The art of probability-of-default curve calibration

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PD curve calibration refers to the transformation of a set of rating grade level probabilities of default (PDs) to another average PD level that is determined by a change of the underlying portfolio-wide PD. This paper presents a framework that allows to explore a variety of calibration approaches and the conditions under which they are fit for purpose. We test the approaches discussed by applying them to publicly available datasets of agency rating and default statistics that can be considered typical for the scope of application of the approaches. We show that the popular ‘scaled PDs’ approach is theoretically questionable and identify an alternative calibration approach (‘scaled likelihood ratio’) that is both theoretically sound and performs better on the test datasets.

Keywords: Probability of default, calibration, likelihood ratio, Bayes’ formula, rating profile, binary classification.

1. Introduction

The best way to understand the subject of this paper is to have a glance at table 1 on page 2 that illustrates the problem studied. Table 1 shows the grade-level and portfolio-wide default rates (third column) that were observed in 2009 for S&P-rated corporate entities together with the rating frequencies that were observed at the beginning of 2009 (second column) and at the beginning of 2010 (fourth column). The question marks in the fifth column indicate the question this paper is intended to answer: How can grade-level default rates for a future time period be forecast on the basis of observations from an earlier period and the known rating profile at the beginning of the future period?

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Table 1: S&P rating frequencies (%) and default rates (%) in 2009 and rating frequencies in 2010. Sources: S&P (2010), tables 51 to 53, and S&P (2011), tables 50 to 52.

| Rating | 2009 Frequency | Default rate | 2010 Frequency | Default rate |
|--------|----------------|--------------|----------------|--------------|
| AAA    | 1.38           | 0            | 1.3            | ?            |
| AA+    | 0.63           | 0            | 0.45           | ?            |
| AA     | 3.21           | 0            | 2.59           | ?            |
| AA-    | 4.18           | 0            | 3.78           | ?            |
| A+     | 5.8            | 0.29         | 6.39           | ?            |
| A      | 8.7            | 0.39         | 8.58           | ?            |
| A-     | 9.32           | 0            | 9.56           | ?            |
| BBB+   | 8.5            | 0.4          | 8.28           | ?            |
| BBB    | 9.23           | 0.18         | 10.56          | ?            |
| BBB-   | 7.83           | 1.09         | 7.79           | ?            |
| BB+    | 4.54           | 0            | 4.6            | ?            |
| BB     | 5.03           | 1.02         | 5              | ?            |
| BB-    | 7.53           | 0.91         | 6.86           | ?            |
| B+     | 7.47           | 5.48         | 7.12           | ?            |
| B      | 8.23           | 9.96         | 7.9            | ?            |
| B-     | 5.17           | 17.16        | 5.25           | ?            |
| CCC-C  | 3.24           | 48.42        | 3.98           | ?            |
| All    | 100            | 3.99         | 100            | 1.14?        |

The question mark in the lower right corner of table 1 indicates that we investigate this question both under the assumption that an independent forecast of the future portfolio-wide default rate is known and under the assumption that also the future portfolio-wide default rate has to be forecast.

We call a forecast of grade-level default rates a *PD curve*. The problem we study in this paper is made more complicated by the fact that for economic reasons PD curves are subject to the constraints that they need to be monotonic and positive – although table 1 shows that this is not necessarily true for empirically observed default rates.

The scope of the concepts and approaches described in this paper is not limited to data from rating agencies but covers any rating system for which data from an estimation period is available. It should, however, be noted that the focus in this paper is on grade-level default rate forecasts while the problem of forecasting the unconditional (or portfolio-wide) default rate is not considered. Forecasting the unconditional default rate is an econometric problem that is beyond the scope of this paper (see Engelmann and Porath, 2012, for an example of how to approach this problem).

This paper appears to be almost unique in that it solely deals with the calibration or recalibration
Calibration of PD curves is a topic that is often mentioned in the literature but mostly only as one aspect of the more general subject of rating model development. For instance, Falkenstein et al. (2000) deployed the approach that is called ‘scaled PDs’ in this paper without any comment about why they considered it appropriate. There are, however, some authors who devoted complete articles or book sections to PD curve calibration. Van der Burgt (2008) suggested a predecessor of the technique that is called quasi moment matching (QMM) in this paper. Bohn and Stein (2009, chapter 4) discussed the conceptual and practical differences between the ‘scaled PDs’ and ‘invariant likelihood ratio’ approaches. More recently, Konrad (2011) investigated in some detail the interplay between the calibration and the discriminatory power of rating models.

In this paper, we revisit the concept of two calibration steps as used by Bohn and Stein (2009). According to Bohn and Stein (2009) the two steps are a consequence of the fact that usually the first calibration of a rating model is conducted on a training sample in which the proportion of good and bad might not be representative of the live portfolio. The second calibration step, therefore, is needed to adjust the calibration to the right proportion of good and bad.

We argue more generally that the two steps actually relate to different time periods (the estimation and the forecast periods) which both can be described by the same type of model. This view encompasses both the situation where a new rating model is calibrated and the situation where an existing rating model undergoes a – possibly periodic – recalibration. The estimation period is used to estimate the model components that are assumed to be invariant (i.e. unchanged) or in a specific way transformed between the estimation and the forecast periods. Calibration approaches for the forecast period are essentially determined by the assumptions of invariance between the periods.

Specifically, the model estimation in the estimation period involves smoothing of the observed default rates in order to create a positive and monotonic PD curve. For this purpose we apply quasi moment matching (QMM) the details of which are described in appendix A.

When in the following we investigate different invariance assumptions that can be made for the forecast period the basic idea is always that the rating system’s discriminatory power is the same or nearly the same both in the estimation and forecast periods. However, discriminatory power can technically be expressed in a number of different ways that correspond to invariance assumptions with a range of different implications. This is why we first study in section 3 in some detail the model components that are related to invariance assumptions:

- Unconditional rating distribution (profile).
- Conditional (on default and survival) rating distributions (profiles).
- Unconditional PD.
- PD curve (grade-level PDs, i.e. PDs conditional on rating grades).
- Accuracy ratio (as a measure of discriminatory power).
- Likelihood ratio.

In particular, we derive a new result (theorem 3.3) on the characterisation of the joint distribution of a borrower’s rating at the beginning of the observation period and his solvency state at the
end of the period by unconditional rating profile and likelihood ratio.

Then, in section 4, we look at different calibration approaches (which may be characterised by invariance assumptions). The suitability of the approaches described depends strongly upon what data (e.g. the unconditional rating profile) can be observed at the time when the forecast exercise takes place. We therefore discuss the different possibilities and assumptions in some detail and examine the performance of the approaches with a real data example. The example is based on the S&P data from table 1 which is presented in more detail in section 2.

In particular, the example in section 4 suggests that the popular ‘scaled PDs’ approach (corresponding to the assumption of an invariant shape of the PD curve) is both theoretically questionable and not very well performing on the example dataset. Two other approaches (‘invariant AR’ and ‘scaled likelihood ratio’) appear to be theoretically sound and better performing when deployed for the numerical example.

However, as the S&P dataset is small the example provides anecdotal evidence only. Its suggestions are therefore backtested and qualified in section 5. The – still rather limited – backtest confirms that the ‘scaled likelihood ratio’ approach performs better than the ‘scaled PDs’ approach. In contrast, the ‘invariant AR’ approach is found to be underperforming in the backtest.

2. Data and context

The numerical examples in section 4 in this paper are based on the S&P rating and default statistics for all corporates as presented in table 2 on page 5. Only with their 2009 default data report S&P (2010) began to make information on modified-grade level issuer numbers readily available. Without issuer numbers, however, there is not sufficient information to calculate rating profiles and conduct goodness-of-fit tests for rating profiles because such tests typically require the occupation frequencies (i.e. the numbers of issuers in each of the rating grades) as input. This explains why we only look at default statistics from 2009 onwards.

For the purposes of this paper, data from Moody’s is less suitable because Moody’s do not provide issuer numbers at alphanumeric grade level and estimate default rates in a way that makes it impossible to infer exact grade-level numbers of defaults (see Hamilton and Cantor, 2006, for details of the estimation approach). Therefore, in order to work with the publicly available Moody’s (2013) data, one has to make assumptions that are likely to make the results less reliable. That is why, in section 5, we use Moody’s default and rating data only for backtesting and qualifying the observations from section 4.

2.1. Observations on the data

S&P’s all corporates default statistics (table 2) represent an example of an interesting, somewhat problematic dataset because it includes some instances of inversions of observed default rates. ‘Inversion of default rates’ means that the default rate observed for a better rating grade is higher than the default rate of the adjacent worse rating grade.

The default rate columns of table 2 show that the corporate grade-level default rates recorded by
Table 2: S&P’s corporate ratings, defaults and default rates (DR, %) in 2009, 2010 and 2011.

Sources: S&P (2010, tables 51 to 53), S&P (2011, tables 50 to 52), S&P (2012, tables 50 to 52).

| Rating grade | 2009 | | | 2010 | | | 2011 |
|--------------|------|--------|------|------|--------|------|--------|------|
|               | rated | defaults | DR   | rated | defaults | DR   | rated | defaults | DR   |
| AAA           | 81    | 0       | 0.00 | 72    | 0       | 0.00 | 51     | 0       | 0.00 |
| AA+           | 37    | 0       | 0.00 | 25    | 0       | 0.00 | 36     | 0       | 0.00 |
| AA            | 188   | 0       | 0.00 | 143   | 0       | 0.00 | 120    | 0       | 0.00 |
| AA-           | 245   | 0       | 0.00 | 209   | 0       | 0.00 | 207    | 0       | 0.00 |
| A+            | 340   | 1       | 0.29 | 353   | 0       | 0.00 | 357    | 0       | 0.00 |
| A             | 510   | 2       | 0.39 | 474   | 0       | 0.00 | 470    | 0       | 0.00 |
| A-            | 546   | 0       | 0.00 | 528   | 0       | 0.00 | 560    | 0       | 0.00 |
| BBB+          | 498   | 2       | 0.40 | 457   | 0       | 0.00 | 473    | 0       | 0.00 |
| BBB           | 541   | 1       | 0.18 | 583   | 0       | 0.00 | 549    | 0       | 0.00 |
| BBB-          | 459   | 5       | 1.09 | 430   | 0       | 0.00 | 508    | 1       | 0.20 |
| BB+           | 266   | 0       | 0.00 | 254   | 2       | 0.79 | 260    | 0       | 0.00 |
| BB            | 295   | 3       | 1.02 | 276   | 1       | 0.36 | 319    | 0       | 0.00 |
| BB-           | 441   | 4       | 0.91 | 379   | 2       | 0.53 | 403    | 0       | 0.00 |
| B+            | 438   | 24      | 5.48 | 393   | 0       | 0.00 | 509    | 2       | 0.39 |
| B             | 482   | 48      | 9.96 | 436   | 3       | 0.69 | 586    | 7       | 1.19 |
| B-            | 303   | 52      | 17.16| 290   | 6       | 2.07 | 301    | 12      | 3.99 |
| CCC-C         | 190   | 92      | 48.42| 220   | 49      | 22.27| 138    | 22      | 15.94 |
| All           | 5860  | 234     | 3.99 | 5522  | 63      | 1.14 | 5847   | 44      | 0.75 |

S&P for 2009, 2010 and 2011 increase in general with deteriorating credit quality as one would expect. However, there are a number of ‘inversions’ in all the default rate columns of the table, i.e. there are some counter-intuitive examples of adjacent rating grades where the less risky grade has a higher default rate than the adjacent riskier grade. Notable for this phenomenon is, in particular, the pair of BBB- and BB+ in 2009 with 1.09% defaults in BBB- and 0% defaults in BB+.

