ARITHMETIC BEHAVIOUR OF FROBENIUS SEMISTABILITY OF SYZYGY BUNDLES FOR PLANE TRINOMIAL CURVES

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Abstract. Here we consider the set of bundles \( \{ V_n \}_{n \in \mathbb{N}} \) associated to the plane trinomial curves \( k[x,y,z]/(h) \). We prove that the Frobenius semistability behaviour of the reduction mod \( p \) of \( V_n \) is a function of the congruence class of \( p \) modulo \( 2\lambda_h \) (an integer invariant associated to \( h \)).

As one of the consequences of this, we prove that if \( V_n \) is semistable in char 0 then its reduction mod \( p \) is strongly semistable, for \( p \) in a Zariski dense set of primes. Moreover, for any given finitely many such semistable bundles \( V_n \), there is a common Zariski dense set of such primes.

1. Introduction

In this paper we discuss the problems regarding Frobenius semistability behaviour of a vector bundle on a nonsingular projective curve.

Recall that a vector bundle \( V \) on a nonsingular projective curve \( X \) is semistable if for any subbundle \( W \subset V \), we have \( \mu(W) \leq \mu(V) \) where \( \mu(W) = \text{deg } W/\text{rank } W \). If \( V \) is not semistable then it has the unique Harder-Narasimhan filtration

\[
0 \subset V_1 \subset \cdots \subset V_n = V \quad \text{such that} \quad \mu(V_1) > \mu(V_2/V_1) \cdots > \mu(V/V_{n-1}),
\]

where \( V_i/V_{i-1} \) is semistable. In this case one defines \( \mu_{\text{max}}(V) = \mu(V_1) \) and \( \mu_{\text{min}}(V) = \mu(V/V_{n-1}) \). Though in characteristic 0, the pull back of a semistable vector under a finite map is semistable, the same is not always true in positive characteristics. On the other hand, the definition of semistability implies that, if \( F : X \to X \) is the Frobenius morphism, and if \( F^*V \) is semistable, then so is \( V \). However, if \( V \) is a semistable and such that \( F^*V \) is not semistable then by the results of Shepherd-Barron [SB] (Corollary 2.5) and X.Sun [S] (Theorem 3.1) there is a bound on \( \mu_{\text{max}}(F^*V) - \mu_{\text{min}}(F^*V) \) in terms of the genus of the curve and the rank of the vector bundle \( V \).

We say a bundle \( V \) is strongly semistable if \( F^s V \) is semistable for all \( s \geq 0 \), where \( F^s \) is the \( s \)-th iterated Frobenius. Recall that unlike semistable bundles, strongly semistable bundles in char \( p > 0 \) behave like semistable bundles in char 0, in many respects. On the other hand there is a result of Langer (Theorem 2.7 of [L]), which says that if \( V \) is a vector bundle (in a fixed char \( p \)) then there is \( s_0 >> 0 \) such that the HN filtration of \( F^{s_0}V \) is the strongly semistable HN filtration, i.e., there is \( s_0 >> 0 \) such that the HN filtration of \( F^{s_0}V \) consists of strongly semistable subquotients.

Now, suppose \( X \) is a nonsingular curve defined over a field of characteristic 0 and \( V \) is a vector bundle on \( X \) and if \( V_p \) denotes the “reduction mod \( p \)” of \( V \), then reduction mod \( p \) of the HN (Harder-Narasimhan) filtration of \( V \) is the HN Filtration of \( V_p \), for \( p >> 0 \). This is a consequence of the openness of the semistability condition (see [Mar]). However such an openness condition does not hold for Frobenius semistability.

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For example, let \( V = Syz(x, y, z) \) be the syzygy bundle on \( X = \text{Proj} \, R \), where \( R = \text{k}[x, y, z]/(x^4 + y^4 + z^4) \) of char \( p \geq d^2 \) then, by [HM] and [T1] it follows that:

\[
\begin{align*}
p \equiv \pm 1 \pmod{8} & \implies F^*V \text{ is semistable for all } s \geq 0 \\
p \equiv \pm 3 \pmod{8} & \implies V \text{ is semistable and } F^*V \text{ has the HN filtration } \\
& \quad \mathcal{L} \subset F^*V \quad \text{and} \quad \mu(\mathcal{L}) = \mu(F^*V) + 2.
\end{align*}
\]

Note that if \( V \) (in characteristic 0) has semistable reduction mod \( p \) for infinitely many primes \( p \) then it is semistable in char 0 to begin with, due to the openness of the semistability property. We look at the following questions.

1. If \( V \) is a semistable vector bundle on \( X \) defined over \( \mathbb{Q} \) then is \( V_p \) (the reduction mod \( p \)) strongly semistable for \( p \) in a Zariski dense set of \( \mathbb{Z} \)?
2. If \( s_0 \) is a number such that \( F^{s_0}V_p \) has strong HN filtration, then can one describe such an \( s_0 \) in terms of the invariants of the curve \( X \), for all but finitely many \( p \)?
3. Is the Frobenius semistability behaviour (i.e., the minimal number \( s_0 \) and the instability degree \( \mu_{\text{max}}(F^{s_0}V_p) - \mu_{\text{min}}(F^{s_0}V_p) \)) a function of the congruence class of \( p \) (modulo) \( N \), for some integer invariant \( N \) of the curve \( X \), for all but finitely many \( p \)? (instead, we may ask if for some finite Galois extension \( K \) of \( \mathbb{Q} \), the Frobenius semistability of \( V_p \) depends only on the splitting behaviour of \( p \) in \( \mathcal{O}_K \) (the ring of integers), for all but finitely many \( p \)).

Here, in this paper, we look at the bundles which arise from the syzygy bundles \( W_n \) of trinomial plane curves \( C \) in \( \mathbb{P}^2 \), defined by the short exact sequences

\[
0 \rightarrow W_n \rightarrow \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C \rightarrow \mathcal{O}_C(n) \rightarrow 0,
\]

where the third map is \((s_1, s_2, s_3) \rightarrow (s_1x^n, s_2y^n, s_3z^n)\). The bundle \( W_n \) is alternatively denoted by \( Syz(x^n, y^n, z^n) \).

Recall that if \( V \) is a rank 2 vector bundle on a nonsingular projective curve \( X \) defined over a field of characteristic \( p > 0 \) then either (a) \( V \) is strongly semistable, i.e., \( F^*V \) is semistable for every \( s \geq 0 \), or (b) for some \( s \geq 0 \), \( F^*V \) is not semistable, and hence it has the nontrivial HN filtration, namely \( \mathcal{L} \subset F^*V \) such that \( \mathcal{L} \) is a line bundle with \( \mu(\mathcal{L}) > \mu(F^*V) \). Note that for such an \( s \), the HN filtration of \( F^*V \) is the strong filtration and \( \mu_{\text{max}}(F^*V) - \mu_{\text{min}}(F^*V) = 2(\mu(\mathcal{L}) - \mu(F^*V)) \).

In this paper we answer the above questions and generalize the above result of Monsky for the set of vector bundles

\[
S_{st} = \{ V_n \mid \pi \circ V_n = \pi^*W_n, \pi : X \rightarrow C \text{ the normalization of } C, \}
\]

\[
W_n \text{ is a syzygy bundle of } C, \quad C \in \{ \text{trinomial curves} \}, \quad n \in \mathbb{N},
\]

where by a trinomial curve \( C \) we mean \( C = \text{Proj} \, \text{k}[x, y, z]/(h) \), for a homogeneous irreducible trinomial \( h \). If a trinomial curve \( C \) is nonsingular then \( V_n = W_n \).

Let \( h \) be a trinomial plane curve of degree \( d \), then following Monsky [Mo2], it is either \text{irregular} or \text{regular} (see beginning of section (2)). For irregular trinomials, the following theorem settles all the above questions.

**Theorem 1.1.** Let \( h \) be a irregular trinomial of degree \( d \) and let \( r \) be the multiplicity of the irregular point (note \( r \geq d/2 \)). Then for all \( n \geq 1 \),

1. If \( r = d/2 \) implies that the bundle \( V_n \) is strongly semistable and
2. If \( r > d/2 \) implies that the bundle \( V_n \) is not semistable to begin with. Moreover it has the HN filtration \( \mathcal{L} \subset V_n \) such that \( \mu(\mathcal{L}) = \mu(\mathcal{L}) + (2r - d)^2n^2/4d \).

In particular the semistability behaviour of \( V_n \) is independent of the characteristic \( p \), (equivalently one can say that it depends on the single congruence class \( p \equiv 1 \pmod{1} \)).

