Magnetic Field Distribution Generated by an Elliptic Electric Current

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Abstract. Because of the change in the diameter of the elliptic current, the magnetic field distribution of its arbitrary spatial position cannot be transformed into an elliptic integral problem. It is very difficult or even impossible to solve this problem by analytical method. Intensity distributions of the magnetic field excited by an elliptic electric current are studied. The magnetic field intensity of the central axis is discussed using analytical and numerical simulation methods. The results obtained by the two methods are in good agreement. Application of numerical simulation method in rectangular coordinate system, spatial distributions of the magnetic field of an elliptic electric current is displayed intuitively, the effect of flattening of ellipse on the spatial magnetic field distribution is analyzed, and the characteristics of spatial magnetic field distribution of elliptic electric current are revealed further.

Keywords. Elliptic electric current; magnetic field; spatial distribution.

1. Introduction

In the study of the field intensity distribution of a circular ring charge and a circular current, because of the circular symmetry of the field distribution, it can be converted into ellipse integrals. Although it is very difficult to solve, we can get the analytical solution of the problem by using elliptic integrals [1-5]. In the circular current plane, the intensity distribution of the magnetic field is discussed in references [6-7]. In reference [8], the uniform magnetic field area of Helmholtz is discussed by calculating the magnetic potential vector first and then the magnetic induction intensity; in reference [9], the small circulation is compared with the electric dipole, and the characteristics of the magnetic field of some small circulation are exhibited; References [10-11] show the magnetic vector potential and the magnetic field distribution of the circular current; Utilizing the characteristics of cylindrical spherical coordinates, Zhu presents the intensity distribution of the magnetic field of the circular current [12].

Because the pole diameter is not constant, the magnetic field intensity of an elliptic electric current does not have the symmetry of circle, which cannot be converted to an elliptic integral, so it is more difficult to solve and discuss. For some special positions, Zhang [13] gives the magnetic induction intensity of the current center point of the elliptic line. Using the elliptic integral theory, Zhu [14] investigates the magnetic field of the central axis and the effect of the flattening of the ellipse. Through the conformal transformation, the field distribution of elliptic electric current is discussed; however, due to the complexity of transformation and inverse transformation, the spatial magnetic field distribution of elliptic electric current cannot be displayed entirely and directly [15]. In the cylindrical coordinates, the magnetic field intensity of elliptic electric current is given by using the
symmetry of the ellipse and the superposition principle of the magnetic field directly from the calculation formula of Biot Savart law. The distribution function of the field intensity of the central axis is given by using the analytical method. Compared with numerical simulation results, we see that results of the two methods are in good agreement. Using numerical simulation method in rectangular coordinates, the spatial magnetic field distribution of the elliptic electric current is displayed intuitively, the effect of flattening of ellipse on spatial magnetic field distribution is analyzed, and the characteristics of spatial magnetic field intensity of elliptic electric current are revealed further.

2. Theoretical Analysis on the Spatial Distribution of Magnetic Field of an Elliptic Electric Current

There is an elliptic electric current ring of which the electric current strength is \( I \), the long half axis is \( a \), and the short half axis is \( b \). In cylindrical coordinates, the elliptic equation is given by

\[
\rho = \frac{ab}{\sqrt{b^2 \cos^2 \phi + a^2 \sin^2 \phi}}
\]

(a) An elliptic electric current ring in a cylindrical coordinate system; (b) Corresponding polar coordinate plane.

A line electric current element \( Idl \) on the elliptic current excites the magnetic field intensity at the spatial point \( P(\rho_0, \phi_0, z_0) \) is given by

\[
d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \vec{r}_0}{r_0^3}
\]

In order to facilitate the integration along the elliptic ring, we first present the relation between the unit vector \((e_{\rho}, e_{\phi}, e_z)\) at point \( A(\rho, \phi, 0) \) and the unit vector \((e_{\rho_0}, e_{\phi_0}, e_{z_0})\) at point \( P \). It is easily obtained from figure 1b.

\[
\begin{align*}
\vec{e}_{\rho} &= \cos(\phi - \phi_0)\vec{e}_{\rho_0} + \sin(\phi - \phi_0)\vec{e}_{\phi_0} \\
\vec{e}_{\phi} &= -\sin(\phi - \phi_0)\vec{e}_{\rho_0} + \cos(\phi - \phi_0)\vec{e}_{\phi_0} \\
\vec{e}_z &= \vec{e}_z
\end{align*}
\]

