ON PATTERN-AVOIDING PARTITIONS

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Abstract. A set partition of the set \([n] = \{1, \ldots, n\}\) is a collection of disjoint blocks \(B_1, B_2, \ldots, B_d\) whose union is \([n]\). We choose the ordering of the blocks so that they satisfy \(\min B_1 < \min B_2 < \cdots < \min B_d\). We represent such a set partition by a canonical sequence \(\pi_1, \pi_2, \ldots, \pi_n\), with \(\pi_i = j\) if \(i \in B_j\). We say that a partition \(\pi\) contains a partition \(\sigma\) if the canonical sequence of \(\pi\) contains a subsequence that is order-isomorphic to the canonical sequence of \(\sigma\). Two partitions \(\sigma\) and \(\sigma'\) are equivalent, if there is a size-preserving bijection between \(\sigma\)-avoiding and \(\sigma'\)-avoiding partitions.

We determine several infinite families of sets of equivalent patterns; for instance, we prove that there is a bijection between \(k\)-noncrossing and \(k\)-nonnesting partitions, with a notion of crossing and nesting based on the canonical sequence. We also provide new combinatorial interpretations of the Catalan numbers and the Stirling numbers. Using a systematic computer search, we verify that our results characterize all the pairs of equivalent partitions of size at most seven.

We also present a correspondence between set partitions and fillings of Ferrers shapes and stack polyominoes. This correspondence allows us to apply recent results on polyomino fillings in the study of partitions, and conversely, some of our results on partitions imply new results on polyomino fillings and ordered graphs.

1. Introduction

A partition of \([n] = \{1, 2, \ldots, n\}\) is a collection \(B_1, B_2, \ldots, B_d\) of nonempty disjoint sets, called blocks, whose union is \([n]\). We will assume that \(B_1, B_2, \ldots, B_d\) are listed in the increasing order of their minimum elements, that is, \(\min B_1 < \min B_2 < \cdots < \min B_d\). In this paper, we will represent a partition of \([n]\) by its canonical sequence, which is an integer sequence \(\pi = \pi_1 \pi_2 \cdots \pi_n\) such that \(\pi_i = k\) if and only if \(i \in B_k\). For instance, \(1231242\) is the canonical sequence of the partition of \([1, 2, \ldots, 7]\) with the four blocks \([1, 4]\), \([2, 5, 7]\), \([3]\) and \([6]\).

Note that a sequence \(\pi\) over the alphabet \([d]\) represents a partition of \([n]\) with \(d\) blocks if and only if it has the following properties:

- Each number from the set \([d]\) appears at least once in \(\pi\).
- For each \(i, j\) such that \(1 \leq i < j \leq d\), the first occurrence of \(i\) precedes the first occurrence of \(j\).

We remark that sequences satisfying these properties are also known as restricted growth functions, and they are often encountered in the study of set partitions \([20, 25]\) as well as other related topics, such as Davenport-Schinzel sequences \([6, 13, 14, 18]\).

Throughout this paper, we identify a set partition with the corresponding canonical sequence, and we use this representation to define the notion of pattern avoidance among set partitions. Let \(\pi = \)

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\(\pi_1 \pi_2 \cdots \pi_n\) and \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_m\) be two partitions represented by their canonical sequences. We say that \(\pi\) contains \(\sigma\), if \(\pi\) has a subsequence that is order-isomorphic to \(\sigma\); in other words, \(\pi\) has a subsequence \(\pi_{f(1)}, \pi_{f(2)}, \ldots, \pi_{f(m)}\), where \(1 \leq f(1) < f(2) < \cdots < f(m) \leq n\), and for each \(i, j \in [m]\), \(\pi_{f(i)} < \pi_{f(j)}\) if and only if \(\sigma_i < \sigma_j\). If \(\pi\) does not contain \(\sigma\), we say that \(\pi\) avoids \(\sigma\). Our aim is to study the set of all the partitions of \([n]\) that avoid a fixed partition \(\sigma\). In such context, \(\sigma\) is usually called a pattern.

Let \(P(n)\) denote the set of all the partitions of \([n]\), let \(P(n; \sigma)\) denote the set of all partitions of \([n]\) that avoid \(\sigma\), and let \(p(n)\) and \(p(n; \sigma)\) denote the cardinality of \(P(n)\) and \(P(n; \sigma)\), respectively. We say that two partitions \(\sigma\) and \(\sigma'\) are equivalent, denoted by \(\sigma \sim \sigma'\), if \(p(n; \sigma) = p(n; \sigma')\) for each \(n\).

The concept of pattern-avoidance described above has been introduced by Sagan [20] and later developed by Goyt [11]. These two papers considered, among other topics, the enumeration of partitions avoiding patterns of size three. In our paper, we extend this study to larger patterns. Rather than studying isolated cases of equivalent pairs of patterns, our aim is to describe infinite families of non-trivial equivalence classes of partitions.

For instance, we define \(k\)-noncrossing and \(k\)-nonnesting partitions as the partitions that avoid the pattern \(12\cdots k 12\cdots k\) and \(12\cdots kk(k-1)\cdots 1\), respectively. We will show that these two patterns are equivalent for every \(k\), by constructing a bijection between \(k\)-noncrossing and \(k\)-nonnesting partitions. It is noteworthy, that a different concept of crossings and nestings in partitions has been considered by Chen et al. [3,4], and this different notion of crossings and nestings also admits a bijection between \(k\)-noncrossing and \(k\)-nonnesting partitions, as has been shown in [4]. There is, in fact, yet another notion of crossings and nestings in partitions that has been extensively studied by Klazar [13,14].

Several of our results are proved using a correspondence between partitions and 0-1 fillings of polyomino shapes. This correspondence allows us to translate recent results on fillings of Ferrers shapes [6,15] and stack polyominoes [19] into the terminology of pattern-avoiding partitions. The correspondence between fillings of shapes and pattern-avoiding partitions works in the opposite way as well: some of our theorems, proved in the context of partitions, imply new results about pattern-avoiding fillings of Ferrers shapes and pattern-avoiding ordered graphs.

Apart from these results, we also present a class of patterns equivalent to the pattern \(12\cdots k\). This result can be viewed as a new combinatorial interpretation of the Stirling numbers of the second kind. Similarly, by providing patterns equivalent to \(1212\), we provide a new combinatorial interpretation of the Catalan numbers.

To test the strength of our general equivalence theorems, we have undertaken systematic computer enumeration of partitions avoiding single patterns of size at most 7; the results of this enumeration are presented in the appendix of this paper. Our methods are able to completely characterize the equivalence of all the patterns of length at most seven. This extends earlier results of Sagan [20], who provided similar characterization of patterns of size three.

The paper is organized as follows: in Section 2 we present basic facts about pattern-avoiding partitions, and we summarize previously known results. Our main results are collected in Section 3, where we present several infinite families of classes of equivalent patterns. In the rest of the paper, we deal with several isolated pairs of patterns which are not covered by any of the general theorems of the previous section. In particular, in Section 4, we prove that the pattern \(1123\) is equivalent to the pattern \(1212\), thus completing the characterization of the patterns of size four and obtaining a new interpretation for the Catalan numbers. In Section 5, we prove the equivalence \(12112 \sim 12212\), and discuss its implications for the theory of pattern-avoiding ordered graphs and polyomino fillings.
2. Basic facts and previous results

Let us first establish some notational conventions that will be applied throughout this paper: for a finite sequence \( S = s_1s_2 \cdots s_p \) and an integer \( k \), we use the notation \( S + k \) to refer to the sequence \( (s_1 + k)(s_2 + k) \cdots (s_p + k) \). For a symbol \( k \) and an integer \( d \), the constant sequence \( (k, k, \ldots, k) \) of length \( d \) is denoted by \( k^d \). To prevent confusion, we will use capital letters \( S, T, \ldots \) to denote arbitrary sequences of positive integers, and we will use lowercase greek symbols \( (\pi, \sigma, \tau, \ldots) \) to denote sequences in their canonical form representing partitions.

An infinite sequence \( a_0, a_1, \ldots \) is often conveniently represented by its exponential generating function (or EGF for short), which is the formal power series \( F(x) = \sum_{n \geq 0} a_n x^n/n! \). We are mostly interested in the generating functions of the sequences of the form \( (p(n; \pi))_{n \geq 0} \), where \( \pi \) is a given pattern. We simply call such a generating function the EGF of the pattern \( \pi \).

2.1. Simple patterns. Several natural classes of partitions can be defined in terms of pattern-avoidance. For instance, the partitions whose every block is smaller than a given constant \( k \) are precisely the partitions that avoid the pattern \( \pi = 11 \cdots 1 = 1^k \). Using standard combinatorial arguments (see, e.g., [9]), we obtain the following well-known formula for the EGF of the pattern \( 1^k \):

\[
F(x) = \exp(\exp^{<k}(x) - 1),
\]

where \( \exp(x) = \sum_{n \geq 0} \frac{x^n}{n!} \) and \( \exp^{<k}(x) = \sum_{n=0}^{k-1} \frac{x^n}{n!} \).

Another class of pattern-avoiding partitions with a similarly natural combinatorial description is the set of the partitions avoiding \( \pi = 12 \cdots k \). Clearly, a partition avoids \( \pi \) if and only if it has less than \( k \) blocks. It is well known (see, e.g., [9]) that the EGF of the pattern \( 12 \cdots k \) is equal to

\[
\exp^{<k}(\exp(x) - 1).
\]

The enumeration of \( \pi \)-avoiding partitions is closely related to the Stirling numbers of the second kind \( S(n, m) \), defined as the number of partitions of \( [n] \) with exactly \( m \) blocks (see sequence A008277 in [21]).

2.2. Patterns of size three. Sagan [20] has described and enumerated the pattern-avoiding classes \( P(n; \pi) \) for the five patterns \( \pi \) of length three. Here we briefly summarize the relevant results (see Table 1). For the sake of completeness, we briefly present the arguments used to obtain these results.

| \( \pi \)     | \( p(n; \pi) \)                |
|-------------|--------------------------------|
| 111         | sequence A000085 in [21]      |
| 112, 121, 122, 123 | \( 2^{n-1} \)        |

Table 1. Number of partitions in \( P(n; \tau) \), where \( \tau \in P(3) \).

Clearly, a partition avoids 111 if and only if each of its blocks has size at most two. This is a special case of the pattern \( 1^k \) discussed above. We remark that the sequence \( p(n; 111) \) enumerating these partitions has several other combinatorial interpretations, such as the number of involutions of the set \( [n] \) (see A000085 in [21]).

The four remaining patterns of size three, namely 123, 122, 121 and 112, are all equivalent and they satisfy \( p(n; \pi) = 2^{n-1} \). To see this, observe first of all that the partitions avoiding 123 correspond
precisely to the restricted growth functions which start with the symbol 1 and any other symbol is
equal to 1 or 2.

Similarly, the partitions avoiding 122 are encoded by sequences whose first symbol is 1, and each of
the following symbols is either equal to 1 or is greater by one than the largest preceding symbol.
The case of the pattern 121 is equally simple: each symbol of the corresponding sequence after the
first one is either equal to the largest preceding symbol or is greater by one than the largest preceding
symbol.

For the pattern 112, the argument is different: note that a partition $\tau$ with $k$ blocks avoids 112 if and
only if $\tau$ consists of the sequence $12\cdots k$ followed by a weakly decreasing sequence, i.e., $\tau = 12\cdots kS$
where $S$ is weakly decreasing. In particular, a 112-avoiding partition is uniquely determined by the
ordered sequence of the sizes of its blocks. Thus, the number of 112-avoiding partitions of $[n]$ is equal
to the number of ordered sequences of positive integers whose sum is $n$, and it is well known that
there are precisely $2^{n-1}$ such sequences.

3. General classes of equivalent patterns

In this section, we introduce the tools that will be useful in our study of pattern-avoidance, and we
prove our key results. We begin by introducing a general relationship between pattern-avoidance in
partitions and pattern-avoidance in fillings of restricted shapes. This approach will provide a useful
tool for dealing with many pattern problems.

3.1. Pattern-avoiding fillings of diagrams. We will use the term diagram to refer to any finite
set of the cells of the two-dimensional square grid. To fill a diagram means to write a non-negative
integer into each cell.

We will number the rows of diagrams from bottom to top, so the “first row” of a diagram is its bottom
row, and we will number the columns from left to right. We will apply the same convention to matrices
and to fillings. We always assume that each row and each column of a diagram is nonempty; thus,
for example, when we refer to a diagram with $r$ rows, it is assumed that each of the $r$ rows contains
at least one cell of the diagram. Note that there is a (unique) empty diagram with no rows and no
columns. Let $r(F)$ and $c(F)$ denote, respectively, the number of rows and columns of $F$, where $F$
is a diagram, or a matrix, or a filling of a diagram.

We will mostly use diagrams of a special shape, namely Ferrers diagrams and stack polyominoes. We
begin by giving the necessary definitions:

Definition 1. A Ferrers diagram, also called Ferrers shape, is a diagram whose cells are arranged
into contiguous rows and columns satisfying the following rules:

- The length of any row is greater than or equal to the length of any row above it.
- The rows are right-justified, i.e., the rightmost cells of the rows appear in the same column.

We admit that our convention of drawing Ferrers diagrams as right-justified rather than left-justified
shapes is different from standard practice; however, our definition will be more intuitive in the context
of our applications.

Definition 2. A stack polyomino $\Pi$ is a collection of finitely many cells of the two-dimensional
rectangular grid, arranged into contiguous rows and columns with the property that for any $i = 1, \ldots, r(\Pi)$, every column intersecting the $i$-th row also intersects all the rows with index smaller
than $i$. 

Clearly, every Ferrers shape is also a stack polyomino. On the other hand, a stack polyomino can be regarded as a union of a Ferrers shape and a vertically reflected copy of another Ferrers shape.

**Definition 3.** A filling of a diagram is an assignment of non-negative integers to the cells of the diagram. A 0-1 filling is a filling that only uses values 0 and 1. In such filling, a 0-cell of a filling is a cell that is filled with value 0, and a 1-cell is filled with value 1. A 0-1 filling is called semi-standard if each of its columns contains exactly one 1-cell. A 0-1 filling is called sparse if every column has at most one 1-cell. A column or row of a 0-1 filling is called zero column (or row) if it contains no 1-cell.

Among several possibilities to define pattern-avoidance in fillings, the following approach seems to be the most useful and most common:

**Definition 4.** Let \( M = (m_{ij}; i \in [r], j \in [c]) \) be a matrix with \( r \) rows and \( c \) columns with all entries equal to 0 or 1, and let \( F \) be a filling of a diagram. We say that \( F \) contains \( M \) if \( F \) contains \( r \) distinct rows \( i_1 < \cdots < i_r \) and \( c \) distinct columns \( j_1 < \cdots < j_c \) with the following two properties:

- Each of the rows \( i_1, \ldots, i_r \) intersects all columns \( j_1, \ldots, j_c \) in a cell that belongs to the underlying diagram of \( F \).
- If \( m_{kl} = 1 \) for some \( k \) and \( l \), then the cell of \( F \) in row \( i_k \) and column \( j_l \) has a nonzero value.

If \( F \) does not contain \( M \), we say that \( F \) avoids \( M \). We will say that two matrices \( M \) and \( M' \) are Ferrers-equivalent (denoted by \( M \simeq M' \)) if for every Ferrers shape \( \Delta \), the number of semi-standard fillings of \( \Delta \) that avoid \( M \) is equal to the number of semi-standard fillings of \( \Delta \) that avoid \( M' \). We will say that \( M \) and \( M' \) are stack-equivalent (denoted by \( M \sim M' \)) if the equality holds even for semi-standard fillings of an arbitrary stack polyomino.

Pattern-avoidance in the fillings of diagrams has received considerable attention lately. Apart from semi-standard fillings, various authors have considered standard fillings with exactly one 1-cell in each row and each column (see [2] or [22]), as well as general fillings with non-negative integers (see [7] or [15]). Also, nontrivial results were obtained for fillings of more general shapes (e.g. moon polyominoes [19]). These results often consider the cases when the forbidden pattern \( M \) is the identity matrix (i.e., the \( r \) by \( r \) matrix, with \( m_{ij} = 1 \) if and only if \( i = j \); this matrix will be denoted by \( I_r \)) or the anti-identity matrix (i.e., the \( r \) by \( r \) matrix with \( m_{ij} = 1 \) if and only if \( i + j = r + 1 \); this matrix will be denoted by \( J_r \)).

Since our next arguments mostly deal with semi-standard fillings, we will drop the adjective ‘semi-standard’ and simply use the term ‘filling’, when there is no risk of ambiguity.

**Remark 5.** In our argument, we will often encounter a mapping \( f \) that transforms a given semi-standard filling of a Ferrers diagram (or a stack polyomino) into another semi-standard filling of the same diagram. It will be convenient to extend such transform \( f \) to act on sparse fillings as well as semi-standard fillings; this is achieved in the following natural way: given a sparse filling \( F \) of a Ferrers diagram \( \Delta \), we ignore all the columns of \( F \) that contain no 1-cell and observe that the remaining columns induce a semi-standard filling of a Ferrers diagram. We then transform the filling \( F \) by letting the mapping \( f \) act on the non-zero columns of \( F \) (i.e. those that contain a 1-cell), while the zero columns are left without change.

In particular, if \( M \) and \( M' \) are two Ferrers-equivalent 0-1 matrices with a 1-cell in every column, the argument above shows that there is a bijection between \( M \)-avoiding and \( M' \)-avoiding sparse fillings of a given Ferrers diagram. To see this, note that a sparse filling \( F \) avoids \( M \) if and only if its subfilling induced by the nonzero columns avoids \( M \), since \( M \) has a 1-cell in every column.

A completely analogous argument can be made for stack polyominoes instead of Ferrers shapes.
We now introduce some more notation, which will be useful for translating the language of partitions to the language of fillings.

**Definition 6.** Let $S = s_1 s_2 \cdots s_m$ be a sequence of positive integers, and let $k \geq \max\{s_i; i \in [m]\}$ be an integer. We let $M(S, k)$ denote the 0-1 matrix with $k$ rows and $m$ columns which has a 1-cell in row $i$ and column $j$ if and only if $s_j = i$.

We now describe the correspondence between partitions and fillings of Ferrers diagrams (recall that $\tau + k$ denotes the sequence obtained from $\tau$ by adding $k$ to every element).

**Lemma 7.** Let $S$ and $S'$ be two sequences over the alphabet $[k]$, let $\tau$ be an arbitrary partition. If $M(S, k)$ is Ferrers-equivalent to $M(S', k)$ then the partition pattern $\sigma = 12 \cdots k(\tau + k)S$ is equivalent to $\sigma' = 12 \cdots k(\tau + k)S'$.

*Proof.* Let $\pi$ be a partition of $[n]$ with $m$ blocks. Let $M$ denote the matrix $M(\pi, m)$. Fix a partition $\pi$ with $t$ blocks, and let $T$ denote the matrix $M(\tau, t)$. We will color the cells of $M$ red and green in the following way: if $\tau$ is nonempty, then a cell in row $i$ and column $j$ is colored green if and only if the submatrix of $M$ induced by the rows $i+1, \ldots, m$ and columns $1, \ldots, j-1$ contains $T$. If $\tau$ is empty, then a cell in row $i$ and column $j$ is green if and only if row $i$ has at least one 1-cell strictly to the left of column $j$. A cell is red if it is not green.

Note that the green cells form a Ferrers diagram, and the entries of the matrix $M$ form a sparse filling $G$ of this diagram. Also note that the leftmost 1-cell of each row is always red, and any 0-cell of the same row to the left of the leftmost 1-cell is red too.

It is not difficult to see that the partition $\pi$ avoids $\sigma$ if and only if the filling $G$ of the ‘green’ diagram avoids $M(S, k)$, and $\pi$ avoids $\sigma'$ if and only if $G$ avoids $M(S', k)$. Since $M(S, k) \sim M(S', k)$, there is a bijection $f$ that maps $M(S, k)$-avoiding fillings of Ferrers shapes onto $M(S', k)$-avoiding fillings of the same shape. By Remark 5, $f$ can be extended to sparse fillings. Using this extension of $f$, we construct the following bijection between $P(n; \sigma)$ and $P(n; \sigma')$: for a partition $\pi \in P(n; \sigma)$ with $m$ blocks, we take $M$ and $G$ as above. By assumption, $G$ is $M(S, k)$-avoiding. Using the bijection $f$ and Remark 5 we transform $G$ into an $M(S', k)$-avoiding sparse filling $f(G) = G'$, while the filling of the red cells of $M$ remains the same. We thus obtain a new matrix $M'$.

Note that if we color the cells of $M'$ red and green using the criterion described in the first paragraph of this proof, then each cell of $M'$ will receive the same color as the corresponding cell of $M$, even though the occurrences of $T$ in $M'$ need not correspond exactly to the occurrences of $T$ in $M$.

By construction, $M'$ has exactly one 1-cell in each column, hence there is a sequence $\pi'$ over the alphabet $[m]$ such that $M' = M(\pi', m)$. We claim that $\pi'$ is a canonical sequence of a partition. To see this, note that for every $i \in [m]$, the leftmost 1-cell of $M$ in row $i$ is red and the preceding 0-cells in row $i$ are red too. It follows that the leftmost 1-cell of row $i$ in $M$ is also the leftmost 1-cell of row $i$ in $M'$, so the first occurrence of the symbol $i$ in $\pi'$ appears at the same place as the leftmost occurrence of $i$ in $\pi'$, hence $\pi'$ is indeed a partition. The green cells of $M'$ avoid $M(S', k)$, so $\pi'$ avoids $\sigma'$. Obviously, the transform $\pi \mapsto \pi'$ is invertible and provides a bijection between $P(n; \sigma)$ and $P(n; \sigma')$. \hfill $\square$

In general, the relation $12 \cdots kS \sim 12 \cdots kS'$ does not imply that $M(S, k)$ and $M(S', k)$ are Ferrers equivalent: in Section 5 we will prove that $12112 \sim 12212$, even though $M(2, 112)$ is not Ferrers equivalent to $M(2, 212)$.

On the other hand, the relation $12 \cdots kS \sim 12 \cdots kS'$ allows us to establish a somewhat weaker equivalence between pattern-avoiding fillings, using the following lemma.
Lemma 8. Let $S$ be an arbitrary sequence over the alphabet $[k]$, and let $\tau = 12 \cdots kS$. For every $n$ and $m$, there is a bijection $f$ that maps the set of $\tau$-avoiding partitions of $[n]$ with $m$ blocks onto the set of all the $M(S, k)$-avoiding fillings $F$ of Ferrers shapes that satisfy $c(F) = n - m$ and $r(F) \leq m$.

Proof. Let $\pi$ be a $\tau$-avoiding partition of $[n]$ with $m$ blocks. Let $M = M(\pi, m)$, and let us consider a red and green coloring of $M$ in the same way as in the proof of Lemma 7, i.e., the green cells of a row $i$ are precisely the cells that are strictly to the right of the leftmost 1-cell in row $i$.

Note that $M$ has exactly $m$ red 1-cells, and each 1-cell is red if and only if it is the leftmost 1-cell of its row. Note also that if $c_i$ is column containing the red 1-cell in row $i$, then either $c_i$ is the rightmost column of $M$, or the column $c_i + 1$ is the leftmost column of $M$ with exactly $i$ green cells.

Let $G$ be the filling formed by the green cells. As was pointed out in the previous proof, the filling $G$ is a sparse $M(S, k)$-avoiding filling of a Ferrers shape. Note that for each $i = 1, \ldots, m - 1$, the filling $G$ has exactly one zero column of height $i$, and this column, which corresponds to $c_{i+1}$, is the rightmost of all the columns of $G$ with height at most $i$.

Let $G^-$ be the subfilling of $G$ induced by all the nonzero columns of $G$. Observe that $G^-$ is a semistandard $M(S, k)$-avoiding filling of a Ferrers shape with exactly $n - m$ columns and at most $m$ rows; we thus define $f(\pi) = G^-$. Let us now show that the mapping $f$ defined above can be inverted. Let $F$ be a filling of a Ferrers shape with $n - m$ columns and at most $m$ rows. We insert $m - 1$ zero columns $c_2, c_3, \ldots, c_m$ into the filling $F$ as follows: each column $c_i$ has height $i - 1$, and it is inserted immediately after the rightmost column of $F \cup \{c_2, \ldots, c_{i-1}\}$ that has height at most $i - 1$. Note that the filling obtained by this operation corresponds to the green cells of the original matrix $M$, so let us color all its cells green, and let us call this sparse filling $G$.