Should we conclude from the existence of such inversions that there is a problem with the rank-ordering capacity of the rating methodology? The long-run average grade-level default rates reported by S&P (2013, table 23) suggest that the observation of default rate inversions as in table 2 might be an exception. By Fisher’s exact test (Fisher, 1922; Casella and Berger, 2002, Example 8.3.30) this explanation can be verified.

A question of similar importance for the estimation of PD curves is the question of whether or not the unconditional (or all-portfolio) rating profile (i.e. the distribution of the rating grades) of a portfolio can be assumed to be unchanged over time. Table 3 on page 6 shows the unconditional rating profiles of the corporate entities for the three years of S&P data used in this paper. It
Table 3: S&P rating profiles for corporates at the beginning of 2009, 2010 and 2011. Sources: S&P (2010, tables 51 to 53), S&P (2011, tables 50 to 52), S&P (2012, tables 50 to 52) and own calculations. All values in %.

| Rating grade | 2009 | 2010 | 2011 |
|--------------|------|------|------|
| AAA          | 1.38 | 1.30 | 0.87 |
| AA+          | 0.63 | 0.45 | 0.62 |
| AA           | 3.21 | 2.59 | 2.05 |
| AA-          | 4.18 | 3.78 | 3.54 |
| A+           | 5.80 | 6.39 | 6.11 |
| A            | 8.70 | 8.58 | 8.04 |
| A-           | 9.32 | 9.56 | 9.58 |
| BBB+         | 8.50 | 8.28 | 8.09 |
| BBB          | 9.23 | 10.56| 9.39 |
| BBB-         | 7.83 | 7.79 | 8.69 |
| BB+          | 4.54 | 4.60 | 4.45 |
| BB           | 5.03 | 5.00 | 5.46 |
| BB-          | 7.53 | 6.86 | 6.89 |
| B+           | 7.47 | 7.12 | 8.71 |
| B            | 8.23 | 7.90 | 10.02|
| B-           | 5.17 | 5.25 | 5.15 |
| CCC-C        | 3.24 | 3.98 | 2.36 |
| All          | 100.00 | 100.00 | 100.00 |

It appears from the percentages that the profiles vary significantly during the three years even if random differences are ignored. Pearson’s $\chi^2$ test for count data (Pearson, 1900; van der Vaart, 1998) can be used to test these observations and also to assess the accuracy of the forecast approaches discussed in the remainder of the paper.

2.2. Consequences for the calibration of PD curves

From the observations in section 2.1 we can draw two conclusions:

- Forcing monotonicity of estimated PD curves can make sense if it is justified by statistical tests or long-run average evidence.

- In general, we cannot assume that the rating profile of a portfolio does not change over time, even if random fluctuation is ignored. However, this assumption can be verified or proven wrong with statistical tests. Depending on the outcome of the tests there are different options for the estimation of PD curves. This will be discussed in detail in section 4.

Although never a default of an AAA-rated corporate was observed within one year after having been rated AAA (Moody’s, 2013; S&P, 2013) we nonetheless try and infer a positive one-year PD
for AAA. This is why in the following we restrict ourselves to only deploy PD curve estimation approaches that guarantee to deliver positive PDs for all rating grades.

On the basis of the data presented in this section, it is also worthwhile to clarify precisely the concept of a two-step (or two-period) approach to the calibration of a rating model as mentioned by Bohn and Stein (2009): The first period is the *estimation* period, the second period is the *calibration and forecast* period. The two periods are determined by their start and end dates and the observation and estimation horizon:

- $h$ is the horizon for the PD estimation, i.e. a borrower’s PD at date $T$ gives the probability that the borrower will default between $T$ and $T + h$.
- The start date $T_0$ of the estimation period is a date in the past.
- $T_1 \geq T_0 + h$ is the date when the calibration or recalibration of the rating model takes place. The calibration is for the current portfolio of borrowers whose ratings at $T_1$ should be known but whose future default status at $T_1 + h$ is still unknown.
- The end date $T_2 = T_1 + h$ of the forecast period is in the future. Then the default status of the borrowers in the current portfolio will be known.

With regard to the two-period concept for calibration, for the remainder of the paper we make the following crucial assumptions:

- For the sample as of date $T_0$ everything is known:
  - The unconditional rating profile at $T_0$,
  - the conditional rating profiles (i.e. conditional on default and conditional on survival respectively) at $T_0$,
  - the unconditional (base) PD (estimated by the observed unconditional default rate) for the time interval between $T_0$ and $T_0 + h$,
  - the conditional PDs (i.e. conditional on the rating grades, estimated by smoothing the observed grade-level default rates) at $T_0$ for the time interval between $T_0$ and $T_0 + h$.
- At date $T_1$ could be known:
  - The unconditional rating profile.
  - A forecast of the unconditional (base) default rate. In general this will be different from the unconditional PD for the estimation period between $T_0$ and $T_0 + h$.

In section 4, we will use the rating and default data for 2009 from table 2 as an example for the estimation period (i.e. $h = 1$ year, $T_0 = January 1, 2009$, $T_0 + h = December 31, 2009$). We will consider both 2010 and 2011 as examples of one-year forecast periods based on the estimation of a model for 2009 (i.e. $T_1 = January 1, 2010$ or $T_1 = January 1, 2011$). The gap of one year between the estimation period 2009 and the forecast period 2011 reflects the gap that is likely to occur in practice when the full data from the estimation period usually becomes available only a couple of months after the end of the period.
3. Description of the model

This section describes a statistical model of a borrower’s beginning of the period rating grade and end of the period state of solvency. This model is applicable to both the estimation and the forecast periods as discussed in section 2.2. In particular, we will consider the following model characteristics and their relationships:

- Unconditional rating distribution (profile).
- Conditional (on default and survival) rating distributions (profiles).
- Unconditional PD.
- PD curve (PDs conditional on rating grades).
- Accuracy ratio (discriminatory power).
- Likelihood ratio.

We rely on the standard binary classification model used for topics like pattern recognition, medical diagnoses, or signal detection (see, e.g., van Trees, 1968). Hand (1997) presents a variety of applications (including credit scoring) for this type of model.

Speaking in technical terms, we study the joint distribution and some estimation aspects of a pair $(X, S)$ of random variables. The variable $X$ is interpreted as the rating grade assigned to a solvent borrower at the beginning of the observation period. Hence $X$ typically takes on values on a discrete scale in a finite set which we describe without loss of generality as $\{1, 2, \ldots, k\}$. This implies that the marginal distribution of $X$ is characterised by the probabilities $\Pr[X = x], x = 1, \ldots, k$, which we call the unconditional rating profile.

**Assumption.** Low values of $X$ indicate low creditworthiness (“bad”), high values of $X$ indicate high creditworthiness (“good”).

The variable $S$ is the borrower’s state of solvency at the end of the observation period, typically one year after the rating grade was observed. $S$ takes on values in $\{0, 1\}$. The meaning of $S = 0$ is “borrower has remained solvent” (solvency or survival), $S = 1$ means “borrower has become insolvent” (default). In particular, $S$ is always observed with a time lag to the observation of $X$. Hence, when $S$ is observed $X$ is already known but when $X$ is observed today $S$ is still unknown. We write $D$ for the event $\{S = 1\}$ and $N$ for the event $\{S = 0\}$. Hence

$$D \cap N = \{S = 1\} \cap \{S = 0\} = \emptyset, \quad D \cup N = \text{whole space}. \quad (3.1)$$

The marginal distribution of the state variable $S$ is characterised by the unconditional probability of default $p$ which is defined as

$$p = \Pr[D] = \Pr[S = 1] \in [0, 1]. \quad (3.2)$$

\footnote{In practice, often a rating model with a small finite number of grades is derived from a score function with values on a continuous scale. This is usually done by mapping score intervals on rating grades. See Tasche (2008, section 3) for a discussion of how such mappings can be defined. Discrete rating models are preferred by practitioners because manual adjustment of results (overriding) is feasible. Moreover, by construction results by discrete rating models tend to be more stable over time.}
p is sometimes also called base probability of default. In the following we assume $0 < p < 1$ as the cases $p = 0$ and $p = 1$ are not of practical relevance.

### 3.1. Model specification

Recall that the two marginal distributions of $X$ and $S$ respectively do not uniquely determine the joint distribution of $X$ and $S$. For easy reference we state in the following proposition the three equivalent standard ways to characterise the joint distribution.

**Proposition 3.1** The joint distribution of the pair $(X, S)$ of the rating variable $X$ and the state of the borrower variable $S$ is fully specified in any of the following three ways:

(i) By the joint probabilities

\[
\begin{align*}
\Pr[X = x, S = 0] &= \Pr[\{X = x\} \cap N], \quad x = 1, \ldots, k, \quad \text{and} \\
\Pr[X = x, S = 1] &= \Pr[\{X = x\} \cap D], \quad x = 1, \ldots, k.
\end{align*}
\]

(3.3a)

(ii) By the unconditional PD $p = \Pr[D] = 1 - \Pr[N]$ and the distributions of $X$ conditional on $D$ and $N$ respectively:

\[
\begin{align*}
\Pr[X = x \mid D] &= \frac{\Pr[\{X = x\} \cap D]}{p}, \quad x = 1, \ldots, k, \quad \text{and} \\
\Pr[X = x \mid N] &= \frac{\Pr[\{X = x\} \cap N]}{1 - p}, \quad x = 1, \ldots, k.
\end{align*}
\]

(3.3b)

\(x \mapsto \Pr[X = x \mid D]\) and \(x \mapsto \Pr[X = x \mid N]\) are called the conditional rating profiles (conditional on default and survival respectively). In a more concise manner \(x \mapsto \Pr[X = x \mid D]\) is also called default (rating) profile and \(x \mapsto \Pr[X = x \mid N]\) is called survival (rating) profile.

(iii) By the unconditional rating profile \(x \mapsto \Pr[X = x]\) and the conditional PDs

\[
\Pr[D \mid X = x] = \frac{\Pr[\{X = x\} \cap D]}{\Pr[X = x]}, \quad x = 1, \ldots, k.
\]

(3.3c)

\(x \mapsto \Pr[D \mid X = x]\) is called the PD curve associated with the grades \(x = 1, \ldots, k\).

