Iso-Geometric Integral Equation Solvers and Their Compression via Manifold Harmonics

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Abstract—The state of the art of electromagnetic integral equations has seen significant growth over the past few decades, overcoming some of the fundamental bottlenecks: computational complexity, low-frequency breakdown, dense-discretization breakdown, preconditioning, and so on. Likewise, the community has seen extensive investment in the development of methods for higher order analysis, in both geometry and physics. Unfortunately, these standard geometric descriptors are continuous, but their normals are discontinuous at the boundary between triangular tessellations of control nodes, or patches, with a few exceptions; as a result, one needs to define additional mathematical infrastructure to define physical basis sets for vector problems. In stark contrast, the geometric representation used for design is second order differentiable almost everywhere on the surfaces. Using these descriptions for analysis opens the door to several possibilities, and is the area we explore in this article. Our focus is on loop subdivision-based isogeometric methods. In this article, our goals are twofold: 1) development of computational infrastructure for isogeometric analysis of electrically large simply connected objects, and 2) introduction of the notion of manifold harmonic transforms and its utility in computational electromagnetics. Several results highlighting the efficacy of these two methods are presented.

Index Terms—Fast multipole method, higher order, integral equations, iso-geometric methods, manifold harmonics, subdivision surfaces.

I. INTRODUCTION

O

VER the past six decades, the state-of-the-art boundary integral equation solvers have grown by leaps and bounds to become a powerful tool for electromagnetic analysis. A sequence of advancements have enabled this transition, starting from the development of integral equations (see [1] and references therein for a more complete historical background), to methods to appropriately discretize them [2], to higher order representations [3], to overcoming computational bottlenecks [4]–[7], to well-conditioned formulations [8]–[10], and more recently, to preconditioning techniques [11], [12]. However, despite the significant recent progress made, the technological drivers demand a more sophisticated and more feature-rich solver, albeit at reduced cost.

Computational analysis typically proceeds in three stages: 1) constructing a geometric model using a computer-aided design (CAD) tool; 2) defining a discrete representation of said geometry; and 3) finally, choosing a representation of the physics on the discrete representation of the geometry. Geometry is typically represented using bi-variate splines [Bezier splines, B-splines, or nonuniform rational B-splines (NURBS)] that can provide higher order continuity on the surface. From this surface representation, a mesh is generated that typically provides low-order continuity on the manifold. As an example, piecewise flat Lagrangian elements are $C^0$, i.e., continuous at interfaces between triangular tessellations of control nodes, or patches, but with discontinuous normals. Furthermore, even higher order meshes are at higher order within a patch/subdomain, but still $C^0$ across patches. As a result, basis functions defined on these meshes must impose additional constraints. In this framework, a number of different approaches to electromagnetic analysis tools have been developed, including RWG basis sets [2], its higher order variants [3], and Buffa–Christansen basis [13]. In addition, there exists an in-depth analysis and study into computational bottlenecks such as ill-conditioning, low-frequency breakdown, dense-mesh breakdown, and topological breakdown [10], [11], [14].

Two more relatively recent methods take a different approach; they still seek to obtain a higher order parameterization of the geometry and, thereby, higher order basis for physics. The first overcomes item (1) mentioned earlier and directly models the object using higher order polynomials [15]. Another approach, the generalized method of moments (GMM), starts with (2) and builds a framework that accommodates both large ($\lambda$) and small patches as well as different functions on each patch [16], [17] all stitched together within a partition of unity framework. This is done using a nonwatertight representation of standard meshes. Other methods rely on different techniques to enrich function spaces to represent physics (for instance, macrobasis sets [18]). All seek to achieve an efficient representation of geometry, physics, or both.

An alternative approach that is gaining currency is equipped with the infrastructure to do physics using the same basis function used to construct the geometry; this is known as isogeometric analysis (IGA). The advantages of such an approach are as follows: 1) they eliminate the error in translating...
between geometry and the mesh; 2) the number of degrees of freedom is limited to that used for geometry representation which is significantly smaller than a corresponding mesh; and 3) the rules used for adaptation and refinement are identical for both geometry and physics; a vivid illustration can be found in [17] and [19]–[21]. One must highlight that, in isogeometric methods, basis functions are co-located on control nodes used to describe the geometry. This is in contrast with parametric methods that require additional infrastructure—for an example of using subdivision for geometry and GMM basis sets (see [17]).

The genesis of IGA methods started with using NURBS for solid mechanics [22], and more recently, in electromagnetics [19], [23] and acoustics [20]. NURBS are geometric descriptions that are topologically a disk, a tube, or a torus. As a result, one of the major difficulties that arise with NURBS is that the patches have to be seamlessly sewed together in order to handle complex surfaces which is time-consuming and complicated. Furthermore, stitching together these patches can result in surfaces that are not watertight and sometimes discontinuous. These complexities have proven to be quite a hindrance when handling complex geometries [24]. Other modalities that have gained currency in geometry representation are T-splines and loop subdivision. While T-splines have been used in an IGA setting (see [24], [25], and references therein), our focus in this article will be on loop subdivision.

Loop subdivision has been extremely popular in the computer graphics industry due to the ease with which one can represent complex topologies, its scalability, inherently multi-resolution features, efficiency, and ease of implementation. More importantly, the surface representation is $C^2$, or continuous twice differentiable surface, almost everywhere making it an attractive candidate for defining physical basis sets, as it avoids the requirement of defining additional mathematical framework that is commonplace in other low-order basis set [2], [14], [26]. There has been a concerted effort to develop IGA methods on subdivision surfaces in a number of fields, including electromagnetics [21], [27], [28], acoustics [20], [29], and shape reconstruction/optimization [30]–[34].