We now add a new red 1-cell on top of each zero column of $G$, and we add a new red 1-cell in front of the bottom row, to obtain a semistandard filling of a diagram with $n$ columns and $m$ rows. which can be completed into a matrix $M = M(\pi, m)$, where $\pi$ is easily seen to be a canonical sequence of a $\tau$-avoiding partition.

Lemma 7 provides a tool to deal with partition patterns of the form $12 \cdots k(\tau + k)S$ where $S$ is a sequence over $[k]$ and $\tau$ is a partition. We now describe a correspondence between partitions and fillings of stack polyominoes, which is useful for dealing with patterns of the form $12 \cdots kS(\tau + k)$.

We use a similar argument as in the proof of Lemma 7.

Lemma 9. If $\tau$ is a partition, and $S$ and $S'$ are two sequences over the alphabet $[k]$ such that $M(S, k) \not\sim M(S', k)$, then the partition $\sigma = 12 \cdots kS(\tau + k)$ is equivalent to the partition $\sigma' = 12 \cdots kS'(\tau + k)$.

Proof. Fix a partition $\tau$ with $t$ blocks. Let $\pi$ be any partition of $[n]$ with $m$ blocks, let $M = M(\pi, m)$. We will color the cells of $M$ red and green as follows: a cell of $M$ in row $i$ and column $j$ is green, if it satisfies both these conditions:

(a) The submatrix of $M$ formed by the intersection of rows $i + 1, i + 2, \ldots, m$ and columns $j + 1, j + 2, \ldots, n$ contains $M(\tau, t)$.

(b) The matrix $M$ has at least one 1-cell in row $i$ appearing strictly to the left of column $j$.

A cell is called red, if it is not green. Note that the green cells form a stack polyomino and the matrix $M$ induces a sparse filling $G$ of this polyomino.

Similarly to Lemma 7, it easy to verify that the partition $\pi$ above avoids the pattern $\sigma$ if and only if the filling $G$ avoids $M(S, k)$, and $\pi$ avoids $\sigma'$ if and only if $G$ avoids $M(S', k)$. 

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The rest of the argument is analogous to the proof of Lemma \[\text{Lemma}\] Assume that $M(S, k)$ and $M(S', k)$ are stack-equivalent via a bijection $f$. By Remark \[\text{Remark}\] we extend $f$ to a bijection between $M(S, k)$-avoiding and $M(S', k)$-avoiding sparse fillings of a given stack polyomino. Consider a partition $\pi \in P(n; \sigma)$ with $m$ blocks, and define $M$ and $G$ as above. Apply $f$ to the filling $G$ to obtain an $M(S', k)$-avoiding filling $G'$; the filling of the red cells of $M$ remains the same. This yields a matrix $M'$ and a sequence $\pi'$ such that $M' = M(\pi', k)$; we may easily check that the green cells of $M'$ are the same as the green cells of $M$. By rule (b) above, the leftmost 1-cell of each row of $M$ is unaffected by this transform. It follows that the first occurrence of $i$ in $\pi'$ is at the same place as the first occurrence of $i$ in $\pi$, and in particular, $\pi'$ is a partition. By the observation of the previous paragraph, $\pi'$ avoids $\sigma'$ and the transform $\pi \mapsto \pi'$ is a bijection from $P(n; \sigma)$ to $P(n; \sigma')$.



The following simple result about pattern-avoidance in fillings will turn out to be useful in the analysis of pattern avoidance in partitions:

**Proposition 10.** If $S$ is a sequence over the alphabet $[k - 1]$, then $M(S, k)$ is stack-equivalent to $M(S + 1, k)$. If $S$ and $S'$ are two sequences over $[k - 1]$ such that $M(S, k - 1) \sim M(S', k - 1)$ then $M(S, k) \sim M(S', k)$, and if $M(S, k - 1) \sim M(S', k - 1)$ then $M(S, k) \sim M(S', k)$.

**Proof.** To prove the first part, let us define $M = M(S, k)$, $M^- = M(S, k - 1)$, and $M' = M(S + 1, k)$. Notice that a filling $F$ of a stack polyomino $\Pi$ avoids $M$ if and only if the filling obtained by erasing the topmost cell of every column of $F$ avoids $M^-$. Similarly, $F$ avoids $M'$, if and only if the filling obtained by erasing the bottom row of $F$ avoids $M^-$. We thus have the following bijection between $M$-avoiding and $M'$-avoiding fillings: take an $M$-avoiding filling $F$, and in every column, move the topmost element into the bottom row, and move every other element into the row directly above it. The second claim of the theorem is proved analogously.

For convenience, we translate the first part of this proposition into the language of pattern-avoiding partitions, using Lemma \[\text{Lemma}\] and Lemma \[\text{Lemma}\].

**Corollary 11.** If $S$ is a sequence over $[k - 1]$ and $\tau$ is an arbitrary partition, then

$$12 \cdots k(\tau + k)S \sim 12 \cdots k(\tau + k)(S + 1) \text{ and } 12 \cdots kS(\tau + k) \sim 12 \cdots k(S + 1)(\tau + k).$$

We now state another result related to pattern-avoidance in Ferrers diagrams, which has important consequences in our study of partitions. Let us first fix the following notation: for two matrices $A$ and $B$, let $(A \ 0 \ B)$ denote the matrix with $r(A) + r(B)$ rows and $c(A) + c(B)$ columns with a copy of $A$ in the top left corner and a copy of $B$ in the bottom right corner.

The idea of the following proposition is not new, it has already been applied by Backelin et al. \[\text{Remark}\] to standard fillings of Ferrers diagrams, and later adapted by de Mier \[\text{Remark}\] for fillings with arbitrary integers. We now apply it to semi-standard fillings.

**Lemma 12.** If $A$ and $A'$ are two Ferrers equivalent matrices, and if $B$ is an arbitrary matrix, then $(B \ 0 \ A) \sim (B \ 0 \ A')$.

**Proof.** Let $F$ be an arbitrary $(B \ 0 \ A)$-avoiding filling of a Ferrers diagram $\Delta$. We say that a cell in row $i$ and column $j$ of $F$ is green if the subfilling of $F$ induced by the intersection of rows $i+1, i+2, \ldots, r(F)$ and columns $1, 2, \ldots, j-1$ contains a copy of $B$. Note that the green cells form a Ferrers shape $\Delta^- \subseteq \Delta$, and that the restriction of $F$ to the cells of $\Delta^-$ is a sparse $A$-avoiding filling $G$. By Remark \[\text{Remark}\] the filling $G$ can be bijectively transformed into a sparse $A'$-avoiding filling $G'$ of $\Delta^-$, which transforms $F$ into a semi-standard $(B \ 0 \ A')$-avoiding filling of $\Delta$. \[\text{Remark}\]
We remark that the argument of the proof fails if the matrices $(B^0 A^0)$ and $(A^0 B^0)$ are replaced with $(A^0 A^0)$ and $(A^0 B^0)$ respectively. Also, the argument fails if Ferrers shapes are replaced with stack polyominoes.

Although Lemma 12 does not directly provide new pairs of equivalent partition patterns, it allows us to prove the following proposition.

**Proposition 13.** Let $s_1 > s_2 > \cdots > s_m$ and $t_1 > t_2 > \cdots > t_m$ be two strictly decreasing sequences over the alphabet $[k]$, let $r_1, \ldots, r_m$ be positive integers. Define weakly decreasing sequences $S = s_1^{r_1} s_2^{r_2} \cdots s_m^{r_m}$ and $T = t_1^{r_1} t_2^{r_2} \cdots t_m^{r_m}$. We have $M(S, k) \leq M(T, k)$, and in particular, if $\tau$ an arbitrary partition, then $12 \cdots k(\tau + k) S \sim 12 \cdots k(\tau + k) T$.

*Proof.* We proceed by induction over minimum $j$ such that $s_i = t_i$ for each $i \leq m - j$. For $j = 0$, we have $S = T$ and the result is clear. If $j > 0$, assume without loss of generality that $s_{m-j+1} - t_{m-j+1} = d > 0$. Consider the sequence $t'_1 > t'_2 > \cdots > t'_m$ such that $t'_i = t_i$ for every $i \leq m - j$ and $t'_i = t_i + d$ for every $i > m - j$. The sequence $(t'_i)_{i=1}^m$ is strictly decreasing, and its first $m - j + 1$ terms are equal to $s_i$. Define $T' = (t'_1)^{r_1}(t'_2)^{r_2} \cdots (t'_m)^{r_m}$. By induction, $M(S, k) \leq M(T', k)$. To prove that $M(T', k) \leq M(T, k)$, first write $T = T_0 T_1$, where $T_0$ is the prefix of $T$ containing all the symbols of $T$ greater than $t_{m-j+1}$ and $T_1$ is the suffix of the remaining symbols. Notice that $T' = T_0(T_1 + d)$. We may write $M(T, k) = (B^0 A^0)$ and $M(T', k) = (B^0 A^0)$, where $A = M(T_1, t_{m-j} - 1)$ and $A' = M(T_1 + d, t_{m-j} - 1)$. By Proposition 10, $A \sim A'$, and by Lemma 12, $M(T, k) \leq M(T', k)$, as claimed. The last claim of the proposition follows from Lemma 7.

### 3.2. Non-crossing and non-nesting partitions

The key application of the framework of the previous subsection is the identity between non-crossing and non-nesting partitions. We define non-crossing and non-nesting partitions in the following way:

**Definition 14.** A partition is $k$-noncrossing if it avoids the pattern $12 \cdots k 12 \cdots k$, and it is $k$-nonnesting if it avoids the pattern $12 \cdots k k(k-1) \cdots 1$.

Let us point out that there are several different concepts of ‘crossings’ and ‘nestings’ used in the literature: for example, Klazar [13] has considered two blocks $X, Y$ of a partition to be crossing (or nesting) if there are four elements $x_1 < y_1 < x_2 < y_2$ (or $x_1 < y_1 < y_2 < x_2$, respectively) such that $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, and similarly for $k$-crossings and $k$-nestings. Unlike our approach, Klazar’s definition makes no assumption about the relative order of the minimal elements of $X$ and $Y$, which allows more general configurations to be considered as crossing or nesting. Thus, Klazar’s $k$-noncrossing and $k$-nonnesting partitions are a proper subset of our $k$-noncrossing and $k$-nonnesting partitions, (except for 2-noncrossing partitions where the two concepts coincide).

Another approach to crossings in partitions has been pursued by Chen et al. [3, 4]. They use the so-called linear representation, where a partition of $[n]$ with blocks $B_1, B_2, \ldots, B_k$ is represented by a graph on the vertex set $[n]$, with $a, b \in [n]$ connected by an edge if they belong to the same block and there is no other element of this block between them. In this terminology, a partition is $k$-crossing (or $k$-nesting) if the representing graph contains $k$ edges which are pairwise crossing (or nesting), where two edges $e_1 = \{a < b\}$ and $e_2 = \{a' < b'\}$ are crossing (or nesting) if $a < a' < b < b'$ (or $a < a' < b' < b$ respectively). Let us call such partitions graph-$k$-crossing and graph-$k$-nesting, to avoid confusion with our own terminology of Definition 14. It is not difficult to see that a partition is graph-2-noncrossing if and only if it is 2-noncrossing, but for nestings and for $k$-crossings with $k > 2$, the two concepts are incomparable. For instance the partition 12121 is graph-2-nonnesting.
but it contains 1221, while 12112 is graph-2-nesting and avoids 1221. Similarly, 1213123 has no graph-3-crossing and contains 123123, while 1232132 has a graph-3-crossing and avoids 123123.

Chen et al. [4] have shown that the number of graph-$k$-noncrossing and graph-$k$-nonnesting partitions of $[n]$ is equal. Below, we prove that the same is true for $k$-noncrossing and $k$-nonnesting partitions as well. It is interesting to note that the proofs of both these results are based on a reduction to theorems on pattern avoidance in the fillings of Ferrers diagrams (this is only implicit in [4], a direct construction is given by Krattenthaler [15]), although the constructions employed in the proofs of these results are quite different.

**Theorem 15.** For every $n$ and $k$, the number of $k$-noncrossing partitions of $[n]$ is equal to the number of $k$-nonnesting partitions of $[n]$.

By Lemma 7, a bijection between $k$-noncrossing and $k$-nonnesting partitions can be constructed from a bijection between $I_k$-avoiding and $J_k$-avoiding semi-standard fillings of Ferrers diagrams.

Krattenthaler [15] has presented a comprehensive summary of the relationships between $I_r$-avoiding and $J_r$-avoiding fillings of a fixed Ferrers diagram under additional constraints for row-sums and column-sums. These relationships are based on a suitable version of the RSK-correspondence (see [10] or [24] for a broad overview of the RSK algorithm and related topics).

We will now state the theorem about the correspondence between $I_k$-avoiding and $J_k$-avoiding fillings of diagrams. The result we will use is a weaker version of Theorem 13 from [15]. Note that in the original paper, it is not explicitly stated that the bijection between $I_k$-avoiding and $J_k$-avoiding fillings preserves the sum of every row and every column; however, this is an immediate consequence of the technique used in the proof. Also, in [15], the result is stated for arbitrary fillings with nonnegative integers; however, the previous remark shows that the result holds even when restricted to semi-standard fillings.

**Theorem 16** (adapted from [15]). For every Ferrers diagram $\Delta$ and every $k$, there is a bijection between the $I_k$-avoiding semi-standard fillings of $\Delta$ and the $J_k$-avoiding semi-standard fillings of $\Delta$. The bijection preserves the number of 1-cells in every row.

Theorem 16 and Lemma 7 give us the result we need.

**Corollary 17.** For every $n$ and every $k$, there is a bijection between $k$-noncrossing and $k$-nonnesting partitions of $[n]$. The bijection preserves the number of blocks, the size of each block, and the smallest element of every block.

Applying Lemma 7 with $S = 12\cdots k$ and $S' = k(k-1)\cdots 1$, and translating it into the terminology of pattern-avoiding partitions, we obtain the following result.

**Corollary 18.** Let $\tau$ be a partition, let $k$ be an integer. The pattern $12\cdots k(\tau + k)12\cdots k$ is equivalent to $12\cdots k(\tau + k)k(k-1)\cdots 1$.

Furthermore, results of Rubey, in particular [19, Proposition 5.3], imply that the matrices $I_k$ and $J_k$ are in fact stack-equivalent, rather than just Ferrers-equivalent. More precisely, Rubey’s theorem deals with fillings of moon polyominoes with prescribed row-sums. However, since a transposed copy of a stack polyomino is a special case of a moon polyomino, Rubey’s general result applies to fillings of stack polyominoes with prescribed column sums as well. Combining this theorem with Lemma 9, we obtain the following result.

**Corollary 19.** For any $k$ and any partition $\tau$, the pattern $12\cdots k12\cdots k(\tau + k)$ is equivalent to $12\cdots kk(k-1)\cdots 1(\tau + k)$.
3.3. Patterns of the form $1(\tau + 1)$. In this subsection, we will establish a general relationship between the partitions that avoid a pattern $\tau$ and the partitions that avoid the pattern $1(\tau + 1)$. The key result is the following theorem.

**Theorem 20.** Let $\tau$ be an arbitrary pattern, and let $F(x)$ be its corresponding EGF. Let $\sigma = 1(\tau + 1)$, and let $G(x)$ be its EGF. For every $n \geq 1$, the following holds:

\begin{equation}
(3)
\quad p(n; \sigma) = \sum_{i=0}^{n-1} \binom{n-1}{i} p(i; \tau).
\end{equation}

In terms of generating functions, this is equivalent to

\begin{equation}
(4)
\quad G(x) = 1 + \int_{0}^{x} F(t)e^t \, dt.
\end{equation}

**Proof.** Fix $\sigma$ and $\tau$ as in the statement of the theorem. Let $\pi$ be an arbitrary partition, and let $\pi^-$ denote the partition obtained from $\pi$ by erasing every occurrence of the symbol 1, and decreasing every other symbol by 1; in other words, $\pi^-$ represents the partition obtained by removing the first block from the partition $\pi$. Clearly, a partition $\pi$ avoids $\sigma$ if and only if $\pi^-$ avoids $\tau$. Thus, for every $\sigma$-avoiding partition $\pi \in P(n; \sigma)$ there is a unique $\tau$-avoiding partition $\rho \in \bigcup_{i=0}^{n-1} P(i; \tau)$ satisfying $\pi^- = \rho$. On the other hand, for a fixed $\rho \in P(i; \tau)$, there are $\binom{n-1}{i}$ partitions $\pi \in P(n; \sigma)$ such that $\pi^- = \rho$. This gives equation (3).

To get equation (4), we multiply both sides of (3) by $\frac{x^{n}}{n!}$ and sum for all $n \geq 1$. This yields

\[
G(x) - 1 = \sum_{n \geq 1} \frac{x^{n}}{n!} \sum_{i=0}^{n-1} \binom{n-1}{i} p(i; \tau) = \int_{0}^{x} \sum_{n \geq 1} \frac{t^{n}}{(n-1)!} \sum_{i=0}^{n-1} \binom{n-1}{i} p(i; \tau) dt
\]

\[
= \int_{0}^{x} \sum_{n \geq 0} \frac{t^{n}}{n!} \sum_{i=0}^{n} \binom{n}{i} p(i; \tau) \frac{t^{n-i}}{(n-i)!} dt
\]

\[
= \int_{0}^{x} \left( \sum_{i \geq 0} \frac{t^{i}}{i!} p(i; \tau) \right) \left( \sum_{k \geq 0} \frac{t^{k}}{k!} \right) dt = \int_{0}^{x} F(t)e^t \, dt,
\]

which is equivalent to equation (4). \qed

The following result is an immediate consequence of Theorem 20.

**Corollary 21.** If $\tau \sim \tau'$ then $1(\tau + 1) \sim 1(\tau' + 1)$, and more generally, $12 \cdots k(\tau + k) \sim 12 \cdots k(\tau' + k)$. In particular, since $123 \sim 122 \sim 112 \sim 121$, we see that for every $m \geq 2$ the patterns $12 \cdots (m-1)m(m+1)$, $12 \cdots (m-1)m$, $12 \cdots (m-1)(m-1)m$ and $12 \cdots (m-1)m(m-1)$ are equivalent. Conversely, if $1(\tau + 1) \sim 1(\tau' + 1)$, then $\tau \sim \tau'$.

**Proof.** To prove the last claim, notice that equation (3) can be inverted to obtain

\[
p(n-1; \tau) = \sum_{i=0}^{n-1} (-1)^{i} \binom{n-1}{i} p(n-i; \sigma).
\]

The other claims follow directly from Theorem 20. \qed
3.4. **Patterns equivalent to** \(12 \cdots m(m+1)\). The partitions that avoid \(12 \cdots m(m+1)\), or equivalently, the partitions with at most \(m\) blocks, are a very natural pattern-avoiding class of partitions. Their number \(p(n; 12 \cdots (m+1))\) is equal to \(\sum_{i=0}^{m} S(n, i)\), where \(S(n, i)\) is the Stirling number of the second kind, which is equal to the number of partitions of \([n]\) with exactly \(i\) blocks.

As an application of the previous results, we will now present two classes of patterns that are equivalent to the pattern \(12 \cdots (m+1)\). From this result, we obtain an alternative combinatorial interpretation of the Stirling numbers \(S(n, i)\).

Our result is summarized in the following theorem.

**Theorem 22.** For every \(m \geq 2\), the following patterns are equivalent:

(a) the pattern \(12 \cdots (m-1)m(m+1)\),

(b) the patterns \(12 \cdots (m-1)m d\), where \(d\) is any number from the set \([m]\),

(c) the patterns \(12 \cdots (m-1)dm\), where \(d\) is any number from the set \([m-1]\).

**Proof.** From Corollary 21 we get the following:

\[12 \cdots m(m+1) \sim 12 \cdots (m-1)mm \sim 12 \cdots (m-1)(m-1)m.\]

The equivalences between

\[12 \cdots (m-1)mm \sim 12 \cdots (m-1)md\] and \(12 \cdots (m-1)(m-1)m \sim 12 \cdots (m-1)dm\)

are obtained by a repeated application of Proposition 10. \(\square\)

3.5. **Binary patterns.** Let us now focus on the avoidance of **binary** patterns, i.e., the patterns that only contain the symbols 1 and 2.

We will first consider the forbidden patterns of the form \(1^k21^l\). We have already seen that \(112 \sim 121\). The following theorem offers a generalization.

**Theorem 23.** For any three integers \(j, k, m\) satisfying \(1 \leq j, k \leq m\), the pattern \(1^j21^{m-j}\) is equivalent to the pattern \(1^k21^{m-k}\).

Before we prove the proof of Theorem 23 we need some preparation. Let \(\pi = \pi_1 \pi_2 \cdots \pi_n\) be a partition. Clearly, \(\pi\) can be uniquely expressed as \(P_1 P_2 1 \cdots P_{p-1} 1 P_p\), where the \(P_i\) are (possibly empty) maximal contiguous subsequences of \(\pi\) that do not contain the symbol 1. The sequence \(P_i\) will be referred to as the \(i\)-th chunk of \(\pi\). By concatenating the chunks into a sequence \(P = P_1 \cdots P_p\) and then subtracting 1 from every symbol of \(P\), we obtain a canonical sequence of a partition; let this partition be denoted by \(\pi^-\). The key ingredient in the proof of Theorem 23 is the following lemma.

**Lemma 24.** Let \(\pi\) be a partition that has \(p\) occurrences of the symbol 1, let \(P_i\) and \(\pi^-\) be as above. Let \(r \geq 1\) and \(s \geq 0\) be two integers. The partition \(\pi\) avoids \(1^r21^s\) if and only if the following two conditions hold:

- \(\pi^-\) avoids \(1^r21^s\).
- For every \(i\) such that \(r \leq i \leq p - s\), the chunk \(P_i\) is empty.

**Proof.** Clearly, the two conditions are necessary. To see that they are sufficient, we argue by contradiction: let \(\pi\) be a partition that satisfies the two conditions, and assume that \(\pi\) has a subsequence \(a'ba^s\) for two symbols \(a < b\). If \(a = 1\) we have a contradiction with the second condition, and if \(a > 1\), then \(\pi^-\) contains the sequence \((a - 1)^r(b - 1)(a - 1)^s\), contradicting the first condition. \(\square\)

We are now ready prove Theorem 23.
Proof of Theorem 25. It is enough to prove that for every $k \geq 1$ and every $m > k$ there is a bijection $f$ from $P(n; 1^k2^{m-k})$ to $P(n; 1^m2)$. To define $f$, we will proceed by induction on the number of blocks of $\pi$. If $\pi = \sigma^n$ then we define $f(\pi) = \sigma$. Assume that $f$ has been defined for all partitions with less than $b$ blocks, and let $\pi \in P(n; 1^k2^{m-k})$ be a partition with $b$ blocks, let $p$ be the size of the first block of $\pi$. Let $P_1, \ldots, P_p$ be the chunks of $\pi$ and let $\pi^-$ be defined as above. Define $\overline{\pi} = f(\pi^-)$; this is well defined, since $\pi^- \in P(n-p; 1^k2^{m-k})$ and $\pi^-$ has $b-1$ blocks. Let $S = \overline{\pi} + 1$. We express $S$ as a concatenation of the form $S = S_1S_2 \cdots S_p$, where the length of $S_i$ is equal to the length of $P_i$. By Lemma 24, the chunk $P_i$ (and hence also $S_i$) is empty whenever $k \leq i \leq p-m+k$. We put $f(\pi) = \sigma$, where $\sigma$ is defined as follows:

- If $p < m$, then $\sigma = 1S_11S_21 \cdots 1S_{p-1}1S_p$.
- If $p \geq m$, then $\sigma = 1S_11S_21 \cdots 1S_{p-m+k+1}1S_{p-m+k+2} \cdots 1S_p1^{p-m+1}$.

Using Lemma 24, we may easily see that $\sigma$ avoids $1^m2$. It is also straightforward to check that $f$ is indeed a bijection from $P(n; 1^k2^{m-k})$ to $P(n; 1^m2)$. Note that $f$ preserves not only the number of blocks of the partition, but also the size of each block. \hfill \qed

Using our results on fillings, we can add another pattern to the equivalence class covered by Theorem 23.

