C*- Algebras and Thermodynamic Formalism

Ruy Exel (*) and Artur O. Lopes (**)

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* Departamento de Matemática, UFSC, Florianópolis, Brasil.

and

** Instituto de Matemática, UFRGS, Porto Alegre, Brasil.

Abstract

We show a relation of the KMS state of a certain C*-Algebra $\mathcal{U}$ with the Gibbs state of Thermodynamic Formalism. More precisely, we consider here the shift $T : X \to X$ acting on the Bernoulli space $X = \{1, 2, ..., k\}^\mathbb{N}$ and $\mu$ a Gibbs (equilibrium) state defined by a Holder continuous normalized potential $p : X \to \mathbb{R}$, and $L^2(\mu)$ the associated Hilbert space.

Consider the C*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, which is a sub-C*-Algebra of the C*-Algebra of linear operators in $L^2(\mu)$ which will be precisely defined later. We call $\mu$ the reference measure. Consider a fixed Holder potential $H > 0$ and the C*-dynamical system defined by the associated homomorphism $\sigma_t$. We are interested in describe for such system the KMS states $\psi_\beta$ for all $\beta \in \mathbb{R}$.

We show a relation of a new Gibbs (eigenprobability of a Ruelle operator) probability $\nu_\beta$ to a KMS state $\psi_\beta = \psi_\beta$, in the C*-Algebra $\mathcal{U} = \mathcal{U}(\mu)$, for every value $\beta \in \mathbb{R}$, where $\beta$ is the parameter that defines the time evolution associated to a homomorphism $\sigma_t = \sigma_{\beta t}$ defined by the potential $H$. We show that for each real $\beta$ the KMS state is unique and we explicit it. The probability $\nu_\beta$ is the eigenprobability of the dual of the Ruelle operator of the non-normalized potential $-\beta \log H$. The purpose of the present work is to explain (for an audience which is more oriented to Dynamical System Theory) part of the content of a previous paper written by the authors.

Introduction

In this paper we show a relation of the KMS state of a certain C*-Algebra $\mathcal{U}$ [BR] [P] [EL2] with the Gibbs state of Thermodynamic Formalism [PP] [Bo]
The purpose of this work is to explain for an audience which is more oriented to Dynamical System Theory part the content of the paper [EL3]. See also [Re1],[Re2] for related material.

R. Bowen, D. Ruelle and Y. Sinai are the founders of what is called in our days Thermodynamic Formalism Theory (see [PP] [R3]).

We will present initially the precise definitions we are going to consider.

We point out that we show here only the uniqueness part of the results in [EL3]. The existence is based on the paper [W] which is of Functional Analysis nature.

We refer the reader to [CL1] for a detailed analysis of the different meanings of the concept of Gibbs state from the point of view of Thermodynamic Formalism.

We consider here an expanding transformation \( T : X \to X \) (to simplify ideas one can consider the particular case where \( T \) is the shift acting on the Bernoulli space \( X = \{1, 2, ..., k\}^\mathbb{N} \)). Consider \( \mu \) the Gibbs state defined by a normalized Holder continuous potential \( p : X \to \mathbb{R} \), and \( \mathcal{L}^2(\mu) \) the associated Hilbert space. The function \( p \) is sometimes called the Jacobian of \( \mu \).

Consider the \( C^* \)-Algebra \( \mathcal{U} = \mathcal{U}(\mu) \), which is a sub-\( C^* \)-Algebra of the \( C^* \)-Algebra of linear operators in \( \mathcal{L}^2(\mu) \) which will be precisely defined later.

We call \( \mu \) the reference measure.

Consider a fixed Holder potential \( H > 0 \) and the \( C^* \)-dynamical system defined by the associated \( \sigma \). We are interested in describe for such system the KMS states \( \psi_\beta \) for all \( \beta \in \mathbb{R} \). We show that for each real \( \beta \) the KMS state is unique in Theorem 2.2. We present the explicit expression of \( \psi_\beta \).

The probability \( \nu_\beta \) is the Gibbs state (eigenprobability) for the potential \(-\beta \log H\) (which is not normalized).

Given a potential \( H \), we say the potential \( \tilde{H} \) is cohomologous to \( H \), if there is \( V \) such that \( \log \tilde{H} = \log H - V + V \circ T \).

After we present our main results for Holder potentials in section 1 and 2 in section 3 we consider a non-Holder potential \( H \) and we will make an analysis of phase transition nature (which do not occur at \( C^* \) Algebra level, in this case) associated to the KMS problem in a case where \( H \) can attain the value 1 (and where there is phase transition at Thermodynamic Formalism level).

Ground states in \( C^* \)-Algebras are also consider in the paper [EL3]. These corresponds to limits of the KMS state \( \psi_\beta \) when \( \beta \to \infty \).

An important contribution in the relation of \( C^* \) algebras and Thermodynamic Formalism appears in chapters I.3 e II.5 in J. Renault Phd thesis [Re] (see also [R0]).

We refer the reader to [CL2] for a detailed description of phase transition in the sense of Thermodynamic Formalism.
Section 1 - KMS and Gibbs states

We denote \( C(X) \) the space of continuous functions on \( X \) taking values on the complex numbers where \( (X,d) \) is a compact metric space.

Consider the Borel sigma-algebra \( \mathcal{B} \) over \( X \) and a continuous transformation \( T : X \to X \). Denote by \( \mathcal{M}(T) \) the set of invariant probabilities for \( T \). We assume that \( T \) is an expanding map.

We refer the reader to [Bo] [R1] [R2] [R3] [L4] for general definitions and properties of Thermodynamic Formalism and expanding maps.

Typical examples of such transformations (for which that are a lot of nice results [R2]) are the shift in the Bernoully space and also \( C^1+\alpha \)-transformations of the circle such that \( \|T'(x)\| > c > 1 \), where \( \| \cdot \| \) is the usual norm (one can associate the circle to the interval \([0,1]\) in a standard way) and \( c \) is a constant.

The geodesic flow in compact constant negative curvature surfaces induces in the boundary of Poincaré disk a Markov transformation \( G \) such that for some \( n \), we have \( G^n = T \), and where \( T \) is continuous expanding and acts on the circle (see [BS]). Our results can be applied for such \( T \).

We denote by \( \mathcal{H} = \mathcal{H}_\alpha \) the set of \( \alpha \)-Holder functions taking complex values, where \( \alpha \) is fixed \( 0 < \alpha \leq 1 \).

For each \( \nu \in \mathcal{M}(T) \), the real non-negative value \( h(\nu) \) denotes the Shannon-Kolmogorov entropy of \( \nu \) and \( h(T) = \sup \{ h(\nu) | \nu \in \mathcal{M}(T) \} \). \( h(T) \) is called the topological entropy of \( T \).

Given a continuous function \( A : X \to \mathbb{R} \) we denote the Ruelle operator by \( \mathcal{L}_A \) (which acts on continuous function \( f \)). More precisely if \( g = \mathcal{L}_A(f) \), then \( g(x) = \mathcal{L}_A(f)(x) = \sum_{T(z) = x} e^{A(z)} f(z) \).

We say that the potential \( A \) is normalized if \( \mathcal{L}_A(1) = 1 \).

Given \( A \), the dual operator \( \mathcal{L}_A^* \) acts on probabilities on \( \mathcal{M}(X) \).

We say that \( \mathcal{L}_A^*(\nu) = \rho \) if for any continuous function \( f \)

\[
\int f \mathcal{L}_A^*(\nu) = \int f \rho = \int \mathcal{L}_A(f) \, d\nu.
\]

We denote by \( \mu \) a fixed Gibbs state for a real \( \alpha \)-Holder potential \( \log p : X \to \mathbb{R} \).

