and $\psi_n^{N(\alpha)}(p)$ are the corresponding eigenvalues and eigenvectors.

An important advantage of Eq. (10) over Eq. (8) is that it gives more weight to events with a larger number of pairs, where the relative uncertainty in the flow vector is smaller. However, there are very important differences in the diagonalization of $V_{n\Delta}$ and $V_n^N$, especially due to the fact that $\langle N_a N_b \rangle \neq \langle N_a \rangle \langle N_b \rangle$. Indeed, we show below that the subleading PCA modes of $V_{n\Delta}$ can be dominated by differential-multiplicity correlations rather than fluctuations of the flow harmonics $V_n$. These particle number fluctuations can be measured directly by performing PCA on the matrix $N_{\Delta}(p_o, p_b) = V_{0\Delta}(p_o, p_b)$ and, thus, they represent an unwanted background for $n \neq 0$ analyses.

**MULTIPlicity FLUCTuations**

The PCA of $V_{0\Delta}$, corresponding to particle number fluctuations, was investigated in Refs. [13-15, 23]. While the leading mode is nearly constant in transverse momentum, the subleading one displays a significant $p_T$ dependence. We shall show that momentum-dependent multiplicity fluctuations have startling consequences for the $n \neq 0$ PCA of Refs. [13-15].

**Fig. 1** displays PCA data from the CMS collaboration, obtained using Eqs. (9) and (10) [18]. The solid curves are the measured subleading PCA modes of the elliptic and triangular flow, while the dashed ones represent the combination $V_0^{N(2)}V_1^{N(1)}$. A striking proximity between the two curves is verified and deserves to be investigated, especially for non-central collisions and $n = 2$. In order to investigate the coincidence in Fig. 4 let us now suppose that the subleading modes of $V_{n\Delta}$ are suppressed for some reason. In that case, $V_n^{N(2)}$ should be dominated by multiplicity fluctuations, as will be now made clear.

Applying Eqs. (7) and (10) to the PCA of $N_{\Delta} = V_{0\Delta}^N$, we can estimate the event-by-event differential multiplicity:

$$N \approx \langle N \rangle \left(1 + V_0^{N(1)} \xi^{N(1)} + V_0^{N(2)} \xi^{N(2)} \right), \quad (11)$$

where, for brevity, we omit the momentum dependence. Substituting Eq. (11) in Eq. (9), we find, up to $O(V_n^{N(2)})$,

$$V_{n\Delta}(p_1, p_2) \approx \langle N(p_1) \rangle \langle N(p_2) \rangle \left(q^{(1)}(p_1) q^{(1)}(p_2) + q^{(2)}(p_1) q^{(2)}(p_2) \right), \quad (12)$$

with

$$q^{(1)}(p) = \sqrt{1 + \left(V_0^{N(1)}(p_b) \right)^2 V_1^{N(1)}(p)}, \quad (13)$$

$$q^{(2)}(p) = V_0^{N(2)}(p) V_1^{N(1)}(p),$$

where we assume that fluctuations of $N$ and $V_n$ are independent and that $V_0^{N(1)}$ is
which might be dominated by fluctuations of most the correction to the leading PCA mode is small, of at severely affect the observables defined in Ref. [13]. While nearly constant, \( V_n^{(1)} \) is approximately parallel to \( V_n^{(1)} \) and contributes to this mode. On the other hand, this is clearly not the case for \( V_0^{(2)} V_n^{(1)} \), which should contribute to the subleading component. Thus, in a first approximation, it is reasonable to expect

\[
V_n^{(1)}(p) \approx q^{(1)}(p), \\
V_n^{(2)}(p) \approx q^{(2)}(p).
\] (14)

Eq. (14) shows that multiplicity fluctuations may severely affect the observables defined in Ref. [13]. While the correction to the leading PCA mode is small, of at most \( \sim 3\% \), this is not the case for subleading modes, which might be dominated by fluctuations of \( N(p) \). Neglecting the corrections to \( V_n^{(1)} \) in Eqs. (13) and (14), we have, for \( V_n^{(2)} \to 0,

\[
V_n^{(2)}(p) \approx V_0^{(2)}(p) V_n^{(1)}(p).
\] (15)

This is precisely the combination shown to approximate the PCA data in Fig. 1. We thus conclude that a large fraction of the observed \( V_2 \) and \( V_3 \) subleading modes are the result of subleading multiplicity fluctuations, rather than fluctuations in anisotropic flow.

