On Multistep Bose-Einstein Condensation in Anisotropic Traps

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Abstract

Multistep Bose-Einstein condensation of an ideal Bose gas in anisotropic harmonic atom traps is studied. In the presence of strong anisotropy realized by the different trap frequency in each direction, finite size effect dictates a series of dimensional crossovers into lower-dimensional excitations. Two-step condensation and the dynamical reduction of the effective dimension can appear in three separate steps. When the multistep behavior occurs, the occupation number of atoms excited in each dimension is shown to behave similarly as a function of the temperature. Multistep behaviors can be easily controlled by changing the degree of anisotropy.

1 Introduction

Although Bose-Einstein condensation (BEC) has been an active topic of research in condensed matter physics for decades [1, 2, 3, 4], recent revival of interests in this field is mainly credited to the achievement in atom molecular optics; most recent laser cooling and evaporative cooling techniques in magnetic and optical atom traps enable us to realize BEC of a weakly interacting gas in a controlled way [5, 6, 7]. The weakly interacting nature of a gas allows us to handle the phenomena and make theoretical predictions with high accuracy. The rapid progress in this field also stimulates other areas in physics such as high energy physics and astrophysics [8].

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For atom trap experiments, since atoms are trapped in a finite geometry, finite size as well as finite number effects play a significant role in the condensation process. The conventional phase transition picture defined in the thermodynamic limit has to be reexamined or modified \[9, 10, 11, 12, 13, 14, 15, 16\]. One significant change due to finiteness is the existence of BEC in low-dimensional systems. Recently, quasi low-dimensional systems prepared by optical or magnetic trapping devices are actively discussed \[17, 18\]. The study of such systems provides an ideal test for the theory of finite size, low-dimensional systems in a controlled environment.

The critical behavior of a finite size system \[19, 20, 21\] is characterized by the effective infrared dimension (EIRD) of the system \[22, 23\]: near the critical point when the contribution of the lowest mode of a system dominates, its symmetry properties can be shown to be equivalent to a lower-dimensional one. The system, in such a case, is said to possess an EIRD. Varying the relative size (or shape) of the potential changes the infrared behavior and hence the EIRD of the system.

Dimensionless parameters \(\eta_i = \beta \bar{h} \omega_i\) \((\bar{\beta} = 1/k_B T)\) for a harmonic oscillator potential with natural frequencies \(\omega_i\) \((i = 1, 2, 3)\) characterize the degree of anisotropy and finite size effects. With respect to \(\eta_i\), we can classify the dynamical behavior of the system into the following four cases dependent on the degree of anisotropy:

- **Case 1:** \(\eta_1, \eta_2, \eta_3 > 1\) \(\rightarrow\) EIRD = 0,
- **Case 2:** \(\eta_1, \eta_2 > 1 > \eta_3\) \(\rightarrow\) EIRD = 1,
- **Case 3:** \(\eta_1 > 1 > \eta_2, \eta_3\) \(\rightarrow\) EIRD = 2,
- **Case 4:** \(1 > \eta_1, \eta_2, \eta_3\) \(\rightarrow\) EIRD = 3.

As the temperature is lowered, dynamical dimensional reduction of the system characterized by the decrease of effective dimension can be observed. Particularly, in the presence of maximal anisotropy \(1 \gg \eta_1 \gg \eta_2 \gg \eta_3\), EIRD decreases one by one from three to zero at the temperature \(k_B T = \bar{h} \omega_1, \bar{h} \omega_2, \bar{h} \omega_3\).

While the picture described above is of generic nature of any quantum system, for particles that obey Bose statistics, the dimension of the dynamics of excited particles can also change in the form of condensation. BEC can also occur in separate steps in the presence of anisotropy reducing the effective dimension attributed to the excited modes of the system by one, two, or three at a time \[10\]. In this sense, the dimensional crossover associated with both the multistep condensation and the reduction of effective dimension due to frozen degrees of freedom can manifest themselves in a similar way in a finite system in spite of their different origin. This is the main subject discussed in this paper.

For the system with finite size and number of atoms, the reduced chemical potential \(\phi \equiv \beta(E_0 - \mu)\) does not strictly vanish at the critical temperature. Only in the thermodynamic limit, \(\phi\) vanishes at the critical temperature and the specific heat develops the discontinuity in the derivative at the critical point \[20\]. In an isotropic system, the thermodynamic limit can be uniquely defined as discussed in many textbooks \[4\]. However, if we allow anisotropy in the system, there are three different ways of taking a thermodynamic limit: (Three-dimensional limit) \(\omega_1, \omega_2, \omega_3 \rightarrow 0\) while keeping \(N \omega_1 \omega_2 \omega_3\) fixed. In this case, the system shows the critical behavior of three dimension and the corresponding three dimensional critical temperature.
can be defined. (Two-dimensional limit) \( \omega_2 \omega_3 \to 0 \) while \( N \omega_2 \omega_3 \) fixed. The system shows the critical behavior of two dimension and the corresponding critical temperature \( T_{2D} \) is the one for the two-dimensional system. (One-dimensional limit) If we simply take \( \omega_3 \to 0 \) while \( N \omega_3 \) fixed, the critical temperature vanishes. However, if we tune \( \omega_3 \) a little slower such that \( \omega_3 \sim \log(2N)/N \to 0 \), the one-dimensional critical temperature \( T_{1D} \) can still be defined in this modified sense. We will discuss this issue again in Sec. 3.1.

In the presence of strong anisotropy, the whole particle spectrum naturally splits into zero, one, two, and three-dimensional excitations. The ground state is viewed as a zero-dimensional excitation. Let us denote the number of modes excited in the corresponding directions as \( N_0, N_1, N_2, \) and \( N_3 \), respectively. An \( n \)-dimensional condensation temperature \( T_{nD} (n = 1, 2, 3) \) can be defined as the temperature at which all the \( n \)-dimensionally excited modes are saturated:

\[
\begin{align*}
3\text{-dimensional; } N &= N_3(T_{3D}), \\
2\text{-dimensional; } N &= N_3(T_{2D}) + N_2(T_{2D}), \\
1\text{-dimensional; } N &= N_3(T_{1D}) + N_2(T_{1D}) + N_1(T_{1D}).
\end{align*}
\]

One can see that the condensation temperatures defined above are equivalent to the critical temperatures if the appropriate \( n \)-dimensional thermodynamic limit is taken. This splitting of the excitation spectrum gives the basis of the rest of our analysis. Similar splitting was proposed in [10] for liquid helium. In a liquid, however, an occupation number of particles excited in a particular direction is extremely difficult to observe. Furthermore, the validity of this splitting is by no means obvious for the strongly interacting system such as the liquid helium. On the other hand, such a quantity is directly observable in atom trap experiments and therefore it deserves careful study. Moreover, as shown in Section 3, occupation numbers \( N_1, N_2, \) and \( N_3 \) behave as if they were independent quantities and show the similar behavior when the multistep crossover occurs. This result indicates the independent nature of each \( N_i \) for strongly anisotropic systems.

For a realistic system where \( N, \omega_1, \omega_2, \omega_3 \), are all finite and fixed, physically observable temperature of interest is the crossover temperature at which the deviation from the bulk critical behavior sets in. The crossover temperature is achieved when the correlation length reaches the size of the system since further ordering in this direction will be suppressed at this point. In the strongly anisotropic system in which \( \omega_1, \omega_2 >> \omega_3 \) holds, \( T_{1D}, T_{2D} << T_{3D} \) gives the necessary (but not sufficient) condition for the multistep condensation: the condensation into two, one-dimensional modes and into the ground state can occur in separate steps. The multistep condensation was discussed for a nonrelativistic ideal gas in a cavity [10], in a harmonic trap [24], and a relativistic ideal gas in a cavity [25].

