VORTEX SOLUTIONS IN BOSE-EINSTEIN CONDENSATION UNDER A TRAPPING POTENTIAL VARYING RANDOMLY IN TIME

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Abstract. The aim of this paper is to perform a theoretical and numerical study on the dynamics of vortices in Bose-Einstein condensation in the case where the trapping potential varies randomly in time. We take a deterministic vortex solution as an initial condition for the stochastically fluctuated Gross-Pitaevskii equation, and we observe the influence of the stochastic perturbation on the evolution. We theoretically prove that up to times of the order of $\varepsilon^{-2}$, the solution having the same symmetry properties as the vortex decomposes into the sum of a randomly modulated vortex solution and a small remainder, and we derive the equations for the modulation parameter. In addition, we show that the first order of the remainder, as $\varepsilon$ goes to zero, converges to a Gaussian process. Finally, some numerical simulations on the dynamics of the vortex solution in the presence of noise are presented.

1. Introduction. We study the Gross-Pitaevskii equation for a two dimensional Bose gas in a randomly varying confinement. The first experimental realization of Bose-Einstein condensation in weakly interacting gases (e.g., [6, 9]) sparked off many theoretical and experimental studies on coherent atomic matter. The Schrödinger equation with cubic nonlinearity and a harmonic potential,

$$i\partial_t u = -\frac{\hbar^2}{2M}\Delta u + V(x)u + \lambda |u|^{2\sigma} u, \quad V(x) = |x|^2, \quad \sigma = 1$$

(1.1)
called Gross-Pitaevskii equation, was initially used as a model equation. The Bose gas is described by $u$, the wave function of the condensate, $\hbar$ is Planck’s constant, $M$ is the atomic mass of atoms in the condensate, and $\lambda$ is an interaction strength...
parameter. For example, in [31] a quantitative argument of the quantum ground state for magnetically trapped Bose gas can be found. Since then, Bose Einstein condensates have been extensively studied and various model equations depending on the experimental situations have been introduced. We will treat mathematically a model of the condensate in, so-called, all-optical far-off resonance laser trap [1].

From the point of view of nonlinear waves, the interesting phenomena is that the Gross-Pitaevskii equation, similarly to other nonlinear dispersive equations, supports various types of solitary wave solutions. In the two-dimensional setting we will study in particular, there are vortex solutions of the form

$$u(t, r, \theta) = e^{-i\mu t} e^{im\theta} \psi(r),$$

(1.2)

where $r, \theta$ are polar coordinates, $m$ is the vortex degree, $\mu$ is the chemical potential and $\psi(r)$ is the radial positive vortex profile. Stability of vortex solutions to diverse forms of nonlinear Schrödinger equations has drawn much attention in recent years. For example, for the case where $V \equiv 0$, Mizumachi in [27, 28] investigated the orbital stability and instability of the vortex solution to (1.1) with $\lambda = -1$, making use of the perturbation with the same symmetry as the vortex, which will be called in this paper “$m$-equivariant” perturbation. Also it was proved in the case of general perturbations that the vortex solution is unstable for any $\sigma > 0$ if $m$ is sufficiently large. Some numerical observations are available too, e.g., see [8] for the computation of the spectrum of the associated linearized operator at the vortex, [30, 32] for the case of other nonlinearities and [18] for the blow-up profile. For the case where $V \neq 0$, especially in the physical important case $V(x) = |x|^2$, a variational approach is used to give stability results in [26, 35]. It was proved in [26] that in the case of $\lambda = 1$, there is a stable vortex solution for any $m \geq 1$ by $m$-equivariant perturbations. The author [35] showed that in the defocusing case $\lambda = 1$, if $m \geq 2$, there is a direction in which the second derivative of the action functional at the vortex is negative. However this is not sufficient to conclude the nonlinear instability. None of these results give a satisfactory answer for the case $m = 1$ where we expect intuitively the stability of the vortex under any sort of perturbation. Here, we note that the vortex solution seems to be stable for $m = 1$ and unstable for $m \geq 2$ in the case $\lambda = 1$ from the numerical studies in [4]. More recent studies can be found in [24, 25], which deal with the spectrum of the linearized operator at the vortex solution in the case $\lambda = 1$. A bound for azimuthal Fourier modes which can cause the instability, is given there. Numerical computations combined with the help of Krein signature allow to detect all unstable eigenvalues and infer a collision process between eigenvalues for $m = 2, 3$. This complexity of the spectrum structure makes it difficult to prove theoretically the stability, and we believe it still challenging and important to push further and complete rigorously all those discussions.

In the present paper, we are interested in the influence of the noise on the vortex solution (1.2), in the Gross-Pitaevskii equation with a stochastic perturbation of the following form

$$i\partial_t u = -\Delta u + V(x)u + \lambda |u|^{2\sigma}u + \varepsilon |x|^2 u\dot{\xi}(t), \quad t \geq 0, \quad x \in \mathbb{R}^d$$

(1.3)

where $\lambda = \pm 1$ and $\dot{\xi}$ is a white noise in time with correlation function $E(\dot{\xi}(t)\dot{\xi}(s)) = \delta_0(t-s)$. Here, $\delta_0$ denotes the Dirac measure at the origin, $\sigma > 0$ and $\varepsilon > 0$. Remark that we may set $h = 2M = 1$ in (1.1) for our analysis without loss of generality. The product arising in the right hand side is interpreted in the Stratonovich sense, since the noise here naturally arises as the limit of processes with nonzero correlation.
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length. We moreover assume that the noise is real valued. The term \( \varepsilon \dot{\xi}(t) \) represents the deviations of the laser intensity \( E(t) \) around its mean value (see [1]). Also, the sign of \( \lambda \) is related to the sign of the atomic scattering length, which may be positive or negative, it may be assumed without loss of generality that \( \lambda = \pm 1 \). This model is proposed in [1], possibly with the addition of a damping term, to describe Bose-Einstein condensate wave function in an all-optical far-off resonance laser trap, arguing that some fluctuations of the laser intensity are observed in this case, and that one should take into account stochasticity in the dynamical behavior of the condensate in real situations (see also [19, 34]). Related equations, for example, the equation (1.3) with \( V \equiv 0 \), are also found in the context of optical fibers [2] in order to model the propagation of the optical soliton in fibers with random inhomogeneities.

Our aim in this paper is to investigate the influence of random perturbations on the propagation of deterministic vortex solutions (1.2) theoretically and numerically. Not only rigorous mathematical results about the stability questions on deterministic vortices in nonlinear Schrödinger equations are rather few, but also the studies on the effect of stochastic perturbations on vortices are quite rare. Because of the presence of noise, a stable vortex would not persist in its form for all time. Thus an interesting question is how long the stable vortex can persist, compared to the noise strength \( \varepsilon \). The method we will use, so called collective coordinates approach, consists in writing that the main part of the solution is given by a modulated vortex and in finding then the modulation equations for the vortex parameters. Such ideas to analyze the asymptotic behavior have been used by many authors in the physics literature, as well as in the study of mathematical problems, but mainly for the ground states (see, for example, in the deterministic case [22, 23, 38] and in the stochastic case [10, 11, 13, 15]). The stability property of the vortex solutions in the deterministic equation is required in order to apply such collective coordinates approach, and we will here take advantage of the fact that the \( m \)-equivariance property of the solutions is preserved by the noise. Finally, we use numerical computations to investigate the sharpness of the bounds that we obtain theoretically, both with respect to time \( T \) and to the strength of the noise \( \varepsilon \).

The paper is organized as follows: in Section 2, we state precisely our results. In Section 3, the existence of the modulation parameter is justified and we give an estimate on the time up to which the modulation procedure is available. In Section 4 we give the equations of the modulation parameter. Section 5 is devoted to some estimates on the remainder term, and using these estimates, we will see the convergence of the remainder term as \( \varepsilon \) goes to zero to a limit process. Note that most arguments follow the ideas of [13], and we will give some technical explanations concerning the differences from this previous works in Section 8 (Appendix). The numerical results are presented in Section 6. We sometimes denote all through the paper by \( C_\theta,... \) a constant which depends on \( \theta \) and so on.

2. Preliminaries and main results. We consider the following stochastic nonlinear Schrödinger equation

\[
{idu + (\Delta u - |x|^2 u - \lambda |u|^{2\sigma} u)dt = \varepsilon |x|^2 u \circ dW},
\]

where \( \circ \) stands for a Stratonovich product in the right hand side of (2.1), \( \sigma > 0 \), \( \varepsilon > 0 \), and \( \lambda = \pm 1 \). The unknown function \( u \) is a random process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a standard filtration \((\mathcal{F}_t)_{t \geq 0}\) such that \( \mathcal{F}_0 \) is complete, \( W(t) \) is a standard real valued Brownian motion on \( \mathbb{R}^+ \) associated with the filtration
\( F_t \), \( t \geq 0 \). We have set \( \dot{\xi} = \frac{d\xi}{dt}, V(x) = |x|^2 \) in the equation (1.3). It will be useful to introduce here the equivalent \( \text{It\'o} \) equation which may be written as

\[ idu + \left( \Delta u - |x|^2 u + \frac{i}{2} \varepsilon^2 |x|^4 u - \lambda |u|^{2\sigma} u \right) dt = \varepsilon |x|^2 u \, dW. \tag{2.2} \]

We define

\[ \Sigma(k) = \{ v \in L^2(\mathbb{R}^d), \sum_{|\alpha| + |\beta| \leq k} |\partial_x^\alpha \partial_t^\beta v|^2_{L^2(\mathbb{R}^d)} = |v|^2_{L^2(\Sigma(k))} < +\infty \} \]

for \( k \in \mathbb{N} \), and write \( \Sigma(-k) \) for the dual space of \( \Sigma(k) \) in the \( L^2 \) sense. In particular we denote \( \Sigma(1) \) by \( \Sigma \).

We define the energy

\[ H(u) = \frac{1}{2} |\nabla u|^2_{L^2} + \frac{1}{2} |xu|^2_{L^2} + \frac{\lambda}{2\sigma + 2} |u|^{2\sigma+2}_{L^2}, \tag{2.3} \]

which is a conserved quantity of the deterministic equation i.e., (2.1) with \( \varepsilon = 0 \). We will consider solutions in the space \( \Sigma \) since \( H(u) \) is well defined in \( \Sigma \), thanks to the Sobolev embedding \( \Sigma \subset H^1(\mathbb{R}^d) \subset L^{2\sigma+2}(\mathbb{R}^d) \), for \( \sigma < \frac{d}{2\sigma+2} \) if \( d \geq 3 \) or \( \sigma < +\infty \) if \( d = 1, 2 \). It is worth adding a remark that there is also another conserved quantity, that is \( L^2 \) norm

\[ Q(u) = \frac{1}{2} |u|^2_{L^2}. \]

In the case where \( \varepsilon = 0 \), it is known that in the energy space \( \Sigma \), equation (2.1) is locally well posed for \( \lambda = \pm 1, \sigma < \frac{2}{d-2} \) if \( d \geq 3 \) or \( \sigma < +\infty \) if \( d = 1, 2 \) and globally well posed if either \( \lambda = 1 \) or \( \lambda = -1 \) and \( \sigma < 2/d \) (see [29]). In the case where \( \varepsilon \neq 0 \), we established a random Strichartz estimate and proved the local and global well-posedness under the same condition for the nonlinear power as the deterministic case \( \varepsilon = 0 \) in [14]. To mention our results exactly, we give some notations. If \( I \) is an interval of \( \mathbb{R} \), \( E \) is a Banach space, and \( 1 \leq r \leq \infty \), then \( L^r(I; E) \) is the space of strongly Lebesgue measurable functions \( v \) from \( I \) into \( E \) such that the function \( t \mapsto |v(t)|_E \) is in \( L^r(I) \). We define similarly the spaces \( C(I; E) \), or \( L^r(Q; E) \). We start by recalling some existence results for equation (2.1) which were proved in [14].

