CLASSIFICATION OF INVARIANT VALUATIONS ON THE QUATERNIONIC PLANE

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Abstract. We describe the orbit space of the action of the group Sp(2) Sp(1) on the real Grassmann manifolds Gr_k(H_2) in terms of certain quaternionic matrices of Moore rank not larger than 2. We then give a complete classification of valuations on the quaternionic plane H^2 which are invariant under the action of the group Sp(2) Sp(1).

1. Introduction and statement of results

1.1. Background. A valuation is a finitely additive map from the space of compact convex subsets of some vector space into an abelian semi-group. Since Hadwiger’s famous characterization of (real-valued) continuous valuations which are euclidean motion invariant, classification results for valuations have long played a prominent role in convex and integral geometry.

Many generalizations of Hadwiger’s theorem were obtained recently. On the one hand, valuations with values in some abelian semi-group other than the reals were characterized. The most important examples are tensor valuations [5, 18, 19, 26], Minkowski valuations [1, 2, 17, 23, 33, 34], curvature measures [15, 32] and area measures [40, 41]. On the other hand, invariance with respect to the euclidean group was weakened to invariance with respect to translations or rotations only [4, 6], or with respect to a smaller group of isometries. Next we briefly describe the main results in this line.

Let V be a finite-dimensional vector space and G a group acting linearly on V. The space of scalar-valued, G-invariant, translation invariant continuous valuations on V will be denoted by Val^G_G. Hadwiger’s theorem applies in the case where V is a euclidean vector space of dimension n, and G = SO(V). It states that Val^G_G is spanned by the so-called intrinsic volumes \( \mu_0, \ldots, \mu_n \). In particular, Val^SO(V)_G is finite-dimensional. From this fact, one can easily derive integral-geometric formulas like Crofton formulas and kinematic formulas [22].

In the same spirit, kinematic formulas with respect to a smaller group G exist provided that Val^G_G is finite-dimensional. Although it is known which groups have this property, much less is known about the explicit form of such formulas. Alesker [9] has shown that Val^G_G is finite-dimensional if and only if G acts transitively on the unit sphere. Such groups were classified

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by Montgomery-Samelson \[27\] and Borel \[16\]. There are six infinite lists
\[
\text{SO}(n), \text{U}(n), \text{SU}(n), \text{Sp}(n), \text{Sp}(n)\text{U}(1), \text{Sp}(n)\text{Sp}(1)
\]
(1)
and three exceptional groups
\[
\text{G}_2, \text{Spin}(7), \text{Spin}(9).
\]
(2)

The euclidean case is \(G = \text{SO}(n)\) where Hadwiger’s theorem applies. In the hermitian case \(G = \text{U}(n)\) or \(G = \text{SU}(n)\), recent results have revealed a lot of unexpected algebraic structures yielding a relatively complete picture \[3, 6, 14, 15, 31, 37\]. Hadwiger-type theorems for the groups \(\text{G}_2\) and \(\text{Spin}(7)\) are also known \[12\]. In the remaining cases, i.e. the quaternionic cases \(G = \text{Sp}(n)\), \(G = \text{Sp}(n)\text{U}(1)\) and \(G = \text{Sp}(n)\text{Sp}(1)\) as well as in the case \(G = \text{Spin}(9)\), only the dimension of \(\text{Val}^G\) is known \[13, 39\].

The combinatorial formulas from \[13\] indicate that the classification of invariant valuations on quaternionic vector spaces will be a rather subtle subject. Note that the case \(n = 1\) can be reduced to the hermitian case, since \(\text{Sp}(1) = \text{SU}(2)\). For higher dimensions, not much is known, except the construction of one example of an \(\text{Sp}(n)\text{Sp}(1)\)-invariant valuation by Alesker \[8\].

1.2. Results of the present paper. In this article, we establish a complete Hadwiger-type theorem for the group \(\text{Sp}(2)\text{Sp}(1)\) acting on the two-dimensional quaternionic space \(\mathbb{H}^2\). More precisely, we find an explicit basis of the space of invariant valuations \(\text{Val}^\text{Sp}(2)\text{Sp}(1)\). The description of the basis is given in terms of Klain functions, which are invariant functions on the real Grassmannians of \(\mathbb{H}^2\).

Our first main theorem concerns the orbit space of the action of \(\text{Sp}(2)\text{Sp}(1)\) on the real Grassmann manifolds \(\text{Gr}_k := \text{Gr}_k(\mathbb{H}^2)\). It is formulated in terms of the Moore rank of hyperhermitian matrices, whose definition will be recalled in the next section. Since taking orthogonal complements commutes with the action of \(\text{Sp}(2)\text{Sp}(1)\), it will be enough to consider the case \(k \leq 4\).

**Theorem 1.** Let \(2 \leq k \leq 4\). Given a tuple of real numbers \(\lambda_{pq}, 1 \leq p < q \leq k\) we define the quaternionic hermitian matrix \(M_\lambda\) by

\[
M_\lambda := \begin{cases}
1 & \lambda_{12}i \\
-\lambda_{12}i & 1 \\
1 & \lambda_{13}j \\
-\lambda_{13}j & \lambda_{23}k \\
1 & \lambda_{14}k \\
-\lambda_{14}k & 1 \\
1 & \lambda_{13}j \\
-\lambda_{13}j & \lambda_{23}k \\
1 & \lambda_{12}i \\
-\lambda_{12}i & 1 \\
1 & \lambda_{14}k \\
-\lambda_{14}k & 1
\end{cases}
\]

Let \(\mathbb{Z}_2^k\) and the permutation group \(S_k\) act on such a tuple by

\[
(\epsilon \cdot \lambda)_{p,q} := \epsilon_p \epsilon_q \lambda_{pq}, \quad \epsilon \in \mathbb{Z}_2^k
\]

\[
(\sigma \cdot \lambda)_{p,q} := \lambda_{\sigma(p)\sigma(q)} = \lambda_{\sigma(q)\sigma(p)}, \quad \sigma \in S_k.
\]

(3)

(4)
Then the quotient $\text{Gr}_k / \text{Sp}(2)\text{Sp}(1)$ is of dimension $(k-1)$ and homeomorphic to the quotient $X_k := \{ \lambda_{pq} \in [-1, 1], 1 \leq p < q \leq k : \text{rank } M_{\lambda} \leq 2 \} / \mathbb{Z}_2^k \times S_k$.

The orbit corresponding to $[\lambda] \in X_k$ contains a plane $V$ admitting a basis $v_1, \ldots, v_k$ such that

$$K(v_i, v_j) = (M_{\lambda})_{i,j} \quad i, j = 1, \ldots, k,$$

where $K$ is the quaternionic hermitian product of $\mathbb{H}^2$.

The construction of this homeomorphism is roughly as follows. Given a plane $V \in \text{Gr}_k(\mathbb{H}^2)$, we construct an orthonormal basis $v_1, \ldots, v_k$ of $V$ such that the matrix $Q = (K(v_i, v_j))$ has a special shape: if $k \in \{3, 4\}$, the pure quaternions $q_{12}, q_{13}, q_{23}$ are pairwise orthogonal, and moreover $q_{12} || q_{24}, q_{14} || q_{23}$ if $k = 4$. Then $Q$ is Sp(1)-conjugate to a matrix $M_{\lambda}$, and $V$ is mapped to $[\lambda]$.

Note that the condition on the Moore rank is a system of polynomial equations in the $\lambda_{pq}$, which can be written down explicitly using equations (7) and (8).

**Corollary 1.1.** Every $\text{Sp}(2)\text{Sp}(1)$-orbit in $\text{Gr}_k$ contains a $k$-plane of the form

\[
\begin{align*}
\text{span}\{ & \cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2 \} & k = 2, \\
\text{span}\{ & \cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2, \cos \theta_3, \sin \theta_3 \} & k = 3, \\
\text{span}\{ & \cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2, \cos \theta_3, \sin \theta_3, \cos \theta_4, \sin \theta_4 \} & k = 4.
\end{align*}
\]

The corresponding $[\lambda] \in X_k$ is given by $\lambda_{pq} = \cos(\theta_p - \theta_q)$.

Let us now describe the Hadwiger-type theorem, which is our second main result. The space of continuous, translation invariant valuations on an $n$-dimensional vector space $V$ is denoted by $\text{Val}(V)$ or just $\text{Val}$ if there is no risk of confusion. A valuation $\phi \in \text{Val}$ is called **even** if $\phi(-B) = \phi(B)$ and **odd** if $\phi(-B) = -\phi(B)$ for each convex body $B$. If $\phi(tB) = t^k \phi(B)$ for all $t > 0$ and all $B$, then $\phi$ is said to be **homogeneous of degree** $k$. The space of even/odd valuations of degree $k$ is denoted by $\text{Val}^\pm_k$. A fundamental result by McMullen [25] is the decomposition

$$\text{Val} = \bigoplus_{k=0}^n \text{Val}^k.$$

An even, continuous and translation invariant valuation can be described by its Klain function, which is defined as follows. Let $\phi \in \text{Val}^+_k$ and $E \in \text{Gr}_k(V)$, the Grassmann manifold of $k$-planes in $V$. Then the restriction of $\phi$ to $E$ is a multiple of the Lebesgue measure, and the corresponding factor is denoted by $K_{\phi}(E)$. The function $K_{\phi} \in C(\text{Gr}_k(V))$ is called the Klain function of $\phi$. The map $K_1 : \text{Val}^+_k \rightarrow C(\text{Gr}_k(V))$ is in fact injective, as was shown by Klain [21].

Let us now specialize to the group $\text{Sp}(2)\text{Sp}(1)$ acting on $V = \mathbb{H}^2$. The dimension of the space of $k$-homogeneous $\text{Sp}(2)\text{Sp}(1)$-invariant valuations
was computed in [13]:

$$\dim \text{Val}_{k}^{\text{Sp}(2)\text{Sp}(1)} = \begin{bmatrix} k \\ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \end{bmatrix}$$

(5)

Since the group $\text{Sp}(2)\text{Sp}(1)$ contains $-\text{Id}$, invariant valuations are even. We will characterize them in terms of their Klain functions. To do so, consider the following invariant functions on $\text{Gr}_{k},0 \leq k \leq 4$, which are defined in terms of the coordinates $\lambda = (\lambda_{ij})$ of $\text{Gr}_{k}/\text{Sp}(2)\text{Sp}(1)$ from Theorem [1]:

$$f_{k,0}(\lambda) := 1, \quad k = 0, \ldots, 4$$

$$f_{2,1}(\lambda) := \lambda_{12}^2$$

$$f_{3,1}(\lambda) := \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2$$

$$f_{3,2}(\lambda) := \lambda_{12}^2 \lambda_{23}^2 + \lambda_{13}^2 \lambda_{23}^2 + \lambda_{12}^2 \lambda_{13}^2$$

$$f_{4,1}(\lambda) := \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{14}^2 + \lambda_{23}^2 + \lambda_{24}^2 + \lambda_{34}^2$$

$$f_{4,2}(\lambda) := \lambda_{12}^2 \lambda_{24}^2 + \lambda_{13}^2 \lambda_{24}^2 + \lambda_{14}^2 \lambda_{24}^2 + \lambda_{12}^2 \lambda_{14}^2 + \lambda_{12}^2 \lambda_{23}^2$$

$$f_{4,3}(\lambda) := \lambda_{12}^2 \lambda_{23}^2 + \lambda_{12}^2 \lambda_{14}^2 + \lambda_{13}^2 \lambda_{14}^2 + \lambda_{23}^2 \lambda_{24}^2 + \lambda_{24}^2 \lambda_{34}^2 + \lambda_{23}^2 \lambda_{34}^2 + \lambda_{12}^2 \lambda_{23}^2$$

$$f_{4,4}(\lambda) := -2\lambda_{12} \lambda_{13}^2 \lambda_{21} \lambda_{24} \lambda_{34} - 2\lambda_{12} \lambda_{13} \lambda_{14} \lambda_{24} \lambda_{34} - 2\lambda_{12} \lambda_{23} \lambda_{13} \lambda_{14} \lambda_{34}$$

Noting that $\text{Gr}_{k} \cong \text{Gr}_{8-k}$ for all $k$, we define $f_{k,i} := f_{8-k,i}$ for $5 \leq k \leq 8$.