We suppose \( \log p \) is already normalized [Bo][R3], in the sense that, if \( \mathcal{L}_{\log p} \) denotes the Ruelle-Perron-Frobenius operator for \( \log p \), that is for any \( f : X \to \mathbb{C} \), and all \( x \in X \), we have \( \mathcal{L}_{\log p}(f)(x) = \sum_{T(z) = x} p(z) f(z) \), then we assume that \( \mathcal{L}_{\log p}(1)(x) = \sum_{T(z) = x} p(z) = 1 \) and \( \mathcal{L}_{\log p}(\mu) = \mu \).

We will show later that the index \( \lambda(x) = p(x)^{-1} \) for the \( C^* \)-algebra associated to \( \mu \).

As an interesting example we mention the case where \( T \) has degree \( k \), that is, for each \( x \in X \) there exists exactly \( k \) different solutions \( z \) for \( T(z) = x \). We call each such \( z \) a pre-image of \( x \).

If \( T \) has degree \( k \) and in the particular case where \( \mu \) is the maximal entropy measure (that is, \( h(\mu) = h(T) = \log k \)), then \( p = 1/k \).

In order to simplify the arguments in our proofs we will assume from now on that \( T \) has degree \( k \).
One can consider alternatively in Thermodynamic Formalism $L_p$ acting on $C(X)$ or on $H$. Different spectral properties for $L_p$ occur in each one of these two cases (see Bo[R2]).

We will consider in the sequel a fixed real Holder-continuous positive potential $H : X \to \mathbb{R}$ and $L_{H,\beta}$, $\beta \in \mathbb{R}$ the Ruelle-Perron-Frobenius operator for $-\beta \log H$, that is, for each continuous $f$ we have by definition

$$L_{-\beta \log H}(f)(x) = L_{H,\beta}(f)(x) = \sum_{T(z)=x} H(z)^{-\beta} f(z).$$

We denote by $\lambda_{H,\beta} \in \mathbb{R}$ the largest eigenvalue of $L_{H,\beta}$. We also denote $\nu_{H,\beta}$ the unique probability such that $L_{H,\beta}^*(\nu_{H,\beta}) = \lambda_{H,\beta} \nu_{H,\beta}$, and $h_{H,\beta}$ the unique function $h \in C(X)$ such that $\int h d\nu_{H,\beta} = 1$ and $L_{H,\beta}(h) = \lambda_{H,\beta} h$.

As $H$ is fixed for good in order to simplify the notation we will sometimes write $L_{\beta}, L^*_{\beta}, \lambda_{\beta}, \nu_{\beta}, h_{\beta}$.

$h_{\beta}$ is a real positive Holder function.

The hypothesis about $H$ and $p$ being Holder in the Statistical Mechanics setting means that in the Bernoulli space the interactions between spins in neighborhoods positions decrease very fast [L2] [L3]. In section 2.3 we will consider a non-Holder potential $H$ where in this case it will appear a phase-transition phenomena. This model is known as the Fisher-Felderhof model [FF], [L2], [L3], [FL]. In this case the interactions do not decrease so fast.

We return now to the Holder case.

It is well known the variational principle for such potential $-\beta \log H$,

$$P_{H}(\beta) = \log \lambda_{H,\beta} = \sup\{h(\nu) + \int (-\beta \log H) d\nu | \nu \in \mathcal{M}(T)\}.$$ 

The probability $\mu_{H,\beta} = h_{H,\beta} \nu_{H,\beta} \in \mathcal{M}(T)$ and satisfies

$$\sup\{h(\nu) + \int (-\beta \log H) d\nu | \nu \in \mathcal{M}(T)\} = h(\mu_{H,\beta}) + \int (-\beta \log H) d\mu_{H,\beta}.$$ 

**Definition 1.1:** The probability $\mu_{\beta} = h_{H,\beta} \nu_{H,\beta}$ is called equilibrium state for the function $-\beta \log H$ where $\beta$ and $H$ are fixed.

**Definition 1.2:** The probability $\nu_{H,\beta}$ is called eigenmeasure or Gibbs state for the function $-\beta \log H$ where $\beta$ and $H$ are fixed. It satisfies

$$L_{H,\beta}^*(\nu_{H,\beta}) = \lambda_{H,\beta} \nu_{H,\beta}.$$ 

The probability $\mu_{H,\beta}$ is unique for the variational problem and $\nu_{H,\beta}$ is unique for the the eigenmeasure problem associated to the value $\lambda_{H,\beta}$, if $p$ and $H$ are Holder. If we do not assume $p$ and $H$ Holder then there exist counterexamples for uniqueness in both cases [L2] [L3]. We will return to this point later.

For some reason the eigen-probabilities have a distinguished role here, but not the equilibrium states.
\( P_H(\beta) \) is called the pressure of \(-\beta \log H\) (or sometimes Free-Energy) and is a convex analytic function of \( \beta \).

If \( T \) has degree \( k \) and in the particular case where \( \mu \) is the maximal entropy measure (that is, \( h(\mu) = h(T) = \log k \)), then \( p = 1/k \).

We consider the \( C^*\)-Algebra \( L(\mathcal{L}^2(\mu)) \) of bounded linear operators acting on \( \mathcal{L}^2(\mu) \) with the strong norm. The operation \( \ast \) on operators is the one induced from the inner product on \( \mathcal{L}^2(\mu) \).

**Definition 1.3:** Denote by \( S : \mathcal{L}^2(\mu) \rightarrow \mathcal{L}^2(\mu) \) the Koopman operator where for \( \eta \in \mathcal{L}^2(\mu) \) we define \((S\eta)(x) = \eta(T(x))\). Such \( S \) defines a linear bounded operator in \( \mathcal{L}^2(\mu) \).

In Thermodynamic Formalism it is usual to consider the Koopman operator acting on \( \mathcal{L}^2(\mu) \) (the space of complex square integrable functions over \( \mathcal{L}^2(\mu) \)), and it is well known that its adjoint (over \( \mathcal{L}^2(\mu) \)) is the operator \( \mathcal{L}_p = S^\ast \) acting on \( \mathcal{L}^2(\mu) \).

As we assume \( X \) is compact, any continuous function \( f \) is in \( \mathcal{L}^2(\mu) \).

**Definition 1.4:** Another important class of linear operators is \( M_f : \mathcal{L}^2(\mu) \rightarrow \mathcal{L}^2(\mu) \), for a given fixed \( f \in C(X) \), and defined by \( M_f(\eta)(x) = f(x)\eta(x) \), for any \( \eta \) in \( \mathcal{L}^2(\mu) \).

In order to simplify the notation, sometimes we denote by \( f \) the linear operator \( M_f \).

Note that for \( M_f \) and \( M_g \), \( f, g \in C(X) \), the product operation satisfies \( M_f \circ M_g = M_{f \ast g} \), where \( \ast \) means multiplication over the complex field \( \mathbb{C} \).

Note that the \( \ast \) operation applied on \( M_f \), \( f \in C(X) \), is given by \( M_f = M_f^\ast \), where \( ^\ast \) is the complex conjugated of \( z \in \mathbb{C} \). In this sense, \( M_f^\ast \) is the adjoint operator of \( M_f \) over \( \mathcal{L}^2(\mu) \).

The main point for our choice of \( \mu \) as eigen-probability for \( \mathcal{L}_p^\ast \), is that in \( \mathcal{L}^2(\mu) \), the dual of the Koopman operator \( S \) is the operator \( \mathcal{L}_p = S^\ast \) acting on \( \mathcal{L}^2(\mu) \). Indeed, for any \( f, g \) we have

\[
\int f(g \circ T) \, d\mu = \int f(g \circ T) \, d\mathcal{L}_p^\ast(\mu) = \int \mathcal{L}_p(f(g \circ T)) \, d\mu = \int \mathcal{L}_p(f) \, g \, d\mu.
\]

It is important not confuse the dual of the Ruelle operator \( \mathcal{L}_p \) in the Hilbert structure sense with the dual of \( \mathcal{L}_p \) as a linear functional on continuous functions.

