Capacity of nonlinear bosonic systems

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We analyze the role of nonlinear Hamiltonians in bosonic channels. We show that the information capacity as a function of the channel energy is increased with respect to the corresponding linear case, although only when the energy used for driving the nonlinearity is not considered as part of the energetic cost and when dispersive effects are negligible.

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Noninteracting massless bosonic systems have been the object of extensive analysis\textsuperscript{1,2}. Their maximum capacity in transmitting information was derived for the noiseless case both in the narrow-band regime (where only few frequency modes are employed) and in the broad-band regime. However, it is still an open question whether nonlinearities in the system may increase these bounds: linearity appears to be the most important assumption in all previous derivations\textsuperscript{2}. Up to now nonlinear effects have been used in fiber optics communications to overcome practical limitations, such as using solitons to beat dispersion or traveling wave amplifiers to beat loss\textsuperscript{3}. The approach adopted in this paper is fundamentally different from these and from others where nonlinearities and squeezing are employed at the coding stage when using linear channels\textsuperscript{3}. We follow the cue of a recent proposal\textsuperscript{3} where interactions were exploited in increasing the capacity of a qubit-chain communication line. In the case of linear bosonic systems, the information storage capacity of a signal divided by the time it takes for it to propagate through the medium gives the transmission capacity of the channel. In the presence of nonlinearities, dispersion can affect the propagation of the signal complicating the analysis, but an increase in the capacity can be shown, at least when the dispersive effects are negligible. A complete analysis of dispersion in nonlinear materials is impossible at this stage, since the quantization of these systems has been solved only perturbatively. The basic idea behind the enhancement we find is that the modification of the system spectrum due to nonlinear Hamiltonians may allow one to better employ the available energy in storing the information: we will present some examples that exhibit such effect. Excluding the down conversion channel (a model sufficiently accurate to include the propagation issue –see Sec. II C), all these examples are highly idealized systems but are still indicative of the possible nonlinearity-induced enhancements in the communication rates. An important caveat is in order. In the physical implementations that we have analyzed, there is no capacity enhancement if we include in the energy balance also the energy required to create the nonlinear Hamiltonians. This is a general characteristic of any system: if one considers the possibility of employing all the available degrees of freedom to encode information, then one cannot do better than the bound obtained in the noninteracting case\textsuperscript{4,7}. However, the enhancement discussed here is not to be underestimated since in most situations many degrees of freedom are not usable to encode information, but can still be employed to augment the capacity of other degrees of freedom. A typical example is when the sender is not able to modulate the signals sufficiently fast to employ the full bandwidth supported by the channel: an external pumping (such as the one involved in the parametric down conversion case) may allow to increase the energy devoted to the transmission modes.

We start by describing a general procedure to evaluate the capacity of a system and we apply it to a linear bosonic system to reobtain some known results (see Sec. I). Such a procedure is instructive since it emphasizes the role of the system spectrum in the capacity calculation. We then analyze a collection of examples of nonlinear bosonic systems in the narrow-band and wide-band regimes (see Sec. III): for each case we describe the capacity enhancements over the corresponding linear systems. General considerations on the energy balance in the information storage and on the information propagation conclude the paper (see Sec. III).

I. INFORMATION CAPACITY

The information capacity of a noiseless channel is defined as the maximum number of bits that can be reliably sent per channel use. From the Holevo bound (proven in Ref.\textsuperscript{1} for the infinite dimensional case), we know that it is given by the maximum of the Von Neumann entropy $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ over all the possible input states $\rho$ of the channel. In our case, since $\rho$ is the state of a massless bosonic field, the associated Hilbert space is infinite dimensional and the maximum entropy is infinite. However, for all realistic scenarios a cut-off must be introduced by constraining the energy required in the storage or in the transmission, e.g. requiring the entropy $S(\rho)$ to be maximized only over those states that have an average energy $E$, i.e.

$$E = \text{Tr}[\rho H],$$

(1)
where $H$ is the system Hamiltonian. The constrained maximization of $S(\rho)$ can be solved by standard variational methods (see [8, 9] for examples), which entail the solution of

$$\delta \left\{ \frac{\lambda}{\ln 2} \text{Tr}[H\rho] - \frac{\lambda'}{\ln 2} \text{Tr}[\rho] \right\} = 0 ,$$