This article builds on our earlier body of work on loop-subdivision-based IGA for the electric field integral equations [21] and construction of Debye sources [28]. In both these cases, the objects analyzed were simply connected and electrically small. Furthermore, they only discretized the electric field integral equation. The key bottleneck is the number of quadrature points required to evaluate all necessary inner products on higher order geometry (fourth) and third-order basis. A principal goal of this article is to alleviate this bottleneck for all methods that use higher order surface representation and higher order basis for physics. It is illustrated here for subdivision basis. To do so, we exploit wideband multilevel fast multipole algorithm to evaluate all interactions (self, near, and far) with leaf boxes as small as 0.025$\Delta$. Furthermore, we pair this approach with a well-conditioned combined field integral equation to analyze objects as large as 120$\Delta$.

Next, we introduce the manifold harmonic basis (MHB) for field computation. This basis is the eigenfunctions of the Laplace–Beltrami operator (LBO) [35] and is computed using finite element on the manifold. MHB is tantamount to the Fourier basis on the manifold [36] and is analogously equipped with a manifold harmonic transform (MHT). It has found numerous applications, ranging from shape analysis [37], [38], dimensionality reduction with spectral embeddings [39], [40], medical imaging applications [41], [42], and shape reconstruction [30]. In this article, we explore the applicability of MHB for electromagnetic analysis, specifically to compress systems resulting from discretization of boundary integral equations in electromagnetics, and demonstrate its numerous benefits. What we do not address, and is outside the scope of this article, is the cost of applying these transformation, remediation of cost, and the other benefits that arise from this transformation; these topics will be addressed in subsequent articles and the direction of our research on these issues is alluded to in Section VII.

\section{Problem Statement}

We consider the analysis of scattered fields $\{E', H'\}$, from a perfectly electrically conducting (PEC) object $\Omega$, due to fields $\{E, H\}$ incident on its boundary $\Gamma \subset \Omega$. It is assumed that this surface is equipped with a unique outward pointing normal denoted by $\hat{n}(r)$. $r \in \Gamma$. The region external to this volume $\mathbb{R}^3 \setminus \Omega$ is occupied by free space. The scattered field at $r \in \mathbb{R}^3 \setminus \Omega$ can be obtained using equivalence theorems leading to the following:

\begin{align}
\hat{n}(r) \times E'(r) &= T_\kappa \circ J(r) \\
\hat{n}(r) \times H'(r) &= K_\kappa \circ J(r)
\end{align}

where

\begin{align}
T_\kappa \circ J(r) &= \frac{1}{\kappa} \int_{\Gamma} G_\kappa(r, r') \cdot \frac{\hat{n}(r') \times \nabla}{\|\hat{n}(r')\|} J(r') \, dr' \\
K_\kappa \circ J(r) &= \hat{n}(r) \times \frac{1}{\mathcal{K}_\kappa} \int_{\Gamma} \nabla G_\kappa(r, r') \cdot J(r') \, dr'
\end{align}

where $G_\kappa(r, r') = \exp[-j\kappa|\mathbf{r} - \mathbf{r}'|] / (4\pi |\mathbf{r} - \mathbf{r}'|)$, $\kappa$ is the free space wavenumber, $\eta$ is the free space impedance, $\mathcal{K}_\kappa$ is taken in the Cauchy principal value sense, and $J(r')$ is the equivalent current that is induced on the surface. In the above-mentioned expressions, and what follows, we assume and suppress $\exp[j\omega t]$ time dependence. Using the above-mentioned equations, one may prescribe the requisite electric field and magnetic field integral equations (EFIE/MFIE) as

\begin{align}
\text{EFIE:} &= \hat{n}(r) \times \nabla \times \hat{n}(r) \times (E'(r) + E'(r)) = 0 \\
\text{MFIE:} &= \hat{n}(r) \times (H'(r) + H'(r)) = \frac{J(r)}{2}
\end{align}

Independently, these equations suffer from nonunique solutions at so-called irregular frequencies, but their linear combination yields a uniquely solvable formulation throughout the frequency spectrum denoted as the combined field integral equation (CFIE)

\begin{align}
(1 - \alpha) \left( \frac{1}{2} T_\kappa - K_\kappa \right) \circ J(r) + \alpha \hat{n} \times T_\kappa \circ J(r) &= (1 - \alpha) \hat{n} \times H' - \alpha \hat{n} \times \hat{n} \times E'
\end{align}
where $\alpha$ is a positive constant. It is well known that these integral equations suffer from several breakdowns (low frequency, dense mesh, topology, etc.) [8], [10], [43]. There has been an extensive body of literature addressing these bottlenecks [11], [21]. In particular, in [17], [21], and [28], the following has been demonstrated for the EFIE: for simply connected objects, employing an isogeometric framework, it is then possible to create a basis that completely satisfy the Helmholtz decomposition, and this basis set can be used in the Calderon setting. While this overcomes a number of problems, a regularized CFIE formulation is still necessary to overcome the nonuniqueness problem. In what follows, we detail a regularized CFIE.

A. Regularized Combined Field Integral Equations (CFIER)

A regularized reformulation of (4) is the CFIER written as follows:

$$\left(\frac{I}{2} - K_\kappa\right) \circ \mathbf{J} + \mathcal{R}_\kappa \circ \mathcal{T}_\kappa \circ \mathbf{J} = \mathbf{\hat{n}} \times \mathbf{H}' - \mathcal{R}_\kappa \circ (\mathbf{\hat{n}} \times \mathbf{E}').$$  \hspace{1cm} (5)

Here, $\mathcal{R}_\kappa$ is chosen as a regularizing operator for $\mathcal{T}_\kappa$ such that the integral operators on the left-hand side of (5) are second-kind Fredholm operators. Typically, the construction of the regularizing operators is based on Calderon identities and complexification techniques. Operator $\mathcal{R}_\kappa$ has been proposed and analyzed in the literature [8], [10], [43]–[46].