Given a regular trinomial \( h \), there are associated positive integers \( \lambda \) and \( \lambda_h \) (see Notations 3.1). Following is the main result of this paper:
Theorem 1.2. Let $h$ be a regular trinomial of degree $d$ then for given $n \geq 1$, there is a well defined set theoretic map
\[
\Delta_{h,n} : \left(\frac{\mathbb{Z}/2\lambda_h \mathbb{Z}}{\{1,-1\}}\right) \rightarrow \left\{0, \frac{1}{\lambda_h}, \frac{2}{\lambda_h}, \ldots, \frac{\lambda_n-1}{\lambda_h}\right\} \times \{0,1,2,\ldots, \phi(2\lambda_h) - 1\} \cup \{(1,\infty)\}
\]
such that, given $p \geq \max\{n, d^2\}$, we have
\[
p \equiv \pm l \pmod{2\lambda_h} \text{ and } \Delta_{h,n}(l) = (1,\infty) \implies V_n \text{ is strongly semistable and}
\]
p \equiv \pm l \pmod{2\lambda_h} \text{ and } \Delta_{h,n}(l) = (t,s) \implies s \text{ is the least integer such that } F^{**}V_n \text{ is not semistable}

and $F^{**}V_n$ has the HN filtration $\mathcal{L} \subset F^{**}V_n$ with $\mu(\mathcal{L}) = \mu(F^{**}V_n) + \frac{\lambda}{2}(1 - t)$.

The existence of such a map has several consequences:

1. The Frobenius semistability behaviour of $V_n$, for a regular trinomial, is a function on the congruence class of $\pm p \pmod{2\lambda_h}$ (which are atmost $\phi(2\lambda_h)/2$ in number).

In Section 4, we compute $\Delta_{h,n}(1)$, for every $h$ and do more elaborate computations for symmetric (Definition 4.2) trinomials.

2. In particular we deduce that (Theorem 4.6) if $p \geq \max\{n, d^2\}$ then a semistable bundle $V_n$ is always strongly semistable for $p \equiv \pm 1 \pmod{2\lambda_h}$, hence given a finite subset $\{V_{n_1}, \ldots, V_{n_m}\}$ of semistable bundles of $S_m$ (see (1.1)), the set of primes $p$, for which every $V_{n_i}$ is strongly semistable, is a Zariski dense set (Corollary 5.7).

Moreover $V_1$ over a regular trinomial is always semistable and hence

(i) for a given finite set of syzygy bundles $V_1$ of regular trinomial curves, there is a Zariski dense set of primes, for which each of the bundles is strongly semistable. On the other hand

(ii) for any symmetric trinomial $h$ of degree $d \geq 4$ and $d \not\equiv 5$, we show that there is a Zariski dense set of primes for which $V_1$ is not strongly semistable.

3. The existence of such a map $\Delta_{h,n}$ also implies that if there is one prime $p \geq \max\{n, d^2\}$ such that $V_n$ is not strongly semistable then (i) there is a Zariski dense set of primes for which $V_n$ fails to be strongly semistable, and infact (ii) (Theorem 5.5) there is a Zariski dense set of primes for which the first Frobenius pull back $F^*V_n$ is not semistable.

4. Since either (i) $V_n$ is strongly semistable or (ii) $F^{**}V_n$ is not semistable for some $0 \leq s < \phi(2\lambda_n)$, to check the strongly semistability of $V_n$, (i.e., to check the semistability of $F^{**}V_n$, for every $s \geq 0$), it is enough to check that $F^{**}V_n$ is semistable for $s = \phi(2\lambda_n)$.

It would be interesting to know if such properties as in (1)-(4) hold in greater generality.

Moreover, because of the bound on $s$ (Theorem 5.5 and Remark 5.7), for any given explicit trinomial curve $\text{Proj } R$ given by $h$, we can compute $\Delta_{h,n}(l)$ (see Remark 5.8). Therefore for any $p \geq \{n, d^2\}$ ($p = \text{char } R$) we get an effective algorithm to compute $e_{HK}(R, (x^n, y^n, z^n))$ and the HN slopes for all the Frobenius pull backs of $V_n$.

We compute some concrete examples. By Corollary 5.9 if $h$ is symmetric trinomial of degree $d$ then it is trivial to check if the bundle $V_n$ is semistable or not, for all $p \equiv \pm 1 \pmod{2\lambda_h}$, $p > \max\{n, d^2\}$.

We give some examples, where $V_n$ need not be strongly semistable and have complicated Frobenius semistability behaviour. In particular we look at the Klein $d$-curve, $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$. Let $d \geq 4$ be even then Monsky’s computation in [Mo2] gives
\[
p \equiv \pm(d-1) \pmod{2\lambda_h} \implies V_2 \text{ is semistable and } F^*V_1 \text{ is not semistable such that } \mu(\mathcal{L}) = \mu(F^*V_1) + (d^2 - 3d)/2
\]

In Corollary 4.7, we prove, for $3.2^{m-2} < d - 1 < 3.2^{m-1}$ if
\[
p \equiv \lambda_h \pm 2 \pmod{2\lambda_h} \implies F^{m-1}V_1 \text{ is semistable and } F^{m*}V_1 \text{ is not semistable and } \mu(\mathcal{L}) = \mu(F^{m*}V_1) + (d-2) \left(2[d-1 - 3.2^{m-2}] + 2\right).
\]
In this paper we crucially used an old result of Monsky for plane trinomial curves which involves the notion of taxicab distance (introduced in [H] and [HM]):

**Theorem** (Monsky) (see Theorem 2.3 for a more precise version) Let $R = k[x, y, z]/(h)$, where $h$ is a regular trinomial of degree $d$ over a field of char $k = p > 0$. Then

$$e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{1}{p^{2s}} \left(\frac{\lambda(1 - t_{pn})}{2}\right)^2,$$

where, either $s = \infty$, or $s < \infty$ and $(1 - t_{pn}) > 0$ with $t_{pn} = T_d(p^s t_n)$.

We combine this with the result from [T1] which gave a dictionary between $e_{HK}(R, (x^n, y^n, z^n))$ and the Frobenius semistability behaviour of the syzygy bundle $V_n$.

**Theorem** (see Theorem 5.1 for the more precise version) of [T1]): If $p \geq \max\{n, d^2\}$ then, $s = \infty$ implies that bundle $V_n$ is strongly semistable. If $0 \leq s < \infty$ then it is the least number such that $F^{**}V_n$ is not semistable. Moreover, for the HN filtration of

$$0 \subset \mathcal{L} \subset F^{**}(V_n),$$

we have $\mu(\mathcal{L}) = \mu(F^{**}(V_n) + \frac{\lambda(1 - t_{pn})}{2}$.

To prove the main theorem 3.5 for a regular trinomial $h$, we define a set $S_h \subset \mathbb{Z} / 2\lambda h \mathbb{Z})^3$, which is a disjoint union of four sets $T_{ijk}$.

We consider the set $L_{odd} = \{(u_1, u_2, u_3) \in \mathbb{Z}^3\},$ (which was introduced in ([H] and [Mo2]) as the disjoint union of four sets $\{L_{odd,}\}$.

For each $\delta$ and $l$, $n \geq 1$, we define a map (Lemma 3.3) $f_{L_n}^\delta: \mathbb{N} \cup \{0\} \to (\mathbb{Z} / 2\lambda h \mathbb{Z})^3$ and characterize the numbers $s$ and $t_{pn}$ (given as in the above theorem of Monsky) in terms of the set $\bigcup_i (f_{1,n}^\delta(\mathbb{Z} / 2\lambda h \mathbb{Z}) \mathbb{Z})^\delta$: The integer $s$ is the minimum element of the set and if $s \in \text{Im}(f_{1,n}^\delta) \cap T_{ijk}$ (and $i, j, k$ will be unique with this property), then $f_{L,n}(s)$ and $i, j, k$ determine $t_{pn}$ for all $p \equiv \pm l$ (mod $2\lambda h$).

The very definition of $f_{L,n}^\delta$ implies that the map factors through $\mathbb{Z} / \phi(2\lambda h) \mathbb{Z}$ and $f_{L,n}^\delta = f_{L^2,n}^\delta$, which gives a well defined map $\Delta_{h,n}$ as in Theorem 3.5.

As a corollary, for all $p \equiv \pm 1$ (mod $2\lambda h$), we get a simple expression for all trinomials $h$ of degree $d$ (Corollaries 6.1 (1) and 6.2 (2):

$$e_{HK}(k[x, y, z]/(h), (x, y, z)) = \frac{3d}{4}$$

if $h$ is a regular trinomial,

$$= \frac{3d}{4} + \frac{2r^2}{4d},$$

if $h$ is a irregular trinomial, where $h$ has the point of multiplicity $r \geq d/2$.

In Corollary [6,2] when $h$ is a Klein $d$-curve defined over a field of char $p > 0$ such that $p \equiv \lambda h \pm 2$ (mod $2\lambda h$), (as expected from the discussion above) we generate a more complex one.

**Remark 1.3.** As stated earlier, for the Fermat quartic, the function $\Delta_{h,1}$ is completely known by the result of [HM]. For the Fermat curve ($h = x^d + y^d + z^d$) and Klein $d$-curve ($h = x^{d-1}y + y^{d-1}z + z^{d-1}x$) Monsky [Mo2] has computed $\Delta_{h,1}(d-1)$, for even $d$, and $\Delta_{h,1}(\lambda \pm (2d-2))$ for odd $d > 5$ where $h$ is a Klein $d$-curve.

