LIMITING FRACTIONAL AND LORENTZ SPACES ESTIMATES OF DIFFERENTIAL FORMS

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Abstract. We obtain estimates in Besov, Lizorkin-Triebel and Lorentz spaces of differential forms on \( \mathbb{R}^n \) in terms of their \( L^1 \) norm.

1. Introduction

The classical Hodge theory states that if \( u \in C_c^\infty(\mathbb{R}^N; \bigwedge^\ell \mathbb{R}^n) \), if \( 1 < p < \infty \), one has

\[
\|Du\|_{L^p} \leq C(\|du\|_{L^p} + \|\delta u\|_{L^p})
\]

where \( du \) is the exterior differential and \( \delta u \) the exterior codifferential. This estimate is known to fail when \( p = 1 \) or \( p = \infty \).

When \( p = 1 \), J. Bourgain and H. Brezis [2, 3], and L. Lanzani and E. Stein [5] have obtained for \( 2 \leq \ell \leq n - 2 \) the estimate

\[
\|u\|_{L^{n/(n-1)}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}),
\]

which would be the consequence that would follow by the Sobolev embedding from (1) with \( p = 1 \). When \( \ell = 1 \) or \( \ell = n - 1 \) one has to assume that \( du \) or \( \delta u \) vanishes.

I. Mitrea and M. Mitrea [6] have in a recent work extended these estimates to homogeneous Besov spaces. Using interpolation theory, they could replace the norm \( \|u\|_{L^{n/(n-1)}} \) by \( \|u\|_{\dot{B}^s_{p,q}} \) with \( \frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n} \) and \( q = \frac{2}{1+q} \). The goal of the present paper is to improve the assumption on \( q \) by relying on previous results and methods.

We follow H. Triebel [10] for the definitions of the function spaces. The first result is the estimate for the Besov spaces \( \dot{B}^s_{p,q}(\mathbb{R}^n) \):

**Theorem 1.** For every \( s \in (0, 1) \), \( p > 1 \) and \( q > 1 \), if

\[
\frac{1}{p} - \frac{s}{n} = 1 - \frac{1}{n},
\]

then there exists \( C > 0 \) such that for every \( u \in C_c^\infty(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \), with moreover, \( \delta u = 0 \) if \( \ell = 1 \) and \( du = 0 \) if \( \ell = n - 1 \), one has

\[
\|u\|_{\dot{B}^s_{p,q}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).
\]

In particular, since \( \|u\|_{W^{s,p}} = \|u\|_{\dot{B}^s_{p,p}} \), one has the estimate

\[
\|u\|_{W^{s,p}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).
\]

In Theorem \( \square \) we assume that \( q > 1 \). If it held for some \( q \in (0, 1] \), then the embedding of \( F_{1,2}^1(\mathbb{R}^n) \subset \dot{B}^{0}_{n/(n-1),q}(\mathbb{R}^n) \) would hold. This can only be the case

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when \( q \geq 1 \). Therefore, the only possible improvement of Theorem 1 would be the limiting case \( q = 1 \):

**Open problem 1.** Does Theorem 1 hold for \( q = 1 \)?

The estimate of Theorem 1 follows from the corresponding estimate for homogeneous Lizorkin–Triebel spaces \( F^s_{p,q}(\mathbb{R}^n) \):

**Theorem 2.** For every \( s \in (0,1), p > 1 \) and \( q > 0 \), if (2) holds, then there exists \( C > 0 \) such that for every \( u \in C^\infty_c(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \), with moreover, \( \delta u = 0 \) if \( \ell = 1 \) and \( du = 0 \) if \( \ell = n-1 \), one has

\[
\|u\|_{F^s_{p,q}} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).
\]

Note that here there is no restriction on \( q > 0 \). Finally, the latter estimate has an interesting consequence for Lorentz spaces.