In particular, we choose the regularization operators provided in [43]. This formulation was found to showcase the superior performance of solvers based on the novel Calderon-Complex CFIER (CC-CFIER) formulations that involve the boundary integral operators

$$\left(\frac{I}{2} - K_\kappa\right) \circ \mathbf{J} - 2\mathcal{T}_\kappa' \circ \mathcal{T}_\kappa \circ \mathbf{J} = \mathbf{\hat{n}} \times \mathbf{H}' + 2\mathcal{T}_\kappa' \circ (\mathbf{\hat{n}} \times \mathbf{E}')$$  \hspace{1cm} (6)

where $\kappa' = \kappa - j0.4\zeta^{2/3}\kappa^{1/3}$ and $\zeta$ is the maximum of the absolute values of mean curvatures on surface $\Gamma$.

To solve (6), we will: 1) represent the surface of the scatterer using isogeometric loop-subdivision basis sets; 2) represent the currents on the surface using the same basis set; and 3) validate solutions to these integral equations solved using the currents on the surface using the isogeometric loop-subdivision basis sets; and

III. SUBDIVISION SURFACES AND FUNCTIONS

In this section, we provide a brief overview of loop subdivision as an isogeometric tool; information provided is purely for completeness and omits details that can be found in [17], [29], [47]–[51], and references therein. Let $T^k$ denote a $k$th refined control mesh, with vertices $V^k := \{v_i, i = 1, \ldots, N_v\}$ and triangular faces $P^k := \{p_i, i = 1, \ldots, N_f\}$. In short, we can represent a $C^2$ (almost everywhere) smooth limit surface $\Gamma$, through an infinite number of iterative refinements of the control mesh $T^0$, following the loop subdivision scheme [52]. In practice, this prescription is not followed. There exist closed-form expressions for computing the limit surface $\Gamma$ for a given control mesh $T^k$ in terms of quantities defined on the given control mesh [48]. Assume that a subdivision surface admits a natural parameterization of the surface $\Gamma$ in terms of the barycentric coordinates defined on each face $e \in P^k$, for some $k$ values. We begin by considering any patch $e \in P^k$ for some $k$ values, as depicted in Fig. 1. We define the zero-ring of a patch (triangle) as the vertices that belong to the patch, and the one-ring as the set of all vertices, $n_{e_i}$, that can be reached by traversing no more than two edges, as shown in Fig. 1. We define the regularity of the triangle by the characterization of its vertices’ valence (zero-ring); the valence of a given vertex is the number of edges incident on itself. A vertex is considered regular if its valence is equal to 6; otherwise, it is called an irregular or extraordinary vertex. A triangle is regular if its vertices are all regular, and irregular otherwise. Using these definitions, we can define the mapping from the barycentric coordinates on a given patch, $\epsilon$, to limit the surface by a weighted average of the effective basis functions associated with its one-ring [48]. As a result, we can define the limit surface as

$$\Gamma(\mathbf{r}) = \sum_{i=1}^{N_v} c_i \tilde{\zeta}_i(\mathbf{r})$$  \hspace{1cm} (7)

where $c_i$ are the vertex locations of the $N_v$ control points, and $\tilde{\zeta}_i$ is the effective basis function that is associated with quantities associated with $c_i$ and has a support $\Gamma_i$; note that $\bigcup \Gamma_i = \Gamma$. The basis functions $\tilde{\zeta}_i$ span an IGA finite-dimensional space $\mathcal{H}^2(\Gamma)$ [50], [51].

To define isogeometric basis sets, we assume that there exists a net of control function values, coincident with the location of the control net. Thus, any scalar function $(f(\mathbf{r}))$ can then be expressed in terms of the loop-subdivision basis set via

$$f(\mathbf{r}) = \sum_{i=1}^{N_v} a_i \tilde{\zeta}_i(\mathbf{r})$$  \hspace{1cm} (8)

where $N_v$ and $\tilde{\zeta}_i(\mathbf{r})$ retain the same definition as those prescribed earlier. The properties of this representation follow from those for subdivision. Henceforth, the functions $\tilde{\zeta}_i(\mathbf{r})$ will be referred to as loop basis. Its properties are: 1) positivity; 2) compact support; 3) forming a partition of unity; and 4) $C^2$ continuity almost everywhere. These properties are critical to the development of both isogeometric analysis as well as defining finite-element spaces on the manifold to obtain MHBs. We note that, in representing a smooth scalar function in terms of MHs on the surface, the error is bounded by a factor of the squared magnitude of its gradient [53]. For a smooth object,
say a sphere, the MHS are the spherical harmonic functions. As a result, you would need $O((\kappa a)^2)$, with some excess due meshing errors. On more complex objects but smooth object, you get a compression rate closer to Nyquist sampling. On objects with sharp corner/edges, it is easier to locally enrich in addition to MHS [54].

IV. CURRENT REPRESENTATION

The loop basis used to define the geometry provides the means to define the current as well. To do so, we begin by representing currents on any closed surface $\Gamma$, via the the Helmholtz decomposition as

$$
\mathbf{J}(\mathbf{r}) = \nabla \phi(\mathbf{r}) + \nabla \times (\hat{n} \psi(\mathbf{r})) + \tilde{\omega}(\mathbf{r})
$$

(9)

where $\tilde{\omega}(\mathbf{r})$ is the harmonic field, $\nabla \Gamma$ is the surface gradient, and $\psi(\mathbf{r})$ and $\phi(\mathbf{r})$ are the scalar potentials (that satisfy the mean-zero constraint). Assuming that $\Gamma$ is simply connected, $\tilde{\omega}(\mathbf{r}) = 0$. While it is possible to develop div-conforming subdivision basis [55], we have chosen to restrict ourselves to simply connected objects; the rationale being that our basis set offers an exact Helmholtz decomposition, enabling us to develop manifold harmonics for simply connected surfaces that are $C^2$. In what follows, we construct currents in terms of the scalar potentials using both the loop-subdivision basis sets and manifold harmonics.