Then we have

\[
\begin{align*}
\vec{r} &= \vec{r}_0 + \rho \vec{e}_\rho + z_0 \vec{e}_z \\
\vec{r}_0 &= \vec{r} - \rho \vec{e}_\rho + z_0 \vec{e}_z \\
\rho &= \rho_0 e_{\rho_0} + e_{\phi_0} + \rho \cos(\phi - \phi_0) e_{\phi_0} + \rho \sin(\phi - \phi_0) e_{\phi_0}
\end{align*}
\]
\[ r_0 = \left[ \rho^2 + \rho_0^2 + z_0^2 - 2 \rho \rho_0 \cos(\phi - \phi_0) \right]^2 \]  
(8)

\[ d\tilde{l} = \rho d\tilde{\phi} - \rho e^\rho d\rho = [\rho d\phi \sin(\phi - \phi_0) + d\rho \cos(\phi - \phi_0)]e^\rho - [\rho d\phi \cos(\phi - \phi_0) + d\rho \sin(\phi - \phi_0)]e^\phi_0 \]  
(9)

\[ d\tilde{l} \times r_0 = \left( \rho \cos(\phi - \phi_0) d\phi + \sin(\phi - \phi_0) d\rho \right) z_0 e_{\rho_0} \]

\[ + \left( \rho \sin(\phi - \phi_0) d\phi + \cos(\phi - \phi_0) d\rho \right) z_0 e_{\phi_0} \]

\[ + ( -\rho \rho_0 \cos(\phi - \phi_0) d\phi + \rho^2 \cos(\phi - \phi_0)^2 d\phi - \rho_0 \sin(\phi - \phi_0) d\rho + \rho^2 \sin(\phi - \phi_0)^2 d\phi ) e_{\phi_0} \]  
(10)

The magnetic field intensity at the point \( P(\rho_0, \phi_0, z_0) \) can be decomposed into three components

\[ B_{\rho_0} = \frac{\mu_0 I}{4\pi} \int d\rho_0 \left( \rho \cos(\phi - \phi_0) d\phi + \sin(\phi - \phi_0) d\rho \right) \left[ \rho^2 + \rho_0^2 + z_0^2 - 2 \rho \rho_0 \cos(\phi - \phi_0) \right]^2 \]  
(11)

\[ B_{\phi_0} = \frac{\mu_0 I}{4\pi} \int d\phi_0 \left( \rho \sin(\phi - \phi_0) d\phi - \cos(\phi - \phi_0) d\rho \right) \left[ \rho^2 + \rho_0^2 + z_0^2 - 2 \rho \rho_0 \cos(\phi - \phi_0) \right]^2 \]  
(12)

and

\[ B_{\phi_0} = \frac{\mu_0 I}{4\pi} \int \rho d\phi_0 \cos(\phi - \phi_0) d\phi + \rho_0 \sin(\phi - \phi_0) d\rho \left[ \rho^2 + \rho_0^2 + z_0^2 - 2 \rho \rho_0 \cos(\phi - \phi_0) \right]^2 \]  
(13)

The vector expression of the magnetic field for an arbitrary spatial point \( P(\rho_0, \phi_0, z_0) \) is presented by

\[ \vec{B} = B_{\rho_0} \vec{e}_{\rho_0} + B_{\phi_0} \vec{e}_{\phi_0} + B_{\phi_0} \vec{e}_{\phi_0} \]  
(14)

For arbitrary space points, because the polar diameter \( \rho \) is not constant above expressions, they cannot be converted into elliptic integrals, and analytical solutions cannot be obtained by analytic methods. The following is a discussion of different methods for various specific situations.

3. Analytical Solution of Magnetic Field Distribution of the Elliptic Electric Current in the Central Axis

From equation (1), we have

\[ d\rho = (b^2 - a^2) \sin \phi \cos \phi \frac{ab d\phi}{\sqrt{b^2 \cos \phi^2 + a^2 \sin \phi^2}} \]  
(15)

where on the central axis \( \rho_0 = 0 \), \( \phi_0 = 0 \), \( d\phi = 0 \), and \( z = z_0 \) has been introduced.

