Maximal Matroids in Weak Order Posets

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Abstract

Let $X$ be a family of subsets of a finite set $E$. A matroid on $E$ is called an $X$-matroid if each set in $X$ is a circuit. We consider the problem of determining when there exists a unique maximal $X$-matroid in the weak order poset of all $X$-matroids on $E$, and characterizing its rank function when it exists.

1 Introduction

1.1 Unique maximality problem and submodularity conjecture

Let $X$ be a family of subsets of a finite set $E$. We will refer to any matroid on $E$ in which each set in $X$ is a circuit as an $X$-matroid on $E$. The set of all $X$-matroids on $E$ forms a poset under the weak order of matroids in which, for two matroids $M_1$ and $M_2$ with the same groundset, we have $M_1 \preceq M_2$ if every independent set in $M_1$ is independent in $M_2$. The main problem addressed in this paper is to determine when this poset has a unique maximal element and to characterise this unique maximal matroid when it exists.

Our key tool is the following upper bound on the rank function of any $X$-matroid on $E$ from [7]. A proper $X$-sequence is a sequence $S = (X_1, X_2, \ldots, X_k)$ of sets in $X$ such that $X_i \notin \bigcup_{j=1}^{i-1} X_j$ for all $i = 2, \ldots, k$. For $F \subseteq E$, let $\text{val}(F, S) = |F \cup (\bigcup_{i=1}^{k} X_i)| - k$.

Lemma 1.1 ([7] Lemma 3.3). Suppose $M$ is an $X$-matroid on $E$ and $F \subseteq E$. Then $r_M(F) \leq \text{val}(F, S)$ for any proper $X$-sequence $S$. Furthermore, if equality holds, then $r_M(F - e) = r_M(F) - 1$ for all $e \in F \setminus (\bigcup_{X \in S} X)$ and $r_M(F + e) = r_M(F)$ for all $e \in \bigcup_{X \in S} X$.

We can use this lemma to derive a sufficient condition for the poset of all $X$-matroids on $E$ to have a unique maximal element. We need to consider a slightly larger poset. We say that a matroid $M$ on $E$ is $X$-cyclic if each $X \in X$ is a cyclic set in $M$ i.e. for every $e \in X$, there is a circuit $C$ of $M$ with $e \in C \subseteq X$.

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Lemma 1.2. Let $\mathcal{X}$ be a family of subsets of a finite set $E$ and define $\text{val}_\mathcal{X} : 2^E \to \mathbb{Z}$ by

$$\text{val}_\mathcal{X}(F) = \min \{ \text{val}(F, S) : S \text{ is a proper } \mathcal{X}\text{-sequence} \} \quad (F \subseteq E).$$

Suppose $\text{val}_\mathcal{X}$ is a submodular set function on $E$. Then $\text{val}_\mathcal{X}$ is the rank function of an $\mathcal{X}$-cyclic matroid $\mathcal{M}_\mathcal{X}$ on $E$. In addition, if the poset of all $\mathcal{X}$-matroids on $E$ is nonempty, then $\mathcal{M}_\mathcal{X}$ is the unique maximal $\mathcal{X}$-matroid on $E$.

Proof. It is straightforward to check that $\text{val}_\mathcal{X}$ is non-decreasing and satisfies $\text{val}_\mathcal{X}(e) \leq 1$ for all $e \in E$. Since $\text{val}_\mathcal{X}$ is also submodular, this implies that $\text{val}_\mathcal{X}$ is the rank function of a matroid $\mathcal{M}_\mathcal{X}$. To see that $\mathcal{M}_\mathcal{X}$ is $\mathcal{X}$-cyclic, we choose $e \in X \in \mathcal{X}$ and let $S$ be a proper $\mathcal{X}$-sequence such that $\text{val}_\mathcal{X}(X - e) = \text{val}(X - e, S)$. If $e \in \bigcup_{X_i \in S} X_i$ then we have $\text{val}_\mathcal{X}(X) = \text{val}(X, S) = \text{val}(X - e, S) = \text{val}_\mathcal{X}(X - e)$. On the other hand, if $e \notin \bigcup_{X_i \in S} X_i$ then we can extend $S$ to a longer proper $\mathcal{X}$-sequence $S'$ by adding $X$ as the last element of $S'$ and we will have $\text{val}_\mathcal{X}(X) \leq \text{val}(X, S') = \text{val}(X - e, S) = \text{val}_\mathcal{X}(X - e)$. In both cases equality must hold throughout since $\text{val}_\mathcal{X}$ is non-decreasing. Since $\text{val}_\mathcal{X}$ is the rank function of $\mathcal{M}_\mathcal{X}$, the equality $\text{val}_\mathcal{X}(X) = \text{val}_\mathcal{X}(X - e)$ implies that $e$ belongs to a circuit of $\mathcal{M}_\mathcal{X}$ which is contained in $X$. Hence $\mathcal{M}_\mathcal{X}$ is $\mathcal{X}$-cyclic.

Lemma 1.1 implies that $\mathcal{M} \preceq \mathcal{M}_\mathcal{X}$ for every $\mathcal{X}$-matroid $\mathcal{M}$ on $E$. If there exists at least one $\mathcal{X}$-matroid on $E$, then this implies that each $X \in \mathcal{X}$ is a circuit in $\mathcal{M}_\mathcal{X}$ and that $\mathcal{M}_\mathcal{X}$ is the unique maximal $\mathcal{X}$-matroid on $E$. \hfill $\Box$

We conjecture that the converse to Lemma 1.2 is also true. The special case when $\mathcal{X}$ is the set of all non-spanning circuits of a matroid on $E$ was previously given in [7].

Conjecture 1.3. Let $\mathcal{X}$ be a family of subsets of a finite set $E$. Suppose there is at least one $\mathcal{X}$-matroid on $E$. Then the poset of all $\mathcal{X}$-matroids on $E$ has a unique maximal element if and only if $\text{val}_\mathcal{X}$ is a submodular set function on $E$. [7]

We will verify this conjecture for various families $\mathcal{X}$ and provide some tools to facilitate further progress on the conjecture.

Conjecture 1.3 is motivated by the polynomial identity testing problem of symbolic determinants (or the Edmonds problem). In this problem, we are given a matrix $A$ with entries in $\mathbb{Q}[x_1, \ldots, x_n]$, and we are asked to decide whether the rank of $A$ over $\mathbb{Q}(x_1, \ldots, x_n)$ is at least a given number. The Schwarz-Zippel Lemma implies that the problem is in the class NP, but it is a long-standing open problem to show that it is also in co-NP. The following experimental approach may aid our understanding of this problem. We first test the linear independence/dependence of small sets of rows of $A$ to obtain a family $\mathcal{X}$ of minimally dependent sets of rows. Then Lemma 1.1 tells us that we can use any $\mathcal{X}$-sequence to obtain a certificate that the rank of $A$ is at most a specified value. In addition, if the "freest" matroid on the groundset $E$ indexed by the rows of $A$ in which each set in $\mathcal{X}$ is a circuit is uniquely determined, then Conjecture 1.3 would imply that its rank is $\text{val}_\mathcal{X}$ and this function has the potential to be the rank function of the row matroid of $A$.

1More generally, if we remove the hypothesis that there is at least one $\mathcal{X}$-matroid on $E$, then we conjecture that the poset of all $\mathcal{X}$-cyclic matroids on $E$ has a unique maximal element if and only if $\text{val}_\mathcal{X}$ is a submodular set function on $E$.  

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1.2 Unique maximality problem on graphs

We will concentrate on the special case of Conjecture 1.3 when $E$ is the edge set of a graph $G$ and $\mathcal{X}$ is the family $\mathcal{H}_G$ of edge sets of all subgraphs of $G$ which are isomorphic to some member of a given family $\mathcal{H}$ of graphs. To simplify terminology we say that a matroid $\mathcal{M}$ is a $\mathcal{H}_G$-matroid on $G$ if it is an $\mathcal{H}_{G'}$-matroid on $E(G)$. We will assume throughout that $G$ contains at least one copy of each $H \in \mathcal{H}$ otherwise we can just consider $\mathcal{H} \setminus \{H\}$. This implies that the edge sets of any two subgraphs of $G$ which are isomorphic to the same subgraph of a graph $H \in \mathcal{H}$ will have the same rank in $\mathcal{M}$, but we do not require $\mathcal{M}$ to be completely symmetric i.e. the edge sets of every pair of isomorphic subgraphs of $G$ have the same rank.

We will simplify notation in the case when $\mathcal{H} = \{H\}$ and refer to a $\mathcal{H}$-matroid on $G$ as a $H$-matroid on $G$. Two examples of $K_3$-matroids on $K_n$ are the graphic matroid of $K_n$ and the rank two uniform matroid on $E(K_n)$.

Chen, Sitharam and Vince previously considered the unique maximality problem for $H$-matroids on $K_n$ for various graphs $H$. They announced at a workshop at BIRS in 2015, see [26], that there is a unique maximal $K_5$-matroid on $K_n$. Sitharam and Vince subsequently released a preprint [27] which claims to show that there is a unique maximal $H$-matroid on $K_n$ for all graphs $H$. Unfortunately their claim is false. Pap [21] pointed out that the poset of $C_5$-matroids on $K_n$ has two maximal elements. We will describe Pap’s counterexample, and give other counterexamples to the Sitharam-Vince claim in Section 5.

Our interest in this topic was motivated by the work of Graver, Servatius, and Servatius [11,12] and Whiteley [30] on maximal abstract rigidity matroids, and that of Chan, Sitharam and Vince [26,27] on maximal $H$-matroids. In two joint papers with Clinch [6,7], we were able to confirm that there is a unique maximal $K_5$-matroid on $K_n$ and, more importantly, give a good characterisation for the rank function of this matroid. The theory of matroid erections due to Crapo [8] is a key ingredient in our proof technique.

In this paper we will use results on matroid erection from [7] to construct a maximal element in the poset of all $\mathcal{X}$-matroids on a set $E$. We will show that this element is the unique maximal element in the poset of all $\mathcal{H}$-matroids on a graph $G$ for various pairs $(\mathcal{H}, G)$, and verify that Conjecture 1.3 holds in each case.

1.3 Weakly saturated sequences

The function val$_X$ defined in (1) is related to the weak saturation number in extremal graph theory. Let $\mathcal{X}$ be a family of subsets of a finite set $E$, and $F_0 \subseteq E$. A proper $\mathcal{X}$-sequence $(X_1, X_2, \ldots, X_m)$ is said to be a weakly $\mathcal{X}$-saturated sequence from $F_0$ if $|X_i \setminus (F_0 \cup \bigcup_{j<i} X_j)| = 1$ for all $i$ with $1 \leq i \leq m$. We say that $E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from $F_0$ if there is a weakly $\mathcal{X}$-saturated sequence $S$ from $F_0$ with $E = F_0 \cup \bigcup_{X \in \mathcal{X}} X$. These sequences were first introduced by Bollobás [4], where he posed the problem of determining the size of a smallest set $F_0$ from which $E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence. The problem has subsequently been studied by several authors, typically in the case when $E$ is the edge set of a complete $k$-uniform hypergraph or a complete bipartite graph, see for example [1,4,15,16,19,22,23]. We will see in Sections 3 and 4 that results on weakly $\mathcal{X}$-saturated sequences can sometimes be used to prove
Lemma 2.1. There exists a circuit \( C \subseteq E \) such that the set of \( M \) neighbors of \( v \) be the number of edges in \( X \) from \([7]\) to construct a maximal element in the poset of all \( X \)-matroids on a given finite set \( E \) by determining its maximal elements.

