Quadratic forms for a 1-form on an isolated complete intersection singularity

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Abstract

We consider a holomorphic 1-form $\omega$ with an isolated zero on an isolated complete intersection singularity $(V,0)$. We construct quadratic forms on an algebra of functions and on a module of differential forms associated to the pair $(V,\omega)$. They generalize the Eisenbud–Levine–Khimshiashvili quadratic form defined for a smooth $V$.

Introduction

Let $g = (g_1, \ldots, g_n)$ be the germ of an analytic map $(\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ which is real (i.e. $g(\mathbb{R}^n) \subset \mathbb{R}^n$) and finite. The latter fact is equivalent to the one that the local algebra $A_g = \mathcal{O}_{\mathbb{C}^n,0}/\langle g_1, \ldots, g_n \rangle$ is a finite dimensional vector space (here $\mathcal{O}_{\mathbb{C}^n,0}$ is the ring of germs at the origin of analytic functions on $\mathbb{C}^n$, $\langle g_1, \ldots, g_n \rangle$ is the ideal generated by the corresponding germs). The degree $\nu$ of the map $g : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is equal to the dimension $\dim_C A_g$ of the algebra $A_g$ (as a $\mathbb{C}$-vector space). Let $A_g^\mathbb{R}$ be the real part of the algebra $A_g$ (it is the ring of germs at 0 of real analytic functions on $\mathbb{R}^n$ factorized by the ideal $\langle g_1, \ldots, g_n \rangle$; $\dim_{\mathbb{R}} A_g^\mathbb{R} = \dim_C A_g$). The famous theorem of D. Eisenbud, H. Levine and G. Khimshiashvili states that there exists a non-degenerate quadratic form $Q_{\text{ELKh}}$ on the algebra $A_g$ such that it is real on $A_g^\mathbb{R}$ and the degree of the real analytic map $g_{\mathbb{R}} = (g_1, \ldots, g_n) : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ is equal to the signature of the quadratic form $Q_{\text{ELKh}}$ on the algebra $A_g^\mathbb{R}$. Moreover this quadratic form can be defined by the formula

$$Q_{\text{ELKh}}(\varphi, \psi) = R(\varphi \cdot \psi)$$

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where \( R \) is an arbitrary real linear function on the space \( \mathcal{A}_g \) which has a positive value on the Jacobian

\[
\text{Jac}_g = \text{Jac}(g_1, \ldots, g_n) = \det \left( \frac{\partial g_i}{\partial x_j} \right)
\]

of the map \( g \).

A linear function with this property can also be defined by the formula

\[
R(\varphi) = \lim_{\varepsilon \to 0} \sum_{i=1}^{\nu} \frac{\varphi(P_i)}{\text{Jac}_g(P_i)}
\]

where the sum is over all points \( P_i \) of the preimage \( g^{-1}(\varepsilon) \subset \mathbb{C}^n \) of a regular value \( \varepsilon \in \mathbb{C}^n \) of the map \( g \). The Eisenbud–Levine–Khimshiashvili theorem can be also formulated in terms of indices of vector fields or of 1-forms on \((\mathbb{R}^n, 0)\).

There were some results which can be considered as attempts to generalize the Eisenbud–Levine–Khimshiashvili theorem to singular varieties: see, e.g., [MvS] [GM1] [GM2] [SZ] [EG1].

Let \( f = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0) \) be a germ of a complex analytic map which defines an isolated complete intersection singularity (ICIS) \((V, 0) = f^{-1}(0) \subset (\mathbb{C}^n, 0)\) and let \( \omega \) be the germ of a holomorphic 1-form on \((\mathbb{C}^n, 0)\) whose restriction to \( V \) has no singular points (zeroes) in a punctured neighbourhood of the origin. Let \( I_{V,\omega} \) be the ideal of the ring \( \mathcal{O}_{\mathbb{C}^n,0} \) generated by the functions \( f_1, \ldots, f_k \) and by the \((k+1) \times (k+1)\)-minors of the matrix

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\
A_1 & \cdots & A_n
\end{pmatrix}
\]

and let \( \mathcal{A}_{V,\omega} \) be the factor algebra \( \mathcal{O}_{\mathbb{C}^n,0}/I_{V,\omega} \). For \( \omega = dh, h \in \mathcal{O}_{\mathbb{C}^n,0} \), this algebra was considered in [Gr1] [Le] in connection with the computation of the Milnor number of an ICIS.

