Bulk Scalar Stabilization of the Radion without Metric Back-Reaction in the Randall-Sundrum Model

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ABSTRACT

Generalizations of the Randall-Sundrum model containing a bulk scalar field $\Phi$ interacting with the curvature $R$ through the general coupling $R f(\Phi)$ are considered. We derive the general form of the effective 4D potential for the spin-zero fields and show that in the mass matrix the radion mixes with the Kaluza-Klein modes of the bulk scalar fluctuations. We demonstrate that it is possible to choose a non-trivial background form $\Phi_0(y)$ (where $y$ is the extra dimension coordinate) for the bulk scalar field such that the exact Randall-Sundrum metric is preserved (i.e. such that there is no back-reaction). We compute the mass matrix for the radion and the KK modes of the excitations of the bulk scalar relative to the background configuration $\Phi_0(y)$ and find that the resulting mass matrix implies a non-zero value for the mass of the radion (identified as the state with the lowest eigenvalue of the scalar mass matrix). We find that this mass is suppressed relative to the Planck scale by the standard warp factor needed to explain the hierarchy puzzle, implying that a mass $\sim 1$ TeV is a natural order of magnitude for the radion mass.

The general considerations are illustrated in the case of a model containing an $R\Phi^2$ interaction term.

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1 Introduction

Although the Standard Model (SM) of electroweak interactions describes successfully almost all existing experimental data, the model suffers from many theoretical drawbacks. One of these is the hierarchy problem: namely, the SM cannot consistently accommodate the weak energy scale $\mathcal{O}(1 \text{ TeV})$ and a much higher scale such as the Planck mass scale $\mathcal{O}(10^{18} \text{ GeV})$. Therefore, it is widely accepted that the SM is only an effective low-energy theory embedded in some more fundamental high-scale theory that presumably could contain gravitational interactions. Recently many models that incorporate gravity have been proposed in the context of higher dimensional space-time. These models have received tremendous attention since they might provide a solution to the hierarchy problem. One of the most attractive attempts has been formulated by Randall and Sundrum [1] who postulated a 5D universe with two 4D surfaces (“3-branes”) with the following action:

$$S = - \int d^4x \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \left\{ \sqrt{|g|} \left( \frac{R}{2\kappa^2} + \Lambda \right) + \sum_{k=1,2} \sqrt{|g_k|} \Lambda_k \delta(y - y_k) \right\},$$

where $R$ is the Ricci scalar, $\kappa^2 = 8\pi G_N^{(5)}$ with $G_N^{(5)}$ the Newton constant in 5D and $\Lambda, \Lambda_1 \equiv \Lambda_{\text{hid}}$ and $\Lambda_2 \equiv \Lambda_{\text{vis}}$ are the cosmological constants in the bulk, on the hidden and visible branes, respectively. In the above, $g_{ij}$ ($i, j = 0, 1, 2, 3, 4$) is the bulk metric and $(g_1)_{\mu\nu} \equiv (g_{\text{hid}})_{\mu\nu}(x) \equiv g_{\mu\nu}(x, y = y_1 \equiv 0)$ and $(g_2)_{\mu\nu} \equiv (g_{\text{vis}})_{\mu\nu}(x) \equiv g_{\mu\nu}(x, y = y_2 \equiv 1/2)$ ($\mu, \nu = 0, 1, 2, 3$) are the induced metrics on the branes.

It turns out that if the bulk and brane cosmological constants are related by

$$\Lambda = -\frac{6m_0^2}{\kappa^2}, \quad \Lambda_{\text{hid}} = -\Lambda_{\text{vis}} = \frac{6m_0}{\kappa^2},$$

and if periodic boundary conditions ($y \to y + 1$) with identification of $(x, y)$ and $(x, -y)$ are imposed, then the following metric is a solution of the 5D Einstein equations:

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu - b_0^2 dy^2,$$

where $\sigma(y) = m_0b_0 [y(2\theta(y) - 1) - 2(y - 1/2)\theta(y - 1/2)]; b_0$ is a constant parameter that is not determined by the equations of motion.
Within the Randall-Sundrum (RS) model all the SM particles as well as the non-gravitational forces are assumed to be present on one of the 3-branes, the “visible brane”. Gravity lives on the visible brane, on the second brane (the “hidden brane”) and in the bulk. All mass scales in the 5D theory are of the order of the Planck mass. By placing the SM fields on the visible brane, the initial 5D electroweak mass scale $O(M_{Pl})$ is rescaled by an exponential suppression factor (the “warp factor”) $\gamma \equiv e^{-m_0 b_0/2}$, down to the weak scale $O(1 \text{ TeV})$ without any severe fine tuning. To achieve the necessary suppression, one needs $m_0 b_0/2 \sim 37$. This is a great improvement compared to the original problem of accommodating both the weak and the Planck scale within a single theory.

2 The radion effective potential

A drawback of the RS model is the presence of a massless degree of freedom called the radion. There have been several attempts (see Refs. [2, 3, 4]) in the literature to generate the radion mass by introducing a bulk scalar field $\Phi$ that would induce an appropriate radion potential. Here we will derive the general form of the potential within a class of models containing the bulk scalar interacting with gravity in the following manner:

$$S = \int d^4x \int_0^{y_{\text{max}}} dy \left\{ \sqrt{|g|} \left[ -\frac{R}{2 \kappa^2} - \Lambda - \alpha Rf(\Phi) + \frac{1}{2} g^{ij} \Phi_i \Phi_j - V(\Phi) \right] \right. $$

$$- \sum_{k=1,2} \sqrt{|g_k|} [\Lambda_k + V_k(\Phi)] \delta(y - y_k) \right\}, \quad (4)$$

where we have introduced the bulk potential $V(\Phi)$ and the brane potentials $V_1(\Phi) \equiv V_{\text{hid}}(\Phi)$ and $V_2(\Phi) \equiv V_{\text{vis}}(\Phi)$. In addition to the standard scalar kinetic-energy term, we have allowed for a general coupling of the bulk scalar to gravity through the $\alpha Rf(\Phi)$ interaction term. Since we would like to preserve the explanation of the hierarchy proposed by Randall and Sundrum, we will also require that the RS metric (3) remain an exact solution\(^1\) of the Einstein equations even in the presence of the bulk scalar $\Phi$. Therefore, it is useful to separate out in the action (4) both

\(^1\) One can, of course, consider slight modifications of the RS metric that would also solve the hierarchy problem in a similar manner. However, in this paper we would like to discuss a scenario with exactly the same metric as in the original RS model.
the bulk ($\Lambda$) and brane ($\Lambda_{hid}$, $\Lambda_{vis}$) cosmological constants that satisfy the RS conditions, Eq. (2).