**Theorem 25.** For every $m \geq 1$, the pattern $12^m$ is equivalent to the pattern $121^{m-1}$.

**Proof.** This is just Proposition 10 with $k = 2$ and $S = 1^{m-1}$. \hfill \qed

**Corollary 26.** Let $m$ be a positive integer, let $\tau$ be any pattern from the set

$$T = \{1^k2^{m-k}; \ 1 \leq k \leq m\} \cup \{12^m\}.$$ 

The EGF $F(x)$ of a pattern $\tau \in T$ is given by

$$F(x) = 1 + \int_0^x \exp \left( t + \sum_{i=1}^{m-1} \frac{t^i}{i!} \right) dt.$$ 

**Proof.** We have seen that all the patterns from the set $T$ are equivalent, so let us pick $\tau = 12^m$. The EGF follows from equation 1 on page 3 and from Theorem 24. \hfill \qed

We now turn to another type of binary patterns, namely the patterns of the form $12^k12^{m-k}$ with $1 \leq k \leq m$. It turns out that these patterns are also equivalent, provided $m$ is fixed. In fact, we will prove several generalizations of this fact.

Our argument will again be based on the analysis of fillings of stack polyominoes. However, to make full use of this approach, we will establish a stronger relation than mere stack equivalence. For this, we need the following definition.

**Definition 27.** Let $F$ be a semi-standard filling of a stack polyomino $\Pi$ and let $t \geq 1$ be an integer. We say that $F$ is $t$-falling if its first $t$ rows all contain at least one 1-cell, and the leftmost 1-cells of these rows form a decreasing chain; formally, $F$ is $t$-falling if for every $i < j \leq t$ the leftmost 1-cell in row $i$ exists and appears to the right of the leftmost 1-cell in row $j$, which must exist as well.

Notice that a $t$-falling filling of $\Pi$ only exists if the leftmost column of $\Pi$ intersects its first $t$ rows.

In the rest of this subsection, $S^p_q$ denotes the sequence $2^p12^q$ and $\overline{S}^p_q$ denotes the sequence $1^p2^q$, where $p, q$ are nonnegative integers.
Lemma 28. For every $p, q \geq 0$, the matrix $M(S_p^q, 2)$ is stack-equivalent to the matrix $M(S_0^{p+q}, 2)$. Furthermore, if $p \geq 1$, then for every stack polyomino $\Pi$, there is a bijection $f$ between the $M(S_p^q, 2)$-avoiding and $M(S_0^{p+q}, 2)$-avoiding semi-standard fillings of $\Pi$ with these two properties:

- $f$ preserves the number of 1-cells in every row.
- Both $f$ and $f^{-1}$ map $t$-falling fillings to $t$-falling fillings, for every $t \geq 1$.

Proof. Let $M = M(S_p^q, 2)$ and $M' = M(S_0^{p+q}, 2)$, for some $p, q \geq 0$. We will proceed by induction over the number of rows of $\Pi$. If $\Pi$ has only one row, then a constant mapping is the required bijection. Assume now that $\Pi$ has $r \geq 2$ rows, and assume that we are presented with a semi-standard filling $F$ of $\Pi$. Let $\Pi^-$ be the diagram obtained from $\Pi$ by erasing the $r$-th row as well as every column that contains a 1-cell of $F$ in the $r$-th row. The filling $F$ induces on $\Pi^-$ a semi-standard filling $F^-$. We claim that for every $p, q \geq 0$, a filling $F$ avoids $M$ if and only if these two conditions are satisfied:

- (a) The filling $F^-$ avoids $M$.
- (b) If the $r$-th row of $F$ contains $m$ 1-cells in columns $c_1 < c_2 < \cdots < c_m$ and if $m \geq p + q$, then for every $i$ such that $p \leq i \leq m - q$, the column $c_i$ is either the rightmost column of the last row of $\Pi$, or it is directly adjacent to the column $c_{i+1}$ (i.e. $c_i + 1 = c_{i+1}$).

Clearly, the two conditions are necessary. To see that they are sufficient, note that the first condition guarantees that $F$ does not contain any copy of $M$ that would be confined to the first $r - 1$ rows, whereas the second condition guarantees that $F$ has no copy of $M$ that would intersect the $r$-th row.

We now define recursively the required bijection between $M$-avoiding and $M'$-avoiding fillings. Let $F$ be an $M$-avoiding filling of $\Pi$, let $F^-$ and $c_1, \ldots, c_m$ be as above. By the induction hypothesis, we already have a bijection between $M$-avoiding and $M'$-avoiding fillings of the shape $\Pi^-$. This bijection maps $F^-$ to a filling $F^-$ of $\Pi^-$. Let $\tilde{F}$ be the filling of $\Pi$ that has the same values as $F$ in the $r$-th row, and the columns not containing a 1-cell in the $r$-th row are filled according to $F^-$. Note that $\tilde{F}$ contains no copy of $M'$ in its first $r - 1$ rows and it contains no copy of $M$ that would intersect the last row.

If $\tilde{F}$ has less than $p + q$ 1-cells in the last row, we define $f(F) = \tilde{F}$, otherwise we modify $\tilde{F}$ as follows: for every $i = 1, \ldots, q$, we consider the columns with indices strictly between $c_{m-q+1}$ and $c_{m-q+i+1}$ (if $i = q$, we take all columns to the right of $c_m$ that intersect the last row), we remove these columns from $\tilde{F}$ and re-insert them between the columns $c_{p+i-1}$ and $c_{p+i}$ (which used to be adjacent by condition (b) above). Note that these transformations preserve the relative left-to-right order of all the columns that do not contain a 1-cell in their $r$-th row; in particular, the resulting filling still has no copy of $M'$ in the first $r - 1$ rows. By construction, the filling also satisfies condition (b) for the values $p' = p + q$ and $q' = 0$ used instead of the original $p$ and $q$; and in particular, it is a $M'$-avoiding filling. This construction provides a bijection $f$ between $M$-avoiding and $M'$-avoiding fillings.

It is clear that $f$ preserves the number of 1-cells in each row. It remains to check that if $p \geq 1$, then $f$ preserves the $t$-falling property. Let us fix $t$, and let $r$ be the number of rows of $\Pi$. If $r < t$ then no filling of $\Pi$ is $t$-falling. If $r > t$, then $F$ is a $t$-falling filling if and only if $F^-$ is $t$-falling, so we obtain the required result from the induction hypothesis and from the fact that the process that transforms the intermediate filling $\tilde{F}$ into the final filling $f(F)$ does not change the relative position of the 1-cells of the first $r - 1$ rows. Finally, if $r = t$, then $F$ is $t$-falling if and only if $F^-$ is $(t - 1)$-falling and the leftmost 1-cell of the $r$-th row of $F$ is in the leftmost column of $\Pi$. Both these conditions are preserved by $f$ and $f^{-1}$, provided $p \geq 1$. \qed
With the help of Lemma \[28\] we are able to prove several results about pattern avoidance in partitions. We first state a direct corollary of Lemmas \[7\], \[9\] and \[28\] and Proposition \[10\].

**Corollary 29.** For any partition \(\tau\), for any \(k \geq 2\), and for any \(p, q \geq 0\), the pattern \(12 \cdots k(\tau + k)S_q^p\) is equivalent to \(12 \cdots k(\tau + k)S_q^{p+q}\) \(12 \cdots kS_q^p(\tau + k)\) is equivalent to \(12 \cdots kS_q^{p+q}(\tau + k)\).

Next, we present two theorems that make use of the \(t\)-falling property. Recall that \(\overline{S}_q^p = 1^p2^q\).

**Theorem 30.** Let \(\tau\) be any partition with \(k\) blocks, let \(p \geq 1\) and \(q \geq 0\). The pattern \(\sigma = \tau(\overline{S}_q^p + k)\) is equivalent to \(\sigma' = \tau(\overline{S}_0^{p+q} + k)\).

**Proof.** Let \(\pi\) be a partition of \([n]\) with \(m\) blocks, let \(M = M(\pi, m)\). We color the cells of \(M\) red and green, where a cell in row \(i\) and column \(j\) is green if and only if the submatrix of \(M\) formed by the intersection of the first \(i - 1\) rows and \(j - 1\) columns of \(M\) contains \(M(\tau, k)\). Let \(\Gamma\) be the diagram formed by the green cells of \(M\), and let \(G\) be the filling of \(\Gamma\) by the values from \(M\). Note that \(\Gamma\) is an upside-down copy of a Ferrers shape. It is easy to see that the partition \(\pi\) avoids \(\sigma\) if and only if \(G\) avoids \(M(\overline{S}_q^p, 2)\), and \(\pi\) avoids \(\sigma'\) if and only if \(G\) avoids \(M(\overline{S}_0^{p+q}, 2)\).

Let us now assume that \(\pi\) is \(\sigma\)-avoiding. We transform \(\pi\) into a \(\sigma'\)-avoiding partition \(\pi'\) by the following procedure. We first turn the filling \(G\) and the diagram \(\Gamma\) upside down, which transforms \(\Gamma\) into a Ferrers shape \(\overline{\Gamma}\), and it also transforms the \(M(\overline{S}_q^p, 2)\)-avoiding filling \(G\) into an \(M(S_q^p, 2)\)-avoiding filling \(\overline{G}\) of \(\overline{\Gamma}\). We may then apply the bijection \(f\) of Lemma \[28\] to \(\overline{G}\), ignoring the zero columns of \(\overline{G}\). We then turn the resulting filling \(\overline{G}' = f(\overline{G})\) of \(\overline{\Gamma}\) upside down again to obtain an \(M(S_0^{p+q}, 2)\)-avoiding filling \(G'\) of \(\Gamma\). We then fill the green cells of \(M\) with the values of \(G'\) while the filling of the red cells remains the same. We thus obtain a matrix \(M'\). The matrix \(M'\) has exactly one 1-cell in each column, so there is a sequence \(\pi'\) over the alphabet \([m]\) such that \(M' = M(\pi', m)\). Note that a cell is green with respect to \(M\) if and only if it is green with respect to \(M'\).

By construction, the sequence \(\pi'\) has no subsequence order-isomorphic to \(\sigma'\). We now need to show that \(\pi'\) is a restricted-growth sequence. For this, we will use the preservation of the \(t\)-falling property. Let \(c_i\) be the leftmost 1-cell of the \(i\)-th row of \(M\), let \(c'_i\) be the leftmost 1-cell of the \(i\)-th row of \(M'\). Let \(s\) be the largest index such that the cell \(c_s\) is red, we set \(s = 0\) if no such cell exists. Note that all the cells \(c_1, \ldots, c_s\) are red and all the cells \(c_{s+1}, \ldots, c_m\) are green. We have \(c_i = c'_i\) for every \(i \leq s\). If \(s > 0\), we also see that all the green 1-cells are in the columns to the right of \(c_s\).

It is easy to see that all the 1-cells above row \(s\) are green; in particular, the filling \(\overline{G}'\) is \(t\)-falling for some \(t \geq m - s\). By Lemma \[28\] the filling \(\overline{G}'\) is \(t\)-falling as well. It follows that the cells \(c'_1, \ldots, c'_m\) form a left-to-right increasing chain, and since all these cells are to the right of \(c'_s\), we see that all the cells \(c'_1, \ldots, c'_m\) form a left-to-right increasing sequence, hence \(\pi'\) is in canonical sequential form, i.e., \(\pi'\) is a partition from \(P(n; \sigma')\).

It is obvious that the above construction can be reversed, which shows that it is indeed a bijection between \(P(n; \sigma)\) and \(P(n; \sigma')\).

The following result is proved by a similar approach, but the argument is slightly more technical.

**Theorem 31.** Let \(T\) be an arbitrary sequence over the alphabet \([k]\), let \(p \geq 1\) and \(q \geq 0\). The partition \(\sigma = 12 \cdots k(\overline{S}_q^p + k)T\) is equivalent to \(\sigma' = 12 \cdots k(\overline{S}_0^{p+q} + k)T\).
Proof. Let $\pi$ be a partition of $[n]$ with $m$ blocks, let $M = M(\pi, m)$. As in the previous proof, we color the cells of $M$ red and green. A cell in row $i$ and column $j$ will be green if the submatrix of $M$ formed by rows $1, \ldots, i - 1$ and columns $j + 1, \ldots, n$ contains $M(T, k)$.

Let $\Gamma$ be the diagram formed by the green cells and $G$ its filling inherited from $M$. The partition $\pi$ contains $\sigma$ (or $\sigma'$) if and only if $G$ contains $M(S_q^p, 2)$ (or $M(S_q^{p+q}, 2)$, respectively). The diagram $\Gamma$ is an upside-down copy of a left-justified stack polyomino. Let $\Gamma^-$ be the diagram obtained by erasing the zero rows and zero columns of $\Gamma$, and let $G^-$ be the corresponding filling of $\Gamma^-$. The upside-down copy of the filling $G^-$ is $r$-falling, where $r$ is the number of rows of $\Gamma^-$. We apply the same construction as in the previous proof and transform $G^-$ into an $M(S_q^{p+q}, 2)$-avoiding filling $G'$ of $\Gamma^-$ and obtain a matrix $M' = M(\pi', n)$ that avoids $M(\sigma', k + 2)$. It remains to argue that $\pi'$ is in canonical sequential form.

Let $c_i$ (or $c'_j$) be the leftmost 1-cell in row $i$ of $M$ (or $M'$, respectively). To prove that $\pi'$ is a partition, we want to show that $c'_1, \ldots, c'_m$ form a left-to-right increasing sequence in $M'$. Let us now fix two row indices $i < j$. We claim that $c'_i$ is left of $c'_j$.

If both $c'_i$ and $c'_j$ are green, then the claim follows from the preservation of the $r$-falling property. If $c'_j$ is red, then all the green cells below row $j$ (including $c'_j$) are left of $c'_j$.

Finally, assume that $c'_i$ is green and $c'_j$ is red. We have $c'_i = c_i$. In the filling $G$, all the green 1-cells that are to the left of $c_i$ are also below row $i$; let $x$ be the number of such 1-cells. Then $x$ corresponds to the number of nonzero columns of $G$ that are to the left of $c_j$. Since the number of these nonzero columns is preserved by the mapping $f$, we see that $G'$ also has $x$ 1-cells left of $c_i$.

Since $f$ preserves the number of 1-cells in each row, both $G$ and $G'$ have exactly $x$ 1-cells below row $i$. All the 1-cells of $G'$ below row $i$ must appear to the left of $c_i$, and since there are only $x$ 1-cells of $G'$ to the left of $c_i$, they must all appear below row $i$, and in particular, all the green 1-cells above row $i$ (including the cell $c'_j$) appear to the right of $c_i$. □

3.6. Patterns equivalent to $12^k13$. Throughout this subsection, we will assume that $t$ is an arbitrary fixed integer, and we will deal with the following sets of patterns:

$$
\Sigma^+ = \{12^{p+1}12^q32^r : p, q, r \geq 0, p + q + r = t\}
$$

$$
\Sigma^- = \{12^{p+1}32^q12^r : p, q, r \geq 0, p + q + r = t\}
$$

$$
\Sigma = \Sigma^+ \cup \Sigma^-
$$

Our aim is to show that all the patterns in $\Sigma$ are equivalent. We will use the following definition.

Definition 32. Let $\sigma$ be a pattern over the alphabet $\{1, 2, 3\}$, let $\pi$ be a partition with $m$ blocks, and let $k \leq m$ be an integer. We say that $\pi$ contains $\sigma$ at level $k$, if there are symbols $l, h \in [m]$ such that $l < k < h$, and the partition $\pi$ contains a subsequence $S$ made of the symbols $\{l, k, h\}$ which is order-isomorphic to $\sigma$.

For example, the partition $\pi = 1231323142221$ contains $\sigma = 121223$ at level 3, because $\pi$ contains the subsequence 131334, but $\pi$ avoids $\sigma$ at level 2, because $\pi$ has no subsequence of the form $l2l2h$ with $l < 2 < h$.

Our plan is to show, for suitable pairs $\sigma, \sigma' \in \Sigma$, that for every $k$ there is a bijection $f_k$ that maps the partitions avoiding $\sigma$ at level $k$ onto the partitions avoiding $\sigma'$ at level $k$, while preserving $\sigma$-avoidance and $\sigma'$-avoidance at all the other levels. Composing the maps $f_k$ for all possible $k$, we will then obtain a bijection between $P(n; \sigma)$ and $P(n; \sigma')$. 
We need more definitions.

**Definition 33.** Consider a partition $\pi$, and fix a level $k \geq 2$. A symbol of $\pi$ is called $k$-low if it is smaller than $k$ and $k$-high if it is greater than $k$. A $k$-low cluster (or $k$-high cluster) is a maximal consecutive sequence of $k$-low symbols (or $k$-high symbols, respectively) in $\pi$. The $k$-landscape of $\pi$ is a word over the alphabet $\{L, k, H\}$ obtained from $\pi$ by replacing each $k$-low cluster with a single symbol $L$ and each $k$-high cluster with a single symbol $H$.

A word $w$ over the alphabet $\{L, k, H\}$ is called $k$-landscape word if it satisfies the following conditions:

- The first symbol of $w$ is $L$, the second symbol of $w$ is $k$.
- No two symbols $L$ are consecutive in $w$, no two symbols $H$ are consecutive in $w$.

Clearly, the landscape of a partition is a landscape word.

Two $k$-landscape words $w$ and $w'$ are said to be compatible, if each of the three symbols $\{L, k, H\}$ has the same number of occurrences in $w$ as in $w'$.

We will often drop the prefix $k$ from these terms, if the value of $k$ is clear from the context.

To give an example, consider $\pi = 1231323142221$: it has five 3-low clusters, namely 12, 1, 2, 1 and 2221, it has one 3-high cluster 4, and its 3-landscape is L3L3L3LHL.

If $w$ and $w'$ are two $k$-landscape words, we have a natural bijection between partitions with landscape $w$ and partitions with landscape $w'$: if $\pi$ has landscape $w$, we map $\pi$ to the partition $\pi'$ of landscape $w'$ which has the same $k$-low clusters and $k$-high clusters as $\pi$, and moreover, the $k$-low clusters appear in the same order in $\pi$ as in $\pi'$, and also the $k$-high clusters appear in the same order in $\pi$ as in $\pi'$. It is not difficult to check that these rules define a unique sequence $\pi'$ and this sequence is indeed a partition. This provides a bijection between partitions of landscape $w$ and partitions of landscape $w'$ which will be called the $k$-shuffle from $w$ to $w'$.

The key property of shuffles is established by the next lemma.

**Lemma 34.** Let $w$ and $w'$ be two compatible $k$-landscape words. Let $\pi$ be a partition with $k$-landscape $w$, let $\sigma$ be a pattern from $\Sigma$, let $\pi'$ be the partition obtained from $\pi$ by the shuffle from $w$ to $w'$, let $j$ be an integer. The following holds:

1. If $\sigma$ does not end with the symbol 1 and $j > k$, then $\pi'$ contains $\sigma$ at level $j$ if and only if $\pi$ contains $\sigma$ at level $j$.
2. If $\sigma$ does not end with the symbol 3 and $j < k$, then $\pi'$ contains $\sigma$ at level $j$ if and only if $\pi$ contains $\sigma$ at level $j$.

**Proof.** We begin with the first claim of the lemma. Let us choose $\sigma \in \Sigma^-$ (the case of $\sigma \in \Sigma^+$ is analogous) and let us fix $j > k$. Let us write $\sigma = 12^{p+1}32^{q}12^r$, with $r > 0$ by the assumption of the lemma. Assume that $\pi$ contains $\sigma$ at level $j$. In particular, $\pi$ has a subsequence $S = lj^{p+1}hj^qlj^r$, with $l < j < h$.

We distinguish two cases: first, if $k < l$, then all the symbols of $S$ are $k$-high. Since the shuffle preserves the relative order of high symbols, $\pi'$ contains the subsequence $S$ as well. Second, if $l \leq k$, then the shuffle preserves the relative order of the symbols $j$ and $h$, which are all high. Let $x$ and $y$ be the two symbols of $S$ directly adjacent to the second occurrence of $l$ in $S$ (if $q > 0$, both these symbols are equal to $j$, otherwise one of them is equal to $h$ and the other to $j$). The two symbols are both high, but they must appear in different $k$-high clusters. After the shuffle, the two symbols $x$ and $y$ will again be in different clusters, separated by a non-high symbol $l' \leq k$, and since the first occurrence of
Let $\pi'$ be a $k$-hybrid with landscape $w$. If $\pi$ has less than $t+1$ occurrences of $k$, then it is also a $(k+1)$-hybrid and we put $f_k(\pi) = \pi$. Otherwise, we write $w = xyz$, where $x$ is the shortest prefix of $w$ that has $p+1$ symbols $k$ and $z$ is the shortest suffix of $w$ that has $r$ symbols $k$. By assumption, $x$ and $z$ do not overlap (although they may be adjacent if $q = 0$). Let $\overline{y}$ be the word obtained by reversing the order of the letters of $y$, define $w' = x\overline{y}z$. Note that $w'$ is a landscape word compatible with $w$, and that $w$ avoids $k^{p+1}Lk^qHk^r$ if and only if $w'$ avoids $k^{p+1}Hk^qLk^r$. We apply to $\pi$ the shuffle from $w$ to $w'$ which transforms it into a partition $\pi' = f_k(\pi)$.

Using Lemma 34 it is easy to check that $\pi'$ is a $(k+1)$-hybrid, which shows that $f_k$ is the required bijection.

Another result in the same spirit is the following lemma.

**Lemma 36.** For every $p, q, r \geq 0$, the pattern $\sigma = 12^{p+1}32^{r+1}$ is equivalent to the pattern $\sigma' = 12^p12^q32^r$.

**Proof.** We follow the same argument as in Lemma 35. As before, a $k$-hybrid is a partition that avoids $\sigma'$ at every level $j < k$ and that avoids $\sigma$ at every level $j \geq k$. We will present a bijection $f_k$ between $k$-hybrids and $(k+1)$-hybrids.

Note that $\pi$ avoids $\sigma$ at level $k$ if and only if its landscape $w$ avoids $k^{p+2}Lk^qHk^r$.

Fix a $k$-hybrid $\pi$ with a landscape $w$. If $\pi$ has less than $p+2+q+r$ occurrences of $k$, then it is also a $(k+1)$-hybrid and we define $f_k(\pi) = \pi$; otherwise, we write $w = xSy$ where $x$ is the shortest prefix of $w$ that has $p+1$ occurrences of $k$, $z$ is the shortest suffix with $r$ occurrences of $k$, $S$ is the subword that starts just after the $(p+1)$-th occurrence of $k$ and ends immediately after the $(p+2)$-th occurrence of $k$. We define $w' = xy\overline{S}z$, where $\overline{S}$ is the reversal of $S$.

Note that in the definition of $w'$, we need to take $w' = xy\overline{S}z$ instead of the seemingly more natural definition $w = xyS$. This is because in general, the string $xyS$ need not be a landscape word, since it may contain to consecutive occurrences of either L or H. Our definition guarantees that $w'$ is a
correct landscape word, and that \( w' \) avoids \( k^{p+1}Lk^qHk^{r+1} \) if and only if \( w \) avoids \( k^{p+2}Lk^qHk^r \) (which is if and only if \( y \) avoids \( Lk^qH \)).

The rest of the argument is the same as in the previous lemma. \( \square \)

We are now ready to state and prove the main result of this subsection.

**Theorem 37.** All the patterns in the set \( \Sigma \) are equivalent.

**Proof.** By Corollary 29, we already know that for any \( p, q \geq 0 \), the pattern \( 12^p12^q3 \) is equivalent to the pattern \( 12^{p+q+1} \). This, together with the two previous lemmas gives the required result by transitivity. \( \square \)

### 3.7. More ‘landscape’ patterns.

We will show that with a little bit of additional effort, the previous argument involving landscapes can be adapted to prove, for every \( p, q \geq 0 \), the following equivalences:

- \( 123^p142^q \sim 123^{p+q}42^q \)
- \( 123^p412^q \sim 123^{p+q}42^q \)
- \( 123^{p+1}413^q \sim 1234^p31^q \)
- \( 123^{p+1}413^q \sim 123^{p+q+1}41^q \)

Throughout this subsection, we will say that \( \tau \) is a 1-2-4 pattern if \( \tau \) has the form \( 123S \) where \( S \) is a sequence that has exactly one occurrence of the symbol 1, exactly one occurrence of the symbol 4, and all its remaining symbols are equal to 2, and furthermore, the symbol 4 is neither the first nor the last symbol of \( S \). Similarly, a 1-3-4 pattern is a pattern of the form \( 123S \) where \( S \) has one occurrence of 1 and of 4, and all its other symbols are equal to 3, and furthermore, the symbol 1 is not the last symbol of \( S \).

