A COMBINATORIAL APPROACH TO MONOTONIC INDEPENDENCE OVER A C*-ALGEBRA

MIHAI POPA

Abstract. The notion of monotonic independence is considered in a more general frame, similar to the construction of operator-valued free probability. The paper presents constructions for maps with similar properties to the $H$ and $K$ transforms from the literature, semi inner-product bimodule analogues for the monotone and weakly monotone product of Hilbert spaces, an ad-hoc version of the Central Limit Theorem, an operator-valued arcsine distribution as well as a connection to operator-valued conditional freeness.

1. Introduction

An important notion in non-commutative probability is monotonic independence, introduced by P. Y. Lu and Naofumi Muraki. Since its beginning, the study of this notion of independence was done by constructions, techniques and developments similar to the theory of free probability. In a 1998 paper published in Mem. AMS (ref. [15]), R. Speicher developed an operator-valued analogue of free independence. The present paper addresses similar problems to ones discussed in [15], but in the context of monotonic independence.

Other motivation is that while for the Free Fock space over a Hilbert space there is a straightforward analogous semi-inner product bimodule construction, (as illustrated in [12] and [15]), there are no similar constructions for various of its deformations, such as the $q$-Fock spaces ([4]). As shown in Section 4 the monotone and weakly monotone Fock-like spaces, which are strongly connected to monotonic independence, admit analogous semi-inner product bimodules.

The paper is structured in six sections. The second section presents the definition of the monotonic independence over an algebra. In the third section there are constructed maps with similar properties to the maps $H$ and $K$ from the theory of monotonic independence, as introduced in [10] and [1]. The fourth section deals with semi-inner product bi-module analogues of the monotone and weakly monotone products of Hilbert spaces and algebras of annihilation operators, as introduced in [10], [11], [16]. The fifth section presents a Central Limit Theorem in the frame of monotonic independence over a C*-algebra and a positivity result concerning it. Since in the scalar-valued case the density of the limit distribution is the arcsine function ([8], [10]), the limit in Theorem 5.3 can be regarded as an "operator-valued arcsine law". The last section introduces a notion of conditionally free product of conditional expectations extending the definition and positivity results from [9] and shows a the connection to monotonic independence analogous to Proposition 3.1 from [5].
2. Preliminaries

Let $\mathcal{B}$ be an algebra (not necessarily unital). Within this paper, the notation $\mathcal{B}_+(\xi_1, \ldots, \xi_n)$ will stand for the free noncommutative algebra generated by $\mathcal{B}$ and the symbols $\xi_1, \ldots, \xi_n$. For the smaller algebra $\mathcal{B}_+((\xi_1, \ldots, \xi_n) \otimes \mathcal{B}$ we will use the notation $\mathcal{B}(\xi_1, \ldots, \xi_n)$.

If $\mathcal{B}$ is a $*$-algebra, we can consider $*$-algebra structures on $\mathcal{B}_+(\xi)$ and $\mathcal{B}(\xi)$ either by letting $(\xi)^* = \xi$ (i.e. the symbol $\xi$ is self-adjoint) or considering $\mathcal{B}_+(\langle \xi, \xi^* \rangle)$ with $(\xi)^* = \xi$.

We need to also consider an extended notion of non-unital complex algebra. A $\mathcal{B}$-algebra will be called an $\mathcal{A}$-algebra if $\mathcal{A}$ is an algebra such that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ or there is an algebra $\tilde{\mathcal{A}}$ containing $\mathcal{B}$ as a subalgebra such that $\tilde{\mathcal{A}} = \mathcal{A} \cup \mathcal{B}$. (The symbol $\cup$ stands for disjoint union).

A map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is said to be $\mathcal{B}$-linear if
\[
\Phi(b_1 xb_2 + y) = b_1 \Phi(x)b_2 + \Phi(y)
\]
for all $x, y \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.

If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $\Phi(b) = b$ for all $b \in \mathcal{B}$, then $\Phi$ will be called a conditional expectation.

**Definition 2.1.** Suppose that $\mathcal{A}$ is a $\mathcal{B}$-algebra and $I$ is a totally ordered set.

A family $\{\mathcal{A}_j\}_{j \in I}$ of subalgebras of $\mathcal{A}$ is said to be monotonically independent over $\mathcal{B}$ if given $X_j \in \mathcal{A}_j (j \in I)$, the following conditions are satisfied:

(a) for all $i < j > k$ in $I$ and $A, B \in \mathcal{A}$:
\[
\Phi(AX_iX_jX_kB) = \Phi(AX_i\Phi(X_j)X_kB)
\]
(b) for all $i_m > \cdots > i_1 < k_1 < \cdots < k_n$ in $I$:
\[
\begin{align*}
\Phi(X_{i_m} \cdots X_{i_1}) &= \Phi(X_{i_m}) \cdots \Phi(X_{i_1}) \\
\Phi(X_{k_1} \cdots X_{k_n}) &= \Phi(X_{k_1}) \cdots \Phi(X_{k_n}) \\
\Phi(X_{i_m} \cdots X_{i_1}X_{k_1} \cdots X_{k_n}) &= \Phi(X_{i_m}) \cdots \Phi(X_{i_1})\Phi(X_{k_1}) \cdots \Phi(X_{k_n})
\end{align*}
\]

The elements $\{X_j\}_{j \in I}$ from $\mathcal{A}$ are said to be monotonically independent over $\mathcal{B}$ if the subalgebras of $\mathcal{A}$ generated by $X_j$ and $\mathcal{B}$, are monotonically independent over $\mathcal{B}$.

Following [10], or [11], one may consider the stricter definition of monotonic independence replacing the first condition with

(a') $X_iX_jX_k = X_j\Phi(X_j)X_k$ whenever $i < j > k$.

Yet, definition 2.1 is sufficient for the results within this paper.

3. The maps $\kappa$, $\rho$ and $\mathfrak{h}$

Two important instruments in monotonic probability are the maps $H_X$ and $K_X$ associated to a selfadjoint element $X$ from a unital $*$-algebra $\mathcal{A}$ with a $C$-linear functional $\varphi$ such that $\varphi(1) = 1$. Namely $H_X$ is reciprocal Cauchy transform $H_X(z) = (G_X(z))^{-1}$, where $G_X$ is the Cauchy transform corresponding to $X$:
\[
G_X(z) = \varphi((z - X)^{-1})
\]
and the map $K_X$ is given by
\[
K_X(z) = \frac{\psi_X(z)}{1 + \psi_X(z)}, \quad \text{where} \quad \psi_X(z) = \varphi\left(zX(1 - zX)^{-1}\right).
\]
Their key properties (see [2], [5]) are that for $X, Y$, respectively $U^{-1}, V$ monotonically independent with respect to $\varphi$, one has:

$$
H_{X+Y} = H_X \circ H_Y \\
\kappa_{UV} = \kappa_{VU} = \kappa_U \circ \kappa_V.
$$

In the scalar-valued case, the moment generating series of $X$, can be recovered from $H$ and $K$. For the $\mathfrak{B}$-valued setting, the $n$-th moment of $X$ is the multilinear function $m_{X,n} : \mathfrak{B}^{n-1} \rightarrow \mathfrak{B}$,

$$
m_{X,n}(b_1, \ldots, b_{n-1}) = \Phi(Xb_1X \cdots Xb_{n-1}X).
$$

The mathematical object replacing the moment generating series is a multilinear function series over $\mathfrak{B}$ (see [3]), that cannot be recovered from a $\mathfrak{B}$-valued analytic map.

The use of analytic tools (such as the Cauchy transform) is strongly impaired by the previous considerations, hence the combinatorial approach is very convenient in the present framework. We will first construct the $\mathfrak{B}$-valued analytic functions $h$ replacing $H$, $\kappa$ and $\rho$ replacing $K$. Based on these constructions, the second part of the section will address the more general framework of multiplicative function series over an algebra.

In this section we require $\mathfrak{A}$ to be a $\ast$-algebra, $\mathfrak{B}$ to be a $C^*$-algebra with norm $\| \cdot \|$, and $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ to be a positive conditional expectation. $\overline{\mathfrak{A}}$ will denote the closure of $\mathfrak{A}$ in the topology given by $X \mapsto \| \Phi(X^\ast X) \|$. For simplicity, we will denote the continuous extension of $\Phi$ to $\overline{\mathfrak{A}}$ also with $\Phi$.

**Definition 3.1.** For $X \in \mathfrak{A}$, consider the $\mathfrak{B}$-valued function $h_X$

$$
\{ z \in \mathfrak{B} : \| z \| < \| X \|^{-1} \} \ni z \mapsto h_X(z) = (1 - z\Phi(X))^{-1}z \in \mathfrak{B}.
$$

Observe that $h$ is an analytic function defined in a neighborhood of $0 \in \mathfrak{B}$ and $h(0) = 0$.

**Theorem 3.2.** If $X, Y \in \mathfrak{A}$ are monotonically independent, then:

$$
h_{X+Y}(z) = h_X \circ h_Y(z)
$$

for $z$ in a neighborhood of $0 \in \mathfrak{B}$.

