Optimal rates of convergence of matrices with applications

Heinz H. Bauschke, J.Y. Bello Cruz, Tran T.A. Nghia, Hung M. Phan and Xianfu Wang

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Abstract

We present a systematic study on the linear convergence rates of the powers of (real or complex) matrices. We derive a characterization when the optimal convergence rate is attained. This characterization is given in terms of semi-simpleness of all eigenvalues having the second-largest modulus after 1. We also provide applications of our general results to analyze the optimal convergence rates for several relaxed alternating projection methods and the generalized Douglas-Rachford splitting methods for finding the projection on the intersection of two subspaces. Numerical experiments confirm our convergence analysis.

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1 Introduction

The focus of this paper is the study of the convergence rate of the powers of a real or complex matrix $A$. Necessary and sufficient conditions for such convergence rates were first established by Hensel [21] and later by Oldenburger [31]. The convergence rate plays a central role in many well-known algorithms for solving linear systems such as Jacobi, Gauss-Seidel, successive over-relaxation methods; see, e.g., [28, 32]. Furthermore, the convergence of the power $A^k$ is linear and the rate is dominated by the second-largest absolute eigenvalue of $A$, $\gamma(A)$, which relates to the subdominant or controlling eigenvalue [23, 30]. Natural questions thus arising are “What is the optimal (smallest) convergence rate?” and “When is $\gamma(A)$ the optimal convergence rate?”. In general, the optimal convergence rate does not exist (see Example 2.11 below). However, many iterative linear methods such as the method of alternating projections (also known as von Neumann’s method) [2, 13] and the Douglas-Rachford splitting algorithm [15, 16, 24, 25] do obtain the optimal linear rates of convergence; see also [5, 12, 22]. We are thus investigating in which case the convergence of the powers $A^k$ admits the optimal linear rate. We will provide complete answers for aforementioned questions in Theorem 2.13 and Theorem 2.15. Furthermore, we then are in a position to...
analyze convergence rates of relaxed alternating projection and generalized Douglas-Rachford algorithms for subspaces.

The rest of the paper is organized as follows. In Section 2 we systematically study convergence rates of matrices. The main result in this section is Theorem 2.15 which gives a necessary and sufficient condition for the powers $A^k$ to converge linearly with the optimal rate $\gamma(A)$ via the semi-simpleness of all the eigenvalues having the second-largest absolute values among the spectrum. Section 3 is devoted to the applications of Section 2 to the relaxed alternating methods and also the generalized Douglas-Rachford splitting methods. In Section 4 we introduce and study a nonlinear map that helps to accelerate the convergence of the alternating projection method. In Section 5, we present some numerical results to illustrate our convergence theory developed in earlier sections. Finally, we present our conclusions in Section 6.

**Notation.** Throughout, we denote by $C^{n \times n}$ and $R^{n \times n}$ the sets of $n \times n$ complex matrices and real matrices, respectively. Let $A$ be a matrix in $C^{n \times n}$ (or $R^{n \times n}$). The notation $A^*$ stands for the adjoint (complex transposed) matrix of $A$. The matrix norm used in this paper is the operator norm, i.e., $\|A\| = \max\{\|Ax\| : x \in C^n, \|x\| \leq 1\}$. We write $\ker A$, $\operatorname{ran} A$, and $\operatorname{rank} A$ as the kernel, range, rank of $A$, respectively. Moreover, $\operatorname{Fix} A := \ker (A - \text{Id})$ is known as the set of fixed points of $A$, where $\text{Id}$ is the identity mapping. We say $A$ is nonexpansive if $\|Ax\| \leq \|x\|$ for all $x \in C^n$; furthermore, $A$ is firmly nonexpansive if $\|Ax\|^2 + \|x - Ax\|^2 \leq \|x\|^2$ for all $x \in C^n$. For any subspace $U$ of $R^n$, the notation $P_U$ is referred to the orthogonal projection operator to $U$, $\dim U$ for the dimension of $U$, and $U^\perp$ for the orthogonal complement of $U$. We denote $I_n, 0_n, 0_{m \times n}$ by the $n \times n$ identity matrix, the $n \times n$ zero matrix, and the $m \times n$ zero matrix, respectively.

## 2 The optimal convergence rate of matrices

In this section we establish conditions under which convergent matrices attain their optimal convergent rate. Let us recall some definitions and facts used in the sequel.

**Definition 2.1 (convergent matrices)** Let $A \in C^{n \times n}$. We say $A$ is convergent\footnote{In the literature, $A$ is called convergent if the power $A^k$ converges to 0; moreover, $A$ is semi-convergent whenever the latter limit $A^k$ exists. To avoid the confusion of these two terminologies, we just say $A$ is convergent in both cases.} to $A^\infty \in C^{n \times n}$ if and only if

$$\|A^k - A^\infty\| \to 0 \quad \text{as} \quad k \to \infty. \tag{1}$$

We say $A$ is linearly convergent to $A^\infty$ with rate $\mu \in [0, 1)$ if there are some $M, N > 0$ such that

$$\|A^k - A^\infty\| \leq M\mu^k \quad \text{for all} \quad k > N, k \in \mathbb{N}. \tag{2}$$

Then $\mu$ is called a convergence rate of $A$. When the infimum of all the convergence rates is also a convergence rate, we say this minimum is the optimal convergence rate.

For any $A \in C^{n \times n}$ we denote by $\sigma(A)$ the spectrum of $A$, the set of all eigenvalues. The spectral radius \footnote{Example 7.1.4} of $A$ is defined by

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}. \tag{3}$$

The next fact is the classical formula of spectral radius.
Remark 2.6  Note from Fact 2.5 (i) and (iv) that

(4) \[ \rho(A) = \lim_{k \to \infty} \|A^k\|^{\frac{1}{k}}. \]

With \( \lambda \in \sigma(A) \), recall from [27] page 587 that index (\( \lambda \)) is the smallest positive integer \( k \) satisfying \( \text{rank} (A - \lambda \text{Id})^k = \text{rank} (A - \lambda \text{Id})^{k+1} \). Furthermore, we say \( \lambda \in \sigma(A) \) is semisimple if index (\( \lambda \)) = 1; see, e.g., [27] Example 7.8.4.

Fact 2.3  For \( A \in \mathbb{C}^{n \times n} \), \( \lambda \in \sigma(A) \) is semisimple if and only if \( \ker(A - \lambda \text{Id}) = \ker(A - \lambda \text{Id})^2 \).

Proof. Note that \( \lambda \in \sigma(A) \) is semisimple if and only if

\[ \dim[\ker(A - \lambda \text{Id})] = n - \text{rank} (A - \lambda \text{Id}) = n - \text{rank} (A - \lambda \text{Id})^2 = \dim[\ker(A - \lambda \text{Id})^2]. \]

Since \( \ker(A - \lambda \text{Id}) \subset \ker(A - \lambda \text{Id})^2 \), the equality \( \dim[\ker(A - \lambda \text{Id})] = \dim[\ker(A - \lambda \text{Id})^2] \) holds if and only if \( \ker(A - \lambda \text{Id}) = \ker(A - \lambda \text{Id})^2 \). This verifies the proof of the fact. ■

The following result taken from [27] gives us a complete characterization of a convergent matrix.

Fact 2.4 (limits of powers)  ([27] page 617-618 and page 630) For \( A \in \mathbb{C}^{n \times n} \), \( A \) is convergent to \( A^\infty \) if and only if

(5) \[ \rho(A) < 1, \text{ or else} \]

(6) \[ \rho(A) = 1 \text{ and } \lambda = 1 \text{ is semisimple and it is the only eigenvalue on the unit circle.} \]

When this happens, we have

(7) \[ A^\infty = \text{the projector onto } \ker(A - \text{Id}) \text{ along } \text{ran}(A - \text{Id}). \]

In particular, when \( \rho(A) < 1 \), we have \( A^\infty = 0 \).

The proof of the above fact is indeed based on the spectral resolution of \( A^k \) stated below.

Fact 2.5 (spectral resolution of \( A^k \))  ([27] page 603 and page 629) For \( k \in \mathbb{N} \) and \( A \in \mathbb{C}^{n \times n} \) with \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\} \) and \( k_i = \text{index} (\lambda_i) \), we have

(8) \[ A^k = \sum_{i=1}^{s} \lambda_i^k G_i + \sum_{i=1}^{s} \sum_{j=1}^{k_i-1} \binom{k}{j} \lambda_i^{k_i-j} (A - \lambda_i \text{Id})^j G_i, \]

where the spectral projector \( G_i \)'s have the following properties:

(i) \( G_i \) is the projector onto \( \ker((A - \lambda_i \text{Id})^{k_i}) \) along \( \text{ran}(A - \lambda_i \text{Id})^{k_i}) \).

(ii) \( G_1 + G_2 + \cdots + G_s = \text{Id}. \)

(iii) \( G_i G_j = 0 \) when \( i \neq j. \)

(iv) \( N_i = (A - \lambda_i \text{Id}) G_i = G_i(A - \lambda_i \text{Id}) \) is nilpotent of index \( k_i \), i.e., \( N_i^{k_i} = 0 \) and \( N_i^{k_i-1} \neq 0 \).

Remark 2.6  Note from Fact 2.5 (i) and (iv) that

\[ 0 \neq N_i^{k_i-1} = (A - \lambda_i \text{Id})^{k_i-1} G_i^{k_i-1} = (A - \lambda_i \text{Id})^{k_i-1} G_i \text{ if } k_i > 1. \]
Corollary 2.7 Suppose that $A \in \mathbb{C}^{n \times n}$ is convergent to $A^\infty \in \mathbb{C}^{n \times n}$. Then the following hold:

(i) $A^\infty = P_{\text{Fix} A}$ if and only if $\text{Fix} A = \text{Fix} A^*$.

(ii) If $A$ is nonexpansive or normal, then $A^\infty = P_{\text{Fix} A}$.

Proof. It follows from (7) that $A^\infty$ is equal to the projector onto $\ker(A - \text{Id})$ along $\text{ran}(A - \text{Id})$. Thanks to the equality [27, (5.9.11)], we have $\text{ran}(A^\infty - \text{Id}) = \text{ran}(A - \text{Id})$. If $A^\infty = P_{\text{Fix} A}$, we obtain

$$\text{ran}(A - \text{Id}) = \text{ran}(A^\infty - \text{Id}) = \text{ran}(P_{\text{Fix} A} - \text{Id}) = \text{ran}(P_{(\text{Fix} A)^\perp}) = (\text{Fix} A)^\perp.$$  

It follows that

$$\text{Fix} A = [(\text{Fix} A)^\perp]^\perp = \text{ran}(A - \text{Id})^\perp = \ker(A^* - \text{Id}) = \text{Fix} A^*.$$  

Conversely, if $\text{Fix} A = \text{Fix} A^*$, we have

$$\ker(A - \text{Id}) = \text{Fix} A = \text{Fix} A^* = \ker(A^* - \text{Id}) = \text{ran}(A - \text{Id})^\perp,$$

which implies in turn that the projector onto $\ker(A - \text{Id})$ along $\text{ran}(A - \text{Id})$ is exactly the orthogonal projection $P_{\text{Fix} A}$. The first part (i) of the corollary is complete.

To justify the second part (ii), suppose in addition that $A$ is nonexpansive. Then $\text{Fix} A = \text{Fix} A^*$ by [6, Lemma 2.1] and thus $A$ is convergent to $P_{\text{Fix} A}$. Moreover, if $A$ is normal, then $A - \text{Id}$ is also normal. Hence for all $x \in \mathbb{C}^n$ we have

$$\|((A - \text{Id})x)^2\| = \langle (A - \text{Id})^*(A - \text{Id})x, x \rangle = \langle (A - \text{Id})(A - \text{Id})^*x, x \rangle = \|((A - \text{Id})^*x\|.$$  

The latter clearly shows that $\text{Fix} A = \text{Fix} A^*$ and thus $A^\infty = P_{\text{Fix} A}$. The proof is complete.

Remark 2.8 (convergence, firmly nonexpansiveness and nonexpansiveness) Let $A \in \mathbb{R}^{n \times n}$. When $A$ is firmly nonexpansive, $A$ is convergent; see, e.g., [3, Example 5.17]. However, the converse implication fails. Indeed, consider, for $n \geq 2$,

$$A = \begin{pmatrix} 0 & n^{-2} \\ n & 0 \end{pmatrix}.$$  

Then $A$ is not (firmly) nonexpansive because $Ae_1 = ne_2$ where $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. On the other hand, the characteristic polynomial is $\lambda \mapsto \lambda^2 - n^{-1}$, which has roots $\pm n^{-1/2}$. Thus $A$ is convergent due to Fact 2.4. Moreover, convergence and nonexpansiveness are independent, e.g., $A = -\text{Id}$ is nonexpansive but not convergent.

We will prove later in this section that whenever $A$ is convergent to $A^\infty$, it is linearly convergent with the rate not smaller than $\rho(A - A^\infty)$. To manipulate this idea, let us take into account the case of diagonalizable matrices as follows.