**Proposition 1.** Assume \( \sigma > 0, \varepsilon > 0 \) and \( \lambda = \pm 1 \). Let \( u_0 \in \Sigma \) and \( \sigma < 2/(d-2) \) if \( d \geq 3 \), \( \sigma < +\infty \) if \( d = 1, 2 \). Then there exist a stopping time \( \tau_{u_0, \omega}^* > 0 \) and a unique solution \( u \) adapted to \( (F_t)_{t \geq 0} \) of (2.1) with \( u(0) = u_0 \) almost surely in \( C([0, \tau_{u_0, \omega}^*); \Sigma) \). Moreover, we have almost surely,

\[ \tau_{u_0, \omega}^* = +\infty \text{ or } \limsup_{t \wedge \tau_{u_0, \omega}^*} |u(t)|_\Sigma = +\infty, \]

and the \( L^2 \) norm is conserved:

\[ Q(u(t)) = Q(u_0), \text{ a.s. in } \omega, \text{ for all } t \in [0, \tau_{u_0, \omega}^*). \]

In particular, if \( \lambda = 1 \), then there exists a unique global solution \( u \) adapted to \( (F_t)_{t \geq 0} \) of (2.1) with \( u(0) = u_0 \) almost surely in \( C(\mathbb{R}^+; \Sigma) \).

The result of Proposition 1 can be applied, not only for a quadratic potential, but for more general potentials \( V(x) \) in (1.3) (see Remark 2.2 in [14] for details). The key point of the proof for Proposition 1 was the following gauge transformation: let \( w \) be a solution of

\[ i\partial_t w = -\left( \nabla - 2ix(t + \varepsilon W(t)) \right)^2 w, \tag{2.4} \]
then
\[
\begin{align*}
  u(t, x) &= \exp\left\{ -i|x|^2(t + \varepsilon W(t)) \right\} w(t, x) \\
&= \exp\left\{ -i|x|^2(t + \varepsilon \tilde{W}(t)) \right\} w(t, x)
\end{align*}
\]  
(2.5)
satisfies the equation (2.1) with \( \lambda = 0 \). Therefore, the Cauchy problem for Eq. (2.1) (or equivalently (2.2)) is deduced to the Cauchy problem for
\[
\begin{align*}
  i\partial_t w &= -\left( \nabla - 2itx(t + \varepsilon W(t)) \right)^2 w + \lambda |w|^{2\sigma} w,
\end{align*}
\]  
(2.6)
which is a deterministic equation for each fixed \( \omega \in \Omega \).

In addition to the results of Proposition 1, in this paper, we will make use of the following fact. This is an explicit representation of the solution of the linear part of Eq. (2.1), which is a derivation from the result in [36]. To introduce this, let \( T_0 > 0 \) be fixed and consider \( (q, v) \in \mathbb{R}^2 \) the solution of the system:
\[
\begin{align*}
  \dot{q}(t) &= v(t) - (t + \varepsilon W(t))q(t), \\
  \dot{v}(t) &= (t + \varepsilon W(t))(v(t) - (t + \varepsilon W(t))q(t)), \quad \text{for } t \in [0, T_0]
\end{align*}
\]
with initial data \( (q(0), v(0)) = (0, 1) \in \mathbb{R}^2 \). There exists a unique solution \( (q, v) \in C([0, T_0], \mathbb{R}^2) \) for each \( \omega \in \Omega \) satisfying \( W(\cdot, \omega) \in C^\alpha([0, T_0]) \) with \( 0 < \alpha < 1/2 \).

Using this solution, we define the following system of ODEs:
\[
\begin{align*}
  \begin{cases}
    2\alpha(t) &= v(t)/q(t), \\
    \dot{\beta}(t) &= (t + \varepsilon W(t) - \alpha(t))\beta(t), \\
    \dot{\gamma}(t) &= -1/2 \beta^2(t)
  \end{cases}
\end{align*}
\]
Remark that \( \alpha(t), \beta(t), \gamma(t) \) are well defined for any \( t \leq \tilde{T} \land T_0 \), where \( \tilde{T} = \inf\{s > 0, q(s) = 0\} \). Note that \( \tilde{T} > 0 \), thanks to the initial condition \( (q(0), v(0)) = (0, 1) \), which implies \( q(0) = 0 \) and \( \dot{q}(0) = 1 \). Then we have the following representation formula for the solution of (2.4).

**Proposition 2.** Let \( T_0 > 0 \) be fixed. For each \( \omega \in \Omega \) satisfying \( W(\cdot, \omega) \in C^\alpha([0, T_0]) \) with \( 0 < \alpha < 1/2 \), the solution of (2.4) with initial data \( w(0) = u_0 \in C_0^\infty(\mathbb{R}^d) \) may be expressed as follows.
\[
w(t, x) = \frac{1}{(2\pi i q(t))^{d/2}} \int_{\mathbb{R}^d} e^{i(\alpha(t)|x|^2 + \beta(t)xy + \gamma(t)|y|^2)} u_0(y)dy,
\]
(2.7)
for any \( t \leq T_0 \land \tilde{T} \).

Remind that our interest is in the vortex solutions. From now on we fix \( \lambda = 1 \), so that we consider the equation:
\[
\begin{align*}
  idu + \left( \Delta u - |x|^2u + \frac{i}{2} \varepsilon^2 |x|^4u - |u|^{2\sigma}u \right) dt &= \varepsilon |x|^2 u dW.
\end{align*}
\]  
(2.8)
This equation can be regarded as a stochastically perturbed version of the following deterministic equation;
\[
\begin{align*}
  i\partial_t u + \Delta u - |x|^2u - |u|^{2\sigma}u &= 0.
\end{align*}
\]  
(2.9)
We also restrict ourselves to the two dimensional case \( d = 2 \), and make use of the polar coordinates: for \( x = (x_1, x_2) \in \mathbb{R}^2 \), we define \( x_1 = r\cos \theta, \ x_2 = r\sin \theta \).

We recall that the vortex solutions are the solutions of (2.9) of the form,
\[
\begin{align*}
  u(t, x) &= e^{-i\mu t} \phi_{\mu, m}(x) = e^{-i\mu t} e^{im\theta} \psi_{\mu, m}(r), \quad \mu \in \mathbb{R}, \ x \in \mathbb{R}^2.
\end{align*}
\]  
(2.10)
where $m \in \mathbb{Z}^*$ is the winding number of the vortex, and we may assume $m \geq 1$ without loss of generality by the reflection symmetry. Substituting (2.10) in (2.9), $\phi_{\mu,m}(x) \in \Sigma$ satisfies

$$- \Delta \phi + |x|^2 \phi - \mu \phi + |\phi|^{2\sigma} \phi = 0.$$  

(2.11)

Before stating more details about the properties of vortex solutions (2.10), we introduce the closed subspace $X_m$ of $\Sigma$ that has the same symmetry, i.e., $m$-equivariant symmetry as the vortex solutions $\phi_{\mu,m}(r \cos \theta, r \sin \theta) = e^{im\theta} \psi_{\mu,m}(r)$:

$$X_m := \{ v \in \Sigma, \ e^{-im\theta} v(r \cos \theta, r \sin \theta) \text{ does not depend on } \theta \}. $$

We remark that, thanks to Proposition 2, one may check that the solution $u(t)$ of (2.8) belongs to $X_m$ if the initial data $u(0) = u_0$ belongs to $X_m$. This preservation of the structure of Eq.(2.8) in $X_m$ leads to the following result.

**Lemma 2.1.** Let $\sigma > 0$ and $u_0 \in X_m$. Then there exists a unique global solution $u$ adapted to $(F_\sigma)_{t \geq 0}$ of (2.8) with $u(0) = u_0$ almost surely in $C(\mathbb{R}^+; X_m)$.

We will give a proof of Lemma 2.1 in Appendix using Proposition 2.

The existence of the vortex solutions is easily proved (see e.g. [26]), with the help of the compact embedding $X_m \subset \Sigma \subset L^2$, for any $\mu > \lambda_m$ where $\lambda_m$ is the solution of the linear eigenvalue problem

$$\lambda_m = \inf \left\{ \int_0^\infty \left( - \frac{d^2}{dr^2} v + r^2 v + \frac{1}{r^2} \left( m^2 - \frac{1}{4} \right) v \right) dr \mid v \in D, \int_0^\infty |v(r)|^2 dr = 1 \right\},$$

(2.12)

where

$$D = \left\{ v \in L^2(0, \infty), \frac{d^2}{dr^2} v, r^2 v, \frac{1}{r^2} v \in L^2(0, \infty) \right\}.$$

It is known that $\lambda_m = 2m + 2$ (see Subsection 4.16 of [37]). More precisely, let $S_\mu$ be the action functional, i.e.,

$$S_\mu(u) = H(u) - \mu Q(u), \ u \in \Sigma.$$

The variational problem

$$\inf \{ S_\mu(u), \ u \in X_m \setminus \{0\} \}$$

(2.13)

is attained by $\phi_{\mu,m}$ on $X_m$ for any $\mu > \lambda_m$. Namely, $\phi_{\mu,m}$ is characterized as a global minimizer of $S_\mu$ on $X_m$. The uniqueness of minimizers is established as in Theorem 9 of [26]. Such a variational characterization allows to show the orbital stability in $X_m$ for any $m \geq 1$. This fact is related with Proposition 3 (i) below.

The solutions $\phi$ of (2.11) have regularity properties by the usual elliptic bootstrap arguments, i.e.,

$$\lim_{|x| \to \infty} |\phi_{\mu,m}(x)| = 0, \ \text{and} \ \phi_{\mu,m} \in \bigcap_{2 \leq q < \infty} W^{2,q} \cap C^2.$$

The radial profile $\psi_{\mu,m}$ is thus of class $C^2(0, \infty)$ and verifies the equation

$$- \frac{d^2}{dr^2} \psi - \frac{1}{r} \frac{d}{dr} \psi + \frac{m^2}{r^2} \psi + r^2 \psi - \mu \psi + |\psi|^{2\sigma} \psi = 0, \ r > 0,$$

(2.14)

with

$$\lim_{r \to \infty} |\psi_{\mu,m}(r)| = 0.$$
In particular, the solution $\psi_{\mu,m}$ associated to the global minimizer $\phi_{\mu,m}$ of (2.13) is non-negative, as seen from the radial form of (2.13), and

$$
\int_0^\infty \left| \frac{d}{dr} |\psi| \right|^2 r dr \leq \int_0^\infty \left| \frac{d}{dr} \psi \right|^2 r dr.
$$

Remark that every regular solution of (2.14) should satisfy $\psi(r) \to 0$, as $r \to 0^+$ (see [27]), which we will see again later in Proposition 3 below.