**NEW SET OF PCA OBSERVABLES**

The results above are clearly critical to the interpretation of the current PCA of flow harmonic data. First, they compromise the interpretation of the subleading PCA modes as revealing entirely new \( p_T \)-dependent anisotropic flow fluctuations. Furthermore, they suggest that these quantities are dominated by redundant information — that is, information that is more directly obtained from existing measurements of the leading anisotropic mode and the multiplicity PCA modes.

It is thus desirable to redefine the PCA observables so as to (at least approximately) remove known contributions from multiplicity fluctuations. One possibility is measuring the PCA of \( V_{n\Delta} \), as defined in Eqs. (5), (8) and (9). This has the disadvantage of giving the same weight to all events, regardless of the number of particles, which may result in an unacceptable statistical uncertainty. The PCA of \( V_{n\Delta} \) was employed in Ref. [17].

As an alternative to \( V_{n\Delta}^N \), we propose the diagonalization of the matrix

\[
V_{n\Delta}^R(p_a, p_b) \equiv \left( \sum_{a \neq b} e^{-i\phi_a - \phi_b} \right) / \langle N_{\text{pairs}}(p_a, p_b) \rangle
\]

\[
\sim \frac{N(p_a) V_n^*(p_a) N(p_b) V_n(p_b)}{\langle N(p_a) N(p_b) \rangle},
\] (16)

so that the average is weighted by the number of pairs. Eq. (16) is the definition of correlation matrix that is typically considered in measurements of the factorization breaking coefficient \( r_n(p_T, p_T^b) \) [19-21, 23]. For our purposes, it will provide a good approximation to \( V_{n\Delta}^R \), as will be shown. The principal components of \( V_{n\Delta}^R \) are
given by

\[ V_n^{R(\alpha)}(p) \equiv \sqrt{\lambda_n^{R(\alpha)}} \psi_n^{R(\alpha)}(p), \]

(17)

where \( \lambda_n^{R(\alpha)} \) and \( \psi_n^{R(\alpha)}(p) \) are the corresponding eigenvalues and eigenvectors.

Thus, each pair of particles is given the same weight, but multiplicity fluctuations are canceled by the denominator if they factor out. That is, if

\[ \langle N(p_a) V_n^+(p_a) N(p_b) V_n(p_b) \rangle \]
\[ \simeq \langle N(p_a) N(p_b) \rangle \langle V_n^+(p_a) V_n(p_b) \rangle, \]

(18)

then the multiplicity factor cancels and Eq. (16) becomes Eq. (2). To test this, we employ a state-of-the-art hybrid model [25], consisting of relativistic viscous hydrodynamics as implemented in MUSIC [20, 21] and evolution of the hadron gas phase according to UrQMD [30, 31]. Initial conditions were provided by TRFENTO [22] and parameter values taken from the Bayesian analysis of Ref. [33].

In Fig. 2, we compare the subleading PCA modes \( V_n^{R(2)}(p) \) to \( V_n^{R(2)}(p) \). A good agreement is found in general, indicating that \( V_n^{R(2)}(p) \) can be used in place of \( V_n^{R(2)}(p) \). While definition (16) does not necessarily remove all effects from multiplicity fluctuations, it does remove the most trivial ones from the corresponding two-point function. The remaining effects come solely from nontrivial, connected three- and four-point functions of \( V_n(p) \) and \( N(p) \), which necessarily involve both \( V_n(p_a) \) and \( V_n(p_b) \). Indeed, \( V_n^{R(2)} \approx V_n^{R(2)} \) to good approximation unless anisotropic flow and radial flow fluctuations are strongly correlated. Fig. 2 suggests, however, that this is not the case, as \( V_n^{R(2)}(p_T) \) closely follows \( V_n^{R(2)}(p_T) \).

ORTHOGONALITY AND ENSURING BIN INDEPENDENCE

Another subtlety regarding a PCA analysis is that the spectral decomposition is not unique. One must define the operation of the covariance matrix as a linear operator on a vector space with a corresponding inner product.

