Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, IR)$ and Character Formulae for $n = 3$

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Abstract. We overview our study of the positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, IR)$. We give more explicitly character formulae for these representations in the case $n = 3$.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. Until recently only those for $D \leq 6$ were studied since in these cases the relevant superconformal algebras satisfy [1] the Haag-Lopuszanski-Sohnius theorem [2]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2,2/N)$ [3] (for $N = 1$), [4, 5, 6, 7] (for arbitrary $N$). More recently, the classification for $D = 3$ (for even $N$), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [8] (some results are conjectural), and then the $D = 6$ case (for arbitrary $N$) was finalized in [9].
On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for \(D > 6\). Most prominent role play the superalgebras \(osp(1|2n)\). Initially, the superalgebra \(osp(1|2n)\) was put forward for \(D = 10\) [10]. Later it was realized that \(osp(1|2n)\) would fit any dimension, though they are minimal only for \(D = 3, 9, 10, 11\) (for \(n = 2, 16, 16, 32\), resp.) [11]. In all cases we need to find first the UIRs of \(osp(1|2n, R)\) which study was started in [12] and [13]. Later, in [14] we finalized the UIR classification of [12] as Dobrev-Zhang-Salom (DZS) Theorem. There we also proved the DZS Theorem for \(osp(1|6)\), while the case \(osp(1|8)\) was proved in [15].

In the present paper we present more explicitly the character formulae for \(osp(1|6)\). For the lack of space we refer for extensive literature on the subject in [12, 14].

2. Preliminaries on representations

Our basic references for Lie superalgebras are [16, 17], although in this exposition we follow [12].

The even subalgebra of \(G = osp(1|2n, R)\) is the algebra \(sp(2n, R)\) with maximal compact subalgebra \(K = u(n) \cong su(n) \oplus u(1)\).

We label the relevant representations of \(G\) by the signature:

\[
\chi = [d; a_1, ..., a_{n-1}] \tag{1}
\]

where \(d\) is the conformal weight, and \(a_1, ..., a_{n-1}\) are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra \(su(n)\) (the simple part of \(K\)).

We present the classification of the positive energy (lowest weight) UIRs of \(G\) following [12, 14] where were used the methods used for the \(D = 4, 6\) conformal superalgebras, cf. [4, 5, 6, 7, 9]. The main tool is an adaptation of the Shapovalov form [18] on the Verma modules \(V^\chi\) over the complexification \(G^C = osp(1|2n)\) of \(G\).

The root system of \(G^C\) are given in terms of \(\delta_1, ..., \delta_n\), \((\delta_i, \delta_j) = \delta_{ij}\), \(i, j = 1, ..., n\). The even and odd roots systems are [16]:

\[
\Delta_0 = \{ \pm \delta_i \pm \delta_j, 1 \leq i < j \leq n, \pm 2\delta_i, 1 \leq i \leq n \}, \tag{2}
\]

\[
\Delta_1 = \{ \pm \delta_i, 1 \leq i \leq n \}
\]

(we remind that the signs \(\pm\) are not correlated). We shall use the following distinguished simple root system [16]:

\[
\Pi = \{ \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n \} \tag{3}
\]
or introducing standard notation for the simple roots:

\[ \Pi = \{ \alpha_1, ..., \alpha_n \} , \]
\[ \alpha_j = \delta_j - \delta_{j+1} , \quad j = 1, ..., n-1 , \quad \alpha_n = \delta_n . \]

The root \( \alpha_n = \delta_n \) is odd, the other simple roots are even. The Dynkin diagram is:

\[ \circ \quad \cdots \quad \circ \quad \Rightarrow \bullet \]

The black dot is used to signify that the simple odd root is not nilpotent. In fact, the superalgebras \( B(0, n) = osp(1|2n) \) have no nilpotent generators unlike all other types of basic classical Lie superalgebras [16].

The corresponding to \( \Pi \) positive root system is:

\[ \Delta^+_0 = \{ \delta_i \pm \delta_j , 1 \leq i < j \leq n , \quad 2\delta_i , 1 \leq i \leq n \} , \quad \Delta^+_I = \{ \delta_i , 1 \leq i \leq n \} \]

Conversely, we give the elementary functionals through the simple roots:

\[ \delta_k = \alpha_k + \cdots + \alpha_n . \]

From the point of view of representation theory more relevant is the restricted root system, such that:

\[ \tilde{\Delta}^+ = \Delta^+_0 \cup \Delta^+_I , \]
\[ \tilde{\Delta}^+_0 = \{ \alpha \in \Delta^+_0 \mid \frac{1}{2} \alpha \notin \Delta^+_I \} = \{ \delta_i \pm \delta_j , 1 \leq i < j \leq n \} \]