\( L(\mathcal{L}^2(\mu)) \), the set of linear operators over \( \mathcal{L}^2(\mu) \), is a very important \( C^*\)-Algebra. We will analyze here a sub-\( C^*\)-Algebra of such \( C^*\)-Algebra (defined with the above operations \( , \ast \)) , more precisely the \( C^*\)-Algebra \( \mathcal{U} \).

**Definition 1.5:** We denote by \( \alpha : C(X) \rightarrow C(X) \) the linear operator such that for any \( f \), we have \( \alpha(f) = f \circ T \).

We have to show how the operators \( S \) and \( M_f \) acting on \( \mathcal{L}^2(\mu) \) interact with the operators \( \mathcal{L}_p \) and \( \alpha \) acting on \( C(X) \).

One can easily see that \( \alpha(M_f) = M_{f \circ T} \). This is the first relation.

In the simplified notation (we identify \( M_f \) with \( f \)), one can read last expression as \( \alpha(f) = f \circ T \).
In this way \( \alpha^n(f) = f \circ T^n \).

If \( \mathcal{B} \) is the Borel sigma-algebra then we denote by \( \mathcal{F}_n \) the Sigma-algebra \( T^{-n} \mathcal{B} \).

It is know that if we consider the probability \( \mu \), then the conditional expected value
\[
E(f \mid \mathcal{F}_n) = E_\mu(f \mid \mathcal{F}_n) = \alpha^n(L^\mu_p(f)).
\]

More precisely
\[
E(f \mid \mathcal{F}_n)(x) = L^n(f)(\sigma^n(x)). \tag{1}
\]

As \( \mathcal{F}_m \subset \mathcal{F}_n \) for \( m \geq n \), we have
\[
E_\mu((E_\mu(f \mid \mathcal{F}_m)) \mid \mathcal{F}_n) = E_\mu(f \mid \mathcal{F}_m),
\]
and
\[
E_\mu((E_\mu(f \mid \mathcal{F}_n)) \mid \mathcal{F}_m) = E_\mu(f \mid \mathcal{F}_m).
\]

**Definition 1.6:** Consider the \( C^* \)-Algebra contained in the set of bounded operators \( L(L^2(\mu)) \) generated by the elements of the form \( M_f \sigma^n(S^*)^n M_g \), where \( n \in \mathbb{N} \) and \( f, g \in C(X) \). We denote such \( C^* \)-Algebra by \( \mathcal{U} = \mathcal{U}(\mu, T) \). We call \( \mathcal{U} \) the \( C^* \)-Algebra associated to \( \mu \).

Each element \( a \) in \( \mathcal{U} \) is the limit of finite sums \( \sum_i M_f \sigma^n(S^*)^n M_g \).

\( C(X) \) is contained in \( \mathcal{U} \), via \( M_f \), where \( f \) is any continuous function \( f : X \to \mathbb{R} \).

Note that \( f \to M_f \) defines a linear injective function of \( C(X) \) on \( \mathcal{U} \).

We denote \( e_n = S^n(S^n)^* = E(f \mid \mathcal{F}_n) \in \mathcal{U} \).

**Important Properties:**

We have basic relations in such \( C^* \)-Algebra \( \mathcal{U} \):

a) \( (S^*)^n S^n = 1 \), for all \( n \in \mathbb{N} \) (it follows from \( S^* S = 1 \)).

proof: for any \( \eta \in L^2(\mu) \), we have
\[
S^* S(\eta)(x) = L_p(\eta(T(.)))(x) = \sum_{T(y)=x} p(y) \eta(T(y)) = \sum_{T(y)=x} p(y) \eta(x) = \eta(x).
\]

That is, \( (S^*)^n S^n \) is the identity operator.

b) \( (S^*)^n M_f S^n = M_{L^\mu_p(f)} \), for all \( n \in \mathbb{N} \), \( f \in C(X) \) (it follows from \( S^* M_f S = M_{L^\mu_p(f)} \)).

proof: for any \( \eta \in L^2(\mu) \), we have
\[
S^* M_f S(\eta)(x) = L_p(f \eta(T(.)))(x) = L_p(f)(x) \eta(x).
\]

c) \( S M_f = \alpha(f) S \) for any continuous \( f \), that is, for any \( \eta \in L^2(\mu) \), \( S M_f(\eta) = f \circ T. \eta \circ T = \alpha(f) \eta \).

d) \( [e^n M_f](\eta) = [S^n(S^*)^n M_f](\eta) = E_\mu((f \eta) \mid \mathcal{F}_n) \).

e) \( e^n(\eta) = S^n(S^*)^n(\eta) = E_\mu(\eta \mid \mathcal{F}_n) \).
f) $M_f e^n(\eta) = [M_f S^n (S^*)^n](\eta) = f \ E_\mu(\eta | \mathcal{F}_n)$.

g) $M_f e^n M_g(\eta) =$

$$[M_f S^n (S^*)^n M_g](\eta) = f \ E_\mu(g \eta | \mathcal{F}_n) = f \ [E_\mu^n(g \eta) (\sigma^n)]. \quad (2)$$

h) $S^n (S^*)^n M_g S^n (S^*)^n = E_\mu(g | \mathcal{F}_n) (S^n (S^*)^n) E_\mu(g | \mathcal{F}_n) e^n$ because $[S^n (S^*)^n M_g S^n (S^*)^n](\eta) = E_\mu(g \ E_\mu(\eta | \mathcal{F}_n) | \mathcal{F}_n) = E_\mu(g | \mathcal{F}_n) \ E_\mu(\eta | \mathcal{F}_n)$.

i) If $n \leq m$ we have

$$[M_f e^n M_g e^m M_h] = [M_f S^n (S^*)^n M_g S^m (S^*)^m M_h] =
[M_f (S^n (S^*)^n M_g S^m (S^*)^m) M_h] = M_f E_\mu(g | \mathcal{F}_n) (S^n (S^*)^m) M_h = M_f E_\mu(g | \mathcal{F}_n) e^m M_h.$$  

j) If $n \geq m$ we have

$$[M_f e^n M_g e^m M_h] = [M_f S^n (S^*)^n M_g S^m (S^*)^m M_h] = M_f e^n E_\mu(g | \mathcal{F}_m) M_h$$

proof: note first that taking adjoint with respect to the $L^2(\mu)$ structure

$$(M_f S^n (S^*)^n M_g)^* = (M_g S^n (S^*)^n M_f).$$

Then,

$$[M_f S^n (S^*)^n M_g S^m (S^*)^m M_h]^* = M_h S^m (S^*)^m M_g S^n (S^*)^n M_f$$

and we can apply item i) to get

$$[M_f S^n (S^*)^n M_g S^m (S^*)^m M_h]^* = M_h E_\mu(g | \mathcal{F}_m) e^n M_f.$$  

Now taking adjoint once more we get

$$[M_f S^n (S^*)^n M_g S^m (S^*)^m M_h] = M_f e^n E_\mu(g | \mathcal{F}_m) M_h.$$  

Example 1:

$$[M_f e^3 M_g e^4 M_h](\eta)(x) =
[M_f S^3 (S^*)^3 M_g S^4 (S^*)^4 M_h](\eta)(x) =
M_f S^3 (S^*)^3 M_g [E_\mu ((f \eta) | \mathcal{F}_4)](x) =
M_f S^3 (S^*)^3 [g(x) E_\mu ((f \eta) | \mathcal{F}_4)](x) =
f(x) E_\mu (g(x) E_\mu ((f \eta) | \mathcal{F}_4) | \mathcal{F}_3) (x).$$

If $u$ is $\mathcal{F}_m$ measurable and $m > n$, then $u$ is $\mathcal{F}_n$ measurable. Then,

$$[M_f S^3 (S^*)^3 M_g S^4 (S^*)^4 M_h](\eta)(x) =$$
\[ f(x) E_\mu((f \eta) | F_4)(x) = E_\mu(g(x) | F_3)(x). \]

By the other hand

\[
\begin{align*}
[M_f e^4 M_g e^3 M_h](\eta) (x) &= [M_f S^4 (S^*)^4 M_g S^3 (S^*)^3 M_h](\eta) (x) \\
&= f(x) E_\mu((f \eta) | F_3)(x) | F_4)(x).
\end{align*}
\]

**Remark 0:** If we consider the \( C^*-\)algebra generated \( M_f S^n(S^*)^n M_g \), where \( n, m \in \mathbb{N} \) and \( f, g \in C(X) \), we have a different setting (which is usually called a Vershik \( C^*-\)algebra) which was consider in another paper by R. Exel [E3]. In this case, the KMS state exists only for one value of \( \beta \).