Questions about Frobenius semistability of $V_n$ for the Fermat curve are also studied extensively in works of Brickmann-Kaid, Brenner, Kaid, Stäbler etc. (see the recent paper [BK] and references given there).

### 2. Preliminaries

Let $h \in k[x, y, z]$ be a homogeneous irreducible trinomial of degree $d$, i.e., $h = M_1 + M_2 + M_3$ where $M_i$ are monomials of degree $d$.

By Lemma 2.2 of [Mo2], one can divide such an $h$ in two types:

1. $h$ is ‘irregular’ if one or more of the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{P}^2$ has multiplicity $\geq d/2$ on the plane curve $h$. 

(2) $h$ is ‘regular’, i.e., the exponents $e_1$ of $x$ in $M_1$, $e_2$ of $y$ in $M_2$ and $e_3$ of $z$ in $M_3$, respectively, are all $> d/2$.

Moreover any regular $h$ is equivalent (i.e., one equation is obtained from the other equation by some permutation of $x$, $y$ and $z$) to one of the following:

(a) Type (I): $h = x^{a_1}y^{a_2} + y^{b_1}z^{b_2} + z^{c_1}x^{c_2}$, where $a_1, b_1, c_1 > d/2$, (here $e_1 = a_1$, $e_2 = b_1$ and $e_3 = c_1$).

(b) Type (II): $h = x^{d} + x^{a_1}y^{a_2}z^{a_3} + y^{b_2}z^{c}$, $a_2, c > d/2$, (here $e_1 = d$, $e_2 = a_2$ and $e_3 = c$).

Given a regular trinomial $h$, Monsky defines a set of positive integers $(\alpha, \beta, \nu, \lambda)$ as follows: $\alpha = c_1 + e_2 - d$, $\beta = c_1 + e_3 - d$ and $\nu = e_2 + e_3 - d$. Moreover $\lambda = \frac{1}{d}\det(A)$, where $A$ is a $3 \times 3$ matrix formed from the exponents of $x$, $y$ and $z$ in $M_1$, $M_2$ and $M_3$.

Notations 2.1. In particular, given a regular trinomial $h$, we can associate positive integers $\alpha, \beta, \nu, \lambda > 0$ as follows:

1. Type (I) $h = x^{a_1}y^{a_2} + y^{b_1}z^{b_2} + z^{c_1}x^{c_2}$, denote $\alpha = a_1 + b_1 - d$, $\beta = a_1 + c_1 - d$, $\nu = b_1 + c_1 - d$, $\lambda = a_1b_1 + a_2c_2 - b_1c_2$.

2. Type (II) $h = x^{d} + x^{a_1}y^{a_2}z^{a_3} + y^{b_2}z^{c}$, denote $\alpha = a_2$, $\beta = c$, $\nu = a_2 + c - d$ and $\lambda = a_2c - a_3b$.

Moreover we denote $t = (t_1, t_2, t_3) = (\alpha/\lambda, \beta/\lambda, \nu/\lambda)$.

Definition 2.2. We recall the following definition given in [HM] and [Mo2], where $p = \text{char} \ k > 0$: Let $L_{odd} = \{ u = (u_1, u_2, u_3) \in \mathbb{Z}^3 \mid \sum_i u_i \text{ odd} \}$. For any $u \in L_{odd}$ and for $s \in \mathbb{Z}$ and $n \geq 1$, the \textit{taxicab distance} between the triples $p^stn = (p^st_1n, p^st_2n, p^st_3n)$, and $u$ is $TD(p^stn, u) = \sum_i |p^st_i - u_i|$. They define $\delta^s(tn) = p^{-s}(1 - Td(p^stn, u))$, and $s$ is the smallest integer such that $Td(p^stn, u) < 1$, for some $u \in L_{odd}$. If there is no such pair then they define $\delta^s(tn) = 0$.

Following is the crucial Theorem 2.3 of [Mo2]

Theorem 2.3. Let $R = k[x, y, z]/(h)$, where $h$ is a regular trinomial of degree $d$. Then
\[
\epsilon_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{\lambda^2}{4d} [\delta^s(tn)]^2 = \frac{3dn^2}{4} + \frac{\lambda^2}{4dp^{2s}} (1 - Td(p^stn))^2,
\]
where $\alpha$, $\beta$, $\nu$ and $\lambda$ are as in Notations 2.1.

We extend the definition of Monsky to every integer $l$, as follows.

Definition 2.4. For an integer $l \geq 1$ we denote $Td(l^st) = Td(l^st, u)$, if there exists a $u \in L_{odd}$ such that $Td(l^st, u) < 1$ (note that such a $u$ is unique if it exists).

Lemma 2.5. (1) The triple $(\alpha, \beta, \nu)$ satisfies the triangle inequalities: $\alpha < \beta + \nu$, $\beta < \alpha + \nu$ and $\nu < \beta + \alpha$. 

(2) and $2\lambda \geq \alpha + \beta + \nu$. Moreover 

(3) the inequality $Td(l^{-stn}, u) < 1$ has no solution, for $s > 0$ and $l \geq n$.

Proof. The triangle inequalities of (1) are obvious as pointed out in [Mo2].

(2) Let $\alpha, \beta, \nu, \lambda$ be the associated integers to the trinomial $h$ of type (I). Then $2\lambda < \alpha + \beta + \nu$ implies $b_1(a_1 - c_2 - 1) + a_2c_2 + d/2 + a_2 < c_1$.

But $a_1 - c_2 - 1 \geq 0$. Now if $a_1 - c_2 - 1 \geq 1$ then $b_1(a_1 - c_2 - 1) + a_2c_2 + d/2 + a_2 > d \geq c_1$, which is a contradiction. (b) If $a_1 - c_2 - 1 = 0$ then $a_1 = t + 1$ and $c_2 = t$, where $d = 2t + 1$ or $d = 2l$. Now

$h_1 = a_1(c_2 + 1) + a_2c_2 + d/2 + a_2 = t^2 + d/2 + t > t + 1 = c_1$,

which is again a contradiction.
(2) (ii) Let $\alpha, \beta, \nu, \lambda$ be associated to the trinomial $h$ of type (II). Then

$$2\lambda < \alpha + \beta + \nu \implies 2a\nu - a\beta < a\beta + c - d/2 \implies (a_2 - 1)(c - 1) - 1 + d/2 < a\beta,$$

which is not possible as $a_2 - 1 \geq a_3$, $c - 1 \geq b$ and $d \geq 2$.

This proves part (2).

(3) Let $t = (t_1, t_2, t_3) = (\alpha/\lambda, \beta/\lambda, \nu/\lambda)$. Note that $s > 0$ and $l \geq n$ implies $0 \leq |t_1n/l^*| = |\alpha n/\lambda| < 1$. Let $u = \{u_1, u_2, u_3\} \in L_{odd}$. Hence for $u_1$ odd we have $|t_1n/l^* - u_1| = 1 - |\alpha n/\lambda|$ and for $u_1$ even we have $|t_1n/l^* - u_1| = \alpha n/\lambda$. Similar assertions hold for $u_2$ and $u_3$. (i) If $u_1$, $u_2$ and $u_3$ are odd then $Td(l^{-*}tn, u) = 3 - (\alpha + \beta + \nu)n/l^*\lambda$. Therefore the existence of a solution for

$$Td(l^{-*}tn, u) < 1 \implies 2\lambda l^* < (\alpha + \beta + \nu)n \implies 2\lambda < (\alpha + \beta + \nu).$$

which is not possible by (2).

(ii) Suppose only one of the $u'_i$'s is odd. Without loss of generality we assume that $u_1$ is odd then $u_2$ and $u_3$ are even. Now $Td(l^*t, u) < 1$ if and only if $\beta + \nu < \alpha$, which contradicts (1). This proves the lemma. $\square$

3. Main theorem

Throughout this section $h$ denotes a regular trinomial.

**Notations 3.1.** Let $\alpha, \beta, \nu, \lambda$ integers associated to $h$ as in Notations 2.1. Let $a = \gcd(\alpha, \beta, \nu, \lambda)$. Then we denote

$$\lambda_h = \frac{\lambda}{a}, \alpha_1 = \frac{\alpha}{a}, \beta_1 = \frac{\beta}{a}, \nu_1 = \frac{\nu}{a}.$$

**Definition 3.2.** Let

$$\delta = (\delta_1, \delta_2, \delta_3) \in \{(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\} \subset L_{odd}.$$

For given such $\delta$, we say $u \in L_{odd}^{\delta}$ if $u = (2v_1 + \delta_1, 2v_2 + \delta_2, 2v_3 + \delta_3)$, where $v_1, v_2, v_3$ are integers.

Thus we can partition $L_{odd}$ into four disjoint sets

$$L_{odd} = \bigcup_{\delta} L_{odd}^{\delta} = L_{odd}^{(1,1,1)} \cup L_{odd}^{(1,0,0)} \cup L_{odd}^{(0,1,0)} \cup L_{odd}^{(0,0,1)}.$$

**Definition 3.3.** Let $R = (\mathbb{Z}/2\lambda_h\mathbb{Z})^3$. We say the element $(w_1, w_2, w_3) \in \mathbb{Z}^3$ represents (or is the representative of) $w \in R$ if $(w_1, w_2, w_3) = w \mod (2\lambda_h\mathbb{Z})^3$ such that $0 \leq w_1, w_2, w_3 < 2\lambda_h$.