**Theorem 2.** For each $0 \leq k \leq 8$ and each $0 \leq i \leq \dim \text{Val}_{k}^{\text{Sp}(2)\text{Sp}(1)} - 1$, there exists a unique valuation $\phi \in \text{Val}_{k}^{\text{Sp}(2)\text{Sp}(1)}$ whose Klain function is $f_{k,i}$. These valuations form a basis of $\text{Val}_{k}^{\text{Sp}(2)\text{Sp}(1)}$.

In the proof of this theorem, we will first use differential geometric methods to show that certain linear combinations of the functions $f_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\text{Gr}_{k}$. Then we will use representation-theoretic tools, in particular the recent computation of the multipliers of the $a$-cosine transform by Olafsson-Pasquale [28], in order to construct valuations with the given Klain functions. As a corollary to their theorem, we prove a formula for the multipliers of the classical cosine transform which might be of independent interest. To see that the so-constructed valuations form a basis, we use the recent computation of $\dim \text{Val}_{k}^{\text{Sp}(2)\text{Sp}(1)}$ in [13].

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## 2. Quaternionic Linear Algebra

The quaternionic skew field $\mathbb{H}$ is defined as the real algebra generated by $1, i, j, k$ with the relations $i^2 = j^2 = k^2 = -1, ijk = -1$. The conjugate of a quaternion $q := a + bi + cj + dk$ is defined by $\bar{q} := a - bi - cj - dk$, its norm by $\sqrt{q\bar{q}}$. The quaternions of norm 1 form the Lie group $\text{Sp}(1)$ which
is isomorphic to SU(2). Conjugation by an element $\xi \in \text{Sp}(1)$ fixes the real line pointwise and acts as a rotation on the pure imaginary part $\text{Im} \mathbb{H} = \mathbb{R}^3$, moreover all rotations are obtained in this way.

Let $V$ be a quaternionic (right) vector space of dimension $n$. We endow $V$ with a quaternionic hermitian form $K$, i.e. an $\mathbb{R}$-bilinear form

$$K : V \times V \to \mathbb{H}$$

such that

i) $K$ is conjugate $\mathbb{H}$-linear in the first and $\mathbb{H}$-linear in the second factor, i.e.

$$K(vq, wr) = \overline{q}K(v, w)r, \quad q, r \in \mathbb{H},$$

ii) $K$ is hermitian in the sense that

$$K(w, v) = \overline{K(v, w)},$$

iii) $K$ is positive definite, i.e.

$$K(v, v) > 0 \quad \forall v \neq 0.$$

The standard example of such a form is given in $V = \mathbb{H}^n$ by

$$K(v, w) = \sum_{i=1}^{n} \overline{v}_i w_i, \quad v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{H}^n.$$

The group $\text{GL}(V, \mathbb{H}) = \text{GL}(n, \mathbb{H})$ is defined as the group of all $\mathbb{H}$-linear automorphisms of $V$. The subgroup of $\text{GL}(V, \mathbb{H})$ of all elements preserving $K$ is called the compact symplectic group and denoted by $\text{Sp}(V, K)$ or $\text{Sp}(n)$. It acts from the left on $V$. An important fact is that this action is transitive on the unit sphere in $V$. In the case $V = \mathbb{H}^n$, the group $\text{Sp}(n)$ consists of all quaternionic matrices $A$ such that $A^* A = \text{Id}$. Here $A^*$ denotes the conjugate transpose of $A$.

The action of $\text{Sp}(n) \times \text{Sp}(1)$ by left and right multiplication on $V$ has kernel $(-\text{Id}, -1)$. The quotient group is denoted by $\text{Sp}(n) \text{Sp}(1)$. It acts effectively on $V$.

Let $Q = (q_{ij})$ be a quaternionic $n \times n$ matrix. Viewing $\mathbb{H}^n$ as a right $\mathbb{H}$-vector space, $Q$ acts as a quaternionic linear map $Q : \mathbb{H}^n \to \mathbb{H}^n$ by multiplication from the left. Writing $\mathbb{H} = \mathbb{R}^4$, we obtain a corresponding real linear map $RQ : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$.

A square matrix $Q$ with quaternionic entries is called hyperhermitian if $Q^* = Q$, i.e. $q_{ji} = \overline{q}_{ij}$ for all $i, j$. In particular, the diagonal entries are real. The determinant of $RQ$ is a polynomial of degree $4n$ in the $n(2n-1)$ real components of $Q$. The Moore determinant is the unique polynomial determinant of degree $n$ in the same variables which satisfies $\det(Q)^4 = \det(RQ)$ and $\det(\text{Id}) = 1$. Note that the Moore determinant is defined only on hyperhermitian matrices. We refer to [7] [8] [11] for more information on the Moore determinant and its relation to other determinants of quaternionic matrices such as the Dieudonné determinant.

If $Q$ is a hyperhermitian matrix, there exists a matrix $A \in \text{Sp}(n)$ and a diagonal matrix $D$ with real entries such that $Q = A^* DA$. Then $\det(Q) =$
det(D). The diagonal entries in D are the (Moore-) eigenvalues of Q. More generally, if Q is hyperhermitian and A is any quaternionic matrix, then
\[ \det(A^* QA) = \det Q \det(A^* A), \]
compare [8], Thm. 1.2.9.

The Moore rank of Q is the quaternionic dimension of the image of Q, or equivalently the number of non-zero eigenvalues. Clearly the Moore rank is maximal if and only if \( \det(Q) \neq 0 \).

We will need explicit formulas for Moore determinants of small size which can be computed using the results from [11]. For \( M \lambda \) as in Theorem 1 the Moore determinant is given by
\[
\begin{align*}
det M & = 1 - \lambda_{12}^2, & k &= 2, \\
det M & = 1 - \lambda_{12}^2 - \lambda_{14}^2 - \lambda_{23}^2 + 2\lambda_{12}\lambda_{13}\lambda_{23}, & k &= 3, \\
det M & = 1 - \lambda_{12}^2 - \lambda_{14}^2 - \lambda_{23}^2 - \lambda_{24}^2 - \lambda_{34}^2 \\
& \quad + 2\lambda_{23}\lambda_{34}\lambda_{24} + 2\lambda_{12}\lambda_{23}\lambda_{13} + 2\lambda_{12}\lambda_{24}\lambda_{14} + 2\lambda_{13}\lambda_{34}\lambda_{14} & \quad + \lambda_{12}\lambda_{34}\lambda_{14} + \lambda_{13}\lambda_{24}^2 \\
& \quad - 2\lambda_{12}\lambda_{23}\lambda_{34}\lambda_{14} - 2\lambda_{12}\lambda_{24}\lambda_{13}\lambda_{34} - 2\lambda_{13}\lambda_{24}\lambda_{23}\lambda_{14}, & k &= 4.
\end{align*}
\]
For \( k = 4 \), the Moore determinants of the diagonal \( 3 \times 3 \) submatrices of \( M \lambda \) can be computed by (7) since \( \det \) is invariant under Sp(1)-conjugation.

3. Grassmann orbits

The aim of this section is the description of the orbit spaces of the action of the group \( G := \text{Sp}(2)\text{Sp}(1) \) on the Grassmann spaces \( \text{Gr}_k \). Note that \( \text{Gr}_k \cong \text{Gr}_{8-k} \), so we may assume \( k \leq 4 \). In the cases \( k = 0, 1 \), the action is transitive, so we are left with \( k = 2, 3, 4 \). Theorem 1 will follow from Theorems 3.4, 3.7 and 3.13 below.

The following propositions will be useful.

**Proposition 3.1.** Let \( Q = (q_{ij}) \) be a \( k \times k \) hyperhermitian matrix with Moore rank at most 2 and non-negative eigenvalues. Then there exist \( u_1, \ldots, u_k \in \mathbb{H}^2 \) such that
\[ K(u_i, u_j) = q_{ij} \quad \forall i, j. \]

**Proof.** We may decompose \( Q = A^* DA \) where \( A = (a_{ij}) \in \text{Sp}(k) \) and \( D = \text{diag}(\delta_1, \delta_2, 0, \ldots, 0) \). Then
\[ u_i = \left( \sqrt{\delta_1} a_{1i}, \sqrt{\delta_2} a_{2i} \right) \in \mathbb{H}^2, \quad i = 1, \ldots, k \]
are such that \( K(u_i, u_j) = q_{ij} \) for all \( i, j \). \( \square \)

**Proposition 3.2.** Let \( u_1, \ldots, u_k \in \mathbb{H}^n \) and \( v_1, \ldots, v_k \in \mathbb{H}^n \) be such that
\[ K(u_i, u_j) = K(v_i, v_j) \quad \forall i, j. \]
Then there exists \( g \in \text{Sp}(n) \) such that \( g(u_i) = v_i \) for all \( i \).

**Proof.** Let \( Q = (q_{ij}) = (K(u_i, u_j)) \), and denote by \( d \) its Moore rank. Then \( \text{span}_\mathbb{H}(u_1, \ldots, u_k) \) and \( \text{span}_\mathbb{H}(v_1, \ldots, v_k) \) have quaternionic dimension
Without loss of generality, we assume that \( u_1, \ldots, u_d \) are \( \mathbb{H} \)-linearly independent, or equivalently that 

\[
P \equiv \begin{pmatrix}
q_{11} & \cdots & q_{1d} \\
\vdots & \ddots & \vdots \\
q_{d1} & \cdots & q_{dd}
\end{pmatrix}
\]
is invertible. Then \( v_1, \ldots, v_d \) are also \( \mathbb{H} \)-linearly independent. Denoting 

\[
P^{-1} = (p^{ij}),
\]
we have for \( r = d + 1, \ldots, k \)

\[
u_r = \sum_{i,j=1}^d v_i p^{ij} q_{jr}, \quad v_r = \sum_{i,j=1}^d v_i p^{ij} q_{jr}.
\]

If \( d = n \), the \( \mathbb{H} \)-linear map \( g \) which sends \( u_i \) to \( v_i \) preserves \( K \) and hence belongs to \( \text{Sp}(n) \). If \( d < n \), we may complete \( u_1, \ldots, u_d \) (resp. \( v_1, \ldots, v_d \)) to a basis of \( \mathbb{H}^n \) by choosing \( K \)-orthonormal vectors in the quaternionic orthogonal complement of \( \text{span}_{\mathbb{H}}(u_1, \ldots, u_d) \) (resp. \( \text{span}_{\mathbb{H}}(v_1, \ldots, v_d) \)). Again, we obtain a map \( g \in \text{Sp}(n) \) which maps \( u_1, \ldots, u_d \) to \( v_1, \ldots, v_d \).

