MEAN VALUE OF REAL CHARACTERS USING A DOUBLE DIRICHLET SERIES

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Abstract. We study the double character sum \( \sum_{m \leq X, \text{odd}} \sum_{n \leq Y, \text{odd}} \left( \frac{m}{n} \right) \) and its smoothly weighted counterpart. An asymptotic formula with power saving error term was obtained in [CFS] by applying the Poisson summation formula. In this paper, we apply the inverse Mellin transform twice and study the resulting double integral that involves a double Dirichlet series. As a result, we recover the asymptotic from [CFS] with an improved error term.

1. Introduction

We study the double character sum

\[
S(X, Y) := \sum_{m \leq X, \text{odd}} \sum_{n \leq Y, \text{odd}} \left( \frac{m}{n} \right).
\]

This sum was studied by Conrey, Farmer and Soundararajan in [CFS], where the authors give an asymptotic formula valid for all large \( X \) and \( Y \).

If \( Y = o(X/\log X) \), then the main term of \( S(X, Y) \) comes from the terms where \( n \) is a square, and the error term can be estimated using the Pólya-Vinogradov inequality. In particular, we get that in this range,

\[
S(X, Y) = \frac{2}{\pi^2} X^{3/2} + O(Y^{3/2} \log Y + Y^{1/2+\varepsilon} + X \log Y),
\]

and similarly for \( X = o(Y/\log Y) \).

Conrey, Farmer and Soundararajan showed that there is a transition in the behavior of \( S(X, Y) \) when \( X, Y \) are of similar size. In particular, they proved the following asymptotic formula, which is valid for all large \( X, Y \):

\[
S(X, Y) = \frac{2}{\pi^2} X^{3/2} C \left( \frac{Y}{X} \right) + O \left( \left( XY^{7/16} + YX^{7/16} \right) \log(XY) \right),
\]

where

\[
C(\alpha) = \alpha + \alpha^{3/2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1/\alpha} \sqrt{y} \sin \left( \frac{\pi k^2 y}{2y} \right) dy.
\]

The size of the main term in this formula is \( XY^{1/2} + YX^{1/2} \), so it is always larger than the error term. The result is interesting, because \( C(\alpha) \) is a non-smooth function. For a heuristic explanation why such functions
arise in this type of problems, see the first section in [Pet] and the references therein. Conrey, Farmer and Soundararajan also gave the following asymptotic estimates for $C(\alpha)$:

\begin{equation}
C(\alpha) = \sqrt{\alpha} + \frac{\pi}{18} \alpha^{3/2} + O\left(\alpha^{5/2}\right) \quad \text{as } \alpha \to 0,
\end{equation}

and

\begin{equation}
C(\alpha) = \alpha + O\left(\alpha^{-1}\right) \quad \text{as } \alpha \to \infty.
\end{equation}

To prove (3), Conrey, Farmer and Soundararajan applied the Poisson summation formula and estimated the sums of Gauss sums which appeared in the computation. Similar techniques were used in the work of Gao and Zhao to compute the mean value in other families of characters, such as cubic and quartic Dirichlet characters [GZ2], and some quadratic, cubic and quartic Hecke characters [GZ1]. Gao used similar methods to compute the mean value of the divisor function twisted by quadratic characters [Gao].

Our approach is to rewrite $S(X,Y)$ as a double integral by using the inverse Mellin transform twice. The integral will then involve the double Dirichlet series

$$A(s,w) = \sum_{m \text{ odd}} \sum_{n \text{ odd}} \frac{(m/n)}{m^w n^s},$$

which was studied by Blomer [Blo], who showed that it admits a meromorphic continuation to the whole $\mathbb{C}^2$ and determined the polar lines. We then shift the integrals to the left and compute the contribution of the residues. The quality of the error term depends whether we assume the truth of the Riemann Hypothesis because the zeros of $\zeta(s)$ appear in the location of the poles of $A(s,w)$, and also in the contribution of the residues.

An interesting feature of our proof is that the 3 polar lines from which our main term arises naturally correspond to the contribution when $n$ is a square (the polar line $w = 1$), when $m$ is a square (the polar line $s = 1$), and the transition term where the non-smooth function appears (the polar line $s + w = 3/2$).

A more general theory of multiple Dirichlet series has been developed by Bump, Chinta, Diaconu, Friedberg, Goldfeld, Hoffstein and others. We refer the reader interested in the theory and its applications to the expository articles [Bum], [BFH], [CFH], the paper [DGH] or the book [BFG].

To state our results, we first define the smooth sum

\begin{equation}
S(X,Y; \varphi, \psi) = \sum_{m,n \text{ odd}} \left(\frac{m}{n}\right) \varphi(m/X) \psi(n/Y),
\end{equation}

where $\varphi, \psi$ are nonnegative smooth functions supported in $(0,1)$.

If we denote by $\hat{f}$ the Mellin transform of $f$ (see (17)), the main result is the following:
Theorem 1.1. Let $\varepsilon > 0$. Then for all large $X,Y$, we have

\begin{equation}
S(X,Y;\varphi,\psi) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X};\varphi,\psi\right) + O_\varepsilon(XY^\delta + YX^\delta),
\end{equation}

where $\delta = \varepsilon$, and

\begin{align*}
D(\alpha;\varphi,\psi) &= \frac{\hat{\varphi}(1)\hat{\psi}(1/2)\alpha^{1/2} + \hat{\psi}(1)\hat{\varphi}(1/2)}{2} + \\
&\quad + \frac{1}{i\sqrt{\pi}} \int_{(3/4)} \left( \frac{\alpha}{2\pi} \right)^s \varphi\left(\frac{3}{2} - s\right) \hat{\psi}(s) \Gamma\left(s - \frac{1}{2}\right) \sin\left(\frac{\pi s}{2}\right) \zeta(2s - 1)ds.
\end{align*}

If we assume the Riemann Hypothesis, then we can take $\delta = -1/4 + \varepsilon$.