**Proof.** First, note that, for $X_1, X_2 \in \mathfrak{A}$ of sufficiently small norm, we have

$$
\sum_{n=0}^{\infty} (X_1 + X_2)^n = \sum_{p=0}^{\infty} \left( \sum_{k=0}^{\infty} X_2^k \right)^p \left( \sum_{m=0}^{\infty} (X_2)^m \right).
$$

Indeed

$$
\sum_{n=0}^{\infty} (X_1 + X_2)^n = \sum_{m=0}^{\infty} \sum_{\alpha_0, \beta_m \geq 0 \ \alpha_j, \beta_j \geq 1} X_1^{\alpha_0} X_2^{\beta_0} \cdots X_1^{\alpha_m} X_2^{\beta_m}
$$

$$
= \sum_{n=0}^{\infty} \sum_{\beta_j \geq 0} \left( \prod_{j=0}^{n} X_2^{\beta_j} X_1 \right) \left( \sum_{m=0}^{\infty} (X_2)^m \right)
$$

$$
= \sum_{p=0}^{\infty} \left( \sum_{k=0}^{\infty} X_2^k \right)^p \left( \sum_{m=0}^{\infty} (X_2)^m \right).
$$
Substituting $X_1 = zX$ and $X_2 = zY$, (1) becomes:

$$\sum_{n=0}^{\infty} (z(X + Y))^n = \sum_{p=0}^{\infty} \left( \left( \sum_{k=0}^{\infty} (zY)^k \right) zX \right)^p \left( \sum_{m=0}^{\infty} (zY)^m \right),$$

and therefore

$$(1 - z(X + Y))^{-1} z = \sum_{p=0}^{\infty} \left( \left( \sum_{k=0}^{\infty} (zY)^k \right) zX \right)^p \left( \sum_{m=0}^{\infty} (zY)^m \right) z.$$

We deduce that

$$\Phi \left( (1 - z(X + Y))^{-1} z \right) = \Phi \left( \sum_{p=0}^{\infty} \left( \left( \sum_{k=0}^{\infty} (zY)^k \right) zX \right)^p \left( \sum_{m=0}^{\infty} (zY)^m \right) z \right).$$

hence

$$h_{X+Y}(z) = \Phi \left( \sum_{p=0}^{\infty} \left[ (1 - zY)^{-1} z \right]^p \left( 1 - zY \right)^{-1} z \right) \Phi \left( Xz(1 - Xz)^{-1} \right)$$

Let $Z = (1 - zY)^{-1} z \in \mathfrak{M}$. $Z$ is in the closure of the algebra generated by $Y$ and $\mathfrak{B}$. If $X, Y$ are monotonically independent over $\mathfrak{B}$, the continuity of $\Phi$ and Definition 2.1(a), imply

$$\Phi \left( (ZX)^p Z \right) = \Phi \left( ZX \Phi(Z) \cdots \Phi(Z) X \right)$$

Since $X \Phi(Z) \cdots \Phi(Z) X$ is in the algebra generated by $X$ and $\mathfrak{B}$, 2.1(b) gives

$$\Phi \left( (ZX)^p Z \right) = \Phi \left( Z \Phi(X)^p \Phi(Z) \right).$$

Therefore

$$h_{X+Y}(z) = \Phi \left( \sum_{p=0}^{\infty} (h_Y(z)X)^p h_Y(z) \right) = (h_X \circ h_Y)(z),$$

as claimed. \(\square\)

**Definition 3.3.** For $X \in \mathfrak{A}$ and $z$ in a neighborhood of $0 \in \mathfrak{B}$, define the maps:

$$\vartheta_X(z) = \Phi \left( (1 - zX)^{-1} zX \right)$$

$$\kappa_X(z) = (1 + \vartheta_X(z))^{-1} \vartheta_X(z)$$

$$\varrho_X(z) = \Phi(Xz(1 - Xz)^{-1})$$

$$\rho_X(z) = \varrho_X(z)(1 + \varrho_X(z))^{-1}.$$

Observe that the above maps are $\mathfrak{B}$-valued analytic maps for which $0$ is a fixed point.

**Theorem 3.4.** Let $U, V \in \mathfrak{A}$ be such that $U - 1$ and $V$ are monotonically independent over $\mathfrak{B}$. Then, for $z$ in some neighborhood of $0 \in \mathfrak{B}$,

$$\kappa_{UV}(z) = (\kappa_U \circ \kappa_V)(z)$$

$$\rho_{UV}(z) = (\rho_U \circ \rho_V)(z).$$
Proof. With the notation $U - 1 = X$, we obtain

$$
\vartheta_{VU}(z) = \Phi((1 - zVU)^{-1}zV)
= \Phi((1 - zVU)^{-1}zV)
= \Phi\left(\sum_{k=0}^{\infty}(zVU)^{k}zV\right)
= \Phi\left(\sum_{k=0}^{\infty}[zV(X + 1)]^{k}zV\right)
= \Phi\left(\sum_{k=0}^{\infty} \sum_{\alpha_{1}+...+\alpha_{p}=k+1} (zV)^{\alpha_{1}}X(zV)^{\alpha_{2}}X... (zV)^{\alpha_{p}}U\right).
$$

As in the proof of 3.2, using Definition 2.1, the above equation becomes

$$
\vartheta_{VU}(z) = \Phi\left(\sum_{k=0}^{\infty} (\vartheta_{V}(z)X)^{k} \vartheta_{V}(z)U\right)
= \Phi\left( (1 - \vartheta_{V}(z)X)^{-1} \vartheta_{V}(z)U\right)
= \Phi\left( (1 + \vartheta_{V}(z) - \vartheta_{V}(z)U)^{-1} (1 + \vartheta_{V}(z) - \vartheta_{V}(z)U)^{-1} \vartheta_{V}(z)U\right)
= \Phi\left( [1 - \vartheta_{V}(z)]^{-1} \vartheta_{V}(z)U\right)^{-1} \vartheta_{V}(z)U
= \Phi\left( [1 - \kappa_{V}(z)U]^{-1} \kappa_{V}(z)U\right)
= \vartheta_{U}(\kappa_{V}(z)).
$$

Therefore:

$$
\kappa_{VU}(z) = [1 + \vartheta_{VU}(z)]^{-1} \vartheta_{VU}(z)
= [1 + \vartheta_{U}(\kappa_{V}(z))]^{-1} \vartheta_{U}(\kappa_{V}(z))
= \kappa_{U}(\kappa_{V}(z)).
$$

The identity for $\rho$ follows analogously. \(\square\)

The proofs of Theorems 3.2 and 3.4 do not use the analyticity of the maps $\vartheta$, $\kappa$, $\rho$, but only properties from Definition 2.1 and some combinatorial identities that are true for any formal series. This leads to a easy reformulation of the results in the more general frame, presented in [3], of multilinear function series over an algebra.

In the following paragraphs we will briefly remaind the reader the construction and several results on multilinear function series.

Let $\mathfrak{B}$ be an algebra. We set $\tilde{\mathfrak{B}}$ equal to $\mathfrak{B}$ if $\mathfrak{B}$ is unital and to the unitalization of $\mathfrak{B}$ otherwise. For $n \geq 1$, we denote by $\mathcal{L}_{n}(\mathfrak{B})$ the set of all multilinear mappings

$$
\omega_{n} : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n \text{ times}} \rightarrow \mathfrak{B}
$$
A formal multilinear function series over $\mathfrak{B}$ is a sequence $\omega = (\omega_0, \omega_1, \ldots)$, where $\omega_0 \in \mathfrak{B}$ and $\omega_n \in \mathcal{L}_n(\mathfrak{B})$ for $n \geq 1$. According to [3], the set of all multilinear function series over $\mathfrak{B}$ will be denoted by $Mul[[\mathfrak{B}]]$.

For $F, G \in Mul[[\mathfrak{B}]]$, the sum $F + G$ and the formal product $FG$ are the elements from $Mul[[\mathfrak{B}]]$ defined by:

$$(F + G)_n(b_1, \ldots, b_n) = F_n(b_1, \ldots, b_n) + G_n(b_1, \ldots, b_n)$$

$$(FG)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} F_k(b_1, \ldots, b_k)G_{n-k}(b_{k+1}, \ldots, b_n)$$

for any $b_1, \ldots, b_n \in \mathfrak{B}$.

If $G_0 = 0$, then the formal composition $F \circ G \in Mul[[\mathfrak{B}]]$ is defined by

$$(F \circ G)_0 = F_0$$

$$(F \circ G)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{p_1 + \cdots + p_k = n \geq 1} F_k(G_{p_1}(b_1, \ldots, b_{p_1}), \ldots, G_{p_k}(b_{q_1}+1, \ldots, b_{q_k}+p_k))$$

where $q_j = p_1 + \cdots + p_{j-1} + 1, n \geq 1$.

With the above operations, $Mul[[\mathfrak{B}]]$ is an algebra, with additional properties similar to ones of power series (see [3], Proposition 2.3 and Proposition 2.6):

**Proposition 3.5.** Let $E, F, G \in Mul[[\mathfrak{B}]]$. Then:

(i) $1 = (1, 0, 0, \ldots) \in Mul[[\mathfrak{B}]]$ is a multiplicative identity element;

(ii) $F = (F_0, F_1, \ldots)$ has a multiplicative inverse if and only if $F_0$ is an invertible element of $\mathfrak{B}$;

(iii) if $F_0 = 0$ and $G_0 = 0$, then $(E \circ F) \circ G = E \circ (F \circ G)$;

(iv) if $G_0 = 0$, then $(E + F) \circ G = E \circ G + F \circ G$ and $(E \circ G)(F \circ G)$;

(v) $F = (0, 0, \ldots) \in Mul[[\mathfrak{B}]]$ is an identity element for the formal composition

(vi) $F = (0, F_1, F_2, \ldots) \in Mul[[\mathfrak{B}]]$ has a compositional inverse, denoted $F^{(-1)}$, if and only if $F_1$ is an invertible element of $\mathcal{L}_1(\mathfrak{B})$;

(vii) if $F = (0, F_1, F_2, \ldots) \in Mul[[\mathfrak{B}]]$, then

$$(1 - F)^{-1} = 1 + \sum_{k=1}^{\infty} F^k$$

For the next definitions and results, $\mathfrak{A}$ will be a $\mathfrak{B}$-algebra ($\mathfrak{B}$ and $\mathfrak{A}$ are not necessarily *-algebras).

