DISSIPATIVE SOLUTIONS AND MARKOV SELECTION TO THE COMPLETE STOCHASTIC EULER SYSTEM

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Abstract. We introduce the concept of stochastic measure-valued solutions to the complete Euler system describing the motion of a compressible inviscid fluid subject to stochastic forcing, where the nonlinear terms are described by defect measures. These solutions are weak in the probabilistic sense (probability space is not a given "a priori", but part of the solution) and analytical sense (derivatives only exists in the sense distributions). In particular, we show that: existence, weak-strong principle; a weak measure-valued solution coincides with a strong solution provided the later exists, all hold true provided they satisfy some form of energy balance. Finally, we show the existence of Markov selection to the associated martingale problem.

1. Introduction

In 1985, Diperna [23] proposed the concept of measure-valued solutions to nonlinear systems of partial differential equations (PDEs) admitting the uncontrollable oscillations (in the context of conservation laws). In the framework of fluid dynamics, Diperna and Majda [22], introduced the concept of measure-valued solutions to the incompressible Euler system. Later on, the concept of measure-valued solutions was extended to compressible fluid dynamics by Neustupa [42], Kröner and Zajączkowski [40], and revisited recently by Breit et al. [3], Feireisl et. al. in [25, 26] and references therein, where they developed the concept of dissipative measure-valued solutions. In this paper, we consider the Complete Euler System describing the motion of a temperature dependent compressible inviscid fluid flow driven by stochastic forcing. The fluid model is described by means of three basic state variables: the mass density \( \rho = \rho(t,x) \), the velocity field \( \mathbf{u} = (t,x) \), and the (absolute) temperature \( \vartheta = \vartheta(t,x) \), where \( t \) is the time, \( x \) is the space variable (Eulerian coordinate system). The time evolution of the fluid flow is governed by a system of partial differential equations (mathematical formulations of the physical principles) given by

\[
\begin{align*}
\frac{d}{dt} \rho + \text{div}_x (\rho \mathbf{u}) &= 0 \quad \text{in } Q, \\
\frac{d}{dt} (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, \vartheta) &= \rho \phi \, dW \quad \text{in } Q, \\
\frac{1}{2} \left( \rho |\mathbf{u}|^2 + \rho e(\rho, \vartheta) \right) + \text{div}_x \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \rho e(\rho, \vartheta) + p(\rho, \vartheta) \right) \mathbf{u} &= \frac{1}{2} \sqrt{\rho \phi} \|\mathbf{u}\|^2_{L^2} \, dt + \rho \phi \cdot \mathbf{u} \, dW,
\end{align*}
\]

describing: the balance of mass, momentum, total energy, respectively. Here, \( p(\rho, \vartheta) \) denotes pressure, the driving force is represented by a cylindrical Wiener process \( W \), and \( \phi \) is a Hilbert-Schmidt operator, see Section 2.2 for details. For completeness, the system (1.1) is supplemented by a set of constitutive relations characterising the physical principles of a compressible inviscid fluid. In particular, we assume that the pressure \( p(\rho, \vartheta) \) and the internal energy \( e = e(\rho, \vartheta) \) satisfy the caloric equation of state

\[
(1.2) \quad p = (\gamma - 1) \rho e,
\]

where \( \gamma > 1 \) is the adiabatic constant. In addition, we suppose that the absolute temperature \( \vartheta \) satisfies the Boyle-Mariotte thermal equation of state:

\[
(1.3) \quad p = \rho \vartheta \quad \text{yielding} \quad e = c_v \vartheta, c_v = \frac{1}{\gamma - 1},
\]
Finally, we suppose that the pressure $p = p(\rho, \vartheta)$, the specific internal energy $e = e(\rho, \vartheta)$, and the specific entropy $s = s(\rho, \vartheta)$ are interrelated through Gibbs’ relation

$$ \vartheta Ds(\rho, \vartheta) = De(\rho, \vartheta) + p(\rho, \vartheta)D \left( \frac{1}{\rho} \right). $$

If $p, e, s$ satisfy (1.4), in context of any smooth solutions to (1.1), the Second law of thermodynamics is enforced through the entropy balance equation

$$ d(\rho s(\rho, \vartheta)) + \text{div}_x(gs(\rho, \vartheta)u)\, dt = 0, $$

where $s(\rho, \vartheta)$ denotes the (specific) entropy and is of the form

$$ s(\rho, \vartheta) = \log(\vartheta c_v) - \log(\rho). $$

For weak solutions, the equality in (1.5) no longer holds, the entropy balance is given as an inequality, for more details see [1]. To circumvent problems from physical boundaries, we impose periodic boundary conditions, the physical domain $\mathbb{T}^3$ can be identified with a flat torus

$$ \mathbb{T}^3 = ([0,1]\times[0,1])^3. $$

Finally, the initial state of fluid emanates from random initial data

$$ \rho(0, \cdot) = \rho_0, \quad \vartheta(0, \cdot) = \vartheta_0, \quad u(0, \cdot) = u_0. $$

For physical relevant solutions, the problem is augmented by the total energy balance

$$ d \int_{\mathbb{T}^3} \left[ \frac{1}{2}\rho |u|^2 + \rho e \right] dx = \int_Q \rho \phi \cdot u \, dW + \int_0^T \frac{1}{2} \| \sqrt{\rho} \phi \|_{L_2}^2 \, dt. $$

The strong solutions of the system (1.1) satisfy (1.8), but in weak solutions it has to be added in the definition.

The deterministic counterpart of the Cauchy problem (1.1) has been extensively studied, and it is well-known that its classical solutions exist only for a finite time after which singularities may develop no matter how smooth or small the initial data are. Consequently, the concept of weak (distributional) solutions is sought to study global-in-time behavior of the system (1.1). Furthermore, the weak solutions may not be uniquely determined by their initial data. Hence, an admissibility criteria condition must be imposed to select physically relevant solutions. In addition, more recently, the results of DeLellis, Székelyhidi and their collaborators [16, 17, 21] show the existence and non-uniqueness of weak solutions to the isentropic Euler system via the method of convex integration. In particular, non-uniqueness was established for weak solutions satisfying the standard entropy admissibility criteria, and to be precise, the deterministic compressible Euler system is ill-posed, see Buckmaster et al [2] and references therein for more details. The existence of weak solutions to Euler system were further extended in [18] to incorporate a compressible heat conducting gas. Finally, results established in [3] show that it is possible to select a system of solutions satisfying the classical semiflow property for the complete Euler system.

The study of stochastically perturbed equations of motion is motivated in two folds: (i) modelling perturbations (numerical, empirical, and physical uncertainties) and thermodynamics fluctuations present in fluid flows; in particular, turbulence, (ii) to circumvent the issue of deterministically ill-posed problems, researchers adopted the use of stochastic perturbation with hope it will provide a regularising effect to the underlying systems. And indeed, recently, the results by Flandoli and Luo [31] showed that a noise of transport type improves the vorticity blow-up control in the Navier-Stokes. In the stochastic context, existence of global-in-time weak solutions for (1.1) were shown in [15] using convex integration, and they identified a large class of initial data for which the complete Euler system is ill-posed; that is, there exist infinitely many global in time solutions.

Our goal in the present paper is to show the existence of dissipative measure-valued solutions to a stochastically driven Euler system (1.1), properties of solutions, that is, weak-strong principle and Markov selection. In particular, measure-valued solutions satisfy the admissibility criterion; they conserve the total energy and
satisfy an appropriate form version of the entropy (entropy admissibility criterion), and they exist global-in-time for any finite initial data. The study of measure-valued solutions is motivated by the following arguments: Inspired by results in Brenier et al [11] in the incompressible Euler system, we see measure-valued solutions as possibly the largest class in which the family of smooth (classical) solutions is stable. Specifically, the weak (measure-valued)-strong uniqueness principle holds, and solutions emanating from numerical schemes can be shown to converge to a measure-valued solutions while the convergence to weak solutions is either not known or computational expensive, see Fjordholm et al [30] and references therein for more details. The concept of measure-valued solutions to fluid model systems driven by stochastic forcing is fairly a new subject area of research. To the best of our knowledge, the study of stochastic measure-valued solutions to the complete Euler system governing the motion of an inviscid, temperature dependent, and compressible fluid subject to stochastic forcing is still an open question. Hence, this is a first attempt to characterise the concept of measure-valued solutions to the full stochastic Euler system. However, it is worth mentioning that, the existence of measure-valued solutions for stochastic incompressible Euler has been established in [9, 36, 39].

In this present work, we prove the existence of measure-valued martingale solutions to the complete Euler system (1.1) following the strategy in [9, 34, 39]. These solutions are weak, in the analytical sense (derivatives only exists in the sense of distributions) and in the probabilistic sense (the probability space is not a given priori, but an integral part of the solution). We start with an approximate system with high order diffusion, see Section 2.4 and show existence of solutions to the original problem in the limit via stochastic compactness arguments based on Jakubowski’s variant of the Skorokhod representation theorem [37]. The latter is needed due to the complicated path space which arises because of the presence of measures describing the oscillations and concentrations in the nonlinearities of the Euler system. Moreover, we deduce the relative entropy inequality (see Section 5); a tool that allows us to establish the weak(measure-valued)-strong uniqueness principle. Although we do not expect solutions to be unique, there is some hope to select solutions which are in a sense continuous with respect to the initial data. This is the Markov property; the memoryless of the stochastic process, the probability law of the future only depends on the current state of the process, but it is independent from the past, see the monograph by Stroock and Varadhan [45] for a thorough exposition. Our work shows the stochastic analog results of [3], that is, the existence of Markov selection to the associated martingale problem following the presentations in [4, 32, 33]. At first sight, the overall proof outline is rather similar, however, we encountered several challenges in this paper. A major challenge originates in the use of defect measures. The defect measures are an equivalence class in time and not stochastic processes in the classical sense. Therefore, it is not clear as to how one applies the Markov selection. To circumvent this problem, we introduce auxiliary continuous stochastic variables \([S, R]\) such that

\[
S = \int_0^t S \, ds, \quad R = \int_0^t (\mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{press}}, V_{t,x}) \, ds,
\]

and this allows us to show Markov selection for \([\theta, m, S, R]\), see Section 6. Here, \(S\) and \(R\) are the defect measures related to entropy balance and momentum equation, respectively. Note, such an approach is reminiscent of [4], where a similar idea was used for the velocity field. It is important to note that, different to [4, 32, 33], we obtain a strong Markov selection. This is due to the energy equality which is a feature of the system (1.1) and is not known to hold for the problems studied in [4, 32, 33]. However, a first result on strong Markov selection has been obtained recently by Hofmanová-Zhu-Zhu [36]. In [36] they study the incompressible stochastic Euler equations and obtain the energy equality by introducing a defect measure for the energy which is included in the Markov selection.

The rest of the paper is organised as follows: we start off with mathematical framework and main results in Section 2. In section 3 we show existence of martingale solutions for the approximate system. In Section 4 we prove the existence of measure-valued martingale solutions using stochastic compactness arguments. Finally, Section 5 is dedicated to showing the weak (measure-valued)-strong uniqueness principle, while Section 6 is dedicated to the Markov selection.
2. Mathematical framework and main results

In this section we present the the probability framework for Markov selection and stochastic framework. In particular we consider tools required for analysis in subsequent sections and state the main results of the paper. Throughout the paper, we assume that $C$ denotes various generic constant.

2.1. Probability framework. Let $X$ be a topological space. We denote by $\mathcal{B}(X)$ the $\sigma$-algebra of Borel subsets of $X$. Let $\mathcal{P}$ be a Borel measure on $X$, the symbol $\mathcal{B}(X)$ denotes the $\sigma$-algebra of all Borel subsets of $X$ augmented by all zero measure sets. Let $\text{Prob}[X]$ denote the set of all Borel probability measures on a topological space $X$. Furthermore, let $(\{0, 1\}, \mathcal{B}[\{0, 1\}], \mathcal{L})$ denote the standard probability space, where $\mathcal{L}$ is the Lebesgue measure.

2.1.1. Trajectory/Path spaces. In regards to the notion of solutions in this paper, let $(X, d_X)$ be a Polish space. For $t > 0$ we introduce the path spaces

$$\Omega_X^{[0,t]} = C([0, t]; X), \quad \Omega_X^{[t,\infty)} = C([t, \infty); X), \quad \Omega_X^{(0,\infty)} = C([0, \infty); X),$$

where the path spaces are Polish as long as $X$ is Polish, and we denote by $\mathcal{B}_t = \mathcal{B}(\Omega_X^{[0,t]})$ the Borel $\sigma$-algebra. Then, for $\omega \in \Omega_X^{[0,t]}$ we define a time shift operator

$$\Phi_\tau : \Omega_X^{[0,\infty)} \to \Omega_X^{[t+\tau,\infty)}, \quad \Phi_\tau[\omega]_s = \omega_{t+s}, s \geq t + \tau,$$

where $\Phi_\tau$ is an isometry mapping from $\Phi_\tau : \Omega_X^{[0,\infty)}$ to $\Omega_X^{[t+\tau,\infty)}$. For a Borel measure $\nu$ on $\Omega_X^{[0,\infty)}$, the time shift $\Phi_{-\tau}[\nu]$ is a Borel measure on the space $\Omega_X^{[t-\tau,\infty)}$ given by

$$\Phi_{-\tau}[\nu](B) = \nu(\Phi_\tau(B)), \quad B \in \mathcal{B}(\Omega_X^{[t-\tau,\infty)}).$$

In the following parts, we recall important results of Stroock and Varadhan [45]. Firstly, from Theorem 1.1.6 in [45] we obtain disintegration results, that is, existence of regular conditional probability law. This implies that a regular

**Theorem 2.1** (Disintegration). Let $X$ be a polish space. Given $\mathcal{P} \in \text{Prob}[\Omega_X^{(0,\infty)}]$, let $T > 0$ be a finite $\mathcal{B}$-stopping time. Then there exists a unique family of probability measures

$$\mathcal{P}|_{\mathcal{B}_T} \in \text{Prob}[\Omega_X^{[T,\infty)}] \text{ for } \mathcal{P}-a.a.\tilde{\omega}$$

such that the mapping

$$\Omega_X^{(0,\infty)} \ni \tilde{\omega} \mapsto \mathcal{P}|_{\mathcal{B}_T} \in \text{Prob}[\Omega_X^{[T,\infty)}]$$

is $\mathcal{B}_T$-measurable and the following properties hold

(a) For $\omega \in \Omega_X^{(0,\infty)}$ we have $\mathcal{P}|_{\mathcal{B}_T} -a.s.$

$$\omega(T) = \tilde{\omega}(T);$$

(b) For any Borel set $A \subset \Omega_X^{[0,T]}$ and any Borel set $B \subset \Omega_X^{[T,\infty)}$,

$$\mathcal{P}(\omega|_{[0,T]} \in A, \omega|_{[T,\infty)} \in B) = \int_{\omega|_{[0,T]} \in A} \mathcal{P}|_{\mathcal{B}_T} \tilde{\omega}(B) d\mathcal{P}(\tilde{\omega}).$$

Note, a conditional probability corresponds to disintegration of probability measure with respect to a $\sigma$-field. Accordingly, reconstruction can be understood as the opposite process of disintegration, that is, some sort of “gluing together” procedure. Based on [45] Lemma 6.1.1 and Theorem 6.1.2 we have the following results on reconstruction.

**Theorem 2.2** (Reconstruction). Let $X$ be a Polish space. Let $\mathcal{P} \in \text{Prob}[\Omega_X^{(0,\infty)}]$ and $T$ be a finite $\mathcal{B}$-stopping time. Suppose that $Q_\omega$ is a family of probability measures, such that

$$\Omega_X^{(0,\infty)} \ni \omega \mapsto Q_\omega \in \text{Prob}[\Omega_X^{[T,\infty)}],$$

is $\mathcal{B}_T$-measurable. Then there exists a unique probability measure $\mathcal{P} \otimes_T Q$ such that :

(a) For any Borel set $A \in \Omega_X^{[0,T]}$ we have

$$(\mathcal{P} \otimes_T Q)(A) = \mathcal{P}(A);$$
(b) For \( \tilde{\omega} \in \Omega \) we have \( \mathcal{P} \)-a.s.
\[
(P \otimes T)|_{\tilde{\omega}\tau} = Q_{\tilde{\omega}}
\]

2.1.2. Markov processes. Following the abstract framework on Markov processes along lines of [4] and references therein we have: Assuming \((X, dX)\) and \((F, dF)\) are two Polish spaces, let the embedding \(F \hookrightarrow X\) be continuous and dense. Moreover, let \(Y\) be a Borel subset of \(F\). Since \((Y, dY)\) is not necessarily a complete space, we assume that the embedding \(Y \hookrightarrow X\) is not dense. A family of probability measures \(\{\mathcal{P}_y\}_{y \in Y}\) on \(\Omega_X^{(0, \infty)}\) is called Markovian if we have for \(y \in Y\) that
\[
\mathcal{P}_\omega(\tau) = \Phi^{-\tau} \mathcal{P}_y|_{\tilde{\omega}\tau} - \text{a.a. } \omega \in \Omega_X^{(0, \infty)} \text{ and all } \tau \geq 0.
\]

Next we define probability measures with support only on certain sub set of a Polish space.

**Definition 2.3.** Let \(Y\) be a Borel subset of \(F\) and let \(U \subseteq \text{Prob}[\Omega_X^{(0, \infty)}]\). A family of probability measures \(U\) is concentrated on the paths with values in \(Y\) if there is some \(A \subseteq \mathcal{B}(\Omega_X^{(0, \infty)})\) such that \(\mathcal{P}(A) = 1\) and \(A \subseteq \{\omega \in \Omega_X^{(0, \infty)} : \omega(\tau) \in Y \forall \tau \geq 0\}\). We write \(U \subseteq \text{Prob}_Y[\Omega_X^{(0, \infty)}]\).

Finally, based on the link between disintegration and reconstruction as observed by [41], we have the following definition.

**Definition 2.4** (Almost Sure Markov property). Let \(y \mapsto \mathcal{P}_y\) be a measurable map defined on a measurable subset \(Y \subseteq F\) with values in \(\text{Prob}_Y[\Omega_X^{(0, \infty)}]\). The family \(\{\mathcal{P}_y\}_{y \in Y}\) has the almost sure Markov property if for each \(y \in Y\) there is a set \(\mathcal{Z} \subseteq (0, \infty)\) with zero Lebesgue measure such that
\[
\mathcal{P}_\omega(\tau) = \Phi^{-\tau} \mathcal{P}_y|_{\tilde{\omega}\tau} \text{ for } \mathcal{P} \text{-a.a. } \omega \in \Omega_X^{(0, \infty)}
\]
and all \(\tau \not\in \mathcal{Z}\).

We generalise the classical Markov property (to a situation where it only holds for a.e time-point) as follows:

**Definition 2.5** (Almost sure pre-Markov family). Let \(Y\) be a Borel subset of \(F\). Let \(\mathcal{C} : Y \to \text{Comp}(\text{Prob}[\Omega_X^{(0, \infty)}])\) be a measurable map, where \(\text{Comp}(\cdot)\) denotes the family of all compact subsets. The family \(\{\mathcal{C}(y)\}_{y \in Y}\) is almost surely pre-Markov if for each \(y \in Y\) and \(U \subseteq \mathcal{C}(y)\) there is a set \(\mathcal{Z} \subseteq (0, \infty)\) with zero Lebesgue measure such that the following holds for all \(\tau \in \mathcal{Z}\):

(a) The disintegration property holds, that is, we have
\[
\Phi^{-\tau} \mathcal{P}_y|_{\tilde{\omega}\tau} \in \mathcal{C}(\omega(\tau)) \text{ for } \mathcal{P} \text{-a.a. } \omega \in \Omega_X^{(0, \infty)};
\]

(b) The reconstruction property holds, that is, for each \(\mathcal{B}_T\)-measurable map \(\omega \mapsto Q_\omega : \Omega_X^{(0, \infty)} \to \text{Prob}(\Omega_X^{(0, \infty)})\) with
\[
\Phi^{-\tau} Q_\omega \in \mathcal{C}(\omega(\tau)) \text{ for } \mathcal{P} \text{-a.a. } \omega \in \Omega_X^{(0, \infty)}
\]
we have \(\mathcal{P} \otimes T \subseteq \mathcal{C}(y)\).

