EQUIVALENCE OF FELL BUNDLES IS AN EQUIVALENCE RELATION

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Abstract. We introduce the notion of groupoid pre-equivalences and prove that they give rise to groupoid equivalences by taking certain quotients. Then, given an equivalence of Fell bundles $B$ and $C$ and another equivalence between $C$ and $D$ over locally compact Hausdorff étale groupoids, we construct an equivalence between $B$ and $D$ out of the tensor product bundle. As a consequence, we obtain that Fell bundle equivalence is indeed an equivalence relation.

1. Introduction

The concept of groupoid equivalence was first introduced in [6] to connect with the notion of strong Morita equivalence of groupoid $C^*$-algebras. Roughly speaking, a groupoid equivalence $\mathcal{X}$ for two groupoids $G$ and $H$ is a left $G$- and right $H$-space that satisfies several traceable algebraic and topological conditions. These conditions allow us to build an imprimitivity bimodule for $C^*(G)$ and $C^*(H)$, proving that they are strongly Morita equivalent. A related notion of Fell bundle equivalence was first outlined in unpublished work by Yamagami and later formalized by Muhly and Williams in [7]. One of the main results in [7], Theorem 6.4, is that an equivalence of Fell bundles, similarly to an equivalence of groupoids, implies the strong Morita equivalence of their associated $C^*$-algebras.

The notion of equivalence for groupoids is easily seen to be a reflexive and symmetric relation: Any groupoid $G$ is an equivalence from itself to itself, and given a $(G,H)$-equivalence $\mathcal{X}$, it is easy to construct a $(H,G)$-equivalence $\mathcal{X}^{\text{op}}$ whose left $H$-action, for example, is defined out of the right $H$-action on $\mathcal{X}$. Showing that equivalence of groupoids is transitive is slightly more involved. Given a second groupoid equivalence $\mathcal{Y}$ between $H$ and $K$, the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}$ is generally not a $(G,K)$-equivalence; instead, one has to take a quotient of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}$ by balancing over the middle groupoid $H$. The resulting balanced product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}$ is a known $(G,K)$-equivalence [6, p. 6], which shows that groupoid equivalence is indeed an equivalence relation.

For equivalence of Fell bundles, it is similarly easy to show reflexivity and symmetry [7, Example 6.6 and 6.7]. For (not necessarily saturated) Fell bundles over locally compact Hausdorff groups, this relation was furthermore shown to be transitive in [1]. However, for Fell bundles over groupoids, an analogue of the balanced product construction in the world of groupoids has so far been missing in the literature, and so, despite its name, it was unclear whether equivalence of Fell bundles over groupoids is a transitive relation. We point out that, while strong Morita equivalence of $C^*$-algebras is known to be an equivalence relation, it may not be true that two strongly Morita equivalent Fell bundle $C^*$-algebras have arisen from Fell bundles that are equivalent in the sense of [7].

Thus, the main motivation behind this paper is to prove that the notion of equivalences of Fell bundles over locally compact Hausdorff étale groupoids is indeed a transitive relation, see Theorem 6.15. To do so, we mimic the construction in the world of groupoids, by taking a quotient of the tensor product of two Fell bundle equivalences. Along the way, we first need

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to introduce two new relations that are better behaved under taking products than the known notions of equivalence; we call these new relations \textit{groupoid pre-equivalence} and \textit{Fell bundle hypo-equivalence} (Definition 3.1 resp. 4.1). We show that the fibre product of two pre-equivalences is another pre-equivalence (Lemma 3.4), and that the tensor product of two hypo-equivalences is another hypo-equivalence (Theorem 5.12). We prove that any groupoid pre-equivalence gives rise to an equivalence by taking an appropriate quotient (Proposition 3.7); this generalizes the above-mentioned result that $X \ast_{\lambda}, \mathcal{Y}$ gives rise to the equivalence $X \ast_{\mu}, \mathcal{Y}$. Ideally, we would like to have a similar result for an appropriate notion of ‘pre-equivalent Fell bundles’, but while the authors conjecture that a slight strengthening of hypo-equivalence may allow such a result, this article will not be concerned with that question. Instead, we will only focus on the case when the hypo-equivalence arises as the tensor product $\mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ of two Fell bundle equivalences $\mathcal{M}$ from $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{N}$ from $\mathcal{C}$ to $\mathcal{D}$, in which case it indeed can be rigged to give an equivalence from $\mathcal{B}$ to $\mathcal{D}$, which is the content of Theorem [6.15]

2. Definitions and notation

\textbf{Notation.} Suppose we have maps $s: M \rightarrow Z$, $r: N \rightarrow Z$. We define the \textit{fibre product} of $M$ and $N$ as

$$M_{\ast, r} N := \{(m, n) \in M \times N : s(m) = r(n)\}.$$ 

If the spaces involved are topological and the maps continuous, we will always equip $M_{\ast, r} N$ with the subspace topology. If $M$ and $N$ are the total spaces of bundles $\mathcal{M}$ resp. $\mathcal{N}$, we will sometimes write $\mathcal{M}_{\ast, r} \mathcal{N}$ for $M_{\ast, r} N$ instead. If the maps carry subscripts, we will refrain from writing those in the fibre product; in other words, we will prefer writing $M_{\ast, r} N$ over $M_{s, \ast, r} N_{s, r}$, as the maps involved should always be clear from context. Given $U \subseteq M$ and $V \subseteq N$, we write $U \ast V$ or $U_{s, r} V$ for $(U \times V) \cap M_{\ast, r} N$.

\textbf{Definition 2.1 ([6, p. 5])}. A locally compact Hausdorff groupoid $\mathcal{G}$ \textit{acts on the left} on a locally compact Hausdorff space $X$ if there is a continuous, open surjection $r_X: X \rightarrow \mathcal{G}^{(0)}$ (called \textit{momentum map}) and a continuous map $\mathcal{G}_{\ast, r} X \rightarrow X$, $(g, x) \mapsto g \cdot x$, such that

\begin{enumerate}
    \item[(GA1)] $r_X(g \cdot x) = r_G(g)$ for all $(g, x) \in \mathcal{G}_{\ast, r} X$,
    \item[(GA2)] if $(g, x) \in \mathcal{G}_{\ast, r} X$ and $(g', g) \in \mathcal{G}^{(2)}$, then $(g'g, x) \in \mathcal{G}_{\ast, r} X$ and $(g'g) \cdot x = g' \cdot (g \cdot x)$, and
    \item[(GA3)] $r_X(x) \cdot x = x$ for all $x \in X$.
\end{enumerate}

We call $X$ a \textit{left $\mathcal{G}$-space}. We further call $X$ \textit{principal} if the action is

\begin{enumerate}
    \item[(GA4)] free, meaning that $g \cdot x = x$ implies $g = r_X(x)$; and
    \item[(GA5)] proper, meaning that, if $(g_\lambda \cdot x_\lambda, x_\lambda) \rightarrow (y, x)$ in $X \times X$, then the net $(g_\lambda)_\lambda$ in $\mathcal{G}$ has a convergent subnet (see [10, Proposition 2.7]).
\end{enumerate}

If we want to exchange left by right, we need to swap range and source maps.

We point out that there are other definitions of groupoid actions in the literature in which the momentum map $r_X$ is not necessarily assumed to be open. In this article, however, openness of $r_X$ will be used frequently.

\textbf{Definition 2.2 ([6, Def. 2.1])}. Two locally compact Hausdorff groupoids $\mathcal{G}$ and $\mathcal{H}$ are \textit{equivalent} if there is a topological space $\mathcal{X}$ such that

\begin{enumerate}
    \item[(GE1)] $\mathcal{X}$ is a principal left $\mathcal{G}$- and principal right $\mathcal{H}$-space,
    \item[(GE2)] the left and right action commute, and
    \item[(GE3)] $r_X$ induces a homeomorphism between $\mathcal{X}/\mathcal{H}$ and $\mathcal{G}^{(0)}$, and $s_X$ induces a homeomorphism between $\mathcal{G}\setminus\mathcal{X}$ and $\mathcal{H}^{(0)}$.
\end{enumerate}

We then call $\mathcal{X}$ a $(\mathcal{G}, \mathcal{H})$-\textit{equivalence}. 

As observed in [6], whenever \( x, x' \in \mathcal{X} \) with \( s_{\mathcal{X}}(x) = s_{\mathcal{X}}(x') \), there exists a unique \( g \in \mathcal{G} \) such that \( x = g \cdot x' \). This element is denoted by \( \{ x \mid x' \} \). Similarly, when \( r_{\mathcal{X}}(x) = r_{\mathcal{X}}(x') \), then \( h = \{ \cdot \mid \cdot \} \) is the unique element in \( \mathcal{H} \) such that \( x \cdot h = x' \).

It is well known (see [6, p. 6]) that groupoid equivalence is indeed an equivalence relation. Let us recall why this relation is transitive: Suppose \( \mathcal{X} \) is a \((\mathcal{G}, \mathcal{H})\)-equivalence and \( \mathcal{Y} \) is an \((\mathcal{H}, \mathcal{K})\)-equivalence. Then the quotient \( \mathcal{X} \times_{\mathcal{H}} \mathcal{Y} \) of \( \mathcal{X} \times \mathcal{Y} \) by forcing the equality
\[
[x \cdot h, h^{-1} \cdot y]_{\mathcal{H}} = [x, y]_{\mathcal{H}} \quad \text{for all} \ h \in \mathcal{H} \ \text{for which} \ \cdot \ \text{makes sense},
\]
is a \((\mathcal{G}, \mathcal{K})\)-equivalence with momentum maps
\[
\xymatrix{ \mathcal{X} \times_{\mathcal{H}} \mathcal{Y} \ar[r]^{r} & \mathcal{G}^{(0)} \quad \text{and} \quad \mathcal{X} \times_{\mathcal{H}} \mathcal{Y} \ar[r]^{s} & \mathcal{K}^{(0)} }
\]
and the actions
\[
g \cdot [x, y]_{\mathcal{H}} := [g \cdot x, y]_{\mathcal{H}} \quad \text{resp.} \quad [x, y]_{\mathcal{H}} \cdot k := [x, y \cdot k]_{\mathcal{H}}.
\]

**Definition 2.3** ([2, Definition 2.1]). Suppose \( X \) and \( M \) are topological spaces and \( q_{\mathcal{M}}: M \to X \) a continuous surjection. We call \( \mathcal{M} = (q_{\mathcal{M}}: M \to X) \) an upper semi-continuous (USC) Banach bundle if its fibres \( \mathcal{M}_x = M(x) = q_{\mathcal{M}}^{-1}(x) \) have the structure of complex Banach spaces and if the following hold:

- **(USC1)** The map \( M \to \mathbb{R}_{\geq 0}, m \mapsto \|m\| \), is upper semi-continuous.
- **(USC2)** The map \( M \times M \to M, (m, m') \mapsto m + m' \), is continuous.
- **(USC3)** For each \( \lambda \in \mathbb{C} \), the map \( M \to M, m \mapsto \lambda m \), is continuous.
- **(USC4)** If \( (m_i) \) is a net in \( M \) such that \( q_{\mathcal{M}}(m_i) \) converges to \( x \in X \) and \( \|m_i\| \to 0 \), then \( (m_i) \) converges to \( 0 \in \mathcal{M}(x) \) in \( M \).

We will not always make a clear distinction between the total space \( M \) and the bundle \( \mathcal{M} \) itself.

**Remark 2.4.** We point out that our USC bundles always contain enough sections, i.e., for any given \( x \in X \) and \( m \in \mathcal{M}_x \), there is a continuous section \( \sigma \) such that \( \sigma(x) = m \); see also [7, Appendix A] and [5, Corollary 2.10].

**Notation 2.5.** If \( \mathcal{B} = (p_{\mathcal{B}}: B \to \mathcal{G}) \) is a bundle over a groupoid \( \mathcal{G} \), we write
\[
s_{\mathcal{B}} := s_{\mathcal{G}} \circ p_{\mathcal{B}}: \mathcal{B} \to \mathcal{G}^{(0)} \quad \text{and} \quad r_{\mathcal{B}} := r_{\mathcal{G}} \circ p_{\mathcal{B}}: \mathcal{B} \to \mathcal{G}^{(0)}.
\]
Similarly, if \( \mathcal{M} = (q_{\mathcal{M}}: M \to X) \) is a bundle over a left \( \mathcal{G} \)-space \( X \) with momentum map \( r_{\mathcal{X}}: X \to \mathcal{G}^{(0)} \), we write
\[
r_{\mathcal{M}} := r_{\mathcal{X}} \circ q_{\mathcal{M}}: M \to \mathcal{G}^{(0)}.
\]

If \( X \) is a space with a right action instead, so that its momentum map is a source map, we analogously define \( s_{\mathcal{M}} \). And if there is no ambiguity, we may drop the subscripts.

**Definition 2.6.** Let \( \mathcal{M} = (q_{\mathcal{M}}: M \to X) \) be a USC Banach bundle and \( \mathcal{G} \) a groupoid. Suppose we are given a continuous map \( t: X \to \mathcal{G}^{(0)} \). Consider the continuous map \( f: \mathcal{G} \times X \to (g, x) \mapsto x \).

The pull-back bundle \( f^*(\mathcal{M}) \) of \( \mathcal{M} \) via \( f \) is the bundle over \( \mathcal{G} \times X \) defined by
\[
f^*(\mathcal{M}) = \{(g, x, m) \in \mathcal{G} \times M \mid f(g, x) = q_{\mathcal{M}}(m)\}.
\]

We will denote this pull-back bundle by \( \mathcal{G} \times \mathcal{M} \) rather than \( f^*(\mathcal{M}) \), and we make an analogous definition for \( \mathcal{M} \times \mathcal{G} \).

We want to give a description of how to topologize these pull-back bundles. To this end, we first need to know how to topologize arbitrary bundles:
Remark 2.7 (see [4 Corollary 3.7]). If we are given a bundle $\mathcal{M} = (q,\mathcal{M}: M \to X)$ of Banach spaces, where $M$ is just an untopologized set, then there is a standard trick of inducing a topology on $M$ via a space of sections. To be more precise, suppose that $\Gamma$ is a vector space of sections of $\mathcal{M}$ such that

1. For each $\gamma \in \Gamma$, the map $X \to \mathbb{R}_{\geq 0}, x \mapsto \|\gamma(x)\|$, is upper semi-continuous, and
2. For each $x \in X$, the set $\Gamma(x) := \{\gamma(x) \mid \gamma \in \Gamma\}$ is dense in $\mathcal{M}_x$.

Then there exists a unique topology on $M$ which makes $\mathcal{M}$ a USC Banach bundle such that all elements of $\Gamma$ are continuous sections; see [3, 13.18] for a proof for continuous Banach bundles and [9, Theorem C.25] for a proof for upper semi-continuous C*-bundles which can be readily adapted to USC Banach bundles. We give a description of net convergence with respect to this topology in Lemma A.3.

Lemma 2.8. The pull-back bundle $\mathcal{G}_{s\ast} \mathcal{M}$ has a unique topology that makes it a USC Banach bundle such that for any continuous cross section $\sigma$ of $\mathcal{M}$, the map

$$\text{id} \ast \sigma: \mathcal{G}_{s\ast} X \to \mathcal{G}_{s\ast} \mathcal{M}, \quad (g, x) \mapsto (g, \sigma(x)),$$

is a continuous cross-section. Moreover, the sets $U_1 \ast U_2$ for $U_1 \subseteq \mathcal{G}, U_2 \subseteq \mathcal{M}$ basic open sets, form a basis for this topology.

Note that, the bundle automatically has enough continuous cross-sections (see Remark 2.4).

Proof. Note that $\mathcal{G}_{s\ast} X$ is locally compact Hausdorff, as it is a closed subspace of the locally compact Hausdorff space $\mathcal{G} \times X$. By construction, the projection map $\pi: \mathcal{G}_{s\ast} \mathcal{M} \to \mathcal{G}_{s\ast} X, (g, m) \mapsto (g, q,\mathcal{M}(m))$, is a surjection and each fibre

$$\pi^{-1}(g, x) = \{(g, m) \mid q,\mathcal{M}(m) = x\} \cong \mathcal{M}_x$$

is a Banach space. We now let $\Gamma$ denote the linear span of all elements $\text{id} \ast \sigma$. If we can show the conditions in Remark 2.7 then it follows that there is such a unique topology.

For (1) first fix $\gamma = \text{id} \ast \sigma$. We have $\|\gamma(g, x)\| = \|\sigma(x)\|$. As $\sigma$ is continuous and as $m \mapsto \|m\|$ is upper semi-continuous since $\mathcal{M}$ is a USC Banach bundle, $(g, x) \mapsto \|\gamma(g, x)\|$ is indeed upper semi-continuous. It follows that the same is true for any complex linear combination of such elements, i.e., any $\gamma \in \Gamma$.

For (2) we compute

$$\pi^{-1}(g, x) \cong \Gamma(g, x) \supseteq \{(g, \sigma(x)) \mid \sigma \in \Gamma_0(X; \mathcal{M})\} \supseteq \{(g, m) \mid m \in \mathcal{M}_x\} = \pi^{-1}(g, x),$$

where $(\ast)$ follows since $\mathcal{M}$ has enough continuous cross-sections.

It now remains to show that sets of the form $U_1 \ast U_2$ form basic open subsets. From [9, Proof of Theorem C.25], we know that basic open sets with respect to the newly constructed topology are of the form

$$W(\text{id} \ast \sigma, V, \epsilon) := \{(g, m) \in \mathcal{G}_{s\ast} \mathcal{M} \mid (g, q,\mathcal{M}(m)) \in V, \|g, m) - (\text{id} \ast \sigma)(g, q,\mathcal{M}(m))\| < \epsilon\}$$

for open $V \subseteq \mathcal{G}_{s\ast} X$, a fixed section $\sigma$ of $\mathcal{M}$, and $\epsilon > 0$. As it suffices to take basic open sets in $\mathcal{G}_{s\ast} X$, we may take $V := U_1 \ast V'$ for some open $V' \subseteq X$. We get

$$W(\text{id} \ast \sigma, U_1 \ast V', \epsilon) = \{(h, \xi) \in \mathcal{G}_{s\ast} \mathcal{M} \mid h \in U_1, q,\mathcal{M}(\xi) \in V', \|\xi - \sigma(q,\mathcal{M}(\xi))\| < \epsilon\}$$

But that set is exactly $U_1 \ast U_2$ for $U_2 := W(\sigma, V', \epsilon)$, a basic open subset of $\mathcal{M}$. \qed
Definition 2.9 ([2, Definition 2.8]). An upper semi-continuous Banach bundle \( \mathcal{B} = (B, p_\mathcal{B}) \) over a (locally compact Hausdorff étale) groupoid \( \mathcal{G} \) is called a Fell bundle if it comes with continuous maps

\[
\cdot : \mathcal{B}^{(2)} := \{(a, b) \in B \times B : (p_\mathcal{B}(a), p_\mathcal{B}(b)) \in \mathcal{G}^{(2)}\} \to B \quad \text{and} \quad ^* : B \to B
\]
such that:

(F1) For each \((x, y) \in \mathcal{G}^{(2)}\), \(B_x \cdot B_y \subseteq B_{xy}\), i.e. \(p_\mathcal{B}(b \cdot c) = p_\mathcal{B}(b)p_\mathcal{B}(c)\) for all \((b, c) \in \mathcal{B}^{(2)}\).

(F2) The multiplication is bilinear.

(F3) The multiplication is associative, whenever it is defined.

(F4) If \((b, c) \in \mathcal{B}^{(2)}\), then \(\|b \cdot c\| \leq \|b\| \|c\|\), where the norm is the Banach norm of the respective fibre.

(F5) For any \(x \in \mathcal{G}\), \(\mathcal{B}_x \subseteq \mathcal{B}_{x^{-1}}\).

(F6) The involution map \(b \mapsto b^*\) is conjugate linear.

(F7) If \((b, c) \in \mathcal{B}^{(2)}\), then \((b \cdot c)^* = c^* \cdot b^*\).

(F8) For any \(b \in B\), \(b^{**} = b\).

(F9) For any \(b \in B\), \(\|b^* \cdot b\| = \|b\|^2 = \|b^*\|^2\).

(F10) For any \(b \in B\), \(b^* \cdot b \geq 0\) in the fibre of \(\mathcal{B}\) over \(s_\mathcal{B}(b)\).

We call \(\mathcal{B}\) saturated if we have an equality of sets in Condition (F1). We will often write \(bc\) for \(b \cdot c\).

Definition 2.10 ([7]). Suppose that \(\mathcal{B} = (p_\mathcal{B} : B \rightarrow \mathcal{G})\) is a Fell bundle over a (locally compact Hausdorff étale) groupoid \(\mathcal{G}\), \(X\) is a left \(\mathcal{G}\)-space, and \(\mathcal{M} = (q_\mathcal{M} : M \rightarrow X)\) is a USC Banach bundle. Then we say that \(\mathcal{B}\) acts on (the left) of \(\mathcal{M}\) if there is a continuous map \(\mathcal{B}_\mathcal{M} : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}\), \((b, m) \mapsto b \cdot m\), such that

(FA1) \(q_\mathcal{M}(b \cdot m) = p_\mathcal{B}(b) \cdot q_\mathcal{M}(m)\),

(FA2) \(a \cdot (b \cdot m) = (ab) \cdot m\) for all appropriate \(a \in B\), and

(FA3) \(\|b \cdot m\| \leq \|b\| \|m\|\).

We point out that in [7], Condition (FA3) had a typo: it should read \(\leq\) instead of =. An analogous definition can be made for a right action of a Fell bundle.

Definition 2.11 ([7, Definition 6.1]). Suppose that \(\mathcal{G}, \mathcal{H}\) are locally compact Hausdorff étale groupoids, that \(\mathcal{X}\) is a \((\mathcal{G}, \mathcal{H})\)-equivalence, that \(\mathcal{B} = (p_\mathcal{B} : B \rightarrow \mathcal{G})\) and \(\mathcal{C} = (p_\mathcal{H} : C \rightarrow \mathcal{H})\) are saturated Fell bundles, and that \(\mathcal{M} = (q_\mathcal{M} : M \rightarrow \mathcal{X})\) is a USC Banach bundle. We say that \(\mathcal{M}\) is a \(\mathcal{B}-\mathcal{C}\)-equivalence if the following conditions hold:

(FE1) There is a left \(\mathcal{B}\)-action and a right \(\mathcal{C}\)-action on \(\mathcal{M}\) such that \(b \cdot (m \cdot c) = (b \cdot m) \cdot c\) for all \(b \in B\), \(m \in M\), and \(c \in C\), wherever it makes sense.

(FE2) There are sesquilinear maps

\[
\mathcal{B}(\cdot, \cdot) : M \times \mathcal{M} \rightarrow B, \quad \langle \cdot, \cdot \rangle_\mathcal{C} : M \times \mathcal{M} \rightarrow C
\]

\[
(m_1, m_2) \mapsto \mathcal{B}(m_1 | m_2), \quad (m_1, m_2) \mapsto \langle m_1 | m_2 \rangle_\mathcal{C}
\]

such that for all appropriately chosen \(m_i \in M\), \(b \in B\), and \(c \in C\), we have

(FE2.a) \(p_\mathcal{G}(\mathcal{B}(m_1 | m_2)) = \langle q_\mathcal{M}(m_1) | q_\mathcal{M}(m_2) \rangle\) and \(p_\mathcal{H}(\langle m_1 | m_2 \rangle_\mathcal{C}) = \{q_\mathcal{M}(m_1) | q_\mathcal{M}(m_2)\}_\mathcal{H}\).

(FE2.b) \(\mathcal{B}(m_1 | m_2)^* = \mathcal{B}(m_2 | m_1)^*\) and \(\langle m_1 | m_2 \rangle_\mathcal{C}^* = \langle m_2 | m_1 \rangle_\mathcal{C}^*\).

(FE2.c) \(\mathcal{B}(b \cdot m_1 | m_2) = b \cdot \mathcal{B}(m_1 | m_2)\) and \(\langle m_1 | m_2 \cdot c \rangle_\mathcal{C} = \langle m_1 | m_2 \rangle_\mathcal{C} \cdot c\), and

(FE2.d) \(\mathcal{B}(m_1 | m_2) \cdot m_3 = \mathcal{B}(m_1 | m_2) \cdot m_3\).

(FE3) With the actions coming from (FE1) and the inner products coming from (FE2), each \(M(x)\) is a \(B(r(x)) \rightarrow C(s(x))\)-imprimitivity bimodule.
Remark 2.12 ([7, Lemma 6.2]). For \((g, x) \in \mathcal{G} \times \mathcal{X}\), we have that \(\mathcal{M}_{g \cdot x}\) is isomorphic to \(\mathcal{B}_{g} \otimes \mathcal{B}_{s(g)} \cdot \mathcal{M}_{x}\) as \(\mathcal{B}_{r(g)} - \mathcal{C}_{s(x)}\)-imprimitivity bimodules. We denote by \(\mathcal{B}_{g} \cdot \mathcal{M}_{x}\) the dense subspace of \(\mathcal{M}_{g \cdot x}\) given by the image of \(\mathcal{B}_{g} \otimes \mathcal{B}_{s(g)} \cdot \mathcal{M}_{x}\) under that isomorphism.

Similarly for \((x, h) \in \mathcal{X} \times \mathcal{H}\), \(\mathcal{M}_{x \cdot h}\) is isomorphic to \(\mathcal{M}_{x} \otimes \mathcal{C}_{s(h)} \cdot \mathcal{H}_{h}\) as \(\mathcal{B}_{r(x)} - \mathcal{C}_{s(h)}\)-imprimitivity bimodules, and we let \(\mathcal{M}_{x} \cdot \mathcal{H}_{h}\) be the corresponding dense subspace of \(\mathcal{M}_{x \cdot h}\).

3. Groupoid Pre-Equivalences

In Section 5, we will construct, out of Fell bundle equivalences, new USC Banach bundles over spaces that are not groupoid equivalences. We therefore need to be able to talk about spaces that are only ‘almost’ equivalences, and so we start with the following definition.