We have decided to exclude the patterns of the form \( 123^p12^q4 \), \( 1234^p12^q4 \) and \( 1233^p43^q1 \) from the set of 1-2-4 and 1-3-4 patterns defined above, because some of the arguments we will need in the following discussion (namely in Lemma 38) would become more complicated if these special types of patterns were allowed. We need not be too concerned about this constraint, because we have already dealt with the patterns of the three excluded types in Corollary 29 and Theorem 31.

For our arguments, we need to extend some of the terminology of the previous subsection to cover the new family of patterns. Let \( \tau \) be a 1-2-4 pattern, let \( k \) be a natural number, and let \( \pi \) be a partition. We say that \( \pi \) contains \( \tau \) at level \( k \), if \( \pi \) has a subsequence \( T \) order-isomorphic to \( \tau \) such that the occurrences of the symbol 2 in \( \tau \) correspond to the occurrences of the symbol 4 in \( T \). Similarly, if \( \tau \) is a 1-3-4 pattern, we say that a partition \( \pi \) contains \( \tau \) at level \( k \) if \( \pi \) has a subsequence \( T \) order-isomorphic to \( \tau \) with the symbol 3 in \( T \) corresponding to the symbol 3 in \( \tau \).

Our aim is to prove an analogue of Lemma 34 for 1-2-4 and 1-3-4 patterns. Unfortunately, general \( k \)-shuffles may behave badly with respect to the avoidance of these patterns. However, we will define special types of \( k \)-shuffles that have the properties we need. We first introduce some new definitions.

Let \( w \) be a \( k \)-landscape word. We say that two occurrences of the symbol \( H \) in \( w \) are separated if there is at least one occurrence of \( L \) between them. Similarly, two symbols \( L \) are separated if there is at least one \( H \) between them. As an example, consider the \( k \)-landscape word \( w = LkLkHkHkHLkH \). In \( w \), neither the first two occurrences of \( L \) nor the first two occurrences of \( H \) are separated; however, the second and third occurrence of \( H \), as well as the second and third occurrence of \( L \) are separated. We also say that two \( k \)-high clusters of a partition are separated if there is at least one low cluster between them and similarly, two low clusters are separated if there is a high cluster between them.
Let $w$ and $w'$ be two $k$-landscape words. We say that $w$ and $w'$ are $H$-compatible if they are compatible, and moreover, they have the property that for any $i,j$, the $i$-th and $j$-th occurrence of $H$ in $w$ are separated if and only if the $i$-th and $j$-th occurrence of $H$ in $w'$ are separated. An L-compatible pair of words is defined analogously.

For example, the two compatible words $w = LkHkkHL$ and $w' = LkHkLHk$ are L-compatible (since the two occurrences of $L$ are separated in both words) but they are not H-compatible (the two symbols $H$ are not separated in $w$ but they are separated in $w'$).

We are now ready to prove the following key lemma.

**Lemma 38.** Let $k$ be an integer. The following holds:

1. Let $w$ and $w'$ be two L-compatible $k$-landscape words, and let $\tau$ be a 1-2-4 pattern. Let $\pi$ be an arbitrary partition, and let $\pi'$ be the partition obtained from $\pi$ by the $k$-shuffle from $w$ to $w'$. For every $j < k$, $\pi$ contains $\tau$ at level $j$ if and only if $\pi'$ contains $\tau$ at level $j$. Moreover, if the last symbol of $\tau$ is equal to 2, then the previous equivalence also holds for every $j > k$.

2. Let $w$ and $w'$ be two H-compatible $k$-landscape words, and let $\tau$ be a 1-3-4 pattern. Let $\pi$ be an arbitrary partition, and let $\pi'$ be the partition obtained from $\pi$ by the $k$-shuffle from $w$ to $w'$. For every $j > k$, $\pi$ contains $\tau$ at level $j$ if and only if $\pi'$ contains $\tau$ at level $j$. Moreover, if the last symbol of $\tau$ is equal to 3, then the previous equivalence also holds for every $j < k$.

**Proof.** We first prove (1). Assume that $\pi$ contains a 1-2-4 pattern $\tau$ at level $j$. If $j > k$, it is easy to see that the occurrence of $\tau$ is preserved by the shuffle as long as $\tau$ does not end with a 1: we may use the same argument as in the proof of the first part of Lemma 34. Assume now that $j < k$. Let us write $\tau = 1232^p42^q12^r$ (the case when $\tau$ has the form $1232^p12^q42^r$ is analogous). By assumption, $\pi$ contains a subsequence $T$ order-isomorphic to $\tau$, with the symbol 2 of $\tau$ corresponding to the symbol $j$ in $T$. Let us label the $1 + p + q + r$ occurrences of $j$ in $T$ by $j_0, j_1, \ldots, j_{p+q+r}$, in their natural left-to-right order. Let $a < b < c$ denote the symbols of $T$ that correspond respectively to the symbols 1, 3 and 4 in $\tau$; we label the two occurrences of $a$ in $T$ by $a_0$ and $a_1$. With this notation, we may write $T$ as follows:

$$T = a_0j_0b_1 \cdots j_p c_{j_{p+1}} \cdots j_{p+q} a_1 j_{p+q+1} \cdots j_{p+q+r}.$$ 

Now, we distinguish several cases, based on the relative order of $b, c$ and $k$: If $c < k$, then all the symbols of $T$ are $k$-low and their relative position is preserved by the shuffle, which means that $T$ is also a subsequence of $\pi'$.

If $c = k$, then the symbols $a < j < b$ are $k$-low. Let $x$ and $y$ be the two symbols adjacent to $c$ in $T$ (typically $x = j_p$ and $y = j_{p+1}$, unless $q$ is zero, in which case $y = a_1$; recall that $c$ cannot directly follow $b$ and it cannot be the last element of $T$ by the definition of 1-2-4 pattern). The elements $x$ and $y$ are low and they appear in two distinct low clusters. After the shuffle, the occurrences of $a, b$ and $j$ in $T$ have the same relative order, and the elements $x$ and $y$ still belong to different clusters, which means that $\pi'$ contains a symbol greater than $b$ between $x$ and $y$. This shows that $\pi'$ has a subsequence order-isomorphic to $\tau$.

If $c > k$ and $b < k$, the argument from the previous paragraph applies as well.

It remains to consider the most complicated case: $c > k$ and $b \geq k$. This is when we first use the L-compatibility assumption. Since $b$ is not $k$-low, we know that $j_1$ does not belong to the leftmost low cluster. Let $x$ and $y$ be the two symbols adjacent to $c$ in $T$; by the definition of 1-2-4 patterns, $x$ and $y$ are both $k$-low.

We know that $x$ and $y$ belong to distinct low clusters, and that their clusters are separated, since $c$ is high. The shuffle preserves these properties; in particular, in $\pi'$, the symbol $j_1$ does not belong to the
leftmost low cluster, which means that there is at least one non-low symbol appearing in \( \pi' \) before \( j_1 \). Since \( \pi' \) is a partition in its canonical sequential form, this implies that all the symbols 1, 2, \ldots, \( k \) appear in \( \pi' \) in this order before \( j_1 \). Let \( a', j' \) and \( k' \) denote respectively the leftmost occurrences of \( a, j \) and \( k \) in \( \pi' \). We also know, from the L-compatibility of \( w \) and \( w' \), that in \( \pi' \) the two symbols \( x \) and \( y \) appear in distinct and separated low clusters. In particular, \( \pi' \) contains a \( k \)-high symbol \( c' \) between \( x \) and \( y \). Putting it all together, we see that \( \pi' \) contains the subsequence

\[ T' = a'j'k'j_1 \cdots j_p c'j_{p+1} \cdots j_{p+q}a_1j_{p+q+1} \cdots j_{p+q+r}, \]

which is order isomorphic to \( \tau \).

In all the cases, we see that if \( \pi \) contains a 1-2-4 pattern \( \tau \) at level \( j \), then \( \pi' \) contains the same pattern at the same level as well. The same proof also applies to the reverse shuffle from \( \pi' \) to \( \pi \). This completes the proof of (1).

Claim (2) is proved by a similar argument. Let \( \tau \) be a 1-3-4 pattern of the form \( 123^{p+1}13^{q}43^{r} \) (the case when \( \tau = 123^{p+1}43^{q}13^{r} \) is analogous and easier). Assume that \( \pi \) contains \( \tau \) at level \( j \), witnessed by a sequence \( T \) of the form

\[ T = a_0bja_1j_1 \cdots j_p a_1j_{p+1} \cdots j_{p+q}c_j j_{p+q+1} \cdots j_{p+q+r}, \]

with \( a < b < j < c \).

If \( j < k \), we apply the same argument as in the proof of the second claim of Lemma 34 to prove that if \( \tau \) does not end with 4, the occurrence of \( \tau \) is preserved by the shuffle.

Next, we assume that \( j > k \) and we distinguish several cases based on the relative order of \( a, b \) and \( k \).

If \( a > k \), then all the symbols of \( T \) are \( k \)-high and their order is preserved by the shuffle.

If \( a = k \), or if \( a < k \) and \( b > k \), we let \( x \) and \( y \) denote the two symbols adjacent to \( a_1 \) in \( T \), and we observe that \( \pi' \) has a non-high element \( a' \) between \( x \) and \( y \). The first occurrence of \( a' \) in \( \pi' \) must appear to the left of any \( k \)-high symbol, hence \( \pi' \) contains a subsequence \( a'b_j^{p+1}a'j^q c_j^r \) order-isomorphic to \( \tau \).

If \( a < k \) and \( b \leq k \), we define \( x \) and \( y \) as in the previous paragraph. This time, \( x \) and \( y \) belong to two separated high clusters, so \( \pi' \) has a \( k \)-low element \( a' \) between \( x \) and \( y \), and in particular, \( \pi' \) contains the subsequence \( a'k_j^{p+1}a'j^q c_j^r \). \( \square \)

With the help of Lemma 35, we may prove all the equivalence relations announced at the beginning of this section. We split the proofs into four lemmas and then summarize the results in a theorem.

**Lemma 39.** Let \( p, q \geq 1 \). The pattern \( \tau = 123^{p+1}412^{q} \) is equivalent to \( \tau' = 123^{q}42^{q}1 \).

**Proof.** For an integer \( k \) we say that a partition \( \pi \) is a \( k \)-hybrid if \( \pi \) avoids \( \tau \) at level \( j \) for every \( j \geq k \) and it avoids \( \tau' \) at level \( j \) for every \( j < k \). To prove the claim, it is enough to establish a bijection \( f_k \) between \( k \)-hybrids and \((k + 1)\)-hybrids.

We say that a \( k \)-high cluster of \( \pi \) is extra-high if it contains a symbol greater than \( k + 1 \). We claim that \( \pi \) contains \( \tau \) at level \( k \) if and only if by scanning \( \pi \) in the left-to-right direction we may find (not necessarily consecutively) the leftmost high cluster, followed by \( p \) occurrences of the symbol \( k \), followed by an extra-high cluster, followed by a low cluster, followed by \( q \) occurrences of \( k \). To see this, it suffices to notice that the leftmost high cluster contains the symbol \( k + 1 \), and to the left of this cluster we may always find all the symbols \( 12 \cdots k \) in the increasing order.

By a similar argument, we see that \( \pi \) contains \( \tau' \) at level \( k \) if and only if it contains, left-to-right, the leftmost high-cluster, \( p \) occurrences of \( k \), an extra-high cluster, \( q \) occurrences of \( k \) and a low cluster.
Now assume that π is a k-hybrid partition. Let us try to find the leftmost extra-high cluster H′ of π with the property that between H′ and the leftmost high cluster of π there are at least p occurrences of k. If no such cluster exists, or if π has less than q symbols equal to k to the right of H′, then π avoids both τ and τ′ at level k, and we define $f_k(\pi) = \pi$.

Otherwise, let $w$ be the k-landscape of π. We will decompose $w$ into the following concatenation:

$$w = xH'yk_qs_1k_{q-1}s_2\cdots s_q,$$

where $H'$ represents the extra-high cluster defined above, $x$ is the prefix of $w$ ending just before $H'$, $k_i$ represents the i-th symbol k in π, counted from the right, $y$ is the subword of $w$ between $H'$ and $k_q$, and $s_i$ is the subword of $w$ between $k_{q-i+1}$ and $k_{q-i}$ with $s_q$ being equal to the suffix of $w$ to the right of $k_1$. By construction, none of the $s_i$'s contains the symbol k, so each of them is an alternating sequence over the alphabet {L, H}, possibly empty. Since π avoids τ at level k, the subword $y$ does not contain the symbol L.

As the next step, we decompose $s_1$ into two parts $s_1 = H'S_1^-$ as follows: if the first letter of $s_1$ is H, then we put $H^* = H$ and $S_1^-$ is equal to $s_1$ with the first letter removed; on the other hand, if $s_1$ does not start with H, then $H^*$ is the empty string and $S_1^- = s_1$.

Now, let us define the word $w'$ as follows:

$$w' = xH'S_1^-k_1s_2k_2s_3k_3\cdots k_{q-1}s_qk_qH^*y.$$

It is not difficult to check that $w'$ is a landscape word (note that neither y nor $s_1^-$ can start with the symbol H), and that $w'$ is L-compatible with w (recall that y contains no L).

Let $\pi'$ be the partition obtained from $\pi$ by the shuffle from $w$ to $w'$.

Note that the prefix of $\pi$ up to the cluster $H'$ (inclusive) is not affected by the shuffle, because the words $w$ and $w'$ share the same prefix up to the symbol $H'$. In particular, the shuffle preserves the property that $H'$ is the leftmost extra-high cluster with at least p symbols k between $H'$ and the leftmost high cluster of $\pi'$. It is routine to check that $\pi'$ avoids $\tau'$ at level k. By Lemma 38, $\pi'$ is a $(k + 1)$-hybrid partition. With these observations, it is easy to see that for any given $(k + 1)$-hybrid partition $\pi'$, we may uniquely invert the procedure above and obtain a k-hybrid partition $\pi$.

Defining $f_k(\pi) = \pi'$, we obtain the required bijection between k-hybrids and $(k + 1)$-hybrids. □

The proofs of the following three lemmas follow the same basic argument as the proof of Lemma 39 above. The only difference is in the decompositions of the corresponding landscape words $w$ and $w'$.

We omit repeating the common parts of the arguments and concentrate on pointing out the differences.

**Lemma 40.** Let $p, q \geq 1$. The pattern $\tau = 123^p2142^q$ is equivalent to $\tau' = 1231^{p+1}42^q$.

**Proof.** A partition $\pi$ contains $\tau$ at level k if and only if it contains, in left-to-right order, the leftmost high cluster, $p$ copies of k, a low cluster, an extra-high cluster, and $q$ copies of k. Similar characterization applies to $\tau'$.

Let $H_1$ denote the leftmost high cluster of $\pi$, let $H'$ denote the rightmost extra-high cluster of $\pi$ that has the property that there are at least q occurrences of k to the right of $H'$. If $H'$ does not exist, or if there are less than p symbols k between $H_1$ and $H'$, then $\pi$ contains neither $\tau$ nor $\tau'$ at level k and we put $f_k(\pi) = \pi$. Otherwise, let $w$ be the landscape of $\pi$, and let us write

$$w = xH_1s_1k_1s_2k_2\cdots s_ps_qyH'z$$

$$w = xH'yk_qs_1k_{q-1}s_2\cdots s_q,$$
where none of the $S_i$ contains $k$, and $y$ avoids $L$. Define $S_p^-$ and $H^*$ by writing $S_p = S_p^- H^*$ where $S_p^-$ does not end with the letter $H$ and $H^*$ is equal either to $H$ or to the empty string, depending on whether $S_p$ ends with $H$ or not.

Now we write
\[ w = xH_1y_1k_1S_1k_2S_2 \ldots k_pS_p^- H'z, \]
where $\overline{y}$ is the reversal of $y$. The rest of the proof is analogous to Lemma 40. \hfill \Box

We now apply the same arguments to 1-3-4 patterns.

**Lemma 41.** For any $p \geq 0$ and $q \geq 1$, the pattern $\tau = 123^{p+1}13^q4$ is equivalent to the pattern $\tau' = 123^{p+1}143^q$.

**Proof.** As usual, a $k$-hybrid is a partition that avoids $\tau$ at every level $j \geq k$ and that avoids $\tau'$ at every level below $k$.

Let us say that a $k$-cluster of a partition $\pi$ is extra-low if it contains a symbol smaller than $k-1$. A partition contains $\tau$ at level $k$ if and only if it has $p+1$ occurrences of $k$ followed by an extra-low cluster, followed $q$ symbols $k$, followed by a high cluster; similarly, a partition contains $\tau'$ at level $k$ if and only if it has $p+1$ copies of $k$, followed by an extra-low cluster, followed by a high cluster, followed by $q$ copies of $k$.

Assume $\pi$ is a $k$-hybrid partition. Let $L'$ denote the leftmost extra-low cluster of $\pi$ that has at least $p+1$ copies of $k$ to its left. If $L'$ does not exist, or if it has less than $q$ copies of $k$ to its right, we put $f_k(\pi) = \pi$; otherwise, we decompose the landscape word $w$ of $\pi$ as follows:
\[ w = xL'S_1k_1S_2k_2 \ldots S_{q-1}k_{q-1}S_qk_0y, \]
where the $S_i$ do not contain $k$, and by assumption, $y$ avoids $H$. Next, we write $y = L' y^-$ where $L^*$ is an empty string or a single symbol $L$, and $y^-$ does not start with $L$. We define $w'$:
\[ w' = xL'y^- k_1L^* S_1k_2 \ldots S_{q-1}k_qS_q. \]

The words $w$ and $w'$ are $H$-compatible. We define the bijection between $k$-hybrids and $(k+1)$-hybrids in the usual way. \hfill \Box

**Lemma 42.** For every $p \geq 0$ and $q \geq 1$, the pattern $\tau = 123^{p+1}1413^q$ is equivalent to the pattern $\tau' = 12343^p13^q$.

**Proof.** As before, take $\pi$ to be a $k$-hybrid partition. Let $L'$ be the rightmost extra-low cluster that has at least $q$ copies of $k$ to its right. If $L'$ has at least $p+1$ copies of $k$ to its left, we perform the following decomposition of the landscape word $w$ of $\pi$:
\[ w = Lk_1S_1k_2S_2 \ldots k_pS_p k_{p+1}yL'z. \]

Next, we write $S_p = S_p^- L^*$ with the usual meaning and define
\[ w' = Lk_1L^* yk_2S_1k_3S_2 \ldots S_{p-1}k_{p+1}S_p^- L'z. \]

The rest is the same as before. \hfill \Box

We summarize our results:

**Theorem 43.** For every $p, q \geq 0$, we have the following equivalences:

1. $123^{p+2}142^q \sim 1231^{p+2}42^q$
2. $123^{p+1}412^q \sim 1232^{p+2}42^q1$
Proof. If \( p \) and \( q \) are both positive, the results follow directly from the four preceding lemmas. If \( p = 0 \), the first and the third claim are trivial, the second one is a special case of Corollary 29, and the fourth is covered by Lemma 41. If \( q = 0 \), the first claim is a special case of Corollary 29, the second and the fourth are trivial, and the third follows from Theorem 31. □

4. The avoidance of four-letter patterns

In this section, we will complete the classification of the equivalence classes of the patterns of length four, by proving the equivalence \( 1212 \sim 1123 \). Unlike in the previous arguments, we do not present a direct bijection between pattern-avoiding classes, but rather we prove that \( p(n;1123) \) is equal to the \( n \)-th Catalan number. Since it is well known that noncrossing partitions are enumerated by the \( n \)-th Catalan number as well, this will yield the desired equivalence. All the other equivalent pairs of patterns of length four are covered by the general theorems proved in the previous parts of the paper (see Table 2).

\[
\begin{array}{|c|c|}
\hline
\tau & p(n;\tau) \\
\hline
1111 & 21 \text{ Sequence A001680} \text{ (see Equation (1))} \\
1112, 1121, 1211, 1222 & 21 \text{ Sequence A005425} \text{ (see Corollary 29)} \\
1122 & 1, 1, 2, 5, 14, 42, 133, 441, \ldots \\
1123, 1212, 1221 & \frac{1}{n+1} \binom{2n}{n} \\
1213, 1223, 1231, 1232, 1233, 1234 & 21 \text{ Sequence A007051} \text{ (see Equation (2))} \\
\hline
\end{array}
\]

Table 2. Number of partitions in \( P(n;\tau) \), where \( \tau \in P(4) \).

4.1. Enumeration of 1123-avoiding partitions. As we said before, our aim is to show that the 1123-avoiding partitions of \([n]\) are enumerated by the \( n \)-th Catalan number, i.e., \( p(n;1123) = \frac{1}{n+1} \binom{2n}{n} \).

We achieve this by proving that 1123-avoiding partitions of \( n \) are in bijection with Dyck paths of semilength \( n \). A Dyck path of semilength \( n \) is a nonnegative path on the two-dimensional integer lattice from \((0,0)\) to \((2n,0)\) composed of up-steps connecting \((x,y)\) to \((x+1,y+1)\) and down-steps connecting \((x,y)\) to \((x+1,y-1)\). It is well known that these paths are enumerated by Catalan numbers.

We will need the following refinement: let \( T(n,k) \) denote the set of 1123-avoiding partitions \( \pi \) of the set \([n]\) with the property that \( \pi_n = k \). Let \( t(n,k) \) be the cardinality of \( T(n,k) \). We will prove that \( t(n,k) \) is equal to the number of Dyck paths of semilength \( n \) whose last up-step is followed by exactly \( k \) down-steps.

Let \( T'(n,k) \) denote the set of Dyck paths of semilength \( n \) such that their last up-step is followed by exactly \( k \) down-steps, let \( t'(n,k) \) be the cardinality of \( T'(n,k) \). We remark that standard bijections between Dyck paths and pattern-avoiding permutations show that \( t'(n,k) \) is also equal to the number of 123-avoiding permutations \( (\pi_1,\pi_2,\ldots,\pi_n) \) such that \( \pi_n = k \). Our aim is to prove the following result:
Theorem 44. For every $n, k$, $t(n, k)$ is equal to $t'(n, k)$.

Before starting the proof of the theorem, we introduce more definitions.

Definition 45. A 123-avoiding sequence is a sequence $s_1, s_2, \ldots, s_\ell$ of positive integers, such that there are no three indices $i < j < k$ that would satisfy $s_i < s_j < s_k$. We define the rank of a sequence to be equal to $\ell + m - 1$, where $\ell$ is the length of the sequence and $m = \max\{s_i, i = 1, \ldots, \ell\}$ is the largest element of the sequence.

For example, there are five 123-avoiding sequences of rank 3: those are the sequences $(1,1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, and $(2,2)$. There are fourteen 123-avoiding sequences of rank 4: $(1,1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(1,3)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$, $(2,2,2)$, $(2,3)$, $(3,1)$, $(3,2)$, $(3,3)$, and $(4)$.

The proof of Theorem 44 is divided into the following two claims:

Claim 46. A 1123-avoiding partition $\pi$ of $[n]$ with $m$ blocks has the following form:

$$\pi = 123 \cdots (m - 2)(m - 1)S$$

where $S$ is a 123-avoiding sequence of rank $n$, with maximum element $m$. Conversely, if $S$ is any 123-avoiding sequence of rank $n$ with maximum element $m$ then $\pi$ defined by the formula (5) is a canonical sequence of a 1123-avoiding partition of $[n]$.

In particular, the number of 123-avoiding sequences of rank $n$ with last element $k$ is equal to $t(n, k)$.

Claim 47. The numbers $t(n, k)$ satisfy the following recurrences:

$$t(1, 1) = 1$$

$$t(n, k) = 0 \text{ if } k < 1 \text{ or } k > n$$

$$t(n, k) = \sum_{j=k-1}^{n-1} t(n - 1, j) \text{ for } n \geq 2, n \geq k \geq 1$$

It is easy to see the recurrences (6),(7) and (8) would all hold if $t$ were replaced by $t'$: given a Dyck path from $T'(n, k)$, we erase its last up-step and the following down-step, to obtain a Dyck path from $\cup_{j=k-1}^{n-1} T'(n - 1, j)$; in particular, Claim 47 implies that $t(n, k) = t'(n, k)$. Thus, the two claims also show that the number of 123-avoiding sequences of rank $n$ is exactly the $n$-th Catalan number.

