Electromagnetic transitions in the algebraic cluster model

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Abstract
We study electromagnetic transition rates in the framework of the algebraic cluster model. The concept of shape-phase transitions is used to propose a mechanism that allows to have interband and intraband quadrupole transitions of comparable strength, as observed in $^{12}$C.

Keywords: alpha-cluster nuclei, algebraic cluster model, $B(EL)$ values

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently there has been a lot of renewed interest in $\alpha$ clustering in light nuclei [1]. Especially, the measurement of new excited states in $^{12}$C has motivated many theoretical studies to understand the structure of $\alpha$-cluster nuclei, like the semi-microscopic algebraic cluster model (ACM) [2], the antisymmetrized molecular dynamics [3], fermionic molecular dynamics [4], BEC-like cluster model [5], ab initio no-core–shell model [6], lattice effective field theory [7], no-core symplectic model [8], and the ACM [9, 10]. Recent reviews on the structure of $^{12}$C can be found in [1, 11] and on $\alpha$-particle clusters in [12].

The ACM for three-body clusters was introduced in 2000 in an application to $^{12}$C [9]. In the ACM, the cluster states in $^{12}$C are interpreted in terms of rotations and vibrations of a (equilateral) triangular configuration of $\alpha$ particles. As a consequence of the geometric symmetry positive and negative parity states merge into a single ground-state rotational band characterized by the sequence $L^P = 0^+, 2^+, 3^-, 4^+, 5^-$, all of which have been observed by now, most recently the $4^-$ state in 2007 [13] and the $5^-$ states in 2014 [14]. In the ACM, the Hoyle state is interpreted as the bandhead of a breathing vibration of the triangular configuration, on top of which an entire rotational band is built. Since the underlying geometric symmetry is the same as that of the ground state, the rotational sequence is expected to be the same as that of the ground-state band. In recent years, evidence has been found for the $2^+$ state of the Hoyle band in three different experiments [15–17], as well as for the $4^+$ Hoyle state [18].

In 2014, the ACM was extended to four-body clusters. An application to the cluster states in $^{16}$O suggested that these can be interpreted in terms of rotations and vibrations of tetrahedral configuration of $\alpha$ particles. The triangular configuration in $^{12}$C and the tetrahedral configuration in $^{16}$O implied by the observed rotational sequences, were confirmed by a study of $B(EL)$ electric transitions along the ground-state band [9, 10].

The main discrepancy concerns the electromagnetic decay of the Hoyle state in $^{12}$C. In the ACM, the quadrupole transition from the Hoyle state to the first excited $2^+$ state is an interband transition which is suppressed with respect to intraband transitions whereas experimentally they are found to be comparable.

The aim of this contribution is to present a possible solution to this problem. As a first step we analyze the spectral properties of a simplified version of the ACM namely for two-body clusters. It will be shown that a Hamiltonian which mixes the spherical and the deformed phases can have intra- and inter-band quadrupole transitions of comparable strength while still retaining, to a good approximation, the rotational structure of the bands.

2. The ACM

For the case of three identical clusters, as is the case for $^{12}$C as a cluster of three $\alpha$ particles, the ACM Hamiltonian has a
The two-body ACM was introduced in applications to diatomic molecules as the vibron model originally, the two-body ACM was introduced in applications

The Hoyle state was interpreted as a breathing vibration of the triangular configuration. As a consequence, the quadrupole transition from the Hoyle state to the ground-state band corresponds to an interband transition which is suppressed with respect to transitions within the same band, see table 1.

The latter scenario was used to study the spectral properties of 12C. Whereas the spherical phase corresponds to a dynamical symmetry of the ACM in which all nuclear properties of interest can be obtained in closed analytic form, the triangular configuration does not. In order to investigate the transition between a spherical and a deformed phase, we analyze the spectral properties of a simplified version of the ACM, namely for two-body clusters, which has the advantage that both the spherical and the deformed phases correspond to a dynamical symmetry. This makes it possible to analyze selection rules and derive closed expressions for quadrupole transitions which can be used a benchmarks for the two phases.