We can remove the smooth weights and obtain the following asymptotic formula for $S(X,Y)$, which improves the error term in (3):

Theorem 1.2. Let $\varepsilon > 0$. Then for all large $X,Y$, we have

\begin{equation}
S(X,Y) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X}\right) + O_\varepsilon(XY^{2/5+\varepsilon} + YX^{2/5+\varepsilon}),
\end{equation}

where

\begin{align*}
D(\alpha) &= \sqrt{\alpha} + \alpha - \frac{1}{i\sqrt{\pi}} \int_{(3/4)} \left( \frac{\alpha}{2\pi} \right)^s \Gamma\left(s - \frac{3}{2}\right) \sin\left(\frac{\pi s}{2}\right) \zeta(2s - 1)ds.
\end{align*}

If we assume the Riemann Hypothesis, then we have

\begin{equation}
S(X,Y) = \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left(\frac{Y}{X}\right) + O_\varepsilon(XY^{7/20+\varepsilon} + YX^{7/20+\varepsilon}).
\end{equation}

We show in Section 7 that $D(\alpha) = C(\alpha)$, so our main term agrees with that of Conrey, Farmer and Soundararajan.

Let us also remark that a similar asymptotic can be obtained if the integers $m,n$ were restricted to lie in a congruence class modulo 8 by working with a suitable combination of the twisted double Dirichlet series, as defined in (23).

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2. Preliminaries and notation

We follow the notation of [Blo]. For integers $m,n$, we denote by $\chi_m(n)$ the Kronecker symbol

$\chi_m(n) = \left(\frac{m}{n}\right)$.
If \( m \) is odd, we can write it as \( m = m_0m_1^2 \) with \( (m_0, m_1) = 1 \) and \( m_0 \) squarefree. Then \( \chi_m \) is a character of conductor \( |m_0| \) if \( m \equiv 1 \) (mod 4) and \( |4m_0| \) if \( m \equiv 3 \) (mod 4). We denote by \( \psi_1, \psi_{-1}, \psi_2, \psi_{-2} \) the four Dirichlet characters modulo 8 given by the Kronecker symbol \( \psi_j(n) = \left( \frac{j}{n} \right) \). We also let

\[
\tilde{\chi}_m = \begin{cases} 
\chi_m, & \text{if } m \equiv 1 \text{ (mod 4)}, \\
\chi_{-m}, & \text{if } m \equiv 3 \text{ (mod 4)}.
\end{cases}
\]

With this notation, quadratic reciprocity tells that if \( m, n \) are odd and positive, then

\[
\chi_m(n) = \tilde{\chi}_n(m).
\]

The fundamental discriminants \( m \) correspond to primitive real characters of conductor \( |m| \). In such cases, the completed L-function is

\[
\Lambda(s, \chi_m) = \left( \frac{|m|}{\pi} \right)^{\frac{s+a}{2}} \Gamma \left( \frac{s+a}{2} \right) L(s, \chi_m),
\]

where \( a = 0 \) or 1 depending on whether the character is even or odd, i.e., whether \( \chi_m(-1) = 1 \) or \(-1\), and we have the functional equation

\[
\Lambda(s, \chi_m) = \Lambda(1-s, \chi_m).
\]

All primitive real characters can be uniquely written as \( \chi_{m_0} \psi_j \) for some positive odd squarefree integer \( m_0 \) and \( j \in \{\pm 1, \pm 2\} \).

When \( m \) is not a fundamental discriminant, then \( \chi_m \) is a character of conductor \( m_0 | 4m \), and we have

\[
L(s, \chi_m) = L(s, \chi_{m_0}) \cdot \prod_{p | |m|/m_0} \left( 1 - \frac{\chi_{m_0}(p)}{p^s} \right).
\]

A subscript 2 of an L-function means that the Euler factor at 2 is removed, so in particular

\[
L_2(s, \chi) = \sum_{n \text{ odd}} \frac{\chi(n)}{n^s}.
\]

We now record two estimates that we will use in our paper. The first holds for Re(\( s \)) \( \geq 1/2 \):

\[
\sum_{\substack{m \leq X, \ \text{\scriptsize odd} \ m}} |L_2(s, \chi_m\psi_j)| \ll X^{1+\varepsilon} |s|^{\frac{1}{2}+\varepsilon}.
\]

It follows after applying Hölder’s inequality on the bound for the fourth moment, proved by Heath-Brown [Hea, Theorem 2].

The second is conditional under RH, and it says that for any fixed \( \sigma > 1/2 \) and any \( \varepsilon > 0 \), we have

\[
\left| \frac{1}{\zeta(\sigma + it)} \right| \ll \varepsilon (1 + |t|)^{\varepsilon}.
\]
It follows from [CC, Theorem 2].

For a function $f(x)$, we denote by $\hat{f}(s)$ its Mellin transform, which is defined as

\begin{equation}
\hat{f}(s) = \int_0^\infty f(x)x^{s-1}dx,
\end{equation}

when the integral converges. If $\hat{f}$ is analytic in the strip $a < \text{Re}(s) < b$, then the inverse Mellin transform is given by

\begin{equation}
f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s}\hat{f}(s)ds,
\end{equation}

where the integral is over the vertical line $\text{Re}(s) = c$, and $a < c < b$ is arbitrary.

We will use the following estimate for the Gamma function, which is a consequence of Stirling’s formula: for a fixed $\sigma \in \mathbb{R}$ and $|t| \geq 1$, we have

\begin{equation}
|\Gamma(\sigma + it)| \asymp e^{-\frac{|t|}{2}}|t|^\sigma - \frac{1}{2}.
\end{equation}

We will also use the formula

\begin{equation}
\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{2^s \sin(\pi s/2)\Gamma(1-s)}{\sqrt{\pi}}.
\end{equation}

Throughout the paper, $\varepsilon$ will denote a sufficiently small positive number which may differ from line to line, and all implied constants are allowed to depend on $\varepsilon$.

3. Outline of the proof and double Dirichlet series

Applying Mellin inversion to $S(X,Y;\varphi,\psi)$ twice, we obtain

\begin{equation}
S(X,Y;\varphi,\psi) = \left(\frac{1}{2\pi i}\right)^2 \int_{(\sigma)}\int_{(\omega)} A(s,w)X^wY^s\hat{\varphi}(w)\hat{\psi}(s)dwds,
\end{equation}

where for $\text{Re}(s) = \sigma$ and $\text{Re}(w) = \omega$ large enough, we have the absolutely convergent double Dirichlet series

\begin{equation}
A(s,w) = \sum_{m \text{ odd}} \sum_{n \text{ odd}} \frac{(m/n)^s}{m^{w}s} = \sum_{m \text{ odd}} \frac{L_2(s,\chi_m)}{m^w}.
\end{equation}

We use the results of Blomer to meromorphically continue $A(s,w)$ to the whole $\mathbb{C}^2$, shift the two integrals to the left and compute the contribution of the crossed polar lines.