Note Definition 2.5 is motivated by results in [32 33]. We conclude our probability framework by stating the following results.

**Theorem 2.6** (Markov Selection ?). Let \(Y\) be a Borel subset of \(F\). Let \(\{\mathcal{C}(y)\}_{y \in Y}\) be an almost sure pre-Markov family (as defined in ) with nonempty convex values. Then there is a measurable map \(y \mapsto U_y\) defined on \(Y\) with values in \(\text{Prob}_Y[\Omega_X^{(0, \infty)}]\) such that \(\mathcal{P}_y \subseteq \mathcal{C}(y)\) for all \(y \in Y\) and \(\{\mathcal{P}_y\}_{y \in Y}\) has the almost sure Markov property (as defined in )

The following proposition is proved in [32].

**Proposition 2.7** ([32], Proposition B.1). Let \(\alpha\) and \(\beta\) be two real-valued continuous and \((\mathcal{B}_t)\)-adapted stochastic processes on \(\Omega\) such that and let \(t_0 \geq 0\). For \(U \subseteq \text{Prob}[\Omega]\) the following conditions are equivalent:

(a) \((\alpha_t)_{t \geq 0}\) is a \((\mathcal{B}_t)_{t \geq 0}, U)\)-square integrable martingale with quadratic variation \((\beta_t)_{t \geq 0}\)

(b) For \(U\)-a.a. \(\omega \in \Omega\) the stochastic process \((\alpha_t)_{t \geq t_0}\) is a \((\mathcal{B}_t)_{t \geq t_0}, U|_{\tilde{\omega} t_0})\)-square integrable martingale with quadratic variation \((\beta_t)_{t \geq t_0}\) and we have \(\mathbb{E}^U[\mathbb{E}_{U|_{\tilde{\omega} t_0}}[\beta_t]] < \infty\) for all \(t \geq t_0\).
2.2. Stochastic analysis. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})\) be a complete stochastic basis with a probability measure \(\mathbb{P}\) on \((\Omega, \mathcal{F})\) and right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\). Let \(\mathcal{U}\) be a separable Hilbert space with orthonormal basis \((e_k)_{k \in \mathbb{N}}\). We denote by \(L_2(\mathcal{U}, L^2(\mathbb{T}^3))\) the set of Hilbert-Schmidt operators from \(\mathcal{U}\) to \(L^2(\mathbb{T}^3)\). The stochastic process \(W\) is a cylindrical Wiener process \(W = (W_t)_{t \geq 0}\) in \(\mathcal{U}\), and is of the form

\[
W(s) = \sum_{k \in \mathbb{N}} e_k \beta_k(s),
\]

where \((\beta_k)_{k \in \mathbb{N}}\) is a sequence of independent real-valued Wiener process relative to \((\mathcal{F}_t)_{t \geq 0}\). To identify the precise definition of the diffusion coefficient, let \(U\) be a separable Hilbert space with orthonormal basis \((e_k)_{k \in \mathbb{N}}\). We denote by \(L_2(\mathcal{U}, L^2(\mathbb{T}^3))\) the set of Hilbert-Schmidt operators from \(\mathcal{U}\) to \(L^2(\mathbb{T}^3)\). The stochastic process \(W\) is a cylindrical Wiener process \(W = (W_t)_{t \geq 0}\) in \(\mathcal{U}\), and is of the form

\[
W(s) = \sum_{k \in \mathbb{N}} e_k \beta_k(s),
\]

where \((\beta_k)_{k \in \mathbb{N}}\) is a sequence of independent real-valued Wiener process relative to \((\mathcal{F}_t)_{t \geq 0}\). To identify the precise definition of the diffusion coefficient, set \(U = \mathbb{R}^2\) and consider \(\varphi \in L^2(\mathbb{T}^3), \varphi > 0\), then the mapping \(\phi \in L_2(\mathcal{U}, L^2(\mathbb{T}^3))\), that is, \(\phi : \mathcal{U} \to L^2(\mathbb{T}^3)\) is defined as follows

\[
\phi(e_k) = \phi_k.
\]

We suppose that \(\phi\) is a Hilbert-Schmidt operator such that

\[
\sum_{k \geq 1} \|\phi(e_k)\|_{L^2(\mathbb{T}^3)}^2 < \infty.
\]

Consequently, since \(\phi_k\) is bounded we deduce

\[
\|\sqrt{\mathbb{E}}(\phi_k)\|_{L^2(\mathcal{U}, L^2(\mathbb{T}^3))} \leq c(\phi)(\|\phi\|_{L^1(\mathbb{T}^3)}).
\]

The stochastic integral

\[
\int_0^T \varphi \phi \, dW = \sum_{k \geq 1} \int_0^T \varphi \phi(e_k) \, d\beta_k,
\]

takes values in the Banach space \(C([0, T]; W^{-k,2}(\mathbb{T}^3))\) in the sense that

\[
\int_0^T \left( \int_0^T \varphi \phi \, dW \right) \, dx = \sum_{k \geq 1} \int_0^T \left( \int_0^T \varphi \phi(e_k) \, d\beta_k \right) \, dx, \quad \varphi \in W^{k,2}(\mathbb{T}^3), \ k > \frac{3}{2}.
\]

Finally, we define the auxiliary space \(U_0\) with \(U \subset U_0\) as

\[
U_0 : = \left\{ e = \sum_k \alpha_k e_k : \sum_k \frac{\alpha_k^2}{k^2} < \infty \right\},
\]

\[
\|e\|_{U_0}^2 : = \sum_k \frac{\alpha_k^2}{k^2}, \quad e = \sum_k \alpha_k e_k,
\]

so that the embedding \(U \hookrightarrow U_0\) is Hilbert-Schmidt, and the trajectories of \(W\) belong \(\mathbb{P}\)-a.s. to the class \(C([0, T]; U_0)\) (see [13]).

2.3. Strong solutions. The concept of weak(measure-valued)-strong uniqueness principle for dissipative measure-valued solutions requires existence of strong solutions. These solutions are strong in the probabilistic and PDE sense, at least locally in time. In particular, the Euler system \([11]\) will be satisfied pointwise (not only in the sense of distributions) on the given stochastic basis associated to the cylindrical Wiener process \(W\).

**Definition 2.8 (Strong Solution).** Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete stochastic basis with right continuous filtration, let \(W\) be an \((\mathcal{F}_t)_{t \geq 0}\)-cylindrical Wiener process. The triplet \([r, \Theta, U]\) and a stopping time \(t\) is called a (local) strong solution to the system \([11]\) provided:

- the density \(r > 0\) \(\mathbb{P}\)-a.s., \(t \mapsto r(t, \cdot) \in W^{3,2}(\mathbb{T}^3)\) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted,

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|r(t, \cdot)\|_{W^{3,2}}^p \right] < \infty \quad \text{for all } 1 \leq p < \infty;
\]

- the temperature \(\Theta > 0\) \(\mathbb{P}\)-a.s., \(t \mapsto \Theta(t, \cdot) \in W^{3,2}(\mathbb{T}^3)\) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted,

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|\Theta(t, \cdot)\|_{W^{3,2}}^p \right] < \infty \quad \text{for all } 1 \leq p < \infty;
\]
• the velocity \( t \mapsto U(t, \cdot) \in W^{3,2}(\mathbb{T}^3) \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted,
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| U(t, \cdot) \|_{W^{3,2}}^p \right] < \infty \quad \text{for all } 1 \leq p < \infty;
\]
• for all \( t \in [0,T] \) there holds \( \mathbb{P}\text{-a.s.} \)
\[
r(t \wedge t) = \varrho(0) - \int_0^{t \wedge t} \text{div}_x(rU) \, dt,
\]
\[
(rU)(t \wedge t) = (rU)(0) - \int_0^{t \wedge t} \text{div}(rU \otimes U) \, dt - \int_0^{t \wedge t} \nabla_x p(r, \Theta) \, dt + \int_0^{t \wedge t} r\varphi \, dW,
\]
\[
(rs(r, \Theta))(t \wedge t) = (rs(r, \Theta))(0) - \int_0^{t \wedge t} \text{div}_x(rs(r, \Theta)U) \, dt,
\]
where \( s \) is the total entropy given by (1.9).

**Remark 2.9.** We expect blow up in finite time for strong solutions as in the deterministic case [44].

2.4. **The approximate system.** To begin, we introduce a cut-off function

\[
\chi \in C^\infty(\mathbb{R}), \chi(z) = \begin{cases} 
1 \text{ for } z \leq 0, \\
\chi'(z) \leq 0 \text{ for } 0 < z < 1, \\
\chi(z) = 0 \text{ for } z \geq 1,
\end{cases}
\]

together with the operator

\[
\phi_\varepsilon = \chi \left( |v| - \frac{1}{\varepsilon} \right) \phi, \quad \varepsilon > 0.
\]

Let \( Q = (0,T) \times \mathbb{T}^3 \) be the periodic space-time cylinder. We consider a stochastic variant of a system introduced in [40], and further refined in [3]. That is, the **complete Euler system** (1.1) is approximated by:

\[
\begin{cases}
\begin{aligned}
d\varrho + \text{div}(\varrho u) & dt = 0 & \quad & \text{in } Q, \\
d\varrho u + \text{div}(\varrho u \otimes u) & dt + \nabla_x p(s) \, dt = \varepsilon \mathcal{L}u \, dt + q \phi_\varepsilon \, dW & \quad & \text{in } Q, \\
dqs + \text{div}(qsu) & dt \geq 0 & \quad & \text{in } Q,
\end{aligned}
\end{cases}
\]

with initial conditions

\[
\varrho(0, \cdot) = \varrho_{0, \varepsilon} \, u(0, \cdot) = u_{0, \varepsilon}, \quad s(0, \cdot) = s_{0, \varepsilon}.
\]

Here, the unknown fields are: the fluid density \( \varrho = \varrho(t, x) \), the velocity \( u = u(t, x) \) and the total entropy \( S = qs \). We denote by \( \mathcal{L} \), the suitable ‘viscosity’ operator.

**Remark 2.10** (Viscosity operator). Let \( W^{3,2}(\mathbb{T}^3) \) be a separable Hilbert space

\[
W^{3,2}(\mathbb{T}^3) = \left\{ u \in W^{3,2}(\mathbb{T}^3) \right\},
\]

complemented with \((; ;)\), a scalar product on \( W^{3,2}(\mathbb{T}^3) \), i.e.

\[
((v; w)) = \sum_{|\alpha|=3} \int_{\mathbb{T}^3} \partial_x^\alpha v \cdot \partial_x^\alpha w \, dx + \int_{\mathbb{T}^3} v \cdot w \, dx, \quad v, w \in W^{3,2}(\mathbb{T}^3).
\]

In reference to [38], we consider a self-adjoint operator \( \mathcal{L} \) on \( W^{3,2}(\mathbb{T}^3) \) associated with the bilinear form \((; ;)\) given by

\[
\mathcal{L}u = \Delta^3 u - u = \sum_{|\alpha|=3} (\partial_x^\alpha)\partial_x^\alpha u - u.
\]
Theorem 2.11. Assume \((2.2)\) holds. Let
\[
\int_0^T \int_{\mathbb{T}^3} \left[ au \otimes u : \nabla_x \phi + g^\gamma \exp \left( \frac{s}{c_v} \right) \div \phi \right] \, dx \, dt
\]
for any \(T > 0\), and any \(\phi \in W^{3,2}(\mathbb{T}^3)\). The continuity equation and total entropy in \((2.7)\) are solved strongly, while the momentum equation is solved in the weak sense. We expect the approximate system \((2.7)\) to have stochastically strong solutions, but in the present work martingale solutions are sufficient for our purposes. In the following we state the existence theorem of martingale solutions to the approximate system \((2.7)\).

**Theorem 2.11.** Assume \((2.2)\) holds. Let \(\Lambda_0\) be a Borel probability measure on \(L^\gamma(\mathbb{T}^3) \times L^\gamma(\mathbb{T}^3) \times L^{2\gamma}(\mathbb{T}^3)\) such that
\[
\Lambda \left\{ (\varrho, S, \mathbf{m}) \in L^\gamma(\mathbb{T}^3) \times L^\gamma(\mathbb{T}^3) \times L^{2\gamma}(\mathbb{T}^3) : 0 < \varrho < \overline{\varrho}, 0 < \underline{\varrho} < \varrho < \overline{\varrho} \right\} = 1,
\]
where \(\varrho, \overline{\varrho}, \underline{\varrho}, \overline{\varrho}\) are deterministic constants. Moreover, the moment estimate
\[
\int_{L^\gamma(\mathbb{T}^3) \times L^\gamma(\mathbb{T}^3) \times L^{2\gamma}(\mathbb{T}^3)} \left\| \frac{1}{2} \frac{|m|^2}{\varrho} + c_v \varrho^\gamma \exp \left( \frac{S}{c_v \varrho} \right) \right\|^p \, d\Lambda \leq \infty,
\]
holds for all \(p \geq 1\). Then there exists a martingale solution to the approximate problem \((2.7)\) subject to initial law \(\Lambda_0\).

2.5. **Measure-valued solutions.** In order to introduce the concept of stochastic measure-valued martingale solutions, we reformulate the complete Euler system using the variables \(\mathbf{m} = au\) and \(S = gs\) so that \((1.1)\) reads
\[
\begin{align*}
(2.8) \quad d\varrho + \div_x \mathbf{m} \, dt &= 0 \\
(2.9) \quad d\mathbf{m} + \div_x \left( \frac{m \otimes m}{\varrho} \right) \, dt + \nabla_x p(\varrho, s) \, dt &= \varrho \phi \, dW \quad \text{in } Q, \\
(2.10) \quad dS + \div_x \left( \frac{Sm}{\varrho} \right) \, dt &\geq 0 \quad \text{in } Q.
\end{align*}
\]

We note that, in general, the admissibility criterion for physically possible solutions is the energy equality and it is the only tool of establishing a priori bounds. However, the ‘a priori’ bounds deduced do not guarantee weak convergences of nonlinear terms \(\frac{m \otimes m}{\varrho} p(\varrho, s) \in L^1(\mathbb{T}^3)\) due to the presence of oscillations and concentrations. Given such scenario, we adopt the characterisation of (nonlinear terms) in the weak formulation as combination of Young measures and defect measures.

- Young measures are probability measures on the phase space, they capture the oscillations of the solution.
- Defect measures are measures on physical space-time and they account for the ‘blow up’ type collapse due to possible concentration points.

We are now ready to introduce the concept of measure-valued martingale solutions to the complete stochastic Euler system \((2.8)-(2.11)\). From here onward, we denote by \(\mathcal{M}\) the space of non-negative radon measures, and we denote by \(A\) the space of “dummy variables”:
\[
A = \left\{ [\varrho', \mathbf{m}', S'] \mid \varrho' \geq 0, \mathbf{m}' \in \mathbb{R}^3, S' \in \mathbb{R} \right\}
\]
By \(\mathcal{P}(A)\) we denote the space of probability measures on \(A\).

**Definition 2.12** (Dissipative measure-valued martingale solution). Let \(\Lambda\) be a borel probability measure on \(L^\gamma \times L^{2\gamma}(\mathbb{T}^3)\) and \(\phi \in L_2(\mathcal{U}; L^2(\mathbb{T}^3))\). Then
\[
((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), \varrho, \mathbf{m}, S, \mathcal{R}_{\text{cons}}, \mathcal{R}_{\text{press}}, \mathcal{V}_{1,x}, W)
\]
is called a dissipative measure-valued solution to (2.8)-(2.10) with initial law \( \Lambda \), provided:

(a) \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) is a stochastic basis with complete right-continuous filtration;
(b) \(W\) is a \((\mathcal{F}_t)_{t \geq 0}\)-cylindrical Wiener process;
(c) The density \( \rho \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted and satisfies \( P\text{-a.s.} \)
\[
\rho \in C_{\text{loc}}([0, \infty), W^{-4,2}(T^3)) \cap L^\infty_{\text{loc}}(0, \infty; L^7(T^3));
\]
(d) The momentum \( m \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted and satisfies \( P\text{-a.s.} \)
\[
m \in C_{\text{loc}}([0, \infty), W^{-4,2}(T^3)) \cap L^\infty_{\text{loc}}(0, \infty; L^\frac{28}{5}(T^3));
\]
(e) The total entropy \( S \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted and satisfies \( P\text{-a.s.} \)
\[
S \in L^\infty([0, \infty), L^7(T^3)) \cap BV_{\text{w,loc}}(0, \infty; W^{-1,2}(T^3)), l > \frac{5}{2};
\]
(f) The parametrised measures \((\mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{press}}, \mathcal{V})\) are \((\mathcal{F}_t)_{t \geq 0}\)-progressively measurable and satisfy \( P\text{-a.s.} \)
\[
\begin{align*}
(2.11) & \quad t \mapsto \mathcal{R}_{\text{conv}}(t) \in L^\infty(0, \infty; M^+(T^3, R^{3 \times 3})); \\
(2.12) & \quad t \mapsto \mathcal{R}_{\text{press}}(t) \in L^\infty(0, \infty; M^+(T^3, R)); \\
(2.13) & \quad (t, x) \mapsto \mathcal{V}_{t,x} \in L^\infty(0, \infty; M^+(Q; P(A)));
\end{align*}
\]
(g) \( \Lambda = P \circ (\rho(0), m(0), S(0))^{-1} \);
(h) For all \( \varphi \in C^\infty(T^3) \) and all \( \tau > 0 \) there holds \( P\text{-a.s.} \)
\[
\int_{T^3} \varphi \, dx \bigg|_{t=0}^{\tau} = \int_0^\tau \int_{T^3} m \cdot \nabla \varphi \, dx \, dt;
\]
(i) For all \( \varphi \in C^\infty(T^3) \) and all \( \tau > 0 \) there holds \( P\text{-a.s.} \)
\[
\int_{T^3} m \cdot \varphi \bigg|_{t=0}^{\tau} = \int_0^\tau \int_{T^3} \frac{m \otimes m}{\rho} \cdot \nabla \varphi + \varphi \exp \left( \frac{S}{c_v \rho} \right) \div \varphi \, dx \, dt
\]
\[
\quad + \int_0^\tau \int_{T^3} \nabla \varphi \cdot d\mathcal{R}_{\text{conv}} \, dt + \int_0^\tau \int_{T^3} \div \varphi \, d\mathcal{R}_{\text{press}} \, dt
\]
\[
\quad + \int_0^\tau \varphi \cdot \varphi \, dW;
\]
(j) The total entropy holds in the sense that
\[
\int_0^\tau \int_{T^3} \left( \mathcal{V}_{t,x}; Z(S') \right) \partial_t \varphi + \left( \mathcal{V}_{t,x}, Z(S') \frac{m'}{\rho'} \right) \cdot \nabla \varphi \, dx \, dt \leq \left( \int_{T^3} \mathcal{V}_{t,x}; Z(S') \right) \varphi \, dx \bigg|_{t=0}^{\tau},
\]
for any \( \varphi \in C^1([0, \infty) \times T^3), \varphi \geq 0, P\text{-a.s.}, \) and any \( Z, \mathcal{Z} \in BC(R) \) non-decreasing.

(k) The total energy satisfies
\[
(2.17) \quad E_t = E_s + \frac{1}{2} \int_s^t \| \sqrt{\rho} \phi \|_{L^2(\mathcal{U}; L^2(T^3))}^2 \, ds + \int_s^t \int_{T^3} m \cdot \phi \, dW,
\]
\( P\text{-a.s. for a.a. } 0 \leq s < t, \)
where
\[
E = \int_{T^3} \frac{1}{2} \frac{m^2}{\rho} + c_v \rho^\gamma \exp \left( \frac{S}{c_v \rho} \right) \, dx + \frac{1}{2} \int_{T^3} \div \mathcal{R}_{\text{conv}}(t) + c_v \int_{T^3} \div \mathcal{R}_{\text{press}}(t)
\]
for \( \tau \geq 0 \) and
\[
E_0 = \int_{T^3} \frac{1}{2} \frac{m_0^2}{\rho_0} + c_v \rho_0^\gamma \exp \left( \frac{S_0}{c_v \rho_0} \right) \, dx.
\]

Remark 2.13. The use of cut-off function \( Z \) in (2.16) is inspired by Chen and Frid [14].