Proof of Claim 46 Let $\pi$ be a 1123-avoiding partition of $[n]$ with $m$ blocks. Observe that for every $i \in [m - 1]$, the symbol $\pi_i$ is equal to $i$, otherwise $\pi$ would contain the forbidden pattern. It follows that $\pi$ can be decomposed according to the equality $\pi = 123 \cdots (m - 2)(m - 1)S$, where the sequence $S$ has length $l = n - m + 1$ and maximum element equal to $m$, hence $S$ has rank $n$. Also, the last element of $S$ is equal to $k$.

It remains to check that $S$ is 123-avoiding: if $S$ contained a subsequence $x y z$ for $x < y < z$ then the original partition would contain a subsequence $x y z$, which is forbidden. It follows that $S$ obtained from a 1123-avoiding partition $\pi$ has all the required properties.

The “converse” part of the claim is equally easy to verify, and we omit it. \hfill \square

Claim 46 motivates the following definition:

Definition 48. Let $\pi$ be a 1123-avoiding partition of $[n]$ with $m$ blocks. The 123-avoiding sequence $S$ obtained from the decomposition according to the formula (5) will be called the tail of $\pi$. Let $T_0(n, k)$ be the set of all the tails of the partitions from $T(n, k)$; equivalently, $T_0(n, k)$ is the set of 123-avoiding sequences of rank $n$ with the last element equal to $k.$
We are now ready to prove Claim 47.

Proof of Claim 47. Only the recurrence (8) is nontrivial. To prove the recurrence, we need a bijection from \( T(n, k) \) to \( \cup_{j=k-1}^{n-1} T(n-1, j) \). It is more convenient to work with the tails of the partitions, i.e., to describe a bijection between \( T_0(n, k) \) and \( \cup_{j=k-1}^{n-1} T_0(n-1, j) \). We will construct the required bijection as the union of two injective maps \( f_1 \) and \( f_2 \) with the property that the domains of \( f_1 \) and \( f_2 \) form a disjoint partition of \( T_0(n, k) \) and their ranges form a disjoint partition of \( \cup_{j=k-1}^{n-1} T_0(n-1, j) \).

Let \( S \in T_0(n, k) \) be a 123-avoiding sequence of length \( \ell \). The sequence \( S \) can be uniquely decomposed into a concatenation of the form \( S = S_01^k, \) where \( S_0 \) is a (possibly empty) prefix of \( S \) whose last element is different from 1.

We distinguish the following two cases:

Case 1. If \( S_0 \) is nonempty and the last element of \( S_0 \) is greater than or equal to \( k \), we define \( S' = f_1(S) = S_0(k-1)^b \). It is easy to see that \( f_1 \) is injective. By construction, the length of \( S' \) is equal to \( \ell - 1 \) and the maximum of \( S' \) has the same value as the maximum of \( S \), so \( S' \) has rank \( n - 1 \). It is also easy to check that \( S' \) is 123-avoiding, and that \( S' \in \cup_{j=k-1}^{n-1} T_0(n-1, j) \).

Case 2. We now deal with the case when \( S_0 \) is empty, or the last element of \( S_0 \) is smaller than \( k \). We first observe that all the elements of \( S_0 \) are greater than 1. Indeed, the last element of \( S_0 \) is never equal to 1 by definition, and if \( S_0 \) contained an element 1 before the last one, then \( S \) would contain a subsequence \( 1jk \), where \( j \) is the last element of \( S_0 \) and \( k \) the last element of \( S \); however, this is impossible, because \( S \) avoids 123. Now, we define \( S' = f_2(S) = (S_0-1)(k-1)^{b+1}, \) where \( S_0-1 \) denotes the sequence \( S_0 \) with all its elements decreased by 1. Note that the last symbol of \( S_0-1 \) is smaller than \( k-1 \), which implies that \( f_2 \) is an injective map. Clearly, the length of \( S' \) is \( \ell \) and the maximum of \( S' \) is one less than the maximum of \( S \), so \( S' \) has rank \( n - 1 \). Also, \( S' \) is easily seen to be 123-avoiding, and \( S' \in T_0(n-1, k-1) \).

To finish the proof, we need to check that every sequence \( S' \in \cup_{j=k-1}^{n-1} T_0(n-1, j) \) is in the range of exactly one of the two injections \( f_1 \) and \( f_2 \). To see this, express \( S' \) as the concatenation \( S' = S_0'(k-1)^c, \) where \( c \geq 0 \) and \( S_0' \) is a (possibly empty) prefix of \( S' \) whose last element is different from \( k-1 \). The sequence \( S_0' \) is in the range of \( f_1 \) if and only if \( S_0' \) is nonempty and the last element of \( S_0' \) is at least \( k; \) in such case, we have \( S' = f_1(S_0'(1)^k). \) On the other hand, if \( S_0' \) is empty or if its last element is smaller than \( k - 1 \), then necessarily \( c \geq 1 \) and \( S' = f_2((S_0' + 1)^{c-1}1). \) This completes the proof.

We may use Theorem 44 to derive the closed-form expression for \( t(n, k) \). Since the number of Dyck paths that end with an up-step followed by \( k \) down-steps is equal to the number of non-negative lattice paths from \( (0, 0) \) to \( (2n - k - 1, k - 1) \), we may apply standard arguments for the enumeration of non-negative lattice paths to obtain the formula

\[
t(n, k) = \frac{k}{n} \binom{2n - k - 1}{n - 1}.
\]

Let us remark that it is possible to obtain a closed form formula for \( t(n, k) \) without Theorem 44 using directly the recurrences from Claim 47. This is achieved by the kernel method techniques as described, e.g., in [16]. We omit the details here.

5. The patterns of size five

For a full characterization of the equivalence of patterns up to size seven, we need to consider one more isolated case, namely the pattern 12112. Our aim is to show that this pattern is equivalent to
the three patterns 12221, 12212, and 12122. Note that the latter three patterns are all equivalent by Corollary 29 it is thus sufficient to show that 12112 \sim 12212. The proof we are about to present is rather long and occasionally technical, however, it is well worth the effort, since the equivalence of 12112 and 12212 has several consequences related to fillings of Ferrers shapes and pattern-avoiding graphs. We will discuss these consequences in greater detail at the end of this section.

We remark that contrary to the case of the three equivalent patterns 12221 \sim 12212 \sim 12122, whose equivalence was obtained as a consequence of the Ferrers-equivalence of the corresponding matrices $M(2,221) \sim M(2,212) \sim M(2,122)$, the proof involving the pattern 12112 does not use the notion of Ferrers equivalence. In fact, the matrix $M(2,112)$ is not Ferrers-equivalent to the three matrices above.

5.1. Introduction. We will first introduce the basic terminology and notation that we will use throughout the proof.

Let $S = s_1 s_2 \ldots s_n$ be a sequence of length $n$ over the alphabet $[m]$, such that every element of $[m]$ appears in $S$ at least once. For $i \in [m]$ let $f_i$ and $l_i$ denote the index of the first and the last symbol of $S$ that is equal to $i$; formally, $f_i = \min \{ j \mid s_j = i \}$ and $l_i = \max \{ j \mid s_j = i \}$.

**Definition 49.** For $k \in [m]$, we say that the sequence $S$ is a $k$-semicanonical sequence (or $k$-sequence for short), if $S$ has the following properties:

- For every $i, i'$ such that $1 \leq i < k$ and $i < i'$, we have $f_i < f_{i'}$.
- For every $i, i'$ such that $k \leq i < i' \leq m$, we have $l_i < l_{i'}$.

Note that $m$-semicanonical sequences are precisely the canonical sequences of partitions of $[n]$ with $m$ blocks (i.e., the sequences satisfying $f_i < f_{i+1}$ for $i \in [m-1]$), while the 1-canonical sequences are precisely the sequences satisfying $l_i < l_{i+1}$ for $i \in [m-1]$. Also, for every fixed $k \in [m]$ and a fixed partition $\pi$ with $m$ blocks, there is exactly one $k$-sequence $S$ with the property $s_i = s_j \iff \pi_i = \pi_j$; this sequence $S$ can be obtained from $\pi$ by preserving the numbering of the first $k-1$ blocks of $\pi$, and by numbering the remaining $m-k+1$ blocks in the increasing order of their largest element.

In particular, assuming $n$ and $m$ are fixed, the number of $k$-sequences is independent of $k$, and each partition of $[n]$ with $m$ blocks is represented by a unique $k$-sequence. To prove the equivalence 12112 \sim 12212, we will exploit a remarkable property of the pattern 12112, described by the following key lemma.

**Lemma 50.** For every fixed $n$ and $m$, the number of 12112-avoiding $k$-sequences is independent of $k$. Thus, for every $k \in [m]$, the number of 12112-avoiding $k$-sequences of length $n$ with $m$ symbols is equal to the number of 12112-avoiding partitions of $n$ with $m$ blocks.

We stress that a 12112-avoiding $k$-sequence can actually represent a partition that contains 12112 in its canonical representation.

Before we present the proof of Lemma 50, let us explain how the lemma implies the equivalence 12112 \sim 12212.

**Theorem 51.** The pattern 12112 is equivalent to 12212. In fact, for every $m$ and $n$, there is a bijection between 12112-avoiding partitions of $[n]$ with $m$ blocks and 12212-avoiding partitions of $[n]$ with $m$ blocks.

**Proof.** Fix $m$ and $n$. We know that the 12112-avoiding partitions of $[n]$ with $m$ blocks correspond precisely to $m$-semicanonical sequences over $[m]$ of length $n$, and by Lemma 50 these sequences are in...
bijection with 1-semicanonical 12112-avoiding sequences of the same length and alphabet. It remains to provide a bijection between the 12112-avoiding 1-sequences and the 12212-avoiding partitions, which is done as follows: take a 1-semicanonical 12112-avoiding sequence $S$ with $m$ symbols and length $n$, reverse the order of letters in $S$, and then replace each symbol $i$ of the reverted sequence by the symbol $m - i + 1$. It is easy to check that this transform is an involution which maps 12112-avoiding 1-sequences onto 12212-avoiding $m$-sequences, which are precisely the 12212-avoiding partitions of $[n]$ with $m$ blocks.

It now remains to prove Lemma 50. For the rest of the proof, let us assume that $m$ and $n$ are fixed, and that each sequence we consider has length $n$ and $m$ distinct symbols, unless otherwise noted.

In the following arguments, it is often convenient to represent a sequence $S = s_1 \cdots s_n$ by the matrix $M(S, m)$ (recall that $M(S, m)$ is the 0-1 matrix with a 1-cell in row $i$ and column $j$ if and only if $s_j = i$). A matrix representing a $k$-sequence will be called $k$-semicanonical matrix (or just $k$-matrix), and a matrix representing a 12112-avoiding sequence will be simply called 12112-avoiding matrix. In accordance with earlier terminology, we will use the term sparse matrix for a 0-1 matrix with at most one 1-cell in each column, and we will use the term semi-standard matrix for a 0-1 matrix with exactly one 1-cell in each column. For a 0-1 matrix $M$, we let $f_i(M)$ and $l_i(M)$ denote the column-index of the first and the last 1-cell in the $i$-th row of $M$. We will write $f_i$ and $l_i$ instead of $f_i(M)$ and $l_i(M)$ if there is no risk of confusion.

Before we formulate the proof of our key lemma, let us present a brief sketch of the main idea. Assume we want to build a bijection that transforms a $(k + 1)$-matrix $M$ into a $k$-matrix (ignoring 12112-avoidance for a while). Such bijection is easy to obtain: assume that the last 1-cell in row $k$ is in column $c$, let us call the row $k$ the key row of $M$. If the last 1-cell in row $k + 1$ appears to the right of column $c$, then $M$ is already a $k$-matrix and we do not need to do anything; on the other hand, if row $k + 1$ has no 1-cell to the right of $c$, we swap the key row $k$ with the row $k + 1$, to obtain a new matrix $M'$ whose key row is now the row $k + 1$. We now repeat the same procedure: we compare the position of the last 1-cell in the key row $k + 1$ (which is in the column $c$, as we know) with the last 1-cell in the row $k + 2$, and if necessary, we swap the key row with the row directly above it, until we reach the situation when the key row is either the topmost row of the matrix, or the row above the key row has a 1-cell to the right of column $c$. This procedure transforms the original $k + 1$ matrix into a $k$-matrix. Also, the procedure is invertible (note that the first 1-cell of the key row is always to the left of any other 1-cell in the rows $k, k + 1, \ldots, m$).

Unfortunately, this simplistic approach does not preserve 12112-avoidance. However, we will present a more sophisticated algorithm, which follows the same basic structure as the procedure above, but instead of simply swapping the key row with the row above it, it performs a more complicated step. The description of this step is the fundamental ingredient of our proof.

To formalize our argument, we need to introduce more definitions. Let $M$ be a 0-1 matrix with exactly one 1-cell in each column and at least one 1-cell in each row, and let us write $f_i = f_i(M)$ and $l_i = l_i(M)$. Let $k, q, p$ be three row-indices of $M$, with $k \leq p \leq q$. We will say that $M$ is a $(k, p, q)$-matrix, if $M$ has the following form:

- The matrix obtained from $M$ by erasing row $p$ is a $k$-semicanonical matrix with $m - 1$ rows.
- For each $i < k$, we have $f_i < f_p$. For every $j \geq k$, $j \neq p$, we have $f_p < f_j$.
- The number $q$ is determined by the relation $q = \max\{j; l_j \leq l_p\}$. By the first condition, this implies that $l_j \leq l_p$ for every $j \in \{k, k + 1, \ldots, q\}$.

In a $(k, p, q)$-matrix, row $p$ will be called the key row.
In particular, a matrix $M$ of the key row in a given step of the procedure, while the number $k$ needs to be swapped with the key row to produce the required $k$-forms a $(M)$ corresponds to the following matrix.

As an example, consider the sequence $S = 1331232431$ with $n = 10$ and $m = 4$. This sequence corresponds to the following matrix $M = M(S, 4)$.

$$M = \begin{pmatrix} 0000000100 \\ 0110010010 \\ 0000101000 \\ 1001000001 \end{pmatrix}$$

$$M' = \begin{pmatrix} 0110010010 \\ 0000000100 \\ 0000101000 \\ 1001000001 \end{pmatrix}$$

The matrix $M$ is a $(2,3,4)$-matrix. If we exchange the third row (which acts as the key row) with the fourth row, we obtain a $(2,4,4)$-matrix $M'$ representing the 2-sequence $S' = 1441242341$. The matrix $M'$ can also be regarded as a $(1,1,4)$-matrix, with the key row at the bottom.

In general, a matrix $M$ is $(k + 1)$-semicanonical if and only if it is a $(k, k, q)$-matrix, and $M$ is $k$-semicanonical if and only if it is a $(k, q, q)$-matrix. To prove Lemma 50, we will prove the following lemma.

**Lemma 52.** For arbitrary $k \leq p < q$, there is a bijection $\phi$ between 12112-avoiding $(k, p, q)$-matrices and 12112-avoiding $(k, p + 1, q)$-matrices.

Clearly, Lemma 52 implies Lemma 50. Before we construct the bijection $\phi$ and prove its correctness, we need to prove several basic properties of the 12112-avoiding $(k, p, q)$-matrices.

### 5.2. Tools of the proof

We will use the following terminology: if $x \in [m]$ is a row of a matrix $M$, then an $x$-column is a column of $M$ that has a 1-cell in row $x$. Similarly, if $X \subseteq [m]$ is a set of rows of $M$, we will say that a column $j$ is an $X$-column if $j$ has a 1-cell in a row belonging to $X$.

If $x, y$ is a pair of rows of $M$ with $x < y$, we will say that $M$ contains 12112 in $(x, y)$ if the submatrix of $M$ induced by the pair of rows $x, y$ contains 12112. If $X$ and $Y$ are two sets of rows, we will say that $M$ contains 12112 in $(X, Y)$ if there is an $x \in X$ and $y \in Y$ such that $x < y$ and $M$ contains 12112 in $(x, y)$.

Throughout this section, we will assume that $k, p, q$ are fixed, and that $k \leq p < q$.

We now state a pair of simple but useful observations. Their proofs are straightforward, and we omit them.

**Observation 53.** Let $M$ be a sparse 0-1 matrix, and let $x < y$ be two rows of $M$, such that $f_x < f_y$. The matrix $M$ avoids 12112 in $(x, y)$ if and only if $M$ has at most one $x$-column $s$ satisfying $f_y < s < l_y$. If such a unique column $s$ exists, we will say that $s$ separates row $y$. The $y$-columns that are to the left of the separating column $s$ will be called front $y$-columns (with respect to row $x$) and their 1-cells will be called front 1-cells, and similarly, the $y$-columns to the right of $s$ will be called rear $y$-columns and their 1-cells are rear 1-cells. If there is no such separating column, then we will assume that all the $y$-columns and their 1-cells are front.

**Observation 54.** Let $M$ be a sparse 0-1 matrix, and let $x < y$ be a pair of rows such that $l_x < l_y$. Let $t$ be the number of 1-cells in row $x$, and let $c_i$ be the $i$-th $x$-column, i.e., $f_x = c_1 < c_2 < \cdots < c_t = l_x$. The matrix $M$ avoids 12112 in $(x, y)$, if and only if every $y$-column appears either to the left of column $l_y$. If such a unique column $l_y$ exists, we will say that $l_y$ separates row $y$. The $y$-columns that are to the left of the separating column $l_y$ will be called front $y$-columns (with respect to row $x$) and their 1-cells will be called front 1-cells, and similarly, the $y$-columns to the right of $l_y$ will be called rear $y$-columns and their 1-cells are rear 1-cells. If there is no such separating column, then we will assume that all the $y$-columns and their 1-cells are front.
$c_1$, or between the columns $c_{t-1}$ and $c_t$, or to the right of column $c_t$. These three types of $y$-columns (and their 1-cells) will be called left, middle, and right $y$-columns (or 1-cells) with respect to row $x$.

Next, we prove a lemma that will greatly simplify our task of constructing the bijection $\phi$ and proving its correctness.

**Lemma 55.** Let $M$ be a 12112-avoiding $(k, p, q)$-matrix, Let $j$ be a row of $M$ with $k \leq j \leq p$. Let $M'$ be a sparse 0-1 matrix of the same size as $M$, with the property that for every $i \notin \{j, j+1, \ldots, q\}$, the $i$-th row of $M'$ is equal to the $i$-th row of $M$. If $M'$ has a copy of the pattern 12112 in a pair of rows $x < y$, then $j \leq x \leq q$.

**Proof.** Let $M$ and $M'$ be as above. We will call the rows $\{j, j+1, \ldots, q\}$ mutable, and the remaining rows will be called constant.

Assume that $M'$ has a copy of the forbidden pattern in the rows $x < y$. Clearly, at least one of the two rows $x, y$ must be mutable, and in particular, we must have $x \leq q$. The lemma claims that $x$ must be mutable. For contradiction, assume that $x < j$. We now distinguish two cases.

**The case** $x < k$: Necessarily, $y$ is one of the mutable rows. From the definition of the $(k, p, q)$-matrix, we obtain that all the columns of $M$ to the left of $f_y(M)$ and to the right of $l_y(M)$ contain a 1-cell in one of the constant rows. Since $M'$ is sparse, we conclude that in $M'$, all the 1-cells in the mutable rows can only appear in the columns $i$ such that $f_p(M) \leq i \leq l_p(M)$.

Now, we apply Observation [53] to the rows $x$ and $p$ in the matrix $M$, to conclude that $M$ (and hence also $M'$) has at most one $x$-column $s$ such that $f_p(M) \leq s \leq l_p(M)$, and therefore $M'$ also has at most one $x$-column between $f_y(M')$ and $l_y(M')$. By Observation [53] this shows that $x$ cannot form the forbidden pattern with any of the mutable rows $y$ of $M'$.

**The case** $k \leq x < j$. As before, we have $y \in \{j, \ldots, q\}$. Let $c_1 < c_2 < \cdots < c_t$ be the $x$-columns of $M$ (and hence of $M'$ as well, since $x$ is constant). For any mutable row $i$, we have $l_x(M) < l_i(M)$ by the definition of $(k, p, q)$-matrix. Observation [54] applied to the pair of rows $x, i$ in $M$, tells us that all the $i$-columns of $M$ appear either to the left of $c_1$ or to the right of $c_{t-1}$. In particular, all the 1-cells between the columns $c_1$ and $c_{t-1}$ belong to the constant rows. This implies that $M'$ can have no occurrence of 12112 in the two rows $x < y$. \qed

We will now describe a simple operation on 12112-avoiding pairs of rows. This operation, which we will call *pseudoswap* will play an important part in the construction of the bijection $\phi$.

Assume that $M$ is a sparse matrix with a pair of adjacent rows $x < y$ (where $y = x + 1$) that avoids 12112 in $(x, y)$. Assume furthermore that $f_x < f_y \leq l_y < l_x$. The pseudoswap of the two rows is performed as follows.

**Easy case:** If the row $y$ is not separated by an $x$-column (in the sense of Observation [53]), or if $M$ has at most one rear $y$-column with respect to row $x$, the pseudoswap is performed by changing the order of the two rows; in other words, each 1-cell in row $x$ moves into row $y$ in the same column, and vice versa.

**Hard case:** Assume $M$ has an $x$-column $s$ separating $y$, and that it has $r > 1$ rear $y$-columns $c_1 < c_2 < \cdots < c_r$ (see Figure 1). In this case, the pseudoswap preserves the position of all the 1-cells in columns $c_1, \ldots, c_{r-1}$ (i.e., the 1-cells in these columns remain in row $y$), and all the other 1-cells of in rows $x, y$ are moved from $x$ to $y$ and vice versa. Note that after the pseudoswap is performed, the columns $s < c_1 < c_2 < \cdots < c_{r-1}$ all contain a 1-cell in row $y$, and these $r$ 1-cells are precisely the middle 1-cells of $y$ with respect to $x$. (in the sense of Observation [54]).
Let $M'$ be the matrix obtained from $M$ by the pseudoswap. It can be routinely checked that $M'$ avoids $12112$ in $(x, y)$. Let us write $f'_i$ for $f_i(M')$ and $l'_i$ for $l_i(M')$. Clearly, $f'_x = f_y$ and $f'_y = f_x$, and also $l'_x = l_y$ and $l'_y = l_x$. Also, if $M$ has $r \geq 0$ rear cells in row $y$, then $M'$ has $r$ middle cells in row $y$.

It is not difficult to see that the pseudoswap can be inverted in the following way: let $M'$ be a sparse matrix avoiding $12112$ in two adjacent rows $x < y$, such that $f'_x < f'_y \leq l'_x < l'_y$. We again distinguish two cases: if $M'$ has less than two middle $y$-columns, we invert the easy case of the pseudoswap by exchanging the two rows; on the other hand, if $M'$ has $r > 1$ middle $y$-columns $m_1 < \cdots < m_r$, we invert the hard case by preserving the position of the 1-cells in columns $m_2, m_3, \ldots, m_r$ and inverting all the other $\{x, y\}$-columns.

We will be mostly interested in the situation when the pseudoswap is applied to the pair of rows $(p, p+1)$ in a $(k, p, q)$-matrix with $p < q$. It is not hard to see that this operation yields a $(k, p+1, q)$-matrix. Let us now look in more detail at the situation related to the hard case of the pseudoswap. Recall that if $X$ and $Y$ are two sets of rows of $M$, we say that $M$ avoids $12112$ in $(X, Y)$, if there is no $x \in X$ and $y \in Y$ such that $x < y$ and the two rows $x, y$ contain a copy of $12112$.

The following technical lemma is illustrated on Figure 2.