For further reference we note how the specification of the joint distribution of $(X, S)$ given in proposition 3.1 (ii) implies the representation provided in proposition 3.1 (iii):

- By the law of total probability, the unconditional rating profile \(\Pr[X = x], x = 1, \ldots, k\) can be calculated as

\[
\Pr[X = x] = p \Pr[X = x \mid D] + (1 - p) \Pr[X = x \mid N].
\]

(3.4a)

- Bayes’ formula implies the following representation of the PD curve \(\Pr[D \mid X = x]\):

\[
\Pr[D \mid X = x] = \frac{p \Pr[X = x \mid D]}{p \Pr[X = x \mid D] + (1 - p) \Pr[X = x \mid N]}.
\]

(3.4b)
Also for further reference, we observe how the specification of the joint distribution of \((X,S)\) given in proposition 3.1 (iii) implies the representation provided in proposition 3.1 (ii):

- Again by the law of total probability, the unconditional PD \(p\) can be calculated as
  \[
  p = \sum_{x=1}^{k} \Pr[D \mid X = x] \Pr[X = x].
  \] (3.5a)

- With regard to the conditional rating profiles, it follows directly from the definition of conditional probability that
  \[
  \Pr[X = x \mid D] = \frac{\Pr[D \mid X = x] \Pr[X = x]}{p}, \quad \text{and}
  \]
  \[
  \Pr[X = x \mid N] = \frac{(1 - \Pr[D \mid X = x]) \Pr[X = x]}{(1 - p)}.
  \] (3.5b)

The equivalence between equations (3.4a) and (3.4b) on the one hand and equations (3.5a), (3.5b) and (3.5c) on the other hand allows the calculation of one set of characteristics once the other set of characteristics is known. But the equivalence also represents a consistency condition that must be kept in mind if one of the characteristics is changed. In particular, if for a given unconditional rating profile there are independent estimates of the unconditional PD and the PD curve, equation (3.5a) becomes a crucial consistency condition.

The following proposition presents another consistency condition based on (3.4a) that proves useful in section 4 below. We omit its easy proof.

**Proposition 3.2** Let \(\pi_x, x = 1, \ldots, k\) and \(q_x, x = 1, \ldots, k\) be probability distributions and fix a number \(p \in (0,1)\).

(i) Define numbers \(u_x, x = 1, \ldots, k\) by solving the following equations for \(u_x\):

\[
\pi_x = pu_x + (1 - p)q_x, \quad x = 1, \ldots, k.
\] (3.6a)

Then \(u_x, x = 1, \ldots, k\) is a proper probability distribution if and only if the following two inequalities hold for all \(x = 1, \ldots, k\):

\[
pq_x \leq \pi_x,
\]

\[
(1 - p)(1 - q_x) \leq 1 - \pi_x.
\] (3.6b)

(ii) Define numbers \(v_x, x = 1, \ldots, k\) by solving the following equations for \(v_x\):

\[
\pi_x = pq_x + (1 - p)v_x, \quad x = 1, \ldots, k.
\] (3.6c)

Then \(v_x, x = 1, \ldots, k\) is a proper probability distribution if and only the following two inequalities hold for all \(x = 1, \ldots, k\):

\[
p(1 - q_x) \leq 1 - \pi_x,
\]

\[
(1 - p)q_x \leq \pi_x.
\] (3.6d)

In this paper, we use quasi moment matching (QMM) as described in appendix A to transform the grade-level empirical default rates into smoothed PD curves. As mentioned in section 2.2, such smoothing of the empirical PD curve is needed in order to
force monotonicity of the PD curve and
force the PDs to be positive.

Matching in this context means fitting a two-parameter curve to the empirically observed unconditional default rate and discriminatory power. The discriminatory power is measured as accuracy ratio whose general formula is given in (A.11a). Using the conditional rating profiles defined by (3.3b) the accuracy ratio can also be described by

\[ \text{AR} = \sum_{x=2}^{k} \Pr[X = x \mid N] \Pr[X \leq x - 1 \mid D] - \sum_{x=1}^{k-1} \Pr[X = x \mid N] \Pr[X \geq x + 1 \mid D]. \] (3.7)

### 3.2. Likelihood ratio

The specification of the model by unconditional rating profile and PD curve (see proposition 3.1 (iii)) may be inappropriate if we want to combine a forecast period profile with an estimation period PD curve. For according to equations (3.4a) and (3.4b) both components depend upon the unconditional PD – which might be different in the estimation and forecast periods. The likelihood ratio is a concept closely related to the PDs but avoids the issue of dependence on the unconditional PD. The natural logarithm of the likelihood ratio is called *weights of evidence* and is an important concept in credit scoring (see Thomas, 2009, for a detailed discussion).

In the context of credit ratings, it can be reasonably assumed that all components of the conditional rating profiles \( \Pr[X = x \mid D] \) and \( \Pr[X = x \mid N], x = 1, \ldots, k \) are positive. For otherwise, there would be rating grades with sure predictions of default and survival – which is unlikely to happen with real-world rating models. We can therefore define the *likelihood ratio* \( \lambda \) associated with the rating model:

\[ \lambda(x) = \frac{\Pr[X = x \mid N]}{\Pr[X = x \mid D]}, \quad x = 1, \ldots, k. \] (3.8)

The likelihood ratio \( \lambda(x) \) specifies how much more (or less) likely it is for a survivors’s rating grade to come out as \( x \) than for a defaulter’s rating grade. Observe that (3.4b) can be rewritten as

\[ \Pr[D \mid X = x] = \frac{p}{p + (1 - p) \lambda(x)}, \quad x = 1, \ldots, k. \] (3.9a)

This is equivalent to an alternative representation of the likelihood ratio:

\[ \lambda(x) = \frac{1 - \Pr[D \mid X = x]}{\Pr[D \mid X = x]} \frac{p}{1 - p}, \quad x = 1, \ldots, k. \] (3.9b)

By (3.9b), the likelihood ratio can alternatively be described as the ratio of the grade \( x \) odds of survival and the unconditional odds of survival. By (3.5b), (3.9a) also implies

\[ \Pr[X = x \mid D] = \frac{\Pr[X = x]}{p + (1 - p) \lambda(x)}, \quad x = 1, \ldots, k, \] (3.10a)

and, by taking the sum of all \( \Pr[X = x \mid D] \)

\[ 1 = \sum_{x=1}^{k} \frac{\Pr[X = x]}{p + (1 - p) \lambda(x)}. \] (3.10b)
This observation suggests that the information borne by the likelihood ratio is very closely related to the information inherent in the PD curve. More specifically, we obtain the following characterisation of (3.10b) which is basically the likelihood ratio version of (3.5a).

**Theorem 3.3** Let \( \pi_x > 0 \), \( x = 1, \ldots, k \) be a probability distribution. Assume that \( x \mapsto \lambda(x) \) is positive for \( x = 1, \ldots, k \). Consider the equation

\[
\sum_{x=1}^{k} \frac{\pi_x}{p + (1-p)\lambda(x)} = 1. \tag{3.11a}
\]

Then with regard to solutions \( p \in [0,1] \) of (3.11a) other than \( p = 1 \) the following statements hold:

(i) Assume that \( x \mapsto \lambda(x) \) is a mapping onto a constant, i.e. \( \lambda(x) = \lambda \) for all \( x = 1, \ldots, k \). Then all \( p \in [0,1] \) are solutions of (3.11a) if \( \lambda = 1 \) and there is no solution \( p \in [0,1) \) if \( \lambda \neq 1 \).

(ii) Assume that \( x \mapsto \lambda(x) \) is not a mapping onto a constant. Then there exists a solution \( p \in [0,1) \) of (3.11a) if and only if

\[
\sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} \geq 1 \quad \text{and} \quad \sum_{x=1}^{k} \pi_x \lambda(x) > 1. \tag{3.11b}
\]

If there exists a solution \( p \in [0,1) \) of (3.11a) then this solution is unique. The unique solution is \( p = 0 \) if and only if

\[
\sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} = 1. \tag{3.11c}
\]

**Proof.** Statement (i) is obvious. With regard to statement (ii), define the function \( f : [0,1] \to (0,\infty) \) by

\[
f(p) = \sum_{x=1}^{k} \frac{\pi_x}{p + (1-p)\lambda(x)}. \tag{3.12a}
\]

Observe that \( f \) is twice continuously differentiable in \( p \) with

\[
f'(p) = \sum_{x=1}^{k} \frac{(\lambda(x) - 1)\pi_x}{(p + (1-p)\lambda(x))^2} \quad \text{and} \quad \tag{3.12b}
\]

\[
f''(p) = 2 \sum_{x=1}^{k} \frac{(\lambda(x) - 1)^2 \pi_x}{(p + (1-p)\lambda(x))^3}. \tag{3.12c}
\]

From (3.12a) and (3.12b) we obtain

\[
f(0) = \sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)}, \quad f(1) = 1, \quad \text{and} \quad f'(1) = \sum_{x=1}^{k} \pi_x \lambda(x) - 1. \tag{3.12d}
\]

(3.12c) implies \( f''(p) > 0 \) because \( \lambda(x) \) is not constant by assumption. Hence \( f \) is strictly convex in \( p \). The strict convexity of \( f \) implies that the shape of the graph of \( f \) is determined by (3.12d).
and that only the following three cases can occur:

- case A: \( f(0) > 1 \) and \( f'(1) \leq 0 \),
- case B: \( f(0) \geq 1 \) and \( f'(1) > 0 \), or
- case C: \( f(0) < 1 \) and \( f'(1) > 0 \).

A stylised illustration of the three different possible shapes of the graph of \( f \) is shown in figure 1 on page 13. Only in case B there is a second (and only one) intersection at a \( p < 1 \) of the horizontal line through 1. By (3.12d), case B is equivalently described by (3.11b). The second intersection of the horizontal line through 1 occurs at \( p = 0 \) if and only if \( f(0) = 1 \) which is equivalent to (3.11c).

q.e.d.

At first glance, theorem 3.3 might appear as an unnessecarily complicated way to describe the interplay of unconditional rating profile, likelihood ratio, and unconditional PD. However, theorem 3.3 becomes interesting when we try to construct the joint distribution of a borrower’s rating \( X \) at the beginning of the observation period and the borrower’s state \( S \) at the end of the period from an unconditional rating profile and a candidate likelihood ratio (which might
have been estimated separately). In this context, theorem 3.3 tells us that the construction will work only if condition (3.11b) is satisfied. In contrast, by proposition 3.1 (iii) the construction is always possible if one combines an unconditional rating profile with a candidate PD curve (assuming that all its components take values between 0 and 1).

Actually, from theorem 3.3 it is not yet clear that it gives indeed rise to a fully specified joint distribution of rating $X$ and default or survival state $S$. This, however, is confirmed by the next proposition whose straight-forward proof is omitted.

**Proposition 3.4** Let $\pi_x > 0$, $x = 1, \ldots, k$ be a probability distribution. Assume that $x \mapsto \lambda(x)$ is positive for $x = 1, \ldots, k$ and that equation (3.11a) has a solution $0 < p < 1$. Then there exists a unique joint distribution of $X$ and $S$ such that $x \mapsto \lambda(x)$ is the likelihood ratio associated with the joint distribution in the sense of equation (3.8).

### 3.3. Smoothing observed default rates

In this section, we illustrate the concepts introduced in sections 3.1 and 3.2 by revisiting the S&P data for 2009 presented in section 2. As the notation introduced at the beginning of the section requires we map the S&P rating symbols CCC-C, B-, B, ..., AA+, AAA onto the numbers 1, ..., 17 (hence grade 17 stands for the least risky grade AAA).

Column 2 of table 3 shows the unconditional profile $x \mapsto \Pr[X = x]$, $\{1, \ldots, 17\} \to [0, 1]$ for the S&P corporate ratings.

Table 4 on page 14 shows the empirical unconditional default rate and accuracy ratio for our estimation data (i.e. the 2009 S&P data). The accuracy ratio was calculated according to (3.7). Table 5 on page 15 presents both the empirically observed grade-level default rates and the smoothed PD curve (according to appendix A, with the values from table 4 as targets) for the 2009 S&P data. It is hard to assess directly from the numbers how well or badly the smoothed curves fit the empirical data. Therefore we calculate an implied default profile and compare it by means of a $\chi^2$ test with the observed default profile. ‘Implied default profile’ means the theoretical rating distribution conditional on default that is derived by means of Equation (3.5b) from the unconditional rating profile, the PD curve and the unconditional PD.