We now cite and sketch the proof of Lemma 2 in [Blo]. For two characters $\psi,\psi'$ of conductor dividing 8, we define

\begin{equation}
Z(s,w;\psi,\psi') := \zeta_2(2s + 2w - 1) \sum_{m,n \text{ odd}} \frac{\chi_m(n)\psi(n)\psi'(m)}{m^{w}s}.
\end{equation}
which converges absolutely if $\text{Re}(s)$ and $\text{Re}(w)$ are large enough, and we let
\begin{equation}
Z(s, w) := Z(s, w; \psi_1, \psi_1) = \zeta_2(2s + 2w - 1)A(s, w).
\end{equation}
We also denote
\[
Z(s, w; \psi) = \begin{pmatrix}
Z(s, w; \psi, \psi_1) \\
Z(s, w; \psi, \psi_{-1}) \\
Z(s, w; \psi, \psi_2) \\
Z(s, w; \psi, \psi_{-2})
\end{pmatrix},
\quad Z(s, w) = \begin{pmatrix}
Z(s, w; \psi_1) \\
Z(s, w; \psi_{-1}) \\
Z(s, w; \psi_2) \\
Z(s, w; \psi_{-2})
\end{pmatrix}.
\]

**Theorem 3.1.** The functions $Z(s, w; \psi, \psi')$ have a meromorphic continuation to the whole $\mathbb{C}^2$ with a polar line $s + w = 3/2$. There is an additional polar line at $s = 1$ with residue $\text{res}_{(1,w)}Z(s, w) = \zeta_2(2w)/2$ if and only if $\psi = \psi_1$, and an additional polar line $w = 1$ with residue $\text{res}_{(s,1)}Z(s, w) = \zeta_2(2s)/2$ if and only if $\psi' = \psi_1$.

The functions $(s - 1)(w - 1)(s + w - 3/2)Z(s, w; \psi, \psi')$ are polynomially bounded in vertical strips, meaning that for fixed $\text{Re}(s)$ and $\text{Re}(w)$, $(s - 1)(w - 1)(s + w - 3/2)Z(s, w; \psi, \psi')$ is bounded by a polynomial in $\text{Im}(s), \text{Im}(w)$. The functions satisfy functional equations relating $Z(s, w)$ with $Z(w, s)$, and $Z(s, w)$ with $Z(1 - s, s + w - 1/2)$.

**Remarks:**
1. Blomer gives explicit $16 \times 16$ matrices $A$ and $B(s)$, such that $Z(s, w) = A \cdot Z(w, s)$, and $Z(s, w) = B(s) \cdot Z(1 - s, s + w - 1/2)$, we will use the explicit form in (41) to compute the residues on the polar line $s + w = 3/2$.
2. We can also iterate the two equations and obtain others, for example relating $Z(s, w)$ with $Z(1 - s, 1 - w)$. Blomer also gives an almost explicit form of this case, which we will use to obtain a convexity bound in Lemma 3.2.
3. For us, a polar line means that if we fix one of the variables, the resulting function of the other variable has a pole on the corresponding line with the given residue. What we state isn’t quite right at the points $(1/2, 1)$ and $(1, 1/2)$, where two of the polar lines intersect, but we will not need to know the behavior at these points.

**Proof sketch.** We write the Dirichlet series for $Z(s, w; \psi, \psi')$ in two ways.

First, writing $m = m_0m_1^2$ with $\mu^2(m_0) = 1$ and $(m_0, m_1) = 1$, we have
\begin{equation}
Z(s, w; \psi, \psi') = \zeta_2(2s + 2w - 1) \sum_{m_0 \text{ odd, } \mu^2(m_0) = 1} L_2(s, \chi_{m_0} \psi) \psi'(m_0) m_0^w \times \sum_{m_1 \text{ odd, } \mu^2(m_0) = 1} \frac{1}{m_1^{2w}} \prod_{p|m_1} \left(1 - \frac{\chi_{m_0} \psi(p)}{p^s}\right)
\end{equation}
\begin{equation}
= \zeta_2(2s + 2w - 1) \sum_{m_0 \text{ odd, } \mu^2(m_0) = 1} L_2(s, \chi_{m_0} \psi) \psi'(m_0) \zeta_2(2w) m_0^w L_2(s + 2w, \chi_{m_0} \psi).
\end{equation}
If $\psi$ is non-trivial, the right-hand side converges absolutely in the region

$$\{(s, w) : \text{Re}(w) > 1 \text{ and } \text{Re}(s + w) > 3/2\},$$

the second condition comes from using the functional equation in the numerator when $\text{Re}(s) < 1/2$. When $\psi$ is the trivial character, the summand corresponding to $m_0 = 1$ is $\frac{\zeta_2(s)\zeta_2(2w)}{\zeta_2(s+2w)}$, so there is a pole at $s = 1$ with residue $\zeta_2(2w)/2$. Note that the other potential polar lines coming from $L_2(s+2w, \overline{\chi_0\psi})$ are outside of the considered region.

The second way to write $Z(s, w; \psi, \psi')$ is by exchanging summations and using the quadratic reciprocity. We obtain

$$Z(s, w; \psi, \psi') = \zeta_2(2s + 2w - 1) \sum_{m, n \text{ odd}} \frac{\chi_m(n)\psi(n)\psi'(m)}{m^w n^s}$$

(26)

$$= \zeta_2(2s + 2w - 1) \sum_{n \text{ odd}} \frac{L_2(w, \overline{\chi_n\psi'}(n))}{n^s}.$$ 

We can again write $n = n_0 n_1^2$ with $\mu(n_0)^2 = 1$ and obtain a series that is absolutely convergent in the region

$$\{(s, w) : \text{Re}(s) > 1 \text{ and } \text{Re}(s + w) > 3/2\},$$

unless $\psi'$ is the trivial character, in which case there is a pole at $w = 1$ coming from the summands when $n$ is a square, and the residue is $\zeta_2(2s)/2$.