2.1. Two-body ACM

Originally, the two-body ACM was introduced in applications to diatomic molecules as the vibron model. The vibron model is an algebraic model to describe the relative motion of two clusters. The building blocks are four bosons, the three components of a vector boson with \( L^p = 1^- \) and a scalar boson with \( L^p = 0^+ \), denoted by

\[
\hat{b}_m^\dagger, \ s^3,
\]

with \( m = \pm 1, 0 \). The group structure is \( U(4) \). The model space is characterized by the symmetric irreducible representation \([N]\) of \( U(4) \) where \( N \) represents the total number of bosons.

The most general one- and two-body ACM Hamiltonian that conserves the total number of bosons, angular momentum and parity, is given by

\[
H = \epsilon_0 s^3 s - \epsilon_1 b^\dagger \cdot \b + u_0 s^3 s x s - u_1 s^1 b^\dagger \cdot \b x + v_0 [b^\dagger \cdot b^\dagger] s^3 s x s + s^1 b^\dagger \cdot \b \]

\[
+ \sum_{L=0,2} a_L (b^\dagger \times b^\dagger)^{(L)} \cdot (\b \times \b)^{(L)},
\]

with \( \b_m = (-1)^{1-m} b_{-m} \). For two identical clusters, the condition of invariance under the interchange of the two clusters is equivalent to parity conservation. Hence, positive parity states are symmetric and negative parity states antisymmetric. Since we do not consider internal excitations of the clusters, the two-body wave functions arise from the relative motion only, and have to be symmetric. Therefore, the allowed states are the ones with positive parity.

The ACM Hamiltonian of equation (2) has two special solutions corresponding to dynamical symmetries called the \( U(3) \) and \( SO(4) \) limit, which can be interpreted as the three-dimensional harmonic oscillator and the deformed oscillator, respectively. In the following, we study the spectral properties of these two cases.

2.2. Harmonic oscillator

The first dynamical symmetry corresponds to the group chain

\[
\begin{align*}
U(4) &\supset U(3) \supset SO(4) \supset L
\end{align*}
\]

The basis states are classified by the total number of bosons \( N \), the number of oscillator quanta \( n = 0, 1, \ldots, N \), and the angular momentum \( L = n, n-2, \ldots, 1 \) or 0 for \( n \) odd or even. The parity of the levels is given by \( P = (-1)^n \). For two identical clusters, the allowed states have \( n \) even, and therefore also \( L \) even.

We consider the one-body Hamiltonian

\[
H_1 = \epsilon \sum_m b_m^\dagger b_m,
\]

with eigenvalues

\[
E_1 = \epsilon n.
\]

The corresponding energy spectrum is that of a three-dimensional harmonic oscillator.

In the ACM, \( E2 \) transitions are described by the quadrupole operator

\[
Q^{(2)} = q_2 [D \times D]^{(2)} , \quad \hat{D}_m = (b^\dagger s - s^1 b_m^{(1)}) .
\]

In the \( U(3) \) limit, the selection rule of the quadrupole operator is given by \( \Delta n = 0, \pm 2 \). The ratio of \( B(E2) \) values can be derived in closed analytic form as

\[
\frac{B(E2; 0^+_2 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{10(2N - 3)^2}{3N(N-1)} = \frac{40}{3} \left[ 1 - \frac{2}{N} + O(1/N^2) \right] ,
\]

\[
\frac{B(E2; 2^+_1 \rightarrow 0^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{6(N - 2)(N - 3)}{N(N-1)} = 6 \left[ 1 - \frac{4}{N} + O(1/N^2) \right] .
\]

2.3. Deformed oscillator

The second dynamical symmetry corresponds to the group chain

\[
\begin{align*}
U(4) &\supset SO(4) \supset SO(3) \supset L
\end{align*}
\]

The corresponding energy spectrum is that of a three-dimensional harmonic oscillator.

In the ACM, \( E2 \) transitions are described by the quadrupole operator

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Q^{(2)} = q_2 [D \times D]^{(2)} , \quad \hat{D}_m = (b^\dagger s - s^1 b_m^{(1)}) .
\]

In the \( U(3) \) limit, the selection rule of the quadrupole operator is given by \( \Delta n = 0, \pm 2 \). The ratio of \( B(E2) \) values can be derived in closed analytic form as

\[
\frac{B(E2; 0^+_2 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{10(2N - 3)^2}{3N(N-1)} = \frac{40}{3} \left[ 1 - \frac{2}{N} + O(1/N^2) \right] ,
\]

\[
\frac{B(E2; 2^+_1 \rightarrow 0^+_1)}{B(E2; 2^+_1 \rightarrow 0^+_1)} = \frac{6(N - 2)(N - 3)}{N(N-1)} = 6 \left[ 1 - \frac{4}{N} + O(1/N^2) \right] .
\]
The basis states are classified by the quantum numbers $N$, $\sigma$ and $L$. In this case, the energy levels are organized into bands labeled by $\sigma$ with $\sigma = N, N - 2, \ldots, 1, 0$ for $N$ odd or even, respectively. For two identical clusters, the rotational excitations are denoted by $L$ even with $L = 0, 2, \ldots, \sigma$.