Suppose that the map \( f \) and the 1-form \( \omega \) are real (i.e. \( f(\mathbb{R}^n) \subset \mathbb{R}^k \) and \( \omega \) has real values on the tangent spaces \( T_x \mathbb{R}^n \)). The 1-form \( \omega \) considered on the real part \((V_\mathbb{R}, 0)\) of the ICIS \((V, 0)\) has an invariant — an index \( \text{ind}_{V,0} \omega \) (which can be called the radial index): [EG1]. In [EG1], there was constructed a family \( Q_\varepsilon \) of quadratic forms on a vector space (the dimension of which coincides with the dimension of the algebra \( \mathcal{A}_{V,\omega} \) as a \( \mathbb{C} \)-vector space) depending on \( \varepsilon \) from a neighbourhood of the origin in the target \( \mathbb{C}^k \) of the map \( f \) such that:

1. the quadratic form \( Q_\varepsilon \) is nondegenerate for \( \varepsilon \) from the complement
to the bifurcation diagram \( \Sigma \subset (\mathbb{C}^k, 0) \) of the map \( f \);

(2) for real \( \varepsilon \), the quadratic form \( Q_\varepsilon \) is real and, for real \( \varepsilon \notin \Sigma \), its signature is equal to

\[
\text{ind}_{V,0} \omega + (\chi(V_\varepsilon) - 1),
\]

where \( V_\varepsilon = f^{-1}(\varepsilon) \cap \mathbb{R}^n \cap B_\delta \) for \( 0 < \varepsilon \ll \delta \) small enough, \( B_\delta \) is the ball of radius \( \delta \) centred at the origin in \( \mathbb{C}^n \).

Here we show that the construction of [EG1] gives a (non-trivial) quadratic form on the algebra \( A_{V,\omega} \). For the curve case (i.e. for \( k = n - 1 \)) this quadratic form is nondegenerate and coincides with the one constructed by J. Montaldi and D. van Straten in [MvS].

We also define a quadratic form on a quotient module \( \Omega_{V,\omega} \) of the module \( \Omega_{\mathbb{C}^n,0}^{n-k} \) of \( (n-k) \)-forms on \( (\mathbb{C}^n, 0) \) which, as a \( \mathbb{C} \)-vector space, has the same dimension as \( A_{V,\omega} \). We describe some relations between these quadratic forms.

We define these quadratic forms by an analytic construction. It would be interesting to understand to which extend it is possible to define them in algebraic terms.

1 Jacobian of a 1-form

Let \( M^m \) be a complex analytic manifold of dimension \( m \) with local coordinates \( y^1, \ldots, y^m \). A vector field \( X = \sum_{i=1}^m X^i \frac{\partial}{\partial y^i} \) on \( M \) is a tensor field of type \((1,0)\). Its "derivative" (Jacobian matrix) \( \left( \frac{\partial X^i}{\partial y^j} \right) \) is generally speaking not a tensor, but it is a tensor of type \((1,1)\) at points where the vector field \( X \) vanishes. At such a point \( P \) it has a well defined (i.e. not dependent on the choice of coordinates) determinant which is the value of the Jacobian \( \text{Jac}(X^1, \ldots, X^m) \) of the vector field \( X \) at the point \( P \).

Now let \( \omega = \sum_{i=1}^m A_i dy^i \) be a 1-form on \( M \), i.e. a tensor field of type \((0,1)\). Again, its Jacobian matrix \( J = \left( \frac{\partial A_i}{\partial y^j} \right) \) defines the tensor \( \sum_{i,j} \frac{\partial A_i}{\partial y^j} dy^i \otimes dy^j \) of type \((0,2)\) (i.e. a bilinear form on the tangent space) only at points where the 1-form \( \omega \) vanishes. The determinant \( J \) of the matrix \( J = \left( \frac{\partial A_i}{\partial y^j} \right) \) is not a scalar (it depends on the choice of local coordinates). Under a change of coordinates it is multiplied by the square of the Jacobian of the change of coordinates (since the Jacobian matrix \( J \) is transformed to \( C^T J C \)). Therefore it should be considered as the coefficient in the tensor

\[
J(dy^1 \wedge \ldots \wedge dy^m)^\otimes 2
\]
of type $(0,2m)$. In this sense the Jacobian of a 1-form is a sort of a "quadratic differential". To get a number, one can divide this tensor by the tensor square of a volume form.

Let $m = n - k$ and let $M$ be a submanifold in $\mathbb{C}^n$ defined by $k$ equations, i.e. let $M = F^{-1}(0)$, where $F = (f_1, \ldots, f_k)$ is a nondegenerate holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^k$ (defined in a neighbourhood of the manifold $M$). Let us fix volume forms on $\mathbb{C}^n$ and $\mathbb{C}^k$, say, the standard ones $\sigma_n = dx_1 \wedge \ldots \wedge dx_n$ and $\sigma_k = dz_1 \wedge \ldots \wedge dz_k$ where $x_1, \ldots, x_n$ and $z_1, \ldots, z_k$ are Cartesian coordinates in $\mathbb{C}^n$ and in $\mathbb{C}^k$ respectively. There exists (at least locally) an $(n-k)$-form $\sigma$ on $\mathbb{C}^n$ such that $\sigma_n = F^* \sigma_k \wedge \sigma$. The restriction $\sigma|_M$ of the form $\sigma$ to the manifold $M$ is well defined and is a volume form on $M$. We also denote it by $\sigma$. After that, for a holomorphic 1-form $\omega$ on the manifold $M$ (say, the restriction to $M$ of a 1-form on the ambient space $\mathbb{C}^n$), the above construction gives well defined numbers $\tilde{J}(P)$ associated with singular points (zeroes) $P$ of the form $\omega$ ("the values of the Jacobian").

