Wave-mechanical phenomena in optical coupled-mode structures

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Abstract. We derive a formal mapping between Schrödinger equations and certain classes of Maxwell equations describing the classical electromagnetic wave’s propagation inside coupled-modes waveguides. This mapping reveals a phenomenon, which is not visible in the original form of Maxwell equations: multiple solutions occur which satisfy same boundary conditions but correspond to different eigenvalues of a certain operator; the latter is analogous to Hamiltonian operators which occur in quantum systems. If one deals with normalized state vectors then a proper analogy with the conventional wave mechanics is established: solutions form a Hilbert space which is somewhat similar to that in the quantum mechanics. Therefore, coupled-mode configurations should possess certain wave-mechanical features, which can be formally studied using a formalism of quantum mechanics or, at least, its mathematical part. We notice also that the occurring Hamiltonian operators always possess a skew-adjoint part if one deals with normalized state vectors – even if permittivity and permeability are real-valued. This leads to the “dressing” effect of propagation constants, which indicates presence of additional gain or loss processes in the coupled-mode systems.

1. Introduction
Notwithstanding the long history of studies [1, 2], the propagation of electromagnetic waves inside dielectric media remains an important and rapidly developing topic [3, 4]. Apart from an obvious theoretical value, it finds numerous applications in the designs of the nanoscale photonic and plasmonic devices, structures and metamaterials.

Among such devices, the coupled-mode waveguides constitute a substantial part, due to their numerous applications in science and industry. One of such special cases is the wave-propagating problems with coupling of modes, whose theory’s history can be traced as far back as to the middle of the previous century [5, 6, 7, 8, 9]. In this paper, we are going to summarize the previous results to create some unified picture, as well as to briefly enumerate the common properties of main physical objects of the theory, such as Hamiltonian operators.

In doing so, we shall be differentiating the coupled-mode computational approach from the phenomenon of coupled modes. The former is a computational approach in a general theory of waveguides, which tries to preserve a mode concept in situations where such modes cannot be found in general but only in some limit or subset. Thus, it can be regarded as a robust approximation which grasps a leading-order description. On the contrary, the phenomenon of coupled modes is an actual effect which occurs in some specifically designed types of waveguides, dubbed as coupled-modes waveguides. One of reasons why such waveguides became popular is
a hope that one can create a lossless waveguide by making coupled waveguides to channel their
dissipating energy towards each other and thus exchange energy between themselves only.

2. Maxwell-Schrödinger analogy for coupled modes
This approach’s formulation begins with a simplification of the Maxwell equations by expanding
the transverse electric and magnetic fields in series with respect to modes, referred as the modal
expansion [5, 10, 11, 12, 13]. Let us begin with the stationary Maxwell equations for a wave-
propagating dielectric system with \( \varepsilon = \varepsilon(x, y, z) \) and \( \mu = 1 \) we wish to study. Let us also
introduce an auxiliary uniform lossless medium, characterized by permittivity \( \tilde{\varepsilon} = \tilde{\varepsilon}(x, y) \)
and permeability \( \tilde{\mu} = 1 \), which has a cylindrical but not necessarily rotational symmetry. We thus
have two sets of the Maxwell equations, for each media’s set of fields: \( \mathbf{E}, \mathbf{H} \) and \( \tilde{\mathbf{E}}, \tilde{\mathbf{H}} \), respectively.

Introducing the value \( \mathbf{F} = \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} \), one can derive the following relations:

\[
\nabla \cdot \mathbf{F} = -\omega (\varepsilon^* - \tilde{\varepsilon}) \mathbf{E} \cdot \mathbf{E}^*,
\]

\[
\int \nabla \cdot \mathbf{F} \, dx_\perp = \frac{d}{dz} \int e_z \cdot \mathbf{F} \, dx_\perp + \oint n \cdot \mathbf{F} \, dl_\perp,
\]

where \( \int dx_\perp \) and \( \oint dl_\perp \) denote the integration over the waveguide’s cross-section and encircling
contour, respectively, and \( n \) is an outward normal to the contour.