The superalgebra \( \mathcal{G} = osp(1|2n, \mathbb{R}) \) is a split real form of \( osp(1|2n) \) and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra \( B_n \) (dropping the distinction between even and odd roots) with Dynkin diagram:

\[ \circ \quad \cdots \quad \circ \quad \Rightarrow \circ \]

and root system:

\[ \Delta^+_{B_n} = \{ \delta_i \pm \delta_j , 1 \leq i < j \leq n , \quad \delta_i , 1 \leq i \leq n \} \cong \tilde{\Delta}^+ \]

This shall be used essentially below.
We need explicitly the lowest weight $\Lambda \in \mathcal{H}^*$ (where $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}^C$) the values of which should be related to the signature (1):

$$ (\Lambda, \alpha_k^\vee) = -a_k , \quad 1 \leq k \leq n , $$

(11)

where $\alpha_k^\vee \equiv 2\alpha_k/\langle \alpha_k, \alpha_k \rangle$, and the minus signs anticipate the fact that we shall use lowest weight Verma modules (instead of the highest weight modules used in [17]) and to Verma module reducibility w.r.t. the roots $\alpha_k$ (this is explained in detail in [6, 12]).

Obviously, $a_n$ must be related to the conformal weight $d$ which is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

$$ a_n = -2d - a_1 - \cdots - a_{n-1} . $$

(12)

The actual Dynkin labelling is given by:

$$ m_k = (\rho - \Lambda, \alpha_k^\vee) $$

(13)

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_0, \rho_1$ of the even, odd, resp., positive roots (cf. (6):

$$ \rho \equiv \rho_0 - \rho_1 = (n - \frac{1}{2})\delta_1 + (n - \frac{3}{2})\delta_2 + \cdots + \frac{3}{2}\delta_{n-1} + \frac{1}{2}\delta_n , $$

$$ \rho_0 = n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n , $$

$$ \rho_1 = \frac{1}{2}(\delta_1 + \cdots + \delta_n) . $$

(14)

Naturally, the value of $\rho$ on the simple roots is 1: $\langle \rho, \alpha_i^\vee \rangle = 1, i = 1, \ldots, n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of $a_n$ is arbitrary. In the cases when $a_n$ is also a non-negative integer, and then $m_k \in \mathbb{N}$ ($\forall k$) the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$.

To introduce Verma modules we use the standard decomposition:

$$ \mathcal{G}^C = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- $$

(15)

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and $v_0$ is a lowest weight vector $v_0$ such that:

$$ Z v_0 = 0 , \quad Z \in \mathcal{G}^- $$

$$ H v_0 = \Lambda(H) v_0 , \quad H \in \mathcal{H} . $$

(16)
Further, for simplicity we omit the sign $\otimes$, i.e., we write $p v_0 \in V^\Lambda$ with $p \in U(G^+)$. Adapting the criterion of [17] to lowest weight modules, one finds that a Verma module $V^\Lambda$ is reducible w.r.t. the positive root $\beta$ iff the following holds [12]:

$$\rho - \Lambda, \beta^\vee = m_\beta, \quad \beta, m_\beta \in \mathbb{N}. \quad (17)$$

If a condition from (17) is fulfilled then $V^\Lambda$ contains a submodule which is a Verma module $V_{\Lambda'}$ with shifted weight given by the pair $m, \beta$:

$$\Lambda' = \Lambda + m\beta.$$ 

Explicitly, $v_{s}^{m,\beta}$ is given by a polynomial in the positive root generators [20, 21]:

$$v_{s}^{m,\beta} = P^{m,\beta} v_0, \quad P^{m,\beta} \in U(G^+). \quad (19)$$

Thus, the submodule $I^\beta$ of $V^\Lambda$ which is isomorphic to $V_{\Lambda'}$ is given by $U(G^+) P^{m,\beta} v_0$.

Certainly, (17) may be fulfilled for several positive roots (even for all of them). Let $\Delta_{\Lambda}$ denote the set of all positive roots for which (17) is fulfilled, and let us denote: $\tilde{I}^\Lambda = \cup_{\beta \in \Delta_{\Lambda}} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of $V^\Lambda$. Let us also denote $I^\Lambda \equiv V^\Lambda / \tilde{I}^\Lambda$.

The Verma module $V^\Lambda$ contains a unique proper maximal submodule $I^\Lambda$ [17, 22]. Among the lowest weight modules with lowest weight $\Lambda$ there is a unique irreducible one, denoted by $L^\Lambda$, i.e., $L^\Lambda = V^\Lambda / I^\Lambda$.

