Cyclicity and Maximal Multiplicity for Zeros of Families of Analytic Functions

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Abstract

Let $f_\lambda$ be a family of holomorphic functions in the unit disk $D \subset \mathbb{C}$, holomorphic in parameter $\lambda \in U \subset \mathbb{C}^n$. We estimate the number of zeros of $f_\lambda$ in a smaller disk via some characteristic of the ideal generated by Taylor coefficients of $f_\lambda$. Our estimate is locally sharp and improve the previous estimate obtained in [RY].

1. Introduction.

1.1. In what follows $D_r := \{z \in \mathbb{C} : |z| < r\}$, $\overline{D}_r$ is the closure of $D_r$ and $D := D_1$. Let $U \subset V \subset \mathbb{C}^n$ be open connected sets, and

$$f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k, \quad a_k \in \mathcal{O}(V),$$  \hspace{1cm} (1.1)

be a family of holomorphic functions in $D$ depending holomorphically on $\lambda \in V$. Let $\mathcal{I}(f; U)$ be the ideal in $\mathcal{O}(U)$ generated by all $a_k(\lambda)$. Following the pioneering work of Bautin [B], we refer to $\mathcal{I}(f; U)$ as the Bautin ideal of $f_\lambda$ in $U$. Further,

$$C(f; U) := \{\lambda \in U : f_\lambda \equiv 0\}$$

is called the central set of $f_\lambda$ in $U$. The Hilbert finiteness theorem states that $\mathcal{I}(f; U)$ is generated by a finite number of coefficients. The Bautin index of $f$ in $U$ is the minimal number $d_f(U)$ such that $a_0, ..., a_{d_f(U)}$ generate $\mathcal{I}(f; U)$. Usually computing $\mathcal{I}(f; U)$ and $d_f(U)$ is not easy. The number of zeros (counted with multiplicities) which $f_\lambda$ can have near 0 for $\lambda$ close to some $\lambda_0 \in C(f; U)$ is called cyclicity, following [R]. The next result was established by Yomdin [Y, Th. 3.1]:

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Assume that for any \( \lambda \in V \) either \( f_\lambda \equiv 0 \), or the multiplicity of zero of \( f_\lambda \) at \( 0 \in \mathbb{C} \) is at most \( N \). Let \( I_N \) be the ideal in \( \mathcal{O}(U) \) generated by the first \( N \) Taylor coefficients \( a_0(\lambda), \ldots, a_N(\lambda) \). Assume that \( I_N \) is radical (i.e. \( g^s \in I_N \) for some \( s \geq 1 \) implies that \( g \in I_N \)). Then the cyclicity of \( f_\lambda \), \( \lambda \in U \), is at most \( N \).

This result follows from some theorems of \([FY]\) based on the fact that \( d_f(U) \leq N \). In particular, the required conditions are satisfied if \( f_\lambda \) depends linearly on \( \lambda \). The main purpose of the present paper is to extend the above result to a general situation.

In \([RY]\) it was shown that one can obtain a local upper bound on the number of zeros of \( f_\lambda \) just in terms of \( d_f(U) \). More precisely, there is some small positive \( r < 1 \) depending on \( d_f(U) \) and \( \mathcal{I}(f; U) \) such that each function \( f_\lambda(z) \) has at most \( d_f(U) \) complex zeros in the disk \( \overline{D}_r \). In our paper we will improve this estimate and will show that another algebraic characteristic of \( \mathcal{I}(f; U) \) is responsible to the estimate of the number of zeros of \( f_\lambda \). Moreover, our local estimate is sharp.

1.2. To formulate the results, let \( K \subset \subset V \) be a compact. For any open \( W \) by \( \mathcal{O}_c(\mathbb{D}; W) \) we denote the set of maps \( \mathbb{D} \mapsto W \) holomorphic in open neighbourhoods of \( \mathbb{D} \). Assume that \( W \subset V \). For any \( \phi \in \mathcal{O}_c(\mathbb{D}; W) \) consider the function

\[
  f_{\phi(w)}(z) = \sum_{k=0}^{\infty} a_k(\phi(w))z^k, \quad w, z \in \mathbb{D}.
\]

Let \( d(\phi) \) be the Bautin index of the Bautin ideal of \( f_{\phi(w)} \) in \( \mathbb{D} \). Further, we fix a sequence of open sets \( \{O_j\} \), \( O_{j+1} \subset \subset O_j \subset \subset V \) for any \( j \), such that \( \cap_j O_j = K \).

**Definition 1.1** The integer number

\[
  \mu_f(K) := \lim_{j \to \infty} \sup_{\phi \in \mathcal{O}_c(\mathbb{D}; O_j)} d(\phi)
\]

will be called the maximal multiplicity on \( K \) of zero of \( f_\lambda \) at \( 0 \in \mathbb{C} \).

Obviously, \( \mu_f(K) \leq d_f(O_j) \), the Bautin index of \( \mathcal{I}(f; O_j) \) in \( O_j \) (for any \( j \)). Therefore the definition is correct and \( \mu_f(K) < \infty \). Also, \( \mu_f(K) \) does not depend of the choice of \( \{O_j\} \).