Let $\omega = \sum_{i=1}^n A_i \, dx_i$ be a holomorphic 1-form on $\mathbb{C}^n$ and let $P$ be a singular point (zero) of the form $\omega$ on $M$. There exists a $(k \times k)$-minor of the matrix $\left( \frac{\partial f_i}{\partial x_j} \right)$ which is different from zero at the point $P$. Let it be

$$\Delta = \left| \frac{\partial f_i}{\partial x_j} \right|_{i,j=1,\ldots,k}.$$ 

In this case $x_{k+1}, \ldots, x_n$ are local coordinates on $M$ in a neighbourhood of the point $P$ and therefore the restriction of the 1-form $\omega$ to the manifold $M$ can be written as

$$\sum_{i=k+1}^n \hat{A}_i \, dx_i.$$ 

The Jacobian determinant of the 1-form $\omega$ in these coordinates is

$$J = \left| \frac{\partial \hat{A}_i}{\partial x_j} \right|_{i,j=k+1,\ldots,n}.$$ 

A precise formula for the determinant $J$ can be found in [EG1, Prop. 4]. For $i = k+1, \ldots, n$, let

$$m_i := \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_1} & \frac{\partial f_1}{\partial x_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_k} & \cdots & \frac{\partial f_k}{\partial x_k} & \frac{\partial f_k}{\partial x_i} \\ A_1 & \cdots & A_k & A_i \end{vmatrix}.$$
One can see that
\[ \omega|_M = \frac{m_{k+1}}{\Delta} dx_{k+1} + \cdots + \frac{m_n}{\Delta} dx_n. \]

From this it was derived that
\[ J = \frac{1}{\Delta^{2+(n-k)}} \begin{vmatrix} \Delta & \frac{\partial \Delta}{\partial x_1} & \cdots & \frac{\partial \Delta}{\partial x_n} \\ 0 & \frac{\partial \Delta}{\partial x_1} & \cdots & \frac{\partial \Delta}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial \Delta}{\partial m_{k+1}} & \cdots & \frac{\partial \Delta}{\partial x_n} \\ m_{k+1} & \frac{\partial m_{k+1}}{\partial x_1} & \cdots & \frac{\partial m_{k+1}}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & \frac{\partial m_n}{\partial x_1} & \cdots & \frac{\partial m_n}{\partial x_n} \end{vmatrix}. \]

At the point \( P \) (a zero of the 1-form \( \omega \) on the manifold \( M \)) the minors \( m_i \) vanish and therefore
\[ J(P) = \frac{1}{\Delta^{1+(n-k)}} \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\ \frac{\partial m_{k+1}}{\partial x_1} & \cdots & \frac{\partial m_{k+1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial m_n}{\partial x_1} & \cdots & \frac{\partial m_n}{\partial x_n} \end{vmatrix} \]
(\text{the expressions have to be evaluated at the point } P \text{ of course}).

One has
\[ F^* \sigma_k = \sum_{1 \leq s_1 < \cdots < s_k \leq n} \left| \frac{\partial f_i}{\partial x_{s_j}} \right|_{i,j=1,\ldots,k} dx_{s_1} \wedge \ldots \wedge dx_{s_k}. \]

Therefore \( \sigma = \frac{1}{\Delta} dx_{k+1} \wedge \ldots \wedge dx_n \) and
\[ \tilde{J}(P) = \Delta^2 J(P). \]

\textbf{Remark.} The number \( \tilde{J}(P) \) was already used in [EG1].

\section{A quadratic form on the algebra of functions.}

Let \( f = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0) \) be a holomorphic map which defines an \((n-k)\)-dimensional isolated complete intersection singularity
(ICIS) \( V = f^{-1}(0) \subset (\mathbb{C}^n, 0) \) and let \( \omega = \sum_{i=1}^{n} A_i \, dx_i \) be the germ of a 1-form on \((\mathbb{C}^n, 0)\). Let \( F : (\mathbb{C}^n \times \mathbb{C}^M_{\varepsilon}, 0) \to (\mathbb{C}^k \times \mathbb{C}^M_{\varepsilon}, 0) \) be a deformation of the map \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0) \) \((F(x, \varepsilon) = (f_x(x), \varepsilon), \, f_0 = f, \, f_\varepsilon = (f_{1\varepsilon}, \ldots, f_{k\varepsilon}))\) and let \( \omega_\varepsilon \) be a deformation of the form \( \omega \) (defined in a neighbourhood of the origin in \((\mathbb{C}^n \times \mathbb{C}^M_{\varepsilon}), 0)\)) such that, for generic \( \varepsilon \in (\mathbb{C}^M_{\varepsilon}, 0)\), the preimage \( f_\varepsilon^{-1}(0) \) is smooth and (the restriction of) the form \( \omega_\varepsilon \) to it has only non degenerate singular points. The number \( \nu \) of these points is the (complex) index of the 1-form \( \omega \) on \((V, 0)\) considered in \([\text{EG1}]\) (it is an analogue of the GSV-index of a vector field). It is equal to the dimension (as a \( \mathbb{C} \)-vector space) of the algebra \( \mathcal{A}_{V, \omega} \) (\([\text{EG1}]\); see also \([\text{EG2}]\) where some inaccuracies in the proof were corrected).