(2)

where $\lambda$ and $\lambda'$ are Lagrange multipliers that take into account the energy constraint (1) and the normalization constraint $\text{Tr}[\rho] = 1$, and where the ln2 factor is introduced so that all subsequent calculations can be performed using natural logarithms. Eq. (2) is solved by the density matrix $\rho = \exp[-\lambda H]/Z(\lambda)$ where

$$Z(\lambda) \equiv \text{Tr}[e^{-\lambda H}]$$

(3)

is the partition function of the system and $\lambda$ is determined from the constraint (1) by solving the equation

$$E = -\frac{\partial}{\partial \lambda} \ln Z(\lambda) .$$

(4)

The corresponding capacity is thus given by

$$C = S[\exp(-\lambda H)/Z(\lambda)] = [\lambda E + \ln Z(\lambda)] / \ln 2 ,$$

(5)

which means that we can evaluate the system capacity only from its partition function $Z(\lambda)$.

In general an explicit expression for $Z(\lambda)$ is difficult to derive, but it proves quite simple for noninteracting bosonic systems, such as the free modes of the electromagnetic field. In fact, in this case the Hamiltonian is given by

$$H = \sum_k \hbar \omega_k a_k^\dagger a_k ,$$

(6)

where $\omega_k$ is the frequency of the $k$th mode and the mode operators $a_k$ satisfy the usual commutation relations $[a_k, a_k^\dagger] = \delta_{k,k'}$. Hence, the partition function is

$$Z(\lambda) = \prod_k \sum_{n_k=0}^\infty e^{-\lambda n_k} \prod_k \frac{1}{1 - e^{-\lambda \omega_k}} .$$

(7)

From Eqs. (5) and (7) it is clear that the capacity $C$ will be ultimately determined by the spectrum $\omega_k$ of the system. In particular, in the narrow-band case (where only a mode of frequency $\omega$ is employed) Eq. (5) gives

$$C_{nb} = g \left( \frac{E}{\hbar \omega} \right) ,$$

(8)

where $g(x) \equiv (1 + x) \log_2(1 + x) - x \log_2 x$ for $x \neq 0$ and $g(0) = 0$. On the other hand, in the homogeneous wide-band case (where an infinite collection of equispaced frequencies $\omega_k = k\delta \omega$ is employed for $k \in \mathbb{N}$) Eq. (5) gives

$$C_{wb} \simeq \frac{\pi}{\ln 2} \sqrt{\frac{2E}{3\hbar \delta \omega}} ,$$

(9)

which is valid in the limit $\hbar \delta \omega \ll E$. When applied to communication channels, this last equation is usually expressed in terms of the rate $R$ (bits transmitted per unit time) and power $P$ (energy transmitted per unit time) as

$$R = \frac{1}{\ln 2} \sqrt{\frac{\pi P}{3\hbar}} ,$$

(10)

by identifying the transmission time with $2\pi/\delta \omega$. In the next section we analyze how the capacities $C_{nb}$ and $C_{wb}$ are modified by introducing nonlinear terms in the system Hamiltonian.

II. NONLINEAR HAMILTONIANS

Nonlinear terms in the Hamiltonian of the electromagnetic field derive from the interactions between the photons and the medium in which they propagate. In this section we will employ the techniques described above to derive the narrow-band and wide-band capacities when quadratic nonlinearities are present. In Secs. II A, II B, and II C we discuss parametric down-conversion type Hamiltonians in the narrow-band and broad-band regimes. In Secs. II D and II E we discuss a mode swapping interaction. All these nonlinearities arise in real-world systems from $\chi^{(2)}$ type couplings, when one of the three fields involved in these kinds of interactions is a strong pump field that can be considered classical [10]. For all the cases analyzed we present the capacity enhancement over the corresponding noninteracting Hamiltonian.