Below we give another characterization of \( \mu_f(K) \). First we choose an open set \( U \) such that \( O_j \subset \subset U \subset \subset V \) for any \( j \). Let \( N_j \) be minimum of integers \( N \) for which there is \( c(N) > 0 \) so that

\[
  |a_k(\lambda)| \leq c(N) \cdot \max_U |a_k| \cdot \max_{i=0,\ldots,N} |a_i(\lambda)| \quad \text{for} \quad k > N, \ \lambda \in O_j.
\] (1.2)

**Theorem 1.2**

\[
  \mu_f(K) = \lim_{j \to \infty} N_j.
\]

We will show that one can take the Bautin index \( d_f(U) \) of \( \mathcal{I}(f; U) \) as one of such \( N \) in (1.2). Let \( c(N_j) \) be the best constant in (1.2) for \( N = N_j \). We set

\[
  c_{\mu_f}(K) = \lim_{j \to \infty} c(N_j).
\] (1.3)
The definition is correct because there is \( i_0 \) such that for any \( i \geq i_0 \), \( N_i = \lim_{j \to \infty} N_j \).

Also, \( c_{\mu_f}(K) \) does not depend of the choice of \( \{O_i\} \). In fact, as follows from the proof of the theorem \( c_{\mu_f}(K) \) depends only on \( K \) and \( a_0, \ldots, a_{\mu_f}(K) \). In general \( c_{\mu_f}(K) \) cannot be estimated effectively. However, in many cases if the maximal multiplicity \( \mu_f(K) \) is known, \( c_{\mu_f}(K) \) can be found by a finite computation (which involves a resolution of singularities type algorithm for the central set of \( f_\lambda \)).

Further, for \( R < 1 \) and \( \lambda \notin C(f; V), \lambda \in V \), we set

\[
m_{f_\lambda}(R) := \sup_{D_R} \log |f_\lambda| \quad \text{and} \quad \mu_f(\lambda, R) := m_{f_\lambda}(R) - m_{f_\lambda}(R/e) .
\]

**Theorem 1.3** Let

\[
\mu_i := \limsup_{R \to 0} \sup_{\lambda \in O_i \setminus C(f, O_i)} \mu_f(\lambda, R).
\]

Then

\[
\mu_f(K) = \lim_{i \to \infty} \mu_i .
\]

1.3. Next we formulate some simple corollaries from Theorems 1.2 and 1.3.

1. There is \( i_0 \) such that the central set \( C(f; O_{i_0}) \) is defined as the set of common zeros of the first \( \mu_f(K) + 1 \) Taylor coefficients \( a_0(\lambda), \ldots, a_{\mu_f(K)}(\lambda) \) of \( f_\lambda (\lambda \in O_{i_0}) \).

2. Let \( f_\lambda \) and \( g_\lambda \) be families of holomorphic functions in \( D \), holomorphic in \( \lambda \in V \), and let \( f', e^j \) be the families \( f'_j(z) = \frac{df_j(z)}{dz} \) and \( e^{i\lambda} \), respectively. Then

\[
\mu_f(K) \leq \mu_f(K) + 1 ; \quad \mu_{e^j}(K) = 0 ; \quad \mu_{f_g}(K) \leq \mu_f(K) + \mu_g(K) .
\]

3. Let \( S \subset V \) be a compact and \( K(S) \) be the space of all compact subsets of \( S \) equipped with the Hausdorff metric. Then the function \( \mu_f : K(S) \to \mathbb{Z}_+ \) is upper-semicontinuous, i.e., if \( \{K_i\} \) is a sequence of compacts in \( S \) converging in the Hausdorff metric to \( K \subset S \), then

\[
\limsup_{i \to \infty} \mu_f(K_i) \leq \mu_f(K) .
\]

4. For any family of compacts \( \{K_i\}_{i \in I} \) in \( V \) such that \( \bigcup_{i \in I} K_i \) is a compact

\[
\mu_f(\bigcup_{i \in I} K_i) = \max_{i \in I} \mu_f(K_i) .
\]

(Here the maximum is taken because by (3) \( \mu_f \) is upper-semicontinuous on \( K(\bigcup K_i) \).)

For \( x \in V \) being a point the number \( \mu_f(x) \) will be called the generalized multiplicity of zero of \( f_x \) at \( 0 \in \mathbb{C} \). From Theorem 1.3 it follows that for \( x \notin C(f; V) \) the number \( \mu_f(x) \) coincides with the usual multiplicity of zero of \( f_x \) at \( 0 \in \mathbb{C} \). Also, for a compact \( K \) from the above identity we have

\[
\mu_f(K) = \max_{x \in K} \mu_f(x) .
\]

We leave proofs of these simple properties as an exercise for the reader.

1.4. According to Theorem 1.2 there is \( i_0 \) such that

\[
N_i = \mu_f(K) \quad \text{and} \quad c(N_i) \leq 2c_{\mu_f}(K) \quad \text{for any} \quad i \geq i_0 .
\]
Assume that

\[ M := \sup_{k > \mu_f(K), \lambda \in U} |a_k(\lambda)| < \infty \tag{1.5} \]

with \(U\) as in Theorem 1.2, and set

\[ R := \frac{1}{4c_{\mu_f(K)} \cdot M \cdot 2^{3\mu_f(K)} + 2} \cdot \]

Let \(P\) be the Taylor polynomial of \(f\) of degree \(\mu_f(K)\). Let \(N_r(f)\) and \(N_r(P)\), \(\lambda \in V\), be the number of zeros of \(f\) and \(P\) in \(\overline{B}_r\).

**Theorem 1.4 (Cyclicity Theorem)**

1. \(N_{r/2}(P) \leq N_r(f) \leq N_{2r}(P)\) for any \(r < R\), \(\lambda \in O_i\), \(i \geq i_0\).
2. For any \(r < R\), \(i \geq i_0\) there is some \(\lambda \in O_i\) such that \(N_r(f) = \mu_f(K)\).