Now we summarize here some properties of $\psi_{\mu,m}$. The inner product in the Hilbert space $L^2(\mathbb{R}^2)$ is denoted by $(\cdot, \cdot)_{L^2(\mathbb{R}^2, dx)}$, i.e.,

$$(u, v)_{L^2(\mathbb{R}^2, dx)} = \int_{\mathbb{R}^2} u(x) \overline{v(x)} \, dx, \quad \text{for } u, v \in L^2(\mathbb{R}^2).$$

Moreover we denote $\langle u, v \rangle = \text{Re}(u, v)_{L^2(\mathbb{R}^2, dx)}$. Taking account of the property of the space $X_m$, we here introduce the radial norm and the radial inner products

$$(u, v)_{L^2} = \int_0^\infty u(r) \overline{v(r)} \, r dr, \quad |u|_{L^2}^2 = \int_0^\infty |u(r)|^2 \, r dr.$$

**Proposition 3.** Let $\sigma > 0$, $m \geq 1$ and $\mu > \lambda_m$. Let $\phi_{\mu,m}$ be the minimizer of (2.13), and consider the associated radial profile $\psi_{\mu,m}(r) := e^{-im\theta}\phi_{\mu,m}(r \cos \theta, r \sin \theta)$. Then the following properties hold.

(i) There exists $\nu = \nu(\mu, m) > 0$, such that for any $v \in X_m$ satisfying $\text{Re}(v, i\phi_{\mu,m})_{L^2(\mathbb{R}^2, dx)} = 0$, we have

$$\langle S''_{\mu}(\phi_{\mu,m})u, v \rangle \geq \nu |v|_{L^2}^2.$$

(ii) The radial profile $\psi_{\mu,m}$ has the following asymptotics

$$\psi_{\mu,m}(r) = O(r^{\mu/2} e^{-r^2/2}), \quad r \to +\infty, \quad \text{and} \quad \psi_{\mu,m}(r) = O(r^m), \quad r \to 0^+.$$

(iii) The radial profile $\psi_{\mu,m}$ has a local maximum for some $r_0 \in (m/\sqrt{\mu}, \sqrt{\mu})$, is increasing on $(0, r_0)$ and is decreasing on $(r_0, \infty)$. Moreover, $|\psi_{\mu,m}(r)|^{2\sigma} < \mu - 2m$ for all $r > 0$.

The positivity of $S''_{\mu}(\phi_{\mu,m})$, that is (i) of Proposition 3, will be discussed in the Appendix, and for the properties (ii), see the proofs of Lemmas 3.1 and 3.2 in [21] and Appendix E of [24]. For (iii), we refer to Appendix B of [24].

Note that the second derivative $S''_{\mu}(\phi_{\mu,m})$ is related to the linearization problem around $e^{-i\mu t}\phi_{\mu,m}$ in (2.9), which is precisely written as

$$
\frac{dy}{dt} = J \mathcal{L}_{\mu,m} y \quad \text{in} \quad \Sigma(-1),
$$

where

$$J = -i : \left( \begin{array}{c} \text{Re } u \\ \text{Im } u \end{array} \right) \mapsto \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \text{Re } u \\ \text{Im } u \end{array} \right),$$

$$\mathcal{L}_{\mu,m} = S''_{\mu}(\phi_{\mu,m}) = \left( \begin{array}{cc} L^1_{\mu,m} & 0 \\ 0 & L^2_{\mu,m} \end{array} \right),$$

$$L^1_{\mu,m} = -\Delta + |x|^2 - \mu + (2\sigma + 1)|\phi_{\mu,m}|^{2\sigma}, \quad L^2_{\mu,m} = -\Delta + |x|^2 - \mu + |\phi_{\mu,m}|^{2\sigma}.$$
In the radial notation, we will use, as the linearized operators corresponding to $L^{1}_{\mu,m}$ and $L^{2}_{\mu,m}$,

\[
L^{1}_{\mu,m} = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + r^2 + \frac{m^2}{r^2} - \mu + (2\sigma + 1)\psi^{2\sigma}_{\mu,m}, \tag{2.16}
\]

\[
L^{2}_{\mu,m} = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + r^2 + \frac{m^2}{r^2} - \mu + \psi^{2\sigma}_{\mu,m}. \tag{2.17}
\]

Our purpose here is to investigate the influence of random perturbations of the form given in equation (2.8) on the phase of vortex solutions (2.10).

We fix $\mu_0 > \lambda_m$ and consider for $\epsilon > 0$ the solution $u^\epsilon$ of equation (2.8) given by Lemma 2.1 with initial data $u^\epsilon(0,x) = \phi_{\mu_0,m}(x)$, which is a stable vortex in $X_m$ of the deterministic equation.

We will show, as a first step, that we can decompose $u^\epsilon$ in $X_m$ as the sum of a modulated vortex solution and a remainder with small $\Sigma$ norm, for $t$ less than some stopping time $\tau^\epsilon$, and that this $\tau^\epsilon$ goes to infinity in probability as $\epsilon$ goes to zero.

We will see then that the remaining part is of order one with respect to $\epsilon$. The proof of this fact is rather similar to Theorem 2.1 in [10], Theorem 2.1 in [11], and Theorem 2 in [13], and we only mention the differences from the previous works. The decomposition is in the form

\[
u^\epsilon(t,x) = e^{-i\xi^\epsilon(t)}(\phi_{\mu_0,m}(x) + \epsilon \eta^\epsilon(t,x)) \tag{2.18}
\]

for a semi-martingale process $\xi^\epsilon$ with values in $\mathbb{R}$, and $\eta^\epsilon$ with values in $X_m$. Here, we use the following orthogonality condition

\[\text{Re}(\eta^\epsilon, i\phi_{\mu_0,m})_{L^2(\mathbb{R}^2,dx)} = 0, \quad \text{a.s.,} \quad t \leq \tau^\epsilon, \tag{2.19}\]

for a suitable stopping time $\tau^\epsilon$ as long as $|\epsilon \eta^\epsilon|_\Sigma$ remains small. Precisely we have the following result.

**Theorem 2.2.** Let $1/2 \leq \sigma < \infty$ and $\mu_0 > \lambda_m$ be fixed. For $\epsilon > 0$, let $u^\epsilon$, as defined above, be the solution of (2.8) with $u(0,x) = \phi_{\mu_0,m}(x)$. Then there exists $\alpha_0 > 0$ such that, for each $\alpha$, $0 < \alpha \leq \alpha_0$, there is a stopping time $\tau^\epsilon_\alpha \in (0,\infty)$ a.s., and there is a semi-martingale process $\xi^\epsilon(t)$, defined a.s. for $t \leq \tau^\epsilon_\alpha$, with values in $\mathbb{R}$, so that if we set $\epsilon \eta^\epsilon(t,x) = e^{i\xi^\epsilon(t)}u^\epsilon(t,x) - \phi_{\mu_0,m}(x)$, then (2.19) holds. Moreover, a.s. for $t \leq \tau^\epsilon_\alpha$, we have

\[|\epsilon \eta^\epsilon(t)|_\Sigma \leq \alpha. \tag{2.20}\]

In addition, there is a constant $C = C_{\alpha,\mu_0} > 0$, such that for any $T > 0$ and any $\alpha \leq \alpha_0$, there is an $\epsilon_0 > 0$, such that for each $\epsilon < \epsilon_0$,

\[\mathbb{P}(\tau^\epsilon_\alpha \leq T) \leq \exp\left(-\frac{C}{\epsilon^2 T}\right). \tag{2.21}\]

This means that the modulation parameter $\xi^\epsilon$ is a semi-martingale process defined up to times of the order $\epsilon^{-2}$, i.e. the shape of the vortex solution is preserved over this time scale. We will detail in the next theorem the behavior of $\eta^\epsilon$, and the convergence of the modulation parameter as $\epsilon$ goes to zero.

**Theorem 2.3.** Let $1 \leq \sigma < \infty$, $\mu_0 > \lambda_m$ be fixed and $\eta^\epsilon$, $\xi^\epsilon$, for $\epsilon > 0$ be given by Theorem 2.2, with $\alpha \leq \alpha_0$ fixed. Then, for any $T > 0$, the $X_m$-valued process $(\eta^\epsilon(t))_{t \in [0,T \wedge \tau^\epsilon_\alpha]}$ converges in probability, as $\epsilon$ goes to zero, to a process...
\( \eta = (\eta(t))_{t \in [0,T]} \) satisfying \( e^{-im\theta} \eta = \tilde{\eta} \) and
\[
d\tilde{\eta} = J \begin{pmatrix} F_{\mu_0,m}^{1,\sigma} & 0 \\ L_{\mu_0,m}^{2,\sigma} & 0 \end{pmatrix} \tilde{\eta} dt - y \begin{pmatrix} 0 \\ \psi_{\mu_0,m} \end{pmatrix} dt - z \begin{pmatrix} 0 \\ \psi_{\mu_0,m} \end{pmatrix} dW,
\]
with \( \tilde{\eta}(0) = 0 \), and
\[
y(t) = \frac{2\sigma (Re \tilde{\eta}(t), \psi_{\mu_0,m}^{2\sigma+1})_{L^2_y}}{||\psi_{\mu_0,m}||_{L^2_y}}, \quad z = \frac{|r\psi_{\mu_0,m}|^2_{L^2_y}}{||\psi_{\mu_0,m}||^2_{L^2_y}},
\]
for all \( t \geq 0 \). The convergence holds in \( C([0, \tau_\alpha^\varepsilon \wedge T], L^2) \).

The above process \( \eta \) satisfies for any \( T > 0 \) the estimate
\[
\mathbb{E} \left( \sup_{t \leq T} |\eta(t)|^2_{L^2} \right) \leq CT
\]
for some constant \( C > 0 \).

Moreover the modulation parameter \( \xi^\varepsilon \) may be written, for \( t \leq \tau_\alpha^\varepsilon \), as
\[
d\xi^\varepsilon = \mu_0 dt + \varepsilon y^\varepsilon dt + \varepsilon z^\varepsilon dW,
\]
for some adapted processes \( y^\varepsilon, z^\varepsilon \) with values in \( \mathbb{R} \) satisfying the following property: as \( \varepsilon \) goes to zero, \( y^\varepsilon, z^\varepsilon \) converge in probability in \( C([0,T]) \) respectively to \( y, z \) given in (2.23).

Since \( z \) is deterministic, this shows that at first order in \( \varepsilon \) the noise acts on the phase of the vortex in a Gaussian way.