**Definition 3.6.** For $X \in \mathfrak{A}$ consider $\mathfrak{F}_X = (\mathfrak{F}_{X,0}, \mathfrak{F}_{X,1}, \ldots) \in Mul[[\mathfrak{B}]]$, where

$\mathfrak{F}_{X,0} = 0$

$\mathfrak{F}_{X,1}(b) = b$

$\mathfrak{F}_{X,n}(b_1, \ldots, b_n) = \Phi(b_1 X b_2 \cdots b_{n-1} X b_n)$

for all $b, b_1, \ldots, b_n \in \mathfrak{B}$, $n \geq 1$.

**Theorem 3.7.** If $X, Y \in \mathfrak{A}$ are monotonically independent over $\mathfrak{B}$, then

$\mathfrak{F}_{X + Y} = \mathfrak{F}_X \circ \mathfrak{F}_Y$
Proof. It suffices to show that $\mathcal{H}_{X+Y,n} = (\mathcal{H}_X \circ \mathcal{H}_Y)_n$ for all $n \geq 0$.

For $n = 0, 1$, the assertion is trivial. For $n \geq 2$,

$$ (\mathcal{H}_X \circ \mathcal{H}_Y)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{p_1, \ldots, p_k \geq 1} \mathcal{H}_{X,k}(\mathcal{H}_{Y,p_1}(b_1, \ldots, b_{p_1}), \ldots, \mathcal{H}_{Y,p_k}(b_{q_k+1}, \ldots, b_n)) $$

where $q_j = p_1 + \cdots + p_{j-1}$ and the second summation is over all $p_1, \ldots, p_k \geq 1$ such that $p_1 + \cdots + p_k = n$

$$ = \Phi \left( \sum_{k=1}^{n} \sum_{p_1, \ldots, p_k \geq 1} \mathcal{H}_{Y,p_1}(b_1, \ldots, b_{p_1}) X \ldots X \mathcal{H}_{Y,p_k}(b_{q_k+1}, \ldots, b_n) \right) $$

Using Definition 2.1, the above relation becomes

$$ (\mathcal{H}_X \circ \mathcal{H}_Y)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{p_1, \ldots, p_k \geq 1} \Phi \left( b_1 X \ldots X b_{p_1} X \ldots X b_{q_k+1} Y \ldots Y b_n \right) $$

On the other hand,

$$ \mathcal{H}_{X+Y,n}(b_1, \ldots, b_n) = \Phi \left( b_1 (X + Y) \ldots (X + Y) b_n \right) $$

hence the conclusion.

If the algebra $\mathfrak{A}$ is unital, there also exist multilinear function series analogous to $\kappa$, $\rho$. First, for $X \in \mathfrak{A}$, define the elements $\beta_X$ and $\gamma_X$ of $\text{Mul}[[\mathfrak{B}]]$ by

$$ \beta_{X,0} = 0 \quad \beta_{X,n}(b_1, \ldots, b_n) = \Phi(b_1 X b_2 \ldots b_n X) \quad \gamma_{X,0} = 0 \quad \gamma_{X,n}(b_1, \ldots, b_n) = \Phi(X b_1 X \ldots X b_n) $$

From the property 3.5(ii), the multilinear function series $\mathfrak{R}_X$ and $\tau_X$ are well-defined, where

$$ \mathfrak{R}_X = (1 + \beta_X)^{-1} \beta_X \quad \tau_X = \gamma_X (1 + \gamma_X)^{-1} $$
Theorem 3.8. Let $U, V \in A$ be such that $U - 1$ and $V$ are monotonically independent over $B$. Then:

$$R_{VU} = R_U \circ R_V$$
$$t_{VU} = t_V \circ t_U.$$  

The proof is a routine (though tedious) verification, using Proposition 3.3 and the techniques from the proof of Theorem 3.7 and Theorem 3.4.

4. Semi-Inner Product Bimodules

The terminology in used in this section is the one from [3]. Let $B$ be a unital $C^*$-algebra. A semi-inner product $B$-bimodule is a linear space $E$ which is a $B$-bimodule, together with a map $(x, y) \mapsto \langle x, y \rangle : E \times E \rightarrow B$ such that:

(i) $\langle x, ay + bz \rangle = a \langle x, y \rangle + b \langle x, z \rangle$ for any $x, y, z \in E, a, b \in C$.

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$, for any $x, y \in E, a \in B$.

(iii) $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in E$.

(iv) $\langle x, x \rangle \geq 0$ for any $x \in E$.

$E$ is called an inner-product $B$-bimodule if $\langle x, x \rangle = 0$ implies $x = 0$ and Hilbert $B$-bimodule if it is complete with respect to the norm $||x|| = ||\langle x, x \rangle||^{1/2}$. (second norm is the $C^*$-algebra norm of $B$.) The algebra of $B$-linear (not necessarily bounded) operators on $E$ will be denoted by $L(E)$.

4.1. Given a family $(E_i)_{i \in I}$ of semi-inner product $B$-bimodules indexed by a totally ordered set $I \subseteq \mathbb{Z}$, we define, following [10] and [12], the monotonic product $E^m$ of $(E_i)_{i \in I}$ to be the semi-inner product $B$-bimodule:

$$E^m = B \oplus \left( \bigoplus_{n \geq 1} \bigoplus_{(i_1, \ldots, i_n) \in [I,n]} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n} \right)$$

where all the tensor products are with amalgamation over $B$ and

$$[I, n] = \{ (i_1, \ldots, i_n) : i_1, \ldots, i_n \in I, i_1 > \cdots > i_n \},$$

with the inner-product given by

$$\langle f_1 \otimes \cdots \otimes f_n, e_1 \otimes \cdots \otimes e_m \rangle = \delta_{m,n} \langle f_n, \langle f_{n-1}, \ldots, \langle f_1, e_1 \rangle \ldots \rangle e_n \rangle.$$

Note that in general $E^m$ is not an inner-product $B$-bimodule even if $E_i$ are inner-product bimodules or Hilbert bimodules. For example, if $\langle f_1, f_1 \rangle = b^*b > 0$ and $bf_2 = 0$, then

$$\langle f_1 \otimes f_2, f_1 \otimes f_2 \rangle = \langle f_2, \langle f_1, f_1 \rangle f_2 \rangle = \langle f_2, b^*bf_2 \rangle = 0;$$

see also [15].

If $i \in I$ is fixed, we have the natural identification

$$E^m = (B \oplus E_i) \otimes B \left( \bigoplus_{n \geq 1} \bigoplus_{(i_1, \ldots, i_n) \in [I,n]_{t_1 < i}} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n} \right)$$

where

$$E^m_{(\leq i)} = \bigoplus_{n \geq 1} \bigoplus_{(i_1, \ldots, i_n) \in [I,n]_{t_1 < i}} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n},$$

$$E^m_{(\geq i)} = \bigoplus_{n \geq 1} \bigoplus_{(i_1, \ldots, i_n) \in [I,n]_{t_1 > i}} E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_n},$$
Based on this decomposition, one also has the (non-unital) $\ast$-representation
\[ \lambda_i : \mathcal{L}(\mathfrak{B} \oplus \mathcal{E}_i) \to \mathcal{L}(\mathcal{E}^m) \]
\[ \lambda_i (A) = \left( A \otimes I_{\mathfrak{B} \oplus \mathcal{E}^m_{(\leq i)}} \right) \oplus 0_{\mathcal{E}^m_{(> i)}} \]

**Theorem 4.1.** With the above notations, \{\lambda_i(\mathcal{L}(\mathfrak{B} \oplus \mathcal{E}_i))\}_{i \in I} are monotonically independent in \( \mathcal{L}(\mathcal{E}^m) \) with respect to the conditional expectation \( \Phi(\cdot) = \langle 1, \cdot \rangle \).

**Proof.** We need to show that the two conditions from the definition of monotonic independence (Definition 2.1) are satisfied. In fact, it will be shown that the family \{\lambda_i(\mathcal{L}(\mathfrak{B} \oplus \mathcal{E}_i))\}_{i \in I} satisfies 2.1(b) and the stricter condition 2.1(a').

The proof is similar to the proof of Theorem 2.1 from [10]. For \( i \in I \), consider \( A_i \in \mathcal{L}(\mathfrak{B} \oplus \mathcal{E}_i) \) and \( X_i = \lambda_i(A_i) \).

We can write:
\[ X_i 1 = \alpha_i + s_i, \ \alpha_i \in \mathfrak{B}, s_i \in \mathcal{E}_i \]
where \( 1 \in \mathfrak{B} \subset \mathfrak{B} \oplus \mathcal{E}_{i,j} \).

If \( k < l \),
\[ X_kX_1 = X_k(\alpha_l + s_l) = X_k1\alpha_l = X_k\langle 1, X_j 1 \rangle \]

therefore
\[ X_jX_{k_1} \ldots X_{k_n} 1 = \langle 1, X_j 1 \rangle \langle 1, X_{k_1} 1 \rangle \ldots \langle 1, X_{k_n} 1 \rangle \]
whenever \( j < k_1 < \cdots < k_n \).

