On Pareto eigenvalue of distance matrix of a graph

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Abstract

In this article, we study Pareto eigenvalues of distance matrix of connected graphs and show that the non zero entries of every distance Pareto eigenvector of a tree forms a strictly convex function on the forest generated by the vertices corresponding to the non zero entries of the vector. Besides we find the minimum number of possible distance Pareto eigenvalue of a connected graph and establish lower bounds for $n$ largest distance Pareto eigenvalues of a connected graph of order $n$. Finally, we discuss some bounds for the second largest distance Pareto eigenvalue and find graphs with optimal second largest distance Pareto eigenvalue.

Keywords: Pareto eigenvalue, Distance matrix, spectral radius.

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1 Introduction and terminology

All graphs considered here are finite, undirected, connected and simple. Let $G$ be a graph on vertices $1, 2, \ldots, n$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of $G$, respectively. For $i, j \in V(G)$, the distance between $i$ and $j$, denoted by $d_G(i, j)$ or simply $d_{ij}$, is the length of a shortest path from $i$ to $j$ in $G$. The distance matrix of $G$, denoted by $D(G)$ is the $n \times n$ matrix with $(i, j)$-th entry $d_{ij}$. For a column vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ we have

$$x^T D(G)x = \sum_{1 \leq i < j \leq n} d_{ij}x_i x_j.$$

If vertices $i$ and $j$ are adjacent, we write $i \sim j$. Edges $e_1$, $e_2$ in a graph $G$ are said to be incident if they have a common vertex. If $V_1 \subseteq V(G)$ and $E_1 \subseteq E(G)$, then by $G - V_1$ and $G - E_1$ we mean the graphs obtained from $G$ by deleting the vertices in $V_1$ and the edges $E_1$ respectively. In particular case when $V_1 = \{u\}$ or $E_1 = \{e\}$, we simply write $G - V_1$ by

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7, we introduce distance Pareto eigenvalues of a connected graph and establish lower bounds for $n$ largest distance Pareto eigenvalues of a connected graph of order $n$, equality conditions have also

$G - u$ and $G - E_1$ by $G - e$ respectively. $K_n - e$ is the graph obtained from $K_n$ by removing any one edge of it. Degree of a vertex $v$ in a graph $G$ will be denoted by $d_v$. By a pendent vertex of a graph we mean a vertex of degree 1. For a connected graph $G$ a block $B$ is said to be a pendent block if exactly one vertex of $B$ is a cut vertex of $G$. A quasipendent vertex is a vertex which is adjacent to a pendent vertex. The transmission, denoted by $Tr(v)$ of a vertex $v$ is the sum of the distances from $v$ to all other vertices in $G$. The Wiener index, denoted by $W$ of a connected graph $G$ is defined as $W = \frac{1}{2} \sum_{v \in V(G)} Tr(v)$.

By $P_n, K_n$ and $C_n$ we mean the usual path graph, complete graph and cycle graph with $n$ vertices respectively. The diameter of a connected graph $G$ denoted by $diam(G)$ is the maximum distance between any two vertices in $G$, i.e. $diam(G)$ is the largest entry of $D(G)$. A clique of a graph is a maximal complete subgraph and clique number of a graph is the order of a maximal clique. We denote clique number of a graph $G$ by $\omega(G)$. For $1 \leq p \leq \omega$ by $K^p_n$ we denote a graph obtained by joining one vertex to $p$ vertices of $K_n$. By $S_n$ we mean the usual star graph $K_{1,n-1}$ and by $S_n^+$ we represent the graph so that $S_n^+ - e = S_n$. The graph obtained from $G$ and $H$ by identifying $u \in G$ and $v \in H$ is denoted by $G_u \ast H_v$ or simply by $G \ast H$ when there is no confusion of the vertices. We write $H_{u,v}$ to denote a graph of order $n$ with $u, v \in V(H_{u,v})$ so that $d_u = n - 1$ and each vertex in $V(H) - \{u, v\}$ has same vertex degree and same transmission.

By spectral radius of a symmetric matrix $M$, we mean its largest eigenvalue and denote it by $\rho(M)$. Note that for a connected graph $G$, $D(G)$ is irreducible nonnegative matrix. Thus by the Perron-Frobenius theorem, $\rho(D)$ is simple, and there is a positive eigenvector of $D(G)$ corresponding to $\rho(D)$. Such eigenvectors corresponding to $\rho(D)$ is called Perron vector of $D(G)$. By an eigenvector we mean a unit eigenvector and by $\mathbb{M}_n$, we denote the class of all real matrices of order $n$. We use the notation $A \geq 0$ to indicate that each component of the matrix $A$ is nonnegative. Furthermore in places we write $A \geq B$ to mean $A - B \geq 0$.

**Definition 1.1.** A real number $\lambda$ is said to be a Pareto eigenvalue of $A \in \mathbb{M}_n$ if there exists a nonzero vector $x(\geq 0) \in \mathbb{R}^n$ such that

$$Ax \geq \lambda x \quad \text{and} \quad \lambda = \frac{x^T Ax}{x^T x},$$

also we call $x$ to be a Pareto eigenvector of $A$ associated with Pareto eigenvalue $\lambda$.

Pareto eigenvalues are also known as complementarity eigenvalues. Fernandes et al. [3] and Seeger [7] studied the Pareto eigenvalues of adjacency matrix of a graph.

We now outline the contents of this article. In Section 2, we introduce distance Pareto eigenvalue (eigenvector) of a connected graph and show that non zero entries of every distance Pareto eigenvector of a tree forms a strictly convex function on the forest generated by vertices corresponding to the non zero entries of the vector. We also find the complete distance Pareto spectrum for some special class of graphs like complete graph, Star graph etc. Partial distance Pareto spectrum of graphs with given diameter or given clique number are also supplied. Besides we find the minimum number of possible distance Pareto eigenvalue of a connected graph and establish lower bounds for $n$ largest distance Pareto eigenvalues of a connected graph of order $n$, equality conditions have also
been established. In Section 3, we discuss the second largest distance Pareto eigenvalue of a connected graph, specially we give some bounds for it and find graphs with optimal second largest distance Pareto eigenvalue.

2 Distance Pareto eigenvalue of a connected graph

Definition 2.1. Distance Pareto eigenvalue of a connected graph $G$ is a Pareto eigenvalue of the distance matrix of $G$.

Multiplicity of Pareto eigenvalue of a matrix is not considered. We denote the $k^{th}$ largest and $k^{th}$ smallest distance Pareto eigenvalue of a connected graph $G$ by $\rho_k(G)$ and $\mu_k(G)$ respectively. We simply write them by $\rho_k$ and $\mu_k$ when the graph under consideration is understood from the context. Besides we use $\Pi(G)$ to denote the set of all distance Pareto eigenvalues of a connected graph $G$.

We write $[n] = \{1, 2, \ldots, n\}$ and for an $n \times n$ matrix $A$ and $S \subset [n]$, we reserve the symbol $A(S)$ for the principal submatrix of $A$ obtained by deleting rows and columns of $A$ corresponding to $S$. In particular if $D(G)$ is the distance matrix of a graph $G$ then by $D(i)$ we will denote the principal submatrix of $D$ obtained by deleting row and column corresponding to vertex $i$ of $G$. By $\mathbb{I}$ we denote the column vector of all ones and by $J$ the matrix of all ones of appropriate size. We now recall some known results which will be used.