In Section 2, we study the excitation spectrum of anisotropic harmonic oscillators. We focus our attention to three different cases in which the equipotential surface has a prolate, oblate, and maximally anisotropic ellipsoidal shape. In Section 3, we show that each case shows qualitatively different condensation behavior. After introducing the condensation temperatures defined in the bulk limit, we focus on the multistep crossover behavior of excited modes between different effective dimensions through BEC or dynamical reduction.
of EIRD. In particular, in a maximally anisotropic potential, the dimensional reduction can occur in three-steps. In such a case, we show that each dimensional component behaves in a similar manner as a function of the reduced temperature defined near the corresponding crossover temperature. The possible effect of interactions is also discussed.

This work deals with a nonrelativistic ideal Bose gas in anisotropic magnetic traps, and a companion paper deals with a relativistic ideal Bose gas in rectangular cavities [25]. Our calculations are focused on the occupation number and condensation temperature for each dimension. The effect of an interaction in condensation process has been discussed in many literatures. In principle, it can affect the dynamics of condensation considerably. For a weakly interacting gas, however, the averaged quantities such as condensation fractions and critical temperatures are relatively insensitive to the presence of interactions and the corrections to bulk ideal-gas value are well-explained by the finite number correction [26, 27]. On the other hand, interaction effects are known to affect higher moments such as specific heat significantly and considered to be essential to explain the observed specific heat data. Throughout the rest of the paper, we use units such that \( k_B = \hbar = 1 \) for brevity. The results in ordinary units can be easily reproduced by replacing \( \omega \rightarrow \bar{\hbar}\omega \) and \( T \rightarrow k_B T \).

2 Anisotropic Harmonic Oscillator and Excitation Spectrum

For an anisotropic harmonic oscillator with oscillator frequencies \( \omega_i (i = 1, 2, 3) \), the Hamiltonian has the form

\[
H = \frac{1}{2} \sum_{i=1}^{3} (p_i^2 + \omega_i^2 x_i^2). \tag{4}
\]

In this paper, we study cases where the frequencies \( \omega_i \) are rationally related. Whence there exist integers \( k_i (i = 1, 2, 3) \) such that \( \omega_i k_i = \omega (i = 1, 2, 3) \) where \( \omega = \Omega(k_1 k_2 k_3)^{1/3} \) and \( \omega_1 \omega_2 \omega_3 = \Omega^3 \).

The energy level of an anisotropic harmonic oscillator is given by

\[
E_n = \sum_{i=1}^{3} \omega_i(n_i + 1/2). \tag{5}
\]

We can also define the energy level modulo \( k_i \) as \( n_i = k_i \nu_i + \lambda_i \) where \( \nu_i \equiv [n_i/k_i] \), and \( [\cdot] \) denotes the integer part of the number inside the bracket. Then Eq. (5) can be written as

\[
E_n = \omega M + \sum_{i=1}^{3} \omega_i \lambda_i + E_0, \tag{6}
\]

where \( M = \nu_1 + \nu_2 + \nu_3 \). The first term in Eq. (6) corresponds to the isotropic harmonic oscillator Hamiltonian (see also Appendix A). The ground state energy \( E_0 \) has the familiar

\footnote{This assumption of rationality is rather for the technical convenience and will not affect the physical results of this paper. Other methods can be found, for example, in [14], [24], and [28].}
form
\[ E_0 = \frac{\omega_1 + \omega_2 + \omega_3}{2}. \] (7)

2.1 Excitation spectrum

2.1.1 Prolate shape potential

First we discuss the case of anisotropy corresponding to \( \omega_1 = \omega_2 > \omega_3 \) (we simply choose here \( k_1 = k_2 = 1 < k_3 \)). In such a case, the equipotential surface has a prolate shape. For a strong anisotropy \( k_3 >> 1 \), two-step condensation can occur. In such a case, \( \omega = \omega_1 = \omega_2 \) and the energy eigenvalue is
\[ E_n = \omega M + \omega_3 \lambda_3 + E_0. \] (8)

For sufficiently large values of \( k_3 \), the whole energy spectrum can be split into the energy level of the ground state (\( E_n = 0 \)), one-dimensionally excited states (\( E_n = n_3 \omega_3; n_3 = 1, 2, \ldots \)), two-dimensionally excited states (\( E_n = n_1 \omega_1 + n_3 \omega_3; n_1 = 1, 2, \ldots, n_3 = 0, 1, \ldots \) and \( n_2 \omega_2 + n_3 \omega_3; n_2 = 1, 2, \ldots, n_3 = 0, 1, \ldots \)), and three-dimensionally excited states (\( E_n = n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3; n_1, n_2 = 1, 2, \ldots, n_3 = 0, 1, \ldots \)), where \( E_n \equiv E_n - E_0 = \omega M + \omega_3 \lambda_3 \) is the energy measured from the ground state. The number of particles excited in these dimensions are given respectively by
\[ N_0 = \frac{z}{1-z}, \] (9)
\[ N_1 = \sum_{n_3=1}^{\infty} \frac{z}{e^{n_3 \eta_3} - z}, \] (10)
\[ N_2 = \sum_{n_1=1, n_3=0}^{k_3-1} \sum_{M=1}^{\infty} \frac{2Mz}{e^{M \eta_1 + \lambda_3 \eta_3} - z}, \] (11)
\[ N_3 = \sum_{n_1=n_2=1, n_3=0}^{k_3-1} \sum_{M=2}^{\infty} \frac{(M-1)Mz}{e^{M \eta_1 + \lambda_3 \eta_3} - z}, \] (12)

where \( z \equiv e^{\beta(\mu - E_0)} \) is the (reduced) fugacity. The factor 2 in Eq. (11) accounts for the symmetry between the first and second axis. These expressions can be further simplified in the following manner.

For one dimension,
\[ N_1 = \sum_{n_3=1}^{\infty} \frac{ze^{-n_3 \eta_3}}{e - ze^{-n_3 \eta_3}} = \sum_{l=1}^{\infty} \frac{z^l e^{-l \eta_3}}{e - e^{-l \eta_3}} \] (13)
\[ = g_1 \left( \frac{ze^{-\eta_3/2}}{\eta_3} \right) + \cdots, \] (14)
where \( g_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p} \) is the Bose-Einstein function. We used Eq. (B4) to obtain the second line from the first line.

For two-dimensional excitations, making use of Eq. (B2), Eq. (11) can be written as

\[
N_2 = 2 \sum_{\lambda_3=0}^{k_1-1} \sum_{M=1}^{\infty} \sum_{l=1}^{\infty} M z^l e^{-l(M\eta_1+\lambda_3\eta_3)}
\]

\[
= 2 \sum_{l=1}^{\infty} \frac{z^l e^{-l\eta_1}}{(1 - e^{-l\eta_1})(1 - e^{-l\eta_3})}
\]

\[
= 2 \sum_{l=1}^{\infty} \frac{z^l e^{-l(m-\eta_3)/2}}{l^2 \eta_1 \eta_3} - \frac{1}{k_3} \sum_{l=1}^{\infty} \frac{z^l e^{-l(m-\eta_3)/2}}{12} + \cdots
\]

\[
= \frac{2g_2(ze^{-(m-\eta_3)/2})}{\eta_1 \eta_3} - \frac{k_3 g_0(ze^{-(m-\eta_3)/2})}{12} + \cdots. \tag{15}
\]

To obtain the third line from the second line, we used Eq. (B5).