Table 6 on page 16 shows the empirically observed and implied default profiles for the 2009 S&P data. Clearly the fit is not perfect. This impression is confirmed by application of the $\chi^2$ test mentioned in section 2.1. To apply the test, choose the implied profile as Null hypothesis distribution and test against the default numbers for 2009 as given in table 2.

The test result (based on Monte-Carlo approximation) is a p-value of 9% for the test of the fit of

Table 4: Unconditional default rate and accuracy ratio for the 2009 corporate data from table 2.

| Default rate | Accuracy ratio |
|--------------|----------------|
| 3.99%        | 82.7%          |
Table 5: Grade-level default rates and smoothed conditional PDs (PD curve) for the 2009 corporate corporate data from table 2. All numbers in %.

| Rating grade | Default rate | Smoothed PD |
|--------------|--------------|-------------|
| AAA          | 0.000        | 0.003       |
| AA+          | 0.000        | 0.006       |
| AA           | 0.000        | 0.012       |
| AA-          | 0.000        | 0.025       |
| A+           | 0.294        | 0.047       |
| A            | 0.392        | 0.091       |
| A-           | 0.000        | 0.173       |
| BBB+         | 0.402        | 0.299       |
| BBB          | 0.185        | 0.495       |
| BBB-         | 1.089        | 0.797       |
| BB+          | 0.000        | 1.138       |
| BB           | 1.017        | 1.518       |
| BB-          | 0.907        | 2.280       |
| B+           | 5.479        | 3.943       |
| B            | 9.959        | 7.999       |
| B-           | 17.162       | 19.557      |
| CCC-C        | 48.421       | 48.355      |

the implied corporates default profile. Hence the fit could be rejected as too poor at 10% type-I error level. However, given the inversions of default rates in the corporates data it might be hard to get a much better fit with any other forced monotonic PD curve estimate. We therefore adopt the corporates smoothed PD curve from table 5 as a starting point for the PD curve calibration examples described in section 4 below.

4. Calibration approaches

The result of the estimation period is a fully specified (and smoothed) model for the joint distribution of a borrower’s beginning of the period rating $X$ and end of the period solvency state $S$. In this section, we discuss how to combine the estimation period model with observations from the beginning of the forecast period in order to predict the grade-level default rates that are observed at the end of the forecast period. This process is often referred to as calibration of the PD curve.

**Notation.** All objects (like probabilities and the likelihood ratio) from the estimation period are labelled with subscript 0. All objects from the forecast period are labelled with subscript 1.

In the following we will make use, in particular, of assumptions on the invariance or specific transformation between estimation and forecast period of
Table 6: Empirical and implied default profiles for the 2009 corporate data from table 2. All numbers in %.

| Rating grade | Empirical profile | Implied profile |
|--------------|-------------------|-----------------|
| AAA          | 0.0000            | 0.0010          |
| AA+          | 0.0000            | 0.0009          |
| AA           | 0.0000            | 0.0094          |
| AA-          | 0.0000            | 0.0261          |
| A+           | 0.4274            | 0.0685          |
| A            | 0.8547            | 0.1989          |
| A-           | 0.0000            | 0.4041          |
| BBB+         | 0.8547            | 0.6355          |
| BBB          | 0.4274            | 1.1436          |
| BBB-         | 2.1368            | 1.5642          |
| BB+          | 0.0000            | 1.2934          |
| BB           | 1.2821            | 1.9143          |
| BB-          | 1.7094            | 4.2968          |
| B+           | 10.2564           | 7.3802          |
| B            | 20.5128           | 16.4769         |
| B-           | 22.2222           | 25.3242         |
| CCC-C        | 39.3162           | 39.2628         |
| All          | 100.0000          | 100.0000        |

- the conditional rating profiles $\Pr_0[X = x \mid D]$ and $\Pr_0[X = x \mid N]$ (for $x = 1, \ldots, k$),
- the PD curve $x \mapsto \Pr_0[D \mid X = x]$, and
- the likelihood ratio $x \mapsto \lambda_0(x)$.

Imagine we are now at the beginning of the forecast period. The borrowers’ states of solvency at the end of the period are yet unknown. The objective of the forecast period is to predict the default rates to be observed at the end of the period for the rating grades $1, \ldots, k$ by conditional PDs (PD curve) $\Pr_1[D \mid X = x]$, $x = 1, \ldots, k$. There are different forecast approaches for the conditional PDs. The selection of a suitable approach, in particular, depends on what we already know at the beginning of the forecast period about the joint distribution of a borrower’s rating $X$ at the beginning of the period and the borrower’s solvency state $S$ at the end of the period. We will look in detail at the following two possibilities:

- The unconditional rating profile $\Pr_1[X = x]$, $x = 1, \ldots, k$ is known. This is likely to be the case for a newly developed rating model if all borrowers can be re-rated with the new model in a big-bang effort before the beginning of the forecast period. It will also be the case if an existing rating model is re-calibrated. Even if in the case of a new rating model no timely re-rating of the whole portfolio is feasible, it might still be possible (and should be tried) to re-rate a representative sample of the borrowers in the portfolio such that a
reliable estimate of the unconditional rating profile is available. Where this is not possible, the rating model should be used in parallel run with the incumbent rating model until such time as the full rating profile of the portfolio has been determined. Only then a PD curve forecast with some chance of being accurate can be made. This might be one of the reasons for the ‘credible track record’ requirement of the Basel Committee (BCBS, 2006, paragraph 445). However, we will see in section 4.3 that as soon as a forecast of the unconditional PD is given a meaningful if not accurate PD curve forecast can be made without knowledge of the actual unconditional rating profile. This forecast could be used for a preliminary calibration during the Basel II ‘track record’ period.

- An estimate of the unconditional PD \( p_1 \) for the forecast period is available. This forecast could be a proper best estimate, a pessimistic estimate for stress testing purposes, or a long-run estimate for the purpose of a through-the-cycle (TTC) calibration\(^2\).

These two possibilities are not exclusive nor do they necessarily occur together. That is why, in the following, we discuss four cases:

- **Case 1.** The unconditional rating profile for the forecast period is known and an independent estimate of the unconditional PD is available.

- **Case 2.** The unconditional rating profile for the forecast period is not known but an independent estimate of the unconditional PD is available.

- **Case 3.** The unconditional rating profile for the forecast period is known but no independent estimate of the unconditional PD is available.

- **Case 4.** Neither the unconditional rating profile nor the unconditional PD for the forecast period are known.

For each of the four cases we will present one or more approaches to estimate a set of model components needed to specify a full model. See proposition 3.1 for the main possibilities to specify a full proper model of a borrower’s beginning of the period rating and end of the period solvency state. We will illustrate the forecast approaches presented with numerical examples based on the S&P data from table 2. In none of the four cases there is sufficient information from the forecast period available to completely specify a model. That is why assumptions about inter-period invariance of model components play an important role in the forecast process.

### 4.1. Invariance assumptions

Forecasting without assuming that some of the features observed in the estimation period are invariant (i.e. unchanged) between the estimation and forecast periods is impossible. Ideally, any assumption of invariance should be theoretically sound, and it should be possible to verify it by backtesting. In this section, we briefly discuss which invariance assumptions for the model from section 3 we should look at closer in the following.

- It is obvious that no invariance assumptions must be made on objects that can be observed or reliably estimated in a separate forecast exercise at the beginning of the forecast period:

\(^a\)See Heitfield (2005) for a detailed discussion of point-in-time (PIT) and TTC rating and PD estimation approaches. See Löffler (2013) for the question of how much TTC agency ratings are.
As explained above, in particular, the actual unconditional rating profile of the portfolio should be known at the beginning of the observation period.

We look both at the case that the forecast unconditional default rate is estimated based on the unconditional forecast period rating profile and at the case where an independent forecast of the forecast period unconditional default rate is available.

- As the future solvency states of the borrowers in the portfolio are not yet known at the beginning of the forecast period, assuming that both conditional rating profiles are invariant could make sense.

- Assuming that the likelihood ratio is invariant is less restrictive than the assumption of invariant conditional rating profiles.

- Instead of assuming that both conditional rating profiles are invariant, one could also assume that only one of the two is invariant. If we assume that the survival profile $x \mapsto \Pr[X = x | N]$ is invariant then proposition 3.2 implies for all rating grades $x$ the restriction

\[(1 - p_1) \Pr_0[X = x | N] \leq \Pr_1[X = x]. \tag{4.1}\]

Probabilities of default are often measured at a one year horizon. In that case the forecast $p_1$ of the unconditional PD for principal portfolios like banks or corporates will hardly ever exceed 5%. This implies, however, that condition (4.1) is easily violated. In practice, therefore, quite often the assumption of an invariant survival rating profile will not result in a proper model. That is why we do not discuss further details of this invariance assumption in this paper.

- Assuming the default rating profile as invariant is a much more promising approach because the conditions for the default profile to generate a proper model are much easier satisfied than condition (4.1) for the survival profile.

- Invariance assumptions may be weakened by restating them as shape invariance assumptions.

  - For instance, a common approach is to assume that the shape of the PD curve is preserved between the estimation and the forecast periods. This can be accomplished by scaling the PD curve with a constant multiplier that is determined at the beginning of the forecast period (see, e.g., Falkenstein et al., 2000, page 67). (3.4b) shows that the scaled PD curve strongly depends on the estimation period unconditional PD. Hence making use of the scaled PD curve for forecasts in the forecast period might ‘contaminate’ the forecast with the estimation period unconditional PD which might be quite different from the forecast period unconditional PD. We include the scaled PDs approach nonetheless in the subsequent more detailed discussion because of its simplicity and popularity.

  - Scaling the likelihood ratio instead of the PD curve avoids the contamination issue we have observed for the scaled PD curve.
4.2. Case 1: Unconditional rating profile and unconditional PD given

In this case, it is assumed that the unconditional rating profile $Pr_1[X = x]$, $x = 1, \ldots, k$ can directly be observed at the beginning of the forecast period, and it is also assumed that a forecast unconditional PD $0 < p_1 < 1$ is given that is likely to differ from the estimation period unconditional PD. There are several approaches to prediction in the forecast period that may lead to proper models for the forecast period:

- **Invariant default profile.** Assume that the default rating profile is invariant, i.e.
  \[ Pr_1[X = x | D] = Pr_0[X = x | D], \quad x = 1, \ldots, k. \]  
  (4.2)

- **Invariant AR.** Assume that the discriminatory power of the model as measured by the accuracy ratio (see (3.7)) is invariant, i.e.
  \[ AR_1 = AR_0. \]  
  (4.3)

- **Scaled PDs.** Assume that the estimation period PD curve can be linearly scaled to become the forecast period PD curve, i.e. there is a constant $c_{PD} > 0$ such that
  \[ Pr_1[D | X = x] = c_{PD} Pr_0[D | X = x], \quad x = 1, \ldots, k. \]  
  (4.4)

- **Scaled likelihood ratio.** Assume that the estimation period likelihood ratio can be linearly scaled to become the forecast period likelihood ratio, i.e. there is a constant $c_{LR} > 0$ such that
  \[ \lambda_1(x) = c_{LR} \lambda_0(x), \quad x = 1, \ldots, k. \]  
  (4.5)

In principle, a fifth approach is cogitable, namely to assume that the survivor rating profile does not change from the estimation period to the forecast period. However, as explained in section 4.1 it is unlikely that this approach results in a proper forecast period model with a proper default rating profile. That is why we do not discuss this approach.