Example 2.9 (diagonalizable case) Suppose that $A \in \mathbb{C}^{n \times n}$ is diagonalizable and that $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ with

$$1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_s|.$$  

By Fact 2.5 and Fact 2.4 we have $A$ is convergent to $A^\infty$ and that

$$A^k = A^\infty + \lambda_2^k G_2 + \cdots + \lambda_s^k G_s,$$  

where $G_j$ is the $j$th Jordan block of $A$. This example shows that the rate of convergence of $A^\infty$ is precisely $\rho(A - A^\infty)$. In other words, for any $x \in \mathbb{C}^n$ we have

$$\|A^k x - A^\infty x\| \leq \rho(A - A^\infty) \|x\| \text{ for all } k \geq 0.$$  

We will show in the next section that this result is sharp.
which yields
\[ A^k - A^\infty = \lambda_2^k G_2 + \cdots + \lambda_s^k G_s. \]

It follows that
\[
\|A^k - A^\infty\| \leq |\lambda_2|^k \left( |\lambda_2| \|G_2\| + \cdots + |\lambda_2|^s |\lambda_2|^s G_s \right) \\
\leq |\lambda_2|^k \left( \|G_2\| + \cdots + \|G_s\| \right).
\]

Hence \( A^k \rightarrow A^\infty \) with the linear rate \(|\lambda_2|\).

In general an eigenvalue having second-largest modulus after 1 is called a subdominant eigenvalue.

**Definition 2.10 (subdominant eigenvalues) ([23, 30])** For \( A \in \mathbb{C}^{n \times n} \), we define
\[
\gamma(A) := \max \| \lambda \| \lambda \in \{0\} \cup \sigma(A) \setminus \{1\}.
\]
An eigenvalue \( \lambda \in \sigma(A) \) satisfying \(|\lambda| = \gamma(A)\) is referred as a subdominant eigenvalue.

When \( A \) is not diagonalizable, \( \gamma(A) \) need not be the convergence rate.

**Example 2.11** Let us consider the following matrix
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & 1 \\
0 & 0 & 1/2
\end{pmatrix},
\]
which gives us that \( \gamma(A) = \frac{1}{2} \). Note also that \( A \) is not diagonalizable. Moreover, by induction it is easy to check that
\[
A^k = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2^k} & \frac{k}{2^k} \\
0 & 0 & \frac{1}{2^k}
\end{pmatrix}
\]
for all \( k \in \mathbb{N} \).

Hence we have \( A^k \rightarrow A^\infty := \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \) as \( k \rightarrow \infty \). However, observe that
\[
\frac{\|A^k - A^\infty\|}{\gamma(A)^k} = 2^k \left\| \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{1}{2^k} & \frac{k}{2^k} \\
0 & 0 & \frac{1}{2^k}
\end{pmatrix} \right\| = \left\| \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 2k \\
0 & 0 & 1
\end{pmatrix} \right\| \rightarrow \infty \text{ as } k \rightarrow \infty.
\]

Hence \( \gamma(A) \) is not a convergence rate. However, observe further that any \( \mu \in (\frac{1}{2}, 1) \) is a convergence rate of \( A \). Thus \( A \) does not obtain the optimal convergence rate.

Our first main result below shows that whenever a matrix \( A \) is convergent, it must be linearly convergent with any rate in \((\gamma(A), 1)\). The theorem can be extended for linear operator in infinite-dimensional spaces by connecting the proof below with those of [1] Theorem 2.1 and 2.2.
Theorem 2.12 (rate of convergence I) Suppose that \( A \in \mathbb{C}^{n \times n} \) is convergent to \( A^\infty \in \mathbb{C}^{n \times n} \). Then we have \( \gamma(A) = \rho(A - A^\infty) < 1 \) and that

\[
(A - A^\infty)^k = A^k - A^\infty \text{ for all } k \in \mathbb{N}.
\]

Moreover, the following two assertions are satisfied:

(i) \( A \) is linearly convergent with any rate \( \mu \in (\gamma(A), 1) \).

(ii) If \( A \) is linearly convergent with rate \( \mu \in [0, 1) \), then \( \mu \in [\gamma(A), 1) \).

Proof. First let us justify that \( \gamma(A) = \rho(A - A^\infty) < 1 \) and (11) by considering the two following cases taken from Fact 2.4:

Case 1. \( \rho(A) < 1 \). In this case we have \( A^\infty = 0 \) by (4). It follows that \( \gamma(A) = \rho(A) = \rho(A - A^\infty) < 1 \). Note also that (11) is trivial, since \( A^\infty = 0 \).

Case 2. \( \rho(A) = 1 \) and \( \lambda = 1 \) is semisimple and the only eigenvalue on the unit circle. Suppose that \( \sigma(A) \setminus \{1\} = \{\lambda_2, \ldots, \lambda_s\} \) with \( 1 > |\lambda_2| \geq \ldots \geq |\lambda_s| \). The Jordan decomposition [27, page 590] of \( A \) allows us to find an invertible matrix \( P \in \mathbb{C}^{n \times n} \) and \( r > 0 \) such that

\[
A = PJP^{-1}
\]

with \( J \) being the Jordan form of \( A \),

\[
J = \begin{pmatrix}
I_r & 0 & \cdots & 0 \\
0 & J(\lambda_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J(\lambda_s)
\end{pmatrix},
\]

\[
J(\lambda_j) = \begin{pmatrix}
J_1(\lambda_j) & 0 & \cdots & 0 \\
0 & J_2(\lambda_j) & \ddots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_t(\lambda_j)
\end{pmatrix},
\]

and

\[
J_*(\lambda_j) = \begin{pmatrix}
\lambda_j & 1 \\
\ldots & \ldots \\
\ldots & \ldots \\
1 & \lambda_j
\end{pmatrix}.
\]

Moreover, it follows from [27, p. 629] that

\[
A^\infty = P \begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix} P^{-1}.
\]

This together with the Jordan decomposition above gives us that

\[
A - A^\infty = P \begin{pmatrix}
0_r & 0 & \cdots & 0 \\
0 & J(\lambda_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J(\lambda_s)
\end{pmatrix} P^{-1},
\]

which readily yields \( \rho(A - A^\infty) = \max\{0, |\lambda_2|\} = \gamma(A) < 1 \). Observe further from (12) and (13) that \( AA^\infty = A^\infty A = (A^\infty)^2 = A^\infty \). For any \( k \in \mathbb{N} \) the latter gives us that

\[
(A^k - A^\infty)(A - A^\infty) = A^{k+1} - A^k A^\infty - A^\infty A + (A^\infty)^2 = A^{k+1} - A^\infty - A^\infty + A^\infty = A^{k+1} - A^\infty.
\]
By using this expression, we may prove by induction \(11\) and thus completes the first part of the theorem.

Now to verify (i), pick any \(\mu \in (\gamma(A), 1) = (\rho(A - A^\infty), 1)\). Employing \(4\) for operator \(A - A^\infty\) allows us to find some \(N \in \mathbb{N}\) such that
\[
\|A^k - A^\infty\| = \|(A - A^\infty)^k\| \leq \mu^k \quad \text{for all} \quad k \geq N,
\]
which verifies the linear convergence of \(A\) with rate \(\mu\).

It remains to prove (ii). Suppose that \(A\) is convergent to \(A^\infty\) with rate \(\mu \in [0, 1)\). Hence there are some \(M, N > 0\) such that
\[
\|A^k - A^\infty\| \leq M\mu^k \quad \text{for all} \quad k > N, k \in \mathbb{N}.
\]
Combining this with the spectral radius formula \(4\) and \(11\) gives us that
\[
\gamma(A) = \rho(A - A^\infty) = \lim_{k \to \infty} \|(A - A^\infty)^k\|^{1/k} = \lim_{k \to \infty} \|A^k - A^\infty\|^{1/k} \leq \lim_{k \to \infty} M^{1/k} \mu = \mu,
\]
which ensures \(\gamma(A) \leq \mu\) and thus completes the proof of the theorem.

A natural question arising from the above theorem is that in which case \(\gamma(A)\) is the optimal convergence rate of \(A\); see our Definition 2.1. By Theorem 2.12, the actual problem is that when \(\gamma(A)\) is a convergence rate of \(A\); see also our Example 2.11. The next theorem gives us a complete answer for this question.

**Theorem 2.13 (rate of convergence II)** Let \(A \in \mathbb{C}^{n \times n}\) be convergent to \(A^\infty \in \mathbb{C}^{n \times n}\). Then \(\gamma(A)\) is the optimal convergence rate of \(A\) if and only if all the subdominant eigenvalues are semisimple.

**Proof.** By Theorem 2.12, we only need to prove that \(\gamma(A)\) is a convergence rate of \(A\) if and only if \(\lambda\) is semisimple for every eigenvalue \(\lambda \in \sigma(A)\) satisfying \(|\lambda| = \gamma(A)\). For the matrix \(A\), denote the set of distinct eigenvalues in \(\sigma(A) \setminus \{1\}\) by \(\{\lambda_2, \ldots, \lambda_s\}\) and \(k_i = \text{index}(\lambda_i), i = 2, \ldots, s\). This set may be empty, but in this case we have \(\{1\} = \sigma(A)\) and \(\text{index}(1) = 1\) by Fact 2.4; thus \(A^k = G_1 = A^\infty\) by \(7\) and \(8\) for all \(k \in \mathbb{N}\), which ensures that \(\gamma(A) = 0\) is a convergence rate of \(A\). From now on we suppose that \(\sigma(A) \setminus \{1\} \neq \emptyset\) and that
\[
1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_s|.
\]
If \(1 \notin \sigma(A)\), we get from Fact 2.4 that \(A^\infty = 0\) and from \(8\) that
\[
A^k = \sum_{i=2}^{s} \lambda_i^k G_i + \sum_{i=2}^{s} \sum_{j=1}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i = A^\infty + \sum_{i=2}^{s} \sum_{j=0}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i.
\]
If \(\lambda_1 := 1 \in \sigma(A)\), Fact 2.4 tells us that its index is 1. Hence, we obtain from \(7\) that \(A^\infty = G_1\). This together with the spectral resolution \(9\) gives us that
\[
A^k = G_1 + \sum_{i=2}^{s} \lambda_i^k G_i + \sum_{i=2}^{s} \sum_{j=1}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i = A^\infty + \sum_{i=2}^{s} \sum_{j=0}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i.
\]
From both cases above, we always have
\[
A^k = A^\infty + \sum_{i=2}^{s} \sum_{j=0}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i.
\]
If \( \gamma(A) = 0 \), then \( \lambda_2 = 0 \) and \( s = 2 \), by (17) we have \( A^k = A^\infty \) for all \( k \geq 1 \). This means that 
\( A = A^\infty \) and \( A^2 = A \), which ensures that \( \lambda_2 \) is semisimple and \( \gamma(A) = 0 \) is a convergence rate. Thus the statement of the theorem is trivial in this case. It remains to prove the theorem when \( \gamma(A) > 0 \). Denote by

\[
E := \{2, \ldots, s\}, \quad F := \{i \in \mathbb{N} \mid |\lambda_i| = |\lambda_2|, 2 \leq l \leq s\},
\]
\( \alpha := \max \{\text{index}(\lambda_i) \mid i \in F\} \), and \( S := \{i \in F \mid \text{index}(\lambda_i) = \alpha\} \).