Transverse momentum \( p_T \) being a continuous variable, a natural form of eigenvalue/eigenvector equation for \( V_n^{R(2)} \) is

\[ \int dp_b V_n^{R(2)}(p_a, p_b) \psi_n(p_b) = \lambda_n \psi_n(p_a). \]

(19)

With this definition, \( V_n^{R(2)} \) is a Hermitian operator, with orthogonal eigenvectors. The orthonormality relation between the resulting eigenvectors in this case is

\[ \int dp \psi_n^{(\alpha)}(p) \psi_n^{(\beta)}(p) = \delta_{\alpha\beta}. \]

(20)

In practice, one must use finite sized bins in momentum space and the integrals become sums over discrete momentum indices, i.e.,

\[ \sum_b \Delta p_b V_n^{R(2)}(p_a, p_b) \psi_n(p_b) = \lambda_n \psi_n(p_a) \]

\[ \sum_b \Delta p_b \psi_n^{(\alpha)}(p_a) \psi_n^{(\beta)}(p_b) = \delta_{\alpha\beta}. \]

(21)

(22)

A few comments are in order. A naive spectral decomposition of the discrete correlation matrix might not have the factor of bin width \( \Delta p_b \) in Eq. (21) (and therefore nor in Eq. (22)). If the bin width is not uniform, the result will not be the same, and will depend on the specific choice of binning. This is especially important for the subleading mode, which is related to the dominant leading mode by a binning-dependent orthogonality relation. This issue was overlooked in Refs. [13, 18]. As written in Eq. (21), on the other hand, the result is stable under any choice of binning.

FINAL REMARKS

In this letter, we discussed the effect of multiplicity fluctuations in the PCA of anisotropic flow. Redundancies found in the CMS data suggest that these particle number fluctuations contribute significantly to subleading components, and may completely dominate over the fluctuations of anisotropic flow that are nominally being measured. The importance of multiplicity fluctuations to the standard PCA of flow fluctuations is a result of the remarkable sensitivity of the subleading PCA mode, the small size of the actual subleading flow \( V_n^{R(2)} \) and, of course, the choice of the covariance matrix of Eq. (9) in [13, 18]. Since particle number fluctuations can be measured separately and directly, they represent a redundant and unwanted background to principal component analyses of anisotropic flow.

This led us to propose the PCA of \( V_n^{R} \) and \( V_n^{R(2)} \), as defined in Eqs. (9) and (10). The new observables are free of trivial contributions from multiplicity fluctuations, so that the new subleading PCA modes actually reveal new sources of flow fluctuations. In Fig. 2 the new subleading modes appear to be relatively stable against changes in centrality, suggesting that they are not driven by the average geometry of the system. Also, \( V_n^{R(2)} \), from \( V_n^{R} \), is seen to reasonably reproduce \( V_n^{R(2)} \) from \( V_n^{R(2)} \). The main advantage of the proposed observables over the factorization breaking measure \( r_n(p_T^2, p_T^2) \) of Ref. [4] is that they isolate linearly uncorrelated modes, making their physical content more transparent. They also allow for better, more compact, visualization, since the modes are functions of a single momentum variable. Furthermore, \( r_n(p_T^2, p_T^2) \) only measures the relative importance of flow.
fluctuations, rendering them nearly imperceptible in non-central collisions, where \( V_n^{(1)}(p) \) is larger \([13, 19, 21]\).

We stress that the positivity of the PCA eigenvalues provides a highly nontrivial check of the hydrodynamic picture, as noted in \([13]\). While a covariance matrix is necessarily positive semidefinite, this is not strictly the case for the matrices defined in Eqs. \((8), (9)\) and \((16)\) outside a hydrodynamic picture. This is due to the absence of self-correlation terms \((a = b)\) and would be aggravated by the implementation of a rapidity gap in an experimental analysis. Within the hydrodynamic picture, on the other hand, \( V_n(\Delta) \) does not depend on individual particles and both \( V_n^\Delta \) and \( V_N^\Delta \) are covariance matrices, with positive or vanishing eigenvalues \([13]\). Even in this picture, \( V_n^\Delta \) is not actually a covariance matrix, which could be seen as its main disadvantage.

Another, less essential test of hydrodynamic models could be obtained from comparing measurements of \( V_n^\Delta \) and \( V_N^\Delta \), which could elucidate nontrivial correlations between anisotropic and radial flow fluctuations. Significant differences would signal incompatibilities among different measurements and calculations of the factorization breaking ratio \( r_n \), since this quantity has been obtained from both \( V_n^\Delta \) and \( V_N^\Delta \) in the past \([4, 7, 9, 12, 19, 21]\).