In particular, from (1) we have \(N_R(f) \leq \mu_f(K)\) for any \(\lambda \in O_i\), \(i \geq i_0\).

**Remark 1.5** The straightforward application of Lemma 2.2.3 of [RY] gives also the following global estimate

\[ N_{1/4}(f) < 4\mu_f(K) + \log_{3/4}(2 + 2c_{\mu_f(K)} M) \quad \text{for} \quad \lambda \in O_i, \ i \geq i_0 . \]

**Example 1.6** (1) Let \(B \subset \mathbb{C}^3\) be a complex ball centered at 0 and

\[ f_\lambda(z) = \lambda^2 + \lambda^2 z^2 + \lambda_1 \lambda_2 z^4 + \lambda_1 \lambda_3 z^6 + \lambda_2 \lambda_3 z^6, \quad (\lambda, z) \in B \times \mathbb{D}. \]

It is easy to see that the Bautin ideal in \(\mathcal{O}(B)\) is generated by all coefficients of the function. Therefore the Bautin index is 5 and according to [RY] the number of zeros of any \(f_\lambda\) is \(\leq 5\) in a small neighbourhood of 0 \(\in \mathbb{C}\). However, from the inequalities

\[ |\lambda_i \lambda_j| \leq \max\{|\lambda_1|^2, |\lambda_2|^2, |\lambda_3|^2\}, \quad 1 \leq i, j \leq 3, \]

it follows that \(\mu_f(B) = 2\). Thus according to the above theorems, the number of zeros of \(f_\lambda\) in \(\overline{B}_r\) with \(r\) small enough is \(\leq 2\). Moreover, for any \(\lambda = (0, 0, \lambda_3) \in B\), \(\lambda_3 \neq 0\), the number of zeros (counted with multiplicities) of \(f_\lambda\) in \(\overline{B}_r\) is exactly 2.

2. Let \(f_\lambda(z) = \lambda(z^{10} - \lambda), \ \lambda \in \mathbb{D}\). In this case the Bautin ideal is not radical and the result of Yomdin do not apply (see [Y, page 363]). However, it is easy to see that \(\mu_f(\lambda) = 0\), \(\lambda \neq 0\), and \(\mu_f(0) = \mu_f(\mathbb{D}) = d_f(\mathbb{D}) = 10\). Thus the cyclicity is 10.

3. Assume that for any \(\lambda \in V \subset \mathbb{C}^n\) the function \(f_\lambda\) satisfies

\[ f_\lambda^{(r)}(z) + p_{r-1}(z)f_\lambda^{(r-1)}(z) + \ldots + p_1(z)f_\lambda(z) = 0, \quad z \in \mathbb{D}. \tag{1.6} \]

Here each \(p_\lambda(z)\) is holomorphic in \((\lambda, z) \in V \times \mathbb{D}\). Then [RY, Corollary 4.2] and Theorem 1.2 above imply that \(\mu_f(K) \leq r - 1\) for any compact \(K \subset V\). Let \(F_\lambda(z) = \sum_{k=1}^m P_k(z)e^{Q_k(z)}\) where \(P_k, Q_k\) are holomorphic polynomials of maximal degree \(p\) and \(q\), respectively, and \(\lambda \in \mathbb{C}^{m(p+q+2)}\) is the vector of coefficients of all
$P_k, Q_k$ ($k = 1, \ldots, m$). First consider $q = 1$ (usual exponential polynomials). Then $F_\lambda$ satisfies (1.6) with $r = m(p + 1)$. Hence $\mu_F(K) \leq m(p + 1) - 1$ for any compact $K \subset \mathbb{C}^{m(p+3)}$. Assume now that $q \geq 2$. It is easy to check (see e.g. [VPT]) that $F_\lambda$ satisfies (1.6) with $r = \frac{(p+1)(q^n-1)}{q-1}$, and so $\mu_F(K) \leq \frac{(p+1)(q^n-1)}{q-1} - 1$. However, one obtains a better estimate using [Br, Lemma 8]. This result says that there is $r_0 > 0$ such that for any $r \leq r_0$ the number of zeros in $\mathbb{D}_r$ of any $F_\lambda$ is $\leq 3 \cdot 2^{m-1}(p+q-1)$. Then by Theorem [1.4], $\mu_F(K) \leq 3 \cdot 2^{m-1}(p+q-1)$.

2. Proofs of Theorems 1.2 and 1.3.

2.1. Resolution Theorem. Our main tool is a version of Hironaka’s theorem on resolution of singularities proved in Theorem 4.4 and Lemma 4.7 of Bierstone and Milman [BM]. As usual, if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index ($\alpha_j \in \mathbb{Z}_+$), $z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and $|\alpha| := \sum_{1 \leq j \leq n} \alpha_j$. If $\alpha$ and $\beta$ are multi-indices we write $\alpha \prec \beta$ to mean that there is a multi-index $\gamma$ such that $\beta = \alpha + \gamma$.

Fix $\lambda_0 \in W$, an open subset of $\mathbb{C}^n$. A dominating family for $W$ at $\lambda_0$ is a finite collection $(W_\alpha, K_\alpha, \phi_\alpha)_{1 \leq \alpha \leq A}$ where, for each $\alpha$, $K_\alpha \subset W$, $W_\alpha$ is an open set in $\mathbb{C}^n$ containing 0, $\phi_\alpha : W_\alpha \to W$ is a holomorphic map satisfying the two conditions

(1) $\det \phi_\alpha \neq 0$ outside a complex analytic variety of codimension $\geq 1$, and

(2) The images $\phi_\alpha(K_\alpha)$ ($\alpha = 1, \ldots, A$) cover a neighbourhood of $\lambda_0$ in $\mathbb{C}^n$.