\(^1\)Some of our variables are not stochastic processes in the classical sense and we use their adaptedness in the sense of random distributions as introduced in [7] (Chap. 2.2).
2.6. Main results. In light of the above discussion, we are now ready to state the main results of the paper. The existence of dissipative measure-valued martingale solutions follows from the following theorem.

**Theorem 2.14.** Assume (2.2) holds. Let $\Lambda$ be a Borel probability measure on $L^\gamma(T^3) \times L^\gamma(T^3) \times L_weak^2(T^3)$ such that
\[
\Lambda\left\{(\varrho, S, \mathbf{m}) \in L^\gamma(T^3) \times L^\gamma(T^3) \times L_weak^2(T^3) : 0 < \varrho < \varrho_0, 0 < \vartheta < \vartheta_0\right\} = 1,
\]
where $\varrho, \vartheta, \varrho_0, \vartheta_0$ are deterministic constants. Moreover, the moment estimate
\[
\int_{L^\gamma(T^3) \times L^\gamma(T^3) \times L_weak^2(T^3)} \left\{\frac{1}{2} \varrho^2 + c_v \vartheta \exp\left(\frac{S}{c_v \vartheta}\right)\right\}^p d\Lambda < \infty,
\]
holds for all $p \geq 1$. Then there exists a dissipative measure-valued martingale solution to the complete stochastic Euler system (2.8)-(2.10) in the sense of Definition (2.12) subject to initial law $\Lambda$.

Furthermore, in view of results in [28], additional to the natural physical principles prescribed in (1.2)-(1.6), we need a purely technical hypothesis
\[
(2.18) \quad |p(\varrho, \vartheta)| \leq (1 + \varrho + v_0(\varrho, \vartheta) + \vartheta|s(\varrho, \vartheta)|),
\]
to establish bounds. Consequently, we then establish the following weak (measure-valued)-strong uniqueness principle:

**Theorem 2.15.** The pathwise weak-strong uniqueness holds true for the system (2.8)-(2.10) in the following sense. Let the thermodynamics functions $e = e(\varrho, \vartheta), s = s(\varrho, \vartheta)$, and $p = p(\varrho, \vartheta)$ satisfy the Gibbs’ relation (1.3), and the technical hypothesis (2.18). Let
\[
\{(\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, \mathbb{P}), \varrho, \mathbf{m}, S, R_{conv}, R_{press}, \mathcal{V}_{t, x}, W\}
\]
be a dissipative measure-valued martingale solution to (2.8)-(2.10) in the sense of Definition (2.12), let $[r, \Theta, \mathbf{U}]$ and a stopping time $t$ be a strong solution of the same problem defined on the stochastic basis with the same Wiener process and with initial data
\[
(2.19) \quad \varrho(0, \cdot) = r(0, \cdot), \quad \mathbf{u}(0, \cdot) = \mathbf{U}(0, \cdot), \quad \vartheta(0, \cdot) = \Theta(0, \cdot) \quad \mathbb{P}\text{-a.s.}
\]
Then
\[
\mathbb{P}\text{-a.s., and for any } (t, x) \in (0, T) \times T^3
\]
\[
\mathcal{V}_{t, x} = \delta_{s(r, t)}
\]
endowed with the Hilbert structure of the Lebesgue space $L^2(Q, \mathbb{R}^3)$. Let
\[ \Pi_m : L^2(Q, \mathbb{R}^3) \rightarrow H_m, \]
be the associated $L^2$-orthogonal projection, and we have $W^{2,2}(T^3, \mathbb{R}^3) \hookrightarrow C(T^3, \mathbb{R}^3)$. Indeed,
\begin{equation}
\|\Pi_m[f]\|_{L^\infty(T^3)} \lesssim \|\Pi_m[f]\|_{W^{2,2}(T^3)} \lesssim \|f\|_{W^{2,2}(T^3)},
\end{equation}
where the associated embedding constants are independent of $m$. Furthermore, since $H_m$ is finite dimensional, all norms are equivalent on $H_m$ for any fixed $m$—a property that will be frequently used at the first level of approximation. Finally, we introduce the operator
\[ [v]_R = \chi(\|v\|_{H_m} - R)v, \]
defined for $v \in H_m$. Let $Q = (0, T) \times T^3$ be the space-time cylinder, we seek to solve the basic approximate system:
\begin{equation}
\begin{aligned}
d\varrho + \text{div}(\varrho(u)_R) dt &= 0, \\
\Pi_m[\varrho u] + \Pi_m[\text{div}(\varrho(u)_R \otimes u)] dt + \Pi_m\left[ \chi(\|u\|_{H_m} - R) \nabla (p(\varrho, \vartheta)) \right] &= 0, \\
dS + [\text{div}(S[\|u\|_R])] dt &= 0,
\end{aligned}
\end{equation}
subject to initial law $\Lambda$, prescribed with random initial data
\begin{equation}
\begin{aligned}
\varrho(0, \cdot) &= \varrho_0 \in C^{2+\nu}(T^3), \quad \varrho_0 > 0, \\
\vartheta(0, \cdot) &= \vartheta_0, \quad \vartheta_0 \in C^{2+\nu}(T^3), \\
u(0, \cdot) &= u_0 \in H_m.
\end{aligned}
\end{equation}
In our basic approximate system (3.2), equations (3.3) and (3.4) are deterministic, that is, they can be solved pathwise, and equation (3.2) involves stochastic integration. The Galerkin projection applied above reduces the problem to a variant of ordinary stochastic differential equation. We solve the system (3.2)-(3.5) using an iteration scheme.

3.1. Iteration Scheme. We construct solutions to (3.2)-(3.5) using a modification of the Cauchy collocation method. Thus, fixing a time step $h > 0$ we set
\begin{equation}
\begin{aligned}
\varrho(t, \cdot) &= \varrho_0, \\
\vartheta(t, \cdot) &= \vartheta_0, \\
u(t, \cdot) &= u_0, \quad \text{for } t \leq 0,
\end{aligned}
\end{equation}
and define recursively, for $t \in [nh, (n+1)h)$
\begin{equation}
\begin{aligned}
d\varrho + \text{div}(\varrho(u(nh, \cdot))_R) dt &= 0, \\
\Pi_m[\varrho u] + \Pi_m\left[ \text{div}(\varrho(u(nh, \cdot))_R \otimes u(nh, \cdot)) \right] dt + \Pi_m\left[ \chi(\|u(nh, \cdot)\|_{H_m} - R) \nabla (p(\varrho, \vartheta)) \right] dt &= 0, \\
dS + [\text{div}(S[u(nh, \cdot)]_R)] dt &= 0,
\end{aligned}
\end{equation}
Here, the unknown quantities $\varrho, \vartheta$ are uniquely deduced from the deterministic equations (3.7) and (3.8) in terms of $u$ and initial data. Now given $\varrho, \vartheta$ we solve
\begin{equation}
\begin{aligned}
d\Pi_m[\varrho u] + \Pi_m\left[ \text{div}(\varrho(u(nh, \cdot))_R \otimes u(nh, \cdot)) \right] dt + \Pi_m\left[ \chi(\|u(nh, \cdot)\|_{H_m} - R) \nabla (p(\varrho, \vartheta)) \right] dt \\
= \Pi_m\left[ \varepsilon \mathbf{L}u \right] dt + \Pi_m[\varrho(\varphi_\epsilon)] dW, \\
t \in [nh, (n+1)h), \\
u(nh, \cdot) &= u(nh-).
\end{aligned}
\end{equation}
To solve (3.9), it is convenient to reformulate the system using $d\mathbf{u}$. We observe that
\[ d\Pi_m(\varrho u) = \Pi_m(d\varrho u) + \Pi_m(\varrho du) = \Pi_m(\partial_t \varrho u) + \Pi_m(\partial_t \varrho du). \]
We adopt the linear mapping $\mathcal{M}[\varrho]$,
\[ \mathcal{M}[\varrho] : H_m \rightarrow H_m, \quad \mathcal{M}[\varrho](u) = \Pi_m(\varrho u), \]
or, equivalently,
\[ \int_Q \mathcal{M}[\varrho] u \cdot \varphi \, dx = \int_Q \varrho u \cdot \varphi \, dx \quad \text{for all } \varphi \in H_m, \]
and its properties as introduced in (29, section 2.2). To be specific, using maximum principle, we take \( \varrho \) to be bounded from below away from zero so that the operator \( \mathcal{M}[\varrho] \) is invertible. Then we reformulate the relation in (3.9) to obtain
\[
\begin{align*}
\mathbf{u}(t) &- \mathbf{u}(nh-) + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[ \text{div} \left( \varrho \mathbf{u}(nh, \cdot) \right) \right] \, dt + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[ \chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \nabla (p(\varrho, \vartheta)) \right] \, dt \\
&= \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m [\varepsilon \mathbf{L} \mathbf{u}] \, dt \\
&+ \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t g \Pi_m [\varphi_{\varepsilon}] \, dW, \quad nh < t < (n + 1)h.
\end{align*}
\]

The constructed iteration scheme (3.7)-(3.9) gives a unique solution \( [\varrho, \vartheta, \mathbf{u}] \) for any initial data (3.6), and the variables \( \varrho, \vartheta \) and \( \mathbf{u} \) are continuous in time \( \mathbb{P} \)-a.s. Indeed, we find solution \( \varrho \) and \( \vartheta \) such that
\[
\varrho \in C([0, T]; \mathbb{C}^{2+r}(\mathbb{T}^3)), \quad \vartheta \in C([0, T]; \mathbb{C}^{2+r}(\mathbb{T}^3)) \quad \text{a.s.}
\]
by applying standard results (see, e.g. [43]) pathwise. Finally, given \( \varrho \) and \( \vartheta \) we can find the velocity
\[
\mathbf{u} \in C([0, T]; H_m), \quad \mathbb{P}\text{-a.s.}
\]
solving (3.9) recursively.

3.1.1. The limit for vanishing time step. The solution \( [\varrho, \vartheta, \mathbf{u}] \) provided by the iteration scheme (3.7)-(3.9) exists for any time step \( h \). Next, we show that as \( h \to 0 \) the iteration scheme yields the basic approximate system (3.2)-(3.4). This essentially follows from establishing uniform bounds for (3.7)-(3.9) independent of \( h \) following the arguments presented in ([5], section 3.2). In particular, we assume the initial data satisfies the bounds
\[
0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho_0} := \overline{\varrho}, \quad 0 < \underline{\varrho} \leq \vartheta_0 \leq \overline{\vartheta}, \quad 0 < \varrho \leq \vartheta \leq \vartheta_0 = \overline{\vartheta} \leq \overline{\vartheta},
\]
for deterministic constants \( \underline{\varrho} \) and \( \overline{\varrho} \) with \( \nu > 0 \). Taking into account the standard results on compressible transport equations and that \( \|\mathbf{u}\|_R \) is bounded in any Sobolev space in terms of \( R \), and the equivalence of norms we have:

- A priori bound for density \( \varrho \) is given by
  \[
  \text{ess sup}_{t \in [0, T]} \|\varrho(t, \cdot)\|_{C^{2+r}} \leq \overline{\varrho}, \quad 0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho_0} := \overline{\varrho}, \quad \mathbb{P}\text{-a.s.}
  \]
  uniformly in \( h \) for deterministic constants \( \underline{\varrho} \) and \( \overline{\varrho} \) with \( \nu > 0 \).

- Similarly, a priori bound for total entropy \( \vartheta \) is
  \[
  \text{ess sup}_{t \in [0, T]} \|\vartheta(t, \cdot)\|_{C^{2+r}} \leq \overline{\vartheta}, \quad 0 < \underline{\varrho} \leq \vartheta_0 \leq \overline{\vartheta}, \quad \mathbb{P}\text{-a.s.}
  \]
  where the same bound of \( \vartheta \) follows immediately from using \( S = \varrho(\log \vartheta - \log(\varrho)) \), for deterministic constants \( \underline{\varrho} \) and \( \overline{\varrho} \) uniform in \( h \).

- Following the lines in ([5] (Section 3.2), that is, establishing bounds for relation (4.9), we use a test function \( \varphi \in H_m \) and take a supremum over \( \varphi \), pass to expectations and apply Burkholder-Davis-Gundy inequality to control the noise term noise. Finally, applying Gronwall’s lemma we deduce the estimate
  \[
  \mathbb{E} \left[ \sup_{r \in [0, T]} \|\Pi_m [\varrho \mathbf{u}(\tau, \cdot)] \|_{H_m}^r + \varepsilon \sup_{r \in [0, T]} \|\mathbf{u}(\tau, \cdot)\|_{H_m}^r \right] \leq c(r, T) \mathbb{E} [1 + \|\mathbf{u}_0\|_{H_m}^r], \quad r > 1.
  \]
To pass to the limit \( h \to 0 \) in the momentum equation (5.10) we require the uniform bound (5.11) and compactness on the velocities in the space \( C([0, T], \mathbb{H}_m) \). Furthermore, we need to control the
difference \((u - u(\alpha h, \cdot))\) uniformly in time. Similarly, following closely the presentations in [5] with appropriate modifications to our particular case we infer
\begin{equation}
\mathbb{E} \left[ \left\| u \right\|_{C^\beta([0,T]; H_m)} \right] \leq \mathbb{E} \left[ \left\| u_0 \right\|_{H_m}^r + 1 \right], \quad r > 2, \beta \in \left(0, \frac{1}{2} - \frac{1}{r}\right),
\end{equation}
uniformly in \(h\). The ‘a priori’ bounds (3.11)-(3.14) are sufficient to take the limit \(h \to 0\) in the iteration scheme (3.7)-(3.9).

We consider the joint law of the basic state variables \([\varrho, \vartheta, u, W]\) in the pathspace
\begin{equation}
\mathfrak{B} = C^l([0,T]; C^2(T^3)) \times C^l([0,T]; C^2(T^3)) \times C^l([0,T]; H_m) \times C([0,T]; U_0), \quad l \in (0, \mathfrak{v}),
\end{equation}
where \(\mathfrak{v}\) is the minimum Hölder exponent in (3.14). Now let \([\varrho_h, \vartheta_h, u_h, W]\) be the unique solution to the iteration scheme (3.7)-(3.9), with initial data being \(\mathcal{F}_0\) measurable and satisfying
\begin{equation}
0 < \varrho \leq \varrho_0, \left\| \varrho_0 \right\|_{C^1([0,T]; C^2(T^3))} \leq \mathfrak{v}, \quad 0 < \vartheta \leq \vartheta_0, \left\| \vartheta_0 \right\|_{C^1([0,T]; C^2(T^3))} \leq \mathfrak{v},
\end{equation}
as well as
\begin{equation}
\mathbb{E} \left[ \left\| u_0 \right\|_{H_m} \right] \leq \mathfrak{v} \text{ for some } r > 2.
\end{equation}

\(\mathbb{P}\)-a.s., Denote by \(\mathcal{L}_h[\varrho, \vartheta_h, u_h, W]\) the joint law of \([\varrho_h, \vartheta_h, u_h, W]\) on \(\mathfrak{B}\), and by \(\mathcal{L}_h[\vartheta_h], \mathcal{L}_h[u_h]\) and \(\mathcal{L}[W]\) the corresponding marginals, respectively. As a consequence of bounds established (3.11)-(3.14), the joint law \(\mathcal{L}[\varrho_h, \vartheta_h, u_h, W]\) is tight on the Quasi-Polish space \(\mathfrak{B}\). By applying Jabusowsk-Skorokhod’s representation theorem \(3.7 \) we get a new probability space with new random variables, a.s convergence of new variables on the pathspace \((w.l.o.g \text{ we keep the same notation})\). Performing the limit \(h \to 0\) in the new probability space yields
\begin{equation}
\partial_t \varrho + \text{div}(\varrho u) = 0,
\end{equation}
\begin{equation}
d\Pi_m[\varrho u] + \Pi_m[\text{div}(\varrho u) R] \text{d}t + \Pi_m\left[\chi(\left\| u \right\|_{H_m} - R) \nabla (p(\varrho, \vartheta))\right] = \mathcal{P} \left[ \varepsilon \mathcal{L} u \right] \text{d}t + \varrho \Pi_m[\phi_e] \text{d}W,
\end{equation}
\begin{equation}
\partial_t S + \text{div}(S u) = 0.
\end{equation}

The system (3.17), (3.19) is still dependent on \(R\) and \(m\). Now our goal is to perform the limits \(R \to \infty\) and \(m \to \infty\), respectively. The procedure is similar to the above discussion for the limit \(h \to 0\). To proceed as discussed above, we start off by deducing uniform bounds enforced by random initial data and the energy balance.

3.1.2. Energy balance. A solution to the approximate system (3.2)- (3.4) satisfies a variant of energy balance. Derivation of total energy balance to the system consist of testing (3.3) with the test function \(u\) and integrating by parts the resultant formulation. Observe that the scalar product
\[\int_{T^3} \Pi_m(\varrho u) \cdot u \text{d}x = \int_{T^3} \varrho |u|^2 \text{d}x\]
and the linear mapping \(\mathcal{M}\) yields
\[\int_{T^3} \mathcal{M}^{-1}[\varrho] \Pi_m[u] \cdot \Pi_m[\varrho u] \text{d}x = \int_{T^3} \varrho \mathcal{M}^{-1}[\varrho] \Pi_m[u] \cdot u \text{d}x = \int_{T^3} u \cdot u \text{d}x.\]

Now we are ready to derive the total energy balance, for this, we consider the following proposition.

**Proposition 3.1.** Let \([\varrho, \vartheta, u, W]\) be a martingale solution of the basic approximate system (3.2)- (3.4). Then the following total energy balance equations
\begin{equation}
d \int_{T^3} \left[\frac{1}{2} |u|^2 + \varrho e \right] \text{d}x + \varepsilon ((u, u)) \text{d}t = \frac{1}{2} \sum_{k=1}^{\infty} \int_{T^3} \varrho |\Pi_m[\varphi_k e_k]|^2 \text{d}x \text{d}t + \int_{T^3} \varrho \Pi_m[\varphi e] \cdot u \text{d}x \text{d}W.
\end{equation}
holds \(\mathbb{P}\)-a.s.
We note, the projections $\Pi_m$ such that the integral with convective term simplifies to

3.1.3. Uniform Bounds.

Applying Itô’s formula to the functional

$$f(q, q \mathbf{u}) = \frac{1}{2} \int_{T^3} q |\mathbf{u}|^2 \, dx = \frac{1}{2} \int_{T^3} \frac{|\mathbf{m}|^2}{q} \, dx,$$

from (3.20) we obtain,

$$d \int_{T^3} \frac{1}{2} q |\mathbf{u}|^2 \, dx = - \int_{T^3} \left[ \text{div}(q[\mathbf{u}] \otimes \mathbf{u}) + \chi(\|\mathbf{u}\|_{H_m} - R) \nabla_q p(q, \vartheta) \right] \cdot \mathbf{u} \, dx \, dt$$

$$+ \int_{T^3} \varepsilon \mathbf{L} \cdot \mathbf{u} \, dx \, dt - \frac{1}{2} \int_{T^3} |\mathbf{u}|^2 \, d\vartheta \, dx$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_{T^3} q[\Pi_m(\varphi_x)\varepsilon_k] |2 \, dx \, dt + \int_{T^3} q\varphi_x \cdot \mathbf{u} \, dx \, dW.$$  

(3.21)

Furthermore, from the continuity equation (3.2), we deduce that;

$$\frac{1}{2} \int_{T^3} |\mathbf{u}|^2 \, d\vartheta \, dx = - \frac{1}{2} \int_{T^3} \text{div}(q[\mathbf{u}] \otimes \mathbf{u}) \cdot |\mathbf{u}|^2 \, dx \, dt,$$

such that the integral with convective term simplifies to

$$\int_{T^3} \text{div}(q[\mathbf{u}] \otimes \mathbf{u}) \cdot \mathbf{u} \, dx = \frac{1}{2} \int_{T^3} \text{div}(q[\mathbf{u}] \otimes \mathbf{u}) \cdot |\mathbf{u}|^2 \, dx,$$

and

$$\int_{T^3} \chi(\|\mathbf{u}\|_{H_m} - R) \nabla_q p(q, \vartheta) \cdot \mathbf{u} \, dx = - \int_{T^3} p(q, \vartheta) \text{div} [\mathbf{u}] \, R \, dx.$$