Lemma 2.2. [Weyl’s Inequalities][4] Let $\lambda_i(M)$ denote the $i^{th}$ largest eigenvalue of a real symmetric matrix $M$. If $A$ and $B$ are two real symmetric matrices of order $n$, then

$$\lambda_1(A) + \lambda_i(B) \geq \lambda_i(A + B) \geq \lambda_n(A) + \lambda_i(B) \quad \text{for} \quad i = 1, 2, \ldots, n.$$  

Lemma 2.3. [5] If $A$ is an irreducible matrix and $A \geq B \geq 0$, $A \neq B$, then $\rho(A) > \rho(B)$.

Lemma 2.4. [Cauchy’s Inequalities][?] Let $A = \begin{pmatrix} B & y \\ y^T & a \end{pmatrix} \in \mathbb{M}_{n+1}$ with $y \in \mathbb{C}^n, a \in \mathbb{R}$ and $B \in \mathbb{M}_n$ be symmetric. Then

$$\lambda_1(A) \leq \lambda_1(B) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n+1}(A),$$

where $\lambda_i(N)$ is the $i^{th}$ smallest eigenvalue of $N$. Also $\lambda_i(A) = \lambda_i(B)$ if and only if there is a non zero vector $z \in \mathbb{C}^n$ such that $Bz = \lambda_i(B)z, y^Tz = 0$ and $Bz = \lambda_i(A)z$.

Lemma 2.5. [4] Let $A, B \in \mathbb{M}_n$ be nonnegative where $A$ is irreducible and $B$ is non zero, then $\rho(A + B) > \rho(A)$.

Lemma 2.6. [4] If $A$ is a symmetric $n \times n$ matrix with $\lambda_1$ as the largest eigenvalue then for any normalized vector $x \in \mathbb{R}^n (x \neq 0)$,

$$x^T Ax \leq \lambda_1.$$  

The equality holds if and only if $x$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_1$. 

Putting $x = \frac{1}{\sqrt{n}} \mathbb{1}$ in Lemma 2.6 we get the following result as a Corollary.

**Corollary 2.7.** If $A$ is a symmetric $n \times n$ matrix, then

$$\rho(A) \geq \bar{R},$$

(2.1)

where $\bar{R}$ is the average row sum of the matrix $A$. The equality in (2.1) holds if and only if all the row sums of $A$ are equal.

**Theorem 2.8.** [6] The scalar $\lambda \in \mathbb{R}$ is a Pareto eigenvalue of $A \in \mathbb{M}_n$ if and only if there exists a nonempty set $J \subset \{1, 2, \ldots, n\}$ and a vector $\xi \in \mathbb{R}^{|J|}$ such that

$$A^J \xi = \lambda \xi,$$

$$\xi_j > 0 \quad \text{for all } j \in J,$$

$$\sum_{j \in J} a_{i,j} \xi_j \geq 0 \quad \text{for all } i \notin J.$$

Furthermore, a Pareto eigenvector $x$ associated to $\lambda$ is constructed by setting

$$x_j = \begin{cases} 
\xi_j & \text{if } j \in J, \\
0 & \text{otherwise.}
\end{cases}$$

From Theorem 2.8, we get the following result similar to that of [7, Theorem 1].

**Theorem 2.9.** The distance Pareto eigenvalues of a connected graph $G$ are given by

$$\Pi(G) = \{\rho(A) : A \in M\},$$

where $M$ is the class of all principal sub-matrices of $D(G)$.

From Theorem 2.9, for any connected graph $G$ we see that every Pareto eigenvalue of any principal sub-matrix of $D(G)$ is the distance Pareto eigenvalue of $G$. Therefore as a consequence we get that if $G$ is a connected graph and $H$ is a block of $G$, then every distance Pareto eigenvalue of $H$ is also a distance Pareto eigenvalue of $G$. Also if $G$ is a connected graph and $H$ is a subgraph of $G$ obtained by removing one or more pendant blocks of $G$, then $\lambda \in \Pi(H)$ implies $\lambda \in \Pi(G)$. Again if $\omega$ is the clique number of the graph $G$, then $\mathbb{J}_\omega - I_\omega$ is a principal sub-matrix of $D(G)$ and eigenvalues of $\mathbb{J}_\omega - I_\omega$ are $0, 1, \ldots, \omega - 2$ and $\omega - 1$. Therefore $0, 1, \ldots, \omega - 1$ are distance Pareto eigenvalues of $G$. Similarly if there are $p$ vertices in a graph $G$ which are at a distance $k$ from each other, then $k(\mathbb{J}_p - I_p)$ is a principal sub-matrix of $D(G)$ and therefore eigenvalues of $k(\mathbb{J}_p - I_p)$ i.e. $0, k, \ldots, k(p - 2)$ and $k(p - 1)$ are distance Pareto eigenvalues of $G$. Besides from Theorem 2.9, we get the following lemma which states that for a connected graph the distance spectral radius and the largest distance Pareto eigenvalue coincide.

**Lemma 2.10.** The largest distance Pareto eigenvalue of a connected graph is the distance spectral radius of the graph.

**Definition 2.11.** [2] Let $T$ be a tree with $V(T) = \{1, \ldots, n\}, n \geq 3$, and let $f : V(T) \rightarrow [0, \infty)$. Then $f$ is said to be convex if for any distinct $i, j, k \in V(T)$ with $i \sim j, j \sim k$ we have $2f(j) \leq f(i) + f(k)$ and strictly convex if $2f(j) < f(i) + f(k)$. Also $f$ is said
to be quasiconvex if for any distinct \(i, j, k \in V(T)\) with \(i \sim j, j \sim k\) we have \(f(j) \leq \max\{f(i), f(k)\}\) and strictly quasiconvex if \(f(j) < \max\{f(i), f(k)\}\). A function defined on the vertices of a forest \(F\) is said to be convex, strict convex, quasiconvex or strictly quasiconvex if it is so in any subtree of \(F\).

**Lemma 2.12.** [2] Let \(T\) be a tree with \(V(T) = \{1, \ldots, n\}\) and let \(f : V(T) \rightarrow [0, \infty)\) be either strictly convex or strictly quasiconvex. Then \(f\) attains its minimum, either at a unique vertex, or at two adjacent vertices. Furthermore,

(i) if \(f\) attains its minimum at the unique vertex \(i\), then for any path \(i = i_1 - i_2 - \cdots - i_k\), starting at \(i\), \(f(i_1) < f(i_2) < \cdots < f(i_k)\).

(ii) if \(f\) attains its minimum at the two adjacent vertices \(i\) and \(j\), then \(f\) is strictly increasing along any path starting at \(i\), and not containing \(j\), or starting at \(j\), and not containing \(i\).

**Lemma 2.13.** [2] Let \(D\) be the distance matrix of a tree \(T\) and let \(f : [0, \infty) \rightarrow [0, \infty)\) be a strictly increasing convex function. Then the Perron vector of \(f(D)\) is strictly convex on \(T\).

**Theorem 2.14.** If \(x\) is a distance Pareto eigenvector of a tree \(T\) and \(V_x = \{v \in T : x_v > 0\}\) then the non zero components of \(x\) form a strictly convex function on the subgraph of \(T\) generated by \(V_x\).