Also, writing \( \mathcal{E}^m = \mathfrak{B} \oplus \mathcal{E}^0 \), note that \( X_i f \in \mathcal{E}^0 \) for any \( f \in \mathcal{E}^0 \) and any \( l \in I \), and that \( (k < l) : \)
\[ X_lX_k = X_l(\alpha_k + s_k) = X_l1\alpha_k + X_l(1 \otimes s_k) = X_l\langle 1, X_k 1 \rangle + (\alpha_l + s_l) \otimes s_k = X_l\langle 1, X_k 1 \rangle + f \text{ for some } f \in \mathcal{E}^0. \]

Iterating the above relations, for \( i_m > \cdots > i_1 > j < k_1 < \cdots < k_n \), we obtain
\[ \langle 1, X_{i_m} \ldots X_{i_1}X_jX_{k_1} \ldots X_{k_n} 1 \rangle = \langle 1, X_{i_m} \ldots X_{i_1} 1 \rangle \langle 1, X_j 1 \rangle \ldots \langle 1, X_{k_n} 1 \rangle 
\]
\[ = \langle (1, X_{i_m} 1) \ldots \langle 1, X_{i_1} 1 \rangle + \langle 1, f \rangle \rangle \langle 1, X_j 1 \rangle \ldots \langle 1, X_{k_n} 1 \rangle 
\]
\[ = \langle 1, X_{i_m} 1 \rangle \ldots \langle 1, X_{k_n} 1 \rangle \]
that is, property (b).

For \( i < j > k \), a direct computation gives
\[ X_iX_jX_k = X_iX_j(\alpha_k + s_k) = X_i(X_j 1)\alpha_l + X_iX_j(1 \otimes s_k) = X_i(\alpha_j + s_j)\alpha_k + X_i(\alpha_j + s_j) \otimes s_k = X_i(\alpha_j \alpha_k) + X_i(\alpha_j s_k) + X_i(s_j \otimes s_k) = X_i\alpha_j(\alpha_k + s_k) = X_i\langle 1, X_j 1 \rangle X_k 1, \]
so it remains to show (a') on elements of the form $\tilde{h} = h_{i_1} \otimes \cdots \otimes h_{i_n}$, $h_{i_i} \in \mathcal{E}_{i_i}$.

If $i_1 > i$, then $X_2 h_{i_1} \otimes \cdots \otimes h_{i_n} = 0$, therefore

$$X_1 Y X_2 \tilde{h} = 0 = X_1 (1, Y 1) X_2 \tilde{h}.$$  

If $i_1 = i$, with the notations $h^0 = h_{i_2} \otimes \cdots \otimes h_{i_n}$ and $X_2 h_{i_1} = \theta \oplus u$ for some $\theta \in \mathfrak{B}, u \in \mathcal{E}_i$, one has

$$X_1 Y X_2 h_{i_1} \otimes \cdots \otimes h_{i_n} = X_1 Y (\theta \oplus u) h^0 = X_1 [\beta \theta \oplus \theta \theta \oplus (\beta \oplus t) \otimes u] \otimes h^0 = X_1 [\beta \theta + \beta u] \otimes h^0 = X_1 \beta (\theta \oplus u) \otimes h^0 = X_1 (1, Y 1) X_2 h_{i_1} \otimes \cdots \otimes h_{i_n}.$$ 

The case $i_1 < i$ is analogous.

\[ \square \]

4.2. The weakly monotone product of the bimodules $\{\mathcal{E}_i\}_{i \in I}$ is the semi-inner product $\mathfrak{B}$-bimodule

$$\mathcal{E}^{wm} = \mathfrak{B} \oplus \bigoplus_{n=1}^{\infty} \left( \bigoplus_{i_1 \geq \cdots \geq i_n} \mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_n} \right)$$

If $I$ has only one element, $i_0$, then $\mathcal{E}^{wm}$ is the full Fock bimodule over $\mathcal{E}_{i_0}, \mathcal{F}(\mathcal{E}_{i_0})$ (see [12], [15]).

For $j \in I$, let $\mathfrak{j} = \{l \in I, 1 \leq j\}$ and let $\mathcal{E}^{wm}(j)$ be the weakly monotonic product of $\{\mathcal{E}_i\}_{i \in \mathfrak{j}}$. We will also use the following notations

$$\mathcal{F}_0(\mathcal{E}) = \mathcal{F}(\mathcal{E}) \oplus \mathfrak{B}$$

$$\mathcal{E}^{wm}_0(\mathcal{E}) = \mathcal{E}^{wm} \oplus \mathfrak{B}$$

$$\mathcal{E}^{wm}_0(j) = \mathcal{E}^{wm}(j) \oplus \mathfrak{B}$$

For $f \in \mathcal{E}_i$, define the $\mathfrak{B}$-linear creation and annihilation maps $a^*(f)$ and $a(f)$ on $\mathcal{E}^{wm}$ by:

$$a^*(f) 1 = f$$

$$a^*(f) f_{i_1} \otimes \cdots \otimes f_{i_n} = \begin{cases} f \otimes f_{i_1} \otimes \cdots \otimes f_{i_n}, & \text{if } i \geq i_1 \\
0, & \text{if } i < i_1 \end{cases}$$

$$a(f) 1 = 0$$

$$a(f) f_{i_1} \otimes \cdots \otimes f_{i_n} = \begin{cases} (f, f_{i_1}) f_{i_2} \otimes \cdots \otimes f_{i_n}, & \text{if } i = i_1 \\
0, & \text{if } i \neq i_1 \end{cases}$$

Note that $a(f)$ and $a^*(f)$ are adjoint to each other. Denote by $G(f)$ their sum, $G(f) = a(f) + a^*(f)$, and by $\mathfrak{A}$, the algebra generated over $\mathfrak{B}$ by $\{G(f), f \in \mathcal{E}_i\}$.

We will use the shorthand notation $\Phi(\cdot)$ for the $\mathfrak{B}$-valued functional $(1, \cdot)$ on the set of all $\mathfrak{B}$-linear maps on $\mathcal{E}^{wm}$. Also, for $\tilde{e} = e_1 \otimes \cdots \otimes e_n$, $(e_l \in \mathcal{E}_l, 1 \leq l \leq n)$
we will use the notations
\[ A^*(\tilde{e}) = a^*(e_1) \cdots a^*(e_n) \]
\[ A(\tilde{e}) = a(e_1) \cdots a(e_n) \]

Lemma 4.2. For any \( f_1, \ldots, f_n \in \mathcal{E}_k \) there are some sequences of elements of \( \mathcal{E}_{0}^{wm}(k), (\tilde{e}_r)_{r=1}^{N_1}, (\tilde{g}_s)_{s=1}^{N_2}, (\tilde{h}_q)_{q=1}^{N_3}, (\tilde{k}_q)_{q=1}^{N_4} \), such that
\[ P = \prod_{l=1}^{n} G(f_l) \]
can be written as:
\[ P = \Phi(P) + \sum_{r=1}^{N_1} A^*(\tilde{e}_r) + \sum_{s=1}^{N_2} A(\tilde{g}_s) + \sum_{q=1}^{N_3} A^*(\tilde{h}_q)A(\tilde{k}_q) \]

Proof. Let \( (2)' \) be the weaker form of \( (2) \) where \( \Phi(P) \) is replaced by some element \( \alpha \in \mathcal{B} \). Note that \( (2)' \) is in fact equivalent to \( (2) \), since
\[ \Phi(P) \quad = \quad \langle 1, P1 \rangle \]
\[ = \quad \langle 1, \alpha + \sum_{r=1}^{N_1} A^*(\tilde{e}_r) + \sum_{s=1}^{N_2} A(\tilde{g}_s) + \sum_{q=1}^{N_3} A^*(\tilde{h}_q)A(\tilde{k}_q) \rangle \]
\[ = \quad \langle 1, \alpha + \sum_{r=1}^{N_1} \tilde{e}_r \rangle \]
\[ = \quad \alpha \]

It remains to prove \( (2)' \).

Note that
\[ P \quad = \quad \prod_{l=1}^{n} G(f_l) = \prod_{l=1}^{n} (a^*(f_l) + a(f_l)) \]
\[ = \quad \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in \{1, 2\}^n} a_{\varepsilon_1}(f_1) \cdots a_{\varepsilon_n}(f_n) \]

where \( a_1 \) stands for \( a \) and \( a_2 \) stands for \( a^* \).

Also, for any \( f, g, h \in \mathcal{E}_k, \alpha \in \mathcal{B} \) and \( \varepsilon \in \{1, 2\} \)
\[ a(f)a^*(g) \quad = \quad \langle f, g \rangle I \]
\[ a_{\varepsilon}(h)(f, g) \quad = \quad a_{\varepsilon}(h)(f, g) \]
\[ a\alpha(f) \quad = \quad a(\alpha^* f) \]
\[ a\alpha^*(f) \quad = \quad a^*(\alpha f) \]

It follows that in the expression of \( a_{\varepsilon_1}(f_1) \cdots a_{\varepsilon_n}(f_n) \) any \( a(f_p)a^*(f_{p+1}) \) can be reduced to \( (f_p, f_{p+1}) \) which can be included in the expression of the previous or following factor. Iterating, after a finite number of steps no summand will have factors of the type \( a(f_q) \) in front of factors of the type \( a^*(f_p) \), so \( (2)' \) is proved. [proved]

Lemma 4.3. Any \( X \in \mathfrak{A}_i \) is satisfying the following properties:
(i) \( \mathcal{E}^{wm}(i) \) is \( X \)-invariant

(ii) \( X1 = \Phi(X) + s \), for some \( s \in \mathcal{F}_0(\mathcal{E}_i) \)

(iii) if \( u \in \mathcal{E}^{wm}(j), j < i, \) then

\[
Xu = \Phi(X)u + t \otimes u, \text{ for some } t \in \mathcal{F}_0(\mathcal{E}_i)
\]

(iv) if \( i < j \) and \( v \in \mathcal{F}_0(\mathcal{E}_j) \otimes \mathcal{E}^{wm}(k), k < j, \) then

\[
Xv = 0
\]

Proof.