We close this section by listing notation used throughout the paper. Let \( M \) be a matroid on a finite set. Its rank function and closure operator are denoted by \( r_M \) and \( cl_M \), respectively. A set \( F \subseteq E \) with \( cl_M(F) = F \) is called a flat.

For a graph \( G \), \( V(G) \) and \( E(G) \) denote its vertex set and its edge set, respectively. Let \( N_G(v) \) be the set of neighbors of \( v \) in \( G \). For \( F \subseteq E(G) \), let \( V(F) \) be the set of vertices incident to \( F \) and let \( G[F] \) be the graph with vertex set \( V(F) \) and edge set \( F \). Let \( d_F(v) \) be the number of edges in \( F \) incident to a vertex \( v \in V(G) \), and let \( N_F(v) \) be the set of neighbors of \( v \) in \( G[F] \).

For disjoint sets \( X \) and \( Y \), let \( K(X) \) be the complete graph with vertex set \( X \) and \( K(X; Y) \) be the complete bipartite graph with vertex partition \((X, Y)\).

2 Maximal Matroids and Matroid Elevations

Let \( X \) be a family of subsets of a finite set \( E \). We first derive a sufficient condition for a given \( X \)-matroid on \( E \) to be the unique maximal such matroid. We then use results from \([7]\) to construct a maximal element in the poset of all \( X \)-matroids on \( E \) (whenever this poset is non-empty).

2.1 A sufficient condition for unique maximality

Recall that a set \( F \) in a matroid \( M \) is connected if, for every pair of elements \( e_1, e_2 \in F \), there exists a circuit \( C \) of \( M \) with \( e_1, e_2 \in C \subseteq F \), and that \( F \) is a connected component of \( M \) if \( F \) is either a coloop of \( M \) or a maximal connected set in \( M \). It is well known that the set \( \{F_1, F_2, \ldots, F_m\} \) of all connected components partitions the ground set of \( M \) and that \( r_M = \sum_{i=1}^m r_M(F_i) \). In addition, \( F \) is connected in \( M \) if and only if \( r_M(F) < r_M(F') + r_M(F'') \) for all partitions \( \{F', F''\} \) of \( F \).

Lemma 2.1. Let \( X \) be a family of subsets of a finite set \( E \) and \( M \) be a loopless \( X \)-matroid on \( E \). Suppose that, for every connected flat \( F \) of \( M \), there is a proper \( X \)-sequence \( S \) with \( r_M(F) = val(F, S) \). Then \( val_X = r_M \) and \( M \) is the unique maximal \( X \)-matroid on \( E \).

Proof. Since \( r_M \leq val_X \) for all \( X \)-matroids on \( E \) by Lemma \([11]\) it will suffice to show that, for each \( F \subseteq E \), there is a proper \( X \)-sequence \( S \) such that \( r_M(F) = val(F, S) \).

Suppose, for a contradiction, that this is false for some set \( F \). We may assume that \( F \) has been chosen such that \( r_M(F) \) is as small as possible and, subject to this condition, \(|F|\) is as large as possible. If \( F \) is not a flat then \( r_M(F + e) = r_M(F) \) for some \( e \in E \setminus F \) and we can now use the maximality of \(|F|\) to deduce that there exists a proper \( X \)-sequence \( S \) such
that $r_M(F + e) = \text{val}(F + e, S)$. By Lemma 1.1 and $r_M(F) = r_M(F + e)$, $e \in \bigcup_{X \in S} X$. Hence, $\text{val}(F + e, S) = \text{val}(F, S) = r_M(F + e) = r_M(F)$. This would contradict the choice of $F$. Hence $F$ is a flat.

Suppose $F$ is not connected. Then we have $r_M(F) = r_M(F_1) + r_M(F_2)$ for some partition $\{F_1, F_2\}$ of $F$. Since $M$ is loopless, $F_i$ is a flat of $M$ and $1 \leq r_M(F_i) < r_M(F)$ for both $i = 1, 2$. The choice of $F$ now implies that there exists a proper $X$-sequence $S_i$ such that $r_M(F_i) = \text{val}(F_i, S_i)$ for $i = 1, 2$. Since each $F_i$ is a flat, we have $X_i \subseteq F_i$ for all $X_i \in S_i$ by Lemma 1.1. This implies that the concatenation $S = (S_1, S_2)$ is a proper $X$-sequence and satisfies

$$\text{val}(F, S) = \text{val}(F_1, S_1) + \text{val}(F_2, S_2) = r_M(F_1) + r_M(F_2) = r_M(F).$$

This contradicts the choice of $F$.

Hence $F$ is a connected flat and we can use the hypothesis of the lemma to deduce that there is a proper $X$-sequence $S$ such that $r_M(F) = \text{val}(F, S)$, as required. \hfill \Box

## 2.2 Matroid elevations

The truncation of a matroid $M_1 = (E, I_1)$ of rank $k$ is the matroid $M_0 = (E, I_0)$ of rank $k - 1$, where $I_0 = \{I \in I_1 : |I| \leq k - 1\}$. Crapo [8] defined matroid erection as the ‘inverse operation’ to truncation. So $M_1$ is an erection of $M_0$ if $M_0$ is the truncation of $M_1$. (For technical reasons we also consider $M_0$ to be a trivial erection of itself.) Note that, although every matroid has a unique truncation, matroids may have several, or no, non-trivial erections.

Crapo [8] showed that the poset of all erections of a matroid $M_0$ is actually a lattice. It is clear that the trivial erection of $M_0$ is the unique minimal element in this lattice. Since this is a finite lattice, there also exists a unique maximal element which Crapo called the \textit{free erection} of $M_0$.

A \textit{partial elevation} of $M_0$ is any matroid $M$ which can be constructed from $M_0$ by a sequence of erections. A \textit{(full) elevation} of $M_0$ is a partial elevation $M$ which has no non-trivial erection. The \textit{free elevation} of $M_0$ is the matroid we get from $M_0$ by recursively constructing a sequence of free erections until we arrive at a matroid which has no non-trivial erection. The set of all partial elevations of $M_0$ forms a poset $P(M_0)$ under the weak order and $M_0$ is its unique minimal element. Every maximal element of $P(M_0)$ will have no non-trivial erection so will be a full elevation of $M_0$. Given Crapo’s result that the poset of all erections of $M_0$ is a lattice, it is tempting to conjecture that $P(M_0)$ will also be a lattice and that the free elevation of $M_0$ will be its unique maximal element. But this is false: Brylawski gives a counterexample based on the Vamos matroid in [4] and we will construct another counterexample using $\mathcal{H}$-matroids on $K_n$ in Section 5. The following weaker result is given in [7].

\textbf{Lemma 2.2 (\cite[Lemma 3.1]{7})}. \textit{Suppose that $M_0$ is a matroid. Then the free elevation of $M_0$ is a maximal element in the poset of all partial elevations of $M_0$.}

Our next result extends Lemma 2.2 to $\mathcal{X}$-matroids. Given a finite set $E$ and an integer $k$, let $\mathcal{U}_k(E)$ be the uniform matroid on $E$ of rank $k$, i.e. the matroid on $E$ in which a set $F \subseteq E$ is independent if and only if $|F| \leq k$.\hfill \Box
Lemma 2.3. Let $E$ be a finite set, $\mathcal{X}$ be a family of subsets of $E$ of size at most $s$, and $\mathcal{M}_0$ be a maximal matroid in the poset of all $\mathcal{X}$-matroids on $E$ with rank at most $s$. Suppose that $\mathcal{M}_0 \neq U_{s-1}(E)$. Then the free elevation of $\mathcal{M}_0$ is a maximal matroid in the poset of all $\mathcal{X}$-matroids on $E$.

Proof. Let $\mathcal{M}$ be the free elevation of $\mathcal{M}_0$. Since $\mathcal{M}_0$ is an $\mathcal{X}$-matroid and $\mathcal{M}_0 \neq U_{s-1}(E)$, every set in $\mathcal{X}$ is a non-spanning circuit of $\mathcal{M}_0$. This implies that every partial elevation of $\mathcal{M}_0$ is an $\mathcal{X}$-matroid. In particular, $\mathcal{M}$ is an $\mathcal{X}$-matroid.

Lemma 2.2 implies that $\mathcal{M}$ is a maximal element in the poset of all partial elevations of $\mathcal{M}_0$. Let $\mathcal{N}$ be an $\mathcal{X}$-matroid on $E$ which is not a partial elevation of $\mathcal{M}_0$. Let $\mathcal{N}_0$ be the truncation of $\mathcal{N}$ to rank $s$ if $\mathcal{N}$ has rank at least $s$, and otherwise let $\mathcal{N}_0 = \mathcal{N}$. Then $\mathcal{N}_0 \neq \mathcal{M}_0$. Since $\mathcal{M}_0$ is a maximal $\mathcal{X}$-matroid in the poset of all $\mathcal{X}$-matroids on $E$ with rank at most $s$, $\mathcal{N}_0 \not\succeq \mathcal{M}_0$ holds. Hence there exists $F \subseteq E$ with the properties that $|F| \leq s$, $F$ is independent in $\mathcal{N}_0$ and $F$ is independent in $\mathcal{M}_0$. This implies that $F$ is dependent in $\mathcal{N}$ and independent in $\mathcal{M}$ so $\mathcal{N} \not\succeq \mathcal{M}$. Hence $\mathcal{M}$ remains as a maximal element in the poset of all $\mathcal{X}$-matroids on $E$.

Lemma 2.3 can be applied whenever there exists at least one $\mathcal{X}$-matroid $\mathcal{M}$ on $E$ since we can truncate $\mathcal{M}$ to obtain an $\mathcal{X}$-matroid of rank at most $s$, and hence the poset of all $\mathcal{X}$-matroids on $E$ with rank at most $s$ will be non-empty.

In the next subsection, we give an explicit construction of a maximal $\mathcal{X}$-matroid in the poset of all $\mathcal{X}$-matroids on $E$ with rank at most $s$ whenever $\mathcal{X}$ is an $s$-uniform families.

We close this subsection by stating a useful property of free elevations. We say that an $\mathcal{X}$-matroid $\mathcal{M}$ on a finite set $E$ has the $\mathcal{X}$-covering property if every cyclic flat in $\mathcal{M}$ is the union of sets in $\mathcal{X}$.

Lemma 2.4 ([7 Lemma 3.6]). Let $\mathcal{M}_0$ be a matroid on a finite set $E$ and $\mathcal{X}$ be the family of non-spanning circuits of $\mathcal{M}_0$. Suppose that $E = \bigcup_{X \in \mathcal{X}} X$. Then the free elevation of $\mathcal{M}_0$ has the $\mathcal{X}$-covering property.