BM Theorem. Let $f_1, \ldots, f_N$ be holomorphic functions defined on a neighbourhood of $\lambda_0 \in \mathbb{C}^n$. Suppose that none of the $f_j$ vanishes identically in any neighbourhood of $\lambda_0$. Then there exists a dominating family $(W_\alpha, K_\alpha, \phi_\alpha)_{1 \leq \alpha \leq A}$ for $W$ at $\lambda_0$, such that for each $\alpha$ we can find multi-indices $\gamma_1, \ldots, \gamma_N$ and functions $h_1, \ldots, h_N$ on $W_\alpha$ with the following properties

(A) Each $h_j$ is a nowhere-vanishing holomorphic function on $W_\alpha$.

(B) $f_j \circ \phi_\alpha(z) = h_j(z) \cdot z^{\gamma_j}$ for all $z \in W_\alpha$, $1 \leq j \leq N$, $1 \leq \alpha \leq A$.

(C) For each $\alpha$, the multi-indices $\gamma_1, \ldots, \gamma_N$ are totally ordered under $\prec$ (i.e. given $1 \leq i, j \leq N$, we either have $\gamma_i \prec \gamma_j$ or $\gamma_j \prec \gamma_i$).

2.2. Proof of Theorem 1.2. Let $f_\lambda(z) = \sum_{k=0}^{\infty} a_k(\lambda) z^k$, $a_k \in \mathcal{O}(V)$, and $K \subset U \subset V$ where $K$ is compact and $U$ is open. Let $d_f(U)$ be the Bautin index of $\mathcal{I}(f; U)$ in $U$. Let $\lambda_0 \in U$ and $W \subset U$ be an open neighbourhood of $\lambda_0$. Assume that $(W_\alpha, K_\alpha, \phi_\alpha)_{1 \leq \alpha \leq A}$ is the dominating family for $W$ at $\lambda_0$ for which $a_0, \ldots, a_{d_f(U)}$ satisfy BM Theorem. Here $a_j \circ \phi_\alpha(z) = h_j(z) \cdot z^{\gamma_j}$ for all $z \in W_\alpha$, $1 \leq j \leq d_f(U)$, $1 \leq \alpha \leq A$. Then from (C) it follows that one can find an index $j_0(\alpha)$ and multi-indices $\tilde{\gamma}_j$ so that $\gamma_{j_0} = \gamma_{j_0(\alpha)} + \tilde{\gamma}_j$, with $\gamma_{j_0(\alpha)} = 0$. The minimal $j_0(\alpha)$ satisfying this condition will be denoted $j(W_\alpha)$. 

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Lemma 2.1 For any \( k \geq 0 \) the function \( g_{ka}(z) := \frac{a_k \circ \phi(z)}{z^{\gamma_j(W_\alpha)\alpha}} \) is holomorphic on \( W_\alpha \). There is a constant \( C(W_\alpha) > 0 \) such that

\[
|a_k \circ \phi_\alpha(z)| \leq C(W_\alpha) \cdot \max_{W} |a_k| \cdot \max_{i=0,\ldots,d_j(W_\alpha)} |a_i \circ \phi_\alpha(z)| \ , \ z \in K_\alpha . \tag{2.1}
\]

Proof. For the first statement it suffices to consider \( k > d_j(U) \). Then \( a_k \) belongs to the ideal generated by \( a_0, \ldots, a_{d_j(U)} \) on \( U \). So there are \( b_j \in \mathcal{O}(U) \), \( 1 \leq j \leq d_j(U) \), such that \( a_k = \sum_{j=1}^{d_j(U)} b_j \cdot a_j \). This implies the first part of the lemma. Further, let us cover the compact \( K_\alpha \) by a finite number of closed polydisks \( \Delta_l := \{ z \in \mathbb{C}^n : \max_{1 \leq i \leq n} |z_i - \xi_{il}| \leq a_l \} \), \( \Delta_l \subset W_\alpha \), \( \min_{1 \leq i \leq n} ||\xi_{il}|| - |a_i| := r_l > 0 \) \( (l = 1, \ldots, s) \). Assume that \( z_{\gamma_j(W_\alpha)\alpha} = z_1^{\gamma_1} \cdots z_n^{\gamma_n} \) with \( |\gamma_j(W_\alpha)\alpha| = \sum_{i=1}^{n} \gamma_i \). Then on the boundary torus \( S^n_l \subset \Delta_l \) we have

\[
\min_{S^n_l} |z^{\gamma_j(W_\alpha)\alpha}| \geq r_l |z^{\gamma_j(W_\alpha)\alpha}| := r_l(W_\alpha) .
\]

Also, for \( z \in \Delta_l \) by the definition we have

\[
|z^{\gamma_j(W_\alpha)\alpha}| \leq |a_j(W_\alpha) \circ \phi_\alpha(z)| \cdot \max_{S^n_l} \frac{1}{|h_j(W_\alpha)\alpha|} \leq k_l(W_\alpha) \cdot \max_{i=0,\ldots,d_j(W_\alpha)} |a_i \circ \phi_\alpha(z)| .
\]