In view of the above observations, (3.21) reduces to

(3.22)$$d \int_{T^3} \frac{1}{2} q |\mathbf{u}|^2 \, dx + \varepsilon ((\mathbf{u}; \mathbf{u})) = \int_{T^3} p(q, \vartheta) \text{div} [\mathbf{u}] \, R \, dx \, dt$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_{T^3} q[\Pi_m(\varphi_x)\varepsilon_k] |2 \, dx \, dt + \int_{T^3} q\varphi_x \cdot \mathbf{u} \, dx \, dW.$$  

Finally, re-writing the entropy equation as an expression of internal energy using Gibb’s relation (1.4) the followings holds

$$\int_{T^3} p(q, s) \text{div}[\mathbf{u}] \, R \, dx = - d \int_{T^3} \vartheta \, dx.$$

Consequently, re-writing (3.22) yields

$$d \int_{T^3} \frac{1}{2} q |\mathbf{u}|^2 + \vartheta \, dx + \varepsilon ((\mathbf{u}; \mathbf{u})) \, dt = \frac{1}{2} \sum_{k=1}^{\infty} \int_{T^3} q[\Pi_m(\varphi_x)\varepsilon_k] |2 \, dx \, dt + \int_{T^3} q\varphi_x \cdot \mathbf{u} \, dx \, dW,$$

as required. □

3.1.3. Uniform Bounds. Keeping $\varepsilon > 0$ fixed, we derive bounds independent of the parameters $R$ and $m$. We note, the projections $\Pi_m$ are bounded by (3.1). In view of (3.20), we deduce the estimate

(3.23)$$\int_{T^3} \left[ \frac{1}{2} q |\mathbf{u}|^2 + \vartheta \right] \, dx + \varepsilon \int_{0}^{T} ((\mathbf{u}; \mathbf{u})) \, dt \lesssim \left( E_0 + c(T, \varphi_x, \mathbf{v}) + M_t \right),$$

where

$$E_0 = \int_{T^3} \left[ \frac{1}{2} q_0 |\mathbf{u}_0|^2 + q_0 \vartheta_0 \right] \, dx, \quad M_t = \int_{T^3} q[\Pi_m(\varphi_x)] \cdot \mathbf{u} \, dx \, dW.$$
Furthermore, taking the exponent of (3.23) and the expectation of the resultant exponent formulation we obtain

\[
\mathbb{E} \left[ \exp \left( \lambda E_t + \lambda \int_0^T ((\mathbf{u}, \mathbf{u})) \, dt \right) \right] \leq c \mathbb{E} \left[ \exp (\lambda M_t) \right] \lesssim c(\lambda) \quad \forall \lambda > 0,
\]

\(\mathbb{P}\)-a.s, the bound follows from applying exponential version of Burkholder-Davis-Gundy inequality to the r.h.s of (3.24), and using \(\chi(|\mathbf{u}| - \frac{1}{2}) \mathbf{u} \leq 1/\varepsilon\) to deduce

\[
\langle \langle M_t \rangle \rangle = \sum_k \int_0^T \left( \int_{\mathbb{T}^3} \frac{\sum_{k} \phi_k \cdot \mathbf{u}}{\theta} \right) \, dx \right)^2 \, dt 
\]

\[
\leq c(\varepsilon) \sum_k \int_0^T \left( \int_{\mathbb{T}^3} \frac{\phi_k \, dx}{\theta} \right)^2 \left\| \Pi_k \phi_k \right\|_{\mathcal{L}_p}^2 \, dt 
\]

\[
\lesssim c(\varepsilon, \phi, \theta).
\]

**Limit \( R \to \infty \).** We assume the parameter \( m \) is fixed. The approximate problem (3.22)-(3.4) admits a martingale solution \([\theta_R, \vartheta_R, \mathbf{u}_R] \) with initial law \( \Lambda \) for any fixed \( R > 0 \). To perform the limit \( R \to \infty \), we establish compactness of the phase variables and use Jakubowski’s variant of the Skorokhod representation theorem [37].

**Compactness.** We recall the standard regularity estimates of compressible transport equations in [43] applied to (3.17):

\[
\| \vartheta(t, \cdot) \|_{L^\infty(0,T;W^{2,2}(\mathbb{T}^3))} \lesssim \| \vartheta_0 \|_{W^{2,2}(\mathbb{T}^3)} \exp \left( \int_0^T \| \mathbf{u} \|_{\mathcal{L}^{3,2}} \, dt \right),
\]

(3.25)

\[
\lesssim \| \vartheta_0 \|_{W^{2,2}(\mathbb{T}^3)} \exp \left( \int_0^T \| \mathbf{u} \|_{W^{3,2}} \, dt \right),
\]

and

\[
\| \nabla \vartheta(t, \cdot) \|_{L^\infty(0,T;W^{1,2}(\mathbb{T}^3))} \lesssim \| \vartheta_0 \|_{W^{2,2}(\mathbb{T}^3)} \exp \left( \int_0^T \| \mathbf{u} \|_{\mathcal{L}^{3,2}} \, dt \right),
\]

(3.26)

\[
\lesssim \| \vartheta_0 \|_{W^{2,2}(\mathbb{T}^3)} \exp \left( \int_0^T \| \mathbf{u} \|_{W^{3,2}} \, dt \right).
\]

We control the right-hand side of (3.25) and (3.26) in expectation by using (3.24) to deduce the estimate

\[
\mathbb{E} \left[ \| \vartheta \|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \right] \lesssim c, \quad \mathbb{E} \left[ \| \nabla \vartheta \|_{L^\infty(0,T;L^3(\mathbb{T}^3))} \right] \lesssim c,
\]

(3.27)

where \( c > 0 \) is dependent on initial data. In view of (3.23), (3.26), (3.27) and (3.17) we deduce that

\[
\mathbb{E} \left[ \| \partial_t \vartheta \|_{L^2(0,T;L^\infty(\mathbb{T}^3))} \right] \lesssim c.
\]

Finally, we obtain the estimate

\[
\mathbb{E} \left[ \| \partial_t \vartheta \|_{L^2(0,T;L^\infty(\mathbb{T}^3))}^2 \right] + \mathbb{E} \left[ \| \vartheta \|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))} \right] \lesssim c,
\]

(3.28)

where \( c > 0 \) is dependent on initial data. The standard regularity estimate of the total entropy for the variable \( S \) follows same arguments as shown for \( \varrho \) and we obtain the \( \vartheta \) estimate via the relation \( s = \log \vartheta - \log(\varrho) \). Consequently, using (3.28), (3.26) and (3.17), the compactness of \( \varrho \mathbf{u} \) with respect to the time variable follows from the bound

\[
\mathbb{E} \left[ \| \varrho \mathbf{u} \|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))} \right] \lesssim c(r),
\]

(3.29)
for all $0 < \alpha(r) < 1/2$. Accordingly, with established uniform bounds necessary to perform the limit, we proceed as in $h$-limit. We consider the joint law of the basic state variables $[\rho, S, u, W]$ in the pathspace

\[
\mathfrak{G} \equiv L^2(0, T; W^{1,2}(T^3)) \times L^2(0, T; W^{1,2}(T^3)) \times C([0, T]; W^{-4,2}(T^3)) \times L^2(0, T; W^{3,2}(T^3)) \times C([0, T]; \mathcal{U}_0).
\]

Let $[\rho_R, \vartheta_R, m_R, W]$ be the unique solution to the iteration scheme \((3.32)-(3.34)\) with respect to initial law $\Lambda$ and assume

\[
0 < \rho \leq \rho_0, \|\rho_0\|_{W^{2,2}(T^3)} \leq \bar{\rho}, \quad 0 < \vartheta \leq \vartheta_0, \|\vartheta_0\|_{W^{2,2}(T^3)} \leq \bar{\vartheta},
\]

as well as

\[
\mathbb{E}\left[\|u_0\|^r_{H_m}\right] \leq \bar{\pi} \quad \text{for some } r > 2,
\]

$\mathbb{P}$-a.s. Arguing similarly as in the $h$-limit (with obvious modifications): We apply the Jakubowski’s variant of Skorokhod representation theorem \([37]\) to deal with Quasi-Polish spaces. We weaken the regularity of initial data in \((3.30)\) by considering a sequence of initial laws that lose regularity (w.l.o.g we keep the same notation). Thus, passing the limit $R \to \infty$ in \((3.17)-(3.19)\) yields

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
d\Pi_m[\rho u] + \Pi_m[\text{div}(\rho u \otimes u)]dt + \Pi_m\left[\nabla (p(\rho, \vartheta))\right] &= \Pi_m \left[\epsilon \mathcal{L} u\right] dt + \rho \Pi_m [\phi] dW, \\
\partial_t S + \text{div}(Su) &= 0.
\end{align*}
\]

3.1.4. Galerkin Limit. Limit $m \to \infty$. The approximate problem \((3.32)-(3.34)\) admits a martingale solution $[\rho_m, \vartheta_m, u_m]$ with initial law $\Lambda$ for any fixed $m > 0$. We proceed step by step as in the $R$-limit, that is, following preceding parts, we establish uniform bounds (compactness) and perform the limit. In this case, the density estimate \((3.28)\) and similarly the temperature estimate continue to hold for $m \to \infty$, and we can weaken the regularity of initial data in \((3.30)\) by considering a sequence of initial laws that lose regularity when $m \to \infty$. Performing the limit $m \to \infty$ yields a martingale solution in to the system

\[
\begin{align*}
\text{d}\rho + \text{div}(\rho u)\text{d}t &= 0, \\
\text{d}\rho u + \text{div}(\rho u \otimes u)\text{d}t + \nabla (p(\rho, \vartheta)) &= \epsilon \mathcal{L} u\text{d}t + \rho \phi_\epsilon \text{d}W, \\
\text{d}S + \text{div}(Su)\text{d}t &= 0.
\end{align*}
\]

this completes the proof of Theorem \(2.11\).

4. Existence results

The proof Theorem \(2.14\) consists of establishing ‘a priori bounds’ from the energy inequality, compactness arguments in space-time variables, and application of Jakubowski’s version of Skorokhod representation theorem \([37]\) to deal with Quasi-Polish spaces.

Remark 4.1. For any $\epsilon > 0$ Theorem \(2.11\) yields the existence of martingale solution

\[
((\Omega_\epsilon, \mathcal{F}_\epsilon, (\mathcal{F}_t^\epsilon), \mathbb{P}_\epsilon), \rho_\epsilon, m_\epsilon, S_\epsilon, W^\epsilon)
\]

to \((2.7)\). We can assume without loss of generality that the probability space does not depend on $\epsilon$ (see, e.g \([37]\)), that is, the solution is given by

\[
((\Omega, \mathcal{F}, (\mathcal{F}_t^\epsilon), \mathbb{P}), \rho_\epsilon, m_\epsilon, S_\epsilon, W^\epsilon).
\]

We are now ready to consider the following proposition of ‘a priori bounds’.
Proof. First, we observe that the energy formulation of the approximate system (2.7) is of the form
\[
\begin{aligned}
E \left( \sup_{t \in (0,T)} \int_{\mathbb{T}^3} \left[ \frac{1}{2} |\mathbf{m}|^2 + c_v \varrho_0 \exp \left( \frac{S_{0 \epsilon}}{c_v \varrho_0} \right) \right] dx + \varepsilon \int_0^T (\varphi, u_\varepsilon) dt \right)^p \\
\leq C \left( 1 + E \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho_0 |u_\varepsilon|^2 + c_v \varrho_0 \exp \left( \frac{S_{0 \epsilon}}{c_v \varrho_0} \right) \right] dx \right)^p \leq C(T, \varphi, \phi, \Lambda),
\end{aligned}
\]
uniformly in \( \varepsilon \), where \( \Lambda \) is the initial law.

Proposition 4.2. Let \( p \in [1, \infty) \). Then the functions \( \varrho, u \) and \( s \) satisfy the following
\[
\begin{aligned}
(4.1)
E \left( \sup_{t \in (0,T)} \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{m}|^2 \right) + c_v \varrho_0 \exp \left( \frac{S_{0 \epsilon}}{c_v \varrho_0} \right) dx + \varepsilon \int_0^T (\varphi, u_\varepsilon) dt \\
\leq C \left( 1 + E \int_{\mathbb{T}^3} \left[ \frac{1}{2} \varrho_0 |u_\varepsilon|^2 + c_v \varrho_0 \exp \left( \frac{S_{0 \epsilon}}{c_v \varrho_0} \right) \right] dx \right)^p \leq C(T, \varphi, \phi, \Lambda),
\end{aligned}
\]

For the estimate to hold, we take the supremum in time first, and complete the proof by taking the expectations. Accordingly, splitting terms and proving them individual in separate steps yields:

- Firstly, considering the correction term we deduce

\[
\begin{aligned}
\frac{1}{2} \sum_k \int_0^T \int_{\mathbb{T}^3} |\varphi_\varepsilon e_k|^2 dx dt = \frac{1}{2} \sum_k \int_0^T \int_{\mathbb{T}^3} |\sqrt{\varrho_\varepsilon} e_k|^2 dx dt \\
\leq \frac{1}{2} \int_0^T \| \sqrt{\varrho_\varepsilon} e_k \|^2_{L^2(\mathbb{T}^3)} dt < \infty.
\end{aligned}
\]

The bound follows from the assumptions of \( \phi \) in (2.2) and using
\[
\| \varrho \|_{L^1} = \| \varrho_0 \|_{L^1} \leq \bar{\varrho}.
\]

- Next, the noise term. Here we take supremum in time and build expectations. Furthermore, we use the Burgholder-Davis-Gundy inequality to obtain

\[
\begin{aligned}
E \left( \sup_{t \in (0,T)} \left| \int_0^t \int_{\mathbb{T}^3} \varphi_\varepsilon u dx dW \right| \right) &= E \left( \sup_{t \in (0,T)} \left| \int_0^t \int_{\mathbb{T}^3} \varphi_\varepsilon e_k u dx \right| \right) \\
&\leq c E \left( \int_0^T \sum_k \left( \int_{\mathbb{T}^3} \varphi_\varepsilon e_k u dx \right)^2 dt \right)^{1/2} \\
&\leq c E \left( \int_0^T \sum_k \left( \int_{\mathbb{T}^3} \varphi_\varepsilon e_k u dx \right)^2 dt \right)^{1/2} \\
&\leq c(\phi) E \left( \sup_{t \in (0,T)} \int_{\mathbb{T}^3} \varrho dx \int_{\mathbb{T}^3} \varphi_\varepsilon u^2 dx \right)^{1/2} \\
&\leq \frac{\delta}{2} E \left( \sup_{t \in (0,T)} \int_{\mathbb{T}^3} \varrho dx \right) + c^2(\phi, \bar{\varrho}, \delta),
\end{aligned}
\]

where the last line follows from Young’s inequality. Now taking delta \( \delta \) small enough, we can absorb the supremum term from the right.

- We note that expectation on initial data is bounded by assumption i.e.

\[
E \left[ \int_{\mathbb{T}^3} \left( \frac{1}{2} \varrho_0 |u_0|^2 + c_v \varrho_0 \exp \left( \frac{S_0}{c_v \varrho_0} \right) \right) dx \right]^p < \infty.
\]

Hence combining the correction and stochastic terms we deduce (4.1).
In view of Proposition 4.2 we establish the following bounds:

Firstly, we consider $Z \in BC(\mathbb{R})$ such that

\[
Z' \geq 0, \quad Z(s) \begin{cases} < 0 & \text{for } s < s_0, \\ = 0 & \text{for } s \geq s_0, \end{cases}
\]

then the total entropy in (2.7) satisfies the minimum principle provided that

\[
S_0 \geq \rho_0 s_0 > -\infty \text{ a.a. in } T^3,
\]

see [25] for details. Since the entropy is bounded below, using (4.1) we deduce

\[
E \left( \sup_{t \in [0,T]} \int_{T^3} \varrho^{\gamma} \, dx \right) \lesssim C(T, \varrho, \phi, \Lambda),
\]

for any $t \in [0,T]$. Now using $m = \rho u$, we observe

\[
|m|^{\frac{2\gamma}{\gamma + 1}} = |\varrho|^{\frac{2\gamma}{\gamma + 1}} \left( \frac{|m|}{\sqrt{\varrho}} \right)^{\frac{2\gamma}{\gamma + 1}} \lesssim \varrho^{\gamma} + \frac{|m|^2}{\varrho},
\]

and we obtain

\[
E \left( \sup_{t \in [0,T]} \int_{T^3} |m|^{\frac{2\gamma}{\gamma + 1}} \, dx \right) \lesssim C(T, \varrho, \phi, \Lambda),
\]

for any $t \in [0,T]$. Bounds on the total entropy $S$. Since $S \geq s_0 \varrho$, for $S \leq 0$

\[
|S| = -S \leq -s_0 \varrho.
\]

If $S \geq 0$, we note

\[
\varrho^{\gamma} \exp \left( \frac{S}{c_v \varrho} \right) = c_v^{-\gamma} \exp \left( \frac{s}{c_v \varrho} \right) S^\gamma \gtrsim S^\gamma;
\]

hence

\[
E \left( \sup_{t \in [0,T]} \int_{T^3} |S|^\gamma \, dx \right) \lesssim C(T, \varrho, \phi, \Lambda),
\]

for any $t \in [0,T]$. Finally, we derive an estimate for the quantity $S/\sqrt{\varrho}$. For $S \leq 0$, we obtain

\[
|S/\sqrt{\varrho}| \leq -s_0 \sqrt{\varrho}.
\]

If $S > 0$, we have

\[
\varrho^{\gamma} \exp \left( \frac{S}{c_v \varrho} \right) = \varrho^{\gamma} \exp \left( \frac{S}{\sqrt{\varrho} c_v \sqrt{\varrho}} \right) = c_v^{-2\gamma} \left( \frac{s}{\sqrt{\varrho} c_v \sqrt{\varrho}} \right)^{2\gamma} \left( \frac{S}{\sqrt{\varrho}} \right)^{2\gamma} \gtrsim \left( \frac{S}{\sqrt{\varrho}} \right)^{2\gamma},
\]

and in view of this result we deduce the bound

\[
E \left( \sup_{t \in [0,T]} \int_{T^3} \left| \frac{S}{\sqrt{\varrho}} \right|^{2\gamma} \, dx \right) \lesssim C(T, \varrho, \phi, \Lambda).
\]

In view of the above bounds (4.2)-(4.5) and the energy inequality (3.20), we deduce the following (uniform) bounds
(4.6) \[ \sqrt{\varepsilon} u_\varepsilon \in L^p(\Omega; L^2([0, T]; W^{3,2}(T^3))) \]
(4.7) \[ \varrho_\varepsilon \in L^p(\Omega; L^\infty([0, T]; L^7(T^3))), \]
(4.8) \[ m_\varepsilon \in L^p(\Omega; L^\infty([0, T]; \mathbb{R}^3(T^3))), \]
(4.9) \[ \frac{m_\varepsilon \otimes m_\varepsilon}{\varrho_\varepsilon} \in L^p(\Omega; L^\infty([0, T]; L^2(T^3))), \]
(4.10) \[ \frac{m_\varepsilon \otimes m_\varepsilon}{\varrho_\varepsilon} \in L^p(\Omega; L^\infty([0, T]; L^1(T^3))), \]
(4.11) \[ S_\varepsilon \in L^p(\Omega; L^\infty([0, T]; L^2(T^3))), \]
(4.12) \[ \sqrt{\varepsilon} \phi_\varepsilon \in L^p(\Omega; L^\infty([0, T]; L^2(T^3))), \]

4.1. **A priori estimates.** Next, we seek to pass to the limit in the nonlinear convective term and this procedure requires compactness arguments. The balance of momentum is given by

\[ \int_{T^3} \varrho u_\varepsilon \cdot \varphi \, dx = \int_{T^3} \varrho_0 u_0 \cdot \varphi \, dx + \int_0^t \int_{T^3} \varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \varphi \, dx \, ds \]
\[ - \varepsilon \int_0^t \int_{T^3} \nabla \Delta u_\varepsilon \cdot \nabla \Delta \varphi \, dx \, ds - \varepsilon \int_0^t u_\varepsilon \varphi \, dx \, ds \]
\[ + \int_0^t \int_{T^3} \varrho_\varepsilon \exp \left( \frac{S_\varepsilon}{c_\varepsilon \varrho_\varepsilon} \right) \cdot \text{div} \varphi \, dx \, ds + \int_0^t \int_{T^3} \varrho_\varepsilon \phi_\varepsilon dW_s \cdot \varphi \, dx, \]

for all \( \varphi \in C^\infty(T^3) \). We show boundedness of the system by considering deterministic and stochastic parts separately. For the deterministic case, we consider the functional

\[ \mathcal{H}_\varepsilon(t, \varphi) := \int_0^t \int_{T^3} \varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla \varphi \, dx \, ds - \varepsilon \int_0^t \langle (u_\varepsilon; \varphi) \rangle \, ds + \int_0^t \int_{T^3} \varrho_\varepsilon \exp \left( \frac{S_\varepsilon}{c_\varepsilon \varrho_\varepsilon} \right) \cdot \text{div} \varphi \, dx \, ds. \]