2.3 Uniform $\mathcal{X}$-matroids

A family $\mathcal{X}$ of sets is $k$-uniform if each set in $\mathcal{X}$ has size $k$. Given a $k$-uniform family $\mathcal{X}$, the $\mathcal{X}$-uniform system $\mathcal{U}_\mathcal{X}$ is defined as the pair $(E, \mathcal{I}_\mathcal{X})$, where $E = \bigcup_{X \in \mathcal{X}} X$ and

$$\mathcal{I}_\mathcal{X} := \{ F \subseteq E : |F| \leq k \text{ and } F \notin \mathcal{X} \}.$$  

We first characterise when $\mathcal{U}_{\mathcal{X}}$ is a matroid. We say that the $k$-uniform family $\mathcal{X}$ is union-stable if, for any $X_1, X_2 \in \mathcal{X}$ and $e \in X_1 \cap X_2$, either $|(X_1 \cup X_2) - e| > k$ or $(X_1 \cup X_2) - e \in \mathcal{X}$.

Lemma 2.5. Suppose that $\mathcal{X}$ is a $k$-uniform family. Then $\mathcal{U}_{\mathcal{X}}$ is a matroid if and only if $\mathcal{X}$ is union-stable.

Proof. Let $C = \mathcal{X} \cup \{ C \subseteq E : |C| = k \text{ and } X \not\subseteq C \text{ for all } X \in \mathcal{X} \}$. It is straightforward to check $\mathcal{U}_{\mathcal{X}}$ is a matroid if and only if $C$ satisfies the matroid circuit axioms and that the latter property holds if and only if $\mathcal{X}$ is union-stable.

Note that in our main motivation, the Edmonds Problem, $\mathcal{X}$ will be a family of minimal row dependencies of a matrix $A$ and hence the row matroid of $A$ will be an $\mathcal{X}$-matroid.
Given an arbitrary $k$-uniform family $\mathcal{X}$, we construct the union-stable closure $\bar{\mathcal{X}}$ of $\mathcal{X}$ by first putting $\bar{\mathcal{X}} = \mathcal{X}$ and then recursively adding $(X_1 \cup X_2) - e$ to $\bar{\mathcal{X}}$ whenever $X_1, X_2 \in \bar{\mathcal{X}}$, $|X_1 \cup X_2| = k + 1$ and $e \in X_1 \cap X_2$. It is straightforward to check that the resulting family $\bar{\mathcal{X}}$ is $k$-uniform and union-stable and that $\mathcal{U}_{\bar{\mathcal{X}}}$ is a maximal matroid in the poset of all $\mathcal{X}$-matroids on $E$ with rank at most $k$. We can now apply Lemma 2.3 to deduce:

**Lemma 2.6.** Let $\mathcal{X}$ be a $k$-uniform family of sets such that $\mathcal{U}_{\bar{\mathcal{X}}} \neq \mathcal{U}_{k-1}(E)$. Then the free elevation of $\mathcal{U}_{\bar{\mathcal{X}}}$ is a maximal $\mathcal{X}$-matroid on $E$.

Note that if $\mathcal{U}_{\bar{\mathcal{X}}} = \mathcal{U}_{k-1}(E)$ then $\mathcal{U}_{\bar{\mathcal{X}}}$ is the unique maximal $\mathcal{X}$-matroid on $E$ but the free-elevation of $\mathcal{U}_{\bar{\mathcal{X}}}$ is the free matroid on $E$ i.e. the matroid in which every subset of $E$ is independent.

Suppose that $G$ and $H$ are graphs with $|E(H)| = k$. We will also assume that every edge of $G$ belongs to a subgraph which is isomorphic to $H$ (we can reduce to this case by deleting all edges of $G$ which do not belong to copies of $H$). Recall that $\{H\}_G$ denotes the $k$-uniform family containing all edge sets of copies of $H$. The graph $H$ is said to be union-stable on $G$ if $\{H\}_G$ is union-stable, i.e., for any two distinct copies $H_1$ and $H_2$ of $H$ in $G$ and any $e \in E(H_1) \cap E(H_2)$, either $H_1 \cup H_2 - e$ is isomorphic to $H$ or $|E(H_1 \cup H_2 - e)| > k$. To simplify notation we denote the uniform $\{H\}_G$-matroid $\mathcal{U}_{\{H\}_G}$ by $\mathcal{U}_H(G)$ when $H$ is union-stable. Examples of union-stable graphs on $K_n$ are stars, cycles, complete graphs, and complete bipartite graphs. Lemmas 2.3 and 2.5 now give:

**Lemma 2.7.** Suppose that $G$ and $H$ are graphs. Then $\mathcal{U}_H(G)$ is a matroid if and only if $H$ is union-stable on $G$. Furthermore, if $\mathcal{U}_H(G)$ is a matroid, then its free elevation is a maximal $H$-matroid on $G$.

## 3 Weakly Saturated Sequences

Let $\mathcal{X}$ be a family of subsets of a finite set $E$, and $F_0 \subseteq E$. Recall that a proper $\mathcal{X}$-sequence $(X_1, X_2, \ldots, X_m)$ is a weakly $\mathcal{X}$-saturated sequence from $F_0$ if $|X_i \setminus (F_0 \cup \bigcup_{j<i} X_j)| = 1$ for all $i$ with $1 \leq i \leq m$. We say that a set $F \subseteq E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from $F_0$ if there is a weakly $\mathcal{X}$-saturated sequence $S$ from $F_0$ with $F = F_0 \cup \bigcup_{X \in S} X$. Note that if this is the case then we will have $\text{val}(F, S) = |F_0|$. We can combine this simple observation with Lemma 2.1 to give several examples of unique maximality.

**Lemma 3.1.** Let $\mathcal{X}$ be a $k$-uniform family of subsets of a finite set $E$. Suppose that $E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from some $X_0 \in \mathcal{X}$. Then the rank $k-1$ uniform matroid $\mathcal{U}_{k-1}(E)$ is the unique maximal $\mathcal{X}$-matroid on $E$ and its rank function is $\text{val}_X$.

**Proof.** We denote $\mathcal{U} = \mathcal{U}_{k-1}(E)$. Since $\mathcal{U}$ is uniform, $E$ is the only connected flat in $\mathcal{U}$ and hence, by Lemma 2.1 it will suffice to show that there is a proper $\mathcal{X}$-sequence $S$ such that $r_\mathcal{U}(E) = \text{val}_X(S, E)$. By hypothesis, there is a weakly saturated $\mathcal{X}$-sequence $S_0$ from $X_0$ to $E$. Let $S$ be the proper $\mathcal{X}$-sequence obtained by inserting $X_0$ at the beginning of $S_0$. Then $\text{val}_X(S, E) = \text{val}_X(S_0, E) - 1 = |X_0| - 1 = k - 1 = r_\mathcal{U}(E)$,
as required.

The same proof technique can handle a slightly more complicated situation.

**Lemma 3.2.** Let $\mathcal{X}$ be a $k$-uniform, union-stable family of sets. Suppose that $E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from some $Y \subseteq E$ with $|Y| = k$ and $Y \notin \mathcal{X}$. Then $\mathcal{U}_\mathcal{X}$ is the unique maximal $\mathcal{X}$-matroid on $E$ and its rank function is $\operatorname{val}_\mathcal{X}$.

**Proof.** By Lemma 2.1, it will suffice to show that there is a proper $\mathcal{X}$-sequence $S$ such that $r_{\mathcal{U}_\mathcal{X}}(F) = \operatorname{val}_\mathcal{X}(S, F)$ for every connected flat in $\mathcal{U}_\mathcal{X}$. Let $F$ be a connected flat in $\mathcal{U}_\mathcal{X}$. Then the definition of $\mathcal{U}_\mathcal{X}$ implies that the rank of $F$ is either $k$ or $k - 1$.

Suppose that the rank of $F$ is $k$. Then we have $F = E$. By hypothesis, $E$ can be constructed from $Y$ by a weakly $\mathcal{X}$-saturated sequence $S$. Then $r_{\mathcal{U}_\mathcal{X}}(E) = k = |Y| = \operatorname{val}(E, S)$ follows.

Hence we may assume that the rank of $F$ is $k - 1$. Since $F$ is a flat in $\mathcal{U}_\mathcal{X}$, every subset of $F$ of size $k$ belongs to $\mathcal{X}$. We will use this fact to define a weakly $\mathcal{X}$-saturated sequence for $F$. Choose a set $F_0$ of $k - 1$ elements in $F$, and let $X_e = F_0 \cup \{e\}$ for each $e \in F \setminus F_0$. Then each $X_e \in \mathcal{X}$, and $\{X_e : e \in F \setminus F_0\}$ (ordered arbitrarily) is a weakly $\mathcal{X}$-saturated sequence $S'$ which constructs $F$ from $F_0$. We have $\operatorname{val}(S', F) = |F_0| = k - 1 = r_{\mathcal{U}_\mathcal{X}}(F)$ as required.

**Applications to matroids on graphs.**

Given graphs $G$ and $H$ and subgraphs $F_0, F \subseteq G$, we say that $F$ can be constructed by a weakly $H$-saturated sequence from $F_0$ if $E(F)$ can be constructed by a weakly $\{H\}_G$-saturated sequence from $E(F_0)$.

**Lemma 3.3.** Let $H_k$ be the vertex-disjoint union of $k$ copies of $K_2$. Then $K_n$ can be constructed by a weakly saturated $H_k$-sequence from any copy of $H_k$ in $K_n$ whenever $n \geq 2k + 1$.

**Proof.** Let $H = \{e_1, e_2, \ldots, e_k\}$ be a copy of $H_k$ in $K_n$ for some $n \geq 2k + 1$. We show that $K_n$ has a weakly saturated $H_k$-sequence starting from $H$ by induction on $n$. Choose a vertex $v \in V(K_n) \setminus V(H)$.

Suppose $n = 2k + 1$. For each edge $f$ from $v$ to $H$ we can choose a $k$-matching $H^f$ containing $f$ and $k - 1$ edges of $H$. Then for each edge $g$ of $(K_n - v) - E(H)$ we can choose a $k$-matching $H^g$ containing $g$ and $k - 1$ edges of $H \cup \bigcup_{f \sim v} H^f$. Concatenating $H$ with $H^f$ for $f \sim v$ and then $H^g$ for the remaining edges $g$ gives a weakly saturated $H_k$-sequence which constructs $K_{2k+1}$ from $H$.

Now suppose $n > 2k + 1$. By induction, $K_n - v$ has a weakly saturated $H_k$-sequence $S$ starting from $H$. For each edge $f$ from $v$ to $H$ we can choose a $k$-matching $H^f$ containing $f$ and $k - 1$ edges of $K_n - v$. Concatenating $H$ with $H^f$ gives the required weakly saturated $H_k$-sequence for $K_n$.

Combining Lemmas 3.1 and 3.3, we immediately obtain:

**Theorem 3.4.** Let $H_k$ be the vertex-disjoint union of $k$ copies of $K_2$. Then $\mathcal{U}_{k-1}(K_n)$ is the unique maximal $H_k$-matroid on $K_n$ for all $n \geq 2k + 1$, and $\operatorname{val}_{H_k}$ is its rank function.
Let $P_k$ be the path with $k$ edges. It is straightforward to show that $K_n$ can be constructed by a weakly saturated $P_k$-sequence starting from a particular copy of $P_k$ in $K_n$ whenever $n \geq k + 1$. Lemma 3.1 now gives:

**Theorem 3.5.** Let $P_k$ be the path of length $k$. Then $U_{k-1}(K_n)$ is the unique maximal $P_k$-matroid on $K_n$ for all $n \geq k + 1$, and $\text{val}_{P_k}$ is its rank function.