Definition 3.1. Two locally compact Hausdorff groupoids \(\mathcal{G}\) and \(\mathcal{H}\) are called \emph{pre-equivalent} if there is a topological space \(X\) such that

(GP1) \(X\) is a proper left \(\mathcal{G}\)- and proper right \(\mathcal{H}\)-space,
(GP2) the left and right action commute, and
(GP3) there exist continuous surjective maps

\[ L_{g}: X \times_{s} X \to \mathcal{G} \quad \text{and} \quad R_{h}: X \times_{r} X \to \mathcal{H} \]

which satisfy the following for \(x, x' \in X, g \in \mathcal{G}, h \in \mathcal{H}\) wherever it makes sense:

(GP3.a) \[ r_{g}(L_{g}(x \mid x')) = r_{X}(x) \quad \text{and} \quad s_{h}(R_{h}(x \mid x')) = s_{X}(x'), \]
(GP3.b) \[ L_{g}(x \mid x')^{-1} = L_{g}(x' \mid x) \quad \text{and} \quad R_{h}(x \mid x')^{-1} = R_{h}(x' \mid x), \]
(GP3.c) \[ L_{g}(g \cdot x \mid x') = gL_{g}(x \mid x') \quad \text{and} \quad R_{h}(x \mid x' \cdot h) = R_{h}(x \mid x')h, \]
(GP3.d) \[ L_{g}(x \cdot h^{-1} \mid x') = L_{g}(x \mid x' \cdot h) \quad \text{and} \quad R_{h}(g^{-1} \cdot x \mid x') = R_{h}(x \mid g \cdot x'), \]
(GP3.e) \[ L_{g}(x \mid x')L_{g}(x' \mid x'') = L_{g}(x \mid x'') \quad \text{and} \quad R_{h}(x \mid x')R_{h}(x' \mid x'') = R_{h}(x \mid x''), \]
(GP3.f) \[ \ker(L_{g}(\cdot \mid \cdot)) = \ker(R_{h}(\cdot \mid \cdot)). \]

We then call \(X\) a \((\mathcal{G}, \mathcal{H})\)-pre-equivalence. Sometimes, we will equip the maps \(L_{g}\) and \(R_{h}\) with a superscript-\(X\) to avoid ambiguity.

One should think of \(L_{g}(\cdot \mid \cdot)\) and \(R_{h}(\cdot \mid \cdot)\) as weakened analogues of the maps \(\varrho(\cdot \mid \cdot)\) and \(\{\cdot \mid \cdot\}_{\mathcal{H}}\), respectively, that a groupoid equivalence carries; however, we allow that

\[ L_{g}(x \mid x') \cdot x' \neq x \quad \text{or} \quad x \cdot R_{h}(x \mid x') \neq x'. \]

In fact, we will see in Corollary 3.10 that this is the only difference between a groupoid equivalence and a groupoid pre-equivalence.

Remark 3.2. The axioms of a groupoid pre-equivalence have some easy consequences; here are the ones we will use frequently.

(GP3.a)' \[ s_{g}(L_{g}(x \mid x')) = r_{X}(x') \quad \text{and} \quad r_{h}(R_{h}(x \mid x')) = s_{X}(x), \]
(GP3.b)' \[ L_{g}(x \mid x) \in \mathcal{G}^{(0)} \quad \text{and} \quad R_{h}(x \mid x) \in \mathcal{H}^{(0)} \]

In fact, \([\text{GP3.a}']\) can be proved using \([\text{GP3.a}]\) and \([\text{GP3.b}]\), and it was implicitly used in \([\text{GP3.e}]\).

Example 3.3. Any \((\mathcal{G}, \mathcal{H})\)-equivalence \(\mathcal{X}\) is a \((\mathcal{G}, \mathcal{H})\)-pre-equivalence if we let

\[ L_{g}(x \mid x') := \varrho(\{x \mid x'\}) \quad \text{and} \quad R_{h}(x \mid x') := \{x \mid x'\}_{\mathcal{H}}. \]

\textit{Proof.} Conditions \([\text{GP1}]\) and \([\text{GP2}]\) hold by assumption (see Definition 2.2). We will only verify continuity and surjectivity of \(L_{g}\); all algebraic properties directly follow from the fact that the element \(g = \varrho(\{x \mid x'\}) \in \mathcal{G}\) uniquely satisfies \(x = g \cdot x'\), and by symmetry, all properties then also follow for \(R_{g}\).
For continuity of $L_g$, suppose we are given a convergent net $(x_\lambda, x_\lambda') \to (x, x')$ in $\mathcal{X} \ast_s \mathcal{X}$. For each $\lambda$, $g_\lambda := \varphi(x_\lambda \mid x_\lambda') \in \mathcal{G}$ satisfies $x_\lambda = g_\lambda \cdot x_\lambda'$, just as $g := \varphi(x \mid x') \in \mathcal{G}$ satisfies $x = g \cdot x'$. Thus,

$$(g_\lambda \cdot x_\lambda', x_\lambda') = (x_\lambda, x_\lambda') \to (x, x') = (g \cdot x', x').$$

Since the actions on an equivalence are proper (see [GA5] in Definition 2.1), this implies that a subnet of

$$L_g(x_\lambda \mid x_\lambda') = \varphi(x_\lambda \mid x_\lambda') = g_\lambda$$

converges to $g = \varphi(x \mid x') = L_g(x \mid x')$.

This suffices to conclude that $L_g$ is continuous. □

**Lemma 3.4.** Suppose $X$ is a $(\mathcal{G}, \mathcal{H})$-pre-equivalence and $Y$ is an $(\mathcal{H}, \mathcal{K})$-pre-equivalence. Then $Z := X \ast_r Y$ is a $(\mathcal{G}, \mathcal{H})$-pre-equivalence: its momentum maps are given by $r_Z(x, y) = r_X(x)$ and $s_Z(x, y) = s_Y(y)$; the actions are defined by $g \cdot (x, y) = (g \cdot x, y)$ and $(x, y) \cdot k = (x, y \cdot k)$; it carries the maps

$$L^\pi_Z((x, y) \mid (x', y')) := L^\pi_X(x \mid x') \cdot L^\pi_Y(y' \mid y) \quad \text{and} \quad R^\pi_Z((x, y) \mid (x', y')) := R^\pi_Y(R^\pi_X(x' \mid x) \cdot y \mid y').$$

**Proof.** Continuity of $L^\pi_Z$ follows from continuity of $L^\pi_X$, of the right $\mathcal{H}$-action on $X$, and of $L^\pi_Y$. Surjectivity follows from surjectivity of both $L^\pi_X$ and of $L^\pi_Y$. The algebraic properties are again easy to verify, using the corresponding properties of $L^\pi_X$ and $L^\pi_Y$. The same is true for $R^\pi_Z$. □

**Remark 3.5.** If $X$ and $Y$ are equivalences with $L^\pi_X = \varphi^X(\cdot \mid \cdot)$ etc. as in Example 3.3, then $Z = X \ast_s Y$ is not automatically an equivalence: if we let

$$g := L^\pi_Z((x, y) \mid (x', y')) = \varphi^X(x \mid x') \cdot \varphi^Y(y' \mid y),$$

then the element $g \cdot (x', y') = (g \cdot x', y')$ does not necessarily equal $(x, y)$. This is exactly our motivation for the definition of groupoid pre-equivalence.

However, we do have that the pair $((g \cdot x', y'), (x, y))$ belongs to the kernel of $L^\pi_Z$: since $X$ is a groupoid equivalence, we have $\varphi^X(x \mid x') \cdot x' = x$, so

$$g \cdot x' = \varphi^X(x \mid x') \cdot \varphi^Y(y' \mid y) \cdot x' = \varphi^X(x \cdot \varphi^Y(y' \mid y)^{-1} \mid x') \cdot x' = x \cdot \varphi^Y(y \mid y').$$

Thus, we see that

$$L^\pi_Z((g \cdot x', y') \mid (x, y)) = \varphi^X(g \cdot x' \mid x \cdot \varphi^Y(y \mid y')) = \varphi^X(g \cdot x' \mid g \cdot x')$$

is an element of $\mathcal{G}^{(0)}$ by Condition [GP3.b]′.

The remark motivates the following:

**Lemma 3.6.** If $X$ is a $(\mathcal{G}, \mathcal{H})$-pre-equivalence, then we define the relation $\mathcal{R}$ on $X$ by

$$x \mathcal{R} x' \text{ if } (x, x') \in \ker(L_g(\cdot \mid \cdot)).$$

This defines a closed equivalence relation; we let $\mathcal{X}$ denote the quotient of $X$ by $\mathcal{R}$. The quotient map $\pi: X \to \mathcal{X}$ is open, and so $\mathcal{X}$ is locally compact Hausdorff. For $x \in X$, we will sometimes write $\tilde{x}$ for $\pi(x)$.

**Proof.** Reflexivity, symmetry, and transitivity of $\mathcal{R}$ follow easily from [GP3.b]′ [GP3.b] and [GP3.e] respectively. To show that $\mathcal{R}$ is closed, notice first that the domain $X_\ast_s X$ of $L_g(\cdot \mid \cdot)$ is closed in $X \times X$, so it suffices to check that $\mathcal{R}$ is closed in $X_\ast_s X$. If $(x_\lambda, x_\lambda') \to (x, x')$ and $x_\lambda \mathcal{R} x_\lambda'$, then by continuity of $L_g(\cdot \mid \cdot)$ and since $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$,

$$L_g(x \mid x') = \lim_\lambda L_g(x_\lambda \mid x_\lambda') \in \mathcal{G}^{(0)},$$

showing that $x \mathcal{R} x'$ also.
To see that the quotient map \( \pi \) is open, note that \( L^{-1}_G(\mathcal{G}(0)) \) is open in \( X \ast_s X \) since \( \mathcal{G} \) is étale and since \( L_g \) is continuous. By definition of the subspace topology, there thus exists an open set \( V \subseteq X \times X \) such that \( L^{-1}_G(\mathcal{G}(0)) = V \cap (X \ast_s X) \). Now, suppose \( U \subseteq X \) is open, so that \( V \cap (U \times X) \) is open in \( X \times X \) and hence \( V \cap (U \ast_s X) = (V \cap (U \times X)) \cap X \ast_s X \) is open in \( X \ast_s X \). We compute

\[
\text{pr}_2(V \cap (U \ast_s X)) = \{ x \in X \mid \exists x' \in U \text{ such that } s_X(x') = s_X(x) \text{ and } (x', x) \in V \} = \{ x \in X \mid \exists x' \in U \text{ such that } s_X(x') = s_X(x) \text{ and } L_g(x' \mid x) \in \mathcal{G}(0) \} = \{ x \in X \mid \exists x' \in U \text{ such that } \pi(x') = \pi(x) \} = \pi^{-1}(\pi(U)).
\]

Since \( s_X \) is open and continuous, the map \( \text{pr}_2 : X \ast_s X \to X \) is open by Lemma 3.6, so that \( \text{pr}_2(V \cap (U \ast_s X)) = \pi^{-1}(\pi(U)) \) is open as claimed.

Since \( X \) is locally compact Hausdorff and \( \pi \) is open, it follows immediately that \( \mathcal{X} \) is also locally compact Hausdorff. \( \square \)

**Proposition 3.7.** Suppose \( X \) is a \((\mathcal{G}, \mathcal{H})\)-pre-equivalence and \( \mathcal{X} \) the quotient constructed in Lemma 3.6. Then \( \mathcal{X} \) is an equivalence of groupoids if we define

\[
\tilde{r} : \mathcal{X} \to \mathcal{G}(0), \quad \tilde{r}(\tilde{x}) := r_X(x); \quad g \cdot \tilde{x} := \pi(g \cdot x),
\]

\[
\tilde{s} : \mathcal{X} \to \mathcal{H}(0), \quad \tilde{s}(\tilde{x}) := s_X(x); \quad \tilde{x} \cdot h := \pi(x \cdot h).
\]

Furthermore, for all \((x, x') \in X \ast_s X\) and all \((y, y') \in X \ast_r X\), we have

\[
\mathcal{X}^\mathcal{X}_g(\pi(x) \mid \pi(x')) = L^\mathcal{X}_g(x \mid x') \quad \text{and} \quad (\pi(y) \mid \pi(y'))^\mathcal{X}_\pi = R^\mathcal{X}_\pi(y \mid y').
\]

**Proof.** We will prove all things only for the left side; the right side will follow by symmetry and since the actions clearly commute. So let us first check that the locally compact Hausdorff space \( \mathcal{X} = X / \mathcal{R} \) is a left \( \mathcal{G} \)-space.

To see that the action is well defined, suppose \( \tilde{x} = \tilde{y} \), i.e., \( L^\mathcal{X}_g(x \mid y) \in \mathcal{G}(0) \). Then

\[
L^\mathcal{X}_g(g \cdot x \mid g \cdot y) = g L^\mathcal{X}_g(x \mid y) g^{-1} = g L^\mathcal{X}_g(\pi(x) \mid \pi(y)) g^{-1} = g L^\mathcal{X}_g(\pi(x') \mid \pi(x')) = \pi(x') \quad \text{for some open sets } V_i \subseteq \mathcal{G} \text{ and } U_i \subseteq X.
\]

Then

\[
f_0^{-1}(U_0) = (\text{id} \times \pi)(f^{-1}(\pi^{-1}(U_0))) = (\text{id} \times \pi)(\bigcup_{i \in I} V_i \ast_s U_i) = \bigcup_{i \in I} V_i \ast_s \pi(U_i),
\]

which is open in \( \mathcal{G}_s \ast_s \mathcal{X} \) since \( \pi \) is an open map by Lemma 3.6.

We now verify that this action satisfies all the axioms listed in Definition 2.1. The items (GAL), (GA2), (GA3) follow immediately from the corresponding property of the \( \mathcal{G} \)-action on \( X \). Now, if \( g \cdot \tilde{x} = \tilde{x} \), then \( L^\mathcal{X}_g(x \mid x) \in \mathcal{G}(0) \) by definition of \( \mathcal{R} \), so by (GP3,c) and (GP3,b), that means \( g \in \mathcal{G}(0) \); in other words, the action on \( \mathcal{X} \) is free. To see that the action is proper, it thus suffices to check that \( \Phi_0 : \mathcal{G}_s \ast_s \mathcal{X} \to \mathcal{X} \times \mathcal{X}, (g, \tilde{x}) \mapsto (g \cdot \tilde{x}, \tilde{x}) \), is a closed map. For any closed \( A \subseteq \mathcal{G}_s \ast_s \mathcal{X} \), we must show that

\[
(\pi \times \pi)^{-1}(\Phi_0(A)) = \{(y, x) \in X \times X \mid \exists g \in \mathcal{G}, \tilde{y} = \pi(g \cdot x) \text{ and } (g, \tilde{x}) \in A \}
\]

is closed in \( X \times X \). So suppose that \( \{(y_\lambda, x_\lambda)\}_\lambda \) is a net in \( (\pi \times \pi)^{-1}(\Phi_0(A)) \) that converges to \( (y, x) \) in \( X \times X \). By the above description of the preimage, it follows that, for every \( \lambda \), there exists
By Conditions (GP3.c) and (GP3.b), that means
\[ \pi \text{ so that } \exists \in G \text{ such that } g \cdot x = y. \]

By continuity of \( L^X_\partial (\cdot | x) \), we conclude that \( g \xrightarrow{\partial} L^X(y | x) = g. \) As \( x \xrightarrow{} x \) by assumption, it follows that the net \( \{ g(x, \tilde{x}) \}_x \) converges to \( (g, \tilde{x}) \). Since \( A \) contains the net and is closed, we conclude \( (g, \tilde{x}) \in A \). By definition of \( g \) and using Conditions (GP3.c) and (GP3.b), we see
\[ L^X_\partial (g \cdot x | y) = g \cdot L^X_\partial (x | y) = gg^{-1} \in G(0), \]
so that \( \pi (g \cdot x) = \tilde{g} \) by definition of \( R \). In particular, the limit \( (g \cdot x, x) = (y, x) \) of \( \{(g, x)_x \}_x \) is an element of \( (\pi \times \pi)^{-1}(\Phi_0(A)) \). This concludes our proof that this set is closed.

Lastly, we need to check that the map \( r : \mathcal{X}/\mathcal{H} \to G(0), \tilde{x} : \mathcal{H} \to r_X(x) \), is well defined, injective, continuous, and open. We let \( q : \mathcal{X} \to \mathcal{X}/\mathcal{H} \) denote the quotient map. Suppose \( \tilde{x} : \mathcal{H} = \tilde{y} : \mathcal{H} \), so there exists \( h \in \mathcal{H} \) such that \( v = L^X_\partial (x | y \cdot h) \in G(0) \). Then \( v = r_{\mathcal{H}}(v) = r_X(x) \) and \( v = s_{\mathcal{H}}(v) = r_X(x \cdot h) = r_X(y) \), showing that \( r_X(x) = r_X(y) \) and thus \( r \) is well defined on \( \mathcal{X}/\mathcal{H} \).

To see that \( r \) is injective, suppose \( r_X(x) = r_X(y) \) and let \( h : R^X_{\mathcal{H}}(y | x) \). Then using Conditions (GP3.a) and (GP3.e) we have
\[ R^X_{\mathcal{H}}(x | y \cdot h) = R^X_{\mathcal{H}}(x | y)h = R^X_{\mathcal{H}}(x | x) \in G(0), \]
showing that \( (x, y \cdot h) \in \ker(R^X_{\mathcal{H}}) = \ker(L^X_\partial) \). In particular, \( \tilde{x} = \pi(x) = \pi(y \cdot h) = \pi(y) \cdot h = \tilde{y} \cdot h \). That means that \( \tilde{x} : \mathcal{H} = \tilde{y} : \mathcal{H} \), as claimed.

For continuity, take \( U \subseteq G(0) \). Then
\[ r^{-1}(U) \text{ is open in } \mathcal{X}/\mathcal{H} \iff q^{-1}(r^{-1}(U)) \text{ is open in } \mathcal{X} \iff \pi^{-1}(q^{-1}(r^{-1}(U))) \text{ is open in } X. \]

But note that
\[ \pi^{-1}(q^{-1}(r^{-1}(U))) = \{ x \in X \mid r(q(\pi(x))) \in U \} = \{ x \in X \mid r_X(x) \in U \} = r^{-1}_X(U), \]
which, if \( U \) is open, is open by continuity of \( r_X \).

Finally, to see that \( r \) is an open map, suppose \( V \subseteq \mathcal{X}/\mathcal{H} \) is open, i.e., \( q^{-1}(V) \) is open in \( \mathcal{X} \), i.e., \( \pi^{-1}(q^{-1}(V)) \) is open in \( X \). Since \( r_X \) is an open map, \( r_X(\pi^{-1}(q^{-1}(V))) \) is open in \( G(0) \). We compute:
\[ r_X(\pi^{-1}(q^{-1}(V))) = \{ u \in G(0) \mid \exists x \in X \text{ such that } q(\pi(x)) \in V \text{ and } r_X(x) = u \} = \{ u \in G(0) \mid \exists \tilde{x} \in \mathcal{X} \text{ such that } q(\tilde{x}) \in V \text{ and } \tilde{r}(\tilde{x}) = u \} = \{ u \in G(0) \mid \exists \tilde{x} : \mathcal{H} \in \mathcal{X}/\mathcal{H} \text{ such that } \tilde{x} : \mathcal{H} \in V \text{ and } \tilde{r}(\tilde{x} : \mathcal{H}) = u \} = r(V). \]

We conclude that \( \mathcal{X} \) is an equivalence between the groupoids \( \mathcal{G} \) and \( \mathcal{H} \).

It remains to show that \( x_\partial \tilde{x} | \tilde{y} = L^X_\partial(x | y) \). We have that
\[ L^X_\partial (L^X_\partial (x | y) \cdot y | x) = L^X_\partial (x | y) L^X_\partial (y | x) = r_{\mathcal{H}}(L^X_\partial (x | y)) \in G(0), \]
which means that \( L^X_\partial (x | y) \cdot y \) is related to \( x \) via \( R \). In other words,
\[ L^X_\partial (x | y) \cdot y = \pi (L^X_\partial (x | y) \cdot y) = \tilde{x}. \]
This means that \( L^X_\partial (x | y) \) satisfies the property that uniquely determines \( x_\partial \tilde{x} | \tilde{y} \).

\[ \square \]

**Remark 3.8.** It is clear that a groupoid equivalence is also a pre-equivalence. Proposition 3.7 proves that any pre-equivalence gives rise to an equivalence, and therefore, two groupoids are equivalent if and only if they allow a groupoid pre-equivalence.
Corollary 3.9 (cf. [3]). If $\mathcal{X}$ is a $(\mathcal{G}, \mathcal{H})$-equivalence and $\mathcal{Y}$ is an $(\mathcal{H}, \mathcal{K})$-equivalence, then their balanced fibre product $\mathcal{X} \ast_\mathcal{H} \mathcal{Y}$ is a $(\mathcal{G}, \mathcal{K})$-equivalence and the quotient map $[\cdot]_\mathcal{H}: \mathcal{X} \ast_\mathcal{H} \mathcal{Y} \to \mathcal{X} \ast_\mathcal{H} \mathcal{Y}$ is open.

**Proof.** We have seen in Example 3.3 that $\mathcal{X}$ and $\mathcal{Y}$ are groupoid pre-equivalences, so by Lemma 3.4 $Z := \mathcal{X} \ast_\mathcal{H} \mathcal{Y}$ is a $(\mathcal{G}, \mathcal{K})$-pre-equivalence. We claim that $\mathcal{R}$ is exactly the equivalence relation that gives rise to $\mathcal{X} \ast_\mathcal{H} \mathcal{Y}$, so let us unravel what exactly $\mathcal{R}$ means here:

$$(x, y) \mathcal{R} (x', y') \iff ((x, y), (x', y')) \in \ker(L_\mathcal{G}(\cdot \mid \cdot))$$

$$\iff x = x' \cdot \mathcal{H}(y' \mid y)$$

$$\iff \exists h \in \mathcal{H}, x = x' \cdot h \text{ and } y' = h \cdot y$$

$$\iff \exists h \in \mathcal{H}, (x, y) = (x' \cdot h, h^{-1} \cdot y')$$

$$\iff [x, y]_\mathcal{H} = [x', y']_\mathcal{H} \text{ in } \mathcal{X} \ast_\mathcal{H} \mathcal{Y}.$$ 

We conclude that $\mathcal{X} \ast_\mathcal{H} \mathcal{Y}$ is exactly $(\mathcal{X} \ast \mathcal{Y})/\mathcal{R}$, which is a $(\mathcal{G}, \mathcal{K})$-equivalence by Proposition 3.7. \qed

**Corollary 3.10.** Suppose $X$ is a $(\mathcal{G}, \mathcal{H})$-pre-equivalence. Then the following are equivalent:

1. For all $x, x' \in X$ such that $s_\mathcal{X}(x) = s_\mathcal{X}(x')$, we have $L_\mathcal{G}(x \mid x') \cdot x' = x$.
2. For all $y, y' \in X$ such that $r_\mathcal{X}(y) = r_\mathcal{X}(y')$, we have $y \cdot R_\mathcal{H}(y \mid y') = y'$.
3. $X$ is a groupoid equivalence for which $\mathcal{G}(\cdot \mid \cdot) = L_\mathcal{G}(\cdot \mid \cdot)$ and $\mathcal{H}(\cdot \mid \cdot)_\mathcal{H} = R_\mathcal{H}(\cdot \mid \cdot)$.

**Proof.** Clearly, (3) implies (1) and (2). By symmetry, it now suffices to prove that (1) implies (3).

Suppose $x \mathcal{R} y$ with $\mathcal{R}$ as defined in Lemma 3.6 so $L_\mathcal{G}(x \mid y) \in \mathcal{G}(0)$. Then in particular, $L_\mathcal{G}(x \mid y) = s_\mathcal{G}(L_\mathcal{G}(x \mid y)) = r_\mathcal{X}(y)$. It follows from our assumption (1) that $y = L_\mathcal{G}(x \mid y) \cdot y = x$. In other words, $\mathcal{R}$ is the diagonal in $X \times X$. Thus, $X$ equals $\mathcal{X}$, which we have shown in Proposition 3.7 to be an equivalence such that $\mathcal{G}(\cdot \mid \cdot) = L_\mathcal{G}(\cdot \mid \cdot)$ and $\mathcal{H}(\cdot \mid \cdot)_\mathcal{H} = R_\mathcal{H}(\cdot \mid \cdot)$.

\qed

4. **Fell Bundle Hypo-Equivalences**

Just like we weakened the notion of groupoid equivalences to account for the fact that their fibre product is not another equivalence, we need to weaken the notion of Fell bundle equivalence as follows.

**Definition 4.1.** Suppose that $X$ is a $(\mathcal{G}, \mathcal{H})$-pre-equivalence, that $\mathcal{B} = (p_\mathcal{G}: B \to \mathcal{G})$ and $\mathcal{C} = (p_\mathcal{H}: C \to \mathcal{H})$ are saturated Fell bundles, and that $\mathcal{M} = (q_\mathcal{M}: M \to X)$ is a USC Banach bundle. We say that $\mathcal{B}$ is a $\mathcal{B}$-$\mathcal{C}$-hypo-equivalence if it satisfies the conditions in Definition 2.11 only that Conditions [FE2.a] and [FE2.d] are replaced by the following:

- [FE2.a]$_0$: $p_\mathcal{G}(m_1 \mid m_2) = L_\mathcal{G}(q_\mathcal{M}(m_1) \mid q_\mathcal{M}(m_2))$ and $p_\mathcal{H}((m_1 \mid m_2)_\mathcal{e}) = R_\mathcal{H}(q_\mathcal{M}(m_1) \mid q_\mathcal{M}(m_2))$.
- [FE2.d]$_0$: $\langle m_1 \mid c \rangle \cdot m_2 = \langle m_1 \mid m_2 \cdot c \rangle$ and $\langle b \cdot m_1 \mid m_2 \rangle_\mathcal{e} = \langle m_1 \mid b^* \cdot m_2 \rangle_\mathcal{e}$

for all appropriately chosen $m_i \in M$, $c \in C$, and $b \in B$. We furthermore require the following condition:

- [FE2.e]: For any $(m_1, m_2), (m_3, m_4) \in M \ast M$ with $q_\mathcal{M}(m_2) = q_\mathcal{M}(m_3)$, we have

$$\langle m_1 \mid m_2 \rangle \cdot \langle m_3 \mid m_4 \rangle = \langle m_1 \cdot m_2 \mid m_3 \cdot m_4 \rangle,$$

and

$$\langle m_1 \mid m_3 \rangle \cdot \langle m_2 \mid m_4 \rangle_\mathcal{e} = \langle m_1 \mid q_\mathcal{M}(m_2) \cdot q_\mathcal{M}(m_3) \rangle_\mathcal{e}.$$  

**Remark 4.2.** Condition [FE2.e] is the Fell bundle analogue of Condition [GP3.e] for groupoid pre-equivalences. Indeed, by Condition [FE2.a]$_0$, we have

$$p_\mathcal{G}(\langle m_i \mid m_j \rangle) = L_\mathcal{G}(q_\mathcal{M}(m_i) \mid q_\mathcal{M}(m_j)).$$
Since \( q_\#(m_2) = q_\#(m_3) \), we have by Condition [GP3.e] that
\[
p_\#(m_1 \mid m_2, m_3, m_4) = L_\#(q_\#(m_1) \mid q_\#(m_2)) L_\#(q_\#(m_3) \mid q_\#(m_4)) = L_\#(q_\#(m_1) \mid q_\#(m_4)).
\]

On the other hand, by Conditions [FA1] and [FE2.a]
\[
q_\#(m_1 \cdot (m_2 | m_3)_e) = q_\#(m_1) \cdot p_\#((m_2 | m_3)_e) = q_\#(m_1) \cdot R_\#(q_\#(m_2) | q_\#(m_3)).
\]
Again, since \( q_\#(m_2) = q_\#(m_3) \), we have by Condition [GP3.b] that \( R_\#(q_\#(m_2) | q_\#(m_3)) = s_\#(m_2) \), which equals \( s_\#(m_1) \) since \( (m_1, m_2) \in M_\# \). Therefore, \( q_\#(m_1 \cdot (m_2 | m_3)_e) = q_\#(m_1) \), and thus by Condition [FE2.a]
\[
p_\#(m_1 \cdot (m_2 | m_3)_e) = L_\#(q_\#(m_1) \mid q_\#(m_4)),
\]
so the equalities in Condition [FE2.e] make sense.