**Lemma 56.**

(a) Let $M$ be a $(k, p, q)$-matrix that has no copy of $12112$ in the two rows $p, p+1$. Let $f_p(M) = b_1 < b_2 < \cdots < b_t = l_p(M)$ be the $p$-columns of $M$. Assume that the row $p+1$ is separated by the column $b_t$, and that it has $r \geq 2$ rear 1-cells. Let $c_1 < c_2 < \cdots < c_s$ be the front $(p+1)$-columns and let $d_1 < d_2 < \cdots < d_r$ be the rear $(p+1)$-columns. By Observation 55, we have the inequalities

$$b_1 < \cdots < b_{t-1} < c_1 < \cdots < c_s < b_i < d_1 < \cdots < d_r < b_{t+1} < \cdots < b_t.$$ 

Let $X = \{p, p+1\}$ and let $Y$ be the set of all the rows above $p+1$ that contain at least one 1-cell to the left of the column $d_{r-1}$; formally,

$$Y = \{y > p + 1; \quad f_y(M) < d_{r-1}\}.$$ 

The matrix $M$ avoids $12112$ in $(X, Y)$ if and only if each $Y$-column $y$ satisfies one of the following three inequalities:

1. $b_{t-1} < y < c_1 = f_{p+1}$
2. $d_{r-1} < y < d_r$
3. $d_r < y < b_{t+1}$

The rows in $Y$ are precisely the rows above $p+1$ that are separated by the $p$-column $b_t$.

(b) Let $M'$ be a $(k, p+1, q)$-matrix that avoids $12112$ in $(p, p+1)$. Let $\alpha_1 < \cdots < \alpha_u < \beta_1 < \cdots < \beta_t < \gamma_1 < \cdots < \gamma_v$ be the $(p+1)$-columns of $M'$, where the $\alpha_i$, $\beta_i$, and $\gamma_i$ denote respectively the left, middle and right $(p+1)$-columns with respect to row $p$. Assume that there are at least
Let us consider part (a).

Proof. Clearly, \( M \) is a \((p + 1, p, q)\)-matrix, we have \( d_r < l_y \). By Observation 54, we see that \( M \) avoids 12112 in \((p + 1, y)\) if and only if every \( y \)-column \( j \) satisfies either \( j < c_1 = f_{p+1} \), \( d_{r-1} < j < d_r \), or \( j > d_r \). The first \( y \)-column satisfies \( f_y < d_{r-1} \) by the definition of \( Y \), and hence \( f_y < c_1 = f_{p+1} < b_1 \). Since \( l_y > l_{p+1} = d_r > b_1 \), we see that if the pair of rows \((p + 1, y)\) avoids 12112, then \( y \) is separated by \( b_1 \) and the two rows \((p, y)\) avoid 12112 if and only if \( b_{i-1} < f_y < l_y < b_{i+1} \). This proves part (a) of the lemma.

The proof of part (b) is analogous and we omit it.

5.3. The bijection. We are now ready to present the bijection \( \phi \). Let \( M \) be a 12112-avoiding \((k, p, q)\)-matrix with \( p < q \), and let us write \( f_1 \) and \( l_i \) for \( f_1(M) \) and \( l_i(M) \). By the definition of \((k, p, q)\)-matrix and by the assumption \( p < q \), we know that \( f_p < f_{p+1} \leq l_{p+1} < l_p \), so we may perform the pseudoswap of the rows \( p \) and \( p + 1 \) in \( M \). Let \( M' \) be the \( m \times n \) matrix obtained from \( M \) by this pseudoswap. Let \( f'_i = f_i(M') \) and \( l'_i = l_i(M') \). Note that \( f'_i = f_i \) and \( l'_i = l_i \) for every \( i \notin \{p, p + 1\} \).

Clearly, \( M' \) is a \((k, p + 1, q)\)-matrix. We now distinguish two cases, depending on whether the pseudoswap we performed was easy or hard.
**Easy case:** If the row $p + 1$ of $M$ has at most one rear 1-cell with respect to row $p$, then $M'$ is 12112-avoiding, and we may define $\phi(M) = M'$. Indeed, from the definition of the pseudoswap we know that $M'$ cannot contain a copy of 12112 in the rows $(p, p + 1)$, and since we are performing the easy case of the pseudoswap, we cannot create any new copy of the forbidden pattern that would intersect the remaining $m - 2$ rows.

**Hard case:** Assume that the row $p + 1$ of $M$ has $r > 1$ rear 1-cells. Let $b_1 < \cdots < b_t$, $c_1 < \cdots < c_s$, $d_1 < d_2 < \cdots < d_r$, and $Y$ have the same meaning as in part (a) of Lemma 56. Let $Y_1$, $Y_2$ and $Y_3$ denote, respectively, the $Y$-columns that lie between $b_{t-1}$ and $c_1$, between $d_{r-1}$ and $d_r$, and between $d_r$ and $b_{t+1}$.

The bijection $\phi$ is now constructed in two steps. In the first step, we perform the pseudoswap of the rows $p$ and $p + 1$. Let $M'$ be the result of this first step. Let us now apply the notation of part (b) of Lemma 56 to the matrix $M'$; see Fig. 2. Note that $d_{r-1} = \beta_r$, and hence $Y = Y'$. Part (b) of Lemma 56 requires that all the $Y'$-columns of a 12112-avoiding $(k, p + 1, q)$-matrix fall into one of the three groups:

- columns between $\delta_w < y < \gamma_1$. In $M'$, we have $\delta_w = d_r$ and $\gamma_1 = b_{t+1}$, so these columns are precisely the columns $Y_3$.
- columns between $\beta_r < y < \delta_w$. In $M'$, these are precisely the columns $Y_2$.
- columns between $\beta_{r-1} < y < \beta_r$. In $M'$, there are no $Y$-columns in this range.

On the other hand, if $Y_1$ is nonempty, then these columns violate the inequalities of part (b) in Lemma 56, showing that $M'$ is not 12112-avoiding. We will now apply the second step of the bijection $\phi$, which will exchange the relative order of the columns $Y_1$ and some of the $(p, p + 1)$-columns of $M'$. Intuitively, we will move the columns $Y_1$ into the gap between $\beta_{r-1}$ and $\beta_r$, to transform $M'$ into a matrix that satisfies the inequalities of Lemma 56. This operation only affects the columns $Y_1$ and the $(p, p + 1)$-columns; all the other cells of the matrix $M'$ remain unchanged.

Formally, the second step of the bijection is performed as follows. Consider the submatrix of $M'$ induced by the columns $Y_1$ and the columns $Z = \{\delta_1 < \cdots < \delta_{w-1} < \beta_1 < \cdots < \beta_{r-1}\}$. Note that the columns $Y_1$ are to the left of any column of $Z$. Now we rearrange the columns inside this submatrix, such that the columns of the set $Z$ precede the columns of $Y_1$. The relative order of the columns in $Z$, as well as the relative order of the columns in $Y_1$, is preserved. This operation transforms the matrix $M'$ into a matrix $M''$ that satisfies the conditions of part (b) of Lemma 56. We now define $\phi(M) = M''$.

Since $M''$ is clearly a $(k, p + 1, q)$-matrix, it remains to check that $M''$ avoids 12112. Let $x < y$ be a pair of rows of $M''$. We want to check that $M''$ avoids 12112 in these two rows. Let us consider the following cases separately.

**The case** $x < p$: Since the rows below row $p$ are unaffected by $\phi$, by Lemma 55 we know that $M''$ avoids 12112 in the rows $(x, y)$.

**The case** $x = p, y = p + 1$: The properties of pseudoswap guarantee that $M''$ avoids 12112 in these two rows.

**The case** $x \in X = \{p, p + 1\}$ and $y \in Y'$: By construction, $M''$ satisfies the inequalities of part (b) of Lemma 56 and thus it avoids 12112 in $(X, Y')$.

**The case** $x \in X = \{p, p + 1\}$, $y \not\in Y'$ and $y > p + 1$: By the definition of $Y'$, we have $f_y(M'') = f_y(M) > d_{r-1} = \beta_r$. In any column to the right of $\beta_r$, the mapping $\phi$ acts by exchanging the rows $p$ and $p + 1$. It is easy to check that this action cannot create a copy of 12112 in $(x, y)$ (note that in any of the three matrices $M$, $M'$ and $M''$, both the rows $p$ and $p + 1$ have a 1-cell to the left of $\beta_r$).
The case \( y > x > p + 1 \): The submatrix of \( M'' \) induced by the rows above \( p + 1 \) only differs from the corresponding submatrix of \( M \) by the position of the zero columns. Thus, it cannot contain any copy of 12112.

This shows that \( \phi (M) \) is indeed a 12112-avoiding \((k, p + 1, q)\)-matrix.

It is routine to check that the mapping \( \phi \) can be inverted, which shows that \( \phi \) is indeed the required bijection.

5.4. Consequences. Theorem \[51\] has several consequences for pattern-avoiding fillings of Ferrers shapes and pattern-avoiding ordered graphs.

By Lemma \[5\] there is a bijection between the 12112-avoiding partitions of \([n]\) with \( m \) blocks and semi-standard fillings of Ferrers shapes with \( n - m \) columns and at most \( m \) rows that avoid \( M(2, 112) \); similarly, there is an analogous bijection between 12212-avoiding partitions and \( M(2, 212) \)-avoiding fillings of Ferrers shapes. Thus, we obtain the following direct consequence of Theorem \[51\].

**Corollary 57.** For every \( r \) and \( c \), there is a bijection between the \( M(2, 112) \)-avoiding semi-standard fillings of all the Ferrers shapes with \( r \) rows and \( c \) columns and the \( M(2, 212) \)-avoiding semi-standard fillings of all the Ferrers shapes with \( r \) rows and \( c \) columns.

It would be tempting to assume that for a given Ferrers shape \( F \), the \( M(2, 112) \)-avoiding semi-standard fillings of \( F \) are in bijection with the \( M(2, 212) \)-avoiding semi-standard fillings of \( F \), i.e., that the two matrices \( M(2, 112) \) and \( M(2, 212) \) are Ferrers-equivalent. However, as we already mentioned in the introduction of Section \[5\] this is not the case. For instance, the Ferrers shape \( F \) with five columns of height 4 and one column of height 2 has 866 \( M(2, 112) \)-avoiding fillings but only 865 \( M(2, 212) \)-avoiding fillings. Thus, the bijection of Corollary \[57\] in general cannot preserve the shape of the underlying graph.

Let us now describe a well-known and useful correspondence between 0-1 fillings of Ferrers shapes, and graphs with linearly ordered vertex sets; the correspondence has been used, e.g., in \[7\] or \[15\].

Every 0-1 filling \( F \) of a Ferrers shape with \( p \) columns and \( q \) rows can be represented by a graph with \( p + q \) linearly ordered vertices, defined in the following way: the graph has two kinds of vertices, called right vertices \( r_1, \ldots , r_p \) and left vertices \( l_1, \ldots , l_q \). The \( i \)-th column of \( F \) is associated with the \( i \)-th right vertex \( r_i \), and the \( j \)-th row of \( F \) is associated with the \( j \)-th left vertex \( l_j \). All the vertices are linearly ordered by a left-to-right relation \(< \) with the properties \( r_1 < \cdots < r_p \), \( l_1 < l_2 < \cdots < l_q \), and furthermore, \( l_j < r_i \) if and only if row \( j \) intersects column \( i \) inside \( F \). The edge-set of the graph is determined by the 1-cells of \( F \) in the natural way: a 1-cell in row \( j \) and column \( i \) corresponds to the edge between \( l_j \) and \( r_i \). Note that if \( l_j \) and \( r_i \) are connected by an edge, then \( l_j < r_i \).

In this representation, the semi-standard fillings of Ferrers shapes correspond precisely to the ordered graphs with the property that every right vertex is connected to precisely one left vertex, whereas the degrees of the left vertices can be arbitrary. In accordance with our terminology for fillings, we will call such graphs semi-standard. Pattern-avoidance of semi-standard graphs has been studied by A. de Mier \[5\], who considered the avoidance of crossings and nestings with prescribed size within this class of graphs. The \( M(2, 112) \) avoiding fillings of \( F \) correspond precisely to ordered graphs which avoid a subgraph \( G_1 \) with five vertices \( l_j < l_j' < r_i < r_i' < r_i'' \) and three edges \( l_j r_i, l_j r_i', \) and \( l_j r_i'' \). Similarly, the fillings avoiding \( M(2, 212) \) correspond to graphs avoiding the subgraph \( G_2 \) with vertices \( l_j < l_j' < r_i < r_i' < r_i'' \) and edges \( l_j r_i', l_j r_i, \) and \( l_j r_i'' \) (see Figure \[3\]).

Theorem \[51\] then immediately yields the following result.
Corollary 58. There is a bijection between semi-standard $G_1$-avoiding graphs and semi-standard $G_2$-avoiding graphs that preserves the number of left vertices and right vertices.

Whether this result can be extended to more general classes of graphs or more general pairs of patterns is at this point an open problem.

6. Concluding remarks

In Appendix A, B and C, we present the results of the computer enumeration of partitions avoiding fixed patterns of size five, six and seven, respectively. Each row of the tables corresponds to one equivalence class.

In Table 3, we present the total number of equivalence classes of patterns of length 1, 2, ..., 7.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| number of classes of patterns of size $n$ | 1 | 1 | 2 | 5 | 21 | 114 | 617 |

Table 3. Number of equivalence classes of patterns of size 1, 2, ..., 7

To provide an accurate asymptotic estimate of the number of equivalence classes of patterns of a given size remains out of reach of our methods.

Let us remark that our computer enumeration has revealed several pairs of non-equivalent patterns $\tau \not\sim \tau'$ whose growth functions $p(n; \tau)$ and $p(n; \tau')$ coincide for several small values of $n$. For instance, the growth functions of the two patterns $\tau = 1234415$ and $\tau' = 1234152$ are equal for $n < 15$; in other words, the value of $n = 15$ is the smallest witness of the non-equivalence of the two patterns. It is an interesting open problem to find, for a given $l$, a common upper bound for all the smallest witnesses demonstrating the non-equivalence of the non-equivalent pairs of patterns of length $l$. Note that for any $k$ and for $\tau, \tau'$ chosen as above, the pair of non-equivalent patterns $12\cdots k(\tau + k)$ and $12\cdots k(\tau' + k)$ of length $k + 7$ requires a witness of length $k + 15$ (this follows from Theorem 20).

Appendix A: Table of patterns of length five
Table 4: Number of partitions in $P(n; \tau)$, where $\tau \in P(5)$.

| $\tau$ | $\{p(n; \tau)\}_{n \geq \tau}^{11}$ |
|--------|----------------------------------|
| 12314, 12324, 12334, 12341, 12342, 12343, 12344, 12345 | 187, 716, 2765, 11051, 43947, 175275 |
| 12313 | 188, 730, 2933, 12061, 50423, 213423 |
| 12323, 12332, 12123, 12132, 12213, 12231, 12312 | 188, 731, 2950, 12235, 51822, 223191 |
| 12321, 12331, 12134 | 188, 732, 2969, 12452, 53769, 238379 |
| 11223, 11232 | 189, 746, 3094, 13371, 59873, 276670 |
| 11234 | 189, 747, 3109, 13507, 60837, 282503 |
| 12131 | 189, 747, 3109, 13517, 61061, 285593 |
| 11233 | 189, 747, 3111, 13550, 61393, 288157 |
| 12232, 12232, 12333, 12311, 12113 | 189, 747, 3111, 13551, 61419, 288543 |
| 11231 | 190, 760, 3222, 14350, 66715, 322218 |
| 11213 | 190, 760, 3223, 14366, 66882, 323663 |
| 11223 | 191, 771, 3310, 14069, 70831, 348887 |
| 12112, 12112, 12212, 12221 | 191, 773, 3336, 15207, 72697, 362447 |
| 12121 | 191, 773, 3337, 15224, 72892, 364317 |
| 1212 | 191, 773, 3337, 15224, 72893, 364341 |
| 12211 | 191, 774, 3351, 15361, 74043, 373270 |
| 11221 | 191, 774, 3353, 15383, 74395, 376555 |
| 11222 | 191, 774, 3354, 15409, 75745, 378365 |
| 11122 | 191, 774, 3355, 15424, 74738, 379805 |
| 11112, 11121, 11211, 12111, 12222 | 192, 789, 3495, 16545, 83142, 441009 |
| 11111 | 196, 827, 3795, 18755, 99146, 556711 |

APPENDIX B: Table of patterns of length six

Table 5: Number of partitions in $P(n; \tau)$, where $\tau \in P(6)$.

| $\tau$ | $\{p(n; \tau)\}_{n \geq \tau}$ |
|--------|-------------------------------|
| 123415, 123425, 123435, 123445, 123451, 123452 | 855, 3845, 18002, 86472 |
| 123453, 123454, 123455, 123456 | 856, 3867, 18286, 89291 |
| 123414 | 856, 3867, 18288, 89348 |
| 123413, 123424 | 856, 3867, 18289, 89375, 447219, 2277477 |
| 123413 | 856, 3868, 18312, 89684, 450407 |
| 123314 | 856, 3868, 18312, 89684, 450408, 2305592, 11978961, 62983208 |
| 123142 | 856, 3868, 18312, 89684, 450408, 2305592, 11978961, 62983209 |
| 123142, 123145, 123214, 123241, 123243, 123245 | 856, 3868, 18313, 89711, 450825, 2310453 |
| 123324, 123341, 123342, 123345, 123412, 123421 | 856, 3869, 18340, 90135, 455917 |
| 123423, 123431, 123432, 123434, 123441, 123442 | 857, 3888, 18555, 92027 |
| 123142 | 857, 3889, 18578, 92339 |
| 122341 | 857, 3889, 18578, 92341 |
| 121324 | 857, 3889, 18579, 92369 |
| 121334, 122334, 121343, 122343 | 857, 3890, 18605, 92767, 478726, 2544145 |
| 123415, 122345 | 857, 3891, 18628, 93074, 481845, 2570867 |
| 123141 | 857, 3891, 18628, 93082 |

continue
Table 5: Number of partitions in $P(n;\tau)$, where $\tau \in P(6)$.

| $\tau$                      | $\{p(n;\tau)\}_{n>n}$ |
|-----------------------------|------------------------|
| 12342                       | 857, 3891, 18628, 93084 |
| 12134, 12234                | 857, 3891, 18630, 93135, 482921, 2585332 |
| 12343, 12334, 12343, 123411, 123422 | 857, 3891, 18630, 93136 |
| 12134, 122134               | 858, 3908, 18801, 94448, 491234, 2628772 |
| 12313                       | 858, 3909, 18821, 94686 |
| 12321                       | 858, 3909, 18822, 94712 |
| 12231                       | 858, 3910, 18844, 95008 |
| 121332                      | 858, 3910, 18845, 95037 |
| 123123, 123312, 123321      | 858, 3910, 18846, 95058 |
| 12323                       | 858, 3910, 18847, 95086 |
| 121323                      | 858, 3910, 18847, 95087 |
| 12233                       | 858, 3911, 18871, 95434 |
| 121341                      | 858, 3911, 18872, 95455 |
| 121314                      | 858, 3911, 18872, 95460 |
| 122133, 121233              | 858, 3911, 18872, 95461, 523161 |
| 121342                      | 858, 3911, 18873, 95485 |
| 122342                      | 858, 3911, 18873, 95486 |
| 122324                      | 858, 3911, 18874, 95511 |
| 121324                      | 858, 3911, 18874, 95513 |
| 121334, 112343              | 858, 3912, 18879, 95828, 506812, 2781704 |
| 121345                      | 858, 3912, 18900, 95904 |
| 121344                      | 858, 3912, 18900, 95909 |
| 121134, 122234              | 859, 3929, 19077, 97377, 518804, 2869604 |
| 121234                      | 859, 3930, 19096, 97599 |
| 121323                      | 859, 3930, 19100, 97700, 522447 |
| 121332                      | 859, 3930, 19100, 97700, 522415 |
| 121313, 112131              | 859, 3931, 19115, 97828, 523161 |
| 123313                      | 859, 3931, 19115, 97828, 523144 |
| 123133                      | 859, 3931, 19115, 97831 |
| 123113                      | 859, 3931, 19116, 97852 |
| 121313                      | 859, 3931, 19117, 97872 |
| 121312                      | 859, 3931, 19117, 97882 |
| 121312                      | 859, 3931, 19118, 97898 |
| 121223, 121232, 121232, 1221123, 122112, 122213 | 859, 3931, 19119997921, 524460, 2921730 |
| 122231, 122321, 122321, 123112, 123112, 123122 | 859, 3931, 19120, 97945 |
| 123123                      | 859, 3931, 19120, 97947, 524870, 2926845 |
| 123123                      | 859, 3931, 19120, 97947, 524870, 2926847 |
| 122323                      | 859, 3931, 19120, 97947, 524871 |
| 121231                      | 859, 3931, 19120, 97948 |
| 121321                      | 859, 3931, 19121, 97972 |
| 121344                      | 859, 3931, 19122, 97987 |
| 122314                      | 859, 3931, 19122, 97992 |
| 122333                      | 859, 3931, 19123, 98023 |
| 122133                      | 859, 3932, 19139, 98173 |
| 122113                      | 859, 3932, 19141, 98222 |
| 121331                      | 859, 3932, 19142, 98242 |
| 122131, 123311, 123211, 123422 | 859, 3932, 19142, 98246, 528141, 2958634 |

continue
Table 5: Number of partitions in $P(n; \tau)$, where $\tau \in P(6)$.

| $\tau$          | $\{p(n; \tau)\}_{n > \tau}$ |
|-----------------|-------------------------------|
| 122332          | 859, 9352, 9144, 98296        |
| 121133, 122333  | 859, 9352, 9145, 98321, 529292, 2972760 |
| 121133, 122233  | 859, 9352, 9146, 98345, 529648, 2976939 |
| 112312          | 860, 9348, 9309, 99685         |
| 112123          | 860, 9348, 9310, 99730540195   |
| 112132          | 860, 9348, 9310, 99730540193   |
| 112134          | 860, 9349, 9327, 99908         |
| 112313          | 860, 9349, 9327, 99914         |
| 112321          | 860, 9349, 9330, 99990         |
| 112223, 112232, 112322 | 860, 9349, 9330, 99993, 543077, 3081145 |
| 112231          | 860, 9349, 9331, 100010        |
| 112331          | 860, 9350, 9350, 100240        |
| 112333          | 860, 9350, 9354, 100332        |
| 112333          | 860, 9350, 9354, 100338        |
| 121131          | 860, 9354, 9434, 101388        |
| 121211          | 860, 9354, 9434, 101342        |
| 121113, 122232, 123222, 123122, 123333 | 860, 9354, 9434, 101350, 557570, 3220754 |
| 112223          | 861, 9364, 9448, 101434, 555332, 3181699 |
| 112123          | 861, 9366, 9516, 101662        |
| 112123          | 861, 9366, 9523, 101837        |
| 112131          | 861, 9370, 9599, 102778        |
| 112113          | 861, 9370, 9599, 102786        |
| 123131          | 861, 9370, 9600, 102802        |
| 112113          | 862, 9384, 9731, 103869        |
| 112131          | 862, 9384, 9733, 103905        |
| 111123          | 863, 9396, 9897, 104726        |
| 122122          | 863, 9399, 9880, 105134, 587479, 3449505 |
| 122121          | 863, 9399, 9880, 105134, 587479, 3449509 |
| 122121          | 863, 9399, 9880, 105135        |
| 122122          | 863, 9399, 9889, 105150        |
| 121122          | 863, 9399, 9889, 105170        |
| 121122          | 863, 9399, 9889, 105188        |
| 122122          | 863, 9399, 9889, 105192        |
| 122121          | 863, 9399, 9885, 105226        |
| 122121          | 863, 9399, 9885, 105233        |
| 112222          | 863, 9399, 9889, 105234        |
| 122221, 121222, 122122, 122212 | 863, 4001, 19917, 105594 |
| 112112          | 863, 4001, 19918, 105614, 592676 |
| 121112          | 863, 4001, 19918, 105614, 592671 |
| 121121          | 863, 4001, 19918, 105618        |
| 111122          | 863, 4001, 19919, 105636, 592976, 3504921 |
| 112121          | 863, 4001, 19919, 105636, 592976, 3504918 |
| 112222          | 863, 4002, 19938, 105878        |
| 122111          | 863, 4002, 19939, 105886        |
| 111122          | 863, 4002, 19939, 105893        |
| 112112          | 863, 4002, 19939, 105895        |
| 111122          | 863, 4002, 19939, 105901        |
| continue         | continue                      |
Table 5: Number of partitions in $P(n; \tau)$, where $\tau \in P(6)$.

| $\tau$ | $\{p(n; \tau)\}_{n \geq s}$ |
|-------|-------------------|
| 111112, 111121, 111211, 112111, 121111, 122222 | 884, 4020, 20150, 107964 |
| 111111 | 809, 4075, 20645, 112124 |