Sitharam and Vince [27] showed that $U_{K_{1,3}}(K_n)$ is the unique maximal $K_{1,3}$-matroid on $K_n$ for all $n \geq 4$. Their result can be deduced from Lemma 3.2 since $K_{1,3}$ is union-stable and $K_n$ can be constructed by a weakly $K_{1,3}$-saturated sequence starting from a copy of $K_3$. We may also deduce that $\text{val}_{K_{1,3}}$ is the rank function of $U_{K_{1,3}}(K_n)$.

It is an open problem to determine whether there is a unique maximal $T_k$-matroid on $K_n$ for any fixed tree $T_k$ with $k$ edges. The following result gives some information on this poset: it implies that the rank of every $T_k$-matroid on $K_n$ is bounded by a quadratic polynomial in $k$.

**Lemma 3.6.** Suppose $H$ is a graph with $s$ vertices and minimum degree $\delta$, and $M$ is a $H$-matroid on $K_n$ with $n \geq s - 1$. Then the rank of $M$ is at most $(\delta - 1)(n - s + 1) + (\binom{s - 1}{2})$.

**Proof.** Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and put $V_i = \{v_1, v_2, \ldots, v_i\}$ and $E_i = E(K[V_i])$ for $1 \leq i \leq n$. Choose a base $B_{s-1}$ of $E(K[V_{s-1}])$ in $M$. Clearly $|B_{s-1}| \leq (\binom{s - 1}{2})$.

We inductively construct a base $B_i$ of $E_i$ in $M$ from $B_{s-1}$, for $i = s, \ldots, n$. Suppose we have a base $B_i$ of $E_i$; and let $B'_{i+1} = B_i \cup \{v_{i+1}v_1, \ldots, v_{i+1}v_{s-1}\}$. Then, for each $j$ with $s \leq j \leq i$, $K(V_j) + \{v_{i+1}v_1, \ldots, v_{i+1}v_{s-1}, v_{i+1}v_j\}$ contains a graph isomorphic to $H$ in which the degree of $v_{i+1}$ is equal to $\delta$. Hence $B'_{i+1}$ spans $E_i$ in $M$. Let $B_{i+1}$ be a base of $B'_{i+1}$ obtained by extending $B_i$. Then $B_{i+1}$ is obtained from $B_i$ by adding at most $\delta - 1$ edges. This implies that $B_n$ has size at most $(\delta - 1)(n - s + 1) + (\binom{s - 1}{2})$. The lemma now follows since $B_n$ is a base of $M$. \hfill $\square$

We close this section with one more application of Lemma 3.2. Consider the graph $G_5$ in Figure 1. It is straightforward to check that $G_5$ is union-stable and that $K_n$ can be constructed by a weakly saturated $G_5$-sequence starting from $K_{2,3}$. Hence $U_{G_5}(K_n)$ is the unique maximal $G_5$-matroid on $K_n$ for all $n \geq 5$, and $\text{val}_{G_5}$ is its rank function.

### 4 Matroids Induced by Submodular Functions

In this section we use weakly saturated sequences and a matroid construction due to Edmonds to give more examples of unique maximal matroids.

**Theorem 4.1** (Edmonds [10]). Let $E$ be a finite set and $f : 2^E \to \mathbb{Z}$ be a non-decreasing, submodular function. Put

$$\mathcal{I}_f := \{F \subseteq E : |I| \leq f(I) \text{ for any } I \subseteq F \text{ with } I \neq \emptyset\}.$$ 

Then $\mathcal{M}_f := (E, \mathcal{I}_f)$ is a matroid with rank function $\hat{f} : 2^E \to \mathbb{Z}$ given by

$$\hat{f}(F) := \min \left\{ |F_0| + \sum_{i=1}^k f(F_i) : F_0 \subseteq F \text{ and } \{F_1, \ldots, F_k\} \text{ is a partition of } F \setminus F_0 \right\}. \quad (2)$$
We refer to the matroid $\mathcal{M}_f$ given by Edmond’s theorem as the matroid induced by $f$. Given a set $F \subseteq E$ it is straightforward to check that:

\begin{align}
F \text{ is a circuit in } \mathcal{M}_f \text{ if and only if } 0 \neq |F| = f(F) + 1 \quad \text{(3)} \\
|F'| \leq f(F') \text{ for all } F' \subseteq F \text{ with } 0 \neq F' \neq F. 
\end{align}

\begin{align}
F \text{ is a flat in } \mathcal{M}_f \text{ if and only if } f(F + e) = f(F) + 1 \text{ for all } e \in E \setminus F. 
\end{align}

The function $f_{a,b} : 2^{E(K_n)} \to \mathbb{Z}$ by $f_{a,b}(F) = a|V(F)| - b$ is submodular and non-decreasing for all $a, b \in \mathbb{Z}$ with $a \geq 0$ and hence induces a matroid $\mathcal{M}_{f_{a,b}}(K_n)$ on $E(K_n)$. These matroids are known as count matroids. It is well known that the cycle matroid of $K_n$ is the count matroid $\mathcal{M}_{f_{1,1}}(K_n)$. Another well-known example is when $a = 2$ and $b = 3$, which gives the rigidity matroid of generic frameworks in $\mathbb{R}^2$. Sitharam and Vince [27] showed that $\mathcal{M}_{f_{1,1}}(K_n)$ and $\mathcal{M}_{f_{2,3}}(K_n)$ are the unique maximal $K_3$-matroid and $K_4$-matroid on $K_n$, respectively. Slightly weaker versions of these results were previously obtained by Graver [11].

We will show that the maximality of both these matroids, as well as that of several other count matroids, follow easily from Lemma 2.1 and Theorem 4.1. We need the following observation on the connected flats of count matroids which follows immediately from (3) and (4).

**Lemma 4.2.** Suppose $a, b \in \mathbb{Z}$ with $a \geq 0$ and $F \subseteq E(G)$ is a connected flat in $\mathcal{M}_{f_{a,b}}(G)$. Then $G[F]$ is the subgraph of $G$ induced by $V(F)$ and $|F| \geq a|V(F)| - b + 1$.

Let $K_n$ be the graph obtained from $K_n$ by removing an edge.

**Theorem 4.3.**
(a) $\mathcal{M}_{f_{1,1}}(K_n)$ is the unique maximal $K_3$-matroid on $K_n$ and its rank function is $\text{val}_{K_3}$.
(b) $\mathcal{M}_{f_{2,3}}(K_n)$ is the unique maximal $K_4$-matroid on $K_n$ and its rank function is $\text{val}_{K_4}$.
(c) $\mathcal{M}_{f_{1,0}}(K_n)$ is the unique maximal $K_4^-$-matroid on $K_n$ and its rank function is $\text{val}_{K_4^-}$.
(d) $\mathcal{M}_{f_{2,2}}(K_n)$ is the unique maximal $K_5^-$-matroid on $K_n$ and its rank function is $\text{val}_{K_5^-}$.
(e) $\mathcal{M}_{f_{3,6}}(K_n)$ is the unique maximal $K_6^-$-matroid on $K_n$ and its rank function is $\text{val}_{K_6^-}$.

**Proof.** In each case $\mathcal{M}_{f_{a,b}}(K_n)$ is loopless and is an $X$-matroid on $K_n$ for $X = K_3$, $K_4$, $K_4^-$, $K_5^-$, $K_6^-$, respectively, by (3). Lemmas 2.1 and 4.2 will now imply that $\mathcal{M}_{f_{a,b}}(K_n)$ is the unique maximal $X$-matroid on $K_n$ once we have shown that, for every $K_m \subseteq K_n$ with $|E(K_m)| > am - b$, there is a proper $X$-sequence $S$ with $r_{\mathcal{M}_{f_{a,b}}}(K_m) = \text{val}_{K_m}(S)$.

We will do this by finding a weakly saturated $X$-sequence which constructs $K_m$ from a subgraph $G \subseteq K_m$ with $|E(G)| = am - b$. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$.

In cases (a) and (b) we can use the well known fact that, for $m \geq d + 1$, $K_m$ can be constructed by a weakly saturated $K_{d+2}$-sequence starting from the spanning subgraph $G$ with $E(G) = \{e_i e_j : 1 \leq i < j \leq d\} \cup \{e_i e_j : 1 \leq i \leq d, d + 1 \leq j \leq m\}$, then taking $d = 3, 4$ for cases (a), (b), respectively.

In cases (c), (d) and (e) we can use a result of Pikhurko [23] that, for $d \geq m + 1$, $K_m$ has a weakly saturated $K_d^-$-sequence starting from a spanning subgraph with $(d - 3)m - \binom{d - 2}{2} + 1$ edges, and then taking $d = 4, 5, 6$ for cases (c), (d), (e), respectively. 

\[\square\]
Lemma 2.1 can also be used to extend Theorem 4.3 to matroids on non-complete graphs. For example, if \( G \) is a chordal graph, then every connected flat of \( \mathcal{M}_{f_{1,1}}(G) \) is a 2-connected chordal graph and we can use Lemma 2.1 and an appropriate weakly saturated \( K_3 \)-sequence to deduce that \( \mathcal{M}_{f_{1,1}}(G) \) is the unique maximal \( K_3 \)-matroid on \( G \) and \( \text{val}_{K_3} \) is its rank function.

Our next result gives another example of uniqueness for matroids on non-complete graphs.

**Theorem 4.4.** The matroid \( \mathcal{M}_{f_{1,3}}(K_{m,n}) \) is the unique maximal \( K_{2,3} \)-matroid on \( K_{m,n} \) and \( \text{val}_{K_{2,3}} \) is its rank function.

**Proof.** By (3) and (4), each copy of \( K_{2,3} \) is a circuit in \( \mathcal{M}_{f_{1,0}}(K_{m,n}) \) and each connected flat is a copy of \( K_{s,t} \) for some \( s \geq 2, t \geq 3 \). By the same argument as in the proof of Theorem 4.3 it will suffice to show that, for any \( K_{s,t} \) with \( s \geq 2, t \geq 3 \), there is a weakly saturated \( K_{2,3} \)-sequence which constructs \( K_{s,t} \) from a subgraph \( G \subset K_{s,t} \) with \( |E(G)| = s + t \). This follows easily by taking \( V(G) = \{u_1, u_2, \ldots, u_s\} \cup \{w_1, w_2, \ldots, w_t\} \) and \( E(G) = \{u_2w_2\} \cup \{u_iw_i : 1 \leq i \leq t\} \cup \{w_iw_1 : 2 \leq i \leq s\} \).

In contrast to this result, we will see in Section 5 that there are two distinct maximal \( K_{2,3} \)-matroids on \( K_n \).

The even cycle matroid is the matroid \( \mathcal{M} \) on \( E(K_n) \), in which a set \( F \) is independent if and only if each connected component of the induced subgraph \( K_n[F] \) contains at most one cycle, and this cycle is odd if it exists. The rank function of \( \mathcal{M} \) is given by \( r_M(F) = |V(F)| - \beta(F) \), where \( \beta(F) \) denotes the number of bipartite connected components in the graph \( K_n[F] \). We can use this fact to define a modified version of count matroids.