**Remark 4.3.** Remark 2.12 still holds in the case of hypo-equivalences. Note further that any Fell bundle equivalence is a hypo-equivalence. Indeed, Condition [FE2.a] is exactly Condition [FE2.a] in the case that \( X \) and \( \mathcal{M} \) are equivalences of groupoids resp. Fell bundles, with \( L_\#(\cdot \mid \cdot) = s_\#_{\mathcal{M}}(\cdot \mid \cdot) \) and \( R_\#(\cdot \mid \cdot) = (\cdot \mid \cdot)_{\mathcal{M}} \). Moreover, we will see in Corollary 4.6 that Condition [FE2.d] is weaker than [FE2.d]. Lastly, Condition [FE2.e] follows as well because
\[
\langle m_1 \mid m_2 \rangle \cdot \langle m_3 \mid m_4 \rangle = \langle m_1 \mid m_2 \rangle \cdot m_3 \cdot m_4 \quad \text{by Condition [FE2.e]}
\]
\[
= \langle m_1 \cdot (m_2 | m_3)_e \mid m_4 \rangle \quad \text{by [FE2.d]}
\]

We will use this section to establish some basic results about hypo-equivalences. In the next section, we will then prove our first main result, namely that we can take the product of two hypo-equivalences \( \mathcal{M} \) and \( \mathcal{N} \) to construct another hypo-equivalence; see Theorem 5.12 for the exact statement. This result is analogous to Lemma 3.4 in the land of groupoids.

Our second main result will then show that, when the original data \( \mathcal{M}, \mathcal{N} \) are actually equivalences, their product gives rise to another equivalence by taking an appropriate quotient of the product hypo-equivalence; see Theorem 6.15. This result, in turn, is analogous to Corollary 3.9 in groupoid land.

One groupoid result that currently does not have an analogue is Proposition 3.7 which states that any groupoid pre-equivalence gives rise to an equivalence. The authors are convinced that hypo-equivalence is not quite strong enough to force the existence of a Fell bundle equivalence, which is the reason for the chosen terminology.

Let us first prove some basic results. For the remainder of this section unless otherwise indicated, \( \mathcal{M} = (q_\#: M \to X) \) will denote a fixed but arbitrary \( (\mathcal{B}, \mathcal{C}) \)-hypo-equivalence.

**Lemma 4.4.** For any \( b \in \mathcal{B} \) with \( \|b\| \neq 0 \), there exists \( m \in M \) with \( r_\#(m) = s_\#(b) \) and \( \|m\| \neq 0 \) in the Banach space norm of \( \mathcal{M} \) for \( x = q_\#(m) \).

**Proof.** Let \( g := p_\#(b) \in \mathcal{G} \). By assumption, \( r_\#: X \to \mathcal{G}^{(0)} \) is onto, so there exist \( x \in X \) such that \( r_\#: x = s_\#(g) \). Now, by Condition [FE3] on \( \mathcal{M} \), we know in particular that \( M(x) \) is a full left Hilbert module over
\[
B(r_\#: x) = B(s_\#: g) = B(s_\#: b).
\]
By fullness, it follows that span \( \{ \langle m \mid m' \rangle : m, m' \in M(x) \} \) is dense in \( B(s_\#: b) \).

By assumption on \( b \), we have \( 0 < \varepsilon := \|b\|^2 = \|b^* b\| \). Since \( b^* b \in B(s_\#: b) \), there exist \( m_i, m_i' \in M(x) \) with \( \|b^* b - \sum_{i=1}^n \langle m_i \mid m_i' \rangle \| < \frac{\varepsilon}{4} \). In particular, \( r_\#: (m_i) = r_\#: (q_\#: (m_i)) = r_\#: (x) = s_\#: (b) \), and further \( \|\sum_{i=1}^n \langle m_i \mid m_i' \rangle \| > \frac{\varepsilon}{2} \) by the reverse triangle inequality. In particular, for some \( 1 \leq i \leq n \), we must...
have \( \| \langle m_i \mid m'_i \rangle \| > \frac{\epsilon}{2n} > 0 \). By the Cauchy–Schwarz inequality for Hilbert modules [\S \text{Lemma } 2.5], we have
\[
\langle m_i \mid m'_i \rangle^* \langle m_i \mid m'_i \rangle \leq \| \langle m_i \mid m_i \rangle \| \langle m'_i \mid m'_i \rangle.
\]
Recall that, if \( a, b \) in some \( C^* \)-algebra \( A \) are self-adjoint with \( a \leq b \), then \( \| a \| \leq \| b \| \), so the above implies that
\[
0 < \| \langle m_i \mid m'_i \rangle \|^2 \leq \| \langle m_i \mid m_i \rangle \| \| \langle m'_i \mid m'_i \rangle \| = \| m_i \|_{M(x)}^2 \| m'_i \|_{M(x)}^2,
\]
where the last equality again stems from the fact that \( M(x) \) is a Hilbert module, so its norm comes from the inner product. As the norm is exactly the norm with respect to which \( \mathcal{M} \) is a USC Banach bundle, this concludes our proof. \( \square \)

**Corollary 4.5.** If \( b \cdot m = b' \cdot m \) for all \( m \in \mathcal{M} \) with \( r_{\mathcal{M}}(m) = s_{\mathcal{M}}(b) = s_{\mathcal{M}}(b') \), then \( b = b' \).

**Proof.** Since \( b \cdot m - b' \cdot m = (b - b') \cdot m \), we have
\[
0 = \| b \cdot m - b' \cdot m \| = \| (b - b') \cdot m \| = \| b - b' \| \| m \|
\]
for all \( m \). By Lemma 4.4, it follows that \( \| b - b' \| = 0 \) in \( \mathcal{B}_2 \), meaning that \( b = b' \) as claimed. \( \square \)

**Corollary 4.6.** If \( \mathcal{M} \) is an equivalence of Fell bundles, then Condition (FE2.d) implies that
\[
\langle m_1 \cdot c \mid m_2 \rangle = \langle m_1 \mid m_2 \cdot c^* \rangle \quad \text{and} \quad \langle b \cdot m_1 \mid m_2 \rangle = \langle m_1 \mid b^* \cdot m_2 \rangle
\]
whenever each side makes sense for \( m_i \in M \), \( c \in C \), and \( b \in B \).

Corollary 4.6 is not as trivial as it seems, despite Condition (FE3), since \( c \) is allowed to be in a fibre that does not live over a unit in \( \mathcal{H} \).

**Proof.** We do the proof for the \( \mathcal{B} \)-valued inner product; the other one follows *mutatis mutandis*.

First, a sanity check:
\[
\langle m_1 \cdot c \mid m_2 \rangle \text{ is defined} \iff s_{\mathcal{M}}(m_1) = r_{\mathcal{M}}(c) \quad \text{and} \quad s_{\mathcal{M}}(m_1 \cdot c) = s_{\mathcal{M}}(m_2)
\]
\[
\iff s_{\mathcal{M}}(m_1) = r_{\mathcal{M}}(c) \quad \text{and} \quad s_{\mathcal{M}}(c) = s_{\mathcal{M}}(m_2)
\]
\[
\iff s_{\mathcal{M}}(m_1) = s_{\mathcal{M}}(c^*) \quad \text{and} \quad r_{\mathcal{M}}(c^*) = s_{\mathcal{M}}(m_2)
\]
\[
\iff \langle m_1 \mid m_2 \cdot c^* \rangle \text{ is defined.}
\]
Furthermore,
\[
p_{\mathcal{M}}(\langle m_1 \cdot c \mid m_2 \rangle) = \langle q_{\mathcal{M}}(m_1 \cdot c) \mid q_{\mathcal{M}}(m_2) \rangle \quad \text{by (FE2.a)}
\]
\[
= \langle q_{\mathcal{M}}(m_1) \cdot p_{\mathcal{M}}(c) \mid q_{\mathcal{M}}(m_2) \rangle \quad \text{by (FA1)}
\]
\[
= \langle q_{\mathcal{M}}(m_1) \mid q_{\mathcal{M}}(m_2) \cdot p_{\mathcal{M}}(c)^{-1} \rangle \quad \text{by (FA1) and (F5)}
\]
\[
= \langle q_{\mathcal{M}}(m_1) \mid q_{\mathcal{M}}(m_2 \cdot c^*) \rangle \quad \text{by (FE2.a)}
\]
so the two inner products live in the same fibre. Condition (FE2.d) implies the first and last equality in the following computation:
\[
\langle m_1 \cdot c \mid m_2 \rangle \cdot m_3 = \langle m_1 \cdot c \mid \{ m_2 \mid m_3 \} \rangle = \{ c \{ m_2 \mid m_3 \} \}
\]
\[
= m_1 \cdot \langle m_2 \cdot c^* \mid m_3 \rangle = \langle m_1 \mid m_2 \cdot c^* \rangle \cdot m_3.
\]
Since \( m_3 \in M \) was arbitrary, it follows from Corollary 4.5 that \( \langle m_1 \cdot c \mid m_2 \rangle = \langle m_1 \mid m_2 \cdot c^* \rangle \), as claimed. \( \square \)
Even the ‘inhomogeneous’ inner product on the hypo-equivalence $\mathcal{M}$, which allows to take in elements from two different fibres, satisfies a Cauchy–Schwarz inequality; one can compare our result below with the classical Cauchy–Schwarz inequality on an inner product module [8 Lemma 2.5].

**Lemma 4.7** (Cauchy–Schwarz). If $(m, m') \in \mathcal{M}^* \times \mathcal{M}$, then

$$\langle m \mid m' \rangle \langle m \mid m' \rangle^* \leq \|m'\|^2 \langle m \mid m \rangle.$$  

**Proof.** First, we want to establish that the inequality makes sense. So note that, if

$$g := p_{\mathcal{B}}(\langle m \mid m' \rangle) = L_g(q_{\mathcal{B}}(m) \mid q_{\mathcal{B}}(m')),$$

then by Item (FE2.b) and by Item (GP3.b) in Definition 3.1

$$p_{\mathcal{B}}(\langle m \mid m' \rangle^*) = p_{\mathcal{B}}(\langle m' \mid m \rangle) = L_g(q_{\mathcal{B}}(m) \mid q_{\mathcal{B}}(m)) = L_g(q_{\mathcal{B}}(m) \mid q_{\mathcal{B}}(m'))^{-1} = g^{-1},$$

so that the product of $\langle m \mid m' \rangle$ and $\langle m \mid m' \rangle^*$ lives in the fibre over $gg^{-1} = r_g(g)$. Furthermore, it follows from the assumptions in Definition 3.1 that

$$p_{\mathcal{B}}(\langle q_{\mathcal{B}}(m) \mid q_{\mathcal{B}}(m) \rangle) = L_g(q_{\mathcal{B}}(m) \mid q_{\mathcal{B}}(m)) = r_g(g).$$

Thus, the claimed inequality makes sense in the C*-algebra $\mathcal{B}_{r(g)}$. To prove it, we follow the proof of [8 Lemma 2.5]. Let $\rho$ be a state on $\mathcal{B}_{r(g)}$; it will suffice to prove that

$$(4.1) \quad \rho(\langle m \mid m' \rangle \langle m \mid m' \rangle^*) \leq \|m'\|^2 \rho(\langle m \mid m \rangle).$$

Let $x = q_{\mathcal{B}}(m)$. The map

$$\mathcal{M}_x \times \mathcal{M}_x \to \mathbb{C}, \quad (m_1, m_2) \mapsto \rho(\langle m_1 \mid m_2 \rangle),$$

is a (semi-definite) positive sesquilinear form on $\mathcal{M}_x$, so the ordinary Cauchy-Schwarz inequality implies that

$$|\rho(\langle m_1 \mid m_2 \rangle)| \leq \rho(\langle m_1 \mid m_1 \rangle)^{\frac{1}{2}} \rho(\langle m_2 \mid m_2 \rangle)^{\frac{1}{2}}.$$  

Notice that

$$q_{\mathcal{B}}(\langle m \mid m' \rangle m') = p_{\mathcal{B}}(\langle m \mid m' \rangle) q_{\mathcal{B}}(m') = g q_{\mathcal{B}}(m') = q_{\mathcal{B}}(m) = x,$$

so we may let

$$m_1 := m \quad \text{and} \quad m_2 := \langle m \mid m' \rangle m',$$

which yields:

$$0 \leq \rho(\langle m \mid m' \rangle \langle m \mid m' \rangle^*) = \rho(\langle m \mid \langle m \mid m' \rangle m' \rangle)$$

$$= \rho(\langle m_1 \mid m_2 \rangle) \leq \rho(\langle m_1 \mid m_1 \rangle)^{\frac{1}{2}} \rho(\langle m_2 \mid m_2 \rangle)^{\frac{1}{2}}$$

$$= \rho(\langle m \mid m \rangle)^{\frac{1}{2}} \rho(\langle m \mid m' \rangle \langle m \mid m' \rangle \langle m \mid m' \rangle^*)^{\frac{1}{2}}$$

$$= \rho(\langle m \mid m \rangle)^{\frac{1}{2}} \rho(\langle m \rangle \langle m' \rangle \langle m' \rangle \langle m \rangle^*)^{\frac{1}{2}}.$$  

With $b := \langle m \mid m' \rangle$ and $c := \langle m' \mid m' \rangle$, it follows from [8 Corollary 2.22] that

$$\rho(\langle m \mid m' \rangle \langle m \mid m' \rangle^*) \leq \rho(\langle m \mid m \rangle)^{\frac{1}{2}} \|m' \mid m'\|^{\frac{1}{2}} \rho(\langle m \mid m \rangle \langle m \mid m' \rangle^*)^{\frac{1}{2}}.$$
which yields the desired Inequality (4.1) after squaring and cancelling a factor of
\[ \rho \left( m | m' \right) \rho \left( m | m' \right)^* \]
on both sides. \qed

5. PRODUCT OF FELL BUNDLE HYPO-EQUIVALENCES

The goal of this section is to show an analogous result of Lemma 3.4 for Fell bundle hypo-equivalences, namely that the tensor product of two hypo-equivalences is again a hypo-equivalence. Throughout this section, we make the following assumptions:

- We fix three saturated Fell bundles \( B = (p_B: B \to G), C = (p_C: C \to H), D = (p_D: D \to K) \) over locally compact Hausdorff étale groupoids \( G, H, K \).
- Let \( X \) be a \((G, H)\)-pre-equivalence and \( Y \) an \((H, K)\)-pre-equivalence; we let \( Z := X \ast Y \), which is a \((G, K)\)-pre-equivalence by Lemma 3.4.
- Let \( M = (q_M: M \to X) \) be a \((B, C)\)-hypo-equivalence and \( N = (q_N: N \to Y) \) a \((C, D)\)-hypo-equivalence.
- We write \( \cdot \) for the left and right actions on \( M, X, N, \) and \( Y \).

Out of \( M \) and \( N \), we will construct a new bundle, denoted \( M \otimes_{(\xi)} N \), that we will show to be a hypo-equivalence between \( B \) and \( D \).

5.1. Tensor product bundle. If \( x \in X, y \in Y \) with \( s_X(x) = u = r_Y(y) \), we may let \( M_x \otimes_{C(u)} N_y \) denote the algebraic balanced tensor product as defined in [8, Proposition 3.16]. On it, we define two inner products with values in a fibre of \( B \) resp. of \( D \); they are determined by
\[
\langle m \otimes n | m' \otimes n' \rangle_B = \langle m | m' \rangle_B \cdot \langle n | n' \rangle_B, \quad \langle m \otimes n | m' \otimes n' \rangle_D = \langle m' | m \rangle_D \cdot \langle n | n' \rangle_B,
\]
and then extended to the algebraic tensor product by linearity. Recall from [8, Remark 3.17] that these are indeed ‘positive definite’, meaning if \( \langle \xi | \xi \rangle = 0 \), then \( \xi = 0 \). Furthermore, \( \| \langle \xi | \xi \rangle_B \| = \| \xi \| \cdot \| \xi \|_B \), so that we can unambiguously denote its completion with respect to this norm by \( M_x \otimes_{C(u)} N_y \).

Note that, since \( u \) is the source of \( x \), it is clear from context, and so we will instead often write \( M_x \otimes_{(\xi)} N_y \).

**Lemma 5.1.** On the set
\[ K := \bigsqcup_{(x,y) \in X \ast Y} M(x) \otimes_{(\xi)} N(y), \]
consider all cross-sections of the form
\[ \sigma \otimes \tau: \quad Z = X \ast Y \to K, \quad (x, y) \mapsto \sigma(x) \otimes \tau(y), \]
for \( \sigma \in \Gamma_0(X; M) \) and \( \tau \in \Gamma_0(Y; N) \). Then there is a unique topology on \( K \) making it a US
Banach bundle over \( Z \) such that all of those cross-sections are continuous. We denote said bundle by \( M \otimes_{(\xi)} N \) or, since \( M \) and \( N \) are fixed, by \( K \); its bundle map will be denoted by \( q_{K} \).

We note that we reserve the notation \( \otimes_{(\xi)} \) for a different construction.

**Proof.** By construction, the obvious map
\[ K = \bigsqcup_{(x,y) \in X \ast Y} M_x \otimes_{(\xi)} N_y \xrightarrow{q_{K}} X \ast Y \]
is a surjection and each fibre \( q_{K}^{-1}(x, y) \) is a Banach space. We now let \( \Gamma \) denote the \( \mathbb{C} \)-linear span of all elements \( \sigma \otimes \tau \). If we can show that \( \Gamma \) satisfies (1) and (2) in Remark 2.7, then there is such a unique topology.
For \(1\) fix an arbitrary element of \(\Gamma\), and write it as \(f = \sum_{i=1}^{k} \sigma_i \otimes \tau_i\). Let 
\[
f' : X \rtimes Y \to B, \quad f'(x, y) := \sum_{i,j=1}^{k} \varphi(\sigma_i(x) \cdot \sigma_j(x) \cdot \tau_j(y) \cdot \tau_i(y))_i.\]
Note that all summands in \(f'(x, y)\) live in the same fibre, so that \(f'\) is well defined. By continuity of \(\sigma, \tau\), of the right \(\mathcal{C}\)-action on \(\mathcal{M}\), of \(\varphi(\cdot | \cdot)\) and of \(\varphi(\cdot \cdot \cdot)\), and by continuity of taking finite sums, we know that \(f'\) is continuous. By definition of the fibrewise norm, we have
\[
\|f(x, y)\| = \left\| \sum_{i=1}^{k} \sigma_i(x) \otimes \tau_i(y)\right\| = \|f'(x, y)\|^{\frac{1}{2}},
\]
so we have written \((x, y) \mapsto \|f(x, y)\|\) as the composition of the continuous map \(f'\) with the map \(b \mapsto \|b\|^{\frac{1}{2}}\), which is upper semi-continuous by assumption on \(B\). As such, \((x, y) \mapsto \|f(x, y)\|\) is upper semi-continuous also.

For \(2\) fix any \(m \in \mathcal{M}_x\) and \(n \in \mathcal{N}_y\). By assumption, there exist elements \(\sigma \in \Gamma_0(X; \mathcal{M})\) and \(\tau \in \Gamma_0(Y; \mathcal{N})\) for which \(\sigma(x) = m\) and \(\tau(y) = n\). In particular, \(\Gamma(x, y)\) contains all elementary tensors \(m \otimes n\) and, by construction, their linear hull, making \(\Gamma(x, y)\) dense. \(\square\)

**Remark 5.2.** As pointed out in Remark 2.4, the bundle \(\mathcal{K}\) automatically has enough continuous cross-sections according to [4] Proposition 3.4 (see also [3] II.13.19 and the discussion in [7] Appendix A). One needs to be cautious, however, that the linear span of sections \(\sigma \otimes \tau\) is by itself not sufficient for that.

**Lemma 5.3.** Suppose we have convergent nets \(\{m_\lambda\}_{\lambda \in \Lambda}\) in \(M\) and \(\{n_\lambda\}_{\lambda \in \Lambda}\) in \(N\), with limits \(m\) resp. \(n\). Suppose \(s_\mathcal{K}(m_\lambda) = r_\mathcal{N}(n_\lambda)\) for all \(\lambda \in \Lambda\). Then \(m_\lambda \otimes n_\lambda \to m \otimes n\) in \(\mathcal{M} \otimes_{\mathcal{C}(0)} \mathcal{N}\).

**Proof.** We will use Lemma A.3. Let \(x_\lambda := q_\mathcal{K}(m_\lambda), y_\lambda := q_\mathcal{N}(n_\lambda)\). Since \(s_\mathcal{K}(x_\lambda) = s_\mathcal{K}(m_\lambda) = r_\mathcal{N}(n_\lambda) = s_\mathcal{N}(y_\lambda)\) by assumption, continuity of the maps involved imply that \(s_\mathcal{K}(x) = r_\mathcal{N}(y)\) for \(x := q_\mathcal{K}(m)\) and \(y := q_\mathcal{N}(n)\) also. Moreover, we automatically have
\[
q_\mathcal{K}(m_\lambda \otimes n_\lambda) = (x_\lambda, y_\lambda) \to (x, y) = q_\mathcal{K}(m \otimes n).
\]
If \(m_\lambda \in \mathcal{M}_x, n_\lambda \in \mathcal{N}_y\), then
\[
\left\| \sum_{i=0}^{k} m_i \otimes n_i \right\|^2 = \left\| \sum_{i,j=0}^{k} (m_i \otimes n_i | m_j \otimes n_j) \right\| = \left\| \sum_{i,j=0}^{k} \varphi(m_i | m_j \cdot \varphi(n_j | n_i)) \right\|.
\]
Now take arbitrary \(\sigma_i \in \Gamma_0(X; \mathcal{M})\) and \(\tau_i \in \Gamma_0(Y; \mathcal{N})\). Their continuity, the continuity of the right \(\mathcal{C}\)-action on \(\mathcal{M}\), of \(\varphi(\cdot | \cdot)\) and of \(\varphi(\cdot \cdot \cdot)\), and of taking finite sums, and the upper semi-continuity of the norm on \(B\) together with Equation 5.2 imply that
\[
\lim_{\lambda} \left\| m_\lambda \otimes n_\lambda - \sum_{i=1}^{k} \sigma_i(x_\lambda) \otimes \tau_i(y_\lambda) \right\|^2 \leq \left\| m \otimes n - \sum_{i=1}^{k} \sigma_i(x) \otimes \tau_i(y) \right\|^2.
\]
Since the sections on \(\mathcal{M}\) and \(\mathcal{N}\) were arbitrary, it follows from Lemma A.3 \(3 \implies 1\) that \(m_\lambda \otimes n_\lambda \to m \otimes n\) in \(\mathcal{M} \otimes_{\mathcal{C}(0)} \mathcal{N}\), as claimed. \(\square\)

**Lemma 5.4.** The map \(q_\mathcal{K} : \mathcal{K} = \mathcal{M} \otimes_{\mathcal{C}(0)} \mathcal{N} \to X \rtimes Y = Z\) is open.

**Proof.** Let \(A \subseteq K\) be any closed subset. To see that \(q := q_\mathcal{K}\) is open, it suffices to prove that the set \(B := \{(x, y) \in Z \mid q^{-1}(x, y) \subseteq A\}\) is closed. To this end, let \((x_\lambda, y_\lambda) \in B\) be a net that converges to \((x, y)\) in \(Z\); we claim that \((x, y) \in B\). In other words, if we fix any \(\xi \in q^{-1}(x, y)\), we need to show that \(\xi \in A\).

For \(\xi\), take any \(\sigma \in \Gamma_0(Z; \mathcal{K})\) for which \(\sigma(x, y) = \xi\). Then, by continuity of \(\sigma\), we have that \(\xi_\lambda := \sigma(x_\lambda, y_\lambda)\) converges to \(\xi\) in \(\mathcal{K}\). Since \(\sigma\) is a section, \(\xi_\lambda \in q^{-1}(x_\lambda, y_\lambda)\). As \((x_\lambda, y_\lambda) \in B\), we
have $q^{-1}(x_\lambda, y_\lambda) \subseteq A$, so that $\xi_\lambda \in A$ and thus $\xi$ is the limit of a net in $A$. As $A$ was closed by assumption, it follows that, indeed, $\xi \in A$ as claimed. \hfill $\Box$

5.2. **Additional structure on the product bundle.** In this subsection, we will study the USC Banach bundle $\mathcal{K} := \mathcal{M} \otimes_{\varphi(0)} \mathcal{N} = (q_x : K \to X \ast Y)$ in more detail. Recall that $Z := X \ast Y$ naturally is a $(\mathcal{G}, \mathcal{K})$-pre-equivalence by Lemma 3.4, which is also where we defined $L^*_g (\cdot | \cdot )$ and $R^*_g (\cdot | \cdot )$.

**Proposition 5.5.** The bundle $\mathcal{K}$ naturally carries a left $\mathcal{B}$- and a right $\mathcal{D}$-action in the sense of Definition 2.10 determined by

$$b \cdot (m \otimes n) = (b \cdot m) \otimes n \quad \text{and} \quad (m \otimes n) \cdot d = m \otimes (n \cdot d)$$

where $b \in \mathcal{B}$, $m \otimes n \in K$, and $d \in D$ are such that $s_\mathcal{B}(b) = r_\mathcal{M}(m)$ and $s_\mathcal{N}(n) = r_\mathcal{D}(d)$. Moreover, these actions commute.