A. Squeezing Hamiltonian

Consider the single mode described by the Hamiltonian

$$H = \hbar \omega a^\dagger a + \hbar \xi \left( (a^\dagger)^2 + a^2 \right) / 2 + \hbar \Omega(\xi) ,$$

(11)

where $\xi$ is the squeezing parameter. We employ $|\xi| < \omega$ to avoid Hamiltonians which are unbounded from below. In Eq. (11) the frequency $\Omega(\xi) \equiv \frac{1}{2} \left( (\omega - \sqrt{\omega^2 - \xi^2} \right)$ has been introduced so that the ground state of the system is null: with this choice, the average energy $E$ is the energy associated to the mode $a$ in the nonlinear medium. By applying the canonical transformation

$$a = A \cosh \theta - A^\dagger \sinh \theta , \quad \theta \equiv \frac{1}{4} \ln \left( \frac{\omega + \xi}{\omega - \xi} \right) ,$$

(12)

the Hamiltonian $H$ is transformed to the free field form $\hbar \sqrt{\omega^2 - \xi^2} A^\dagger A$, so that the derivation of the previous section can be employed to calculate the capacity. It is thus immediate [see Eq. (8)] to find the capacity of this system as

$$C = g \left( \frac{E}{\hbar \sqrt{\omega^2 - \xi^2}} \right) .$$

(13)
This quantity measures the amount of information that can be encoded when \( E \) is the total energy associated with the mode \( a \), and a nonlinear squeezing-generating term is present in the Hamiltonian: this result is quite different from the one obtained by using squeezed states as inputs to a linear system (see for example \([1]\)). The capacity \( C \) of Eq. (14) is higher than the capacity \( C_{nb} \) of the linear case \( \xi = 0 \) since \( g(x) \) is an increasing function (see Fig. 1). The reason behind this enhancement is that the nonlinearity reduces the effective frequencies of the modes (from \( \omega \) to \( \sqrt{\omega^2 - \xi^2} \)), so that more energy levels can now be populated with the same energy.

**FIG. 1:** Capacity increase of the squeezing Hamiltonian of Eq. (11) as a function of the energy parameter \( E/\hbar\omega \) and of the squeezing ratio \( \xi/\omega \). The increase is evident from the positivity of \( C - C_{nb} \), where \( C \) is given in Eq. (13) and \( C_{nb} \) is the free space narrow-band capacity of Eq. (8). Notice that for high energy, the increase tends asymptotically to \( -\log_2 \left[ 1 - \xi^2/\omega^2 \right]/2 \).

**B. Two mode parametric down conversion**

Consider the two interacting modes \( a \) and \( b \) evolved by the Hamiltonian

\[
H = \hbar\omega (a^\dagger a + b^\dagger b) + \hbar\xi (a^\dagger b^\dagger + a b) + \hbar\Omega(\xi) ,
\]

where \( |\xi| < \omega \) is the coupling constant and \( \Omega(\xi) \equiv \omega - \sqrt{\omega^2 - \xi^2} \) has again been introduced to ensure that the energy ground state is null. [This Hamiltonian, as the previous one, possesses the structure of the algebra of the SU(1,1) group, in the Holstein-Primakoff realization]. We can again morph the Hamiltonian to the free field form using the two-field canonical transformation

\[
\begin{align*}
A &= a \cosh \theta + b^\dagger \sinh \theta , \\
B &= a^\dagger \sinh \theta + b \cosh \theta,
\end{align*}
\]

\( \theta = \frac{1}{4} \ln \left[ \frac{\omega + \xi}{\omega - \xi} \right] \) (15),

which transforms the Hamiltonian (14) to the form \( \hbar \sqrt{\omega^2 - \xi^2} (A^\dagger A + B^\dagger B) \). The capacity in this case is

\[
C = 2 \log g \left( \frac{E}{2\hbar \sqrt{\omega^2 - \xi^2}} \right) ,
\]

which is higher than the capacity of two independent single-mode bosonic systems, given by Eq. (16) for \( \xi = 0 \). [The factors 2 in (10) derive from the presence of the two modes \( A \) and \( B \).] The enhancement is again a consequence of the fact that the nonlinearity reduces the effective frequency of the two modes: \( \omega \rightarrow \sqrt{\omega^2 - \xi^2} \).