**Theorem 3.** For every \( q > 1 \), then there exists \( C > 0 \) such that for every \( u \in C^\infty_c(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \), with moreover, \( \delta u = 0 \) if \( \ell = 1 \) and \( du = 0 \) if \( \ell = n-1 \), one has

\[
\|u\|_{L^{p+1-\frac{1}{q}}(\mathbb{R}^n)} \leq C(\|du\|_{L^1} + \|\delta u\|_{L^1}).
\]

In Theorem 3 the case \( q = 1 \) and \( \ell = 0 \) is equivalent to the embedding of \( W^{1,1}(\mathbb{R}^n) \) in \( L^{p+1-\frac{1}{q}}(\mathbb{R}^n) \) which was obtained by J. Peetre [8] (see also [15]). This raises the question

**Open problem 2.** Does Theorem 3 hold for \( q = 1 \) and \( \ell \geq 1 \)?

The proof of the theorems rely on the techniques developed by the author [11,12], and on classical embeddings and regularity theory in fractional spaces.

## 2. The main tool

Our main tool is a generalization of an estimate for divergence-free \( L^1 \) vector fields of the author [11]:

**Proposition 2.1.** For every \( s \in (0,1), p > 1 \) and \( q > 0 \) with \( sp = n \), there exists \( C > 0 \) such that for every \( f \in (C^\infty_c \cap L^1)(\mathbb{R}^n; \bigwedge^{n-1} \mathbb{R}^n) \) and \( \varphi \in C^\infty_c(\mathbb{R}^n; \bigwedge \mathbb{R}^n) \), with \( df = 0 \),

\[
\int_{\mathbb{R}^n} f \wedge \varphi \leq C\|f\|_{L^1}\|\varphi\|_{F^s_{p,q}}.
\]

The proof of this proposition follows the method introduced by the author [4, 11,12,14] and followed subsequently by L. Lanzani and E. Stein [5] and I. Mitrea and M. Mitrea [6]. The extension to the case \( q = p \) in a previous work of the author [11, Remark 5] (see also [14, Remark 2] and [4]); the proposition can be deduced therefrom by following a remark in a subsequent paper [13, Remark 4.2].

**Proof.** Write \( \varphi = \varphi_1 dx_1 + \varphi^n dx_n \) and \( f = f_1 dx_2 \wedge \ldots \wedge dx_n + \ldots + f_n dx_1 \wedge \ldots \wedge dx_{n-1} \).

Without loss of generality, we shall estimate

\[
\int_{\mathbb{R}^n} f_1 \varphi^1.
\]

Fix \( t \in \mathbb{R} \), and consider the function \( \psi : \mathbb{R}^{n-1} \to \mathbb{R} \) defined by \( \psi(y) = \varphi^1(t,y) \). Choose \( \rho \in C^\infty_c(\mathbb{R}^n) \) such that \( \int_{\mathbb{R}^{n-1}} \rho = 1 \) and set \( \rho_\varepsilon(y) = \frac{1}{\varepsilon^n} \rho(y/\varepsilon) \). For every \( \alpha \in (0,1) \), there is a constant \( C > 0 \) that only depends on \( \rho \) and \( \alpha \) such that

\[
\|\nabla \rho_\varepsilon \ast \psi\|_{L^\infty} \leq C\varepsilon^{\alpha-1}\|\psi\|_{C^{0,\alpha}(\mathbb{R}^{n-1})}
\]

and

\[
\|\psi - \rho_\varepsilon \ast \psi\|_{L^\infty} \leq C\varepsilon^\alpha\|\psi\|_{C^{0,\alpha}(\mathbb{R}^{n-1})},
\]
where \( |\psi|_{C^{0,\alpha}(\mathbb{R}^{n-1})} \) is the \( C^{0,\alpha} \) seminorm of \( \psi \), i.e.,

\[
|\psi|_{C^{0,\alpha}(\mathbb{R}^{n-1})} = \sup_{y, z \in \mathbb{R}^{n-1}} \frac{|\psi(z) - \psi(y)|}{|z - y|}.
\]

One has on the one hand

\[
\int_{\mathbb{R}^{n-1}} f_1(t, \cdot)(\psi - \rho_x \ast \psi) \leq C \|f_1(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} \varepsilon^{\alpha} |\psi|_{C^{0,\alpha}(\mathbb{R}^{n-1})}.
\]