4.2.1. On assumption (4.2)

This assumption is not necessarily viable as (3.4a) must be satisfied. It follows from proposition 3.2 that assumption (4.2) makes for a proper model of a borrower’s rating and state of solvency if and only if we have for all $x = 1, \ldots, k$

\[ p_1 Pr_0[X = x | D] \leq Pr_1[X = x] \quad \text{and} \quad p_1 (1 - Pr_0[X = x | D]) \leq (1 - Pr_1[X = x]). \]  
(4.6a)

If (4.6a) holds then by (3.5b) we obtain the following equation for the PD curve:

\[ Pr_1[D | X = x] = \frac{p_1 Pr_0[X = x | D]}{Pr_1[X = x]}. \]  
(4.6b)

Actually, there are two slightly different approaches to implement assumption (4.2):
(i) Use a smoothed version of the estimation period default profile that could be derived via equation (3.5b) from a smoothed PD curve – which in turn might have been determined by QMM as described in appendix A.

(ii) Use the observed estimation period default profile and the given forecast period unconditional profile to determine by means of (3.5c) an implied raw survivor profile. Based on this survivor profile and the observed estimation period default profile deploy equation (3.7) to compute a forecast accuracy ratio. Apply then QMM as described in appendix A to determine a smoothed PD curve for the forecast period.

Compared with approach (i), approach (ii) has the advantage of always delivering a monotonic PD curve. That is why for the purpose of this paper we implement assumption (4.2) in the shape of (ii) although anecdotal evidence shows that the performance of (ii) is not necessarily better than the performance of (i).

4.2.2. On assumption (4.3)

Actually, even with unconditional rating profile, unconditional PD, and accuracy ratio given the joint distribution of a borrower’s beginning of the period rating and end of the period state is not uniquely determined. We suggest applying QMM as in the estimation period (see section 3.3) and described in appendix A to compute a PD curve as a forecast of the grade-level default rates. There is, however, the problem that QMM requires the rating profile conditional on survival as an input – which cannot be observed or implied at this stage. But QMM is fairly robust with regard to the frequencies of the rating grades used as input to the algorithm. That is why approximating the rating profile conditional on survival with the unconditional rating profile (known by assumption) seems to work reasonably well.

4.2.3. On assumption (4.4)

The constant $c_{PD}$ is determined by equation (3.5a):

$$c_{PD} = \frac{p_1}{\sum_{x=1}^{k} \Pr_0[D \mid X = x] \Pr_1[X = x]} \tag{4.7a}$$

However, if $c_{PD} > 1$ the resulting model could be improper because by (4.4) it could turn out that $\Pr_1[D \mid X = x] > 1$ for some $x$. If the resulting model under assumption (4.4) is proper the implied default profile is as follows:

$$\Pr_1[X = x \mid D] = c_{PD} \Pr_0[D \mid X = x] \Pr_1[X = x] / p_1, \quad x = 1, \ldots, k. \tag{4.7b}$$

4.2.4. On assumption (4.5)

By (3.10b) we obtain an equation that determines the constant $c_{LR}$:

$$1 = \sum_{x=1}^{k} \frac{\Pr_1[X = x]}{p_1 + (1 - p_1) c_{LR} \lambda_0(x)}. \tag{4.8a}$$
Table 7: 2010 and 2011 grade-level forecast default rates for S&P corporates ratings. P-values are for the $\chi^2$-tests of the implied default profiles. All values in %.

|       | Default rate | Invariant default profile (4.2) | Invariant AR (4.3) | Scaled PDs (4.4) | Scaled likelihood ratio (4.5) |
|-------|--------------|---------------------------------|--------------------|-----------------|-------------------------------|
|       |              | 2010: Unconditional default rate 1.141 |                    |                 |                               |
| AAA   | 0            | 0.0012                          | 0.0004             | 0.0007          | 0.0005                        |
| AA+   | 0            | 0.0023                          | 0.0009             | 0.0015          | 0.0012                        |
| AA    | 0            | 0.0041                          | 0.0018             | 0.0031          | 0.0023                        |
| AA-   | 0            | 0.0083                          | 0.0040             | 0.0066          | 0.0049                        |
| A+    | 0            | 0.0163                          | 0.0086             | 0.0125          | 0.0093                        |
| A     | 0            | 0.0319                          | 0.0183             | 0.0241          | 0.0180                        |
| A-    | 0            | 0.0593                          | 0.0366             | 0.0458          | 0.0342                        |
| BBB+  | 0            | 0.0995                          | 0.0652             | 0.0789          | 0.0590                        |
| BBB   | 0            | 0.1647                          | 0.1145             | 0.1307          | 0.0979                        |
| BBB-  | 0            | 0.2660                          | 0.1955             | 0.2107          | 0.1581                        |
| BB+   | 0.7874       | 0.3706                          | 0.2827             | 0.3006          | 0.2263                        |
| BB    | 0.3623       | 0.4847                          | 0.3806             | 0.4012          | 0.3029                        |
| BB-   | 0.5277       | 0.6907                          | 0.5631             | 0.6024          | 0.4576                        |
| B+    | 0.0000       | 1.1043                          | 0.9460             | 1.0417          | 0.8023                        |
| B     | 0.6881       | 2.0554                          | 1.8843             | 2.1134          | 1.6844                        |
| B-    | 2.0690       | 4.5380                          | 4.5164             | 5.1671          | 4.5716                        |
| CCC-C | 22.2727      | 12.9712                         | 14.5179            | 12.7755         | 15.5760                       |
| P-value | Exact       | 4.6                             | 8.0                | 4.0             | 11.3                          |

|       |              | 2011: Unconditional default rate 0.752 |                    |                 |                               |
| AAA   | 0            | 0.0006                          | 0.0003             | 0.0006          | 0.0004                        |
| AA+   | 0            | 0.0013                          | 0.0006             | 0.0012          | 0.0009                        |
| AA    | 0            | 0.0024                          | 0.0013             | 0.0024          | 0.0018                        |
| AA-   | 0            | 0.0048                          | 0.0027             | 0.0050          | 0.0039                        |
| A+    | 0            | 0.0095                          | 0.0058             | 0.0095          | 0.0074                        |
| A     | 0            | 0.0186                          | 0.0120             | 0.0183          | 0.0143                        |
| A-    | 0            | 0.0345                          | 0.0236             | 0.0347          | 0.0271                        |
| BBB+  | 0            | 0.0579                          | 0.0416             | 0.0599          | 0.0468                        |
| BBB   | 0            | 0.0923                          | 0.0691             | 0.0992          | 0.0777                        |
| BBB-  | 0.1969       | 0.1468                          | 0.1147             | 0.1600          | 0.1256                        |
| BB+   | 0            | 0.2065                          | 0.1662             | 0.2282          | 0.1797                        |
| BB    | 0            | 0.2694                          | 0.2219             | 0.3046          | 0.2405                        |
| BB-   | 0            | 0.3828                          | 0.3248             | 0.4573          | 0.3635                        |
| B+    | 0.3929       | 0.6291                          | 0.5567             | 0.7909          | 0.6378                        |
| B     | 1.1945       | 1.3483                          | 1.2710             | 1.6046          | 1.3414                        |
| B-    | 3.9867       | 3.7460                          | 3.7941             | 3.9231          | 3.6627                        |
| CCC-C | 15.9420      | 12.1942                         | 13.3871            | 9.6998          | 12.7721                       |
| P-value | Exact       | 78.5                            | 89.4               | 36.6            | 82.2                          |
Note that
\[\lim_{c \to \infty} \sum_{x=1}^{k} \frac{\Pr_{1}[X = x]}{p_{1} + (1 - p_{1}) c \lambda_{0}(x)} = 0 \quad \text{and} \quad \lim_{c \to 0} \sum_{x=1}^{k} \frac{\Pr_{1}[X = x]}{p_{1} + (1 - p_{1}) c \lambda_{0}(x)} = 1/p_{1} > 1.\]

Hence equation (4.8a) has always a unique solution \(c_{LR} > 0\). By proposition 3.4 then we know that under assumption (4.5) we have a proper model of a borrower’s rating and default state. In addition, by theorem 3.3 the resulting forecast likelihood ratio \(\lambda_{1}(x) = c_{LR} \lambda_{0}(x)\) satisfies the inequalities
\[\sum_{x=1}^{k} \frac{\Pr_{1}[X = x]}{\lambda_{1}(x)} > 1 \quad \text{and} \quad \sum_{x=1}^{k} \Pr_{1}[X = x] \lambda_{1}(x) > 1.\]

This implies the following inequalities for \(c_{LR}\):
\[\frac{1}{\sum_{x=1}^{k} \Pr_{1}[X = x] \lambda_{0}(x)} < c_{LR} < \frac{k}{\sum_{x=1}^{k} \Pr_{1}[X = x] \lambda_{0}(x)}.\] (4.8b)

(4.8b) is useful because it provides initial values for the numerical solution of (4.8a) for \(c_{LR}\). Once \(c_{LR}\) has been determined (3.10a) and (3.9a) imply the following equations for the default profile and the PD curve under assumption (4.5):
\[\Pr_{1}[X = x | D] = \frac{\Pr_{1}[X = x]}{p_{1} + (1 - p_{1}) c_{LR} \lambda_{0}(x)}, \quad x = 1, \ldots, k,\] (4.8c)
\[\Pr_{1}[D | X = x] = \frac{p_{1}}{p_{1} + (1 - p_{1}) c_{LR} \lambda_{0}(x)}, \quad x = 1, \ldots, k.\] (4.8d)

### 4.2.5. Summary of section 4.2

Table 7 on page 21 shows the results of an application of the approaches presented above to forecasting the 2010 and 2011 grade-level default rates of the S&P corporates portfolio, based on estimates made with data from 2009. To allow for a fair performance comparison, we have made use of prophetic estimates of the 2010 and 2011 unconditional default rates, by setting the value of \(p_{1}\) to the observed unconditional default rate of the respective year and sample.

In order to express the performance of the different approaches in one number for each approach, we have used the forecast PD curves to derive forecast default profiles by means of (3.5b). The forecast default profiles can be \(\chi^{2}\) tested against the observed grade-level default numbers from table 2. The p-values of these tests are shown in the last rows of the panels of table 7. Recall that higher p-values mean better goodness of fit.

Table 7 hence indicates that under the constraints of this section (unconditional rating profile and default rate are given) the scaled likelihood ratio approach (4.5) and the invariant accuracy ratio approach (4.3) work best, followed by the invariant default profile approach (4.2). This anecdotal evidence, however, does not allow an unconditional conclusion that ‘scaled likelihood ratio’ or ‘invariant accuracy ratio’ are the best approaches to PD curve calibration. We will test this conclusion on a larger dataset in section 5.

But also from a conceptual angle there might be good reasons to prefer the ‘invariant default profile’ approach. When a new rating model is developed one has often to combine data from
several observation periods in order to create a sufficiently large training sample. Estimating the likelihood ratio from such a combined sample would implicitly be based on the assumption of an invariant likelihood ratio. Hence it would be strange to modify the likelihood ratio via scaling in the forecast period. This consistency issue is obviously avoided with the ‘invariant default profile’ and the ‘invariant accuracy ratio’ approaches. As we have seen, to implement the ‘invariant accuracy ratio’ approach we need to approximate the forecast period survivor profile by the forecast period unconditional rating profile. This approximation could be poor if the forecast period unconditional default rate is high. Hence, depending on what approach had been followed in the estimation period and how big the forecast period unconditional default rate is, the ‘invariant default profile’ approach (4.2) could be preferable for the forecast period despite its only moderate performance in our numerical examples.