We define

$$S_h = T_{000} \cup T_{100} \cup T_{010} \cup T_{001} \cup T_{110} \cup T_{011} \cup T_{101} \cup T_{111} \subset R,$$

where, for $(i, j, k) \in \{0, 1\}^3$,

$$T_{ijk} = \{w \in R \mid 2\lambda_h i + (1)^i w_1 + 2\lambda_h j + (1)^j w_2 + 2\lambda_h k + (1)^k w_3 < \lambda_h, \ (w_1, w_2, w_3) \text{ represents } w\}.$$

For example

$$T_{000} = \{w \in R \mid w_1 + w_2 + w_3 < \lambda_h, \ \text{where } (w_1, w_2, w_3) \text{ represents } w\} \quad \text{and}$$

$$T_{100} = \{w \in R \mid 2\lambda_h - w_1 + w_2 + w_3 < \lambda_h, \ \text{where } (w_1, w_2, w_3) \text{ represents } w\}, \ etc..$$

Note that

$$(\text{the representatives of } T_{ijk}) \subset [i\lambda_h, (i + 1)\lambda_h) \times [j\lambda_h, (j + 1)\lambda_h) \times [k\lambda_h, (k + 1)\lambda_h] \subset \mathbb{Z}^3.$$

In particular the set $S_h$ is a disjoint union of $\{T_{ijk}\}_{i,j,k \in \{0,1\}}$.

**Lemma 3.4.** For a given $\delta$ as given in Equation 2.7 and for given integers $n, l \geq 1$, let

$$f_{l, \delta}^h : \mathbb{N} \cup \{0\} \rightarrow (\mathbb{Z}/2\lambda_h\mathbb{Z})^3,$$

given by $s \rightarrow (l^s \alpha_1 n - \delta_1 \lambda_h, l^s \beta_1 n - \delta_2 \lambda_h, l^s \nu_1 n - \delta_3 \lambda_h) = \lambda_h (l^s tn - \delta) \mod (2\lambda_h\mathbb{Z})^3$ be a set theoretic map. Then
(1) for an integer \( s \geq 0 \), the element \( f_{l,n}^\delta(s) \in S_h \) if and only if \( Td(l^tn, u) < 1 \) has a solution for some \( u \in L_{odd}^\delta \). Moreover,

(2) in this case, \( f_{l,n}^\delta(s) \) determines \( Td(l^tn, u) \):

\[
Td(l^tn) = 2(i + j + k)(-1)^i \frac{w_1}{\lambda_h} + (-1)^j \frac{w_2}{\lambda_h} + (-1)^k \frac{w_3}{\lambda_h},
\]

where \((w_1, w_2, w_3)\) is the representative of \( f_{l,n}^\delta(s) \) and \((i, j, k)\) is the triple such that \( f_{l,n}^\delta(s) \in T_{ijk} \).

Proof. Suppose \( Td(l^tn, u) < 1 \) has a solution for some \( u \in L_{odd}^\delta \). Then we have \( u = (2v_1 + \delta_1, 2v_2 + \delta_2, 2v_3 + \delta_3) \), for some integers \( v_1, v_2 \) and \( v_3 \). Therefore \( Td(l^tn, u) < 1 \) implies

\[
|l^\alpha_1n - \delta_1\lambda_h - 2v_1\lambda_h| + |l^\beta_1n - \delta_2\lambda_h - 2v_2\lambda_h| + |l^\nu_1n - \delta_3\lambda_h - 2v_3\lambda_h| < \lambda_h
\]

Let \((w_1, w_2, w_3)\) be the representative of \( f_{l,n}^\delta(s) \). Then

\[
(w_1, w_2, w_3) = (l^\alpha_1n - \delta_1\lambda_h, l^\beta_1n - \delta_2\lambda_h, l^\nu_1n - \delta_3\lambda_h, l^\nu_2n - \delta_3\lambda_h + 2k_3\lambda_h)
\]

for some integers \( k_1, k_2 \) and \( k_3 \). Hence by Equation (3.3),

\[
\frac{w_1}{2\lambda_h} - (v_1 + k_1) + \frac{w_2}{2\lambda_h} - (v_2 + k_2) + \frac{w_3}{2\lambda_h} - (v_3 + k_3) < \frac{1}{2}
\]

Now

\[
w_1 \in [0, \lambda_h] \implies v_1 + k_1 = 0 \quad \text{and} \quad \frac{w_1}{2\lambda_h} - (v_1 + k_1) = \frac{w_1}{2\lambda_h}.
\]

If \( w_1 \in [\lambda_h, 2\lambda_h) \) then \( v_1 + k_1 = 1 \) and \( \frac{w_1}{2\lambda_h} - (v_1 + k_1) = 1 - \frac{w_1}{2\lambda_h} \). In other words

\[
w_1 \in [i\lambda_h, (i + 1)\lambda_h) \implies v_1 + k_1 = i \quad \text{and} \quad \frac{w_1}{2\lambda_h} - (v_1 + k_1) = i - \frac{w_1}{2\lambda_h}.
\]

Similar statements hold for \( w_2 \) and \( w_3 \). Now Equation (3.3) gives

\[
i + \frac{1}{2\lambda_h} + j + \frac{1}{2\lambda_h} + k + \frac{1}{2\lambda_h} < \frac{1}{2},
\]

which implies \( f_{l,n}^\delta(s) \in T_{ijk} \subset S_h \).

Conversely, let \( f_{l,n}^\delta(s) \in S_h \) then there exists a unique \( T_{ijk} \) such that \( f_{l,n}^\delta(s) \in T_{ijk} \). Therefore \( f_{l,n}^\delta(s) \) is represented by \((w_1, w_2, w_3) \in \mathbb{Z}^3 \) such that

\[
2\lambda hi + (-1)^i w_1 + 2\lambda hj + (-1)^j w_2 + 2\lambda hk + (-1)^k w_3 < \lambda_h.
\]

Let

\[
(w_1, w_2, w_3) = (l^\alpha_1n - \delta_1\lambda_h + 2k_1\lambda_h, l^\beta_1n - \delta_2\lambda_h + 2k_2\lambda_h, l^\nu_1n - \delta_3\lambda_h + 2k_3\lambda_h).
\]

Then, by inequality (3.4), we have

\[
Td(l^tn, u) = \left| \frac{l^\alpha_1n}{\lambda_h} - u_1 \right| + \left| \frac{l^\beta_1n}{\lambda_h} - u_2 \right| + \left| \frac{l^\nu_1n}{\lambda_h} - u_3 \right| < 1,
\]

where

\[
u = (\delta_1 - 2k_1 + (-1)^i2i, \delta_2 - 2k_2 + (-1)^j2j, \delta_3 - 2k_3 + (-1)^k2k) \in L_{odd}^\delta.
\]

This also proves that \( Td(l^tn) = 2(i + j + k)(-1)^i w_1/\lambda_h + (-1)^j w_2/\lambda_h + (-1)^k w_3/\lambda_h \), which proves part (2) of the lemma and hence the lemma.

\[\square\]

**Theorem 3.5.** Let \( h \in k[x, y, z] \) be a regular trinomial over a field of char \( p > 0 \). Consider the set theoretic map

\[
\Delta_{h,n} : \left( \frac{\mathbb{Z}/2\lambda_h\mathbb{Z}}{\{1, -1\}} \right) \times \left\{ \frac{1}{\lambda_h}, \frac{2}{\lambda_h}, \ldots, \frac{\lambda_h - 1}{\lambda_h} \right\} \times \{0, 1, \ldots, \phi(2\lambda_h) - 1\} \cup \{(1, \infty)\},
\]

given by \( l \to (Td(l), Ds(l)) \), where \( Ds(l) = s \geq 0 \) is the smallest integer, for which \( Td(l^tn, u) < 1 \) has a solution for some \( u \in L_{odd} \) and \( Td(l) := Td(l^tn, u) \). If there is no such \( s \) then \( \Delta_{h,n}(l) = (1, \infty) \).
Remark 3.7. 

Proof. (1) By Lemma 3.4, the inequality $T_d(l^*tn, u) < 1$ has a solution if and only if $f^\delta_{l,n}(s) \in S_h$, for some $\delta$ (if it does then $f^\delta_{l,n}(s) \in S_h$, for a unique $\delta$). Hence $D_s(l) = \min\{s' \mid s' \in (\bigcup_0^\infty \text{Im}(f^\delta_{l,n})) \cap S_h\}

Let $B = \{0, 1, \frac{2}{\lambda_h}, \ldots, \frac{\lambda_h - 1}{\lambda_h}\} \times \{0, 1, \ldots, \phi(2\lambda_h) - 1\} \cup \{(1, \infty)\}$, and let $\mathbb{Z}_{\geq 0} \to B$ be the map given by $l \mapsto (T_d(l), D_s(l))$. By the definition of $f^\delta_{l,n}$, it follows that $f^\delta_{l,n}(s) = f^\delta_{l+2\lambda_h,n}(s)$, for all $s \geq 0$. Therefore, by Lemma 3.4(2), $D_s(l) = D_s(l + 2\lambda_h)$ and $T_d(l) = T_d(l + 2\lambda_h)$. Hence the above map factors through $\mathbb{Z}/2\lambda_h\mathbb{Z} \to B$, which gives a well defined map $(\mathbb{Z}/2\lambda_h\mathbb{Z})^* \to \mathbb{Z}/2\lambda_h\mathbb{Z} \to B$.