(i) It is enough to verify that the property holds true for \( X = G(f), f \in \mathcal{E}_i \).

Indeed,

\[
G(f)1 = f \in \mathcal{F}(\mathcal{E}_i) \subset \mathcal{E}^{wm}(i)
\]

and for any \( f_1 \otimes \cdots \otimes f_n \in \mathcal{E}^{wm}(i) \),

\[
G(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n + \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n \in \mathcal{E}^{wm}(i).
\]

(ii) If \( f, f_1, \ldots, f_n \in \mathcal{F}(\mathcal{E}_i) \)

\[
G(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n + \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n \in \mathcal{F}(\mathcal{E}_i).
\]

It follows that \( \mathcal{F}(\mathcal{E}_i) \) is invariant to \( A_i \). Since \( 1 \in \mathcal{F}(\mathcal{E}_i) \), we have \( X1 \in \mathcal{F}(\mathcal{E}_i) \), and the conclusion follows from the orthogonality of \( \mathfrak{B} \) and \( \mathcal{F}_0(\mathcal{E}_i) \).

(iii) It is enough to prove the relation for \( X = G(f_1) \ldots G(f_n), f_i \in \mathcal{E}_i \). First note that for any \( \tilde{f} \in \mathcal{F}_0(\mathcal{E}_i) \)

\[
A^*(\tilde{f})u = \tilde{f} \otimes u
\]

\[
A(\tilde{f})u = 0
\]

From Lemma 4.2, there are some sequences \((\tilde{e}_r)_{r=1}^{N_1}, (\tilde{g}_s)_{s=1}^{N_2}, (\tilde{h}_q)_{q=1}^{N_3}, (\tilde{k}_q)_{q=1}^{N_3}\), of elements of \( \mathcal{F}(\mathcal{E}_i) \), such that

\[
Xu = \Phi(X)u + \sum_{r=1}^{N_1} A^*(\tilde{e}_r)u + \sum_{s=1}^{N_2} A(\tilde{g}_s)u + \sum_{q=1}^{N_3} A^*(\tilde{h}_q)A(\tilde{k}_q)u
\]

\[
= \Phi(X)u + \sum_{r=1}^{N_1} \tilde{e}_r \otimes u
\]

\[
= \Phi(X)u + \left( \sum_{r=1}^{N_1} \tilde{e}_r \right) \otimes u
\]

(iv) Similarly, it is enough to prove the relation for \( X = G(f_1) \ldots G(f_n), f_i \in \mathcal{E}_i \). First note that for any \( \tilde{f} \in \mathcal{F}_0(\mathcal{E}_i) \)

\[
A^*(\tilde{f})v = A(\tilde{f})v = 0
\]

From Lemma 4.2, there are some sequences \((\tilde{e}_r)_{r=1}^{N_1}, (\tilde{g}_s)_{s=1}^{N_2}, (\tilde{h}_q)_{q=1}^{N_3}, (\tilde{k}_q)_{q=1}^{N_3}\), of elements of \( \mathcal{F}(\mathcal{E}_i) \), such that
\[ Xv = \Phi(X)v + \sum_{r=1}^{N_1} A^*(u_r)u + \sum_{s=1}^{N_2} A(g_s)v + \sum_{q=1}^{N_3} A^*(h_q)A(\tilde{k}_q)v = \Phi(X)v \]

\[ \square \]

**Theorem 4.4.** The algebras \( \{A_i\}_{i \in I} \) are monotonically independent with respect to the \( \mathcal{B} \)-valued functional \( \Phi(\cdot) = (1, 1) \).

**Proof.** Let \( X_i \in A_i, i \in I \). We will prove that they satisfy the relations (b) and (a') from the definition of the monotonic independence. If \( k < l \), from Lemma 4.3 we have

\[ X_k X_l = X_k(\Phi(X_i) + t) \text{ for some } t \in \mathcal{F}_0(\mathcal{E}_l) \]

\[ = X_k 1 \Phi(X_l), \text{ since } X t = 0, \]

therefore

\[ X_j X_l \ldots X_k n = X_j 1 \Phi(X_k) \ldots \Phi(X_{k_n}) \]

whenever \( j < k_1 < \ldots k_n \). Similarly, \( k < l \):

\[ X_k X_l = X_k(\Phi(X_k) + t), \text{ for some } t \in \mathcal{F}_0(\mathcal{E}_k) \]

\[ = \Phi(X_k)\Phi(X_l) + X_k t_l + t \otimes (\Phi(X_k) + t_k) \text{ for some } t_l \in \mathcal{F}_0(\mathcal{E}_l) \]

\[ = \Phi(X_k)\Phi(X_l) + t, \text{ for some } t \in \mathcal{E}_0^{\text{wm}}(l). \]

Using the above relations, we obtain

\[ \Phi(X_i \ldots X_{1a} X_j X_{k_1} \ldots X_{k_m}) = \Phi(X_i \ldots X_{1a} X_j)\Phi(X_{k_1}) \ldots \Phi(X_{k_m}) \]

\[ = \Phi(\Phi(X_i) \ldots \Phi(X_j) + s)\Phi(X_{k_1}) \ldots \Phi(X_{k_m}) \]

for some \( s \in \mathcal{E}_0^{\text{wm}}(k_m) \)

\[ = \Phi(X_i) \ldots \Phi(X_j) + s)\Phi(X_{k_1}) \ldots \Phi(X_{k_m}), \]

that is, property (b).

Also, for \( i < j > k \),

\[ X_i X_j X_k = X_i X_j (\Phi(X_k) + s) \text{ for some } s \in \mathcal{F}_0(\mathcal{E}_k) \]

\[ = (X_i X_j)\Phi(X_k) + X_i X_j s \]

\[ = X_i (\Phi(X_j) + t_1)\Phi(X_k) + X_i (\Phi(X_j) + t_2 \otimes s), \text{ for some } t_1, t_2 \in \mathcal{F}_0(\mathcal{E}_j) \]

\[ = X_i \Phi(X_j)\Phi(X_k) + X_i \Phi(X_j)s \]

\[ = X_i \Phi(X_j) (\Phi(X_k) + s) \]

\[ = X_i \Phi(X_j) X_k 1. \]

It remains to prove that \( X_i X_j X_k \) and \( X_i \Phi(X_j) X_k \) coincide on vectors of the form \( \vec{f} = f_{i_1} \otimes \cdots \otimes f_{i_m} \), where \( f_{i_k} \in \mathcal{E}_{i_k}, i_1 \geq \cdots \geq i_m \).

If \( i_1 > k \), then \( X_k 1 \vec{f} = 0 \), so the equality is trivial.

If \( i_1 \leq k \), then Lemma 4.3 implies that \( X_k 1 \vec{f} \in \mathcal{E}^{\text{wm}}(k) \), therefore:

\[ \Phi(X_i) X_k 1 \vec{f} = X_i \left( (\Phi(X_j) X_k \vec{f} + t \otimes X_k \vec{f}) \right), \text{ for some } t \in \mathcal{F}_0(\mathcal{E}_j) \]

\[ = X_i \Phi(X_j) X_k 1 \vec{f}. \]
Remark 4.5. An analogous construction can be done for creation and annihilation maps on $E^m$, and similar computations will lead to the monotonic independence of the correspondent algebras (see [11]).

5. CENTRAL LIMIT THEOREM

In this section $\mathfrak{A}$ will be a $\ast$-algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is also a $C^*$-algebra and $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ a conditional expectation. $\mathfrak{B}_+(\xi)$ will be the $\ast$-algebra generated by $\mathfrak{B}$ and the selfadjoint symbol $\xi$, as described in Introduction.

As discussed in Section 2, given $X$ a selfadjoint element of $\mathfrak{A}$, the $n$-th moment of $X$ is the multilinear function $m_{X,n} : \mathfrak{B}^{n-1} \rightarrow \mathfrak{B}$,

$$m_{X,n}(b_1, \ldots, b_{n-1}) = \Phi(Xb_1X \cdots Xb_{n-1}X).$$

We define the moment function of $X$ as

$$\mu_X = \bigoplus_{m=1}^{\infty} \mu_X^{(m)}$$

Before stating the main theorem of this section, we will begin with some combinatorial considerations on the joint moments of the family of selfadjoint elements $(X_n)_{n \geq 1}$ of $\mathfrak{A}$ with the properties:

(1) for any $i < j$, $X_i$ and $X_j$ are monotonically independent over $\mathfrak{B}$;
(2) all $X_k$ have the same moment function, denoted by $\mu$.

Let $NC(m)$ be the set of all non-crossing partitions of the ordered set $\{1, 2, \ldots, m\}$. For $\gamma \in NC(m)$, let $B = \{b_1, b_2, \ldots, b_p\}$ and $C = \{c_1, c_2, \ldots, c_q\}$ be two blocks of $\gamma$. We say that $C$ is interior to $B$ if there is an index $k \in \{1, \ldots, p - 1\}$ such that $b_k < c_1, c_2, \ldots, c_q < b_{k+1}$. $B$ and $C$ will be called adjacent if $c_1 = b_p + 1$ or $b_1 = c_q + 1$. The block $B$ will be called outer if it is not interior to any other block of $\gamma$.