**Proof.** If \(x\) is a distance Pareto eigenvector of tree \(T\) with corresponding distance Pareto eigenvalue \(\lambda\) then from **Theorem 2.9** and **Theorem 2.8**, \(\lambda\) is a spectral radius for some principal sub-matrix \(M\) of \(D(T)\) and \(x_v > 0\) if and only if \(u\)th row(column) of \(D(T)\) is in \(M\). If \(\lambda = 0\), then cardinality of \(V_x\) is 1 and therefore the result is trivially true. So we assume \(\lambda \neq 0\). Now if the subgraph of \(T\) generated by \(V_x\) is again a tree, then the result follows from **Lemma 2.13**.

Otherwise we consider a subtree \(T'\) of the subgraph (forest) of \(T\) generated by \(V_x\). Let \(u, v, w\) be vertices of \(T'\) such that \(u \sim v \sim w\). Let \(e_1 = \{u, v\}, e_2 = \{v, w\}\) be edges of \(T'\). Let \(T_u, T_v\) and \(T_w\) be the components of \(T - \{e_1, e_2\}\), containing \(u, v\) and \(w\), respectively. From eigenequations of \(M\), we have

\[
\lambda x_u = \sum_{y \in T_1 \cap V_x} d_{uy} x_y + \sum_{y \in T_2 \cap V_x} (d_{vy} + 1) x_y + \sum_{y \in T_3 \cap V_x} g(d_{wy} + 2) x_y \tag{2.2}
\]

\[
\lambda x_v = \sum_{y \in T_1 \cap V_x} (d_{uy} + 1) x_y + \sum_{y \in T_2 \cap V_x} d_{vy} x_y + \sum_{y \in T_3 \cap V_x} (d_{wy} + 1) x_y \tag{2.3}
\]

\[
\lambda x_w = \sum_{y \in T_1 \cap V_x} (d_{uy} + 2) x_y + \sum_{y \in T_2 \cap V_x} (d_{vy} + 1) x_y + \sum_{y \in T_3 \cap V_x} d_{wy} x_y. \tag{2.4}
\]

It follows from (2.2)–(2.4) that

\[
\lambda(x_u + x_w - 2x_v) = \sum_{y \in T_2 \cap V_x} 2x_y
\]

\[
= 2x_v
\]

\[
> 0 \quad \text{as} \quad x_v > 0
\]

Thus \(x_u + x_w > 2x_v\) as \(\lambda > 0\).
As $T'$ is an arbitrary subtree of the subgraph of $T$ generated by $V_x$, hence the result follows. ■

**Theorem 2.15.** For a connected graph of diameter $d$, the integers $0, 1, \ldots, d$ are always its distance Pareto eigenvalues.

**Proof.** If $G$ is a graph with diameter $d$, then $\mathcal{D}(P_d)$ is a principal sub-matrix of $\mathcal{D}(G)$ and therefore by Theorem 2.9 every $\lambda \in \Pi(P_d)$ implies $\lambda \in \Pi(G)$. Let

$$A_k = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$ 

Then for $1 \leq k \leq d$, $A_k$ is a principal sub-matrix of $\mathcal{D}(P_d)$ and $\rho(A_k) = k$. Therefore $k \in \Pi(P_d)$ for $1 \leq k \leq d$. Besides 0 being a diagonal element of $\mathcal{D}(P_d)$ is in $\Pi(P_d)$. Hence the result follows. ■

As for a complete graph the distance matrix and the adjacency matrix coincide, therefore from [7] we have the following.

**Lemma 2.16.** [7] For any positive integer $n$, $\Pi(K_n) = \{0, 1, \ldots, n-1\}$.

**Definition 2.17.** If $A$ and $B$ are two nonnegative matrices then we say that $A$ dominates $B$ if either of the following two cases hold

1. $A$ and $B$ are of same size and upto permutation similarity $A \geq B$, $A \neq B$.
2. $A$ is permutation similar to $\begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and at least one of $C, D$ and $E$ is a nonzero matrix.

From Lemma 2.3, Lemma 2.4 and Lemma 2.5 we get the following.

**Lemma 2.18.** If $A$ and $B$ are two symmetric nonnegative irreducible matrices, then $A$ dominates $B$ implies $\rho(A) > \rho(B)$.

**Lemma 2.19.** [1] Distance spectral radius of the star $S_n$ is

$$n - 2 + \sqrt{(n-2)^2 + n - 1}$$

**Theorem 2.20.** There are exactly $2(n-1)$ distance Pareto eigenvalues of $S_n$ and they are

$$\mu_{2k} = 2(k-1), \mu_{2k-1} = k - 1 + \sqrt{k^2 - 3k + 3}$$ where $k = 1, \ldots, n - 1$.

**Proof.** Upto permutation similarity there are exactly two distinct principal sub-matrices of $\mathcal{D}(S_n)$ of order $k = 2, \ldots, n-1$ and they are $\mathcal{D}(S_k)$ and $2(J_k - I_k)$. Clearly the later always dominates the former one. Besides $\rho(2(J_k - I_k)) = 2(k-1)$ and from Lemma 2.19, we have $\rho(\mathcal{D}(S_k)) = k - 2 + \sqrt{(k-2)^2 + k - 1}$. By routine calculation it can be easily shown that

$$\rho(\mathcal{D}(S_{k+1})) > \rho(2(J_k - I_k)) > \rho(\mathcal{D}(S_k)) \quad \text{for} \quad k = 2, \ldots, n - 1 \quad \text{(2.5)}$$

As 0 is always a distance Pareto eigenvalue of a connected graph and $\rho(\mathcal{D}(S_n))$ is the largest distance Pareto eigenvalue of $S_n$, using (2.5) we get all the distance Pareto eigenvalues of $S_n$ as needed. ■
Theorem 2.21. If $G$ is a connected graph of order $n$ and diameter $d$ then $|\Pi(G)| \geq n + d - 1$, with equality if and only if $G = P_3$ or $K_n$.

Proof. From Theorem 2.15, we have $0, 1, \ldots, d$ as distance Pareto eigenvalues of $G$. Let $A_2 = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$. For $n \leq 2$ there is nothing to prove. If $n \geq 3$ then for $i = 3, \ldots, n$ we can have sub-matrices $A_i$ of $D(G)$ of order $i$ such that $A_i$ dominates $A_{i-1}$. Therefore by Lemma 2.18, $\rho(A_i) > \rho(A_{i-1})$. Thus we get $(d + 1) + (n - 2) = n + d - 1$ distance Pareto eigenvalues of $G$ as follows

$$0 < 1 < \cdots < d < \rho(A_3) < \cdots < \rho(A_n).$$

Hence

$$|\Pi(G)| \geq n + d - 1. \tag{2.6}$$

Now from Lemma 2.16, we have $|\Pi(K_n)| = n$ and by direct calculation $|\Pi(P_3)| = 4$. Therefore for $G = P_3$, $K_n$ the equality holds in (2.6).

If $G \neq P_3, K_n$ then $n \geq 4$ and $d \geq 2$ and thus $D(P_3)$ is a principal sub-matrix of $D(G)$. For $d \geq 3$ we see that $\rho(D(P_3)) = 1 + \sqrt{3} < \rho(A_3)$ as minimum row sum of $A_3$ is at least $d + 1 \geq 4$. Again if $d = 2$ then either $P = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ or $Q = 2(J_3 - I_3)$ must be a principal submatrix of $D(G)$. So we can choose $A_3 = P$ or $A_3 = Q$, whichever be the case. Also $1 + \sqrt{3} < \min\{\rho(P), \rho(Q)\}$.