Now let $l' = 2\lambda_h - l$ then $l'^* = 2\lambda_hk - (-l)^*$, for some integer $k$. If $s$ is even then $f^\delta_{l',n}(s) = f^\delta_{l,n}(s)$. If $s$ is odd then $l'^* = 2\lambda_hk - l^*$. Let $u = (u_1, u_2, u_3) \in L^\delta_{odd}$ such that $T_d(l'^*tn, u) < 1$ has a solution. Then

$$T_d(l'^*tn, u) = \left|\frac{l'^*\alpha_1n}{\lambda_h} - u_1\right| + \left|\frac{l'^*\beta_1n}{\lambda_h} - u_2\right| + \left|\frac{l'^*\nu_1n}{\lambda_h} - u_3\right| = T_d(l'^*tn, u') < 1,$$

where $u' = (2k\alpha_1n - u_1, 2k\beta_1n - u_2, 2k\nu_1n - u_3) \in L^\delta_{odd}$. This implies that for any $s \geq 0$, $f^\delta_{l,n}(s) \in S_h$ if and only if $f^\delta_{l,n}(s) \in S_h$ and $T_d(l'^*tn) = T_d(l'^*tn)$. Hence $(T_d(l), D_s(l)) = (T_d(l'), D_s(l'))$. This gives the well defined map $(\mathbb{Z}/2\lambda_h\mathbb{Z})^* \to \mathbb{Z}/2\lambda_h\mathbb{Z} \to B$, which is $\Delta_{h,n}$.

This proves assertion (1) of the theorem.

(2) Since $f^\delta_{l,n} = f^\delta_{l,n+2\lambda_h}$, assertion (2) follows.

(3) If $D_s(l) < \infty$ then $f^\delta_{l,n}(s) \in S_h$, for some $s \in \mathbb{Z}_{\geq 0}$. Let order of $t$ in $(\mathbb{Z}/2\lambda_h\mathbb{Z})^*$ be $t$. We can write $s = kt + r$, for some integers $k$ and $r$ such that $0 \leq r < t$. Then $l'^* = l'^*t = (2\lambda_hk + 1)t'$, for some $k \in \mathbb{Z}$. This implies $f^\delta_{l,n}(s) = f^\delta_{l,n}(r)$, as $\alpha_1, \beta_1, \nu_1, \lambda_h$ are integers. Hence $D_s(l) \leq r < O(l)$. This proves the assertion (3).

(4) Note that $s$ is the minimal integer such that $f^\delta_{l,n}(s) \in S_h$ if and only if $s_1$ is the minimal integer such that $f^\delta_{l'^{s_1},n}(s_1) \in S_h$. Moreover $f^\delta_{l,n}(s) = f^\delta_{l'^{s_1},n}(s_1)$. Therefore $\Delta_{h,n}(l'^{s_1}) = (t, s_1)$. This proves the assertion (4) and hence the theorem.

Corollary 3.6. Let $s \geq 0$ and $1 \leq l < 2\lambda_h$ be integers. Then

$$T_d(l'^*tn) = T_d(p'^*tn) \quad \text{for} \quad p \geq n \quad \text{where} \quad p \equiv \pm l \pmod{2\lambda_h}.$$ 

Moreover, in that case

$$T_d(l'^*tn) = T_d((2\lambda_h - l)^*tn) = T_d(l'^*tn + 2\lambda_h)) = T_d(p'^*tn).$$ 

Proof. It follows from Theorem 3.5.
Remark 3.8. Given an explicit trinomial $h$ of degree $d$ over a field of char $p > 0$, let $p \equiv l \pmod{2\lambda_h}$ then we can compute $\Delta_{h,n}(l)$ in an effective way: Let $O(l)$ be the order of $l$ in $\mathbb{Z}/2\lambda_h \mathbb{Z}$ (in fact can take $O(l)$ to be the order of $l$ in $\mathbb{Z}/2\lambda_h \mathbb{Z}$), we look for the first $0 \leq s \leq O(l) - 1$, where $T_d(l^* t_n u) = \sum_i |l^* t_n u_i| < 1$ has a solution for some $u \in L_{od}$. If there is such a solution then $\Delta_{h,n}(l) = (s, t, s)$. Otherwise $\Delta_{h,n}(l) = (1, \infty)$.

4. Computations of some values of $\Delta_{h,n}$

We will see that $\Delta_{h,n}(l \mod 2\lambda_h)$ determines the Frobenius data (Lemma 2.3) of $V_n$ over the trinomial $h$, for $p \equiv \pm l \pmod{2\lambda_h}$ and also Hilbert-Kunz multiplicity (Theorem 5.3) of $k[x,y,z]/(h)$ with respect to the ideal $(x^n, y^n, z^n)$, we compute some of them.

Theorem 4.1. Let $h$ be a regular trinomial then

1. For $n = 1$, $\Delta_{h,n}(1 \mod 2\lambda_h) = (1, \infty)$.
2. In general, for $n > 1$,
   
   $\begin{align*}
   &\text{either } \Delta_{h,n}(1 \mod 2\lambda_h) = (0, \infty), \\
   &\text{or } \Delta_{h,n}(1 \mod 2\lambda_h) = (T_d(1), 0).
   \end{align*}$

Proof. Since the order of the element $l = 1$ is $1$ in $\mathbb{Z}/2\lambda_h \mathbb{Z}^*$. Assertion (2) follows from Theorem 4.1 (3).

To prove Assertion (1), it is enough to show that $T_d(t, u) < 1$ has no solution. Note that $\alpha, \beta, \nu < \lambda$. Therefore $\alpha/\lambda, \beta/\lambda, \nu/\lambda < 1$.

Let $u \in L_{od}$ be a solution for $T_d(t, u) < 1$. Then $u_1$ odd implies $u_1 = 1$ which implies $|\alpha/\lambda - u_1| = 1 - \alpha/\lambda$, and $u_1$ even implies $u_1 = 0$ and $|\alpha/\lambda - u_1| = \alpha/\lambda$.

(i) Suppose only one of the $u_i$’s is odd. Without loss of generality we assume that $u_1$ is odd then $u_2$ and $u_3$ are even. Now $T_d(t, u) = 1 - \alpha/\lambda + \beta/\lambda + \nu/\lambda < 1$ if and only if $\beta + \nu < \alpha$, which contradicts Lemma 2.3 (1).

(ii) Suppose $u_1, u_2$ and $u_3$ are odd. Then $T_d(t, u) = 1 - \alpha/\lambda + 1 - \beta/\lambda + 1 - \nu/\lambda < 1$ implies $2\lambda < \alpha + \beta + \nu$, which is not true by Lemma 2.3 (2). This proves that $T_d(t, u) < 1$ has no solution for any $s \in \mathbb{Z}$ and $u \in L_{od}$. Hence $\Delta_{h,n}(1) = (1, \infty)$. This proves (1). □

4.1. Some Computations of $\Delta_{h,n}$ for symmetric trinomial curves.

Definition 4.2. A trinomial curve $h$ of degree $d$ is symmetric if $h = x^a y^p z^r + y^a z^p x^r + z^a x^p y^r$.

Remark 4.3. A trinomial curve is symmetric if and only if $\alpha = \beta = \nu$. One can easily check that if $T_d(l^* t_n u) < 1$ has a solution for some $(u_1, u_2, u_3) \in L_{od}$ then $u_1 = u_2 = u_3$ and $u_1$ is odd.

Corollary 4.4. Let $h$ be a symmetric curve of degree $d$. For a given $n \geq 1$, we have

1. $\Delta_{h,n}(1 \mod 2\lambda_h) = (T_d(1), D_s(1)) = \left(3|m_1 - \frac{\alpha n}{\lambda}|, 0\right)$, if $|m_1 - \frac{\alpha n}{\lambda}| < 1/3,$
2. $\Delta_{h,n}(1 \mod 2\lambda_h) = (1, \infty)$ otherwise,

where $m_1$ is one of the nearest odd integer to $\alpha n/\lambda$.

Proof. By Theorem 4.1 it is enough to compute $T_d(t n, u)$.

If $\alpha n/\lambda$ is an even integer then one of the $u_i$’s, say $u_1$, is equal to $\alpha n/\lambda \pm 1$, which implies $T_d(t n) \geq 1$. So $D_s(1) = \infty$. On the other hand for any nearest odd integer $m_1$ to $\alpha n/\lambda$, we have $|m_1 - \frac{\alpha n}{\lambda}| \geq 1$. This proves the corollary for when $\alpha n/\lambda$ is an even integer.