A. Loop-Subdivision Basis Sets

Using (8), we can define the scalar potentials $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ on the limit surface as

$$
\phi(\mathbf{r}) \approx \tilde{\phi}(\mathbf{r}) = \sum_{n=1}^{N_n} a_n^1 \xi_n(\mathbf{r})
$$

$$
\psi(\mathbf{r}) \approx \tilde{\psi}(\mathbf{r}) = \sum_{n=1}^{N_n} a_n^2 \eta_n(\mathbf{r}).
$$

(10)

It follows from (9) that it is possible to define the approximation of the current on a simply connected limit surface as

$$
\mathbf{J}(\mathbf{r}) \approx \mathbf{J}_\beta(\mathbf{r}) = \sum_{n} \left[ a_n^1 \mathbf{J}_n^1(\mathbf{r}) + a_n^2 \mathbf{J}_n^2(\mathbf{r}) \right]
$$

(11a)

$$
\mathbf{J}_n^1(\mathbf{r}) = \nabla \xi_n(\mathbf{r})
$$

$$
\mathbf{J}_n^2(\mathbf{r}) = \hat{n}(\mathbf{r}) \times \nabla \tilde{\psi}_n(\mathbf{r}).
$$

(11b)

Finally, since the representation is constructed using conditions on currents that rely on derivatives of the potentials $\tilde{\phi}(\mathbf{r})$ and $\tilde{\psi}(\mathbf{r})$, leading to the existence of nontrivial solutions to (11), we must enforce uniqueness. In order to ensure uniqueness, we impose an additional zero-mean constraint on the finite-dimensional space $\Psi$, leading to

$$
\Psi = H^2(\Gamma) \cap \left\{ \int_{\Gamma} f(\mathbf{r}) d\mathbf{r} = 0 \right\}.
$$

(12)

A more thorough explanation and several properties of the basis functions can be found in [21] and [28].

B. Manifold Harmonics

While the loop-subdivision basis sets are local basis sets, what we explore next is the possibility of developing a global representation for the potentials $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$. In effect, we are seeking the smoothest possible way to interpolate $\psi(\mathbf{r})$ and $\phi(\mathbf{r})$; it is well known that the Laplace–Beltrami operator (LBO) is an ideal candidate [35], [56]. Consider a real-valued function $\chi(\mathbf{r})$ defined on a compact 2D Riemannian manifold $\Gamma$ embedded in $\mathbb{R}^3$. The Laplace–Beltrami operator $\Delta_{\Gamma}$ is defined as

$$
\Delta_{\Gamma} \chi(\mathbf{r}) = \nabla \cdot (\nabla \chi(\mathbf{r})).
$$

(13)

The LBO $\Delta_{\Gamma}$ admits a complete and countable sequence of eigenfunctions which form an orthonormal basis in $L^2(\Gamma)$ [35], denoted by $\{ H_m \}$ such that

$$
- \Delta_{\Gamma} H_m = \lambda_m H_m.
$$

(14)

These eigenfunctions, known as manifold harmonic basis (MHB), are the building blocks for a complete system of eigenfunctions of the vector Laplace–Beltrami operator (or Hodge Laplace operator) $\Delta_{\Gamma} = \nabla_{\Gamma} \cdot - curl_{\Gamma} curl_{\Gamma}$. Indeed, the system $\{ \nabla_{\Gamma} H_m, curl_{\Gamma} H_m \}$ forms a system of orthogonal nontrivial eigenvectors for $\Delta_{\Gamma}$ with the same eigenvalues $\lambda_m$

$$
- \Delta_{\Gamma} \nabla_{\Gamma} H_m = \lambda_m \nabla_{\Gamma} H_m
$$

$$
- \Delta_{\Gamma} curl_{\Gamma} H_m = \lambda_m curl_{\Gamma} H_m.
$$

(15)

(16)

Therefore, given $\mathbf{J} \in L^2(\Gamma)$, we have

$$
\mathbf{J} = \sum_{m=1}^{\infty} u_m \nabla_{\Gamma} H_m + v_m curl_{\Gamma} H_m
$$

(17)

so that $\{ \nabla_{\Gamma} H_m, curl_{\Gamma} H_m \}$ is an orthonormal basis for the space of square integrable tangential vector field.

C. Computing the Manifold Harmonics

In order to numerically compute MHBs, we employ the loop subdivision FEM Galerkin method. This is akin to similar efforts using the Lagrangian surface descriptions [38], [57] that have shown both $h$- and $p$-convergence [28], [38], [57]. The numerics necessary for computing eigenfunctions of the LBO rely on casting the Laplacian eigenvalue problem in a variational setting. The solution of this variational problem is approximated using the finite-element Galerkin technique on the surface. We begin by evaluating an inner product of (14) with some test function $\nu(\mathbf{r}) \in \{ \xi_i(\mathbf{r}) \}$ and then use Green’s theorems to arrive to the following:

$$
\langle \nabla_{\Gamma} \nu(\mathbf{r}), \nabla_{\Gamma} H_m(\mathbf{r}) \rangle_{\Gamma} = - \lambda_m \nu(\mathbf{r}), H_m(\mathbf{r}) \rangle_{\Gamma}
$$

(18)

where $\langle f(\mathbf{r}), g(\mathbf{r}) \rangle_{\Gamma} = \int_{\Gamma} f(\mathbf{r}) \cdot g(\mathbf{r}) d\mathbf{r}$ follows the standard inner product definition. The MH $H_m(\mathbf{r})$ is represented in the same fashion as (8) leading to

$$
H_m \approx \tilde{H}_m(\mathbf{r}) = \sum_{i} h_m^i \xi_i(\mathbf{r})
$$

(19)

for $h_m^i \in \mathbb{R}$. This leads to a generalized eigenvalue problem

$$
[A][H] = -[A][B][H],
$$

(20)
where
\[
[A]_{ij} = \int_{\Gamma \cap \Gamma_J} \nabla r_i(r) \cdot \nabla r_j(r) dr,
\]
\[
[B]_{ij} = \int_{\Gamma \cap \Gamma_J} r_i(r) r_j(r) dr.
\] (21a)