Let \( \Sigma \subset (\mathbb{C}^M_{\varepsilon}, 0) \) be the germ of the set of the values of the parameters \( \varepsilon \) from \((\mathbb{C}^M_{\varepsilon}, 0)\) such that either the preimage \( f_\varepsilon^{-1}(0) \) is singular or the restriction of the 1-form \( \omega_\varepsilon \) to it has degenerate singular points. The bifurcation diagram \( \Sigma \) is a germ of a hypersurface in \((\mathbb{C}^M_{\varepsilon}, 0)\). For \( \varepsilon \notin \Sigma \), let \( P_1, \ldots, P_\nu \) be the (non-degenerate) singular points of the 1-form \( \omega_\varepsilon \) on the \((n-k)\)-dimensional manifold \( f_\varepsilon^{-1}(0) \). For \( \varepsilon \notin \Sigma \) and a germ \( \varphi \in \mathcal{O}_{\mathbb{C}^n, 0} \), let

\[
R(\varphi, \varepsilon) := \sum_{i=1}^{\nu} \frac{\varphi(P_i)}{J_\varepsilon(P_i)},
\]

(1)

For a fixed \( \varphi \) the function \( R_\varphi(\varepsilon) := R(\varphi, \varepsilon) \) is holomorphic in the complement of the bifurcation diagram \( \Sigma \).

**Theorem 1** The function \( R_\varphi(\varepsilon) \) has removable singularities on the bifurcation diagram \( \Sigma \).

**Proof.** Without any loss of generality we can suppose that the deformation \((F, \omega_\varepsilon)\) is "as big as we would like". In particular we can assume that it includes a versal deformation of the map \( f \) with the trivial deformation of the 1-form \( \omega \) and also that \( n \) parameters of it (marked by elements of \( \mathbb{C}^{n*} \)) correspond to the trivial deformation of the map and adding the corresponding linear form (i.e. a 1-form with constant coefficients) to the 1-form.

It is sufficient to prove that the function \( R_\varphi(\varepsilon) \) has removable singularities outside of a set of codimension 2 in \( \mathbb{C}^M_{\varepsilon} \). Therefore it suffices to prove this for points of the bifurcation diagram \( \Sigma \) of two types:

1. Those points \( \varepsilon \in \Sigma \) for which the preimage \( f_\varepsilon^{-1}(0) \) is smooth (but a singular point of the 1-form \( \omega_\varepsilon \) on it is degenerate). In this case the proof can be found, e.g., in \([\text{AGV}]\) §5.
2. Those points $\varepsilon \in \Sigma$ for which the preimage $f_{\varepsilon}^{-1}(0)$ has one singular point $P$ of type $A_1$, the 1-form $\omega_\varepsilon$ does not vanish at this point in $\mathbb{C}^n$, and its kernel is not tangent to the tangent cone of the variety $f_{\varepsilon}^{-1}(0)$. In this case the proof is essentially already contained in \cite{EG1}. For the sake of completeness we repeat it in an appropriate way.

Without loss of generality (excluding equations which are non-degenerate), we can suppose that $k = 1$, $n \geq 2$. Changing local coordinates in a neighbourhood of the point $P$, we can suppose that $P$ is the origin in $\mathbb{C}^n$, $f_1 = x_1^2 + \ldots + x_n^2$, $\omega(0) = dx_1$ (we omit $\varepsilon$ in the notations). The last equation means that $\omega = (1 + C_1)dx_1 + C_2dx_2 + \ldots C_n dx_n$ where $C_i(0) = 0$. It is sufficient to show that the function $R_\varphi$ has a finite limit when $\eta \to 0$ for the deformation $f_1\eta = f_1 - \eta^2$, $\eta \in \mathbb{C}$, of the equation keeping the 1-form $\omega$ constant. For $\eta \neq 0$ small enough the 1-form $\omega$ has two singular points $P_+$ and $P_-$ on the level manifold $F_1\eta = 0$ with coordinates $x_1 = \pm \eta + o(\eta)$, $x_i = o(\eta)$ for $i \geq 2$. The formula (1) gives

$$\tilde{J}(P_\pm) = (-1)^{n-1}4(\pm \eta)^{3-n}(1 + o(\eta)).$$

The corresponding summands in the expression for $R_\varphi$ are

$$\frac{(-1)^{n-1}}{4} \left( \varphi(P_+)(\eta^{n-3} + o(\eta^{n-3})) + \varphi(P_-)((-\eta)^{n-3} + o(\eta^{n-3})) \right).$$

For $n \geq 3$ or for $\varphi(0) = 0$, each summand has a finite limit when $\eta \to 0$. For $n = 2$ and $\varphi(x) \equiv 1$, $\varphi(P_+)(-\eta)^{-1} + \varphi(P_-)(-\eta)^{-1} = 0$. $\square$

Theorem \ref{thm:1} means that $R_\varphi(\varepsilon)$ can be extended to a holomorphic function on $(\mathbb{C}^M,0)$ (which we denote by the same symbol). Let

$$R(\varphi) := R_\varphi(0).$$

This defines a linear function $R$ on $\mathcal{O}_{\mathbb{C}^n,0}$. It is clear that, if the map $f$ and the 1-form $\omega$ are real, the function $R$ is real as well.