It is clear that \( F \supset S \neq \emptyset \) and \( \alpha \geq 1 \). By (17) we have

\[
A^k - A^\infty = \sum_{i \in S} \sum_{j=0}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i + \sum_{i \in E \setminus S} \sum_{j=0}^{k-1} \binom{k}{j} \lambda_i^{k-j} (A - \lambda_i \text{Id})^j G_i.
\]

Note that

\[
\frac{k}{|\lambda_2|^k \left( \frac{k}{\alpha - 1} \right)} = \sum_{i \in E \setminus S} \sum_{j=0}^{k-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_2|^k} \frac{1}{\left( \frac{k}{\alpha - 1} \right)} (A - \lambda_i \text{Id})^j G_i
\]

\[
= \sum_{i \in (E \setminus S) \cup F} \sum_{j=0}^{k-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_2|^k} \frac{1}{\left( \frac{k}{\alpha - 1} \right)} (A - \lambda_i \text{Id})^j G_i + \sum_{i \in E \setminus F} \sum_{j=0}^{k-1} \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_2|^k} \frac{1}{\left( \frac{k}{\alpha - 1} \right)} (A - \lambda_i \text{Id})^j G_i.
\]

For \( i \in (E \setminus S) \cup F \) and \( 0 \leq j \leq k_i - 1 \), observe from the definition of \( \alpha \) in (18) that \( j \leq \alpha - 2 \). It follows that

\[
\frac{\binom{k}{j} \lambda_i^{k-j}}{|\lambda_2|^k \left( \frac{k}{\alpha - 1} \right)} = \frac{(\alpha - 1)!(k - \alpha + 1)!}{j!(k - j)!} \frac{1}{|\lambda_2|^j} \leq \frac{(\alpha - 1)!}{j!(k - \alpha + 2)} \frac{1}{|\lambda_2|^j} := \epsilon_1(k) \to 0
\]

as \( k \to \infty \). For \( i \in E \setminus F \) and \( 0 \leq j \leq k_i - 1 \), we have

\[
\left| \binom{k}{j} \frac{\lambda_i^{k-j}}{|\lambda_2|^k} \right| \leq k^j \left( \frac{|\lambda_i|}{|\lambda_2|} \right)^{k-j} := \epsilon_2(k) \to 0,
\]

since \( k^j \) is polynomial in \( k \) and (\( |\lambda_i|/|\lambda_2| \))\(^{k-j} \) is exponential with \( |\lambda_i|/|\lambda_2| < 1 \) for \( i \notin F \). It follows from (20), (21), and (22) that

\[
\frac{\|K\|}{|\lambda_2|^k \left( \frac{k}{\alpha - 1} \right)} \leq \sum_{i \in (E \setminus S) \cup F} \sum_{j=0}^{k_i-1} \epsilon_1(k) \| (A - \lambda_i)^j G_i \| + \sum_{i \in E \setminus F} \sum_{j=0}^{k_i-1} \epsilon_2(k) \| (A - \lambda_i)^j G_i \| \to 0.
\]
Next let us justify the “⇒” part by supposing that \( A \) is convergent to \( A^\infty \) with the rate \( \gamma(A) = |\lambda_2| \in (0, 1) \). Hence there are some \( M, N > 0 \) such that

\[
\|A^k - A^\infty\| \leq M|\lambda_2|^k \quad \text{for all} \quad k > N, k \in \mathbb{N}.
\]

We will prove that \( \alpha = 1 \). Assume by contradiction that \( \alpha > 1 \) and note from (19) that

\[
\frac{H}{|\lambda_2|^k (\frac{k}{\alpha - 1})} = \sum_{i \in S} \sum_{j=0}^{\alpha-1} \binom{k}{j} \frac{\lambda_i^k}{|\lambda_2|^k} (A - \lambda_i \text{Id})^j G_i
\]

\[
= \sum_{i \in S} \frac{\lambda_i^{k-(\alpha-1)}}{|\lambda_2|^k} (A - \lambda_i \text{Id})^{\alpha-1} G_i + \sum_{i \in S} \sum_{j=0}^{\alpha-2} \binom{k}{j} \frac{\lambda_i^k}{|\lambda_2|^k} (A - \lambda_i \text{Id})^j G_i
\]

\[
= \sum_{i \in S} \frac{\lambda_i^k}{|\lambda_2|^k} \lambda_i^{-(\alpha-1)} (A - \lambda_i \text{Id})^{\alpha-1} G_i + \sum_{i \in S} \sum_{j=0}^{\alpha-2} \binom{k}{j} \frac{\lambda_i^k}{|\lambda_2|^k} (A - \lambda_i \text{Id})^j G_i.
\]

Furthermore, for \( i \in S \) and \( j \leq \alpha - 2 \) similarly to (21) we may prove that

\[
\frac{\binom{k}{j}}{\binom{k}{\alpha-1}} \frac{\lambda_i^{k-j}}{|\lambda_2|^k} := \varepsilon_3(k) \to 0 \quad \text{when} \quad k \to \infty,
\]

which implies in turn that

\[
\|H_1\| \leq \sum_{i \in S} \sum_{j=0}^{\alpha-2} \varepsilon_3(k) \| (A - \lambda_i \text{Id})^j G_i \| \to 0 \quad \text{as} \quad k \to \infty.
\]

By dividing (24) by \( |\lambda_2|^k \left( \frac{k}{\alpha - 1} \right) \) and taking \( k \to \infty \), we get from (19), (23), (25), and (26) tells us that

\[
\lim_{k \to \infty} \sum_{i \in S} \frac{\lambda_i^k}{|\lambda_2|^k} \lambda_i^{-(\alpha-1)} (A - \lambda_i \text{Id})^{\alpha-1} G_i = \lim_{k \to \infty} \frac{M}{\left( \frac{k}{\alpha - 1} \right)} = 0.
\]

Since \( \frac{\lambda_i}{|\lambda_i|} = 1 \) for all \( i \in S \), by passing to subsequences we may assume without loss of generality that for each \( i \in S \) the sequence \( \left( \frac{\lambda_i}{|\lambda_i|} \right)^k \to x_i \) with \( |x_i| = 1 \) as \( k \to \infty \). Hence, it follows from (27) that

\[
\sum_{i \in S} x_i \lambda_i^{-(\alpha-1)} (A - \lambda_i \text{Id})^{\alpha-1} G_i = 0.
\]
By Fact 2.5 (i) and (iii), we have $G_i G_j = 0$ when $i \neq j$ and $G_i G_i = G_i$. For any $l \in S$, multiplying both sides of (28) by $G_l$ yields

$$x_l \lambda_l^{-(a-1)} (A - \lambda_l \text{Id})^{a-1} G_l = 0,$$

which is impossible since $x_l \neq 0$, $|\lambda_l| = |\lambda_2| \neq 0$, $(A - \lambda_l \text{Id})^{a-1} G_l \neq 0$ by Remark 2.6. Thus, $a = 1$, thanks to the definition of $a$ in (18) we get that all $\lambda_i$, $i \in F$ has the same index 1 and complete the first part of the proof.

Conversely, suppose that all $\lambda_i \in \sigma(A)$ satisfying $|\lambda_i| = \gamma(A) = |\lambda_2| > 0$ are semisimple, which implies $a$ in (18) is 1. Hence we observe that from (19) that

$$\|H\| = \|\sum_{i \in S} \frac{\lambda_i^k}{|\lambda_2|^k} G_i\| \leq \sum_{i \in S} \|G_i\|.$$

Moreover, the term $\|K\|_{|\lambda_2|}$ still converges to 0 as proved in (23). Combining this with (29) and (19) gives us that $A$ is convergent to $A^\infty$ with the linear rate $|\lambda_2|$. The proof of the theorem is complete.

Remark 2.14 It is worth mentioning that Example 2.9 is also a direct consequence of Theorem 2.13, since all the eigenvalues of $A$ are semisimple when $A$ is diagonalizable. Moreover, $\gamma(A)$ is not the convergence rate in Example 2.11, since $\frac{1}{2} = \gamma(A)$ is not semisimple in this case.

Next let us summarize Fact 2.4, Theorem 2.12, and Theorem 2.13 in the following result, which provides a complete characterization for obtaining the optimal convergence rate.

**Theorem 2.15 (optimal convergence rate)** Let $A \in \mathbb{C}^{n \times n}$. Then $A$ is convergent with the optimal convergence rate, which is $\gamma(A)$ if and only if one of the following holds:

(i) $\rho(A) < 1$ and all $\lambda \in \sigma(A)$ satisfying $|\lambda| = \gamma(A)$ are semisimple.

(ii) $\rho(A) = 1$, $\lambda = 1$ is the only eigenvalue on the unit circle, $\lambda = 1$ is semisimple, and all $\lambda \in \sigma(A)$ satisfying $|\lambda| = \gamma(A)$ are semisimple.

Proof. If $A$ is convergent with the optimal convergence rate, Theorem 2.12 tells us that $\gamma(A)$ is the optimal convergence rate. Moreover, (i) and (ii) follow from Fact 2.4 and Theorem 2.13. Conversely, if (i) and (ii) hold, we also get from Fact 2.4 and Theorem 2.13 that $A$ is convergent with the optimal rate $\gamma(A)$. ■

**Theorem 2.16** Let $A \in \mathbb{C}^{n \times n}$ be convergent to $A^\infty$. Then we have

$$\|A^k - A^\infty\| \leq \|A - A^\infty\|^k$$

and thus $\gamma(A) \leq \|A - A^\infty\|$. Furthermore, if $A$ is normal then we have

$$\|A^k - A^\infty\| = \|A - A^\infty\|^k$$

and $\gamma(A) = \|A - A^\infty\|$ is the optimal convergence rate of $A$.

Proof. First, observe from (11) in Theorem 2.12 that

$$\|A^k - A^\infty\| = \|(A - A^\infty)^k\| \leq \|A - A^\infty\|^k,$$
which together with (4) for \( A - A^\infty \) clearly ensures (30) and thus \( \gamma(A) = \rho(A - A^\infty) \leq \|A - A^\infty\| \) by Theorem 2.12.

To justify the second part, suppose that \( A \) is convergent and normal. We claim that \( A - A^\infty \) is also normal. This is trivial when \( A^\infty = 0 \). It remains to take into account the case \( A^\infty \neq 0 \). Since \( A \) is normal, we can find a diagonal matrix \( J = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n) \) with \( |\lambda_1| \geq \cdots \geq |\lambda_n| \) and a unitary matrix \( P \) such that \( A = PJP^* \). Fact 2.4 tells us that \( 1 \in \sigma(A) \) and \( 1 = \lambda_1 = \cdots = \lambda_r > |\lambda_{r+1}| \geq \cdots \geq |\lambda_n| \) for some \( r \in \mathbb{N} \). It follows that

\[
A^\infty = \lim_{k \to \infty} A^k = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^*.
\]

Hence we obtain

\[
A - A^\infty = P \begin{pmatrix} 0_{r \times r} & 0 & \cdots & 0 \\ 0 & \lambda_{r+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P^*,
\]

which is a normal matrix. The latter formula together with (11) also gives us that

\[
\|A^k - A^\infty\| = \|(A - A^\infty)^k\| = \|A - A^\infty\|^k = \|\lambda_{r+1}\|^k = [\rho(A - A^\infty)]^k = \gamma(A)^k,
\]

which ensures (31) and completes the proof of the theorem.

\[\blacklozenge\]

3 Applications to relaxed alternating projection and generalized Douglas-Rachford methods

In this section, using results in Section 2 and principal angles between two subspaces, we will analyze convergence rates of relaxed alternating projections and generalized Douglas-Rachford methods for two subspaces comprehensively. Throughout the section we suppose that \( U \) and \( V \) are two subspaces of \( \mathbb{R}^n \) with \( 1 \leq p := \dim U \leq \dim V := q \leq n - 1 \). Note that the whole section will be not interesting if \( \dim U = 0 \) or \( \dim V = n \). Let us recall the principal angles and the Friedrichs angles between \( U \) and \( V \) as follows, which are crucial for our quantitative analysis of convergence rates.

**Definition 3.1 (principal angles)\([8], [27] \text{ page 456}\)** The principal angles \( \theta_k \in [0, \frac{\pi}{2}] \), \( k = 1, \ldots, p \) between \( U \) and \( V \) are defined by

\[
\cos \theta_k := \langle u_k, v_k \rangle = \max \left\{ \langle u, v \rangle \bigg| u \in U, v \in V, \|u\| = \|v\| = 1, \langle u, u_j \rangle = \langle v, v_j \rangle = 0, j = 1, \ldots, k - 1 \right\} \quad \text{with} \quad u_0 = v_0 := 0.
\]

It is worth mentioning that the vectors \( u_k, v_k \) are not uniquely defined, but the principal angles \( \theta_k \) are unique with \( 0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_p \leq \frac{\pi}{2} \); see \([27] \text{ page 456}\)).

**Definition 3.2 (Friedrichs angle)** The cosine of the Friedrichs angle \( \theta_F \in (0, \frac{\pi}{2}] \) between \( U \) and \( V \) is

\[
c_F(U, V) := \max \left\{ \langle u, v \rangle \bigg| u \in U \cap (U \cap V)^\perp, v \in V \cap (U \cap V)^\perp, \|u\| = \|v\| = 1 \right\}.
\]
In the following proposition we show that the Friedrichs angle is exactly the \((s + 1)\)-th principal angle \(\theta_{s+1}\) where \(s := \dim(U \cap V)\).

**Proposition 3.3 (principal angles and Friedrichs angle)** Let \(s := \dim(U \cap V)\). Then we have \(\theta_k = 0\) for \(k = 1, \ldots, s\) and \(\theta_{s+1} = \theta_F > 0\).

**Proof.** Let \(x_1, \ldots, x_s\) be an orthonormal basis of the subspace \(U \cap V\). We may choose \(u_k = v_k = x_k, k = 1, \ldots, s\) from (33). It follows that \(\cos \theta_k = \langle x_k, x_k \rangle = 1\) and thus \(\theta_k = 0\) for all \(k = 1, \ldots, s\). Moreover, since \(\text{span} \{u_1, \ldots, u_s\} = \text{span} \{v_1, \ldots, v_s\} = U \cap V\), we obtain from (33) that

\[
\cos \theta_{s+1} = \max \{ \langle u, v \rangle \mid u \in U, v \in V, \|u\| = \|v\| = 1, u, v \in (U \cap V) \}.
\]

This together with (34) tells us that \(\theta_{s+1} = \theta_F\). The proof is complete. \(\blacksquare\)

The following result follows the idea of [8, 12] to construct the orthogonal projections \(P_U\) and \(P_V\) with the appearance of the principal angles.