Combining these inequalities we obtain (for \( z \in \Delta_l \))

\[
|a_k \circ \phi_\alpha(z)| = |g_{ka}(z)| \cdot |z^{\gamma_j(W_\alpha)\alpha}| \leq k_l(W_\alpha) \cdot \max_{S^n_l} |g_{ka}| \cdot \max_{i=0,\ldots,d_j(W_\alpha)} |a_i \circ \phi_\alpha(z)| \leq \frac{k_l(W_\alpha)}{r_l(W_\alpha)} \cdot \max_{S^n_l} |a_k| \cdot \max_{i=0,\ldots,d_j(W_\alpha)} |a_i \circ \phi_\alpha(z)| .
\]

From here it follows the required inequality (2.1) with

\[
C(W_\alpha) := \max_{1 \leq l \leq s} \frac{k_l(W_\alpha)}{r_l(W_\alpha)} . \quad \square
\]

Set \( j(W) = \max_{\alpha} j(W_\alpha) \). Consider the family \( \{O_l\} \) as in Theorem 1.2, i.e., \( K \subset O_i \), \( O_{i+1} \subset O_i \subset O \subset U \) for any \( i \). Let \( i_0 \) be such that \( \lim_{j \to \infty} N_j = N_i \) for any \( i \geq i_0 \).

Lemma 2.2 For any \( i \geq i_0 \) there are finite number of points \( \lambda_{i\alpha} \in \overline{O}_{i+1} \), open sets \( W_{i\alpha} \subset O_i \), \( \lambda_{i\alpha} \in W_{i\alpha} \), and dominating families \( (W_{\alpha,i}; K_{\alpha,i}; \phi_{\alpha,i})_{1 \leq \alpha \leq A_i} \) for \( W_{i\alpha} \) at \( \lambda_{i\alpha} \), satisfying BM Theorem for the functions \( a_0, \ldots, a_{d_j(U)} \) \( (1 \leq i \leq t_i) \), such that

\[
\overline{O}_{i+1} \subset \bigcup_{l=1}^{t_i} \bigcup_{\alpha=1}^{A_i} \phi_{\alpha,i}(K_{\alpha,i}) .
\]

Proof. The proof follows from the fact that the closure \( \overline{O}_{i+1} \) is compact. \( \square \)
Let $\mathcal{W}_i = (W_{i})_{i=1}^{t_i}$ be the corresponding open cover of $\overline{O}_{i+1}$. Then we set

$$U(\mathcal{W}_i) = \bigcup_{i=1}^{t_i} W_i \quad \text{and} \quad j(\mathcal{W}_i) = \max_{1 \leq i \leq t_i} j(W_i).$$

Clearly, $U(\mathcal{W}_i) \subset O_i$, $U(\mathcal{W}_{i+1}) \subset U(\mathcal{W}_i)$, and $j(\mathcal{W}_i) \leq d_f(U)$. Now inequality (2.1) applied to the elements of dominating families of a cover $\mathcal{W}_i$ as above implies

Lemma 2.3 There is $C_i > 0$ such that

$$|a_k(\lambda)| \leq C_i \cdot \max_U |a_j| \cdot \max_{i=0,\ldots,j(\mathcal{W}_i)} |a_i(\lambda)| \quad \text{for} \quad k \geq 0, \ \lambda \in O_{i+1}. \quad \square$$

Next we will prove

Lemma 2.4 For any $i \geq i_0$,

$$\lim_{j \to \infty} N_j = j(\mathcal{W}_i).$$

Proof. By definition, $O_{i+1} \subset \subset O_{i_0}$. Thus from Lemma 2.3 and from the definition of $N_{i+1}$ we have $L := \lim_{j \to \infty} N_j \leq j(\mathcal{W}_i)$. Assume, to the contrary, that $L < j(\mathcal{W}_i)$.

By definition, there is an element $(W_{\alpha}, K_{\alpha}, \phi_{\alpha})$ of one of the dominating families for $\mathcal{W}_i$ (as in Lemma 2.2) such that $j(W_{\alpha}) = j(\mathcal{W}_i)$. Since $U(\mathcal{W}_i) \subset \subset O_i \subset \subset O_{i_0}$, we have from our assumption

$$|a_j(\mathcal{W}_i) \circ \phi_{\alpha}(z)| \leq c(L) \cdot \max_U |a_j(\mathcal{W}_i)| \cdot \max_{j=0,\ldots,L} |a_j \circ \phi_{\alpha}(z)|, \quad z \in W_{\alpha}. \quad (2.2)$$

But by BM Theorem, $a_j \circ \phi_{\alpha}(z) = h_{j\alpha}(z) \cdot z^{\gamma_{j\alpha}}$ where each $h_{j\alpha}$ is nowhere-vanishing on $W_{\alpha}$, and $j(\mathcal{W}_i)$ is the minimal number such that $\gamma_{j(\mathcal{W}_i)\alpha} < \gamma_{j\alpha}$ ($j = 1, \ldots, d_f(U)$). Then (2.2) gives a contradiction with minimality of $j(\mathcal{W}_i)$ (since $0 \in W_{\alpha}$). \quad \square