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1. Jean-Yves Ollitrault, “Anisotropy as a signature of transverse collective flow,” Phys. Rev. D46, 229–245 (1992)
2. Derek Teaney and Li Yan, “Triangularity and Dipole Asymmetry in Heavy Ion Collisions,” Phys. Rev. C83, 064904 (2011) arXiv:1010.1876 [nucl-th]
3. Fernando G. Gardim, Frederique Grassi, Matthew Luzum, and Jean-Yves Ollitrault, “Mapping the hydrodynamic response to the initial geometry in heavy-ion collisions,” Phys. Rev. C85, 024908 (2012) arXiv:1111.6538 [nucl-th]
4. Fernando G. Gardim, Frederique Grassi, Matthew Luzum, and Jean-Yves Ollitrault, “Breaking of factorization of two-particle correlations in hydrodynamics,” Phys. Rev. C87, 031901 (2013) arXiv:1211.0989 [nucl-th]
5. Ulrich Heinz, Zhi Qiu, and Chun Shen, “Fluctuating flow angles and anisotropic flow measurements,” Phys. Rev. C87, 034913 (2013) arXiv:1302.3535 [nucl-th]
6. Fernando G. Gardim, Frederique Grassi, Pedro Ishida, Matthew Luzum, Pablo S. Magalhães, and Jacqueyln Noronha-Hostler, “Sensitivity of observables to coarse-graining size in heavy-ion collisions,” Phys. Rev. C97, 064919 (2018) arXiv:1712.03912 [nucl-th]
7. Igor Kozlov, Matthew Luzum, Gabriel Denicol, Sangyong Jeon, and Charles Gale, “Transverse momentum structure of pair correlations as a signature of collective behavior in small collision systems,” (2014), arXiv:1405.3976 [nucl-th]
8. Jacquelyn Noronha-Hostler, Jorge Noronha, and Miklos Gyulassy, “Sensitivity of flow harmonics to subnucleon scale fluctuations in heavy ion collisions,” Phys. Rev. C93, 024909 (2016) arXiv:1508.02455 [nucl-th]
9. I. Kozlov, Matthew Luzum, Gabriel S. Denicol, Sangyong Jeon, and Charles Gale, “Signatures of collective behavior in small systems,” Proceedings, 24th International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions (Quark Matter 2014): Darmstadt, Germany, May 19-24, 2014, Nucl. Phys. A931, 1045–1050 (2014) arXiv:1412.3147 [nucl-th]
10. Chun Shen, Zhi Qiu, and Ulrich Heinz, “Shape and flow fluctuations in ultracentral Pb + Pb collisions at the energies available at the CERN Large Hadron Collider,” Phys. Rev. C92, 014901 (2015) arXiv:1502.04636 [nucl-th]
11. Wenbin Zhao, Hao-jie Xu, and Huichao Song, “Collective flow in 2.76 A TeV and 5.02 A TeV Pb+Pb collisions,” Eur. Phys. J. C77, 645 (2017) arXiv:1703.10792 [nucl-th]
12. Piotr Bozek, “Angle and magnitude decorrelation in the factorization breaking of collective flow,” Phys. Rev. C98, 064906 (2018) arXiv:1808.04248 [nucl-th]
13. Rajeev S. Bhalerao, Jean-Yves Ollitrault, Subrata Pal, and Derek Teaney, “Principal component analysis of event-by-event fluctuations,” Phys. Rev. Lett. 114, 152301 (2015) arXiv:1410.7739 [nucl-th]
14. Alekskas Mazeliauskas and Derek Teaney, “Subleading harmonic flows in hydrodynamic simulations of heavy ion collisions,” Phys. Rev. C91, 044902 (2015)
15. Alekskas Mazeliauskas and Derek Teaney, “Fluctuations of harmonic and radial flow in heavy ion collisions with principal components,” Phys. Rev. C93, 024913 (2016) arXiv:1509.07492 [nucl-th]
16. P. Cirkovic, D. Devetak, M. Dordevic, J. Milosevic, and M. Stojanovic, “Sub-leading flow modes in PbPb collisions at \( \sqrt{s_{NN}} = 2.76 \) TeV from HYDJET++ model,” Chin. Phys. C41, 074001 (2017) arXiv:1611.06602 [nucl-ex]
17. Piotr Bozek, “Principal component analysis of the non-linear coupling of harmonic modes in heavy-ion collisions,” Phys. Rev. C97, 034905 (2018) arXiv:1711.07773 [nucl-th]
18. A. M. Sirunyan et al. (CMS), “Principal-component analysis of two-particle azimuthal correlations in PbPb and pPb collisions at CMS,” Phys. Rev. C96, 064902 (2017) arXiv:1708.07113 [nucl-ex]
[19] Vardan Khachatryan et al. (CMS), “Evidence for transverse momentum and pseudorapidity dependent event plane fluctuations in PbPb and pPb collisions,” Phys. Rev. C92, 034911 (2015) [arXiv:1503.01692 [nucl-ex]]