**C. Broadband parametric down-conversion**

Here we apply the above results to the case of two wide-band modes (signal and idler) coupled by a parametric down conversion interaction. The interaction is mediated by a nonlinear crystal with second order susceptibility \( \chi^{(2)} \) pumped with an intense coherent field of amplitude \( \xi_p \) at frequency \( \omega_p \). Assuming undepleted pumping and perfect phase matching, the Hamiltonian at the first order in the interaction is given by

\[
H = \sum_k \left[ \hbar \omega_k a_k^\dagger a_k + \hbar (\omega_p - \omega_k) b_k^\dagger b_k \right. \\
+ \hbar \xi_k (a_k^\dagger b_k^\dagger + a_k b_k) + \hbar \Omega_k \right] ,
\]

(17)

where \( a_k \) and \( b_k \) are the mode operators of the downconverted modes of frequency \( \omega_k \) and \( \omega_p - \omega_k \) respectively. Their interaction is described by the coupling parameter

\[
\xi_k = \frac{\chi^{(2)}(\omega_p)\xi_p}{c \epsilon_0 n_a(\omega_k) n_b(\omega_p - \omega_k)} \Phi(\omega_k) ,
\]

(18)

with \( n_a \) and \( n_b \) the refractive indices of the signal and idler, and \( \Phi(\omega_k) \) the phase matching function that takes into account the spatial matching of the modes in the crystal (11). As usual, the frequency \( \Omega_k = |\omega_p - \sqrt{\omega^2 - 4\xi_k^2}|/2 \) has been introduced in the Hamiltonian to appropriately rescale the ground state energy. Notice that, in contrast to the case described in Sec. III B the Hamiltonian (17) couples nondegenerate modes whose frequencies sum up to the pump frequency \( \omega_p \). Canonical transformations analogous to (15) allow us to rewrite the Hamiltonian in the free field form

\[
H = \sum_k \left[ \hbar (\omega_k - \Omega_k) A_k^\dagger A_k + \hbar (\omega_p - \omega_k - \Omega_k) B_k^\dagger B_k \right] .
\]

(19)

With this Hamiltonian, the partition function is given by

\[
\ln Z(\lambda) = \sum_k \ln \left[ \frac{1}{1 - e^{-\lambda(\omega_k - \Omega_k)}} \right] \\
+ \sum_k \ln \left[ \frac{1}{1 - e^{-\lambda(\omega_p - \omega_k - \Omega_k)}} \right] \),
\]

(20)

where the two contributions are due to the signal and idler modes respectively and the sum over \( k \) is performed on all the frequencies up to \( \omega_p \). For ease of calculation, we will assume that the coupling \( \xi_k \) is acting only over a frequency band \( \zeta \omega_p \) (\( \xi < 1 \)) centered around \( \omega_p/2 \),
where it assumes the constant value $\xi$. This choice gives a rough approximation of the crystal phase matching function $\Phi(\omega_k)$ that prevents the coupling of signal and idler photons when their frequencies are too mismatched [15]. In the high energy regime $E \gg \hbar \omega$, the sums in Eq. (22) can be replaced by frequency integrals, obtaining

$$\ln Z(\lambda) = \frac{2}{\Delta \omega} \int_0^{\omega_p(1-\zeta)/2} d\omega \ln \left( \frac{1}{1 - e^{-\lambda \omega}} \right)$$

where $\Omega \equiv [\omega_p - \sqrt{\omega_p^2 - 4\xi^2}] / 2$. The integrals in Eq. (21) have no simple analytical solution, but we can give a perturbative expansion in the low-interaction regime i.e. $\epsilon \equiv 4\xi^2/\omega_p^2 \ll 1$. In this limit, the result is derived in App. A and is given by

$$C = \frac{2\omega_p}{\Delta \omega} \left[ c_0 \left( \frac{E \delta \omega}{\hbar \omega_p^2} + \epsilon c_1 \left( \frac{E \delta \omega}{\hbar \omega_p^2}, \zeta \right) + O(\epsilon^2) \right) \right] \equiv C_{\text{asymp}} \equiv \frac{2\pi}{\ln 2} \frac{\sqrt{2\gamma}}{3}$$

where the functions $c_0$ and $c_1$ are plotted in Fig. 2. The zeroth order term $c_0$ in Eq. (22) gives the capacity of two broad-band non-interacting modes with cutoff frequency $\omega_p$: in the limit of infinite bandwidth ($\omega_p \to \infty$), this function reaches the asymptotic behavior $C \to C_{\text{asymp}} \equiv \frac{2\pi}{\ln 2} \frac{\sqrt{2\gamma}}{3}$ (continuous line) and its correction $c_1(\gamma, \zeta)$ (dashed line) of Eq. (22) with fractional coupling bandwidth $\zeta = 0.5$ and $\gamma \equiv E \delta \omega / (2 \hbar \omega_p^2)$.