We observe that

\[ \begin{align*}
\partial_t \mathcal{H}_\varepsilon(t, \varphi) &\in L^1(\Omega; L^2(0, T; W^{-3,2}(T^3))), \\
\mathcal{H}_\varepsilon(t, \varphi) &\in L^1(\Omega; W^{1,2}(0, T; W^{-3,2}(T^3))),
\end{align*} \]

uniformly in \( \varepsilon \). Then we deduce the estimate

\[ \mathbb{E} \left[ \| \mathcal{H}_\varepsilon \|_{W^{1,2}(0, T; W^{-3,2}(T^3))} \right] \leq C. \]

The stochastic term yields

\[ \mathbb{E} \left[ \left\| \int_0^T \varrho_\varepsilon \phi_\varepsilon dW \right\|^p_{C^\infty([0, T]; L^2(T^3))} \right] \leq c \mathbb{E} \int_0^T \| \sqrt{\phi} \|_{L^2(T^3)}^p \, dt \leq c(T, p, \phi, T), \]

for all \( \alpha \in (1/p, 1/2) \) and \( p > 2 \), see [8, Lemma 9.1.3. b)] or [35, Lemma 4.6]). Now combining the deterministic and stochastic estimates, and using the embedding \( W_{t, x}^{1,2} \rightarrow C_t^0 \) and \( L^2 \rightarrow W_x^{-3,2} \) shows

(4.13) \[ \mathbb{E} \left[ \| \varrho_\varepsilon u_\varepsilon \|_{C^\infty([0, T]; W^{-3,2}(T^3))} \right] \leq c(T), \]

for all \( \alpha < 1/2 \). On the regularity of mass continuity we have

\[ \int_0^T \int_{T^3} \partial_t \varrho \varphi \, dx = \int_0^T \int_{T^3} [\varrho \nabla_x \varphi] \, dx \, dt, \]

so that

\[ \sup_t \| \partial_t \varrho \|_{W^{-3,2}} \leq C \sup_t \| \varrho_\varepsilon u_\varepsilon \|_{1}. \]

Hence, \( \partial_t \varrho_\varepsilon \in L^\infty(0, T; W^{-3,2}(T^3)) \) a.s. and in view of (4.1) we obtain
\[ E \left[ \| \varrho \|_{\mathcal{C}_{\infty}([0,T]; W^{-3,2}(T^3))} \right] \leq C. \]

Similarly, for the entropy balance we have
\[ \int_0^T \int_{T^3} \partial_t S \varphi \, dx = \int_0^T \int_{T^3} \left[ S \frac{m}{\varrho} \nabla \varphi \right] \, dx \, dt, \]
and we deduce
\[ E \left[ \| S \|_{\mathcal{C}_{\infty}([0,T]; W^{-3,2}(T^3))} \right] \leq C. \]

4.2. Compactness Argument. Our goal here is to show tightness of the approximate solutions using the following compact embeddings.

\begin{align*}
(4.14) \quad C^\alpha([0,T]; W^{-3,2}(T^3)) \cap L^\infty(L^\frac{2n}{n+1}(T^3)) & \hookrightarrow C([0,T]; W^{-4,2}(T^3)) \cap C_w(L^\frac{2n}{n+1}(T^3)), \\
(4.15) \quad C^\alpha([0,T]; W^{-\frac{2n}{n+1}}) \cap L^\infty(0,T; L^\gamma(T^3)) & \hookrightarrow C([0,T]; W^{-4,2}(T^3)) \cap C_w(0,T; L^\gamma(T^3)).
\end{align*}

We set the spaces:

\begin{align*}
\mathcal{X}_0 & := L^7(T^3), & \mathcal{X}_m & := L^\frac{2n}{n+1}(T^3), \\
\mathcal{X}_u & := C([0,T]; W^{-4,2}(T^3)) \cap C_w(L^\frac{2n}{n+1}(T^3)), & \mathcal{X}_w & := C([0,T]; U_0), \\
\mathcal{X}_g & := C([0,T]; W^{-4,2}(T^3)) \cap C_w([0,T]; L^7(T^3)), & \mathcal{X}_C & := L^\infty(0,T; M^\infty(T^3, \mathbb{R}^{3x3})), \\
\mathcal{X}_B & := L^\infty(0,T; M^\infty(T^3, \mathbb{R})), & \mathcal{X}_U & := L^2(0,T; W^{3,2}(T^3)), \\
\mathcal{X}_s & := C([0,T]; W^{-4,2}) \cap C_w([0,T]; L^7(T^3)), & \mathcal{X}_S & := L^\infty(Q; \mathcal{P}(A)), \\
\mathcal{X}_Q & := L^\infty(U_0; \mathcal{P}(A)).
\end{align*}

with respect to weak-* topology for all spaces with $L^\infty(\cdot, M(\cdot))$. Furthermore, for $T > 0$, we choose the product path space

\[ \mathcal{X}_T := \mathcal{X}_0 \times \mathcal{X}_m \times \mathcal{X}_s \times \mathcal{X}_u \times \mathcal{X}_B \times \mathcal{X}_C \times \mathcal{X}_U \times \mathcal{X}_S \times \mathcal{X}_Q \times \mathcal{X}_w, \]

with the following laws:

\begin{align*}
\mu_{(u_0)} & \quad \text{is the law of } u_0 \text{ on } C([0,T]; W^{-4,2}(T^3)) \cap C_w(L^\frac{2n}{n+1}(T^3)), \\
\mu_{\varrho} & \quad \text{is the law of } \varrho \text{ on } C([0,T]; W^{-4,2}) \cap C_w(0,T; L^7(T^3)), \\
\mu_{S_\varrho} & \quad \text{is the law of } S_\varrho \text{ on } C([0,T]; W^{-4,2}) \cap C_w(0,T; L^7(T^3)), \\
\mu_{W} & \quad \text{is the law of } W \text{ on } C([0,T], U_0). 
\end{align*}

In addition, let $\mu_{U_0}, \mu_{C_w}, \mu_{P_\varrho}$ and $\mu_{Q_\varrho}$ denote the laws of

\[ U_\varepsilon := \sqrt{\varepsilon} u, \quad C_\varepsilon := \varrho \varepsilon u_\varepsilon \otimes u_\varepsilon, \quad P_\varepsilon := \varrho_\varepsilon \exp \left( \frac{S_\varepsilon}{c_0 \varepsilon} \right), \quad Q_\varepsilon := S_\varepsilon \frac{m_\varepsilon}{\varrho_\varepsilon}, \]

respectively. Let $\mathcal{R}_T$ be the restriction operator which restricts measurable functions (or space-time distributions) defined on $(0, \infty)$ to $(0, T)$. We denote by $\mathcal{L}_T[u_0, \varrho_0 u_0, S_0, \varrho_\varepsilon, \varepsilon u_\varepsilon, S_\varepsilon, U_\varepsilon, P_\varepsilon, C_\varepsilon, Q_\varepsilon, W]$ the probability law on $\mathcal{X}_T$. Note, tightness on $\mathcal{L}_T[u_0, \varrho_0 u_0, S_0, \varrho_\varepsilon, \varepsilon u_\varepsilon, S_\varepsilon, U_\varepsilon, P_\varepsilon, C_\varepsilon, Q_\varepsilon, W]$ for any $T > 0$ implies tightness of $\mathcal{L}_T[u_0, \varrho_0 u_0, S_0, \varrho_\varepsilon, \varepsilon u_\varepsilon, S_\varepsilon, U_\varepsilon, P_\varepsilon, C_\varepsilon, Q_\varepsilon, W]$ for any $T > 0$ on $\mathcal{X} = \cap_T \mathcal{X}_T$. For $\varrho_0 u_0$ we fix $T > 0$ and consider the ball $B_{R_0}$ in the space

\[ C^\alpha([0,T]; W^{-3,2}(T^3)) \cap L^\infty(L^\frac{2n}{n+1}(T^3)). \]

In view of (4.14), (4.15), and using Markov inequality for the complement $B_{R_0}^c$, we deduce that
\[
\mu_{(\omega)_{\varepsilon}}(B_{R_1}^{C}) = \mathbb{P}\left( \left\| \partial u_{\varepsilon} \right\|_{C^{\infty}([0,T];W^{-3,2}_B(T^3))} + \left\| \partial u_{\varepsilon} \right\|_{L^\infty(L^{\frac{2}{3}}(T^3))} \geq R \right) \\
\leq \frac{E}{R_1} \left( \left\| \partial u_{\varepsilon} \right\|_{C^{\infty}([0,T];W^{-3,2}_B(T^3))} + \left\| \partial u_{\varepsilon} \right\|_{L^\infty(L^{\frac{2}{3}}(T^3))} \right) \\
\leq \frac{C}{R_1}.
\]

Therefore, for a fixed \( \eta > 0 \) we find \( R_1(\eta) \) with

\[
\mu_{(\omega)_{\varepsilon}}(B_{R_1}) \geq 1 - \eta.
\]

Hence, the law \( \mu_{(\omega)_{\varepsilon}} \) is tight. Using similar arguments as shown above, we deduce that the laws: \( [\mu_{\phi_{\varepsilon}}, \mu_{S_{\varepsilon}}, \mu_{u_{\varepsilon}}] \) are tight.

**Proposition 4.3.** The law \( \mu_{C_{\varepsilon}} \) is tight.

**Proof.** We consider a ball \( B_R \in L^\infty(0,T;\mathcal{M}^+(T^3,\mathbb{R}^{3 \times 3})) \) that is relatively compact with respect to weak*-topology. Now taking the complement ( \( B_R^{C} \)) of the ball and using Markov-inequality we deduce

\[
\mathcal{L}[C_{\varepsilon}](B_{R}^{C}) = \mathbb{P}\left( \int_0^T \int_{T_3} d|C_{\varepsilon}|dt > R \right) = \mathbb{P}\left( \int_0^T \int_{T_3} \left| \frac{m_{\varepsilon} \otimes m_{\varepsilon}}{\theta_{\varepsilon}} \right| dxdt > R \right) \leq \frac{1}{R} \mathbb{E}\left\| \frac{m_{\varepsilon} \otimes m_{\varepsilon}}{\theta_{\varepsilon}} \right\|_{L^\infty(0,T;L^1(T^3))} \leq \frac{C}{R},
\]

where the last line follows from Proposition 4.2. Therefore, for a fixed \( \eta > 0 \) we find \( R(\eta) \) with

\[
\mathcal{L}[C_{\varepsilon}](B_{R}) \geq 1 - \eta.
\]

The proof is complete. \( \square \)

Similarly, arguing as shown above, the laws: \( \mu_{\phi_{\varepsilon}} \) and \( \mu_{Q_{\varepsilon}} \) are tight. The laws \( \mu_{W}, \mu_{\phi_{0}}, \mu_{\phi_{0}u_{0}} \) and \( \mu_{S_0} \) are tight since they are Radon measures on the Polish spaces. Therefore, we can infer that the law \( \mathcal{L}_T[\phi_{0}, \phi_{0}u_{0}, S_0, \phi_{\varepsilon}, \phi_{\varepsilon}u_{\varepsilon}, S_{\varepsilon}, U_{\varepsilon}, P_{\varepsilon}, C_{\varepsilon}, Q_{\varepsilon}, W] \) is a sequence of tight measures on \( (\mathcal{X}_T) \). Consequently, its weak-* limit is tight as well and hence a Radon measure. Since \( T \) was arbitrary chosen we deduce that \( \mathcal{L}[\phi_{0}, \phi_{0}u_{0}, S_0, \phi_{\varepsilon}, \phi_{\varepsilon}u_{\varepsilon}, S_{\varepsilon}, U_{\varepsilon}, P_{\varepsilon}, C_{\varepsilon}, Q_{\varepsilon}, W] \) is tight on \( \mathcal{X} \). In view of the Jakubowskis version of Skorokhod representation theorem \([37]\) (see also Brzeziak et al.\([12]\)), we have the following proposition.

**Proposition 4.4.** There exists a nullsequence \( (\varepsilon_m)_{m \in \mathbb{N}} \), a complete probability space \((\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})\) with \( (\mathcal{X}, \mathcal{B}_X)\)-valued random variables

\[
(\tilde{\phi}_{0,\varepsilon_m}, \tilde{\phi}_{0,\varepsilon_m} \tilde{u}_{0,\varepsilon_m}, \tilde{S}_{0,\varepsilon_m}, \tilde{\phi}_{\varepsilon_m} \tilde{u}_{\varepsilon_m}, \tilde{S}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{Q}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m}), m \in \mathbb{N},
\]

and \((\tilde{\phi}_{0}, \tilde{\phi}_{0} \tilde{u}_{0}, \tilde{S}, \tilde{\phi}_{\varepsilon} \tilde{u}_{\varepsilon}, \tilde{S}_{\varepsilon}, \tilde{U}_{\varepsilon}, \tilde{P}_{\varepsilon}, \tilde{C}_{\varepsilon}, \tilde{Q}_{\varepsilon}, \tilde{W}_{\varepsilon})\) such that

(a) For all \( m \in \mathbb{N} \) the law of \((\tilde{\phi}_{0,\varepsilon_m}, \tilde{\phi}_{0,\varepsilon_m} \tilde{u}_{0,\varepsilon_m}, \tilde{S}_{0,\varepsilon_m}, \tilde{\phi}_{\varepsilon_m} \tilde{u}_{\varepsilon_m}, \tilde{S}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{Q}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m})\) on \( \mathcal{X} \) is given by (coincides with) \( \mathcal{L}[\phi_{0,\varepsilon_m}, \phi_{0,\varepsilon_m} u_{0,\varepsilon_m}, S_{0,\varepsilon_m}, \phi_{\varepsilon_m} u_{\varepsilon_m}, S_{\varepsilon_m}, U_{\varepsilon_m}, P_{\varepsilon_m}, C_{\varepsilon_m}, Q_{\varepsilon_m}, W] \);

(b) The law of \((\tilde{\phi}_{0}, \tilde{\phi}_{0} \tilde{u}_{0}, \tilde{S}, \tilde{\phi}_{\varepsilon} \tilde{u}_{\varepsilon}, \tilde{S}_{\varepsilon}, \tilde{U}_{\varepsilon}, \tilde{P}_{\varepsilon}, \tilde{C}_{\varepsilon}, \tilde{Q}_{\varepsilon}, \tilde{W}_{\varepsilon})\) is a Radon measure on \( (\mathcal{X}, \mathcal{B}_X) \);
(c) \((\tilde{\varrho}_{0,\varepsilon,m}, \tilde{\varrho}_0, \tilde{u}_{0,\varepsilon,m}, \tilde{S}_{0,\varepsilon,m}, \tilde{\varrho}_{\varepsilon,m}, \tilde{u}_{\varepsilon,m}, \tilde{S}_{\varepsilon,m}, \tilde{U}_{\varepsilon,m}, \tilde{P}_{\varepsilon,m}, \tilde{C}_{\varepsilon,m}, \tilde{Q}_{\varepsilon,m}, \tilde{W}_{\varepsilon,m}), m \in \mathbb{N}\), converges \(\tilde{P}\)-almost surely to \((\tilde{\varrho}_0, \tilde{\varrho}_0, \tilde{u}_0, \tilde{S}_0, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{S}, \tilde{U}, \tilde{P}, \tilde{C}, \tilde{Q}, \tilde{W})\) in the topology of \(\mathcal{X}\), i.e.

\[
\begin{align*}
\tilde{\varrho}_{0,\varepsilon,m} & \to \tilde{\varrho}_0 \text{ in } L^\infty(T^3), \\
\tilde{\varrho}_{0,\varepsilon,m} & \to \tilde{\varrho}_0 \text{ in } L^{\frac{2N}{N+1}}(T^3), \\
\tilde{S}_{0,\varepsilon,m} & \to \tilde{S}_0 \text{ in } C([0,T]; W^{-4,2}) \cap C_w(0,T; L^\infty(T^3)), \\
\tilde{\varrho}_{\varepsilon,m} & \to \tilde{\varrho} \text{ in } C([0,T]; W^{-4,2}) \cap C_w(0,T; L^\infty(T^3)), \\
\tilde{S}_{\varepsilon,m} & \to \tilde{S} \text{ in } C([0,T]; W^{-4,2}) \cap C_w(0,T; L^\infty(T^3)), \\
\tilde{U}_{\varepsilon,m} & \to \tilde{U} \text{ in } L^2([0,T]; W^{3,2}(T^3)), \\
\tilde{\varrho}_{\varepsilon,m} & \to \tilde{\varrho} \text{ in } C([0,T]; W^{-4,2}(T^3)) \cap C_w(L^{\frac{2N}{N+1}}(T^3)), \\
\tilde{P}_{\varepsilon,m} & \to \tilde{P} \text{ in } L^\infty([0,T]; M^+(T^3, \mathbb{R})), \\
\tilde{C}_{\varepsilon,m} & \to \tilde{C} \text{ in } L^\infty([0,T]; M^+(T^3, \mathbb{R}^{3 \times 3})), \\
\tilde{Q}_{\varepsilon,m} & \to \tilde{Q} \text{ in } L^\infty([0,T]; \mathbb{R}^3), \\
\tilde{W}_{\varepsilon,m} & \to \tilde{W} \text{ in } C([0,T]; L^2_0).
\end{align*}
\]

\(\tilde{P}\)-a.s.

To guarantee adaptedness of random variables and to ensure that the stochastic integral continues to hold in the new probability space we introduce filtration for correct measurability. We simplify notation as follows, set

\[
\mathcal{X}_0 := [\tilde{\varrho}_0, \tilde{\varrho}_0, \tilde{u}_0, \tilde{S}_0], \quad \mathcal{X} := [\tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{S}, \tilde{U}, \tilde{P}, \tilde{C}, \tilde{Q}, \tilde{W}].
\]

Let \(\tilde{\mathcal{F}}_t\) and \(\tilde{\mathcal{F}}^\varepsilon_t\) be the \(\tilde{P}\)-augmented filtration of random variables \((\tilde{\varrho}_0, \tilde{\varrho}_0, \tilde{u}_0, \tilde{S}_0, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{\varrho}, \tilde{S}, \tilde{U}, \tilde{P}, \tilde{C}, \tilde{Q}, \tilde{W})\) and \((\tilde{\varrho}_{0,\varepsilon,m}, \tilde{\varrho}_{0,\varepsilon,m} \tilde{u}_{0,\varepsilon,m}, \tilde{S}_{0,\varepsilon,m}, \tilde{\varrho}_{\varepsilon,m}, \tilde{\varrho}_{\varepsilon,m}, \tilde{u}_{\varepsilon,m}, \tilde{S}_{\varepsilon,m}, \tilde{U}_{\varepsilon,m}, \tilde{P}_{\varepsilon,m}, \tilde{C}_{\varepsilon,m}, \tilde{Q}_{\varepsilon,m}, \tilde{W}_{\varepsilon,m})\) respectively, i.e.

\[
\begin{align*}
\tilde{\mathcal{F}}_t &= \sigma(\theta(\mathcal{X}_0, \mathcal{X}_t), \mathcal{X}_t) \cup \{N \in \tilde{\mathcal{F}}; \tilde{P}(N) = 0\}, t \geq 0, \\
\tilde{\mathcal{F}}^\varepsilon_t &= \sigma(\theta(\mathcal{X}_{0,\varepsilon,m}, \mathcal{X}_t), \mathcal{X}_t) \cup \{N \in \tilde{\mathcal{F}}; \tilde{P}(N) = 0\}, t \geq 0.
\end{align*}
\]

Here \(\theta_t\) denotes the restriction operator to the interval \([0,t]\) on the path space and \(\sigma(\cdot)\) denotes the history of a random distribution.