For \( a, b, c \in \mathbb{Z} \), define \( g_{a,b,c} : 2^{E(K_n)} \rightarrow \mathbb{Z} \) by \( g_{a,b,c}(F) = a|V(F)| - b\beta(F) - c \). Then \( g_{a,b,c} \) is submodular and non-decreasing for all \( a, b \in \mathbb{Z} \) with \( a \geq 0 \) since the functions \( F \mapsto |V(F)| \) and \( F \mapsto |V(F)| - \beta(F) \) are both submodular and non-decreasing. Hence \( g_{a,b,c} \) induces a matroid \( \mathcal{M}_{g_{a,b,c}}(K_n) \) on \( E(K_n) \) whenever \( a \geq 0 \). We will give examples of families \( \mathcal{H} \) for which \( \mathcal{M}_{g_{a,b,c}}(K_n) \) is the unique maximal \( \mathcal{H} \)-matroid on \( K_n \). We need the following observation on the connected flats of \( \mathcal{M}_{g_{a,b,c}}(K_n) \) which follows immediately from (3) and (4).

**Lemma 4.5.** Suppose \( a, b, c \in \mathbb{Z} \) with \( a \geq 0 \), \( c \geq 0 \), and \( F \subseteq E(K_n) \) is a connected flat in \( \mathcal{M}_{g_{a,b,c}}(K_n) \). Then \( K_n[F] \) is either a complete graph with \( |F| \geq a|V(F)| - c + 1 \) or a complete bipartite graph with \( |F| \geq a|V(F)| - b - c + 1 \).

The hypothesis of Lemma 4.5(c) that \( c \geq 0 \) is needed to ensure that the circuits of \( \mathcal{M}_{g_{a,b,c}}(K_n) \) induce connected subgraphs of \( K_n \), which in turn implies that the same property holds for the connected flats of \( \mathcal{M}_{g_{a,b,c}}(K_n) \). This is not true when \( c \leq -1 \), for example the disjoint union of two copies of \( C_4 \) is both a circuit and a connected flat in \( \mathcal{M}_{g_{1,1,-1}}(K_n) \).

**Theorem 4.6.** (a) The even cycle matroid \( \mathcal{M}_{g_{1,1,0}}(K_n) \) is the unique maximal \( C_4 \)-matroid on \( K_n \) and its rank function is \( \text{val}_{C_4} \).

(b) \( \mathcal{M}_{g_{2,1,2}} \) is the unique maximal \( \{K_5^-, K_{3,4}\} \)-matroid on \( K_n \) and its rank function is \( \text{val}_{\{K_5^-, K_{3,4}\}} \).

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Proof. In each case, $M_{g_{a,b,c}}(K_n)$ is loopless and is an $\mathcal{X}$-matroid on $K_n$ for $\mathcal{X} = \{C_4\}_K$, and $\mathcal{X} = \{K_5^-,K_{3,3}\}_K$, respectively, by [3]. Lemmas 2.1 and 4.5 will now imply that $M_{g_{a,b,c}}(K_n)$ is the unique maximal $\mathcal{X}$-matroid on $K_n$ once we have shown that: for every $K_m \subseteq K_n$ with $|E(K_m)| \geq am - c + 1$, there is a weakly saturated $\mathcal{X}$-sequence which constructs $K_m$ from a subgraph $G \subseteq K_m$ with $|E(G)| = am - c$; and for every $K_{s,t} \subseteq K_n$ with $|E(K_{s,t})| \geq am - b - c + 1$, there is a weakly saturated $\mathcal{X}$-sequence which constructs $K_{s,t}$ from a subgraph $G \subseteq K_{s,t}$ with $|E(G)| = am - b - c$. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(K_{s,t}) = \{u_1, u_2, \ldots, u_s\} \cup \{w_1, w_2, \ldots, w_t\}$.

(a) For $m \geq 4$, $K_m$ can be constructed by a weakly saturated $C_4$-sequence starting from a spanning subgraph $G$ with $g_{1,1,0}(E(K_m)) = m$ edges by taking $E(G) = \{v_2v_3 \cup \{v_1v_i: 2 \leq i \leq m\}$.

(b) For $m \geq 5$, $K_m$ can be constructed by a weakly saturated $K_5^-$-sequence starting from a spanning subgraph $G$ with $g_{2,1,2}(E(K_m)) = 2m - 2$ edges by taking $E(G) = \{v_1v_2, v_3v_4 \cup \{v_iw_j: 1 \leq i \leq 2, 3 \leq j \leq m\}$. For $s \geq 3$ and $t \geq 4$, $K_{s,t}$ can be constructed by a weakly saturated $K_{3,4}$-sequence starting from a spanning subgraph $G$ with $g_{2,1,2}(E(K_{s,t})) = 2(s + t) - 3$ edges by taking $E(G) = \{u_iw_j: 1 \leq i \leq 3, 1 \leq j \leq 3\} \cup \{u_iw_j: 1 \leq i \leq 2, 4 \leq j \leq t\} \cup \{u_iw_j: 4 \leq i \leq s, 1 \leq j \leq 2\}$. \hfill \Box

The matroid $M_{g_{2,1,2}}(K_n)$ in Theorem 4.6(b) is the Dilworth truncation of the union of the graphic matroid and the even cycle matroid. It appears in the context of the rigidity of symmetric frameworks in $\mathbb{R}^2$, see for example [28].

The concept of count matroids has been extended to hypergraphs [22] and to group-labeled graphs [13]. The technique in this section can be adapted to both settings.

We close this section with a remark on the poset of all $\{K_4, K_{2,3}\}$-matroids on $K_n$. It is straightforward to check that $M_{g_{1,1,1}}(K_n)$ is a $\{K_4, K_{2,3}\}$-matroid on $K_n$. But we cannot show it is the unique maximal such matroid by using the same proof technique as Theorem 4.6 since Lemma 4.5 does not hold when $c < 0$. In fact, we will see in Theorem 5.4 below that $M_{g_{1,1,1}}(K_n)$ is not the unique maximal $\{K_4, K_{2,3}\}$-matroid on $K_n$.

5 Examples of Non-uniqueness

We will give three examples of posets of $\mathcal{H}$-matroids on $K_n$ in which there is not a unique maximal matroid. We will frequently use the following fact, which follows from the procedure for constructing the free erection of a matroid due to Duke [9], see for example [7, Algorithm 1].

Lemma 5.1. If $M_0$ is a symmetric matroid on $K_n$, then the free elevation of $M_0$ is a symmetric matroid on $K_n$.

Gyula Pap [21] observed that the cycle matroid of $K_n$ and the uniform $C_5$-matroid on $K_n$ are two distinct maximal $C_5$-matroids on $K_n$. We can use Lemma 2.6 to show that Pap’s example extends to $C_k$ for all $k \geq 5$. 


**Theorem 5.2.** There are two distinct maximal $C_k$-matroids on $K_n$ for all $k \geq 5$ and $n \geq \left(\frac{k-1}{2}\right) + 2$.

*Proof.* It is straightforward to check that $C_k$ is union-stable, and hence $U_{C_k}(K_n)$ is a matroid. Consider the free elevation $\mathcal{M}$ of $U_{C_k}(K_n)$. Lemmas 2.6 and 5.1 imply that $\mathcal{M}$ is a maximal $C_k$-matroid on $K_n$ and is symmetric. We will show that $\mathcal{M}$ contains a circuit $Z$ such that $K_n[Z]$ has minimum degree one. To see this, consider two distinct copies $X$ and $Y$ of $C_k$ such that $X \cap Y$ forms a path of length $k-2$. By the circuit elimination axiom, $(X \cup Y) - e$ contains a circuit $Z$ of $\mathcal{M}$ for any $e \in X \cap Y$. Since the copy of $C_3$ in $K_n[(X \cup Y) - e]$ is not a circuit in $U_{C_3}(K_n)$, it cannot be a circuit in $\mathcal{M}$. Hence $(X \cup Y) - e$ contains a circuit $Z$ such that $K_n[Z]$ has minimum degree one. We may now apply Lemma 3.6 with $H = Z$ to deduce that the rank of $\mathcal{M}$ is at most $(\frac{k-1}{2})$.

The facts that $\mathcal{M}$ is a maximal $C_k$-matroid and the cycle matroid of $K_n$ is a $C_k$-matroid of rank $n-1$, imply that there are at least two maximal $C_k$-matroids on $K_n$ whenever $n \geq \left(\frac{k-1}{2}\right) + 2$. \hfill $\Box$

We saw in Theorem 1.4 that the poset of all $K_{2,3}$-matroids on $K_{m,n}$ has a unique maximal element. We next show that this statement becomes false if we change the ground set to $E(K_n)$.

**Theorem 5.3.** There are two distinct maximal $K_{2,3}$-matroids on $K_n$ for all $n \geq 7$.

*Proof.* Since $K_{2,3}$ is union-stable, $U_{K_{2,3}}(K_n)$ is a matroid by Lemma 2.7, and its free elevation $\mathcal{M}$ is a maximal $K_{2,3}$-matroid on $K_n$ and is symmetric by Lemmas 2.6 and 5.1.

We will show that $U_{K_{2,3}}(K_n)$ has no non-trivial erection and hence $\mathcal{M} = U_{K_{2,3}}(K_n)$.

We first show that $\mathcal{M}$ contains a circuit $Z$ such that $K_n[Z]$ has minimum degree one. To see this consider the graphs $G_1$ and $G_2$ given in Figure 1. Both are isomorphic to $K_{2,3}$, and hence, by the circuit elimination axiom, the edge set of $G_3 = (G_1 \cup G_2) - v_2v_5$, is dependent in $\mathcal{M}$. Since every set of six edges which does not induce a copy $K_{2,3}$ is independent in $\mathcal{M}$, the edge set of $G_3$ is a circuit in $\mathcal{M}$. Since the graph $G_4$ in Figure 1 is isomorphic to $G_3$, the circuit elimination axiom now implies the edge set of $G_5 = (G_3 \cup G_4) - v_4v_5$ is dependent in $\mathcal{M}$. Again, since every set of six edges which does not induce a copy $K_{2,3}$ is independent in $\mathcal{M}$, the edge set of $G_5$ is a circuit in $\mathcal{M}$.

This implies that $\mathcal{M}$ is a $G_5$-matroid on $K_n$ and hence, by Lemma 5.1, the rank of $\mathcal{M}$ is at most 6. Since $U_{K_{2,3}}(K_n)$ has rank 6, this gives $\mathcal{M} = U_{K_{2,3}}(K_n)$, and hence $U_{K_{2,3}}(K_n)$ is a maximal $K_{2,3}$-matroid on $K_n$. Since the bicircular matroid $\mathcal{M}_{1,0}$ is a $K_{2,3}$-matroid on $K_n$ of rank $n$, we have at least two maximal $K_{2,3}$-matroids on $K_n$ whenever $n \geq 7$. \hfill $\Box$

Our final example of this section shows that the unique maximality property may not hold even if we restrict our attention to the poset of all partial elevations of a given $\mathcal{H}$-matroid on $K_n$ (and hence provides another example, in addition to that given by Brylawski [5], which shows that the free elevation may not be the unique maximal matroid in the poset of all partial elevations of a given matroid). Note that the matroids described in Theorem 5.2 and 5.3 do not give such an example.

**Theorem 5.4.** There are two distinct maximal matroids in the poset of all partial elevations of the uniform $\{K_4, K_{2,3}\}$-matroid $U_{\{K_4, K_{2,3}\}}(K_n)$ whenever $n \geq 36$. 