We now return to our setting.

An extremely important result will be shown in expression (*1) and (*2) in Lemma 2.1 which claims that there exists functions \( u_i, i \in \{1, 2, \ldots, k\} \), such that

\[
\sum_{i=1}^{k} M_{u_i} S S^* M_{u_i} = 1.
\]

A bijective linear transformation \( K : \mathcal{U} \rightarrow \mathcal{U} \) which preserves the composition and the \( * \) operation is called an automorphism of \( \mathcal{U} \).

**Definition 1.7:** Given a positive function \( H \) we define the group homomorphism \( \sigma_t \), where for each \( t \in \mathbb{R} \) we have \( \sigma_t \in \text{Aut}(\mathcal{U}) \) [BP] [P], is defined by:

a) for each fixed \( t \in \mathbb{R} \) and any \( M_f \), we have \( \sigma_t(M_f) = M_f \),

b) for each fixed \( t \in \mathbb{R} \), we have \( \sigma_t(S) = M_{H^{*t}} \circ S \), in the sense that \( (\sigma_t(S)(\eta))(x) = H^{it}(x) \eta(T(x)) \in L^2(\mu) \), for any \( \eta \in L^2(\mu) \).

The value \( t \) above is related to temperature and not time, more precisely we are going to consider below \( t = \beta i \) where \( \beta \) is related to the inverse of temperature in Thermodynamic Formalism (or Statistical Mechanics).

It can be shown that for each \( t \) fixed, we just have to define \( \sigma_t \) over the generators of \( \mathcal{U} \) in order to define \( \sigma_t \) uniquely on \( \mathcal{U} \). In this way a) and b) above define \( \sigma_t \).

We will assume in this section from now on that \( H \) is Holder in order we can use the strong results of Thermodynamic Formalism.

**Remark 1:** Note that for \( \eta \in L^2(\mu) \), we have

\[
(\sigma_t(S^2)(\eta))(x) = \sigma_t(M_{H^{*t}}(\eta \circ T))(x) = M_{H^{*t} M_{H^{*t} \circ T}}(\eta \circ T^2)(x),
\]

therefore \( \sigma_t(S^2) = H^{it}(H \circ T)^{it} S^2 \). It follows easily by induction that

\[
\sigma_t(S^n) = \Pi_{j=0}^{n-1}(H \circ T)^{it} S^n.
\]
Taking dual in both sides of the above expression we get other important relation

\[ \sigma_t((S^*)^n) = (S^*)^n \Pi_{j=0}^{n-1}(H \circ T^i)^{-t_i}. \]

Finally,

\[ \sigma_t(M_{t_2}^m(S^*)^n M_{g_2}) = M_{t_2}^H \Pi_{i=0}^{m-1} H(T^i(x))^{t_i}. \]

From f) above we get for \( t = i \)

\[ \sigma_i(M_{t_2}^m(S^*)^n M_{g_2})(\eta)(x) = \]

\[ [ M_{t_2}^H \Pi_{i=0}^{m-1} S^m(S^*)^n H^{-i} \cdot i[m] M_{g_2} ](\eta)(x) = \]

\[ [ M_{t_2}^H \Pi_{i=0}^{m-1} S^m(S^*)^n H^{-i} \cdot i[m] M_{g_2} ](\eta)(x) = \]

\[ f_2(x) H^{-m}(x) E_{\mu}(H[g_2 \eta]|F_m)(x). \]

In terms of the formalism of \( C^* \)-dynamical systems, the positive function \( H \) defines the dynamics of the evolution with time \( t \in \mathbb{R} \) of a \( C^* \)-dynamical system. Our purpose is to analyze such system for each pair \( (H, \beta) \).

**Definition 1.8:** An element \( a \) in a \( C^* \)-Algebra is positive, if it is of the form \( a = b \cdot b^* \) with \( b \) in the \( C^* \)-Algebra.

**Definition 1.9:** By definition a "\( C^* \)-dynamical system state" is a linear functional \( \psi : \mathcal{U} \to \mathbb{C} \) such that

a) \( \psi(M_1) = 1 \)

b) \( \psi(a) \) is a positive real number for each positive element \( a \) on the \( C^* \)-Algebra \( \mathcal{U} \).

A "\( C^* \)-dynamical system state" \( \psi \) in \( C^* \)-dynamical systems plays the role of a probability \( \nu \) in Thermodynamic Formalism. For a fixed \( H \), we have a dynamic temporal evolution defined by \( \sigma_t \) where \( t \in \mathbb{R} \).

**Definition 1.10:** An element \( a \in \mathcal{U} \) is called analytic for \( \sigma \) if \( \sigma_t(a) \) has an analytic extension from \( t \in \mathbb{R} \) to all \( t \in \mathbb{C} \).

**Definition 1.11:** For a fixed \( \beta \in \mathbb{R} \) and \( H \), by definition, \( \psi \) is a KMS state associated to \( H \) and \( \beta \) in the \( C^* \)-Algebra \( \mathcal{U}(\mu, T) \), if \( \psi \) is a \( C^* \)-dynamical system state, such that for any \( b \in \mathcal{U} \) and any analytic \( a \in \mathcal{U} \) we have

\[ \psi(a \cdot b) = \psi(b \cdot \sigma_\beta(a)). \]

For \( H \) and \( \beta \) fixed, we denote a KMS state by \( \psi_{H, \beta} = \psi_\beta \) and we leave \( \psi \) for a general \( C^* \)-dynamical system state.

It is easy to see that for \( H \) and \( \beta \) fixed, the condition

\[ \psi(a \cdot b) = \psi(b \cdot \sigma_\beta(a)), \]

is equivalent to \( \forall \tau \in \mathbb{C} \),

\[ \psi(\sigma_\tau(a) \cdot b) = \psi(b \cdot \sigma_{\tau + \beta}(a)). \]
It follows from section 8.12 in [P] that if $\psi_\beta$ is a KMS state for $H$, $\beta$, then for any analytic $a \in \mathcal{U}$, we have that $\tau \to \psi_\beta(\sigma_\tau(a))$ is a bounded entire function and therefore constant. In this sense $\psi$ is stationary for the continuous time evolution defined by the flow $\sigma_t$.

Note that the KMS state, in principle, could depend of the initially chosen $\mu$ because we are considering $L^2(\mu)$ when defining $\mathcal{U}$, but in the end it will be defined by a measure that depends only in $\beta$ and $H$.

We point out that it can be shown that in order to characterize $\psi$ as a KMS state we just have to check the condition $\psi(a,b) = \nu(b,\sigma_\beta(a))$ for $a,b$ the linear generators of $\mathcal{U}$, that is, $a$ of the form $M_f S^n(S^*)^\alpha M_g$, and $b$ of the form $M_f S^n(S^*)^\beta M_g$.

A natural question is: for a given $\beta$ and $H$, when the KMS state $\psi_{H,\beta}$ exist and when it is unique?