**Proof.** We will do the proof for $\mathcal{B}$; the proof for $\mathcal{D}$ will follow *mutatis mutandis*, and it is clear from the given description that the actions commute. Suppose $g \in \mathcal{G}$ and $(x, y) \in Z$ are such that $s_\mathcal{G}(g) = r_\mathcal{X}(x)$. For any fixed $b \in \mathcal{B}_g$, define

$$\tilde{\phi}_b^{(x, y)} = \tilde{\phi} : \mathcal{M}_x \times \mathcal{N}_y \to \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y, \quad (m, n) \mapsto (b \cdot m) \otimes n.$$ 

By the properties of the left $\mathcal{B}$-action on $\mathcal{M}$, one quickly verifies that $\tilde{\phi}$ is bilinear, and since

$$(b \cdot (m \cdot c)) \otimes n = ((b \cdot m) \cdot c)) \otimes n = (b \cdot m) \otimes (c \cdot n)$$

for any $c \in \mathcal{C}_{(x)}$, we conclude that the induced linear map on $\mathcal{M}_x \otimes \mathcal{N}_y$ factors through the $\mathcal{C}(0)$-balancing. In other words, there exists a map

$$\tilde{\phi}_b^{(x, y)} : \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y \to \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y \quad \text{determined by} \quad m \otimes n \mapsto (b \cdot m) \otimes n.$$ 

By Conditions (FE2.b) and (FE2.c) on $\mathcal{B}$ (linearity of the inner products), we have

$$\mathcal{B} \langle (b' \cdot m') \otimes n' | (b \cdot m) \otimes n \rangle = \mathcal{B} \langle b' \cdot m' | (b \cdot m) \otimes (n | n') \rangle = b' \mathcal{B} \langle m' | m \otimes (n | n') \rangle b^*$$

$$= b' \mathcal{B} \langle m' | m \otimes n' \rangle b^*,$$

so we conclude that for any $\xi \in \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y$,

$$\| \tilde{\phi}_b^{(x, y)}(\xi) \|_\mathcal{K} = \| \mathcal{B} \langle \tilde{\phi}_b^{(x, y)}(\xi) | \tilde{\phi}_b^{(x, y)}(\xi) \rangle \|_\mathcal{B} = \| b \mathcal{B} \langle \xi | \xi \rangle b^* \|_\mathcal{B} \leq \| b \|_\mathcal{B}^2 \| \mathcal{B} \langle \xi | \xi \rangle \|_\mathcal{B} = \| b \|_\mathcal{B}^2 \| \xi \|_{\mathcal{K}(x, y)}^2,$$

where we added the subscripts for clarity. It is now clear that $\tilde{\phi}_b^{(x, y)}$ extends to a continuous linear map

$$\phi_b^{(x, y)} : \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y \to \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y \quad \text{determined by} \quad m \otimes n \mapsto (b \cdot m) \otimes n,$$

which satisfies

$$\| \phi_b^{(x, y)}(\xi) \| \leq \| b \|_\mathcal{K} \| \xi \| \quad \text{for all} \quad \xi \in \mathcal{K}(x, y) = \mathcal{M}_x \otimes_{\varphi(0)} \mathcal{N}_y.$$ 

Other properties of the $\mathcal{B}$-action on $\mathcal{M}$ imply that

$$\phi_{b + b'}^{(x, y)} = \phi_b^{(x, y)} + \lambda \phi_{b'}^{(x, y)} \quad \text{and} \quad \phi_b^{(x, y)} \circ \phi_{b_2}^{(x, y)} = \phi_{b_1 + b_2}^{(x, y)} \quad \text{where} \quad g = p_{\mathcal{B}}(b_2)$$

for all $b, b' \in \mathcal{B}$ with $p_{\mathcal{B}}(b) = p_{\mathcal{B}}(b')$ and all $(b_1, b_2) \in \mathcal{B}^2(2)$. We define

$$\mathcal{B} \ast \ast, \mathcal{K} \to \mathcal{K}, \quad (b, \xi) \mapsto b \cdot \xi := \phi_b^{(x, y)}(\xi) \quad \text{where} \quad (x, y) = q_{\mathcal{K}}(\xi).$$

By construction, we have $q_{\mathcal{K}}(b \cdot \xi) = p_{\mathcal{B}}(b) \cdot q_{\mathcal{K}}(\xi)$. The first equality in (5.4) implies that the action is linear and the second that $a \cdot (b \cdot \xi) = (ab) \cdot \xi$ for appropriate $a, b \in \mathcal{B}$. Inequality (5.3) implies that $\| b \cdot \xi \| \leq \| b \|_\mathcal{K} \| \xi \|$.
It only remains to show that \((b, \xi) \mapsto b \cdot \xi\) is continuous, so suppose that we have a convergent net in \(\mathcal{B}_*, \mathcal{H}\), say \((b_\lambda, \xi_\lambda) \to (b, \xi)\). In particular, \((g_\lambda, x_\lambda, y_\lambda) := (p_\mathcal{H}(b_\lambda), q_\mathcal{H}(\xi_\lambda)) \to (p_\mathcal{H}(b), q_\mathcal{H}(\xi)) =: (g, x, y)\). Fix \(\epsilon > 0\). Because of Lemma A.3 \(\text{(2) } \implies \text{(1)}\) it suffices to find a section \(\varphi\) of \(\mathcal{H}\) and some \(\lambda_0\) such that for all \(\lambda \geq \lambda_0\), we have

\[
\|b_\lambda \cdot \xi_\lambda - \varphi(g_\lambda x_\lambda, y_\lambda)\| < \epsilon \quad \text{and} \quad \|b \cdot \xi - \varphi(gx, y)\| < \epsilon.
\]

Pick finitely many \(\tau_j \in \Gamma_0(X; \mathcal{M}), \kappa_j \in \Gamma_0(Y; \mathcal{N})\) such that

\[
\left\| \xi - \sum_j \tau_j(x) \otimes \kappa_j(y) \right\| < \frac{\epsilon}{2(\|b\| + 1)}.
\]

Because of Inequality (5.3), this implies

\[
\left\| b \cdot \xi - \sum_j \tau_j(x) \otimes \kappa_j(y) \right\| \leq \left\| \xi - \sum_j \tau_j(x) \otimes \kappa_j(y) \right\| < \frac{\epsilon}{2}.
\]

Since \((b_\lambda, \xi_\lambda) \to (b, \xi)\), we know from Lemma A.3 \(\text{(1) } \implies \text{(3)}\) that

\[
\lim_{\lambda} \|b_\lambda\| \leq \|b\| \quad \text{and} \quad \lim_{\lambda} \left\| \xi_\lambda - \sum_j \tau_j(x_\lambda) \otimes \kappa_j(y_\lambda) \right\| \leq \left\| \xi - \sum_j \tau_j(x) \otimes \kappa_j(y) \right\|.
\]

Together with Inequalities (5.3) and (5.5), this implies that

\[
\lim_{\lambda} \left\| b_\lambda \cdot \xi_\lambda - b_\lambda \cdot \sum_j \tau_j(x_\lambda) \otimes \kappa_j(y_\lambda) \right\| \leq \lim_{\lambda} \left( \|b_\lambda\| \left\| \xi_\lambda - \sum_j \tau_j(x_\lambda) \otimes \kappa_j(y_\lambda) \right\| \right) 
\leq \left( \lim_{\lambda} \|b_\lambda\| \right) \left( \lim_{\lambda} \left\| \xi_\lambda - \sum_j \tau_j(x_\lambda) \otimes \kappa_j(y_\lambda) \right\| \right) \leq \|b\| \left\| \xi - \sum_j \tau_j(x) \otimes \kappa_j(y) \right\| < \frac{\epsilon}{2}.
\]

So there exists \(\lambda_1\) such that for \(\lambda \geq \lambda_1\)

\[
\left\| b_\lambda \cdot \xi_\lambda - b_\lambda \cdot \sum_j \tau_j(x_\lambda) \otimes \kappa_j(y_\lambda) \right\| < \frac{\epsilon}{2}.
\]

Since the \(\mathcal{B}\)-action on \(\mathcal{M}\) is continuous, we have that \(\lim_{\lambda} b_\lambda \cdot \tau_j(x_\lambda) = b \cdot \tau_j(x)\). By Lemma 5.3 it follows that \(\lim_{\lambda} \left[ b \cdot \tau_j(x_\lambda) \right] \otimes \kappa_j(y_\lambda) = [b \cdot \tau_j(x)] \otimes \kappa_j(y)\) in \(\mathcal{H}\). By Lemma A.3 \(\text{(1) } \implies \text{(2)}\) there exists a section \(\varphi\) of \(\mathcal{H}\) and some \(\lambda_2\) such that for all \(\lambda \geq \lambda_2\)

\[
\left\| \sum_j [b_\lambda \cdot \tau_j(x_\lambda)] \otimes \kappa_j(y_\lambda) - \varphi(g_\lambda x_\lambda, y_\lambda) \right\| < \frac{\epsilon}{2}
\]

and

\[
\left\| \sum_j [b \cdot \tau_j(x)] \otimes \kappa_j(y) - \varphi(gx, y) \right\| < \frac{\epsilon}{2}.
\]

Fix any \(\lambda_0 \geq \lambda_1, \lambda_2\). Combining Equations (5.7) and (5.8) respectively Equations (5.6) and (5.9) with the triangle inequality, we see that for all \(\lambda \geq \lambda_0\)

\[
\|b_\lambda \cdot \xi_\lambda - \varphi(g_\lambda x_\lambda, y_\lambda)\| < \epsilon \quad \text{and} \quad \|b \cdot \xi - \varphi(gx, y)\| < \epsilon.
\]

This concludes our proof of continuity. \(\square\)

**Lemma 5.6.** Suppose we are given elements \(m, m' \in \mathcal{M}\) and \(n, n' \in \mathcal{N}\) such that \((x, y) := (q, \mathcal{M}(m), q, \mathcal{N}(n)), (x', y') := (q, \mathcal{M}(m'), q, \mathcal{N}(n'))\) are elements of \(Z = X_\ast, Y\).

1. We have \(s_\mathcal{X}(m \otimes n) = s_\mathcal{X}(m' \otimes n')\) if and only if \(s_\mathcal{X}(n) = s_\mathcal{X}(n')\). In that case, the element \(s_\mathcal{X}(m \mid m' \mid \varphi(n' \mid n))\) is well defined and lives in the fibre over \(L^+_Z((x, y) \mid (x', y'))\) of \(\mathcal{B}\).
(2) We have \( r_x(m \otimes n) = r_x(m' \otimes n') \) if and only if \( r_{\cdot \cdot}(m) = r_{\cdot \cdot}(m') \). In that case, the element \( \langle m' | m \cdot n \cdot n' \rangle \) is well defined and lives in the fibre over \( R_x^2((x,y) \mid (x',y')) \) of \( \mathcal{D} \).

**Proof.** We will only deal with mutatis mutandis, one proves We compute
\[
\begin{align*}
  s_x(m \otimes n) &= s_x(m' \otimes n') \\
  &\iff s_Z(q_x(m \otimes n)) = s_Z(q_x(m' \otimes n')) \\
  &\iff s_Z(x, y) = s_Z(x', y') \\
  &\iff s_Y(y) = s_Y(y') \iff s_{\cdot \cdot}(n) = s_{\cdot \cdot}(n'),
\end{align*}
\]
which shows that \( \phi(n' \mid n) \) makes sense. Further, as \( m \otimes n, m' \otimes n' \in \mathcal{X} \), we have \( s_{\cdot \cdot}(m) = s_{\cdot \cdot}(n) \) and \( s_{\cdot \cdot}(m') = s_{\cdot \cdot}(n') \) in \( \mathcal{H}(0) \), so
\[
r_{\phi}(\phi(n' \mid n)) = (r_H \circ p_{\phi})(\phi(n' \mid n)) = r_H(L_h^\gamma(q_{\cdot \cdot}(n') \mid q_{\cdot \cdot}(n))) = r_H(L_h^\gamma(y' \mid y)),
\]
where the second equality follows by Condition (FE2.a) of a property of the \( \mathcal{C} \)-valued inner product on \( \mathcal{N} \). By Items (GP3.a) and (GP3.a) in Definition 3.1, the range of \( L_h^\gamma(y' \mid y) \) in \( \mathcal{H}(0) \) is \( r_Y(y') \) and its source is \( r_Y(y) \). Thus,
\[
r_{\phi}(\phi(n' \mid n)) = r_H(L_h^\gamma(y' \mid y)) = r_Y(y') = r_{\cdot \cdot}(n') = s_{\cdot \cdot}(m'),
\]
so that \( m' \cdot \phi(n' \mid n) \) makes sense, and
\[
s_{\cdot \cdot}(m' \cdot \phi(n' \mid n)) = s_{\cdot \cdot}(\phi(n' \mid n)) = (s_{\cdot \cdot} \circ p_{\phi})(\phi(n' \mid n))
\]
so that \( m \mid m' \cdot \phi(n' \mid n) \) makes sense. To see over which fibre that element lives, we compute
\[
p_{\cdot \cdot}(m \mid m' \cdot \phi(n' \mid n)) = L_0^\gamma(q_{\cdot \cdot}(m) \mid q_{\cdot \cdot}(m' \cdot \phi(n' \mid n)))
\]
and
\[
L_0^\gamma(x \mid q_{\cdot \cdot}(m') \cdot p_{\phi}(\phi(n' \mid n)))
\]
which satisfies
\[
\langle m \cdot n \mid m' \cdot \phi(n' \mid n) \rangle = \langle m \mid m' \cdot \phi(n' \mid n) \rangle
\]
This concludes our proof. \( \square \)

**Lemma 5.7.** Assume \( (x, y), (x', y') \in Z = X \star Y \).

(1) If \( s_Y(y) = s_Y(y') \) and \( g := L_0^\gamma((x, y) \mid (x', y')) \), then there exists a sesquilinear map
\[
\phi \langle \cdot \mid \cdot \rangle : (\mathcal{M}_z \otimes \mathcal{N}_y) \times (\mathcal{M}_z \otimes \mathcal{N}_y) \rightarrow \mathcal{B}_g
\]
determined by
\[
\phi \langle m \otimes n \mid m' \otimes n' \rangle := \langle m \mid m' \cdot \phi(n' \mid n) \rangle
\]
which satisfies
\[
\phi \langle m \otimes n \mid m' \otimes n' \rangle^* = \phi \langle m' \otimes n' \mid m \otimes n \rangle.
\]

(2) If \( r_X(x) = r_X(x') \) and \( k := R_x^2((x, y) \mid (x', y')) \), then there exists a sesquilinear map
\[
\phi \langle \cdot \mid \cdot \rangle : (\mathcal{M}_z \otimes \mathcal{N}_y) \times (\mathcal{M}_z \otimes \mathcal{N}_y) \rightarrow \mathcal{D}_k
\]
determined by
\[
\phi \langle m \otimes n \mid m' \otimes n' \rangle := \langle \langle m' \mid m \cdot n \rangle \mid n' \rangle_{\phi}
\]
which satisfies
\[
\phi \langle m \otimes n \mid m' \otimes n' \rangle^* = \phi \langle m' \otimes n' \mid m \otimes n \rangle_{\phi}.
We point out that, while the formulas of the above forms look identical to those for the inner products on a fibre $\mathcal{H}(x,y)$ as defined in [5.1], there is one major difference: These new forms can be evaluated at two elements that live in different fibres; this is hinted at in the codomains of the map, since the elements $g$ and $k$ are not necessarily units in $G$ resp. $K$. We will soon see that the forms can be induced from the algebraic tensor product to the balanced completions, $\mathcal{H}(x,y)$ and $\mathcal{H}(x',y')$.

Proof. We will prove [1] the other claim is done analogously. We have seen in Lemma 5.6 that the formula makes sense for elementary tensors. We now follow the ideas in [8, Proof of Prop. 3.16]. Fix $m' \in \mathcal{M}_{x'}$, $n' \in \mathcal{N}_{y'}$, and consider the map

$$\mathcal{M}_x \times \mathcal{N}_y \ni (m, n) \mapsto \varrho(m \cdot \varphi(n' \mid n)) \in \mathcal{B}_y.$$

As $\varrho(\cdot \mid \cdot)$ and $\varphi(\cdot \mid \cdot)$ are both linear in the first and conjugate linear in the second component, this map is bilinear and hence induces a linear map

$$F_{m', n'} : \mathcal{M}_x \otimes \mathcal{N}_y \to \mathcal{B}_y$$

determined by $m \otimes n \mapsto \varrho(m \cdot \varphi(n' \mid n))$.

Again, sesquilinearity of $\varrho(\cdot \mid \cdot)$ and $\varphi(\cdot \mid \cdot)$ imply that the map

$$(m', n') \mapsto F^*(m', n') := (\xi \mapsto F_{m', n'}(\xi)^*)$$

is a bilinear map from $\mathcal{M}_x \times \mathcal{N}_{y'}$ into the vector space $\text{CL}$ of conjugate linear maps $\mathcal{M}_x \otimes \mathcal{N}_y \to \mathcal{B}_{y'}$. Thus, there exists a linear map $F^* : \mathcal{M}_x \otimes \mathcal{N}_y \to \text{CL}$ determined by $m' \otimes n' \mapsto F^*(m', n')$. We can thus define

$$\langle \langle \cdot \mid \cdot \rangle \rangle : (\mathcal{M}_x \otimes \mathcal{N}_y) \times (\mathcal{M}_x \otimes \mathcal{N}_y) \to \mathcal{B}_y, \quad \langle \langle \xi \mid \xi' \rangle \rangle := [F^*(\xi')(\xi)]^*.$$

The map is sesquilinear: it is linear in the first component since $F^*(\xi')$ is conjugate linear, and it is conjugate linear in the second component by linearity of $F^*$.

For the claim about the adjoint, we compute

$$\langle \langle m \otimes n \mid m' \otimes n' \rangle \rangle^* = \varrho(m \cdot \varphi(n' \mid n))^*$$

$$= \varrho(m' \cdot \varphi(n' \mid n) \mid m) \quad \text{by [FE2.b] for $\mathcal{M}$}$$
$$= \varrho(m' \cdot m \cdot \varphi(n' \mid n)^* \mid n) \quad \text{by [FE2.d] for $\mathcal{M}$}$$
$$= \varrho(m' \cdot m \cdot \varphi(n \mid n') \mid n') \quad \text{by [FE2.b] for $\mathcal{N}$}$$
$$= \langle \langle m' \otimes n' \mid m \otimes n \rangle \rangle.$$

This concludes our proof. □

We now verify that the products satisfy Condition [FE2.e].

Lemma 5.8. Suppose we are given elements $\xi_i \in \mathcal{M}_{x_i} \otimes \mathcal{N}_{y_i} \subseteq \mathcal{H}(x_i, y_i)$ for $i = 1, 2, 3, 4$. If $(x_2, y_2) = (x_3, y_3)$, then we have

$$\langle \langle \xi_1 \mid \xi_2 \rangle \langle \xi_3 \mid \xi_4 \rangle = \langle \langle \xi_1 \cdot \xi_2 \mid \xi_3 \cdot \xi_4 \rangle \rangle, \quad \text{and}$$

$$\langle \langle \xi_1 \mid \xi_2 \rangle \langle \xi_3 \mid \xi_4 \rangle = \langle \langle \xi_1 \cdot \xi_2 \mid \xi_3 \cdot \xi_4 \rangle \rangle,$$

wherever the inner products and products on the left-hand side make sense.

Proof. It suffices to prove the claim for elementary tensors $\xi_i = m_i \otimes n_i$. Applying the definition of $\langle \langle \cdot \mid \cdot \rangle \rangle$ and $\langle \langle \cdot \mid \cdot \rangle \rangle$, we have:

$$\langle \langle m_1 \otimes n_1 \mid m_2 \otimes n_2 \rangle \langle m_3 \otimes n_3 \mid m_4 \otimes n_4 \rangle$$

$$= \varrho(m_1 \cdot \varphi(n_2 \mid n_1)) \varrho(m_3 \cdot \varphi(n_4 \mid n_3))$$

$$= \varrho(m_1 \cdot \varphi(n_2 \mid n_1) \mid m_3 \cdot \varphi(n_4 \mid n_3)) \quad \text{by Condition [FE2.e]}.$$
Lemma 5.9. Let
\[ (m_1 \otimes n_1) \cdots (m_2 \otimes n_2) (m_3 \otimes n_3) \cdots (m_4 \otimes n_4) = m_1 \cdot (\langle n_1 | n_2 \rangle (m_2 | m_3) \cdot m_4 \cdot (\langle n_4 | n_3 \rangle) ) \]
by Conditions \([\text{FE2.e}]\) and \([\text{FE2.b}]\).

On the other hand,
\[
= \langle m_1 \otimes n_1 | n_2 \rangle (m_2 | m_3) \cdot m_4 \cdot (\langle n_4 | n_3 \rangle) = m_1 \cdot (\langle n_1 | n_2 \rangle (m_2 | m_3) \cdot m_4 \cdot (\langle n_4 | n_3 \rangle) )
\]
by Conditions \([\text{FE2.d_0}]\) and \([\text{FE2.b}]\).

\[ = \langle m_1 | m_4 \cdot (\langle n_3 | m_3 \cdot m_2 \rangle \cdot n_2 | n_1 \rangle) \]
by Condition \([\text{FE2.e}]\).

\[ = \langle m_1 | m_4 \cdot (\langle n_3 | m_3 \cdot m_2 \rangle \cdot n_2 | n_1 \rangle) \]
by Condition \([\text{FE2.c}]\).

\[ = \langle m_1 | m_4 \cdot (\langle n_3 | m_3 \cdot m_2 \rangle \cdot n_2 | n_1 \rangle) \]
by Condition \([\text{FE2.d_0}]\).

\[ = \langle m_1 | m_4 \cdot (\langle n_3 | m_3 \cdot m_2 \rangle \cdot n_2 | n_1 \rangle) \]
by Condition \([\text{FE2.c}]\).

\[ = \langle m_1 | m_4 \cdot (\langle n_3 | m_3 \cdot m_2 \rangle \cdot n_2 | n_1 \rangle) \]
by our computation above.

The other equality follows similarly. \(\square\)

Lemma 5.9. Suppose we are given elements \(\xi \in \mathcal{M}_x \otimes \mathcal{N}_y \subseteq \mathcal{H}(x,y), \xi' \in \mathcal{M}_x \otimes \mathcal{N}_y \subseteq \mathcal{H}(x',y')\).

1. If \(s_Y(y) = s_Y(y')\), then the following inequality of positive elements in the \(C^*\)-algebra \(\mathcal{B}(r(x))\) holds:
\[
\langle \xi | \xi' \rangle_{\mathcal{B}} \leq \langle \xi | \xi' \rangle_{\mathcal{B}}^* \mathcal{B} \langle \xi | \xi \rangle_{\mathcal{B}}.
\]

2. If \(r_X(x) = r_X(x')\), then the following inequality of positive elements in the \(C^*\)-algebra \(\mathcal{D}(s(y))\) holds:
\[
\langle \xi | \xi' \rangle_{\mathcal{D}} \leq \langle \xi | \xi' \rangle_{\mathcal{D}}^* \mathcal{D} \langle \xi | \xi \rangle_{\mathcal{D}}.
\]

In both cases, the norm on the right-hand side is on the bi-Hilbert module \(\mathcal{M}_x \otimes \mathcal{N}_y\) resp. \(\mathcal{M}_x \otimes \mathcal{N}_y\).

Proof. We will prove the claim for the \(\mathcal{B}\)-valued inner product. By Lemma 5.8, we can rewrite the left-hand side of our inequality as
\[
\langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi | \xi' \rangle_{\mathcal{B}}^* = \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}} = \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}},
\]
so our claim becomes
\[
\langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi | \xi' \rangle_{\mathcal{B}}^* \leq \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}}.
\]
Let \(z := (x, y)\) and \(z := (x', y')\). Notice that by Lemma 5.8(2) and Condition \([\text{GP3.b}]\), \(\langle \xi' | \xi' \rangle_{\mathcal{B}}\) is an element of the fibre of \(\mathcal{B}\) over \(R^z(\xi' | \xi') = s_Z(z')\). Moreover, since
\[
q_X \left( \langle \xi | \xi' \rangle_{\mathcal{B}}^* \right) = q_X(\xi),
\]
we similarly have that \(\langle \xi | \xi' \rangle_{\mathcal{B}}^* \leq \langle \xi | \xi' \rangle_{\mathcal{B}}\) is an element of the fibre of \(\mathcal{B}\) over \(L^z(\xi | \xi) = r_Z(z)\). Thus, we have identified that the inner products in question are simply the inner products as in Equation \([\text{FE3}]\) on the spaces \(K(z) = \mathcal{M}_x \otimes C_{s_X(z)}(x) \mathcal{N}_y\) and \(K(z')\), which are imprimitivity bimodules by Assumption \([\text{FE3}]\) on \(\mathcal{M}\) and \(\mathcal{N}\):
\[
\langle \xi | \xi' \rangle_{\mathcal{B}} \leq \langle \xi | \xi' \rangle_{\mathcal{B}}^* \leq \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}}.
\]
The claim now follows from \([\text{S} \ Cor. 2.22]\) and \([\text{S} \ Prop. 3.16]\). \(\square\)

Corollary 5.10. If \(\xi, \xi' \in \mathcal{H}\), then \(\langle \xi | \xi' \rangle_{\mathcal{B}} \leq \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}}\) and \(\langle \xi | \xi' \rangle_{\mathcal{B}} \leq \langle \xi | \xi' \rangle_{\mathcal{B}} \langle \xi' | \xi \rangle_{\mathcal{B}}\), whenever the left-hand side of the inequality makes sense.
Proposition 5.11. There exist continuous sesquilinear maps $\langle \cdot | \cdot \rangle$ on $\mathcal{H}_\cdot \mathcal{H}$ and $\langle \cdot | \cdot \rangle_\sigma$ on $\mathcal{H}_\cdot \mathcal{K}$ which are determined by

$$\langle m \otimes n | m' \otimes n' \rangle := \langle m | m' \cdot \varphi(n' | n) \rangle$$

and

$$\langle m \otimes n | m' \otimes n' \rangle_\sigma := \langle (m' | m) \cdot n' | n \rangle_\sigma.$$ 

These maps further satisfy for all $\xi \in K$, $b \in B$, $d \in D$ where it makes sense:

(a) $p_\mathcal{H}(\langle \xi_1 | \xi_2 \rangle) = L_2^x(q_x(\xi_1) | q_x(\xi_2))$ and $p_\mathcal{H}(\langle \xi_1 | \xi_2 \rangle_\sigma) = R_2^x(q_x(\xi_1) | q_x(\xi_2))$,

(b) $\langle \xi_1 | \xi_2 \rangle^* = \langle \xi_2 | \xi_1 \rangle$ and $\langle \xi_1 | \xi_2 \rangle_\sigma^* = \langle \xi_2 | \xi_1 \rangle_\sigma$, and

(c) $\langle b \cdot \xi_1 | \xi_2 \rangle = b \cdot \langle \xi_1 | \xi_2 \rangle$ and $\langle \xi_1 | \xi_2 \cdot d \rangle = \langle \xi_1 | \xi_2 \rangle d$.