It may happen that the maximal submodule $I^\Lambda$ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if $\text{rank } \mathcal{G} \leq 2$, or when (17) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of $I^\Lambda$. We need the following notion.

**Definition:** [22, 23, 24] Let $V^\Lambda$ be a reducible Verma module. A vector $v_{\text{suv}} \in V^\Lambda$ is called a **subsingular vector** if $v_{\text{suv}} \notin \tilde{I}^\Lambda$ and the following holds:

$$X v_{\text{suv}} \in \tilde{I}^\Lambda, \quad \forall X \in \mathcal{G}^- \quad (20)$$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [12] it was established that from (17) follows that the Verma module $V^{\Lambda(\chi)}$ is
reducible if one of the following relations holds:

\[
\begin{align*}
\mathbb{N} \ni m_{ij} &= j - i + a_i + \cdots + a_{j-1} \\
\mathbb{N} \ni m_{ij} &= 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d \\
\mathbb{N} \ni m_i &= 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d \\
\mathbb{N} \ni m_{ii} &= n - i + \frac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) - d
\end{align*}
\]  

(21a)

(21b)

(21c)

(21d)

corresponding to the roots \( \delta_i - \delta_j, \ \delta_i + \delta_j, \ (i < j), \ \delta_i, \ 2\delta_i, \) resp. Further we shall use the fact from [12] that we may eliminate the reducibilities and embeddings related to the roots \( 2\delta_i \). Indeed, since \( m_i = 2m_{ii} \), whenever (21d) is fulfilled also (21c) is fulfilled.

For further use we introduce notation for the root vector \( X^+_j \in G^+ \), \( j = 1, \ldots, n \), corresponding to the simple root \( \alpha_j \).

Further, we notice that all reducibility conditions in (21a) are fulfilled.

In particular, for the simple roots from those condition (21a) is fulfilled with \( \beta \rightarrow \alpha_i = \delta_i - \delta_{i+1}, \ i = 1, \ldots, n - 1 \) and \( m_i = m_{i,i+1} = 1 + a_i \). The corresponding submodules \( \mathcal{I}^\alpha_i = U(G)^+ v_i \), where \( \Lambda_i = \Lambda + m_i^\alpha \alpha_i \) and \( v_i = (X_i^+)^{1+a_i} v_0 \). These submodules generate an invariant submodule which we denote by \( \mathcal{I}^\alpha_i \subset \mathcal{I}^{\Lambda} \). Since these submodules are nontrivial for all our signatures in the question of unitarity instead of \( V^\Lambda \) we shall consider also the factor-modules:

\[
F^\alpha_c = V^\Lambda / \mathcal{I}^\alpha_c \supset F^{\Lambda}.
\]

(22)

We shall denote the lowest weight vector of \( F^\alpha_c \) by \( \vert \Lambda_c \rangle \) and the singular vectors above become null conditions in \( F^\alpha_c : \)

\[
(X_i^+)^{1+a_i} \vert \Lambda_c \rangle = 0, \ \ i = 1, \ldots, n - 1.
\]

(23)

If the Verma module \( V^\Lambda \) is not reducible w.r.t. the other roots, i.e., (21b,c,d) are not fulfilled, then \( F^\alpha_c = F^{\Lambda} \) is irreducible and is isomorphic to the irrep \( L_\Lambda \) with this weight.

In fact, for the factor-modules reducibility is controlled by the value of \( d \), or in more detail:

The maximal \( d \) coming from the different possibilities in (21b) are obtained for \( m_{ij}^+ = 1 \) and they are:

\[
d_{ij} \equiv n + \frac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j), \ \ i < j,
\]

(24)

the corresponding root being \( \delta_i + \delta_j \).
The maximal \( d \) coming from the different possibilities in (21c) are obtained for \( m_i = 1 \) and they are:

\[
d_i \equiv n - i + \frac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) ,
\]

the corresponding roots being \( \delta_i \).

There are some orderings between these maximal reduction points [12]:

\[
\begin{align*}
d_1 & > d_2 > \cdots > d_n , \\
d_{i,i+1} & > d_{i,i+2} > \cdots > d_{in} , \\
d_{1,j} & > d_{2,j} > \cdots > d_{j-1,j} , \\
d_i & > d_{jk} > d_\ell , \quad i \leq j < k \leq \ell .
\end{align*}
\]

Obviously the first reduction point is:

\[
d_1 = n - 1 + \frac{1}{2}(a_1 + \cdots + a_{n-1}) .
\]

Below we shall use the following notion. The singular vector \( v_1 \) is called descendant of the singular vector \( v_2 \notin \mathbb{C}v_1 \) if there exists a homogeneous polynomial \( P_{12} \) in \( U(G^+) \) so that \( v_1 = P_{12} v_2 \). Clearly, in this case we have: \( I^1 \subset I^2 \), where \( I^k \) is the submodule generated by \( v_k \). Thus, when we factor the submodule \( I_2 \) this means factoring also the submodule \( I_1 \).