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Proof. Let $\mathcal{X} = \{K_4, K_{2,3}\}$. Since $\mathcal{X}$ is 6-uniform and union-stable, $\mathcal{U}_\mathcal{X}$ is a matroid. Hence the free elevation $\mathcal{M}$ of $\mathcal{U}_\mathcal{X}$ is a symmetric $\mathcal{X}$-matroid on $K_n$ by Lemma 5.1. We will show that $\mathcal{M}$ has bounded rank.

Claim 5.5. The rank of $\mathcal{M}$ is at most 36.

Proof. Let $D_1$ be the edge set of the union of a vertex-disjoint 3-cycle and 4-cycle in $K_n$ and $D_2$ be the edge set of the union of two vertex-disjoint 4-cycles in $K_n$. We split the proof into three cases.

Case 1: $D_1$ is dependent in $\mathcal{M}$. Since $|D_1| = 7$ and neither $K_4$ nor $K_{2,3}$ is contained in $K_n[D_1]$, every proper subset of $D$ is independent in $\mathcal{M}$, and hence $D_1$ is a circuit in $\mathcal{M}$. By Lemma 2.4, the closure $\text{cl}_\mathcal{M}(D_1)$ of $D_1$ is the union of copies of $K_4$ and $K_{2,3}$. Hence $\text{cl}_\mathcal{M}(D_1) \neq D_1$ and, for each $e \in \text{cl}_\mathcal{M}(D_1) \setminus D_1$, there exists a circuit $C$ with $e \in C \subsetneq D_1 + e$. Since $K_n[D_1 + e]$ cannot contain $K_4$ or $K_{2,3}$ and $|D_1 + e| = 8$ we have $|C| = 7$. Observe that any 7-element subset of $D_1 + e$ containing $e$ has a vertex of degree one. We may now use Lemma 3.6 and the fact that $|V(C)| \leq 9$ to deduce that $\mathcal{M}$ has rank at most $\binom{9}{2} = 28$.

Case 2: $D_2$ is independent in $\mathcal{M}$ and $D_2$ is dependent in $\mathcal{M}$. If some proper subset $C$ of $D_2$ is a circuit in $\mathcal{M}$ then $K_n[C]$ would contain a vertex of degree one and we could again use Lemma 3.6 and the fact that $|V(C)| \leq 8$ to deduce that $\mathcal{M}$ has rank at most $\binom{9}{2} = 21$. Hence we may assume that $D_2$ is a circuit.

By Lemma 2.4, $\text{cl}_\mathcal{M}(D_2)$ is the union of copies of $K_4$ and $K_{2,3}$. Hence $\text{cl}_\mathcal{M}(D_2) \neq D_2$ and, for each $e \in \text{cl}_\mathcal{M}(D_2) \setminus D_2$, there exists a circuit $C'$ with $e \in C' \subsetneq D_2 + e$. Since $K_n[D_2 + e]$ cannot contain $K_4$ or $K_{2,3}$, and $|D_2 + e| = 9$, we have $7 \leq |C'| \leq 8$. Observe that every subset of $D_2 + e$ of size 7 or 8 which contains $e$ and is distinct from $D_1$, has a vertex of degree one. Hence, we may use Lemma 3.6 and the fact that $|V(C')| \leq 10$ to deduce that $\mathcal{M}$ has rank at most $\binom{9}{2} = 36$.

Figure 1: The graphs $G_1, G_2, \ldots, G_5$ in the proof of Theorem 5.3.
Case 3: $D_1, D_2$ are independent in $\mathcal{M}$. By Theorem 4.4 if $F \subseteq E(K_n)$ induces a bipartite subgraph in $K_n$, then $F$ has rank at most $|V(F)|$ in any $K_{2,3}$-matroid. We can use this fact, to compute the rank of the edge set $D_3$ of the graph $G$ in Figure 2, in $\mathcal{M}$.

Let $B$ be a base of $D_3$ which contains the independent subset $D_2$. Since $D_2 + e$ induces a bipartite subgraph with $|V(D_2 + e)| + 1$ edges for each $e \in D_3 \setminus D_2$, $D_2 + e$ is dependent.

Hence $B = D_2$ and $r_{\mathcal{M}}(D_3) = |D_2| = 8$.

On the other hand the edge set $D_4$ of the 5-cycle with a chord in $G$ is independent in $\mathcal{M}$ (since $D_4$ is independent in $U_K$). Since $r_{\mathcal{M}}(D_3) = 8$, this implies that $D_3 - e$ contains a circuit $C$ of $\mathcal{M}$ with $1 \leq |C \setminus D_4| \leq 3$. Then $K_n[C]$ has a vertex of degree 1, and we may again use Lemma 3.6 and the fact that $|V(C)| \leq 8$ to deduce that $\mathcal{M}$ has rank at most $\binom{n}{3} = 21$.

We can now complete the proof by observing that the matroid $\mathcal{M}_{g_{1,1,-1}}(K_n)$ from Section 4 is a partial elevation of $U_K$ and has rank $n + 1$. Since $\mathcal{M}$ is a maximal partial elevation of $U_K$ by Lemma 2.2 and has rank at most 36 by Claim 5.5, $\mathcal{M}$ is not the unique maximal partial elevation of $U_K$ for all $n \geq 36$. This completes the proof.

The above proof also implies that there are two distinct maximal $\{K_4, K_{2,3}\}$-matroids on $K_n$ for all $n \geq 36$. The only modification is that we use Lemma 2.6 in the final paragraph to deduce that $\mathcal{M}$ is a maximal $\{K_4, K_{2,3}\}$-matroid on $K_n$.

6 Matroids from Rigidity, Hyperconnectivity and Matrix Completion

6.1 Rigidity matroids, cofactor matroids and $K_{d+2}$-matroids

We are given a generic realisation $p : V(K_n) \rightarrow \mathbb{R}^d$ and we would like to know when a subgraph $G \subseteq K_n$ is $d$-rigid i.e., every continuous motion of the vertices of $(G,p)$ which preserves the distances between adjacent pairs of vertices must preserve the distances between all pairs of vertices. The (edge sets of the) minimal $d$-rigid spanning subgraphs of $K_n$ are the bases of a matroid $\mathcal{R}_d(K_n)$ which is referred to as the $d$-dimensional generic rigidity matroid. It is well known that $\mathcal{R}_d(K_n)$ is a $K_{d+2}$-matroid on $K_n$ and $\mathcal{R}_1(K_n)$ is the cycle matroid of $K_n$. Pollaczeck-Geiringer [24] and subsequently Laman [18] showed that $\mathcal{R}_2(K_n) = \mathcal{M}_{f_{2,3}}(K_n)$. Charactising $\mathcal{R}_d(K_n)$ for $d \geq 3$ is an important open problem in discrete geometry.

Graver [11] suggested we may get a better understanding of $\mathcal{R}_d(K_n)$ by studying the poset of all abstract $d$-rigidity matroids on $K_n$. This can be defined, using a result of Nguyen [20], as the poset of all $K_{d+2}$-matroids on $K_n$ of rank $dn - \binom{d+1}{2}$. Graver conjectured that $\mathcal{R}_d(K_n)$ is the unique maximal element in this poset and verified his
conjecture for the cases when \( d = 1, 2 \). The same proofs yield the slightly stronger results given in Theorem 4.3 (a) and (b).

Whiteley [30] showed that Graver’s conjecture is false when \( d \geq 4 \) by showing that the cofactor matroid \( c_{d-1}^{d-2}(K_n) \) from the theory of bivariate splines is an abstract \( d \)-rigidity matroid for all \( d \geq 1 \), and that \( R_d(K_n) \neq c_{d-1}^{d-2}(K_n) \) for all \( d \geq 4 \) and sufficiently large \( n \). He offered the revised conjecture that \( c_{d-1}^{d-2}(K_n) \) is the unique maximal element in the poset of all abstract \( d \)-rigidity matroids on \( K_n \). We recently verified the case \( d = 3 \) of this conjecture in joint work with Clinch [6].

**Theorem 6.1** ([6]). The cofactor matroid \( C_d^1(K_n) \) is the unique maximal \( K_5 \)-matroid on \( K_n \) and \( \text{val}_{K_5} \) is its rank function.

We propose the following strengthening of Whiteley’s conjecture.

**Conjecture 6.2.** The cofactor matroid \( C_{d-2}^d(K_n) \) is the unique maximal \( K_{d+2} \)-matroid on \( K_n \) for all \( d \geq 1 \) and \( \text{val}_{K_{d+2}} \) is its rank function.

### 6.2 Birigidity and rooted \( K_{s,t} \)-matroids on \( K_{m,n} \)

Let \( H \) be a bipartite graph with bipartition \((A, B)\) and \( K_{m,n} \) be a copy of the complete bipartite graph with bipartition \((U, W)\) where \(|U| = m\) and \(|W| = n\). We say that a subgraph \( H' \) of \( K_{m,n} \) is a **rooted copy of \( H \)** in \( K_{m,n} \) if there is an isomorphism \( \theta \) from \( H \) to \( H' \) with \( \theta(A) \subseteq U \) and \( \theta(B) \subseteq W \). Let \( \{H\}^*_K \) be the set of all rooted-copies of \( H \) in \( K_{m,n} \). A matroid \( \mathcal{M} \) on \( K_{m,n} \) is said to be a **rooted \( H \)-matroid** if it is a \( \{H\}^*_K \)-matroid. Note that the given ordered bipartition \((A, B)\) of \( H \) plays a significant role in this definition - we do not require that an isomorphic image \( \theta(H) \) of \( H \) in \( K_{m,n} \) is a circuit in \( \mathcal{M} \) when \( \theta(A) \not\subseteq U \). On the other hand, if \( H \) has an automorphism which maps \( A \) onto \( B \), then we will get the same matroid for each ordering of the bipartition of \( H \) and this matroid will be equal to the (unrooted) \( H \)-matroid on \( K_{m,n} \).

#### 6.2.1 Birigidity matroids

As a primary example of matroids on complete bipartite graphs, we shall introduce the birigidity matroids of Kalai, Nevo, and Novik [17].

Let \( G = (U \cup W, E) \) be a bipartite graph with \( m = |U| \) and \( n = |W| \), \( p : U \to \mathbb{R}^k \), and \( q : W \to \mathbb{R}^\ell \). We assume that the vertices of \( U \) and \( W \) are ordered as \( u_1, u_2, \ldots, u_m \) and \( v_1, v_2, \ldots, v_n \), respectively. We define the \((k, \ell)\)-**rigidity matrix** of \((G, p, q)\), denoted by \( R_{k,\ell}(G, p, q) \), to be the matrix of size \(|E| \times (\ell m + kn)\) in which each vertex in \( U \) labels a set of \( \ell \) consecutive columns from the first \( \ell m \) columns, each vertex in \( W \) labels a set of \( k \) consecutive columns from the last \( kn \) columns, each row is associated with an edge, and the row labelled by the edge \( e = u_i w_j \) is

\[
e = u_i w_j \begin{bmatrix} u_i & 0 & \cdots & 0 & q(w_j) & 0 & \cdots & 0 & p(u_i) & 0 & \cdots & 0 \end{bmatrix}
\]

The generic \((k, \ell)\)-rigidity matroid \( R_{m,n}^{k,\ell} \) is the row matroid of \( R_{k,\ell}(K_{m,n}, p, q) \) for any generic \( p \) and \( q \). It can be checked that the rank of \( R_{m,n}^{k,\ell} \) is equal to \( \ell m + kn - k \ell \), from which it follows that \( K_{k+1,\ell+1} \) is a circuit and \( R_{m,n}^{k,\ell} \) is a rooted \( K_{k+1,\ell+1} \)-matroid.
As pointed in \[17\], $\mathcal{R}_{m,n}^{k,\ell}$ coincides with the picture lifting matroids extensively studied by Whiteley \[29\] when $\min\{k, \ell\} = 1$. We will show that this matroid is the unique maximal rooted $K_{k+1,\ell+1}$-matroid in this case.