4.3. The new probability space. In this section we will use the elementary method from [13] to show that the approximated equations hold in the new probability space. The essence of this elementary method is to identify the quadratic and cross variations corresponding to the martingale with limit Wiener process obtained via compactness. Now in view of proposition 4.3 we note that \(\tilde{\mathcal{W}}\) has the same law as \(\tilde{W}\). And as result, there exist a collection of independent real-valued \((\tilde{\mathcal{F}}_t)_{t \geq 0}\) - Wiener process \(\tilde{\mathcal{W}}^\varepsilon_k\) such that \(\tilde{\mathcal{W}}^\varepsilon_k = \sum_k \tilde{\mathcal{W}}^\varepsilon_k e_k\). To be specific, there exist a collection of independent real-valued \((\tilde{\mathcal{F}}_t)_{t \geq 0}\) - Wiener process \(\tilde{\mathcal{W}}^\varepsilon_k\) such that \(\tilde{\mathcal{W}} = \sum_k \tilde{\mathcal{W}}^\varepsilon_k e_k\). For all \(t \in [0,T]\) and \(\varphi \in C^\infty(T^3)\) define the functionals:

\[
\begin{align*}
\mathcal{M}^\varepsilon_m(\theta_0, m_0, \varrho, m, \mathcal{U}, \mathcal{P})_t &= \int_{T^3} (m - m_0) \cdot \varphi \, dx - \int_0^t \int_{T^3} \nabla \varphi \, dCds \\
&\quad - \nabla \left( \int_0^t \int_{T^3} \nabla \mathcal{U} \cdot \nabla \varphi \, dxds \right) \\
&\quad - \int_0^t \int_{T^3} \nabla \varphi \, dPds,
\end{align*}
\]

The family of \(\sigma\)-fields \(\sigma(\mathcal{V})_{t \geq 0}\) given as random distribution history of

\[
\sigma(\mathcal{V}) := \bigcap_{s > t} \sigma \left( \bigcup_{\varphi \in C^\infty(Q; \mathbb{R}^3)} \{ (\mathcal{V}, \varphi) < 1 \} \cup \{ N \in \tilde{\mathcal{F}}; \tilde{P}(N) = 0 \} \right)
\]

is called the history of \(\mathcal{V}\). In fact, any random distribution is adapted to its history, see[7] (Chap. 2.2).
\[
\Psi_t = \sum_{k=1}^{\infty} \int_0^t \left( \int_{\mathbb{R}^3} \rho \phi e_k \cdot \varphi \, dx \right)^2 \, ds,
\]

\[\langle \Psi_k \rangle = \int_0^t \int_{\mathbb{R}^3} \rho \phi e_k \cdot \varphi \, dx ds.
\]

Now, let \( \mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m})_{s,t} \) denote the increment \( \mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}) = \mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m})_{s,t} \) and similarly for \( \Psi_{s,t} \) and \( \langle \Psi_k \rangle_{s,t} \). In the new probability space, completeness of proof follows from showing that

\[
\mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m})_{t} = \int_0^t \int_{\mathbb{R}^3} \tilde{\varrho}_{\varepsilon_m} \varphi \cdot dW_{\tilde{W}^{\varepsilon_m}}.
\]

For \( \text{(4.13)} \) to hold, it suffices to show that \( \mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m})_{t} \) is an \((\mathcal{F}^{\varepsilon_m} t)_{t \geq 0}\)-martingale process and its corresponding quadratic and cross variations satisfy, respectively,

\[
\mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}) = \langle \Psi_k \rangle_{t} = \int \int_{\mathbb{R}^3} \tilde{\varrho}_{\varepsilon_m} \varphi \cdot dW_{\tilde{W}^{\varepsilon_m}}.
\]

and consequently

\[
\mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}) = \int \int_{\mathbb{R}^3} \tilde{\varrho}_{\varepsilon_m} \varphi \cdot dW_{\tilde{W}^{\varepsilon_m}}.
\]

which implies the desired equation on the new probability space. We note that \( \text{(4.19)} \) and \( \text{(4.20)} \) hold based on the following observation: the mapping

\[
(\varrho_0, m_0, \varrho, m, U, C, P) \mapsto \mathcal{M} (\varrho_0, m_0, \varrho, m, U, C, P)_{t}
\]

is well-defined and continuous on the path space. Using proposition \( \text{(4.4)} \) we infer that

\[
\mathcal{M}_m^{\varepsilon_m} (\varrho_{0,\varepsilon_m}, m_{0,\varepsilon_m}, \varrho_{\varepsilon_m}, m_{\varepsilon_m}, U_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}) \sim d \mathcal{M}_m^{\varepsilon_m} (\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}).
\]

Fixing times \( s, t \in [0, T] \), with \( s < t \) we consider a continuous function \( h \) such that

\[
h : V |_{[0, s]} \to [0, 1].
\]

The process

\[
\mathcal{M}_m^{\varepsilon_m} (\varrho_{0,\varepsilon_m}, m_{0,\varepsilon_m}, \varrho_{\varepsilon_m}, m_{\varepsilon_m}, U_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}) = \int \int_{\mathbb{R}^3} \varrho_{\varepsilon_m} \varphi \cdot dW_{\tilde{W}^{\varepsilon_m}},
\]

is a square integrable \((\mathcal{F}_t)_{t \geq 0}\)-martingale, consequently, we infer

\[
\mathcal{M}_m^{\varepsilon_m} (\varrho_{0,\varepsilon_m}, m_{0,\varepsilon_m}, \varrho_{\varepsilon_m}, m_{\varepsilon_m}, U_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}) - \mathcal{M}_m^{\varepsilon_m} (\varrho_{0,\varepsilon_m}, m_{0,\varepsilon_m}, \varrho_{\varepsilon_m}, m_{\varepsilon_m}, U_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}) = \Psi_k,
\]

are \((\mathcal{F}_t)_{t \geq 0}\)-martingales. Now we set

\[
X := [\varrho_{0,\varepsilon_m}, \varrho, m, U, C, P], \quad X_{\varepsilon_m} := [\varrho_{0,\varepsilon_m}, m_{0,\varepsilon_m}, \varrho_{\varepsilon_m}, m_{\varepsilon_m}, U_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}],
\]

and

\[
\tilde{X} := [\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}], \quad \tilde{X}_{\varepsilon_m} := [\tilde{\varrho}_{0,\varepsilon_m}, \tilde{m}_{0,\varepsilon_m}, \tilde{\varrho}_{\varepsilon_m}, \tilde{m}_{\varepsilon_m}, \tilde{U}_{\varepsilon_m}, \tilde{C}_{\varepsilon_m}, \tilde{P}_{\varepsilon_m}].
\]

Let \( r \) be a restriction function to the interval \([0, s]\). In view of proposition \( \text{(4.4)} \) and the equality of laws we obtain:
\begin{align}
(4.22) \quad \bar{\mathbb{E}} \left[ h( r_s \dot{X}_{\varepsilon,m}, r_s W^{\varepsilon}) \right. & \left. \mathcal{M}^\varepsilon_m (\dot{X}_{\varepsilon,m}) \right]_{s,t} = \mathbb{E} \left[ h( r_s X_{\varepsilon,m}, r_s W^{\varepsilon}) \right. \\
& \left. \mathcal{M}^\varepsilon_m (X_{\varepsilon,m}) \right]_{s,t} = 0 \\
(4.23) \quad \bar{\mathbb{E}} \left[ h( r_s \dot{X}_{\varepsilon,m}, r_s W^{\varepsilon}) \right. & \left. \mathcal{M}^\varepsilon_m (\dot{X}_{\varepsilon,m}) \right]_{s,t} \mathcal{M}^\varepsilon_m (X_{\varepsilon,m}) \right]_{s,t} = 0 \\
(4.24) \quad \bar{\mathbb{E}} \left[ h( r_s \dot{X}_{\varepsilon,m}, r_s W^{\varepsilon}) \right. & \left. \mathcal{M}^\varepsilon_m (\dot{X}_{\varepsilon,m}) \right]_{s,t} \mathcal{M}^\varepsilon_m (X_{\varepsilon,m}) \right]_{s,t} = 0 \\
\int_0^T \int_{T^3} (\tilde{m}_{\varepsilon,m} \cdot \varphi) \mathrm{d}x & = \int_0^T \int_{T^3} (\tilde{m}_{0,\varepsilon,m} \cdot \varphi) \mathrm{d}x + \int_0^T \int_{T^3} \nabla \varphi \cdot \tilde{c}_{\varepsilon,m} \mathrm{d}s \\
& - \sqrt{\varepsilon_m} \int_0^T \int_{T^3} \nabla \Delta \tilde{U}_{\varepsilon,m} \cdot \nabla \Delta \varphi \mathrm{d}x \mathrm{d}s - \sqrt{\varepsilon_m} \int_0^T \tilde{U}_{\varepsilon,m} \varphi \mathrm{d}x \mathrm{d}s \\
& + \int_0^T \int_{T^3} \nabla \varphi \cdot \tilde{P}_{\varepsilon,m} \mathrm{d}x + \int_0^T \int_{T^3} \tilde{\varphi}_{\varepsilon,m} \phi \mathrm{d}W^{\varepsilon}_s \cdot \varphi \mathrm{d}x, \\
\end{align}

holds \( \tilde{P} \)-a.s in new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\). We note that, the terms in the continuity equation and entropy balance are continuous on the path-space and as such, both equations continue to hold on the new probability space \( \tilde{P} \)-a.s as well.

**4.4. Passage to the limit.** To identify the limits in the nonlinear terms we first introduce defect measures. For this, we adopt notion of measures as presented in [6]. In view of Proposition 4.4 we have

\[ p(\tilde{\varphi}_{\varepsilon,m}, \tilde{S}_{\varepsilon,m}) \rightarrow p(\tilde{\varphi}, \tilde{S}) \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^3,\mathbb{R})]. \]

Noting that \( p(\tilde{\varphi}, \tilde{S}) = \tilde{\varphi}^\gamma \exp \left( \frac{\tilde{S}}{e_{\varepsilon,m}} \right) \) is a convex functional, we deduce

\[ 0 \leq p(\tilde{\varphi}, \tilde{S}) \leq \bar{p}(\tilde{\varphi}, \tilde{S}), \quad \bar{\mathcal{R}}_{\text{press}} \equiv \bar{p}(\tilde{\varphi}, \tilde{S}) - p(\tilde{\varphi}, \tilde{S}) \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^3,\mathbb{R})]. \]

Arguing similarly for the convective term,

\[ \frac{\bar{\mathbb{E}}}{\tilde{\varphi}_{\varepsilon,m}} \mathcal{M}^\varepsilon_m \mathcal{M}^\varepsilon_m \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times3})], \]

setting

\[ \bar{\mathcal{R}}_{\text{conv}} \equiv \frac{\mathcal{M}^\varepsilon_m \mathcal{M}^\varepsilon_m}{\tilde{\varphi}} - \frac{\mathcal{M} \mathcal{M}}{\tilde{\varphi}}, \]

for \( \xi \in \mathbb{R}^3 \), convexity implies

\[ \bar{\mathcal{R}}_{\text{conv}} : (\xi \otimes \xi) \rightarrow \lim_{\varepsilon_m \rightarrow 0} \left[ \frac{\mathcal{M}^\varepsilon_m \mathcal{M}^\varepsilon_m}{\rho^\varepsilon_m} : (\xi \otimes \xi) \right] - \frac{\mathcal{M} \mathcal{M}}{\tilde{\varphi}} : (\xi \otimes \xi) \]

\[ = \lim_{\varepsilon_m \rightarrow 0} \left[ \frac{\mathcal{M}^\varepsilon_m \cdot \xi^2}{\tilde{\varphi}_{\varepsilon,m}} - \frac{\mathcal{M} \cdot \xi^2}{\tilde{\varphi}} \right] \geq 0, \]

so that \( \bar{\mathcal{R}}_{\text{conv}} \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times3})) \). To perform the stochastic limit term we use Lemma 2.1 in [19]. On the account of convergences in Proposition [14] Lemma 2.1 in [19] and the higher moments from (4.22 - 4.24) we can pass to the limit \( \varepsilon_m \rightarrow 0 \) in the momentum equation in (2.7) and obtain

\begin{align}
(4.25) \quad \bar{\mathbb{E}} \left[ \int_0^T \int_{T^3} \bar{m} \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \right] & = \mathbb{E} \left[ \int_0^T \int_{T^3} \bar{m}_0 \cdot \varphi \, \mathrm{d}x + \int_0^T \int_{T^3} \frac{\bar{m} \otimes \bar{m}}{\tilde{\varphi}} : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}s \\
& + \int_0^T \int_{T^3} \bar{\varphi}^\gamma \exp \left( \frac{\tilde{S}}{e_{\varepsilon,m}} \right) \cdot \nabla \varphi \, \mathrm{d}x + \int_0^T \int_{T^3} \tilde{\varphi}_{\varepsilon,m} \phi \mathrm{d}W^{\varepsilon}_s \cdot \varphi \, \mathrm{d}x \right] \, \mathrm{d}t. \\
\end{align}
Consequently, the momentum equation in the sense of (2.15) follows from rewriting (4.25) using defect measures. Similarly, using Proposition 4.4 we perform \( \varepsilon_m \to 0 \) limit in the mass continuity and total entropy to deduce the equivalence of (2.17) and (2.16) in the new probability space, respectively.

### 4.4.1. On the Energy inequality

Finally, we consider the energy equality. In the original probability space, the approximate system (2.7) has an energy equality of the form

\[
E_t^\varepsilon_m = E_s^\varepsilon_m + \frac{1}{2} \int_s^t \| \sqrt{\varepsilon_m} \phi \|_{L^2_2(U,L^2(T^3))}^2 \, d\sigma + \int_s^t \int_{T^3} m_{\varepsilon_m} \phi \, dx dW_{\varepsilon_m}^m,
\]

\( \mathbb{P} \)-a.s for a.a. \( 0 \leq s < t \), where

\[
E_t^\varepsilon_m = \int_{T^3} \left[ \frac{1}{2} \frac{|m_{\varepsilon_m}|^2}{\sqrt{\varepsilon_m}} + c_v \theta_{\varepsilon_m} \exp \left( \frac{S_{\varepsilon_m}}{c_v \theta_{\varepsilon_m}} \right) \right] \, dx + \varepsilon_m \int_0^t \left( (u_{\varepsilon_m},u_{\varepsilon_m}) \right) \, dt,
\]

for a.a. \( t \geq 0 \). For any fixed \( s \) this is equivalent to

\[
- \int_s^\infty \partial_t \varphi E_t^\varepsilon_m \, dt - \varphi(s) E_s^\varepsilon_m = \frac{1}{2} \int_s^\infty \| \sqrt{\varepsilon_m} \phi \|_{L^2_2(U,L^2(T^3))}^2 \, dt + \int_s^\infty \varphi \int_{T^3} m_{\varepsilon_m} \cdot \phi \, dx dW_{\varepsilon_m}^m,
\]

\( \mathbb{P} \)-a.s for all \( \varphi \in C_0^\infty([s, \infty)) \). By virtue of Theorem 2.9.1 in [10], and in view of Proposition 4.4 the energy equality continues to hold in the new probability space and reads

\[
\tilde{E}_t^\varepsilon_m = \tilde{E}_s^\varepsilon_m + \frac{1}{2} \int_s^t \| \sqrt{\varepsilon_m} \phi \|_{L^2_2(U,L^2(T^3))}^2 \, d\sigma + \int_s^t \int_{T^3} m_{\varepsilon_m} \phi \, dx dW_{\varepsilon_m}^m,
\]

\( \tilde{\mathbb{P}} \)-a.s for a.a. \( s \) (including \( s = 0 \)) and all \( t \geq s \). Averaging in \( t \) and \( s \), as in [9], the above expression becomes continuous on the path space. Furthermore, fixing \( s = 0 \) and we use Lemma 2.1 in [20], the bounds established in Proposition 4.4 and higher moments to perform the limit \( \varepsilon_m \to 0 \) and obtain

\begin{equation}
(4.26)
\tilde{E}_t \leq \tilde{E}_0 + \frac{1}{2} \int_s^t \| \sqrt{\varepsilon_m} \phi \|_{L^2_2(U,L^2(T^3))}^2 \, d\sigma + \int_s^t \int_{T^3} \tilde{m} \cdot \phi \, dx d\tilde{W},
\end{equation}

\( \mathbb{P} \)-a.s. for a.a. \( t \in [0,T] \), where

\[
\tilde{E}_t = \int_{T^3} \left[ \frac{1}{2} \frac{|\tilde{m}|^2}{\tilde{\theta}} + c_v \tilde{\theta}^7 \exp \left( \frac{\tilde{S}}{c_v \tilde{\theta}} \right) \right] \, dx + \frac{1}{2} \int_{T^3} d\tilde{R}_{\text{conv}}(t) + c_v \int_{T^3} d\tilde{R}_{\text{press}}(t),
\]

and

\[
\tilde{E}_0 = \int_{T^3} \left[ \frac{1}{2} \frac{|\tilde{m}_0|^2}{\tilde{\theta}_0} + c_v \tilde{\theta}_0^7 \exp \left( \frac{S_0}{c_v \tilde{\theta}_0} \right) \right] \, dx.
\]

Performing the limit \( \varepsilon_m \to 0 \) yields an energy inequality. Our goal now is to convert (4.26) to equality, for this, we argue as in [9]. The entropy balance in the approximate system (2.7) holds as an equality. Hence, to convert (4.26) to equality, it is sufficient to augment the term contributing to the internal energy \( \mathbb{R}_{\text{press}}(t) \) by \( h(t) dx \) with suitable spatially homogeneous \( h \geq 0 \). And \( \mathbb{R}_{\text{press}}(t) \) acts on \( \operatorname{div}_x \phi \) in a periodic domain \( T^3 \), therefore,

\[
\int_{T^3} h(t) \operatorname{div}_x \phi \, dx = 0.
\]

Finally, for any \( s \) we have

\[
- \int_s^\infty \partial_t \varphi \tilde{E}_t \, dt - \varphi(s) \tilde{E}_s = \frac{1}{2} \int_s^\infty \varphi \| \sqrt{\varepsilon_m} \phi \|_{L^2_2(U,L^2(T^3))}^2 \, dt + \int_s^\infty \varphi \int_{T^3} \tilde{m} \cdot \phi \, dx d\tilde{W},
\]

\( \mathbb{P} \)-a.s for all \( \varphi \in C_0^\infty([s, \infty)) \).
5. Weak-strong Uniqueness

In this section we prove that a stochastic measure-valued martingale solution to (2.8)-(2.10) coincides with a strong solution so long as the later exists. In order to do this, we need to introduce a relative entropy inequality; a tool that allows us to compare two solutions. In the following analysis, it is more convenient to express the variable \( S \) as \( gS(\varrho, E) \) where \( E = gE(\varrho, \vartheta) \) and to work with new state variables: the density \( \varrho \), the momentum \( m \) and the internal energy \( E \), see [25].

We begin by stating the following auxiliary results, a variant of ([7], Thm. A.4.1), to which we refer to for more details.

**Lemma 5.1.** Let \( q \) be a stochastic process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) such that for some \( \alpha \in \mathbb{R} \),
\[
q \in C_{\text{weak}}([0, T]; W^{-\alpha, p}(\mathbb{T}^3)) \cap L^\infty(0, \infty; L^1(\mathbb{T}^3)) \quad \mathbb{P}\text{-a.s.},
\]
where
\[
D_t^q q = D_t^d q \, dt + D_t^s q \, dW,
\]
for some \( k > 1 \) and some \( m \in \mathbb{N} \).

Let \( w \) be a stochastic process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying
\[
w \in C([0, T]; W^\alpha, k' \cap C(\mathbb{T}^3)) \quad \mathbb{P}\text{-a.s.},
\]
where
\[
dw = D_t^d w + D_t^s wdW
\]
for some \( k > 1 \) and some \( m \in \mathbb{N} \).