Note that (26) gives a link between $Z(s, w; \psi, \psi')$ and $Z(w, s; \psi', \psi)$, which gives us a functional equation relating $Z(s, w)$ with $Z(w, s)$. To finish the proof and obtain the meromorphic continuation to the whole $\mathbb{C}^2$, we use the functional equation in the numerator of (25), which gives a functional equation relating $Z(s, w)$ and $Z(1-s, s+w-1/2)$, where the change in the second coordinate comes from the conductor in the functional equation for $L(s, \chi_m)$. Notice that this change of variables interchanges $2s + 2w - 1$ and $2w$, leaves $s + 2w$ fixed, and maps the line $w = 1$ to $s + w = 3/2$, which becomes a new polar line.

We can iterate the two transformations coming from (25) and (26) and obtain a function meromorphic on a tube region of the form $\{(s, w) : \text{Re}(s)^2 + \text{Re}(w)^2 > c\}$ for some $c$. During this process, we obtain some additional potential polar lines, but these will be canceled by the gamma factors coming from the functional equations. To obtain a continuation to the region $\{(s, w) : \text{Re}(s)^2 + \text{Re}(w)^2 \leq c\}$, we use Bochner’s theorem from multivariable complex analysis, which states that a function that is holomorphic on a tube region can be continued to its convex hull (see [Boc]).

The proof that the function is polynomially bounded in vertical strips is similar to the proof of Proposition 4.11 in [DGH].

We will also use the following convexity bound for $Z(s, w)$:
Lemma 3.2. Let $\varepsilon > 0$. Then for $\frac{1}{3} \leq \text{Re}(s) = \sigma < 1 - \varepsilon$ and $0 < \text{Re}(w) = \omega < \frac{1}{2}$, we have

$$|Z(s, w; \psi, \psi')| \ll |s|^{\frac{\sigma+2}{1+\omega} + \varepsilon} |w|^{1-\omega + \varepsilon}. \tag{27}$$

We note that taking $\sigma = \omega = 1/2$ gives the bound $|sw|^{\frac{3}{2} + \varepsilon}$, which was improved by Blomer. However, the stronger bound wouldn’t improve upon our final result.

Proof. Note that in the considered range for $s, w$, we stay away from the polar lines of $Z(s, w)$. For $s$ with $\frac{1}{2} \leq \sigma < 1 - \varepsilon$, and $\omega > 1 + \varepsilon$, (15) gives

$$|Z(s, w; \psi, \psi')| = |\zeta_2(2s + 2w - 1)| \left| \sum_{m \text{ odd}} \frac{L_2(s, \chi_m \psi) \psi'(m)}{m^w} \right| \ll |s|^{\frac{1}{2} + \varepsilon}. \tag{28}$$

In the range $\varepsilon < \sigma < \frac{1}{2}$, and $\sigma + \omega > \frac{3}{2} + \varepsilon$, we use (25) and apply the functional equation in the numerator. Depending whether $\chi_m \psi$ is even or odd, we let $a_{m, \psi} = 0$ or 1, so (12), (15) and (19) give

$$|Z(s, w; \psi, \psi')| \ll \sum_{m_0 \text{ odd, } \mu(m_0)^2 = 1} \frac{L_2(1-s, \chi_m \psi) \psi'(m_0) \zeta_2(2w)}{m_0^{w + s}} \frac{\Gamma \left( \frac{1-s+a_{m_0, \psi}}{2} \right)}{\Gamma \left( \frac{s+a_{m_0, \psi}}{2} \right)}. \tag{29}$$

Finally, using the functional equation relating $Z(s, w)$ with $Z(1-s, 1-w)$, which is (37) in [Blo], we have that for $\sigma + \omega < 1$ and away from the poles,

$$|Z(s, w; \psi, \psi')| \ll (1 + |s|)^{\frac{1}{2} - \sigma} (1 + |w|)^{1 - \sigma - \omega} \max_{\psi, \psi'} |Z(1-s, 1-w; \psi, \psi')| \ll (1 + |s|)^{\frac{1}{2} - \sigma} (1 + |w|)^{\frac{1}{2} - \omega} \times \max_{\psi, \psi'} |Z(1-s, 1-w; \psi, \psi')| \ll (1 + |s|)^{\frac{3}{2} - 2\sigma - \omega} (1 + |w|)^{\frac{3}{2} - 2\omega - \sigma} \max_{\psi, \psi'} |Z(1-s, 1-w; \psi, \psi')|. \tag{30}$$

We finish the proof by using the Phragmén-Lindelöf principle (see [IK, Theorem 5.53]) in the variable $w$ with fixed $s$ with $\frac{1}{2} \leq \sigma < 1 - \varepsilon$. For $\omega = 1 + \varepsilon$, (28) gives

$$|Z(s, w; \psi, \psi')| \ll |s|^{1/4 + \varepsilon}. \tag{31}$$

For $\omega = \frac{1}{2} - \sigma - \varepsilon$, we use (30) and (29) and obtain

$$|Z(s, w; \psi, \psi')| \ll |s|^{\frac{3}{2} - 2\sigma - \omega} |w|^{\frac{3}{2} - 2\omega - \sigma} \max_{\psi, \psi'} |Z(1-s, 1-w; \psi, \psi')| \ll |s|^{\frac{3}{2} + \varepsilon} |w|^{\frac{1}{2} + \sigma + \varepsilon}. \tag{32}$$
The linear functional $\ell(x)$ with $\ell(1/2 - \sigma - \varepsilon) = 1$ and $\ell(1 + \varepsilon) = 0$ is given by $\ell(x) = \frac{x^{1/2 + \sigma + 2\varepsilon}}{2^{1/2 + \sigma + 2\varepsilon}}$, so the Phragmen-Lindelöf principle gives

$$|Z(s, w; \psi, \psi')| \ll |s| \Bigl( \frac{\varepsilon}{4} + \varepsilon + \ell(\delta) \bigl|w\bigr| \bigr) \Bigl( \frac{1}{2} + \frac{\sigma}{2} + \varepsilon \Bigr)$$

(33)

$$\ll |s| \frac{\varepsilon}{4 + \sigma} + \varepsilon \bigl|w\bigr|^{1-\omega+\varepsilon}.$$
We again estimate this integral using the residue theorem. The integrand has the following poles:

• At $s = 1/2$ with residue

$$\frac{Y^{1/2}\hat{\psi}(\frac{1}{2})}{8\zeta_2(2)} = \frac{Y^{1/2}\hat{\psi}(\frac{1}{2})}{\pi^2}.$$ 

• Zeros of

$$\zeta_2(2s + 1) = \left(1 - \frac{1}{2^{2s+1}}\right)\zeta(2s + 1).$$

These are at the points $s = \frac{k\pi}{\log 2} - \frac{1}{2}$, $k \in \mathbb{Z}$.