Therefore we can assume that $\alpha n/\lambda$ is not an even integer, and $m_1$ is the unique nearest odd integer $m_1$.

Let $u = (u_1, u_2, u_3) \in L_{od}$ be a solution for $T_d(t n, u) < 1$ then $u_1 = m_1$ and hence $T_d(t n, u) = 3|m_1 - \alpha n/\lambda|$, which is $< 1$ if and only if $|m_1 - \alpha n/\lambda| < 1/3$.

This implies $\Delta_{h,n}(1 \mod 2\lambda_h) = (3|m_1 - \alpha n/\lambda|, 0)$ if $|m_1 - \alpha n/\lambda| < 1/3$. Otherwise $\Delta_{h,n}(1) = (1, \infty)$. □
Lemma 4.6. Let $h$ be a symmetric curve of degree $d \geq 4$ and $d \neq 5$. Then there is $l' \in (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ such that $\Delta_{h,1}(l') \neq (1, \infty)$. In fact there is $l' \in (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ such that

\[
\begin{align*}
\Delta_{h,1}(l') &= (6/\lambda h, 1) \quad \text{if } d \text{ is odd} \\
\Delta_{h,1}(l') &= (3/\lambda h, 1) \quad \text{if } d \text{ is even and } \lambda h \text{ is even} \\
\Delta_{h,1}(l') &= (t, m) \quad \text{if } d \text{ is even and } \lambda h \text{ is odd},
\end{align*}
\]

where $1 \leq m < \infty$ and $(t, m)$ is given as in Lemma 4.7.

Proof. (1) Note that $d$ odd implies $\alpha, \lambda$ and hence $\alpha_1, \lambda h$ are both odd. Since $\gcd(d, 2\lambda h) = 1$, the map $\mathbb{Z}/2\lambda h\mathbb{Z} \rightarrow (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ given by $l \bmod 2\lambda h \mapsto l\alpha_1 \bmod 2\lambda h$ is bijective.

Let $l = \lambda h + 2$ then there is $l' \in (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ such that $l = l'\alpha_1 \bmod 2\lambda h$. If $\lambda h > 6$ then $\Delta_{h,n}(l') = (3|\alpha n/\lambda h - 2| - 1, 1)$.

If $d > 12$ then $\lambda h \geq (a_1 - a_2) + a_1 a_2/(a_1 - a_2) > 6$. One can check that for $d = 7, 9, 11$ also $\lambda h > 6$. In particular $\Delta_{h,1}(l') = (6/\lambda h, 1)$, if $d > 5$ is odd.

(2) (a) If $d$ and $\lambda h$ are even then $\alpha_1$ is odd, which implies $\gcd(d, 2\lambda h) = 1$. Let $l = \lambda h + 1$ then there is $l' \in (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ such that $l = l'\alpha_1 \bmod 2\lambda h$. If $\lambda h > 3$ (which holds for $d \geq 4$) then $3|l'\alpha/\lambda - 1| = 3/\lambda h < 1$. Therefore $\Delta_{h,n}(l') = (3/\lambda h, 1)$.

(2) (b) Let $d$ be even and $\lambda h$ be odd. In Lemma 4.8 for $n = 1$ we have $m_1 = 1$, which implies $|\alpha/\lambda - 1| > 1/3$. In particular, there is $1 \leq m \infty$ such that $\Delta_{h,1}(\lambda h \pm 2) = (t, m) \neq (1, \infty)$. This proves the theorem.

\[\square\]

Lemma 4.6. Let $h$ be a symmetric trinomial of even degree such that $\lambda h$ is odd. Let $m_{1}$ denote a nearest odd integer to $\alpha n/\lambda$. Then the number

\[
|m_1 - \alpha n/\lambda| \in \{1\} \bigcup \left(0, \frac{1}{3}\right] \bigcup \left[1 - \frac{4}{3.2^m}, 1 - \frac{2}{3.2^m}\right),
\]

and

\[
|\alpha n/\lambda - m_1| = 1 \quad \text{or} \quad \frac{1}{3} \Rightarrow \Delta_{h,n}(\lambda h \pm 2) = (1, \infty).
\]

(2) $|\alpha n/\lambda - m_1| \in \left(0, \frac{1}{3}\right] \Rightarrow \Delta_{h,n}(\lambda h \pm 2) = \left(3|\alpha n/\lambda - m_1|, 0\right).$

(3) If $m \geq 1$ then $|\alpha n/\lambda - m_1| \in \left(1 - \frac{4}{3.2^m}, 1 - \frac{2}{3.2^m}\right) \Rightarrow \Delta_{h,n}(\lambda h \pm 2) = (3.2^m \left|\alpha n/\lambda - m_1\right| - (1 - \frac{1}{2^m}), m).$

Proof. Assertions (1) and (2) can be easily checked. Assertion (3) can be checked by dividing it into two cases:

(1) $|\alpha n/\lambda - m_1| \in \left(1 - \frac{4}{3.2^m}, 1 - \frac{1}{2^m}\right)$ and

(2) $|\alpha n/\lambda - m_1| \in \left(1 - \frac{1}{2^m}, 1 - \frac{2}{3.2^m}\right).
\[\square\]

Corollary 4.7. Let $h$ be symmetric trinomial of degree $d \geq 4$. If for $l \in (\mathbb{Z}/2\lambda h\mathbb{Z})^*$ there is an integer $s \geq 0$ such that

\[
3l^s/4 \leq \lambda/\alpha < 3l^s/2 \quad \text{then} \quad \Delta_{h,1}(l) = (3|l^s\alpha/\lambda - 1|, s).
\]

In particular if $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$, where $d \geq 4$.

(1) Suppose $d$ is an even integer. Then (such an $m \geq 2$ always exists)

\[
3.2^{m-2} \leq d - 1 < 3\cdot 2^{m-1} \quad \Rightarrow \quad \Delta_{h,1}(\lambda \pm 2) = (3|1 - 2^m\alpha/\lambda|, \{m\}).
\]
Consider the canonical sequence of $\mathcal{O}$s where the third map is given by $(\varepsilon)$. Let $C$ be an irreducible curve of degree $d$ over a field of characteristic $p$. Now it is easy to check the rest.

$\Delta_{h,1}(\lambda + 2) = (1, \infty)$.

$(1/\lambda, \{3\})$.

$(6\alpha/\lambda, \{1\})$.

Proof. First part of the corollary can be checked by considering two case (1) $3l^s/4 \leq \lambda/\alpha < l^s$ and (2) $l^s \leq \lambda/\alpha < 3l^s/4$.

For the second part note that $d$ even implies $\alpha = d - 2$ even and $\lambda = (d - 1)(d - 2) + 1$ odd. Hence $\lambda \pm 2 \in (\mathbb{Z}/2\mathbb{Z})^\ast$.

Hence the assertion follows from the first part of the corollary.

Now if $d$ is odd then for any $s \geq 0$, we have $(\lambda \pm 2)^s/\lambda = \text{odd integer} + (\pm 2)^s \alpha/\lambda$ as $\lambda$ and $\alpha$ are both odd. Now it is easy to check the rest.

$\Box$

5. Semistability of Syzygy Bundles

Let $C = \text{Proj} \ R$, where $R$ is an irreducible plane curve given by a homogeneous polynomial $h$ of degree $d$ over a field of characteristic $p$. Let $\pi : X \to C$ be the normalization of $C$. Consider the canonical sequence of $\mathcal{O}_X$-modules

$$0 \to W_n \to \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{O}_C(n) \to 0,$$

where the third map is given by $(s_1, s_2, s_3) \mapsto (s_1x^n, s_2y^n, s_3z^n)$.

We recall the following Theorem 5.3 of [T1],

Theorem 5.1. Let $C$ be an irreducible curve of degree $d \geq 4$. Let $\pi : X \to C$ be the normalization of $C$. Consider the canonical sequence of $\mathcal{O}_X$-modules

$$0 \to W_1 \to \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{O}_C \to \mathcal{O}_C(1) \to 0.$$

Then

1. either $e_{HK}(R, (x, y, z)) = 3d/4$ and $V_1 = \pi^*W_1$ is strongly semistable, or

2. $e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{\tilde{t}^2}{4dp^2s^s}$,

where $\tilde{t}$ is an integer such that $0 < \tilde{t} \leq d(d - 3)$ and $s \geq 0$ is the least number such that $F^{ss}(V_1)$ is not semistable. Moreover, for the HN filtration of

$$0 \subset \mathcal{L} \subset F^{ss}(V_1), \quad \mu(\mathcal{L}) = \mu(F^{ss}(V_1)) + \frac{\tilde{t}}{2}.$$

Remark 5.2. If $V_1$ is replaced by $V_n$, then the same argument (see Lemma 4.7 and Corollary 4.11 of [T1], to justify the appearance of $n^2$ in the expression) shows that

$e_{HK}(R, x^n, y^n, z^n) = \frac{3dn^2}{4} + \frac{\tilde{t}^2}{4dp^2s^s}$,

where $0 \leq \tilde{t} \leq d(d - 3)$ and $s$ is the least integer for which $F^{ss}V_n$ is not semistable.