(d) $\langle \xi_1 | \xi_2 \rangle = \langle \xi_1 | \xi_2 \cdot d \rangle = \langle \xi_1 | \xi_2 \rangle_\sigma$ and $\langle b \cdot \xi_1 | \xi_2 \rangle = \langle \xi_1 | b \cdot \xi_2 \rangle$.

(e) If $q_x(\xi_2) = q_x(\xi_3)$, $\langle \xi_1 | \xi_2 \rangle_\sigma$ and $\langle \xi_1 | \xi_2 \rangle_\sigma^*$.

$\langle \xi_1 | \xi_2 \rangle_\sigma = \langle \xi_1 | \xi_2 \rangle_\sigma$, and $\langle \xi_1 | \xi_2 \rangle_\sigma = \langle \xi_1 | \xi_2 \rangle_\sigma$.

Proof. As always, we will only deal with the left (pre-)inner product $\langle \cdot | \cdot \rangle$. We have seen in Lemma 5.7 for $(x, y), (x', y') \in Z$ with $s_Y(y) = s_Y(y')$ that there exists a map

$$\langle \cdot | \cdot \rangle : (\mathcal{M}_x \otimes \mathcal{N}_y) \times (\mathcal{M}_x \otimes \mathcal{N}_y) \rightarrow B_g$$

with the defining property. Note that the maps $F_{m', n'}$ that we used in its definition through the $\mathcal{C}^{(0)}$-balancing because of Condition (FE2.d) for $\mathcal{N}$:

$$F_{m', n'}((m \cdot c) \otimes n) = \langle m \cdot c | m' \cdot \varphi(n' | n) \rangle = \langle m | (m' \cdot \varphi(n' | n) \cdot c^*) \rangle = \langle m | (m' \cdot \varphi(c \cdot n) | n) \rangle = F_{m', n'}(m \otimes (c \cdot n)),$$

Similarly, the map $F^*$ factors through the $\mathcal{C}^{(0)}$-balancing because of Condition (FE2.c) for $\mathcal{N}$:

$$F^*((m \cdot c) \otimes n') = \langle m | (m' \cdot \varphi(n' | n) \rangle^* = \langle m | m' \cdot \varphi(c \cdot n' | n) \rangle^* = (m' \cdot (c \cdot n' | n)) (m \otimes n)$$

Consequently, we may define the pre-inner product on the balanced spaces,

$$\langle \cdot | \cdot \rangle : (\mathcal{M}_x \otimes \mathcal{N}_y) \times (\mathcal{M}_x \otimes \mathcal{N}_y) \rightarrow B_g.$$

By Corollary 5.10, we have $\| \langle \xi | \xi' \rangle \| \leq \| \xi \|_\mathcal{H} \| \xi' \|_\mathcal{H}$, which shows that this map then extends to a sesquilinear map

$$\langle \cdot | \cdot \rangle : \mathcal{H}_\cdot \mathcal{H} \rightarrow B_g$$

which also satisfies $\| \langle \xi | \xi' \rangle \| \leq \| \xi \| \| \xi' \|$. Since $(x, y), (x', y')$ were arbitrary with $s_Y(y) = s_Y(y')$, we get the map

$$\langle \cdot | \cdot \rangle : \mathcal{H}_\cdot \mathcal{H} \rightarrow B_g.$$ 

To see that this map is continuous, assume that $(\xi_\lambda, \xi'_\lambda)$ is a net in $\mathcal{H}_\cdot \mathcal{H}$ converging to $(\xi, \xi')$. Let $q_x(\xi_\lambda, \xi'_\lambda) = (x_\lambda, y_\lambda, x'_\lambda, y'_\lambda)$ and $q_x(\xi, \xi') = (x, y, x', y')$, and write $g_\lambda := L_2^x(x_\lambda | x'_\lambda \cdot L_2^y(y'_\lambda | y_\lambda))$ and $g := L_2^x(x | x' \cdot L_2^y(y' | y))$. By continuity of $q_x^*$, of $L_2(\cdot | \cdot)$, of the $\mathcal{H}$-action on $X$, and of $L_2^x(\cdot | \cdot)$, it follows that $g_\lambda \rightarrow g$. Fix $\epsilon > 0$. By Lemma 3.2, it suffices to fix a section $\sigma$ of $\mathcal{H}$ for which

$$(5.10) \quad \| \langle \xi | \xi' \rangle - \sigma(g) \| < \frac{\epsilon}{3}$$

and to show that there exists $\lambda_0$ such that for all $\lambda \geq \lambda_0$, we have $\| \langle \xi_\lambda | \xi'_\lambda \rangle - \sigma(g_\lambda) \| < \epsilon$. The proof will follow the same ideas as that of continuity of the left $\mathcal{B}$-action on $\mathcal{H}$ (see Proposition 5.5). Fix sections $\tau_1, \tau'_1$ of $\mathcal{M}$ and $\kappa_i, \kappa'_i$ of $\mathcal{N}$ such that for the sections $\alpha := \sum_i \tau_i \otimes \kappa_i$ and $\alpha' := \sum_i \tau'_i \otimes \kappa'_i$ of $\mathcal{H}$, we have

$$(5.11) \quad \| \xi - \alpha(x, y) \| < \frac{\epsilon}{6(1 + \| \xi' \|)} \text{ and } \| \xi' - \alpha'(x', y') \| < \frac{\epsilon}{6(1 + \| \alpha(x, y) \|)}.$$
We write

$$\langle \alpha(x, y) \mid \alpha'(x', y') \rangle = \langle \alpha(x, y) \mid \alpha'(x', y') - \xi' \rangle + \langle \alpha(x, y) - \xi \mid \xi' \rangle + \langle \xi \mid \xi' \rangle.$$  

Using the triangle inequality and that $$\| \langle \eta \mid \eta' \rangle \| \leq \| \eta \| \| \eta' \|$$, the choice of $$\sigma$$ in Equation (5.10) and of $$\alpha$$ and $$\alpha'$$ in Equation (5.11) then implies

$$\| \langle \alpha(x, y) \mid \alpha'(x', y') \rangle - \sigma(g) \| \leq \| \alpha(x, y) \| \| \alpha'(x', y') - \xi' \| + \| \alpha(x, y) - \xi \| \| \xi' \| + \| \langle \xi \mid \xi' \rangle - \sigma(g) \|$$

(5.12)

$$< \frac{2\epsilon}{3}.$$  

As $$\xi_\lambda \to \xi, \xi'_\lambda \to \xi'$$ in $$\mathcal{X}$$ and as $$\alpha, \alpha'$$ are continuous, we have by Lemma A.3 [1] that there exists $$\lambda$$ such that for all $$\lambda \geq \lambda_1$$, we have

$$\| \xi_\lambda \| \leq \| \xi \|$$  and  $$\| \xi'_\lambda \| \leq \| \xi' \|$$

(5.13)

$$\| \alpha(x_\lambda, y_\lambda) \| \leq \| \alpha(x, y) \|$$  and  $$\| \alpha(x'_\lambda, y'_\lambda) \| \leq \| \alpha(x', y') \|$$

(5.14)

$$\| \xi_\lambda - \alpha(x_\lambda, y_\lambda) \| \leq \| \xi - \alpha(x, y) \|$$  and  $$\| \xi'_\lambda - \alpha(x'_\lambda, y'_\lambda) \| \leq \| \xi' - \alpha(x', y') \|.$$  

This time, we write

$$\langle \xi_\lambda \mid \xi'_\lambda \rangle = \langle \xi_\lambda - \alpha(x_\lambda, y_\lambda) \mid \xi'_\lambda \rangle + \langle \alpha(x_\lambda, y_\lambda) \mid \xi'_\lambda - \alpha(x'_\lambda, y'_\lambda) \rangle + \langle \alpha(x_\lambda, y_\lambda) \mid \alpha(x'_\lambda, y'_\lambda) \rangle,$$

so that similarly to before in Inequality (5.12), we get

$$\| \langle \xi_\lambda \mid \xi'_\lambda \rangle - \sigma(g_\lambda) \| \leq \| \xi_\lambda - \alpha(x_\lambda, y_\lambda) \| \| \xi'_\lambda \| + \| \alpha(x_\lambda, y_\lambda) \| \| \xi'_\lambda - \alpha(x'_\lambda, y'_\lambda) \| + \| \alpha(x_\lambda, y_\lambda) \| \| \alpha(x'_\lambda, y'_\lambda) - \sigma(g_\lambda) \|.$$  

By the second inequality in (5.13), the first in (5.15), and by choice of $$\alpha$$ in (5.11), the first summand for $$\lambda \geq \lambda_1$$ is bounded by

$$\| \xi_\lambda - \alpha(x_\lambda, y_\lambda) \| \| \xi'_\lambda \| \leq \| \xi - \alpha(x, y) \| \| \xi' \| < \frac{\epsilon}{6}.$$  

Similarly, by the first inequality in (5.14), the second in (5.13), and by choice of $$\alpha'$$ in (5.11), the second summand for $$\lambda \geq \lambda_2$$ is bounded by

$$\| \alpha(x_\lambda, y_\lambda) \| \| \xi'_\lambda - \alpha(x'_\lambda, y'_\lambda) \| \leq \| \alpha(x, y) \| \| \xi' - \alpha(x', y') \| < \frac{\epsilon}{6}.$$  

Using the last two inequalities in the preceding one, we have for $$\lambda \geq \lambda_1$$

$$\| \langle \xi_\lambda \mid \xi'_\lambda \rangle - \sigma(g_\lambda) \| \leq \frac{\epsilon}{3} + \| \alpha(x_\lambda, y_\lambda) \mid \alpha(x'_\lambda, y'_\lambda) \rangle - \sigma(g_\lambda) \|.$$  

(5.16)

By definition of the inner product and of $$\alpha$$ and $$\alpha'$$, we have

$$\langle \alpha(x_\lambda, y_\lambda) \mid \alpha(x'_\lambda, y'_\lambda) \rangle = \sum_{i,j} \langle \tau_i(x_\lambda) \mid \tau'_j(x'_\lambda) \cdot \varphi(k'_j(y'_\lambda) \mid k_i(y)) \rangle.$$  

By continuity of all maps involved in the right-hand side, we know that this converges to

$$\langle \alpha(x, y) \mid \alpha(x', y') \rangle = \sum_{i,j} \langle \tau_i(x) \mid \tau'_j(x') \cdot \varphi(k'_j(y') \mid k_i(y)) \rangle$$

in $$\mathcal{B}$$. Thus, there exists $$\lambda_2$$ such that for all $$\lambda \geq \lambda_2$$, we have

$$\| \langle \alpha(x_\lambda, y_\lambda) \mid \alpha(x'_\lambda, y'_\lambda) \rangle - \sigma(g_\lambda) \| \leq \| \langle \alpha(x, y) \mid \alpha(x', y') \rangle - \sigma(g) \| < \frac{2\epsilon}{3}.$$  

Combining the above with Inequality (5.16), we arrive at $$\| \langle \xi_\lambda \mid \xi'_\lambda \rangle - \sigma(g_\lambda) \| < \epsilon$$ for all $$\lambda \geq \lambda_1, \lambda_2$$, as claimed. The map $$\langle \cdot, \cdot \rangle$$ is hence continuous.
It remains to check Conditions \([a], [e]\). Condition \([a]\) follows directly from Lemma \(5.6\). Condition \([b]\) follows from Lemma \(5.7\) and Condition \([e]\) from Lemma \(5.8\). For the other two, it suffices to restrict our attention to elementary tensors \(\xi = m \otimes n\) and \(\xi' = m' \otimes n'\): For \([c]\) we compute
\[
\langle b \cdot \xi \mid \xi' \rangle = \langle (b \cdot m) \otimes n \mid m' \otimes n' \rangle \quad \text{by def. of the action on } K
\]
\[
= \langle b \cdot m \mid m' \cdot \varphi(n') \rangle \quad \text{by def. of the inner product on } K
\]
\[
= b_g \langle m \mid m' \cdot \varphi(n') \rangle \quad \text{by } (FE2.c) \text{ for } M
\]
\[
= b_g \langle \xi \mid \xi' \rangle.
\]
Lastly, for \([d]\) we compute
\[
\langle \xi \cdot d \mid \xi' \rangle = \langle m \otimes (n \cdot d) \mid m' \otimes n' \rangle \quad \text{by def. of the action on } K
\]
\[
= \langle m \mid m' \cdot \varphi(n \cdot d) \rangle \quad \text{by def. of the inner product on } K
\]
\[
= \langle m \mid m' \cdot \varphi(n' \cdot d^*) \rangle \quad \text{by Cond. } (FE2.d) \text{ for } N
\]
\[
= \langle m \otimes n \mid m' \otimes (n' \cdot d^*) \rangle \quad \text{by def. of the inner product on } K
\]
\[
= \langle \xi \mid \xi' \cdot d^* \rangle.
\]
This concludes our proof. \(\square\)

We can now come to our first main theorem:

**Theorem 5.12.** Assume we are given

- three saturated Fell bundles \(B, C, D\) over locally compact Hausdorff étale groupoids \(G, H, K\), respectively;
- a \((G, H)\)-pre-equivalence \(X\) and an \((H, K)\)-pre-equivalence \(Y\);
- a \((B, C)\)-hypo-equivalence \(M\) over \(X\) and a \((C, D)\)-hypo-equivalence \(N\) over \(Y\).

Let \(K\) be the bundle over the \((G, H)\)-pre-equivalence \(X \ast_y Y\) defined in Lemma \(5.1\). Then \(K\) is a \((B, D)\)-hypo-equivalence when equipped with the actions defined in Proposition \(5.5\) and the inner products defined in Proposition \(5.11\).

**Proof.** In this proof, all Items are references to Definition \(2.11\) unless otherwise specified.

We have seen in Lemma \(3.4\) that \(Z = X \ast_y Y\) is a groupoid pre-equivalence, and in Lemma \(5.1\) that \(K\) is a USC Banach bundle over \(Z\). By Proposition \(5.5\), \(K\) carries commuting \(B\)- and \(D\)-actions, so Item \((FE1)\) is satisfied. Conditions \([b]\) and \([c]\) in Proposition \(5.11\) prove Items \((FE2.b)\) and \((FE2.c)\), respectively, while Conditions \([a]\), \([d]\), and \([e]\) in the same proposition prove Items \((FE2.a)\), \((FE2.d)\), and \((FE2.e)\) of Definition \(4.4\).

Lastly, the fibre of \(K\) over \((x, y) \in Z\) is \(M_x \otimes_{C(u)} N_y\) with \(u = s_X(x) = r_Y(y)\); as \(M_x\) is a \(B(r_X(x)) - C(u)\)-imprimitivity bimodule and \(N_y\) is a \(C(u) - D(s_Y(y))\)-imprimitivity bimodule by Item \((FE3)\) for \(M\) resp. \(N\), we conclude that \(K_{(x, y)}\) is a \(B(r_X(x)) - D(s_Y(y))\)-imprimitivity bimodule, as needed for \(K\) for Item \((FE3)\). \(\square\)

**Remark 5.13.** One may be tempted to say that, if \(X = \mathcal{X}\) and \(Y = \mathcal{Y}\) are equivalences of groupoids and \(M\) and \(N\) equivalences of Fell bundles, then \(K\) is a \((B, D)\)-equivalence. Unfortunately, that is not the case: Just like one has to take a quotient of \(\mathcal{X} \ast_{\mathcal{Y}} \mathcal{Y}\) to make the groupoid equivalence \(\mathcal{X} \ast_{\mathcal{Y}} \mathcal{Y}\), we will have to take a quotient of \(K\) to get a Fell bundle equivalence.

As \(Z = \mathcal{X} \ast_{\mathcal{Y}} \mathcal{Y}\) is not a \(G - K\) equivalence as pointed out in Remark \(3.5\), it does not admit a map \(\mathcal{Y}_h\{ \cdot | \cdot \}\); in particular, we cannot ask for \(K\) to satisfy Item \((FE2.a)\). But we can remedy that to some extent: Suppose we are given \(m \otimes n \in K_{(x, y)}\) and \(m' \otimes n' \in K_{(x', y')}\) where \(r_X(x) = r_X(x')\) and \(s_Y(y) = s_Y(y')\). Let \(h := \mathcal{Y}_h\{ y' \mid y \}\), the unique element of \(H\) such that \(y' = h \cdot y\), and let \(g := \mathcal{X}_g\{ x \mid x' \cdot h\}\), the unique element of \(G\) such that \(x = g \cdot (x' \cdot h)\). Then
\[
(q_M(m), q_K(n)) = (x, y) = (g \cdot (x' \cdot h), h^{-1} \cdot y').
\]
If $[\cdot]_\mathcal{H}$ denotes the quotient map $Z = \mathcal{X} \ast \mathcal{Y} \to \mathcal{X} \ast \mathcal{H} \mathcal{Y} =: Z$, then we at least have

$$\gamma([m \otimes n])_\mathcal{H} = \gamma([(x, y)])_\mathcal{H} = \gamma([(x', y')])_\mathcal{H} = \gamma([x, y]_\mathcal{H} [x', y']_\mathcal{H})_\mathcal{H} = \gamma([x, y]_\mathcal{H} [x', y']_\mathcal{H})_\mathcal{H},$$

Remark 5.14. Just as with balanced tensor products of Hilbert $C^*$-modules, one can show that the above construction of taking a balanced tensor product of Fell bundle hypo-equivalences is associative. To be precise, if $V, X, Y$ are groupoid pre-equivalences such that $V \ast X$ and $X \ast Y$ make sense, and if $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are hypo-equivalences over those spaces, then there is a natural isomorphism

$$(\mathcal{L} \otimes_{\mathcal{G}(0)} \mathcal{M}) \otimes_{\mathcal{E}(0)} \mathcal{N} \simeq \mathcal{L} \otimes_{\mathcal{G}(0)} (\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N})$$

of USC Banach bundles.

5.3. Some isomorphic bundles. As before, we will let $\mathcal{K}$ denote $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$, the Fell bundle hypo-equivalence described in Theorem 5.12. The goal of this subsection is to construct a map that allows us to ‘collapse’ certain fibres of $\mathcal{K}$: We intend to mimic the balancing over $C(u)$ in the tensor product $M_x \otimes N_y$, where $s_x(x') = u = r_Y(y')$, and extend it to a balancing over $C(h)$ for any $h \in \mathcal{H}$. In other words, we would like to identify the fibres $M_x \otimes C(u)N_y$ and $M_{xh} \otimes C(v)N_y$, where $u = r_h(h)$ and $v = s_h(h)$. A naïve approach would be to mod out by a space spanned by all elements of the form

$$(m \cdot c) \otimes n - m \otimes (c \cdot n),$$

where $c \in C(h)$ for some $h$. However, this is doomed to fail: the displayed difference of elementary tensors does not make sense since the two summands live in distinct vector spaces, $M_x \otimes C(u)N_y$ resp. $M_{xh} \otimes C(v)N_y$. Therefore, we must construct a function on $\mathcal{K}$ that allows us to identify these fibres; see Theorem 5.17.

Definition 5.15. Let $t_Z: Z = X \ast Y \to \mathcal{H}(0)$ be defined by $t_Z(x, y) = s_Z(x) = r_Y(y)$, and let $t_{\mathcal{K}} := t_Z \circ q_{\mathcal{K}}$. Note that $t_Z(g \cdot x, y) = t_Z(x, y) = t_Z(x, y \cdot k)$ for all $g \in G$ and $k \in K$ where it makes sense.

Lemma 5.16. There exists an isomorphism

$$(\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{G}) \otimes_{\mathcal{E}(0)} \mathcal{N} \xrightarrow{\Omega} \mathcal{H} \ast \mathcal{K}$$

of USC Banach bundles determined by

$$(m \otimes c) \otimes n \mapsto (p_\mathcal{E}(c), (m \cdot c) \otimes n)$$

covering the homeomorphism

$$\omega: (X \ast \mathcal{H}) \ast Y \to \mathcal{H} \ast (X \ast Y), \ (x, h, y) \mapsto (h, x \cdot h, y).$$

Here, $\mathcal{H} \ast \mathcal{K}$ is as defined in Definition 2.6.

In the above, we have tacitly used Theorem 5.12. Since $\mathcal{M}$ and $\mathcal{G}$ are both hypo-equivalences of Fell bundles, it follows that $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{G}$ is also a hypo-equivalence, namely over the $(G, \mathcal{H})$-pre-equivalence $X \ast \mathcal{H}$; otherwise, we would not be able to consider $(\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{G}) \otimes_{\mathcal{E}(0)} \mathcal{N}$.  

Proof. For this proof, let \( \mathcal{L} := (\mathcal{M} \otimes \mathcal{E}(0)) \otimes \mathcal{N} \). The fibre of \( \mathcal{L} \) at some element \((x, h, y)\) of \((X_s \ast \mathcal{H})_s \ast Y\) is by definition (Lemma \[\text{5.1}\]) exactly

\[
\mathcal{L}_{(x, h, y)} = (\mathcal{M} \otimes \mathcal{E}(0))_{(x, h)} \otimes \mathcal{N}_y.
\]

By another application of the same lemma (this time with \( \mathcal{N} := \mathcal{E} \) and \( Y := \mathcal{H} \)), we further have

\[
(\mathcal{M} \otimes \mathcal{E}(0))_{(x, h)} = \mathcal{M}_x \otimes \mathcal{E}(0)_{C_h}.
\]

As a consequence of Remark \[\text{4.3}\] the fibrewise map determined by \((m \otimes c) \otimes n \mapsto (m \cdot c) \otimes n\) is not only well defined but induces an isomorphism

\[
\Omega_{(x, h, y)} : \mathcal{L}_{(x, h, y)} = (\mathcal{M} \otimes \mathcal{E}(0))_{(x, h)} \otimes \mathcal{N}_y \cong \{h\} \times \mathcal{M}_x \otimes \mathcal{E}(0)_{C_h} = (\mathcal{H}_s \ast \mathcal{K})_{\omega(x, h, y)}
\]

of imprimitivity bimodules. In particular, \( \|\Omega_{(x, h, y)}(\xi)\| = \|\xi\| \) for all \( \xi \in \mathcal{L}_{(x, h, y)} \). These fibrewise maps can be ‘stitched together’ to a global, bijective map \( \Omega \). To see that \( \Omega \) is an isomorphism of Banach bundles, we will show that \( \Omega \) is continuous; from Proposition \[\text{A.8}\] and from the fact that \( \Omega \) is isometric, it will follow that \( \Omega \) is isometric, proving that \( \Omega \) is the claimed isomorphism.

For continuity of \( \Omega \), Proposition \[\text{A.7}\] asserts that it suffices to find a family \( \Gamma \) of continuous cross-sections of \( \mathcal{L} \) such that

1. for each \((x, h, y) \in (X_s \ast \mathcal{H})_s \ast Y\), the linear span of \( \{\gamma(x, h, y) \mid \gamma \in \Gamma\} \) is dense in \( \mathcal{L}_{(x, h, y)} \), and

2. for all \( \gamma \in \Gamma \), \( \Omega \circ \sigma \circ \omega^{-1} \) is a continuous cross-section of \( \mathcal{H}_s \ast \mathcal{K} \).