We set $j(K) := j(\mathcal{W}_i)$ ($i \geq i_0$). It remains to prove that $\mu_f(K) = j(K)$. Assume also that the above $i_0$ is so big that for any $i \geq i_0$,

$$\mu_f(K) = \sup_{\phi \in O_c(\mathbb{D}; O_{i_0})} d(\phi).$$

Let $\phi \in O_c(\overline{\mathbb{D}}; O_{i_0})$ be such that $d(\phi) = \mu_f(K)$. By definition $\phi$ maps some $\overline{\mathbb{D}}_r$, $r > 1$, into $O_{i_0}$. Then the definition of $N_{i_0} = j(K)$ implies that

$$|a_k \circ \phi(z)| \leq c(j(K)) \cdot \max_U |a_k| \cdot \max_{j=0,\ldots,j(K)} |a_j \circ \phi(z)|, \quad k > j(K), \quad z \in \overline{\mathbb{D}}_r. \quad (2.3)$$

Let $Z \subset \overline{\mathbb{D}}_r$ be the set of common zeros of $a_j \circ \phi$, $0 \leq j \leq j(K)$, counted with multiplicities, and $B_Z$ be the Blaschke product in $\overline{\mathbb{D}}_r$, whose set of zeros is $Z$. For any $i \geq 0$ we set $h_i := \frac{a_k \circ \phi}{B_Z}$. From the above inequality it follows that $h_i$ is holomorphic on $\overline{\mathbb{D}}_r$. Then for some $r_1$, $1 < r_1 < r$,

$$\max_{j=0,\ldots,j(K)} |h_j(z)| \geq C > 0, \quad z \in \overline{\mathbb{D}}_{r_1}. \quad (2.4)$$
Hence, by the corona theorem, there are bounded holomorphic on \( \mathbb{D}_{r_1} \) functions \( g_0, \ldots, g_{j(K)} \) such that

\[
\sum_{i=0}^{j(K)} g_i \cdot h_j \equiv 1 \quad \text{on} \quad \mathbb{D}_{r_1}.
\]

From here it follows (for any \( k \))

\[
a_k \circ \phi \equiv \sum_{i=0}^{j(K)} (a_i \circ \phi) \cdot (h_k \cdot g_i) \quad \text{on} \quad \mathbb{D}.
\]

This means that the Bautin index \( d(\phi) (= \mu_f(K)) \) is \( \leq j(K) \).

Conversely, let \( W_i \) and \( W_\alpha, K_\alpha, \phi_\alpha \) be the same as in the proof of Lemma 2.4. Here \( j(W_\alpha) = j(K) \). Then from BM Theorem we have \( a_j \circ \phi_\alpha(z) = h_{j\alpha}(z) \cdot z^{j\alpha}, \)

\( z \in W_\alpha, \) with a nowhere-vanishing \( h_{j\alpha} \) \( (1 \leq j \leq d_f(U)) \). Let us consider the closed disk \( D_s := \{(z, \ldots, z) \in \mathbb{C}^n : z \in \mathbb{C}, |z| \leq s\} \) with \( s \) so small that \( D_s \subset W_\alpha \). Then by the definition of \( j(K) \), the multiplicity of zero of each \( (a_k \circ \phi_\alpha)|_{D_s}, \)

\( 0 \leq k < j(K), \) at \( (0, \ldots, 0) \in D_s \) is greater than \( |j_{j(K)\alpha}| \), but the multiplicity of zero of \( (a_j \circ \phi_\alpha)|_{D_s} \) at \( (0, \ldots, 0) \in D_s \) equals \( |j_{j(K)\alpha}| \). Now, for some \( r > 1 \) we still have \( \tau(w) := (sw, \ldots, sw) \in W_\alpha \) for \( w \in \mathbb{D}_r \). Let us define \( \phi \in \mathcal{O}_c(\mathbb{D}; O_{i_0}) \) as \( \phi := \phi_\alpha \cdot \tau \). According to the above argument for multiplicities, the Bautin index \( d(\phi) \) of \( f_{\phi(w)} \) in \( \mathbb{D} \) is \( \geq j(K) \). This shows that \( \mu_f(K) = j(K) \).

The proof of the theorem is complete. \( \Box \)

2.3. Proof of Theorem 1.3. Let \( i_0 \) be the same as in the proof of Theorem 1.2. We will prove that for any \( i \geq i_0 + 1, \mu_i = \mu_f(K) \).

Let \( W_{i-1} \) be a cover from Lemma 2.2. Here \( U(W_{i-1}) \subset O_{i-1} \subset O_{i_0} \) and \( j(W_{i-1}) = \mu_f(K) \). By definition there is a sequence of points \( w_s \in O_i \setminus C(f; O_i) \) and numbers \( R_s > 0, \lim_{s \to \infty} R_s = 0 \), such that \( \mu_i = \lim_{s \to \infty} \mu_f(w_s, R_s) \). Without loss of generality we may assume that \( \lim_{s \to \infty} w_s = w \in \mathcal{O}_i \). Then from Lemma 2.2 it follows that there is a dominating family \( (W_\alpha, K_\alpha, \phi_\alpha)_{1 \leq \alpha \leq A} \) for one of the open sets of the cover \( W_{i-1} \) such that images \( \phi_\alpha(K_\alpha) \) \( (1 \leq \alpha \leq A) \) cover a neighbourhood of \( w \). In particular, we can find some \( \alpha \) and a sequence \( \{\tilde{w}_k\} \subset K_\alpha \) such that \( \phi_\alpha(\tilde{w}_k) = w_{s_k} \) for some subsequence \( \{w_{s_k}\} \subset \{w_s\} \) and \( \lim_{k \to \infty} \tilde{w}_k := \tilde{w} \in W_\alpha \). Now, according to BM Theorem and Lemma 2.3 the function

\[
h_z(y) := \frac{f_{\phi_\alpha(z)}(y)}{z^{j\alpha(w_\alpha)}}, \quad z \in W_\alpha, \quad y \in \mathbb{D},
\]