For three dimensional excitations, Eq. (12) gives

\[
N_3 = \sum_{\lambda_3=0}^{k_1-1} \sum_{M=2}^{\infty} \sum_{l=1}^{\infty} \frac{(M-1)M}{2} z^l e^{-l(M\eta_1+\lambda_3\eta_3)}
\]

\[
= \sum_{l=1}^{\infty} \frac{z^l e^{-l\eta_1}}{(1 - e^{-l\eta_1})^2(1 - e^{-l\eta_3})}
\]

\[
= \sum_{l=1}^{\infty} \frac{z^l e^{-l(m-\eta_3)/2}}{l^3 \eta_1^2 \eta_3} - \sum_{l=1}^{\infty} \frac{z^l e^{-l(m-\eta_3)/2}}{12 \eta_3} + \cdots
\]

\[
= \frac{g_3(ze^{-(m-\eta_3)/2})}{\eta_1^2 \eta_3} - \frac{g_1(ze^{-(m-\eta_3)/2})}{12 \eta_3} + \cdots. \tag{16}
\]

2.1.2 Oblate shape potential

Next we discuss the case of anisotropy corresponding to an oblate shape potential \( \omega_1 > \omega_2 = \omega_3 \) ( \( k_1 = 1 < k_2 = k_3 \)). In this case, \( \omega = \omega_1 \) and

\[
\bar{E}_n = \omega M + \omega_2(\lambda_2 + \lambda_3). \tag{17}
\]

The number of particles excited in these dimensions are given respectively by

\[
N_0 = \frac{z}{1 - z}, \tag{18}
\]

\[
N_1 = \sum_{n_2=1}^{\infty} \frac{2z}{e^{n_2 \eta_2} - z}, \tag{19}
\]

\[
N_2 = \sum_{n_2=1, n_3=1}^{\infty} \frac{z}{e^{n_2 \eta_2 + n_3 \eta_3} - z}
\]
\[ N_3 = \sum_{n_1=1, n_2=0, n_3=0}^{\infty} \frac{z}{e^{n_1 \eta_1 + n_2 \eta_2 + n_3 \eta_3} - z} = \sum_{n_1=1}^{\infty} \sum_{M=0}^{\infty} \frac{(M + 1) z}{e^{n_1 \eta_1 + M \eta_2} - z}, \]  

where the factor 2 in Eq. (19) is due to the symmetry between the second and third axis. Compared to Eq. (14), we obtain

\[ N_1 = \frac{2 g_1 (ze^{-\eta_2/2})}{\eta_2} + \ldots \]  
in the present case.

The number of particles excited two-dimensionally on \( x_2 - x_3 \) plane can be written as

\[ N_2 = \sum_{l=1}^{\infty} \sum_{M=2}^{\infty} (M - 1) z^l e^{-lM \eta_2} = \sum_{l=1}^{\infty} \frac{z^l e^{-2l \eta_2}}{(1 - e^{-l \eta_2})^2} = \frac{g_2 (ze^{-\eta_2})}{\eta_2^2} + \ldots. \]  

For three dimensional excitations, Eq. (21) gives

\[ N_3 = \sum_{l=1}^{\infty} \frac{z^l e^{-l \eta_1}}{(1 - e^{-l \eta_1})(1 - e^{-l \eta_2})^2} = \frac{g_3 (ze^{-(\eta_1/2 - \eta_2)})}{\eta_1 \eta_2^2} - \frac{g_1 (ze^{-(\eta_1/2 - \eta_2)})}{24 \eta_3^2} (\frac{\eta_1^2 + 2 \eta_2^2}{\eta_1 \eta_2} + \ldots). \]

### 2.1.3 Maximally anisotropic potential

For anisotropies \( \omega_1 > \omega_2 > \omega_3 \) with \( k_1 = 1 << k_2 << k_3 \), \( \omega = \omega_1 \) and

\[ \bar{E}_n = \omega M + \omega_2 \lambda_2 + \omega_3 \lambda_3. \]  

The number of excited modes in the corresponding dimensions can be defined by

\[ N_1 = \sum_{n_3=1}^{\infty} \frac{z}{e^{n_3 \eta_3} - z}, \]

\[ N_2 = \sum_{n_2=1, n_3=0}^{\infty} \frac{z}{e^{n_2 \eta_2 + n_3 \eta_3} - z}. \]
\[ N_3 = \sum_{n_1=1, n_2=0, n_3=0}^{\infty} \frac{z}{e^{n_1 \eta_1 + n_2 \eta_2 + n_3 \eta_3} - z} \]
\[ = \sum_{\lambda_2=0}^{k_2-1} \sum_{\lambda_3=0}^{k_3-1} \sum_{M=1}^{\infty} \frac{M(M+1)z}{2 e^{M \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3} - z}. \]  

(28)

For the two dimensional case, following Eq. (15),

\[ N_2 = \sum_{\lambda_3=0}^{k_3-1} \sum_{M=1}^{\infty} \sum_{l=1}^{\infty} M z e^{-l(M \eta_2 + \lambda_3 \eta_3)} \]
\[ = \sum_{l=1}^{\infty} \frac{z e^{-l \eta_2}}{(1 - e^{-l \eta_2})(1 - e^{-l \eta_3})} \]
\[ = \sum_{l=1}^{\infty} \frac{z e^{-l / 2 \eta_2}}{l^2 \eta_2 \eta_3} - \left( \kappa + \frac{1}{\kappa} \right) \sum_{l=1}^{\infty} \frac{z e^{-l / 2 \eta_2}}{24} + \cdots \]
\[ = \frac{g_2(z e^{-\eta_2 / 2})}{\eta_2 \eta_3} - \frac{\kappa g_0(z e^{-\eta_2 / 2})}{\eta_2 \eta_3} + \cdots, \]  

(29)

where \( \kappa \equiv k_3/k_2 \).

For three dimensional excitations, Eq. (28) gives

\[ N_3 = \sum_{l=1}^{\infty} \frac{z e^{-l \eta_2}}{l^3 \eta_1 \eta_2 \eta_3} \]
\[ = \sum_{l=1}^{\infty} \frac{z e^{-l (\eta_1 - \eta_2 - \eta_3) / 2}}{l^3 \eta_1 \eta_2 \eta_3} \]
\[ = \frac{g_3(z e^{-(\eta_1 - \eta_2 - \eta_3) / 2})}{\eta_1 \eta_2 \eta_3} - \frac{g_1(z e^{-(\eta_1 - \eta_2 - \eta_3) / 2})}{\eta_1 \eta_2 \eta_3} \]
\[ + \cdots. \]  

(30)

3 Finite Size Effects and Dimensional Crossover Behavior

3.1 Bulk behavior

The bulk three-dimensional condensation temperature is defined in the thermodynamic limit \( \eta_i \to 0 \) \( (i = 1, 2, 3) \) and \( N \to \infty \) while \( \eta_1/\eta_3 \) and \( \eta_2/\eta_3 \) fixed. The dominant term is given by the first term in \( N_3 \) and the critical temperature satisfies

\[ N = \frac{T_{3D}^3}{\omega_1 \omega_2 \omega_3} \zeta(3). \]  