These poles have $\text{Re}(s) < 0$ and if we assume RH, they all have $\text{Re}(s) \leq -1/4$.

Therefore, we have the following:

$$\hat{\varphi}(1)X \int \frac{Y^s\hat{\psi}(s)\zeta_2(2s)}{2\zeta_2(2s + 1)} ds = \frac{\hat{\varphi}(1)\hat{\psi}(\frac{1}{2}) XY^{1/2}}{\pi^2} + \frac{\hat{\varphi}(1)X}{2\pi i} \int_{(\delta)} \frac{Y^s\hat{\psi}(s)\zeta_2(2s)}{2\zeta_2(2s + 1)} ds.$$ (36)

Depending whether we assume RH or not, we take $\delta = -\frac{1}{4} + \varepsilon$ or $\delta = \varepsilon$, bound the integral trivially (we use (16) when $\delta = -1/4 + \varepsilon$) and get

$$\hat{\varphi}(1)X \int \frac{Y^s\hat{\psi}(s)\zeta_2(2s)}{2\zeta_2(2s + 1)} ds = \frac{\hat{\varphi}(1)\hat{\psi}(\frac{1}{2}) XY^{1/2}}{\pi^2} + \mathcal{O}(XY^\delta).$$ (37)

Using this in (34), we obtain

$$S(X,Y; \varphi, \psi) = \frac{\hat{\varphi}(1)\hat{\psi}(\frac{1}{2}) XY^{1/2}}{\pi^2} +$$

$$+ \left(\frac{1}{2\pi i}\right)^2 \int \int \mathcal{A}(s,w)X^w Y^s \hat{\varphi}(w)\hat{\psi}(s) dw ds + \mathcal{O}(XY^\delta),$$ (38)

Note that when $\varphi = \psi = 1_{[0,1]}$, the first term is $\frac{2}{\pi^2}XY^{1/2}$ and corresponds to the contribution when $n$ is a square.

Next, we exchange the integrals and shift the integral over $\text{Re}(s) = 2$ to $\text{Re}(s) = 3/4$, crossing the polar line at $s = 1$. The computation of the residues coming from this polar line is completely analogous to the previous case, and the result is stated in the following theorem:
Theorem 4.1. Let $\varepsilon > 0$. Then we have:

\[
\sum_{m \text{ odd}} \sum_{n \text{ odd}} \frac{(m/n)}{\varphi(m/n)} \psi(n) = \frac{\hat{\varphi}(1)\hat{\psi} \left( \frac{1}{2} \right) XY^{1/2} + \hat{\psi}(1)\hat{\varphi} \left( \frac{1}{2} \right) YX^{1/2}}{\pi^2} + \\
+ \left( \frac{1}{2\pi i} \right)^2 \int_{(3/4)} \int_{(3/4+\varepsilon)} A(s,w)X^wY^s \hat{\varphi}(w)\hat{\psi}(s)dwds + \\
+ O_{\varepsilon} \left( YX^{\delta} + XY^{\delta} \right),
\]

where $\delta = \varepsilon$. If we assume the Riemann Hypothesis, then we can take $\delta = -1/4 + \varepsilon$.

5. Contribution of the polar line $s + w = 3/2$

Before further shifting the integrals, we need to compute the residues on the polar line $s + w = 3/2$, which is done in the following lemma.

Lemma 5.1. For all $s \in \mathbb{C}$

\[
\text{res}(s, \frac{3}{2} - s) Z(s,w) = \frac{\sqrt{\pi} \sin \left( \frac{\pi s}{2} \right) \Gamma \left( s - \frac{1}{2} \right) \zeta(2s - 1)}{2(2\pi)^s},
\]

Proof. We use the functional equation (28) in [Blo], from which it follows that

\[
Z(1 - u, u + v - 1/2) = \\
= \frac{\pi^{-u+\frac{3}{2}}\Gamma \left( \frac{u}{2} \right)}{(4^{1-u} - 4) \Gamma \left( \frac{1-u}{2} \right)} \cdot \left( -4^uZ(u,v;\psi_1,\psi_1) + (4^u - 2) Z(u,v;\psi_1,\psi_{-1}) + \right. \\
\left. + \left( 2^u - 2^{1-u} \right) (Z(u,v;\psi_1,\psi_2) + Z(u,v;\psi_1,\psi_{-2}) \right). 
\]

Under the change of variables $(s,w) = (1 - u, u + v - 1/2)$, the line $v = 1$ transforms to the line $s + w = 3/2$. Since $v = 1$ is a polar line of $Z(u,v;\psi,\psi')$ if and only if $\psi' = \psi_1$, the residue comes only from the first term in the parenthesis on the right-hand side of (41), and is given by

\[
\text{res}_{(1-u,u+\frac{1}{2})} Z(u,v;\psi_1,\psi_1) = \frac{\pi^{-u+\frac{3}{2}}\Gamma \left( \frac{u}{2} \right) (-4^u) \zeta(2u)}{2(4^{1-u} - 4) \Gamma \left( \frac{1-u}{2} \right)} = \frac{\pi^{-u+\frac{3}{2}}\Gamma \left( \frac{u}{2} \right) \zeta(2u)}{2 \cdot 4^{1-u} \Gamma \left( \frac{1-u}{2} \right)},
\]

so we have

\[
\text{res}(s, \frac{3}{2} - s) Z(s,w) = \frac{\pi^{-\frac{s}{2}}\Gamma \left( \frac{1-s}{2} \right) \zeta(2 - 2s)}{2 \cdot 4^s \Gamma \left( \frac{s}{2} \right)} \\
= \frac{\sqrt{\pi} \sin \left( \frac{\pi s}{2} \right) \Gamma \left( s - \frac{1}{2} \right) \zeta(2s - 1)}{2(2\pi)^s},
\]
where the last equality follows after using the functional equation for \( \zeta(s) \) and the formula (20).