**Remark.** Just in the same way we can assume that the function $\varphi$ depends on the parameters $\varepsilon \in \mathbb{C}^M$ as well: $\varphi = \varphi(x,\varepsilon)$, $\varphi : (\mathbb{C}^n \times \mathbb{C}^M,0) \to (\mathbb{C},0)$. The statement and the proof are the same. Moreover, the limit $R(\varphi) = \lim_{\varepsilon \to 0} R_{\varphi}(\varepsilon)$ is also the same, i.e. $R(\varphi(x,\varepsilon)) = R(\varphi(x,0))$ (this was the reason why we did not include the statement into the theorem).

To show the latter fact we can enlarge the deformation space including one with the function $\varphi$ depending on $\varepsilon$ and the identical one with the function $\varphi$ not changing with $\varepsilon$.\[7]
As above let $I_{V,\omega} \subset \mathcal{O}_{\mathbb{C}^n,0}$ be the ideal generated by the equations $f_1, \ldots, f_k$ of the ICIS $(V,0)$ and by the $(k+1) \times (k+1)$-minors of the matrix
\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\
A_1 & \cdots & A_n
\end{pmatrix}.
\] (3)

Proposition 1 The linear function $R$ vanishes on the ideal $I_{V,\omega}$.

Proof. To prove that $R$ vanishes on a germ $\varphi = h f_i$ ($h \in \mathcal{O}_{\mathbb{C}^n,0}$), consider the function $\varphi(x,\varepsilon) = h(f_i + \varepsilon_i)$. On a nonsingular level set $f + \varepsilon = 0$ the function $\varphi(x,\varepsilon)$ vanishes identically and therefore the expression (1) tends to zero. To prove that $R$ vanishes on a germ $\varphi = h m_I$ ($h \in \mathcal{O}_{\mathbb{C}^n,0}$) where $m_I$ is one of the $(k+1) \times (k+1)$-minors of the matrix (3), consider the function $\varphi(x,\alpha) = h \cdot m_I(x,\alpha)$ where $m_I(x,\alpha)$ is the corresponding minor of the matrix
\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\
A_1 - \alpha_1 & \cdots & A_n - \alpha_n
\end{pmatrix}.
\] (4)

The function $\varphi(x,\alpha)$ vanishes at all the singular points of the 1-form $\omega_\alpha$ on the level set $\{f_\varepsilon = 0\}$ and therefore the expression (1) tends to zero. □

Corollary 1 The formula (1) defines a linear function $R$ on the (finite dimensional) algebra $\mathcal{A}_{V,\omega} = \mathcal{O}_{\mathbb{C}^n,0}/I_{V,\omega}$.

Remark. The function $R$ is well-defined up to multiplication by a unit in the algebra $\mathcal{A}_{V,\omega}$ (which depends on the choice of volume forms on $\mathbb{C}^n,0$ and on $(\mathbb{C}^k,0)$).

At the moment, the definition of the linear function $R$ on $\mathcal{A}_{V,\omega}$ uses the map $f$ (i.e. equations of the ICIS $(V,0)$) and a 1-form $\omega$ defined on the ambient space $\mathbb{C}^n,0$. It is clear that up to multiplication by a unit the function $R$ depends only on the ICIS $(V,0)$ itself. It fact it merely depends on the restriction of the 1-form $\omega$ to the ICIS $(V,0)$.

Proposition 2 The linear function $R$ is defined by the class of the 1-form $\omega$ in the module
\[
\Omega^1_{V,0} = \Omega^1_{\mathbb{C}^n,0}/(f_i \Omega^1_{\mathbb{C}^n,0}, \mathcal{O}_{\mathbb{C}^n,0} df_i)
\]
of germs of 1-forms on the ICIS $(V,0)$. 