is holomorphic and its central set in \( W_\alpha \) is empty. Let \( \mu \) be the multiplicity of zero of \( h_z \) at 0. Then the function \( g_z(y) := \frac{h_z(y)}{y^\mu} \) is nowhere-vanishing in a small neighbourhood \( O \) of \( (\tilde{w}, 0) \in W_\alpha \times \mathbb{D} \). Without loss of generality we may assume that all pairs \( (\tilde{w}_k, y) \in \mathbb{D}_{R_{s_k}}, \) belong to \( O \). Then we have

\[
\mu_i = \lim_{k \to \infty} \mu_f(w_{s_k}, R_{s_k}) = \lim_{k \to \infty} \mu_h(\tilde{w}_k, R_{s_k}) = \lim_{k \to \infty} \mu_g(\tilde{w}_k, R_{s_k}) + \mu = \mu . \quad (2.3)
\]

But in \( W_\alpha \) the maximal multiplicity of zero of \( h_z \) at 0 is \( j(W_\alpha) \) because by BM Theorem the Taylor coefficient of \( h_z \) whose number is \( j(W_\alpha) + 1 \) is nowhere-vanishing.
and the previous coefficients have common zero at \( 0 \in W_\alpha \). This shows that
\[
\mu_i \leq j(W_{i-1}) = \mu_f(K).
\]

Let us prove the opposite inequality. Recall that from Theorem \( \square \) applied to the cover \( \mathcal{W}_i \) it follows that there is an element \((W_\alpha, K_\alpha, \phi_\alpha)\) of a dominating family of the cover \( \mathcal{W}_i \) (with \( U(\mathcal{W}_i) \subset O_i \)) such that \( j(W_\alpha) = \mu_f(K) \). In particular, there is a point \( z_0 \in W_\alpha \) such that the multiplicity \( \mu \) of zero of \( h_{z_0} \) (defined as above) at \( 0 \) is \( j(W_\alpha) \). Then from (2.3) with \( \lim_{k \to \infty} \hat{w}_k = z_0 \) and from maximality of \( \mu_i \) we have
\[
\mu_i \geq \mu = j(W_\alpha) = \mu_f(K).
\]
Combining these inequalities we get
\[
\mu_i = \mu_f(K) \quad \text{and} \quad \lim_{i \to \infty} \mu_i = \mu_f(K)
\]
which completes the proof of the theorem. \( \square \)

3. Proof of the Cyclicity Theorem.

3.1. Cartan’s Lemma. In the proof we use a version of the Cartan Lemma proved in Levin’s book [L, p.21].

**Cartan’s Lemma.** Let \( f(z) \) be a holomorphic function on \( \mathbb{D}_{2\pi R} \), \( f(0) = 1 \), and \( \eta \) be a positive number \( \leq \frac{3\pi}{2} \). Then there is a set of disks \( \{D_j\} \) with \( \sum j r_j \leq 4\eta R \), where \( r_j \) is radius of \( D_j \) such that
\[
\log |f(z)| > -H(\eta) \log \max_{\mathbb{D}_{2\pi R}} |f|
\]
for any \( z \in \mathbb{D}_R \setminus (\cup_j D_j) \). Here \( H(\eta) = 2 + \log \frac{3\pi}{2\eta} \).

Let \( g(z) \) be a holomorphic function on \( \mathbb{D}_{(6\pi+1)r/2} \). Let \( m_1 := \max_{\mathbb{D}_{r/2}} |g| \) and \( m_2 := \max_{\mathbb{D}_{(6\pi+1)r/2}} |g| \). In what follows \( S_t := \{z \in \mathbb{C} : \eta \in C\} \). From Cartan’s Lemma we have

**Lemma 3.1** There is a number \( t_r \), \( r/2 \leq t_r \leq r \), such that
\[
\min_{S_{t_r}} \log |g| > \log m_1 - 7 \cdot \log \frac{m_2}{m_1}.
\]

**Proof.** Let \( w \in S_{r/2} \) be such that \( |g(w)| = m_1 \). We set \( f(z) := \frac{g(z+w)}{g(w)} \). Then \( f \) is defined on \( \mathbb{D}_{2\pi R} \) with \( R = \frac{3\pi}{2} \), and \( f(0) = 1 \), \( \max_{\mathbb{D}_{2\pi R}} |f| \leq \frac{m_2}{m_1} \). For any \( t \), \( r/2 \leq t \leq r \), we set
\[
K_t := \{z \in S_t : |f(z)| = \min_{S_{t_r}} |f|\} \quad \text{and} \quad K = \cup_t K_t.
\]
Clearly one cannot cover \( K \) by a set of disks \( \{D_j\} \) with \( \sum_j r_j < 4\eta R \) where \( \eta = 1/24 \). In particular, by Cartan’s lemma there is \( t_r \in [r/2, r] \) such that
\[
\min_{S_{t_r}} \log |f| \geq -(2 + \log 36e) \log \frac{m_2}{m_1} > -7 \log \frac{m_2}{m_1}.
\]
Going back to \( g \) gives the required inequality. \( \square \)

Assume that \( g \) as above is a polynomial of degree \( d \). Then we have
Corollary 3.2 There is a number $t_r$, $r/2 \leq t_r \leq r$, such that
\[
\min_{S_{t_r}} |g| > \frac{1}{(6e + 1)^d} \cdot \max_{B_{t_r/2}} |g| > \frac{1}{2\cdot 2^{2^d}} \cdot \max_{B_{t_r/2}} |g| .
\]