We let \( \Gamma \) be the collection of elements of the form \( \sigma := (\sigma_\mathcal{M} \otimes \sigma_\mathcal{E}) \otimes \sigma_\mathcal{N} \) for \( \sigma_\mathcal{M} \in \Gamma_0(X; \mathcal{M}) \) etc.; these cross-sections are, by construction of the topology on \( \mathcal{L} \), indeed continuous and satisfy \[\text{[1]}\]

For \(\text{[2]}\), suppose that \((h_\lambda, x_\lambda, y_\lambda) \to (h, x, y)\) is a convergent net in \((X_s \ast \mathcal{H})_s \ast Y\). We have to show that, for any fixed \( \sigma \) as above, the net

\[
s_\lambda := (\Omega \circ \sigma \circ \omega^{-1})(h_\lambda, x_\lambda, y_\lambda)
\]

converges to

\[
s := (\Omega \circ \sigma \circ \omega^{-1})(h, x, y).
\]

We compute:

\[
s = (\Omega \circ \sigma \circ \omega^{-1})(h, x, y) = (\Omega \circ \sigma)(x \cdot h^{-1}, h, y) = \Omega\big((\sigma_\mathcal{M}(x \cdot h^{-1}) \otimes \sigma_\mathcal{E}(h)) \otimes \sigma_\mathcal{N}(y)\big)
\]

and similarly

\[
s_\lambda = (h, [\sigma_\mathcal{M}(x \cdot h_\lambda^{-1}) \cdot \sigma_\mathcal{E}(h_\lambda)] \otimes \sigma_\mathcal{N}(y_\lambda)).
\]

We know by continuity of \( \sigma_\mathcal{M} \), \( \sigma_\mathcal{E} \), and the right-action of \( \mathcal{H} \) on \( X \) that

\[
m_\lambda := \sigma_\mathcal{M}(x_\lambda \cdot h_\lambda^{-1}) \cdot \sigma_\mathcal{E}(h_\lambda) \to \sigma_\mathcal{M}(x \cdot h^{-1}) \cdot \sigma_\mathcal{E}(h) =: m \quad \text{in } \mathcal{M}, \quad \text{and}
\]

\[
n_\lambda := \sigma_\mathcal{N}(y_\lambda) \to \sigma_\mathcal{N}(y) =: n \quad \text{in } \mathcal{N}.
\]

By Lemma \[\text{5.3}\] it follows that \( m_\lambda \otimes n_\lambda \to m \otimes n \) in \( \mathcal{K} \). By Lemma \[\text{A.3} \quad \text{[1]} \quad \implies \quad \text{[3]}\] we conclude that for any \( \tau_\mathcal{M} \in \Gamma_0(X; \mathcal{M}) \) and \( \tau_\mathcal{N} \in \Gamma_0(X; \mathcal{N}) \),

\[
\lim \|m_\lambda \otimes n_\lambda - (\tau_\mathcal{M} \otimes \tau_\mathcal{N})(x_\lambda, y_\lambda)\| \leq \|m \otimes n - (\tau_\mathcal{M} \otimes \tau_\mathcal{N})(x, y)\|.
\]

But note that

\[
\|m \otimes n - (\tau_\mathcal{M} \otimes \tau_\mathcal{N})(x, y)\|_{\mathcal{K}} = \|s - [\id(\tau_\mathcal{M} \otimes \tau_\mathcal{N})](h, x, y)\|_{\mathcal{H}_s \ast \mathcal{K}},
\]

and similarly for the version with subscript \( \lambda \). Thus, Equation \[\text{[5.17]}\] is exactly a proof of convergence \( s_\lambda \to s \) in \( \mathcal{H}_s \ast \mathcal{K} \). This proves continuity of \( \Omega \).

\[\square\]
We arrive at the following commutative diagram:

$$\Lambda: \mathcal{M} \otimes_{\mathcal{E}(0)} (\mathcal{C} \otimes_{\mathcal{E}(0)} \mathcal{N}) \to \mathcal{K} \otimes_{\mathcal{E}(0)} \mathcal{H}$$

$$m \otimes (c \otimes n) \mapsto (m \otimes (c \cdot n), p_{\mathcal{E}(0)}(c))$$

of USC Banach bundles, covering the homeomorphism

$$\lambda: (X \otimes_{\mathcal{H}} Y) \to (X \otimes_{\mathcal{H}} Y)^{\mathbb{T}} \otimes_{\mathcal{H}} \mathcal{H}, \quad (x, h, y) \mapsto (x, h \cdot y, h).$$

We arrive at the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_{\mathbb{T}}(X \otimes_{\mathcal{H}} Y) & \xrightarrow{\omega} & (X \otimes_{\mathcal{H}} Y) \\
\downarrow \psi & & \downarrow \\
\mathcal{H}_{\mathbb{T}}(\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}) & \xrightarrow{\Psi} & (\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N})_{\mathbb{T}} \\
\downarrow \sigma & & \downarrow \\
\mathcal{H}_{\mathbb{T}}^*(X \otimes_{\mathcal{H}} Y) & \xrightarrow{\lambda} & X \otimes_{\mathcal{H}} Y
\end{array}$$

In particular, since the vertical maps and the horizontal map at the top are isomorphisms, it follows that there is a unique isomorphism on the bottom making the diagram commute. To sum up:

**Theorem 5.17.** Suppose \( \mathcal{M} \) and \( \mathcal{N} \) are hypo-equivalences and let \( \mathcal{K} = \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N} \) be the hypo-equivalence constructed in Theorem 5.12. Then the map

$$\Psi: \mathcal{H}_{\mathbb{T}}^*(\mathcal{K}) \to \mathcal{H}_{\mathbb{T}}^*(\mathcal{H})$$

$$(h, \xi) \mapsto (\Psi_h(\xi), h)$$

determined by \( \Psi_h((m \cdot c) \otimes n) = m \otimes (c \cdot n) \) for \( c \in \mathcal{C}_h \),

is an isomorphism of USC Banach bundles covering the homeomorphism

$$\psi := \lambda \circ \omega^{-1}: \mathcal{H}_{\mathbb{T}}(X \otimes_{\mathcal{H}} Y) \to (X \otimes_{\mathcal{H}} Y)^{\mathbb{T}} \otimes_{\mathcal{H}} \mathcal{H}$$

$$(h, x, y) \mapsto (xh^{-1}, hy, h).$$

The map \( \Psi \) has the following additional properties:

1. **(Ψ1)** If \( (h, \xi) \to (h, \xi) \) is a convergent net in \( \mathcal{H}_{\mathbb{T}}(\mathcal{K}) \), then \( \Psi_h(\xi) \to \Psi_h(\xi) \) in \( \mathcal{K} \).
2. **(Ψ2)** For appropriate \( x \in X, y \in Y \), and \( (h', h) \in \mathcal{H}(2) \), the composition of

$$\begin{align*}
\mathcal{M}_{xh'} \otimes_{\mathcal{E}(0)} \mathcal{N}_y & \xrightarrow{\Psi_{h'}} \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy'} \\
\mathcal{M}_{xh} \otimes_{\mathcal{E}(0)} \mathcal{N}_y & \xrightarrow{\Psi_h} \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy}
\end{align*}$$

coincides with

$$\begin{align*}
\mathcal{M}_{xh'} \otimes_{\mathcal{E}(0)} \mathcal{N}_y & \xrightarrow{\Psi_{h'h'}} \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy'} \\
\mathcal{M}_{xh} \otimes_{\mathcal{E}(0)} \mathcal{N}_y & \xrightarrow{\Psi_{h'h}} \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy}.
\end{align*}$$

3. **(Ψ3)** If \( h = v \in \mathcal{H}(0) \) is a unit, then \( \Psi_h \) is the identity on \( \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y \).
4. **(Ψ4)** For appropriate \( x, h, y \), the map

$$\Psi_h: \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y = \mathcal{M}_{(xh^{-1})h} \otimes_{\mathcal{E}(0)} \mathcal{N}_y \to \mathcal{M}_{xh^{-1}} \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy}$$

is inverse to

$$\Psi_{h^{-1}}: \mathcal{M}_{xh^{-1}} \otimes_{\mathcal{E}(0)} \mathcal{N}_{hy} \to \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_{h^{-1}(hy)} = \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y.$$
(Ψ5) The map $\Psi$ is bi-equivariant in the following sense: if $(h, \xi) \in H \ast, K$ and if $b \in B$ and $d \in D$ are such that $s \cdot (b) = r \cdot (\xi)$ and $s \cdot (\xi) = r \cdot (d)$, then

$$\Psi_h(b \cdot \xi) = b \cdot \Psi_h(\xi) \quad \text{and} \quad \Psi_h(\xi \cdot d) = \Psi_h(\xi) \cdot d.$$ 

In other words, if $p \cdot (b) = g \in G$ and $p \cdot (d) = k \in K$, then the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{M}_{x_k} \otimes N_y & \xrightarrow{\Psi_h} & \mathcal{M}_x \otimes N_{h_k} \\
\mathcal{M}_{(g \cdot x_k)} \otimes N_y & \xrightarrow{\Psi_h} & \mathcal{M}_{g \cdot x_k} \otimes N_{(g \cdot h_k)}
\end{array}$$

In Item (Ψ5), we made use of the fact that our computations in Definition 5.13 imply

$$t \cdot (b \cdot \xi) = t \cdot (p \cdot (b) \cdot q \cdot (\xi)) = t \cdot (q \cdot (\xi)) = t \cdot (\xi),$$

so that $\Psi_h(b \cdot \xi)$ makes sense if and only if $b \cdot \Psi_h(\xi)$ makes sense; similarly for $\xi \cdot d$.

Proof. The existence of the isomorphism follows from our prior discussion, so we only have to check the additional properties.

Re (Ψ1): $\Psi$ is continuous by construction as the concatenation of continuous functions. The claim now follows directly from the definition of the topology of the bundles involved.

Re (Ψ2): For $m \in M_x, c_1 \in C_h, c_2 \in C_h, n \in N_y$, we have

$$\Psi_h((mc_1) c_2 \otimes n) = mc_1 \otimes c_2 n.$$ 

As this is an element of $(M_x \cdot C_h) \otimes (C_h \cdot N_y)$, we know how $\Psi_{h'}$ acts:

$$\Psi_{h'}(mc_1 \otimes c_2 n) = m \otimes c_1 c_2 n.$$ 

On the other hand,

$$\Psi_{h \cdot h'}(mc_1 c_2 \otimes n) = m \otimes (c_1 c_2) n = \Psi_{h'} \circ \Psi_h((mc_1) c_2 \otimes n).$$

Using linearity, we conclude that $\Psi_{h \cdot h'}$ and $\Psi_{h'} \circ \Psi_h$ coincide on the subspace $(M_x \cdot C_h) \otimes (C_h \cdot N_y)$ of $M_x \otimes C_h \otimes C(v) \otimes N_y$, which is dense by Remark 4.3. They thus agree everywhere.

Re (Ψ3): Follows since the module is balanced over $C(v) = C(v)$.

Re (Ψ4): If $v = h^{-1} h$ and $u = g h^{-1}$, then $\Psi_{h^{-1} \circ h} = \Psi_v$ and $\Psi_h \circ \Psi_{h^{-1}} = \Psi_u$ by (Ψ2), both of which are the identity map according to (Ψ3).

Re (Ψ5): Since the actions on $\mathcal{M}$ and on $\mathcal{N}$ commute, we have

$$b \cdot \Psi_h ((mc) \otimes n) = b \cdot [m \otimes (c \cdot n)] = [b \cdot m] \otimes (c \cdot n) = \Psi_h([b \cdot m] \cdot c) \otimes n) = \Psi_h((b \cdot (mc) \otimes n),$$

and similarly for $d \in D$. 

6. Transitivity of Fell bundle equivalence

In this section, we prove that Fell bundle equivalence is transitive and hence an equivalence relation. Throughout this section, we make slightly stronger assumptions than in Section 5.

- We still fix three saturated Fell bundles $\mathcal{B} = (p \cdot B \to G), \mathcal{C} = (p \cdot C \to H), \mathcal{D} = (p \cdot D \to K)$ over locally compact Hausdorff étale groupoids $G, H, K$.
- This time, let $\mathcal{X}$ be a $(G, H)$- and $\mathcal{Y}$ be an $(H, K)$-equivalence; let $Z := X \ast, Y$, which is just a $(G, K)$-pre-equivalence.
- Similarly, we let $\mathcal{M} = (q \cdot M \to \mathcal{X})$ be a $(\mathcal{B}, \mathcal{C})$- and $\mathcal{N} = (q \cdot N \to \mathcal{Y})$ be a $(\mathcal{C}, \mathcal{D})$-equivalence.
- As before, we write $\cdot$ for the left and right actions on $\mathcal{M}, \mathcal{X}, \mathcal{N}$, and $\mathcal{Y}$.
Using the techniques developed in the last section, we will build a USC Banach bundle $\mathcal{P}$ over the known $(\mathcal{G}, \mathcal{K})$-groupoid equivalence

$$Z := \mathcal{X} \ast_{\mathcal{H}} \mathcal{Y} = \{[x, y]_\mathcal{H} : s_x(x) = r_y(y)\}$$

where $[\cdot]_\mathcal{H}: Z \to Z$ is the quotient map, defined by setting $[xh, h^{-1}y]_\mathcal{H} = [x, y]_\mathcal{H}$ for all $h \in \mathcal{H}^{s(x)}$. We will show that $\mathcal{P}$ is the alleged Fell bundle equivalence between $\mathcal{B}$ and $\mathcal{D}$.

### 6.1. The quotient bundle.

**Lemma 6.1.** Let $\xi, \xi' \in \mathcal{H} = \mathcal{M} \otimes_{\xi(0)} \mathcal{N}$ with $q_{\mathcal{X}}(\xi) = (x, y)$ and $q_{\mathcal{X}}(\xi') = (x', y')$. We define the relation $\mathcal{R}$ by

$$\xi \mathcal{R} \xi' : \iff \exists h \in \mathcal{H}^{s(x')}_{s(x)} \text{ such that } x = x'h, \ y = h^{-1}y', \ \Psi_h(\xi) = \xi'.$$

This defines a closed equivalence relation on $\mathcal{H}$.

**Proof.** For reflexivity, fix any $\xi$ with $q_{\mathcal{X}}(\xi) = (x, y)$. Take $h = s_x(x) \in \mathcal{H}^{(0)}$. As units act trivially on both $\mathcal{X}$ and $\mathcal{Y}$, we automatically have $x = xh$ and $y = h^{-1}y$. Furthermore, by Theorem 5.17 [ψ3] we have $\Psi_h(\xi) = \xi$ and thus $\xi \mathcal{R} \xi$.

For symmetry, suppose we have some $h$ such that $x = x'h$, $y = h^{-1}y'$, and $\Psi_h(\xi) = \xi'$. Then in particular, $xh^{-1} = x'$ and $h^{-1}y = y'$. By Theorem 5.17 [ψ4] we further have that $\Psi_h(\xi) = \xi'$ implies $\xi = \Psi_{h^{-1}}(\xi')$. This shows that $\xi \mathcal{R} \xi'$ implies $\xi' \mathcal{R} \xi$.

For transitivity, suppose that $\xi \mathcal{R} \xi'$ and $\xi' \mathcal{R} \xi''$; let $h$ implement the first and $h'$ implement the second. Since $s_x(x) = s(h) = r_y(y)$, $s_x(x') = r(h) = r_y(y')$, and similarly $s_x(x'') = r(h') = r_y(y'')$, we see that $r(h) = s(h')$, so $(h', h) \in \mathcal{H}^{(2)}$; let $k = h'h$. Then

$$x = x'h = (x''h')h = x''k, \quad \text{and}$$

$$y = h^{-1}y' = h^{-1}((h')^{-1}y'') = k^{-1}y''.$$

Since $\Psi_{h'} \circ \Psi_h = \Psi_k$ by Theorem 5.17 [ψ2], we lastly conclude that

$$\Psi_h(\xi) = \Psi_{h'} \circ \Psi_h(\xi) = \Psi_h(\xi') = \xi'',$$

as required.

To see that $\mathcal{R}$ is closed, suppose $\xi_\lambda, \xi'_\lambda \in \mathcal{M} \otimes_{\xi(0)} \mathcal{N}$ are such that $\xi_\lambda \mathcal{R} \xi'_\lambda$ and $(\xi_\lambda, \xi'_\lambda) \to (\xi, \xi')$; let $q_{\mathcal{X}}(\xi_\lambda) = (x_\lambda, y_\lambda)$, $q_{\mathcal{X}}(\xi'_\lambda) = (x', y')$, and analogously for the primed versions. For each $\lambda$, there exists $h_\lambda \in \mathcal{H}^{s(x'_\lambda)}_{s(x_\lambda)}$ such that $x_\lambda = x'_\lambda \cdot h_\lambda$ and $\Psi_{h_\lambda}(\xi_\lambda) = \xi'_\lambda$. Being an equivalence, $\mathcal{X}$ is a proper $\mathcal{H}$-space, and so the fact that $(x_\lambda' \cdot h_\lambda, x'_\lambda) = (x_\lambda, x'_\lambda)$ converges to $(x, x')$ implies that $h_\lambda$ converges to some $h \in \mathcal{H}$ which satisfies $x = x' \cdot h$. By uniqueness, this $h$ also satisfies $y = h^{-1} \cdot y'$. It follows from [ψ1] that $\xi'_\lambda = \Psi_{h_\lambda}(\xi_\lambda) \to \Psi_h(\xi)$. Since $\mathcal{M} \otimes_{\xi(0)} \mathcal{N}$ is Hausdorff, limits are unique, so that $\xi'_\lambda \to \xi'$ implies that $\Psi_h(\xi) = \xi'$. In other words, $\xi \mathcal{R} \xi'$.

**Definition 6.2.** Let $\mathcal{P}$ be the quotient of $\mathcal{M} \otimes_{\xi(0)} \mathcal{N}$ by the above equivalence relation, equipped with the quotient topology, and let $Q: \mathcal{M} \otimes_{\xi(0)} \mathcal{N} \to (\mathcal{M} \otimes_{\xi(0)} \mathcal{N})/\mathcal{R}$ denote the quotient map. The equivalence class $Q(\xi)$ of $\xi$ will be written as $[\xi]$.

**Remark 6.3.** By construction, if $\xi \in \mathcal{M} \otimes_{\xi(0)} \mathcal{N}$ with $q_{\mathcal{X}}(\xi) = (x, y)$, then for any $h \in \mathcal{H}$ such that $s_x(x) = r_H(h) = r_Y(y)$, we have $[\xi] = [\Psi_h(\xi)]$ in $\mathcal{P}$. In particular, for any $m \in \mathcal{M}, c \in \mathcal{C}, n \in \mathcal{N}$ with $s_m(m) = r_{\mathcal{C}}(c)$ and $s_n(n) = r_{\mathcal{N}}(n)$, we have $[(m \cdot c) \otimes n] = [m \otimes (c \cdot n)]$ in $\mathcal{P}$.

**Lemma 6.4.** The map $q_{\mathcal{P}}: \mathcal{P} \to Z := \mathcal{X} \ast_{\mathcal{H}} \mathcal{Y}$ given by $q_{\mathcal{P}}[\xi] := [q_{\mathcal{X}}(\xi)]$ is well defined, surjective, continuous, and open.
Proof. If \( \xi, \xi' \) with \( q_x(\xi) = (x, y) \) and \( q_x(\xi') = (x', y') \), then there exists \( h \in \mathcal{H} \) such that
\[ x = x'h, \quad \text{and} \quad y = h^{-1}y', \]
which exactly means that
\[ [q_x(\xi)]_h = [x, y]_h = [x'h, h^{-1}y']_h = [x', y']_h = [q_x(\xi')]_h \]
in \( Z \), which shows that \( q_\mathcal{P} \) is well defined. Surjectivity is clear. For continuity, assume \( U \) is an open set in \( Z \). In order to show that \( q_\mathcal{P}^{-1}(U) \) is open in \( \mathcal{H} \), we have to show that \( Q^{-1}(q_\mathcal{P}^{-1}(U)) \) is open in \( \mathcal{M} \otimes_{\mathcal{K}(0)} \mathcal{N} \). Since \( q_\mathcal{P}(Q(\xi)) = [q_x(\xi)]_h \), we have \( Q^{-1}(q_\mathcal{P}^{-1}(U)) = q_x^{-1}([\cdot]_h^{-1}(U)) \). As \([\cdot]_h^{-1}\) is continuous by definition of the topology on \( Z \), and as \( q_x: \mathcal{M} \otimes_{\mathcal{K}(0)} \mathcal{N} \to \mathcal{X}_* \ast \mathcal{Y} \) is continuous by construction of the topology on \( \mathcal{M} \otimes_{\mathcal{K}(0)} \mathcal{N} \), we see that \( Q^{-1}(q_\mathcal{P}^{-1}(U)) \) is indeed open.

Lastly, to see that \( q_\mathcal{P} \) is an open map, take any \( V \subseteq \mathcal{P} \) open; we have to show that \( q_\mathcal{P}(V) \) is open in \( Z \), i.e., that \([\cdot]_h^{-1}(q_\mathcal{P}(V)) \) is open in \( \mathcal{X}_* \ast \mathcal{Y} \). We claim that
\[ ([\cdot]_h^{-1}(q_\mathcal{P}(V)) \ni q_x^{-1}(V) ; \] this is sufficient for our claim because \( Q \) is continuous and \( q_x \) is open (see Lemma 5.4).

For \( \geq \), take \((x', y') \in q_x^{-1}(V) \), i.e., there exists some \( \xi' \in Q^{-1}(V) \) such that \((x', y') = q_x(\xi') \). As \( p' := Q(\xi') \in V \), it follows from
\[ [x', y']_h = [q_x(\xi')]_h = q_\mathcal{P}(p') \in q_\mathcal{P}(V) \]
that \((x', y') \in [\cdot]_h^{-1}(q_\mathcal{P}(V)) \). For \( \leq \), take \((x, y) \in [\cdot]_h^{-1}(q_\mathcal{P}(V)) \), i.e., there exists \( p \in V \) such that \( q_\mathcal{P}(p) = [x, y]_h \). By definition of \( q_\mathcal{P} \), any \( \xi \in Q^{-1}(\{p\}) \subseteq Q^{-1}(V) \) then has the property \([q_x(\xi)]_h = [x, y]_h \). If \( (x', y') := q_x(\xi) \), then that implies that there exists \( h \in \mathcal{H} \) with \( (xh, h^{-1}y) = (x', y') \), so \( \xi \in \mathcal{M}_{xh} \otimes_{\mathcal{K}(0)} \mathcal{N}_{h^{-1}y} \) and we may let \( \eta := \Psi_h(\xi) \in R x \otimes_{\mathcal{K}(0)} \mathcal{M}_y \). By construction we have \( \xi \otimes \eta \), so that \( Q(\eta) = Q(\xi) \in V \), i.e., \( \eta \in Q^{-1}(V) \) also. We have shown that
\[ (x, y) = q_x(\eta) \in q_x^{-1}(V), \]
proving Assertion (6.1) and hence the claim that \( q_\mathcal{P} \) is open. \( \square \)

**Proposition 6.5.** Any fibre \( \mathcal{P}_z := q_\mathcal{P}^{-1}(z) \) of \( \mathcal{P} \) is naturally a vector space. In fact, if \([\xi_1] \in \mathcal{P}_z \) for \( i = 1, 2 \), then there exists a unique \( h \in \mathcal{H} \) such that \( q_x(\xi_1) = q_x(\Psi_h(\xi_2)) \) and we can define
\[ [\xi_1] + [\xi_2] := [\xi_1 + \Psi_h(\xi_2)] \quad \text{and} \quad \lambda[\xi_1] := [\lambda \xi_1] \]
where \( \lambda \in \mathbb{C} \).

**Proof.** Let \( \xi_1, \xi_2 \) be any representatives in \( \mathcal{M} \otimes_{\mathcal{K}(0)} \mathcal{N} \) of the given elements of \( \mathcal{P} \). Let us first note that \( \lambda[\xi_1] = [\lambda \xi_1] \) is well defined as \( \Psi_h \) is \( \mathbb{C} \)-linear.

As \( [\xi_1] \in \mathcal{P}_z \), we have \([x_1, y_1]_h = z = [x_2, y_2]_h \), where \( q_x(\xi_1) = (x_1, y_1) \). Thus, there exists \( h \in \mathcal{H}_{x_1}(z) \) such that \( x_2 = x_1h \), and \( y_2 = h^{-1}y_1 \). By Remark 6.3, we have \([\xi_2] = [\Psi_h(\xi_2)] \). As \( \xi_1 \) and \( \Psi_h(\xi_2) \) are both elements of \( \mathcal{M}_{x_1} \otimes_{\mathcal{K}(0)} \mathcal{M}_{y_1} = \mathcal{M}_{x_1h^{-1}} \otimes_{\mathcal{K}(0)} \mathcal{M}_{h^{-1}y_2} \), it makes sense to consider the element \( \xi_1 + \Psi_h(\xi_2) \). We now want to define
\[ [\xi_1] + [\xi_2] := [\xi_1 + \Psi_h(\xi_2)] := [\xi_1 + \Psi_h(\xi_2)], \]
and to this end, we need to check that it does not depend on the choice of representatives. So suppose that \( \xi_3, \xi_4 \) are such that \([\xi_1] = [\xi_3] \) and \([\xi_2] = [\xi_4] \). With \( x_i \in \mathcal{X} \) and \( y_i \in \mathcal{Y} \) the elements such that \( q_x(\xi_i) = (x_i, y_i) \) for \( i = 3, 4 \), we then know
\[ \exists h \in \mathcal{H}_{s(x_3)} \text{ such that } \Psi_h(\xi_3) = \xi_1 \]
and
\[ \exists h \in \mathcal{H}_{s(x_4)} \text{ such that } \Psi_h(\xi_4) = \xi_2. \]

\[ \exists h \in \mathcal{H}_{s(x_3)} \text{ such that } \Psi_h(\xi_3) = \xi_1 \]
and
\[ \exists h \in \mathcal{H}_{s(x_4)} \text{ such that } \Psi_h(\xi_4) = \xi_2. \]
We point out that, since \( h \in \mathcal{H}_{s(x_1)} \), we can consider
\[
h' := h_0^{-1}hh_e \in \mathcal{H}.
\]
We claim that \( h' \) is the unique element such that \( \xi_3 \) and \( \Psi_{h'}(\xi_4) \) are in the same fibre. Indeed, by Theorem 5.17 (\( \Psi_2 \)), we have
\[
(6.3) \quad \Psi_{h'}(\xi_4) = \Psi_{h_0^{-1}hh_e}(\xi_4) = \Psi_{h_0^{-1}}(\Psi_h(\xi_2)).
\]
Now \( \Psi_h(\xi_2) \) is in the same fibre as \( \xi_1 = \Psi_{h_0}(\xi_3) \), so that by Theorem 5.17 (\( \Psi_4 \)) we indeed have that \( \Psi_{h'}(\xi_4) \) is in the same fibre as \( \xi_3 \). Thus, our definition requires
\[
[\xi_3] + [\xi_4] = [\xi_3 + \Psi_{h'}(\xi_4)],
\]
and we therefore only need to show that
\[
[\xi_1 + \Psi_h(\xi_2)] = [\xi_3 + \Psi_{h'}(\xi_4)].
\]
We have
\[
[\xi_1 + \Psi_h(\xi_2)] = [\Psi_{h_0}(\xi_3) + \Psi_{h_0}(\Psi_{h'}(\xi_4))] \quad \text{by Equations (6.2) and (6.3)}
\]
\[
= [\Psi_{h_0}(\xi_3 + \Psi_{h'}(\xi_4))] \quad \text{by linearity}
\]
\[
= [\xi_3 + \Psi_{h'}(\xi_4)] \quad \text{by Remark 6.3}.
\]
To see that this defines the structure of a vector space, we need to check some axioms. First, addition is associative: for appropriate \( h, h' \), we have
\[
([\xi_1] + [\xi_2]) + [\xi_3] = [\xi_1 + \Psi_h(\xi_2)] + [\xi_3] = [(\xi_1 + \Psi_h(\xi_2)) + \Psi_{h'}(\xi_3)]
\]
\[
= [\xi_1 + \Psi_h(\xi_2 + \Psi_{h^{-1}h'}(\xi_3))]
\]
\[
= [\xi_1] + [\xi_2 + \Psi_{h^{-1}h'}(\xi_3)] (\ast) = [\xi_1] + ([\xi_2] + [\xi_3]).
\]
As both \( h \) and \( h' \) are unique, we do not need to dwell on the fact that \( h^{-1}h' \) is the unique element that makes (\( \ast \)) true.

Second, addition is also commutative: using that \([\xi] = [\Psi_h(\xi)]\) for appropriate \( h \), we get with linearity of \( \Psi_h \),
\[
[\xi_1] + [\xi_2] = [\xi_1 + \Psi_h(\xi_2)] = [\Psi_h(\xi_2) + \xi_1]
\]
\[
= [\xi_2 + \Psi_{h^{-1}}(\xi_1)] = [\xi_2] + [\xi_1].
\]
The other axioms are even easier:
- the identity element of addition is \([0] = [\Psi_h(0)]\);
- the additive inverse of \([\xi] \) is \([-\xi]\);
- for \( \lambda, \nu \in \mathbb{C} \), we have \((\lambda \nu)[\xi] = \lambda(\nu[\xi]), \lambda([\xi_1] + [\xi_2]) = [\lambda \xi_1] + [\lambda \xi_2], \) and \((\lambda + \nu)[\xi] = \lambda[\xi] + \nu[\xi]\); and lastly
- \( 1[\xi] = [\xi] \).

All of these follow from the corresponding properties of the vector space \( \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y \).