In either situation we get $1 + \sqrt{3} \in \Pi(G)$ in addition to the above listed $n + d - 1$ distance Pareto eigenvalues of $G$. Thus $|\Pi(G)| \geq n + d$.

Note: From Theorem 2.21 we see that for any connected graph $G$ with $n$ vertices, $|\Pi(G)| \geq n$ and the equality is achieved if and only if $G = K_n$. But the exact number of distance Pareto eigenvalues of a connected graph of order $n$ is not known. Now for any $n \times n$ matrix there are exactly $2^n - 1$ principal sub-matrices, so by Theorem 2.9 we can not have more than $2^n - 1$ distance Pareto eigenvalues of any connected graph with $n$ vertices. Also we can see that for any connected graph with $n \geq 2$ vertices, all the principal sub-matrices of $D(G)$ are not distinct. Besides we may have some principal sub-matrices which are distinct but having same spectral radius. Therefore for any connected graph $G$ of order $n$ if there are $s(G)$ distinct principal sub-matrices of $D(G)$ then $|\Pi(G)| \leq s(G)$, with equality if and only if all the distinct sub-matrices of $D(G)$ have distinct spectral radii. In this regard another question arises that among all connected graphs of given order which graph(s) will have maximum number of distance Pareto eigenvalues. By direct calculation we have seen that among all connected graphs of order $n = 2, 3, 4$ the path graph $P_n$ has the maximum number of distance Pareto eigenvalues. The numbers are respectively 2, 4 and 7. But for graphs of order 5, we see that the path graph $P_5$ together with graphs $G_1$ and $G_2$ of Figure 1 have maximum (here 13) number of distance Pareto eigenvalues. Again for the class of graphs of order 6, the graph $G_3$ in Figure 1 attains uniquely the maximum (30) number of distance Pareto eigenvalues. Therefore this seems to be an interesting problem to characterize all graphs for which the maximum number of distance Pareto eigenvalues occur among all connected graphs of given order.
We here pose a question which perhaps require deep investigation. Can distance Pareto eigenvalues of a connected graph uniquely determine the graph? In addition to that it is worth studying how fast distance Pareto eigenvalues grow when number of vertices increases. We leave these problems for future research scope.

**Theorem 2.22.** If $G$ is a graph with $n$ vertices, then

$$\rho_k(G) \geq n - k \text{ for } k = 1, 2, \ldots, n.$$  

Equality holds if and only if $G = K_n$.

**Proof.** If $G = K_n$, then using Lemma 2.16 we are done.

If $G \neq K_n$, then for $i = 1, 2, \ldots, n - 2$, we can have $A_i$ of order $n - i + 1$ as principal sub-matrix of $\mathcal{D}(G)$ such that $A_1 = \mathcal{D}(G)$, $A_{n-2} = \mathcal{D}(P_3)$, $A_{i+1}$ is a principal sub-matrix of $A_i$ for $i = 1, 2, \ldots, n - 3$.

Besides $A_i$ dominates $\mathcal{J}_{n-i+1} - I_{n-i+1}$ for $i = 1, 2, \ldots, n - 3$.

But

$$\rho(\mathcal{J}_{n-i+1} - I_{n-i+1}) = n - i.$$  

Therefore

$$\rho_i(G) > n - i \text{ for } i = 1, 2, \ldots, n - 3. \quad (2.7)$$

Now since $A_{n-2} = \mathcal{D}(P_3)$ and $\Pi(P_3) = \{0, 1, 2, 1 + \sqrt{3}\}$, we get

$$\rho_{n-2}(G) \geq 1 + \sqrt{3} > 2, \rho_{n-1}(G) \geq 2 > 1 \text{ and } \rho_n(G) \geq 1 > 0. \quad (2.8)$$

Combining (2.7) and (2.8), we get the result as desired.

3 Second largest distance Pareto eigenvalue

The largest distance Pareto eigenvalue of a connected graph is nothing but the distance spectral radius of the graph. Also in last few decades distance spectral radius have been extensively studied. In this section we study some bounds for the second largest distance Pareto eigenvalue.
Theorem 3.1. If $G$ is a connected graph with at least two vertices, then
\[ \rho_2(G) = \max\{\rho(A) : A \in P\}, \]
where $P = \{(\mathcal{D}(G))(v) : v \in V(G), d_v > 1\}$

Proof. Since $\mathcal{D}(G)$ dominates $A$ for every $A \in P$, we have
\[ \rho_2(G) \leq \rho(A) < \rho_1(G) \text{ for every } A \in P. \]

Also for every principal sub-matrix $B$ of $\mathcal{D}(G)$ of order $n - 2$ or less, there exist $A \in P$ which dominates $B$ and therefore $\rho(B) < \rho(A)$. Thus $\rho_2(G)$ must be equal to the largest $\rho(A)$ for all possible $A \in P$.

Now if $u$ is a pendent vertex in $G$ and $v$ is a quasi-pendent vertex in $G$ adjacent to $u$, then the principal sub-matrix of $\mathcal{D}(G)$ obtained by removing row and column corresponding to vertex $v$ dominates the principal sub-matrix of $\mathcal{D}(G)$ obtained by removing row and column corresponding to vertex $u$. Therefore in calculating $\rho_2(G)$ we can ignore those principal sub-matrices of $\mathcal{D}(G)$ which are obtained by removing row and column corresponding to pendant vertex. Hence the result follows.

\[ \square \]

Theorem 3.2. If $G$ is a graph of order $n$ with a vertex of degree $n - 1$, then
\[ n - 2 \leq \rho_2(G) \leq 2(n - 2), \]
the right hand equality holds if and only if $G = S_n$ and the left hand equality holds if and only if $G = K_n$.

Proof. From Theorem 2.22, we see that $\rho_2(G) \geq n - 2$ and equality holds if and only if $G = K_n$.

Now suppose $G \neq K_n$. Let $v \in V(G)$ with $d_G(v) = n - 1$, and $A$ be the principal sub-matrix of $\mathcal{D}(G)$ obtained by deleting row and column of $\mathcal{D}(G)$ corresponding to vertex $v$. Then Clearly $\rho_2(G) = \rho(A)$, as $A$ dominates any other principal sub-matrix of $\mathcal{D}(G)$ of order $n - 1$.

From Theorem 2.20, $\rho_2(S_n) = 2(n - 1)$. Now if $G \neq S_n$, then we can find two vertices $u, w(\neq v) \in V(G)$ such that $u \sim w$ in $G$. Thus $2(\mathbb{I}_{n-1} - I_{n-1})$ dominates $A$. Therefore $\rho_2(G) = \rho(A) < 2(n - 1)$.

\[ \square \]

Theorem 3.3. If $G$ is a connected graph of order $n$ and diameter 2 then $\rho_2(G) \leq 2(n - 2)$, with equality if and only if $G = S_n$.