To each $m$-tuple $(i_1, \ldots, i_m)$ of indices from $\{1, 2, \ldots\}$ we associate a unique non-crossing partition $nc[i_1, \ldots, i_m] \in NC(m)$ as follows:

(1) if $m = 1$, then $nc[i_1] = (1)$
(2) if $m > 1$, put

$$B = \{k, i_k = \min\{i_1, \ldots, i_m\}\} = \{k_1, \ldots, k_p\}$$

and define

$$nc[i_1, \ldots, i_m] = B \sqcup nc[i_1, \ldots, i_{k_1} - 1] \sqcup nc[i_{k_1} + 1, \ldots, i_{k_2} - 1] \sqcup \cdots \sqcup nc[i_{k_p} + 1, \ldots, i_m]$$

Reciprocally, the $m$-tuple $(i_1, \ldots, i_m)$ will be called an admissible configuration for $\gamma \in NC(m)$ if $nc[i_1, \ldots, i_m] = \gamma$.

Lemma 5.1. Suppose $(i_1, \ldots, i_m)$ is an admissible configuration for $\gamma \in NC(m)$ and $B = \{k_1, \ldots, k_p\}$ is an outer block of $\gamma$. Then, for any $b_1, \ldots, b_{m-1} \in \mathfrak{B}$, $\mu$ the common moment function of $\{X_n\}_n$, we have

$$\Phi(X_{i_1}b_1X_{i_2} \cdots b_{m-1}X_{i_m})$$
$$= \mu(\Phi(X_{i_1}b_1 \cdots X_{i_{k_1} - 1}b_{k_1 - 1}), \Phi(b_{k_1}X_{i_{k_1} + 1} \cdots b_{k_2}), \ldots, \Phi(b_{k_p} \cdots X_{i_m}))$$
Lemma 5.2. If \( \gamma \) has only one block, then the result is trivial. If \( \gamma \) has more than one block, but only one outer block, \( B \), then \( B = \{ k : i_k = \min \{ i_1, \ldots, i_m \} \} = \{ k_1, \ldots, k_p \} \), since the last set forms always an outer block. Also this block must contain 1 and \( m \). The monotonic independence of \( (X_n)_{n \geq 1} \) over \( \mathcal{B} \) implies

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) \\
= \Phi(X_{i_1} \Phi(b_1 X_{i_2} b_2 \ldots X_{i_{k_2-1}} b_{k_2-1}) X_{i_{k_2}} \Phi(b_{k_2} X_{i_{k_2+1}} \ldots) \ldots X_{i_m}) \\
= \mu(\Phi(b_1 X_{i_2} \ldots b_{k_2-1}), \ldots, \Phi(b_{k_{p-1}} X_{i_{k_{p-1}+1}} \ldots b_{m-1}))
\]

If \( \gamma \) has more than one outer block, the result comes by induction on the number of blocks of \( \gamma \). Suppose the result is true for less than \( r \) blocks and that \( \gamma \) has exactly \( r \) blocks. Consider again \( B_0 = \{ k, i_k = \min \{ i_1, \ldots, i_m \} \} = \{ k_1, \ldots, k_p \} \).

Using again Definition 2.1 we obtain

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) \\
= \Phi(\Phi(X_{i_1} b_1 \ldots b_{k_1-1}) X_{i_{k_1}} \ldots X_{i_m}, \Phi(b_{k_m} X_{i_{k_m+1}} \ldots X_{i_m})) \\
= \mu(\Phi(X_{i_1} b_1 \ldots b_{k_1-1}), \ldots, \Phi(b_{k_m} X_{i_{k_m+1}} \ldots X_{i_m}))
\]

If \( B = B_0 \), then the result is proved above. If \( B \neq B_0 \), then without losing generality we can suppose that \( B \) is at the right of \( B_0 \), hence

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) \\
= \Phi(\Phi(X_{i_1} b_1 \ldots b_{k_1-1}) X_{i_{k_1}} \ldots X_{i_m}, \Phi(b_{k_m} X_{i_{k_m+1}} \ldots X_{i_m})) \\
= \Phi(X_{i_1} b_1 \ldots X_{i_{k_1-1}}, \Phi(b_{k_1} X_{i_{k_1}} \ldots X_{i_m}),
\]

and the result follows applying the induction hypothesis to \( \Phi(X_{i_1} b_1 \ldots X_{i_{k_1-1}}) \).

\[\square\]

Lemma 5.2. If \( (i_1, \ldots, i_m) \) and \( (l_1, \ldots, l_m) \) are two admissible configurations for \( \gamma \in NC(m) \), then for any \( b_1, \ldots, b_{m-1} \in \mathcal{B} \), one has:

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) = \Phi(X_{l_1} b_1 X_{l_2} \ldots b_{m-1} X_{l_m})
\]

Proof. Again, if \( \gamma \) is the partition with a single block, then \( i_1 = \cdots = i_m \) and \( l_1 = \cdots = l_m \) and

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) = \Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_1}) = \mu(b_1, \ldots, b_{m-1}) = \Phi(X_{l_1} b_1 X_{l_2} \ldots b_{m-1} X_{l_m})
\]

The conclusion follows now by induction on \( m \).

If \( m = 1 \), then \( \Phi(X_{i_1}) = \Phi(X_{i_1}) \).

Suppose the result is true for \( m \leq N - 1 \) and that \( \gamma \in NC(N) \) has more than one block. Let then \( B = \{ k_1, \ldots, k_p \} \) be a outer block of \( \gamma \). From Lemma 5.1

\[
\Phi(X_{i_1} b_1 X_{i_2} \ldots b_{m-1} X_{i_m}) \\
= \mu(\Phi(X_{i_1} b_1 \ldots X_{i_{k_1-1}} b_{k_1-1}), \Phi(b_{k_1} X_{i_{k_1+1}} \ldots b_{k_2}), \ldots, \Phi(b_{k_p} X_{i_m})), \\
= \mu(\Phi(X_{l_1} b_1 \ldots X_{l_{k_1-1}} b_{k_1-1}), \Phi(b_{k_1} X_{l_{k_1+1}} \ldots b_{k_2}), \ldots, \Phi(b_{k_p} X_{l_m})), \\
= \Phi(X_{l_1} b_1 X_{l_2} \ldots b_{m-1} X_{l_m})
\]

\[\square\]
Since the value $\Phi((X_i, b_1 X_i, \ldots, b_{m-1} X_i))$ is the same for all the admissible configurations $(i_1, \ldots, i_m)$, we will denote it by $V(\gamma, b_1, \ldots, b_{m-1})$.

**Theorem 5.3.** Let $(X_n)_{n=1}^\infty$ be a sequence of selfadjoint elements from $\mathfrak{A}$ such that:

1. $\{X_n\}$ is a monotonically independent family
2. $\mu_{X_i} = \mu_{X_j}$ for any $i, j \geq 1, n \geq 0$
3. $\Phi(X_k) = 0$

Then there exists a conditional expectation $\nu : \mathfrak{B}_+\langle \xi \rangle \to \mathfrak{B}$ with the property:

$$\lim_{N \to \infty} \Phi\left(f \left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right)\right) = \nu(f)$$

for any $f \in \mathfrak{B}_+\langle \xi \rangle$. Moreover $\nu(f)$ depends only on the second order moments of $X_i$.

**Proof.** For convenience, we will use the notations $\mu$ for $\mu_{X_i}$ $(i \geq 1)$, $a(\gamma)$ for the set of all admissible configurations of $\gamma$, $a(\gamma, N)$ for the set of all admissible configurations of $\gamma$ with indices from $\{1, 2, \ldots, N\}$, and $PP(m)$ for the set of all non-crossing pair partitions (partitions where each block has exactly two elements) of $\{1, \ldots, m\}$.

It is enough to show the property for some arbitrary $b_1, \ldots, b_{m-1} \in \mathfrak{B}$ and

$$f = \xi b_1 \xi \ldots b_{m-1} \xi$$

From Lemma 5.2 one has:

$$\Phi\left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right) b_1 \ldots b_{m-1} \left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right) = \frac{1}{N^{m-2}} \sum_{(i_1, \ldots, i_m)} \Phi(X_{i_1}, b_1 \ldots b_{m-1} X_{i_m})$$

$$= \frac{1}{N^{m-2}} \sum_{\gamma \in NC(m)} V(\gamma, b_1, \ldots, b_{m-1}) \text{card}(a(\gamma, N))$$

If $\gamma$ contains blocks with only one element, the condition $\Phi(X_i) = 0$ $(i \geq 1)$ and Lemma 5.1 imply that $V(\gamma, b_1, \ldots, b_{m-1}) = 0$.