(31)

Therefore

\[ T_{3D} = \left( \frac{N \omega_1 \omega_2 \omega_3}{\zeta(3)} \right)^{1/3}. \]  

(32)
Two-dimensional limit is given by $\eta_2, \eta_3 \to 0, N \to \infty$, but $\eta_1 >> 1$. In such a case, the dominant term in particle number is $N_2$ and the critical temperature $T_{2D}$ is defined by

$$N = \frac{T_{2D}^2}{\omega_2 \omega_3} g_2(e^{-(\omega_2+\omega_3)/2T_{2D}}) = \frac{T_{2D}^2}{\omega_2 \omega_3} \zeta(2) + \cdots. \quad (33)$$

Thus we have

$$T_{2D} = \left( \frac{N\omega_2 \omega_3}{\zeta(2)} \right)^{1/2}. \quad (34)$$

For one-dimensional limit $\eta_3 \to 0, N \to \infty$, but $\eta_1, \eta_2 >> 1$, the dominant term in particle number is $N_1$. The condensation temperature is defined by

$$N = \frac{T_{1D}}{\omega_3} g_1(e^{-\omega_3/2T_{1D}}). \quad (35)$$

To leading order in $\eta_3^{-1}$, Eq. (14) can be approximated as $N_1 \sim \frac{T_{1D}}{\omega_3} \log \left( \frac{2\omega_3}{\omega_3} \right)$. Thus the one-dimensional condensation temperature $T_{1D}$ is defined by

$$N = \frac{T_{1D}}{\omega_3} \log \left( \frac{2T_{1D}}{\omega_3} \right), \quad (36)$$

which gives

$$T_{1D} = \frac{N\omega_3}{\log(2N)} \quad (37)$$

for large $N$. Note that in the thermodynamic limit $N \to \infty$ while $N\omega_3$ fixed, $T_{1D}$ vanishes.

In Figure 1, we plot $T_{1D}$, $T_{2D}$, and $T_{3D}$ as a function of the anisotropy parameter $k_3$. In an isotropic and weakly isotropic case ($k_3 \sim 1$), $T_{3D} < T_{1D}, T_{2D}$ and the condensation is directly into the ground state. As anisotropy is increased ($k_3 >> 10^3$), $T_{1D}, T_{2D} < T_{3D}$ is achieved. This is the regime where various multistep behaviors can take place.

In terms of bulk condensation temperatures we obtained in Section 3.1 as $T_{1D} = N\omega_3/\log(2N)$, $T_{2D} = (N\omega_2 \omega_3/2\zeta(2))^{1/2}$, and $T_{3D} = (N\omega_1 \omega_2 \omega_3 / \zeta(3))^{1/3}$, the conditions (A) $T_{1D} << T_{2D}$, (B) $T_{2D} << T_{3D}$, and (C) $T_{1D} << T_{3D}$ give constraints for $k_3$ and $\kappa$ as

$$\begin{align*}
(A) \quad \kappa & \gg \frac{N\zeta(2)}{(\log(2N))^2}, \\
(B) \quad k_3^2 / \kappa & \gg \frac{N\zeta(3)^2}{\zeta(2)^3}, \\
(C) \quad k_3 \kappa & \gg \frac{N^2}{(\log(2N))^3}.
\end{align*} \quad (38)$$

Since (B) $T_{2D} << T_{3D}$ implies $T_{3D} << \omega_1$, three-step BEC never occurs in harmonic traps.

In Figure 2, different condensation behaviors corresponding to various anisotropy parameters
ω₁/ω₂ and ω₂/ω₃ are shown. The vertical axis corresponds to the prolate-shape potential studied in Section 3.2.1 (this case was studied in [24]). In such a potential, two-step BEC can be seen. The horizontal axis corresponds to the oblate-shape potential discussed in Section 3.2.2, where we show that there is no multistep condensation in this case. The more general class of anisotropic case will be discussed in Section 3.2.3. The combined effect of dynamical dimensional reduction and two-step BEC in such a potential can appear in three steps.

3.2 Dimensional crossover and condensation

For a highly anisotropic trap, the three dimensional crossover temperature \(T_{3D}^*\) should be reached when the correlation length is in the order of the size of the ground state wave function in the most confining direction. Spreading of the wave function can be characterized by \(L_i \equiv \sqrt{\hbar/m\omega_i}\) (for \(i = 1, 2, 3\)) [29, 30]. Hence the above condition is equivalent to \(\xi(T_{3D}^*) \sim \lambda_{dB}/\sqrt{T_3} \sim L_1\), where \(\lambda_{dB} \equiv h/\sqrt{2\pi mkT}\) is the thermal de Broglie wavelength, and \(t_3 \equiv |T_{3D}^* - T_{3D}|/T_{3D}\). This will give us the crude estimate of \(T_{3D}^*\) as

\[|T_{3D}^* - T_{3D}| \sim \left(\frac{k_3 \zeta(3)}{N}\right)^{1/3} T_{3D}. \tag{39}\]

3.2.1 Two-step condensation

For a prolate shape potential discussed in Section 2.1.1, we expand the whole particle spectrum with respect to \(\eta_1\) and \(\eta_3\) and obtain

\[
N = N_0 + N_1 + N_2 + N_3 = g_0(z) + \frac{g_1(z)}{\eta_3} + \frac{2g_2(z)\eta_1}{\eta_3^2} + \frac{g_3(z)\eta_1^2}{\eta_3^2} + \ldots
\]

\[
= g_0(z) + \frac{g_1(z)}{\eta_3} + \frac{2g_2(z)}{\eta_1\eta_3} + \frac{g_3(z)}{\eta_1^2}\eta_3
\]

\[
- \frac{g_0(z)}{\eta_3} - \frac{2g_1(z)}{\eta_3^2} - \frac{g_2(z)}{\eta_3^2}\eta_3
\]

\[
+ \frac{g_0(z)}{\eta_1\eta_3} \left(\frac{\eta_1}{2}\right)^2 + \frac{g_1(z)}{\eta_1^2\eta_3} \frac{\eta_1^2}{2} - \frac{g_2(z)}{\eta_1^2\eta_3} \frac{\eta_1^3}{6} + \ldots. \tag{40}\]

This expression can be simplified to give

\[N\eta_3 = \frac{g_3(z)}{\eta_1^2} + \frac{g_2(z)}{\eta_1} + \frac{g_1(z)}{2} + \ldots. \tag{41}\]

Writing \(z = e^{-\phi}\) and expanding Eq. (41) with respect to \(\phi\) give

\[N\eta_3 = \frac{\zeta(3)}{\eta_1^2} + \frac{\zeta(2)}{\eta_1} - \frac{\zeta(2)\phi}{\eta_1^2} + \ldots. \tag{42}\]
where we used an asymptotic expansion of the Bose-Einstein function \( g_3(e^{-\alpha}) \sim \zeta(3) - \zeta(2)\alpha + 1/2(3/2 - \log \alpha)\alpha^2 + \cdots \) and \( g_2(e^{-\alpha}) \sim \zeta(2) + (\log \alpha - 1)\alpha + \cdots \) for small \( \alpha \) [4].