On the other hand, by integration by parts, and since \( \sum_{i=1}^{n} \partial_i f_i = 0 \),

\[
\int_{\mathbb{R}^{n-1}} f_1(t, \cdot) \rho_x \ast \psi = - \sum_{i=2}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^+} f_i(t, y) \partial_i (\rho_x \ast \psi)(y) \, dt \, dy 
\leq C \|f_1(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} \varepsilon^{\alpha-1} |\psi|_{C^{0,\alpha}(\mathbb{R}^{n-1})}.
\]

Taking \( \varepsilon = \|f\|_{L^1(\mathbb{R}^n)}/\|f(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} \), one obtains

\[
(4) \quad \int_{\mathbb{R}^{n-1}} f_1 \psi \leq C \|f\|_{L^1(\mathbb{R}^n)} \|f_1(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} |\psi|_{C^{0,\alpha}(\mathbb{R}^{n-1})}.
\]

Now, by the embedding theorem for Lizorkin–Triebel spaces, one has the estimate

\[
|\psi|_{C^{0,\alpha}} \leq C \|\psi\|_{F^{\alpha}_p,q(\mathbb{R}^{n-1})};
\]

with \( \alpha = \frac{1}{p} \); hence from (1) we deduce the inequality

\[
\int_{\mathbb{R}^{n-1}} f_1 \psi \leq C \|f\|_{L^1} \|f_1(t, \cdot)\|_{L^1}^{1 - \frac{1}{p}} \|\psi\|_{F^{\alpha}_p,q(\mathbb{R}^{n-1})}.
\]

Now, recalling that, as a direct consequence of the Fubini property that is stated in [10, Theorem 2.5.13] [1, Théorème 2], [9, Theorem 2.3.4/2]

\[
\left( \int_{\mathbb{R}} \|\varphi(t, \cdot)\|_{F^{\alpha,p}(\mathbb{R}^{n-1})}^p \, dt \right)^{\frac{1}{p}} \leq C \|\varphi\|_{F^{\alpha,p}(\mathbb{R}^n)}
\]

one concludes, using Hölder’s inequality that

\[
\int_{\mathbb{R}} f_1 u^1 \leq C \|f\|_{L^1} \left( \int_{\mathbb{R}} \left( \|f_1(t, \cdot)\|_{L^1}^{1 - \frac{1}{p}} \|\varphi(t, \cdot)\|_{F^{\alpha,p}(\mathbb{R}^{n-1})} \right) \, dt \right) \leq C' \|f\|_{L^1} \|\varphi\|_{F^{\alpha,p}(\mathbb{R}^n)}. \tag*{\Box}
\]

**Proposition 2.2.** For every \( s \in (0, 1) \), \( p > 1 \) with \( \frac{1}{p} + \frac{s}{n} = 1 \), \( q > 1 \) and \( 1 \leq \ell \leq n-1 \), there exists \( C > 0 \) such that for every \( f \in C_c^\infty(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \) with \( df = 0 \), one has

\[
\|f\|_{\dot{F}^{-s,p}_q} \leq C \|f\|_{L^1}.
\]

**Proof.** The proposition will be proved by downward induction. The proposition is true for \( \ell = n-1 \) by Proposition 2.1. Assume now that it holds for \( \ell + 1 \), and let \( f \in C_c^\infty(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \). Since \( df \wedge dx_i = 0 \), Proposition 2.1 is applicable and

\[
\|f\|_{\dot{F}^{-s,p}_q} \leq \sum_{i=1}^{n} \|f \wedge dx_i\|_{\dot{F}^{-s,p}_q} \leq C \sum_{i=1}^{n} \|f\|_{L^1} = Cn \|f\|_{L^1}. \tag*{\Box}
\]

A useful corollary of the previous proposition is

**Corollary 2.3.** For every \( s \in (0, 1) \), \( p > 1 \) with \( \frac{1}{p} + \frac{s}{n} = 1 \), \( q > 1 \) and \( 1 \leq \ell \leq n-1 \), there exists \( C > 0 \) such that for every \( f \in C_c^\infty(\mathbb{R}^n; \bigwedge^\ell \mathbb{R}^n) \) with \( df = 0 \), one has

\[
\|f\|_{B_q^{-s,p}} \leq C \|f\|_{L^1}.
\]