3. Unitarity
The first results on the unitarity were given in [12], and then improved in [14]. Thus, the statement below should be called Dobrev-Zhang-Salom Theorem:

**Theorem DZS:** All positive energy unitary reducible representations of the superalgebras \( osp(1|2n, \mathbb{R}) \) characterized by the signature \( \chi \) in (1)
are obtained for real \( d \) and are given as follows:

\[
\begin{align*}
  d & \geq n - 1 + \frac{1}{2}(a_1 + \cdots + a_{n-1}) = d_1, \quad a_1 \neq 0, \\
  d & \geq n - \frac{3}{2} + \frac{1}{2}(a_2 + \cdots + a_{n-1}) = d_{12}, \quad a_1 = 0, \ a_2 \neq 0, \\
  d & = n - 2 + \frac{1}{2}(a_2 + \cdots + a_{n-1}) = d_2 > d_{13}, \quad a_1 = 0, \ a_2 \neq 0, \\
  d & \geq n - 2 + \frac{1}{2}(a_3 + \cdots + a_{n-1}) = d_2 = d_{13}, \quad a_1 = a_2 = 0, \ a_3 \neq 0, \\
  d & = n - \frac{5}{2} + \frac{1}{2}(a_3 + \cdots + a_{n-1}) = d_{23} > d_{14}, \quad a_1 = a_2 = 0, \ a_3 \neq 0, \\
  d & = n - 3 + \frac{1}{2}(a_3 + \cdots + a_{n-1}) = d_3 = d_{24} > d_{15}, \quad a_1 = a_2 = 0, \ a_3 \neq 0.
\end{align*}
\]

\[
\begin{align*}
  \ldots \\
  d & \geq n - 1 - \kappa + \frac{1}{2}(a_{2k+1} + \cdots + a_{n-1}), \quad a_1 = \ldots = a_{2k} = 0, \ a_{2k+1} \neq 0, \\
  \kappa & = \frac{1}{2}, 1, \ldots, \frac{1}{2}(n - 1), \\
  d & = n - \frac{3}{2} - \kappa + \frac{1}{2}(a_{2k+1} + \cdots + a_{n-1}), \quad a_1 = \ldots = a_{2k} = 0, \ a_{2k+1} \neq 0, \\
  \ldots \\
  d & = n - 1 - 2\kappa + \frac{1}{2}(a_{2k+1} + \cdots + a_{n-1}), \quad a_1 = \ldots = a_{2k} = 0, \ a_{2k+1} \neq 0, \\
  \ldots \\
  d & \geq \frac{1}{2}(n - 1), \quad a_1 = \ldots = a_{n-1} = 0 \\
  d & = \frac{1}{2}(n - 2), \quad a_1 = \ldots = a_{n-1} = 0 \\
  \ldots \\
  d & = \frac{1}{2}, \quad a_1 = \ldots = a_{n-1} = 0 \\
  d & = 0, \quad a_1 = \ldots = a_{n-1} = 0
\end{align*}
\]

Parts of the \textit{Proof} were given in [12], while in [14] was given a detailed sketch of the Proof. In [14] was given also the Proof for the case \( n = 3 \), while the proof for \( n = 4 \) was given in [15].

4. Character formulae

Let \( \mathcal{G} \) be a simple Lie algebra of rank \( \ell \) with Cartan subalgebra \( \mathcal{H} \), root system \( \Delta \), simple root system \( \hat{\pi} \). Let \( \Gamma \), (resp. \( \Gamma_+ \)), be the set of all integral, (resp. integral dominant), elements of \( \mathcal{H}^* \), i.e., \( \lambda \in \mathcal{H}^* \) such that \( (\lambda, \alpha_i^\vee) \in \mathbb{Z} \), (resp. \( \mathbb{Z}_+ \)), for all simple roots \( \alpha_i \), \( (\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i, \alpha_i)) \). Let \( V \) be a lowest weight module with lowest weight \( \Lambda \) and lowest weight vector \( v_0 \). It has the following decomposition:

\[
\begin{align*}
  V & = \bigoplus_{\mu \in \Gamma_+} V_\mu, \\
  V_\mu & = \{ u \in V \mid Hu = (\Lambda + \mu)(H)u, \ \forall \ H \in \mathcal{H} \}
\end{align*}
\]
(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu \in \mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for $\mu$ outside the union of a finite number of sets of the form $D(\lambda) = \{ \mu \in \mathcal{H}^* | \mu \geq \lambda \}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of $V$ is defined by:

$$
\text{ch}_0 V = \sum_{\mu \in \Gamma^+} (\dim V_\mu) e(\Lambda + \mu) = \sum_{\mu \in \Gamma^+} (\dim V_\mu) e(\mu) \quad (31)
$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has $\dim V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \# \text{ of ways } \mu \text{ can be presented as a sum of positive roots } \beta$, each root taken with its multiplicity $\dim G_\beta (= 1 \text{ here})$, $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$
\text{ch}_0 V^\Lambda = \sum_{\mu \in \Gamma^+} P(\mu) e(\mu) = \sum_{\mu \in \Gamma^+} (\dim V_\mu) e(\mu) \quad (32)
$$

Further we recall the standard reflections in $\tilde{\mathcal{H}}^*$:

$$
\alpha \cdot \Lambda = \Lambda + n_\alpha \alpha, \quad \alpha \in \hat{\Delta}. \quad (33)
$$

The Weyl group $W$ is generated by the simple reflections $s_\alpha \equiv s_{\alpha_\iota}$, $\alpha_\iota \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that $w$ is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of $w$ is called the length of $w$, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM $L_\Lambda$ over $\hat{G}$, i.e., when $\Lambda \in -\Gamma^+$, has the form:

$$
\text{ch}_0 L_\Lambda = \sum_{w \in W} (-1)^{\ell(w)} \text{ch}_0 V^{w \cdot \Lambda}, \quad \Lambda \in -\Gamma^+ \quad (34)
$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda - \rho) + \rho$. For future reference we note:

$$
\alpha_\iota \cdot \Lambda = \Lambda + n_\alpha \alpha \quad (35)
$$
where
\[ n_{\alpha} = n_{\alpha}(\Lambda) = (\rho - \Lambda, \alpha^\vee) = (\rho - \Lambda)(H_{\alpha}), \quad \alpha \in \Delta^+. \quad (36) \]

In the case of basic classical Lie superalgebras the first character formulae were given by Kac. They are more complicated than the bosonic case, except for the algebras we consider. Actually, for \(osp(1/2n)\) the Verma module character formula is the same as (32):
\[ ch V^\Lambda = e(\Lambda) \prod_{\alpha \in \Delta^+} \frac{1}{1 - e(\alpha)} \quad (37) \]
using the restricted root system \(\tilde{\Delta}^+\). Naturally, the character formula for the finite-dimensional irreducible LWM \(L_\Lambda\) is again (34) using the Weyl group \(W_n\) of \(B_n\).

**Multiplets**

A Verma module \(V^\Lambda\) may be reducible w.r.t. to many positive roots, and thus there maybe many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

One main ingredient of the approach of [20] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called *multiplets* [20]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

If a Verma module \(V^\Lambda\) is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., \(m_k \in \mathbb{N}\) for all \(k\), then the irreducible submodules are isomorphic to the finite-dimensional irreps of \(G^C\) [20]. (Actually, this is a condition only for \(m_n\) since \(m_k \in \mathbb{N}\) for \(k = 1, \ldots, n - 1\).)
In these cases we have the *main multiplets* which are isomorphic to the Weyl group of \(G^C\) [20].

In the cases of non-dominant weight \(\Lambda\) the character formula for the irreducible LWM is [25] :
\[ ch L_\Lambda = \sum_{w \in W} (-1)^{\ell(w_{\Lambda}w)} P_{w, w_{\Lambda}}(1) \ ch V^{w_{\Lambda}^{-1}(\Lambda)} \ \Lambda \in \Gamma \quad (38) \]
where \(P_{y,w}(u)\) are the Kazhdan–Lusztig polynomials \(y,w \in W\) [25] (for an easier exposition see [24]), \(w_{\Lambda}\) is a unique element of \(W\) with minimal
length such that the signature of \( \Lambda_0 = w^{-1}_\Lambda \cdot \Lambda \) is anti-dominant or semi-anti-dominant:

\[
\chi_0 = (m'_1, \ldots, m'_n), \quad m'_k = 1 - \Lambda_0(H_k) \in \mathbb{Z}_-. \tag{39}
\]

Note that \( P_{y,w}(1) \in \mathbb{N} \) for \( y \leq w \).

When \( \Lambda_0 \) is semi-anti-dominant, i.e., at least one \( m'_k = 0 \), then in fact \( W \) is replaced by a reduced Weyl group \( W_R \).