We are interested mainly in uniqueness and explicitly. We will explain this point more carefully later.

**Remark 2:** Note that when $\psi$ is a KMS state, $\psi(f.a.g) = \psi(\sigma_\beta(g) f a) = \psi(g.f.a) = \psi(f.g.a)$, for any $f,g \in C(X)$ and $a \in \mathcal{U}$.

Our purpose here is to show how to associate in a unique way each KMS state $\psi_{H,\beta} = \psi_\beta$ to the eigenmeasure $\nu_{H,\beta} = \nu_\beta$ defined before.

Remember that over $L^2(\mu)$ the operator $L_p = S^*$ is adjoint of the operator $f \to S(f) = f \circ T$.

We call $\lambda(x) = p(x)^{-1}$ the index and we denote by

$$\lambda^{[n]}(x) = (p(x)p(T(x))...p(T^{n-1}(x)))^{-1}.$$ 

We denote $H^{[n]}(x) = \Pi_{i=0}^{n-1} H(T^i(x))^{\beta}$ and $\Lambda_n = H^{-\beta}[n] \lambda^{[n]}$.

From this follows that for any continuous function $f$ we have $L_p^\beta(f) = L_p^\beta(\Lambda_n f)$.

Remember that for any continuous function $k$ we have $L_p^n(k \circ T^n)(x) = k(x)$ because $L_p^1(1) = 1$.

**Lemma 1.1** For any $\beta$ and continuous function $f$

$$\int f \, d\nu_\beta = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n f | F_n) \, d\nu_\beta. \quad (3)$$

**Proof:**

Note that

$$\int L_p^n(\Lambda_n f) \, d\nu_\beta = \int L_p^n(f) \, d\nu_\beta = \lambda^{[n]}_\beta \int f \, d\nu_\beta.$$

Now taking $f = (\Lambda_n)^{-1} \alpha^n(g) = (\Lambda_n)^{-1} (g \circ T^n)$ we get from above

$$\int g \, d\nu_\beta = \int L_p^n(\Lambda_n (\Lambda_n)^{-1} (g \circ T^n)) \, d\nu_\beta = \lambda^{[n]}_\beta \int (\Lambda_n)^{-1} (g \circ T^n) \, d\nu_\beta.$$

Now taking $g = L_p^n(\Lambda_n f)$ we get
\[
\int (\Lambda_n)^{-1} E_\mu(\Lambda_n f | \mathcal{F}_n) \, d\nu_\beta = \int (\Lambda_n)^{-1} \alpha^n (\mathcal{L}_\mu^n(\Lambda_n f)) \, d\nu_\beta = \\
\lambda_\beta^{-n} \int \mathcal{L}_\mu^n(\Lambda_n f) \, d\nu_\beta = \lambda_\beta^{-n} \int \mathcal{L}_\mu^n(f) \, d\nu_\beta = \int f \, d\nu_\beta.
\]

Section 2 - The main result

We define \(G : \mathcal{U} \rightarrow C(X)\) by \(G(M_f e_n M_g) = f \chi^{\lfloor n \rfloor} g\) where \(e_n = S^n(S^*)^n\). Moreover, \(G(M_f M_g) = f g\)

Note that we define \(G\) in the elements of the form \(M_f e_n M_g, n \geq 0\), and then we define \(G\) in \(\mathcal{U}\) by linear combinations and limits.

Suppose \(\phi = \phi_\nu : C(X) \rightarrow \mathbb{C}\) is of the form \(\phi(f) = \int f \, d\nu\) where \(\nu\) is a probability on \(X\).

There is a canonical way to define a \(C^*\)-dynamical system state \(\psi_\nu : \mathcal{U} \rightarrow \mathbb{C}\) by

\[
\psi_\nu(M_f e_n M_g) = \phi_\nu(G(M_f e_n M_g)) = \int f \chi^{\lfloor n \rfloor} g \, d\nu.
\]

In this way if \(n \leq m\) (by item i)

\[
\psi_\nu(M_f S^n(S^*)^n M_g S^m(S^*)^m M_h) = \\
\psi_\nu(M_f E_\mu(g | \mathcal{F}_n)(S^m(S^*)^m) M_h) = \\
\psi_\nu(M_f E_\mu(g | \mathcal{F}_n) e^m M_h) = \int E_\mu(g | \mathcal{F}_n) f h \chi^{\lfloor m \rfloor} \, d\nu.
\]

In this way if \(n \geq m\) (by item j)

\[
\psi_\nu(M_f S^n(S^*)^n M_g S^m(S^*)^m M_h) = \\
\psi_\nu(M_f e^n E_\mu(g | \mathcal{F}_m) M_h) = \int E_\mu(g | \mathcal{F}_m) f h \chi^{\lfloor n \rfloor} \, d\nu.
\]

**Theorem 2.1:** Given \(\phi_\nu\) and \(\psi_\nu = \phi_\nu \circ G\) we get that \(\psi_\nu\) is KMS for temperature \(\beta\), if and only if, \(\phi_\nu\) satisfies

\[
\phi_\nu(f) = \phi_\nu((\Lambda_n)^{-1} E_\mu(\Lambda_n f | \mathcal{F}_n)),
\]

which is the same that to say that \(\nu\) satisfies

\[
\int f \, d\nu = \int (\Lambda_n)^{-1} E_\mu(\Lambda_n f | \mathcal{F}_n) \, d\nu.
\]

**Proof:**
In order to simplify the notation we call $E_n(f) = E_\mu(f \mid \mathcal{F}_n)$.
Suppose that $\psi$ is a KMS state. Then for all $a, b, c, d \in C(X)$ and all $n$ we have

$$
\psi((ae_n b)\sigma_{i\beta}(ce_n d)) = \psi((ce_n d)(ae_n b)). \quad (*3)
$$

The left hand side is equals to

$$
\psi(a e_n b \, cH^{-\beta[n]}e_n H^{\beta[n]}d) = \psi(a S^n (S^*)^n b \, cH^{-\beta[n]}e_n H^{\beta[n]}d) = \\
\psi(aE_n(bcH^{-\beta[n]})e_n H^{\beta[n]}d) = \phi(aE_n(bcH^{-\beta[n]} \lambda^{-[n]} H^{\beta[n]}d).
$$

The right hand side of (*3) is equals to

$$
\psi(cE_n(da)e_n b) = \phi(cE_n(da)\lambda^{-[n]}b).
$$

Now take $b = 1, c = H^{\beta[n]}$, and $d = H^{-\beta[n]}\lambda^{[n]}$ and from (*3) we get

$$
\phi(a) = \phi(H^{\beta[n]}E_n(H^{-\beta[n]}\lambda^{[n]}a)\lambda^{-[n]}) = \phi(\Lambda^{-[n]}E_n(\lambda^{[n]}a)).
$$

Now, we want to prove the other implication.
Note that $\phi_{\nu}(ab) = \phi_{\nu}(ba)$ for continuous functions $a$ and $b$.
We would like to prove that

$$
\psi((ae_n b)\sigma_{i\beta}(ce_m d)) = \psi((ce_m d)(ae_n b)), \quad (*4)
$$

for all $a, b, c, d \in A$ and $n, m \in \mathbb{N}$.
Suppose first the case $n \leq m$.
By the important property i) we get that the left hand side of (*4) is equals to

$$
\psi(ae_n bcH^{-\beta[m]}e_m H^{\beta[m]}d) = \psi(aE_n(bcH^{-\beta[m]}e_m H^{\beta[m]}d) = \\
\phi(aE_n(bcH^{-\beta[m]}\lambda^{-[m]} H^{\beta[m]}d) = \phi(E_n(bcH^{-\beta[m]} H^{\beta[m]}\lambda^{-[m]} da).
$$