In order to identify the radion, it is sufficient to consider scalar excitations of the metric around the background RS solutions. Hereafter, we will adopt the following parameterization (see Refs. [5, 6]) of the metric fluctuations:

$$ds^2 = e^{-2\sigma(y) - 2b(x)e^{2\sigma(y)}(\eta_{\mu\nu} + h_{\mu\nu}(x, y))}dx^\mu dx^\nu - b_0^2 \left(1 + 2b(x)e^{2\sigma(y)}\right)^2 dy^2,$$

where $h_{\mu\nu}(x, y)$ and $b(x)$ are related to the graviton\(^2\) and the radion, respectively. Then from $-\sqrt{|g|} R/(2\kappa^2)$ in the action (4) one obtains [after expanding in powers of $b(x)$] the kinetic term for the radion:\(^3\)

$$S = \frac{6}{\kappa^2 m_0} \left(e^{m_0 b_0} - 1\right) \int d^4x \frac{1}{2} (\partial_\mu b)(\partial^\mu b) + \cdots$$

It is easy to verify that if interactions of the scalar field $\Phi$ are switched off, then there is no potential for $b(x)$ and consequently the radion would be massless.

The bulk scalar has been introduced here in order to generate a non-trivial potential for the radion. However, in general the presence of the scalar leads to a non-trivial interaction potential between the radion and the scalar in addition to the appearance of a radion potential. Therefore, the strategy that we will follow here will be to determine the background scalar configuration $\Phi(y)$ such that the RS background metric is preserved and then to expand the action (4) around it. First, one has to solve the Einstein equations together with the equation of motion for $\Phi$. Let us start with the Einstein equations, keeping in mind that we would like to preserve the RS metric as a vacuum solution. We write

$$G_{ij} = \kappa^2 \left[T_{ij}^{(RS)} + (\delta T)_{ij}(\Phi)\right],$$

where $G_{ij}$ is the Einstein tensor, $T_{ij}^{(RS)}$ denotes the RS contribution to the energy-momentum tensor and $(\delta T)_{ij}$ contains all new contributions emerging from interactions of the scalar $\Phi$. It is useful to first calculate all the extra (compared to the

\(^2\)In the following we will not discuss interactions with gravitons as they will not influence the potential for scalar degrees of freedom.

\(^3\)Hereafter, the flat metric $\eta^{\mu\nu}$ will be assumed whenever repeated indices are summed.
pure RS model) contributions to the energy-momentum tensor. It is easy to show that:

$$\left(\delta T\right)_{ij} = T^{(\Phi)}_{ij} + 2\alpha \{D_{ij}[f(\Phi)] - G_{ij}f(\Phi)\} + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi)(g_k)_{\mu\nu} \delta_i^\mu \delta_j^\nu \delta(y - y_k),$$

where

$$T^{(\Phi)}_{ij} \equiv \nabla_i \Phi \nabla_j \Phi - g_{ij} \left[ \frac{1}{2} g^{kl} \nabla_k \Phi \nabla_l \Phi - V(\Phi) \right]$$

$$D_{ij}[X] \equiv \nabla_i \nabla_j X - g_{ij} g^{kl} \nabla_k \nabla_l X .$$

For the RS background metric we obtain:

$$G_{ij} = \left( \begin{array}{cc} -3b_0^{-2} e^{-2\sigma} (2\sigma'{}^2 - \sigma''\sigma'') & 0 \\ 0 & 6\sigma'{}^2 \end{array} \right)$$

and

$$D_{ij}[f(\Phi)] = \left( \begin{array}{cc} b_0^{-2} e^{-2\sigma} (-3\sigma' f(\Phi)' + f(\Phi)'') & 0 \\ 0 & 4\sigma' f(\Phi)' \end{array} \right) ,$$

where here, and in what follows, the prime denotes differentiation with respect to the 5th coordinate, $y$. Since we demand that the RS metric be preserved even when the scalar is present (no back-reaction from the scalar), we have to require that the extra contributions to the energy momentum tensor (calculated using the RS metric) vanish:

$$\left(\delta T\right)_{ij}(\Phi) = 0 .$$

Since we want to find a background solution for $\Phi$ that satisfies 4D Lorentz invariance, we will assume that the solution is only a function of the extra dimension coordinate, $y$. The ($\mu,\nu$) and (5,5) components of Eq. (13) read, respectively:

$$(\Phi')^2 + 12\alpha (2\sigma'{}^2 - \sigma'') f - 12\alpha \sigma' f' + 4\alpha f'' +$$

$$2b_0^2 \left[ V(\Phi) + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi) \delta(y - y_k) \right] = 0$$

$$(\Phi')^2 - 24\alpha \sigma'{}^2 f + 16\alpha \sigma' f' - 2b_0^2 V(\Phi) = 0$$

Note that since $\Phi(y)$ should be a continuous function, the above equations imply $\Phi'(y) = 0$ and $V(\Phi) = V_{vis}(\Phi) = V_{hid}(\Phi) = 0$ for $\alpha = 0$. Therefore, introduction
of the extra coupling $\alpha R f(\Phi)$ is essential in order to obtain a no-back reaction solution, $(\delta T)_{ij}(\Phi) = 0$.

In addition, $\Phi$ must satisfy the following equation of motion:

$$
-\Phi'' + 4\sigma' \Phi' + 4\alpha (5\sigma''^2 - 2\sigma''') \frac{d f}{d \Phi} + b_0^2 \left[ \frac{dV}{d\Phi} + \frac{1}{b_0} \sum_{k=1,2} \frac{dV_k}{d\Phi} \delta(y - y_k) \right] = 0, \quad (16)
$$

where the RS metric was used.\(^4\) Once the vacuum solution ($\Phi_0$) is determined, we expand the action, Eq. (4), adopting the parameterization of the scalar fluctuations of the metric given in Eq. (5) and the following definition for the $\Phi$ quantum fluctuation:

$$
\Phi(x, y) \rightarrow \Phi_0(y) + \phi(x, y). \quad (17)
$$

Then, in order to determine the effective 4D potential for the scalar degrees of freedom, we collect all non-derivative contributions to the $d^4x$ integrand in the action of Eq. (4) containing $b(x)$ and $\phi(x, y)$.

In other words, we expand the theory defined by the action (4) around the vacuum solutions for the metric (the RS solution) and the bulk scalar field $\Phi$ [the solution of Eqs. (13) and (16)] in terms of the scalar fluctuation of the metric, $b(x)$, and fluctuation of the scalar field, $\phi(x, y)$.

First, let us calculate the Ricci scalar for the metric (5) and collect all the terms containing derivatives with respect the the extra component $y$:

$$
R = \frac{20}{b_0^2} \sigma'^2 - \frac{8}{b_0^2} \frac{\sigma''}{1 + 2b_0^2} + \cdots \quad (18)
$$

where ellipses contain only $(x, y)$-derivatives of the graviton and $x$-derivatives of the radion. Since we are going to calculate the potential, derivatives of $b(x)$ will be dropped hereafter. As has already been mentioned, we will not consider fluctuations of the $\eta_{\mu\nu}$ part of the metric. Therefore, we will also neglect all terms containing $h_{\mu\nu}(x, y)$.\(^5\)

\(^4\)Both the Einstein equations (13) and the equation of motion (16) constrain the scalar $\Phi$. However, it can be verified that the equations are not independent; a certain linear combination of derivatives with respect to $y$ of the $(\mu, \nu)$ and $(5, 5)$ components of Eq. (13) is proportional to Eq. (16). Since we have not specified the potential $V(\Phi)$, we can find a consistent solution both for $\Phi$ and the potential. Details of the derivations for $f(\Phi) = 3/32 \Phi^2$ will be presented in Sec. 3.