4.3. Case 2: No unconditional rating profile but unconditional PD given

In this case, we assume that a forecast unconditional PD \( 0 < p_1 < 1 \) is given that is likely to differ from the estimation period unconditional PD. But the unconditional current rating profile is assumed not to be known. This would typically be the case if a rating model was newly developed and it was not possible to rate all the borrowers in the portfolio in one big-bang effort. The new ratings would then only become available in the course of the regular annual rating process. This is clearly suboptimal, in particular with a view on the validation of the new rating model, but sometimes unavoidable due to limitation of resources.

In this situation, proposition 3.1 suggests the assumption that both conditional rating profiles are invariant as the only possibility to infer a full model of a borrower’s beginning of the period rating and end of the period state of solvency.

**Invariant conditional profiles:**

\[
\begin{align*}
\Pr_1[X = x \mid D] &= \Pr_0[X = x \mid D], \quad x = 1, \ldots, k, \quad \text{and} \\
\Pr_1[X = x \mid N] &= \Pr_0[X = x \mid N], \quad x = 1, \ldots, k.
\end{align*}
\]

(4.9)

Note that (4.9) is a stronger assumption than (4.3) because (4.3) is implied by (4.9).

If, however, it is sufficient to obtain an estimate of the forecast period PD curve then it is solely the estimation period likelihood ratio \( x \mapsto \lambda_0(x) \) that one needs to know in addition to the unconditional PD \( p_1 \). Formally, the assumption of an **invariant likelihood ratio** is used here:

\[
\lambda_1(x) = \lambda_0(x), \quad x = 1, \ldots, k.
\]

(4.10)

From equation (3.9a) it follows that we can then calculate the PD curve as follows:

\[
\Pr_1[D \mid X = x] = \frac{p_1}{p_1 + (1 - p_1) \lambda_0(x)}, \quad x = 1, \ldots, k.
\]

(4.11)

Elkan (2001, theorem 2) stated this observation as ‘change in base rate’ theorem. It is also often mentioned in the specific context of logistic regression (see, for instance Cramer, 2003, section 6.2).
4.4. Case 3: Unconditional rating profile but no unconditional PD given

In this case, the unconditional rating profile $Pr_1[X = x]$, $x = 1, \ldots, k$ can directly be observed at the beginning of the forecast period. Like in case 1, we consider several approaches to prediction in the forecast period that may lead to proper models for the forecast period:

- **Invariant PD curve.** Assume that the PD curve is invariant, i.e. the following equation holds:
  \[
  Pr_1[D | X = x] = Pr_0[D | X = x], \quad x = 1, \ldots, k. \tag{4.12}
  \]

- **Invariant conditional profiles.** Assume that both conditional rating profiles are invariant, i.e. assumption (4.9).

- **Invariant likelihood ratio.** Assume that the likelihood ratio is invariant, i.e. (4.10) holds. Note that (4.10) is implied by (4.9) and, hence, is a weaker assumption.

4.4.1. On assumption (4.12)

As mentioned in section 4.1, it might not be the best idea to work under this assumption because there is a risk to ‘contaminate’ the forecast with the estimation period unconditional default rate $p_0$. However, by proposition 3.1 the combination of unconditional rating profile with any PD curve creates a unique proper model of a borrower’s beginning of the period rating and end of the period state of solvency. In particular, by (3.5a) this approach implies a forecast of the unconditional default rate in the forecast period:

\[
p_1 = \sum_{x=1}^{k} Pr_0[D | X = x] Pr_1[X = x]. \tag{4.13}
\]

4.4.2. On assumption (4.9)

Equation (3.4a) functions here as a constraint. The unknown unconditional PD $p_1$ and the two conditional profiles therefore must satisfy

\[
Pr_1[X = x] = p_1 Pr_0[X = x | D] + (1 - p_1) Pr_0[X = x | N], \quad x = 1, \ldots, k. \tag{4.14a}
\]

Hence, as all three profiles $Pr_1[X = x]$, $Pr_0[X = x | D]$, and $Pr_0[X = x | N]$ are known, we have $k$ equations for the one unknown $p_1$. In general, it seems unlikely that all the $k$ equations can be simultaneously satisfied if only one variable can be freely chosen. However, we can try and compute a best fit by solving the following least squares optimisation problem:

\[
p_1^* = \arg \min_{p_1 \in [0,1]} \sum_{x=1}^{k} (Pr_1[X = x] - p_1 Pr_0[X = x | D] - (1 - p_1) Pr_0[X = x | N])^2
\]

\[
\Rightarrow \quad p_1^* = \sum_{x=1}^{k} \frac{(Pr_1[X = x] - Pr_0[X = x | N]) (Pr_0[X = x | D] - Pr_0[X = x | N])}{(Pr_0[X = x | D] - Pr_0[X = x | N])^2}. \tag{4.14b}
\]

Observation (4.14b) is interesting because it indicates a technique to extract a forecast of the unconditional PD from the unconditional rating profile at the beginning of the forecast period.
that also avoids the contamination issue observed for the invariant PD curve assumption. It should be checked whether the forecast PD $p_1^*$ is indeed in line with the profile $x \mapsto \Pr_1[X = x]$. This can readily be done because with $p_1^*$ from (4.14b) we obtain an implied unconditional rating profile

$$\Pr_1^*[X = x] = p_1^* \Pr_0[X = x | D] + (1 - p_1^*) \Pr_0[X = x | N], \quad x = 1, \ldots, k. \quad (4.15)$$

This can be $\chi^2$-tested against the grade-level frequencies of borrowers at the beginning of the forecast period. If the hypothesis that $x \mapsto \Pr_1[X = x]$ is just a random realisation of $x \mapsto \Pr_1^*[X = x]$ cannot be rejected we can proceed to predict the PD curve on the basis of $x \mapsto \Pr_1^*[X = x]$ by using (3.5b):

$$\Pr_1[D | X = x] = \frac{p_1^* \Pr_0[X = x | D]}{\Pr_1^*[X = x]}, \quad x = 1, \ldots, k. \quad (4.16)$$

The optimisation problem (4.14b) is convenient for deriving a forecast of $p_1$ from the unconditional rating profile because it yields a closed-form solution. In principle, there is no reason why the least squares should not be replaced with a – say – least absolute value optimisation. This would result in a slightly different forecast of $p_1$. However, as we will check the appropriateness of the $p_1$ forecast by applying a $\chi^2$ test as mentioned in section 2.1, it seems natural to also look at the variant of (4.14b) where the $\chi^2$ statistic is directly minimised. It is easy to show that this minimisation problem is well-posed and has a unique solution.

4.4.3. On assumption (4.10)

Like for assumption (4.9), it is not a priori clear that a proper model of a borrower’s rating profile and solvency state can be based on the unconditional profile $x \mapsto \Pr_1[X = x]$ and the likelihood ratio $\lambda(x).$ The necessary and sufficient condition for the likelihood ratio to match the rating profile is provided in equation (3.11b) of theorem 3.3, with $\pi_x = \Pr_1[X = x]$ and $\lambda(x) = \lambda_0(x).$

If condition (3.11b) is satisfied then proposition 3.4 implies that there is a unique model of a borrower’s rating and solvency state with characteristics $\Pr_1[X = x]$ and $\lambda_0(x).$ The unconditional PD in this model is determined as the unique solution $p_1$ of equation (3.11a), and we can calculate the PD curve by (4.11).

4.4.4. Summary of section 4.4

Table 8 on page 26 displays some forecast results that were calculated with the approaches described in this section. Forecast values for the 2010 and 2011 S&P unconditional default rates are presented together with assessments of the goodness of fit of the actual unconditional rating profiles by the implied or assumed unconditional rating profiles. It is immediately clear from table 8 that the forecasts of the unconditional default rates are much too high in all cases. That is why we did not bother to show the grade-level forecast default rates or any other model characteristics as the fit would have been equally poor.
Table 8: Forecasts of 2010 and 2011 S&P unconditional default rates and p-values for goodness of fit tests of 2010 and 2011 unconditional rating profiles. The forecast approaches are described in section 4.4.

| Forecast for | 2010       | 2011       |
|--------------|------------|------------|
| Observed default rate | 1.14%     | 0.75%     |
| Forecast DR | p-value    | Forecast DR | p-value    |
| Invariant PDs (4.12) | 4.32% | Exact | 3.75% | Exact |
| Least squares (4.14b) | 4.80% | 0.0037 | 3.50% | $< 10^{-10}$ |
| Least $\chi^2$ | 5.35% | 0.0051 | 2.84% | $< 10^{-10}$ |
| Invariant LR (4.10) | 5.38% | Exact | 2.79% | Exact |

We have argued above that assumption (4.12) is suboptimal for risking undesirable impact on the forecast of the estimation period unconditional default rate. Assumption (4.10) is the most promising of the three assumptions we have explored because it guarantees exact fit of the unconditional rating profile and avoids contamination of the forecast. Assumption (4.9) is stronger than (4.10) because it implies (4.10). In principle, assumption (4.9) will hardly ever provide a proper model because it is rather unlikely that the overdetermined equation (4.14a) has an exact solution. By (4.14b) or minimisation of the $\chi^2$ Pearson statistic, however, we could try and determine an approximate fit that could turn out to be statistically indistinguishable from the rating profile at the beginning of the forecast period – which would make assumption (4.9) a viable approach, too.

As ‘contamination’ by the 2009 unconditional default rate is prevented under assumptions (4.9) and (4.10), it is interesting to speculate why the implied default rate forecasts are so poor nonetheless. The natural conclusion is that the assumptions are simply wrong for the S&P data. Indeed, as table 9 on page 27 demonstrates for the likelihood ratio, with hindsight it is clear that the invariance assumptions made in this sections do not hold. An alternative and complementary explanation could however be that the S&P ratings made in 2009 and 2010 were over-pessimistic and for this reason generate too high default rate forecasts. This explanation is supported by the observation that in 2009 the downgrade-to-upgrade ratio for the S&P corporate ratings was 3.99 (S&P, 2012, table 6) – which could presumably not even be compensated by the 2010 downgrade-to-upgrade ratio of 0.74. Possibly, the truth is a mixture of these two explanations.