D. Swapping Hamiltonian

Consider $N$ modes that are pairwise coupled through the Hamiltonian

$$H = \sum_{j=1}^{N} \hbar \omega_j a_j + \sum_{j \neq j'} \Lambda_{jj'} a_j^\dagger a_{j'}$$

where $\vec{a}$ is the column vector containing the annihilation operators $a_j$ of the $N$ modes and $\Lambda$ is an $N \times N$ symmetric real matrix with null diagonal. This Hamiltonian describes $N$ modes $a_j$ whose photons have a probability amplitude $\Lambda_{jj'}$ to be swapped into the mode $a_{j'}$. By performing a canonical transformation on all the mode operators, it is possible to rewrite Eq. (23) in the free field form

$$H = \sum_{j=1}^{N} \hbar \omega_j + \Lambda_j A_j^\dagger A_j$$

where the $\lambda_j$'s are the $N$ eigenvalues of $\Lambda$. Two conditions must be satisfied: the positivity of $H$ requires that $\lambda_j \leq \omega$ for all $j$, and, since the diagonalization of $\Lambda$ must preserve its trace, $\sum_j \lambda_j = 0$. The capacity of this system can now be easily computed as [11 2 3 6]

$$C = \max_{\epsilon_1, \epsilon_2, \ldots, \epsilon_N} \left[ \sum_{j=1}^{N} g \left( \frac{\epsilon_j}{\hbar (\omega + \lambda_j)} \right) \right]$$

where $g$ is the positivity of the term $H$, so that in the infinite bandwidth regime no improvement is obtained from the interaction. In fact, an infinite continuous spectrum is invariant under the transformation $\omega \to \sqrt{\omega^2 - 2\xi^2}$. As in the non-interacting broad-band case of Eqs. (6) and (11), the transmission time $\tau$ of the signal can be estimated as $2\pi / \Delta \omega$. The validity of this assumption (which implies negligible dispersion) rests on the fact that the Hamiltonian (17) is valid to first order in the interaction term $\xi k$ and the small dispersion which derives from such term does not play any role in the capacity [11 11].

FIG. 2: a) Capacity function $c_0(\gamma)$ (continuous line) and its correction $c_1(\gamma, \zeta)$ (dashed line) of Eq. (22) with fractional coupling bandwidth $\zeta = 0.5$ and $\gamma \equiv E \delta \omega / (2 \hbar \omega_p^2)$. The capacity increase accomplished by the nonlinearity is evident from the positivity of the term $c_1$. b) Comparison between the capacity $C$ of Eq. (22) with $\epsilon \equiv 4\xi^2/\omega_p^2 = 0.1$ (continuous line) and the asymptotic two mode rate $C_{\text{asymp}} \equiv \frac{2\pi}{\ln 2} \frac{\sqrt{2\gamma}}{3}$, obtained using an infinite frequency band (dotted line).
where the maximum must be evaluated under the requirement that \( \sum_j e_j = E \). In the simple case of \( N = 2 \) (where \( \lambda_1 = -\lambda_2 \equiv \xi \)), the maximization \((26)\) can be easily performed numerically; the increase in capacity over the two mode non interacting case is presented in Fig. 3. In this case, in the strong coupling regime \( (|\xi| \to \omega) \) the capacity diverges as \( \log_2 [E/(\hbar(\omega - |\xi|))] \); this corresponds to employing for the information storage only the lowest frequency mode among \( A_1 \) and \( A_2 \).

\[ H = (1 - r) \sum_k \sum_{j=1}^{N-1} \hbar \omega_k A_{jk}^\dagger A_{jk} + 1 + (N - 1) r \sum_k \hbar \omega_k A_{Nk}^\dagger A_{Nk}. \]  

Essentially we have chosen a coupling that contracts by a factor \( 1 - r \) the frequencies of the first \( N - 1 \) normal-modes and stretches by a factor \( 1 + (N - 1) r \) the frequency of the \( N \)th one. Choosing \( r \to 1 \), we can increase the capacity \textit{ad libitum} over the case of \( N \) non-interacting systems. In fact, in this limit a straightforward application of the wideband calculation of Eq. (9), gives \( C \simeq \sqrt{NC_{wb}} \). This result must be compared with the case in which the \( N \) wideband modes are independent where the capacity scales as \( \sqrt{N}C_{wb} \). Clearly an arbitrary increase in capacity is gained as \( r \to 1 \).