Remark that in [13] we assumed \( \sigma = 1 \) in order to show a similar theorem. In the present paper, Theorem 2.3 above gives an improvement in this respect, although \( \sigma \geq 1 \) is still essential due to the treatment of the estimate for the remainder \( \eta^\varepsilon \) in \( L^2(\Omega, C([0,T], \Sigma)) \). We will need the following proposition, whose proof will be given in the Appendix.

**Proposition 4.** Let \( \lambda = 1 \) and \( 1/2 \leq \sigma < \infty \). Let \( u_0 \in \Sigma(2) \). Then there exists a unique global solution \( w \) of (2.6) with \( w(0) = u_0 \), adapted to \( (\mathcal{F}_t)_{t \geq 0} \), in \( C(\mathbb{R}^+, \Sigma(2)) \) almost surely.

3. **Proof of Theorem 2.2.** In this section, we give the outline of proof for the existence of the modulation parameter and the estimate on the exit time (2.21). The arguments are similar to those in [11, 13]. The following lemma shows the equalities of the charge \( Q \) and of the energy \( H \) by the evolution of (2.8). For the proof, we refer to Theorem 3 (i) in [12].

**Lemma 3.1.** Assume \( 0 < \sigma < \infty \) and \( \mu_0 > \lambda_m \). Let \( \phi_{\mu_0,m} \) be the vortex solution and \( w^\varepsilon \) be the solution of (2.8) given by Proposition 1, with \( w^\varepsilon(0, x) = \phi_{\mu_0,m} \). Then for any stopping time \( \tau \) we have
\[
|u^\varepsilon(\tau)|_{L^2} = |\phi_{\mu_0,m}|_{L^2}, \quad a.s.,
\]
\[
H(u^\varepsilon(\tau)) = H(\phi_{\mu_0,m}) - 2\varepsilon \text{Im} \int_0^\tau \int_{\mathbb{R}^d} \nabla u^\varepsilon \cdot xu^\varepsilon dxdW(s)
\]
\[
+ 2\varepsilon^2 \int_0^\tau |xu^\varepsilon|_{L^2}^2 ds, \quad a.s.
\]
The implicit function theorem ensures the existence of the modulation parameter satisfying the orthogonality condition (2.19). Indeed, let \( B_{\phi_{0,\alpha}}(2\alpha) = \{ v \in X_m, |v - \phi_{0,\alpha}| \leq 2\alpha \} \) for \( \alpha \) with \( 0 < \alpha < \mu_0 / 4 \). We then consider a \( C^2 \) mapping \( I \), from \((-2\alpha, 2\alpha) \times B_{\phi_{0,\alpha}}(2\alpha) \) into \( \mathbb{R} \), defined by

\[
I(\xi, u) = |e^{i\xi} u - \phi_{0,\alpha}|^2.
\]

One easily obtains

\[
\partial_\xi I(0, \phi_{0,\alpha}) = 0, \quad \partial^2_\xi I(0, \phi_{0,\alpha}) = 2|\phi_{0,\alpha}|^2 > 0.
\]

Applying the implicit function theorem shows that for \( \alpha \leq \alpha_0 \) where \( \alpha_0 \) is sufficiently small, there exists a \( C^2 \) mapping \( \xi(u) \) defined for \( u \in B_{\phi_{0,\alpha}}(2\alpha) \), such that

\[
\partial_\xi I(\xi(u), u) = 0.
\]

We now define a stopping time \( \tau^\alpha \) given by Lemma 2.1, we get the existence of \( \xi^\alpha(t) = \xi(u^\alpha(t)) \) such that the orthogonality condition (2.19) holds with \( \varepsilon \eta^\alpha(t) = e^{i\xi^\alpha(t)} u^\alpha(t) - \phi_{\mu_0, \alpha} \). Then the Itô formula shows, as in [13], that \( \xi^\alpha \) is a semi-martingale process. Moreover, since it is clear that \( \partial_\xi I(0, e^{i\xi^\alpha(t)} u^\alpha(t)) = 0 \), the existence of \( \xi^\alpha \) holds as long as

\[
|e^{i\xi^\alpha(t)} u^\alpha(t) - \phi_{\mu_0, \alpha}| \leq \alpha.
\]

We now define a stopping time

\[
\tau^\alpha = \inf\{ t \geq 0, |e^{i\xi^\alpha(t)} u^\alpha(t) - \phi_{\mu_0, \alpha}| \geq \alpha \}.
\]

The process \( \xi^\alpha(t) \) is defined for all \( t \leq \tau^\alpha \), and satisfies (2.20) for all \( t \leq \tau^\alpha \) and \( \alpha \leq \alpha_0 \), together with the orthogonality condition (2.19), where again \( \varepsilon \eta^\alpha(t) = e^{i\xi^\alpha(t)} u^\alpha(t) - \phi_{\mu_0, \alpha} \).

Let us fix \( T > 0 \). Setting \( \tau = \tau^\alpha \wedge T \), remark that

\[
P(\tau^\alpha \leq T) \leq P(|\varepsilon \eta^\alpha(\tau)| \geq \alpha).
\]

Thus, similar, even simpler arguments than in [10, 15, 13] lead to the exponential estimate (2.21). Indeed, by Taylor expansion,

\[
S_{\mu_0}(u^\alpha(t, \cdot) - S_{\mu_0}(\phi_{\mu_0, \alpha}) = S_{\mu_0}(e^{i\xi^\alpha(t)} u^\alpha(t, \cdot)) - S_{\mu_0}(\phi_{\mu_0, \alpha})
= \langle S_{\mu_0}(\phi_{\mu_0, \alpha}), \varepsilon \eta^\alpha(t) \rangle + \langle S_{\mu_0}(\phi_{\mu_0, \alpha}) \varepsilon \eta^\alpha(t), \varepsilon \eta^\alpha(t) \rangle + o(|\varepsilon \eta^\alpha(t)|^2).
\]

Note that \( o(|\varepsilon \eta^\alpha(t)|^2) \) is uniform in \( \varepsilon, \omega \) and \( t \) for \( t \leq \tau^\alpha \), since \( |e^{i\xi^\alpha(t)} u^\alpha(t, \cdot) - \phi_{\mu_0, \alpha}| \leq \alpha \) for all \( t \leq \tau^\alpha \). We then assume \( \alpha_0 \) small enough so that the last term is less than \( 2^\alpha |\varepsilon \eta^\alpha(t)|^2 \) for all \( t \leq \tau^\alpha \). Since, the positivity of \( S''_{\mu_0}(\phi_{\mu_0, \alpha}) \) is satisfied in \( X_m \) (see Proposition 3), for any \( \mu_0 > \lambda_m \),

\[
\langle S''_{\mu_0}(\phi_{\mu_0, \alpha}) \varepsilon \eta^\alpha, \varepsilon \eta^\alpha \rangle \geq \nu |\varepsilon \eta^\alpha|^2
\]
holds a.s. for \( t \leq \tau^\alpha \), noticing that \( \eta^\alpha \in X_m \) and satisfies the orthogonality condition (2.19). It thus follows, since \( S'_{\mu_0}(\phi_{\mu_0, \alpha}) = 0 \),

\[
S_{\mu_0}(u^\alpha(t, \cdot)) - S_{\mu_0}(\phi_{\mu_0, \alpha}) \geq \frac{\nu}{2} |\varepsilon \eta^\alpha(t)|^2.
\]
Again we choose \( \alpha_0 \) sufficiently small, then make use of (3.1), (3.2) to get, for any \( \tau \leq \tau_\alpha^\varepsilon \)

\[
\frac{\nu_1}{2} |\dot{\eta}^\varepsilon(\tau)|^2_{L^2_\alpha^\varepsilon} \leq S_{\mu_0}(u^\varepsilon(\tau, x)) - S_{\mu_0}(\phi_{\mu_0,m}) = H(u^\varepsilon(\tau, x)) - H(\phi_{\mu_0,m})
\]

\[
= -2\varepsilon \text{Im} \int_0^\tau \int_{\mathbb{R}^\varepsilon} \nabla u^\varepsilon(s, x) \cdot x\tilde{u}^\varepsilon(s, x) dx dW(s)
\]

\[
+ 2\varepsilon^2 \int_0^\tau |xu^\varepsilon(s, \cdot)|^2_{L^2_\alpha^\varepsilon} ds.
\]

(3.4)

Then the estimate of the right hand side of (3.3) follows from exactly the same arguments as in Theorem 2 of [13], thanks to the classical exponential tail estimates for 1D stochastic integrals, once we have noticed that for any \( t \in [0, \tau] \),

\[
\left| \int_{\mathbb{R}^\varepsilon} \nabla u^\varepsilon \cdot x\tilde{u}^\varepsilon(t, x) dx \right|^2 \leq \sup_{t \in [0, \tau]} |u^\varepsilon(t)|^2_{L^2_\alpha^\varepsilon} \leq C_{\alpha_0, \mu_0,m}, \quad \text{a.s.,}
\]

which concludes (2.21).

\[\square\]

4. **Modulation equations and SDE for the remainder.** In this section we derive formally the equation of the modulation parameter \( \xi^\varepsilon \), and the remaining term \( \eta^\varepsilon \). A justification of these formal computations is obtained similarly to [13] and thus we omit it. We fix \( \alpha \) so that the conclusion of Section 3 holds and we write \( \tau^\varepsilon \) for \( \tau_{\alpha}^\varepsilon \) from now on. Since \( \xi^\varepsilon \) is a semi-martingale process, adapted to the filtration \( (\mathcal{F}_t)_{t \geq 0} \) generated by \( (W(t))_{t \geq 0} \), we may thus write a priori the equation for \( \xi^\varepsilon \) in the form

\[
d\xi^\varepsilon = \mu_0 dt + \varepsilon y^\varepsilon dt + \varepsilon z^\varepsilon dW
\]

(4.1)

where \( y^\varepsilon \) is a real valued adapted process with paths in \( L^1(0, \tau^\varepsilon) \) a.s., \( z^\varepsilon \) is a real valued predictable process, with paths in \( L^2(0, \tau^\varepsilon) \) a.s. We write \( \phi_{\mu_0,m}(r \cos \theta, r \sin \theta) = e^{i\beta \theta} \psi_{\mu_0,m}(r) \) and \( \eta^\varepsilon(t, r \cos \theta, r \sin \theta) = e^{i\beta \theta} \tilde{\eta}^\varepsilon(t, r) \). Recall that \( \psi_{\mu_0,m} \) is a real-valued function verifying (2.14) with \( \mu = \mu_0 \), and note that \( \tilde{\eta}^\varepsilon \) is a complex-valued radial function since \( \eta^\varepsilon(t, \cdot) \in X_m \). Note that the orthogonality condition (2.19) may be written as

\[
(\text{Im} \tilde{\eta}^\varepsilon, \psi_{\mu_0,m})_{L^2_\alpha} = 0.
\]

(4.2)