In this case, the Hamiltonian is given by

$$H_2 = \xi P^T P + \kappa \tilde{L} \cdot \tilde{L},$$

where $P^T$ denotes a generalized pairing operator

$$P^T = s^+ s^+ - b^+ b^+,$$

and $\tilde{L}$ is the angular momentum in coordinate space

$$\tilde{L}_m = \sqrt{2} (b^+ \times \tilde{b})^{(1)}_m.$$

The energy eigenvalues are given by

$$E_2 = \xi (N - \sigma)(N - \sigma + 2) + \kappa L(L + 1),$$

which corresponds to the rotation–vibration spectrum of a three-dimensional deformed oscillator, consisting of a series of vibrational excitations labeled by $\sigma$, and rotational states labeled by $L$.

In the SO(4) limit, the selection rule of the quadrupole operator $Q^{(2)}$ is given by $\Delta \sigma = 0$, i.e. only intraband transitions are allowed. The ratios of $B(E2)$ values can be derived as

$$\frac{B(E2; 0^+_1 \rightarrow 2^+_1)}{B(E2; \tilde{2}^+_1 \rightarrow 0^+_1)} = 0,$$

$$\frac{B(E2; 4^+_1 \rightarrow 2^+_1)}{B(E2; \tilde{2}^+_1 \rightarrow 0^+_1)} = \frac{10(N - 2)(N - 3)(N + 4)(N + 5)}{7N(N - 1)(N + 2)(N + 3)} = \frac{10}{7} \left[ 1 - \frac{20}{N^2} + O(1/N^4) \right].$$

In this case, the quadrupole transition $B(E2; 0^+_1 \rightarrow 2^+_1)$ corresponds to an interband transition which is forbidden by the $\Delta \sigma = 0$ selection rule.

The characteristics of the two dynamical symmetries are summarized in figure 1 and table 2. In the $U(3)$ limit the spectrum is harmonic, and the ratio of the excitation energies of the first excited $4^+$ and the first excited $2^+$ states is $E(4^+_1)/E(2^+_1) = 2$, whereas in the SO(4) limit these two states belong to a rotational band with energy ratio $10/3$.

The main difference between the two dynamical symmetries concerns the quadrupole transition between the first excited $0^+$ and the first excited $2^+$ state. For the harmonic oscillator both states belong to the same $n = 2$ multiplet and therefore the $B(E2; 0^+_1 \rightarrow 2^+_1)$ is allowed by the $\Delta n = 0$ selection rule. For the deformed oscillator, the first excited $2^+$ state belongs to the ground-state rotational band with $\sigma = N$, whereas the first excited $0^+$ state corresponds is the bandhead of the breathing vibration with $\sigma = N - 2$. As a consequence, the quadrupole transition is forbidden by the $\Delta \sigma = 0$ selection rule.

The ratio of $B(E2)$ values changes from 0 in the deformed oscillator to 40/3 (to leading order in $N$) for the harmonic oscillator, in comparison to the value of $1.72 \pm 0.25$ as measured in $^{12}$C, see table 1.

The situation for $^{12}$C is inbetween the results for the $U(3)$ and SO(4) limits. The spectrum is to a good approximation harmonic, whereas the strengths of the quadrupole transitions $B(E2; 0^+_2 \rightarrow 2^+_1)$ and $B(E2; 2^+_1 \rightarrow 0^+_1)$ are comparable.

In the next section we will mix the two scenarios to investigate the transitional region between the $U(3)$ and SO(4) limits in a systematic manner, to see whether there exists an intermediate Hamiltonian that reproduces the ratios of energies and $B(E2)$ values as observed in $^{12}$C.