We are now ready to prove Theorem 1.1:

**Proof of Theorem 1.1.** We move the integral from (39) further to the left. According to the discussion at the end of Section 3, we know that except the line \( s + w = 3/2 \), the integrand has no poles with \( \text{Re}(s+w) \geq 1 \), or with \( \text{Re}(s+w) > 3/4 \) if we assume RH. Hence we have

\[
\left( \frac{1}{2\pi i} \right)^2 \int_{(3/4)} \int_{(3/4+\epsilon)} A(s,w)X^wY^s\hat{\varphi}(w)\hat{\psi}(s)dwds
\]

\[
= \left( \frac{1}{2\pi i} \right)^2 \int_{(3/4)} \int_{(3/4)} A(s,w)X^wY^s\hat{\varphi}(w)\hat{\psi}(s)dwds + \]

\[
+ \frac{1}{2\pi i} \int_{(3/4)} X^{3/2-s}Y^s\hat{\varphi}\left( \frac{3}{2} - s \right)\hat{\psi}(s)\text{res}(s,\frac{3}{2}-s)A(s,w)ds,
\]

where \( \delta' = 1/4 + \epsilon \), or \( \epsilon \) under RH. Using Lemma 5.1 and (24), we have

\[
\text{res}(s,\frac{3}{2}-s)A(s,w) = \frac{\text{res}(s,\frac{3}{2}-s)Z(s,w)}{\zeta(2)} = \frac{\sqrt{\pi}\sin\left( \frac{\pi s}{2} \right)\Gamma\left( s - \frac{1}{2} \right)\zeta(2s-1)}{2\zeta(2)(2\pi)^s}.
\]

Therefore the second integral on the right-hand side of (42) is

\[
\frac{2X^{3/2}}{i\pi^{3/2}} \int_{(3/4)} \left( \frac{Y}{2\pi X} \right)^s \hat{\varphi}\left( \frac{3}{2} - s \right)\hat{\psi}(s)\Gamma\left( s - \frac{1}{2} \right)\sin\left( \frac{\pi s}{2} \right)\zeta(2s-1)ds.
\]

Hence we have

\[
S(X,Y;\varphi,\psi) = \frac{\hat{\varphi}(1)\hat{\psi}\left( \frac{3}{2} \right)XY^{1/2} + \hat{\psi}(1)\hat{\varphi}\left( \frac{1}{2} \right)YX^{1/2}}{\pi^2} +
\]

\[
+ \frac{2X^{3/2}}{i\pi^{3/2}} \int_{(3/4)} \left( \frac{Y}{2\pi X} \right)^s \hat{\varphi}\left( \frac{3}{2} - s \right)\hat{\psi}(s)\Gamma\left( s - \frac{1}{2} \right)\sin\left( \frac{\pi s}{2} \right)\zeta(2s-1)ds +
\]

\[
+ \left( \frac{1}{2\pi i} \right)^2 \int_{(3/4)} \int_{(3/4)} A(s,w)X^wY^s\hat{\varphi}(w)\hat{\psi}(s)dwds + O\left( XY^\delta + YX^\delta \right)
\]

\[
= \frac{2}{\pi^2} \cdot X^{3/2} \cdot D\left( \frac{Y}{X};\varphi,\psi \right) + O\left( XY^\delta + YX^\delta \right),
\]

where the last equality follows after trivially bounding the second integral, and using \( X^\delta Y^{3/2} \ll XY^\delta + YX^\delta \). \( \square \)
6. Removing the smooth weights

In this section, we show how to remove the smooth weights from Theorem 1.1 and prove Theorem 1.2. We choose the weights \( \varphi = \psi \) to be a smooth function which is 1 on the interval \( \left[ \frac{1}{U}, 1 - \frac{1}{U} \right] \) for some \( U \) to be chosen later, and 0 outside of \((0, 1)\), and which satisfies

\[
|\hat{\varphi}(\sigma + it)| \ll j, \sigma \frac{U^{j-1}}{1 + |t|^j}
\]

for all positive integers \( j \). Then using the Pólya-Vinogradov inequality (see (3.1) in [CFS]), we have

\[
|S(X, Y) - S(X, Y; \varphi, \psi)| \ll \frac{X^{3/2} + Y^{3/2}}{U} \log(XY).
\]

We now compute the dependence on \( U \) of the error term in Theorem 1.1 coming from evaluating the residues and the shifted integral. The error term from the polar line \( s = 1 \) in \((36)\) is

\[
\ll XY^\delta \int_{(\delta)} |\hat{\psi}(s)\zeta_2(2s)| \left| \frac{\hat{\psi}(s)\zeta_2(2s + 1)}{\zeta_2(2s + 1)} \right| ds.
\]

For \( \delta = \varepsilon \) or \(-1/4 + \varepsilon\) under RH, the integral converges when we take \( j = 2 \) in \((45)\) (using \((16)\) when we assume RH). Similarly for the error coming from the polar line \( w = 1 \), so these two contribute

\[
\ll \left( XY^\delta + YX^\delta \right) U.
\]

The error term from the shifted integral in \((44)\) is

\[
\ll XY^{\delta'} \int_{(3/4)} \int_{(\delta')} |A(s, w)| \left| \hat{\varphi}(w)\hat{\psi}(s) \right| dwds.
\]

Using \((24)\), Lemma 3.2 with \( \sigma = 3/4 \) and \( \omega = \delta' = \delta + 1/4 \), we see that the double integral will converge if we take \( j = 2 \) in \((45)\) for both \( \hat{\varphi}(w) \) and \( \hat{\psi}(s) \). Therefore \((49)\) is

\[
\ll XY^{\delta'} U^2 \ll \left( XY^\delta + YX^\delta \right) U^2.
\]
Hence Theorem 1.1 gives
\[(51)\]
\[
S(X, Y) = S(X, Y; \varphi, \psi) + O \left( \frac{X^{3/2} + Y^{3/2}}{U} \log(XY) \right)
\]
\[
= \frac{2}{\pi^2} \cdot X^{3/2} \cdot D \left( \frac{Y}{X}; \varphi, \psi \right) + O \left( \frac{X Y^\delta + Y X^\delta U^2 + X^{3/2} + Y^{3/2}}{U} \log(XY) \right)
\]
\[
= \frac{2}{\pi^2} \cdot X^{3/2} \cdot D \left( \frac{Y}{X} \right) + E(X, Y; \varphi, \psi) +
\]
\[
+ O \left( \frac{X Y^\delta + Y X^\delta}{U^2} + \frac{X^{3/2} + Y^{3/2}}{U} \log(XY) \right),
\]
where
\[
E(X, Y; \varphi, \psi) = \frac{2}{\pi^2} X^{3/2} \left( D \left( \frac{Y}{X}; \varphi, \psi \right) - D \left( \frac{Y}{X} \right) \right)
\]
\[
\ll XY^{-1/2} \left| \hat{\varphi}(1) \hat{\psi} \left( \frac{1}{2} \right) - 2 \right| + YX^{1/2} \left| \hat{\psi}(1) \hat{\varphi} \left( \frac{1}{2} \right) - 2 \right| +
\]
\[
+ X^{3/2} \left| \int_{(3/4)} \left( \frac{Y}{2\pi X} \right)^s \Gamma \left( s - \frac{1}{2} \right) \sin \left( \frac{\pi s}{2} \right) \zeta(2s - 1) \times
\]
\[
\times \left( \hat{\varphi} \left( \frac{3}{2} - s \right) \hat{\psi}(s) - \frac{1}{s \left( \frac{3}{2} - s \right)} \right) ds \right|.
\]