Correlation length \( \xi \) of an ideal Bose gas is given by \( \xi = \lambda_{dB}/2\sqrt{\pi\phi} \) [3]. In terms of scaling parameters \( x_i(T) \equiv \phi(T)/\eta_i(i = 1, 2, 3) \), the above argument implies that \( T_{3D}^{*} \) is achieved when \( x_1(T_{3D}^{*}) \equiv c_1 \), where \( c_1 \) is some constant in the order of unity. Inserting this into Eq. (42) gives

\[
N = \frac{\zeta(3)}{\eta_1^2\eta_3} + \frac{\zeta(2)(1 - c_1)}{\eta_1\eta_3} + \cdots \quad \text{at} \quad T = T_{3D}^{*}. \tag{43}
\]

Thus we have

\[
\frac{T_{3D}^{*}}{T_{3D}} = 1 + \frac{c_1 - 1}{3} \frac{\zeta(2)}{\zeta(3)} \left(\frac{k_3}{N}\right)^{1/3}. \tag{44}
\]

This result gives the same correction term proportional to \((k_3/N)^{1/3}\) as in Eq. (39) obtained by heuristic arguments.

The one-dimensional condensation temperature \( T_{1D} \) is defined in Eq. (35) in the limit of small \( \eta_3 \) and the vanishing reduced chemical potential \( \phi \); finite size effects on \( T_{1D} \) originate in finiteness of both \( \eta_3 \) and \( \phi \). At one-dimensional crossover temperature \( T_{1D}^{*} \), correlation length reaches the size of the ground state wave function in the least confining direction, namely, along the third axis in the present case. Thus \( \xi(T_{1D}^{*}) \sim L_3 \) or equivalently \( x_3(T_{1D}^{*}) \equiv c_3 = O(1) \). Then from the second line in Eq. (40), we obtain

\[
N = \frac{g_1(e^{-(1+2c_3)\eta_3/2})}{\eta_3} + \frac{2g_2(e^{-(k_3+2c_3)\eta_3/2})}{\eta_1\eta_3} + \frac{g_3(e^{-(k_3+c_3)\eta_3})}{\eta_1^2\eta_3} + \cdots. \tag{45}
\]

In the limit \( \eta_3 \to 0 \), only the first term dominates and we obtain the crossover temperature as

\[
N = -\frac{T_{1D}^{*}}{\omega_3} \log(1 - e^{-(1+2c_3)\omega_3/2T_{1D}^{*}}) = \frac{T_{1D}^{*}}{\omega_3} \log \frac{2T_{1D}^{*}}{1 + 2c_3\omega_3}. \tag{46}
\]

For large \( N \), this gives

\[
T_{1D}^{*} = \frac{N\omega_3}{\log[2N/(1 + 2c_3)]}. \tag{47}
\]

Note that \( T_{1D}^{*} > T_{1D} \) holds.

The conditions (C) in Eq. (38) and \( \omega_1 < T_{1D} \) are satisfied if

\[
\frac{N}{(\log(2N))^{3/2}} < k_3 < \frac{N}{(\log(2N))}. \tag{48}
\]
In such a case, two-step condensation leading to the condensation into the ground state can be seen whereas the system is effectively still three-dimensional. In Figure 3-6, the condensation fractions $N_i/N(i = 0,1,2,3)$ as a function of temperature are plotted. At high temperature, three-dimensionally excited states dominate, as expected from the density of states which grows as $M^2$, where $M$ is the number of degeneracy of an isotropic harmonic oscillator appeared in Eq. (6).

In the isotropic case (Figure 3), condensation is only into the ground state. Due to the finite size effects, condensation already starts before the critical temperature is reached. In a strongly anisotropic case, as in Figure 4, two-step condensation can be seen. $T_{3D}^*$ determines the onset of condensation into one-dimensionally excited states. At $T_{3D}^*$ the ground state fraction is negligibly small. Condensation into the ground state will not start until $T_{1D}^*$ is reached.

In the multistep process peculiar to the highly anisotropic system, when the correlation length reaches the size of the system, the dynamics shows the crossover to the low-dimensional one before the actual phase transition occurs. In this sense, the critical temperature is never observed in such a process and the directly relevant quantity to the observation is the crossover temperature, the temperature at which the finite size correction sets in. For practical purposes, this is often replaced by including the finite size correction as the term proportional to the power of $1/N$ whereas the chemical potential is set to the ground state energy \[32\]. Strictly speaking, however, since the chemical potential never reaches the ground state energy in the finite system, the meaning of this correction has some ambiguity. The difference between the crossover temperature and the finite size-corrected critical temperature in the present axially symmetric trap case is given by

$$\frac{\Delta T}{T_{3D}} = c_1 \frac{\zeta(2)}{3} \frac{\zeta(3)}{2/3} \left(\frac{k_3}{N}\right)^{1/3}.$$  

(49)

While this is fairly small for an isotropic or weakly anisotropic case: $\Delta T/T_{3D} \sim 0.024$ for $k_3 = 1$, it is no longer so for a strongly anisotropic case: $\Delta T/T_{3D} \sim 0.24$ for $k_3 = 10^3$ as used in Figure 4, where $c_1 = 1, N = 10^4$ for both cases. The ordinary finite size correction significantly underestimates the results in the latter case.\(^2\) For these reasons, we focus our discussions on crossover temperatures in the present work. We should also note that there is a slight amount of ambiguity in the choice of $c_i(i = 1,2,3)$. In general, the correlation length $\xi$ is a complicated function of the temperature away from the critical value and the reliable choice is obtained by the numerical fitting. We will simply put $c_i(i = 1,2,3) = 1$ for our comparison with numerical data for brevity.

### 3.2.2 Two-dimensional condensation

For an oblate shape potential discussed in Section 2.1.2, assuming $\eta_1 >> 1$ and we obtain

$$N = N_0 + N_1 + N_2 + N_3$$

\(^2\)Since the boundary condition in the harmonic potential corresponds to the one in the Neumann boundary condition, the surface correction increases the density of states and hence decreases the critical temperature.
\[
N = g_0(z) + \frac{2g_1(ze^{-\eta_2/2})}{\eta_2} + \frac{g_2(ze^{-\eta_2})}{\eta_2^2} + \ldots. \tag{50}
\]

Two-dimensional crossover temperature \( T_{2D}^* \) can be defined when the correlation length reaches the size of the ground state wave function along the second axis. Thus we have \( \xi(T_{2D}) \sim L_2 \) or equivalently \( x_2(T_{2D}^*) \equiv c_2 = O(1) \). Then from Eq. (50) we obtain

\[
N = \frac{g_2(e^{-(1+c_2)\eta_2})}{\eta_2^2} + \frac{2g_1(e^{-(1+2c_2)\eta_2/2})}{\eta_2} + \ldots \tag{51}
\]
at \( T = T_{2D}^* \). Expanding in terms of \( \eta_2 \) and we obtain \( T_{2D}^* \) as

\[
\begin{align*}
N &= \frac{\zeta(2)}{\eta_2^2} - \frac{1 + c_2}{\eta_2} + \left[ \frac{(1 + c_2) \log[\eta_2(1 + c_2)]}{\eta_2} - \frac{2 \log(\eta_2 c_2)}{\eta_2} \right] \\
&= \frac{T_{2D}^* \zeta(2)}{\omega_2^2} - \frac{T_{2D}^*}{\omega_2} (1 + c_2) \\
&\quad + \left( 1 + c_2 \right) \log \left( \frac{T_{2D}^*}{\omega_2 (1 + c_2)} \right) + 2 \log \left( \frac{T_{2D}^*}{\omega_2 c_2} \right) + \ldots \tag{52}
\end{align*}
\]

for \( \eta_2, \eta_3 << 1 \). For large \( N \) and \( c_2 = 1 \), this gives

\[
\frac{T_{2D}^*}{T_{2D}} = 1 + \left( \frac{\kappa}{N \zeta(2)} \right)^{1/2} \log \left( \frac{N}{\kappa \zeta(2)} \right). \tag{53}
\]

As explained in Section 3.1, condensation into \( \tilde{N}_2 \) does not occur in harmonic traps. For an oblate shape potential, the system dynamics freezes out along the first axis at \( T = \omega_1 \). Therefore the dynamics of the system at \( T < \omega_1 \) is two-dimensional. Ordinary two-dimensional BEC can still be observed as long as \( T_{2D} < \omega_1 \). This condition requires \( k_3 > (N/\zeta(2))^{1/2} \). In Figure 5, two-dimensional BEC in this parameter regime is shown.