Therefore we can write

\[ B_{\rho} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} z_0 \cos \phi [1 + (b^2 - a^2) \sin^2 \phi] ab(b^2 \cos^2 \phi + a^2 \sin^2 \phi) d\phi \]

\[ \sqrt{(z_0^2 (b^2 \cos^2 \phi + a^2 \sin^2 \phi) + a^2 b^2)} \]
\[
\int_0^\pi z_0 \cos \phi [1 + (b^2 - a^2) \sin^2 \phi] ab(b^2 \cos^2 \phi + a^2 \sin^2 \phi) d\phi \\
\int_0^\pi z_0 \cos \phi [1 + (b^2 - a^2) \sin^2 \phi] ab(b^2 \cos^2 \phi + a^2 \sin^2 \phi) d\phi = 0
\]

Equation (16)

Similarly,\n\[
B_\rho = \frac{\mu_0 I}{4\pi} \int_0^\pi \rho \sin \phi d\phi - \cos \phi d\rho = 0
\]

Equation (17)

and\n\[
B_z = \frac{\mu_0 I}{4\pi} \int_0^\pi \rho b^2 \sqrt{\rho (b^2 \cos^2 \phi + a^2 \sin^2 \phi)} d\rho
\]

Equation (18)

In equation (18), taking the transformation \( \theta = \phi + \pi/2 \), we obtain\n\[
B_z = \frac{\mu_0 I}{4\pi} \int_0^\pi \rho \sin^2 \phi d\phi = 0
\]

Equation (19)

where \( b < a \), \( e^2 = 1 - b^2/a^2 < 1 \), and \( e^2 = e^2 z^2 / (b^2 + z^2) < e^2 < 1 \).

Equation (19) is an elliptic integral, using the elliptic integral theory we obtain [16].\n\[
B_z = \frac{\mu_0 I ab}{\pi(b^2 + z^2) \sqrt{a^2 + z^2}} E(k)
\]

Equation (20)

where \( k = \sqrt{(1-b^2/a^2)/(1+z^2/a^2)} \), and \( E(k) \) is a second kind of complete elliptic integral.

Similarly, when \( a < b \), we obtain\n\[
B_z = \frac{\mu_0 I ab}{\pi(a^2 + z^2) \sqrt{b^2 + z^2}} E(k)
\]

Equation (21)

where \( k = \sqrt{(1-a^2/b^2)/(1+z^2/b^2)} \).

Equations (20) and (21) are the analytical distribution function of the magnetic field on the central axis of the elliptic electric current.

Figure 2. Comparisons of theoretical results and numerical simulation results of the field intensity distribution of the elliptic electric current on the central axis line: (a) the central magnetic field \( B_z \) as a function of the variable \( z \) for \( \mu_0 I / \pi = 1, a > b, a = 10, \) and \( b = 3 \); (b) the central magnetic field \( B_z \) as a function of the variable \( b \) for \( \mu_0 I / \pi = 1, z = 0.5, a < b, \) and \( b = 10 \).
In Figure 2, the circles represent results of the numerical simulation directly from the type (18), and the solid lines represent results of the theoretical analysis from equations (20) and (21). Obviously, the results of the two methods are in good agreement.

In Figure 3, the effects of the coordinate $z$, the long half axis $a$, and the short half axis $b$ on the magnetic field strength of the central axis of the ellipse electric current with $\mu_0 I / \pi = 1$.

In Figure 3a, the magnetic field as a function of the coordinate $z$ and the long half axis $a$ when the short half axis $b = 10$ is fixed exhibit effects of the coordinate $z$ and the elliptic flattening $b/a$. The closer the point on the central axis is to the origin and the flatter the ellipse is, the stronger the magnetic field intensity will be. Whether $a < b$ or $a > b$, the flatness of the ellipse makes the magnetic field intensity exhibit a maximum structure. The flatter the ellipse is, the stronger the magnetic field is, which is also clearly seen in Figure 2b. The value of the flattening of the ellipse corresponding to the maximum value of the field intensity can be obtained by numerical solution, the analytical result of which cannot be obtained.

4. Magnetic Field Distribution in Space of Elliptic Electric Current

For the spatial magnetic field of elliptic line current at any point in space, the analytical expressions cannot be obtained theoretically. In order to show and investigate the spatial magnetic field distribution of elliptic current more intuitively, numerical simulation can be used in a rectangular coordinate system.