For the auxiliary structure, the spatial modal amplitudes of EM fields can be written in the
form: \( \tilde{\mathbf{E}}_p(x, y, z) = \tilde{\mathbf{E}}_p(x, y) \exp (i\tilde{\beta}_p z), \) \( \tilde{\mathbf{H}}_p(x, y, z) = \tilde{\mathbf{H}}_p(x, y) \exp (i\tilde{\beta}_p z), \) where \( p \) is a nonzero
number enumerating the modes, and \( \tilde{\beta} \) is the \( p \)th modal propagation constant of the auxiliary
optical structure \( \tilde{\varepsilon} \). The modal values have the following properties: \( \tilde{\beta}_p = -\tilde{\beta}_p, \) \( \tilde{\mathbf{E}}_{-p} = \tilde{\mathbf{E}}_p, \) and
\( \tilde{\mathbf{H}}_{-p} = -\tilde{\mathbf{H}}_p, \) and the modal fields obey the orthonormality condition

\[
\int e_z \cdot \tilde{\mathbf{E}}_p \times \tilde{\mathbf{H}}_q^* \, dx_\perp = \text{sign}(q) \delta_{pq},
\]

where \( \delta \) is the Kronecker symbol.

Substituting the amplitudes’ expansions into eqs. (1) and (2), we obtain

\[
\left( \frac{d}{dz} + i\tilde{\beta}_p \right) \int e_z \cdot \mathbf{F}_p \, dx_\perp = i\omega \int (\varepsilon^* - \tilde{\varepsilon}) \tilde{\mathbf{E}}_p \cdot \mathbf{E}^* \, dx_\perp - \oint n \cdot \mathbf{F}_p \, dl_\perp,
\]

where we denoted \( \mathbf{F}_p = \tilde{\mathbf{E}}_p \times \mathbf{H}^* + \mathbf{E}^* \times \tilde{\mathbf{H}}_p \).

Due to the completeness of the auxiliary modal fields, we can use them as a basis for series
expansion of the main system. For the transverse fields we correspondingly obtain:

\[
\mathbf{E}_\perp^* = \sum_p a_p \tilde{\mathbf{E}}_{\perp p}^*, \quad \mathbf{H}_\perp^* = \sum_p a_p \tilde{\mathbf{H}}_{\perp p}^*,
\]

where the modal expansion coefficients \( a_p = a_p(z) \) are functions of the longitudinal coordinate,
whereas the auxiliary medium’s transverse fields \( \tilde{\mathbf{E}}_{\perp p} = \tilde{\mathbf{E}}_{\perp p}(x, y) \) and
\( \tilde{\mathbf{H}}_{\perp p} = \tilde{\mathbf{H}}_{\perp p}(x, y) \) are functions of the transverse coordinates only; for continuous modes the summation is replaced
by integration. Substituting this expansion into eq. (4), we obtain

\[
\left( i \frac{d}{dz} - \tilde{\beta}_p \right) a_p = \frac{1}{2} \text{sign}(p) \omega \int (\tilde{\varepsilon} - \varepsilon^*) \tilde{\mathbf{E}}_p \cdot \mathbf{E}^* \, dx_\perp,
\]

where \( a_p = a_p(z) \) [12].
Furthermore, using the original Maxwell equations, one can write the longitudinal projections of the fields:

\[ E^*_z = \frac{\epsilon^*}{\varepsilon} \sum_p a_p \hat{E}^*_p, \quad H^*_z = \sum_p a_p \hat{H}^*_p, \]

where \( \hat{E}_{zp} \) and \( \hat{H}_{zp} \) are the longitudinal fields of a \( p \)th mode in the auxiliary medium. Therefore, eq. (6) can be further simplified to a matrix differential equation

\[ i \frac{d}{dz} a_p = \sum_{q=1}^{N} C_{pq} a_q, \tag{7} \]

where \( N = \max (p) \), and the matrix

\[ C_{pq} = \tilde{\beta}_q \delta_{pq} + \frac{1}{2} \text{sign}(p) \omega \int \left( 1 - \frac{\varepsilon^*}{\varepsilon} \right) \left( \tilde{\varepsilon} \hat{E}_{\perp p} \cdot \hat{E}^*_{\perp q} + \varepsilon^* \hat{E}_{zp} \hat{E}^*_{zq} \right) d \mathbf{x}_\perp, \tag{8} \]

is complex-valued and \( z \)-dependent, in general.

One can see that physically allowed solutions of eq. (7) form a \( N \)-dimensional Hilbert space of normalizable functions of \( z \). Therefore, one can consider this equation as representing a standalone wave-mechanical system depending only on one variable, \( z \).