\(^5\)Adopting the traceless gauge, $h_{\mu}^\mu = 0$, one can eliminate possible mixing between $b$ or $\phi$ and $h_{\mu\nu}$. Consequently, graviton interactions cannot influence scalar masses. Thus, $h_{\mu\nu}$ will be suppressed in the following.
Using the contributions to the Ricci scalar displayed in Eq. (18), one gets the following form of the effective 4D potential from the action (4):

\[
V_{\text{eff}}(b, \phi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \ e^{-4\sigma - 4be^{2\sigma}} \times
\left\{ \frac{1}{2} \left[ \frac{1}{b_0(1 + 2be^{2\sigma})}(\Phi_0' + \phi')^2 + 2b_0(1 + 2be^{2\sigma})V(\Phi_0 + \phi) \right] + \frac{2}{\kappa^2 b_0} \left[ 5(1 + 2be^{2\sigma})\sigma' - 2\sigma'' \right] \left[ 1 + 2\kappa^2 \alpha f(\Phi_0 + \phi) \right] + \Lambda b_0(1 + 2be^{2\sigma}) + \sum_{k=1,2} \left[ \Lambda_k + V_k(\Phi_0 + \phi) \right] \delta(y - y_k) \right\}
\]  

(19)

where \( \Phi_0 = \Phi_0(y) \) denotes the vacuum solution (that preserves the RS metric) for the scalar \( \Phi \). Note that \( \Phi_0 \) is determined as a solution of the equations of motion for the RS background metric. As a result, it does not contain any dependence on the radion field \( b(x) \). It is easy to verify that contributions from the pure RS model to \( V_{\text{eff}}(b, \phi) \) vanish when the relations (2) are satisfied: a non-trivial potential requires an extension of the minimal RS model.

Next, it is easy to show from Eq. (8) that if \( \Phi \) is independent of \( x \) then the following identity holds:

\[
(\delta T)_{\mu}^{\mu}(\Phi) = \frac{2}{b_0} \left[ \frac{1}{b_0}(\Phi')^2 + 2b_0V(\Phi) \right] + \frac{24\alpha}{b_0^2} \left( 2\sigma' - \sigma'' \right) f(\Phi)
\]

\[
+ \frac{8\alpha}{b_0^2} \left( -3\sigma' f(\Phi)' + f(\Phi)'' \right) + \frac{4}{b_0} \sum_{k=1,2} V_k(\Phi)\delta(y - y_k).
\]

(20)

Multiplying the above equation by \( \exp\{-4\sigma\}(b_0/4) \), integrating the \( (-3\sigma'f + f'') \) term by parts and using the RS relations, Eq. (2), one obtains the following simple relation between \( (\delta T)_{\mu}^{\mu}(\Phi_0) \) and the minimum of the effective potential of Eq. (19) at its \([b(x) = 0, \phi(x, y) = 0] \) minimum:

\[
V_{\text{eff}}(0, 0) = \frac{b_0}{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \ e^{-4\sigma}(\delta T)_{\mu}^{\mu}(\Phi_0).
\]

(21)

Note that in Eq. (21) we employ the trace from Eq. (20) calculated for the background solution \( \Phi_0 \). Since the no-back-reaction requirement, Eq. (13), implies \( (\delta T)_{\mu}^{\mu}(\Phi_0) = 0 \), the relation (21) shows that the effective potential must vanish at the minimum

\[
V_{\text{eff}}(0, 0) = 0.
\]

(22)
It is straightforward to verify that linear terms in $b(x)$ and $\phi(x, y)$ disappear by virtue of Eq. (13) and Eq. (16), respectively.

In order to determine scalar masses one has to expand the action (4) up to terms quadratic in $b$ and $\phi$. First, let us define the K-K modes of the scalar fluctuations:

$$\phi(x, y) = \sum_n \varphi_n(x) J_n(y),$$

with orthonormal functions $J_n(y)$:

$$\int_{\frac{1}{2}}^{\frac{1}{2}} dy e^{-2\sigma(y)} J_n(y) J_m(y) = \delta_{nm}. \tag{24}$$

The resulting mass terms are the following:

$$\frac{1}{2} \left( \begin{array}{c} r \\ \varphi_m \end{array} \right) \left( \begin{array}{cc} M_r^2 & (M^2_{r\phi})_n \\ (M^2_{r\varphi})_m & (M^2_{\varphi})_{mn} \end{array} \right) \left( \begin{array}{c} r \\ \varphi_n \end{array} \right), \tag{25}$$

where $r$ is the canonically normalized radion [see Eq. (6)]:

$$r(x) = \left( \frac{6}{\kappa^2 m_0} \right)^{1/2} e^{m_0 b(x)}/b(x) \tag{26}$$

Inputing the equation of motion (16), the elements of the mass matrix read:

$$M_r^2 = \frac{2}{3} \kappa^2 m_0 e^{-m_0 b_0} \int_{\frac{1}{2}}^{\frac{1}{2}} (\Phi_0')^2 dy \tag{27}$$

$$(M^2_{r\phi})_n = \left( \frac{2}{3} \kappa^2 m_0 e^{-m_0 b_0} \right)^{1/2} \frac{1}{b_0} \int_{\frac{1}{2}}^{\frac{1}{2}} e^{-2\sigma(y)} J_n(y) \times$$

$$\left[ \Phi_0'' + 2\sigma' \Phi_0' + 20 \alpha \sigma'^2 \frac{d^2 f}{d \Phi^2} (\Phi_0) + b_0^2 \frac{d V}{d \Phi} (\Phi_0) \right] \tag{28}$$

$$(M^2_{\varphi})_{mn} = \frac{1}{b_0^2} \int_{\frac{1}{2}}^{\frac{1}{2}} e^{-4\sigma(y)} \left\{ J_n(y) J_m(y) \times$$

$$\left[ 4\alpha (5\sigma'' - 2\sigma') \frac{d^2 f}{d \Phi^2} + b_0^2 \left( \frac{d^2 V}{d \Phi^2} + \frac{1}{b_0} \sum_{k=1,2} \frac{d^2 V_k}{d \Phi^2} \delta(y - y_k) \right) \right] +$$

$$J_n'(y) J_m'(y) \right\}. \tag{29}$$

Before we can estimate the size of the elements of the mass matrix, we must discuss first the constraint that is imposed on the model by the requirement of maintaining the standard strength of classical 4D gravity. Adopting the metric defined by Eq. (5), one can calculate the coefficient of the 4D Ricci scalar obtained
for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In order to reproduce the standard result, the coefficient should be $M_{Pl}^2/2$. The resulting constraint is:

$$M_{Pl}^2 = \frac{1 - \gamma^2}{\kappa^2 m_0} + 2\alpha b_0 \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, e^{-2\sigma f(\Phi_0(y))},$$  