### Table 2. Ratios of excitation energies and $B(E2)$ values to leading order in $N$ in the $U(3)$ and SO(4) limits of the two-body ACM.

| $U(3)$ | SO(4) |
|--------|--------|
| $E(4^+_1)/E(2^+_1)$ | $10/3$ |
| $B(E2; 0^+_1 \rightarrow 2^+_1)$ | $40/3$ |
| $B(E2; 2^+_1 \rightarrow 0^+_1)$ | $0$ |

### 3. Shape-phase transition

The transitional region between the harmonic and the deformed oscillator limits can be described by the schematic Hamiltonian $^{[21, 22]}$

$$\mathcal{H} = (1 - \chi) \sum_m h_m \rho_m + \chi \left[ \frac{1}{4(N - 1)} P^T P + \kappa \tilde{L} \cdot \tilde{L} \right],$$

with $0 \leq \chi \leq 1$. For $\chi = 0$ it reduces to the harmonic oscillator or $U(3)$ limit and for $\chi = 1$ to the deformed oscillator or SO(4) limit. Under the assumption that three-cluster data might be used in this two-body cluster model, the value
values as a function of $^{12}\text{C}$ values) and the ratio of two quadrupole transitions within the ground-state band $B(E2; 4_{1}^{+} \rightarrow 2_{1}^{+})/B(E2; 2_{1}^{+} \rightarrow 0_{1}^{+})$. In the large $N$ limit, this ratio varies between the vibrational value 6 and the rotational value of 10/7.

of $\kappa$ is chosen such as to reproduce for $\chi = 1$ the observed energy ratio $E(0_{2}^{+})/E(2_{1}^{+}) = 1.72$ in $^{12}\text{C}$.

The schematic Hamiltonian of equation (15) exhibits a second-order phase transition at the critical point $\chi_{c} = 1/2$ [21, 22]. Figures 2 and 3 show the results for the energy ratio $E(4_{1}^{+})/E(2_{1}^{+})$ and the ratio of $B(E2)$ values $B(E2; 0_{2}^{+} \rightarrow 2_{1}^{+})/B(E2; 2_{1}^{+} \rightarrow 0_{1}^{+})$ in the transitional region for three different values of $N$: $N = 6$ (green), $N = 10$ (blue) and $N = 20$ (red). The value of $N = 10$ was used in a earlier study of $^{12}\text{C}$ in the three-body ACM [9].

Inspection of figures 2 and 3 shows that the conditions for $^{12}\text{C}$ are met approximately for a value of $\chi$ relatively close to the critical value, $\chi \sim 0.6$. For example, for $N = 10$ bosons (the number used in [9]) and $\chi = 0.59$ the results are $R_{E} = 3.00$ for the energy ratio and $R_{B} = 1.74$ for the ratio of $B(E2)$ values. For this value of $\chi$ the spectrum is, to a good approximation, still rotational whereas the $B(E2)$ values for intraband and interband quadrupole transitions are comparable.

Finally, in figure 4 we show the results for a ratio of two quadrupole transitions within the ground-state band $B(E2; 4_{1}^{+} \rightarrow 2_{1}^{+})/B(E2; 2_{1}^{+} \rightarrow 0_{1}^{+})$. In the large $N$ limit, this ratio varies between the vibrational value 6 and the rotational value of 10/7.

4. Summary and conclusions

In this article, we studied the quadrupole transitions in the ACM. We used a schematic Hamiltonian for two-body clusters which describes the transitional region between the spherical and the deformed phases. Particular attention was paid to the behavior of the ratio of inter- and intra-band transitions. In the deformed limit, interband transitions are forbidden, whereas in the spherical limit, the transition between the $0_{2}^{+}$ and $2_{1}^{+}$ states is much larger than the one between the $2_{1}^{+}$ and the ground state. It was shown that by mixing the two phases with a value of the control parameter $\chi \sim 0.6$ which is a bit larger than the critical value $\chi_{c} = 1/2$, the two transitions are of the same order while the energy spectrum is still close to rotational. The first results are very promising, and may offer a mechanism to explain the observed electromagnetic properties of the Hoyle state in $^{12}\text{C}$ in the framework of the ACM by mixing the deformed triangular phase with a spherical term, such that the quadrupole transition between the Hoyle state and the $2_{1}^{+}$ state is large and comparable to quadrupole transitions inside the ground-state band [23].

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