4.5. Case 4: No unconditional rating profile and no unconditional PD given

From a risk management point of view it is undesirable to have no current data at all. In a stable economic environment, this approach might be justifiable nonetheless. One could assume that the model from the estimation period works without any adaptations also for the forecast period. Of course, at the end of the forecast period, we can then backtest the default profile from the observation period against the grade-level default frequencies observed. Formally, the
Table 9: S&P grade-level smoothed likelihood ratios (defined by (3.8)) for corporates in 2009, 2010 and 2011. Source: Own calculations.

| Rating grade | 2009    | 2010    | 2011    |
|--------------|---------|---------|---------|
| AAA          | 1,501.55| 53,720.74| 20,221.19|
| AA+          | 715.28  | 20,477.36| 7,734.58 |
| AA           | 353.69  | 8,583.13 | 3,410.45 |
| AA-          | 167.12  | 3,099.52 | 1,365.00 |
| A+           | 88.18   | 1,161.49 | 547.00  |
| A            | 45.53   | 436.37  | 227.20  |
| A-           | 23.98   | 177.13  | 100.78  |
| BBB+         | 13.89   | 83.35   | 51.03   |
| BBB          | 8.37    | 40.00   | 27.65   |
| BBB-         | 5.17    | 19.85   | 15.00   |
| BB+          | 3.61    | 12.23   | 9.58    |
| BB           | 2.70    | 8.28    | 6.74    |
| BB-          | 1.78    | 4.93    | 4.24    |
| B+           | 1.01    | 2.46    | 2.19    |
| B            | 0.48    | 0.96    | 0.79    |
| B-           | 0.17    | 0.27    | 0.20    |
| CCC-C        | 0.04    | 0.05    | 0.04    |

assumption made in case 4 may be described by (4.12) and

\[ \Pr_1[X = x] = \Pr_0[X = x], \quad x = 1, \ldots, k. \]  

(4.17)

Table 2, combined with table 5, indicates that it would not have been a good idea to try and predict the grade-level S&P default rates of 2010 and 2011 with the PD curve from 2009. Alternatively, one might try and come up with plausible assumptions on the forecast period unconditional rating profile or unconditional PD – which would bring us back into case 1, case 2, or case 3.

5. Backtest

In this section, we describe a backtest of the observations from section 4.2 on a relatively long time series of rating and default data for the years 1986 to 2012, as published in Moody’s (2013, Exhibit 41). As mentioned in section 2, this data is not optimal for the purpose of this paper. Table 10 illustrates the issue at the example of the data for the year 1986. The first three

---

3We discard the years 1970 to 1985 from the dataset because only from 1986 on there were at least 10 issuers in each rating grade at the beginning of the year. Working with rating frequencies of less than 10 would risk to make the results over-sensitive to random variation.
Table 10: Moody’s reported rating frequencies and one-year default rates for 1986. Source: Moody’s (2013, Exhibit 41). “Def rate” stands for “default rate”.

| Rating grade | Issuers | Def rate | Implied defaults | Rounded defaults | ‘Rounded’ def rate |
|--------------|---------|----------|-----------------|-----------------|-------------------|
| Aaa          | 108     | 0.00%    | 0               | 0               | 0.00%             |
| Aa           | 290     | 0.00%    | 0               | 0               | 0.00%             |
| A            | 569     | 0.00%    | 0               | 0               | 0.00%             |
| Baa          | 307     | 1.34%    | 4.11            | 4               | 1.30%             |
| Ba           | 357     | 2.87%    | 10.25           | 10              | 2.80%             |
| B            | 187     | 11.57%   | 21.63           | 22              | 11.76%            |
| Caa-C        | 10      | 22.22%   | 2.22            | 2               | 20.00%            |

Columns of Table 10 have been extracted from Moody’s (2013, Exhibit 41). The fourth column ‘Implied defaults’ has been determined by element-wise multiplication of the second and third columns. The fact that the entries of the fourth column are not even approximately integers indicates that the default rates from the third column were computed by a non-trivial method that involved information which is not presented in Moody’s (2013). However, in order to be able to apply the $\chi^2$ test for the goodness of fit of our estimates we need integer default numbers. In the following we adopt the obvious solution to this problem by making use of rounded values as shown in the fifth column of Table 10. A minor corruption of the data as shown in the sixth column of the table is the price to pay for this solution.

We repeat the calculations from section 4.2 on the Moody’s dataset but for lack of space do not present the detailed results. Table 11 illustrates the results of the calculations with the example of the grade-level default rates forecast for 1987 based on observations in 1986 (see Table 10). We use again the p-values of the $\chi^2$ tests to compare the goodness of fit achieved by the four different calibration methods. A method with a higher p-value is considered a better fit because the risk of a wrong decision by rejecting the resulting calibration (i.e. 100% minus p-value) is lower for such a method.

For a comparison of goodness of fit on one sample, inspecting the p-values is appropriate for ranking the different calibration methods. This is, however, not the case if the comparison involves several samples as it does in our backtesting exercise on the Moody’s data. The issue with this is the sample-size dependence of the p-values. On small samples, p-values are usually higher because the random variation is stronger. Hence, in years with low default rates one would in general expect higher p-values than in years with high default rates. As a consequence, it does not make sense to directly compare p-values that were calculated for different years.

What we can compare across several years, however, is the ranking of the different calibration methods. The last row of Table 11 shows the ranks of the four calibration methods with regard to forecasting the grade-level default rates for 1987. For that year, ‘scaled PDs’ is best with the highest p-value and hence receives rank 4 while ‘invariant default profile’ is worst and receives rank 1. We determined such rankings for all the 26 years from 1987 to 2012 (each forecast was
Table 11: 1987 grade-level forecast default rates for Moody’s corporates ratings based on observations in 1986. P-values are for the $\chi^2$-tests of the implied default profiles. All values but the ranks are in %.

| Rating grade | Default rate | Invariant default profile (4.2) | Invariant AR (4.3) | Scaled PDs (4.4) | Scaled likelihood ratio (4.5) |
|--------------|--------------|---------------------------------|-------------------|-----------------|-------------------------------|
| Aaa          | 0            | 0.0099                          | 0.0035            | 0.0039          | 0.0037                        |
| Aa           | 0            | 0.0446                          | 0.0204            | 0.0222          | 0.0213                        |
| A            | 0            | 0.1732                          | 0.0997            | 0.1183          | 0.1132                        |
| Baa          | 0            | 0.51                            | 0.3519            | 0.4514          | 0.4323                        |
| Ba           | 2.8139       | 1.4717                          | 1.2083            | 1.4611          | 1.4047                        |
| B            | 6.6667       | 6.7539                          | 7.1427            | 7.2558          | 7.1365                        |
| Caa-C        | 20           | 51.6781                         | 63.4415           | 44.2962         | 51.0725                       |
| P-value      | Exact        | 8.7716                          | 7.5917            | 13.2315         | 10.5396                       |
| Rank         |              | 2                               | 1                 | 4               | 3                             |

Table 12: Average ranks of calibration methods with respect to their $\chi^2$-test p-values for the years 1987 to 2012.

| Approach       | Invariant default profile (4.2) | Invariant AR (4.3) | Scaled PDs (4.4) | Scaled likelihood ratio (4.5) |
|----------------|---------------------------------|-------------------|-----------------|-------------------------------|
| Average Rank   | 2.58                            | 2.05              | 2.61            | 2.76                          |

Based on observations from the previous year) and then calculated the average rank for each of the four calibration methods. Results are shown in table 12.

According to table 12 the ‘scaled likelihood ratio’ approach performs best on average, followed by the ‘scaled PDs’ and ‘invariant default profile’ approaches (with little difference), while the average performance of ‘invariant accuracy ratio’ is worst. Hence, when compared with the observations from section 4.2, the ‘scaled PDs’ and ‘invariant accuracy ratio’ approaches have swapped ranking positions. The poor performance of ‘invariant accuracy ratio’ in the backtest could be a sign that the underlying assumption of a constant accuracy ratio over time is simply not right\(^4\). The stronger than expected performance of the ‘scaled PDs’ approach could be owed to its conceptual similarity to the strong performing ‘scaled likelihood ratio’.

\(^4\)An anonymous referee pointed out that there is a strong negative correlation between unconditional default rate and accuracy ratio over time. Indeed, for the data from Moody’s (2013, Exhibit 41) considered here the rank correlation of default rate and accuracy ratio for the years 1986 to 2012 is -64.2%.
6. Conclusions

Accurate (re-)calibration of a rating model requires careful consideration of a number of questions that include, in particular, the question of which model components can be assumed to be invariant between the estimation period of the model and the forecast period. Looking at PD curve calibration as a problem of forecasting rating-grade level default rates, we have discussed a model framework that is suitable for the description of a variety of different forecasting approaches.

We have then proceeded to present a number of PD curve calibration approaches and explored the conditions under which the approaches are fit for purpose. We have tested the approaches introduced by applying them to publicly available datasets of S&P and Moody’s rating and default statistics that can be considered typical for the scope of application of the approaches.

One negative and one positive finding are the main results of our considerations:

- The popular ‘scaled PDs’ approach for (re-)calibrating a rating model to a different target unconditional PD is not likely to deliver the best calibration results because it implicitly mixes up the unconditional PD of the estimation period and the target PD.
- As shown by example, the ‘scaled likelihood ratio’ approach to PD curve calibration avoids mixing up the unconditional PDs from the estimation and the forecast periods and, on average, performs better than ‘scaled PDs’ and other approaches discussed in the paper. ‘Scaled likelihood ratio’ is, therefore, a promising alternative to ‘scaled PDs’.

A. Appendix

In this paper, we apply quasi moment matching (QMM) as suggested by Tasche (2009) for the smoothing of PD curves. QMM requires the numerical solution of a two-dimensional system of non-linear equations. The solution of such an equation system in general is much facilitated if a meaningful initial guess of the solution can be provided. The binormal model we discuss in the following subsection delivers such a guess. In addition, the binormal model provides the main motivation of the QMM technique. In subsection A.2 we describe the QMM technique itself.

A.1. The binormal model with equal variances

Formally, the binormal model with equal variances is based on the following assumption.

Assumption A.1 $X$ denotes the continuous score of a borrower at the beginning of the observation period.

- The distribution of $X$ conditional on the event $D$ (the borrower defaults during the observation period) is normal with mean $\mu_D$ and variance $\sigma^2 > 0$.
- The distribution of $X$ conditional on the event $N$ (the borrower remains solvent during the whole observation period) is normal with mean $\mu_N > \mu_D$ and variance $\sigma^2 > 0$. 
• $p \in (0,1)$ is the borrower’s unconditional PD (i.e. the unconditional probability that the borrower defaults during the observation period).

Denote by $f_D$ and $f_N$ respectively the conditional densities of the binormal score $X$. Hence by assumption A.1 we have

$$f_D(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x - \mu_D)^2}{2 \sigma^2}\right),$$

$$f_N(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(x - \mu_N)^2}{2 \sigma^2}\right).$$

In the continuous case specified by assumption A.1, Bayes’ formula implies a PD curve $x \mapsto \Pr[D | X = x]$ similar to the discrete formula (3.4b):

$$\Pr[D | X = x] = \frac{pf_D(x)}{pf_D(x) + (1 - p) f_N(x)}$$

$$= \frac{1}{1 + \exp(\alpha + \beta x)},$$

$$\alpha = \frac{\mu^2 D - \mu^2 N}{\sigma^2} + \log\left(\frac{1 - p}{p}\right),$$

$$\beta = \frac{\mu_N - \mu_D}{\sigma^2}.\quad (A.2d)$$

Note that from $(A.2b)$ it follows that

$$\frac{d \Pr[D | X = x]}{dx} = -\beta \Pr[D | X = x] (1 - \Pr[D | X = x]).\quad (A.3)$$

Hence the absolute value of the slope of the PD curve (A.2b) attains its maximum if and only if $\Pr[D | X = x] = 1/2$ and then the maximum absolute slope is $\beta/4$.