**Proposition 3.3.** Let \( V \in \text{Gr}_k \). Denote by \( \pi_V: \mathbb{H}^2 \to V \) the orthogonal projection. Given an orthonormal basis \( u_1, \ldots, u_k \) of \( V \), we define the endomorphism \( \psi_V \in \text{End}(V) \) by

\[
\psi_V(y) := \pi_V \sum_{r=1}^k u_r K(u_r, y)
\]
and set \( Q = (q_{ij})_{i,j} := (K(u_i, u_j))_{i,j} \). Then

i) \( \psi_V \) is independent of the choice of the orthonormal basis \( u_1, \ldots, u_k \) of \( V \).

ii) \( \psi_V \) is self-adjoint with respect to the euclidean scalar product on \( V \).

iii) If \( g \in \text{Sp}(2) \text{Sp}(1) \), then \( \psi_{gV} = g \psi_V g^{-1} \). In particular, the eigenvalues of \( \psi_V \) only depend on the orbit of \( V \).

iv) The matrix of \( \psi_V \) with respect to the basis \( u_1, \ldots, u_k \) is \( \text{Re} Q^2 \).

**Proof.** All claims follow from a straightforward computation. \( \square \)

We remark that the endomorphism \( \psi_V \) admits the following interpretation:

\[
\langle x, \psi_V(y) \rangle = c \int_{\text{Sp}(1)} \langle \pi_V(x \xi), \pi_V(y \xi) \rangle d\xi, \quad x, y \in V,
\]

where \( d\xi \) is the Haar measure on \( \text{Sp}(1) \) and \( c \) is a non-zero constant.

### 3.1. The quotient space \( \text{Gr}_2 / \text{Sp}(2) \text{Sp}(1) \)

**Theorem 3.4.** The quotient \( \text{Gr}_2 / \text{Sp}(2) \text{Sp}(1) \) can be homeomorphically identified with the quotient

\[
X_2 := \{ \lambda \in [-1, 1] / \{ \pm 1 \}
\]
in such a way that \( \lambda \in X_2 \) corresponds to the orbit of

\[
V = \text{span}\{ (\cos \theta_1, \sin \theta_1), (\cos \theta_2, \sin \theta_2)i \}
\]
with \( \lambda = \cos(\theta_1 - \theta_2) \).
8 ANDREAS BERNIG AND GIL SOLANES

Proof. Let \( V \subset \mathbb{H}^2 \) be a two-plane. Choose an orthonormal basis \( u_1, u_2 \) of \( V \). Then \( K(u_1, u_2) \) is purely quaternionic and its norm is bounded by \( 1 \). By using conjugation by an element \( \xi \in \text{Sp}(1) \), we may assume that \( K(u_1, u_2) = \lambda i \) for some \( \lambda \in [-1, 1] \). We send the orbit of \( V \) to \( \lambda \). It is easily checked that this map is well-defined, a homeomorphism, and fulfills the condition of the statement. \( \square \)

3.2. The quotient space \( \text{Gr}_3 / \text{Sp}(2) \text{Sp}(1) \).

Lemma 3.5. Under the hypotheses of Proposition 3.3 with \( k = 3 \), the following statements are equivalent:

i) \( u_1, u_2, u_3 \) is a basis consisting of eigenvectors.

ii) \( q_{12}, q_{13}, q_{23} \) are pairwise orthogonal.

iii) \( \text{Re} \, Q^2 \) is diagonal.

In this case, the diagonal entries of \( \text{Re} \, Q^2 \) are the eigenvalues of \( \psi_V \).

Proof. This follows easily from claim iv) in Proposition 3.3. \( \square \)

For each triple \( \lambda = (\lambda_{12}, \lambda_{13}, \lambda_{23}) \in [-1, 1]^3 \), we denote by \( M_\lambda \) the quaternionic \( 3 \times 3 \)-matrix

\[
M_\lambda := \begin{pmatrix}
1 & \lambda_{12}i & \lambda_{13}j \\
-\lambda_{12}i & 1 & -\lambda_{23}k \\
-\lambda_{13}j & \lambda_{23}k & 1
\end{pmatrix}.
\]

Let

\[
X_3 := \{ \lambda_{pq} \in [-1, 1], 1 \leq p < q \leq 3 : \text{rank } M_\lambda \leq 2 \}/(\mathbb{Z}_2^3 \times S_3),
\]

where the action of \( \mathbb{Z}_2^3 \times S_3 \) is given by equations (9). (4).

Proposition 3.6. Given \( V \in \text{Gr}_3 \), there is a unique \([\lambda] \in X_3\) such that

\[
K(u_i, u_j) = (M_\lambda)_{i,j}, \quad i, j = 1, 2, 3,
\]

for some \( u_1, u_2, u_3 \) spanning an element of the orbit of \( V \).

Proof. Let \( u_1, u_2, u_3 \in V \) be an orthonormal basis of eigenvectors of \( \psi_V \), and denote \( q_{ij} = K(u_i, u_j) \). By the previous lemma, the pure quaternions \( q_{12}, q_{13}, q_{23} \) are pairwise orthogonal. Hence there exist \( \lambda_{12}, \lambda_{13}, \lambda_{23} \in [-1, 1] \) such that \( \lambda_{12}i, \lambda_{13}j, -\lambda_{23}k \in \text{Im} \mathbb{H} \) may be mapped to \( q_{12}, q_{13}, q_{23} \) by a rotation. Let this rotation be \( q \mapsto \xi q \xi \) with \( \xi \in \text{Sp}(1) \), and let us replace \( u_i \) by \( u_i \xi \) (without changing the notation). Then, equation (9) holds. Since \( u_1, u_2, u_3 \) are linearly dependent over \( \mathbb{H} \), the hyperhermitian matrix \( M_\lambda \) has Moore rank at most 2. Hence \( \lambda = (\lambda_{12}, \lambda_{13}, \lambda_{23}) \) defines a class in \( X_3 \). This shows the existence of \([\lambda]\).

In order to show uniqueness, note that \( \text{Re} \, M_\lambda^2 \) is diagonal. Hence, by iv) of Proposition 3.3 the orthonormal basis \( u_1, u_2, u_3 \) in the statement must consist of eigenvectors of \( \psi_V \) (or of \( \psi_g \psi_V \) for some \( g \in \text{Sp}(2) \text{Sp}(1) \)).

If \( \psi_V \) has three different eigenvalues, then the only freedom in choosing these vectors is to permute them or to reflect some of them. This results in the action of the group \( \mathbb{Z}_2^3 \times S_3 \) on \( \lambda \), so \([\lambda]\) does not depend on the basis.

If, however, \( \psi_V \) has repeated eigenvalues, there are different orthonormal bases consisting of eigenvectors. Let \( u_i, u_i' \) be two such bases, related by \( u_i = a_{ij}u_j' \) with \( A = (a_{ij}) \in \text{SO}(3) \). Take \( Q = (K(u_i, u_j))_{i,j} \) and \( Q' = \cdots \)
We will show that $Q, Q'$ are Sp(1)-conjugate to each other. Hence, the corresponding matrices $M_\lambda, M_{\lambda'}$ are Sp(1)-conjugate. It is easy to check that this implies $[\lambda] = [\lambda']$.

We distinguish two cases depending on the multiplicities of the eigenvalues of $\psi_V$.

Case 1. Suppose that $\psi_V$ has exactly one double eigenvalue. By reordering the bases, we may assume that the corresponding eigenspace is $\text{span}\{u_1, u_2\} = \text{span}\{v'_1, v'_2\}$, and

$$A = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.$$ 

Then $Q' = QAQ^t$ has entries $q'_{12} = q_{12} = \lambda_{12}i$, and

$$\begin{pmatrix}
q'_{13} \\
q'_{23}
\end{pmatrix} = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix} \begin{pmatrix}
\lambda_{12} & \lambda_{13} \\
\lambda_{13} & -\lambda_{23}
\end{pmatrix}.$$

On the other hand, repetition of the eigenvalues means

$$1 + \lambda^2_{12} + \lambda_{13}^2 = 1 + \lambda^2_{12} + \lambda_{23}^2$$

which yields $\lambda_{13} = \epsilon \lambda_{23}$ for some $\epsilon = \pm 1$. Let $\zeta = \cos \frac{\pi}{2} + \epsilon \sin \frac{\pi}{2}i$. Then $Q'' = \zeta Q'\zeta$ has entries $q''_{12} = q_{12}, q''_{13} = q_{13}. q''_{23} = \epsilon q_{23}$. Since the Moore determinants of $Q, Q''$ vanish, it follows from (7) that $\epsilon = 1$ or $\lambda_{12} \lambda_{13} = 0$ or $\lambda_{13}, \lambda_{23}$. The latter case can also be reduced to $\epsilon = 1$ by changing the sign of $\lambda_{12}, \lambda_{23}$. Hence, $Q', Q$ are Sp(1)-conjugate to each other, so $[\lambda] = [\lambda']$.

Case 2. Suppose that $\psi_V$ has one triple eigenvalue. Then

$$\lambda^2_{12} + \lambda^2_{13} = \lambda^2_{12} + \lambda^2_{23} = \lambda^2_{13} + \lambda^2_{23},$$

so $\lambda_{12} = \lambda_{13} = \lambda_{23}^2$. By changing signs of $\lambda_{13}, \lambda_{23}$, we can assume that $\lambda_{12} = \lambda_{13}$. Then

$$q'_{12} = (a_{11}a_{22} - a_{12}a_{21})i + (a_{11}a_{23} - a_{13}a_{21})j + (a_{13}a_{22} - a_{12}a_{23})k.$$ 

Since $A \in \text{SO}(3)$, the wedge product of the first two rows equals the third one, hence

$$q'_{12} = a_{33}i - a_{32}j - a_{31}k.$$ 

Similarly,

$$q'_{13} = -a_{23}i + a_{22}j + a_{21}k,$$

$$q'_{23} = a_{13}i - a_{12}j - a_{11}k.$$ 

Hence, each $q'_{ij}$ with $i \neq j$ is the image of $q_{ij}$ under a common rotation of $\mathbb{R}^3 \equiv \text{Im} \mathbb{H}$. Therefore, $Q'$ is an Sp(1)-conjugate of $Q$, and $[\lambda] = [\lambda']$.

\[ \square \]

**Theorem 3.7.** There exists a homeomorphism $X_3 \cong \text{Gr}_3 / \text{Sp}(2)\text{Sp}(1)$ mapping $[\lambda] \in X_3$ to the orbit of a plane spanned by $v_1, v_2, v_3$ such that

$$K(v_i, v_j) = (M_\lambda)_{i,j}, \quad i, j = 1, 2, 3.$$
Proof. Given $V \in \text{Gr}_3$, let $[\lambda] \in X_3$ be given by Proposition 3.6. Clearly $[\lambda]$ only depends on the $\text{Sp}(2)\text{Sp}(1)$-orbit of $V$ in $\text{Gr}_3$. Hence, $V \mapsto [\lambda]$ defines a map $\Phi : \text{Gr}_3 / \text{Sp}(2)\text{Sp}(1) \to X_3$.

Let us show that $\Phi$ is bijective. To show injectivity, suppose that $U, V \in \text{Gr}_3$ are mapped to the same $[\lambda] \in X_3$. This means that $U$ and $V$ admit respective bases $u_1, u_2, u_3$ and $v_1, v_2, v_3$, such that

$$K(u_\zeta, u_\xi) = K(v_\zeta, v_\xi) = M_\lambda$$

for certain $\zeta, \xi \in \text{Sp}(1)$. By Proposition 3.2, there exists $g \in \text{Sp}(2)$ such that $g(u_\zeta) = u_\xi$. Hence $V = g(U)\zeta\xi$, so $U$ and $V$ belong to the same $\text{Sp}(2)\text{Sp}(1)$-orbit.