Most often the value of \( P_{y,w}(1) \) is equal to 1 (as in the character formula for the finite-dimensional irreps), while the cases \( P_{y,w}(1) > 1 \) are related to the appearance of subsingular vectors, though the situation is more subtle, see [24].

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. In the case of \( d_{ij} \) using (24) we have:

\[
m_n(d_{ij}) = 1 - 2m_j - \cdots - 2m_{n-1} - m_i - \cdots - m_{j-1}. \tag{40}
\]

In the case of \( d_i \) (25) we have:

\[
m_n(d_i) = 1 - 2m_i - \cdots - 2m_{n-1}. \tag{41}
\]

As expected the weights related to positive energy \( d \) are not dominant \((m_n(d_{ij}) \in \mathbb{Z}_-, \quad m_n(d_i) \in -\mathbb{N}, \quad (i < n))\), since the positive energy UIRs are infinite-dimensional. (Naturally, \( m_n(d_n) = 1 \) falls out of the picture since \( d_n < 0 \).)

Thus, the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the \( w_{\Lambda} \) in the character formula (38). This we do in the next subsection in the case \( n = 3 \).

5. The case of \( \text{osp}(1|6) \)

For \( n = 3 \) formula (26) simplifies to:

\[
d_1 > d_{12} > d_2 > d_{23} > d_3
\]

\[\leftrightarrow > d_{13} > \uparrow\]
The Theorem now reads:

\[
\begin{align*}
    d &\geq 2 + \frac{1}{2}(a_1 + a_2) = d_1, \quad a_1 \neq 0, \\
    d &\geq \frac{3}{2} + \frac{1}{2}a_2 = d_{12}, \quad a_1 = 0, \quad a_2 \neq 0, \\
    d &\geq 1 + \frac{1}{2}a_2 = d_2 > d_{13}, \quad a_1 = 0, \quad a_2 \neq 0, \\
    d &\geq 1 = d_{13}, \quad a_1 = a_2 = 0, \\
    d &\geq \frac{1}{2} = d_{23}, \quad a_1 = a_2 = 0, \\
    d &\geq 0 = d_3, \quad a_1 = a_2 = 0.
\end{align*}
\]

The Weyl group $W_n$ of $B_n$ has $2^n n!$ elements, i.e., 48 for $B_3$. Let $S = \{s_1, s_2, s_3\}$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

\[
s_1^2 = s_2^2 = s_3^2 = e, \quad (s_1s_2)^3 = e, \quad (s_2s_3)^4 = e, \quad s_1s_3 = s_3s_1,
\]

$e$ being the identity of $W_3$. The 48 elements may be listed as:

\[
e, \quad s_1, \quad s_2, \quad s_3, \quad s_1s_2, \quad s_1s_3, \quad s_2s_3, \quad s_3s_2, \quad s_1s_2s_1, \quad s_1s_2s_3, \quad s_1s_3s_2, \quad s_2s_1s_3, \quad s_2s_3s_2, \quad s_3s_1s_2, \quad s_3s_1s_3, \quad s_3s_2s_3, \quad s_1s_2s_1s_3, \quad s_1s_2s_3s_2, \quad s_1s_3s_2s_1, \quad s_1s_3s_2s_3, \quad s_2s_1s_2s_3, \quad s_2s_1s_3s_2, \quad s_2s_3s_1s_2, \quad s_2s_3s_1s_3, \quad s_3s_1s_2s_3, \quad s_3s_2s_1s_3, \quad s_3s_2s_3s_2, \quad s_3s_2s_3s_1, \quad s_3s_2s_3s_1s_3, \quad s_3s_2s_3s_1s_2, \quad s_3s_2s_3s_1s_2s_3, \quad s_3s_2s_3s_1s_2s_3s_2, \quad s_3s_2s_3s_1s_2s_3s_1, \quad s_3s_2s_3s_1s_2s_3s_2s_1, \quad s_3s_2s_3s_1s_2s_3s_2s_3, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_3, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_1, \quad s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2s_1.
\]

The character formula for the Verma modules in our case is given
explicitly by:

\[
\text{ch } V^\Lambda = \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)} \times \frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}
\]

(45)

where \( t_j \equiv e(\alpha_j) \).