Proof. By Theorem 2.20, we have $\rho_2(S_n) = 2(n - 2)$. Now if $G \neq S_n$, then there is at least two non pendent vertices $u, v \in V(G)$. So there are at least two 1’s in each of $u - th$ and $v - th$ rows(columns) of $\mathcal{D}(G)$. Therefore if $A$ is any principal submatrix of $\mathcal{D}(G)$ of order $n - 1$, then $2(\mathbb{I}_{n-1} - I_{n-1})$ dominates $A$. Thus $2(n - 2) > \rho_2(G)$.

\[ \square \]

Theorem 3.4. If $G$ is a connected graph with $n$ vertices and $\omega(G) \geq n - 1$, then
\[ n - 2 \leq \rho_2(G) \leq \frac{n - 3 + \sqrt{n^2 + 10n - 23}}{2}, \]
with equality in the left hand side if and only if $G = K_n$ and equality in the right hand side if and only if $G = K_{n-1}^1$.
Proof. From Lemma 2.16 we have \( \rho_2(G) \geq n - 2 \) with equality if and only if \( G = K_n \). Now as \( \omega(G) \geq n - 1 \), we can take \( H = K_{n-1} \) to be a subgraph of \( G \) and \( v \in V(G) - V(H) \). Then upto permutation similarity we have

\[
D(G) = \begin{pmatrix} D(K_{n-1}) & x \\ x^T & 0 \end{pmatrix}, \quad \text{where } x_u = 1 \text{ if } u \sim v \\
= 2 \text{ otherwise.}
\]

Since \( G \) is connected, there exists \( w \in V(H) \) with \( w \sim v \). If \( B(G) \) is the principal submatrix of \( D(G) \) obtained by deleting row and column corresponding to \( w \) then clearly \( \rho_2(G) = \rho(B) \), and it can be easily observed that \( B(K_{n-1}^p) \) dominates \( B(K_{n-1}^{p-1}) \) for \( p = 2, \ldots, n \). Therefore

\[
\rho_2(K_{n-1}^1) > \rho_2(K_{n-1}^2) > \cdots > \rho_2(K_{n-1}^{n-1}) > \rho_2(K_{n-1}^n)
\]

Thus we get \( \rho_2(G) \leq \rho(D(K_{n-1}^1)) \), with equality if and only if \( G = K_{n-1}^1 \).

Now let \( \rho, y \) be the eigenpair of \( B(K_{n-1}^1) \). Then due to symmetry \( y_2 = y_3 = \cdots = y_{n-1} \).

From eigenequations we have

\[
2(n-1)y_2 = \rho y_1 \tag{3.9}
\]
\[
2y_1 + (n-3)y_2 = \rho y_2 \tag{3.10}
\]

From (3.9) and (3.10) we get \( \rho^2 - (n-3)\rho - 4(n-2) = 0 \). Which gives

\[
\rho = \frac{n - 3 + \sqrt{n^2 + 10n - 23}}{2}.
\]

Hence the result follows.

**Theorem 3.5.** For any non complete connected graph \( G \) with \( n \) vertices,

\[
\rho_2(G) \geq \frac{n - 2 + \sqrt{n^2 - 4n + 12}}{2},
\]

with equality if and only if \( G = K_n - e \).

Proof. First we find the expression for \( \rho(D(K_n - e)) \). If \( (\rho, x) \) is the eigenpair for \( D(K_n - e) \), where \( e = \{1, 2\} \). Then due to symmetry we have

\[
x_1 = x_2 \text{ and } x_3 = \cdots = x_n.
\]

Also we have

\[
D(K_n - e) = \begin{pmatrix} 0 & 2 & \mathbb{I}^T_{n-2} \\ 2 & 0 & 1 \\ \mathbb{I}_{n-2} & 1 & D(K_{n-2}) \end{pmatrix}
\]

(3.12)
Using (3.11) and (3.12) in eigenequations of \( \mathcal{D}(K_n - e) \), we get

\[
\begin{align*}
2x_1 + (n - 2)x_3 &= \rho x_1 \\
2x_1 + (n - 3)x_3 &= \rho x_3
\end{align*}
\]

(3.13) (3.14)

From (3.13) and (3.14) we have \( \rho^2 - (n - 1)\rho - 2 = 0 \). Thus we get

\[
\rho(\mathcal{D}(K_n - e)) = \frac{n - 1 + \sqrt{(n - 1)^2 + 8}}{2}.
\]

(3.15)

Clearly \( \mathcal{D}(K_{n-1} - e) \) and \( \mathcal{J}_{n-1} - I_{n-1} \) are the only two distinct sub-matrices of \( \mathcal{D}(K_n - e) \) of order \( n - 1 \) and the former dominates the later one. Therefore

\[
\rho_2(K_n - e) = \rho(\mathcal{D}(K_{n-1} - e))
\]

\[= \frac{n - 2 + \sqrt{n^2 - 4n + 12}}{2} \quad \text{[using (3.15)]} \]

Now if \( G \neq K_n - e \), then any principal sub-matrix of \( \mathcal{D}(G) \) of order \( n - 1 \) dominates \( \mathcal{D}(K_{n-1} - e) \) and therefore \( \rho_2(G) > \frac{n - 2 + \sqrt{n^2 - 4n + 12}}{2} \).

Hence the result follows.

**Corollary 3.6.** For any non complete connected graph \( G \) with \( n \) vertices,

\[
\rho_2(G) \geq n - 2 + \frac{2}{n - 1}
\]

equality holds if and only if \( G = P_3 \).

**Proof.** As \( G \) is connected and non complete, therefore \( n \geq 3 \). Which implies that

\[
n^2 - 4n + 12 \geq \left( n - 2 + \frac{4}{n - 1} \right)^2.
\]

Thus

\[
\frac{n - 2 + \sqrt{n^2 - 4n + 12}}{2} \geq n - 2 + \frac{2}{n - 1} \quad \text{(3.16)}
\]

It can easily be shown that equality in (3.16) holds if and only if \( n = 3 \). Therefore by Theorem 3.5 we get

\[
\rho_2(G) \geq n - 2 + \frac{2}{n - 1},
\]

with equality if and only if \( G = K_3 - e = P_3 \).

**Theorem 3.7.** If \( G \) is a connected graph of order \( n \) so that minimum transmission occur at a vertex \( v \in G \) and \( x \) is the normalized distance Pareto eigenvector corresponding to \( \rho_2 \), then

\[
\rho_2(G) \geq \frac{2[W - Tr(v)]}{n - 1},
\]

with equality if and only if \( x_u = \frac{1}{\sqrt{n-1}} \) for \( u \neq v \).
Proof. Let $D_v$ be the sub-matrix of $\mathcal{D}(G)$ obtained by deleting row and column corresponding to vertex $v$. Then the average row sum of $D_v$ equals $\frac{2(W - Tr(v))}{n-1}$. Therefore using Corollary 2.7 we get

$$
\rho_2(G) \geq \rho(D_v) \\
\geq \frac{2(W - Tr(v))}{n-1}.
$$

Now equality holds in the above expression if and only if $\rho_2(G) = \rho(D_v)$ and all the row sums of $D_v$ are equal which is the case if and only if $x_v = 0$ and $x_u$ is constant for $u \neq v$. Thus we get the required condition from the facts that $x$ is normalized and exactly one component of $x$ is zero. 

Note: The equality in Theorem 3.7 holds for several graphs like complete graph, star graph, wheel graph etc.

**Theorem 3.8.** For any connected graph $G$ of order $n$ other than $K_n$ and $K_n - e$

$$
\rho_2(G) \geq n - 2 + \sqrt{n^2 - 4n + 20},
$$

with equality if and only if $G = K_n - \{e_1, e_2\}$, where $e_1$ and $e_2$ are not incident in $K_n$.