For this generalized symmetric eigenvalue problem, \([A] \in \mathbb{R}^{N \times N}\) is positive semi-definite, \([B] \in \mathbb{R}^{N \times N}\) is positive definite, \([\Lambda] \in \mathbb{R}^{N \times N}\) contains \(N\) eigenvalues along its diagonal, and \([H] \in \mathbb{R}^{N \times N}\) contains the solution vectors, i.e., the coefficients of each eigenvector defined in (19), in its column space. For this symmetric generalized eigenvalue problem, we have \([H]^T[A][H] = [\Lambda]\) and \([H]^T[B][H] = [I]\), where \([I]\) is the identity matrix. From the previous relations, it follows that the eigenfunctions are orthogonal with respect to the \([B]\)-based scalar product (i.e., \([H, H']_B = H^T_B H'\)). The eigenvectors with corresponding eigenvalues can be calculated with a direct eigensolver or by using the efficient band-by-band computation method presented in [36]. There is an extensive body of literature on efficient computation of these functions, largely applied to computer graphics [58].

Given the representation of each of the eigenfunction, it follows that functions defined on the manifold can be written in terms of these eigenfunctions, as can its divergence and rotational. Specifically
\[
\nabla r_i H_m(r) \approx \nabla r_i \tilde{H}_m(r) = \sum_{i} h_i^m \nabla r_i \tilde{\xi}_i(r) \quad (22a)
\]
\[
\text{curl}_r H_m(r) \approx \text{curl}_r \tilde{H}_m(r) = \sum_{i} h_i^m \hat{n} \times \nabla r_i \tilde{\xi}_i(r). \quad (22b)
\]

Using these expressions, the currents may alternatively be written in terms of this basis as
\[
\mathbf{J}(r) \approx \mathbf{J}_M(r) = \sum_{m=1}^{N} \left\{ v_m \mathbf{J}_m^1(r) + \omega_m \mathbf{J}_m^2(r) \right\} \quad (23a)
\]
\[
\tilde{\mathbf{J}}_m^1(r) = \frac{\nabla r_i \tilde{H}_m(r)}{\sqrt{\lambda_m}},
\]
\[
\tilde{\mathbf{J}}_m^2(r) = \frac{\text{curl}_r \tilde{H}_m(r)}{\sqrt{\lambda_m}} \quad (23b)
\]
such that \(\{\mathbf{J}_m^1, \mathbf{J}_m^2\}\) is an orthonormal basis of the space of \(L^2\) tangential vector fields. Similarly, as stated earlier, we impose uniqueness of (23) using a zero-mean constraint.

### D. Illustration of Manifold Harmonic Transforms

While the manifold harmonic transform has been commonplace in the computer graphics literature for an array of applications, to the authors’ knowledge, it has not been utilized in the analysis of physics on manifolds. In particular, one of its many attractive features is its ability to rigorously compress the system. In what follows, we illustrate some of the features of this approach within the framework of this article. To wit, we consider representation of currents on two different objects: a bumpy cube and a jet airliner. Our goal is to examine the convergence of the representation of the current to a bandwidth of \(M\) harmonics.

In both the instances, we reconstruct a surface current generated by a 1-GHz plane wave incident in \(\hat{-z}\). In Fig. 2, we visualize the manifold harmonic representation of the current for a bumpy cube, with 5124 DoFs, and in Fig. 3 for a jet airliner, with 12132 DoFs. As can be seen in both the figures, the first \(\mathbf{J}_m\) functions capture the coarse features of the current and the next high-frequency ones corresponding to
we use a Galerkin prescription to discretize these equations. In this section, we detail the discretization of MHs. Our metric for validation is the reconstruction error decreases as the number of MH increases, eventually approaching machine precision.

V. FIELD SOLVERS

Thus far, we have discussed loop subdivision basis and its mapping to MHBs. In this section, we detail the discretization of (6), in terms of both loop basis and MHB; in particular, we use a Galerkin prescription to discretize these equations.

A. Discretization Using Loop Subdivision Basis

Note that discretizing Calderón-type operators requires intermediate spaces, effected through a Gram matrix. We define the required Gram matrix [G] using

$$[G]_n^m = \delta_{nk} [J_n^m]_{\Gamma_x \cap \Gamma_w}$$

where $\delta_{nk}$ is the Kronecker delta [21]. In effect, the system of matrices to be solved can be written as

$$[Z][I] = [V]$$

with

$$[Z] = [G]^{-1}[[L] + [K']]$$

where

$$[L] = 2\sum_{\gamma} [T_{\gamma}] [G]^{-1} [T_{\gamma}] c.$$
\( \hat{k} = \hat{k}(\theta, \phi) \). Note that the gradient on Green’s function is evaluated spectrally; furthermore, a traditional approach to using MLFMA would ensure that the entire support domain \( \Gamma_n \) lies within a leaf box, i.e., for loop basis functions, the size of the leaf box is \( \Delta_0 \approx 0.9 \). This means that each leaf box has approximately \( N_p \) quadrature points, and the cost of computing the near-field interactions in the MLFMA scheme is \( \mathcal{O}(4N_p^2 N_n^2) \). As \( N_p \) is relatively high, this is still untenable. We would like \( \Delta_0 \) to be as small as possible, such that it contains far fewer quadrature points.