(30)

where $M_{Pl} \sim 2 \times 10^{18}$ GeV is the reduced Planck mass and $\gamma = \exp(-m_0 b_0/2)$. In order to solve the hierarchy problem, one needs $m_0 b_0/2 \sim 37$. Therefore, terms of order $\gamma^2$ can be safely neglected in Eq. (30). It is clear that the most natural scenario\(^6\) emerges when all the mass parameters of the 5D theory are of the order of $M_{Pl}$. In this case, the elements of the scalar mass matrix defined by Eq. (25) are of the following order of magnitude:

$$\sim M_{Pl}^2 \left( \begin{array}{ccc} a \gamma^2 & b^{1/2} \gamma & \gamma \\ b^{1/2} \gamma & 1 & \\ \gamma & 1 & \end{array} \right),$$  

(31)

where $a$ and $b$ are calculable coefficients of the order of 1. It is clear that for $m_0 b_0/2 \sim 37$ the lowest scalar mass is of order:

$$\sim \frac{\gamma M_{Pl}(a-b)^{1/2}}{\sqrt{2}} \sim \frac{246 \text{ GeV}}{\sqrt{2}}.$$  

(32)

There are two essential conclusions. First, we see that the lowest scalar mass receives the standard suppression from the warp factor $\gamma = \exp(-m_0 b_0/2)$ that is necessary for the solution of the hierarchy problem. As a result, the mass expected in the presence of a bulk scalar $R f(\Phi)$ interaction is of the order of the electroweak scale. Second, it is clear that in order to find precise values for scalar masses for any particular choice of the interaction function $f(\Phi)$, one has to take into account the mixing between the radion and the Kaluza-Klein modes of the bulk scalar fluctuation. To fully explore the phenomenology of the theory, it would be necessary to calculate all the entries of the mass matrix; however, this is beyond the scope of this paper.

Finally, we close this section by reiterating the fact that if there is no $R f(\Phi)$ interaction, i.e. if $\alpha = 0$, then necessarily $\Phi'_0(y) = 0$ and $V(\Phi_0) = V_{vis}(\Phi_0) = V_{hid}(\Phi_0) = 0$, which in turn would lead to a vanishing mass matrix.

\(^6\)Of course, an appropriate cancellation between contributions coming from parameters that differ even by many orders of magnitude is in principle also possible. However, since we would like to preserve the solution of the hierarchy problem proposed by Randall and Sundrum, we should assume that all the mass parameters in the fundamental 5D theory are of the same order. Then, the only necessary fine tuning is to keep $m_0 b_0/2 \sim 37$. 

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3 The $R\Phi^2$ interaction

In this section, we will illustrate the general discussion from Section 2, choosing a specific form of the interaction between the bulk scalar and gravity:

$$f(\Phi) = \frac{3}{32} \Phi^2.$$ (33)

The function $f(\Phi)$ has been normalized such that $\alpha = -1$ corresponds to a 5D conformally invariant interaction. This coupling was discussed in various contexts by many authors in the past, see e.g. Ref. [7].

In this case, the conditions for $(\delta T)_{ij}(\Phi) = 0$, Eq. (13), as written out in Eqs. (14) and (15), read:

$$-3\sigma' (\Phi^2)' + (\Phi^2)'' + \frac{8}{3\alpha} (\Phi')^2 + 3(2\sigma^2 - \sigma'')\Phi^2$$
$$+ \frac{16b_0^2}{3\alpha} \left[ V(\Phi) + \frac{1}{b_0} \sum_{k=1,2} V_k(\Phi) \delta(y - y_k) \right] = 0;$$ (34)

$$\sigma'(\Phi^2)' + \frac{2}{3\alpha} (\Phi')^2 - \frac{4b_0^2}{3\alpha} V(\Phi) - \frac{3\sigma'}{2} \Phi^2 = 0.$$ (35)

Eliminating $V(\Phi)$ from Eqs. (34,35), one obtains the following equation for $\Phi$:

$$\sigma'(\Phi^2)' + (\Phi^2)'' + \frac{16}{3\alpha} (\Phi')^2 - 3\sigma'' \Phi^2 + \frac{16b_0}{3\alpha} \sum_{k=1,2} V_k(\Phi) \delta(y - y_k) = 0.$$ (36)

Away from the branes, we find the solution:

$$\Phi_0(y) = d \left[ 1 - ce^{-\sigma(y)} \right]^\frac{1}{\beta},$$ (37)

where $c, d$ are integration constants and $\beta \equiv 2 + 8/(3\alpha)$ is required for consistency. If $1/\beta$ is not an integer, we must also demand that $1 - ce^{-\sigma(y)} > 0$ in order that $\Phi_0(y)$ be well-defined. Recalling that $\sigma'' = 2m_0b_0 [\delta(y) - \delta(y - 1/2)]$, and noting that $(\Phi^2)''$ will contain a term proportional to $\sigma''$, the conditions that the coefficients of the $\delta$-functions in Eq. (36) vanish reduce to:

$$m_0 d^2 g(c) + \beta(\beta - 2) V_{\text{hid}}(0) = 0$$ (38)

$$m_0 d^2 g(c\gamma) - \beta(\beta - 2) V_{\text{vis}}\left(\frac{1}{2}\right) = 0.$$ (39)
where \( \gamma \equiv \exp(-m_0 b_0/2) \), \( g(x) \equiv (1-x)^{\beta-1} [x(2+3\beta) - 3\beta] \), \( V_{hid}(0) \equiv V_{hid}[\Phi_0(0)] \) and \( V_{vis} \left( \frac{1}{2} \right) \equiv V_{vis} \left[ \Phi_0 \left( \frac{1}{2} \right) \right] \), and we have introduced the notation \( V_{vis} = V_2 \) and \( V_{hid} = V_1 \).

Insertion of the solution \( \Phi_0(y) \) into, for example, Eq. (35) fixes the form of the bulk potential:

\[
V(\Phi) = \frac{3}{\beta - 2} m_0^2 \Phi^2 \left\{ \frac{4}{3\beta} \left[ \left( \frac{\Phi}{d} \right)^{-\beta} - 1 \right] + \frac{\beta - 2}{6\beta^2} \left[ \left( \frac{\Phi}{d} \right)^{-\beta} - 1 \right]^2 - 1 \right\} .
\] (40)

In addition, \( \Phi \) must satisfy its equation of motion as obtained from Eq. (16) for the form Eq. (33):

\[
4\sigma' \Phi' - \Phi'' + \frac{3\alpha}{4} (5\sigma'^2 - 2\sigma'')\Phi + b_0^2 \left[ \frac{dV}{d\Phi} + \frac{1}{b_0} \sum_{k=1,2} \frac{dV_k(\Phi)}{d\Phi} \delta(y - y_k) \right] = 0 .
\] (41)

It is easily verified that the bulk form for \( \Phi_0(y) \), Eq. (36), also satisfies the equation of motion in the bulk, Eq. (41). However, cancellation of the \( \delta(y) \) and \( \delta(y - 1/2) \) brane delta function pieces yields matching conditions that are different from Eqs. (38,39). For consistency of Eq. (41) we require:

\[
2m_0 dh(c) + \beta(\beta - 2)V'_{hid}(0) = 0 , \tag{42}
\]
\[
2m_0 dh(c\gamma) - \beta(\beta - 2)V'_{vis} \left( \frac{1}{2} \right) = 0 , \tag{43}
\]

where \( h(x) \equiv (1-x)^{\beta-1}[x(2+3\beta) - 4\beta] \), \( V'_{hid}(0) \equiv dV_{hid}/d\Phi|_{\Phi=\Phi_0(0)} \) and \( V'_{vis} \left( \frac{1}{2} \right) \equiv dV_{vis}/d\Phi|_{\Phi=\Phi_0(4)} \).