Also, if $\gamma$ has less than $\frac{m}{2}$ blocks, since

$$\text{card}(a(\gamma, N)) < N^{\text{card}(\gamma)} < N^{\frac{m}{2}}$$

we have that

$$\lim_{N \to \infty} \frac{1}{N^{m-2}} V(\gamma, b_1, \ldots, b_{m-1}) \text{card}(a(\gamma, N)) = 0.$$ 

If follows that only the pair partitions contribute to the limit, that is

$$\lim_{N \to \infty} \Phi\left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right) b_1 \ldots b_{m-1} \left(\frac{X_1 + \ldots + X_N}{\sqrt{N}}\right) = \lim_{N \to \infty} \frac{1}{N^{m-2}} \sum_{\gamma \in PP(m)} V(\gamma, b_1, \ldots, b_{m-1}) \text{card}(a(\gamma, N))$$

In particular, for $m$ odd, the limit exists and it is equal to zero.
If $m$ is even, note first that

$$a(\gamma, N) = \prod_{k=1}^{m/2} a(\gamma, N, k)$$

and that

$$\text{card}(a(\gamma, N, k)) = \binom{N}{k} \text{card}(a(\gamma, k, k))$$

therefore

$$\lim_{N \to \infty} \Phi \left( \left( \frac{X_1 + \ldots + X_N}{\sqrt{N}} \right) b_1 \ldots b_{m-1} \left( \frac{X_1 + \ldots + X_N}{\sqrt{N}} \right) \right) = \lim_{N \to \infty} \frac{1}{N^{m/2}} \sum_{\gamma \in PP(m)} V(\gamma, b_1, \ldots, b_{m-1}) \sum_{k=1}^{m/2} \binom{N}{k} \text{card}(a(\gamma, k, k)) = \lim_{N \to \infty} \frac{1}{N^{m/2}} \sum_{\gamma \in PP(m)} V(\gamma, b_1, \ldots, b_{m-1}) \left( \frac{N}{m/2} \right) \text{card}(a(\gamma, \frac{m}{2}, \frac{m}{2}))$$

since

$$\lim_{N \to \infty} \frac{1}{N^{m/2}} \binom{N}{k} = \begin{cases} 0 & \text{if } k < \frac{m}{2} \\ \frac{1}{(\frac{m}{2})!} & \text{if } k = \frac{m}{2} \end{cases}$$

For the last part, note that $V(\gamma, b_1, \ldots, b_{m-1})$ is computed iterating the result from Lemma 5.1, so for $\gamma \in PP(m)$ it depends only on the moments of order 2 of $X_i$ ($i \geq 1$). □

In the following paragraph we will suppose, without loss of generality, that $\mathcal{B}$ is unital.

**Corollary 5.4.** The functional $\nu$ is positive if and only $\Phi(X_k b^* b X_k) \geq 0$ for any $b \in \mathcal{B}$ ($k \geq 1$).

**Proof.** One implication is trivial: if $\nu \geq 0$, then $\Phi(X_k b^* b X_k) = \nu((b \xi)^* b \xi) \geq 0$.

For the other implication, consider the set of symbols $\{\zeta_i\}_{i \geq 1}$ and the linear spaces

$$\mathcal{B} \zeta_i \mathcal{B} = \{ b_1 \zeta_i b_2, \ b_1, b_2 \in \mathcal{B} \}$$

with the $\mathcal{B}$-bimodule structure given by

$$a_1 (b_1 \zeta_i b_2) a_2 = (a_1 b_1) \zeta_i (b_2 a_2)$$

and with the $\mathcal{B}$-valued pairing $\langle , \rangle$ given by

$$\langle a \zeta_i, b \zeta_i \rangle = \nu(\xi a^* b \xi)$$

for any $a, b \in \mathcal{B}$.

The pairing $\langle , \rangle$ is positive, since $\nu(\xi b^* b \xi) \geq 0$ for any $b \in \mathcal{B}$.
Let $\mathcal{E}$ be the weakly monotone product of $\{B_\zeta B\}_{\zeta \geq 1}$. As shown in part 4.2, the mappings $G(\zeta_i)$ form a monotonic independent family in $L(\mathcal{E})$, therefore, from 5.3, one has that

$$\nu(p^*(\xi)p(\xi)) = \lim_{N \to \infty} \langle 1, p(\frac{G(\zeta_1) + \ldots + G(\zeta_N)}{\sqrt{N}}) * p(\frac{G(\zeta_1) + \ldots + G(\zeta_N)}{\sqrt{N}}) \rangle$$

for any $p(\xi) \in \mathcal{B}(\xi)$.

The conclusion follows from the positivity of the functional $\langle 1, 1 \rangle$. □

6. Positivity results and connection to operator-valued conditionally free products

**Definition 6.1.** Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two algebras containing the subalgebra $\mathfrak{B}$ such that $\mathfrak{A}_1$ has the decomposition $\mathfrak{A}_1 = \mathfrak{B} \oplus \mathfrak{A}_0^1$ for $\mathfrak{A}_0^1$ a subalgebra of $\mathfrak{A}$ which is also a $\mathfrak{B}$-algebra. If $\Phi_1, \Phi_2$ are conditional expectations, $\Phi_j: \mathfrak{A}_j \to \mathfrak{B}$, $j = 1, 2$, we define $\Phi = \Phi_1 \triangleright \Phi_2$, the monotonic product of $\Phi_1$ and $\Phi_2$ to be the conditional expectation on the algebraic free product with amalgamation over $\mathfrak{B}$,

$$\mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$$

given by

$$\Phi(aa_1ba_2\beta) = \Phi(aa_1\Phi_2(b)a_2\beta),$$
$$\Phi(ba_2\beta) = \Phi_2(b)\Phi(a_2\beta),$$
$$\Phi(aa_1b) = \Phi(aa_1)\Phi_2(b)$$

for all $a_1, a_2 \in \mathfrak{A}_0^1$, $b \in \mathfrak{A}_2$ and $\alpha, \beta \in \mathfrak{A}$.

The map $\Phi$ is well-defined, since any element of $\mathfrak{A}$ can be written as a sum of finite products in which the elements from $\mathfrak{A}_0^1$ and $\mathfrak{A}_2$ and the conditions above imply

$$\Phi(b_0a_1b_1 \ldots a_nb_n) = \Phi_1(\Phi_2(b_0)a_1\Phi_2(b_1) \ldots a_n\Phi_2(b_n))$$

for all $b_0, b_1, \ldots, b_n \in \mathfrak{B}_2, a_1, \ldots, a_n \in \mathfrak{A}_0^1$, and all the analogues for the other types of such products.

**Proposition 6.2.** If, in the above setting, $\mathfrak{A}_1, \mathfrak{A}_2$ are $*$-algebras, $\mathfrak{B}$ is a $C^*$-algebra, and $\Phi_1, \Phi_2$ are positive (i.e. $\Phi_j(a^*a) \geq 0$, for all $a \in \mathfrak{A}_j$, $j = 1, 2$), then $\Phi_1 \triangleright \Phi_2$ is also positive.

**Proof.** First, remember that the positivity of the conditional expectations $\Phi_j$ implies that $\Phi_j(x^*) = (\Phi_j(x))^*$, for all $x \in \mathfrak{A}_j$.

Also, the map $\Phi_2$ is completely positive, and for any $b_1, \ldots, b_n \in \mathfrak{A}_2$, the element $(\Phi_2(b_i^*b_j))_{i,j=1}^n \in M_n(\mathfrak{B})$ is positive (see [13], Section 3.5).

We have to show that

$$\Phi(a^*a) \geq 0 \quad \text{for all } a \in \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2.$$
Any such \( a \) can be written as a finite sum of elements of the form \( b_0a_1b_1 \ldots a_nb_n \) with \( b_0, \ldots, b_n \in \mathfrak{A}_2, \ a_1, \ldots, a_n \in \mathfrak{A}_1^0 \). Hence:

\[
\Phi(a^* a) = \Phi \left( \sum_{i=1}^{N} b_{0,i}a_{1,i}b_{1,i} \ldots a_{n(i),i}b_{n(i),i}^* \left( \sum_{j=1}^{N} b_{0,j}a_{1,j}b_{1,j} \ldots a_{n(j),j}b_{n(j),j} \right) \right)
\]

\[
= \Phi \left( \sum_{i,j=1}^{N} b_{n(i),i}^*a_{n(i),i}^* \ldots a_{1,i}^*b_{0,i}^*b_{0,j}a_{1,j}b_{1,j} \ldots a_{n(j),j}b_{n(j),j} \right)
\]

Since \( (\Phi_2(b_{0,i}^*,b_{0,j}))_{i,j=1}^{N} \in M_N(\mathfrak{B}) \subset M_N(\mathfrak{A}_1) \) is positive, there exists a matrix \( T \in M_N(\mathfrak{A}_1) \) such that

\[
(\Phi_2(b_{0,i}^*,b_{0,j}))_{i,j=1}^{N} = T^* T.
\]

With the notation

\[
a_i = a_{1,i}\Phi_2(b_{1,j}) \ldots a_{n(j),j}\Phi_2(b_{n(j),j}) \in \mathfrak{A}_1
\]

we obtain:

\[
\Phi(a^* a) = \Phi_1 \left( [(a_1 \ldots a_N)^*]T^* T(a_1 \ldots a_N) \right) \geq 0
\]

\[\square\]

Let \( \mathfrak{B}(\xi, \xi^*) \) be the *-algebra of polynomials in \( \xi \) and \( \xi^* \) described in Section 2. For \( X \in \mathfrak{A} \), consider \( \mathfrak{A}_X \) the *-subalgebra of \( \mathfrak{A} \) generated by \( X \) and \( \mathfrak{B} \). Define the mapping \( \tau_X : \mathfrak{B}(\xi, \xi^*) \rightarrow \mathfrak{A}_X \) to be the algebra *-homomorphism given by \( \tau_X(\xi) = X \) and the \( \mathfrak{B} \)-functional \( \nu_X : \mathfrak{B}(\xi, \xi^*) \rightarrow \mathfrak{B} \) to be given by \( \nu_X = \Phi \circ \tau_X \).

**Corollary 6.3.** If \( X, Y \in \mathfrak{A} \) are monotonically independent over \( \mathfrak{B} \) and \( \nu_X, \nu_Y \) are positive, then \( \nu_Z \) is also positive for any element \( Z \) in the *-algebra generated by \( X \) and \( Y \). In particular \( \nu_{X+Y} \) and \( \nu_{XY} \) are positive.