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Proof. Let
\[
\omega' = \omega + \sum_{i=1}^{k} f_i \eta_i + \sum_{i=1}^{k} h_i df_i \quad (\eta_i \in \Omega^1_{\mathbb{C}^n,0}, \ h_i \in \mathcal{O}_{\mathbb{C}^n,0}).
\]

Consider the deformation
\[
\omega'_\varepsilon = \omega + \sum_{i=1}^{k} f_i \eta_i + \sum_{i=1}^{k} h_i df_i
\]
of the 1-form \(\omega'\). On a (non-singular) level set \(\{f_i = 0\}\) the 1-form \(\omega'_\varepsilon\) is equal to \(\omega\) and therefore the expressions (1) for \(\omega\) and \(\omega'_\varepsilon\) coincide. □

The linear function \(R\) on the algebra \(\mathcal{A}_{V;\omega}\) defines in the natural way a quadratic form \(Q_{\mathcal{A}_{V;\omega}}\) on it:
\[
Q_{\mathcal{A}_{V;\omega}}(\varphi, \psi) := R(\varphi \cdot \psi).
\]

**Example 1** Let \(k = 1\), \(f_1(x) = f(x) = \sum_{i=1}^{n} x_i^2\), \(\omega = \sum_{i=1}^{n} a_i x_i dx_i (= \frac{1}{2} d(\sum a_i x_i^2))\), where \(a_i\) are pairwise different. The algebra
\[
\mathcal{A}_{V;\omega} = \mathcal{O}_{\mathbb{C}^n,0}/\left\langle \sum_{i=1}^{n} x_i^2, x_i x_j \right\rangle
\]
has dimension \(2n\) as a vector space over \(\mathbb{C}\). It is generated by the classes of the monomials 1, \(x_1\), \ldots, \(x_n\), \(x_2\), \ldots, \(x_{n-1}\). On the level set \(\{f = \varepsilon^2\}\) the singular points of the 1-form \(\omega\) are \(P_i^\pm = (0, \ldots, 0, \pm \varepsilon, 0, \ldots, 0)\). They are non-degenerate. The value of \(J\) at the point \(P_i^\pm\) is equal to \(\varepsilon^2 \prod_{j \neq i} (a_j - a_i)\). One has
\[
R(1) = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2} \left( \sum_{i=1}^{n} \frac{1}{\prod_{j \neq i} (a_j - a_i)} \right) = 0
\]
since the expression in parentheses is identically equal to zero; \(R(x_i) = 0\) (since the values of \(x_i\) at the points \(P_i^+\) and \(P_i^-\) differ only by their signs), and finally
\[
R(x_i^2) = \frac{2}{\prod_{j \neq i} (a_j - a_i)}.
\]
Therefore
\[
Q^A_{V;\omega}(1, x_i^2) = Q^A_{V;\omega}(x_i, x_i) = \frac{2}{\prod_{j \neq i} (a_j - a_i)}
\]
and \(Q^A_{V;\omega}(\varphi, \psi) = 0\) for all other pairs of the generators listed above. Hence \(\text{rk} Q^A_{V;\omega} = n + 2\).
Example 2 Let $k = 1$, $f_1 = f$, $\omega = dx_1$. One has

$$A_{V,\omega} = \mathcal{O}_{\mathbb{C}^n,0}/\langle f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle,$$

$$\tilde{J} = \text{Jac} \left( f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) \left( \frac{\partial f}{\partial x_1} \right)^{2-n}. \tag{5}$$

One can consider the map

$$\left( f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right): (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0).$$

The linear function $R_{\text{ELKh}}$ (and the corresponding (non-degenerate) quadratic form $Q_{\text{ELKh}}$) on the algebra $A_{V,\omega}$ which participate in the Eisenbud-Levine-Khimshiashvili theory can be defined just by the formula (1) applied to the 1-form $fdx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n$ on $(\mathbb{C}^n, 0)$ (with $\tilde{J} = \text{Jac}(f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$). With (5) this implies that

$$R_{V,\omega}(\varphi) = R_{\text{ELKh}}((\frac{\partial f}{\partial x_1})^{n-2}\varphi),$$

$$Q_{V,\omega}^A(\varphi_1, \varphi_2) = Q_{\text{ELKh}}((\frac{\partial f}{\partial x_1})^{n-2}\varphi_1, \varphi_2).$$

Therefore the rank of $Q_{V,\omega}^A$ is equal to the rank of the multiplication by $(\partial f/\partial x_1)^{n-2}$ and

$$Q_{V,\omega}^A = 0 \iff \left( \frac{\partial f}{\partial x_1} \right)^{n-2} \in \langle f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \rangle.$$

In particular, for $n = 2$ (i.e. for plane curves) these two quadratic forms coincide.

For a germ of a reduced curve $C$ and for a meromorphic 1-form $\omega$ on it, Montaldi and van Straten [MvS] introduced two quadratic forms $\psi_+^\omega$ on certain modules $R^+(\omega)$. If the 1-form $\omega$ is holomorphic, one has $R^-(\omega) = 0$. If, in addition, the curve $C$ is an ICIS defined by a map $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0)$ then the module $R^+(\omega)$ coincides with the algebra $A_{C,\omega}$ (cf. [MvS (1.5)]) and the quadratic form $\psi_+^\omega$ coincides with $Q_{V,\omega}^A$. (This can be derived, e.g., from the fact that the dimension of $R^+(\omega)$ satisfies the law of conservation of number [MvS Theorem (1.7)] and the quadratic forms $\psi_+^\omega$ and $Q_{V,\omega}^A$ coincide for a smooth curve.)