### 3.2.3 Three-step dimensional reduction

For a general class of anisotropic potential discussed in Section 2.1.3, three-step process can be observed. We expand each \( N_i \) with respect to \( \eta_1, \eta_2, \eta_3 \) and obtain

\[
N = N_0 + N_1 + N_2 + N_3
\]

\[
\begin{align*}
&= g_0(z) + \frac{g_1(ze^{-\eta_3/2})}{\eta_3} + \frac{g_2(ze^{-\eta_3})}{\eta_2 \eta_3} - \frac{\kappa}{24} g_0(ze^{-\eta_2/2}) + \frac{g_3(ze^{-(\eta_1-\eta_2-\eta_3)/2})}{\eta_1 \eta_2 \eta_3} \\
&\quad - \frac{g_1(ze^{-(\eta_1-\eta_2-\eta_3)/2})}{24} \left( \eta_1^2 + \eta_2^2 + \eta_3^2 \right) + \ldots \\
&= \frac{g_3(z)}{\eta_1 \eta_2 \eta_3} + \frac{g_2(z)}{\eta_2 \eta_3} - \frac{g_2(z)}{\eta_1 \eta_2 \eta_3} \frac{\eta_1 - \eta_2 - \eta_3}{2} + \ldots \tag{54}
\end{align*}
\]
Expanding Eq. (54) with respect to $\phi = -\log z$ gives

$$N = \frac{\zeta(3)}{\eta_1 \eta_2 \eta_3} + \frac{\zeta(2)}{2} \left( \frac{1}{\eta_1 \eta_2} + \frac{1}{\eta_2 \eta_3} + \frac{1}{\eta_3 \eta_1} \right) = \frac{\zeta(2)\phi}{\eta_1 \eta_2 \eta_3} \ldots. \quad (55)$$

As discussed in Section 3.2.1, $x_1(T_{3D}^*) \equiv c_1 = O(1)$ holds at $T = T_{3D}^*$. Inserting this into Eq. (55) gives

$$N = \frac{\zeta(3)}{\eta_1 \eta_2 \eta_3} + \frac{\zeta(2)}{2} \left[ \frac{1}{\eta_1 \eta_2} + \frac{1}{\eta_2 \eta_3} \left( \frac{1}{2} - c_1 \right) + \frac{1}{\eta_3 \eta_1} \right] + \ldots \text{ at } T = T_{3D}^*. \quad (56)$$

Thus we have

$$\frac{T_{3D}^*}{T_{3D}} = 1 + \frac{c_1 - 1/2}{3} \frac{\zeta(2)}{\zeta(3)} \left( \frac{k_2 k_3}{N} \right)^{1/3}. \quad (57)$$

The crossover temperature $T_{2D}^*$ associated with two-dimensional BEC can be defined if $T_{2D}^* < \omega_1$. Then from Eq. (54) we obtain

$$N = \frac{g_2(e^{-\eta_2(1/2+c_2)})}{\eta_2 \eta_3} + \frac{g_1(e^{-\eta_2(1/2+c_2)})}{\eta_3} + \frac{e^{-\eta_2/2}}{\eta_1 \eta_2 \eta_3}$$

$$+ \frac{\log(1 - e^{-\eta_2/2})\eta_1}{24\eta_2 \eta_3} + \ldots \quad (58)$$

at $T = T_{2D}^*$. When $\eta_1$ becomes large, only the first two terms dominate and we obtain the crossover temperature as

$$N = \frac{\zeta(2)}{\eta_2 \eta_3} \frac{1}{\eta_3} + \frac{1}{2 + c_2} + \frac{(1/2 + c_2) \log[\eta_2(1/2 + c_2)]}{\eta_3} - \frac{\log(\eta_2 c_2)}{\eta_3} + \ldots$$

$$= \frac{T_{2D}^* \zeta(2)}{\omega_2 \omega_3} - \frac{T_{2D}^*}{\omega_3} \left[ (1/2 + c_2) \right] + \frac{(1/2 + c_2) \log \left( \frac{T_{2D}^*}{\omega_2 (1/2 + c_2)} \right) + \log \left( \frac{T_{2D}^*}{\omega_2 c_2} \right) }{\eta_3} \ldots \quad (59)$$

for $\eta_2, \eta_3 << 1$. For large $N$ and $c_2 = 1$, we simply have

$$\frac{T_{2D}^*}{T_{2D}} = 1 + \frac{5}{8} \left( \frac{\kappa}{N \zeta(2)} \right)^{1/2} \log \left( \frac{N}{\kappa \zeta(2)} \right). \quad (60)$$

One-dimensional crossover temperature $T_{1D}^*$ has the same form as in the prolate shape potential case.

---

3 Thus the system behaves effectively two-dimensional (EIRD=2) at $T = T_{2D}^*$. 
The conditions (A) and (B) in Eq. (38) give the constraints for anisotropy parameters for three-step behavior to be observed:

\[ \kappa > \frac{N\zeta(2)}{(\log(2N))^2}, \]

\[ k_3 > \frac{\zeta(3)N}{\zeta(2)\log(2N)}. \]  

If \( T_{2D} \ll T_{3D} \) holds, since it also implies \( T_{2D} \ll \omega_1 \), excitations along the first axis will be dynamically suppressed at \( T < \omega_1 \) making the system effectively two-dimensional before BEC sets in. Furthermore, if \( T_{1D} \ll T_{2D} \) is also satisfied, BEC occurs in two-steps, one at \( T_{2D} \) and the other at \( T_{1D} \). In Figure 6, above senario of three-step dimensional reduction is numerically realized. Three-dimensionally excited modes dominant in higher temperature are dynamically suppressed at \( T < \omega_1 \) followed by two-step BEC. When the condensation into the ground state sets in, the effective dimension of the system is still two. From the form of critical temperatures in Eqs. (44) and (53), we see that high anisotropy and the small number of atoms have similar consequences.

It is useful to see how the weak interaction effect modifies the above arguments. The shift of the critical temperature due to the interaction effect is given in [33] as \( \Delta T_{int} \sim \frac{aN^{1/6}}{L} \), where \( a \) is the s-wave scattering length, \( L \equiv \sqrt{\hbar/m\Omega} \). The shift due to finite size is given in Eq. (57) as \( \Delta T_{3D}^{int} \sim (k_3/N)^{1/3} \). For a reasonable choice of parameters, \( N = 5000, k_3 = 10, a/L = 0.001 \), we obtain \( \Delta T_{3D}^{int} \sim 0.003 \) and \( \Delta T_{3D}^{int} \sim 0.13 \). Thus the interaction effect is negligible compared to the finite size correction.