Equations (38) and (42) can be solved for the parameter \( c \) in terms of \( V_{hid}(0) \) and \( V'_{hid}(0) \). Two solutions are possible for \( c \), specified by

\[
c_i = f_i(\beta, -R_{hid}) \quad i = 1, 2 , \tag{44}
\]

where

\[
R_{hid} \equiv \frac{V'_{hid}(0)}{4m_0 V_{hid}(0)} . \tag{45}
\]

The functions \( f_i \) denote the two possible solutions of the quadratic equations for \( c \):

\[
f_i(\beta, R) \equiv \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} , \tag{46}
\]
where \( i = 1, 2 \) corresponds to the + and − signs in front of the square root, respectively. The quantities \( A, B, C \) in the above are given by:

\[
A = -(2 + 3\beta) \left[ 1 + \frac{2 + 3\beta}{R\beta(\beta - 2)} \right]
\]
\[
B = (2 + 3\beta) \left[ 1 + \frac{8}{R(\beta - 2)} \right] + 3\beta
\]
\[
C = -\beta \left[ 3 + \frac{16}{R(\beta - 2)} \right].
\]

Positivity of \( B^2 - 4AC \) leads to the requirement \( 1 - (2 + 3\beta)/R > 0 \).

Once \( c \) is determined, we can compute \( c\gamma \) in terms of \( V_{\text{vis}}' \left( \frac{1}{2} \right) \) and \( V_{\text{vis}} \left( \frac{1}{2} \right) \) using Eqs. (39) and (43). One finds

\[
c_i\gamma_i = f_j(\beta, R_{\text{vis}}) \quad \text{for} \quad i = 1, 2, \quad (47)
\]

where

\[
R_{\text{vis}} \equiv \frac{V_{\text{vis}}'^2 \left( \frac{1}{2} \right)}{4m_0 V_{\text{vis}} \left( \frac{1}{2} \right)}. \quad (48)
\]

and the appropriate branch \( j = 1 \) or \( j = 2 \) is determined by the need to obtain a very small value for the warp factor \( \gamma_i \), i.e. \( f_j(\beta, R_{\text{vis}}) \sim 0 \). The latter is most straightforwardly achieved by requiring \( C_{\text{vis}} \simeq 0 \) and choosing the \( j = 1, + \) (\( j = 2, - \)) solution for \( f_j(\beta, R_{\text{vis}}) \) [see Eq. (46)] for \( B_{\text{vis}} > 0 \) (\( B_{\text{vis}} < 0 \)), respectively.

From Eq. (47), the requirement that \( C_{\text{vis}} \simeq 0 \) is equivalent to

\[
\beta \simeq -\frac{2}{3} \left( \frac{8}{R_{\text{vis}} - 3} \right) \quad \text{or} \quad R_{\text{vis}} \simeq \frac{8}{3(1 - \frac{1}{2}\beta)}. \quad (49)
\]

For \( R_{\text{vis}} \) as above,\(^7\) the positivity of \( \sqrt{B_{\text{vis}}^2 - 4A_{\text{vis}}C_{\text{vis}}} \) is automatic so long as \( B_{\text{vis}} \) is not extremely tiny. For \( R_{\text{vis}} \) as given in Eq. (49), one finds \( B_{\text{vis}} \simeq 3\beta/2 - 1 \), so that we must use the \( j = 1, + \) solution for \( \beta > 2/3 \) and the \( j = 2, - \) solution for \( \beta < 2/3 \). For \( \beta = 2/3 \), the choice becomes ambiguous.

For convenience, in what follows we denote by \( A_{\text{vis}}, B_{\text{vis}}, C_{\text{vis}} \) the values of \( A, B, C \) for \( R = R_{\text{vis}} \) and by \( A_{\text{hid}}, B_{\text{hid}}, C_{\text{hid}} \) the values of \( A, B, C \) for \( R = -R_{\text{hid}} \).

\(^7\)For \( R_{\text{vis}} = \frac{8}{3(1 - \frac{1}{2}\beta)} \equiv r(\beta) \) exactly, \( \gamma = 0 \). As \( R_{\text{vis}} \) changes (for a fixed \( R_{\text{hid}} \)) from a value slightly larger than this to a value slightly smaller, \( \gamma \) will change sign. Whether a positive or negative shift relative to \( r(\beta) \) is required to give \( \gamma_i > 0 \) is determined by the sign of \( c_i \), which can vary.
From Eqs. (44) and (47), the warp factor $\gamma_i$ (and hence the distance between the branes) for a given solution $c_i$ is given by

$$\gamma_i \equiv e^{-\frac{m_0 c_i}{2}} = \frac{f_j(\beta, R_{vis})}{f_i(\beta, -R_{hid})} \quad \text{for} \quad i = 1, 2,$$

(50)

with $j = 1$ for $\beta > 2/3$ and $j = 2$ for $\beta < 2/3$, as discussed earlier. In practice, we will require that $\gamma_i = \gamma \equiv e^{-37}$ for either choice of $i$. Further, one can use (for example) Eq. (42) to determine $d_i$:

$$d_i = -\frac{\beta(\beta - 2) V_{hid}'(0)}{2m_0 h[f_i(\beta, -R_{hid})]}.$$

(51)

Once $d_i$, $c_i$ and $\gamma_i$ have been fixed as specified above, Eqs. (38) and (39) imply a consistency constraint on the model parameters:

$$\frac{V_{hid}(0)}{V_{vis}(\frac{1}{2})} = -g[f_i(\beta, -R_{hid})] \approx \frac{g[f_i(\beta, -R_{hid})]}{3\beta},$$

(52)

where $\gamma_i \sim 0$ has been used to obtain the last approximate form. Using $\gamma_i \sim 0$, Eqs. (39) and (43) also simplify to

$$V_{vis}(\frac{1}{2}) \sim -3 \frac{m_0 d_i^2}{\beta - 2}, \quad V'_{vis}(\frac{1}{2}) \sim -8 \frac{m_0 d_i}{\beta - 2},$$

(53)

with $d_i$ as given in Eq. (51) for solution branch $i$. Note that, for a given value of $\beta$ and choice of solution branch $i$, fixing $V_{vis}(\frac{1}{2})$ corresponds to fixing the normalization $d_i$ of $\Phi_0$ in terms of $m_0$ and that fixing also $V'_{vis}(\frac{1}{2})$ then fixes both $m_0$ and $d_i$.