3 A quadratic form on the module of differential forms

There is a module which as a $\mathbb{C}$-vector space has the same dimension as $A_{V,\omega}$. This is the $\mathcal{O}_{V,\omega}$-module

$$\Omega_{V,\omega} := \Omega_{\mathbb{C}^n,0}^{n-k}/(f_i\Omega_{\mathbb{C}^n,0}^{n-k}, df_i \wedge \Omega_{\mathbb{C}^n,0}^{n-k-1}, \omega \wedge \Omega_{\mathbb{C}^n,0}^{n-k-1}) = \Omega_{V,0}^{n-k}/\omega \wedge \Omega_{V,0}^{n-k-1}. \tag{6}$$
The fact that \( \dim_{\mathbb{C}} \Omega_{V;\omega} = \dim_{\mathbb{C}} A_{V;\omega} \) is proved in \([\text{Gr1}]\). (There it is proved for \( \omega = df \), but G.-M. Greuel informed us that there is no difference for the general case.)

One can define a quadratic form \( Q_{V;\omega}^{\Omega} \) on \( \Omega_{V;\omega} \) in the spirit of the definition above. Let \( \eta_1, \eta_2 \in \Omega_{V;\omega} \) and let \( \eta_1 \) and \( \eta_2 \) be representatives of the elements \( \eta_1 \) and \( \eta_2 \) in \( \Omega_{n-k}^{C_n,0} \). We consider a deformation \( F_{\varepsilon} \) of the map \( f: (\mathbb{C}^n,0) \to (\mathbb{C}^k,0) \) with a deformation \( \omega_{\varepsilon} \) of the 1-form \( \omega \). For \( \varepsilon \not\in \Sigma \), on the smooth level set \( \{ f_{\varepsilon} = 0 \} \) the tensor product \( \eta_1 \otimes \eta_2 \) of the \((n-k)\)-forms \( \eta_1 \) and \( \eta_2 \) is a tensor of the same type as \( \tilde{J}(dy_1 \wedge \ldots \wedge dy^{n-k})^2 \).

Since all the singular points \( P \) of the 1-form \( \omega_{\varepsilon} \) on the level set \( \{ f_{\varepsilon} = 0 \} \) are non-degenerate, the latter tensor (the Jacobian) does not tend to zero at these points. Now one can define \( Q_{V;\omega}^{\Omega}(\eta_1, \eta_2) \) as

\[
Q_{V;\omega}^{\Omega}(\eta_1, \eta_2) := \lim_{\varepsilon \to 0} \sum_{P \in \text{Sing}_{V;\omega_{\varepsilon}}} \frac{\eta_1 \otimes \eta_2|_P}{\tilde{J}(dy_1 \wedge \ldots \wedge dy^{n-k})^2}. \tag{7}
\]

It is possible to show (in the same way as above) that this definition makes sense (i.e. that the limit exists) and that the result does not depend on the choice of representatives of the elements \( \eta_1, \eta_2 \in \Omega_{V;\omega} \). On the other hand, this follows directly from another description of the quadratic form \( Q_{V;\omega}^{\Omega} \).

Let \( \Lambda: \Omega_{V;\omega} \to A_{V;\omega} \) be the map which sends an \((n-k)\)-form \( \eta \in \Omega^{n-k}_{C_n,0} \) to the function

\[
df_1 \wedge \ldots \wedge df_k \wedge \eta
\]

\[
dx_1 \wedge \ldots \wedge dx_n
\]

(one can easily see that this mapping is well-defined). Now one has

\[
Q_{V;\omega}^{\Omega}(\eta_1, \eta_2) = Q_{V;\omega}^{A}(\Lambda \eta_1, \Lambda \eta_2).
\]

Remarks. 1. Again, if the map \( f \) and the 1-form \( \omega \) are real, the quadratic form \( Q_{V;\omega}^{\Omega} \) is real as well.

2. We have indicated that the linear function \( R \) and therefore the quadratic form \( Q_{\omega}^{\Omega} \) on \( A_{V;\omega} \) are defined only up to multiplication by a unit in the algebra \( A_{V;\omega} \). The mapping \( \Lambda \) is also defined up to multiplication by a unit (in fact the same one). However, the quadratic form \( Q_{V;\omega}^{\Omega} \) is absolutely well-defined. This follows from the representation \([\text{Gr1}]\) where the denominator is a well-defined tensor at the singular points \( P \).

Corollary 2 \( \text{rk} Q_{\varepsilon;V;\omega}^{\Omega} \leq \text{rk} Q_{V;\omega}^{A} \).

Let us show that the difference \( \text{rk} Q_{V;\omega}^{A} - \text{rk} Q_{V;\omega}^{\Omega} \) is bounded from above by a constant which does not depend on the 1-form \( \omega \).
Let $T\Omega^{n-k}_{V,0}$ be the torsion module of the module

$$\Omega^{n-k}_{V,0} = \Omega^{n-k}_{C^n,0}/(f_i\Omega^{n-k}_{C^n,0}, df_i \wedge \Omega^{n-k-1}_{C^n,0})$$

of differentiable $(n - k)$-forms on the ICIS $(V,0)$. The dimension of $T\Omega^{n-k}_{V,0}$ as a $\mathbb{C}$-vector space is denoted by $\tau'$ (cf. [Gr2]). (This dimension is finite since the corresponding sheaf is a coherent one concentrated at the origin.)