Furthermore, one can introduce an inner product with the norm

\[ \mathcal{N}^2_{(a)} = \sum_{p=1}^{N} a_p a^*_p, \tag{9} \]

where we took into account that \( a_p \)'s themselves do not depend on transverse coordinates, therefore the cross-section integration results merely in an overall constant factor which can be omitted. With these definitions in hand, we can define the following state vector:

\[ |\Psi\rangle \equiv \frac{1}{\mathcal{N}_{(a)}} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \tag{10} \]

which is automatically normalized to one \( \langle \Psi | \Psi \rangle = \frac{1}{\mathcal{N}_{(a)}^2} \sum_{p=1}^{N} a_p a^*_p = 1 \), i.e., it is a ray in the \( N \)-dimensional Hilbert space endowed with an inner product with respect to conventional matrix multiplication. In terms of this state vector, eq. (7) acquires the Schrödinger form

\[ i \hbar \frac{\partial}{\partial z} \Psi = \hat{H} \Psi, \tag{11} \]

with the components of the Hamiltonian matrix being given by

\[ (\hat{H})_{pq} = \hbar \omega U_{pq} + \left( \tilde{\beta}_q - i \gamma_{(a)} \right) \delta_{pq}, \tag{12} \]

where \( U_{pq} = \frac{1}{2} \text{sign}(p) \omega \int \left( 1 - \frac{\varepsilon^*}{\varepsilon} \right) \left( \tilde{\varepsilon} \hat{E}_{\perp p} \cdot \hat{E}^*_{\perp q} + \varepsilon^* \hat{E}_{zp} \hat{E}^*_{zq} \right) d \mathbf{x}_\perp \), and \( \gamma_{(a)} = \frac{d}{dz} \ln |\mathcal{N}_{(a)}| \) is in general a real-valued function of \( z \). Here and below, the “Planck” constant \( \hbar \) is an effective scale constant of the dimensionality energy \( \times \) time, which is introduced for a purpose of preserving the correct dimensionality of the relevant terms in eq. (11).

One can immediately notice a number of interesting phenomena which become apparent in the representation (11). Firstly, the anti-Hermitian part of the matrix \( \hat{U} \), \( \left( \hat{U} - \hat{U}^\dagger \right)_{pq} = \frac{1}{2} \text{sign}(p) \omega \int \varepsilon_+ \left[ \hat{E}_{\perp p} \cdot \hat{E}^*_{\perp q} - (1 - \frac{\varepsilon^*}{\varepsilon}) \hat{E}_{zp} \hat{E}^*_{zq} \right] d \mathbf{x}_\perp \), where \( \varepsilon_\pm = \varepsilon \pm \varepsilon^* \), does not vanish in general.
This indicates that the Hamiltonian (12) is non-Hermitian. From a theory of open quantum systems, we know that non-Hermiticity effectively describes the presence of dissipative effects [14].

Secondly, one of general effects discussed in ref. [15] occurs also here, for the coupled-mode structures: due to the state vector normalization, modal propagation constants \( \tilde{\beta}_p \) become “dressed” by adding the imaginary component and move into a complex domain:

\[
\tilde{\beta}_p \rightarrow \tilde{\beta}_p^{(\text{eff})} = \tilde{\beta}_p - i\gamma(a). \tag{13}
\]

In physical terms, this indicates presence of additional gain or loss in the coupled-mode systems. Notice that this additional loss persists even if \( \varepsilon \) is real-valued, which could be one of reasons why a lossless waveguide, based on mutual loss compensation between coupled modes, has yet not been created, despite substantial theoretical and experimental efforts.

3. Coupled-mode waveguides

Let us we discuss here a case of materials and devices, for which the phenomenon of coupling modes occurs directly. These are a large class of optical waveguides including optical fibers, couplers and amplifiers. Its theory relies on a number of approximations which are not always obvious for optical waveguides but nevertheless working [5, 16, 17, 18]. In particular, the theory uses non-orthogonal modal expansions which makes its formalism somewhat different from the conventional coupled-mode theory [10, 12]. From the viewpoint of the Maxwell-Schrödinger analogy, this non-orthogonality is naturally connected with non-Hermitian Hamiltonian operators, as we shall see below.

Let us consider typical parallel optical waveguides. We begin with Maxwell equations in a separated form with decoupled time and \( z \)-dependence:

\[
\nabla \times \mathbf{E} + i\omega \mu_0 \mathbf{H} = i\beta \mathbf{e}_z \times \mathbf{E},
\]

\[
\nabla \times \mathbf{H} - i\omega \epsilon \mathbf{E} = i\beta \mathbf{e}_z \times \mathbf{H},
\tag{15}
\]

where \( \mathbf{E} = \mathbf{E}(x, y) \) and \( \mathbf{H} = \mathbf{H}(x, y) \).