Finally, it is important to note that the definition of $R_{hid}$, Eq. (45), yields the following constraint on the relative signs of $R_{hid}$ and $V_{hid}(0)$:

$$V_{hid}'(0) = 4m_0 R_{hid} V_{hid}(0) > 0.$$

(54)

Using Eqs. (53) and (52), the condition Eq. (54) can be converted to a requirement expressed entirely in terms of $R_{hid}$ and $\beta$ for a given solution branch $i$:

$$\frac{R_{hid} g[f_i(\beta, -R_{hid})]}{\beta(\beta - 2)} < 0.$$

(55)
The conformal limit of $\alpha = -1$ ($\beta = -2/3$) requires special treatment, since for this choice $A_{hid} = 0$. In this case, $g(x) = 2(1-x)^{-4}$ and $h(x) = \frac{8}{3}(1-x)^{-5/2}$. Eqs. (38) and (42) then yield

$$m_0d^2(1-c)^{-4} = -\frac{8}{9}V_{hid}(0), \quad m_0d(1-c)^{-5/2} = -\frac{1}{3}V'_{hid}(0),$$

respectively, from which we conclude that $V_{hid}(0) < 0$ and $V'_{hid}(0) < 0$ are required, which also implies that [see Eqs. (45) and (54)] $R_{hid} < 0$. By combining Eq. (56) and Eq. (45) we find

$$c = 1 + \frac{2}{R_{hid}}.$$  \hspace{1cm} (57)

Note that $R_{hid} < -2$ is required for $0 < c < 1$, but that $c$ is negative for $-2 < R_{hid} < 0$. There is nothing obvious to forbid this latter choice since $(1 - ce^{-\sigma(y)})$ will automatically be positive for all $y$ if $c < 0$. In an analogous spirit, utilizing Eqs. (39) and (43), one can show that

$$c\gamma = 1 - \frac{2}{R_{vis}}.$$ \hspace{1cm} (58)

Combining Eqs. (57) and (58), one obtains the following result for the warp factor:

$$\gamma = \frac{R_{hid}}{R_{vis}} \frac{R_{vis} - 2}{R_{hid} + 2}.$$ \hspace{1cm} (59)

In order to have a phenomenologically acceptable small value for the warp factor $\gamma$, either $R_{vis} \sim 2$ or $R_{hid} \sim 0$ is required. The remaining constraint [the analog of Eq.(52)] in this case reads:

$$\frac{V_{hid}(0)}{V_{vis} \left(\frac{1}{2}\right)} = - \left(\frac{R_{hid}}{R_{vis}}\right)^4.$$ \hspace{1cm} (60)

Combining this equation with the earlier-noted constraint that $V_{hid}(0) < 0$ results in the requirement that $V_{vis} \left(\frac{1}{2}\right) > 0$, implying that $R_{vis} > 0$. Combined with Eq. (59) and the requirement that $\gamma > 0$, the only allowed choices are:

$$0 < R_{vis} < 2 \quad \text{and} \quad -2 < R_{hid} < 0 \quad \text{or} \quad R_{vis} > 2 \quad \text{and} \quad R_{hid} < -2.$$ \hspace{1cm} (61)

\[8\]In addition to the general solutions discussed in the main text for this case, there exists a special background solution, $\Phi_0(y) \propto e^{3/2\sigma(y)}$, for which there are no matching conditions since the solution satisfies all the necessary equations everywhere, including the boundaries. For this particular special conformally symmetric case, substituting the form of $\Phi_0$, as given above, into Eq. (35) leads to a vanishing bulk potential, $V(\Phi_0) = 0$. Similarly, Eq. (36) gives $V_{1,2}(\Phi) = 0$. Because all the potentials are zero, one finds $M^2 = 0$. We are only interested in cases for which a non-zero mass is generated for the radion.
Note that Eq. (49) does not apply for the conformal choice of $\alpha$. For $R_{\text{vis}} \sim 2$, as generally needed for small $\gamma$, one finds

$$V_{\text{vis}} \left( \frac{1}{2} \right) = \frac{9}{8} m_0 d^2, \quad V_{\text{hid}}(0) = -\frac{9}{144} m_0 d^2 R_{\text{hid}}^4. \quad (62)$$

In Fig. 3, we display $\Phi_0(y)/d$ as a function of $y$ for three cases. In all cases, we have chosen input parameters so that $9m_0b_0/2 \simeq 37$ as required for the warp factor $\gamma = e^{-m_0b_0/2} \sim 1 \text{ TeV}/M_{\text{Pl}}$. In the first case, we have taken $\beta \sim +2/3$, equivalent to $R_{\text{vis}} = 4$ from Eq. (49), and $R_{\text{hid}} = +1$. For this choice, $c \sim 0.7835$. In the second (third) cases, we make the conformal choice of $\beta = -2/3$, take $R_{\text{vis}} \sim 2$ (for small $\gamma$) and employ $R_{\text{hid}} = -4$ ($R_{\text{hid}} = -1$) for which $c = 1/2$ ($c = -1$). In the $\beta = 2/3$ case (which is representative of cases with $\beta > 0$), we see that $\Phi_0(y)$ is repulsed from the hidden brane located at $y = 0$. The second (third) case is representative of a $\beta = -2/3$ case for which $\Phi_0(y)$ is strongly peaked on (strongly repulsed from) the hidden brane.

A useful cross-check is to adopt the explicit form of the solution (37) to verify that indeed $V(0,0)$ vanishes as predicted by Eq. (22). In order to calculate the radion potential at the minimum we use Eq. (35) to eliminate $V(\Phi)$ in the general formula (19). The result is the simplified expression

$$V_{\text{eff}}(0,0) = \frac{10m_0^2b_0}{\beta - 2} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, e^{-4a(\Phi_0(y))^2} + \frac{1}{b_0} \int_{-\frac{1}{2}}^{\frac{1}{2}} dy \, e^{-4a(\Phi_0'(y))^2} + \frac{8m_0}{\beta - 2} \left[ \Phi_0(0)^2 - \Phi_0 \left( \frac{1}{2} \right)^2 e^{-2m_0b_0} \right] + V_{\text{hid}}(0) + V_{\text{vis}} \left( \frac{1}{2} \right) e^{-2m_0b_0}. \quad (63)$$