Denote by $X_D$ and $X_N$ independent random variables with $X_D \sim \mathcal{N}(\mu_D, \sigma)$ and $X_N \sim \mathcal{N}(\mu_N, \sigma)$. Then, under assumption A.1, we also obtain a simple formula\(^5\) for the discriminatory power of the score $X$ if it is measured as accuracy ratio (see, for instance, Tasche, 2009, section 3.1.1):

$$AR = \Pr[X_D < X_N] - \Pr[X_D > X_N]$$

$$= 2 \Phi\left(\frac{\mu_N - \mu_D}{\sqrt{2} \sigma}\right) - 1.\quad (A.4b)$$

In addition, it is easy to show how the unconditional mean $\mu$ and variance $\tau^2$ of the score $X$ can be described in terms of the means and variances of $X$ conditional on default and survival respectively:

$$\mu = p \mu_D + (1 - p) \mu_N,\quad (A.5a)$$

$$\tau^2 = \sigma^2 + p (1 - p) (\mu_D - \mu_N)^2.\quad (A.5b)$$

\(^5\)\(\Phi\) denotes the standard normal distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-1/2 y^2} dy$. 

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A close inspection of equations (A.4b), (A.5a) and (A.5b) shows that the conditional variance \( \sigma^2 \) and the conditional means \( \mu_D \) and \( \mu_N \) can be written as functions of the unconditional mean, the unconditional variance and the accuracy ratio:

\[
\begin{align*}
    c &= \sqrt{2} \Phi^{-1}(\frac{\Delta R + 1}{2}) , \\
    \sigma^2 &= \frac{\tau^2}{1 + p(1 - p)c^2} , \\
    \mu_N &= \mu + p \sigma c , \\
    \mu_D &= \mu - (1 - p) \sigma c .
\end{align*}
\]

(A.6)

From this, it follows by (A.2c) and (A.2d) that also the coefficients \( \alpha \) and \( \beta \) in (A.2b) can be represented in terms of the unconditional mean \( \mu \) of \( X \), the unconditional variance \( \tau^2 \) of \( X \), and the discriminatory power \( AR \) of \( X \). In particular, we have the following representation of \( \beta \) in terms of the accuracy ratio and the dispersion of the conditional score distributions:

\[
\beta = \frac{\sqrt{2} \Phi^{-1}(\frac{\Delta R + 1}{2})}{\sigma} .
\]

(A.7)

These observations suggest the following three steps approach to identifying initial values for the QMM approach to PD curve smoothing:

1) Calculate the mean \( \mu \) and the standard deviation \( \tau \) of the unconditional rating profile.

2) Use \( \mu \) and \( \tau \) together with the unconditional PD \( p \) and the accuracy ratio \( AR \) implied by the rating profile and the observed grade-level default rates to calculate the conditional standard deviation \( \sigma \) and the conditional means \( \mu_D \) and \( \mu_N \) according to (A.6).

3) Use equations (A.2c) and (A.2d) to determine initial values for \( \alpha \) and \( \beta \).

The initial values found by this approach will be the closer to the true values, the closer the conditional rating profiles are to normal distributions.

**A.2. Quasi moment matching**

Equation (A.2a) shows that the unconditional PD has a direct primary impact on the level of the PD curve. Equation (A.7) suggests that the AR of a rating model has a similar impact on the maximum slope of the PD curve. The two observations together suggest that in general a two-parameter PD curve can be fitted to match given unconditional PD and AR.

It may be argued that for a suitably developed rating model based on carefully selected risk factors, the associated PD curve must be monotonic for economic reasons. Under the assumption that the PD curve is monotonic, Tasche (2009, section 5.2) suggested the following robust version of the logistic curve (A.2b) for fitting the PD curve:

\[
\begin{align*}
    \Pr[D \mid X = x] &\approx \frac{1}{1 + \exp(\alpha + \beta \Phi^{-1}(F_N(x)))} , \\
    F_N(x) &= \Pr[X \leq x \mid N].
\end{align*}
\]

(A.8)

This approach may be considered a variant of the “sigmoid model” suggested by Platt (2000). The term \( \Phi^{-1}(F_N(x)) \) in (A.8) transforms the in general non-normal distribution of the ratings
conditional on survival into another distribution that is approximately normal even if the rating distribution is not continuous. However, in the discontinuous case \( F_N(x) = 1 \) may occur which would entail \( \Pr[D \mid X = x] = 0 \). A suitable work-around to avoid this is to replace the distribution function \( F_N \) by the average \( \tilde{F}_N \) of \( F_N \) and its left-continuous version:

\[
\tilde{F}_N(x) = \frac{\Pr[X < x \mid N] + \Pr[X \leq x \mid N]}{2}.
\] (A.9)

Define, in addition to \( F_N \), the distribution function \( F_D \) of the rating variable \( X \) conditional on default by

\[
F_D(x) = \Pr[X \leq x \mid D].
\] (A.10)

and denote by \( X_D \) and \( X_N \) independent random variables that are distributed according to \( F_D \) and \( F_N \) respectively.

For quantifying discriminatory power, we apply again the notion of accuracy ratio (AR) as specified in Tasche (2009, eq. (3.28b)):

\[
\text{AR} = 2 \Pr[X_D < X_N] + \Pr[X_D = X_N] - 1
= \Pr[X_D < X_N] - \Pr[X_D > X_N]
= \int \Pr[X < x \mid D] dF_N(x) - \int \Pr[X < x \mid N] dF_D(x).
\] (A.11a)

See Hand and Till (2001, section 2) for a discussion of why this definition of accuracy ratio (or the related definition of the area under the ROC curve) is more expedient than the also common definition in geometric terms. Definition (A.11a) of AR takes an ‘ex post’ perspective by assuming the obligors’ states \( D \) or \( N \) at the end of the observation period are known and hence can be used for estimating the conditional (on default and survival respectively) rating distributions \( F_D \) and \( F_N \).

In the case where \( X \) is realised as one of a finite number of rating grades \( x = 1, \ldots, k \), the accuracy ratio can be calculated from the PD curve as follows:

\[
\text{AR} = \frac{1}{p(1 - p)} \left( 2 \sum_{x=1}^{k} (1 - \Pr[D \mid X = x]) \Pr[X = x] \sum_{t=1}^{x-1} \Pr[D \mid X = t] \Pr[X = t] 
+ \sum_{x=1}^{k} \Pr[D \mid X = x] (1 - \Pr[D \mid X = x]) \Pr[X = x]^2 \right) - 1,
\] (A.11b)

where \( p \) stands for the unconditional PD as given by (3.5a).

Then quasi moment matching, for the purpose of this paper means the following procedure:

1) Fix target values \( p_{\text{target}} \) and \( \text{AR}_{\text{target}} \) for the unconditional portfolio PD and the accuracy ratio of the rating model.

2) Substitute \( p_{\text{target}} \) and \( \text{AR}_{\text{target}} \) for the left-hand sides of equations (3.5a) and (A.11b) respectively.

3) Represent \( \Pr[D \mid X = x] \) in (3.5a) and (A.11b) by the robust logistic curve (A.8) (with \( F_N \) replaced by \( \tilde{F}_N \)). Determine \( \tilde{F}_N \) by means of (3.5c) from the empirical unconditional rating profile, the grade-level default rates and the unconditional default rate.
4) Choose initial values for the parameters \( \alpha \) and \( \beta \) according to (A.2c), (A.2d) and (A.6), with 
\[
\mu = \mathbb{E}[\Phi^{-1}(\tilde{F}_N(X))] \quad \text{and} \quad \tau^2 = \text{var}[\Phi^{-1}(\tilde{F}_N(X))].
\]
5) Solve numerically the equation system for \( \alpha \) and \( \beta \).

References

BCBS. *International Convergence of Capital Measurement and Capital Standards. A Revised Framework, Comprehensive Version*. Basel Committee on Banking Supervision, June 2006.

J.R. Bohn and R.M. Stein. *Active Credit Portfolio Management in Practice*. John Wiley & Sons, Inc., 2009.

G. Casella and R.L. Berger. *Statistical Inference*. Duxbury Press, second edition, 2002.

J.S. Cramer. *Logit Models From Economics and Other Fields*. Cambridge University Press, 2003.

C. Elkan. The foundations of cost-sensitive learning. In B. Nebel, editor, *Proceedings of the Seventeenth International Joint Conference on Artificial Intelligence, IJCAI 2001*, pages 973–978. Morgan Kaufmann, 2001.

B. Engelmann and D. Porath. Do not Forget the Economy when Estimating Default Probabilities. *Willmott magazine*, pages 70–73, February 2012.

E. Falkenstein, A. Boral, and L.V. Carty. Riskcalc\(^TM\) for Private Companies: Moody’s Default Model. Rating methodology, Moody’s Investors Service, May 2000.

R.A. Fisher. On the interpretation of \( \chi^2 \) from contingency tables, and the calculation of \( p \). *Journal of the Royal Statistical Society*, 85(1):87–94, 1922.

D.T. Hamilton and R Cantor. Measuring Corporate Default Rates. Special comment, Moody’s Investors Service, November 2006.

D.J. Hand. *Construction and Assessment of Classification Rules*. John Wiley & Sons, Chichester, 1997.

D.J. Hand and R.J. Till. A simple generalisation of the area under the ROC curve for multiple class classification problems. *Machine Learning*, 45(2):171–186, 2001.

E.A. Heitfield. Dynamics of rating systems. In *Studies on the Validation of Internal Rating Systems. Revised version*, pages 10–27. Basel Committee on Banking Supervision, Working Paper No. 14, 2005.

M.P. Konrad. *The Calibration of Rating Models*. PhD thesis, Wirtschaftswissenschaftliche Fakultät der Westfälischen Wilhelms-Universität Münster, 2011.

G. Löffler. Can rating agencies look through the cycle? *Review of Quantitative Finance and Accounting*, 40(4):623–646, 2013.
Moody’s. Annual Default Study: Corporate Default and Recovery Rates, 1920-2012. Special comment, Moody’s Investors Service, February 2013.

K. Pearson. On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *Philosophical Magazine, Series 5, 50*(302):157–175, 1900.

J.C. Platt. Probabilities for SV Machines. In P.J. Bartlett, B. Schölkopf, D. Schuurmans, and A.J. Smola, editors, *Advances in Large-Margin Classifiers*, pages 61–74. MIT Press Cambridge, 2000.

S&P. Default, Transition, and Recovery: 2009 Annual Global Corporate Default Study And Rating Transitions. Report, Standard & Poor’s, March 2010.

S&P. Default, Transition, and Recovery: 2010 Annual Global Corporate Default Study And Rating Transitions. Report, Standard & Poor’s, March 2011.

S&P. Default, Transition, and Recovery: 2011 Annual Global Corporate Default Study And Rating Transitions. Report, Standard & Poor’s, March 2012.

S&P. Default, Transition, and Recovery: 2012 Annual Global Corporate Default Study And Rating Transitions. Report, Standard & Poor’s, March 2013.

D. Tasche. Validation of internal rating systems and PD estimates. In G. Christodoulakis and S. Satchell, editors, *The Analytics of Risk Model Validation*, pages 169–196. Academic Press, 2008.

D. Tasche. Estimating discriminatory power and PD curves when the number of defaults is small. Working paper, Lloyds Banking Group, 2009.

L.C. Thomas. *Consumer Credit Models: Pricing, Profit and Portfolios*. Oxford University Press, 2009.

M. van der Burgt. Calibrating low-default portfolios, using the cumulative accuracy profile. *Journal of Risk Model Validation*, 1(4):17–33, 2008.

A.W. van der Vaart. *Asymptotic Statistics*. Cambridge University Press, 1998.

H.L. van Trees. *Detection, Estimation, and Modulation Theory, Part I*. John Wiley & Sons, 1968.