**Proposition 3.4 (principal angles and orthogonal projections)** Suppose further that \(p + q < n\). Then we may find a orthogonal matrix \(D \in \mathbb{R}^{n \times n}\) such that

\[
P_U = D \left( \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0_n & 0_{n-p-q} \end{pmatrix} \right) D^* \quad \text{and} \quad P_V = D \left( \begin{pmatrix} C^2 & CS & 0 & 0 \\ CS & S^2 & 0 & 0 \\ 0 & 0 & I_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{pmatrix} \right) D^*,
\]

where \(C\) and \(S\) are two \(p \times p\) diagonal matrices defined by

\[
C := \text{diag} \left( \cos \theta_1, \ldots, \cos \theta_p \right) \quad \text{and} \quad S := \text{diag} \left( \sin \theta_1, \ldots, \sin \theta_p \right)
\]

with the principal angles \(\theta_1, \ldots, \theta_p\) between \(U\) and \(V\) found in Definition 3.1. Consequently, we have

\[
P_U P_V = D \left( \begin{pmatrix} C^2 & CS & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0_n & 0_{n-p-q} \end{pmatrix} \right) D^* \quad \text{and} \quad P_U P_V\perp = D \left( \begin{pmatrix} 0_p & 0 & 0 & 0 \\ -CS & C^2 & 0 & 0 \\ 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{pmatrix} \right) D^*.
\]

Furthermore, the orthogonal projection \(P_{U \cap V}\) is computed by

\[
P_{U \cap V} = D \left( \begin{pmatrix} I_s & 0 & 0 & 0 \\ 0 & 0_n & 0_{n-s} \end{pmatrix} \right) D^* \quad \text{with} \quad s := \dim(U \cap V).
\]

**Proof.** Let \(Q_U \in \mathbb{R}^{p \times p}, Q_{U\perp} \in \mathbb{R}^{p \times (n-p)}\) and \(Q_V \in \mathbb{R}^{p \times q}\) be three matrices such that their columns form three orthonormal bases for \(U, U\perp\) and \(V\), respectively. It follows from [27, page 430] that \(P_U = Q_U Q_{U\perp}^*, I - P_U = P_{U\perp} = Q_{U\perp} Q_U^*\) and \(P_V = Q_V Q_V^*\). Furthermore, by [8, Theorem 1] we have the Singular Valued Decomposition (SVD) of the \(p \times q\) matrix \(Q_U^* Q_V\) is

\[
Q_U^* Q_V = A C B^* \quad \text{with} \quad C = \text{diag} \left( \cos \theta_1, \ldots, \cos \theta_p \right) \in \mathbb{R}^{p \times p},
\]

where \(A \in \mathbb{R}^{p \times p}\) and \(B \in \mathbb{R}^{q \times q}\) satisfy \(AA^* = A^* A = B^* B = I_p\). Since all \(p\) columns of \(B\) are orthonormal and \(p \leq q\), we may find a \(q \times (q - p)\) matrix \(B'\) such that \(B := (B, B') \in \mathbb{R}^{q \times q}\) is
Moreover, we get from (40) that
\[ P_U Q_V = Q_U Q_U^* Q_V = Q_U A^* B = D_1 C B^*. \]
Moreover, we get from (40) that
\[
[Q_{U^\perp}^* Q_V] [Q_{U^\perp}^* Q_V] = Q_{U^\perp}^* Q_{U^\perp} Q_V = Q_{U^\perp}^* (\text{Id} - P_U) Q_V = \text{Id} - Q_{U^\perp}^* P_U Q_V \\
= \text{Id} - Q_{U^\perp}^* Q_{U^\perp} Q_V = \text{Id} - (ACB^*)^* (ACB^*) = \text{Id} - B C A^* A C B^* \\
= \text{Id} - B C^2 B^* = B \overline{B} - B \begin{pmatrix} C^2 & 0 & 0 \\ 0 & 0_{q-p} & 0 \end{pmatrix} B^* = B \begin{pmatrix} I_p - C^2 & 0 & 0 \\ 0 & 0_{q-p} & 0 \end{pmatrix} B^*
\]
Hence the columns of $\overline{B}$ are eigenvectors of $[Q_{U^\perp}^* Q_V] [Q_{U^\perp}^* Q_V]$. It follows that the SVD of $Q_{U^\perp}^* Q_V$ has the form
\[ Q_{U^\perp}^* Q_V = A_1 \begin{pmatrix} S & 0 \\ 0_{I_{q-p}} & \end{pmatrix} B^* \]
for some $A_1 \in \mathbb{R}^{(n-p)\times q}$ with $A_1^t A_1 = I_q$. Define $D_2 := Q_{U^\perp} A_1 \in \mathbb{R}^{n\times q}$, we have $D_2 D_2^* = A_1^t Q_{U^\perp}^* Q_{U^\perp} A_1 = A_1^t A_1 = I_q$. Moreover, it follows from (43) that
\[ (I - P_U) Q_V = Q_{U^\perp} Q_{U^\perp}^* Q_V = D_2 \begin{pmatrix} S & 0 \\ 0_{I_{q-p}} & \end{pmatrix} B^*. \]
Note further that $D_1^t D_2 = A^* Q_{U^\perp}^* Q_{U^\perp} A_1 = 0$, since the columns of $Q_{U^\perp} Q_{U^\perp}^*$ are two basis of $U$ and $U^\perp$, respectively. Thus there is an $n \times (n - p - q)$ matrix $D_3$ such that $D := (D_1, D_2, D_3) \in \mathbb{R}^{n\times n}$ is orthogonal. Combining (41) and (44) gives us that
\[ Q_V = D_1 C B^* + D_2 \begin{pmatrix} S & 0 \\ 0_{I_{q-p}} & \end{pmatrix} B^* = D_1 \begin{pmatrix} C & 0_{p\times(q-p)} \end{pmatrix} B^* + D_2 \begin{pmatrix} S & 0 \\ 0_{I_{q-p}} & \end{pmatrix} B^*. \]
Hence we have
\[ P_V = Q_V Q_V^* = \left[ D_1 \begin{pmatrix} C & 0_{p\times(q-p)} \end{pmatrix} B^* + D_2 \begin{pmatrix} S & 0 \\ 0_{I_{q-p}} & \end{pmatrix} B^* \right] \cdot \left[ B \begin{pmatrix} C & \end{pmatrix} D_1^* + B \begin{pmatrix} S & \end{pmatrix} D_2^* \right] \\
= D_1 C^2 D_1^* + D_1 \begin{pmatrix} C & 0_{p\times(q-p)} \end{pmatrix} D_2^* + D_2 \begin{pmatrix} S C & 0_{I_{q-p}} \end{pmatrix} D_1^* + D_2 \begin{pmatrix} S^2 & 0 \\ 0 & 0_{I_{q-p}} \end{pmatrix} D_2^* \\
= D \begin{pmatrix} C^2 & C S & 0 & 0 \\ C S & S^2 & 0 & 0 \\ 0 & 0 & I_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{pmatrix} D^*, \]
which ensures the the second part of (36). Note further that $D_1 D_1^* = Q_U A (Q_U A)^* = Q_U A A^* Q_U^* = Q_U Q_U^* = P_U$. It follows that
\[ P_U = D \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & I_{p} & 0 & 0 \\ 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{pmatrix} D^*, \]

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Throughout this subsection let us denote the classical alternating projection mapping 

\[ (P_U P_V)^k = D \begin{pmatrix} C^k & C^{2(k-1)}CS & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{n-2p} \end{pmatrix} D^* \to D \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0_{n-s} & 0 \end{pmatrix} D^* \quad \text{as} \quad k \to \infty. \]

It is worth noting that the case \( \mu > 0 \) when \( \mu \in \mathbb{R} \).

Remark 3.5 When \( p + q < n \), observe from (38), (34), and Proposition 3.3 that \( \gamma(P_U P_V) = \gamma(P_{U \cap V}) = c_F^2(U, V) \). These equalities is also true when \( p + q \geq n \) by applying the trick used in Case 2 in the proof of Theorem 3.6. It follows that \( c_F(U, V) = c_F(U^+, V^+) \) by replacing \( U, V \) by \( U^+, V^+ \), respectively. This equality is known as Solmon’s formula; see [13, Theorem 16] and also [29, Theorem 3] for different proofs.

### 3.1 Convergence rate of relaxed alternating projection methods

Throughout this subsection let us denote the classical alternating projection mapping by \( T := P_U P_V \), which is well-known to be convergent to \( P_{U \cap V} \) with the linear rate \( c_F^2(U, V) = \cos^2 \theta_{s+1} \) with \( s = \dim(U \cap V) \); see [13, 22]. We will study some relaxations of this operator and show that a better optimal rate can be obtained. We say the relaxed alternating projection mapping defined by

\[ T_{\mu} := (1 - \mu) \text{Id} + \mu P_U P_V \quad \text{with} \quad \mu \in \mathbb{R}. \]

It is worth noting that the case \( \mu = 0 \) is not interesting, since \( T_0 = \text{Id} \) is the identity map. Let us analyze the convergence of \( T_{\mu} \) in the following result mainly for the case \( \mu \neq 0 \). When \( \mu = 1 \), it recovers the classical result aforementioned.

**Theorem 3.6 (relaxed alternating projection)** Let \( \theta_{s+1} = \theta_F \) be defined in Proposition 3.3 with \( s = \dim(U \cap V) \). Then the mapping \( T_{\mu} := (1 - \mu) \text{Id} + \mu P_U P_V, \mu \in \mathbb{R} \) is convergent if and only if \( \mu \in [0, 2) \).

Moreover, the following assertions hold:

(i) If \( \mu \in (0, \frac{2}{1 + \sin^2 \theta_{s+1}}) \), then \( T_{\mu} \) is convergent to \( P_{U \cap V} \) with the optimal rate \( \gamma(T_{\mu}) = 1 - \mu \sin^2 \theta_{s+1} \).

(ii) If \( \mu \in (\frac{2}{1 + \sin^2 \theta_{s+1}}, 2) \), then \( T_{\mu} \) is convergent to \( P_{U \cap V} \) with the optimal rate \( \gamma(T_{\mu}) = \mu - 1 \).

Consequently, when \( \mu \neq 0 \), \( T_{\mu} \) is convergent to \( P_{U \cap V} \) with rate smaller than \( \cos^2 \theta_{s+1} \) and only if \( \mu \in (1, 2 - \sin^2 \theta_{s+1}) \). Furthermore, \( T_{\mu} \) attains the smallest convergence rate \( \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}} \) at \( \mu = \frac{2}{1 + \sin^2 \theta_{s+1}} \).

**Proof.** Let us justify the theorem by considering two main cases as follows.

**Case 1:** \( p + q < n \), where \( 1 \leq p = \dim U \leq q = \dim V \leq n - 1 \). By Proposition 3.3 (36) and (38), we may find some orthogonal matrix \( D \) such that

\[ T_{\mu} = (1 - \mu) \text{Id} + \mu P_U P_V = D \begin{pmatrix} (1 - \mu)I_p + \mu C^2 & \mu CS & 0 \\ 0 & (1 - \mu)I_p & 0 \\ 0 & 0 & (1 - \mu)I_{n-2p} \end{pmatrix} D^* \]

\[ = D \begin{pmatrix} I_p - \mu S^2 & \mu CS & 0 \\ 0 & (1 - \mu)I_p & 0 \\ 0 & 0 & (1 - \mu)I_{n-2p} \end{pmatrix} D^*. \]
It follows that
\[
\sigma(T_\mu) = \{1 - \mu \sin^2 \theta_k \mid k = 1, \ldots, p\} \cup \{1 - \mu\}.
\]

Suppose first that $T_\mu$ is convergent, we get from Fact 2.4 that $\rho(T_\mu) \leq 1$ and $-1 \not\in \sigma(T_\mu)$. Thus we have $|1 - \mu| \leq 1$ and $-1 \neq 1 - \mu$, which yield $0 \leq \mu < 2$. Conversely, suppose that $0 \leq \mu < 2$ and observe from Proposition 3.3 that
\[
1 = 1 - \mu \sin^2 \theta_1 = \ldots = 1 - \mu \sin^2 \theta_s > 1 - \mu \sin^2 \theta_{s+1} \geq \ldots \geq 1 - \mu \sin^2 \theta_p \geq 1 - \mu > -1.
\]

If $\mu = 0$ then $T_\mu = \text{Id}$ is always convergent. If $\mu > 0$ and $s = 0$, it is clear that $1 \not\in \sigma(T_\mu)$ by (49). Thus $T_\mu$ is convergent by Fact 2.4. If $\mu > 0$ and $s > 0$, we claim that $1 \in \sigma(T_\mu)$ is semisimple. Indeed, observe from (48) that
\[
\ker(T_\mu - \text{Id}) = D\left(\ker(-\mu S^2)\right) = D\left(\mathbb{R}^s \oplus \mathbb{R}^{n-p} \times \mathbb{R}^1\right).
\]

Similarly we also have
\[
\ker(T_\mu - \text{Id})^2 = D\left(\ker(-\mu^2 S^4)\right) = D\left(\mathbb{R}^s \oplus \mathbb{R}^{n-p} \times \mathbb{R}^1\right).
\]

It follows from (50) and (51) that $1$ is semisimple to $T_\mu$ due to Fact 2.3. This tells that $T_\mu$ is convergent by Fact 2.4. Thus $T_\mu = (1 - \mu) \text{Id} + \mu P_\nu P_\nu$, $\mu \in \mathbb{R}$ is convergent if and only if $\mu \in [0, 2)$.