**Lemma 4.1.** Let \( \tilde{\eta}^\varepsilon = \tilde{\eta}_{R}^\varepsilon + i\tilde{\eta}_{I}^\varepsilon \), where \( \tilde{\eta}_{R}^\varepsilon = \text{Re} \tilde{\eta}^\varepsilon \) and \( \tilde{\eta}_{I}^\varepsilon = \text{Im} \tilde{\eta}^\varepsilon \). With the above notations, for \( \sigma \geq 1/2 \), \( \tilde{\eta}_{R}^\varepsilon \) and \( \tilde{\eta}_{I}^\varepsilon \) satisfy the equations

\[
d\tilde{\eta}_{R}^\varepsilon = L_{\mu_0,m}^{2,r_{\mu_0,m}}\tilde{\eta}_{R}^\varepsilon dt - \varepsilon y^\varepsilon \tilde{\eta}_{R}^\varepsilon dt - \varepsilon z^\varepsilon r^2 \psi_{\mu_0,m} dt + \varepsilon z^\varepsilon r^2 \psi_{\mu_0,m} dt - \varepsilon r^4 \psi_{\mu_0,m} dt
\]

\[
+ \varepsilon h_{I}^\varepsilon dt + \varepsilon z^\varepsilon r^2 \tilde{\eta}_{R}^\varepsilon dt - \varepsilon^2 r^2 \tilde{\eta}_{R}^\varepsilon dt - \varepsilon^2 r^4 \tilde{\eta}_{R}^\varepsilon dt - \varepsilon^2 r^4 \tilde{\eta}_{R}^\varepsilon dt - \varepsilon z^\varepsilon \tilde{\eta}_{I}^\varepsilon dW
\]

\[
+ \varepsilon r^2 \tilde{\eta}_{I}^\varepsilon dW\]

(4.3)

\[
d\tilde{\eta}_{I}^\varepsilon = -L_{\mu_0,m}^{1,r_{\mu_0,m}}\tilde{\eta}_{I}^\varepsilon dt + y^\varepsilon \psi_{\mu_0,m} dt + \varepsilon y^\varepsilon \tilde{\eta}_{I}^\varepsilon dt - \varepsilon h_{I}^\varepsilon dt - \varepsilon^2 \tilde{\eta}_{I}^\varepsilon dt - \varepsilon^2 r^2 \tilde{\eta}_{I}^\varepsilon dt + \varepsilon^2 z^\varepsilon r^2 \tilde{\eta}_{I}^\varepsilon dt
\]

\[
- \varepsilon^2 r^4 \tilde{\eta}_{I}^\varepsilon dt - \varepsilon r^2 \psi_{\mu_0,m} dW + \varepsilon z^\varepsilon \psi_{\mu_0,m} dW - \varepsilon r^2 \tilde{\eta}_{I}^\varepsilon dW - \varepsilon z^\varepsilon \tilde{\eta}_{R}^\varepsilon dW.
\]

(4.4)
identifying the real and imaginary parts, we deduce the equations (4.3) and (4.4). Using these facts, (2.14), replacing \( u_t \) make use of the orthogonality condition (4.2), we obtain the equations for the \( \sigma \)

\[
\sigma |\psi_{\mu_0,m} + \varepsilon \tilde{\eta}^\varepsilon|^{2\sigma} (\psi_{\mu_0,m} + \varepsilon \tilde{\eta}^\varepsilon) = \psi_{\mu_0,m}^{2\sigma+1} + \varepsilon (2\sigma + 1) \tilde{\eta} R \psi_{\mu_0,m}^{2\sigma} + i\varepsilon \tilde{\eta} R \psi_{\mu_0,m}^{2\sigma} + \varepsilon^2 h_R^{\sigma} + i\varepsilon^2 h_T^{\sigma},
\]

where

\[
\varepsilon^2 h_R^{\sigma} + i\varepsilon^2 h_T^{\sigma} = \int_0^1 (1-s) \frac{\partial^2}{\partial s^2} \left( |\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon|^{2\sigma} (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon) \right) ds.
\]

Using these facts, (2.14), replacing \( e^{i\xi^\varepsilon(t)} v^\varepsilon(t) \) by \( \psi_{\mu_0,m} + \varepsilon \tilde{\eta}^\varepsilon(t,r) \) in (4.5), and identifying the real and imaginary parts, we deduce the equations (4.3) and (4.4). \( \square \)

Remark that if we write \( h_R^{\sigma} \) and \( h_T^{\sigma} \) as functions of \( \psi_{\mu_0,m} \) and \( \varepsilon \tilde{\eta}^\varepsilon \), we obtain what follows.

\[
h_R^{\sigma} = 2\sigma \int_0^1 (1-s) \left( |\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon|^2 + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right)^{\sigma-1}
\times \left\{ \left( (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (\varepsilon \tilde{\eta}^\varepsilon)^2 \right) \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\} ds
+4\sigma(\sigma - 1) \int_0^1 (1-s) \left( |\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon|^2 + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right)^{\sigma-2}
\times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\}^2 \times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\} \times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\} ds,
\]

\[
h_T^{\sigma} = 2\sigma \int_0^1 (1-s) \left( |\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon|^2 + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right)^{\sigma-1}
\times \left\{ 2(\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \varepsilon \tilde{\eta} R \right\} ds
+4\sigma(\sigma - 1) \int_0^1 (1-s) \left( |\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon|^2 + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right)^{\sigma-2}
\times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\}^2 \times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\} \times \left\{ (\psi_{\mu_0,m} + s\varepsilon \tilde{\eta}^\varepsilon)^2 \tilde{\eta} R + (s\varepsilon \tilde{\eta}^\varepsilon)^2 \right\} ds.
\]

We now take the radial \( L^2 \) inner product \( (\cdot, \cdot)_{L^2_R} \) of Eq.(4.4) with \( \psi_{\mu_0,m}(r) \) and make use of the orthogonality condition (4.2), we obtain the equations for the modulation parameters \( y^\varepsilon, z^\varepsilon \) from the identification of drift parts and that of martingale parts.

**Lemma 4.2.** Under the assumptions of Theorem 2.2, the modulation parameters satisfy the system of the equations, for any \( t \leq \tau^\varepsilon \),

\[
z^\varepsilon(t) \left\{ |\psi_{\mu_0,m}|_{L^2}^2 + \varepsilon (\tilde{\eta}^\varepsilon(t), \psi_{\mu_0,m})_{L^2} \right\} = |\tau \psi_{\mu_0,m}|_{L^2}^2 + \varepsilon (\tilde{\eta}^\varepsilon(t), r^2 \psi_{\mu_0,m})_{L^2} (4.6)
\]
and
\[ y^\varepsilon(t) \left\{ |\psi_{\mu_0,m}|^2_{L^2} + \varepsilon(\bar{\eta}_R(t), \psi_{\mu_0,m})_{L^2} \right\} = -\varepsilon^2 z^\varepsilon(t)(\bar{\eta}_R(t), r^2 \psi_{\mu_0,m})_{L^2} \]
\[ + (L^1_{\mu_0,m} \bar{\eta}_R(t), \psi_{\mu_0,m})_{L^2} + \varepsilon(h_R(t), \psi_{\mu_0,m})_{L^2} + \varepsilon^2 (\bar{\eta}_R(t), r^4 \psi_{\mu_0,m})_{L^2}. \]

We deduce from the modulation equations obtained in Lemma 4.2 the following estimates for the modulation parameters. Indeed, it suffices to note that for \( t \leq \tau^\varepsilon \), choosing \( \alpha_0 \) again sufficiently small if necessary such that \( 2\alpha_0 \leq \sqrt{2\pi} |\psi_{\mu_0,m}|_{L^2} \), we have
\[ \left| |\psi_{\mu_0,m}|^2_{L^2} + \varepsilon(\bar{\eta}_R, \psi_{\mu_0,m})_{L^2} \right| \geq |\psi_{\mu_0,m}|^2_{L^2} - |\varepsilon \bar{\eta}_R|_{L^2} |\psi_{\mu_0,m}|_{L^2} \geq (1/2) |\psi_{\mu_0,m}|^2_{L^2} \]
for \( \alpha \leq \alpha_0 \). Other estimates for the right hand sides of (4.7)-(4.6) follow exactly as in Corollary 4.3 of [13], using the expression of \( h_R^1 \) and \( h_R^2 \) above, the Sobolev embedding, and the properties of \( \psi_{\mu_0,m} \) given in Proposition 3.

**Corollary 1.** Under the assumptions of Theorem 2.2, there is a \( \alpha_1 > 0 \) such that for any \( \alpha \leq \alpha_1 \), there is a constant \( C_{\mu_0,\alpha,m} \) with
\[ |z^\varepsilon(t)| \leq C_{\mu_0,\alpha,m}, \text{ a.s. for all } t \leq \tau^\varepsilon, \ \varepsilon \leq \varepsilon_0. \] Moreover, there are constants \( C_1 \) and \( C_2 \) depending only on \( \sigma, \alpha, m \) and \( \mu_0 \) such that
\[ |y^\varepsilon(t)| \leq C_1 \bar{\eta}_R(t)_{L^2} + \varepsilon C_2 \text{ a.s. for all } t \leq \tau^\varepsilon, \ \varepsilon \leq \varepsilon_0. \]
where
\[ \bar{\tau}_N = \inf\{t \leq \tau^\varepsilon \land T, \, |\varepsilon t^\varepsilon|_{L_\infty} \geq N\}, \quad \text{for any } N > 0. \]

In fact, in order to prove (i) of Lemma 5.1, we need a bound \( |\varepsilon t^\varepsilon(t)|_{L_\infty} \leq C' \) for some constant \( C' > 0 \) for any \( t \leq \bar{\tau}_N \). This is realized by the above definition of the stopping time \( \bar{\tau}_N \), because \( |\varepsilon t^\varepsilon(t)|_{L_\infty} \leq C|\varepsilon t^\varepsilon(t)|_{L_\infty(t)} \leq CN \) for any \( t \leq \bar{\tau}_N \).

**Lemma 5.2.** Let \( T > 0 \) be fixed and \( 1 \leq \sigma \). Let \( \mu_0 > \lambda_0 \) be fixed and \( \eta^\varepsilon, \xi^\varepsilon \) \((\varepsilon > 0)\) be given by Theorem 2.2 with \( \alpha \leq \alpha_0 \) fixed. Put \( \eta(t, t^\varepsilon \cos \theta, t^\varepsilon \sin \theta) = e^{im\theta} \tilde{\eta}(t, r) \). There exists a constant \( C_3 \) depending only on \( T, \alpha, \mu_0, m, N \) such that
\[ \mathbb{E}\left( \sup_{t \leq \bar{\tau}_N \land T} |\tilde{\eta}(t)|_{L_\infty}^2 \right) \leq C_3, \]
where \( \bar{\tau}_N \) is defined in Lemma 5.1.

