Applications of the local RBF collocation method and the fictitious time integration method for Burgers’ equations

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Abstract

In this paper, a meshfree numerical scheme, which is a combination of the local RBF (radial basis function) collocation method (LRBFCM) and the fictitious time integration method (FTIM), is proposed to efficiently and accurately analyze the two-dimensional Burgers’ equations. The Burgers’ equations, which can be regarded as the simplified form of the Navier-Stokes equations, are a system of time- and space-dependent nonlinear partial differential equations, so it is very difficult and important to be accurately solved. The LRBFCM, one of newly-developed meshfree methods, is adopted for spatial discretization and, at the meantime, the implicit Euler method is used for temporal discretization. Then, the system of nonlinear algebraic equations at every time step will be efficiently resolved by the FTIM, which can avoid the calculation of the Jacobian matrix. The numerical results and comparisons of one numerical example will be provided to validate the accuracy and the efficiency of the proposed meshfree numerical scheme.

1. Introduction

The Burgers’ equations are used to describe the phenomena of shock wave, acoustic transmission, turbulence,
etc. Therefore, there are many numerical methods proposed in the past to accurately analyze the well-known Burgers’ equations [1-2]. Since the Burgers’ equations are a system of non-linear time- and space-dependent partial differential equations and can depict the balance between the convective term and the viscous term, it is also used as the simplified form of the Navier-Stokes equations. Thus, in this paper, a meshfree numerical scheme, based on the implicit Euler method, the local RBF (radial basis function) collocation method (LRBFCM) and the fictitious time integration method (FTIM), is adopted to form a reliable numerical method to accurately and efficiently acquire the solutions of the two-dimensional Burgers’ equations.

In the past decades, there are many meshless/meshfree methods proposed to analyze the partial differential equations and applied to various engineering applications, such as the method of fundamental solutions [3], the collocation Trefftz method [4-5], the generalized finite difference method [6], the LRBFCM [7-8], etc. Among them, the LRBFCM is one of the most promising domain-type meshfree methods. By using the RBF to approximate the numerical solution in local domain, the LRBFCM is truly free from time-consuming mesh generation and numerical quadrature. Furthermore, the concept of localization in the LRBFCM can avoid the problems of ill-conditioning matrix and form a system of sparse matrix. The idea of localization enables the LRBFCM to be extended to large-scale engineering problems. Therefore, we will use the LRBFCM for spatial discretization of the Burgers’ equations. At the meantime, the implicit Euler methods will be responsible for the temporal discretization to release the strict limitation of size of time step.

Thus, after the space and time discretizations by the LRBFCM and the implicit Euler method, a system of nonlinear algebraic equations will be formed at every time step. The FTIM [9], a newly-developed solver for solutions of nonlinear algebraic equations, will be adopted for obtaining the convergent solutions at every time step. When a system of non-linear algebraic equations is considered, the Newton’s method is the most popular method for such system. Unfortunately, the need for calculation of the inverse of Jacobian matrix will increase the necessary computational time and cost huge computer resource. Therefore, it is very inefficient for Newton’s method when large-scale system of non-linear algebraic equations is considered.

In 2008, the FTIM is proposed by Liu and Atluri [8] for efficiently solving large-scale problems. By introducing a time-like variable, the system of nonlinear algebraic equations will be converted to a system of ordinary differential equations. Then the convergent solutions can be obtained by numerically integrating the system of ordinary differential equations. In their study, the efficiency and the accuracy of the FTIM outperform the Newton’s method in some cases. Frankly speaking, the FTIM is very suitable for large system of non-linear algebraic equations and parallel computation. So, we adopted the FTIM for solving the system of non-linear algebraic equations at every time step. To be brief, the LRBFCM and the implicit Euler method are used for space and time discretization of the Burgers’ equations. Then the FTIM is used for acquiring the solutions for the system of non-linear algebraic equations at every time step.

2. Burgers’ equation

In this paper, the considered two-dimensional Burgers’ equations are in the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{1}{\text{Re}} \nabla^2 u \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{1}{\text{Re}} \nabla^2 v
\end{align*}
\]

where \( u \) and \( v \) are unknown variables. \( \text{Re} \) is the Reynolds number, one important dimensionless parameter. By carefully observing, the above system of partial differential equations describes the balance between the convective terms and the diffusive terms. Depending on the magnitude of \( \text{Re} \), the behavior of the above system may be either hyperbolic or parabolic. Therefore, it is very difficult to analyze the Burgers’ equations. In this paper, a meshfree numerical scheme, which is the combination of the implicit Euler method, the LRBFCM and the FTIM, is adopted to efficiently and accurately solve the Burgers’ equations.
3. Numerical method

3.1. Implicit Euler method

In order to release the strict limitation for the size of time step, the implicit Euler method is used for temporal discretization of the Burgers’ equations.

\[
\frac{u^{n+1} - u^n}{\Delta t} + u^{n+1} \frac{\partial u^{n+1}}{\partial x} + v^{n+1} \frac{\partial u^{n+1}}{\partial y} = \frac{1}{Re} \nabla^2 u^{n+1}
\]

(3)

\[
\frac{v^{n+1} - v^n}{\Delta t} + u^{n+1} \frac{\partial v^{n+1}}{\partial x} + v^{n+1} \frac{\partial v^{n+1}}{\partial y} = \frac{1}{Re} \nabla^2 v^{n+1}
\]

(4)

where the superscript \( n \) and \( n+1 \) denote the \( n \)th and the \( (n+1) \)th time steps. After the temporal discretization by the implicit Euler method, the derivatives with respect to space coordinates in the above equations will be handled by the LRBFCM.