Consider instead Fig. 5 which shows three leaf level boxes within \( \Gamma_n \). Furthermore, assume that we have to compute the self-interaction of basis \( n \) using MLFMA; boxes (1, 3) are in the far-field of each other, and (2, 3) and (1, 2) are in the near field of each other. Let us reexamine the evaluation of \( \Phi(\mathbf{r}) \) for \( \mathbf{r} \in \Gamma_{3,n} \). Using (28)

\[
\Phi(\mathbf{r}) = \frac{-j}{\omega \epsilon} \int_{\Gamma_{1,n}} \nabla' G_\epsilon(\mathbf{r}, \mathbf{r}') \cdot J_0^1(\mathbf{r}') d\mathbf{r}'
- \frac{1}{\omega \epsilon} \int_{\Gamma_{1,n}} \nabla' G_\epsilon(\mathbf{r}, \mathbf{r}') \cdot J_0^1(\mathbf{r}') d\mathbf{r}'
= \Phi_{13}(\mathbf{r}) + \Phi_{23}(\mathbf{r}).
\]  

(30)

Here, \( \Gamma_{i,n} \) denotes the intersection of box-\( i \) with \( \Gamma_n \). There are two possible ways of evaluating the far-field interaction, \( \Phi_{13}(\mathbf{r}) \), using a variation of (29) with the understanding that the the domain of integration in (29b) is confined to \( \Gamma_{1,n} \). Specifically

\[
\Phi_{13}(\mathbf{r}) = \frac{-j \hat{k}^2}{16\pi^2 \omega \epsilon} \int_{S_2} \hat{k} \cdot \mathcal{M}_1(\mathbf{k}, \mathbf{r}_s) T(\mathbf{k}, \mathbf{r}_s - \mathbf{r}) d^2\mathbf{k}
- \frac{\hat{k}}{16\pi^2 \omega \epsilon} \int_{S_2} \mathcal{M}_2(\mathbf{k}, \mathbf{r}_s) T(\mathbf{k}, \mathbf{r}_s - \mathbf{r}) d^2\mathbf{k}
\]  

(31a)

where

\[
\mathcal{M}_1(\mathbf{k}, \mathbf{r}_s) = \int_{\Gamma_{1,n}} J_n^1(\mathbf{r}') e^{-j\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r})} d\mathbf{r}'
\]  

(31b)

\[
\mathcal{M}_2(\mathbf{k}, \mathbf{r}_s) = -\int_{\Gamma_{1,n}} \nabla' \cdot J_0^1(\mathbf{r}') e^{-j\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r})} d\mathbf{r}'
+ \int_{\Gamma_{1,n}} \hat{u}_{\Gamma_1,n}(\mathbf{r}') \cdot J_0^1(\mathbf{r}') e^{-j\mathbf{k} \cdot (\mathbf{r}_s - \mathbf{r})} d\mathbf{r}'
\]  

(31c)

and \( \hat{u}_{\Gamma} \) is outward pointing normal to the boundary \( \partial \Gamma \).

Consider next the near-field evaluation of \( \Phi_{23}(\mathbf{r}) \). As we want the leaf box size to be small, the minimum distance between the box centers becomes very small, and as such, the order of singularity, due to the gradient on Green’s function, introduces near-singular integration challenges. The remedy that is typically taken is to transfer the derivative onto the basis function. Specifically

\[
\Phi_{23}(\mathbf{r}) = \frac{j}{\omega \epsilon} \int_{\partial \Gamma_{2,n}} G_\epsilon(\mathbf{r}, \mathbf{r}') \hat{u}_{\partial \Gamma_{2,n}}(\mathbf{r}', \mathbf{r}) \cdot J_0^1(\mathbf{r}') d\mathbf{r}'
- \frac{j}{\omega \epsilon} \int_{\partial \Gamma_{2,n}} G_\epsilon(\mathbf{r}, \mathbf{r}') \nabla' \cdot J_0^1(\mathbf{r}') d\mathbf{r}'.
\]  

(32)

The lessons we take from the above-mentioned equations are as follows: 1) the aforementioned line integrals have to be accounted for in (32), as they are implicitly included in (31b) and should cancel on the shared interface; 2) unfortunately, finding intersections between higher order surfaces and boxes is nontrivial; and 3) with all challenges considered, we have to use (32) to evaluate \( \Phi_{23}(\mathbf{r}) \) and \( \Phi_{23}(\mathbf{r}) \), respectively. Note that this example is illustrative. Further complication arises in the evaluation of the electric field, as it calls for the gradient of the scalar potential. As a result, one needs additional line integrals to reduce the singularity. As is evident from the earlier discussion, using a mixed potential formulation together with wideband MLFMA permits evaluation of all integrals, near and far, without the consideration of the troublesome line integrals, but at the cost of more tree traversals [7]. Indeed, the size of the leaf box can now be as small as computationally expedient. Leaf box sizes can be chosen such that it contains \( \mathcal{O}(1) \) quadrature points, reducing the cost of near-field evaluation to \( \mathcal{O}(2N_p N_n) \).

We elucidate this process by applying it to (25d), as depicted in Fig. 5: (25d) contains four independent terms that must be computed in the inner integral; three corresponding to the
vector potential, and one corresponding to the scalar potential. It follows that any matrix element can be computed in terms of its partial contributions such that

\[ [T]_{hk,nm}^l = \sum_\gamma \sum_\zeta -j\omega\mu [J_n^l, \tilde{n}(r) \times S_k^m \circ J_m^l]_{\Gamma_h} \]

\[ + j\delta_{mk}\delta_{l1} \langle \nabla_{\Gamma} \cdot J_n^l, S_k^m \circ \nabla_{\Gamma} \cdot J_m^l \rangle_{\Gamma_h} \]  

(33)

where

\[ S_k^m \circ J_m^l = \int_{\Gamma_m} G_s(r, r') J_m^l(r') dr' \]  

(34)

and the indices \( \zeta \) and \( \gamma \) are subpatches of \( \Gamma_m \) and \( \Gamma_n \); subpatches within each other's far-field are constructed via MLFMA, whereas nearfield patches are constructed via direct integration (see Fig. 5).