As for the solution of Eq.(2.22), we have

**Lemma 5.3.** Let \( T > 0 \) be fixed and \( 1/2 \leq \sigma \). Let \( \mu_0 > \lambda_0 \) be fixed and \( \eta^\varepsilon, \xi^\varepsilon \) \((\varepsilon > 0)\) be given by Theorem 2.2 with \( \alpha \leq \alpha_0 \) fixed. Put \( \eta(t, t^\varepsilon \cos \theta, t^\varepsilon \sin \theta) = e^{im\theta} \tilde{\eta}(t, r) \). there exist \( C_4 \) and \( C_5 \) depending only on \( T, \alpha, \mu_0, m \) such that
\[
(i) \quad \mathbb{E}\left( \sup_{t \leq T} |\tilde{\eta}(t)|_{L_\infty}^2 \right) \leq C_4, \quad (ii) \quad \mathbb{E}\left( \sup_{t \leq T} |(1 + r^4)\tilde{\eta}(t)|_{L_\infty}^2 \right) \leq C_5.
\]

We remark that the assumption \( \sigma \geq 1 \) is needed only for Lemma 5.2. The reason is the same as in [13], due to the terms where \( \nabla h_\delta \) and \( \nabla h_\delta^r \) are involved. Using these lemmas we obtain the following convergence.

**Lemma 5.4.** Let \( T > 0 \) be fixed. Under the assumptions of Theorem 2.3, \( \tilde{\eta}^\varepsilon \) converges to \( \tilde{\eta} \) as \( \varepsilon \) tends to 0 in \( L^2(\Omega; C([0, \bar{\tau}_N^\varepsilon \land T], L^2)) \).

The convergence in probability in the time interval \([0, \tau^\varepsilon \land T]\) follows from Lemma 5.4:

**Corollary 2.** Let \( T > 0 \) be fixed. Under the assumptions of Theorem 2.3, \( \tilde{\eta}^\varepsilon \) converges to \( \tilde{\eta} \) as \( \varepsilon \) tends to 0 in probability, in \( C([0, \tau^\varepsilon \land T], L^2) \).

We give here, assuming Lemma 5.4, a different proof from [13] for this corollary, which improves the range of nonlinear power \( \sigma \).

**Proof.** We prove that for any \( \beta > 0, \, \delta > 0, \)
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tau^\varepsilon \land T]} \tilde{\eta}^\varepsilon - 1_{[0, T]} \tilde{\eta}|_{L_2} > \delta \right) \leq \beta, \tag{5.2}
\]
provided that \( \varepsilon \) is sufficiently small. We note that
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tau^\varepsilon \land T]} \tilde{\eta}^\varepsilon - 1_{[0, T]} \tilde{\eta}|_{L_2} > \delta \right) \leq \mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tau^\varepsilon \land T]}(\tilde{\eta}^\varepsilon - \tilde{\eta})|_{L_2} > \delta \right) + \mathbb{P}(\tau^\varepsilon \land T < T).
\]
It follows from (2.21) that for any \( \beta > 0 \) there exists \( \varepsilon_0 > 0 \) such that \( \mathbb{P}(\tau^\varepsilon \land T < T) \leq \beta/3 \) for any \( \varepsilon \leq \varepsilon_0 \). On the other hand,
\[
\mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tau^\varepsilon \land T]}(\tilde{\eta}^\varepsilon - \tilde{\eta})|_{L_2} > \delta \right) \leq \mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tau^\varepsilon \land T]}(\tilde{\eta}^\varepsilon - \tilde{\eta})|_{L_2} > \delta \right) + \mathbb{P}(\bar{\tau}_N^\varepsilon \land T < T).
\]
Concerning the second term, we first show that for any \( \beta > 0 \) there exist \( N_0 \) and \( \varepsilon_0 \) such that for any \( \varepsilon \leq \varepsilon_0 \) and \( N \geq N_0 \),

\[
\mathbb{P}\left( \sup_{t \in [0, \tau^* \wedge T]} |\varepsilon \eta^\varepsilon|_{\Sigma(2)} \geq N \right) \leq \beta/6,
\]

and then use the fact that

\[
\mathbb{P}(\tilde{r}_N^* \wedge T < T) \leq \mathbb{P}\left( \sup_{t \in [0, \tau^* \wedge T]} |\varepsilon \eta^\varepsilon|_{\Sigma(2)} \geq N \right) + \mathbb{P}(\tau^* \wedge T < T).
\]

Remarking \( \varepsilon \eta^\varepsilon(t, r) = e^{\i \xi^\varepsilon(t)} u^\varepsilon(t, r) - \phi_{\mu_0, m} \), it suffices to show that

\[
\lim_{N \to \infty} \mathbb{P}\left( \sup_{t \in [0, \tau^* \wedge T]} |u^\varepsilon(t)|_{\Sigma(2)} \geq N \right) = 0,
\]

uniformly for \( \varepsilon \leq 1 \). Let \( \varepsilon \leq 1 \) and \( w^\varepsilon(t) \) be the solution of (2.6) with \( w^\varepsilon(0) = \phi_{\mu_0, m} \in \Sigma(2) \); let moreover \( \tilde{r}_M := \inf\{ t \geq 0, |w^\varepsilon(t)|_{\Sigma(2)} \geq M \} \) for \( M > 0 \). By Proposition 4 (and Remark 1 in the Appendix),

\[
\lim_{M \to \infty} \tilde{r}_M^* = +\infty, \quad a.s., \text{ uniformly in } \varepsilon \leq 1,
\]

that is, for any finite time \( T \in (0, +\infty) \),

\[
0 = \lim_{M \to \infty} \mathbb{P}(\tilde{r}_M^* < T) = \lim_{M \to \infty} \mathbb{P}\left( \sup_{t \in [0, T]} |w^\varepsilon(t)|_{\Sigma(2)} \geq M \right),
\]

uniformly for \( \varepsilon \leq 1 \). The gauge transformation (2.5) keeps the equivalence of the \( \Sigma(2) \) norm between \( u^\varepsilon \) and associated solution \( w^\varepsilon \) uniformly for \( \varepsilon \leq 1 \), thus (5.3) follows.

On the other hand, using Lemma 5.4, we have for any \( \beta > 0 \),

\[
\mathbb{P}\left( \sup_{t \in [0, T]} |1_{[0, \tilde{r}_N^* \wedge T]}(\tilde{\theta}^\varepsilon - \tilde{\eta})|_{L^2} > \delta \right) \leq \frac{\beta}{6},
\]

for \( \varepsilon \leq \varepsilon_0 \) sufficiently small, and we deduce that (5.2) holds for \( \varepsilon \leq \varepsilon_0 \).

\[\square\]

6. **Numerical observations.** The main goals of this section are to retrieve numerically the result of Theorem 2.2 and to give numerical evidence that, to some extent, the estimate of Theorem 2.2 is optimal. We also compute the numerical evolution of vortices in the presence of noise. We will first present the numerical scheme we use to solve equation (2.1) under the assumption of \( m \)-equivariance symmetry. Then, we will briefly explain how we can compute the radial profile \( \psi_{\mu, m} \) given by equation (2.14) and eventually we will use a classical Monte-Carlo method to estimate the left-hand side of equation (2.21).

We use a relaxed Crank-Nicolson scheme. We limit the resolution domain to \([0, R]\), where \( R \) needs to be chosen large enough to avoid reflections. We set a spatial discretization step \( \Delta r = \frac{R}{N_r} \) where \( N_r \in \mathbb{N} \). We set \( r_j = j \Delta r \) for \( 0 \leq j \leq N_r \), and \( r_{j+1/2} = (j + 1/2) \Delta r \) for \( 0 \leq j \leq N_r \). We set a final time \( T > 0 \) and a time discretization step \( \Delta t = \frac{T}{N_t} \) where \( N_t \in \mathbb{N} \).

Because of the \( m \)-equivariance symmetry we solve equation (2.1) for \( f(t, r) \) where \( u(t, r \cos \theta, r \sin \theta) = e^{-i m \theta} u(t, r) \) (with the notations of the previous sections).
This decomposition is motivated by Proposition 3. The numerical scheme, derived from a scheme used in [3] is the following:

\[
\begin{align*}
&\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}} + \varphi_{j-\frac{1}{2}}^{n-\frac{1}{2}} = |f_{j+\frac{1}{2}}^{n}|^{2\sigma}, \\
f_{j+\frac{3}{2}}^{n+1} = f_{j+\frac{3}{2}}^{n} \quad \text{for all } j, n \\
-\frac{1}{\Delta r} \left( r_{j+\frac{3}{2}} f_{j+\frac{3}{2}}^{n+\frac{1}{2}} - 2r^{m+1} f_{j+\frac{3}{2}}^{n+\frac{1}{2}} + r f_{j+\frac{1}{2}}^{n+\frac{1}{2}} \right) \Delta t \\
\Delta W^{n} = \sqrt{\delta t} \mathcal{N} \quad \text{with } (\mathcal{N}^{n})_{0 \leq n \leq N}, \text{ a family of independent standard Gaussian random variables.}
\end{align*}
\]

\[
\begin{align*}
&f_{N+\frac{1}{2}}^{n+1} = 0 \quad \text{or } f_{N+\frac{1}{2}}^{n} = f_{N-r-\frac{1}{2}}^{n}, \\
&\varphi_{j}^{n} = |f_{j}^{n}|^{2\sigma},
\end{align*}
\]

for all \(0 \leq n \leq N\), where \(f_{j+\frac{3}{2}}^{n+1}\) is an approximation of \(f(n \Delta t, (j+\frac{1}{2}) \Delta r)\), \(f_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \int_{j}^{j+\frac{1}{2}} f_{j+\frac{3}{2}}^{n+\frac{1}{2}} \Delta W^{n} = \sqrt{\delta t} \mathcal{N}^{n}\) with \((\mathcal{N}^{n})_{0 \leq n \leq N}\), a family of independent standard Gaussian random variables.

The boundary condition at \(r = R\) can be either homogeneous Dirichlet condition or homogeneous Neumann condition. Numerical experiments show that the impact is not significant. In this scheme, \(\varphi_{n+\frac{1}{2}}^{n}\) is an approximation of \(|f_{n+\frac{1}{2}}^{n+1}|^{2\sigma}\). This scheme does not conserve exactly the \(L^{2}\) norm, but with a sufficiently good spatial discretization, the fluctuations are negligible.

In order to compute the radial profile of the vortex \(\psi_{\mu,m}\), which is the initial state in our numerical simulations, we use a shooting method inspired by [16] (see also [17]). In our case, setting \(\psi_{\mu,m}(r) = r^{m} f_{\mu,m}(r)\), we look for a \(C^{2}(0, \infty) \cap C([0, \infty))\) real non-negative function \(f_{\mu,m}\) satisfying the following equation, for \(r > 0\),

\[
- f''_{\mu,m}(r) - \frac{2m+1}{r} f'_{\mu,m}(r) - \mu f_{\mu,m}(r) + r^{2\sigma m} f_{\mu,m}(r)^{2\sigma + 1} + r^{2} f_{\mu,m}(r) = 0 \quad (6.2)
\]

with

\[
\lim_{r \to \infty} |f_{\mu,m}(r)| = 0.
\]

This equation can be written as a first degree ODE. We set \(X(r) = (f(r), f'(r))\), and equation (6.2) becomes:

\[
\begin{align*}
X'(r) &= F(r, X(r)), \quad r > 0 \\
X(0) &= (\alpha, \beta),
\end{align*}
\]

with

\[
F(r, (x, y)) := \left( y, -\frac{2m+1}{r} y - \mu x + r^{2\sigma m} x^{2\sigma + 1} + r^{2} x \right),
\]

\(\forall r > 0, \quad \forall x, y \in \mathbb{R}.
\]

In order to solve this equation numerically, we limit the domain to \([0, R]\) and we use a Runge-Kutta scheme of order four. Since the nonlinear term \(f''_{\mu,m}(r)\) vanishes at \(r = 0\), it is easily seen by the argument of [21], Section 2 that we must set \(\beta = 0\). The idea is to find \(\alpha\) thanks to a dichotomy method, looking at the behavior of \(f_{\mu,m}\) for large \(r\). We thus search a value of \(\alpha\) that enables \(f_{\mu,m}\) to be positive on \((0, R]\) and that minimizes \(f_{\mu,m}(r)\) for large \(r\).
We show in figures 1, 2 and 3 some profiles $\psi_{\mu,m}$, computed from equation (6.3) for different $m$ and $\mu$.