Let \( Q \) be \([\alpha + 2]\)-continuously differentiable function satisfying
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| Q^{(j)}(w) \|^p_{W^\alpha, k' \cap C(\mathbb{T}^3)} \right] < \infty \quad j = 0, 1, 2, \quad 1 \leq p < \infty.
\]
Then
\[
d \left( \int_{\mathbb{T}^3} qQ(w) \, dx \right) = \left( \int_{\mathbb{T}^3} \left[ q \left( Q'(w)D_t^d w + \frac{1}{2} \sum_{k \geq 1} Q''(w)\| D_t^s (e_k) \|^2 \right) \right] \, dx + \left\langle Q(w), D_t^d q \right\rangle \right) \, dt
\]
\[
+ \left( \sum_{k \geq 1} \int_{\mathbb{T}^3} D_t^s q(e_k)D_t^d w(e_k) \, dx \right) \, dt + d\mathbb{M},
\]
where
\[
\mathbb{M} = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^3} [qQ'(w)D_t^d w(e_k) + Q(r)D_t^s q(e_k)] \, dx \, dW_k.
\]
Now following the presentation in \cite{27}, we introduce the (thermodynamic potential) \textit{ballistic free energy}

\begin{equation}
H_\Theta(\varrho, \vartheta) = \varrho \varphi(\varrho, \vartheta) - \Theta \varrho \vartheta, \tag{5.10}
\end{equation}

introduced by Gibbs and more recently by Erickson \cite{24}. In addition to Lemma 5.1 we consider the \textit{relative energy} functional in the context of measure-valued martingale solutions to the complete Euler system given by

\[
\mathcal{E}\left(\varrho, E, m \mid r, \Theta, U\right) = \int_{\mathbb{T}^3} \left(\frac{1}{2} \frac{|m|^2}{\varrho} + E\right) dx + \frac{1}{2} \int_{\mathbb{T}^3} d\mathcal{R}_{\text{conv}} + c_v \int_{\mathbb{T}^3} d\mathcal{R}_{\text{press}}
\]

\begin{equation}
- \int_{\mathbb{T}^3} \varrho \vartheta \Theta dx + \int_{\mathbb{T}^3} \frac{1}{2} \varrho |U|^2 dx
\end{equation}

\begin{equation}
- \int_{\mathbb{T}^3} \vartheta \Theta dx - \int_{\mathbb{T}^3} \varrho \partial_\varrho H_\Theta dx + \int_{\mathbb{T}^3} \vartheta \partial_\vartheta H_\Theta(r, \Theta)(r) - H_\Theta(r, \Theta) dx,
\end{equation}

where the relative functional (5.11) is defined for all \( t \in [0, T] \). Now, having stated Lemma 5.1 and the relative energy functional, we are in a position to derive the \textit{relative entropy inequality}.

\textbf{Proposition 5.2 (Relative Entropy Inequality).} Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), \varrho, m, S, \mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{entr}}, W\) be a dissipative measure-valued martingale solution to the system (2.8)-(2.10). Let \((r, \Theta, U)\) be a trio of stochastic processes defined on the same probability space and adapted to the filtration \((\mathcal{F}_t)_{t \geq 0}\) such that

\[
\begin{align*}
\frac{dr}{dt} &= D^d r dt, \\
\frac{dU}{dt} &= D^d U dt + D^d_t U dW, \\
\frac{d\Theta}{dt} &= D^d \Theta dt, \\
d[\partial_\varrho H_\Theta(r, \Theta)] &= D^d_t [\partial_\varrho H_\Theta(r, \Theta)] dt,
\end{align*}
\]

and

\[
r \in C([0, T]; C^1(\mathbb{T}^3)), \quad \Theta \in C([0, T]; C^1(\mathbb{T}^3)), \quad U \in C([0, T]; C^2(\mathbb{T}^3)), \quad \mathbb{P}\text{-a.s.,}
\]

\[
\mathbb{E}\left[\sup_{t \in [0, T]} \|r\|^2_{W^{1, q}(\mathbb{T}^3)}\right]^k + \mathbb{E}\left[\sup_{t \in [0, T]} \|U\|^2_{W^{1, q}(\mathbb{T}^3)}\right]^q \leq c(q), \quad \text{for all } 2 \leq q < \infty, \quad 0 < r \leq r(t, x) \leq T \quad \mathbb{P}\text{-a.s.,}
\]

\[
\mathbb{E}\left[\sup_{t \in [0, T]} \|\Theta\|^2_{W^{1, q}(\mathbb{T}^3)}\right]^k \leq c(q), \quad \text{for all } 2 \leq q < \infty, \quad 0 < \Theta \leq \Theta(t, x) \leq \Theta \quad \mathbb{P}\text{-a.s.}
\]

Furthermore, \( r, \Theta, U, \) satisfy

\[
D^d r, D^d \Theta, D^d U \in L^q(\Omega; C(0, T; C^1(\mathbb{T}^3))) \quad \mathbb{D}^d U \in L^2(\Omega; L^2(0, T; L^2(\mathbb{T}^3))),
\]

\[
\left(\sum_{k \geq 1} \|\mathbb{D}^d U(\varepsilon_k)\|^q\right)^{\frac{1}{q}} \in L^q(\Omega; L^q(0, T; L^q(\mathbb{T}^3))),
\]

respectively. Then the relative entropy inequality:

\[
\mathcal{E}\left(\varrho, E, m \mid r, \Theta, U\right) \leq \mathcal{E}\left(\varrho, E, m \mid r, \Theta, U\right)(0) + \int_0^T \mathcal{Q}\left(\varrho, E, m \mid r, \Theta, U\right) dt + M_t,
\]

\textsuperscript{3}Note, the moment bound for \( \Theta \) below implies the same for \( S(r, \Theta) \) by \textbf{1.10} since \( r \) and \( \Theta \) are bounded below and above.
holds \( P \)-a.s for all \( \tau \in (0, T) \), where
\[
Q(\varrho, \vartheta, u | r, \Theta, U) = \int_{T} \varrho \left( \frac{m}{\varrho} - U \right) \cdot \nabla_x U \cdot \left( U - \frac{m}{\varrho} \right) dx
+ \int_{T} \left[ (D_{t}^{\vartheta} U + U \cdot \nabla_x U) \cdot (\varrho U - m) - p(\varrho, \vartheta) \text{div}_x U \right] dx
\]
\[
- \int_{0}^{T} \int_{T} \left[ \{\nabla_{\vartheta x}; \varrho s(\varrho, E)\} D_{t}^{\vartheta} \Theta + \{\nabla_{\vartheta x}; (\varrho, E) m \} \cdot \nabla_x \Theta \right] dxdt
+ \int_{T} \int_{T} \left[ \varrho s(r, \Theta) \partial_t \Theta + m s(r, \Theta) \cdot \nabla_x \Theta \right] dxdt
+ \int_{0}^{T} \left( \left( 1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{m}{r} \cdot \nabla_x p(r, \Theta) \right) dx.,
\]
where
\[
\int_{T} \int_{T} \int_{T} \left[ \varrho D_{t}^{\vartheta} U \cdot \nabla \varphi + \varrho (e_k - \vartheta) \right] dx - \int_{T} \nabla U : d\mathcal{R}_{\text{conv}} - \int_{T} \text{div} U d\mathcal{R}_{\text{press}}
\]
and
\[
\mathcal{M} = \int_{0}^{T} \int_{T} \varrho \mathcal{M} dW
- \int_{0}^{T} \int_{T} \left[ \varrho D_{t}^{\vartheta} U + U \varrho \varphi \right] dxdt + \int_{0}^{T} \int_{T} \varrho U : D_{t}^{\vartheta} U dxdt + \frac{1}{2} \sum_{k \geq 1} \int_{T} \varrho |D_{t}^{\vartheta} U(e_k)|^2 dx,
\]
where we set \( Z(s(\varrho, E)) = s(\varrho, E) \) for convenience, see \[25\] section 3.2 for properties of \( Z \).

Proof. We note that, the right-hand-side of the formulation \[5.11\] follows from energy inequality. Therefore, using the energy inequality and Lemma \[5.1\] we proceed in several steps as follows:

**Step 1:** To compute \( d \int_{T} \varrho \mathcal{M} dW \) we recall that \( \varrho = m \) satisfies hypotheses \[5.1\] , \[5.3\] with some \( k < \infty \). Applying Lemma \[5.1\] we deduce
\[
d \left( \int_{T} \varrho \mathcal{M} dW \right) = \left( \int_{T} \left[ \varrho \cdot D_{t}^{\vartheta} U + \left( \frac{m \cdot m}{\varrho} \right) \cdot \nabla U + \varrho p(\varrho, s) \text{div}_x U \right] dx \right) dt
\]
\[
+ \sum_{k \geq 1} \int_{T} \varrho D_{t}^{\vartheta} U(e_k) \cdot \varrho \varphi(e_k) dxdt + \int_{T} \nabla U : d\mathcal{R}_{\text{conv}} dt + \int_{T} \text{div} U d\mathcal{R}_{\text{press}} dt
+ dM_1,
\]
where
\[
M_1 = \int_{0}^{T} \int_{T} \left[ \varrho D_{t}^{\vartheta} U + U \varrho \varphi \right] dxdt.
\]
Similarly to \[5.15\], we compute
\[
d \left( \int_{T} \frac{1}{2} \varrho |U|^2 dx \right) = \int_{T} \varrho U \cdot \nabla U \cdot U dxdt + \int_{T} \varrho U \cdot D_{t}^{\vartheta} U dxdt
+ \frac{1}{2} \sum_{k \geq 1} \varrho |D_{t}^{\vartheta} U(e_k)|^2 dxdt + dM_2,
\]
where
\[
M_2 = \int_{0}^{T} \int_{T} \varrho U \cdot D_{t}^{\vartheta} U dxdt.
\]
Testing the entropy balance \[2.16\] with \( \Theta \) we deduce
\[
d \left( \int_{T} \{\nabla_{\vartheta x}; \varrho \Theta \} dx \right) \geq \int_{T} \varrho \{\nabla_{\vartheta x}; m \Theta \} \cdot \nabla_x \Theta dxdt + \int_{T} \{\nabla_{\vartheta x}; \varrho s \} D_{t}^{\vartheta} \Theta dxdt,
\]
where \( D_{t}^{\vartheta} \Theta = \partial_t \Theta \). Similarly, testing the continuity equation \[2.14\] with \( \partial_x H_{\vartheta}(r, \Theta) \) yields
of the form (5.12), with
(5.18) \[ d \left( \int_{T^3} \varrho \partial_x H_\Theta(r, \Theta) \, dx \right) = \int_{T^3} \varrho \cdot \nabla_x \left( \partial_x H_\Theta(r, \Theta) \right) \, dx + \int_{T^3} g D^\ell_t (\partial_x H_\Theta(r, \Theta)) \, dx. \]
where \( D^\ell_t (\partial_x H_\Theta(r, \Theta)) = \partial_t (\varrho \partial_x H_\Theta(r, \Theta)) \).

**Step 2:** Finally, we collect and sum the resulting expressions (5.15)-(5.18), and add (2.17) to the sum to obtain (5.14), see [27] for more details.

**Remark 5.3.** The relative entropy inequality is satisfied for any trio \( [r, \Theta, U] \) provided \( p, e \) and \( s \) satisfy the Gibbs’ relation.

We are now ready to prove Theorem 2.15, accordingly, we use Proposition 5.2 and a Gronwall type argument to prove pathwise weak-strong uniqueness claim as follows.

**Proof of the claim.**

**Step 1**

We begin by introducing a stopping time
\[ \tau_M = \inf \left\{ t \in (0, T) \mid \| U(s, \cdot) \|_{H^1(\mathbb{T}^3)} > M \right\} \]
Since \([r, \Theta, U] \) is a strong solution,
\[ P \left( \lim_{M \to \infty} \tau_M = t \right) = 1; \]
therefore, it is enough to show results for a fixed \( M \). Furthermore, \([r, \Theta, U] \equiv [\varrho, \vartheta, u] \) satisfies an equation of the form (5.12), with
\[ D^\ell_t = -U \cdot \nabla_x U - \frac{1}{r} \nabla_x p(r, \Theta), \quad \nabla^\ast U = \phi, \quad D^\ell_t r = -\text{div}_x (r U). \]

**Remark 5.4.** Note that the Itô correction term in (5.12) vanishes for our choice of \( D^\ell_t U \).

**Step 2**

For \( M > 0 \), we have
\[ \sup_{t \in [0, \tau_M]} \| \nabla U \|_{L^\infty(\mathbb{T}^3)} \leq c(M). \]

Since \( r \) satisfies the continuity equation and hypothesis (2.19), then from maximum and minimum principle we have
\[ 0 < \underline{r} \leq r(t \land t) \leq \overline{r} \]
for some deterministic constants \( \underline{r}, \overline{r} \). Similarly, for \( \Theta \) we have
\[ 0 < \underline{\Theta} \leq \Theta(t \land t) \leq \overline{\Theta}. \]
The relative energy [5.11] can be re-written as
\[ E \left( \varrho, E, m \mid r, \Theta, U \right) = \int_{T^3} \left( \frac{1}{2} \frac{m}{\varrho} - U \right)^2 + E - \Theta \delta s (\varrho, E) - \partial_x H_\Theta(r, \Theta)(s - r) - H_\Theta(r, \Theta) \, dx \]
\[ + \frac{1}{2} \int_{T^3} \text{dtr} \text{conv} (t) + c_v \int_{T^3} \text{dtr} \text{press} (t). \]
where \( E = \varrho e (\varrho, \vartheta) \).
Moreover, we consider a function \( \Phi (\varrho, E) \),
\[ \Phi \in C^\infty_c (0, \infty)^2, 0 \leq \Phi \leq 1. \]

For a measurable function \( G(\varrho, E, m) \), we set
\[ G = G_{\text{ess}} + G_{\text{res}}, \quad G_{\text{ess}} = \Phi (\varrho, E) G(\varrho, E, m), \quad G_{\text{res}} = (1 - \Phi (\varrho, E)) G(\varrho, E, m). \]
Following the presentation in [25] [28], \( G_{\text{ess}} \) accounts for the ‘essential part’ that describes the behaviour of the non-linearity in the non-degenerate area where both \( \varrho \) and \( \vartheta \) are bounded below and above. On the other hand, \( G_{\text{res}} \) accounts for the ‘residual part’ that captures the behaviour in the singular regime \( \varrho, \vartheta \to 0 \).
or/and \( \varrho, \vartheta \to \infty \).

In view of (5.21), we recall the coercivity properties of \( E \) proved in ([28], Chapter 3, Proposition 3.2),

\[(5.22)\]
\[
E (\varrho, E, m | r, \Theta, U) \geq \int_{T^3} \left[ |\varrho - r|^2 + |E - r e(r, \Theta)|^2 + \frac{m}{\varrho} - U \right] \text{ess} \ dx + \int_{T^3} \left[ 1 + \varrho + \varrho s(\varrho,E) + E + \frac{|m|}{\varrho} \right] \text{res} \ dx.
\]

**Step 3:**

In view of (5.21) and Proposition 5.2 we apply the relative entropy inequality (5.14) on the time interval \([0, \tau_M]\)

\[(5.23)\]
\[
\mathcal{E} (\varrho, E, m | r, \Theta, U) (t \wedge \tau_M) \leq \mathcal{E} (\varrho, E, m | r, \Theta, U) (0) + \int_0^{t \wedge \tau_M} \mathcal{Q} (\varrho, E, m | r, \Theta, U) \ dt + \mathcal{M} (t \wedge \tau_M),
\]

with

\[
\mathcal{Q} (\varrho, E, m | r, \Theta, U) = \int_{T^3} \varrho \left( \frac{m}{\varrho} - U \right) \cdot \nabla x U \cdot \left( U - \frac{m}{\varrho} \right) \ dx + \int_{T^3} \left[ (\varrho U - m)(D^4 U + U \cdot \nabla x U) - p(\varrho,E) \text{div}_x U \right] \ dx
\]

\[
- \int_0^T \int_{T^3} [\varrho s(\varrho,E)] D^4 \Theta + \langle \nabla x ; s(\varrho,E) \rangle \cdot \nabla x \Theta \right] \ dx d\tau
\]

\[
+ \int_0^T \int_{T^3} [\varrho s(r, \Theta)] \partial_t \Theta + m s(r, \Theta) \cdot \nabla x \Theta \right] \ dx d\tau
\]

\[
- \int_{T^3} \nabla U : dR_{\text{conv}} - \int_{T^3} \text{div} U dR_{\text{press}}.
\]

\[(5.24)\]

To apply Gronwall’s argument, we need to show the estimate

\[(5.25)\]
\[
\mathcal{Q} (\varrho, E, m | r, \Theta, U) \lesssim c \mathcal{E} (\varrho, E, m | r, \Theta, U).
\]

for some constant \( c > 0 \). In view of (5.19), the estimate on defect measures is given by

\[(5.26)\]
\[
\int_0^{t \wedge \tau_m} \int_{T^3} \nabla x U : d[ R_{\text{conv}} + R_{\text{press}}] \ dx d\tau \lesssim c(M) \frac{1}{2} \int_0^{t \wedge \tau_m} \int_{T^3} \text{dtrace} [ R_{\text{conv}} + R_{\text{press}}] \ dx d\tau.
\]

Similarly, using (5.19) we obtain

\[
\left| \int_{T^3} \varrho \left( \frac{m}{\varrho} - U \right) \cdot \nabla x U \cdot \left( U - \frac{m}{\varrho} \right) \ dx \right| \leq \int_{T^3} \varrho \left( \frac{m}{\varrho} - U \right)^2 |\nabla x U| \ dx,
\]

\[
\lesssim c(M) \int_{T^3} \varrho \left( \frac{m}{\varrho} - U \right)^2 \ dx.
\]

Furthermore, we observe that

\[(5.27)\]
\[
(\varrho U - m)(D^4 U + U \cdot \nabla x U) - \frac{m}{r} \cdot \nabla x p(r, \Theta) = -\frac{\varrho U}{r} \cdot \nabla x p(r, \Theta).
\]
Consequently, (5.24) reduces to
\[
\mathcal{Q}\left(\rho, E, m \bigg| r, \Theta, U\right) \lesssim - \int_0^T \int_{\mathbb{T}^3} \left[ (\mathcal{L}_{t,x} \rho, D_t^4 \Theta + (\mathcal{L}_{t,x} s) \rho \Theta) + m_s(r, \Theta) \cdot \nabla_x \Theta \right] \, dx \, dt \\
+ \int_0^T \int_{\mathbb{T}^3} [p(r, \Theta) \partial_t \Theta + m_s(r, \Theta) \cdot \nabla_x \Theta] \, dx \\
+ \int_{\mathbb{T}^3} \left( (r + \frac{1}{r}) \frac{\rho}{r} \partial_t \rho \right) \left( \frac{m}{r} \cdot \nabla_x \rho \right) - p(r, \Theta) \partial_t \rho \right) \, dx \\
+ c_1 \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right) .
\]
(5.27)

Finally, adopting the notation in (5.21) and manipulating terms in (5.27), as in [25], we obtain
\[
\mathcal{Q}\left(\rho, E, m \bigg| r, \Theta, U\right) \lesssim c \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right) .
\]
(5.28)

**Step 4**

In view of the above estimates, the relative entropy inequality (5.23) reduces to
\[
\mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(t \wedge \tau_M) \lesssim \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(0) + \int_0^{t \wedge \tau_M} c \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(t) \, dt + M(t \wedge \tau_M).
\]
(5.29)

Now, taking the expectation in \([t \wedge \tau_M]\) and applying Gronwall's lemma yields
\[
\mathbb{E} \left[ \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(t \wedge \tau_M) \right] \leq c(M) \mathbb{E} \left[ \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(0) \right] ,
\]
where
\[
\mathbb{E} \left[ \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(0) \right] = 0
\]
by assumptions. Therefore, we observe that
\[
\mathbb{E} \left[ \mathcal{E}\left(\rho, E, m \bigg| r, \Theta, U\right)(t \wedge \tau_M) \right] = 0
\]
for all \(t \in (0, T)\), yielding the claim. \(\Box\)