For the first term, we have
\[
\hat{\varphi}(1) \hat{\psi}(1/2) = \int_0^\infty \varphi(t) dt \int_0^\infty \psi(v) v^{-1/2} dv
\]
\[
= \int_{1/U}^{1-1/U} dt \int_{1/U}^{1-1/U} v^{-1/2} dv + O \left( U^{-3/2} \right)
\]
\[
= 2 + O \left( U^{-1/2} \right),
\]
and similarly for \( \hat{\psi}(1) \hat{\varphi}(1/2) \), so
\[
(54)\quad XY^{1/2} \left| \hat{\varphi}(1) \hat{\psi} \left( \frac{1}{2} \right) - 2 \right| + YX^{1/2} \left| \hat{\psi}(1) \hat{\varphi} \left( \frac{1}{2} \right) - 2 \right| \ll \frac{XY^{1/2} + YX^{1/2}}{\sqrt{U}}.
\]

Now we bound the integral in (52). Let
\[
g(s) = \Gamma \left( s - \frac{1}{2} \right) \sin \left( \frac{\pi s}{2} \right) \zeta(2s - 1),
\]
and
\[
h(s) = \hat{\varphi} \left( \frac{3}{2} - s \right) \hat{\psi}(s) - \frac{1}{s \left( \frac{3}{2} - s \right)}.
\]
Using (19) and a subconvexity estimate of the form $|\zeta(1/2 + it)| \ll (1 + |t|)^{1/6}$, we obtain

$$|g(3/4 + it)| \ll (1 + |t|)^{-1/4}|\zeta(1/2 + it)| \ll (1 + |t|)^{-1/12}.$$  

Hence Hölder’s inequality with $q = 12 + \varepsilon$ yields

$$X^{3/2} \left| \int_{\mathcal{Y}^2} \left( \frac{Y}{X} \right)^s g(s)h(s)ds \right|$$

$$\ll (XY)^{3/4} \left( \int_{\mathcal{Y}^2} |g(s)|^q ds \right)^{1/q} \left( \int_{\mathcal{Y}^2} |h(s)|^p ds \right)^{1/p},$$

where $p = \frac{1}{1-1/q} = \frac{12}{11} - \varepsilon$. The first integral is $\ll 1$ by (55), so we need to bound the second one. To do that, we use the following lemma:

**Lemma 6.1.** Let $\theta$ be a smooth function supported in $(0, 1)$ such that $\theta(x) = 1$ when $x \in [1/U, 1 - 1/U]$. Then for any $0 < \sigma < 1$ and $p > 1$, we have

$$\left( \int_{\sigma} \frac{1}{|s|^p} \cdot \hat{\theta}(s) - \frac{1}{s} \right)^{1/p} \ll_{p, \sigma} \frac{1}{U^{\sigma}}.$$ 

**Proof.** Using Minkowski integral inequality, we have

$$\left( \int_{\sigma} \frac{1}{|s|^p} \cdot \hat{\theta}(s) - \frac{1}{s} \right)^{1/p}$$

$$= \left( \int_{\sigma} \left( \int_0^{\infty} \left( \theta(x) - 1_{[0,1]}(x) \right) \frac{x^{s-1}}{s} dx \right)^p ds \right)^{1/p}$$

$$\ll \int_0^{\infty} \left( \int_{\sigma} \left( \theta(x) - 1_{[0,1]}(x) \right) \frac{x^{s-1}}{s} ds \right)^p dx$$

$$\ll \left( \int_{\sigma} x^{\sigma-1} dx + \int_{1-1/U}^{1/U} x^{\sigma-1} dx \right) \left( \int_{\sigma} \frac{1}{|s|^p} ds \right)^{1/p}$$

$$\ll_{p, \sigma} \frac{1}{U^{\sigma}}.$$  

\[\square\]
The triangle inequality gives

\[(57) \quad |h(s)| \leq \left| \hat{\varphi} \left( \frac{3}{2} - s \right) \right| \left| \hat{\psi}(s) - \frac{1}{s} \right| + \left| \frac{1}{s} \right| \left| \hat{\varphi} \left( \frac{3}{2} - s \right) - \frac{1}{\frac{3}{2} - s} \right|,
\]

so using Minkowski inequality and Lemma 6.1 yields

\[(58) \quad \left( \int_{(3/4)} |h(s)|^p ds \right)^{1/p} \ll \left( \int_{(3/4)} \left| \hat{\varphi} \left( \frac{3}{2} - s \right) \right|^p \left| \hat{\psi}(s) - \frac{1}{s} \right|^p ds \right)^{1/p} + \left( \int_{(3/4)} \left| \hat{\varphi}(3/2 - s) - \frac{1}{\frac{3}{2} - s} \right|^p ds \right)^{1/p}.
\]

From (46), (50)–(54), (56) and (58), we obtain

\[(59) \quad S(X, Y) = \frac{2}{\pi^2} X^{\frac{3}{2}} D \left( \frac{Y}{X} \right) + O \left\{ \left( XY^{\delta} + YX^{\delta} \right) U^2 + \frac{X^{\frac{3}{2}} + Y^{\frac{3}{2}}}{U} \log(XY) + \frac{XY^{\frac{3}{2}} + YX^{\frac{3}{2}}}{\sqrt{U}} + \left( \frac{XY}{U} \right)^{\frac{3}{4}} \right\}.
\]

Then choosing \( U = \left( \frac{XY^{1/2} + YX^{1/2}}{XY^{\delta} + YX^{\delta}} \right)^{2/5} \) gives Theorem 1.2 in the range \( Y^{5/6} < X < Y^{6/5} \). In the remaining range, it follows from (2) and the asymptotic estimates for \( C(\alpha) \) (5) and (6), together with the fact that \( C'(\alpha) = D(\alpha) \), which will be proved in the next section.