### III. General Considerations

In the previous sections we have derived the capacity of various types of nonlinear bosonic systems. The common feature of all these results is that the nonlinearities were used to reshape the spectrum by compressing it to lower frequencies, where it is energetically cheaper to encode information. A couple of remarks are in order. First of all, in our calculations the mean energy \( E \) represents the energy of the modes in the system, whereas customarily one considers the input energy. Our choice is motivated by the fact that nonlinear systems dispense energy to the information bearing modes, so that the input energy does not necessarily coincide with the amount of energy that the medium where information is encoded needs to sustain. This last quantity is a practically relevant one for the cases in which the degrees of freedom used to encode information cannot handle high energies (as for example optical fibers\( \left[12^{\infty}\right] \)). Notice that, in some of the cases we have studied (the example of Sec. II C in particular), the nonlinearity is achieved by supplying the system with an external energy source in the form of an intense coherent beam. If one were to take into account also this contribution in the energy balance, then no capacity enhancement would be evident, since the pumping energy is used only indirectly to store the information. However, it is not unwarranted to exclude the pump from the energy balance,
since it is only used to set up the required Hamiltonian and not directly employed in the information processing. Finally, our analysis is limited to the noiseless case. In the presence of noise in place of the von Neumann entropy in Eq. (21), one would have to consider the Holevo information \[12\]. This is a highly demanding problem because of the yet unknown additivity properties of this quantity. At least in the case of linear bosonic systems, the capacity in the presence of noise was studied in \[4, 14\].

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**APPENDIX A**

In this appendix we derive formula \[22\] for the broad-band parametric down-conversion Hamiltonian. In the low-interaction regime \(\epsilon \ll 1\), Eq. (21) becomes

\[
\ln Z(\lambda) = \frac{2\omega}{\delta \omega} \left[ f_0(\beta) + \epsilon f_1(\beta, \zeta) + O(\epsilon^2) \right],
\]

(A1)

where \(\beta \equiv \lambda \hbar \omega_p\), and

\[
f_0(\beta) \equiv \frac{1}{\beta} \int_0^\beta dx \ln \left[ \frac{1}{1 - e^{-x}} \right]
\]

(A2)

\[
f_1(\beta, \zeta) \equiv \frac{1}{4 \beta} \ln \left[ \frac{1 - e^{-\beta(1+\zeta)/2}}{1 - e^{-\beta(1-\zeta)/2}} \right].
\]

(A3)

Notice that the zeroth order term \(f_0\) in the expansion \[A1\] corresponds to the partition function of two broad-band modes (signal and idler) with cut-off frequency \(\omega_p\). Replacing Eq. \[A1\] into the energy constraint (1) we can find the value of the Lagrange multiplier \(\lambda\), contained in the parameter \(\beta\), by solving the equation

\[
\frac{\partial f_0(\beta)}{\partial \beta} + \epsilon \frac{\partial f_1(\beta, \zeta)}{\partial \beta} = -\gamma,
\]

(A4)

where \(\gamma \equiv E \delta \omega/(2 \hbar \omega_p^2)\) is a dimensionless quantity. By expanding the solution \(\beta\) for small \(\epsilon\) as \(\beta = \beta_0 + \epsilon \beta_1\), it follows that

\[
\frac{\partial f_0(\beta_0)}{\partial \beta} = -\gamma,
\]

(A5)

\[
\beta_1 = -\frac{\partial f_1(\beta_0, \zeta)}{\partial \beta} \left/ \frac{\partial^2 f_0(\beta_0)}{\partial \beta^2} \right.
\]

(A6)

These two equations can be numerically solved for any value of \(\gamma\). Replacing the solution in \[A1\] and using the capacity formula \[13\] we can evaluate the parametric down-conversion capacity as reported in Eq. \[22\] where

\[
c_0(\gamma) \equiv \beta_0 \gamma + f_0(\beta_0)/\ln 2,
\]

(A7)

\[
c_1(\gamma, \zeta) \equiv f_1(\beta_0, \zeta)/\ln 2.
\]

(A8)

Both these functions depend on the system energy only through the quantity \(\gamma\).