Fix \( z \in \mathcal{Z} \) and let \((x, y) \in \mathcal{X}_z, \mathcal{Y} \) with \( v := s(x) = r(y) \) be any representative, i.e., \( z = [x, y]_H \). If we restrict the quotient map \( Q: \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N} \to \mathcal{P} \) to the fibre \( \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y = \mathcal{M}_x \otimes_{\mathcal{C}(v)} \mathcal{N}_y \), it is \( \mathcal{P}_z \)-valued and linear by construction of the linear structure on \( \mathcal{P}_z \). In fact, more is true:

**Lemma 6.6.** The restricted quotient map \( Q: \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y \to \mathcal{P}_z \) is a linear homeomorphism onto \( \mathcal{P}_z \). In particular, the Banach space norm that \( \mathcal{P}_z \) inherits from \( \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y \) induces the subspace topology that \( \mathcal{P}_z \) inherits from \( \mathcal{P} \).
Proof. First note that the map is indeed $\mathcal{P}_z$-valued: if we take $m \in \mathcal{M}_x$ and $n \in \mathcal{N}_y$, we have

$$q_{\mathcal{P}}([m \otimes n]) = [q_{\mathcal{N}}(m \otimes n)]_H = [q_{\mathcal{N}}(m), q_{\mathcal{N}}(n)]_H = [x, y]_H = z.$$  

As $Q|_z$ maps elementary tensors to $\mathcal{P}_z$, it does so for all other elements of $\mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y$ also, since $\mathcal{P}_z$ is closed in $\mathcal{P}$. Linearity is now obvious in light of the definition of $\mathcal{P}$ and its fibre $\mathcal{P}_z$. For continuity, just note that $Q$ is continuous and $\mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y$ is closed in $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$.

Note that $[\xi] = 0$ in $\mathcal{P}$ means $\xi = \Psi_h(0) = 0$ for some $h \in \mathcal{H}$, i.e., only the zero vector in any of the fibres of $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$ gives rise to the zero vector in the corresponding fibre of $\mathcal{P}$. In particular, $Q|_z$ can only send 0 to 0, i.e., the linear map $Q|_z$ is injective.

To see that $Q|_z$ is surjective, take an arbitrary $p \in \mathcal{P}_z$ and let $\xi' \in \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$ be any representative, i.e., $p = [\xi']$. As

$$[x, y]_H = z = q_{\mathcal{P}}(p) = [q_{\mathcal{N}}(\xi')]_H,$$

we know that, if $q_{\mathcal{N}}(\xi') = (x', y')$, there exists $h \in \mathcal{H}$ such that $xh = x'$ and $h^{-1}y = y'$. Let $\xi := \Psi_h(\xi')$, an element of $\mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y$. Then

$$Q|_z(\xi) = [\xi] = [\Psi_h(\xi')] = [\xi'] = p.$$  

To see that $Q|_z$ is a homeomorphism, it now suffices to check that it is closed, so suppose that $A \subseteq \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y$ is some closed set. We have to show that $Q|_z(A)$ is closed in $\mathcal{P}_z$. Since $\mathcal{P}_z$ is closed in $\mathcal{P}$, this is equivalent to showing that $Q(A)$ is closed in $\mathcal{P}$, which, by the quotient topology, means that $Q^{-1}(Q(A))$ is closed in $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$. So assume that $\xi_1 \in Q^{-1}(Q(A))$ converges to some $\xi \in \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$; we have to show that $\xi \in Q^{-1}(Q(A))$.

As $\xi_1 \in Q^{-1}(Q(A))$, there exist $a_\lambda \in A$ such that $\xi_\lambda \mathcal{R} a_\lambda$. Since $A \subseteq \mathcal{M}_x \otimes_{\mathcal{E}(0)} \mathcal{N}_y$, this implies that there exist $h_\lambda \in \mathcal{H}^s(x)$ such that $q_{\mathcal{N}}(\xi_\lambda) = (xh_\lambda, h_\lambda^{-1}y)$ and $a_\lambda = \Psi_{h_\lambda}(\xi_\lambda)$. Since $\xi_\lambda \to \xi$, we have

$$(xh_\lambda, h_\lambda^{-1}y) = q_{\mathcal{N}}(\xi_\lambda) \to q_{\mathcal{N}}(\xi) =: (x_0, y_0).$$

In particular, $r_\chi(x_0) = r_\chi(x)$ and $s_\psi(y_0) = s_\psi(y)$. Since $\chi$ and $\psi$ are equivalences of groupoids, it follows from the assumption $\chi/\mathcal{H} \cong \mathcal{G}(0)$ and $\mathcal{H}\psi \cong \mathcal{K}(0)$ that there exist $h \in \mathcal{H}^s(x), h' \in \mathcal{H}_{r(y)}$ such that $x_0 = xh$ and $y_0 = h'y$. Since

$$xh_\lambda \to xh \quad \text{and} \quad h_\lambda^{-1}y \to h'y,$$

it follows that $h_\lambda \to h$ and $h_\lambda^{-1} \to h'$, so that $h' = h^{-1}$ because $\mathcal{H}$ is Hausdorff. We may let $a := \Psi_h(\xi)$, and it follows from Theorem 5.17 [\[\Psi\]] that $a_\lambda \to a$ since $h_\lambda \to h$ and $\xi_\lambda \to \xi$. Since $A$ is closed, this implies that $a \in A$. As $\xi \mathcal{R} a$ by construction of $a$, it follows that $\xi \in Q^{-1}(Q(A))$ as claimed. 

Proposition 6.7. The quotient map $Q: \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N} \to \mathcal{P}$ is open. Consequently, $\mathcal{P}$ is locally compact Hausdorff.

To prove Proposition 6.7, we need two lemmas; see Definition 5.15 for the map $t$ and Definition 2.6 for that of pull-back bundles.

Lemma 6.8. Any continuous section $\kappa$ of $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$ induces a continuous section $\Psi(\kappa)$ of $\mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$. Define $\Psi(\kappa): (\mathcal{K}, t, \mathcal{H}) \to \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$ by

$$(x, y, h) \mapsto \Psi_h(\kappa(xh, h^{-1}y)).$$

Proof. As $(x, y, h) \in (\mathcal{K}, t, \mathcal{H})$, we have $r_\mathcal{H}(h) = t_Z(x, y)$, so $xh$ and $h^{-1}y$ make sense. Moreover, since $\kappa$ is a section of $\mathcal{K} = \mathcal{M} \otimes_{\mathcal{E}(0)} \mathcal{N}$, we have

$$t(q_{\mathcal{K}}(\kappa(xh, h^{-1}y))) = t(xh, h^{-1}y) = s(h),$$
so \((h, \kappa(xh, h^{-1}y))\) is in the domain \(\mathcal{H}_x^* (\mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N})\) of \(\Psi\). In other words, the definition of \(\Psi(\kappa)\) makes sense. To see that \(\Psi(\kappa)\) is a section, we note that \(\kappa(xh, h^{-1}y) \in \mathcal{M}_x \otimes_{\mathcal{G}(0)} \mathcal{N}_{h^{-1}y}\), which is being mapped to an element of \(\mathcal{M}_x \otimes_{\mathcal{G}(0)} \mathcal{N}_y\) by \(\Psi_h\), so that

\[
\Psi(\kappa)(x, y, h) = \Psi(h, \kappa(xh, h^{-1}y)) = (\Psi_h(\kappa(xh, h^{-1}y)), h)
\]

\[
\in \mathcal{M}_x \otimes_{\mathcal{G}(0)} \mathcal{N}_y \times \{h\} = [(\mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N})_x^* \mathcal{H}]_{(x, y, h)}.
\]

To see that \(\Psi(\kappa)\) is continuous, just notice that it is a concatenation of continuous functions: inversion in \(\mathcal{H}\), the \(\mathcal{H}\)-action on \(\mathcal{X}\) resp. \(\mathcal{Y}\), and the continuous map \(\Psi\).

**Lemma 6.9.** The projection map \(\text{pr}_{\mathcal{X}} : \mathcal{H}^* \times \mathcal{H} \to \mathcal{X}, (\xi, h) \mapsto \xi\), is open.

**Proof.** As explained in Lemma 2.8, subsets of the form \(U_1^*, U_2\) for \(U_1, U_2 \subseteq \mathcal{H} = \mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N}\) and \(U_2 \subseteq \mathcal{H}\) basic open sets form a basis of the topology of \(\mathcal{X}^* \times \mathcal{Y}\), so it suffices to check that \(\text{pr}_{\mathcal{X}}(U_1^*, U_2)\) is open. We compute:

\[
\text{pr}_{\mathcal{X}}(U_1^*, U_2) = \{\xi \in U_1 \mid t(q_{\mathcal{X}}(\xi)) \in r_{\mathcal{H}}(U_2)\}
\]

\[
= U_1 \cap q_{\mathcal{X}}^{-1}\left(t^{-1}(r_{\mathcal{H}}(U_2))\right).
\]

Note that, since \(\mathcal{H}\) is étale, \(r_{\mathcal{H}}(U_2)\) is open in \(\mathcal{H}(0)\). Since \(t\) and \(q_{\mathcal{X}}\) are continuous, we therefore have shown that \(\text{pr}_{\mathcal{X}}(U_1^*, U_2)\) is indeed open.

**Proof of Proposition 6.7.** It suffices to show that \(Q\) maps basic open sets to open sets. Recall [9] that a basic open set of \(\mathcal{X} = \mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N}\) is of the form

\[
W := W(K, V, \epsilon) := \{\xi \in \mathcal{X} \mid q_{\mathcal{X}}(\xi) \in V, \|\xi - \kappa(q_{\mathcal{X}}(\xi))\| < \epsilon\}
\]

for \(\kappa\) a continuous section of \(\mathcal{X}\), \(V \subseteq Z = \mathcal{X}^* \times \mathcal{Y}\) open, and \(\epsilon > 0\). We have to show that \(Q(W)\) is open in \(\mathcal{X}\), meaning that \(Q^{-1}(Q(W))\) is open in \(\mathcal{X}\). We claim that

\[
Q^{-1}(Q(W)) = \text{pr}_{\mathcal{X}}\left(W(\Psi(\kappa), \psi(\mathcal{H}^* \times \mathcal{Y}), \epsilon)\right),
\]

where \(\text{pr}_{\mathcal{X}}\) is the open map from Lemma 6.9 and \(\psi\) the homeomorphism from Theorem 5.17.

For “\(\subseteq\)”, let \(\eta\) be such that there exists \(\xi \in W\) with \(\eta(\mathcal{X})\). In other words, there exists \(h \in \mathcal{H}\) with \(s_{\mathcal{H}}(h) = t_{\mathcal{X}}(\xi)\) such that \(\eta = \Psi_h(\xi)\). We claim that \((\eta, h) \in W(\Psi(\kappa), \psi(\mathcal{H}^* \times \mathcal{Y}), \epsilon)\). If \(\pi\) denotes the projection map \((\mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N})^* \times \mathcal{H} \to (\mathcal{X}^* \times \mathcal{Y})^* \mathcal{H}\) of the pull-back bundle, then we have

\[
\pi(\eta, h) = \psi(\mathcal{H}^* \times \mathcal{Y})
\]

\[
\iff \exists k \in \mathcal{H}, (x, y) \in V \text{ such that } s_{\mathcal{H}}(k) = t(x, y) \text{ and } (q_{\mathcal{X}}(\eta), h) = (xk^{-1}, ky, k)
\]

\[
\iff \exists (x, y) \in V \text{ such that } s_{\mathcal{H}}(h) = t(x, y) \text{ and } q_{\mathcal{X}}(\eta) = (xh^{-1}, hy).
\]

Such a tuple indeed exists: by choice of the element \(\xi\) in \(W\), we have that \((x, y) := q_{\mathcal{X}}(\xi)\) is in \(V\), and the equality \(\eta = \Psi_h(\xi)\) implies \(q_{\mathcal{X}}(\eta) = (xh^{-1}, hy)\). Furthermore,

\[
\|(\eta, h) - \Psi(\kappa)(\pi(\eta, h))\| = \|(\eta, h) - \Psi(\kappa)(xh^{-1}, hy, h)\|
\]

\[
= \|(\eta, h) - \Psi(h, \kappa(x, y))\|
\]

\[
= \|\Psi_h(\xi) - \Psi_h(\kappa(x, y))\|
\]

\[
= \|\xi - \kappa(q_{\mathcal{X}}(\xi))\| < \epsilon, \text{ since } \xi \in W.
\]

This proves that \((\eta, h) \in W(\Psi(\kappa), \psi(\mathcal{H}^* \times \mathcal{Y}), \epsilon)\), so that \(\eta\) is in the right-hand side of the alleged equation.

For “\(\supseteq\)”, assume that \(\eta\) is in the right-hand side, so there exists \(h \in \mathcal{H}\) such that \(t(q_{\mathcal{X}}(\eta)) = r_{\mathcal{H}}(h)\), \(\pi(\eta, h) \in \psi(\mathcal{H}^* \times \mathcal{Y})\), and

\[
\|(\eta, h) - \Psi(\kappa)(\pi(\eta, h))\| < \epsilon.
\]
As argued above, \( \pi(\eta, h) \in \psi(N, \ast, V) \) implies that there exists \((x, y) \in V\) such that \( s_H(h) = t(x, y) \) and \( q_N(\eta) = (xh^{-1}, hy) \). We claim that \( \xi := \Psi_{h^{-1}}(\eta) \) is an element of \( W \), so that, since \( \eta R \xi \), we can conclude \( \eta \in Q^{-1}(Q(W)) \). We have \( q_N(\xi) = (x, y) \in V \) by construction, and the same computation as above yields
\[
\|\xi - \kappa(q_N(\xi))\| = \|\eta, h - \Psi(\kappa)(\pi(\eta, h))\|,
\]
and the right-hand side is smaller than \( \epsilon \) by choice of \( \eta \). This proves that \( \xi \in W \), as claimed. All in all, we have shown Equation (6.4).

By Lemma 6.8 \( \Psi(\kappa) \) is a continuous section of \( \mathcal{K}^{\ast}_{\ast} \mathcal{H} \). Since \( \psi \) is a homeomorphism and \( V \) is open in \( Z \), \( \psi(\mathcal{K}^{\ast}_{\ast} \mathcal{H}) \) is open in \( Z^{\ast}_{\ast} \mathcal{H} \). Thus, \( W(\Psi(\kappa), \psi(\mathcal{K}^{\ast}_{\ast} \mathcal{H}), \epsilon) \) is a basic open set in \( \mathcal{K}^{\ast}_{\ast} \mathcal{H} \). Since \( pr_\mathcal{K} \) is open by Lemma 6.9 we conclude that \( Q^{-1}(Q(W)) \) is open, as claimed.

As \( \mathcal{P} \) is a bundle over a groupoid pre-equivalence, we will follow our convention in Notation 2.5:
\[
r_{\mathcal{P}} : \mathcal{P} \to G^{(0)} \quad \text{and} \quad s_{\mathcal{P}} : \mathcal{P} \to K^{(0)}
\]
are open. As argued above, \( \Psi^{-1} \) is a bundle over a groupoid pre-equivalence, we will follow our convention in Notation 2.5:
\[
r_{\mathcal{P}}([\xi]) := (r_Z \circ q_{\mathcal{P}})([\xi]) = r_Z([q_N(\xi)]_K) = r_Z(q_N(\xi))
\]
respectively \( s_{\mathcal{P}}([\xi]) = s_Z(q_N(\xi)) \). Note that \( r_{\mathcal{P}} \) and \( s_{\mathcal{P}} \) are continuous with respect to the quotient topology on \( \mathcal{P} \): for any \( U \subseteq G^{(0)} \), we have that
\[
r^{-1}_{\mathcal{P}}(U) = Q\left((r_Z \circ q_N)^{-1}(U)\right).
\]

Thus, if \( U \) is open, then \( r^{-1}_{\mathcal{P}}(U) \) is open since \( r_Z \) and \( q_N \) are continuous and since \( Q \) is open by Proposition 6.7.

Remark 6.10. It now follows from Proposition 6.1 and Lemma 3.1 that the maps
\[
id \ast Q : \mathcal{B}^{\ast}_{\ast} \left( \mathcal{M} \otimes_{G^{(0)}} \mathcal{N} \right) \to \mathcal{B}^{\ast}_{\ast} \mathcal{P} , \quad (b, \xi) \mapsto (b, [\xi]), \quad \text{and}
\]
\[
Q \ast \text{id} : \left( \mathcal{M} \otimes_{G^{(0)}} \mathcal{N} \right)_{\ast}^{\ast} \mathcal{P} \to \mathcal{P}^{\ast}_{\ast} \mathcal{P} , \quad (\xi, d) \mapsto ([\xi], d),
\]
are open.

6.2. The quotient bundle is USC.

Proposition 6.11. Let \( \mathcal{B} = (q_{\mathcal{B}} ; B \to X) \) and \( \mathcal{C} = (q_{\mathcal{C}} ; C \to Y) \) be bundles of Banach spaces over locally compact Hausdorff spaces. Assume we have maps
\[
\begin{align*}
B & \xrightarrow{\Phi} C \\
qu_{\mathcal{B}} \downarrow & \quad \downarrow q_{\mathcal{C}} \\
X & \xrightarrow{\varphi} Y
\end{align*}
\]
and topologies on \( B \) and \( C \) with the following properties:

(1) \( \Phi \) and \( \varphi \) are open quotient maps; in particular, they are surjective and continuous.

(2) \( \Phi \) is a linear isometry when restricted to the map \( B_x \to C_{\varphi(x)} \).

(3) For every \( c \in C \) and \( x \in X \) with \( q_{\mathcal{C}}(c) = \varphi(x) \), there exists \( b \in B(x) \) such that \( \Phi(b) = c \).

If \( \mathcal{B} \) is an USC Banach bundle, then \( \mathcal{C} \) is upper semi-continuous as well.

Proof. If \( U \subseteq Y \) is open, then \( q_{\mathcal{C}}^{-1}(U) \) is open in \( C \) exactly when \( \Phi^{-1}(q_{\mathcal{C}}^{-1}(U)) \) is open in \( B \), but
\[
\Phi^{-1}(q_{\mathcal{C}}^{-1}(U)) = q_{\mathcal{B}}^{-1}(\varphi^{-1}(U))
\]
by commutativity of the diagram. As \( q_{\mathcal{B}} \) and \( \varphi \) are continuous, \( q_{\mathcal{C}} \) is continuous. Likewise, surjectivity of \( q_{\mathcal{B}} \) and \( \varphi \) implies surjectivity of \( q_{\mathcal{C}} \): if \( y \in Y \), pick any \( x \in \varphi^{-1}(y) \subseteq X \) and then \( b \in q_{\mathcal{B}}^{-1}(x) \subseteq B \), so that \( \Phi(b) \in q_{\mathcal{C}}(y) \).
We now need to check Conditions [USC1]–[USC4] all of which follow from $\mathcal{B}$ satisfying these conditions and from $\Phi$ being open. To be more precise:

For [USC1] we need to show that the map $C \to \mathbb{R}_{\geq 0}$, $c \mapsto \|c\|$, is upper semi-continuous, or in other words, that $\{c \in C \mid \|c\| < r\}$ is open for any fixed $r \in \mathbb{R}_{\geq 0}$. By Assumption [2] we have $\|b\|_{\mathcal{B}} = \|\Phi(b)\|_{\mathcal{C}}$, so that

$$\{c \in C \mid \|c\| < r\} = \Phi(\{b \in B \mid \|b\| < r\}).$$

Since $\mathcal{B}$ is a USC Banach bundle and $\Phi$ is an open map, the claim follows.

For [USC2] let $U \subseteq C$ be an open neighborhood. Since $\Phi$ is continuous, $\Phi^{-1}(U)$ is an open subset of $B$. Since $\mathcal{B}$ is a USC Banach bundle, $+_B$ is continuous on $\mathcal{B}$. Consequently, $V := +^1_B(\Phi^{-1}(U))$ is an open subset of $B_{c^*}B$. As $\Phi$ is an open map, so is the map $\Phi_{c^*} \Phi$ by Lemma A.2 and hence $\Phi_{c^*} \Phi(V)$ is an open set in $C_{c^*}C$. We compute

$$\Phi_{c^*} \Phi(V) = \{(\Phi(b), \Phi(b')) \mid (b, b') \in +^1_B(\Phi^{-1}(U))\}$$

$$= \{(\Phi(b), \Phi(b')) \mid b + b' \in \Phi^{-1}(U)\} = \{(\Phi(b), \Phi(b')) \mid \Phi(b + b') \in U\} = +^1_C(U),$$

so $+_C(U)$ is open in $C_{c^*}C$, meaning that $+_C$ is indeed continuous on $\mathcal{C}$.

For [USC3] take an arbitrary but fixed $\lambda \in \mathbb{C}$; we have to show that the map $f: C \to C$, $c \mapsto \lambda c$, is continuous. Let $U \subseteq C$ be open, so $\Phi^{-1}(U)$ is open in $B$. Since $\mathcal{B}$ is a USC Banach bundle,

$$V := \{b \in B \mid \lambda b \in \Phi^{-1}(U)\}$$

is open in $B$. Since $\Phi$ is open, $\Phi(V)$ is open in $C$. As

$$\Phi(V) = \{\Phi(b) \mid \lambda b \in \Phi^{-1}(U)\} = \{\Phi(b) \mid \lambda \Phi(b) = \Phi(\lambda b) \in U\} = f^{-1}(U),$$

using that $\Phi$ is fibrewise linear by [2] it follows that $f$ is continuous.

For [USC4] suppose $\{c_i\}_{i \in I}$ is a net in $C$ such that $q_\varphi(c_i)$ converges to some $y \in Y$ and $\|c_i\| \to 0$. Assume for a contradiction that $\{c_i\}_{i \in I}$ does not converge to $0_y \in \mathcal{C}_y$ in $C$. In other words, there exists an open neighborhood $U$ of $0_y$ in $C$ such that

$$\text{(6.5)} \quad \text{for all } i \in I, \text{ there exists } j \geq i \text{ such that } c_j \notin U.$$

The outline of the proof is as follows. We will first construct a subnet $\{c_{h(\beta)}\}_{\beta \in J}$ of $\{c_i\}_{i \in I}$ which lies completely outside of $U$. For a subnet $\{c_{f(\gamma)}\}_{\gamma \in \Gamma}$ of $\{c_{h(\beta)}\}_{\beta \in J}$ and any fixed representative $x \in \varphi^{-1}(y) \subseteq X$, we will then find lifts $b_\gamma \in \Phi^{-1}(c_{f(\gamma)}) \subseteq B$ which satisfy both $\text{lim}_\gamma q_\mathcal{B}(b_\gamma) = x$ and $\text{lim}_\gamma \|b_\gamma\| = 0$. Since $\mathcal{B}$ is upper semi-continuous, this implies $b_\gamma \to 0_x$ in $B$. Since $\Phi^{-1}(U)$ is an open neighborhood of $0_x$ in $B$ by continuity of $\Phi$, this contradicts that $\Phi(b_\gamma) = c_{f(\gamma)} \notin U$ for all $\gamma$.

Let

$$J := \{(i, j) \in I \times I \mid i \leq j \text{ and } c_j \notin U\},$$

which is non-empty by Assumption (6.5) on our net. Define a preorder on $J$ by

$$(i, j) \preceq (i', j') : \iff i \leq i' \text{ and } j \leq j'.$$

To see that $(J, \preceq)$ is directed, take two elements $(i_1, j_1), (i_2, j_2) \in J$. As $I$ is directed, there exist $i_3, j_3 \in I$ such that $i_i \leq i_3$ and $j_i \leq j_3$ for $i = 1, 2$ and also $i_i \leq j_i$ for $i = 1, 2, 3$. By Assumption (6.3), there exists $j_3 \geq j_3'$ such that $c_{j_3} \notin U$. Note that $j_3$ is greater than or equal to all other elements we have considered, so that $(i_3, j_3) \in J$ and $(i_3, j_3)$ dominates both $(i_1, j_1)$ and $(i_2, j_2)$.

The map $h: J \to I$, $h(i, j) = j$, is monotone and final. Thus, $(c_{h(\beta)})_{\beta \in J}$ is a subnet of $(c_i)_{i \in I}$ that lives completely outside of $U$ by construction. The assumptions that $\text{lim}_i \|c_i\| = 0$ and $\text{lim}_i q_\varphi(c_i) = y$ imply that $\text{lim}_\beta \|c_{h(\beta)}\| = 0$ and that $y_\beta := q_\varphi(c_{h(\beta)})$ converges to $y$ also.
Now fix any \( x \in \varphi^{-1}(y) \subseteq X \). Since \( \varphi \) is open and surjective, \( \text{[3, II.13.2]} \) asserts that we can find a subnet set \( \{ y_{f(\gamma)} \}_{\gamma \in \Gamma} \) of \( \{ y_{b} \}_{b \in B} \) and a net \( \{ x_{\gamma} \}_{\gamma \in \Gamma} \) in \( X \), indexed by the same set, such that \( x_{\gamma} \rightarrow x \) and \( \varphi(x_{\gamma}) = y_{f(\gamma)} \) for all \( \gamma \in \Gamma \).

For each \( \gamma \in \Gamma \), let \( b_{\gamma} \) be an element of \( \Phi^{-1}(c_{f(\gamma)}) \cap q_{\beta}^{-1}(x_{\gamma}) \), which exists because of \( \text{(3)} \) in particular,
\[
\lim_{\gamma} q_{\beta}(b_{\gamma}) = \lim_{\gamma} x_{\gamma} = x \quad \text{and} \quad \lim_{\gamma} \| b_{\gamma} \| = \lim_{\gamma} \| c_{f(\gamma)} \| = \lim_{\gamma} \| c_{i} \| = 0,
\]
since, by construction, the net \( \{ c_{f(\gamma)} \}_{\gamma \in \Gamma} \) is a subnet of \( \{ c_{i} \}_{i \in I} \). Since \( \mathcal{B} \) is a USC Banach bundle, it follows from Condition \( \text{(USC4)} \) of \( \mathcal{B} \) that \( \lim_{\gamma} b_{\gamma} = 0_{x} \) in \( B \), implying that \( \lim_{\gamma} \Phi(b_{\gamma}) = 0_{y} \) in \( C \) by fibrewise linearity of \( \Phi \). But by construction, \( \Phi(b_{\gamma}) = c_{f(\gamma)} \notin U \) for all \( \gamma \in \Gamma \), which is a contradiction as \( U \) is a neighborhood of \( 0_{y} \) in \( C \).

**Corollary 6.12.** With the described structure on the bundle \( \mathcal{P} = (q_{\mathcal{P}}: P \rightarrow Z) \), it is a USC Banach bundle.