Let us assume that these fields can be written in a series form, similarly to eq. (5):

\[
\mathbf{E} = \sum_{p=1}^{N} a_p \mathbf{E}_p, \quad \mathbf{H} = \sum_{p=1}^{N} a_p \mathbf{H}_p,
\]

where \( N \) is a number of modes in an \( N \)-guide system, and the modal fields obey the equations:

\[
\nabla \times \mathbf{E}_p + i\omega \mu_0 \mathbf{H}_p = i\beta_p \mathbf{e}_z \times \mathbf{E}_p,
\tag{16}
\]

\[
\nabla \times \mathbf{H}_p = i\omega \epsilon \mathbf{E}_p + i\beta_p \mathbf{e}_z \times \mathbf{H}_p,
\tag{17}
\]

where \( \varepsilon_p \) is a profile of the configuration in which all but \( p \)th waveguide are removed.

The equations for modal expansion coefficients thus become

\[
\sum_{q=1}^{N} iP_{pq} \frac{d}{dz} a_q - H_{pq} a_q = 0,
\tag{18}
\]

where \( W_{pq} = \frac{1}{2} \int e_z \cdot (\mathbf{E}_q \times \mathbf{H}_p^* + \mathbf{E}_p^* \times \mathbf{H}_q) \, dx_\perp, \quad H_{pq} = W_{qp} \beta_q + \frac{1}{2} \int (\varepsilon - \varepsilon_q) \mathbf{E}_q \cdot \mathbf{E}_p^* \, dx_\perp \). If the matrix \( W \) is invertible then eqs. (18) can be rewritten in the form (7):

\[
i \frac{d}{dz} a_p = \sum_{q=1}^{N} C_{pq} a_q,
\tag{19}
\]
where \( C'_{pq} = \sum_{r=1}^{N} (W^{-1})_{pr} H_{rq} \), and \( W^{-1} \) denotes an inverse of the matrix \( W \).

As in the previous section, one can consider eq. (19) as representing a standalone wave-mechanical system depending on the variable \( z \), introduce an inner product of the form (9) and a state vector of the form (10). In terms of this state vector, eq. (19) acquires the form (11) where the components of the Hamiltonian matrix are

\[
(H)_{pq} = h \omega U'_{pq} + (\beta_q - i\gamma(a)) \delta_{pq},
\]

where \( U'_{pq} = \frac{1}{4} \sum_{r=1}^{N} (W^{-1})_{pr} \int (\varepsilon - \varepsilon_q) \tilde{E}_q \cdot \tilde{E}_r d\mathbf{x}_{\perp} \).

Similarly to the previous section, propagation constants \( \beta_p \) also become “dressed” here: \( \beta_p \rightarrow \beta_p^{(\text{eff})} = \beta_p - i\gamma(a) \); therefore the remarks following eq. (13) become applicable for this case too.

4. Conclusion

We formulated a formal mapping between Schrödinger equations and certain classes of Maxwell equations describing the classical electromagnetic wave’s propagation inside coupled-modes waveguides. We work with normalized state vectors which is important for establishing a proper analogy with the conventional wave mechanics. This mapping revealed a phenomenon, which was hitherto not visible in the original form of Maxwell equations. Namely, multiple solutions occur which satisfy same boundary conditions but correspond to different eigenvalues of a Hamiltonian operator analogous to the one which occurs in quantum systems.

These solutions form a Hilbert space which is somewhat similar to that in wave mechanics - except that the Planck constant is replaced by a constant of the same dimensionality but different (undefined) value. From a notion of Hilbert space, it is natural to assume that selection between such solutions must have a probabilistic nature.

This means that classical coupled-mode systems should possess some wave-mechanical features, which can be formally studied using a formalism of quantum mechanics or its mathematical part, at least. For instance, the Hamiltonian operators, which emerge in this approach, should always have a skew-adjoint part if one deals with normalized values – even if a refractive index is real-valued. This leads to the “dressing” of propagation constants, cf. eq. (13), which indicates an additional process of gain or loss in the coupled-mode structures, and brings the whole theory into the realm of a theory of open systems described by non-Hermitian Hamiltonians [19, 20, 21, 22, 23].

In future, we plan to use the mapping between Schrödinger equations and coupled-mode configurations to develop the quantum-statistical approach for studying the coupled waveguides, along the lines of our previous work [15, 24]. It is expected that due to above-mentioned features, quantum-statistical equations will have a simpler form than those described in a general theory [15].

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