Next let us justify (i) and (ii) under the assumption that $\mu \in (0, 2)$. We claim first that $T_\mu$ is convergent to $P_{U \cap V}$. Indeed, note that
\[
\text{Fix } T_\mu = \ker[\mu(P_\nu P_\nu - \text{Id})] = \ker(P_\nu P_\nu - \text{Id}) = \text{Fix}(P_\nu P_\nu) = U \cap V.
\]

Furthermore, we have
\[
\text{Fix } T_\mu^* = \ker[\mu(P_\nu P_\nu - \text{Id})] = \ker(P_\nu P_\nu - \text{Id}) = \text{Fix}(P_\nu P_\nu) = V \cap U,
\]

which yields in turn the equality $\text{Fix } T_\mu = \text{Fix } T_\mu^*$. By Corollary 2.7 the mapping $T_\mu$ is convergent to $P_{U \cap V}$.

Now we justify the quantitative characterizations in (i) and (ii). Observe from (49) that the subdominant eigenvalue of $T_\mu$ is
\[
\gamma(T_\mu) = \max\{|1 - \mu \sin^2 \theta_{s+1}|, |1 - \mu|\}.
\]

Note also that
\[
(1 - \mu \sin^2 \theta_{s+1})^2 - (1 - \mu)^2 = \mu \cos^2 \theta_{s+1}[2 - \mu(1 + \sin^2 \theta_{s+1})].
\]

**Subcase a:** $\cos^2 \theta_{s+1} = 0$. Then we have $\theta_{s+1} = \ldots = \theta_p = \pi$ and $\gamma(T_\mu) = |1 - \mu|$. In this case it is easy to see that $CS = 0$ and thus $T_\mu$ is diagonalizable by (48). Thanks to Example 2.9 we have $T_\mu$ is convergent with optimal rate $|1 - \mu|$. Both (i) and (ii) are valid in this case.

**Subcase b:** $\cos^2 \theta_{s+1} > 0$. Let us consider the following three subsubcases:

**Subsubcase b1:** $\mu \in (0, \frac{2}{\sin^2 \theta_{s+1}+1})$. Then we have $|1 - \mu \sin^2 \theta_{s+1}| > |1 - \mu|$ by (53) and thus $\gamma(T_\mu) = |1 - \mu \sin^2 \theta_{s+1}|$. Observe that
\[
1 > a_\mu := 1 - \mu \sin^2 \theta_{s+1} > 1 - \frac{2 \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}} = \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}} \geq 0.
\]
Hence we have \( \gamma(T_\mu) = 1 - \mu \sin^2 \theta_{s+1} \). Suppose further that \( \theta_{s+1} = \ldots = \theta_k \) and \( \theta_{s+1} \neq \theta_{k+1} \) with some \( k \in \{s+1, \ldots, p\} \), we easily check from (48) that
\[
\ker(T_\mu - a_\mu \text{Id}) = \ker(T_\mu - a_\mu \text{Id})^2 = D (0_{1 \times s} \times (\mathbb{R}^{k-s})^* \times 0_{1 \times (n-k)})^*,
\]
which shows that \( a_\mu \) is semisimple by Fact 2.3. Thanks to Theorem 2.13, \( T_\mu \) is convergent with the optimal rate \( a_\mu \).

Subsubcase b2: \( \mu = \frac{2}{1 + \sin^2 \theta_{s+1}} > 1 \). Then we obtain from (52) that
\[
\gamma(T_\mu) = |1 - \mu \sin^2 \theta_{s+1}| = |1 - \mu| = 1 - \mu \sin^2 \theta_{s+1} = \mu - 1 = \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}}.
\]
It is similar to the above subcase that \( a_\mu \in \sigma(T_\mu) \) is semisimple. Furthermore, \( 1 - \mu \in \sigma(T_\mu) \) is also semisimple. Indeed, observe that
\[
T_\mu - (1 - \mu) \text{Id} = D \begin{pmatrix} \mu C^2 & \mu CS \sin 2\theta_{s+1} \sin \theta_{s+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} D^*, \quad (T_\mu - (1 - \mu) \text{Id})^2 = D \begin{pmatrix} \mu^2 C^4 & \mu^2 C^3 S \sin 2\theta_{s+1} \sin \theta_{s+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} D^*.
\]
By using these two expressions, we may check that
\[
\ker(T_\mu - (1 - \mu) \text{Id}) = \ker(T_\mu - (1 - \mu) \text{Id})^2,
\]
which yields that \( (1 - \mu) \) is also semisimple by Fact 2.3. By Theorem 2.13 again, we obtain that \( \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}} \) is the optimal convergent rate of \( T_\mu \).

Subsubcase b3: \( \mu > \frac{2}{1 + \sin^2 \theta_{s+1}} > 1 \). It follows from (53) that \( |1 - \mu \sin^2 \theta_{s+1}| < |1 - \mu| \). And thus we get from (52) that
\[
\gamma(T_\mu) = |1 - \mu| = \mu - 1 > \frac{2}{1 + \sin^2 \theta_{s+1}} - 1 = \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}}.
\]
Similarly to the above case, \( 1 - \mu \in \sigma(T_\mu) \) is semisimple. Thus Theorem 2.13 tells us that \( \mu - 1 \) is the convergent rate of \( T_\mu \) in this subcase.

Combining Subsubcase b1 and Subsubcase b2 ensures (i), and (ii) is exactly the Subsubcase b3. Thus (i) and (ii) are verified.

Let us complete the proof by verifying the last part of the theorem. When \( \mu \in (0, \frac{2}{1 + \sin^2 \theta_{s+1}}) \), we have \( 1 - \mu \sin^2 \theta_{s+1} < \cos^2 \theta_{s+1} \) if and only if \( \mu > 1 \), since \( \sin^2 \theta_{s+1} > 0 \) by Proposition 3.3. Furthermore, when \( \mu \in \left( \frac{2}{1 + \sin^2 \theta_{s+1}}, 2 \right) \), we have \( \mu - 1 < \cos^2 \theta_{s+1} \) if and only if \( \mu < 1 + \cos^2 \theta_{s+1} = 2 - \sin^2 \theta_{s+1} \). Combining these two observations with (i) and (ii) in the theorem tells us that \( T_\mu \) is convergent to \( P_{U' \cap V'} \) with a rate smaller than \( \cos^2 \theta_{s+1} \) if and only if \( \mu \in (1, 2 - \sin^2 \theta_{s+1}) \). Moreover, the optimal rate \( \frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}} \) of \( T_\mu \) is obtained at \( \mu = \frac{2}{1 + \sin^2 \theta_{s+1}} \) due to (54), (55), and (56).

**Case 2:** \( p + q \geq n \). We may find some \( k \in \mathbb{N} \) such that \( n' := n + k > p + q \). Define \( U' := U \times \{0_k\} \subset \mathbb{R}^n', \quad V' := V \times \{0_k\} \subset \mathbb{R}^n', \) and \( T'_\mu := (1 - \mu) \text{Id} + \mu P_{U'} P_{V'} \). It is clear that \( 1 \leq p = \dim U' \leq \dim V' = q \) and \( p + q < n' \). Observe from Definition 3.1 that the principal angles between \( U' \) and \( V' \) are the same with the ones between \( U \) and \( V \). Moreover, we have \( P_{U'} = \begin{pmatrix} P_U & 0 \\ 0 & 0_k \end{pmatrix} \), \( P_{V'} = \begin{pmatrix} P_V & 0 \\ 0 & 0_k \end{pmatrix} \), and thus
\[
T'_\mu = \begin{pmatrix} T_\mu & 0 \\ 0 & (1 - \mu) I_k \end{pmatrix}.
\]
Since \( q \leq n - 1 \), there is some \( x \in \mathbb{R}^n \setminus \{0\} \) such that \( P_V x = 0 \). It follows that \( Tx = 0 \), and thus we have \( 0 \in \sigma(T) \) and then \( 1 - \mu \in \sigma(T_p) \). If \( T_p \) is convergent, Fact 2.4 tells us that \(-1 < 1 - \mu \leq 1\), i.e., \( \mu \in [0, 2) \). Conversely, if \( \mu \in [0, 2) \) we have \( T_p' \) is convergent due to Case 1. This together with (57) ensures that \( T_p \) is also convergent. Hence \( T_p \) is convergent if and only if \( \mu \in [0, 2) \).

To verify the convergence rate of \( T_p' \), suppose further that \( \mu \in (0, 2) \). We note that \( \sigma(T_p) = \sigma(T_p') \), which implies in turn that \( \gamma(T_p) = \gamma(T_p') \). It follows from Case 1 that \( T_p' \) in (57) is convergent to \( P_{U \cap V'} = \begin{pmatrix} P_{U \cap V} & 0 \\ 0 & 0_k \end{pmatrix} \) with the convergence rate \( \gamma(T_p') \). This together with (57) yields

\[
\|T_p' - P_{U \cap V'}\| \leq \|(T_p')^n - P_{U \cap V'}\|.
\]

Thus \( \gamma(T_p') = \gamma(T_p) \) is the convergence rate of \( T_p \), which will give us a better optimal rate. Since the proof is similar, we only sketch the main steps.

Next we study another kind of relaxation of the map \( T = P_U P_V \), that is

\[
S_\mu := P_U((1 - \mu) I + \mu P_V) = (1 - \mu) P_U + \mu P_{U \cap V'},
\]

see also [26] for a similar form, which will give us a better optimal rate. Since the proof is similar to the one of Theorem 3.6 above, we only sketch the main steps.

**Theorem 3.7 (partial relaxed alternating projection)** The map \( S_\mu := P_U((1 - \mu) I + \mu P_V) = (1 - \mu) P_U + \mu P_{U \cap V} \) is convergent if and only if \( \mu \in \left(0, \frac{2}{\sin^2 \theta_p} \right] \) with the convention \( \frac{1}{0} = \infty \). Moreover, the following assertions hold:

(i) If \( \mu \in \left(0, \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} \right] \), then \( S_\mu \) is convergent to \( P_{U \cap V} \) with the optimal convergence rate \( \gamma(S_\mu) = 1 - \mu \sin^2 \theta_{s+1} \).

(ii) If \( \mu \in \left(\frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}, \frac{2}{\sin^2 \theta_p} \right) \), then \( S_\mu \) is convergent to \( P_{U \cap V} \) with the optimal convergence rate \( \gamma(S_\mu) = \mu \sin^2 \theta_p - 1 \).

Consequently, when \( \mu \neq 0 \), \( S_\mu \) is convergent to \( P_{U \cap V} \) with the optimal convergence rate smaller than \( \cos^2 \theta_{s+1} = c_F^2(U, V) \) if and only if \( \mu \in \left(1, \frac{2 - \sin^2 \theta_{s+1}}{\sin^2 \theta_p} \right) \). Furthermore, \( S_\mu \) attains the smallest convergence rate \( \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p} \) at \( \mu = \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} \).

**Proof.** We separate the proof into two main cases as below:

**Case 1:** \( p + q < n \) with \( 1 \leq p = \dim U \leq q = \dim V \leq n - 1 \). It follows from (36) and (38) that there is some orthogonal matrix \( D \in \mathbb{R}^{n \times n} \) such that

\[
S_\mu = D \begin{pmatrix} (1 - \mu) I_p + \mu C^2 & \mu CS & 0 \\ 0 & 0_p & 0 \\ 0 & 0 & 0_{n-2p} \end{pmatrix} D^* = D \begin{pmatrix} I_p - \mu S^2 & \mu CS & 0 \\ 0 & 0_p & 0 \\ 0 & 0 & 0_{n-2p} \end{pmatrix} D^*.
\]

Hence we have

\[
\sigma(S_\mu) = \{1 - \mu \sin^2 \theta_k | k = 1, \ldots, p\} \cup \{0\}.
\]

Suppose that \( S_\mu \) is convergent, we get from Fact 2.4 that

\[
-1 < 1 - \mu \sin^2 \theta_p \quad \text{and} \quad 1 - \mu \sin^2 \theta_{s+1} \leq 1.
\]
Since $\theta_{s+1} = \theta_F \neq 0$ by Proposition 3.3, the latter gives us that $\mu \in [0, \frac{2}{\sin^2 \theta_p}]$. Conversely, suppose that $\mu \in [0, \frac{2}{\sin^2 \theta_p})$, we have

(62) \quad 1 = 1 - \mu \sin^2 \theta_1 = \cdots = 1 - \mu \sin^2 \theta_s \geq 1 - \mu \sin^2 \theta_{s+1} \geq \cdots \geq 1 - \mu \sin^2 \theta_p > -1.