**Proposition 3** One has

$$\text{Ker } \Lambda \cong T\Omega^{n-k}_{V,0}$$

and therefore $\dim_\mathbb{C} \text{Ker } \Lambda = \tau'$.

**Proof.** There exists the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \omega \wedge \Omega^{n-k-1}_{V,0} & \rightarrow & \Omega^{n-k}_{V,0} & \rightarrow & \Omega_{V,\omega} & \rightarrow & 0 \\
\downarrow \lambda' & & \downarrow \lambda = df / dx & & \downarrow \Lambda & & \\
0 & \rightarrow & \langle m_I \rangle & \rightarrow & \mathcal{O}_{V,0} & \rightarrow & \mathcal{A}_{V,\omega} & \rightarrow & 0 \\
\end{array}
$$

Here $m_I$ is the $(k + 1) \times (k + 1)$-minor of the matrix

$$
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\
A_1 & \cdots & A_n
\end{pmatrix}
$$

consisting of columns with indices from $I \subset \{1, \ldots, n\}$, $\langle m_I \rangle \subset \mathcal{O}_{V,0}$ is the ideal generated by the minors $m_I$. The restriction $\lambda' : \omega \wedge \Omega^{n-k-1}_{V,0} \rightarrow \langle m_I \rangle$ of the mapping $\lambda = df / dx$ to the submodule $\omega \wedge \Omega^{n-k-1}_{V,0}$ is surjective (since $\omega \wedge \wedge_j dx_j$ maps to $\pm m_I$). One has $\dim_\mathbb{C} \Omega_{V,\omega} = \dim_\mathbb{C} \mathcal{A}_{V,\omega}$ and therefore $\dim_\mathbb{C} \text{Ker } \Lambda = \dim_\mathbb{C} \text{Coker } \Lambda$. Moreover, one has

$$\text{Coker } \lambda = \mathcal{O}_{V,0}/\langle \text{det} \left( \frac{\partial f_i}{\partial x_j} \right) \rangle, 0 \leq s_1 < \ldots < s_k \leq n.$$ 

This implies that $\dim_\mathbb{C} \text{Coker } \lambda = \tau'$ (cf. [Gr2, 0.1 Satz]). On the other hand $\text{Ker } \lambda = T\Omega^{n-k}_{V,0}$ [Gr1, Proof of Proposition 1.11]. Now the Snake lemma (see, e.g., [Fa, Exercise A3.10]) yields the statement. \hfill $\square$

**Corollary 3** One has

$$\text{cork } Q^{\Omega}_{V,\omega} \geq \tau', \quad 0 \leq \text{rk } Q^{A}_{V,\omega} - \text{rk } Q^{\Omega}_{V,\omega} \leq 2\tau'.$$
Examples.

1. In the situation of Example 1, the module \( \Omega_{V;\omega} \) (as a \( \mathbb{C} \)-vector space) is generated by the forms

\[
\alpha_i := dx_1 \wedge \ldots \wedge \hat{dx}_i \wedge \ldots \wedge dx_n,
\beta_i := x_i dx_1 \wedge \ldots \wedge \hat{dx}_i \wedge \ldots \wedge dx_n
\]

\((i = 1, \ldots, n)\). We have \( \Lambda(\alpha_i) = \pm x_i, \Lambda(\beta_i) = \pm x_i^2 \) (recall that \( \sum x_i^2 = 0 \) in \( \mathcal{A}_{V;\omega} \)). Therefore

\[
Q^\Omega_{V;\omega}(\alpha_i, \alpha_i) = \frac{2}{\prod_{j \neq i}(a_j - a_i)}
\]

and \( Q^\Omega_{V;\omega}(\eta_1, \eta_2) = 0 \) for all other pairs of the generators listed above. Hence \( \text{rk} Q^\Omega_{V;\omega} = n \).

2. In Example 2, there is a natural isomorphism between \( \Omega_{V;\omega} \) and \( \mathcal{A}_{V;\omega} \) (as \( \mathcal{O}_{\mathbb{C}^n,0} \)-modules). Under this isomorphism the map \( \Lambda \) becomes the multiplication by \( \partial f/\partial x_1 \). Therefore

\[
Q^\Omega_{V;\omega}(\varphi_1, \varphi_2) = Q_{\text{ELKh}}\left((\partial f/\partial x_1)^{n-1}\varphi_1, \varphi_2\right),
\]

the rank of \( Q^\Omega_{V;\omega} \) is equal to the rank of the multiplication by \( (\partial f/\partial x_1)^{n-1} \) and

\[
Q^\Omega_{V;\omega} = 0 \Leftrightarrow \left(\partial f/\partial x_1\right)^{n-1} \in \left\langle f, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle.
\]

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