As we pointed out in [T1], the bound on $\tilde{t}$ in terms of $d$ (which was obtained in [T1], using result from [SB] and [S]), gave a dictionary between $s$ and $\tilde{t}$ and $e_{HK}$ (although for $p > d(d - 3)$).
For example in 1993 Hans-Monsky [HM] have explicitly compute $e_{HK}$ for the plane curve $h = z^4 + y^4 + z^4$:

$$e_{HK}(k[x, y, z]/(h), (x, y, z)) = 3 + (1/p^2) \text{ if } p \equiv \pm 3 \pmod{8}$$

$$= 3 \text{ if } p \equiv \pm 1 \pmod{8}.$$ 

Now, by Theorem 5.3 it is immediate that, for $p \geq 5$, $\tilde{l} = 4$ and $s = 1$, for $p \equiv \pm 3 \pmod{8}$. This means $V$ is semistable but $F^*V$ is not semistable. On the other hand, it says that $V_1$ is strongly semistable if $p \equiv \pm 1 \pmod{8}$.

**Theorem 5.3.** Let $R = k[x, y, z]/(h)$, where $h$ is a regular trinomial of degree $d$ and $k$ is a field of characteristic $p > 0$. If $p \geq n$ and $p \equiv \pm l \pmod{2\lambda_h}$ then

$$e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{\lambda^2}{4d} \left[ \frac{1-t}{p^s} \right]^2,$$

where $\Delta_{h, n}(l) = (Td(l), Ds(l)) = (t, s)$ is as given in Definition 2.4.

**Proof.** If $p \geq n$, then by Lemma 2.3 (3), $\text{Td}(p^sn, u) < 1$ has no solution for any $s < 0$. Hence the minimum integer $s$, for which $\text{Td}(p^sn, u) < 1$ has a solution for some $u \in L_{odd}$, is nonnegative. Therefore, by Theorem 5.3 and Corollary 3.6

$$\delta^*(an/\lambda, \beta n/\lambda, \nu n/\lambda) = p^{-s}(1-t),$$

where $\delta^*(an/\lambda, \beta n/\lambda, \nu n/\lambda)$ is given as in Theorem 2.3. Now the theorem follows from Theorem 2.3.

The following Lemma explicitly relates $\Delta_{h, n}(l \mod 2\lambda_h)$ and the Frobenius semistability data of the syzygy bundle $V_n$ over $h$, for the set of primes $p \equiv \pm l \pmod{2\lambda_h}$, where $p \geq \{n, d^2\}$.

**Lemma 5.4.** Let $R = k[x, y, z]/(h)$, where $h$ is a regular trinomial of degree $d$ over an algebraically closed field of characteristic $p > 0$. Let $p \geq \{n, d^2\}$ and let $p \equiv \pm l \pmod{2\lambda_h}$. For $\Delta_{h, n}$ as in Theorem 5.3

1. If $\Delta_{h, n}(l) = (1, \infty)$ then $V_n$ is a strongly semistable bundle.
2. If $\Delta_{h, n}(l) = (t, s) \neq (1, \infty)$ then $s$ is the least integer for which $F^sV_n$ is not semistable. Moreover $F^s(V_n)$ has the HN filtration

\[0 \subset \mathcal{L}_n \subset F^s(V_n), \quad \text{where} \quad \deg \mathcal{L}_n = \mu(F^sV_n) + \frac{\lambda}{2}(1-t).\]

**Proof.** (1) If $\Delta_{h, n}(l) = (1, \infty)$, then $e_{HK}(R, (x^n, y^n, z^n)) = 3dn^2/4$ and therefore $V_n$ is strongly semistable.

(2) Let $\Delta_{h, n}(l) = (t, s) \neq (1, \infty)$. By Theorem 5.3 and Equation 5.1, we have

$$e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{\lambda^2}{4d} \left[ \frac{(1-t)}{p^s} \right]^2 = \frac{3dn^2}{4} + \frac{l^2}{4dp^{2s}},$$

where $0 \leq \tilde{l} \leq d(d-3)$ and $s_1 \geq 0$ is the least integer for which $F^{s_1}V_n$ is not semistable. Note, by Lemma 2.3 the integer $s \geq 0$. This implies that

$$\frac{\tilde{l}}{p^{s_1}} = \frac{\lambda}{p^s}(1-t).$$

Let $(u_1, u_2, u_3) \in L_{odd}$ such that $t = \text{Td}(p^su, u) < 1$. Therefore $0 < \lambda(1-t) < \lambda$. On the other hand

$$\lambda(1-t) = a\lambda_h(1-t) = a(\lambda_h - |p^s\alpha_1 n - \lambda_h u_1| - |p^s\beta_1 n - \lambda_h u_2| - |p^s\nu_1 n - \lambda_h u_3|) \in \mathbb{Z}.$$ 

This implies $\lambda(1-t) \leq \lambda$ is a positive integer. This with the fact that $0 \leq \tilde{l} \leq d(d-3)$ implies that, for $p \geq d^2$, we have $s_1 = s$ and hence $\tilde{l} = \lambda(1-t)$. This proves the lemma. \(\square\)
Recall that a trinomial curve is irregular or regular. For the irregular trinomials the semistability behaviour is very explicit and independent of the char p as stated in Theorem 4.1 a proof of which is along the same line as in Theorem 4.9 of [T2].

In the light of Lemma 5.3 all the results in this section are immediate consequence of the results of the previous sections.

Following result gives the periodicity in the behaviour of \{V_n\}_{n \in \mathbb{N}} where \(V_n\) are syzygy bundles on a fixed trinomial \(h\).

**Theorem 5.5.** For a regular trinomial defined over a field of characteristic \(p\), if \(p \geq n + 2\lambda_h\), then for any \(s \geq 0\),

1. the bundle \(F^{*s}V_n\) is semistable if and only if \(F^{*s}V_{n+2\lambda_h}\) is semistable. Moreover,
2. \(F^{*s}V_n\) has the HN filtration \(0 \subset L_n \subset F^{*s}V_n\) if and only if \(F^{*s}V_{n+2\lambda_h}\) has the HN filtration \(0 \subset L_{n+2\lambda_h} \subset F^{*s}V_{n+2\lambda_h}\) and in that case we have \(\deg L_{n+2\lambda_h} = \deg L_n - 3\lambda_h dp^s\).

**Proof.** Follows from Corollary 3.6. □

Following theorem implies that every semistable bundle \(V_n\) over a trinomial is strongly semistable for a Zariski dense set of primes.

**Theorem 5.6.** Let \(R = k[x, y, z]/(h)\) be a regular trinomial, where \(k\) is an algebraically closed field of characteristic \(p > 0\). Let \(p \equiv \pm 1 \pmod{2\lambda_h}\) and \(p \geq d^2\) then

1. \(V_1\) is strongly semistable and
2. if, in addition, \(p \geq n\) then
   a. either \(V_n\) is strongly semistable or
   b. \(V_n\) itself is not semistable and has the HN filtration
   \[
   0 \subset L_n \subset \pi^*(V_n) \text{ where } \deg L_n = -\frac{3nd}{2} + \frac{\lambda}{2}(1-t),
   \]
   where \(t = |\alpha n/\lambda - u_1| + |\beta n/\lambda - u_2| + |\gamma n/\lambda - u_3| < 1\) for a unique \((u_1, u_2, u_3) \in \mathbb{L}_{odd}\).

**Corollary 5.7.** If \(V_{n_1}, \ldots, V_{n_s}\) are semistable syzygy bundles on trinomials \(h_1, \ldots, h_s\) respectively then they are all strongly semistable for primes \(p\) in a Zariski dense set

**Proof.** Let \(\lambda_h = 1\) if \(h\) is an irregular trinomial. If \(\lambda = l.c.m.(\lambda_{h_1}, \ldots, \lambda_{h_s})\) then for \(p \equiv \pm 1 \pmod{2\lambda}\) the assertion holds. □

Following theorem asserts that to check the strong semistability property of a syzygy bundle \(V_n\) over a trinomial \(h\), it is sufficient to check the semistability of \(V_n, F^{*s}V_n, \ldots, F^{*d}V_n\), where \(s < \phi(2\lambda_h)\).

**Theorem 5.8.** If \(p \geq \max\{n, d^2\}\), then either

1. \(V_n\) is strongly semistable or
   a. there is \(s < \phi(2\lambda_h)\) such that \(F^{*s}(V_n)\) is not semistable.
   In fact if \(p \equiv \pm 1 \pmod{2\lambda_h}\) then \(F^{*s}(V_n)\) is not semistable for some \(s < \text{order of } l \in (\mathbb{Z}/2\lambda_h\mathbb{Z})^*\)
   b. If there is a prime \(p \geq \max\{n, d^2\}\) such that \(V_n\) is not strongly semistable then there is a Zariski dense set for which \(F^{*s}V_n\) is not semistable.