**Theorem 6.3.** $\mathcal{R}_{m,n}^{k,1}$ is the unique maximal rooted $K_{k+1,2}$-matroid on $K_{m,n}$.

**Proof.** Whiteley \[29\] showed that the picture lifting matroid is the matroid induced by the submodular, non-decreasing function $h : 2^{E(K_{m,n})} \to \mathbb{Z}$ defined by

$$h(F) := |U(F)| + k|W(F)| - k \quad (F \subseteq E(K_{m,n})),$$

where $U(F)$ and $W(F)$ denote the sets of vertices in $U$ and $W$, respectively, that are incident to $F$. Since every connected flat in $\mathcal{M}_h(K_{m,n})$ is a complete bipartite graph $K_{m',n'}$ for some $m' \geq 1$ and $n' \geq 2$, we may deduce the theorem from Lemma 2.1 by showing that $K_{m',n'}$ can be constructed by a weakly saturated, rooted $K_{k+1,2}$-sequence from a subgraph $G$ with $m' + kn' - k$ edges. Such a sequence is easily obtained by taking

$$E(G) = \{u_iw_1 : 1 \leq i \leq m'\} \cup \{u_iw_j : 1 \leq i \leq k \text{ and } 2 \leq j \leq n'\}.$$

We refer the reader to \[1\] for more details on weakly saturated, rooted $K_{s,t}$-sequences in $K_{m,n}$.

Lemma 2.1 also tells us that the rank function of $\mathcal{R}_{m,n}^{k,1}$ is determined by proper, rooted $K_{k+1,2}$-sequences. We conjecture that this extends to $\mathcal{R}_{m,n}^{k,\ell}$ for all $k, \ell \geq 1$.

**Conjecture 6.4.** $\mathcal{R}_{m,n}^{k,\ell}$ is the unique maximal rooted $K_{k+1,\ell+1}$-matroid on $K_{m,n}$ and the rank of any $F \subseteq E(K_{n,m})$ is given by

$$r(F) = \min\{\text{val}(F, S) : S \text{ is a proper, rooted } K_{k+1,\ell+1}\text{-sequence in } K_{m,n}\}.$$

The special case of this conjecture for $\mathcal{R}_{m,n}^{2,2}$ is equivalent to a conjecture on the rank function of $\mathcal{R}_{m,n}^{2,2}$ given in \[14\, Section 8\]. Bernstein \[2\] gave an NP-type combinatorial characterization for independence in $\mathcal{R}_{m,n}^{2,2}$, but no co-NP-type characterization is known. The special case $k = \ell = 2$ of Conjecture 6.4 would provide such a certificate but even this special case seems challenging. As some evidence in support of the conjecture, we can show that Conjecture 1.3 holds for the poset of $K_{3,3}$-matroids on $K_{m,n}$.

**Theorem 6.5.** The following statements are equivalent.

(a) There is a unique maximal $K_{3,3}$-matroid on $K_{m,n}$.

(b) $\text{val}_{K_{3,3}}$ is submodular on $K_{m,n}$.

We will sketch a proof of Theorem 6.5 after Theorem 6.9 below (which gives an analogous result for $\{K_4, K_{3,3}\}$-matroids on $K_n$).
6.3 Hyperconnectivity matroids, matrix completion and \( \{K_d, K_{s,t}\} \)-matroids on \( K_n \)

Let \( p : V(K_n) \to \mathbb{R}^d \) be a generic map. We assume that the vertices of \( K_n \) are ordered as \( v_1, v_2, \ldots, v_n \). Kalai [14] defined the \( d \)-hyperconnectivity matroid, \( \mathcal{H}_{dn}^d \), to be the row matroid of the matrix of size \( \left( \binom{n}{2} \times dn \right) \) in which each vertex of \( K_n \) labels a set of \( d \) consecutive columns, each row is labelled by an edge of \( K_n \), and the row labelled by the edge \( e = v_i v_j \) with \( i < j \) is

\[
e = v_i v_j \quad \begin{bmatrix} v_i & 0 & \cdots & 0 & v_j \\ 0 & p(v_j) & \cdots & p(v_i) & 0 & \cdots & 0 \end{bmatrix},
\]

He showed that, when \( n \geq 2d + 2 \), this matroid is a \( \{K_{d+2}, K_{d+1,d+1}\} \)-matroid of rank \( dn - \binom{d+1}{2} \).

As a variant of \( \mathcal{H}_{dn}^d \), Kalai [15] also introduced the matroid \( \mathcal{I}_{dn}^d \), which is the row matroid of the \( \left( \binom{n}{2} \times dn \right) \)-matrix with rows

\[
e = v_i v_j \quad \begin{bmatrix} v_i & 0 & \cdots & 0 & v_j \\ 0 & p(v_j) & \cdots & p(v_i) & 0 & \cdots & 0 \end{bmatrix}
\]

instead of (5). He showed that, when \( n \geq 2d + 2 \) and \( d \geq 2 \), \( \mathcal{I}_{dn}^d \) is a \( \{K_{d+1,d+1}\} \)-matroid on \( K_n \). In the special case when \( d = 2 \), this rank constraint implies that \( \mathcal{I}_{n}^2 \) is a \( \{K_5, K_{3,3}\} \)-matroid.

The matroids \( \mathcal{H}_{dn}^d \) and \( \mathcal{I}_{dn}^d \) arise naturally in the context of the rank \( d \) completion problem for partially filled \( n \times n \) matrices which are skew-symmetric and symmetric, respectively, see [4, 25]. The restriction of either \( \mathcal{I}_{dn}^d \) or \( \mathcal{H}_{dn}^d \) to the complete bipartite graph \( K_{m,n} \) is the birigidity matroid \( \mathcal{R}_{m,n}^{d,d} \), and this matroid arises in the context of the rank \( d \) completion problem for partially filled \( m \times n \) matrices, see [25].

When \( d = 1 \), \( \mathcal{H}_{n}^1 \) is the cycle matroid (and hence is the unique maximal \( \{K_3, K_{2,2}\} \)-matroid on \( K_n \) by Theorem 4.3(a)) and \( \mathcal{I}_{n}^1 \) is the even cycle matroid (and hence is the unique maximal \( K_{2,2} \)-matroid on \( K_n \) by Theorem 4.6(a)).

We can find one more example of a \( \{K_d, K_{s,t}\} \)-matroid in rigidity theory. Bolker and Roth [3] showed that \( K_{d+2,d+2} \) is a circuit in the \( d \)-dimensional rigidity matroid \( \mathcal{R}_d(K_n) \) when \( d \geq 3 \). Hence \( \mathcal{R}_d(K_n) \) is a \( \{K_{d+2}, K_{d+2,d+2}\} \)-matroid on \( K_n \) for all \( d \geq 3 \).

We conjecture that each of \( \mathcal{H}_{dn}^d, \mathcal{I}_{dn}^d \) and \( \mathcal{R}_{dn}^d \) is the unique maximal matroid in its respective poset.

**Conjecture 6.6.** (a) For \( n \geq 2d + 2 \), \( \mathcal{H}_{dn}^d \) is the unique maximal \( \{K_{d+2}, K_{d+1,d+1}\} \)-matroid on \( K_n \) and its rank function is \( \text{val}(\{K_{d+2}, K_{d+1,d+1}\}) \).

(b) For \( d = 2 \) and \( n \geq 6 \), \( \mathcal{I}_{n}^2 \) is the unique maximal \( \{K_5, K_{3,3}\} \)-matroid on \( K_n \) and its rank function is \( \text{val}(\{K_5, K_{3,3}\}) \).

(c) For \( d \geq 3 \) and \( n \geq 2d + 4 \), \( \mathcal{R}_{dn}^d \) is the unique maximal \( \{K_{d+2}, K_{d+2,d+2}\} \)-matroid on \( K_n \) and its rank function is \( \text{val}(\{K_{d+2}, K_{d+2,d+2}\}) \).

We close this section by considering the special case of Conjecture 6.6(a) when \( d = 2 \).
6.3.1 \( \{K_4, K_{3,3}\} \)-matroids on \( K_n \)

Understanding the poset of all \( \{K_4, K_{3,3}\} \)-matroids on \( K_n \) is important since these matroids appear in applications such as the rank two completion of partially filled skew-symmetric matrices and partially-filled rectangular matrices, see \([4, 23]\). We shall prove that, if this poset has a unique maximal element, then the rank function of the maximal element is \( \text{val}_{\{K_4, K_{3,3}\}} \). This confirms Conjecture \([1, 3]\) for \( \{K_4, K_{3,3}\} \)-matroids. We will need two general results for a matroid on the edge set of a graph. The first was proved for the special case of abstract rigidity matroids in \([7]\). The same proof gives:

**Lemma 6.7.** Let \( \mathcal{M} \) be a matroid defined on the edge set of a graph \( G \). Suppose that \( G[C] \) is 2-connected for every circuit \( C \) in \( \mathcal{M} \). Then, for every connected set \( X \) in \( \mathcal{M} \),

\[
\sum_{v \in V(X)} \min\{d_B(v) : B \text{ is a basis of } X\} \leq 2(r_M(X) - 1) - |V(X)|.
\]

Our second lemma, concerns a well known graph operation from rigidity theory. Given a graph \( G \), the 0-extension operation constructs a new graph by adding a new vertex \( v_0 \) and two edges \( v_0v_1 \) and \( v_0v_2 \) with distinct \( v_1, v_2 \in V(G) \). We say that a matroid \( M \) on \( K_n \) has the 0-extension property if every 0-extension preserves independence in \( M \), i.e. \( E(G') \) is independent if \( E(G) \) is independent and \( G' \) is obtained from \( G \) by a 0-extension operation for all \( G, G' \subseteq K_n \).

**Lemma 6.8.** Let \( \mathcal{M} \) be a \( K_4 \)-matroid on \( K_n \) with the 0-extension property. Then, every circuit in \( \mathcal{M} \) induces a 2-connected subgraph of \( K_n \).

**Proof.** Suppose, for a contradiction, that some circuit \( C \) in \( \mathcal{M} \) does not induce a 2-connected subgraph of \( K_n \).

We first consider the case when \( C \) is connected. Then \( C \) can be partitioned into two sets \( X \) and \( Y \) such that \( |V(X) \cap V(Y)| = 1 \). Let \( K \) be the edge set of the complete graph on \( V(Y) \). Since \( \mathcal{M} \) is a \( K_4 \)-matroid, Theorem \([4, 3] \)b) gives \( r_M(K) \leq 2|V(Y)| - 3 \). The fact that \( X \cup Y \) is a circuit now gives \( r_M(X \cup K) \leq r_M(X) + r_M(K) - 1 \leq |X| + 2|V(Y)| - 4 \).