If $\mu = 0$ then $S_\mu = D$ is always convergent. If $\mu > 0$ and $s = 0$, it is clear that $1 \not\in \sigma(S_\mu)$ by (60). Thanks to Fact 2.4, we have $S_\mu$ is convergent. If $\mu > 0$ and $s > 0$, it is similar to the corresponding part of Theorem 3.6 that $1 \not\in \sigma(S_\mu)$ is semisimple. Combining (62) with Fact 2.4 gives us that $S_\mu$ is convergent. Thus $S_\mu$ is convergent if and only if $\mu \in [0, \frac{2}{\sin^2 \theta_p})$.

To verify (i) and (ii), assume further that $\mu \in (0, \frac{2}{\sin^2 \theta_p})$. Let us claim that $S_\mu$ is convergent to $P_{U\cap V}$. Via the explicit form of $S_\mu$ in (59), we can easily check that

$$\text{Fix } S_\mu = \ker(S_\mu - \text{Id}) = D((\mathbb{R}^s)^* \times \mathbb{R}^{(n-s)}) = \ker(S_\mu^* - \text{Id}) = \text{Fix } S_\mu^*.$$ 

Note also from (59) that

$$U \cap V = \text{Fix } P_{U\cap V} = D((\mathbb{R}^s)^* \times \mathbb{R}^{(n-s)})^*.$$ 

It follows that $\text{Fix } S_\mu = \text{Fix } S_\mu^* = U \cap V$. Thanks to Corollary 2.7, we have $S_\mu$ is convergent to $P_{U\cap V}$.

Next we justify the qualitative characterizations in (i) and (ii). Observe from (60) and (62) that

(63) \quad \gamma(S_\mu) = \max\{|1 - \mu \sin^2 \theta_{s+1}|, |1 - \mu \sin^2 \theta_p|\}.

Note also that

(64) \quad (1 - \mu \sin^2 \theta_{s+1})^2 - (1 - \mu \sin^2 \theta_p)^2 = \mu(\sin^2 \theta_p - \sin^2 \theta_{s+1})[2 - \mu(\sin^2 \theta_{s+1} + \sin^2 \theta_p)].

**Subcase a:** $\sin \theta_p = \sin \theta_{s+1}$, i.e., $\theta_{s+1} = \theta_{s+2} = \cdots = \theta_p$. Hence we have $\sigma(S_\mu) = \{1 - \mu \sin^2 \theta_p, 1 - \mu \sin^2 \theta_{s+1}, 0\}$ and $\gamma(S_\mu) = |1 - \mu \sin^2 \theta_{s+1}|$. Moreover, it is easy to check that $c_\mu := 1 - \mu \sin^2 \theta_{s+1}$ is semisimple by showing that $\ker(S_\mu - c_\mu \text{Id}) = \ker(S_\mu - c_\mu \text{Id})^2$.

**Subcase b:** $\sin \theta_p \neq \sin \theta_{s+1}$, i.e., $\sin \theta_p > \sin \theta_{s+1}$. We continue the proof by taking into account three different cases as follows.

**Subsubcase b1:** $\mu \in (0, \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p})$. Then we have from (64) that $|1 - \mu \sin^2 \theta_{s+1}| > |1 - \mu \sin^2 \theta_p|$, which gives us that $\gamma(S_\mu) = |1 - \mu \sin^2 \theta_{s+1}|$ by (63). Moreover, note that

(65) \quad c_\mu = 1 - \mu \sin^2 \theta_{s+1} > 1 - \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} \sin^2 \theta_{s+1} = \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} > 0.

Thanks to the structure of $S_\mu$ in (59), we may check that $c_\mu$ is semisimple. Thus $c_\mu = \gamma(S_\mu)$ is the optimal convergence rate of $S_\mu$ by Theorem 2.13.

**Subsubcase b2:** $\mu = \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$. Thus

(66) \quad \gamma(S_\mu) = |1 - \mu \sin^2 \theta_{s+1}| = |1 - \mu \sin^2 \theta_p| = \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} > 0.$
We can check that $c_\mu = 1 - \mu \sin^2 \theta_{s+1}$ and $d_\mu := 1 - \mu \sin^2 \theta_p$ are semisimple in this case via Fact 2.3. This together with Theorem 2.13 tells us that $\gamma(S_\mu) = 1 - \mu \sin^2 \theta_{s+1} = \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$ is the optimal linear rate of $S_\mu$.

Subsubcase b3: $\mu \in \left(\frac{\sin \theta_{s+1} + \sin \theta_p}{\sin \theta_{s+1}}\right)$. It follows from (64) that $|1 - \mu \sin^2 \theta_{s+1}| < |1 - \mu \sin^2 \theta_p|$, which yields $\gamma(S_\mu) = |1 - \mu \sin^2 \theta_p|$ by (63). Moreover, observe that

$$\mu \sin^2 \theta_p - 1 > \frac{2}{\sin \theta_{s+1} + \sin \theta_p} \sin^2 \theta_p - 1 = \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} > 0.$$  

We also have $d_\mu = 1 - \mu \sin^2 \theta_p$ is semisimple via Fact 2.3. Thanks to Theorem 2.13, $\gamma(S_\mu) = \mu \sin^2 \theta_p - 1$ is the optimal convergence rate of $S_\mu$.

Combining Subsubcase b1 and Subsubcase b2 gives us (i). Furthermore, Subsubcase b3 exactly verifies (ii). The last part of the theorem is indeed a direct consequence of (i) and (ii). The proof of the theorem for Case 1 is complete.

**Case 2:** $p + q \geq n$. Then we find some $k \in \mathbb{N}$ such that $n' := n + k > p + q$ and define $U' := U \times \{0\} \subset \mathbb{R}^n$, $V' := V \times \{0\} \subset \mathbb{R}^n$, and $S'_\mu = (1 - \mu)P_{U'} + \mu P_{V'}P_{V'}$. It is clear that $1 \leq p = \dim U' \leq \dim V' = q$ and $p + q < n'$. Moreover, we also have

$$S'_\mu = \begin{pmatrix} S_\mu & 0 \\ 0 & 0_k \end{pmatrix},$$

which shows that $S'_\mu$ is convergent if and only if $S_\mu$ is convergent. The rest of the proof is quite similar to the corresponding one in Theorem 3.6. ■

**Remark 3.8** It is clear that the optimal rate $\frac{\sin \theta_p - \sin \theta_{s+1}}{\sin \theta_{s+1} + \sin \theta_p}$ of $S_\mu$ is smaller than the one $\frac{1 - \sin^2 \theta_{s+1}}{1 + \sin^2 \theta_{s+1}}$ of $T_\mu$ in Theorem 3.6. Note further from the above theorem that $S_2 = P_U P_V$ with $R_V := 2P_V - \text{Id}$, which is known as the **reflection-projection** method [7, 9] is convergent to $P_{U \cap V}$ if and only if $2 < \frac{2}{\sin \theta_p}$, i.e., $\theta_p < \frac{\pi}{2}$. When this case is fulfill, the optimal rate of the reflection-projection method is $\max\{|1 - 2 \sin^2 \theta_{s+1}|, |1 - 2 \sin^2 \theta_p|\}$ by (63). Besides the definition of $\theta_{s+1}, \theta_p$ in Definition 3.1 and Definition 3.2, we may also obtain $\phi_{s+1}, \phi_p$ in following formulas

$$\cos^2 \theta_{s+1} = \|P_UP_V - P_{U \cap V}\|$$ and $\sin^2 \theta_p = \|P_U - P_UP_V\|^2 = \|P_U - P_UP_VP_U\|$ from (36), (38), and (39).

**Remark 3.9 (finite termination)** From Theorem 3.6, observe that the map $T_\mu$ has the convergence rate 0, i.e., it will always terminate after finite powers if and only if $\theta_{s+1} = \frac{\pi}{2}$ and $\mu = 1$. Similarly, we get from Theorem 3.7 that $S_\mu$ has the convergence rate 0 if and only if $\mu = \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$ and $\phi_{s+1} = \phi_p$. The latter condition is clearly satisfied when $\dim(U \cap V) = p - 1$ and $\mu = \frac{1}{\sin \theta_{s+1}}$; e.g., $U$ and $V$ are two different lines passing the origin in $\mathbb{R}^2$, or $U$ is a line in $\mathbb{R}^3$ and $V$ is a hyperplane in $\mathbb{R}^3$ with $U \not\subset V$, or $U$ and $V$ are two different hyperplanes in $\mathbb{R}^3$, etc.

### 3.2 Convergence rate of the generalized Douglas-Rachford method

Convergence rate of many specific matrices relating to Douglas-Rachford operator

$$R := P_UP_V + P_{U \perp}P_{V \perp} = \frac{R_U R_V + \text{Id}}{2} = \frac{R_{U \perp} R_{V \perp} + \text{Id}}{2}$$
has been discussed in [12]. One of the particular cases there is the so-called generalized Douglas-Rachford operator $R_\mu$, defined by

$$R_\mu := (1 - \mu) \text{Id} + \mu R.$$  

Convergence rate of this mapping has been obtained in [12] under an additional condition $U \cap V = \{0\}$. In the following result we give a complete characterization of the convergence of this map and also show that the condition $U \cap V = \{0\}$ can be relaxed.

**Theorem 3.10** The map $R_\mu$ is convergent if and only if $\mu \in [0, 2)$. Moreover, the following assertions hold:

(i) $R_\mu$ is normal.

(ii) If $\mu \in (0, 2)$ then $R_\mu$ is convergent to $P_{\text{Fix} R} = P_{(U \cap V) \oplus (U^\perp \cap V^\perp)}$ with the optimal convergence rate $\gamma(R_\mu) = \sqrt{\mu(2 - \mu)\cos^2 \theta_{s+1} + (1 - \mu)^2}$, where $s := \dim(U \cap V)$.

**Proof.** As proceeded in the proof of Theorem 3.6 and Theorem 3.7 we consider two major cases as below.

**Case 1.** $p + q < n$. By using the expressions of (38), we easily establish that

$$R_\mu = D \begin{pmatrix} C^2 + (1 - \mu)S^2 & \mu CS & 0 & 0 \\ -\mu CS & C^2 + (1 - \mu)S^2 & 0 & 0 \\ 0 & 0 & (1 - \mu)I_{q-p} & 0 \\ 0 & 0 & 0 & I_{n-p-q} \end{pmatrix} D^*$$

(70)

$$= D \begin{pmatrix} I_p - \mu S^2 & \mu CS & 0 & 0 \\ -\mu CS & I_p - \mu S^2 & 0 & 0 \\ 0 & 0 & (1 - \mu)I_{q-p} & 0 \\ 0 & 0 & 0 & I_{n-p-q} \end{pmatrix} D^*;$$

see also a similar form on [12, page 14]. It is easy to check that $R_\mu^* R_\mu = R_\mu R_\mu^*$, i.e., $R_\mu$ is normal. Thus (i) is satisfied. We may get from the above format and the block determinant formula, c.f., [27, page 475] that

$$\sigma(R_\mu) = \{ \cos^2 \theta_k + (1 - \mu) \sin^2 \theta_k \pm i \mu \cos \theta_k \sin \theta_k \mid k = 1, \ldots, p \} \cup \{1\} \quad \text{if} \quad q = p,$n

$$\{ \cos^2 \theta_k + (1 - \mu) \sin^2 \theta_k \pm i \mu \cos \theta_k \sin \theta_k \mid k = 1, \ldots, p \} \cup \{1\} \cup \{1 - \mu\} \quad \text{if} \quad q > p,$n

where $i := \sqrt{-1}$. For any $k = 1, \ldots, p$, we have

$$|1 - \mu \sin^2 \theta_k \pm i \mu \cos \theta_k \sin \theta_k| = \sqrt{(1 - \mu \sin^2 \theta_k)^2 + [\mu \cos \theta_k \sin \theta_k]^2} \leq \sqrt{\mu \cos^2 \theta_k + (1 - \mu)^2 + \mu^2 \cos^2 \theta_k (1 - \cos^2 \theta_k)} = \sqrt{\mu(2 - \mu) \cos^2 \theta_k + (1 - \mu)^2}.$$n

Suppose further that $R_\mu$ is convergent. Then we get from Fact 2.4 that

$$\mu(2 - \mu) \cos^2 \theta_{s+1} + (1 - \mu)^2 \leq 1,$n

which yields $\mu(2 - \mu)(1 - \cos^2 \theta_{s+1}) \geq 0$ and thus $\mu \in [0, 2]$, since $\cos^2 \theta_{s+1} < 1$. Next let us consider three particular subcases of $\mu$.

**Subcase a.** $\mu = 2$. Then all eigenvalues of $R_\mu$ have magnitude 1. By Fact 2.4, we have

$$1 - \mu \sin^2 \theta_k \pm i \mu \cos \theta_k \sin \theta_k = 1 \quad \text{for all} \quad k = 1, \ldots, p,$n

(71)
which implies in turn that \( \sin \theta_{s+1} \cos \theta_{s+1} = 0 \) and thus \( \theta_{s+1} = \frac{\pi}{2} \), since \( \sin \theta_{s+1} > 0 \) by Proposition 3.3. It follows that

\[
1 - \mu \sin^2 \theta_{s+1} \pm i \mu \cos \theta_{s+1} \sin \theta_{s+1} = -1,
\]

which contradicts (71). Hence when \( \mu = 2 \), \( R_\mu \) is not convergent.