Then, inserting the solution Eq. (37) into Eq. (63), one obtains:

$$\beta(\beta - 2)V_{\text{eff}}(0,0)$$

$$= \left[ m_0 d^2 (1 - c)^{\frac{1}{\beta} - 1} [c(2 + 3\beta) - 3\beta] + \beta(\beta - 2)V_{\text{hid}}(0) \right] - \gamma^4 \left[ m_0 d^2 (1 - c\gamma)^{\frac{1}{\beta} - 1} [c\gamma(2 + 3\beta) - 3\beta] - \beta(\beta - 2)V_{\text{vis}}(1/2) \right]$$

$$= \left[ m_0 d^2 g(c) + \beta(\beta - 2)V_{\text{hid}}(0) \right] - \gamma^4 \left[ m_0 d^2 g(c\gamma) - \beta(\beta - 2)V_{\text{vis}}(1/2) \right]. \quad (64)$$

\footnote{We reemphasize that $m_0b_0/2$ is calculable in terms of the input parameters using Eq. (50) or Eq. (59).}
Figure 1: The normalized scalar background function $\Phi_0(y)/d = \left[ 1 - ce^{-\sigma(y)} \right]^{\frac{1}{\beta}}$ for $m_0b_0/2 \sim 37$ in three cases: (top) $R_{\text{hid}} = 1$ and $\beta \sim +2/3$ (equivalent to $R_{\text{vis}} = 4$); (middle) $\beta = -2/3$ and $R_{\text{hid}} = -4$ (implying $c = +0.5$); (bottom) $\beta = -2/3$ and $R_{\text{hid}} = -1$ (implying $c = -1$). In the latter two (conformal) cases, $m_0b_0/2 = 37$ requires $R_{\text{vis}} \sim 2$. 
The two bracketed expressions on the right hand side of the above equation vanish by virtue of the fact that they correspond precisely to the expressions appearing in the matching conditions that emerge from the requirement of \((\delta T)_{\mu\nu} = 0\), Eqs. (38,39).

Finally, if we insert the solution Eq. (37) into the formula for the radion mass, Eq. (27), one obtains [after employing Eqs. (38) and (39)] the following result for \(M_r^2\):

\[
M_r^2 = \frac{2}{3} \kappa^2 m_0 \gamma^2 \left[ -V_{\text{hid}}(0) \frac{c(2 - \beta) + \beta}{c(2 + 3\beta) - 3\beta} - V_{\text{vis}} \left( \frac{1}{2} \right) \frac{c\gamma(2 - \beta) + \beta}{c\gamma(2 + 3\beta) - 3\beta} \right],
\]

where \(c\) and \(c\gamma\) are given by Eqs. (44) and (47) or, in the conformal case, Eqs. (57) and (58). Note the presence of the warp factor \(\gamma\) that reduces the radion mass from the typical 5D scale [a natural choice for which would be \(O(M_{Pl})\) if \(V_{\text{hid}}(0)\) and \(V_{\text{vis}} \left( \frac{1}{2} \right)\) are \(O(M_{Pl}^4)\)] down to the electroweak scale. Of course, Eq. (27) guarantees that \(M_r^2 \geq 0\) for any given form of \(f(\Phi)\) and any background solution \(\Phi_0(y)\) that is real and fully consistent. The conditions for reality and consistency are: (a) the constraint Eq. (52) or, in the conformal case, Eq. (60) is satisfied; (b) the positivity condition Eq. (54) is satisfied; and (c) \(1 - c > 0\) when \(1/\beta\) is not an integer. However, substantial variation of \(M_r^2\) is possible. In particular, \(M_r^2 = 0\) at special points when considered as a function of \(R_{\text{hid}}\) at fixed \(\beta\). A final form for \(M_r^2\) can be derived in the \(\gamma \ll 1\) limit by using Eqs. (52) and (53) in Eq. (65):

\[
M_r^2 = \frac{2}{9} \kappa^2 m_0 \gamma^2 V_{\text{vis}} \left( \frac{1}{2} \right) \left[ 1 - \frac{g(c)}{\beta} \frac{c(2 - \beta) + \beta}{c(2 + 3\beta) - 3\beta} \right] = \frac{2}{3} (\kappa m_0 \gamma d)^2 \frac{1}{\beta - 2} \left[ (1 - c)^{\frac{1}{2}\beta - 1} \left( 1 - c + \frac{2}{\beta} \right) - 1 \right].
\]

This shows the fundamental importance of the scale of \(V_{\text{vis}} \left( \frac{1}{2} \right)\) in determining \(M_r^2\). Of course, one should continue to keep in mind the relation Eq. (53) between \(V_{\text{vis}} \left( \frac{1}{2} \right)\) and \(m_0 d^2\) as well as the relation Eq. (52) between \(V_{\text{vis}} \left( \frac{1}{2} \right)\) and \(V_{\text{hid}}(0)\).

To illustrate how \(M_r^2\) depends upon the parameters and how other constraints enter, consider first the conformal case with \(\beta = -2/3\). For \(R_{\text{vis}} \sim 2\) (so that \(\gamma \ll 1\)) and using Eqs. (57) and (60), one can rewrite Eq. (66) in terms of \(V_{\text{vis}} \left( \frac{1}{2} \right)\) (which, as noted earlier, fixes the value of \(d\) in terms of \(m_0\)). The result is:

\[
M_r^2 = \frac{2}{9} \kappa^2 m_0 \gamma^2 V_{\text{vis}} \left( \frac{1}{2} \right) K(R_{\text{hid}}) = \frac{1}{4} (\gamma \kappa m_0 d)^2 K(R_{\text{hid}}),
\]

\[\text{(67)}\]
where

\[ K(R_{hid}) \equiv \frac{3}{16} R_{hid}^4 + \frac{1}{2} R_{hid}^3 + 1 > 0. \]  \hspace{1cm} (68)

In Fig. 2, we plot \( K(R_{hid}) \) as a function of \( R_{hid} \). We observe that \( M_r^2 > 0 \) everywhere except at \( R_{hid} = -2 \) (a point where \( c = 0 \) is required by Eq. (57) and \( \Phi_0(y) \) becomes trivial). Our two earlier \( \beta = -2/3 \) plots of the wave function thus correspond to choices for which \( M_r^2 > 0 \).

![Figure 2: The function \( K(R_{hid}) \) is plotted as a function of \( R_{hid} \) in the (allowed) \( R_{hid} < 0 \) region.](image)