Observe that $H^{\beta[m]}(x) = H^{\beta[n]}(x) H^{\beta[m-n]}(T^n(x))$ so the above is equals to

$$
\phi(E_n(bcH^{-\beta[n]} H^{-\beta[m-n]}(T^n)) H^{\beta[m-n]}(T^n) H^{\beta[n]}\lambda^{-[m]} da) = \\
\phi(E_n(bcH^{-\beta[n]} H^{\beta[n]}\lambda^{-[m]} da) = \\
\phi(\Lambda^{-[n]}E_n(\Lambda^{[n]}E_n(bcH^{-\beta[n]} H^{\beta[n]}\lambda^{-[m]} da)) = \\
\phi(\Lambda^{-[n]}E_n(bcH^{\beta[n]}\lambda^{[n]}\lambda^{-[m]} E_n(da))) = \\
\phi(\Lambda^{-[n]}E_n(bcH^{\beta[n]}\lambda^{[n]}\lambda^{-[m]} E_n(da)))
$$

where in the last equality we use the fact that $\lambda^{-[m]} \lambda^{[n]} = \lambda^{-[m-n]}(T^n)$.
By the other hand the right hand side of (*4) is equals to

$$
\psi(ce_m E_n(da)b) = \phi(c\lambda^{-[m]} E_n(da)b) = \phi(bc\lambda^{-[m]} E_n(da)) = \\
$$
\[ \phi(\Lambda^{-n}\nu E_n(\Lambda^{[n]}bc\lambda^{-[m]}E_n(da))) = \phi(\Lambda^{-n}\nu E_n(bc\lambda^{-[m]}\Lambda^{[n]}H^{-\beta[n]}E_n(da))) = \phi(\Lambda^{-n}\nu E_n(bcH^{-\beta[n]}\lambda^{-[m]}\lambda^{[n]}E_n(da))) \]

where in the last equality we use once more the fact that \(\lambda^{-[m]}\lambda^{[n]} = \lambda^{-[m-n]}(T^n)\).

In this way we showed the KMS condition in the case \(n \leq m\).

For the case \(n \geq m\), using the important property j) we note that the left hand side of (*4) is

\[
\psi(ac_n b\ c\ H^{-\beta[m]}e_{m}H^{\beta[m]} d) = \\
\psi(ac_n E_m(bcH^{-\beta[m]}H^{\beta[m]}d)) = \\
\phi(a\lambda^{-[m]}E_m(bcH^{-\beta[m]}H^{\beta[m]}d)) = \\
\phi(\Lambda^{-[m]}E_m(\Lambda^{[m]}\lambda^{-[n]}E_m(bcH^{-\beta[m]}H^{\beta[m]}da\Lambda^{[m]}\lambda^{-[n]})) = \\
\phi(\Lambda^{-[m]}E_m(bcH^{-\beta[m]}E_m(da\lambda^{[m]}\lambda^{-[n]})).
\]

The right hand side of (*4) equals

\[
\psi(cE_m(da)e_{n}b) = \phi(cE_m(da)\lambda^{-[n]}b) = \\
\phi(\Lambda^{-[m]}bcE_m(da)) = \phi(\Lambda^{-[m]}E_m(!\Lambda^{[m]}\lambda^{-[n]}bcE_m(da))) = \\
\phi(\Lambda^{-[m]}E_m(bcH^{-\beta[m]}\lambda^{[m]}\lambda^{-[n]}E_m(da)),
\]

The conclusion follows at once because \(\lambda^{[m]}\lambda^{-[n]} \in \mathcal{F}_m\).

□

**Corollary 2.1.** Suppose \(\nu_\beta\) is an eigenprobability for the Ruelle operator of the potential \(-\beta \log H\). If the \(C^*\)-dynamical system state \(\psi_{\nu_\beta} : \mathcal{U} \to \mathbb{C}\) is defined by

\[
\psi_{\nu_\beta}(Mf e_n M_g) = \phi_{\nu_\beta}(G(Mf e_n M_g)) = \int f \lambda^{-[n]} g d\nu_\beta,
\]

then, \(\psi_{\nu_\beta}\) is a KMS state for temperature \(\beta\).

**Proof:** This follows from last theorem and Lemma 1.1

□

Note that when \(H\) is constant then \(\mu\) is an eigenprobability for the associated Ruelle operator for any \(\beta > 0\). From expression (*5) we can see that \(\sigma_t\) in this case is the identity for any \(t\). Moreover, by the KMS relation \(\psi_\mu(a b) = \psi_\mu(b a)\).

We can ask about uniqueness of the KMS state. To address this question is the purpose of the next results.

Our main theorem says:
**Theorem 2.2:** If $H$ is Holder positive and $\mu$ is a Gibbs state for $p$ Holder, then for any given $\beta \in \mathbb{R}$, a KMS state $\psi$ in $\mathcal{U}(\mu)$ exists, it is unique and of the form

$$
\psi_\beta(b) = \int \frac{f g}{\lambda^n} d\nu_{H,\beta}, \quad \forall b = M_f e^n M_g \in \mathcal{U},
$$

where $\nu_\beta$ is the eigenmeasure for $\mathcal{L}_{-\beta \log H}$.

**Proof of Theorem 2.2:**

The existence of a KMS follows from the results from above. We fixed $\beta$.

Now we want to show precisely how one can associate a Gibbs measure to a KMS state. We denote such KMS state by $\psi$. We will denote $\psi_\beta$ the KMS state obtained from $\nu_\beta$.

Suppose $\psi$ is a KMS state, where the $H$ is fixed and defines the semigroup $\sigma_t$.

Given the KMS state $\psi$, then $\psi(M_f) = \psi(f)$ defines a continuous positive linear functional over $C(X)$ such that $\psi(M_1) = 1$. Therefore by Riesz Theorem, there exists a probability $\nu$ such that for any $f \in C(X)$ we have $\psi(f) = \int f d\nu = \int f S^n(S^*)^n d\nu$.

The above definition takes in account just $n = 0$ in a) above. Remains the question: what conditions are imposed on $\nu$ (defined from $\psi$ as above) due to the fact that $\psi$ is a KMS state for $H, \beta$?

This $\nu$ is our candidate to be the one associated to $\psi$ via $\psi = \psi_\nu = \phi_\nu \circ G$ where hopefully $\nu$ satisfies

$$
\int f d\nu = \int (\Lambda_n)^{-1} E_{\mu}(\Lambda_n f | F_n) d\nu.
$$

for all continuous $f$, and also

$$
\psi_\nu(M_f e^n M_g) = \phi_\nu(G(M_f e^n M_g)) = \int f \lambda^{-[n]} g d\nu.
$$

Now we will show a recurrence relation which do not assume any KMS state condition for $\psi$.

**Lemma 2.1:** Suppose that the $C^*$-state $\psi$ is such that $\psi(f) = \int f d\nu$, for any $f \in C(X)$.

Then, for any $f \in C(X)$ and $n \in \mathbb{N}$

$$
\psi(f e^n) = \psi(f S^n(S^*)^n) = \psi(S^n(S^*)^n f) = \int f \lambda^{-[n]} d\nu.
$$

In other words, if $G(f S^n(S^*)^n) = G(M_f e^n) = f \lambda^{-[n]}$ for any continuous function $f$, and $\phi_\nu(f) = \int f d\nu = \psi(f)$, then

$$
\psi(f e^n) = \phi_\nu(G(f e^n)).
$$

**Proof:**
The first claim of the lemma will follow from

$$\psi(f S^n(S^*)^n) = \psi(f (\lambda \circ T^n) S^{n+1}(S^*)^{n+1}).$$

Indeed, for instance we get $\psi(f) = \psi(f \lambda S(S^*))$, and $\psi(f S(S^*)) = \psi(f (\lambda \circ T) S^2(S^*)^2)$, and so on.