**Figure 1.** Amplitudes of vortices $|\psi_{\mu,m}|$ for $\mu$ close to $\lambda_m$, for different values of $m$.

**Figure 2.** Amplitudes of vortices $|\psi_{\mu,m}|$ for $m = 1$ for different values of $\mu$.

**Figure 3.** Amplitudes of vortices $|\psi_{\mu,m}|$ for $\mu = 11$ for different values of $m$.

We show in figures 4, 5 and 6 some trajectories (computed from the same realization of a Brownian motion) of $|u^\varepsilon(t,r)|$ for different values of $m$ and $\mu = 2m + 3$. Here, $t$ is represented along the ordinate axis, and $r$ along the abscissa.

These figures have been plotted with $\varepsilon$ of order $10^{-1}$. For this order of magnitude, we can observe that the wave function keeps the same structure over time and that it oscillates with an almost periodic rhythm. The simulations that will be presented in the sequel of the paper have been computed for $\varepsilon$ of order $10^{-2}$. In this context, the oscillations of $|u^\varepsilon|$ are of course much smaller.

The last step is to compute $\xi^\varepsilon(t)$ and $\varepsilon\eta^\varepsilon(t,x)$ from the simulated trajectories. The computation of $\xi^\varepsilon(t)$ for $t \leq \tau^\varepsilon$ is given by

$$
\xi^\varepsilon(t) = - \arg \left( \int_{\mathbb{R}^2} u^\varepsilon(t) \overline{\phi_{\mu_0,m}} \, dx \right),
$$

thanks to the orthogonality condition (2.19). Then, the computation of $\varepsilon\eta^\varepsilon$ is immediately given by $\varepsilon\eta^\varepsilon(t,x) = u^\varepsilon(t,x)e^{i\xi^\varepsilon(t)} - \phi_{\mu_0,m}(x)$. We represent in figure 7 a trajectory of $\xi^\varepsilon(t) - \mu_0 t$ and $|\varepsilon\eta^\varepsilon(t)|$. 

We now estimate the left-hand side of inequality (2.21). We use a classical Monte-Carlo method. We set $\sigma \geq 1/2$, $m \in \mathbb{N}$, $\mu_0 > \lambda_m$, and $\varepsilon > 0$. According to Theorem 2.2, we choose $\alpha > 0$ and assume $\alpha \leq \alpha_0$. Taking $T > 0$, $N \in \mathbb{N}$, $\Delta t > 0$ and $\Delta r > 0$, we estimate $P(\tau_{\alpha}^\varepsilon \leq T)$ by the following estimator:

$$\hat{Y}_{\Delta r, \Delta t, N}^{\alpha, \varepsilon} = \frac{1}{N} \sum_{k=1}^{N} Y_{\Delta r, \Delta t}^{\alpha, \varepsilon, (k)}$$

(6.5)

with

$$Y_{\Delta r, \Delta t}^{\alpha, \varepsilon, (k)} = 1\{|(\varepsilon \eta_{\Delta r, \Delta t}^{(k)})|_{L^\infty((0, T), \mathbb{R})} > \alpha \}$$

(6.6)

where $(\varepsilon \eta_{\Delta r, \Delta t}^{(k)})$ for $1 \leq k \leq N$ are remainders in the decomposition of independent solutions of the numerical scheme with the discretization steps $\Delta r$ and $\Delta t$. Thus the $Y_{\Delta r, \Delta t}^{\alpha, \varepsilon, (k)}$ for $1 \leq k \leq N$ are i.i.d. This estimator is meaningful if

$$\lim_{\Delta r, \Delta t, N \to 0} \mathbb{E}\left(\hat{Y}_{\Delta r, \Delta t, N}^{\alpha, \varepsilon} - P(\tau_{\alpha}^\varepsilon \leq T)\right)^2 = 0.$$

This mean square error can be written:

$$\mathbb{E}\left(\hat{Y}_{\Delta r, \Delta t, N}^{\alpha, \varepsilon} - P(\tau_{\alpha}^\varepsilon \leq T)\right)^2 = \text{Var}\left[\hat{Y}_{\Delta r, \Delta t, N}^{\alpha, \varepsilon}\right] + [\mathbb{E}Y_{\Delta r, \Delta t}^{\alpha, \varepsilon, (1)} - P(\tau_{\alpha}^\varepsilon \leq T)]^2.$$

The first term of the right-hand side converges to zero as $N$ tends to infinity. The convergence to zero of the second term of the right-hand side is an expected property.
of the numerical scheme. This latter property of the scheme is not theoretically proved, but we will assume that taking a discretization small enough enables this term to be as small as we wish.

We made the experiment taking $\alpha = 2.5 \cdot 10^{-3}$ and $\sigma = 1$. The results are given in Figure 8 and 9. We can observe in Figure 8 that for $\epsilon$ small enough the curve becomes a straight line of slope $-2$, which agrees with the upper bound given by (2.21). For larger $\epsilon$, the curve goes below the straight line given by the lowest $\epsilon$. Thus for larger $\epsilon$, estimate (2.21) is not valid anymore. This shows the existence of the $\epsilon_0$ given by Theorem 2.2.
In Figure 9 we fix $\varepsilon = 9 \cdot 10^{-3}$ and we estimate $\mathbb{P}(\tau^\varepsilon_\alpha \leq T)$ for $T \leq 0.1$. Figure 8 ensures that $\varepsilon \leq \varepsilon_0$ for this choice of $\alpha$ and $T$. We observe that the curve is a straight line of slope $-1$, which validates the upper bound given by (2.21).

The classical Monte-Carlo approach we used here becomes unreasonable to evaluate $\mathbb{P}(\tau^\varepsilon_\alpha \leq T)$ when this probability becomes too small. For example, with this method it will be extremely expensive to add points to Figure 8 for smaller $\varepsilon$. In order to do this, it would be necessary to develop a smarter Monte-Carlo method adapted to the evaluation of these very small probabilities.

**Appendix.** In this section we will verify (i) of Proposition 3, Lemma 2.1 and Proposition 4.

**Proof of Proposition 3 (i).** Since $v \in X_m$, we write $v(x) = e^{i m \theta} f(r)$ with $f(r) = f_R(r) + i f_I(r)$ where $f_R = \text{Re} f$, and $f_I = \text{Im} f$. With this notation, we may describe $S''_\mu(\phi_{\mu,m})$ as follows.

$$
(S''_\mu(\phi_{\mu,m})v, v) = 2\pi \left[ (L^{1,r}_{\mu,m}f_R, f_R)_{L^2} + (L^{2,r}_{\mu,m}f_I, f_I)_{L^2} \right],
$$

where $L^{1,r}_{\mu,m}$ and $L^{2,r}_{\mu,m}$ are defined by (2.16) and (2.17). First, the self-adjointness of $L^{1,r}_{\mu,m}$ and $L^{2,r}_{\mu,m}$ for $m \geq 1$ with domain $D$ follows from similar arguments as in Appendix X-1 of [33] via the use of the unitary transform

$$
U : L^2((0, +\infty), dr) \rightarrow L^2((0, +\infty), dr)
$$

and the whole spectrum of both operators is discrete since

$$
\frac{m^2}{r^2} + r^2 - \mu + |\psi_{\mu,m}(r)|^{2\sigma} \rightarrow +\infty, \text{ as } r \rightarrow +\infty
$$

(see Chapter 2 of [5]). Remark that $\psi_{\mu,m}(r)$ is positive for $r > 0$ and satisfies $L^{2,r}_{\mu,m}\psi_{\mu,m} = 0$, thus $\psi_{\mu,m}$ is the simple eigenfunction corresponding to the eigenvalue 0 (see Chapter 3-3 of [5] for details). This concludes that there exists $\delta > 0$ such that

$$
(L^{2,r}_{\mu,m}h, h)_{L^2^2} \geq \delta |h|_{L^2^2}^2
$$

for any $h \in \Sigma_r$ satisfying $(h, \psi_{\mu,m})_{L^2^2} = 0$. Here, recall that the norm $\Sigma_r$ is defined in (5.1). Note that for any $h \in \Sigma_r$,

$$
(L^{1,r}_{\mu,m}h, h)_{L^2^2} = (L^{2,r}_{\mu,m}h, h)_{L^2^2} + 2\sigma \int_0^\infty |\psi_{\mu,m}(r)|^{2\sigma}|h(r)|^2 r dr.
$$

(6.8)

Therefore, if we denote the first eigenvalues of both operators by

$$
\mu^{(1)}_j = \inf\{(L^{j,r}_{\mu,m}h, h)_{L^2^2}, \ h \in \Sigma_r, \ |h|_{L^2^2} = 1\}, \ j = 1, 2
$$

it turns out from the relation (6.8) that

$$
\mu^{(1)}_1 \geq \mu^{(2)}_1 + 2\sigma \inf\{ \int_0^\infty |\psi_{\mu,m}(r)|^{2\sigma}|h(r)|^2 r dr, \ h \in \Sigma_r, \ |h|_{L^2^2} = 1\}.
$$

Since $\psi_{\mu,m}(r)$ is strictly positive for $r > 0$, we see that $\mu^{(1)}_1 > \mu^{(2)}_1 = 0$, and we get

$$
(L^{1,r}_{\mu,m}h, h)_{L^2^2} \geq \mu^{(1)}_1 |h|_{L^2^2}^2
$$

for any $h \in \Sigma_r$. Finally going back to (6.7), we may see that there exists $\nu > 0$ such that

$$
(S''_\mu(\phi_{\mu,m})v, v) \geq 2\pi \int_0^\infty \nu(|f_I(r)|^2 + |f_R(r)|^2) r dr
$$
for any $f_{1} \in \Sigma_{r}$ satisfying $(f_{1}, \psi_{,m})_{L_{2}}^{r} = 0$, i.e.,

$$(S_{\mu}^{n}(\phi_{,m})v, v) \geq \nu|v|_{L_{2}}^{2},$$

for any $v \in X_{m}$ with $\text{Re}(v, i\phi_{,m})_{L^{2}(\mathbb{R}^{2}, dx)} = 0$. This implies the statement (i). □