APPENDIX C: TABLE OF PATTERNS OF LENGTH SEVEN

Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$ | $\{p(n; \tau)\}_{n \geq s}$ |
|-------|-------------------|
| 1234516, 1234526, 1234536, 1234546 | 411, 20648, 109299, 601492 |
| 1234556, 1234561, 1234562, 1234563 | 411, 20678, 109817, 608258 |
| 1234564, 1234565, 1234566, 1234567 | 411, 20678, 109817, 608261 |
| 1234515 | 411, 20678, 109818, 608300, 3478443 |
| 1234514, 1234525 | 411, 20678, 109818, 608300, 3478444 |
| 1234513, 1234524, 1234535 | 411, 20678, 109819, 608338, 3479249 |
| 1234513 | 411, 20678, 109819, 608338, 3479251 |
| 123415, 1234245, 1234254 | 411, 20678, 109820, 608375 |
| 1234251 | 411, 20679, 109852, 608957, 3487954, 20485468 |
| 1234415 | 411, 20679, 109852, 608957, 3487954, 20485475, 122666770, 745713106 |
| 1234152 | 411, 20679, 109852, 608957, 3487954, 20485475, 122666770, 745713111 |
| 1234351, 1234352 | 411, 20679, 109852, 608959, 3488036, 20487341 |
| 1234315, 1234425 | 411, 20679, 109852, 608959, 3488037, 20487386, 122699078, 746161492 |
| 1234253 | 411, 20679, 109852, 608959, 3488037, 20487386, 122699078, 746161493 |
| 1234125, 1234156, 1234215, 1234235 | 411, 20679, 109853, 608996, 3488806 |
| 1234256, 1234325, 1234345, 1234354 | 411, 20680, 109889, 609735 |
| 1234356, 1234435, 1234451, 1234452 | 411, 20707, 110309, 614064 |
| 1234435, 1234456, 1234512, 1234521 | 411, 20707, 110311, 614747 |
| 123453, 1234531, 1234532, 1234534 | 411, 20707, 110311, 614752 |
| 1234541, 1234542, 1234543, 1234545 | 411, 20707, 110313, 614824 |
| 1234551, 1234552, 1234553, 1234554 | 411, 20708, 110342, 615304, 3558058 |
| 1234555, 1234556, 1234557, 1234558 | 411, 20708, 110342, 615304, 3558067 |
| 1234515 | 411, 20708, 110342, 615306 |
| 12343415 | 411, 20708, 110343, 615337 |
| 12343425 | 411, 20708, 110343, 615339 |
| 12343451, 1234352 | 411, 20708, 110343, 615341 |
| 1234245 | 411, 20708, 110344, 615379 |
| 1234145, 1234154, 1234245, 1234254 | 411, 20709, 110379, 616082 |
| 1234345, 1234345 | 411, 20709, 110344, 615379 |
| 1234156, 1234256, 1234356 | 411, 20710, 110411, 616664 |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$ | \(\{p(n; \tau)\}_{n \geq 4}\) |
|--------|---------------------------------|
| 1234151| 4113, 20710, 110411, 616672, 3578613 |
| 1234252| 4113, 20710, 110411, 616672, 3578615 |
| 1234353| 4113, 20710, 110411, 616674 |
| 1234555, 1232455, 1233455 | 4113, 20710, 110413, 616745 |
| 1234415, 1234225, 1234335, 1234445 | 1234454, 1234511, 1234522, 1234533 |
| 1234544, 1234555 | 4113, 20710, 110413, 616746 |
| 1231345 | 4114, 20734, 110743, 620127 |
| 1233145 | 4114, 20735, 110772, 620616 |
| 1231245, 1232145, 1232345, 1233245 | 4114, 20735, 110773, 620653 |
| 1234214 | 4114, 20736, 110799, 621037 |
| 1234143 | 4114, 20736, 110800, 621064 |
| 1234142 | 4114, 20736, 110800, 621065 |
| 1232413 | 4114, 20736, 110800, 621066 |
| 1231342 | 4114, 20736, 110800, 621070 |
| 1234314 | 4114, 20736, 110802, 621134 |
| 1234132, 1234243 | 4114, 20736, 110802, 621136 |
| 1232143 | 4114, 20736, 110802, 621137 |
| 1231432 | 4114, 20736, 110802, 621138 |
| 1234124 | 4114, 20736, 110802, 621145 |
| 1234134 | 4114, 20736, 110803, 621172, 3622245 |
| 1234213, 1234324 | 4114, 20736, 110803, 621172, 3622246 |
| 1231324 | 4114, 20736, 110803, 621173 |
| 1232134 | 4114, 20736, 110804, 621207 |
| 1234241 | 4114, 20736, 110804, 621214 |
| 1231243 | 4114, 20736, 110805, 621243, 3623689 |
| 1234352 | 4114, 20736, 110805, 621243, 3623710 |
| 1231424 | 4114, 20736, 110805, 621246 |
| 1231444 | 4114, 20736, 110807, 621319 |
| 1231425 | 4114, 20736, 110808, 621351 |
| 1232414 | 4114, 20737, 110833, 621694 |
| 1234413 | 4114, 20737, 110833, 621698 |
| 1234414 | 4114, 20737, 110834, 621729 |
| 1233142 | 4114, 20737, 110834, 621730 |
| 1233424 | 4114, 20737, 110834, 621733 |
| 1231443 | 4114, 20737, 110834, 621737 |
| 1233241 | 4114, 20737, 110835, 621766 |
| 1231423 | 4114, 20737, 110835, 621767 |
| 1233124 | 4114, 20737, 110835, 621768, 3630754 |
| 1231442 | 4114, 20737, 110835, 621768, 3630761 |
| 1232443 | 4114, 20737, 110835, 621772 |
| 1233412, 1233421, 1234123, 1234234 | 1234312, 1234321, 1234412, 1234421 |
| 1234423, 1234431, 1234432 | 4114, 20737, 110836, 621803, 3631456, 21211085 |
| 1231234, 1233214 | 4114, 20737, 110836, 621803, 3631456, 21933850 |
| 1232314 | 4114, 20737, 110836, 621804 |
| 1232341 | 4114, 20737, 110837, 621841, 3632280 |
| 1234231, 1234341, 1234432 | 4114, 20737, 110837, 621841, 3632281 |
| 1233241 | 4114, 20737, 110837, 621842, 3632524 |
| 1232434 | 4114, 20737, 110837, 621842, 3632525 |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

$$
\begin{array}{|c|c|}
\hline
\tau & \{p(n; \tau)\}_{n \geq s} \\
\hline
123451 & 4114, 20737, 110838, 621877 \\
123415 & 4114, 20737, 110838, 621875 \\
123344 & 4114, 20737, 110841, 621987 \\
123241 & 4114, 20738, 110868, 622399 \\
123441, 1233442 & 4114, 20738, 110870, 622474 \\
1231451 & 4114, 20738, 110871, 622504 \\
1232452 & 4114, 20738, 110871, 622505 \\
1234144 & 4114, 20738, 110871, 622507 \\
1231445 & 4114, 20738, 110871, 622508 \\
1232425 & 4114, 20738, 110871, 622510 \\
1231244, 1232144, 1232344, 1233244 & 4114, 20738, 110871, 622511 \\
1213453, 1223453 & 4114, 20738, 110872, 622545 \\
1233435 & 4114, 20738, 110872, 622546 \\
1234435 & 4114, 20738, 110873, 622581 \\
1213445, 1223435 & 4114, 20738, 110873, 622583 \\
1213445, 1213454, 1223445, 1223454 & 4114, 20739, 110906, 623173 \\
1213456, 1223456 & 4114, 20739, 110908, 623279 \\
1213455, 1223455 & 4114, 20739, 110908, 623284 \\
1213423 & 4115, 20762, 111212, 626275 \\
1223413 & 4115, 20763, 111238, 626659 \\
1223143 & 4115, 20763, 111239, 626702 \\
1213424 & 4115, 20763, 111239, 626706 \\
1223432 & 4115, 20763, 111240, 626729 \\
1223342 & 4115, 20763, 111240, 626730 \\
1233134 & 4115, 20763, 111240, 626733 \\
1233324 & 4115, 20763, 111240, 626737, 3684077 \\
1233243 & 4115, 20763, 111243, 626837, 3686012 \\
1213234 & 4115, 20763, 111243, 626837, 3686013 \\
1233145 & 4115, 20764, 111269, 627221 \\
1223414 & 4115, 20764, 111270, 627257 \\
1213442 & 4115, 20764, 111270, 627260 \\
1213245 & 4115, 20764, 111270, 627261 \\
1223431 & 4115, 20764, 111272, 627332 \\
1223314 & 4115, 20764, 111274, 627365 \\
1223341 & 4115, 20764, 111274, 627402 \\
1231145, 1232245, 1233345 & 4115, 20764, 111274, 627407 \\
1213345, 1223345 & 4115, 20765, 111302, 627864 \\
1223441 & 4115, 20765, 111303, 627898 \\
1223144 & 4115, 20765, 111306, 627908 \\
1213244 & 4115, 20765, 111306, 628002 \\
1213443, 1232443 & 4115, 20765, 111306, 628005, 3074775 \\
1213434, 1223434 & 4115, 20765, 111306, 628005, 3074777 \\
1123452 & 4115, 20765, 111311, 628168 \\
1123425 & 4115, 20765, 111311, 628173 \\
1234141 & 4115, 20766, 111330, 628320 \\
1234414 & 4115, 20766, 111330, 628321 \\
1234144 & 4115, 20766, 111330, 628322 \\
1231413 & 4115, 20766, 111330, 628325 \\
123314 & 4115, 20766, 111330, 628327 \\
1234313, 1234424, 1233413 & 4115, 20766, 111330, 628328, 3705907 \\
\hline
\end{array}
$$

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$ | \( \{p(n; \tau)\}_{n \geq 1} \) |
|--------|----------------------------------|
| 123431, 123424 | 4115, 20766, 111330, 628328, 3705924 |
| 123341, 1233143 | 4115, 20766, 111330, 628329, 3705940, 2270849 |
| 123433, 1234244 | 4115, 20766, 111330, 628331 |
| 123143 | 4115, 20766, 111330, 628333 |
| 123134, 1231433 | 4115, 20766, 111330, 628335, 3706167, 22714756 |
| 123141 | 4115, 20766, 111331, 628354 |
| 1234114 | 4115, 20766, 111331, 628355 |
| 1231431 | 4115, 20766, 111331, 628358 |
| 1231143 | 4115, 20766, 111331, 628361 |
| 1234113, 1234224 | 4115, 20766, 111331, 628362 |
| 1231242, 1232142 | 4115, 20766, 111331, 628364, 3706656, 22721090 |
| 1231414 | 4115, 20766, 111332, 628387 |
| 1232424 | 4115, 20766, 111332, 628392 |
| 1232412, 1232421 | 4115, 20766, 111332, 628395, 3707209, 22728608 |
| 1231142 | 4115, 20766, 111332, 628400 |
| 1232243 | 4115, 20766, 111332, 628402 |
| 1231412 | 4115, 20766, 111333, 628426 |
| 1232423 | 4115, 20766, 111333, 628428 |
| 1231241 | 4115, 20766, 111333, 628435 |
| 1232241 | 4115, 20766, 111334, 628454 |
| 1233314 | 4115, 20766, 111334, 628456 |
| 1231422 | 4115, 20766, 111334, 628457 |
| 1231224, 1232124, 1232214, 1232344 | 4115, 20766, 111334, 628461 |
| 1232343, 1232433, 1233234, 1233243 | 4115, 20766, 111335, 628495 |
| 1233324, 1233341, 1233342, 1233423 | 4115, 20766, 111335, 628497, 3709217, 22758862 |
| 1234331, 1233432, 1234112, 1234122 | 4115, 20766, 111335, 628497, 3709217, 22758864 |
| 1234212, 1234221, 123423, 1234233 | 4115, 20766, 111335, 628497, 3709218 |
| 1234323, 1234331, 1234332, 1234334 | 4115, 20766, 111335, 628497, 3709218 |
| 1234344, 1234434, 1234441, 1234442 | 4115, 20766, 111335, 628497, 3709218 |
| 1234443 | 4115, 20766, 111335, 628498 |
| 1231124, 1232234 | 4115, 20766, 111335, 628499 |
| 1234121, 1234232, 1234333 | 4115, 20766, 111336, 628523, 3709547 |
| 1231214, 1232324 | 4115, 20766, 111336, 628523, 3709569 |
| 1234344 | 4115, 20766, 111336, 628532 |
| 1232432 | 4115, 20766, 111337, 628557 |
| 1234352 | 4115, 20766, 111337, 628562 |
| 1231344, 1232344 | 4115, 20766, 111338, 628603 |
| 1123453 | 4115, 20766, 111342, 628731 |
| 1123435 | 4115, 20766, 111342, 628735 |
| 1123445, 1123454 | 4115, 20766, 111345, 628837 |
| 1123456 | 4115, 20766, 111345, 628846, 3716242 |
| 1123455 | 4115, 20766, 111345, 628846, 3716256 |
| 1231341 | 4115, 20767, 111363, 628953 |
| 1232141 | 4115, 20767, 111363, 628954 |
| continue | |
| | |
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$          | \{p(n; \tau)\}_{n \geq 4}                  |
|-----------------|--------------------------------------------|
| 1233242         | 4115, 20767, 111366, 629058                |
| 123314          | 4115, 20767, 111366, 629024                |
| 1232114, 1233224| 4115, 20767, 111366, 629027                |
| 1231441         | 4115, 20767, 111366, 629055                |
| 1232411         | 4115, 20767, 111366, 629057, 3717013      |
| 1232412         | 4115, 20767, 111366, 629057, 3717017      |
| 1234411, 1233422, 1234211, 1234311 | 4115, 20767, 111366, 629061 |
| 1234322, 1234411, 1234422, 1234433 | 4115, 20767, 111366, 629131 |
| 1234433         | 4115, 20767, 111368, 629166                |
| 1231444, 1232444, 1233444 | 4115, 20767, 111370, 629200 |
| 1212344, 1212344, 1212344, 1212345 | 4116, 20788, 111626, 631531 |
| 1212345, 1221345 | 4116, 20790, 111675, 632216 |
| 1212344, 1221344 | 4116, 20790, 111682, 632466 |
| 1123424         | 4116, 20790, 111686, 632609                |
| 1213422         | 4116, 20791, 111706, 632807                |
| 1213242         | 4116, 20791, 111707, 632831                |
| 1213422         | 4116, 20791, 111708, 632868                |
| 1213422         | 4116, 20791, 111709, 632895                |
| 1213242         | 4116, 20791, 111709, 632900                |
| 1213422         | 4116, 20791, 111710, 632907                |
| 1223142         | 4116, 20791, 111710, 632925, 3754896      |
| 1223413         | 4116, 20791, 111710, 632925, 3754918      |
| 1223412         | 4116, 20791, 111710, 632929                |
| 1213422         | 4116, 20791, 111710, 632932                |
| 1213422         | 4116, 20791, 111710, 632933                |
| 1223421         | 4116, 20791, 111711, 632962                |
| 1223414         | 4116, 20791, 111711, 632964, 3755736      |
| 1223414         | 4116, 20791, 111711, 632964, 3755740      |
| 1213143         | 4116, 20791, 111711, 632968                |
| 1213134         | 4116, 20791, 111711, 632969                |
| 1223423         | 4116, 20791, 111712, 633000                |
| 1123342         | 4116, 20791, 111712, 633005                |
| 1123423         | 4116, 20791, 111712, 633007                |
| 1123432         | 4116, 20791, 111712, 633008                |
| 1123334         | 4116, 20791, 111712, 633009, 3756838      |
| 1123234         | 4116, 20791, 111712, 633009, 3756839      |
| 1123234         | 4116, 20791, 111712, 633009, 3756840      |
| 1213241         | 4116, 20791, 111713, 633029                |
| 1213241         | 4116, 20791, 111713, 633032                |
| 1223423         | 4116, 20791, 111714, 633065, 3757738      |
| 1223234         | 4116, 20791, 111714, 633065, 3757740      |
| 1123424         | 4116, 20791, 111714, 633070                |
| 1123425         | 4116, 20791, 111715, 633098                |
| 1223141         | 4116, 20792, 111737, 633365                |
| 1213145         | 4116, 20792, 111738, 633395                |
| 1223141         | 4116, 20792, 111739, 633435                |
| 1213414         | 4116, 20792, 111740, 633454                |
| 1223425         | 4116, 20792, 111740, 633458                |
| 1223424         | 4116, 20792, 111740, 633464                |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$          | $\{p(n; \tau)\}_{n \geq 8}$                                |
|-----------------|-------------------------------------------------------------|
| 1223411         | 4116, 20702, 111740, 633468                                 |
| 1213314         | 4116, 20702, 111742, 633529                                 |
| 1213431         | 4116, 20702, 111743, 633561                                 |
| 1213341         | 4116, 20702, 111743, 633562                                 |
| 1223432         | 4116, 20702, 111743, 633570                                 |
| 1213343, 121343, 1213433, 1223344 | 4116, 20702, 111743, 633573 |
| 1223343, 1223433 | 4116, 20702, 111744, 633595                                 |
| 123334          | 4116, 20702, 111744, 633600                                 |
| 1123433         | 4116, 20702, 111744, 633605, 3765380                         |
| 1123434         | 4116, 20702, 111745, 633605, 3765382                         |
| 1223342         | 4116, 20702, 111745, 633631                                 |
| 1123244         | 4116, 20702, 111745, 633640                                 |
| 1123451         | 4116, 20702, 111749, 633753, 3767990                         |
| 1123455         | 4116, 20702, 111749, 633753, 3767916                         |
| 1213441         | 4116, 20703, 111772, 634059                                 |
| 1223442         | 4116, 20703, 111772, 634065                                 |
| 1213144         | 4116, 20703, 111774, 634137                                 |
| 1123344         | 4116, 20703, 111774, 634138                                 |
| 1223444         | 4116, 20703, 111776, 634197                                 |
| 1213444, 1223444 | 4116, 20703, 111776, 634203                                |
| 1231141         | 4116, 20707, 111892, 636183, 3800334                         |
| 1223242         | 4116, 20707, 111892, 636183, 3800350                         |
| 1231411         | 4116, 20707, 111892, 636187, 3800368                         |
| 1223422         | 4116, 20707, 111892, 636187, 3800476                         |
| 1231114, 1232244, 123334, 1233343 | 4116, 20707, 111892, 636195 |
| 1223444, 1234111, 1234222, 1234333 | 4117, 20814, 112031, 636507 |
| 1223432         | 4117, 20814, 112033, 636564                                 |
| 1213432         | 4117, 20814, 112033, 636567, 3791466                         |
| 1213242         | 4117, 20814, 112033, 636567, 3791468                         |
| 1213142         | 4117, 20814, 112034, 636593                                 |
| 1213241         | 4117, 20814, 112034, 636598                                 |
| 1213241         | 4117, 20814, 112034, 636610                                 |
| 1223114         | 4117, 20814, 112035, 636623                                 |
| 1223241         | 4117, 20814, 112036, 636655                                 |
| 1223241         | 4117, 20814, 112037, 636686                                 |
| 1221341         | 4117, 20815, 112059, 636978                                 |
| 1221341         | 4117, 20815, 112061, 637033                                 |
| 1211343, 1211343, 1222344, 1222343 | 4117, 20815, 112072, 637389 |
| 1122343, 1122343 | 4117, 20816, 112092, 637612                                |
| 1123412         | 4117, 20816, 112094, 637657                                 |
| 1123142         | 4117, 20816, 112095, 637686                                 |
| 1123124         | 4117, 20816, 112095, 637694                                 |
| 1211345, 1222345 | 4117, 20817, 112118, 637987                                |
| 1123242         | 4117, 20817, 112119, 638047                                 |
| 1122345         | 4117, 20817, 112120, 638059                                 |
| 1123413         | 4117, 20817, 112120, 638067                                 |
| 1123143         | 4117, 20817, 112121, 638103                                 |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$                  | $\{p(n; \tau)\}_{n \geq s}$                                                                 |
|-------------------------|-----------------------------------------------------------------------------------------------|
| 1123134                 | 4117, 20817, 112121, 638105                                                                  |
| 1123422                 | 4117, 20817, 112121, 638107                                                                  |
| 1123224                 | 4117, 20817, 112121, 638109                                                                  |
| 1123244                 | 4117, 20817, 112122, 638140                                                                  |
| 1123421                 | 4117, 20817, 112123, 638166                                                                  |
| 1123214                 | 4117, 20817, 112123, 638172                                                                  |
| 1123241                 | 4117, 20817, 112124, 638199                                                                  |
| 1211344, 122344         | 4117, 20817, 112125, 638232                                                                  |
| 1123145                 | 4117, 20818, 112126, 638255                                                                  |
| 1123314                 | 4117, 20818, 112150, 638607                                                                  |
| 1123431                 | 4117, 20818, 112151, 638635                                                                  |
| 1123341                 | 4117, 20818, 112151, 638640                                                                  |
| 1123334, 1123343, 1123433| 4117, 20818, 112152, 638673                                                                  |
| 1123441                 | 4117, 20818, 112155, 638729                                                                  |
| 1123344                 | 4117, 20818, 112155, 638779                                                                  |
| 1123144                 | 4117, 20818, 112156, 638802                                                                  |
| 1212234, 1221234, 1222134| 4118, 20835, 112311, 639591                                                                  |
| 1212134                 | 4118, 20835, 112312, 639622                                                                  |
| 1211234                 | 4118, 20835, 112312, 639624                                                                  |
| 1221134                 | 4118, 20836, 112338, 640044                                                                  |
| 1213141                 | 4117, 20821, 112237, 640107                                                                  |
| 1223242                 | 4117, 20821, 112237, 640113, 3839906                                                         |
| 1213114                 | 4117, 20821, 112237, 640113, 3839908                                                         |
| 1223224                 | 4117, 20821, 112237, 640121                                                                  |
| 1213411                 | 4117, 20821, 112238, 640143                                                                  |
| 1223422                 | 4117, 20821, 112238, 640147                                                                  |
| 1213432                 | 4118, 20839, 112419, 641372                                                                  |
| 1213424                 | 4118, 20839, 112423, 641478                                                                  |
| 1231332                 | 4118, 20840, 112437, 641524                                                                  |
| 1231332                 | 4118, 20840, 112437, 641538, 3845007                                                        |
| 1213132                 | 4118, 20840, 112437, 641538, 3845023                                                        |
| 1231312                 | 4118, 20840, 112437, 641528, 3845028                                                        |
| 1232313                 | 4118, 20840, 112437, 641529                                                                  |
| 1222133                 | 4118, 20840, 112437, 641530                                                                  |
| 1222133                 | 4118, 20840, 112437, 641537                                                                  |
| 1223133                 | 4118, 20840, 112438, 641563                                                                  |
| 1231132                 | 4118, 20840, 112439, 641589                                                                  |
| 1231323                 | 4118, 20840, 112439, 641590                                                                  |
| 1231213                 | 4118, 20840, 112439, 641591                                                                  |
| 1231322                 | 4118, 20840, 112439, 641592, 3846251                                                        |
| 1232133                 | 4118, 20840, 112439, 641592, 3846236                                                        |
| 1232131                 | 4118, 20840, 112439, 641595                                                                  |
| 1212313                 | 4118, 20840, 112439, 641601                                                                  |
| 1231223                 | 4118, 20840, 112440, 641620                                                                  |
| 1232123                 | 4118, 20840, 112440, 641622                                                                  |
| 1231323                 | 4118, 20840, 112440, 641623, 3846806                                                        |
| 1231232                 | 4118, 20840, 112440, 641623, 3846810                                                        |
| 1223132                 | 4118, 20840, 112440, 641623, 3846811                                                        |
| 1231213                 | 4118, 20840, 112440, 641623, 3846814                                                        |
| 1231321                 | 4118, 20840, 112440, 641623, 3846829                                                        |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$ | $\{p(n; \tau)\}_{n \geq 4}$ |
|--------|---------------------------|
| 1232132 | 4118, 20840, 112440, 641624 |
| 1232132 | 4118, 20840, 112440, 641625 |
| 1232332 | 4118, 20840, 112440, 641634 |
| 1232332 | 4118, 20840, 112440, 641652, 3847223 |
| 1232332 | 4118, 20840, 112440, 641652, 3847318 |
| 1232332 | 4118, 20840, 112441, 641653 |
| 1232332 | 4118, 20840, 112441, 641654, 3847386 |
| 1232332 | 4118, 20840, 112441, 641654, 3847388 |
| 1232332 | 4118, 20840, 112444, 641654, 3847397 |
| 1232331 | 4118, 20840, 112444, 641655, 3847421, 24108094 |
| 1232331 | 4118, 20840, 112444, 641655, 3847421, 24108095 |
| 1232313 | 4118, 20840, 112444, 641682, 3847867 |
| 1233312, 1233321 | 4118, 20840, 112442, 641682, 3847869 |
| 1232321 | 4118, 20840, 112442, 641684, 3847930 |
| 1232321 | 4118, 20840, 112442, 641684, 3847931 |
| 1232321 | 4118, 20840, 112442, 641684, 3847935 |
| 1232321 | 4118, 20840, 112442, 641684, 3847937 |
| 1232321 | 4118, 20840, 112442, 641684, 3847939 |
| 1232321 | 4118, 20840, 112442, 641686 |
| 1231323 | 4118, 20840, 112443, 641696 |
| 1223313 | 4118, 20840, 112443, 641708 |
| 1233322 | 4118, 20840, 112443, 641712 |
| 1233231 | 4118, 20840, 112444, 641749, 3849207 |
| 1233233 | 4118, 20840, 112444, 641749, 3849214 |
| 1123442 | 4118, 20840, 112445, 641778 |
| 1123334 | 4118, 20840, 112446, 641807 |
| 1122334 | 4118, 20840, 112448, 641860 |
| 1123341 | 4118, 20840, 112448, 641868 |
| 1233131 | 4118, 20841, 112468, 642087, 3852874 |
| 1223131 | 4118, 20841, 112468, 642087, 3852877 |
| 1213132 | 4118, 20841, 112468, 642092 |
| 1233133 | 4118, 20841, 112468, 642095 |
| 1233333 | 4118, 20841, 112468, 642098 |
| 1233231 | 4118, 20841, 112468, 642099 |
| 1233133 | 4118, 20841, 112469, 642117 |
| 1233322 | 4118, 20841, 112469, 642124 |
| 1213332 | 4118, 20841, 112469, 642126 |
| 1233122, 1233221 | 4118, 20841, 112469, 642128, 3853761, 24182582 |
| 1234112 | 4118, 20841, 112469, 642128, 3853761, 2418255 |
| 1234112 | 4118, 20841, 112469, 642131 |
| 1213322 | 4118, 20841, 112469, 642132 |
| 1223322 | 4118, 20841, 112470, 642155 |
| 1223321 | 4118, 20841, 112471, 642189 |
| 1213331 | 4118, 20841, 112471, 642198 |
| 1223331 | 4118, 20841, 112471, 642199 |
| 1233211 | 4118, 20841, 112472, 642224 |
| 1233311 | 4118, 20841, 112472, 642231 |
| 1223331 | 4118, 20841, 112473, 642250 |
| 1212233, 1221233, 1222133 | 4118, 20841, 112473, 642253 |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$                  | $(p(n; \tau))_{n \geq 8}$                                      |
|-------------------------|---------------------------------------------------------------|
| 1211233                 | 6118, 20841, 112473, 642257                                    |
| 1212133                 | 6118, 20841, 112474, 642283                                    |
| 1212333, 1221333        | 6118, 20841, 112475, 642320, 3857455, 24238859                |
| 1121344, 1121343        | 6118, 20841, 112480, 642458                                    |
| 1212341                 | 6118, 20842, 112498, 642629                                    |
| 1223311                 | 6118, 20842, 112500, 642701, 3862079, 24288261                |
| 1221333                 | 6118, 20842, 112500, 642701, 3862079, 24288297                |
| 1121345                 | 6118, 20842, 112506, 642847                                    |
| 1112344                 | 6118, 20842, 112508, 642928                                    |
| 1211314                 | 6118, 20843, 112531, 643262                                    |
| 1222324                 | 6118, 20843, 112531, 643274                                    |
| 1211341                 | 6118, 20843, 112533, 643325                                    |
| 1222342                 | 6118, 20843, 112533, 643330                                    |
| 1123141                 | 6118, 20845, 112588, 644220                                    |
| 1123114                 | 6118, 20845, 112588, 644226                                    |
| 1123411                 | 6118, 20845, 112588, 644230                                    |
| 1121234                 | 6119, 20860, 112699, 644440                                    |
| 1122234                 | 6119, 20861, 112724, 644829                                    |
| 1122134                 | 6119, 20861, 112725, 644859                                    |
| 1123132                 | 6119, 20862, 112751, 645259                                    |
| 1123213                 | 6119, 20862, 112752, 645297                                    |
| 1123123                 | 6119, 20862, 112753, 645319                                    |
| 1122323                 | 6119, 20862, 112757, 645441                                    |
| 1112342                 | 6119, 20862, 112758, 645467                                    |
| 1112324                 | 6119, 20862, 112760, 645522                                    |
| 1123312                 | 6119, 20863, 112778, 645710                                    |
| 1123132                 | 6119, 20863, 112778, 645721                                    |
| 1122332                 | 6119, 20863, 112779, 645747                                    |
| 1123232                 | 6119, 20863, 112780, 645776, 3892647                            |
| 1123232                 | 6119, 20863, 112780, 645776, 3892648                            |
| 1123321                 | 6119, 20863, 112781, 645796                                    |
| 1123321                 | 6119, 20863, 112781, 645796                                    |
| 1123232                 | 6119, 20863, 112781, 645807, 3893233, 24584926                |
| 1123232                 | 6119, 20863, 112781, 645807, 3893233, 24584935                |
| 1211134, 1222334        | 6119, 20863, 112781, 645808                                    |
| 1121233                 | 6119, 20863, 112783, 645862                                    |
| 1123322                 | 6119, 20864, 112805, 646169                                    |
| 1122344, 1112343        | 6119, 20864, 112807, 646204                                    |
| 1122332                 | 6119, 20864, 112807, 646223                                    |
| 1122332                 | 6119, 20864, 112807, 646223                                    |
| 1123133                 | 6119, 20864, 112810, 646306                                    |
| 1122333                 | 6119, 20864, 112810, 646307                                    |
| 1112345                 | 6119, 20864, 112814, 646384                                    |
| 1121344                 | 6119, 20864, 112814, 646403                                    |
| 1231313                 | 6119, 20866, 112848, 646718                                    |
| 1231331                 | 6119, 20866, 112850, 646778                                    |
| 1231331                 | 6119, 20866, 112850, 646779                                    |
| continue                |                                                               |
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