We need a preliminary estimate before proving the lemma.

For the transformation $T$, consider a partition $A_1, ..., A_k$ of $X$ such that $T$ is injective in each $A_i$. Our proof below is for the shift in the Bernoulli space.

In the case of the Bernoulli space with $k$ symbols $A_i$ is the cylinder $T$ with first coordinate $i$. Now we continue the argument:

$$\lambda = \lambda^1 \cdot \ldots \cdot \lambda^k$$

Indeed, for instance we get $f = f \lambda S(S^*)$, and, $f S(S^*) = f (\lambda \circ T) S^2(S^*)^2$, and so on.

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$$\lambda = \lambda^1 \cdot \ldots \cdot \lambda^k$$

Indeed, for instance we get $f = f \lambda S(S^*)$, and, $f S(S^*) = f (\lambda \circ T) S^2(S^*)^2$, and so on.
Now we use the relations \( S^n M_g = M_{\alpha^n(g)} S^n \) and \( M_g (S^*)^n = (S^*)^n M_{\alpha^n(g)} \)
in last expression and we get

\[
S^n(S^*)^n = \sum_{i=1}^k M_{\alpha^n(u_i)} S^n S (S^*)^n M_{\alpha^n(u_i)} = \sum_{i=1}^k M_{\alpha^n(u_i)} S^{n+1}(S^*)^{n+1} M_{\alpha^n(u_i)}
\]

Now we will prove the lemma.

Using last expression and then Remark 2 for \( g = \alpha^n(u_i) \in C(X) \) and \( a = S^{n+1}(S^*)^{n+1} \) we get

\[
\psi(M_f S^n (S^*)^n) = \psi(M_f \sum_{i=1}^k M_{\alpha^n(u_i)} S^{n+1}(S^*)^{n+1} M_{\alpha^n(u_i)}) = \\
\psi(M_f \sum_{i=1}^k M_{\alpha^n(u_i)} M_{\alpha^n(u_i)} S^{n+1}(S^*)^{n+1}) = \\
\psi(M_f \sum_{i=1}^k M_{\alpha^n(u_i)} S^{n+1}(S^*)^{n+1}) = \\
\psi(M_f M_{\alpha^n(\sum_{i=1}^k u_i)} S^{n+1}(S^*)^{n+1}) = \\
\psi(M_f (\lambda \circ T^n) S^{n+1}(S^*)^{n+1})
\]

This shows the claim of the lemma.

We denote \( E_m(f) = E_\mu(f | F_m) \).

**Corollary 2.1** If \( \psi \) is Gibbs for \( H \) at temperature zero, and \( \nu \) is such that for any continuous function \( f \) we have \( \psi(f) = \int f d\nu \), then

\[
\phi_\nu(f) = \phi_\nu(\Lambda_n)^{-1} E_\mu(\Lambda_n f | F_n),
\]

which is the same that to say that \( \nu \) satisfies

\[
\int f d\nu = \int \Lambda_n^{-1} E_\mu(\Lambda_n f | F_n) d\nu. \quad (\ast 6)
\]

**Proof:** We get from last lemma that \( \psi(f e^n) = \phi_\nu(G(f e^n)) \) where \( \phi_\nu(f) = \int f d\nu = \psi(f) \). Now, from Theorem 2.1 we get that (\ast 6) is true.

Now we will show the uniqueness of the KMS state:

**Theorem 2.3:** Given any KMS \( \psi \), then \( \psi = \psi_\beta \) where \( \psi_\beta \) is the KMS state associated to the Gibbs probability \( \nu_\beta \).

**Proof:**

In order to do that we will show that any possible \( \nu \) as defined above from the KMS \( \psi \) is equal to \( \nu_\beta \).
Take $\nu$ a probability associated to $\psi$, then for each $n$, and $f \in C(X)$ we have

$$\int f d\nu = \int E_n(f \Lambda_n^{-1}) \Lambda_n d\nu = \int \alpha^n(\mathcal{L}_\beta^n(f)) \Lambda_n d\nu. \quad \text{(7)}$$

We claim that

$$\lim_{n \to \infty} \int E_n(f \Lambda_n^{-1}) \Lambda_n d\nu = \int f d\nu_\beta,$$

and this shows that $\nu = \nu_\beta$, and therefore $\psi = \psi_\beta$.

Now we show the claim. Note that

$$\int f d\nu = \int E_n(f \Lambda_n^{-1}) \Lambda_n d\nu = \int \alpha^n(\mathcal{L}_\beta^n(f)) \Lambda_n d\nu = \int \alpha^n(\frac{\mathcal{L}_\beta^n(f)}{\lambda_\beta^n}) \Lambda_n \lambda_\beta^n d\nu,$$

where $\lambda_\beta$ is the eigenvalue associated to $\mathcal{L}_\beta$.

Applying the above expression to $f = h_\beta$ (we can assume $h_\beta$ is such that $\int h_\beta d\nu_\beta = 1$) and using the fact that $\mathcal{L}_\beta^n(h_\beta) = \lambda_\beta^n h_\beta$ we get

$$0 < d = \int h_\beta d\nu = \int \alpha^n(h_\beta) \Lambda_n \lambda_\beta^n d\nu.$$

As $h_\beta$ is continuous and positive, there exists $c > 0$ such for all $x \in X$ we have $h_\beta(x) > c$.

From this follow that

$$d = \int \alpha^n(h_\beta) \Lambda_n \lambda_\beta^n d\nu > c \int \lambda_\beta^n \Lambda_n d\nu.$$

Therefore,

$$\int \lambda_\beta^n \Lambda_n d\nu < d/c$$

Denote $I = \int f d\nu_\beta$.

It is known (see [Bo]) that uniformly in $z \in X$, we have

$$\lim_{n \to \infty} \frac{\mathcal{L}_\beta^n(f)(z)}{\lambda_\beta^n} = h_\beta(z) I = h_\beta(z) \int f d\nu_\beta.$$

Therefore, given $\epsilon > 0$, we can find $N > 0$ such that for all $n > N$ we have

for all $z \in X$

$$|\frac{\mathcal{L}_\beta^n(f)(z)}{\lambda_\beta^n} - Ih_\beta(z)| \leq \epsilon.$$

Then, for $n > N$

$$|\int \frac{\alpha^n(\mathcal{L}_\beta^n(f))}{\lambda_\beta^n} \Lambda_n \lambda_\beta^n d\nu - \int I \alpha^n(h_\beta) \Lambda_n \lambda_\beta^n d\nu| \leq$$
\[
\int \frac{\alpha^n(L^n(f))}{\lambda^n_{\beta}}(y) - I\alpha^n(h_{\beta})(y)\Lambda_n(y)\Lambda^n_{\beta}(y)d\nu =
\int \frac{L^n(f)}{\lambda^n_{\beta}}(T^n(y)) - Ih_{\beta}(T^n(y))\Lambda_n(y)\Lambda^n_{\beta}(y)d\nu \leq \frac{ed}{c}
\]

The conclusion from (*7) is that for any \( f \in C(X) \)

\[
\lim_{n \to \infty} I \int \alpha^n(h_{\beta})\Lambda_n\lambda^n_{\beta}d\nu = \int fd\nu.
\]

Consider now \( f = 1 \) and we get

\[
\lim_{n \to \infty} \int \alpha^n(h_{\beta})\Lambda_n\lambda^n_{\beta}d\nu = 1.
\]

From this we conclude that \( \int fd\nu = \int f d\nu_{\beta} \) for all \( f \in C(X) \).

This shows the uniqueness and that \( \nu = \nu_{\beta} \).

\(
\square
\)

The final conclusion is that any KMS \( \psi \) for \( H, \beta \) is equal to the \( \psi_{\beta} \) associated to \( \nu_{\beta} \).