**Proof.** Part (1) (a) and (b) follow from Theorem 5.5 (3). For part (2) suppose \(p \geq \max\{n, d^2\}\) such that \(V_n\) is not strongly semistable. There is \(l \in (\mathbb{Z}/2\lambda_h\mathbb{Z})^*\) such that \(p \equiv \pm 1 \pmod{2\lambda_h}\). By Corollary 3.6 we have \(\Delta_{h,n}(l) = (t, s) \neq (1, \infty)\). Therefore there exists \(u \in L_{odd}\) such that \(Td(l^tn, u) < 1\). Now \(l^t \in (\mathbb{Z}/2\lambda_h\mathbb{Z})^*\) such that \(\Delta_{h,n}(l^t) = (t, 1)\). Therefore for \(p \equiv \pm l^t \pmod{2\lambda_h}\), the bundle \(F^{*s}V_n\) is not semistable. □

By the following corollary it is trivial to check if a syzygy bundle \(V_n\), of a symmetric (see Definition 4.2) regular trinomial, is semistable or not.
Corollary 5.9. Let $p \equiv \pm 1 \pmod{2\lambda_n}$ and $p \geq \max\{n, d^2\}$ and let $h$ be a symmetric trinomial of degree $d$. Let $m_1$ denote any of the nearest odd integer to $an/\lambda$.

1. If $\frac{an}{\lambda} - m_1 \geq \frac{1}{3}$ then $V_n$ is semistable (and hence strongly semistable), and
2. if $|m_1 - an/\lambda| < \frac{1}{3}$, then $V_n$ is not semistable and has the HN filtration

$$0 \subset L_n \subset V_n \text{ where } \deg L_n = -\frac{3nd}{2} + \frac{3\lambda}{2} \left(\frac{1}{3} - \frac{|an}{\lambda} - m_1\right).$$

Corollary 5.10. Let $h$ be a symmetric trinomial of degree $d \geq 4$ but $d \neq 5$ then, for $p > d^2$,

1. $V_1$ is strongly semistable for a Zariski dense set of primes and
2. $V_1$ is semistable but not strongly semistable for a Zariski dense set of primes.

Proof. Follows from Theorem 4.11 (1) and Theorem 4.5

Corollary 5.11. Let $X$ be the plane curve given by $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$ where $k$ is a field of characteristic $p \geq d^2$. Let

$$0 \to V \to H^0(X, O_X(1)) \otimes O_X \to O_X \to 0,$$

be the canonical map.

1. If $p \equiv \pm 1 \pmod{\lambda}$. Then $V$ is strongly semistable.
2. If $p \equiv \pm 2 \pmod{\lambda}$, $d$ is even and
   a. If $d = 4$ then $F^*V$ is semistable and the HN filtration of $F^{2*}V$ is given by
   $$0 \subset L \subset F^2(V) \text{ with } \mu(L) = \mu(F^{2*}V) + 2$$
   b. If $d \geq 6$ and $m \geq 2$ such that (1) $3.2^{m-2} \leq d - 1 < 2^m$ then $F^{m-1*}V$ is semistable and the HN filtration of $F^{m*}V$ is given by
   $$0 \subset L \subset F^{m*}(V) \text{ with } \mu(L) = \mu(F^{m*}V) + 2\alpha(d - 1 - 3.2^{m-2}) + 2,$$
   c. If $2^m \leq d - 1 < 3.2^{m-1}$ then $F^{m-1*}V$ is semistable and the HN filtration of $F^{m*}V$ is given by
   $$0 \subset L \subset F^{m*}(V) \text{ with } \mu(L) = \mu(F^{m*}V) + \alpha(3.2^{m-1} - (d - 1)) - 1.$$
3. If $p \equiv \pm 2 \pmod{\lambda}$ and $d$ is odd then
   a. For $d \geq 7$, the bundle $V$ is semistable and for the HN filtration $0 \subset L \subset F^*V$, we have
   $$\mu(L) = \mu(F^*V) + \left(\frac{\lambda - 6\alpha}{2}\right).$$
   b. If $d = 5$, $F^{2*}V$ is semistable and the HN filtration $0 \subset L \subset F^{3*}V$, we have
   $$\mu(L) = \mu(F^{3*}V) + \frac{7}{2}.$$

Proof. Follows from Corollary 4.17

6. Hilbert-Kunz multiplicity

Throughout this section $R = k[x, y, z]/(h)$, where $h$ is trinomial of degree $d$ and $k$ is a field of char $k = p > 0$.

Corollary 6.1. Let $R = k[x, y, z]/(h)$, where $h$ is a trinomial of degree $d$. Let $n \geq 1$.

1. If $h$ is an irregular trinomial then

$$e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{(2r - d)^2n^2}{4d},$$

where $r$ is the multiplicity of the irregular point.
(2) If $h$ is a regular trinomial then
\[e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{\lambda^2}{4dp^2} (1 - t)^2,\]
where $\lambda(1 - t) \leq \lambda$ is a nonnegative integer and $0 \leq s < \phi(2\lambda h)$ and $t$ and $s$ are constant on the congruence classes of $p \mod (2\lambda h)$.

**Corollary 6.2.** If $h$ is a regular trinomial then

1. for all $p \geq n + 2\lambda h$,
\[e_{HK}(R, (x^{n+2\lambda h}, y^{n+2\lambda h}, z^{n+2\lambda h})) = e_{HK}(R, (x^n, y^n, z^n)) + 3d(n\lambda h + 1).\]

2. if $p \equiv \pm 1 \pmod{2\lambda h}$ then we have
\[e_{HK}(R, (x, y, z)) = \frac{3d}{4},\]

(a) \[\frac{\alpha n}{\lambda} - m_1 \geq \frac{1}{3} \implies e_{HK}(R, (x^n, y^n, z^n)) = \frac{3dn^2}{4},\]

(b) \[\frac{\alpha n}{\lambda} - m_1 < \frac{1}{3} \implies e_{HK}(R_{p}, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \frac{9\lambda^2}{4d} \left[\frac{1}{3} - \frac{\alpha n}{\lambda} - m_1\right]^2.\]

2. If $d$ is odd and $\geq 5$ then there is $l' \in (\mathbb{Z}/2\lambda h\mathbb{Z})^\ast$ such that for $p \equiv \pm l' \pmod{2\lambda h}$
\[e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{\lambda^2}{4dp^2} \left[1 - \frac{6}{\lambda h}\right]^2.\]

3. If $d \geq 4$ is even such that
(a) $\lambda h$ is even then
\[e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{\lambda^2}{4dp^2} \left[1 - \frac{6}{\lambda h}\right]^2.\]

(b) If $\lambda h$ is odd. Then
\[e_{HK}(R_{p}, (x^n, y^n, z^n)) = \frac{3dn^2}{4} + \lambda^2 \left[\frac{1}{4} - t\right]^2,\]
where $\Delta_{h,n}(\lambda \pm 2) = (t, s) \neq (1, \infty)$ (hence $0 < t < 1$ and $0 \leq s < \infty$) is given as in Lemma 4.6.

**Corollary 6.4.** Let $p \equiv \lambda \pm 2 \pmod{2\lambda}$ and let $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$, where $d \geq 4$ (in this case $\lambda = \lambda_h$).

1. Suppose $d$ is an even integer.
\[d = 4 \implies e_{HK}(R, (x, y, z)) = 3 + \frac{7}{p^3}.\]

Let $d \geq 6$. Let $m \geq 2$ such that $3.2^{m-2} < d - 1 < 2^m$. Then
\[e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{4}{dp^2m} \left[\alpha (d - 1 - 3.2^{m-2}) + 1\right]^2.\]
If $2^m < d - 1 < 3 \cdot 2^{m-1}$. Then
\[ e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{1}{dp^{2m}} \left[ \alpha (3.2^{m-1} - (d - 1)) - 1 \right]^2. \]

(2) Suppose $d$ is odd then
\[ d = 5 \implies e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{1}{dp^2} \left[ \frac{49}{4} \right]^2. \]
\[ d \geq 7 \implies e_{HK}(R, (x, y, z)) = \frac{3d}{4} + \frac{1}{4dp^2} [(d - 2)(d - 7) + 1]^2. \]

Remark 6.5. If $h$ is a regular trinomial of degree $d = 3$ then it is an elliptic plane curve. Note that $e_{HK}$ with respect to the maximal ideal was first computed in [BC] and [Mo3]. Also on an elliptic curve every semistable bundle is strongly semistable by [MR] (Theorem 2.1).

Remark 6.6. For $R$ as in Corollary 6.4 Monsky in [M1] had computed $e_{HK}$ in the following situation:

(1) If $d \geq 4$ is even and $p \equiv \pm (d - 1) \pmod{2\lambda}$ then
\[ e_{HK}(R) = \frac{3d}{4} + \frac{(d^2 - 3d)^2}{4dp^2}. \]

(2) If $d \geq 5$ is odd and $p \equiv \lambda \pm (2d - 2) \pmod{2\lambda}$, $p \neq 2$ then
\[ e_{HK}(R) = \frac{3d}{4} + \frac{(d^2 - 3d - 3)^2}{4dp^2}. \]

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