Comparing the zero point energy in the most confining direction with the interaction energy, if \( \omega_1 > n_0U \), where \( n_0 \) is the ground state density and \( U \equiv 4\pi\hbar^2a/m \), the criteria for the system to behave as effectively two-dimensional is still given by \( \omega_1 > T \). For the choice of value \( n_0U/k_B \sim 110 \text{ [nK]} \) for a sodium atom, this gives \( \omega_1 > n_0U \sim 10^4 \text{ [1/sec]} \). We should note that the collision effect even in such a case cannot be strictly two-dimensional; for high energy atoms, the interaction vertex has the form of the three-dimensional one. Nevertheless, these effects will be still suppressed for \( a \ll L_1 \) and a small number of particles whereas the finite size correction behaves as \( \Delta T_{2D}^{int}/T_{2D} \sim (\kappa/N)^{1/2} \) and \( \Delta T_{1D}^{int}/T_{1D} \sim 1/\log N \). Therefore the effect of interaction to the particle number and the critical temperature remains small and the system is still expected to show the multistep behavior under these conditions. At \( T < \omega_1 \), the most notable difference would be the absence of long range order due to interactions in this effectively two-dimensional system. In this case, the system can show quasi-condensation, the condensation with the fluctuating phase at near \( T_{2D} \). The true condensation with the constant phase will be acheived at lower temperature. One example of quasi two-dimensional system, a gas of spin-polarized hydrogen in liquid helium, is known to exhibit Kosterlitz-Thouless transition [35, 36]. Manifestation of the crossover from BEC to Kosterlitz-Thouless transition [37] during the multistep process is of particular interest to study. But even in the presence of such a transition, we expect that

\[ \text{We recover ordinary units in this discussion for quantitative comparison with other literatures.} \]
the particle occupation number which is insensitive to the phase information still behaves similar to the ideal gas. The interaction contributions used above are calculated within the framework of the mean field theory. Thus we conclude, apart from the critical regime where the mean field theory fails, interactions do not essentially alter the multistep process for a small number of atoms in highly anisotropic traps.

From the arguments given in this section, the sum of most relevant terms in Eq. (54) around each crossover temperature will give the simple expression of the total number of atoms as

\[ N = N_0 + N_1 + N_2 + N_3 \]

\[ \sim g_0(e^{-\phi}) + \frac{g_1(e^{-\phi})}{\eta_3} + \frac{g_2(e^{-\phi})}{\eta_2 \eta_3} + \frac{g_3(e^{-\phi})}{\eta_1 \eta_2 \eta_3}. \] (62)

Making use of the fact that \( \phi \) varies as a nonvanishing function of the temperature for the finite system, we define

\[ N_3(\lambda) \equiv \frac{g_3(e^{\lambda \phi})}{\eta_1 \eta_2 \eta_3}. \]

The relation for the Bose function \( \partial_{\phi} g_n(e^{-\phi}) = -g_n-1(e^{-\phi}) \) allows us to write

\[ N_2(\lambda) \equiv \frac{g_2(e^{\lambda \phi})}{\eta_2 \eta_3} = \frac{1}{x_1} \frac{dN_3(\lambda)}{d\lambda}, \]

\[ N_1(\lambda) \equiv \frac{g_1(e^{\lambda \phi})}{\eta_3} = \frac{1}{x_1 x_2} \frac{d^2N_3(\lambda)}{d\lambda^2}, \]

\[ N_0(\lambda) \equiv g_0(e^{\lambda \phi}) = \frac{1}{x_1 x_2 x_3} \frac{d^3N_3(\lambda)}{d\lambda^3}. \] (63)

Thus the total number of atoms is given by

\[ N = \frac{1}{x_1 x_2 x_3} \frac{d^3N_3(-1)}{d\lambda^3} + \frac{1}{x_1 x_2} \frac{d^2N_3(-1)}{d\lambda^2} + \frac{1}{x_1} \frac{dN_3(-1)}{d\lambda} + N_3(-1). \] (64)

At \( T \sim T_{3D}^*, x_1 \sim 1 \) and \( x_2, x_3 >> 1 \), then we have

\[ N \sim \frac{d}{d\lambda} N_3(-1) + N_3(-1). \] (65)

At \( T \sim T_{2D}^*, x_2 \sim 1, x_1 << 1, \) and \( x_3 >> 1 \), we have

\[ N \sim \frac{d}{d\lambda} N_2(-1) + N_2(-1). \] (66)

At \( T \sim T_{1D}^*, x_3 \sim 1 \) and \( x_1, x_2 << 1 \), we have

\[ N \sim \frac{d}{d\lambda} N_1(-1) + N_1(-1). \] (67)

These results suggest that each condensation fraction \( N_i \) behaves similarly as a function of the temperature in the neighborhood of its characteristic temperature \( T_{iD}^* \). We simply write
this fact as \( N_i(T) \sim F(T/T_{iD}) \) for \( i = 1, 2, 3 \), where \( F \) is a function independent of \( i \). It also implies that each component has a similar shape in the logarithmic \( T \) scale. The occupation number of each dimension is plotted in the logarithmic \( T \) scale in Figure 7. Note that this derivation relies on the special property of the Bose function which determines the density of states of an ideal gas trapped in the harmonic potential. Whether the same result holds or not in other systems with different density of states is not obvious.

In conclusion, finite size effects on the Bose-Einstein condensation of an ideal gas in a strongly anisotropic trap give rise to various different types of multistep behavior depending on the degree of anisotropy. In an isotropic trap, BEC into the ground state always begins while the system is effectively three-dimensional, i.e. \( T_{3D} > \omega_1 = \omega_2 = \omega_3 \). In an anisotropic trap, in addition to the BEC which may occur in multisteps, EIRD will also decrease in steps as the temperature is lowered. The combined effect of these leads to the apparent multistep behavior. The existence of the intermediate condensation into one-dimensional space can be traced back to the logarithmic divergence of the one-dimensional occupation number in Eq. \((36)\). This means that, when the trap is loosened in one direction, the particles tend to occupy quantum states along this direction with more likelihood than along other directions. Thus one-dimensionally excited modes in this direction will dominate multidimensional excitations spread in other directions \( (N_1 >> N_2, N_3) \) and the thermodynamic behavior of such a system is characterized by \( T_{1D} \) even though effective dimension of the system is still three. Note that the same mechanism is responsible for the nonexistence of BEC in one-dimensional harmonic trap in the ordinary thermodynamic limit. For the same reason, the intermediate condensation into two-dimensionally excited modes can be observed in a rectangular cavity \([10, 24]\), where \( T_{2D} \) does not exist in the naive thermodynamic limit. Three-step BEC can take place only in such a system. Away from the thermodynamic limit, the temperature dependence of the chemical potential around \( T_{1D}, T_{2D}, \) and \( T_{3D} \) causes similar crossover behaviors in condensation fractions \( N_1, N_2, \) and \( N_3 \) as a function of the reduced temperature.

Atom trap experiments probing the two-step BEC are realizable in Ioffe-Pritchard type magnetic traps or in optical dipole traps \([24]\). The similar type of device can be used to study multistep behavior discussed here, although it is difficult to achieve BEC in a maximally anisotropic trap with our current cooling technique. Further progress in a trapping device may be required. The basic mechanism of multistep dimensional crossovers discussed here can be applied to many other bosonic systems and should be amenable to future experimental verification. Quasi low-dimensional systems realized in the optical lattice or waveguide are the promising option for testing such processes \([17, 18]\). Also of interest is the kinetics of multistep behavior where the correlation length and the thermal de Broglie wave length are related in a nontrivial manner. This is currently under investigation.