For fixed \( \beta \neq -2/3 \), and \( \gamma \ll 1 \), \( M_r^2 \) is approximately a function of \( R_{hid} \) only, where \( R_{hid} \) is to be restricted to those values such that a given solution \( c_1 \) or \( c_2 \) satisfies the other consistency constraints. We explore the behavior of \( M_r^2 \) as follows. First, recall that \( V_{vis}(\frac{1}{2}), V_{hid}(0), R_{vis}, R_{hid} \) and \( \beta \) are the input model parameters. At fixed \( \beta \), in order to obtain \( \gamma \ll 1 \), as desired on phenomenological grounds, we adjust \( R_{vis} \) according to Eq. (49). Then, for a chosen \( R_{hid} \) we calculate \( c_i \) using Eq.(44). In fact, there are the two solution branches, \( c_{1,2}(R_{hid}) = f_{1,2}(\beta, -R_{hid}) \). Positivity of \( B_{hid}^2 - 4A_{hid}C_{hid} \), as determined by the sign of \( 1 - (2 + 3\beta)/(\beta, -R_{hid}) \), requires \( R_{hid} < \text{Min}[-(2 + 3\beta), 0] \) or \( R_{hid} > \text{Max}[0, -(2 + 3\beta)] \) (the limit cross over taking place at the conformal point of \( \beta = -2/3 \)). Once \( c_1 \) (\( c_2 \)) is chosen, if \( 1/\beta \) is not an integer we check to see if \( 1 - c_1 > 0 \) (\( 1 - c_2 > 0 \)), as required for a well-defined \( \Phi_0(y) = d(1 - ce^{-2\sigma(y)})^{1/\beta} \). Finally, for any values of \( \beta \) and \( R_{hid} \), and given choice of branch \( i \), we check the positivity requirement of Eq. (54). To
compute $M_r^2$, we adopt Eq. (66), which takes into account the consistency constraint (52) such that $V_{hid}(0)$ is expressed in terms of $V_{vis}(\frac{1}{2})$. Eq. (66) makes it clear that, for a given value of $\beta$, $M_r^2$ is proportional to $V_{vis}(\frac{1}{2})$ [which fixes $m_0d_i^2$ through Eq. (53)] and depends non-trivially on $R_{hid}$, the other parameters being held fixed. Remarkably, one finds $(M_r^2)_i > 0$ (as expected) so long as: (a) $R_{hid}$ is such that $1 + (2 + 3\beta)/R_{hid} > 0$ (so that $c_{1,2}$ are real); (b) $1 - c_i > 0$ when $1/\beta$ is not an integer; and (c) the positivity condition, Eq. (54), which we abbreviate as $R_{hid}V_{hid} > 0$, is satisfied. There are many different cases. Here we simply describe a couple of illustrative possibilities.

Figure 3: For $\beta = 2/3$, we plot $[K']_2 \equiv [M_r^2/(\frac{2}{3}(km_0d\gamma)^2)]_2$, see Eq. (66), as a function of $R_{hid}$ in the $R_{hid} \leq -(2 + 3\beta) = -4$ region. Both $1 - c_2$ and $(R_{hid}V_{hid})_2$ display the same sign changes and singular behavior as $[M_r^2]_2$. Only the $[K']_2 > 0$ region corresponds to a solution consistent with all constraints.

Consider first two choices of $\beta$ such that $2 + 3\beta > 0$.

- For $\beta = 2/3$, any value of $R_{hid} > 0$ gives $1 - c_{1,2} > 0$ and $(R_{hid}V_{hid})_{1,2} > 0$ and substantial values for the corresponding $(M_r^2)_1$ and $(M_r^2)_2$. For $R_{hid} < -(2 + 3\beta) = -4$, any value of $R_{hid}$ gives $1 - c_1 > 0$ and substantial $(M_r^2)_1$. In all these cases, $M_r^2$ is a relatively smooth function of $R_{hid}$.

However, in the $R_{hid} \leq -4$ region, $1 - c_2$ and $(R_{hid}V_{hid})_2$ are both only positive for $-4.5 \lesssim R_{hid} \lesssim -4$ and $(M_r^2)_2$ varies rapidly, as illustrated in Fig. 3. This case illustrates the extreme sensitivity that $M_r^2$ can have to $R_{hid}$ and shows
that very large and very small values of $M^2_r$ are quite possible.

- For $\beta = 1/2$, since $1/\beta$ is an integer, the sign of $1 - c$ is unconstrained. One finds $(R_{hid}V_{hid})_{1,2} > 0$ for all $R_{hid} > 0$ and $(M^2_r)_{1,2} > 0$ and behaves smoothly. For all $R_{hid} < -7/2$, $(R_{hid}V_{hid})_1 < 0$ and this solution branch fails the positivity constraint. In the small region $-4.65 \lesssim R_{hid} < -7/2$, $(R_{hid}V_{hid})_2 > 0$ and $(M^2_r)_2 > 0$, blowing up at $R_{hid} \sim -4.65$.

We also briefly describe one interesting $2 + 3\beta < 0$ case.

- For $\beta = -1$, again the $1 - c > 0$ constraint is not necessary. For all $R_{hid} \geq -(2 + 3\beta) = 1$, both $(R_{hid}V_{hid})_1 < 0$ and $(R_{hid}V_{hid})_2 < 0$ so that the $c_1$ and $c_2$ solutions both fail the positivity constraint. In contrast, for all $R_{hid} < 0$ one finds $(R_{hid}V_{hid})_1 > 0$ and, of course, $(M^2_r)_1 > 0$. However, $(R_{hid}V_{hid})_2 > 0$ only for $-0.32 \lesssim R_{hid} < 0$ and $R_{hid} \lesssim -1.78$ — in these two ranges $(M^2_r)_2 > 0$ aside from a zero at the top end, $R_{hid} = -1.78$, of the 2nd range.

Overall it is clear that there is a large range of possible models that satisfy all the constraints necessary for exact Randall-Sundrum metric with positive radion mass-squared. (Some particular choices are somewhat more fine-tuned than others.) We have not understood how to choose between the various models; possibly, the conformal models with $\beta = -2/3$ should be regarded as the more attractive.

4 Conclusions

We have considered a class of generalizations of the Randall-Sundrum model containing a bulk scalar field $\Phi$. We demonstrated that no-back reaction from the scalar on the Randall-Sundrum metric solution requires the existence of an extra interaction between gravity and the scalar. Here, we considered the coupling $R f(\Phi)$. A general form of the potential for the fluctuation of the compactification volume (the radion) and the Kaluza-Klein modes of the excitation of the bulk scalar was derived and the mass matrix was determined. In order to obtain the values of scalar masses, one has to take into account the mixing between the radion and the Kaluza-Klein modes of the fluctuation of the bulk scalar. We demonstrated
that a non-zero mass for the lowest eigenstate (which we identify with the physical radion) can be generated using a choice of the background bulk field, $\Phi_0(y)$, that preserves the RS metric (no back-reaction). We found that the radion mass receives the same suppression from the warp factor that is necessary to explain the hierarchy puzzle. Thus, $\sim 1$ TeV is a natural order of magnitude for the radion mass. Finally, we illustrated the general scenario for the case of $f(\Phi) \propto \Phi^2$, for which the scalar background solution that preserves the Randall-Sundrum metric was explicitly found.

Since the mass-squared matrix for the radion and KK bulk scalar excitations is non-diagonal, it is clear that the introduction of Higgs-radion mixing on the visible brane through a term in the Lagrangian of the form $\xi \sqrt{|g_{vis}|} R(g_{vis}) \hat{H}^\dagger \hat{H}$, as considered for example in [8], would result in a complicated situation where the Higgs field, the radion and the KK excitations of the bulk scalar would all mix. A phenomenological study of the magnitudes of the various mixings, as a function of the available parameters, would be required to understand the extent to which the phenomenology of the physical Higgs boson eigenstate can be modified. Such a study is beyond the scope of this paper.

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