To see surjectivity, it is enough to apply Proposition 3.1 with $Q = M_\lambda$.

Since $\text{Gr}_3$ is compact and $X_3$ is Hausdorff, it remains only to prove that $\Phi$ is continuous.

Let $(V^m)$ be a sequence of 3-planes converging to the 3-plane $V$ in $\text{Gr}_3$. Let $(u_1^m, u_2^m, u_3^m)$ be an orthonormal basis of $V^m$ and $\lambda^m = (\lambda_{12}^m, \lambda_{13}^m, \lambda_{23}^m)$ as in Proposition 3.6. By compactness, there exists a subsequence $m_1, m_2, \ldots$ such that $(u_1^{m_i}, u_2^{m_i}, u_3^{m_i})$ converges to an orthonormal basis $(u_1, u_2, u_3)$ of $V$. Hence $\lambda^m \to \lambda$ for some $\lambda = (\lambda_{12}, \lambda_{13}, \lambda_{23})$. Then $\Phi(V) = [\lambda]$ and it follows that $\Phi(V^{m_i})$ converges to $\Phi(V)$.

Since we may apply the same argument to any subsequence of a given sequence, we obtain the following: every subsequence of $(V_m)$ contains a subsequence such that the images under $\Phi$ converge to $\Phi(V)$. But this implies that the images under $\Phi$ of the original sequence converge to $\Phi(V)$.

Corollary 3.8. Given $[\lambda] \in X_3$, there exist $\theta_1, \theta_2, \theta_3$ such that

$$\lambda_{ij} = \cos(\theta_i - \theta_j),$$

and the orbit corresponding to $[\lambda]$ contains the plane

$$V = \text{span}\{ \cos \theta_1, \sin \theta_1, \cos \theta_2, \sin \theta_2, \cos \theta_3, \sin \theta_3 \}. \jmath.$$

Proof. By Theorem 3.7, the orbit corresponding to $[\lambda]$ contains a plane $V$ admitting an orthonormal basis $v_1, v_2, v_3$ such that $K(v_1, v_2) = (M_\lambda)_{i,j}$. Since $\text{Sp}(2)$ acts transitively on the unit sphere of $\mathbb{H}^2$, we can assume $v_1 = (1, 0)$. From $K(v_1, v_2) = \lambda_{12}i$, we deduce that $v_2 = (\lambda_{12}i, w)$ for some $w \in \mathbb{H}$. By applying an element of $\text{Sp}(1)$ to the second component of $\mathbb{H}^2$, we may assume that $w$ and $i$ are parallel, $w \parallel i$. Together with $K(v_2, v_3) = \lambda_{23}j$, this implies that $v_3 = (aj, bj)$ for some $a, b \in \mathbb{R}$. Therefore, $V$ agrees with the given description. \hfill \square

3.3. The quotient space $\text{Gr}_4 / \text{Sp}(2)\text{Sp}(1)$.

Let $V \subset \mathbb{H}^2$ be a 4-plane. Given an orthonormal basis $u_1, \ldots, u_4$ of $V$, we set $Q := (K(u_p, u_q))_{p,q}$. Clearly the Moore rank of $Q$ is at most 2 and $\text{tr} Q = 4$. We call $V$ degenerated if $Q$ has Moore eigenvalues $(2, 2, 0, 0)$ and non-degenerated otherwise. This notion is independent of the choice of the orthonormal basis.

Note that if $\text{Re} Q^2 = 2 \text{Id}$ (which is equivalent to $\psi_V = 2 \text{Id}$), then $Q$ is degenerated. Indeed, if $\lambda, 4 - \lambda$ are the non-zero Moore eigenvalues of $Q$, then $\lambda^2 + (4 - \lambda)^2 = \text{tr} Q^2 = 8$ which implies that $\lambda = 2$. 

□
**Lemma 3.9.** Non-degenerated planes are dense in $\text{Gr}_4$.

**Proof.** Consider the continuous map which sends $g \in \text{SO}(8)$ to the plane $V$ spanned by the first four columns in $\mathbb{R}^8 \cong \mathbb{H}^2$. Let $u_1, \ldots, u_8$ be the columns of $g$ and $Q := (K(u_p, u_q))_{p,q}$. Then $V$ is non-degenerated if and only if $\text{tr}Q^2 \neq 8$. Clearly the function $\text{tr}Q^2 - 8$ is a polynomial function on the irreducible algebraic variety $\text{SO}(8)$. Since this function does not vanish identically on $\text{SO}(8)$, its zero set does not contain any open set. \hfill \Box

**Proposition 3.10.** In each $\text{Sp}(2)\text{Sp}(1)$-orbit of $\text{Gr}_4$ there is an element with an orthonormal basis $v_1, v_2, v_3, v_4$ such that each $v_i = (v_{i1}, v_{i2}) \in \mathbb{H}^2$ has parallel components; i.e., $v_{i1} \| v_{i2}$ as vectors of $\mathbb{H} \equiv \mathbb{R}^4$ for $i = 1, \ldots, 4$.

**Proof.** By Lemma 3.9 non-degenerated 4-planes are dense in $\text{Gr}_4$. By continuity it is enough to prove the statement for non-degenerated planes.

Let $V \in \text{Gr}_4$ be non-degenerated and let $u_1, \ldots, u_4$ be a basis consisting of eigenvectors of $\psi_V$. Define

$$Q := (K(u_m, u_l))_{m,l=1,\ldots,4}.$$ 

Since $u_1, \ldots, u_4$ are eigenvectors of $\psi_V$, the matrix $\text{Re}Q^2$ is diagonal. Moreover, $\text{tr}Q = 4$ and the Moore rank of $Q$ is at most 2. We can therefore write $Q = A^*D A$, where $A = (a_{ij}) \in \text{Sp}(2)$ and $D = \text{diag}(\delta, 4 - \delta, 0, 0)$, $\delta \in [0, 4]$. Since $V$ is non-degenerated, we have $\delta \neq 2$, hence $\text{Re}Q^2 \neq 2 \text{Id}$.

We claim that $a_{1m}, m = 1, \ldots, 4$ are pairwise orthogonal in $\mathbb{H}$, and the same holds for $a_{2m}, m = 1, \ldots, 4$. For instance, we have

$$q_{12} = \delta a_{11}a_{12} + (4 - \delta)\bar{a}_{21}a_{22}$$
and

$$(Q^2)_{12} = \delta^2 a_{11}a_{12} + (4 - \delta)^2\bar{a}_{21}a_{22}.$$  
The real part of these two quaternions vanishes if and only if $a_{11}a_{12}$ and $\bar{a}_{21}a_{22}$ are pure quaternions (here we use that $\delta \neq 2$).

The matrix $A$ can be left multiplied by a diagonal matrix with entries in $\text{Sp}(1)$ and $Q$ remains unchanged. Since this action is transitive on the unit sphere in each summand of $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$, we can assume that $a_{14}, a_{24} \in \mathbb{R}^+$. Also, we can conjugate $A$ by an element $\xi \in \text{Sp}(1)$. The effect is that also $Q$ is conjugated by $\xi$, which is equivalent to multiplying $V$ by $\xi$ from the right.

The vectors $(\sqrt{\delta}a_{1m}, \sqrt{4 - \delta}a_{2m}), m = 1, \ldots, 4$ form an orthonormal basis of a 4-plane in the same orbit as $V$. We may therefore assume that $V$ is spanned by the vectors

$$u_1 = (\sqrt{\delta}a_{11}, \sqrt{4 - \delta}a_{21}) := (\cos \theta_1 \mathbf{i}, \sin \theta_1 w_1), \quad (10)$$
$$u_2 = (\sqrt{\delta}a_{12}, \sqrt{4 - \delta}a_{22}) := (\cos \theta_2 \mathbf{j}, \sin \theta_2 w_2), \quad (11)$$
$$u_3 = (\sqrt{\delta}a_{13}, \sqrt{4 - \delta}a_{23}) := (\cos \theta_3 \mathbf{k}, \sin \theta_3 w_3), \quad (12)$$
$$u_4 = (\sqrt{\delta}a_{14}, \sqrt{4 - \delta}a_{24}) := (\cos \theta_4, \sin \theta_4), \quad (13)$$

where $w_1, w_2, w_3$ is an orthonormal basis of $\mathbb{R}^3 \equiv \text{Im} \mathbb{H}$.

By changing the sign of some $u_m$, $w_m$ we can suppose that $0 \leq \theta_1, \ldots, \theta_4 \leq \frac{\pi}{2}$. 

Since $A \in \text{Sp}(2)$, we have $\sum \bar{a}_{1m}a_{2m} = 0$, i.e.
\[
\sin(2\theta_4) - \sin(2\theta_1)i \cdot w_1 - \sin(2\theta_2)j \cdot w_2 - \sin(2\theta_3)k \cdot w_3 = 0. \tag{14}
\]

Considering the imaginary part we deduce
\[
\sin(2\theta_m)w_{mn} = \sin(2\theta_n)w_{nm}, \quad m, n = 1, 2, 3.
\]
where $w_{mn}$ are the coordinates of $w_m$ with respect to the basis $i, j, k$ of $\mathbb{R}^3$; i.e., the matrix $M = (\sin(2\theta_m)w_{mn})_{m,n=1,2,3}$ is symmetric. Let $d_m := \sin 2\theta_m$, $D := \text{diag}(d_1, d_2, d_3)$ and $O := (w_1, w_2, w_3) \in O(3)$. Then $M = DO$ and hence $DO = O'D, OD = DO'$.

Therefore $OD^2 = DO' D = D^2O$, i.e.
\[
(d_i^2 - d_j^2)a_{ij} = 0.
\]

We consider three cases according to the multiplicities of the entries in $D$.

**Case 1.** If $\#\{d_i\} = 3$ then $O$ is diagonal and the statement is trivial.

**Case 2.** $\#\{d_i\} = 2$ and $O$ contains a row with zeros outside the diagonal position, i.e. up to a simultaneous reordering of rows and columns, $D$ and $O$ have the form
\[
D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad O = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = \pm 1.
\]

After reordering $u_1, u_2, u_3$ and conjugating by a suitable element of $\text{Sp}(1)$ we have
\[
\begin{align*}
    u_1 &= (\cos \theta_1 i, \sin \theta_1 (\cos \alpha i + \sin \alpha j)) \\
    u_2 &= (\cos \theta_2 j, \sin \theta_2 (\sin \alpha i - \cos \alpha j)) \\
    u_3 &= (\cos \theta_3 k, \varepsilon \sin \theta_3 k) \\
    u_4 &= (\cos \theta_4, \sin \theta_4)
\end{align*}
\]
with $\sin 2\theta_1 = \sin 2\theta_2$. Thus, either $\theta_2 = \theta_1$ or $\theta_2 = \frac{\pi}{2} - \theta_1$.

By considering the real part of (14) we deduce $\sin 2\theta_3 = \sin 2\theta_4$ and $\varepsilon = -1$.

We consider three cases.

- **If** $\theta_3 = \theta_1$, **we set** $u'_i := \cos \frac{\pi}{2} u_1 + \sin \frac{\pi}{2} u_2, u'_2 := -\sin \frac{\pi}{2} u_1 + \cos \frac{\pi}{2} u_2, u'_3 = u_3, u'_4 = u_4$. Then, the first and second components of $u'_i \in \mathbb{H}^2$ are parallel for each $1 \leq i \leq 4$.