We may construct an independent subset of \( X \cup K \) by extending the independent set \( X \) using 0-extensions. Let \( e \) be an edge in \( K \) incident to the vertex in \( V(X) \cap V(Y) \). Then \( X + e \) is independent by the 0-extension property. Repeatedly applying the 0-extension operation, we can extend \( X + e \) to an independent set \( B \) of size \( |X| + 1 + 2(|V(Y)| - 2) = |X| + 2|V(Y)| - 3 \) by adding edges in \( K \). This contradicts the fact that the rank of \( X \cup K \) is at most \( |X| + 2|V(Y)| - 4 \).

The case when \( C \) is not connected can be proved similarly. \( \square \)

We need one more graph operation. Given a vertex \( v_1 \) of a graph \( G \), the diamond splitting operation at \( v_1 \) (with respect to a fixed partition \( \{U_0, U^*, U_1\} \) of \( N_G(v_1) \) with \( |U^*| = 2 \)) removes the edges between \( v_1 \) and the vertices in \( U_0 \), adds a new vertex \( v_0 \), and adds new edges \( v_0u \) for all \( u \in U_0 \cup U^* \). We say that a matroid \( M \) on \( K_n \) has the diamond splitting property if any diamond splitting operation preserves independence in \( \mathcal{M} \). It was shown in \([17]\) that \( \mathcal{H}_n^2 \) has both the 0-extension property and the diamond splitting property.

We can now prove our main result on \( \{K_4, K_{3,3}\} \)-matroids.

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Theorem 6.9. Let $\mathcal{X} = \{K_4, K_{3,3}\}_{K_n}$. Then the following statements are equivalent.

(a) There is a unique maximal $\mathcal{X}$-matroid on $K_n$.

(b) The free elevation of $\mathcal{U}_X$ has the 0-extension property and the diamond splitting property.

(c) There is an $\mathcal{X}$-matroid on $K_n$ that has the 0-extension property, the diamond splitting property, and the $\mathcal{X}$-covering property.

(d) $\text{val}_X$ is submodular on $E(K_n)$.

Proof. (d) $\Rightarrow$ (a): This follows from Lemma [1.1].

(a) $\Rightarrow$ (b): Since $\mathcal{X}$ is 6-uniform and union-stable, $\mathcal{U}_X(K_n)$ is a maximal matroid in the poset of all $\mathcal{X}$-matroids on $K_n$ of rank at most 6. Clearly $\mathcal{U}_X(K_n) \neq \mathcal{U}_5(K_n)$. Lemma 2.3 and (a) now imply that the free elevation of $\mathcal{U}_X(K_n)$ is the unique maximal $\mathcal{X}$-matroid on $K_n$. Since $\mathcal{H}_n^2$ is an $\mathcal{X}$-matroid on $K_n$ with the 0-extension property and the diamond splitting property, the free elevation of $\mathcal{U}_X(K_n)$ also has the 0-extension property and the diamond splitting property.

(b) $\Rightarrow$ (c): This follows from Lemma 2.4.

(c) $\Rightarrow$ (d): Suppose that (c) holds for some $\mathcal{X}$-matroid $\mathcal{M}$ on $K_n$. We prove that $r_{\mathcal{M}} = \text{val}_X$. By Lemma 2.3 it suffices to show that, for each connected flat $F$ of $\mathcal{M}$, there is a proper $\mathcal{X}$-sequence $S$ such that $r_{\mathcal{M}}(F) = \text{val}(F, S)$. We prove this by induction on the rank of $F$.

Since $\mathcal{M}$ has the 0-extension property, Lemma 6.8 implies that every circuit in $\mathcal{M}$ induces a 2-connected subgraph of $K_n$. Since $\mathcal{M}$ is a $K_4$-matroid, $r_{\mathcal{M}}(F) \leq 2|V(F)| - 3$ and Lemma 6.7 now implies that there exists a base $B$ of $F$ and a vertex $v \in V(B)$ such that $d_B(v) \leq 2$. Let $F_v$ and $B_v$ be the set of edges in $F$ and $B$, respectively, which are not incident to $v$. We first show that

$$F_v \text{ is a flat in } \mathcal{M}, \quad d_B(v) = 2 \text{ and } r_{\mathcal{M}}(F_v) = r_{\mathcal{M}}(F) - 2. \quad (6)$$

To verify (6) we first note that, since $\mathcal{M}$ has the 0-extension property, every circuit in $\mathcal{M}$ has minimum degree at least three. Since $d_B(v) \leq 2$, this implies that $cl_{\mathcal{M}}(B_v) = F_v$ and hence $F_v$ is a flat in $\mathcal{M}$. In addition, since $F$ is connected in $\mathcal{M}$, we have $d_{\mathcal{M}}(v) \geq 3$. The facts that $d_B(v) \leq 2$ and $\mathcal{M}$ has the 0-extension property, now give $|B| = r_{\mathcal{M}}(F) \geq r_{\mathcal{M}}(F_v) + 2 = |B_v| + 2 \geq |B|$. Hence equality holds throughout and (6) holds.

Claim 6.10. Let $x, y \in N_F(v)$ and $z \in V(F_v)$ be three distinct vertices. Suppose that $xz, yz \in F_v$. Then $uz \notin F_v$ for all $u \in N_F(v) \setminus \{z\}$.

Proof. Suppose, for a contradiction, that $uz \notin F_v$ for some $u \in N_F(v) \setminus \{z\}$. Since $F_v$ is a flat, $B_v + uz$ is independent. We may construct $B' = B_v \cup \{vx, vy, vu\}$ from $B_v + uv$ by applying the diamond splitting operation to $z$ in such a way that the new vertex $v$ has degree three and is adjacent to $x, y, u$. Then $B'$ is contained in $F$ and is independent in $\mathcal{M}$. Since $|B'| > |B|$, this contradicts the fact that $B$ is a base of $F$. \qed
We may apply induction to each connected component of $F_v$ in $\mathcal{M}$ to obtain a proper $\mathcal{X}$-sequence $S' = (X_1, X_2, \ldots, X_t)$ such that $r_M(F_v) = \text{val}(F_v, S')$. Since $\mathcal{M}$ has the $\mathcal{X}$-covering property, $F$ is the union of copies of $K_4$ and $K_{3,3}$. Let $N_F(v) = \{u_1, u_2, \ldots, u_k\}$.

Suppose that some edge of $F$ which is incident to $v$ is contained in a copy $K_4$ in $F$. Relabelling if necessary, we may suppose that the complete graph $K(v, u_1, u_2, u_3)$ satisfies $K(v, u_1, u_2, u_3) \subseteq F$. We will show that $K(v, u_1, u_2, \ldots, u_t) \subseteq F$. For each $i = 4, 5, \ldots, t$, we may apply Claim 6.10 with $x = u_1, y = u_2, z = u_3, u = u_i$ to deduce that $u_1 u_1, u_1 u_2 \in F$.

Since $\mathcal{M}$ has the 0-extension property, this implies that $F$ contains an independent set of size $2|N_F(v)| - 3$ on $N_F(v)$. Since $F$ is a flat and every $A \subseteq E(K_n)$ has rank at most $2|V(A)| - 3$, this implies that $K(v, u_1, u_2, \ldots, u_t) \subseteq F$. Let $X_{t+i}$ be a copy of $K_4$ on $\{v, u_i, u_{i+1}, u_{i+2}\}$ for $i = 1, \ldots, k - 2$, and let $S = (X_1, \ldots, X_t, X_{t+1}, \ldots, X_{t+k-2})$ be obtained by appending $(X_{t+1}, \ldots, X_{t+k-2})$ to $S'$. Then we have $\text{val}(F, S) = \text{val}(F_v, S') + 2 = r_M(F_v) + 2 = r_M(F)$, as required.

It remains to consider the case when no edge of $F$ incident to $v$ is contained in a copy of $K_4$ in $F$. Then every edge in $F$ which is incident to $v$ is contained in a copy of $K_{3,3}$. Relabelling if necessary we may suppose that the complete bipartite graph $K(v, u_1, w_1; u_2, w_2)$ is contained in $F$. Then $w_i \notin N_F(v)$ for $i = 1, 2$, since otherwise the facts that $F$ is a flat and $\mathcal{M}$ is a $K_4$-matroid would imply that $K(v, u_1, u_2, w_i) \subseteq F$. By Claim 6.10 $F$ contains $u_1 w_1$ and $u_1 w_2$ for all $u_i \in N_F(v)$. Hence, $F$ contains the complete bipartite graph $K(N_F(v); \{w_1, w_2\})$. Let $X_{t+i} = K(v, w_1, w_2; u_i, u_{i+1}, u_{i+2})$ for $i = 1, \ldots, k - 2$, and let $S = (X_1, \ldots, X_t, X_{t+1}, \ldots, X_{t+k-2})$ be obtained by appending $(X_{t+1}, \ldots, X_{t+k-2})$ to $S'$. Then we have $\text{val}(F, S) = \text{val}(F_v, S') + 2 = r_M(F_v) + 2 = r_M(F)$, as required.

This completes the proof of the theorem. □

We can prove Theorem 6.5 by restricting the above argument to complete bipartite graphs. We need the following counterpart to Lemma 6.8.

Lemma 6.11. Let $\mathcal{M}$ be a $K_{3,3}$-matroid on $K_{m,n}$ with the 0-extension property. Then, every circuit in $\mathcal{M}$ induces a 2-connected subgraph of $K_{m,n}$.

Proof. Suppose, for a contradiction, that some circuit $C$ in $\mathcal{M}$ does not induce a 2-connected subgraph of $K_{m,n}$.

We first consider the case when $K_{m,n}[C]$ is connected. Then $C$ can be partitioned into two sets $X$ and $Y$ such that $|V(X) \cap V(Y)| = 1$.

Suppose that $|U(Y)| = 1$ or $|W(Y)| = 1$. Then $Y$ is a tree. Since $X$ is independent, the 0-extension property implies that $X \cup Y$ is independent, which is a contradiction. Hence, $|U(Y)| \geq 2$ and $|W(Y)| \geq 2$.

Let $K = K(U(Y); W(Y))$. Since $\mathcal{M}$ is a $K_{3,3}$-matroid on $K_{m,n}$, $r_M(E(K)) \leq 2|V(Y)| - 4$. The fact that $X \cup Y$ is a circuit now gives $r_M(X \cup E(K)) \leq r_M(X) + r_M(E(K)) - 1 \leq |X| + 2|V(Y)| - 5$.

We may construct an independent subset of $X \cup E(K)$ by extending the independent set $X$ using 0-extensions. Without loss of generality, we can assume $X$ and $Y$ share a vertex $u$ in $U$. By $|U(Y)| \geq 2$, we can choose $v \in U(Y) \setminus \{u\}$. Let $G$ be the graph obtained from $K_{m,n}[X]$ by appending $v$ as an isolated vertex. Then, repeatedly applying the 0-extension operation, we can extend $G$ to a graph $G'$ of size $|X| + 2(|V(Y)| - 2) = |X| + 2|V(Y)| - 4$. This contradicts the fact that the rank of $X \cup E(K)$ is at most $|X| + 2|V(Y)| - 5$.

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The case when $K_{m,n}[C]$ is disconnected is similar.

**Proof of Theorem 6.7** The proof proceeds in the same way as that of Theorem 6.9 by using Lemma 6.11 instead of Lemma 6.8.

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