Proof. Let $G_1 = K_n - \{e_1, e_2\}$, where $e_1, e_2 \in E(K_n)$ are not incident and $G_2 = K_n - \{f_1, f_2\}$, where $f_1, f_2 \in E(K_n)$ are incident. Upto permutation similarity there are exactly two distinct principal sub-matrices of $\mathcal{D}(G_1)$ of order $n - 1$ and they are given by

$$
M = \begin{pmatrix}
2(J_2 - I_2) & J_2 - I_2 & J - I \\
J_2 - I_2 & 2(J_2 - I_2) & J_2 - I_2 \\
J - I & J_2 - I_2 & J - I
\end{pmatrix}
$$

and

$$
N = \begin{pmatrix}
2(J_2 - I_2) & J - I \\
J - I & J - I
\end{pmatrix}.
$$

Clearly $M$ dominates $N$ and therefore $\rho_2(G_1) = \rho(M)$. If $(\rho, x)$ be the eigenpair of $M$, then due to symmetry

$$
x_1 = x_2 = x_3 = x_4 = a \quad \text{(say)} \quad (3.17)
$$

and

$$
x_5 = \cdots = x_n = b \quad \text{(say)} \quad (3.18)
$$

Using (3.17) and (3.18) in eigenequations of $M$ we get

$$
2a + 2a + (n - 5)b = \rho a
$$

$$
4a + (n - 6)b = \rho b
$$

From (3.19) and (3.20) we get

$$
\rho_2(G_1) = \rho(M) = \frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}.
$$

Now if $G \neq G_1$, then we consider the following two cases.
Case I: $G$ has at most $\binom{n}{2} - 3$ edges. In this case there are principal sub-matrices of $D(G)$ of order $n - 1$ which dominates $M$. Therefore we get

$$\rho_2(G) > \rho(M) = \frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}.$$ 

Case II: If $G = G_2$, then it can be observed that $\rho_2(G_2) = \rho(A)$, where

$$A = \begin{pmatrix}
0 & 2 & 2 & 1 & 1 \\
2 & 0 & 1 & 1 \\
2 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\mathbb{I} & \mathbb{I} & \mathbb{I} & \mathbb{I} - I
\end{pmatrix}.$$ 

If $(\rho, x)$ is the eigenpair of $A$, then due to symmetry,

$$x_2 = x_3 \quad \text{and} \quad x_4 = x_5 = \cdots = x_n.$$ 

Therefore from eigenequations we have

\begin{align*}
2x_2 + 2x_2 + (n - 4)x_4 &= \rho x_1 \quad (3.21) \\
2x_1 + x_2 + (n - 4)x_4 &= \rho x_2 \\
x_1 + 2x_2 + (n - 5)x_4 &= \rho x_4 \quad (3.23)
\end{align*}

From (3.21)–(3.23), we have

$$\begin{vmatrix}
\rho & -4 & -(n - 4) \\
-2 & \rho - 1 & -(n - 4) \\
-1 & -2 & \rho - n + 5
\end{vmatrix} = 0$$

Which implies $(\rho + 2)[\rho^2 - (n - 1)\rho + (n - 8)] = 0$

Therefore $\rho(A)$ is the largest root of $y^2 - (n - 1)y + n - 8 = 0$.

Let

$$f(y) = y^2 - (n - 1)y + n - 8.$$ 

Then it can be verified that

$$f\left(\frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}\right) = \frac{n - \sqrt{n^2 - 4n + 20}}{2} - 3.$$ 

Now $G \neq K_n, K_n - e$ gives $n \geq 4$. Which in turn implies that

$$\frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2} < 3$$

Thus

$$f\left(\frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}\right) < 0 \quad \text{for all} \quad n \geq 3.$$ 

Hence the largest root of $f(y) = 0$ must be greater than $\frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}$.
Lemma 2.4

\[ \rho_2(G) \geq \frac{n - 2 + \sqrt{(n - 2)^2 + 16}}{2}, \]

with equality if and only if \( G = G_1 \).

Theorem 3.9. If \( \lambda_2(G) \) is the second largest distance eigenvalue of a connected graph \( G \), then \( \rho_2(G) > \lambda_2(G) \).

Proof. Let

\[ D(G) = \begin{pmatrix} 0 & x^T \\ x & B \end{pmatrix} \]

such that \( \rho_2(G) = \rho(B) \).

From Cauchy’s inequality (Lemma 2.4), we have

\[ \rho_2(G) \geq \lambda_2(G), \]

with equality if and only if there exists \( z \in \mathbb{C}^{n-1} - \{0\} \) such that

\[ Bz = \rho(B)z \text{ and } x^Tz = 0. \]

Here \( x \) is always a positive vector and \( z \) being an eigenvector corresponding to the Perron value of a nonnegative irreducible matrix is real and is either positive or negative. In either case \( x^Tz = 0 \) can never hold. Hence \( \rho_2(G) > \lambda_2(G) \).

Theorem 3.10. If \( G \) and \( G' = G - e \) are connected graphs then \( \rho_2(G') \geq \rho_2(G) \).

Proof. Suppose \( \rho_2(G) = \rho(A) \), where \( A \) is the sub-matrix of \( D(G) \) obtained by deleting row and column of \( D(G) \) corresponding to vertex \( v \in G \). Let \( B \) be the sub-matrix of \( D(G') \) obtained by deleting row and column corresponding to vertex \( v \).

Then clearly either \( B = A \) or \( B \) dominates \( A \). Thus \( \rho_2(G') \geq \rho_2(G) \).

Note: The inequality in the Theorem 3.10 is not always strict. For example we can consider the graphs \( G_1 \) and \( G_2 = G_1 - e \) as in Figure 2 with \( \rho_2(G_1) = 6 = \rho_2(G_2) \). In Theorem 3.10 if \( e \) is not incident with \( v \) then clearly \( B \) dominates \( A \) and therefore \( \rho_2(G') > \rho_2(G) \). Besides if \( e \) connects \( v \) to \( u \in V(G) \) and for some \( i, j (\neq v) \in V(G) \), if \( d_{ij} \) increases from \( G \) to \( G' \) then again \( B \) dominates \( A \) and therefore \( \rho_2(G') > \rho_2(G) \). In this regard the problem of classifying the edges (if any) in a graph \( G \) whose removal do not increase \( \rho_2(G) \) seems to be an interesting problem.

From Theorem 3.10 we get the following result as immediate corollary.

Corollary 3.11. Among all connected graphs of given order, second largest distance Pareto eigenvalue is maximum for some tree.

Lemma 3.12. If \( a \leq b \), then \( \rho_2(K_{a,b}) = a + b - 3 + \sqrt{a^2 + b^2 + b - ab - 2a + 1} \).