- **If** $\theta_3 = \theta_4$, **we set** $u'_1 := u_1, u'_2 := u_2, u'_3 := \cos \frac{\pi}{2} u_3 + \sin \frac{\pi}{2} u_3, u'_4 := -\sin \frac{\pi}{2} u_3 + \cos \frac{\pi}{2} u_4$. Again we obtain an orthonormal basis of $V$ that satisfies the statement.

- **If** $\theta_2 = \frac{\pi}{2} - \theta_1$ and $\theta_4 = \frac{\pi}{2} - \theta_3$, **then one checks that** $\text{Re}(Q^2) = 2 \text{Id}$, contradicting our assumption.

**Case 3.** $D$ is a multiple of the identity.

Then $\sin 2\theta_m = c \neq 0$ for $m = 1, 2, 3$. The real part of (14) is
\[
\sin 2\theta_4 + c \text{tr}O = 0.
\]

Since $O$ is orthogonal and diagonalizable, it has eigenvalues $1, 1, 1$ or $1, -1, -1$ or $1, -1, 1$ or $-1, -1, -1$. In the first and last cases, $O$ is diagonal and we are done. Otherwise $\text{tr}O = \pm 1$. Since $\sin 2\theta_m \geq 0$, we deduce that $\text{tr}O = -1$, i.e. $O$ has eigenvalues $1, -1, -1$, and $\sin 2\theta_4 = c$. 


Therefore every two angles \( \theta_m, \theta_n, 1 \leq m, n \leq 4 \) are equal or complementary. If \( \theta_1, \ldots, \theta_4 \) contain exactly two pairs of equal angles, then one checks that \( \text{Re}(Q^2) = 2 \text{Id} \), again contradicting our assumption. Hence at least three angles \( \theta_m \) are equal. By reordering, we may assume that \( \theta_1 = \theta_2 = \theta_3 \). Then we write

\[
O = P^t \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} P,
\]

where \( P \in O(3) \) and set

\[
\begin{pmatrix}
 u'_1 \\
 u'_2 \\
 u'_3
\end{pmatrix} := P \begin{pmatrix}
 u_1 \\
 u_2 \\
 u_3
\end{pmatrix} \quad u'_4 := u_4.
\]

Then, the first and second components of each \( u'_i \) are parallel vectors in \( \mathbb{H} \). \( \square \)

**Corollary 3.11.** Every \( V \in \text{Gr}_4 \) admits an orthonormal basis \( u_1, \ldots, u_4 \) such that \( q_{ij} = K(u_i, u_j) \) satisfy

- \( q_{12}, q_{13}, q_{23} \) are pairwise orthonormal
- \( q_{12} \parallel q_{34}, q_{13} \parallel q_{24}, q_{14} \parallel q_{23} \).

**Proof.** It is enough to check the statement for one plane in each \( \text{Sp}(2) \text{Sp}(1) \)-orbit of \( \text{Gr}_4 \). By the previous proposition, we may assume that \( V \) admits an orthonormal basis \( u_1, \ldots, u_4 \) with \( u_1, u_2 \) both parallel to some \( \xi_i \in \mathbb{H} \setminus \{0\} \) for each \( i \). Since \( u_1, \ldots, u_4 \) are orthogonal, so are \( \xi_1, \ldots, \xi_4 \). Since \( q_{ij} \parallel \xi_i \xi_j \), we get \( q_{ij} \perp q_{ik} \) if \( j \neq k \). The statement follows. \( \square \)

Given \( \lambda_{pq} \in [-1, 1], 1 \leq p < q \leq 4 \), we define the quaternionic matrix

\[
M_\lambda := \begin{pmatrix}
1 & \lambda_{12}i & \lambda_{13}j & \lambda_{14}k \\
-\lambda_{12}i & 1 & -\lambda_{23}k & \lambda_{24}j \\
-\lambda_{13}j & \lambda_{23}k & 1 & -\lambda_{34}i \\
-\lambda_{14}k & -\lambda_{24}j & \lambda_{34}i & 1
\end{pmatrix}
\]

Let

\[
X_4 := \{ \lambda_{pq} \in [-1, 1], 1 \leq p < q \leq 4 : \text{rank } M_\lambda \leq 2 \}/(\mathbb{Z}_2^4 \times S_4),
\]

where the action of \( \mathbb{Z}_2^4 \times S_4 \) is given by equations (2),(4).

**Proposition 3.12.** Given \( V \in \text{Gr}_4 \), there is a unique \( [\lambda] \in X_4 \) such that \( K(u_i, u_j) = (M_\lambda)_{i,j} \), \( i, j = 1, 2, 3, 4 \), for some \( u_1, \ldots, u_4 \) spanning an element of the orbit of \( V \).

**Proof.** Let \( u_1, \ldots, u_4 \) be given by the previous corollary. Using a rotation \( q \mapsto \xi q \xi \), we may map \( q_{12} \) to a multiple of \( i \), \( q_{13} \) to a multiple of \( j \) and \( q_{14} \) to
a multiple of \( k \). For \( i = 1, \ldots, 4 \) take \( u_i \xi \) and denote it again by \( u_i \). Then,

\[
K(u_1, u_2) = \lambda_{12}i \\
K(u_1, u_3) = \lambda_{13}j \\
K(u_1, u_4) = \lambda_{14}k \\
K(u_2, u_3) = -\lambda_{23}k \\
K(u_2, u_4) = \lambda_{24}j \\
K(u_3, u_4) = -\lambda_{34}i
\]

for real numbers \( \lambda_{pq} \in [-1, 1], 1 \leq p < q \leq 4 \). Since any 3 vectors in \( \mathbb{H}^2 \) are linearly dependent over \( \mathbb{H} \), the rank of the matrix \( Q := M_\lambda \) is at most 2. This shows the existence part of the statement.

In order to prove uniqueness, let \( A = (a_{ij}) \in SO(4) \) and suppose that \( u'_i = a_{ij}u_j \) is another basis of \( V \) such that \( Q' = AQ A'^t \) is Sp(1)-conjugate to \( (M_\lambda')_{ij} \) for some \( |\lambda'| \in X_4 \). Then \( \text{Re}Q'^2, \text{Re}(Q')^2 \) are both diagonal. By Proposition 3.3, the orthonormal bases \( u_1, \ldots, u_4 \) and \( u'_1, \ldots, u'_4 \) consist both of eigenvectors of \( \psi_V \). We need to show that \( Q, Q' \) are Sp(1)-conjugates of each other, which will imply that \( |\lambda| = |\lambda'| \).

If \( \psi_V \) has no multiple eigenvalues, then the two bases coincide up to signs and order. Hence \( |\lambda| = |\lambda'| \).

Next we consider different cases according to the multiplicities of the eigenvalues of \( \psi_V \).

**Case 1.** Suppose that \( \psi_V \) has exactly one double eigenvalue. By re-ordering the bases, we may assume that the corresponding eigenspace is \( \text{span}\{u_1, u_2\} = \text{span}\{u'_1, u'_2\} \), and

\[
A = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Then \( Q' = AQ A'^t \) has entries \( q'_{12} = q_{12}, q'_{34} = q_{34} \), and

\[
\begin{pmatrix}
q'_{13} & q'_{14} \\
q'_{23} & q'_{24}
\end{pmatrix}
= \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\lambda_{13}j & \lambda_{14}k \\
-\lambda_{23}k & \lambda_{24}j
\end{pmatrix}
\]

Our assumption is that each row and each column in \( Q' \) has orthogonal entries. This implies that either \( \sin \alpha \cos \alpha = 0 \), in which case everything follows trivially, or \( \lambda_{13} = \epsilon \lambda_{23}, \lambda_{14} = \epsilon \lambda_{24} \) for some \( \epsilon = \pm 1 \). Since the \( 3 \times 3 \) upper left minors of \( Q, Q' \) vanish, we have \( \epsilon = 1 \) (except if \( \lambda_{13} \lambda_{23} = 0 \), in which case we may choose \( \epsilon = 1 \) as well). It follows that \( Q' = \zeta Q \zeta \) with \( \zeta = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} i \).

**Case 2.** Suppose that \( \psi_V \) has two different double eigenvalues. We may assume that \( A \) has the form

\[
A = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 & 0 \\
-\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & \sin \beta \\
0 & 0 & -\sin \beta & \cos \beta
\end{pmatrix}
\]

Then \( \lambda_{13}^2 + \lambda_{14}^2 = \lambda_{23}^2 + \lambda_{24}^2 \) as well as \( \lambda_{13}^2 + \lambda_{14}^2 = \lambda_{14}^2 + \lambda_{14}^2 \), which implies that \( \lambda_{13}^2 = \lambda_{24}^2 \) and \( \lambda_{14}^2 = \lambda_{23}^2 \).
By changing some sign if necessary, we may assume that $\lambda_{13} = \lambda_{24}$. The
rank 2 condition of $Q$ leads to $\lambda_{14} = \lambda_{23}$ or $\lambda_{13}\lambda_{14} = 0$ or $\lambda_{12} = \lambda_{14} = 0$.
The third possibility is excluded by the assumption that the eigenvalues are
different, and the second one also allows to suppose $\lambda_{14} = \lambda_{23}$.

The upper right square of $Q$ is thus given by

$$
\begin{pmatrix}
q_{13} & q_{14} \\
q_{23} & q_{24}
\end{pmatrix} = \lambda
\begin{pmatrix}
\cos(\theta)j & \sin(\theta)k \\
-\sin(\theta)k & \cos(\theta)j
\end{pmatrix},
$$

where $\lambda := \sqrt{\lambda_{13}^2 + \lambda_{14}^2}$. The upper right square of $Q'$ is

$$
\lambda
\begin{pmatrix}
\cos(\theta)\cos(\alpha - \beta)j - \sin(\theta)\sin(\alpha - \beta)k & \cos(\theta)\sin(\alpha - \beta)j + \sin(\theta)\cos(\alpha - \beta)k \\
-\cos(\theta)\sin(\alpha - \beta)j - \sin(\theta)\cos(\alpha - \beta)k & \cos(\theta)\cos(\alpha - \beta)j - \sin(\theta)\sin(\alpha - \beta)k
\end{pmatrix}.
$$

The assumption that rows and columns have orthogonal entries implies that
either $2\alpha - 2\beta$ is a multiple of $\pi$, or $\sin^2 \theta = \cos^2 \theta$. In the first case, one
checks easily that $Q'$ is related to $Q$ by an element of $\mathbb{Z}_2 \times S_3$.

Next, suppose that $\sin^2 \theta = \cos^2 \theta = \frac{1}{2}$. In this case $Q$ and $Q'$ differ only
by a rotation in the plane spanned by $\{j, k\}$.

**Case 3.** Suppose that $\psi_V$ has a triple eigenvalue, say corresponding to
the first three vectors of each basis. Then $A \in SO(3) \subset SO(4)$, and

$$
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \lambda_4^2 + \lambda_2^2 + \lambda_3^2.
$$

Putting $P = (q_{14}, q_{24}, q_{34})^t = (\lambda_{14}k, \lambda_{24}j, -\lambda_{34}i)^t$ we have

$$
PP^* = \begin{pmatrix}
\lambda_{14}^2 & 0 & 0 \\
0 & \lambda_{24}^2 & 0 \\
0 & 0 & \lambda_{34}^2
\end{pmatrix} =: D.
$$

By assumption, $P' = (q'_{14}, q'_{24}, q'_{34})^t$ has orthogonal entries. Since $P' = AP$
we deduce that $D' := P'(P'^*)^t = ADA'$ is diagonal. After multiplication of
$A$ by a permutation matrix, we can assume $D' = D$.