**Proof.** Consider \( Z = Z(X,Y) \) a polynomial in \( X \) and \( Y \). Since the maps

\[
\nu_X : \mathfrak{B}(\xi_1, \xi_1^*) \rightarrow \mathfrak{B} \\
\nu_Y : \mathfrak{B}(\xi_2, \xi_2^*) \rightarrow \mathfrak{B}
\]

are positive, from Proposition 6.2 so is

\[
\nu_x \triangleright \nu_Y : \mathfrak{B}(\xi_1, \xi_1^*) \ast_{\mathfrak{B}} \mathfrak{B}(\xi_2, \xi_2^*) = \mathfrak{B}(\xi_1, \xi_1^*, \xi_2, \xi_2^*) \rightarrow \mathfrak{B}
\]

Remark also that

\[
i_Z : \mathfrak{B}(\xi, \xi^*) \ni f(\xi) \rightarrow f(Z(\xi_1, \xi_2)) \in \mathfrak{B}(\xi_1, \xi_1^*, \xi_2, \xi_2^*)
\]

is a positive \( \mathfrak{B} \)-functional.

The conclusion follows from the fact that the monotonic independence over \( \mathfrak{B} \) of \( X \) and \( Y \) is equivalent to

\[
\nu_Z = (\nu_X \triangleright \nu_Y) \circ i_Z.
\]

\[\square\]
Lemma 6.4. Let \( \mathfrak{A}_1, \mathfrak{A}_2 \) be two *-algebras containing the \( C^* \)-algebra \( \mathfrak{B} \), and \( \Phi_j : \mathfrak{A}_j \to \mathfrak{B}, j = 1, 2 \) positive conditional expectations. Let \( a_1, \ldots, a_n \in \mathfrak{A}_1, a_{n+1}, \ldots, a_{n+m} \in \mathfrak{A}_2 \) and

\[
A = (A_{i,j}) \in M_{n+m}(\mathfrak{B}) \text{ be the matrix with the entries }
\]

\[
A_{i,j} = \begin{cases} 
\Phi_1(a_i^*a_j) & \text{if } i, j \leq n \\
\Phi_1(a_i^*)\Phi_2(a_j) & \text{if } i \leq n, j > n \\
\Phi_2(a_i^*)\Phi_1(a_j) & \text{if } i > n, j \leq n \\
\Phi(a_i^*a_j) & \text{if } i, j > n
\end{cases}
\]

Then \( A \) is positive.

Proof. As shown in [15], Theorem 3.5.6, the \( \mathfrak{B} \)-functional \( \Phi_1 *_{\mathfrak{B}} \Phi_2 \) is completely positive on \( \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 \). Also, note that \( A_{i,j} = ((\Phi_1 *_{\mathfrak{B}} \Phi_2)(a_i^*a_j)) \) for all \( 1 \leq i, j = 1 \leq n + m \), and the conclusion follows from [15], Lemma 3.5.2. \( \square \)

Consider now \( \mathfrak{A}_1, \mathfrak{A}_2 \) two *-algebras over the \( C^* \)-algebra \( \mathfrak{B} \), each endowed with two positive conditional expectations \( \Phi_j : \mathfrak{A}_j \to \mathfrak{B} \), \( j = 1, 2 \). We define \( (\mathfrak{A}, \Phi, \Psi) \), the conditionally free product with amalgamation over \( \mathfrak{B} \) of the triples \( (\mathfrak{A}_1, \Phi_1, \Psi_1) \) and \( (\mathfrak{A}_2, \Phi_2, \Psi_2) \), by:

1. \( \mathfrak{A} = \mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2 \)
2. \( \Psi = \Psi_1 *_{\mathfrak{B}} \Psi_2 \) and \( \Phi = \Phi_1 *_{(\Psi_1, \Psi_2)} \Phi_2 \), i.e. the functionals \( \Psi \) and \( \Phi \) are determined by the relations

\[
\Psi(a_1a_2 \ldots a_n) = 0 \\
\Phi(a_1a_2 \ldots a_n) = \Phi(a_1)\Phi(a_2) \ldots \Phi(a_n)
\]

for any \( a_i \in \mathfrak{A}_{\varepsilon(i)}, \varepsilon(i) \in \{1, 2\} \), such that \( \varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n) \) and

\[
\Psi_{\varepsilon(i)}(a_i) = 0.
\]

Theorem 6.5. In the above setting, \( \Phi \) and \( \Psi \) are positive \( \mathfrak{B} \)-functionals.

Proof. The positivity of \( \Psi \) is proved in [15], Theorem 3.5.6.

For the positivity of \( \Phi \) we have to show that \( \Phi(a^*a) \geq 0 \) for any \( a \in \mathfrak{A} \).

Since any element of \( \mathfrak{A} \) can be written

\[
a = \sum_{k=1}^{N} s_{1,k} \ldots s_{n(k),k},
\]

where \( s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}, \varepsilon(1, k) \neq \varepsilon(2, k) \neq \cdots \neq \varepsilon(n(k), k) \)

\[
= \sum_{k=1}^{N} \prod_{j=1}^{n(k)} (s_{j,k}) - \Psi(s_{j,k})) + \Psi(s_{j,k}))
\]

we can consider \( a \) of the form

\[
a = \alpha + \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k}
\]

such that

\[
\alpha \in \mathfrak{B} \\
a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}, \varepsilon(1, k) \neq \varepsilon(2, k) \neq \cdots \neq \varepsilon(n(k), k) \\
\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0.
\]
Therefore

\[ \Phi(a^*a) = \Phi \left( \left( \alpha + \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right)^* \left( \alpha + \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right) \right) \]

\[ = \Phi \left( \alpha^* + \alpha^* \left( \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right) + \left( \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right)^* \alpha + \right. \]

\[ \left. \left( \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right)^* \left( \sum_{k=1}^{N} a_{1,k} \ldots a_{n(k),k} \right) \right) \]

\[ = \Phi(\alpha^* \alpha) + \sum_{k=1}^{N} \Phi(\alpha^* a_{1,k} \ldots a_{n(k),k}) \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \]

\[ + \sum_{k=1}^{N} \Phi(a_{n(k),k})^* \ldots \Phi(a_{2,k})^* \Phi(a_{1,k}^* \alpha) \]

\[ + \sum_{k,l=1}^{N} \left( \Phi(a_{n(k),k})^* \ldots \Phi(a_{2,k})^* \Phi(a_{1,k}^* a_{1,l}) \Phi(a_{2,l}) \ldots \Phi(a_{n(l),l}) \right) \]

that is

\[ \Phi(a^*a) = \Phi(\alpha^* \alpha) + \sum_{k=1}^{N} \Phi(\alpha^* a_{1,k}) \left[ \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \right] \]

\[ + \sum_{k=1}^{N} \left[ \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^* \alpha) \]

\[ + \sum_{k,l=1}^{N} \left[ \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^* a_{1,l}) \left[ \Phi(a_{2,l}) \ldots \Phi(a_{n(l),l}) \right] \]

Denote now \( a_{1,N+1} = \alpha \) and \( \beta_k = \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \).

From Lemma 6.4, the matrix \( S = \left( \Phi(a_{1,j}^* a_{1,j}) \right)_{j=1}^{N+1} \) is positive in \( M_{N+1}(\mathfrak{B}) \), therefore

\[ S = T^*T, \text{ for some } T \in M_{N+1}(\mathfrak{B}) \]

The identity for \( \Phi(a^*a) \) becomes:

\[ \Phi(a^*a) = (\beta_1, \ldots, \beta_N, 1)^* T^*T(\beta_1, \ldots, \beta_N, 1) \geq 0, \]
as claimed.

Suppose now that the $\ast$-algebra $A_1$ has the decomposition $A_1 = B \oplus A_1^0$, such that $A_1^0$ is a $\ast$-subalgebra of $A_1$ which is also a $B$-algebra. Define the $B$-valued conditional expectation

$$\delta : A_1 \ni (\lambda + a_0) \mapsto \lambda \in B$$

for all $a_0 \in A_1^0$.

**Theorem 6.6.** With the notations above,

$$\Phi_1 \triangleright \Phi_2 = \Phi_1 \ast_{(\delta, \Phi_2)} \Phi_2$$

**Proof.** First remark that $\delta(a) = 0$ implies $a \in A_1^0$, from the definition of $\delta$.

For $\varepsilon(1), \ldots, \varepsilon(n) \in \{1, 2\}$ such that $\varepsilon(1) \neq \cdots \neq \varepsilon(n)$ and $a_j \in A_{\varepsilon(j)}$ such that $\delta(a_j) = 0$ if $\varepsilon(j) = 1$ and $\Phi_2(a_j) = 0$ if $\varepsilon(j) = 2$, one has ($\chi_{A_j}$ denotes the characteristic function of $A_j$):

$$(\Phi_1 \triangleright \Phi_2)(a_1 \ldots a_n) = \Phi_1 \left( \prod_{j=1}^{n} \left[ \chi_{A_1}(a_j) + \Phi_2(\chi_{A_2}(a_j)) \right] \right)$$

$$= \Phi_1 \left( \prod_{j=1}^{n} \chi_{A_1}(a_j) \right)$$

$$= 0$$ since there is at least one $a_j \in A_2$

The conclusion follows from the above equality, since the conditional expectation $\Phi_1 \ast_{(\delta, \Phi_2)} \Phi_2$ is generated by

$$(\Phi_1 \ast_{(\delta, \Phi_2)} \Phi_2)(a_1 \ldots a_n) = 0.$$

\[\square\]

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1Indiana University at Bloomington, Department of Mathematics, Rawles Hall, 931 E 3rd St, Bloomington, IN 47405
E-mail address: mi@math.indiana.edu
2Institute of Mathematics, Romanian Academy, P.O.Box 1-764, Bucharest, RO-70700, Romania