**6. Martingale solutions as measures on the space of trajectories**

Firstly, we observe that from the proof of Theorem [existence], the natural filtration associated to a dissipative measure-valued martingale solution in the sense of Definition 2.12 is the joint canonical filtration of \([\rho, m, S, R_{\text{conv}}, R_{\text{press}}, \mathcal{L}_{t,x}, W]\). However, the canonical processes \([S, R_{\text{conv}}, R_{\text{press}}, \mathcal{L}_{t,x}]\) are class of equivalences in time and not a stochastic processes in the classical sense. Therefore, it is not obvious as to how one should formulate the Markovianity of the system (2.8)-(2.10). To circumvent this problem, we shall introduce new variables \(S, R\) (time integrals) such that
\[
S = \int_0^t S \, ds , \quad R = \int_0^t (R_{\text{conv}}, R_{\text{press}}, \mathcal{L}_{t,x}) \, ds .
\]
Consequently, the notion of new variables allows us to establish the Markov selection for the joint law of \([\rho, m, S, R]\). In this case, the stochastic process has continuous trajectories and contains all necessary information. The initial data for \([S, R]\) is superfluous and only needed for technical reasons in the selection process. To study Markov selection, it is desirable to consider the martingale solutions as probability measures \(P \in \text{Prob} [\Omega]\) such that
\[
\Omega = C_{\text{loc}}([0, \infty); W^{-k,2}(\mathbb{T}^3)) ,
\]
where $k > 3/2$. Adopting the set-up of Section 2.1.1 we set $X = W^{-k,2}(T^3)$. Accordingly, let $B$ denote the Borel $\sigma$-field on $\Omega$. Let $\xi = (\xi^1, \xi^2, \xi^3, \xi^4)$ denote the canonical process of projections such that

$$\xi : \Omega = (\xi^1, \xi^2, \xi^3, \xi^4) : \Omega \to \Omega, \quad \xi_t \omega = (\xi^1_t, \xi^2_t, \xi^3_t, \xi^4_t)(\omega) = \omega_t \in W^{-k,2}(T^3),$$

for any $t \geq 0$

where the notation $\omega$ indicates that our random variable is time dependent. In addition, let $(B_t)_{t \geq 0}$ be the filtration associated to canonical process given by

$$B_t := \sigma(\xi|0, t), \quad t \geq 0,$$

which coincides with the Borel $\sigma$-field on $\Omega^{[0, t]} = ([0, t]; W^{-k,2}(T))$. For analysis, the dissipative martingale solutions

$$((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), \varrho, m, S, R_{\text{conv}}, R_{\text{press}}, \nu_{t,x}, W),$$

in the sense of Definition 6.1

we observe that

$$\rho \cdot \mathcal{P} = L \left[ \varrho, m = \varrho u, \int_0^t S ds, \int_0^t (R_{\text{conv}}, R_{\text{press}}, \nu_{t,x}) ds \right] \in \text{Prob}[\Omega].$$

Consequently, we obtain the probability space $(\Omega, B, (B_t)_{t \geq 0}, U)$. Furthermore, let $\mathcal{P}$ be probability measure and introduce the space

$$F = \left\{ [\varrho, m, S, R] \in \tilde{F} \bigg| \int_{T^3} \frac{|m|^2}{\varrho} dx < \infty \right\},$$

$$\tilde{F} = L^\gamma(T^3) \times L^{\frac{2\gamma}{2\gamma - k}}(T^3) \times (L^\gamma(T^3)) \cap BV^2_{w,\text{loc}}(W^{-l,2}(T^3)) \times (W^{-k,2}(T^3, \cdot))^2 \times (W^{-k,2}(T^3, A)).$$

where $A = R \times R^3 \times R$. We augment $F$ with the points of the form $(0, 0, S, R)$ for $S \in W^{-l,2}(T^3) \cap L^\gamma(T^3)$ and $R \in W^{-k,2}(T^3, R^{15})$. Therefore, $F$ is Polish space with metric

$$d_F(y, z) = d_Y((y^1, y^2, y^3, y^4), (z^1, z^2, z^3, z^4)) = \|y - z\|_F + \left\| \frac{y^2}{\sqrt{|y|^2}} - \frac{z^2}{\sqrt{|z|^2}} \right\|_{L^2}.$$
Proposition 6.2. In the following we state the relation between Definition 2.12 and Definition 6.1.

(c) it holds $\mathcal{P}$-a.s.
\[
\left[ \int_{T_3} \xi_1^2 \psi \right]_{t=0}^{t=\tau} - \int_{0}^{\tau} \int_{T_3} \xi_1^2 \cdot \nabla \psi \, dx \, dt = 0
\]
for any $\psi \in C^1(T^3)$ and $\tau \geq 0$;

(d) for any $\varphi \in C^1(T^3, \mathbb{R})$, the stochastic process
\[
\mathcal{M}(\varphi) : [\omega, \tau] \mapsto \left[ \int_{T_3} \xi_1^2 \cdot \varphi \right]_{t=0}^{t=\tau} - \int_{0}^{\tau} \int_{T_3} \left[ \xi_1^2 \otimes \xi_1^2 : \nabla \varphi + \xi_1 \exp \left( \frac{\xi_1^3}{c_0 \xi_1^2} \right) \text{div} \varphi \right] \, dx \, dt
\]
\[
- \int_{0}^{\tau} \nabla \varphi : d\xi_4^4 dt - \int_{0}^{\tau} \text{div} \varphi : d\xi_4^4 dt
\]
is a square integrable $(\mathcal{B}_t)_{t \geq 0, \mathcal{P})$-martingale with quadratic variation
\[
\frac{1}{2} \int_{0}^{\tau} \sum_{k=1}^{\infty} \left( \int_{T_3} \xi_1^2 \varphi e_k \cdot \varphi \right)^2 \, dt;
\]

(e) It holds $\mathcal{P}$-a.s.
\[
\int_{0}^{\tau} \int_{T_3} \left[ (\xi_1^4)_{t,x} ; Z(S(\hat{\omega})) \right] \, dx \, dt \leq \left[ \int_{T_3} (\xi_1^4)_{t,x} ; Z(S(\hat{\omega})) \, dx \right]_{t=0}^{t=\tau}
\]
for any $\varphi \in C^1([0, \infty) \times T^3), \varphi \geq 0$;

(f) The stochastic process
\[
\mathcal{E} : [\omega, \tau] \mapsto \xi_1 - \xi_0 - \frac{1}{2} \int_{0}^{\tau} \| \sqrt{\xi_1} \phi \|_{L^2(T^3 \times T^3)}^2 \, d\sigma
\]
is a square integrable $(\mathcal{B}_t)_{t \geq 0, U}$-martingale with quadratic variation
\[
\frac{1}{2} \int_{0}^{\tau} \sum_{k=1}^{\infty} \left( \int_{T_3} \xi_1^2 \cdot \varphi e_k \right)^2 \, dt
\]
for $\tau \geq 0$.

In the following we state the relation between Definition 2.12 and Definition 6.1.

Proposition 6.2. The following statement holds true

(1) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0, \mathbb{P}), \varrho, m, S, \mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{press}}, \nu_{t,x}, W)$ be a dissipative martingale solution to (2.8)-(2.10) in the sense of Definition 2.12. Then for every $\mathcal{F}_0$-measurable random variables $[\mathcal{S}_0, \mathcal{R}_0]$ with values in $\mathcal{S} \in W^{-1,2}(T^3) \cap L^2(T^3)$ and $\mathcal{R} \in W^{-k,2}(T^3, \mathbb{R}^{15})$ we have that
\[
\mathcal{P} = \mathcal{L} \left[ \rho, m = \rho u, \mathcal{S}_0 + \int_{0}^{\tau} \mathcal{S} \, ds, \mathcal{R}_0 + \int_{0}^{\tau} \mathcal{R} \, ds \right] \in \text{Prob}[\Omega]
\]
is a solution to the martingale problem associated to (2.8)-(2.10) in the sense of Definition 6.1.

(2) Let $\mathcal{P}$ be a solution to the martingale problem associated to (2.8)-(2.10) in the sense of Definition 6.1. Then there exists a dissipative martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0, \mathbb{P}), \varrho, m, S, \mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{press}}, \nu_{t,x}, W_1)$ to the system (2.8)-(2.10) in the sense of Definition 2.12 satisfying properties (a)-(j), furthermore there exists $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0, \mathbb{P}), \varrho, m, S, \mathcal{R}_{\text{conv}}, \mathcal{R}_{\text{press}}, \nu_{t,x}, W_2)$ in the sense of Definition 2.12 satisfying property (k) and an $\mathcal{F}_0$-measurable random variables $[\mathcal{S}_0, \mathcal{R}_0]$ for $\mathcal{S} \in W^{-1,2}(T^3) \cap L^2(T^3)$ and $\mathcal{R} \in W^{-k,2}(T^3, \mathbb{R}^{15})$ such that
\[
\mathcal{P} = \mathcal{L} \left[ \rho, m = \rho u, \mathcal{S}_0 + \int_{0}^{\tau} \mathcal{S} \, ds, \mathcal{R}_0 + \int_{0}^{\tau} \mathcal{R} \, ds \right] \in \text{Prob}[\Omega],
\]
where $W_1$ and $W_2$ correspond to Wiener process generated by momentum equation and energy equality, respectively.
The proof of Proposition 6.2 follows the same arguments as presented in [1] with appropriate adjustment to our system (2.8)-(2.10). Moreover, on showing part (2) implies part (1) in Proposition 6.2 we need two Wiener processes since we apply the martingale representation theorem (see [19], Theorem 8.2) twice.

7. Markov selection

In this section we state and prove the strong Markov selection to the complete stochastic Euler system (2.8)-(2.10). Let \( y \in Y \) be an admissible initial data (condition), we denote by \( \mathcal{P}_y \) a solution to the martingale problem associated with (2.8)-(2.10) starting on \( y \) at time \( t = 0 \); that is, the marginal of \( \mathcal{P}_y \) at \( t = 0 \) is \( \Lambda_y \.

We start off with the definition of strong Markov family:

**Definition 7.1.** A family \( \{\mathcal{P}_y\}_{y \in Y} \) of probability measures is called a strong Markov selection family provided

1. For every \( A \in \mathcal{B} \), the mapping \( y \mapsto \mathcal{P}_y(A) \) is \( \mathcal{B}(Y)/\mathcal{B}([0,1]) \)-measurable.
2. For every finite \( (\mathcal{B}_t)_{t \geq 0} \)-stopping time \( T \), every \( y \in Y \) and \( \mathcal{P}_y \)-a.s. \( \omega \in \Omega \)

\[
\mathcal{P}_y|_{\mathcal{F}_T} = \mathcal{P}_y \circ \Phi_T^{-1}.
\]

Accordingly, a strong Markov family follows from the so-called pre-Markov family via a selection procedure. Finally, we have all we need to state the following theorem.

**Theorem 7.2.** Assume (2.8) and (2.9) holds. Then there exists a family \( \{\mathcal{P}_y\}_{y \in Y} \) of solutions to the martingale problem associated to (2.8)-(2.10) in the sense of Definition 6.1 with a.s. Markov property (as defined in Definition 2.4).

We set \( y = (y^1, y^2, y^3, y^4) \in Y \) and denote by \( C(y) \) the set of probability laws \( \mathcal{P}_y \in \text{Prob}[\Omega] \) solving the martingale problem associated to (2.8)-(2.10) with initial law \( \Lambda_y \).

The proof of Theorem 7.2 follows from applying abstract result of Theorem 2.6. In particular, we show that the family \( \{C(y)\}_{y \in Y} \) of solutions to the martingale problem satisfies the disintegration and reconstruction properties in Definition 2.5.

**Lemma 7.3.** For each \( y = (y^1, y^2, y^3, y^4) \in Y \). The set \( C(y) \) is non-empty and convex. Furthermore, for every \( \mathcal{P} \in C(y) \), the marginal at every time \( t \in (0, \infty) \) is supported on \( Y \).

**Proof.** Assuming \( y \in Y \), application of Theorem [existence] yields existence of a martingale solution to the problem (2.8)-(2.10) in the sense of Definition 6.1. Consequently, by Proposition 6.2 we infer that for each \( y \in Y \) the set \( C(y) \) is non-empty. For some \( \lambda \in (0, 1) \), let \( \mathcal{P}_1, \mathcal{P}_2 \in C(y) \) such that \( \mathcal{P} = \lambda \mathcal{P}_1 + (1 - \lambda) \mathcal{P}_2 \). Then convexity follows from noting that properties of Definition 2.12 involve integration with respect to the elements of \( C(y) \). In view of Definition 6.1 property (f) (energy equality), the marginal \( \mathcal{P} \in C(y) \) at every \( t \in (0, \infty) \) is supported in \( Y \).

For compactness we consider the following Lemma.

**Lemma 7.4.** Let \( y \in Y \). Then \( C(y) \) is a compact set and the map \( C : Y \to \text{Comp(Prob}[\Omega]) \) is Borel measurable.

**Proof.** The lemma follows from the claim: Let \( (y_n = (\eta_n, m_n, S_n, R_n))_{n \in \mathbb{N}} \subset Y \) be a sequence converging in \( Y \) to some \( (y = (\eta, m, S, R)) \) with respect to the metric \( d_F \) in (ref). Let \( \mathcal{P}_n \in C(y_n), n \in \mathbb{N} \). Then for each \( \{\mathcal{P}_n\}_{n \in \mathbb{N}} \), the sequence converges to some \( \mathcal{P} \in C(y) \) weakly in \( \text{Prob}[\Omega] \). While measurability of the map \( y \mapsto C(y) \) follows from using Theorem 12.1.8 in [45] for the metric space \( (Y, d_F) \). Accordingly, the claim follows from Theorem 2.1. Consequently, by Proposition 2.7 \( \mathcal{P} \) is a solution to a martingale problem with initial law \( \Lambda \). Therefore, \( \mathcal{P} \in C(y) \) as required.

Finally, we verify that \( C(y) \) has disintegration and reconstruction property in sense of Definition 2.5.

**Lemma 7.5.** The family \( \{C(y)\}_{y \in Y} \) satisfies the disintegration property of Definition 2.5.

**Proof.** Fix \( y \in Y, \mathcal{P} \in C(y) \) and let \( T \) be \( \mathcal{B}_t \)-stopping time. In view of Theorem 2.1 we know there exists a family of probability measures:

\[
\Omega \ni \omega \mapsto \mathcal{P}|_{\mathcal{F}_T} \in \text{Prob}[\Omega^{[T, \infty]}]
\]

such that

\[
\omega(T) = \tilde{\omega}(T), \mathcal{P}|_{\mathcal{F}_T}-\text{a.s., } \quad \mathcal{P}(\omega|_{[0,T]} \in A, \omega|_{[T,\infty]} \in B) = \int_{\omega|_{[0,T]} \in A} \mathcal{P}|_{\mathcal{F}_T}(B) d\mathcal{P}(\tilde{\omega}).
\]
for any Borel sets: $A \subset \Omega^{[0,T]}$ and $B \subset \Omega^{[T,\infty]}$. Here, we want to show that

$$\Phi_{-\tau}^{\omega^\infty_{[T,T]}} \in C(\omega(T)) \quad \text{for } \omega \in \Omega, \mathcal{P}\text{-a.s.}$$

Thus we are seeking an $\mathcal{P}|_{[\tilde{\Omega},T]}$ null-set $N$ outside of which properties (a)-(f) of Definition 6.1 holds for $\mathcal{P}|_{[\tilde{\Omega},T]}$.

To begin with, set $N_\nu$, ..., $N_\tau$ for each of the properties (a)-(f) of Definition 6.1 respectively, and let $N = N_\nu \cup \cdots \cup N_\tau$. Arguing similarly along the lines of Lemma 4.4 in [22] and [31] we have the following observations:

1. Set

$$H_T = \left\{ \omega \in \Omega : \tau \in \Omega^[[0,T]] \right\} \times \left\{ \omega : \tau \in \Omega^{[0,T]} \right\} \times \left\{ \omega : \tau \in \Omega^{[T,\infty]} \right\}$$

2. Similarly, we assign for each $\omega \in \Omega$ and $\tau \in \Omega$ a nullset $N_\nu$ such that $\mathcal{P}|_{[\tilde{\Omega},T]}(H_T) = 1$ holds for $\mathcal{P}$-a.a. $\omega \in \Omega$ (i.e. the remaining $\omega \in \Omega$ are contained in nullset $N_\nu$).

3. For property (c), let $(\psi_n)_{n \in \mathbb{N}}$ be a dense subset of $W^{1,2}(\mathbb{R}^3)$ and fix $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we assign an $\mathcal{P}$-nullset $N_\nu^n$ and set $N_\nu = \bigcup_{n \in \mathbb{N}} N_\nu^n$. To proceed, we split the continuity equation as follows:

$$\int_{\mathbb{T}^3} \xi^1 \psi_n \, dx = 0 \quad \forall \nu \leq T,$$

$$\int_{\mathbb{T}^3} \xi^2 \psi_n \, dx = 0 \quad \forall \nu \leq T < \infty,$$

and consider the sets

$$\mathcal{A}_\nu = \{ \omega : \omega |_{[0,T]} \text{ satisfies (7.2)} \}$$

$$\mathcal{A}_\nu = \{ \omega : \omega |_{[0,T]} \text{ satisfies (7.3)} \}.$$
(6) Similarly, for (f) we can argue as in the proof of (d) to obtain the nullset $N_f$.

Choosing $N = N_a \cup \cdots \cup N_f$ completes the proof.

□

Lemma 7.6. The family $\{C(y)\}_{y \in Y}$ satisfies the reconstruction property of Definition 2.6.

Proof. Fix $y \in Y$, $\mathcal{P} \in \mathcal{C}(y)$ and let $T$ be $\mathcal{B}_T$-stopping time. In view of Theorem 2.7, suppose that $Q_\omega$ is a family of probability measures such that $Q_\omega$ is a family of probability measures, such that

$$\Omega \ni \omega \mapsto Q_\omega \in \text{Prob}[\Omega^{[T,\infty]}],$$

is $\mathcal{B}_T$-measurable. Then there exists a unique probability measure $\mathcal{P} \otimes_T Q$ such that:

(a) For any Borel set $A \in \Omega^{[0,T]}$ we have

$$\mathcal{P} \otimes_T Q(A) = \mathcal{P}(A);$$

(b) For $\tilde{\omega} \in \Omega$ we have $\mathcal{P}$-a.s.

$$\mathcal{P} \otimes_T Q|_{\mathcal{B}_T} = Q_{\tilde{\omega}}$$

We aim to prove that for a $\mathcal{P}_\omega : \Omega \to \text{Prob}[\Omega^{[T,\infty]}]$-$\mathcal{B}_T$-measurable map such that there is $N \in \mathcal{B}_T$ with $\mathcal{P}(N) = 0$ and for all $\omega \notin N$ it holds

$$\omega(T) \in Y \quad \text{and} \quad \Phi_{-T}Q_\omega \in \mathcal{C}(\omega(T));$$

then $(\mathcal{P} \otimes_T Q) \in \mathcal{C}(y)$. In order to do this we have to verify properties (a)-(f) in Definition 6.1. The proof follows along the lines of Lemma 4.5. Adopting the notation introduced in Lemma 7.5 we argue as follows:

Here we note that $Q_\omega$ is a regular conditional probability distribution of $(\mathcal{P} \otimes_T Q)$ with respect to $\mathcal{B}_T$.

(1) Since (a) holds for $Q_\omega$ we have $Q_\omega(H_T) = 1$ such that

$$\mathcal{P} \otimes_T Q(H_T \cap H_T) = \int_{H_T} Q_\omega[S_T] \, d\mathcal{P}(\omega) = 1.$$

(2) For properties (b), (c) and (e) of Definition 6.1 we argue as in property (a) (Using the notation developed for each property, respectively).

(3) In the case of property (d), we proceed as follows:

Since (d) holds for $Q_\omega$ we know that $(\mathcal{M}_t(\varphi_n))_{t \geq T}$ is a $(\mathcal{B}_T)_{t \geq T}$-square integrable martingale for all $\varphi \in C^1(T^+)$. Consequently, by Proposition 2.7 we deduce that $(\mathcal{M}_t(\varphi_n))_{t \geq T}$ is a $(\mathcal{B}_T)_{t \geq T}$, $\mathcal{P} \otimes_T Q$-square integrable martingale as well. Observing that $\mathcal{P}$ and $\mathcal{P} \otimes_T Q$ coincides on $\mathcal{B}(\Omega^{[0,T]})$ and $(\mathcal{M}_t(\varphi_n))_{0 \leq t \leq T}$ is a $(\mathcal{B}_T)_{0 \leq t \leq T}, \mathcal{P}$-martingale (since $\mathcal{P}$ satisfies property (d)) we infer that

$(\mathcal{M}_t(\varphi_n))_{t \geq 0}$ is a $(\mathcal{B}_T)_{t \geq 0}, \mathcal{P} \otimes_T Q$-martingale.

(4) property (f) follows by the same argument as in property (d) (with obvious modifications).

□

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