Proof. If \( a = 1 \) then by Theorem 2.20 we know that \( \rho_2(K_{a,b}) = 2(b - 1) \). Now if \( a \geq 2 \) then up to permutation similarity \( D(K_{a,b}) \) has exactly two distinct principal submatrices of order \( n - 1 \) namely \( D(K_{a-1,b}) \) and \( D(K_{a,b-1}) \). It can be observed that

\[ \rho_1(K_{a,b}) = a + b - 2 + \sqrt{a^2 + b^2 - ab}. \]
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Therefore we get
\[
\rho(D(K_{a-1,b})) = a + b - 3 + \sqrt{a^2 + b^2 + b - ab - 2a + 1}
\]
and
\[
\rho(D(K_{a,b-1})) = a + b - 3 + \sqrt{a^2 + b^2 + a - ab - 2b + 1}
\]

Now \(a \leq b\) implies that \(\rho(D(K_{a-1,b})) > \rho(D(K_{a,b-1}))\) and the lemma follows.

\[\square\]

**Theorem 3.13.** If \(G\) is a connected bipartite graph of order \(n\), then

\[
\rho_2(G) \geq n - 3 + \sqrt{n^2 + n + 1 + 3\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor - n - 1\right)}
\]

equality is attained if and only if \(G = K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil}\)

**Proof.** Let \(G\) be a bipartite graph with \(V(G) = V_1 \cup V_2\) as vertex bipartition such that \(|V_1| = p\) and \(|V_2| = q\). Then from Theorem 3.10 we have

\[
\rho_2(G) \geq \rho_2(K_{p,q}). \tag{3.24}
\]

Now from the proof of Lemma 3.12 we see that for any edge \(e = (u,v)\) in \(K_{p,q}\), where \(p \geq 2\) we can have a vertex \(w\) in \(K_{p,q}\) different from \(u\) and \(v\) so that \(\rho_2(K_{p,q}) = \rho(A)\) where \(A = (D(K_{p,q}))(w)\). Since \(e\) is not incident with \(w\), therefore \((D(K_{p,q} - e))(w)\) dominates \(A\) and therefore \(\rho_2(K_{p,q} - e) > \rho(K_{p,q})\). Thus the equality in (3.24) holds if and only if \(G = K_{p,q}\). Again for \(p \leq \left\lfloor\frac{n}{2}\right\rfloor\) writin \(q = n - p\) we get from Lemma 3.12

\[
\rho_2(K_{p,n-p}) = n - 3 + \sqrt{n^2 + n + 1 + 3p(p - n - 1)}
\]

which is a strictly decreasing function for \(p \leq \left\lfloor\frac{n}{2}\right\rfloor\). Thus we get

\[
\rho_2(K_{1,n-1}) > \rho_2(K_{2,n-2}) > \cdots > \rho_2(K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil})
\]

Considering all the above arguments we can say that \(\rho_2(G) \geq \rho_2(K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil})\) and equality holds if and only if \(G = K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil}\). This proves the theorem.
Lemma 3.14. Let $G$ be a tree, $G'$ be any connected graph and $G^v = G_v \ast G^v_w$ with $v \in V(G), w \in V(G')$ and $\rho^u(G^v) = \rho(A_u)$ where $A_u = D(G^v)(u)$. If $i, j, k \in V(G)$ with $i \sim j \sim k$ then for any $u \in V(G) \cup V(G')$

$$\rho^u(G^i) + \rho^u(G^k) \geq 2 \rho^u(G^j).$$

Furthermore, either $\rho^u(G^i) > \rho^u(G^j)$ or $\rho^u(G^k) > \rho^u(G^j)$.

Proof. Without loss of generality we have

$$D(G^i) + D(G^k) - 2D(G^j) = \begin{pmatrix} G' & G_i & G_j & G_k \\ 0 & 0 & 2 \mathbb{I} & 0 \\ 0 & 0 & 0 & 0 \\ 2 \mathbb{I}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $G_i, G_j, G_k$ are components of $G - \{ij, jk\}$ containing $i, j, k$ respectively. Now since $|V(G_j)| \geq 1$ and $|V(G')| \geq 2$, for any $u \in V(G) \cup V(G')$ we get

$$D(G^i)(u) + D(G^k)(u) - 2D(G^j)(u) \geq 0.$$ 

Therefore if $x$ is the Perron vector of $D(G^j)(u)$, then

$$x^T \{D(G^i)(u) + D(G^k)(u) - 2D(G^j)(u)\}x \geq 0$$

Thus

$$\rho^u(G^i) + \rho^u(G^k) - 2 \rho^u(G^j) \geq x^T \{D(G^i)(u) + D(G^k)(u) - 2D(G^j)(u)\}x \geq 0. \quad \text{[Figure 3]}$$

(3.25)

Now if $|V(G_j)| \geq 2$ or if $u \neq j$, then $D(G^i)(u) + D(G^k)(u) - 2D(G^j)(u) \neq 0$ and therefore Inequality (3.25) is strict and we are done. But if $u = j$, then $D(G^i)(u) + D(G^k)(u) - 2D(G^j)(u) = 0$. In this case to the contrary we assume that $\rho^u(G^i) = \rho^u(G^k) = \rho^u(G^j) = \rho$ (say). Then we must have $D(G^i)(u)x = \rho x, D(G^j)(u)x = \rho x$ and $D(G^k)(u)x = \rho x$. But we see that
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\[ D(G^i)(u) - D(G^j)(u) = \begin{pmatrix} G' & G_i & G_k \\ 0 & -J & J \\ -J^T & 0 & 0 \\ J^T & 0 & 0 \end{pmatrix} \]

Therefore as \((D(G^i)(u) - D(G^j)(u))x = 0\) we observe from the above matrix that

\[ \sum_{\ell \in V(G_i)} x_\ell = 0 \quad \text{and} \quad \sum_{\ell \in V(G_k)} x_\ell = 0 \]

which are not possible as \(x\) is a positive vector and \(V(G_i), V(G_k)\) are non empty. Hence either \(\rho_u(G^i) \neq \rho_u(G^j)\) or \(\rho_u(G^k) \neq \rho_u(G^j)\) and therefore we get the result as desired.

**Corollary 3.15.** Let \(T\) be a tree and \(G'\) be a graph. Then \(\rho_2(G^i)\) is strictly quasiconvex on \(T\), where \(G^i\) as defined in Lemma 3.14.

**Proof.** Let \(u\) be the vertex of \(G^i\), so that \(\rho_2(G^i) = \rho(D(G^i)(u))\). Then by Lemma 3.14, we get either \(\rho_u(G^i) > \rho_2(G^i)\) or \(\rho_u(G^k) > \rho_2(G^j)\). But by Theorem 3.1 we have \(\rho_2(G^i) \geq \rho_u(G^i)\) and \(\rho_2(G^k) \geq \rho_u(G^k)\). Hence the result follows.

As a result of repeated application of Lemma 2.12 and Corollary 3.15 we get the following results.

**Theorem 3.16.** Among all trees of given order, the second largest distance Pareto eigenvalue is maximized in the path graph.

**Theorem 3.17.** Among all trees of given order, the second largest distance Pareto eigenvalue is minimized in the star graph.

From Theorem 2.22 we see that among all connected graphs of given order \(n\) the minimum value of the second largest distance Pareto eigenvalue is \(n - 2\) and is uniquely achieved by the complete graph \(K_n\). Now using Corollary 3.11 and Theorem 3.16 we have the following result regarding the unique graph with maximum second largest distance Pareto eigenvalue among all connected graphs of given order.

**Theorem 3.18.** Among all connected graphs of given order, the second largest distance Pareto eigenvalue is maximized in the path graph.

Let us reserve the following notations for the rest of this article. For \(S \subset [n]\),

\[ X_n(S) = \{ x \in \mathbb{R}^n : x \geq 0, x^T x = 1, x_i = 0 \iff i \in S \} \]

and \(X_n^k = \{ x \in X_n(S) : S \subset [n], |S| = k \} \).