From $AD = DA$ we get three possibilities: either $\lambda_{14}^2, \lambda_{24}^2, \lambda_{34}^2$ has no
repetitions and $A$ is the identity, or $\# \{\lambda_{14}^2, \lambda_{24}^2, \lambda_{34}^2\} = 2$ and $A$ is a rotation
in some 2-plane (this case can be handled as Case 1), or $\lambda_{14}, \lambda_{24}, \lambda_{34}$ have the
same absolute value $\mu$. From the equations above it follows that $\lambda_{12}, \lambda_{13}, \lambda_{23}$
also have the same absolute value $\tau$. We may assume that $\lambda_{12}, \lambda_{13}, \lambda_{14} \geq 0$.
Then $\lambda_{23} = \pm \tau, \lambda_{24} = \pm \mu, \lambda_{34} = \pm \mu$.

Since the upper $3 \times 3$ minor of $Q$ must vanish, we obtain from (7) that
$\tau \in \{\pm 1, \pm \frac{1}{2}\}$. Checking all possible combinations, the only matrices of this
type of rank 2 are

$$
Q = \begin{pmatrix}
1 & i & j & \mu k \\
-\mu k & 1 & -k & \mu j \\
-i & \mu j & 1 & -\mu i \\
-j & k & \mu i & 1
\end{pmatrix},
$$

where $\mu$ is arbitrary. The rest of the proof in this case is analogous to Case
2 in the proof of Proposition 3.6.

**Case 4.** Suppose that all eigenvalues of $\psi_V$ are the same. Then

$$
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = \lambda_4^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + \lambda_2^4 + \lambda_3^2,
$$

which implies that $\lambda_{23} = \epsilon_1 \lambda_{14}, \lambda_{24} = \epsilon_2 \lambda_{13}, \lambda_{34} = \epsilon_3 \lambda_{12}$ with $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3$. Using the fact that $Q$ has Moore rank 2 yields two possibilities
The conjugation of a matrix of this form by $A \in SO(4)$ can be described as follows. Let $\Lambda^2 \mathbb{R}^4$ be the $(-1)$-eigenspace of the Hodge operator $\ast : \Lambda^2 \mathbb{R}^4 \to \Lambda^2 \mathbb{R}^4$. We identify $\Lambda^2 \mathbb{R}^4$ with $\mathbb{R}^3$ by choosing the orthonormal basis $e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3$. The action of $SO(4)$ on $\Lambda^2 \mathbb{R}^4$ preserves $\Lambda^2 \mathbb{R}^4 \cong \mathbb{R}^3$, which yields a map $\rho : SO(4) \to SO(3)$.

Now consider real $4 \times 4$-matrices of the form

$$P := \begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} \\
-x_{12} & 1 & x_{23} & x_{24} \\
-x_{13} & -x_{23} & 1 & x_{34} \\
-x_{14} & -x_{24} & -x_{34} & 1
\end{pmatrix}$$

and set $\iota(P) := \sum_{1 \leq i < j \leq 4} x_{ij} e_i \wedge e_j \in \Lambda^2 \mathbb{R}^4$. Then $\iota(P) \in \Lambda^2 \mathbb{R}^4$ if and only if $x_{34} = -x_{12}, x_{24} = x_{13}, x_{23} = -x_{14}$. In this case, $\iota(APA^t) = \rho(A)(\iota(P))$ for $A \in SO(4)$.

Tensorizing everything with $\mathbb{R}^3 = \text{Im } \mathbb{H}$ we conclude that $Q' = AQ A^t$ has the same form as $Q$ and

$$\begin{pmatrix}
q'_{12} \\
q'_{13} \\
q'_{14}
\end{pmatrix} = \rho(A) \begin{pmatrix}
q_{12} \\
q_{13} \\
q_{14}
\end{pmatrix}.$$

Hence, $Q'$ is obtained by applying a rotation of $\mathbb{R}^3$ to the purely quaternionic coefficients of $Q$; i.e. $Q$ and $Q'$ are $\text{Sp}(1)$-conjugates of each other.

In case ii), after reordering indices we may suppose $\lambda_{12} = \lambda_{34} = 0$. From the rank 2 condition we also have

$$\lambda_{13}^2 + \lambda_{14}^2 = 1, \quad (\epsilon_1 \lambda_{14}^2 - \epsilon_2 \lambda_{13}^2)^2 = 1.$$

Hence, $\lambda_{13} = \cos \theta, \lambda_{14} = \sin \theta$ for some $\theta$. Moreover, the second equation yields $\epsilon_1 \epsilon_2 = -1$ or $\sin \theta \cos \theta = 0$. In both cases, after the action of $\mathbb{Z}_2^3$ we can assume $\lambda_{13} = \lambda_{24} = \cos \theta$ and $\lambda_{14} = -\lambda_{23} = \sin \theta$. The matrix $M_\lambda$ is then given by

$$M_\lambda = \begin{pmatrix}
1 & 0 & \cos \theta j & \sin \theta k \\
0 & 1 & \sin \theta k & \cos \theta j \\
-\cos \theta j & -\sin \theta k & 1 & 0 \\
-\sin \theta k & -\cos \theta j & 0 & 1
\end{pmatrix}.$$

Up to permutations, $M_\lambda$ has the same form possibly with a different $\theta$.

The function

$$W \mapsto \min_{u \in W, |u| = 1} \max_{\xi \in S^3 \cap \text{Im } \mathbb{H}} |\pi_W(u \cdot \xi)|$$

is a $\text{Sp}(2)\text{Sp}(1)$-invariant function on $Gr_4$. It is easily checked that it assumes the value $\max\{|\cos \theta|, |\sin \theta|\}$ on the plane $V$. The proof is completed.
by noting that the equivalence class of $|\lambda|$ only depends on $\max\{|\cos \theta|,|\sin \theta|\}$.

\begin{proposition}
There exists a homeomorphism $X_4 \cong \text{Gr}_4 / \text{Sp}(2) \text{Sp}(1)$ mapping $[\lambda] \in X_4$ to the orbit of a plane spanned by $v_1, \ldots, v_k$ such that

$$K(v_i, v_j) = (M_\lambda)_{i,j}, \quad i, j = 1, \ldots, 4.$$ 

The proof is exactly as in Theorem 3.7.
\end{proposition}

\begin{corollary}
Given $[\lambda] \in X_k$, there exist $\theta_1, \ldots, \theta_4$ such that $\lambda_{ij} = \cos(\theta_i - \theta_j)$, and the orbit corresponding to $[\lambda]$ contains the plane $V = \text{span}\{(\cos \theta_1, \sin \theta_1), (\cos \theta_2, \sin \theta_2)i, (\cos \theta_3, \sin \theta_3)j, (\cos \theta_4, \sin \theta_4)k\}$.

The proof is analogous to that of Corollary 3.8.
\end{corollary}

\section{Irreducible representations of $\text{SO}(n)$}

It is well-known that equivalence classes of complex irreducible (finite-dimensional) representations of $\text{SO}(n)$ are indexed by their highest weights. The possible highest weights are tuples $(\lambda_1, \lambda_2, \ldots, \lambda_{[n/2]})$ of integers such that

i) $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{[n/2]} \geq 0$ if $n$ is odd,

ii) $\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_{n/2}| \geq 0$ if $n$ is even.

We will write $\Gamma_\lambda$ for any isomorphic copy of an irreducible representation with highest weight $\lambda$. As in [10], if $n$ is even and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{[n/2]})$ then we set $\lambda' := (\lambda_1, \lambda_2, \ldots, -\lambda_{[n/2]})$. It will be useful to use the following notation:

$$\tilde{\Gamma}_\lambda := \begin{cases} 
\Gamma_\lambda & n \text{ odd or } \lambda_{[n/2]} = 0 \\
\Gamma_\lambda \oplus \Gamma_{\lambda'} & n \text{ even and } \lambda_{[n/2]} \neq 0.
\end{cases}$$

The following proposition is well-known, compare [35, 36] and ([28, Lemma 5.3]).

\begin{proposition}
Let $\text{Gr}_k(\mathbb{R}^n)$ denote the Grassmann manifold consisting of all $k$-dimensional subspaces in $\mathbb{R}^n$. The $\text{SO}(n)$-module $L^2(\text{Gr}_k(\mathbb{R}^n))$ decomposes as

$$L^2(\text{Gr}_k(\mathbb{R}^n)) \cong \bigoplus_{\lambda} \Gamma_\lambda,$$

where $\lambda$ ranges over all highest weights such that $\lambda_i = 0$ for $i > \min\{k, n-k\}$ and such that all $\lambda_i$ are even. In particular, it is multiplicity-free.

Let $\Gamma_\lambda$ be an irreducible representation of $\text{SO}(n)$ appearing in $L^2(\text{Gr}_k(\mathbb{R}^n))$. By Schur's lemma, the Laplacian $\Delta$ acts by multiplication by some scalar, which was computed by James-Constantine [20]. We will follow the convention $\Delta f := -\text{div} \circ \nabla f$. 

Proposition 4.2. The Laplace-Beltrami operator $\Delta$ of $\text{Gr}_k(\mathbb{R}^n)$ acts on $\Gamma_\lambda$ by the scalar

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \lambda_i (\lambda_i - 2i + n).$$

We will also need the decomposition of $\text{Val}_k$ as a sum of irreducible $\text{SO}(n)$-modules, which was obtained recently in [10].

Proposition 4.3. The $\text{SO}(n)$-module $\text{Val}_k$ decomposes as

$$\text{Val}_k \cong \bigoplus_{\lambda} \Gamma_{\lambda},$$

where $\lambda$ ranges over all highest weights such that $|\lambda_2| \leq 2$, $|\lambda_i| \neq 1$ for all $i$ and $\lambda_i = 0$ for $i > \min\{k, n-k\}$. In particular, it is multiplicity-free.

5. The Laplacian on the Grassmann manifold

In this section $\pi : \text{SO}(8) \rightarrow \text{Gr}_k$ denotes the projection mapping each matrix to the plane spanned by its first $k$ columns. We also let $S^1$ be the unit circle and define $\Phi : (S^1)^4 \rightarrow \text{SO}(8)$ by

$$\Phi(\theta_1, \ldots, \theta_4) := \begin{pmatrix} C & S \\ S & C \end{pmatrix} \in \text{SO}(8),$$

where

$$C := \begin{pmatrix} \cos \theta_1 & 0 & 0 & 0 \\ 0 & \cos \theta_2 & 0 & 0 \\ 0 & 0 & \cos \theta_3 & 0 \\ 0 & 0 & 0 & \cos \theta_4 \end{pmatrix}, \quad S := \begin{pmatrix} \sin \theta_1 & 0 & 0 & 0 \\ 0 & \sin \theta_2 & 0 & 0 \\ 0 & 0 & \sin \theta_3 & 0 \\ 0 & 0 & 0 & \sin \theta_4 \end{pmatrix}.$$  \hfill (21)

The image of $\Phi$ is a maximal torus of $\text{SO}(8)$. We denote by $T$ the projection of this torus to $\text{Gr}_k$, which is a flat totally geodesic submanifold of dimension $k$. By Corollaries 3.8 and 3.14 each $\text{Sp}(2) \text{Sp}(1)$-orbit has non-empty intersection with $T$.