Proof. The Bernstein Doubling inequality for polynomials implies $\frac{\max}{\min} \leq (6e + 1)^d$. Then we apply Lemma 3.1. \qed

3.2. Proof of the Cyclicity Theorem. (1) The definition of $i_0$, estimate (1.3) and Theorem 1.2 imply for $i \geq i_0$,
\[
|a_k(\lambda)| \leq 2c_{\mu_f(K)} \cdot M \cdot \max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)|, \quad k > \mu_f(K), \quad \lambda \in O_i.\]

Then in a disk $\mathbb{D}_R$ with $R < 1$ for $\lambda \in O_i$ we have
\[
|f_\lambda(z) - P_\lambda(z)| = \left| \sum_{i > \mu_f(K)} a_i(\lambda) \cdot z^i \right| \leq 2c_{\mu_f(K)} \cdot M \cdot \frac{R^{\mu_f(K)+1}}{1 - R} \cdot \max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)| . \quad (3.1)
\]

Also, by the Cauchy inequality we have
\[
\max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)| \leq \max_{\mathbb{D}} |P_\lambda| .
\]

The last inequality, Corollary 3.2 and the Bernstein Doubling inequality imply that there is $t_R$, $R/2 \leq t_R \leq R$, such that
\[
\min_{S_{t_R}} |P_\lambda| \geq \frac{\max_{\mathbb{D} \cap B_{t_R/2}} |P_\lambda|}{2^{2^d} \cdot \mu_f(K)} \geq \frac{R^{\mu_f(K)}}{2^{2^d} \cdot \mu_f(K)} \cdot \max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)| . \quad (3.2)
\]

We set
\[
R_0 = \frac{1}{2c_{\mu_f(K)} \cdot M \cdot 2^{2^d} \cdot \mu_f(K) + 1} .
\]

Combining (3.1) and (3.2) we have for any $R < R_0$, $z \in S_{t_R}$, $\lambda \in O_i$,
\[
|f_\lambda(z) - P_\lambda(z)| \leq 2c_{\mu_f(K)} \cdot M \cdot \frac{R^{\mu_f(K)+1}}{1 - R} \cdot \max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)| < \frac{R^{\mu_f(K)}}{2^{2^d} \cdot \mu_f(K)} \cdot \max_{j=0,\ldots,\mu_f(K)} |a_j(\lambda)| < \min_{S_{t_R}} |P_\lambda| \leq |P_\lambda(z)| .
\]

From here by the Rouche theorem it follows that $f_\lambda$ and $P_\lambda$ have the same number of zeros in $\mathbb{D}_{t_R}$. If we apply the last statement to any $R < R_0/2$ we obtain
\[
\mathcal{N}_{R/2}(P_\lambda) \leq \mathcal{N}_{t_R}(P_\lambda) = \mathcal{N}_{t_R}(f_\lambda) \leq \mathcal{N}_R(f_\lambda) \leq \mathcal{N}_{t_{2R}}(f_\lambda) = \mathcal{N}_{t_{2R}}(P_\lambda) \leq \mathcal{N}_{2R}(P_\lambda) .
\]

This proves the first part of the theorem.

(2) Let $r < R_0/2$ and $i \geq i_0$ with $R_0$ and $i_0$ as above. From the proof of Theorem 1.3 we know that there is a point $\lambda \in O_i$, an open set $W \subset O_i$, $\lambda \in W$, and an
element \((W_\alpha, K_\alpha, \phi_\alpha)\) of the dominating family for \(W\) at \(\lambda\) such that \(j(W_\alpha) = \mu_f(K)\). Moreover from (2.3) it follows that for the function

\[
h_z(y) := \frac{f_\alpha(z)(y)}{z^{\lambda}(W_\alpha)}, \quad z \in W_\alpha, \; y \in \mathbb{D},
\]

there is \(z_0 \in W_\alpha\) such that the multiplicity of zero of \(h_{z_0}\) at 0 equals \(\mu_f(K)\). Further, we can find \(r', r \leq r' < R_0/2\), such that \(h_{z_0}|_{S_{r'}}\) is nowhere-vanishing. In particular, there is an open connected neighbourhood \(O \subset W_\alpha\) of \(z_0\) such that for any \(z \in O\) we still have that \(h_z\) is nowhere-vanishing on \(S_{r'}\). For \(z \in O\) consider the integral

\[
N_{r'}(h_z) = \frac{1}{2\pi i} \int_{S_{r'}} \frac{\partial}{\partial y} h_z(y) h_z(y) dy
\]

which counts the number of zeros of \(h_z\) in \(\mathbb{D}_{r'}\). Clearly the function \(N_{r'}(h_z), z \in O\), is continuous and integer-valued. Thus it is constant on \(O\). From here by the definition of the dominating family it follows that there is a point \(w \in O\) such that \(\lambda_0 := \phi_\alpha(w) \in O \setminus C(f; O_i)\) and

\[
N_{r'}(f_{\lambda_0}) = N_{r'}(h_w) = N_{r'}(h_{z_0}) \geq \mu_f(K).
\]

But according to part (1) of the Cyclicity Theorem, for any \(\lambda \in O_i\)

\[
N_{R_0/2}(f_\lambda) \leq \mu_f(K).
\]

These inequalities imply

\[
N_{r'}(f_{\lambda_0}) = \mu_f(K).
\]

The proof of the theorem is complete. \(\Box\)

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