**Proof.** This follows from classical embeddings between Besov and Lizorkin–Triebel spaces; see the proof of Theorem 2.1 below. \( \Box \)
3. Proofs of the main results

We begin by proving Theorem 2:

Proof of Theorem 2: To fix ideas, assume that $2 \leq \ell \leq n - 1$. Recall that one has

$$u = d(K*(\delta u)) + \delta(K*(du)),$$

where the Newton kernel is defined by $K(x) = \frac{\Gamma(\frac{\ell}{2})}{2\pi^{\frac{n}{2}}|x|^{n-\ell}}$. By the classical elliptic estimates for Lizorkin–Triebel spaces,

$$\|K*(\delta u)\|_{\dot{F}^{s+1}_{p,q}} \leq C\|\delta u\|_{\dot{F}^{s-1}_{p,q}}$$

and

$$\|K*(\delta u)\|_{\dot{F}^{s+1}_{p,q}} \leq C\|\delta u\|_{\dot{F}^{s-1}_{p,q}}.$$

Now, since $d(\delta u) = 0$, Proposition 2.2 is applicable and yields

$$\|K*(du)\|_{\dot{F}^{s+1}_{p,q}} \leq C\|\delta u\|_{L^1}.$$

Since $\delta(\delta u) = 0$, one can by the Hodge duality between $d$ and $\delta$ treat $\|K*(du)\|_{\dot{F}^{s+1}_{p,q}}$ similarly.

□

We can now deduce Theorem 1 from Theorem 2:

Proof of Theorem 1: First assume that $q \geq p$. Then one has

$$\|u\|_{\dot{B}^{s}_{r,q}} \leq C\|u\|_{\dot{F}^{s}_{p,q}},$$

and Theorem 1 follows from Theorem 2. Otherwise, if $q < p$, then by the embedding theorems of Besov spaces,

$$\|u\|_{\dot{B}^{s}_{r,q}} \leq C\|u\|_{\dot{B}^{r}_{q,q}} = C\|u\|_{\dot{F}^{r}_{q,q}}$$

with $r = s + n(\frac{1}{q} - \frac{1}{p})$ and Theorem 1 also follows from Theorem 2.

□

We finish with the proof of Theorem 3. It relies on the

Lemma 3.1. For every $s > 0$, $p > 1$ and $q > 1$ with $sq < n$ and

$$\frac{1}{p} = \frac{1}{q} - \frac{s}{n},$$

there exists $C > 0$ such that for every $u \in C^\infty_c(\mathbb{R}^n)$,

$$\|u\|_{L^{p,s}} \leq C\|u\|_{\dot{F}^{s}_{p,q}}.$$

Proof. One has

$$u = I_s * ((-\Delta)^{\frac{s}{2}}u),$$

where the Riesz kernel $I_s$ is defined for $x \in \mathbb{R}^n$ by

$$I_s(x) = \frac{\Gamma(\frac{n-s}{2})}{\pi^{\frac{n}{2}}2^s\Gamma(\frac{s}{2})|x|^{n-s}}.$$

One has then by Sobolev inequality for Riesz potentials in Lorentz spaces of R. O’Neil [7] (see also e.g. [15, Theorem 2.10.2]),

$$\|u\|_{L^{r,p}} \leq C\|(-\Delta)^{\frac{s}{2}}u\|_{L^p}.$$

One concludes by noting that $\|(-\Delta)^{\frac{s}{2}}u\|_{L^p}$ and $\|u\|_{\dot{F}^{s}_{p,q}}$ are equivalent norms [10, Theorem 2.3.8 and section 5.2.3].

□

Proof of Theorem 3: Choose $s$ so that (5) holds with $p = \frac{n}{n-\frac{s}{n}}$. Since $\frac{1}{q} - \frac{s}{n} = 1 - \frac{1}{n}$, one can combine Theorem 2 and Lemma 3.1 to obtain the conclusion.

□
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