Proof of Lemma 2.1. First of all, we note that the formula (2.7) is well defined as an oscillatory integral until the time $T_{0} \land T$, and using Proposition 6 in [14], we see that if the initial data $u_{0} \in \Sigma$, then $w \in \Sigma$. Also, if $u_{0}$ is written in the form

$$u_{0}(r \cos \theta, r \sin \theta) = e^{i\mu h}(r)$$

for some radial function $h(r)$, $w(t, x)$ defined by (2.7) is in such form too; indeed, for any $x \in \mathbb{R}^{2}$, define for any $g \in L^{1}(\mathbb{R}^{2})$,

$$\bar{g}(x) := \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot y} g(y)dy.$$  \hspace{1cm} (6.9)

With the following argument, it suffices to show that for any phase $\phi \in \mathbb{R}$, and any $f \in L^{1}(\mathbb{R}^{2})$,

$$\tilde{f}(e^{i\phi}x) = \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot y} f(e^{i\phi}y)dy$$  \hspace{1cm} (6.10)

holds. Suppose $f(x) = e^{i\mu h}(r)$ where $m \in \mathbb{Z}$, $\theta \in \mathbb{R}$ and $x = re^{i\theta}$ ($r = |x|$). Then,

$$f(e^{i\phi}x) = f(x)e^{i\mu \phi}$$

for any $x \in \mathbb{R}^{2}$ and $\phi \in \mathbb{R}$. We operate the transformation (6.9) on both sides and we get by (6.10), $\tilde{f}(e^{i\phi}x) = e^{i\mu \phi}\tilde{f}(x)$. In particular, taking $x = r \geq 0$, we have $\tilde{f}(e^{i\phi}r) = e^{i\mu \phi}\tilde{f}(r)$. Since $r$ and $\phi$ are arbitrary, putting $z = re^{i\phi}$, for any $z \in \mathbb{R}^{2}$, the relation $\tilde{f}(z) = e^{i\mu \phi}\tilde{f}(r)$ is satisfied. This means that any function in the form $e^{i\mu h}(r)$ is preserved in the same form by the transformation (6.9).

Now we verify the equality (6.10). Noting that

$$U = e^{i\phi} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

is unitary and its determinant equals 1,

$$\int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot y} f(e^{i\phi}y)dy = \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot y} f(Uy)dy = \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot y} f(U^{-1}w)dw = \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot w} f(w)dw = \int_{\mathbb{R}^{2}} e^{i\beta(t)x \cdot w} f(Uw)dw = \tilde{f}(Ux) = \tilde{f}(e^{i\phi}x).$$

We recall Theorem 1 of [14]. For fixed $T_{0} > 0$ and $\alpha \in (0, 1/2)$, and for $\omega \in \Omega$ such that $W(\omega, \omega) \in C^{\alpha}([0, T_{0}])$, there exist $T_{\omega} > 0$ and a propagator $\{U^{\omega}(t, s), t, s \in [0, T_{0}], t - s \leq T_{\omega}\}$ corresponding to Eq.(2.4). By the uniqueness of solutions, the solution of (2.4), constructed in (3.21) of [14], with the kernel of this propagator $U^{\omega}(t, 0)$ is the same as $w(t, x)$ defined by (2.7) for a small time interval $[0, T]$ with $T \leq T_{0} \land T_{\omega} \land T$.

On the other hand, to consider the solution of the nonlinear equation (2.6), let $u_{0} \in X_{m}$. Using the integral form for (2.6) with $w(0) = u_{0}$, for $t \in [0, T]$,

$$w(t) = U^{\omega}(t, 0)u_{0} - i\lambda \int_{0}^{t} U^{\omega}(t, s)|w|^{2\gamma} w(s)ds.$$  \hspace{1cm} (6.11)

We see that if $w \in X_{m}$, then $|w|^{2\gamma} \in X_{m}$, and so, by the above argument, $U^{\omega}(t, s)|w|^{2\gamma} w(s) \in X_{m}, too$. In particular, the initial data $u_{0} \in X_{m}$ belongs to $\Sigma$ and thus it follows from Proposition 1 that there is a unique solution $w(\cdot) \in$
The local existence in $\Sigma(2)$ is proved by a fixed point method applied to the map $U$ where

4.10.1 in [7]: let $u$

Proof of Proposition 4.

Local existence follows from the arguments of Theorem 4.10.1 in [7], one may prove that $\omega$ dependence of the estimates. Setting $\epsilon$

$\tau$ it follows

This solution exists in fact globally. To see this, we follow the argument in Proposition 2.2 of [20]. Remind that $\omega,T$ such that, $\omega,T > 0$ comes from the $\Sigma(2)$-bound for the propagator $U(t,0)$, which was shown in Proposition 6(3) of [14]. This allows us to show the local existence and blow up alternative in $\Sigma(2)$.

This solution exists in fact globally. To see this, we follow the argument in Proposition 2.2 of [20]. Remind that $u_0 \in \Sigma(2)$, then in particular $u_0 \in \Sigma$. Thus there exists a unique solution $w(\cdot) \in C([0, \tau_{u_0,\omega}), \Sigma) \cap (2.6)$ with the maximal time $\tau_{u_0,\omega}$. Moreover, since $\lambda = 1$, $\tau_{u_0,\omega} = +\infty$ a.s. We may suppose that there exists a uniform constant $K_{\omega,T_0} > 0$ such that

$$\sup_{0 \leq t \leq T_0} |w(t)|_\Sigma \leq K_{\omega,T_0} < \infty, \quad a.s.$$  \hspace{1cm} (6.12)

This uniform bound in $\Sigma$ implies that for any $q > 2$ there exists a constant $K_{\omega,T_0,q} > 0$ s.t.

$$\sup_{0 \leq t \leq T_0} \{ |\nabla w(t)|_{L^q(\mathbb{R}^2)} + |w(t)|_{L^q(\mathbb{R}^2)} \} \leq K_{\omega,T_0,q} < \infty, \quad a.s.$$  \hspace{1cm} (6.13)

Indeed, let $u_0 \in \Sigma(2)$; Using (2) of Proposition 6, and Lemma 4.1 in [14], we obtain the existence of bounded real-valued functions $a_{jk,lm}$ for $j,k,l,m \in \{1,2\}$ such that, for $t,s \in [0,T]$ with $|t-s| \leq T_0$, where $T_0$ is given by Lemma 4.1 in [14],

$$x_j U^\omega(t,s) = U^\omega(t,s)x_j - (t-s)U^\omega(t,s)(i\partial_{x_j})$$

$$+(t-s)\sum_{k=1}^2 \{ I(t,s,a_{jk,11})x_k + I(t,s,a_{jk,12})(i\partial_{x_k}) \},$$

$$i\partial_{x_j} U^\omega(t,s) = U^\omega(t,s)i\partial_{x_j} + \sum_{k=1}^2 \{ I(t,s,a_{jk,21})x_k + (t-s)I(t,s,a_{jk,22})(i\partial_{x_k}) \},$$

where we have set

$$I(t,s,a)f(x) = (2\pi i(t-s))^{-1}a(t,s) \int_{\mathbb{R}^2} e^{iS(t,s,x,y)} f(y) dy, \quad \text{for } f \in C_0^\infty(\mathbb{R}^2),$$

and $S(t,s,x,y)$ is a real valued continuous function of all its arguments (see Proposition 4 of [14]). Using then the integral equation (6.11), we easily deduce that for
where we have used (6.13). The uniform bound in $\Sigma(2)$ follows a.s. Once (6.13) is proved, by the Sobolev embedding, we conclude that $I(t, s, a)$ is integrable near $\partial\Sigma(2)$ and the $L^q$ norm is also uniformly bounded in time since $q > 2 = d$.

Thus, we get in summary, by (6.12), for $t \in [0, T]$

$$\|\nabla w(t)\|_{L^q} + |xw(t)|_{L^q} \leq C_{\omega, T_0} |u_0|_{\Sigma(2)} + C_{\omega, T_0} \int_0^t \left( |t-s|^{-(1-\sigma)/2} + |t-s|^{\sigma/2} \right)$$

Thus, we get in summary, by (6.12), for $t \in [0, T]$

$$\|\nabla w(t)\|_{L^q} + |xw(t)|_{L^q} \leq C_{\omega, T_0} |u_0|_{\Sigma(2)} + C_{\omega, T_0} \int_0^t \left( |t-s|^{-(1-\sigma)/2} + |t-s|^{\sigma/2} \right)$$

Note that $|t|^{-(1-\sigma)/2}$ is integrable near $t = 0$, and then by Gronwall’s inequality, together with an iteration argument on $[T, 2T], \ldots, [(k-1)T, kT \land T_0]$, we obtain (6.13). Once (6.13) is proved, by the Sobolev embedding, we conclude that $w \in L^\infty(\mathbb{R}^2)$ and the $L^\infty$ norm is also uniformly bounded in time since $q > 2 = d$.

We finally estimate the solution of (6.11) in the $\Sigma(2)$ norm, using the fact that for $\sigma \geq 1/2$,

$$\|w(t)\|_{\Sigma(2)} \leq \|U^\omega(t, 0)u_0\|_{\Sigma(2)} + \int_0^t \|U^\omega(t, s)\|_{L^2} \|w|^{2\sigma} w(s)|_{\Sigma(2)} ds$$

$$\leq C_{\omega, T_0} \left( |u_0|_{\Sigma(2)} + \int_0^t \|w|^{2\sigma} w(s)|_{\Sigma(2)} ds \right)$$

$$\leq C_{\omega, T_0} \left( |u_0|_{\Sigma(2)} + \int_0^t \left\{ \|w|^{2\sigma}_L \|w\|_{\Sigma} + \|x\nabla w\|_{L^2} + \|x^2 w\|_{L^2} \right\} ds \right)$$

$$\leq C_{\omega, T_0} \left( |u_0|_{\Sigma(2)} + \int_0^t \|w(s)|_{\Sigma(2)} ds \right),$$

where we have used (6.13). The uniform bound in $\Sigma(2)$ follows a.s. $\omega$ on any interval $[0, T_0]$ again from the Gronwall inequality. \qed
Remark 1. It follows from the above computations, together with the fact that all the constants appearing in Proposition 6 and Lemma 4.1 in [14] depend only on $|W|_{C^0([0,T_0])}$, for some $\alpha > 0$, that if we replace $W$ by $\varepsilon W$, and if the constant in (6.12) is uniform for $\varepsilon \leq 1$, then the bound on $\sup_{t \in [0,T_0]} |u^\varepsilon(t)|_\Sigma(2)$ is also uniform for $\varepsilon \leq 1$. Now, since $\sup_{t \in [0,\tau^\varepsilon \wedge T_0]} |u^\varepsilon(t)|_\Sigma$ is uniformly bounded for $\varepsilon \leq 1$ by the definition of $\tau^\varepsilon$, we deduce that

$$\lim_{N \to +\infty} \mathbb{P} \left( \sup_{t \in [0,\tau^\varepsilon \wedge T_0]} |u^\varepsilon(t)|_\Sigma(2) \geq N \right) = 0,$$

uniformly for $\varepsilon \leq 1$.

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