[20] Serguei Chatrchyan et al. (CMS), “Studies of azimuthal dihadron correlations in ultra-central PbPb collisions at $\sqrt{s_{NN}} = 2.76$ TeV,” JHEP 02, 088 (2014) [arXiv:1312.1845 [nucl-ex]].

[21] Shreyasi Acharya et al. (ALICE), “Searches for transverse momentum dependent flow vector fluctuations in Pb-Pb and p-Pb collisions at the LHC,” JHEP 09, 032 (2017) [arXiv:1707.05690 [nucl-ex]].

[22] Ian Jolliffe, “Principal component analysis,” in Encyclopedia of Statistics in Behavioral Science (American Cancer Society, 2005).

[23] Fernando G. Gardim, Frederique Grassi, Pedro Ishida, Matthew Luzum, and Jean-Yves Ollitrault, “$p_T$-Dependent Particle Number Fluctuations From Principal Component Analyses in Hydrodynamic Simulations of Heavy-Ion Collisions,” (2019) [arXiv:1906.03045 [nucl-th]].

[24] Ulrich Heinz and Raimond Snellings, “Collective flow and viscosity in relativistic heavy-ion collisions,” Annu. Rev. Nucl. Part. Sci. 63, 123–151 (2013) [arXiv:1301.2826 [nucl-th]].

[25] Tiago Nunes da Silva, David Dobrigkeit Chinellato, Rafael Derradi de Souza, Mauricio Hippert, Matthew Luzum, Jorge Noronha, and Jun Takahashi, “Testing a best-fit hydrodynamical model using PCA,” in Hot Quarks 2018: Workshop for Young Scientists on the Physics of Ultrarelativistic Nucleus-Nucleus Collisions (HQ2018) De Krim, Texel Island, Netherlands, September 7-14, 2018 (2018) [arXiv:1811.05048 [nucl-th]].

[26] Bjoern Schenke, Sangyong Jeon, and Charles Gale, “(3+1)D hydrodynamic simulation of relativistic heavy-ion collisions,” Phys. Rev. C82, 041903 (2010) [arXiv:1004.1408 [hep-ph]].

[27] Bjoern Schenke, Sangyong Jeon, and Charles Gale, “Elliptic and triangular flow in event-by-event (3+1)D viscous hydrodynamics,” Phys. Rev. Lett. 106, 042301 (2011) [arXiv:1009.3244 [hep-ph]].

[28] Bjoern Schenke, Sangyong Jeon, and Charles Gale, “Higher flow harmonics from (3+1)D event-by-event viscous hydrodynamics,” Phys. Rev. C85, 024901 (2012) [arXiv:1109.6289 [hep-ph]].

[29] Jean-François Paquet, Chun Shen, Gabriel S. Denicol, Matthew Luzum, Bjoern Schenke, Sangyong Jeon, and Charles Gale, “Production of photons in relativistic heavy-ion collisions,” Phys. Rev. C93, 044906 (2016) [arXiv:1506.06738 [hep-ph]].

[30] S. A. Bass et al., “Microscopic models for ultrarelativistic heavy ion collisions,” Prog. Part. Nucl. Phys. 41, 255–369 (1998) [arXiv:nucl-th/9909035 [nucl-th]].

[31] M. Bleicher et al., “Relativistic hadron hadron collisions in the ultrarelativistic quantum molecular dynamics model,” J. Phys. G25, 1859–1896 (1999) [arXiv:hep-ph/9909407 [hep-ph]].

[32] J. Scott Moreland, Jonah E. Bernhard, and Steffen A. Bass, “Alternative ansatz to wounded nucleon and binary collision scaling in high-energy nuclear collisions,” Phys. Rev. C92, 011901 (2015) [arXiv:1412.4708 [nucl-th]].

[33] Jonah E. Bernhard, Bayesian parameter estimation for relativistic heavy-ion collisions, Ph.D. thesis, Duke U. (2018-04-19) [arXiv:1804.06469 [nucl-th]].