3.2. Local RBF collocation method

The LRBFCM is used to approximate the space derivative at every node in computational domain. While the sub-domain at the \( i \)th node is considered, we can find the \( n_s \) nearest nodes around the \( i \)th node centered at the sub-domain. The approximation of interpolation in the sub-domain can be expressed as:

\[
u(x, y) = \sum_{j=1}^{s} \beta_j \Phi(r_j),
\]

(5)

where \( j \) is the local index in the sub-domain. \( \{\beta_j\}_{j=1}^{s} \) are the unknown coefficients corresponding to the \( i \)th node. \( r_j = |x - \bar{x}| \) is the distance between nodes in the sub-domain. In this paper, we used the multiquadrics (MQ) function \( \Phi(r) = (1 + |r|)^b \) in the numerical approximation. In the MQ function, \( b \) is the shape parameter. If the first-order derivation with respect to \( x \) at the \( i \)th node is needed, the derivative can be expressed by

\[
\frac{\partial u}{\partial x} = \sum_{j=1}^{s} w_{xj} u'_j,
\]

(6)

where \( \{w_{xj}\}_{j=1}^{s} \) are the weighting coefficients corresponding to the \( i \)th node for the first-order derivative with respect to \( x \). For readers who are interested in the derivations of Eq.(6), the readers can refer to [7-8] for more details. Similarly, the expressions for first-order derivative with respect to \( y \) and for the second-order derivatives with respect to \( x \) and \( y \) can be acquired. The expressions of the spatial derivatives in the LRBFCM are used to replace the derivatives at the system of equations, Eqs.(3)-(4). Then, a system of nonlinear algebraic equations will be formed and efficiently solved by the FTIM.

3.3. Fictitious time integration method

When a system of nonlinear algebraic equations in the following form is considered,

\[
F(x) = 0
\]

(7)

where \( F \in \mathbb{R}^m \) and \( x \in \mathbb{R}^m \) are column vectors of \( m \) equations and \( m \) unknowns. After introducing a time-like variable and performing a series of mathematical derivations [6,9], the system of nonlinear algebraic equations will be transformed to the system of ordinary differential equations,

\[
\frac{dx}{d\tau} = F(x)
\]

(8)

where \( \tau \) is the time-like variable. For the sake of simplicity, the explicit Euler method is used to integrate Eq.(8) to obtain the convergent solutions at every time step.
4. Numerical results and comparisons

4.1. Example 1

For the first example, the computational domain is a unit square and the analytical solution is acquired from [1]. The following conditions are used in this example: \( Re=100, N=441, n_s=13, b=2, dt=0.005 \) and \( d\tau=10^{-4} \). The numerical results of \( u \) and \( v \) at \( t=0.01, t=0.5 \) and \( t=2 \) are shown in figures 1 and 2. From these figures, it is obvious that the sharp fronts in \( u \) and \( v \) are moving towards the same direction and the sharp fronts can be very clearly simulated. These solutions are in good agreements with the analytical solutions [1].

![Fig. 1. The distributions of u at (a)t=0.01, (b)t=0.5 and (c)t=2.](image1.jpg)

![Fig. 2. The distributions of v at (a)t=0.01, (b)t=0.5 and (c)t=2.](image2.jpg)

In order to check carefully, the maximum absolute errors for \( u \) and \( v \) at \( t=0.01, t=0.5 \) and \( t=2 \) are demonstrated at figures 3 and 4. The maximum absolute errors for \( u \) and \( v \) are extremely small even by using such few nodes. Therefore, the great accuracy and the merits of meshfree scheme can be found. From figures 3 and 4, it can be noticed that the errors for \( u \) and \( v \) appeared at the sharp fronts and will moved with these fronts. In addition, the effects of different numbers of used nodes are numerically examined in figures 5. In this figure, the maximum absolute errors for \( u \) and \( v \) by using different numbers of nodes are depicted. Three different numbers of nodes are used and they are 121, 441 and 961. From the figures, the better solutions can be obtained by using more nodes. The stability and the consistency of the proposed meshfree numerical scheme are preliminarily validated.

![Fig. 3. The distributions of error of u at (a)t=0.01, (b)t=0.5 and (c)t=2.](image3.jpg)
5. Conclusions

In this paper, a meshfree numerical scheme is proposed to analyze the quasi-linear two-dimensional Burgers’ equations. The Burgers’ equations are used to describe the balance of the convective term and the viscous term; thus it is nontrivial to find the efficient numerical scheme for accurately solving the Burgers’ equations. In this study, the combination of the implicit Euler method, the LRBFCM and the FTIM is adopted to efficiently solve the Burgers’ equations. The proposed numerical scheme is free from time-consuming mesh generation and numerical quadrature. In addition, the calculation of inverse of Jacobian matrix can be avoided by using the FTIM and then the computation efficiency can be greatly improved. One numerical example is used to validate the accuracy and the efficiency of the proposed numerical scheme. More numerical examples, which are used to verify the ability of the proposed method, are the ongoing research. In addition, this approach will be combined with suitable parallel computation for solving large-scale engineering problems in the future.

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