7. Proving that \( C(\alpha) = D(\alpha) \).

In this section, we show that \( C(\alpha) = D(\alpha) \). Recall that

\[(60) \quad D(\alpha) = \sqrt{\alpha} + \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(3/4)} \left( \frac{\alpha}{2\pi} \right)^s \cdot \frac{\Gamma \left( s - \frac{3}{2} \right) \sin \left( \frac{\pi s}{2} \right) \zeta(2s - 1)}{s} ds.
\]

We shift the integral to the left, capturing the pole at \( s = \frac{1}{2} \), which contributes

\[2\sqrt{\pi} \left( \frac{\alpha}{2\pi} \right)^{1/2} \cdot -\frac{\sin(\pi/4)\zeta(0)}{1/2} = \alpha^{1/2}.
\]
MEAN VALUE OF REAL CHARACTERS

The horizontal integrals vanish by (19) and a convexity estimate for \( \zeta(s) \), so

\[
D(\alpha) = \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(1/4)} \left( \frac{\alpha}{2\pi} \right)^s \Gamma(s - \frac{3}{2}) \frac{\sin \left( \frac{\pi s}{2} \right)}{s} \zeta(2s - 1) ds
\]
(61)

\[
= \alpha - \frac{2\sqrt{\pi}}{2\pi i} \int_{(1/4)} \alpha^{-s} \cdot (2\pi)^s \Gamma(-s - \frac{3}{2}) \frac{\sin \left( \frac{\pi s}{2} \right)}{s} \zeta(-2s - 1) ds.
\]

This integral is an inverse Mellin transform, so we rewrite \( C(\alpha) \) using Mellin inversion. We have

\[
C(\alpha) = \alpha + \frac{\alpha^{3/2}}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1/\alpha} \sqrt{y} \sin \left( \frac{\pi k^2}{2y} \right) dy
\]
(62)

\[
= \alpha + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left( \frac{\pi k^2 \alpha}{2u} \right) du
\]

\[
= \alpha + \frac{2}{\pi} \cdot \frac{1}{2\pi i} \int \alpha^{-s} \hat{f}(s) ds,
\]

where

\[
f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left( \frac{\pi k^2 x}{2u} \right) du.
\]

When \( 0 < \text{Re}(s) < 1 \), we have

\[
\hat{f}(s) = \int_0^\infty \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left( \frac{\pi k^2 x}{2u} \right) du \ x^{s-1} dx
\]
(64)

\[
= \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \sqrt{u} \int_0^\infty \sin \left( \frac{\pi k^2 x}{2u} \right) x^{s-1} dx \ du,
\]

which isn’t obvious because the double integral doesn’t converge absolutely, but we will justify the interchange of summation and integrals in Lemma 7.1. We can now make a change of variables \( y = \frac{\pi k^2 x}{2u} \) and obtain

\[
\hat{f}(s) = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 \sqrt{u} \int_0^{\infty} \sin(y) y^{s-1} dy \ \left( \frac{2u}{\pi k^2} \right)^s du
\]
(65)

\[
= \left( \frac{2}{\pi} \right)^s \zeta(2 + 2s) \int_0^1 u^{s+1/2} du \int_0^{\infty} \sin(y) y^{s-1} dy
\]

\[
= \frac{2^s \zeta(2 + 2s) \Gamma(s) \sin \left( \frac{\pi s}{2} \right)}{\pi^s (s + 3/2)},
\]
which holds when \(0 < \text{Re}(s) < 1\), so we can take \(c = 1/4\) in (62). It therefore suffices to show that
\[
-2\sqrt{\pi} \frac{(2\pi)^s \Gamma \left(-s - \frac{3}{2}\right) \sin \left(\frac{\pi s}{2}\right) \zeta(-2s - 1)}{s} = \frac{2}{\pi} \frac{2^s \zeta(2 + 2s) \Gamma(s) \sin \left(\frac{\pi s}{2}\right)}{\pi^s(s + 3/2)}.
\]

The functional equation for the zeta function gives
\[
\zeta(2s + 2) = \pi^{2s+3/2} \cdot \frac{\Gamma \left(-s - \frac{1}{2}\right)}{\Gamma(s + 1)} \zeta(-2s - 1),
\]
so using \(s \Gamma(s) = \Gamma(s + 1)\) and \((-s - 3/2) \Gamma(-s - 3/2) = \Gamma(-s - 1/2)\) gives the result.

It remains to prove that we can interchange the order of summation and integrations in (64):

**Lemma 7.1.** When \(0 < \text{Re}(s) < 1\), it holds that
\[
\int_0^\infty \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left(\frac{\pi k^2 x}{2u}\right) du x^{s-1} dx
\]
\[
= \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \int_0^\infty \sin \left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx du.
\]

**Proof.** We have
\[
\int_0^\infty \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left(\frac{\pi k^2 x}{2u}\right) du x^{s-1} dx
\]
\[
= \lim_{A \to \infty} \int_0^1/4 \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left(\frac{\pi k^2 x}{2u}\right) du x^{s-1} dx.
\]

We can now interchange the integrals and summation, because
\[
\sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \int_{1/4}^A \left| \sqrt{u} \sin \left(\frac{\pi k^2 x}{2u}\right) x^{s-1} \right| dx du \ll A,
\]
so
\[
\lim_{A \to \infty} \int_0^1/4 \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \sin \left(\frac{\pi k^2 x}{2u}\right) du x^{s-1} dx
\]
\[
= \lim_{A \to \infty} \sum_{k=1}^\infty \frac{1}{k^2} \int_0^1 \sqrt{u} \int_{1/4}^A \sin \left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx du.
\]

To insert the limit inside the sum and integral, we use the Dominated convergence theorem with the bound \(\sqrt{u} \left| \int_{1/4}^A \sin \left(\frac{\pi k^2 x}{2u}\right) x^{s-1} dx \right| \leq K \sqrt{u}\) for an absolute constant \(K\) independent of \(u\) and \(A\), because \(\int_0^\infty \sin(y) y^{s-1} dy\) converges. □
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