B. Manifold Harmonic Transform of CC-CFIER

As presented earlier, the MHs are constructed as a linear combination of the loop-subdivision basis functions, and can thus be seen as global basis functions built on top of loop-subdivision basis set. As an aside, these basis would be an excellent candidate to create a reduced-order representation of currents. Consider reduced \( M \) orthogonal MHBs that span \( W^{MH}(\Gamma) \subset \Psi(\Gamma) \). This is tantamount to using \( M < n_n \) for both the representation and measurement space in (23). As a result, one obtains a compressed impedance matrix. As such, we can reconstruct (25a) as

\[ [H] [Z_H] [I_H] = [H] [V_H] \]  

(35)

where

\[ Z_H = [L_H] + [K_H] \]  

(36a)

\[ I_H = [H]^T [I] \]  

(36b)

\[ V_H = [H]^T \left[-2[T]_\epsilon [H] [H]^T [V_T] + [V_K] \right] \]  

(36c)

and

\[ L_H = -2[H]^T [T_\epsilon] [H] [H]^T [T_\epsilon] [H] \]  

(36d)

\[ K_H = -2[H]^T [K] [H]. \]  

(36e)

Here, \([H]^T\) denotes the transpose of \([H]\).

VI. NUMERICAL EXAMPLES

In this section, we present a collection of numerical results to demonstrate the efficacy of the proposed approach. All examples in this section were run serially using a single 2.4 GHz Intel Xeon Gold 6148 CPU core on the HPC Center at Michigan State University. As parallelization is used to generate neither the near-field matrix elements nor MATVECs, RWG data are only generated for the number of DoFs that is within the reach of the available resources. Note that the number of degrees of freedom necessary to model the structure using piecewise flat triangles (and indirectly, the number of RWG basis) is significantly higher. As alluded to in Section I, the two main contributions are: 1) a fast method for evaluating matrix vector products for higher order geometries and higher order physical basis illustrated via application to subdivision-based isogeometric formulation for simply connected objects and 2) employing manifold harmonics for EM analysis. To this end, the data presented in this section highlight the following: 1) the accuracy of the two proposed approaches when compared against analytical data; 2) the improved spectral properties of the CC-CFIER by means of the reduced numbers of iterations required for convergence of the GMRES iterative solver for loop and MHB; 3) the high accuracy and reduced DOF under the MHB; and 4) application of both to analyzing complex targets.

Unless otherwise stated, we compute scattering due to a plane-wave field propagating in \( \hat{k} = -\hat{x} \) and polarized along \( \hat{x} \) axis. Furthermore, we compare radar cross sections (RCSs) in the \( \phi = 0 \) plane, using the proposed methods against either analytical data or a validated method of moments code that is based on RWG basis functions, otherwise referred to as RWG-CFIE. For every scattering experiment presented in the tables, the maximum relative far-field error, denoted by \( \epsilon_\infty \), is defined as

\[ \epsilon_\infty = \frac{\max_{x} |E_{calc}(\hat{x}) - E_{ref}^{\infty}(\hat{x})|}{\max_{x} E_{ref}^{\infty}(\hat{x})} \]  

(37)

where the reference solutions \( E_{ref}^{\infty} \) were computed by Mie series in the case of spherical scatterers, otherwise, by a loop-subdivision-based CC-CFIER. All of the numerical results presented in the tables and graphs in this section were obtained by prescribing a GMRES residual tolerance equal to \( 10^{-5} \) for the overall system and \( 10^{-11} \) for inverting the gram matrix with a diagonal preconditioner. Finally, we note that we provide the iteration count to reach the specified GMRES tolerance, the time taken to reach the prescribed tolerance, and the error relative to the benchmark data.

A. Accuracy of Wideband MLFMA for Adaptive Interactions

Herein, we study the accuracy of using wideband MLFMA to alleviate the computational complexity associated with nearfield computations. To test the controllable accuracy of the aforementioned scheme, we conduct a controlled test. As an aside, the support of a basis function is a one-ring associated with a control vertex; three basis functions are defined on a patch. The most efficient assembly of interactions is computing these in a patchwise manner. To that end, consider two patches that share an edge. The edge length of each patch is approximately \( 0.25\lambda \). We compute the patch-to-patch interaction by using an integration rule developed by subdividing each patch into 16 subpatches and using a 3-point rule in each. Note that we are not computing self-patch interactions. Next, we compute the same interaction, but through an MLFMA tree with leaf box sizes \( \Delta_0 = 0.125\lambda \) and \( \Delta_o = 0.0625\lambda \) which results in one-level and two-level computation of the interactions. The standard tree partitioning of interactions is used; the leaf box of size \( 0.125\lambda \) has about four subpatches, whereas \( 0.0625\lambda \) has approximately one subpatch. Given the size of leaf box, interactions are computed using wideband MLFMA which invokes accelerated Cartesian expansions (ACEs) for leaf box sizes smaller than \( 0.2\lambda \). Fig. 7 demonstrates the controllable accuracy of computing these interactions as a function of \( p \), the expansion coefficient for our wideband MLFMA scheme. As is evident from this figure, one can control the accuracy to very fine precision.
B. Accuracy of CC-CFIER

In the first set of numerical results, we aim to compare the accuracy and high-order nature of the proposed approaches for the analysis of EM scattering against an analytical solution, as well as the number of iterations required by the GMRES solver to reach the prescribed tolerance. To this end, we consider a sphere of diameter \(8 \lambda\) that is modeled using an initial control mesh comprising of 642 vertices and 1280 faces. We consider two meshes generated by refining the initial control mesh once and thrice, respectively, using loop subdivision. Note that, unlike typical mesh refinement, under the rules of subdivision, the limit surface that all meshes point to is identical. More to the point, all the required numerics are carried out on the limit surface, NOT the Lagrangian geometric approximation. This refinement process leads to a coarser sphere of 2562 vertices and 5120 faces and a finer one composed of 40482 vertices and 80960 faces. The main benefit in refining a mesh is better approximation of the physics on the limit surface.