**Proof.** We will apply Proposition \( \text{[6.11]} \) to the following diagram which is commutative by construction; here, \( K \) is the total space of the bundle \( \mathcal{K} = \mathcal{M} \otimes_{\mathcal{E}(\mathcal{O})} \mathcal{N} \).

\[
\begin{array}{ccc}
K & \xrightarrow{Q} & P \\
\downarrow{\varphi} & & \downarrow{q_{\mathcal{P}}} \\
\mathcal{X}_{\mathcal{M}} \ast \mathcal{Y} & \xrightarrow{[\cdot]_{\mathcal{K}}} & \mathcal{Z}
\end{array}
\]

First recall that each fibre \( \mathcal{P} \) has the structure of a complex Banach space by Lemma \( \text{[6.6]} \) and that \( \mathcal{K} \) is a USC Banach bundle by Lemma \( \text{[5.1]} \). By Proposition \( \text{[6.7]} \) and Corollary \( \text{[3.9]} \), the quotient maps \( Q \) and \( [\cdot]_{\mathcal{K}} \) are open, so we have Assumption \( \text{(1)} \). By definition of the Banach space structure on the fibres of \( \mathcal{C} \) (see Lemma \( \text{[6.6]} \)), we have both Assumption \( \text{(2)} \) and \( \text{(3)} \). \( \square \)

**Lemma 6.13.** The left \( \mathcal{B} \)-action on \( \mathcal{M} \) induces a left \( \mathcal{B} \)-action on \( \mathcal{P} \). To be precise, the action is given by \( b \cdot [\xi] := [b \cdot \xi] \) for \( b \in B \) and \( \xi \in \mathcal{K} = \mathcal{M} \otimes_{\mathcal{E}(\mathcal{O})} \mathcal{N} \) with \( s_{\mathcal{B}}(b) = r_{\mathcal{K}}(\xi) \).

Similarly, the right \( \mathcal{D} \)-action on \( \mathcal{N} \) induces a right \( \mathcal{D} \)-action on \( \mathcal{P} \) given by \( [\xi] \cdot d := [\xi \cdot d] \) for \( d \in \mathcal{D} \) and \( \xi \in \mathcal{M} \otimes_{\mathcal{E}(\mathcal{O})} \mathcal{N} \) with \( s_{\mathcal{D}}(\xi) = r_{\mathcal{K}}(d) \).

**Proof.** We will only deal with the left action; the same proof, replacing left- with right-arguments and vice versa, shows the claim about the right action.

We have seen in Proposition \( \text{[5.5]} \) that \( \mathcal{K} \) has a left \( \mathcal{B} \)-action determined by \( b \cdot (m \otimes n) = (b \cdot m) \otimes n \), which explains what \( b \cdot \xi \) means. Let us first note that our map is well defined, so suppose \( [\xi] = [\eta] \). Then there exists \( h \in \mathcal{H} \) such that \( \xi = \Psi_{h}(\eta) \). We have seen in Theorem \( \text{[5.17]} \) \( \text{(Ψ5)} \) that \( b \cdot \xi = b \cdot \Psi_{h}(\eta) = \Psi_{h}(b \cdot \eta) \). Thus, \( [b \cdot \xi] = [\Psi_{h}(b \cdot \eta)] = [b \cdot \eta] \), so the definition of \( b \cdot [\xi] \) does not depend on the chosen representative \( \xi \).

To see that the map
\[
f: \mathcal{B}_{s} \ast \mathcal{P} \rightarrow \mathcal{P}, \quad (b, p) \mapsto b \cdot p,
\]
is continuous, let \( U \subseteq \mathcal{P} \) be an open set, i.e., \( Q^{-1}(U) \) is open in \( \mathcal{K} \). As the left \( \mathcal{B} \)-action on \( \mathcal{K} \) is continuous (see Proposition \( \text{[5.5]} \)), we know that
\[
V := \{(b, \xi) \in \mathcal{B}_{s} \ast \mathcal{K} \mid b \cdot \xi \in Q^{-1}(U)\}
\]
is open in \( \mathcal{B}_{s} \ast \mathcal{K} \). By Remark \( \text{[6.10]} \) the set \( (\text{id} \ast Q)(V) \) is thus open in \( \mathcal{B}_{s} \ast \mathcal{P} \). Since
\[
(\text{id} \ast Q)(V) = \{(b, [\xi]) \in \mathcal{B}_{s} \ast \mathcal{P} \mid b \cdot [\xi] \in Q^{-1}(U)\}
= \{(b, [\xi]) \in \mathcal{B}_{s} \ast \mathcal{P} \mid b \cdot [\xi] = b \cdot [\xi] \in U\}
= \{(b, p) \in \mathcal{B}_{s} \ast \mathcal{P} \mid b \cdot p \in U\} = f^{-1}(U),
\]
it follows that $f$ is continuous. It remains to check the numbered conditions of an action in Definition 2.11.

For (FA1) we compute:
\[
q_{\mathcal{S}}(b \cdot [\xi]) = q_{\mathcal{S}}(b \cdot [\xi]) = [q_{\mathcal{S}}(b \cdot \xi)]_{\mathcal{H}}
\]
\[
= [p_{\mathcal{S}}(b) \cdot q_{\mathcal{S}}(\xi)]_{\mathcal{H}}, \quad \text{by the same property for } \mathcal{H}.
\]

By definition of the left $\mathcal{G}$-action on $\mathcal{Z}$, we have for $g \in \mathcal{G}$ and compatible $[x, y]_{\mathcal{H}}$ in $\mathcal{Z}$ that $[g \cdot x, y]_{\mathcal{H}} = g \cdot [x, y]_{\mathcal{H}}$. Both combined yield:
\[
q_{\mathcal{S}}(b \cdot [\xi]) = p_{\mathcal{S}}(b) \cdot [q_{\mathcal{S}}(\xi)]_{\mathcal{H}} = p_{\mathcal{S}}(b) \cdot q_{\mathcal{S}}([\xi]), \quad \text{as claimed.}
\]

Property (FA2) i.e., that $b' \cdot (b \cdot [\xi]) = (b'b) \cdot [\xi]$ for all compatible $b', b \in \mathcal{B}$ and $[\xi] \in \mathcal{P}$, follows from the same property of $\mathcal{H}$.

For (FA3) we compute
\[
\|b \cdot [\xi]\| = \|[b \cdot \xi]\| = \|b \cdot \xi\| \quad \text{by Lemma 6.6}
\]
\[
\leq \|b\| \|\xi\| \quad \text{by (FA3) for } \mathcal{H}
\]
\[
= \|b\| \|[\xi]\| \quad \text{by Lemma 6.6}
\]
This proves our claim. \qed

Clearly, the left- and right-actions on $\mathcal{P}$ commute, i.e., Condition (FE1) in Definition 2.11 is satisfied. In the following, we will use superscripts to provide more clarity.

**Proposition 6.14.** There exist sesquilinear, continuous maps $\mathcal{S}(\cdot | \cdot)$ on $\mathcal{P} \times \mathcal{P}$ and $\langle \cdot | \cdot \rangle_{\mathcal{S}}$ on $\mathcal{P} \times \mathcal{P}$ defined for $([\xi], [\xi'])$ in the appropriate set by
\[
\mathcal{S}([\xi] | [\xi']) := \mathcal{S}(\xi | \xi') \quad \text{resp. } ([\xi] | [\xi'])_{\mathcal{S}} := \langle \xi | \xi' \rangle_{\mathcal{S}}.
\]
These maps further satisfy the conditions in (FE2) of Definition 2.11.

**Proof.** As always, we will only deal with the left (pre-)inner product $\mathcal{S}(\cdot | \cdot)$. Let us first check that it is well defined for elementary tensors: suppose $\xi = (m \cdot c) \otimes n$ and $\xi' = (m' \cdot c') \otimes n'$ with $c \in \mathcal{C}_h, c' \in \mathcal{C}_{h'}$, so that $\xi$ is equivalent to $m \otimes (c \cdot n)$ and $\xi'$ to $m' \otimes (c' \cdot n')$. We have
\[
\mathcal{S}(\xi | \xi') = \mathcal{S}(m \cdot c \otimes n' | (m' \cdot c') \otimes n')
\]
\[
= \mathcal{S}(m \cdot c | (m' \cdot c') \cdot n' | n)
\]
\[
= \mathcal{S}(m | (m' \cdot c') \cdot (\Psi_{h'}(n' | n) c^*)) \quad \text{by Corollary 4.6}
\]

Note that $c^*(n' | n)c^* = \psi(\psi'(n' | c \cdot n))$, so that all in all:
\[
\mathcal{S}(\xi | \xi') = \mathcal{S}(m | (m' \cdot c') \cdot n' | c \cdot n)
\]
\[
= \mathcal{S}(m \otimes (c \cdot n) | m' \otimes (c' \cdot n'))
\]
\[
= \mathcal{S}(\Psi_h(\xi) | \Psi_{h'}(\xi')).
\]
By linearity, we conclude that the same equation holds for all $\xi \in \mathcal{M}_h \otimes \mathcal{N}_y$ and $\xi' \in \mathcal{M}_{h'} \otimes \mathcal{N}_{y'}$ for which $s_{\mathcal{S}}(y) = s_{\mathcal{S}}(y')$ and for all compatible $h, h'$. By continuity of the inner product on $\mathcal{H}$ and continuity of $\Psi$ (see Proposition 5.11 resp. Theorem 5.17), uniqueness of limits implies that the equality holds for all $(\xi, \xi') \in \mathcal{H} \times \mathcal{H}$ and compatible $h, h'$. Consequently, the (pre-)inner product on $\mathcal{P}$ is well defined.

To see that $f := \mathcal{S}(\cdot | \cdot)$ is continuous, let $U \subseteq B$ be open. By continuity of $\tilde{f} := \mathcal{S}(\cdot | \cdot)$, we know that $\tilde{f}^{-1}(U)$ is open in $\mathcal{H} \times \mathcal{H}$. As the quotient map $Q$ is open by Proposition 6.7 $(Q_\times Q)(\tilde{f}^{-1}(U))$ is open in $\mathcal{P} \times \mathcal{P}$. As
\[
(Q_\times Q)(\tilde{f}^{-1}(U)) = \{[[\xi], [\xi']] \in \mathcal{S}([\xi] | [\xi']) = \mathcal{S}(\xi | \xi') \in U = \tilde{f}^{-1}(U),
\]
Theorem 6.15. Assume we are given

on \( P \) commuting \( B \). In this proof, all Items refer to Definition 2.11. We have seen in Corollary 6.12 that

\[
\text{Proof.}
\]

where \( f \) is continuous.

For the remaining conditions of Definition 2.11 note that sesquilinearity follows from the

\( \xi \) is an elementary tensor \( \xi = m_i \otimes n_i \). We compute:

\[
\xi \langle \xi_2 | \xi_3 \rangle = \langle m_1 | m_2 \cdot \xi(n_2 | n_1) \rangle \cdot \xi_3 = (\langle m_1 | m_2 \cdot \xi(n_2 | n_1) \rangle \cdot m_3) \otimes n_3
\]

where

\[
c := \langle m_2 \cdot \xi(n_2 | n_1) | m_3 \rangle = \langle m_2 | m_3 \rangle \cdot \xi(n_1 | n_2) = \langle n_1 | m_3 \rangle \cdot \xi(n_2 | n_1).
\]

On the other hand,

\[
\xi_1 \cdot \langle \xi_2 | \xi_3 \rangle = \xi_1 \cdot (\langle m_3 | m_2 \rangle \cdot \xi(n_2 | n_3)) = m_1 \otimes (\langle m_3 | m_2 \rangle \cdot \xi(n_2 | n_3)) = m_1 \otimes (c \cdot n_3).
\]

Since \( (m_1 \cdot c) \otimes n_3 = [m_1 \otimes (c \cdot n_3)] \) in \( P \) (see Remark 6.3), we have proved the claim.

We can now prove our second main theorem:

**Theorem 6.15.** Assume we are given:

- three saturated Fell bundles \( \mathcal{B}, \mathcal{C}, \mathcal{D} \) over locally compact Hausdorff étale groupoids \( \mathcal{G}, \mathcal{H}, \mathcal{K} \), respectively;
- a \( (\mathcal{G}, \mathcal{H}) \)-equivalence \( \mathcal{X} \) and an \( (\mathcal{H}, \mathcal{K}) \)-equivalence \( \mathcal{Y} \);
- a \( (\mathcal{B}, \mathcal{C}) \)-equivalence \( \mathcal{M} \) over \( \mathcal{X} \) and a \( (\mathcal{C}, \mathcal{D}) \)-equivalence \( \mathcal{N} \) over \( \mathcal{Y} \).

Then the USC Banach bundle \( \mathcal{P} = (q_{\mathcal{P}} : P \to \mathcal{X} \ast_{\mathcal{K}} \mathcal{Y}) \), defined as the quotient of \( \mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N} \) in Definition 6.2, is a \( (\mathcal{B}, \mathcal{D}) \)-Fell bundle equivalence when equipped with the actions defined in Lemma 6.13 and the inner products defined in Proposition 6.14.

The following diagram gives an overview of our construction, where dotted arrows are the maps we have constructed in the current section.

\[
\begin{array}{ccc}
\mathcal{M} \otimes_{\mathcal{G}(0)} \mathcal{N} & \xrightarrow{q_{\mathcal{P}}} & \mathcal{P} \\
\mathcal{B} & \xrightarrow{q_{\mathcal{P}}} & \mathcal{D} \\
\mathcal{G} & \xrightarrow{q_{\mathcal{P}}} & \mathcal{K} \\
\end{array}
\]

\[
X \ast_{\mathcal{H}} Y
\]

**Proof.** In this proof, all Items refer to Definition 2.11. We have seen in Corollary 6.12 that \( \mathcal{P} \) is a USC Banach bundle over the \( (\mathcal{G}, \mathcal{K}) \)-equivalence \( \mathcal{Z} = \mathcal{X} \ast_{\mathcal{K}} \mathcal{Y} \). By Lemma 6.13, \( \mathcal{P} \) carries commuting \( \mathcal{B} \)- and \( \mathcal{D} \)-actions, so Item (FE1) is satisfied. By Proposition 6.14, the inner products on \( \mathcal{P} \) satisfy exactly the conditions in Item (FE2).
For Item (FE3) we recall that each fibre $\mathcal{P}_{[x,y]}$ is isomorphic as bimodule to $\mathcal{M}_x \otimes_{E(0)} \mathcal{N}_y$ by Lemma 6.6. As argued in the proof of Theorem 5.12, this is a $B(r_X(x)) - D(s_Y(y))$-imprimitivity bimodule, as needed for $\mathcal{P}$ for Item (FE3). This concludes our proof. □

**Corollary 6.16.** Fell bundle equivalence is transitive and hence an equivalence relation.

### 7. Acknowledgements

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Appendix A. Some lemmas on topology and USC Banach bundles

Lemma A.1. Suppose $X, Y, Z$ are topological spaces, $p: X \to Z$ is continuous, and $q: Y \to Z$ is open. Then the map $\text{pr}_X: X \to Y \times_{q} Y \to X, (x, y) \mapsto x$, is an open map.

Proof. We have to show that, for any open $U \subseteq X \times_{q} Y$ and $(x, y) \in U$, there exists an open neighborhood $V$ of $x$ in $X$ such that $V \subseteq \text{pr}_X(U)$. It suffices to consider a basic open set $U$, so assume $U = V_1 \ast V_2$ where $V_1$ is an open neighborhood of $x$ and $V_2$ of $y$. Let $V := V_1 \cap p^{-1}(q(W_2))$. This set is open, since $p$ is continuous and $q$ is open. It is further a neighborhood of $\text{pr}_X(x, y) = x$, because $(x, y) \in V_1 \times V_2$ and $p(x) = q(y)$. Now take an arbitrary $a \in V$, so that $a \in V_1$ and there exists $b \in V_2$ such that $p(a) = q(b)$. In particular, $(a, b) \in V_1 \ast V_2 \subseteq U$ and $\text{pr}_X(a, b) = a$, proving that $a \in \text{pr}_X(U)$.

Lemma A.2. Suppose we are given bundles $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ over the same topological space $Z$ and maps $f, g$ as follows:

\[
\begin{array}{ccc}
A & \rightarrow & Z \\
\downarrow f & & \downarrow g \\
A' & \rightarrow & Z \\
\end{array}
\]

If $f, g$ are open bundle maps, then the map

\[
f \ast g: \mathcal{A} \ast q \mathcal{B} \rightarrow \mathcal{A}' \ast q \mathcal{B}', \quad (a, b) \mapsto (f(a), g(b)),
\]

is also an open bundle map.

Proof. The map is a bundle map since

\[
(q \circ [f \ast g])(a, b) = q(f(a), g(b)) \quad \overset{\text{(1)}}{=} \quad (q(f(a)), q(g(b))) \quad \overset{\text{(1)}}{=} \quad (q(a), q(b)) \quad \overset{\text{(1)}}{=} \quad q(a, b),
\]

where we dropped the subscripts on the bundle maps and where both equations labeled (1) follow from the definition of the bundle map on a fibre product and (1) follows since $f$ and $g$ are bundle maps. Next fix open sets $V \subseteq A$ and $U \subseteq B$, and consider $V \ast U := (V \times U) \cap (\mathcal{A} \ast q \mathcal{B})$. We have $(f \ast g)(V \ast U) = f(V) \ast g(U)$, since $f$ and $g$ are bundle maps. Since $f$ and $g$ are open, $f(V) \ast g(U)$ is an open set in $\mathcal{A}' \ast q \mathcal{B}'$. This proves that $f \ast g$ maps basic open sets to (basic) open sets and is hence open.

The following lemmas give an upper semi-continuous version of [3] II.13.16 and 17 and should be compared to [2] Proposition 2.4. We first prove a strengthened condition on the convergence in an USC Banach bundle.

Lemma A.3 ([2] Proposition 2.4). Let $\mathcal{M} = (q, \mathcal{M}: M \to X)$ be an USC Banach bundle over some Hausdorff space $X$. Let $\Gamma \subseteq \Gamma(X; \mathcal{M})$ be a vector space of continuous sections such that $\{f(x) : f \in \Gamma\}$ is dense in $M_x$ for each $x \in X$. Suppose $(m_i)_{i}$ is a net in $M$ and $m \in M$. Then the following are equivalent:

1. $m_i \to m$ in $M$.
2. We have $q_\mathcal{M}(m_i) \to q_\mathcal{M}(m)$ and for every $\epsilon > 0$, there exists $g \in \Gamma$ such that $\|m - g(q_\mathcal{M}(m))\| < \epsilon$ and $\|m_i - g(q_\mathcal{M}(m_i))\| < \epsilon$ for $i$ large enough.
3. We have $q_\mathcal{M}(m_i) \to q_\mathcal{M}(m)$ and for all $f \in \Gamma$,

\[
\lim \|m_i - f(q_\mathcal{M}(m_i))\| \leq \|m - f(q_\mathcal{M}(m))\|.
\]
Proof. Let \( x_i := q_\mathcal{M}(m_i) \) and \( x := q_\mathcal{M}(m) \). For \( (1) \implies (3) \) continuity of \( q_\mathcal{M} \) implies \( x_i \to x \). Since any \( f \in \Gamma \) is continuous, we thus have \( f(x_i) \to f(x) \). By Condition \( (\text{USC}2) \) and \( (\text{USC}3) \) it follows that \( m_i - f(x_i) \to m - f(x) \). By Condition \( (\text{USC}4) \) we conclude

\[
\lim \|m_i - f(x_i)\| \leq \|m - f(x)\|.
\]

For \( (3) \implies (2) \) note that for every \( \epsilon > 0 \), there exists \( g \in \Gamma \) such that \( \|m - g(x)\| < \epsilon \), since \( \{f(x) : f \in \Gamma\} \) is dense in \( M_x \). By \( (3) \)

\[
\lim \|m_i - g(x_i)\| \leq \|m - g(x)\|.
\]

Therefore, there exists \( i_0 \) such that for all \( i \geq i_0 \), \( \|m_i - g(x_i)\| \leq \|m - g(x)\| < \epsilon \).

Finally, \( (2) \implies (1) \) follows from \cite{9} Proposition C.20. \( \square \)

Remark A.4. For continuous Banach bundles, the \( \lim \) in Condition \( (3) \) can be replaced by \( \lim \) (see \cite{3} II.13.12). While \cite{9} Appendix C focuses on upper semi-continuous \( C^* \)-bundles, several of its proofs, including the proof of \cite{9} Proposition C.20, do not use the \( C^* \)-identity and thus work for general USC Banach bundle.

If we pick a continuous section \( f \in \Gamma(X; \mathcal{M}) \) such that \( f(q_\mathcal{M}(m)) = m \), then \( m_i \to m \) if and only if \( q_\mathcal{M}(m_i) \to q_\mathcal{M}(m) \) and by \( (3) \)

\[
0 \leq \lim \|m_i - f(q_\mathcal{M}(m_i))\| \leq \|m - f(q_\mathcal{M}(m))\| = 0.
\]

In this case, the \( \lim \) can likewise be replaced by \( \lim \) (see \cite{7} Lemma A.3).)

Lemma A.5 (cf. \cite{9} Lemma C.18). Let \( \mathcal{M} = (q_\mathcal{M} : M \to X) \) be a USC Banach bundle over some Hausdorff space \( X \). Then \( m_i \to 0_x \) if and only if \( q_\mathcal{M}(m_i) \to x \) and \( \|m_i\| \to 0 \).

Proof. It follows from the upper semi-continuity of \( m \mapsto \|m\| \) and the continuity of \( q_\mathcal{M} \) that \( m_i \to 0_x \) implies \( q_\mathcal{M}(m_i) \to x \) and \( \lim \|m_i\| \leq 0 \); in particular, \( \|m_i\| \to 0 \). The converse follows from Condition \( (\text{USC}4) \). \( \square \)

Remark A.6. For continuous Banach bundles, Lemma A.5 is a direct consequence of continuity of the norm. As our proof shows, however, upper semi-continuity is in fact sufficient.

The above results imply that a USC analogue of \cite{3} II.13.16 and II.13.17 holds: their proofs go through verbatim, except where continuity of the norm needs to be replaced by \( \lim \)-approximation of the norm. To be more precise:

Proposition A.7 (cf. \cite{3} II.13.16). Let \( \mathcal{M} = (q_\mathcal{M} : M \to X) \) and \( \mathcal{M}' = (q_\mathcal{M}' : M' \to X') \) be two USC Banach bundles and let \( \omega : X \to X' \) be a homeomorphism. Let \( \Gamma \) be a vector space of continuous sections of \( \mathcal{M} \) such that the \( \mathcal{C} \)-linear span of \( \{f(x) : f \in \Gamma\} \) is dense in each \( M_x \). Let \( \Phi : M \to M' \) be a map which satisfies the following:

(a) For each \( x \in X \), \( \Phi(M_x) \subseteq M_{\omega(x)} \) and \( \Phi|_{M_x} \) is linear.

(b) There exists a constant \( K > 0 \) such that \( \|\Phi(m)\| \leq K \|m\| \) for all \( m \in M \).

(c) For each \( f \in \Gamma \), \( \Phi \circ f \circ \omega^{-1} \) is a continuous section of \( \mathcal{M}' \).

Then \( \Phi \) is continuous.

Proof. Suppose we have a convergent net in \( \mathcal{M} \), say \( m_i \to m \). Let \( x_i = q_\mathcal{M}(m_i) \), \( x = q_\mathcal{M}(m) \), and \( x'_i = \omega(q_\mathcal{M}(m_i)) \), \( x' = \omega(q_\mathcal{M}(m)) \); by continuity of \( q_\mathcal{M} \) and \( \omega \), we have \( x_i \to x \) in \( X \) and \( x'_i \to x' \) in \( X' \). By (a) we have \( \Phi(m_i) = x'_i \) and \( \Phi(m) = x' \).

Now fix \( \epsilon > 0 \). By density of \( \Gamma \) in \( M_x \), we can pick some \( g_1, \ldots, g_n \in \Gamma \) such that we have \( \|m - g(x)\| < \frac{\epsilon}{K} \) for \( g = \sum_{j=1}^n g_j \) and \( K \) the constant in (b). Since \( g \) is continuous, Lemma A.3 \( (1) \implies (3) \) asserts that there exists \( i_0 \) such that for all \( i \geq i_0 \),

\[
\|m_i - g(x_i)\| \leq \|m - g(x)\| < \frac{\epsilon}{K}.
\]
Using Assumptions \[\text{(a)}\] and \[\text{(b)}\] we see that not only
\[
\| \Phi(m) - \Phi \circ g(x) \| \leq K \| m - g(x) \| < \epsilon,
\]
but also for \(i \geq i_0\),
\[
\| \Phi(m_i) - \Phi \circ g(x_i) \| \leq K \| m_i - g(x_i) \| < \epsilon.
\]
If we let \(f := \Phi \circ g \circ \omega^{-1}\), then the above inequalities can be rephrased to
\[
\| \Phi(m) - f(x') \| < \epsilon \quad \text{and} \quad \| \Phi(m_i) - f(x_i') \| < \epsilon
\]
for all \(i \geq i_0\). By Assumption \[\text{(c)}\], each \(\Phi \circ g_j \circ \omega^{-1}\) is a continuous section of \(\mathcal{M}'\), and hence so is \(f = \sum_{j=1}^n \Phi \circ g_j \circ \omega^{-1}\). Since \(\epsilon > 0\) was arbitrary, it follows from [9, Proposition C.20] that \(\Phi(m_i) \to \Phi(m)\), proving the continuity of \(\Phi\). \(\square\)

**Proposition A.8** (cf. [3, II.13.17]). Let \(\mathcal{M} = (q, \mathbb{M}: M \to X)\) and \(\mathcal{M}' = (q', \mathbb{M}': M' \to X')\) be two USC Banach bundles and let \(\omega: X \to X'\) be a homeomorphism. Let \(\Phi: M \to M'\) be a map such that:

1. For each \(x \in X\), \(\Phi(M_x) \subseteq M'_\omega(x)\) and \(\Phi|_{M_x}\) is linear.
2. \(\Phi\) is continuous.
3. There exists a constant \(k > 0\) such that \(\| \Phi(m) \| \geq k \| m \|\) for all \(m \in M\).

Then \(\Phi^{-1}: \Phi(M) \to M\) is continuous.

Note that \(\Phi^{-1}\) exists because \(\Phi\) is injective as a fibrewise linear map that is bounded below in norm.

**Proof.** The proof of [3, II.13.17] can be adapted almost verbatim: They invoke continuity only once to argue that \(m_i \to 0_x\) if and only if \(q, \mathbb{M}(m_i) \to x\) and \(\| m_i \| \to 0\). But we have seen in Lemma A.5 that that condition is also true in USC Banach bundles. Moreover, adapting from \(X = X'\) to the more general case involving \(\omega\) is straightforward as done in Proposition A.7. We will omit the details. \(\square\)

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