Subcase b: \( \mu = 0 \). It is obvious that \( R_\mu = \text{Id} \) is convergent to \( \text{Id} \) with rate 0.

Subcase c: \( 0 < \mu < 2 \). By Proposition 3.3, we have

\[
1 = \sqrt{\mu (2 - \mu) \cos^2 \theta_1 + (1 - \mu)^2} = \ldots = \sqrt{\mu (2 - \mu) \cos^2 \theta_s + (1 - \mu)^2}
\]

(72)

\[
> \sqrt{\mu (2 - \mu) \cos^2 \theta_{s+1} + (1 - \mu)^2} \geq \sqrt{\mu (2 - \mu) \cos^2 \theta_{s+2} + (1 - \lambda)^2}
\]

\[
\geq \ldots \geq \sqrt{\mu (2 - \mu) \cos^2 \theta_p + (1 - \mu)^2} \geq |1 - \mu|.
\]

Since \( R_\mu \) is normal, it follows from Fact 2.4 and Corollary 2.7 that \( R_\mu \) is convergent. Hence we have \( R_\mu \) is convergent if and only if \( \mu \in (0, 2) \).

It remains to verify (ii) in this case. Suppose that \( \mu \in (0, 2) \), we get from the normality of \( R_\mu \) and Theorem 2.16 that \( \gamma (R_\mu) = \sqrt{\mu (2 - \mu) \cos^2 \theta_{s+1} + (1 - \mu)^2} \) (by (72)) is the optimal convergence rate of \( R_\mu \) and that \( R_\mu \) is convergent to \( P_{\text{Fix } R_\mu} = P_{\text{Fix } R} \). Moreover, we have \( \text{Fix } R = (U \cap V) \oplus (U^\perp \cap V^\perp) \) by [5] Proposition 3.6. This ensures (ii) and thus completes the proof of the theorem for Case 1.

Case 2: \( p + q \geq n \). Similarly to the proof of Theorem 3.6 and Theorem 3.7, we find \( k > 0 \) such that \( n + k := n' > p + q \). Define further that \( U' := U \times \{0_k\} \subset \mathbb{R}^{n'} \), \( V' := V \times \{0_k\} \subset \mathbb{R}^{n'} \), and \( R'_\mu = (1 - \mu) \text{Id} + \mu [P_{U'} P_{V'} + P_{(V')^\perp} P_{(V')^\perp}] \). It is easy to verify that

\[
R'_\mu = \begin{pmatrix} R_\mu & 0 \\ 0 & I_k \end{pmatrix}.
\]

(73)

Note from Case 1 that \( R'_\mu \) is normal, and so is \( R_\mu \). Moreover, we get from (73) that \( R_\mu \) is convergent if and only if \( R'_\mu \) is convergent with the same rate. The analysis of the convergence of \( R'_\mu \) in Case 1 justifies all the statement of the theorem in this case. The proof is complete. \( \blacksquare \)

Remark 3.11 (1). Unlike the relaxed alternating projection methods studied in Theorem 3.6 and 3.7, convergence rate of the (over and under) relaxation of the Douglas-Rachford algorithm is always bigger than the original one due to

\[
\gamma (R_1) = \cos \theta_{s+1} \leq \sqrt{\mu (2 - \mu) \cos^2 \theta_{s+1} + (1 - \mu)^2} = \gamma (R_\mu) \quad \text{for all } \mu \in [0, 2).\]

Moreover, it is worth mentioning here that Theorem 3.10 also tells us that \( R_2 = R_{U} R_{V} \), which is known as reflection-reflection method will never be convergent in the case of two nontrivial subspaces with \( 1 \leq \dim U, \dim V \leq n-1 \).

(2). For the convergence rate of the Douglas-Rachford method on a general Hilbert space, see [5].

4 A nonlinear approach to the alternating projection method

Throughout this section, we also suppose that \( U \) and \( V \) are two subspaces of \( \mathbb{R}^n \) with \( 1 \leq p = \dim U \leq \dim V = q \leq n - 1 \). From Theorem 3.7, we know that the map \( S_\mu \) (58) obtains its
smallest rate \( \frac{\sin^2 \theta_p - \sin^2 \theta_{p+1}}{\sin^2 \theta_{p+1} + \sin^2 \theta_p} \) at \( \mu = \frac{2}{\sin^2 \theta_{p+1} + \sin^2 \theta_p} \). This rate is smaller than the optimal rate of \( T_\mu \) and \( T \). However, it is not trivial to determine \( \theta_{p+1} \) and \( \theta_p \) to construct \( \mu = \frac{2}{\sin^2 \theta_{p+1} + \sin^2 \theta_p} \) for \( S_\mu \) especially with big dimensions of \( U \) and \( V \); see Definition 3.1 Definition 3.2 and (68). In this section we introduce a simple nonlinear mapping, by using the idea of a line search [6, 18, 20] for the map \( S_\mu \), so that the iterative sequence given by this nonlinear mapping is linearly convergent to the projection on \( U \cap V \) with the same optimal rate mentioned above. One may think of this mapping as the partial relaxed alternating projection with an adaptive parameter \( \mu(x) \) depending on each iteration period. This is a technique employed for other iterative methods, see, e.g., [3, 4, 9, 10, 11, 17].

**Definition 4.1** Define the map \( B_T \) with \( T = P_UP_V \) by

\[
B_T(x) := P_U((1 - \mu)x + \mu P_Vx) = (1 - \mu)pUx + \mu pUP_Vx,
\]

where

\[
\mu_x := \begin{cases} 
\frac{(p_Ux - p_UP_Vx)}{||p_Ux - p_UP_Vx||^2} & \text{if } p_Ux - p_UP_Vx \neq 0 \\
1 & \text{if } p_Ux - p_UP_Vx = 0.
\end{cases}
\]

**Remark 4.2** In [4, 6, 18], an accelerated mapping of \( T \) is introduced by using the line-search [20] as

\[
A_T(x) := (1 - \lambda_x)x + \lambda_x p_UP_Vx,
\]

where

\[
\lambda_x = \begin{cases} 
\frac{x - p_UP_Vx}{||x - p_UP_Vx||^2} & \text{if } x - p_UP_Vx \neq 0 \\
1 & \text{if } x - p_UP_Vx = 0.
\end{cases}
\]

It is worth noting that \( \mu_x = \lambda_x \) and \( B_Tx = A_Tx \) when \( x \in U \).

Set \( M := U \cap V \). The proof of the following convenient fact can be found in [14, Lemma 9.2]

\[
P_UP_VP_M = P_MP_U = P_VP_M = P_MP_V = P_M.
\]

The main result in this section is Theorem 4.3 before proving it we provide two useful lemmas.

**Lemma 4.3** For each \( x \in \mathbb{R}^n \) and \( y \in U \cap V \) we have

\[
\min_{\mu \in \mathbb{R}} \|(1 - \mu)p_Ux + \mu p_UP_Vx - y\| = \|B_Tx - y\|.
\]

Moreover, \( \mu_x \) given in (75) is the unique minimizer when \( p_Ux - p_UP_Vx \neq 0 \).

**Proof.** When \( p_Ux = p_UP_Vx \), inequality (79) is trivial. Now suppose that \( p_Ux \neq p_UP_Vx \) and note that

\[
\|(1 - \mu)p_Ux + \mu p_UP_Vx - y\|^2 = \|(1 - \mu)(p_Ux - y) + \mu (p_UP_Vx - y)\|^2
\]

\[
= (1 - \mu)\|p_Ux - y\|^2 + \mu\|p_UP_Vx - y\|^2 - \mu(1 - \mu)\|p_Ux - p_UP_Vx\|^2.
\]

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This is a quadratic in $\mu$ and thus attains its minimum at the following unique minimizer

\[
\mu = \frac{1}{2} \left( \frac{\|P_Ux - y\|^2 - \|P_UP_Vx - y\|^2 + \|P_Ux - P_UP_Vx\|^2}{\|P_Ux - P_UP_Vx\|^2} \right).
\]

(81)

Since $y \in U \cap V$, we derive that

\[
\langle y, P_Ux - P_UP_Vx \rangle = \langle y, P_UP_Vx \rangle = \langle P_Uy, P_Vx \rangle = \langle y, P_Vx \rangle = 0.
\]

Moreover, note that $\langle P_Ux, P_Ux - P_UP_Vx \rangle = \langle x, P_Ux - P_UP_Vx \rangle$. This together with (80), (81) and (82) tells us that the left-hand side of (79) attains its minimum at $\mu_x$ in (75). We verify (79) and complete the proof of the lemma.

**Lemma 4.4** For any $\mu \in \mathbb{R}$ and $x \in U$ we have

\[
(1 - \mu)P_U + \mu P_UP_VP_U - P_{U\cap V} = \left( (1 - \mu)P_U + \mu P_UP_VP_U - P_M \right)(x - P_Mx),
\]

where $S_\mu = (1 - \mu)P_U + \mu P_UP_V$ defined in Theorem 3.7. Moreover, when $\mu = \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$ with $s = \dim(U \cap V)$ and $\theta_{s+1}, \theta_p$ found in Definition 3.1 we have

\[
\|P_Ux - P_UP_VP_U - P_{U\cap V}\| = \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}.
\]

Proof. Set $M := U \cap V$. For any $x \in U$ we get from (78) that

\[
((1 - \mu)P_U + \mu P_UP_VP_U - P_M)(x - P_Mx) = ((1 - \mu)P_U + \mu P_UP_VP_U)x - P_Mx - P_Mx + P^2_Mx
\]

\[
= (1 - \mu)P_Ux + \mu P_UP_Vx - P_Mx
\]

\[
= S_\mu x - P_Mx,
\]

which verifies (83). To justify (84), without loss of generality, suppose that $p + q < n$ (otherwise, we follow the trick used in the proof of Case 2 of Theorem 3.7). It is easy to check from (36) and (39) that

\[
(1 - \mu)P_U + \mu P_UP_VP_U - P_M = D \left( (1 - \mu)I_p + \mu C^2 - \begin{pmatrix} I_s & 0 \\ 0 & 0_{p-s} \end{pmatrix} \right) D^*
\]

\[
= D \begin{pmatrix} 0_s & 1 - \mu \sin^2 \theta_{s+1} & \cdots & 1 - \mu \sin^2 \theta_p \\ 0 & 0_{n-p} \end{pmatrix} D^*
\]

for some orthogonal matrix $D \in \mathbb{R}^{n \times n}$. When $\mu = \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$, we get that

\[
\|P_Ux - P_UP_VP_U - P_{U\cap V}\| = \max \left\{ 1 - \mu \sin^2 \theta_p, 1 - \mu \sin^2 \theta_{s+1} \right\}
\]

\[
= \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}.
\]

This ensures (84) and completes the proof of the lemma. ■

We are ready to establish the main result of this section as follows.
Theorem 4.5 For any $n \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we have

$$
\|B^n_T(x) - P_{U \cap V}x\| \leq \left[ \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p + \sin^2 \theta_{s+1}} \right]^{n+1} \|x - P_{U \cap V}x\|,
$$

where $\theta_{s+1}$ and $\theta_p$ are the principal angles found in Definition 3.1. Hence the algorithm $B^n_T(x) \to P_{U \cap V}(x)$ is at least as fast as the partial relaxed projection \[58\]. Furthermore, when $x \in U$ we obtain a sharper inequality

$$
\|B^n_T(x) - P_{U \cap V}x\| \leq \left[ \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p + \sin^2 \theta_{s+1}} \right]^{n+1} \|x - P_{U \cap V}x\|.
$$

Proof. For any $x \in \mathbb{R}^n$, define $y = P_{U \cap V}x$ and $M = U \cap V$, note that $y = P_M B_T x$. Fix $\mu := \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$ and $\gamma := \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$. We obtain

$$
\|B^n_T(x) - P_{U \cap V}x\| \leq \|B_T(B^n_T(x)) - y\| \leq \|S_\mu(B^n_T(x)) - y\| \quad \text{(by } 79\text{)}
$$

$$
= \|(1 - \mu)P_U + \mu P_U P_V P_U - P_M)(B^n_T(x) - y)\| \quad \text{(by } 83\text{ and } B^n_T x \in U\text{)}
$$

$$
\leq \|(1 - \mu)P_U + \mu P_U P_V P_U - P_M\| \cdot \|B^n_T(x) - y\| \quad \text{(by } 84\text{)}
$$

$$
= \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} \|B^n_T(x) - y\| \quad \text{(by } 84\text{)}
$$

$$
\leq \ldots
$$

$$
\leq \gamma^n \|B_T(x) - y\| \leq \gamma^n \|S_1(x) - y\| \quad \text{(by } 79\text{ again)}
$$

$$
= \gamma^n \|(P_U P_V - P_M)(x - P_M x)\| \quad \text{(by } 78\text{)}
$$

$$
\leq \gamma^n \|P_U P_V - P_M\| \cdot \|x - P_M x\| = \gamma^n \cos^2 \theta_{s+1} \|x - P_M x\| \quad \text{(by } 68\text{)}.
$$

This verifies \[85\]. To justify \[86\], suppose further that $x \in U$, note that

$$
B_T x - P_M x = [(1 - \mu_x)P_U + \mu_x P_U P_V P_U](x - P_M x).
$$

With $y = P_M x$, following the above inequalities gives us that

$$
\|B^n_T(x) - P_{U \cap V}x\| \leq \gamma^n \|B_T(x) - y\| \leq \gamma^n \|S_\mu x - y\| \quad \text{(by } 79\text{)}
$$

$$
= \gamma^n \|(1 - \mu)P_U + \mu P_U P_V P_U - P_M)(x - y)\| \quad \text{(by } 83\text{)}
$$

$$
\leq \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_{s+1} + \sin^2 \theta_p} \|x - y\| \quad \text{(by } 84\text{),}
$$

which ensures \[86\] and completes the proof of the theorem. ■

Remark 4.6 As discussed at the beginning of this section, though the map $S_\mu$ obtains the optimal convergence rate at $\mu_0 := \frac{2}{\sin^2 \theta_{s+1} + \sin^2 \theta_p}$, computing $\theta_{s+1}$ and $\theta_p$ may be expensive when the dimensions of $U$ and $V$ are big. Our nonlinear map $B_T$ indeed has a similar form to $S_\mu$ and also obtains the same rate with $S_{\mu_0}$, but it is easier to compute $\mu_x$ in \[75\] and hence $B_T(x)$ for any $x \in \mathbb{R}^n$.