In the experiments discussed next, the finer discretization was used with RWG basis (together with the Lagrangian geometry description). We ensured that the surface areas of the Lagrangian mesh agree within 99% to the subdivision mesh. In Fig. 8, we compare RCS data on an \(8 \lambda\) sphere for all three candidates.

For the CC-CFIER: MH, we use 1000 MHs leading to 2000 DoFs which converges in seven iterations for a total solve time of 33 s; RWG-CFIE requires 122880 DoFs, converges in 36 iterations in 166 s, and CC-CFIER: loop contains 5124 DoFs, converges in seven iterations in 35 s. As it is evident from Fig. 8, the agreement between the three sets of numerical data to analytical solutions is excellent. In addition, we have analyzed a series of electrically larger spheres. These geometries are obtained via refinement of the initial mesh, such that at any frequency, the edge length is approximately \(0.3 \lambda\). The details of these experiments are presented in Table II. As is evident from this table, there is excellent agreement between the proposed methods and analytic data. The convergence of loop and MH implementations of CC-CFIER is approximately the same as the total solve time. The approximately fourfold compression is not sufficient to affect the overall solve time due to the well-conditioned gram matrix for the sphere.

C. EM Scattering From Complex Objects

In this section, we provide several examples to demonstrate the viability of using the formulations presented here for EM scattering on complex objects. We do so by comparing our results obtained from CC-CFIER: MH against those obtained using the CC-CFIER: loop and RWG-CFIE.

First, we consider the bumpy cube shown in Fig. 9 that fits in an \(8 \lambda \times 8 \lambda \times 8 \lambda\) box. The number of DoFs for the RWG-CFIE is 122880, converges in 47 iterations for a total of 277 s, whereas CC-CFIER: loop and CC-CFIER: MH require 5124 and 2400 degrees of freedom, respectively.
converge in 11 iterations for a solve time of 60 seconds. Fig. 9 illustrates excellent agreement between all three.

As before, we use mesh refinement to generate electrically larger structures. The results of these runs are presented in Table III, specifically, iteration count for CC-CFIER: loop and CC-CFIER: MH formulation. We report that the iteration count is low, approximately the same for both loop and MH, and both took approximately the same time for the matrix solve. The agreement between loop and the compressed MH system is also excellent.

Next, we consider a shuttle that fits in a $20\lambda \times 12.22\lambda \times 7.22\lambda$ box. The number of DoFs for the RWG-CFIE is 190,080, converges in 273 iterations for a total time of 1202 s. The CC-CFIER: loop uses 31,684 DoFs, converges in 78 iterations that takes 684 s. Finally, for CC-CFIER: MH, it uses 6000 DoFs, converges in 39 iterations that take 311 s. Fig. 10 shows excellent agreement between all three. Again, we refine the geometry to consider electrically larger scatterers, in this case up to $80\lambda$. Table IV reports the iteration count, for CC-CFIER: loop and CC-CFIER: MH basis, as we increase the frequency. We find that the iteration count is stable for both formulation, and they are in excellent agreement. Furthermore, we note the significant compression achieved via MHBs.

Finally, we consider a Jet airliner that fits in an $18\lambda \times 17\lambda \times 5\lambda$ box. In this example, the plane wave propagating in the $\hat{y}$-direction (incident on the nose) and polarized along $\hat{x}$-direction. The number of DoFs for the RWG-CFIE is 72,768, converging in 243 iterations in 1 h, whereas for the CC-CFIER: loop is 12,132,68 iterations in 12 min and 45 s and the CC-CFIER: MH is 5000 and reaches tolerance within 45 iterations in 5 min. It is evident from Fig. 11 that all three datasets agree well with each other. In Table V, we report the iteration count, for CC-CFIER: loop and CC-CFIER: MH basis, as we increase the electrical size of the object. We find that the iteration count is stable for both formulation, as well as excellent agreement. Also, note the excellent compression produced by MHBs.

VII. CONCLUSION

In this article, we have presented the isogeometric analysis method for subdivision surface; in presenting this approach, we assumed a simply connected structure, and used a complete surface Helmholtz decomposition to effect the Calderón oper-
ator. The latter is possible, because the surface is $C^2$ almost everywhere. To evaluate inner products, which are the main bottleneck for higher order basis functions on higher order surfaces, we use wideband MLFMA to evaluate all interactions. Finally, we introduce the notion of manifold harmonics as a means to represent the currents on the surface. These geometry basis can be used for compression of both the manifold and physics on the manifold. We present numerous results using both the subdivision and MH basis, on a collection of electrically large geometries. Two salient points that are evident: 1) subdivision basis are excellent candidates for analysis and Debye potentials on the surface. While one can get the compression expected due to a global eigenstructure, a problem that we have not addressed in this article is the cost of effecting this transformation [61]. One avenue in particular that aims to mitigate the costs of the MHT is the use of a set of MHs generated by a pointwise product of a small subset of the original MHB [62]. Furthermore, there are a number of capabilities that are still missing; the two most significant are: 1) extension to multiply connected structures due to using an exact Helmholtz decomposition and 2) open structures. We have made significant progress on the former, and an exact Helmholtz decomposition and 2) open structures.

| Size   | $N_U / N_M H$ | CC-CFIHER-Loop | CC-CFIHER-MH |
|--------|---------------|----------------|--------------|
| 30Ω    | 12/37/2/1000  | 57/14m 59s     | 39/5m 43s    | 3.9E-3    |
| 60Ω    | 48/56/2/1000  | 42/6m 12s      | 39/52m 10s   | 4.7E-3    |
| 120Ω   | 19/4052/63000 | 41/739m 46s    | 40/725m 28s  | 1.3E-2    |

**TABLE V** Data for jetliner geometries from 30Ω to 120Ω.

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