In a rectangular coordinate system, three components of the magnetic field for an arbitrary point $P(x_0, y_0, z_0)$ in space can be expressed as

$$B_{x_0} = \oint dB_{y_0} = \frac{\mu_0 I}{4\pi} \left\{ \int_{-a}^{a} \frac{z_0 \, dy}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} + \int_{-\pi/2}^{\pi/2} \frac{z_0 \, dy}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} \right\}$$

(22)

$$B_{y_0} = \oint dB_{z_0} = \frac{\mu_0 I}{4\pi} \left\{ \int_{-a}^{a} \frac{-z_0 \, dx}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} + \int_{-\pi/2}^{\pi/2} \frac{-z_0 \, dx}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} \right\}$$

(23)

$$B_{z_0} = \oint dB_{x_0} = \frac{\mu_0 I}{4\pi} \left\{ \int_{-a}^{a} \frac{(y_0 - y)dx - (x_0 - x)dy}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} + \int_{-\pi/2}^{\pi/2} \frac{(y_0 - y)dx - (x_0 - x)dy}{\left[ (x-x_0)^2 + (y-y_0)^2 + z_0^2 \right]^{3/2}} \right\}$$

(24)

where $a > b$, when the integral interval is $[-a, a]$, we have $y = \sqrt{b^2 - b^2 x^2 / a^2}$; when the integral interval is $[-a, a]$, we have $y = -\sqrt{b^2 - b^2 x^2 / a^2}$.
The magnetic field of at any point \( P \) in space is given by

\[
\mathbf{B} = B_0 \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}
\]  

and the magnitude of magnetic field strength is

\[
B = \sqrt{B_x^2 + B_y^2 + B_z^2}
\]  

Figure 4. The intensity distribution of the magnetic field of the elliptical current in the plane \( z=0 \) for \( \mu_0 I / \pi = 1 \). (a) The field intensity as a function of variables \( x \) and \( y \) for \( a = 6 \), and \( b = 4 \); (b) The field intensity as a function of variables \( y \) for \( x = 0 \), \( a = 6 \), and \( b = 4 \); (c) The contour map of the field intensity of an elliptic circular current in the plane \( z = 0 \).

On the plane \( z = 0 \), \( z_0 = 0 \), \( B_x = 0 \), \( B_y = 0 \), and \( B_z \neq 0 \). Then we have \( \mathbf{B} = B_z \mathbf{k} \). From figure 4, it is clearly seen that the points on the current position of the elliptic loop are singularity, the magnetic field inside the elliptic ring possesses the same direction, and the minimum value that is not zero at the origin point; the closer to the elliptical ring is, the stronger the magnetic field will be.

Figure 5. The intensity distribution of the spatial magnetic field of the elliptical current in the plane \( z = 0.6 \) for \( \mu_0 I / \pi = 1 \). (a) The spatial distribution of the field intensity versus variables \( x \) and \( y \) for \( a = 6 \), and \( b = 4 \); (b) The distribution of the field intensity versus variables \( y \) for \( x = 0 \), \( a = 6 \), and \( b = 4 \); (c) The contour map of the spatial field intensity of an elliptic circular current in the plane \( z = 0.6 \).

Figure 5 presents spatial distributions of the magnetic field intensity on the planes \( z = \pm 0.6 \). On the planes, there is no singularity of the magnetic field and the field intensity continuously distribute. The central magnetic field takes the minimum value, and the maximum value of the field intensity appears in the interior of the elliptical ring.
Figure 6. Effects of the elliptic flattening on the magnetic field intensity of the elliptic electric current with \( \mu_0 I / \pi = 1, a > b, x = 0, a = 10, \) and \( b/a \) taking 0.33, and 0.67, respectively. (a) The plane \( z = 0 \); (b) The planes \( z = \pm 0.6 \).

In order to further show effects of elliptic flattening on the magnetic field intensity, the distribution diagrams of the field intensity for different values of flattening are drawn in figure 6. The plane \( z = 0 \) is in Figure 6a and the planes \( z = \pm 0.6 \) are in Figure 6b. When \( a > b, a = 10, \) and \( x = 0 \) are fixed, the central magnetic field inside the elliptic ring decreases with the elliptic flattening \( b/a \) and the maxima of the magnetic field remains unchanged.

5. Conclusions
The magnetic field excited by an elliptic electric current is studied. The magnetic field of the central axis is discussed using the analytical and the numerical simulation method. The results of two methods are in good agreement. Whether \( a < b \) or \( a > b \), the flatness of the ellipse makes the field intensity show a maximum structure. The flatter the ellipse is, the stronger the magnetic field is. On the plane of \( z \neq 0 \), there is no singularity of the magnetic field and the field intensity continuously distribute. The central magnetic field takes the minimum value, and the maximum value of the field intensity appears in the interior of the elliptical ring. The central magnetic field inside the elliptic ring decreases with the elliptic flattening \( b/a \) and the maxima of the field intensity remains unchanged.

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