**Section 3 - no phase transitions**

We consider here an interesting example of a KMS state associated with the reference measure \( \mu \) given by the maximal entropy measure for the shift in 2 symbols \( \{0, 1\} \). In this case \( p = 1/2 \) is constant. We will define a special potential \( H \) and we will consider specifically the special value \( \beta = 1 \).

We refer the reader to [H] [L2] [L3] [FL] [Y] [L] for references and results about the topics discussed in this section.

We are going to introduce the Fisher-Fedenhorf model of Statistical Mechanics in the terminology of Bernoulli spaces and Thermodynamic Formalism [H].

We define \( \Sigma^+ \) to be the shift space \( \Sigma^+ = \Pi^\infty_1 \{0, 1\} \) and denote by \( T : \Sigma^+ \to \Sigma^+ \) the left shift map. We write \( z = (z_0z_1\ldots) \) for a point in \( \Sigma^+ \) and \([w_0w_1\ldots w_k] = \{ z : z_0 = w_0, z_1 = w_1, \ldots z_k = w_k \} \) for a cylinder set of \( \Sigma^+ \).

We denote by \( M_k \subset \Sigma^+ \), for \( k > 1 \), the cylinder set \( [11\ldots110] \) and by \( M_0 \) the cylinder set \([0]\). The ordered collection \( (M_k)_{k=0}^\infty \) is a partition of \( \Sigma^+ \); in other words these sets are disjoint and their union is the whole space (minus the point \((11\ldots)) \). Note that \( T \) maps \( M_k \) bijectively onto \( M_{k-1} \) for \( k \geq 1 \), and onto \( \Sigma^+ \) for \( k = 0 \).

The point \((111\ldots)\) is fixed for \( T \).

For \( \gamma > 1 \) a fixed real constant, we consider the potential \( g(x) \) such that \( g(111111\ldots) = 0 \),

\[
g(x) = a_k = -\gamma \log \left( \frac{k+1}{k} \right),
\]

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for $x \in M_{k}$, for $k \neq 0$, and
\[
a_0 = -\log(\zeta(\gamma)),
\]
for $x \in M_0$, where $\zeta$ is the Riemann zeta function.

By definition,
\[
\zeta(\gamma) = (1^{-\gamma} + 2^{-\gamma} + \ldots)
\]
and so the reason for defining $a_0$ in such way is that, if we define $s_k = a_0 + a_1 + \cdots + a_k$, then $\Sigma e^{s_k} = 1$.

From now on we assume $\gamma > 2$, otherwise we have to consider sigma-finite measures and not probabilities in our problem.

The potential $1 < (\frac{k-1}{k})^\gamma = H(x) = e^{-g(x)}$, for $x \in M_k$, is not Hölder and in fact is not of summable variation. Note that $H(111\ldots) = 1$, The pressure $P(-\log H) = P(g) = P(\log p + \log 2 - 1 \log H) = 0$ and one can show that there exist two equilibrium states for such a potential $g$ (in the sense of minimizing measures for the variational problem): a point mass (the Dirac delta $\delta(111\ldots)$) at $(111\ldots)$, and a second measure which we shall denote by $\tilde{\mu}$ (see [H]).

The existence of two probabilities $\tilde{\mu}$ and $\delta(111\ldots)$ for the variational problem of pressure defines what is called a phase transition in the sense of Statistical Mechanics [H] [L3].

We will describe bellow how to define this measure $\tilde{\mu}$.

Consider as in [H] $\mathcal{L}_g^*$, the dual of the Ruelle-Perron-Frobenius operator $\mathcal{L}_g$ associated to $g$, where the action of $\mathcal{L}_g$ on continuous functions is given by
\[
\mathcal{L}_{\beta=1}(\psi)(y) = \sum_{T(x)=y} e^{g(x)} \psi(x).
\]

The function $P(-\beta \log H) = P(\beta g)$ is strictly monotone for $\beta < 1$ and constant equal zero for $\beta > 1$ [H].

We claim that there is a unique probability measure $\nu$ on $\Sigma^+$ which satisfies $\mathcal{L}_g^* \nu = \nu$ [FL] [H]. To prove this, note first that $\nu$ cannot have any mass at $(11\ldots)$; it follows that $M_0$ has positive mass, and the stipulation that $\nu$ be an eigenmeasure then gives a recurrence relation for the masses of $M_k$. Since $T(M_k) = M_{k-1}$ for $k \geq 1$, we have that the masses of the sets in this partition are
\[
\nu(k) = \nu(M_k) = e^{s_k} = e^{(k+1)^{-\gamma}}\zeta(\gamma), k \geq 0;
\]
in particular,
\[
\nu(0) = \nu(M_0) = e^{s_0} = e^{a_0} = \frac{1}{\zeta(\gamma)}.
\]

By the same reasoning, $\nu$ is determined on all higher cylinder sets for the partition $(M_k)_{k=0}^\infty$. Hence $\nu$ exists and is unique.

The measure $\nu$ defined above is the unique eigenmeasure for $\mathcal{L}^*_{\beta=1}$ and denoted by $\nu_1$.

The measure defined by the delta-Dirac on $(111\ldots)$ is invariant but is not a fixed eigenmeasure for $\mathcal{L}^*_g$.
This measure \( \nu_1 \) defines a KMS state \( \psi_{\nu_1} \) for such \( H, \beta = 1 \) and \( \mathcal{U}(\mu) \).

We conjecture that there is another KMS state \( \psi \) different from \( \psi_{\nu_1} \) but not associated to a measure. Note that such \( H \) assumes the value 1 in just one point.

We define \( \hat{h}(x) \) for \( x \in \mathcal{M} \) by

\[
\hat{h}_t = \hat{h}(x) = \nu(t)^{-1} \sum_{i=t}^{\infty} \nu(i).
\]

The function \( \hat{h} \) satisfies \( \mathcal{L}_\beta(\hat{h}) = \hat{h} \).

The integral \( \int \hat{h}(x) d\nu_1(x) \) is finite if and only if \( \gamma > 2 \). One can normalize \( \hat{h} \), multiplying by a constant \( u \) to get \( \tilde{h} = uh \) with \( \int \tilde{h} d\nu_1 = 1 \).

This constant is

\[
\frac{1}{\sum_{i=1}^{\infty} t\nu(t-1)} = \frac{\zeta(\gamma)}{\sum_{i=1}^{\infty} t^{1-\gamma}} = \frac{\zeta(\gamma)}{\zeta(\gamma - 1)}.
\]

The probability \( \tilde{\mu} \) has positive entropy and its support is all \( \Sigma^+ \) (see [H] or [L3] [FL]).

Consider now the invariant probability measure \( \hat{\nu} = h\nu_1 \). It is known that \( \hat{\nu} \) is an equilibrium state for \( -\log H \) in the variational sense (\( \beta = 1 \)) [H]. It is easy to see (because \( -\log H(11111...) = -\log 1 = 0 \)) that the Dirac-delta measure \( \delta_{(11111...)} \) is also an equilibrium state for \( -\log H \) in the variational sense (\( \beta = 1 \)).

The probability \( \tilde{\mu} \) has positive entropy and its support is all \( \Sigma^+ \) (see [H] or [L3] [FL]).

We can conclude from the above considerations that not always an equilibrium probability \( \rho \) for the pressure is associated to a KMS state \( \psi_{\rho} \) without the hypothesis of \( H \) and \( p \) been Hölder. In the present example, this happen because \( \rho = \delta_{(11111...)} \) is not an eigenmeasure of the dual of the Ruelle-Perron-Frobenius operator \( \mathcal{L}_\beta \) but it is an equilibrium measure for \( \beta = 1 \).

In [L2] and [L3] the lack of differentiability of the Free energy is analyzed and in [L3] [FL] [Y] it is shown that such systems present polynomial decay of correlation. In [L1] it is presented a dynamical model with three equilibrium states.

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