$$\tau \quad \{p(n; \tau)\}_{n \geq 20}$$

| $\tau$  | 4119, 20866, 112850, 646794 |
|---------|-------------------------------|
| 122132  | 4119, 20866, 112850, 646795   |
| 122131  | 4119, 20866, 112850, 646796   |
| 122132  | 4119, 20866, 112850, 646797   |
| 122131  | 4119, 20866, 112851, 646823   |
| 123121, 123323 | 4119, 20866, 112851, 646824, 3905223, 24711200, 163188860 |
| 123123  | 4119, 20866, 112851, 646824, 3905223, 24711200, 163188863 |
| 123121  | 4119, 20866, 112851, 646824, 3905223, 24711200, 163188865 |
| 123121, 123323 | 4119, 20866, 112851, 646824, 3905223, 24711204 |
| 122133  | 4119, 20866, 112851, 646824, 3905223 |
| 121232  | 4119, 20866, 112851, 646824, 3905224 |
| 122132  | 4119, 20866, 112851, 646824, 3905230 |
| 121232  | 4119, 20866, 112851, 646824, 3905236 |
| 121322  | 4119, 20866, 112851, 646824, 3905254 |
| 123121, 123332 | 4119, 20866, 112851, 646825 |
| 122133  | 4119, 20866, 112851, 646826, 3905284 |
| 122132  | 4119, 20866, 112851, 646826, 3905292 |
| 123113  | 4119, 20866, 112852, 646837 |
| 122132  | 4119, 20866, 112852, 646842, 3905432 |
| 123133  | 4119, 20866, 112852, 646842, 3905461 |
| 121333  | 4119, 20866, 112852, 646845 |
| 122112  | 4119, 20866, 112852, 646847 |
| 121133  | 4119, 20866, 112852, 646848 |
| 123211, 123323 | 4119, 20866, 112852, 646850 |
| 122123  | 4119, 20866, 112852, 646851 |
| 121123  | 4119, 20866, 112852, 646856 |
| 111333  | 4119, 20866, 112853, 646874 |
| 122333  | 4119, 20866, 112853, 646886 |
| 122131  | 4119, 20866, 112854, 646902 |
| 122112, 123223 | 4119, 20866, 112854, 646908 |
| 121123  | 4119, 20866, 112854, 646909 |
| 122333  | 4119, 20866, 112854, 646912 |
| 122133  | 4119, 20866, 112856, 646963 |
| 122311, 122321, 123221, 123331 | 4119, 20866, 112856, 646966 |
| 123322  | 4119, 20866, 112856, 646967 |
| 122333  | 4119, 20866, 112856, 646973 |
| 121333, 122333 | 4119, 20866, 112860, 647094 |
| 112131  | 4119, 20867, 112883, 647408 |
| 112134  | 4119, 20867, 112884, 647438 |
| 123313  | 4119, 20868, 112906, 647727 |
| 123333  | 4119, 20868, 112906, 647729, 3917135 |
| 123133  | 4119, 20868, 112906, 647729, 3917150 |
| 121132  | 4119, 20868, 112906, 647739 |
| 121223, 121232, 121322, 121322 | 4119, 20868, 112906, 647739 |
| 122123, 122132, 122132, 122213 | 4119, 20868, 112906, 647739 |
| 122212, 122213, 122321, 122321 | 4119, 20868, 112906, 647739 |
| 122321, 122312, 122321, 122321 | 4119, 20868, 112906, 647739 |
| 123233, 123323, 123333, 123333 | 4119, 20868, 112906, 647739 |

**continue**
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$ | $(p(n; \tau))_{n \geq 8}$ |
|--------|-----------------------------|
| 123332 | 4119, 20868, 112906, 647744, 3917573 |
| 123131 | 4119, 20868, 112907, 647757 |
| 121313 | 4119, 20868, 112907, 647763 |
| 123113 | 4119, 20868, 112907, 647764 |
| 121312 | 4119, 20868, 112907, 647767 |
| 121132 | 4119, 20868, 112907, 647769 |
| 123112, 123223 | 4119, 20868, 112907, 647774, 3918115 |
| 122223 | 4119, 20868, 112907, 647774, 3918120 |
| 121131 | 4119, 20868, 112907, 647776 |
| 123112, 123223 | 4119, 20868, 112907, 647778 |
| 121123 | 4119, 20868, 112907, 647779 |
| 123131 | 4119, 20868, 112908, 647788, 3918150 |
| 123131 | 4119, 20868, 112908, 647788, 3918180 |
| 121331 | 4119, 20868, 112908, 647796 |
| 121231 | 4119, 20868, 112908, 647797 |
| 121231 | 4119, 20868, 112908, 647801 |
| 123211 | 4119, 20868, 112908, 647802 |
| 123121 | 4119, 20868, 112908, 647804 |
| 121113 | 4119, 20868, 112908, 647806, 3918705 |
| 123121, 123232 | 4119, 20868, 112908, 647806, 3918717, 24875277 |
| 122322 | 4119, 20868, 112908, 647806, 3918717, 24875279 |
| 122323 | 4119, 20868, 112908, 647806, 3918717, 24875282 |
| 121311 | 4119, 20868, 112909, 647832 |
| 121321 | 4119, 20868, 112909, 647836 |
| 122131 | 4119, 20869, 112936, 648262 |
| 122131 | 4119, 20869, 112936, 648263 |
| 122213 | 4119, 20869, 112936, 648270 |
| 121333, 122333 | 4119, 20869, 112936, 648283 |
| 121331 | 4119, 20869, 112937, 648299 |
| 122311, 123311, 123211, 123222 | 4119, 20869, 112937, 648301, 3925257 |
| 121331 | 4119, 20869, 112937, 648301, 3925333 |
| 122332 | 4119, 20869, 112937, 648308 |
| 123332 | 4119, 20869, 112937, 648310 |
| 1211133, 1222333 | 4119, 20869, 112937, 648316 |
| 1122334 | 4120, 20883, 113036, 648480 |
| 1122332 | 4120, 20883, 113048, 648817 |
| 1122332 | 4120, 20883, 113048, 648818 |
| 1122333 | 4120, 20884, 113074, 649231 |
| 1123122 | 4120, 20886, 113110, 649636 |
| 1123212 | 4120, 20886, 113110, 649640 |
| 112322 | 4120, 20886, 113110, 649650 |
| 112322 | 4120, 20886, 113111, 649672 |
| 112223 | 4120, 20886, 113111, 649673 |
| 1122312 | 4120, 20886, 113112, 649692 |
| 1122312 | 4120, 20886, 113112, 649694, 3934278 |
| 1122312 | 4120, 20886, 113112, 649694, 3934279 |
| 1122321 | 4120, 20886, 113114, 649746 |
| 1122231 | 4120, 20886, 113114, 649751 |
| 1122231 | 4120, 20886, 113114, 649752, 3935305 |
| 1122321 | 4120, 20886, 113114, 649752, 3935315 |

continue
Table 6: Number of partitions in \( P(n; \tau) \), where \( \tau \in P(7) \).

| \( \tau \) | \( \{ p(n; \tau) \}_{n \geq 2} \) |
|---|---|
| 1121134 | 4120, 20887, 113134, 650020 |
| 1123133 | 4120, 20887, 113137, 650074, 3938938 |
| 1123313 | 4120, 20887, 113137, 650074, 3938943 |
| 1123331 | 4120, 20887, 113141, 650187 |
| 1123333 | 4120, 20887, 113145, 650310 |
| 112314 | 4120, 20888, 113162, 650462 |
| 112341 | 4120, 20888, 113162, 650471 |
| 1123112 | 4120, 20888, 113163, 650505 |
| 1121132 | 4120, 20888, 113163, 650509 |
| 1121312 | 4120, 20888, 113164, 650531 |
| 112131 | 4120, 20888, 113164, 650537 |
| 1121123 | 4120, 20888, 113164, 650539 |
| 112123 | 4120, 20888, 113164, 650540 |
| 1121321 | 4120, 20888, 113165, 650564 |
| 112132 | 4120, 20888, 113165, 650568 |
| 1123113 | 4120, 20889, 113188, 650884 |
| 1123222, 1122322, 1123222 | 4120, 20889, 113188, 650904 |
| 112331 | 4120, 20889, 113189, 650907 |
| 1123313 | 4120, 20889, 113189, 650913 |
| 112311 | 4120, 20889, 113191, 650978 |
| 1122113 | 4120, 20889, 113191, 650985 |
| 112321 | 4120, 20889, 113192, 651006 |
| 112311 | 4120, 20889, 113193, 651015 |
| 112331 | 4120, 20890, 113216, 651360 |
| 112311 | 4120, 20890, 113216, 651362 |
| 112313 | 4120, 20890, 113216, 651375 |
| 112333 | 4120, 20890, 113216, 651379 |
| 112222, 1112232, 1112332 | 4121, 20904, 113336, 652122 |
| 1121312 | 4121, 20906, 113380, 652756 |
| 1121322 | 4121, 20906, 113381, 652784, 3967043, 25325818 |
| 1121323 | 4121, 20906, 113381, 652784, 3967043, 25325832 |
| 112333 | 4121, 20906, 113385, 652864 |
| 112134 | 4121, 20908, 113412, 653041 |
| 1121213 | 4121, 20907, 113406, 653173 |
| 112321 | 4121, 20907, 113407, 653198 |
| 112231 | 4121, 20907, 113407, 653204 |
| 112313 | 4121, 20908, 113425, 653385 |
| 121131 | 4120, 20905, 113346, 653419, 3982042 |
| 121331 | 4120, 20905, 113346, 653419, 3982043 |
| 121311 | 4120, 20905, 113346, 653419, 3982063 |
| 121111, 1222223, 1222232, 1222322 | 4120, 20905, 113346, 653419, 3982093 |
| 1223222, 1231111, 1232222, 1233333 | 4121, 20909, 113450, 653784 |
| 1112331 | 4121, 20909, 113450, 653791 |
| 1121332, 1111232 | 4122, 20923, 113586, 654964 |
| 111234 | 4122, 20926, 113627, 655233 |
| 1112333 | 4122, 20926, 113643, 655673 |
| 112131 | 4121, 20914, 113574, 655691, 4002872 |
| 112113 | 4121, 20914, 113574, 655691, 4002915 |
| 1121311 | 4121, 20914, 113574, 655692 |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$          | $\{p(n; \tau)\}_{n \geq 8}$                          |
|-----------------|-------------------------------------------------------|
| 112111          | 412, 20914, 113575, 655720                            |
| 1112113         | 412, 20931, 113763, 657478                            |
| 1112131         | 412, 20931, 113763, 657480                            |
| 1112311         | 412, 20931, 113763, 657533                            |
| 1111213         | 412, 20946, 113917, 658862                            |
| 1111231         | 412, 20946, 113920, 658937                            |
| 1111123         | 412, 20959, 114044, 660003                            |
| 1221122         | 412, 20966, 114159, 661276                            |
| 1221222         | 412, 20966, 114159, 661277                            |
| 1221122         | 412, 20966, 114159, 661279, 4052524, 26151249, 176986852 |
| 1221221         | 412, 20966, 114159, 661279, 4052524, 26151249, 176986866 |
| 1221122         | 412, 20966, 114159, 661279, 4052526                     |
| 1221122         | 412, 20966, 114159, 661280                            |
| 1221122         | 412, 20966, 114159, 661282                            |
| 1221122         | 412, 20966, 114159, 661284                            |
| 1221122         | 412, 20966, 114159, 661286                            |
| 1221122         | 412, 20966, 114161, 661324                            |
| 1221122         | 412, 20966, 114161, 661326                            |
| 1221122         | 412, 20966, 114161, 661328                            |
| 1221122         | 412, 20966, 114161, 661329, 4053312, 26162919, 176986866 |
| 1221122         | 412, 20966, 114161, 661329, 4053324                     |
| 1221122         | 412, 20966, 114161, 661330                            |
| 1112121         | 412, 20966, 114161, 661331, 4053377, 26162919, 176986866 |
| 1112122         | 412, 20966, 114161, 661331, 4053377, 26162926           |
| 1112122         | 412, 20966, 114161, 661332, 4053398                     |
| 1112122         | 412, 20966, 114161, 661332, 4053400                     |
| 1112122         | 412, 20966, 114161, 661333, 4053404                     |
| 1112122         | 412, 20966, 114161, 661333, 4053421                     |
| 1112122         | 412, 20966, 114161, 661333, 4053424                     |
| 1112122         | 412, 20966, 114161, 661334, 4053429                     |
| 1112122         | 412, 20966, 114161, 661333, 4053431                     |
| 1112122         | 412, 20966, 114161, 661334, 4053447                     |
| 1112122         | 412, 20966, 114161, 661337                            |
| 1222111         | 412, 20966, 114161, 661339                            |
| 1112122         | 412, 20966, 114161, 661340, 4053579                     |
| 1122212         | 412, 20966, 114161, 661340, 4053593                     |
| 1122111         | 412, 20966, 114165, 661427                            |
| 1122111         | 412, 20966, 114165, 661429                            |
| 1122111         | 412, 20966, 114165, 661432                            |
| 1112222         | 412, 20966, 114165, 661439                            |
| 1112222         | 412, 20966, 114165, 661444                            |
| 1222222, 1221222, 1222122, 1222212, 1222221         | 412, 20968, 114209, 662046                            |
| 1222222         | 412, 20968, 114210, 662074, 4062705                     |
| 1222222         | 412, 20968, 114210, 662074, 4062709, 26270000           |
| 1222222         | 412, 20968, 114210, 662074, 4062709, 26270389           |
| 1121212         | 412, 20968, 114210, 662075, 4062730                     |
| 1221211         | 412, 20968, 114210, 662075, 4062742                     |
| 1221211         | 412, 20968, 114210, 662075, 4062747                     |

continue
Table 6: Number of partitions in $P(n; \tau)$, where $\tau \in P(7)$.

| $\tau$       | $\{p(n; \tau)\}_{n \geq 4}$ |
|--------------|-----------------------------|
| 111211       | 4124, 20968, 114211, 662099, 4063105, 26275156 |
| 121211       | 4124, 20968, 114211, 662099, 4063105, 26275158, 178364080 |
| 111212       | 4124, 20968, 114211, 662099, 4063105, 26275158, 178364100 |
| 112211       | 4124, 20968, 114211, 662099, 4063106 |
| 112222       | 4124, 20969, 114233, 662412 |
| 122111       | 4124, 20969, 114234, 662438, 4067231 |
| 111121       | 4124, 20969, 114234, 662438, 4067256 |
| 111211       | 4124, 20969, 114234, 662438, 4067265 |
| 111122       | 4124, 20969, 114234, 662438, 4067287 |
| 1111111111   | 4125, 20990, 114516, 665004 |
| 1111111     | 4131, 21065, 115274, 672673 |

References

[1] A. V. Aho, R. Sethi, and J. D. Ullman, Compilers: principles, techniques, and tools, Addison-Wesley, Reading, Mass., 1986

[2] J. Backelin, J. West, and G. Xin, Wilf-equivalence for singleton classes, Adv. Appl. Math. 32:2 (2007) 133–148.

[3] William Y. C. Chen, Eva Y. P. Deng, and Rosena R. X. Du, Reduction of $m$-regular noncrossing Partitions, Europ. J. Combin. 26:2 (2005) 237–243.

[4] William Y. C. Chen, Eva Y. P. Deng, Rosena R. X. Du, R. P. Stanley, and Catherine H. Yan, Crossings and Nestings of Matchings and Partitions, Trans. Amer. Math. Soc. 359(4) (2007) 1555–1575.

[5] L. Comtet, Advanced Combinatorics, Presses Universitaires de France Io8, Boulevard Saint-Germain, Paris, 1970.

[6] H. Davenport and A. Schinzel, A combinatorial problem connected with differential equations, Amer. J. Math. 87 (1965) 684–694.

[7] A. de Mier, $k$-noncrossing and $k$-nonnesting graphs and fillings of Ferrers diagrams, arXiv: math.CO/0602195.

[8] A. de Mier, On the Symmetry of the Distribution of $k$-crossings and $k$-nestings in Graphs, Elect. J. Combin. 13 (2006) #N21.

[9] P. Flajolet and R. Sedgewick Analytic Combinatorics, Part A, Chapter II; available at http://algo.inria.fr/flajolet/Publications/books.html, 2006.

[10] W. Fulton, Young Tableaux, London Mathematical Society Student Texts 35; Cambridge University Press, 1997.

[11] A. M. Goyt, Avoidance of partitions of a three-element set, arXiv: math.CO/0603481.

[12] J. E. Hopcroft, R. Motwani, and J. D. Ullman, Introduction to automata theory, languages, and computation, Addison-Wesley, 2001.

[13] M. Klazar, On abab-free and abba-free set partitions, Europ. J. Combin. 17 (1996) 53–68.

[14] M. Klazar, On trees and noncrossing partitions, Discr. Appl. Math. 82 (1998) 263–269.

[15] C. Krattenthaler, Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes, Adv. Appl. Math. 37:3 (2006) 404–431.

[16] T. Mansour, Combinatorial methods and recurrence relations with two indices, J. Diff. Eq. Appl. 12:6 (2006) 555–563.

[17] T. Mansour and S. Severini, Enumeration of $(k, 2)$-noncrossing partitions, preprint.

[18] R. C. Mullin and R. G. Stanton, A map-theoretic approach to Davenport-Schinzel sequences, Pacific J. Math. 40 (1972) 167–172.

[19] M. Rubey, Increasing and decreasing sequences in fillings of moon polyominoes, arXiv: math.CO/0604140.

[20] B. Sagan, Pattern avoidance in set partitions, arXiv: math.CO/0604292.

[21] N. J. A. Sloane (ed.), The On-Line Encyclopedia of Integer Sequences (http://www.research.att.com/~njas/sequences/).

[22] Z. Stankova and J. West, A new class of Wilf-equivalent permutations, J. Alg. Combin. 15 (2002) 271–290.

[23] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge University Press, Cambridge, UK, 1996.

[24] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999.
[25] M. Wachs and D. White, p,q-Stirling numbers and set partition statistics, *J. Combin. Theory, Series A*, **56**:1 (1991) 27–46.

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