**Theorem 3.19.** If \(A \in M_n\) is symmetric, non negative and irreducible (or positive), then

\[ \rho_2(A) = \max_{x \in X_n^1} x^T Ax. \]
Proof. From Theorem 2.8, it can be observed that $\rho_2(A)$ is the largest possible eigenvalue among all principal submatrices of $A$ of order $n - 1$.

For any $i \in [n]$ let $B_i = A(i)$ and $x \in \mathbb{R}^{n-1}$ be an arbitrary normalized vector. If we take $z \in \mathbb{R}^n$ such that $z_i = 0$ and $z_j = x_j$ for $j \neq i$, then $z^T A z = x^T B_i x$. Now

$$
\rho(B_i) = \max_{x \in \mathbb{R}^{n-1}} x^T B_i x \quad \text{subject to} \quad x^T x = 1
$$

$$
\rho(B_i) = \max_{x \in \mathbb{R}^{n-1}} x^T B_i x
$$

$$
\rho(B_i) = \max_{x \in \mathbb{R}^{n-1}} x^T B_i x
$$

Therefore

$$
\rho_2(A) = \max_{i \in [n]} \rho(B_i)
$$

$$
\rho_2(A) = \max_{i \in [n]} \max_{z \in X_n({\{i}\})} z^T A z
$$

$$
\rho_2(A) = \max_{i \in [n]} \max_{z \in X_n({\{i}\})} z^T A z
$$

$$
\rho_2(A) = \max_{z \in X_n^1} z^T A z
$$

This completes the proof. 

Corollary 3.20. If $z \in X_n^1$ and $A \in M_n$ is symmetric, non negative irreducible (or positive) then

$$
\min_{z \in X_n^1} z^T A z \leq \rho_2(A) \leq \max_{z \in X_n^1} z^T A z,
$$

with left hand (right hand) equality if and only if $z$ is the normalized Pareto eigenvector of $A$ corresponding to $\rho_2(A)$.

Corollary 3.21. If $A \in M_n$ is symmetric, non negative irreducible (or positive) then

$$
\rho_2(A) = \max_{x \in T_n^k} \frac{x^T A x}{x^T x}
$$

where $T_n^k = \{ x \in \mathbb{R}^n : x \geq 0 \text{ and } x_i = 0 \text{ for exactly } k \text{ values of } i \in [n] \}$.

Theorem 3.22. If $G$ is a connected graph of diameter $d$ and $T_{\min} = \min_{v \in V(G)} Tr(v)$, then

$$
\rho_2(G) \geq \frac{T_{\min} - 2d + \sqrt{(T_{\min} - 2d)^2 + 4(n - d - 1)}}{2}
$$

Equality holds if and only if $G = K_n$.

Proof. Let $x = (x_1, x_2, \ldots, x_n)^T$ be the Pareto eigenvector of $D(G)$ corresponding to $\rho_2(G)$ with $x_i = 0$ i.e. $\rho_2(G) = \rho(D(G)(i))$.

Let

$$
x_j = \min_{\ell \neq i} x_\ell \quad \text{and} \quad x_k = \min_{\ell \neq i,j} x_\ell.
$$
For $j \in V(G)$ if $T_j = Tr(j)$, then from Pareto eigenequations we have

$$\rho_2 x_j = \sum_{\ell \in V(G)} d_{j\ell} x_{\ell}$$

$$\geq (T_j - d_j)x_k$$

$$\geq (T_j - d)x_k$$  \hspace{1cm} (3.26)$$

$$\rho_2 x_k = \sum_{\ell \in V(G)} d_{k\ell} x_{\ell}$$

$$\geq d_{kj}x_j + (T_k - d_{ki} - d_{kj})x_k$$

$$\geq x_j + (T_k - 2d)x_k$$  \hspace{1cm} (3.27)

$$\rho_2 x_k = \sum_{\ell \in V(G)} d_{k\ell} x_{\ell}$$

$$\geq d_{kj}x_j + (T_k - d_{ki} - d_{kj})x_k$$

$$\geq x_j + (T_k - 2d)x_k$$  \hspace{1cm} (3.28)

From (3.27) and (3.28) we get

$$\rho_2 (\rho_2 - T_k + 2d) - T_j + d \geq 0$$

$$\Rightarrow \rho_2 \geq \frac{T_k - 2d + \sqrt{(T_k - 2d)^2 + 4(T_j - d)}}{2}$$

$$\geq \frac{T_k - 2d + \sqrt{(T_k - 2d)^2 + 4(n - d - 1)}}{2}.$$  \hspace{1cm} (3.29)

If $G = K_n$, then $T_k = n - 1$ and $d = 1$. Therefore by Lemma 2.16 equality holds in (3.29).

Now suppose the equality holds in (3.29) then equality must hold in all the inequalities (3.26)–(3.29). Equality in (3.29) gives $d_j = n - 1$. Again equality in (3.28) implies $d_{jk} = d$. Therefore we must have $d = 1$ i.e. $G = K_n$.

**Theorem 3.23.** Let $G$ is a connected graph and $(\rho_2(G), \mathbf{x})$ be a distance Pareto eigenpair of $G$. If the second largest component of $\mathbf{x}$ corresponds to $j \in V(G)$ then

$$\rho_2(G) \leq \frac{T_j - 2 + \sqrt{(T_j - 2)^2 + 4(n - 2)}}{2}.$$  

If equality holds, then $G \cong H_{u,v}$ for some $u, v \in V(G)$. 

---

**Figure 4:** Graph $H_{i,u}$ as in Theorem 3.23
**Proof.** Let \( x = (x_1, x_2, \ldots, x_n)^T \) be the Pareto eigenvector of \( D(G) \) corresponding to \( \rho_2(G) \) with \( x_u = 0 \) i.e. \( \rho_2(G) = \rho(D(G)(u)) \).

Let

\[
x_i = \max_{k \in [n]} x_k \quad \text{and} \quad x_j = \max_{k \neq i} x_k
\]

From Pareto eigenequations we have

\[
\rho_2 x_i = \sum_{\ell \in V(G)} d_{i\ell} x_\ell \\
\leq (T_i - d_{iu}) x_j \\
\leq (T_i - 1) x_j
\]

(3.30)

and

\[
\rho_2 x_j \leq d_{ij} x_i + (T_j - d_{ju} - d_{ij}) x_j \\
\leq x_i + (T_j - 2) x_j
\]

(3.31)

Now (3.30) and (3.31) together implies \( \rho_2^2 - (T_j - 2)\rho_2 - (T_i - 1) \leq 0 \). Which in turn gives

\[
\rho_2 \leq \frac{T_j - 2 + \sqrt{(T_j - 2)^2 + 4(n - 2)}}{2}
\]

(3.32)

Now suppose the equality holds in (3.32), then equality must hold in (3.30) and (3.31) as well.

Equality in (3.30) implies

\[
i \sim u \quad \text{and} \quad x_k = x_j \quad \text{for all} \quad k \neq i, u
\]

Thus \( T_k = T_j \quad \text{for all} \quad k \neq i, u \quad [\text{using (3.31)}] \) (3.33)

Equality in (3.31) implies \( i \sim j \sim u \) and equality in (3.32) gives \( d_i = n - 1 \) and so \( \text{diam}(G) \leq 2 \). Hence using (3.33) we get \( T_k = d_k + 2(n - d_k - 1) = 2(n - 1) - d_k \) for all \( k \neq i, u \).

Combining all the above arguments, we get the required result.

\[\blacksquare\]

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