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A Dynamical Symmetry in Anisotropic Harmonic Oscillator

It is known that dynamical symmetry of anisotropic harmonic oscillators has the reducible representation which can be written as the multiple of irreducible representation of $SU(3)$ symmetry, the symmetry of isotropic harmonic oscillators.

For a given $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, we can define a set of many boson annihilators [38] as

$$A^\lambda_i = a^k_i \sqrt{\frac{\hat{n}_i (\hat{n}_i - k_i)!}{\hat{n}_i!}} \text{ for } i = 1, 2, 3,$$  

(A1)

where $\hat{n}_i = a_i^\dagger a_i$ are boson number operators. Many boson annihilators and creators satisfy the following commutation relations

$$[A^\lambda_i, A^\lambda_j^\dagger] = \delta_{ij}.$$  

(A2)

Rewriting the Hamiltonian (4) as

$$H = \frac{\omega}{2} \sum_{i=1}^{3} \{A^\lambda_i, A^\lambda_i^\dagger\} - \frac{1}{2} \sum_{i=1}^{3} \omega_i (k_i - 2\lambda_i - 1),$$  

(A3)

the corresponding energy eigenvalue becomes

$$E_n = \omega (M + \frac{3}{2}) - \frac{1}{2} \sum_{i=1}^{3} \omega_i (k_i - 2\lambda_i - 1).$$  

(A4)

This gives the alternative method to derive energy eigenvalues given in Eq. (6).

Thus the reducible representation (4) of the original Hamiltonian leads to the cluster of isotropic harmonic oscillators. With this decomposition, it is possible to understand the mechanism of condensation in anisotropic traps in terms of condensation in isotropic traps [39].

B Mathematical Formulas

By taking derivatives of

$$\sum_{M=0}^{\infty} e^{-M\eta} = \frac{1}{1 - e^{-\eta}}$$  

(B1)

by $\eta$ on both sides, we obtain

$$\sum_{M=1}^{\infty} Me^{-M\eta} = \frac{e^{-\eta}}{(1 - e^{-\eta})^2}$$  

(B2)
and
\[ \sum_{M=1}^{\infty} M^2 e^{-M\eta} = \frac{e^{-\eta}(1 + e^{-\eta})}{(1 - e^{-\eta})^3}. \] (B3)

For a small \( \eta \), we have an expansion
\[ \frac{e^{-\eta}}{1 - e^{-\eta}} = \frac{e^{-\eta/2}}{2 \sinh(\eta/2)} = \frac{e^{-\eta/2}}{\eta} - \frac{\eta e^{-\eta/2}}{24} + \cdots. \] (B4)

Similarly, we have
\[ \frac{e^{-k\eta}}{(1 - e^{-k\eta})(1 - e^{-\eta})} = \frac{e^{-(k-1)\eta/2}}{k\eta^2} - (k + \frac{1}{k}) \frac{e^{-(k-1)\eta/2}}{24} + \cdots \] (B5)

and
\[ \frac{e^{-\eta}}{(1 - e^{-\eta})(1 - e^{-\eta_2})(1 - e^{-\eta_3})} = \frac{e^{-(\eta - \eta_2 - \eta_3)/2}}{\eta_1 \eta_2 \eta_3} - \frac{8 \sinh(\eta_1/2) \sinh(\eta_2/2) \sinh(\eta_3/2)}{e^{-(\eta - \eta_2 - \eta_3)/2}} \frac{\eta_1^2 + \eta_2^2 + \eta_3^2}{\eta_1 \eta_2 \eta_3} + \cdots. \] (B6)

From Eq. (B4),
\[ \sum_{i=1}^{\infty} \frac{z^i e^{-l\eta}}{1 - e^{-l\eta}} \sim \sum_{i=1}^{\infty} \frac{z^i e^{-ln/2}}{ln} = -\frac{1}{\eta} \log(1 - ze^{-\eta/2}). \] (B7)

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Figure Captions

Figure 1 The condensation temperatures $T_{1D}$ (solid curve), $T_{2D}$ (dashed curve), and $T_{3D}$ (dot-dashed curve) are shown as a function of $k_3$. $N = 10^4$, $\omega_1 = 0.5$, $k_2$ is also varied as $k_2^2 = k_3$. The logarithmic scale is used for both axis.

Figure 2 Different condensation behaviors corresponding to different anisotropy parameters $\omega_1/\omega_2$ and $\omega_2/\omega_3$ are indicated. $N = 10^4$ is chosen. The logarithmic scale is used for both horizontal and vertical axis. $\zeta(2)$ and $\zeta(3)$ are approximated as one for simplicity.

Figure 3 The condensation fractions $N_0/N$ (solid curve), $N_1/N$ (dashed curve), $N_2/N$ (dot-dashed curve), $N_3/N$ (dotted curve) as a function of the temperature for an isotropic trap are plotted. The same symbols are used in Figures 4-7. $\omega_1 = 0.1$, $\omega_2 = 0.1$, $\omega_3 = 0.1$, and $N = 1000$ are chosen. $T_c = 0.94$ is the three-dimensional critical temperature in Eq. (32).

Figure 4 The condensation fractions for a prolate shape anisotropic trap are plotted. $\omega_1 = 0.3$, $\omega_2 = 0.3$, $\omega_3 = 0.0003$, and $N = 10^4$ are chosen. Two-step condensation occurs in this case. $T_c = T_{3D}^* = 0.61$ from Eq. (44) and $T_{2D}^* = 0.34$ from Eq. (47). $c_1$ and $c_3$ are set equal to one. Note that EIRD=3 during the whole two-step BEC process.

Figure 5 The condensation fractions for an oblate shape anisotropic trap are plotted. $\omega_1 = 0.3$, $\omega_2 = 0.002$, $\omega_3 = 0.002$, and $N = 10^3$ are chosen. $T_c = T_{2D}^* = 0.057$ is the two-dimensional crossover temperature in Eq. (53). $c_2$ is set equal to one. The system shows the dimensional reduction from three to two dimension at $T = \omega_1 = 0.3$. Hence the accumulation of particles into $N_2$ from $N_3$ is not the result of condensation. At $T < \omega_1$, the system show the ordinary two-dimensional condensation into the ground state.

Figure 6 The condensation fractions for a maximally anisotropic trap are plotted. Three-step dimensional reduction can be seen. $\omega_1 = 0.3$, $\omega_2 = 0.02$, $\omega_3 = 0.0004$, and $N = 5 \times 10^3$ are used. $T_{3D}^* = 0.17$, $T_{2D}^* = 0.06$, and $T_{1D}^* = 0.02$ are crossover temperatures defined in Eqs. (57), (60), and (47), respectively. $c_1 = c_2 = c_3 = 1$ are used. The system shows the dimensional reduction at $T = \omega_1 = 0.3$ similar to Figure 5, and behaves two-dimensionally near $T < 0.3$. Therefore, the accumulation of particles into $N_2$ from $N_3$ is not the result of condensation. For $T < \omega_1$, the system shows two-step BEC at $T_{2D}^*$ into one-dimensionally excited states and at $T_{1D}^*$ into the ground state.

Figure 7 The logarithmic $T$ scale is used for the three-step behavior shown in Figure 6. The same parameters as in Figure 6 are used.