Proposition 5.1. Each $\text{Sp}(2) \text{Sp}(1)$-orbit intersects $T$ orthogonally along a curve of the form $c(t) = \pi \circ \Phi(\theta_1 + t, \ldots, \theta_4 + t)$; i.e. the tangent space to $T$ at $c(t)$ is spanned by $c'(t)$ and a collection of vectors orthogonal to the orbit $\text{Sp}(2) \text{Sp}(1) \cdot c(t)$.

Proof. By Corollary 1.1 the curve $c$ is contained in a single orbit. It remains to show that the intersection of an orbit with $T$ is orthogonal. Let us take the following basis of $\mathfrak{g} = T_e \text{Sp}(2) \text{Sp}(1)$, viewed as a subspace of $\mathfrak{so}_8$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} L_q & 0 \\ 0 & L_q \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_q & 0 \\ L_q & 0 & R_q \end{pmatrix}, \begin{pmatrix} R_q & 0 \\ 0 & 0 \\ R_q \end{pmatrix}, \quad q = i, j, k$$

where $L_q, R_q \in \text{End}_\mathbb{R}(\mathbb{H}) = \text{End}_\mathbb{R}(\mathbb{R}^4)$ correspond to left and right multiplication by $q$ respectively. Let $N_i = \frac{\partial \Phi}{\partial \theta_i} - \frac{\partial \Phi}{\partial \theta_{i+1}}$, $1 \leq i \leq 3$ be bi-invariant vector fields defined on the maximal torus of $\text{SO}(8)$. These vectors, together with the vector $\sum \frac{\partial \Phi}{\partial \theta_i}$, span the tangent space at each point of the maximal torus.
It is straightforward to check that \((N_i)_e\) is orthogonal to \(g\), with respect to the Killing form of \(\mathfrak{so}_8\). By right-invariance, \((N_i)_g \perp g\) for every \(g\) in the maximal torus. Since \(N_i \perp \ker d\pi\), and \(\pi\) is a riemannian submersion, we deduce that \((d\pi)_g N_i\) is orthogonal to the orbit \(\text{Sp}(2)\ \text{Sp}(1) \cdot \pi(g)\). Since these vectors, together with \(c'(t)\), span the tangent space of \(T\) at \(\pi(g)\), the statement follows.

Let \(\text{vol} : T \to \mathbb{R}\) be the function which assigns to \(t \in T\) the volume of the orbit \(\text{Sp}(2)\ \text{Sp}(1) \cdot t\). By [30, Corollary 1 and Proposition 1], this function is positive and smooth on a dense subset of \(T\).

**Proposition 5.2.** Let \(f\) be a smooth function on \(\text{Gr}_k\) which is invariant under \(\text{Sp}(2)\ \text{Sp}(1)\). Let \(\Delta\) be the Laplace-Beltrami operator acting on smooth functions on \(\text{Gr}_k\). Let \(\Delta_T\) be the Laplacian acting on functions on \(T\). Then, at all points where \(\text{vol}\) is strictly positive,

\[
(\Delta f)|_T = \Delta_T f|_T - \langle \nabla (f|_T), \nabla (\log \text{vol}) \rangle.
\]

**Proof.** By the previous proposition, there exists an orthonormal moving frame \(E_1, \ldots, E_N\) on \(\text{Gr}_k\) such that \(E_1, \ldots, E_d\) are orthogonal to the \(\text{Sp}(2)\ \text{Sp}(1)\) orbits, and \(E_1, \ldots, E_{k-1}\) span the tangent spaces of \(T\). Since \(T\) is flat, we can assume that \(\nabla_{E_i} E_j|_T = 0\) for \(i, j = 1, \ldots, k\). Since \(f\) is constant on the orbits,

\[
\nabla f = \sum_{i=1}^{k-1} E_i(f) E_i.
\]

Hence, on \(T\),

\[
\Delta (f) = -\text{div}(\nabla f)
\]

\[
= -\sum_{j=1}^{k-1} \sum_{i=1}^{k-1} \langle E_j, \nabla_{E_j}(E_i(f) E_i) \rangle
\]

\[
= -\sum_{i=1}^{k-1} E_i \circ E_i(f) + \sum_{i=1}^{k-1} E_i(f) \sum_{j=k}^{N} \langle \nabla_{E_j} E_j, E_i \rangle
\]

\[
= \Delta_T f + \langle \nabla f, \vec{H} \rangle,
\]

where \(\vec{H}\) denotes the mean curvature vector of the \(\text{Sp}(2)\ \text{Sp}(1)\)-orbits. The result follows from the identity (cf. e.g. [30])

\[
\vec{H} = -\nabla \log \text{vol}.
\]

**Proposition 5.3.** Let \(g = \Phi(\theta_1, \ldots, \theta_k)\). The orbit \(\text{Sp}(2)\ \text{Sp}(1) \cdot \pi(g) \subset \text{Gr}_k\) has volume

\[
\text{vol} = c_2 |\sin(\theta_1 - \theta_2)|^3 \cos(\theta_1 - \theta_2)^2 \quad \text{if } k = 2,
\]

\[
= c_3 \prod_{1 \leq i < j \leq 3} |\sin(\theta_i - \theta_j)| \prod_{m \in \mathbb{Z}_4} |\sin(\theta_{m+1} + \theta_{m+2} - 2\theta_m)| \quad \text{if } k = 3,
\]

\[
= c_4 \prod_{1 \leq i < j \leq 4} |\sin(\theta_i - \theta_j)| \prod_{\{h,l\},\{m,n\}} |\sin(\theta_h + \theta_l - \theta_m - \theta_n)| \quad \text{if } k = 4,
\]
where the last product runs over all unordered partitions \(\{h,l\}, \{m,n\}\) of \(\{1,2,3,4\}\) into two disjoint pairs, and \(c_k\) is a constant depending only on \(k\).

Proof. We sketch the computation for \(k = 4\), the cases \(k = 2,3\) being similar. We just need to find the jacobian of the natural map \(\psi : \text{Sp}(2) \text{Sp}(1) \to \text{Sp}(2) \text{Sp}(1) \cdot \pi(g)\). By left-invariance, it is enough to compute \(\text{jac}(\psi)\) at \(g = T_e \text{Sp}(2) \text{Sp}(1)\). We will use again the basis \((21)\) of \(g\). The tangent space at \(\pi(g)\) of \(\text{Gr}_4\) is identified using \(d\pi \circ g^\prime\) with the horizontal part \(m\) of \(\mathfrak{so}_8\). This way, for \(X \in g\)

\[
d\psi(X) = \pi_m(g^\prime X g)
\]

where \(\pi_m : \mathfrak{so}_8 \to m \cong M_{4\times 4}(\mathbb{R})\) consists of taking the lower left block of the matrix. After identifying \(m\) with \(\mathbb{R}^{16}\), the matrix \(A \in M_{13\times 16}(\mathbb{R})\) associated with \(d\psi\) is easily computed. The jacobian of \(\psi\) is (up to constants) the determinant of \(A\), with three rows of zeros removed. By suitably reordering the rows of \(A\), one gets a structure of \(4 \times 4\) diagonal blocks, which makes the computation of the determinant an elementary task. \(\square\)

**Proposition 5.4.** Let \(f_{k,i}\) be the \(\text{Sp}(2) \text{Sp}(1)\)-invariant functions on \(\text{Gr}_k\) defined in the introduction. Then

\[
\begin{align*}
\Delta(f_{k,0}) &= 0, \quad k = 0, \ldots, 4 \\
\Delta(f_{2,1}) &= 28f_{2,1} - 12, \\
\Delta(f_{3,1}) &= 28f_{3,1} - 36, \\
\Delta(f_{3,2}) &= 60f_{3,2} - 34f_{3,1} + 18, \\
\Delta(f_{4,1}) &= 28f_{4,1} - 72, \\
\Delta(f_{4,2}) &= 40f_{4,2} - 2f_{4,1} - 12, \\
\Delta(f_{4,3}) &= 60f_{4,3} + 8f_{4,2} - 68f_{4,1} + 48, \\
\Delta(f_{4,4}) &= 96f_{4,4} + 64f_{4,1} - 92f_{4,3} - 152f_{4,2} + 24.
\end{align*}
\]

Proof. It is enough to prove the identities on \(T\). By continuity, it suffices to prove them on the dense subset of points corresponding to orbits of strictly positive volume. By Propositions 5.2 and 5.3 and using \(\lambda_{ij} = \cos(\theta_i - \theta_j)\), this is a straightforward but tedious computation. \(\square\)
Corollary 5.5. In each $\tilde{\Gamma}_\lambda$, there exists a unique (up to scale) invariant eigenfunction of the Laplace-Beltrami operator on $\text{Gr}_k$:

| $k$ | eigenfunction | eigenvalue | $\tilde{\Gamma}_\lambda$ |
|-----|---------------|------------|--------------------------|
| 0   | $f_{0,0}$     | 0          | $(0,0,0,0)$              |
| 1   | $f_{1,0}$     | 0          | $(0,0,0,0)$              |
| 2   | $f_{2,0}$     | 0          | $(0,0,0,0)$              |
| 2   | $7f_{2,1} - 3f_{2,0}$ | 28  | $(2,2,0,0)$          |
| 3   | $f_{3,0}$     | 0          | $(0,0,0,0)$              |
| 3   | $7f_{3,1} - 9f_{3,0}$ | 28  | $(2,2,0,0)$          |
| 3   | $16f_{3,2} - 17f_{3,1} + 15f_{3,0}$ | 60  | $(4,2,2,0)$          |
| 4   | $f_{4,0}$     | 0          | $(0,0,0,0)$              |
| 4   | $7f_{4,1} - 18f_{4,0}$ | 28  | $(2,2,0,0)$          |
| 4   | $6f_{4,2} - f_{4,1}$ | 40  | $(2,2,2,2)$          |
| 4   | $20f_{4,3} + 8f_{4,2} - 43f_{4,1} + 66f_{4,0}$ | 60  | $(4,2,2,0)$          |
| 4   | $63f_{4,4} - 161f_{4,3} - 194f_{4,2} + 226f_{4,1} - 210f_{4,0}$ | 96  | $(6,2,2,2)$          |

Proof. To check that these functions are eigenvectors of the Laplacian with the given eigenvalues is easy using the previous proposition.

Let us show that these functions belong to $\tilde{\Gamma}_\lambda$ as stated in the last column. It follows from Proposition 4.2 that the eigenspaces corresponding to the eigenvalues 28 and 60 are given by $\tilde{\Gamma}_{(2,2,0,0)}$ and $\tilde{\Gamma}_{(4,2,2,0)}$.

The eigenspace corresponding to the eigenvalue 40 is given by $\tilde{\Gamma}_{(2,2,2,2)} \oplus \tilde{\Gamma}_{(4,0,0,0)}$. The irreducible representation $\tilde{\Gamma}_{(4,0,0,0)}$ does not contain any $\text{Sp}(2) \text{Sp}(1)$-invariant vector (otherwise $\dim \text{Val}_{\text{Sp}(2) \text{Sp}(1)}^1$ would be larger than 1, e.g. by Proposition 4.3). Therefore an invariant eigenvector corresponding to the eigenvalue 40 must belong to $\tilde{\Gamma}_{(2,2,2,2)}$.