Now we give the character formulae of the five boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight \( \Lambda_0 \) which determines the main multiplet by \( (m'_1, m'_2, m'_3) \), \( m'_k \in \mathbb{N} \), using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some \( m'_k = 0 \).

\begin{itemize}
  \item In the case of \( d = d_1 = 2 + \frac{1}{2}(a_1 + a_2) \) there are twelve members of the multiplet which is a submultiplet of a main multiplet. (Remember that \( m_1 > 1 \) since \( a_1 \neq 0 \).) They are grouped into two standard \( sl(3) \) submultiplets of six members. The first submultiplet starts from \( V^{\Lambda_0^d} \), where \( \Lambda_0^d = w \cdot \Lambda_0 \), \( w = w_{\Lambda_0^d} = s_2s_1s_3s_2s_3 \), with signature:

\[
\Lambda_0^d : \quad (m_1, m_2, m_3' = 1 - 2m_{12}) ,
\]

\( m_1, m_2 \in \mathbb{N} \), \( m_{12} \equiv m_1 + m_2 \).

The other submultiplet starts from \( V^{\Lambda'_0} \) with \( \Lambda'_0 = \Lambda_0^d + \delta_1 = \Lambda_0^d + \alpha_1 + \alpha_2 + \alpha_3 \), with signature: \( \Lambda'_0 : \quad (m_1 - 1, m_2, m'_3 = 1 - 2m_{12}) \), \( m_1 > 1 \).

The character formula is (38) with \( w_\Lambda = w_{\Lambda_0^d} \):

\[
\text{ch } \Lambda_0^d = \frac{e(\Lambda_0^d)}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \{ \text{ch } \Lambda_{m_1,m_2}(t_1,t_2) - t_1t_2t_3 \text{ch } \Lambda_{m_1-1,m_2}(t_1,t_2) \} , \quad m_1 > 1
\]

where \( \text{ch } \Lambda_{m_1,m_2}(t_1,t_2) \) is the normalized character of the finite-dimensional \( sl(3) \) irrep with Dynkin labels \( (m_1, m_2) \) (and dimension \( m_1m_2(m_1 + m_2)/2 \)):

\[
\text{ch } \Lambda_{m_1,m_2}(t_1,t_2) = \frac{1-t_1^{m_1} - t_2^{m_2} + t_1^{m_1}t_2^{m_2} + t_1^{m_1}t_2^{m_2} - t_1^{m_1}t_2^{m_2}}{(1-t_1)(1-t_2)(1-t_1t_2)}
\]

(48)
Naturally, the latter formula is a polynomial in \( t_1, t_2 \), e.g.,
\[ \text{ch} \, \Lambda_{1,1}(t_1, t_2) = 1, \quad \text{ch} \, \Lambda_{2,1}(t_1, t_2) = 1 + t_1 + t_2. \]

In the case \( m_1 = 2, m_2 = 1 \) the character formula (47) simplifies to:
\[
\text{ch} \, \Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3)} \times \left(1 + \frac{t_1(1 + t_2)}{1-t_1t_2t_3}\right), \quad m_1 = 2, m_2 = 1 \tag{49}
\]

- In the case of \( d = d_{12} = \frac{1}{2}(3 + a_2) \) which is relevant for unitarity, i.e., \( m_1 = 1 \), there are again twelve members of the multiplet. Omitting the details [14] the character \( f\text{-la} \) is:
\[
\text{ch} \, \Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3)} \times \{ \text{ch} \, \Lambda_{1,m_2}(t_1, t_2) - (t_1t_2t_3^2)^{m_2} \text{ch} \, \Lambda_{1,m_2-1}(t_1, t_2) \} \tag{50}
\]

In the case \( m_2 = 2 \) it simplifies to:
\[
\text{ch} \, \Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2^2t_3)} \times \{1 + t_1t_2t_3^2 + t_2(1 + t_1) \frac{t_2}{1-t_1t_2t_3^{2}}\} \tag{51}
\]

- In the case \( d = d_2 = d_{13} = 1 \) and \( a_1 = a_2 = 0, \ m_1 = m_2 = 1 \), the signature is:
\[
\Lambda_0^{d_2 = d_{13}} : (1, 1, -1). \tag{52}
\]

Again there are twelve members of the multiplet which has two \( sl(3) \) submultiplets. First there is a \( sl(3) \) sextet starting from \( \Lambda_0^{d_2 = d_{13}} \) with parameters \( (1, 1) \). Then there is a \( sl(3) \) sextet starting from \( \Lambda_0^{d_2 = d_{13}} + \alpha_1 + 2\alpha_2 + 3\alpha_3 \) with parameters \( (1, 1) \). Note that that \( \alpha_1 + 2\alpha_2 + 3\alpha_3 = \delta_1 + \delta_2 + \delta_3 \) is the weight of a subsingular vector [14], yet the corresponding KL polynomial \( P_{y,w}(1) \) is equal to 1. Thus, the character formula is [14]:
\[
\text{ch} \, \Lambda_0^{d_2 = d_{13}} = \frac{e(\Lambda_0^{d_2 = d_{13}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \{1 + t_1t_2t_3^2 + t_2(1 + t_1) \frac{t_2}{1-t_1t_2t_3^{2}}\} \tag{53}
\]
Note that the above formula may be rewritten as:

\[
\text{ch } \Lambda_{0}^{d_{2}=d_{13}} = \frac{e(\Lambda_{0}^{d_{2}=d_{13}})}{(1-t_{2}t_{3})(1-t_{1}t_{2}t_{3})(1-t_{2}t_{3}^{2})(1-t_{1}t_{2}t_{3}^{2})} \times
\]

\[
\times \left( \frac{1}{1-t_{1}t_{2}t_{3}^{2}} + \frac{t_{3}}{1-t_{3}} \right) \tag{54}
\]

- In the case of \( d = d_{2} = 1 + \frac{1}{2}a_{2} > d_{13} = 1 \), i.e., \( m_{1} = 1, \ m_{2} = 1 + a_{2} > 1 \). The multiplet has 24 members for \( m_{2} > 2 \). Omitting the details [14] the character f-la is:

\[
\text{ch } \Lambda_{0}^{d_{2}} = \frac{e(\Lambda_{0}^{d_{2}})}{(1-t_{3})(1-t_{2}t_{3})(1-t_{1}t_{2}t_{3})} \times
\]

\[
\times \frac{1}{(1-t_{2}t_{3}^{2})(1-t_{1}t_{2}t_{3}^{2})(1-t_{1}t_{2}t_{3}^{2})} \times
\]

\[
\times \left\{ \text{ch } \Lambda_{1,m_{2}}(t_{1}, t_{2}) - t_{2}t_{3} \text{ch } \Lambda_{2,m_{2}-1}(t_{1}, t_{2}) +
\right.
\]

\[
+ t_{1}t_{2}t_{3}^{3} \text{ch } \Lambda_{2,m_{2}-2}(t_{1}, t_{2}) - t_{1}t_{2}^{2}t_{3}^{4} \text{ch } \Lambda_{1,m_{2}-2}(t_{1}, t_{2}) \} \tag{55}
\]

When \( m_{2} = 2 \) (\( a_{2} = 1 \)) the multiplet reduces to only 12 members, and the character formula simplifies to:

\[
\text{ch } \Lambda_{0}^{d_{2}} = \frac{e(\Lambda_{0}^{d_{2}})}{(1-t_{2}t_{3}^{2})(1-t_{1}t_{2}t_{3}^{2})(1-t_{2}t_{3}^{2})} \times
\]

\[
\times \left\{ \frac{1}{(1-t_{3})(1-t_{1}t_{2}t_{3})} + \frac{t_{2}}{(1-t_{3})(1-t_{2}t_{3})}
\right.
\]

\[
+ \frac{t_{1}t_{2}}{(1-t_{2}t_{3})(1-t_{1}t_{2}t_{3})} \} \tag{56}
\]

- In the case of \( d = d_{23} = \frac{1}{2} \), \( a_{1} = a_{2} = 0 \), i.e., \( m_{1} = m_{2} = 1 \), again we have a multiplet with 24 members. Omitting the details [14] the character
The formula is:

\[
\ch \Lambda^d_{23} = \frac{e(\Lambda^d_{23})}{(1-t_3)(1-t_2 t_3)(1-t_1 t_2 t_3)} \times \\
\times \frac{1}{(1-t_2 t_3^2)(1-t_1 t_2 t_3^2)(1-t_1^2 t_2^2 t_3^2)} \times \\
\times \{ 1 - t_2 t_3^2 \ch \Lambda_{2,1}(t_1, t_2) + t_1^2 t_2^4 \ch \Lambda_{1,2}(t_1, t_2) - t_1^2 t_2 t_3^6 \} = \\
= \frac{e(\Lambda^d_{23})}{(1-t_3)(1-t_2 t_3)(1-t_1 t_2 t_3)(1-t_1 t_2 t_3^2)(1-t_1^2 t_2^2 t_3^2)} \times \\
\times \{ 1 - t_2 t_3^2 (1 + t_1 + t_1 t_2) + t_1 t_2^6 t_3^4 (1 + t_2 + t_1 t_2) - t_1^2 t_2 t_3^6 \}
\]

Note that the above formula may be rewritten as:

\[
\ch \Lambda^d_{23} = \frac{e(\Lambda^d_{23})}{(1-t_3)(1-t_2 t_3)(1-t_1 t_2 t_3)} \quad (58)
\]

Note that formulae (49),(51),(54),(56),(58) are new w.r.t. [14].

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