The following corollary suggests a convergence rate for the accelerated map $A_T$ in \[76\]. This is actually a counterpart of \[6\] Theorem 3.28 when $T = P_UP_V$, which is not selfadjoint as required in \[6\] Theorem 3.28.
Corollary 4.7 Let \( T = P_U P_V \). Then for any \( n \in \mathbb{N} \) and \( x \in \mathbb{R}^n \), we have

\[
\| A^n_T (Tx) - P_{U \cap V} x \| \leq \left[ \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p + \sin^2 \theta_{s+1}} \right]^n \cos^2 \theta_{s+1} \| x - P_{U \cap V} x \|,
\]

where \( A_T \) is defined in (76) and where \( \theta_{s+1} \) and \( \theta_p \) are the principal angles found in Definition 3.1.

Proof. For any \( x \in \mathbb{R}^n \), note that \( Tx \in U \), \( P_{U \cap V} Tx = TP_{U \cap V} x = P_{U \cap V} x \) by (78), and that \( A^n_T (Tx) = B^n_T (Tx) \). Thus we get from (86) that

\[
\| A^n_T (Tx) - P_{U \cap V} x \| \leq \left[ \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p + \sin^2 \theta_{s+1}} \right]^n \| Tx - P_{U \cap V} x \| \leq \left[ \frac{\sin^2 \theta_p - \sin^2 \theta_{s+1}}{\sin^2 \theta_p + \sin^2 \theta_{s+1}} \right]^n \cos^2 \theta_{s+1} \| x - P_{U \cap V} x \| \quad \text{(by (68)),}
\]

which completes the proof of the corollary.

5 Numerical experiments

In this section, we compare several algorithms developed in previous sections with some classic methods for finding \( P_{U \cap V} x_0 \). Our test algorithms are the following

- \( B_T \) defined in (74);
- \( S_p \) with \( \mu_1 = \frac{2}{\sin^2 \theta_p + \sin^2 \theta_p} \) (the best parameter); \( \mu_2 = \frac{1}{\sin^2 \theta_p} \in [0, \frac{2}{\sin^2 \theta_p}] \); and \( \mu_3 = \frac{1}{2} + \frac{1}{\sin^2 \theta_p} \in [1, \frac{2}{\sin^2 \theta_p}) \) (see Theorem 3.7);
- \( T_p \) with \( \mu_1 = \frac{2}{1 + \sin^2 \theta_p} \) (the best parameter) and \( \mu_2 = 1.5 \in [0, 2) \) (see Theorem 3.6);
- the classic method of alternating projections (MAP);
- the classic Douglas-Rachford method (DR).

There are (at least) two angles that might affect the convergence: \( \theta_F \) (the Friedrichs angle, see Proposition 3.3) and \( \theta_p \), thus we will use them to categorize the pairs of subspaces. Our numerical set up is as follows. We assume that \( X = \mathbb{R}^{100} \) and define \( \mathcal{X} \) to be the set of all subspaces of \( X \). First, we define our primary categories based on the Friedrichs angle (in radians) as follows

\[
\begin{align*}
(88a) \quad W_1 & := \{(U, V) \in \mathcal{X}^2 \mid 0 < \theta_F < 0.05\}; \\
(88b) \quad W_2 & := \{(U, V) \in \mathcal{X}^2 \mid 0.05 \leq \theta_F < 0.1\}; \\
(88c) \quad W_3 & := \{(U, V) \in \mathcal{X}^2 \mid 0.1 \leq \theta_F < 0.5\}; \\
(88d) \quad \text{and} \quad W_4 & := \{(U, V) \in \mathcal{X}^2 \mid 0.5 \leq \theta_F < 1\}.
\end{align*}
\]
Since we always have $\theta_F \leq \theta_p \leq \pi/2$, we define our secondary categories as follows:\footnote{We do not test the case $\theta_F = \theta_s + 1 = \theta_p$: in such case, it is proved in Theorems \ref{thm:3.7} and \ref{thm:4.5} that $S_{\mu_1}$ and $B_T$ converge after a single step, i.e., they are the clear winners!}

\begin{equation}
Z_j := \{(U, V) \in X^2 \mid \theta_p > \theta_F \text{ and } \frac{\theta_p - \theta_F}{\frac{\pi}{2} - \theta_F} \in [\frac{j-1}{5}, \frac{j}{5})\}, \quad j = 1, \ldots, 5.
\end{equation}

Thus, there are 20 induced categories $W_i \cap Z_j$ for $i = 1, \ldots, 4$ and $j = 1, \ldots, 5$. In each $W_i \cap Z_j$, we randomly generated 5 pairs of subspaces $U$ and $V$ of $X$ such that $\dim U \leq \dim V$ and $U \cap V \neq \{0\}$. So there are 100 pairs of subspaces. For each pair of subspaces, we choose randomly 10 starting points, each with Euclidean norm 10. This results in a total of 1,000 instances for each algorithm. Note that the sequences to monitor are as follows:

| Algorithm  | sequence $(z_n)$ to monitor  |
|------------|------------------------------|
| $B_T$ (see (74)) | $(B_T)^n(x_0)$               |
| $S_{\mu}$ (see Theorem 3.7) | $(S_{\mu})^n(x_0)$          |
| $T_{\mu}$ (see Theorem 3.6) | $(T_{\mu})^n(x_0)$          |
| MAP        | $(P_{UI\!P_{V}})^n(x_0)$    |
| DR (see 69) | $P_{V}(\frac{\text{Id} + R_{U\!R_{V}}}{2})^n(x_0)$ |

We terminate the algorithm when the current iterate of the monitored sequence $(z_n)_{n \in \mathbb{N}}$ satisfies

\begin{equation}
d_{U \cap V}(z_n) \leq 0.01
\end{equation}

for the first time or when the number of iterations reaches 100,000 (i.e., problem unsolved). In applications, we in general would not have access to this information but here we use it to see the true performance of these algorithms.

In Figures 1, 2, and 3, the horizontal axis represents the Friedrichs angle between two subspaces; and the vertical axis represents the (median) number of iterations, more specifically, the median is computed over 10 instances of one pair of subspaces.

In Figure 1, we compare $B_T$, the "best" versions $S_{\mu_1}$ and $T_{\mu_1}$, MAP, and DR. We see that $B_T$ is generally the fastest when $\theta_F > 0.02$. This can be interpreted by the fact that $B_T$ optimizes its parameter $\mu_x$ at each iteration. While when $\theta_F \leq 0.02$, DR seems to be the fastest, this phenomenon has been previously observed in [5]. In Figure 2, we compare $S_{\mu_i}$, $i = 1, 2, 3$. The results suggest that the "best" version $S_{\mu_1}$ is somewhat faster than $S_{\mu_2}$ and $S_{\mu_3}$. In Figure 3, we compare $T_{\mu_i}$, $i = 1, 2$. On the contrary, it is not clear that the "best" version $T_{\mu_1}$ is more favorable than $T_{\mu_2}$.\footnote{We do not test the case $\theta_F = \theta_s + 1 = \theta_p$: in such case, it is proved in Theorems \ref{thm:3.7} and \ref{thm:4.5} that $S_{\mu_1}$ and $B_T$ converge after a single step, i.e., they are the clear winners!}
Figure 1: $B_T$ is the fastest for large $\theta_F$, while DR is the fastest for small $\theta_F$.

Figure 2: $S_\mu$ with $\mu_1 = \frac{2}{\sin^2 \delta + \sin^2 \delta_p}$ (“best”); $\mu_2 = \frac{1}{\sin^2 \delta_p}$; and $\mu_3 = \frac{1}{2} + \frac{1}{\sin^2 \delta_p}$. 
Finally, in Table [1] for each primary category $W_i$, we record the median, the mean, and the standard deviation of the number of iterations required for the algorithms to terminate. The table clearly supports these observations above. In general, the results suggest that all algorithms are more preferable than MAP.

Figure 3: $T_\mu$ with $\mu_1 = \frac{1}{1 + \sin^2 \theta_F}$ ("best"); and $\mu_2 = 1.5$. 
| Primary category | $W_1$ | $W_2$ | $W_3$ | $W_4$ |
|------------------|-------|-------|-------|-------|
| Number of instances | 250   | 250   | 250   | 250   |
| $B_1$ Median     | 1139  | 169   | 13.5  | 5     |
| $B_1$ Mean       | 6002.5| 206.7 | 22.9  | 5.1   |
| $B_1$ Std        | 19437.1| 163.8 | 21.6  | 2     |
| $S_{\mu_1}$ Median | 1404 | 226.5 | 16    | 5     |
| $S_{\mu_1}$ Mean | 6586.8| 260.1 | 27.8  | 5.8   |
| $S_{\mu_1}$ Std  | 19396.5| 195.7 | 26    | 2.1   |
| $S_{\mu_2}$ Median | 2318.5| 359.5 | 24.5  | 7     |
| $S_{\mu_2}$ Mean | 8096.6| 417.5 | 43.5  | 7.6   |
| $S_{\mu_2}$ Std  | 19657.3| 326.9 | 42.5  | 3.5   |
| $S_{\mu_3}$ Median | 1697.5| 272   | 18.5  | 7     |
| $S_{\mu_3}$ Mean | 6980.3| 307.4 | 32    | 6.7   |
| $S_{\mu_3}$ Std  | 19426.9| 221.5 | 30.3  | 1.5   |
| $T_{\mu_1}$ Median | 3636.5| 611   | 42    | 9     |
| $T_{\mu_1}$ Mean | 11571 | 684.7 | 64.5  | 8.8   |
| $T_{\mu_1}$ Std  | 21298.1| 265.9 | 59.3  | 3.3   |
| $T_{\mu_2}$ Median | 2704.5| 481.5 | 32.5  | 10    |
| $T_{\mu_2}$ Mean | 9788.2| 528.2 | 48.8  | 10.2  |
| $T_{\mu_2}$ Std  | 20599 | 223.1 | 44.2  | 0.6   |
| MAP Median       | 4058.5| 722.5 | 49    | 10    |
| MAP Mean         | 12683 | 793.1 | 74    | 10.2  |
| MAP Std          | 22531.1| 334.7 | 66.4  | 4.1   |
| DR Median        | 1231  | 448.5 | 83.5  | 17.5  |
| DR Mean          | 1395.2| 511.3 | 92    | 17.4  |
| DR Std           | 847.9 | 203.6 | 56.1  | 7.1   |

Table 1: Median, mean, and standard deviation of number of iterations.

The data and figures in this section were computed with the help of Julia (see [33]) and Gnuplot (see [19]).
6 Conclusion

This paper presents a constructive study on the optimal convergence linear rate of a matrix. We give a complete characterization when the matrix has the optimal convergence rate in term of semi-simpleness of all the subdominant eigenvalues and the unit eigenvalue. Combined with the principal angles between two subspaces, this allows us to provide convergence analysis for relaxed alternating projection, partial relaxed alternating projection and generalized Douglas-Rachford methods for two subspaces. It turns out that the partial relaxed alternating projection method and its nonlinear version could obtain the smallest convergence rate among these ones, which are demonstrated by numerical performances. Our results not only recover but also significantly extend currently known results in the literature. In future research one may similarly investigate Jacobi, Gauss-Seidel, and especially successive over-relaxation methods. Understanding further the partial relaxed alternating projection method for two sets is also an intriguing project.

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