The eigenspace corresponding to the eigenvalue 96 is given by $\tilde{\Gamma}_{(6,2,2,2)} \oplus \tilde{\Gamma}_{(4,4,4,0)}$. The representation $\tilde{\Gamma}_{(4,4,4,0)}$ does not contain any $\text{Sp}(2) \text{Sp}(1)$-invariant vector. This can be checked using Weyl’s character formula or a computer algebra system like LiE [38]. An invariant eigenvector corresponding to the eigenvalue 96 must thus belong to $\tilde{\Gamma}_{(6,2,2,2)}$.

Finally, to see that each $\tilde{\Gamma}_\lambda$ contains only one invariant function on $\text{Gr}_k$, it is enough to remark that each such function is the Klain function of an invariant valuation by Proposition 4.3. By comparing dimensions (see table (5)), the claim follows.

Theorem 2 follows from Corollary 5.5 and Proposition 4.3. More precisely, each $\text{SO}(8)$-representation $\tilde{\Gamma}_\lambda$ from the last column of the table enters the decomposition of $\text{Val}_k$ by Proposition 4.3. By Schur’s lemma and the injectivity of the Klain embedding, $\text{Val}_k$ contains an $\text{Sp}(2) \text{Sp}(1)$-invariant valuation with the Klain function given in the second column. Since these functions are linearly independent, we deduce from the dimensions in Table 5 that these valuations form a basis of $\text{Val}_k^{\text{Sp}(2) \text{Sp}(1)}$. 

\[\]
Since we want to construct these valuations as explicit as possible, we
follow however a different path which allows to compute Crofton measures
associated to the constructed valuations.

6. Multipliers of the cosine transform

Let $V \cong \mathbb{R}^n$ be a euclidean vector space. Set $\rho := \frac{n}{2}$. The $\alpha$-cosine transform $T_{k,k}^\alpha$ is defined for $\alpha \in \mathbb{C}$ with $\Re\alpha > \rho$ by

$$L^2(\text{Gr}_k(\mathbb{R}^n)) \to L^2(\text{Gr}_k(\mathbb{R}^n))$$

$$f \mapsto \left[ E \mapsto \int_{\text{Gr}_k} f(F) |\cos(E,F)|^{\alpha-\rho} dF \right]$$

and by meromorphic continuation for all $\alpha \in \mathbb{C}$.

The case $\alpha = \rho + 1$ yields the classical cosine transform [24], also denoted by $T_{k,k}$.

Since $T_{k,k}^\alpha$ intertwines the $\text{SO}(n)$-action, it acts as a scalar on each irreducible representation of $\text{SO}(n)$ which enters the decomposition of $L^2(\text{Gr}_k(\mathbb{R}^n))$.

The precise value of this constant was computed by Ölafsson and Pasquale [28] (compare also [29] and [42]).

Let $\Gamma_k(\lambda) := \prod_{j=1}^k \Gamma \left( \lambda_j - \frac{j-1}{2} \right)$, $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ be the Siegel $\Gamma$-function.

**Theorem 6.1** (Ölafsson-Pasquale). Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a highest weight for $\text{SO}(n)$ such that $\Gamma_{\lambda}$ enters the decomposition of $L^2(\text{Gr}_k(\mathbb{R}^n))$. Then $T_{k,k}^\alpha$ acts on $\Gamma_{\lambda}$ by the scalar

$$c_{n,k}^\alpha := (-1)^\frac{|\lambda|}{2} \frac{\Gamma_k(\rho) \Gamma_k \left( \frac{\alpha-\rho+k}{2} \right) \Gamma_k \left( \frac{-\alpha+\rho+\lambda}{2} \right)}{\Gamma_k \left( \frac{k}{2} \right) \Gamma_k \left( \frac{n-k+1}{2} \right) \Gamma_k \left( \frac{n+1+\lambda}{2} \right) \Gamma_k \left( \frac{n+1+a}{2} \right)}.$$  

In this formula, a complex number $z$ is identified with the vector $(z, \ldots, z) \in \mathbb{C}^k$.

**Corollary 6.2.** Let $\lambda = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ be a highest weight of $\text{SO}(n)$ such that $\Gamma_{\lambda}$ enters the decomposition of $\text{Val}_k$ with $1 \leq k \leq \frac{n}{2}$. Then $T_{k,k}$ acts on $\Gamma_{\lambda}$ by the scalar

$$c_{n,k} := (-1)^{\frac{a}{2} - 1} \frac{b(n-b+1)! \Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k+1}{2} \right) \Gamma \left( \frac{n-1}{2} \right)}{2\pi n! \Gamma \left( \frac{n+1+a}{2} \right)}.$$  

Here $a := \lambda_1$ and $b$ is the depth of $\lambda$ (i.e. $\lambda_b \neq 0$, $\lambda_{b+1} = 0$).

**Proof.** Clearly $\Gamma_k(\alpha)$ is well-defined and non-zero for $\alpha \in \mathbb{R}, \alpha > \frac{k-1}{2}$. We thus have

$$c_{n,k} = \lim_{\alpha \to \rho+1} c_{n,k}^\alpha = (-1)^{\frac{|\lambda|}{2}} \frac{\Gamma_k(\rho) \Gamma_k \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k+1}{2} \right) \Gamma \left( \frac{n-1}{2} \right)}{\Gamma_k \left( \frac{n+1+a}{2} \right) \Gamma_k \left( \frac{n+1+a}{2} \right) \Gamma_k \left( \frac{n+1+a}{2} \right)}.$$
Recall that, if \( n \) is odd, we have \( \lambda_j \in \{0, 2\} \) for all \( j > 1 \). If \( n \) is even, then \( \lambda_j \in \{0, 2\} \) for \( 1 < j < \frac{n}{2} \) and \( \lambda_{\frac{n}{2}} \in \{0, 2, -2\} \).

Let us consider the first factor. Clearly
\[
\frac{\Gamma_k \left( \frac{k+1}{2} \right)}{\Gamma_k \left( \frac{k}{2} \right)} = \frac{\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{k}{2} \right)}.
\]

Next, we compute
\[
\frac{\Gamma_k(\rho)}{\Gamma_k \left( \frac{n+1+\lambda}{2} \right)} = \frac{\Gamma \left( \frac{n-k+1}{2} \right)}{\Gamma \left( \frac{n-\rho+1}{2} \right)} \prod_{j=2}^{k} \frac{\Gamma \left( \frac{n-j+2}{2} \right)}{\Gamma \left( \frac{n-j+2+\lambda_j}{2} \right)}.
\]

If \( \lambda_j = 0 \), then the corresponding factor in the product equals 1, while it equals \( \frac{1}{n-j+2} \) if \( \lambda_j = 2 \). If \( n \) is odd or \( \lambda_{\frac{n}{2}} \neq -2 \), the product thus equals \( \frac{1}{n!} (n-b+1)! \).

The last factor may be rewritten as
\[
\lim_{\alpha \to \rho+1} \frac{\Gamma_k \left( -\alpha+\rho+\lambda \right)}{\Gamma_k \left( \frac{-\alpha+\rho}{2} \right)} = \frac{\Gamma \left( \frac{a-1}{2} \right)}{\Gamma \left( \frac{-1}{2} \right)} \prod_{j=2}^{k} \lim_{x \to 0} \frac{\Gamma \left( \frac{x+j-\lambda_j}{2} \right)}{\Gamma \left( \frac{x-j}{2} \right)}.
\]

If \( \lambda_j = 0 \), then the corresponding term is 1. If \( \lambda_j = 2 \), then the corresponding term equals
\[
\lim_{x \to 0} \frac{\Gamma \left( \frac{x+2-\lambda_j}{2} \right)}{\Gamma \left( \frac{x-\lambda_j}{2} \right)} = \frac{-\lambda_j}{2}.
\]

If \( \lambda_{\frac{n}{2}} \neq -2 \), we thus get that
\[
\lim_{\alpha \to \rho+1} \frac{\Gamma_k \left( -\alpha+\rho+\lambda \right)}{\Gamma_k \left( \frac{-\alpha+\rho}{2} \right)} = \frac{\Gamma \left( \frac{a-1}{2} \right)}{\Gamma \left( \frac{-1}{2} \right)} (-1)^{b-1} b! \frac{\Gamma \left( \frac{a-1}{2} \right) b! (-1)^b}{\sqrt{\pi} 2^b}.
\]

Putting these pieces together yields for \( \lambda_{\frac{n}{2}} \neq -2 \)
\[
c_{n,k} = (-1)^{\frac{n}{2}-1} \frac{b! (n-b+1)! \Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k+1}{2} \right) \Gamma \left( \frac{a-1}{2} \right)}{2 \pi n! \Gamma \left( \frac{n+1+a}{2} \right)}.
\]

Finally, if \( n \) is even, let us compare the cases \((a, 2, \ldots, 2)\) and \((a, 2, \ldots, 2, -2)\).

The first factor gets multiplied by \( \frac{\Gamma \left( \frac{a}{2} + 1 \right)}{\Gamma \left( \frac{a}{2} \right)} \), while the second factor gets multiplied by \( \frac{\Gamma \left( \frac{a}{2} \right)}{\Gamma \left( \frac{a}{2} + 2 \right)} \). Hence the constant \( c_{n,k} \) is the same in both cases, which completes the proof. \( \square \)
Corollary 6.3. The cosine transform acts by the following scalars

| \( k \) | \( \tilde{\Gamma}_\lambda \) | \( c \) |
|---|---|---|
| 2 | \( (0, 0, 0, 0) \) | \( \frac{9}{7} \) |
| 2 | \( (2, 2, 0, 0) \) | \( \frac{1}{272} \) |
| 3 | \( (0, 0, 0, 0) \) | \( \frac{96\pi}{35} \) |
| 3 | \( (2, 2, 0, 0) \) | \( \frac{8}{3} \) |
| 3 | \( (4, 2, 2, 0) \) | \( -\frac{8}{2125\pi} \) |
| 4 | \( (0, 0, 0, 0) \) | \( \frac{27}{35} \) |
| 4 | \( (2, 2, 0, 0) \) | \( \frac{1}{120} \) |
| 4 | \( (2, 2, 2, 2) \) | \( \frac{1}{1470} \) |
| 4 | \( (4, 2, 2, 0) \) | \( -\frac{1}{10780} \) |
| 4 | \( (6, 2, 2, 2) \) | \( \frac{1}{70070} \) |

7. Construction of invariant valuations

Proposition 7.1. There exist valuations in \( \text{Val}^{\text{Sp}(2)\text{Sp}(1)} \) whose Klain functions are given by the eigenfunctions from Corollary 5.5. These valuations form a basis of \( \text{Val}^{\text{Sp}(2)\text{Sp}(1)} \).

Proof. Let \( g \in C(\text{Gr}_k) \) and define a valuation in \( \mu \in \text{Val}_k^+ \) by

\[
\mu(K) := \int_{\text{Gr}_k} g(E) \text{vol}(\pi_E K) dE,
\]

where \( \pi_E : \mathbb{H}^2 \to E \) is the orthogonal projection. Then \( \text{Kl}_\mu = T_{k,k} g \).

If \( f \) is an eigenfunction from the table in Corollary 5.5, then the cosine transform \( T_{k,k} \) acts by a non-zero scalar \( c \). Setting \( g := e^{-f} \) we get \( \text{Kl}_\mu = f \).

By looking at their Klain functions, we deduce that the so-constructed valuations are linearly independent in each degree of homogeneity. By comparing with the dimensions in (5), they actually must form a basis. \( \square \)

Proof of Theorem 2. The theorem follows from Proposition 7.1 by noting that the transformation matrix between the \( f_{k,i} \) and the eigenvectors is invertible. \( \square \)

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VALUATIONS ON THE QUATERNIONIC PLANE

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